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*Miskolc Mathematical Notes*

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# ON THE METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF CAYLEY GRAPHS $Cay(Z_n \oplus Z_m)$

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*Received 03 April, 2014*

*Abstract.* Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The *representation* $r(v|W)$ of $v$ with respect to $W$ is the k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. $W$ is called a *resolving set* or a *locating set* if every vertex of $G$ is uniquely identified by its distances from the vertices of $W$, or equivalently, if distinct vertices of $G$ have distinct representations with respect to $W$. A resolving set of minimum cardinality is called a *metric basis* for $G$ and this cardinality is the *metric dimension* of $G$, denoted by $dim(G)$.
Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).
In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of Cayley graphs $Cay(Z_n \oplus Z_m)$.

2010 *Mathematics Subject Classification:* 05C12

*Keywords:* metric dimension, basis, resolving set, barycentric subdivision, Cayley graph

## 1. Introduction and preliminary results

Metric dimension is a parameter that has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [3], robot navigation [13], combinatorial optimization [15] and sonar and coast guard Loran [16], to name a few. Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists).
In a connected graph $G$, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The *representation* $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$. $W$ is called a *resolving set* [3] or *locating set* [16] if every vertex of $G$ is uniquely identified by its distances from the vertices of $W$, or equivalently, if distinct vertices of $G$ have distinct representations with respect to $W$. A resolving set of minimum cardinality is

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called a *basis* for $G$ and this cardinality is the *metric dimension* or *location number* of $G$, denoted by $\beta(G)$ [2].
For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph $G$, the $i$th component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that $W$ is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.
A useful property in finding $\beta(G)$ is the following lemma:

**Lemma 1** ([17]). *Let $W$ be a resolving set for a connected graph $G$ and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \varnothing$.*

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\geq 1}$ depending on $n$ as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$, then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.
If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on $n$), $\mathcal{F}$ is called a family with constant metric dimension [12]. Some classes of *regular graphs* with constant metric dimension have been studied in [1, 10] recently while metric dimension of some classes of *convex polytopes* has been determined in [7] and [9].
Other families of graphs have unbounded metric dimension: if $W_n$ denotes a *wheel* with $n$ spokes and $J_{2n}$ the graph deduced from the wheel $W_{2n}$ by alternately deleting $n$ spokes, then $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ [2] and $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ [18] for every $n \geq 4$. The generalized Petersen graphs $P(n,3)$ have bounded metric dimension [8].
The graphs having metric dimension 1 are characterized in the following theorem.

**Theorem 1** ([3]). *The metric dimension of a graph $G$ is 1 if and only if $G \cong P_n$, where $P_n$ denotes a path of length $n-1$ or $G$ is one-way infinite path.*

The next theorem gives a nice property of the graphs with metric dimension 2.

**Theorem 2** ([14]). *Let $G$ be a graph with metric dimension 2 and let $\{v_1, v_2\} \subseteq V(G)$ be a metric basis in $G$, then the degree of both $v_1$ and $v_2$ is at most 3.*

Geometrically, subdividing an edge is an operation that inserts a new vertex into the interior that results in splitting that edge into two edges. *Subdividing a graph $G$* means performing a sequence of edge-subdivision operations. The resulting graph is called a *subdivision of the graph $G$*. The operation of subdivision can be used to convert a general graph into a simple graph. The *barycentric subdivision* of a graph $G$ is the subdivision in which one new vertex is inserted in the interior of each edge. The following propositions give some nice results related to barycentric subdivision of a graph (see [6]).

- The barycentric subdivision of any graph is a bipartite graph.
- The barycentric subdivision of any graph yields a loopless graph.

- The barycentric subdivision of any loopless graph yields a simple graph.

A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivision of graphs play a very important role in characterization of planar graphs. A graph $G$ is planar if and only if every subdivision of $G$ is planar. Two graphs are said to be homeomorphic if they are subdivisions of same graph $G$. The next theorem gives a nice characterization of planar graphs.

**Theorem 3** ([6]). *A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.*

Note that the problem of determining whether $\beta(G) < k$ is an $NP$-complete problem [5].
In this paper, we study the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$. We prove that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$.

## 2. The metric dimension of Barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$

Let $G$ be a semigroup, and let $S$ be a nonempty subset of $G$. The Cayley graph $Cay(G,S)$ of $G$ relative to $S$ is defined as the graph with vertex set $G$ and edge set $E(S)$ consisting of those ordered pairs $(x,y)$ such that $sx = y$ for some $s \in S$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $Cay(G,S)$ of a group $G$ is symmetric or undirected if and only if $S = S^{-1}$.
The Caylay graphs $Cay(Z_n \oplus Z_m), n \geq 3, m \geq 2$, is a graph which can be obtained as the cartesian product $P_m \square C_n$ of a path on $m$ vertices with a cycle on $n$ vertices. The vertex set and edge set of $Cay(Z_n \oplus Z_m)$ defined as: $V(Cay(Z_n \oplus Z_m)) = \{(x_i, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(Cay(Z_n \oplus Z_m)) = \{(x_i, y_j)(x_{i+1}, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(x_i, y_j)(x_i, y_{j+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. We have $|V(Cay(Z_n \oplus Z_m)| = mn$, $|E(Cay(Z_n \oplus Z_m)| = (2m-1)n$, where $|V(Cay(Z_n \oplus Z_m)|, |E(Cay(Z_n \oplus Z_2)|$ denote the number of vertices, edges of the Cayley graphs $Cay(Z_n \oplus Z_m)$, respectively.

The metric dimension of Cayley graphs $Cay(Z_n \oplus Z_2)$ has been determined in [4] while the metric dimension of Cayley graphs $Cay(\mathbb{Z}_n : S)$ for all $n \geq 7$ and $S = \{\pm 1, \pm 3\}$ hase been determined in [11].
The barycentric subdivision graph $S(Cay(Z_n \oplus Z_m))$ can be obtained by adding a new vertex $u_i^j$ between $(x_i, y_j)$ and $(x_{i+1}, y_j)$ and adding a new vertex $v_i^j$ between $(x_i, y_j)$ and $(x_i, y_{j+1})$. Clearly, $S(Cay(Z_n \oplus Z_m))$ has $3nm - n$ vertices and $4nm - 2n$ edges.
The metric dimension of $P_m \square C_n$ has been determined in [4] and Cayley graphs $Cay(Z_n \oplus Z_2)$ is actually the cartesian product of $P_2 \square C_n$. In the next theorem, we

prove that the metric dimension of the barycentric subdivision $S(Cay(Z_n \oplus Z_m))$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $S(Cay(Z_n \oplus Z_m))$. Note that the choice of appropriate basis vertices (also refereed to as landmarks in [14]) is the core of the problem. For our purpose, we call the sets of points as $S_1 = \{(x_i, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$, $S_2 = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $S_3 = \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m-1\}$.

**Theorem 4.** *Let $S(Cay(Z_n \oplus Z_m))$ be the barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$; then $\beta(S(Cay(Z_n \oplus Z_m))) = 3$ for every $n \geq 6$.*

*Proof.* We will prove the above equality by double inequalities.
**Case 1.** When $n$ is even.
Let $W = \{(x_1, y_1), (x_2, y_1), (x_{\frac{n}{2}+1}, y_1)\} \subset V(S(Cay(Z_n \oplus Z_m)))$, we show that $W$ is a resolving set for $S(Cay(Z_n \oplus Z_m))$ in this case. For this we give representations of any vertex of $V(S(Cay(Z_n \oplus Z_m)))W$ with respect to $W$.
Representations for the vertices of $S(Cay(Z_n \oplus Z_m))$ are

$$r((x_i, y_j)|W) = \begin{cases} (2(j-1), 2j, n+2(j-1)), i = 1 \ \& \ 1 \leq j \leq m \\ (2j, 2(j-1), n+2(j-2)), i = 2 \ \& \ 1 \leq j \leq m \\ (2(i+j-2), 2(i+j-3), 2(\frac{n}{2}+j-i)), 3 \leq i \leq \frac{n}{2} \& 1 \leq j \leq m \\ (n+2(j-1), n+2(j-2), 2(j-1)), i = \frac{n}{2}+1 \ \& \ 1 \leq j \leq m \\ (2(n+j-i), 2(n+j+1-i), 2(j+i-2)-n), \frac{n}{2}+2 \leq i \leq n \\ \qquad \& \ 1 \leq j \leq m \end{cases}$$

and

$$r(u_i^j|W) = \begin{cases} (2j-1, 2j-1, n+2j-3), i = 1 \ \& \ 1 \leq j \leq m \\ (2(i+j)-3, 2(i+j)-5, n-1-2(i-j)), 2 \leq i \leq \frac{n}{2} \\ \qquad \& \ 1 \leq j \leq m \\ (n+2j-3, n+2j-3, 2j-1), i = \frac{n}{2}+1 \ \& \ 1 \leq j \leq m \\ (2(n+j-i)-1, 2(n+j-i)+1, 2(j+i)-n-3), \frac{n}{2}+2 \leq i \leq n \\ \qquad \& \ 1 \leq j \leq m \end{cases}$$

and

$$r(v_i^j|W) = \begin{cases} (2j-1, 2j+1, n+2j-1), i = 1 \ \& \ 1 \leq j \leq m-1 \\ (2(i+j)-3, 2(i+j)-5, n+1-2(i-j)), 2 \leq i \leq \frac{n}{2}+1 \\ \qquad \& \ 1 \leq j \leq m-1 \\ (2(n+j-i)+1, 2(n+j-i)+3, 2(j+i)-n-3), \frac{n}{2}+2 \leq i \leq n \\ \qquad \& \ 1 \leq j \leq m-1 \end{cases}$$

We note that there are no two vertices having the same representations implying that $\beta(S(Cay(Z_n \oplus Z_m))) \leq 3$.
On the other hand, we show that $\beta(S(D_n)) \geq 3$. Suppose on contrary that $\beta(S(Cay(Z_n \oplus Z_m))) = 2$, then there are the following possibilities to be discussed. The behavior of the outmost cycle and innermost cycle are same.
**(1)** Both vertices are in the same set. Here are the following subcases.
• Both vertices belong to the set $S_1$. When Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_1)$. Suppose that the second resolving vertex

is $(x_p, y_j)$ $(2 \leq p \leq \frac{n}{2}+1, 1 \leq j \leq m)$. Then for $2 \leq p \leq \frac{n}{2}, 1 \leq j \leq m-1$, we have

$$r(u_n^j|\{(x_1,y_j),(x_p,y_j)\}) = r(v_1^j|\{(x_1,y_j),(x_p,y_j)\}) = (1,2p-1).$$

For $j = m$, we have

$$r(u_n^m|\{(x_1,y_m),(x_p,y_m)\}) = r(v_1^{m-1}|\{(x_1,y_m),(x_p,y_m)\}) = (1,2p-1)$$

and for $p = \frac{n}{2}+1, 1 \leq j \leq m$ we have

$$r(u_1^j|\{(x_1,y_j),(x_p,y_j)\}) = r(u_n^j|\{(x_1,y_j),(x_p,y_j)\}) = (1,n-1),$$

a contradiction.

• Both vertices belong to the set $S_2$. Without loss of generality, we can suppose that one resolving vertex is $u_1^j$. There are two possibilities. When second resolving vertex on the same level. Suppose that the second resolving vertex is $u_p^j$ $(2 \leq p \leq \frac{n}{2}+1)$. Then for $2 \leq p \leq \frac{n}{2}$, we have

$$r(u_n^j|\{u_1^j,u_p^j\}) = r(v_1^j|\{u_1^j,u_p^j\}) = (2,2p),$$

for $j = m$, we have

$$r(u_n^m|\{u_1^m,u_p^m\}) = r(v_1^{m-1}|\{u_1^m,u_p^m\}) = (2,2p)$$

and for $p = \frac{n}{2}+1$, we have

$$r((x_1,y_j)|\{u_1^j,u_{\frac{n}{2}+1}^j\}) = r((x_2,y_j)|\{u_1^j,u_{\frac{n}{2}+1}^j\}) = (1,n-1),$$

a contradiction. When second resolving vertex on the different level. Suppose that the second resolving vertex is $u_p^s$ $(1 \leq p \leq \frac{n}{2}+1, s > j)$. Then for $p = 1$, we have

$$r((x_1,y_j)|\{u_1^j,u_1^s\}) = r((x_2,y_j)|\{u_1^j,u_1^s\})) = (1,2(s-j)+1),$$

for $p = 2$, we have

$$r(v_3^{s-1}|\{u_1^j,u_2^s\}) = r(u_1^s|\{u_1^j,u_2^s\}) = (2(s+1-j),2)$$

and when $3 \leq p \leq \frac{n}{2}+1$, we have

$$r(v_3^j|\{u_1^j,u_p^s\}) = r(u_2^{j+1}|\{u_1^j,u_p^s\}) = (4,2(p+s-j-3)),$$

a contradiction.

• Both vertices belong to the set $S_3$. Without loss of generality, we can suppose that one resolving vertex is $v_1^j$, $1 \leq j \leq m-1$. Suppose that the second resolving vertex is $v_p^s$ $(2 \leq p \leq \frac{n}{2}+1, 1 \leq j,s \leq m-1, j = s)$. Then for $2 \leq p \leq \frac{n}{2}+1$, we have

$$r((x_1,y_j)|\{v_1^j,v_p^j\}) = r((x_1,y_{j+1})|\{v_1^j,v_p^j\}) = (1,2p-1),$$

a contradiction. If one resolving vertex is $v_p^j$ and second resolving vertex is $v_p^s$ for $1 \leq p \leq \frac{n}{2}+1, 1 \leq j,s \leq m-1, s > j$, then we have

$$r(u_p^j|\{v_p^j,v_p^s\}) = r(u_{p-1}^j|\{v_p^j,v_p^s\}) = (2j,2s),$$

a contradiction. Now if we can suppose that one resolving vertex is $v_1^j, 1 \leq j \leq m-1$. Suppose that the second resolving vertex $v_p^s, (2 \leq p \leq \frac{n}{2}+1, 1 \leq s \leq m-1, 1 \leq j \leq m-2, s > j)$. For $2 \leq p \leq \frac{n}{2}$, we have

$$r(u_1^j|\{v_1^j, v_p^s\}) = r(u_n^{j+1}|\{v_1^j, v_p^s\}) = (2, 2(s+p-j-1)),$$

and for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{v_1^j, v_p^s\}) = r(u_n^j|\{v_1^j, v_p^s\}) = (2, n+2s-2j),$$

a contradiction.

**(2)** Both vertices are not in the same set. Here are the following subcases.

• One vertex in the set $S_1$ and the other one in the set $S_2$ but in the same level. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \leq j \leq m$. Suppose that the second resolving vertex is $u_p^j$ $(1 \leq p \leq \frac{n}{2}+1, 1 \leq j \leq m)$. Then for $1 \leq p \leq \frac{n}{2}, 1 \leq j \leq m-1$, we have

$$r(u_n^j|\{(x_1, y_j), u_p^j\}) = r(v_1^j|\{(x_1, y_j), u_p^j\}) = (1, 2p)$$

and for $1 \leq p \leq \frac{n}{2}, j = m$, we have

$$r(u_n^m|\{(x_1, y_m), u_p^m\}) = r(v_1^{m-1}|\{(x_1, y_m), u_p^m\}) = (1, 2p),$$

a contradiction. For $p = \frac{n}{2}+1, 1 \leq j \leq m-1$ we have

$$r(u_1^j|\{(x_1, y_j), u_{\frac{n}{2}+1}^j\}) = r(v_1^j|\{(x_1, y_j), u_{\frac{n}{2}+1}^j\}) = (1, n)$$

and for $p = \frac{n}{2}+1, j = m$ we have

$$r(u_1^m|\{(x_1, y_m), u_{\frac{n}{2}+1}^m\}) = r(v_1^{m-1}|\{(x_1, y_m), u_{\frac{n}{2}+1}^m\}) = (1, n),$$

a contradiction. Now one vertex in the set $S_1$ and the other one in the set $S_2$ but in the different level $(1 \leq p \leq \frac{n}{2}+1, 1 \leq j \leq m-1, 1 \leq s \leq m, s > j)$. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \leq j \leq m$. Suppose that the second resolving vertex is $u_p^j$ $(1 \leq p \leq \frac{n}{2}+1, 1 \leq j \leq m)$. Then for $p = 1$, we have

$$r(u_1^j|\{(x_1, y_j), u_1^s\}) = r(u_n^j|\{(x_1, y_j), u_1^s\}) = (1, 2(s-j+1)),$$

and for $2 \leq p \leq \frac{n}{2}$, we have

$$r(u_1^j|\{(x_1, y_j), u_p^s\}) = r(v_1^j|\{(x_1, y_j), u_p^s\}) = (1, 2(s+p-j-1)).$$

For $p = \frac{n}{2}+1$, we have

$$r(u_n^j|\{(x_1, y_j), u_p^s\}) = r(v_1^j|\{(x_1, y_j), u_p^s\}) = (1, 2(s-j-1)+n),$$

a contradiction.

• One vertex in the set $S_1$ and the other one in the set $S_3$ but in the same level. Without loss of generality, we can suppose that one resolving vertex is $(x_1, y_j), 1 \leq j \leq m$.

Suppose that the second resolving vertex is $v_p^s$ $(1 \leq p \leq \frac{n}{2}+1, 1 \leq s \leq m-1)$. Then for $p = 1, 1 \leq j, s \leq m-1, s \geq j$, we have

$$r(u_1^j|\{(x_1, y_j), v_p^s\}) = r(u_n^j|\{(x_1, y_j), v_p^s\}) = (1, 2(s-j+1)),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j, s \leq m-1, s = j$, we have

$$r(u_n^j|\{(x_1, y_j), v_p^j\}) = r(v_1^j|\{(x_1, y_j), v_p^j\}) = (1, 2p),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{(x_1, y_j), v_p^j\}) = r(u_n^j|\{(x_1, y_j), v_p^j\}) = (1, n),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j, s \leq m-1, s > j$, we have

$$r(u_1^j|\{(x_1, y_j), v_p^s\}) = r(v_1^j|\{(x_1, y_j), v_p^s\}) = (1, 2(s+p-j-1)),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_1^j|\{(x_1, y_j), v_p^s\}) = r(u_n^j|\{(x_1, y_j), v_p^s\}) = (1, 2(s+p-j-1)),$$

a contradiction.

• One vertex in the set $S_2$ and the other one in the set $S_3$ but in the same level. Without loss of generality, we can suppose that one resolving vertex is $u_1^j, 1 \leq j \leq m$. Suppose that the second resolving vertex is $v_1^s$ $(1 \leq p \leq \frac{n}{2}+1, 1 \leq s \leq m-1)$. Then for $p = 1, 1 \leq j, s \leq m-1, s \geq j$, we have

$$r(u_1^{s+1}|\{u_1^j, v_1^s\}) = r(u_n^{s+1}|\{u_1^j, v_1^s\}) = (2(s-j+2), 2),$$

a contradiction. For $2 \leq p \leq \frac{n}{2}, 1 \leq j \leq m, 1 \leq s \leq m-1, s = j$, we have

$$r(u_n^j|\{u_1^j, v_p^j\}) = r(v_1^j|\{u_1^j, v_p^j\}) = (2, 2p),$$

for $p = \frac{n}{2}+1$, we have

$$r(u_n^j|\{u_1^j, v_p^j\}) = r(v_2^j|\{u_1^j, v_p^j\}) = (2, n),$$

a contradiction. For $p = 2, 1 \leq j, s \leq m-1, s > j$, we have

$$r(u_2^j|\{u_1^j, v_p^s\}) = r(v_n^j|\{u_1^j, v_p^s\}) = (2, 2(s-j+1)),$$

for $3 \leq p \leq \frac{n}{2}+1$, we have

$$r(u_2^j|\{u_1^j, v_p^s\}) = r(v_2^j|\{u_1^j, v_p^s\}) = (2, 2(s+p-j-2)),$$

a contradiction.

**Case 2.** When $n$ is even.

Let $W = \{(x_1, y_1), (x_2, y_1), u_{\frac{n}{2}+1}\} \subset V(S(Cay(Z_n \oplus Z_m)))$, we show that $W$ is a resolving set for $S(Cay(Z_n \oplus Z_m))$ in this case. For this we give representations of any vertex of $V(S(Cay(Z_n \oplus Z_m)))W$ with respect to $W$.

Representations for the vertices of $S(Cay(Z_n \oplus Z_m))$ are

$$r((x_i,y_j)|W)=\begin{cases}(2(j-1),2j,n+2j-2), i=1 \;\&\; 1\le j\le m\\ (2j,2(j-1),n+2j-4), i=2 \;\&\; 1\le j\le m\\ (2(i+j-2),2(i+j-3),n-2(i-j)), 3\le i\le \frac{n+1}{2}\\ \qquad\qquad \&\; 1\le j\le m\\ (n+2j-3,n+2j-3,2j-1), i=\frac{n+3}{2} \;\&\; 1\le j\le m\\ (2(n+j-i),2(n+j+1-i),2(j+i)-n-3), \frac{n+5}{2}\le i\le n\\ \qquad\qquad \&\; 1\le j\le m\end{cases}$$

$$r(u_i^j|W)=\begin{cases}(2j-1,2j-1,n+2j-3), i=1 \;\&\; 1\le j\le m\\ (2(i+j)-3,2(i+j)-5,n-1+2(j-i)), 2\le i\le \frac{n-1}{2}\\ \qquad\qquad \&\; 1\le j\le m\\ (n+2j-2,n+2j-4,2j-2), i=\frac{n+1}{2} \;\&\; 1\le j\le m\\ (2(n+j-i)-1,2(n+j-i)+1,2(j+i)-n-3), \frac{n+3}{2}\le i\le n\\ \qquad\qquad \&\; 1\le j\le m\end{cases}$$

and

$$r(v_i^j|W)=\begin{cases}(2j-1,2j+1,n+2j-1), i=1 \;\&\; 1\le j\le m-1\\ (2(i+j)-3,2(i+j)-5,n-2i+2j+1)), 2\le i\le \frac{n+1}{2}\\ \qquad\qquad \&\; 1\le j\le m-1\\ (n+2(j-1),n+2(j-1),2j), i=\frac{n+3}{2} \;\&\; 1\le j\le m-1\\ (2(n+j-i)+1,2(n+j-i+1)+1,2j+2i-n-3),\\ \qquad\qquad \frac{n+5}{2}\le i\le n \& \; 1\le j\le m-1\end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $\beta(S(Cay(Z_n\oplus Z_m)))\le 3$.
On the other hand, suppose that $\beta(S(Cay(Z_n\oplus Z_m)))=2$, then there are the same possibilities as in case (1) and contradictions can be deduced analogously. This implies that $\beta(S(Cay(Z_n\oplus Z_m)))=3$ in this case, which completes the proof. □

## 3. Conclusion

The problem of determining whether $\beta(G)<k$ is an $NP$-complete problem. In this paper, we have studied the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n\oplus Z_m)$. We proved that these subdivisions of Cayley graphs have constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of subdivisions of Cayley graphs $Cay(Z_n\oplus Z_m)$. It is natural to ask for characterization of graph classes with constant metric dimension.

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# GENERALIZED EQUILIBRIUM PROBLEMS IN ABSTRACT CONVEX SPACES

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*Abstract.* In this paper, the existence of solution for generalized equilibrium problems under compactness or non-compactness assumptions in abstract convex spaces is proved. The closedness and the convexity of the solution set are also obtained. Our results generalize and improve some recent results in the literature.

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*Keywords:* equilibrium problem, abstract convex space, $\Gamma-$convex, set-valued map, existence results

## 1. Introduction

Let $K$ be a convex subset of topological vector space $X$ and let $F: X \times X \to \mathbb{R}$ be a scalar bifunction such that $F(x, x) \geq 0$, for all $x \in K$. By an equilibrium problem, Blum and Oettli understood the problem of finding $\overline{x} \in K$ such that $F(\overline{x}, y) \geq 0$ for all $y \in K$, which is called equilibrium problem (for short EP). Many authors extended EP to vector case in different ways, see for example [4, 9, 10, 13].

The equilibrium problem contains many important problems like optimization, variational inequalities, minimax inequalities, Nash equilibrium problems, and complementarity problems. Recently, many researchers investigated different models of equilibrium problem and established general results on the existence of the equilibrium problem. For more details see [1–3, 5–7, 11, 17].
In order to establish existence results, convexity plays an important role, which is a rather restrictive assumption. On the other hand, the concept of abstract convex spaces, which include convex subsets of topological vector space, C-spaces and G-convex spaces, was introduced by Sehie Park [14] in 2006.

In this paper, using KKM theorem in abstract convex spaces, the convexity assumption is relaxed to $\Gamma-$convexity, in order to obtain existence of the solution of

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generalized equilibrium problem (for short GEP) in abstract convex space is presented. Our results extend and generalize various existence theorems for similar problems. Although in general the solution set fails to be convex, here a sufficient condition such that the solution set is to be convex is stated. A characterization of the closedness of the solution set is also given. Meanwhile, Theorem 3.1 of [10] is extended.

## 2. Preliminaries

Let $X$ and $Y$ be topological vector spaces and $K$ be a convex subset of $X$. Let $F : K \times K \multimap Y$ be a set-valued function and $W \subseteq Y$. There are several possible ways to generalized equilibrium problems. The following problem was extensively studied:

$$Find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \subseteq W \quad \forall y \in Y.$$

Consider a closed convex cone $C$ in $Y$. If $W = Y \backslash -intC$ (where $intC \neq \varnothing$) then the generalized vector equilibrium problem may be formulated as one of the following three problems:

$$(GVEP):\ Find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \subseteq (Y \backslash -intC) \quad \forall y \in Y,$$

or

$$Find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \nsubseteq -intC \quad \forall y \in Y,$$

or

$$Find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \cap (Y \backslash -intC) \neq \varnothing \quad \forall y \in Y.$$

Furthermore, if $W = C$ then the strong version of $GVEP$ is considered as follows:

$$(SGVEP):\ Find\ \overline{x} \in X\ such\ that\ F(\overline{x},\ y) \subseteq C \quad \forall y \in Y,$$

$GVEP$ has been extensively studied by many authors in recent years (see [1–3, 5–7, 11, 17]).

We give now some notation and recall some definitions, which are intended to be used in the sequel.

Let $A$ be a subset of topological space $X$. We denote by $\overline{A}$ the closure of $A$ in $X$. Let $< D >$ be the set of all nonempty finite subsets of a set $D$. A multimap (or a set-valued mapping) $F :\ X \multimap Y$ is a mapping $F :\ X \to 2^Y$ to the power set of $Y$, while $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$.

An abstract convex space $(E,\ D;\ \Gamma)$ (see [16]) consists of a topological space $E$, a nonempty set $D$ and a multimap $\Gamma : < D > \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in < D >$. For any nonempty $D' \subseteq D$, the $\Gamma-$convex hull of $D'$ is denoted and defined by

$$co_\Gamma D' := \bigcup \{\Gamma_A \mid A \in < D' >\} \subseteq E.$$

A subset $X$ of $E$ is called a $\Gamma-$ convex subset of $(E,\ D;\ \Gamma)$ relative to $D'$ if for any $N \in < D' >$, we have $\Gamma_N \subseteq X$, that is, $co_\Gamma D' \subseteq X$.
When $D \subseteq E$, a subset $X$ of $E$ is said to be $\Gamma-$convex if $co_\Gamma(X \cap D) \subseteq X$; in other words, $X$ is $\Gamma-$convex relative to $D' := X \cap D$. In case $E = D$, let $(E;\ \Gamma) := (E,\ E;\ \Gamma)$. If $E$ is compact, then $(E,\ D;\ \Gamma)$ is called a compact abstract convex space.

It is obvious that any vector space $E$ is an abstract convex space with $\Gamma = co$, where $co$ is the convex hull in a vector space. In specially, $(\mathbb{R};\ co)$ is an abstract convex space. Several examples of abstract convex spaces are given in [15] and the references therein.

**Definition 1** ([16])**.** Let $(E, D; \Gamma)$ be an abstract convex space. If $G:\ D \multimap E$ satisfies

$$\Gamma_A \subseteq G(A) := \bigcup_{y \in A} G(y)\ for\ all\ A \in < D >,$$

then $G$ is called a KKM map.

**Definition 2** ([16])**.** The partial KKM principle for an abstract convex space $(E,\ D;\ \Gamma)$ means that, for any closed-valued KKM mapping $G:\ D \multimap E$, the family $\{G(z)\}_{z \in D}$ has the finite intersection property. The KKM principle means that the same property also holds for any open-valued KKM map.

**Definition 3.** Let $(E, D; \Gamma)$ be an abstract convex space. The set-valued mapping $G:\ D \multimap E$ is called:
(a) Intersectionally closed-valued if $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$;

(b) Transfer closed-valued if $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$.

Luc et al. [12] noted that $(a) \Leftarrow (b)$ and gave examples of multimaps satisfying $(a)$ but not $(b)$.

**Definition 4** ([16])**.** For an abstract convex space $(E,\ D;\ \Gamma)$, a subset $X$ of $E$ is said to be intersectionally closed (resp., transfer closed) if there is an intersectionally (respectively, transfer) closed-valued map $G:\ D \multimap E$ such that $X = G(z)$ for some $z \in D$.

We state now a theorem proved by Park in [15] which will be used in the main section.

**Theorem 1** ([15], Generalized Partial KKM Principle)**.** *Let $(E, D; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle and $G : D \multimap E$ a mapping such that:*
(1) *$G$ is closed-valued;*

(2) *G is a KKM map.(that is,* $\Gamma_A \subseteq G(A)$ *for all* $A \in < D >$*).*
*Then*

$$\bigcap \{G(z) \mid z \in D\} \neq \emptyset.$$

## 3. Main results

Let $(X;\ \Gamma)$ be an abstract convex space, $Y$ be a topological space and $W \subseteq Y$. Suppose that $K$ is a nonempty $\Gamma-$ convex subset of $X$ and $F : K \times K \multimap Y$. We consider the system of generalized equilibrium problem:

$$Find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \subseteq W \quad \forall\ y \in Y.$$

We give an existence theorem for the above generalized equilibrium problem in compact abstract convex space, which is a generalization of Theorem 3.1. in [10] (indeed, if $(K;\Gamma) = (\mathbb{R}^n; co)$ and $F$ is a vector-valued function, then we get Theorem 3.1 in [10]).

**Theorem 2.** *Let* $(X;\Gamma)$ *be an abstract convex space satisfying the partial KKM principle and* $K$ *be a compact and* $\Gamma-$*convex subset of* $X$*. Let* $Y$ *be a topological space and* $W :\ K \multimap Y$ *be any set-valued mapping with nonempty values. Let* $F : K \times K \multimap Y$ *be a set-valued mapping satisfying the following conditions:*
$(i)$ $F(x,\ x) \subseteq W(x)$*, for every* $x \in K$*;*
$(ii)$ *for all* $x,\ y \in K,\ F(x,\ y) \subseteq W(x)$ *implies* $F(y,\ x) \subseteq -W(y)$*;*
$(iii)$ *for all* $x \in K$*, the set* $\{y \in K \mid F(x,\ y) \subseteq -W(x)\}$ *is closed;*
$(iv)$ *for all* $x \in K$*, the set* $\{y \in K \mid F(x,\ y) \nsubseteq W(x)\}$ *is* $\Gamma-$*convex;*
$(v)$ *for all* $y \in K,\ F(x,\ y) \subseteq -W(x)$ *for all* $x \in K$ *implies* $F(y,\ x) \subseteq W(y)$ *for all* $x \in K$*;*
*Then, the solution set to the problem*

$$find\ \overline{x} \in K\ such\ that\ F(\overline{x},\ y) \subseteq W(\overline{x})\ for\ all\ y \in K,$$

*and that of the problem*

$$find\ \overline{x} \in K\ such\ that\ F(y,\ \overline{x}) \subseteq -W(y)\ for\ all\ y \in K,$$

*are nonempty and closed. Moreover, they both coincide.*

*Proof.* Set $G(y) := \{x \in K \mid -F(y,\ x) \subseteq W(y)\}$. For the beginning, we prove that $\Gamma(A) \subseteq G(A)$ for all $A \in < K >$. If there exists $N \in < K >$ such that $\Gamma(N) \nsubseteq G(N)$ then there exists $x \in \Gamma(N)$ and $x \notin\ G(N)$ ( in other word $x \notin\ G(y),\ \forall y \in N$). Thus $-F(y,\ x) \nsubseteq W(y),\ \forall y \in N$ , which implies $F(x,\ y) \nsubseteq W(x)$ for all $y \in N$ by assumption $(ii)$. So, $N \in < \{y \in K \mid F(x,\ y) \nsubseteq W(x)\} >$. By assumption $(iv)$, $\Gamma_N \subseteq \{y \in K \mid F(x,\ y) \nsubseteq W(x)\}$. Since $x \in \Gamma_N$, we have

$$x \in \{y \in K \mid F(x,\ y) \nsubseteq W(x)\}$$

or $F(x, x) \nsubseteq W(x)$ this contradicts $(i)$. This proves that $\Gamma(A) \subseteq G(A)$ for all $A \in < K >$. Now we find $\overline{x} \in K$ such that

$$\overline{x} \in \bigcap_{y \in K} \{x \in K \mid -F(y, x) \subseteq W(y)\}.$$

Because of assumption $(iii)$ each set $G(y)$ is closed. An application of the generalized partial KKM principle (see Theorem 1) yields the existence of $\overline{x} \in K$ such that $\overline{x} \in \bigcap_{y \in K} G(y)$, i. e, $F(y, \overline{x}) \subseteq -W(y)$ for all $y \in K$. In other words, the second problem has a solution. By the assumption $(v)$ we conclude that such a solution is also a solution of the first problem. Both solution sets coincide by $(ii)$. The closedness is a consequence of $(iii)$. □

*Example* 1. Let $E$ denote $\mathbb{R}^2$ with metric $d_r$ which defined as follow for any $x, y \in E$:

$$d_r(x, y) = \begin{cases} d(x, y) & \exists \lambda \in \mathbb{R} : x = \lambda y \\ d(x, 0) + d(0, y) & otherwise \end{cases},$$

where $d$ denote the Euclidean metric and 0 denote the origin. If there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$, then segment joining $x$ and $y$ ($[x, y]$) is the geodesic segment, otherwise, union of the segments $[x, 0]$ and $[0, y]$ is the geodesic segment. So, $E$ is a complete $\mathbb{R}-$tree space.
Also, we recall that a subset $A$ in $\mathbb{R}-$tree space $E$ is convex if for any $x, y \in A$, all geodesic segments joining $x$ and $y$ are contained in $A$.
Let $\Gamma_A := conv(A)$ for each $A \in E$, where $conv(A)$ is intersection of all closed convex subsets of the $\mathbb{R}-$tree $E$, that contains $A$. $(E, \Gamma)$ is an abstract convex space which satisfies the partial KKM principle, see [15].

Let $X = (E, \Gamma)$, $Y = \mathbb{R}^2$, $K = \{(x, y) \in [-1, 1] \times [-1, 1] \mid x = y \vee x = -y\}$ and $W = \{(r, s) \in \mathbb{R}^2 \mid r \geq 0 \vee s \geq 0\}$. Suppose $F : K \times K \multimap \mathbb{R}^2$ defined by

$$F((x_1, y_1), (x_2, y_2)) = \begin{cases} (-1, -1) & x_1, y_1 < 0 \; and \; (x_2 \leq 0, y_2 \geq 0 \; or \; x_2 \geq 0, y_2 \geq 0) \\ (1, 1) & (x_1 \leq 0, y_1 \geq 0 \; or \; x_1 \geq 0, y_1 \geq 0) \; and \; x_2, y_2 < 0 \\ (0, 0) & otherwise \end{cases}.$$

Then $F$ satisfies the assumptions of Theorem 2 and so the solution set of

$$"find \; \overline{x} \in K \; such \; that \; F(\overline{x}, y) \subseteq W(\overline{x}) \; for \; all \; y \in K",$$

is nonempty and closed. In fact the solution set is

$$\{(x, y) \in K \mid y = -x \vee y = x \geq 0\}.$$

Note that we can not apply Theorem 3.1 in [10] for the set-valued $F$, because in spite of convexity of $K$ the set $\{y \in K \mid F(x, y) \nsubseteq W(x)\}$ in $(E, \Gamma)$, they are not convex in Euclidean space $\mathbb{R}^2$.

To deal with the unbounded case, inspired by Theorem 5.1 in [16], we state now the following lemma.

**Lemma 1.** *Let $(E,\ D;\ \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $G:\ D \multimap E$ is intersectionally closed-valued mapping. Suppose there exists a mapping $H:\ E \multimap E$ satisfying:*
*$(i)$ For each $x \in E$, $x \in H(x)$;*
*$(ii)$ For each $x \in E$, $co_\Gamma(D \setminus G^-(x)) \subseteq E \setminus H^-(x)$.*
*Then*

$$\bigcap\{G(z) \mid z \in D\} \neq \varnothing.$$

*Proof.* It is sufficient to show that $(i)$ and $(ii)$ imply that $G$ is a KKM map. Suppose that there exists an $N \in< D >$ such that $\Gamma_N \nsubseteq G(N)$. That is, there exists an $x \in \Gamma_N$ such that $x \notin G(y)$ for all $y \in N$. In other words, $N \in< D \setminus G^-(x) >$. By $(ii)$, $\Gamma_N \subseteq E \setminus H^-(x)$. Since $x \in \Gamma_N$, we have $x \notin H^-(x)$ or $x \notin H(x)$. This contradicts $(i)$. Thus $G$ is a KKM mapping so $\Gamma_N \subseteq G(N) \subseteq \overline{G(N)}$ for any $N \in< D >$. Therefore, $\overline{G(N)}$ is a KKM mapping. Since $(E, D; \Gamma)$ satisfies the partial KKM principle and $\overline{G}$ has closed values, the family $\{\overline{G(z)}\}_{z\in D}$ has the finite intersection property. Since $G$ is intersectionally closed-valued, we have that

$$\overline{\bigcap_{z\in D} G(z)} = \bigcap_{z\in D} \overline{G(z)} \neq \varnothing.$$

Since $\overline{\bigcap_{z\in D} G(z)} \neq \varnothing$ we get that $\bigcap\{G(z) \mid z \in D\} \neq \varnothing$, which proves the lemma. □

**Theorem 3.** *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle, $K$ be a $\Gamma-$convex subset of $X$ and $Y$ be a topological space. Let $F:\ K \times K \multimap Y$ be a set-valued map, $W$ be a nonempty subset of $Y$ and suppose that:*
*$(i)$ $F(x,\ x) \subseteq W$, for every $x \in K$.*
*$(ii)$ For each $z \in K$, $\{y \in K \mid F(y,\ z) \subseteq W\}$ is intersectionally closed.*
*$(iii)$ For every fixed $y \in K$, $co_\Gamma\{z \in K \mid F(y,\ z) \nsubseteq W\} \subseteq \{x \in K \mid F(x,\ y) \nsubseteq W\}$.*
*Then, there exists a point $x_0 \in K$ such that $F(x_0,\ y) \subseteq W$ for every $y \in K$.*

*Proof.* For each $z \in K$, let $G(z) := \{y \in K \mid F(y,\ z) \subseteq W\}$ (we proof that $\bigcap_{z\in K} G(z) \neq \varnothing$). By $(ii)$, $G(z)$ is intersectionally closed. For each $x \in K$, let $H(x) := \{y \in K \mid F(x,\ y) \subseteq W\}$. Then, $(i)$ and $(iii)$ imply conditions $(i)$ and $(ii)$ of Lemma 1 respectively. By $(i)$, $x \in H(x)$ for every $x \in K$. By $(iii)$,

$$co_\Gamma(D \setminus G^-(x)) \subseteq K \setminus H^-(x)$$

So, by Lemma 1, we get that

$$\bigcap\{G(z) \mid z \in K\} \neq \varnothing.$$

□

**Theorem 4.** *Let $(X;\ \Gamma)$ be an abstract convex space, $K$ be a $\Gamma-$convex subset of $X$ and $Y$ be a topological space. Let $F:\ K\times K \multimap Y$ be a set-valued mapping and $W$ be a nonempty subset of $Y$. If for each $z\in K$, $\{y\in K \mid F(y,\ z)\subseteq W\}$ be transfer closed-valued, then the solution set is closed.*

*Proof.* Let $G(z) := \{y\in K \mid F(y,\ z)\subseteq W\}$. Then $G:\ K\multimap K$ is a transfer closed-valued mapping. So, we have

$$\bigcap_{z\in K}\overline{G(z)} = \bigcap_{z\in K} G(z).$$

Since $G$ also is an intersectionally closed-valued map, we have

$$\bigcap_{z\in K}\overline{G(z)} = \overline{\bigcap_{z\in K} G(z)}.$$

So, we conclude that

$$\bigcap_{z\in K} G(z) = \overline{\bigcap_{z\in K} G(z)}.$$

Therefore, the solution set is closed. □

Motivated by Balaj and Lin [8], we define the following concept.

**Definition 5.** Let $X$ be a $\Gamma-$convex set in an abstract convex space and $Y$ be a convex set in a vector space. A mapping $F:\ X\multimap Y$ is said to be:
$(a)$ quasiconvex if $F(x_1)\cap C\neq\varnothing$ and $F(x_2)\cap C\neq\varnothing$ then $F(\overline{x})\cap C\neq\varnothing$ for all convex sets $C\subseteq Y$, $x_1,x_2\in X$ and for any $\overline{x}\in co_\Gamma\{x_1,x_2\}$.
$(b)$ quasiconcave if $F(x_1)\subseteq C$ and $F(x_2)\subseteq C$ then $F(\overline{x})\subseteq C$ for all convex sets $C\subseteq Y$, $x_1,x_2\in X$ and for any $\overline{x}\in co_\Gamma\{x_1,x_2\}$.

Then, we can prove the following result.

**Theorem 5.** *Let $(X;\Gamma)$ be an abstract convex space, $K$ be a $\Gamma-$convex subset of $X$, $Y$ be a vector space and $W$ be a convex subset of $Y$. Let $F:\ K\times K\multimap Y$ be a set-valued map. If $F(\cdot,y)$ is a quasiconcave mapping for all $y\in K$, then solution set is $\Gamma-$convex set.*

*Proof.* If $x_1,\ x_2\in E_w := \cap_{y\in K}\{x\in K \mid F(x,\ y)\subseteq W\}$ with $x_1\neq x_2$, then $F(x_1,\ y)\subseteq W$ and $F(x_2,\ y)\subseteq W$ for all $y\in K$. For any $y\in K$, the quasiconcaveness of $F(.,\ y)$ implies $F(co_\Gamma\{x_1,\ x_2\},\ y)\subseteq W$. Hence, $co_\Gamma\{x_1,\ x_2\}\subseteq E_w$. Since $x_1$ and $x_2$ were arbitrary, we obtain that

$$co_\Gamma E_w = \bigcup\{\Gamma_A \mid A\in < E_w >\}\subset E_w.$$

This means that $E_w$ is convex. □

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# $g(x)$-FULL CLEAN RINGS

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*Abstract.* Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial of ring $C(R)[x]$. An element $r \in R$ is called "$g(x)$-clean" if $r = s + u$ where $g(s) = 0$ and $u$ is a unit of $R$ and $R$ is $g(x)$-clean if every element is $g(x)$-clean. In this paper, we introduce the concept of $g(x)$-full clean rings and study various properties of them.

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*Keywords:* clean ring, $g(x)$-clean ring, $g(x)$-full clean ring.

## 1. Introduction

Clean rings were introduced by Nicholson [4]. A ring $R$ is called clean if for every element $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = e + u$. Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial in $C(R)[x]$. Following Camillo and Simon [2], an element $r \in R$ is called $g(x)$-clean if $r = u + s$ where $g(s) = 0$ and $u$ is a unit of $R$, and $R$ is $g(x)$-clean if every element is $g(x)$-clean. Moreover, Fan and Yang have studied $g(x)$-clean rings and their generalizations in [3]. Ashrafi and Ahmadi also studied weakly $g(x)$-clean rings [1].
In this paper, we extend $g(x)$-clean rings and introduce the concept of $g(x)$-full clean rings and study various properties of them. Also we prove that $M_n(R)$ is $g(x)$-full clean rings for any $g(x)$-full clean rings $R$ and get a condition under which the definitions of $g(x)$-cleanness and $g(x)$-full cleanness are equivalent.

Throughout this paper all rings are assumed to be associative with identity and modules are unitary. $J(R)$ always stands for the Jacobson radical of a ring $R$, $U(R)$ is the set of all invertible elements of a ring $R$, $M_n(R)$ denotes the $n \times n$ matrix ring over the ring $R$ and $\mathbb{T}_n(R)$ stands for $n \times n$ upper triangular matrix ring. Recall that:

**Definition 1.** Let $I$ be an ideal of a ring $R$, we say that:

(1) Idempotents can be lifted modulo $I$ if, whenever $a^2 - a \in I$ , there exists $e^2 = e \in R$ such that $e - a \in I$.
(2) The root $\bar{s}$ of the polynomial $\bar{g}(x) \in (R/I)[X]$ can be lifted modulo $I$, if there exists $a \in R$ such that $g(a) = 0$ and $s - a \in I$.

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**Definition 2.** An element $x \in R$ is said to be a full element if there exist $s, t \in R$ such that $sxt = 1$. The set of all full elements of a ring $R$ will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$.

**Definition 3.** A ring $R$ is called full-clean if every element of $R$ is a sum of a full element and an idempotent.

**Definition 4.** Let $C(R)$ denote the center of a ring $R$ and $g(x)$ be a polynomial of ring $C(R)[x]$. An element in $R$ is said to be $g(x)$-full clean if it can be written as the sum a root of $g(x)$ and a full element. A ring $R$ is called a $g(x)$-full clean ring if each element in $R$ is a $g(x)$-full clean element.

## 2. $g(x)$-FULL CLEAN RINGS

Firstly, we get some basic properties of $g(x)$-full clean rings.

Let $R$ and $S$ be rings and $\theta : C(R) \longrightarrow C(S)$ be a ring homomorphism with $\theta(1) = 1$. Then $\theta$ induces a map $\theta'$ from $C(R)[x]$ to $C(S)[x]$ such that for $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$, $\theta'(g(x)) := \sum_{i=0}^{n} \theta(a_i)x^i \in C(S)[x]$. We should note that if $n \in \mathbb{Z}$, then $\theta(n) = \theta(1 + \cdots + 1) = n\theta(1) = n$. So if $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then clearly $\theta'(g(x)) = g(x)$.

Here we give some properties of $g(x)$-full clean rings which are similar to those of $g(x)$-clean rings.

**Proposition 1.** *Let $\theta : R \longrightarrow S$ be a ring epimorphism. If $R$ is $g(x)$-full clean, then $S$ is $\theta'(g(x))$-full clean.*

*Proof.* Let $g(x) = a_0 + a_1x + \cdots + a_nx^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in C(S)[x]$. As $\theta$ is a ring epimorphism so for any $s \in S$, there exist $r \in R$ such that $\theta(r) = s$. Since $R$ is $g(x)$-full clean, there exist $w \in K(R)$ and $s_0 \in R$ such that $r = w + s_0$ where $g(s_0) = 0$ and $swt = 1$ for some $s, t \in R$. Then $s = \theta(r) = \theta(w + s_0) = \theta(w) + \theta(s_0)$. But as $swt = 1$ we have $\theta(s)\theta(w)\theta(t) = \theta(swt) = \theta(1) = 1$. Therefore $\theta(w) \in K(S)$. But $\theta'(g(\theta(s_0))) = \theta(a_0) + \theta(a_1)\theta(s_0) + \cdots + \theta(a_n)\theta(s_0^n) = \theta(a_0 + a_1s_0 + \cdots + a_ns_0^n) = \theta(g(s_0)) = \theta(0) = 0$, so $s$ is $\theta'(g(x))$-full clean. Therefore $S$ is $\theta'(g(x))$-full clean. □

**Corollary 1.** *If $R$ is $g(x)$-full clean, then for any ideal $I$ of $R$, $R/I$ is $\bar{g}(x)$-full clean where $\bar{g}(x) \in C(R/I)[x]$.*

*Proof.* Let $\theta$: $R \longrightarrow R/I$ be the canonical epimorphism. Note that if $a \in C(R)$ then $\bar{a} \in C(R/I)$, so the result follows from previous proposition. □

**Proposition 2.** *Let $I \leq J(R)$ be an ideal of $R$, $\eta : R \longrightarrow R/I$ with $\eta(r) = r + I = \bar{r}$, and $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ with $\bar{g}(x) = \sum_{i=0}^{n} \bar{a}_i x^i \in C(R/I)[x]$. If $R/I$ is $\bar{g}(x)$-full clean and roots of $g(x)$ lift modulo $I$, then $R$ is $g(x)$-full clean.*

*Proof.* For any $r \in R$, Let $r + I = \bar{r} = \bar{s} + \bar{w}$ be the $\bar{g}(x)$-full clean expression, i.e., $\bar{g}(\bar{s}) = 0$, $\bar{w} \in K(R/I)$ and $\bar{s'}\bar{w}\bar{t} = \bar{1}$ for some $s', t \in R$. Since roots of $\bar{g}(x)$ lift modulo $I$, there exist $e \in R$ such that $g(e) = 0$ and $\bar{e} = \bar{s}$. So, $r - e - w = i$ for some $i \in I$ and $r = e + (w + i)$. Hence $\bar{s'}\bar{w}\bar{t} = \bar{1}$, we have $s'wt = 1 + h \in 1 + I \subseteq 1 + J(R) \subseteq U(R)$ for some $h \in I$. Therefore, there exist $a \in R$ where $(s'wt)a = 1$ and $s_1, t_1 \in R$ such that $s_1 w t_1 = 1$. Hence $s_1(w + i)t_1 = 1 + s_1 i t_1 \in 1 + J(R) \subseteq U(R)$. We have $s_1(w + i)t_1 u = 1$ for some $u \in U(R)$, hence $w + i$ is a full element. Therefore, $r$ is $g(x)$-full clean, as asserted. □

**Proposition 3.** *Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is $g(x)$-full clean if and only if for all $i \in I$, $R_i$ is $g(x)$-full clean.*

*Proof.* Let $\prod_{i \in I} R_i$ is $g(x)$-full clean. Define $\pi_j : \prod_{i \in I} R_i \longrightarrow R_j$ by $\pi_j(\{a_i\}_{i \in I}) = a_j$. Since for all $j \in I$, $\pi_j$ is a ring epimorphism, so by Proposition 1, for every $i \in I$, each $R_i$ is $g(x)$-full clean ring.
For the contrary, suppose that for every $i \in I$, $Ri$ is a $g(x)$-full clean ring. For any $x = \{x_i\}_{i \in I} \in \prod_{i \in I} R_i$, we write $xi = s_i + w_i$ with $g(s_i) = 0$ and $s_i w_i t_i = 1$ for some $s_i, t_i \in R$. Then $x = s + w$, where

$$\begin{aligned} g(s = \{s_i\}_{i \in I}) &= a_0\{1_{R_i}\}_{i \in I} + a_1\{s_i\}_{i \in I} + \cdots + a_n\{s_i^n\}_{i \in I} \\ &= \{a_0\}_{i \in I} + \{a_1 s_i\}_{i \in I} + \cdots + \{a_n s_i^n\}_{i \in I} \\ &= \{a_0 + a_1 s_i + \cdots + a_n s_i^n\}_{i \in I} \\ &= \{g(s_i)\}_{i \in I} = \{0\}_{i \in I} \end{aligned}$$

and $w = \{wi\}_{i \in I} \in K(\prod_{i \in I} R_i)$ with $\{s_i\}_{i \in I}\{wi\}_{i \in I}\{ti\}_{i \in I} = \{1\}_{i \in I}$. Hence $x$ is $g(x)$-full clean, as required. □

Recall that for a ring $R$ with a ring endomorphism $\alpha : R \longrightarrow R$, the skew power series ring $R[[x;\alpha]]$ of $R$ is the ring obtained by giving the formal power series ring over $R$ with this property that $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x, id_R]]$.

**Proposition 4.** *Let $\alpha$ be an endomorphism of $R$ and $g(x) \in C(R)[x]$. Then the following statements are equivalent.*

(1) *$R$ is a $g(x)$-full clean ring.*

(2) *The formal power series ring* $R[[x]]$ *of* $R$ *is a* $g(x)$*-full clean ring.*
(3) *The skew power series ring* $R[[x;\alpha]]$ *of* $R$ *is a* $g(x)$*-full clean ring.*

*Proof.* Being homomorphic image of $R[[x]]$ and $R[[x;\alpha]]$, $R$ is $g(x)$-full clean when $R[[x]]$ or $R[[x;\alpha]]$ is $g(x)$-full clean.

Now, suppose $R$ is a $g(x)$-full clean ring. For any $h = a_0 + a_1x + \ldots \in R[[x,\alpha]]$, write $a_0 = e_0 + u_0$ such that $g(e_0) = 0$ and $u_0 \in K(R)$. Assume that $s_0u_0t_0 = 1$ for some $s_0, t_0 \in R$ and let $h' = h - e_0 = u_0 + a_1x + \ldots$ . The equation $u = (s_0 + 0 + \ldots)h'(t_0 + 0 + \ldots) = 1 + s_0a_1\alpha(t_0)x + \ldots$ shows that $u \in U(R[[x,\alpha]])$, since $U(R[[x;\alpha]]) = \{a_0 + a_1x + \ldots : a_0 \in U(R)\}$ without any assumption on the endomorphism $\alpha$. Hence $h' \in K(R[[x,\alpha]])$ and $h = e_0 + h'$ where $e_0 \in R[[x,\alpha]]$ and $g(e_0) = 0$. so, $R[[x;\alpha]]$ is a $g(x)$-full clean ring.

Since $R[[x]] = R[[x, id_R]]$, the proof is similar to that of ((1) $\Rightarrow$ (3)), as desired. □

*Remark* 1. Generally, the polynomial ring $R[t]$ is not $g(x)$-clean for an arbitrary nonzero polynomial $g(x) \in C(R)[x]$. For example let $R$ be a commutative ring, then the polynomial ring $R[t]$ is not $g(x)$-clean ring [3]. Full elements and invertible elements are the same when the ring $R$ is commutative, so the concept of $g(x)$-clean and $g(x)$-full clean are equivalent for commutative rings. Now, let $g(x) = x$, we show that $t$ is not $g(x)$-full clean. If $t = w + s$ then it must be that $s = 0$, so $t = w$. As, $w$ is a full element then $fth = 1$ for $f, h \in R$. Since $R$ is commutative, $t \in U(R[t])$. But clearly $t \notin U(R[t])$, therefore $R[t]$ is not $g(x)$ full-clean.

Next we will investigate some cases in which the concept of $g(x)$-full cleanness and $g(x)$-cleanness are equivalence. Yu [5] called a ring $R$ to be a left quasi-duo ring if every maximal left ideal of $R$ is a two-sided ideal. Commutative rings, local rings, rings in which every nonunit has a power that is central are all belong to this class of rings [5].

**Theorem 1.** *For a left quasi-duo ring* $R$ *and* $g(x) \in C(R)[x]$*, the followings are equivalent:*

(1) $R$ *is a* $g(x)$*-clean ring;*
(2) $R$ *is a* $g(x)$*-full clean ring.*

*Proof.* If $R$ is $g(x)$-clean, then this is trivial that $R$ is $g(x)$-full clean.

Now, let $R$ be a $g(x)$-full clean ring and $r \in R$. So $r = w + s$ such that $g(s) = 0$ and $w \in K(R)$. It suffices to show that $w \in K(R)$ implies that $w \in U(R)$. Let $swt = 1$ for some $s, t \in R$, so $s$ is right invertible. Assume that $s$ is not left invertible. Then $Rs \subsetneqq R$, and there exists a maximal left ideal $M$ of $R$ such that $Rs \subseteq M \subsetneqq R$. But since $R$ is a left quasi-duo ring, so $M$ is a two sided ideal and $s \in M$. Therefore $sR \subseteq M$. But as $s$ is right invertible, so $M$ is not a proper ideal and this is a contradiction. So $s$ should have left inverse as well and therefore $s$ is invertible. Thus $wts = 1$. In a similar way, we get that $w \in U(R)$, and the result follows. □

In Fan and Yang [3], proved that if $R$ is $g(x)$-clean, then so is $M_n(R)$ for all $n \geq 1$. Here we have a similar result for $g(x)$-full clean. Define $\pi_n : C(R) \longrightarrow M_n(R)$ by $a \longmapsto aI_n$ where $I_n$ is the identity matrix of $M_n(R)$ and $a \in C(R)$. Then $M_n(R)$ is a $C(R)$-algebra for all $n \geq 1$.

**Theorem 2.** *Let $R$ be a ring and $g(x) \in C(R)[x]$. If $R$ is $g(x)$-full clean, then $M_n(R)$ is also $g(x)$-full clean ring for all $n \geq 1$.*

*Proof.* Suppose that $R$ is $g(x)$-full clean. Given any $x \in R$, there exist $e \in R$ and $w \in K(R)$ such that $x = e + w$ and $g(e) = 0$. We write $swt = 1$ for some $s, t \in R$. Assume that theorem holds for the matrix ring $M_k(R)$, $k \geq 1$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{k+1}(R)$$

with $A_{11} \in R$, $A_{12} \in R^{1\times k}$, $A_{21} \in R^{k\times 1}$ and $A_{22} \in M_k(R)$.
We have $A_{11} = e + w$ where $g(e) = 0$ and $swt = 1$ for some $s, t \in R$. There also exist matrix $E$ and a full matrix $W$ such that $A_{22} - A_{21}tsA_{12} = E + W$ where $g(E) = 0$ by induction. We write $SWT = I_k$ for some $S, T \in M_k(R)$. Therefore, we have $A = diag(e, E) + \begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}$. Let $g(x) = a_0 + a_1x + \cdots + a_mx^m$, then we have:

$$\begin{aligned} g(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix}) &= a_0I_{k+1} + a_1(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix}) + \cdots + a_m(\begin{pmatrix} e & 0 \\ 0 & E \end{pmatrix})^m \\ &= \begin{pmatrix} a_01_R & 0 \\ 0 & a_0I_k \end{pmatrix} + \begin{pmatrix} a_1e & 0 \\ 0 & a_1E \end{pmatrix} + \cdots + \begin{pmatrix} a_me^m & 0 \\ 0 & a_mE^m \end{pmatrix} \\ &= \begin{pmatrix} g(e) & 0 \\ 0 & g(E) \end{pmatrix} = 0 \end{aligned}$$

Also, let $P = \begin{pmatrix} s & 0 \\ -SA_{21}ts & S \end{pmatrix}, Q = \begin{pmatrix} t & -tsA_{12}T \\ 0 & T \end{pmatrix} \in M_{k+1}(R)$ and the equation

$$P\begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}Q = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix} = I_{k+1}$$

shows that $\begin{pmatrix} w & A_{12} \\ A_{21} & W + A_{21}tsA_{12} \end{pmatrix}$ is a full matrix, hence $A$ is $g(x)$-full clean, as desired. □

**Proposition 5.** *Let $a \in R$ be a $g(x)$-full clean element and $g(x) \in C(R)[x]$ where $g(1) = 0$, then $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is always $g(x)$-full clean in $M_2(R)$ for any $b \in R$.*

*Proof.* If $a = e + w$ where $g(e) = 0$ and $swt = 1$ for some $s, t \in R$, then we can write $A$ as

$$A = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} w & b \\ 0 & -1 \end{pmatrix}$$

We also have $\begin{pmatrix} s & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} w & b \\ 0 & -1 \end{pmatrix}\begin{pmatrix} t & -tsb \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now if $g(x) = \sum_{i=0}^{n} a_i x^i$ we have,

$$\begin{aligned} g(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}) &= a_0 I_2 + a_1(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}) + \cdots + a_m(\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix})^m \\ &= \begin{pmatrix} a_0 1_R & 0 \\ 0 & a_0 1_R \end{pmatrix} + \begin{pmatrix} a_1 e & 0 \\ 0 & a_1 \end{pmatrix} + \cdots + \begin{pmatrix} a_m e^m & 0 \\ 0 & a_m \end{pmatrix} \\ &= \begin{pmatrix} g(e) & 0 \\ 0 & g(1) \end{pmatrix} = 0 \end{aligned}$$

Therefore, $A$ is a $g(x)$-full clean element. □

**Theorem 3.** *Let* $C = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ *where* $A, B$ *and* ${}_A V_B, {}_B W_A$ *are respectively two rings and bimodules. Also let* $g(x) \in \mathbb{Z}[x]$. *Then* $C$ *is* $g(x)$*-full clean if and only if* $A$ *and* $B$ *are* $g(x)$*-full clean.*

*Proof.* Assume that $C$ is $f$-($g(x)$-clean). Let $I = \begin{pmatrix} 0 & V \\ W & B \end{pmatrix}$ and $J = \begin{pmatrix} A & V \\ W & 0 \end{pmatrix}$. One can check that $I, J$ are ideals of $C$ and $C/I \simeq A$, $C/J \simeq B$ (it is enough to consider the epimorphism $\varphi : C \longrightarrow A$ by $\varphi(\begin{pmatrix} a & v \\ w & b \end{pmatrix}) = a$ and the epimorphism $\psi : C \longrightarrow B$ by $\varphi(\begin{pmatrix} a & v \\ w & b \end{pmatrix}) = b$, respectively with kernel $I$ and $J$). Clearly $g(x)$-full cleanness of $A, B$ follows from Corollary 1.

Conversely, let $A$ and $B$ be both $g(x)$-full clean rings. For any $r = \begin{pmatrix} a & v \\ w & b \end{pmatrix} \in C$, we have $a = e_1 + u_1$ and $b = e_2 + u_2$ for some $e_1, e_2 \in R$ where $g(e_1) = g(e_2) = 0$ and $u_1, u_2 \in K(R)$. Assume that $s_1 u_1 t_1 = 1$, $s_2 u_2 t_2 = 1$ for some $s_1, t_1, s_2, t_2 \in R$. So we have $r = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} u_1 & v \\ w & u_2 \end{pmatrix} = E + U$. Now $g(x) = \sum_{i=0}^{n} a_i x^i$. Hence

$$\begin{aligned} g(E) &= a_0 I_0 + a_1 E + \cdots + a_n E^n \\ &= \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 e_1 & 0 \\ 0 & a_1 e_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_n e_1^n & 0 \\ 0 & a_n e_2^n \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} g(e_1) & 0 \\ 0 & g(e_2) \end{pmatrix} = 0$$

and the equation

$$\begin{pmatrix} s_1 & 0 \\ -s_2 w t_1 s_1 & s_2 \end{pmatrix} \begin{pmatrix} u_1 & v \\ w & u_2 \end{pmatrix} \begin{pmatrix} t_1 & -t_1 s_1 v t_2 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

implies that $U$ is a full matrix. Hence $r$ is $g(x)$-full clean, as required. □

**Proposition 6.** *Let $R$ and $S$ be two rings, $M$ be an $(R,S)$-bimodule and $g(x) \in \mathbb{Z}[x]$.*

(1) *Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the formal triangular matrix ring. Then $E$ is $g(x)$-full clean ring if and only if $R$ and $S$ are $g(x)$-full clean rings.*
(2) *For any $n \geq 1$, $R$ is $g(x)$-full clean if and only if the $n \times n$ upper triangular matrix ring $T_n(R)$ are $g(x)$-full clean.*

*Proof.* Formal triangular matrix rings are special cases of $C$ in Theorem 3.

Let $R$ be $g(x)$-full clean and $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ for $1 \leq j < i \leq n$. Since $R$ is $g(x)$-full clean, for any $1 \leq i \leq n$, there exist $e_{ii} \in R$ and $w_{ii} \in K(R)$ such that $a_{ii} = w_{ii} + e_{ii}$ with $g(e_{ii}) = 0$. Also assume that $s_{ii} w_{ii} t_{ii} = 1$ for some $s_{ii}, t_{ii} \in R$. So we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} w_{11}+e_{11} & a_{12} & \dots & a_{1n} \\ 0 & w_{22}+e_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{nn}+e_{nn} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} w_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & w_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w_{nn} \end{pmatrix}}_{W} + \underbrace{\begin{pmatrix} e_{11} & 0 & \dots & 0 \\ 0 & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{nn} \end{pmatrix}}_{E}.$$

Suppose $g(x) = \sum_{i=0}^{m} a_i x^i \in C(R)[x]$, so we have

$$g(E) = a_0 I_n + a_1 E + \dots + a_n E^n$$
$$= \begin{pmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 \end{pmatrix} + \begin{pmatrix} a_1 e_{11} & 0 & \dots & 0 \\ 0 & a_1 e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_1 e_{nn} \end{pmatrix} + \dots$$

$$+\begin{pmatrix} a_m e_{11}^m & 0 & \dots & 0 \\ 0 & a_m e_{22}^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m e_{nn}^m \end{pmatrix}$$

$$= \begin{pmatrix} g(e_{11}) & 0 & \dots & 0 \\ 0 & g(e_{22}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g(e_{nn}) \end{pmatrix} = 0.$$

Also, it is straightforward with induction on $n$, to prove calculate that $W \in K(\mathbb{T}_n(R))$. So $\mathbb{T}_n(R)$ is $g(x)$-full clean.

Now let $\mathbb{T}_n(R)$ is $g(x)$-full clean. Define $\theta : \mathbb{T}_n(R) \longrightarrow R$ by $\theta(A) = a_{11}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

then it is clear that $\theta$ is a ring epimorphism. For any $a \in R$, let $B$ be the diagonal matrix diog$(a, ...a)$. Then $a = \theta(B) = \theta(W + S) = \theta(W) + \theta(S)$ where $\theta(W) = w_{11} \in K(R)$ and

$$\begin{aligned} g(\theta(S)) &= a_0 + a_1\theta(S) + \cdots + a_n\theta(S^n) \\ &= \theta(B_0) + \theta(B_1)\theta(S) + \cdots + \theta(B_n)\theta(S^n) \\ &= \theta(B_0 + B_1 S + \cdots + B_n S^n) \\ &= \theta(a_0 I_n + (a_1 I_n)S + \cdots + (a_n I_n)S^n) \\ &= \theta(g(S)) = 0. \end{aligned}$$

Thus $a$ is $g(x)$-full clean, i.e., $R$ is $g(x)$-full clean ring. □

**Proposition 7.** *Let $R$ be a ring, $n \in \mathbb{N}$ and $2 \in U(R)$. Then the followings are equivalent:*

(1) *$R$ is full-clean;*
(2) *$R$ is $(x^2 - 2^n x)$-full clean;*
(3) *$R$ is $(x^2 + 2^n x)$-full clean;*
(4) *$R$ is $(x^2 - 2x)$-full clean;*
(5) *$R$ is $(x^2 + 2x)$-full clean;*
(6) *$R$ is $(x^2 - 1)$-full clean;*
(7) *every element of $R$ is the sum of a full element and a square root of* 1.

*Proof.* (1) ⇒ (7) Suppose $R$ is full-clean and $x \in R$. Then $(x+1)/2 = e + u$ for some $e^2 = e$ and $u \in K(R)$. So $x = (2e-1) + 2u$ with $(2e-1)^2 = 1$ and $2u \in K(R)$.

(7) $\Rightarrow$ (1) Let every element of $R$ is the sum of a full element and a square root of 1. Then given $x \in R$, we have $2x - 1 = t + w$ with $t^2 = 1$ and $w$ a full element in $R$. So $x = (t+1)/2 + w/2$ with $((t+1)/2)^2 = (t+1)/2$ and $w/2$ is a full element in $R$, as asserted.

(1) $\Rightarrow$ (2) Since $2 \in U(R)$, $2^n \in U(R)$. Let $a \in R$, then $a/2^n = e + u$ such that $e^2 = e$ and $u \in K(R)$. So, $a = 2^n e + 2^n u$ where $(2^n e)^2 - 2^n(2^n e) = 0$ and $2^n u \in K(R)$. Therefore, $R$ is $(x^2 - 2^n x)$-full clean.

(2) $\Rightarrow$ (1) Let $r \in R$. Since $R$ is $f$-($(x^2 - 2^n x)$-clean), $r2^n = s + w$ such that $s$ is a root of $(x^2 - 2^n x)$ and $w \in K(R)$. Thus, $r = s/2^n + w/2^n$ where $w/2^n \in K(R)$ and $(s/2^n)^2 = s(s - 2^n + 2^n)/(2^n)^2 = s2^n/(2^n)^2 = s/2^n$. So $R$ is $f$-clean.

Similarly, we can prove (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (5). $\square$

Let $R$ be a ring and ${}_RV_R$ be an $R-R-$bimodule which is a ring possibly without a unity in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. The ideal extension of $R$ by $V$ is defined to be the additive abelian group $I(R,V) = R \bigoplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

**Proposition 8.** *Let $R$ be a ring and ${}_RV_R$ be an $R-R-$bimodule, $g(x) \in \mathbb{Z}[x]$. An ideal-extension $E = I(R,V)$ of $R$ by $V$ is $g(x)$-full clean if $R$ is $g(x)$-full clean and for any $v \in R$, there exists $w \in R$ such that $v + w + wv = 0$.*

*Proof.* Let $t = (r,v) \in E$. Then $r = s + u$ where $g(s) = 0$ and $u \in K(R)$. Therefore $t = (s,0) + (u,v)$. Let $g(x) = \sum_{i=0}^{n} a_i x^i$, we have

$$\begin{aligned} g((s,0)) &= a_0(1,0) + a_1(s,0) + \cdots + a_n(s,0)^n \\ &= a_0(1,0) + a_1(s,0) + \cdots + a_n(s^n,0) \\ &= (a_0,0) + (a_1 s,0) + \cdots + (a_n s^n,0) \\ &= (a_0 + a_1 s + \cdots + a_n s^n, 0) = (g(s),0) = (0,0) \end{aligned}$$

and we will show that $(u,v) \in K(E)$. Assume that $sut = 1$. For $svt \in V$ , there exists $w \in V$ such that $svt + w + wsvt = 0$ by assumption. Also, one can check that $(s, ws)(u,v)(t,0) = (1,0)$. Hence $(u,v) \in K(E)$ and $E$ is a $g(x)$-full clean ring. $\square$

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# APPROXIMATION PROPERTIES OF BASKAKOV-BALAZS TYPE OPERATORS FOR FUNCTIONS OF TWO VARIABLES

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*Abstract.* In this paper, we study Baskakov type positive operators in polynomial weighted spaces of functions of two variables. We obtain some well known operators by using our operators which are special cases of them. We give theorems on approximation, on the degrees of approximations of functions and the Vornovskaya type theorem for these operators. Finally, we present an open problem concerning the $q$ analogue of these operators.

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## 1. Introduction

Let $p \in \mathbb{N}_0$. We define $\omega_p(u) = (1+u^p)^{-1}$, $u \in \mathbb{R}_0 = [0,\infty)$ and for fixed $p,q \in \mathbb{N}_0$, we define the weighted function

$$\omega_{p,q}(x,y) = \omega_p(x)\omega_q(y), \quad (x,y) \in \mathbb{R}_0^2 = [0,\infty)\times[0,\infty).$$

We denote by the weighted space $C_{\omega_{p,q}}$ the space of all real-valued functions $f$ continuous on $\mathbb{R}_0^2$ for which $\omega_{p,q} f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$ with the norm

$$\|f\|_{\omega_{p,q}} = \sup_{(x,y)\in\mathbb{R}_0^2} \omega_{p,q}(x,y)\,|f(x,y)|.$$

The modulus of continuity of $f \in C_{\omega_{p,q}}$ defined by

$$\Omega(f;t,s) = \sup_{0\le h\le t}\ \sup_{0\le\delta\le s} \left\|\Delta_{h,\delta} f\right\|_{\omega_{p,q}}, \quad t,s \ge 0,$$

where $\Delta_{h,\delta} f(x,y) = f(x+h,y+\delta) - f(x,y)$. Also we denote by $C^m_{\omega_{p,q}}$ is the set of all functions $f \in C_{\omega_{p,q}}$ with the partial derivatives $f^{(k)}_{x^j,y^{k-j}}$, $k = 1,2,\dots,m$, belonging to $C_{\omega_{p,q}}$.

Let $\{\varphi_n\}$ $(n = 1,2,\dots)$ is a sequence of functions $\varphi_n : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:

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$(i)$ $\varphi_n$ $(n=1,2,...)$ is analytic on a domain $D$ containing the disk $B=\{z\in C:\ |z-b|\leq b\}\subset D$;

$(ii)$ $\varphi_n(0)=1$ $(n=1,2,...)$;

$(iii)$ for any $x\geq 0$, $\varphi_n(x)>0$ and $\varphi_n^{(k)}(0)\geq 0$ for any $n=1,2,...$ and $k=1,2,....$

$(iv)$ for every $n=1,2,...$

$$\frac{\varphi_n^{(\upsilon)}(a_n x)}{n^{\upsilon}\varphi_n(a_n x)}=1+o\left(\frac{1}{b_n}\right)\quad \upsilon=0,1,2,3,4 \tag{1.1}$$

where $a_n=\dfrac{b_n}{n}\to 0$ and $b_n\to\infty$ as $n\to\infty$.

For a real valued function $f$ defined on the interval $[0,\infty)$, generalized Balazs type operator defined by (see [8])

$$L_n(f;x)=\frac{1}{\varphi_n(a_n x)}\sum_{k=0}^{\infty}f\left(\frac{k}{b_n}\right)\frac{\varphi_n^{(k)}(0)}{k!}(a_n x)^k \qquad (n=1,2,...). \tag{1.2}$$

In (1.2), choosing $a_n=1$ we obtain following operators

$$L_n(f;x)=\frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty}f\left(\frac{k}{b_n}\right)\frac{\varphi_n^{(k)}(0)}{k!}x^k$$

which are known as the Baskakov type operator. It can be easily verified that in case $\varphi_n(x)=(1+x)^n$, the operators $L_n(f;x)$ reduce to the well known Bernstein type rational function introduced by K. Balazs[3] as

$$L_n(f;x)=\frac{1}{(1+a_n x)^n}\sum_{k=0}^{n}\binom{n}{k}(a_n x)^k f\left(\frac{k}{b_n}\right).$$

In the present work, inspired by operator (1.2), for $f\in C_{\omega_{p,q}}$ and $p,q\in\mathbb{N}_0$ we introduce the following operators

$$L_{n,m}(f;a_n,b_n,c_m,d_m;x,y)\equiv L_{n,m}(f;x,y)$$

defined by

$$\begin{aligned}&L_{n,m}(f;x,y)\\&=\frac{1}{\varphi_n(a_n x)}\frac{1}{\varphi_m(c_m y)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\varphi_n^{(j)}(0)}{j!}(a_n x)^j\frac{\varphi_m^{(k)}(0)}{k!}(c_m y)^k f\left(\frac{j}{b_n},\frac{k}{d_m}\right),\end{aligned} \tag{1.3}$$

$$(x,y)\in\mathbb{R}_0^2,\ \ n,m\in\mathbb{N}$$

where $(a_n),(b_n),(c_m),(d_m)$ are given increasing and unbounded sequences of positive numbers such that

$$\frac{\varphi_n^{(\upsilon)}(a_n x)}{n^{\upsilon}\varphi_n(a_n x)} = 1 + o\left(\frac{1}{b_n}\right), \quad \frac{\varphi_m^{(\upsilon)}(c_m x)}{m^{\upsilon}\varphi_m(c_m x)} = 1 + o\left(\frac{1}{d_m}\right). \tag{1.4}$$

From (1.3) and (1.4) we decide that $L_{n,m}(f)$ are well-defined in every space $f \in C_{\omega_{p,q}}$ and $p,q \in \mathbb{N}_0$. Moreover, for $(x,y) \in \mathbb{R}_0^2$, $n,m \in \mathbb{N}$, we have

$$L_{n,m}(1;a_n,b_n,c_m,d_m;x,y) = 1. \tag{1.5}$$

If $f \in C_{\omega_{p,q}}$ and $f(x,y) = f_1(x) f_2(y)$ ; for all $(x,y) \in \mathbb{R}_0^2$ and $n,m \in \mathbb{N}$ then

$$L_{n,m}(f;a_n,b_n,c_m,d_m;x,y) = L_n(f_1;a_n,b_n;x) L_m(f_2;c_m,d_m;y). \tag{1.6}$$

This paper is devoted to a study aimed at obtaining approximation results by using the modulus of continuity and obtaining a Voronovskaya type theorem for the Baskakov type operators defined by (1.3) in polynomial weighted spaces. Approximation results for some different operators in the weighted spaces have been investigated in some papers (e.g. [2],[6], [5], [7],[4], [8], [9],[10], [12], [11]).

## 2. Auxiliary results

In this section, we give some lemmas, which are essential to prove our main theorems.

**Lemma 1.** *Let $L_n(f;x)$ be defined by (1.2). For $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, we have*

$$L_n(1;x,y) = 1, \tag{2.1}$$

$$L_n(t-x;x,y) = \left(\frac{a_n}{b_n}\frac{\varphi_n'(a_n x)}{\varphi_n(a_n x)} - 1\right)x, \tag{2.2}$$

$$L_n\left((t-x)^2;x\right) = \left(\frac{a_n^2}{b_n^2}\frac{\varphi_n''(a_n x)}{\varphi_n(a_n x)} - 2\frac{a_n}{b_n}\frac{\varphi_n'(a_n x)}{\varphi_n(a_n x)} + 1\right)x^2 + \frac{a_n x}{b_n^2}\frac{\varphi_n'(a_n x)}{\varphi_n(a_n x)} \tag{2.3}$$

*where $a_n = \dfrac{b_n}{n} \to 0$, $b_n \to \infty$ and $\dfrac{\varphi_n^{(\upsilon)}(a_n x)}{n^{\upsilon}\varphi_n(a_n x)} = 1 + o\left(\dfrac{1}{b_n}\right)$.*

**Lemma 2.** *For every fixed $p \in \mathbb{N}_0$, for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, we have*

$$\omega_p(x) L_n\left(\frac{1}{\omega_p(t)};x\right) \le M_1, \tag{2.4}$$

$$\omega_p(x) L_n\left(\frac{(t-x)^2}{\omega_p(t)};x\right) \le M_2\left[\left(\frac{a_n^2}{b_n^2}\frac{\varphi_n''(a_n x)}{\varphi_n(a_n x)} - 2\frac{a_n}{b_n}\frac{\varphi_n'(a_n x)}{\varphi_n(a_n x)} + 1\right)x^2\right. \tag{2.5}$$

$$+\frac{a_n x}{b_n^2}\frac{\varphi_n'(a_n x)}{\varphi_n(a_n x)}\Bigg]$$

*where $M_1, M_2$ are positive constants.*

**Lemma 3.** *For every $x \in \mathbb{R}_0$, one has*

$$\lim_{n\to\infty} b_n L_n\left((t-x)^k; x\right) = \begin{cases} 0, & if\, k=1 \\ x, & if\, k=2 \end{cases}$$

$$\lim_{n\to\infty} b_n^2 L_n\left((t-x)^k; x\right) = \begin{cases} x, & if\, k=3 \\ 3x^2, & if\, k=4 \end{cases}.$$

**Lemma 4.** *For every $p,q \in \mathbb{N}_0$, $m,n \in \mathbb{N}$ and every $f \in C_{\omega_{p,q}}$, we have*

$$\left\| L_{n,m}\left(\frac{1}{\omega_{p,q}(t,z)}\right)\right\|_{C_{\omega_{p,q}}} \le M_4, \tag{2.6}$$

$$\|L_{n,m}(f)\|_{C_{\omega_{p,q}}} \le M_4 \|f\|_{C_{\omega_{p,q}}} \tag{2.7}$$

*where $M_4$ is a positive constant.*

**Lemma 5.** *Let $f \in C_{\omega_{p,q}}$ and $p,q \in \mathbb{N}_0$.For all $m,n \in \mathbb{N}$, we get*

$$\left\|\left(L_{n,m} f\right)_x'\right\|_{C_{\omega_{p,q}}} \le M_5 \|f\|_{C_{\omega_{p,q}}} a_n, \tag{2.8}$$

$$\left\|\left(L_{n,m} f\right)_y'\right\|_{C_{\omega_{p,q}}} \le M_5 \|f\|_{C_{\omega_{p,q}}} c_m \tag{2.9}$$

*where $M_5$ is a positive constant.*

## 3. Main results

Now, we give firstly following two theorems on the degree of approximation of functions by $L_{n,m}$ defined by (1.3).

**Theorem 1.** *Let $f \in C^1_{\omega_{p,q}}$ and fixed $p,q \in \mathbb{N}_0$. For all $(x,y) \in \mathbb{R}_0^2$ and $m,n \in \mathbb{N}$, we have*

$$\omega_{p,q}(x,y)\left|L_{n,m}(f;x,y) - f(x,y)\right| \le M_6\left\{\|f_x'\|_{p,q}\sqrt{\frac{x}{b_n}} + \|f_y'\|_{p,q}\sqrt{\frac{y}{d_m}}\right\} \tag{3.1}$$

*where $M_6$ is a positive constant.*

*Proof of Theorem 1.* Let $(x,y) \in \mathbb{R}_0^2$ be a fixed point. Then $f \in C^1_{\omega_{p,q}}$ we have the formula

$$f(t,z) - f(x,y) = \int_x^t f_u'(u,z)\,du + \int_y^z f_v'(x,v)\,dv \qquad (t,z) \in \mathbb{R}_0^2.$$

Thus, by (1.5), we obtain

$$L_{n,m}\left(f\left(t,z\right);x,y\right)-f\left(x,y\right) \tag{3.2}$$
$$=L_{n,m}\left(\int_x^t f_u'\left(u,z\right)du;x,y\right)+L_{n,m}\left(\int_y^z f_v'\left(x,v\right)dv;x,y\right)$$

On the other hand, we have

$$\begin{aligned}\left|\int_x^t f_u'\left(u,z\right)du\right| &= \left|\int_x^t \omega_{p,q}\left(u,z\right)f_u'\left(u,z\right)\frac{1}{\omega_{p,q}\left(u,z\right)}du\right| \\ &\le \left\|f_x'\right\|_{p,q}\left|\left|\int_x^t \frac{1}{\omega_{p,q}\left(u,z\right)}du\right|\right| \\ &\le \left\|f_x'\right\|_{p,q}\left(\frac{1}{\omega_{p,q}\left(t,z\right)}+\frac{1}{\omega_{p,q}\left(x,z\right)}\right)\left|t-x\right|,\end{aligned}$$

which implies (1.2) and (1.5) that

$$\begin{aligned}&\omega_{p,q}\left(x,y\right)\left|L_{n,m}\left(\int_x^t f_u'\left(u,z\right)du;x,y\right)\right| \\ &\le \omega_{p,q}\left(x,y\right)L_{n,m}\left(\left|\int_x^t f_u'\left(u,z\right)du\right|;x,y\right) \\ &\le \left\|f_x'\right\|_{p,q}\omega_{p,q}\left(x,y\right)\left\{L_{n,m}\left(\frac{\left|t-x\right|}{\omega_{p,q}\left(t,z\right)};x,y\right)+L_{n,m}\left(\frac{\left|t-x\right|}{\omega_{p,q}\left(x,z\right)};x,y\right)\right\} \\ &\le \left\|f_x'\right\|_{p,q}\omega_q\left(y\right)L_m\left(\frac{1}{\omega_q\left(z\right)};y\right)\left\{\omega_p\left(x\right)L_n\left(\frac{\left|t-x\right|}{\omega_p\left(t\right)};x\right)+L_n\left(\left|t-x\right|;x\right)\right\}.\end{aligned}$$

Applying the Hölder inequality and (2.1), (2.2), (2.3), we get the inequalities

$$\begin{aligned}L_n\left(\left|t-x\right|;x\right) &\le \left\{L_n\left(\left(t-x\right)^2;x\right)L_n\left(1;x\right)\right\}^{1/2} \\ &= \left\{\left(\frac{a_n^2}{b_n^2}\frac{\varphi_n''\left(a_nx\right)}{\varphi_n\left(a_nx\right)}-2\frac{a_n}{b_n}\frac{\varphi_n'\left(a_nx\right)}{\varphi_n\left(a_nx\right)}+1\right)x^2+\frac{a_nx}{b_n^2}\frac{\varphi_n'\left(a_nx\right)}{\varphi_n\left(a_nx\right)}\right\}^{1/2} \\ &= \left\{\left(\left(1+o\left(\frac{1}{b_n}\right)\right)-2\left(1+o\left(\frac{1}{b_n}\right)\right)+1\right)x^2+\frac{x}{b_n}\left(1+o\left(\frac{1}{b_n}\right)\right)\right\}^{1/2}\end{aligned}$$

for a sufficiently large $n$, we have

$$L_n\left(\left|t-x\right|;x\right)\le M_7\sqrt{\frac{x}{b_n}},$$

and so

$$\omega_p\left(x\right)L_n\left(\frac{\left|t-x\right|}{\omega_p\left(t\right)};x\right)$$

$$\leq \left\{ \omega_p(x) L_n\left( \frac{(t-x)^2}{\omega_p(t)}; x \right) \right\}^{1/2} \left\{ \omega_p(x) L_n\left( \frac{1}{\omega_p(t)}; x \right) \right\}^{1/2}$$
$$\leq M_8 \sqrt{\frac{x}{b_n}}.$$

Consequently

$$\omega_{p,q}(x,y) \left| L_{n,m}\left( \int_x^t f'_u(u,z)\, du; x, y \right) \right| \leq M_9 \left\| f'_x \right\|_{p,q} \sqrt{\frac{x}{b_n}},$$
$$\omega_{p,q}(x,y) \left| L_{n,m}\left( \int_y^z f'_v(x,v)\, dv; x, y \right) \right| \leq M_{10} \left\| f'_y \right\|_{p,q} \sqrt{\frac{y}{d_m}}.$$

Finally, the last two inequalities, for all $m,n \in \mathbb{N}$, we derive from (3),

$$\omega_{p,q}(x,y) \left| L_{n,m}(f;x,y) - f(x,y) \right| \leq M_6 \left\{ \left\| f'_x \right\|_{p,q} \sqrt{\frac{x}{b_n}} + \left\| f'_y \right\|_{p,q} \sqrt{\frac{y}{d_m}} \right\}$$

Thus the proof of the theorem is completed. □

**Theorem 2.** *Let $f \in C_{\omega_{p,q}}$ and $p,q \in \mathbb{N}_0$. For all $(x,y) \in \mathbb{R}_0^2$ and $m,n \in \mathbb{N}$, we have*

$$\omega_{p,q}(x,y) \left| L_{n,m}(f;x,y) - f(x,y) \right| \leq M_{13} \Omega\left( f; \sqrt{\frac{x}{b_n}}, \sqrt{\frac{y}{d_m}} \right) \tag{3.3}$$

*where $M_{13}$ is a positive constant and $\Omega$ is the modulus of continuity of $f$.*

*Proof of Theorem 2.* For $f \in C_{\omega_{p,q}}$, applying the Stieklov function $f_{h,\delta}$

$$f_{h,\delta} = \frac{1}{h\delta} \int_0^h du \int_0^\delta f(x+u, y+v)\, dv, \quad (x,y) \in \mathbb{R}_0^2, \quad h,\delta > 0. \tag{3.4}$$

From (3.4), we can write

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta} \int_0^h du \int_0^\delta \Delta_{u,v} f(x,y)\, dv,$$

and therefore

$$\left(f_{h,\delta}\right)'_x(x,y) = \frac{1}{h\delta} \int_0^h du \int_0^\delta \frac{\partial f}{\partial u}(x+u, y+v)\, dv \tag{3.5}$$
$$= \frac{1}{h\delta} \int_0^\delta \{ (f(x+h, y+v) - f(x,y)) - (f(x, y+v) - f(x,y)) \}\, dv$$
$$= \frac{1}{h\delta} \int_0^\delta \left( \Delta_{h,v} f(x,y) - \Delta_{0,v} f(x,y) \right) dv,$$

Similarly, we can obtain

$$\left(f_{h,\delta}\right)'_y(x,y)=\frac{1}{h\delta}\int_0^h\left(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y)\right)du. \tag{3.6}$$

Thus we have

$$\begin{aligned}\left\|f_{h,\delta}-f\right\|_{C_{\omega_{p,q}}}&=\sup\omega_{p,q}(x,y)\left|f_{h,\delta}(x,y)-f(x,y)\right| \\ &=\sup\omega_{p,q}(x,y)\left|\frac{1}{h\delta}\int_0^h du\int_0^\delta\Delta_{u,v}f(x,y)\,dv\right| \\ &\leq\Omega(f;h,\delta),\end{aligned} \tag{3.7}$$

and from (3.5) and (3.6), we get

$$\left\|\left(f_{h,\delta}\right)'_x\right\|_{C_{\omega_{p,q}}}\leq\frac{2}{h}\Omega(f;h,\delta), \tag{3.8}$$

$$\left\|\left(f_{h,\delta}\right)'_y\right\|_{C_{\omega_{p,q}}}\leq\frac{2}{\delta}\Omega(f;h,\delta). \tag{3.9}$$

Now, for $L_{n,m}$ defined by (1.3), we can write

$$\begin{aligned}&\omega_{p,q}(x,y)\left|L_{n,m}(f;x,y)-f(x,y)\right| \\ &\leq\omega_{p,q}(x,y)\left\{\left|L_{n,m}\left(f(t,z)-f_{h,\delta}(t,z);x,y\right)\right|\right. \\ &+\left|L_{n,m}\left(f_{h,\delta}(t,z);x,y\right)-f_{h,\delta}(x,y)\right|+\left|f_{h,\delta}(x,y)-f(x,y)\right| \\ &=\lambda_1+\lambda_2+\lambda_3.\end{aligned}$$

Firstly, we consider $\lambda_1=\omega_{p,q}(x,y)\left|L_{n,m}\left(f(t,z)-f_{h,\delta}(t,z);x,y\right)\right|$. By (2.6) and (3.7) we have

$$\begin{aligned}\lambda_1&\leq\left\|L_{m,n}\left(f-f_{h,\delta}\right)\right\|_{C_{\omega_{p,q}}} \\ &\leq M_4\left\|f-f_{h,\delta}\right\|_{C_{\omega_{p,q}}} \\ &\leq M_4\Omega(f;h,\delta),\end{aligned}$$

and for $\lambda_3$, we get

$$\begin{aligned}\lambda_3&=\omega_{p,q}(x,y)\left|f_{h,\delta}(x,y)-f(x,y)\right| \\ &\leq\left\|f_{h,\delta}-f\right\|_{C_{\omega_{p,q}}} \\ &\leq\Omega(f;h,\delta).\end{aligned}$$

Finally, for $\lambda_2$, Applying Theorem 1 and (3.8) and (3.9), we obtain

$$\begin{aligned}\lambda_2&=\omega_{p,q}(x,y)\left|L_{n,m}\left(f_{h,\delta}(t,z);x,y\right)-f_{h,\delta}(x,y)\right| \\ &\leq M_{11}\left\{\left\|\left(f_{h,\delta}\right)'_x\right\|_{C_{\omega_{p,q}}}\sqrt{\frac{x}{b_n}}+\left\|\left(f_{h,\delta}\right)'_y\right\|_{C_{\omega_{p,q}}}\sqrt{\frac{y}{d_m}}\right\}\end{aligned}$$

$$\leq M_{11}\left\{\frac{2}{h}\Omega\left(f;h,\delta\right)\sqrt{\frac{x}{b_n}}+\frac{2}{\delta}\Omega\left(f;h,\delta\right)\sqrt{\frac{y}{d_m}}\right\}$$
$$\leq \frac{2}{M_{11}}\Omega\left(f;h,\delta\right)\left\{\frac{1}{h}\sqrt{\frac{x}{b_n}}+\frac{1}{\delta}\sqrt{\frac{y}{d_m}}\right\}.$$

Consequently, for $(x,y)\in\mathbb{R}_0^2$, $m,n\in\mathbb{N}$ and $h,\delta>0$, there exists $M_{13}$ we get

$$\omega_{p,q}\left(x,y\right)\left|L_{n,m}\left(f;x,y\right)-f\left(x,y\right)\right|\leq M_{13}\Omega\left(f;h,\delta\right)\left\{1+\frac{1}{h}\sqrt{\frac{x}{b_n}}+\frac{1}{\delta}\sqrt{\frac{y}{d_m}}\right\}. \tag{3.10}$$

For fixed $x,y>0$, taking $h=\sqrt{\frac{x}{b_n}}$ and $\delta=\sqrt{\frac{y}{d_m}}$ in (3.10), we obtain desired result. □

From Theorem 2, we obtain following approximation theorem for $L_{n,m}$ operators defined by (1.3):

**Theorem 3.** *Let $f\in C_{\omega_{p,q}}$ and $p,q\in\mathbb{N}_0$. Then*

$$\lim_{n,m\to\infty}L_{n,m}\left(f;x,y\right)=f\left(x,y\right)\quad \textit{for all}\quad (x,y)\in\mathbb{R}_0^2. \tag{3.11}$$

*Moreover (3.11) holds uniformly on every rectangle $0\leq x\leq x_0,\ \ 0\leq y\leq y_0$ (see Example 1).*

*Remark* 1. Some particular cases of the operators (1.3) are defined as follows: If $\varphi_n\left(x\right)=e^{a_nx}$, $\varphi_m\left(y\right)=e^{c_my}$, then the operators $L_{n,m}\left(f;x,y\right)$ reduce to the Szász type operators of two variables defined by

$$L_{n,m}(f;x,y)=S_{n,m}(f;x,y)$$
$$=e^{-(a_nx+c_my)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_nx)^k}{k!}\frac{(c_my)^j}{j!}f(\frac{k}{b_n},\frac{j}{d_m}),\ (x,y)\in\mathbb{R}_0^2.$$

If $\varphi_n\left(x\right)=(1+x)^n$, $\varphi_m\left(y\right)=(1+y)^m$, then the operators $L_{n,m}\left(f;x,y\right)$ reduce to Bernstein Balazs type operators of two variables defined by

$$L_{n,m}(f;x,y)=R_{n,m}(f;x,y)$$
$$=\frac{1}{(1+a_nx)^n}\frac{1}{(1+c_my)^m}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}(a_nx)^k\left(c_my\right)^j f\left(\frac{k}{b_n},\frac{j}{d_m}\right),$$
$$(x,y)\in\mathbb{R}_0^2.$$

*Example* 1. For $\varphi_n\left(x\right)=e^{a_nx}$, $\varphi_m\left(y\right)=e^{c_my}$ and $a_n=\sqrt{n}+1, b_n=\sqrt{n}$ ; $c_m=\sqrt{m}+1$, $d_m=\sqrt{m}$; the convergence of $L_{n,m}(f;x,y)=S_{n,m}(f;x,y)$ to $f(x,y)=xye^{-(x+y)}$ will be illustrated following Fig.(1).

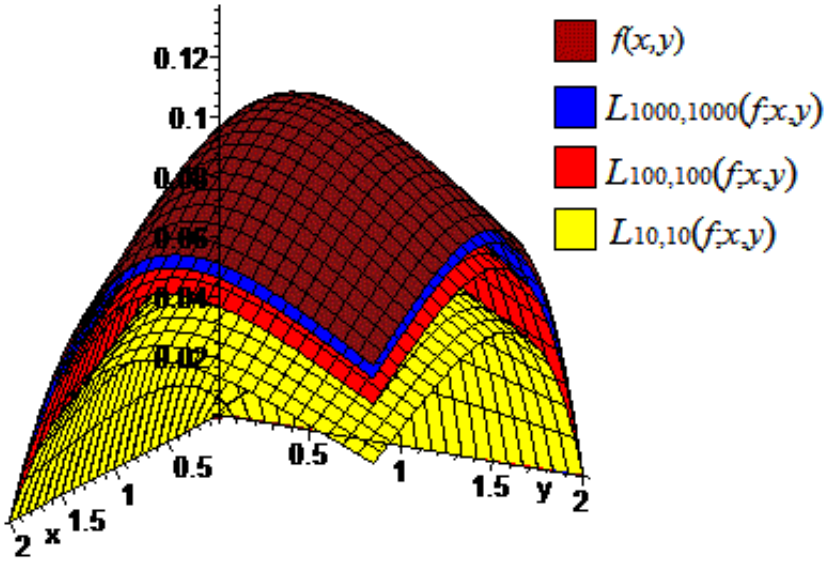


FIGURE 1. Convergenge of the $L_{n,m} = S_{n,m}$ operators for $n, m = 10, 100, 1000$.

*Example* 2. For $a_n = \sqrt{n}+1, b_n = \sqrt{n}$ ; $c_m = \sqrt{m}+1$, $d_m = \sqrt{m}$; the convergence of $S_{n,m}(f;x,y)$ to $f(x,y) = \frac{x}{1+x^2}e^{-y}$ and the convergence of $R_{n,m}(f;x,y)$ to $f(x,y) = \frac{x}{1+x^2}e^{-y}$ will be illustrated following Fig.(2).

Now, we give the Voronovskaya type theorem for the following operators given by (1.3) for $n = m$:

$$L_{n,n}(f(t,z);x,y) \tag{3.12}$$
$$= \frac{1}{\varphi_n(a_n x)} \frac{1}{\varphi_n(c_n y)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varphi_n^{(j)}(0)}{j!} (a_n x)^j \frac{\varphi_n^{(k)}(0)}{k!} (c_n y)^k f\left(\frac{j}{b_n}, \frac{k}{b_n}\right),$$

$(x,y) \in \mathbb{R}_0^2$, $n \in \mathbb{N}$, where $a_n = \dfrac{b_n}{n} \to 0$ and $b_n \to \infty$ and $\dfrac{\varphi_n^{(\upsilon)}(a_n x)}{n^{\upsilon}\varphi_n(a_n x)} = 1 + o\left(\dfrac{1}{b_n}\right)$.

**Theorem 4.** *Let $f \in C^2_{\omega_{p,q}}$ and $p, q \in \mathbb{N}_0$. Then for every $(x,y) \in \mathbb{R}_+^2 = (0,\infty) \times (0,\infty)$,*

$$\lim_{n\to\infty} b_n \{L_{n,n}(f;x,y) - f(x,y)\} = \frac{x}{2} f''_{xx}(x,y) + \frac{y}{2} f''_{yy}(x,y). \tag{3.13}$$

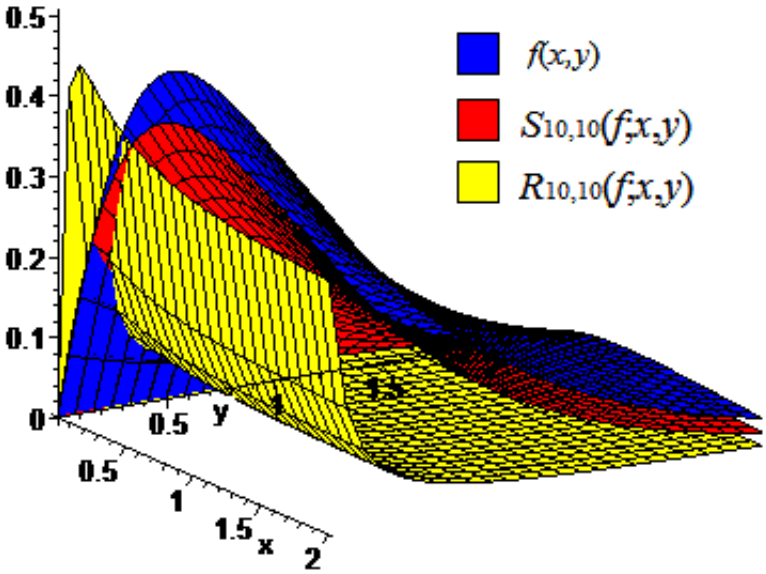


FIGURE 2. Convergenge comparation of the Szasz operators and Bernstein Balazs operators of two varibles for $n, m = 10$.

*Proof of Theorem 4.* Choosing $(x, y) \in \mathbb{R}_+^2$, by the Taylor formula for $f \in C_{\omega_{p,q}}^2$, we have

$$\begin{aligned} f(t,z) &= f(x,y) + f_x'(x,y)(t-x) + f_y'(x,y)(z-y) \\ &\quad + \frac{1}{2}\left\{f_{xx}''(x,y)(t-x)^2 + 2f_{xy}''(x,y)(t-x)(z-y) + f_{yy}''(z-y)^2\right\} \\ &\quad + \varepsilon_1(t,z)\sqrt{(t-x)^4 + (z-y)^4} \end{aligned}$$

where $\varepsilon_1(t,z) = \varepsilon_1(t,z;x,y)$ is a function belong to $C_{p,q}$ and $\varepsilon_1(x,y) = 0$. From this and by (1.1), (1.3), (1.5)-(2.1) and (3.12), we can write the following equality

$$\begin{aligned} &L_{n,n}(f(t,z);x,y) \\ &= f(x,y) + f_x'(x,y)L_n((t-x);x) + f_y'(x,y)L_n((z-y);y) \\ &+ \frac{1}{2}\left\{f_{xx}''(x,y)L_n\left((t-x)^2;x\right) + 2f_{xy}''(x,y)L_n((t-x);x)L_n((z-y);y)\right. \\ &\left. + f_{yy}''L_n\left((z-y)^2;y\right)\right\} + L_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4 + (z-y)^4};x,y\right) \end{aligned}$$

by using (2.2), (2.3) and Lemma 2, for $n \in \mathbb{N}$ we have

$$\lim_{n\to\infty} b_n\{L_{n,n}(f;x,y) - f(x,y)\} = \frac{x}{2}f_{xx}''(x,y) + \frac{y}{2}f_{yy}''(x,y) \tag{3.14}$$

$$+\lim_{n\to\infty} b_n L_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right).$$

in view of (1.1), (1.5)- (2.1) and using the Hölder inequality, we get

$$\begin{aligned}&\left|L_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right)\right| \qquad (3.15)\\ &\leq \left\{L_{n,n}\left(\varepsilon_1^2(t,z);x,y\right)\right\}^{1/2}\left\{L_{n,n}\left((t-x)^4+(z-y)^4;x,y\right)\right\}^{1/2}\\ &\leq \left\{L_{n,n}\left(\varepsilon_1^2(t,z);x,y\right)\right\}^{1/2}\left\{L_n\left((t-x)^4;x\right)+L_n\left((z-y)^4;y\right)\right\}^{1/2}\end{aligned}$$

From Theorem 3 , we obtain

$$\lim_{n\to\infty} L_{n,n}\left(\varepsilon_1^2(t,z);x,y\right)=\varepsilon_1^2(x,y)=0, \qquad (3.16)$$

and considering (3.15), (3.16) and Lemma 3 , we get

$$\lim_{n\to\infty} b_n L_{n,n}\left(\varepsilon_1(t,z)\sqrt{(t-x)^4+(z-y)^4};x,y\right)=0. \qquad (3.17)$$

Using the equality (3.17) in (3.14), the proof is completed. □

*Remark* 2. Recently many generalizations of well-known positive linear operators, based on $q-$integers were introduced and studied widely by several authors (we refer the reader to [1]). One can define and study $q-$analogue of the operators (1.3) .

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# SOME FIXED POINT RESULTS FOR $A$-CONTRACTIONS IN 2-METRIC SPACES AND THEIR APPLICATIONS

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*Abstract.* In this paper, we have given a criteria for the existence of common fixed point for pair of mappings in 2-metric spaces and establish common fixed point theorems for certain class of contractive type mappings. The results presented here, generalize some well known fixed point theorems given in the literature.

*Keywords:* 2-metric space, A-contractions, common fixed point, weakly compatible maps

## 1. Introduction

In the 1960's, Gahler introduced the notion of 2-metric space [9–11], whose abstract properties were suggested by the area of function in Euclidean space. Iseki [16] set out the tradition of proving fixed point theorems in 2-metric spaces employing various contractive conditions. Akram et al. [3, 4] have introduced a large class of mappings called $A$-contractions, which is a proper superclass of Kannan's [18], Bianchini [6] and Reich [19] type contractions. Ahmad [1], Debashis Dey and Mantu Saha [7, 20], Fathollahi [8] and Vishal Gupta [12] proved very useful common fixed point results in complete 2-metric spaces. Akram et al. [2, 5] gave results for self maps satisfying $A$-contractions in the setting of $G$-metric spaces. Some other results [13–15] in different type of metric spaces are also proved as a generalization of Banach contraction principal.

## 2. Preliminaries

**Definition 1** ([9])**.** Let $X$ be a non-empty set. A real valued function $d$ on $X \times X \times X$ is said to be a 2-metric on $X$ if,

(i) for given distinct elements $x, y$ of $X$, there exist an element $z$ of $X$ such that $d(x, y, z) \neq 0$;
(ii) $d(x, y, z) = 0$, when at least two of $x, y, z$ are equal;
(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z$ in $X$;
(iv) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w$ in $X$.

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When $d$ is a 2-metric on $X$, then the ordered pair $(X,d)$ is called a 2-metric space.

**Definition 2** ([16])**.** A sequence $\{x_n\}$ in 2-metric space $X$ is said to be a Cauchy sequence, if for each $a \in X$,

$$\lim_{n,m\to\infty} d(x_n, x_m, a) = 0.$$

**Definition 3** ([16])**.** A sequence $\{x_n\}$ in 2-metric space $X$ is said to be convergent to an element $x \in X$, if for each $a \in X$,

$$\lim_{n\to\infty} d(x_n, x, a) = 0.$$

**Definition 4** ([17])**.** Two mappings $A$, $S$ are said to be weakly compatible in $X$ if they commute at their coincidence points i.e if for some $x \in X$, $Ax = Sx$, then $ASx = SAx$.

Akram et al. [3] defined $A$-contractions as follows:

**Definition 5** ([3])**.** Let a non-empty set $A$ consisting of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying:

(i) $\alpha$ is continuous on the set $\mathbb{R}_+^3$ of all triplets of non-negative reals (with respect to the Euclidean metric on $\mathbb{R}^3$);

(ii) $a \leqslant kb$ for some $k \in [0,1)$, whenever $a \leqslant \alpha(a,b,b)$, $a \leqslant \alpha(b,a,b)$ or $a \leqslant \alpha(b,b,a)$ for all $a,b$.

**Definition 6** ([20])**.** A self map $T$ on a 2-metric space $X$ is said to be $A$-contraction if for each $u \in X$,

$$d(Tx,Ty,u) \leqslant \alpha\,[d(x,y,u), d(x,Tx,u), d(y,Ty,u)]$$

holds for any $x, y \in X$ and some $\alpha \in A$.

In the present paper, we are proving some fixed point theorems for $A$-contraction mapping in a 2-metric spaces.

## 3. Main result

**Theorem 1.** *The self map $T$ on a 2-metric space $X$ satisfying*

$$d(Tx,Ty,u) \leqslant \beta \max\{d(Tx,x,u) + d(Ty,y,u), d(Ty,y,u) + d(x,y,u), d(Tx,x,u) + d(x,y,u)\}$$

*for all $x,y,u$ in $X$ and some $\beta \in [0,\frac{1}{2})$, is an $A$-contraction.*

*Proof.* Define the map $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ as,

$$\alpha(u,v,w) \leqslant \beta \max\{u+v, v+w, u+w\} \tag{3.1}$$

for all $u,v,w$ in $\mathbb{R}_+$, where $\beta$ is any fixed number in $[0,\frac{1}{2})$. Then $\alpha \in A$, because

(1) $\alpha$ is continuous;

(2) for $u \leqslant \alpha(u,v,v) = \beta \max\{u+v, v+u, v+v\}$.

We consider the following cases:

**Case I:** Let $\max\{u+v, v+u, v+v\} = u+v$. In this case by the virtue of equation (3.1)

$$u \leqslant \beta(u+v),$$

that is,
$$u \leqslant kv \text{ with } k = \frac{\beta}{1-\beta} \in [0,1).$$

**Case II:** Let $\max\{u+v, v+u, v+v\} = v+v$. In this case $u \leqslant kv$, with $k = 2\beta \in [0,1)$.

Similarly, when we have either $u \leqslant \alpha(v,u,v)$ or $u \leqslant (v,v,u)$, then $u \leqslant kv$ for some $k \in [0,1)$. Hence,

$$\begin{aligned} d(Tx,Ty,u) &\leqslant \beta \max\{d(Tx,x,u)+d(Ty,y,u), d(Ty,y,u) \\ &\quad + d(x,y,u), d(Tx,x,u)+d(x,y,u)\} \\ &= \alpha\{d(x,y,u), d(Tx,x,u), d(Ty,y,u)\}. \end{aligned}$$

Thus $T$ is an $A$-contraction. □

**Theorem 2.** *Let $\alpha \in A$ and $\{T_n\}_{n=1}^{\infty}$ be a sequence of self maps on complete 2-metric space $(X,d)$ such that*

$$d(T_i x, T_j y, u) \leqslant \alpha\left[d(x,y,u), d(T_i x, x, u), d(T_j y, y, u)\right] \tag{3.2}$$

*for all $x,y,u \in X$. Then $\{T_n\}_{n=1}^{\infty}$ has a unique common fixed point in $X$.*

*Proof.* Consider $x_0 \in X$ arbitrarily, then for each $n \in \mathbb{N}$, we define $x_n = T_n x_{n-1}$. Now,

$$\begin{aligned} d(x_1,x_2,u) &= d(T_1x_0, T_2x_1, u) \\ &\leqslant \alpha\,[d(x_0,x_1,u), d(T_1x_0,x_0,u), d(T_2x_1,x_1,u)] \\ &= \alpha\,[d(x_0,x_1,u), d(x_1,x_0,u), d(x_2,x_1,u)] \\ &\leqslant k d(x_0,x_1,u), \end{aligned} \tag{3.3}$$

for some $k \in [0,1)$ and $\alpha \in A$.
Again,

$$\begin{aligned} d(x_2,x_3,u) &= d(T_2x_1, T_3x_2, u) \\ &\leqslant \alpha\,[d(x_1,x_2,u), d(T_2x_1,x_1,u), d(T_3x_2,x_2,u)] \\ &= \alpha\,[d(x_1,x_1,u), d(x_2,x_1,u), d(x_3,x_2,u)] \\ &\leqslant k d(x_1,x_2,u). \end{aligned} \tag{3.4}$$

We get from (3.3) and (3.4),

$$d(x_2,x_3,u) \leqslant k^2 d(x_0,x_1,u).$$

Proceeding in this way, we get

$$d(x_n, x_{n+1}, u) \leqslant k^n d(x_0, x_1, u). \tag{3.5}$$

Next we prove that

$$\begin{aligned} d(x_n, x_{n+2}, u) &\leqslant d(x_n, x_{n+2}, x_{n+1}) + d(x_n, x_{n+1}, u) + d(x_{n+1}, x_{n+2}, u) \\ &\leqslant d(x_n, x_{n+2}, x_{n+1}) + \sum_{r=0}^{1} d(x_{n+r}, x_{n+r+1}, u). \end{aligned} \tag{3.6}$$

Now,

$$\begin{aligned} d(x_n, x_{n+2}, x_{n+1}) &= d(x_{n+1}, x_{n+2}, x_n) = d(T_{n+1}(x_n), T_{n+2}(x_{n+1}), x_n) \\ &\leqslant \alpha\,[d(x_n, x_{n+1}, x_n), d(x_n, T_{n+1}(x_n), x_n), \\ &\quad\; d(x_{n+1}, T_{n+2}(x_{n+1}), x_n)] \\ &= \alpha\,[d(x_n, x_{n+1}, x_n), d(x_n, x_{n+1}, x_n), d(x_{n+1}, x_{n+2}, x_n)] \\ &\leqslant k\,d(x_n, x_{n+1}, x_n) \text{ for some } k \in [0, 1). \end{aligned}$$

Since $\alpha \in A$, so it follows that

$$d(x_n, x_{n+2}, x_{n+1}) = 0. \tag{3.7}$$

From (3.6) and (3.7), we get,

$$d(x_n, x_{n+2}, u) \leqslant \sum_{r=0}^{1} d(x_{n+r}, x_{n+r+1}, u).$$

Again, by using the property (iv) of Definition 1,

$$d(x_n, x_{n+3}, u) \leqslant \sum_{r=0}^{1} d(x_{n+3}, x_{n+r}, x_{n+r+1}) + \sum_{r=0}^{2} d(x_{n+r}, x_{n+r+1}, u).$$

Similarly, we can show that,

$$d(x_{n+3}, x_n, x_{n+1}) = 0 \quad \text{and} \quad d(x_{n+3}, x_{n+1}, x_{n+2}) = 0.$$

Hence, $d(x_n, x_{n+3}, u) \leqslant \sum_{r=0}^{2} d(x_{n+2}, x_{n+r+1}, u)$. Continuing in the same sense, we get,

$$d(x_n, x_{n+p}, u) \leqslant \sum_{r=0}^{p-1} d(x_{n+r}, x_{n+r+1}, u)$$

for any integer $p$. So we have, for any integer $p > 0$,

$$d(x_n, x_{n+p}, u) \leqslant \frac{k^n}{1-k} d(x_0, x_1, u).$$

Hence $\{x_n\}$ is a Cauchy sequence in $X$ and so by completeness of $X$, $\{x_n\}$ converges to a point $z \in X$.
Now,

$$\begin{aligned} d(x_{n+1}, T_n z, u) &= d(T_{n+1}(x_n), T_n(z), u) \\ &\leqslant \alpha\left[d(x_n, z, u), d(x_n, T_{n+1}(x_n), u), d(z, T_n z, u)\right] \\ &= \alpha\left[d(x_n, z, u), d(x_n, x_{n+1}, u), d(z, T_n z, u)\right]. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} d(z, T_n z, u) &\leqslant \alpha\left[d(z, z, u), d(z, z, u), d(z, T_n z, u)\right] \\ &\leqslant k\, d(z, z, u) \\ &= 0. \end{aligned}$$

This gives $T_n z = z$.
To prove the uniqueness, let $w$ be another fixed point of $T_n$ for all $n$, then

$$\begin{aligned} d(z, w, u) &= d(T_i z, T_j w, u) \\ &\leqslant \alpha\left[d(z, w, u), d(z, T_i z, u), d(w, T_j w, u)\right] \\ &= \alpha\left[d(z, w, u), d(z, z, u), d(w, w, u)\right] \\ &= \alpha\left[d(z, w, u), 0, 0\right] \\ &\leqslant k \cdot 0 \\ &= 0. \end{aligned}$$

This gives $z = w$. □

**Theorem 3.** *Let $X$ be a 2-metric space with 2-metrics $d$, and $\delta$ satisfying the following conditions,*

*(i)* $d(x, y, z) \leqslant \delta(x, y, z)$;
*(ii)* *$X$ is complete 2-metric space with respect to $d$;*
*(iii)* *$S, T$ are self maps on $X$ such that $T$ is continuous with respect to $d$ and*

$$\delta(Tx, Sy, u) \leqslant \alpha\left[\delta(x, y, u), \delta(x, Tx, u), \delta(y, \delta y, u)\right]$$

*for all $x, y, u \in X$ and for some $\alpha \in A$.*

*Then $S$ and $T$ have a unique common fixed point.*

*Proof.* For an arbitrary point $x_0 \in X$, define a sequence $\{x_n\} \subset X$ as follows:

$$Tx_{2n} = x_{2n+1}, \quad Sx_{2n+1} = x_{2n+2}.$$

For every non-negative integer $n$, and for all $u \in X$,

$$\delta(x_{2n}, x_{2n+1}, u) = \delta(Sx_{2n-1}, Tx_{2n}, u) = \delta(Tx_{2n}, Sx_{2n-1}, u),$$

then by the above inequality (iii)

$$\leqslant \alpha\left[\delta(x_{2n}, x_{2n-1}, u), \delta(x_{2n}, Tx_{2n}, u), \delta(x_{2n-1}, Sx_{2n-1}, u)\right]$$

$$\begin{aligned}&= \alpha\,[\delta(x_{2n}, x_{2n-1}, u), \delta(x_{2n}, x_{2n+1}, u), \delta(x_{2n-1}, x_{2n}, u)]\\&\leqslant k\delta(x_{2n-1}, x_{2n}, u). \qquad (3.8)\end{aligned}$$

Again consider,

$$\begin{aligned}\delta(x_{2n-1}, x_{2n}, u) &= \delta(Tx_{2n-2}, Sx_{2n-1}, u)\\&\leqslant \alpha\,[\delta(x_{2n-2}, x_{2n-1}, u), \delta(x_{2n-2}, Tx_{2n-2}, u),\\&\quad \delta(x_{2n-1}, Sx_{2n-1}, u)]\\&= \alpha\,[\delta(x_{2n-2}, x_{2n-1}, u), \delta(x_{2n-2}, x_{2n-1}, u),\\&\quad \delta(x_{2n-1}, x_{2n}, u)].\\\delta(x_{2n-1}, x_{2n}, u) &\leqslant k\delta(x_{2n-2}, x_{2n-1}, u) \qquad \text{for some } k \in [0,1). \qquad (3.9)\end{aligned}$$

From (3.8) and (3.9), we conclude

$$\delta(x_{2n}, x_{2n+1}, u) \leqslant k\delta(x_{2n-1}, x_{2n}, u) \leqslant k^2\delta(x_{2n-2}, x_{2n-1}, u).$$

Continuing in this way, we get

$$\delta(x_{2n}, x_{2n+1}, u) \leqslant k^{2n}\delta(x_0, x_1, u).$$

In general,

$$\delta(x_m, x_{m+1}, u) \leqslant k^m\delta(x_0, x_1, u) \text{ for all } u \in X. \qquad (3.10)$$

Now, we claim that $\delta(x_0, x_1, x_n) = 0$ for all $n \in \mathbb{N}$. Clearly it is true for $n = 0$ and $n = 1$. Based on mathematical induction, let it be true for $1 \leqslant n \leqslant l-1$, then

$$\begin{aligned}\delta(x_0, x_1, x_l) &\leqslant \delta(x_0, x_1, x_{l-1}) + \delta(x_0, x_{l-1}, x_l) + \delta(x_{l-1}, x_1, x_l)\\&= 0 + \delta(x_{l-1}, x_l, x_0) + \delta(x_{l-1}, x_l, x_1)\\&\leqslant k^{l-1}\,[\delta(x_0, x_1, x_0) + \delta(x_0, x_1, x_1)]\\&= 0.\end{aligned}$$

Thus we have

$$\delta(x_0, x_1, x_n) = 0 \text{ for all } n \in \mathbb{N}. \qquad (3.11)$$

Hence (3.10) and (3.11) shows $\delta(x_m, x_{m+1}, x_n) = 0$ for all $n, m \in \mathbb{N}$. Now, for $n > m$, we have

$$\begin{aligned}\delta(x_m, x_n, u) &\leqslant \delta(x_m, x_{m+1}, u) + \delta(x_m, x_{m+1}, x_n) + \delta(x_{m+1}, x_n, u)\\&= \delta(x_m, x_{m+1}, u) + \delta(x_{m+1}, x_n, u)\\&\leqslant \delta(x_m, x_{m+1}, u) + \delta(x_{m+1}, x_{m+2}, u) + \cdots + \delta(x_{n-1}, x_n, u)\\&\leqslant (k^m + k^{m+1} + \cdots + k^{n-1})\delta(x_0, x_1, u)\\&\leqslant k^m(1 + k + \cdots + k^{n-m-1})\delta(x_0, x_1, u)\\&\leqslant k^m\frac{(1-k^{n-m})}{1-k}\delta(x_0, x_1, u).\end{aligned}$$

Thus by condition (i),

$$d(x_n, x_m, u) \leqslant \delta(x_n, x_m, u) \leqslant \frac{k^m(1-k^{n-m})}{1-k}\delta(x_0, x_1, u),$$

for all $n, m \in \mathbb{N}$ with $k \in [0,1)$.

So $\{x_n\}$ is a Cauchy sequence in $X$ with respect to $d$ and hence by condition (ii), $d(x_n, x', u) \to 0$ for some $x' \in X$ and for each $u \in X$.
Since $T$ is given to be continuous with respect to $d$, we have,

$$0 = \lim_{n\to\infty} d(x_{2n+1}, x', u) = \lim_{n\to\infty} d(Tx_{2n}, x', u) = d(Tx', x', u)$$

for all $u \in X$. So $Tx' = x'$.

Now by condition (iii), for each $u \in X$

$$\begin{aligned}
\delta(x', Sx', u) &= \delta(Tx', Sx', u) \\
&\leqslant \alpha\left[\delta(x', x', u), \delta(x', Tx', u), \delta(x', Sx', u)\right] \\
&\leqslant \alpha\left[0, 0, \delta(x', Sx', u).\right] \\
&< k \cdot 0 \\
&= 0.
\end{aligned}$$

Hence, $x' = Sx'$.

Thus $x'$ is a common fixed point of $S$ and $T$. For the uniqueness, let $y$ be another common fixed point of $S$ and $T$ in $X$.
Then by condition (iii)

$$\begin{aligned}
\delta(x', y, u) &= \delta(Tx', Sy, u) \\
&\leqslant \alpha\left[\delta(x', y, u), \delta(x', Tx', u), \delta(y, Sy, u)\right] \\
&\leqslant 0.
\end{aligned}$$

Thus, $x' = y$. □

## 4. Applications

Here, we give some applications related to our results. For this we use a Lebesgue integrable function as a summable for each compact $R^+$.
Let us define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \int_0^t \varphi(t)$, $\forall t > 0$ be a non-decreasing and continuous function. Moreover, for each $\epsilon > 0$, $\varphi(\epsilon) > 0$ and $\varphi(t) = 0$ iff $t = 0$.

**Theorem 4.** *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of self maps on the complete 2-metric space $(X,d)$ satisfying the following condition:*

$$\int_0^{d(T_i x, T_j y, u)} \varphi(t)\,dt \tag{4.1}$$
$$\leq \alpha\left(\int_0^{d(x,y,u)} \varphi(t)\,dt, \int_0^{d(T_i x, x, u)} \varphi(t)\,dt, \int_0^{d(T_j y, y, u)} \varphi(t)\,dt\right)$$

*for some $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which is summable(i.e with finite integral) on each compact subset of $[0,\infty)$, non-negative and such that for each $\epsilon > 0$,*

$$\int_0^{\epsilon} \varphi(t)\,dt > 0. \tag{4.2}$$

*Then $\{T_n\}_{n=1}^{\infty}$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be arbitrary and for each $n \in N$ define $x_n = T_n x_{n-1}$. Now

$$\int_0^{d(x_1,x_2,u)} \varphi(t)\,dt = \int_0^{d(T_1x_0, T_2x_1, u)} \varphi(t)\,dt$$
$$\leq \alpha\left(\int_0^{d(x_0,x_1,u)} \varphi(t)\,dt, \int_0^{d(T_1x_0, x_0, u)} \varphi(t)\,dt, \int_0^{d(T_2x_1, x_1, u)} \varphi(t)\,dt\right)$$
$$= \alpha\left(\int_0^{d(x_0,x_1,u)} \varphi(t)\,dt, \int_0^{d(x_1, x_0, u)} \varphi(t)\,dt, \int_0^{d(x_2, x_1, u)} \varphi(t)\,dt\right).$$

By definition of $\alpha$, we get

$$\int_0^{d(x_1,x_2,u)} \varphi(t) \leq k \int_0^{d(x_0,x_1,u)} \varphi(t) \tag{4.3}$$

for some $k \in [0,1)$ as $\alpha \in A$.

Again,

$$\int_0^{d(x_2,x_3,u)} \varphi(t) = \int_0^{d(T_2x_1, T_3x_2, u)} \varphi(t)$$
$$\leq \alpha\left(\int_0^{d(x_1,x_2,u)} \varphi(t)\,dt, \int_0^{d(T_2x_1, x_1, u)} \varphi(t)\,dt, \int_0^{d(T_3x_2, x_2, u)} \varphi(t)\,dt\right)$$
$$= \alpha\left(\int_0^{d(x_1,x_2,u)} \varphi(t)\,dt, \int_0^{d(x_2, x_1, u)} \varphi(t)\,dt, \int_0^{d(x_3, x_2, u)} \varphi(t)\,dt\right).$$

Therefore,

$$\int_0^{d(x_2,x_3,u)} \varphi(t) \leq k \int_0^{d(x_1,x_2,u)} \varphi(t), \tag{4.4}$$

from equation (4.3) and (4.4), we get

$$\int_0^{d(x_2,x_3,u)} \varphi(t) \leq k^2 \int_0^{d(x_0,x_1,u)} \varphi(t).$$

Proceeding in the same way, we have

$$\int_0^{d(x_n,x_{n+1},u)} \varphi(t) \leq k^n \int_0^{d(x_0,x_1,u)} \varphi(t). \tag{4.5}$$

Taking limit as $n \to \infty$, we get $\lim_n \int_0^{d(x_n,x_{n+1},u)} \varphi(t)\,dt = 0, \quad$ as $k \in [0,1)$,

which from (4.2) implies that $\lim_n d(x_n, x_{n+1}, u) = 0$.

We now show that $\{x_n\}$ is a Cauchy sequence.

Consider,

$d(x_n, x_{n+2}, u) \leq d(x_n, x_{n+2}, x_{n+1}) + d(x_n, x_{n+1}, u) + d(x_{n+1}, x_{n+2}, u)$.

Therefore,

$$d(x_n, x_{n+2}, u) \leq d(x_n, x_{n+2}, x_{n+1}) + \sum_{r=0}^{1} d(x_{n+r}, x_{n+r+1}, u). \tag{4.6}$$

Now

$$\begin{aligned}\int_0^{d(x_n,x_{n+2},x_{n+1})} \varphi(t)\,dt &= \int_0^{d(x_{n+1},x_{n+2},x_n)} \varphi(t)\,dt \\ &= \int_0^{d(T_{n+1}(x_n),T_{n+2}(x_{n+1}),x_n)} \varphi(t)\,dt \\ &\leq \alpha \left( \int_0^{d(x_n,x_{n+1},x_n)} \varphi(t)\,dt, \right.\end{aligned}$$

$$\int_0^{d(x_n,T_{n+1}(x_n),x_n)} \varphi(t)\,dt, \int_0^{d(x_{n+1},T_{n+2}(x_{n+1}),x_n)} \varphi(t)\,dt\Bigg)$$

$$= \alpha\left(\int_0^{d(x_n,x_{n+1},x_n)} \varphi(t)\,dt, \int_0^{d(x_n,x_{n+1},x_n)} \varphi(t)\,dt, \int_0^{d(x_{n+1},x_{n+2},x_n)} \varphi(t)\,dt\right)$$

$$\leq k\int_0^{d(x_n,x_{n+1},x_n)} \varphi(t)\,dt \quad for\, k \in [0,1) \;\; and\;\; \alpha \in A.$$

It follows that $\int_0^{d(x_n,x_{n+2},x_{n+1})} \varphi(t)\,dt = 0$,

and by (4.2), $d(x_n, x_{n+2}, x_{n+1}) = 0$.

Again, by using the property (iv) of Definition 1,

$d(x_n, x_{n+3}, u) = \sum_{r=0}^{1} d(x_{n+3}, x_{n+r}, x_{n+r+1}) + \sum_{r=0}^{2} d(x_{n+r}, x_{n+r+1}, u)$.

As above, we can show that $d(x_{n+3}, x_n, x_{n+1}) = 0$ and $d(x_{n+3}, x_{n+1}, x_{n+2}) = 0$, hence

$d(x_n, x_{n+3}, u) = \sum_{n=0}^{2} d(x_{n+r}, x_{n+r+1}, u)$.

Continuing in the same manner, we get for any integer p,

$d\left(x_n, x_{n+p}, u\right) \leq \sum_{r=0}^{p-1} d(x_{n+r}, x_{n+r+1}, u) \leq \frac{k^n}{1-k} d(x_0, x_1, u) \to 0$

as $n \to \infty$, since $k \in [0,1)$.

Hence $\{x_n\}$ is a Cauchy sequence in $X$ and so by completeness of $X$, $\{x_n\}$ converges to a point $z \in X$.

Now, $\int_0^{d(x_{n+1},T_n z,u)} \varphi(t)\,dt = \int_0^{d(T_{n+1}(x_n),T_n z,u)} \varphi(t)\,dt$

$\leq \alpha\left(\int_0^{d(x_n,z,u)} \varphi(t)\,dt, \int_0^{d(x_n,T_{n+1}(x_n),u)} \varphi(t)\,dt, \int_0^{d(z,T_n z,u)} \varphi(t)\,dt\right)$

$= \alpha\left(\int_0^{d(x_n,z,u)} \varphi(t)\,dt, \int_0^{d(x_n,x_{n+1},u)} \varphi(t)\,dt, \int_0^{d(z,T_n z,u)} \varphi(t)\,dt\right)$.

Taking limit as $n \to \infty$, we have

$\int_0^{d(z,T_n z,u)} \varphi(t)\,dt$

$\leq \alpha\left(\int_0^{d(z,z,u)}\varphi(t)\,dt, \int_0^{d(z,z,u)}\varphi(t)\,dt, \int_0^{d(z,T_nz,u)}\varphi(t)\,dt\right)$
$\leq k\int_0^{d(z,z,u)}\varphi(t)$

for some $k \in [0,1)$ as $\alpha \in A \Rightarrow \int_0^{d(z,T_nz,u)}\varphi(t)\,dt = 0$,

and by (4.2), $d(z,T_nz,u) = 0$ and thus $T_nz = z \ \ \forall n$.

To prove uniqueness of $z$, let $w$ be another fixed point of $T_n$, Then

$\int_0^{d(z,w,u)}\varphi(t)\,dt = \int_0^{d(T_iz,T_jw,u)}\varphi(t)\,dt$

$\leq \alpha\left(\int_0^{d(z,w,u)}\varphi(t)\,dt, \int_0^{d(z,T_iz,u)}\varphi(t)\,dt, \int_0^{d(w,T_jw,u)}\varphi(t)\,dt\right)$
$= \alpha\left(\int_0^{d(z,w,u)}\varphi(t)\,dt, \int_0^{d(z,z,u)}\varphi(t)\,dt, \int_0^{d(w,w,u)}\varphi(t)\,dt\right)$
$= \alpha\left(\int_0^{d(z,w,u)}\varphi(t)\,dt, 0, 0\right) \leq k.0 = 0$

for some $k \in [0,1)$ as $\alpha \in A$. This gives $z = w$. □

**Theorem 5.** *Let $X$ be a set with 2-metrics $d$ and $\delta$ satisfying the following conditions:*

*(i)* $d(x,y,z) \leq \delta(x,y,z)$;
*(ii)* *$X$ is complete 2-metric space with respect to $d$;*
*(iii)* *$S$ and $T$ are self maps on $X$ such that $T$ is continuous with respect to $d$ and satisfying*

$$\int_0^{\delta(Tx,Sy,u)}\varphi(t)\,dt \leq \alpha\left[\int_0^{\delta(x,y,u)}\varphi(t)\,dt, \int_0^{\delta(x,Tx,u)}\varphi(t)\,dt, \int_0^{\delta(y,Sy,u)}\varphi(t)\,dt\right] \tag{4.7}$$

*for all $x,y,u \in X$ and for some $\alpha \in A$, where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable (i.e finite integral) on each compact subset of $[0,\infty)$, non-negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon}\varphi(t)\,dt > 0$.*
*Then $S$ and $T$ have a unique common fixed point.*

*Proof.* For an arbitrary point $x_0 \in X$ , define a sequence $\{x_n\} \subset X$ as follows:-

$Tx_{2n} = x_{2n+1}$, $Sx_{2n+1} = x_{2n+2}$ for every non-negative integer $n$ and for all $u \in X$.

Now $\int_0^{\delta(x_{2n},x_{2n+1},u)}\varphi(t)\,dt = \int_0^{\delta(Sx_{2n-1},Tx_{2n},u)}\varphi(t)\,dt$

$= \int_0^{\delta(Tx_{2n},Sx_{2n-1},u)} \varphi(t)\,dt$

$\leq \alpha\left[\int_0^{\delta(x_{2n},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n},Tx_{2n},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n-1},Sx_{2n-1},u)} \varphi(t)\,dt\right]$

$\leq \alpha\left[\int_0^{\delta(x_{2n},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt\right].$

Thus,

$$\int_0^{\delta(x_{2n},x_{2n+1},u)} \varphi(t)\,dt \leq k\int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt \tag{4.8}$$

for some $k \in [0,1)$ as $\alpha \in A$.

Consider, $\int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt = \int_0^{\delta(Tx_{2n-2},Sx_{2n-1},u)} \varphi(t)\,dt$

$\leq \alpha\left[\int_0^{\delta(x_{2n-2},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n-2},Tx_{2n-2},u)} \varphi(t)\,dt,\right.$
$\left.\int_0^{\delta(x_{2n-1},Sx_{2n-1},u)} \varphi(t)\,dt\right]$

$\leq \alpha\left[\int_0^{\delta(x_{2n-2},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n-2},x_{2n-1},u)} \varphi(t)\,dt, \int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt\right].$

By definition of function $\alpha$,

$$\int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt \leq k\int_0^{\delta(x_{2n-2},x_{2n-1},u)} \varphi(t)\,dt \tag{4.9}$$

for some $k \in [0,1)$.

From (4.8) and (4.9), we conclude that

$\int_0^{\delta(x_{2n},x_{2n+1},u)} \varphi(t)\,dt \leq k\int_0^{\delta(x_{2n-1},x_{2n},u)} \varphi(t)\,dt \leq k^2\int_0^{\delta(x_{2n-2},x_{2n-1},u)} \varphi(t)\,dt.$

Proceeding in the same way, we get

$\int_0^{\delta(x_{2n},x_{2n+1},u)} \varphi(t)\,dt \leq k^{2n}\int_0^{\delta(x_0,x_1,u)} \varphi(t)\,dt.$

In general,

$$\int_0^{\delta(x_m,x_{m+1},u)} \varphi(t)\,dt \leq k^m\int_0^{\delta(x_0,x_1,u)} \varphi(t)\,dt, \quad \forall u \in X. \tag{4.10}$$

Now we claim that $\delta(x_0,x_1,x_n) = 0$ for all $n \in N$. Clearly it is true for $n = 0$ and $n = 1$, let it be true for $1 \leq n \leq l-1$, then

$\delta(x_0,x_1,x_l) \leq \delta(x_0,x_1,x_{l-1}) + \delta(x_0,x_{l-1},x_l) + \delta(x_{l-1},x_1,x_l)$

$= 0 + \delta(x_{l-1}, x_l, x_0) + \delta(x_{l-1}, x_l, x_1)$

$\leq k^{l-1}[\delta(x_0, x_1, x_0) + \delta(x_0, x_1, x_1)].$

Thus we have,

$$\delta(x_0, x_1, x_n) = 0 \quad for \ \ all \ \ n \in N. \tag{4.11}$$

Hence (4.10) and (4.11) implies, $\delta(x_m, x_{m+1}, x_n) = 0$.

Consider, $\int_0^{\delta(x_m,x_n,u)} \varphi(t)\,dt$
$\leq \int_0^{\delta(x_m,x_{m+1},u)} \varphi(t)\,dt + \int_0^{\delta(x_m,x_{m+1},x_n)} \varphi(t)\,dt + \int_0^{\delta(x_{m+1},x_n,u)} \varphi(t)\,dt$

$= \int_0^{\delta(x_m,x_{m+1},u)} \varphi(t)\,dt + \int_0^{\delta(x_{m+1},x_n,u)} \varphi(t)\,dt$

$\leq \int_0^{\delta(x_m,x_{m+1},u)} \varphi(t)\,dt + \int_0^{\delta(x_{m+1},x_{m+2},u)} \varphi(t)\,dt + \ldots + \int_0^{\delta(x_{n-1},x_n,u)} \varphi(t)\,dt$

$\leq \left(k^m + k^{m-1} + \ldots + k^{n-1}\right) \int_0^{\delta(x_0,x_1,u)} \varphi(t)\,dt$

$= \frac{k^m(1-k^{n-m})}{1-k} \int_0^{\delta(x_0,x_1,u)} \varphi(t)\,dt \quad for \ some \ \ k \in [0,1).$

Taking $n, m \to \infty$, we have $\int_0^{\delta(x_n,x_m,u)} \varphi(t)\,dt = 0$, and then $\delta(x_n, x_m, u) = 0$ as $n, m \to \infty$.

Thus by condition (i), $d(x_n, x_m, u) \leq \delta(x_n, x_m, u) \to 0 \ \ as \ n, m \to \infty$.

So $\{x_n\}$ is a Cauchy sequence in $X$ with respect to $d$ and hence $d\left(x_n, x', u\right) \to 0$ for some $x' \in X$ as $n \to \infty$ and for each $u \in X$. Since $T$ is given to be continuous with respect to $d$, we have

$0 = \lim_{n\to\infty} d\left(x_{2n+1}, x', u\right) = \lim_{n\to\infty} d\left(Tx_{2n}, x', u\right) = d\left(Tx', x', u\right)$ for all $u \in X$, so that $Tx' = x'$.

From condition (iii), for each $u \in X$ we have,

$\int_0^{\delta\left(x', Sx', u\right)} \varphi(t)\,dt = \int_0^{\delta\left(Tx', Sx', u\right)} \varphi(t)\,dt$

$\leq \alpha \left[ \int_0^{\delta\left(x', x', u\right)} \varphi(t)\,dt, \int_0^{\delta\left(x', Tx', u\right)} \varphi(t)\,dt, \int_0^{\delta\left(x', Sx', u\right)} \varphi(t)\,dt \right]$

$$\leq \alpha\left[0,0,\int_0^{\delta\left(x',Sx',u\right)}\varphi(t)\right] \leq k.0 = 0 \quad for\, some\, k \in [0,1)$$

and hence $\delta\left(x',Sx',u\right) = 0$ for all $u \in X$.

Therefore $x' = Sx'$. Thus $x'$ is the common fixed point of $S$ and $T$.

For uniqueness, let $y$ be any other common fixed point of $S$ and $T$ in $X$, then by condition (iii),

$$\int_0^{\delta\left(x',y,u\right)}\varphi(t)\,dt = \int_0^{\delta\left(Tx',Sy,u\right)}\varphi(t)\,dt$$
$$\leq \alpha\left[\int_0^{\delta\left(x',y,u\right)}\varphi(t)\,dt, \int_0^{\delta\left(x',Tx',u\right)}\varphi(t)\,dt, \int_0^{\delta(y,Sy,u)}\varphi(t)\,dt\right].$$

Hence $x' = y$. □

*Example* 1. Consider $X = \{1,2,3,4\}$ and define $d : X \times X \times X \to R$ by $d(x,y,z) = 0$ for at least two of any $x,y,z$ are zero and $d(x,y,z) = d(y,x,z) = d(z,y,x)$ for $x \neq y \neq z$ be such that $d(1,2,3) = 6, d(1,2,4) = 7, d(1,3,4) = 8, d(2,3,4) = 9$. Then $(X,d)$ is 2-metric space.
Let $T : X \to X$ be defined by $T(1) = 2, T(2) = 3, T(3) = 4, T(4) = 1$.
Then, clearly the condition of $A$-contraction
$d(Tx,Ty,u) \leq \alpha[d(x,y,u), d(x,Tx,u), d(y,Ty,u)]$ is not satisfied.

Take $x = 1, y = 2$ and $u = 4$ L.H.S of condition becomes $d(T(1),T(2),4) = d(2,3,4) = 9$ and R.H.S is

$$\alpha[d(1,2,4), d(1,T(1),4), d(2,T(2),4)] = \alpha[7,7,9].$$

Then $9 \leq k.7$, this is a contradiction, as $k \in [0,1]$. Hence $T$ is not an $A$-contraction and $T$ has no fixed point.

*Example* 2. Consider 2-metric space as in Example 1, and define
$T(1) = 1, T(2) = 1, T(3) = 1, T(4) = 1$. Clearly, the map $T$ satisfies all the conditions of Theorem 1, in all cases. Therefore $T$ is an $A$-contraction.

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# TAUBERIAN CONDITIONS UNDER WHICH $\lambda$-STATISTICAL CONVERGENCE FOLLOWS FROM STATISTICAL SUMMABILITY $(V,\lambda)$

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*Abstract.* In this paper, we will show Tauberian conditions under which $\lambda-$statistical convergence follows from $(V,\lambda)-$statistical convergence. Our results generalize the ones given in [3].

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*Keywords:* statistical convergence, $\lambda-$statistical convergence, statistical summability $(V,\lambda)$, one-sided and two-sided Tauberian conditions

## 1. Introduction

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of $K$ is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [1, 6]), i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely.

The idea of $\lambda$-statistical convergence was introduced in [5] as follows: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, n - 1 < \lambda_n$. The generalized de la Vallée-Poussin mean is defined by $T_n(x) =: \frac{1}{\lambda_n} \sum_{j\in I_n} x_j$, where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_j)$ is said to be $(V,\lambda)$-summable to a number $L$ (see [2]) if $T_n(x) \to L$ as $n \to \infty$. In this case $L$ is called the $\lambda$-limit of $x$. And we say that $x = (x_n)$ is $\lambda$-statistical convergent to $L$, if

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

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for every given $\epsilon > 0$. And will write $st_\lambda - \lim_n x_n = L$.

In paper of Mursaleen et al. [4], the definition of the statistically $\lambda-$ convergent sequences was given as follows:

A sequence $x = (x_n)$ is said to be statistically $\lambda-$convergent to $L$ if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{n} |\{k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.1}$$

holds. In this case we write that $st - \lim_n T_n = L$.

In what follows we will define the following type of the statistical convergence. A sequence $x = (x_n)$ is said to be $(V, \lambda)-$statistically convergent to $L$ if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.2}$$

holds. In this case we write that $st_\lambda - \lim_n T_n = L$.

In the sequel we will show conditions under which for every bounded sequence $(x_k)$ the implication

$$st_\lambda - \lim_k x_k = L \quad \text{implies} \quad st_\lambda - \lim_k T_k = L$$

holds.

**Theorem 1.** *Let us suppose that $(x_k)$ is a bounded sequence such that exists $st_\lambda - \lim_k x_k = L$, then it follows that $st_\lambda - \lim_k T_k = L$, but not conversely.*

*Proof.* Let us suppose that $st_\lambda - \lim_k x_k = L$. Let $\epsilon > 0$ be any given number, and $B_\epsilon = \{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}$. Then

$$|T_k(x) - L| = \left| \frac{1}{\lambda_k} \sum_{j \in I_k} x_j - L \right| = \left| \frac{1}{\lambda_k} \sum_{j \in I_k} (x_j - L) \right| \leq \left| \frac{1}{\lambda_k} \sum_{j \in B_\epsilon} (x_j - L) \right| \leq$$

$$\frac{1}{\lambda_k} (\sup_j |x_j - L|) \cdot B_\epsilon \to 0,$$

as $k \to \infty$. Hence $T_k(x) \to L$, as $k \to \infty$, respectively $st_\lambda - \lim_k T_k = L$.

*Example* 1. Let us consider that $\lambda_n = n$ and $x = (x_n)$ defined as follows:

$$x_n = \begin{cases} 1 & \text{if} \quad k \quad \text{is odd} \\ -1 & \text{if} \quad k \quad \text{is even} \end{cases}$$

Of course this sequence is not $st_\lambda$- convergent. On the other hand, $x$ is $(V, \lambda)$-summable to 0 and hence $(V, \lambda)-$statistically convergent to 0. □

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for the defined type of convergence. Exactly, we will prove under which conditions $\lambda-$statistical convergence follows from $(V,\lambda)$-statistically convergence. This method generalize the method given in [3], as it is shown by the following example.

*Example* 2. In case where $\lambda_n = n$, then $(V,\lambda)-$ summability method is the Cesaro summability method $(C,1)$ as given in [3].

## 2. MAIN RESULTS

**Theorem 2.** *Let $(\lambda_n)$ be a sequence of real numbers defined as above and*

$$st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} > 1, \quad t > 1 \tag{2.1}$$

*where $t_n$, denotes the integral parts of the $[tn]$ for every $n \in \mathbb{N}$, and let $(T_k)$ be a sequence of real numbers such that $st_\lambda - \lim_k T_k = L$. Then $(x_k)$ is $st_\lambda-$ convergent to the same number $L$ if and only if the following to conditions holds:*

$$\inf_{t>1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| = 0 \tag{2.2}$$

*and*

$$\inf_{0<t<1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} (x_k - x_j) \leq -\epsilon \right\} \right| = 0. \tag{2.3}$$

*Remark* 1. Let us suppose that $st_\lambda - \lim_k x_k = L$; $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies, then for every $t > 1$, the following relation is valid:

$$st_\lambda - \lim_n \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} x_j = 0 \tag{2.4}$$

and in case where $0 < t < 1$,

$$st_\lambda - \lim_n \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} x_j = 0. \tag{2.5}$$

In what follows we will show some auxiliary lemmas which are going to be used in the sequel.

**Lemma 1.** *For the sequence of real numbers $\lambda = (\lambda_n)$, condition given by relation (2.1) is equivalent to this one:*

$$st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} > 1, \quad 0 < t < 1. \tag{2.6}$$

*Proof.* Let us suppose that relation (2.1) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n$$

now by the nondecreasing property of the sequence $\lambda = (\lambda_n)$, we get:

$$\frac{\lambda_n}{\lambda_{t_n}} \geq \frac{\lambda_{[\frac{m}{t}]}}{\lambda_m} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} \geq st_\lambda - \liminf_n \frac{\lambda_{[\frac{m}{t}]}}{\lambda_{t_n}} > 1.$$

Conversely, let use suppose that relation (2.6) is valid. Let $t > 1$ be given and let $t_1$ be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is a case where if $n$ is large enough. Under this conditions we have:

$$\frac{\lambda_{t_n}}{\lambda_n} \geq \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} \geq st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} > 1.$$

□

**Lemma 2.** *Let us suppose that relation (2.1) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is $(V, \lambda)-$statistically convergent to $L$. Then for every $t > 0$,*

$$st_\lambda - \lim_n T_{t_n} = L.$$

*Proof.* Let us consider that $t > 1$, then from construction of the sequence $\lambda = (\lambda_n)$, we get:

$$\lim_n (n - \lambda_n) = \lim_n (t_n - \lambda_{t_n}), \tag{2.7}$$

and for every $\epsilon > 0$ we have:

$$\{k \in I_{t_n} : |T_{t_k} - L| \geq \epsilon\} \subset \{k \in I_n : |T_k - L| \geq \epsilon\} \cup$$
$$\left\{k \in I_n : \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^{k} x_j \neq \frac{1}{\lambda_{t_k}} \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j\right\}.$$

Now proof of the lemma in this case follows from relation (2.7) and $st_\lambda - \lim_n T_{t_n} = L$.

(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number $n$, we can conclude that $(T_{t_n})$ does not appears more than $[1 + t^{-1}]$ times in the sequence $(T_n)$. In fact if there exists a integers $k, l$ such that

$$n \leq t \cdot k < t(k+1) < \cdots < t(k+l-1) < n+1 \leq t(k+l),$$

then

$$n+t(l-1)\leq t(k+l-1)<n+1\Rightarrow l<1+\frac{1}{t}.$$

And we have this estimation:

$$\frac{1}{\lambda_{t_n}}\left|\{k\in I_{t_n}:|T_{t_k}-L|\geq\epsilon\}\right|\leq\left(1+\frac{1}{t}\right)\frac{1}{\lambda_{t_n}}|\{k\in I_n:|T_k-L|\geq\epsilon\}|\leq$$

$$2(1+t)\frac{1}{\lambda_n}|\{k\in I_n:|T_k-L|\geq\epsilon\}|.$$

From the last relation it follows that $st_\lambda-\lim_n T_{t_n}=L$. □

**Lemma 3.** *Let us suppose that relation (2.1) is satisfied and let $x=(x_k)$ be a sequence of complex numbers which is $(V,\lambda)-$statistically convergent to $L$. Then for every $t>1$,*

$$st_\lambda-\lim_n(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=L; \tag{2.8}$$

*and for every $0<t<1$,*

$$st_\lambda-\lim_n(\lambda_n-\lambda_{t_n})^{-1}\sum_{j=t_n+1}^{n}x_j=L. \tag{2.9}$$

*Proof.* Let us suppose that $t>1$. After some calculations we get:

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=T_n+\lambda_{t_n}(\lambda_{t_n}-\lambda_n)^{-1}(T_{t_n}-T_n)+$$

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n-\lambda_n+1}^{t_n}x_j-(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j,$$

respectively

$$(\lambda_{t_n}-\lambda_n)^{-1}\sum_{j=n+1}^{t_n}x_j=T_n+\lambda_{t_n}(\lambda_{t_n}-\lambda_n)^{-1}(T_{t_n}-T_n)+$$

$$(\lambda_{t_n}-\lambda_n)^{-1}\left(\sum_{j=n-\lambda_n+1}^{t_n}x_j-\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j\right). \tag{2.10}$$

From definition of the sequence $\lambda=(\lambda_n)$, we obtain

$$st_\lambda-\limsup_n\sum_{j=n-\lambda_n+1}^{t_n}x_j=st_\lambda-\limsup_n\sum_{j=t_n-\lambda_{t_n}+1}^{t_n}x_j. \tag{2.11}$$

Really, let us suppose that $st_\lambda - \lim_n \sup \sum_{j=n-\lambda_n+1}^{t_n} x_j = L$, and for every $\epsilon > 0$, we get:

$$\frac{\left|\left\{k \in I_{t_n} : \left|\sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j - L\right| \geq \epsilon\right\}\right|}{\lambda_{t_n}}$$
$$\leq \frac{\left|\left\{k \in I_n : \left|\sum_{j=k-\lambda_k+1}^{t_k} x_j - L\right| \geq \epsilon\right\}\right|}{\lambda_n}$$
$$+ \frac{\left|\left\{k \in I_n : \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j \neq \sum_{j=k-\lambda_k+1}^{t_k} x_j\right\}\right|}{\lambda_n}.$$

The first summand in the right side of the inequality tends statistically to zero as $n \to \infty$ and second summands tends to zero, too (from relations (2.7)). And this means that $st_\lambda - \lim_n \sup \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j = L$.

Since by (2.1)

$$st_\lambda - \limsup_n \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1} < \infty, st_\lambda - \limsup_n (\lambda_{t_n} - \lambda_n)^{-1} < \infty, \tag{2.12}$$

now relation (2.8) follows from (2.10), (2.11), (2.12), Lemma 2 and statistical convergence of $T_n$.

Case where $0 < t < 1$. In this case we have:

$$(\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} x_j = T_n + \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) +$$

$$(\lambda_n - \lambda_{t_n})^{-1} \sum_{j=n-\lambda_n+1}^{n} x_j - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=t_n-\lambda_{t_n}+1}^{n} x_j.$$

Following Lemma 2 and the conclusions like in the previous case, we get that relation (2.9) is valid. □

In what follows we will prove Theorem 1.

*Proof of Theorem 2.* Necessity. Let us suppose that $st_\lambda - \lim_k x_k = L$, and $st_\lambda - \lim_k T_k = L$. For every $t > 1$ following Lemma 2, we get relation (2.2) and in case where $0 < t < 1$, again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that $st_\lambda - \lim_n T_n = L$, and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st_\lambda - \lim_n x_n = L$. Or equivalently, $st_\lambda - \lim_n (T_n - x_n) = 0$.

First we consider the case where $t > 1$. We will start from this estimation:

$$x_n - T_n = \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n).$$

For any $\epsilon > 0$, we obtain:

$$\{k \in I_n : x_n - T_n \geq \epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) \geq \frac{\epsilon}{2}\right\} \cup$$
$$\left\{k \in I_n : (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n) \leq -\frac{\epsilon}{2}\right\}.$$

From relation (2.2), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| \leq \gamma.$$

By Lemma 2 and relation (2.12) we get:

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : |\lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n)| \geq \frac{\epsilon}{2} \right\} \right| = 0.$$

Combining last three relations we have:

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| \leq \gamma,$$

and $\gamma$ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| = 0. \tag{2.13}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$x_n - T_n = \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) + (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} (x_n - x_j).$$

For any $\epsilon > 0$,

$$\{k \in I_n : x_n - T_n \leq -\epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) \leq -\frac{\epsilon}{2}\right\} \cup$$
$$\left\{k \in I_n : (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} (x_k - x_j) \leq -\frac{\epsilon}{2}\right\}.$$

For same reasons as in the case where $t > 1$, by Lemma 2, we have that for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \leq -\epsilon\}| = 0. \tag{2.14}$$

Finally from relations (2.13) and (2.14) we get:

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_n - T_n| \geq \epsilon\}| = 0.$$

□

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

**Theorem 3.** *Let $(\lambda_n)$ be a sequence of complex numbers defined above,which satisfied relation (2.1) and let us consider that $st_\lambda - \lim T_k = L$. Then $(x_k)$ is $st_\lambda-$ statistically convergent to the same number L if and only if the following to conditions holds: for every $\epsilon > 0$,*

$$\inf_{t>1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \geq \epsilon \right\} \right| = 0 \tag{2.15}$$

*and*

$$\inf_{0<t<1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} (x_k - x_j) \geq \epsilon \right\} \right| = 0. \tag{2.16}$$

*Remark* 2. Let us suppose that $st_\lambda - \lim_k x_k = L$, $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies. Then for every $t > 1$, relation (2.15) holds, and in case where $0 < t < 1$, relation (2.16) is valid.

*Proof of Theorem 3.* We omit it, because it is similar to that of Theorem 1. □

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# STRUCTURE OF CESARO SECOND ORDER FUNCTION SPACES

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*Abstract.* We will define the second order Cesaro operator spaces and we will show that those spaces are Banach spaces, are separable, fulfill dominant Lebesgue theorem, are not rearrangement invariant, not reflexive and strictly convex.

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*Keywords:* Second Cesaro operator, dominant theorem, rearrangement, reflexive, strictly convex

## 1. Introduction

In [3] the Cesaro function space is defined as follows:

$$Ces_p(X) = \left\{ f \in L^0(X) : \int_X \left( \frac{1}{x} \int_0^x |f(t)| dt \right)^p d\mu(x) < \infty \right\}.$$

And for this space several properties are shown, such as separability, dominant Lebesgue theorem etc. In [4] the second order Cesaro sequence space is defined and some of the topological properties are given . In this paper we will define the Cesaro second order function space.

Let $(X, s, \mu)$ be a $\sigma-$finite measure space and let $L^0 = L^0(X)$ denote the set of all equivalence classes of complex valued measurable functions defined on $X$, where $X = [0,1]$ or $X = [0,\infty)$. Then, for $1 \leq p < \infty$, the Cesaro second function space is denoted by $Ces_p^2(X)$, where $X = [0,1]$ or $X = [0,\infty)$ and is defined as follows for $1 \leq p < \infty$,

$$Ces_p^2(X) = \left\{ f \in L^0(X) : \int_X \left( \frac{1}{x^2} \int_0^x |(2x-t) f(t)| dt \right)^p d\mu(x) < \infty \right\} \tag{1.1}$$

and for $p = \infty$

$$Ces_\infty^2(X) = \left\{ f \in L^0(X) : \sup_{x \in X, x > 0} \frac{1}{x^2} \int_0^x |(2x-t) f(t)| dt < \infty \right\}. \tag{1.2}$$

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In what follows we will use this inequality: For any numbers $A, B$ and for any $1 < p < \infty$, we have

$$|A+B|^p \leq 2^{p-1}(|A|^p + |B|^p). \tag{1.3}$$

**Proposition 1.** *The Cesaro function space $Ces_p(X)$ is a subset of the Cesaro second order function space $Ces_p^2(X)$, for $1 \leq p \leq \infty$.*

*Proof.* Let us suppose that $f \in Ces_p(X)$, then we get:

$$\int_X \left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx < \infty.$$

On the other side, taking into consideration (1.3), we have:

$$\begin{aligned}&\int_X \left(\frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt\right)^p dx\\ &\leq 2^{p-1}\int_X \left(\frac{1}{x^2}\int_0^x |2xf(t)|dt\right)^p dx + 2^{p-1}\int_X \left(\frac{1}{x^2}\int_0^x |tf(t)|dt\right)^p dx.\end{aligned}$$

Hence, inclusion follows from the last relation. □

Based in the above Proposition and in Hardy inequalities (see [1])

$$||f||_{Ces_p(X)} \leq \left(\frac{p}{p-1}\right)||f||_{L_p},$$

we get the following result.

**Proposition 2.** *$Ces_p^2(X)$, for $2 < p \leq \infty$ is a generalization of $ces_p(X)$, in the sense of inclusion and inclusion is strict.*

*Proof.* The first part follows directly from Hardy's inequality and Proposition 1. In what follows we will prove that inclusion is strict. Let

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \chi_{n^2-1,n^2}(x),$$

where $\chi$ is the characteristic function of the interval subscripted and for $2 < p < \infty$, set $a_n = (0,0,\cdots, \underbrace{n}_{n-th \text{ position}}, 0, \cdots)$. Then

$$\int_X \left(\frac{1}{x}\int_0^x |f(t)|dt\right)^p dx > \int_X \left(\frac{1}{x}\sum_{n=1}^{N-1} n\right)^p dx,$$

for $(N-1)^2 < x < N^2$. Also,

$$\int_X \left(\frac{1}{x}\sum_{n=1}^{N-1} n\right)^p dx = \sum_{N=1}^{\infty}\int_{(N-1)^2}^{N^2} \frac{dx}{x^p}\left(\sum_{n=1}^{N-1} n\right)^p \sim \sum_{N=1}^{\infty} \frac{N^{4p-3}}{2^p(1-p)N^{4p-4}} = \infty.$$

Which proves that $f(x) \notin ces_p(X)$. On the other side $f(x) \in ces^2(p)$, for $2 < p \leq \infty$. In fact

$$\int_X \left( \frac{1}{x^2} \int_0^x |(2x-t) f(t)| dt \right)^p dx < \int_X \left( \frac{N(N+1)}{(N-1)^2} - \frac{(N-1)^3}{2N^3} \right)^p dx \sim$$

$$\sim \int_X \left( \frac{1}{2N} \right)^p dx,$$

for $0 < t < x$ and $(N-1)^2 < x < N^2$.

$$\int_X \left( \frac{1}{2N} \right)^p dx = \sum_{N=1}^{\infty} \int_{(N-1)^2}^{N^2} \left( \frac{1}{2N} \right)^p dx \sim \sum_{N=1}^{\infty} \frac{1}{N^{p-1}} < \infty,$$

for $p > 2$. Hence, $f(x) \in ces_p^2(X)$.

For $p = \infty$, set $a_n = (0, 0, \cdots, \underbrace{n^2}_{n-th \text{ position}}, 0, \cdots)$. Then

$$\begin{aligned} \sup_{x>0, x\in X} \frac{1}{x} \int_0^x |f(t)| dt &> \sup_{x>0, x\in X, (N-1)^2<x<N^2} \frac{1}{x} \sum_{n=1}^{N-1} n^2 \\ &= \sup_{x>0, x\in X, (N-1)^2<x<N^2} \frac{(N-1)N(2N-1)}{6x} \\ &> \sup_{N>1} \frac{(N-1)N(2N-1)}{6N^2} = \infty, \end{aligned}$$

which implies that $f(x) \notin Ces_\infty(X)$. On the other side,

$$\sup_{x>0, x\in X} \frac{1}{x^2} \int_0^x |2x-t||f(t)| dt < \sup_{N>1} \frac{1}{2N} < \infty,$$

which shows that $f(x) \in Ces_\infty^2(X)$. □

**Theorem 1.** *The Cesaro second order function space $Ces_p^2(X)$ is a normed linear metric space with norm given by the relation:*

$$||f||_{C(p)} = \left( \int_X \left[ \frac{1}{x^2} \int_0^x |(2x-t) f(t)| dt \right]^p dx \right)^{\frac{1}{p}}$$

*and*

$$||f||_{C(\infty)} = \sup_{x\in X, x>0} \frac{1}{x^2} \int_0^x |(2x-t) f(t)| dt \quad \textit{for} \quad p = \infty.$$

**Lemma 1.** *If $f \in Ces_p^2(X)$, $1 < p \leq \infty$, then $f \in L_1(0,k)$ for any fixed positive number $k$.*

*Proof.* To prove this Lemma, we will follow ideas given in [2]. Let $1 < p < \infty$ and $k$ any fixed positive number, then using into consideration relation (1.3), we get:

$$\begin{aligned} ||f||^p_{C(p)} &= \int_0^\infty \left[\frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt\right]^p dx \\ &\geq \int_k^\infty \left[\frac{1}{x^2}\int_0^k |(2x-t)f(t)|dt\right]^p dx. \end{aligned} \tag{1.4}$$

On the other side

$$\begin{aligned} &2^{p-1}\int_k^\infty \left[\frac{1}{x^2}\int_0^k |(2x-t)f(t)|dt\right]^p dx \\ &\geq \int_k^\infty \left[\frac{1}{x^2}\int_0^k |(2x)f(t)|dt\right]^p dx - 2^{p-1}\int_k^\infty \left[\frac{1}{x^2}\int_0^k |tf(t)|dt\right]^p dx \\ &= \frac{2^p||f||_1^p}{(p-1)k^{p-1}} + \frac{2^{p-1}}{(2p-1)k^{2p-1}}\left[-\left(\int_0^k t|f(t)|dt\right)^p\right] \\ &\geq \frac{2^p||f||_1^p}{(p-1)k^{p-1}} - \frac{2^{p-1}||f||_1^p}{(2p-1)k^{p-1}} \\ &= \frac{(3p-1)2^{p-1}||f||_1^p}{(p-1)(2p-1)k^{p-1}}. \end{aligned} \tag{1.5}$$

Hence, from relations (1.4) and (1.5), we get the desired result.
Second case where $p = \infty$.

Let us denote by $E = \left\{x : \frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt > ||f||_{C(\infty)}\right\}$. From definition of the space $Ces_p^2(X)$ it follows that $\mu(E) = 0$. In $E^c$, we can choose real number $x_k$ such that $x_k > k$ and

$$\frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt \leq ||f||_{C(\infty)} \Rightarrow \int_0^x |(2x-t)f(t)|dt \leq x^2 \cdot ||f||_{C(\infty)}.$$

On the other side we have this estimation:

$$\int_0^x |(2x-t)f(t)|dt \geq \int_0^x |2xf(t)|dt - \int_0^x |tf(t)|dt \geq x\int_0^x |f(t)|dt.$$

Hence,

$$||f||_1 \leq x_k \cdot ||f||_{C(\infty)}.$$

□

**Theorem 2.** *The Cesaro second order function space $Ces_p^2(X)$, for $1 < p < \infty$, is a Banach space under the norm:*

$$||f||_{C(p)} = \left( \int_X \left[ \frac{1}{x^2} \int_0^x |(2x-t)f(t)|dt \right]^p dx \right)^{\frac{1}{p}}.$$

*Proof.* Let $(f_n)$ be any Cauchy sequence of functions belonging to $Ces_p^2(X)$. Then for every $\epsilon > 0$, there exists an integer $n(\epsilon)$ such that

$$||f_m - f_n|| \leq \epsilon \quad \text{for all} \quad m,n \geq n(\epsilon).$$

By Lemma 1 every $(f_n) \in L_1(0,k)$ for any fixed $k \in \mathbb{N}$. Let us denote by $f = f_m - f_n$, from Lemma 1 it follows that $(f_n)$ is a Cauchy sequence related to the norm in the $L_1(0,k)$. In particular, for $k = 1$ the sequence $(f_n)$ has a subsequence converging a.e., on $(0,1)$. Let $f_n^0 = f_n$, and for any positive integer $k$, let $(f_n^k)$ be a subsequence of $f_n^{k-1}$ converging a.e., on $(0,k)$. Then diagonal sequence $(f_k^k)$ converges a.e., on $(0,k)$ to some measurable function $f$. Now we will estimate the difference $||f - f_n||_{C(p)}$ as $n \to \infty$. For any fixed $x$ and Fatou's Lemma we have:

$$\begin{aligned}\int_0^x |(2x-t)(f(t)-f_n(t))|dt &= \int_0^x \lim_{k\to\infty} |(2x-t)(f(t)-f_n(t))|dt \\ &\leq \liminf_k \int_0^x |(2x-t)(f_k^k(t)-f_n(t))|dt.\end{aligned}$$

Let $\phi_k(x) = \left( \frac{1}{x^2} \int |(2x-t)(f^k(t)-f_n(t))|dt \right)^p$. Then by applying Fatou's Lemma again we get:

$$\begin{aligned}\int_0^\infty \liminf \phi_k(x)dx &\leq \liminf \int_0^\infty \phi_k(x)dx \\ &= \liminf ||f_k^k - f_n||_{C(p)}^p < \epsilon^p, \quad \text{for} \quad n > N(\epsilon).\end{aligned}$$

Thus $||f - f_n||_{C(p)} < \epsilon$ for every $n > N(\epsilon)$ and fixing one such $n$, the inequalities

$$||g||_{C(p)} \leq ||f - f_n||_{C(p)} + ||f_n||_{C(p)} < \infty,$$

shows that $f \in Ces_p^2(X)$. □

**Theorem 3.** *Let $(f_n)$ be a sequence of measurable functions on $Ces_p^2(X)$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

*exists for almost every $x \in (0,\infty)$. If there is a function $g \in Ces_p^2(X)$, $1 < p < \infty$, such that*

$$|f_n(x)| \leq g(x) \quad a.e., n = 1,2,\cdots$$

*then $f \in Ces_p^2(X)$ and $||f_n - f||_{C(p)} \to 0$, as $n \to \infty$*

We omit the proof of the Theorem, because it is similar to Theorem 4 in [2].

**Theorem 4.** *$Ces_p^2(X)$ is a separable space for $1 < p < \infty$.*

*Proof.* Proof of the Theorem is similar to the Theorem 5 given in [2]. For this reason we omit it. □

**Theorem 5.** *The Cesaro second order function space $Ces_p^2(X)$, for $p = \infty$, is a Banach space under the norm:*

$$||f||_{Ces^2_{(\infty)}}(X) = \sup_{x\in X, x>0} \frac{1}{x^2}\int_0^x |(2x-t)f(t)|dt, \quad for \quad p=\infty.$$

**Theorem 6.** *The Cesaro second order function space $Ces_\infty^2(X)$ is not a separable space.*

**Theorem 7.** *The Cesaro second order function spaces $Ces_\infty^2(X)$, are:*

(1) *Not rearrangement invariant*

(2) *Not reflexive*

(3) *Strictly convex, that is, if $||f||_{Ces_p^2(X)} = ||g||_{Ces_p^2(X)} = 1$ and $f \neq g$, then*

$$\left\| \frac{f+g}{2} \right\|_{Ces_p^2(X)} < 1.$$

*Proof.* (1) Let us consider the function $f(x) = \frac{1}{1-x}$, for $x \in [0,1)$. Then from Theorem 1-c, in [5], it follows that $f \in Ces_p[0,1]$, for $1 \leq p < \infty$. And from Proposition 2 we get $f(x) \in Ces_p^2[0,1]$. In other hand, rearrangement function $f^* = \frac{1}{t}$, for $t \in (0,1]$ is not in $Ces_p^2[0,1]$. And therefore $Ces_p^2[0,1]$ is not rearrangement invariant. In case where $p = \infty$, we can take the function $g(x) = \frac{1}{\sqrt{1-x}}$, for $x \in [0,1)$ and $||g||_{Ces_\infty[0,1]} = 2$, respectively from Proposition 2 if follows that $g(x) \in Ces_\infty^2[0,1]$. The rearrangement function for $g$ is $g^* = \frac{1}{\sqrt{t}}$, for $t \in [0,1)$ and after some calculations we have that $||g||_{Ces^2\infty[0,1]} = \infty$. Hence, $g^* \notin Ces_\infty^2[0,1]$.
(2) It follows directly from Proposition 2, and fact that $Ces_p(I)$ contains a copy of the space $L_1(I)$. It is known that $L_1(I)$ are not reflexive spaces and therefore $Ces_p^2(I)$ are not reflexive spaces.
(3) Proof is similar to the proof of Theorem 1-h given in [5]. For this reason we omit it. □

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# ON THE ZERO OF A RADIAL MINIMIZER OF $p$-GINZBURG-LANDAU TYPE

YUZE CAI, YUTIAN LEI, AND BEI WANG

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*Abstract.* Zeros of minimizers of $p$-Ginzburg-Landau functional are helpful to understand the location of vortices of superconductivity and superfluid. When $p = 2$ and the degree of the boundary data around the boundary is $\pm 1$, there exists only one zero of the $p$-Ginzburg-Landau minimizers in the bounded domain. When $p > 2$, this becomes a more complicated problem. This paper is concerned with the location of the zeros of a radial minimizer of a $p$-Ginzburg-Landau type functional. The authors use the method of moving planes and the idea of the proof of Pohozaev's identity to verify that the origin is the unique zero of the radial minimizer in the domain.

2010 *Mathematics Subject Classification:* 35J92; 35Q56

*Keywords:* location of zeros, method of moving planes, Pohozaev's identity, $p$-Ginzburg-Landau type functional

## 1. Introduction

Let $B = \{x \in \mathbf{R}^2; |x| < 1\}$ and $p > 2$. Consider the $p$-Ginzburg-Landau type functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p + \frac{1}{2p\varepsilon^p}\int_B (1 - |u|^p)^2.$$

When $p = 2$, the minimizer $u_\varepsilon$ of $E_\varepsilon(u, B)$ in the class $H^1_g(B, \mathbf{R}^2)$ was well-studied by many papers (see [4], [5] and the references therein), where $g : \partial B \to \partial B$ is a smooth map. In particular, the paper [4] pointed out that there is only one zero of $u_\varepsilon$ in $B$ if $\deg(g, \partial B) = \pm 1$. When $p \neq 2$, the minimizer in the class $W^{1,p}_g(B, \mathbf{R}^2)$ was also researched in [2], [3], [10], [11] and [16]. The paper [10] proved that when $p > 2$, the zeros of the minimizer $u_\varepsilon$ are located near the finite singularities of a $p$-harmonic map. The same result was generalized to higher dimensions (cf. [9]).

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However, the relation between the number of singularities and the degree $\deg(g, \partial B)$ is still not clear.

In order to investigate this problem, we consider a special case. Investigate the minimization problem of $E_\varepsilon(u, B)$ in the class

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbf{R}^2); f(1) = 1\},$$

where $r = |x|$. By the direct method in the calculus of variations, the minimizer $u_\varepsilon$ exists, and it is called the *radial minimizer*. When $p = 2$, many papers investigated the properties of the radial minimizer. Papers [13] and [14] discussed the stability properties of $f_\varepsilon(r)$, the modulus of $u_\varepsilon$. The paper [8] also pointed out that

$$f_\varepsilon(r) > 0, \quad (f_\varepsilon(r))_r > 0, \quad in \quad (0, 1].$$

Here $(f_\varepsilon)_r$ is the first order derivative of $f_\varepsilon$. This means the origin is the unique zero of $f_\varepsilon$ in $B$.

In this paper, we will also prove that the origin is the unique zero of $f_\varepsilon$ in the case of $p > 2$. This improves the result in [12], which was only showed that the zeros of $u_\varepsilon$ are located near the origin.

If we define

$$V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\},$$

then, $V \subset \{f \in C[0, 1]; f(0) = 0\}$ (cf. Proposition 1.1 in [12]). Clearly, $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, B)$ in $W$ if and only if $f_\varepsilon$ is a minimizer of

$$E_\varepsilon(f) = \int_0^1 [\frac{1}{p}A^{p/2} + \frac{1}{2p\varepsilon^p}(1 - f^p)^2]rdr$$

in $V$, where $A = (f_r)^2 + (f/r)^2$. According to Proposition 2.2 in [12], the Euler-Lagrange equation which the minimizer $f_\varepsilon$ satisfies is the following ODE

$$-(A^{\frac{p-2}{2}} f_r r)_r + A^{\frac{p-2}{2}}\frac{f}{r} = \frac{r}{\varepsilon^p} f^{p-1}(1 - f^p). \tag{1.1}$$

Similar to the argument of regularity results in [3], we see that the minimizer $f_\varepsilon$ is a classical solution of the equation above. By the proof of Proposition 2.1 in [3], we can always assume

$$0 \le f_\varepsilon(r) \le 1, \quad r \in [0, 1].$$

*Remark* 1. Theorem 3.5 in [12] shows that for any given $\eta \in (0, 1)$, there exists $h = h(\eta) > 0$ and $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,

$$f_\varepsilon(r) > 1 - \eta, \quad in \quad [h\varepsilon, 1].$$

This means that all zeros of $f_\varepsilon$ are located in $[0, h\varepsilon)$ for each $\varepsilon \in (0, \varepsilon_0)$. Therefore, we can search the zeros only in this interval. On the other hand, Theorem IV.1 in [5] shows the relation between $h$ and $\eta$:

$$\lambda \le h \le \lambda 9^{Card\ J},$$

where $Card\ J$ is a positive integer which is independent of $\varepsilon$ (cf. Proposition 3.4 in [12]), and $\lambda$ is defined in the proof of Proposition 3.3 in [12] as

$$\lambda = (\frac{\eta}{2C})^{\frac{p}{p-2}}.$$

These results mean that $h$ can be suitably small as long as $\eta$ is chosen small properly.

The main result of this paper, which will be proved in §2, is read as follows

**Theorem 1.** *Let $f_\varepsilon$ be the modulus of a radial minimizer of $E_\varepsilon(u, B)$. Then there exists $\varepsilon_0 \in (0, 1)$, such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$f_\varepsilon > 0, \quad in \quad (0, h\varepsilon], \tag{1.2}$$

$$(f_\varepsilon)_r > 0, \quad in \quad [0, h\varepsilon]. \tag{1.3}$$

*Remark* 2. Clearly, Remark 1, (1.2) and (1.3) show that the origin is the unique zero of the radial minimizer $u_\varepsilon$ in $B$.

An analogous result for minimizer of $E(u)$ was proved in [3], where $E(u) = \int_B [p^{-1}|\nabla u|^p + (4\varepsilon^p)^{-1}(1-|u|^2)^2]dx$ is equipped with a different penalization. Here we can also locate the zero of minimizers of $E_\varepsilon(u, B)$ by another technique-the method of moving planes. Such a method was first proposed by Alexandrov [1] and developed by Serrin [15], Gidas-Ni-Nirenberg [7], and Chen-Li [6]. Now, this method has been a powerful tool to handle the monotonicity of solutions of differential equations.

## 2. Proof of Theorem 1

Let $f = f_\varepsilon$ be a minimizer of $E_\varepsilon(f)$ in $V$. In view of $f(0) = 0$ and $f(h\varepsilon) > 1 - \eta > 0$, we can set

$$\sigma = \sup\{R \in [0, h\varepsilon); f(r) \equiv 0, \quad in \quad [0, R]\}.$$

**Proposition 1.** *We have*

$$f > 0, \quad in \quad (\sigma, 1];$$

$$f_r > 0, \quad in \quad [\sigma, h\varepsilon].$$

*Proof.* The idea of the method of moving planes is used here. Thus, consider the equation (1.1) multiplied with $r^{-1}$,

$$-A^{(p-2)/2} f_{rr} - (A^{(p-2)/2})_r f_r - A^{(p-2)/2}\frac{f_r}{r} + A^{(p-2)/2}\frac{f}{r^2} = \frac{f^{p-1}}{\varepsilon^p}(1 - f^p). \tag{2.1}$$

*Step 1. We claim that $f_r(h\varepsilon - 0) > 0$.*

In fact, if the claim is not true, then we can consider two cases:

$$Case \quad I: \quad f(r) \equiv f(h\varepsilon), \quad in \quad (h\varepsilon - \delta, h\varepsilon);$$

$$Case \quad II: \quad f(r) > f(h\varepsilon), \quad in \quad (h\varepsilon - \delta, h\varepsilon)$$

when $\delta$ is sufficiently small. In Case I, (2.1) becomes

$$\frac{f^{p-1}}{r^p} = \frac{f^{p-1}}{\varepsilon^p}(1-f^p), \quad in \quad (h\varepsilon-\delta, h\varepsilon).$$

By virtue of $r < h\varepsilon$, the result above implies

$$\frac{f^{p-1}}{\varepsilon^p}(h^{-p}-(1-f^p)) < 0.$$

Noting $r > h\varepsilon-\delta > \sigma$, we see that $f > 0$. Therefore,

$$h^{-p}-(1-f^p) < 0.$$

It is impossible if we let $\eta$ in $h$ be sufficiently small (cf. Remark 1 in \$1).

In Case II, we can find a maximizer $r_0 \in (\sigma, h\varepsilon)$ of $f$. Then,

$$f(r_0) > 0, \quad f_r(r_0) = 0, \quad and \quad f_{rr}(r_0) < 0.$$

Hence, (2.1) with $r = r_0$ leads to

$$\frac{f^{p-1}(r_0)}{\varepsilon^p}[h^{-p}-(1-f^p(r_0))] < 0.$$

This is also impossible when we choose $\eta$ sufficiently small such that $h < 1$.

*Step 2. Moving planes from $\sigma$ to $h\varepsilon$.* For $\mu \in [\sigma, h\varepsilon]$, let $r_\mu = 2\mu - r$ be the reflection point of $r$ about the point $\mu$. Define

$$F_\mu(r) = f(r_\mu) - f(r), \quad as \quad r \in (\mu-\delta, \mu).$$

We claim that the set

$$\begin{aligned} S := \quad & \{\mu \in [\sigma, h\varepsilon]; \quad \text{there is} \quad r' \in (\mu-\delta, \mu) \\ & \text{such that} \quad F_\mu(r') = 0\} = \varnothing. \end{aligned}$$

Otherwise, we can suppose

$$\mu_* = \min_{\mu \in S} \mu.$$

Then, we claim $\mu_* > \sigma$. In fact, if $\mu_* = \sigma$, then there exists $r' \in (\sigma-\delta, \sigma)$ such that

$$f(r'_\sigma) - f(r') = 0.$$

On the other hand, in view of the definition of $\sigma$, we get

$$f(r') = 0, \quad f(r'_\sigma) > 0.$$

Therefore, we see the contradiction and hence the claim means

$$\mu_* \in (\sigma, h\varepsilon].$$

This consequence implies either $f \equiv Constant$ in some open interval $I \in (r', r'_\mu)$, or a maximizer of $f$ in $(r', r'_\mu)$. Similar to the argument in Cases I and II of Step 1, we can also deduce the contradiction. Thus, $S = \varnothing$.

The result $S = \varnothing$ shows that $f(r') < f(r'_\mu)$, $\forall r' \in (\mu-\delta, \mu)$. After $\mu$ moves from $\sigma$ to $h\varepsilon$, it follows that $f(r_1) < f(r_2)$ as long as $\sigma < r_1 < r_2 \le h\varepsilon$. Therefore,

$$f_r(r) > 0, \quad \forall r \in (\sigma, h\varepsilon]. \tag{2.2}$$

By virtue of the definition of $\sigma$, we have

$$f(\sigma) = 0; \quad f(r) > 0 \quad \forall r \in (\sigma, h\varepsilon]. \tag{2.3}$$

This implies $f_r(\sigma+0) > 0$. Combining with (2.2) yields

$$f_r(r) > 0, \quad \forall r \in [\sigma, h\varepsilon].$$

Combining (2.3) with Remark 1 in §1, we can see $f(r) > 0$ in $(\sigma, 1]$.

Thus, we complete the proof of Proposition 1. □

**Proposition 2.** *Let $\sigma \in [0, h\varepsilon)$ satisfy*

$$f(r) \equiv 0, \quad in \quad [0, \sigma];$$

$$f_r(r) > 0, \quad in \quad [\sigma, h\varepsilon].$$

*Then $\sigma = 0$.*

*Proof.* Suppose $\sigma > 0$. Since the Pohozaev identity can show the properties of two end points of an interval by investigating the integrals on this interval, we use the idea of proof of Pohozaev's identity here to deduce a contradiction.

Multiplying (1.1) by $f_r$ and integrating on $(0, \sigma)$, we have

$$\begin{aligned} &-\int_0^\sigma (A^{\frac{p-2}{2}} f_r r)_r f_r dr + \int_0^\sigma A^{\frac{p-2}{2}} \frac{f}{r^2} f_r r dr \\ &= \frac{1}{\varepsilon^p} \int_0^\sigma f^{p-1}(1-f^p) f_r r dr. \end{aligned} \tag{2.4}$$

By $f(r) = 0$ in $[0, \sigma]$, (2.4) becomes

$$\int_0^\sigma (A^{\frac{p-2}{2}} f_r r)_r f_r dr = 0. \tag{2.5}$$

Calculating the left hand side of (2.5), we obtain that

$$\begin{aligned} &-\int_0^\sigma (A^{\frac{p-2}{2}} f_r r)_r f_r dr \\ &= -\int_0^\sigma A^{\frac{p-2}{2}} f_r f_{rr} r dr - \int_0^\sigma (A^{\frac{p-2}{2}})_r (f_r)^2 r dr - \int_0^\sigma A^{\frac{p-2}{2}} (f_r)^2 dr \\ &= -\int_0^\sigma A^{\frac{p-2}{2}} f_r f_{rr} r dr - A^{\frac{p-2}{2}} (f_r)^2 r|_{r=\sigma} \\ &\quad + \int_0^\sigma A^{\frac{p-2}{2}} ((f_r)^2 r)_r dr - \int_0^\sigma A^{\frac{p-2}{2}} (f_r)^2 dr \\ &= \int_0^\sigma A^{\frac{p-2}{2}} f_r f_{rr} r dr - A^{\frac{p-2}{2}} (f_r)^2 r|_{r=\sigma}. \end{aligned}$$

This result, together with (2.5), implies

$$\int_0^\sigma A^{\frac{p-2}{2}} f_r f_{rr} r dr = A^{\frac{p-2}{2}} (f_r)^2 r|_{r=\sigma}. \tag{2.6}$$

Noting $f(r) = 0$ in $(0, \sigma)$, we can deduce that the left hand side of (2.6) is zero. In view of $f_r(\sigma) > 0$, we obtain that the right hand side of (2.6) is nonzero, which leads to a contradiction. □

Combining Propositions 1 and 2, we can complete the proof of Theorem 1.

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# ON SUBSEQUENTIAL CONVERGENCE OF BOUNDED SEQUENCES

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*Abstract.* In this paper we establish some Tauberian-like conditions in terms of the weighted general control modulo of integer order to retrieve subsequential convergence of a sequence from its boundedness.

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*Keywords:* subsequential convergence, weighted means, weighted general control modulo, slowly oscillating sequence

## 1. Introduction

Let $p = (p_n)$ be a sequence of nonnegative numbers with $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

The weighted means of a sequence $(u_n)$ of real numbers are defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k$$

for all nonnegative integers $n$.

We define the weighted mean method as follows:

**Definition 1.** If $\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s$, then we say that $(u_n)$ is said to be limitable to $s$ by the weighted mean method $(\overline{N}, p_n)$ and we write $u_n \to s(\overline{N}, p_n)$.

If $p_n = 1$ for all $n$, then the corresponding weighted mean method is the $(C, 1)$ method of Cesàro.

The sequence $\Delta u = (\Delta u_n)$ of the backward differences of $(u_n)$ is defined by $\Delta u_n = u_n - u_{n-1}$, and $\Delta u_0 = u_0$ for $n = 0$.

The identity

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u), \tag{1.1}$$

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where $V_{n,p}^{(0)}(\Delta u) = \frac{1}{P_n}\sum_{k=1}^{n} P_{k-1}\Delta u_k$, is known as the weighted Kronecker identity.

We define $\sigma_{n,p}^{(m)}(u)$ and $V_{n,p}^{(m)}(\Delta u)$ by

$$\sigma_{n,p}^{(m)}(u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k \sigma_{k,p}^{(m-1)}(u) & ,m \geq 1 \\ u_n & ,m = 0 \end{cases}$$

and

$$V_{n,p}^{(m)}(\Delta u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k V_{k,p}^{(m-1)}(\Delta u) & ,m \geq 1 \\ V_{n,p}^{(0)}(\Delta u) & ,m = 0 \end{cases}$$

respectively (see [4]).

The weighted classical control modulo of $(u_n)$ is denoted by $\omega_{n,p}^{(0)}(u) = \dfrac{P_{n-1}}{p_n}\Delta u_n$ and the weighted general control modulo of integer order $m \geq 1$ of $(u_n)$ is defined by $\omega_{n,p}^{(m)}(u) = \omega_{n,p}^{(m-1)}(u) - \sigma_{n,p}^{(1)}(\omega^{m-1}(u))$ (see [2]).

A new kind of convergence was defined by Dik [5] as follows.

**Definition 2.** A sequence $u = (u_n)$ is said to be subsequentially convergent if there exists a finite interval $I(u)$ such that all accumulation points of $(u_n)$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $(u_n)$.

It is clear from the definition that subsequential convergence implies boundedness. But the converse is not true in general. For example, $((-1)^n)$ is bounded, but it is not subsequentially convergent. The converse implication is true under some conditions imposed on the sequence.

The following theorem is a more general one stating that under which condition bounded sequences are subsequential convergent.

**Theorem 1** ([5])**.** *Let $(u_n)$ be a bounded sequence. If $\Delta u_n = o(1)$, then $(u_n)$ is subsequentially convergent.*

Throughout this paper, we write $u_n = o(1)$ or $u_n = O(1)$ rather than $\lim_n u_n = 0$ or $(u_n)$ is bounded for large enough $n$.

*Example* 1. Using Theorem 1, one can easily show that the sequence $(u_n) = (\sin\sqrt{n})$ is subsequentially convergent. Indeed, the sequence $(u_n)$ is bounded. From the fact that

$$\begin{aligned} |\Delta u_n| = |\Delta \sin(\sqrt{n})| &= |\sin(\sqrt{n}) - \sin(\sqrt{n-1})| \leq |\sqrt{n} - \sqrt{n-1}| \\ &= \left|\frac{1}{\sqrt{n}+\sqrt{n-1}}\right| = o(1), \end{aligned}$$

it follows that $\Delta u_n = o(1)$. Thus by Theorem 1, the sequence $(u_n)$ is subsequentially convergent.

**Definition 3** ([9])**.** A sequence $(u_n)$ is slowly oscillating if

$$\lim_{\lambda\to 1+} \overline{\lim_{n}} \max_{n+1\le k\le[\lambda n]} \left| \sum_{j=n+1}^{k} \Delta u_j \right| = 0.$$

It is clear from Definition 3 that slow oscillation of $(u_n)$ implies $\Delta u_n = o(1)$.

By Littlewood's condition $n\Delta u_n = O(1)$ [8] and slow oscillation of $(u_n)$, Dik et al. [6] obtained some Tauberian-like theorems to recover subsequential convergence of $(u_n)$ from its boundedness. The condition $n\Delta u_n = O(A_n)$, where $(A_n)$ is an unbounded sequence, is used as a Tauberian-like condition for the recovery of subsequential convergence of $(u_n)$ from its boundedness by Dik et al. [6]. Later in [1], Çanak and Dik introduced some Tauberian-like conditions $\omega_{n,1}^{(m)}(u) = O(A_n)$, where $(A_n)$ is an unbounded sequence, and slow oscillation of $(\Delta\sigma_{n,1}(\omega^{(m)}(u)))$ for recovering subsequential convergence of the sequence of $(u_n)$ from its boundedness.

In this paper we prove that subsequential convergence of $(u_n)$ follows from its boundedness under the condition given in terms of the weighted general control modulo of the oscillatory behavior of integer order of the sequence $(u_n)$.

## 2. Main results

The main theorem of this paper involves the concepts of a regularly varying sequence of index $\alpha > -1$ and slowly oscillating sequence.

**Definition 4** ([7])**.** A positive sequence $(R(n))$ is said to be regularly varying of index $\alpha > -1$ if

$$\lim_{n\to\infty} \frac{R([\lambda n])}{R(n)} = \lambda^{\alpha},\ \lambda > 0,\ \alpha > -1. \tag{2.1}$$

The following theorems generalize some theorems in [1] that are exactly given in terms of the weighted general control modulo of the oscillatory behavior of the sequence $(u_n)$.

**Theorem 2.** *Let $(u_n)$ be a bounded sequence and*

$$\frac{P_{n-1}}{p_n} = O(n). \tag{2.2}$$

*Let $(A_n)$ be a sequence such that*

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k |A_k|^{\mathfrak{p}} = O(1), \mathfrak{p} > 1 \tag{2.3}$$

*for some regularly varying sequence* $(P_n)$ *of index* $\alpha > -1$. *If*

$$\omega_{n,p}^{(m)}(u) = O(A_n) \tag{2.4}$$

*then* $(u_n)$ *is subsequentially convergent.*

**Theorem 3.** *Let* $(u_n)$ *be a bounded sequence and let* $(P_n)$ *be regularly varying of index* $\alpha > -1$. *If* $(\Delta\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ *is slowly oscillating for some nonnegative integer* $m$, *then* $(u_n)$ *is subsequentially convergent.*

## 3. Identities and a Lemma

In this section, we present some identities and a lemma to be used in the proof of the main theorems.

The identities

$$\frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(m)}(u) = V_{n,p}^{(m-1)}(\Delta u)$$

and

$$\sigma_{n,p}^{(1)}\left(\frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u)\right) = \frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(1)}(\Delta u).$$

are proved by Totur and Çanak [10].

For a sequence $u = (u_n)$, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n = \left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right) = \frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1} u_n\right),$$

where $\left(\frac{P_{n-1}}{p_n}\Delta\right)_0 u_n = u_n$, and $\left(\frac{P_{n-1}}{p_n}\Delta\right)_1 u_n = \frac{P_{n-1}}{p_n}\Delta u_n$.

A different representation of the weighted general control modulo of integer order $m \geq 1$ of a sequence $(u_n)$ is given by the identity

$$\omega_{n,p}^{(m)}(u) = \left(\frac{P_{n-1}}{p_n}\Delta\right)_m V_{n,p}^{(m-1)}(\Delta u) \tag{3.1}$$

in ([10]).

We note that any $(P_n)$ with $\lim_{n\to\infty}\frac{P_n}{n} = 1$ is regularly varying of index 1.

**Lemma 1.** *Let* $(P_n)$ *be regularly varying of index* $\alpha > -1$. *If* $(u_n)$ *is limitable to* $s$ *by the weighted mean method* $(\overline{N}, p_n)$ *and slowly oscillating, then* $(u_n)$ *converges to* $s$.

*Proof.* Assume that $(P_n)$ be regularly varying of index $\alpha > -1$. If $(u_n)$ is slowly oscillating, then $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating (see [3]). Since $u_n \to s(\overline{N}, p_n)$ and $(V_{n,p}^{(0)}(\Delta u))$ is slowly oscillating, the proof is completed by [2, Theorem 6]. □

## 4. Proofs

In this section, proofs of theorems and corollaries are given.

*Proof of Theorem 2.* By (2.3), it follows that the sequence $\left(\sum_{j=1}^{n} \frac{p_j A_j}{P_{j-1}}\right)$ is slowly oscillating. Indeed,

$$\begin{aligned}
\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\frac{p_jA_j}{P_{j-1}}\right| &\le \max_{n+1\le k\le[\lambda n]}\sum_{j=n+1}^{k}\left|\frac{p_jA_j}{P_{j-1}}\right| \\
&\le \sum_{j=n+1}^{[\lambda n]}\frac{p_j|A_j|}{P_{j-1}} \le \frac{1}{P_n}\sum_{j=n+1}^{[\lambda n]}p_j|A_j| \\
&= \frac{P_{[\lambda n]}-P_n}{P_n}\frac{1}{P_{[\lambda n]}-P_n}\sum_{j=n+1}^{[\lambda n]}p_j|A_j| \\
&\le \frac{P_{[\lambda n]}-P_n}{P_n}\frac{1}{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{p}}}}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{1-\frac{1}{\mathfrak{p}}}}{P_n}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{P_n}\left(\sum_{j=n+1}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}}, \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{(P_n)^{\frac{1}{\mathfrak{q}}}}\left(\frac{P_{[\lambda n]}}{P_n}\frac{1}{P_{[\lambda n]}}\sum_{j=0}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \\
&= \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{(P_n)^{\frac{1}{\mathfrak{q}}}}\frac{(P_{[\lambda n]})^{\frac{1}{\mathfrak{p}}}}{(P_n)^{\frac{1}{\mathfrak{p}}}}\left(\frac{1}{P_{[\lambda n]}}\sum_{j=0}^{[\lambda n]}p_j|A_j|^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}},
\end{aligned}$$

where $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$. Taking lim sup of both sides as $n\to\infty$, we get

$$\limsup_{n}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\frac{p_jA_j}{P_{j-1}}\right|$$

$$\leq \limsup_n \frac{(P_{[\lambda n]}-P_n)^{\frac{1}{\mathfrak{q}}}}{(P_n)^{\frac{1}{\mathfrak{q}}}} \frac{(P_{[\lambda n]})^{\frac{1}{\mathfrak{p}}}}{(P_n)^{\frac{1}{\mathfrak{p}}}} \limsup_n \left( \frac{1}{P_{[\lambda n]}} \sum_{j=0}^{[\lambda n]} p_j |A_j|^{\mathfrak{p}} \right)^{\frac{1}{\mathfrak{p}}}$$

$$= \lim_n \left( \frac{P_{[\lambda n]}-P_n}{P_n} \right)^{\frac{1}{\mathfrak{q}}} \lim_n \left( \frac{P_{[\lambda n]}}{P_n} \right)^{\frac{1}{\mathfrak{p}}} \limsup_n \left( \frac{1}{P_{[\lambda n]}} \sum_{j=0}^{[\lambda n]} p_j |A_j|^{\mathfrak{p}} \right)^{\frac{1}{\mathfrak{p}}}$$

Since $(P_n)$ is regularly varying of index $\alpha > -1$, we have

$$\limsup_n \max_{n+1\leq k\leq [\lambda n]} \left| \sum_{j=n+1}^{k} \frac{p_j A_j}{P_{j-1}} \right| \leq C(\lambda^\alpha - 1)^{\frac{1}{\mathfrak{q}}} (\lambda^\alpha)^{\frac{1}{\mathfrak{p}}}.$$

for some $C > 0$.

Finally, taking lim sup of both sides as $\lambda \to 1+$, we obtain

$$\lim_{\lambda\to 1^+} \limsup_n \max_{n+1\leq k\leq [\lambda n]} \left| \sum_{j=n+1}^{k} \frac{p_j A_j}{P_{j-1}} \right| \leq C \lim_{\lambda\to 1^+} (\lambda^\alpha - 1)^{\frac{1}{\mathfrak{q}}} (\lambda^\alpha)^{\frac{1}{\mathfrak{p}}} = 0.$$

Slow oscillation of $\left( \sum_{j=1}^{n} \frac{p_j A_j}{P_{j-1}} \right)$ implies that $\frac{p_j A_j}{P_{j-1}} = o(1)$. Since $\omega_{n,p}^{(m)}(u) = O(A_n)$, it follows from

$$\omega_{n,p}^{(m)}(u) = \frac{P_{n-1}}{p_n} \Delta \left( (\frac{P_{n-1}}{p_n} \Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right) = O(A_n)$$

that

$$\Delta \sigma_{n,p}^{(1)}(\omega^{(m-1)}(u)) = \Delta \left( (\frac{P_{n-1}}{p_n} \Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right) = o(1).$$

Since $(u_n)$ is bounded, $(\sigma_{n,p}^{(1)}(\omega^{(m-2)}(u)))$ is bounded for every nonnegative integer $m$. From the identity

$$\Delta \left( (\frac{P_{n-1}}{p_n} \Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u) \right) = \frac{p_n (\frac{P_{n-1}}{p_n} \Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u)}{P_{n-1}} + \Delta \left( (\frac{P_{n-1}}{p_n} \Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u) \right),$$

it follows that

$$\sigma_{n,p}^{(1)}(\omega^{(m-2)}(u)) = \Delta \left( (\frac{P_{n-1}}{p_n} \Delta)_{m-2} V_{n,p}^{(m-2)}(\Delta u) \right) = o(1), n \to \infty.$$

Continuing in this manner, we obtain that

$$\Delta\sigma_{n,p}^{(1)}(\omega^{(0)}(u)) = \Delta V_{n,p}^{(0)}(\Delta u) = o(1).$$

Using (1.1), we have

$$\Delta u_n = \frac{p_n V_{n,p}^{(0)}(\Delta u)}{P_{n-1}} + \Delta V_n^{(0)}(\Delta u) = o(1).$$

By Theorem 1 we have the proof. □

**Corollary 1.** *Let $(u_n)$ be a bounded sequence and $(A_n)$ be a sequence such that*

$$\frac{1}{n}\sum_{k=1}^{n} |A_k|^{\mathfrak{p}} = O(1),\ \mathfrak{p} > 1. \tag{4.1}$$

*If $\omega_n^{(m)} = O(A_n)$, then $(u_n)$ is subsequentially convergent.*

*Proof.* Take $p_n = 1$ for every nonnegative integer $n$ in Theorem 2. □

Corollary 1 was given by Çanak and Dik [1].

*Proof of Theorem 3.* Since $(u_n)$ is bounded, then $(\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ is bounded for every nonnegative integer $m$. Thus, we get

$$\sigma_{n,p}^{(1)}(\Delta\sigma^{(1)}(\omega^{(m)}(u))) = \frac{p_n (n\Delta)_m V_{n,p}^{(m)}(\Delta u)}{P_{n-1}} = o(1).$$

Since $(\Delta\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ is slowly oscillating, applying Lemma 1 to the sequence $(\Delta\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ we obtain that

$$\Delta\sigma_{n,p}^{(1)}(\omega^{(m)}(u)) = \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_m V_{n,p}^{(m)}(\Delta u)\right) = o(1).$$

Since $(\sigma_{n,p}^{(1)}(\omega^{(m-1)}(u)))$ is bounded for every nonnegative integer $m$, by the identity

$$\Delta\left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u)\right) = \frac{p_n(\frac{P_{n-1}}{p_n}\Delta)_m V_{n,p}^{(m)}(\Delta u)}{P_{n-1}} + \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_m V_{n,p}^{(m)}(\Delta u)\right),$$

it follows that

$$\Delta\sigma_{n,p}^{(1)}(\omega^{(m-1)}(u)) = \Delta\left((\frac{P_{n-1}}{p_n}\Delta)_{m-1} V_{n,p}^{(m-1)}(\Delta u)\right) = o(1).$$

The rest of the proof is as in the proof of Theorem 2. □

**Corollary 2.** *Let $(u_n)$ be a bounded sequence. If $(\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ is slowly oscillating for some nonnegative integer $m$, then $(u_n)$ is subsequentially convergent.*

*Proof.* Proof follows from the fact that $\lim_n \Delta u_n = 0$ for every slowly oscillating sequence $(u_n)$. □

**Corollary 3.** *Let $(u_n)$ be a bounded sequence. If $(\Delta\sigma_n^{(1)}(\omega^{(m)}(u)))$ is slowly oscillating for some nonnegative integer $m$, then $(u_n)$ is subsequentially convergent.*

*Proof.* Take $p_n = 1$ for every nonnegative integer $n$ in Theorem 3. □

Corollary 3 was given by Çanak and Dik [1].

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# ASYMPTOTIC DECOMPOSITION OF LINEAR SINGULARLY PERTURBED MULTISCALE SYSTEMS

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*Abstract.* We study a scheme which decomposes a singularly perturbed linear system with several small parameters into a block-diagonal form. The algorithm of the scheme contains $k$ steps of a successive splitting based on the integral manifold of quick and slow variables method. The splitting substitution is defined constructively in the form of expansions by powers of small parameters.

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*Keywords:* linear singularly perturbed systems, decomposition, asymptotic decomposition, integral manifolds

## 1. Introduction

The theory of singularly perturbed systems of differential equations has been intensively developed recently, and its methods have been widely used for the study of various problems of science and technology. Singularly perturbed models describe dynamic processes in physics, theory of automatic control, hydroscopic systems, economical and biological processes [2, 3]. Singularly perturbed systems have boundary layer and stiffness property in numerical calculations. In this connection the main results for high-dimensional systems with fast and slow variables boil down to the construction of asymptotic expansions of the initial and boundary problems. Let us note the surveys [6, 8, 13], where a great amount of works dedicated to this subject were analyzed.

For qualitative studies of entire systems it is expedient to apply decomposition algorithms. Effective methods of decomposition of singularly perturbed systems, which are based on the ideas of the integral manifold method, were developed in the works [11, 12]. For linear singularly perturbed systems such an approach allows us to make the decomposition of an input system to the independent fast and slow subsystems [7, 10]. Linear non-autonomous three-time and n-time scale systems were studied in [1, 4, 9].

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The aim of this paper is to justify the decomposition method of a linear singularly perturbed system with several small parameters to the collection of independent subsystems. Such an approach is effective if we can find the coefficients of the splitting transformation. The suggested approach combines both asymptotic and qualitative methods of analysis. The splitting transformation is constructed with an arbitrary precision as asymptotic expansions in a series of small parameters.

## 2. The scheme of decomposition

Consider a linear singularly perturbed system given by

$$\prod_{j=0}^{i} \varepsilon_j \dot{x}_i = \sum_{j=0}^{k} A_{ij} x_j,\ i = \overline{0,k}, \tag{2.1}$$

where $t \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$, $A_{ij} = A_{ij}(t)$, $i, j = \overline{0,k}$, are $n_i \times n_j$ matrices, $\varepsilon_0 = 1$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ are small positive parameters.

Let the following conditions be true:

1) matrices $A_{ij}(t)$, $i, j = \overline{0,k}$, are uniformly bounded in $t \in \mathbb{R}$,

2) eigenvalues of the matrix $A_{kk}(t)$ satisfy the inequality

$$\mathrm{Re}\lambda_j(A_{kk}) \le -2\beta < 0,\ j = \overline{1,n_k}.$$

The decomposition of system (2.1) will be performed in $k$ steps. At the first step the substitution is made as follows

$$\begin{cases} x_i = y_i^1 + \prod\limits_{j=i+1}^{k} \varepsilon_j H_i^1 y_k^1,\ i = \overline{0,k-1}, \\ x_k = y_k^1 + \sum\limits_{j=0}^{k-1} P_j^1 x_j, \end{cases} \tag{2.2}$$

where $H_i^1, P_i^1$, $i = \overline{0,k-1}$ are matrix functions of appropriate dimensions.

If $H_i^1, P_i^1$, $i = \overline{0,k-1}$ are bounded solutions of the systems

$$\prod_{j=0}^{k} \varepsilon_j \dot{P}_i^1 = A_{ki} + A_{kk} P_i^1 - \sum_{j=0}^{k-1} \prod_{m=j+1}^{k} \varepsilon_m P_j^1 \left(A_{ji} + A_{jk} P_i^1\right), \tag{2.3a}$$

$$\prod_{j=0}^{k} \varepsilon_j \dot{H}_i^1 = A_{ik} - H_i^1 \left( A_{kk} - \sum_{j=0}^{k-1} \prod_{m=j+1}^{k} \varepsilon_m P_j^1 A_{jk} \right) + \\ + \sum_{j=0}^{k-1} \prod_{m=j+1}^{k} \varepsilon_m \left(A_{ij} + A_{ik} P_j^1\right) H_j^1, \tag{2.3b}$$

then system (2.1) has the form

$$\prod_{j=0}^{i} \varepsilon_j \dot{y}_i^1 = \sum_{j=0}^{k-1} B_{ij}^1 y_j^1, \quad i = \overline{0, k-1}, \tag{2.4a}$$

$$\prod_{j=0}^{k} \varepsilon_j \dot{y}_k^1 = B_{kk}^1 y_k^1, \tag{2.4b}$$

where $B_{ij}^1 = A_{ij} + A_{ik} P_j^1$, $i, j = \overline{0, k-1}$, $B_{kk}^1 = A_{kk} - \sum_{j=0}^{k-1} \prod_{m=j+1}^{k} \varepsilon_m P_j^1 A_{jk}$.

To perform the second step of decomposition, we make such assumption:

3) eigenvalues of $B_{k-1,k-1}^1$ satisfy the inequality

$$\mathrm{Re}\lambda_i(B_{k-1,k-1}^1) \le -2\gamma < 0.$$

Then by making the substitution

$$\begin{cases} y_i^1 = y_i^2 + \prod_{j=i+1}^{k-1} \varepsilon_j H_i^2 y_{k-1}^2, \; i = \overline{0, k-2}, \\ y_{k-1}^1 = y_{k-1}^2 + \sum_{j=0}^{k-2} P_j^2 y_j^1 \end{cases} \tag{2.5}$$

two independent subsystems are obtained from system (2.4a)

$$\prod_{j=0}^{i} \varepsilon_j \dot{y}_i^2 = \sum_{j=0}^{k-2} B_{ij}^2 y_j^2, \quad i = \overline{0, k-2}, \tag{2.6a}$$

$$\prod_{j=0}^{k-1} \varepsilon_j \dot{y}_{k-1}^2 = B_{k-1,k-1}^2 y_{k-1}^2, \tag{2.6b}$$

where $B_{ij}^2 = B_{ij}^1 + B_{i,k-1}^1 P_j^2$, $i, j = \overline{0, k-2}$, $B_{k-1,k-1}^2 = B_{k-1,k-1}^1 -$
$- \sum_{j=0}^{k-2} \prod_{m=j+1}^{k-1} \varepsilon_m P_j^2 B_{jk}^1$.

Matrices $P_i^2, H_i^2, i = \overline{0, k-2}$, are uniformly bounded solutions of the systems

$$\prod_{j=0}^{k-1} \varepsilon_j \dot{P}_{i,2} = B_{k-1,i}^1 + B_{k-1,k-1}^1 P_i^2 - \sum_{j=0}^{k-2} \prod_{m=j+1}^{k-1} \varepsilon_m P_j^2 \left( B_{ji}^1 + B_{j,k-1}^1 P_{i,2} \right),$$

$$\prod_{j=0}^{k-1} \varepsilon_j \dot{H}_i^2 = B_{i,k-1}^1 - H_i^2 (B_{k-1,k-1}^1 - \sum_{j=0}^{k-2} \prod_{m=j+1}^{k-1} \varepsilon_m P_j^2 B_{j,k-1}^1) +$$

$$+\sum_{m=0}^{k-2}\prod_{n=m+1}^{k-1}\varepsilon_n\left(B^1_{im}+B^1_{i,k-1}P^2_m\right)H^2_m,\quad i=\overline{0,k-2}. \tag{2.7}$$

At the step $k$ by making the substitution

$$\begin{cases} y_0^{k-1}=y_0^k+\varepsilon_1 H_0^k y_1^k,\\ y_1^{k-1}=y_1^k+P_0^k y_0^{k-1},\end{cases} \tag{2.8}$$

we obtain the block-diagonal system

$$\begin{cases} \dot y_0^k=B_{00}^k y_0^k,\\ \prod\limits_{j=0}^{i}\varepsilon_j\dot y_i^{k-i+1}=B_{ii}^{k-i+1}y_i^{k-i+1}, i=\overline{1,k},\end{cases} \tag{2.9}$$

where $B_{00}^k=B_{00}^{k-1}+B_{01}^{k-1}P_0^k$, $B_{11}^k=B_{11}^{k-1}-\varepsilon_1 P_0^k B_{01}^{k-1}$.
Here the matrices $H_0^k, P_0^k$ of appropriate dimensions satisfy the equations

$$\begin{aligned} &\varepsilon_1\dot P_0^k=B_{10}^{k-1}+B_{11}^{k-1}P_0^k-\varepsilon_1 P_0^k\left(B_{00}^{k-1}+B_{02}^{k-1}P_0^k\right),\\ &\varepsilon_1\dot H_0^k=B_{01}^{k-1}-H_0^k\left(B_{11}^{k-1}-\varepsilon_1 P_0^k B_{01}^{k-1}\right)+\\ &\quad+\varepsilon_1\left(B_{00}^{k-1}+B_{02}^{k-1}P_0^k\right)H_0^k.\end{aligned} \tag{2.10}$$

At each step of decomposition we assume that
$3_i$) eigenvalues of matrices $B_{ii}^{k-i+1}, i=\overline{1,k-1}$, satisfy the inequality

$$\mathrm{Re}\lambda_j(B_{ii}^{k-i+1})\le -2\gamma<0.$$

For the transformation of system (2.1) to block-diagonal form (2.9) the existence of the corresponding solutions of systems (2.3a)-(2.3b), (2.7), ..., (2.10) is necessary. Then, we have the following lemma.

**Lemma 1.** *Let conditions 1)-2) be true. Then there exists $\varepsilon_k^\star>0$ such that for any $\varepsilon_k\in(0,\varepsilon_k^\star)$ system (2.3a)-(2.3b) has a unique bounded solution for $t\in\mathbb{R}$.*

*Proof.* Write the subsystem (2.3a) in the equivalent form

$$P_i(t,s,\varepsilon)=\frac{1}{\prod\limits_{j=0}^{k}\varepsilon_j}\int\limits_{-\infty}^{t}Q(t,s,\varepsilon)(A_{ki}(s,\varepsilon)-\sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m P_j(s,\varepsilon)(A_{ij}(s,\varepsilon)+$$
$$+A_{jk}(s,\varepsilon)P_i(s,\varepsilon)))ds,\qquad i=\overline{0,k-1}, \tag{2.11}$$

where $Q(t,s,\varepsilon)$ is the fundamental matrix of the equation

$$\prod_{j=0}^{k}\varepsilon_j\dot x_k=A_{kk}x_k, \tag{2.12}$$

which by 1)-2) satisfies the estimation

$$\|Q(t,s,\varepsilon)\| \le K e^{-\frac{3\beta}{2\prod_{j=0}^{k}\varepsilon_j}(t-s)}, \tag{2.13}$$

for some $K > 0$ and any $-\infty < s \le t < \infty$.

We shall seek a bounded solution of system (2.11) on the entire real axis by the method of successive approximations

$$P_i^{n+1} = \frac{1}{\prod_{j=0}^{k}\varepsilon_j}\int_{-\infty}^{t} Q\Big(A_{ki} - \sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m P_j^n\left(A_{ji} + A_{jk}P_i^n\right)\Big)ds, \tag{2.14}$$

$$i = \overline{0,k-1},\ \ n = 0,1,2,\dots,$$

putting $P_i^0 = 0,\ i = \overline{0,k-1}$.

Using inequality (2.13) and conditions 1)-2) in the way analogous to [5] it is easy to show that there exists $\varepsilon_k^\star > 0$ such that for all $\varepsilon_k \in (0,\varepsilon_k^\star)$ sequences $P_i^n(t,\varepsilon), i = \overline{0,k-1}$, are uniformly convergent as $n \to \infty$ and bounded for $t \in \mathbb{R}$.

Put $P_i(t,\varepsilon) = \lim_{n\to\infty} P_i^n(t,\varepsilon), i = \overline{0,k-1}$.

Considering the correlation for matrices $B_{ij}, i,j = \overline{0,k}$ we shall rewrite (2.3b) in the form

$$\prod_{j=0}^{k}\varepsilon_j \dot{H}_i^1 = A_{ik} + \sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m B_{ij}H_j^1 - H_i^1 B_{kk},\ i = \overline{0,k-1}. \tag{2.15}$$

Denote by $Q_{H_i}(t,s,\varepsilon)$ the fundamental matrices of the equations

$$\prod_{j=0}^{i}\varepsilon_j \dot{x}_i = B_{ij}(t,\varepsilon)x_i, \qquad i = \overline{0,k-1},$$

and by $\overline{Q}(t,s,\varepsilon)$ the fundamental matrix of the equation

$$\prod_{i=0}^{k}\varepsilon_i \dot{y}_k^1 = \left(A_{kk} - \sum_{i=0}^{k-1}\prod_{j=i+1}^{k}\varepsilon_j P_i^1 A_{ik}\right)y_k^1. \tag{2.16}$$

We can equivalently rewrite system (2.15) as follows

$$H_i^1 = -\frac{1}{\prod_{j=1}^{k}\varepsilon_j}\int_{-\infty}^{t}\overline{Q}(A_{ik} + \sum_{\substack{j=0,\\ j\neq i}}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m B_{ij}H_j^1)Q_{H_i}ds,\ i = \overline{0,k-1}. \tag{2.17}$$

If conditions 1)-2) are satisfied and $\varepsilon_k$ are sufficiently small then the fundamental matrix $\overline{Q}(t,s,\varepsilon)$ satisfies the estimation [5]

$$\left|\overline{Q}(t,s,\varepsilon)\right| \le Ke^{-\frac{\beta}{\prod_{i=0}^{k}\varepsilon_i}(t-s)}. \tag{2.18}$$

Using inequality (2.18) we can prove the existence of a bounded on the entire real axis solution of system (2.17) in a similar way as for (2.11).

The lemma is proved. □

Similar statements for systems (2.7), ..., (2.10) are true.

Let us obtain now a general decomposing transformation.

**Theorem 1.** *Let conditions 1)-3) be true. Then for sufficiently small parameters $\varepsilon_i, i=\overline{1,k}$, there exists a nonsingular substitution*

$$(x_0,x_1,x_2,\dots,x_k)^T = \Phi\left(y_0^k, y_1^k, y_2^{k-1},\dots,y_k^1\right)^T,$$

*which transforms system (2.1) to $(k+1)$ independent subsystems (2.9).*

*Proof.* The structure of elements of decomposition transformation

$$\Phi[i,j] = \begin{cases} R_j^{k-j} + \sum\limits_{m=0}^{j-1}\prod\limits_{n=m+1}^{j}\varepsilon_n R_m^{k-j+1}H_m^{k-j+1}, \quad (i\ge j)\wedge(i>1), \\ R_i^{k-i}=E,\ R_m^{k-i+1}=P_m^{k-i+1}, \\ R_m^{n+1}=R_m^n+R_{k-n}^n P_m^{n+1}, \\ \prod\limits_{m=i+1}^{j}\varepsilon_m H_i^{k+1-j}y_j^{k+1-j}, \quad (i<j)\wedge(j>1), \\ E,\ i=j=0, \\ \varepsilon_1 H_0^k,\ i=0,\ j=1, \\ P_0^k,\ i=1,\ j=0, \\ E+\varepsilon_1 P_0^k H_0^k,\ i=j=1, \end{cases}$$

follows from the representation of the substitution at each step of the decomposition. Let us show that this transformation is nonsingular.

From (2.4a), (2.4b) we obtain

$$\begin{cases} y_i^1 = x_i - \prod\limits_{j=i+1}^{k}\varepsilon_j H_i^1 y_k^1, \quad i=\overline{0,k-1}, \\ y_k^1 = x_k - \sum\limits_{j=0}^{k-1}P_j^1 x_j. \end{cases}$$

Now by making the next substitution we obtain

$$y_0^k = \sum_{j=0}^{k} D_{0j}^{k+1} x_j,$$
$$y_i^{k+1-i} = \sum_{j=0}^{k} D_{ij}^{k+1-i} x_j,$$

where for $n = \overline{0,k-1},\ j = \overline{0,k}$ one has

$$\begin{aligned}
&D_{k-n,j}^{n+1} = D_{k-n,j}^{n} - \sum_{m=0}^{k-n-1} P_m^{n+1} D_{k-n,j}^{n+1},\\
&D_{ij}^{n+1} = D_{ij}^{n} - \prod_{m=i+1}^{k-n} \varepsilon_m H_i^{n+1} D_{k-n,j}^{n+1}, i = \overline{0,k-n-1},\\
&D_{ij}^{0} = 0,\ i \neq j,\ D_{ii}^{0} = E,\\
&D_{0j}^{k+1} = \left(E + \varepsilon_1 H_0^k P_0^k\right) D_{0j}^{k-1} - \varepsilon_1 H_0^k D_{1j}^{k-1}.
\end{aligned}$$

Therefore, there exists the inverse matrix $\Phi^{-1}[i,j] = D_{ij}^{k+1-i},\ i = \overline{0,k}$.

The theorem is proved. □

*Remark* 1. Condition $3_i$) is difficult to verify, because matrices $P_j^i$ and $H_j^i$ can be found only in the simplest cases. It is easy to make sure that condition $3_i$) will be verified for small $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}$, if system (2.1) is of such form

$$\begin{cases}
\dot{x}_0 = \sum_{j=0}^{k} A_{0j} x_j,\\
\prod_{j=0}^{i} \varepsilon_j \dot{x}_i = \sum_{j=0, j\neq i+1}^{k} A_{ij} x_j + \varepsilon_i A_{i,i+1} x_{i+1}, \quad i = \overline{1,k},
\end{cases} \tag{2.19}$$

and the eigenvalues of the matrices $A_{ii}, i = \overline{1,k}$, satisfy the inequality

$$\mathrm{Re}\lambda_j(A_{ii}) \leq -2\gamma < 0.$$

## 3. Asymptotic decomposition of the splitting transformation.

We can find the exact form for the coefficients of the asymptotic decomposition only in the simplest cases, so we are interested in finding asymptotic decomposition of these coefficients and obtaining corresponding split systems.

Let us make the following important assumption:

4) matrices $A_{ij}(t), i, j = \overline{0,k}, A_{kk}^{-1}(t)$ and their derivatives up to $(n+1)$ order are uniformly bounded in $t \in \mathbb{R}$.

Consider the differential expression

$$T^0(u^0) = \sum_{i=0}^{k-1} A_{ki} x_i + A_{kk} u^0 - \prod_{i=0}^{k} \varepsilon_i \frac{d}{dt} u^0. \tag{3.1}$$

We show that the function $u^0(t,x_0(t),x_1(t),\dots,x_{k-1}(t),\varepsilon_k)$, which can be represented in the form

$$u^0=\sum_{i=0}^{k-1}\overline{P}_i^1(t,\varepsilon_k)x_i=\sum_{i=0}^{k-1}\left(\sum_{j=0}^{n}\varepsilon_k^j P_{i,j}^1(t)\right)x_i, \tag{3.2}$$

where matrices $P_{i,j}^1(t), j=\overline{0,n}, i=\overline{0,k-1}$, and their $(n-j+1)$ derivatives are uniformly bounded, is such that for bounded solutions of system (2.1) one has $T^0(u^0)=o(\varepsilon_k^{n+1})$.

We substitute (3.2) to (3.1) and choose the functions

$$P_{i,j}^1(t), j=\overline{0,n}, i=\overline{0,k-1},$$

so that in (3.1) all the terms that contain powers of $\varepsilon_k$ less than $(n+1)$, are vanishing. For the coefficients in representation (3.2) we obtain

$$\begin{aligned}
&P_{i,0}^1(t)=-A_{kk}^{-1}(t)A_{ki}(t),\\
&P_{i,j}^1(t)=A_{kk}^{-1}(t)\left(\prod_{m=1}^{k-1}\varepsilon_m\dot{P}_{i,j-1}^1(t)+\sum_{m=0}^{k-1}\prod_{l=m+1}^{k-1}\varepsilon_l P_{m,j-1}^1(t)A_{mi}(t)+\right.\\
&\qquad\left.+\sum_{m=0}^{k-1}\prod_{l=m+1}^{k-1}\varepsilon_l\sum_{l=0}^{j-1}P_{m,l}^1(t)A_{mk}(t)P_{i,j-l-1}^1(t)\right),\quad j=\overline{1,n}.
\end{aligned}\tag{3.3}$$

Observe that 4) implies that $P_{i,j}^1(t), j=\overline{0,n}, i=\overline{0,k-1}$, and their partial derivatives up to $(n-j+1)$ order, are bounded. If the functions $P_{i,j}^1$ are chosen by formulaes (3.3), then (3.1) has the form

$$T^0(u^0)=\varepsilon_k^{n+1}\left(\sum_{i=0}^{k-1}\eta_i^0(t,\varepsilon_1,\dots,\varepsilon_{k-1})x_i\right),$$

where $\eta_i^0, i=\overline{0,k-1}$, are uniformly bounded functions.

If we make in input system (2.1) the substitution $x_k=\sum_{i=0}^{k-1}\overline{P}_i^1x_i+\varepsilon_k^{n+1}z$, then for the variables $x_0,x_1,\dots,x_{k-1},z$ we obtain the system

$$\begin{cases}
\prod_{j=0}^{i}\varepsilon_j\dot{x}_i=\sum_{j=0}^{k-1}\left(A_{ij}+A_{ik}\overline{P}_j^1\right)x_j+\varepsilon_k^{n+1}A_{ik}z, \quad i=\overline{0,k-1},\\
\prod_{j=0}^{k}\varepsilon_j\dot{z}=\sum_{j=0}^{k-1}\eta_j^0(t,\varepsilon_1,\dots,\varepsilon_{k-1})x_j+A_{kk}z.
\end{cases}\tag{3.4}$$

System (3.4) is a system of type (2.1), for which there exists an integral manifold [12]

$$z = \sum_{i=0}^{k-1} P_{i,\star}^{1}(t,\varepsilon_k)x_i, \tag{3.5}$$

where $P_{i,\star}^{1}, i = \overline{0,k-1}$ are uniformly bounded functions.

If system (3.4) has integral manifold (3.5), then system (2.1) has the integral manifold

$$x_k = \sum_{i=0}^{k-1}\left(\overline{P}_i^1 + \varepsilon_k^{n+1}P_{i,\star}^1\right)x_i = \sum_{i=0}^{k-1} P_i^1 x_i$$

for which the asymptotic decomposition

$$x_k = \sum_{i=0}^{k-1}\left(\sum_{i=0}^{k-1}\varepsilon_k^j P_{i,j}^1 + \varepsilon_k^{n+1}P_{i,\star}^1\right)x_i \tag{3.6}$$

is valid.

By making in system (2.1) the substitution $x_k = \sum_{i=0}^{k-1} P_i^1 x_i y_k^1$, we can get

$$\begin{aligned}
&\prod_{j=0}^{i}\varepsilon_j\dot{x}_i = \sum_{j=0}^{k-1}\left(A_{ij} + A_{ik}P_j^1\right)x_j + A_{ik}y_k^1, \quad i = \overline{0,k-1},\\
&\prod_{j=0}^{k}\varepsilon_j\dot{y}_k^1 = \left(A_{kk} - \sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m P_j^1 A_{jk}\right)y_k^1.
\end{aligned} \tag{3.7}$$

Consider now the differential expressions

$$\begin{aligned}
T_i^0(u_0^0,u_1^0,\dots,u_{k-1}^0) = \sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m(A_{ij} + A_{ik}P_j^1)u_j^0 -\\
-\prod_{j=1}^{k}\varepsilon_j\frac{d}{dt}u_i^0 + A_{ik}y_k^1, i = \overline{0,k-1}.
\end{aligned} \tag{3.8}$$

Let us show, that there exist functions $u_i^0, i = \overline{0,k-1}$, which can be presented in the form

$$u_i^0 = \overline{H}_i^1 y_k^1 = \sum_{j=0}^{n}\varepsilon_k^j H_{i,j}^1 y_k^1, \quad i = \overline{0,k-1}, \tag{3.9}$$

where the matrices $H_{i,j}^1, j = \overline{0,n}$, and their $(n-j+1)$ derivatives, are uniformly bounded, and these functions are such that

$$T_i^0(u_0^0,u_1^0,\dots,u_{k-1}^0) = o(\varepsilon_k^{n+1}), i = \overline{0,k-1}.$$

We substitute (3.9) to (3.8) and choose functions $H^1_{i,j}, j=\overline{0,n}$, so that in (3.8) all the terms that contain powers of $\varepsilon_k$ less than $(n+1)$, are vanishing. For the coefficients $H^1_{i,j}$ we obtain

$$\begin{aligned}
&H^1_{i,0} = A_{ik}A^{-1}_{kk},\\
&H^1_{i,j} = \Bigg(\sum_{m=0}^{k-1}\prod_{l=m+1}^{k-1}\varepsilon_l A_{im}H^1_{m,j-1} + \sum_{m=0}^{k-1}\prod_{l=m+1}^{k-1}\varepsilon_l\sum_{l=0}^{j-1}H^1_{i,l}P^1_{m,j-1-l}A_{mk}+\\
&+\sum_{m=0}^{k-1}\prod_{l=m+1}^{k-1}\varepsilon_l A_{ik}\sum_{l=0}^{j-1}P^1_{m,l}H^1_{m,j-1-l} - \prod_{m=0}^{k-1}\varepsilon_m\dot{H}^1_{i,j-1}\Bigg)A^{-1}_{kk}, \quad j=\overline{1,n}.
\end{aligned} \tag{3.10}$$

Now 4) yields that $H^1_{i,j}$ and their partial derivatives are bounded. In this case differential expressions (3.8) have the form

$$T^0_i(u^0_0,u^0_1,\dots,u^0_{k-1}) = \varepsilon_k^{n+1}\mu^0_i(t,\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k)y^1_k,$$

where $\mu^0_i, i=\overline{0,k-1}$, are uniformly bounded functions.

If we make the substitution in system (3.7) now

$$x_i = \varepsilon_k^{n+1}y^1_i + \prod_{j=i+1}^{k}\varepsilon_j\overline{H}^1_i y^1_k, \quad i=\overline{0,k-1},$$

then for the variables $y^1_i, i=\overline{0,k}$, we obtain the system

$$\begin{aligned}
&\prod_{j=0}^{i}\varepsilon_j\dot{y}^1_i = \sum_{j=0}^{k-1}\left(A_{ij}+A_{ik}P^1_j\right)y^1_j + \mu^0_i y^1_k, \quad i=\overline{0,k-1},\\
&\prod_{j=0}^{k}\varepsilon_j\dot{y}^1_k = \left(A_{kk} - \sum_{j=0}^{k-1}\prod_{m=j+1}^{k}\varepsilon_m P^1_j A_{jk}\right)y^1_k,
\end{aligned} \tag{3.11}$$

of type (2.1), for which there exist integral manifolds

$$y^1_i = \prod_{j=i+1}^{k}\varepsilon_j H^1_{i,\star}y^1_k, i=\overline{0,k-1}. \tag{3.12}$$

If system (3.11) has integral manifolds (3.12), then system (3.7) has integral manifolds

$$x_i = \prod_{j=i+1}^{k}\left(\overline{H}^1_i + \varepsilon_k^{n+1}H^1_{i,\star}\right)y^1_k = \prod_{j=i+1}^{k}H^1_i y^1_k, \quad i=\overline{0,k-1},$$

for which the asymptotic decomposition

$$x_i = \prod_{j=i+1}^{k}\left(\sum_{j=0}^{n}\varepsilon_k^j H^1_{i,j} + \varepsilon_k^{n+1}H^1_{i,\star}\right)y^1_k, \quad i=\overline{0,k-1}$$

is valid.

By making in system (3.7) the substitution

$$x_i = y_i^1 + \prod_{j=i+1}^{k} \varepsilon_j H_i^1 y_k^1,$$

we obtain system (2.4a)-(2.4b) and the first stage of decomposition of (2.1) is finished.

**Theorem 2.** *Let the conditions 1), 2), 4) be true. Then for a sufficiently small $\varepsilon_k$ there exists a substitution (2.2), which transforms system (2.1) to form (2.4a)-(2.4b), and the coefficients of the asymptotic decomposition of transformation (2.2) can be uniquely found from algebraic relations (3.3), (3.10).*

Consider now $l+1 (l \geq 1)$ stage of the decomposition of the system

$$\prod_{j=0}^{i} \varepsilon_j \dot{y}_i^l = \sum_{j=0}^{k-l} B_{ij}^l y_j^l, \quad i = \overline{0, k-l}, \tag{3.13a}$$

$$\prod_{j=0}^{i} \varepsilon_j \dot{y}_l^{k-i+1} = B_{ii}^{k-i+1} y_i^{k-i+1}, i = \overline{k-l+1, k}. \tag{3.13b}$$

Assume the following condition to be true:

$5_l$) matrices $B_{ij}^l (i, j = \overline{0, k-l}), (B_{k-l,k-l}^l)^{-1}$ and their $(n+1)$ derivatives are uniformly bounded in $t \in \mathbb{R}$ .

Consider the differential expression

$$T^l(u^l) = \sum_{i=0}^{k-l-1} B_{k-l,i}^l y_i^l + B_{k-l,k-l} u^l - \prod_{i=0}^{k-l} \varepsilon_i \frac{d}{dt} u^l. \tag{3.14}$$

Let us show that the function $u^l(t, y_0^l(t), y_1^l(t), \ldots, y_{k-l-1}^l(t), \varepsilon_{k-l})$, which can be presented in the form

$$u^l = \sum_{i=0}^{k-l-1} \overline{P}_i^{l+1}(t, \varepsilon_{k-l}) = \sum_{i=0}^{k-l-1} \left( \sum_{j=0}^{n} \varepsilon_{k-l}^j P_{i,j}^{l+1}(t) \right) y_i^l, \tag{3.15}$$

where the matrices $P_{i,j}^{l+1}(t), j = \overline{0, n}, i = \overline{0, k-l-1}$, and their $(n-j+1)$ derivatives are uniformly bounded, is such that for bounded solutions of the system (3.13a) we have

$$T^l(u^l) = o(\varepsilon_{k-l}^{n+1}).$$

We substitute (3.15) to (3.14) and choose functions

$$P_{i,j}^{l+1}(t), j = \overline{0, n}, i = \overline{0, k-l-1},$$

so that in (3.14) all the terms that contain powers of $\varepsilon_{k-l}$ less than $(n+1)$, are vanishing. For the coefficients in representation (3.2) we obtain

$$\begin{aligned}
&P_{i,0}^{l+1}(t) = -(B_{k-l,k-l}^{l})^{-1}(t)B_{k-l,i}^{l}(t),\\
&P_{i,j}^{l+1}(t) = (B_{k-l,k-l}^{l})^{-1}(t)\left(\prod_{m=1}^{k-l-1}\varepsilon_m \dot{P}_{i,j-1}^{l+1}(t) + \sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s P_{m,j-1}^{l+1}B_{mi}^{l} + \right.\\
&\left. + \sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s \sum_{s=0}^{j-1} P_{m,s}^{l+1}B_{m,k-l}^{l}P_{i,j-s-1}^{l+1}\right), \quad j=\overline{1,n}.
\end{aligned} \tag{3.16}$$

Observe that $5_l$) implies that $P_{i,j}^{l+1}(t), j=\overline{0,n}, i=\overline{0,k-l-1}$, and their partial derivatives up to $(n-j+1)$ order, are bounded. If the functions $P_{i,j}^{l+1}$ are chosen by formulaes (3.16), then (3.14) will have the form

$$T^l(u^l) = \varepsilon_{k-l}^{n+1}\left(\sum_{i=0}^{k-l-1}\eta_i^l(t,\varepsilon_1,\dots,\varepsilon_{k-l-1})y_i^l\right),$$

where $\eta_i^l, i=\overline{0,k-l-1}$, are uniformly bounded functions.

If we make in input system (3.13a) the substitution

$$y_{k-l}^l = \sum_{i=0}^{k-l-1}\overline{P}_i^{l+1}y_i^l + \varepsilon_{k-l}^{n+1}z,$$

then for the variables $y_0^l, y_1^l, \dots, y_{k-l-1}^l, z$ we obtain the system

$$\begin{cases}
\prod_{j=0}^{i}\varepsilon_j\dot{y}_i^l = \sum_{j=0}^{k-l-1}\left(B_{ij}^l + B_{i,k-l}^l\overline{P}_j^{l+1}\right)y_j^l + \varepsilon_{k-l}^{n+1}B_{i,k-l}^l z,\\
\qquad i=\overline{0,k-l-1},\\
\prod_{j=0}^{k-l}\varepsilon_j\dot{z} = \sum_{j=0}^{k-l-1}\eta_j^l(t,\varepsilon_1,\dots,\varepsilon_{k-l-1})y_j^l + B_{k-l,k-l}^l z.
\end{cases} \tag{3.17}$$

System (3.17) is a system of type (3.13a), for which there exists an integral manifold

$$z = \sum_{i=0}^{k-l-1}P_{i,\star}^{l+1}(t,\varepsilon_{k-l})y_i^l, \tag{3.18}$$

where $P_{i,\star}^{l+1}, i=\overline{0,k-l-1}$ are uniformly bounded functions.

If system (3.17) has integral manifold (3.18), then system (3.13a) has integral manifold

$$y_{k-l}^l = \sum_{i=0}^{k-l-1}\left(\overline{P}_i^{l+1} + \varepsilon_{k-l}^{n+1}P_{i,\star}^{l+1}\right)y_i^l = \sum_{i=0}^{k-l-1}P_i^{l+1}y_i^l,$$

for which the asymptotic decomposition

$$y_{k-l}^{l} = \sum_{i=0}^{k-l-1} \left( \sum_{i=0}^{k-l-1} \varepsilon_{k-l}^{j} P_{i,j}^{l+1} + \varepsilon_{k-l}^{n+1} P_{i,\star}^{l+1} \right) y_i^l. \tag{3.19}$$

holds.

By making in system (3.13a) the substitution

$$y_k^l = \sum_{i=0}^{k-l-1} P_i^{l+1} y_i^l + y_{k-l}^{l+1}, \tag{3.20}$$

we can get

$$\begin{aligned} &\prod_{j=0}^{i} \varepsilon_j \dot{y}_i^l = \sum_{j=0}^{k-l-1} \left( B_{ij}^l + B_{i,k-l}^l P_j^{l+1} \right) y_j^l + B_{i,k-l} y_{k-l}^{l+1}, \\ &\qquad i = \overline{0, k-l-1}, \\ &\prod_{j=0}^{k-l} \varepsilon_j \dot{y}_{k-l}^{l+1} = \left( B_{k-l,k-l}^l - \sum_{j=0}^{k-l-1} \prod_{m=j+1}^{k-l} \varepsilon_m P_j^{l+1} B_{j,k-l}^l \right) y_{k-l}^{l+1}. \end{aligned} \tag{3.21}$$

Consider now the differential expressions

$$\begin{aligned} T_i^l(u_0^l, u_1^l, \ldots, u_{k-l-1}^l) = &\sum_{j=0}^{k-l-1} \prod_{m=j+1}^{k-l} \varepsilon_m (B_{ij}^l + B_{i,k-l} P_j^{l+1}) u_j^l - \\ &- \prod_{j=1}^{k-l} \varepsilon_j \frac{d}{dt} u_i^l + B_{i,k-l} y_{k-l}^{l+1}, i = \overline{0, k-l-1}. \end{aligned} \tag{3.22}$$

Let us show, that there exist functions $u_i^l, i = \overline{0, k-l-1}$, which can be presented in the form

$$u_i^l = \overline{H}_i^{l+1} y_{k-l}^{l+1} = \sum_{j=0}^{n} \varepsilon_{k-l}^{j} H_{i,j}^{l+1} y_{k-l}^{l+1}, \quad i = \overline{0, k-l-1}, \tag{3.23}$$

where matrices $H_{i,j}^{l+1}, j = \overline{0,n}$, and their $(n-j+1)$ derivatives, are uniformly bounded, and these functions satisfy

$$T_i^l(u_0^l, u_1^l, \ldots, u_{k-l-1}^l) = o(\varepsilon_{k-l}^{n+1}), i = \overline{0, k-l-1}.$$

We substitute (3.23) to (3.22) and choose functions $H_{i,j}^{l+1}, j = \overline{0,n}$, so that in (3.22) all the terms that contain powers of $\varepsilon_{k-l}$ less than $(n+1)$, are vanishing. For

the coefficients $H_{i,j}^{l+1}$ we obtain

$$\begin{aligned}
&H_{i,0}^{l+1} = B_{i,k-l}(B_{k-l,k-l}^{l})^{-1},\\
&H_{i,j}^{l+1}\\
&=\Bigg(\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-2}\varepsilon_s B_{im}^{l} H_{m,j-1}^{l+1}+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s\sum_{s=0}^{j-1}H_{i,s}^{l+1}P_{m,j-s-l}^{l+1}B_{m,k-l}^{l}+\\
&+\sum_{m=0}^{k-l-1}\prod_{s=m+1}^{k-l-1}\varepsilon_s B_{i,k-l}^{l}\sum_{s=0}^{j-1}P_{m,s}^{l+1}H_{m,j-1-s}^{l+1}-\prod_{m=0}^{k-l-1}\varepsilon_m\dot{H}_{i,j-1}^{l+1}\Bigg)(B_{k-l,k-l}^{l})^{-1},\\
&j=\overline{1,n}.
\end{aligned}\tag{3.24}$$

Now $5_l$) yields that $H_{i,j}^{l+1}$ and their partial derivatives are bounded. In this case differential expressions (3.22) will have the form

$$T_i^l(u_0^l,u_1^l,\dots,u_{k-l-1}^l)=\varepsilon_{k-l}^{n+1}\mu_i^l(t,\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k-l)y_{k-l}^{l+1},$$

where $\mu_i^l, i=\overline{0,k-l-1}$, are uniformly bounded functions.

If we make the substitution

$$y_i^l=\varepsilon_{k-l}^{n+1}y_i^{l+1}+\prod_{j=i+1}^{k-l}\varepsilon_j\overline{H}_i^{l+1}y_{k-l}^{l+1},\quad i=\overline{0,k-l-1},$$

to system (3.21) then for variables $y_i^{l+1}, i=\overline{0,k-l}$, we obtain the system

$$\begin{aligned}
&\prod_{j=0}^{i}\varepsilon_j\dot{y}_i^{l+1}=\sum_{j=0}^{k-l-1}\left(B_{ij}^{l}+B_{i,k-l}^{l}P_j^{l+1}\right)y_j^{l+1}+\mu_i^l y_{k-l}^{l+1}, i=\overline{0,k-l-1},\\
&\prod_{j=0}^{k-l}\varepsilon_j\dot{y}_{k-l}^{l+1}=\left(B_{k-l,k-l}^{l}-\sum_{j=0}^{k-l-1}\prod_{m=j+1}^{k-l}\varepsilon_m P_j^{l+1}B_{j,k-l}\right)y_{k-l}^{l+1}
\end{aligned}\tag{3.25}$$

of type (3.13a), for which there exist integral manifolds

$$y_i^{l+1}=\prod_{j=i+1}^{k-l}\varepsilon_j H_{i,\star}^{l+1}y_{k-l}^{l+1}, i=\overline{0,k-l-1}.\tag{3.26}$$

If system (3.25) has integral manifolds (3.26), then system (3.21) has the integral manifolds

$$y_i^l=\prod_{j=i+1}^{k-l}\left(\overline{H}_i^{l+1}+\varepsilon_{k-l}^{n+1}H_{i,\star}^{l+1}\right)y_{k-l}^{l+1}=\prod_{j=i+1}^{k-l}H_i^{l+1}y_{k-l}^{l+1},\quad i=\overline{0,k-l-1},$$

for which the asymptotic decomposition

$$y_i^l = \prod_{j=i+1}^{k-l} \left( \sum_{j=0}^{n} \varepsilon_{k-l}^j H_{i,j}^{l+1} + \varepsilon_{k-l}^{n+1} H_{i,\star}^{l+1} \right) y_{k-l}^{l+1}, \quad i = \overline{0, k-l-1}$$

holds.

By making in system (3.21) the substitution

$$y_i^l = y_i^{l+1} + \prod_{j=i+1}^{k-l} \varepsilon_j H_i^{l+1} y_{k-l}^{l+1}, \tag{3.27}$$

we obtain system with $l+1$ decomposed variables and $l+1$ stage of decomposition of the system (2.1) is finished.

**Theorem 3.** *Let the conditions 1), 2) and $5_l$), $l = \overline{1, k-1}$, be true. Then for sufficiently small $\varepsilon_l$ there exists a substitution (3.20),(3.27), which transforms system (3.13a) to the form*

$$\prod_{j=0}^{i} \varepsilon_j \dot{y}_i^{l+1} = \sum_{j=0}^{k-l-1} B_{ij}^{l+1} y_j^{l+1}, \quad i = \overline{0, k-l-1}, \tag{3.28a}$$

$$\prod_{j=0}^{k-l} \varepsilon_j \dot{y}_{k-l}^{l+1} = B_{k-l,k-l}^{l+1} y_{k-l}^{l+1}, \tag{3.28b}$$

*and the coefficients of the asymptotic decomposition of the transformation can be uniquely found by (3.16) and (3.24).*

*Remark* 2. By completing $k$ steps of the decomposition of system (2.1) by the scheme described above, we obtain block-diagonal system (2.9). Moreover, the coefficients of the asymptotic decomposition of the transformation can be found by recurrent algebraic relations analogous to (3.16), (3.24).

## 4. Example.

Consider the singularly perturbed system of differential equations with constant coefficients and two small parameters

$$\begin{aligned} \dot{x}_0 &= x_0 + x_1 + x_2, \\ \varepsilon_1 \dot{x}_1 &= x_0 + x_1 + x_2, \\ \varepsilon_1 \varepsilon_2 \dot{x}_2 &= -x_0 - x_1 - x_2. \end{aligned} \tag{4.1}$$

Let us make the substitution in (4.1) analogous to (2.2)

$$x_0 = y_0 + \varepsilon_1 \varepsilon_2 h_0 w, x_1 = y_1 + \varepsilon_2 h_1 w, x_2 = w + p_0 x_0 + p_1 x_1, \tag{4.2}$$

where $p_0, p_1, h_0, h_1$ are the solutions of the algebraic systems

$$\begin{cases} -1-p_0-\varepsilon_1\varepsilon_2 p_0-\varepsilon_1\varepsilon_2 p_0^2-\varepsilon_2 p_1-\varepsilon_2 p_0 p_1=0, \\ -1-p_1-\varepsilon_1\varepsilon_2 p_0-\varepsilon_1\varepsilon_2 p_0 p_1-\varepsilon_2 p_1-\varepsilon_2 p_1^2=0, \end{cases}$$

$$\begin{cases} h_0(-1+\varepsilon_1\varepsilon_2+\varepsilon_2)=1, \\ h_1(-1+\varepsilon_1\varepsilon_2+\varepsilon_2)=1 \end{cases}$$

By solving of these systems we obtain

$$p_0=p_1=-1, \quad h_0=h_1=\frac{1}{-1+\varepsilon_1\varepsilon_2+\varepsilon_2}.$$

Thus the substitution has the form

$$x_0=y_0+\frac{\varepsilon_1\varepsilon_2}{-1+\varepsilon_1\varepsilon_2+\varepsilon_2}w, \quad x_1=y_1+\frac{\varepsilon_2}{-1+\varepsilon_1\varepsilon_2+\varepsilon_2}w,$$
$$x_2=w-x_0-x_1.$$

In this case system (4.1) at the first step is split to the collection of independent equations

$$\begin{cases} \dot{y}_0=0, \\ \varepsilon_1\dot{y}_1=0, \\ \varepsilon_1\varepsilon_2\dot{w}=(-1+\varepsilon_1\varepsilon_2+\varepsilon_2)w. \end{cases} \tag{4.3}$$

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# THE INITIAL JUMPS OF SOLUTIONS AND INTEGRAL TERMS IN SINGULAR BVP OF LINEAR HIGHER ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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*Abstract.* We consider the two-point boundary value problem for the singularly perturbed higher order linear integro-differential equation. The initial jumps of solutions and integral terms are proved. The constructive formulae and asymptotic estimations for solutions and their derivatives are obtained.

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*Keywords:* singularly perturbation, integro-differential equation, small parameter, initial jump, asymptotic method, asymptotic expansion, boundary layer

## 1. Introduction

Interests to singularly perturbed equations are due to the fact that they are mathematical models of many applied problems associated with processes of diffusion, heat and mass transfers, chemical kinetics and combustion, problems of heat propagation in subtle bodies, theory of semiconductors, dynamics of gyroscopes, quantum mechanics, biology and biophysics, and many others. Various asymptotic methods exist to approximate solutions of certain singularly perturbed problems. Investigations by Lyusternik, Vishik, Vasil'eva and Butuzov [21, 22, 24], led to the development the method of boundary functions. This method is also known as the method of boundary layer corrections (O'Malley [14], [15], Smith [2]). Let us name some other asymptotic methods: matching of asymptotic expansions (Kevorkian and Cole [8], Lagerstrom [9], Eckhaus [2]), matching applied to relaxation oscillations (Mischenko and Rozov [11], Grasman [4]), WKB-method and matching for turning point problems (Fedoriuk [3], Olver [13]), multiple scales techniques (Kevorkian and Cole [8], Kapila [5]), averaging method (Bogoliubov and Mitropolskii [1], Sanders and Verhulst [19]), regularization of singularly perturbed problems (Lomov [10]).

Development of different asymptotic methods can be found, e.g., in O'Malley [15] and Vasil'eva [20].

For a nonlinear ordinary differential equation of the second order with a small parameter the initial value problem with singular initial conditions was studied by

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M. I. Vishik and L. A. Lyusternik [23] and K. A. Kasymov [6]. They show that the solution of the original initial value problem tends to the solution of the degenerate equation with *modified* initial conditions, when the small parameter approaches zero. Such problems became known as the Cauchy problems with *initial jumps*. The most general cases of the Cauchy problem for singularly perturbed nonlinear systems of ordinary differential and integro-differential equations, as well as for partial differential equations of hyperbolic type were studied by K. A. Kasymov [7]. Boundary problems for singularly perturbed ordinary differential equations of higher order with the initial jump was considered in [12].

In this work we consider boundary value problems for singularly perturbed n-th order linear integro-differential equations, when fast variables of solution $y^{(i)}(t,\varepsilon), i = \overline{m_1+1, n-1}$ satisfy $y_0^i(\varepsilon) \to \infty$ as $\varepsilon \to 0$ near to the boundary of the domain. It is shown that there is the phenomenon of initial jumps not only of solutions, but also of the integral terms. The boundary-value problems were investigated in [16], [17] by using parametrization and the numerical analytic method based upon successive approximations [18].

## 2. Statement of the problem and Preliminary materials

Consider the following integro-differential equation

$$L_\varepsilon y \equiv \varepsilon y^{(n)} + A_1(t)y^{(n-1)} + \ldots + A_n(t)y = F(t) + \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x)y^{(i)}(x)dx \tag{2.1}$$

with boundary conditions

$$h_i y(t,\varepsilon) \equiv \sum_{j=0}^{m_i} \alpha_{ij} y^{(j)}(0,\varepsilon) + \sum_{j=0}^{l_i} \beta_{ij} y^{(j)}(1,\varepsilon) = a_i, \ \ i = \overline{1,n}, \tag{2.2}$$

where $\varepsilon > 0$ is small parameter, $\alpha_{ij}, \beta_{ij}, a_i \in R$ are known constants independent of $\varepsilon$ .

Assume that the following conditions hold:

C1) $A_i(t), F(t), \ i = \overline{1,n}$, *are sufficiently smooth functions defined on the interval [0,1];*

C2) $A_1(t) \geq \gamma = const > 0, \ 0 \leq t \leq 1$;

C3) $\alpha_{1,m_1} \neq 0, \ n-1 > m_1 > m_2 \geq \ldots \geq m_n, \ n-1 > l_1 > l_2 \geq \ldots \geq l_n$, where $m_i, l_i$ are nonnegative integers;

C4) $\overline{\Delta} = \begin{vmatrix} h_2 y_{10}(t) & \dots & h_2 y_{n-1,0}(t) \\ \dots & \dots & \dots \\ h_n y_{10}(t) & \dots & h_n y_{n-1,0}(t) \end{vmatrix} \neq 0,$
*where* $y_{i0}(t)$, $i = 1,\dots,n-1$, *is the fundamental set of solutions of the following homogeneous degenerate differential equation*

$$L_0 y_0(t) \equiv A_1(t) y_0^{(n-1)} + \dots + A_n(t) y_0(t) = 0; \tag{2.3}$$

C5) *Functions* $H_i(t,x)$, $i = 0,\dots,m_1+1$, *are defined in the domain* $D = (0 \le t \le 1,\ 0 \le x \le 1)$ *and sufficiently smooth.*

Some other conditions will be imposed later. We consider the following homogeneous singularly perturbed equation associated with (2.1)

$$L_\varepsilon y \equiv \varepsilon y^{(n)} + A_1(t) y^{(n-1)} + \dots + A_n(t) y = 0. \tag{2.4}$$

**Lemma 1.** *If conditions (C1) and (C2) are satisfied, then the fundamental set of solutions* $y_i(t,\varepsilon)$, $i = 1,\dots,n$, *of (2.4) in the interval* $0 \le t \le 1$, *has the following asymptotic representation as* $\varepsilon \to 0$.*:*

$$\begin{cases} y_i^{(j)}(t,\varepsilon) = y_{i0}^{(j)}(t) + O(\varepsilon), i = 1,\dots,n-1, j = 0,\dots,n-1, \\ y_n^{(j)}(t,\varepsilon) = \dfrac{1}{\varepsilon^j} exp\left(\dfrac{1}{\varepsilon}\displaystyle\int_0^t \mu(x)dx\right)(\mu^j(t) y_{n0}(t) + O(\varepsilon)), j = 0,\dots,n-1, \end{cases} \tag{2.5}$$

*where* $\mu(t) = -A_1(t) < 0$, $y_{i0}(t)$, $i = 1,\dots,n-1$, *are solutions of the problem*

$$L_0 y_0 \equiv A_1(t) y_0^{(n-1)}(t) + \dots + A_n(t) y_0(t) = 0,$$

$$y_{i0}^{(j)}(0) = \begin{cases} 1, & j = i-1, \\ 0, & j \neq i-1, \end{cases} j = 0,\dots,n-2, i = 1,\dots,n-1,$$

*and*

$$y_{n0}(t) = (A_1(0)/A_1(t))^{n-1} exp\left(\int_0^t (A_2(x)/A_1(t))dx\right). \tag{2.6}$$

The proof of the Lemma 1 is readily obtained from the known theorems of L. Schlesinger, G. D. Birkhoff, and P. Noaillon [10].

**Definition 1.** Let the function $K(t,s,\varepsilon), 0 \le s \le t \le 1$, be a solution for the following problem

$$L_\varepsilon K(t,s,\varepsilon) = 0,\ K^{(j)}(s,s,\varepsilon) = 0,\ j = 0,\dots,n-2,\ K^{(n-1)}(s,s,\varepsilon) = 1. \tag{2.7}$$

The Cauchy function, $K(t,s,\varepsilon)$, can be represented as

$$K(t,s,\varepsilon)=\frac{W_n(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{2.8}$$

where $W(s,\varepsilon)$ is the Wronskian of the fundamental set of solutions $y_1(s,\varepsilon),\dots,$ $y_n(s,\varepsilon)$ of equation (2.4), $W_n(t,s,\varepsilon)$ is the n-th order determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the n-th row with the fundamental set of solutions $y_1(t,\varepsilon),\dots,y_n(t,\varepsilon)$.

**Lemma 2.** *If conditions C1) and C2) are satisfied, then for sufficiently small $\varepsilon$ the Cauchy function $K(t,s,\varepsilon)$ in the domain $0\le s\le t\le 1$ can be represented as followings*

$$\begin{gathered}K^{(j)}(t,s,\varepsilon)=-\varepsilon\frac{\overline{W}^{(j)}_{n-1}(t,s)}{\mu(s)\overline{W}(s)}+\varepsilon^{n-1-j}exp\left(\frac{1}{\varepsilon}\int\limits_s^t\mu(x)dx\right)\times\\ \times\frac{\mu^j(t)y_{n0}(t)}{\mu^{n-1}(s)y_{n0}(s)}+O\left(\varepsilon^2+\varepsilon^{n-j}exp\left(\frac{1}{\varepsilon}\int\limits_s^t\mu(x)dx\right)\right),\\ j=0,\dots,n-1,\end{gathered} \tag{2.9}$$

*where $y_{n0}(t)$ is given by (2.6), the determinant $\overline{W}(s)$ is the Wronskian of the fundamental set of solutions $y_{10}(s),\dots,y_{n-1,0}(s)$ of equation (2.3), and $\overline{W}^{(j)}_{n-1}(t,s)$ is the determinant obtained from $\overline{W}(s)$ by replacing the $(n-1)$-th row with $y_{10}(t),\dots,$ $y_{n-1,0}(t)$ .*

The proof of the Lemma 2 immediately follows by means of (2.5) and (2.8).

**Definition 2.** The functions $\Phi_i(t,\varepsilon)$, $i=1,\dots,n$, are called boundary functions for the boundary value problem (2.1) and (2.2), if they satisfy the following problem

$$L_\varepsilon\Phi_i(t,\varepsilon)=0,\ i=1,\dots,n,\ h_k\Phi_i(t,\varepsilon)=\begin{cases}1, & i=k,\\ 0, & i\ne k, k=1,\dots,n.\end{cases} \tag{2.10}$$

Let us consider the determinant

$$\Delta(\varepsilon)=\begin{vmatrix}h_1y_1(t,\varepsilon) & \dots & h_1y_n(t,\varepsilon)\\ \dots & \dots & \dots\\ h_ny_1(t,\varepsilon) & \dots & h_ny_n(t,\varepsilon)\end{vmatrix}.$$

For the determinant $\Delta(\varepsilon)$ using formulas (2.2) and (2.5), we obtain the following asymptotic representation as $\varepsilon\to 0$

$$\Delta(\varepsilon)=(-1)^{1+n}\frac{\mu^{m_1}(0)}{\varepsilon^{m_1}}(\alpha_{1,m_1}\overline{\Delta}+O(\varepsilon)), \tag{2.11}$$

where $\overline{\Delta}$ is the determinant given by the condition C4.

The boundary functions $\Phi_i(t,\varepsilon)$, $i=1,\dots,n$, on the interval [0,1] can be represented in the form

$$\Phi_i(t,\varepsilon)=\frac{\overline{\Delta}_i(t,\varepsilon)}{\Delta(\varepsilon)},\quad i=1,\dots,n, \tag{2.12}$$

where $\Delta_i(t,\varepsilon)$ is the determinant obtained from $\Delta(\varepsilon)$ by replacing the $i$-th row by the fundamental set of solutions $y_1(t,\varepsilon),\dots,y_n(t,\varepsilon)$ of the equation (2.4).

**Lemma 3.** *If conditions (C1)-(C4) are satisfied, then for boundary functions $\Phi_i(t,\varepsilon)$, $i=1,\dots,n$, in the interval $0\le t\le 1$ we obtain the following asymptotic representation as $\varepsilon\to 0$*

$$\Phi_1^{(j)}(t,\varepsilon) \tag{2.13}$$

$$=-\varepsilon^{m_1-m_2}\frac{\alpha_{2,m_2}\overline{\Delta}_1^{(j)}(t)}{\mu^{m_1-m_2}(0)\alpha_{1,m_1}\overline{\Delta}}+\varepsilon^{m_1-j}\cdot\frac{y_{n0}(t)\mu^j(t)}{\mu^{m_1}(0)\alpha_{1,m_1}}exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)$$

$$+O\left(\varepsilon^{m_1-m_2+1}+\varepsilon^{m_1+1-j}exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right),\quad j=0,\dots,n-1,$$

$$\Phi_i^{(j)}(t,\varepsilon)=\frac{\overline{\Delta}_{i-1}^j(t)}{\overline{\Delta}}+(-1)^{i-1}\varepsilon^{m_1-j}\cdot\frac{y_{n0}(t)\mu^j(t)\overline{\Delta}_i}{\mu^{m_1}(0)\alpha_{1,m_1}\overline{\Delta}}\times$$

$$\times exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx+O\left(\varepsilon+\varepsilon^{m_1+1-j}exp\left(\frac{1}{\varepsilon}\int_0^t\mu(x)dx\right)\right)\right),$$

$$j=0,\dots,n-1;\ i=2,\dots,n,$$

where $\overline{\Delta}_i^{(j)}(t)$ is the determinant obtained from $\overline{\Delta}$ by replacing the $i$-th row with the fundamental set of solutions $y_{10}^{(j)}(t),\dots,y_{n-1,0}^{(j)}(t)$ of the equation (2.3), and $\overline{\Delta}_i$, $i=2,,\dots,n$, are determinants obtained from the matrix

$$\begin{pmatrix} h_1y_{10}(t) & \dots & h_1y_{n-1,0}(t)\\ \dots & \dots & \dots\\ h_ny_{10}(t) & \dots & h_ny_{n-1,0}(t)\end{pmatrix}$$

after deleting the $i$-th row.

The proof of the Lemma 3 follows from (2.12) and by taking into account (2.2), (2.5), and (2.11).

## 3. Main results

We seek to find the solution of the boundary value problem (2.1) and (2.2) in the form

$$y(t,\varepsilon) = C_1\Phi_1(t,\varepsilon) + \ldots + C_n\Phi_n(t,\varepsilon) + \frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{3.1}$$

where $K(t,s,\varepsilon)$ is the Cauchy function, $\Phi_i(t,\varepsilon)$, $i = 1,\ldots,n$, are the boundary functions, $C_i$, $i = 1,\ldots,n$, are unknown constants, and $z(s,\varepsilon)$ is an unknown function. Use (3.1) in (2.1), to obtain that $z(s,\varepsilon)$ satisfies the following Fredholm integral equation of the second kind

$$z(s,\varepsilon) = F(t) + \sum_{i=1}^{n} C_i\varphi_i(t,\varepsilon) + \int_0^1 H(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{3.2}$$

where

$$\varphi_i(t,\varepsilon) = \int_0^1 \sum_{j=0}^{m_1+1} H_j(t,x)\Phi_i^{(j)}(x,\varepsilon)dx, \quad i = 1,\ldots,n,$$

$$H(t,s,\varepsilon) = \frac{1}{\varepsilon}\int_s^1 \sum_{j=0}^{m_1+1} H_j(t,x)K^{(j)}(x,s,\varepsilon)dx. \tag{3.3}$$

Assume that the following condition is valid.

C6) $\lambda = 1$ is not an eigenvalue of the kernel $H(t,s,\varepsilon)$.

In view of condition (C6), the integral equation (3.2) has a unique solution [22] that can be represented in the form

$$z(s,\varepsilon) = \overline{F}(t,\varepsilon) + \sum_{i=1}^{n} C_i\overline{\varphi}_i(t,\varepsilon), \tag{3.4}$$

where

$$\begin{aligned} \overline{F}(t,\varepsilon) &\equiv F(t) + \int_0^1 R(t,s,\varepsilon)F(s)ds, \\ \overline{\varphi}_i(t,\varepsilon) &= \varphi_i(t,\varepsilon) + \int_0^1 R(t,s,\varepsilon)\varphi_i(s,\varepsilon)ds, \quad i = 1,\ldots,n, \end{aligned} \tag{3.5}$$

and $R(t,s,\varepsilon)$ is a resolvent of the kernel $H(t,s,\varepsilon)$ such that

$$R(t,s,\varepsilon) = \overline{R}(t,s) + O(\varepsilon), \tag{3.6}$$

where $\overline{R}(t,s)$ is a part of $R(t,s,\varepsilon)$ independent of $\varepsilon$. The last representation follows easily by means of (2.9), (3.3) and boundedness of the kernel $H(t,s,\varepsilon)$ as $\varepsilon \to 0$. Use (3.4) in the right-hand side of (3.1) to obtain the solution of the boundary value problem (2.1) and (2.2) in the form

$$y(t,\varepsilon) = \sum_{i=1}^{n} C_i Q_i(t,\varepsilon) + P(t,\varepsilon), \tag{3.7}$$

where

$$Q_i(t,\varepsilon) = \Phi_i(t,\varepsilon) + \frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)\overline{\varphi}_i(s,\varepsilon)ds, \tag{3.8}$$

$$P(t,\varepsilon) = \frac{1}{\varepsilon}\int_0^t K(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds,$$

where $\overline{F}(t,\varepsilon)$ and $\overline{\varphi}_i(t,\varepsilon)$ are defined in (3.5). Now, we determine the unknown constants $C_i, i = 1,\dots,n$, in (3.7) which satisfy the boundary conditions (2.2). Thus, we need to solve the system of algebraic equations

$$\begin{cases} C_1(1+d_{11}(\varepsilon)) + C_2 d_{12}(\varepsilon) + \ldots + C_n d_{1n}(\varepsilon) & = a_1 - e_1(\varepsilon) \\ \ldots\ldots\ldots\ldots & \\ C_1 d_{n1}(\varepsilon) + C_2 d_{n2}(\varepsilon) + \ldots + C_n(1+d_{nn}(\varepsilon) & = a_n - e_n(\varepsilon), \end{cases} \tag{3.9}$$

where

$$d_{ik}(\varepsilon) = \sum_{j=0}^{l_i} \frac{\beta_{ij}}{\varepsilon}\int_0^1 K^{(j)}(1,s,\varepsilon)\overline{\varphi}_k(s,\varepsilon)ds, \quad k,i = 1,\dots,n,$$

$$e_i(\varepsilon) = \sum_{j=0}^{l_i} \frac{\beta_{ij}}{\varepsilon}\int_0^1 K^{(j)}(1,s,\varepsilon)\overline{F}(s,\varepsilon)ds, \quad i = 1,\dots,n.$$

In view of (2.9), (3.5) and (3.6) we have following asymptotic representations

$$d_{ik}(\varepsilon) = \overline{d_{ik}} + O(\varepsilon), \;\; e_i(\varepsilon) = \overline{e_i} + O(\varepsilon) \quad i = 1,\dots,n, \tag{3.10}$$

where

$$\overline{d}_{i1}(\varepsilon) = \sum_{j=0}^{l_i} \beta_{ij}\int_0^1 \frac{\overline{W}_{n-1}^{(j)}(1,s)}{\mu(s)\alpha_{1,m_1}\overline{W}(s)}\overline{H}_{m_1+1}(s,0)ds, \quad i = 1,\dots,n,$$

$$\overline{d}_{ik}(\varepsilon)=\sum_{j=0}^{l_i}\beta_{ij}\int_0^1\frac{\overline{W}_{n-1}^{(j)}(1,s)}{\mu(s)\alpha_{1,m_1}\overline{W}(s)}\overline{H}_k(s)ds,\quad i=1,\dots,n,\ k=2,\dots,n, \tag{3.11}$$

$$\overline{e}_i=-\sum_{j=0}^{l_i}\beta_{ij}\int_0^1\frac{\overline{W}_{n-1}^{(j)}(1,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds,\quad i=1,\dots,n,$$

the functions $\overline{H}_k(t)$, $\overline{H}_i(t,x)$, and $\overline{F}(t)$ have the form

$$\overline{H}_k(t)\equiv(-1)^k\frac{\overline{\Delta}_k}{\alpha_{1,m_1}\overline{\Delta}}\overline{H}_{m_1}+1(t,0)+\int_0^1\sum_{i=0}^{m_1+1}\overline{H}_i(t,x)\frac{\overline{\Delta}_{k-1}^{(i)}(x)}{\overline{\Delta}}dx,\ k=2,\dots,n, \tag{3.12}$$

$$\overline{H}_i(t,x)\equiv H_i(t,x)+\int_0^1\overline{R}(t,s)H_i(s,x)ds,\quad i=0,\dots,m_1+1,$$

$$\overline{F}(t)\equiv F(t)+\int_0^1\overline{R}(t,s)F(s)ds.$$

For the main determinant $\omega(\varepsilon)$ of the system (3.9), we have the following asymptotic representation: $\omega(\varepsilon)=\overline{\omega}+O(\varepsilon)$, where

$$\overline{\omega}=\begin{vmatrix}1+\overline{d}_{11} & \overline{d}_{12} & \dots & \overline{d}_{1n}\\ \overline{d}_{21} & 1+\overline{d}_{22} & \dots & \overline{d}_{2n}\\ \dots & \dots & \dots & \dots\\ \overline{d}_{n1} & \overline{d}_{n2} & \dots & 1+\overline{d}_{nn}\end{vmatrix}$$

.

Assume now that the following condition is valid:

C7) $\overline{\omega}\neq 0$.

Thus, in view of conditions (C1)-(C7), the singularly perturbed boundary value problem (2.1) and (2.2) has a unique solution $y(t,\varepsilon)$, which can be presented in the following form

$$y(t,\varepsilon)=\sum_{i=1}^{n}C_iQ_i(t,\varepsilon)+P(t,\varepsilon), \tag{3.13}$$

where $Q_i(t,\varepsilon)$, $i=1,\dots,n$, and $P(t,\varepsilon)$, defined by the formula (3.8), $C_i(\varepsilon)$, $i=1,\dots,n$, are the solutions of the system (3.9). Then the following Theorem is valid.

**Theorem 1.** *Under conditions (C1)-(C7), for the solution $y(t,\varepsilon)$ of the boundary value problem (2.1) and (2.2), and its derivatives in the interval $0 \le t \le 1$ the following asymptotic estimations hold as $\varepsilon \to 0$*

$$
\begin{aligned}
|y^{(j)}(t,\varepsilon)| \le \frac{c}{\overline{\omega}} \Bigg[ & \frac{1}{\alpha_{1,m_1}} \sum_{k=1}^{n} |\alpha_k| \cdot \max_{0\le t\le 1} |H_{m_1+1}(t,0)| + \sum_{k=2}^{n} |\alpha_k| + \\
& + \sum_{k=1}^{n} |\alpha_k| \cdot \varepsilon^{m_1-j} \exp\left(-\gamma \frac{t}{\varepsilon}\right) + \max_{0\le t\le 1} |F(t)| \Bigg], \\
& j = \overline{0,n-1}, \ \ 0 \le t \le 1,
\end{aligned}
\tag{3.14}
$$

*where $C > 0, \gamma > 0$ are constants independent of $\varepsilon$.*

*Proof of Theorem 1.* By means of (2.9), (2.13), (3.3), (3.5), (3.6) and (3.12), we obtain the following asymptotic representation for functions $Q_i(t,\varepsilon)$, $i = 1,\dots,n$, defined in (3.8), as $\varepsilon \to 0$

$$
Q_1^{(j)}(t,\varepsilon) = \int_0^t \frac{\overline{W}_{n-1}^{(j)}(t,s)}{\alpha_{1,m_1}\mu(s)\overline{W}(s)} \overline{H}_{m_1+1}(s,0)ds +
$$

$$
+\varepsilon^{m_1-j} \frac{y_{n0}(t)\mu^j(t)}{\mu^{m_1}(0)\alpha_{1,m_1}} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right) +
$$

$$
+O\left(\varepsilon + \varepsilon^{m_1+1-j} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right), j = 0,\dots,n-2;
$$

$$
Q_1^{(n-1)}(t,\varepsilon) = \int_0^t \frac{\overline{W}_{n-1}^{(n-1)}(t,s)}{\alpha_{1,m_1}\mu(s)\overline{W}(s)} \overline{H}_{m_1+1}(s,0)ds +
$$

$$
+\frac{\overline{H}_{m_1+1}(t,0)}{\mu(t)\alpha_{1,m_1}} + \frac{1}{\varepsilon^{n-1-m_1}} \frac{y_{n0}(t)\mu^{n-1}(t)}{\mu^{m_1}(0)\alpha_{1,m_1}} \times
$$

$$
\times exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right) + O\left(\varepsilon + \frac{1}{\varepsilon^{n-2-m_1}} exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right),
$$

$$
Q_i^{(j)}(t,\varepsilon) = \frac{\overline{\Delta}_{i-1}^{(j)}(t)}{\overline{\Delta}} - \int_0^t \frac{\overline{W}_{n-1}^{(j)}(t,s)}{\mu(s)\overline{W}(s)} \overline{H}_i(s)ds + \varepsilon^{m_1-j} \frac{(-1)^{i-1} y_{n0}(t)\mu^j(t)\overline{\Delta}_i}{\mu^{m_1}(0)\alpha_{1,m_1}\overline{\Delta}} \times
$$

$$\times exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)+O\left(\varepsilon+\varepsilon^{m_1+1-j}exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right),$$

$$i=2,\dots,n;\, j=0,\dots,n-2;$$

$$Q_i^{(n-1)}(t,\varepsilon)=\frac{\overline{\Delta}_{i-1}^{(n-1)}(t)}{\overline{\Delta}}-\int_0^t \frac{\overline{W}_{n-1}^{(n-1)}(t,s)}{\mu(s)\overline{W}(s)}\overline{H}_i(s)ds-\frac{\overline{H}_i(t)}{\mu(t)}+$$

$$+\frac{1}{\varepsilon^{n-1-m_1}}\frac{(-1)^{i-1}y_{n0}(t)\mu^{n-1}(t)\overline{\Delta}_i}{\mu^{m_1}(0)\alpha_{1,m_1}\overline{\Delta}}exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)+$$

$$+O\left(\varepsilon+\frac{1}{\varepsilon^{n-2-m_1}}exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)\right), i=2,\dots,n,$$

$$P^{(j)}(t,\varepsilon)=-\int_0^t \frac{\overline{W}_{n-1}^{(j)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds+O(\varepsilon),\, j=0,\dots,n-2,$$

$$P^{(n-1)}(t,\varepsilon)=-\int_0^t \frac{\overline{W}_{n-1}^{(n-1)}(t,s)}{\mu(s)\overline{W}(s)}\overline{F}(s)ds-\frac{\overline{F}(t)}{\mu(t)}+ \tag{3.15}$$

$$+\frac{y_{n0}(t)\mu^{n-1}(t)\overline{F}(0)}{\mu^n(0)}exp\left(\frac{1}{\varepsilon}\int_0^t \mu(x)dx\right)+O(\varepsilon).$$

In view of (3.10), $C_i(\varepsilon)$ can be asymptotically represented in the form

$$C_i(\varepsilon)=\frac{\overline{\omega_i}}{\overline{\omega}}+O(\varepsilon),\qquad i=1,\dots,n, \tag{3.16}$$

where $\overline{\omega_i}$ is the determinant obtained from $\overline{\omega}$ by substituting the $i$-th column by $\begin{pmatrix} a_1-\overline{e}_1 \\ \dots \\ a_n-\overline{e}_n \end{pmatrix}$. By using presentations of (3.16), (3.15) and condition (C2), we derive the asymptotic estimations (3.14) from (3.13). The Theorem is proved. □

**Definition 3.** The solution of singularly perturbed integro-differential equation at the point $t=t_0$ has the phenomenon of the $m$-th order initial jump, if at the point $t=t_0$ the solution of boundary value problem (2.1) and (2.2) have the pole of $\varepsilon$ :

$$y^{(j)}(t_0,\varepsilon)=O(1),\, j=\overline{0,m},\;\; y^{(m+j)}(t_0,\varepsilon)=O\left(\frac{1}{\varepsilon^j}\right),\, j=\overline{1,n-1-m},\varepsilon\to 0.$$

By Theorem 1, we have that

$$y^{(j)}(0,\varepsilon) = O(1), j = \overline{0,m_1}, y^{(m_1+j)}(0,\varepsilon) = O\left(\frac{1}{\varepsilon^j}\right), j = \overline{1,n-1-m_1}, \varepsilon \to 0.$$

Hence, according to Definition 3 one can easily see that the solution of the problem (2.1) and (2.2) has the $m_1$-th order initial jump at point $t = 0$.

Consider the *modified* degenerate integro-differential equation:

$$\begin{aligned} L_0\overline{y} &\equiv A_1(t)\overline{y}^{(n-1)}(t) + \sum_{i=2}^{n} A_i(t)\overline{y}^{(n-i)}(t) \\ &= F(t) + \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x)\overline{y}^{(i)}(x)dx + \Delta(t), \end{aligned} \tag{3.17}$$

with boundary conditions

$$\begin{aligned} h_1\overline{y}(t) &\equiv \sum_{j=0}^{m_1} \alpha_{1j}\overline{y}^{(j)}(0) + \sum_{j=0}^{l_1} \beta_{1j}\overline{y}^{(j)}(1) = a_1 - \alpha_{1,m_1}\Delta_0, \\ h_i\overline{y}(t) &\equiv \sum_{j=0}^{m_i} \alpha_{ij}\overline{y}^{(j)}(0) + \sum_{j=0}^{l_i} \beta_{ij}\overline{y}^{(j)}(1) = a_i, \ \ i = \overline{2,n}, \end{aligned} \tag{3.18}$$

where

$$\Delta(t) = \Delta_0 H_{m_1+1}(t,0). \tag{3.19}$$

It is seen that, the modified degenerate problem (3.17) and (3.18) has additional terms $\Delta(t)$ and $\Delta_0$, where $\Delta(t)$ is called *the initial jump of the integral term* and $\Delta_0$ is called *the initial jump of the $m_1$-th order derivative of the solution.* The following condition is needed in the paper.

C8) *The modified degenerate problem (3.17) and (3.18) has a unique solution $\overline{y}(t)$.*

**Theorem 2.** *If conditions (C1)-(C8) and (3.19) are satisfied, then in the interval $0 \le t \le 1$ the following asymptotic estimation is valid as $\varepsilon \to 0$*

$$\left|y^{(j)}(t,\varepsilon) - \overline{y}^{(j)}(t)\right| \le C\left[\varepsilon + \varepsilon^{m_1-j} exp\left(-\gamma\frac{t}{\varepsilon}\right)\right], \ j = 0,\dots,n-1, \tag{3.20}$$

*where $C > 0, \gamma > 0$ are constants independent of $\varepsilon$.*

*Proof of Theorem 2.* Let $u(t,\varepsilon) = y(t,\varepsilon) - \overline{y}(t)$ where $y(t,\varepsilon)$ is the solution of the problem (2.1) and (2.2) and $\overline{y}(t)$ is the solution of the problem (3.17) and (3.18). We substitute the variable $y(t,\varepsilon) = u(t,\varepsilon) + \overline{y}(t)$ to obtain the following problem

$$L_\varepsilon u = \int_0^1 \sum_{i=0}^{m_1+1} H_i(t,x)u^{(i)}(x,\varepsilon)dx - \varepsilon\overline{y}^{(n)} - \Delta(t), \tag{3.21}$$
$$h_1 u(t,\varepsilon) = \alpha_{1,m_1}\Delta_0,\ h_i u(t,\varepsilon) = 0,\ i = 2,\dots,n.$$

The problem (3.21) is of the same type as the problem (2.1) and (2.2). By applying Theorem 1 and estimates (3.14), we obtain the following asymptotic estimation for the solution $u(t,\varepsilon)$ of the problem (3.21) as $\varepsilon \to 0$

$$|u^{(j)}(t,\varepsilon)| \le C[\varepsilon + \max_{0\le t\le 1}|\Delta(t) - \Delta_0 H_{m_1+1}(t,0)| + \varepsilon^{m_1-j} exp(-\gamma\frac{t}{\varepsilon})], \tag{3.22}$$
$$j = 0,\dots,n-1.$$

From (3.22) and (3.19) one can obtain (3.20). The Theorem is proved. □

According to (3.22), it is true that

$$\lim_{\varepsilon\to 0} y^{(j)}(t,\varepsilon) = \overline{y}^{(j)}(t),\ j = \overline{0,m_1-1},\ 0 \le t \le 1,$$
$$\lim_{\varepsilon\to 0} y^{(m_1+j)}(t,\varepsilon) = \overline{y}^{(m_1+j)}(t),\ j = \overline{0,n-1-m_1},\ 0 < t \le 1. \tag{3.23}$$

From (3.23) it follows that the solution $y(t,\varepsilon)$ of the boundary value problem (2.1) and (2.2) converges to the solution $\overline{y}(t)$ of the modified degenerate problem (3.17) and (3.18) as $\varepsilon \to 0$ . We note that the limits for $y^{(m_1+j)}$, $j = \overline{0,n-1-m_1}$ of (3.23) are not uniform on the interval $0 \le t \le 1$ . They are uniform an interval $0 < t_0 \le t \le 1$ , where $t_0$ is sufficiently small but fixed number as $\varepsilon \to 0$.

## 4. Example

Let us consider the following singularly perturbed boundary value problem

$$\varepsilon y''' + ay'' = b + \int_0^1 \delta y''(x,\varepsilon)dx,\ a > 0, \delta \ne 0, \delta \ne a, 0 < t < 1,$$
$$h_1 y(t,\varepsilon) \equiv y(0,\varepsilon) + y'(0,\varepsilon) = a_1, \tag{4.1}$$
$$h_2 y(t,\varepsilon) \equiv y(0,\varepsilon) = a_2,$$
$$h_3 y(t,\varepsilon) \equiv y(1,\varepsilon) - y'(1,\varepsilon) = a_3,$$

where $a, b, \delta, a_i, i = 1,2,3$, are known constants. The exact solution of the problem (4.1) has the form

$$
\begin{aligned}
y(t,\varepsilon) = a_2 - \varepsilon \frac{2(a_2-a_3)(a-\delta)-b}{a\delta+2(a-\delta)\varepsilon-[2a^2-a\delta+2(a-\delta)\varepsilon]e^{-\frac{a}{\varepsilon}}} + \\
+\left(a_1-a_2+\frac{2a(a_2-a_3)(a-\delta)-ab}{a\delta+2(a-\delta)\varepsilon-[2a^2-a\delta+2(a-\delta)\varepsilon]e^{-\frac{a}{\varepsilon}}}\right)t + \\
+\frac{a\delta(a_2-a_3)+b\varepsilon-[ab-\delta a(a_3-a_2)+b\varepsilon]e^{-\frac{a}{\varepsilon}}}{a\delta+2(a-\delta)\varepsilon-(2a^2-a\delta+2(a-\delta)\varepsilon)e^{-\frac{a}{\varepsilon}}}t^2 + \\
+\varepsilon\frac{2(a_2-a_3)(a-\delta)-b}{a\delta+2(a-\delta)\varepsilon-[2a^2-a\delta+2(a-\delta)\varepsilon]e^{-\frac{a}{\varepsilon}}}e^{-\frac{at}{\varepsilon}}.
\end{aligned}
\tag{4.2}
$$

It is seen that $y^{(i)}(0,\varepsilon) = O(1)$, $y''(0,\varepsilon) = O\left(\frac{1}{\varepsilon}\right)$, $i = 0,1$, as $\varepsilon \to 0$. That is, the solution of problem (4.1) has the first order initial jump at t=0. Next, one can obtain that

$$
\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t),\ o \le t \le 1, \lim_{\varepsilon\to 0} y^{(j)}(t,\varepsilon) = \overline{y}^{(j)}(t),\ i = 1,2,\ o < t \le 1,
$$

where $\overline{y}(t) = a_2 + \left(a_1 - a_2 + \frac{2(a_2-a_3)(a-\delta)+b}{\delta}\right)t + (a_2-a_3)t^2$ is the solution of the modified degenerated problem

$$
\begin{aligned}
a\overline{y}'' = b + \int_0^1 \delta\overline{y}''(x)dx + \Delta(t),\ a > 0, \delta \ne 0, \delta \ne a, 0 < t < 1, \\
h_1\overline{y}(t) \equiv \overline{y}(0) + \overline{y}'(0) = a_1 + \Delta_0,\ h_2\overline{y}(t) \equiv \overline{y}(0) = a_2, \\
h_3\overline{y}(t) \equiv \overline{y}(1) - \overline{y}'(1) = a_3,
\end{aligned}
\tag{4.3}
$$

where

$$
\Delta(t) = \delta\Delta_0. \tag{4.4}
$$

By applying (4.4) to (4.3), we obtain the initial jump of the first order derivative $\Delta_0 = \frac{2(a-\delta)(a_2-a_3)-b}{\delta}$, and then from (4.4) the initial jump of integral part $\Delta(t) = 2(a-\delta)(a_2-a_3)-b$.

## 5. Conclusion

The method of initial jumps has been applied to the singular linear integro-differential equation. The modified degenerated equation with the initial jump of the integral term is constructed such that the solution of the original singular problem converges to the solution of the modified equation with the jumps in the initial values. It is verified that the order of the jump derivative depends on the highest order of derivatives in the right-hand side of the original equation.

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# A NOTE ABOUT ITERATED ARITHMETIC FUNCTIONS

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*Abstract.* Let $f:\mathbb{N}\to\mathbb{N}_0$ be a multiplicative arithmetic function such that for all primes $p$ and positive integers $\alpha$, $f(p^\alpha)<p^\alpha$ and $f(p)|f(p^\alpha)$. Suppose also that any prime that divides $f(p^\alpha)$ also divides $pf(p)$. Define $f(0)=0$, and let $H(n)=\lim_{m\to\infty} f^m(n)$, where $f^m$ denotes the $m^{th}$ iterate of $f$. We prove that the function $H$ is completely multiplicative.

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*Keywords:* arithmetic function, iteration, multiplicative, Euler totient, Schemmel totient

## 1. Introduction

The study of iterated arithmetic functions, especially functions related to the Euler totient function $\varphi$, has burgeoned over the past century. In 1943, H. Shapiro's monumental work on a function $C(n)$, which counts the number of iterations of $\varphi$ needed for $n$ to reach 2, paved the way for subsequent number-theoretic research [5]. In this paper, we study a problem concerning the limiting behavior of iterations of functions related to the Euler totient function.

Throughout this paper, we let $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{P}$ denote the set of positive integers, the set of nonnegative integers, and the set of prime numbers, respectively. We will let $f:\mathbb{N}_0\to\mathbb{N}_0$ be a multiplicative arithmetic function which has the following properties for all primes $p$ and positive integers $\alpha$.

I. $f(p^\alpha)<p^\alpha$.
II. $f(p)|f(p^\alpha)$.
III. If $q$ is prime and $q|f(p^\alpha)$, then $q|pf(p)$.
IV. $f(0)=0$.

First, note that property IV does not effectively restrict the choice of $f$. Indeed, we may let $f$ be any multiplicative arithmetic function that satisfies properties I, II, and III and then simply define $f(0)=0$. One class of arithmetic functions which satisfy I, II, and III are the Schemmel totient functions. For each positive integer $r$, the

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Schemmel totient function $S_r$ is a multiplicative arithmetic function which satisfies

$$S_r(p^\alpha) = \begin{cases} 0, & \text{if } p \le r; \\ p^{\alpha-1}(p-r), & \text{if } p > r \end{cases}$$

for all primes $p$ and positive integers $\alpha$ [4]. These interesting generalizations of the Euler totient function have applications in the study of magic squares [3, page 184] and in the enumeration of cliques in certain graphs [1].

Because $f$ is multiplicative, properties I and II of $f$ are equivalent to the following properties, which we will later reference.

A. For all integers $n > 1$, $f(n) < n$.
B. If $p$ is a prime divisor of a positive integer $n$, then $f(p) | f(n)$.

Let $f^0(n) = n$ and $f^{k+1}(n) = f(f^k(n))$ for all nonnegative integers $k$ and $n$. Observe that, for any $n \in \mathbb{N}$, $f^n(n) \in \{0,1\}$. Furthermore, $f^n(n) = \lim_{m\to\infty} f^m(n)$, so we will define $H(n) = \lim_{m\to\infty} f^m(n)$. The author has shown that the function $H: \mathbb{N} \to \{0,1\}$ is completely multiplicative for the case in which $f$ is a Schemmel totient function [2]. Our purpose is to prove that $H$ is completely multiplicative for any choice of a multiplicative arithmetic function $f$ that satisfies properties I, II, III, and IV. To help do so, we define the following sets.

$$P = \{p \in \mathbb{P}: H(p) = 1\}$$
$$Q = \{q \in \mathbb{P}: H(q) = 0\}$$
$$S = \{n \in \mathbb{N}: q \nmid n \ \forall\, q \in Q\}$$

We define $T$ to be the unique set of positive integers defined by the following criteria:

- $1 \in T$.
- If $p$ is prime, then $p \in T$ if and only if $f(p) \in T$.
- If $x$ is composite, then $x \in T$ if and only if there exist $x_1, x_2 \in T$ such that $x_1, x_2 > 1$ and $x_1 x_2 = x$.

Note that $T$ is a set of *positive* integers; in particular, $0 \notin T$. We may now establish a couple of lemmas that should make the proof of the desired theorem relatively painless.

**Lemma 1.** *Let $k \in \mathbb{N}$. If all the prime divisors of $k$ are in $T$, then all the positive divisors of $k$ (including $k$) are in $T$. Conversely, if $k \in T$, then every positive divisor of $k$ is an element of $T$.*

*Proof.* First, suppose that all the prime divisors of $k$ are in $T$, and let $d$ be a positive divisor of $k$. Then all the prime divisors of $d$ are in $T$. Let $d = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical prime factorization of $d$. As $p_1 \in T$, the third defining criterion of $T$ tells us that $p_1^2 \in T$. Then, by the same token, $p_1^3 \in T$. Eventually, we find that

$p_1^{\alpha_1} \in T$. As $p_1^{\alpha_1}, p_2 \in T$, we have $p_1^{\alpha_1} p_2 \in T$. Repeatedly using the third criterion, we can keep multiplying by primes until we find that $d \in T$. This completes the first part of the proof. Now we will prove that if $k \in T$, then every positive divisor of $k$ is an element of $T$. The proof is trivial if $k$ is prime, so suppose $k$ is composite. We will induct on $\Omega(k)$, the number of prime divisors (counting multiplicities) of $k$. If $\Omega(k) = 2$, then, by the third defining criterion of $T$, the prime divisors of $k$ must be elements of $T$. Therefore, if $\Omega(k) = 2$, we are done. Now, suppose the result holds whenever $\Omega(k) \leq h$, where $h > 1$ is an integer. Consider the case in which $\Omega(k) = h + 1$. By the third defining criterion of $T$, we can write $k = k_1 k_2$, where $1 < k_1, k_2 < k$ and $k_1, k_2 \in T$. By the induction hypothesis, all of the positive divisors of $k_1$ and all of the positive divisors of $k_2$ are in $T$. Therefore, all of the prime divisors of $k$ are in $T$. By the first part of the proof, we conclude that all of the positive divisors of $k$ are in $T$. □

**Lemma 2.** *The sets $S$ and $T$ are equal.*

*Proof.* First, note that $1 \in S \cap T$. Let $m > 1$ be an integer such that, for all $k \in \{1, 2, \ldots, m-1\}$, either $k \in S \cap T$ or $k \notin S \cup T$. We will show that $m \in S$ if and only if $m \in T$. First, we must show that if $k \in \{1, 2, \ldots, m-1\}$, then $k \in S$ if and only if $f(k) \in S$. Suppose, by way of contradiction, that $f(k) \in S$ and $k \notin S$. As $k \notin S$, we have that $k > 1$ and $k \notin T$. Lemma 1 then guarantees that there exists a prime $q$ such that $q|k$ and $q \notin T$. As $q \notin T$, the second defining criterion of $T$ implies that $f(q) \notin T$. As $f(k) \in S$, $f(k) \neq 0$. By property B of $f$, $f(q)|f(k)$, so $f(q) \neq 0$. Therefore, $f(q) \in \{1, 2, \ldots, m-1\}$, and $f(q) \notin T$. By the induction hypothesis, $f(q) \notin S$. Therefore, there exists some $q_0 \in Q$ such that $q_0|f(q)$. Thus, $q_0|f(q)|f(k)$, which contradicts the assumption that $f(k) \in S$.

Conversely, suppose that $f(k) \notin S$ and $k \in S$. The fact that $f(k) \notin S$ implies that $k > 1$, and the fact that $k \in S$ implies (by the induction hypothesis) that $k \in T$. By Lemma 1, all prime divisors of $k$ are elements of $T$. The second criterion defining $T$ then implies that $f(p) \in T$ for all prime divisors $p$ of $k$. Using Lemma 1 again, we conclude that, for any prime divisor $p$ of $k$, all prime divisors of $f(p)$ are in $T$. By property III of $f$, all prime divisors of $f(k)$ are elements of $T$. Therefore, Lemma 1 guarantees that $f(k) \in T$. From property A of $f$ and the fact that $0 \notin T$, we see that $f(k) \in \{1, 2, \ldots, m-1\}$. The induction hypothesis then implies that $f(k) \in S$, which is a contradiction. Thus, we have established that if $k \in \{1, 2, \ldots, m-1\}$, then $k \in S$ if and only if $f(k) \in S$.

We are now ready to establish that $m \in S$ if and only if $m \in T$. Assume, first, that $m$ is prime. By the second criterion defining $T$, $m \in T$ if and only if $f(m) \in T$. By the induction hypothesis and property A of $f$, $f(m) \in T$ if and only if $f(m) \in S$. From the preceding argument, we see that $f(m) \in S$ if and only if $f^2(m) \in S$. Similarly, $f^2(m) \in S$ if and only if $f^3(m) \in S$. Continuing this pattern, we eventually find that $m \in T$ if and only if $f^m(m) \in S$. Observe that $f^m(m) = H(m)$ and that $0 \notin S$ and $1 \in S$. Hence, $m \in T$ if and only if $H(m) = 1$. Because $m$ is prime, $H(m) = 1$ if and

only if $m \notin Q$. Finally, it follows from the definition of $S$ that $m \notin Q$ if and only if $m \in S$. This completes the proof of the case in which $m$ is prime.

Assume, now, that $m$ is composite. By Lemma 1, $m \in T$ if and only if all the prime divisors of $m$ are in $T$. Because $m$ is composite, all the prime divisors of $m$ are elements of $\{1, 2, \ldots, m-1\}$. Therefore, by the induction hypothesis, all the prime divisors of $m$ are in $T$ if and only if all the prime divisors of $m$ are in $S$. It should be clear from the definition of $S$ that all the prime divisors of $m$ are in $S$ if and only if $m \in S$. Hence, $m \in T$ if and only if $m \in S$. □

We may now use the sets $S$ and $T$ interchangeably. In addition, part of the above proof gives rise to the following corollary.

**Corollary 1.** *Let $k, r \in \mathbb{N}$. Then $f^r(k) \in S$ if and only if $k \in S$.*

*Proof.* The proof follows from the argument in the above proof that $f(k) \in S$ if and only if $k \in S$ whenever $k \in \{1, 2, \ldots, m-1\}$. As we now know that we can make $m$ as large as we need, it follows that $f(k) \in S$ if and only if $k \in S$. Repeating this argument, we see that $f^2(k) \in S$ if and only if $f(k) \in S$. The proof then follows from repeated application of the same argument. □

**Corollary 2.** *For any positive integer $k$, $H(k) = 1$ if and only if $k \in S$.*

*Proof.* It is clear that $H(k) = 1$ if and only if $H(k) \in S$. Therefore, the proof follows immediately from setting $r = k$ in Corollary 1. □

Notice that Corollary 2, Lemma 2, and the defining criteria of $T$ provide a simple recursive means of constructing the set of positive integers $x$ that satisfy $H(x) = 1$. We also have the following theorem.

**Theorem 1.** *The function $H : \mathbb{N} \to \{0, 1\}$ is completely multiplicative.*

*Proof.* Corollary 2 tells us that $H$ is the characteristic function of the set $S$ of positive integers that are not divisible by primes in $Q$. If $x, y \in \mathbb{N}$, then it is clear that $xy \in S$ if and only if $x \in S$ and $y \in S$. The proof follows immediately. □

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# LIE IDEALS AND ACTION OF GENERALIZED DERIVATIONS IN RINGS

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*Abstract.* Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$ and $F, G, H$ be the generalized derivations with associated derivations $d, \delta, h$ of $R$ respectively. In the present paper, we study the situations if one the follwoing holds (1) $F(u)G(v) \pm H(uv) \in Z(R)$, (2) $F(u)F(v) \pm H(vu) \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

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## 1. Introduction

Let $R$ be a prime ring with center $Z(R)$. For any pair of elements $x, y \in R$, we shall write $[x, y]$ for the commutator $xy - yx$. An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$, if $[U, R] \subseteq U$. The centralizer of $U$ is denoted by $C_R(U)$ and defined by $C_R(U) = \{x \in R \mid [x, U] = 0\}$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a generalized inner derivation on $R$, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping $F$, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$, where $I_b$ is an inner derivation determined by $b$. This observation leads to the definition given in [8] : an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation $d$ if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously any derivation is a generalized derivation. Other basic examples of generalized derivations are the following: $(i)$ $F(x) = ax + xb$ for $a, b \in R$; $(ii)$ $F(x) = ax$ for some $a \in R$. Clearly, if $d = 0$, then $F$ is a left multiplier map of $R$. An additive subgroup $U$ of $R$ is said to be a Lie ideal if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal $U$ of $R$ is said to be square closed if $u^2 \in U$ for all $u \in U$.

In [5], Ashraf and Rehman established that a prime ring $R$ with a nonzero ideal $I$ must be commutative, if $R$ admits a nonzero derivation $d$ satisfying $d(xy) + xy \in Z(R)$ for all $x, y \in I$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$. Recently in [4] Ashraf et al. studied the case by replacing derivation $d$ with a generalized derivation $F$ in a

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prime ring $R$. More precisely, they proved that the prime ring $R$ with a nonzero ideal $I$ must be commutative, if $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ satisfying any one of the following situations: (i) $F(xy)-xy \in Z(R)$, (ii) $F(xy)+xy \in Z(R)$, (iii) $F(xy)-yx \in Z(R)$, (iv) $F(xy)+yx \in Z(R)$, (v) $F(x)F(y)-xy \in Z(R)$, (vi) $F(x)F(y)+xy \in Z(R)$; for all $x, y \in I$. In several papers, all these identities are also investigated in some appropriate subsets of prime and semiprime rings. For further details, we refer to [1, 3, 13, 14, 17, 18, 20]. Golbasi and Koc [13] studied all the cases (i) - (vi) in a square closed Lie ideal $U$ in a 2-torsion free prime ring $R$ and obtained that if $d \neq 0$, then $U \subseteq Z(R)$. It is natural to consider the situation $F(x)F(y) \pm yx \in Z(R)$ for all $x, y$ in some suitable subset of $R$. Recently, in [11], Dhara et al. considered this situation in a square closed Lie ideal $U$ in a 2-torsion free prime ring $R$ and obtained that if $d \neq 0$, then $U \subseteq Z(R)$.

The present paper is motivated by the previous results and our aim is to generalizes all the above results by considering three generalized derivations.

## 2. Preliminaries

Let $U$ be a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. Therefore, for any $u, v \in U$, we get $uv + vu = (u+v)^2 - u^2 - v^2 \in U$. Again in the same way, we have $uv - vu \in U$. Combining these two we get $2uv \in U$ for all $u, v \in U$.

Following results are needed for the proof of our main results.

**Lemma 1** ([19, Lemma 2.6]). *Let $R$ be a prime ring with char $(R) \neq 2$. If $U$ is a commutative Lie ideal of $R$, then $U \subseteq Z(R)$.*

**Lemma 2** ([7, Lemma 4]). *Let $R$ be a prime ring with char $(R) \neq 2$. If $U \nsubseteq Z(R)$ is a Lie ideal of $R$ and $aUb = 0$, then either $a = 0$ or $b = 0$.*

**Lemma 3** ([15, Theorem 5]). *Let $R$ be a prime ring with char $(R) \neq 2$. If $d$ be a nonzero derivation of $R$ and $U$ be a nonzero Lie ideal of $R$ such that $[u, d(u)] \in Z(R)$ for all $u \in U$, then $U \subseteq Z(R)$.*

**Lemma 4** ([12, Theorem 1])). *Let $R$ be a prime ring with char $(R) \neq 2$. If $d$ be a nonzero derivation of $R$ and $U$ be a nonzero Lie ideal of $R$ such that $u[[d(u), u], u] = 0$ for all $u \in U$, then $U \subseteq Z(R)$.*

**Lemma 5** ([9, Lemma 2]). *If $R$ is prime with a nonzero central ideal, then $R$ is commutative.*

**Lemma 6** ([6, Theorem 4]). *Let $R$ be a prime ring and $I$ be a nonzero left ideal of $R$. If $R$ admits a nonzero derivation $d$ which is centralizing on $I$, then $R$ is commutative.*

**Lemma 7** ([16, Theorem 2]). *Let $R$ be a prime ring with a nonzero derivation $d$ of $R$ and $I$ a nonzero ideal of $R$. If $x^p[d(x^q), x^r]_k = 0$ for all $x \in I$, where $p, q, r, k$ are fixed positive integers, then $R$ must be commutative.*

## 3. Results on Lie ideals in prime rings

**Theorem 1.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$, and $F, G$ and $H$ generalized derivations associated to the derivations $d$, $\delta$ and $h$ of $R$ respectively. Suppose that $F(u)G(v) - H(uv) \in Z(R)$ for all $u, v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* We assume that $U \nsubseteq Z(R)$ and prove that a contradiction. Now by the given hypothesis we have

$$F(u)G(v) - H(uv) \in Z(R) \text{ for all } u, v \in U. \tag{3.1}$$

Replacing $v$ by $2vw$ in (3.1) we get

$$2(F(u)(G(v)w + v\delta(w)) - H(uv)w - uvh(w)) \in Z(R) \text{ for all } u, v, w \in U.$$

Since char $(R) \neq 2$, this gives $(F(u)(G(v)w + v\delta(w)) - H(uv)w - uvh(w)) \in Z(R)$ that is,

$$(F(u)G(v) - H(uv))w + F(u)v\delta(w) - uvh(w) \in Z(R) \text{ for all } u, v, w \in U. \tag{3.2}$$

Commuting with $w$, we get

$$[(F(u)G(v) - H(uv))w, w] + [F(u)v\delta(w) - uvh(w), w] = 0 \text{ for all } u, v, w \in U. \tag{3.3}$$

Since $F(u)G(v) - H(uv) \in Z(R)$ for all $u, v \in U$, above relation reduces to

$$[F(u)v\delta(w) - uvh(w), w] = 0 \text{ for all } u, v, w \in U. \tag{3.4}$$

Now, replacing $u$ by $2ux$ in (3.4) and then using the restriction on characteristic, we obtain

$$[(F(u)x + ud(x))v\delta(w) - uxvh(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.5}$$

Again, putting $v = 2xv$ in (3.4) we get

$$[F(u)xv\delta(w) - uxvh(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.6}$$

Subtracting (3.6) from (3.5), we have

$$[ud(x)v\delta(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.7}$$

Replacing $u$ by $2tu$ and using (3.7) and char $(R) \neq 2$, we get

$$\begin{aligned} 0 &= [tud(x)v\delta(w), w] \\ &= t[ud(x)v\delta(w), w] + [t, w]ud(x)v\delta(w) \\ &= [t, w]ud(x)v\delta(w) \text{ for all } u, v, w, x, t \in U. \end{aligned} \tag{3.8}$$

By Lemma 2, for each $w \in U$, either $[t, w] = 0$ for all $t \in U$ or $d(x)v\delta(w) = 0$ for all $x, v \in U$. Let $T_1 = \{w \in U | [U, w] = (0)\}$ and $T_2 = \{w \in U | d(U)U\delta(w) = (0)\}$. Then $T_1$ and $T_2$ are two additive subgroups of $U$ such that $T_1 \cup T_2 = U$. Since a group cannot be union of its two proper subgroups, therefore either $T_1 = U$ or $T_2 = U$.

Let $T_1 = U$. Then $[U,U] = 0$ implying by Lemma 1 that $U \subseteq Z(R)$, a contradiction. Now let $T_2 = U$. Then $d(U)U\delta(U) = 0$. Again by Lemma 2, either $d(U) = 0$ or $\delta(U) = 0$. By Lemma 3, both of these imply $U \subseteq Z(R)$, a contradiction. □

**Theorem 2.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$ and $F,G$ and $H$ generalized derivations associated to the derivations $d$, $\delta$ and $h$ of $R$ respectively. Suppose that $F(u)G(v) + H(uv) \in Z(R)$ for all $u,v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* We note that $-H$ is a generalized derivations of $R$ with associated derivations $-h$. Hence replacing $H$ by $-H$ in Theorem 1, we have $F(u)G(v) - (-H)uv \in Z(R)$ for all $u,v \in U$, that is $F(u)G(v) + H(uv) \in Z(R)$ for all $u,v \in U$ implies $U \subseteq Z(R)$. □

In particular, when $F = d$ and $G = \delta$ are two nonzero derivations of $R$, then we have the following corollary:

**Corollary 1.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$, $d,\delta$ two nonzero derivation of $R$ and $H$ a generalized derivation associated to the derivation $h$ of $R$. If $d(u)\delta(v) \pm H(uv) \in Z(R)$ for all $u,v \in U$, then $U \subseteq Z(R)$.*

In particular, when $H$ is an identity map, then we have the following:

**Corollary 2.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$ and $F,G$ generalized derivations associated with the derivations $d$ and $\delta$ of $R$ respectively. Suppose that $F(u)G(v) \pm uv \in Z(R)$ for all $u,v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

**Theorem 3.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$ and $F,G$ generalized derivations associated with the derivations $d$ and $\delta$ of $R$ respectively. Suppose that $F(u)F(v) - H(vu) \in Z(R)$ for all $u,v \in U$. If $d \neq 0$, then $U \subseteq Z(R)$.*

*Proof.* On contrary assume that $U \nsubseteq Z(R)$. To prove our theorem we have to prove that this assumption leads to a contradiction. By the hypothesis, we have

$$F(u)F(v) - H(vu) \in Z(R) \text{ for all } u,v \in U. \tag{3.9}$$

Putting $v = 2vw$ in (3.9)and using char$(R) \neq 2$, we have

$$F(u)(F(v)w + vd(w)) - H(v)wu - v\delta(wu) \in Z(R) \tag{3.10}$$

which gives

$$F(u)F(v)w + F(u)vd(w) - H(v)wu - v\delta(wu) \in Z(R). \tag{3.11}$$

Commuting with $w$, we have

$$[F(u)F(v)w + F(u)vd(w) - H(v)wu - v\delta(wu), w] = 0 \tag{3.12}$$

i.e.,

$$[F(u)F(v),w]w+[F(u)vd(w),w]-[H(v)wu,w]-[v\delta(wu),w]=0. \tag{3.13}$$

From (3.9), we can write that $[F(u)F(v)-H(vu),w]=0$ for all $u,v,w\in U$, that is, $[F(u)F(v),w]=[H(vu),w]$ for all $u,v,w\in U$. Thus (3.13) reduces to

$$[H(vu),w]w+[F(u)vd(w),w]-[H(v)wu,w]-[v\delta(wu),w]=0. \tag{3.14}$$

Putting $u=w^2$ in (3.14), we have

$$\begin{aligned}[H(v)w^2+v\delta(w^2),w]w+[(F(w)w+wd(w))vd(w),w]\\ -[H(v)w^3,w]-[v\delta(w^3),w]=0,\end{aligned} \tag{3.15}$$

i.e.,

$$[(F(w)w+wd(w))vd(w),w]-[vw^2\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.16}$$

Putting $v=2wv$ and $u=w$ in (3.14), then using char$(R)\neq 2$, we have

$$[H(wvw),w]w+[F(w)wvd(w),w]-[H(wv)w^2,w]-[wv\delta(w^2),w]=0 \tag{3.17}$$

i.e.,

$$[F(w)wvd(w),w]-[wvw\delta(w),w]=0 \text{ for all } u,v,w\in U. \tag{3.18}$$

Subtracting (3.18) from (3.16), we get

$$[wd(w)vd(w),w]-[vw^2\delta(w),w]+[wvw\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.19}$$

Now putting $v=2wv$ in (3.19) and using char$(R)\neq 2$ we get

$$[wd(w)wvd(w),w]-w[vw^2\delta(w),w]+w[wvw\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.20}$$

Left multiplying (3.19) by $w$ and then subtracting from (3.20), we get

$$[w[d(w),w]vd(w),w]=0 \text{ for all } v,w\in U. \tag{3.21}$$

Replacing $v$ with $2vw$ in (3.21) and using char$(R)\neq 2$, we have

$$[w[d(w),w]vwd(w),w]=0 \text{ for all } v,w\in U. \tag{3.22}$$

Now right multiplying (3.21) by $w$ and then subtracting from (3.22), we have

$$[w[d(w),w]v[d(w),w],w]=0 \tag{3.23}$$

and again replacing $v$ with $2vw$, we get

$$[w[d(w),w]vw[d(w),w],w]=0 \text{ for all } v,w\in U, \tag{3.24}$$

i.e.,

$$w[d(w),w]vw[d(w),w]w-w^2[d(w),w]vw[d(w),w]=0 \text{ for all } v,w\in U. \tag{3.25}$$

Now we put $v=8vw[d(w),w]u$ in (3.25) and using char$(R)\neq 2$, obtain

$$\begin{aligned}&w[d(w),w]vw[d(w),w]uw[d(w),w]w\\ &\quad -w^2[d(w),w]vw[d(w),w]uw[d(w),w]=0 \text{ for all } u,v,w\in U.\end{aligned}$$

By (3.25), this can be written as

$$w[d(w),w]vw^2[d(w),w]uw[d(w),w]-w[d(w),w]vw[d(w),w]wuw[d(w),w]=0$$

i.e.,

$$w[d(w),w]vw[[d(w),w],w]uw[d(w),w]=0 \text{ for all } u,v,w\in U.$$

By Lemma 2, this implies that $w[[d(w),w],w]=0$ for all $w\in U$. Then, by Lemma 4, we have $U\subseteq Z(R)$, a contradiction. □

**Theorem 4.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$, and $F$, $G$ are two generalized derivations associated to the derivations $d$ and $\delta$ of $R$ respectively. Suppose that $F(u)F(v)+H(vu)\in Z(R)$ for all $u,v\in U$. If $d\neq 0$, then $U\subseteq Z(R)$.*

*Proof.* Replacing $H$ by $-H$ and $h$ by $-h$ in Theorem 3, we get our conclusion. □

In particular, when $F=d$ is a nonzero derivation of $R$, then we have the following corollary:

**Corollary 3.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$, $d$ a nonzero derivation of $R$ and $H$ a generalized derivation associated to the derivation $h$ of $R$. If $d(u)d(v)\pm H(vu)\in Z(R)$ for all $u,v\in U$, then $U\subseteq Z(R)$.*

In particular, when $H$ is identity map of $R$, then we have the following corollary.

**Corollary 4.** *Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero square closed Lie ideal of $R$ and $F$ a generalized derivation of $R$ associated to the nonzero derivation $d$ of $R$. If $F(u)F(v)\pm vu\in Z(R)$ for all $u,v\in U$, then $U\subseteq Z(R)$.*

We know that any both sided ideal is also a Lie ideal of $R$. If $R$ is a prime ring and $I$ is a nonzero ideal of $R$, then $aIb=0$ implies either $a=0$ or $b=0$. Moreover, similar Lemmas are holds for both sided ideals (see Lemma 5, Lemma 6 and Lemma 7) in prime rings without assumption of char $(R)\neq 2$, therefore, we see that if we replace Lie ideal with a both sided ideal of $R$ in the above Theorems, then the conclusion remain valid even without assumption of characteristic on $R$. Thus the following corollaries are straightforward.

**Corollary 5.** *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $F,G$ and $H$ generalized derivations associated to the derivations $d$, $\delta$ and $h$ of $R$ respectively. Suppose that $F(x)G(y)\pm H(xy)\in Z(R)$ for all $x,y\in I$. If $d\neq 0$ and $\delta\neq 0$, then $R$ must be commutative.*

**Corollary 6.** *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $F$ and $H$ generalized derivations associated to the derivations $d$ and $\delta$ of $R$ respectively. Suppose that $F(x)F(y)\pm H(yx)\in Z(R)$ for all $x,y\in I$. If $d\neq 0$, then $R$ must be commutative.*

## 4. Results on semiprime rings with identity element

In this section we discussed the identity $F(x^n y^m) = F(x^n)F(y^m)$ for all $x, y \in R$. Let us introduce some well known and elementary definitions for the sake of completeness. For any nonempty subset $S$ of $R$. If $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in S$, then $F$ is called a generalized derivation which acts as a homomorphism or an anti-homomorphism on $S$, respectively.

Before the beginning our proofs, we would like to recall Ali et al. results, more precisely we refer to Theorem 4.1 and Theorem 4.3 in [2]. All that we need here is to remind the conclusions contained in [2] in the case $F$ is a generalized derivation associated with derivation $d$ in semiprime ring, because for $x, y, z \in R$, $F((xy)z) = F(x(yz))$ implies $F(xy)z + xyd(z) = F(x)yz + xd(yz)$, that is, $F(x)yz + xd(y)z + xyd(z) = F(x)yz + xd(yz)$, implying $R(d(yz) - d(y)z - yd(z)) = (0)$. Since $R$ is semiprime ring, this implies that $d$ is a derivation of $R$ and hence $F$ is a generalized derivation of $R$.

We summarize these reduced results in the following lemmas:

**Lemma 8** ([2, Theorem 4.1])**.** *Let $R$ be an $n!$-torsion free semiprime ring with identity 1, where $n \geq 2$ is a fixed integer and let $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F((xy)^n) = F(x^n y^n)$ holds for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

*Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.*

**Lemma 9** ([2, Theorem 4.3])**.** *Let $R$ be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where $m$ and $n$ are positive integers and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F(x^m y^n) = F(y^n x^m)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

*Moreover, if $R$ is prime and $d$ is a nonzero derivation of $R$, then $R$ is commutative.*

**Lemma 10** ([10, Theorem 2.2])**.** *Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $F$ a nonzero generalized derivation of $R$ associated with a derivation $d$. If $F(xy) = F(x)F(y)$ for all $x, y \in I$, then $d(I) = 0$ and $F$ is a commuting left multiplier mapping on $I$.*

*In particular, if $R$ is a prime ring, then $d = 0$ and $F$ is identity mapping of $R$.*

**Lemma 11** ([10, Theorem 2.4])**.** *Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $F$ a nonzero generalized derivation of $R$ associated with a derivation $d$. If $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = 0$ or $R$ contains a nonzero central ideal.*

*In particular, if $R$ is a prime ring, then $R$ is commutative and and $F$ is left multiplier mapping of $R$.*

We are now ready to prove our theorems.

**Theorem 5.** *Let $R$ be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where $m$ and $n$ are two fixed positive integers and $F$ a nonzero generalized derivation associated with a derivation $d$ of $R$. If $F(x^n y^m) = F(x^n)F(y^m)$ for all $x, y \in R$, then $d = 0$. In particular, if $R$ is a prime ring, then $d = 0$ and $F$ is a commuting left multiplier mapping of $R$.*

*In particular, if $R$ is a prime ring, then $F$ is identity mapping of $R$.*

*Proof.* We have the relation

$$F(x^n y^m) = F(x^n)F(y^m) \tag{4.1}$$

for all $x, y \in R$. In particular, when $x = 1$, we have from above that

$$F(y^m) = F(1)F(y^m) \tag{4.2}$$

for all $y \in R$. Now replacing $x$ by $x + k1$ in (4.1), where $k$ is any positive integer, we get

$$F((x+k1)^n y^m) = F((x+k1)^n)F(y^m)$$

for all $x, y \in R$. Expanding the power values of $(x + k1)$, we have

$$\begin{aligned} F\left(\left\{x^n + \binom{n}{1}kx^{n-1} + \binom{n}{2}k^2x^{n-2} + \cdots + \binom{n}{n-1}k^{n-1}x + k^n 1\right\} y^m\right) \\ = F\left(\left\{x^n + \binom{n}{1}kx^{n-1} + \binom{n}{2}k^2x^{n-2} + \right.\right. \\ \left.\left. \cdots + \binom{n}{n-1}k^{n-1}x + k^n 1\right\}\right)F(y^m) \end{aligned} \tag{4.3}$$

for all $x, y \in R$. Using relation (4.1) and (4.2), this can be written as

$$kf_1(x,y) + k^2 f_2(x,y) + \cdots + k^{n-1} f_{n-1}(x,y) = 0 \tag{4.4}$$

for all $x, y \in R$. Now, replacing $k$ by $1, 2, 3, \ldots, n-1$ in turn, and considering the resulting system of $n-1$ homogeneous equations, we see that the coefficient matrix of the system is a Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (n-1) & (n-1)^2 & (n-1)^3 & \cdots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n-1$, and since $R$ is $(n-1)!$-torsion free, it follows immediately that

$$f_1(x,y) = f_2(x,y) = \cdots = f_{n-1}(x,y) = 0$$

for all $x \in R$. Now, $f_{n-1}(x,y) = 0$ implies that

$$F\left(\binom{n}{n-1} xy^m\right) = F\left(\binom{n}{n-1} x\right) F(y^m) \tag{4.5}$$

for all $x, y \in R$. Which gives

$$nF(xy^m) = nF(x)F(y^m) \tag{4.6}$$

for all $x, y \in R$. Since $R$ is $n$-torsion free, we have

$$F(xy^m) = F(x)F(y^m) \tag{4.7}$$

for all $x, y \in R$. Again, since $R$ is $m!$-torsion free, by applying the same argument for $y$ as above for $x$, we can write that

$$F(xy) = F(x)F(y) \tag{4.8}$$

for all $x, y \in R$. Then by Lemma 10, $d = 0$ and $F$ is a commuting left multiplier mapping of $R$.

In particular, if $R$ is a prime ring, then $F$ is identity mapping of $R$. □

By the similar proof of Theorem 5, following theorem is straight forward by using Lemma 11.

**Theorem 6.** *Let $R$ be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where $m$ and $n$ are two fixed positive integers and $F$ a nonzero generalized derivation of $R$ associated with a derivation $d$. If $F(x^n y^m) = F(y^n)F(x^m)$ for all $x, y \in R$, then $d = 0$ or $R$ contains a nonzero central ideal.*

*In particular, if $R$ is a prime ring, then $R$ is commutative and and $F$ is left multiplier mapping of $R$.*

## 5. Some examples

This section contains two examples which shows that the main results are not true in the case of arbitrary rings.

*Example* 1. Let $\mathbb{Z}$ be the ring of integers. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad U = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Clearly, $R$ is a ring with identity under the natural operations which is not prime. Define the maps on $R$ as follows

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}, \quad d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};$$

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & b+a \\ 0 & 0 \end{pmatrix}, \quad \delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix};$$

$$H\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}, \quad h\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

Then, it is easy to see that $U$ is a nonzero square closed Lie ideal of $R$ and $F, G$, and $H$ are generalized derivations associated with nonzero derivations $d, \delta$, and $h$ of $R$ respectively. Moreover, $F, G$ and $H$ satisfies the requirements of Theorems 1, 2, 3, and 4, but $U \not\subseteq Z(R)$. Hence, the hypothesis of primeness is crucial.

*Example* 2. Let $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}\right\}$. Clearly, $R$ is a ring with identity which is not semiprime as $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0)$ for $b \neq 0$. Define $F, d : R \to R$ such that $F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and $d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. Then, it is easy to see that $F$ is a generalized derivation associated with derivation $d$ of $R$. Further, for any $x, y \in R$ the following conditions: $F(x^n y^m) = F(x^n)F(y^m)$, $F(x^n y^m) = F(y^n)F(x^m)$ are satisfied, where $m, n$ are positive integers. However, $d \neq 0$. Hence, in Theorems 5 and 6, the hypothesis of semiprimeness can not be omitted.

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# GENERALIZED DERIVATIONS ACTING AS HOMOMORPHISM OR ANTI-HOMOMORPHISM WITH CENTRAL VALUES IN SEMIPRIME RINGS

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*Abstract.* Let $R$ be a semiprime ring with center $Z(R)$. A mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper, our main object is to study the situations: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$; for all $x, y$ in some suitable subset of $R$.

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## 1. Introduction

Let $R$ be an associative ring with center $Z(R)$. For $x, y \in R$, $[x, y]$ denotes the commutator element $xy - yx$. We use the notation to define the Engel type polynomial $[x, y]_{n+1} = [[x, y]_n, y]$ instead of $[x, y, y, \ldots, y]$ for $n \geq 1$ and $[x, y]_1 = [x, y]$. Recall that a ring $R$ is called prime if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. An additive mapping $F : R \to R$ is called a generalized derivation of $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for any $x, y \in R$. If $d = 0$, then $F$ is said to be a left centralizer map of $R$. For any subset $S$ of $R$, $r_R(S)$ denotes the right annihilator of $S$ in $R$, that is, $r_R(S) = \{x \in R | Sx = 0\}$ and $l_R(S)$ denotes the left annihilator of $S$ in $R$ that is, $l_R(S) = \{x \in R | xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of $R$ and is written as $ann_R(S)$.

Let $S$ be a nonempty subset of a ring $R$. The mapping $F : R \to R$ is said to be a homomorphism ( anti-homomorphism) acting on $S$ if $F(xy) = F(x)F(y)$ holds for all $x, y \in S$ (respectively $F(xy) = F(y)F(x)$ holds for all $x, y \in S$).

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A series of papers in literature studied the homomorphism or anti-homomorphism of some specific type of additive mappings in prime and semiprime rings under certain conditions (see [1–4, 7, 10–12, 14, 15]).

In [4], Bell and Kappe showed that if a derivation $d$ of a prime ring $R$ can act as homomorphism or anti-homomorphism on a nonzero right ideal of $R$, then $d = 0$ on $R$. Then Ali, Rehman and Ali in [2] proved a similar result to Lie ideal case. They proved that if $R$ is a 2-torsion free prime ring, $L$ a nonzero Lie ideal of $R$ such that $u^2 \in L$ for all $u \in L$ and $d$ acts as a homomorphism or anti-homomorphism on $L$, then either $d = 0$ or $L \subseteq Z(R)$.

On the other hand, the authors developed above results, replacing the derivation $d$ with a generalized derivation $F$ of $R$. In this view, Rehman [14] proved the following:

*Let $R$ be a 2-torsion free prime ring and $I$ be a nonzero ideal of $R$. Suppose $F : R \to R$ is a nonzero generalized derivation with $d$.*

*(i) If $F$ acts as a homomorphism on $I$ and if $d \neq 0$, then $R$ is commutative.*

*(ii) If $F$ acts as an anti-homomorphism on $I$ and if $d \neq 0$, then $R$ is commutative.*

Recently, in [3] Ali and Huang studied the case when a generalized Jordan $(\alpha, \beta)$-derivation $F$ acts as homomorphism or anti-homomorphism on a square closed Lie ideal $U$ in prime ring $R$.

It is natural to investigate the above situations in semiprime rings. Recently, in [7] the first author of this article has studied the situations, when a generalized derivation $F$ of a semiprime ring $R$ acts as homomorphism or anti-homomorphism in a nonzero left ideal of $R$.

From above results, it is natural to consider the situations, when the generalized derivations $F$ satisfies the identities: (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$; for all $x, y$ in some suitable subset of $R$.

Recently, Albas [1] studied the above mentioned identities in prime rings. Albas proved the following theorems:

**Theorem A.** *Let $R$ be a prime ring with center $Z(R)$ and $I$ be a nonzero ideal of $R$. If $R$ admits a nonzero generalized derivation $F$ of $R$, with associated derivation $d$ such that $F(xy) - F(x)F(y) \in Z(R)$ or $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in I$, then either $R$ is commutative or $F = I_{id}$ or $F = -I_{id}$, where $I_{id}$ denotes the identity map of the ring $R$.*

**Theorem B.** *Let $R$ be a prime ring with center $Z(R)$ and $I$ be a nonzero ideal of $R$. If $R$ admits a nonzero generalized derivation $F$ of $R$, with associated derivation $d$ such that $F(xy) - F(y)F(x) \in Z(R)$ or $F(xy) + F(y)F(x) \in Z(R)$ for all $x, y \in I$, then $R$ is commutative.*

In the present paper our main object is to investigate the situations in semiprime rings.

## 2. Preliminaries

We shall use following basic identities which will be used frequently: for $x, y, z \in R$,

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z.$$

We need the following facts which will be used to prove our theorems:

**Fact-1.** [5, Theorem 3] *Let $R$ be a semiprime ring and $U$ a nonzero left ideal of $R$. If $R$ admits a derivation $d$ which is nonzero on $U$ and $[d(x), x] \in Z(R)$ for all $x \in U$, then $R$ contains a nonzero central ideal.*

**Fact-2.** [8, Fact-4]*Let $R$ be a semiprime ring, $d$ a nonzero derivation of $R$ such that $x[[d(x), x], x] = 0$ for all $x \in R$. Then $d$ maps $R$ into its center.*

**Fact-3.** [13, Corollary 2] *If $R$ is a semiprime ring and $I$ is an ideal of $R$, then $I \cap ann_R(I) = 0$.*

**Fact-4.** [6, Lemma 2] *(a) If $R$ is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of $R$; in particular, any commutative one-sided ideal is contained in the center of $R$.*

*(b) If $R$ is a prime ring with a nonzero central ideal, then $R$ must be commutative.*

**Fact-5.** [9, Corollary 2.6] *Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $F : R \to R$ a nonzero left centralizer map. (1) If $F(x)F(y) - F(xy) \in Z(R)$ for all $x, y \in I$, then either $R$ is commutative or $F(r) = r$ for all $r \in R$. (2) If $F(x)F(y) + F(xy) \in Z(R)$ for all $x, y \in I$, then either $R$ is commutative or $F(r) = -r$ for all $r \in R$.*

## 3. Main results

**Theorem 1.** *Let $R$ be a semiprime ring with center $Z(R)$ and $I$ a nonzero ideal of $R$. Let $F : R \to R$ be a generalized derivation associated with the derivation $d : R \to R$. If $F(xy) - F(x)F(y) \in Z(R)$ for all $x, y \in I$, then one of the following holds:*

*(1) $R$ contains a nonzero central ideal;*

*(2) $d(I) = (0)$ and $F$ is a left centralizer map on $I$ such that $[F(x), x] = 0$ for all $x \in I$.*

*Proof.* By our assumption, we have

$$F(xy) - F(x)F(y) \in Z(R) \tag{3.1}$$

for all $x, y \in I$. Replacing $y$ with $yz$, where $z \in I$, we get

$$F(xyz) - F(x)F(yz) \in Z(R) \tag{3.2}$$

which implies

$$F(xy)z + xyd(z) - F(x)\{F(y)z + yd(z)\} \in Z(R) \tag{3.3}$$

that is

$$(F(xy) - F(x)F(y))z + (x - F(x))yd(z) \in Z(R). \tag{3.4}$$

Commuting both sides with $z$, we get

$$[(F(xy) - F(x)F(y))z + (x - F(x))yd(z), z] = 0 \tag{3.5}$$

for all $x, y, z \in I$. By using (3.1), above relation yields

$$[(x - F(x))yd(z), z] = 0 \tag{3.6}$$

for all $x, y, z \in I$. Now we put $x = xz$, and then obtain that

$$[(xz - F(x)z - xd(z))yd(z), z] = 0 \tag{3.7}$$

which is

$$[(x - F(x))zyd(z), z] - [xd(z)yd(z), z] = 0 \tag{3.8}$$

for all $x, y, z \in I$. In (3.6), replacing $y$ with $zy$, we get

$$[(x - F(x))zyd(z), z] = 0 \tag{3.9}$$

for all $x, y, z \in I$. Using (3.9), (3.8) implies

$$[xd(z)yd(z), z] = 0 \tag{3.10}$$

for all $x, y, z \in I$. Now we put $x = d(z)x$ in (3.10), and then we see that

$$0 = [d(z)xd(z)yd(z), z] = d(z)[xd(z)yd(z), z] + [d(z), z]xd(z)yd(z) \tag{3.11}$$

for all $x, y, z \in I$. As an application of (3.10), (3.11) reduces to

$$[d(z), z]xd(z)yd(z) = 0 \tag{3.12}$$

for all $x, y, z \in I$. Replacing $x$ with $xz$ and $y$ with $zy$ respectively in (3.12), we get

$$[d(z), z]xzd(z)yd(z) = 0 \tag{3.13}$$

and

$$[d(z), z]xd(z)zyd(z) = 0 \tag{3.14}$$

for all $x, y, z \in I$. Subtracting one from another yields

$$[d(z), z]x[d(z), z]yd(z) = 0 \tag{3.15}$$

for all $x, y, z \in I$. Replacing $y$ with $yz$ in (3.15) and right multiplying (3.15) by $z$ respectively and then subtracting one from another yields

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \tag{3.16}$$

for all $x, y, z \in I$, which implies $(I[d(z), z])^3 = (0)$ for all $z \in I$. Since $R$ is semiprime, it contains no nonzero nilpotent left ideal, implying $I[d(z), z] = (0)$ for all $z \in I$. Thus, $[d(z), z] \in Ann_R(I)$ for all $z \in I$. Since $I$ is an ideal, we conclude that $[d(z), z] \in I$ for all $z \in I$. This implies that $[d(z), z] \in I \cap Ann_R(I)$ for all $z \in I$. In view of Fact-3, $[d(z), z] = 0$ for all $z \in I$. Further, if $d$ is derivation such that $d(I) \neq (0)$, then by Fact-1, $R$ contains a nonzero central ideal.

Let $d(I) = (0)$. Then $F(xy) = F(x)y + xd(y) = F(x)y$ for all $x, y \in I$, i.e., $F$ is a left centralizer map on $I$. Then by our hypothesis, we have

$$F(x)(y - F(y)) \in Z(R) \tag{3.17}$$

for all $x, y \in I$. Replacing $y$ with $yu$, where $u \in I$, we get

$$F(x)(y - F(y))u \in Z(R) \tag{3.18}$$

for all $x, y, u \in I$. Commuting both sides with $v$, where $v \in I$, we get

$$F(x)(y - F(y))uv - vF(x)(y - F(y))u = 0. \tag{3.19}$$

By using (3.18), it reduces to

$$vF(x)(y - F(y))u \in Z(R) \tag{3.20}$$

for all $u, v, x, y \in I$. We choose $x, y \in I$ such that $a = F(x)(y - F(y)) \neq 0$. Then from above, we have $IaI \subseteq Z(R)$, that is, $R$ contains a central ideal. If this ideal is zero ideal, then

$$I(F(x)(y - F(y))) = (0)$$

for all $x, y \in I$. Replacing $x$ with $xz$, $z \in I$, this gives

$$I(F(x)z(y - F(y))) = (0)$$

for all $x, y, z \in I$. Thus $F(x)z(y - F(y)) \in I \cap ann_R(I) = (0)$ for all $x, y, z \in I$. This gives

$$[F(x), x]z(y - F(y)) = 0$$

for all $x, y, z \in I$. Putting $y = y^2$ and $z = zy$ respectively and then subtracting one from another, we get

$$[F(x), x]z[F(y), y] = 0$$

for all $x, y, z \in I$. Since $I$ is an ideal of $R$, it follows that $([F(x), x]I)^2 = (0)$ for all $x \in I$. Since semiprime ring contains no nonzero nilpotent ideal, we have $[F(x), x]I = (0)$ for all $x \in I$. Thus by Fact-3, $[F(x), x] \in I \cap ann_R(I) = (0)$ for all $x \in I$. Thereby, the proof is completed. □

**Theorem 2.** *Let $R$ be a semiprime ring with center $Z(R)$ and $I$ a nonzero ideal of $R$. Let $F : R \to R$ be a generalized derivation associated with the derivation $d : R \to R$. If $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in I$, then one of the following holds:*

*(1) $R$ contains a nonzero central ideal;*

*(2) $d(I) = (0)$ and $F$ is a left centralizer map on $I$ such that $[F(x), x] = 0$ for all $x \in I$.*

*Proof.* If we replace $F$ with $-F$ and $d$ with $-d$ in Theorem 1, we conclude that $(-F)(xy) - (-F)(x)(-F)(y) \in Z(R)$ for all $x, y \in I$, implies $[(-d)(x), x] = 0$ for all $x \in I$, that is, $F(xy) + F(x)F(y) \in Z(R)$ for all $x, y \in I$, implies $[d(x), x] = 0$ for all $x \in I$. Hence conclusion follows by Theorem 1. □

The following corollary is immediate consequences of the Theorem 1 and Theorem 2 by using Fact-4 and Fact-5.

**Corollary 1.** *Let $R$ be a prime ring with center $Z(R)$ and $F : R \to R$ be a generalized derivation associated with the derivation $d : R \to R$.*

*(1) If $R$ satisfies $F(xy) + F(x)F(y) \in Z(R)$, then either $R$ is commutative or $F(x) = -x$ for all $x \in R$.*

*(2) If $R$ satisfies $F(xy) - F(x)F(y) \in Z(R)$, then either $R$ is commutative or $F(x) = x$ for all $x \in R$.*

**Theorem 3.** *Let $R$ be a semiprime ring with center $Z(R)$. Let $F : R \to R$ be a generalized derivation associated with the derivation $d : R \to R$. If $F(xy) - F(y)F(x) \in Z(R)$ for all $x, y \in R$, then one of the following holds: (1) $R$ contains a nonzero central ideal; (2) $F$ is a left centralizer map of $R$ such that $F : R \to Z(R)$.*

*Proof.* By hypothesis, we have

$$F(xy) - F(y)F(x) \in Z(R) \tag{3.21}$$

for all $x, y \in R$. Putting $x = xz$, we have

$$F(xzy) - F(y)(F(x)z + xd(z)) \in Z(R) \tag{3.22}$$

which gives

$$F(x)zy + xd(zy) - F(y)F(x)z - F(y)xd(z) \in Z(R). \tag{3.23}$$

Commuting both sides with $z$, we have

$$[F(x)zy - F(y)F(x)z - F(y)xd(z) + xd(zy), z] = 0 \tag{3.24}$$

that is

$$[F(x)zy, z] - [F(y)F(x), z]z - [F(y)xd(z) - xd(zy), z] = 0 \tag{3.25}$$

for all $x, y, z \in R$. From (3.21), we can write that $[F(xy) - F(y)F(x), z] = 0$ for all $x, y, z \in R$, that is, $[F(xy), z] = [F(y)F(x), z]$ for all $x, y, z \in R$. Thus (3.25) reduces to

$$[F(x)zy, z] - [F(xy), z]z - [F(y)xd(z) - xd(zy), z] = 0 \tag{3.26}$$

for all $x, y, z \in R$. Putting $y = z^2$ in (3.26), we have

$$[F(x)z^3, z] - [F(x)z^2 + xd(z^2), z]z - [(F(z)z + zd(z))xd(z) - xd(z^3), z] = 0 \tag{3.27}$$

that is,

$$[F(z)zxd(z)+zd(z)xd(z)-xd(z^3)+xd(z^2)z,z]=0 \tag{3.28}$$

for all $x,z\in R$. Putting $x=zx$ in (3.26), we have

$$[(F(z)x+zd(x))zy,z]-[F(z)xy+zd(xy),z]z-[F(y)zxd(z)-zxd(zy),z]=0 \tag{3.29}$$

that is,

$$[F(z)xzy,z]-[F(z)xy,z]z- \\ [F(y)zxd(z)-zxd(zy)-zd(x)zy+zd(xy)z,z]=0 \tag{3.30}$$

for all $x,y,z\in R$. Assuming $y=z$, we have

$$[F(z)zxd(z)-zxd(z^2)-zd(x)z^2+zd(xz)z,z]=0 \tag{3.31}$$

for all $x,z\in R$. Subtracting (3.31) from (3.28), we get

$$[zd(z)xd(z)-xd(z^3)+xd(z^2)z+zxd(z^2)+zd(x)z^2-zd(xz)z,z]=0 \tag{3.32}$$

for all $x,z\in R$. This reduces to

$$[zd(z)xd(z),z]+[-xd(z^3)+xd(z^2)z+zxd(z^2)-zxd(z)z,z]=0 \tag{3.33}$$

for all $x,z\in R$. Now putting $x=zx$ in (3.33), we get

$$[zd(z)zxd(z),z]+z[-xd(z^3)+xd(z^2)z+zxd(z^2)-zxd(z)z,z]=0 \tag{3.34}$$

for all $x,z\in R$. Left multiplying (3.33) by $z$ and then subtracting from (3.34), we get

$$[z[d(z),z]xd(z),z]=0 \tag{3.35}$$

for all $x,z\in R$. Again putting $x=xz$ in above relation, we get

$$[z[d(z),z]xzd(z),z]=0 \tag{3.36}$$

for all $x,z\in R$. Now right multiplying (3.35) by $z$ and then subtracting from (3.36), we obtain

$$[z[d(z),z]x[d(z),z],z]=0 \tag{3.37}$$

and hence

$$[z[d(z),z]xz[d(z),z],z]=0 \tag{3.38}$$

for all $x,z\in R$. This implies

$$z[d(z),z]xz[d(z),z]z-z^2[d(z),z]xz[d(z),z]=0 \tag{3.39}$$

for all $x,z\in R$. In (3.39), replacing $x$ with $xz[d(z),z]u$, we obtain

$$z[d(z),z]xz[d(z),z]uz[d(z),z]z-z^2[d(z),z]xz[d(z),z]uz[d(z),z]=0 \tag{3.40}$$

for all $x,u,z\in R$. Using (3.39), (3.40) gives

$$z[d(z),z]xz^2[d(z),z]uz[d(z),z]-z[d(z),z]xz[d(z),z]zuz[d(z),z]=0 \tag{3.41}$$

that is

$$z[d(z),z]x[z[d(z),z],z]uz[d(z),z]=0 \tag{3.42}$$

for all $x,u,z\in R$. This implies $[z[d(z),z],z]x[z[d(z),z],z]u[z[d(z),z],z]=0$ for all $x,u,z\in R$, which is $(R[z[d(z),z],z])^3=(0)$ for all $z\in R$. Since $R$ is semiprime, we conclude that $R[z[d(z),z],z]=(0)$ for all $z\in R$. Hence, $z[[d(z),z],z]=0$ for all $z\in R$. Then by Fact-2, either $d(R)=(0)$ or $d(R)\subseteq Z(R)$. If $d(R)\neq(0)$, then the second case implies $[d(x),x]=0$ for all $x\in R$. Hence in view of Fact-1, $R$ contains a nonzero central ideal.

Let $d(R)=(0)$. Then $F(xy)=F(x)y+xd(y)=F(x)y$ for all $x,y\in R$, i.e., $F$ is a left centralizer map of $R$. Then by our hypothesis, we have

$$F(x)y-F(y)F(x)\in Z(R) \tag{3.43}$$

for all $x,y\in R$. Replacing $y$ with $yu$, where $u\in R$, we get

$$F(x)yu-F(y)uF(x)\in Z(R) \tag{3.44}$$

that is

$$(F(x)y-F(y)F(x))u+F(y)[F(x),u]\in Z(R) \tag{3.45}$$

for all $x,y,u\in R$. Commuting both sides with $u$, we get

$$[(F(x)y-F(y)F(x))u,u]+[F(y)[F(x),u],u]=0 \tag{3.46}$$

for all $x,y,u\in R$. Since $F(x)y-F(y)F(x)\in Z(R)$ for all $x,y\in R$, we have from (3.46) that

$$[F(y)[F(x),u],u]=0 \tag{3.47}$$

for all $x,y,u\in R$. We put $y=yr$ in above and get

$$[F(y)r[F(x),u],u]=0 \tag{3.48}$$

for all $x,y,u,r\in R$. Now putting $y=yu$ in above, we have

$$[F(y)ur[F(x),u],u]=0 \tag{3.49}$$

for all $x,y,u,r\in R$. Left multiplying (3.48) by $u$, we get

$$[uF(y)r[F(x),u],u]=0 \tag{3.50}$$

for all $x,y,u,r\in R$. Subtracting (3.50) from (3.49), we obtain that

$$[[F(y),u]r[F(x),u],u]=0 \tag{3.51}$$

for all $x,y,u,r\in R$. In particular, above relation yields

$$[[F(x),u]r[F(x),u],u]=0 \tag{3.52}$$

that is

$$[F(x),u]r[F(x),u]u-u[F(x),u]r[F(x),u]=0 \tag{3.53}$$

for all $x,u,r\in R$. In (3.53), replacing $r$ with $r[F(x),u]v$ we get

$$[F(x),u]r[F(x),u]v[F(x),u]u-u[F(x),u]r[F(x),u]v[F(x),u]=0. \tag{3.54}$$

By using (3.53), (3.54) becomes

$$[F(x),u]ru[F(x),u]v[F(x),u]-[F(x),u]r[F(x),u]uv[F(x),u]=0, \tag{3.55}$$

which is

$$[F(x),u]r[[F(x),u],u]v[F(x),u]=0 \tag{3.56}$$

for all $x,r,u,v \in R$. Replacing $v$ with $vu$ in (3.56) and right multiplying (3.56) by $u$ respectively and then subtracting one from another, we have

$$[F(x),u]r[[F(x),u],u]v[[F(x),u],u]=0. \tag{3.57}$$

Similarly, just from above relation, we can write

$$[[F(x),u],u]r[[F(x),u],u]v[[F(x),u],u]=0. \tag{3.58}$$

Thus $([[F(x),u],u]R)^3=(0)$ for all $x,u \in R$. Since $R$ is semiprime, $R$ contains no nonzero nilpotent ideals. Hence $[[F(x),u],u]R=(0)$ and so $[[F(x),u],u]=0$ for all $x,u \in R$. Then by Fact-1, we conclude that either $[F(x),u]=0$ for all $x,u \in R$ or $R$ contains a nonzero central ideal. If $[F(x),u]=0$ for all $x,u \in R$, then $F$ maps $R$ into its center. Thus we obtain our all conclusions. □

**Theorem 4.** *Let $R$ be a semiprime ring with center $Z(R)$. Let $F:R\to R$ be a generalized derivation associated with the derivation $d:R\to R$. If $F(xy)+F(y)F(x)\in Z(R)$ for all $x,y\in R$, then one of the following holds: (1) $R$ contains a nonzero central ideal; (2) $F$ is a left centralizer map of $R$ such that $F:R\to Z(R)$.*

*Proof.* If we replace $F$ with $-F$ and $d$ with $-d$ in Theorem 3, we conclude that $(-F)(xy)-(-F)(y)(-F)(x)\in Z(R)$ for all $x,y\in R$ implies $x[(-d)(x),x]_2=0$ for all $x\in I$, that is, $F(xy)+F(y)F(x)\in Z(R)$ for all $x,y\in R$, implies $x[d(x),x]_2=0$ for all $x\in R$. Hence the conclusion follows by Theorem 3. □

We conclude our paper with the following example which shows that the above theorems do not hold for arbitrary rings.

**Example:** Consider the ring $R=\left\{\begin{pmatrix} a & b \\ 0 & 0\end{pmatrix} | a,b\in\mathbb{Z}\right\}$. Obviously, $R$ is not semiprime, because $\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}=(0)$.

We define maps $F,d:R\to R$ by $F\begin{pmatrix} a & b \\ 0 & 0\end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix}$ and $d\begin{pmatrix} a & b \\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & -b \\ 0 & 0\end{pmatrix}$. Then $F$ is a generalized derivation of $R$ associated with the derivation $d$ of $R$. For $I=R$, we have that $F(xy)-F(x)F(y)\in Z(R)$ for all $x,y\in I$ and $F(xy)-F(y)F(x)\in Z(R)$ for all $x,y\in I$. Since $d(R)\neq(0)$ and $R$ contains no nonzero central ideal for $Z(R)=\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\right\}$, the semiprimeness hypothesis in Theorem 1 and Theorem 3 is not superfluous.

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# OPTIMAL CONTROL OF A TWO TEAMS PREY-PREDATOR INTERACTION MODEL

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*This paper is dedicated to the memory of Professor Miklós Farkas .*

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*Abstract.* In this paper, a model of two teams of predators interacting with two teams of preys is proposed. The predator teams help each other and so do the prey teams. Therefore, some stability analysis are carried out on the model to detect the stability and instability of the system's equilibrium solutions. Also, the persistence of the solutions is investigated. Finally, the optimal control is applied to the unstable equilibrium solutions. Some numerical simulations are carried out to verify the analytic results.

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*Keywords:* predator, prey, equilibrium solutions, asymptotic stability, persistence, optimal control

## 1. Introduction

The work of Malthus [10] and Verhulst [13] is the base to the field of the growth and decline of a population [3, 16]. In the nature, species do not exist alone. So, there will be interactions with the other species. When species interact, the population dynamics of each species is affected. There are three main types of interaction [7, 11]:

(1) If the growth rate of one population is increased while the growth rate of the other population decreased, then the two populations are in a predator–prey situation.
(2) If the growth rate of each population is decreased affected by the other species, then the two populations are in a competition case.
(3) If the growth rate of each population is enhanced affected by the other species, then it is a mutualism situation.

The work that was proposed independently by Lotka [9] and Volterra [14] opened the gate for studying the predator–prey interaction. Lotka-Volterra model is the simplest model of these interactions. Since their pioneering work, many other notable contributions were made [12, 15].

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Many creatures form teams. This has, at least, two main advantages: the first is the improvement in foraging, since looking for food in a team is more efficient than doing it alone. The second advantage is that living in a team reduces predation risk due to early spotting of predators and that existing in a team gives a higher probability that the predator will attack another member of the team. In this paper, a model is given where two teams of predators interact with two teams of preys. The teams of each group (predators or preys) help each other.

A biological realization of our study is represented by herds of zebras and gazelles living side by side and attacked by two types of predators (like lions and tigers). Multi-team game theory has been studied previously in [1,2].

The model is presented in Section 2. The equilibrium points, their local stability and persistence are studied in Section 3. In Section 4, an optimal control to the model is investigated.

## 2. The model

Consider two teams of preys with densities $x_1,\ x_2$ interacting with two teams of predators with densities $y_1,\ y_2$, respectively. We assume that, in the absence of the predators, each team of preys grows logistically, $a_i\, x_i\,(1-x_i)$. Each team of the preys will contribute to the growth of the other team, $a_i\, x_j,\ \ i \neq j$. Also, the two teams of preys help each other against the predator. The effect of the predation is to reduce the prey growth rate by a term proportional to the prey and predator populations, $x_i\,(y_1+y_2)$. Finally, in the absence of any prey for sustenance, the predators death rate results in inverse decay, $c_i\, y_i$. Using these assumptions, the following model is proposed [6];

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1\,[x_1(1-x_1)+x_2]-x_1(y_1+y_2),\\
\frac{dx_2}{dt} &= a_2\,[x_2(1-x_2)+x_1]-x_2(y_1+y_2),\\
\frac{dy_1}{dt} &= -c_1 y_1+(x_1+x_2)(y_1+y_2),\\
\frac{dy_2}{dt} &= -c_2 y_2+(x_1+x_2)(y_1+y_2),
\end{aligned} \tag{2.1}$$

where the positive coefficients $a_1$ and $a_2$ represent the intrinsic growth rates of the preys densities $x_1$ and $x_2$, respectively. Both the positive parameters $c_1$ and $c_2$ are the death rates of the predator $y_1$ and $y_2$, respectively. It is clear that the two preys (predators) help each other e.g. in foraging and (in the case of preys) in early warning against predation.

## 3. THE ANALYSIS OF THE MODEL

The steady state (equilibrium) solutions are important [4]. The system (2.1) has the following equilibria. The trivial one $E_1 = (0,0,0,0)$, the boundary one $E_2 = (2,2,0,0)$ and the coexistence (interior) $E_3 = (x_1, x_2, y_1, y_2)$. The components of the interior solutions are given by the relations;

$$y_2 = \frac{c_1}{c_2} y_1, \; y_1 = \frac{a_1 c}{x_1 c_1}\left(c - x_1^2\right), \; x_2 = (c - x_1), \tag{3.1}$$

and

$$x_1^3 - c\,(1 + A_2)\, x_1^2 - c\,(1 - c\,A_2)\, x_1 + c^2 A_1 = 0, \tag{3.2}$$

where

$$A_1 = \frac{a_1}{a_1 + a_2}, \;\; A_2 = \frac{a_2}{a_1 + a_2} \;\text{ and }\; c = \frac{c_1\, c_2}{c_1 + c_2}.$$

*Proposition* 1. The system (2.1) has at least one and at most two interior equilibrium solutions [6].

*Proposition* 2. Local stability analysis [4] shows that the equilibrium state $E_1 = (0,0,0,0)$ is unstable, while $E_2 = (2,2,0,0)$ is stable under the conditions $c_1 + c_2 > 8$, and $4\,(c_1 + c_2) < c_1\, c_2$.

*Proof.* The Jacobian matrix of the system (2.1) is given by $J =$

$$\begin{bmatrix} a_1\,(1-2x_1) - (y_1 + y_2) & a_1 & -x_1 & -x_1 \\ a_2 & a_2\,(1-2x_2) - (y_1 + y_2) & -x_2 & -x_2 \\ (y_1 + y_2) & (y_1 + y_2) & (x_1 + x_2) - c_1 & (x_1 + x_2) \\ (y_1 + y_2) & (y_1 + y_2) & (x_1 + x_2) & (x_1 + x_2) - c_2 \end{bmatrix}.$$

At the equilibrium point $E_1 = (0,0,0,0)$, the above Jacobian matrix becomes;

$$J = \begin{bmatrix} a_1 & a_1 & 0 & 0 \\ a_2 & a_2 & 0 & 0 \\ 0 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & -c_2 \end{bmatrix},$$

whose eigenvalues are $\lambda = 0, -c_1, -c_1, a_1 + a_2$, which has one positive eigenvalue, $a_1 + a_2 > 0$. So, it is unstable equilibrium point.

Similarly, at the equilibrium point $E_2 = (2,2,0,0)$, The Jacobian matrix becomes;

$$J = \begin{bmatrix} -3a_1 & a_1 & -2 & -2 \\ a_2 & -3a_2 & -2 & -2 \\ 0 & 0 & 4 - c_1 & 4 \\ 0 & 0 & 4 & 4 - c_2 \end{bmatrix},$$

whose eigenvalues satisfy the characteristic equations;

$$\lambda^2 + 3\,(a_1 + a_2)\,\lambda + 8\,a_1\, a_2 = 0,$$

and

$$\lambda^2-(8-c_1-c_2)\,\lambda+c_1\,c_2-4\,(c_1+c_2)=0.$$

The first equation has two negative eigenvalues since $3\,(a_1+a_2)>0$, $8\,a_1\,a_2>0$ and the two other eigenvalues come from the second equation which are negative under the conditions

$$c_1+c_2>8, \quad 4\,(c_1+c_2)<c_1\,c_2.$$

□

**Definition 1.** A system is persistent if there exists a compact region $V$ subset of the interior of the state space such that all solutions with positive initial conditions are attracted to $V$ [8].

*Proposition* 3. The system (2.1) is persistent if at least one of the above conditions is not satisfied [6].

The characteristic equation of the eigenvalue of the internal equilibrium point $E_3$ is given by, $\lambda^4+D_1\lambda^3+D_2\lambda^2+D_3\lambda+D_4=0$, where

$$\begin{aligned}
D_1 &= (-2\,a_2)\,x_1+(c_1+c_2-a_1-a_2+2\,a_2\,c-2c)+(2a_1\,c)\,x_1^{-1}\\
D_2 &= (2\,a_1-a_1^2-2\,a_1\,a_2)\,x_1^2\\
&\quad+(4\,a_2\,c+2\,a_1\,a_2\,c+a_1\,a_2+a_1^2-2\,a_1\,c-2\,a_2\,c_1-2\,a_2\,c_2)\,x_1\\
&\quad-(4\,a_1a_2c+4\,a_2c^2+a_1c_1+a_1c_2+a_2c_1+a_2c_2-2\,a_2cc_1-2\,a_2cc_2-2\,a_2c)\\
&\quad+(a_1\,c)\,(2\,c_1+2\,c_2+2\,a_2\,c-a_1-a_2-2\,c)\,x_1^{-1}+(a_1^2c^2)\,x_1^{-2}\\
D_3 &= (2\,a_1^2)\,x_1^3+a_1\,(4\,a_2\,c+2\,c_1-2\,a_2c_1-2\,a_2c_2-a_1c_1-a_1c_2)\,x_1^2\\
&\quad-a_1\,(c\,(c_1+c_2)+(a_1+a_2+2\,a_2\,c)(2\,c-c_1-c_2))\,x_1+(2\,a_1\,c)\\
&\quad(2\,a_2\,(2\,c-c_1-c_2)-c_1)+(a_1\,c)((2\,c-c_1-c_2)(a_1+a_2-2\,a_2\,c)\\
&\quad+c\,(c_1+c_2)-2\,a_1\,c)\,x_1^{-1}+(a_1^2\,c\,c_1\,c_2)\,x_1^{-2}\\
D_4 &= (2\,a_1)(a_1\,c_2-a_2\,c_1+a_2\,c_2)\,x_1^3-(a_1\,c)(a_1\,(c_1+c_2)+2\,a_2\,(c_2-c_1))\,x_1^2\\
&\quad+(2\,a_1\,c)(a_1+a_2)(c_1-c_2)\,x_1+(2\,a_1\,a_2\,c^2)(c_2-c_1)-(2\,a_1^2\,c^2\,c_1)\,x_1^{-1}\\
&\quad+(a_1^2\,c^3)(c_1+c_2)\,x_1^{-2}
\end{aligned}$$

The Routh-Hurwitz criteria [4] shows that the interior point $E_3(x_1,x_2,y_1,y_2)$ is locally asymptotically stable if the determinants of all Hurwitz matrices are positive where the Hurwitz matrices are given by;

$$H_1=(D_1) \qquad H_2=\begin{pmatrix} D_1 & 1\\ D_3 & D_2\end{pmatrix}$$

$$H_3=\begin{pmatrix} D_1 & 1 & 0\\ D_3 & D_2 & D_1\\ 0 & D_4 & D_3\end{pmatrix} \quad H_4=\begin{pmatrix} D_1 & 1 & 0 & 0\\ D_3 & D_2 & D_1 & 1\\ 0 & D_4 & D_3 & D_2\\ 0 & 0 & 0 & D_4\end{pmatrix}$$

Using different values of the constants $a_1, a_2, c_1$ and $c_2$ to find the roots of equation (3.1), one gets that one of them is negative and one of the other two positive roots is greater than $c$ i.e $x_2$ will be negative. So, we have only one interior equilibrium point.

Let the initial values of the preys and the predators are, respectively, $x_1(0) = 0.1$, $x_2(0) = 0.4$, $y_1(0) = 0.3$, $y_2(0) = 0.7$, and the values of the parameters of the model are $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$ and $c_2 = 0.5$. Substituting by these values in equations (3.2, 3.1), we get the interior equilibrium point $E_3 = (0.079,\ 0.1085,\ 1.7208,\ 1.0325)$, which is asymptotically stable since there are two negative real eigenvalues $\lambda = -3.019684381,\ -0.4022762114$ and another two conjugate complex eigenvalues with negative real parts $\lambda = -0.08409198332 \pm 1.009538530\,I$. We simulate the model (2.1) for the mentioned initial values of the densities, we found that the system converges to the asymptotically stable point $E_3$ as shown in Fig. 1 and Fig. 2.

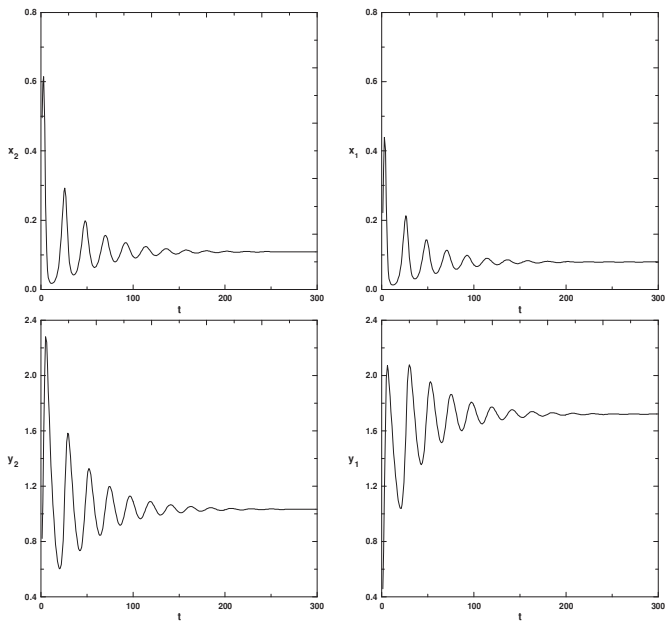


FIGURE 1. The densities of the preys and predators with time at the parameter values $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $x_1(0) = 0.1$, $x_2(0) = 0.4$, $y_1(0) = 0.3$, $y_2(0) = 0.7$.

## 4. OPTIMAL CONTROL PROBLEM

In some cases the presence of the predators is harmful. So, we should keep the situation in which the predator does not exist. Such as the case when the predators are viruses or tumors. Also, there are some animals that present in few numbers, like

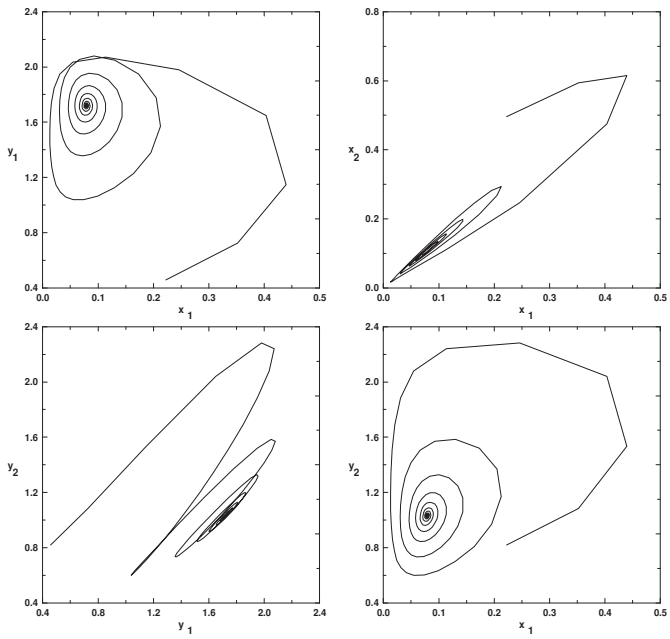


FIGURE 2. The system converges to the stable point $E_3$ for the parameter values $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $x_1(0) = 0.1,\ x_2(0) = 0.4,\ y_1(0) = 0.3,\ y_2(0) = 0.7$.

Panda, we have to keep them from extinction or predation. In cases like these, it is required to make these equilibrium solutions stable.

We saw that the first equilibrium solution $E_1$ is unstable and the second equilibrium solution $E_2$ is unstable if one of the conditions $c_1 + c_2 > 8$ or $4\,(c_1 + c_2) < c_1\, c_2$ is not satisfied. Then, we will use the optimal control technique to stabilize the second solution $E_2$ in this section.

Here, we will use the same feedback control approach which is presented in [5] for a different model. Consider the above system in the following form;

$$\begin{aligned}
\frac{dx_1}{dt} &= a_1\,[x_1(1-x_1)+x_2]-x_1(y_1+y_2)+u_1,\\
\frac{dx_2}{dt} &= a_2\,[x_2(1-x_2)+x_1]-x_2(y_1+y_2)+u_2,\\
\frac{dy_1}{dt} &= -c_1 y_1+(x_1+x_2)(y_1+y_2)+u_3,\\
\frac{dy_2}{dt} &= -c_2 y_2+(x_1+x_2)(y_1+y_2)+u_4,
\end{aligned} \tag{4.1}$$

where $u_i$, $(i = 1, 2, 3, 4)$, are the control inputs. Some performance measure is used to determine these inputs from the conditions of asymptotic stability of the system (2.1) about its unstable equilibrium states.

Let

$$z_1 = x_1 - \overline{x}_1, \;\; z_2 = x_2 - \overline{x}_2, \;\; w_1 = y_1 - \overline{y}_1, \;\; w_2 = y_2 - \overline{y}_2, \tag{4.2}$$

where $z_1, z_2, w_1, w_2$ represent perturbed states about its unstable equilibrium states and $(\overline{x}_1, \overline{x}_2, \overline{y}_1, \overline{y}_2)$ denote to the coordinates of the unstable equilibrium states of system (2.1). Substituting by (4.2) into the system (4.1) taking into account the identities that satisfied by the equilibrium states, we get the following system,

$$\begin{aligned}
\dot{z}_1 &= -(2a_1\overline{x}_1 + \overline{y}_1 + \overline{y}_2)z_1 + a_1(z_1 + z_2 - z_1^2) - (\overline{x}_1 + z_1)(w_1 + w_2) + u_1,\\
\dot{z}_2 &= -(2a_2\overline{x}_2 + \overline{y}_1 + \overline{y}_2)z_2 + a_2(z_1 + z_2 - z_2^2) - (\overline{x}_2 + z_2)(w_1 + w_2) + u_2,\\
\dot{w}_1 &= -c_1 w_1 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2) + u_3,\\
\dot{w}_2 &= -c_2 w_2 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2) + u_4.
\end{aligned} \tag{4.3}$$

The system (4.3) admits the trivial solution,

$$z_1 = z_2 = w_1 = w_2 = 0, \;\; u_1 = u_2 = u_3 = u_4 = 0. \tag{4.4}$$

This solution (from equation (4.2)) represents the equilibrium states of the system (2.1).

**Theorem 1.** *Using the optimal nonlinear control inputs;*

$$\begin{aligned}
u_1^* &= -a_1(z_1 + z_2 - z_1^2) + (\overline{x}_1 + z_1)(w_1 + w_2),\\
u_2^* &= -a_2(z_1 + z_2 - z_2^2) + (\overline{x}_2 + z_2)(w_1 + w_2),\\
u_3^* &= -(z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) - (\overline{x}_1 + \overline{x}_2)(w_1 + w_2),\\
u_4^* &= -(z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) - (\overline{x}_1 + \overline{x}_2)(w_1 + w_2),
\end{aligned} \tag{4.5}$$

*the unstable equilibrium states of the system (2.1) will be optimally asymptotically stable with respect to the performance measure;*

$$\begin{aligned}
I &= \int_0^\infty \Omega \, dt\\
&= \int_0^\infty \Big\{ [u_1 + a_1(z_1 + z_2 - z_1^2) - (\overline{x}_1 + z_1)(w_1 + w_2)]^2\\
&\quad + [u_2 + a_2(z_1 + z_2 - z_2^2) - (\overline{x}_2 + z_2)(w_1 + w_2)]^2\\
&\quad + [u_3 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2)]^2\\
&\quad + [u_4 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2)]^2\\
&\quad + (2a_1\overline{x}_1 + \overline{y}_1 + \overline{y}_2)z_1^2 + (2a_2\overline{x}_2 + \overline{y}_1 + \overline{y}_2)z_2^2 + c_1 w_1^2 + c_2 w_2^2 \Big\} \, dt.
\end{aligned} \tag{4.6}$$

*Proof.* We will use Bellman dynamic programming technique to obtain the optimal control inputs and Liapunov asymptotic stability theorem to prove our theorem.

Suppose that we have the nonlinear system (4.3) with the four perturbed state variables $z_1, z_2, w_1, w_2$ and the four control inputs $u_i$, $(i = 1, 2, 3, 4)$ with $\dot{z}_i, \dot{w}_i$, are definite functions of $z_i, w_i, i = 1, 2$, and $u_j,\ j = 1, 2, 3, 4$. If we consider the minimization of an integral of the form (4.6) we can write the condition for minimization of the integral of the form;

$$\begin{aligned}\min_{\vec{u}} \Bigg\{ & \frac{\partial V}{\partial z_1}\dot{z}_1 + \frac{\partial V}{\partial z_2}\dot{z}_2 + \frac{\partial V}{\partial w_1}\dot{w}_1 + \frac{\partial V}{\partial w_2}\dot{w}_2 \\ & + [u_1 + a_1(z_1 + z_2 - z_1^2) - (\overline{x}_1 + z_1)(w_1 + w_2)]^2 \\ & + [u_2 + a_2(z_1 + z_2 - z_2^2) - (\overline{x}_2 + z_2)(w_1 + w_2)]^2 \\ & + [u_3 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2)]^2 \\ & + [u_4 + (z_1 + z_2)(w_1 + w_2 + \overline{y}_1 + \overline{y}_2) + (\overline{x}_1 + \overline{x}_2)(w_1 + w_2)]^2 \\ & + (2a_1\overline{x}_1 + \overline{y}_1 + \overline{y}_2)z_1^2 + (2a_2\overline{x}_2 + \overline{y}_1 + \overline{y}_2)z_2^2 + c_1 w_1^2 + c_2 w_2^2 \Bigg\} = 0.\end{aligned} \tag{4.7}$$

Now, the function $V$ is a function of $z_i, w_i, i = 1, 2$, and it represents the minimum value of the integral performance measure (4.6) achievable by starting from time $t$ and state $(z_1, z_2, w_1, w_2)$. Let us assume that $V$ is the solution of the (4.7) takes the form,

$$V = z_1^2 + z_2^2 + w_1^2 + w_2^2, \tag{4.8}$$

which is a positive definite function of the variables $(z_1, z_2, w_1, w_2)$ and so will be used to prove the asymptotic stability of the solution (4.4).

Substituting by (4.3) and (4.8) in (4.7) and differentiating the resulting expression with respect to $u_i$ and setting it to zero, we obtain the optimal control inputs $u_i^*$ as given by (4.5).

Differentiating the function $V$ along the trajectory of the system (4.3) taking into account the control inputs (4.5) we get;

$$\begin{aligned}\dot{V} &= 2z_1\dot{z}_1 + 2z_2\dot{z}_2 + 2w_1\dot{w}_1 + +2w_1\dot{w}_1 \\ &= -2\Big\{(2a_1\overline{x}_1 + \overline{y}_1 + \overline{y}_2)z_1^2 + (2a_2\overline{x}_2 + \overline{y}_1 + \overline{y}_2)z_2^2 + c_1 w_1^2 + c_2 w_2^2\Big\} \\ &= -2\Omega^*\end{aligned} \tag{4.9}$$

where $\Omega^*$ is the value of the objective function $\Omega$ at the optimal control inputs.

Obviously, the function $\dot{V}$ is a negative definite function of $(z_1, z_2, w_1, w_2)$ only for positive equilibrium states $\bar{x}_i, \bar{y}_i > 0, (i = 1, 2)$ and so the positive equilibrium states (4.4) are asymptotically stable in the Liapunov sense if the conditions of stability are satisfied. □

Some numerical analysis are achieved to show the stabilization behavior of the system (2.1) at the second unstable equilibrium solution $E_2$. This numerical analysis supports the above analytical analysis. The model parameters are fixed as follows: $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$. Note that the conditions of stability of $E_2$ are not satisfied. The initial densities of preys are $z_1 = 0.1$ and $z_2 = 0.4$ while the initial densities of predators are $w_1 = 0.3$ and $w_2 = 0.7$. A lot of simulations for different values of the parameters and the initial values of the densities are done. Fig. 3 shows that the preys and predator densities perturbations decay to zero. In Fig. 4, the components of the control inputs tend to zero. So, we conclude that the unstable equilibrium solution $E_2$ of system (2.1) can be stabilized by using nonlinear control parameters (4.1)

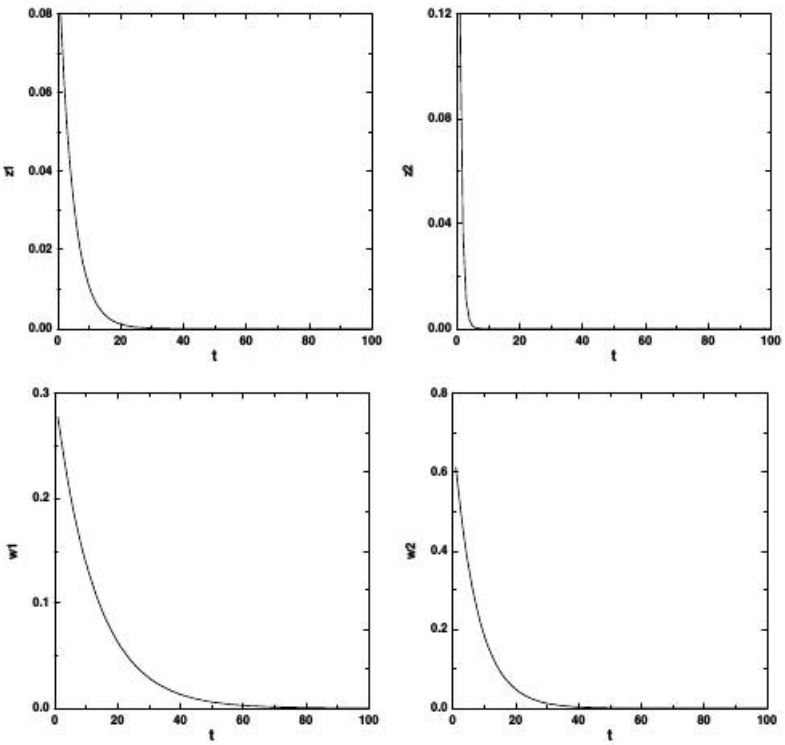


FIGURE 3. The densities perturbations of the preys and predators with time converges to zero at the parameter values $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $z_1(0) = 0.1$, $z_2(0) = 0.4$, $w_1(0) = 0.3$, $w_2(0) = 0.7$.

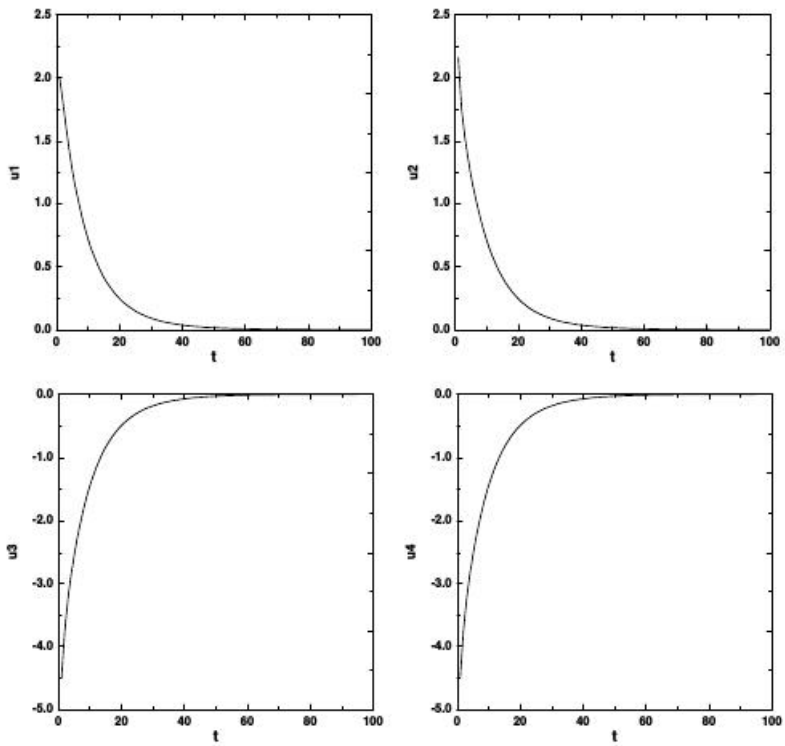


FIGURE 4. The components of the control inputs converges to the stable point state $(0,0,0,0)$ at the parameter values $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $z_1(0) = 0.1$, $z_2(0) = 0.4$, $w_1(0) = 0.3$, $w_2(0) = 0.7$.

## 5. CONCLUSION

Many animals live in groups. Different groups share one habitat hence these groups may cooperate, compete with each other or form predator-prey system. In this work we present a model for the interaction between predator-prey teams. Equilibrium solutions are derived, their local stability and persistence. An optimal control is used to stabilized the unstable equilibrium solution. The boundary unstable equilibrium solution $E_2$ is stabilized using nonlinear control inputs.

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# SOME SETS OF DOUBLE LACUNARY INVARIANT SEQUENCES DEFINIED BY FOUR DIMENSIONAL SUMMABLE MATRICES AND ORLICZ FUNCTIONS

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*Abstract.* The paper introduces some sets of double lacunary invariant sequences defined by four dimensional $RH$-regular matrices and Orlicz functions. First we study some basic properties of such sets by equipping them with linear topological structure. Further, we investigate some relationships among such sets under different conditions and finally compare with a set of double lacunary statistically convergent sequences.

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*Keywords:* Orlicz function, double sequence, $P$-convergence, invariant mean, double lacunary sequence

## 1. Introduction

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function $M$ is said to satisfy the $\Delta_2 - condition$ for all values of $u$, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$,then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$.

In 1971 Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space for single sequences as follows:

$$l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

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The Orlicz sequence spaces was further investigated from sequence space point of view and related summability theory by Esi [3], Dutta [2], Tripathy and Dutta [15], Altin et.al. [1] and many others.

Let $\sigma$ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional $\phi$ on $\ell_\infty$ is said to be an invariant mean or a $\sigma - mean$ provided that

$(a)$ $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all $k \in \mathbb{N}$,

$(b)$ $\phi(e) = 1$, where $e = (1, 1, 1, ...)$, and

$(c)$ $\phi(x) = \phi\left(x_{\sigma(k)}\right)$ for all $x = (x_k) \in \ell_\infty$.

For certain class of mapping $\sigma$ every invariant mean $\varphi$ extends the limit functional on space $c$, the space of all convergent sequences, in sense that $\varphi(x) = \lim x$ for all $x = (x_k) \in c$.

The space $[V_\sigma]$ is of strongly $\sigma - convergent$ sequence space was introduced by Mursaleen [8] as follows: A sequence $x = (x_k)$ is said to be strongly $\sigma - convergent$ if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|x_{\sigma^i(m)} - L\right| \to 0 \text{ as } k \to \infty, \text{ uniformly in m.}$$

If $\sigma(m) = m + 1$, then $[V_\sigma] = [\widehat{c}]$, the space of strongly almost convergent sequences, was introduced by Maddox in [7].

A double sequence $x = \left(x_{k,l}\right)$ has a $Pringsheim\ limit\ L$ [10] (denoted by $P - limx = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = \left(x_{k,l}\right)$ more briefly as "$P - convergent$".

The four dimensional matrix $A$ is said to be $RH - regular$ if it maps every bounded $P - convergent$ sequence into a $P - convergent$ sequence with the same $P - limit$. The assumption of boundedness was made because a double sequence which is $P - convergent$ is not necessarily bounded. Using this definition Robison and Hamilton, independently, both presented the following Silverman-Toeplitz type characterization of $RH - regularity$.

**Proposition 1** ([5, 11])**.** *The four dimensional matrix $A$ is $RH - regular$ if and only if*

$RH_1 : P - \lim_{m,n} a_{m,n,k,l} = 0$ *for each $k$ and $l$;*

$RH_2 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

$RH_3 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right| = 0$ *for each $l$;*

$RH_4 : P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right| = 0$ *for each $k$;*

$RH_5 : \sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right|$ *is $P - convergent$;*

$RH_6$ : *There exist finite positive integers $A$ and $B$ such that* $\sum_{k,l>B} \left|a_{m,n,k,l}\right| < A$.

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$k_o = 0,\ h_r = k_r - k_{r-1} \to \infty \ \text{ as } r \to \infty$$

and

$$l_o = 0,\ \overline{h_s} = l_s - l_{s-1} \to \infty \ \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) :\ k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q_s} = \frac{l_s}{l_{s-1}} \ and\ q_{r,s} = q_r \overline{q_s}.$$

(see [7]).

The set of all double lacunary sequences denoted by $N_{\theta_{r,s}}$ and defined by Savaş and Patterson [13] as follows:

$$N_{\theta_{r,s}} = \left\{ x = \left(x_{k,l}\right) :\ P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left|x_{k,l} - L\right| = 0, \text{ for some L} \right\}.$$

For some relevant literature, we also refer to Savaş and Patterson [14].

Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, $M$ be an Orlicz function, $p = \left(p_{k,l}\right)$ be a factorable double sequence of strictly positive real numbers and $A = \left(a_{m,n,k,l}\right)$ be a nonnegative $RH - regular$ summability matrix method. We now define the following new sets of double sequences:

$$w_o^2(A, M, p)_{\theta_{r,s}}^{\sigma} =$$

$$\left\{ \begin{array}{c} x = \left(x_{k,l}\right) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[ M\left( \frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho} \right) \right]^{p_{k,l}} = 0, \\ \text{uniformly in } (p,q), \text{for some } \rho > 0 \end{array} \right\},$$

$$w^2(A, M, p)_{\theta_{r,s}}^{\sigma} =$$

$$\left\{ \begin{array}{c} x = \left(x_{k,l}\right) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[ M\left( \frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0, \\ \text{uniformly in } (p,q), \text{for some } \rho > 0 \text{ and } L \end{array} \right\},$$

and

$$w_\infty^2(A, M, p)_{\theta_{r,s}}^{\sigma} =$$

$$\left\{ \begin{array}{c} x = \left(x_{k,l}\right) : \sup_{m,n,r,s,p,q} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[ M\left( \frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho} \right) \right]^{p_{k,l}} < \infty, \\ \text{for some } \rho > 0 \end{array} \right\}.$$

When $M(x) = x$, for all $x \in [0, \infty)$ we have the following sets of double sequences:

$$w_o^2(A, p)_{\theta_{r,s}}^{\sigma} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left|x_{\sigma^k(p),\sigma^l(q)}\right|^{p_{k,l}}=0, \\ \text{uniformly in } (p,q) \end{array} \right\},$$

$$w^2(A,p)^{\sigma}_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|^{p_{k,l}}=0, \\ \text{uniformly in } (p,q), \text{ for some } L \end{array} \right\},$$

and

$$w^2_{\infty}(A,p)^{\sigma}_{\theta_{r,s}} =$$

$$\left\{ x=\left(x_{k,l}\right): \sup_{m,n,r,s,p,q} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left|x_{\sigma^k(p),\sigma^l(q)}\right|^{p_{k,l}} < \infty \right\}.$$

When $\sigma(p)=p+1$ and $\sigma(q)=q+1$, we have the following sets of double sequences:

$$\widehat{w}^2_o(A,M,p)_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k+p,l+q}\right|}{\rho}\right)\right]^{p_{k,l}}=0, \\ \text{uniformly in } (p,q), \text{ for some } \rho>0 \end{array} \right\},$$

$$\widehat{w}^2(A,M,p)_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k+p,l+q}-L\right|}{\rho}\right)\right]^{p_{k,l}}=0, \\ \text{uniformly in } (p,q), \text{ for some } \rho>0 \text{ and } L \end{array} \right\},$$

and

$$\widehat{w}^2_{\infty}(A,M,p)_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): \sup_{m,n,r,s,p,q} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k+p,l+q}\right|}{\rho}\right)\right]^{p_{k,l}}<\infty, \\ \text{for some } \rho>0 \end{array} \right\}.$$

When $p_{k,l}=1$, for all $k,l\in\mathbb{N}$, we obtain the following sets of double sequences:

$$w^2_o(A,M)^{\sigma}_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]=0, \\ \text{uniformly in } (p,q), \text{for some } \rho>0 \end{array} \right\},$$

$$w^2(A,M)^{\sigma}_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): P-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]=0, \\ \text{uniformly in } (p,q), \text{for some } \rho>0 \text{ and } L \end{array} \right\},$$

and

$$w^2_{\infty}(A,M)^{\sigma}_{\theta_{r,s}} =$$

$$\left\{ \begin{array}{c} x=\left(x_{k,l}\right): \sup_{m,n,r,s,p,q} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]<\infty, \\ \text{for some } \rho>0 \end{array} \right\}.$$

## 2. Main results

In this section we shall investigate relevant properties of the sets defined in previous sections.

**Theorem 1.** *Let $p=\left(p_{k,l}\right)$ be bounded. The classes of sequences $w_o^2(A,M,p)_{\theta_{r,s}}^{\sigma}$, $w^2(A,M,p)_{\theta_{r,s}}^{\sigma}$ and $w_\infty^2(A,M,p)_{\theta_{r,s}}^{\sigma}$ are linear spaces.*

*Proof.* It is easy, so we omit it. □

**Theorem 2.** *Let $0<h=\inf p_{k,l}\le \sup p_{k,l}=H<\infty$, and let $A$ is a nonnegative $RH-regular$ summability matrix method. For any Orlicz function $M$, if $\beta=\lim_{t\to\infty}\frac{M(t)}{t}\ge 1$, then $w^2(A,p)_{\theta_{r,s}}^{\sigma}=w^2(A,M,p)_{\theta_{r,s}}^{\sigma}$.*

*Proof.* Let $0<h=\inf p_{k,l}\le \sup p_{k,l}=H<\infty$ and $x=\left(x_{k,l}\right)\in w^2(A,p)_{\theta_{r,s}}^{\sigma}$ and let $0<\varepsilon<1$ and $\delta$ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\le t<\delta$. We can write for each $m$ and $n$

$$\begin{aligned} &\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]^{p_{k,l}} \\ &= \sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|\le\delta}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]^{p_{k,l}} \\ &+ \sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|>\delta}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|\le\delta}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]^{p_{k,l}} \\ &\le \varepsilon^h \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} \end{aligned} \tag{2.1}$$

On the other hand, we use the fact that

$$\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|<1+\left[\left|\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right|\right]$$

where $[|t|]$ denotes the integer part of $t$. Since $M$ is Orlicz function we have

$$M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\geq M(1).$$

Now, let us consider the second part where the sum is taken over $\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|>\delta$.
Thus

$$\begin{aligned}
&\sum_{\substack{(k,l)\in I_{r,s}\\ \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|>\delta}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)\right]^{p_{k,l}}\\
&\leq \sum_{\substack{(k,l)\in I_{r,s}\\ \left|x_{\sigma^k(p),\sigma^l(q)}-L\right|>\delta}} a_{m,n,k,l}\left[M\left(1+\left[\left|\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right|\right]\right)\right]^{p_{k,l}}\\
&\leq \left(2M(1)\delta^{-1}\right)^H \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}-L\right|}{\rho}\right)^{p_{k,l}}
\end{aligned}$$

This and from (2.1) and $RH-regularity\ of\ A$, we are granted that $x=\left(x_{k,l}\right)\in w^2(A,M,p)^{\sigma}_{\theta_{r,s}}$. Observe that in this part of the proof we did not use $\beta\geq 1$. Let $\beta\geq 1$ and $x=\left(x_{k,l}\right)\in w^2(A,M,p)^{\sigma}_{\theta_{r,s}}$. Since $\beta\geq 1$ we have $M(t)\geq\beta t$ for all $t\geq 0$. It follows that $x=\left(x_{k,l}\right)\in w^2(A,M,p)^{\sigma}_{\theta_{r,s}}$ implies $x=\left(x_{k,l}\right)\in w^2(A,p)^{\sigma}_{\theta_{r,s}}$ and this completes the proof. □

**Theorem 3.** *$w^2_o(A,M,p)^{\sigma}_{\theta_{r,s}}$, $w^2(A,M,p)^{\sigma}_{\theta_{r,s}}$ and $w^2_\infty(A,M,p)^{\sigma}_{\theta_{r,s}}$ are complete linear topological spaces with the paranorm*

$$g\left(\left(x_{k,l}\right)\right)=$$

$$\inf\left\{\rho^{\frac{p_{k,l}}{T}}>0:\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\leq 1\right\}$$

*where $T=\max(1,H)$, $H=\sup_{k,l}p_{k,l}<\infty$.*

*Proof.* Clearly $g(0)=0$, $g(-x)=g(x)$.

Let $x = (x_{k,l}),\ y = (y_{k,l}) \in w_{\infty}^{2}(A, M, p)_{\theta_{r,s}}^{\sigma}$. Then there exist some $\rho_1$and $\rho_2$ such that

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}$$

and

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} + y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}$$

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} + y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_1+\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}$$

$$\leq \sup_{m,n,r,s,p,q}\Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\Big[\frac{\rho_1}{\rho_1+\rho_2}M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_1}\right)$$

$$+\frac{\rho_2}{\rho_1+\rho_2}M\left(\frac{\left|y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_2}\right)\Big]^{p_{k,l}}\Big)^{\frac{1}{T}}$$

By Minkowsky's inequality

$$\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_1}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}$$

$$+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1.$$

Now

$$g\left((x_{k,l}) + (y_{k,l})\right) = \inf\{\rho^{\frac{p_{k,l}}{T}} > 0:$$

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} + y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1\}$$

$$\leq$$

$$\inf\left\{\rho_1^{\frac{p_{k,l}}{T}} > 0: \sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_1}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1\right\}$$

$$+$$

$$\inf\left\{\rho_2^{\frac{p_{k,l}}{T}}>0:\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}a_{m,n,k,l}\left[M\left(\frac{\left|y_{\sigma^k(p),\sigma^l(q)}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\leq 1\right\}$$

$$=g\left(\left(x_{k,l}\right)\right)+g\left(\left(y_{k,l}\right)\right).$$

Let $\lambda\in\mathbb{C}$, then the continuity of the product follows from the following equality:

$$g\left(\lambda\left(x_{k,l}\right)\right)=\inf\{\rho^{\frac{p_{k,l}}{T}}>0:$$

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}a_{m,n,k,l}\left[M\left(\frac{\left|\lambda x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\leq 1,\rho>0\}$$

$$=\inf\{(|\lambda|r)^{\frac{p_{k,l}}{T}}>0:$$

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}a_{m,n,k,l}\left[M\left(\frac{\left|\lambda x_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\leq 1,r>0\}$$

$$=|\lambda|\,g\left(\left(x_{k,l}\right)\right),$$

where $\frac{1}{r}=\frac{|\lambda|}{\rho}$.

Now $\left(x_{k,l}^s\right)$ is a Cauchy sequence in $w_\infty^2(A,M,p)_{\theta_{r,s}}^\sigma$. Then

$$g\left(\left(x_{k,l}^s-x_{k,l}^t\right)\right)\to 0 \text{ as } s,t\to\infty.$$

For given $\varepsilon>0$, choose $r>0$ and $x_o>0$ be such that $\frac{\varepsilon}{rx_o}>0$ and $M\left(\frac{rx_o}{2}\right)\geq 1$. Now $g\left(\left(x_{k,l}^s-x_{k,l}^t\right)\right)\to 0$ as $s,t\to\infty$ implies that there exists $n_o\in\mathbb{N}$ such that

$$g\left(\left(x_{k,l}^s-x_{k,l}^t\right)\right)<\frac{\varepsilon}{rx_o}\quad\text{for all } s,t\geq n_o.$$

This implies

$$\inf\{\rho^{\frac{p_{k,l}}{T}}>0:$$

$$\sup_{m,n,r,s,p,q}\left(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}a_{m,n,k,l}\left[M\left(\frac{\left|x^s_{\sigma^k(p),\sigma^l(q)}-x^t_{\sigma^k(p),\sigma^l(q)}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\leq 1\}$$

$$<\frac{\varepsilon}{rx_o}\tag{2.2}$$

Now from (2.2) we have,

$$M\left(\frac{\left|x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}-x^{t}_{\sigma^{k}(p),\sigma^{l}(q)}\right|}{\rho}\right)\leq 1\leq M\left(\frac{rx_{o}}{2}\right)$$

$$\Rightarrow \frac{\left|x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}-x^{t}_{\sigma^{k}(p),\sigma^{l}(q)}\right|}{g\left(\left(x^{s}_{k,l}-x^{t}_{k,l}\right)\right)}\leq\frac{rx_{o}}{2}$$

$$\Rightarrow \left|x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}-x^{t}_{\sigma^{k}(p),\sigma^{l}(q)}\right|<\frac{rx_{o}}{2}.\frac{\varepsilon}{rx_{o}}=\frac{\varepsilon}{2}.$$

This implies that $\left(x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}\right)$ is a Cauchy sequence of real numbers.
Let $\lim_{s\to\infty} x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}=x_{\sigma^{k}(p),\sigma^{l}(q)}$ for all $k,l\in N$. Using continuity of $M$, we have

$$\lim_{t\to\infty} M\left(\frac{\left|x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}-x^{t}_{\sigma^{k}(p),\sigma^{l}(q)}\right|}{\rho}\right)\leq 1$$

$$\Rightarrow M\left(\frac{\left|x^{s}_{\sigma^{k}(p),\sigma^{l}(q)}-x_{\sigma^{k}(p),\sigma^{l}(q)}\right|}{\rho}\right)\leq 1.$$

Let $s\geq n_{o}$, then taking infimum of such $\rho's$ we have $g\left(\left(x^{s}_{k,l}-x_{k,l}\right)\right)<\varepsilon$. Thus $\left(x^{s}_{k,l}-x_{k,l}\right)\in w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$. By linearity of the space $w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$ we have $\left(x_{k,l}\right)\in w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$. Hence $w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$ is a complete space. □

**Proposition 2.** *We have the following inclusions*
(*a*) $w^{2}(A,M,p)^{\sigma}_{\theta_{r,s}}\subset w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$,
(*b*) $w^{2}_{o}(A,M,p)^{\sigma}_{\theta_{r,s}}\subset w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$.

*Proof.* It is easy, so we omit it. □

**Theorem 4.** *The spaces* $w^{2}_{o}(A,M,p)^{\sigma}_{\theta_{r,s}}$ *and* $w^{2}(A,M,p)^{\sigma}_{\theta_{r,s}}$ *are nowhere dense subsets of* $w^{2}_{\infty}(A,M,p)^{\sigma}_{\theta_{r,s}}$.

*Proof.* The proof is clear in view of Theorem 3 and Proposition 2. □

**Theorem 5.** (*a*) *If* $0<h=\inf p_{k,l}<p_{k,l}\leq 1$, *then*

$$w^{2}(A,M,p)^{\sigma}_{\theta_{r,s}}\subset w^{2}(A,M)^{\sigma}_{\theta_{r,s}}.$$

(*b*) *If* $1\leq p_{k,l}\leq\sup p_{k,l}<\infty$, *then*

$$w^{2}(A,M)^{\sigma}_{\theta_{r,s}}\subset w^{2}(A,M,p)^{\sigma}_{\theta_{r,s}}.$$

*Proof.* (a) Let $x = (x_{k,l}) \in w^2(A, M, p)_{\theta_{r,s}}$, since $0 < h = \inf p_{k,l} < p_{k,l} \leq 1$, we obtain the following:

$$\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)$$
$$\leq \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)\right]^{p_{k,l}}$$

thus $x = (x_{k,l}) \in w^2(A, M)^{\sigma}_{\theta_{r,s}}$.

$(b)$ Let $p_{k,l} \geq 1$ for each $k$, $l$ and $\sup p_{k,l} < \infty$ and let $x = (x_{k,l}) \in w^2(A, M)^{\sigma}_{\theta_{r,s}}$. Then for each $0 < \varepsilon < 1$ there exists a positive integer $K$ such that

$$\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right) \leq \varepsilon < 1$$

for all $n, m \geq K$. This implies that

$$\sum_{(k,l)\in I_{r,s}} a_{m,n,k,l}\left[M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)\right]^{p_{k,l}}$$
$$\leq \sum_{(k,l)\in I_{r,s}} a_{m,n,k,l} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right).$$

Thus $x = (x_{k,l}) \in w^2(A, M, p)^{\sigma}_{\theta_{r,s}}$. This completes the proof. □

## 3. Double statistical convergence

The concept of statistical convergence for single sequences was introduced by Fast [4] in 1951. Later, Mursaleen and Edely [9] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be $P-statistical\ convergence\ to\ L$ provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn}\left|\left\{(k,l) \in \mathbb{N}\times\mathbb{N} :\ k \leq m, l \leq n;\ \left|x_{k,l} - L\right| \geq \varepsilon\right\}\right| = 0$$

where the vertical bars indicate the numbers of elements in the enclosed set. In this case, we write $st_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-statistical\ convergent\ double\ sequences$ by $st_2$.

**Definition 1** ([12])**.** Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence, a real double sequence $x = (x_{k,l})$ is said to be uniformly $S^2_{(\theta,\sigma)} - convergent$ or uniformly $(\theta,\sigma) - convergence\ to\ L$ provided that for each $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \max_{p,q}\left|\left\{(k,l) \in I_{r,s} :\ \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $S^2_{(\theta,\sigma)} - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-$ *statistical* $S^2_{(\theta,\sigma)}-$*convergent double sequences* by $S^2_{(\theta,\sigma)}$.

In this section we give some relationship between the double sequence spaces $S^2_{(\theta,\sigma)}$ and $w^2(M)^{\sigma}_{\theta_{r,s}}$

**Theorem 6.** *If $M$ be an Orlicz function, then $w^2(M)^{\sigma}_{\theta_{r,s}} \subset S^2_{(\theta,\sigma)}$, where*

$$w^2(M)^{\sigma}_{\theta_{r,s}} = \left\{ \begin{array}{c} x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right) = 0, \\ \textit{uniformly in } (p,q) \textit{, for some } \rho > 0 \textit{ and } L \end{array} \right\}$$

*Proof.* Suppose that $x = (x_{k,l}) \in w^2(M)^{\sigma}_{\theta_{r,s}}$ and $\varepsilon > 0$, then we obtain the following for every $p$ and $q$

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right) \geq \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| \geq \varepsilon}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)$$

$$\geq \frac{M(\varepsilon)}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| \geq \varepsilon\right\}\right|.$$

Hence $x = (x_{k,l}) \in S^2_{(\theta,\sigma)}$. □

**Theorem 7.** *Let $M$ be an Orlicz function. Then $S^2_{(\theta,\sigma)} \cap \ell^2_\infty \subset w^2(M)^{\sigma}_{\theta_{r,s}}$.*

*Proof.* Let $x = (x_{k,l}) \in S^2_{(\theta,\sigma)} \cap \ell^2_\infty$. Since $x \in \ell^2_\infty$, we can find a positive number $K$ such that $M(x) \leq K$, for all $x \geq 0$. Then for each $p$ and $q$, we have

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)$$

$$= \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| \geq \varepsilon}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)$$

$$+ \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| < \varepsilon}} M\left(\frac{\left|x_{\sigma^k(p),\sigma^l(q)} - L\right|}{\rho}\right)$$

$$\leq \frac{K}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \left|x_{\sigma^k(p),\sigma^l(q)} - L\right| \geq \varepsilon\right\}\right| + M(\varepsilon)$$

and thus the Pringsheim's limit on $r$ and $s$ grant us the result. □

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# SOME LIMIT THEOREMS FOR GENERALIZED ALLOCATION SCHEMES

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*Abstract.* In this paper two modifications of Kolchin's generalized allocation scheme are studied. Results known for Kolchin's scheme are extended to the new models. Representation theorems, strong laws of large numbers and local limit theorems are obtained. In the proofs some general inequalities are used.

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*Keywords:* Generalized allocation scheme, law of large numbers, normal limit theorem

## 1. Introduction

The generalized allocation scheme was introduced by V. F. Kolchin. Its properties and applications are described in the monographies [12] and [14]. The usual allocation scheme (see [1,2,13]), the random forests and other random structures are special cases of Kolchin's scheme (see [12] and [14]). The scheme itself is defined as follows (we call it Model 1).

*Model 1*

Let $\eta_1, \eta_2, \dots, \eta_N$ be nonnegative integer-valued random variables. If there exist independent identically distributed random variables $\xi_1, \xi_2, \dots, \xi_N$ such that the joint distribution of $\eta_1, \eta_2, \dots, \eta_N$ admits the representation

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \tag{1.1}$$
$$= \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \;\Big|\; \sum\nolimits_{i=1}^{N} \xi_i = n\Big\},$$

where $k_1, k_2, \dots, k_N$ are arbitrary non-negative integers with $\sum_{i=1}^{N} k_i = n$, we say that the distribution of $\eta_1, \eta_2, \dots, \eta_N$ is represented by the generalized allocation scheme with parameters $n$ and $N$, and independent random variables $\xi_1, \xi_2, \dots, \xi_N$.

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This scheme can be considered as the allocation of $n$ balls into $N$ boxes. The random variable $\eta_i$ can be interpreted as the number of balls in the $i$th box. The detailed description of the properties of scheme (1.1) are given in [12] and [14]. The importance of scheme (1.1) is its connection with random graphs. Further properties of scheme (1.1) can be found in the papers of V. F. Kolchin and A. V. Kolchin (see e.g. [10,11]). In [5] a strong law of large numbers (SLLN) was proved for Model 1. Using that SLLN one can obtain SLLN-s for usual random allocations and random forests. We mention that the original proofs of SLLN-s for usual allocations and random forests needed very long elementary calculations (see [2] and [3]).

In [7] the following analogue of Kolchin's scheme was studied.

*Model 2*

Consider random variables $\eta_1, \eta_2, \dots, \eta_N$ with joint distribution

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \tag{1.2}$$

$$= \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \;\Big|\; \sum\nolimits_{i=1}^{N} \xi_i \le n\Big\},$$

where $k_1, \dots, k_N$ are arbitrary non-negative integers with $\sum_{i=1}^{N} k_i \le n$. In this case, we place at most $n$ balls into $N$ boxes. In [7] strong laws of large numbers, normal and Poisson limit theorems for the number of boxes containing a fixed number of balls were obtained. The distribution of the maximum number of balls contained in the boxes was studied in [6].

In [8], another version of Kolchin's scheme was studied (see Model 3 below). In that modification, the condition in (1.1) was changed for $\sum_{i=1}^{N} \xi_i \ge n$. For Model 3 normal and Poisson local limit theorems were obtained in [8].

In this paper our aim is to study two modifications of Kolchin's model. We extend certain results of [7] and [6] for these models. In Section 2.1 we review Model 3 and we present some new results for it. Actually, we obtain strong laws (Theorems 1 and 2) and a local limit theorem (Theorem 3). In Section 2.2 we introduce a new version of Kolchin's scheme (Model 4). Model 4 is a very general scheme. It contains Model 1, Model 2 and Model 3 as special cases. We find representation theorems and some limit theorems for Model 4. Finally, in Section 3 we present our proofs and some auxiliary results. In forthcoming research we intend to apply Model 4 to describe certain properties of random graphs.

## 2. Notations and main results

We shall apply o(.) and O(.) in the usual sense, that is $k_n = \mathrm{o}(l_n)$ if $\lim_{n\to\infty} k_n / l_n = 0$ and $k_n = \mathrm{O}(l_n)$ if the sequence $k_n / l_n$ is bounded.

Let us denote by $\mu_{nN}$ the number of cases when $\{\eta_i = r\}$, where $r$ is a fixed integer, $r \in \{0, 1, \dots, n\}$. Then

$$\mu_{nN} = \sum_{i=1}^{N} \mathbb{I}_{\{\eta_i = r\}}$$

can be interpreted as the number of boxes containing $r$ balls. Here $\mathbb{I}_A$ denotes the indicator of the set $A$.

Let $\xi_0$ be a random variable, $\mathbb{P}\{\xi_0 = r\} = p_r$ and $\mathbb{E}\xi_0 = a$. Let $\xi_1, \xi_2, \dots, \xi_N$ be independent copies of $\xi_0$. Introduce notation $S_N = \sum_{i=1}^{N} \xi_i$ and $S_N^c = \sum_{i=1}^{N} (\xi_i - a)$.

Denote by $\xi_0^{(r)}$ a random variable with distribution

$$\mathbb{P}\{\xi_0^{(r)} = k\} = \mathbb{P}\{\xi_0 = k \mid \xi_0 \neq r\}.$$

Let $\mathbb{E}\xi_0^{(r)} = a_r$ and let $\xi_1^{(r)}, \dots, \xi_N^{(r)}$ be independent copies of $\xi_0^{(r)}$. Let $S_N^{(r)} = \sum_{i=1}^{N} \xi_i^{(r)}$.

We see that the connection between the expectations $a$ and $a_r$ is $a_r = \frac{a - rp_r}{1 - p_r}$.

### 2.1. *Model* 3

In this section we consider Model 3 that is we assume that (2.1) is satisfied.

Let $\eta_1, \eta_2, \dots, \eta_N$ be random variables with joint distribution

$$\mathbb{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \tag{2.1}$$
$$= \mathbb{P}\Big\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \sum\nolimits_{i=1}^{N} \xi_i \geq n\Big\},$$

where $k_1, \dots, k_N$ are arbitrary non-negative integers with $\sum_{i=1}^{N} k_i \geq n$. Then we say that $\eta_1, \eta_2, \dots, \eta_N$ obey the assumption of Model 3. In this case we place at least $n$ balls into $N$ boxes.

Model 3 was introduced in [8]. We shall see that appropriate versions of certain results of [7] are true for Model 3.

First we present SLLN-s for Model 3.

The following theorem is a version of Theorem 1 of [7].

**Theorem 1** (Model 3.)**.** *Let $\mathbb{E}\xi_0 = a < \infty$. Suppose that there exists a sequence $B_N$, $N = 1, 2, \dots$, such that $B_N \to \infty$, $N \geq B_N > 0$ for all $N = 1, 2, \dots$,*

$$\lim_{N \to \infty} \frac{B_{N+1}}{B_N} = 1, \tag{2.2}$$

*and*

$$\frac{S_N^c}{B_N} \to \xi' \ \text{ as } \ N \to \infty, \tag{2.3}$$

*in distribution, where $\xi'$ is a random variable with distribution function $F$. Let $C$ be a point of continuity of $F$ such that $F(C) = \mathbb{P}\{\xi' < C\} = q_1 < 1$. Then we have*

$$\lim_{n,N\to\infty,\, 0<\frac{n}{N}<C\frac{B_N}{N}+a} \frac{1}{N}\mu_{nN} = p_r \qquad \textit{almost surely.} \tag{2.4}$$

**Corollary 1.** *(Model 3.) Let $\mathbb{E}\xi_0 = a < \infty$. Let $C < 0$. Then we have*

$$\lim_{n,N\to\infty,\, 0<\frac{n}{N}<a+C} \frac{1}{N}\mu_{nN} = p_r \qquad \textit{almost surely.}$$

**Corollary 2.** *(Model 3.) Suppose that $\mathbb{E}\xi_0^2 < \infty$, $\mathbb{E}\xi_0 = a$. Let $C \in \mathbb{R}$. Then we have*

$$\lim_{n,N\to\infty,\, 0<\frac{n}{N}<\frac{C}{\sqrt{N}}+a} \frac{1}{N}\mu_{nN} = p_r \qquad \textit{almost surely.}$$

Recall that a random variable $\xi$ satisfies the Cramér's condition, if there exists a positive constant $H$ such that $\mathbb{E}e^{\lambda\xi} < \infty$ for all $|\lambda| < H$.

We have the following analogue of Theorem 2 of [7].

**Theorem 2** (Model 3.)**.** *Let $\mathbb{E}\xi_0 = a < \infty$.*

*(1) Let $\alpha < a$. Then*

$$\lim_{n,N\to\infty,\, \frac{n}{N}\to\alpha} \frac{1}{N}\mu_{nN} = p_r \qquad \textit{almost surely.} \tag{2.5}$$

*(2) Assume that $\xi_0$ satisfies the Cramér condition. Then*

$$\lim_{n,N\to\infty,\, \frac{n}{N}\to a} \frac{1}{N}\mu_{nN} = p_r \qquad \textit{almost surely.} \tag{2.6}$$

Consider the random variable $\xi_0$ with the following power series distribution. Let $b_0, b_1, b_2, \dots$ be a sequence of non-negative numbers and let $R$ denote the radius of convergence of the series

$$B(\theta) = \sum\nolimits_{k=0}^{\infty} \frac{b_k\theta^k}{k!}.$$

Assume that $R > 0$. We assume that $\xi_0 = \xi_0(\theta)$ has distribution

$$p_k = p_k(\theta) = \mathbb{P}\{\xi_0(\theta) = k\} = \frac{b_k\theta^k}{k!B(\theta)},\ k = 0, 1, 2, \dots. \tag{2.7}$$

We will assume that the distribution of the random variable $\xi_0(\theta)$ satisfies

$$b_0 > 0,\ b_1 > 0. \tag{2.8}$$

(For more details, see e.g. [12].)

The following theorem is a version of Theorem 8 of [7].

**Theorem 3** (Model 3.)**.** *Suppose that the random variable $\xi_0 = \xi_0(\theta)$ has distribution (2.7) and condition (2.8) is satisfied. Suppose that $r = 1$ and $n \geq 1$ is fixed. Let $N \to \infty$ such that $Np_1(\theta) \to \lambda$ for some $0 < \lambda < \infty$. Then for all $k \in \mathbb{N}_0$ we have*

$$\mathbb{P}\{\mu_{nN} = k\} = \frac{\frac{\lambda^k e^{-\lambda}}{k!}}{\sum_{l=n}^{\infty} \frac{\lambda^l e^{-\lambda}}{l!}}(1 + o(1)). \tag{2.9}$$

2.2. *Model* 4

In this section we will study the following very general modification of Kolchin's scheme.

Consider random variables $\eta_1, \eta_2, \ldots, \eta_N$ with joint distribution

$$\mathbb{P}\{\eta_1 = k_1, \ldots, \eta_N = k_N\} = \tag{2.10}$$
$$= \mathbb{P}\{\xi_1 = k_1, \ldots, \xi_N = k_N \mid \xi_1 + \cdots + \xi_N \in B_n\},$$

where $k_1, \ldots, k_N$ are arbitrary non-negative integers with $\sum_{i=1}^N k_i \in B_n$ and $B_n$ is a nonempty subset of $\mathbb{R}_+$. Then we say that $\eta_1, \ldots, \eta_N$ obey a generalized allocation scheme of type Model 4.

*Remark* 1. Here we assume that $\mathbb{P}\{\sum_{i=1}^n \xi_i \in B_n\} > 0$.

We have the following analogue of Kolchin's formula in Model 4 (see Lemma 1.2.1 of [12]).

**Lemma 1** (Model 4.)**.** *For all $k = 0, 1, 2, \ldots, N$ we have*

$$\mathbb{P}\{\mu_{nN} = k\} = \binom{N}{k} p_r^k (1 - p_r)^{N-k} \frac{\mathbb{P}\{S_{N-k}^{(r)} \in B_n - kr\}}{\mathbb{P}\{S_N \in B_n\}}. \tag{2.11}$$

Let us denote by $\eta_{(N)}$ the maximal number of balls contained by any of the boxes, that is $\eta_{(N)} = \max_{1 \leq i \leq N} \eta_i$.

Consider the random variable $\xi_0^{(\leq r)}$ with distribution

$$\mathbb{P}\{\xi_0^{(\leq r)} = k\} = \mathbb{P}\{\xi_0 = k \mid \xi_0 \leq r\}.$$

Let $\xi_i^{(\leq r)}$, $i = 1, \ldots, N$, be independent copies of $\xi_0^{(\leq r)}$. Let $S_N^{(\leq r)} = \sum_{i=1}^N \xi_i^{(\leq r)}$ and $\mathbb{E}\xi_0^{(\leq r)} = a_{\leq r}$.

We have the following representation of the distribution of $\eta_{(N)}$. This lemma is a version of Theorem 1 of [6].

**Lemma 2** (Model 4.)**.** *We have*

$$\mathbb{P}\{\eta_{(N)} \leq r\} = (1 - P_r)^N \frac{\mathbb{P}\{S_N^{(\leq r)} \in B_n\}}{\mathbb{P}\{S_N \in B_n\}}, \tag{2.12}$$

*for all $r \in \mathbb{N}$, where $P_r = \mathbb{P}\{\xi_0 > r\}$.*

In the following theorem we shall consider the case when the set $B_n$ is of the following form: $B_n = f(n)B$, where $n \in \mathbb{N}$, $B \subset \mathbb{R}_+$ and $f(n)$ is a real-valued function of $n$.

*Remark* 2. If $f(n) = n$, then we have the following examples for Model 4.

- If $B = \{1\}$, then $B_n = \{n\}$, so we obtain the original Kolchin's model (Model 1).
- If $B = [0,1]$, then $B_n = [0,n]$, so we get Model 2.
- If $B = [1,\infty)$, then $B_n = [n,\infty)$, so we get Model 3.

Let $B^\circ$ be the set of inner points of $B$. We have the following local limit theorem for $\mu_{nN}$. This theorem is an appropriate version of Theorem 4 of [7].

**Theorem 4** (Model 4 with $B_n = f(n)B$.)**.** *Let* $0 < \frac{N}{f(n)} < \infty$, *for* $n, N \in \mathbb{N}$. *Let* $\frac{N}{f(n)} \to \widetilde{\alpha_1}$ *as* $n, N \to \infty$ , *where* $\widetilde{\alpha_1} \in \mathbb{R}_+$. *Assume that* $\widetilde{\alpha_1} a \in B^\circ$. *Let* $s_r^2 = p_r(1-p_r)$. *Then*

$$\mathbb{P}\{\mu_{nN} = k\} = \frac{1}{\sqrt{2\pi N} s_r} e^{-u^2/2}(1+\mathrm{o}(1)) \tag{2.13}$$

*as* $n, N \to \infty$ *so that* $u = \frac{k - Np_r}{s_r N^{1/2}}$ *belongs to an arbitrary bounded fixed interval.*

Using Lemma 2, we can prove the following result. This theorem is an appropriate version of Theorem 2 of [6].

**Theorem 5** (Model 4 with $B_n = f(n)B$.)**.** *Let* $0 < \frac{N}{f(n)} < \infty$ *for* $n, N \in \mathbb{N}$. *Let* $\frac{N}{f(n)} \to \widetilde{\alpha_1}$ *as* $n, N \to \infty$ , *where* $\widetilde{\alpha_1} \in \mathbb{R}_+$. *Assume that* $\widetilde{\alpha_1} a \in B^\circ$ *and* $\widetilde{\alpha_1} a_{\le r} \in B^\circ$. *Then for all* $r \in \mathbb{N}$, *as* $n, N \to \infty$, *we have*

$$\mathbb{P}\{\eta_{(N)} \le r\} = (1-P_r)^N(1+\mathrm{o}(1)). \tag{2.14}$$

## 3. Proofs and auxiliary results

Let $A$ be a fixed event, $\mathbb{P}(A) > 0$. Let $\mathbb{P}^A$ denote the conditional probability with respect to the event $A$ and let $\mathbb{E}^A$ denote the expectation with respect to $\mathbb{P}^A$. Let us denote by $\mathbb{I}\{A\}$ the indicator variable of the event $A$. Let us denote by $A_{nN}$ the following event: $A_{nN} = \{\sum_{i=1}^N \xi_i \ge n\}$.

**Lemma 3** (Model 3.)**.** *Assume that* (2.1) *is satisfied. Let* $\frac{4\sqrt{2}s_r}{\sqrt{N}} < \varepsilon < \frac{s_r^3}{\sqrt{2}}$ *be fixed. Then for* $N$ *large enough, we have*

$$\mathbb{P}\{|\mu_{nN} - \mathbb{E}\mu_{nN}| \ge \varepsilon N\} \le \frac{K}{\mathbb{P}(A_{nN})} \frac{\varepsilon^4 N^2}{s_r^4} e^{-\frac{N\varepsilon^2}{32 s_r^2}}, \tag{3.1}$$

*where* $K$ *is an absolute constant and* $s_r^2 = p_r(1-p_r)$.

*Proof.* Let $\tau_i = \mathbb{I}\{\xi_i = r\}$, $\nu = \sum_{i=1}^N \tau_i$, $i = 1,\dots,n$. The variance of $\tau_i$ is $s_r^2 = p_r(1-p_r)$, where $p_r = \mathbb{P}(\xi_0 = r)$. Let $\tau_i'$ be an independent copy of $\tau_i$, $i = 1,\dots,N$. Let us denote by $d_i$ the variance of $(\tau_i - \tau_i')^2$. Let $d = \frac{1}{N}\sum_{i=1}^N d_i$.

In Theorem 2.1 (i) of [4] it was proved that for any fixed event $A$ with $\mathbb{P}(A) > 0$ and $\varepsilon' \geq 4\sqrt{2}s_r$

$$\mathbb{P}^A\left\{\frac{|\nu - \mathbb{E}^A\nu|}{\sqrt{N}} \geq \varepsilon'\right\} \leq \frac{\sqrt{2}}{\mathbb{P}(A)} e^{-\frac{\varepsilon'^2}{16s_r^2}}(1+B), \tag{3.2}$$

where

$$B = B(N,\sigma) = \frac{d}{32}\left(\frac{\varepsilon'^2}{8s_r^4\sqrt{N}}\right)^2 f_2\left(\frac{2\varepsilon'^2}{8s_r^4\sqrt{N}}\right) + \mathrm{O}\left(\frac{8s_r^2}{\varepsilon'^2}\right) \tag{3.3}$$

and

$$f_2(x) = 4\left(e^{\frac{x^2}{2}}\left(x^2+1\right)\Phi(x) + \frac{x}{\sqrt{2\pi}}\right) - 1. \tag{3.4}$$

Here $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.

We shall apply Theorem 2.1 (i) of [4] with $A = A_{nN}$ and $\varepsilon' = \sqrt{N}\varepsilon$. By (3.2) and (2.1), we obtain

$$\mathbb{P}\{|\mu_{nN} - \mathbb{E}\mu_{nN}| \geq N\varepsilon\} = \mathbb{P}^{A_{nN}}\left\{|\nu - \mathbb{E}^{A_{nN}}\nu| \geq N\varepsilon\right\} \leq \frac{K}{\mathbb{P}(A_{nN})}\frac{\varepsilon^4 N^2}{s_r^4} e^{-\frac{N\varepsilon^2}{32s_r^2}},$$

where $K$ is an absolute constant. □

*Proof of Theorem 1.* It follows from equation (2.3), that either $\xi'$ is a (nondegenerate) $p$-stable random variable or $\xi'$ is a constant. If $\xi'$ is a $p$-stable random variable then its distribution function $F$ is arbitrary many times differentiable (see [9]), therefore it is uniformly continuous. If $\xi'$ is a constant, then $F(x) = 0$, $x \leq C + \delta$ for some $0 < \delta < \infty$. Therefore, in both cases there exists $0 < \delta < \infty$ such that $F(x)$, $x \in (-\infty, C+\delta]$ is a uniformly continuous function and $F(C+\delta) < 1$. Let $\frac{n}{N} \in \left(0, a + C\frac{B_N}{N}\right]$. Then, using (2.2), we have

$$\frac{N-1}{B_{N-1}}\left(\frac{n}{N-1} - a - \frac{r}{N-1}\right) \leq C + \delta \quad \text{and} \quad \frac{N}{B_N}\left(\frac{n}{N} - a\right) \leq C + \delta \tag{3.5}$$

for $n$ and $N$ large enough. Consider the quotient $\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}}$. If $\xi'$ is a constant, then, using (3.5),

$$\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}} = \frac{\mathbb{P}\left\{\frac{1}{B_{N-1}}\sum_{i=1}^{N-1}(\xi_i - a) \geq \frac{n-r-(N-1)a}{B_{N-1}}\right\}}{\mathbb{P}\left\{\frac{1}{B_N}\sum_{i=1}^{N}(\xi_i - a) \geq \frac{n-Na}{B_N}\right\}}$$

$$= \frac{1-\mathbb{P}\left\{\frac{1}{B_{N-1}}\sum_{i=1}^{N-1}(\xi_i-a) < \frac{n-r-(N-1)a}{B_{N-1}}\right\}}{1-\mathbb{P}\left\{\frac{1}{B_N}\sum_{i=1}^{N}(\xi_i-a) < \frac{n-Na}{B_N}\right\}} = \frac{1-\mathrm{o}(1)}{1-\mathrm{o}(1)}. \tag{3.6}$$

Now, consider the case when $\xi'$ is a $p$-stable random variable. By (2.3), we have

$$\begin{aligned}\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}} &= \frac{1-\mathbb{P}\left\{\frac{1}{B_{N-1}}\sum_{i=1}^{N-1}(\xi_i-a) < \frac{n-r-(N-1)a}{B_{N-1}}\right\}}{1-\mathbb{P}\left\{\frac{1}{B_N}\sum_{i=1}^{N}(\xi_i-a) < \frac{n-Na}{B_N}\right\}} \\ &= \frac{1-F\left(\frac{N}{B_{N-1}}(\frac{n}{N}-a-\frac{r-a}{N})\right)+\mathrm{o}(1)}{1-F\left(\frac{N}{B_N}(\frac{n}{N}-a)\right)+\mathrm{o}(1)}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\left|1-\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}}\right| \\ &\leq \frac{\left|F\left(\frac{N}{B_{N-1}}(\frac{n}{N}-a-\frac{r-a}{N})\right)-F\left(\frac{N}{B_N}(\frac{n}{N}-a)\right)\right|+\mathrm{o}(1)}{1-F(C+\delta)+\mathrm{o}(1)}.\end{aligned} \tag{3.7}$$

Using either (3.6) or (3.7) and the uniform continuity property of $F$, we obtain

$$\lim_{n,N\to\infty,\,0<\frac{n}{N}<a+C\frac{B_N}{N}} \frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}} = 1. \tag{3.8}$$

Consequently

$$\begin{aligned}&\lim_{n,N\to\infty,\,0<\frac{n}{N}\leq a+C\frac{B_N}{N}} \frac{1}{N}\mathbb{E}\mu_{nN} \\ &= \lim_{n,N\to\infty,\,0<\frac{n}{N}\leq a+C\frac{B_N}{N}} p_r\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}} = p_r.\end{aligned} \tag{3.9}$$

Let $\varepsilon > 0$. By (2.3) and the continuity property of $F$, the above calculation shows that there exists $K_0 > 0$, $n_0 \in \mathbb{N}$ such that $\mathbb{P}\{A_{nN}\} > K_0$ for $n, N > n_0$, $0 < \frac{n}{N} \leq a + C\frac{B_N}{N}$. Using this inequality and Lemma 3, we have for $\varepsilon > 0$

$$\begin{aligned}&\sum_{n,N>n_0,\,0<\frac{n}{N}<a+C\frac{B_N}{N}} \mathbb{P}\{|\mu_{nN}-\mathbb{E}\mu_{nN}| \geq N\varepsilon\} \\ &\leq \sum_{n,N>n_0,\,0<\frac{n}{N}<a+C\frac{B_N}{N}} \frac{K}{\mathbb{P}(A_{nN})}\frac{\varepsilon^4N^2}{s_r^4}e^{\frac{-\varepsilon^2N}{32s_r^2}}\end{aligned}$$

$$\leq \sum_{n,N>n_0,0<\frac{n}{N}<a+C\frac{B_N}{N}} \frac{K}{K_0}\frac{\varepsilon^4 N^2}{s_r^4} e^{\frac{-\varepsilon^2 N}{32 s_r^2}}$$
$$= \sum_{n,N>n_0,0<n<aN+CB_N} \frac{K}{K_0}\frac{\varepsilon^4 N^2}{s_r^4} e^{\frac{-\varepsilon^2 N}{32 s_r^2}}$$
$$\leq \sum_{N>n_0} \frac{K}{K_0}(a+C)N\frac{\varepsilon^4 N^2}{s_r^4} e^{\frac{-\varepsilon^2 N}{32 s_r^2}} < \infty.$$

By the Borel-Cantelli Lemma,

$$\frac{1}{N}\mu_{nN} - \mathbb{E}\frac{1}{N}\mu_{nN} \to 0 \qquad \text{almost surely} \tag{3.10}$$

as $n, N \to \infty$ such that $0 < \frac{n}{N} \leq a + C\frac{B_N}{N}$. By (3.9) and (3.10) we see that $\frac{1}{N}\mu_{nN} \to p_r$ almost surely as $n, N \to \infty$ such that $0 < \frac{n}{N} \leq a + C\frac{B_N}{N}$. The proof is complete. □

Let $\xi_0$ satisfy the Cramér condition. The variance of $\xi_0$ is $\sigma^2$ and $\Phi$ denotes the standard normal distribution function. Then, by Petrov's large deviation theorem (see Theorem 5.23 in [15]) we have

$$\frac{\mathbb{P}\{S_N^c \geq y\sqrt{N}\sigma\}}{1-\Phi(y)} = \exp\left\{\frac{y^3}{\sqrt{N}}\lambda\left(\frac{y}{\sqrt{N}}\right)\right\}\left[1+\mathrm{O}\left(\frac{y+1}{\sqrt{N}}\right)\right] \tag{3.11}$$

as $y \to \infty$ and $y = \mathrm{o}(\sqrt{N})$. Here $\lambda(t) = \sum_{k=0}^{\infty} a_k t^k$ is a power series with coefficients depending on the cumulants of $\xi_0 - a$ and with radius of convergence $R_1 > 0$.

**Lemma 4.** *Let $\xi_0$ satisfy the Cramér condition.*
*(1) Then we have*

$$\lim_{n,N\to\infty,\ \frac{n}{N}\to a,\ \sqrt{N}(\frac{n}{N}-a)\to\infty} \frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i \geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i \geq n\}} = 1. \tag{3.12}$$

*(2) As $n, N \to \infty$ such that $\frac{n}{N} \to a$ and $\sqrt{N}(\frac{n}{N}-a) \to \infty$, then we have*

$$\mathbb{P}\left\{\sum_{i=1}^{N}\xi_i \geq n\right\} \tag{3.13}$$
$$= \frac{\sigma}{\sqrt{2\pi N}(\frac{n}{N}-a)}\exp\left(-N\left(\frac{(\frac{n}{N}-a)^2}{2\sigma^2} - \frac{(\frac{n}{N}-a)^3}{\sigma^3}\lambda\left(\frac{\frac{n}{N}-a}{\sigma}\right)\right)\right)(1+\mathrm{o}(1)).$$

*Proof.* Let $x = \frac{n}{N} - a > 0$. Then we have

$$\frac{\mathbb{P}\left\{\sum_{i=1}^{N-1} \xi_i \geq n-r\right\}}{\mathbb{P}\left\{\sum_{i=1}^{N} \xi_i \geq n\right\}}$$
$$= \frac{\mathbb{P}\left\{\frac{1}{\sigma\sqrt{N-1}} \sum_{i=1}^{N-1} (\xi_i - a) \geq \frac{\sqrt{N}}{\sigma\sqrt{N-1}} \sqrt{N}\left(x + \frac{a-r}{N}\right)\right\}}{\mathbb{P}\left\{\frac{1}{\sigma\sqrt{N}} \sum_{i=1}^{N} (\xi_i - a) \geq \frac{1}{\sigma} x\sqrt{N}\right\}} = \frac{A}{B}.$$

Using Petrov's large deviation theorem, we obtain

$$\begin{aligned} A &= \left(1 - \Phi\left(\frac{\sqrt{N}}{\sigma\sqrt{N-1}} \sqrt{N}\left(x + \frac{a-r}{N}\right)\right)\right) \\ &\quad \times \exp\left\{N\left(\frac{\sqrt{N}}{\sigma\sqrt{N-1}}\left(x + \frac{a-r}{N}\right)\right)^3 \lambda\left(\frac{\sqrt{N}}{\sigma\sqrt{N-1}}\left(x + \frac{a-r}{N}\right)\right)\right\} \\ &\quad \times \left(1 + \mathrm{O}\left(\frac{\frac{\sqrt{N}}{\sigma\sqrt{N-1}} \sqrt{N}\left(x + \frac{a-r}{N}\right) + 1}{\sqrt{N}}\right)\right) \\ &= A_1 A_2 A_3 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} B &= \left(1 - \Phi\left(\frac{1}{\sigma} x\sqrt{N}\right)\right) \exp\left\{N\left(\frac{1}{\sigma}x\right)^3 \lambda\left(\frac{1}{\sigma}x\right)\right\} \times \left(1 + \mathrm{O}\left(\frac{\frac{1}{\sigma}x\sqrt{N} + 1}{\sqrt{N}}\right)\right) \\ &= B_1 B_2 B_3. \end{aligned} \tag{3.15}$$

Using the approximation

$$1 - \Phi(y) = \frac{1}{\sqrt{2\pi}\, y} e^{-\frac{y^2}{2}} \left(1 + \mathrm{O}\left(\frac{1}{y^2}\right)\right) \quad \text{as} \quad y \to \infty, \tag{3.16}$$

we obtain

$$\begin{aligned} \frac{A_1}{B_1} &= \frac{\frac{1}{\sigma} x\sqrt{N}}{\frac{\sqrt{N}}{\sigma\sqrt{N-1}} \sqrt{N}\left(x + \frac{a-r}{N}\right)} \\ &\quad \times \exp\left(\frac{1}{2}\left(\frac{1}{\sigma}x\sqrt{N}\right)^2 - \frac{1}{2}\left(\frac{\sqrt{N}}{\sigma\sqrt{N-1}} \sqrt{N}\left(x + \frac{a-r}{N}\right)\right)^2\right)(1 + \mathrm{o}(1)) \\ &= \frac{1}{1 - \frac{a-r}{Nx}} \exp\left(\frac{1}{2}\frac{1}{\sigma^2} N\left(x^2 - \frac{N}{N-1}\left(x + \frac{a-r}{N}\right)^2\right)\right)(1 + \mathrm{o}(1)) \end{aligned}$$

$$= \exp\left(\frac{1}{2\sigma^2} N \left(2x + \frac{a-r}{N}\right)\left(-\frac{a-r}{N}\right)\right)(1-\mathrm{o}(1))$$
$$= 1-\mathrm{o}(1). \tag{3.17}$$

The power series $\lambda(t)$ is convergent, if $n$ and $N$ is large enough, therefore

$$\ln\left(\frac{A_2}{B_2}\right) = N\sum_{k=0}^{\infty} a_k \left(\left(\frac{1}{\sigma}\sqrt{\frac{N}{N-1}}\left(x+\frac{a-r}{N}\right)\right)^{k+3} - \left(\frac{1}{\sigma}x\right)^{k+3}\right)$$
$$= N\sum_{k=0}^{\infty} a_k \left(\frac{1}{\sigma}\left(\sqrt{\frac{N}{N-1}}\left(x+\frac{a-r}{N}\right)-x\right)\right)$$
$$\times \sum_{i=0}^{k+2}\left(\frac{1}{\sigma}\sqrt{\frac{N}{N-1}}\left(x+\frac{a-r}{N}\right)\right)^{k+2-i}\left(\frac{1}{\sigma}x\right)^{i}.$$

Therefore

$$\left|\ln\left(\frac{A_2}{B_2}\right)\right| \le \frac{1}{\sigma}\left|\sqrt{\frac{N}{N-1}}(a-r) + x\frac{\frac{N}{N-1}}{\sqrt{\frac{N}{N-1}}+1}\right|$$
$$\times \sum_{k=0}^{\infty}|a_k|\left(\frac{1}{\sigma}\sqrt{\frac{N}{N-1}}\left(|x|+\frac{|a-r|}{N}\right)\right)^{k+2}(k+3).$$

The radius of convergence of the above series is positive, so we obtain

$$\left|\ln\left(\frac{A_2}{B_2}\right)\right| = \mathrm{o}(1), \text{ and } \frac{A_2}{B_2} = 1+\mathrm{o}(1) \text{ as } n, N \to \infty,$$

because $|x| \to 0$ and $\frac{|a-r|}{n} \to 0$.

Finally, it is easy to see that

$$\frac{A_3}{B_3} \to 1.$$

(2) Let $x = (\frac{n}{N} - a)$. Using (3.15) and approximation (3.16), we obtain

$$\mathbb{P}\left\{\sum_{i=1}^{N}\xi_i \ge n\right\} = B$$
$$= \frac{1}{\sqrt{2\pi}\frac{1}{\sigma}x\sqrt{N}} e^{-\frac{1}{2}\left(\frac{1}{\sigma}x\sqrt{N}\right)^2} \exp\left(N\left(\frac{1}{\sigma}x\right)^3 \lambda\left(\frac{1}{\sigma}x\right)\right)(1+\mathrm{o}(1))$$
$$= \frac{\sigma}{\sqrt{2\pi}(\frac{n}{N}-a)\sqrt{N}}$$

$$\times \exp\left(-\frac{1}{2\sigma^2}N(\frac{n}{N}-a)^2+N\frac{1}{\sigma^3}(\frac{n}{N}-a)^3\lambda\left(\frac{(\frac{n}{N}-a)}{\sigma}\right)\right)(1+o(1))$$

as $n,N\to\infty$, $\frac{n}{N}\to\alpha$ and $\sqrt{N}(\frac{n}{N}-a)\to\infty$. □

*Proof of Corollary 1.* We apply Theorem 1 with $B_N=N$. We see that, by Kolmogorov's law of large numbers, (2.3) is true with $\xi'=0$. So Theorem 1 implies the result. □

*Proof of Corollary 2.* We apply Theorem 1 with $B_N=\sqrt{N}$. By the central limit theorem, (2.3) is true if $\xi'$ is a Gaussian random variable with expectation 0 and variance $\sigma^2$. So Theorem 1 implies the result. □

*Proof of Theorem 2.* (1) (2.5) follows from Corollary 1.

(2) Let $n,N\to\infty$ such that $\frac{n}{N}\to a$, $\sqrt{N}(\frac{n}{N}-a)\to\infty$. By Lemma 4 (1), we have

$$\frac{1}{N}\mathbb{E}\mu_{nN}=p_r\frac{\mathbb{P}\{\sum_{i=1}^{N-1}\xi_i\geq n-r\}}{\mathbb{P}\{\sum_{i=1}^{N}\xi_i\geq n\}}\to p_r. \tag{3.18}$$

Let $0<\varepsilon<1$. If $n,N\to\infty$ such that $\frac{n}{N}\to a$, $\sqrt{N}(\frac{n}{N}-a)\to\infty$, then there exists $n_0\in\mathbb{N}$, such that

- $|\frac{n}{N}-a|<\varepsilon$ for all $n,N>n_0$;
- $\sqrt{N}(\frac{n}{N}-a)>1$ for all $n,N>n_0$;
- $\frac{(\frac{n}{N}-a)^2}{2\sigma^2}-\frac{(\frac{n}{N}-a)^3}{\sigma^3}\lambda\left(\frac{\frac{n}{N}-a}{\sigma}\right)<\frac{\varepsilon^2}{64s_r^2}$ for all $n,N>n_0$.

As before, $A_{nN}$ denotes the following event: $A_{nN}=\{\sum_{i=1}^{N}\xi_i\geq n\}$. Under the above relations, by Lemma 3 and Lemma 4 (2), for $n,N>n_0$ we have

$$\begin{aligned}
&\sum_{n,N>n_0}\mathbb{P}\left\{\left|\frac{1}{N}\mu_{nN}-\mathbb{E}\frac{1}{N}\mu_{nN}\right|\geq\varepsilon\right\}\leq K\sum_{n,N>n_0}\frac{1}{\mathbb{P}(A_{nN})}\frac{\varepsilon^4N^2}{s_r^4}e^{-\frac{N\varepsilon^2}{32s_r^2}}\\
&\leq K\sum_{n,N>n_0,|\frac{n}{N}-a|<\varepsilon}\frac{\sqrt{2\pi}(\frac{n}{N}-a)\sqrt{N}}{\sigma\exp\left(-N\left(\frac{(\frac{n}{N}-a)^2}{2\sigma^2}-\frac{(\frac{n}{N}-a)^3}{\sigma^3}\lambda\left(\frac{\frac{n}{N}-a}{\sigma}\right)\right)\right)}\frac{\varepsilon^4N^2}{s_r^4}e^{\frac{-\varepsilon^2N}{32s_r^2}}\\
&\leq K_1\sum_{n,N>n_0,N(a-\varepsilon)<n<(a+\varepsilon)N}\frac{\sqrt{N}\varepsilon}{\sigma}\frac{1}{e^{\frac{-\varepsilon^2N}{64s_r^2}}}\frac{\varepsilon^4N^2}{s_r^4}e^{\frac{-\varepsilon^2N}{32s_r^2}}\\
&\leq K_1\sum_{N=n_0}^{\infty}\frac{C\varepsilon^5}{\sigma}\frac{\sqrt{N}}{s_r^4}2N^3\varepsilon e^{\frac{-\varepsilon^2N}{64s_r^2}}<\infty.
\end{aligned}$$

Above we used that the number of $n$'s in the sum in bounded by $2NC$. Consequently, by the Borel-Cantelli Lemma, $\frac{1}{N}\mu_{nN}-\mathbb{E}\frac{1}{N}\mu_{nN}\to 0$ almost surely, as $n,N\to\infty$

such that $\frac{n}{N} \to a$, $\sqrt{N}\left(\frac{n}{N} - a\right) \to \infty$. Therefore, by (3.18),

$$\frac{1}{N}\mu_{nN} = \frac{1}{N}\mu_{nN} - \mathbb{E}\frac{1}{N}\mu_{nN} + \mathbb{E}\frac{1}{N}\mu_{nN} \to p_r \qquad \text{almost surely,} \tag{3.19}$$

as $n, N \to \infty$ such that $\frac{n}{N} \to a$, $\sqrt{N}(\frac{n}{N} - a) \to \infty$. By (3.19), there exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}\{\Omega_1\} = 1$ and for all $\omega \in \Omega_1$ we have

$$\frac{1}{N}\mu_{nN}(\omega) \to p_r \quad \text{as} \quad n, N \to \infty \quad \text{such that} \quad \frac{n}{N} \to a, \ \sqrt{N}\left(\frac{n}{N} - a\right) \to \infty.$$

By Corollary 2, there exists $\Omega_2 \subset \Omega$ such that $\mathbb{P}\{\Omega_2\} = 1$ and for all $\omega \in \Omega_2$ we have

$$\frac{1}{N}\mu_{nN}(\omega) \to p_r \quad \text{as} \quad n, N \to \infty \quad \text{such that} \quad \frac{n}{N} \to a, \ \sqrt{N}\left(\frac{n}{N} - a\right) \le m \tag{3.20}$$

for an arbitrary fixed $m \in \mathbb{N}$.

Let $\omega \in \Omega_3 = \Omega_1 \cap \Omega_2$. Suppose that $\frac{n}{N} \to a$, as $n, N \to \infty$. Let $\varepsilon > 0$.

Since $\omega \in \Omega_1$, there exist $m = m(\omega), n_1 = n_1(\omega) \in \mathbb{N}$ and $\delta_1 = \delta_1(\omega) > 0$ depending on $\omega$ such that $\left|\frac{1}{N}\mu_{nN}(\omega) - p_r\right| < \varepsilon$, if $n, N > n_1$, $\sqrt{N}(\frac{n}{N} - a) > m$ and $|\frac{n}{N} - a| < \delta_1$.

Since $\omega \in \Omega_2$, there exist $n_2 = n_2(\omega, m) \in \mathbb{N}$ and $\delta_2 = \delta_2(\omega, m) > 0$ depending on $\omega$ and $m$, such that $\left|\frac{1}{N}\mu_{nN}(\omega) - p_r\right| < \varepsilon$, if $n, N > n_2$, $\sqrt{N}(\frac{n}{N} - a) \le m$ and $|\frac{n}{N} - a| < \delta_2$.

Introduce notation $\delta = \min(\delta_1, \delta_2)$, $n_0 = \max(n_1, n_2)$. Consequently, if $n, N > n_0$ and $|\frac{n}{N} - a| < \delta$, then $\left|\frac{1}{N}\mu_{nN}(\omega) - p_r\right| < \varepsilon$. The proof is complete. □

*Proof of Theorem 3.* Consider the following represention of the distribution of $\mu_{nN}$ in Model 3 (see Theorem 2.2 of [8]).

For all $k = 0, 1, 2, \ldots, N$,

$$\mathbb{P}\{\mu_{nN} = k\} = \binom{N}{k} p_1^k (1 - p_1)^{N-k} \frac{\mathbb{P}\{S_{N-k}^{(1)} \ge n - k\}}{\mathbb{P}\{S_N \ge n\}}. \tag{3.21}$$

Let $k \in \mathbb{N}_0$. By the Poisson limit theorem, one has

$$\binom{N}{k} p_1^k (1 - p_1)^{N-k} = \frac{\lambda^k e^{-\lambda}}{k!}(1 + \mathrm{o}(1)). \tag{3.22}$$

$Np_1(\theta) \to \lambda$ implies that $\theta \to 0$, therefore $B(\theta) = b_0 + \mathrm{o}(1)$ and $\theta = \frac{b_0\lambda + \mathrm{o}(1)}{Nb_1}$. Using Theorem 2 in [11], we have

$$\mathbb{P}\{S_N \ge n\} = \left(\sum_{l=n}^{\infty} \frac{\lambda^l e^{-\lambda}}{l!}\right)(1 + \mathrm{o}(1)). \tag{3.23}$$

Since $\mathbb{P}\{\xi_0^{(1)} = 1\} = 0$, we obtain that

$$\mathbb{P}\{S_{N-k}^{(1)} < n-k\} \le \mathbb{E} S_{N-k}^{(1)} \le N\mathbb{E}\xi_0^{(1)} = N\frac{\frac{b_2}{b_0}\left(\frac{b_0\lambda+o(1)}{Nb_1}\right)^2}{1-\frac{b_1}{b_0}\frac{b_0\lambda+o(1)}{Nb_1}}(1+o(1)) = o(1).$$

Therefore

$$\{S_{N-k}^{(1)} \ge n-k\} = 1+o(1). \tag{3.24}$$

Using (3.22), (3.23) and (3.24) in (3.21), we obtain the desired result. □

*Proof of Lemma 1.* The proof is a modification of the proof of Lemma 1.2.1 in [12]. Let us denote by $B-kr = \{x-kr \,|\, x \in B\}$ for any $B \subset \mathbb{R}_+$, $k,r \in \mathbb{R}_+$ fixed. Let $A_k^{(r)}$ denote the event that exactly $k$ of the random variables $\xi_1,\dots,\xi_N$ being equal to $r$. By (2.10), we have

$$\mathbb{P}\{\mu_{nN} = k\} = \mathbb{P}(A_k^{(r)} \,|\, S_N \in B_n) = \frac{\mathbb{P}(A_k^{(r)}, S_N \in B_n)}{\mathbb{P}(S_N \in B_n)}\,.$$

Therefore, using that $\xi_1,\dots,\xi_N$ are independent random variables and the event $A_k^{(r)}$ can occur $\binom{N}{k}$ differents ways, we have

$$\begin{aligned}\mathbb{P}(A_k^{(r)}, S_N \in B_n) &= \mathbb{P}(S_N \in B_n \,|\, A_k^{(r)})\mathbb{P}(A_k^{(r)})\\ &= \binom{N}{k} p_r^k (1-p_r)^{N-k}\\ &\quad \times \mathbb{P}(S_N \in B_n \,|\, \xi_1 \ne r,\dots,\xi_{N-k} \ne r, \xi_{N-k+1} = r,\dots,\xi_N = r)\\ &= \binom{N}{k} p_r^k (1-p_r)^{N-k}\mathbb{P}(S_{N-k}^{(r)} \in B_n - kr)\,.\end{aligned}$$

□

*Proof of Theorem 4.* The proof of our limit theorem is based on representation (2.11). Using the Moivre-Laplace theorem, we have

$$\binom{N}{k} p_r^k (1-p_r)^{N-k} = \frac{1}{\sqrt{2\pi N}s_r} e^{-u^2/2}(1+o(1)) \tag{3.25}$$

as $N \to \infty$ and $u = \frac{k-Np_r}{s_r N^{1/2}}$ belongs to a bounded fixed interval, where $s_r^2 = p_r(1-p_r)$.

Let $a < \infty$, $\widetilde{\alpha_1} a \in B^\circ$. Apply Kolmogorov's law of large numbers. Then we have

$$\mathbb{P}\{S_N \in B_n\} = \mathbb{P}\left(\frac{N}{f(n)}\frac{S_N}{N} \in B\right) \xrightarrow{n,N\to\infty} 1\,, \tag{3.26}$$

and

$$\mathbb{P}\{S_{N-k}^{(r)} \in B_n - kr\} = \mathbb{P}\left(\frac{N-k}{f(n)}\frac{S_{N-k}^{(r)}}{N-k} \in B - \frac{kr}{f(n)}\right) \xrightarrow{n,N\to\infty} 1\,. \tag{3.27}$$

To obtain (3.27), we used that

$$\frac{N-k}{f(n)} = \frac{N}{f(n)}\left(1 - \frac{us_r}{\sqrt{N}} - p_r\right) \xrightarrow{n,N\to\infty} \widetilde{\alpha_1}(1-p_r),$$

and

$$\frac{kr}{f(n)} = \frac{N}{f(n)}\left(\frac{us_r r}{\sqrt{N}} + rp_r\right) \xrightarrow{n,N\to\infty} \widetilde{\alpha_1} rp_r.$$

□

*Proof of Lemma 2.* The proof of Lemma 2 is a modification of the proof of Lemma 1.2.2 in [12].

$$\mathbb{P}\{\eta_{(N)} \le r\} = \mathbb{P}\{\eta_1 \le r, \dots, \eta_N \le r\} = \mathbb{P}\left\{\xi_1 \le r, \dots, \xi_N \le r \;\middle|\; \sum_{i=1}^N \xi_i \in B_n\right\}$$
$$= \frac{\mathbb{P}\left\{\xi_1 \le r, \dots, \xi_N \le r\,, S_N \in B_n\right\}}{\mathbb{P}\left\{S_N \in B_n\right\}} = (1-P_r)^N \frac{\mathbb{P}\{S_N^{(\le r)} \in B_n\}}{\mathbb{P}\{S_N \in B_n\}}.$$

□

*Proof of Theorem 5.* Consider representation (2.12) and apply Kolmogorov's law of large numbers for $S_N$ and $S^{(\le r)}$. Then we have

$$\mathbb{P}\{S_N \in B_n\} = \mathbb{P}\left(\frac{N}{f(n)}\frac{S_N}{N} \in B\right) \xrightarrow{n,N\to\infty} 1\,, \tag{3.28}$$

and

$$\mathbb{P}\{S_N^{(\le r)} \in B_n\} = \mathbb{P}\left(\frac{N}{f(n)}\frac{S_N^{(\le r)}}{N} \in B\right) \xrightarrow{n,N\to\infty} 1\,. \tag{3.29}$$

□

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# REFINEMENTS OF THE LOWER BOUNDS OF JENSEN'S FUNCTIONAL REVISITED

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*Abstract.* The functional defined as the difference of the right-hand and the left-hand side of Jensen's inequality is called Jensen's functional. The aim of this paper is to study its lower bounds. More precisely, some refinements of its lower bounds are recaptured under more relaxed conditions on the $n$-tuple $\mathbf{x}$.

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*Keywords:* Jensen's inequality, Jensen's functional, lower bounds, refinement

## 1. Introduction

Jensen's inequality is possibly one of the most famous and most explored inequalities. It has been generalized in numerous directions and it has found its application in various areas; see for example [8] and the references therein. Classical Jensen's inequality states as follows.

**Theorem 1** (Jensen, 1905. [8])**.** *Let $I$ be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a positive $n$-tuple, that is, such that $p_i > 0$ for $i = 1, \ldots, n$. Then*

$$f \quad \frac{1}{P_1^n} \sum_{i=1}^{n} p_i x_i \Bigg) \leq \frac{1}{P_1^n} \sum_{i=1}^{n} p_i f(x_i),$$

*where $P_1^n = \sum_{i=1}^{n} p_i$. If $f$ is strictly convex, then the inequality is strict, unless $x_1 = x_2 = \ldots = x_n$.*

The functional defined as the difference of the right-hand and the left-hand side of Jensen's inequality:

$$J(\mathbf{x}, \mathbf{p}, f) = \frac{1}{P_1^n} \sum_{i=1}^{n} p_i f(x_i) - f \quad \frac{1}{P_1^n} \sum_{i=1}^{n} p_i x_i \Bigg), \tag{1.1}$$

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is called Jensen's functional. What will interest us here in particular are its lower bounds. Some related results can be found, for example, in [1], [4], [5], [7], but the papers of special interest at the moment are [3], [2] and [6].

In [2], the author presented some lower bounds for Jensen's functional which were later recaptured in [6] under relaxed conditions on the $n$-tuple $\mathbf{x}$. The results from [6] are the basis for the results which are to be presented in this paper. Without any loss of generality, we assume $P_1^n = 1$ since for positive $n$-tuples such that $P_1^n \neq 1$ results follow easily after substituting $p_j$ by $p_j/P_1^n$. Furthermore, for $1 \leq i < k \leq n$, we introduce notation:

$$J_{ik}(\mathbf{x},\mathbf{p},f) = P_1^i f(x_i) + \sum_{m=i+1}^{k-1} p_m f(x_m) + P_k^n f(x_k) - f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right),$$

where

$$P_i^j = \sum_{m=i}^{j} p_m.$$

Note that $J_{ik}(\mathbf{x},\mathbf{p},f)$ can be obtained from $J(\mathbf{x},\mathbf{p},f)$ upon taking $x_1 = x_2 = \ldots = x_i$ and $x_k = x_{k+1} = \ldots = x_n$ and that $J_{1n}(\mathbf{x},\mathbf{p},f) = J(\mathbf{x},\mathbf{p},f)$. In what follows, $I$ is an interval in $\mathbb{R}$ and

$$\bar{X}_i^j = \frac{1}{P_i^j}\sum_{m=i}^{j} p_m x_m$$

**Theorem 2** (Theorem 2.1, [6])**.** *Let $f$ be a convex function on $I$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 2$. Let $1 \leq i < k \leq n$ and $x_1, x_2, \ldots, x_k \in I$. If $x_i$ is such that*

$$\bar{X}_1^i \leq x_i \leq \frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \tag{1.2}$$

$$\text{or} \quad \frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \leq x_i \leq \bar{X}_1^i, \tag{1.3}$$

*then we have*

$$J_{1k}(\boldsymbol{x},\boldsymbol{p},f) \geq J_{ik}(\boldsymbol{x},\boldsymbol{p},f). \tag{1.4}$$

**Theorem 3** (Theorem 2.3, [6])**.** *Let $f$ be a convex function on $I$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 2$. Let $1 \leq i < k \leq n$ and $x_j, x_{j+1}, \ldots, x_n \in I$.*

*If $x_k$ is such that*

$$\frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right) \le x_k \le \bar{X}_k^n \tag{1.5}$$

$$\text{or} \quad \bar{X}_k^n \le x_k \le \frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right), \tag{1.6}$$

*then we have*

$$J_{in}(\boldsymbol{x},\boldsymbol{p},f) \ge J_{ik}(\boldsymbol{x},\boldsymbol{p},f). \tag{1.7}$$

In the same paper, in Theorems 2.2 and 2.4, alternative sets of conditions under which (1.4) and (1.7) hold were given, respectively, but it can be seen from the proofs that these conditions are actually more restricting than those given in Theorem 2 and Theorem 3, respectively. Finally, another related result, providing a somewhat different type of conditions, was given in the same paper in the following theorem.

**Theorem 4** (Theorem 2.7, [6])**.** *Let $f$ be a convex function on $I$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \ge 2$. Let $\boldsymbol{x} \in I^n$ be a real $n$-tuple and let $1 \le i < k \le n$. If $x_i$ and $x_k$ are such that*

$$\bar{X}_1^i \le x_i \le \bar{X}_1^n \le x_k \le \bar{X}_k^n \tag{1.8}$$

$$\text{or} \quad \bar{X}_k^n \le x_k \le \bar{X}_1^n \le x_i \le \bar{X}_1^i, \tag{1.9}$$

*then we have*

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge J_{ik}(\boldsymbol{x},\boldsymbol{p},f).$$

These results dealt with obtaining the lower bound for Jensen's functional with two fixed variables. In [3], the author went one step further and gave lower bounds for Jensen's functional with three fixed variables.

**Theorem 5.** *Let $f$ be a convex function on $I$ and let $x_1, x_2, \ldots, x_n \in I$ $(n \ge 4)$ be such that*

$$x_1 \le x_2 \le \ldots \le x_n$$

*and let $p_1, p_2, \ldots, p_n$ be positive weights such that $P_1^n = 1$. For fixed $x_i, x_j, x_k$ $(1 \le i < j < k \le n)$, we have*

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge F_{ijk}(\boldsymbol{x},\boldsymbol{p},f) \quad \text{for } x_j \le \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n}$$

$$\text{and} \quad J(\boldsymbol{x},\boldsymbol{p},f) \ge G_{ijk}(\boldsymbol{x},\boldsymbol{p},f) \quad \text{for } x_j \ge \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n},$$

*where*

$$F_{ijk}(\boldsymbol{x},\boldsymbol{p},f) = P_1^i f(x_i) + P_{i+1}^j f(x_j) + P_k^n f(x_k)$$

$$-(P_1^j + P_k^n)\, f\left(\frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n}\right) \tag{1.10}$$

$$G_{ijk}(\boldsymbol{x},\boldsymbol{p},f) = P_1^i f(x_i) + P_j^{k-1} f(x_j) + P_k^n f(x_k)$$
$$-(P_1^i + P_j^n)\, f\left(\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n}\right) \tag{1.11}$$

*In addition, we have* $J(\boldsymbol{x},\boldsymbol{p},f) = F_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ *for*

$$\begin{aligned}
&x_1 = x_2 = \ldots = x_i,\\
&x_{i+1} = x_{i+2} = \ldots = x_j,\\
&x_{j+1} = x_{j+2} = \ldots = x_{k-1} = \frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n},\\
&x_k = x_{k+1} = \ldots = x_n
\end{aligned}$$

*and* $J(\boldsymbol{x},\boldsymbol{p},f) = G_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ *for*

$$\begin{aligned}
&x_1 = x_2 = \ldots = x_i,\\
&x_{i+1} = x_{i+2} = \ldots = x_{j-1} = \frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n},\\
&x_j = x_{j+1} = \ldots = x_{k-1},\\
&x_k = x_{k+1} = \ldots = x_n.
\end{aligned}$$

The principal aim of this paper is to recapture results from Theorem 5 under relaxed conditions on the $n$-tuple $\mathbf{x}$. We shall do so by employing Theorems 2, 3 and 4.

The key step in proving the results in [3], [2] and [6] was the following lemma which was presented in [2]. It will play a vital role here as well.

**Lemma 1.** *Let $f$ be a convex function on $I$ and let $p_1$, $p_2$ be non-negative real numbers. If $a_1, a_2, b_1, b_2 \in I$ are such that $a_1, a_2 \in [b_1, b_2]$ and*

$$p_1 a_1 + p_2 a_2 = p_1 b_1 + p_2 b_2,$$

*then*

$$p_1 f(a_1) + p_2 f(a_2) \le p_1 f(b_1) + p_2 f(b_2).$$

## 2. Main results

We present a series of theorems providing several different sets of conditions on the $n$-tuple $\mathbf{x}$ under which the inequalities stated in Theorem 5 hold.

**Theorem 6.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If $x_i$ is such that*

$$\bar{X}_1^i \leq x_i \leq \bar{X}_{i+1}^n \qquad or \qquad \bar{X}_{i+1}^n \leq x_i \leq \bar{X}_1^i, \tag{2.1}$$

*$x_j$ is such that*

$$\bar{X}_{i+1}^{j-1} \leq x_j \leq \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \tag{2.2}$$

*or*

$$\frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \leq x_j \leq \bar{X}_{i+1}^{j-1} \tag{2.3}$$

*and $x_k$ is such that either (1.5) or (1.6) holds, then we have*

$$J(\boldsymbol{x}, \boldsymbol{p}, f) \geq F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.4}$$

*where $J(\boldsymbol{x}, \boldsymbol{p}, f)$ is defined in (1.1) and $F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ in (1.10).*

*Proof.* We present the proof in three steps.

*First step*. We have

$$\begin{aligned} J(\mathbf{x}, \mathbf{p}, f) \geq & \sum_{m=1}^{i} p_m f(x_m) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) \\ & + \sum_{m=k}^{n} p_m f(x_m) - f\left(\bar{X}_1^n\right). \end{aligned} \tag{2.5}$$

Namely, this is equivalent to

$$\sum_{m=i+1}^{j-1} p_m f(x_m) \geq P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) \quad \text{and} \quad \sum_{m=j+1}^{k-1} p_m f(x_m) \geq P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}),$$

which are both immediate consequences of Jensen's inequality. Note that the right-hand side of (2.5) is equal to the value of $J(\mathbf{x}, \mathbf{p}, f)$ for an n-tuple $\mathbf{x}$ such that $x_{i+1} = \ldots = x_{j-1} = \bar{X}_{i+1}^{j-1}$ and $x_{j+1} = \ldots = x_{k-1} = \bar{X}_{j+1}^{k-1}$.

*Second step*. We claim that

$$\begin{aligned} & \sum_{m=1}^{i} p_m f(x_m) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) \\ & \quad + \sum_{m=k}^{n} p_m f(x_m) - f\left(\bar{X}_1^n\right) \\ & \geq P_1^i f(x_i) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + P_k^n f(x_k) \end{aligned}$$

$$- f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right). \tag{2.6}$$

This inequality follows immediately after first applying Theorem 2 for $k = n$ and then Theorem 3.

*Third step*. The final claim is that

$$P_1^i f(x_i) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + P_k^n f(x_k)$$
$$- f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \geq F_{ijk}(\mathbf{x}, \mathbf{p}, f),$$

which is equivalent to

$$P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^j + P_k^n)\, f\left(\frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n}\right)$$
$$\geq P_{i+1}^{j-1} f(x_j) + f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right).$$

Jensen's inequality implies that we have

$$f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right)$$
$$\leq P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^j + P_k^n) f\left(\frac{P_1^i x_i + \sum_{m=i+1}^{j} p_m x_m + P_k^n x_k}{P_1^j + P_k^n}\right),$$

so we now only need to prove that

$$P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + (P_1^j + P_k^n)\, f\left(\frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n}\right)$$
$$\geq P_{i+1}^{j-1} f(x_j) + (P_1^j + P_k^n) f\left(\frac{P_1^i x_i + \sum_{m=i+1}^{j} p_m x_m + P_k^n x_k}{P_1^j + P_k^n}\right),$$

Here we employ Lemma 1 for

$$a_1 = x_j, \;\; a_2 = (P_1^i x_i + \sum_{m=i+1}^{j} p_m x_m + P_k^n x_k)/(P_1^j + P_k^n),$$

$b_1 = \bar{X}_{i+1}^{j-1}$ and $b_2 = (P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k)/(P_1^j + P_k^n)$ (or the obvious rearrangement) to complete the proof. Condition (2.2) (that is, (2.3)) ensures that the conditions of Lemma 1 are satisfied. This is easily checked. □

**Theorem 7.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If $x_i$ is such that either (1.2) or (1.3) holds, $x_j$ is such that either (2.2) or (2.3) holds and $x_k$ is such that*

$$\bar{X}_1^{k-1} \leq x_k \leq \bar{X}_k^n \qquad \text{or} \qquad \bar{X}_k^n \leq x_k \leq \bar{X}_1^{k-1}, \tag{2.7}$$

*then the inequality (2.4) is valid.*

*Proof.* Proof is analogous to the proof of Theorem 6. Namely, the first and the third step of the proof are completely the same, but we have a slight difference in the second step. To obtain inequality (2.6), first apply Theorem 3 for $i = 1$ and then Theorem 2. □

*Remark* 1. Note that since

$$\frac{1}{P_{i+1}^n}\left(\sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \leq \bar{X}_{i+1}^n \quad \text{if} \quad x_k \leq \bar{X}_k^n$$

and similarly

$$\bar{X}_1^{k-1} \leq \frac{1}{P_1^{k-1}}\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m\right) \quad \text{if} \quad \bar{X}_1^i \leq x_i,$$

Theorem 6 imposes a more restricting condition on $x_k$, while Theorem 7 imposes a more restricting condition on $x_i$.

*Remark* 2. An increasing $n$-tuple $\mathbf{x}$ satisfies the first condition in (2.1), the left-hand side of the condition (2.2) and the condition (1.5), and also the condition (1.2) and the first condition in (2.7). Thus, Theorems 6 and 7 both provide generalizations of Theorem 5.

**Theorem 8.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. Let $x_i$ and $x_k$ be such that either (1.8) or (1.9) holds and $x_j$ be such that either (2.2) or (2.3) holds. Then the inequality (2.4) is valid.*

*Proof.* In respect to the proof of Theorem 6, there is a difference again only in the second step. To obtain inequality (2.6), apply Theorem 4. □

*Remark* 3. Note that an increasing $n$-tuple $\mathbf{x}$ does not necessarily satisfy either (1.8) or (1.9).

**Theorem 9.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If $x_i$ is such that either one of the conditions in (2.1) hold, $x_j$ is such that*

$$\bar{X}_{j+1}^{k-1} \leq x_j \leq \frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \tag{2.8}$$

*or*

$$\frac{P_1^i x_i + P_k^n x_k}{P_1^i + P_k^n} \le x_j \le \bar{X}_{j+1}^{k-1} \tag{2.9}$$

*and* $x_k$ *is such that either* (1.5) *or* (1.6) *holds, then we have*

$$J(\boldsymbol{x},\boldsymbol{p},f) \ge G_{ijk}(\boldsymbol{x},\boldsymbol{p},f), \tag{2.10}$$

*where* $J(\boldsymbol{x},\boldsymbol{p},f)$ *is defined in* (1.1) *and* $G_{ijk}(\boldsymbol{x},\boldsymbol{p},f)$ *in* (1.11).

*Proof.* The proof follows in the footsteps of the proof of Theorem 6. Namely, the first and the second step are the same (note that the conditions on $x_i$ and $x_k$ are the same). The difference is in the third step. Here we need to prove that

$$\begin{aligned} &P_1^i f(x_i) + P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + p_j f(x_j) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + P_k^n f(x_k) \\ &\quad - f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \ge G_{ijk}(\mathbf{x},\mathbf{p},f), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^i + P_j^n)\, f\left(\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n}\right) \\ &\ge P_{j+1}^{k-1} f(x_j) + f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right). \end{aligned}$$

Jensen's inequality implies that

$$\begin{aligned} &f\left(P_1^i x_i + \sum_{m=i+1}^{k-1} p_m x_m + P_k^n x_k\right) \\ &\quad \le P_{i+1}^{j-1} f(\bar{X}_{i+1}^{j-1}) + (P_1^i + P_j^n) f\left(\frac{P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k}{P_1^i + P_j^n}\right), \end{aligned}$$

so there is only left to prove that

$$\begin{aligned} &P_{j+1}^{k-1} f(\bar{X}_{j+1}^{k-1}) + (P_1^i + P_j^n)\, f\left(\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n}\right) \\ &\ge P_{j+1}^{k-1} f(x_j) + (P_1^i + P_j^n) f\left(\frac{P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k}{P_1^i + P_j^n}\right), \end{aligned}$$

Now we apply Lemma 1 for $a_1 = x_j$, $a_2 = (P_1^i x_i + \sum_{m=j}^{k-1} p_m x_m + P_k^n x_k)/(P_1^i + P_j^n)$, $b_1 = \bar{X}_{j+1}^{k-1}$ and $b_2 = (P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k)/(P_1^i + P_j^n)$ (or the obvious rearrangement) to complete the proof. This time condition (2.8) (that is, (2.9)) ensures that the conditions of Lemma 1 are satisfied. □

**Theorem 10.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. If $x_i$ is such that either (1.2) or (1.3) holds, $x_j$ is such that either (2.8) or (2.9) holds and $x_k$ is such that either one of the conditions in (2.7) holds, then the inequality (2.10) is valid.*

*Proof.* The first and the second step of the proof are the same as in Theorem 7, while the third step is the same as in the proof of Theorem 9. □

*Remark* 4. An increasing $n$-tuple $\mathbf{x}$ satisfies the first condition in (2.1), the right-hand side of the condition (2.9) and the condition (1.5), and also the condition (1.2) and the first condition in (2.7). Thus, Theorems 9 and 10 both provide generalizations of Theorem 5.

**Theorem 11.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$. Let $1 \leq i < j < k \leq n$. Let $x_i$ and $x_k$ be such that either (1.8) or (1.9) holds and $x_j$ be such that either (2.8) or (2.9) holds. Then the inequality (2.10) is valid.*

*Proof.* The first and the second step of the proof are the same as in Theorem 8, while the third step is the same as in the proof of Theorem 9. □

**Proposition 1.** *Let $f$ be a convex function on $I$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\boldsymbol{p}$ be a positive $n$-tuple such that $P_1^n = 1$ for some $n \geq 3$.*
*If $P_{j+1}^{k-1} \geq P_{i+1}^{j-1}$, then we have*

$$G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f) \geq F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.11}$$

*while if $P_{i+1}^{j-1} \geq P_{j+1}^{k-1}$, then we have*

$$F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f) \geq G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f), \tag{2.12}$$

*where $F_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ is defined as in (1.10) and $G_{ijk}(\boldsymbol{x}, \boldsymbol{p}, f)$ in (1.11).*

*Proof.* Note that

$$\begin{aligned}
&\frac{P_1^i x_i + P_j^{k-1} x_j + P_k^n x_k}{P_1^i + P_j^n} \\
&= \frac{P_1^j + P_k^n}{P_1^i + P_j^n} \cdot \frac{P_1^i x_i + P_{i+1}^j x_j + P_k^n x_k}{P_1^j + P_k^n} + \frac{P_{j+1}^{k-1} - P_{i+1}^{j-1}}{P_1^i + P_j^n} \cdot x_j .
\end{aligned}$$

Inequality (2.11) now follows directly from Jensen's inequality, while the obvious rearrangement yields (2.12). □

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# CASTELNUOVO-MUMFORD REGULARITY OF FIBER CONES OF FILTERED MODULES

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*Abstract.* We obtain upper bounds for the Castelnuovo-Mumford regularity of the fiber cones which depend on the length of certain local cohomology modules. The bounds are the analogue of the ones proved by Dung and Hoa for the associated graded module of a filtered module.

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*Keywords:* Castelnuovo-Mumford regularity, associated graded module, fiber cone, good filtration

## 1. Introduction

Let $(A, \mathfrak{m})$ be a commutative Noetherian local ring, $I$ an $\mathfrak{m}$-primary ideal, and $M$ a finitely generated $A$-module. A chain of submodules

$$\mathbb{M} : M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is called an *$I$-filtration* if $IM_i \subseteq M_{i+1}$ for all $i$, and a *good $I$-filtration* if $IM_i = M_{i+1}$ for all sufficiently large $i$. Thus $\{I^n M\}$ is a good $I$-filtration. A module $M$ with a filtration is called a *filtered module*.

The *associated graded module* to the filtration $\mathbb{M}$ is defined by

$$G(\mathbb{M}) = \bigoplus_{n \geq 0} M_n / M_{n+1}.$$

We also say that $G(\mathbb{M})$ is the associated graded ring of the filtered module $M$. This is a finitely generated graded module over the standard graded ring $G_I(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$. In the particular case, when $\mathbb{M}$ is the $I$-adic filtration $\{I^n M\}$, $G(\mathbb{M})$ is just the usual associated graded module $G_I(M)$.

It is well known that the Castelnuovo-Mumford regularity is a measure for the complexity of the structure of graded modules and several important invariants of graded modules can be estimated by means of the Castelnuovo-Mumford regularity (see e.g. [7], [9], [10], [3]). Therefore, bounding the Castelnuovo-Mumford

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regularity $\operatorname{reg}(G_I(M))$ is an important problem. This problem has been studied by Rossi-Trung-Valla [7] for the case $M = A$ and $I = \mathfrak{m}$, Linh [6] for arbitrary finitely generated $A$-modules with respect to an arbitrary $\mathfrak{m}$-primary ideal and by Dung-Hoa [4] for good filtrations of $M$, where bounds on $\operatorname{reg}(G(\mathbb{M}))$ were given in terms of the extended degree $D(I, M)$ of $M$ with respect to $I$ and $r(\mathbb{M})$, where

$$r(\mathbb{M}) = \min\{t \geq 0 | M_{n+1} = IM_n \text{ for all } n \geq t\}.$$

Recently, Dung [2] and Dung-Hoa [4] gave bounds for $\operatorname{reg}(G(\mathbb{M}))$ in terms of the Hilbert coefficients.

Another important subject associated to $I$ is the fiber cone $F_{\mathfrak{q}}(\mathbb{M}) = \bigoplus_{n\geq 0} M_n/\mathfrak{q}M_n$ of $\mathbb{M}$ with respect to an ideal $\mathfrak{q}$ containing $I$. Dung-Hoa [4] bounded for $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}))$ in terms of the dimension, $r(\mathbb{M})$ and of the extended degree $D(I, M)$.

The aim of this paper is to give upper bounds for $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}))$ in terms of $r(\mathbb{M})$ and certain lengths associated to an $\mathbb{M}$-superficial sequence for $I$, which are easier be computed than $D(I, M)$. From that we can get upper bounds for $\operatorname{reg}(F_{\mathfrak{m}}(I))$, which is the same as the ones in [4, Corollary 3.4].

## 2. Preliminaries

**Definition 1** ([4, Definition 1.2])**.** Let $\mathbb{M} = \{M_n\}$ be a good $I$-filtration of $M$. We set

$$r = r(\mathbb{M}) = \min\{t \geq 0 | M_{n+1} = IM_n \text{ for all } n \geq t\}.$$

In particular, in the $I$-adic case, $r(\{I^n M\}) = 0$. Note that $r$ is always finite, $M_{r+j} = I^j M_r$ for all $j \geq 0$, and $r$ is the largest generating degree of the graded module $G(\mathbb{M})$ as a graded module over $G_I(A)$.

Let $R = \oplus_{n\geq 0} R_n$ be a Noetherian standard graded ring over a local Artinian ring $(R_0, \mathfrak{m}_0)$. Let $E = \oplus_{n\geq 0} E_n$ be a finitely generated graded $R$-module of dimension $d$. For $0 \leq i \leq d$, put

$$a_i(E) = \sup\{n | H^i_{R_+}(E)_n \neq 0\}$$

where $R_+ = \oplus_{n>0} R_n$. The *Castelnuovo-Mumford regularity* of $E$ is defined by

$$\operatorname{reg}(E) = \max\{a_i(E) + i | 0 \leq i \leq d\}.$$

and the *Castelnuovo-Mumford 1-regularity* of $E$, is defined by

$$\operatorname{reg}^1(E) = \max\{a_i(E) + i | 1 \leq i \leq d\}.$$

We will write $\operatorname{reg}(G(\mathbb{M}))$( resp. $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}))$) to mean the Castelnuovo-Mumford regularity of $G(\mathbb{M})$( resp. $F_{\mathfrak{q}}(\mathbb{M})$) being a graded module over the standard graded ring $G_I(A)$.

**Definition 2.** ([8, Definition 1.1]) An element $x \in I$ is called $\mathbb{M}$-superficial element for $I$ if there exists a non-negative integer $c$ such that $(M_{n+1} :_M x) \cap M_c = M_n$

for all $n \geq c$ and we say that a sequence of elements $x_1, \ldots, x_r$ is an $\mathbb{M}$-superficial sequence for $I$ if, for $i = 1, 2, \ldots, r$, $x_i$ is an $\mathbb{M}/(x_1, \ldots, x_{i-1})M$-superficial sequence for $I$.

An element $x \in R_1$ is called *filter-regular* on $E$ if $(0 :_E x)_n = 0$ for all $n \gg 0$. From this one can show that $(0 :_E x)_n = 0$ for all $n > \operatorname{reg}(E)$.

For $x \in I$, let $x^*$ and $x^0$ denote the initial form in degree one componenet of $G_I(A)$ and $F_{\mathfrak{m}}(I)$, respectively. It is easy to see that $x$ is $\mathbb{M}$-superficial if and only if $(0 :_{G(\mathbb{M})} x^*)_n = 0$ for all $n \gg 0$. Further, if $x_1, \ldots, x_d$ is an $\mathbb{M}$-superficial sequence for $I$, then $(x_1, \ldots, x_d)$ is a minimal reduction of $I$ with respect to $M$.

We denote the Hilbert function $\lambda_{R_0}(E_n)$ and the Hilbert polynomial of $E$ by $h_E(n)$ and $p_E(n)$, respectively. Writing $p_E(n)$ in the form:

$$p_E(n) = \sum_{i=0}^{d-1} (-1)^i e_i(E) \binom{n+d-1-i}{d-1-i},$$

we call the numbers $e_i(E)$ *Hilbert coefficients* of $E$.

We denote by $H_{\mathbb{M}}(n) = \lambda(M/M_{n+1})$ the Hilbert-Samuel function of the filtration $\mathbb{M}$. This function agrees with a polynomial-called the Hilbert-Samuel polynomial and denoted by $P_{\mathbb{M}}(n)$ for $n \gg 0$. We write

$$P_{\mathbb{M}}(n) = \sum_{i=0}^{d} (-1)^i e_i(\mathbb{M}) \binom{n+d-i}{d-i}.$$

The coefficients $e_i(\mathbb{M})$ are integers and they are called the *Hilbert coefficients* of $\mathbb{M}$. In particular, $e_0(\mathbb{M})$ is the multiplicity of $\mathbb{M}$ (see [8, Section 1.3]). Note that $e_i(\mathbb{M}) = e_i(G(\mathbb{M}))$ for $0 \leq i \leq d-1$. When $\mathbb{M} = \{I^n M\}$, we denote by $H_{I,M}(n) = \lambda(M/I^{n+1}M)$ Hilbert-Samuel function of $M$ with respect to $I$, $e_i(I, M)$ are called the *Hilbert coefficients* of $M$ with respect to $I$, and we set $e(I, M) = e_0(I, M)$.

We call $H_{F_{\mathfrak{q}}(\mathbb{M})}(n) = \lambda(M_n/\mathfrak{q}M_n)$ the Hilbert function of $F_{\mathfrak{q}}(\mathbb{M})$, its corresponding polynomial is

$$p_{F_{\mathfrak{q}}(\mathbb{M})}(n) = \sum_{i=0}^{d-1} (-1)^i e_i(F_{\mathfrak{q}}(\mathbb{M})) \binom{n+d-i-1}{d-i-1}.$$

The coefficients $e_i(F_{\mathfrak{q}}(\mathbb{M}))$ are integers and they called the *Hilbert coefficients* of $F_{\mathfrak{q}}(\mathbb{M})$(see [8, Section 5.2]).

We call $H_{\mathfrak{q}\mathbb{M}}(n) = \lambda(M/\mathfrak{q}M_n)$ the Hilbert-Samuel function of the filtration $\mathfrak{q}\mathbb{M}$, its corresponding polynomial is

$$p^1_{\mathfrak{q}\mathbb{M}}(n) = \sum_{i=0}^{d} (-1)^i e_i(\mathfrak{q}\mathbb{M}) \binom{n+d-i}{d-i}.$$

The coefficients $e_i(\mathfrak{q}\mathbb{M})$ are integers and they are called the *Hilbert coefficients* of $\mathfrak{q}\mathbb{M}$. Since $\lambda(M/M_n)+\lambda(M_n/\mathfrak{q}M_n)=\lambda(M/\mathfrak{q}M_n)$, then $e_0(\mathbb{M})=e_0(\mathfrak{q}\mathbb{M})$ and $e_i(F_{\mathfrak{q}}(\mathbb{M}))=e_i(\mathbb{M})+e_{i+1}(\mathbb{M})-e_{i+1}(\mathfrak{q}\mathbb{M})$ for all $0\le i\le d-1$ (see [8, Section 5.2]).

## 3. Bounds for the Castelnuovo-Mumford regularity of $F_{\mathfrak{q}}(\mathbb{M})$

Throughout this paper, we assume that $(A,\mathfrak{m})$ is a commutative Noetherian local ring with an infinite residue field $K=A/\mathfrak{m}$, $I$ is an $\mathfrak{m}$-primary and $\mathbb{M}=\{M_n\}$ is a good $I$-filtration of a finitely generated module $M$ of dimension $d$, the ideal $\mathfrak{q}$ containing $I$.

**Definition 3** ([6, Definition])**.** An *extend degree* $D(I,M)$ of $M$ with respect to $I$ is a numerical function satisfying the following properties:

(i) $D(I,M)=D(I,M/L)+\lambda(L)$, where $L=H^0_{\mathfrak{m}}(M)$,
(ii) $D(I,M)\ge D(I,M/xM)$ for a generic element $x\in I\setminus\mathfrak{m}I$ on $M$,
(iii) $D(I,M)=e(I,M)$ if $M$ is a Cohen-Macaulay $A$-module.

Any extended degree $D(I,M)$ will satisfy $D(I,M)\ge e(I,M)$, with equality holding if and only if $M$ is a Cohen-Macaulay module.

In this section, we always assume that $x_1,\dots,x_d$ is an $\mathbb{M}$-superficial sequence for $I$. Set

$$B(\mathbf{x},M)=\lambda(M/(x_1,\dots,x_d)M)$$

and

$$\kappa(\mathbf{x},M)=\max\{h^0(M/(x_1,\dots,x_i)M)\mid 0\le i\le d-1\},$$

where $\mathbf{x}$ denotes the superficial sequence $x_1,\dots,x_d$. For a finitely generated module $N$, $h^0(N)=\lambda(H^0_{\mathfrak{m}}(N))=\lambda(\cup_{n\ge 0}(0_N:\mathfrak{m}^n))$.

The next result is proved in [3, Theorem 1.2], where the authors gave bounds for $\operatorname{reg}(G(\mathbb{M}))$ in terms of $r(\mathbb{M})$, $B(\mathbf{x},M)$ and $\kappa(\mathbf{x},M)$.

**Theorem 1** ([3, Theorem 1.2])**.**

(i) $\operatorname{reg}(G(\mathbb{M}))\le B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M})-1$ *if* $d=1$*;*
(ii) $\operatorname{reg}(G(\mathbb{M}))\le (B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M}))+1)^{3(d-1)!-1}-d$ *if* $d\ge 2$.

The following two lemmas will be crucial in the proof of the main result of this paper.

**Lemma 1.** *We have*

(i) $e_0(\mathbb{M})=e(I,M)\le B(\mathbf{x},M)$*;*
(ii) $|e_1(\mathbb{M})|\le B(\mathbf{x},M)(B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M})-1)$*;*
(iii) $|e_i(\mathbb{M})|\le (B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M})+1)^{3i!-i+1}$ *if* $i\ge 2$.

*Proof.* (i) By [1, Proposition 11.4(iii)], $e_0(\mathbb{M}) = e(I, M)$. Let $J = (x_1, \dots, x_d)$ and $J' = (x_1, \dots, x_{d-1})$. By [8, Proposition 2.1], we have

$$\begin{aligned} e_0(\mathbb{M}) &= e_0(\mathbb{M}/J'M) = e_0(\mathbb{M}/JM) - \lambda(J'M : x_d/J'M) \\ &= \lambda(M/JM) - \lambda(J'M : x_d/J'M). \end{aligned}$$

Then (i) is clear.

(ii)-(iii) From the Grothendieck-Serre formula

$$h_{G(\mathbb{M})}(n) - p_{G(\mathbb{M})}(n) = \sum_{j=0}^{d} (-1)^j H^j_{G_+}(G(\mathbb{M}))_n,$$

it follows that $h_{G(\mathbb{M})}(n) = p_{G(\mathbb{M})}(n)$ for all $n > \operatorname{reg}(G(\mathbb{M}))$. Then

$$\lambda(M/M_{m+1}) = \sum_{i=0}^{d} (-1)^i e_i(\mathbb{M}) \binom{m+d-i}{d-i} \tag{3.1}$$

for all $m \geq \operatorname{reg}(G(\mathbb{M}))$. For short, we set $r = r(\mathbb{M})$, $B = B(\mathbf{x}, M)$, $\kappa = \kappa(\mathbf{x}, M)$ and $e_i = e_i(\mathbb{M})$.

If $d = 1$, by Theorem 1 (i), we can put $m = B + \kappa + r - 1$ into the equality (3.1), it leads to

$$e_1 = (B + \kappa + r)e_0 - \lambda(M/M_{B+\kappa+r})$$

We know that $M_n = I^{n-r} M_r$ for $n \geq r$ and $M_r \neq 0$,

$$\begin{aligned} &\lambda(M/M_{B+\kappa+r}) \\ &\geq \lambda(M_r/IM_r) + \lambda(IM_r/I^2M_r) + \cdots + \lambda(I^{B+\kappa-1}M_r/I^{B+\kappa}M_r) \geq B + \kappa, \end{aligned}$$

so

$$e_1 \leq (B + \kappa + r)e_0 - (B + \kappa) \leq (B + \kappa + r)B - (B + \kappa) \leq B(B + \kappa + r - 1).$$

On the other hand, since $r \geq 0$,

$$\begin{aligned} -e_1 &= -(B + \kappa + r)e_0 + \lambda(M/M_{B+\kappa+r}) \leq -(B + \kappa + r) + \lambda(M/I^{B+\kappa+r}M) \\ &\leq -(B + \kappa + r) + B(B + \kappa + r) \leq B(B + \kappa + r - 1), \end{aligned}$$

where the second inequality is due to [4, Lemma 1.7(i)]. Hence $|e_1| \leq B(B + \kappa + r - 1)$ and the case $d = 1$ is settled.

Assume that $d \geq 2$. Next, we denote $M/H^0_{\mathfrak{m}}(M)$ by $\overline{M}$ and the filtration $\mathbb{M}/H^0_{\mathfrak{m}}(M)$ of $\overline{M}$ by $\overline{\mathbb{M}}$. Let $N = \overline{M}/x_1\overline{M}$, $\mathbb{N} = \overline{\mathbb{M}}/x_1\overline{M}$ and $\overline{\mathbf{x}}$ denote the superficial sequence $x_2, \dots, x_d$. Then $e_i(\mathbb{M}) = e_i(\overline{\mathbb{M}}) = e_i(\mathbb{N})$ for all $i < d$.

Note that $0 \leq r(\overline{\mathbb{M}}) \leq r$, $\kappa(\overline{\mathbf{x}}, N) \leq \kappa$ and $B(\overline{\mathbf{x}}, N) \leq B$. By the inductive hypothesis, we get

$$|e_1| \leq B(B + \kappa + r - 1) \quad \text{and} \quad |e_i| \leq (B + \kappa + r + 1)^{3i!-i+1} \quad \text{for } 2 \leq i \leq d-1. \tag{3.2}$$

It remains to show the inequality for $e_d$. Set $\mu = (B+\kappa+r+1)^{3(d-1)!-1}$. By Theorem 1, we get $\operatorname{reg}(G(\mathbb{M})) \le \mu - d$. Since $\operatorname{reg}(G(\mathbb{M})) \ge r \ge 0$, $\mu \ge d$, we can take $m = \mu - d$ in the equality (3.1), this implies that

$$\begin{aligned}|e_d| &= |\lambda(M/M_{\mu-d+1}) - e_0\binom{\mu-d+d}{d} - \sum_{i=1}^{d-1}(-1)^i e_i\binom{\mu-d+d-i}{d-i}| \\ &\le \max\{\lambda(M/M_{\mu-d+1}), e_0\binom{\mu}{d}\} + \sum_{i=1}^{d-1}|e_i|\binom{\mu-i}{d-i}\end{aligned}$$

Since $\binom{\mu}{d} \le \mu^d$, $\max\{\lambda(M/M_{\mu-d+1}), e_0\binom{\mu}{d}\} \le B\mu^d$ by [4, Lemma 1.7(i)] and (i). From (3.2) we see that

$$|e_1|\binom{\mu-1}{d-1} \le B(B+\kappa+r-1)\mu^{d-1},$$

and

$$\sum_{i=2}^{d-1}|e_i|\binom{\mu-i}{d-i} \le \sum_{i=2}^{d-1}(B+\kappa+r+1)^{3i!-i+1}\mu^{d-i} \le \mu^{d-1}\sum_{i=0}^{d-2}\frac{1}{2^i} < 2\mu^{d-1}.$$

Since $B(B+\kappa+r-1)+2 < (B+\kappa+r+1)^2 \le \mu$, we finally obtain

$$|e_d| \le B\mu^d + \mu^d = (B+1)(B+\kappa+r+1)^{3d!-d} \le (B+\kappa+r+1)^{3d!-d+1}.$$

□

**Lemma 2.** *We have*

$$a_0(F_{\mathfrak{q}}(\mathbb{M})) \le \max\{\operatorname{reg}(G(\mathfrak{q}\mathbb{M})), r(\mathbb{M})\}.$$

*Proof.* Set $r = r(\mathbb{M})$. First, we have the exact sequence of $G_I(A)$-modules

$$0 \longrightarrow \oplus_{n>r}F_{\mathfrak{q}}(M)_n \longrightarrow \oplus_{n>r}G(\mathfrak{q}M)_n \longrightarrow \oplus_{n>r}\mathfrak{q}M_n/IM_n \longrightarrow 0.$$

Thus $a_0(\oplus_{n>r}F_{\mathfrak{q}}(M)_n) \le a_0(\oplus_{n>r}G(\mathfrak{q}M)_n)$. On the other hand, from the exact sequence $0 \to \oplus_{n>r}F_{\mathfrak{q}}(M)_n \to F_{\mathfrak{q}}(M) \to \oplus_{n\le r}F_{\mathfrak{q}}(M)_n \to 0$, we get

$$a_0(F_{\mathfrak{q}}(M)) \le \max\{a_0(\oplus_{n>r}F_{\mathfrak{q}}(M)_n), r\}.$$

From the injective map $\oplus_{n>r}G(\mathfrak{q}M)_n \to G(\mathfrak{q}M)$, we get

$$a_0(\oplus_{n>r}G(\mathfrak{q}M)_n) \le a_0(G(\mathfrak{q}M)) \le \operatorname{reg}(G(\mathfrak{q}\mathbb{M})).$$

Therefore, we can conclude that $a_0(F_{\mathfrak{q}}(\mathbb{M})) \le \max\{\operatorname{reg}(G(\mathfrak{q}\mathbb{M})), r\}$.

□

Next, we give the main result of this paper. The techniques we use are similar to that in [4, Theorem 3.3], where we replace $D(I,M)$ by $B(\mathbf{x},M)$ and $\kappa(\mathbf{x},M)$.

**Theorem 2.** *We have*

(i) $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) \leq 2B(\mathbf{x},M)(B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M}))+r(\mathbb{M})-1$ *if* $d=1$*;*
(ii) $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) \leq (B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M})+2)^2+B(\mathbf{x},M)^2-3$ *if* $d=2$*;*
(iii) $\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) \leq (B(\mathbf{x},M)+\kappa(\mathbf{x},M)+r(\mathbb{M})+2)^{3(d-1)!-1}-d$ *if* $d \geq 3$.

*Proof.* We proceed by induction on $d$. Set $B=B(\mathbf{x},M), \kappa=\kappa(\mathbf{x},M)$ and $r=r(\mathbb{M})$.

Let $d=1$. Then $a_1(F_{\mathfrak{q}}(\mathbb{M}))+1 \leq e_0(F_{\mathfrak{q}}(\mathbb{M}))+r-1$ by [6, Lemma 2.2]. It is proved in [8, Section 5.2] that

$$e_i(F_{\mathfrak{q}}(\mathbb{M}))=e_i(\mathbb{M})+e_{i+1}(\mathbb{M})-e_{i+1}(\mathfrak{q}\mathbb{M})$$

for all $0 \leq i \leq d-1$. In virtue of $r(\mathfrak{q}\mathbb{M}) \leq r+1$, we get

$$\begin{aligned} e_0(F_{\mathfrak{q}}(\mathbb{M})) &\leq |e_0(\mathbb{M})|+|e_1(\mathbb{M})|+|e_1(\mathfrak{q}\mathbb{M})| \\ &\leq B+B(B+\kappa+r-1)+B(B+\kappa+r)=2B(B+\kappa+r) \end{aligned}$$

by Lemma 1, thus $a_1(F_{\mathfrak{q}}(\mathbb{M}))+1 \leq 2B(B+\kappa+r)+r-1$. By Theorem 1 and Lemma 2, we get $a_0(F_{\mathfrak{q}}(\mathbb{M})) \leq B+\kappa+r$. Hence

$$\begin{aligned} \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) &\leq \max\{B+\kappa+r, 2B(B+\kappa+r)+r-1\} \\ &= 2B(B+\kappa+r)+r-1. \end{aligned}$$

Now let $d \geq 2$. By [5, Proposition 2.2], we can choose $x_1$ such that $x_1^* \in I/I^2$ is a filter-regular element on $G(\mathbb{M})$ and $x_1^0 \in I/\mathfrak{m}I$ is a filter-regular element on $F_{\mathfrak{q}}(\mathbb{M})$. Then

$$F_{\mathfrak{q}}(\mathbb{M})/x_1^0F_{\mathfrak{q}}(\mathbb{M}) \cong \frac{M}{\mathfrak{q}M} \oplus (\oplus_{n\geq 1}\frac{M_n}{\mathfrak{q}M_n+x_1M_{n-1}}),$$

and

$$F_{\mathfrak{q}}(\mathbb{M}/x_1M)=\oplus_{n\geq 0}\frac{M_n}{\mathfrak{q}M_n+x_1M\cap M_n}.$$

Let

$$K=\oplus_{n\geq 1}\frac{\mathfrak{q}M_n+x_1M\cap M_n}{\mathfrak{q}M_n+x_1M_{n-1}}.$$

Then we have an exact sequence of $G_I(A)$-module:

$$0 \longrightarrow K \longrightarrow F_{\mathfrak{q}}(\mathbb{M})/x_1^0F_{\mathfrak{q}}(\mathbb{M}) \longrightarrow F_{\mathfrak{q}}(\mathbb{M}/x_1M) \longrightarrow 0. \tag{3.3}$$

By [4, Lemma 1.3(ii)], we get $x_1M\cap M_n=x_1M_{n-1}$ for all $n>\operatorname{reg}(G(\mathbb{M}))$. Then $K$ has finite length and $\operatorname{reg}(K) \leq \operatorname{reg}(G(\mathbb{M}))$. The exact sequence (3.3) implies

$$\begin{aligned} \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})/x_1^0F_{\mathfrak{q}}(\mathbb{M})) &= \max\{\operatorname{reg}(K), \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}/x_1M))\} \\ &\leq \max\{\operatorname{reg}(G(\mathbb{M})), \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}/x_1M))\}. \end{aligned}$$

By [9, Proposition 3.2], we get

$$\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}))=\max\{a_0(F_{\mathfrak{q}}(\mathbb{M})), \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})/x_1^0F_{\mathfrak{q}}(\mathbb{M}))\},$$

so that

$$\operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) \leq \max\{a_0(F_{\mathfrak{q}}(\mathbb{M})), \operatorname{reg}(G(\mathbb{M})), \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M}/x_1M))\}.$$

Note that $r(\mathbb{M}/x_1M) \leq r$, $B(\overline{\mathbf{x}}, M/x_1M) \leq B$ and $\kappa(\overline{\mathbf{x}}, M/x_1M) \leq \kappa$, where $\overline{\mathbf{x}}$ denotes the superficial sequence $x_2, \ldots, x_d$.

If $d = 2$, by the inductive hypothesis, we have

$$\begin{aligned} \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) &\leq \max\{(B+\kappa+r+2)^2-2, (B+\kappa+r+1)^2-2, \\ &\quad 2B(B+\kappa+r)+r-1\} \\ &< (B+\kappa+r+2)^2+B^2-3 \end{aligned}$$

from Theorem 1.

If $d \geq 3$,

$$\begin{aligned} \operatorname{reg}(F_{\mathfrak{q}}(\mathbb{M})) &\leq \max\{(B+\kappa+r+2)^{3(d-1)!-1}-d, (B+\kappa+r+1)^{3(d-1)!-1}-d, \\ &\quad (B+\kappa+r+2)^{3(d-2)!-1}-d+1\} \\ &= (B+\kappa+r+2)^{3(d-1)!-1}-d. \end{aligned}$$

□

The following result can be also deduced from [4, Corollary 3.4].

**Corollary 1.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and $I$ a $\mathfrak{m}$-primary ideal of $A$. Then*

(i) $\operatorname{reg}(F_{\mathfrak{m}}(I)) \leq 2e(I,A)^2-1$ *if* $d = 1$;
(ii) $\operatorname{reg}(F_{\mathfrak{m}}(I)) \leq 2e(I,A)^2+4e(I,A)+1$ *if* $d = 2$;
(iii) $\operatorname{reg}(F_{\mathfrak{m}}(I)) \leq (e(I,A)+2)^{3(d-1)!-1}-d$ *if* $d \geq 3$.

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# ON RICCI SYMMETRIC GENERALIZED QUASI EINSTEIN SPACETIMES

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*Abstract.* The object of the present paper is to prove the existence of a generalized quasi Einstein spacetime, briefly $G(QE)_4$, by constructing a non-trivial Lorentzian metric and to study such spacetime. First, we prove that every $W_2$-Ricci pseudosymmetric $G(QE)_4$ is an $N(k)$-quasi Einstein spacetime which can be considered as a model of perfect fluid, in general relativity. Then, we consider Ricci symmetric $G(QE)_4$ and we prove that in such spacetime satisfying Einstein's field equations, the energy density and the isotropic pressure are constants. As a consequence of this result, the expansion scalar and the acceleration vector vanish and also the possible local cosmological structures of this spacetime obeying Einstein's field equations are of Petrov I, D or O.

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## 1. Introduction

The notion of quasi-Einstein manifold was introduced by Chaki and Maity [3]. A non-flat $n$-dimensional Riemannian manifold $(M_n, g)$, $(n \geqslant 3)$ is said to be quasi-Einstein manifold if its Ricci tensor of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

where for all vector fields $X$,

$$g(X,U) = A(X), \quad g(U,U) = 1 \tag{1.2}$$

i.e., $A$ is a non-zero associated 1-form, metrically equivalent to the unit vector field $U$ and $a, b$ are associated non-zero scalar functions. The vector field $U$ is usually called the generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. Obviously, if $b = 0$, then this manifold reduces to an Einstein manifold. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g. Schwarzschild spacetime) is quasi-Einstein.

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Quasi Einstein manifolds arose during the study of exact solutions of Einstein's field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi Einstein manifolds. Also, quasi Einstein manifolds can be taken as a model of perfect fluid spacetime in general relativity, [6]. Thus quasi Einstein manifolds are very important in general relativity and cosmology. These manifolds have been studied by several authors, such as M.C. Chaki [3], U.C. De and G. C. Ghosh [6] and S. Guha [8]. Also, A.A. Shaikh, Y.H. Kim and S.K. Hui [17] have been studied on Lorentzian quasi Einstein spacetimes. Many authors have been generalized the notion of quasi Einstein manifolds, in different ways such as nearly quasi Einstein manifolds [5], generalized quasi Einstein manifolds [2], $N(k)$-quasi Einstein manifolds [4, 20] and so on.

In 2001, Chaki [2] introduced the notion of generalized quasi Einstein manifold. An n-dimensional ($n \geqslant 3$) Riemannian manifold $(M_n, g)$ is said to be generalized quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \tag{1.3}$$

where $a$, $b$, $c$ are real valued, non-zero scalar functions of which $b \neq 0$, $c \neq 0$, $A$ and $B$ are non-zero 1-forms such that

$$g(X,U) = A(X) \ , \ g(X,V) = B(X) \ , \ g(U,V) = 0 \tag{1.4}$$

for all $X, Y \in \chi(M)$. Also, $U$ and $V$ are two unit vector fields perpendicular to each other. Similarly, $a$, $b$ and $c$ are called associated scalars, $A$ and $B$ are called associated 1-forms and $U$ and $V$ are generators of the manifold. Such an $n$-dimensional manifold has been denoted by $G(QE)_n$. If $c = 0$, then (1.3) takes the form (1.1) and so the manifold reduces to a $(QE)_n$. Thus a quasi Einstein manifold is a special case of a generalized quasi Einstein manifold. If $b = c = 0$, then the manifold becomes an Einstein manifold. Hence, in a $G(QE)_n$, we may assume that the associated scalar functions $a$ and $b$ are nowhere zero, but $c$ can be any smooth function on the manifold, even constant zero.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal frame field at any point of the manifold. Then, setting $X = Y = e_i$ in (1.3) and taking summation over $i$; $(1 \leq i \leq n)$, we obtain

$$r = an + b \tag{1.5}$$

where $r$ is the curvature function of the manifold.

Let $R$ denote the Riemannian curvature tensor of $M$. The k-nullity distribution $N(k)$ [19] of a Riemannian manifold $M$ is defined by

$$N(k) : p \to N_p(k) = \\ \Big\{ Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]; \ \forall X, Y \in T_p(M) \Big\} \tag{1.6}$$

where $k$ is some smooth function. In a quasi Einstein manifold $M$, if the generator $U$ belongs to some k-nullity distribution $N(k)$, then $M$ said to be $N(k)$-quasi Einstein manifold [20]. Özgür and Triphati [13] proved that in an $n$-dimensional $N(k)$-quasi Einstein manifold, $k = \frac{a+b}{n-1}$.

A spacetime is a time oriented, 4-dimensional manifold $(M^4, g)$ with Lorentz metric whose signature $(+,+,+,-)$. A Lorentzian 4-dimensional manifold is said to be generalized quasi Einstein spacetime with generator $U$ as the unit timelike vector field if its Ricci tensor of type $(0,2)$ is not identically zero and satisfies the equation (1.3), where again $a$, $b$, $c$ are real valued, non-zero scalar functions of which $b \neq 0$, $c \neq 0$, $A$ and $B$ are non-zero 1-forms such that $V$ being the heat flux vector field orthogonal to the velocity vector field $U$. That is, for all $X$, we have

$$g(X,U) = A(X),\ g(X,V) = B(X),\ g(U,U) = -1,\ g(V,V) = 1,\ g(U,V) = 0 \tag{1.7}$$

The energy momentum tensor for a perfect fluid spacetime [12] is given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) \tag{1.8}$$

together with $g(X,U) = A(X)$, $A(U) = -1$,for all $X,Y$. Here, $\sigma$ and $p$ are respectively the energy density and isotropic pressure and $U$ is the unit timelike velocity vector field. For a fluid matter distribution, the energy momentum tensor is given by Ellis [7] and Landau and Lifschitz [11] as

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) + A(Y)B(X) \tag{1.9}$$

for all $X,Y \in TM$ where $g(X,U) = A(X)$, $A(U) = -1$, $g(X,V) = B(V)$, $B(V) > 0$, $g(U,V) = 0$, $\sigma$ and $p$ are respectively the energy density and isotropic pressure, $U$ is the velocity vector field orthogonal to the heat conduction vector field $V$.

Ricci tensor controls the geometry of spacetime whereas energy momentum tensor $T$ signifies the physical aspects of spacetime and in general relativity they are related by Einstein's field equations [12]

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y)$$

for all vector fields $X,Y$ where $S$ is the Ricci tensor of type $(0,2)$, $r$ is the scalar curvature, $\lambda$ is the cosmological constant and $k$ is the gravitational constant. Einstein's field equation implies that the energy momentum tensor is a symmetric $(0,2)$-tensor with divergence zero [12].

This paper is organized as follows:

After introduction, in Section 2, we prove the existence of a generalized quasi Einstein spacetime, briefly $G(QE)_4$, by constructing a non-trivial Lorentzian metric. In Section 3, we prove that every $W_2$-Ricci pseudosymmetric $G(QE)_4$ is an $N(\frac{a-b}{3})$-quasi Einstein spacetime which can be considered as a model of perfect fluid, in general relativity. In Section 4, a Ricci symmetric $G(QE)_4$ has been considered and a detailed study on this spacetime has been made. We prove that in such spacetime

satisfying Einstein's field equations, the energy density and the isotropic pressure are constants. Moreover, the expansion scalar and the acceleration vector vanish, also the possible local cosmological structures of this spacetime obeying the Einstein's field equation are of Petrov I, D or O.

## 2. Existence of a generalized quasi Einstein spacetime

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was still not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible and the matter content of the universe may be assumed to be perfect fluid. The study of $(QE)_4$ is important because such spacetime represents the third phase in the evolution of the universe. In addition to this, the importance of $G(QE)_4$ lies in the fact that such a 4-dimensional semi-Riemannian manifold is related to the study of general relativistic fluid spacetime admitting heat flux represents the intermediate phase in the evolution of the universe, [16]. Thus, in this section, we prove the existence of a generalized quasi Einstein spacetime with non-zero and non-constant scalar curvature by constructing a new Lorentzian metric.

We consider a Lorentzian metric $g$ on the 4-dimensional real number space $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^2(dx^1)^2 + (e^{2x^4})[(dx^2)^2 + (dx^3)^2] - (dx^4)^2 \tag{2.1}$$

where $\frac{1}{2} < x^4 < 1$ and $x^1, x^2, x^3, x^4$ are the standard coordinates of $\mathbb{R}^4$. Then the only non vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are obtained as follows:

$$\Gamma^1_{14} = \frac{1}{(x^4)}, \;\; \Gamma^4_{11} = (x^4), \;\; \Gamma^2_{24} = \Gamma^3_{34} = 1, \;\; \Gamma^4_{22} = \Gamma^4_{33} = e^{2(x^4)} \tag{2.2}$$

$$R_{1221} = R_{1331} = -e^{2(x^4)}(x^4), \tag{2.3}$$

$$R_{2332} = -e^{4(x^4)}, \;\; R_{2442} = R_{3443} = e^{2(x^4)} \tag{2.4}$$

$$R_{11} = -2(x^4), \;\; R_{22} = R_{33} = -e^{2(x^4)}\frac{[1+2(x^4)]}{(x^4)}, \;\; R_{44} = 2 \tag{2.5}$$

and the components which can be obtained from these by symmetry properties. Also it can be shown that the scalar curvature is given by

$$r = -2[\frac{2}{(x^4)} + 3] \tag{2.6}$$

which is non-zero and non-constant.

In terms of local coordinate system, let us define associated scalar functions as

$$a = -2 - \frac{1}{(x^4)}, \quad b = -2, \quad c = \frac{\sqrt{2(x^4)-1}}{(x^4)} \tag{2.7}$$

and the 1-forms

$$A_i(x) = \begin{cases} (x^4)\sqrt{\frac{2(x^4)-1}{2(1-(x^4))}} & \text{if } i = 1 \\ 0 & \text{if } i = 2,3 \\ \frac{-1}{\sqrt{2(1-(x^4))}} & \text{if } i = 4 \end{cases} \tag{2.8}$$

and

$$B_i(x) = \begin{cases} \frac{(x^4)}{\sqrt{2(1-(x^4))}} & \text{if } i = 1 \\ 0 & \text{if } i = 2,3 \\ -\sqrt{\frac{2(x^4)-1}{2(1-(x^4))}} & \text{if } i = 4 \end{cases} \tag{2.9}$$

Then, we can show that

(1) $R_{11} = ag_{11} + bA_1A_1 + 2cA_1B_1$
(2) $R_{22} = ag_{22} + bA_2A_2 + 2cA_2B_2$
(3) $R_{33} = ag_{33} + bA_3A_3 + 2cA_3B_3$
(4) $R_{44} = ag_{44} + bA_4A_4 + 2cA_4B_4$

Since all the cases other than (1)-(4) are trivial, we obtain

$$R_{ij} = ag_{ij} + bA_iA_j + c(A_iB_j + A_jB_i)\,, \quad for \;\; i,j = 1,2,3,4 \tag{2.10}$$

Moreover, we find

$$g^{ij}A_iA_j = -1\,, \;\; g^{ij}B_iB_j = 1\,, \; and \; g^{ij}A_iB_j = 0 \tag{2.11}$$

and so

$$r = 4a - b = -2[\frac{2}{(x^4)} + 3] \tag{2.12}$$

Therefore, we can say that the manifold under consideration is a generalized quasi Einstein spacetime with non-zero and non-constant scalar curvature. Hence we can state that:

**Theorem 1.** *Let $M^4 = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 : \frac{1}{2} < x^4 < 1\}$ be an open subset of $\mathbb{R}^4$ endowed with the Lorentzian metric given by*

$$ds^2 = g_{ij}dx^idx^j = (x^4)^2(dx^1)^2 + (e^{2x^4})[(dx^2)^2 + (dx^3)^2] - (dx^4)^2$$

*where $x^1,x^2,x^3,x^4$ are the standard coordinates of $\mathbb{R}^4$. Then $(M^4,g)$ is a generalized quasi Einstein spacetime with non-zero and non-constant scalar curvature $r = -2[\frac{2}{(x^4)} + 3]$.*

## 3. Some properties of generalized quasi Einstein spacetimes

In 1970, G. P. Pokhariyal and R. S. Mishra [15] were introduced the new tensor called $W_2$-curvature tensor and studied its properties. According to them, an $W_2$-curvature tensor on a manifold $(M^n, g)$, $(n > 3)$ is defined by

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY] \tag{3.1}$$

where $R$ is the curvature tensor and $Q$ denotes the Ricci operator defined by $g(QX,Y) = S(X,Y)$, for all $X,Y$. Note that for all $X,Y,Z,W \in TM$, the $W_2$ curvature tensor satisfies the following symmetry properties:

- $W_2(X,Y,Z,W) = -W_2(Y,X,Z,W)$
- $W_2(X,Y,Z,W) \neq -W_2(X,Y,W,Z)$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leqslant i \leqslant n$. Now, from (3.1), we have

$$\sum_{i=1}^{n} W_2(Y,e_i,e_i,U) = 0 \tag{3.2}$$

In this section, a comprehensive study of $W_2$-curvature tensor has been made on $G(QE)_4$. Firstly, we consider a generalized quasi Einstein spacetime satisfying the condition $W_2 \cdot S = L_S Q(g,S)$ where $L_S$ is a certain function on the set $U_S = \{x \in M : S \neq \frac{r}{n} g \ at \ x\}$ and $Q(g,S)$ is the Tachibana tensor of the metric tensor and the Ricci tensor. Recall that this spacetime is said to be *$W_2$-Ricci pseudosymmetric* $G(QE)_4$. Then for all $X,Y,Z \in \mathfrak{X}(M^4)$;

$$\begin{aligned} &S(W_2(X,Y)Z,W) + S(Z,W_2(X,Y)W) \\ &= L_S\Big[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)\Big] \end{aligned} \tag{3.3}$$

Combining (1.3) and (3.3), we get

$$\begin{aligned} &-(\frac{a}{3} + L_S)[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(X,Z) \\ &- g(X,W)S(Y,Z)] + b[A(W_2(X,Y)Z)A(W) + A(Z)A(W_2(X,Y)W)] \\ &+ c[A(W_2(X,Y)Z)B(W) + A(W)B(W_2(X,Y)Z) + A(Z)B(W_2(X,Y)W) \\ &+ A(W_2(X,Y)W)B(Z)] = 0 \end{aligned} \tag{3.4}$$

Putting $Z = U$ and $W = V$ in (3.4), we get

$$\begin{aligned} &-(\frac{a}{3} + L_S)[A(Y)S(X,V) - A(X)S(Y,V) + B(Y)S(X,U) - B(X)S(Y,U)] \\ &- bW_2(X,Y,V,U) + c[W_2(X,Y,U,U) - W_2(X,Y,V,V)] = 0 \end{aligned} \tag{3.5}$$

In a generalized quasi Einstein spacetime, from (1.7) we have

$$S(X,U) = (a-b)A(X) - cB(X), \quad S(X,V) = aB(X) + cA(X) \tag{3.6}$$

In view of (3.6) and (3.1), (3.5) yields

$$bR(X,Y,U,V) = b(\frac{2a-b}{3} + L_S)[A(Y)B(X) - A(X)B(Y)] \tag{3.7}$$

Since $b \neq 0$, we obtain

$$R(X,Y,U,V) = (\frac{2a-b}{3} + L_S)[A(Y)B(X) - A(X)B(Y)] \tag{3.8}$$

Contracting (3.4) over $X$ and $W$, we get

$$-(\frac{a}{3} + L_S)[rg(Y,Z) - 4S(Y,Z)] + bW_2(U,Y,Z,U) \\ + c[W_2(V,Y,Z,U) + W_2(U,Y,Z,V)] = 0 \tag{3.9}$$

Putting $Z = U$ in (3.9), we get

$$-(\frac{a}{3} + L_S)[rA(Y) - 4S(Y,U)] + bW_2(U,Y,U,U) \\ + c[W_2(V,Y,U,U) - W_2(U,Y,U,V)] = 0 \tag{3.10}$$

In view of (3.6) and (3.1), (3.10) yields

$$\Big[-3b(\frac{a}{3} + L_S) + \frac{c^2}{3}\Big]A(Y) \\ + \Big[-4c(\frac{a}{3} + L_S) + \frac{bc}{3} + c(\frac{2a-b}{3} + L_S) - \frac{ac}{3}\Big]B(Y) = 0 \tag{3.11}$$

Putting $Y = U$ in (3.11), we get:

$$L_S = \frac{-3ab + c^2}{9b} \tag{3.12}$$

Putting $Y = V$ in (3.11), we get:

$$c(a + 3L_S) = 0 \tag{3.13}$$

Then, $c = 0$ or $L_S = \frac{-a}{3}$. If $c = 0$, then by (3.12), as $b \neq 0$, we get $L_S = \frac{-a}{3}$. Otherwise, if $b = 0$, then the spacetimes reduces to an Einstein manifold. On the other hand, if $c \neq 0$, then $L_S = \frac{-a}{3}$. Then by (3.12), again $c$ must be zero.

Thus, in each case $c = 0$ and $L_S = \frac{-a}{3}$. That is, the spacetime reduces to a quasi Einstein spacetime and also by (3.8),

$$R(X,Y)U = (\frac{a-b}{3})[A(Y)X - A(X)Y] \tag{3.14}$$

which means that the generator vector field $U$ belongs to $(\frac{a-b}{3})$-nullity distribution. Therefore, this manifold becomes an $N(\frac{a-b}{3})$-quasi Einstein spacetime. Hence, we can state the following theorem:

**Theorem 2.** *Every $W_2$-Ricci pseudosymmetric generalized quasi Einstein spacetime (i.e. satisfying the condition $W_2 \cdot S = L_S Q(g,S)$ for some certain function $L_S$) is an $N(\frac{a-b}{3})$-quasi Einstein spacetime where $L_S = \frac{-a}{3}$.*

Similarly, if we consider $W_2$-Ricci semisymmetric generalized quasi Einstein spacetime (i.e. satisfying the condition $W_2 \cdot S = 0$), from above we obtain $c = 0$ and $L_S = \frac{-a}{3} = 0$. Hence, in this case the Ricci tensor can be written as

$$S(X,Y) = bA(X)A(Y) \tag{3.15}$$

Also, contracting (3.15) over $X$ and $Y$, we obtain $r = -b$. Thus, we get

$$S(X,Y) = -rA(X)A(Y) \tag{3.16}$$

where $r$ is the scalar curvature of the spacetime. Such a spacetime is said to be *special quasi Einstein spacetime*.

**Theorem 3.** *Every $W_2$-Ricci semisymmetric generalized quasi Einstein spacetime (i.e. satisfying the condition $W_2 \cdot S = 0$) is a special quasi Einstein spacetime.*

Combining (3.16) and Einstein's field equations [12]

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{3.17}$$

we obtain the energy-momentum tensor of this spacetime as follows:

$$T(X,Y) = \frac{2\lambda - r}{2k}g(X,Y) - \frac{r}{k}A(X)A(Y) \tag{3.18}$$

which means that this spacetime can be considered as model of perfect fluid spacetime. Hence we can state that the following:

**Corollary 1.** *Every generalized quasi Einstein spacetime satisfying the condition $W_2 \cdot S = 0$ can be considered as model of perfect fluid spacetime, in general relativity.*

Now, we consider a fluid spacetime admitting heat flux with the energy momentum tensor of the form (1.9) and satisfying Einstein's field equations with cosmological constant. In this case, the Ricci tensor can be expressed as

$$\begin{aligned} S(X,Y) = \Big(kp + \frac{r}{2} - \lambda\Big)g(X,Y) \\ + k(\sigma + p)A(X)A(Y) + k[A(X)B(Y) + A(Y)B(X)] \end{aligned} \tag{3.19}$$

for all vector fields $X, Y$ where $r$ is the scalar curvature, $k$ is the gravitational constant, $T$ is the energy momentum tensor of type $(0,2)$ and $\sigma$ and $p$ are the matter density and pressure of the fluid, respectively. Letting

$$a = \frac{r}{2} + kp - \lambda, \quad b = k(\sigma + p), \quad c = k \tag{3.20}$$

we get the following result:

**Theorem 4.** *A fluid spacetime admitting heat flux with the energy-momentum tensor of the form* (1.9) *and satisfying Einstein's field equations with cosmological constant is a generalized quasi Einstein spacetime.*

Now, we consider quasi Einstein spacetime satisfying Einstein's field equations and we assume that the matter distribution of this spacetime is a fluid with the basic vector field. Then comparing the equations (1.1) and (3.19), we get

$$\left(a-\frac{r}{2}-kp+\lambda\right)g(X,Y)+[b-k(\sigma+p)]A(X)A(Y)=k[A(X)B(Y)+A(Y)B(X)] \tag{3.21}$$

Putting $Y=U$ in (3.21), we get

$$\left(a-\frac{r}{2}-kp+\lambda-b+k(\sigma+p)\right)A(X)=-kB(X) \tag{3.22}$$

Putting $X=U$ in (3.22) and as $B(U)=0$, we get

$$a-\frac{r}{2}-kp+\lambda-b+k(\sigma+p)=0 \tag{3.23}$$

In view of (3.23), as $k\neq 0$, (3.22) yields $B(X)=0$, for all vector field $X$. Hence we obtain:

**Theorem 5.** *In a quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant, if the matter distribution is fluid with the basic vector field of the spacetime, then this spacetime can not admit heat flux.*

*Remark* 1. This means that such spacetime describes a universe which has already attained thermal equilibrium.

On the other hand, in a generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant, if the matter distribution is a perfect fluid, then comparing the equations (1.3) and (3.19), we get $c=0$, which is a contradiction. Hence we can state the following:

**Theorem 6.** *There exists no generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant whose matter content is perfect fluid.*

In the sequel, we shall determine some physical properties of $W_2$-flat spacetime satisfying Einstein's field equations with cosmological constant. That is, in this spacetime, the $W_2$-curvature tensor vanishes. Then by (3.1), we have

$$R(X,Y,Z,W)=\frac{1}{3}[g(Y,Z)S(X,W)-g(X,Z)S(Y,W)] \tag{3.24}$$

Contracting (3.24) over $X$ and $W$, we get

$$S(Y,Z)=\frac{r}{4}g(Y,Z) \tag{3.25}$$

which gives the scalar curvature $r$ is constant.

By virtue of (3.25), (3.24) yields

$$R(X,Y,Z,W) = \frac{r}{12}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \tag{3.26}$$

which also shows that $W_2$-flat spacetimes are of constant curvature. From (3.26), we have

$$R(X,Y)Z = \frac{r}{12}[g(Y,Z)X - g(X,Z)Y] \tag{3.27}$$

Let $U^{\perp}$ denote the 3-dimensional distribution in an $W_2$-flat spacetime orthogonal to $U$. Then for all $X,Y,Z \in U^{\perp}$, the equation (3.27) holds. Moreover, for every $X \in U^{\perp}$, we obtain

$$R(X,U)U = \frac{r}{12}X \tag{3.28}$$

According to H. Karchar [9], a Lorentzian manifold is called *infinitesimally spatially isotropic* relative to timelike unit vector field $U$ if its curvature tensor $R$ satisfies the relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y]; \ \ \forall X,Y,Z \in U^{\perp} \tag{3.29}$$

and

$$R(X,U)U = mX; \ \ \forall X \in U^{\perp} \tag{3.30}$$

where $l,m$ are real valued function on the manifold. Hence, (3.27) and (3.28) lead to following result:

**Theorem 7.** *Every $W_2$-flat spacetime satisfying Einstein's field equations with the velocity vector field $U$ is infinitesimally spatially isotropic relative to timelike unit vector field $U$.*

The geometrical symmetries of a spacetime are expressed through the equation

$$\mathcal{L}_{\xi}A - 2\Omega A = 0 \tag{3.31}$$

where $A$ represents a geometrical or physical quantity, $\mathcal{L}_{\xi}$ denotes the Lie derivative with respect to a vector field $\xi$ and $\Omega$ is a scalar, [10].

One of the most simple and widely used example is the *metric inheritance symmetry* for which $A = g_{ij}$ in (3.31). That is,

$$(\mathcal{L}_{\xi}g)(X,Y) = 2\Omega g(X,Y) \tag{3.32}$$

In this case, $\xi$ is the *Killing vector field* if $\Omega = 0$ and is the *conformal Killing vector field* if $\Omega$ is a scalar function.

Afterwards of this section, the existence of Killing and conformal Killing vector fields has been established. By Einstein's field equations (3.17) and (3.25) we obtain

$$(\lambda - \frac{r}{4})g(X,Y) = kT(X,Y) \tag{3.33}$$

Taking the Lie derivative of both sides of (3.33) and remembering that the scalar curvature $r$ is constant, we get

$$(\lambda - \frac{r}{4})(\mathcal{L}_\xi g)(X,Y) = k(\mathcal{L}_\xi T)(X,Y) \tag{3.34}$$

Now, if $\xi$ is conformal Killing vector field, from (3.34) and (3.32), we get

$$2(\lambda - \frac{r}{4})\Omega g(X,Y) = k(\mathcal{L}_\xi T)(X,Y) \tag{3.35}$$

where $\Omega$ is a scalar function. In a $W_2$ flat spacetime satisfying Einstien's field equation with cosmological constant, from (3.33) and (3.35), as $k \neq 0$ we get

$$(\mathcal{L}_\xi T)(X,Y) = 2\Omega T(X,Y) \tag{3.36}$$

It follows from (3.31) that the energy momentum tensor has the symmetry inheritance property.

Conversely, if (3.36) holds, then by (3.33) and (3.34), as $\lambda, k$ and $r$ are constant, (3.32) holds for some scalar function $\Omega$. Thus $\xi$ is the conformal Killing vector field. If we take $\Omega = 0$, then from (3.35) and (3.36), a necessary and sufficient condition for $\xi$ to be a Killing vector field is that Lie derivative of the energy momentum tensor with respect to $\xi$ be zero. Hence we can state that:

**Theorem 8.** *An $W_2$-flat spacetime satisfying Einstein's field equations with cosmological constant admits:*

(1) *a conformal Killing vector field if and only if the energy momentum tensor has the symmetry inheritance property.*
(2) *a Killing vector field $\xi$ if and only if Lie derivative of the energy momentum tensor with respect to $\xi$ vanishes.*

## 4. Ricci symmetric generalized quasi Einstein spacetimes

In this section, a Ricci symmetric generalized quasi Einstein spacetime has been considered. Then, in this spacetime, the Ricci tensor of $S$ satisfies $\nabla S = 0$ where $\nabla$ denotes the Riemannian connection. It follows from (1.3), we have

$$\begin{aligned}(\nabla_Z S)(X,Y) =& da(Z)g(X,Y) + db(Z)A(X)A(Y) + b[(\nabla_Z A)(X)A(Y) \\ &+ A(X)(\nabla_Z A)(Y)] + dc(Z)[A(X)B(Y) + A(Y)B(X)] \\ &+ c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) \\ &+ (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] = 0\end{aligned} \tag{4.1}$$

Putting $X = Y = U$ in (4.1), we get

$$-da(Z) + db(Z) - 2c(\nabla_Z B)(U) = 0 \tag{4.2}$$

and again putting $X = Y = V$ in (4.1), we get

$$da(Z) + 2c(\nabla_Z A)(V) = 0 \tag{4.3}$$

Since the generators $U$ and $V$ are mutually orthogonal, $g(U,V)=0$ and so $Z(g(U,V))=g(\nabla_Z U,V)+g(U,\nabla_Z V)=0$. This yields,

$$(\nabla_Z B)(U)=-(\nabla_Z A)(V) \tag{4.4}$$

Substracting (4.3) from (4.2) and using the last relation, we obtain

$$d(b-2a)(Z)=0,\ \forall Z \tag{4.5}$$

which means that $b-2a$ is a constant.

Moreover, contracting (4.1) over $X$ and $Y$, it follows from (4.4) that we obtain

$$d(4a-b)(Z)=0,\ \forall Z \tag{4.6}$$

and so $4a-b$ is also a constant. From these results, we conclude that the scalar functions $a$ and $b$ are constants. Then, the equation (4.3) yields $c(\nabla_Z A)(V)=0$. Thus, we have either $c=0$ or $(\nabla_Z A)(V)=0$. If $c=0$, then by (4.1), we get

$$b[(\nabla_Z A)(X)A(Y)+A(X)(\nabla_Z A)(Y)]=0 \tag{4.7}$$

In this case, $b$ must be different than zero. Otherwise, $b=c=0$ which is a contradiction. Thus, we get

$$(\nabla_Z A)(X)A(Y)+A(X)(\nabla_Z A)(Y)=0 \tag{4.8}$$

Putting $X=U$ in (4.8), we obtain

$$(\nabla_Z A)(Y)=0,\ \forall Y,Z \tag{4.9}$$

On the other hand, if $c\neq 0$, then we have $(\nabla_Z A)(V)=0$. Using the last relation and putting $X=U$, $Y=V$ in (4.1), we obtain

$$d(c)(Z)=0,\ \forall Z \tag{4.10}$$

and so $c$ is constant. Then, putting $X=V$ in (4.1) and using $(\nabla_Z A)(V)=0$, we get $c(\nabla_Z A)(Y)=0$. Since in this case $c\neq 0$, again we obtain (4.9).

Hence, in each case $(\nabla_Z A)(Y)=g(\nabla_Z U,Y)=0$; for all $Y,Z$. That is, $\nabla_Z U=0$, for all $Z$ which implies that the generator vector field $U$ is parallel. Moreover, putting $Z=U$ in the last eqution, we get

$$\nabla_U U=0 \tag{4.11}$$

Thus, the integral curves of $U$ are geodesics. Hence we can state that:

**Theorem 9.** *In a Ricci symmetric generalized quasi Einstein spacetime, generator vector field $U$ is parallel and the integral curves of $U$ are geodesics.*

By virtue of the above theorem, the Riemannian curvature tensor satisfies

$$R(X,Y)U=\nabla_X\nabla_Y U-\nabla_Y\nabla_X U-\nabla_{[X,Y]}U=0 \tag{4.12}$$

for all $X,Y$. Contacting (4.12), in a Ricci symmetric $G(QE)_4$, we obtain

$$S(X,U)=(a-b)A(X)-cB(X)=0 \tag{4.13}$$

Putting $X = U$ and $X = V$ in (4.13), we get $a = b$ and $c = 0$, respectively. Hence the Ricci tensor can be written as in the following form:

$$S(X,Y) = a[g(X,Y) + A(X)A(Y)] \tag{4.14}$$

This leads to the following theorem:

**Theorem 10.** *Every Ricci symmetric generalized quasi Einstein spacetime is a quasi Einstein spacetime whose associated scalar functions are constant and equal.*

Combining Einstein's field equations (3.17) and (4.14), in a Ricci symmetric generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant, the energy momentum tensor can be written as in the following form

$$T(X,Y) = \frac{2\lambda - a}{2k} g(X,Y) + \frac{a}{k} A(X)A(Y) \tag{4.15}$$

Thus, this spacetime can be considered as a model of perfect fluid spacetime, in general relativity. Since the velocity vector field $U$ of this spacetime is parallel and the associated scalar $a$ is constant, the energy momentum tensor given by (4.15) is of Codazzi type (even covariantly constant). Hence we can state that:

**Theorem 11.** *In a Ricci symmetric generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant, the energy momentum tensor is of Codazzi type.*

From the equations (3.19), (3.20) and Theorem (10), we get

$$\sigma = \frac{3a - 2\lambda}{2k}\ ,\ p = \frac{2\lambda - a}{2k} \tag{4.16}$$

Additionally, since $a$ is constant, $\sigma$ and $p$ are constants. Therefore, we can state that:

**Theorem 12.** *In a Ricci symmetric generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant, the energy density and the isotropic pressure are constants.*

If we assume $\sigma > 0$ and $p > 0$, from the equation (4.16), we have

$$\frac{a}{2} < \lambda < \frac{3a}{2} \tag{4.17}$$

so we obtain:

**Corollary 2.** *In a Ricci symmetric generalized quasi Einstein spacetime satisfying Einstein's field equations with cosmological constant $\lambda$, if the isotropic pressure and energy density are positive, then $\lambda$ satisfies the relation $\frac{a}{2} < \lambda < \frac{3a}{2}$.*

Further, we have the energy and force equations for a perfect fluid [12], as follows:

$$U\sigma = g(grad\sigma, U) = -(\sigma + p)divU \tag{4.18}$$

and

$$(\sigma+p)(\nabla_U U) = -grad_{\perp}p = -gradp - g(gradp, U)U$$
$$= -gradp - (Up)U \tag{4.19}$$

where the spatial pressure gradient $grad_{\perp}p$ is the component of $gradp$ orthogonal to $U$. From Theorem (12), in a Ricci symmetric generalized quasi Einstein spacetime satisfying Einstein's field equations, both $\sigma$ and $p$ are constants. Then by (4.18) and (4.19), we obtain

$$U\sigma = -(\sigma+p)divU = 0 \;\; and \;\; (\sigma+p)(\nabla_U U) = 0 \tag{4.20}$$

Since $\sigma + p \neq 0$, we have

$$divU = 0 \;\; and \;\; \nabla_U U = 0 \tag{4.21}$$

*Remark* 2. $\sigma + p = 0$ means that the matter contents of the spacetime satisfy the vacuum like equation of state.

It is known that $divU$ represents the expansion scalar and $\nabla_U U$ represents the acceleration vector. Thus in view of (4.21), both of them vanish. This leads to following result:

**Theorem 13.** *In a Ricci symmetric generalized quasi Einstein spacetime obeying Einstein's field equation with (or without) cosmological constant, the expansion scalar and the acceleration vector vanish.*

Shaikh et. al. [18] proved that in a perfect fluid quasi-Einstein spacetime with Codazzi type energy momentum tensor, the fluid has vanishing vorticity and vanishing shear. Combining this argument, Theorem (10) and Theorem (11), we get the following result:

**Theorem 14.** *In a Ricci symmetric generalized quasi Einstein spacetime obeying the Einstein's field equations, the vorticity and shear tensors vanish.*

According to Petrov classification [14] a spacetime can be divided into six types denoted by I, II, III, D, N and O. A. Barnes [1] has been proved that if a perfect fluid spacetime is shear-free, vorticity-free and the velocity vector field of the fluid is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Thus by using the Theorem (12) and Theorem (14), we can state the following:

**Theorem 15.** *The possible local cosmological structures of a Ricci symmetric generalized quasi Einstein spacetime obeying the Einstein's field equation are of Petrov I, D or O.*

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# PARAMETRIC MARCINKIEWICZ INTEGRALS ON WEIGHTED HERZ SPACES

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*Abstract.* Let $0 < \rho < n$ and $\mu_\Omega^\rho$ be the parametric Marcinkiewicz integral. In this paper we shall obtain the strong type and weak type estimates of $\mu_\Omega^\rho$ on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ with general weights. The boundedness of the commutators generated by $BMO$ functions and parametric Marcinkiewicz integral is also obtained.

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## 1. Introduction and results

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. For $0 < \rho < n$, Hörmander in [6] defined the parametric Marcinkiewicz integral operator $\mu_\Omega^\rho$ of higher dimension as follows.

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(x)|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2},$$

where

$$F_{\Omega,t}^\rho(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy.$$

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Let $b$ be a locally integrable function, the commutator generated by parametric Marcinkiewcz integral $\mu_\Omega^\rho$ and $b$ is defined by

$$[b,\mu_\Omega^\rho](f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}(b(x)-b(y))f(y)dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{\frac{1}{2}}.$$

When $\rho=1$, we shall denote $\mu_\Omega^1$ simply by $\mu_\Omega$.

The area of the Marcinkiewcz integrals have been under intensive research. This operator $\mu_\Omega$ was first introduced by Stein in [15]. He proved that if $\Omega \in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, then $\mu_\Omega$ is the operator of strong type $(p,p)$ for $1<p\le 2$ and of weak type $(1,1)$. Here, we say that $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ if

$$|\Omega(x')-\Omega(y')|\le|x'-y'|^\alpha,\ \ x',y'\in\mathbb{S}^{n-1}.$$

In 1990, Torchinsky and Wang in [16] considered the weighted case and proved that If $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, and $b\in BMO(\mathbb{R}^n)$, then for all $1<p<\infty$, and $\omega\in A_p$ (Muckenhoupt weight class), $\mu_\Omega$ and $[b,\mu_\Omega]$ are all bounded in $L^p(\omega)$. On the other hand, in 1960, Hörmander [6] showed that if $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)$, then for $0<\rho<n$, $\mu_\Omega^\rho$ is of strong type $(p,p)$ for all $1<p<\infty$. In [14] Shi and Jiang obtained the weighted $L^p$-boundedness of parametric Marcinkiewicz integral and its commutator.

**Theorem A** ([14]). *Let $0<\rho<n$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$. If $\omega\in A_p(1<p<\infty)$ and $b\in BMO(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of $f$ such that*

$$\|\mu_\Omega^\rho(f)\|_{L^p(\omega)}\le C\|f\|_{L^p(\omega)}$$

*and*

$$\|[b,\mu_\Omega^\rho](f)\|_{L^p(\omega)}\le C\|b\|_*\|f\|_{L^p(\omega)}.$$

Notice that $Lip_\alpha(\mathbb{S}^{n-1})(0<\alpha\le 1)\subsetneqq L^\infty(\mathbb{S}^{n-1})\subsetneqq L^1(\mathbb{S}^{n-1})$, then the results of Theorem A are extend and improve the results of Torchinsky and Wang in [16].

Let us recall the definition of Littlewood-Paley function

$$S_\Phi(f)(x)=\left(\int_{\mathbb{R}^n}|\Phi_t*f(x)|^2\frac{dt}{t}\right)^{1/2},$$

where $\Phi_t(x)=t^{-n}\Phi(x/t)$ and $\Phi\in L^1(\mathbb{R}^n)$ which satisfies $\int_{\mathbb{R}^n}\Phi(x)dx=0$. It is well known that the Littlewood-Paley function have long played important roles in harmonic analysis. When $\Phi$ is given by

$$\Phi(x)=|x|^{-n+\rho}\Omega(x)\chi_{[0,1]}(|x|),$$

where $\Omega$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1), then $\Phi(x)$ becomes the parametric Marcinkiewicz integral operator. Therefore, many authors have been interested in studying the boundedness properties of $\mu_\Omega^\rho$ on various function spaces, it can be seen in [1–3, 18].

In [13], Lu, Yabuta and Yang obtained the boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators on weighted Herz spaces with general Muckenhoupt weights. In [8], Komori and Matsuoka showed the boundedness of singular integral operators and fractional integrals on weighted Herz spaces. In [5], Guo and Jiang discussed the boundedness of commutators of singular integral operators on weighted Herz spaces, and as an application, they obtained the interior estimates on weighted Herz spaces for the solutions of some nondivergence elliptic equations. Hu , He and Wang [19] studied the boundedness of commutators of fractional integrals in generalized Herz spaces. More results concerning the boundedness of operators on Herz spaces can be seen in [12, 17].

The main purpose of this paper is to consider the boundedness of parametric Marcinkiewicz integral on weighted Herz spaces with $A_p$ weights. At the extreme case, we will also prove that $\mu_\Omega^\rho$ is bounded from the weighted Herz spaces to the weighted weak Herz spaces. The boundedness of commutators generated by parametric Marcinkiewicz integral and $BMO$ functions on weighted Herz spaces is also considered.

Our main results in the paper are formulated as follows.

**Theorem 1.** *Let $0<\rho<n$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$. Suppose $0<p<\infty, 1<q<\infty, \omega_1\in A_{q_1}$ and $\omega_2\in A_{q_2}$. Then $\mu_\Omega^\rho$ is bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ provided that $\omega_1$ and $\omega_2$ satisfy either of the following*

(i) $\omega_1=\omega_2, 1\le q_1=q_2\le q$ *and* $-nq_1/q<\alpha q_1<n(1-q_1/q)$;
(ii) $\omega_1\ne\omega_2, 1\le q_1<\infty, 1\le q_2\le q$ *and* $0<\alpha q_1<n(1-q_2/q)$.

**Theorem 2.** *Let $0<\rho<n$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$. Suppose $0<p<1, 1<q<\infty, \omega_1\in A_{q_1}$ and $\omega_2\in A_{q_2}$. If $1\le q_1<\infty, 1\le q_2\le q$ and $\alpha q_1=n(1-q_2/q)$, then $\mu_\Omega^\rho$ is bounded from $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ to $W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.*

**Theorem 3.** *Let $0<\rho<n$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$. Suppose $1<q<\infty, \omega_1\in A_{q_1}$, $\omega_2\in A_{q_2}$ and $b\in BMO$. Then $[b,\mu_\Omega^\rho]$ is bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ provided that $\omega_1$ and $\omega_2$ satisfy either of the following*

(i) $\omega_1=\omega_2, 1\le q_1=q_2\le q$ *and* $-nq_1/q<\alpha q_1<n(1-q_1/q)$;
(ii) $\omega_1\ne\omega_2, 1\le q_1<\infty, 1\le q_2\le q$ *and* $0<\alpha q_1<n(1-q_2/q)$.

Throughout this paper, unless otherwise indicated, $C$ will be used to denote a positive constant that is not necessarily the same at each occurrence.

## 2. Definitions and preliminaries

We begin this section with some properties of $A_p$ weights which play important role in the proofs of our main results.

A weight $\omega$ is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B=B(x_0,r)$ denote the ball with the center $x_0$ and radius $r$, and let $\lambda B=B(x_0,\lambda r)$ for any $\lambda>0$.

For a given weight function $\omega$ and a measurable set $E$, we also denote the Lebesgue measure of $E$ by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$.

**Definition 1.** A weight $\omega$ is said to belong to $A_p$ for $1 < p < \infty$, if there exists a constant $C$ such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} \le C, \tag{2.1}$$

where $s'$ is the dual of $s$ such that $1/s + 1/s' = 1$.

The class $A_1$ is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B \omega(y)dy \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x).$$

By (2.1), we have

$$\left(\int_B \omega(x)dx\right)\left(\int_B \omega(x)^{1-p'}dx\right)^{p-1} \le C|B|^p. \tag{2.2}$$

The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$-boundedness of Hardy-Littlewood maximal function in [10].

**Lemma 1** ([4, 10])**.** *Let $1 \le p < \infty$ and $\omega \in A_p$. Then the following statements are true:*

(i) *There exists constant $C$ such that*

$$\frac{\omega(B_k)}{\omega(B_j)} \le C2^{np(k-j)} \ \text{for}\ k > j; \tag{2.3}$$

(ii) *For any $1 < p < \infty$, there exists $1 < q < p$ such that $\omega \in A_q(\mathbb{R}^n)$;*

(iii) *There exist two constants $C$ and $\delta > 0$ such that for any measurable set $E \subset B$,*

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{\delta}; \tag{2.4}$$

(iv) *For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.*

If $\omega$ satisfies (2.4), we say $\omega \in A_\infty(\mathbb{R}^n)$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p(\mathbb{R}^n).$$

Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where $\chi_E$ is the characteristic function of the set $E$. In [9], Lu and Yang introduced weighted Herz spaces.

**Definition 2.** Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$ and $\omega_1, \omega_2$ be two weight functions on $\mathbb{R}^n$. The homogenous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \left\{ f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, \omega_2) : \ \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \left( \sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|^p_{L^q(\omega_2)} \right)^{1/p}.$$

For any $k \in \mathbb{Z}, \lambda > 0$ and any measurable function $f$ on $\mathbb{R}^n$, we write $E_k(\lambda, f) = \{x \in C_k : |f(x)| > \lambda\}$.

**Definition 3** ([13])**.** Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$ and $\omega_1, \omega_2$ be two weight function on $\mathbb{R}^n$. A measurable function $f$ on $\mathbb{R}^n$ is said to belong to the homogeneous weighted weak Herz space $W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \sup_{\lambda>0} \lambda \left( \sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} [\omega_2(E_k(\lambda, f))]^{p/q} \right)^{1/p} < \infty.$$

Obviously, if $\alpha = 0$, then $\dot{K}_p^{0,p}(\omega_1,\omega_2) = L^p(\omega_2)$ and $W\dot{K}_p^{0,p}(\omega_1,\omega_2) = WL^p(\omega_2)$ for any $0 < p < \infty$. Thus, weighted Herz spaces are generalizations of the weighted Lebesgue spaces.

**Definition 4.** A locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx = \|b\|_* < \infty,$$

where $b_B = \frac{1}{|B|} \int_B b(y) dy$.

**Lemma 2.** *(John-Nirenberg inequality, see [7]) Let $b \in BMO(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$, there exist constant $C_1, C_2$ such that for all $\lambda > 0$,*

$$|\{x \in B : |b(x) - b_B| > \lambda\}| \le C_1 |B| exp(-C_2 \lambda / \|b\|_*).$$

**Lemma 3** ([11])**.** *Let $\omega \in A_\infty$. Then the norm of $BMO(\omega, \mathbb{R}^n)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where*

$$BMO(\omega, \mathbb{R}^n) = \left\{ b : \|b\|_{*,\omega} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}| \omega(x) dx \right\},$$

*and*

$$b_{B,\omega} = \frac{1}{\omega(B)} \int_B b(z)\omega(z) dz.$$

**Lemma 4.** *Suppose $\omega \in A_\infty(\mathbb{R}^n), b \in BMO(\mathbb{R}^n)$. Then for any $p \ge 1$ we have*

$$\left( \frac{1}{\omega(B)} \int_B |b(x) - b_{B,\omega}|^p \omega(x) dx \right)^{1/p} \le C \|b\|_*.$$

*Proof.* Since $\omega(x) \in A_\infty(\mathbb{R}^n)$, then (iii) of Lemma 1 and Lemma 2 imply

$$\omega(\{x \in B : |b(x) - b_B| > \lambda\}) \le C\omega(B) exp(-C_2\delta\lambda/\|b\|_*).$$

So

$$\begin{aligned}\int_B |b(x)-b_B|^p \omega(x)dx &\le \int_0^\infty p\lambda^{p-1}\omega(\{x \in B : |b(x)-b_B| > \lambda\})\,d\lambda \\ &\le C\omega(B)\int_0^\infty p\lambda^{p-1} exp(-C_2\delta\lambda\|b\|_*)d\lambda \\ &\le C\omega(B)\|b\|_*^p.\end{aligned}$$

Thus

$$\frac{1}{\omega(B)}\int_B |b(x)-b_{B,\omega}|^p\omega(x)dx \le \frac{C}{\omega(B)}\int_B |b(x)-b_B|^p\omega(x)dx \le C\|b\|_*^p.$$

□

## 3. Proof of Theorem 1

Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. Then

$$\begin{aligned}&\|\mu_\Omega^\rho(f)\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \\ \le\ & C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p \\ &+ C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p \\ &+ C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p \\ =\ & E_1+E_2+E_3.\end{aligned}$$

By the fact that $\mu_\Omega^\rho$ is a bounded operator on $L^q(\omega_2)$, we get

$$\begin{aligned}E_2 =& C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|(\mu_\Omega^\rho(f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p \\ \le& C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|f\chi_k\|_{L^q(\omega_2)}\right)^p \\ \le& C\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.\end{aligned}$$

For any $x \in C_j$ and $y \in C_k \cap \{y : |x-y| < t\}$ with $j \geq k+2$, we have $t > |x-y| \geq |x|-|y| \geq |x|/2 \geq C2^{jn}$. So

$$\begin{aligned}\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| &= \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f_k(y)dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2} \\ &\leq C2^{-j(n-\rho)}\|f\chi_k\|_{L^1}\left(\int_{C2^{jn}}^\infty \frac{dt}{t^{2\rho+1}}\right)^{1/2}.\end{aligned}$$

Then

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| = C2^{-jn}\|f\chi_k\|_{L^1}, \tag{3.1}$$

and

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \leq C2^{-jn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}.$$

By Hölder's inequality,

$$\|f\chi_k\|_{L^1} \leq C\|f\chi_k\|_{L^q(\omega_2)}\left(\int_{B_k}\omega_2(x)^{1-q'}dx\right)^{1/q'}. \tag{3.2}$$

Since $\omega_2 \in A_{q_2}(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$, by (2.2) and (2.3) we get,

$$\begin{aligned}&\left(\int_{B_k}\omega_2(x)^{1-q'}dx\right)^{1/q'}\left(\int_{B_j}\omega_2(x)dx\right)^{1/q} \\ &= \left(\int_{B_k}\omega_2(x)^{1-q'}dx\right)^{1/q'}\left(\int_{B_k}\omega_2(x)dx\right)^{1/q}\left(\frac{\omega_2(B_j)}{\omega_2(B_k)}\right)^{1/q} \\ &\leq C2^{kn+(j-k)nq_2/q}.\end{aligned} \tag{3.3}$$

Then, for $j \geq k+2$ we have

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \leq C\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}.$$

Thus

$$\begin{aligned}E_1 &= C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)}\right)^p \\ &\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}\left(\omega_1(B_j)\right)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}\right)^p \\ &\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}\left(\omega_1(B_k)\right)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)(\alpha q_1+q_2n/q-n)}\right)^p.\end{aligned}$$

When $0 < p \le 1$, we get

$$\begin{aligned} E_1 &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k+2} 2^{(j-k)p(\alpha q_1+q_2 n/q-n)} \\ &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \\ &\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}^p. \end{aligned}$$

When $p > 1$, by Hölder's inequality we get

$$\begin{aligned} E_1 &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{j-2} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)(\alpha q_1+q_2 n/q-n)} \right) \\ &\qquad \left( \sum_{k=-\infty}^{j-2} 2^{(j-k)(\alpha q_1+q_2 n/q-n)} \right)^{p/p'} \\ &\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \\ &\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}^p. \end{aligned}$$

Let us now turn to estimate the last term $E_3$. In the case $k \ge j+2$, for any $x \in C_j$ and $y \in C_k \cap \{y : |x-y| < t\}$, we have $t > |x-y| \ge |y| - |x| \ge |y|/2 \ge C2^{kn}$. Then, similar to the estimates of (3.1),

$$\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le C2^{-kn}\|f\chi_k\|_{L^1}. \tag{3.4}$$

So,

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C2^{-kn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}. \tag{3.5}$$

By (2.2) and (2.4),

$$\Big(\int_{B_k} \omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j} \omega_2(x)dx\Big)^{1/q} \le C2^{kn+(j-k)\delta_2 n/q}. \tag{3.6}$$

Combining with (3.2), (3.5) and (3.6), we have

$$\left\|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}$$

for $j \le k-2$.

When $0 < p \le 1$, we get

$$\begin{aligned}
E_3 &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \omega_1(B_j)^{\frac{\alpha p}{n}} \sum_{k=j+2}^{\infty} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)p\delta_2 n/q} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} \left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha p}{n}} 2^{(j-k)p\delta_2 n/q} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} 2^{(j-k)p(\delta_1\alpha+\delta_2 n/q)} \\
&\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}$$

When $q > 1$, by Hölder's inequality we get

$$\begin{aligned}
E_3 &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=j+2}^{\infty} (\omega_1(B_k))^{\frac{\alpha}{n}} \|f\chi_k\|_{L^q(\omega_2)} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)} \right)^p \\
&\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} \left( \sum_{k=j+2}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)} \right) \\
&\qquad \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)} \right)^{p/p'} \\
&\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)} \\
&\le C\|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}$$

Combining the above estimates for $E_1, E_2$ and $E_3$, the proof of Theorem 1 is completed.

## 4. Proof of Theorem 2

Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. Then

$$\begin{aligned}
&\lambda^p \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \big(\omega_2(\{x \in C_j : |\mu_\Omega^\rho(f)(x)| > \lambda\})\big)^{p/q} \\
&\le \lambda^p \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left( \omega_2\Big(\Big\{x \in C_j : \sum_{k=-\infty}^{j-2} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\Big\}\Big) \right)^{p/q}
\end{aligned}$$

$$
\begin{aligned}
&+\lambda^p \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left(\omega_2\Big(\Big\{x\in C_j : \sum_{k=j-1}^{j+1} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\Big\}\Big)\right)^{p/q}\\
&+\lambda^p \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left(\omega_2\Big(\Big\{x\in C_j : \sum_{k=j-2}^{\infty} |\mu_\Omega^\rho(f_k)(x)| > \lambda/3\Big\}\Big)\right)^{p/q}\\
=\ & F_1+F_2+F_3.
\end{aligned}
$$

Applying Chebyshev's inequality [4] and Theorem A, we obtain

$$
\begin{aligned}
F_2 &\le C\lambda^p \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left(\frac{1}{\lambda^q}\sum_{k=j-1}^{j+1} \|\mu_\Omega^\rho(f_k)\|_{L^q(\omega_2)}^q\right)^{p/q}\\
&\le C \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{j+1} \|f\chi_k\|_{L^q(\omega_2)}\right)^{p}\\
&\le C\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}
$$

For any $x\in C_j$ and $y\in C_k\cap\{y:|x-y|<t\}$ with $j\ge k+2$, by the inequalities (3.1), (3.2) and (3.3) we have

$$
\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le C\left(\omega_2(B_j)\right)^{-1/q} 2^{-jn+kn+(j-k)nq_2/q}\|f\chi_k\|_{L^q(\omega_2)}.
$$

Noting the fact $\alpha q_1 = n(1-q_2/q)$, then

$$
\left|\mu_\Omega^\rho(f\chi_k)\chi_j(x)\right| \le C\left(\omega_2(B_j)\right)^{-1/q} 2^{(k-j)\alpha q_1}\|f\chi_k\|_{L^q(\omega_2)}.
$$

Moreover, since $0<p\le 1$, then for any $x\in C_j$,

$$
\begin{aligned}
&\sum_{k=-\infty}^{j-2} \left|\mu_\Omega^\rho(f_k)(x)\right|\\
&\le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\\
&\quad \sum_{k=-\infty}^{j-2} 2^{(k-j)\alpha q_1}\left(\omega_1(B_k)\right)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha}{n}}\\
&\le C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\sum_{k=-\infty}^{j-2}\left(\omega_1(B_k)\right)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}
\end{aligned}
$$

$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q(\omega_2)}\right)^{1/p}$$
$$\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}.$$

If

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu^\rho_\Omega(f_k)(x)|>\lambda/3\right\}=\varnothing,$$

then

$$F_1\leq C\|f\|^p_{\dot{K}^{\alpha,p}_p(\omega_1,\omega_2)}$$

holds is trivially. Now we suppose

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu^\rho_\Omega(f_k)(x)|>\lambda/3\right\}\neq\varnothing.$$

Let

$$S_j=\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}.$$

Since $\alpha>0$, it is easy to see that

$$\lim_{j\to\infty}S_j=0.$$

Then for any $\lambda>0$, we can find a maximal positive integer $j_\lambda$ such that

$$\lambda/3\leq CS_{j_\lambda}\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}.$$

So

$$\begin{aligned}F_1\leq\ &\lambda^p\sum_{j=-\infty}^{j_\lambda}\left(\omega_2(B_j)\right)^{p/q}\left(\omega_1(B_j)\right)^{\alpha p/n}\\ \leq\ &C\|f\|^p_{\dot{K}^{\alpha,p}_p(\omega_1,\omega_2)}\sum_{j=-\infty}^{j_\lambda}\left(\frac{\omega_1(B_j)}{\omega_1(B_{j_\lambda})}\right)^{\frac{\alpha p}{n}}\left(\frac{\omega_2(B_j)}{\omega_2(B_{j_\lambda})}\right)^{\frac{p}{q}}\\ \leq\ &C\|f\|^p_{\dot{K}^{\alpha,p}_p(\omega_1,\omega_2)}\sum_{j=-\infty}^{j_\lambda}2^{(j-j_\lambda)(\delta_1\alpha p+\delta_2 pn/q)}\\ \leq\ &C\|f\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}.\end{aligned}$$

Let us now estimate $F_3$. From (3.2), (3.4) and (3.6) we have

$$\left|\mu^\rho_\Omega(f\chi_k)\chi_j(x)\right|\leq\ C\left(\omega_2(B_j)\right)^{-1/q}2^{(j-k)\delta_2 n/q}\|f\chi_k\|_{L^q(\omega_2)}$$

for $j \geq k-2$. So

$$\begin{aligned}
&\sum_{k=j-2}^{\infty} |\mu_\Omega^\rho(f_k)(x)| \\
&\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n} \\
&\quad \sum_{k=j-2}^{\infty} (\omega_1(B_k))^{\alpha/n}\|f\chi_k\|_{L^q(\omega_2)} 2^{(j-k)\delta_2 n/q}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\alpha/n} \\
&\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\sum_{k=j-2}^{\infty} (\omega_1(B_k))^{\alpha/n}\|f\chi_k\|_{L^q} 2^{(j-k)(\delta_2 n/q+\alpha\delta_1)} \\
&\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\sum_{k=j-2}^{\infty} (\omega_1(B_k))^{\alpha/n}\|f\chi_k\|_{L^q}.
\end{aligned}$$

Noting that $0 < p \leq 1$, we have

$$\begin{aligned}
&\sum_{k=j-2}^{\infty} |\mu_\Omega^\rho(f_k)(x)| \\
&\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\left(\sum_{k=j-2}^{\infty} |(\omega_1(B_k))^{\alpha p/n}\|f\chi_k\|_{L^q}^p\right)^{1/p} \\
&\leq C\left(\omega_2(B_j)\right)^{-1/q}\left(\omega_1(B_j)\right)^{-\alpha/n}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.
\end{aligned}$$

Repeating the arguments used for the term $F_1$, we can also obtain

$$F_3 \leq C\,\dot{K}_q^{\alpha,p}(\omega_1,\omega_2).$$

Combining the above estimates for $F_1, F_2$ and $F_3$, and taking the supremum for all $\lambda > 0$, the proof of Theorem 2 is finished.

## 5. Proof of Theorem 3

Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$, then

$$\begin{aligned}
&\|[b,\mu_\Omega^\rho](f)\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p \\
&\leq C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\|([b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p \\
&\quad + C\sum_{j=-\infty}^{\infty}\left(\omega_1(B_j)\right)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|([b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\|_{L^q(\omega_2)}\right)^p
\end{aligned}$$

$$
\begin{aligned}
&+ C \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left( \sum_{k=j+2}^{\infty} \big\| ([b, \mu_\Omega^\rho](f\chi_k)\chi_j(x) \big\|_{L^q(\omega_2)} \right)^p \\
&= G_1 + G_2 + G_3.
\end{aligned}
$$

By the fact that $[b, \mu_\Omega^\rho]$ is a bounded operator on $L^q(\omega_2)$, we obtain

$$
G_2 \le C \sum_{j=-\infty}^{\infty} \big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}} \left( \sum_{k=j-1}^{j+1} \big\| f\chi_k \big\|_{L^q(\omega_2)} \right)^p \le C \|b\|_*^p \|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.
$$

Obviously,

$$
\begin{aligned}
&\Big\| [b, \mu_\Omega^\rho](f\chi_k)\chi_j(x) \Big\|_{L^q(\omega_2)} \\
&\le C \left( \int_{C_j} \big| (b(x) - b_{B_k}) \mu_\Omega^\rho(f_k)(x) \big|^q \omega_2(x) dx \right)^{1/q} \\
&\quad + C \left( \int_{C_j} \big| \mu_\Omega^\rho((b(\cdot) - b_{B_k}) f_k)(x) \big|^q \omega_2(x) dx \right)^{1/q} \\
&= H_1 + H_2.
\end{aligned}
$$

For the term $G_1$, since $j \ge k+2$, by (3.1) we have

$$
\begin{aligned}
H_1 &= C \left( \int_{C_j} \big| (b(x) - b_{B_k}) \mu_\Omega^\rho(f_k)(x) \big|^q \omega_2(x) dx \right)^{1/q} \\
&\le C 2^{-jn} \|f\chi_k\|_{L^1} \left( \int_{B_j} |b(x) - b_{B_k}|^q \omega_2(x) dx \right)^{1/q} \\
&\le C 2^{-jn} \|f\chi_k\|_{L^1} \Big( \Big( \int_{B_j} |b(x) - b_{B_j,\omega_2}|^q \omega_2(x) dx \Big)^{1/q} \\
&\qquad + (|b_{B_j} - b_{B_j,\omega_2}| + |b_{B_j} - b_{B_k}|) \Big( \int_{B_j} \omega_2(x) dx \Big)^{1/q} \Big).
\end{aligned}
$$

By Lemma 4 and the definition of $BMO(\mathbb{R}^n)$, we have

$$
\Big( \int_{B_j} |b(x) - b_{B_j,\omega_2}|^q \omega_2(x) dx \Big)^{1/q} \le C \|b\|_* \Big( \int_{B_j} \omega_2(x) dx \Big)^{1/q}, \tag{5.1}
$$

$$
|b_{B_j} - b_{B_j,\omega_2}| \le C \|b\|_*, \tag{5.2}
$$

and

$$
|b_{B_j} - b_{B_k}| \le C(j-k) \|b\|_*. \tag{5.3}
$$

From (3.2),(3.3) and (5.1)-(5.3), we get

$$\begin{aligned} H_1 &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{-jn}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'} \\ &\le C\|b\|_\star\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)} \end{aligned}$$

for $j \ge k+2$.

Similar to the estimate of (3.1),

$$\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right| \le C2^{-jn}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

So

$$\begin{aligned} &\left(\int_{C_j}\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q} \\ &\le C2^{-jn}\left(\omega_2(B_j)\right)^{1/q}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right). \end{aligned}$$

Since $\omega_2 \in A_{q_2}$, by Lemma 1 we know that $\omega_2^{1-q_2'} \in A_{q_2'}$. Therefore by Lemma 4 we have

$$\left(\int_{B_k}|b(x)-b_{B_k}|^{q'}\omega_2(x)^{1-q'}dx\right)^{1/q'} \le C\|b\|_*\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}. \tag{5.4}$$

Using Hölder's inequality , (3.3) and (5.4) we get

$$\begin{aligned} H_2 &\le C2^{-jn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q} \\ &\le C2^{-jn}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega_2(y)^{1-q'}dy\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q} \\ &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{-jn}\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q} \\ &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}. \end{aligned}$$

Summarizing the above estimates, we have that for $j \ge k+2$,

$$\left\|[b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)}. \tag{5.5}$$

Using (5.5) and repeating the estimation process of $E_1$, we obtain

$$G_1 \le C\|b\|_*^p\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}$$

for $0 < p < \infty$.

Finally, let us estimate $G_3$. Since $j \le k-2$, by (3.4) we have

$$\begin{aligned} H_1 &= C\left(\int_{C_j} \left|(b(x)-b_{B_k})\mu_\Omega^\rho(f_k)(x)\right|^q \omega_2(x)dx\right)^{1/q} \\ &\le C2^{-kn}\|f\chi_k\|_{L^1}\left(\int_{B_j}|b(x)-b_{B_k}|^q\omega_2(x)dx\right)^{1/q}. \end{aligned}$$

From (3.2),(5.1)-(5.3) and (3.6), we get

$$\begin{aligned} H_1 &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{-jn}(\int_{B_j}\omega_2(x)dx)^{1/q}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'} \\ &\le C\|b\|_\star\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)\delta_2 n/q} \end{aligned}$$

for $j \le k-2$.

Similar to the estimate of (3.4),

$$\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right| \le C2^{-kn}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).$$

So

$$\begin{aligned} &\left(\int_{C_j}\left|\mu_\Omega^\rho((b(\cdot)-b_{B_k})f_k)(x)\right|^q\omega_2(x)dx\right)^{1/q} \\ &\le C2^{-kn}\left(\omega_2(B_j)\right)^{1/q}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right). \end{aligned}$$

Using Hölder's inequality, (5.4) and (3.6),

$$\begin{aligned} H_2 &\le C2^{-kn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy\left(\int_{B_j}\omega_2(x)dx\right)^{1/q} \\ &\le C2^{-kn}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega_2(y)^{1-q'}dy\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}\left(\int_{B_j}\omega_2(x)dx\right)^{1/q} \\ &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{-kn}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}\left(\int_{B_j}\omega_2(x)dx\right)^{1/q} \\ &\le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}. \end{aligned}$$

Thus

$$\left\|[b,\mu_\Omega^\rho](f\chi_k)\chi_j(x)\right\|_{L^q(\omega_2)} \le C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(k-j)2^{(j-k)\delta_2 n/q}$$

for $j \ge k-2$. Repeating the estimation process of $E_3$, we obtain

$$G_3 \le C\|b\|_*^p\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}$$

for $0 < p < \infty$.

Summing up the estimates of $G_1, G_2$ and $G_3$, it completes the proof of Theorem 3.

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# OPTIMALITY CONDITIONS AND DUALITY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING INVOLVING RIGHT UPPER-DINI-DERIVATIVE FUNCTIONS

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*Abstract.* In this paper, we introduce $\rho$-generalized pseudo and $\rho$-genera-lized quasi with the tool right upper-Dini-derivative and illustrated these by non-trivial examples. Necessary and sufficient optimality conditions are obtained for a nonlinear multiobjective fractional programming problem involving some classes of generalized convexities with the tool-right upper-Dini-derivative. Furthermore, usual duality theorems are proved for a general dual problem using the concept of generalized convex functions. Our results generalize and extend the several results appeared in the literature.

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*Keywords:* multiobjective programming, fractional programming, upper-Dini-derivative, optimality conditions, duality

## 1. Introduction

The weak minimum (weakly efficient, weak Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control and game theory. During the last decades the analysis of duality in multiobjective theory has been a focal issue. For brief accounts of the evolution of convex theory, with and without differentiability assumptions, and in the framework of multiobjective theory, the reader may consult [1–3, 5–8, 10–12, 16, 19] and the references cited therein.

In the past few years extensive literature relative to the other families of more general functions to substitute the convex functions in the mathematical programming has grown immensely. Such functions are called generalized convex functions. In fact the concept of invexity existed earlier in the literature, e.g., Hanson [9] used it to give Kuhn-Tucker sufficient optimality conditions in nonlinear programming. For the most part, the study of invexity has been in the context of differentiable functions. But the corresponding conclusions cannot be obtained for nondifferentiable programming with the help of invex because the derivative is required in the definition of invex. However, in the recent years, the concept of invexity, previously introduced for differentiable functions, was generalized to the case of nondifferentiable functions.

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Ye [17] defined a generalized convexity, called $d$-invexity, and discussed some theoretical problems for nondifferentiable programming. Nahak and Mohapatra [13] introduced the notation of $d$-$\rho$-$(\eta,\theta)$-invexity and obtained optimality conditions and duality results for a multiobjective programming problem. Jayswal et al. [11] established sufficient conditions and duality theorems for a multiobjective fractional programming problem under $(p,r)$-$\rho$-$(\eta,\theta)$-invex functions. In [16], Sharma and Ahmad used the concept of $(F,\rho,\sigma)$-type I functions to establish Karush-Kuhn-Tucker type sufficient optimality conditions for a nonsmooth multiobjective programming problem.

Making use of arcwise connected set, as defined by Ortega and Rheinboldt [15], Avriel and Zang [4] extended the concept of convex functions to the corresponding forms of arcwise connected functions and presented some interrelations between them. Zhang [20] introduced B-arcwise connected and strictly B-arcwise connected functions based on arcwise connected functions. Optimality and duality results are also obtained for a nonlinear semi-infinite programming problem in [20]. Recently, Yuan and Liu [18] introduced some new generalized convexity notations using right-upper Dini derivative, and established optimality conditions and duality theorems for two types of dual programming.

Inspired and motivated by above works, the purpose of this paper is to investigate the following multiobjective nonlinear fractional programming problem involving generalized convex functions, in terms of the right upper-Dini-derivative:

$$\text{(VFP)} \qquad \min\left(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\dots,\frac{f_p(x)}{g_p(x)}\right)$$

$$\text{subject to}\begin{cases} h_j(x)\leqq 0, \quad j=1,2,\dots,m,\\ x\in X_0,\end{cases}$$

where $X_0$ be a nonempty subset of $R^n$ and $g_i(x)>0$ for all $x\in X_0$ and each $i=1,2,\dots,p$. Let $f=(f_1,f_2,\dots,f_p)$, $g=(g_1,g_2,\dots,g_p)$, $h=(h_1,h_2,\dots,h_m)$. We denote by $X=\{x\in X_0 : h_j(x)\leqq 0,\ j=1,2,\dots,m\}$ the set of all feasible solutions to problem (VFP). For $\bar{x}\in X$ we denote $M(\bar{x})=\{j\in\{1,2,\dots,m\}: h_j(\bar{x})=0\}$, $h^0=(h_j)_{j\in M(\bar{x})}$ and $N(\bar{x})=\{1,2,\dots,m\}\setminus M(\bar{x})$.

This paper is divided into five sections. Section 2 recalls some definitions and related results which will be used in later sections. Non-trivial examples are also discussed to support our functions. In Section 3, necessary and sufficient optimality conditions are established for a nonlinear fractional multiobjective programming problem involving generalized convex functions. In Section 4, we establish appropriate duality theorems for a general dual problem. Finally, conclusion and further developments are given in Section 5.

## 2. Preliminaries

Let $R^n$ be the $n$-dimensional Euclidean space and $R^n_+$ its non-negative orthant. If $x, y \in R^n$ then $x < y \Leftrightarrow x_i < y_i, i = 1,2,\ldots,n$; $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1,2,\ldots,n$ and $x \leq y \Leftrightarrow x_i \leqq y_i, i = 1,2,\ldots,n$ and $x \neq y$.
In this section, we recall some well known results and concepts.

**Definition 1.** A point $\bar{x} \in X$ is said to be a weak minimum to (VFP) if there exists no other feasible point $x$ such that $\frac{f(\bar{x})}{g(\bar{x})} > \frac{f(x)}{g(x)}$.

**Definition 2** ( [20]). A set $C \subset R^n$ is said to be an arcwise connected set if, for every $x_1 \in C$, $x_2 \in C$, there exists a continuous vector-valued function $H_{x_1,x_2} : [0,1] \to C$, called an arc, such that

$$H_{x_1,x_2}(0) = x_1, \quad H_{x_1,x_2}(1) = x_2.$$

**Definition 3** ([18]). Let $\varphi$ be a real-valued function defined on an arcwise connected set $C \subset R^n$. Let $x_1, x_2 \in C$ and $H_{x_1,x_2}$ be the arc connecting $x_1$ and $x_2$ in $C$. The right upper-Dini-derivative of $\varphi$ with respect to $H_{x_2,x_1}(t)$ at $t = 0$ is defined as follows:

$$(d\varphi)^+(H_{x_2,x_1}(0^+)) = \lim_{t\to 0^+} \sup \frac{\varphi(H_{x_2,x_1}(t)) - \varphi(x_2)}{t}. \tag{2.1}$$

Using this upper-Dini-derivative concept, Yuan and Liu [18] introduced a class of functions, which called $(\alpha,\rho)$-right upper-Dini-derivative function. For convenience, we recall the following definitions.

**Definition 4** ([18]). A set $X_0 \subset R^n$ is said to be locally arcwise connected at $\bar{x}$ if for any $x \in X_0$ and $x \neq \bar{x}$ there exist a positive number $a(x,\bar{x})$, with $0 < a(x,\bar{x}) \leqq 1$, and a continuous arc $H_{\bar{x},x}$ such that $H_{\bar{x},x}(t) \in X_0$ for any $t \in (0, a(x,\bar{x}))$.
The set $X_0$ is locally arcwise connected on $X_0$ if $X_0$ is locally arcwise connected at any $x \in X_0$.

**Definition 5** ([18]). Let $X_0 \subset R^n$ be a locally arcwise connected set and $\varphi : X_0 \to R$ be a real function defined on $X_0$. The function $\varphi$ is said to be $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to $H$ at $\bar{x}$, if there exist real functions $\alpha : X_0 \times X_0 \to R$, $\rho : X_0 \times X_0 \to R$ such that

$$\varphi(x) - \varphi(\bar{x}) \geqq \alpha(x,\bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) + \rho(x,\bar{x}), \ \forall x \in X_0.$$

If $\varphi$ is $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to $H$ at $\bar{x}$ for any $\bar{x} \in X_0$, then $\varphi$ is called $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to $H$ on $X_0$.

*Remark* 1. It revealed by an example given in [18] that there exists a function, which is $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected but neither $d$-$\rho$-$(\eta,\theta)$-invex [13] nor $d$-invex [17], nor directional differentially B-arcwise connected [20].

Now we define the notions of $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected, strictly $\rho$-generalized-pseudo-right upper-Dini-deriv-ative locally arcwise connected and $\rho$-generalized-quasi-right upper-Dini-deriva-tive locally arcwise connected functions.

**Definition 6.** The function $\varphi : X_0 \to R$ is said to be $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) \geqq -\rho(x,\bar{x}) \Rightarrow \varphi(x) \geqq \varphi(\bar{x}),\ \forall x \in X_0,$$

equivalently

$$\varphi(x) < \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) < -\rho(x,\bar{x}),\ \forall x \in X_0.$$

The function $\varphi : X_0 \to R$ is said to be $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) on $X_0$ if it is $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at any $\bar{x} \in X_0$.

It is noted that, not every $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to $H$ is $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to $H$ [18]. We have the following counter-example, which shows that the function $\varphi$ is $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to $H$ but not $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to $H$.

*Example* 1. Let $\varphi : R \to R$ be defined by

$$\varphi(x) = \begin{cases} x\cos^2\dfrac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

For any, $x, y \in R$, defining the arc $H : [0,1] \to R$ by

$$H_{y,x}(t) = tx + (1-t)y, t \in [0,1].$$

Now, by the definition of right upper-Dini-derivative defined by (2.1), we have

$$(d\varphi)^+(H_{0,x}(0^+)) = \lim_{t\to 0^+}\sup\frac{\varphi(H_{0,x}(t)) - \varphi(0)}{t} = \lim_{t\to 0^+}\sup\frac{\varphi(tx)}{t}$$

$$= \lim_{t\to 0^+}\sup\frac{tx\cos^2\dfrac{1}{tx}}{t} = x.$$

Consider the function $\rho : R \times R \to R$ to be defined by

$$\rho(x,y) = \begin{cases} -x + 3x\cos^2\frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

For $\bar{x} = 0$, we have

$$\varphi(x) - \varphi(\bar{x}) = \begin{cases} x\cos^2\frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0, \end{cases}$$

while

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) + \rho(x,\bar{x}) = \begin{cases} 3x\cos^2\frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

So, clearly $\varphi$ is $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x} = 0$. But, $\varphi$ is not a $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected with respect to same arc $H$ and $\rho$ at $\bar{x} = 0$ because for $x = 0.3$ and $\alpha(x,\bar{x}) = 1$, we get

$$\varphi(x) - \varphi(\bar{x}) - \alpha(x,\bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) - \rho(x,\bar{x}) = -0.5782 < 0.$$

**Definition 7.** The function $\varphi : X_0 \to R$ is said to be strictly $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) \geqq -\rho(x,\bar{x}) \Rightarrow \varphi(x) > \varphi(\bar{x}),\ \forall x \in X_0, x \neq \bar{x},$$

equivalently

$$\varphi(x) \leqq \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) < -\rho(x,\bar{x}),\ \forall x \in X_0, x \neq \bar{x}.$$

The function $\varphi : X_0 \to R$ is said to be strictly $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) on $X_0$ if it is strictly $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at any $\bar{x} \in X_0$.

**Definition 8.** The function $\varphi : X_0 \to R$ is said to be $\rho$-generalized-quasi-right upper-Dini-derivative locally arcwise connected with respect to $H$ at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$\varphi(x) \leqq \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) \leqq -\bar{\rho}(x,\bar{x}),\ \forall x \in X_0,$$

equivalently

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) > -\bar{\rho}(x,\bar{x}) \Rightarrow \varphi(x) > \varphi(\bar{x}),\ \forall x \in X_0.$$

The function $\varphi : X_0 \to R$ is said to be $\rho$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to $H$) on $X_0$ if it is $\rho$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at any $\bar{x} \in X_0$.

Now, we present an example which shows that there exists a function which is $\rho$-generalized-quasi-right upper-Dini-derivative locally arcwise connected but neither $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected nor $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to the arc $H$.

*Example* 2. Let $\varphi : R \to R$ be a function defined by

$$\varphi(x) = \begin{cases} x \sin\frac{1}{x} - 3x; & \text{if } 0 < x < 1 \\ 0; & \text{otherwise .} \end{cases}$$

For any, $x, y \in R$, defining the arc $H : [0,1] \to R$ by

$$H_{y,x}(t) = tx + (1-t)y, t \in [0,1].$$

Clearly, $(d\varphi)^+(H_{0,x}(0^+)) = -2x$.
Let $\rho : R \times R \to R$ be defined by

$$\rho(x,y) = \begin{cases} 2x; & \text{if } 0 < x < 1 \\ 0; & \text{otherwise .} \end{cases}$$

Then, it can be easily seen that $\varphi$ is $\rho$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x} = 0$. However, for $0 < x < 1, \alpha(x,\bar{x}) = 1$ and $\bar{x} = 0$, we have

$$\varphi(x) - \varphi(\bar{x}) - \alpha(x,\bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) - \rho(x,\bar{x}) < 0,$$

and

$$\varphi(x) - \varphi(\bar{x}) < 0, \text{ but } (d\varphi)^+(H_{\bar{x},x}(0^+)) + \bar{\rho}(x,\bar{x}) = 0.$$

Hence, $\varphi$ is neither $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected nor $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to same arc $H$ and $\rho$ at $\bar{x} = 0$.

**Definition 9** ([18])**.** A function $f : X_0 \to R^p$ is called preinvex (with respect to $\eta : X_0 \times X_0 \to R^n$) on $X_0$ if there exists a vector valued function $\eta$ such that,

$$f(u + t\eta(x,u)) \leqq tf(x) + (1-t)f(u)$$

holds for all $x, u \in X_0$ and any $t \in [0,1]$.

**Definition 10** ([18])**.** A function $f : X_0 \to R^p$ is said to be convexlike if for any $x, y \in X_0$ and $0 \leqq \lambda \leqq 1$, there is $z \in X_0$ such that

$$f(z) \leqq \lambda f(x) + (1-\lambda)f(y).$$

*Remark* 2. The convex and the preinvex functions are convexlike functions.

**Lemma 1** ([18])**.** *Let $S$ be a nonempty set in $R^n$ and $\psi : S \to R^p$, a convexlike function. Then either*

$\psi(x) < 0$ *has a solution* $x \in S$

*or*

$\lambda^T \psi(x) \geqq 0$ *for all* $x \in S$, *for some* $\lambda \in R^p, \lambda \geq 0$,

*but both alternatives are never true simultaneously, and the symbol $T$ denotes the transpose of a matrix.*

In the next section we will use the following version of Theorem 3.2 from [18].

**Lemma 2.** *Let $\bar{x} \in X$ be a (local) weak minimum solution for the following problem:*

$$\min\ (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)),$$
$$\text{subject to} \begin{cases} h_j(x) \leqq 0, & j \in \{1, 2, ..., m\}, \\ x \in X_0, \end{cases}$$

*where $\varphi = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)) : X_0 \to R^p$ and $h_1, h_2, \ldots, h_m$ are right upper-Dini-derivative functions of $x$ with respect to the arc $H_{\bar{x},x}(0^+)$. Assume that $h_j$ ($j \in N(\bar{x})$) is a upper semi-continuous function at $\bar{x}$ and $(d\varphi)^+$ $(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ are convexlike functions of $x$ on $X_0$. If $(dh^0)^+(H_{\bar{x},x}(0^+))$ satisfies a regularity condition at $\bar{x}$ [18], then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^m_+$ such that*

$$\lambda^{0^T}(d\varphi)^+(H_{\bar{x},x}(0^+)) + u^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0, \ for\ all\ x \in X_0,$$
$$u^{0^T} h(\bar{x}) = 0,$$
$$\sum_{i=1}^{p} \lambda_i^0 = 1.$$

## 3. Optimality conditions

In this section, we first derive Fritz-John type and Karush-Kuhn-Tucker type necessary conditions for (VFP) and then using the concept of (local) weak optimality, we also establish some sufficient optimality conditions.

**Theorem 1** (Fritz-John type necessary conditions)**.** *Assume that $\bar{x}$ is a (local) weak minimum solution to* (VFP)*. Let $(df)^+(H_{\bar{x},x}(0^+))$, $(dg)^+(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ be convexlike functions of $x$ on $X_0$ and also let $h_j$ be upper semi-continuous at $\bar{x}$ for $j \in N(\bar{x})$. Then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^p_+$ and $v^0 \in R^m_+$, $(\lambda^0, u^0, v^0) \neq 0$ such that*

$$\lambda^{0^T}(df)^+(H_{\bar{x},x}(0^+)) - u^{0^T}(dg)^+(H_{\bar{x},x}(0^+)) + v^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$
$$\text{for all } x \in X_0,$$
$$v^{0^T} h(\bar{x}) = 0.$$

*Proof.* Let $\bar{x}$ be a (local) weak minimum solution for (VFP) and suppose there exists $x^* \in X_0$ such that

$$(df)^+(H_{\bar{x},x^*}(0^+)) < 0, \tag{3.1}$$
$$(dg)^+(H_{\bar{x},x^*}(0^+)) > 0, \tag{3.2}$$
$$(dh^0)^+(H_{\bar{x},x^*}(0^+)) < 0. \tag{3.3}$$

By the relation (3.1) we have

$$\lim_{t \to 0^+} \sup \frac{f_i(H_{\bar{x},x^*}(t)) - f_i(\bar{x})}{t} < 0, \forall\ i \in \{1, 2, ..., p\},$$

which implies that there exists $\delta_i > 0$ such that

$$f_i(H_{\bar{x},x^*}(t)) < f_i(\bar{x}), \text{ for } t \in (0,\delta_i).$$

Similarly (by the relations (3.2), (3.3)), for each $i \in \{1,2,...,p\}$, $j \in M(\bar{x})$ there exist $\xi_i > 0, \sigma_j > 0$ such that

$$g_i(H_{\bar{x},x^*}(t)) > g_i(\bar{x}), \text{ for } t \in (0,\xi_i),$$

$$h^0(H_{\bar{x},x^*}(t)) < h^0(\bar{x}) = 0, \text{ for } t \in (0,\sigma_j).$$

Now, for $j \in N(\bar{x})$, $h_j(\bar{x}) < 0$ and $h_j$ is semi-continuous at $\bar{x}$, then $h_j(H_{\bar{x},x^*}(t))$ is semi-continuous at $t = 0$. Hence, for $\epsilon = \frac{1}{2}h_j(\bar{x}) > 0$, there exists $\sigma_j$ such that

$$h_j(H_{\bar{x},x^*}(t)) < h_j(\bar{x}) + \epsilon = \frac{1}{2}h_j(\bar{x}), \text{ for } t \in (0,\sigma_j).$$

Let $\delta^* = \min\{\delta_i, \xi_i, \sigma_j\}$, then for $t \in (0,\delta^*)$, we have

$$f_i(H_{\bar{x},x^*}(t)) < f_i(\bar{x}), \text{ for } i \in \{1,2,...,p\}, \tag{3.4}$$

$$g_i(H_{\bar{x},x^*}(t)) > g_i(\bar{x}), \text{ for } i \in \{1,2,...,p\}, \tag{3.5}$$

$$h_j(H_{\bar{x},x^*}(t)) < h_j(\bar{x}) = 0, \text{ for } j \in \{1,2,...,m\}. \tag{3.6}$$

Using (3.4) and (3.5), for $\frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_p(x)}{g_p(x)}\right)$ we get $\frac{f(\bar{x})}{g(\bar{x})} > \frac{f(H_{\bar{x},x^*}(t))}{g(H_{\bar{x},x^*}(t))}$, which contradicts the assumption that $\bar{x}$ is a local weak minimum solution of (VFP). By Lemma 1 and the hypothesis that $(df)^+, (dg)^+$ and $(dh^0)^+$ are convexlike functions of $x$ on $X_0$, we obtain the required result. □

For each $u = (u_1, u_2, \ldots, u_p)^T \in R^p_+$, where $R^p_+$ denotes the positive orthant of $R^p$, we consider the parametric problem

$$(\text{VFP}_u) \quad \text{minimize } (f_1(x) - u_1 g_1(x), f_2(x) - u_2 g_2(x), \ldots, f_p(x) - u_p g_p(x)),$$

$$\text{subject to} \begin{cases} h_j(x) \leqq 0, \quad j \in \{1,2,...,m\}, \\ x \in X_0. \end{cases}$$

**Lemma 3** ([14]). *If $\bar{x}$ is a (local) weak minimum to* (VFP)*, then $\bar{x}$ is a (local) weak minimum to* ($\text{VFP}_{u^0}$)*, where $u^0 = \frac{f(\bar{x})}{g(\bar{x})}$.*

Using this lemma we can derive a Karush-Kuhn-Tucker type necessary optimality criterion for the problem (VFP).

**Theorem 2** (Karush-Kuhn-Tucker type necessary conditions)**.** *Assume that $\bar{x}$ is a (local) weak minimum solution to* (VFP)*. Let $(df)^+(H_{\bar{x},x}(0^+))$, $(dg)^+(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ be convexlike functions of $x$ on $X_0$ and also let $h_j$ be upper semi-continuous at $\bar{x}$ for $j \in N(\bar{x})$, further $(dg)^+(H_{\bar{x},x}(0^+))$ satisfies the slater*

*constraint qualification (i.e., there exists $\hat{x} \in X$ such that $(dg)^+(H_{\bar{x},\hat{x}}(0^+)) < 0$). Then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^p_+$ and $v^0 \in R^m_+$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\right) + v^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$for\ all\ x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1.$$

*Proof.* Let $\bar{x}$ be a (local) weak minimum solution for (VFP). According to Lemma 3, we have that $\bar{x}$ is a (local) weak minimum solution for (VFP$_{u^0}$), where $u^0 = (u_1^0, u_2^0, ..., u_p^0)$, $u_i^0 = f_i(\bar{x})/g_i(\bar{x}), i \in \{1, 2, ..., p\}$. Now, applying Lemma 2 to problem (VFP$_{u^0}$), we get that there exist $\lambda^0 \in R^p_+$, $v^0 \in R^m_+$ such that

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\right) + v^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$for\ all\ x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1,$$

and the theorem is proved. □

*Remark* 3. In the above theorem we can suppose, for any $i = 1, 2, ..., p$, that $(df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))$ is convexlike on $X_0$, where $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ instead of considering that $(df_i)^+(H_{\bar{x},x}(0^+))$ and $(dg_i)^+(H_{\bar{x},x}(0^+))$ are convexlike on $X_0$, for any $i = 1, 2, ..., p$.

Now, we establish some sufficient optimality conditions for the problem (VFP) using the concept of (local) weak minimum. As we go on weakening the assumptions of convexity, we get a weaker conclusion of the (weak) efficient solution of (VFP). Let $\alpha : X_0 \times X_0 \to R$, $\bar{\rho}, \hat{\rho} : X_0 \times X_0 \to R^p$ and $\breve{\rho} : X_0 \times X_0 \to R^m$.

**Theorem 3** (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let $\bar{x} \in X$ and $f$ be $(\alpha, \bar{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, $-g$ be $(\alpha, \hat{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, and $h$ be $(\alpha, \breve{\rho})$-right upper-Dini-derivative locally arcwise connected (with*

*respect to $H$) at $\bar{x}$. Assume also that there exist $\lambda^0 \in R^p$, $u^0 \in R^p$ and $v^0 \in R^m$ such that $\alpha(\tilde{x},\bar{x}) > 0$, $\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0$, $\hat{\rho}(\tilde{x},\bar{x}) \geqq 0$ and*

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))\big) + \sum_{j=1}^{m} v_j^0(dh_j)^+(H_{\bar{x},x}(0^+)) \geqq 0, \ \ for\ all\ x \in X, \tag{3.7}$$

$$(dg_i)^+(H_{\bar{x},x}(0^+)) \leqq 0, \ \forall x \in X, \ i = 1,2,...,p, \tag{3.8}$$

$$\sum_{j=1}^{m} v_j^0 h_j(\bar{x}) = 0, \tag{3.9}$$

$$\sum_{i=1}^{p}\lambda_i^0 = 1, \tag{3.10}$$

$$\lambda^0 \geqq 0, u^0 \geqq 0, v^0 \geqq 0. \tag{3.11}$$

*Then $\bar{x}$ is a weak minimum solution to* (VFP).

*Proof.* Suppose contrary to the result. Hence there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad i = 1,2,...,p. \tag{3.12}$$

Since $f$ is $(\alpha,\bar{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, and $h$ is $(\alpha,\breve{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, we get

$$f_i(\tilde{x}) - f_i(\bar{x}) \geqq \alpha(\tilde{x},\bar{x})(df_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \bar{\rho}_i(\tilde{x},\bar{x}), \ i = 1,2,...,p, \tag{3.13}$$

$$h_j(\tilde{x}) - h_j(\bar{x}) \geqq \alpha(\tilde{x},\bar{x})(dh_j)^+(H_{\bar{x},\tilde{x}}(0^+)) + \breve{\rho}_j(\tilde{x},\bar{x}), \ j = 1,2,...,m. \tag{3.14}$$

Multiplying (3.13) by $\lambda_i^0 \geqq 0$, $i \in P$, $\lambda^0 \in R_+^p$, (3.14) by $v_j^0 \geqq 0$, $j = 1,2,...,m$, and then summing the obtained relations, we get

$$\left[\sum_{i=1}^{p}\lambda_i^0(f_i(\tilde{x}) - f_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0(h_j(\tilde{x}) - h_j(\bar{x}))\right]$$

$$\geqq \alpha(\tilde{x},\bar{x})\sum_{i=1}^{p}\lambda_i^0(df_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \alpha(\tilde{x},\bar{x})\sum_{j=1}^{m} v_j^0(dh_j)^+(H_{\bar{x},\tilde{x}}(0^+))$$

$$+\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0,$$

where the last inequality is according to (3.7), $\alpha(\tilde{x},\bar{x}) > 0$ and

$$\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0.$$

Hence

$$\left[\sum_{i=1}^{p}\lambda_i^0(f_i(\tilde{x}) - f_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0(h_j(\tilde{x}) - h_j(\bar{x}))\right] \geqq 0. \tag{3.15}$$

Since $x \in X$, $v^0 \geqq 0$, by (3.9) and (3.15), we get

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(\tilde{x}) - f_i(\bar{x})) \geqq 0. \tag{3.16}$$

Using (3.11) and (3.16), we obtain that there exists $i_0 \in \{1, 2, ..., p\}$ such that

$$f_{i_0}(\tilde{x}) \geqq f_{i_0}(\bar{x}). \tag{3.17}$$

Since $-g$ is $(\alpha, \hat{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, we get

$$-g_i(\tilde{x}) + g_i(\bar{x}) \geqq -\alpha(\tilde{x}, \bar{x})(dg_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \hat{\rho}_i(\tilde{x}, \bar{x}),\ i = 1, 2, ..., p,$$

equivalently,

$$g_i(\tilde{x}) - g_i(\bar{x}) \leqq \alpha(\tilde{x}, \bar{x})(dg_i)^+(H_{\bar{x},\tilde{x}}(0^+)) - \hat{\rho}_i(\tilde{x}, \bar{x}),\ i = 1, 2, ..., p. \tag{3.18}$$

By (3.8), (3.18) and using $\alpha(\tilde{x}, \bar{x}) > 0$, $\hat{\rho}(\tilde{x}, \bar{x}) \geqq 0$, it follows

$$g_i(\tilde{x}) \leqq g_i(\bar{x}), \quad i = 1, 2, ..., p. \tag{3.19}$$

Now, using (3.17), (3.19), $f \geqq 0$ and $g > 0$, we obtain

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geqq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which contradicts (3.12). Thus $\bar{x}$ is a weak minimum solution to (VFP). □

**Theorem 4** (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let $\bar{x} \in X$ and let $\sum_{i=1}^{p} \lambda_i^0 [f_i(.) - u_i^0 g_i(.)] + \sum_{j=1}^{m} v_j^0 h_j(.)$ be $(\alpha, \rho)$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$. Assume also that there exist $\lambda^0 \in R^p$, $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1, 2, ..., p$, and $v^0 \in R^m$ such that $\alpha(x, \bar{x}) > 0$, $\rho(x, \bar{x}) \geqq 0$,*

$$\sum_{i=1}^{p} \lambda_i^0 \big((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\big)$$

$$+ \sum_{j=1}^{m} v_j^0 (dh_j)^+(H_{\bar{x},x}(0^+)) \geqq 0, \quad \forall x \in X, \tag{3.20}$$

$$\sum_{j=1}^{m} v_j^0 h_j(\bar{x}) = 0, \tag{3.21}$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1, \tag{3.22}$$

$$\lambda^0 \geqq 0,\ u^0 \geqq 0,\ v^0 \geqq 0. \tag{3.23}$$

*Then $\bar{x}$ is a weak minimum solution to* (VFP).

*Proof.* Suppose contrary to the result. Hence there exists a $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \text{ for any } i = 1,2,\ldots,p.$$

That is,

$$f_i(x) < u_i^0 g_i(x), \text{ for any } i = 1,2,\ldots,p. \tag{3.24}$$

From the $(\alpha,\rho)$-right upper-Dini-derivative locally arcwise connected (with respect to $H$) of $\sum_{i=1}^{p} \lambda_i^0[f_i(.) - u_i^0 g_i(.)] + \sum_{j=1}^{m} v_j^0 h_j(.)$ at $\bar{x}$, we have

$$\left[\sum_{i=1}^{p} \lambda_i^0(f_i(x) - f_i(\bar{x}))\right] - \left[\sum_{i=1}^{p} \lambda_i^0 u_i^0(g_i(x) - g_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0(h_j(x) - h_j(\bar{x}))\right]$$
$$\geqq \alpha(x,\bar{x})\left[\sum_{i=1}^{p} \lambda_i^0\Big((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))\Big)\right.$$
$$\left. + \sum_{j=1}^{m} v_j^0(dh_j)^+(H_{\bar{x},x}(0^+))\right] + \rho(x,\bar{x}) \geqq 0,$$

where the last inequality is according to (3.20), $\alpha(\tilde{x},\bar{x}) > 0$ and $\rho(x,\bar{x}) \geqq 0$. Hence,

$$\left[\sum_{i=1}^{p} \lambda_i^0(f_i(x) - f_i(\bar{x}))\right] - \left[\sum_{i=1}^{p} \lambda_i^0 u_i^0(g_i(x) - g_i(\bar{x}))\right]$$
$$+ \left[\sum_{j=1}^{m} v_j^0(h_j(x) - h_j(\bar{x}))\right] \geqq 0. \tag{3.25}$$

Now, $x \in X$, (3.21) and (3.25), gives

$$\sum_{i=1}^{p} \lambda_i^0(f_i(x) - f_i(\bar{x})) - \sum_{i=1}^{p} \lambda_i^0 u_i^0(g_i(x) - g_i(\bar{x})) \geqq 0.$$

That is,

$$\sum_{i=1}^{p} \lambda_i^0((f_i(x) - u_i^0 g_i(x)) - (f_i(\bar{x}) - u_i^0 g_i(\bar{x}))) \geqq 0.$$

Since $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1,2,\ldots,p$, we obtain

$$\sum_{i=1}^{p} \lambda_i^0(f_i(x) - u_i^0 g_i(x)) \geqq 0.$$

From $\lambda^0 \geqq 0$, $\sum_{i=1}^{p} \lambda_i^0 = 1$, we obtain that there exists $i_0 \in \{1,2,\ldots,p\}$ such that

$$f_{i_0}(x) - u_{i_0}^0 g_{i_0}(x) \geqq 0,$$

That is,

$$f_{i_0}(x) \geqq u^0_{i_0} g_{i_0}(x), \text{ for some } i_0 \in \{1,2,...,p\},$$

which contradicts (3.24). Thus $\bar{x}$ is a weak minimum solution for (VFP). □

**Theorem 5** (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let $\bar{x} \in X$, $\lambda^0 \in R^p$, $u^0_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1,2,...,p$ and $v^0 \in R^m$ such that the conditions (3.20)-(3.23) of Theorem 3.4 hold. Assume also that for any $i = 1,2,...,p$, $f_i(.) - u^0_i g_i(.)$ is $\bar{\rho}$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, and for any $j \in M(\bar{x})$, $h_j(.)$ is $\breve{\rho}$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to $H$) at $\bar{x}$, further the inequality $\sum_{i=1}^p \lambda^0_i \bar{\rho}_i(x,\bar{x}) + \sum_{j=1}^m v^0_j \breve{\rho}_j(x,\bar{x}) \geqq 0$ holds. Then $\bar{x}$ is a weak minimum solution to* (VFP).

*Proof.* Suppose contrary to the result. Hence there exists $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \text{ for any } i = 1,2,...,p.$$

That is,

$$f_i(x) - u^0_i g_i(x) < 0, \text{ for any } i = 1,2,...,p,$$

which is equivalent to

$$f_i(x) - u^0_i g_i(x) < f_i(\bar{x}) - u^0_i g_i(\bar{x}), \text{ for any } i = 1,2,...,p.$$

Now, by the $\bar{\rho}$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to $H$) of $f_i(.) - u^0_i g_i(.)$, $i = 1,2,...,p$, at $\bar{x}$, we get

$$(df_i)^+(H_{\bar{x},x}(0^+)) - u^0_i (dg_i)^+(H_{\bar{x},x}(0^+)) < -\bar{\rho}_i(x,\bar{x}), \text{ for any } i = 1,2,...,p.$$

Using $\lambda^0_i \in R^p_+$, $\sum\limits_{i=1}^p \lambda^0_i = 1$, we obtain

$$\sum_{i=1}^p \lambda^0_i \big((df_i)^+(H_{\bar{x},x}(0^+)) - u^0_i (dg_i)^+(H_{\bar{x},x}(0^+))\big) < -\sum_{i=1}^p \lambda^0_i \bar{\rho}_i(x,\bar{x}). \tag{3.26}$$

For $x \in X$ we have $h_j(x) \leqq 0$. But for $j \in M(\bar{x})$, $h_j(\bar{x}) = 0$. Hence

$$h_j(x) \leqq h_j(\bar{x}), \text{ for any } j \in M(\bar{x}),$$

which by using the $\breve{\rho}$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to $H$) of $h_j(.)$, $j \in M(\bar{x})$ at $\bar{x}$, we get

$$(dh_j)^+(H_{\bar{x},x}(0^+)) \leqq -\breve{\rho}_j(x,\bar{x}), \ \forall j \in M(\bar{x}).$$

But $v^0 \in R^m_+$ and $v^0_j = 0$ for $j \in N(\bar{x})$, we obtain

$$\sum_{j=1}^m v^0_j (dh_j)^+(H_{\bar{x},x}(0^+)) \leqq -\sum_{j=1}^m v^0_j \breve{\rho}_j(x,\bar{x}). \tag{3.27}$$

On adding (3.26), (3.27) and using $\sum_{i=1}^{p}\lambda_i^0\bar{\rho}_i(x,\bar{x})+\sum_{j=1}^{m}v_j^0\breve{\rho}_j(x,\bar{x})\geqq 0$, we get

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))-u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))\big)$$
$$+\sum_{j=1}^{m}v_j^0(dh_j)^+(H_{\bar{x},x}(0^+))<-\left(\sum_{i=1}^{p}\lambda_i^0\bar{\rho}_i(x,\bar{x})+\sum_{j=1}^{m}v_j^0\breve{\rho}_j(x,\bar{x})\right)\leqq 0.$$

That is,

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))-u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))\big)$$
$$+\sum_{j=1}^{m}v_j^0(dh_j)^+(H_{\bar{x},x}(0^+))<0,$$

which contradicts (3.20). Thus $\bar{x}$ is a weak minimum to (VFP). □

## 4. Duality

We consider, for (VFP), a general Mond-Weir dual (FMWD) as

$$\max\ \Psi(y,\lambda,u,v)=u-v_{I_0}^T h_{I_0}(y)e,$$

subject to

$$\sum_{i=1}^{p}\lambda_i\big((df_i)^+(H_{y,x}(0^+))-u_i(dg_i)^+(H_{y,x}(0^+))\big)+v^T(dh)^+(H_{y,x}(0^+))\geqq 0,$$
$$\text{for all } x\in X, \tag{4.1}$$
$$f_i(y)-u_ig_i(y)\geqq 0,\ \text{for any } i=1,2,\dots,p, \tag{4.2}$$
$$v_{I_s}^T h_{I_s}(y)\geqq 0,\ (1\leqq s\leqq\gamma), \tag{4.3}$$
$$\lambda^T e=1,\ \lambda\geqq 0,\ \lambda\in R^p, \tag{4.4}$$
$$u\geqq 0,\ u\in R^p,\ v\geqq 0,\ y\in X_0, \tag{4.5}$$

where $e$ is $p$-tuple of $1'$s, $\gamma\geqq 1$, $I_s\cap I_t=\emptyset$ for $s\neq t$ and $\bigcup_{s=0}^{\gamma}I_s=\{1,2,\dots,m\}$, $v_{I_s}=(v_j)_{j\in I_s}$ and $h_{I_s}=(h_j)_{j\in I_s}$.

Let $W$ denote the set of all feasible solutions to (FMWD). Also, we define the following sets

$$A=\{(\lambda,u,v)\in R^p\times R^p\times R^m:(y,\lambda,u,v)\in W \text{ for some } y\in X_0\}$$

and for $(\lambda,u,v)\in A$,

$$B(\lambda,u,v)=\big\{y\in X_0:(y,\lambda,u,v)\in W\big\}.$$

We put $B = \bigcup_{(\lambda,u,v)\in A} B(\lambda,u,v)$ and note that $B \subset X_0$. Also, we note that if $(y,\lambda,u,v) \in W$ then $(\lambda,u,v) \in A$ and $y \in B(\lambda,u,v)$.

Now we establish certain duality results between (VFP) and (FMWD). Assume that $f$, $g$ and $h$ are right upper-Dini-derivative functions of $x$ with respect to the arc $H$ on $X$.

**Theorem 6** (Weak duality). *Let $x \in X$ and $(y,\lambda,u,v) \in W$ be feasible solutions for* (VFP) *and* (FMWD) *respectively. Assume also that $\sum_{i=1}^{p} \lambda_i \big(f_i(.) - u_i g_i(.)\big) + v_{I_0}^T h_{I_0}(.)$ is $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected at $y$ on $B(\lambda,u,v)$ and for $1 \leqq s \leqq \gamma$, $v_{I_s}^T h_{I_s}(.)$ are $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected at $y$ on $B(\lambda,u,v)$, further $\rho(x,y) + \sum_{s=1}^{\gamma} \breve{\rho}_s(x,y) \geqq 0$. Then the following cannot hold:*

$$f_i(x) - u_i g_i(x) \leqq v_{I_0}^T h_{I_0}(y) \ \ for\ any\ i = 1,2,...,p, \tag{4.6}$$

*and*

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{I_0}^T h_{I_0}(y) \ \ for\ some\ i_0 \in \{1,2,...,p\}. \tag{4.7}$$

*Proof.* Using the feasibility of $x$ for (VFP) and of $(y,\lambda,u,v)$ for (FMWD), we have

$$v_{I_S}^T h_{I_S}(x) \leqq v_{I_S}^T h_{I_S}(y),\ (1 \leqq s \leqq \gamma),$$

which by using $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected of $v_{I_s}^T h_{I_s}(.)$, for $1 \leqq s \leqq \gamma$ at $y$ on $B(\lambda,u,v)$, we obtain

$$\sum_{j\in I_s} v_j (dh_j)^+ (H_{y,x}(0^+)) \leqq -\breve{\rho}_s(x,y),\ 1 \leqq s \leqq \gamma. \tag{4.8}$$

Now, we suppose to the contrary of the result of the theorem that (4.6) and (4.7) holds. Then by (4.4), we get

$$\sum_{i=1}^{p} \lambda_i \big(f_i(x) - u_i g_i(x)\big) < v_{I_0}^T h_{I_0}(y). \tag{4.9}$$

On the other hand, by using (4.2), (4.4), (4.5) and the feasibility of $x$ for (VFP), we have

$$v_{I_0}^T h_{I_0}(x) \leqq 0 \leqq \sum_{i=1}^{p} \lambda_i \big(f_i(y) - u_i g_i(y)\big). \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\sum_{i=1}^{p} \lambda_i \big(f_i(x) - u_i g_i(x)\big) + v_{I_0}^T h_{I_0}(x) < \sum_{i=1}^{p} \lambda_i \big(f_i(y) - u_i g_i(y)\big) + v_{I_0}^T h_{I_0}(y),$$

which by using $\rho$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected of $\sum_{i=1}^{p} \lambda_i \big(f_i(.) - u_i g_i(.)\big) + v_{I_0}^T h_{I_0}(.)$ at $y$ on $B(\lambda, u, v)$, we obtain

$$\sum_{i=1}^{p} \lambda_i \big((df_i)^+(H_{y,x}(0^+)) - u_i (dg_i)^+(H_{y,x}(0^+))\big) + \sum_{j \in I_0} v_j (dh_j)^+(H_{y,x}(0^+)) < -\rho(x, y).$$

Now, by (4.1) and $\rho(x, y) + \sum_{s=1}^{\gamma} \breve{\rho}_s(x, y) \geqq 0$, we obtain

$$\sum_{s=1}^{\gamma} \sum_{j \in I_s} v_j (dh_j)^+(H_{y,x}(0^+)) > \rho(x, y) \geqq -\sum_{s=1}^{\gamma} \breve{\rho}_s(x, y).$$

That is,

$$\sum_{s=1}^{\gamma} \sum_{j \in I_s} v_j (dh_j)^+(H_{y,x}(0^+)) > -\sum_{s=1}^{\gamma} \breve{\rho}_s(x, y),$$

which contradicts (4.8). Thus, the theorem is proved. □

**Theorem 7** (Strong duality)**.** *Let $\bar{x}$ be a (local) weak minimum solution for* (VFP)*. Assume that the hypotheses of Theorem 2 hold. Then, there exist $(\bar{\lambda}^0, \bar{u}^0, \bar{v}^0) \in R^p \times R^p \times R^m$ such that $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is feasible for dual* (FMWD) *and the two objectives are equal. Also, if the weak duality Theorem 6 holds for all feasible solutions of the problems* (VFP) *and* (FMWD)*, then $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is a (local) weak minimum solution of* (FMWD)*.*

*Proof.* Since $\bar{x}$ is a (local) weak minimum solution of (VFP) and all the assumptions of Theorem 2 are satisfied, therefore, there exist $\bar{\lambda}^0 \in R_+^p$, $\bar{u}^0 \in R_+^p$ and $\bar{v}^0 \in R_+^m$ such that

$$\sum_{i=1}^{p} \bar{\lambda}_i^0 \big((df_i)^+(H_{\bar{x},x}(0^+)) - \bar{u}_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\big) + \bar{v}^{0^T} (dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$\text{for all } x \in X_0,$$

$$\bar{v}^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \bar{\lambda}_i^0 = 1,$$

where $\bar{u}_i^0 = f_i(\bar{x})/g_i(\bar{x}), i = 1, 2, ..., p$. Thus, $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is feasible for (FMWD) and the two objectives are equal. The (local) weak minimum solution of $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ for (FMWD) follows from weak duality Theorem 6. This completes the proof. □

**Theorem 8** (Strict converse duality)**.** *Let $\bar{x}$ and $(\bar{y},\bar{\lambda},\bar{u},\bar{v})$ be (local) weak minimum solutions for* (VFP) *and* (FMWD)*, respectively such that $\bar{u}_i^0 = f_i(\bar{x})/g_i(\bar{x}), i = 1,2,\ldots,p$ and*

$$\sum_{i=1}^{p} \bar{\lambda}_i \big(f_i(\bar{x}) - \bar{u}_i g_i(\bar{x})\big) \leqq \bar{v}_{I_0}^T h_{I_0}(\bar{y}). \tag{4.11}$$

*Assume also that $\sum_{i=1}^{p} \bar{\lambda}_i \big(f_i(.) - \bar{u}_i g_i(.)\big) + \bar{v}_{I_0}^T h_{I_0}(.)$ is strictly $\rho$-generalized pseudo-right upper-Dini-derivative locally arcwise connected at $\bar{y}$ on $B(\bar{\lambda},\bar{u},\bar{v})$ and for $1 \leqq s \leqq \gamma$, $\bar{v}_{I_s}^T h_{I_s}(.)$ are $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected at $\bar{y}$ on $B(\bar{\lambda},\bar{u},\bar{v})$, further the inequality $\rho(\bar{x},\bar{y}) + \sum_{s=1}^{\gamma} \breve{\rho}_s(\bar{x},\bar{y}) \geqq 0$ holds. Then $\bar{x} = \bar{y}$; that is, $\bar{y}$ is an efficient solution for* (VFP)*.*

*Proof.* Suppose that $\bar{x} \neq \bar{y}$. Using the feasibility of $\bar{x}$ for (VFP) and of $(\bar{y},\bar{\lambda},\bar{u},\bar{v})$ for (FMWD), we have

$$\bar{v}_{I_S}^T h_{I_S}(\bar{x}) \leqq \bar{v}_{I_S}^T h_{I_S}(\bar{y}),\ (1 \leqq s \leqq \gamma),$$

which by using $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected of $\bar{v}_{I_s}^T h_{I_s}(.)$, for $1 \leqq s \leqq \gamma$ at $\bar{y}$ on $B(\bar{\lambda},\bar{u},\bar{v})$, we obtain

$$\sum_{j \in I_s} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \leqq -\breve{\rho}_s(\bar{x},\bar{y}),\ 1 \leqq s \leqq \gamma. \tag{4.12}$$

Now, by (4.1) and (4.12), we get

$$\begin{aligned}
0 &\leqq \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+))\big) + \bar{v}^T (dh)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&= \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+))\big) + \sum_{j \in I_0} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\qquad + \sum_{s=1}^{\gamma} \sum_{j \in I_s} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\leqq \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+))\big) + \sum_{j \in I_0} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\qquad - \sum_{s=1}^{\gamma} \breve{\rho}_s(\bar{x},\bar{y}).
\end{aligned}$$

Since, $\rho(\bar{x},\bar{y})+\sum_{s=1}^{\gamma}\breve{\rho}_s(\bar{x},\bar{y})\geqq 0$, we have

$$\sum_{i=1}^{p}\bar{\lambda}_i\left((df_i)^+(H_{\bar{y},\bar{x}}(0^+))-\bar{u}_i(dg_i)^+(H_{\bar{y},\bar{x}}(0^+))\right)$$
$$+\sum_{j\in I_0}\bar{v}_j(dh_j)^+(H_{\bar{y},\bar{x}}(0^+))\geqq -\rho(\bar{x},\bar{y}),$$

which by using strictly $\rho$-generalized pseudo-right upper-Dini-derivative locally arc-wise connected of $\sum_{i=1}^{p}\bar{\lambda}_i\left(f_i(.)-\bar{u}_i g_i(.)\right)+\bar{v}_{I_0}^T h_{I_0}(.)$ at $\bar{y}$ on $B(\bar{\lambda},\bar{u},\bar{v})$, we obtain

$$\sum_{i=1}^{p}\bar{\lambda}_i\left(f_i(\bar{x})-\bar{u}_i g_i(\bar{x})\right)+\bar{v}_{I_0}^T h_{I_0}(\bar{x})>\sum_{i=1}^{p}\bar{\lambda}_i\left(f_i(\bar{y})-\bar{u}_i g_i(\bar{y})\right)+\bar{v}_{I_0}^T h_{I_0}(\bar{y}).$$

Using (4.2), (4.4), (4.5) and the feasibility of $\bar{x}$ for (VFP), the above inequality yields

$$\sum_{i=1}^{p}\bar{\lambda}_i\left(f_i(\bar{x})-\bar{u}_i g_i(\bar{x})\right)>\bar{v}_{I_0}^T h_{I_0}(\bar{y}),$$

which contradicts (4.11). Thus, the theorem is proved. □

## 5. Conclusion

In this paper, we established necessary and sufficient optimality conditions and duality results for a class of multiobjective fractional programming problem under generalized convexity using right-upper Dini derivative. The methods used here can be extended to the study of nonsmooth variational and nonsmooth control problems, which will orient the future research of the author.

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# SOME NEW PARANORMED DIFFERENCE SEQUENCE SPACES DERIVED BY FIBONACCI NUMBERS

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*Abstract.* In this study, we define new paranormed sequence spaces derived by the sequences of Fibonacci numbers. Furthermore, we compute the $\alpha-, \beta-$ and $\gamma-$ duals and obtain bases for these sequence spaces. Besides this, we characterize the matrix transformations from the new paranormed sequence spaces to the Maddox's spaces $c_0(q), c(q), \ell(q)$ and $\ell_\infty(q)$.

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## 1. Preliminaries, background and notation

By $\omega$, we shall denote the space of all real valued sequences. Any vector subspace of $\omega$ is called as a *sequence space*. We shall write $\ell_\infty, c$ and $c_0$ for the spaces of all bounded, convergent and null sequences, respectively. Also by $bs, cs, \ell_1$ and $\ell_p$; we denote the spaces of all bounded, convergent, absolutely and $p-$ absolutely convergent series, respectively; where $1 < p < \infty$.

A linear topological space $X$ over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha'$s in $\mathbb{R}$ and all $x$'s in $X$, where $\theta$ is the zero vector in the linear space $X$.

Assume here and after that $(p_k)$ be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $c(p), c_0(p)$, $\ell_\infty(p)$ and $\ell(p)$ were defined by Maddox [20,21] (see also Simons [27] and Nakano [25]) as follows:

$$c(p) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C} \right\},$$

$$c_0(p) = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\},$$

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$$\ell_\infty(p) = \left\{ x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty \right\}$$

and

$$\ell(p) = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$h_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M} \text{ iff } \inf_{k\in\mathbb{N}} p_k > 0 \qquad \text{and} \qquad h_2(x) = \left( \sum_k |x_k|^{p_k} \right)^{1/M},$$

respectively. We shall assume throughout that $p_k^{-1} + (p_k^{'})^{-1} = 1$ provided $1 < \inf p_k < H < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. We write by $\mathcal{U}$ for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in \mathcal{U}$, let $1/u = (1/u_n)$.

For the sequence spaces $X$ and $Y$, define the set $S(X,Y)$ by

$$S(X,Y) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in Y \ \text{ for all } \ x \in X\}. \tag{1.1}$$

With the notation of (1.1), the $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space $X$, which are respectively denoted by $X^\alpha, X^\beta$ and $X^\gamma$, are defined by

$$X^\alpha = S(X,\ell_1), \ \ X^\beta = S(X,cs) \text{ and } \ X^\gamma = S(X,bs).$$

Let $(X,h)$ be a paranormed space. A sequence $(b_k)$ of the elements of $X$ is called a basis for $X$ if and only if, for each $x \in X$, there exists a unique sequence $(\alpha_k)$ of scalars such that

$$h\left( x - \sum_{k=0}^{n} \alpha_k b_k \right) \to 0 \ \ as \ \ n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$ and written as $x = \sum \alpha_k b_k$.

Let $X, Y$ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers $a_{nk}$, where $n,k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $X$ into $Y$, and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = ((Ax)_n)$, the $A$-transform of $x$, is in $Y$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \ \ (n \in \mathbb{N}). \tag{1.2}$$

By $(X : Y)$, we denote the class of all matrices $A$ such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right-hand side of (1.2) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$. A

sequence $x$ is said to be $A$- summable to $\alpha$ if $Ax$ converges to $\alpha$ which is called as the $A$- limit of $x$.

For a sequence space $X$, the matrix domain $X_A$ of an infinite matrix $A$ is defined by

$$X_A = \{x = (x_k) \in \omega : \quad Ax \in X\}. \tag{1.3}$$

The approach constructing a new paranormed sequence space by means of the matrix domain of a particular limitation method has recently been employed by Malkowsky [23], Et and Çolak [15], Malkowsky and Savaş [24], Altay and Başar [3,4], F. Başar et al., [9], Aydın and Başar [5,6].

Define the sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers given by the linear recurrence relations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}, \quad n \geq 2.$$

In modern science and particularly physics, there is quite an interest in the theory and applications of Fibonacci numbers. The ratio of the successive Fibonacci numbers is as known golden ratio. There are many applications of golden ratio in many places of mathematics and physics, in general theory of high energy particle theory [26]. Also, some basic properties of Fibonacci numbers [19] are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \qquad \text{(golden ratio)}$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \qquad (n \in \mathbb{N}) \quad \text{and} \quad \sum_k \frac{1}{f_k} \quad \text{converges}$$

$$f_{n-1} f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \quad \text{(Cassini formula)}.$$

Substituting for $f_{n+1}$ in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Let $f_n$ be the $n$th Fibonacci number for every $n \in \mathbb{N}$. Then, the infinite Fibonacci matrix $\widehat{F} = (\widehat{f}_{nk})$ is defined by

$$\widehat{f}_{nk} = \begin{cases} -\dfrac{f_{n+1}}{f_n} & (k = n-1), \\ \dfrac{f_n}{f_{n+1}} & (k = n), \\ 0 & (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

where $n, k \in \mathbb{N}$ [17]. Recently, a lot of papers have been studying by many researchers on Fibonacci sequences. For instance, see [11–13].

The main purpose of this study is to introduce the sequence spaces $c_0(\widehat{F}, p)$, $c(\widehat{F}, p)$, $\ell_\infty(\widehat{F}, p)$ and $\ell(\widehat{F}, p)$ which are the set of all sequences whose $\widehat{F}-$trans forms are in the spaces $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$, respectively. Also, we have investigated some topological structures, which have completeness, the $\alpha-, \beta-$ and $\gamma-$ duals, and the bases of these sequence spaces. Besides this, we characterize some matrix mappings on these spaces.

## 2. The paranormed Fibonacci difference sequence spaces

In this section, we define the new sequence spaces $c_0(\widehat{F},p), c(\widehat{F},p)$, $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ by using the sequences of Fibonacci numbers, and prove that these sequence spaces are the complete paranormed linear metric spaces and compute their $\alpha-,\beta-$ and $\gamma-$ duals. Moreover, we give the basis for the spaces $c_0(\widehat{F},p), c(\widehat{F},p)$ and $\ell(\widehat{F},p)$.

In [14], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S$-transforms are in $\ell(p)$, where $S=(s_{nk})$ is defined by

$$s_{nk}=\begin{cases} 1, & (0\le k\le n),\\ 0, & (k>n).\end{cases}$$

Başar and Altay [8] have recently examined the space $bs(p)$ which is formerly defined by Başar in [7] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$. More recently, Altay and Başar have studied the sequence spaces $r^t(p), r^t_\infty(p)$ in [1] and $r^t_c(p), r^t_0(p)$ in [2] which are derived by the Riesz means from the sequence spaces $\ell(p), \ell_\infty(p), c(p)$ and $c_0(p)$ of Maddox, respectively. With the notation of (1.3), the spaces $\overline{\ell(p)}, bs(p), r^t(p), r^t_\infty(p), r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)}=[\ell(p)]_S,\ \ bs(p)=[\ell_\infty(p)]_S,\ \ r^t(p)=[\ell(p)]_{R^t},$$

$$r^t_\infty(p)=[\ell_\infty(p)]_{R^t},\ \ r^t_c(p)=[c(p)]_{R^t},\ \ r^t_0(p)=[c_0(p)]_{R^t}.$$

Following Choudhary and Mishra [14], Başar and Altay [8], Altay and Başar [1,2], we define the sequence spaces $c_0(\widehat{F},p), c(\widehat{F},p)$, $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ by

$$c_0(\widehat{F},p)=\left\{x=(x_k)\in\omega:\lim_{n\to\infty}\left|\frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1}\right|^{p_n}=0\right\}$$

$$c(\widehat{F},p)=\left\{x=(x_k)\in\omega:\exists l\in\mathbb{C}\ni\lim_{n\to\infty}\left|\frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1}-l\right|^{p_n}=0\right\}$$

$$\ell_\infty(\widehat{F},p)=\left\{x=(x_k)\in\omega:\sup_{n\in\mathbb{N}}\left|\frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1}\right|^{p_n}<\infty\right\}$$

and

$$\ell(\widehat{F},p)=\left\{x=(x_k)\in\omega:\sum_n\left|\frac{f_n}{f_{n+1}}x_n-\frac{f_{n+1}}{f_n}x_{n-1}\right|^{p_n}<\infty\right\}.$$

In the case $(p_k)=e=(1,1,1,...)$, the sequence spaces $c_0(\widehat{F},p), c(\widehat{F},p)$, $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ are , respectively, reduced to the sequence spaces $c_0(\widehat{F}), c(\widehat{F})$, $\ell_\infty(\widehat{F})$ and $\ell_p(\widehat{F})$ which are introduced by E.E.Kara [17] and M. Başarır et al. [10].

With the notation (1.3), we may redefine the spaces $c_0(\widehat{F},p), c(\widehat{F},p)$, $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ as follows:

$$c_0(\widehat{F},p)=\{c_0(p)\}_{\widehat{F}},\quad c(\widehat{F},p)=\{c(p)\}_{\widehat{F}},$$

$$\ell_\infty(\widehat{F},p)=\{\ell_\infty(p)\}_{\widehat{F}},\quad \ell(\widehat{F},p)=\{\ell(p)\}_{\widehat{F}}.$$

Define the sequence $y=(y_k)$, which will be frequently used as the $\widehat{F}-$transform of a sequence $x=(x_k)$, i.e.,

$$y_k=\widehat{F}_k(x)=\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1};\quad (k\in\mathbb{N}_0). \tag{2.1}$$

Since the proof may also be obtained in the similar way as for the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces. Now, we may begin with the following theorem which is essential in the study.

**Theorem 1.** *(i) The sequence spaces $c_0(\widehat{F},p), c(\widehat{F},p)$ and $\ell_\infty(\widehat{F},p)$ are the complete linear metric spaces paranormed by $g$, defined by*

$$g(x)=\sup_{k\in\mathbb{N}}\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|^{p_k/M}.$$

*$g$ is a paranorm for the spaces $c(\widehat{F},p)$ and $\ell_\infty(\widehat{F},p)$ only in the trivial case* $\inf p_k>0$ *when $c(\widehat{F},p)=c(\widehat{F})$ and $\ell_\infty(\widehat{F},p)=\ell_\infty(\widehat{F})$.*
*(ii) $\ell_p(\widehat{F})$ is a complete linear metric space paranormed by*

$$g^*(x)=\left(\sum_k\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|^{p_k}\right)^{1/M}.$$

*Proof.* We prove the theorem for the space $c_0(\widehat{F},p)$. The linearity of $c_0(\widehat{F},p)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x,z\in c_0(\widehat{F},p)$ (see [22, p.30]):

$$\begin{aligned}&\sup_{k\in\mathbb{N}}\left|\frac{f_k}{f_{k+1}}(x_k+z_k)-\frac{f_{k+1}}{f_k}(x_{k-1}+z_{k-1})\right|^{p_k/M}\\&\le\sup_{k\in\mathbb{N}}\left|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\right|^{p_k/M}+\sup_{k\in\mathbb{N}}\left|\frac{f_k}{f_{k+1}}z_k-\frac{f_{k+1}}{f_k}z_{k-1}\right|^{p_k/M}\end{aligned} \tag{2.2}$$

and for any $\alpha\in\mathbb{R}$ (see [20]),

$$|\alpha|^{p_k}\le\max\{1,|\alpha|^M\}. \tag{2.3}$$

It is clear that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x\in c_0(\widehat{F},p)$. Again the inequalities (2.2) and (2.3) yield the subadditivity of $g$ and

$$g(\alpha x)\leq \max\{1,|\alpha|\}g(x).$$

Let $\{x^n\}$ be any sequence of the points $x^n\in c_0(\widehat{F},p)$ such that $g(x^n-x)\to 0$ and $(\alpha_n)$ also be any sequence of scalars such that $\alpha_n\to\alpha$. Then, since the inequality

$$g(x^n)\leq g(x)+g(x^n-x)$$

holds by the subadditivity of $g$, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha_n x^n-\alpha x) &= \sup_{k\in\mathbb{N}}\left|\frac{f_k}{f_{k+1}}(\alpha_n x_k^n-\alpha x_k)-\frac{f_{k+1}}{f_k}(\alpha_n x_{k-1}^n-\alpha x_{k-1})\right|^{p_k/M}\\ &\leq |\alpha_n-\alpha|\ \ g(x^n)+|\alpha|\ \ g(x^n-x), \end{aligned}$$

which tends to zero as $n\to\infty$. That is to say that the scalar multiplication is continuous. Hence, $g$ is a paranorm on the space $c_0(\widehat{F},p)$.

It remains to prove the completeness of the space $c_0(\widehat{F},p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $c_0(\widehat{F},p)$, where $x^i=\{x_0^{(i)},x_1^{(i)},\ldots\}$. Then, for a given $\varepsilon>0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i-x^j)<\frac{\varepsilon}{2}$$

for all $i,j\geq n_0(\varepsilon)$. We obtain by using definition of $g$ for each fixed $k\in\mathbb{N}$ that

$$\left|\{\widehat{F}x^i\}_k-\{\widehat{F}x^j\}_k\right|^{p_k/M}\leq \sup_{k\in\mathbb{N}}\left|\{\widehat{F}x^i\}_k-\{\widehat{F}x^j\}_k\right|^{p_k/M}<\frac{\varepsilon}{2} \tag{2.4}$$

for every $i,j\geq n_0(\varepsilon)$, which leads us to the fact that $\{(\widehat{F}x^0)_k,\ (\widehat{F}x^1)_k,\ldots\}$ is a Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say

$$\{\widehat{F}x^i\}_k\to\{\widehat{F}x\}_k$$

as $i\to\infty$. Using these infinitely many limits $(\widehat{F}x)_0,(\widehat{F}x)_1,\ldots$, we define the sequence $\{(\widehat{F}x)_0,(\widehat{F}x)_1,\ldots\}$. We have from (2.4) with $j\to\infty$ that

$$\left|\{\widehat{F}x^i\}_k-\{\widehat{F}x\}_k\right|^{p_k/M}\leq\frac{\varepsilon}{2}\quad (i\geq n_0(\varepsilon)) \tag{2.5}$$

for every fixed $k\in\mathbb{N}$. Since $x^i=\{x_k^{(i)}\}\in c_0(\widehat{F},p)$,

$$\left|\{\widehat{F}x^i\}_k\right|^{p_k/M}<\frac{\varepsilon}{2}$$

for all $k\in\mathbb{N}$. Therefore, we obtain (2.5) that

$$\begin{aligned} \left|\{\widehat{F}x\}_k\right|^{p_k/M} &\leq \left|\{\widehat{F}x\}_k-\{\widehat{F}x^i\}_k\right|^{p_k/M}+\left|\{\widehat{F}x^i\}_k\right|^{p_k/M}\\ &<\varepsilon\quad (i\geq n_0(\varepsilon)). \end{aligned}$$

This shows that the sequence $\{\widehat{F}x\}$ belongs to the space $c_0(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $c_0(\widehat{F},p)$ is complete and this concludes the proof. □

Therefore, one can easily check that the absolute property does not hold on the spaces $c_0(\widehat{F},p), c(\widehat{F},p)$ , $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ that is $h(x) \neq h(|x|)$ for at least one sequence in those spaces, and this says that $c_0(\widehat{F},p), c(\widehat{F},p)$, $\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ are the sequence spaces of non-absolute type; where $|x| = (|x_k|)$.

**Theorem 2.** *The sequence spaces $c_0(\widehat{F},p), c(\widehat{F},p)$ ,$\ell_\infty(\widehat{F},p)$ and $\ell(\widehat{F},p)$ are linearly isomorphic to the spaces $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

*Proof.* We establish this for the space $\ell_\infty(\widehat{F},p)$. To prove the theorem, we should show the existence of a linear bijection between the spaces $\ell_\infty(\widehat{F},p)$ and $\ell_\infty(p)$ for $0 < p_k \leq H < \infty$. With the notation of (2.1), define the transformation $T$ from $\ell_\infty(\widehat{F},p)$ to $\ell_\infty(p)$ by $x \mapsto y = Tx$. The linearity of $T$ is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence $T$ is injective.

Let $y = (y_k) \in \ell_\infty(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j; \quad (k \in \mathbb{N}).$$

Then, we get that

$$\begin{aligned} g(x) &= \sup_{k\in\mathbb{N}} \left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} \left| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^2}{f_j f_{j+1}} y_j \right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} |y_k|^{p_k/M} = h_1(y) < \infty. \end{aligned}$$

Thus, we deduce that $x \in \ell_\infty(\widehat{F},p)$ and consequently $T$ is surjective and paranorm preserving. Hence, $T$ is a linear bijection and this says us that the spaces $\ell_\infty(\widehat{F},p)$ and $\ell_\infty(p)$ are linearly isomorphic, as desired. □

We shall quote some lemmas which are needed in proving related to the duals our theorems.

**Lemma 1** ([16], Theorem 5.1.1 with $q_n = 1$)**.** *$A \in (c_0(p) : \ell(q))$ if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} B^{-1/p_k} \right| < \infty, \quad (\exists B \in \mathbb{N}_2). \tag{2.6}$$

**Lemma 2** ([16], Theorem 5.1.9 with $q_n = 1$)**.** *$A \in (c_0(p) : c(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{-1/p_k} < \infty \quad (\exists B \in \mathbb{N}_2), \tag{2.7}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} |a_{nk} - \alpha_k| = 0 \quad for \quad all \ k \in \mathbb{N}, \tag{2.8}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \sup_{n\in\mathbb{N}} \sum_k |a_{nk} - \alpha_k| B^{-1/p_k} < \infty. \quad (\exists B \in \mathbb{N}_2) \tag{2.9}$$

**Lemma 3** ([16], Theorem 5.1.13 with $q_n = 1$)**.** *$A \in (c_0(p) : \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| B^{-1/p_k} < \infty. \quad (\exists B \in \mathbb{N}_2) \tag{2.10}$$

**Lemma 4** ([16], Theorem 5.1.0 with $q_n = 1$)**.** *(i) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{K\in\mathcal{F}} \sum_k \left| \sum_{n\in K} a_{nk} B^{-1} \right|^{p_k'} < \infty. \tag{2.11}$$

*(ii) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in K} a_{nk} \right|^{p_k} < \infty. \tag{2.12}$$

**Lemma 5** ([16], Theorem 1 (i)-(ii))**.** *(i) Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk} B^{-1}|^{p_k'} < \infty. \tag{2.13}$$

*(ii) Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{2.14}$$

**Lemma 6** ([16], Corollary for Theorem 1)**.** *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : c)$ if and only if (2.13), (2.14) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \quad (k \in \mathbb{N}) \tag{2.15}$$

*also holds.*

**Theorem 3.** *Let $K^* = \{k \in \mathbb{N} : 0 \leq k \leq n\} \cap K$ for $K \in \mathcal{F}$ and $B \in \mathbb{N}_2$. Define the sets $\widehat{F}_1(p), \widehat{F}_2(p), \widehat{F}_3(p), \widehat{F}_4(p), \widehat{F}_5(p), \widehat{F}_6(p), \widehat{F}_7(p)$ and $\widehat{F}_8(p)$ as follows:*

$$\widehat{F}_1(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K^*} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n B^{-1/p_k} \right| < \infty \right\}$$

$$\widehat{F}_2(p) = \left\{ a = (a_k) \in \omega : \sum_n \left| \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right| < \infty \right\}$$

$$\widehat{F}_3(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| B^{-1/p_k} < \infty \right\}$$

$$\widehat{F}_4(p) = \left\{ a = (a_k) \in \omega : \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| < \infty \quad \textit{for all} \quad k \in \mathbb{N} \right\}$$

$$\widehat{F}_5(p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \exists (\alpha_k) \subset \mathbb{R} \right.$$
$$\left. \ni \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j - \alpha_k \right| B^{-1/p_k} < \infty \right\}$$

$$\widehat{F}_6(p) = \left\{ a = (a_k) \in \omega : \exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} \left| \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j - \alpha \right| = 0 \right\}$$

$$\widehat{F}_7(p) = \left\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| < \infty \right\}$$

*Then*

*(i)* $\{c_0(\widehat{F}, p)\}^\alpha = \widehat{F}_1(p)$

*(ii)* $\{c(\widehat{F}, p)\}^\alpha = \widehat{F}_1(p) \cap \widehat{F}_2(p)$

*(iii)* $\{c_0(\widehat{F}, p)\}^\beta = \widehat{F}_3(p) \cap \widehat{F}_4(p) \cap \widehat{F}_5(p)$

*(iv)* $\{c(\widehat{F}, p)\}^\beta = \{c_0(\widehat{F}, p)\}^\beta \cap \widehat{F}_6(p)$

*(v)* $\{c_0(\widehat{F}, p)\}^\gamma = \widehat{F}_3(p)$ *(vi)* $\{c(\widehat{F}, p)\}^\gamma = \widehat{F}_3(p) \cap \widehat{F}_7(p)$

*Proof.* We give the proof for the space $c_0(\widehat{F}, p)$. Let us take any $a = (a_n) \in \omega$ and define the matrix $C = (c_{nk})$ via the sequence $a = (a_n)$ by

$$c_{nk} = \begin{cases} \dfrac{f_{n+1}^2}{f_k f_{k+1}} a_n, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Bearing in mind (2.1) we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = (Cy)_n; \quad (n \in \mathbb{N}). \tag{2.16}$$

We therefore observe by (2.16) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in c_0(\widehat{F}, p)$ if and only if $Cy \in \ell_1$ whenever $y \in c_0(p)$. This means that $a = (a_n) \in \{c_0(\widehat{F}, p)\}^\alpha$ whenever $x = (x_n) \in c_0(\widehat{F}, p)$ if and only if $C \in (c_0(p) : \ell_1)$. Then, we derive by Lemma 1 that

$$\{c_0(\widehat{F}, p)\}^\alpha = \widehat{F}_1(p).$$

Consider the following equations for $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left( \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \right) \\ &= \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right) y_k \\ &= (Dy)_n \end{aligned} \tag{2.17}$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \displaystyle\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Thus, we deduce from Lemma 2 with (2.17) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0(\widehat{F}, p)$ if and only if $Dy \in c$ whenever $y = (y_k) \in c_0(p)$. This means that $a = (a_n) \in \{c_0(\widehat{F}, p)\}^\beta$ whenever $x = (x_n) \in c_0(\widehat{F}, p)$ if and only if $D \in (c_0(p) : c)$. Therefore we derive from Lemma 2 that

$$\{c_0(\widehat{F}, p)\}^\beta = \widehat{F}_3(p) \cap \widehat{F}_4(p) \cap \widehat{F}_5(p).$$

As this, we deduce from Lemma 3 with (2.17) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in c_0(\widehat{F}, p)$ if and only if $Dy \in \ell_\infty$ whenever $y = (y_k) \in c_0(p)$. This means that $a = (a_n) \in \{c_0(\widehat{F}, p)\}^\gamma$ whenever $x = (x_n) \in c_0(\widehat{F}, p)$ if and only if $D \in (c_0(p) : \ell_\infty)$. Therefore we obtain Lemma 3 that

$$\{c_0(\widehat{F}, p)\}^\gamma = \widehat{F}_3(p)$$

and this completes the proof. □

**Theorem 4.** *Let $K^* = \{k \in \mathbb{N} : 0 \le k \le n\} \cap K$ for $K \in \mathcal{F}$ and $B \in \mathbb{N}_2$. Define the sets $\widehat{F}_8(p), \widehat{F}_9(p), \widehat{F}_{10}(p)$ and $\widehat{F}_{11}(p)$ as follows:*

$$\widehat{F}_8(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K^*} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{1/p_k} \right| < \infty \right\}$$

$$\widehat{F}_9(p) = \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| B^{1/p_k} < \infty \right\}$$

$$\widehat{F}_{10}(p) = \bigcap_{B>1} \Bigg\{ a = (a_k) \in \omega : \exists (\alpha_k) \subset \mathbb{R}$$
$$\ni \lim_{n\to\infty} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j - \alpha_k \right| B^{1/p_k} = 0 \Bigg\}$$

$$\widehat{F}_{11}(p) = \bigcap_{B>1} \Bigg\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| B^{1/p_k} < \infty \Bigg\}$$

*Then*

*(i)* $\{\ell_\infty(\widehat{F}, p)\}^\alpha = \widehat{F}_8(p)$

*(ii)* $\{\ell_\infty(\widehat{F}, p)\}^\beta = \widehat{F}_9(p) \cap \widehat{F}_{10}(p)$

*(iii)* $\{\ell_\infty(\widehat{F}, p)\}^\gamma = \widehat{F}_{11}(p)$.

*Proof.* This may be obtained in the similar way, as mentioned in the proof of Theorem 3 with Lemmas 4(i), 5(i), 6 instead of Lemmas 1-3. So, we omit the details. □

**Theorem 5.** *Let* $K^* = \{k \in \mathbb{N} : 0 \le k \le n\} \cap K$ *for* $K \in \mathcal{F}$ *and* $B \in \mathbb{N}_2$. *Define the sets* $\widehat{F}_{12}(p), \widehat{F}_{13}(p), \widehat{F}_{14}(p), \widehat{F}_{15}(p)$ *and* $\widehat{F}_{16}(p)$ *as follows:*

$$\widehat{F}_{12}(p) = \Bigg\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left| \sum_{n\in K^*} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right|^{p_k} < \infty \Bigg\}$$

$$\widehat{F}_{13}(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in \omega : \sup_{K\in\mathcal{F}} \sum_{k} \left| \sum_{n\in K} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{-1} \right|^{p_k'} < \infty \Bigg\}$$

$$\widehat{F}_{14}(p) = \bigcup_{B>1} \Bigg\{ a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{-1} \right|^{p_k'} < \infty \Bigg\}$$

$$\widehat{F}_{15}(p) = \Bigg\{ a = (a_k) \in \omega : \sup_{n,k\in\mathbb{N}} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right|^{p_k} < \infty \Bigg\}$$

$$\widehat{F}_{16}(p) = \Bigg\{ a = (a_k) \in \omega : \lim_{n\to\infty} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \quad \textit{exists} \Bigg\}$$

*Then*

*(i)*

$$\{\ell(\widehat{F}, p)\}^\alpha = \begin{cases} \widehat{F}_{12}(p), & 0 < p_k \le 1 \\ \widehat{F}_{13}(p), & 1 < p_k \le H < \infty \end{cases}$$

*(ii)*

$$\{\ell(\widehat{F},p)\}^{\gamma} = \begin{cases} \widehat{F}_{15}(p), & 0 < p_k \leq 1 \\ \widehat{F}_{14}(p), & 1 < p_k \leq H < \infty. \end{cases}$$

*(iii) Let* $0 < p_k \leq H < \infty$. *Then*

$$\{\ell(\widehat{F},p)\}^{\beta} = \widehat{F}_{14}(p) \cap \widehat{F}_{15}(p) \cap \widehat{F}_{16}(p).$$

*Proof.* This may be obtained in the similar way, as mentioned in the proof of Theorem 3 with Lemmas 4(ii), 5(ii), 6 instead of Lemmas 1-3. So, we omit the details. □

Now, we may give the sequence of the points of the spaces $c_0(\widehat{F},p)$, $\ell(\widehat{F},p)$ and $c(\widehat{F},p)$ which forms a Schauder basis for those spaces. Because of the isomorphism $T$, defined in the proof of Theorem 2, between the sequence spaces $c_0(\widehat{F},p)$ and $c_0(p)$, $\ell(\widehat{F},p)$ and $\ell(p)$, $c(\widehat{F},p)$ and $c(p)$ is onto, the inverse image of the basis of the spaces $c_0(p), \ell(p)$ and $c(p)$ is the basis for our new spaces $c_0(\widehat{F},p)$, $\ell(\widehat{F},p)$ and $c(\widehat{F},p)$, respectively. Therefore, we have:

**Theorem 6.** *Let* $\mu_k = (\widehat{F}x)_k$ *for all* $k \in \mathbb{N}$. *We define the sequence* $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)} = \begin{cases} \dfrac{f_{n+1}^2}{f_k f_{k+1}}, & n \geq k, \\ 0, & n < k. \end{cases}$$

*Then*
*(a) The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $c_0(\widehat{F},p)$ *and any* $x \in c_0(\widehat{F},p)$ *has a unique representation in the form*

$$x = \sum_k \mu_k b^{(k)}.$$

*(b) The sequence* $\{b^{(k)}\}_{k\in\mathbb{N}}$ *is a basis for the space* $\ell(\widehat{F},p)$ *and any* $x \in \ell(\widehat{F},p)$ *has a unique representation in the form*

$$x = \sum_k \mu_k b^{(k)}.$$

*(c) The set* $\{z, b^{(k)}\}$ *is a basis for the space* $c(\widehat{F},p)$ *and any* $x \in c(\widehat{F},p)$ *has a unique representation in the form*

$$x = lz + \sum_k (\mu_k - l) b^{(k)}$$

*where* $l = \lim_{k\to\infty}(\widehat{F}x)_k$ *and* $z = (z_k)$ *with*

$$z_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}}.$$

## 3. Some matrix mappings on the sequence spaces $c_0(\hat{F}, p), c(\hat{F}, p)$, $\ell_\infty(\hat{F}, p)$ and $\ell(\hat{F}, p)$

In this section, we characterize some matrix mappings on the spaces $c_0(\widehat{F}, p)$, $c(\widehat{F}, p), \ell_\infty(\widehat{F}, p)$ and $\ell(\widehat{F}, p)$. Firstly, we may give the following theorem which is useful for deriving the characterization of the certain matrix classes.

**Theorem 7** ([18], Theorem 4.1)**.** *Let* $\lambda$ *be an FK-space,* $U$ *be a triangle,* $V$ *be its inverse and* $\mu$ *be arbitrary subset of* $\omega$*. Then we have* $A \in (\lambda_U : \mu)$ *if and only if*

$$E^{(n)} = (e_{mk}^{(n)}) \in (\lambda : c) \quad \textit{for all} \quad n \in \mathbb{N} \tag{3.1}$$

*and*

$$E = (e_{nk}) \in (\lambda : \mu) \tag{3.2}$$

*where*

$$e_{mk}^{(n)} = \begin{cases} \displaystyle\sum_{j=k}^{m} a_{nj} v_{jk}, & 0 \le k \le m, \\ 0, & k > m, \end{cases}$$

*and*

$$e_{nk} = \sum_{j=k}^{\infty} a_{nj} v_{jk} \quad \textit{for all} \quad k, m, n \in \mathbb{N}.$$

Now, we may quote our theorems on the characterization of some matrix classes concerning with the sequence spaces $c_0(\widehat{F}, p), c(\widehat{F}, p)$ and $\ell_\infty(\widehat{F}, p)$. The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann [16]. Let $N$ and $K$ denote the finite subset of $\mathbb{N}$, $L$ and $M$ also denote the natural numbers. Prior to giving the theorems, let us suppose that $(q_n)$ is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_{m\to\infty} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} = e_{nk}, \tag{3.3}$$

$$\forall L, \quad \sum_k |e_{nk}| L^{1/p_k} < \infty, \tag{3.4}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \lim_{m\to\infty} \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} - \alpha_k \right| = 0 \quad \text{for all} \quad k \in \mathbb{N}, \tag{3.5}$$

$$\exists M, \quad \sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| M^{-1/p_k} < \infty, \tag{3.6}$$

$$\forall L, \exists M, \sup_{m\in\mathbb{N}} \sum_{k=0}^{m} \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| L^{1/q_n} M^{-1/p_k} < \infty, \tag{3.7}$$

$$\lim_{m\to\infty} \sum_{k} \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} - \alpha \right| = 0, \tag{3.8}$$

$$\forall L, \quad \sup_{n\in\mathbb{N}} \sum_{k} |e_{nk}| L^{1/p_k} < \infty, \tag{3.9}$$

$$\lim_{n\to\infty} e_{nk} = \alpha_k \qquad \text{for all} \quad k \in \mathbb{N}, \tag{3.10}$$

$$\forall L, \quad \lim_{n\to\infty} \sum_{k} |e_{nk}| L^{1/p_k} < \infty, \tag{3.11}$$

$$\forall L, \quad \lim_{n\to\infty} \sum_{k} |e_{nk}| L^{1/p_k} = 0, \tag{3.12}$$

$$\exists M, \quad \sup_{n\in\mathbb{N}} \left( \sum_{k\in K} |e_{nk}| M^{-1/p_k} \right)^{q_n} < \infty, \tag{3.13}$$

$$\lim_{n\to\infty} |e_{nk}|^{q_n} = 0, \quad \text{for all} \quad k \in \mathbb{N}, \tag{3.14}$$

$$\forall L, \exists M, \quad \sup_{n\in\mathbb{N}} \sum_{k} |e_{nk}| L^{1/q_n} M^{-1/p_k} < \infty, \tag{3.15}$$

$$\lim_{n\to\infty} |e_{nk} - \alpha_k|^{q_n} = 0, \quad \text{for all} \quad k \in \mathbb{N}, \tag{3.16}$$

$$\exists M, \quad \sup_{n\in\mathbb{N}} \sum_{k} |e_{nk}| M^{-1/p_k} < \infty, \tag{3.17}$$

$$\forall L, \exists M, \quad \sup_{n\in\mathbb{N}} \sum_{k} |e_{nk} - \alpha_k| L^{1/q_n} M^{-1/p_k} < \infty, \tag{3.18}$$

$$\sup_{n\in\mathbb{N}} \left| \sum_{k} e_{nk} \right|^{q_n} < \infty, \tag{3.19}$$

$$\lim_{n\to\infty} \left| \sum_{k} e_{nk} \right|^{q_n} = 0, \tag{3.20}$$

$$\lim_{n\to\infty} \left| \sum_{k} e_{nk} - \alpha \right|^{q_n} = 0, \tag{3.21}$$

**Theorem 8.** *(i)* $A \in (\ell_\infty(\widehat{F}, p) : \ell_\infty)$ *if and only if (3.3), (3.4) and (3.9) hold.*

*(ii)* $A \in (\ell_\infty(\widehat{F}, p) : c)$ *if and only if (3.3), (3.4), (3.10) and (3.11) hold.*

*(iii)* $A \in (\ell_\infty(\widehat{F}, p) : c_0)$ *if and only if (3.3), (3.4) and (3.12) hold.*

**Theorem 9.** *(i)* $A \in (c_0(\widehat{F}, p) : \ell_\infty(q))$ *if and only if (3.5), (3.6), (3.7) and (3.13) hold.*

*(ii)* $A \in (c_0(\widehat{F}, p) : c_0(q))$ *if and only if (3.5), (3.6), (3.7), (3.14) and (3.15) hold.*

*(iii)* $A \in (c_0(\widehat{F}, p) : c(q))$ *if and only if (3.5), (3.6), (3.7), (3.16), (3.17) and (3.18) hold.*

**Theorem 10.** *(i)* $A \in (c(\widehat{F}, p) : \ell_\infty(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.13) and (3.19) hold.*

*(ii)* $A \in (c(\widehat{F}, p) : c_0(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.14), (3.15) and (3.20) hold.*

*(iii)* $A \in (c(\widehat{F}, p) : c(q))$ *if and only if (3.5), (3.6), (3.7), (3.8), (3.16), (3.17), (3.18) and (3.21) hold.*

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# ON THE EQUATIONS $U_n = 5\square$ AND $V_n = 5\square$

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*Abstract.* Let $P \geq 3$ be an integer and let $(U_n)$ and $(V_n)$ denote the generalized Fibonacci and Lucas sequences defined by $U_0 = 0, U_1 = 1$; $V_0 = 2, V_1 = P$, and $U_{n+1} = PU_n - U_{n-1}$, $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. The purpose of this study, assuming $P$ is odd, is to determine the values of $n$ such that $V_n = 5\square$ and $U_n = 5\square$. Moreover, we solve the equations $V_n = 5V_m\square$ and $U_n = 5U_m\square$.

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## 1. Introduction

Let $P$ and $Q$ be nonzero integers such that $P^2 + 4Q \neq 0$. The generalized Fibonacci sequence $(U_n)$ and Lucas sequence $(V_n)$ are given recursively according to the following relations for $n \geq 1$.

$$U_0 = 0,\ U_1 = 1,\ U_{n+1} = PU_n + QU_{n-1}$$

and

$$V_0 = 2,\ V_1 = P, V_{n+1} = PV_n + QV_{n-1}.$$

Both sequences depend on the initial choice of pair $(P, Q)$, hence we sometimes use $U_n(P, Q)$ and $V_n(P, Q)$ in order to emphasize their dependence on the parameters $(P, Q)$. $U_n$ and $V_n$ are called the $n$th generalized Fibonacci number and the $n$th generalized Lucas number, respectively. Furthermore, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = -(-Q)^{-n}U_n \text{ and } V_{-n} = (-Q)^{-n}V_n \ (n \geq 1),$$

respectively. It is well known that

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n$$

where $\alpha = \left(P + \sqrt{P^2 + 4Q}\right)/2$ and $\beta = \left(P - \sqrt{P^2 + 4Q}\right)/2$, which are the roots of the equation $x^2 - Px - Q = 0$. The above formulas are known as Binet's formulas.

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We will assume that $P^2+4Q>0$. Special cases of the sequences $(U_n)$ and $(V_n)$ are known. For example, the generalized Fibonacci sequence $(U_n(1,1))$ consist of the familiar Fibonacci numbers, whereas its companion, $(V_n(1,1))$ gives so called Lucas numbers. When $P=2$ and $Q=1$, $(U_n)=(P_n)$ and $(V_n)=(Q_n)$ are the familiar sequences of Pell and Pell-Lucas numbers. For more information about generalized Fibonacci and Lucas sequences, see [8].

There has been much interest in when the terms of generalized Fibonacci and Lucas sequences are perfect square($=\square$) or $k\square$. When $P$ is odd and $Q=\pm 1$, by using elementary arguments, many authors solved the equations $U_n=k\square$ and $V_n=k\square$ for some specific values of $k$ (see [2–4, 9, 10]). Interested readers can also consult [12] and [6] for a brief history of this subject.

In [6], the authors determined all indices $n$ such that $U_n(P,1)=5\square$ and $U_n(P,1)=5U_m(P,1)\square$ under some assumptions on $P$. When $P$ is odd, the authors solved the equation $V_n(P,1)=5\square$. Moreover, they showed that the equation $V_n(P,1)=5V_m(P,1)\square$ has no solutions. In this study, using congruences, with extensive reliance upon the Jacobi symbol, we determine that the five times square terms of the generalized Fibonacci sequence $(U_n(P,Q))$ for which $P\geq 3$ is odd and $Q=-1$ may occur only for $n=2$ or 3. We obtain a similar result for the generalized Lucas sequence $(V_n(P,Q))$. Moreover, when $P\geq 3$ is odd and $Q=-1$, we solve the equations $V_n=5V_m\square$ and $U_n=5U_m\square$.

In section 2, we give some identities, lemmas, and theorems needed later. Then in section 3, we present our main theorems. Throught this study, $\left(\frac{*}{*}\right)$ will denote the Jacobi symbol. Our method of proof is similar to that presented by Cohn, McDaniel and Ribenboim [2–4, 9].

## 2. Preliminary facts

From now on, we assume that $Q=-1$. We omit the proofs of the following two lemmas, as they are based a straightforward induction.

**Lemma 1.** *If $n$ is even, then $V_n\equiv\pm 2\ (mod\ P^2)$ and if $n$ is odd, then $V_n\equiv\pm nP\ (mod\ P^2)$.*

**Lemma 2.** *If $n$ is even, then $U_n\equiv\pm\frac{n}{2}P\ (mod\ P^2)$ and if $n$ is odd, then $U_n\equiv\pm 1\ (mod\ P^2)$.*

**Lemma 3.**

$$3|U_n \Leftrightarrow \begin{cases} n\equiv 0\ (mod\ 2)\ if\ 3|P, \\ n\equiv 0\ (mod\ 3)\ if\ 3\nmid P. \end{cases}$$

One can see the proofs of the following two theorems in [5].

**Theorem 1.** *Let $P\geq 3$ be odd. If $V_n=kx^2$ for some $k|P$ with $k>1$, then $n=1$.*

**Theorem 2.** *Let $P\geq 3$ be odd. If $U_n=kx^2$ for some $k|P$ with $k>1$, then $n=2$ or $n=6$ and $3|P$.*

The proofs of the following two theorems can be found in [11].

**Theorem 3.** *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and $m$ be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \ (mod\ U_m), \tag{2.1}$$

$$V_{2mn+r} \equiv V_r \ (mod\ U_m). \tag{2.2}$$

**Theorem 4.** *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^n U_r \ (mod\ V_m), \tag{2.3}$$

$$V_{2mn+r} \equiv (-1)^n V_r \ (mod\ V_m). \tag{2.4}$$

Now we state the following theorem from [9].

**Theorem 5.** *Let $P \geq 3$ be odd. If $V_n = x^2$ for some integer $x$, then $n = 1$. If $V_n = 2x^2$ for some integer $x$, then $n = 3$, $P = 3, 27$.*

We state the following theorem due to Ribenboim and McDaniel [9].

**Theorem 6.** *Let $P \geq 3$ be odd. If $U_n = x^2$, then $n = 1$ or $n = 6$ and $P = 3$.*

The following theorem can be obtained from Theorem 9 given in [4].

**Theorem 7.** *Let $P \geq 3$ be odd, $m, n > 1$ be integers. The equation $U_n = 2U_m x^2$ has no solutions except for the cases $n = 6$, $m = 3$, $P = 3, 27$.*

The following two theorems can be obtained from Theorems 14 and 15 given in [4].

**Theorem 8.** *The equation $V_n = V_m x^2$, where $P \geq 3$, and $P$ is odd, and $n \geq m > 0$ has only the trivial solution $n = m$.*

**Theorem 9.** *The equation $V_n = 2V_m x^2$, where $P \geq 3$, and $P$ is odd, and $m, n > 0$ has no solutions.*

Now we give some identities concerning generalized Fibonacci and Lucas numbers:

$$U_{-n} = -U_n \text{ and } V_{-n} = V_n, \tag{2.5}$$

$$U_{2n} = U_n V_n, \tag{2.6}$$

$$V_{2n} = V_n^2 - 2, \tag{2.7}$$

$$V_{3n} = V_n(V_n^2 - 3), \tag{2.8}$$

$$U_{3n} = U_n\left((P^2-4)U_n^2 + 3\right) = U_n(V_n^2 - 1), \tag{2.9}$$

$$V_n^2 - \left(P^2 - 4\right)U_n^2 = 4, \tag{2.10}$$

$$\text{if } P \text{ is odd, then } 2|V_n \Leftrightarrow 2|U_n \Leftrightarrow 3|n, \tag{2.11}$$

$$V_m | V_n \Leftrightarrow m|n \text{ and } n/m \text{ is odd}, \tag{2.12}$$

$$U_m | U_n \Leftrightarrow m|n. \tag{2.13}$$

Let $m = 2^a k$, $n = 2^b l$, $k$ and $l$ are odd, $a, b \geq 0$, and $d = (m,n)$. Then

$$(U_m, V_n) = \begin{cases} V_d \text{ if } a > b, \\ 1 \text{ or } 2 \text{ if } a \leq b. \end{cases} \tag{2.14}$$

$$U_{5n} = U_n \left( (P^2-4)^2 U_n^4 + 5(P^2-4)U_n^2 + 5 \right). \tag{2.15}$$

If $5|U_n$, then from (2.15), we have

$$U_{5n} = 5U_n(5a+1) \tag{2.16}$$

for some $a \geq 0$.

$$V_{5n} = V_n(V_n^4 - 5V_n^2 + 5). \tag{2.17}$$

If $5|P$ and $n$ is odd, then $5|V_n$ and therefore from (2.17), it follows that

$$V_{5n} = 5V_n(5a+1), \tag{2.18}$$

for some $a \geq 0$.

From Lemma 1 and the identity (2.7), we have

$$5|V_n \Leftrightarrow 5|P \text{ and } n \text{ is odd}. \tag{2.19}$$

When $P$ is odd, it is clear that

$$\left( \frac{-1}{V_{2^r}} \right) = -1. \tag{2.20}$$

If $P$ is odd and $r \geq 2$, then $V_{2^r} \equiv -1 \left( mod \ \frac{P^2-3}{2} \right)$ and thus

$$\left( \frac{(P^2-3)/2}{V_{2^r}} \right) = \left( \frac{P^2-3}{V_{2^r}} \right) = 1. \tag{2.21}$$

$$V_{2^r} \equiv \begin{cases} -2 \ (mod \ P), \text{ if } r = 1, \\ 2 \ (mod \ P), \text{ if } r \geq 2. \end{cases} \tag{2.22}$$

If $3 \nmid P$ and $P$ is odd, then $V_{2^r} \equiv -1 \ (mod \ 3)$ for $r \geq 1$ and therefore

$$\left( \frac{3}{V_{2^r}} \right) = 1. \tag{2.23}$$

If $3|P$ and $P$ is odd, then $V_{2^r} \equiv -1 \ (mod \ 3)$ for $r \geq 2$ and therefore

$$\left( \frac{3}{V_{2^r}} \right) = 1. \tag{2.24}$$

Let $P$ be odd. Then

$$\left(\frac{5}{V_{2^r}}\right) = \begin{cases} -1, \text{ if } 5|P, \\ 1, \text{ if } P^2 \equiv 1 \ (mod\ 5), \\ -1, \text{ if } P^2 \equiv -1 (mod\ 5), \end{cases} \tag{2.25}$$

for every $r \geq 1$.
Most of the properties above are well–known; properties between (2.5)–(2.10) can be found in [8], [9], [10], [2]; properties between (2.11)–(2.14) can be found in [7], [9], [10], [2]. Since the others are fairly easy to prove, we omit their proofs.

The following lemma can be proved by using (2.1).

**Lemma 4.**

$$5|U_n \Leftrightarrow \begin{cases} 2|n, \text{ if } 5|P, \\ 3|n, \text{ if } P^2 \equiv 1 \ (mod\ 5), \\ 5|n, \text{ if } P^2 \equiv -1 \ (mod\ 5). \end{cases}$$

## 3. Main theorems

From now on, we assume that $n$ and $m$ are positive integers, $P \geq 3$, and $P$ is odd.

**Theorem 10.** *The equation $V_n = 5x^2$ has a solution only if $n = 1$.*

*Proof.* Assume that $V_n = 5x^2$ for some integer $x$. Since $5|V_n$, it follows from (2.19) that $5|P$. This implies by Theorem 1 that $n = 1$. This completes the proof. □

**Theorem 11.** *There is no integer $x$ such that $V_n = 5V_m x^2$.*

*Proof.* Assume that $V_n = 5V_m x^2$. Then by (2.19), it is seen that $5|P$ and $n$ is odd. Moreover, since $V_m|V_n$, there exists an odd integer $t$ such that $n = mt$ by (2.12). Since $n$ and $t$ are odd and $n = mt$, $m$ is also odd. Hence, we have from Lemma 1 that

$$V_n \equiv \pm nP \ (mod\ P^2) \text{ and } V_m \equiv \pm mP \ (mod\ P^2).$$

This implies that

$$\pm nP \equiv \pm 5mPx^2 \ (mod\ P^2),$$

i.e.,

$$n \equiv 5mx^2 \ (mod\ P).$$

Using the fact that $5|P$, it follows that $5|n$. Firstly, assume that $5|t$. Then $t = 5s$ for some positive odd integer $s$ and therefore $n = mt = 5ms$. By (2.17), we immediately have

$$V_n = V_{5ms} = V_{ms}(V_{ms}^4 - 5V_{ms}^2 + 5).$$

Since $ms$ is odd and $5|P$, it follows that $5|V_{ms}$ by (2.19) and therefore

$$\frac{V_{ms}}{V_m}\left(\frac{V_{ms}^4-5V_{ms}^2+5}{5}\right)=x^2.$$

Clearly,

$$\left(V_{ms}/V_m,(V_{ms}^4-5V_{ms}^2+5)/5\right)=1.$$

This implies that

$$V_{ms}^4-5V_{ms}^2+5=5b^2$$

for some $b\geq 0$. But the integral points on $5Y^2=X^4-5X^2+5$ are immediately determined by using MAGMA [1] to be $(X,\pm Y)=(0,1)$, which gives $V_{ms}=0$, which is impossible. Secondly, assume that $5\nmid t$. Since $n=mt$ and $5|n$, it is seen that $5|m$. Then we can write $m=5^r a$ with $5\nmid a$ and $r\geq 1$. By (2.18), we obtain

$$V_m=V_{5^r a}=5V_{5^{r-1}a}(5a_1+1)$$

for some positive integer $a_1$. Thus, we conclude that

$$V_m=V_{5^r a}=5^r V_a(5a_1+1)(5a_2+1)...(5a_r+1)$$

for some positive integers $a_i$ with $1\leq i\leq r$. Let $A=(5a_1+1)(5a_2+1)...(5a_r+1)$. Thus, we have $V_m=5^r V_a A$, where $5\nmid A$. In a similar manner, we see that

$$V_n=V_{5^r at}=5^r V_{at}(5b_1+1)(5b_2+2)...(5b_r+1)$$

for some positive integers $b_j$ with $1\leq j\leq r$. Thus, we have $V_n=5^r V_{at}B$, where $5\nmid B$. As a consequence, we get

$$5^r V_{at}B=5\cdot 5^r V_a Ax^2,$$

implying that

$$V_{at}B=5V_a Ax^2.$$

By Lemma 1, it is seen that

$$\pm atPB\equiv\pm 5aPAx^2\ (mod\ P^2),$$

i.e.,

$$atB\equiv 5aAX^2\ (mod\ P).$$

Since $5|P$, it follows that $5|atB$. However, this is impossible since $5\nmid a$, $5\nmid t$, and $5\nmid B$. This completes the proof. □

**Theorem 12.** *If $P\geq 3$ is odd, then the equation $U_n=5x^2$ has the solution $n=2$ when $5|P$ and $n=3$ when $P^2\equiv 1\ (mod\ 5)$. The equation $U_n=5x^2$ has no solutions when $P^2\equiv -1\ (mod\ 5)$.*

*Proof.* Assume that $U_n = 5x^2$ for some integer $x$. Now we distinguish three cases.
**Case** $I$ : Let $5|P$. Then by Theorem 2, we see that $n = 2$ or $n = 6$ and $3|P$. But, it can be easily shown that for the case when $n = 6$ and $3|P$, the equation $U_n = 5x^2$ has no solutions.
**Case** $II$ : Let $P^2 \equiv 1 \ (mod\ 5)$. Since $5|U_n$, it follows from Lemma 4 that $3|n$. Hence, $n = 3m$ for some positive integer $m$. Assume that $m$ is even. Then $m = 2s$ for some positive integer $s$ and therefore $n = 6s$. And so by (2.6), we get $U_n = U_{6s} = U_{3s}V_{3s} = 5x^2$. Clearly, $(U_{3s}, V_{3s}) = 2$ by (2.14) and (2.11). Then either

$$U_{3s} = 2a^2,\ V_{3s} = 10b^2 \tag{3.1}$$

or

$$U_{3s} = 10a^2,\ V_{3s} = 2b^2 \tag{3.2}$$

for some positive integers $a$ and $b$. Assume that (3.1) is satisfied. Since $5|V_{3s}$, it follows from (2.19) that $5|P$. But this contradicts the fact that $P^2 \equiv 1(mod\,5)$. Now assume that (3.2) is satisfied. Then by Theorem 5, we have $3s = 3$ and $P = 3, 27$. Therefore $s = 1$. If $P = 3$, then $U_3 = P^2 - 1 = 8 = 10a^2$, which is impossible. If $P = 27$, then $U_3 = P^2 - 1 = 27^2 - 1 = 10a^2$, which is also impossible. Now assume that $m$ is odd. Then by (2.9), we get $U_{3m} = U_m\left((P^2-4)U_m + 3\right)$. Clearly, $\left(U_m, (P^2-4)U_m^2 + 3\right) = 1$ or 3. Then it follows that $(P^2-4)U_m^2 + 3 = wa^2$ for some $w \in \{1, 3, 5, 15\}$. Since $(P^2-4)U_m^2 + 3 = V_{2m} + 1$ by (2.7) and (2.10), it is seen that $V_{2m} + 1 = wa^2$. Assume that $m > 1$. Then $m = 4q \pm 1 = 2^r a \pm 1$ with $a$ odd and $r \geq 2$. Thus,

$$wa^2 = V_{2m} + 1 \equiv 1 - V_2 \equiv -(P^2 - 3) \ (mod\ V_{2^r})$$

by (2.4). This shows that

$$\left(\frac{w}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right).$$

By using (2.23), (2.24), and (2.25), it can be seen that $\left(\frac{w}{V_{2^r}}\right) = 1$ for $w = 3, 5, 15$. Moreover, $\left(\frac{-1}{V_{2^r}}\right) = -1$ and $\left(\frac{P^2-3}{V_{2^r}}\right) = 1$ by (2.20) and (2.21), respectively. Thus, we get

$$1 = \left(\frac{w}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

which is impossible. Therefore $m = 1$ and thus $n = 3$.
**Case** $III$ : Let $P^2 \equiv -1 \ (mod\ 5)$. Since $5|U_n$, it follows that $5|n$ by Lemma 4. Thus $n = 5t$ for some positive integer $t$. Since $P^2 \equiv -1 \ (mod\ 5)$, it is obvious that $5|P^2 - 4$ and therefore there exists a positive integer $A$ such that $P^2 - 4 = 5A$. By (2.15), we get $U_n = U_{5t} = U_t\left((P^2-4)^2U_t^4 + 5(P^2-4)U_t^2 + 5\right)$. Substituting $P^2 - 4 = 5A$

into the preceding equation gives $U_n = U_{5t} = 5U_t\left(5A^2U_t^4 + 5AU_t^2 + 1\right)$. Let $B = A^2U_t^4 + AU_t^2$. As a consequence, we have

$$U_n = U_{5t} = 5U_t(5B+1) = 5x^2,$$

implying that

$$U_t(5B+1) = x^2.$$

It can be easily seen that $(U_t, 5B+1) = 1$. This shows that $U_t = a^2$ and $5B+1 = b^2$ for some $a, b > 0$. By Theorem 6, we see that the only possible values of $t$ and $P$ in which $U_t = a^2$ are $t = 1$ or $t = 6$ and $P = 3$. If $t = 1$, then $n = 5$ and therefore we get $U_n = U_{5t} = U_5 = P^4 - 3P^2 + 1 = 5x^2$. With MAGMA [1], we get $P = 2$, which is impossible since $P$ is odd. If $t = 6$, then $n = 30$. A simple computation shows that there is no integer $x$ such that $U_{30} = 5x^2$ for $P = 3$. □

**Theorem 13.** *Let $P \geq 3$ and $m > 1$. The equation $U_n = 5U_m x^2$ has no solutions in any of the following cases:*

$(i)$ : $P^2 \equiv -1 \ (mod\ 5)$;
$(ii)$ : $P$ is odd and $5|P$;
$(iii)$ : $P^2 \equiv 1 \ (mod\ 5)$, $n$ is odd, and $P$ is odd;
$(iv)$ : $P^2 \equiv 1 \ (mod\ 5)$, $n$ is even, and $P$ is odd.

*Proof.* Assume that $U_n = 5U_m x^2$ for some $x > 0$. Since $U_m | U_n$, it follows that $m|n$ by (2.13). Thus, $n = mt$ for some $t > 0$. Since $n \neq m$, we have $t > 1$.
**Case** $I$ : Let $P^2 \equiv -1 \ (mod\ 5)$. It is obvious that $5|P^2 - 4$. On the other hand, since $5|U_n$, it follows that $5|n$ by Lemma 4. Dividing the proof into two subcases, we have
**Subcase** $(i)$ : Assume that $5|t$. Then $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.15), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right) = 5U_m x^2. \tag{3.3}$$

Since $5|P^2-4$, it is seen that $5|(P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5$. Also, we have $(P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.10). Rearranging the equation (3.3), we readily obtain

$$x^2 = (U_{ms}/U_m)\left(\left(V_{ms}^4 - 3V_{ms}^2 + 1\right)/5\right),$$

where $\left(U_{ms}/U_m, \left(V_{ms}^4 - 3V_{ms}^2 + 1\right)/5\right) = 1$. Hence, $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some $b > 0$. But the integral points on $5Y^2 = X^4 - 3X^2 + 1$ are immediately determined by using MAGMA [1] to be $(\pm X, \pm Y) = (2, 1)$, which gives $V_{ms} = 2$, implying that $ms = 0$, which is impossible.
**Subcase** $(ii)$ : Assume that $5 \nmid t$. Since $5|n$, it follows that $5|m$. Then we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.16), it is sen that $U_m = U_{5^r a} = 5U_{5^{r-1}a}(5a_1 + 1)$ for some positive integer $a_1$. Thus, we conclude that $U_m = U_{5^r a} = 5^r U_a(5a_1 + 1)(5a_2 + 1)...(5a_r + 1)$ for some positive integers $a_i$ with $1 \leq i \leq r$. Let $A = (5a_1 +$

$1)(5a_2+1)...(5a_r+1)$. Then, we have $U_m = 5^r U_a A$, where $5 \nmid A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at}(5b_1+1)(5b_2+1)...(5b_r+1)$ for some positive integers $b_i$ with $1 \le i \le r$. Let $B = (5b_1+1)(5b_2+1)...(5b_r+1)$. Hence, we have $U_n = 5^r U_{at} B$, where $5 \nmid B$. As a consequence, we get

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2$$

i.e.,

$$U_{at} B = 5 U_a A x^2.$$

Since $5 \nmid B$, it follows that $5|U_{at}$, implying that $5|at$ by Lemma 4. This contradicts the fact that $5 \nmid a$ and $5 \nmid t$. This concludes the proof of the case when $P^2 \equiv -1$ $(mod\ 5)$.

**Case** $II$ : Let $P$ be odd and $5|P$. Since $5|U_n$, it is seen from Lemma 4 that $n$ is even. On the other hand, we have $n = mt$. So, we first assume that $t$ is even. Then $t = 2s$ for some $s > 0$. By (2.6), we get $U_n = U_{2ms} = U_{ms} V_{ms} = 5U_m x^2$, implying that $(U_{ms}/U_m) V_{ms} = 5x^2$. Clearly, $d = (U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.14). If $d = 1$, then

$$U_{ms} = U_m a^2,\ V_{ms} = 5b^2 \tag{3.4}$$

or

$$U_{ms} = 5U_m a^2,\ V_{ms} = b^2 \tag{3.5}$$

for some $a, b > 0$. If (3.4) holds, then the only possible value of $ms$ in which $V_{ms} = 5b^2$ is 1 by Theorem 1, which contradicts the fact that $m > 1$. If (3.5) holds, then by Theorem 5, we have $ms = 1$, which is impossible since $m > 1$.

If $d = 2$, then

$$U_{ms} = 2U_m a^2,\ V_{ms} = 10b^2 \tag{3.6}$$

or

$$U_{ms} = 10U_m a^2,\ V_{ms} = 2b^2 \tag{3.7}$$

for some $a, b > 0$. Suppose (3.6) holds. Then by Theorem 7, we get $ms = 6$, $m = 3$, $P = 3, 27$. There is no integer $b$ such that $V_6 = 10b^2$ for the case when $P = 3$ or 27. Suppose (3.7) holds. Then by Theorem 7, the only possible values of $ms$ and $P$ in which $V_{ms} = 2b^2$ are $ms = 3$ and $P = 3, 27$. Since $m > 1$, it follows that $m = 3$ and therefore we obtain $U_3 = 10U_3 a^2$, which is impossible.

Now assume that $t$ is odd. Since $n = mt$ and $n$ is even, it follows that $m$ is even. Hence, we have $U_n \equiv \pm(n/2)P(mod\ P^2)$ and $U_m \equiv \pm(m/2)P\ (mod\ P^2)$ by Lemma 2. This shows that $\pm\frac{n}{2}P \equiv \pm 5\frac{m}{2}Px^2\ (mod\ P^2)$, i.e., $\frac{n}{2} \equiv 5\frac{m}{2}x^2\ (mod\ P)$. Since $5|P$, it is seen that $5|n$. Dividing remainder of the proof into two subcases, we have

**Subcase** $(i)$ : Let $5|t$. Then $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.15), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2 U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right). \tag{3.8}$$

Since $ms$ is even and $5|P$, it is seen that $5|U_{ms}$ by Lemma 4. Also, we have $(P^2-4)^2U_{ms}^4+5(P^2-4)U_{ms}^2+5=V_{ms}^4-3V_{ms}^2+1$ by (2.10). Hence, rearranging the equation (3.8) gives

$$x^2=(U_{ms}/U_m)\left(\left(V_{ms}^4-3V_{ms}^2+1\right)/5\right),$$

where $\left((U_{ms}/U_m),\left(V_{ms}^4-3V_{ms}^2+1\right)/5\right)=1$. This implies that $V_{ms}^4-3V_{ms}^2+1=5b^2$ for some $b>0$. But the integral points on $5Y^2=X^4-3X^2+1$ are immediately determined by using MAGMA [1] to be $(\pm X,\pm Y)=(2,1)$, which gives $V_{ms}=2$, implying that $ms=0$, which is impossible.

**Subcase** $(ii)$ : Let $5\nmid t$. Since $5|n$, it follows that $5|m$. Then we can write $m=5^ra$ with $5\nmid a$ and $r\geq 1$. By (2.16), it is sen that $U_m=U_{5^ra}=5U_{5^{r-1}a}(5a_1+1)$ for some positive integer $a_1$. Thus, we conclude that $U_m=U_{5^ra}=5^rU_a(5a_1+1)(5a_2+1)...(5a_r+1)$ for some positive integers $a_i$ with $1\leq i\leq r$. Let $A=(5a_1+1)(5a_2+1)...(5a_r+1)$. Then, we have $U_m=5^rU_aA$, where $5\nmid A$. In a way similar, we get $U_n=U_{5^rat}=5^rU_{at}(5b_1+1)(5b_2+1)...(5b_r+1)$ for some positive integers $b_i$ with $1\leq i\leq r$. Let $B=(5b_1+1)(5b_2+1)...(5b_r+1)$. Hence, we have $U_n=5^rU_{at}B$, where $5\nmid B$. Substituting the new values of $U_n$ and $U_m$ into $U_n=5U_mx^2$ gives

$$5^rU_{at}B=5\cdot 5^rU_aAx^2$$

i.e.,

$$U_{at}B=5U_aAx^2.$$

On the other hand, since $a$ is even and $at$ is even, it follows from Lemma 2 that $U_{at}\equiv\pm\frac{at}{2}P\ (mod\ P^2)$ and $U_a\equiv\pm\frac{a}{2}P\ (mod\ P^2)$. Hence, we have

$$\pm\frac{at}{2}PB\equiv\pm 5\frac{a}{2}PAx^2\ (mod\ P^2),$$

implying that

$$\frac{at}{2}B\equiv 5\frac{a}{2}Ax^2\ (mod\ P^2).$$

Since $5|P$, it follows that $5|\frac{at}{2}B$, which shows that $5|atB$. This contradicts the fact that $5\nmid a$, $5\nmid b$, and $5\nmid B$. This concludes the proof for the case when $5|P$.

**Case** $III$ : Let $P^2\equiv 1\ (mod\ 5)$, $n$ is odd, and $P$ is odd. Then, both $m$ and $t$ are odd. Since $5|U_n$, it follows immediately from Lemma 4 that $3|n$. Using the fact that $n=mt$, we have

**Subcase** $(i)$ : Assume that $3|m$. Since $t$ is odd, we can write $t=4q\pm 1$ for some $q>0$. If $t=4q+1$, then $t=2\cdot 2^ra+1$ with $a$ odd and $r>0$. And so by (2.3), we get $U_n=U_{mt}=U_{2\cdot 2^ram+m}\equiv -U_m\ (mod\ V_{2^r})$, implying that $5U_mx^2\equiv -U_m\ (mod\ V_{2^r})$. Since $(U_m,V_{2^r})=1$ by (2.14), it follows that $5x^2\equiv -1\ (mod\ V_{2^r})$, which is impossible since $\left(\frac{5}{V_{2^r}}\right)=1$ by (2.25) and $\left(\frac{-1}{V_{2^r}}\right)=-1$ by (2.20). If $t=4q-1$, then by (2.1), we get $U_n=U_{m(4q-1)}=U_{2\cdot 2mq-m}\equiv -U_m\ (mod\ U_{2m})$. This shows that $5U_mx^2\equiv -U_m\ (mod\ U_{2m})$, implying that $5x^2\equiv -1\ (mod\ V_m)$ by

(2.6). Since $3|m$, it is seen by (2.12) that $V_3|V_m$. Hence, we obtain $5x^2 \equiv -1$ $(mod$ $V_3)$, i.e., $5x^2 \equiv -1$ $(mod$ $P^2-3)$. But this is impossible since

$$\left(\frac{5}{(P^2-3)/2}\right) = \left(\frac{(P^2-3)/2}{5}\right) = \left(\frac{-1}{5}\right) = 1$$

and

$$\left(\frac{-1}{(P^2-3)/2}\right) = (-1)^{\frac{P^2-5}{4}} = -1.$$

**Subcase** $(ii)$ : Assume that $3 \nmid m$. Since $n = mt$ and $3|n$, it follows that $3|t$ and therefore $t = 3s$ for some $s > 0$. Then by (2.9), we get

$$U_n = U_{3ms} = U_{ms}\left((P^2-4)U_{ms}^2+3\right) = 5U_m x^2,$$

implying that

$$(U_{ms}/U_m)\left((P^2-4)U_{ms}^2+3\right) = 5x^2.$$

Clearly,

$$d = (U_{ms}/U_m, \left((P^2-4)U_{ms}^2+3\right) = 1 \text{ or } 3.$$

If $d = 1$, then either

$$U_{ms} = U_m a^2,\ (P^2-4)U_{ms}^2+3 = 5b^2 \tag{3.9}$$

or

$$U_{ms} = 5U_m a^2,\ (P^2-4)U_{ms}^2+3 = b^2 \tag{3.10}$$

for some $a, b > 0$. Suppose (3.9) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 5b^2$ and this gives by (2.7) that $V_{2ms} = 5b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2^r a \pm 1$ with $a$ odd and $r > 0$. By using (2.4), we get $5b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2$ $(mod$ $V_{2^r})$. This shows that $5b^2 - 1 \equiv -(P^2-2)$ $(mod$ $V_{2^r})$, implying that $5b^2 \equiv -(P^2-3)$ $(mod$ $V_{2^r})$. By using (2.20), (2.25), and (2.21), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

a contradiction. Suppose (3.10) holds. It can be easily seen by combining two equations that $b^2 \equiv 3$ $(mod$ $5)$, which is impossible.

If $d = 3$, then either

$$U_{ms} = 3U_m a^2,\ (P^2-4)U_{ms}^2+3 = 15b^2 \tag{3.11}$$

or

$$U_{ms} = 15U_m a^2,\ (P^2-4)U_{ms}^2+3 = 3b^2 \tag{3.12}$$

for some $a, b > 0$. If we combine two equations given in (3.11), then we readily obtain $b^2 \equiv 2$ $(mod$ $3)$, which is impossible. Suppose (3.12) holds. Then by (2.10), we get $V_{ms}^2 - 1 = 3b^2$ and this gives by (2.7) that $V_{2ms} = 3b^2 - 1$. Since $ms > 1$ is odd, $ms = 4q \pm 1$ for some $q > 0$. Thus $ms = 2 \cdot 2^r a \pm 1$ with $a$ odd and $r > 0$. By using (2.4), we get $3b^2 - 1 = V_{2ms} \equiv -V_{\pm 2} \equiv -V_2$ $(mod$ $V_{2^r})$. This shows that

$3b^2-1 \equiv -(P^2-2) \ (mod \ V_{2^r})$, implying that $3b^2 \equiv -(P^2-3) \ (mod \ V_{2^r})$. By (2.23), (2.24), (2.20), and (2.21), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right) = -1,$$

a contradiction.

**Case** $IV$ : Let $P^2 \equiv 1 \ (mod \ 5)$, $n$ is even, and $P$ is odd. Since $n = mt$, we divide the proof into two subcases:

**Subcase** $(i)$ : Assume that $t$ is even. Then $t = 2s$ for some $s > 0$. Hence, we immediately have $U_n/U_m = U_{2ms}/U_m = (U_{ms}/U_m)V_{ms} = 5x^2$. Clearly, $d = (U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.14). If $d = 1$, then

$$U_{ms} = U_m a^2, \ V_{ms} = 5b^2 \tag{3.13}$$

or

$$U_{ms} = 5U_m a^2, \ V_{ms} = b^2 \tag{3.14}$$

for some $a, b > 0$. Suppose (3.13) is satisfied. Since $5|V_{ms}$, it follows from (2.19) that $5|P$, which contradicts the fact that $P^2 \equiv 1 \ (mod \ 5)$. Now suppose (3.14) is satisfied. By Theorem 5, the only possible value of $ms$ in which $V_{ms} = b^2$ is 1, which is impossible since $m > 1$.

If $d = 2$, then

$$U_{ms} = 2U_m a^2, \ V_{ms} = 10b^2 \tag{3.15}$$

or

$$U_{ms} = 10U_m a^2, \ V_{ms} = 2b^2 \tag{3.16}$$

for some $a, b > 0$. Obviously, (3.15) is not satisfied because of the same reason given above for (3.13). If (3.16) holds, then it is seen by Theorem 5 that the only possible values of $ms$ and $P$ in which $V_{ms} = 2b^2$ are $ms = 3$ and $P = 3, 27$. But this is impossible since $P^2 \equiv 1 \ (mod \ 5)$.

**Subcase** $(ii)$ : Assume that $t$ is odd. Since $t > 1$, we can write $t = 4q+1$ for some $q > 0$ or $t = 4q+3$ for some $q \geq 0$. On the other hand, since $n$ is even and $n = mt$, it follows that $m$ is even. Therefore we can write $m = 2^r a$ with $a$ odd and $r > 0$. Assume that $t = 4q+1$. Then $n = mt = 4qm + m = 2 \cdot 2^{r+k} b + m$ with $b$ odd. Hence, we get

$$5U_m x^2 = U_n = U_{2 \cdot 2^{r+k} b + m} \equiv -U_m \ (mod \ V_{2^{r+k}})$$

by (2.3). Since $(U_m, V_{2^{r+k}}) = (U_{2^r a}, V_{2^{r+k}}) = 1$ by (2.14), it follows that

$$5x^2 \equiv -1 \ (mod \ V_{2^{r+k}}).$$

This is impossible. Because $\left(\frac{5}{V_{2^{r+k}}}\right) = 1$ and $\left(\frac{-1}{V_{2^{r+k}}}\right) = -1$ by (2.25) and (2.20), respectively. Now assume that $t = 4q+3$. Then we have $n = mt = 4qm + 3m$. And so by (2.1), we get

$$5U_m x^2 = U_n = U_{4qm+3m} \equiv U_{3m} \ (mod \ U_{2m}).$$

By using (2.6) and (2.9), we readily obtain

$$5x^2 \equiv V_m^2 - 1 \ (mod \ V_m),$$

which implies that

$$5x^2 \equiv -1 (mod \ V_m).$$

Using the fact that $m = 2^r a$ with $a$ odd, we have

$$5x^2 \equiv -1 \ (mod \ V_{2^r a}),$$

implying that

$$5x^2 \equiv -1 \ (mod \ V_{2^r})$$

by (2.12). But this is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.25) and (2.20), respectively. This completes the proof. □

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# FIXED POINT THEOREMS IN BIMETRIC SPACE ENDOWED WITH BINARY RELATION AND APPLICATIONS

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*Abstract.* In this paper, we obtained some fixed point results for continuous mappings satisfying a generalized contractive condition in the setting of two metrics space endowed with a binary relation. Our theorems generalize and extend several known results in the literature. As application, we establish an existence theorem for the solution of a nonlinear first order differential equation.

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## 1. Introduction

The Banach contraction mapping is one of the pivotal results of analysis. It is very popular tool for solving existence and uniqueness problems in many different fields of mathematics. Due to its importance and applications potential, the Banach Contraction Mapping Principle has been investigated heavily by many authors. Consequently, a number of generalizations of this celebrated principle have appeared in the literature (see [1–19]).

Recently, Samet and Turinici [17] introduced the notion of contractive mapping in a metric space endowed with amorphous binary relation. They showed a theorem subsumes many known results in the literature. For further study about contractive mappings in a metric space endowed with binary relation, we refer the reader to [3–7], [13], [17] and [19].

The purpose of this paper is to establish some fixed point results in the setting of two metrics space; called bimetric space; endowed with a binary relation. Our results generalized and extended many existing fixed point theorems, for generalized contractive and quasi-contractive mappings, in a metric space endowed with binary relation. Also, an application to the study the existence of solution to a nonlinear first order differential equation has been given.

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## 2. Preliminaries

We first recall Maia's fixed point theorem:

**Theorem 1** ([14]). *Let $(X,d,\delta)$ be a bimetric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(i) *$d(x,y) \leq \delta(x,y)$ for all $x,y \in X$;*
(ii) *$X$ is complete with respect to $d$;*
(iii) *$T$ is continuous with respect to $d$;*
(iv) *there exists a constant $\alpha \in [0,1)$ such that*

$$\delta(Tx,Ty) \leq \alpha\delta(x,y).$$

*Then $T$ has a unique fixed point in $X$.*

In the sequel, let $(X,d,\delta)$ be a bimetric space, and $T : X \to X$ be a mapping. Denote

$$Fix(T) := \{x^* \in X : x^* = Tx^*\}.$$

Let $\mathcal{R}$ be a binary relation on $X$, and let $\mathcal{S}$ be the *symmetric binary relation* defined by

$$x,y \in X, \quad x\mathcal{S}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

From $x_0 \in X$, we define the sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

**Definition 1.** Let $(X,\delta)$ be metric space and $n \in \mathbb{N} \cup \{0\}$. For $A \subset X$ we denote by $D(A) := \sup\{\delta(a,b) : a,b \in A\}$. For each $x_0 \in X$, the orbit sets of $T$ at $x_0$ are defined as following

$$O_n(x_0) = \{x_0, x_1, \ldots, x_n\} \quad \text{and} \quad O_\infty(x_0) = \{x_0, x_1, x_2, \ldots\}.$$

We say that $(X,\delta)$ is *$T$-orbitally complete* iff every $\delta$-Cauchy sequence from $O_\infty(x)$ for some $x \in X$ converges in $X$.

**Definition 2** ([17]). A subset $D$ of $X$ is called $\mathcal{R}$-directed if for every $x,y \in D$, there exists $z \in X$ such that $x\mathcal{R}z$ and $y\mathcal{R}z$.

**Definition 3.** A mapping $T : X \to X$ is called $\mathcal{R}$-preserving mapping if

$$x,y \in X, \quad x\mathcal{R}y \Longrightarrow Tx\mathcal{R}Ty.$$

Next, we define the set $\Phi$ of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying:

(I) $\varphi$ is nondecreasing;
(II) $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where $\varphi^n$ is the $n$-th iterate of $\varphi$.

*Remark* 1. Let $\varphi \in \Phi$. We have $\varphi(t) < t$ for all $t > 0$.

*Remark* 2. Let $\varphi \in \Phi$. We have $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

**Definition 4.** Assume that for $T : X \to X$, there exists $\varphi \in \Phi$ such that

$$\delta(Tx, Ty) \leq \varphi(M_\delta(x, y)), \quad \text{for all } x, y \in X \text{ with } x\S y.$$

A mapping $T$ is called a *generalized contractive* with respect to $\delta$, if

$$M_\delta(x, y) = \max\{\delta(x, y), \frac{1}{2}[\delta(x, Tx) + \delta(y, Ty)], \frac{1}{2}[\delta(x, Ty) + \delta(y, Tx)]\}).$$

A mapping $T$ is called a *generalized quasi-contractive* with respect to $\delta$, if

$$M_\delta(x, y) = \max\{\delta(x, y), \delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(y, Tx)\}).$$

## 3. Main results

To begin with, we have the following result.

**Theorem 2.** *Let $(X, d, \delta)$ be a bimetric space, and $\mathcal{S}$ be a binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(A1) *$d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;*
(A2) *$(X, d)$ is $T$-orbitally complete;*
(A3) *$T$ is continuous with respect to $d$;*
(A4) *$T$ is $\mathcal{S}$-preserving;*
(A5) *there exists $x_0 \in X$ with $x_0 \mathcal{S} T x_0$;*
(A6) *$T$ is a generalized contractive with respect to $\delta$.*

*Then $T$ has a fixed point $x^*$ in $X$. Moreover, if in addition, $Fix(T)$ is $\mathcal{S}$-directed, then $x^*$ is the unique fixed point of $T$ in $X$.*

*Proof.* From (A5), there exists $x_0 \in X$ with $x_0 \mathcal{S} T x_0$, and from (A4), $T$ is $\mathcal{S}$-preserving, we get

$$x_n \mathcal{S} T x_n \text{ for all } n \geq 0. \tag{3.1}$$

If $x_n = Tx_n$, then $x_n$ is a fixed point of $T$. Suppose that $x_n \neq Tx_n$ for all $n$. Since (3.1) is satisfied for all $n \geq 1$, by applying the contraction condition (A6), we have

$$\delta(x_n, Tx_n) \leq \varphi(\max\{\delta(x_{n-1}, x_n), \frac{1}{2}[\delta(x_{n-1}, x_n) + \delta(x_n, x_{n+1})], \frac{1}{2}[\delta(x_{n-1}, x_{n+1}) + \delta(x_n, x_n)]\}))$$

Hence, we get

$$\delta(x_n, x_{n+1}) \leq \varphi(\max\{\delta(x_{n-1}, x_n), \delta(x_n, x_{n+1})\}).$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence in $(X, \delta)$. If for some $n \geq 1$, we have $\delta(x_{n-1}, x_n) \leq \delta(x_n, x_{n+1})$, then from Remark 1, we get

$$\delta(x_n, x_{n+1}) \leq \varphi(\delta(x_n, x_{n+1})) < \delta(x_n, x_{n+1}),$$

which is a contradiction. Thus, $\delta(x_{n-1}, x_n) > \delta(x_n, x_{n+1})$, and

$$\delta(x_n, x_{n+1}) \leq \varphi(\delta(x_{n-1}, x_n)), \quad \text{for all } n \geq 1.$$

Hence, by induction, we obtain

$$\delta(x_n, x_{n+1}) \leq \varphi^n(\delta(x_0, x_1)), \quad \text{for all } n \geq 1. \tag{3.2}$$

Fix $\varepsilon > 0$ and let $n_\varepsilon$ be a positive integer such that

$$\sum_{n \geq n_\varepsilon} \varphi^n(\delta(x_1, x_0)) < \varepsilon.$$

Using the triangular inequality and (3.2), for some $m > n > n_\varepsilon$, we get

$$\delta(x_n, x_m) \leq \sum_{k=n}^{m-1} \delta(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi^k(\delta(x_1, x_0)) \leq \sum_{n \geq n_\varepsilon} \varphi^n(\delta(x_1, x_0)) < \varepsilon.$$

Consequently, the sequence $\{x_n\}$ is $\delta$-Cauchy, so by (A1), $\{x_n\}$ is $d$-Cauchy too. Since from (A2), we have metric space $(X, d)$ is $T$-orbitally complete, then there exists $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{3.3}$$

From (A3), we have that $T$ is continuous with respect to $d$, and, so it follows that $x^* = \lim_{n \to \infty} Tx_n = T(\lim_{n \to \infty} x_n) = Tx^*$, that is, $x^*$ is a fixed point of $T$.

Next, suppose that $Fix(T)$ is $\mathcal{S}$-directed, and we will show that $x^*$ is the unique fixed point of $T$ in $X$. Suppose that $y^* \in Fix(T)$ is another fixed point of $T$. Then, there exists $z_0 \in X$ such that $z_0 \mathcal{S} x^*$ and $z_0 \mathcal{S} y^*$. Define the sequence $\{z_n\}$ in $X$ by $z_{n+1} = Tz_n$ for all $n \geq 0$. Since $T$ is $\mathcal{S}$-preserving, for all $n \geq 0$, we have $z_n \mathcal{S} x^*$ and $z_n \mathcal{S} y^*$. Applying (A6), for all $n \geq 0$, we get

$$\begin{aligned}
\delta(z_{n+1}, x^*) &= \delta(Tz_n, Tx^*) \\
&\leq \varphi(M(z_n, x^*)) \\
&= \varphi(\max\{\delta(z_n, x^*), \tfrac{1}{2}[\delta(z_n, z_{n+1}) + \delta(x^*, x^*)], \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})]\}) \\
&\leq \varphi(\max\{\delta(z_n, x^*), \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})], \tfrac{1}{2}[\delta(z_n, x^*) + \delta(x^*, z_{n+1})]\}) \\
&\leq \varphi(\max\{\delta(z_n, x^*), \delta(x^*, z_{n+1})\}).
\end{aligned}$$

Now, we will show that $\lim_{n \to \infty} \delta(z_n, x^*) = 0$. Without the loss of generality, suppose that $\delta(z_n, x^*) > 0$ for all $n$. Assume that for some $n$, we have $\delta(z_n, x^*) \leq \delta(x^*, z_{n+1})$. Hence,

$$\delta(z_{n+1}, x^*) \leq \varphi(\delta(x^*, z_{n+1})) < \delta(x^*, z_{n+1}),$$

which is a contradiction. Then, for all $n \geq 0$, we have $\delta(z_n, x^*) > \delta(x^*, z_{n+1})$. Consequently, for all $n$, we obtain

$$\delta(z_{n+1}, x^*) \leq \varphi(\delta(x^*, z_n)).$$

By induction, for all $n$, we get

$$\delta(z_n, x^*) \leq \varphi^n(\delta(x^*, z_0)).$$

Which implies that $\lim_{n\to\infty} \delta(z_n, x^*) = 0$. Similarly, we can prove that

$$\lim_{n\to\infty} \delta(z_n, y^*) = 0.$$

Hence, $x^* = y^*$. □

To prove our next result, we need the following lemmas which will be used in the sequel.

**Lemma 1.** *Let $n \in \mathbb{N}$, $(X, \delta)$ be a metric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(a1) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$;*
(a2) *$T$ is $\mathcal{R}$-preserving mapping;*
(a3) *$T$ is generalized quasi-contractive with respect to $\delta$.*

*Then, $\delta(x_i, x_j) \leq \varphi(D(O_n(x_0)))$ for all $i, j \in \{1, \dots, n\}$.*

*Proof.* From (a1) there exists $x_0 \in X$ such that $x_0 \mathcal{R} x_1$. Hence, by (a2) we get $x_k \mathcal{R} x_{k+1}$ for all $k$. Since, $\mathcal{R}$ is transitive, then

$$x_{i-1} \mathcal{R} x_{j-1} \text{ for all } 1 \leq i < j \leq n. \tag{3.4}$$

Without loss of generality, we can suppose that $1 \leq i < j \leq n$ for which we have $x_{i-1}, x_i, x_{j-1}, x_j \in O_n(x_0)$. Now, using (a3) and (3.4), we get

$$\delta(Tx_{i-1}, Tx_{j-1}) \leq$$
$$\varphi\big(\max\{\delta(x_{i-1}, x_{j-1}), \delta(x_{i-1}, x_i), \delta(x_{j-1}, x_j), \delta(x_{i-1}, x_j), \delta(x_i, x_{j-1})\}\big),$$

which implies that

$$\delta(x_i, x_j) \leq \varphi(D(O_n(x_0))).$$ □

*Remark* 3. From the previous lemma and Remark 1, we have

$$\delta(x_i, x_j) \leq \varphi(D(O_n(x_0))) < D(O_n(x_0)) \text{ for all } 1 \leq i < j \leq n,$$

which implies the existence of an integer $k \leq n$ such that $\delta(x_0, T^k x_0) = D(O_n(x_0))$.

**Lemma 2.** *Let $(X, \delta)$ be a metric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(b1) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$;*
(b2) *$T$ is $\mathcal{R}$-preserving mapping;*
(b3) *$T$ is generalized quasi-contractive with respect to $\delta$.*

*Then,*

$$D(O_\infty(x_0)) \leq \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)).$$

*Proof.* At first, we note that we have

$$D(O_1(x_0)) \leq D(O_2(x_0)) \leq \dots,$$

which implies that $D(O_\infty(x)) = \sup\{D(O_n(x)) : n \in \mathbb{N}\}$. By (b1), there exists $x_0 \in X$ such that $x_0 \mathcal{R} x_1$, then (3.4) holds. Let $n \in \mathbb{N}$ so by Lemma 1, there exists $T^k x_0 \in O_n(x_0)$ for $1 \leq k \leq n$, such that $\delta(x_0, T^k x_0) = D(O_n(x_0))$. Using the triangular inequality, (3.4) and (b3), we get

$$\begin{aligned} D(O_n(x_0)) = \delta(x_0, T^k x_0) &\leq \delta(x_0, Tx_0) + \delta(Tx_0, T^k x_0) \\ &\leq \delta(x_0, Tx_0) + \varphi(D(O_n(x_0))), \end{aligned}$$

that is,

$$(id - \varphi)(D(O_n(x_0))) \leq \delta(x_0, Tx_0)$$

where $id : [0, +\infty) \to [0, +\infty)$ is the identity function. It follows from the monotony of $\varphi$ that $\sum_{\ell=0}^{k} \varphi^\ell$ is monotone too for all $k \in \mathbb{N}$. Thus, by applying $\sum_{\ell=0}^{k} \varphi^\ell$ to both side of the previous inequality, we get

$$\sum_{\ell=0}^{k} \varphi^\ell((id - \varphi)(D(O_n(x_0)))) \leq \sum_{\ell=0}^{k} \varphi^\ell(\delta(x_0, Tx_0))$$

which implies

$$D(O_n(x_0)) - \varphi^{k+1}(D(O_n(x_0))) \leq \sum_{\ell=0}^{k} \varphi^\ell(\delta(x_0, Tx_0))$$

Since $\sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)) < \infty$, then by letting $k \to \infty$ in the above inequality, we get

$$D(O_n(x_0)) \leq \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)), \quad \text{for all } n \geq 1.$$

Hence, we get

$$D(O_\infty(x_0)) \leq \sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0)). \qquad \square$$

Now, we are ready to state our second main result.

**Theorem 3.** *Let $(X, d, \delta)$ be a bimetric space, and $\mathcal{R}$ a transitive binary relation over $X$. Assume that for $T : X \to X$, the following conditions are satisfied:*

(B1) *$d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;*
(B2) *$(X, d)$ is $T$-orbitally complete;*
(B3) *$T$ is continuous with respect to $d$;*
(B4) *$T$ is $\mathcal{R}$-preserving;*

(B5) *there exists* $x_0 \in X$ *with* $x_0 \mathcal{R} T x_0$*;*

(B6) $T$ *is a generalized quasi-contractive with respect to* $\delta$*.*

*Then* $T$ *has a fixed point* $x^*$ *in* $X$*. Moreover, if in addition,* $\mathcal{R}$ *is symmetric and* $Fix(T)$ *is* $\mathcal{R}$*-directed, then* $x^*$ *is the unique fixed point of* $T$ *in* $X$*.*

*Proof.* Let $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$. Define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$, for all $n \geq 0$. Since $T$ is an $\mathcal{R}$-preserving, then $x_n \mathcal{R} x_{n+1}$ for all $n$. Let $n$ and $m$ $(n < m)$ be any positive integers. From (B6) and Lemma 1 it follows

$$\delta(T^n x_0, T^m x_0) = \delta(TT^{n-1}x_0, T^{m-n+1}T^{n-1}x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1}x_0))).$$

From Remark 1, there exists an integer $k_1$, $1 \leq k_1 \leq m-n+1$, such that

$$D(O_{m-n+1}(T^{n-1}x_0)) = \delta(T^{n-1}x, T^k T^{n-1}x_0).$$

Using Lemma 1 again, we get

$$\begin{aligned}\delta(T^{n-1}x_0, T^k T^{n-1}x_0) &= \delta(TT^{n-2}x_0, T^{k_1+1}T^{n-2}x_0)\\ &\leq \varphi(D(O_{k_1+1}(T^{n-2}x_0)))\\ &\leq \varphi(D(O_{m-n+2}(T^{n-2}x_0)))\end{aligned}$$

Combining, the above inequalities, we get

$$\delta(T^n x_0, T^m x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1}x_0))) \leq \varphi^2(D(O_{m-n+2}(T^{n-2}x_0))).$$

Continue this process, we obtain

$$\delta(T^n x_0, T^m x_0) \leq \varphi(D(O_{m-n+1}(T^{n-1}x_0))) \leq \cdots \leq \varphi^n(D(O_m(x_0))).$$

By Lemma 2, we get

$$\delta(T^n x_0, T^m x_0) \leq \varphi^n(\sum_{\ell=0}^{\infty} \varphi^\ell(\delta(x_0, Tx_0))).$$

Since $\lim_{n\to\infty} \varphi^n(s) = 0$ for all $s \geq 0$, it follows that the sequence $\{T^n x_0\}$ is a $\delta$-Cauchy sequence. Therefore, by (B1) the sequence $\{T^n x_0\}$ is a $d$-Cauchy sequence too. Now, since the metric space $(X, d)$ is $T$-orbitally complete, we deduce that the sequence $\{T^n x_0\}$ converges to some $x^*$ in $X$. From (B3), we have that $T$ is continuous with respect to $d$, and, so it follows that $x^* = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx^*$, that is, $x^*$ is a fixed point of $T$.

Now, suppose that $Fix(T)$ is $\mathcal{R}$-directed, and we claim that the fixed point is unique. Let $x^*$ and $y^*$ two fixed points of $T$. Suppose that $x^* \neq y^*$. Since $Fix(T)$ is $\mathcal{R}$-directed, then there exists $z \in X$ such that $z \mathcal{R} x^*$ and $z \mathcal{R} y^*$, which implies by the transitivity of $\mathcal{R}$, that $x^* \mathcal{R} y^*$. Then, we apply the contraction condition (B6), we get

$$\begin{aligned}&\delta(x^*, y^*) = \delta(Tx^*, Ty^*) \leq\\ &\varphi(\max\{\delta(x^*, y^*), \delta(x^*, Tx^*), \delta(y^*, Ty^*), \delta(x^*, Ty^*), \delta(y^*, Tx^*)\})\end{aligned}$$

which implies, by Remark 1, that

$$\delta(x^*, y^*) \leq \varphi(\delta(x^*, y^*)) < \delta(x^*, y^*). \tag{3.5}$$

which is a contradiction. Consequently, $x^* = y^*$, and our claim holds. □

The following results are an immediate consequences of Theorems 2 and 3.

**Corollary 1.** *Theorem 1 is a particular case of Theorem 2.*

**Corollary 2.** *Let $(X, \leq)$ be a partially ordered set and $(X, d, \delta)$ be a bimetric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(C1) *$d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;*
(C2) *$X$ is complete with respect to $d$;*
(C3) *$T$ is continuous with respect to $d$;*
(C4) *$T$ is monotone nondecreasing mapping;*
(C5) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$;*
(C6) *there exists $\varphi \in \Phi$ such that*

$$\delta(Tx, Ty) \leq \varphi(M_\delta(x, y)) \text{ for all } x, y \in X \text{ with } x \leq y.$$

*Then $T$ has a unique fixed point in $X$.*

**Corollary 3.** *Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Assume that for $T : X \to X$, the following conditions are satisfied:*

(D1) *$T$ is continuous;*
(D2) *$T$ is monotone nondecreasing mapping;*
(D3) *there exists $x_0 \in X$ with $x_0 \leq Tx_0$;*
(D4) *there exists a constant $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha M_d(x, y), \quad \text{for all } x, y \in X \text{ with } x \leq y.$$

*Then $T$ has a unique fixed point in $X$.*

**Corollary 4.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be continuous mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$d(Tx, Ty) \leq \varphi\left(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\right), \ \forall x, y \in X.$$

*Then $T$ has a unique fixed point in $X$.*

## 4. Application to Cauchy problem

In this section, we study the Cauchy problem for a class of nonlinear differential equations, using the results obtained in the previous section.

**Theorem 4.** *Let $(X, \leq)$ is a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $T : X \to X$ be a*

*nondecreasing and continuous mappings with respect to $d$ and consider the following power series*

$$\sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y). \tag{4.1}$$

*Suppose there exists $\lambda \in (0,1)$, for which the power series (4.1) converges for all $x, y \in X$. For such $\lambda$ we define a function $\delta$ by*

$$\delta(x,y) = \sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y). \tag{4.2}$$

*Suppose that also that*

$$\delta(x,y) \leq \frac{1}{1-\lambda^2} d(x,y) \quad \textit{for all } x \leq y. \tag{4.3}$$

*Then,*

(i) *$\delta$ is a metric satisfies $d(x,y) \leq \delta(x,y)$ for all $x, y \in X$;*
(ii) *$T$ has a unique fixed point in $X$.*

*Proof.* The function $\delta$ is a metric. Indeed, If $x = y$, then $\delta(x,x) = 0$. On the other hand, if $\delta(x,y) = 0$, so $\sum_{n=0}^{\infty} \lambda^n d(T^n, x, T^n y) = 0$ which implies that $x = y$ since $d(T^n, x, T^n y) \geq 0$ for all $n$.
Next, it is easy to prove that $\delta(x,y) = \delta(y,x)$ and that $\delta$ satisfies also the triangle inequality.

Now, we shall prove (ii). From hypothesis, there exists some $\lambda \in (0,1)$ such that

$$\begin{aligned} \delta(x,y) &= \sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y) \\ &= d(x,y) + \lambda \sum_{0}^{\infty} \lambda^n d(T^{n+1} x, T^{n+1} y) \\ &= d(x,y) + \lambda \delta(Tx, Ty) \end{aligned} \tag{4.4}$$

On the other hand, from condition (4.3) we have

$$\delta(x,y) \leq \frac{1}{1-\lambda^2} d(x,y) \quad \text{for all } x \leq y.$$

Therefore, from (4.4) we get

$$\delta(Tx, Ty) \leq \lambda \delta(x,y) \quad \text{for all } \ x \leq y.$$

All conditions of Theorem 2 are satisfied, then we conclude (ii). □

**Theorem 5.** *Let $(X,\leq)$ is a partially ordered set and suppose that there exist a metric $d$ on $X$ such that $(X,d)$ is a complete metric space. Let $T : X \to X$ be a self mappings. Suppose that the condition (4.3) holds, further suppose that there exists a constant $L \geq 1$ and $k \geq 1$ such that*

$$d(T^k x, T^k y) \leq L d(x,y) \quad \text{for all } x \leq y. \tag{4.5}$$

*Then the power series*

$$S = \sum_{n=0}^{\infty} \lambda^n d(T^n x, T^n y), \tag{4.6}$$

*converges in $X$ for all $x, y \in X$ with $x \leq y$.*

*Proof.* From hypothesis, there exists some $L \geq 1$ and $k \leq 1$ such that:

$$d(T^k x, T^k y) \leq L d(x,y) \quad \text{for all } x \leq y.$$

By induction we get

$$d(T^{nk} x, T^{nk} y) \leq L^n d(x,y) \quad \text{for all } n \geq 1 \text{ and } x \leq y. \tag{4.7}$$

Hence, we get

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \lambda^{nk} d(T^{nk} x, T^{nk} y) + \lambda^{nk+1} d(T^{nk+1} x, T^{nk+1} y) + \dots \\ &\qquad + \lambda^{nk+(k-1)} d(T^{nk+(k-1)} x, T^{nk+(k-1)} y) \\ &\leq \sum_{n=0}^{\infty} \lambda^{nk} L^n d(x,y) + \lambda^{nk+1} L^{n+1} d(Tx, Ty) + \dots \\ &\qquad + \lambda^{nk+(k-1)} L^{n+(k-1)} d(T^{(k-1)} x, T^{(k-1)} y) \\ &\leq \sum_{p=0}^{k-1} \lambda^p L^{2p} \sum_{n=0}^{\infty} (\lambda^k L)^n d(x,y) \end{aligned}$$

for all $x \leq y$. We can choose $0 < \lambda^k < \frac{1}{L}$ to obtain the convergence of the power series (4.6) for all $x, y \in X$ with $x \leq y$. □

Now, consider the nonlinear differential equation

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in [a,b] \\ x(t_0) &= x_0, \end{cases} \tag{4.8}$$

where $a, b, t_0 \in \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$. Let $X = C([a,b], \mathbb{R})$ denotes the space of all continuous $\mathbb{R}$-valued functions on $[a,b]$. We endow this space with the metric $d$ given by

$$d(u,v) = \sup_{t \in [a,b]} |u(t) - v(t)|, \quad \text{for all } u, v \in X.$$

It is well known that $(X,d)$ is a complete metric space. We define an order relation $\leq$ on $X$ by

$$u,v \in X, \quad u \leq v \Longleftrightarrow u(t) \leq v(t), \text{ for all } t \in [a,b].$$

Consider the mapping $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s,x(s)))ds; \quad t \in [a,b],$$

for all $x \in C([a,b],\mathbb{R})$. Clearly, $x^* \in C([a,b],\mathbb{R})$ is a solution of (4.8) if and only if $x^*$ is a fixed point of $T$.
Furthermore, we consider the following assumptions:

(H1) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous;
(H2) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to the second variable;
(H3) $|f(t,x(t)) - f(t,y(t))| \leq L|x(t) - y(t)|$ for all $x(t) \leq y(t)$ and $t \in [a,b]$.

It is worth noting that condition (H3) is weaker compared to those used by Maia for studying Cauchy problem in [14], that is, $f$ is $L$-Lipschitzien function on the whole space.

The next result shows that $T$ satisfies the conditions of Corollary 2.
Let $d_r(x,y) = \sup_{a \leq t \leq r} |x(t) - y(t)|$ where $r \in [a,b]$. From condition (H3) it follows

$$d_r(T^n x, T^n y) \leq \frac{L^n (b-a)^n}{n!} d_r(x,y) \quad \text{for all } x \leq y, \quad n \geq 0. \tag{4.9}$$

Indeed, obviously (4.9) holds for $n = 0$. Suppose that (4.9) holds for some integer $n$, then using (H2) and (H3), we get

$$\begin{aligned} d_r(T^{n+1}x, T^{n+1}y) &= \sup_{a \leq t \leq r} \left| \int_{t_0}^{t} (f(s,T^n x(s)) - f(s,T^n y(s)))\, ds \right| \\ &\leq \int_a^r L d_r(T^n x(s), T^n y(s))\, ds \\ &\leq \frac{L^{n+1}}{n!} \int_a^r (s-a)^n\, ds\, d_r(x,y) = \frac{L^{n+1}(r-a)^{n+1}}{(n+1)!} d_r(x,y) \end{aligned}$$

This implies that (4.9) holds. Next, by taking $r = b$ in (4.9), we obtain

$$d(T^n x, T^n y) \leq \frac{L^n (b-a)^n}{n!} d(x,y) \quad \text{for all } x \leq y \text{ and } n \geq 0. \tag{4.10}$$

**Theorem 6.** *Suppose that (H1)–(H3) hold. Then (4.8) has at least one solution* $x^* \in C([a,b],\mathbb{R})$.

*Proof.* We shall show the existence of $\lambda \in (0,1)$ such that (4.3) holds. First, we have

$$d(T^n x, T^n y) \leq \frac{L^n (b-a)^n}{n!} d(x,y) \quad \text{for all } x \leq y, \quad n \geq 0.$$

Thus, we get

$$\delta(x,y) \leq \exp(L(b-a))d(x,y) \quad \text{for all } x \leq y.$$

If we take $\lambda = \sqrt{1-\exp(L(a-b))}$, then the condition (4.3) holds.
Since, we have

$$d(T^n x, T^n y) \leq \exp(L(b-a))d(x,y) \quad \text{for all } x \leq y, \quad n \geq 0,$$

then the condition (4.5) holds too. We deduce by using Corollary 2 that $T$ has a unique fixed point $x^* \in C([a,b],\mathbb{R})$. □

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# ASSOCIATED CURVES OF FRENET CURVES IN THREE DIMENSIONAL COMPACT LIE GROUP

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*Abstract.* General definition of associated curves of a Frenet curve is given in a three dimensional compact Lie group $G$. The principal normal direction curve and principal normal donor curve are introduced and some characterizations for these curves are obtained in $G$. Later, the relationships between a principal normal direction curve and some special curves such as helix, slant helix or curve with a special torsion are obtained.

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*Keywords:* Lie group, associated curve, donor curve

## 1. Introduction

The curve theory in the Euclidean space or in a Riemannian manifold has been developed by many mathematicians so far by the view of local or global differential geometry. The most fascinating subject of curve theory is to obtain some characterizations for a regular curve or curve pairs. The well-known characterizations obtained for a single curve have allowed to define some special curves such as helix, slant helix, plane curve, spherical curve, etc. [1, 11, 13, 20, 23] and these special curves, especially helices, are used in many applications [2, 9, 10, 19]. Similarly, by considering two curves, some special curve pairs such as involute-evolute curves, Bertrand curves, Mannheim curves have been defined and studied so far [4, 12, 14, 15, 18, 21, 22].

Furthermore, Çöken and Çiftçi have studied the degenerate semi-Riemannian geometry of a Lie group [6]. They have obtained a naturally reductive homogeneous semi-Riemannian space using the Lie group. Later, some of subjects given above have been considered in three dimensional Lie groups and some characterizations for these curves have been obtained in a three dimensional Lie group [3, 5, 16–18].

Recently, Choi and Kim have defined a new curve pair in the Euclidean 3-space $E^3$ [7]. They have considered a unit vector field $X$ defined in the Frenet basis of a curve $\alpha$ and an integral curve $\gamma$ of $X$ and given the definition and characterizations

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of principal-direction curve and principal-donor curve in $E^3$. They have also given some applications of these curves related to a helix, a slant helix or a plane curve.

In the present paper, we give the general definition of associated curves of a Frenet curve in a three dimensional compact Lie group $G$. Then, we define principal-direction curve and principal-donor curve in $G$ and obtain some characterizations for these curves. In the last section of the paper, we give the relationships between principal direction curves and some special curves such as helix, slant helix or rectifying curve in $G$.

## 2. Preliminaries

Let $G$ be a Lie group with a bi-invariant metric $\langle , \rangle$ and $D$ be the Levi-Civita connection of $G$. It is well-known that the Lie algebra $g$ of $G$ is isomorphic to $T_eG$ where $e$ is neutral element of $G$. Let $\langle , \rangle$ be a bi-invariant metric on $G$. Then we have

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle , \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X, Y], \tag{2.2}$$

for all $X, Y, Z \in g$ [5].

Let $\alpha : I \subset \mathrm{R} \to G$ be an arc-lenghted curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $g$. In this case, we can write any two vector fields $W$ and $Z$ along the curve $\alpha$ as $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ where $w_i : I \to \mathrm{R}$ and $z_i : I \to \mathrm{R}$ are smooth functions. Also the Lie bracket of two vector fields $W$ and $Z$ is given by

$$[W, Z] = \sum_{i=1}^{n} w_i z_i [X_i, X_j],$$

and the covariant derivative of $W$ along the curve $\alpha$ with the notation $D_{\alpha'} W$ is given as follows

$$D_{\alpha'} W = \dot{W} + \frac{1}{2}[T, W], \tag{2.3}$$

where $T = \alpha'$ and $\dot{W} = \sum_{i=1}^{n} \dot{w}_i X_i$ [8]. If $W$ is the left-invariant vector field to the curve $\alpha$ then it is important to point out that $\dot{W} = 0$. Let $G$ be a three dimensional Lie group and $(T, N, B, \kappa, \tau)$ denote the Frenet apparatus of the curve $\alpha$. Then the Frenet formulae of $\alpha$ are given as follows

$$D_T T = \kappa N, \ D_T N = -\kappa T + \tau B, \ D_T B = -\tau N$$

where $\kappa = \|\dot{T}\|$. If $\kappa = \|\dot{T}\| \neq 0$, then $\alpha$ is called a Frenet curve in $G$ [16].

**Proposition 1** ([17]). *Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B)$. Then the following equalities hold*

$$\begin{cases} [T, N] = \langle [T, N], B \rangle B = 2\tau_G B, \\ [T, B] = \langle [T, B], N \rangle N = -2\tau_G N. \end{cases} \tag{2.4}$$

Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve with arc length parameter $s$. Then from equation (2.3) and Proposition 2.1, the following Frenet formulae are obtained

$$\begin{pmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau - \tau_G \\ 0 & -(\tau - \tau_G) & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\{T, N, B\}$ is the Frenet frame $\tau_G = \frac{1}{2}\langle [T, N], B \rangle$ and $\kappa, \tau$ are curvature and torsion of $\alpha$ in three dimensional Lie group $G$ , respectively [17].

**Definition 1** ([17]). Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve in three dimensional compact Lie group with the Frenet apparatus $(T, N, B, \kappa, \tau)$. Then the harmonic curvature function of the curve $\alpha$ is defined by

$$H = \frac{\tau - \tau_G}{\kappa}. \tag{2.5}$$

**Theorem 1** ([5]). *Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve in three dimensional compact Lie group with the Frenet apparatus $(T, N, B, \kappa, \tau)$. The curve $\alpha$ is a general helix if and only if $\tau = c\kappa + \tau_G$ , where $c$ is a constant.*

From Definition 1 and Theorem 1, we have the following corollary:

**Corollary 1.** *The curve $\alpha : I \subset \mathrm{R} \to G$ is a general helix in $G$ if and only if $H =$ constant.*

**Theorem 2** ([17]). *Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$. Then $\alpha$ is a slant helix if and only if the function*

$$\sigma = \frac{\kappa(1 + H^2)^{3/2}}{H'}, \tag{2.6}$$

*is a constant where $H$ is the harmonic curvature function of the curve $\alpha$.*

**Theorem 3** ([16]). *Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized Bertrand curve whose curvature function $\kappa$ and harmonic curvature function $H$ satisfy $\lambda\kappa(s) + \mu\kappa(s)H(s) = 1$, for all $s \in I$, where $\lambda, \mu$ are constants. If the curve $\beta(s)$ given by $\beta(s) = \alpha(s) + \lambda N(s)$ for all $s \in I$ is a special Frenet curve, then $(\alpha, \beta)$ is the Bertrand curve couple.*

## 3. The principal-direction curves and the principal-donor curves in a compact Lie group

In this section, we study the principal-direction curves and the principal-donor curves in three dimensional compact Lie group $G$ with a bi-invariant metric. First, we give the existence and definition of an $W$-direction curve of a curve in a Riemannian manifold.

**Theorem 4.** *Let $M$ be an $n$-dimensional Riemannian manifold and $\alpha(t) : I \to U \subset M$ a curve on a local chart $U$ of $M$. Then, for a continuous vector field $W = W(t)$ on $U \subset M$ along the curve $\alpha$, there exists a curve $\gamma(t) : I \subset \mathrm{R} \to M$ on the neighborhood $U$ such that $\gamma'(t) = W(t)$.*

*Proof.* Let $x = (x_1, x_2, ...x_n)$ be a local coordinate on $U$. Then the vector field $W = W(t)$ on $U$ along $\gamma(t)$ is expressed by $\sum_{i=1}^{n} W_i(t) \frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$ where $W_i(t)$, $(1 \leq i \leq n)$ are continuous real valued functions on the interval $I$. Let $x_i : I \to \mathrm{R}$ be a solution of the linear ordinary differential equation $\frac{\partial}{\partial t} x_i(t) = W_i(t)$, for each $i$. Then, the curve $\gamma$ is clearly $x^{-1} \circ (x_1(t), x_2(t), ..., x_n(t))$. □

*Remark* 1. Let $\alpha(t) : I \to U \subset M$ be a curve in an $n$-dimensional Riemannian manifold $M$ and $W = W(t)$ a vector field on $U \subset M$ along the curve $\alpha$. Then, for a point $p = \alpha(t_0)$, there are a sub-interval $J \subset I$ containing $t_0$ such that the unique curve $\gamma(t)$ on a local neighborhood $U$ satisfying $\gamma(t_0) = p$ and $\frac{d}{dt}\gamma(t) = W(t)|_U$ for all $t \in J$.

**Definition 2.** For a given curve in an $n$-dimensional Riemannian manifold $M$ and a point $p$ on $\alpha$ , the curve $\gamma$ of Remark 3.2 is called the $W$-direction curve passing the point $p$ of $\alpha$. In this case, $\alpha$ is called the $W$-donor curve passing $p$ of $\gamma$.

Moreover, it is well-known that a Lie group is also a differentiable manifold, with the property that the group operations are compatible with the smooth structure. Then, Definition 2 can be considered in a compact Lie group $G$. For this purpose, let consider a Frenet curve $\alpha$ in three dimensional compact Lie group $G$ with the Frenet frame $\{T, N, B\}$ and a unit vector field $W$ given by

$$W(s) = \eta(s)T(s) + \vartheta(s)N(s) + \xi(s)B(s), \tag{3.1}$$

where $\eta$, $\vartheta$ and $\xi$ are arbitrary differentiable functions of $s$ which is the arc length parameter of $\alpha$. Let

$$\eta^2(s) + \vartheta^2(s) + \xi^2(s) = 1, \tag{3.2}$$

holds. Then, we define a $W$-direction curve and a $W$-donor curve in three dimensional compact Lie group as follows.

**Definition 3.** Let $\alpha$ be a Frenet curve in a three dimensional compact Lie group $G$ and $W$ be a unit vector field satisfying the equations (3.1) and (3.2). The integral curve $\gamma : I \subset \mathrm{R} \to G$ of $W$ is called a $W$-direction curve of $\alpha$. The curve $\alpha$ whose $W$-direction curve is $\gamma$ is called the $W$-donor curve in three dimensional compact Lie group $G$.

**Definition 4.** An integral curve of $N(s)$ (resp. $B(s)$) in (3.1) is called the principal-direction curve (resp. binormal-direction curve) of $\alpha$ in three dimensional compact Lie group $G$.

*Remark* 2. A principal-direction (resp. the binormal-direction) curve is an integral curve of $W(s)$ with $\eta(s) = \xi(s) = 0,\ \vartheta(s) = 1$ (resp. $\eta(s) = \vartheta(s) = 0,\ \xi(s) = 1$) for all $s$ in (3.1).

**Proposition 2.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve and an integral curve of $W(s)$ be the curve $\gamma : I \subset \mathrm{R} \to G$. Then, the principal-direction curve of $\gamma$ equals to $\alpha$ if and only if*

$$\eta(s) = 0,\ \ \vartheta(s) = \sin\left(\int H\kappa ds\right) \neq 0,\ \ \xi(s) = \cos\left(\int H\kappa ds\right) \neq 0, \tag{3.3}$$

*hold, where $H$ is harmonic curvature function of the curve $\alpha$.*

*Proof.* Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B\}$. From (3.2) we have

$$\eta^2(s) + \vartheta^2(s) + \xi^2(s) = 1. \tag{3.4}$$

Differentiating (3.4) with respect to $s$ gives us

$$\eta\eta' + \vartheta\vartheta' + \xi\xi' = 0. \tag{3.5}$$

Also, differentiating (3.1) with respect to $s$ and using (2.3) we get

$$\begin{aligned} \tfrac{d}{ds}W(s) = {} & \eta'(s)T(s) + \vartheta'(s)N(s) + \xi'(s)B(s) \\ & + \eta(s)\left(D_T T - \tfrac{1}{2}[T,T]\right) + \vartheta(s)\left(D_T N - \tfrac{1}{2}[T,N]\right) \\ & + \xi(s)\left(D_T B - \tfrac{1}{2}[T,B]\right) \end{aligned} \tag{3.6}$$

By using the Frenet formulas and with the aid of equation (2.5), we obtain

$$\frac{d}{ds}W = (\eta' - \vartheta\kappa)T + (\vartheta' + \eta\kappa - \xi H\kappa)N + (\xi' + \vartheta H\kappa)B, \tag{3.7}$$

where $H$ is the harmonic curvature function and $\kappa$ is the curvature of $\alpha$. Since

$$\frac{d}{ds}W(s) = \frac{d^2}{ds^2}\gamma(s) = \dot{\bar{T}}(s) = D_{\bar{T}}\bar{T} - \frac{1}{2}[\bar{T},\bar{T}] = \bar{\kappa}\bar{N},$$

$\alpha$ is a principal-direction curve of $\gamma$ if and only if the following system hold

$$\begin{cases} \eta' - \vartheta\kappa \neq 0, \\ \vartheta' + \eta\kappa - \xi H\kappa = 0, \\ \xi' + \vartheta H\kappa = 0. \end{cases}$$

Multiplying the last equation of this system by $\xi$ and substituting the obtained result in (3.5) gives us

$$\eta\eta' + \vartheta\vartheta' = \xi\vartheta H\kappa.$$

Similarly, multiplying the second equation of the system by $\vartheta$ and using the last equality we have $\eta(\eta' - \vartheta\kappa) = 0$. Since $\eta' - \vartheta\kappa \neq 0$, it follows that $\eta = 0$. Then, from the system we have

$$\begin{cases} \xi' + \vartheta H\kappa = 0, \\ \vartheta' - \xi H\kappa = 0, \end{cases}$$

and it has the following solution

$$\vartheta(s) = \sin\left(\int H\kappa ds\right) \neq 0, \ \ \xi(s) = \cos\left(\int H\kappa ds\right) \neq 0.$$

□

**Definition 5.** An integral curve $\gamma$ of

$$W(s) = \sin\left(\int H\kappa ds\right) N(s) + \cos\left(\int H\kappa ds\right) B(s)$$

is called a principal-donor curve of $\alpha$.

**Theorem 5.** *Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $(T, N, B, \kappa, \tau)$ and $\gamma$ the principal-direction curve of $\alpha$. Then the curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of $\gamma$ are given by*

$$\bar{\kappa} = \sqrt{\kappa^2(1+H^2)}, \ \ \bar{\tau} - \bar{\tau}_g = -\frac{H'}{1+H^2}, \tag{3.8}$$

*where $H$ is harmonic curvature function of the curve $\alpha$.*

*Proof.* By the definition of principal direction curve of $\alpha$,

$$\frac{d\gamma}{ds} = \bar{T} = N. \tag{3.9}$$

Differentiating (3.9) with respect to $s$ and considering Frenet formulas, we get

$$\bar{\kappa}\bar{N} = -\kappa T + \kappa HB. \tag{3.10}$$

From the norms of both sides in (3.10) we have

$$\bar{\kappa} = \sqrt{\kappa^2(1+H^2)}. \tag{3.11}$$

Substituting (3.11) into (3.10), we obtain

$$\bar{N} = -\frac{1}{\sqrt{1+H^2}}T + \frac{H}{\sqrt{1+H^2}}B,$$

and then

$$\bar{B} = \bar{T} \times \bar{N} = N \times \bar{N} = \frac{H}{\sqrt{1+H^2}}T - \frac{1}{\sqrt{1+H^2}}B. \tag{3.12}$$

Differentiating (3.12) with respect to $s$ we have

$$\dot{\bar{B}} = \frac{H'}{\left(1+H^2\right)^{\frac{3}{2}}}(T - HB). \tag{3.13}$$

Since $\dot{\bar{B}} = D_{\bar{T}}\bar{B} - \frac{1}{2}[\bar{T}, \bar{B}] = (\bar{\tau}_g - \bar{\tau})\bar{N}$, from (3.13) it follows

$$(\bar{\tau}_g - \bar{\tau})\bar{N} = \frac{H'}{\left(1+H^2\right)^{\frac{3}{2}}}(T - HB), \tag{3.14}$$

which gives us

$$\bar{\tau} - \bar{\tau}_g = -\frac{H'}{1+H^2}.$$

□

From Theorem 5, we have the following corollaries:

**Corollary 2.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve in three dimensional compact Lie Group $G$. If $\gamma$ is the principal-direction curve of $\alpha$, then the relationships between the Frenet frames of the curves are given as follows*

$$\bar{T} = N, \quad \bar{N} = -\frac{1}{\sqrt{1+H^2}}T + \frac{H}{\sqrt{1+H^2}}B, \quad \bar{B} = \frac{H}{\sqrt{1+H^2}}T - \frac{1}{\sqrt{1+H^2}}B.$$

**Corollary 3.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve in three dimensional compact Lie Group $G$. If $\gamma$ is the principal-direction curve of $\alpha$, then*

$$\bar{H} = -\frac{H'}{\kappa\left(1+H^2\right)^{3/2}}.$$

**Theorem 6.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve. If $\alpha$ is principal donor curve of the curve $\gamma$, then the curvature $\kappa$ and the torsion $\tau - \tau_G$ of the curve $\alpha$ are given by*

$$\kappa = \bar{\kappa}\left|\sin\left(\int \bar{H}\bar{\kappa}ds\right)\right|, \quad \tau - \tau_G = \bar{\kappa}\left|\cos\left(\int \bar{H}\bar{\kappa}ds\right)\right|, \tag{3.15}$$

*recpevtively.*

*Proof.* Since $\alpha$ is a principal-donor curve of $\gamma$, from Definition 3, we have

$$\frac{d\alpha(s)}{ds} = \sin\left(\int \bar{H}\bar{\kappa}ds\right)\bar{N}(s) + \cos\left(\int \bar{H}\bar{\kappa}ds\right)\bar{B}(s). \tag{3.16}$$

Differentiating (3.16) with respect to $s$ we have

$$\frac{d^2\alpha(s)}{ds^2} = -\bar{\kappa}\sin\left(\int \bar{H}\bar{\kappa}ds\right)\bar{T}(s). \tag{3.17}$$

Since $\left\|\frac{d^2\alpha(s)}{ds^2}\right\| = \kappa$, we get

$$\kappa = \bar{\kappa}\left|\sin\left(\int \bar{H}\bar{\kappa}ds\right)\right|. \tag{3.18}$$

Then substituting (3.18) in (3.10), it follows that

$$\tau - \tau_G = \bar{\kappa}\left|\cos\left(\int \bar{H}\bar{\kappa}ds\right)\right|.$$

Hence our theorem is proved. □

## 4. Applications of principal-direction curves and principal donor curves to some special curves in a compact Lie group

In this section, we give relationships between the principal-direction curve, the principal donor curve and some special curves such as helix or slant helix in a three dimensional compact Lie group $G$.

Let $\gamma$ be a principal-direction curve of $\alpha$ and $\bar{\gamma}$ principal-direction curve of $\gamma$ in $G$. Then, $\bar{\gamma}$ is called a second principal-direction curve of $\alpha$ and $\alpha$ is called a second principal-donor curve of $\bar{\gamma}$.

**Theorem 7.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve in $G$. Then the followings are equivalent:*

*i) A Frenet curve $\alpha$ is a general helix in $G$.*

*ii) $\alpha$ is a principal-donor curve of a curve $\gamma$ with curvature $\bar{\tau} = \bar{\tau}_g$ and torsion $\bar{\kappa} = \frac{\sqrt{1+c^2}}{c}|\tau - \tau_G|$ in $G$ where $c = H$ is a constant*

*iii) A principal-direction curve of $\alpha$ is a curve $\gamma$ with curvature $\bar{\tau} = \bar{\tau}_g$ in $G$.*

Let $\bar{\gamma} = (x(s), y(s), 0)$ be a unit speed curve with curvature $\bar{\tau} = \bar{\tau}_g$ in $G$. Then $\bar{\gamma}$ can be expressed by

$$x(s) = \int_0^s \cos\left(\int_0^\sigma \bar{\bar{\kappa}}(t)dt\right) d\sigma, \;\; y(s) = \int_0^s \sin\left(\int_0^\sigma \bar{\bar{\kappa}}(t)dt\right) d\sigma,$$

where $\bar{\bar{\kappa}} = \sqrt{[x''(s)]^2 + [y''(s)]^2}$ is the curvature of $\bar{\gamma}$. Denote the Frenet frame of the curve $\bar{\gamma}$ by $\left\{\bar{\bar{T}}, \bar{\bar{N}}, \bar{\bar{B}}\right\}$. Then we have

$$\begin{aligned}
\bar{\bar{T}}(s) &= \left(\cos\left(\textstyle\int_0^s \bar{\bar{\kappa}}(t)dt\right),\ \sin\left(\textstyle\int_0^s \bar{\bar{\kappa}}(t)dt\right),\ 0\right),\\
\bar{\bar{N}}(s) &= \left(-\sin\left(\textstyle\int_0^s \bar{\bar{\kappa}}(t)dt\right),\ \cos\left(\textstyle\int_0^s \bar{\bar{\kappa}}(t)dt\right),\ 0\right),\\
\bar{\bar{B}}(s) &= (0,0,1).
\end{aligned}$$

Thus, we construct a principal-donor curve of $\bar{\gamma}$ as:

$$\gamma = \int_0^s \gamma'(s)ds = \left(a\int_0^s \sin\left(\int_0^\sigma \bar{\bar{\kappa}}(t)dt\right) d\sigma,\ -a\int_0^s \cos\left(\int_0^\sigma \bar{\bar{\kappa}}(t)dt\right) d\sigma,\ bs\right)$$

where $a \neq 0$ and $b$ are constant with $a^2 + b^2 = 1$. Consequently, we have the following corollary:

**Corollary 4.** *Let $\bar{\gamma} = (x(s), y(s), 0)$ be a unit speed curve with curvature $\bar{\tau} = \bar{\tau}_g$ in $G$. Then, the principal-donor curve $\gamma$ of $\bar{\gamma}$ is given by*

$$\gamma = (ay(s),\ -ax(s),\ bs),$$

*where $a \neq 0$ and $b$ are constant with $a^2 + b^2 = 1$.*

From Theorem 2 and Corollary 2 we have the following theorem

**Theorem 8.** *Let $\alpha : I \subset \mathrm{R} \to G$ be a Frenet curve in $G$. Then the followings are equivalent:*

*i) A frenet curve $\alpha$ is a slant helix in $G$.*
*ii) $\alpha$ is a principal-donor curve of a general helix in $G$.*
*iii) $\alpha$ is a second principal-donor curve of a curve $\gamma$ with curvature $\bar{\tau} = \bar{\tau}_g$ in $G$.*
*iv) A principal-direction curve of $\alpha$ is a general helix in $G$.*
*v) A second principal-direction curve of $\alpha$is a curve $\gamma$ with curvature $\bar{\tau} = \bar{\tau}_g$ in $G$.*

**Definition 6** ([3])**.** Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve in three dimensional compact Lie group $G$. If the position vector of $\alpha$ always lies in its rectifying plane, then it is called a rectifying curve in $G$.

**Definition 7.** Let $\alpha : I \subset \mathrm{R} \to G$ be an arc length parametrized curve in three dimensional compact Lie group $G$ and the curve $\gamma$ the principal-direction curve of $\alpha$. If the position vector of $\gamma$ always lies in rectifying plane of $\alpha$, then $\gamma$ is called principal direction rectifying (or $PD$-rectifying) curve in $G$.

By this definition, the position vector of $PD$-rectifying curve $\gamma$ in $G$ satisfies the following equation

$$\gamma = \omega(s)T(s) + \rho(s)B(s), \tag{4.1}$$

where $\omega(s)$, $\rho(s)$ are non-zero smooth functions and $T(s)$, $B(s)$ are Frenet vectors of $\alpha$. Since $\alpha$ is a principal-donor curve of $\gamma$ in $G$, from Definition (3.8), we have

$$\begin{cases} \alpha'(s) = T(s) = \sin\left(\int \bar{H}\bar{\kappa}ds\right)\bar{N}(s) + \cos\left(\int \bar{H}\bar{\kappa}ds\right)\bar{B}(s), \\ B(s) = -\cos\left(\int \bar{H}\bar{\kappa}ds\right)\bar{N}(s) + \sin\left(\int \bar{H}\bar{\kappa}ds\right)\bar{B}(s). \end{cases} \tag{4.2}$$

Then substituting (4.2) in (4.1) it follows

$$\begin{aligned} \gamma = &\left[\omega(s)\sin\left(\int \bar{H}\bar{\kappa}ds\right) - \rho(s)\cos\left(\int \bar{H}\bar{\kappa}ds\right)\right]\bar{N}(s) \\ &+ \left[\omega(s)\cos\left(\int \bar{H}\bar{\kappa}ds\right) + \rho(s)\sin\left(\int \bar{H}\bar{\kappa}ds\right)\right]\bar{B}(s). \end{aligned} \tag{4.3}$$

Differentiating (4.3) with respect to $s$ gives us

$$\begin{aligned} \bar{T} = &\bar{\kappa}\left[-\omega\sin\left(\int \bar{H}\bar{\kappa}ds\right) - \rho\cos\left(\int \bar{H}\bar{\kappa}ds\right)\right]\bar{T}(s) \\ &+ \left[\omega'\sin\left(\int \bar{H}\bar{\kappa}ds\right) - \rho'\cos\left(\int \bar{H}\bar{\kappa}ds\right)\right]\bar{N}(s) \\ &+ \left[\omega'\cos\left(\int \bar{H}\bar{\kappa}ds\right) + \rho'\sin\left(\int \bar{H}\bar{\kappa}ds\right)\right]\bar{B}(s). \end{aligned}$$

From the last equality we have

$$\omega'(s) = \rho'(s) = 0, \ \omega\bar{\kappa}\sin\left(\int \bar{H}\bar{\kappa}ds\right) + \rho\bar{\kappa}\cos\left(\int \bar{H}\bar{\kappa}ds\right) = -1. \tag{4.4}$$

Since $\omega$ and $\rho$ are non-zero constants, the last equation implies that

$$\bar{\tau} - \bar{\tau}_G = \frac{\bar{\kappa}'}{\bar{\kappa}\sqrt{(\omega^2+\rho^2)\bar{\kappa}^2 - 1}}. \tag{4.5}$$

Moreover, by writing (3.15) in (28) we have

$$\lambda\kappa + \mu\kappa H = 1.$$

where $\lambda = -\omega$, $\mu = -\rho$ are constants. Then we can give the following theorem characterizing $PD$-rectifying curves in $G$.

**Theorem 9.** *Let $\gamma$ be a Frenet curve in three dimensional compact Lie group $G$ and $\alpha$ a principal-donor curve of $\gamma$. Then, the followings are equivalent,*

*i) gamma is $PD$-rectifying curve in $G$.*

*ii) $\gamma$ is a Frenet curve with non-zero curvature and non-zero torsion satisfying equation (4.4).*

*iii) The position vector of $\gamma$ lies in its normal plane in $G$.*

*iv) There exists a Bertrand couple $(\alpha, \beta)$ such that $\beta(s) = \alpha(s) + \lambda N(s)$ for all $s \in I$ is a special Frenet curve in $G$.*

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# CONSIMILARITY OF COMMUTATIVE QUATERNION MATRICES

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*Abstract.* In this paper, the consimilarity of complex matrices is generalized for commutative quaternion matrices. In this regard, the coneigenvalue and coneigenvector for commutative quaternion matrices are defined. Also, the existence of solution to the some commutative quaternion matrix equations is characterized and solutions of these matrix equations are derived by means of real representations of commutative quaternion matrices.

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*Keywords:* commutative quaternion, commutative quaternion matrix, consimilarity, coneigenvalue

## 1. Introduction

In the middle of 19th century, Sir William Hamilton defined the set of real quaternions which are denoted by [3]

$$\mathbb{K} = \{q = q_0 + q_1 i + q_2 j + kz; q_s \in \mathbb{R}, s = 0,1,2,3\}$$

where

$$i^2 = j^2 = k^2 = -1, \;\; ij = -ji = k, \;\; ik = -ki = -j, \;\; jk = -kj = i.$$

There are many applications of these quaternions. One of them is also about matrix theory. The study of the real quaternion matrices began in the first half of the 20th century, [13]. So, Baker discussed right eigenvalues of the real quaternion matrices with a topological approach in [1]. On the other hand, Huang and So introduced left eigenvalues of real quaternion matrices [6]. After that Huang discussed the consimilarity of the real quaternion matrices and obtained the Jordan canonical form of the real quaternion matrices under consimilarity [5]. Jiang and Wei studied the real quaternion matrix equation $X - A\widetilde{X}B = C$ by means of real representation of the real quaternion matrices, [8]. Also, Jiang and Ling studied the problem of solution of the quaternion matrix equation $A\widetilde{X} - XB = C$ via real representation of a quaternions matrix [7].

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After Hamilton had discovered the real quaternions, Segre defined the set of commutative quaternions, [11]. Commutative quaternions are decomposable into two complex variables [2]. The set of commutative quaternions is 4-dimensional like set of quaternions. But this set contains zero-divisor and isotropic elements. There are a lot of works associate with commutative quaternions. Catoni *et al.* gave a brief survey on commutative quaternions [2]. They introduced functions of commutative quaternionic variables and obtained generalized Cauchy-Riemann conditions. Geometrical introduction of commutative quaternions were considered by Severi in association with functions of two complex variable [12]. Differential properties of commutative quaternion functions were studied by Scorza-Drogoni [10]. Also, Kosal and Tosun considered commutative quaternion matrices. Moreover, they investigated commutative quaternion matrices using properties of complex matrices [9].

## 2. Algebraic properties of commutative quaternions

A set of commutative quaternions is denoted by [2]

$$\mathbb{H} = \{q = t + xi + yj + zk : \ t, x, y, z \in \mathbb{R} \text{ and } i, j, k \notin \mathbb{R}\}$$

where

$$i^2 = k^2 = -1, \ j^2 = 1, \ ij = ji = k, \ ik = ki = -j, \ jk = kj = i.$$

There exist three kinds of conjugate of $q = t + xi + yj + zk$,

$$\overline{\overline{q}} = t - xi + yj - zk, \ \widetilde{\overline{q}} = t + xi - yj - zk, \ \widetilde{\overline{q}} = t - xi - yj + zk$$

and the norm is defined by

$$\begin{aligned} \|q\|^4 &= q.\overline{\overline{q}}.\widetilde{\overline{q}}.\widetilde{\overline{q}} \\ &= \left[(t+y)^2 + (x+z)^2\right]\left[(t-y)^2 + (x-z)^2\right]. \end{aligned}$$

Multiplication of any commutative quaternions $q = t + xi + yj + zk$ and $q_1 = t_1 + x_1 i + y_1 j + z_1 k$ are defined in the following ways,

$$\begin{aligned} qq_1 = &(tt_1 - xx_1 + yy_1 - zz_1) + (xt_1 + tx_1 + zy_1 + yz_1)\, i \\ &+ (ty_1 + yt_1 - xz_1 - zx_1)\, j + (zt_1 + tz_1 + xy_1 + yx_1)\, k \end{aligned}.$$

It is nearby to identify a commutative quaternion $q \in \mathbb{H}$ with a real vector $\boldsymbol{q} \in \mathbb{R}^4$. Such an identification is denoted by

$$q = t + xi + yj + zk \cong \boldsymbol{q} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

Then multiplication of $q$ and $q_1$ can be shown by using ordinary matrix multiplication

$$qq_1 = q_1 q \cong L_q \mathbf{q_1} = \begin{pmatrix} t & -x & y & -z \\ x & t & z & y \\ y & -z & t & -x \\ z & y & x & t \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

where $L_q$ is called the fundamental matrix of $q$.

**Theorem 1** ([9]). *If $q$ and $q_1$ are commutative quaternions and $\lambda_1, \lambda_2$ are real numbers, then the following identities hold:*

1) $q = q_1 \Leftrightarrow L_q = L_{q_1}$
2) $L_{q+q_1} = L_q + L_{q_1}$
3) $L_{\lambda_1 q + \lambda_2 q_1} = \lambda_1 L_q + \lambda_2 L_{q_1}$
4) $Trace\left(L_q\right) = q + \overline{\overline{q}} + \widetilde{\overline{q}} + \widetilde{\overline{q}} = 4t$, $\|q\| = \left[\det\left(L_q\right)\right]^{\frac{1}{4}}$.

**Theorem 2** ([9]). *Every commutative quaternion can be represented by a 2x2 matrix with complex entries.*

*Proof.* Let $q = t + xi + yj + zk \in \mathbb{H}$, then every commutative quaternion can be uniquely expressed as $q = c_1 + c_2 j$ where $c_1 = t + xi$, $c_2 = y + zi$ are complex numbers. The linear map $\varphi_q : \mathbb{H} \to \mathbb{H}$ is defined by $\varphi_q(p) = qp$ for all $p \in \mathbb{H}$. This map is bijective and

$$\varphi_q(1) = c_1 + c_2 j$$
$$\varphi_q(j) = c_2 + c_1 j$$

with this transformation, the commutative quaternions can be seen as subsets of the matrix ring $M_2(\mathbb{C})$, the set of 2x2 matrices

$$\mathbb{N} = \left\{ \begin{pmatrix} c_1 & c_2 \\ c_2 & c_1 \end{pmatrix} : c_1, c_2 \in \mathbb{C} \right\}.$$

Then, $\mathbb{H}$ and $\mathbb{N}$ are essentially same. □

**Definition 1.** Two commutative quaternions $q$ and $q_1$ are said to be consimilar if there exists a commutative quaternion $p$, $\|p\| \neq 0$ such that $\overline{\overline{p}} q\, p^{-1} = q_1$; this is written as $q \overset{c}{\sim} q_1$.

**Theorem 3.** *For three commutative quaternion $q, q_1, q_2 \in \mathbb{H}$, the following statements hold:*
*Reflexive: $q \overset{c}{\sim} q$.*
*Symmetric: if $q \overset{c}{\sim} q_1$, then $q_1 \overset{c}{\sim} q$,*
*Transitive: if $q \overset{c}{\sim} q_1$, $q_1 \overset{c}{\sim} q_2$, then $q \overset{c}{\sim} q_2$.*

*Proof.* Reflexive $1\,q\,1^{-1} = q$ trivially, for $q \in \mathbb{H}$. So, consimilarity is reflexive. Symmetric: Let $\overline{\overline{p}}\,q\,p^{-1} = q_1$. As $p$ is nonsingular, we have

$$\left(\overline{\overline{p}}\right)^{-1} q_1\, p = \left(\overline{\overline{p}}\right)^{-1} \overline{\overline{p}}\,q\,p^{-1}\,p = q.$$

So, consimilarity is symmetric.
Transitive: Let $\overline{\overline{p_1}}\,q\,p_1^{-1} = q_1$ and $\overline{\overline{p_2}}\,q_1\,p_2^{-1} = q_2$. Then

$$\begin{aligned} q_2 &= \overline{\overline{p_2}}\,\overline{\overline{p_1}}\,q\,p_1^{-1}\,p_2^{-1} \\ &= \overline{\overline{(p_2\,p_1)}}\,q\,(p_2\,p_1)^{-1}. \end{aligned}$$

So, consimilarity is transitive. □

Then, $\overset{c}{\sim}$ is an equivalence relation on commutative quaternions. Obviously the consimilar commutative quaternions have the same norm.

## 3. Consimilarity of commutative quaternion matrices

The set of all $m \times n$ commutative quaternion matrices, which is denoted by $\mathbb{H}^{m\times n}$, with ordinary matrix addition and multiplication is a ring with unit. For $A = \left(a_{ij}\right)_{m\times n} \in \mathbb{H}^{m\times n}$, the matrices $\overline{\overline{A}} = \left(\overline{\overline{a_{ij}}}\right)_{m\times n}$, $\widetilde{\overline{A}} = \left(\widetilde{\overline{a_{ij}}}\right)_{m\times n}$ and $\overline{\widetilde{A}} = \left(\overline{\widetilde{a_{ij}}}\right)_{m\times n}$ are conjugates of $A$ and $A^T$ is transpose matrix of $A$.

**Theorem 4** ([9]). *For any $A \in \mathbb{H}^{m\times n}$ and $B \in \mathbb{H}^{n\times s}$, the followings are satisfied:*

*i.* $\left(\overline{\overline{A}}\right)^T = \overline{\overline{\left(A^T\right)}}$, $\left(\widetilde{\overline{A}}\right) = \overline{\widetilde{\left(A^T\right)}}$, $\left(\overline{\widetilde{A}}\right)^T = \overline{\widetilde{\left(A^T\right)}}$,

*ii.* $(AB)^T = B^T A^T$,

*iii. If $A, B \in \mathbb{H}^{n\times n}$ are nonsingular, then* $(AB)^{-1} = B^{-1}A^{-1}$,

*iv.* $\left(\overline{\overline{AB}}\right) = \overline{\overline{A}}\,\overline{\overline{B}}$, $\left(\widetilde{\overline{AB}}\right) = \widetilde{\overline{A}}\,\widetilde{\overline{B}}$, $\left(\overline{\widetilde{AB}}\right) = \overline{\widetilde{A}}\,\overline{\widetilde{B}}$.

**Definition 2.** A matrix $A \in \mathbb{H}^{n\times n}$ is said to be similar to a matrix $B \in \mathbb{H}^{n\times n}$ if there exists a nonsingular matrix $P \in \mathbb{H}^{n\times n}$ such that $P^{-1}AP = B$. The relation, $A$ is similar to $B$, is denoted $A \sim B$. $\sim$ is an equivalence relation on $\mathbb{H}^{n\times n}$.

**Definition 3.** A matrix $A \in \mathbb{H}^{n\times n}$ is said to be consimilar a matrix $B \in \mathbb{H}^{n\times n}$ if there exists a nonsingular matrix $P \in \mathbb{H}^{n\times n}$ such that $\overline{\overline{P}}\,A\,P^{-1} = B$. The relation, $A$ is consimilar to $B$, is denoted $A \overset{c}{\sim} B$. $\overset{c}{\sim}$ is an equivalence relation on $\mathbb{H}^{n\times n}$.

Clearly if $A \in \mathbb{C}^{n\times n}$, then $\overline{A} = \overline{\overline{A}}$ Thus, if $A \in \mathbb{C}^{n\times n}$ is consimilar to $B \in \mathbb{C}^{n\times n}$ as complex matrices, $A$ is consimilar to $B$ as commutative quaternion matrices. Then, consimilarity relation in $\mathbb{H}^{n\times n}$ is a natural extension of complex consimilarity in $\mathbb{C}^{n\times n}$ (for complex consimilarity see reference [4]).

**Definition 4.** Let $A \in \mathbb{H}^{n\times n}$, $\lambda \in \mathbb{H}$. If there exists $0 \neq x \in \mathbb{H}^{n\times 1}$ such that

$$A\overline{\overline{x}} = x\lambda$$

then $\lambda$ is called a coneigenvalues of $A$ and $x$ is called a coneigenvector of $A$ associate with $\lambda$. The set of all coneigenvalues is defined as

$$\overline{\sigma} = \left\{\lambda \in \mathbb{H} : A\overline{\overline{x}} = x\lambda, \text{ for some } x \neq 0\right\}.$$

Recall that if $x \in \mathbb{H}^{n\times 1}(x \neq 0)$, and $\lambda \in \mathbb{H}$ satisfying $Ax = x\lambda$, we call $x$ an eigenvector of $A$, while $\lambda$ is an eigenvalue of $A$.

**Theorem 5.** *If $A \in \mathbb{H}^{n\times n}$ is consimilar to $B \in \mathbb{H}^{n\times n}$ then, coneigenvalues of $A$ and $B$ are the same.*

*Proof.* Let $A \overset{c}{\sim} B$. Then, there exists a nonsingular matrix $P \in \mathbb{H}^{n\times n}$ such that $B = \overline{\overline{P}}AP^{-1}$. Let $\lambda \in \mathbb{H}$ be a coneigenvalue for the matrix $A$, then we can find a matrix $x \in \mathbb{H}^{n\times 1}$ such that $A\overline{\overline{x}} = x\lambda$, $x \neq 0$. Let $y = P\overline{\overline{x}}$. Then $By = \overline{\overline{P}}AP^{-1}y = \overline{\overline{P}}AP^{-1}P\overline{\overline{x}} = \overline{\overline{P}}A\overline{\overline{x}} = \overline{\overline{P}}x\lambda = \overline{\overline{y}}\lambda$. □

**Theorem 6.** *Let $A \in \mathbb{H}^{n\times n}$, then $\lambda$ is coneigenvalue of $A$ if and only if for any $0 \neq \|\beta\|$, $\overline{\overline{\beta}}\lambda\beta^{-1}$ is a coneigenvalue of $A$.*

*Proof.* From $A\overline{\overline{x}} = x\lambda$, we get $A\left(\overline{\overline{x}}\beta^{-1}\right) = x\left(\overline{\overline{\beta}}\right)^{-1}\left(\overline{\overline{\beta}}\right)\lambda\beta^{-1}$. □

**Definition 5** ([9])**.** Let $A = A_0 + A_1 j \in \mathbb{H}^{n\times n}$ where $A_s \in \mathbb{C}^{n\times n}$, $s = 0, 1$. The $2n \times 2n$ matrix

$$\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$$

is called the complex adjoint matrix of $A$ and denoted by $\eta_A$.

It is nearby to identify a commutative quaternion matrix $A \in \mathbb{H}^{n\times n}$ with a complex matrix $\mathbf{A} \in \mathbb{C}^{2n\times n}$. By the $\cong$ symbol, we will denote

$$A = A_0 + A_1 j \cong \mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} \in \mathbb{C}^{2n\times n}.$$

Then, the multiplication of $A \in \mathbb{H}^{n\times n}$ and $B \in \mathbb{H}^{n\times n}$ can be represented by an ordinary matrix product $AB \cong \eta(A)\,\mathbf{B}$.

**Theorem 7.** *Let $A, B \in \mathbb{H}^{n\times n}$, the followings are satisfied:*
*i.* $\eta(I_n) = I_{2n}$,
*ii.* $\eta(A + B) = \eta(A) + \eta(B)$,
*iii.* $\eta(AB) = \eta(A)\eta(B)$,
*iv. If $A$ is nonsingular, then* $(\eta(A))^{-1} = \eta\left(A^{-1}\right)$.

**Theorem 8.** *For every* $A \in \mathbb{H}^{n\times n}$,

$$\overline{\sigma}(A) \cap \mathbb{C} = \overline{\sigma}(\eta(A))$$

*is the set of coneigenvalues of* $\eta(A)$. *where*

$$\overline{\sigma}(\eta(A)) = \{\lambda \in \mathbb{C} : \ \eta(A)\overline{y} = y\lambda, \ for\ some\ y \neq 0\},$$

*Proof.* Let $A = A_0 + A_1 j \in \mathbb{H}^{n\times n}$ where $A_s \in \mathbb{C}^{n\times n}$, $s = 0,1$ and $\lambda \in \mathbb{C}$ be a coneigenvalue of $A$. Therefore there exists $0 \neq x \in \mathbb{H}^{n\times 1}$ such that $A\overline{\overline{x}} = x\lambda$. This implies

$$(A_0 + A_1 j)(\overline{x_0} + \overline{x_1} j) = (x_0 + x_1 j)\lambda,$$
$$(A\overline{x_0} + A_1\overline{x_1}) = x_0\lambda \quad \text{and} \quad (A_0\overline{x_1} + A_1\overline{x_0}) = x_1\lambda.$$

Using these equations, we can write

$$\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix} \begin{pmatrix} \overline{x_0} \\ \overline{x_1} \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \lambda.$$

Therefore, the complex coneigenvalue of the commutative quaternion matrix $A$ is equal to the coneigenvalue of the adjoint matrix $\eta(A)$ that is

$$\overline{\sigma}(A) \cap \mathbb{C} = \overline{\sigma}(\eta(A)).$$

□

## 4. Real representation of commutative quaternion matrices

Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}^{m\times n}$ where $A_s \in \mathbb{R}^{m\times n}$, $s = 0,1,2,3$. We will define the linear transformation $\phi_A(X) = A\overline{\overline{X}}$. We can write

$$\begin{aligned} \phi_A(1) &= A_0 + A_1 i + A_2 j + A_3 k \\ \phi_A(i) &= A_1 - A_0 i + A_3 j - A_2 k \\ \phi_A(j) &= A_2 + A_3 i + A_0 j + A_1 k \\ \phi_A(k) &= A_3 - A_2 i + A_1 j - A_0 k. \end{aligned}$$

Then, we obtain

$$\phi_A = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & -A_0 & A_3 & -A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & -A_2 & A_1 & -A_0 \end{pmatrix} \in \mathbb{R}^{4m\times 4n}.$$

Here $\phi_A$ is called the representation of $A$ corresponding to the linear transformation $\phi_A(X) = A\overline{\overline{X}}$.

It is nearby to identify a commutative quaternion matrix $A \in \mathbb{H}^{m\times n}$ with a real matrix $\mathbf{A} \in \mathbb{R}^{4m\times n}$. By the $\cong$ symbol,we will denote

$$A = A_0 + A_1 i + A_2 j + A_3 k \cong \mathbf{A} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Then, multiplication of $A \in \mathbb{H}^{m\times n}$ and $\overline{\overline{B}} \in \mathbb{H}^{n\times k}$ can be represented by an ordinary matrix product $A\overline{\overline{B}} \cong \phi_A \mathbf{B}$.

**Theorem 9.** *For commutative quaternion matrix A, the following identities are satisfied:*
*i. If $A \in \mathbb{H}^{m\times n}$, then*

$$({}^1P_m)^{-1}\phi_A({}^1P_n) = \phi_{\overline{\overline{A}}},\ Q_m^{-1}\phi_A Q_n = -\phi_A,$$
$$R_m^{-1}\phi_A R_n = \phi_A,\quad S_m^{-1}\phi_A S_n = -\phi_A;$$

*where*

$${}^1P_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix},\quad Q_m = \begin{pmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{pmatrix},$$

$$R_m = \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix},\qquad S_m = \begin{pmatrix} 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \\ 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

*ii. If $A, B \in \mathbb{H}^{m\times n}$ then $\phi_{A+B} = \phi_A + \phi_B$,*
*iii. If $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{n\times r}$, in that case $\phi_{AB} = \phi_A({}^1P_n)\phi_B = \phi_A\phi_{\overline{\overline{B}}}({}^1P_r)$,*
*iv. If $A \in H^{m\times m}$, then $A$ is nonsingular if and only if $\phi_A$ is nonsingular and $(\phi_A)^{-1} = ({}^1P_m)\phi_{A^{-1}}({}^1P_m)$,*
*v. If $A \in \mathbb{H}_S^{m\times m}$,*

$$\phi_{\widetilde{\overline{A}}} = {\varepsilon_1}^{-1}\phi_A\varepsilon_1$$
$$\phi_{\widetilde{\overline{A}}} = {\varepsilon_2}^{-1}\phi_A\varepsilon_2$$

*where* $\varepsilon_1 = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix},\quad \varepsilon_2 = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix},$

*vi. If $A \in \mathbb{H}_S^{m\times m}$,*

$$\overline{\sigma}(A) \cap \mathbb{C} = \sigma(\phi_A)$$

*where* $\sigma(\phi_A) = \{\lambda \in \mathbb{C} : \ \phi_A y = \lambda y, \ for\ some\ y \neq 0\}$, *is the set of all eigenvalues of* $\phi_A$.

*Proof.* By direct calculation, i., iii. and v. can be easily shown. For now we will prove ii., iv. and vi.
ii. Let $A = A_0 + A_1 i + A_2 j + A_3 k,\ B = B_0 + B_1 i + B_2 j + B_3 k \in \mathbb{H}^{m\times n}$ where $A_s, B_s \in \mathbb{R}^{m\times n}$, $s = 0,1,2,3$. Then, we have

$$\begin{aligned}\phi_{A+B} &= \begin{pmatrix} A_0+B_0 & A_1+B_1 & A_2+B_2 & A_3+B_3 \\ A_1+B_1 & -A_0-B_0 & A_3+B_3 & -A_2-B_2 \\ A_2+B_2 & A_3+B_3 & A_0+B_0 & A_1+B_1 \\ A_3+B_3 & -A_2-B_2 & A_1+B_1 & -A_0-B_0 \end{pmatrix} \\ &= \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & -A_0 & A_3 & -A_2 \\ A_2 & A_3 & A_0 & A_1 \\ A_3 & -A_2 & A_1 & -A_0 \end{pmatrix} + \begin{pmatrix} B_0 & B_1 & B_2 & B_3 \\ B_1 & -B_0 & B_3 & -B_2 \\ B_2 & B_3 & B_0 & B_1 \\ B_3 & -B_2 & B_1 & -B_0 \end{pmatrix} \\ &= \phi_A + \phi_B\end{aligned}$$

iv. Suppose that $A \in \mathbb{H}^{m\times m}$ is nonsingular, From $AA^{-1} = I_4$, we have

$$\phi_{AA^{-1}} = \phi_A{}^1 P_m \phi_{A^{-1}} = \phi_{I_4}$$

and

$$\phi_A{}^1 P_m \phi_{A^{-1}}{}^1 P_m = I_{4m}.$$

Then, $\phi_A$ is nonsingular and $(\phi_A)^{-1} = {}^1 P_m \phi_{A^{-1}}{}^1 P_m$.
vi. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}^{m\times m}$ where $A_s \in \mathbb{R}^{m\times m}$, $s = 0,1,2,3$ and $\lambda \in \mathbb{C}$ be a coneigenvalue of $A$. Therefore, there exists a nonzero column vector $x \in \mathbb{H}^{m\times 1}$ so that $A\overline{\overline{x}} = x\lambda$. Then, we can write $\phi_A \mathbf{x} = \mathbf{x}\lambda$. Then complex coneigenvalue of commutative quaternion matrix $A$ is equivalent to the eigenvalue of $\phi_A$ that is

$$\overline{\sigma}(A) \cap \mathbb{C} = \sigma(\phi_A).$$

□

## 5. The commutative quaternion matrix equation $X - A\overline{\overline{X}}B = C$

In this part, we take into consideration the commutative quaternion matrix equation

$$X - A\overline{\overline{X}}B = C \tag{5.1}$$

via the real representation, where $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{n\times n}$ and $C \in \mathbb{H}^{m\times n}$. We define the real representation matrix equation of the matrix equation (5.1) by

$$Y - \phi_A Y \phi_B = \phi_C. \tag{5.2}$$

**Proposition 1.** *The equation (5.1) has a solution if and only if the equation (5.2) has a solution* $Y = \phi_X$.

**Theorem 10.** *Let $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{n\times n}$ and $C \in \mathbb{H}^{m\times n}$. Then the equation (5.1) has a solution $X \in \mathbb{H}^{m\times n}$ if and only if the equation (5.2) has a solution $Y \in R^{4m\times 4n}$, in which case, if $Y$ is a solution to (5.2), then the matrix:*

$$X = \frac{1}{16}(I_m \;\; i I_m \;\; j I_m \;\; k I_m)\left(Y - Q_m^{-1} Y Q_n + R_m^{-1} Y R_n - S_m^{-1} Y S_n\right)\begin{pmatrix} I_n \\ i I_n \\ j I_n \\ k I_n \end{pmatrix} \tag{5.3}$$

*is a solution to (5.1).*

*Proof.* We show that if the real matrix

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix}, \; Y_{ab} \in \mathbb{R}^{m\times n}, \; a,b = 1,2,3,4$$

is a solution to (5.2), then the matrix represented in (5.3) is a solution to (5.1). Since $Q_m^{-1} Y Q_n = -Y$, $R_m^{-1} Y R_n = Y$, $S_m^{-1} Y S_n = -Y$, we have

$$\begin{aligned} &-Q_m^{-1} Y Q_n - \phi_A\left(-Q_m^{-1} Y Q_n\right)\phi_B = \phi_C \\ &R_m^{-1} Y R_n - \phi_A\left(R_m^{-1} Y R_n\right)\phi_B = \phi_C \\ &-S_m^{-1} Y S_n - \phi_A\left(-S_m^{-1} Y S_n\right)\phi_B = \phi_C. \end{aligned} \tag{5.4}$$

The last equation shows that if $Y$ is a solution to (5.2), then $-Q_m^{-1} Y Q_n$, $R_m^{-1} Y R_n$ and $-S_m^{-1} Y S_n$ are also solutions to (5.2). Thus the undermentioned real matrix:

$$Y' = \frac{1}{4}\left(Y - Q_m^{-1}\phi_A Q_n + R_m^{-1}\phi_A R_n - S_m^{-1}\phi_A S_n\right) \tag{5.5}$$

is a solution to (5.2). After calculation , we easily obtain

$$Y' = \begin{pmatrix} Y_0' & Y_1' & Y_2' & Y_3' \\ Y_1' & -Y_0' & Y_3' & -Y_2' \\ Y_2' & Y_3' & Y_0' & Y_1' \\ Y_3' & -Y_2' & Y_1' & -Y_0' \end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned} &Y_0' = \tfrac{1}{4}(Y_{11} - Y_{22} + Y_{33} - Y_{44}), \;\; Y_1' = \tfrac{1}{4}(Y_{12} + Y_{21} + Y_{34} + Y_{43}), \\ &Y_2' = \tfrac{1}{4}(Y_{13} - Y_{24} + Y_{31} - Y_{42}), \;\; Y_3' = \tfrac{1}{4}(Y_{14} + Y_{23} + Y_{32} + Y_{41}). \end{aligned} \tag{5.7}$$

From (5.7), we formulate a matrix as follows:

$$X = Y_0' + Y_1' i + Y_2' j + Y_3' k = \frac{1}{4}(I_m \;\; i I_m \;\; j I_m \;\; k I_m) Y' \begin{pmatrix} I_m \\ i I_m \\ j I_m \\ k I_m \end{pmatrix}.$$

Clearly the real representation of the commutative quaternion matrix $X$ is $Y'$. By Proposition (1), $X$ is a solution to equation (5.1).

□

*Example* 1. Solve matrix equation

$$X - \begin{pmatrix} 1 & i \\ i & j \end{pmatrix} \overline{\overline{X}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2i - j & 1 + j \\ -1 + i + k & i + j \end{pmatrix}$$

by using its real representation.

Real representation of given equation is

$$\phi_X - \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} \phi_X \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

If we solve this equation, we have

$$\phi_X = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then,

$$X = \frac{1}{4}\begin{pmatrix} I_2 & i I_2 & j I_2 & k I_2 \end{pmatrix} \phi_X \begin{pmatrix} I_2 \\ i I_2 \\ j I_2 \\ k I_2 \end{pmatrix} = \begin{pmatrix} i & 1 + j \\ k & i + j \end{pmatrix}.$$

## 6. Appendix A. Commutative quaternion matrix equation $X - A\overline{\tilde{X}}B = C$

Let

$$\mu_A = \begin{pmatrix} A_0 & -A_1 & -A_2 & A_3 \\ A_1 & A_0 & -A_3 & -A_2 \\ A_2 & -A_3 & -A_0 & A_1 \\ A_3 & A_2 & -A_1 & -A_0 \end{pmatrix} \in \mathbb{R}^{4m\times 4n},$$

Then $\mu_A$ is called a real representation of $A$ corresponding to the linear transformation $L_A(X) = A\overline{\widetilde{X}}$. For $A \in \mathbb{H}^{m\times n}$ and $B \in \mathbb{H}^{n\times r}$ the following equalities are easy to confirm.

$$\text{i.} \quad \left({}^2P_m\right)^{-1}\phi_A\left({}^2P_m\right) = \phi_{\overline{\widetilde{A}}},\ Q_m^{-1}\mu_A Q_n = \phi_A, \tag{6.1}$$

$$R_m^{-1}\mu_A R_n = -\phi_A,\ S_m^{-1}\mu_A S_n = -\phi_A, \tag{6.2}$$

$$\text{ii.} \quad \mu_{AB} = \mu_A\left({}^2P_n\right)\mu_B = \mu_A\mu_{\overline{\widetilde{B}}}\left({}^2P_r\right),$$

where $Q, R, S$ are given by equation (4.1) and

$$\left({}^2P_m\right) = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}.$$

Now, we investigate the solution of the matrix equation

$$X - A\overline{\widetilde{X}}B = C \tag{6.3}$$

by its real representation, where $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{n\times n}$ and $C \in \mathbb{H}^{m\times n}$. We first define the real representation matrix equation (6.3) by

$$Y - \mu_A Y \mu_B = \mu_C. \tag{6.4}$$

In the same manner, we have $Y = \mu_X$.

**Theorem 11.** *Let $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{n\times n}$ and $C \in \mathbb{H}^{m\times n}$. Then the equation (6.3) has a solution $X \in \mathbb{H}^{m\times n}$ if and only if the equation (6.4) has a solution $Y \in \mathbb{R}^{4m\times 4n}$, in which case, if $Y$ is a solution to (6.4), then the matrix:*

$$X \tag{6.5}$$

$$= \frac{1}{16}\left(I_m \ \ iI_m \ \ jI_m \ \ kI_m\right)\left(Y + Q_m^{-1}YQ_n - R_m^{-1}YR_n - S_m^{-1}YS_n\right)\begin{pmatrix} I_m \\ -iI_m \\ -jI_m \\ kI_m \end{pmatrix}$$

*is a solution to (6.3).*

*Proof.* The proof is a routine process as was performed in Theorem 10. □

## 7. Appendix B. Commutative quaternion matrix equation $X - A\widetilde{\overline{X}}B = C$

In the same manner,

$$\varphi_A = \begin{pmatrix} A_0 & A_1 & -A_2 & -A_3 \\ A_1 & -A_0 & -A_3 & A_2 \\ A_2 & A_3 & -A_0 & -A_1 \\ A_3 & -A_2 & -A_1 & A_0 \end{pmatrix} \in \mathbb{R}^{4m\times 4n}$$

is called real representation of $A$ corresponding to the linear transformation $L_A(X) = A\widetilde{\overline{X}}$.

For $A \in \mathbb{H}^{m\times n}$ and $B \in \mathbb{H}^{n\times r}$ the following equalities are easy to confirm.

$$\text{i.} \quad \left({}^3P_m\right)^{-1}\varphi_A\left({}^3P_m\right) = \phi_{\widetilde{\overline{A}}},\ Q_m^{-1}\varphi_A Q_n = -\varphi_A,$$
$$R_m^{-1}\varphi_A R_n = -\varphi_A,\ S_m^{-1}\varphi_A S_n = \varphi_A,$$
$$\text{ii.} \quad \varphi_{AB} = \varphi_A\left({}^3P_n\right)\varphi_B = \varphi_A\varphi_{\widetilde{\overline{B}}}\left({}^3P_r\right),$$

where $Q, R, S$ are given by equation (4.1) and

$$\left({}^3P_m\right) = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}.$$

Now, we define the real representation matrix equation of the matrix equation

$$X - A\widetilde{\overline{X}}B = C \tag{7.1}$$

by

$$Y - \varphi_A Y \varphi_B = \varphi_C \tag{7.2}$$

where $Y = \varphi_X$.

**Theorem 12.** *Let $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{n\times n}$ and $C \in \mathbb{H}^{m\times n}$. Then the equation (7.1) has a solution $X \in \mathbb{H}^{m\times n}$ if and only if the equation (7.2) has a solution $Y \in \mathbb{R}^{4m\times 4n}$, in which case, if $Y$ is a solution to (7.2), then the matrix:*

$$X \tag{7.3}$$
$$= \frac{1}{16}\left(I_m \ \ iI_m \ \ jI_m \ \ kI_m\right)\left(Y - Q_m^{-1}YQ_n - R_m^{-1}YR_n + S_m^{-1}YS_n\right)\begin{pmatrix} I_m \\ -iI_m \\ -jI_m \\ kI_m \end{pmatrix}$$

*is a solution to (7.1) .*

*Proof.* The proof is a routine process as was performed in Theorem 10. □

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# TG-SUPPLEMENTED MODULES

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*Abstract.* In this work, we define tg-supplemented modules and investigate some properties of these modules. We prove that the finite t-sum of tg-supplemented modules is tg-supplemented. We also prove that the homomorphic image of a distributive tg-supplemented module is tg-supplemented. We give some examples separating tg-supplemented modules from supplemented and generalized $\oplus$-supplemented modules.

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## 1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$-module. We will denote a submodule $K$ of $M$ by $K \leq M$. Let $M$ be an $R$-module and $K \leq M$. If $T = M$ for every submodule $T$ of $M$ such that $K + T = M$, then $K$ is called a *small submodule* of $M$ and denoted by $K \ll M$. Let $M$ be an $R$-module and $K \leq M$. If there exists a submodule $T$ of $M$ such that $K + T = M$ and $K \cap T = 0$, then $K$ is called a *direct summand* of $M$ and it is denoted by $M = K \oplus T$. For any module $M$, the intersection of maximal submodules of $M$ is called the *radical* of $M$ and denoted by $RadM$. If $M$ have no maximal submodules, then we define $RadM = M$. A module $M$ is called *distributive* [8] if for every submodules $K, L, T$ of $M$, $K \cap (L+T) = K \cap L + K \cap T$ or equivalently $(K + L) \cap (K + T) = K + L \cap T$. Let $U$ and $V$ be submodules of a module $M$. If $U + V = M$ and $V$ is minimal with respect to this property, or equivalently, $U + V = M$ and $U \cap V \ll V$, then $V$ is called a *supplement* [10] of $U$ in $M$. $M$ is called a *supplemented module* if every submodule of $M$ has a supplement in $M$. $M$ is called ([5],[6]) $\oplus$-*supplemented module* if every submodule of $M$ has a supplement that is a direct summand of $M$. Let $M$ be an $R$-module and $U$, $V$ be submodules of $M$. $V$ is called a *generalized supplement* ([1],[9],[11]) of $U$ in $M$ if $M = U + V$ and $U \cap V \leq RadV$. $M$ is called *generalized supplemented* or briefly a $GS$-*module* if every submodule of $M$ has a generalized supplement in $M$. Clearly

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we see that every supplemented module is a generalized supplemented module. $M$ is called a *generalized $\oplus$-supplemented* ([2],[4],[7],[8]) module if every submodule of $M$ has a generalized supplement that is a direct summand of $M$. In this paper we generalize these modules.

**Lemma 1.** *Let $V$ be a supplement of $U$ in $M$ and $L, K \leq V$. Then $K$ is a supplement of $L$ in $V$ if and only if $K$ is a supplement of $U + L$ in $M$.* (*[3], Exercise 20.39*)

*Proof.* ($\Rightarrow$) Let $U + L + T = M$, for some $T \leq K$. Since $V$ is a supplement of $U$ in $M$ and $L + T \leq V$, $L + T = V$ and by $K$ being a supplement of $L$ in $V$, $T = K$. Hence $K$ is a supplement of $U + L$ in $M$.

($\Leftarrow$) Let $L + T = V$, for some $T \leq K$. Then $U + L + T = M$, and by $K$ being a supplement of $U + L$ in $M$, $T = K$. Hence $K$ is a supplement of $L$ in $V$. □

**Lemma 2.** *Let $M$ be a $\pi$-projective module and $K, L$ be two submodules of $M$. If $K$ and $L$ are mutual supplements in $M$, then $K \cap L = 0$ and $M = K \oplus L$.*

*Proof.* See ([10], 41.14(2)). □

## 2. TG-SUPPLEMENTED MODULES

**Definition 1.** Let $M$ be an $R$-module and $K, L$ be two submodules of $M$. If $K$and $L$ are mutual supplements in $M$, then $M$ is called a *t-sum* of $K$ and $L$. This equivalent to $M = K + L$, $K \cap L \ll K$ and $K \cap L \ll L$. This case $K$ and $L$ are called *t-summands* of $M$.

**Definition 2.** Let $M$ be an $R$-module and $\{A_i\}_{i \in I}$ be a family of submodules of $M$. $M$ is called a *t-sum* of $\{A_i\}_{i \in I}$, if $A_k$ and $\sum_{j \neq k} A_j$ are mutual supplements in $M$ for every $k \in I$.

**Lemma 3.** *Let $M$ be an $R$-module, $V$ be a t-summand of $M$ and $K \leq V$. Then $K \ll M$ if and only if $K \ll V$.*

*Proof.* Clear from ([12], Lemma 1.1). □

**Lemma 4.** *Let $M$ be a t-sum of $U$ and $V$. If $K$ is a supplement of $S$ in $U$ and $L$ is a supplement of $T$ in $V$, then $K + L$ is a supplement of $S + T$ in $M$.*

*Proof.* Since $U$ is a supplement of $V$ in $M$ and $K$ is a supplement of $S$ in $U$, by Lemma 1.1, $K$ is a supplement of $V + S$ in $M$. Hence $(V + S) \cap K \ll K$. Similarly, we prove that $(U + T) \cap L \ll L$. Then $(S + T) \cap (K + L) \leq (S + T + K) \cap L + (S + T + L) \cap K = (U + T) \cap L + (V + S) \cap K \ll K + L$, and by $M = U + V = S + K + T + L = S + T + K + L$, $K + L$ is a supplement of $S + T$ in $M$. □

**Lemma 5.** *Let $M$ be a t-sum of $U$ and $V$, and $L, T \leq V$. Then $V$ is a t-sum of $L$ and $T$ if and only if $M$ is a t-sum of $U+L$ and $T$, and $M$ is a t-sum of $U+T$ and $L$.*

*Proof.* ($\Rightarrow$) Let $V$ be a t-sum of $L$ and $T$. Since $T$ is a supplement of $L$ in $V$ and $V$ is a supplement of $U$ in $M$, then by Lemma 1, $T$ is a supplement of $U+L$ in $M$. Then $(U+L)\cap T \ll T$. Similarly, we can prove that $(U+T)\cap L \ll L$. Then by $U\cap V \ll U$, $(U+L)\cap T \leq U\cap(L+T)+L\cap(U+T) = U\cap V+(U+T)\cap L \ll U+L$. Since $(U+L)\cap T \ll T, (U+L)\cap T \ll U+L$ and $M = U+V = U+L+T$, then by Definition 1 $M$ is a t-sum of $U+L$ and $T$. Similarly, we prove that $M$ is a t-sum of $U+T$ and $L$.

($\Leftarrow$)Clear from Lemma 1. □

**Corollary 1.** *Let $M$ be a t-sum of $U_1, U_2, \ldots, U_n$. If $K_i$ is a supplement of $T_i$ in $U_i$ $(i = 1, 2, \ldots, n)$, then $K_1 + K_2 + \cdots + K_n$ is a supplement of $T_1 + T_2 + \cdots + T_n$ in $M$.*

*Proof.* Clear from Lemma 5. □

**Corollary 2.** *Let $M$ be a t-sum of $U_1, U_2, \ldots, U_n$. If $U_i$ is a t-sum of $K_i$ and $T_i$ $(i = 1, 2, \ldots, n)$, then $M$ is a t-sum of $K_1 + K_2 + \cdots + K_n$ and $T_1 + T_2 + \cdots + T_n$.*

*Proof.* Clear from Corollary 1. □

**Corollary 3.** *Let $M$ be a t-sum of $U_1, U_2, \ldots, U_n$. If $K_i$ is a supplement in $U_i$ $(i = 1, 2, \ldots, n)$, then $K_1 + K_2 + \cdots + K_n$ is a supplement in $M$.*

*Proof.* Clear from Corollary 1. □

**Corollary 4.** *Let $M$ be a t-sum of $U_1, U_2, \ldots, U_n$. If $K_i$ is a t-summand of $U_i$ $(i = 1, 2, \ldots, n)$, then $K_1 + K_2 + \cdots + K_n$ is a t-summand of $M$.*

*Proof.* Clear from Lemma 5. □

**Lemma 6.** *Let $M$ be a distributive $R$-module and $N \leq M$. Then $(K+N)/N$ is a t-summand of $M/N$ for every t-summand $K$ of $M$.*

*Proof.* Let $K$ be a t-summand of $M$. Then there exists a submodule $L$ of $M$ such that $M = L+K, L\cap K \ll L$ and $L\cap K \ll K$. Since $M = L+K$, then $M/N = (L+N)/N + (K+N)/N$. Since $M$ is distributive, then we have $(L+N)\cap(K+N) = L\cap K+N$. Since $L\cap K \ll L$ and $L\cap K \ll K$, then we have $((L+N)/N)\cap((K+N)/N) = (L\cap K+N)/N \ll (L+N)/N$ and $((L+N)/N)\cap((K+N)/N) = (L\cap K+N)/N \ll (K+N)/N$. Hence $(K+N)/N$ is a t-summand of $M/N$. □

**Theorem 1.** *Let $M$ be a t-sum of $\{A_i\}_{i\in I}$ . Then $RadM = \sum_{i\in I} RadA_i$.*

*Proof.* Let $x \in RadM$. Since $x \in M = \sum_{i\in I} A_i$, there exist $i_1, i_2, \ldots, i_n \in I$ and $x_{i_1} \in A_{i_1}, x_{i_2} \in A_{i_2}, \ldots, x_{i_n} \in A_{i_n}$ such that $x = x_{i_1} + x_{i_2} + \cdots + x_{i_n}$. Let $k \in \{1, 2, \ldots, n\}, T \leq A_{i_k}$ and $Rx_{i_k} + T = A_{i_k}$. Let $a \in M$. Since $a \in M = \sum_{i\in I, i\neq i_k} A_i + A_{i_k}$, we can write $a = b + c$ for some $b \in \sum_{i\in I, i\neq i_k} A_i$ and $c \in A_{i_k}$. Since $c \in A_{i_k} = Rx_{i_k} + T$, there exist $r \in R$ and $t \in T$ such that $c = rx_{i_k} + t$. Then $a = b + c = b + rx_{i_k} + t = b + r\left(x - \sum_{s=1, s\neq i_k}^{n} x_{i_s}\right) + t = rx + b - \sum_{s=1, s\neq i_k}^{n} rx_{i_s} + t \in Rx + \sum_{i\in I, i\neq i_k} A_i + T$. Hence $M = Rx + \sum_{i\in I, i\neq i_k} A_i + T$ and since $Rx \ll M, M = \sum_{i\in I, i\neq i_k} A_i + T$. Since $M = \sum_{i\in I, i\neq i_k} A_i + T$ and $M$ is a t-sum of $\{A_i\}_{i\in I}, T = A_{i_k}$. Thus $Rx_{i_k} \ll A_{i_k}$ and $x_{i_k} \in RadA_{i_k}$. Consequently, $x \in \sum_{i\in I} RadA_i$ and $RadM \leq \sum_{i\in I} RadA_i$. $\sum_{i\in I} RadA_i \leq RadM$ is clear. Thus $RadM = \sum_{i\in I} RadA_i$. □

**Definition 3.** Let $M$ be an $R$-module. $M$ is called a *tg-supplemented* module if every submodule of $M$ has a generalized supplement that is a t-summand of $M$. Clearly generalized $\oplus$-supplemented modules are tg-supplemented. But the converse is not true in general (See Example 4).

We can also clearly see that every supplemented module is tg-supplemented. But the converse of this statement is not always true (See Example 1, 2, 3). Since hollow and local modules are supplemented, they are tg-supplemented modules. Clearly, every tg-supplemented module is generalized supplemented.

**Lemma 7.** *Let $M$ be an $R$-module. If $RadM = M$, then $M$ is tg-supplemented.*

*Proof.* Let $N$ be any submodule of $M$. Since $N + M = M$ and $N \cap M \leq M = RadM$, we get that $M$ is a generalized supplement of $N$ in $M$. On the other hand $M$ and 0 are mutual supplements in $M$. Hence $M$ is tg-supplemented. □

**Lemma 8.** *Let $M$ be a tg-supplemented $R$-module and $N \ll M$. Then $M/N$ is tg-supplemented.*

*Proof.* Let $U/N \leq M/N$. Since $M$ is tg-supplemented, $U$ has a generalized supplement $V$ that is a t-summand in $M$. Then by ([9], the proof of Proposition 2.6), $(V + N)/N$ is a generalized supplement of $U/N$ in $M/N$. Since $V$ is a t-summand of $M$, there exists a submodule $L$ of $M$ such that $L$ and $V$ are mutual supplements in $M$. Since $L$ is a supplement of $V$ in $M$ and $N \ll M$, by ([10], 41.1(4)) $L$ is a supplement of $V + N$ in $M$. Then by ([10], 41.1(7)) $(L + N)/N$ is a supplement of $(V + N)/N$ in $M/N$. Similarly, we can prove that $(V + N)/N$ is a supplement of $(L + N)/N$ in $M/N$. Hence $(L + N)/N$ and $(V + N)/N$ are mutual supplements in $M/N$. Thus $M/N$ is tg-supplemented. □

**Corollary 5.** *Any small homomorphic image of a tg-supplemented module is tg-supplemented.*

*Proof.* Clear from Lemma 8. □

**Lemma 9.** *Let $M$ be a tg-supplemented module and $N \leq M$. If $(K+N)/N$ is a t-summand of $M/N$ for every t-summant $K$ of $M$, then $M/N$ is tg-supplemented.*

*Proof.* Let $U/N \leq M/N$. Since $M$ is tg-supplemented, $U$ has a generalized supplement $K$ in $M$ such that $K$ is a t-summand of $M$. Since $K$ is a generalized supplement of $U$ in $M$ and $N \leq U$, we can see that $(K+N)/N$ is a generalized supplement in $M/N$. Since $K$ is a t-summand of $M$, then by hypothesis $(K+N)/N$ is a t-summand of $M/N$. Hence every submodule of $M/N$ has a generalized supplement that is a t-summand of $M/N$, and $M/N$ is tg-supplemented. □

**Lemma 10.** *Let $M$ be a distributive tg-supplemented $R$-module. Then every factor module of $M$ is tg-supplemented.*

*Proof.* Clear from Lemma 6 and Lemma 9. □

**Corollary 6.** *Let $M$ be a distributive tg-supplemented $R$-module. Then every homomorphic image of $M$ is tg-supplemented.*

*Proof.* Clear from Lemma 10. □

**Lemma 11.** *Let $M$ be an $R$-module and $RadM \ll M$. The following assertions are equivalent.*
*$(i)$ $M$ is supplemented.*
*$(ii)$ $M$ is tg-supplemented.*

*Proof.* $(i)\Rightarrow(ii)$ Clear from definitions.
$(ii)\Rightarrow(i)$ Let $U \leq M$. Since $M$ is tg-supplemented, there exists a generalized supplement $V$ of $U$ that is a t-summand of $M$. Since $V$ is supplement in $M$, then $V \cap RadM = RadV$. Since $RadM \ll M$, $RadV \ll M$ and, by Lemma 3, $U \cap V \leq RadV \ll V$. Thus $V$ is a supplement of $U$ in $M$ and $M$ is supplemented. □

**Corollary 7.** *Let $M$ be a finitely generated $R$-module. The following assertions are equivalent.*
*$(i)$ $M$ is supplemented.*
*$(ii)$ $M$ is tg-supplemented.*

*Proof.* Since $M$ is finitely generated, $RadM \ll M$. Then clearly this assertions is derived from Lemma 11. □

**Lemma 12.** *Let $M$ be a t-sum of $M_1$and $M_2$. If $M_1$and $M_2$ are tg-supplemented, then $M$ is tg-supplemented.*

*Proof.* Let $U \leq M$. Since $M_1$ is tg-supplemented, $(M_2 + U) \cap M_1$ has a generalized supplement $X$ that is a t-summand in $M_1$. Since $M_2$ is tg-supplemented, $(U + X) \cap M_2$ has a generalized supplement $Y$ that is a t-summand in $M_2$. Then we get $M = M_1 + M_2 = M_2 + U + X = U + X + Y$ and $U \cap (X + Y) \leq (U + Y) \cap X + (U + X) \cap Y \leq RadX + RadY \leq Rad(X + Y)$. Hence $X + Y$ is a generalized supplement of $U$ in $M$. Since $M$ is a t-sum of $M_1$ and $M_2$, and $X$ is a t-summand of $M_1$, and $Y$ is a t-summand of $M_2$, then by Corollary 3, $X + Y$ is a t-summand of $M$. Thus $M$ is tg-supplemented. □

**Corollary 8.** *Let $M$ be a t-sum of $M_1, M_2, \ldots, M_n$. If $M_i$ is tg-supplemented ($i = 1, 2, \ldots, n$), then $M$ is tg-supplemented.*

*Proof.* Clear from Lemma 12. □

*Example* 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathbb{Q}$ has no maximal submodule, we have $Rad\mathbb{Q} = \mathbb{Q}$. By Lemma 2.13, $\mathbb{Q}$ is a tg-supplemented module. But it is well known that $\mathbb{Q}$ is not supplemented (See [3], Example 20.12 ).

*Example* 2. Let $M$ be a non-torsion $\mathbb{Z}$-module with $RadM = M$. Since $RadM = M$, then by Lemma 2.13, $M$ is tg-supplemented. But $M$ is not supplemented ([12]).

*Example* 3. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$, for any prime $p$. In this case $RadM \neq M$. Since $\mathbb{Q}$ and $p\mathbb{Z}$ are tg-supplemented, then by Lemma 12, $M$ is tg-supplemented. But $M$ is not supplemented.

*Example* 4. Let $R$ be a commutative local ring which is not a valuation ring. Let $a$ and $b$ be elements of $R$, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$ where $m$ is the maximal ideal of $R$. Let $F$ be a free $R$-module with generators $x_1, x_2$ and $x_3$, $K$ be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$. Here $M$ is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented ([5]).

Since $F$ is completely $\oplus$-supplemented, $F$ is supplemented. Since a factor module of a supplemented module is supplemented, we have $M$ is supplemented. So $M$ is tg- supplemented. But since $M$ is finitely generated and not $\oplus$-supplemented, $M$ is not generalized $\oplus$-supplemented.

**Lemma 13.** *Let $M$ be a t-sum of $M_1$ and $M_2$. Then $M_2$ is tg-supplemented if and only if for every submodule $N$ of $M$ such that $M_1 \leq N \leq M$, there exists a t-summand $K$ of $M_2$ such that $M = K + N$ and $N \cap K \leq RadM$.*

*Proof.* ($\Rightarrow$) Let $M_1 \leq N \leq M$. Since $M_2$ is tg-supplemented, there exists a generalized supplement $K$ of $N \cap M_2$ in $M_2$ such that $K$ is a t-summand of $M_2$. Then $M = M_1 + M_2 = N + N \cap M_2 + K = K + N$ and $N \cap K = N \cap M_2 \cap K \leq RadK \leq RadM$.

($\Leftarrow$) Let $L \leq M_2$ and $N = M_1 + L$. By hypothesis, there exists a t-summand $K$ of $M_2$ such that $M = K + N$ and $N \cap K \leq RadM$. Since $K, L \leq M_2$, by Modular law, $M_2 = M_2 \cap M = M_2 \cap (K + N) = K + M_2 \cap N = K + M_2 \cap (M_1 + L) = L + K + M_2 \cap M_1$, and then by $M_2 \cap M_1 \ll M_2$, $M_2 = L + K$. Since $K$ is a t-summand of $M_2$, then by Corollary 3, $K$ is a t-summand of $M$. Then $RadK = K \cap RadM$ and by $N \cap K \leq RadM, L \cap K \leq N \cap K = K \cap (N \cap K) \leq K \cap RadM = RadK$. Hence $K$ is a generalized supplement of $L$ in $M_2$. Thus, $M_2$ is tg-supplemented. □

**Theorem 2.** *Let $M$ be a tg-supplemented module. Assume that $M$ is a t-sum of $M_1$ and $M_2$. If $K \cap M_2$ is a t-summand of $M_2$ for every t-summand $K$ of $M$ such that $M = K + M_2$, then $M_2$ is tg-supplemented.*

*Proof.* Let $M_1 \leq N \leq M$. Since $M$ is tg-supplemented, $N \cap M_2$ has a generalized supplement $K$ in $M$ such that $K$ is a t-summand of $M$. From this we have $M = N \cap M_2 + K$ and $N \cap M_2 \cap K \leq RadK \leq RadM$. Since $M = N \cap M_2 + K$, then by Modular law $M_2 = N \cap M_2 + M_2 \cap K$. Since $M_1 \leq N$, $M = M_1 + M_2 = M_1 + N \cap M_2 + M_2 \cap K = N + M_2 \cap K$. Since $M = K + M_2$ and $K$ is a t-summand of $M$, then by hypothesis $M_2 \cap K$ is a t-summant of $M_2$. Hence by Lemma 13, $M_2$ is tg-supplemented. □

**Lemma 14.** *Let $M$ be a $\pi$-projective module. Then $M$ is tg-supplemented if and only if $M$ is generalized $\oplus$-supplemented.*

*Proof.* Clear from Lemma 2. □

**Theorem 3.** *Let $M$ be a projective module. The following assertions are equivalent.*

($i$) *$M$ is semiperfect.*
($ii$) *$M$ is generalized $\oplus$-supplemented.*
($iii$) *$M$ is tg-supplemented.*

*Proof.* ($i$)$\Leftrightarrow$($ii$) Clear from ([10], 42.1).
($ii$)$\Leftrightarrow$($iii$) Clear from Lemma 14. □

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# A NEW CRITERION FOR THE ROUGHNESS OF EXPONENTIAL DICHOTOMY ON $\mathbb{R}$

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*Abstract.* It is well known (see [Palmer, K.J.: Exponential dichotomies and transversal homoclinic points, J. Differential Equations 55 (1984), 225-256]) that system $x' = A(t)x$ possesses exponential dichotomy on $\mathbb{R}^+$ iff so does perturbed system $x' = (A(t) + B(t))x$, where perturbation $B(t)$ is "small", that is $\lim_{t\to+\infty} \|B(t)\| = 0$. However, it is also known that corresponding statement for exponential dichotomy on $\mathbb{R}$ fails. In this work we show that under additional hypothesis of commutativity of $A(s)$ and $B(t)$ for all $s,t \in \mathbb{R}$ it can indeed be shown that dichotomy on $\mathbb{R}$ is unaffected by "small" perturbations (while "small" now has meaning of "being in $L^1(\mathbb{R})$ class"). We also provide some examples and particular corollaries of theory to show that hypotheses are natural and apply to wide range of problems.

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*Keywords:* exponential dichotomy, perturbation theory, linear differential equations

## 1. Introduction

Consider the system

$$x'(t) = A(t)x(t),$$

where $x(t)$ and $A(t)$ are respectively vector-valued and square-matrix-valued functions defined for $t \in S \subseteq \mathbb{R}$. In subsequent, we will impose conditions on $A(t)$, so that any initial value problem $x(t_0) = x_0$, $t_0 \in S$ has a unique continuous solution that extends to the whole $S$. This will allow us to speak of the *fundamental matrix* $\Omega_a^b(A)$, which is a linear map that maps $x_0$ to $x(b)$, where $x(t)$ satisfies the aforementioned system and initial conditions $x(a) = x_0$ (linearity follows from the homogeneity of a system). Now, the system is said to possess **"exponential dichotomy"** on $S$ (or simply "dichotomy" in subsequent), if the phase space $\mathbb{R}^n$ can be decomposed into the direct sum of subspaces $E^+ \oplus E^-$ and for arbitrary $x^+ \in E^+$, $x^- \in E^-$ and $a, b \in S$ the following estimates hold:

$$\begin{aligned} \|\Omega_0^b(A)x^+\| &\le K\|\Omega_0^a(A)x^+\| \cdot e^{-\gamma|a-b|},\ a \le b, \\ \|\Omega_0^b(A)x^-\| &\le K\|\Omega_0^a(A)x^-\| \cdot e^{-\gamma|a-b|},\ b \le a \end{aligned} \tag{1.1}$$

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for some constants $K,\ \gamma > 0$, that do not depend on $x^+,\ x^-,\ a,\ b$.

*Remark* 1. The alternative definition, as given in [3] can be formulated as follows: there exist projection-operator $P$ and numbers $K,\ \gamma > 0$, such that for $\forall a, b \in S$

$$\|\Omega_0^a(A)P\left(\Omega_0^b(A)\right)^{-1}\| \le Ke^{-\gamma(a-b)},\ a \ge b,$$

$$\|\Omega_0^a(A)(I-P)\left(\Omega_0^b(A)\right)^{-1}\| \le Ke^{-\gamma(b-a)},\ b \ge a.$$

The concept of exponential dichotomy plays an important role both in theory and applications. For instance in [1], it has been utilized for the development of sufficient conditions for the existence of mean square bounded solutions to linear stochastic Ito system. In [6], exponential dichotomy properties have been utilized for the investigation of multi-frequency oscillations and qualitative behavior of solutions of linear extensions of dynamical systems on torus. Roughness of exponential dichotomy (that is, its persistence under "small" perturbations) is the main property that allows to apply this concept to the investigations of a variety of real-world phenomena and processes. This causes a particular relevance of research in this field.

One of the first monographs entirely devoted to the subject of exponential dichotomy was the book of Coppel [3]. It contains the exposition of how the dichotomy relates to basic concepts in systems of ODE, such as roughness, Lyapunov functions and functional analysis aspects. It also encompasses the criteria for dichotomy and roughness conditions, in particular that exponential dichotomy on any interval $J$ is *not* affected when the perturbation $B(t)$ satisfies $\|B\|_\infty := \sup_{t\in J}\|B(t)\| < \delta$, where $\delta$ is some fixed constant, depending on function $A$. From there, this roughness result was worked upon by numerous researchers. For example, Palmer in [9] has shown (among other results) that dichotomy on $[t_0, +\infty)$ is preserved under perturbation $B(t)$ if $\lim_{t\to\infty}\|B(t)\| = 0$. The constant $\delta$ mentioned above was sharpened in [8], where authors also have extended Coppel's roughness statement to the case when dichotomy constants $K$ and $\gamma$ (see above) can be different for $E^+$ and $E^-$. Besides, in [4], authors tackled the case when $A$ is assumed only locally integrable, not necessarily continuous or even bounded on $J$.

Although in this work we confine ourselves to the finite-dimensional setting, one can easily introduce the corresponding exponential dichotomy theory to the case of operators on Banach spaces with many results and proofs carrying through. Then it is fruitful to study the preservation of dichotomy under "small" (in some sense) perturbations in this new generalised setting. Anyhow, we refer the interested reader to [2, 5, 10] and references therein.

In this work we show that under additional hypothesis of commutativity of $A(s)$ and $B(t)$ for all $s, t \in \mathbb{R}$ it can indeed be shown that dichotomy on $\mathbb{R}$ is unaffected by "small" perturbations (while "small" now has meaning of "being in $L^1(\mathbb{R})$ class"). We also provide some examples and particular corollaries of theory to show that

hypotheses are natural and apply to wide range of problems. Some comparison with previously known results will follow.

The rest of the paper is organized as follows: In Section 2 we state our main result, Section 3 is devoted to its proof. In Section 4 we have collected some corollaries and examples to highlight how our result relates to the previously known ones. Finally, the last section concludes with a short discussion of a further work.

## 2. Results

In subsequent, all matrix functions will be assumed to be continuous. In particular, continuity assumption will let us immediately assume that Cauchy problem for every ODE system $x' = A(t)x$ has a unique solution on $\mathbb{R}$ for any initial conditions.

**Theorem 1.** *Assume that for the two systems*

$$x' = A(t)x(t) \tag{2.1}$$

*and*

$$x' = (A(t) + B(t))x(t) \tag{2.2}$$

*the following two conditions hold:*

(1) *For any $t, s \in \mathbb{R}$ matrices $A(s)$ and $B(t)$ commute;*

(2) *$B(t) \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, that is $\int\limits_{-\infty}^{\infty} \|B(s)\|\, ds < \infty$.*

*Then, (2.2) has exponential dichotomy on $\mathbb{R}$ if (2.1) does.*

*Remark* 2. The hypothesis of the theorem 1 about the commutativity (the first condition) holds in particular if $\forall t,\ B(t) \in \bigcap\limits_{s\in\mathbb{R}} Z_{\mathfrak{gl}_n}(A(s))$, where $Z_{\mathfrak{gl}_n}(A)$ denotes the (Lie algebra) centralizer of $A$. If furthermore $A(t) \equiv A$, this condition can be simply rewritten as $\forall t,\ B(t) \in Z_{\mathfrak{gl}_n}(A)$.

## 3. Proofs

*Proof of Theorem 1.* The desired result will follow from two lemmas, which we shall state now and prove later. It should be noted that the lemmas have weaker hypothesis than that of the theorem, and are useful results on their own.

**Lemma 1.** *Let $A(t)$ and $B(t)$ be continuous (not necessary bounded matrices-valued functions on $\mathbb{R}$). Assume further that for every $s, t \in \mathbb{R}$ $A(s)$ and $B(t)$ commute. Then, for the system $x' = B(t)x$, every Cauchy problem has a unique solution on $\mathbb{R}$, in other words, the fundamental matrix (we shall denote it by $X_B(t)$) is well-defined on $\mathbb{R}$ and for arbitrary $t, s \in \mathbb{R}$, $A(t)$ and $X_B(s)$ commute.*

**Lemma 2.** *Let $B(t)$ be continuous matrix-valued function, such that $B(t) \in L^1(\mathbb{R})$. Then, the fundamental matrix $X_B(t)$ of $x' = B(t)x$ is well-defined on $\mathbb{R}$, and both $\|X_B(t)\|$ and $\|X_B^{-1}(t)\|$ are uniformly bounded in $t \in \mathbb{R}$.*

Assuming these two for the moment, let us see how the proof of the theorem can be obtained from them. Consider the system $x' = B(t)x$, whose fundamental matrix (existing by Lemma 2) we shall denote by $X_B(t)$. Let us make the variable change $x = X_B(t)z$. Then, we have

$$BX_B z + X_B z' = (X_B z)' = x' = (A+B)X_B z$$

Now, as by Lemma 1 $A$ and $X_B$ commute, we have

$$X_B z' = X_B A z$$

thus $z$ is the solution to $z' = Az$, hence the fundamental matrix of $x' = (A+B)x$ has the structure

$$X_{A+B} = X_B X_A.$$

Now, from [3] we know, that the exponential dichotomy on $\mathbb{R}$ can be defined as the existence of projection operator $P$ and numbers $\gamma, K > 0$, such that $\forall s, t \in \mathbb{R}$ we have

$$\|X(t)PX^{-1}(s)\| \le Ke^{-\gamma(t-s)},\ s \le t,$$

$$\|X(t)(I-P)X^{-1}(s)\| \le K^{-\gamma(s-t)},\ t \le s.$$

We shall show, that this estimation remains the same for perturbed system 2.2 with the same $P$ and $\gamma$, but perhaps different $K$ . Indeed, we shall have

$$\|X_{A+B}(t)PX_{A+B}^{-1}(s)\| = \|X_B(t)X_A(s)PX_A^{-1}(t)X_B^{-1}(t)\| \le$$

$$\le \|X_B(t)\| \cdot \|X_A(s)PX_A^{-1}(t)\| \cdot \|X_B^{-1}(t)\| \le \underbrace{KC^2}_{=:\tilde{K}} e^{-\gamma(t-s)}.$$

The second inequality is worked out in the same way, so it just remains to prove the lemmas stated above. □

*Proof of Lemma 2.* The existence of fundamental matrix follows immediately from the continuity assumption on $B(t)$. The uniform boundedness of $\|X_B(t)\|$ follows from [12, Corollary 3.24] (which states that if the system $x' = Cx$ with $C$ constant is uniformly stable and $B(t) \in L^1(\mathbb{R} \cap C(\mathbb{R})$, then $x' = (C + B(t))x$ is stable as well; in our case we take obviously stable system $x' = 0 \cdot x$ as $x' = Cx$). We shall denote the upper bound obtained for $X_B(t)$ by $M$. It is sufficient to show that $\|X_B^{-1}(t)\| \le M$. Let us fix $\tau \in \mathbb{R}$. $X_B^{-1}(\tau)$ can be characterized as $X_B^{-1}(\tau)x(\tau) = x(0)$, where $x$ is the solution to $x' = B(t)x$ or as $X^{-1}(\tau)y(0) = y(-\tau)$, where $y' = \tilde{B}y$ and $\tilde{B}(t) := B(t+\tau)$. Thus, it's enough to show that the norm of $X_{\tilde{B}}$ is uniformly bounded by $M$ as well. The latter, however, follows from the proof of [12, Corollary 3.24], as the upper bound obtained there really depends only on the $\int_{-\infty}^{\infty} \|\tilde{B}(s)\|\, ds = \int_{-\infty}^{\infty} \|B(s)\|\, ds.$ □

*Proof of Lemma 1.* We shall employ the theory of Magnus expansion, described in [7]. For convenience, we collect the properties we need in the following proposition

**Proposition** (**Magnus Expansion**, see [7]). *Assume that we have a system $x' = B(t)x$ with continuous $B(t)$ and fundamental matrix $X(t)$. Assume further that for some $T$ we have*

$$\left| \int_0^T \|B(s)\| \, ds \right| < \pi$$

*then, on $t \in [0,T)$ $X(t) = \exp(\Omega(t))$, where $\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$ and $\Omega_k(t)$ is an integral of commutators of increasing length, such as*

$$\Omega_1(t) = \int_0^t B(t_1)\, dt_1,$$

$$\Omega_2(t) = \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \; [B(t_1), B(t_2)],$$

$$\Omega_3(t) = \frac{1}{6} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \Big( [B(t_1), [B(t_2), B(t_3)]] + [B(t_3), [B(t_2), B(t_1)]] \Big),$$

$$\begin{aligned} \Omega_4(t) = \frac{1}{12} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \Big( & [[[B_1, B_2], B_3], B_4] \\ & + [B_1, [[B_2, B_3], B_4]] + [B_1, [B_2, [B_3, B_4]]] + [B_2, [B_3, [B_4, B_1]]] \Big) \end{aligned}$$

Having this, we shall fix $\tau \in \mathbb{R}$ and show that for any $t \in \mathbb{R}$,

$$[A(\tau), X_B(t)] = 0,$$

where $[\cdot,\cdot]$ denotes the Lie bracket. As for $t = 0$ the equality holds (as $X(0) = I$ commutes with all matrices), it is enough to show that the set where equality holds is both open and closed. As both $A$, $X$ and Lie bracket are continuous, closedness follows and it remains thus to show, that if $[A(\tau), X(t_0)] = 0$, then equality also holds on a small neighborhood of $t_0$.

It is sufficient to show that $A(\tau)$ commutes with $\tilde{X}(s) := X(t_0 + s)X^{-1}(t_0)$ for small $s$. Now, $\tilde{X}$ can be realized as a fundamental matrix of $x' = \tilde{B}(t)x$, where $\tilde{B}(t) = B(t + t_0)$. Thus, without loss of generality we may assume that $t_0 = 0$. Now, on the small neighborhood of 0 the hypothesis of Proposition is satisfied, hence we may apply Magnus expansion.

Now, if $A$ and $B$ commute with $C$, then so do $A \pm B$, $AB$ and $[A, B]$. Also, if $B(t)$ commutes with $A$ for all $t$, then so does $\int_a^b B(s)\, ds$. Thus, we see that every $\Omega_k(t)$ commutes with $A(\tau)$, as former is the integral of commutators of $B$. Finally, as Lie bracket is continuous, we also have that $\Omega(t)$ commutes with $A(\tau)$ whenever well-defined, and so does $X(t) = \exp(\Omega(t)) = \sum_{n=0}^{\infty} \frac{\Omega^n(t)}{n!}$. □

## 4. Corollaries and examples

We recall the often used in analysis space of test functions, called *Schwartz space*. These are characterized as

$$\left\{ f \in C^\infty(\mathbb{R}) \middle| \forall \alpha, \beta \in \mathbb{Z}, \quad \sup_x \left| x^\alpha D^\beta f(x) \right| < \infty \right\}.$$

The definition can be simply adapted to matrix-valued functions. Schwartz space is easily shown to be strictly bigger than the more often used space of $C^\infty(\mathbb{R})$ functions with compact support. As it is a well-known fact that Schwartz functions are $L^1(\mathbb{R})$, we have the following.

**Corollary 1.** *Let $x' = A(t)x(t)$ be the system with an exponential dichotomy on $\mathbb{R}$, $B(t)$ be the matrix valued Schwartz function, such that $\forall s, t \in \mathbb{R}$ $A(t)$ and $B(s)$ commute. Then the perturbed system $x' = (A(t) + B(t))x(t)$ also possesses exponential dichotomy.*

As Schwartz functions are so commonly encountered nowadays in analysis, we hope this corollary to be of some interest and use.

Another way to produce interesting corollaries may be on contrary, to *strengthen* commutativity hypothesis of Theorem 1. In this direction the following may be of interest

**Corollary 2.** *If $x' = cx$ is system with exponential dichotomy, where $c$ is a (scalar) constant (that is the same as require $\operatorname{Re} c \neq 0$) , then for any matrix-valued $L^1(\mathbb{R})$ function $B(t)$ perturbed system has also dichotomy.*

It should be noted that this corollary is *not* a direct corollary of well-known result [12, Corollary 3.21]. Indeed, that result states that asymptotical stability is unaffected by $o(1)$-perturbations (that is, such that $\lim_{t \to \pm\infty} \|B(t)\| = 0$), while previous corollary states that it is unaffected by $L^1(\mathbb{R})$ perturbations (indeed, for $x' = cx$ exponential dichotomy is the same as forward/backward asymptotic stability). As neither of classes $o(1)$ and $L^1(\mathbb{R})$ is not a subset of other, last corollary is no direct implication of [12, Corollary 3.21].

To conclude, we shall demonstrate one example, illustrating how in some particular cases our derived results allow to make more precise conclusions on dichotomy behaviour and persistence than the original results of Palmer in [9] or Coppel in [3].

*Example* 1. Consider the two-dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The eigenvalues of the matrix are real, so system has a dichotomy on $\mathbb{R}$. Now, perturb it to get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 + P(t)e^{-t^2} & 1 + (P-Q)(t)e^{-t^2}/2 \\ 0 & -1 + Q(t)e^{-t^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $P(t)$ and $Q(t)$ are arbitrary polynomials in $t$.

Note, that both the original matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ and perturbation we've applied $\left[\begin{smallmatrix} P(t) & (P-Q)(t)/2 \\ 0 & Q(t) \end{smallmatrix}\right] e^{-t^2}$ commute. Then, our theorem shows that perturbed system still possess the dichotomy on $\mathbb{R}$ (as we've perturbed by Schwartz function, see above), while Palmer's theorem only allows one to conclude that the dichotomy on $\pm\mathbb{R}$ is preserved. Neither Coppel's classical result from [3], nor any of the latter developments ([4,8]) *cannot* be applied for general $P$ and $Q$, as any of them requires the supremum-norm $\|B(t)\|_\infty$ of perturbation be not bigger than some particular value, while evidently by adjusting $P$ and $Q$ latter can be made arbitrarily big.

## 5. Further work

The theorem we have proven together with its corollaries listed can be possibly extended to other classes of differential equations, whenever the concept of exponential dichotomy is of relevance to them. In particular new results and corresponding proof techniques could be applied to the investigations of the existence of invariant toroidal manifolds of multi-frequency systems, for which the exponential dichotomy plays an essential role (see [6, 11] for details).

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# NONEXISTENCE OF $2-(v,k,1)$ DESIGNS ADMITTING AUTOMORPHISM GROUPS WITH SOCLE $E_8(q)$

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*Abstract.* One of the outstanding problems in combinatorial design theory is concerning the existence of $2-(v,k,1)$ designs. In particular, the existence of $2-(v,k,1)$ designs admitting an interesting group of automorphisms is of great interest. Thirty years ago, a six-person team classified $2-(v,k,1)$ designs which have flag-transitive automorphism groups. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. This paper is a contribution to this program and we prove there is nonexistence of $2-(v,k,1)$ designs admitting a point-primitive block-transitive but not flag-transitive automorphism group $G$ with socle $E_8(q)$.

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*Keywords:* block-transitive, point-primitive, design, socle

## 1. Introduction

This paper is part of a project to classify groups and $2-(v,k,1)$ designs where the group acts transitively on the blocks of the design. A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. Traditionally one defined $v=:|\mathcal{P}|$ and $b=:|\mathcal{B}|$. We will always assume that $2<k<v$.

One of the outstanding problems in combinatorial design theory is concerning the existence of $2-(v,k,1)$ designs. In particular, the existence of $2-(v,k,1)$ designs admitting an interesting group of automorphisms is of great interest. Thirty years ago, a six-person team [2] classified the pairs $(\mathcal{D},G)$ where $\mathcal{D}$ is a $2-(v,k,1)$ design and $G$ is a flag-transitive automorphism group of $\mathcal{D}$, with the exception of those in which $G$ is a one-dimensional affine group. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. These fall naturally into two classes, those where the action on points is primitive and those where the action on points is imprimitive. The primitive ones are now subdivided,

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according to the O'Nan-Scotte theorem and some further work by Camina, into the socles which are either elementary abelian or non-abelian simple. As a result of [6] it is known that the second only occur finitely times for a given line size. This paper contributes to the program for determining the pairs $(\mathcal{D}, G)$ in which $\mathcal{D}$ has a point-primitive block-transitive subgroup, $G$, of automorphisms. From the assumption that $G$ is transitive on the set $\mathcal{B}$ of blocks, it follows that $G$ is also transitive on the point set $\mathcal{P}$. This is a consequence of the theorem of Block in [1].

The classification of block-transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [4]). In [3] Camina and Siemons classified $2-(v,4,1)$ designs with a block-transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block-transitive, unsolvable group of automorphisms (see [11]). Tong and Li classified $2-(v,5,1)$ designs with a block-transitive, solvable group of automorphisms in [19]. Liu classified $2-(v,k,1)$ (where $k = 6,7,8,9,10$) designs with a block-transitive, solvable group of automorphisms in [16]. Ding [8] considered $2-(v,k,1)$ designs admitting block-transitive automorphism groups in $AGL(1,q)$ and prove the existence of $2-(v,6,1)$ designs which have block-transitive but not flag-transitive automorphism groups in $AGL(1,q)$ (see [7]). Dai and Zhao consider $2-(v,13,1)$ designs with point-primitive block-transitive unsolvable group of automorphisms whose socle is $Sz(2^{2n+1})$ in [5]. Recently, there have been a number contributions to this classification (see [13, 14]). Here we focus on the existence problem of $2-(v,k,1)$ ($k \leq 2793$) designs with a point-primitive block-transitive automorphism group of almost simple type and prove the following theorem:

**Theorem 1.** *Suppose that $E_8(q) \trianglelefteq G \leq Aut(E_8(q))$ for $q > 5$. Then there is nonexistence of $2-(v,k,1)$ ($k \leq 2793$) design $\mathcal{D}$ admitting a point-primitive block-transitive but not flag-transitive automorphism group $G$.*

We introduce some notation below. Let $X$ and $Y$ be arbitrary finite groups. The expression $X \cdot Y$ denotes an extension of $X$ by $Y$ and $X : Y$ denotes the split extension. If $Y$ is a subgroup of $X$, then the symbol $|X : Y|$ denotes the index of $Y$ in $X$. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and $G$ be an automorphism group of $\mathcal{D}$. If $B$ is a block, then $G_B$ denotes the setwise stabilizer of $B$ in $G$ and $G_{(B)}$ is the pointwise stabilizer of $B$ in $G$. In addition, $G^B$ denotes the permutation group induced by the action of $G_B$ on the points of $B$. Then $G^B \cong G_B/G_{(B)}$. We will write $\alpha$ to be a point of $\mathcal{D}$ and $G_\alpha$ to be the stabilizer of $\alpha$ under the action of $G$. Other notation for group structure is standard.

The paper is organized as follows. Section 2 describes several preliminary results concerning the group $E_8(q)$ and $2-(v,k,1)$ designs. Section 3 gives the proof of the theorem.

## 2. Preliminary results

Suppose that $G$ is a block-transitive automorphism group of a $2-(v,k,1)$ design. It is well-known that:

$$v = r(k-1)+1; \tag{2.1}$$

$$v(v-1) = bk(k-1). \tag{2.2}$$

Then we have $r = (v-1)/(k-1)$. We can show that $b \geq v$ and so $k \leq r$. If $k = r$ then $v = k^2-k+1$; if $r \geq k+1$, then $v \geq k^2$.

We use a result of W. Fang and H. Li [9]. Define the following constants:

$$b_1 = (b,v), b_2 = (b,v-1), k_1 = (k,v), \text{ and } k_2 = (k,v-1).$$

Using the basic equalities 2.1 and 2.2, we get the Fang-Li Equations:

$$k = k_1k_2, b = b_1b_2, r = b_2k_2, \text{ and } v = b_1k_1.$$

We shall state a number of basic results which will be used repeatedly throughout the paper. Liebeck and Saxl have determined the maximal subgroups of $Soc(G) = E_8(q)$ in [15].

**Lemma 1** ([15]). *Suppose that $T = E_8(q) \trianglelefteq G \leq Aut(T)$. Let $M$ be a maximal subgroup of $G$ not containing $T$. Then one of the following holds*

(1) $|M| < q^{110}|G:T|$;
(2) *$M \cap T$ is a parabolic group;*
(3) *$M \cap T$ is isomorphic to $(SL_2(q) \circ E_7(q)).d$, $D_8(q).d$, or $E_8(q^{\frac{1}{2}})$ with $q$ square, where $d = (2,q-1)$.*

**Lemma 2** ([18]). *Let $G = T : \langle x \rangle$ and act block-transitively on a $2-(v,k,1)$ design $\mathcal{D} = (\mathcal{P},\mathcal{B})$, where $x \in Out(T)$. Then $T$ acts transitively on $\mathcal{P}$.*

**Lemma 3** ([17]). *Let $G$ be a solvable block-transitive automorphism group of a $2-(v,k,1)$ design. If $G$ is point-primitive, then*

(1) *there exists a prime number $p$ and a positive integer $n$ such that $v = p^n$;*
(2) *if there exists a $p$-primitive prime divisor $e$ of $p^n-1$, such that $e||G|$, then either $G \leq A\Gamma L(1,p^n)$ or $k|v$.*

**Lemma 4** ([10]). *Let $\mathcal{D}$ be a $2-(v,k,1)$ design admitting a block-transitive and point-primitive but not flag-transitive automorphism group $G$. Assume that $T = Soc(G)$ and $T_\alpha = T \cap G_\alpha$ where $\alpha \in \mathcal{P}$. Then the following hold:*

(1) $\frac{v}{z} < (k_2k-k_2+1)|G:T|$, *where $z$ is the size of a $T_\alpha$−orbit in $\mathcal{P} \setminus \{\alpha\}$;*
(2) *if $(v-1,q) = 1$, then there exists a $T_\alpha$-orbit with size $y$ in $\mathcal{P} \setminus \{\alpha\}$ such that $y||T_\alpha|_{p'}$.*

**Lemma 5.** *Let $\mathcal{D}$ be a $2-(v,k,1)$ design admitting a block-transitive automorphism group $G$. Assume that $T = Soc(G)$ and $T_\alpha = T \cap G_\alpha$ where $\alpha \in \mathcal{P}$. Then*

(1) $v = k_2(k-1)b_2+1$;

(2) $b_2 || T_\alpha|_{v'} |G:T|$ *and* $v \le 1 + k(k-1)|T_\alpha|_{v'}|G:T|$;
(3) *If $G$ is not flag-transitive and non-solvable, then* $\frac{|T|}{|T_\alpha|^2} \le \frac{k(k-1)+1}{2}|G:T|$.

*Proof.* (1) Since $k(k-1)b = v(v-1)$ and $k = k_1k_2, b = b_1b_2, v = b_1k_1$, we obtain $k_2(k-1)b_2 = v-1$ and hence $v = 1 + k_2(k-1)b_2$.

(2) Since $rv = bk$, it follows that $r|G:G_\alpha| = k|G:G_B|$, where $\alpha \in \mathcal{P}, B \in \mathcal{B}$. Recall that $k = k_1k_2, r = b_2k_2$. It is clear that $b_2|G_B| = k_1|G_\alpha|$. Note that $(b_2, k_1) = 1$ and hence $b_2$ divides $|G_\alpha|$. Since $(b_2, v) = 1$, then $b_2 || G_\alpha|_{v'}$. Since $G$ is block-transitive, by Lemma 2, $T$ is point-transitive. We conclude that $v = |G:G_\alpha| = |T:T_\alpha|$. Hence $|G_\alpha| = |T_\alpha||G:T|$ and so $b_2 || T_\alpha|_{v'}|G:T|$. Together with (1), it deduces that $v \le 1 + k_2(k-1)|T_\alpha|_{v'}|G:T|$ and hence $v \le 1 + k(k-1)|T_\alpha|_{v'}|G:T|$.

(3) Let $B$ be a block of $\mathcal{D}$. Since $G$ is non-solvable, the following possibility for the structure of $G^B$, the rank and subdegree of $G$ does not occur:

| Type of $G^B$ | Rank of $G$ | Subdegree of $G$ |
|---|---|---|
| $\langle 1 \rangle$ | $1 + k_2(k-1)$ | $1, \overbrace{b_2, b_2, \cdots, b_2}^{k_2(k-1)}$ |

Otherwise, $|G^B|$ is odd, whence $|G|$ is odd and so $G$ is solvable, which contradicts the fact that $G$ is non-solvable. Then by the proof of Proposition 3.1 in [10] the conclusion holds. □

**Lemma 6** ([12]). *Suppose that $\mathcal{D}$ is a $2-(v,k,1)$ design and $G$ is an almost simple group acting on $\mathcal{D}$ block-transitively. Let $G_\alpha$ be the stabilizer in $G$ of a point $\alpha$ of $\mathcal{D}$ and suppose the socle $T$ of $G$ is a simple group of Lie type. If the intersection of $G_\alpha$ and $T$ is a parabolic subgroup of $T$, then $G$ acts on $\mathcal{D}$ flag-transitively.*

## 3. Proof of Theorem 1

Suppose that there exists a $2-(v,k,1)$ ($k \le 2793$) design $\mathcal{D}$ satisfying the conditions of the Main Theorem. We will derive contradictions to prove the Main Theorem.

Since $T = E_8(q) \trianglelefteq G \le Aut(E_8(q))$, then $G = T : \langle x \rangle$ and $|Out(T)| = a$, where $x \in Out(T)$. Let $o(x) = m$. Then we obtain that $m|a$ and $|G| = q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)m$. Since $G$ is point-primitive, $G_\alpha$ is the maximal subgroup of $G$ for any $\alpha \in \mathcal{P}$. Then $M = G_\alpha$ satisfies one of the three cases in Lemma 1. If $G_\alpha \cap T$ is a parabolic subgroup of $T$, then by Lemma 6 we see that $G$ is flag-transitive, which is a contradiction. Therefore, the case (2) in Lemma 1 does not occur and it suffices to consider the following two cases.

**Case 3.1:** $|G_\alpha| < q^{110}|G:T|$.

Since $G$ is block-transitive, by Lemma 2, $T$ is point-transitive. Hence $|G_\alpha| = |T_\alpha||G:T|$ and so $|T_\alpha| < q^{110}$. Then $v = |T:T_\alpha|$ is not a prime power and by Lemma 3 we have that $G$ is non-solvable. Note that $m = |G:T|$. It follows by

Lemma 5 (3) that

$$|T| \leq \frac{k(k-1)+1}{2}|T_\alpha|^2|G:T| \leq \frac{7798057}{2}q^{220}m.$$

This gives,

$$\frac{|T|}{q^{220}} = \frac{(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)}{q^{100}}$$
$$< \frac{7798057}{2}m.$$

Since

$$(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1) > \frac{7}{10}q^{128},$$

it implies that

$$\frac{7}{10}q^8 < \frac{7798057}{2}m.$$

Recall that $m|a$, $q = p^a$, $p \geq 2$. We can conclude therefore that

$$\frac{7}{10}\cdot 2^{8a} \leq \frac{7}{10}\cdot p^{8a} = \frac{7}{10}q^8 < \frac{7798057}{2}a, \tag{3.1}$$

which forces $a \leq 2$. We calculate to obtain all possibilities for the values of $p$ and $a$ satisfying the inequality 3.1: (1) $a = 1$, $p \leq 5$, a prime; (2) $a = 2$, $p = 2$. This contradicts $q > 5$.

**Case 3.2:** $G_\alpha \cap T$ is case (3) in Lemma 1.

Now we consider three cases.

**Subcase 3.2.1:** $T_\alpha = (SL_2(q) \circ E_7(q)).d$ where $d = (2, q-1)$.

We observe that

$$|T_\alpha| = q^{64}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)^2$$

and

$$v = \frac{q^{56}(q^{30}-1)(q^{24}-1)(q^{20}-1)}{(q^{10}-1)(q^6-1)(q^2-1)}.$$

Hence

$$|T_\alpha|_{v'} \leq (q^2-1)^8(q^{12}+q^6+1)(1+q^2+q^4+q^6+q^8+q^{10}+q^{12}) < \frac{7}{5}q^{40}.$$

Since

$$v = \frac{q^{56}(q^{30}-1)(q^{24}-1)(q^{20}-1)}{(q^{10}-1)(q^6-1)(q^2-1)} > \frac{1}{50}q^{112},$$

we can appeal to Lemma 5 (2) to observe that

$$\frac{1}{50}q^{112} < v \leq 1 + k(k-1)|T_\alpha|_{v'}|G:T| < 1 + 7798056\cdot\frac{7}{5}\cdot q^{40}a.$$

This implies the following inequality

$$\frac{1}{50}\cdot 2^{72a} \le \frac{1}{50}\cdot q^{72} < \frac{1}{2^{40a}} + 7798056\cdot\frac{7}{5}\cdot a < \frac{4}{5}\cdot 2^{24}a,$$

which is impossible.

**Subcase 3.2.2:** $T_\alpha = D_8(q).d$, where $d = (2, q-1)$.

We calculate that

$$|T_\alpha| = \frac{dq^{56}(q^8-1)\prod_{i=1}^{7}(q^{2i}-1)}{d_1}$$

and

$$v = \frac{d_1 q^{64}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)}{d(q^{10}-1)(q^8-1)(q^6-1)(q^4-1)},$$

where $d_1 = (4, q^8-1)$. Since $(v-1, q) = 1$, by Lemma 4 (2), there exists in $\mathcal{P}\setminus\{\alpha\}$ a $T_\alpha$−orbit of size $y$ such that $y \mid |T_\alpha|_{p'}$. Hence

$$y \le |T_\alpha|_{p'} \le 2(q^8-1)\prod_{i=1}^{7}(q^{2i}-1).$$

Thus

$$\begin{aligned}\frac{v}{y} &\ge \frac{d_1 q^{64}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)}{2d(q^{14}-1)(q^{12}-1)(q^{10}-1)^2(q^8-1)^3(q^6-1)^2(q^4-1)^2(q^2-1)}\\ &> \frac{\frac{1}{10}\cdot q^{108}}{4\cdot\frac{15}{2}\cdot q^{44}} = \frac{1}{300}q^{64}.\end{aligned}$$

Note that $k_2 \le k$. We now apply Lemma 4 (1) to conclude that

$$\frac{1}{300}\cdot 2^{64a} \le \frac{1}{300}q^{64} < \frac{v}{y} < (k(k-1)+1)|G:T| \le 7798057a < \frac{19}{20}\cdot 2^{23}a,$$

which is a contradiction.

**Subcase 3.2.3:** $T_\alpha = E_8(q^{\frac{1}{2}})$.

We obtain that

$$|T_\alpha| = q^{60}(q^{15}-1)(q^{12}-1)(q^{10}-1)(q^9-1)(q^7-1)(q^6-1)(q^4-1)(q-1)$$

and

$$v = q^{60}(q^{15}+1)(q^{12}+1)(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)(q^4+1)(q+1).$$

Then it deduces that

$$\begin{aligned}|T_\alpha|_{v'} &\le (q-1)^8(q^2+q+1)^4(q^6+q^3+1)(1+q+q^2+q^3+q^4)^2\\ &\quad\cdot(1+q+q^2+q^3+q^4+q^5+q^6)(1-q+q^3-q^4+q^5-q^7+q^8) < 48q^{44}.\end{aligned}$$

Since

$$v = q^{60}(q^{15}+1)(q^{12}+1)(q^{10}+1)(q^9+1)(q^7+1)(q^6+1)(q^4+1)(q+1) > q^{124},$$

by Lemma 5 (2) this implies that

$$q^{124} < v \leq 1 + k(k-1)|T_\alpha|_{v'}|G:T| < 1 + 7798056 \cdot 48 \cdot q^{44} \cdot a.$$

This leads to the following result

$$2^{80a} \leq q^{80} < \frac{1}{2^{44a}} + 7798056 \cdot 48a < \frac{4}{5} \cdot 2^{29}a,$$

which gives a contradiction.

This completes the proof of **Theorem 1**. □

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# ON $\Lambda$-STATISTICAL CONVERGENCE OF ORDER $\alpha$ IN RANDOM 2-NORMED SPACE

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*Abstract.* Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to $\infty$ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. In the present paper, we introduce the notion of $\Lambda$-statistical convergence of order $\alpha$, $\Lambda$-statistical Cauchy sequences of order $\alpha$ in random 2-normed spaces and obtain some results. We display examples which show that our method of convergence is more general in random 2-normed space.

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## 1. Introduction and background

The idea of the statistical convergence was given by Zygmund [25] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Fast [5] and Steinhaus [23] and then reintroduced by Schoenberg [20] independently. Over the years, statistical convergence has been developed in [3, 6, 7, 12, 16, 17, 24] and turned out very useful to resolve many convergence problems arising in Analysis.

**Definition 1** ([5])**.** A sequence $x = (x_k)$ of numbers is said to be statistically convergent to a number $L$ if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| > \epsilon\}| = 0,$$

where vertical bars denotes the cardinality of enclosed set. In this case, we write $S - \lim_{k\to\infty} x_k = L$.

In literature, several interesting generalizations of statistical convergence have been appeared. One among these is $\lambda$-statistical convergence given by Mursaleen [14] with the help of a non-decreasing sequence $\lambda = (\lambda_n)$. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to $\infty$ with

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1.$$

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The idea of $\lambda$-statistical convergence can be connected to the generalized de la Vallée-Poussin mean. It is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \quad \text{where } I_n = [n - \lambda_n + 1, n] \text{ for } n = 1, 2, \ldots$$

**Definition 2** ([14])**.** A sequence $x = (x_k)$ of numbers is said to be $\lambda$-statistically convergent to a number $L$ provided that for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, the number $L$ is called $\lambda$-statistical limit of the sequence $x = (x_k)$ and we write $S_\lambda - \lim_{k \to \infty} x_k = L$.

Recently, for $\alpha \in (0, 1]$ Çolak and Bektaş[2] generalized Definition 2 in terms of $\lambda$-statistical convergence of order $\alpha$ and obtained some analogous results.

**Definition 3** ([2])**.** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers as defined above and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ of numbers is said to be $\lambda$-statistically convergent of order $\alpha$ if there is a number $L$ such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda^\alpha - \lim_{k \to \infty} x_k = L$.

We next quote the following definition due to Mursaleen and Noman [19] on $\mu$-convergent series.

**Definition 4** ([19])**.** Let $\mu = (\mu_k)$ be a sequence of positive real numbers tending to infinity such that

$$0 < \mu_0 < \mu_1 < \mu_2 \ldots \text{ and } \mu_k \to \infty \text{ as } k \to \infty.$$

Then a sequence $x = (x_k)$ of numbers is said to be $\mu$-convergent to a number $l$ if $\Lambda x_k \to l$ as $k \to \infty$, where

$$\Lambda x_k = \frac{1}{\mu_k} \sum_{i=0}^{k} (\mu_i - \mu_{i-1}) x_i$$

Esi and Braha [4] used Definition 4 to introduce a new notion called $\Lambda$-statistical convergence in random 2-normed spaces and studied some of its properties. Before we proceed further it would be better to recall the ideas of probabilistic and random 2-normed spaces which are of much interest in the study of random operator equations. The concept of probabilistic normed spaces was initially introduced by A. N. Sherstnev [22] in 1962. Menger [13] introduced the notion of probabilistic metric spaces in 1942. The idea of Menger [13] was to use distribution function instead of non-negative real numbers as values of the metric. In last few years these spaces

are grown up rapidly and many detereministic results of linear normed spaces are obtained for probabilistic normed spaces. For a detailed study on probabilistic functional analysis, we refer [1, 10, 11, 18, 21]. In 2005, Goleţ [9] used the concept of 2-norm of Gähler [8] and presented generalized probabilistic normed space which he called random 2-normed space.

Let $\mathbb{R}$ denotes the set of reals and $\mathbb{R}_0^+ = [0,\infty)$. A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We will denote the set of all distribution functions by $\mathfrak{D}$.

Also, a distance distribution function is a non decreasing function $F$ defined on $\mathbb{R}^+ = [0,\infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0,\infty)$. Let $\mathfrak{D}^+$ denotes the set of all distance distribution functions.

A triangular norm, briefly t-norm, is a binary operation $*$ on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that for all $a,b,c \in [0,1]$:

(1) $a * 1 = a$,
(2) $a * b = b * a$,
(3) $c * d \geq a * b$ if $c \geq a$ and $d \geq b$,
(4) $(a * b) * c = a * (b * c)$.

The $*$ operations $a * b = max\{a+b-1,0\}$, $a * b = ab$, and $a * b = min\{a,b\}$ on $[0,1]$ are t-norms.

In following, we quote some needful definitions.

**Definition 5** ([8])**.** Let X be a real vector space of dimension $d > 1$ ($d$ may be infinite). A real valued function $||\cdot,\cdot|| : X^2 \to \mathbb{R}$ satisfying the following conditions:

(1) $||x_1,x_2|| = 0$, if and only if $x_1,x_2$ are linearly dependent.
(2) $||x_1,x_2|| = ||x_2,x_1||$ for all $x_1,x_2 \in X$,
(3) $||\alpha x_1,x_2|| = |\alpha|||x_1,x_2||$, for any $\alpha \in \mathbb{R}$ and
(4) $||x_1+x_2,x_3|| \leq ||x_1,x_3|| + ||x_2,x_3||$

is called a 2-norm and the pair $(X,||\cdot,\cdot||)$ is called a 2-normed space.

For example, if we take $X = \mathbb{R}^2$ with 2-norm $||x_1,x_2||$ = area of parallelogram spanned by the vectors $x_1,x_2$ which may be given explicitly by the formula

$$||x_1,x_2|| = |det(x_{ij})| = abs.\left(det\left(x_i,x_j\right)\right)$$

where $x_i = (x_{i1},x_{i2}) \in \mathbb{R}^2$ for each $i = 1,2$. Then $(X,||\cdot,\cdot||)$ is a 2-normed space.

**Definition 6** ([9])**.** Let X be a real linear space of dimension $d > 1$ ($d$ may be infinite), $\tau$ be a triangle function(a binary operation on $\mathfrak{D}^+$ which is associative, commutative, nondecreasing and $\epsilon_0$ as a unit) and $\mathcal{F} : X\times X \to \mathfrak{D}^+$ (for $x,y \in X$, $\mathcal{F}(x,y;t)$ is the value of $\mathcal{F}(x,y)$ at $t \in \mathbb{R}$). Then $\mathcal{F}$ is called a probabilistic norm and $(X,\mathcal{F},\tau)$ a probabilistic 2-normed space if the following conditions are satisfied:

(1) $\mathcal{F}(x,y;t) = H_0(t)$ if $x,y$ are linearly dependent, where $H_0(t) = 0$ if $t \leq 0$ and $H_0(t) = 1$ if $t > 0$.
(2) $\mathcal{F}(x,y;t) \neq H_0(t)$ if $x,y$ are linearly independent,
(3) $\mathcal{F}(x,y;t) = \mathcal{F}(y,x;t)$ for all $x,y \in X$,
(4) $\mathcal{F}(\alpha x,y;t) = \mathcal{F}(x,y;\frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x,y \in X$,
(5) $\mathcal{F}(x+y,z;t) \geq \tau(\mathcal{F}(x,z;t),\mathcal{F}(y,z;t))$, where $x,y,z \in X$.

If (5) is replaced by $\mathcal{F}(x+y,z;t_1+t_2) \geq \mathcal{F}(x,z;t_1) * \mathcal{F}(y,z;t_2)$ for all $x,y,z \in X$ and $t_1,t_2 \in \mathbb{R}_0^+$ then $(X,\mathcal{F},*)$ is called a random 2-normed space.

*Example* 1. Every 2-normed space $(X,||\cdot,\cdot||)$ can be made a random 2-normed space by setting $\mathcal{F}(x,y;t) = H_0(t-||x,y||)$ where

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a \end{cases}$$

for all $x,y \in X$, $t > 0$ and $a * b = ab$; $a,b \in [0,1]$.

*Example* 2. Let $(X,||\cdot,\cdot||)$ be a 2-normed space with $||x,z|| = |x_1z_2 - x_2z_1|$; $x = (x_1,x_2)$, $z = (z_1,z_2)$ and $a * b = ab$ for all $a,b \in [0,1]$. For every $x,y \in X$ and $t > 0$ we define $\mathcal{F}(x,y;t) = \frac{t}{t+||x,y||}$, then $(X,\mathcal{F},*)$ is a random 2-normed space.

**Definition 7** ([15])**.** Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be convergent to $x_0 \in X$ with respect to the norm $\mathcal{F}$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, there exists a positive integer $k_0$ such that $\mathcal{F}(x_k - x_0,z;\epsilon) > 1-t$ whenever $k \geq k_0$. It is denoted by $\mathcal{F} - \lim x_k = x_0$.

**Definition 8.** p[15]] Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be statistically convergent or $S^{R2N}$ convergent to $x_0 \in X$ with respect to the norm $\mathcal{F}$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\delta(\{k \in \mathbb{N} : \mathcal{F}(x_k - x_0,z;\epsilon) \leq 1-t\}) = 0.$$

In this case, we write $S^{R2N} - \lim x_k = x_0$.

**Definition 9** ([4])**.** Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be $\Lambda$-statistically convergent with respect to the norm $\mathcal{F}$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\delta_\Lambda(\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0,z;\epsilon) \leq 1-t\}) = 0,$$

*i.e.*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0,z;\epsilon) \leq 1-t\}| = 0.$$

In this case, we write $S_\Lambda^{R2N} - \lim x_k = x_0$.

**Definition 10** ([4])**.** Let $(X, \mathcal{F}, *)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be $\Lambda$-statistically cauchy with respect to the norm $\mathcal{F}$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, there exists a positive integer $k_0(\epsilon)$ such that for all $k, l \geq k_0$

$$\delta_\Lambda(\{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) \leq 1 - t\}) = 0.$$

In present paper, we study quite natural and new notion of $\Lambda$-statistical convergence of order $\alpha$ in random 2-normed spaces.

## 2. Main results

In this section, we begin with the following definitions of statistical and $\Lambda$-statistical convergence of order $\alpha$ in random 2-normed spaces.

**Definition 11.** A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be statistically convergent of order $\alpha$ $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1.$$

In this case, we write $S^\alpha - \lim_{k\to\infty} x_k = x_0$.

Let $S^\alpha(X)$ denotes the set of all statistically convergent sequences of order $\alpha$ in a random 2-normed space $(X, \mathcal{F}, *)$.

**Definition 12.** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to $\infty$ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be $\Lambda$-statistically convergent of order $\alpha$ $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1,$$

where $\lambda_n^\alpha$ denote the $\alpha$th power of $\lambda_n$, *i.e.*, $(\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha, \cdots\cdots)$. In this case, we write $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$.

Let $S_\Lambda^\alpha(X)$ denotes the set of all $\Lambda$-statistically convergent sequences of order $\alpha$ in a random 2-normed space $(X, \mathcal{F}, *)$.
For the particular choice $\alpha = 1$, Definition 12 coincides with the notion of $\Lambda$-statistical

convergence of [4]; For $\lambda_n = n$, Definition 12 coincides with the notion of statistical convergence of order $\alpha$ in random 2-normed space; For $\lambda_n = n$ and $\alpha = 1$, Definition 12 coincides with the notion of statistical convergence in random 2-normed space[15].
We next give an example that shows Definition 12 is well defined for $(0 < \alpha \leq 1)$ but not for $\alpha > 1$. In view of this, we need the following theorem with lemma.

**Lemma 1.** *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence as defined above and $(X, \mathcal{F}, *)$ be a random 2-normed space. Let $0 < \alpha \leq 1$ and $x = (x_k)$ be a sequence in $X$. Then, for $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, the following statements are equivalent:*

(1) $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$,
(2) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1-t\}| = 0$,
(3) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1-t\}| = 1$,
(4) $S_\Lambda^\alpha - \lim_{k\to\infty} \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) = 1$.

**Theorem 1.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. If $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$, then $x_0$ must be unique.*

*Proof.* Suppose $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = y_0$ where $y_0 \neq x_0$. Given $\epsilon > 0$ and $t > 0$, choose $\eta > 0$ such that $(1-\eta) * (1-\eta) > 1-\epsilon$. For $\theta \neq z \in X$, define

$$K_1(\eta, t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1-\eta \right\};$$
$$K_2(\eta, t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right) \leq 1-\eta \right\}.$$

Since $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$ and $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = y_0$, it follows for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_1(\eta, t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_2(\eta, t)| = 0$$

Let $K(\eta, t) = K_1(\eta, t) \cup K_2(\eta, t)$, then clearly $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\eta, t)| = 0$ which immediately implies $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\eta, t)| = 1$. Let $k \in K^c(\eta, t) = K_1^c(\eta, t) \cap K_2^c(\eta, t)$. Now one can write,

$$\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) \geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right) > (1-\eta) * (1-\eta) > 1-\epsilon.$$

Since $\epsilon$ is arbitrary, it follows that $\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) = 1$, for $t > 0$ and $\theta \neq z \in X$. This shows that $x_0 = y_0$. □

*Example* 3. Let $X = \mathbb{R}^2$ with the 2-norm $||x, z|| = ||x_1 z_2 - x_2 z_1||$ where $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0,1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t + ||x,z||}$

where $x \in X, t \in (0,1)$ and $\theta \neq z \in X$. Then $(\mathbb{R}^2, \mathcal{F}, *)$ is a random 2-normed space. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1,0), & \text{if } k \text{ is even,} \\ (0,0), & \text{if } k \text{ is odd.} \end{cases}$$

For $\epsilon > 0$, $t \in (0,1)$, if we define

$$\begin{aligned} K(\epsilon,t) &= \{k \in I_n : \mathcal{F}(\Lambda x_k - \theta, z; t) \leq 1-\epsilon\}, \theta = (0,0) \\ &= \left\{k \in I_n : \frac{t}{t + ||\Lambda x_k - \theta, z||} \leq 1-\epsilon\right\} \\ &= \left\{k \in I_n : ||\Lambda x_k - \theta, z|| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\} \\ &= \{k \in I_n : \Lambda x_k = (1,0)\} \\ &= \{k \in I_n : k \text{ is even}\}; \end{aligned}$$

then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : k \text{ is even}\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.

Similarly, for $\epsilon > 0$ and $t \in (0,1)$ if we define

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}, x_0 = (1,0)$$

then

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |B(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \text{k is odd }\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.

This shows that $S_\Lambda^\alpha - \lim_k x_k$ is not unique and we obtain a contradiction to Theorem 1.

**Theorem 2.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. For a sequence $x = (x_k)$ in $X$ if $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, then $S_\Lambda^\alpha - \lim_k x_k = x_0$. However, the converse need not be true in general.*

*Proof.* Since $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, so for $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$ there exists a positive integer $n_0$ such that $\mathcal{F}(\Lambda x_k - x_0, z; t) > 1-\epsilon$ for all $k \geq n_0$. Hence the set

$$K(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\} \subset \{1,2,3,\ldots,n_0-1\},$$

for which we have,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}| = 0.$$

This shows that $S_\Lambda^\alpha - \lim_k x_k = x_0$. We next give the following example which shows that the converse need not be true.

*Example* 4. Consider the random 2-normed space as in Example 3. Define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (k,0), & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ (0,0), & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$ and $t > 0$ if we define $K(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k, z; t) \le 1-\epsilon\}$, then one can write as in Example 3 $K(\epsilon,t) = \left\{k \in I_n : n-[\sqrt{\lambda_n}]+1 \le k \le n\right\}$

Thus,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left| \left\{k \in I_n : n-[\sqrt{\lambda_n}]+1 \le k \le n\right\} \right|$$

$$\le \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n^\alpha} = 0$$

for $\frac{1}{2} < \alpha \le 1$. This shows that $S_\Lambda^\alpha - \lim_k x_k = 0$. But $\mathcal{F}_\Lambda - \lim_k x_k \ne 0$, since

$$\mathcal{F}(\Lambda x_k, z; t) = \frac{t}{t + ||\Lambda x_k, z||} = \begin{cases} \frac{t}{t+kz_2}, & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ 1, & \text{otherwise.} \end{cases}$$

which implies

$$\lim_{k\to\infty} \mathcal{F}(\Lambda x_k, z; t) = \begin{cases} 0, & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ 1, & \text{otherwise.} \end{cases}$$

□

**Theorem 3.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \le 1$ be given. Let $x = (x_k)$ and $y = (y_k)$ be two sequences in $X$.*
*(i) If $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$ and $0 \ne c \in \mathbb{R}$, then $S_\Lambda^\alpha - \lim_{k\to\infty} c x_k = c x_0$.*
*(ii) If $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$ and If $S_\Lambda^\alpha - \lim_{k\to\infty} y_k = y_0$, then $S_\Lambda^\alpha - \lim(x_k + y_k) = x_0 + y_0$.*

*Proof.* The proof of the Theorem is not so hard so is omitted here. □

**Theorem 4.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \le \beta \le 1$ be given. Then $S_\Lambda^\alpha(X) \subset S_\Lambda^\beta(X)$ and the inclusion is strict for some $\alpha$ and $\beta$ such that $\alpha < \beta$.*

*Proof.* If $0 < \alpha \le \beta \le 1$, then for every $\epsilon > 0$, $t > 0$ and $\theta \ne z \in X$, we have

$$\frac{1}{\lambda_n^\beta} |\{k \in I_n : \mathcal{F}(\Lambda x_k - l, z; t) \le 1-\epsilon\}|$$

$$\le \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - l, z; t) \le 1-\epsilon\}|;$$

which immediately implies the inclusion $S_\Lambda^\alpha(X) \subset S_\Lambda^\beta(X)$. We next give an example that shows the inclusion in $S_\Lambda^\alpha(X) \subset S_\Lambda^\beta(X)$ is strict for some $\alpha$ and $\beta$ with $\alpha < \beta$.

*Example* 5. Let $(\mathbb{R}^2, \mathcal{F}, *)$ be a random 2-normed space as defined above. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1,0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \le k \le n, \\ (0,0), & \text{otherwise.} \end{cases}$$

Then one can easily see $S_\Lambda^\beta - \lim_k x_k = 0$, *i.e.*, $x \in S_\Lambda^\beta(X)$ for $\frac{1}{2} < \beta \le 1$ but $x \notin S_\Lambda^\alpha(X)$ for $0 < \alpha \le \frac{1}{2}$. This shows that the inclusion in $S_\Lambda^\alpha(X) \subset S_\Lambda^\beta(X)$ is strict.

□

**Theorem 5.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \le 1$ be given. If $x = (x_k)$ be a sequence in $X$, then $S_\Lambda^\alpha - \lim_k x_k = x_0$ if and only if there exists a subset $K = \{k_m : k_1 < k_2 < \ldots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K| = 1$ and $\mathcal{F}_\Lambda - \lim_k x_k = x_0$.*

*Proof.* First suppose that $S_\Lambda^\alpha - \lim_k x_k = x_0$. For $t > 0$, $\theta \neq z \in X$ and $p \in \mathbb{N}$, if we define

$$K(p,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) \le 1 - \frac{1}{p} \right\}$$
$$M(p,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) > 1 - \frac{1}{p} \right\};$$

then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(p,t)| = 0.$$

Also, for $p = 1, 2, 3, \ldots$

$$M(1,t) \supset M(2,t) \supset \ldots M(i,t) \supset M(i+1,t) \supset \ldots \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |M(p,t)| = 1. \tag{2.2}$$

Now, to prove the result in one way, it is sufficient to prove that $\mathcal{F}_\Lambda - \lim_k x_k = x_0$ over $M(p,t)$. Suppose $x_k$ is not convergent to $x_0$ over $M(p,t)$ with respect to the norm $\mathcal{F}_\Lambda$. Then, there exists some $\eta > 0$ such that

$$\{k \in \mathbb{N} : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) \le 1 - \eta\}$$

for infinitely many terms $x_k$. Let

$$M(\eta, t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) > 1 - \eta\}$$

and $\eta > \frac{1}{p}$ for $p = 1,2,3\cdots$. This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|M(\eta,t)| = 0$. Also from (2.1), we have $K_1(p,t) \subset M(\eta,t)$ which gives that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|M(p,t)| = 0$, this contradicts (2.2). Hence $\mathcal{F}_\Lambda - \lim_k x_k = x_0$.

Conversely, suppose that there exists a subset $K = \{k_m : k_1 < k_2 < \ldots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K| = 1$ and $\mathcal{F}_\Lambda - \lim_k x_k = x_0$. Then for every $t > 0$, $\epsilon > 0$ and $\theta \neq z \in X$ there exists a positive integer $k_0$ such that

$$\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) > 1-\epsilon\}$$

for all $k \geq k_0$. Since the set $\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}$ is contained in $\mathbb{N} - \{k_0+1, k_0+2, k_0+3, \cdots\}$ therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}| = 0.$$

Hence, $S_\Lambda^\alpha - \lim_k x_k = x_0$. □

**Definition 13.** Let $(X, \mathcal{F}, *)$ be a random 2-normed space. A sequence $x = (x_k)$ is said to be $\Lambda$-statistically Cauchy of order $\alpha$ $(0 < \alpha \leq 1)$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$ there exists a positive integer $k_0$ such that for all $k, l \geq k_0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) \leq 1-t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) > 1-t\}| = 1.$$

**Theorem 6.** *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. Then a sequence $x = (x_k)$ is said to be $\Lambda$-statistically convergent of order $\alpha$ iff it is $\Lambda$-statistically Cauchy of order $\alpha$.*

*Proof.* Let $x = (x_k)$ be a $\Lambda$-statistically convergent sequence of order $\alpha$. Suppose that $S_\Lambda^\alpha - \lim_k x_k = x_0$. Let $\epsilon > 0$. Choose $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$. If we define

$$K(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1-r\right\},$$

then

$$K^c(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r\right\};$$

which gives by virtue of $S_\Lambda^\alpha - \lim_k x_k = x_0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K(r,t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha}|K^c(r,t)| = 1.$$

Let $m \in K^c(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1-r$. If we take

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon\},$$

then to prove the first part it is sufficient to prove that $B(\epsilon,t) \subset K(r,t)$. Let $k \in B(\epsilon,t)$, which gives $\mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon$. Suppose $k \notin K(r,t)$, then $\mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r$. Now, we can observe that

$$\begin{aligned} 1-\epsilon \geq \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) &\geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ &\geq (1-r) * (1-r) > 1-\epsilon. \end{aligned}$$

This contradiction shows that $B(\epsilon,t) \subset K(r,t)$ and therefore, one way of the theorem is proved.

Conversely, suppose that $x = (x_k)$ is $\Lambda$-statistically Cauchy sequence of order $\alpha$ but not $\Lambda$-statistically convergent of order $\alpha$ with respect to $\mathcal{F}$. Then for every $t > 0$, $\epsilon > 0$ and $\theta \neq z \in X$ there exists a positive integer $m$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon,t)| = 0 \text{ where } K(\epsilon,t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon\}.$$

This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\epsilon,t)| = 1$. Choose $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$. Let

$$B(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r\right\}.$$

Let $m \in B(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1-r$.
Since

$$\begin{aligned} \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) &\geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ &> (1-r) * (1-r) > 1-\epsilon; \end{aligned}$$

therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) > 1-\epsilon\}| = 0.$$

*i.e.* $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\epsilon,t)| = 0$ which leads to a contradiction. Hence $x = (x_k)$ is $\Lambda$-statistically convergent of order $\alpha$. □

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# SHARP BOUNDS FOR THE FIRST ZAGREB INDEX AND FIRST ZAGREB COINDEX

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*Abstract.* Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges with vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. Lower and upper bounds of a graph invariants $M_1 = \sum_{i=1}^{n} d_i^2$, referred to as the first Zagreb index, and $\bar{M}_1 = \sum_{i \nsim j}(d_i + d_j)$, named the first Zagreb coindex, depending on parameters $n$, $m$, $\Delta$ and $\delta$ are obtained. The obtained results represent improvement of the results reported in the literature.

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*Keywords:* first Zagreb index, first Zagreb coindex, degree (of vertex)

## 1. Introduction

Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges, and let $\Delta = d_1 \geq d_2 \geq \ldots \geq d_n = \delta > 0$ be the sequence of its vertex degrees.

In graph theory, a graph invariant is a property of graphs that does not depend on graph representations, such as particular labeling of its vertices or drawings of the graph. A number of different invariants have been introduced so far. One of the oldest was introduced by Gutman and Trinajstić in 1972, known under the name *first Zagreb index*, defined as the sum of squares of vertex degrees of graph, i.e. as

$$M_1 = \sum_{i=1}^{n} d_i^2$$

In [8] Došlić proposed a new graph invariant named *first Zagreb coindex*, defined as

$$\bar{M}_1 = \sum_{i \nsim j}(d_i + d_j).$$

These graph invariants play an important role in many scientific areas, notably in chemistry and network theory (see for example [12] and [1, 18]). In earlier works [2, 3, 5–7, 9–17, 19], several bounds for $M_1$ and $\bar{M}_1$ were reported. These depend on usual structural parameters (number of vertices, number of edges, vertex degrees, and similar).

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In this paper we consider the problem of finding lower and upper bounds of a graph invariants $M_1$ and $\bar{M}_1$. The obtained results represent improvement of the results reported in the literature.

## 2. Preliminaries

Here we recall some results from spectral graph theory, and state a few analytical inequalities needed for our work.

**Lemma 1** ([9, 16])**.** *Let $G$ be an undirected connected graph with $n$ , $n \geq 2$, vertices and $m$ edges. Then*

$$M_1 \geq \frac{4m^2}{n} \tag{2.1}$$

*Equality holds if and only if $G$ is isomorph with a regular graph.*

**Lemma 2** ([10])**.** *Let $G$ be simple graph with $n$, vertices and $m$ edges. Then*

$$M_1 \leq \frac{4m^2}{n} + \frac{n}{4}(\Delta - \delta)^2 \tag{2.2}$$

**Lemma 3** ([10, 11, 15, 17])**.** *Let $G$ be simple graph with $n$ vertices and $m$ edges. Then*

$$M_1 \leq \frac{m^2}{n}\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right)^2 \tag{2.3}$$

*with equality if and only if $G$ is regular graph, or $G$ is bidegreed graph such that $\Delta + \delta$ divides $\delta n$ and there are exactly $p = \frac{\delta n}{\Delta+\delta}$ vertices of degree $\Delta$, and $q = \frac{\Delta n}{\Delta+\delta}$ vertices of degree $\delta$.*

Note that complete conditions when equality in (2.3) occurs were given only in [15].

**Lemma 4** ([2])**.** *Suppose $G$ is a connected graph with exactly $n$ vertices and $m$ edges. Then we have*

$$\bar{M}_1 = 2m(n-1) - M_1. \tag{2.4}$$

**Lemma 5** ([14])**.** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\bar{M}_1 \leq -\frac{4m^2}{n} + 2m(n-1), \tag{2.5}$$

*and*

$$\bar{M}_1 \geq 2m(n-1) - \frac{(\Delta+\delta)^2}{n\Delta\delta}m^2. \tag{2.6}$$

*The equality holds if and only if $G$ is regular.*

**Lemma 6** ([4]). *Let $p_1, p_2, \dots, p_n$ be non-negative real numbers and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, real numbers with the properties*

$$0 < r_1 \le a_i \le R_1 < +\infty \quad \text{and} \quad 0 < r_2 \le b_i \le R_2 < +\infty$$

*for each $i = 1, 2, \dots, n$. Further, let $S$ be a subset of $I_n = \{1, 2, \dots, n\}$ which minimizes the expression*

$$\left| \sum_{i \in S} p_i - \frac{1}{2} \sum_{i=1}^{n} p_i \right|. \tag{2.7}$$

*Then*

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \le (R_1 - r_1)(R_2 - r_2) \sum_{i \in S} p_i \left( \sum_{i=1}^{n} p_i - \sum_{i \in S} p_i \right). \tag{2.8}$$

## 3. Main results

### 3.1. *The first Zagreb index*

The following theorem establishes bound for $M_1$ depending on parameters $n$, $m$, $\Delta$ and $\delta$.

**Theorem 1.** *Let $G$ be an undirected connected graph with $n$, $n \ge 2$, vertices and $m$ edges. Then*

$$M_1 \ge \frac{4m^2}{n} + \frac{1}{2}(\Delta - \delta)^2. \tag{3.1}$$

*Equality holds if and only if $G$ is isomorph with $k$-regular graph, $1 \le k \le n-1$.*

*Proof.* From equality

$$n M_1 - 4m^2 = n \sum_{i=1}^{n} d_i^2 - \left( \sum_{i=1} d_i \right)^2 = \sum_{1 \le i \le j \le n} (d_i - d_j)^2$$

and inequality

$$\begin{aligned} \sum_{1 \le i < j \le n} (d_i - d_j)^2 &\ge \sum_{i=2}^{n-1} ((d_1 - d_i)^2 + (d_i - d_n)^2) + (d_1 - d_n)^2 \\ &\ge \sum_{i=2}^{n-1} \frac{1}{2}(d_1 - d_n)^2 + (d_1 - d_n)^2 \\ &= \frac{n-2}{2}(\Delta - \delta)^2 + (\Delta - \delta)^2 = \frac{n}{2}(\Delta - \delta)^2 \end{aligned} \tag{3.2}$$

the inequality (3.1) is obtained.

Since equality in (3.2) holds if and only if $d_1 = d_2 = \dots = d_n$, equality in (3.1) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \le k \le n-1$. □

*Remark* 1. Since $(\Delta-\delta)^2 \geq 0$, the inequality (3.1) is stronger than inequality (2.1).

**Theorem 2.** *Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$M_1 \leq \frac{4m^2}{n} + n(\Delta-\delta)^2\alpha(n) \tag{3.3}$$

*where*

$$\alpha(n) = \frac{1}{4}\left(1 - \frac{1+(-1)^{n+1}}{2n^2}\right).$$

*Equality in (3.3) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \leq k \leq n-1$.*

*Proof.* Suppose that $S$ is a subset of the form $S = \{1,2,\dots,k\} \subset I_n = \{1,2,\dots,n\}$ and that $p_i = 1$, $i = 1,2,\dots,n$. Then the expression (2.7) reaches the minimum if $k = \lfloor \frac{n}{2} \rfloor$, i.e. if $S = \{1,2,\dots,\lfloor \frac{n}{2} \rfloor\}$. Now, for $p_i = 1, a_i = d_i, b_i = d_i, i = 1,2,\dots,n$, $r_1 = r_2 = \delta$, $R_1 = R_2 = \Delta$ and $S = \{1,2,\dots,\lfloor \frac{n}{2} \rfloor\}$, the equality (2.8) transforms into

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \leq (\Delta-\delta)^2 \lfloor \frac{n}{2} \rfloor \left(n - \lfloor \frac{n}{2} \rfloor\right), \tag{3.4}$$

i.e.

$$nM_1 - 4m^2 \leq n^2(\Delta-\delta)^2 \frac{1}{n}\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right).$$

Since

$$\alpha(n) = \frac{1}{n}\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right) = \frac{1}{4}\left(1 - \frac{1+(-1)^{n+1}}{2n^2}\right)$$

from the above inequality we obtain (3.3).

Since equality in (3.4) holds if and only if $d_1 = d_2 = \cdots = d_n$, equality in (3.3) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \leq k \leq n-1$. □

*Remark* 2. If $n$ is even number, $n \geq 2$, then $\alpha(n) = \frac{1}{4}$, and if $n$ is odd, $n \geq 3$, then $\alpha(n) = \frac{(n-1)(n+1)}{4n^2} < \frac{1}{4}$. This means that inequality (3.3) is stronger than inequality (2.2) for each odd $n$, $n \geq 3$.

**Theorem 3.** *Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Further, let $S$ be a subset of $I_n = \{1,2,\dots,n\}$ that minimizes the expression*

$$\left|\sum_{i\in S} d_i - m\right|. \tag{3.5}$$

*Then*

$$M_1 \leq \frac{4m^2\left(1 + \left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2 \beta(S)\right)}{n} \tag{3.6}$$

*where*

$$\beta(S) = \frac{1}{2m}\sum_{i\in S} d_i \left(1 - \frac{1}{2m}\sum_{i\in S} d_i\right).$$

*Equality in (3.6) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \le k \le n-1$, or bidegreed graph such that $\Delta+\delta$ divides $n\delta$ and there are exactly $p = \frac{n\delta}{\Delta+\delta}$ vertices of degree $\Delta$ and $q = \frac{n\Delta}{\Delta+\delta}$ vertices of degree $\delta$.*

*Proof.* For $p_i = d_i$, $i = 1,2,\dots,n$, the expression (2.7) transforms into (3.5). Suppose that $S$ is subset of $I_n = \{1,2,\dots,n\}$ for which the expression (3.5) reaches a minimum. Now for $p_i = d_i$, $a_i = d_i$, $b_i = \frac{1}{d_i}$, $i = 1,2,\dots,n$, $r_1 = \delta$, $R_1 = \Delta$, $r_2 = \frac{1}{\Delta}$ and $R_2 = \frac{1}{\delta}$ the inequality (2.8) becomes

$$n\sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} d_i\right)^2 \le (\Delta-\delta)\left(\frac{1}{\delta} - \frac{1}{\Delta}\right)\sum_{i\in S} d_i \left(2m - \sum_{i\in S} d_i\right) \tag{3.7}$$

i.e.

$$nM_1 - 4m^2 \le 4m^2\left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2 \beta(S)$$

wherefrom the inequality (3.6) is obtained.

Equality in (3.7) holds if and only if $d_1 = d_2 = \cdots = d_n$, therefore the equality in (3.6) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \le k \le n-1$.

Suppose $\Delta+\delta$ divides $n\delta$ and that graph $G_1$ has exactly $p = \frac{n\delta}{n+\delta}$ vertices of degree $\Delta$ and $q = \frac{n\Delta}{\Delta+\delta}$ vertices of degree $\delta$, where $p+q = n$. In that case $m = \frac{n\delta\Delta}{\Delta+\delta}$ and expression (3.5) reaches the minimum if $S = \{1,2,\dots,p\}$ and $\beta(s) = \frac{1}{4}$. This means that equality in (3.5) holds if and only if $G$ is isomorph with bidegreed graph $G_1$. □

*Remark* 3. Since for each set $S \subset I_n = \{1,2,\dots,n\}$ holds $\beta(S) \le \frac{1}{4}$, we have that

$$\frac{4m^2\left(1 + \left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2 \beta(S)\right)}{n} \le \frac{m^2\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right)^2}{n}.$$

This means that inequality (3.6) is stronger than (2.3).

**Corollary 1.** *Let $G$ be an undirected connected graph with $n$ , $n \ge 2$, vertices and $m$ edges. If $\delta = 1$ then*

$$M_1 \le \frac{4m^2\left(1 + \frac{(n-2)^2}{n-1}\beta(S)\right)}{n}. \tag{3.8}$$

*Equality holds if and only if $G$ is isomorph with $K_{1,n-1}$. If $\delta \geq 2$, then*

$$M_1 \leq \frac{4m^2\left(1+\frac{(n-3)^2}{2(n-1)}\beta(S)\right)}{n}. \tag{3.9}$$

*Equality holds if and only if $G$ is isomorph with graph $K_3$.*

*Remark* 4. Since $\beta(S) \leq \frac{1}{4}$, it follows that inequalities (3.8) and (3.9) are stronger of the corresponding proved in [15] (their Corollary 2.1) and [17] (their Corollary 2.3).

3.2. *First Zagreb coindex*

Now we give a theorems which provides a lower and upper bounds for $\bar{M}_1$ in terms of parameters $n$, $m$, $\Delta$ and $\delta$.

**Theorem 4.** *Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\bar{M}_1 \leq \frac{2m}{n}(n(n-1)-2m)-\frac{1}{2}(\Delta-\delta)^2. \tag{3.10}$$

*Equality holds if and only if $G$ is isomorph with $k$-regular graph, $1 \leq k \leq n-1$.*

*Proof.* The proof immediately follows by Lemma 4 and Theorem 1. □

*Remark* 5. The inequality (3.10) is stronger than (2.5).

**Theorem 5.** *Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\bar{M}_1 \geq \frac{2m}{n}(n(n-1)-2m)-n(\Delta-\delta)^2\alpha(n).$$

*Equality holds if and only if $G$ is isomorph with $k$-regular graph, $1 \leq k \leq n-1$.*

*Proof.* The proof immediately follows by Lemma 4 and Theorem 2. □

**Theorem 6.** *Let $G$ be an undirected connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Further, let $S$ be a subset of $I_n = \{1,2,\ldots,n\}$ that minimizes the expression*

$$\left|\sum_{i\in S} d_i - m\right|.$$

*Then*

$$\bar{M}_1 \geq 2m(n-1)-\frac{4m^2\left(1+\left(\sqrt{\frac{\Delta}{\delta}}-\sqrt{\frac{\delta}{\Delta}}\right)^2\beta(S)\right)}{n} \tag{3.11}$$

*where $\beta(s) = \frac{1}{2m}\sum_{i\in S} d_i\left(1-\frac{1}{2m}\sum_{i\in S} d_i\right)$.*

*Equality in (3.11) holds if and only if $G$ is isomorph with $k$-regular graph, $1 \leq k \leq n-1$, or bidegreed graph such that $\Delta + \delta$ divides $n\delta$ and there are exactly $p = \frac{n\delta}{\delta+\delta}$ vertices of degree $\Delta$ and $q = \frac{n\Delta}{\Delta+\delta}$ vertices of degree $\delta$.*

*Proof.* The proof immediately follows by Lemma 4 and Theorem 3. □

*Remark* 6. Inequality (3.11) is stronger than (2.6).

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# RICCI COEFFICIENTS OF ROTATION OF GENERALIZED FINSLER SPACES

SVETISLAV M. MINČIĆ, MIĆA S. STANKOVIĆ, AND MILAN LJ. ZLATANOVIĆ

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*Abstract.* Generalized Finsler space $GF_n$ is a differentiable $N$-dimensional manifold with non-symmetric basic tensor $g_{ij}(x,\dot{x})$ defined by condition (1.2). Using the basic tensor, by (1.4) a non-symmetric connection $P^*$ is defined, and also four kinds of covariant derivative in the Rund's sense and five curvature tensors are obtained (Section 1).

In Section 2 two kinds of Ricci coefficients of rotation are defined and their properties are exposed. Also, integrability conditions of the equation expressing covariant derivatives of the congruence vector by means of coefficients of rotation, are obtained.

In Section 3 a geodesic mapping of two spaces $GF_n$ and $G\overline{F}_n$ is defined and some it's properties are proved.

In Section 4 some invariants of such mappings in relation with the coefficients of rotation are studied.

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## 1. Introduction

The generalized Finsler space $(GF_n)$ is a differentiable manifold with non-symmetric basic tensor field $g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N) \equiv g_{ij}(x,\dot{x})$, where

$$g_{ij}(x,\dot{x}) \neq g_{ji}(x,\dot{x}), \quad (g = det(g_{ij}) \neq 0,\ \dot{x} = dx/dt). \tag{1.1}$$

Based on (1.1), the symmetric and anti-symmetric part of $g_{ij}$ are defined

$$g_{\underline{ij}} = \frac{1}{2}(g_{ij} + g_{ji}), \qquad g_{\underset{\vee}{ij}} = \frac{1}{2}(g_{ij} - g_{ji}).$$

Following [18], in our notation, hold

$$a)\ g_{\underline{ij}}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j}, \qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}}{\partial\dot{x}^k} = 0, \tag{1.2}$$

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where $F(x,\dot{x})$ is a metric function in $GF_n$, having the properties known from the theory of usual Finsler spaces $(F_n)$. A lot of research papers (see for example [1–4, 6, 11, 14–19, 21, 22]) are dedicated to the theory of Finsler spaces and their generalizations. The discussed structure is a particular case of the Eisenhart-Lagrange approach which is studied in [12] (Chap. 8).

The lowering and the raising of indices are defined by the tensors $g_{\underline{ij}}$ and $h^{ij}$ respectively, where $h^{ij}$ satisfy equation

$$g_{\underline{ij}}h^{jk} = \delta_i^k, \quad (\underline{g} = det(g_{\underline{ij}}) \neq 0).$$

**Generalized Cristoffel symbols** of the $1^{st}$ and the $2^{nd}$ kind are defined:

$$\gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i.kj},$$
$$\gamma^i_{jk} = h^{ip}\gamma_{p.jk} = \frac{1}{2}h^{ip}(g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj},$$

where, e.g., $g_{ji,k} = \partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk}g_{\underline{ip}} = \gamma_{s.jk}h^{ps}g_{\underline{ip}} = \gamma_{s.jk}\delta_i^{*s} = \gamma_{i.jk}.$$

Introducing a tensor $C_{ijk}$ like as at space $F_n$, we have

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2}g_{ij,\dot{x}^k} = \frac{1}{2}g_{\underline{ij},\dot{x}^k} = \frac{1}{4}F^2_{\dot{x}^i\dot{x}^j\dot{x}^k}.$$

We see that $C_{ijk}$ is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip}C_{pjk} = h^{ip}C_{jpk} = h^{ip}C_{jkp}. \tag{1.3}$$

With help of coefficients

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^{*s} \neq P^i_{kj}$$

one obtains coefficients of a non-symmetric affine connections in the Rund's sence (see [17, 18]):

$$P^{*i}_{jk} = \gamma^i_{jk} - h^{iq}(C_{jqp}P^p_{ks} + C_{kqp}P^p_{js} - C_{jkp}P^p_{qs})\dot{x}^{*s} \neq P^{*i}_{kj}, \tag{1.4}$$

$$P^*_{i.jk} = P^{*r}_{jk}g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^{*s} \neq P^*_{i.kj}.$$

In $GF_n$ we denote anti-symmetric and symmetric part for a connection $P^*$ respectively:

$$a)\ T^{*i}_{jk}(x,\dot{x}) = \frac{1}{2}(P^{*i}_{jk} - P^{*i}_{kj}) = \frac{1}{2}(\gamma^i_{jk} - \gamma^i_{kj}), \quad b)\ P^{*i}_{\underline{jk}} = \frac{1}{2}(P^{*i}_{jk} + P^{*i}_{kj}),$$

where $T^{*i}_{jk}$ is the **torsion tensor** of the connection $P^{*i}_{jk}$.

We define four kinds of covariant derivative of a tensor in the space $GF_n$. For example, for a tensor $a^i_j(x,\xi)$:

$$\begin{aligned} a^i_{j\underset{1}{|}m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot{p}}\xi^p_{,m} + P^{*i}_{pm}a^p_j - P^{*p}_{jm}a^i_p, \\ a^i_{j\underset{2}{|}m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot{p}}\xi^p_{,m} + P^{*i}_{mp}a^p_j - P^{*p}_{mj}a^i_p, \end{aligned} \tag{1.5}$$

$$\begin{aligned} a^i_{j\underset{3}{|}m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot{p}}\xi^p_{,m} + P^{*i}_{pm}a^p_j - P^{*p}_{mj}a^i_p, \\ a^i_{j\underset{4}{|}m}(x,\xi) &= a^i_{j,m} + a^i_{j,\dot{p}}\xi^p_{,m} + P^{*i}_{mp}a^p_j - P^{*p}_{jm}a^i_p, \end{aligned} \tag{1.6}$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_n(x)$, and $a^i_{j,\dot{p}} = \partial a^i_j/\partial \dot{x}^p$.

In the work [10] we obtain 10 Ricci type identities in the general case for a tensor $a^{r_1\dots r_u}_{t_1\dots t_v}(x,\xi)$ and three curvature tensors of $GF_n$:

$$\underset{1}{\widetilde{K}}{}^i_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^{*s}_{,n} - P^{*p}_{jn,\dot{s}}\xi^{*s}_{,m}, \tag{1.7}$$

$$\underset{2}{\widetilde{K}}{}^i_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^{*s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{*s}_{,m}, \tag{1.8}$$

$$\begin{aligned} \underset{3}{\widetilde{K}}{}^i_{jmn} &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{nm}(P^{*i}_{pj} - P^{*i}_{jp}) \\ &\quad + P^{*p}_{jm,\dot{s}}\xi^{*s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{*s}_{,m}. \end{aligned} \tag{1.9}$$

The magnitudes $\underset{t}{\widetilde{K}}{}^i_{jmn}$, $t = 1,2,3$ are tensors and we call them curvature tensors of the first, the second and the third kind respectively.

In the work [9] we use the third and the fourth kind of covariant derivative (1.4), and in that manner one gets 10 new Ricci type identities. In these identities appear the same quantities $\underset{t}{\widetilde{K}}{}^i_{jmn}$, $t = 1,2,3$; $\underset{t}{\widetilde{A}}{}^i_{jmn}$, $t = 1,\dots,15$, but in different distribution. Only in the last case appears a new curvature tensor (of the fourth kind) $\underset{4}{\widetilde{K}}$ :

$$a^i_{j\underset{3}{|}m\underset{4}{|}n} - a^i_{j\underset{4}{|}n\underset{3}{|}m} = \underset{4}{\widetilde{K}}{}^i_{pmn}a^p_j + \underset{3}{\widetilde{K}}{}^p_{jnm}a^i_p\ ,$$

where

$$\begin{aligned} \underset{4}{\widetilde{K}}{}^i_{jmn} &= P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^{*s}_{,n} - P^{*p}_{nj,\dot{s}}\xi^{*s}_{,m} \\ &\quad + P^{*p}_{mn}(P^{*i}_{pj} - P^{*i}_{jp}). \end{aligned} \tag{1.10}$$

Denoting by semicolon (;) the covariant derivative with respect to the symmetric connection $P^{*i}_{\underline{jk}}$, then using (1.3) we have

$$\underset{1}{\widetilde{K}}{}^i_{jmn} = \widetilde{K}^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{jn;m} + T^{*p}_{jm}T^{*i}_{pn} - T^{*p}_{jn}T^{*i}_{pm}, \tag{1.11}$$

$$\underset{2}{\widetilde{K}}{}^{i}{}_{jmn} = \widetilde{K}^{i}{}_{jmn} + T^{*i}_{mj;n} - T^{*i}_{nj;m} + T^{*p}_{mj}T^{*i}_{np} - T^{*p}_{nj}T^{i}_{mp}, \tag{1.12}$$

$$\underset{3}{\widetilde{K}}{}^{i}{}_{jmn} = \widetilde{K}^{i}{}_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm}T^{*i}_{np} - T^{*p}_{nj}T^{i}_{pm} + T^{*p}_{mn}T^{*i}_{jp}, \tag{1.13}$$

$$\underset{4}{\widetilde{K}}{}^{i}{}_{jmn} = \widetilde{K}^{i}{}_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm}T^{*i}_{np} - T^{*p}_{nj}T^{*i}_{pm} - T^{*p}_{mn}T^{*i}_{jp}, \tag{1.14}$$

where $\widetilde{K}^{i}_{jmn}$ is the curvature tensor formed by symmetric connection $P^{*i}_{\underline{jk}}$

$$\widetilde{K}^{i}{}_{jmn} = P^{*i}_{\underline{jm},n} - P^{*i}_{\underline{jn},m} + P^{*p}_{\underline{jm}}P^{*i}_{\underline{pn}} - P^{*p}_{\underline{jn}}P^{*i}_{\underline{pm}} + P^{*i}_{\underline{jm},\dot{s}}\xi^{*s}_{,n} - P^{*i}_{\underline{jn},\dot{s}}\xi^{*s}_{,m}.$$

In the work [7] we find five linearly independent curvature tensors $\underset{1}{\widetilde{K}}, \dots, \underset{5}{\widetilde{K}}$, where $\underset{1}{\widetilde{K}}, \dots, \underset{4}{\widetilde{K}}$ given by equations (1.7-1.10), (1.11-1.14), and

$$\underset{5}{\widetilde{K}}{}^{i}{}_{jmn} = \widetilde{K}^{i}{}_{jmn} + T^{*p}_{jm}T^{*i}_{pn} + T^{*p}_{jn}T^{*i}_{pm}, \tag{1.15}$$

$$\begin{aligned}\underset{5}{\widetilde{K}}{}^{i}{}_{jmn} = \frac{1}{2}(&P^{*i}_{\underline{jm},n} - P^{*i}_{\underline{jn},m} + P^{*p}_{jm}P^{*i}_{pn} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp} \\ &- P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{\underline{jm},\dot{s}}\xi^{s}_{,n} - P^{*p}_{\underline{jn},\dot{s}}\xi^{s}_{,m}),\end{aligned} \tag{1.16}$$

where $\underline{jm}$ denotes symmetrization by indices $j$ and $m$.

Applying two kinds of covariant differentiation (1.5), we get

$$\begin{aligned}a^{i}_{j\underset{\underset{2}{1}}{|}m}(x,\xi) &= a^{i}_{j,m} + a^{i}_{j,\dot{p}}\xi^{p}_{,m} + P^{*i}_{\underset{mp}{pm}}a^{p}_{j} - P^{*p}_{\underset{mj}{jm}}a^{i}_{p} \\ &= a^{i}_{j,m} + a^{i}_{j,\dot{p}}\xi^{p}_{,m} + (P^{*i}_{\underline{pm}} \pm T^{*i}_{pm})a^{p}_{j} - (P^{*p}_{\underline{jm}} \pm T^{*p}_{jm})a^{i}_{p} \\ &= a^{i}_{j;m} + T^{*i}_{\underset{mp}{pm}}a^{p}_{j} - T^{*p}_{\underset{mj}{jm}}a^{i}_{p}.\end{aligned}$$

## 2. Ricci coefficients of rotation in a generalized Finsler space

On Ricci coefficients of rotation in $R_n$ the reader can find e.g. in [20], §54. On that matter in $GR_n$ is written in [8]. On Ricci coefficients of rotation in $F_n$ see [13].

### 2.1. *Congruence of curves and orthogonal ennuple*

**Definition 1.** A congruence of curves in a $GF_n$ is such a family of curves that though each point of $GF_n$ passes one curve of the family. N mutually orthogonal congruences of curves constitute an **orthogonal ennuple**. Instead of congruences of curves we shall sometimes speak about congruences of the corresponding tangent vectors.

If $\lambda_{(h)}$, $(h = 1, \dots, N)$ are unit tangent vectors of congruences of curves of an orthogonal ennuple, then in virtue of the previous definition

$$g_{\underline{ij}}(x, \dot{x})\lambda^i_{(h)}\lambda^j_{(k)} = e_{(k)}\delta_{hk},\ e_{(k)} = \pm 1, \tag{2.1}$$

or

$$e_{(k)}\lambda^i_{(h)}\lambda_{(k)i} = \delta_{hk}, \tag{2.2}$$

where $\delta_{hk}$ are Kronecker symbols. (of course, we do not suppose summation w.r.t. (k) in (2.1), (2.2) and in similar formulas later on.) The next theorem expresses the basic properties of orthogonal ennuples.

**Theorem 1.** *For the unit tangent vectors $\lambda_{(h)}$, $(h = 1, \dots, N)$ of congruences of curves of an orthogonal ennuple the relations*

$$\begin{aligned} &a)\ \sum_{k=1}^{N} e_{(k)}\lambda_{(k)i}\lambda^j_{(k)} = \delta^j_i, \quad b)\ \sum_{k=1}^{N} e_{(k)}\lambda_{(k)i}\lambda_{(k)j} = g_{\underline{ij}} \\ &c)\ \sum_{k=1}^{N} e_{(k)}\lambda^i_{(k)}\lambda^j_{(k)} = g^{\underline{ij}} \end{aligned} \tag{2.3}$$

*are valid.*

*Proof.* In the determinant $det(\lambda^i_{(k)})$, whose value is 1, we can regard $e_{(k)}\lambda_{(k)i}$ as cofactor of the element $\lambda^i_{(k)}$. Developing the determinant either by rows or by columns (2.3$a$) follows.
Further, we have

$$\sum_k e_{(k)}\lambda_{(k)i}\lambda_{(k)j} = g_{\underline{jl}}\sum_k e_{(k)}\lambda_{(k)i}\lambda^l_{(k)} = g_{\underline{jl}}\delta^l_i \Rightarrow (2.3b).$$

The equation (2.3$c$) can be obtained in the same manner. □

2.2. *Definition and basic properties of the coefficient of rotation*

Using the two kinds of covariant derivative (1.5) of a vector in a $GF_n$, we can define two kinds of coefficients of rotation, as two systems of invariants (for $\theta = 1, 2$).

**Definition 2.** The invariants

$$\underset{\theta}{Y}_{(hkm)}(x, \dot{x}) \overset{def}{=} \lambda_{(h)i\underset{\theta}{|}j}\lambda^i_{(k)}\lambda^j_{(m)} = \lambda^i_{(h)\underset{\theta}{|}j}\lambda_{(k)i}\lambda^j_{(m)}, \quad \theta = 1, 2. \tag{2.4}$$

are said to be coefficients of rotation of the given orthogonal ennuple.

From (1.5, 1.6) it is evidently that

$$\lambda^i_{(h)\underset{4}{|}j} = \lambda^i_{(h)\underset{2}{|}j}, \quad \lambda_{(h)i\underset{3}{|}j} = \lambda_{(h)i\underset{2}{|}j}, \quad \lambda^i_{(h)\underset{3}{|}j} = \lambda^i_{(h)\underset{1}{|}j}, \quad \lambda_{(h)i\underset{4}{|}j} = \lambda_{(h)i\underset{1}{|}j},$$

and it is easy to prove that there exist two kinds of coefficients of rotation in the non-symmetric case.

**Theorem 2.** *Both kinds of coefficients of rotation are antisymmetric in their first two indices, i.e.*

$$\underset{\theta}{Y}_{(hkm)} = -\underset{\theta}{Y}_{(khm)} \Rightarrow \underset{\theta}{Y}_{(hkm)} = 0. \tag{2.5}$$

*Proof.* By covariant differentation we get from (2.2) the relation

$$e_{(k)}(\lambda^i_{(h)\underset{\theta}{|}j}\lambda_{(k)i} + \lambda^i_{(h)}\lambda_{(k)i\underset{\theta}{|}j}) = 0,$$

from where, transvecting by $\lambda^j_{(m)}$,

$$e_{(k)}(\lambda^i_{(h)\underset{\theta}{|}j}\lambda_{(k)i}\lambda^j_{(m)} + \lambda^i_{(h)}\lambda_{(k)i\underset{\theta}{|}j}\lambda^j_{(m)}) \underset{(2.4)}{=} e_{(k)}(\underset{\theta}{Y}_{(hkm)} + \underset{\theta}{Y}_{(khm)}) = 0$$

that is

$$\underset{\theta}{Y}_{(hkm)} + \underset{\theta}{Y}_{(khm)} = 0 \Rightarrow (2.5).$$

□

**Theorem 3.** *If*

$$Y_{(hkm)} = \lambda_{(h)i;j}\lambda^i_{(k)}\lambda^j_{(m)} \tag{2.6}$$

*are coefficients of rotation in the associated Finsler space $F_n$ (see [13]), then*

$$\underset{\theta}{Y}_{(hkm)} = Y_{(hkm)} + (-1)^\theta T^{*p}_{ij}\lambda_{(h)p}\lambda^i_{(k)}\lambda^j_{(m)}, \quad \theta = 1,2, \tag{2.7}$$

$$Y_{(hkm)} = (\underset{1}{Y}_{(hkm)} + \underset{2}{Y}_{(hkm)})/2,$$

$$\underset{\theta}{Y}_{(hkh)} = Y_{(hkh)}, \quad \underset{\theta}{Y}_{(hhk)} = Y_{(hhk)}, \quad \underset{\theta}{Y}_{(hkk)} = Y_{(hkk)}, \quad \theta = 1,2. \tag{2.8}$$

*Proof.* In virtue of (2.4) and

$$\lambda_{i\underset{\theta}{|}n} = \lambda_{i;n} + (-1)^\theta T^{*p}_{in}\lambda_p, \quad \theta = 1,2,$$

we have

$$\underset{\theta}{Y}_{(hkm)} = (\lambda_{(h)i;j} + (-1)^\theta T^{*p}_{ij}\lambda_{(h)p})\lambda^i_{(k)}\lambda^j_{(m)} \underset{(2.6)}{\Rightarrow} (2.7) \quad \theta = 1,2.$$

In virtue of (2.7) for two indices there follows (2.8) because for example (for $h = m$) is

$$\begin{aligned} T^{*p}_{ij}\lambda_{(h)p}\lambda^i_{(k)}\lambda^j_{(m)} = T^*_{p.ij}\lambda^p_{(h)}\lambda^i_{(k)}\lambda^j_{(h)} &= T^*_{j.ip}\lambda^j_{(h)}\lambda^i_{(k)}\lambda^p_{(h)} \\ &= -T^*_{p.ij}\lambda^p_{(h)}\lambda^i_{(k)}\lambda^j_{(h)} = 0, \end{aligned}$$

where $T^*_{p.ij} = T^{*s}_{ij}g_{\underline{sp}}$. Here we applied the fact that $T^*_{p.ij}$ is antisymmetric in all pairs of indices. □

2.3. *Expression of the derivative of the vectors of a congruence by the coefficients of rotation*

**Theorem 4.** *In $GF_n$ the relation*

$$\begin{aligned} a)\ \lambda_{(h)i\underset{\theta}{|}j} &= \sum_{k,m=1}^{N} e_{(k)}e_{(m)}\underset{\theta}{Y}_{(hkm)}\lambda_{(k)i}\lambda_{(m)j}, \\ b)\ \lambda^{i}_{(h)\underset{\theta}{|}j} &= \sum_{k,m=1}^{N} e_{(k)}e_{(m)}\underset{\theta}{Y}_{(hkm)}\lambda^{i}_{(k)}\lambda_{(m)j} \end{aligned} \tag{2.9}$$

*are valid.*

*Proof.* *a)* Multiplying the relation (2.4) by $e_{(k)}e_{(m)}\lambda_{(k)p}\lambda_{(m)q}$ and summing with respect to $m,k$ get

$$\begin{aligned} \sum_{k,m} e_{(k)}e_{(m)}\underset{\theta}{Y}_{(hkm)}\lambda_{(k)p}\lambda_{(m)q} &= \sum_{k,m}\lambda_{(h)i\underset{\theta}{|}j}\lambda^{i}_{(k)}\lambda^{j}_{(m)}e_{(k)}e_{(m)}\lambda_{(k)p}\lambda_{(m)q} \\ &= \lambda_{(h)i\underset{\theta}{|}j}\{\sum_{k} e_{(k)}\lambda_{(k)p}\lambda^{i}_{(k)}\}\cdot\{\sum_{m} e_{(m)}\lambda_{(m)q}\lambda^{j}_{(m)}\} \\ &\underset{(2.3a)}{=} \lambda_{(h)i\underset{\theta}{|}j}\delta^{i}_{p}\delta^{j}_{q} = \lambda_{(h)p\underset{\theta}{|}q}. \end{aligned}$$

□

**Theorem 5.** *The covariant derivatives of vectors $\lambda_{(h)i}$ and $\lambda^{i}_{(h)}$ in the direction of the vector $\lambda^{j}_{(p)}$ can be expressed by the coefficients of rotation as linear combination of the vectors of the ennuple as follows:*

$$a)\ \lambda_{(h)i\underset{\theta}{|}j}\lambda^{j}_{(p)} = \sum_{k} e_{(k)}\underset{\theta}{Y}_{(hkp)}\lambda_{(k)i} \quad b)\ \lambda^{i}_{(h)\underset{\theta}{|}j}\lambda^{j}_{(p)} = \sum_{k} e_{(k)}\underset{\theta}{Y}_{(hkp)}\lambda^{i}_{(k)} \tag{2.10}$$

*Proof.* Transvecting the equation (2.9) by $\lambda^{j}_{(p)}$, we obtain

$$\begin{aligned} a)\ \lambda_{(h)i\underset{\theta}{|}j}\lambda^{j}_{(p)} &= \sum_{k,m} e_{(k)}e_{(m)}\underset{\theta}{Y}_{(hkp)}\lambda_{(k)i}\lambda_{(m)j}\lambda^{j}_{(p)} \\ &= \sum_{k,m} e_{(k)}\underset{\theta}{Y}_{(hkp)}\lambda_{(k)i}\delta_{mp} = \sum_{k} e_{(k)}\underset{\theta}{Y}_{(hkp)}\lambda_{(k)i} \Rightarrow (2.10). \end{aligned}$$

□

2.4. *Integrability conditions of the equation (2.9)*

The relation (2.9) is a partial differential equation with respect to the unknown functions $\lambda_{(h)i}$. Now we are going to examine its integrability condition.

In [10] we have obtained 10 Ricci-type identities in $GF_n$. In three of these identities appear the curvature tensors $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{3}{\widetilde{K}}$, and in the others appear the quantities $\underset{1}{\widetilde{A}},\dots,\underset{15}{\widetilde{A}}$ which have the form and the role of the curvature tensors, but they are not tensors. In [5] we obtained combined Ricci-type identities, in which appear "derived" curvature tensors $\underset{1}{\widetilde{K}}^*,\dots,\underset{8}{\widetilde{K}}^*$. In [7] is proved that only five are independent among the mentioned curvature tensors, for example $\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}, \underset{3}{\widetilde{K}}, \underset{4}{\widetilde{K}}, \underset{5}{\widetilde{K}}$ while the others are linear combinations of these five tensors and $\widetilde{K}$. We shall use further those of the Ricci-type identities in which appear the above tensors (the tensor $\underset{1}{\widetilde{K}}^*$ is linear combination of $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, while the tensor $\underset{2}{\widetilde{K}}^*$ does not appear in identities which we need.)

**Theorem 6.** *In $GF_n$ the first two integrability conditions ($\theta = 1,2$) of equations (2.9) are*

$$\begin{aligned}
&(-\underset{\theta}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s} + 2(-1)^\theta T^{*s}_{jr}\lambda_{(h)i\underset{\theta}{|}s})\lambda^i_{(p)}\lambda^j_{(q)}\lambda^r_{(t)} = (\underset{\theta}{Y}_{(hpq),j} + \underset{\theta}{Y}_{(hpq),\dot{s}}\xi^s_j)\lambda^j_{(t)} \\
&-(\underset{\theta}{Y}_{(hpt),j} + \underset{\theta}{Y}_{(hpt),\dot{s}}\xi^s_j)\lambda^j_{(q)} + \sum_{k=1}^{N} e_{(k)}\{\underset{\theta}{Y}_{(hkq)}\underset{\theta}{Y}_{(kpt)} - \underset{\theta}{Y}_{(hkt)}\underset{\theta}{Y}_{(kpq)} \\
&+\underset{\theta}{Y}_{(hpk)}[\underset{\theta}{Y}_{(kqt)} - \underset{\theta}{Y}_{(ktq)}]\}, \quad \theta = 1,2
\end{aligned} \tag{2.11}$$

*where $\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}$ are given by (1.7).*

*Proof.* Applying the Ricci-type identities from [10], we have

$$\lambda_{(h)i\underset{\theta}{|}jr} - \lambda_{(h)i\underset{\theta}{|}rj} = -\underset{\theta}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s} + 2(-1)^\theta T^{*s}_{jr}\lambda_{(h)i\underset{\theta}{|}s}, \quad \theta = 1,2,$$

By repeated differentiation of (2.9) one can form the difference on the left side of this equation, and then (2.11) easily follows. □

**Theorem 7.** *The third integrability condition of the equation (2.9) in $GF_n$ is*

$$\begin{aligned}
&-\underset{3}{\widetilde{K}}{}^s_{ijr}\lambda_{(h)s}\lambda^i_{(p)}\lambda^j_{(q)}\lambda^r_{(t)} = (\underset{1}{Y}_{(hpq),j} + \underset{1}{Y}_{(hpq)\dot{s}}\xi^s_j)\lambda^j_{(t)} \\
&-(\underset{2}{Y}_{(hpt),j} + \underset{2}{Y}_{(hpt)\dot{s}}\xi^s_j)\lambda^j_{(q)} + \sum_{k=1}^{N} e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)} - \underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)} \\
&+\underset{1}{Y}_{(hpk)}\underset{2}{Y}_{(kqt)} - \underset{2}{Y}_{(hpk)}\underset{1}{Y}_{(ktq)}].
\end{aligned} \tag{2.12}$$

*where $\underset{3}{\widetilde{K}}$ is given by (1.9).*

*Proof.* Applying the corresponding identity from [10] we get

$$\lambda_{(h)i\underset{1}{|}j\underset{2}{|}r} - \lambda_{(h)i\underset{2}{|}r\underset{1}{|}j} = -\underset{3}{\widetilde{K}}{}^{s}_{ijr}\lambda_{(h)s}.$$

Further, we use (2.9) to form the difference on the left side. □

**Theorem 8.** *The fourth integrability condition of the equation* (2.9) *is*

$$\begin{aligned}
&\underset{4}{\widetilde{K}}{}^{i}_{sjr}\lambda^{s}_{(h)}\lambda_{(p)i}\lambda^{j}_{(q)}\lambda^{r}_{(t)} = (\underset{1}{Y}_{(hpq),j} + \underset{1}{Y}_{(hpq)\dot{s}}\xi^{s}_{j})\lambda^{j}_{(t)} \\
&-(\underset{2}{Y}_{(hpt),j} + \underset{2}{Y}_{(hpt)\dot{s}}\xi^{s}_{j})\lambda^{j}_{(q)} + \sum_{k=1}^{N} e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)} + \underset{1}{Y}_{(hpk)}\underset{1}{Y}_{(kqt)} \\
&-\underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)} - \underset{2}{Y}_{(hpk)}\underset{2}{Y}_{(ktq)}].
\end{aligned} \tag{2.13}$$

*where* $\underset{4}{\widetilde{K}}$ *is given by* (1.10).

*Proof.* Using by equations (1.10) we see that

$$\lambda^{i}_{(h)\underset{3}{|}j\underset{4}{|}r} - \lambda^{i}_{(h)\underset{4}{|}r\underset{3}{|}j} = \underset{4}{\widetilde{K}}{}^{i}_{sjr}\lambda^{s}_{(h)},$$

and the use of (2.9) yields the integrability condition (2.13). □

**Theorem 9.** *The fifth integrability condition of the equation* (2.9) *is*

$$\begin{aligned}
&\underset{2}{\widetilde{K}}{}^{*i}_{sjr}\lambda^{s}_{(h)}\lambda_{(p)i}\lambda^{j}_{(q)}\lambda^{r}_{(t)} = (\underset{1}{Y}_{(hp[q),j} + \underset{1}{Y}_{(hp[q)\dot{s}}\xi^{s}_{j})\lambda^{j}_{(t)]} \\
&-(\underset{2}{Y}_{(hp[q),j} + \underset{2}{Y}_{(hp[q)\dot{s}}\xi^{s}_{j})\lambda^{j}_{(t)]} + \sum_{k} e_{(k)}[\underset{1}{Y}_{(hkq)}\underset{1}{Y}_{(kpt)} + \underset{1}{Y}_{(hpk)}\underset{1}{Y}_{(kqt)} \\
&-\underset{1}{Y}_{(hkt)}\underset{2}{Y}_{(kpq)} - \underset{1}{Y}_{(hkt)}\underset{2}{Y}_{(kpq)} + \underset{2}{Y}_{(hkq)}\underset{2}{Y}_{(kpt)} + \underset{2}{Y}_{(hpk)}\underset{2}{Y}_{(kqt)} \\
&-\underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)} - \underset{2}{Y}_{(hkt)}\underset{1}{Y}_{(kpq)}],
\end{aligned}$$

*where* $[qt]$ *we denoted symmetrization without division by indices* $q$ *and* $t$ *and* $\underset{2}{\widetilde{K}}{}^{*i}_{sjr} \equiv \underset{5}{\widetilde{K}}{}^{i}_{sjr}$ *given by* (1.16).

*Proof.* We use that Ricci-type identity in which the curvature $\underset{2}{\widetilde{K}}{}^{*}$ appears. From corresponding identities in [5] we have

$$\lambda^{i}_{(h)\underset{1}{|}jr} - \lambda^{i}_{(h)\underset{1}{|}r\underset{2}{|}j} + \lambda^{i}_{(h)\underset{2}{|}jr} - \lambda^{i}_{(h)\underset{2}{|}r\underset{1}{|}j} = 2\underset{2}{\widetilde{K}}{}^{*i}_{sjr}\lambda^{s}_{(h)},$$

and then use (2.9). □

## 3. Geodesic mappings

**Definition 3.** Geodesic in $GF_n$ is given by

$$\frac{d^2x^i}{ds^2} + P^{*i}_{jk}(x, \frac{dx}{ds})\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{3.1}$$

Consider two $N$-dimensional spaces of non-symmetrical affine connection: $GF_n$ and $G\overline{F}_n$. So, if connection coefficients of these spaces are respectively $P^{*i}_{jk}$ and $\overline{P}^{*i}_{jk}$, we suppose that in general the symmetry with respect to indices $j,k$ is not in force.

One says that reciprocal one valued mapping $f : GF_n \to G\overline{F}_n$ is *geodesic*, if geodesics of the space $GF_n$ pass to geodesics of the space $G\overline{F}_n$. We can consider these spaces in the common by this mapping system of local coordinates, i.e. for $f : M \mapsto \overline{M}$ we have $M(x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n) \equiv M(x,\dot{x})$ and $\overline{M}(x^1,\dots,x^n,x^1,\dots,x^n,\dot{x}^1,\dots,\dot{x}^n) \equiv \overline{M}(x,\dot{x})$, where $M \in GF_n$, $\overline{M} \in G\overline{F}_n$. In the corresponding points $M(x,\dot{x})$ and $\overline{M}(x,\dot{x})$ we can put

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + D^i_{jk}, \qquad (i,j,k = 1,\dots,N), \tag{3.2}$$

where $D^i_{jk}(x)$ is **the deformation tensor** of the connection $P^*$ of $GF_n$ according to the mapping $f : GF_n \to G\overline{F}_n$.

**Definition 4.** The curve

$$l : x^i = x^i(t)$$

is **geodesic** of $GF_n$ if and only if is:

$$\ddot{x}^i + P^{*i}_{pq}\dot{x}^p\dot{x}^q = \rho(t)\dot{x}^i(t), \tag{3.3}$$

where $\rho(t)$ is an invariant and $t$ an arbitrary parameter.

If $f : l \to \overline{l}$, then in the common by the mapping $f$ coordinates $\overline{x}^i \equiv x^i$ it is $\dot{\overline{x}}^i = d\overline{x}^i/dt = \dot{x}^i$, and $\overline{l}$ is geodesic in $G\overline{F}_n$ too, from where we get

$$\ddot{x}^i + \overline{P}^{*i}_{pq}\dot{x}^p\dot{x}^q = \overline{\rho}(t)\dot{x}^i(t) \tag{3.4}$$

Subtracting (3.3) and (3.4), we obtain

$$(\overline{P}^{*i}_{pq} - P^{*i}_{pq})\dot{x}^p\dot{x}^q = (\overline{\rho}(t) - \rho(t))\dot{x}^i(t),$$

and, because of (3.2):

$$D^i_{pq}\dot{x}^p\dot{x}^q = 2\psi(t)\dot{x}^i(t), \tag{3.5}$$

where $\psi(t) = (\overline{\rho}(t) - \rho(t))/2$. Denoting by $D^i_{\underline{jk}}$, $D^i_{\underset{\vee}{jk}}$ the symmetric and antisymmetric part of $D^i_{jk}$ respectively, we get

$$D^i_{jk} = D^i_{\underline{jk}} + D^i_{\underset{\vee}{jk}},$$

and (3.5) reduces to

$$D^i_{\underline{pq}}\dot{x}^p\dot{x}^q = 2\psi(t)\dot{x}^i(t). \tag{3.6}$$

As in the case of a symmetric connection (see for ex. [4]) one concludes that

$$\psi(t) = \psi_p(x^1(t),\dots,x^n(t),\dot{x}^1(t),\dots,\dot{x}^n(t))\dot{x}^p(t),$$

and from (3.6):

$$D^i_{\underline{pq}}\dot{x}^p\dot{x}^q = 2\psi_p\dot{x}^p\dot{x}^i = \psi_p\dot{x}^p\dot{x}^q\delta^i_q + \psi_q\dot{x}^p\dot{x}^q\delta^i_p = (\psi_p\delta^i_q + \psi_q\delta^i_p)\dot{x}^p\dot{x}^q,$$

wherefrom

$$D^i_{\underline{jk}} = \delta^i_j\psi_k + \delta^i_k\psi_j. \tag{3.7}$$

Denoting also

$$D^i_{\underset{\vee}{jk}} = \xi^i_{jk} = -\xi^i_{kj},$$

by substitution in (3.2) we obtain

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{3.8}$$

and the deformation tensor is

$$D^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}. \tag{3.9}$$

So, the condition (3.8) is necessary that the mapping $f$ be geodesic. It is easily to prove that this condition is sufficient too, and we have

**Theorem 10.** *A necessary and sufficient condition that the mapping $f : GF_n \to G\overline{F}_n$ be geodesic is the deformation tensor $D^i_{jk}$ from (3.2) at the mapping $f$ to has the form (3.9), where $\psi_j(x,\dot{x})$ is a covariant vector, and $\xi^i_{jk}(x,\dot{x})$ an antisymmetric tensor.*

For $k = i$, we obtain from (3.7):

$$D^i_{\underline{ji}} = \delta^i_j\psi_i + \delta^i_i\psi_j = \psi_j + N\psi_j,$$

wherefrom

$$\psi_i = \frac{1}{N+1}D^p_{\underline{ip}}, \tag{3.10}$$

which, by substitution in (3.8), gives

$$\overline{P}^{*i}_{jk} = P^{*i}_{jk} + \frac{1}{N+1}(\delta^i_j D^p_{\underline{kp}}(x,\dot{x}) + \delta^i_k D^p_{\underline{jp}}(x,\dot{x})) + D^i_{\underset{\vee}{jk}}(x,\dot{x}), \tag{3.11}$$

where $D^i_{jk}(x)$ is the deformation tensor.

On the base of above explained, we get

**Theorem 11.** *Let a space $GF_n$ be given, i.e. on a differentiable manifold $M_n$ let be defined non-symmetric connection coefficients $P^{*i}_{jk}(x,\dot{x})$. If on $M_n$ is given a tensor $D^i_{jk}(x,\dot{x})$ too and we determine $\overline{P}^{*i}_{jk}(x,\dot{x})$ according to (3.11), then on $M_n$ will be defined a space $G\overline{F}_n$, with connection coefficients $\overline{P}^{*i}_{jk}$, and then $GF_n$ and $G\overline{F}_n$ have common geodesics. We obtain the same result (on the base of (3.8) by choice of a vector $\psi_i(x,\dot{x})$ and antisymmetric tensor $\xi^i_{jk}(x,\dot{x}) = D^i_{\underset{\vee}{jk}}(x,\dot{x})$.*

The question forces itself: Is it possible a geodesic mapping of a space $GF_n$ with a non-symmetric affine connection on to a space $\overline{F}_n$ with a symmetric affine connection? It is easy to see that the next theorem is valid:

**Theorem 12.** *A necessary and sufficient condition that a mapping $f : GF_n \to \overline{F}_n$ of a non-symmetric affine connection space $GF_n$ onto a symmetric affine connection space $\overline{F}_n$ be geodesic, is*

$$D^i_{\underset{\vee}{jk}}(x,\dot{x}) = -P^{*i}_{\underset{\vee}{jk}}(x,\dot{x}),$$

*where $D^i_{\underset{\vee}{jk}}(x,\dot{x})$, $P^{*i}_{\underset{\vee}{jk}}(x,\dot{x})$, are antisymmetric parts of the deformation tensor and connection coefficients of the $GF_n$ respectively.*

*Remark* 1. It is easy to check that a set of geodesic mappings of a space $GF_n$ makes a group.

## 4. Some projective invariants of geodesic mappings

Putting $D$ into (3.11) in accordance with (3.2) we get

$$\overline{P}^{*i}_{jk} - \overline{P}^{*i}_{\underset{\vee}{jk}} - \frac{1}{N+1}(\delta^i_j \overline{P}^{*p}_{\underline{kp}} + \delta^i_k \overline{P}^{*p}_{\underline{jp}}) = P^{*i}_{jk} - P^{*i}_{\underset{\vee}{jk}} - \frac{1}{N+1}(\delta^i_j P^{*p}_{\underline{kp}} + \delta^i_k P^{*p}_{\underline{jp}}),$$

Denoting

$$\mathcal{T}^i_{jk} = P^{*i}_{\underline{jk}} - \frac{1}{N+1}(\delta^i_j P^{*p}_{\underline{kp}} + \delta^i_k P^{*p}_{\underline{jp}}) = \mathcal{T}^i_{kj}, \tag{4.1}$$

we see that

$$\overline{\mathcal{T}}^i_{jk} = \mathcal{T}^i_{jk}. \tag{4.2}$$

The magnitudes $\mathcal{T}^i_{jk}$ we call **generalized Thomas's projective parameters** at the mapping $f : GF_n \to G\overline{F}_n$. Accordingly, these magnitudes are invariant at a geodesic mapping. Starting from (4.1, 4.2), one obtains (3.11), and we conclude that the next theorem is valid:

**Theorem 13.** *A necessary and sufficient condition that a mapping $f : GF_n \to G\overline{F}_n$ be geodesic is that generalized Thomas's projective parameters (4.1) are invariant, that is (4.2) to be valid.*

**Theorem 14.** *If $\lambda^i_{(h)}(x)$ and $\lambda_{(h)i}(x)$ are the contravariant, respectively covariant components of an orthogonal ennuple, then the following geometric entities are invariant under the geodesic mapping:*

$$\underset{\theta}{\mathcal{M}}{}^i_k(x,\dot{x}) = \lambda^i_{(h)\underset{\theta}{|}}(x) - \frac{1}{n+1}\lambda^p_{(h)}(x)\sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)\underset{\theta}{|}k)},\ \theta = 1,2, \tag{4.3}$$

*and*

$$\underset{\theta}{\mathcal{M}}{}^*{}_k(x,\dot{x}) = \sum_h e_{(h)}\lambda_{(h)p}\lambda^p_{(h)\underset{\theta}{|}k} - D^q_{\underline{kq}},\ \theta = 1,2, \tag{4.4}$$

*where $(p\ldots k)$ denotes symmetrization by respect of indices $p$ and $k$.*

*Proof.* If we denote by $\lambda^i_{(h)\underset{\theta}{|}\bar{k}}$, $\theta = 1,2$ the covariant derivative wrt the connection $\overline{P}^*$ in sense of (1), we have

$$\lambda^i_{(h)\underset{\substack{1\\2}}{|}\bar{k}} = \lambda^i_{(h),k} + \lambda^p_{(h)}\overline{P}^{*i}_{\underset{kp}{pk}}. \tag{4.5}$$

Using (3.8, 4.1) we have

$$\begin{aligned}\lambda^i_{(h)\underset{1}{|}\bar{k}} - \lambda^i_{(h)\underset{1}{|}k} &= \lambda^p_{(h)}(\delta^i_p\psi_k + \delta^i_k\psi_p + \xi^i_{pk})\\ \lambda^i_{(h)\underset{2}{|}\bar{k}} - \lambda^i_{(a)\underset{2}{|}k} &= \lambda^p_{(h)}(\delta^i_k\psi_p + \delta^i_p\psi_k + \xi^i_{kp}).\end{aligned} \tag{4.6}$$

Multiplying (4.6) by $\lambda_{(h)i}$ and summing with respect to indices $h$ and using orthogonality condition, we have

$$\begin{aligned}\sum_h e_{(h)}\lambda_{(h)p}(\lambda^p_{(h)\underset{1}{|}\bar{k}} - \lambda^p_{(h)\underset{1}{|}k}) &= (N+1)\psi_k\\ \sum_h e_{(h)}\lambda_{(h)p}(\lambda^p_{(h)\underset{2}{|}\bar{k}} - \lambda^p_{(h)\underset{2}{|}k}) &= (N+1)\psi_k.\end{aligned} \tag{4.7}$$

Eliminating the $\psi_k$ from (4.6) and (4.7) we get

$$\lambda^i_{(h)\underset{\substack{1\\2}}{|}\bar{k}} - \lambda^i_{(h)\underset{\substack{1\\2}}{|}k} = \frac{1}{n+1}\lambda^p_{(h)}\left(\sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)\underset{\substack{1\\2}}{|}\bar{k})} - \sum_h e_{(h)}\lambda_{(h)q}\delta^q_{(p}\lambda_{(h)\underset{\substack{1\\2}}{|}k)}\right).$$

With help (3.10) we get (4.3). □

**Theorem 15.** *When $GF_n$ and $G\overline{F}_n$ are in geodesic correspondence, we have the following projective invariant entities:*

$$\underset{\theta}{S}_{hkm}(x,\dot{x}) = \underset{\theta}{Y}_{(hkm)} - \frac{1}{n+1}(\delta_{hk}\lambda^k_{(m)}P^{*i}_{ki} + \delta_{km}\lambda^j_{(h)}P^{*i}_{ji}),\ \theta = 1,2,$$

*where* $\underset{\theta}{Y}_{(hkm)}$ $\theta = 1,2$, *are Ricci coefficients of rotation.*

*Proof.* Using (4.5) we get

$$\lambda^{i}_{(h)\underset{\substack{1\\2}}{|}\overline{k}} - \lambda^{i}_{(h)\underset{\substack{1\\2}}{|}k} = \frac{1}{N+1}\lambda^{j}_{(h)}(\delta^{i}_{j} D^{p}_{\underline{kp}}(x,\dot{x}) + \delta^{i}_{k} D^{p}_{\underline{jp}}(x,\dot{x})). \tag{4.8}$$

Multiplying (4.8) by $\lambda_{(k)i}\lambda^{k}_{(m)}$ and using orthogonality condition, we have

$$\begin{aligned}&\underset{\substack{1\\2}}{\overline{Y}}_{hkm} - \frac{1}{n+1}(\delta_{hk}\lambda^{k}_{(m)}\overline{P}^{*i}_{ki} + \delta_{km}\lambda^{q}_{(h)}\overline{P}^{*p}_{qp})\\ &= \underset{\substack{1\\2}}{Y}_{hkm} - \frac{1}{n+1}(\delta_{hk}\lambda^{k}_{(m)}P^{*p}_{kp} + \delta_{km}\lambda^{j}_{(h)}P^{*p}_{jp}),\end{aligned}$$

where the projectively transformed Ricci coefficients of rotation are given by

$$\underset{\theta}{\overline{Y}}_{hkm} = \lambda_{(h)p\underset{\theta}{|}\overline{q}}\lambda^{p}_{(k)}\lambda^{q}_{(m)} = \lambda^{p}_{(h)\underset{\theta}{|}\overline{q}}\lambda_{(k)p}\lambda^{q}_{(m)}, \qquad \theta = 1,2.$$

□

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# THE EXPLICIT FORMULAS FOR REPRODUCING KERNEL OF SOME HILBERT SPACES

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*Abstract.* This paper is devoted to finding explicit formulas for two kinds of reproducing kernels in some reproducing kernel spaces. Usually for calculating the reproducing kernel of space $W_2^m[a,b]$, one must solve a large system of ordinary differential equations with $4m$ equations in $4m$ unknowns. Hence, it is worthy to have explicit formulas to save time and computational efforts. The advantage of using these formulas is clearly seen in the plotted graphs.

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## 1. Introduction

In recent decades, the reproducing kernel Hilbert space method has been considered by many authors. Reproducing kernel Hilbert space method is an effective method for solving many problems, including ordinary, partial, integro-differential, integral equations, etc. [1–25].

To obtain reproducing kernel we have to perform complex calculations at high volumes. So we decided to introduce, in this paper, explicit formulas to calculate unknown coefficients of the reproducing kernel. That will save time and reduce the high cost of computing. First, we divide reproducing kernel spaces in terms of the types of problems, different initial and boundary conditions, the inner product, and the necessary norm on the environment. The first class (i.e. $W_2^m[a,b]$) is intended for problems without initial and boundary conditions and the second class (i.e. $W_{2,a}^{n+1}[a,b]$) is devoted to the problems with the initial conditions. The findings of this article for readers, especially for users of reproducing kernel method, is helpful in time saving.

The structure of this paper is as follows. In Section 2 we introduce some reproducing kernel spaces. We devote Section 3 to finding explicit formulas for reproducing kernels. We end the paper with conclusions.

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## 2. Reproducing kernel Hilbert spaces

In this section, we first define the reproducing kernel and reproducing kernel space, and then we introduce some reproducing kernel spaces.

**Definition 1** ([1,4])**.** Let

$$\mathcal{H} = \{f \mid f \text{ is a real valued function on } \mathcal{X},\ \mathcal{X} \text{ is an abstract set}\}$$

be a Hilbert space, with inner product

$$\langle f(.), g(.)\rangle_{\mathcal{H}}, \quad f,\ g \in \mathcal{H}.$$

If there exists a function $K(x,.) \in \mathcal{H}$ for each fixed $x \in \mathcal{X}$ and for any $f \in \mathcal{H}$

$$\langle f(.), K(x,.)\rangle_{\mathcal{H}} = f(x),$$

then $K(x,.)$ is called the reproducing kernel of $\mathcal{H}$ and Hilbert space $\mathcal{H}$ is called the reproducing kernel space.

For solving some functional equations such as integral and differential equations using the reproducing kernel Hilbert space method, we use the reproducing kernel space $W_2^m[a,b]$ defined as follows.

**Definition 2** ([4])**.** For $m \in \mathbb{N}$,

$$W_2^m[a,b] = \{u \mid u^{(m-1)} \text{ is an absolutely continuous function}, u^{(m)} \in L^2[a,b]\}.$$

The inner product and the norm in the function space $W_2^m[a,b]$ are defined as

$$\langle u,v\rangle_m = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(\tau)v^{(m)}(\tau)\,d\tau, \quad u,v \in W_2^m[a,b]$$

$$||u||_m = \sqrt{\langle u,u\rangle_m}, \quad u \in W_2^m[a,b].$$

To solve initial value problems including ordinary differential equations, integro-differential equations and differential equations of fractional order, we define the reproducing kernel space $W_{2,a}^{n+1}[a,b]$ as follows.

**Definition 3.** For $n \in \mathbb{N}$,

$$W_{2,a}^{n+1}[a,b] = \{u \in W_2^{n+1}[a,b] \mid u^{(j)}(a) = 0,\ j = 0,\dots,n-1\}.$$

The inner product and the norm in the function space $W_{2,a}^{n+1}[a,b]$ are defined as

$$\langle u,v\rangle_{n,a} = u^{(n)}(a)v^{(n)}(a) + \int_a^b u^{(n+1)}(\tau)v^{(n+1)}(\tau)\,d\tau, \quad u,v \in W_{2,a}^{n+1}[a,b]$$

$$||u||_{n,a} = \sqrt{\langle u,u\rangle_{n,a}}, \quad u \in W_{2,a}^{n+1}[a,b].$$

## 3. Explicit formulas for reproducing kernels

In this section we investigate explicit formula for each one of the reproducing kernel Hilbert spaces introduced in Definitions 2 and 3.

### 3.1. *Reproducing kernel of the space $W_2^m[a,b]$*

Assume that function $K^{\{m\}}(x,t) \in W_2^m[a,b]$ satisfies the following generalized differential equation

$$\begin{cases} (-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = \delta(t-x), \\ \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=a} = (-1)^{m-j-1} \frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}|_{t=a}, \\ \frac{\partial^{m+j} K^{\{m\}}(x,t)}{\partial t^{m+j}}|_{t=b} = 0, \ 0 \le j < m. \end{cases} \tag{3.1}$$

where $\delta$ is the Dirac delta function. The following theorem holds.

**Theorem 1** ([4]). *Under the assumptions of Eq. (3.1), Hilbert space $W_2^m[a,b]$ is a reproducing kernel Hilbert space with the reproducing kernel function $K^{\{m\}}(x,.)$, namely for each $u \in W_2^m[a,b]$ and any fixed $x \in [a,b]$, it follows that*

$$\langle u(.), K^{\{m\}}(x,.)\rangle_m = u(x).$$

While $x \neq t$, function $K^{\{m\}}(x,t)$ is the solution of the following linear homogeneous differential equation of order $2m$,

$$(-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = 0, \tag{3.2}$$

with the boundary conditions:

$$\begin{cases} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=a} = (-1)^{m-j-1} \frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}|_{t=a}, \\ \frac{\partial^{m+j} K^{\{m\}}(x,t)}{\partial t^{m+j}}|_{t=b} = 0, \ 0 \le j < m. \end{cases} \tag{3.3}$$

We know that Eq. (3.2) has characteristic equation $\lambda^{2m} = 0$, and the eigenvalue $\lambda = 0$ is a root with multiplicity $2m$. Hence, the general solution of the system (3.1) is

$$K^{\{m\}}(x,t) = \begin{cases} \sum_{i=1}^{2m} c_i(x)(t-a)^{i-1}, & t \le x, \\ \sum_{i=1}^{2m} d_i(x)(t-a)^{i-1}, & t > x. \end{cases} \tag{3.4}$$

Now, we are ready to calculate the coefficients $c_i(x)$ and $d_i(x)$, $i = 1,\dots,2m$. Since

$$(-1)^m \frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = \delta(t-x),$$

we have

$$\begin{cases} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^-}^{t=x^+} = 0, & j = 0,\dots,2m-2, \\ \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^-}^{t=x^+} = (-1)^m, & j = 2m-1. \end{cases} \tag{3.5}$$

Therefore, using Eqs. (3.1) and (3.5), we have a system of $4m$ equations with $4m$ unknowns ($c_i(x)$ and $d_i(x)$, $i = 1,\dots,2m$). By solving this system the unknown coefficients of Eq. (3.4) are uniquely determined. Solving the above system needs high CPU time.

We have obtained the following formula for calculating the reproducing kernel of $W_2^m[a,b]$.

**Theorem 2.** *Let*

$$K^{\{m\}}(x,t) = \begin{cases} R^{\{m\}}(x,t), & t \le x, \\ L^{\{m\}}(x,t), & t > x, \end{cases}$$

*be the reproducing kernel function of the reproducing kernel Hilbert space $W_2^m[a,b]$. Then, $R^{\{m\}}(x,t)$ is given by the following formula*

$$R^{\{m\}}(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{1}\left((-1)^{ij}\frac{(x-a)^{i+(m-2i-1)j}(t-a)^{mj+i}}{(i+(m-2i-1)j)!(mj+i)!}\right),$$

*and $L^{\{m\}}(x,t) = R^{\{m\}}(t,x)$.*

*Proof of Theorem 2.* It is enough to prove that the function $K^{\{m\}}(x,t)$ satisfies Eqs. (3.2), (3.3) and (3.5). For this purpose we first rewrite $R^{\{m\}}(x,t)$ as

$$R^{\{m\}}(x,t) = \sum_{i=0}^{m-1}\frac{(x-a)^i}{i!}\frac{(t-a)^i}{i!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^i}{i!}.$$

For $j = 0,\dots,2m$, the $j^{th}$-derivative of the function $R^{\{m\}}(x,t)$ with respect to $t$ is computed as

$$\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j} = \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^i}{i!}\frac{(t-a)^{i-j}}{(i-j)!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < m, \\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & m \le j < 2m, \\ 0, & j = 2m. \end{cases} \tag{3.6}$$

Hence,

$$\left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} = \begin{cases} \frac{(x-a)^j}{j!}, & j = 0,\dots,m-1, \\ (-1)^{j-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!}, & j = m,\dots,2m-1, \end{cases} \tag{3.7}$$

$$\left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=x} = \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^{2i-j}}{i!(i-j)!} + \sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & 0 \le j < m, \\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & m \le < 2m. \end{cases} \tag{3.8}$$

Also, we rewrite $L^{\{m\}}(x,t)$ as

$$L^{\{m\}}(x,t) = R^{\{m\}}(t,x) = \sum_{i=0}^{m-1}\left[\frac{(x-a)^i}{i!} + (-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right]\frac{(t-a)^i}{i!}.$$

Furthermore, for $j = 0,\dots,2m$, the $j^{th}$-derivative of the function $L^{\{m\}}(x,t)$ with respect to $t$ is

$$\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^i}{i!} + (-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right)\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < m, \\ 0, & m \le j \le 2m. \end{cases} \tag{3.9}$$

Therefore,

$$\left.\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}\right|_{t=b} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^i}{i!}+(-1)^{m-i-1}\frac{(x-a)^{2m-i-1}}{(2m-i-1)!}\right)\frac{(b-a)^{i-j}}{(i-j)!}, & 0\le j<m,\\ 0, & m\le j<2m. \end{cases} \tag{3.10}$$

$$\left.\frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}\right|_{t=x} = \begin{cases} \sum_{i=j}^{m-1}\left(\frac{(x-a)^{2i-j}}{i!(i-j)!}+(-1)^{m-i-1}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}\right), & 0\le j<m,\\ 0, & m\le j<2m. \end{cases} \tag{3.11}$$

Thus, by Eqs. (3.6) and (3.9), $\frac{\partial^{2m} K^{\{m\}}(x,t)}{\partial t^{2m}} = 0$. Also, Eq. (3.7) implies

$$\begin{aligned} &\left.\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} - (-1)^{m-j-1}\left.\frac{\partial^{2m-j-1} K^{\{m\}}(x,t)}{\partial t^{2m-j-1}}\right|_{t=a}\\ &\quad= \left.\frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}\right|_{t=a} - (-1)^{m-j-1}\left.\frac{\partial^{2m-j-1} R^{\{m\}}(x,t)}{\partial t^{2m-j-1}}\right|_{t=a}\\ &\quad= \frac{(x-a)^j}{j!} - (-1)^{m-j-1}(-1)^{(2m-j-1)-m}\frac{(x-a)^{2m-(2m-j-1)-1}}{(2m-(2m-j-1)-1)!}\\ &\quad= \frac{(x-a)^j}{j!} - (-1)^{m-j-1}(-1)^{m-j-1}\frac{(x-a)^j}{j!} = 0, \quad j=0,\dots,m-1, \end{aligned} \tag{3.12}$$

and Eq. (3.10) implies

$$\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=b} = \frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}|_{t=b} = 0, j=m,\dots,2m-1. \tag{3.13}$$

According to Eqs. (3.12) and (3.13), Eq. (3.3) is proved. Also, using Eqs. (3.8) and (3.11) we have

$$\begin{aligned} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=x^+} &= \frac{\partial^j L^{\{m\}}(x,t)}{\partial t^j}|_{t=x}\\ &= \begin{cases} 0, & m\le j<2m,\\ \sum_{i=j}^{m-1}\left(\frac{(x-a)^{2i-j}}{i!(i-j)!}+(-1)^{m-i-1}\frac{(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}\right), & 0\le j<m, \end{cases} \end{aligned} \tag{3.14}$$

$$\begin{aligned} \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}|_{t=x^-} &= \frac{\partial^j R^{\{m\}}(x,t)}{\partial t^j}|_{t=x}\\ &= \begin{cases} \sum_{i=j}^{m-1}\frac{(x-a)^{2i-j}}{i!(i-j)!}+\sum_{i=m}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!(i-j)!}, & 0\le j<m,\\ \sum_{i=j}^{2m-1}(-1)^{i-m}\frac{(x-a)^{2m-j-1}}{(2m-j-1)!(i-j)!}, & m\le j<2m. \end{cases} \end{aligned} \tag{3.15}$$

TABLE 1. Explicit expressions of $R^{\{m\}}(x,t)$ for $m = 1,\dots,5$

| $m$ | $R^{\{m\}}(x,t)$ |
|---|---|
| 1 | $1-a+t$ |
| 2 | $1+(x-a)(t-a)+(x-a)(t-a)^2/2-(t-a)^3/6$ |
| 3 | $1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^2(t-a)^3/12$ $-(x-a)(t-a)^4/24+(t-a)^5/120$ |
| 4 | $1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^3(t-a)^3/36+(x-a)^3(t-a)^4/144$ $-(x-a)^2(t-a)^5/240+(x-a)(t-a)^6/720-(t-a)^7/5040$ |
| 5 | $1+(x-a)(t-a)+(x-a)^2(t-a)^2/4+(x-a)^3(t-a)^3/36+(x-a)^4(t-a)^4/576$ $+(x-a)^4(t-a)^5/2880-(x-a)^3(t-a)^6/4320+(x-a)^2(t-a)^7/10080$ $-(x-a)(t-a)^8/40320+(t-a)^9/362880$ |

So, Eqs. (3.14) and (3.15) imply that

$$\begin{aligned}
&\frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^+} - \frac{\partial^j K^{\{m\}}(x,t)}{\partial t^j}\Big|_{t=x^-} \\
&= \begin{cases} \sum_{i=j}^{m-1} \frac{(-1)^{m+i-1}(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!} + \sum_{i=m}^{2m-1} \frac{(-1)^{m+i-1}(x-a)^{2m-j-1}}{(2m-i-1)!(i-j)!}, & 0 \le j < m, \\ (x-a)^{2m-j-1} \sum_{i=j}^{2m-1} \frac{(-1)^{m+i-1}}{(2m-i-1)!(i-j)!}, & m \le j < 2m-1, \\ (-1)^m, & j = 2m-1. \end{cases} \\
&= \begin{cases} (-1)^{m-1}(x-a)^{2m-j-1} \sum_{i=j}^{2m-1} \frac{(-1)^i}{(2m-i-1)!(i-j)!}, & 0 \le j \le 2m-2, \\ (-1)^m, & j = 2m-1. \end{cases} \\
&= \begin{cases} \frac{(-1)^{m+j-1}(x-a)^{2m-j-1}}{(2m-j-1)!} \sum_{i=0}^{2m-j-1} (-1)^i \binom{2m-j-1}{i}, & 0 \le j \le 2m-2, \\ (-1)^m, & j = 2m-1. \end{cases} \\
&= \begin{cases} (-1)^{m+j-1}(x-a)^{2m-j-1}(1-1)^{2m-j-1}, & 0 \le j \le 2m-2, \\ (-1)^m, & j = 2m-1. \end{cases} \\
&= \begin{cases} 0, & 0 \le j \le 2m-2, \\ (-1)^m, & j = 2m-1. \end{cases}
\end{aligned}$$

□

Explicit expressions of the reproducing kernel functions in $W_2^m[a,b]$ for $1 \le m \le 5$ are shown in Table 1.

### 3.2. *Reproducing kernel of the space* $W_{2,a}^{n+1}[a,b]$

Suppose that function $K_a^{\{n+1\}}(x,t) \in W_{2,a}^{n+1}[a,b]$ satisfies the following generalized differential equations

$$\begin{cases} (-1)^{n+1} \frac{\partial^{2n+2} K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = \delta(t-x), \\ \frac{\partial^{n+1} K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}\Big|_{t=a} = \frac{\partial^n K^{\{n+1\}}(x,t)}{\partial t^n}\Big|_{t=a}, & 0 \le j \le n, \\ \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=a} = 0, & 0 \le j \le n, \\ \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=b} = 0, & n < j < 2n, \end{cases} \tag{3.16}$$

where $\delta$ is the Dirac delta function. Similar to Theorem 1, we have the following theorem.

**Theorem 3.** *Under the assumptions of Eq. (3.16), Hilbert space $W_{2,a}^{n+1}[a,b]$ is a reproducing kernel Hilbert space with the reproducing kernel function $K_a^{\{n+1\}}(x,.)$, namely for each $u \in W_{2,a}^{n+1}[a,b]$ and any fixed $x \in [a,b]$, it follows that*

$$\langle u(.), K_a^{\{n+1\}}(x,.)\rangle_{n,a} = u(x).$$

While $x \neq t$, function $K_a^{\{n+1\}}(x,t)$ is the solution of the following linear homogeneous differential equation of order $2n+2$,

$$(-1)^{n+1}\frac{\partial^{2n+2}K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = 0, \tag{3.17}$$

with the boundary conditions:

$$\begin{cases} \left.\frac{\partial^{n+1}K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}\right|_{t=a} = \left.\frac{\partial^{n}K^{\{n+1\}}(x,t)}{\partial t^{n}}\right|_{t=a}, & 0 \le j \le n, \\ \left.\frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\right|_{t=a} = 0, & 0 \le j \le n, \\ \left.\frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\right|_{t=b} = 0, & n < j < 2n. \end{cases} \tag{3.18}$$

We know that Eq. (3.17) has characteristic equation $\lambda^{2n+2} = 0$, and the eigenvalue $\lambda = 0$ is a root with multiplicity $2n+2$. Hence, the general solution of the system (3.16) is

$$K_a^{\{n+1\}}(x,t) = \begin{cases} \sum_{i=1}^{2n+2}\alpha_i(x)(t-a)^{i-1}, & t \le x, \\ \sum_{i=1}^{2n+2}\beta_i(x)(t-a)^{i-1}, & t > x. \end{cases} \tag{3.19}$$

Now, we are ready to calculate the coefficients $\alpha_i(x)$ and $\beta_i(x)$, $i = 1,\dots,2n+2$. Since

$$(-1)^{n+1}\frac{\partial^{2n+2}K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = \delta(t-x),$$

we have

$$\begin{cases} \left.\frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\right|_{t=x^-}^{t=x^+} = 0, & j = 0,\dots,2n, \\ \left.\frac{\partial^{j}K_a^{\{n+1\}}(x,t)}{\partial t^{j}}\right|_{t=x^-}^{t=x^+} = (-1)^{n+1}, & j = 2n+1. \end{cases} \tag{3.20}$$

Therefore, using Eqs. (3.16) and (3.20), we have a system of $4n+4$ equations with $4n+4$ unknowns ($\alpha_i(x)$ and $\beta_i(x)$, $i = 1,\dots,2n+2$). By solving this system the unknown coefficients of Eq. (3.19) are uniquely obtained. Solving the above system needs high CPU time.

Now, we present the following formula for calculating the reproducing kernel of $W_{2,a}^{n+1}[a,b]$.

**Theorem 4.** *Let*

$$K_a^{\{n+1\}}(x,t) = \begin{cases} R_a^{\{n+1\}}(x,t), & t \le x, \\ L_a^{\{n+1\}}(x,t), & t > x, \end{cases}$$

*be the reproducing kernel function of the reproducing kernel Hilbert space* $W_{2,a}^{n+1}[a,b]$. *Then,* $R_a^{\{n+1\}}(x,t)$ *is given by the following formula.*

$$R_a^{\{n+1\}}(x,t) = \frac{(x-a)^n(t-a)^n}{n!n!} + \sum_{i=n+1}^{2n+1}\left((-1)^{i-n-1}\frac{(x-a)^{2n-i+1}(t-a)^i}{(2n-i+1)!i!}\right),$$

*and* $L_a^{\{n+1\}}(x,t) = R_a^{\{n+1\}}(t,x)$.

*Proof of Theorem 4.* It is enough to prove that the function $K_a^{\{n+1\}}(x,t)$ satisfies in Eqs. (3.17), (3.18) and (3.20). For $j = 0,\dots,2n+2$, the $j^{th}$-derivative of the function $R_a^{\{n+1\}}(x,t)$ with respect to $t$ is computed as follows:

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(t-a)^{n-j}}{(n-j)!} + \\ \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & n < j < 2n+2, \\ 0, & j = 2n+2. \end{cases} \tag{3.21}$$

Then,

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=a} = \begin{cases} \frac{(x-a)^j}{j!}, & j = n, \\ (-1)^{j-n-1}\frac{(x-a)^{2n-j+1}}{(2n-j+1)!}, & n < j < 2n, \end{cases} \tag{3.22}$$

$$\frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=x} = \begin{cases} \frac{(x-a)^{2n-j}}{n!(n-j)!} + \\ \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n, \\ \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & n < j < 2n. \end{cases} \tag{3.23}$$

For $L_a^{\{n+1\}}(x,t)$ we have

$$L_a^{\{n+1\}}(x,t) = R_a^{\{n+1\}}(t,x) = \frac{(x-a)^n}{n!}\frac{(t-a)^n}{n!} + \sum_{i=0}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^i}{i!}.$$

Furthermore, for $j = 0,\dots,2n+2$, the $j^{th}$-derivative of the function $L_a^{\{n+1\}}(x,t)$ with respect to $t$ is computed as follows:

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(t-a)^{n-j}}{(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(t-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2. \end{cases} \tag{3.24}$$

Therefore,

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}\Big|_{t=b} = \begin{cases} \frac{(x-a)^n}{n!}\frac{(b-a)^{n-j}}{(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-i+1}}{(2n-i+1)!}\frac{(b-a)^{i-j}}{(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2, \end{cases} \tag{3.25}$$

$$\frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} = \begin{cases} \frac{(x-a)^{2n-j}}{n!(n-j)!} + \\ \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j < n+1, \\ 0, & n < j \le 2n+2. \end{cases} \tag{3.26}$$

Thus, Eqs. (3.21) and (3.24), it is clear that $\frac{\partial^{2n+2} K_a^{\{n+1\}}(x,t)}{\partial t^{2n+2}} = 0$. Also, Eq. (3.22) implies that

$$\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=a} = \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=a} = 0, \quad j = 0,\dots,n-1,$$

$$\begin{aligned} &\frac{\partial^{n+1} K_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}|_{t=a} - \frac{\partial^n K_a^{\{n+1\}}(x,t)}{\partial t^n}|_{t=a} = \\ &\frac{\partial^{n+1} L_a^{\{n+1\}}(x,t)}{\partial t^{n+1}}|_{t=a} - \frac{\partial^n L_a^{\{n+1\}}(x,t)}{\partial t^n}|_{t=a} = \frac{(x-a)^n}{n!} - \frac{(x-a)^n}{n!} = 0, \end{aligned} \tag{3.27}$$

and Eq. (3.25) implies that

$$\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=b} = \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=b} = 0, n < j < 2n. \tag{3.28}$$

According to Eqs. (3.27) and (3.28), Eq. (3.18) is proved. Also, using Eqs. (3.23) and (3.26) we have

$$\begin{aligned} \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^+} &= \frac{\partial^j L_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} 0, & n < j < 2n, \\ \frac{(x-a)^{2n-j}}{n!(n-j)!} + \sum_{i=j}^{n}(-1)^{n-i}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n, \end{cases} \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^-} &= \frac{\partial^j R_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x} \\ &= \begin{cases} \sum_{i=j}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & n < j < 2n, \\ \frac{(x-a)^{2n-j}}{n!(n-j)!} + \sum_{i=n+1}^{2n+1}(-1)^{i-n-1}\frac{(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j \le n. \end{cases} \end{aligned} \tag{3.30}$$

So, Eqs. (3.29) and (3.30) imply that

$$\begin{aligned} &\frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^+} - \frac{\partial^j K_a^{\{n+1\}}(x,t)}{\partial t^j}|_{t=x^-} \\ &= \begin{cases} \sum_{i=j}^{n}\frac{(-1)^{n+i}(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!} + \sum_{i=n+1}^{2n+1}\frac{(-1)^{n+i}(x-a)^{2n-j+1}}{(2n-i+1)!(i-j)!}, & 0 \le j < n+1, \\ (x-a)^{2n-j+1}\sum_{i=j}^{2n+1}\frac{(-1)^{n+i}}{(2n-i+1)!(i-j)!}, & n < j < 2n+1, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} (-1)^n(x-a)^{2n-j+1}\sum_{i=j}^{2n+1}\frac{(-1)^i}{(2n-i+1)!(i-j)!}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} \frac{(-1)^{n+j}(x-a)^{2n-j+1}}{(2n-j+1)!}\sum_{i=0}^{2n-j+1}(-1)^i\binom{2n-j+1}{i}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} (-1)^{n+j}(x-a)^{2n-j+1}(1-1)^{2n-j+1}, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \\ &= \begin{cases} 0, & 0 \le j \le 2n, \\ (-1)^{n+1}, & j = 2n+1. \end{cases} \end{aligned}$$

TABLE 2. Explicit expressions of $R_a^{\{n+1\}}(x,t)$ for $n = 1,\ldots,4$

| $n$ | $R_a^{\{n+1\}}(x,t)$ |
|---|---|
| 1 | $(x-a)(t-a)+(x-a)(t-a)^2/2-(t-a)^3/6$ |
| 2 | $(x-a)^2(t-a)^2/4+(x-a)^2(t-a)^3/12-(x-a)(t-a)^4/24+(t-a)^5/120$ |
| 3 | $(x-a)^3(t-a)^3/36+(x-a)^3(t-a)^4/144-(x-a)^2(t-a)^5/240$ $+(x-a)(t-a)^6/720-(t-a)^7/5040$ |
| 4 | $(x-a)^4(t-a)^4/576+(x-a)^4(t-a)^5/2880-(x-a)^3(t-a)^6/4320$ $+(x-a)^2(t-a)^7/10080-(x-a)(t-a)^8/40320+(t-a)^9/362880$ |

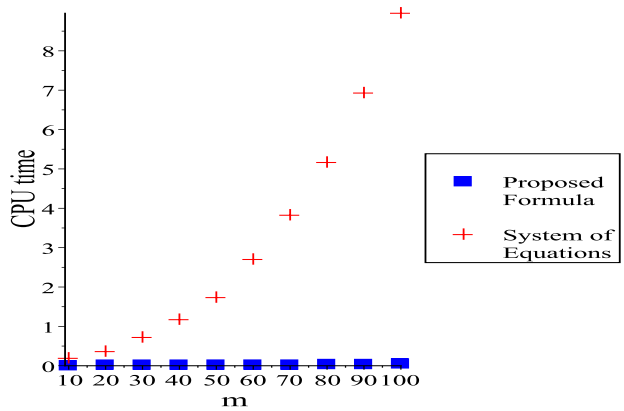


FIGURE 1. Comparing CPU time to compute the reproducing kernel for space $W_2^m[a,b]$

□

Explicit expressions of reproducing kernel functions in $W_{2,a}^{n+1}[a,b]$ for $n = 1,\ldots,4$, are shown in Table 2.

*Remark* 1. For some values of $m$ (or $n$), we have calculated the reproducing kernel, using two methods: the proposed explicit formula, and also by solving the system of $4m$ (or $4n+4$) equations in $4m$ (or $4n+4$) unknowns. Figures 1 and 2, represent the advantage of the explicit formulas.

## 4. CONCLUSIONS

In this paper we examined some reproducing kernel spaces and we obtained explicit formulas for the reproducing kernel of these spaces. Usually finding the reproducing kernel for solving large systems of differential equations is required (especially for high-order differential equations). For an $n$th-order differential equation, we must solve a system of $4(n+1)$ equations in $4(n+1)$ unknowns which is very time-consuming and therefore an explicit formula for these reproducing kernels is

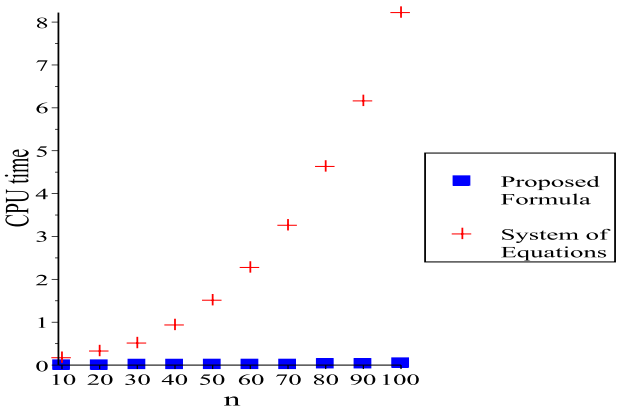


FIGURE 2. Comparing CPU time to compute the reproducing kernel for space $W_2^{n+1}[a,b]$

very useful (see Figures 1 and 2).

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# BEST PROXIMITY POINTS FOR GENERALIZED MULTIVALUED CONTRACTIONS IN METRIC SPACES

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*Abstract.* In the present paper, we prove a best proximity point theorem for multivalued non-self-contractive type mappings which is a generalization of recent best proximity point theorems and some famous fixed point theorems.

2010 *Mathematics Subject Classification:* 47H10; 47H06

*Keywords:* best proximity point, fixed point, optimal approximate solution, multivalued mapping

## 1. Introduction

Let $A$, $B$ be nonempty subsets of a metric space $(X,d)$ and $T : A \to B$ be a non-self-mapping. Clearly, the set of fixed points of $T$ can be empty. Therefore, it is of primary importance to seek an element $x$ that in some sense is closest to $Tx$. That is, if there is no solution to the fixed point equation $Tx = x$, one tries to determine an approximate solution $x$ subject to the condition that the distance between $x$ and $Tx$ is minimal. A classical best approximation theorem was introduced by Fan [4]. It states that if $A$ is a non-empty compact convex subset of a Hausdorff locally convex topological vector space $X$ and $T : A \to X$ is a continuous mapping, Then there exists $x \in A$ such that $d(x,Tx) = d(Tx,A)$. Recently, there have been many subsequent extensions of Fan's theorem, see [7, 8, 12] and references therein. A point $x \in A$ is called a best proximity point for $T$ if distance of $x$ to $Tx$ is equal to the distance of $A$ to $B$. In fact best proximity point theorems have been studied to find necessary conditions such that the minimization problem,

$$\min_{x\in A} d(x,Tx) \tag{1.1}$$

has at least one solution. Investigation of several variants of contractions for the existence of a best proximity point can be found in [2, 3, 5, 9–11, 13, 14].

In this article, we consider a classes of multivalued non-self-mapping which called $(\phi,\theta)$ contractive mappings and we present some best proximity point theorems for these classes of non-self-mappings in metric spaces.

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Let $A$ and $B$ be two nonempty subsets of a metric space. We will use the following notations:

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x,y) = d(A,B) \quad \text{for some} \quad y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \quad \text{for some} \quad y \in A\},$$
$$D(x,B) = \inf\{d(x,y) : y \in B\}, \quad \forall x \in X,$$
$$H(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\}.$$

Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. Assume that $T : A \to 2^B$ is a multivalued non-self-mapping. A point $x \in A$ is said to be a fixed point of $T$ if $x \in Tx$. In case $A \cap B = \varnothing$, the multifunction $T$ has not fixed point. Then $D(x,Tx) > 0$ for all $x \in A$. Therefore, we can explore to find necessary conditions so that the minimization problem

$$\min_{x\in A} D(x,Tx) \tag{1.2}$$

has at least one solution. Since $D(x,Tx) \geq d(A,B)$ for all $x \in A$, the optimal solution to the problem (1.2) is obtained in some points of $A$ for which the value $d(A,B)$ is attained. A point $x \in A$ is called a best proximity point of a multivalued non-self-mapping $T$, if $D(x,Tx) = d(A,B)$. We note that if $d(A,B) = 0$, then we get a fixed point of $T$.

**Definition 1** ([11])**.** Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \varnothing$. Then the pair $(A,B)$ is said to have the $P$-property iff

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \Rightarrow d(x_1,x_2) = d(y_1,y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$

**Definition 2** ([15])**.** Let $(A,B)$ be a pair of nonempty subsets of a metric space $(X,d)$ with $A_0 \neq \varnothing$. Then the pair $(A,B)$ is said to have the weak $P$-property iff

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \Rightarrow d(x_1,x_2) \leq d(y_1,y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

**Definition 3.** We say that $\varphi : [0,\infty[ \to [0,\infty[$ is a (c)-comparison function if and only if the following conditions hold:
(i) $\varphi$ is a nondecreasing function,
(ii) for any $t > 0$, $\sum_{n=0}^{\infty} \varphi^n(t)$ is a convergent series.

In what follows, we will denote:

$$\Theta = \{\theta : [0,+\infty)^4 \to [0,+\infty) : \\ \theta \text{ is continuous and } \theta(t_1,t_2,t_3,t_4) = 0 \Leftrightarrow t_1t_2t_3t_4 = 0\}.$$

*Example* 1. The following functions belong to $\Theta$ :
(1) $\theta(t_1,t_2,t_3,t_4) = L\min\{t_1,t_2,t_3,t_4\}, L > 0$
(2) $\theta(t_1,t_2,t_3,t_4) = t_1t_2t_3t_4$,
(3) $\theta(t_1,t_2,t_3,t_4) = \ln(1+t_1t_2t_3t_4)$,
(4) $\theta(t_1,t_2,t_3,t_4) = \exp(t_1t_2t_3t_4) - 1$.

The notion of almost $(\varphi,\theta)$-contraction for single valued non-self mapping was introduced by Bessem Samet as follows.

**Definition 4** ([10]). A mapping $T : A \to B$ is said to be an almost $(\varphi,\theta)$-contraction if and only if there exist $\varphi \in \Phi$ and $\theta \in \Theta$ such that, for all $x, y \in A$,

$$d(Tx,Ty) \le \varphi\Big(d(x,y)\Big) + \theta\Big(d(y,Tx) - d(A,B), d(x,Ty) \\ - d(A,B), d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\Big)$$

He proved the following result.

**Theorem 1** ([10]). *Let $A$ and $B$ be closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty. Suppose that $T : A \to B$ satisfies the following conditions:*
*(i) $T$ is an almost $(\varphi,\theta)$-contraction,*
*(ii) $T(A_0) \subseteq B_0$,*
*(iii) the pair $(A,B)$ has the $P$-property.*

*Then, there exists a unique element $x^* \in A$ such that*

$$d(x^*,Tx^*) = d(A,B)$$

*Moreover, for any fixed element $x_0 \in A_0$, any iterative sequence $\{x_n\}$ satisfying*

$$d(x_{n+1},Tx_n) = d(A,B)$$

*converges to $x^*$.*

Now, in the following we defined the notion of $(\varphi,\theta)$- contraction for multivalued mappings.

**Definition 5.** A mapping $T : A \to 2^B$ is said to be an almost $(\varphi, \theta)$-contraction if and only if there exist $\varphi \in \Phi$ and $\theta \in \Theta$ such that, for all $x, y \in A$,

$$H(Tx,Ty) \leq \varphi\Big(d(x,y)\Big) + \theta\Big(D(y,Tx) - d(A,B), D(x,Ty) - d(A,B), D(x,Tx) - d(A,B), D(y,Ty) - d(A,B)\Big)$$

## 2. Main Results

Our first main result is the following theorem.

**Theorem 2.** *Let $A$ and $B$ be closed subsets of a complete metric space $(X,d)$ such that $A_0 \neq \varnothing$ and the pair $(A,B)$ satisfies the weak P-property. Suppose that $T : A \to 2^B$ be a multi-valued almost $(\varphi,\theta)$-contraction non-self mapping. If $T(x)$ is bounded and closed in $B$ for all $x \in A$, and $T(x_0) \subseteq B_0$ for each $x_0 \in A_0$, then $T$ has a best proximity point in $A$.*

*Proof.* Select $x_0 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$. By the definition of the set $B_0$, we can fined an element $x_1$ in $A_0$ such that $d(x_1,y_0) = d(A,B)$. If $y_0 \in Tx_1$, then $d(A,B) \leq D(x_1,Tx_1) \leq d(x_1,y_0) = d(A,B)$, therefore $D(x_1,Tx_1) = d(A,B)$ and $x_1$ is a best proximity point of $T$. If $y_0 \notin Tx_1$ and $q > 1$ be given. Then

$$0 < d(y_0,Tx_1) \leq H(Tx_0,Tx_1) < qH(Tx_0,Tx_1).$$

Hence, there exists $y_1 \in Tx_1$ such that

$$0 < d(y_0,y_1) < qH(Tx_0,Tx_1) \leq q\varphi\Big(d(x_0,x_1)\Big) + q\theta\Big(D(x_1,Tx_0) - d(A,B), D(x_0,Tx_1) - d(A,B), D(x_0,Tx_0) - d(A,B), D(x_1,Tx_1) - d(A,B)\Big)$$

Since $D(x_1,Tx_0) = d(A,B)$, we have

$$\begin{aligned} 0 < d(y_0,y_1) &< q\varphi\Big(d(x_0,x_1)\Big) + q\theta\Big(0, D(x_0,Tx_1) - d(A,B), \\ &\qquad D(x_0,Tx_0) - d(A,B), D(x_1,Tx_1) - d(A,B)\Big) \qquad (2.1) \\ &= q\varphi\Big(d(x_0,x_1)\Big). \end{aligned}$$

One the other hand since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2,y_1) = d(A,B)$. By using the weak P-property of $(A,B)$ we obtain $d(x_2,x_1) \leq d(y_0,y_1)$. Now, put $t_0 = d(x_0,x_1)$, then $t_0 > 0$ and by (2.1) we have $d(x_1,x_2) < q\varphi(t_0)$. Since $\varphi$ is strictly increasing, $\varphi\Big(d(x_1,x_2)\Big) < \varphi\Big(q\varphi(t_0)\Big)$. Set $q_1 = \dfrac{\varphi\Big(q\varphi(t_0)\Big)}{\varphi\Big(d(x_1,x_2)\Big)} > 1$. If $y_1 \in Tx_2$ then $x_2$ is a best proximity point of $T$. suppose that $y_1 \notin Tx_2$, then

$$0 < d(y_1,Tx_2) \leq H(Tx_1,Tx_2) < qH(Tx_1,Tx_2).$$

Therefore, there exits $y_2 \in Tx_2$ such that

$$\begin{aligned} 0 < d(y_2, y_1) &< q_1 H(Tx_2, Tx_1) \\ &\leq q_1\varphi\Big(d(x_1, x_2)\Big) + q_1\theta\Big(D(x_2, Tx_1) - d(A, B), D(x_1, Tx_2) \\ &\quad - d(A, B), D(x_1, Tx_1) - d(A, B), D(x_2, Tx_2) - d(A, B)\Big) \end{aligned}$$

Since $D(x_2, Tx_1) = d(A, B)$, we have

$$\begin{aligned} 0 < d(y_2, y_1) &< q_1\varphi\Big(d(x_1, x_2)\Big) + q_1\theta\Big(0, D(x_1, Tx_2) - d(A, B), D(x_1, Tx_1) \\ &\quad - d(A, B), D(x_2, Tx_2) - d(A, B)\Big) \\ &= q_1\varphi\Big(d(x_1, x_2)\Big) = \varphi\Big(q\varphi(t_0)\Big). \end{aligned} \tag{2.2}$$

Again, since $y_2 \in Tx_2 \subseteq B_0$, there exist $x_3 \in A_0$ such that $d(x_3, y_2) = d(A, B)$. By using the weak P-property of $(A, B)$ we obtain $d(x_3, x_2) \leq d(y_2, y_1)$. Since $\varphi$ is in strictly increasing by using (2.2) we have $\varphi\Big(d(x_3, x_2)\Big) < \varphi^2(q\varphi(t_0))$. Set $q_2 = \dfrac{\varphi^2\Big(q\varphi(t_0)\Big)}{\varphi\Big(d(x_3,x_2)\Big)} > 1$. If $y_2 \in Tx_3$ then $x_3$ is a best proximity point of $T$. Suppose that $y_2 \notin Tx_3$ then we have,

$$0 < d(y_2, Tx_3) \leq H(Tx_2, Tx_3) < q_2 H(Tx_2, Tx_3).$$

Then there is $y_3 \in Tx_3$ such that

$$\begin{aligned} 0 < d(y_3, y_2) &< q_2 H(Tx_3, Tx_2) \leq q_2\varphi\Big(d(x_3, x_2)\Big) \\ &\quad + q_2\theta\Big(D(x_3, Tx_2) - d(A, B), d(x_2, Tx_3) - d(A, B), D(x_3, Tx_3) \\ &\quad - d(A, B), D(x_2, Tx_2) - d(A, B)\Big) \end{aligned}$$

Since $D(x_3, Tx_2) = d(A, B)$ we have

$$\begin{aligned} 0 < d(y_3, y_2) &< \varphi\Big(d(x_3, x_2)\Big) + q_2\theta\Big(0, d(x_2, Tx_3) - d(A, B), D(x_3, Tx_3) \\ &\quad - d(A, B), D(x_2, Tx_2) - d(A, B)\Big) \\ &= q_2\varphi\Big(d(x_3, x_2)\Big) = \varphi^2(q\varphi(t_0)) \end{aligned}$$

By continuing this process, for each $n \in N$, we can find a sequences $\{x_n\}$ and $\{y_n\}$ in $A_0$ and $B_0$ respectively, such that,
(1) $y_n \in Tx_n \subseteq B_0$,
(2) $d(x_{n+1}, y_n) = d(A, B)$

(3) $d(y_{n+1}, y_n) \leq \varphi^n\Big(q\varphi(t_0)\Big)$.
Since $(A, B)$ satisfies the weak p-property, we conclude that

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \qquad \forall n \in N$$

we now have

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n) \leq \varphi^{n-1}\Big(q\varphi(t_0\Big)$$

Let $m > n$. Then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \varphi^{i-1}\Big(q\varphi(t_0)\Big)$$

and so $\{x_n\}$ is a Cauchy sequence in $A$. Hence, there exists $x^* \in A$ such that $x_n \to x^*$. Similarly, by using (3) we can show that the sequence $\{y_n\}$ in $B$ is Cauchy and hence is convergent. Suppose that $y_n \to y^*$. By the relation $d(x_{n+1}, y_n) = d(A, B)$, for all $n \in N$, we conclude that $d(x^*, y^*) = d(A, B)$. Now we show that $y^* \in Tx^*$. Since $y_n \in Tx_n$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} D(y_n, Tx^*)\\
&\leq \lim_{n\to\infty} H(Tx_n, Tx^*)\\
&\leq \lim_{n\to\infty} \Big[\varphi\Big(d(x_n, x^*)\Big) + \theta\Big(D(x^*, Tx_n)\\
&- d(A, B), D(x_n, Tx^*) - d(A, B), D(x_n, Tx_n) - d(A, B),\\
&D(x^*, Tx^*) - d(A, B)\Big)\Big]\\
&= 0 + \theta\Big(\lim_{n\to\infty} d(x^*, y_n) - d(A, B), \lim_{n\to\infty} \big(D(x_n, Tx^*)\\
&- d(A, B)\big), \lim_{n\to\infty} \big(D(x_n, Tx_n) - d(A, B)\big), D(x^*, Tx^*) - d(A, B)\Big)\\
&= 0 + \theta\Big(0, \lim_{n\to\infty} \big(D(x_n, Tx^*)\\
&- d(A, B)\big), \lim_{n\to\infty} \big(D(x_n, Tx_n) - d(A, B)\big), D(x^*, Tx^*) - d(A, B)\Big)\\
&= 0.
\end{aligned}$$

Thus, we have

$$\lim_{n\to\infty} D(y_n, Tx^*) = 0.$$

Hence $D(y^*, Tx^*) = 0$. Since $Tx^*$ is closed, We conclude that $y^* \in Tx^*$. Now we have,

$$d(A, B) \leq D(x^*, Tx^*) \leq d(x^*, y^*) = d(A, B),$$

which implies that $D(x^*, Tx^*) = d(A, B)$, that is $x^* \in A$ is a best proximity point of $T$. This completes the proof of theorem. □

Taking $\varphi(t) = \alpha t$ we have the following result which an extension of theorem 2.1 in [1].

**Corollary 1.** *Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0 \neq \varnothing$ and $(A, B)$ satisfies the weak $P$-property. Let $T : A \to 2^B$ be a multivalued non-self-mapping, for which there exist a constant $\alpha \in [0, 1)$ and $\theta \in \Theta$ such that for all $x, y \in X$*

$$H(Tx, Ty) \leq \alpha d(x, y) + \theta\Big(D(y, Tx) - d(A, B), D(x, Ty) - d(A, B), D(x, Tx) - d(A, B), D(y, Ty) - d(A, B)\Big)$$

*Suppose also that $T(x)$ is bounded and closed in $B$ for all $x \in A$, and $T(x_0) \subseteq B_0$ for each $x_0 \in A_0$, then $T$ has a best proximity point in $A$.*

*Example* 2. Let $X = \Re$ with the usual metric. Suppose $A := \{0, 3, 6, 9\}$ and $B := \{-1, 2, 5, 8\}$. Then, $A$ and $B$ are nonempty and closed subsets of $X$ and $A_0 = A$ and $B_0 = B$. We note that, $d(A, B) = 1$. It is easy to show that the pair $(A, B)$ has the weak $P$-property. Let $T : A \to 2^B$ ba a mapping defined by $T0 = \{8\}$ and $Tx = \{5, 8\}$, if $x \neq 0$. Consider the functions $\theta(t_1, t_2, t_3, t_4) = t_1 t_2 t_3 t_4$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$. Then $T$ is $(\varphi, \theta)$- multivalued contraction. Thus $T$ has a best proximity point Note that $x = 6$ and $x = 9$ are best proximity point of $T$. It is interesting to note that the non-self mapping T is not a non-self contraction.

Taking $B = A$ in Theorem 2, we obtain the following result.

**Corollary 2.** *Let $(X, d)$ be a complete metric space, and $A$ be a nonempty and closed subset of $X$. Let $T : A \to 2^A$ be an almost $(\varphi, \theta)$-contraction self-mapping. Then $T$ has a fixed point $x \in A$.*

Taking $\varphi(t) = \alpha t$ and $\theta(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\}$, we obtain from Corollary 2 the following result which is a generalization of Nadler fixed point theorem [6].

**Corollary 3.** *Let $(X, d)$ be a complete metric space, and $A$ be a nonempty closed subset of $X$. Let $T : A \to 2^A$ be a mapping such that there exist $\alpha \in [0, 1)$ and $L > 0$ such that, for all $x, y \in A$,*

$$H(Tx, Ty) \leq \alpha d(x, y) + L \min\{D(y, Tx), D(x, Ty), D(x, Tx), D(y, Ty)\}$$

*Then $T$ has a unique fixed point $x \in A$.*

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# REFINED ALMOST DOUBLE DERIVATIONS AND LIE $*$-DOUBLE DERIVATIONS

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*Abstract.* In this paper, our approach allows to refine the results announced by Ebadian et al. [Results Math., 36 (2013), 409–423]. Namely, we reduce the distance between approximate and exact double derivations on Banach algebras and Lie $C^*$-algebras up to $\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-2}}$ for $n \geq 2$. Indeed, we give a correct utilization of fixed point theory in the sense of Diaz and Margolis [Bull. Amer. Math. Soc., 74 (1968), 305–309] concerning the stability of double derivations.

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*Keywords:* Hyers-Ulam stability, double derivation, $*$-double derivation, Lie $C^*$-algebra

## 1. Introduction

A classical question in the theory of functional equations is the following:

*when is it true that a mapping which approximately satisfies a functional equation $\xi$ must be somehow near to an exact solution of $\xi$?*

In 1940, Ulam [7] gave a wide ranging talk and discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

*Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

Generally, the concept of stability for a functional equation comes up when the functional equation is replaced by an inequality which acts as a perturbation of that equation. The case of approximately additive functions was solved by D. Hyers [5] under certain assumptions. In 1950, Hyers' Theorem was generalized by Aoki [1] for additive mappings and independently, in 1978, by Rassias [6] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. For the history and various aspects of this theory we refer the reader to [3] and the references therein. Note that a functional equation $\zeta$ is stable if any function g satisfying the equation $\zeta$ approximately is near to true solution of $\zeta$.

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Recently, Ebadian et al. [3] used the fixed point alternative method to establish the Hyers–Ulam stability of double derivations on Banach algebras and Lie $*$-double derivations on Lie $C^*$-algebras associated with the following additive functional equation

$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n}x_i-\sum_{r=1}^{n-k+1}x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}x_i\Big)=2^{n-1}f(x_1).$$

Throughout this paper following [3], we assume that $\mathcal{A}$ is a Banach algebra (Lie $C^*$-algebra). For given mapping $f:\mathcal{A}\to\mathcal{A}$, we define the difference operator $D_\mu f:\mathcal{A}^n\to\mathcal{A}$ by

$$D_\mu f(x_1,\dots,x_n):=$$
$$\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big)f\Big(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n}\mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}\Big)$$
$$+f\Big(\sum_{i=1}^{n}\mu x_i\Big)-2^{n-1}f(\mu x_1)$$

for all $\mu\in\{\lambda\in\mathbb{C}:|\lambda|=1\}$.

In this paper, we improve main results of [3] and reduce the distance between approximate and exact double derivations on Banach algebras and Lie $C^*$-algebras up to $\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-2}}$ for $n\geq 2$.

In section 2, we discuss on main results of [3] and improve some theorems and corollaries including Theorem 2.3, 2.5 and Corollary 2.4, 2.6. In section 3, we also refine some results of [3] including Theorem 3.2, 3.4 and Corollary 3.3, 3.5. Indeed, we are going to weaken their assumptions and giving a correct utilization of fixed point theory in the sense of Diaz and Margolis [2].

## 2. Almost double derivation

Throughout this section, we assume that $\mathcal{A}$ is a Banach algebra, $f(0)=g(0)=h(0)=0$, and for given mappings $f,g,h:\mathcal{A}\longrightarrow\mathcal{A}$, we define

$$C_{f,g,h}(a,b):=f(ab)-f(a)b-af(b)-g(a)h(b)-h(a)g(b)$$

for all $a,b\in\mathcal{A}$.

**Definition 1.** Let $\mathcal{A}$ be a Banach algebra and let $\delta,\varepsilon:\mathcal{A}\longrightarrow\mathcal{A}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f:\mathcal{A}\longrightarrow\mathcal{A}$ is called a $(\delta,\varepsilon)$-double derivation if

$$f(ab)=f(a)b+af(b)+\delta(a)\varepsilon(b)+\varepsilon(a)\delta(b)$$

for all $a, b \in \mathcal{A}$.

A fundamental result in fixed point theory is the following theorem. We recall the following theorem by Diaz and Margolis [2].

**Theorem 1.** *Let $(X, d)$ be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all $n \geq 0$, or there exists a natural number $n_0$ such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$,*
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$,*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$,*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

The following theorem is a refined version of [3, Theorem 2.3]:

**Theorem 2.** *Let $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings for which there exist functions $\varphi : \mathcal{A}^n \longrightarrow [0, \infty)$ and $\psi : \mathcal{A}^2 \longrightarrow [0, \infty)$ such that*

$$\lim_{m \to \infty} 2^m \varphi(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m}) = 0, \tag{2.1}$$

$$\lim_{m \to \infty} 4^m \psi(\frac{a}{2^m}, \frac{b}{2^m}) = 0, \tag{2.2}$$

$$||D_\mu f(x_1, ..., x_n)|| \leq \varphi(x_1, ..., x_n), \tag{2.3}$$

$$||C_{f,g,h}(a, b)|| \leq \psi(a, b) \tag{2.4}$$

*for all $a, b, x_1, ..., x_2 \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, ..., x_n) \leq \frac{L}{2} \varphi(2x_1, ..., 2x_n)$ for all $x_1, ..., x_n \in \mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that*

$$||f(x) - d(x)|| \leq \frac{L}{\alpha(1-L)} \varphi(x, x, 0, ..., 0), \tag{2.5}$$

$$||g(x) - \delta(x)|| \leq \frac{L}{\alpha(1-L)} \varphi(x, x, 0, ..., 0), \tag{2.6}$$

$$||h(x) - \varepsilon(x)|| \leq \frac{L}{\alpha(1-L)} \varphi(x, x, 0, ..., 0) \tag{2.7}$$

*for all $x \in \mathcal{A}$, where $\alpha = 2^{n-1}$ and $n \geq 2$. Moreover, $d$ is a $(\delta, \varepsilon)$-double derivation on $\mathcal{A}$.*

*Proof.* Put $\mu = 1, x_1 = x_2 = x$, and $x_3 = x_4 = ... = x_n = 0$ in (2.3) to reach

$$||\frac{\alpha}{2} f(2x) - \alpha f(x)\| \leq \varphi(x, , x, 0, ..., 0)$$

for all $x \in \mathcal{A}$ and so

$$||2f(\frac{x}{2}) - f(x)\| \leq \frac{2}{\alpha} \varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0) \leq \frac{L}{\alpha} \varphi(x, , x, 0, ..., 0). \tag{2.8}$$

Define $F := \{f : \mathcal{A} \longrightarrow \mathcal{A}\}$. The metric defined on $F$ by

$$\rho(f,g) := \inf\{c \in [0,\infty] : ||f(x)-g(x)|| \leq c\varphi(x,x,0,...,0), \forall x \in \mathcal{A}\}.$$

is a generalized metric and $(F,\rho)$ is a generalized complete metric space. Consider the mapping $(Jf)(x) := 2f(\frac{x}{2})$ for all $f \in F$ and $x \in \mathcal{A}$. Use [4, Lemma 1.3] to see that $J$ is a strictly contractive mapping with the Lipschitz constant $L$. It follows from (2.8) that $\rho(Jf,f) \leq \frac{L}{\alpha}$. Therefore according to Theorem 1, the sequence $\{J^m f\}$ converges to a fixed point $d$ such that $d(x) = \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ and $d(2x) = 2d(x)$. Note that $d$ is the unique fixed point of $J$ and

$$\rho(d,f) \leq \frac{1}{1-L}\rho(Jf,f) \leq \frac{L}{\alpha(1-L)}.$$

This means that inequality (2.5) holds for all $x \in \mathcal{A}$. The proof of the linearity of $d$ and also the rest of the proof is similar to that of [3, Theorem 2.3] and we omit it. □

The importance of our result becomes clear when we take

$$\varphi(x_1,...,x_n) = \theta_1 \sum_{i=1}^{n} ||x_i||^p, \ \psi(a,b) = \theta_2(||a||^q + ||b||^q).$$

In this situation, by choosing $L = 2^{1-p}$, we can get strong and close approximations of the functions $f,g,h$ with linear mappings $d,\delta,\varepsilon$, where $d$ is a $(\delta,\varepsilon)$-double derivation on $\mathcal{A}$. Thus, we improve [3, Corollary 2.4] up to $\frac{1}{2^{n-1}}$ as follows:

**Corollary 1.** *Let $p,q,\theta_1,\theta_2$ be non-negative real numbers with $p,q > 1$. Suppose that $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings such that*

$$||D_\mu f(x_1,...,x_n)|| \leq \theta_1 \sum_{i=1}^{n} ||x_i||^p,$$

$$||C_{f,g,h}(a,b)|| \leq \theta_2(||a||^q + ||b||^q)$$

*for all $a,b,x_1,...,x_n \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ such that*

$$||f(x)-d(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p,$$

$$||g(x)-\delta(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p,$$

$$||h(x)-\varepsilon(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p$$

*for all $x \in \mathcal{A}$. Moreover, $d$ is a $(\delta,\varepsilon)$-double derivation on $\mathcal{A}$.*

In the following theorem we give an improved version of [3, Theorem 2.5]:

**Theorem 3.** *Suppose that $f,g,h:\mathcal{A}\longrightarrow\mathcal{A}$ are mappings satisfying (2.3) and (2.4) for which there exist functions $\varphi:\mathcal{A}^n\longrightarrow[0,\infty)$ and $\psi:\mathcal{A}^2\longrightarrow[0,\infty)$ such that*

$$\lim_{m\to\infty}\frac{1}{2^m}\varphi(2^m x_1,\dots,2^m x_n)=0, \tag{2.9}$$

$$\lim_{m\to\infty}\frac{1}{2^m}\psi(2^m a,2^m b)=0 \tag{2.10}$$

*for all $a,b,x_1,\dots,x_2\in\mathcal{A}$. If there exists a constant $0<L<1$ such that $\varphi(x_1,\dots,x_n)\le 2L\varphi(\frac{x_1}{2},\dots,\frac{x_n}{2})$ for all $x_1,\dots,x_n\in\mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ such that*

$$||f(x)-d(x)||\le\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,\dots,0), \tag{2.11}$$

$$||g(x)-\delta(x)||\le\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,\dots,0), \tag{2.12}$$

$$||h(x)-\varepsilon(x)||\le\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,\dots,0) \tag{2.13}$$

*for all $x\in\mathcal{A}$, where $\beta=\frac{\alpha}{2}$ and $n\ge 2$. Moreover, $d$ is a $(\delta,\varepsilon)$-double derivation on $\mathcal{A}$.*

*Proof.* It follows from (2.8) that

$$||\frac{1}{2}f(2x)-f(x)\|\le\frac{1}{\alpha}\varphi(x,,x,0,\dots,0)\le\frac{2L}{\alpha}\varphi(\frac{x}{2},\frac{x}{2},0,\dots,0) \tag{2.14}$$

for all $x\in\mathcal{A}$. Consider the generalized complete metric $(F,\rho)$ with the generalized metric $\rho$ defined by

$$\rho(f,g):=\inf\{c\in[0,\infty]:||f(x)-g(x)||\le c\varphi(\frac{x}{2},\frac{x}{2},0,\dots,0),\forall x\in\mathcal{A}\}.$$

Define the mapping $(Jf)(x):=\frac{1}{2}f(2x)$ for all $f\in F$ and $x\in\mathcal{A}$. Apply [4, Lemma 1.2]) to find that $J$ is a strictly contractive mapping with the Lipschitz constant $L$. It follows from (2.14) that $\rho(Jf,f)\le\frac{2L}{\alpha}$. Applying Theorem 1, we get the sequence $\{J^m f\}$ converges to a unique fixed point $d$ of $J$ such that

$$\rho(d,f)\le\frac{1}{1-L}\rho(Jf,f)\le\frac{2L}{\alpha(1-L)}=\frac{L}{\beta(1-L)},$$

i. e., inequality (2.11) holds for all $x\in\mathcal{A}$. The rest of the proof is similar to that of [3, Theorem 2.3]. □

As we mentioned in Corollary 1, the importance of Theorem 3 becomes also clear when we put $L=2^{p-1}$ and

$$\varphi(x_1,\dots,x_n)=\theta_1+\theta_2\sum_{i=1}^{n}||x_i||^p,\ \psi(a,b)=\theta_1+\theta_2(||a||^q+||b||^q).$$

However, we can improve [3, Corollary 2.6] up to $\frac{1}{2^{n-2}}$ as follows:

**Corollary 2.** *Let $p,q,\theta_1,\theta_2$ be non-negative real numbers with $p,q \in (0,1)$. Suppose that $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings such that*

$$||D_\mu f(x_1,...,x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,$$

$$||C_{f,g,h}(a,b)|| \leq \theta_1 + \theta_2(||a||^q + ||b||^q)$$

*for all $a,b,x_1,...,x_n \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ such that*

$$||f(x) - d(x)|| \leq \frac{1}{2^{n-2}}(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p),$$

$$||g(x) - \delta(x)|| \leq \frac{1}{2^{n-2}}(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p),$$

$$||h(x) - \varepsilon(x)|| \leq \frac{1}{2^{n-2}}(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p)$$

*for all $x \in \mathcal{A}$. Moreover, $d$ is a $(\delta,\varepsilon)$-double derivation on $\mathcal{A}$.*

## 3. Almost Lie $*$-double derivation

A unital $C^*$-algebra $\mathcal{A}$, endowed with the Lie product $[x,y] = xy - yx$ on $\mathcal{A}$, is called a Lie $C^*$-algebra. In this section, we assume that $\mathcal{A}$ is a Lie $C^*$-algebra and $U(\mathcal{A}) = \{u \in \mathcal{A} : uu^* = u^*u = e\}$. For given mappings $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$, we let $f(0) = g(0) = h(0) = 0$ and define

$$J_{f,g,h}(a,b) := f([a,b]) - [f(a),b] - [a,f(b)] - [g(a),h(b)] - [h(a),g(b)]$$

for all $a,b \in \mathcal{A}$.

**Definition 2.** Let $\mathcal{A}$ be a Lie $C^*$-algebra and let $\delta,\varepsilon : \mathcal{A} \longrightarrow \mathcal{A}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is called a Lie $(\delta,\varepsilon)$-double derivation if

$$f([a,b]) = [f(a),b] + [a,f(b)] + [\delta(a),\varepsilon(b)] + [\varepsilon(a),\delta(b)]$$

for all $a,b \in \mathcal{A}$.

The presented results in this section are refinements of [3, Theorem 3.2, 3.4] and [3, Corollary 3.3, 3.5]:

**Theorem 4.** *Let $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings for which there exist functions $\varphi : \mathcal{A}^n \longrightarrow [0,\infty)$ and $\psi : \mathcal{A}^2 \longrightarrow [0,\infty)$ such that*

$$\lim_{m\to\infty} 2^m \varphi(\frac{x_1}{2^m},...,\frac{x_n}{2^m}) = 0, \tag{3.1}$$

$$\lim_{m\to\infty} 4^m \psi(\frac{a}{2^m},\frac{b}{2^m}) = 0, \tag{3.2}$$

$$||D_\mu f(x_1,...,x_n)|| \leq \varphi(x_1,...,x_n), \tag{3.3}$$

$$||J_{f,g,h}(a,b)|| \leq \psi(a,b) \tag{3.4}$$

$$\max\{f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*, g(\frac{u^*}{2^k}) - g(\frac{u}{2^k})^*, h(\frac{u^*}{2^k}) - h(\frac{u}{2^k})^*\} \leq \varphi(\frac{u}{2^k}, ..., \frac{u}{2^k}) \tag{3.5}$$

*for all* $a, b, x_1, ..., x_2 \in \mathcal{A}$, $k = 0, 1, 2, ...$, $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *If there exists a constant* $0 < L < 1$ *such that* $\varphi(x_1, ..., x_n) \leq \frac{L}{2}\varphi(2x_1, ..., 2x_n)$ *for all* $x_1, ..., x_n \in \mathcal{A}$, *then there exist unique* $\mathbb{C}$*-linear mappings* $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\} \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0)$$

*for all* $x \in \mathcal{A}$. *Moreover,* $d$ *is a Lie* $*$*-*$(\delta, \varepsilon)$*-double derivation on* $\mathcal{A}$.

*Proof.* Using the same methods as in the proof of [3, Theorem 2.3, 3.2], we can obtain the desired results. □

**Corollary 3.** *Let* $p, q, \theta_1, \theta_2$ *be non-negative real numbers with* $p, q > 1$. *Suppose that* $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ *are mappings such that*

$$||D_\mu f(x_1, ..., x_n)|| \leq \theta_1 \sum_{i=1}^{n} ||x_i||^p,$$

$$||J_{f,g,h}(u,b)|| \leq \theta_2(1 + ||b||^q)$$

$$\max\{f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*, g(\frac{u^*}{2^k}) - g(\frac{u}{2^k})^*, h(\frac{u^*}{2^k}) - h(\frac{u}{2^k})^*\} \leq \frac{\theta_1 + \theta_2}{2^{kp}}$$

*for all* $a, b, x_1, ..., x_n \in \mathcal{A}$, $k = 0, 1, 2, ...$, $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *There exist unique* $\mathbb{C}$*-linear mappings* $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\} \leq \frac{2\theta_1}{2^{n-1}(2^{p-1} - 1)}||x||^p$$

*for all* $x \in \mathcal{A}$. *Moreover,* $d$ *is a* $(\delta, \varepsilon)$*-double derivation on* $\mathcal{A}$.

*Proof.* The results follows from above theorem by taking $L = 2^{1-p}$ and

$$\varphi(x_1, ..., x_n) = \theta_1 \sum_{i=1}^{n} ||x_i||^p, \ \psi(a,b) = \theta_2(1 + ||b||^q).$$

□

**Theorem 5.** *Suppose that* $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ *are mappings satisfying* (3.3) *and* (3.4) *for which there exist functions* $\varphi : \mathcal{A}^n \longrightarrow [0, \infty)$ *and* $\psi : \mathcal{A}^2 \longrightarrow [0, \infty)$ *such that*

$$\lim_{m\to\infty} \frac{1}{2^m}\varphi(2^m x_1, ..., 2^m x_n) = 0,$$

$$\lim_{m\to\infty} \frac{1}{4^m}\psi(2^m a, 2^m b) = 0$$

$$\max\{f(2^k u^*) - f(2^k u)^*, g(2^k u^*) - g(2^k u)^*, h(2^k u^*) - h(2^k u)^*\} \leq \varphi(2^k u, ...2^k u)$$

*for all* $a,b,x_1,...,x_2 \in \mathcal{A}$, $k=0,1,2,...$, $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *If there exists a constant* $0<L<1$ *such that* $\varphi(x_1,...,x_n) \leq 2L\varphi(\frac{x_1}{2},...,\frac{x_n}{2})$ *for all* $x_1,...,x_n \in \mathcal{A}$, *then there exist unique* $\mathbb{C}$*-linear mappings* $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x)-d(x)||,||g(x)-\delta(x)||,||h(x)-\varepsilon(x)||\} \leq \frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,...,0)$$

*for all* $x \in \mathcal{A}$, *where* $\beta = \frac{\alpha}{2}$ *and* $n \geq 2$. *Moreover,* $d$ *is a Lie* $*$*-*$(\delta,\varepsilon)$*-double derivation on* $\mathcal{A}$.

*Proof.* The proof is similar to that of [3, Theorem 2.5, 3.2]. □

**Corollary 4.** *Let* $p,q,\theta_1,\theta_2$ *be non-negative real numbers with* $p,q \in (0,1)$. *Suppose that* $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$ *are mappings such that*

$$||D_\mu f(x_1,...,x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,$$

$$||J_{f,g,h}(u,b)|| \leq \theta_1 + \theta_2(1+||b||^q)$$

$$\max\{f(2^k u^*)-f(2^k u)^*, g(2^k u^*)-g(2^k u)^*, h(2^k u^*)-h(2^k u)^*\} \leq \frac{\theta_1+\theta_2}{2^{kp}}$$

*for all* $a,b,x_1,...,x_n \in \mathcal{A}$, $k=0,1,2,...$, $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *Then there exist unique* $\mathbb{C}$*-linear mappings* $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x)-d(x)||,||g(x)-\delta(x)||,||h(x)-\varepsilon(x)||\}$$
$$\leq \frac{1}{2^{n-2}}(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p)$$

*for all* $x \in \mathcal{A}$. *Moreover,* $d$ *is a Lie* $*$*-*$(\delta,\varepsilon)$*-double derivation on* $\mathcal{A}$.

*Proof.* Apply above theorem by putting $L = 2^{p-1}$ and

$$\varphi(x_1,...,x_n) = \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,\ \psi(a,b) = \theta_1 + \theta_2(1+||b||^q).$$

□

## 4. Conclusion

Our results can give the results proved by Ebadian et al. [3]. For instance, under the hypotheses of Theorem 2 we can conclude [3, Theorem 2.3], but not vice versa. In other words, if there exists a constant $0<L<1$ such that $\varphi(x_1,...,x_n) \leq \frac{L}{2}\varphi(2x_1,...,2x_n)$ for all $x_1,...,x_n \in \mathcal{A}$, then $\varphi(x_1,...,x_n) \leq \frac{\alpha}{2}L\varphi(2x_1,...,2x_n)$, i.e, all of the hypotheses of [3, Theorem 2.3] hold. On the other hand, Theorem 2 says that there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$||f(x)-d(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x,x,0,...,0),$$

$$||g(x)-\delta(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x,x,0,...,0),$$
$$||h(x)-\varepsilon(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x,x,0,...,0)$$

for all $x \in \mathcal{A}$, where $\alpha = 2^{n-1}$ and $n \geq 2$. Since $\frac{L}{\alpha(1-L)} \leq \frac{L}{1-L}$, we have

$$||f(x)-d(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$
$$||g(x)-\delta(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$
$$||h(x)-\varepsilon(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$

which coincide with the results of [3, Theorem 2.3]. The same arguments can be applied for Theorem 2.5, 3.2, 3.4 and Corollary 2.4, 2.6, 3.3, 3.5 of [3].

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# DIRECT AND INVERSE THEOREMS FOR MULTIVARIATE BERNSTEIN-SCHURER-STANCU OPERATORS

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*Abstract.* In this paper, we introduce the multivariate Bernstein-Schurer-Stancu operators. Then, we state the Volkov-type theorem and investigate the order of convergence by means of modulus of continuity and by Lipschitz class functionals. Moreover, the inverse theorems are studied for the multivariate Stancu variant of Bernstein operators.

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*Keywords:* multivariate Bernstein-Schurer-Stancu operators, modulus of continuity, direct theorem, inverse theorems

## 1. Introduction

The celebrated Bernstein polynomials [16]

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^k, \qquad x\in[0,1]$$

have been the most attractive approximating operators which were studied by various mathematicians and have many applications in mathematics, physics, engineering, economy, etc (see [3], [4], [7], [8], [9], [10],[11], [12], [14], [15], [17], [18], [19], [20], [21], [23]) .
In the year 1962, Schurer [22] defined the Schurer operators as:

$$B_{n,p}(f;x)=\sum_{k=0}^{n+p} f\left(\frac{k}{n}\right)\binom{n+p}{k}x^k(1-x)^{n+p-k}, \qquad x\in[0,1]. \tag{1.1}$$

We should notice that if $p=0$ in (1.1), we have $B_{n,0}(f;x)=B_n(f;x)$.
On the other hand, Barbosu [1] estimated the Schurer-Stancu operators as:

$$\left(\tilde{S}_{n,p}^{(\alpha,\beta)}f\right)(x)=\sum_{k=0}^{n+p} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n+p}{k}x^k(1-x)^{n+p-k}, \quad x\in[0,1] \tag{1.2}$$

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where $\alpha$ and $\beta$ are real parameters such that $0 \leq \alpha \leq \beta$ and $p$ is a fixed non-negative integer.
Letting $\alpha = \beta = 0$ in (1.2), we have

$$\left(\tilde{S}_{n,p}^{(0,0)} f\right)(x) = B_{n,p}(f;x).$$

Afterwards, Barbosu introduced the bivariate operators of Schurer-Stancu type in [2] as:

$$\left(\tilde{S}_{m,p,n,q}^{(\alpha_1,\alpha_2,\beta_1,\beta_2)} f\right)(x,y) = \sum_{k=0}^{n+p}\sum_{j=0}^{m+q} f\left(\frac{k+\alpha_1}{n+\beta_1}, \frac{j+\alpha_2}{m+\beta_2}\right)\binom{n+p}{k}\binom{m+q}{j} \times x^k (1-x)^{n+p-k} y^k (1-y)^{m+q-j}, \tag{1.3}$$

where $m,\ n \in \mathbb{N},\ p,q \in \mathbb{N}_0$ and $\alpha_1,\beta_1,\alpha_2,\beta_2$ are real numbers which satisfy the conditions $0 \leq \alpha_1 \leq \beta_1$ and $0 \leq \alpha_2 \leq \beta_2$ and $(x,y) \in [0,1]\times[0,1]$. Then he obtained the Volkov-theorem (bivariate Bohman-Korovkin theorem) and studied the order of convergence of these operators by means of the first modulus of continuity for two variables functions. It should be noted that, taking $p = q = 0$ in (1.3), we get the operators investigated in [5].

In 2008, Büyükyazıcı and İbikli introduced the multivariate generalized Bernstein polynomials [6] as:

$$B_n(f;x_1,\ldots,x_m) := \sum_{k_1,\ldots,k_m=0}^{n} f\left(\frac{k_1}{b_n},\cdots,\frac{k_m}{b_n}\right)\prod_{j=1}^{m}\left[\binom{n}{k_j} x_j^{k_j}(1-x_j)^{n-k_j}\right] \tag{1.4}$$

where $(b_n)$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n}{n} = 1$. Then, they studied direct and inverse theorems for the multivariate generalized Bernstein operators.

We organize the paper in the following way:
In section two, the multivariate Bernstein-Schurer-Stancu operators are introduced and the moments of the operators are calculated. In section three, the Volkov-type theorems are obtained. In section four, we give quantitative results for the error in terms of the modulus of continuiy and the Lipschitz class functionals. In section five, we study inverse theorems of the multivariate Bernstein-Schurer-Stancu operators.

## 2. Construction of the operators

We introduce the multivariate Bernstein-Schurer-Stancu operators by

$$B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m) \tag{2.1}$$

$$:= \sum_{k_1,\ldots,k_m=0}^{n+p} f\left(\frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n}\right)\prod_{j=1}^{m}\left[\binom{n+p}{k_j}x_j^{k_j}\left(1-x_j\right)^{n+p-k_j}\right]$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)$ be $m$-tuples real parameters satisfying the conditions $\alpha_i\geq 0$ $(i=1,\ldots,m)$, $f$ be a continuous function on $D^p:=\bigotimes_{j=1}^{m}[0,p+1]$, for fixed $p\in\mathbb{N}_0$ and $(b_n)$ is a sequence of positive real numbers satisfying
(1°) For each $n\in\mathbb{N}$, $n+\max\{\alpha_1,\cdots,\alpha_m\}\leq b_n$,
(2°) $\lim_{n\to\infty}\frac{b_n}{n}=1$.

It is clear that $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}:C_{\mathbb{R}^m}(D^p)\to C_{\mathbb{R}^m}\left(D^0\right)$. Note that, if we choose $p=0$ and $\alpha=(\alpha_1,\ldots,\alpha_m)=(0,\ldots,0)$ in (2.1), we get the operators in (1.4).
In giving the approximation properties of the operators
$B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m)$, we need the following lemma:

**Lemma 1.** *Let $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m)$ be given in (2.1). The first few moments of the operators are*
*i)* $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(1;x_1,\ldots,x_m)=1$,
*ii)* $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(t_j;x_1,\ldots,x_m\right)=\frac{n+p}{b_n}x_j+\frac{\alpha_j}{b_n},\quad (j=1,2,\ldots,m)$,
*iii)* $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(|\mathbf{t}|^2;x_1,\ldots,x_m\right)$

$$=\frac{(n+p)^2}{b_n^2}|\mathbf{x}|^2-\frac{n+p}{b_n^2}|\mathbf{x}|^2+\frac{|\boldsymbol{\alpha}|^2}{b_n^2}+\frac{n+p}{b_n^2}\textstyle\sum_{j=0}^{m}\left(2\alpha_j+1\right)x_j$$

*iv)* $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left((\mathbf{t}-\mathbf{x})^2;x_1,\ldots,x_m\right)=\left(\left(\frac{n+p}{b_n}-1\right)^2-\frac{n+p}{b_n^2}\right)|\mathbf{x}|^2$

$+\sum_{j=1}^{m}\left[\frac{n+p}{b_n^2}\left(2\alpha_j+1\right)-2\frac{\alpha_j}{b_n}\right]x_j+\frac{|\boldsymbol{\alpha}|^2}{b_n^2}\ (j=1,2,\ldots,m)$,
*where* $|\mathbf{x}|=\sum_{i=1}^{m}x_i$, $|\mathbf{t}|=\sum_{i=1}^{m}t_i$, $|\boldsymbol{\alpha}|=\sum_{i=1}^{m}\alpha_i$.

*Proof.* *i)* It is clear that $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(1;x_1,\ldots,x_m)=1$.
*ii)* For the proof of (ii), we have

$$B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(t_j;x_1,\ldots,x_m\right)$$
$$=\sum_{k_1,\ldots,k_m=0}^{n+p}\left(\frac{k_j+\alpha_j}{b_n}\right)\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}\left(1-x_i\right)^{n+p-k_i}\right]$$
$$=\frac{n+p}{b_n}\sum_{k_1,\ldots,k_m=0}^{n+p}\frac{k_j}{n+p}\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}\left(1-x_i\right)^{n+p-k_i}\right]+\frac{\alpha_j}{b_n}.$$

Then after some straightforward calculations, we obtain the result directly.
$iii)$ Using the operator $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m)$, we have

$$\begin{aligned}
&B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(|\mathbf{t}|^2;x_1,\ldots,x_m\right)\\
&=\sum_{k_1,\ldots,k_m=0}^{n+p}\sum_{j=1}^{m}\left(\frac{k_j+\alpha_j}{b_n}\right)^2\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}(1-x_i)^{n+p-k_i}\right]\\
&=\sum_{k_1,\ldots,k_m=0}^{n+p}\sum_{j=1}^{m}\left(\frac{k_j^2+2k_j\alpha_j+\alpha_j^2}{b_n^2}\right)\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}(1-x_i)^{n+p-k_i}\right]
\end{aligned}$$

$$\begin{aligned}
&=\frac{(n+p)^2}{b_n^2}\sum_{j=1}^{m}\sum_{k_1,\ldots,k_m=0}^{n+p}\left(\frac{k_j^2}{(n+p)^2}\right)\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}(1-x_i)^{n+p-k_i}\right]\\
&+\frac{2(n+p)}{b_n^2}\sum_{j=1}^{m}\alpha_j\sum_{k_1,\ldots,k_m=0}^{n+p}\left(\frac{k_j}{n+p}\right)\prod_{i=1}^{m}\left[\binom{n+p}{k_i}x_i^{k_i}(1-x_i)^{n+p-k_i}\right]+\frac{|\boldsymbol{\alpha}|^2}{b_n^2}.
\end{aligned}$$

Since $|\mathbf{x}|=\sum_{i=1}^{m}x_i$ and $|\boldsymbol{\alpha}|=\sum_{i=1}^{m}\alpha_i$ then we have

$$\begin{aligned}
&B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(|\mathbf{t}|^2;x_1,\ldots,x_m\right)\\
&=\frac{(n+p)^2}{b_n^2}|\mathbf{x}|^2-\frac{n+p}{b_n^2}|\mathbf{x}|^2+\frac{|\boldsymbol{\alpha}|^2}{b_n^2}+\frac{n+p}{b_n^2}\sum_{j=1}^{m}\left(2\alpha_j+1\right)x_j.
\end{aligned}$$

Then, the proof is completed.
$iv)$ Using (i), (ii) and (iii) we have

$$\begin{aligned}
&B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left((\mathbf{t}-\mathbf{x})^2;x_1,\ldots,x_m\right)=B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(|\mathbf{t}|^2;x_1,\ldots,x_m\right)\\
&-2B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(\sum_{j=1}^{m}x_jt_j;x_1,\ldots,x_m\right)+B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}\left(|\mathbf{x}|^2;x_1,\ldots,x_m\right)
\end{aligned}$$

Hence, we get the desired result. □

## 3. Volkov-type approximation

In this section, the Volkov-type approximation theorem is proved for the operators $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m)$. Let $C_{\mathbb{R}^m}\left(D^0\right)$ be a linear normed space endowed with the following norm

$$\|g\|_{C_{\mathbb{R}^m}(D^0)}=\max_{(x_1,x_2,\ldots,x_m)\in D^0}|g(x_1,x_2,\ldots,x_m)|.$$

We will use the following Volkov-type approximation theorem to give our main results of this section.

**Theorem 1** ([13]). *Let $T_n : C_{\mathbb{R}^m}\left(D^0\right) \to C_{\mathbb{R}^m}\left(D^0\right)$ be a sequence of linear positive operators. If the following $(m+2)$ conditions are satisfied*

$$\lim_{n\to\infty} \|T_n(1;x)-1\|_{C_{\mathbb{R}^m}(D^0)} = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \left\|T_n(t_j;x)-x_j\right\|_{C_{\mathbb{R}^m}(D^0)} = 0, \ (j=1,2,\ldots,m) \tag{3.2}$$

$$\lim_{n\to\infty} \left\|T_n(|\mathbf{t}|^2;x)-|\mathbf{x}|^2\right\|_{C_{\mathbb{R}^m}(D^0)} = 0, \tag{3.3}$$

*where $\mathbf{t}=(t_1,\ldots,t_m)$, $\mathbf{x}=(x_1,\ldots,x_m)$, then for any function $f \in C_{\mathbb{R}^m}\left(D^0\right)$, we have*

$$\|T_n(f;\cdot)-f(\cdot)\|_{C_{\mathbb{R}^m}(D^0)} = 0, \ n\to\infty.$$

Note that, similar theorem can be stated for the operator $T_n : C_{\mathbb{R}^m}(D^p) \to C_{\mathbb{R}^m}\left(D^0\right)$.

**Theorem 2.** *Let the sequence of real numbers $(b_n)$ satisfies $(1^\circ)$ and $(2^\circ)$. For all $f \in C_{\mathbb{R}^m}(D^p)$, $B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_m)$ is uniformly convergent to $f$ as $n\to\infty$.*

*Proof.* Using Lemma 1 and taking into account $(1^\circ)$ and $(2^\circ)$, we have

$$\left\|B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(1;x)-1\right\|_{C_{\mathbb{R}^m}(D^0)} = 0,$$

$$\lim_{n\to\infty} \left\|B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(t_j;x)-x_j\right\|_{C_{\mathbb{R}^m}(D^0)} = 0, \ (j=1,2,\ldots,m)$$

$$\lim_{n\to\infty} \left\|B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(|\mathbf{t}|^2;x)-|\mathbf{x}|^2\right\|_{C_{\mathbb{R}^m}(D^0)} = 0.$$

Thus, the conditions (3.1), (3.2) and (3.3) are satisfied and the proof follows from Theorem 1. □

*Remark* 1. If we choose $m=2$ in (2.1), the multivariate Bernstein-Schurer-Stancu operators for $f(x_1,x_2)$ have the following form:

$$B_{n,p}^{(\alpha_1,\alpha_2)}(f;x_1,x_2) := \sum_{k_1=0}^{n+p}\sum_{k_2=0}^{n+p} f\left(\frac{k_1+\alpha_1}{b_n},\frac{k_2+\alpha_2}{b_n}\right)\binom{n+p}{k_1}\binom{n+p}{k_2} \tag{3.4}$$
$$\times x_1^{k_1}(1-x_1)^{n+p-k_1}x_2^{k_2}(1-x_2)^{n+p-k_2}.$$

Furthermore, we set the following,

$$f(x_1,x_2) = \cos(x_1x_2),$$
$$b_n = 1/(\log(n+p+1)-\log(n+p))$$
$$n = 5,10,30,60,80,100,$$

$$p = 1,2,3,4,5,6,$$
$$\alpha_1 = \alpha_2 = \frac{1}{100}, \frac{1}{90}, \frac{1}{80}, \frac{1}{70}, \frac{1}{60}, \frac{1}{50},$$

in (3.4). According to the above parameters, the convergence of $B_{n,p}^{(\alpha_1,\alpha_2)}(f;x_1,x_2)$ to $f$ is shown in Figure 1.

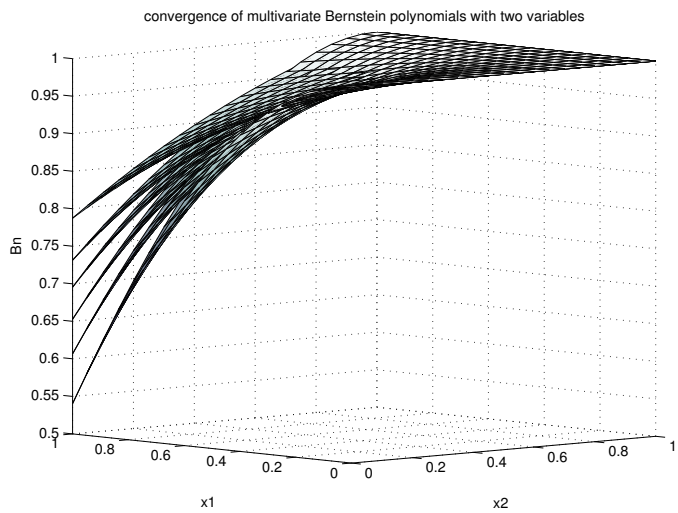


FIGURE 1. The convergence of multivariable Bernstein-Schurer-Stacu operators with two variables

*Remark* 2. If we choose $m = 3$ in (2.1), the multivariate Bernstein-Schurer-Stancu operators for $f(x_1,x_2,x_3)$ have the following form:

$$\begin{aligned} &B_{n,p}^{(\alpha_1,\alpha_2,\alpha_3)}(f;x_1,x_2,x_3) \\ &:= \sum_{k_1=0}^{n+p}\sum_{k_2=0}^{n+p}\sum_{k_3=0}^{n+p} f\left(\frac{k_1+\alpha_1}{b_n}, \frac{k_2+\alpha_2}{b_n}, \frac{k_3+\alpha_3}{b_n}\right)\binom{n+p}{k_1}\binom{n+p}{k_2}\binom{n+p}{k_3} \\ &\times x_1^{k_1}(1-x_1)^{n+p-k_1} x_2^{k_2}(1-x_2)^{n+p-k_2} x_3^{k_3}(1-x_3)^{n+p-k_3}. \end{aligned} \tag{3.5}$$

Moreover, let us choose

$$f(x_1,x_2,x_3) = \left(x_1-\frac{1}{3}\right)^2 + \left(x_2-\frac{1}{3}\right)^2 + \left(x_3-\frac{1}{3}\right)^2 - \frac{1}{9},$$
$$b_n = 1/(\log(n+p+1) - \log(n+p)),$$

$$n = 5, 30, 60, 80,$$
$$p = 1,$$
$$\alpha_1 = \alpha_2 = \frac{1}{100}, \frac{1}{90}, \frac{1}{80}, \frac{1}{70}$$

in (3.5). According to the above parameters, the convergence of $B_{n,p}^{(\alpha_1,\alpha_2,\alpha_3)}(f; x_1, x_2, , x_3)$ to $f$ is illustrated in Figure 2.

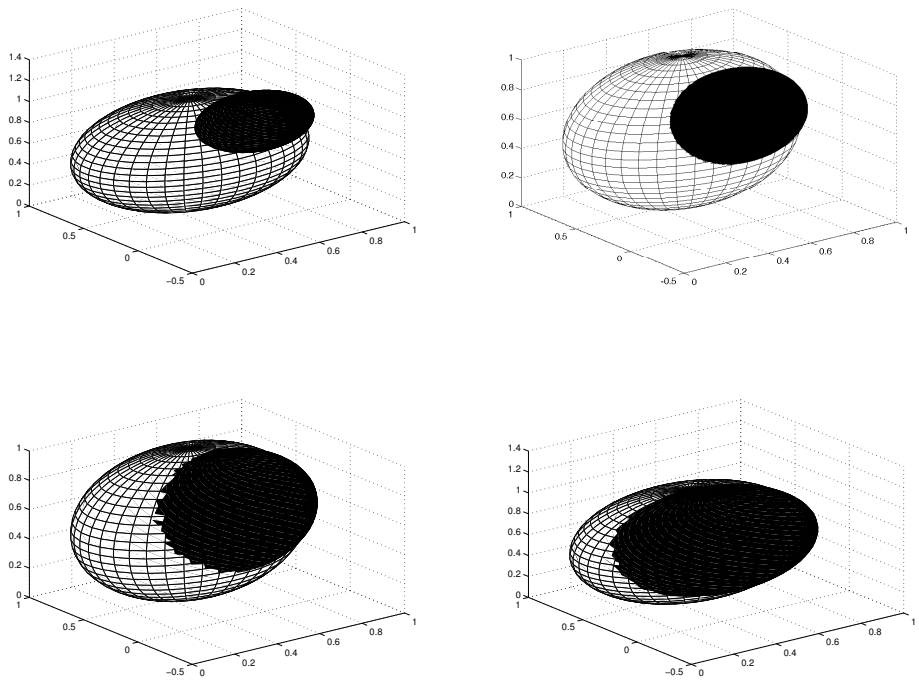

FIGURE 2. The convergence of multivariable Bernstein-Schurer-Stancu operators with three variables

## 4. RATE OF CONVERGENCE

In this section, we study the order of convergence of multivariate Bernstein-Schurer-Stancu operators by means of modulus of continuity and Lipschitz class functionals.

Firstly, let us recall the definitions of modulus of continuity and partial modulus of continuity, recpectively.

The full modulus of continuity of the function $f \in C_{\mathbb{R}^m}(D^p)$ is defined by

$$\omega(f;\delta) := \max_{\substack{\sqrt{\sum_{j=1}^{m}(x_j-y_j)^2}\le\delta \\ (x_1,\dots,x_m),\,(y_1,\dots,y_m)\in D^p}} |f(x_1,\dots,x_m)-f(y_1,\dots,y_m)|$$

where $f \in C_{\mathbb{R}^m}(D^p)$ and $\delta > 0$. For each fixed $i = 1,\dots,m$, the partial modulus of continuity of $f$ with respect to $x_i$ is defined by

$$\omega^{(i)}(f;\delta) := \max_{0\le x_1,\dots,x_m\le 1} \max_{\substack{|x_i-y_i|\le\delta \\ x_i,y_i\in[0,1+p]}} \left|f\left(x_1,\dots,x_{i-1},x_i,x_{i+1,\dots,}x_m\right)-f\left(x_1,\dots,x_{i-1},y_i,x_{i+1,\dots,}x_m\right)\right|.$$

Furthermore, it is well known that

$$\lim_{\delta\to 0}\omega(f;\delta) = 0$$

and for any $\lambda > 0$,

$$\omega(f;\lambda\delta) \le (\lambda+1)\,\omega(f;\delta). \tag{4.1}$$

Note that the same properties are satisfied by partial modulus of continuities. Our quantitative results in terms of full and partial modulus of continuies are presented in the following theorem.

**Theorem 3.** *Let $f \in C_{\mathbb{R}^m}(D^p)$, then for all $\mathbf{x} = (x_1,\dots,x_m) \in D^0$, the following inequalities*

$$i)\ \left|B_{n,p}^{(\alpha_1,\dots,\alpha_m)}(f;\mathbf{x})-f(\mathbf{x})\right| \le 2\omega\left(f;\sqrt{\delta_n(\mathbf{x})}\right),$$

$$ii)\ \left|B_{n,p}^{(\alpha_1,\dots,\alpha_m)}(f;\mathbf{x})-f(\mathbf{x})\right| \le 2\sum_{i=1}^{m}\omega^{(i)}\left(f;\sqrt{\delta_n^{(i)}(x_i)}\right) \tag{4.2}$$

*holds, where*

$$\delta_n(\mathbf{x}) = \left(\left(\frac{n+p}{b_n}-1\right)^2-\frac{n+p}{b_n^2}\right)|\mathbf{x}|^2+\sum_{j=1}^{m}\left[\frac{n+p}{b_n^2}(2\alpha_j+1)-2\frac{\alpha_j}{b_n}\right]x_j+\frac{|\boldsymbol{\alpha}|^2}{b_n^2}$$

*and*

$$\delta_n^{(i)}(x_i) = \left(\left(\frac{n+p}{b_n}-1\right)^2-\frac{n+p}{b_n^2}\right)x_i^2+\left[\frac{n+p}{b_n^2}(2\alpha_i+1)-2\frac{\alpha_i}{b_n}\right]x_i+\frac{\alpha_i^2}{b_n^2}.$$

*Proof.* $i)$ From (2.1) we get

$$B_{n,p}^{(\alpha_1,\dots,\alpha_m)}(f;\mathbf{x})-f(\mathbf{x})$$

$$= \sum_{k_1,\ldots,k_m=0}^{n+p} \left( f\left(\frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n}\right) - f(\mathbf{x}) \right) \prod_{j=1}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right],$$

then

$$\begin{aligned}
&\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \\
&\leq \sum_{k_1,\ldots,k_m=0}^{n+p} \left| f\left(\frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n}\right) - f(\mathbf{x}) \right| \prod_{j=1}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right] \\
&\leq \sum_{k_1,\ldots,k_m=0}^{n+p} \omega\left( f; \sqrt{\sum_{i=1}^{m} \left( \frac{k_i+\alpha_i}{b_n} - x_i \right)^2} \right) \prod_{j=1}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right].
\end{aligned}$$

Then by using (4.1), and then applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \leq \omega(f;\delta_n(x)) \\
&\times \{ 1 + \frac{1}{\delta_n(x)(\mathbf{x})} ( \sum_{k_1,\ldots,k_m=0}^{n+p} \sum_{i=1}^{m} \left( \frac{k_i+\alpha_i}{b_n} - x_i \right)^2 \\
&\prod_{j=1}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right] )^{1/2} \} \\
&= \omega(f;\delta_n(x)) + \frac{\omega(f;\delta_n(x))}{\delta_n(\mathbf{x})} \left( B_{n,p}^{(\alpha_1,\ldots,\alpha_m)} \left( (\mathbf{t}-\mathbf{x})^2;\mathbf{x} \right) \right)^{1/2}.
\end{aligned}$$

By choosing

$$\delta_n(\mathbf{x}) := \sqrt{B_{n,p}^{(\alpha_1,\ldots,\alpha_m)} \left( (\mathbf{t}-\mathbf{x})^2;\mathbf{x} \right)},$$

then we can write the following inequality

$$\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \leq 2\omega\left( f; \sqrt{\delta_n(\mathbf{x})} \right).$$

*ii*) In a similar manner, we can obtain the inequality (4.2). □

Now, we want to give the quantitative result in terms of the Lipschitz class functionals. It is known that $f \in D^p$ belongs to $Lip_M(\gamma)$ if

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M \left( \sum_{i=1}^{m} (x_i - y_i)^2 \right)^{\gamma/2}$$

and $f \in Lip_M(\gamma_1, \ldots, \gamma_m)$, if

$$\left| f(\mathbf{x}) - f\left(x_1, \ldots, x_{i-1}, y_i, x_{i+1,\ldots,} x_m\right) \right| \leq \sum_{i=1}^{m} M_i |x_i - y_i|^{\gamma_i}, \; i = 1, 2, \ldots, m$$

where $0 < \gamma \leq 1$ and $0 < \gamma_i \leq 1$ $(i = 1, 2, \ldots, m)$, $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$.

**Theorem 4.** *For all $f \in Lip_M(\gamma)$, we have*

$$\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \leq M \left( \left( \left( \frac{n+p}{b_n} - 1 \right)^2 - \frac{n+p}{b_n^2} \right) |\mathbf{x}|^2 \right.$$
$$\left. + \sum_{j=1}^{m} \left[ \frac{n+p}{b_n^2} (2\alpha_j + 1) - 2\frac{\alpha_j}{b_n} \right] x_j + \frac{|\boldsymbol{\alpha}|^2}{b_n^2} \right)^{\gamma/2}.$$

**Theorem 5.** *For all $f \in Lip_M(\gamma_1, \ldots, \gamma_m)$, we have*

$$\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \leq \sum_{i=1}^{m} M_i \left( \left( \left( \frac{n+p}{b_n} - 1 \right)^2 - \frac{n+p}{b_n^2} \right) x_i^2 \right.$$
$$\left. + \left[ \frac{n+p}{b_n^2} (2\alpha_i + 1) - 2\frac{\alpha_i}{b_n} \right] x_i + \frac{\alpha_i^2}{b_n^2} \right)^{\gamma_i/2}.$$

*Remark* 3. In Table 1, the error of approximation of the function $f(x_1, x_2) = \cos(x_1 x_2)$ by the $B_{n,p}^{(\alpha_1,\alpha_2)}(f; x_1, x_2)$ is given, where we choose

$$alpha_1 = \alpha_2 = \frac{1}{100}, \frac{1}{90}, \frac{1}{80}, \frac{1}{70}, \frac{1}{60}$$

and $b_n = 1/(\log(n+p+1) - \log(n+p))$.

TABLE 1. Error estimation table for full moduli of continuity

| (for p=1) | $n = 20$ | $n = 30$ | $n = 50$ | $n = 100$ | $n = 150$ |
|---|---|---|---|---|---|
| $x_1 = x_2 = 0.1$ | 4.4180e-005 | 2.8893e-005 | 1.7007e-005 | 8.3609e-006 | 5.5395e-006 |
| $x_1 = x_2 = 0.25$ | 1.5877e-004 | 1.5877e-004 | 1.5877e-004 | 8.0214e-005 | 5.3661e-005 |
| $x_1 = x_2 = 0.5$ | 3.8730e-005 | 2.9857e-005 | 2.0044e-005 | 1.0882e-005 | 7.4546e-006 |
| $x_1 = x_2 = 0.75$ | 0.091 | 0.0630 | 0.0039 | 0.0020 | 0.0013 |
| $x_1 = x_2 = 0.99$ | 0.0356 | 0.0246 | 0.0152 | 0.0770 | 0.0520 |

## 5. Inverse theorems

In this section, we obtain the inverse theorems of multivariate Bernstein-Schurer-Stancu operators.

**Theorem 6.** *If $f \in C_{\mathbb{R}^m}(D^p)$ such that*

$$\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f(\mathbf{x}) \right| \leq M \left( \frac{1}{n+p} \right)^{\gamma}, \quad (0 < \gamma \leq 1)$$

*for some positive constant $M$, then $f \in Lip_M(\gamma)$.*

*Proof.* For fixed $i = 1,2,\ldots,m$, taking partial derivative of (2.1) with respect to $x_i$ we get

$$\begin{aligned}
&\frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x_1,\ldots,x_i,\ldots,x_m)}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\left[ \sum_{k_1,\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n} \right) \right. \\
&\left. \times \prod_{j=1}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right] \right] \\
&= \frac{\partial}{\partial x_i}\left[ \sum_{k_1,\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n} \right) \right. \\
&\left. \times \left[ \binom{n+p}{k_i} x_i^{k_i} (1-x_i)^{n+p-k_i} \right] \prod_{\substack{j=1 \\ j\neq i}}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right] \right] \\
&= \sum_{k_i=1}^{n+p} \sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n} \right) \\
&\times \frac{(n+p)!}{(k_i-1)!\,(n+p-k_i)!} x_i^{k_i-1} (1-x_i)^{n+p-k_i} \prod_{\substack{j=1 \\ j\neq i}}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right] \\
&- \sum_{k_i=0}^{n+p-1} \sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n} \right)
\end{aligned}$$

$$\times \frac{(n+p)!}{k_i!\,(n+p-k_i-1)!} x_i^{k_i} (1-x_i)^{n+p-k_i-1} \prod_{\substack{j=1\\ j\neq i}}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right]$$

$$= \sum_{k_i=0}^{n+p-1} \sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n}, \cdots, \frac{k_i+1+\alpha_i}{b_n}, \cdots, \frac{k_m+\alpha_m}{b_n} \right)$$

$$\times \frac{(n+p)\,(n+p-1)!}{k_i!\,(n+p-k_i-1)!} x_i^{k_i} (1-x_i)^{n+p_i-k_i-1} \prod_{\substack{j=1\\ j\neq i}}^{m} \left[ \binom{n+p_j}{k_j} x_j^{k_j} (1-x_j)^{n+p_j-k_j} \right]$$

$$- \sum_{k_i=0}^{n+p-1} \sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p} f\left( \frac{k_1+\alpha_1}{b_n}, \cdots, \frac{k_i+\alpha_i}{b_n}, \cdots, \frac{k_m+\alpha_m}{b_n} \right)$$

$$\times \frac{(n+p)\,(n+p-1)!}{k_i!\,(n+p-k_i-1)!} x_i^{k_i} (1-x_i)^{n+p_i-k_i-1} \prod_{\substack{j=1\\ j\neq i}}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right].$$

Now, we get

$$\begin{aligned} &\frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i} \\ &= (n+p) \sum_{k_i=0}^{n+p-1} \sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p} \\ &\qquad \left( f\left( \frac{k_1+\alpha_1}{b_n}, \cdots, \frac{k_i+1+\alpha_i}{b_n}, \cdots, \frac{k_m+\alpha_m}{b_n} \right) \right. \\ &\qquad \left. - f\left( \frac{k_1+\alpha_1}{b_n}, \cdots, \frac{k_i+\alpha_i}{b_n}, \cdots, \frac{k_m+\alpha_m}{b_n} \right) \right) \end{aligned}$$

$$\times \frac{(n+p-1)!}{k_i!\,(n+p-k_i-1)!} x_i^{k_i} (1-x_i)^{n+p-k_i-1} \prod_{\substack{j=1\\ j\neq i}}^{m} \left[ \binom{n+p}{k_j} x_j^{k_j} (1-x_j)^{n+p-k_j} \right].$$

Then, by triangle inequality,

$$\begin{aligned} &\left| \frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i} \right| \\ &\leq (n+p) \sum_{k_i=0}^{n+p-1} \sum_{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_m=0}^{n+p} \end{aligned}$$

$$\left| f\left(\frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+1+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n}\right) - f\left(\frac{k_1+\alpha_1}{b_n},\cdots,\frac{k_i+\alpha_i}{b_n},\cdots,\frac{k_m+\alpha_m}{b_n}\right)\right|$$

$$\times\frac{(n+p-1)!}{k_i!\,(n+p-k_i-1)!}x_i^{k_i}\,(1-x_i)^{n+p-k_i-1}\prod_{\substack{j=1\\ j\neq i}}^{m}\left[\binom{n+p}{k_j}x_j^{k_j}\,(1-x_j)^{n+p-k_j}\right].$$

Using (4.2), we have

$$\begin{aligned}
&\left|\frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i}\right| \\
&\leq (n+p)\sum_{k_i=0}^{n+p-1}\sum_{k_1,\ldots,k_{i-1},k_{i+1}\ldots,k_m=0}^{n+p}\omega^{(i)}\left(f;\frac{1}{b_n}\right)\frac{(n+p-1)!}{k_i!\,(n+p-k_i-1)!} \\
&\times x_i^{k_i}\,(1-x_i)^{n+p-k_i-1}\prod_{\substack{j=1\\ j\neq i}}^{m}\left[\binom{n+p}{k_j}x_j^{k_j}\,(1-x_j)^{n+p-k_j}\right] \\
&= (n+p)\sum_{k_i=0}^{n+p-1}\omega^{(i)}\left(f;\frac{1}{b_n}\right)\frac{(n+p-1)!}{k_i!\,(n+p-k_i-1)!}x_i^{k_i}\,(1-x_i)^{n+p-k_i-1} \\
&\leq (n+p)\,\omega^{(i)}(f;\delta)\sum_{k_i=0}^{n+p-1} \\
&\qquad\left(\left(1+\frac{1}{\delta b_n}\right)\frac{(n+p-1)!}{k_i!\,(n+p-k_i-1)!}x_i^{k_i}\,(1-x_i)^{n+p-k_i-1}\right) \\
&= (n+p)\,\omega^{(i)}(f;\delta)\left\{1+\frac{1}{\delta b_n}\right\} = \omega^{(i)}(f;\delta)\left\{n+p+\frac{n+p}{\delta b_n}\right\}.
\end{aligned}$$

For any pair of points $x$, $y$ in $[0,1]$, we can write

$$\begin{aligned}
\int_x^y\left|\frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i}\right|dx_i &\leq \omega^{(i)}(f;\delta)\left\{n+p+\frac{n+p}{\delta b_n}\right\}\int_x^y dx_i \\
\int_x^y\left|\frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i}\right|dx_i &\leq \omega^{(i)}(f;\delta)\,|x-y|\left\{n+p+\frac{n+p}{\delta b_n}\right\}.
\end{aligned}$$

By choosing $\delta_n := \dfrac{1}{n+p}$, we obtain

$$\int_x^y \left| \frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i} \right| dx_i \leq \omega^{(i)}(f;\delta)\,|x-y| \left\{ \frac{1}{\delta_n} + \frac{n+p}{\delta b_n} \right\}. \tag{5.1}$$

Then the sequences $\delta_n$ decreases to zero as $n \to \infty$. Also $\delta_{n-1} \leq 2\delta_n$ is satisfied for $n = 2, 3, \cdots$. Hence for a given $0 < \delta \leq 1$ there exists $n \in \mathbb{N}$ such that

$$\delta_n \leq \delta \leq \delta_{n-1} \leq 2\delta_n. \tag{5.2}$$

Using (5.2) in (5.1), we have

$$\int_x^y \left| \frac{\partial B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})}{\partial x_i} \right| dx_i \leq \left\{ 2 + \frac{n+p}{b_n} \right\} \frac{1}{\delta} \omega^{(i)}(f;\delta)\,|x-y|. \tag{5.3}$$

On the other hand, since we have

$$\begin{aligned} &|f(\mathbf{x}) - f(\mathbf{y})| \leq |f(\mathbf{x}) - f(y_1, x_2, \ldots, x_m)| \\ &+ |f(y_1, x_2, \ldots, x_m) - f(y_1, y_2, \ldots, x_m)| + \cdots \\ &+ |f(y_1, \ldots, y_{m-1}, \ldots, x_m) - f(\mathbf{y})| \end{aligned}$$

and

$$\begin{aligned} &|f(\mathbf{x}) - f(y_1, x_2, \ldots, x_m)| \leq \left| f(\mathbf{x}) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) \right| \\ &+ \left| f(y_1, x_2, \ldots, x_m) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, x_2, \ldots, x_m) \right| \\ &+ \int_{x_1}^{y_1} \left| \frac{\partial}{\partial x} B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; x, x_2, \ldots, x_m) \right| dx, \\ &|f(y_1, x_2, \ldots, x_m) - f(y_1, y_2, x_3 \ldots, x_m)| \\ &\leq \left| f(y_1, x_2, \ldots, x_m) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, x_2, \ldots, x_m) \right| \\ &\quad + \left| f(y_1, y_2, \ldots, x_m) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, y_2, \ldots, x_m) \right| \\ &\quad + \int_{x_2}^{y_2} \left| \frac{\partial}{\partial x} B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, x, \ldots, x_m) \right| dx \\ &\quad \vdots \\ &|f(y_1, y_2, \ldots, x_m) - f(y_1, y_2, \ldots, y_m)| \\ &\leq \left| f(y_1, \ldots, y_{m-1}, x_m) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, \ldots, y_{m-1}, x_m) \right| \\ &+ \left| f(y_1, y_2, \ldots, y_m) - B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f; y_1, y_2, \ldots, y_m) \right| \end{aligned}$$

$$+\int_{x_m}^{y_m}\left|\frac{\partial}{\partial x}B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,y_2,\ldots,x)\right|dx$$

we get, for all fixed $n\in\mathbb{N}$

$$\begin{aligned}
&|f(\mathbf{x})-f(y_1,\ldots,y_m)|\\
&\leq\left|f(\mathbf{x})-B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x})\right|\\
&+\left|f(y_1,x_2,\ldots,x_m)-B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,x_2,\ldots,x_m)\right|\\
&+\left|f(y_1,x_2,\ldots,x_m)-B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,x_2,\ldots,x_m)\right|\\
&+\left|f(y_1,y_2,\ldots,x_m)-B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,y_2,\ldots,x_m)\right|\\
&+\cdots+\left|f(y_1,\ldots,y_{m-1},y_m)-B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,\ldots,y_m)\right|\\
&\quad+\int_{x_1}^{y_1}\left|\frac{\partial}{\partial x}B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;x,x_2\ldots,x_m)\right|dx \qquad (5.4)\\
&\quad+\int_{x_2}^{y_2}\left|\frac{\partial}{\partial x}B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,x,\ldots,x_m)\right|dx\\
&\quad+\cdots+\int_{x_m}^{y_m}\left|\frac{\partial}{\partial x}B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;y_1,y_2,\ldots,x)\right|dx. \qquad (5.5)
\end{aligned}$$

Using (5.2) and (5.3) in (5.5), we obtain

$$\begin{aligned}
|f(\mathbf{x})-f(\mathbf{y})|&\leq 2mM\delta^{\gamma}+\sum_{i=1}^{m}\left(2+\frac{n+p}{b_n}\right)\frac{1}{\delta}\omega^{(i)}(f;\delta)\left(\sum_{j=1}^{m}(x_j-y_j)^2\right)^{1/2},\\
&\leq 2mM\delta^{\gamma}+\left(2+\frac{n+p}{b_n}\right)\frac{m}{\delta}\omega(f;\delta)\left(\sum_{j=1}^{m}(x_j-y_j)^2\right)^{1/2}\\
&\leq K_1\left(\delta^{\gamma}+\frac{\omega(f;\delta)}{\delta}\left(\sum_{j=1}^{m}(x_j-y_j)^2\right)^{1/2}\right),
\end{aligned}$$

where $K_1=\max\left\{2mM,m\left(2+\frac{n+p}{b_n}\right)\right\}$.

Therefore, taking $\left(\sum_{j=1}^{m}(x_j-y_j)^2\right)^{1/2}\leq h$ with $0<h\leq 1$ and considering the definition of $\omega$, we write

$$\omega(f;h)\leq K_1\left(\delta^{\gamma}+\frac{h}{\delta}\omega(f;\delta)\right),\quad 0<h,\delta\leq 1. \qquad (5.6)$$

Then, by (5.6) we have $K_2 h^{\gamma}$. Hence, we get

$$\omega(f;h) \leq K_2 h^{\gamma} \quad (0<\gamma<1). \tag{5.7}$$

From (5.7), we obtain $f \in Lip\gamma$. We get the desired result. □

In a similar manner, we can state the following theorem:

**Theorem 7.** *If the inequality*

$$\left| B_{n,p}^{(\alpha_1,\ldots,\alpha_m)}(f;\mathbf{x}) - f\left(x_1,\ldots,x_{j-1},y_j,x_{j+1},\ldots,x_m\right)\right| \leq M_j \left(\frac{1}{n+p}\right)^{\gamma_j}$$

*is satisfied for* $0<\gamma_j \leq 1$ *and for some constants* $M_j$, $(j=1,2,\ldots,m)$, *then*

$$f \in \bigcap_{j=1}^{m} Lip_{M_j}\gamma_j.$$

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# COMPLEX $B$-SPLINE COLLOCATION METHOD FOR SOLVING WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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*Abstract.* In this paper we propose a new collocation type method for solving Volterra integral equations of the second kind with weakly singular kernels. In this method we use the complex $B$-spline basics in collocation method for solving Volterra integral. We compare the results obtained by this method with exact solution. A few numerical examples are presented to demonstrate the effectiveness of the proposed method.

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*Keywords:* Volterra integral equation, complex $B$-spline, collocation method

## 1. Introduction

In this paper we consider the Volterra integral equation with the second kind weakly singular kernel, namely

$$u(t) = f(t) + \int_0^t k(t,s)u(s)\,ds, \qquad t \in (a,b], \tag{1.1}$$

where $k(t,s)$ and $f(t)$ are known and $u(t)$ is unknown. The function $k(t,s)$ is called a polar kernel if

$$k(t,s) = \frac{g(t,s)}{(t-s)^{\alpha}}, \quad \alpha \in (0,1),$$

where $g$ is bounded on $s$, $g(t,t) \neq 0$ and for all

$$t,s \in C[a,b]; g(t,s) \in C([a,b]\times[a,b]).$$

We rewrite the equation (1.1) in the following operator form:

$$(I-K)u = f,$$

where the operator $K$ is assumed to be compact on a Banach Space $X$ to $X$.

During the past few decades, this equation has been used to study various problems of mathematical chemistry and physics, such as reactions including stereology, heat conduction with mixed boundary conditions [10], crystal growth, electrochemistry,

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super fluidity and the radiation of heat [7], electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics [12, 13, 27], the particle transport problems of astrophysics, potential theory and Dirichlet problem, electrostatic and radiative heat transfer problems and in some engineering fields [29, 30], astronomy, optics, computational electromagnetic, quantum mechanics, seismology image processing [1, 6].

In addition a method with exponential order convergence rate has been developed by Riley [23] for Volterra integral equations of the form

$$u(t)-\int_a^t (t-s)^{p-1}k(t,s)u(s)ds = f(t), \qquad a \le t \le b, \tag{1.2}$$

where the kernel is also assumed to be weakly singular and the solution $u$ is generally not differentiable at $t = a$. In [16], the equation (1.2) has been solved with fractional B-spline basics.

In most of the cases, it is difficult to obtain analytical solution of integral equations, therefore many numerical methods such as collocation method with different basics [2,19,20], orthogonal bases and wavelets [17,21], Galerkin methods have been developed to solve equation (1.1) [4–6, 14].

Recently, many different basic functions have been utilized to estimate the result of integral equations, such as modified quadrature [24], optimal homotopy asymptotic method [15], Tau approximate method [18].

Spline functions are very efficient and useful in signal processing, mathematical and computer graphics [8, 9, 22, 25, 26]. In [3], Blu and Unser gave an extension of $B$-splines to fractional orders and later in [11], Forster *et. al.* gave an extension to complex power.

In this paper, we solve equation (1.1) by using complex $B$-spline to obtain approximate solution. The paper is organized as follows: In Section 2, we recall some basic definitions and theorems of complex $B$-splines and its properties. Section 3 is devoted to the solution of weakly singular integral equation of second kind using collocation methods with complex $B$-spline basics. In Section 4, by considering numerical examples reported in our work, the accuracy of the proposed scheme is demonstrated.

## 2. Complex $B$-splines

In this section we state some definitions and theorems [11, 20] that will be used later in our work.

**Definition 1.** The inner product $\int \overline{f(x)}g(x)dx$ between two complex $L^2$ functions $f$, $g$ is denoted by $(f,g)$, and the associated Euclidean norm is written as $\|.\|_2$.

**Definition 2.** The Riemann zeta function is defined as $\xi(s) = \sum_{n\ge 1} n^{-s}$ for all real $s > 1$.

**Definition 3.** The basic functions for Schoenberg's polynomial splines with uniform knots [3, 20] are defined as

$$\beta^n(x) = \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(x-j)_+^n \quad x \in \mathbb{R}, n \in \mathbb{N},$$

where

$$(x-j)_+^n = \begin{cases} (x-j)^n & \text{if} \quad x > j, \\ 0 & \text{if} \quad x \leq j. \end{cases}$$

**Definition 4.** $x_+^z$ denotes the truncated power function of complex degree $z$ with knot zero:

$$x_+^z = \begin{cases} x^{\Re z} e^{i\Im z \ln x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

**Definition 5.** The complex $B$-spline $\beta^z$ of complex degree $z$ is defined in $L^2(\mathbb{R})$ via its Plancherel transform as

$$\hat{\beta}^z(\omega) = (\frac{1-e^{-i\omega}}{i\omega})^{z+1}, \tag{2.1}$$

where $z = \alpha + i\gamma$ with parameters $\alpha, \gamma \in \mathbb{R}$ and $\alpha > \frac{-1}{2}$.

**Theorem 1** (cf. [11])**.** *The complex $B$-spline $\beta^z$ is well-defined, uniformly continuous and belongs to the space $L^2(\mathbb{R})$.*

**Theorem 2** (cf. [11])**.** *The time domain representation of the complex $B$-spline $\beta^z$ is given by*

$$\beta^z(x) = \frac{1}{\Gamma(z+1)}\sum_{k\geq 0}(-1)^k\binom{z+1}{k}(x-k)_+^z. \tag{2.2}$$

*This equation is valid pointwise for all $x \in \mathbb{R}$ and $L^2(\mathbb{R})$.*

The complex $B$-splines generate dyadic multiresolution analysis; *i.e.* they generate a sequence of spaces:

$$\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R})$$

with the following properties:

(1) $\cap_i V_i = \{0\}$ and $\overline{\cup_i V_i} = L^2(\mathbb{R})$,
(2) $f(\bullet) \in V_i$ if and only if $f(2^{-i}\bullet) \in V_0$,
(3) $f(\bullet) \in V_0$ if and only if $f(\bullet - k) \in V_0$ for all $k \in \mathbb{Z}$,
(4) there exists a function $\varphi \in V_0$, called a scaling function, such that $\varphi(\bullet - k)_{k\in\mathbb{Z}}$ forms an orthonormal basis of $V_0$.

$V_i$ is the complex $B$-spline of order $z \in \mathbb{C}$ with knot points $k.2^i, k \in \mathbb{Z}$.

**Theorem 3** (cf. [11]). *Let $Re\, z > 0$. Then the spaces*

$$V_i = \overline{span\{\beta^z(\frac{x-2^i k}{2^i})\}^{L^2(\mathbb{R})}} \quad , i \in \mathbb{Z} \tag{2.3}$$

*form a dyadic multiresolution analysis with scaling function $\beta^z$.*

The complex $B$-spline spaces at scale $a$ is defined as

$$S_a^z = \{s_a : \exists c \in \ell^2, s_a(x) = \sum_{k\in\mathbb{Z}} c(k)\beta^z(\frac{x}{a}-k)\}. \tag{2.4}$$

Then given an arbitrary function $f \in L^2(\mathbb{R})$, we determine its least-squares approximation in $S_a^z$ by applying the following orthogonal projection operator

$$P_a f = \sum_{k\in\mathbb{Z}} (f, \frac{1}{a}\tilde{\beta}^z(\frac{\bullet}{a}-k))\beta^z(\frac{\bullet}{a}-k). \tag{2.5}$$

This defines a projector because the functions $\beta^z$ and $\tilde{\beta}^z$ are biorthonormal [11], where $\tilde{\beta}^z \in S_a^z$ is the dual $B$-spline whose Fourier transform is

$$\hat{\tilde{\beta}}^z(\bullet) = \frac{\hat{\beta}^z(\bullet)}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{\beta}^z(\bullet + 2\pi k)|^2}}. \tag{2.6}$$

**Theorem 4** (cf. [28]). *The complex $B$-splines have a fractional order of approximation $\alpha+1$. Specifically, the least-squares approximation error is bounded by:*

$$\forall f \in W_2^{\alpha+1}, \|f - P_a f\|_2 \le \frac{\sqrt{2\xi(\alpha+2)-\frac{1}{2}}}{\pi^{\alpha+1}} \|D^{\alpha+1}\|_2 a^{\alpha+1}. \tag{2.7}$$

## 3. The complex $B$-spline collocation method

To solve approximately the integral equation equation (1.1), we assume that $K$ is compact on a Banach space $X$ to $X$. We choose a finite dimensional family of functions $\tilde{u}(x)$ which is close to the exact solution $u(x)$. In practice, we choose a sequence of dimensional subspaces $X_n \subset X, n \ge 1$, with $X_n$ having dimension $d_n$. Let $X_n$ have a basis $\{\varphi_1, ..., \varphi_d\}$ with $d \equiv d_n$ for notational simplicity. We seek $u_n(x) \in X_n$, which can be written as

$$u_n(x) = \sum_{j=0}^{d} c_j \varphi_j(x), \quad x \in D. \tag{3.1}$$

This is substituted into equation (1.1), and coefficients $\{c_1, ..., c_d\}$ are determined by forcing the equation to be exact in some sense. For later use, we introduce

$$r_n(x) = u_n(x) - \int_D k(x,s)u_n(s)ds - f(x),$$

$$= \sum_{j=1}^{d} c_j \{\varphi_j(x) - \int_D k(x,s)\varphi_j(s)ds\} - f(x), \quad x \in D. \tag{3.2}$$

We pick distinct node $x_1, \cdots, x_d \in D$, and require

$$r_n(x_i) = 0, \quad i = 1, \cdots, d. \tag{3.3}$$

This leads to determining $\{c_1, ..., c_d\}$ as the solution of the linear system

$$\sum_{j=1}^{d} c_j \{u(x_i) - \int_D k(x_i,s)u(s)ds\} = f(x_i), \quad i = 1, \cdots, d. \tag{3.4}$$

In following we show that this method can be used to solve equation (1.1), In this regard we give the following Lemma 1 and Theorem 5.

**Lemma 1** (cf. [2]). *Let $X$ be a Banach space and $P_n$ be a family of bounded projections on $X$ with*

$$P_n u \longrightarrow u \quad as \quad n \longrightarrow \infty, u \in X$$

*and $K : X \longrightarrow X$ be compact. Then*

$$\|K - P_n K\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

**Theorem 5** (cf. [11]). *If $G \in \mathbb{R}$ be an integral equation with a weakly singular kernel then it is a compact operator on $C(G)$, where $C(G)$ is space of continuous real or complex valued functions on compact subsets $G \in \mathbb{R}$.*

**Theorem 6.** *Equation (1.1) can be solved with collocation method by using complex $B$-spline basis.*

*Proof.* If we introduce equation (3.1) to projection operator $P_n$ that maps $X$ onto $X_n$, define $P_n u(x)$ to be that element of $X_n$ that interpolates $X$ at the nodes $\{x_1, ..., x_d\}$. This means writing

$$P_n u(x) = \sum_{j=1}^{d} c_j \varphi_j(x)$$

with the coefficients $\{c_j\}$ determined by solving the linear system

$$\sum_{j=1}^{d} c_j \varphi_j(x_i) = u(x_i), \quad i = 1, \cdots, d$$

Then this linear system has a unique solution if

$$det[\varphi_j(x_j)] \neq 0.$$

From Theorem 1, complex B-spline basis belong to $L^2(\mathbb{R})$ and with the help Theorem 2 this method is convergent. Then in view of Lemma 1 and Theorem 5 we can use collocation method for these type of integral equations. □

Now we can use collocation methods for solving weakly singular integral equation of second kind with complex B-spline basis.

In equation (1.1), let $X = L^2(\mathbb{R})$ and $V_n = X_n$. Then if $u(x) \in L^2(\mathbb{R})$ and $u_n(x) \in V_n$, where

$$u_n(x) = \sum_{j\in\mathbb{Z}} c(j)\beta^z(2^n x - j), \quad j \in \mathbb{Z},$$

with $0 \le t \le b$ and $n \in \mathbb{N}$ then we have

$$u_n^{2^n}(x) = \sum_{j=1-2^n}^{b} c(j)\beta^z(2^n x - j), \quad b \in \mathbb{R}, \tag{3.5}$$

with nodes $x_i = \frac{bi}{2^n}$. Then

$$r_n^{2^n}(x_i) = \sum_{j=1-2^n}^{b} c_j\{u(x_i) - \int_D k(x_i,s)u(s)ds\} - f(x_i) = 0 \quad i = 0,\cdots,b. \tag{3.6}$$

We define the absolute error

$$E_n^{2^n}(u(x)) = \|u(x) - u_n^{2^n}(x)\|_2 = \left(\int_0^b |u(x) - u_n^{2^n}(x)|^2\right)^{\frac{1}{2}}, \tag{3.7}$$

and note that when $n \to \infty$ and $d \to \infty$ then $u_n^{2^n}(x) \to u(x)$.

Using the Theorem 1 and Lemma 1, the relatively error is defined as

$$e_n = \frac{\max_{0\le i\le 2^n} |E_n^{2^n}(u(x_i))|}{\max_{0\le x\le b} |u(x)|}. \tag{3.8}$$

## 4. Illustrative examples

In order to show better the theoretical results of the previous sections, we now consider the numerical solution of the equation (1.1), with various choices of $f(x)$ for $x \in [0,1] = D$. By using equation (3.6), we obtain $\{c_1,...,c_d\}$. Then in view of (3.7) and (3.8) at several points of interval $D$ we obtain the absolute and the relative errors.

*Example* 1. Let $b = 1$, $g(t,s) = ts$ and $f(x) = x(1-x) + \frac{16}{105}x^{\frac{7}{2}}(7-6x)$ with the exact solution $u(x) = x(1-x)$. Table 1 shows the absolute errors obtained by the knot points $x_i = \frac{i}{2^n}; i = 0,...,2^n$ with $z = 0.5 + i$.

In Figure 1, the horizontal axis represents the $n$ index's $V_n$ the vertical axis represents the relative error $e_n$ is intentional, as can be seen, by increasing the index of $n$, the relative error decreases.

From Table 1 we see that the maximum error occurs at point $x = 1$. We now show the relative error for different interval of $z$ this point ($x = 1$) in Tables 3, 4 and 5.

TABLE 1. Absolute Errors

| $x$ | $E_0^{2^0}(u(x))$ | $E_1^{2^1}(u(x))$ | $E_2^{2^2}(u(x))$ | $E_3^{2^3}(u(x))$ | $E_4^{2^4}(u(x))$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.180375 | 0.119014 | 0.0013166 | 0.0008720 | 0.0005532 |
| 0.5 | 0.1676936 | 0.126606 | 0.019329 | 0.013126 | 0.00670011 |
| 0.75 | 0.1676936 | 0.126606 | 0.019329 | 0.0131267 | 0.00670011 |
| 1 | 0.0720055 | 0.0578148 | 0.0425467 | 0.0263254 | 0.0111375 |

TABLE 2. Relatively Errors

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $e_n$ | 0.837556 | 0.506424 | 0.1701868 | 0.10530168 | 0.04455 |

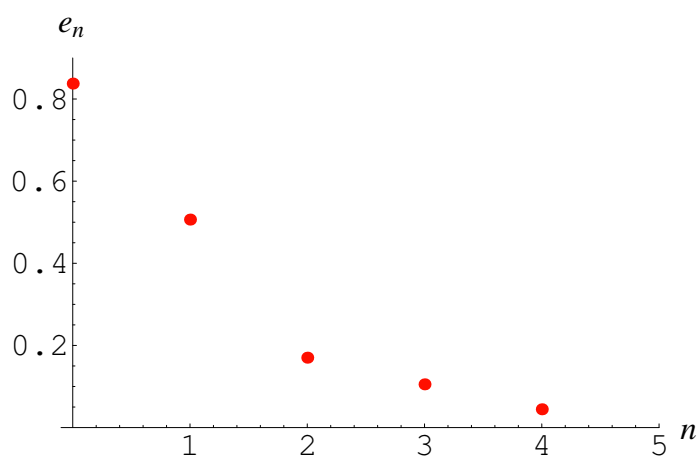


FIGURE 1. Points relative error in spaces $V_n$.

TABLE 3. The relative error for $|z| < 1$

| $z$ | 0.1+0.01i | 0.5+0.1i | 0.2+0.9i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.00913271 | 0.00301154 | 0.01920683 |

TABLE 4. The relative error for $1 \leq |z| < 2$

| $z$ | 1+i | 1+0.5i | 1+0.9i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.00745215 | 0.00248121 | 0.00842375 |

TABLE 5. The relative error for $2 \leq |z| < 3$

| $z$ | 2+0.1i | 2+0.5i | 2+1i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.0054914 | 0.00949278 | 0.0301505 |

We note that when $1 \leq |z| < 2$ , the error is minimum.

*Example* 2. Let $b = 1$, $g(x,s) = 1$ and $f(x) = \frac{1}{2}\pi x + \sqrt{x}$ with the exact solution $u(x) = \sqrt{x}$ and $z = 0.5 + 0.5i$. Table 6 shows the absolute error obtained by the knot points $x_i = \frac{i}{2^n}, \quad i = 0, ..., 2^n$ with $z = 0.5 + 0.5i$.

TABLE 6. The absolute errors

| x | $E_0^{2^0}(u(x))$ | $E_1^{2^1}(u(x))$ | $E_2^{2^2}(u(x))$ | $E_3^{2^3}(u(x))$ | $E_4^{2^4}(u(x))$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.290563 | 0.126959 | 0.0413826 | 0.0199285 | 0.000803727 |
| 0.5 | 0.168809 | 0.0702198 | 0.0333131 | 0.0130709 | 0.0044490 |
| 0.75 | 0.0356813 | 0.0508203 | 0.00297352 | 0.00907127 | 0.00297352 |
| 1 | 0.1156 | 0.0540076 | 0.0205252 | 0.0067599 | 0.00221725 |

TABLE 7. The relatively error

| $m$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $e_m$ | 0.290563 | 0.1269598 | 0.0413826 | 0.0199285 | 0.00803727 |

In Figure 2, the horizontal axis represents the $n$ index's $V_n$ and vertical axis represents the relative error $e_n$ is intentional, as can be seen by increasing the index of $n$ relative error decreases.

From Table 6 we see that the maximum error occurs at point $x = 0.5$. We now show the relative error for different interval of $z$ this point($x = 0.5$) in Tables 8, 9 and 10.

TABLE 8. The relative error for $|z| < 1$

| $z$ | 0.01+0.1i | 0.5+0.01i | 0.9+0.1i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.0156468 | 0.000083763 | 0.00115488 |

Here we note that when $|z| < 1$, the error is minimum.

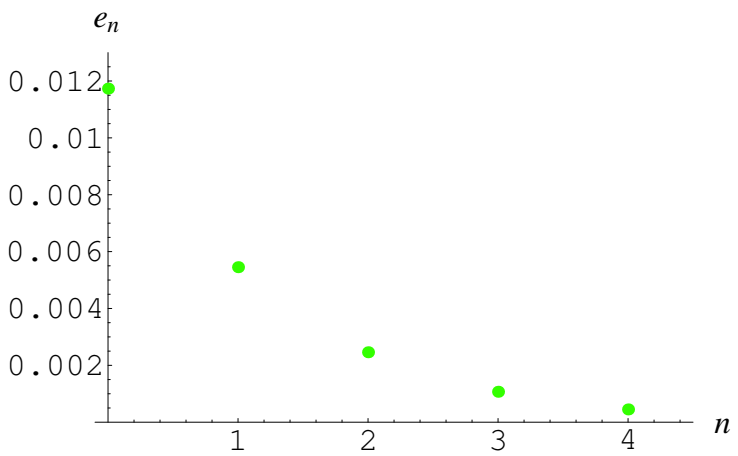


FIGURE 2. Points relative error in spaces $V_n$.

TABLE 9. The relative error for $1 \leq |z| < 2$

| $z$ | 1+0.1i | 0.5+i | 0.9+i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.00112826 | 0.019533 | 0.00526669 |

TABLE 10. The relative error for $2 \leq |z| < 3$

| $z$ | 1.5+i | 2+0.1i | 2+0.5i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.022596 | 0.0175306 | 0.0224039 |

*Example* 3. Let $a = -1$ , $b = 1$, $g(x,s) = 1$ and $f(x) = \sqrt{x+1} + \frac{1}{9}(x+1)^{\frac{3}{2}}(3\ln(x+1)^2 - 16 + \ln(4096))$ with the exact solution $u(x) = \sqrt{x+1}$.
Table 11 shows the absolute error obtained by the knot points $x_i = \frac{i}{2^n} - 1, \quad i = 0, ..., 2^n$ with $z = 1 + 0.5i$.

TABLE 11. The absolute errors

| $x$ | $E_0^{2^0}(u(x))$ | $E_1^{2^1}(u(x))$ | $E_2^{2^2}(u(x))$ | $E_3^{2^3}(u(x))$ | $E_4^{2^4}(u(x))$ |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 0 | 0 | 0 |
| -0.5 | 0.016592 | 0.0056849 | 0.0026678 | 0.0012071 | 0.00052264 |
| 0 | 0.016534 | 0.0077106 | 0.0034793 | 0.0015144 | 0.00063639 |
| 0.5 | 0.007966 | 0.00063781 | 0.0030579 | 0.0013167 | 0.00060392 |
| 1 | 0.019279 | 0.00068 | 0.0004755 | 0.00002275 | 0.00001357 |

TABLE 12. The relative errors

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $e_n$ | 0.0117325 | 0.0054522 | 0.0024602 | 0.0010708 | 0.00044999 |

In Figure 3, the horizontal axis represents the $n$ index's $V_n$, axis represents the relative error $e_n$ is intentional, as can be seen by increasing the index of $n$ relative error decreases.

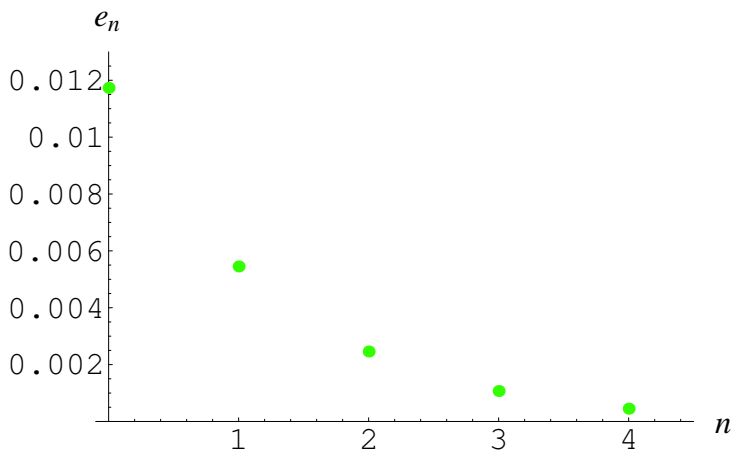


FIGURE 3. Points relative error in spaces $V_n$.

From Table 12 see that the maximum error occurs at point $x = 0.5$. Now show the relative error for different interval of $z$ this point($x = 0.5$) in Tables 13, 14 and 15.

TABLE 13. The relative error for $|z| < 1$

| $z$ | 0.1+0.1i | 0.5+0.1i | 0.2+0.9i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.080492 | 0.00864855 | 0.086332 |

TABLE 14. The relative error for $1 \leq |z| < 2$

| $z$ | 0.1+i | 1+0.5i | 0.9+i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.100855 | 0.0415961 | 0.0482324 |

We see that when $|z| < 1$, the error is minimum.

TABLE 15. The relative error for $2 \leq |z| < 3$

| $z$ | 1+i | 2+0.5i | 2+i |
|---|---|---|---|
| $E_4^{2^4}(u(x))$ | 0.0298086 | 0.0398539 | 0.0419062 |

## 5. CONCLUSION

In this paper, we proposed an efficient algorithm for solving Volterra integral equations of second kind with weakly singular kernels by collocation type method. We used complex $B$-spline basics as basic functions in the collocation method. This approach gives better solution with respect to ordinary $B$-spline basics function. We presented three numerical examples which demonstrated That our proposal method is very attractive. *Mathematica* has been used in this paper for computation.

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# PI-PROPERTIES OF SOME MATRIX ALGEBRAS WITH INVOLUTION

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*Abstract.* We define the nilpotency index of the $\flat$-variables in second order matrix algebras with Grassmann entries and involution $\flat$. Identities of minimal degree are found for a concrete subalgebra of the matrix algebra $M_4(K)$. When it has an involution $\phi$ as well some of its $\phi$-identities are given. For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution $(\flat)$ is introduced and its $(\flat)$-identities are discussed.

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*Keywords:* PI-algebras, matrix algebra over Grassmann algebras, algebras with involution $\phi$, $\phi$-variables, $\phi$-identities

## 1. Introduction

The classical PI-theory (the theory of the polynomial identities) has its development for algebras with involution as well. The contributions of Amitsur [1], Levchenko [9], Rowen [14], Wenxin and Racine [17], Giambruno and Valenti [6], Drensky and Giambruno [5], Rashkova [11], La Mattina and Misso [8] are only a part of it.

In 1973 Krasovski and Regev [7] described completely the $T$-ideal of the identities of the Grassmann algebra $E$ and it was a natural step to investigate the PI-structure of algebras not only over fields (with any characteristic) but over algebras as well, especially Grassmann algebras [4, 12, 16].

In the paper we consider mainly finite dimensional Grassmann algebras and special matrix algebras over them.

We recall the definition of the Grassmann algebra $E$ as:

$$E = K\langle e_1, e_2, \ldots | e_i e_j + e_j e_i = 0,\ i, j = 1, 2, \ldots\rangle,$$

where $K$ is a field of characteristic zero.

We cite basic propositions from [3,7]. The notation $[x, y, z] = [[x, y], z] = [x, y]z - z[x, y]$ will be used.

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**Proposition 1** ([7, Corollary, p. 437]). *The $T$ - ideal of the Grassmann algebra $E$ is generated by the identity $[x,y,z]=0$.*

**Proposition 2** ([3, Lemma 6.1]). *For any $n,k\geq 2$ in the algebra $E$ the identity $S_n^k(x_1,...,x_n)=0$ holds, where*

$$S_n(x_1,...,x_n)=\sum_{\sigma\in Sym(n)}(-1)^{\sigma}x_{\sigma(1)}...x_{\sigma(n)}$$

*is the $n$-th standard polynomial.*

**Proposition 3** ([3, Lemma 6.6]). *The matrix algebra $M_n(E)$ does not satisfy the identity*

$$S_m^n(x_1,...,x_m)=0$$

*for any $m$.*

There are subalgebras of $M_n(E)$ however being counter examples of Proposition 3 for concrete $m$.

We use the notation $E_n^{'}$ for a non unitary Grassmann algebra with generators $e_1,...,e_n$.

The existence of nilpotent elements of minimal nilpotency index both in finite dimensional Grassmann algebras and in matrix algebras over them was investigated in [12, 13]. We state some of the results needed:

**Proposition 4** ([13, Proposition 13]). *The identity $x^3=0$ holds for the algebra $E_4^{'}$.*

**Proposition 5** ([13, Proposition 16]). *The algebra $M_2(E_4^{'})$ satisfies the identity $X^4=0$.*

In [13] examples were given as well of subalgebras $\mathfrak{A}_i, i=1,2$ of $M_n(\mathfrak{R})$ such that the identities $x^4=0$ and $[x,y,z]=0$ in $\mathfrak{R}$ imply the identity $X^4=0$ in $\mathfrak{A}_i, i=1,2$.

An involution $\psi$ on the Grassmann algebras $E_2^{'}$ and $E_3^{'}$ defines an involution $\phi$ on the corresponding $2\times 2$ matrix algebra over any of them. In that case the classes of symmetric and of skew-symmetric to the involution $\phi$ matrices of nilpotency indices 2 and 3 were described in [12].

In the present paper we continue the investigations started in [12]:

We define the nilpotency index of the $\flat$-variables in the considered algebras with involution $\phi=\flat$.

For a concrete subalgebra of the matrix algebra $M_4(K)$ identities of minimal degree are found. When additionally the algebra has an involution $\phi$ some of its $\phi$-identities are given.

For an analogue of this subalgebra over finite dimensional Grassmann algebras a new involution $\phi=(\flat)$ is introduced and some $(\flat)$-identities are discussed.

## 2. Results

### 2.1. *PI-properties of involution second order matrix algebras with Grassmann entries*

We recall the definition of an involution on an algebra $R$: it is a second order antiautomorphism $\psi$ such that $\psi(ab) = \psi(b)\psi(a)$ for all $a, b \in R$.

By $R^-$ we denote the skew-symmetric due to the involution elements of $R$, namely $z_1, .., z_i, ...$ and by $R^+$ we denote the symmetric due to the involution elements $y_1, .., y_j, ....$ It is important to consider $\psi$-variables (symmetric and skew-symmetric) as the elements of $R^+$ form a Jordan algebra due to the multiplication $y_1 \circ y_2 = y_1 y_2 + y_2 y_1$ and the elements of $R^-$ form a Lie algebra due to the operation $[z_1, z_2]$.

**Definition 1.** Let $f = f(x_1, ..., x_m) \in K\langle x_1, ..., x_n\rangle$, the free associative algebra on $n$ generators over $K$. We say that $f$ is a $\psi$-identity in skew variables for the algebra $R$ over $K$ if $f(z_1, ..., z_m) = 0$ for all $z_1, ..., z_m \in R^-$. Accordingly $f$ is a $\psi$-identity in symmetric variables for the algebra $R$ over $K$ if $f(y_1, ..., y_m) = 0$ for all $y_1, ..., y_m \in R^+$.

We say that $f$ is a $\psi$-identity if $f(z_1, ..., z_i, y_{i+1}, ..., y_m) = 0$ for any $z_1, ..., z_i \in R^-$ and any $y_{i+1}, ..., y_m \in R^+$.

We denote an involution on the basic field or algebra as $\psi$ while $\phi$ will mean an involution on the corresponding matrix algebra.

If a ring $R$ has an involution $\psi = *$ two involutions $\phi_1 = \sharp$ and $\phi_2 = \flat$ on $M_2(R)$ are defined as follows [15]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\flat} = \begin{pmatrix} d^* & b^* \\ c^* & a^* \end{pmatrix}.$$

It is known [2] that two involutions play an important role in the Grassmann algebra: the involution $\psi_1$ acting on the generators $e_i$ of $E$ as $\psi_1(e_{2k}) = e_{2k-1}$, $\psi_1(e_{2k-1}) = e_{2k}$ and the trivial on the generators involution $\psi_2$ for which $\psi_2(e_i) = e_i$ for all $e_i$.

Here we consider the algebra $(M_2(E_4', \psi_2), \flat)$ and continue some of the investigations made in [12] by finding the nilpotency index of the $\flat$-variables of $(M_2(E_4', \psi_2), \flat)$.

**Theorem 1.** *The algebra $(M_2(E_4', \psi_2), \flat)$ satisfies the $\flat$-identity $Y^4 = 0$ in $\flat$-symmetric variables and the $\flat$-identity $Z^3 = 0$ in $\flat$-skew symmetric variables.*

*Proof of Theorem 1.* As Proposition 5 holds we have to prove only that $Z^3 = 0$ in $\flat$-skew symmetric variables.

Let $Z = \begin{pmatrix} y_1 & z_1 \\ z_2 & y_2 \end{pmatrix}$. The condition $\phi_2(Z) = -Z$ means that $\psi_2(z_1) = -z_1$, $\psi_2(z_2) = -z_2$, $\psi_2(y_1) = -y_2$ and $\psi_2(y_2) = -y_1$. Thus we get that

$$z_1 = \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 + \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4$$

$$
\begin{aligned}
&+\alpha_{11}e_1e_2e_3+\alpha_{12}e_1e_2e_4+\alpha_{13}e_1e_3e_4+\alpha_{14}e_2e_3e_4;\\
z_2 &= \beta_5e_1e_2+\beta_6e_1e_3+\beta_7e_1e_4+\beta_8e_2e_3+\beta_9e_2e_4+\beta_{10}e_3e_4\\
&+\beta_{11}e_1e_2e_3+\beta_{12}e_1e_2e_4+\beta_{13}e_1e_3e_4+\beta_{14}e_2e_3e_4;\\
y_1 &= \gamma_1e_1+\gamma_2e_2+\gamma_3e_3+\gamma_4e_4\\
&+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4\\
&+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4+\gamma_{15}e_1e_2e_3e_4;\\
y_2 &= -\gamma_1e_1-\gamma_2e_2-\gamma_3e_3-\gamma_4e_4\\
&+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4\\
&+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4-\gamma_{15}e_1e_2e_3e_4.
\end{aligned}
$$

As in $z_iz_j$ the least degree of the summands is 4 we have $xz_jz_k=0$, $z_jxz_k=0$, $z_jz_kx=0$ for any entry $x$ of the matrix $Z$. As the least degree of the summands in $y_iz_j$ is 3 we get that $y_iz_jz_k=0$. The least degree in $y_i^2$ is 3 and we have $y_i^2z_j=0$ and $z_iy_j^2=0$ as well. Thus for the matrix $Z^3=(a_{ij})$ we get $a_{11}=a_{22}=0$, $a_{12}=y_1z_1y_2$ and $a_{21}=y_2z_2y_1$.

We consider the four summands of degree 3 (the minimal one) in $y_1z_1$:

$$
\begin{array}{lcl}
\alpha e_1e_2e_3 & \to & \alpha=\gamma_1\alpha_8-\gamma_2\alpha_6+\gamma_3\alpha_5\\
\beta e_1e_2e_3 & \to & \beta=\gamma_1\alpha_9-\gamma_2\alpha_7+\gamma_4\alpha_5\\
\gamma e_1e_2e_3 & \to & \gamma=\gamma_1\alpha_{10}-\gamma_3\alpha_7+\gamma_4\alpha_6\\
\delta e_1e_2e_3 & \to & \delta=\gamma_2\alpha_{10}-\gamma_3\alpha_9+\gamma_4\alpha_8.
\end{array}
$$

Now we define the coefficient of the only summand (of degree 4) in $a_{12}=y_1z_1y_2$. It is equal to

$$
\begin{aligned}
&-\gamma_4(\gamma_1\alpha_8-\gamma_2\alpha_6+\gamma_3\alpha_5)+\gamma_3(\gamma_1\alpha_9-\gamma_2\alpha_7+\gamma_4\alpha_5)\\
&-\gamma_2(\gamma_1\alpha_{10}-\gamma_3\alpha_7+\gamma_4\alpha_6)+\gamma_1(\gamma_2\alpha_{10}-\gamma_3\alpha_9+\gamma_4\alpha_8)\equiv 0.
\end{aligned}
$$

The same is valid for $a_{21}=y_2z_2y_1$ as well. Thus $Z^3$ is the zero matrix. □

If we change the involution $\psi_2$, considered in $E_4'$, with the involution $\psi_1$, the $\flat$-variables of $(M_2(E_4',\psi_1),\flat)$ do not have a lower nilpotency index, namely

**Theorem 2.** *The algebra $(M_2(E_4',\psi_1),\flat)$ satisfies the $\flat$-identity $A^4=0$ for $A$ being any $\flat$-variable.*

*Proof of Theorem 2.* We mach only the crucial steps of the proof.

In this case $\psi_1(e_1)=e_2$ ($\psi_1(e_2)=e_1$) and $\psi_1(e_3)=e_4$ ($\psi_1(e_4)=e_3$).

We have to consider only the case when $A=Z$ is a $\flat$-skew symmetric variable. The conditions $\psi_1(z_i)=-z_i$ and $\psi_1(y_1)=-y_2$ give that

$$
\begin{aligned}
z_1 &= \alpha_1(e_1-e_2)+\alpha_3(e_3-e_4)+\alpha_6(e_1e_3+e_2e_4)+\alpha_7(e_1e_4+e_2e_3)\\
&+\alpha_{11}(e_1e_2e_3-e_1e_2e_4)+\alpha_{13}(e_1e_3e_4-e_2e_3e_4);\\
z_2 &= \beta_1(e_1-e_2)+\beta_3(e_3-e_4)+\beta_6(e_1e_3+e_2e_4)+\beta_7(e_1e_4+e_2e_3)
\end{aligned}
$$

$$
\begin{aligned}
&+\beta_{11}(e_1e_2e_3-e_1e_2e_4)+\beta_{13}(e_1e_3e_4-e_2e_3e_4);\\
y_1 &= \gamma_1e_1+\gamma_2e_2+\gamma_3e_3+\gamma_4e_4\\
&+\gamma_5e_1e_2+\gamma_6e_1e_3+\gamma_7e_1e_4+\gamma_8e_2e_3+\gamma_9e_2e_4+\gamma_{10}e_3e_4\\
&+\gamma_{11}e_1e_2e_3+\gamma_{12}e_1e_2e_4+\gamma_{13}e_1e_3e_4+\gamma_{14}e_2e_3e_4;\\
y_2 &= -\gamma_2e_1-\gamma_1e_2-\gamma_4e_3-\gamma_3e_4\\
&-\gamma_5e_1e_2+\gamma_9e_1e_3+\gamma_8e_1e_4+\gamma_7e_2e_3+\gamma_6e_2e_4-\gamma_{10}e_3e_4\\
&-\gamma_{12}e_1e_2e_3-\gamma_{11}e_1e_2e_4-\gamma_{14}e_1e_3e_4-\gamma_{13}e_2e_3e_4.
\end{aligned}
$$

We follow the coefficient of $e_1e_2e_3$ in the entry $a_{11}=z_1z_2y_1+y_1z_1z_2+z_1y_2z_2$ of the matrix $Z^3=(a_{ij})$. Forming $z_1z_2$ we find the coefficient of $e_1e_2e_3$ in the product $y_1(z_1z_2)$, namely $-(\gamma_1+\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$.

The same holds for the coefficient of $e_1e_2e_3$ in the products $z_1z_2y_1$ and in $z_1y_2z_2$. Thus $Z^3$ is not a zero matrix.

Taking into account the conditions on the entries of a $\flat$-symmetric matrix $Y$ we see that the coefficient of $e_1e_2e_3$ in the entry $b_{11}$ of the matrix $Y^3=(b_{ij})$ is $3(\gamma_1-\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$. □

2.2. *PI-properties of some fourth order matrix algebras*

We define the 8-th dimensional matrix algebra $AM_4(K)$ as the algebra of the matrices of type
$\begin{pmatrix} a_{11} & 0 & a_{13} & 0\\ 0 & a_{22} & 0 & a_{24}\\ a_{31} & 0 & a_{33} & 0\\ 0 & a_{42} & 0 & a_{44}\end{pmatrix}$, $a_{ij}\in K$. The following theorem holds:

**Theorem 3.** *The algebra $AM_4(K)$ satisfies the Hall identity* $[[X_1,X_2]^2,X_3]=0$.

*Proof of Theorem 3.* For $X_1,X_2\in AM_4(K)$ in $[X_1,X_2]=(c_{ij})$ we have $c_{33}=-c_{11}$ and $c_{44}=-c_{22}$. The matrix $[X_1,X_2]^2=(d_{ij})$ is a diagonal matrix with $d_{33}=d_{11}$ and $d_{44}=d_{22}$. Thus $[[X_1,X_2]^2,X_3]=0$. □

By the system for computer algebra *Mathematica* we see that $AM_4(K)$ satisfies the identity $S_4(X_1,X_2,X_3,X_4)=0$ as well.

The n-th analogue of $AM_4(K)$ is the algebra $AM_{2n}(K)$. Its elements are of type $(a_{ij})$ with non-zero entries only among $a_{ii}$ for $i=1,...,2n$, $a_{j,n+j}$ and $a_{n+j,j}$ for $j=1,...,n$. The two identities in $AM_4(K)$ hold in $AM_{2n}(K)$ as well.

It is known that in a matrix algebra over a field $K$ of characteristic zero up to isomorphism there are two types of involutions - the transpose one $t$ and the symplectic involution $*$, the latter defined on an even $2k$ order matrix algebra as

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}^* = \begin{pmatrix} D & -B^t\\ -C^t & A\end{pmatrix},$$

where $A, B, C, D$ are $k \times k$ matrices.

We recall that the Hall identity $[[Y_1, Y_2]^2, Y_3] = 0$ is a $*$-identity of minimal degree in $*$-symmetric variables for the algebra $(M_4(K), *)$ [5].

Next we consider the matrix algebra $AM_4(K)$ with the symplectic involution $*$.

**Theorem 4.** *The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Y_1, Y_2] = 0$ in $*$-symmetric variables.*

*Proof of Theorem 4.* From

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}^*$$

$$= \begin{pmatrix} a_{33} & 0 & -a_{13} & 0 \\ 0 & a_{44} & 0 & -a_{24} \\ -a_{31} & 0 & a_{11} & 0 \\ 0 & -a_{42} & 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}$$

we see that the $*$-symmetric elements of $(AM_4(K), *)$ are diagonal matrices. □

As $z^2$ is $*$-symmetric we come to

**Corollary 1.** *The algebra $(AM_4(K), *)$ satisfies the $*$-identity $[Z_1^2, Z_2^2] = 0$ in $*$-skew symmetric variables.*

Now the matrix algebras considered will have entries that are elements of a Grassmann algebra. In the statements below we use Proposition 4. As it was proved in [13] using the system for computer algebra *Mathematica* we give here its analytic proof.

*Proof of Proposition 4.* Without loss of generality we consider $x \in E_4'$ with summands of length 1 and 2 only (the other ones will give zeros either in $x^2$ or in $x^3$). Thus

$$\begin{aligned} x &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_1 e_2 + \alpha_6 e_1 e_3 \\ &+ \alpha_7 e_1 e_4 + \alpha_8 e_2 e_3 + \alpha_9 e_2 e_4 + \alpha_{10} e_3 e_4. \end{aligned}$$

We define the coefficients of the four summands of length 3 in $x^2$. They are:

$$\begin{aligned} \alpha e_1 e_2 e_3 &\mapsto \alpha = 2(\alpha_1\alpha_8 - \alpha_2\alpha_6 + \alpha_3\alpha_5) \\ \beta e_1 e_2 e_4 &\mapsto \beta = 2(\alpha_1\alpha_9 - \alpha_2\alpha_7 + \alpha_4\alpha_5) \\ \gamma e_1 e_3 e_4 &\mapsto \gamma = 2(\alpha_1\alpha_{10} - \alpha_3\alpha_7 + \alpha_4\alpha_6) \\ \delta e_2 e_3 e_4 &\mapsto \delta = 2(\alpha_2\alpha_{10} - \alpha_3\alpha_9 + \alpha_4\alpha_8). \end{aligned}$$

The coefficient of the only summand (which is of length 4) of $x^3$ is proportional to

$$\begin{aligned} &-\alpha_1(\alpha_2\alpha_{10} - \alpha_3\alpha_9 + \alpha_4\alpha_8) + \alpha_2(\alpha_1\alpha_{10} - \alpha_3\alpha_7 + \alpha_4\alpha_6) \\ &-\alpha_3(\alpha_1\alpha_9 - \alpha_2\alpha_7 + \alpha_4\alpha_5) + \alpha_4(\alpha_1\alpha_8 - \alpha_2\alpha_6 + \alpha_3\alpha_5) \equiv 0. \end{aligned}$$

□

The identity $[y,x,x]=0$ and the linearization of $x^3=0$ lead to

**Corollary 2.** *In $E_4'$ the following identities hold:*

$$x^2y+yx^2=0,\ xyx=0,\ xyz+zyx=0,$$
$$xy^2z=-zyxy=0,\ y^2xz=-zyxy=0,\ zxy^2=-yxyz=0.$$

**Theorem 5.** *The algebra $AM_4(E_4')$ is a nil algebra with nil index* 4.

*Proof of Theorem 5.* For a matrix $A\in AM_4(E_4')$, where

$$A=\begin{pmatrix} y_1 & 0 & z_1 & 0\\ 0 & y_2 & 0 & z_2\\ z_3 & 0 & y_3 & 0\\ 0 & z_4 & 0 & y_4\end{pmatrix}$$

and $A^3=(a_{ij})$ we get

$$\begin{aligned}
a_{11}&=z_1z_3y_1+y_1z_1z_3+z_1y_3z_3,\\
a_{13}&=y_1^2z_1+z_1z_3z_1+y_1z_1y_3+z_1y_3^2,\\
a_{22}&=z_2z_4y_2+y_2z_2z_4+z_2y_4z_4,\\
a_{24}&=y_2^2z_2+z_2z_4z_2+y_2z_2y_4+z_4y_4^2,\\
a_{31}&=z_3y_1^2+y_3z_3y_1+z_3z_1z_3+y_3^2z_3,\\
a_{33}&=z_3y_1z_1+y_3z_3z_1+z_3z_1y_3,\\
a_{42}&=z_4y_2^2+y_4z_4y_2+z_4z_2z_4+y_4^2z_4,\\
a_{44}&=z_2y_2z_2+y_4z_4z_1+z_4z_2y_4.
\end{aligned}$$

Now we investigate the entries of $A^4=(b_{ij})$:

$$\begin{aligned}
b_{11}&=z_1z_3y_1^2+y_1z_1z_3y_1+z_1y_3z_3y_1+y_1^2z_1z_3\\
&\quad+z_1z_3z_1z_3+y_1z_1y_1z_3+z_1y_3^2z_3
\end{aligned}$$

Applying Corollary 2 we simplify $b_{11}$ and get $b_{11}=z_1y_3z_3y_1+y_1z_1y_3z_3$. The identity $xyz=-zyx$ gives

$$z_1y_3z_3y_1=-z_3y_1y_3z_1=y_3z_1y_1z_3=-y_1z_1y_3z_3.$$

Thus $b_{11}=0$.

In an analogous way we investigate the other entries of $A^4$:

$$\begin{aligned}
b_{13}&=z_1z_3y_1z_1+y_1z_1z_3z_1+z_1y_3z_3z_1+y_1^2z_1y_3\\
&\quad+z_1z_3z_1y_3+y_1z_1y_3^2+z_1y_3^3
\end{aligned}$$

According to Corollary 2 we have $b_{13}=0$.

Now we consider

$$b_{22} = z_2z_4y_2^2 + y_2z_2z_4y_2 + z_2y_4z_4y_2 + y_2^2z_2z_4 \\ + z_2z_2z_2z_4 + y_2z_2y_4z_4 + z_4y_4^2z_4.$$

The same Corollary leads to $b_{22} = z_2y_4z_4y_2 + y_2z_2y_4z_4$. As

$$z_2y_4z_4y_2 = -z_4y_2y_4z_2 = y_4z_2y_2z_4 = -y_2z_2y_4z_4$$

we get $b_{22} = 0$.

Applying Corollary 2 we get $b_{24} = b_{31} = 0$. In $b_{33}$ we have to consider only the part $y_3z_3y_1z_1 + z_3y_1z_1y_3$. As

$$y_3z_3y_1z_1 = -y_1z_1z_3y_3 = z_3z_1y_1y_3 = -y_3y_1z_3z_1 = z_1y_1z_3y_3 = -z_3y_1z_1y_3$$

we get $b_{33} = 0$.

The identities in Corollary 2 immediately lead to $b_{42} = 0, b_{44} = 0$. Thus $A^4 = 0$. □

Now we consider the subalgebra $ASM_4(E)$ of the matrices of type $\begin{pmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \\ c & 0 & c & 0 \\ 0 & d & 0 & d \end{pmatrix}$ and prove that it is a PI-algebra.

**Theorem 6.** *The algebra $ASM_4(E)$ satisfies the identity $U[X,Y,Z] = 0$.*

*Proof of Theorem 6.* Let $X, Y, Z$ be matrices from $ASM_4(E)$ denoting its entries by $a_i, b_i, c_i, d_i$ for $i = 1,2,3$ respectively. We form the diagonal entries of $[X,Y] = (a_{ij})$, namely

$$\begin{aligned} a_{11} &= [a_1,a_2] + a_1c_2 - a_2c_1, \\ a_{22} &= [b_1,b_2] + b_1d_2 - b_2d_1, \\ a_{33} &= [c_1,c_2] + c_1a_2 - c_2a_1, \\ a_{44} &= [d_1,d_2] + d_1b_2 - d_2b_1. \end{aligned}$$

For the matrix $[X,Y,Z] = (b_{ij})$ we have modulo $[x,y,z] = 0$ for $x,y,z \in E$ that

$$\begin{aligned} & b_{11} + b_{33} \\ &= [a_1c_2 - a_2c_1, a_3] + ([a_1,a_2] + a_1c_2 - a_2c_1)c_3 - a_3([c_1,c_2] + c_1a_2 - c_2a_1) \\ &+ [c_1a_2 - c_2a_1, c_3] + ([c_1,c_2] + c_1a_2 - c_2a_1)a_3 - c_3([a_1,a_2] + a_1c_2 - a_2c_1) \\ &= [a_1c_2 - a_2c_1, a_3] + (a_1c_2 - a_2c_1)c_3 - a_3(c_1a_2 - c_2a_1) \\ &+ [c_1a_2 - c_2a_1, c_3] + (c_1a_2 - c_2a_1)a_3 - c_3(a_1c_2 - a_2c_1) \\ &= [a_1c_2 - a_2c_1, a_3] + [c_1a_2 - c_2a_1, c_3] + [a_1c_2 - a_2c_1, c_3] + [c_1a_2 - c_2a_1, a_3] \\ &= [[a_1,c_2] + [c_1,a_2], a_3] + [[c_1,a_2] + [a_1,c_2], c_3] \equiv 0. \end{aligned}$$

Analogously we get that $b_{22}+b_{44}=0$. Thus $U[X,Y,Z]=0$ for any matrix $U\in ASM_4(E)$. □

The analogue of $ASM_4(E)$ in the general case is the matrix algebra $ASM_{2n}(E)$. Its elements are of type $(a_{ij})$, where $a_{ii}=a_{i,n+i}$ for $i=1,\dots,n$ and $a_{jj}=a_{j,j-n}$ for $j=n+1,\dots,2n$. The algebra $ASM_{2n}(E)$ satisfies the same identity $U[X,Y,Z]=0$.

For now we are able to find involutions in $M_n(E)$ for $n>2$ only considering an involution in $E$. We generalize the case $n=2$, namely

**Proposition 6.** *The mapping* $(\flat)$, *defined as*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{(\flat)}$$

$$=\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{(\flat)}=\begin{pmatrix} (D)^{\flat} & (B)^{\flat} \\ (C)^{\flat} & (A)^{\flat} \end{pmatrix}=\begin{pmatrix} a_{44}^* & a_{34}^* & a_{24}^* & a_{14}^* \\ a_{43}^* & a_{33}^* & a_{23}^* & a_{13}^* \\ a_{42}^* & a_{32}^* & a_{22}^* & a_{12}^* \\ a_{41}^* & a_{31}^* & a_{21}^* & a_{11}^* \end{pmatrix}$$

*is an involution on* $M_4(E,\psi=*)$.

*Proof of Proposition 6.* Considering in details the entries of the two matrices $(AB)^{(\flat)}$ and $(B)^{(\flat)}(A)^{(\flat)}$ we see that their corresponding entries are equal i.e. the mapping $(\flat)$ is an involution. □

We cover the following special case: Let $E_3'$ be the non-unitary finite dimensional Grassmann algebra with generators $e_1,e_2,e_3$ and $AM(2)(E_3')$ be the subalgebra of $AM_4(E_3')$ defined by the matrices of type

$$\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix},$$

where $y_i$ are even elements (of even length) of $E_3'$, while $z_i$ are odd elements (of odd length) of $E_3'$, $i=1,\dots,4$. We equip the algebra $AM(2)(E_3',\psi_2)$ with the involution $(\flat)$ as defined in Proposition 6.

We characterize the $(\flat)$-symmetric elements $Y_i$ and the $(\flat)$-skew symmetric elements $Z_j$ of the algebra $(AM(2)(E_3',\psi_2),(\flat))$.

**Theorem 7.** *The algebra* $(AM(2)(E_3',\psi_2),(\flat))$ *satisfies the* $(\flat)$*-identity* $Y^3=0$ *in* $(\flat)$*-symmetric variables.*

*Proof of Theorem 7.* Let consider a $(\flat)$-symmetric element $Y$. Denoting for short $\psi_2$ as $*$ in the equality $\begin{pmatrix} y_4^* & 0 & z_2^* & 0 \\ 0 & y_3* & 0 & z_1^* \\ z_4* & 0 & y_2^* & 0 \\ 0 & z_3* & 0 & y_1^* \end{pmatrix} = \begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_2 \\ z_3 & 0 & y_3 & 0 \\ 0 & z_4 & 0 & y_4 \end{pmatrix}$ we get the following conditions on the entries of $Y$: $\psi_2(y_4) = y_1$, $\psi_2(y_3) = y_2$, $\psi_2(z_2) = z_1$ and $\psi_2(z_4) = z_3$.

Let $y_1 = s_1e_1e_2 + s_2e_1e_3 + s_3e_2e_3$. Then $y_4 = \psi_2(y_1) = -y_1$. For $y_2 = t_1e_1e_2 + t_2e_1e_3 + t_3e_2e_3$ we get $y_3 = \psi_2(y_2) = -y_2$. Obviously $y_1^2 = y_2^2 = 0$.

As the entries are from $E_3^{'}$ we could work with odd entries having summands of degree 1 only. Let $z_1 = \alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3$ and $z_3 = m_1e_1 + m_2e_2 + m_3e_3$. Then $z_2 = \psi_2(z_1) = z_1$, $z_4 = \psi_2(z_3) = z_3$. Considering $Y^3 = Y^2Y = (a_{ij})$ as

$$\begin{pmatrix} z_1z_3 & 0 & y_1z_1 - z_1y_2 & 0 \\ 0 & z_1z_3 & 0 & y_2z_1 - z_1y_1 \\ z_3y_1 - y_2z_3 & 0 & z_3z_1 & 0 \\ 0 & z_3y_2 - y_1z_3 & 0 & z_3z_1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 & 0 & z_1 & 0 \\ 0 & y_2 & 0 & z_1 \\ z_3 & 0 & -y_2 & 0 \\ 0 & z_3 & 0 & -y_1 \end{pmatrix}$$

we see that manipulating with the generators $e_1, e_2, e_3$, probably nontrivial entries could be only

$$a_{13} = a_{24} = z_1z_3z_1 = \beta_1e_1e_2e_3,\ a_{31} = a_{42} = z_3z_1z_3 = \beta_2e_1e_2e_3.$$

Applying Corollary 2 we get that both of them are zero. □

**Theorem 8.** *The algebra* $(AM(2)(E_3^{'}, \psi_2), (\flat))$ *satisfies the* $(\flat)$*-identity* $Z^3 = 0$ *in* $(\flat)$*-skew symmetric variables.*

*Proof of Theorem 8.* Using the same notations for the matrix entries of $Z$ as in the previous theorem, in this case we have

$$\begin{aligned} &y_4 = -\psi_2(y_1) = y_1,\ y_3 = -\psi_2(y_2) = y_2, \\ &y_1^2 = y_2^2 = 0, \\ &z_2 = -\psi_2(z_1) = -z_1,\ z_4 = -\psi_2(z_3) = -z_3. \end{aligned}$$

In $Z^3 = Z^2Z = (b_{ij})$ nonzero could be only the entries $b_{13} = -b_{24} = z_1z_3z_1$ and $b_{31} = -b_{42} = z_3z_1z_3$. Corollary 2 proves they both are zero. □

We consider the subalgebra $(AM(2)(E_3^{'}, \psi_2), (\flat))$ instead of the algebra $(AM_4(E_3^{'}, \psi_2), (\flat))$ itself as if $A^n = 0$ for a $\flat$-variable $A$ of $(AM_4(E_4^{'}, \psi_2), (\flat))$ we

have $n > 3$. Thus the algebras $(AM_4(E_4^{'}, \psi_2), (\flat))$ and $AM_4(E_4^{'})$ have equal nil indices.

We give an example of another matrix algebra with involution $(\flat)$ having lower nilpotency index of its $(\flat)$-skew symmetric variables:

Let $BM(2)(E_3^{'})$ be the algebra defined by the matrices of type

$$\begin{pmatrix} y_1 & 0 & 0 & z_1 \\ 0 & y_2 & z_2 & 0 \\ 0 & z_3 & y_3 & 0 \\ z_4 & 0 & 0 & y_4 \end{pmatrix}$$

, where $y_i$ are even elements of $E_3^{'}$, while $z_i$ are odd elements of $E_3^{'}$, $i = 1, ..., 4$. We equip the algebra $BM(2)(E_3^{'}, \psi_2)$ with the involution $(\flat)$ as defined in Proposition 6.

**Theorem 9.** *The algebra* $(BM(2)(E_3^{'}, \psi_2), (\flat))$ *satisfies the* $(\flat)$*-identity* $Y^3 = 0$ *in* $(\flat)$*-symmetric variables and the* $(\flat)$*-identity* $Z^2 = 0$ *in* $(\flat)$*-skew symmetric variables.*

*Proof of Theorem 9.* In the algebra $(BM(2)(E_3^{'}, \psi_2), (\flat))$ any $(\flat)$-skew symmetric variable $Z$ is a diagonal matrix and $Z^2 = 0$ as $y_i^2 = 0$ for $i = 1, ..., 4$. □

There is a package written in the system for computer algebra *Mathematica* [10] for manipulating in finite dimensional Grassmann algebras. Using it a programme was written by the author giving an alternative way of confirming the validity of the corresponding theorems in the paper.

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# REGIONS OF VARIABILITY FOR JANOWSKI FUNCTIONS

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*Abstract.* Let $A \in \mathbb{C}$, $B \in [-1,0)$. Then $P[A,B]$ denotes the class of analytic functions $p$ in the open unit disk with $p(0) = 1$ such that

$$p(z) = \frac{1 + Aw(z)}{1 + Aw(z)},$$

where $w(0) = 0$ and $|w(z)| < 1$. In this article we find the regions of variability $V(z_0, \lambda)$ for $\int_0^{z_0} p(\rho)\, d\rho$ when $p$ ranges over the class $P_\lambda[A,B]$ defined as

$$P_\lambda[A,B] = \left\{ p \in P[A,B] : p'(0) = (A - B)\lambda \right\}$$

for any fixed $z_0 \in E$ and $\lambda \in \overline{E}$. As a consequence, the regions of variability are also illustrated graphically for different sets of parameters.

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*Keywords:* region of variability, Janowski functions

## 1. Introduction

Let $A$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disk $E = \{z : |z| < 1\}$. Also Consider $A$ as a topological vector space endowed with the topology of uniform convergence over a compact subsets of $E$. Also let $\mathcal{B}$ denote the class of analytic functions $w$ in $E$ such that $|w(z)| < 1$ and $w(0) = 0$. A function $f$ is said to be subordinate to a function $g$ written as $f \prec g$, if there exists a Schwarz function $w$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular if $g$ is univalent in $E$, then $f(0) = g(0)$ and $f(E) \subset g(E)$. Let $P[A,B]$ denote the class of analytic functions $p$ such that $p(0) = 1$ and

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \; w \in B, z \in E, \tag{1.2}$$

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where $A \in \mathbb{C}$, $B \in [-1,0)$ with $A \neq B$. Note that $P[1,-1] = P$.

Let $p \in P[A,B]$. Then using the Herglotz representation, there exists a unique positive unit measure $\mu$ in $(-\pi,\pi]$ such that

$$p(z) = \int_{-\pi}^{\pi} \frac{1+Ae^{it}z}{1+Be^{it}z} d\mu(t), z \in E. \tag{1.3}$$

It can easily be seen from (1.2) that

$$w_p(z) = \frac{p(z)-1}{A-Bp(z)}, z \in E, \tag{1.4}$$

and conversely, we have

$$p'(0) = (A-B)\, w_p'(0) \tag{1.5}$$

Let $p \in P[A,B]$. Then by using classical Schwarz lemma, that is $\left|w_p'(0)\right| \leq 1$ (see [1],) we obtain

$$p'(0) = (A-B)\lambda$$

for some $\lambda \in \overline{E}$. By using (1.4) one can compute

$$\frac{w_p''(0)}{2} = \frac{p''(0)}{2(A-B)} + B\lambda^2.$$

Now, if we let

$$g(z) = \begin{cases} \frac{\frac{w_p(z)}{z}-\lambda}{1-\bar{\lambda}\frac{w_p(z)}{z}}, & |\lambda| < 1, \\ 0 & |\lambda| = 1. \end{cases}$$

This implies that

$$g'(0) = \begin{cases} \frac{1}{1-|\lambda|^2}\left(\frac{w_p(z)}{z}\right)' |_{z=0} = \frac{1}{1-|\lambda|^2}\frac{w_p''(z)}{2}, & |\lambda| < 1, \\ 0 & |\lambda| = 1. \end{cases}$$

By Schwarz lemma for $|\lambda| < 1$, we see that $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$. Equality is attained in both cases if $g(z) = e^{i\alpha}z$ for some $\alpha \in \mathbb{R}$. Now the condition $|g'(0)| \leq 1$ shows that there exists an $a \in \overline{E}$ such that $g'(0) = a$. Thus, we have

$$p''(0) = 2(A-B)\left[\left(1-|\lambda|^2\right)a - B\lambda^2\right].$$

Consequently for $\lambda \in \overline{E}$ and $z_0 \in E$, we have $V(z_0,\lambda)$ for $\int_0^{z_0} p(\rho)\, d\rho$ when $p$ ranges over the class $P_\lambda[A,B]$ defined as

$$P_\lambda[A,B] = \left\{p \in P[A,B] : p'(0) = (A-B)\lambda\right\}$$

and

$$V_\lambda(z_0,A,B) = \left\{\int_0^{z_0} p(\rho)\, d\rho, \qquad p \in P_\lambda[A,B]\right\}.$$

For related study we refer to [2–9] and the references therein. The aim of this paper is to investigate explicitly the region of variability $V_\lambda(z_0,A,B)$ for the class $P_\lambda[A,B]$.

## 2. Basic properties of $V_\lambda(z_0, A, B)$

**Proposition 1.** *(i) $V_\lambda(z_0, A, B)$ is a compact subset of $\mathbb{C}$.*

*(ii) $V_\lambda(z_0, A, B)$ is convex subset of $\mathbb{C}$.*

*(iii) If $|\lambda| = 1$ or $z_0 = 0$, then $V_\lambda(z_0, A, B) = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ and if $|\lambda| < 1$ and $z_0 \neq 0$, then $\left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ is the interior point of the set $V_\lambda(z_0, A, B)$.*

*Proof. (i)* Since $P_\lambda[A, B]$ is a compact subset of $\mathbb{C}$, therefore $V_\lambda(z_0, A, B)$ is also compact.

*(ii)* Let $p_1, p_2 \in P_\lambda[A, B]$. Then

$$p(z) = (1-t)\, p_1(z) + t p_2(z), t \in [0, 1]$$

is also in $P_\lambda[A, B]$, therefore $V_\lambda(z_0, A, B)$ is convex.

*(iii)* Since if $|\lambda| = |w_f'(0)| = 1$, then from Schwarz lemma, we obtain $w_f(z) = \lambda z$, which yields $p(z) = \frac{1+\lambda A z}{1+\lambda B z}$. This implies that

$$p(z) = 1 + \left(\frac{A-B}{B}\right)\left\{1 - \frac{1}{1 + B\lambda z}\right\}.$$

Therefore

$$V_\lambda(z_0, A, B) = \int_0^{z_0} p(\rho)\, d\rho = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}.$$

This also trivially holds true when $z_0 = 0$. For $\lambda \in E$ and $\alpha \in \overline{E}$, set

$$\delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda} z},$$

$$\int_0^z H_{a,\lambda}(\rho)\, d\rho = \int_0^z \frac{1 + A\rho\delta(a\rho, \lambda)}{1 + B\rho\delta(a\rho, \lambda)} d\rho,\ z \in E. \tag{2.1}$$

Then $\int_0^z H_{a,\lambda}(\rho)\, d\rho$ is in $P_\lambda[A, B]$ and $w_f(z) = z\delta(az, \lambda)$. For fixed $\lambda \in E$ and $z_0 \in E\backslash\{0\}$ the function

$$E \ni a \mapsto \int_0^{z_0} H_{a,\lambda}(\rho)\, d\rho = \int_0^{z_0} \frac{1 + (\overline{\lambda} a + A\lambda)\rho + A a \rho^2}{1 + (\overline{\lambda} a + B\lambda)\rho + B a \rho^2} d\rho$$

is a non-constant analytic function of $a \in E$, and therefore is an open mapping. Hence $\int_0^{z_0} H_{0,\lambda}(\rho)\, d\rho = \left\{z_0 - \frac{A-B}{B}\left(z_0 + \frac{1}{B\lambda}\log(1 + B\lambda z_0)\right)\right\}$ is an interior point of $\left\{\int_0^{z_0} H_{a,\lambda}(\rho)\, d\rho : a \in E\right\} \subset V_\lambda[z_0, A, B]$. □

Keeping in view the above proposition, it is sufficient to find $V_\lambda[z_0, A, B]$ for $0 \leq \lambda < 1$ and $z_0 \in E\backslash\{0\}$.

## 3. Main results

In this section, we state and prove some results which are needed in the proof of our main theorems.

**Proposition 2.** *For $p \in P_\lambda[A,B]$, we have*

$$|p(z)-q(z,\lambda)| \le r(z,\lambda),\ z \in E, \lambda \in \overline{E}, \tag{3.1}$$

*where*

$$q(z,\lambda) = \frac{(1+\lambda Az)\left(1+B\overline{\lambda}\overline{z}\right)-|z|^2(\lambda+B\overline{z})\left(\overline{\lambda}+Az\right)}{1-B^2|z|^4+2B(1-|z|^2)\Re(\lambda z)+|\lambda|^2|z|^2(B^2-1)}, \tag{3.2}$$

$$r(z,\lambda) = \frac{|A-B|(1-|\lambda|^2)|z|^2}{1-B^2|z|^4+2B(1-|z|^2)\Re(\lambda z)+|\lambda|^2|z|^2(B^2-1)}.$$

*The inequality is sharp for $z_0 \in E\backslash\{0\}$ if and only if $f(z) = H_{e^{i\theta},\lambda}(z)$ for some $\theta \in \mathbb{R}$.*

*Proof.* Let $p \in P_\lambda[A,B]$. Then there exists $w_p \in \mathcal{B}$ such that

$$\left|\frac{\frac{w_p(z)}{z}-\lambda}{1-\overline{\lambda}\frac{w_p(z)}{z}}\right| \le |z|. \tag{3.3}$$

From (1.4) this can be written equivalently as

$$\left|\frac{p(z)-b(z,\lambda)}{p(z)+c(z,\lambda)}\right| \le |z|\,|\tau(z,\lambda)|, \tag{3.4}$$

where

$$b(z,\lambda) = \frac{1+\lambda Az}{1+\lambda Bz},\ \ c(z,\lambda) = -\frac{\overline{\lambda}+Az}{Bz+\overline{\lambda}},\ \ \tau(z,\lambda) = \frac{\overline{\lambda}+Bz}{1+\lambda Bz}. \tag{3.5}$$

Simple computations show that the inequality (3.4) can be written as

$$\left|p(z)-\frac{b(z,\lambda)+|z|^2|\tau(z,\lambda)|^2c(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}\right| \le \frac{|z|\,|\tau(z,\lambda)|\,|b(z,\lambda)+c(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}. \tag{3.6}$$

Now we have

$$1-|z|^2|\tau(z,\lambda)|^2 = \frac{1-B^2|z|^4+2B(1-|z|^2)\Re(\lambda z)+|\lambda|^2|z|^2(B^2-1)}{|1+\lambda zB|^2},$$

$$b(z,\lambda)+c(z,\lambda) = \frac{(B-A)(1-|\lambda|^2)z}{(1+\lambda Bz)(Bz+\overline{\lambda})},$$

$$b(z,\lambda)+|z|^2|\tau(z,\lambda)|^2c(z,\lambda)$$

$$= \frac{(1+\lambda Az)\left(1+B\overline{\lambda}\overline{z}\right)}{|1+\lambda zB|^2} - \frac{|z|^2(\lambda+B\overline{z})\left(\overline{\lambda}+Az\right)}{|1+\lambda zB|^2}.$$

By simple calculations, we obtain

$$q(z,\lambda) = \frac{b(z,\lambda)+|z|^2|\tau(z,\lambda)|^2 c(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2},$$
$$r(z,\lambda) = \frac{|z||\tau(z,\lambda)||b(z,\lambda)+c(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}.$$

All these relations together with (3.6) give (3.1). Equality occurs in (3.1) when $p = H_{i\theta,\lambda}(z)$, for some $z \in E$. Conversely if equality occurs in (3.1) for some $z \in E\backslash\{0\}$ , then equality must hold in (3.3). Thus by Schwarz lemma there exists $\theta \in \mathbb{R}$ such that $w_p(z) = z\delta(e^{i\theta}z,\lambda)$ for all $z \in E$. This implies $P = H_{i\theta,\lambda}$. □

Geometrically the above lemma means that the functional $p$ lies in the closed disk centred at $q(z,\lambda)$ with radius $r(z,\lambda)$.

For $\lambda = 0$, we have the following result

**Corollary 1.** *For $p \in P_0[A,B]$, we have*

$$\left|p(z) - \frac{1-AB|z|^4}{1-B^2|z|^4}\right| \le \frac{(A-B)|z|^2}{1-B^2|z|^4},\ z \in E\backslash\{0\}.$$

Equality is attained if and only if $p = H_{i\theta,0}$.

**Corollary 2.** *Let $\gamma : z(t), 0 \le t \le 1$, be a $C^1$−curve in $E$ with $z(0) = 0$ and $z(1) = z_0$. Then*

$$V_\lambda(z_0,A,B) \subset \{w \in \mathbb{C} : |w - Q(\lambda,\gamma)| \le R(\lambda,\gamma)\},$$

*where*

$$Q(\lambda,\gamma) = \int_0^1 q(z(t),\lambda)z'(t)dt,\ \ R(\lambda,\gamma) = \int_0^1 r(z(t),\lambda)z'(t)dt.$$

*Proof.* Since $p$ is in $P_\lambda[A,B]$, therefore by using Proposition 2, we get

$$\begin{aligned}\left|\int_0^1 p(z(t))z'(t)dt - Q(\lambda,\gamma)\right| &= \left|\int_0^1 p(z(t))z'(t)dt - \int_0^1 q(z(t),\lambda)z'(t)dt\right| \\ &= \left|\int_0^1 (p(z(t)) - q(z(t),\lambda))z'(t)dt\right| \\ &\le \int_0^1 r(z(t),\lambda)|z'(t)|dt = R(\lambda,\gamma).\end{aligned}$$

This implies the required result. □

For our next result we need the following lemma:

**Lemma 1.** *For $\theta \in \mathbb{R}$ and $|\lambda| < 1$, the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta^2}{\left(1 + (\overline{\lambda}e^{i\theta} + B\lambda)\zeta + Be^{i\theta}\zeta^2\right)^2} d\zeta, \quad z \in E,$$

*has zeros of order 3 at the origin and no zero elsewhere in $E$. Moreover, there exists a starlike normalized univalent function $s$ in $E$ such that $G(z) = 3^{-1}e^{i\theta}s^3(z)$.*

The above lemma is due to Ponnusamy et al. [4].

**Proposition 3.** *Let $\theta \in (-\pi, \pi]$, $z_0 \in E\backslash\{0\}$. Then, $\int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\, d\rho \in \partial V_\lambda(z_0, A, B)$. Moreover, $\int_0^{z_0} p(\rho)\, d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\, d\rho$ implies $p = H_{e^{i\theta},\lambda}$ for some $p \in P_\lambda[A, B]$ and $\theta \in (-\pi, \pi]$.*

*Proof.* It follows from (2.1) that

$$\begin{aligned} H_{a,\lambda}(z) &= \frac{1 + Az\delta(az, \lambda)}{1 + Bz\delta(az, \lambda)} \\ &= \frac{1 + (\overline{\lambda}a + A\lambda)z + Aaz^2}{1 + (\overline{\lambda}a + B\lambda)z + Baz^2}. \end{aligned}$$

Thus from (3.5), it follows that

$$\begin{aligned} H_{a,\lambda}(z) - b(z, \lambda) &= \frac{(A - B)(1 - |\lambda|^2)az^2}{\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)(1 + \lambda Bz)}, \\ H_{a,\lambda}(z) + c(z, \lambda) &= \frac{(B - A)(1 - |\lambda|^2)z}{\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)(Bz + \overline{\lambda})}. \end{aligned}$$

Therefore

$$\begin{aligned} &H_{a,\lambda}(z) - q(z, \lambda) \\ &= H_{a,\lambda}(z) - \frac{b(z, \lambda) + |z|^2 |\tau(z, \lambda)|^2 c(z, \lambda)}{1 - |z|^2 |\tau(z, \lambda)|^2} \\ &= \frac{1}{1 - |z|^2 |\tau(z, \lambda)|^2}\left[H_{a,\lambda}(z) - b(z, \lambda) - |z|^2 |\tau(z, \lambda)|^2 \left(H_{a,\lambda}(z) + c(z, \lambda)\right)\right] \\ &= \frac{(A - B)(1 - |\lambda|^2)\left[az(1 + B\overline{\lambda z}) + |z|^2(B\overline{z} + \lambda)\right]}{\left(\begin{array}{c} 1 - B^2|z|^4 + 2B(1 - |z|^2)\Re(\lambda z) \\ +|\lambda|^2|z|^2(B^2 - 1) \end{array}\right)\left(1 + (\overline{\lambda}a + B\lambda)z + Baz^2\right)}. \end{aligned}$$

Putting $a = e^{i\theta}$, we obtain

$$H_{e^{i\theta},\lambda}(z) - q(z, \lambda)$$

$$= \frac{r(z,\lambda)e^{i\theta}z^2}{|z|^2} \frac{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)\overline{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)}}{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)^2}$$

$$= \frac{r(z,\lambda)e^{i\theta}z^2}{|z|^2} \frac{\left|1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right|^2}{\left(1+(\overline{\lambda}e^{i\theta}+B\lambda)z+Be^{i\theta}z^2\right)^2}.$$

Now using $G(z)$ defined in Lemma 1, it follows that

$$H_{e^{i\theta},\lambda}(z) - q(z,\lambda) = r(z,\lambda)\frac{G'(z)}{|G'(z)|}. \tag{3.7}$$

Using the argument of Lemma 1 that $G = 3^{-1}e^{i\theta}s^3$, where $s$ is starlike in $E$ with $s(0) = s'(0) - 1 = 0$, for any $z_0 \in E\backslash\{0\}$ the linear segment joining 0 and $s(z_0)$ lies entirely in $s(E)$. Let $\gamma_0$ be the curve defined by

$$\gamma_0 : z(t) = s^{-1}(ts(z_0)), t \in [0,1].$$

Since $G(z(t)) = 2^{-1}e^{i\theta}(s(z(t)))^2 = 3^{-1}e^{i\theta}(ts(z_0))^3 = t^3G(z_0)$. Differentiation w.r.t $t$ gives us

$$G'(z(t))z'(t) = 3t^2G(z_0), \quad t \in [0,1]. \tag{3.8}$$

This relation together with (3.7), leads to

$$\begin{aligned}
\int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho - Q(\lambda,\gamma_0) &= \int_0^1 \left(H_{e^{i\theta},\lambda}(z(t)) - q(z(t),\lambda)\right)z'(t)dt \\
&= \int_0^1 r(z(t),\lambda)\frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|}|z'(t)|dt \\
&= \frac{G(z_0)}{|G(z_0)|}\int_0^1 r(z(t),\lambda)|z'(t)|dt \\
&= \frac{G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0).
\end{aligned} \tag{3.9}$$

This implies that $\int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho \in \partial\overline{E}(Q(\lambda,\gamma_0), R(\lambda,\gamma_0))$, where $Q(\lambda,\gamma_0)$ and $R(\lambda,\gamma_0)$ are defined as in Corollary 2. Hence from Corollary 2, we have $\int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho \in \partial V_\lambda(z_0,A,B)$. For uniqueness, we suppose that

$$\int_0^{z_0} p(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$$

for some $\theta \in (-\pi,\pi]$ and $p \in P_\lambda[A,B]$. Let

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|}(p(z(t)) - q(z(t),\lambda))z'(t),$$

where $\gamma_0 : z(t),\ 0 \le t \le 1$. Then the function $h$ is continuous and

$$|h(t)| = \frac{\left|\overline{G(z_0)}\right|}{|G(z_0)|} |(p(z(t)) - q(z(t),\lambda))| \left|z'(t)\right|.$$

Now using Proposition 2, we get $|h(t)| \le r(z(t),\lambda)|z'(t)|$. Further from (3.9), we have

$$\begin{aligned}\int_0^1 \Re(h(t))dt &= \int_0^1 \Re\left(\left[\frac{\overline{G(z_0)}}{|G(z_0)|}(p(z(t)) - q(z(t),\lambda))z'(t)\right]\right)dt \\ &= \Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}\int_0^{z_0}\left\{H_{e^{i\theta},\lambda}(\rho)\,d\rho - Q(z(t),\lambda)\right\}\right] \\ &= \int_0^1 \Re(r((z(t),\lambda)))|z'(t)|dt.\end{aligned}$$

Thus $h(t) = r(z(t),\lambda)|z'(t)|$, for all $t \in [0,1]$. From (3.7) and (3.8) we have $\int_0^{z_0} p(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$ on $\gamma_0$. The identity theorem for analytic functions yields us $p = H_{e^{i\theta},\lambda}$, $z \in E$. □

**Main Theorem**

**Theorem 1.** *Let $\lambda \in E$ and $z_0 \in E\backslash\{0\}$. Then boundary $\partial V_\lambda(z_0, A, B)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho = \int_0^{z_0} \frac{1 + A\rho\delta(e^{i\theta}\rho,\lambda)}{1 + B\rho\delta(e^{i\theta}\rho,\lambda)}d\rho$$

*If $\int_0^{z_0} p(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$ for some $p \in P_\lambda[A,B]$ and $\theta \in (-\pi,\pi]$, then $p(z) = H_{e^{i\theta},\lambda}(z)$.*

*Proof.* First we have to show that the curve

$$(-\pi,\pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\rho)\,d\rho$$

is simple. Let us assume that

$$\int_0^{z_0} H_{e^{i\theta_1},\lambda}(\rho)\,d\rho = \int_0^{z_0} H_{e^{i\theta_2},\lambda}(\rho)\,d\rho$$

for some $\theta_1, \theta_2 \in (-\pi,\pi]$ with $\theta_1 \ne \theta_2$. Then the use of Proposition 3 yields us that $H_{e^{i\theta_1},\lambda}(z_0) = H_{e^{i\theta_2},\lambda}(z_0)$, which further gives the following relation

$$\tau\left(\frac{w_{H_{e^{i\theta_1},\lambda}}(z)}{z},\lambda\right) = \tau\left(\frac{w_{H_{e^{i\theta_2},\lambda}}(z)}{z},\lambda\right).$$

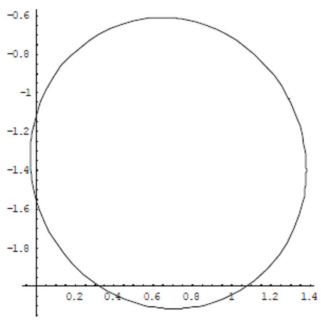
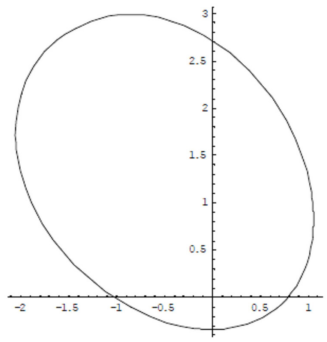
This implies that

$$\frac{B(ze^{i\theta_1}+\lambda)+\overline{\lambda}(1+\overline{\lambda}e^{i\theta_1}z)}{1+\overline{\lambda}e^{i\theta_1}z+\lambda B(ze^{i\theta_1}+\lambda)}=\frac{B(ze^{i\theta_2}+\lambda)+\overline{\lambda}(1+\overline{\lambda}e^{i\theta_2}z)}{1+\overline{\lambda}e^{i\theta_2}z+\lambda B(ze^{i\theta_2}+\lambda)}.$$

After some simplification, we obtain $ze^{i\theta_1}=ze^{i\theta_2}$, which leads us to a contradiction. Hence the curve is simple. Since $V_\lambda(z_0,A,B)$ is compact convex subset of $\mathbb{C}$ and has non-empty interior, therefore the boundary $\partial V_\lambda(z_0,A,B)$ is a simple closed curve. From Proposition 3 the curve $\partial V_\lambda(z_0,A,B)$ contains the curve $(-\pi,\pi]\ni\theta\mapsto\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho$. Since a simple closed curve cannot contain any simple closed curve other than itself. Thus $\partial V_\lambda(z_0,A,B)$ is given by $(-\pi,\pi]\ni\theta\mapsto\int_0^{z_0}H_{e^{i\theta},\lambda}(\rho)\,d\rho$. □

**Geometric View of Theorem 1**

By using Mathematica, we have the following views for different parameters.

| | | |
|---|---|---|
| Values of parameters | $z_0 = 0.335192 - 0.787333i$<br>$\lambda = 0.0737292 + 0.466706i$<br>$A = 1,\ B = -1$ | $z_0 = -0.261209 + 0.926935i$<br>$\lambda = 0.0737292 + 0.466706i$<br>$\beta = -1.991244,\ \gamma = 0.383292$<br>$A = 2(1-\beta)\cos\gamma e^{-i\gamma} - 1,\ B = -1$ |
| Geometric View of Theorem 1 |  |  |

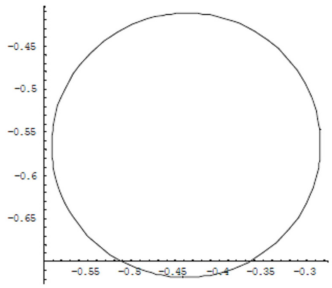
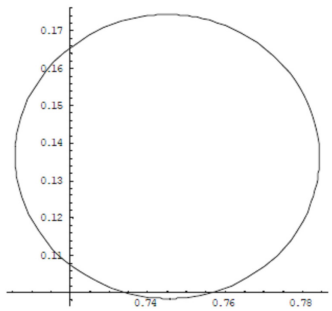
| | | |
|---|---|---|
| Values of parameters | $z_0 = -0.41227 - 0.521734i$<br>$\lambda = -0.0875648 + 0.0714166i$<br>$A = 0.9868233 + 0.00835453i$<br>$B = -0.50$ | $z_0 = 0.771264 + 0.1512040i$<br>$\lambda = -0.391149 - 0.294747i$<br>$A = -0.2346400 - 0.180560i$<br>$B = -0.50$ |
| Geometric View of Theorem 1 |  |  |

FIGURE 1. For $A = 2(1-\beta)(\cos\gamma)e^{-i\gamma} - 1,\ B = -1$, where $\beta < 1, |\gamma| < \pi/2$, we have the known result proved by Ponnusamy and Vasudevarao [3] as special cases of our results.

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# SUCCESSIVE APPROXIMATIONS AND INTERVAL HALVING FOR INTEGRAL BOUNDARY VALUE PROBLEMS

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*Abstract.* We show how a suitable interval halving and parametrization technique can help to essentially improve the sufficient convergence condition for the successive approximations dealing with solutions of nonlinear non-autonomous systems of ordinary differential equations under integral boundary conditions.

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## 1. Introduction

Recently, boundary value problems (BVPs) with integral conditions for non-linear differential equations have attracted much attention, see, e.g. [1],[2] and the references therein. However mainly scalar non-linear differential equations of special kinds have been studied. According to our best knowledge, there are only a few works dealing with a constructive investigation of systems of non-linear differential equations of a general form with linear or non-linear integral boundary restrictions. So, in [10] the following mixed integral BVP was studied

$$\frac{dx(t)}{dt} = f\left(t, x(t)\right), t \in [0, T],$$

$$Ax(0) + \int_0^T P(s)x(s)ds + Cx(T) = d.$$

The paper [6] deals with the problem of general form

$$\frac{du(t)}{dt} = f\left(t, u(t)\right), t \in [a, b],$$

$$\Phi(u) = d, \tag{1.1}$$

where $\Phi : C\left([a, b], \mathbb{R}^n\right)$ is a vector functional (possibly non-linear), $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the Caratheodory condition in a certain bounded set and

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$d$ is a given vector. In [9] the authors investigated the problem for which the integral boundary restrictions also depend upon the derivative of the solution

$$\frac{du(t)}{dt} = f\left(t, u(t)\right), \; t \in [a,b],$$

$$\int_a^b \left[g(s,x(s)) + h(s, f(s,x(s)))\right] ds = d \tag{1.2}$$

and in [5] it was given a constructive existence analysis of solutions of the problem above.

Note, that the investigation of the solutions of problems (1.1) and (1.2) was connected with the properties of the following special sequence of functions well posed on the interval $t \in [a,b]$

$$x_0\left(t,z,\eta\right) := z + \frac{t-a}{b-a}\left[\eta - z\right] = \left[1 - \frac{t-a}{b-a}\right] z + \frac{t-a}{b-a}\eta, \; t \in [a,b],$$

$$x_m\left(t,z,\eta\right) := z + \int_a^t f\left(s, x_{m-1}\left(s,z,\eta\right)\right) ds - \tag{1.3}$$

$$-\frac{t-a}{b-a}\int_a^b f\left(s, x_{m-1}\left(s,z,\eta\right)\right) ds + \frac{t-a}{b-a}\left[\eta - z\right], \; t \in [a,b],$$

where $z$ and $\eta$ are some vector-parameters.

The aim of this paper is to show how a natural interval halving and parametrization technique can help to essentially improve the sufficient convergence conditions mentioned in papers [6], [9].

More precisely we consider the problem

$$\frac{du(t)}{dt} = f\left(t, u(t)\right), \; t \in [a,b], \tag{1.4}$$

$$\int_a^b g\left(s, x(s)\right) ds = d, \tag{1.5}$$

where $f : [a,b] \times D \to \mathbb{R}^n$ , $g : [a,b] \times D \to \mathbb{R}^n$ are a continuous functions in a certain bounded set $D$ and $d \in \mathbb{R}^n$ is a given vector.

We use an appropriate numerical-analytic approach and a natural interval halving technique which was suggested in [6], [4], [7], [8],[11], [3]. At first, we reduce

the given problem (1.4), (1.5) to two "model-type" two-point BVPs with separated parameterized conditions

$$\frac{dx}{dt} = f(t,x), t \in \left[a, a + \frac{b-a}{2}\right], \tag{1.6}$$

$$x(a) = z, \ x\left(a + \frac{b-a}{2}\right) = \lambda \tag{1.7}$$

and

$$\frac{dy}{dt} = f(t,y), t \in \left[a + \frac{b-a}{2}, b\right], \tag{1.8}$$

$$y\left(a + \frac{b-a}{2}\right) = \lambda, \ y(b) = \eta, \tag{1.9}$$

where $z, \lambda, \eta \in \mathbb{R}^n$ are parameters. Note, that the length of the interval in problems (1.6) ,(1.7) and (1.8), (1.9) is $\frac{b-a}{2}$ opposite to $b-a$ in the case of original BVP (1.4), (1.5).

To study the solutions of BVPs (1.6), (1.7) and (1.8), (1.9) we use the special modified form of parameterized successive approximations $x_m(t,z,\lambda)$ and $y_m(t,\lambda,\eta)$ of type (1.3) constructed in analytic form and well defined on the intervals $t \in \left[a, a + \frac{b-a}{2}\right]$ and $t \in \left[a + \frac{b-a}{2}, b\right]$, respectively.

We give sufficient conditions for the uniform convergence of these successive approximations to some limit functions $x_\infty(t,z,\lambda)$ and $y_\infty(t,\lambda,\eta)$, respectively. The main limitation in order to guarantee the convergence of the introduced sequences $\{x_m(t,z,\lambda)\}_{m=0}^{\infty}$ and $\{y_m(t,z,\lambda)\}_{m=0}^{\infty}$ is that one has to assume a certain smallness of the eigenvalue $r(Q)$ of the matrix $Q := \frac{3(b-a)}{20}K$, where $K$ stands in the Lipschitz condition $|f(t,u_1) - f(t,u_2)| \leq K|u_1 - u_2|$, provided that

$$r(Q) < 1.$$

Note that in [6], [9], [5], [10] where the numerical-analytic approach was applied, instead of the previous inequality the convergence condition

$$2r(Q) < 1.$$

has appeared. Thus, the convergence condition is weakened by its half. We show that the limit functions $x_\infty(t,z,\lambda)$ and $y_\infty(t,\lambda,\eta)$ are solutions of some additively modified equations of form (1.6) and (1.8). The functional perturbation term, by which the modified equation differs from the original one, essentially depends on the parameters $z, \lambda, \eta$ and generates finitely many *determining* equations together with (1.5) from which the numerical values of these parameters should be found.

## 2. Notation and definitions

We fix an $n \in \mathbb{N}$ and a bounded set $D \subset \mathbb{R}^n$. The following symbols are used in the sequel:

1. For vectors $x = \mathrm{col}(x_1, ..., x_n) \in \mathbb{R}^n$ the obvious notation $|x| = \mathrm{col}(|x_1|, ..., |x_n|)$ is used and the inequalities between vectors are understood component-wise.

The same convention is adopted implicitly for operations $'\max', '\min', '\sup', '\inf'$, so that e.g. $\max\{h(z) : z \in Q\}$ for any $h = (h_i)_{i=1}^n : Q \to \mathbb{R}^n$, where $Q \subset \mathbb{R}^m$, $m \leq n$, is defined as the column vector with components $\max\{h_i(z) : z \in Q\}$, $i = 1, 2, \ldots, n$.

2. $1_n$ is the unit matrix of dimension $n$.
3. $0_n$ is the zero matrix of dimension $n$.
4. $r(K)$ is the maximal eigenvalue (in modulus) of a matrix $K$.

**Definition 1.** For any non-negative vector $\rho \in \mathbb{R}^n$ under the component-wise $\rho-$neighborhood of a point $z \in \mathbb{R}^n$ we understand

$$B(z, \rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}. \tag{2.1}$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise $\rho-$neighborhood by putting

$$B(\Omega, \rho) := \bigcup_{\xi \in \Omega} B(\xi, \rho). \tag{2.2}$$

**Definition 2.** For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a, \eta \in D_b, \theta \in [0,1] \tag{2.3}$$

and its component-wise $\rho-$neighborhood

$$D := B(D_{a,b}, \rho). \tag{2.4}$$

For a set $D \subset \mathbb{R}^n$, closed interval $[a,b] \subset \mathbb{R}$, continuous function $f : [a,b] \times D \to \mathbb{R}^n$, $n \times n$ matrix $K$ with non-negative entries, we write

$$f \in Lip(K, D) \tag{2.5}$$

if the inequality

$$|f(t,u) - f(t,v)| \leq K|u - v| \tag{2.6}$$

holds for all $\{u, v\} \subset D$ and $t \in [a,b]$.

Finally, on the base of function $f : [a,b] \times D \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D}(f) := \frac{1}{2}\left[\max_{(t,x)\in[a,b]\times D} f(t,x) - \min_{(t,x)\in[a,b]\times D} f(t,x)\right]. \tag{2.7}$$

## 3. Problem setting and reduction to model-type, subsidiary statements

Let us consider the nonlinear integral BVP (1.4), (1.5).

Let us fix certain closed bounded sets $D_a$ , $D_{\frac{a+b}{2}}$ , $D_b \subset \mathbb{R}^n$ and focus on the continuously differentiable solutions $u$ of problem (1.4), (1.5) with values

$$u(a) \in D_a,\ u\left(\frac{a+b}{2}\right) \in D_{\frac{a+b}{2}} \text{ and } u(b) \in D_b. \tag{3.1}$$

Without loss of generality, we shall choose $D_a$ , $D_{\frac{a+b}{2}}$ , $D_b$ to be convex.

Based on the sets $D_a$ and $D_{\frac{a+b}{2}}$ we introduce the set

$$D_{a,\frac{a+b}{2}} := (1-\theta)z + \theta\lambda,\ z \in D_a,\ \lambda \in D_{\frac{a+b}{2}},\ \theta \in [0,1], \tag{3.2}$$

according to (2.3) and its component-wise $\rho^x-$neighborhood

$$D^x := B(D_{a,\frac{a+b}{2}}, \rho^x) \tag{3.3}$$

as in (2.4). Similarly, based on the sets $D_{\frac{a+b}{2}}$ and $D_b$ we introduce the set

$$D_{\frac{a+b}{2},b} := (1-\theta)\lambda + \theta\eta,\ \lambda \in D_{\frac{a+b}{2}},\ \eta \in D_b,\ \theta \in [0,1], \tag{3.4}$$

according to (2.3) and its component-wise $\rho^y-$neighborhood

$$D^y := B(D_{\frac{a+b}{2},b}, \rho^y) \tag{3.5}$$

as in (2.4).

It is important to emphasize that $D^x$, $D^y$ and $D$ are bounded sets and, thus, the Lipschitz conditions of type (2.5), (2.6) are not assumed globally. The problem is to find a continuously differentiable solution $u : [a,b] \to D$ of the problem (1.4), (1.5) for which inclusions (3.1) hold.

At first we simplify the boundary integral conditions (1.5) and reduce it to some two-point separated conditions. To replace the boundary conditions (1.5) by certain linear two-point linear separated ones, similarly to [6], [10], [11],[9] we apply a certain "freezing" technique. Namely, we introduce the vectors of parameters

$$z = \mathrm{col}(z_1, z_2, ..., z_n),\ \lambda = \mathrm{col}(\lambda_1, \lambda_2, ..., \lambda_n),\ \eta = \mathrm{col}(\eta_1, \eta_2, ..., \eta_n) \tag{3.6}$$

by formally putting

$$z := u(a),\ \lambda := u\left(\frac{a+b}{2}\right),\ \eta := u(b). \tag{3.7}$$

Now, instead of boundary value problem (1.4), (1.5) using a natural interval halving technique, we will consider on the intervals $t \in \left[a, \frac{a+b}{2}\right]$ and $t \in \left[a + \frac{b-a}{2}, b\right]$, respectively, the following two "model-type" two-point BVPs with separated parameterized conditions (1.6), (1.7) and (1.8), (1.9).

The parametrization technique that we are going to use suggests that instead of the original boundary value problem (1.4) (1.5), we study the family of parameterized boundary value problems (1.6), (1.7) and (1.8), (1.9) where the boundary restrictions are linear and separated. We then go back to the original problem by choosing the values of the introduced parameters appropriately.

*Remark* 1. The set of solutions of the boundary value problem (1.4), (1.5) coincides with the set of the solutions of the parameterized problems (1.6), (1.7) and (1.8), (1.9) with separated restrictions, satisfying additional conditions (3.7).

We recall some subsidiary statements which are needed below in the following form.

**Lemma 1** ([3], Lemma 3.13)**.** *Let $f:[\tau,\tau+I]\to\mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t\in[\tau,\tau+I]$, the inequality*

$$\left|\int_\tau^t\left[f(\tau)-\frac{1}{I}\int_\tau^{\tau+I}f(s)ds\right]d\tau\right|\le\alpha_1(t,\tau,I)\,\delta_{[\tau,\tau+I]}(f) \tag{3.8}$$

*holds, where*

$$\alpha_1(t,\tau,I)=2(t-\tau)\left(1-\frac{t-\tau}{I}\right),\ |\alpha_1(t,\tau,I)|\le\frac{I}{2},\ t\in[\tau,\tau+I], \tag{3.9}$$

*and*

$$\delta_{[\tau,\tau+I]}(f)=\frac{\max\limits_{t\in[\tau,\tau+I]}f(t)-\min\limits_{t\in[\tau,\tau+I]}f(t)}{2}.$$

**Lemma 2** ([3], Lemma 3.16)**.** *Let the sequence of continuous functions $\{\alpha_m(t,\tau,I)\}_{m=0}^{\infty}$, for $t\in[\tau,\tau+I]$ be defined by the recurrence relation*

$$\alpha_{m+1}(t,\tau,I)=\left(1-\frac{t-\tau}{I}\right)\int_\tau^t\alpha_m(s,\tau,I)ds+$$
$$+\frac{t-\tau}{I}\int_t^{\tau+I}\alpha_m(s,\tau,I)ds, m=0,1,2,\ldots, \tag{3.10}$$

*where*

$$\alpha_0(t,\tau,I)=1.$$

*Then the following estimates hold for $t\in[\tau,\tau+I]$:*

$$\alpha_{m+1}(t,\tau,I)\le\frac{10}{9}\left(\frac{3I}{10}\right)^m\alpha_1(t,\tau,I),\ m\geqslant 0, \tag{3.11}$$

$$\alpha_{m+1}(t,\tau,I)\le\frac{3I}{10}\alpha_m(t,\tau,I), m\geqslant 2,$$

*where $\alpha_1(t,\tau,I)$ is given in (3.9).*

## 4. Interval halving and successive approximations

So, our approach to the integral BVP (1.4), (1.5) requires that we first study the auxiliary problems (1.6), (1.7) and (1.8), (1.9) separately, for which purpose appropriate iteration processes were introduced .

Let us put that the domain of the space of variables of the problem (1.6), (1.7) is $D^x$.

We suppose that $f \in Lip(K_x, D^x)$ with $\rho^x$ satisfies the inequality

$$\rho^x \geq \frac{b-a}{4}\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f) \tag{4.1}$$

and

$$r(Q_x) < 1, Q_x = \frac{3(b-a)}{20}K_x. \tag{4.2}$$

Let us define for the parameterized problem (1.6), (1.7) the recurrence parameterized sequence of functions $x_m : \left[a,\frac{a+b}{2}\right] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ , $m = 0,1,2,...$, by putting

$$x_0(t,z,\lambda) := z + \frac{2(t-a)}{b-a}[\lambda - z] = \tag{4.3}$$

$$= \left[1 - \frac{2(t-a)}{b-a}\right] z + \frac{2(t-a)}{b-a}\lambda, t \in \left[a, \frac{a+b}{2}\right],$$

$$x_m(t,z,\lambda) := z + \int_a^t f(s, x_{m-1}(s,z,\lambda))\,ds - \tag{4.4}$$

$$-\frac{2(t-a)}{b-a}\int_a^{\frac{a+b}{2}} f(s, x_{m-1}(s,z,\lambda))\,ds + \frac{2(t-a)}{b-a}[\lambda - z],\ t \in \left[a, \frac{a+b}{2}\right],$$

for all $m = 1,2,...,z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$ .

In a similar manner, for the parameterized problem (1.8), (1.9) on the interval $\left[\frac{a+b}{2}, b\right]$, we put that the domain of the space variables for the problem is $D^y$ and suppose that $f \in Lip(K_y, D^y)$ with $\rho^y$ satisfying the inequality

$$\rho^y \geq \frac{b-a}{4}\delta_{\left[\frac{a+b}{2},b\right],D^y}(f) \tag{4.5}$$

and

$$r\left(\frac{3(b-a)}{20}K_y\right) < 1. \tag{4.6}$$

Let us define for the problem (1.8), (1.9) the recurrence parameterized sequence of functions $y_m : \left[\frac{a+b}{2}, b\right] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $m = 0, 1, 2, \ldots$, by putting

$$y_0(t, \lambda, \eta) := \lambda + \frac{2t - a - b}{2b - a - b}[\eta - \lambda] = \tag{4.7}$$

$$= \left[1 - \frac{2t - a - b}{2b - a - b}\right]\lambda + \frac{2t - a - b}{2b - a - b}\eta, t \in \left[\frac{a+b}{2}, b\right],$$

$$y_m(t, \lambda, \eta) := \lambda + \int_{\frac{a+b}{2}}^{t} f(s, y_{m-1}(s, \lambda, \eta))\, ds - \tag{4.8}$$

$$-\frac{2t - a - b}{2b - a - b}\int_{\frac{a+b}{2}}^{b} f(s, y_{m-1}(s, z, \lambda))\, ds + \frac{2t - a - b}{2b - a - b}[\eta - \lambda],\ t \in \left[\frac{a+b}{2}, b\right],$$

for all $m = 1, 2, \ldots, \lambda \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$.

We note that all members of the sequences (4.4), (4.8) satisfy the two-point boundary conditions (1.7) and (1.9) for any $z$, $\lambda$ and $\eta$ from $\mathbb{R}^n$.

## 5. Convergence of successive approximations

We would like to use the sequences $x_m : \left[a, \frac{a+b}{2}\right] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $m = 0, 1, 2, \ldots$, and $y_m : \left[\frac{a+b}{2}, b\right] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $m = 0, 1, 2, \ldots$from (4.4) and (4.8) for the investigation of the solutions of the given boundary value problem (1.4), (1.5).

The following statement shows that the sequence (4.4) is uniformly convergent and its limit is a solution of a certain additively perturbed problem for all $(z, \lambda) \in D_a \times D_{\frac{a+b}{2}}$.

**Theorem 1.** *Let there exist a non negative vector $\rho^x$ such that $f \in Lip(K_x, D^x)$ on the interval $t \in \left[a, \frac{a+b}{2}\right]$ with $\rho^x$ satisfying the inequality (4.7) and for the matrix*

$$Q_x := \frac{3(b-a)}{20} K_x, \tag{5.1}$$

*the inequality (5.2) is satisfied*

$$r(Q_x) < 1. \tag{5.2}$$

*Then, for arbitrary fixed pair of vectors $(z, \lambda) \in D_a \times D_{\frac{a+b}{2}}$:*

*1. All members of sequence (4.4) are continuously differentiable functions on the interval $t \in \left[a, \frac{a+b}{2}\right]$ satisfying the two-point separated parameterized boundary conditions (1.7).*

*2. The sequence of functions (4.4) in* $t \in \left[a, \frac{a+b}{2}\right]$ *converges uniformly as* $m \to \infty$ *to the limit function*

$$x_\infty(t,z,\lambda) = \lim_{m\to\infty} x_m(t,z,\lambda). \tag{5.3}$$

*3. The limit function satisfies the conditions*

$$x_\infty(a,z,\lambda) = z,\ x_\infty\left(\frac{a+b}{2}, z, \lambda\right) = \lambda. \tag{5.4}$$

*4. The function* $x_\infty(t,z,\lambda)$ *is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_a^t f(s,x(s))ds -$$

$$-\frac{2(t-a)}{b-a}\int_a^{\frac{a+b}{2}} f(s,x(s))ds + \frac{2(t-a)}{b-a}[\lambda - z],\ t \in \left[a, \frac{a+b}{2}\right] \tag{5.5}$$

*in the domain* $D^x$*. In other words,* $x_\infty(t,z,\lambda)$ *is a solution of the following Cauchy problem for the modified system of integro-differential equations:*

$$\frac{dx}{dt} = f(t,x) + \frac{2}{b-a}\Delta(z,\lambda),\ t \in \left[a, \frac{a+b}{2}\right], \tag{5.6}$$

$$x(a) = z, \tag{5.7}$$

*where* $\Delta(z,\lambda) : D_a \times D_{\frac{a+b}{2}} \to \mathbb{R}^n$ *is a mapping given by the formula*

$$\Delta(z,\lambda) = \lambda - z - \int_a^{\frac{a+b}{2}} f(s, x_\infty(s,z,\lambda))ds. \tag{5.8}$$

*5. The following estimate holds:*

$$|x_\infty(t,z,\lambda) - x_m(t,z,\lambda)| \leqslant$$

$$\leqslant \frac{10}{9}\alpha_1\left(t, a, \frac{b-a}{2}\right) Q_x^m (1_n - Q_x)^{-1} \delta_{\left[a,\frac{a+b}{2}\right]\times D^x}(f),\ t \in \left[a, \frac{a+b}{2}\right], m \geq 0, \tag{5.9}$$

*where* $\delta_{\left[a,\frac{a+b}{2}\right]\times D^x}(f)$ *is given in (2.7) and* $\alpha_1(t,a,\frac{b-a}{2})$ *is defined by (3.9).*

*Proof.* The validity of assertion 1 is verified by direct computation. To obtain the other required properties, similarly to [6],[9] we will prove that under the condition assumed for fixed $z \in D_a$, $\lambda \in D_{\frac{a+b}{2}}$ and $t \in \left[a, \frac{a+b}{2}\right]$ the functions of the sequence (4.4) are contained in the domain $D^x$ and (4.4) is a Cauchy sequence in the Banach space $C\left(\left[a, \frac{a+b}{2}\right], \mathbb{R}^n\right)$ equipped with the standard uniform norm. Indeed, using the

estimate (3.8) of Lemma 1 for $\tau = a, I = \frac{b-a}{2}$, relation (4.4) for $m = 0, t \in \left[a, \frac{a+b}{2}\right]$ implies that

$$|x_1(t,z,\lambda) - x_0(t,z,\lambda)| \leq$$

$$\leq \frac{1}{2}\alpha_1(t,a,\frac{a+b}{2})\left[\max_{t\in[a,b]} f(t,x_0(t,z,\lambda)) - \min_{t\in[a,b]} f(t,x_0(t,z,\lambda))\right]$$

$$\leq \alpha_1\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f) \leq \frac{b-a}{4}\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f), \tag{5.10}$$

which means taking into account (4.1) , that $x_1(t,z,\lambda) \in D^x$, whenever $(t,z,\lambda) \in \left[a, \frac{a+b}{2}\right] \times D_a \times D_{\frac{a+b}{2}}$ .

Using this and arguing by induction according to Lemma 1 we can easily establish that

$$|x_m(t,z,\lambda) - x_0(t,z,\lambda)| \leq \alpha_1\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f) \leq$$

$$\leq \frac{b-a}{4}\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f),\ m = 2,3,..., \tag{5.11}$$

which means that all functions in (4.4) are also contained in the domain $D^x$, for all $m = 1,2,3,..$and $(t,z,\lambda) \in \left[a, \frac{a+b}{2}\right] \times D_a \times D_{\frac{a+b}{2}}$ .

Now, consider the difference of functions

$$x_{m+1}(t,z,\lambda) - x_m(t,z,\lambda) = \tag{5.12}$$

$$= \int_a^t [f(s,x_m(s,z,\lambda)) - f(s,x_{m-1}(s,z,\lambda))]\,ds -$$

$$-\frac{2(t-a)}{b-a}\int_a^{\frac{a+b}{2}} [f(s,x_m(s,z,\lambda)) - f(s,x_{m-1}(s,z,\lambda))]\,ds,\ m = 1,2,...$$

and introduce the notation

$$r_m(t,z,\lambda) := |x_m(t,z,\lambda) - x_{m-1}(t,z,\lambda)|, m = 1,2,.... \tag{5.13}$$

According to the recurrence relation (3.10) of Lemma 2, using the Lipschitz condition $f \in Lip(K_x, D^x)$ and estimation (3.11), for $m = 1$ it follows from (5.12) and (5.10) that

$$r_2(t,z,\lambda) \leq K_x\left[\left(1 - \frac{2(t-a)}{b-a}\right)\int_a^t \alpha_1\left(s,a,\frac{a+b}{2}\right)ds +\right.$$

$$+\frac{2(t-a)}{b-a}\int_t^b \alpha_1\left(s,a,\frac{a+b}{2}\right)ds\Bigg]\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f)\leq \quad (5.14)$$

$$\leq K_x\alpha_2\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f)\leq\frac{10}{9}Q_x\alpha_1\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f),$$

where the matrix $Q_x$ has the form given in (5.1). By induction we can easily establish that

$$r_{m+1}(t,z,\lambda)\leq K_x^m\alpha_{m+1}\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f)\leq$$
$$\leq\frac{10}{9}Q_x^m\alpha_1\left(t,a,\frac{a+b}{2}\right)\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f), \quad (5.15)$$

Therefore, in view of (5.15)

$$\left|x_{m+j}(t,z,\lambda)-x_m(t,z,\lambda)\right|\leq$$
$$\leq\left|x_{m+j}(t,z,\lambda)-x_{m+j-1}(t,z,\lambda)\right|+\left|x_{m+j-1}(t,z,\lambda)-x_{m+j-2}(t,z,\lambda)\right|+\ldots+$$
$$+\left|x_{m+1}(t,z,\lambda)-x_m(t,z,\lambda)\right|=\sum_{i=1}^{j}r_{m+i}(t,z,\lambda)\leq$$
$$\leq\frac{10}{9}\alpha_1\left(t,a,\frac{a+b}{2}\right)\sum_{i=1}^{j}Q_x^{m+i-1}\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f)=$$
$$=\frac{10}{9}\alpha_1\left(t,a,\frac{a+b}{2}\right)Q_x^m\sum_{i=0}^{j-1}Q^i\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f), \quad (5.16)$$

where $\delta_{\left[a,\frac{a+b}{2}\right],D^x}(f)$ is given according to (2.7). Since, due to (5.2), the maximum eigenvalue of the matrix $Q_x$ does not exceed the unity, we have

$$\sum_{i=0}^{j-1}Q_x^i\leq(1_n-Q_x)^{-1},\ \lim_{m\to\infty}Q_x^m=0_n. \quad (5.17)$$

Therefore, we conclude from (5.16) that, according to Cauchy criterium, the sequence $\{x_m(t,z,\lambda)\}_{m=0}^{\infty}$ of the form (4.4) uniformly converges in the domain $(t,z,\lambda)\in\left[a,\frac{a+b}{2}\right]\times D_a\times D_{\frac{a+b}{2}}$ to the limit function $x_\infty(t,z,\lambda)$. Since all functions of the sequence (4.4) satisfy the boundary conditions (1.7) for all values of the introduced parameters $z\in D_a$, $\eta\in D_{\frac{a+b}{2}}$ the limit function $x_\infty(t,z,\lambda)$ also satisfies these conditions. Passing to the limit as $m\to\infty$ in equality (4.4) we show that the limit function satisfies both the integral equation (5.5) and the Cauchy problem (5.6),

(5.7), where $\Delta(z,\lambda)$ is given by (5.8). Passing to the limit as $j \to \infty$ in (5.16) we get the estimation (5.9). □

By analogy with Theorem 1, under similar conditions, we can establish the uniform convergence of sequence (5.8).

**Theorem 2.** *Let there exist a non negative vector $\rho^y$ such that $f \in Lip(K_y, D^y)$ on the interval $t \in \left[\frac{a+b}{2}, b\right]$ with $\rho^y$ satisfying the inequality (4.5) and for the matrix*

$$Q_y := \frac{3T}{20} K_y \tag{5.18}$$

*the condition*

$$r(Q_y) < 1 \tag{5.19}$$

*is satisfied.*

*Then, for arbitrary fixed pair of vectors $(\lambda, \eta) \in D_{\frac{a+b}{2}} \times D_b$:*

*1. All members of sequence (4.8) are continuously differentiable functions on the interval $t \in \left[\frac{a+b}{2}, b\right]$ satisfying the two-point separated parameterized boundary conditions (1.9).*

*2. The sequence of functions (4.8) in $t \in \left[\frac{a+b}{2}, b\right]$ converges uniformly as $m \to \infty$ to the limit function*

$$y_\infty(t,\lambda,\eta) = \lim_{m\to\infty} y_m(t,\lambda,\eta) \tag{5.20}$$

*3. The limit function satisfies the conditions*

$$y_\infty\left(\frac{a+b}{2},\lambda,\eta\right) = \lambda, \; y_\infty(b,\lambda,\eta) = \eta. \tag{5.21}$$

*4. The function $y_\infty(t,\lambda,\eta)$ is a unique continuously differentiable solution of the integral equation*

$$y(t) = \lambda + \int_{\frac{a+b}{2}}^{t} f(s,y(s))ds - \frac{2t-a-b}{2b-a-b}\int_{\frac{a+b}{2}}^{b} f(s,y(s))ds + \\ + \frac{2t-a-b}{2b-a-b}[\eta-\lambda], \; t \in \left[\frac{a+b}{2}, b\right] \tag{5.22}$$

*in the domain $D^y$. In other words, $y_\infty(t,\lambda,\eta)$ is a solution of the following Cauchy problem for the modified system of integro-differential equations:*

$$\frac{dy}{dt} = f(t,y) + \frac{2}{2b-a-b}H(\lambda,\eta), \; t \in \left[\frac{a+b}{2}, b\right] \tag{5.23}$$

$$y\left(\frac{a+b}{2}\right)=\lambda, \tag{5.24}$$

*where* $H(\lambda,\eta): D_{\frac{a+b}{2}} \times D_b \to \mathbb{R}^n$ *is a mapping given by the formula*

$$H(\lambda,\eta)=\eta-\lambda-\int_{\frac{a+b}{2}}^{b} f(s,y_\infty(s,\lambda,\eta))ds. \tag{5.25}$$

*5. The following estimate holds*

$$\begin{aligned}&|y_\infty(\cdot,\lambda,\eta)-y_m(\cdot,\lambda,\eta)| \\ &\leqslant \frac{10}{9}\alpha_1\left(t,\frac{a+b}{2},\frac{b-a}{2}\right)Q_y^m\left(1_n-Q_y\right)^{-1}\delta_{\left[\frac{a+b}{2},b\right]\times D^y}(f), \\ &t\in\left[\frac{a+b}{2},b\right], m\geq 0,\end{aligned} \tag{5.26}$$

*where* $\delta_{\left[\frac{a+b}{2},b\right]\times D^y}(f)$ *is given in (2.7) and* $\alpha_1(t,\frac{a+b}{2},\frac{b-a}{2})$ *is defined by (3.9).*

*Proof.* The proof can be carried out similarly to the proof of Theorem 1. □

## 6. Limit functions and determining equations

It is natural to expect that the limit functions $x_\infty(t,z,\lambda)$ and $y_\infty(t,\lambda,\eta)$ of the iterations (4.4) and (4.8) on the half-intervals will help one to formulate criteria of solvability of the integral BVP (1.4), (1.5). It turns out that the functions

$$\Delta(z,\lambda): D_a\times D_{\frac{a+b}{2}} \to \mathbb{R}^n \text{ and } H(\lambda,\eta): D_{\frac{a+b}{2}}\times D_b \to \mathbb{R}^n \tag{6.1}$$

defined according to equalities (5.8) and (5.25) provide such conclusion.

Indeed, Theorems 1 and 2 guarantee that under the conditions assumed, the functions

$$x_\infty(t,z,\lambda):\left[a,\frac{a+b}{2}\right]\to\mathbb{R}^n \text{ and } y_\infty(t,\lambda,\eta):\left[\frac{a+b}{2},b\right]\to\mathbb{R}^n \tag{6.2}$$

are well defined for all $(z,\lambda)\in D_a\times D_{\frac{a+b}{2}}$ and $(\lambda,\eta)\in D_{\frac{a+b}{2}}\times D_b$. Therefore, by putting

$$u_\infty(t,z,\lambda,\eta):=\begin{cases} x_\infty(t,z,\lambda), \text{ if } t\in\left[a,\frac{a+b}{2}\right] \\ y_\infty(t,\lambda,\eta), \text{ if } t\in\left[\frac{a+b}{2},b\right]\end{cases} \tag{6.3}$$

we obtain a function $u_\infty(\cdot,z,\lambda,\eta):[a,b]\to\mathbb{R}^n$, which is well defined for the same values $(z,\lambda)\in D_a\times D_{\frac{a+b}{2}}$ and $(\lambda,\eta)\in D_{\frac{a+b}{2}}\times D_b$. This function is obviously

continuous, because at the point $t = \frac{a+b}{2}$ we have

$$x_\infty\left(\frac{a+b}{2}, z, \lambda\right) = y_\infty\left(\frac{a+b}{2}, \lambda, \eta\right) = \lambda. \tag{6.4}$$

Along with equations (1.6) and (1.8) defined on the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ respectively, consider the following equations with the additive perturbation of the right side

$$\frac{dx}{dt} = f(t,x) + \frac{2}{b-a}\mu^x,\ t \in \left[a, \frac{a+b}{2}\right] \tag{6.5}$$

with initial condition

$$x(a) = z, \tag{6.6}$$

and

$$\frac{dy}{dt} = f(t,y) + \frac{2}{2b-a-b}\mu^y,\ t \in \left[\frac{a+b}{2}, b\right] \tag{6.7}$$

with initial condition

$$y\left(\frac{a+b}{2}\right) = \lambda, \tag{6.8}$$

where

$$\mu^x = \operatorname{col}\left(\mu_1^x, \dots, \mu_n^x\right),\ \mu^y = \operatorname{col}\left(\mu_1^y, \dots, \mu_n^y\right) \in \mathbb{R}^n \tag{6.9}$$

are some control parameters.

**Theorem 3.** *Let $z \in D_a$ and $\lambda \in D_{\frac{a+b}{2}}$ be fixed. Suppose that all conditions of Theorems 1,2 hold.*

*Then, for the solutions $x(\cdot, a, z)$ and $y(\cdot, \frac{a+b}{2}, \lambda)$ of the Cauchy problems (6.5), (6.6) and (6.7),(6.8), to have the properties*

$$x\left(\frac{a+b}{2}, a, z\right) = \lambda \tag{6.10}$$

*and*

$$y\left(b, \frac{a+b}{2}, \lambda\right) = \eta \tag{6.11}$$

*respectively, by other words, to satisfy the parameterized separated two-point boundary conditions (1.7) and (1.9), respectively, it is necessary and sufficient that the control parameters $\mu^x$ and $\mu^y$ be given by the formulas*

$$\mu^x := \lambda - z - \int_a^{\frac{a+b}{2}} f(s, x_\infty(s,z,\lambda))ds \tag{6.12}$$

*and*

$$\mu^y := \eta - \lambda - \int_{\frac{a+b}{2}}^{b} f(s, y_\infty(s,\lambda,\eta))ds, \tag{6.13}$$

*where* $x_\infty(\cdot, z, \lambda)$ *and* $y_\infty(\cdot, \lambda, \eta)$ *are the limit functions of the sequences (4.4) and (4.8), respectively. Moreover, in this case*

$$x(\cdot, a, z) = x_\infty(\cdot, z, \lambda) \tag{6.14}$$

*and*

$$y\left(\cdot, \frac{a+b}{2}, \lambda\right) = y_\infty(\cdot, \lambda, \eta). \tag{6.15}$$

*Proof.* The proof can be carried out similarly to the proof of Theorem 2 from [9]. □

The following theorem establishes a relation of function (6.3) to the solution of integral BVP (1.4), (1.5) in terms of the zeros of functions $\Delta(z,\lambda) : D_a \times D_{\frac{a+b}{2}} \to \mathbb{R}^n$ and $H(\lambda,\eta) : D_{\frac{a+b}{2}} \times D_b \to \mathbb{R}^n$, defined according to (5.8) and (5.25).

**Theorem 4.** *Let the conditions of Theorem 1,2 hold. Then :*

*1. The function* $u_\infty(\cdot, z, \lambda, \eta) : [a,b] \to \mathbb{R}^n$ *defined by (6.3) is a continuously differentiable solution of the boundary value problem (1.4), (1.5) if and only if the triplet* $(z, \lambda, \eta)$ *satisfies the system of* $3n$ *algebraic equations*

$$\Delta(z,\lambda) = \lambda - z - \int_{a}^{\frac{a+b}{2}} f(s, x_\infty(t,z,\lambda))ds = 0, \tag{6.16}$$

$$H(\lambda,\eta) = \eta - \lambda - \int_{\frac{a+b}{2}}^{b} f(s, y_\infty(s,\lambda,\eta))ds = 0, \tag{6.17}$$

$$P(z,\lambda,\eta) := \int_{a}^{\frac{a+b}{2}} g(s, x_\infty(s,z,\lambda))\,ds + \int_{\frac{a+b}{2}}^{b} g(s, y_\infty(s,\lambda,\eta))\,ds - d = 0. \tag{6.18}$$

*2. For every solution* $U(\cdot)$ *of problem (1.4), (1.5) with* $\left(U(a), U\left(\frac{a+b}{2}\right), U(b)\right) \in D_a \times D_{\frac{a+b}{2}} \times D_b$*, there exist a triplet* $(z_0, \lambda_0, \eta_0)$ *such that* $U(\cdot) = u_\infty(t, z_0, \lambda_0, \eta_0)$*, where the function* $u_\infty(t, z_0, \lambda_0, \eta_0)$ *is given in (6.3).*

*Proof.* The proof can be carried out similarly to the proof of Theorem 4 from [4]. The continuous differentiability of the solution $u_\infty(\cdot,z,\lambda,\eta):[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ at the point $t=\frac{a+b}{2}$ follows from the equations (6.4), (5.6), (5.23), (6.16), (6.17) and the continuous differentiability of this function at other points is obvious from its definition. □

Equations (6.16) are usually referred to as *determining* or *bifurcation* equations because their roots determine solutions of the original problem.

## 7. Approximate determining equations

Although Theorem 4 provides a theoretical answer to the question on the construction of a solution of the boundary value problem (1.4), (1.5), its application faces difficulties due to the fact that the explicit form of $x_\infty(t,z,\lambda)$, $y_\infty(t,\lambda,\eta)$ and the functions

$$\Delta(z,\lambda):D_a\times D_{\frac{a+b}{2}}\to\mathbb{R}^n,\ H(\lambda,\eta):D_{\frac{a+b}{2}}\times D_b\to\mathbb{R}^n,$$
$$P(z,\lambda,\eta):D_a\times D_{\frac{a+b}{2}}\times D_b\to\mathbb{R},$$

appearing in (6.16), (6.17), (6.18) are usually unknown. This complication can be overcome by using $x_m(s,z,\lambda)$, $y_m(s,\lambda,\eta)$ for a fixed $m$, which will lead one to the so-called *approximate determining equations:*

$$\Delta_m(z,\lambda)=\lambda-z-\int_a^{\frac{a+b}{2}}f(s,x_m(t,z,\lambda))ds=0, \tag{7.1}$$

$$H_m(\lambda,\eta)=\eta-\lambda-\int_{\frac{a+b}{2}}^{b}f(s,y_m(s,\lambda,\eta))ds=0, \tag{7.2}$$

$$P_m(z,\lambda,\eta):=\int_a^{\frac{a+b}{2}}g(s,x_m(s,z,\lambda))\,ds+\int_{\frac{a+b}{2}}^{b}g(s,y_m(s,\lambda,\eta))\,ds-d=0. \tag{7.3}$$

Note that, unlike system (6.16), (6.17), (6.18), the $m$-th approximate determining system (7.1), (7.2), (7.3) contains only terms involving the functions $x_m(\cdot,z,\lambda)$, $y_m(\cdot,\lambda,\eta)$, and thus it is known explicitly.

It is natural to expect that approximations to the unknown solution of problems (1.4), (1.5) can be obtained by using the function

$$u_m(t,z,\lambda,\eta):=\begin{cases} x_m(t,z,\lambda), \text{ if } t\in\left[a,\frac{a+b}{2}\right] \\ y_m(t,\lambda,\eta), \text{ if } t\in\left[\frac{a+b}{2},b\right]\end{cases} \tag{7.4}$$

which is an "approximate " version of (6.3) and well defined for all $t \in [a,b]$ and $(z,\lambda) \in D_a \times D_{\frac{a+b}{2}}$ ,$(\lambda,\eta) \in D_{\frac{a+b}{2}} \times D_b$.

**Lemma 3.** *If $z$ ,$\lambda$ and $\eta$ satisfy equations (7.1), (7.2) and (7.3) for a certain $m$, then the function $u_m(t,z,\lambda,\eta)$ determined by equality (7.4) is continuously differentiable on $[a,b]$.*

*Proof.* We recall that the functions of the sequences (4.4) and (4.8) have the property

$$x_m\left(\frac{a+b}{2},z,\lambda\right) = y_m\left(\frac{a+b}{2},\lambda,\eta\right) = \lambda. \tag{7.5}$$

It follows immediately from (4.4), (4.8) that

$$\frac{dx_{m+1}\left(\frac{a+b}{2},z,\lambda\right)}{dt} = f\left(\frac{a+b}{2},x_m\left(\frac{a+b}{2},z,\lambda\right)\right) - \\ -\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f\left(s,x_m\left(s,z,\lambda\right)\right)ds + \frac{2}{b-a}\left[\lambda - z\right] \tag{7.6}$$

and

$$\frac{dy_{m+1}\left(\frac{a+b}{2},\lambda,\eta\right)}{dt} = f\left(\frac{a+b}{2},y_m\left(\frac{a+b}{2},\lambda,\eta\right)\right) - \\ -\frac{2}{2b-a-b}\int_{\frac{a+b}{2}}^{b} f\left(s,y_m\left(s,z,\lambda\right)\right)ds + \frac{2}{2b-a-b}\left[\eta - \lambda\right]. \tag{7.7}$$

In view of (7.1) and (7.2) it follows from (7.6) and (7.7) that

$$\frac{dx_{m+1}\left(\frac{a+b}{2},z,\lambda\right)}{dt} = f\left(\frac{a+b}{2},x_m\left(\frac{a+b}{2},z,\lambda\right)\right) \tag{7.8}$$

and

$$\frac{dy_{m+1}\left(\frac{a+b}{2},\lambda,\eta\right)}{dt} = f\left(\frac{a+b}{2},y_m\left(\frac{a+b}{2},\lambda,\eta\right)\right). \tag{7.9}$$

In view of (7.5) it follows from (7.8), (7.9) that

$$\frac{dx_{m+1}\left(\frac{a+b}{2},z,\lambda\right)}{dt} = \frac{dy_{m+1}\left(\frac{a+b}{2},\lambda,\eta\right)}{dt}$$

and therefore according to (7.4) $\frac{d\overline{u}_{m+1}(\cdot,z,\lambda,\eta)}{dt}$ is continuous at $\frac{a+b}{2}$. The continuous differentiability of the function $u_m(\cdot,z,\lambda,\eta)$ in other points is obvious from its definition. □

## 8. Approximation of the solutions

Theorem 4 can be complemented by the following natural observation. Let $(\widetilde{z},\widetilde{\lambda},\widetilde{\eta})\in D_a\times D_{\frac{a+b}{2}}\times D_b$ be a root of the approximate determining system (7.1), (7.2), (7.3) for a certain $m$. Then the function

$$U_m(t)=u_m(t,\widetilde{z},\widetilde{\lambda},\widetilde{\eta}):=\begin{cases} x_m\left(t,\widetilde{z},\widetilde{\lambda}\right), \text{ if } t\in\left[a,\frac{a+b}{2}\right] \\ y_m\left(t,\widetilde{\lambda},\widetilde{\eta}\right), \text{ if } t\in\left[\frac{a+b}{2},b\right]\end{cases}, \tag{8.1}$$

defined according to (7.4) can be regarded as the $m-$th approximation to a solution of the two-point boundary value problem (1.4), (1.5). This is justified by Lemma 3. The following estimates follow directly from Theorems 1,2

$$|\widetilde{x}_\infty(t,\widetilde{z},\widetilde{\lambda})-x_m\left(t,\widetilde{z},\widetilde{\lambda}\right)|\leq$$

$$\leq\frac{10}{9}\alpha_1\left(t,a,\frac{b-a}{2}\right)Q_x^m\left(1_n-Q_x\right)^{-1}\delta_{\left[a,\frac{a+b}{2}\right]\times D^x}(f),\ t\in\left[a,\frac{a+b}{2}\right], \tag{8.2}$$

$$|\widetilde{y}_\infty(t,\widetilde{\lambda},\widetilde{\eta})-y_m\left(t,\widetilde{\lambda},\widetilde{\eta}\right)|\leq$$

$$\leq\frac{10}{9}\alpha_1\left(t,\frac{a+b}{2},\frac{b-a}{2}\right)Q_y^m\left(1_n-Q_y\right)^{-1}\delta_{\left[\frac{a+b}{2},b\right]\times D^y}(f),\ t\in\left[a,\frac{a+b}{2}\right]. \tag{8.3}$$

## 9. Example

Let us apply the numerical-analytic approach described above to the system of differential equations

$$\begin{cases} x_1'(t)=\frac{t}{4}x_2^2(t)-\frac{7t}{2}x_1(t)+\frac{27}{64}t^3+\frac{3}{5}t, \\ x_2'(t)=t^2x_1(t)-\frac{t}{4}x_2(t)-\frac{1}{8}t^4-\frac{3}{80}t^2+\frac{1}{4},\ \ t\in[0,1],\end{cases} \tag{9.1}$$

considered with the integral boundary conditions

$$\begin{cases} \int_0^1 sx_1(s)x_2(s)ds=\frac{7}{480} \\ \int_0^1 s^2x_2^2(s)ds=\frac{1}{80}\end{cases}. \tag{9.2}$$

Clearly, (9.1) is a particular case of (1.4), (1.5) with $a:=0$, $b:=1$,

$$f(t,x_1,x_2):=\begin{pmatrix}\frac{t}{4}x_2^2-\frac{7t}{2}x_1+\frac{27}{64}t^3+\frac{3}{5}t, \\ t^2x_1-\frac{t}{4}x_2-\left(\frac{1}{8}t^2-\frac{3}{80}\right)t^2+\frac{1}{4}\end{pmatrix},$$

$g(t,x_1,x_2):=\begin{pmatrix}tx_1x_2\\ t^2x_2^2\end{pmatrix}$, and $d:=\begin{pmatrix}7/480\\ 1/80\end{pmatrix}$.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} + \frac{1}{10},\ x_2^*(t) = \frac{t}{4} \tag{9.3}$$

is a continuously differentiable solution of the problem (9.1), (9.2).

Following (1.7) and (1.9), introduce parameters

$$\begin{gathered} z := x(0) = col(x_1(0), x_2(0)) = col(z_1, z_2), \\ \lambda := x\left(\frac{1}{2}\right) = col\left(x_1\left(\frac{1}{2}\right), x_2\left(\frac{1}{2}\right)\right) = col\left(\lambda_1, \lambda_2\right), \\ \eta := x(1) = col\left(x_1(1), x_2(1)\right) = col\left(\eta_1, \eta_2\right). \end{gathered} \tag{9.4}$$

Let us choose the sets $D_a$ and $D_{\frac{a+b}{2}}$, where one looks for the values $x(a)$ and $x(\frac{a+b}{2})$, as follows:

$$D_a = D_{\frac{a+b}{2}} = \{(x_1, x_2) : 0.001 \le x_1 \le 0.14,\ -0.1 \le x_2 \le 0.14\}.$$

Let us choose the sets $D_{\frac{a+b}{2}}$ and $D_b$, where one looks for the values $x(\frac{a+b}{2})$ and $x(b)$ and as follows:

$$D_{\frac{a+b}{2}} = D_b = \{(y_1, y_2) : 0.12 \le y_1 \le 0.23,\ 0.1 \le y_2 \le 0.26\}.$$

In this case, a convex linear combination $D^x$ of form (3.1) of vectors $z \in D_a$ and $\lambda \in D_{\frac{a+b}{2}}$ will be $D^x = D_a = D_{\frac{a+b}{2}}$.

In this case, a convex linear combination $D^y$ of form (3.3) of vectors $\lambda \in D_{\frac{a+b}{2}}$ and $\eta \in D_b$ will be $D^y = D_{\frac{a+b}{2}} = D_b$.

For $\rho^x, \rho^y$ involved in (3.3), (3.5) and (4.1), (4.5) we choose the value

$$\rho^x = \rho^y := col\,(0.7;\ 0.7).$$

Consequently $\rho^x$−neighbourhood $D^x$ of the set $D_{a,\frac{a+b}{2}}$ is given as follows

$$D^x = \{(x_1, x_2) : -0.701 \le x_1 \le 0.84,\ -0.8 \le x_2 \le 0.84\}$$

Consequently $\rho^y$−neighbourhood $D^y$ of the set $D_{\frac{a+b}{2},b}$ is given as follows

$$D^y = \{(y_1, y_2) : -0.58 \le y_1 \le 0.93,\ -0.6 \le y_2 \le 0.96\}.$$

Direct computations show that the Lipschitz condition (2.5) for the right hand side in (9.1) in the domain $D$ holds with matrix

$$K = \begin{bmatrix} 7/2 & 1/8 \\ 1 & 1/4 \end{bmatrix}$$

and

$$Q = \frac{3}{10}\begin{bmatrix} 7/2 & 1/8 \\ 1 & 1/4 \end{bmatrix},\ r(Q) = 1.061405 > 1,$$

$$Q_x := \frac{3}{20}K_x = \frac{3}{20}\begin{bmatrix} 7/4 & 1/32 \\ 1/4 & 1/8 \end{bmatrix}, \quad r(Q_x) = 0.2632 < 1,$$
$$Q_y := \frac{3}{20}K_y = \frac{3}{20}\begin{bmatrix} 7/2 & 1/8 \\ 1 & 1/4 \end{bmatrix}, \quad r(Q_y) = 0.5307 < 1,$$

$$\delta_{[a,\frac{a+b}{2}],D^x}(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x) - \min_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x)\right]$$
$$= \begin{bmatrix} 1.392475, \\ 0.295125 \end{bmatrix},$$
$$\rho^x = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix} \geq \frac{b-a}{4}\delta_{[a,\frac{a+b}{2}],D^x}(f) = \begin{bmatrix} 0.34811875, \\ 0.07378125 \end{bmatrix}.$$

$$\delta_{[\frac{a+b}{2},b],D^y}(f) := \frac{1}{2}\left[\max_{(t,x)\in[\frac{1}{2},1]\times D^y} f(t,x) - \min_{(t,x)\in[\frac{1}{2},1]\times D^y} f(t,x)\right]$$
$$= \begin{bmatrix} 2.7577, \\ 0.95 \end{bmatrix},$$
$$\rho^y = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix} \geq \frac{b-a}{4}\delta_{[\frac{a+b}{2},b],D^y}(f) = \begin{bmatrix} 0.689425, \\ 0.2375 \end{bmatrix}.$$

So, we check that all conditions of Theorems 1, 2 are fulfill, and the sequence of functions (4.1) for this example is convergent.

Using (4.4),(4.7) and applying Maple 14 at the first iteration ($m = 1$) we obtain the following results for the first and the second component:

$$\begin{aligned} x_{11}(t,z,\lambda) &:= z_1 + 1/4(1/4(-2z_2+2\lambda_2)^2 + 27/64)t^4 + 1/3(7z_1 - 7\lambda_1 + \\ &+ 1/2z_2(-2z_2+2\lambda_2))t^3 + 1/2(1/4z_2^2 + 3/5 - 7/2z_1)t^2 - 2t((1/256)\times \\ &(-2z_2+2\lambda_2)^2 + 1671/20480 - (7/48)z_1 - (7/24)\lambda_1 + \\ &+ (1/48)z_2(-2z_2+2\lambda_2) + (1/32)z_2^2) + 2t(\lambda_1 - z_1), \end{aligned}$$

$$\begin{aligned} x_{12}(t,z,\lambda) &:= z_2 + 1/4t - 1/40t^5 + 1/4(-2z_1+2\lambda_1)t^4 + \\ &+ 1/3(1/2z_2 - 1/2\lambda_2 + z_1 - 3/80)t^3 - 1/8z_2t^2 - 2t(157/1280 + \\ &+ (1/96)z_1 + (1/32)\lambda_1 - (1/96)z_2 - (1/48)\lambda_2) + 2t(\lambda_2 - z_2), \end{aligned} \tag{9.5}$$

$$\begin{aligned} y_{11}(t,\lambda,\eta) &:= \lambda_1 + 1/4(1/4(-2\lambda_2+2\eta_2)^2 + 27/64)(t^4 - 1/16) + \\ &+ 1/3(7\lambda_1 - 7\eta_1 + 1/2(2\lambda_2 - \eta_2)(-2\lambda_2+2\eta_2))t^3 - 1/8 + (1/2((1/4)\times \\ &(2\lambda_2-\eta_2)^2 + 3/5 - 7\lambda_1 + (7/2)\eta_1))(t^2 - 1/4) - (2t-1)((15/256)(-2\lambda_2 + \end{aligned}$$

$$+2\eta_2)^2+6633/20480-(7/12)\lambda_1-(35/48)\eta_1+(7/48(2\lambda_2-\eta_2))\times$$
$$(-2\lambda_2+2\eta_2)+(3/32)(2\lambda_2-\eta_2)^2)+(2t-1)(\eta_1-\lambda_1),$$

$$\begin{aligned}y_{12}(t,\lambda,\eta) := {} & \lambda_2+1/4t-159/1280-1/40t^5+1/4(-2\lambda_1+2\eta_1)(t^4-1/16)+\\ & +1/3(1/2\lambda_2-1/2\eta_2+2\lambda_1-\eta_1-3/80)(t^3-1/8)+\\ & +(1/2(-(1/2)\lambda_2+(1/4)\eta_2))(t^2-1/4)-(2t-1)(23/256+(11/96)\lambda_1+\\ & +(17/96)\eta_1-(1/24)\lambda_2-(5/96)\eta_2)+(2t-1)(\eta_2-\lambda_2).\end{aligned}$$

The numerical computations show that the components of the solution of equations (7.1)–(7.3) for $m=1$ are:

$$\begin{aligned} z_1 &:= z_{11}\approx 0.1000871447, & z_2 &:= z_{12}\approx -0.00003539691,\\ \lambda_1 &:= \lambda_{11}\approx 0.1312562435, & \lambda_2 &:= \lambda_{12}\approx 0.1250395576,\\ \eta_1 &:= \eta_{11}\approx 0.2248977438, & \eta_2 &:= \eta_{12}\approx 0.2500433008.\end{aligned} \tag{9.6}$$

By putting (9.6) into (9.5), we obtain the first and the second components of the first approximation to the solution of the given integral BVP (9.1), (9.2).

The graphs of the first approximation and the exact solution (9.3) of the original boundary-value problem are shown on Figure 1.

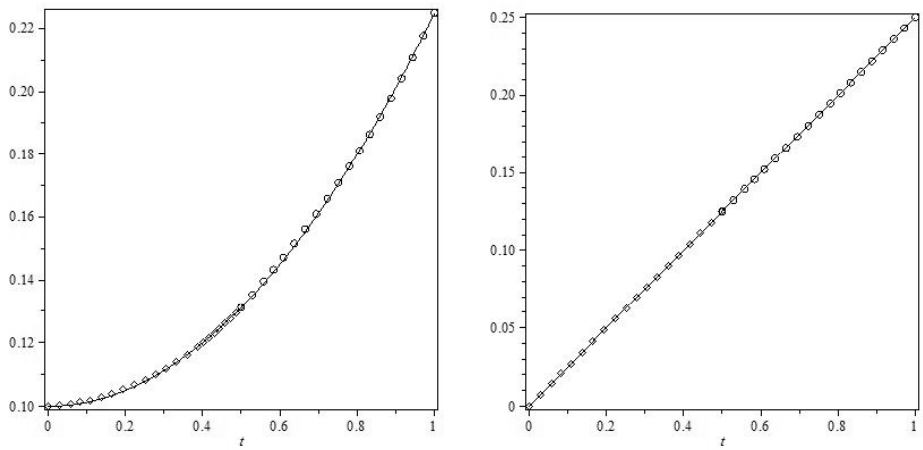


FIGURE 1. The components of the exact solution (solid line) and its first approximation (drawn with dots)

The number of the solutions of the algebraic determined system (5.23) is coincide with the number of the solutions of the given integral BVP.

Let us choose the sets $D_a$ and $D_{\frac{a+b}{2}}$, where one looks for the values $x(a)$ and $x(\frac{a+b}{2})$ as follows:

$$D_a = D_{\frac{a+b}{2}} = \{(x_1, x_2) : -0.47 \le x_1 \le -0.2,\ -0.36 \le x_2 \le -0.23\}.$$

Let us choose the sets $D_{\frac{a+b}{2}}$ and, $D_b$ where one looks for the values $x(\frac{a+b}{2})$ and $x(b)$ and as follows:

$$D_{\frac{a+b}{2}} = D_b = \{(y_1, y_2) : -0.23 \le y_1 \le 0.13,\ -0.25 \le y_2 \le -0.12\}.$$

$$D^x = D_a = D_{\frac{a+b}{2}},\ D^y = D_{\frac{a+b}{2}} = D_b.$$

Consequently $\rho^x-$neighhourhood $D^x$ of the set $D_{a,\frac{a+b}{2}}$ is given as follows

$$D^x = \{(x_1, x_2) : -1.17 \le x_1 \le 0.5,\ -1.06 \le x_2 \le 0.47\}.$$

Consequently $\rho^y-$neighhourhood $D^y$ of the set $D_{\frac{a+b}{2},b}$ is given as follows

$$D^y = \{(y_1, y_2) : -0.93 \le y_1 \le 0.83,\ -0.95 \le y_2 \le 0.58\},$$

$$\delta_{[a,\frac{a+b}{2}],D^x}(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x) - \min_{(t,x)\in[0,\frac{1}{2}]\times D^x} f(t,x)\right]$$
$$= \begin{bmatrix} 1.531475, \\ 0.30437 \end{bmatrix},$$

$$\rho^x = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix} \ge \frac{b-a}{4}\delta_{[a,\frac{a+b}{2}],D^x}(f) = \begin{bmatrix} 0.3828687, \\ 0.07609375 \end{bmatrix},$$

$$\delta_{[\frac{a+b}{2},b],D^y}(f) := \frac{1}{2}\left[\max_{(t,x)\in[\frac{1}{2},1]\times D^y} f(t,x) - \min_{(t,x)\in[\frac{1}{2},1]\times D^y} f(t,x)\right]$$
$$= \begin{bmatrix} 2.6678125 \\ 0.92125 \end{bmatrix},$$

$$\rho^y = \begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix} \ge \frac{b-a}{4}\delta_{[\frac{a+b}{2},b],D^y}(f) = \begin{bmatrix} 0.6669531250, \\ 0.2303125 \end{bmatrix}.$$

Computations show that the approximate determined system of algebraic equations (7.1)–(7.3) side by side with the solution (9.6) for $m = 1$ has an another solution

$$\begin{aligned} &\hat{z}_1 := z_{11} \approx -0.4600459223, && \hat{z}_2 := z_{12} \approx -0.3588100487, \\ &\hat{\lambda}_1 := \lambda_{11} \approx -0.2288301475, && \hat{\lambda}_2 := \lambda_{12} \approx -0.2405391230, \\ &\hat{\eta}_1 := \eta_{11} \approx 0.1278495443, && \hat{\eta}_2 := \eta_{12} \approx -0.1346356173. \end{aligned} \tag{9.7}$$

By substituting (9.7) into first approximation (9.5) we obtain the first approximation to the second solution of given integral BVP (9.1), (9.2).

By analogy we obtain the second, third and fourth approximations ($m = 2, m = 3, m = 4$)

The graphs of the first and the fourth approximations to the second solution of the given BVP are shown on Figure 2.

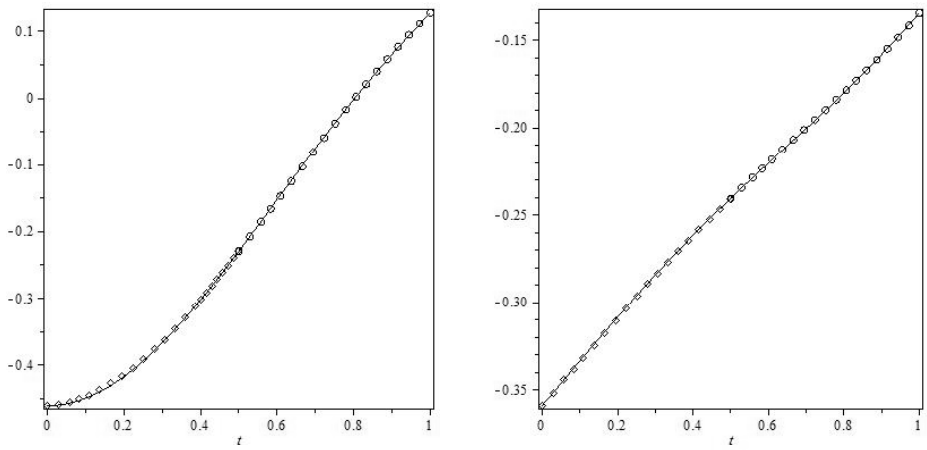


FIGURE 2. The components of the first (∘) and the fourth (solid line) approximations to the second solution

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# RECURSION FORMULAS FOR $G_1$ AND $G_2$ HORN HYPERGEOMETRIC FUNCTIONS

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*Abstract.* The aim of this paper is to present various recursion formulas for Horn hypergeometric functions by the contiguous relations of hypergeometric series.These recursion formulas allow us to state the functions $G_1$ and $G_2$ Horn hypergeometric functions as a combination of themselves.

2010 *Mathematics Subject Classification:* 33C20; 11J72

*Keywords:* recursion formulas, Horn hypergeometric functions, contiguous relation

## 1. Introduction

The first $G_1$ and $G_2$ Horn hypergeometric functions were defined by the series [3–5]

$$G_1\left(\alpha,\beta,\beta';x,y\right)=\sum_{m,p=0}^{\infty}(\alpha)_{m+p}(\beta)_{p-m}\left(\beta'\right)_{m-p}\frac{x^m}{m!}\frac{y^p}{p!}$$
$$|x|<r\ \ ,\ \ |y|<s\ \ ,\ \ r+s=1$$

and

$$G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)=\sum_{m,p=0}^{\infty}(\alpha)_m\left(\alpha'\right)_p(\beta)_{p-m}\left(\beta'\right)_{m-p}\frac{x^m}{m!}\frac{y^p}{p!}$$
$$|x|<1\ \ ,\ \ |y|<1,$$

respectively. Recently, Opps *et al*. [1] have obtained some recursion formulas for the function $F_2$ by the contiguous relation of the Gauss hypergeometric function ${}_2F_1$. In [6], Wang gave some recursion formulas for Appell hypergeometric functions. The aim of our present investigation is to construct various recursion formulas for each of Horn hypergeometric functions $G_1$ and $G_2$.

Recall that gamma function is defined in [2,3] by

$$\Gamma(n)=\int_0^{\infty}t^{n-1}e^{-t}dt\ \ ,\ \ \mathrm{Re}(n)>0.$$

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The Pochhammer symbol $(\lambda)_n$ is denoted by

$$(\lambda)_n := \lambda(\lambda+1)\ldots(\lambda+n-1)\ ,\ (n \in \mathbb{N} := \{1,2,3,\ldots\}) \text{ and } (\lambda)_0 := 1$$

and its well known form is also given in [1] as

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}\ ,\ (n \in \mathbb{N} := \{1,2,3,\ldots\}). \tag{1.1}$$

It easily follows from (1.1) that

$$(\lambda)_{m-n} = (\lambda)_m (\lambda+m)_{-n}, \tag{1.2}$$

for $m,n \in \mathbb{N}$.

## 2. Recursion formulas of $G_1$

In this section, we give some recursion formulas for the function $G_1$. We start the following theorem.

**Theorem 1.** *Recursion formulas for the function $G_1$ are as follows :*

$$\begin{aligned} G_1(\alpha+n,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &+ (\beta)_{-1}\beta' x \sum_{k=1}^{n} G_1(\alpha+k,\beta-1,\beta'+1;x,y) \\ &+ \beta(\beta')_{-1} y \sum_{k=1}^{n} G_1(\alpha+k,\beta+1,\beta'-1;x,y) \end{aligned} \tag{2.1}$$

*and*

$$\begin{aligned} G_1(\alpha-n,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &- (\beta)_{-1}\beta' x \sum_{k=0}^{n-1} G_1(\alpha-k,\beta-1,\beta'+1;x,y) \\ &- \beta(\beta')_{-1} y \sum_{k=0}^{n-1} G_1(\alpha-k,\beta+1,\beta'-1;x,y) \end{aligned} \tag{2.2}$$

*Proof.* From the definition of the function $G_1$ and transformation

$$(\alpha+1)_{m+p} = (\alpha)_{m+p}\left(1+\frac{m}{\alpha}+\frac{p}{\alpha}\right)$$

we can get the following relation:

$$\begin{aligned} G_1(\alpha+1,\beta,\beta';x,y) &= G_1(\alpha,\beta,\beta';x,y) \\ &+ (\beta)_{-1}\beta' x G_1(\alpha+1,\beta-1,\beta'+1;x,y) \\ &+ \beta(\beta')_{-1} y G_1(\alpha+1,\beta+1,\beta'-1;x,y). \end{aligned} \tag{2.3}$$

By applying this contiguous relation to function $G_1$ with the parameter $\alpha+2$ , we have

$$\begin{aligned}
&G_1\left(\alpha+2,\beta,\beta';x,y\right)\\
&=G_1\left(\alpha+1,\beta,\beta';x,y\right)+(\beta)_{-1}\beta' xG_1\left(\alpha+2,\beta-1,\beta'+1;x,y\right)\\
&\quad+\beta\left(\beta'\right)_{-1}yG_1\left(\alpha+2,\beta+1,\beta'-1;x,y\right)\\
&=G_1\left(\alpha,\beta,\beta';x,y\right)\\
&\quad+(\beta)_{-1}\beta' x\left[G_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right)+G_1\left(\alpha+2,\beta-1,\beta'+1;x,y\right)\right]\\
&\quad+\beta\left(\beta'\right)_{-1}y[G_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right)+G_1\left(\alpha+2,\beta+1,\beta'-1;x,y\right)]
\end{aligned}$$

$$\begin{aligned}
G_1\left(\alpha+n,\beta,\beta';x,y\right)=G_1\left(\alpha,\beta,\beta';x,y\right)&\\
+(\beta)_{-1}\beta' x\sum_{k=1}^{n}G_1\left(\alpha+k,\beta-1,\beta'+1;x,y\right)&\\
+\beta\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_1\left(\alpha+k,\beta+1,\beta'-1;x,y\right)&
\end{aligned}$$

If we compute the function $G_1$ with the parameter $\alpha+n$ by relation (2.3) for $n$ times, we find the formula given by (2.1). Replacing $\alpha$ by $\alpha-1$ in the contiguous relation (2.3), we get

$$\begin{aligned}
G_1\left(\alpha-1,\beta,\beta';x,y\right)=G_1\left(\alpha,\beta,\beta';x,y\right)-(\beta)_{-1}\beta' xG_1\left(\alpha,\beta-1,\beta'+1;x,y\right)&\\
-\beta\left(\beta'\right)_{-1}yG_1\left(\alpha,\beta+1,\beta'-1;x,y\right).&
\end{aligned}$$

If we apply this relation to the function $G_1$ with the parameter $\alpha-n$ for $n$ times, we obtain the recursion formula (2.2) similar to the proof of formula (2.1). □

By the same contiguous relations (2.1) and (2.2) we can express the functions $G_1$ in the above theorem in other forms.

**Theorem 2.** *The function $G_1$ satisfies the recursion formulas* :

$$\begin{aligned}
G_1\left(\alpha+n,\beta,\beta';x,y\right)=\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}&\\
\times x^k y^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right)&
\end{aligned}\tag{2.4}$$

$$\begin{aligned}
G_1\left(\alpha-n,\beta,\beta';x,y\right)=\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n}{i}\binom{n-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}&\\
\times(-x)^k(-y)^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right).&
\end{aligned}\tag{2.5}$$

*Proof.* By the induction method, we only prove the recursion formula given by (2.4). For $n=1$, formula (2.4) is satisfied. Assume that the result (2.4) is true for $n=t$. Then, we need to show that the relation (2.4) is satisfied for $n=t+1$. Setting $n=t$ in (2.4), we have

$$G_1\left(\alpha+t,\beta,\beta';x,y\right)=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i \\ G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right).$$

Replacing $\alpha$ by $\alpha+1$ in the above relation, we obtain

$$G_1\left(\alpha+t+1,\beta,\beta';x,y\right)=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i \\ G_1\left(\alpha+i+k+1,\beta+i-k,\beta'+k-i;x,y\right).$$

In the above equality, we apply the contiguous relation (2.3) with the transformations $\alpha\to\alpha+i+k$, $\beta\to\beta+i-k$, $\beta'\to\beta'+k-i$. Using the relations

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$$

and

$$\binom{n}{k}=0\ ,\ \ when\ \ k>n\ \ or\ \ k<0$$

and with some simplifications, we deduce

$$\begin{aligned}
&G_1\left(\alpha+t+1,\beta,\beta';x,y\right)\\
&=\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right)\\
&\quad+\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^{k+1}y^i(\beta+i-k)_{-1}\left(\beta'+k-i\right)_1\\
&\quad G_1\left(\alpha+1+i+k,\beta-1+i-k,\beta'+1+k-i;x,y\right)\\
&\quad+\sum_{i=0}^{t}\sum_{k=0}^{t-i}\tbinom{t}{i}\tbinom{t-i}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^{i+1}(\beta+i-k)_1\left(\beta'+k-i\right)_{-1}\\
&\quad G_1\left(\alpha+1+i+k,\beta+1+i-k,\beta'-1+k-i;x,y\right)\\
&=\sum_{i=0}^{t+1}\sum_{k=0}^{t+1-i}\tbinom{t+1}{i}\tbinom{t-i+1}{k}(\beta)_{i-k}\left(\beta'\right)_{k-i}x^k y^i
\end{aligned}$$

$$\times G_1\left(\alpha+i+k,\beta+i-k,\beta'+k-i;x,y\right),$$

where we replace $k$ by $k-1$ in the second summation term and $i$ by $i-1$ in the third summation term in the first equality. So, we obtain the recursion formula (2.4). In a similar manner, the relation (2.5) can be easily proved. □

**Theorem 3.** *For the function $G_1$, we have*

$$\begin{aligned} G_1\left(\alpha,\beta+n,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} y\sum_{k=1}^{n} G_1\left(\alpha+1,\beta+k,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)} G_1\left(\alpha+1,\beta+k-2,\beta'+1;x,y\right) \end{aligned} \tag{2.6}$$

*and*

$$\begin{aligned} G_1\left(\alpha,\beta-n,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad -\alpha\left(\beta'\right)_{-1} y\sum_{k=1}^{n} G_1\left(\alpha+1,\beta-k+1,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta-k)_{-1}}{(\beta-k)} G_1\left(\alpha+1,\beta-k-1,\beta'+1;x,y\right). \end{aligned} \tag{2.7}$$

*Proof.* Using the definition of the function $G_1$ and the equality

$$(\beta+1)_{p-m}=(\beta)_{p-m}\left(1+\frac{p}{\beta}-\frac{m}{\beta}\right)$$

we can easily obtain the contiguous function

$$\begin{aligned} G_1\left(\alpha,\beta+1,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} y G_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right) \\ &\quad -\alpha\frac{(\beta)_{-1}}{\beta}\beta' x G_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right). \end{aligned} \tag{2.8}$$

If we apply this contiguous relation for two times for the function $G_1$ with the parameter $\beta+2$, we get

$$\begin{aligned} G_1\left(\alpha,\beta+2,\beta';x,y\right) &= G_1\left(\alpha,\beta+1,\beta';x,y\right) \\ &\quad +\alpha\left(\beta'\right)_{-1} y G_1\left(\alpha+1,\beta+2,\beta'-1;x,y\right) \\ &\quad -\alpha\frac{(\beta+1)_{-1}}{(\beta+1)}\left(\beta'\right)_1 x G_1\left(\alpha+1,\beta,\beta'+1;x,y\right) \\ &= G_1\left(\alpha,\beta,\beta';x,y\right) \end{aligned}$$

$$
\begin{aligned}
&+\alpha\left(\beta'\right)_{-1} y G_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right)\\
&-\alpha\frac{(\beta)_{-1}}{\beta}\beta' x G_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right)\\
&+\alpha\left(\beta'\right)_{-1} y G_1\left(\alpha+1,\beta+2,\beta'-1;x,y\right)\\
&-\alpha\frac{(\beta+1)_{-1}}{(\beta+1)}\beta' x G_1\left(\alpha+1,\beta,\beta'+1;x,y\right).
\end{aligned}
$$

By iterating this method on $G_1$ with $\beta+n$ for $n$ times, we find

$$
\begin{aligned}
G_1\left(\alpha,\beta+n,\beta';x,y\right) &= G_1\left(\alpha,\beta+1,\beta';x,y\right)\\
&+\alpha\left(\beta'\right)_{-1} y\sum_{k=1}^{n} G_1\left(\alpha+1,\beta+k,\beta'-1;x,y\right)\\
&-\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)} G_1\left(\alpha+1,\beta+k-2,\beta'+1;x,y\right).
\end{aligned}
$$

Replacing $\beta$ by $\beta-1$ in contiguous relation (2.8), we obtain

$$
\begin{aligned}
G_1\left(\alpha,\beta-1,\beta';x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)-\alpha\left(\beta'\right)_{-1} y G_1\left(\alpha+1,\beta,\beta'-1;x,y\right)\\
&+\alpha\frac{(\beta-1)_{-1}}{(\beta-1)}\beta' x G_1\left(\alpha+1,\beta-2,\beta'+1;x,y\right).
\end{aligned}
$$

If we apply this relation to the function $G_1$ with the parameter $\beta-n$ for n times, we obtain the recursion formulas (2.7). □

**Theorem 4.** *For the function $G_1$, the equalities*

$$
\begin{aligned}
G_1\left(\alpha,\beta,\beta'+n;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)\\
&+\alpha\left(\beta\right)_{-1} x\sum_{k=1}^{n} G_1\left(\alpha+1,\beta-1,\beta'+k;x,y\right)\\
&-\alpha\beta y\sum_{k=1}^{n}\frac{(\beta'+k-1)_{-1}}{(\beta'+k-1)} G_1\left(\alpha+1,\beta+1,\beta'+k-2;x,y\right)
\end{aligned}
\tag{2.9}
$$

*and*

$$
\begin{aligned}
G_1\left(\alpha,\beta,\beta'-n;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right)\\
&-\alpha\left(\beta\right)_{-1} x\sum_{k=1}^{n} G_1\left(\alpha+1,\beta-1,\beta'-k+1;x,y\right)\\
&+\alpha\beta y\sum_{k=1}^{n}\frac{(\beta'-k)_{-1}}{(\beta'-k)} G_1\left(\alpha+1,\beta+1,\beta'-k-1;x,y\right)
\end{aligned}
\tag{2.10}
$$

*hold.*

*Proof.* If we use following equalities

$$\begin{aligned} G_1\left(\alpha,\beta,\beta'+1;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad +\alpha\left(\beta\right)_{-1}xG_1\left(\alpha+1,\beta-1,\beta'+1;x,y\right) \\ &\quad -\alpha\beta\frac{(\beta')_{-1}}{\beta'}yG_1\left(\alpha+1,\beta+1,\beta'-1;x,y\right) \\ G_1\left(\alpha,\beta,\beta'-1;x,y\right) &= G_1\left(\alpha,\beta,\beta';x,y\right) \\ &\quad -\alpha\left(\beta\right)_{-1}xG_1\left(\alpha+1,\beta-1,\beta';x,y\right) \\ &\quad +\alpha\beta\frac{(\beta'-1)_{-1}}{(\beta'-1)}yG_1\left(\alpha+1,\beta+1,\beta'-2;x,y\right) \end{aligned}$$

we obtain recursion formulas given by (2.9) and (2.10). □

## 3. Recursion formulas of $G_2$

In this section, we give some recursion formulas for the function $G_2$. We first present the recursion formulas for the function $G_2$ about the parameter $\alpha$ and $\alpha'$.

**Theorem 5.** *The function $G_2$ satisfies the recursion formulas :*

$$\begin{aligned} G_2\left(\alpha+n,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +\left(\beta\right)_{-1}\beta'x\sum_{k=1}^{n}G_2\left(\alpha+k,\alpha',\beta-1,\beta'+1;x,y\right) \end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned} G_2\left(\alpha-n,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad -\left(\beta\right)_{-1}\beta'x\sum_{k=0}^{n-1}G_2\left(\alpha-k,\alpha',\beta-1,\beta'+1;x,y\right). \end{aligned} \tag{3.2}$$

*Proof.* By the definition of the function $G_2$, we get

$$\begin{aligned} G_2\left(\alpha+1,\alpha',\beta,\beta';x,y\right) &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) \\ &\quad +\left(\beta\right)_{-1}\beta'xG_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right). \end{aligned}$$

Applying this relation $n$ times recursively, as the same as we have done in the proof of Theorem 1, we immediately have complete the proof (3.1). □

The function $G_2$ in the above theorem can be expressed in other forms as follows:

**Theorem 6.** *For the function $G_2$, the equalities*

$$G_2\left(\alpha+n,\alpha',\beta,\beta';x,y\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\beta)_{-k} (\beta')_k x^k G_2(\alpha+k, \alpha', \beta-k, \beta'+k; x, y) \quad (3.3)$$

*and*

$$G_2(\alpha-n, \alpha', \beta, \beta'; x, y)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\beta)_{-k} (\beta')_k (-x)^k G_2(\alpha, \alpha', \beta-k, \beta'+k; x, y), \quad (3.4)$$

*hold.*

*Proof.* By the inductive method as we have done in the proof of Theorem 2, the proof can be easily seen. □

**Theorem 7.** *The function $G_2$ satisfies the following recursion formulas*

$$G_2(\alpha, \alpha'+n, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) \quad (3.5)$$
$$+ \beta (\beta')_{-1} y \sum_{k=1}^{n} G_2(\alpha, \alpha'+k, \beta+1, \beta'-1; x, y)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\beta)_k (\beta')_{-k} y^k G_2(\alpha, \alpha'+k, \beta+k, \beta'-k; x, y)$$

*and*

$$G_2(\alpha, \alpha'-n, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y) \quad (3.6)$$
$$- \beta (\beta')_{-1} y \sum_{k=0}^{n-1} G_2(\alpha, \alpha'-k, \beta+1, \beta'-1; x, y)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\beta)_k (\beta')_{-k} (-y)^k G_2(\alpha, \alpha', \beta+k, \beta'-k; x, y)$$

*Proof.* By the definition of the function $G_2$, we get

$$G_2(\alpha, \alpha'+1, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y)$$
$$+ \beta (\beta')_{-1} y G_2(\alpha, \alpha'+1, \beta+1, \beta'-1; x, y)$$

and

$$G_2(\alpha, \alpha'-1, \beta, \beta'; x, y) = G_2(\alpha, \alpha', \beta, \beta'; x, y)$$
$$- \beta (\beta')_{-1} y G_2(\alpha, \alpha', \beta+1, \beta'-1; x, y).$$

By applying this relation for $n$ times, as the same as we have done in the proof of Theorem 1, we complete the proof. □

Now, we present the recursion formulas of the function $G_2$ about the parameter $\beta$ and $\beta'$.

**Theorem 8.** *Recursion formulas for the function $G_2$ are as follows*

$$\begin{aligned} &G_2\left(\alpha,\alpha',\beta+n,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha'\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_2\left(\alpha,\alpha'+1,\beta+k,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta+k-1)_{-1}}{(\beta+k-1)}G_2\left(\alpha+1,\alpha',\beta+k-2,\beta'+1;x,y\right) \end{aligned} \tag{3.7}$$

*and*

$$\begin{aligned} &G_2\left(\alpha,\alpha',\beta-n,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)-\alpha'\left(\beta'\right)_{-1}y\sum_{k=1}^{n}G_2\left(\alpha,\alpha'+1,\beta-k+1,\beta'-1;x,y\right) \\ &\quad +\alpha\beta' x\sum_{k=1}^{n}\frac{(\beta-k)_{-1}}{(\beta-k)}G_2\left(\alpha+1,\alpha',\beta-k-1,\beta'+1;x,y\right). \end{aligned} \tag{3.8}$$

*Proof.* From the definition of the function $G_2$, we have the following relation:

$$\begin{aligned} &G_2\left(\alpha,\alpha',\beta+1,\beta';x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha'\left(\beta'\right)_{-1}yG_2\left(\alpha,\alpha'+1,\beta+1,\beta'-1;x,y\right) \\ &\quad -\alpha\beta' x\frac{(\beta)_{-1}}{\beta}G_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right). \end{aligned}$$

If we apply this relation for $n$ times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. □

**Theorem 9.** *Recursion formulas for the function $G_2$ are as follows*

$$\begin{aligned} &G_2\left(\alpha,\alpha',\beta,\beta'+n;x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right)+\alpha\left(\beta\right)_{-1}x\sum_{k=1}^{n}G_2\left(\alpha+1,\alpha',\beta-1,\beta'+k;x,y\right) \\ &\quad -\alpha'\beta y\sum_{k=1}^{n}\frac{(\beta'+k-1)_{-1}}{(\beta'+k-1)}G_2\left(\alpha,\alpha'+1,\beta+1,\beta'+k-2;x,y\right) \end{aligned} \tag{3.9}$$

*and*

$$G_2\left(\alpha,\alpha',\beta,\beta'-n;x,y\right) \tag{3.10}$$

$$= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) - \alpha\left(\beta\right)_{-1} x \sum_{k=1}^{n} G_2\left(\alpha+1,\alpha',\beta-1,\beta'-k+1;x,y\right)$$
$$+ \alpha'\beta y \sum_{k=1}^{n} \frac{(\beta'-k)_{-1}}{(\beta'-k)} G_2\left(\alpha,\alpha'+1,\beta+1,\beta'-k-1;x,y\right).$$

*Proof.* From the definition of the function $G_2$, we have the following relation:

$$\begin{aligned} &G_2\left(\alpha,\alpha',\beta,\beta'+n;x,y\right) \\ &= G_2\left(\alpha,\alpha',\beta,\beta';x,y\right) + \alpha\left(\beta\right)_{-1} x G_2\left(\alpha+1,\alpha',\beta-1,\beta'+1;x,y\right) \\ &\quad - \alpha'\beta y \frac{(\beta')_{-1}}{\beta'} G_2\left(\alpha,\alpha'+1,\beta+1,\beta'-1;x,y\right). \end{aligned}$$

If we apply this relation for $n$ times, we can easily prove the theorem by the same method as we have done in the proof of Theorem 3. □

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# EXTREMAL UNICYCLIC GRAPHS WITH RESPECT TO ADDITIVELY WEIGHTED HARARY INDEX

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*Abstract.* In this paper we define cycle-star graph $CS_{k,n-k}$ to be a graph on $n$ vertices consisting of the cycle of length $k$ and $n-k$ leafs appended to the same vertex of the cycle. Also, we define cycle-path graph $CP_{k,n-k}$ to be a graph on $n$ vertices consisting of the cycle of length $k$ and of path on $n-k$ vertices whose one end is linked to a vertex on a cycle. We establish that cycle-star graph $CS_{3,n-3}$ is the only maximal graph with respect to additively weighted Harary index among all unicyclic graphs on $n$ vertices, while cycle-path graph $CP_{3,n-3}$ is the only minimal unicyclic graph (here $n$ must be at least 5). The values of additively weighted Harary index for extremal unicyclic graphs are established, so these values are the upper and the lower bound for the value of additively weighted Harary index on the class of unicyclic graphs on $n$ vertices.

2010 *Mathematics Subject Classification:* 05C35

*Keywords:* additively weighted Harrary index, unicyclic graph, extremal graph

## 1. Introduction

A topological index of a graph is a number attributed to a graph in a way that it is derived from the structure of the graph but so that it doesn't depend on the labeling of vertices in a graph. Chemical graph theory is a branch of graph theory whose focus of interest is finding topological indices of chemical graphs (i.e. graphs that represent chemical molecules) which correlate well with chemical properties of the corresponding molecules. One of the most famous topological indices is Wiener index, defined as the sum of all distances between different vertices of a molecular graph, introduced by Wiener in 1947 (see [10]) in a paper concerned with boiling points of alkanes. In a research that followed many other useful properties of Wiener index were discovered (for a survey of mathematical properties and chemical applications of Wiener index one can look into [3,4,7]). But, contrary to chemical intuition, the contribution of close pairs of vertices to the overall value of the index was much smaller than that of distant vertices. To deal with this inconsistency, the new index was proposed which was named Harary index ([6, 8]). Harary index is defined as the

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sum of all reciprocal values of distances between different vertices of a molecular graph. Properties of Harary index were then extensively researched (see for example [5, 9, 11, 12]). But it turned out that this modification of Wiener index has not solved the inconsistency as expected. In order to improve the performance of Harary-type indices, several modifications were proposed recently. In [2] the authors increased the attenuation of contributions of vertex pairs with their distance. In [1] an attempt was made in a different direction, the authors introduced a correction that gives more weight to the contributions of pairs of vertices of high degrees. This modification was named additively weighted Harary index. The aim of this paper is to establish the upper and lower bound for the value of additively weighted Harary index on the class of unicyclic graphs and to characterize all extremal graphs in that class.

The present paper is organized as follows. In Section 2 we introduce necessary definitions and preliminary results. In Section 3 we characterize all maximal unicyclic graphs by introducing several graph transformations which increase the value of additively weighted Harary index and which, when applied combined finitely many times, lead to maximal unicyclic graphs. In Section 4 we do the same for minimal unicyclic graphs. Finally, in Section 5 we give the conclusion and directions for further research. The paper is completed with acknowledgements and references.

## 2. Preliminaries

Let $G = (V, E)$ be a graph with set of vertices $V$ and set of edges $E$. In this paper all graphs are finite, simple and connected. For a pair of vertices $u, v \in V$ the *distance* $d_G(u, v)$ is defined as the length of the shortest path between $u$ and $v$. The *degree* $\delta_G(v)$ of a vertex $v \in V$ is defined as the number of vertices in $V$ neighboring to $v$. A *leaf* in a graph $G$ is every vertex in $G$ of degree 1. *Additively weighted Harary index* of a graph $G$ is defined as

$$H_A(G) = \sum_{u \neq v} \frac{\delta_G(u) + \delta_G(v)}{d_G(u, v)}.$$

We say that graph $G$ is a *tree* if $G$ doesn't contain a cycle. We say that $G$ is a *unicyclic* graph if $G$ contains exactly one cycle. The set of all unicyclic graphs on $n$ vertices will be denoted by $\mathcal{U}(n)$. The only cycle in a unicyclic graph $G$ will be denoted by $C$ and the length of $C$ will be denoted by $c$. Usually, we will suppose $C$ consists of path $P = w_i w_{i+1} \dots w_{i+c-1}$ and an edge $w_i w_{i+c-1}$. Starting index $i$ will not always be the same. Vertex $w_{i-j}$ will denote $j-$th vertex on $C$ from $w_j$ in negative direction, while $w_{i+j}$ will denote $j-$th vertex on $C$ from $w_j$ in positive direction. Note that in this way we possibly introduce alternative labels of vertices on $C$. For example, if $C = w_0 w_1 \dots w_5$, then $w_{-2} = w_4$ and $w_7 = w_1$. In other words, indices are added modulo $c$. We say $v \in G \backslash C$ is *branching* if $\delta_G(v) \geq 2$, we say $v \in C$ is *branching* if $\delta_G(v) \geq 3$.

A *cycle-star* graph is a unicyclic graph consisting only of a cycle and leafs appended to vertices of the cycle. Note that a cycle-star graph can be obtained from stars and a cycle by identifying central vertex of stars with different vertices on the cycle, hence the name. A *cycle-path* graph is a unicyclic graph consisting only of a cycle and at most one path appended to each vertex of the cycle. We denote by $CS_{k,n-k}$ a cycle-star graph on $n$ vertices consisting of the cycle of length $k$ and $n-k$ leafs appended to the same vertex of the cycle. We denote by $CP_{k,n-k}$ a cycle-path graph on $n$ vertices consisting of the cycle of length $k$ and the path on $n-k$ vertices whose end vertex is linked to a vertex on the cycle. These notions are illustrated in Figures 1 and 2.

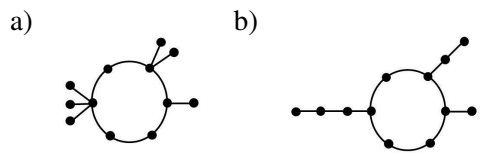


FIGURE 1. An example of: a) cycle-star, b) cycle path.

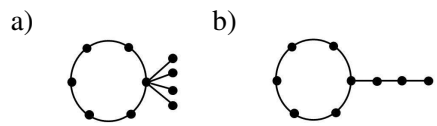


FIGURE 2. Graphs: a) $CS_{6.4}$, b) $CP_{6,3}$.

The cycle on $n$ vertices will be denoted by $C_n$. Obviously, it holds that $C_n \in \mathcal{U}(n)$. It is also convenient to consider that $C_n$ is both a cycle-star and a cycle-path graph, i.e. $C_n = CS_{n,0} = CP_{n,0}$.

Finally, we have to introduce numbers which are quite useful when expressing the values of Harary index. *Harmonic number* $H_n$, where $n \geq 1$ is an integer, is defined as $H_n = \sum_{i=1}^{n} \frac{1}{i}$. We define also $H_0 = 0$.

Before we proceed to our main results, we will prove two simple lemmas which will be useful later.

**Lemma 1.** *For $n \geq 3$ it holds that:*

(1) $H_A(CS_{3,n-3}) = \frac{3}{2}(n^2 - n + 2)$,

(2) $H_A(CP_{3,n-3}) = 4\sum_{i=1}^{n-2} H_{n-i-1} + H_{n-3} + 3H_{n-2} + \frac{6n-13}{n-2}$.

*Proof.* By direct calculation. □

Also, we introduce the following lemma, which is the direct consequence of the fact that $n$-th Harmonic number $H_n$ is roughly equal to the natural logarithm of $n$.

**Lemma 2.** *For odd $n \geq 5$ it holds that $\frac{3}{2}(n^2 - n + 2) - 4n \cdot H_{\frac{n-1}{2}} > 0$.*

## 3. Maximal unicyclic graphs

We will find maximal unicyclic graphs by introducing transformations of a graph $G$ to $G'$ which **increase** the value of $H_A$, therefore $H_A(G') - H_A(G) > 0$. Since $H_A(G') - H_A(G)$ is a sum over all pairs of vertices in $G$, it is very convenient to denote by $\Delta(u,v)$ the contribution of a pair $u, v \in G$ to the sum $H_A(G') - H_A(G)$. The problem to solve will be negative contributions of certain pairs of vertices, for which we will have to find enough pairs with positive contribution to compensate in the sum.

**Lemma 3.** *For an odd $n \geq 5$ it holds that $H_A(C_n) < H_A(CS_{3,n-3})$.*

*Proof.* For an odd $n$, it is easily verified that $H_A(C_n) = 4n \cdot H_{\frac{n-1}{2}}$. From Lemma 1 we have $H_A(CS_{3,n-3}) = \frac{3}{2}(n^2 - n + 2)$. Now the claim follows from Lemma 2. □

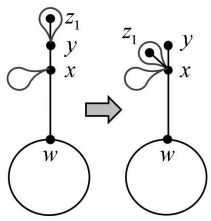


FIGURE 3. Graph transformation from the proof of Lemma 4.

**Lemma 4.** *For every $G \in \mathcal{U}(n)$ which is not a cycle-star graph, there is a cycle-star graph $G' \in \mathcal{U}(n)$ such that $H_A(G) < H_A(G')$.*

*Proof.* Let $C$ be the only cycle in $G$ and let $w \in C$ be the branching vertex. Let $T$ be the connected component of $G \setminus C$ containing $w$. Note that $T$ must be a tree. Let $z_1$ be a leaf in $T$ furthest from $w$. Let $P = w \ldots xyz_1$ be the shortest path between $w$ and $z_1$. Let $z_1, z_2, \ldots, z_k$ be all leafs neighboring to $y$. Since $z_1$ is a leaf furthest from $w$, it follows that $k = \delta_G(y) - 1$. Let $G'$ be a graph obtained from $G$ by deleting edges $z_i y$ for $i = 1, \ldots, k$ and adding edges $z_i x$ instead. This transformation is illustrated in Figure 3. We have to prove that $H_A(G) < H_A(G')$. Because of the definition of the index $H_A$, the problem are pairs of vertices whose distance increases and vertices whose degree decreases. Note that in this transformation the only pairs $u, v$ whose distance increases are pairs $u = y$ and $v = z_i$. Also, the only vertex for which degree decreases is $y$. Therefore, contributions $\Delta(y, z_i)$ will be negative due to increase in distance, while contributions $\Delta(y, v)$, where $v \in G$, will (possibly) be negative due

to decrease in degree of vertex $y$. Note that contributions $\Delta(y,v)$ are not necessarily negative, since degree of $v$ can increase or the distance $d_G(y,v)$ can also decrease.

Let us first consider the problem with the increase in distance. We will show that the negative contribution $\Delta(y,z_i)$ is compensated by the positive contribution $\Delta(x,z_i)$. More formally, we have

$$\begin{aligned}\Delta(y,z_i)+\Delta(x,z_i) &= \frac{1+1}{2}-\frac{1+k+1}{1}+\frac{\delta_G(x)+k+1}{1}-\frac{\delta_G(x)+1}{2}\\ &= \frac{\delta_G(x)-1}{1}-\frac{\delta_G(x)-1}{2}>0.\end{aligned}$$

Let us now consider the problem with degrees. We have to consider contributions $\Delta(y,v)$, where $v\in G$. First, note that we have already considered and compensated contributions $\Delta(y,z_i)$. Further, note that not all of these contributions are necessarily negative, since there can be increase in degree of the other vertex. So, for $v=x$ we have

$$\Delta(y,x)=\frac{1+(k+\delta(x))}{1}-\frac{(k+1)+\delta_G(x)}{1}=0.$$

For $v\in G\backslash\{x,y,z_1,\dots,z_k\}$, negative contribution $\Delta(y,v)$ can be compensated by the positive contribution $\Delta(x,v)$. More formally, we have

$$\begin{aligned}\Delta(y,v)+\Delta(x,v) &= \frac{1+\delta_G(x)}{d_G(y,v)}-\frac{k+1+\delta_G(x)}{d_G(y,v)}\\ &\quad+\frac{k+\delta_G(x)+\delta_G(v)}{d_G(y,v)-1}-\frac{\delta_G(x)+\delta_G(v)}{d_G(y,v)-1}\\ &= \frac{k}{d_G(y,v)-1}-\frac{k}{d_G(y,v)}>0.\end{aligned}$$

Therefore, we have proved $H_A(G)<H_A(G')$. If $G'$ is a cycle-star graph, then the proof is over, else this transformation must be repeated finitely many times in order to obtain a cycle-star graph and then the proof is over. □

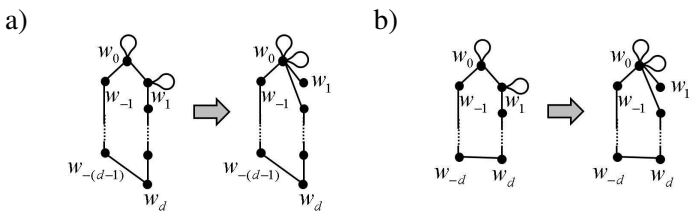


FIGURE 4. Graph transformation from the proof of Lemma 5: a) even $c$, b) odd $c$.

**Lemma 5.** *Let $G \in \mathcal{U}(n)$ be a cycle-star graph with the length of the only cycle being $c \geq 4$. Then there is a cycle-star graph $G' \in \mathcal{U}(n)$ with the length of the only cycle being 3 for which $H_A(G) < H_A(G')$.*

*Proof.* Let $C$ be the only cycle in $G$ and let $c$ be the length of $C$. Let $d = \lfloor c/2 \rfloor$. In the case of even $c$ we will denote vertices on $C$ by $C = w_{-(d-1)} \dots w_d$, in the case of odd $c$ we will denote $C = w_{-d} \dots w_d$. Without loss of generality we may assume that $w_{-d}$ (if it exists, since it exists only for odd $c$) is of minimum degree among vertices on $C$. It is convenient to introduce the notation $k_i = \delta_G(w_i)$. Now, leafs neighboring to $w_i$ will be denoted by $x_{i,j}$ ($j = 1, \dots, k_i - 2$) and we define $V_i = \{w_i, x_{i,j} : j = 1, \dots, k_i - 2\}$. Also, let $V^+ = (V_1 \cup \dots \cup V_d) \setminus \{w_1\}$ and $V^- = \big(V_0 \cup \dots \cup V_{-(d-1)}\big) \setminus \{w_0\}$. Obviously, in the case of even $c$ it holds that $V = \{w_0, w_1\} \cup V^+ \cup V^-$, while in the case of odd $c$ it holds that $V = \{w_0, w_1\} \cup V^+ \cup V^- \cup V_{-d}$.

Now, let $G'$ be a graph obtained from $G$ by deleting all edges $vw_1$ incident to $w_1$, except $w_0w_1$, and adding the edge $vw_0$ instead. This transformation is illustrated in Figure 4. Obviously, $G'$ is a cycle-star graph in which the only cycle is of the length $c-1$. We have to prove that $H_A(G) < H_A(G')$. Note that in this transformation the only pairs $u, v$ whose distance increases are pairs $u = w_1$ and $v \in V^+ \cup V_{-d}$ (recall that $V_{-d}$ only exists for odd $c$). Also, the only vertex whose degree decreases is $w_1$. Therefore, we have to consider all contributions $\Delta(w_1, v)$ where $v \in G$.

If $v = w_0$ we have $\Delta(w_1, w_0) = 0$, else if $v \in V^+$ we have

$$\Delta(w_1, v) + \Delta(w_0, v) = \frac{k_0 - 1}{d_G(w_1, v)} - \frac{k_0 - 1}{d_G(w_1, v) + 1} > 0,$$

else if $v \in V^-$ we have

$$\Delta(w_1, v) + \Delta(w_0, v) = \frac{k_1 - 1}{d_G(w_1, v) - 1} - \frac{k_1 - 1}{d_G(w_1, v)} > 0.$$

Note that in the case of even $c$ this completes the proof. In the case of odd $c$ we still have to consider contributions $\Delta(w_1, v)$ where $v \in V_{-d}$. First, note that for $G = C_n$ the claim of this lemma follows from Lemma 3. So, we will suppose $G \neq C_n$. Now, for negative contributions $\Delta(w_1, v)$, where $v \in V_{-d}$, positive contributions $\Delta(w_0, v)$ will not suffice to compensate, so we will have to find more pairs with positive contribution for compensation. To prove this more formally, we will distinguish cases $v = w_{-d} \in V_{-d}$ and $v = x_{-d,j} \in V_{-d}$.

If $v = w_{-d} \in V_{-d}$ we have

$$\Delta(w_1, w_{-d}) + \Delta(w_0, w_{-d}) = \frac{1 + k_{-d}}{d + 1} - \frac{1 + k_{-d}}{d} < 0.$$

Therefore, we have to find more pairs with positive contributions to compensate. Since $G \neq C_n$ there has to be at least one leaf $x_{i,1}$ in $G$. Without loss of generality

we can assume $i \leq 0$. Then for $x_{i,1}$ and $w_j$ where $j = d - i$, recalling that $w_{-d}$ is of minimum degree on $C$ (i.e. $k_j \geq k_{-d}$), we have

$$\Delta(x_{i,1}, w_j) = \frac{1+k_j}{d} - \frac{1+k_j}{d+1} \geq [k_j \geq k_{-d}] \geq \frac{1+k_{-d}}{d} - \frac{1+k_{-d}}{d+1} > 0.$$

So, obviously $\Delta(w_1, w_{-d}) + \Delta(w_0, w_{-d}) + \Delta(x_{i,1}, w_j) \geq 0$.

Finally, if $v = x_{-d,j} \in V_{-d}$ we have

$$\Delta(w_1, x_{-d,i}) + \Delta(w_0, x_{-d,i}) = \frac{2}{d+2} - \frac{2}{d+1} < 0.$$

Again, it follows that we have to find more pairs with positive contribution to compensate. Since $w_{-d}$ is of minimum degree on $C$, it follows that every $w_i$ has at least as many leafs as $w_{-d}$. Let us now consider pair of leafs $x_{0,1}$ and $x_{d,i}$. We have

$$\Delta(x_{0,1}, x_{d,i}) = \frac{2}{d+1} - \frac{2}{d+2} > 0.$$

Obviously, we have $\Delta(w_1, x_{-d,i}) + \Delta(w_0, x_{-d,i}) + \Delta(x_{0,1}, x_{d,i}) = 0$ which completes the proof. □

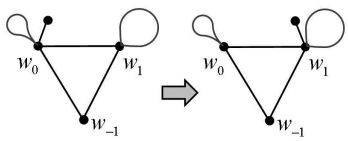


FIGURE 5. Graph transformation from the proof of Lemma 6.

**Lemma 6.** *Let $G \in \mathcal{U}(n)$, where $G \neq CS_{3,n-3}$, be a cycle-star graph with the length of the only cycle being $c = 3$. Then $H_A(G) < H_A(CS_{3,n-3})$.*

*Proof.* Let $C = w_{-1}w_0w_1$ be a cycle in $G$ and let $k_i = \delta_G(w_i)$. Let $x_{i,j}$ be all leafs attached to $w_i$ and let $V_i = \{w_i, x_{i,j} : j = 1, \ldots, k_i - 2\}$. Since $G \neq CS_{3,n-3}$, it follows that at least two vertices on $C$ are branching. Without loss of generality we can suppose that $w_0$ has minimum degree and $w_1$ maximum degree among branching vertices on $C$. Let $G'$ be a graph obtained from $G$ by deleting the edge $x_{0,1}w_0$ and adding the edge $x_{0,1}w_1$ instead. We have to prove $H_A(G) < H_A(G')$.

Note that in this transformation the only pairs $u, v$ whose distance increases are pairs $u = x_{0,1}$ and $v \in V_0$. The only vertex whose degree decreases is $w_0$. Let us first consider the problem with the increase in distances. It is easily verified that $\Delta(x_{0,1}, x_{0,j}) + \Delta(x_{0,1}, x_{1,j}) = 0$. Also, we have

$$\Delta(x_{0,1}, w_0) + \Delta(x_{0,1}, w_1) = \frac{k_1 + 1 - k_0}{1} - \frac{k_1 + 1 - k_0}{2} > [k_1 \geq k_0] > 0.$$

Let us now consider the problem with the increase in degree. It is easily verified that for $v \in V_{-1}$ we have $\Delta(w_0, v) + \Delta(w_1, v) = 0$ and also that $\Delta(w_0, w_1) = 0$. Further, we have

$$\Delta(w_0, x_{0,j\geq 2}) + \Delta(w_1, x_{0,j\geq 2}) = -\frac{1}{2} < 0.$$

$$\Delta(w_0, x_{1,j}) + \Delta(w_1, x_{1,j}) = \frac{1}{2} > 0.$$

Note that the negative contribution $\Delta(w_0, x_{0,j\geq 2})$ is still not compensated, since the positive contribution $\Delta(w_1, x_{0,j\geq 2})$ is not enough. But since $k_0 \leq k_1$, it follows that for each $x_{0,j}$ there is the corresponding $x_{1,j}$. So, for $j = 2, \dots, k_{0-2}$ we have

$$\Delta(w_0, x_{0,j}) + \Delta(w_1, x_{0,j}) + \Delta(w_0, x_{1,j}) + \Delta(w_1, x_{1,j}) = 0,$$

which means we have finally compensated for the negative contribution $\Delta(w_0, x_{0,j\geq 2})$. By applying this transformation finitely many times, we will obtain the desired graph $CS_{3,n-3}$, so the proof is over. □

**Theorem 1.** *Let $G \in \mathcal{U}(n)$. Then*

$$H_A(G) \leq \frac{3}{2}(n^2 - n + 2)$$

*with equality if and only if $G = CS_{3,n-3}$.*

*Proof.* Using Lemmas 4, 5 and 6 we first transform $G$ to a cycle-star graph, then we reduce the length of the cycle to 3, so that we can finally transform it to $C_{3,n-3}$. In each transformation the value of $H_A$ increases, so $G = C_{3,n-3}$ is the only maximal unicyclic graph. Note that the case of $G = C_n$ is covered by Lemma 3 for $n$ odd, while for even $n$ it is covered by Lemma 5. Now the bound for $H_A$ follows from Lemma 1. □

## 4. Minimal unicyclic graphs

As in previous section, we will find minimal graphs by introducing transformations of a graph $G$ to $G'$, but which now **decrease** the value of $H_A$. Therefore, in this section $\Delta(u, v)$ will denote the contribution of a pair of vertices $u, v \in G$ to the sum $H_A(G) - H_A(G')$. The problem will again be pairs of vertices with negative contribution, for which we will have to find enough pairs with positive contribution to compensate in the sum.

**Lemma 7.** *For every $G \in \mathcal{U}(n)$ which is not a cycle-path graph, there is a cycle-path graph $G' \in \mathcal{U}(n)$ such that $H_A(G) > H_A(G')$.*

*Proof.* Let $w$ be a branching vertex on $C$, let $T$ be the connected component of $G \backslash C$ containing $w$, let $x \in T$ be a branching vertex in $T$ furthest from $w$. If $x = w$ let $k = \delta_G(x) - 2$, else if $x \neq w$ let $k = \delta_G(x) - 1$. For $i = 1, \dots, k$, let $P_i = xy_i \dots z_i$ be a path in $T$ starting in $x$ such that $d(w, x) < d(w, y_i)$. Since $x$ is the branching

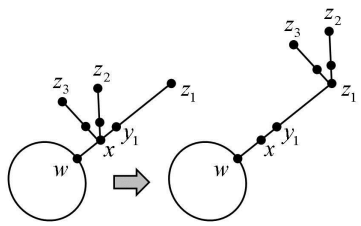


FIGURE 6. Graph transformation from the proof of Lemma 7.

vertex furthest from $w$, these paths are obviously vertex disjoint except for the vertex $x$. Without loss of generality we may assume that $P_1$ is the longest among these paths. Let $G'$ be a graph obtained from $G$ by deleting edges $xy_i$ for $i = 2,\dots,k-1$, and adding edges $z_1y_i$ instead. This transformation is illustrated in Figure 6. Note that $G'$ is a unicyclic graph. We have to prove that $H_A(G) > H_A(G')$. Since the value of the index $H_A$ has to decrease, the problem are the pairs of vertices whose distance decreases or vertices whose degree increases. Distances possibly decrease only for pairs $u,v$ where $u \in P_i$ $(i = 2,\dots,k)$ and $v \in P_1$. The only vertex for which the degree increases in this transformation is $z_1$.

Let us first consider the problem with the decrease in distances. Let $v \mapsto v'$ be the automorphism of the path $P_1$ such that $x' = z_1$. For $u \in P_{i\geq 2}$ we have

$$\Delta(u,x) + \Delta(u,z_1) = \frac{1}{d_G(u,x)} - \frac{1}{d_G(u,x)+|P_1|} > 0.$$

For $u \in P_{i\geq 2}$ and $v \in P_1\backslash\{x,z_1\}$, if $v = v'$ then $d_G(u,v) = d_{G'}(u,v)$ so $\Delta(u,v) = 0$, else if $v \neq v'$ then we have $d_G(x,v) = d_G(z,v')$ and $d_G(z,v) = d_G(x,v')$ so it is easily verified that $\Delta(u,v) + \Delta(u,v') = 0$.

Let us now consider the problem with the increase in degree. Let again $v \mapsto v'$ be the automorphism of path $P_1$ such that $x' = z_1$. For $v \in P_i$ we have already considered and compensated negative contributions $\Delta(z_1,v)$. If we now consider $v \in P_1$, then for $v = x$ we have

$$\Delta(z_1,x) = \frac{1+\delta_G(x)}{d_G(z_1,x)} - \frac{(1+k-1)+(\delta_G(x)-k+1)}{d_G(z_1,x)} = 0,$$

while for $v \in P_1\backslash\{z_1,x\}$, we have

$$\Delta(z_1,v) + \Delta(x,v') = \frac{\delta_G(x)-2}{d_G(x,v')} - \frac{\delta_G(x)-2}{d_G(z_1,v)} = \left[d_G(z_1,v) = d_G(x,v')\right] = 0.$$

Finally, we have to consider $v \in G\backslash(\cup_{i=1}^{k} P_i)$. We have

$$\Delta(z_1,v) + \Delta(x,v) = \frac{k-1}{d_G(x,v)} - \frac{k-1}{d_G(z_1,v)} > [d_G(x,v) < d_G(z_1,v)] > 0.$$

Therefore, we have proved that $H_A(G) > H_A(G')$. If $G'$ is a cycle-path graph, then the proof is completed. If not, then by repeating this transformation finitely many times we obtain a cycle path graph $G'$ for which $H_A(G) > H_A(G')$, so the proof is complete. □

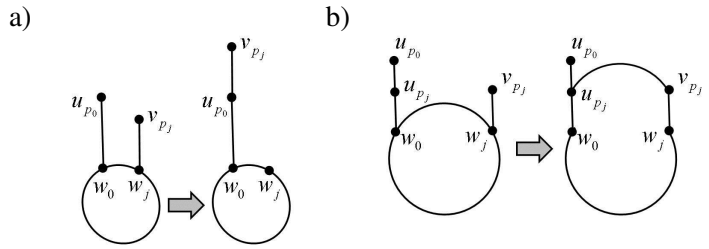


FIGURE 7. Graph transformation from the proof of Lemma 8: a) case 1, b) case 2a.

**Lemma 8.** *Let $G \in \mathcal{U}(n)$ be a cycle-path graph with at least 2 branching vertices. Then there is a cycle-path graph $G' \in \mathcal{U}(n)$ with only one branching vertex such that $H_A(G) > H_A(G')$.*

*Proof.* Let $C$ be the only cycle in $G$ with vertices denoted by $w_i$. Let $P_i$ be the path appended to the branching vertex $w_i \in C$ (here $P_i$ includes $w_i$) and let $p_i = |P_i|$. We distinguish several cases.

CASE 1. There are two consecutive branching vertices $w_i, w_j$ on $C$ such that $d(w_i, w_j) \le p_i$ or $d(w_i, w_j) \le p_j$.

Without loss of generality we may assume that $0 = i < j \le \lfloor \frac{c}{2} \rfloor$, $d(w_0, w_j) \le p_0$ and $p_0 \ge p_j$. Let $P_0 = w_0 u_1 \dots u_{p_0}$ and $P_j = w_j v_1 \dots v_{p_j}$. Let $G'$ be a graph obtained from $G$ by deleting the edge $w_j v_1$ and adding the edge $u_{p_0} v_1$ instead. This transformation is illustrated in Figure 7 a). Note that $G'$ is a cycle-path graph with one branch less than $G$. We have to prove $H_A(G) > H_A(G')$. Let $P_A$ be the shortest path in $G$ connecting vertices $u_{p_0}$ and $v_{p_j}$. Note that in this transformation distances possibly decrease only for pairs of vertices $u, v \in P_A$. The only vertex for which the degree increases is $u_{p_0}$.

Let us first consider the problem with distances. Let $P'_A$ be a path in $G'$ connecting vertices $v_{p_j}$ and $w_j$. Let $v \mapsto v'$ be an isomorphism of paths $P_A$ and $P'_A$ such that $v'_{p_j} = v_{p_j}$. Note that $u'_{p_0} = w_j$ (and vice versa) and also $w'_0 \in P_0$ since $d(w_0, w_j) \le p_0$. Now, we first consider $u = v_a \in P_j$. If $v = v_b \in P_j$ we have $\Delta(v_a, v_b) = 0$, else if $v = w_j$ and $v' = u_{p_0}$ we have

$$\Delta(v_a, w_j) + \Delta(v_a, u_{p_0}) = \frac{1}{d_G(v_a, w_j)} - \frac{1}{d_G(v_a, w_j) + p_0 + j} > 0,$$

else if $v = v'$ then from $d_{G'}(v_a, v) = d_G(v_a, v)$ we have $\Delta(v_a, v) = 0$, else if $v = w_0$ and $v' = w_0'$ we have

$$\Delta(v_a, w_0) + \Delta(v_a, w_0') = \frac{1}{d_G(v_a, w_j) + j} - \frac{1}{d_G(v_a, w_j) + p_0} \geq [j \leq p_0] \geq 0.$$

else if $v \neq v'$ then from $d_G(w_j, v') = d_G(u_{p_0}, v)$ and $d_G(u_{p_0}, v') = d_G(w_j, v)$ we have $\Delta(v_a, v) + \Delta(v_a, v') = 0$. Therefore we have covered all pairs $u, v \in P_A$ for which $u \in P_j$. Let now $u = u_{p_0}$. We have already considered pairs with $v \in P_j \backslash \{w_j\}$ in previous text. If $v = w_j$ we have $\Delta(u_{p_0}, w_j) = 0$, else

$$\Delta(u_{p_0}, v) + \Delta(w_j, v') = \frac{1}{d_G(w_j, v')} - \frac{1}{d_G(u_{p_0}, v)} = [d_G(w_j, v') = d_G(u_{p_0}, v)] = 0$$

Finally, for $u, v \in P_A \backslash (P_j \cup \{u_{p_0}, w_j\})$ both the degrees and the distances have not changed in the transformation, so we have $\Delta(u, v) = 0$. Therefore, we have considered and compensated all negative contributions due to decrease in distances.

Let us now consider all negative contributions due to increase in degree of $u_{p_0}$. Note that we have already considered and compensated pairs $u_{p_0}, v$ where $v \in P_A$. Let now $v \in G \backslash P_A$. We have

$$\Delta(u_{p_0}, v) + \Delta(w_j, v) = \frac{1}{d_G(w_j, v)} - \frac{1}{d_G(u_{p_0}, v)} = [d_G(w_j, v) \leq d_G(u_{p_0}, v)] \geq 0.$$

CASE 2. For every two consecutive branching vertices $w_i, w_j$ on $C$ it holds that $d(w_i, w_j) > p_i$ and $d(w_i, w_j) > p_j$.

SUBCASE 2a. There are at least 4 branching vertices on $C$.

Let $w_i$ and $w_j$ be the pair of branching vertices on minimum distance. Without loss of generality we may assume that $0 = i < j \leq \lfloor \frac{c}{4} \rfloor$. Let $P_0 = w_0 u_1 \ldots u_{p_0}$ and $P_j = w_j v_1 \ldots v_{p_j}$ be paths appended to $w_0$ and $w_j$. The condition of Case 2 is now read as $j > p_0$ and $j > p_j$. Without loss of generality we may assume $p_0 \geq p_j$. Now, let $G'$ be a graph obtained from $G$ by deleting edges $w_0 w_1$ and $w_{j-1} w_j$ and adding edges $u_{p_j} w_1$ and $w_{j-1} v_{p_j}$ instead. This transformation is illustrated in Figure 7 b). Graph $G'$ is obviously a cycle-path graph with one branch less than $G$. We have to prove $H_A(G) > H_A(G')$. Let $P_A$ be the shortest path in $G$ connecting vertices $u_{p_0}$ and $v_{p_j}$. Note that distances possibly decrease only for pairs of vertices $u, v \in P_A$. The only vertices for which degree increases are $u_{p_j}$ and $v_{p_j}$.

Let us first consider the problem with distances. Let $P_B$ be the shortest path in $G$ connecting vertices $u_{p_j}$ and $v_{p_j}$ ($P_B$ is subpath of $P_A$). Let $P_B'$ be the shortest path in $G'$ connecting vertices $w_0$ and $w_j$. It is important to note that because of the condition of subcase it holds that $2p_j + j < 3\lfloor \frac{c}{4} \rfloor$, so the shortest path between $w_0$ and $w_j$ both before and after the transformation goes through the same side of cycle (i.e. paths $P_B$ and $P_B'$ are of the same length and contain the same vertices though not in the same order). Let $v \mapsto v'$ be the isomorphism of paths $P_B$ and $P_B'$ such that $u_{p_j}' = w_0$. We first consider pairs $u, v \in P_B$. For pairs $u, v \in P_B$ such that $u, v \in \{u_{p_j}, v_{p_j}, w_0, w_j\}$,

it is easily verified that $\Delta(u_{p_j}, w_j) = \Delta(v_{p_j}, w_0) = \Delta(u_{p_j}, w_0) = \Delta(v_{p_j}, w_j) = 0$ and

$$\Delta(u_{p_j}, v_{p_j}) + \Delta(w_0, w_j) = \frac{3-\delta_G(u_{p_j})}{j} - \frac{3-\delta_G(u_{p_j})}{2p_j + j} > 0.$$

Now, of the remaining pairs $u, v \in P_B$, let us consider pairs where $u \in P_0$. First, let $u = u_{p_j} \in P_0$. Then, for $v = u_b \in P_0$ and $v = v_b \in P_j$ $(b < p_j)$ we have

$$\begin{aligned}\Delta(u_{p_j}, u_b) + \Delta(w_0, u'_b) &= \frac{1}{d_G(w_0, u'_b)} - \frac{1}{d_G(u_{p_j}, u_b)} \\ &= \left[d_G(u_{p_j}, u_b) = d_G(w_0, u'_b)\right] = 0, \\ \Delta(u_{p_j}, v_b) + \Delta(w_0, v'_b) &= \frac{2-\delta_G(u_{p_j})}{j + p_j - b} - \frac{2-\delta_G(u_{p_j})}{p_j + j + b} > 0,\end{aligned}$$

and for $v = w_b \in C$ $(0 < b < j)$ we have

$$\Delta(u_{p_j}, w_b) + \Delta(w_0, w_b) = \frac{2-\delta_G(u_{p_j})}{b} - \frac{2-\delta_G(u_{p_j})}{p_j + b} > 0.$$

Now, let $u = u_a \in P_0$ for $a < p_j$. Then for $b < p_j$, if $u_a = u'_a$ and $v_b = v'_b$ (i.e. $a = b = \frac{p_j}{2}$) we have $\Delta(u_a, v_b) = 0$, else we have $\Delta(u_a, v_b) + \Delta(u'_a, v'_b) = 0$. Also, for $0 < b < j$, if $u_a = u'_a$ (i.e. $a = \frac{p_j}{2}$) then $\Delta(u_a, w_b) = 0$, else $\Delta(u_a, w_b) + \Delta(u'_a, w_b) = 0$. Completely analogously one can obtain analogous results for pairs $u, v \in P_B$ where $u \in P_j$. Now, for all unconsidered pairs $u, v \in P_B$ neither degrees change, nor the distance, therefore for those pairs it holds that $\Delta(u, v) = 0$. Hence, we have considered all pairs $u, v \in P_B$.

It remains to consider pairs $u, v \in P_A$ where $u \in P_A \backslash P_B$. Now, let $u = u_{p_j + a} \in P_A \backslash P_B$. If $v \in P_A \backslash P_B$ then obviously $\Delta(u, v) = 0$, else if $v \in \{w_0, u_{p_j}\}$ we have

$$\begin{aligned}\Delta(u_{p_j+a}, w_0) + \Delta(u_{p_j+a}, u_{p_j}) &= \frac{1}{p_j + a} - \frac{1}{a} < 0, \\ \Delta(w_{-a}, w_0) + \Delta(w_{-a}, u_{p_j}) &= \frac{1}{a} - \frac{1}{a + p_j} > 0,\end{aligned}$$

so we obtain $\Delta(u_{p_j+a}, w_0) + \Delta(u_{p_j+a}, u_{p_j}) + \Delta(w_{-a}, w_0) + \Delta(w_{-a}, u_{p_j}) = 0$. Else if $v \in \{w_j, v_{p_j}\}$ we have

$$\begin{aligned}\Delta(u_{p_j+a}, w_j) + \Delta(u_{p_j+a}, v_{p_j}) &= \frac{1}{j + a + p_j} - \left(\frac{\delta_G(u_{p_j+a}) + 2}{a + j} - \frac{\delta_G(u_{p_j+a}) + 1}{p_j + a + j + p_j}\right), \\ \Delta(w_{j+a}, w_0) + \Delta(w_{j+a}, u_{p_j}) &= \left(\frac{5}{a + j} - \frac{4}{j + a + p_j + p_j}\right) - \frac{1}{j + a + p_j},\end{aligned}$$

so we obtain

$$\Delta(u_{p_j+a}, w_j) + \Delta(u_{p_j+a}, v_{p_j}) + \Delta(w_{j+a}, w_0) + \Delta(w_{j+a}, u_{p_j})$$

$$= \frac{3-\delta_G(u_{p_j+a})}{a+j} - \frac{3-\delta_G(u_{p_j+a})}{j+a+p_j+p_j} > 0.$$

Else if $v \in \{u_0, \dots, u_{p_j-1}\}$ we have $\Delta(u_{p_j+a}, v) = 0$. Else if $v \in \{w_1, \dots, w_{j-1}\}$ we have

$$\Delta(u_{p_j+a}, w_b) + \Delta(w_{-a}, w_b) = \frac{2-\delta_G(u_{p_j+a})}{a+b} - \frac{2-\delta_G(u_{p_j+a})}{a+b+p_j} > 0.$$

Else if $v \in \{v_1, \dots, v_{p_j-1}\}$ we have

$$\Delta(u_{p_j+a}, v_b) + \Delta(w_{-a}, v'_b) = \frac{2-\delta_G(u_{p_j+a})}{a+j+p_j-b} - \frac{2-\delta_G(u_{p_j+a})}{a+p_j+j+b} > 0.$$

Therefore, we have considered and compensated all negative contributions due to decrease in distances.

Let us now consider all negative contributions due to increase in degree of $u_{p_j}$ and $v_{p_j}$. First, note that we have already considered pairs $u_{p_j}, v$ and $v_{p_j}, v$ for which $v \in P_A$. For $u = u_{p_j}$ we have also already considered pairs where $v = w_{-a}$ or $v = w_{j+a}$ $(a = 1, \dots, p_0 - p_j)$. Now, let $v$ be an unconsidered vertex, then from $d_G(w_0, v) \le d_{G'}(w_0, v)$ and $d_G(u_{p_j}, v) = d_{G'}(u_{p_j}, v)$ we have

$$\Delta(u_{p_j}, v) + \Delta(w_0, v) \ge \frac{1}{d_G(w_0, v)} - \frac{1}{d_G(u_{p_j}, v)} > \left[d_G(w_0, v) < d_G(u_{p_j}, v)\right] > 0.$$

Completely analogously one obtains $\Delta(v_{p_j}, v) + \Delta(w_j, v) > 0$ and the proof of this subcase is over.

SUBCASE 2b. There are exactly 3 branching vertices on $C$.

Let $w_0, w_j$ and $w_k$ be three branching vertices on $C$. Let us denote $d_1 = d_G(w_0, w_j)$, $d_2 = d_G(w_j, w_k)$, $d_3 = d_G(w_k, w_0)$. Without loss of generality we may assume that $d_3 \ge \max\{d_1, d_2\}$. We will also need $d_4 = \min\{d_1 + d_2, d_3\} \ge d_2$. Let $P_0 = w_0 u_1 \dots u_{p_0}$, $P_j = w_j v_1 \dots v_{p_j}$, $P_k = w_k z_1 \dots z_{p_k}$ be paths appended to branching vertices on $C$. Without loss of generality we may assume $p_0 \le p_k$. Let $G'$ be the graph obtained from $G$ by deleting edge $w_j v_1$ and adding the edge $u_{p_0} v_1$ instead. Note that $G'$ is a cycle-path graph with one branch less than $G$. We have to prove $H_A(G) > H_A(G')$. Note that distances possibly decrease only for pairs $u, v$ for which $u \in P_j$ and $v \in G$. The only vertex whose degree increases is $u_{p_0}$.

We will first consider the problem with distances, so let $u = v_a \in P_j$. Let $P_A$ be the shortest path in $G$ connecting vertices $v_{p_j}$ and $u_{p_0}$, let $P'_A$ be the shortest path in $G'$ connecting vertices $v_{p_j}$ and $w_j$. Let $v \mapsto v'$ be an isomorphism of paths $P_A$ and $P'_A$ such that $v_{p_j} = v'_{p_j}$. Note that $w'_0 \in \{w_1, \dots, w_{j-1}\}$ since $p_0 < j$ (supposition of Case 2). First, we will consider cases where $v \in P_A$. If $v = v_b \in P_j$, then we have $\Delta(v_a, v_b) = 0$. Else if $v \in \{w_j, u_{p_0}, w'_0, w_0\}$ we have

$$\Delta(v_a, u_{p_0}) + \Delta(v_a, w_j) = \frac{1}{a} - \frac{1}{a + d_1 + p_0} > 0,$$

$$\Delta(v_a, w_0) + \Delta(v_a, w_0') = \frac{1}{a+d_1} - \frac{1}{a+p_0} < [d_1 > p_0] < 0,$$

so obviously

$$\begin{aligned}\Delta(v_a, u_{p_0}) + \Delta(v_a, w_j) + \Delta(v_a, w_0) + \Delta(v_a, w_0') \\ = \frac{1}{a} - \frac{1}{a+p_0} + \frac{1}{a+d_1} - \frac{1}{a+d_1+p_0} > 0.\end{aligned}$$

Else if $v = v'$ from $d_G(w_j, v) = d_G(u_{p_0}, v)$ we have $\Delta(v_a, v) = 0$. Else if $v \neq v'$ from $d_G(v_a, v') = d_{G'}(v_a, v)$ and $d_{G'}(v_a, v') = d_G(v_a, v)$ we have $\Delta(v_a, v) + \Delta(v_a, v') = 0$. Therefore, we have considered all cases where $v \in P_A$. Let us now consider cases where $v \in C \setminus P_A$. Let $v \mapsto v'$ be the automorphism of cycle $C$ such that $w_0' = w_j$ and $w_j' = w_0$. If $v = v'$ then from $d_G(w_j, v) = d_G(u_{p_0}, v)$ we have

$$\Delta(v_a, v) = \frac{\delta_G(v_a) + \delta_G(v)}{d_G(v_a, w_j) + d_G(w_j, v)} - \frac{\delta_G(v_a) + \delta_G(v)}{d_G(v_a, w_j) + p_0 + d_G(w_j, v)} > 0,$$

else if $v \in \{w_k, w_k'\}$ we have

$$\begin{aligned}\Delta(v_a, w_k) + \Delta(v_a, w_k') &= \frac{\delta_G(v_a)+3}{a+d_2} - \frac{\delta_G(v_a)+3}{a+p_0+d_4} + \frac{\delta_G(v_a)+2}{a+d_4} - \frac{\delta_G(v_a)+2}{a+p_0+d_2} \\ &> \left[\frac{\delta_G(v_a)+3}{a+d_4} - \frac{\delta_G(v_a)+3}{a+p_0+d_4} > 0\right] \\ &> \frac{\delta_G(v_a)+3}{a+d_2} - \frac{\delta_G(v_a)+2}{a+p_0+d_2} - \frac{1}{a+d_4} \geq [d_4 \geq d_2] \\ &\geq \frac{\delta_G(v_a)+2}{a+d_2} - \frac{\delta_G(v_a)+2}{a+p_0+d_2} > 0,\end{aligned}$$

else from $d_G(w_j, v') = d_G(w_0, v)$ and $d_G(w_0, v') = d_G(w_j, v)$ we have

$$\begin{aligned}\Delta(v_a, v) + \Delta(v_a, v') = \frac{\delta_G(v_a)+2}{a+d_G(w_0,v)} - \frac{\delta_G(v_a)+2}{a+p_0+d_G(w_0,v)} + \\ + \frac{\delta_G(v_a)+2}{a+d_G(w_j,v)} - \frac{\delta_G(v_a)+2}{a+p_0+d_G(w_j,v)} > 0.\end{aligned}$$

Finally, we have to consider $v = z_b \in P_k$. We have

$$\Delta(v_a, z_b) = \frac{\delta_G(v_a) + \delta_G(z_b)}{a+d_2+b} - \frac{\delta_G(v_a) + \delta_G(z_b)}{a+d_4+b} \geq [d_2 \leq d_4] \geq 0.$$

Therefore, we have considered and compensated all negative contributions due to decrease in distances.

Let us now consider all negative contributions due to increase in degree of $u_{p_0}$, so let $u = u_{p_0}$. We have already considered pairs $u, v$ where $v \in P_j \setminus \{w_j\}$. For the

remaining possibilities for $v$, we will again use already introduced isomorphisms of $P_A$ and $C$. If $v = w_j$ then $\Delta(u_{p_0}, w_j) = 0$, else if $v \in P_A \setminus P_j$ we have

$$\Delta(u_{p_0}, v) + \Delta(w_j, v') = \frac{1}{d_G(w_j, v')} - \frac{1}{d_G(u_{p_0}, v)} = \left[d_G(u_{p_0}, v) = d_G(w_j, v')\right] = 0,$$

else if $v \in C \setminus P_A$ we have

$$\Delta(u_{p_0}, v) + \Delta(w_j, v') = \frac{1}{d_G(w_0, v')} - \frac{1}{d_G(u_{p_0}, v)} = \left[d_G(w_0, v') = d_G(u_{p_0}, v)\right] = 0,$$

else if $v = z_a \in P_k \setminus \{w_k\}$ we have

$$\Delta(u_{p_0}, z_a) + \Delta(w_j, z_a) = \frac{1}{d_2 + a} - \frac{1}{p_0 + d_4 + a} > 0.$$

SUBCASE 2c. There are exactly 2 branching vertices on $C$.

Let $w_0$ and $w_j$ be branching vertices on $C$ and let $P_0 = w_0 u_1 \dots u_{p_0}$, $P_j = w_j v_1 \dots v_{p_j}$ be paths appended to branching vertices. Let $G'$ be a graph obtained from $G$ by deleting the edge $w_j v_1$ and adding the edge $u_{p_0} v_1$ instead. Graph $G'$ is obviously a cycle-path graph with only one branch. Proof that $H_A(G) > H_A(G')$ is completely analogous to the proof of subcase 2b, one just doesn't have to consider vertex $w_k$ separately and there are no vertices $z_a$.

So, in all cases we have proved $H_A(G) > H_A(G')$. Since in all cases $G'$ is a cycle-path graph with one branch less than in $G$, we have either obtained a cycle-path graph $G'$ which has only one branch, or by repeating the transformation finitely many times we will obtain such graph. Therefore, the lemma is proved. □

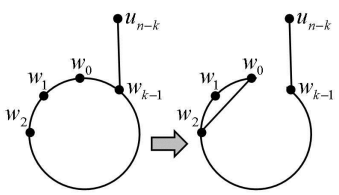


FIGURE 8. Graph transformation from the proof of Lemma 9.

**Lemma 9.** *Let $G = C_{k,n-k}$ where $k \geq 4$ and $n \geq 5$. Then $H_A(G) > H_A(C_{3,n-3})$.*

*Proof.* Let us denote vertices in $G$ so that for the only cycle in $G$ holds $C = w_0 w_1 \dots w_{k-1}$ where $4 \leq k \leq n$. If there is a branching vertex in $G$, without loss of generality we may assume that it is $w_{k-1}$ and that $P_{k-1} = w_{k-1} u_1 \dots u_{n-k}$ is the only path appended to $w_{k-1}$. Let $G'$ be a graph obtained from $G$ by deleting the edge $w_0 w_{k-1}$ and adding the edge $w_0 w_2$ instead. This transformation is illustrated in Figure 8. Note that $G' = CP_{3,n-3}$. We have to prove $H_A(G) > H_A(G')$. Note that

distances in this transformation decrease only for $u, v$ where $u = w_0$ and $v = w_i$ ($i = 2, \dots, \lfloor k/2 \rfloor$). The only vertex whose degree increases is $w_2$. For the ease of proving the lemma, we introduce $d_m = \min\{3, k-3\}$ and the path $P_A = w_2 w_3 \dots w_{k-1}$ in $G$ with the automorphism $v \mapsto v'$ of $P_A$ such that $w_2' = w_{k-1}$. Now, we distinguish two cases with respect to whether $G = C_n$ ($k = n$) or $G = G = C_{k,n-k}$ ($4 \le k \le n-1$).

CASE 1. Let $G = C_n$ ($k = n$). We will first consider the problem with distances. Let $u = w_0$. If $v = w_2$ we have

$$\Delta(w_0, w_2) + \Delta(w_0, w_{k-1}) = \frac{1}{2}\frac{7k-20}{k-2} > 0,$$

else if $v = w_i$ ($i = 3, \dots, \lfloor k/2 \rfloor$) we have

$$\Delta(w_0, w_i) + \Delta(w_0, w_{k-i+1}) = \frac{4}{i} - \frac{4}{k-i} \ge [i \le k-i] \ge 0.$$

Let us now consider the problem with the increase in degree of $w_2$, so let $u = w_2$. Note that we have already considered pairs $u, v$ where $v = w_0$. If $v = w_1$ we have

$$\Delta(w_2, w_1) + \Delta(w_{k-1}, w_1) = \frac{k-5}{k-2} \ge [k = n \ge 5] \ge 0,$$

else if $v = w_{k-1}$ we have

$$\Delta(w_2, w_{k-1}) = \frac{2+2}{d_m} - \frac{3+1}{k-3} \ge [k-3 \le d_m] \ge 0,$$

else using the automorphism of $P_A$ (and supposing $d_{P_A}(w_2, v) \le d_{P_A}(w_2, v')$) from $d_G(w_2, v) = d_G(w_{k-1}, v')$ we have $\Delta(w_2, v) + \Delta(w_{k-1}, v') = 0$.

CASE 2. Let $G = C_{k,n-k}$. Again, we first consider the problem with distances. Let $u = w_0$, we have to consider $v = w_2$ and $v \in \{w_3, \dots, w_{\lfloor k/2 \rfloor}\}$. We have

$$\Delta(w_0, w_2) + \Delta(w_0, w_{k-1}) = \frac{2k-8}{k-2} \ge [k \ge 4] \ge 0$$
$$\Delta(w_0, w_i) + \Delta(w_0, w_{k-i+1}) = \frac{4}{i} - \frac{4}{k-i} \ge [i \le k-i] \ge 0.$$

Now, let us consider the problem with the increase in degree of $w_2$, so let $u \in w_2$. Note that we have already considered $v = w_0$. We have to consider $v = w_1$, $v = w_{k-1}$, $v \in P_A \backslash \{w_1, w_{k-1}\}$, $v = u_a \in P_{k-1}$. We have

$$\Delta(w_2, w_1) + \Delta(w_{k-1}, w_1) = \frac{1}{2}\frac{3k-14}{k-2} > 0 \text{ for } k \ge 5,$$
$$\Delta(w_2, w_{k-1}) = \frac{2+3}{d_m} - \frac{3+2}{k-3} \ge [d_m \le k-3] \ge 0.$$

Further, for $v \in P_A \backslash \{w_1, w_{k-1}\}$, assuming $d_{P_A}(w_v, v) \le d_{P_A}(w_v, v')$, we have $\Delta(w_2, v) + \Delta(w_{k-1}, v') = 0$. Finally, using $d_m \le k-3$ we obtain

$$\Delta(w_2, u_a) + \Delta(w_{k-1}, u_a) \ge \frac{1}{a} - \frac{1}{d_m + a} > 0.$$

Therefore, the only problem is $\Delta(w_2,w_1)+\Delta(w_{k-1},w_1)$ for $k=4$. But note that in that case $d_m=\min\{4,1\}=1$, so we have

$$\Delta(w_2,w_1)+\Delta(w_{k-1},w_1)+\Delta(w_2,u_1)+\Delta(w_{k-1},u_1)$$
$$=\frac{1}{2}\frac{12-14}{4-2}+\frac{1}{1}-\frac{1}{1+1}=0.$$

Note that in this case we have not proved the strict inequality if there is only one $u_a$, i.e. if $n=5$. But in that case it is easily verified that $H_A(C_{4,1})>H_A(C_{3,2})$. The positive contribution which makes the difference is $\Delta(w_0,u_a)$ which was not considered in the proof. □

Note that $C_4$ and $C_{3,1}$ are the only unicyclic graphs on $n=4$ vertices. It holds that

$$H_A(C_4)=20<21=H_A(C_{3,1}).$$

So, for $n=4$ graph $C_4$ is the only minimal unicyclic graph, while for $n\geq 5$ the answer to the question of minimal unicyclic graph is given by the following theorem.

**Theorem 2.** *Let $G\in\mathcal{U}(n)$ for $n\geq 5$. Then*

$$H_A(G)\geq 4\sum_{i=1}^{n-2}H_{n-i-1}+H_{n-3}+3H_{n-2}+\frac{6n-13}{n-2}$$

*with equality if and only if $G=CP_{3,n-3}$.*

*Proof.* Using Lemmas 7, 8 and 9 we first transform a unicyclic graph to cycle-path graph, then we decrease the number of branches in obtained cycle-path graph, so that finally we can transform it to $CP_{3,n-3}$. In each of these transformations the value of $H_A$ strictly decreases, so $CP_{3,n-3}$ is the only extremal graph. Now, the bound follows from Lemma 1. Note that the case of $C_n$ is covered by Lemma 9. □

## 5. Conclusion

In this paper we defined cycle-star graph $CS_{k,n-k}$ to be a graph consisting of cycle of length $k$ and $n-k$ leafs appended to the same vertex of the cycle. Also, we defined cycle-path graph to be a graph consisting of cycle of length $k$ and of path on $n-k$ vertices whose one end is linked to a vertex on a cycle. We establish that $CS_{3,n-3}$ is the only maximal unicyclic graph (see Theorem 1), while $CP_{3,n-3}$ is the only minimal unicyclic graph (see Theorem 2), with respect to additively weighted Harary index. The values of additively weighted Harary index of $CS_{3,n-3}$ and $CP_{3,n-3}$ are established in Lemma 1, so these values are the upper and the lower bound for the value of Harary index on the class of unicyclic graphs. For further research it would be interesting to investigate the values of Harary index on classes of graphs with given parameters, the relation of this variant of Harary index with other topological indices and similar.

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# CONVERGENCE AND SUBSEQUENTIAL CONVERGENCE OF REGULARLY GENERATED SEQUENCES

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*Abstract.* In this paper we recover convergence and subsequential convergence of a sequence of real numbers regularly generated by another sequence in some sequence spaces under certain conditions. We also give some information about the behavior of a sequence whose generator is given in terms of a moderately divergent sequence.

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*Keywords:* regularly generated sequences, slow oscillation, moderate oscillation, moderate divergence, subsequential convergence, weighted means

## 1. Introduction

Throughout this paper, $\mathbb{N}_0$ will denote the set of all nonnegative integers. Let $u = (u_n)$ be a sequence of real numbers and any term with a negative index be zero. Let $p = (p_n)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty, \ n \to \infty. \tag{1.1}$$

The $n^{\text{th}}$ weighted mean of the sequence $(u_n)$ is defined by

$$\sigma_{n,p}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k \tag{1.2}$$

for all $n \in \mathbb{N}_0$.

The sequence $(u_n)$ is said to be summable by the weighted mean method determined by the sequence $p$; in short, $(\overline{N}, p)$ summable to a finite number $s$ if

$$\lim_{n\to\infty} \sigma_{n,p}(u) = s.$$

The difference between $u_n$ and its $n^{\text{th}}$ weighted mean $\sigma_{n,p}(u)$, which is called the weighted Kronecker identity, is given by

$$u_n - \sigma_{n,p}(u) = V_{n,p}(\Delta u), \tag{1.3}$$

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where

$$V_{n,p}(\Delta u) := \frac{1}{P_n}\sum_{k=0}^{n} P_{k-1}\Delta u_k \tag{1.4}$$

and

$$\Delta u_n = u_n - u_{n-1}. \tag{1.5}$$

The $(\overline{N}, p)$ summability method is regular if and only if $P_n \to \infty$ as $n \to \infty$. If $p_n = 1$ for all $n \in \mathbb{N}_0$, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

A sequence $(u_n)$ is slowly oscillating [14] if

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n+1\le k\le[\lambda n]} \mid u_k - u_n \mid = 0, \tag{1.6}$$

where $[\lambda n]$ denotes the integer part of $\lambda n$.

The space of all slowly oscillating sequences is denoted by $\mathcal{SO}$. Dik [9] proved that if a sequence $(u_n)$ is slowly oscillating, then $(V_{n,1}(\Delta u))$ is bounded and slowly oscillating.

A generalization of slow oscillation is given as follows.

A sequence $(u_n)$ is moderately oscillating [14] if

$$\limsup_{n\to\infty}\max_{n+1\le k\le[\lambda n]} \mid u_k - u_n \mid < \infty \tag{1.7}$$

for $\lambda > 1$. The space of all moderately oscillating sequences is denoted by $\mathcal{MO}$.

Set

$$t_n = n\frac{p_n}{P_{n-1}}, \tag{1.8}$$

for $n \in \mathbb{N}_0$. We say that $(u_n)$ is regularly generated by a sequence $\alpha = (\alpha_n)$ in some sequence space $\mathcal{A}$ and $\alpha$ is called a generator of $(u_n)$ if

$$u_n = \alpha_n + \sum_{k=1}^{n}\frac{t_k}{k}\alpha_k. \tag{1.9}$$

The space of all sequences which are regularly generated by sequences in $\mathcal{A}$ is denoted by $U(\mathcal{A})$.

If $(u_n)$ is regularly generated by a sequence $(\alpha_n)$ where $(\Delta\alpha_n) \in \mathcal{SO}$, we write $(u_n) \in U(\mathcal{SO}_\Delta)$. If $(u_n)$ is regularly generated by a sequence $(\alpha_n)$ where $(\alpha_n) \in \mathcal{SO}$, we write $(u_n) \in U(\mathcal{SO})$.

A positive sequence $(u_n)$ is O-regularly varying [12] if

$$\limsup_{n\to\infty}\frac{u_{[\lambda n]}}{u_n} < \infty \tag{1.10}$$

for $\lambda > 1$ and it is slowly varying if

$$\lim_{n\to\infty}\frac{u_{[\lambda n]}}{u_n} = 1. \tag{1.11}$$

It was proved by [11] that if a positive sequence $(u_n)$ is O-regularly varying, then $(\log u_n)$ is slowly varying.

A positive sequence $(u_n)$ is moderately divergent [13] if for every $\lambda > 1$

$$u_n = o(n^{\lambda-1}),\ n \to \infty \tag{1.12}$$

and

$$\sum_{n=1}^{\infty} \frac{u_n}{n^{\lambda}} < \infty. \tag{1.13}$$

We denote the space of all moderately divergent sequences by $\mathcal{MD}$. Note that every slowly oscillating sequence of positive numbers is moderately divergent.

The convergence of a sequence $(u_n)$ implies that $(u_n)$ is bounded and $\Delta u_n = o(1)$ as $n \to \infty$. But it is clear that the converse of this implication is not true in general. In the case where $(u_n)$ is bounded with $\Delta u_n = o(1)$ as $n \to \infty$, we may not recover convergence of $(u_n)$ but we may have convergence of some subsequences of $(u_n)$. A new kind of convergence is defined as follows (See [8] for more details on subsequentially convergent sequences):

A sequence $u = (u_n)$ is said to be subsequentially convergent if there exists a finite interval $I(u)$ such that all accumulation points of $(u_n)$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $(u_n)$.

Recently, several results in terms of regularly generated sequences for different purposes have been obtained by Dik et al. [10], Çanak et al. [1], Çanak and Totur [3], Çanak et al. [2], Çanak et al. [7], Çanak and Totur [5] and many more. In this paper, we first recover convergence and subsequential convergence of a sequence which is regularly generated by another sequence in some sequence spaces under certain conditions. Secondly, we give some information about the behavior of a sequence whose generator is given in terms of a moderately divergent sequence.

## 2. The preliminary results

We need the following lemmas for the proof of our results.

**Lemma 1** ([8])**.** *Let $(u_n)$ be a bounded sequence of real numbers. If $\Delta u_n = o(1)$ as $n \to \infty$, then $(u_n)$ converges subsequentially.*

**Lemma 2.** *If $(\sum_{k=1}^{n} t_k \alpha_k)$ is moderately oscillating, then $(\sum_{k=1}^{n} \frac{t_k}{k} \alpha_k)$ converges.*

*Proof.* Set $R_n := \exp\left(\left|\sum_{k=1}^{n} t_k \alpha_k\right|\right)$. Then we have

$$\frac{R_{[\lambda n]}}{R_n} \le \exp\left(\left|\sum_{k=n+1}^{[\lambda n]} t_k \alpha_k\right|\right). \tag{2.1}$$

Taking limsup of both sides of (2.1) as $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{R_{[\lambda n]}}{R_n} \leq \exp\left(\limsup_{n\to\infty} \left|\sum_{k=n+1}^{[\lambda n]} t_k \alpha_k\right|\right). \tag{2.2}$$

Since $(\sum_{k=1}^{n} t_k \alpha_k)$ is moderately oscillating, we have

$$\limsup_{n\to\infty} \frac{R_{[\lambda n]}}{R_n} \tag{2.3}$$

is finite for $\lambda > 1$. This says that $(R_n)$ is O-regularly varying. Since $(R_n)$ is O-regularly varying, $(\log R_n)$ is slowly varying. It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \left|\sum_{k=1}^{n} t_k \alpha_k\right|^p < \infty \tag{2.4}$$

for $p > 1$. This implies that

$$\sum_{n=1}^{\infty} \frac{t_n}{n} \alpha_n < \infty. \tag{2.5}$$

□

**Lemma 3.** *If $(\sum_{k=1}^{n} \frac{t_k}{k} \alpha_k)$ converges, then $\sigma_{n,p}(\alpha) = o(1),\ n \to \infty$.*

*Proof.* Set $\gamma_n := \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k$. Then we obtain

$$\alpha_n = \frac{P_{n-1}}{p_n} \Delta \gamma_n \tag{2.6}$$

and

$$\sigma_{n,p}(\alpha) = V_{n,p}(\Delta \gamma) \tag{2.7}$$

for $n \in \mathbb{N}_0$. Since $(\gamma_n)$ converges, it follows by the weighted Kronecker identity

$$\gamma_n - \sigma_{n,p}(\gamma) = V_{n,p}(\Delta \gamma) \tag{2.8}$$

that

$$V_{n,p}(\Delta \gamma) = o(1),\ \ n \to \infty.$$

This completes the proof. □

**Lemma 4** ([6])**.** *Let $(p_n)$ satisfy the condition*

$$1 \leq \frac{P_n}{n} \to 1,\ n \to \infty. \tag{2.9}$$

*If $(u_n)$ is slowly oscillating, then $(V_{n,p}(\Delta u))$ is slowly oscillating and bounded.*

**Lemma 5** ([15])**.** *Let $(u_n)$ be Cesàro summable to $s$. If $(u_n)$ is slowly oscillating, then $(u_n)$ converges to $s$.*

## 3. The main results

**Theorem 1.** *Suppose that*

$$\left(\sum_{k=1}^{n} t_k \alpha_k\right) \in \mathcal{MO}, \tag{3.1}$$

$$1 \leq \frac{P_n}{n} \to 1, \ n \to \infty, \tag{3.2}$$

$$t_n = O(1), n \to \infty. \tag{3.3}$$

*If* $(u_n) \in U(\mathcal{SO}_\Delta)$*, then* $(u_n)$ *converges subsequentially.*

*Proof.* Since $(u_n) \in U(\mathcal{SO}_\Delta)$, $(u_n)$ can be written as

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k} \alpha_k, \tag{3.4}$$

where $(\Delta\alpha_n) \in \mathcal{SO}$. Moderate oscillation of $\left(\sum_{k=1}^{n} t_k \alpha_k\right)$ implies convergence of $(\gamma_n) = \left(\sum_{k=1}^{n} \frac{t_k}{k} \alpha_k\right)$ by Lemma 2 and $\sigma_{n,p}(\alpha) = o(1)$ as $n \to \infty$ by Lemma 3. Hence, $(u_n)$ is $(\overline{N}, p)$ summable to the limit of $(\gamma_n)$. By the condition (3.3), it follows that

$$\frac{\alpha_n}{n} \to 0, n \to \infty \tag{3.5}$$

by Lemma 3. Since $(\Delta\alpha_n) \in \mathcal{SO}$, we have that

$$\Delta\alpha_n \to 0, n \to \infty \tag{3.6}$$

by Lemma 5. Taking the backward difference of (3.4), we have

$$\Delta u_n = \Delta\alpha_n + \alpha_n \frac{p_n}{P_{n-1}} \tag{3.7}$$

for $n \in \mathbb{N}_0$.

It follows by (3.3), (3.5) and (3.6) that

$$\Delta u_n = o(1), n \to \infty. \tag{3.8}$$

To complete the proof, it suffices to prove that $(u_n)$ is bounded. Applying Lemma 4 to $(v_n) = \left(\sum_{k=1}^{n} \alpha_k t_k\right)$, and taking $(v_n) \in \mathcal{SO}$ into account, we obtain $(V_{n,p}(\alpha t))$ is bounded and slowly oscillating, where $\alpha t = (\alpha_n t_n)$.

From the weighted Kronecker identity

$$S_n(\alpha) - \sigma_{n,p}(S(\alpha)) = V_{n,p}(\alpha) \tag{3.9}$$

where $S(\alpha) = (S_n(\alpha)) = (\sum_{k=0}^{n} \alpha_k)$, we have

$$\alpha_n - \frac{p_n}{P_{n-1}} V_{n,p}(\alpha) = \Delta V_{n,p}(\alpha). \tag{3.10}$$

Replacing $\alpha_n$ by $\alpha_n t_n$ in (3.10) and then dividing by $t_n$, we have

$$\alpha_n = \frac{V_{n,p}(\alpha t)}{n} + \frac{P_{n-1}}{np_n}\Delta V_{n,p}(\alpha t). \tag{3.11}$$

It follows from (3.11) that $(\alpha_n)$ is bounded. Hence, $(u_n)$ is bounded. By Lemma 1, $(u_n)$ is subsequentially convergent. □

**Theorem 2.** *Suppose that*

$$\left(\sum_{k=1}^{n} t_k \alpha_k\right) \in \mathcal{MO}, \tag{3.12}$$

$$1 \leq \frac{P_n}{n} \to 1, \ n \to \infty, \tag{3.13}$$

$$1 < \liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \limsup_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \infty, \quad \textit{for} \quad \lambda > 1, \tag{3.14}$$

$$1 < \liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \limsup_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \infty, \quad \textit{for} \quad 0 < \lambda < 1, \tag{3.15}$$

$$t_n = O(1), \ n \to \infty. \tag{3.16}$$

*If $(u_n) \in U(\mathcal{SO})$, then $(u_n)$ converges.*

*Proof.* Assume that $(u_n) \in U(\mathcal{SO})$. Then, $(u_n)$ can be written as

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{t_k}{k}\alpha_k \tag{3.17}$$

where $(\alpha_n) \in \mathcal{SO}$. From (3.17), we have

$$V_{n,p}(\Delta u) = V_{n,p}(\Delta\alpha) + \sigma_{n,p}(\alpha). \tag{3.18}$$

Moderate oscillation of $\left(\sum_{k=1}^{n} t_k\alpha_k\right)$ implies convergence of $(\gamma_n) = \left(\sum_{k=1}^{n} \frac{t_k}{k}\alpha_k\right)$ by Lemma 2 and $\sigma_{n,p}(\alpha) = o(1)$ as $n \to \infty$ by Lemma 3. Therefore, $(u_n)$ is $(\overline{N}, p)$ summable to the limit of $(\gamma_n)$.

Since $(\alpha_n)$ is slowly oscillating, $(V_{n,p}(\Delta\alpha))$ is bounded and slowly oscillating by Lemma 4.

It follows from (3.18) that $(V_{n,p}(\Delta u)) \in \mathcal{SO}$ and bounded. Since $(u_n)$ is $(\overline{N}, p)$ summable, $(u_n)$ converges to $\lim_{n\to\infty} \sigma_{n,p}(u)$ by Theorem 6 in [4]. □

**Theorem 3.** *Suppose that $(u_n)$ is regularly generated by $(\alpha_n)$ and*

$$\frac{p_n}{P_{n-1}} - \frac{p_{n+1}}{P_n} = O\left(\frac{1}{n^2}\right), n \to \infty. \tag{3.19}$$

*If*

$$\sum_{k=1}^{n} \alpha_k = n^{\gamma} m_n \tag{3.20}$$

*for some* $(m_n) \in \mathcal{MD}$ *and some* $\gamma \in (0,1)$, *then*

i) $(u_n)$ *is* $(\overline{N}, p)$ *summable.*

ii) $u_n = \Delta(n^\gamma m_n) + \beta_n$, *where* $\beta_n = o(1), n \to \infty$.

iii) $u_n = o(n), n \to \infty$ *and* $\sum_{n=1}^{\infty} \frac{u_n}{n^2} < \infty$.

*Proof.* i) By Abel's partial summation formula, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k &= \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} (S_k(\alpha) - S_{k-1}(\alpha)) \\ &= \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_k(\alpha) - \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} S_{k-1}(\alpha) \\ &= \sum_{k=1}^{n} \left( \frac{p_k}{P_{k-1}} S_k(\alpha) - \frac{p_{k+1}}{P_k} S_k(\alpha) \right) + \frac{p_n}{P_{n-1}} S_n(\alpha) - \frac{p_1}{P_0} S_0 \\ &= \frac{p_n}{P_{n-1}} S_n(\alpha) + \sum_{k=1}^{n-1} \left( \frac{p_k}{P_{k-1}} - \frac{p_{k+1}}{P_k} \right) S_k(\alpha)\end{aligned} \tag{3.21}$$

Since $S_n(\alpha) = n^\gamma m_n$ for some $(m_n) \in \mathcal{MD}$, we have

$$\frac{p_n}{P_{n-1}} S_n(\alpha) = O\left(\frac{m_n}{n^{1-\gamma}}\right), \quad n \to \infty. \tag{3.22}$$

By moderate divergence of $(m_n)$, we have

$$\frac{p_n}{P_{n-1}} S_n(\alpha) = o(1), \quad n \to \infty. \tag{3.23}$$

The second term on the right of (3.21) converges by (3.19). It follows from the representation

$$u_n = \alpha_n + \sum_{k=1}^{n} \frac{p_k}{P_{k-1}} \alpha_k, \tag{3.24}$$

that $(u_n)$ is $(\overline{N}, p)$ summable.

ii) Note that the sequence $(\beta_n)$ defined by $\beta_n = \frac{t_n \alpha_n}{n}$ for $n \in \mathbb{N}_0$ converges to zero. From the representation and the condition (3.20) it follows that

$$u_n = \Delta(n^\gamma m_n) + \beta_n \tag{3.25}$$

where $\beta_n = \frac{t_n}{n} \alpha_n$.

iii) By ii), we have

$$u_n = n^\gamma m_n - (n-1)^\gamma m_{n-1} + \beta_n. \tag{3.26}$$

Dividing (3.26) by $n$, we have

$$\frac{u_n}{n} = \frac{m_n}{n^{1-\gamma}} - \frac{m_{n-1}}{(n-1)^{1-\gamma}} + \frac{\beta_n}{n}. \tag{3.27}$$

Since $(m_n) \in \mathcal{MD}$ and $\beta_n = o(1)$, we have

$$\frac{u_n}{n} = o(1), n \to \infty. \tag{3.28}$$

By (3.26), we obtain

$$\sum_{k=2}^{n} \frac{u_k}{k^2} = \sum_{k=2}^{n} \frac{m_k}{k^{2-\gamma}} - \sum_{k=2}^{n} \frac{m_{k-1}}{(k-1)^{2-\gamma}} + \sum_{k=2}^{n} \frac{\beta_k}{k^2}. \tag{3.29}$$

Taking the limit of both sides of (3.29) as $n \to \infty$, we obtain $\sum_{n=1}^{\infty} \frac{u_n}{n^2} < \infty$. □

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# ON THE SATURATION EFFECT FOR LINEAR SHAPE-PRESERVING APPROXIMATION IN SOBOLEV SPACES

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*Abstract.* We show that if a linear finite-dimensional operator defined in Sobolev space preserves $k$-monotonicity then the error of approximation of the operator does not decrease with the increase of smoothness of approximated functions. In other words, there is a saturation effect for linear finite-rank operators defined in Sobolev space and preserving $k$-monotonicity.

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*Keywords:* shape-preserving approximation, saturation effect, spline approximation

## 1. Introduction

In various applications of computer-aided geometric design it is necessary to approximate functions preserving such properties as monotonicity, convexity, concavity, etc. The part of the approximation theory, which is devoted to these kind of problems, is known as the theory of shape preserving approximation (see e.g. [3, 9] for a survey).

A continuous function $f : [0,1] \to \mathbb{R}$ is said to be $k$-monotone, $k \geq 1$, on $[0,1]$ if and only if for all choices of $k+1$ distinct $t_0, \dots, t_k$ in $[0,1]$ the inequality $[t_0, \dots, t_k]f \geq 0$ holds, where $[t_0, \dots, t_k]f$ denotes the $k$-th divided difference of $f$ at $0 \leq t_0 < t_1 < \dots < t_k \leq 1$. Note that 2-monotone functions are just convex functions.

Let $\Delta^k$ denote the set of all $k$-monotone functions defined on [0,1]. If $f$ is a real-valued and $k$-times continuously differentiable function defined on $[0,1]$, then $f \in \Delta^k$ iff $f^{(k)}(t) \geq 0$, $t \in [0,1]$.

Let $W_\infty^{(k)}[0,1]$ be the Sobolev space of all real-valued, $(k-1)$-times differentiable functions whose derivative of order $(k-1)$ is absolutely continuous and whose derivative of order $k$ is in $L^\infty[0,1]$, $\|f\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]}|f(x)|$.

It is said that a linear operator $L$ of $W_\infty^{(k)}[0,1]$ into $W_\infty^{(k)}[0,1]$ preserves $k$-monotonicity, if $L(\Delta^k) \subset \Delta^k$.

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Recall that a linear operator mapping a linear normed space into a linear space of finite dimension $n$ is called an *operator of finite rank* $n$.

Let $D^i$ denote the $i$-th differential operator, $D^i f(x) = d^i f(x)/dx^i$, and $D^0 = I$ is the identity operator. Denote $e_k(x) = x^k$, $k = 1, 2, \ldots$.

Denote $B_\infty^{(k)} := \{W_\infty^{(k)}[0,1] : \|D^k f\|_\infty \le 1\}$.

Let $k \ge 1$. The main goal of the paper is to estimate the asymptotic value of relative linear $n$-width

$$\inf_{L_n(\Delta^k)\subset\Delta^k} \sup_{f\in B_\infty^{(k)}} \|f - L_n f\|_\infty,$$

where infimum is taken over all linear continuous operators $L_n$ of $W_\infty^{(k)}[0,1]$ into $W_\infty^{(k)}[0,1]$ of finite rank $n$, such that $L_n(\Delta^k) \subset \Delta^k$ and $D^k L_n e_k = D^k e_k$. Note that some estimates of shape-preserving $n$-widths have been obtained in papers [4, 7, 8].

T. Popoviciu [12] proved in 1937 that Bernstein operators

$$B_n f(x) := \sum_{i=0}^{n} f\left(\frac{i}{n}\right) C_n^i x^i (1-x)^{n-i}$$

preserve $k$-monotonicity of approximated functions, i.e. if $f$ is $k$-monotone on [0,1], then $B_n f$ is $k$-monotone on [0,1] as well. However, it is well known [2, 13] that the order of approximation by Bernstein operators is low. In this paper we present the example of $k$-monotonicity preserving operator with the optimal order of approximation $n^{-2}$ ($k \ge 2$).

## 2. Lemmas

**Lemma 1.** *Let* $L : W_\infty^{(k)}[0,1] \to W_\infty^{(k)}[0,1]$ *be such that*

(1) $L(\Delta^k) \subset \Delta^k$;

(2) $D^k L e_k = D^k e_k$.

*If* $f \in W_\infty^{(k)}[0,1]$ *satisfies* $\|D^k f\|_\infty \le 1$, *then* $\|D^k L f\|_\infty \le 1$.

*Proof.* For every $f \in W_\infty^{(k)}[0,1]$ such that $\|D^k f\|_\infty \le 1$, we have

$$|D^k f(x)| \le 1 = \frac{1}{k!} D^k e_k(x)$$

for almost all $x \in [0,1]$. Since $L(\Delta^k) \subset \Delta^k$, for almost all $x \in [0,1]$ the following inequality holds:

$$|D^k L f(x)| \le \frac{1}{k!} D^k L e_k(x).$$

Therefore, we have

$$\|D^k L f\|_\infty \le \frac{1}{k!} \|D^k L e_k\|_\infty.$$

It follows from $D^k L e_k = D^k e_k$ that $\|D^k L f\|_\infty \le 1$. □

Lemma 1 says that if $L : W_\infty^{(k)}[0,1] \to W_\infty^{(k)}[0,1]$ satisfies conditions (i) and (ii) then $L$ preserves the ball $B_\infty^{(k)}$, i.e. $L(B_\infty^{(k)}) \subset B_\infty^{(k)}$.

Denote $z_j := j/n,\ j = 0,\dots,n$. Let the linear operator $M_{k,n} : W_\infty^{(k)}[0,1] \to W_\infty^{(k)}[0,1]$, $k \geqslant 2$, $n \geq 2$, be defined by

$$\begin{aligned} M_{k,n} f(x) = {} & \sum_{l=0}^{k-1} \frac{1}{l!} x^l D^l f(0) + \frac{2}{k!} x^k [z_0, z_1, z_2] D^{k-2} f \\ & + \frac{2n}{(k+1)!} x^{k+1} \Big( [z_1, z_2, z_3] D^{k-2} f - [z_0, z_1, z_2] D^{k-2} f \Big), \qquad (2.1) \\ & \text{if } x \in [z_0, z_1]; \end{aligned}$$

$$\begin{aligned} M_{k,n} f(x) = {} & \sum_{l=0}^{k-1} \frac{1}{l!} (x - z_j)^l D^l M_{k,n} f(z_j) \\ & + \frac{2}{k!} (x - z_j)^k [z_j, z_{j+1}, z_{j+2}] D^{k-2} f + \frac{2n}{(k+1)!} (x - z_j)^{k+1} \\ & \times \Big( [z_{j+1}, z_{j+2}, z_{j+3}] D^{k-2} f - [z_j, z_{j+1}, z_{j+2}] D^{k-2} f \Big), \qquad (2.2) \\ & \text{if } x \in (z_j, z_{j+1}],\ j = 1, 2, \dots, n-3; \end{aligned}$$

$$\begin{aligned} M_{k,n} f(x) = {} & \sum_{l=0}^{k-1} \frac{1}{l!} (x - z_{n-2})^l D^l M_{k,n} f(z_{n-2}) \\ & + \frac{2}{k!} (x - z_{n-2})^k [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-2} f + \frac{2n}{(k+1)!} (x - z_{n-2})^{k+1} \\ & \times \Big( [z_{n-2}, z_{n-1}, z_n] D^{k-2} f - [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-2} f \Big), \qquad (2.3) \\ & \text{if } x \in (z_{n-2}, z_{n-1}]; \end{aligned}$$

$$\begin{aligned} M_{k,n} f(x) = {} & \sum_{l=0}^{k-1} \frac{1}{l!} (x - z_{n-1})^l D^l M_{k,n} f(z_{n-1}) \\ & + \frac{2}{k!} (x - z_{n-1})^k [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-2} f + \frac{2n}{(k+1)!} (x - z_{n-1})^{k+1} \\ & \times ([z_{n-2}, z_{n-1}, z_n] D^{k-2} f - [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-1} f), \qquad (2.4) \\ & \text{if } x \in (z_{n-1}, 1]. \end{aligned}$$

**Lemma 2.** *Let $k \geq 2$ and let the operator $M_{k,n} : W_\infty^{(k)}[0,1] \to W_\infty^{(k)}[0,1]$ be defined by (2.1), (2.2), (2.3), (2.4). Then*

(1) $M_{k,n}(\Delta^k) \subset \Delta^k$;

(2) $D^k M_{k,n} e_k = D^k e_k$;

(3) *there exists a real $c > 0$ such that $\|f - M_{k,n} f\|_\infty \leq cn^{-2}$ for all $f \in B_\infty^{(k)}$.*

*Proof.* Note that $D^k M_{k,n} f$ is a piecewise linear function and $D^k M_{k,n} f$ is linear on $[z_j, z_{j+1}]$ for each $0 \leq j \leq n-1$. If $x \in (z_j, z_{j+1}]$, then

$$D^k M_{k,n} f(x) = 2(j+1-nx)[z_j, z_{j+1}, z_{j+2}] D^{k-2} f + 2(nx-j)[z_{j+1}, z_{j+2}, z_{j+3}] D^{k-2} f, \ j = 0, 1, \ldots, n-3, \quad (2.5)$$

$$D^k M_{k,n} f(x) = 2(n-1-nx) \cdot [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-2} f + 2(nx-n+2) \cdot [z_{n-2}, z_{n-1}, z_n] D^{k-2} f, \ j = n-2. \quad (2.6)$$

$$D^k M_{k,n} f(x) = 2(n-nx) \cdot [z_{n-3}, z_{n-2}, z_{n-1}] D^{k-2} f + 2(nx-n+1) \cdot [z_{n-2}, z_{n-1}, z_n] D^{k-2} f, \ j = n-1. \quad (2.7)$$

If $f \in \Delta^k$, then it follows from (2.5), (2.6), (2.7) that $D^k M_{k,n} f \geq 0$ on each $(x_j, x_{j+1})$, $j = 0, \ldots, n-1$, i.e. $M_{k,n} f \in \Delta^k$. Therefore, $M_{k,n}(\Delta^k) \subset \Delta^k$.

A direct verification shows that $D^k M_{k,n} e_k = D^k e_k$.

Since $f \in B_\infty^{(k)}$, the $(k-2)$-th derivative of $f$ can be represented as

$$D^{k-2} f(x) = D^{k-2} f(0) + \frac{D^{k-1} f(0)}{1!} x + \int_0^1 (x-t)_+ D^k f(t) dt,$$

where $\|D^k f\|_\infty \leq 1$ and $y_+ = y$ if $y \geq 0$, and 0 otherwise.

Take an arbitrary $x \in [0, \frac{1}{n}]$ (the case $x \in (z_j, z_{j+1})$, $j = 1, \ldots, n-3$, can be examined analogously), then

$$\begin{aligned}
&D^{k-2} M_{k,n} f(x) - D^{k-2} f(x) \\
&= x^2 [z_0, z_1, z_2] D^{k-2} f + \frac{1}{3} n x^3 \Big( [z_1, z_2, z_3] D^{k-2} f - [z_0, z_1, z_2] D^{k-2} f \Big) \\
&\quad - \int_0^1 (x-t)_+ D^k f(t) dt \\
&= \int_0^1 \big[ x^2 ((z_0-t)_+ - 2(z_1-t)_+ + (z_2-t)_+) \\
&\quad + \frac{1}{3} n x^3 ((z_1-t)_+ - 2(z_2-t)_+ + (z_3-t)_+) \\
&\quad - \frac{1}{3} n x^3 ((z_0-t)_+ - 2(z_1-t)_+ + (z_2-t)_+) - (x-t)_+ \big] D^k f dt,
\end{aligned}$$

and we can obtain

$$|D^{k-2}M_{k,n}f(x)-D^{k-2}f(x)|\le an^{-2},\ x\in[z_j,z_{j+1}],\ j=1,\dots,n-3,$$

for some $a>0$. In the same way it can be shown that

$$|D^{k-2}M_{k,n}f(x)-D^{k-2}f(x)|\le n^{-2},\ x\in[z_{n-2},z_{n-1}],\ \text{and } x\in[z_{n-1},1].$$

Therefore, for every $f\in B_\infty^{(k)}$ and any $x\in[0,1]$

$$|D^{k-2}f(x)-D^{k-2}M_{k,n}f(x)|\le c_1n^{-2}. \tag{2.8}$$

If $k=2$ then the proposition (3) of Lemma 2 is proved.

Let $k\ge 3$. Since $D^i(M_{k,n}f-f)(0)=0$ holds for $i=0,\dots,k-3$, we have

$$\begin{aligned}D^i(M_{k,n}f-f)(x)&=D^i(M_{k,n}f-f)(0)+\int_0^x D^{i+1}(M_{k,n}f-f)(t)dt\\&=\int_0^x D^{i+1}(M_{k,n}f-f)(t)dt\end{aligned}$$

for all $x\in[0,1]$. Then

$$|D^i(M_{k,n}f-f)(x)|\le\frac{1}{n^2}\frac{x^{k-1-i}}{(k-1-i)!}. \tag{2.9}$$

We have used the fact that if $f\in L_\infty[0,1]$ and there exists a real $a\in\mathbb{R}$ such that $|f|\le a$ on [0,1], then

$$\left|\int_0^x\int_0^{t_{p-1}}\dots\int_0^{t_1}f(t_1)dt_1\dots dt_p\right|\le\int_0^x\int_0^{t_{p-1}}\dots\int_0^{t_1}|f(t_1)|dt_1\dots dt_p\le a\frac{x^p}{p!}$$

for every $p\in\mathbb{N}$.

The proposition (3) of Lemma 2 follows from (2.9) with $i=0$. □

## 3. The main result

**Theorem 1.** *Let $k\in\mathbb{N}$. Then*

$$\inf_{L_n(\Delta^k)\subset\Delta^k}\ \sup_{f\in B_\infty^{(k)}}\|f-L_nf\|_\infty\asymp\begin{cases}n^{-2},\ k\geqslant 2,\\ n^{-1},\ k=1,\end{cases} \tag{3.1}$$

*where infimum is taken over all linear continuous operators $L_n$ of $W_\infty^{(k)}[0,1]$ into $W_\infty^{(k)}[0,1]$ of finite rank $n$, such that*

(1) $L_n(\Delta^k)\subset\Delta^k$;
(2) $D^kL_ne_k=D^ke_k$.

*Proof.* We have (by definition)

$$\inf_{L_n(\Delta^k)\subset\Delta^k}\sup_{f\in B_\infty^{(k)}}\|f-L_nf\|_\infty=\inf_{X_n}\inf_{\substack{L_n(\Delta^k)\subset\Delta^k\\ L_n:W_\infty^{(k)}[0,1]\to X_n}}\sup_{f\in B_\infty^{(k)}}\|f-L_nf\|_\infty, \tag{3.2}$$

where the first infimum is taken over all $n$-dimensional subspaces $X_n$ of $W_\infty^{(k)}[0,1]$, and the second one is taken over all continuous linear operators $L_n$ of $W_\infty^{(k)}[0,1]$ into $X_n$ satisfying $L_n(\Delta^k)\subset\Delta^k$ and $D^kL_ne_k=D^ke_k$.

Since $f\in B_\infty^{(k)}$, it follows from Lemma 1 that $L_nf\in B_\infty^{(k)}$. Then Eq. (3.2) implies

$$\inf_{L_n(\Delta^k)\subset\Delta^k}\sup_{f\in B_\infty^{(k)}}\|f-L_nf\|_\infty\geqslant\inf_{X_n}\sup_{f\in B_\infty^{(k)}}\inf_{g\in B_\infty^{(k)}\cap X_n}\|f-g\|_\infty.$$

It was shown in the paper of Konovalov [6] that

$$\inf_{X_n}\sup_{f\in B_\infty^{(k)}}\inf_{g\in B_\infty^{(k)}\cap X_n}\|f-g\|_\infty\asymp\begin{cases}n^{-2}, & k\geqslant 2,\\ n^{-1}, & k=1.\end{cases}$$

To complete the proof it is sufficient to show that there exists a linear finite-dimensional method, that is optimal for $n$-widths (3.1). If $k\geqslant 2$, then as it follows from Lemma 2, the linear operator $M_{k,n}:W_\infty^{(k)}[0,1]\to W_\infty^{(k)}[0,1]$ satisfies the conditions of Theorem and it is optimal for $n$-width (3.1). If $k=1$, then the piecewise linear interpolation method with breakpoints $z_j$, $j=0,\dots,n$, is optimal for $n$-width (3.1). □

Theorem 1 shows that the shape-preserving property of operators is negative in the sense that the error of approximation of such operators does not decrease with the increase of smoothness of approximated functions. In other words, there is a saturation effect for linear finite-rank operators preserving $k$-monotonicity. For the first time the problem was mentioned by Ronald A. De Vore in his paper [1]. It is worth noting that non-linear approximation preserving $k$-monotonicity does not have this shortcoming [10]. On the other hand, for sequences of linear operators preserving $k$-monotonicity (as well as intersections of cones) there are [5, 11] simple convergence conditions (Korovkin type results).

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# BOUNDS FOR THE GEOMETRIC-ARITHMETIC INDEX OF A GRAPH

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*Abstract.* The first geometric-arithmetic index $GA_1(G)$, which was introduced by D. Vukičević and B. Furtula recently, is a graph-based molecular structure descriptor. The aim of this paper is to obtain new inequalities involving the geometric-arithmetic index $GA_1$ and other well known topological indices. In particular, we relate the geometric-arithmetic index of a graph $G$ with its first Zagreb index $M_1(G)$, its second Zagreb index $M_2(G)$, its modified Zagreb index $Z_{-1}(G)$ and its Randić index $R(G)$.

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## 1. Introduction

The study of topological indices is a subject of increasing interest, both in pure and applied mathematics. Topological indices are interesting since they capture some of the properties of a molecule (or a graph) in a single number(see, e.g., [17, 22]). Hundreds of topological indices have been introduced and studied, starting with the seminal work by Wiener [19] in which he used the sum of all shortest-path distances of a (molecular) graph for modeling physical properties of alkanes. Probably, the Randić index $R(G)$ is the best know topological index (see, e.g., [5, 9, 10, 14, 15] and the references cited therein). The Randić index $R(G)$ is defined in [14] as

$$R = R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

During many years, scientists were trying to improve the predictive power of the Randić index. This led to the introduction of a large number of new topological

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descriptors resembling the original Randić index. The first geometric-arithmetic index $GA_1(G)$, defined in [18] as

$$GA_1 = GA_1(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u + d_v)},$$

where $uv$ denotes the edge of the graph $G$ connecting the vertices $u$ and $v$, and $d_u$ is the degree of the vertex $u$, is one of the successors of the Randić index. Although $GA_1$ was introduced just five years ago, there are many papers dealing with this index. There are other geometric-arithmetic indices, like $Z_{p,q}$ ($Z_{0,1} = GA_1$), but the results in [2] show empirically that the $GA_1$ index gathers the same information on observed molecules as other $Z_{p,q}$ indices.

Topological indices based on end-vertex degrees of edges have been used over thirty years. Among them, several indices are recognized to be useful tools in chemical researches. The graph based molecular descriptors called Zagreb indices were introduced by Gutman and Trinajstić. Since then, several results concerning Zagreb indices have been communicated in the chemical and mathematical literature (see [1,4,6,13,21] and the references cited therein). The variable Zagreb index is defined in [11] as

$$Z_\alpha = Z_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha,$$

and was used in the structure-boiling point modeling of benzenoid hydrocarbons. The obtained model is practically identical to the model based on the variable vertex-connectivity index and this is due to close relationship between the formulas for the two indices.

The reason for introducing a new index is to gain prediction of some property of molecules somewhat better than obtained by already presented indices. Therefore, a test study of predictive power of a new index must be done. As a standard for testing new topological descriptors, the properties of octanes are commonly used. We can find 16 physico-chemical properties of octanes at www.moleculardescriptors.eu. The $GA_1$ index gives better correlation coefficients than Randić index for these properties, but the differences between them are not significant. However, the predicting ability of the $GA_1$ index compared with Randić index is reasonably better (see [2, Table 1]).

We begin by stating some notation and terminology. Let $G = (V, E) = (V(G), E(G))$ denotes a (non-oriented) finite simple connected graph of order $n = |V(G)|$ and size $m = |E(G)|$ with $E(G) \neq \varnothing$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a vertex $u \in V$ we denote $N(v) = \{u \in V : u \sim v\}$. The degree of a vertex $v \in V$ will be denoted by $d_v = |N(v)|$. We denote by $\delta$ and $\Delta$ the minimum and maximum degree of the graph, respectively. We use the classical notation $uv$ for the edge of a graph joining the vertices $u$ and $v$. Note that the connectivity of $G$ is not an important restriction, since if $G$ has connected components $G_1, \dots, G_r$, then $GA_1(G) = GA_1(G_1) + \cdots + GA_1(G_r)$.

## 2. Results

We start with the following basic results. In [18] (see also [2]) the following inequalities appear :

$$GA_1(G) \geq \frac{2(n-1)^{3/2}}{n}.$$

$$GA_1(G) \geq \frac{2m}{n}.$$

In [2] we find the bounds

$$\frac{2m\sqrt{\Delta\delta}}{\Delta+\delta} \leq GA_1(G) \leq m.$$

In [16] the authors obtain a lower bound of $GA_1(G)$, depending just on $n$ and $m$, for every graph $G$.

$$GA_1(G) \geq \frac{2m\sqrt{n-1}}{n}.$$

Moreover, [16] provides a lower bound of $GA_1(G)$ for every graph $G$ with $\delta \geq k$, for any fixed $k \geq 2$.

(1) If $n \leq 10$, then $GA_1(G) \geq \frac{nk}{2}$.

(2) If $n \geq 11$, then $GA_1(G) \geq \min\left\{\frac{nk}{2}, \frac{(k+1)\sqrt{k}(n-1)^{3/2}}{n-1+k}\right\}$.

We will denote by $M_1(G)$ and $M_2(G)$ the first and the second Zagreb indices of the graph $G$, respectively, defined in [7] as

$$M_1(G) = \sum_{u\in V(G)} d_u^2, \qquad M_2(G) = \sum_{uv\in E(G)} d_u d_v.$$

These indices have attracted growing interest, see e.g., [20] and [7].

In [3] (see also [2]) we find the bound:

$$GA_1(G) \leq \frac{\sqrt{mM_2(G)}}{\delta}.$$

A similar lower bound is obtained in [16]:

$$GA_1(G) \geq \frac{2}{\Delta+\delta}\sqrt{\frac{\delta m M_2(G)}{\Delta}}.$$

Theorem 1 below improves these lower bounds. In order to prove it, we need the following lemmas (see [16] and [8]).

**Lemma 1.** *Let $g$ be the function $g(x,y) = \frac{2\sqrt{xy}}{x+y}$ with $0 < a \leq x, y \leq b$. Then*

$$\frac{2\sqrt{ab}}{a+b} \leq g(x,y) \leq 1.$$

*The equality in the lower bound is attained if and only if either $x = a$ and $y = b$, or $x = b$ and $y = a$, and the equality in the upper bound is attained if and only if $x = y$. Besides, $g(x, y) = g(x', y')$ if and only if $x/y$ is equal to either $x'/y'$ or $y'/x'$.*

*Proof.* Let $f$ be the function $f(t) = \frac{2t}{1+t^2}$ on the interval $[0, \infty)$. Then $f$ strictly increases in $[0, 1]$, strictly decreases in $[1, \infty)$, $f(t) = 1$ if and only if $t = 1$ and $f(t) = f(t_0)$ if and only if either $t = t_0$ or $t = t_0^{-1}$. The statements follow from $f'(t) = \frac{2(1-t^2)}{(1+t^2)^2}$. Now, taking $g(x, y) = f(t)$ with $t = \sqrt{\frac{x}{y}}$, and $\sqrt{\frac{a}{b}} \le t \le \sqrt{\frac{b}{a}}$, the result follows. □

The following result provides a converse of Cauchy-Schwarz inequality.

**Lemma 2.** *If $0 < n_1 \le a_j \le N_1$ and $0 < n_2 \le b_j \le N_2$ for $1 \le j \le k$, then*

$$\Big(\sum_{j=1}^{k} a_j^2\Big)^{1/2}\Big(\sum_{j=1}^{k} b_j^2\Big)^{1/2} \le \frac{1}{2}\left(\sqrt{\frac{N_1N_2}{n_1n_2}} + \sqrt{\frac{n_1n_2}{N_1N_2}}\right)\Big(\sum_{j=1}^{k} a_jb_j\Big).$$

**Theorem 1.** *We have for any graph $G$*

$$\frac{2\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)mM_2(G)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta})^2} \le GA_1(G) \le \frac{M_2(G)}{\delta^2},$$

*and the equality in each inequality is attained if and only if $G$ is a regular graph.*

*Proof.* By Lemma 1, taking $a = \delta$ and $b = \Delta$, we have

$$\frac{2\sqrt{\Delta\delta}}{\Delta+\delta} \le \frac{\sqrt{d_ud_v}}{\frac{1}{2}(d_u+d_v)} \le 1.$$

Lemma 2 gives

$$\begin{aligned} GA_1(G) &= \sum_{uv\in E(G)} \frac{\sqrt{d_ud_v}}{\frac{1}{2}(d_u+d_v)} \ge \frac{\Big(\sum_{uv\in E(G)} \frac{4d_ud_v}{(d_u+d_v)^2}\Big)^{1/2}\big(\sum_{uv\in E(G)} 1\big)^{1/2}}{\frac{1}{2}\Big(\sqrt{\frac{\Delta+\delta}{2\sqrt{\Delta\delta}}} + \sqrt{\frac{2\sqrt{\Delta\delta}}{\Delta+\delta}}\Big)} \\ &\ge \frac{2\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)m}}{(\sqrt{\Delta}+\sqrt{\delta})^2}\Big(\sum_{uv\in E(G)} \frac{1}{\Delta^2}d_ud_v\Big)^{1/2} \\ &= \frac{2\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)mM_2(G)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta})^2}. \end{aligned}$$

Therefore,

$$\frac{2\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)mM_2(G)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta})^2} \le GA_1(G).$$

On the other hand,

$$GA_1(G) = \sum_{uv\in E(G)} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u+d_v)} = \sum_{uv\in E(G)} \frac{d_u d_v}{\frac{1}{2}(d_u+d_v)\sqrt{d_u d_v}} \leq \sum_{uv\in E(G)} \frac{d_u d_v}{\delta\delta}.$$

Thus, we have

$$GA_1(G) \leq \frac{M_2(G)}{\delta^2}.$$

If the graph is regular, then $M_2(G) = \delta^2 m$, the lower and upper bound are the same, and they are equal to $GA_1(G) = m$. If the equality holds in the lower bound, then $4(d_u+d_v)^{-2} = \Delta^{-2}$ for every $uv \in E(G)$; hence, $d_u = \Delta$ for every $u \in V(G)$ and the graph is regular. If the equality is attained in the upper bound, then $\frac{1}{2}(d_u+d_v) = \delta$ for every $uv \in E(G)$ and we conclude $d_u = \delta$ for every $u \in V(G)$. □

Theorem 1 improves the previous lower bound, as the following result shows.

*Remark* 1. For any $0 < \delta \leq \Delta$ we have

$$\frac{\sqrt{2\sqrt{\Delta\delta}(\Delta+\delta)}}{\Delta(\sqrt{\Delta}+\sqrt{\delta})^2} \geq \frac{1}{\Delta+\delta}\sqrt{\frac{\delta}{\Delta}}.$$

*Proof.* First of all, note that the polynomial $p(x) = 2x^6 - x^5 + 2x^4 - 6x^3 + 2x^2 - x + 2$ verifies $p(x) \geq 0$ if $x \geq 1$, since $p(x) = (x-1)^2(2x^4+3x^3+6x^2+3x+2)$. Hence,

$$2x^6 + 2x^4 + 2x^2 + 2 \geq x(x^4+6x^2+1), \quad if \;\; x \geq 1,$$

$$2\Big(\frac{\Delta}{\delta}\Big)^3 + 2\Big(\frac{\Delta}{\delta}\Big)^2 + 2\frac{\Delta}{\delta} + 2 \geq \sqrt{\frac{\Delta}{\delta}}\Big(\Big(\frac{\Delta}{\delta}\Big)^2 + 6\frac{\Delta}{\delta} + 1\Big),$$

$$2\Delta^3 + 2\Delta^2\delta + 2\Delta\delta^2 + 2\delta^3 \geq \Delta^2\sqrt{\Delta\delta} + 6\Delta\delta\sqrt{\Delta\delta} + \delta^2\sqrt{\Delta\delta},$$

$$2(\Delta+\delta)^3 \geq \sqrt{\Delta\delta}\big(\sqrt{\Delta}+\sqrt{\delta}\big)^4,$$

and this implies the inequality. □

In [2] appears the inequality

$$GA_1(G) \leq \frac{1}{2}M_1(G).$$

In [16] appears the following result

$$GA_1(G) \leq \frac{1}{2\delta}M_1(G).$$

Our next result improves both inequalities and also gives a lower bound of $GA_1$ involving the first Zagreb index.

**Theorem 2.** *We have for any graph $G$*

$$\frac{2\delta M_1(G)}{(\Delta+\delta)^2} \le GA_1(G) \le \sqrt{\frac{mM_1(G)}{2\delta}}.$$

*Furthermore, the equality in each inequality is attained if and only if $G$ is regular.*

*Proof.* By Lemma 1, taking $a=\delta$ and $b=\Delta$, we have

$$\frac{2\sqrt{\Delta\delta}}{\Delta+\delta} \le \frac{2\sqrt{d_u d_v}}{d_u+d_v} \le 1.$$

Hence,

$$1 \le \frac{(d_u+d_v)^2}{4d_u d_v} \le \frac{(\Delta+\delta)^2}{4\Delta\delta}, \qquad \frac{d_u+d_v}{2\sqrt{d_u d_v}} \le \frac{(\Delta+\delta)^2}{4\Delta\delta}\,\frac{2\sqrt{d_u d_v}}{d_u+d_v}.$$

Note that

$$M_1(G) = \sum_{v\in V(G)} d_v^2 = \sum_{uv\in E(G)} (d_u+d_v).$$

Thus, we have

$$\frac{M_1(G)}{2\Delta} = \sum_{uv\in E(G)} \frac{d_u+d_v}{2\Delta} \le \sum_{uv\in E(G)} \frac{d_u+d_v}{2\sqrt{d_u d_v}} \le \frac{(\Delta+\delta)^2}{4\Delta\delta} GA_1(G).$$

Therefore,

$$\frac{2\delta M_1(G)}{(\Delta+\delta)^2} \le GA_1(G).$$

Cauchy-Schwarz inequality and $\frac{1}{d_u+d_v} \le \frac{1}{2\delta}$ give

$$\begin{aligned} GA_1(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v} \le \sum_{uv\in E(G)} \frac{\sqrt{d_u+d_v}}{\sqrt{d_u+d_v}} \\ &\le \Big(\sum_{uv\in E(G)} (d_u+d_v)\Big)^{1/2} \Big(\sum_{uv\in E(G)} \frac{1}{d_u+d_v}\Big)^{1/2} \le \sqrt{\frac{mM_1(G)}{2\delta}}. \end{aligned}$$

If the graph is regular, then $M_1(G) = n\delta^2 = 2m\delta$, the lower and upper bound are the same, and they are equal to $GA_1(G) = m$. If the equality holds in the lower bound, then $\sqrt{d_u d_v} = \Delta$ for every $uv \in E(G)$; hence, $d_u = \Delta$ for every $u \in V(G)$ and $G$ is regular. If the equality is attained in the upper bound, then $d_u + d_v = 2\delta$ for every $uv \in E(G)$ and we conclude $d_u = \delta$ for every $u \in V(G)$. □

Note that Theorem 2 improves the upper bound $\frac{1}{2\delta} M_1(G)$: since $2\delta m \le M_1(G)$, we have

$$\sqrt{\frac{mM_1(G)}{2\delta}} \le \frac{1}{2\delta} M_1(G).$$

Recall that the variable Zagreb index is defined as $Z_\alpha(G) = \sum_{uv\in E(G)}(d_u d_v)^\alpha$. Note that $Z_{-1/2}$ is the usual Randić index $R(G)$, $Z_1$ is the second Zagreb index $M_2$, $Z_{-1}$ is the modified Zagreb index [12], etc.

Next, we obtain new inequalities relating the geometric-arithmetic index with the modified Zagreb index. We will need also the following lemmas.

**Lemma 3.** *We have for any graph $G$ and every $uv \in E(G)$*

$$\delta \le \frac{2d_u d_v}{d_u + d_v} \le \Delta,$$

*and the lower (or upper) bound is attained for every $uv \in E(G)$ if and only if $G$ is a regular graph.*

*Proof.* We know that

$$d_u(d_v - \delta) + d_v(d_u - \delta) \ge 0, \qquad d_u d_v + d_v d_u \ge d_u\delta + d_v\delta.$$

Therefore,

$$\delta \le \frac{2d_u d_v}{d_u + d_v}.$$

On the other hand, using the fact that the geometric mean is at most the arithmetic mean, we obtain

$$\frac{2d_u d_v}{d_u + d_v} \le \frac{(d_u + d_v)^2}{2(d_u + d_v)} = \frac{d_u + d_v}{2} \le \Delta.$$

If $G$ is a regular graph, then it is clear that both inequalities are attained for every $uv \in E(G)$. Assume that the lower bound is attained for every $uv \in E(G)$. Then $d_v - \delta = d_u - \delta = 0$ for every $uv \in E(G)$, and $d_v = \delta$ for every $v \in V(G)$. Hence, $G$ is a regular graph. If the upper bound is attained for every $uv \in E(G)$, then $4d_u d_v = (d_u + d_v)^2$ for every $uv \in E(G)$, $d_u = d_v$ for every $uv \in E(G)$, and we conclude that $G$ is regular since it is connected. □

In order to prove our next result we need the following particular case of Jensen's inequality.

**Lemma 4.** *If $f$ is a convex function in $\mathbb{R}_+$ and $x_1, \dots, x_m > 0$, then*

$$f\Big(\frac{x_1 + \cdots + x_m}{m}\Big) \le \frac{1}{m}\big(f(x_1) + \cdots + f(x_m)\big).$$

**Theorem 3.** *We have for any graph $G$*

$$\sqrt{\frac{m^3}{\Delta^2 Z_{-1}(G)}} \le GA_1(G) \le \Delta^2 Z_{-1}(G).$$

*Furthermore, the equality in each inequality is attained if and only if $G$ is regular.*

*Proof.* Using Lemma 3, we have

$$\delta \leq \frac{2d_u d_v}{d_u + d_v} \leq \Delta,$$

$$\frac{\delta}{d_u d_v} \leq \frac{2}{d_u + d_v} \leq \frac{\Delta}{d_u d_v}.$$

Moreover,

$$\sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \leq \Delta \sum_{uv \in E(G)} \frac{2}{d_u + d_v} \leq \Delta \sum_{uv \in E(G)} \frac{\Delta}{d_u d_v} = \Delta^2 Z_{-1}(G).$$

Since $f(x) = 1/x^2$ is a convex function in $\mathbb{R}_+$, Lemma 4 gives

$$\frac{m^2}{\left(\sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}\right)^2} \leq \frac{1}{m} \sum_{uv \in E(G)} \frac{(d_u + d_v)^2}{4 d_u d_v} \leq \frac{\Delta^2}{m} \sum_{uv \in E(G)} \frac{1}{d_u d_v},$$

$$\frac{m^2}{(GA_1(G))^2} \leq \frac{\Delta^2}{m} Z_{-1}(G).$$

If the graph is regular, then $Z_{-1}(G) = m\Delta^{-2}$, the lower and upper bound are the same, and they are equal to $GA_1(G) = m$. If the equality holds in the upper bound, then $\sqrt{d_u d_v} = \Delta$ for every $uv \in E(G)$; hence, $d_u = \Delta$ for every $u \in V(G)$ and the graph is regular. If the equality is attained in the lower bound, then $(d_u + d_v)^2 = 4\Delta^2$ for every $uv \in E(G)$ and we conclude $d_u = \Delta$ for every $u \in V(G)$. □

The following result gives a lower and upper bounds for $GA_1$ involving the Zagreb indices $M_2(G)$ and $Z_{-1}(G)$.

**Theorem 4.** *We have for any graph $G$*

$$\frac{2\delta^2 \sqrt{M_2(G) Z_{-1}(G)}}{\Delta^2 + \delta^2} \leq GA_1(G) \leq \sqrt{M_2(G) Z_{-1}(G)},$$

*and the equality in each inequality is attained if and only if $G$ is a regular graph.*

*Proof.* Lemma 3 gives

$$\delta \leq \frac{2d_u d_v}{d_u + d_v} \leq \Delta.$$

It is easy to see that

$$\frac{1}{\Delta} \leq \frac{1}{\sqrt{d_u d_v}} \leq \frac{1}{\delta}.$$

Thus Lemma 2 gives

$$\begin{aligned}
GA_1(G) &= \sum_{uv\in E(G)} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u+d_v)} = \sum_{uv\in E(G)} \frac{2d_u d_v}{d_u+d_v}\frac{1}{\sqrt{d_u d_v}} \\
&\geq \frac{\left(\sum_{uv\in E(G)} \frac{4(d_u d_v)^2}{(d_u+d_v)^2}\right)^{1/2}\left(\sum_{uv\in E(G)} \frac{1}{d_u d_v}\right)^{1/2}}{\frac{1}{2}\left(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\right)} \\
&\geq \frac{2\Delta\delta\sqrt{Z_{-1}(G)}}{\Delta^2+\delta^2}\Big(\sum_{uv\in E(G)} \frac{4(d_u d_v)^2}{(d_u+d_v)^2}\Big)^{1/2} \\
&\geq \frac{2\Delta\delta\sqrt{Z_{-1}(G)}}{\Delta^2+\delta^2}\frac{2\delta}{2\Delta}\Big(\sum_{uv\in E(G)} d_u d_v\Big)^{1/2} \\
&= \frac{2\delta^2\sqrt{M_2(G)Z_{-1}(G)}}{\Delta^2+\delta^2}.
\end{aligned}$$

Since

$$4d_u d_v \leq (d_u+d_v)^2,$$

Cauchy-Schwarz inequality gives

$$\begin{aligned}
GA_1(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v} \leq \Big(\sum_{uv\in E(G)} d_u d_v\Big)^{1/2}\Big(\sum_{uv\in E(G)} \frac{4}{(d_u+d_v)^2}\Big)^{1/2} \\
&\leq \Big(\sum_{uv\in E(G)} d_u d_v\Big)^{1/2}\Big(\sum_{uv\in E(G)} \frac{1}{d_u d_v}\Big)^{1/2} = \sqrt{M_2(G)Z_{-1}(G)}.
\end{aligned}$$

If the graph is regular, then the lower and upper bound are the same, and they are equal to $GA_1(G)$. If the equality holds in the lower bound, then $d_u d_v = \delta^2$ for every $uv \in E(G)$; hence, $d_u = \delta$ for every $u \in V(G)$ and the graph is regular. If the equality is attained in the upper bound, then $4d_u d_v = (d_u+d_v)^2$ for every $uv \in E(G)$, $d_u = d_v$ for every $uv \in E(G)$, and we conclude that $G$ is regular since it is connected. □

Next, we obtain inequalities relating the geometric-arithmetic index with the Zagreb indices $M_1(G)$, $M_2(G)$ and $Z_{-1}(G)$. Recall that a $(\Delta,\delta)$-biregular graph is a bipartite graph for which any vertex in one side of the given bipartition has degree $\Delta$ and any vertex in the other side of the bipartition has degree $\delta$.

**Theorem 5.** *For any graph $G$, $\frac{\Delta\delta\sqrt{(nM_1(G)+4M_2(G)-4m^2)Z_{-1}(G)}}{\Delta^2+\delta^2} \leq GA_1(G) \leq \frac{\sqrt{(nM_1(G)+4M_2(G)-4m^2)Z_{-1}(G)}}{2}$. The equality in the lower bound is attained if and only if $G$ is either regular or $(\Delta,\delta)$-biregular; the equality in the upper bound is attained if and only if $G$ is regular.*

*Proof.* We have by Lemma 1

$$\frac{4\Delta\delta}{(\Delta+\delta)^2} \leq \frac{4d_u d_v}{(d_u+d_v)^2} \leq 1.$$

Hence,

$$\frac{2\sqrt{d_u d_v}}{d_u+d_v} \leq \frac{d_u+d_v}{2\sqrt{d_u d_v}} \tag{2.1}$$

and

$$\frac{2\sqrt{d_u d_v}}{d_u+d_v} \geq \frac{4\Delta\delta}{(\Delta+\delta)^2}\frac{d_u+d_v}{2\sqrt{d_u d_v}}. \tag{2.2}$$

We have by (2.1)

$$GA_1(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u+d_v} \leq \sum_{uv\in E(G)} \frac{d_u+d_v}{2\sqrt{d_u d_v}}.$$

Moreover,

$$\begin{aligned}\sum_{uv\in E(G)} (d_u+d_v)^2 &= \sum_{uv\in E(G)} d_u^2 + 2\sum_{uv\in E(G)} d_u d_v + \sum_{uv\in E(G)} d_v^2 \\ &= \sum_{uv\in E(G)} (d_u^2+d_v^2) + 2\sum_{uv\in E(G)} d_u d_v \\ &= \sum_{uv\in E(G)} (d_u^2+d_v^2) + 2M_2(G).\end{aligned}$$

Since

$$\sum_{uv\in E(G)} (d_u^2+d_v^2) = \sum_{v\in V(G)} d_v \cdot d_v^2 = \sum_{v\in V(G)} d_v^3,$$

we have

$$\sum_{uv\in E(G)} (d_u+d_v)^2 = \sum_{v\in V(G)} d_v^3 + 2M_2(G). \tag{2.3}$$

Using Cauchy-Schwarz inequality and the formula

$$\sum_{v\in V(G)} d_v^3 = nM_1(G) + 2M_2(G) - 4m^2 \tag{2.4}$$

(see, e.g., [21]), we obtain

$$\begin{aligned}\sum_{uv\in E(G)}\frac{d_u+d_v}{2\sqrt{d_ud_v}} &\le \Big(\sum_{uv\in E(G)}\frac{(d_u+d_v)^2}{4}\Big)^{1/2}\Big(\sum_{uv\in E(G)}\frac{1}{d_ud_v}\Big)^{1/2}\\ &= \frac{1}{2}\Big(\sum_{v\in V(G)}d_v^3+2M_2(G)\Big)^{1/2}\sqrt{Z_{-1}(G)}\\ &= \frac{\sqrt{(nM_1(G)+4M_2(G)-4m^2)Z_{-1}(G)}}{2}.\end{aligned}$$

Let us prove the lower bound. We have by (2.2)

$$GA_1(G)\ge\frac{4\delta\Delta}{(\delta+\Delta)^2}\sum_{uv\in E(G)}\frac{d_u+d_v}{2\sqrt{d_ud_v}}.$$

Since

$$\delta\le\frac{d_u+d_v}{2}\le\Delta,\qquad \frac{1}{\Delta}\le\frac{1}{\sqrt{d_ud_v}}\le\frac{1}{\delta},$$

Lemma 2, (2.3) and (2.4) give

$$\begin{aligned}\sum_{uv\in E(G)}\frac{d_u+d_v}{2\sqrt{d_ud_v}} &\ge \frac{\Big(\sum_{uv\in E(G)}\frac{(d_u+d_v)^2}{4}\Big)^{1/2}\Big(\sum_{uv\in E(G)}\frac{1}{d_ud_v}\Big)^{1/2}}{\frac{1}{2}\Big(\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\Big)}\\ &\ge \frac{\Delta\delta\sqrt{\big(\sum_{v\in V(G)}d_v^3+2M_2(G)\big)Z_{-1}(G)}}{\Delta^2+\delta^2}\\ &= \frac{\Delta\delta\sqrt{(nM_1(G)+4M_2(G)-4m^2)Z_{-1}(G)}}{\Delta^2+\delta^2}.\end{aligned}$$

If the graph is regular, then the lower and upper bound are the same, and they are equal to $GA_1(G)$.
If the equality in the lower bound is attained, then (2.2) holds for every $uv\in E(G)$; by Lemma 1 we have either $d_u=\Delta$ and $d_v=\delta$, or viceversa, for each $uv\in E(G)$. This happens if and only if $G$ is a regular graph if $\Delta=\delta$ or a $(\Delta,\delta)$-biregular graph otherwise.
If the equality in the upper bound is attained, then (2.1) holds for every $uv\in E(G)$; by Lemma 1 we have $d_u=d_v$ for each $uv\in E(G)$, and $G$ is a regular graph. □

In [16] we find the bound

$$GA_1(G)\le\Delta R(G), \tag{2.5}$$

where $R(G)$ is the Randić index. We recall that $R(G)$ is the best know topological index.

The following result provides lower and upper bounds of $GA_1$ involving the Randić index. Furthermore, it improves (2.5).

**Theorem 6.** *We have for any graph $G$*

$$\delta R(G) \leq GA_1(G) \leq \sqrt{\Delta m R(G)},$$

*and the equality in each inequality holds if and only if $G$ is regular.*

*Proof.* Since $f(x) = x^2$ is a convex function in $\mathbb{R}_+$, Lemma 4 gives

$$\frac{1}{m^2} GA_1(G)^2 = \frac{1}{m^2}\Big(\sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}\Big)^2 \leq \frac{1}{m} \sum_{uv\in E(G)} \frac{4 d_u d_v}{(d_u + d_v)^2}.$$

Lemma 3 gives

$$\begin{aligned}
\frac{2 d_u d_v}{d_u + d_v} &\leq \Delta, \\
\frac{2 d_u d_v (2\sqrt{d_u d_v})}{(d_u + d_v)^2} &\leq \frac{2 d_u d_v (d_u + d_v)}{(d_u + d_v)^2} = \frac{2 d_u d_v}{d_u + d_v} \leq \Delta, \\
\frac{4 d_u d_v}{(d_u + d_v)^2} &\leq \frac{\Delta}{\sqrt{d_u d_v}}, \\
\frac{1}{m^2} GA_1(G)^2 &\leq \frac{1}{m} \sum_{uv\in E(G)} \frac{4 d_u d_v}{(d_u + d_v)^2} \leq \frac{\Delta}{m} \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}.
\end{aligned}$$

Hence,

$$GA_1(G) \leq \sqrt{\Delta m R(G)}.$$

In order to prove the lower bound, notice that Lemma 3 gives

$$\delta \leq \frac{2 d_u d_v}{d_u + d_v} = \frac{2\sqrt{d_u d_v}\sqrt{d_u d_v}}{d_u + d_v}.$$

Hence,

$$\begin{aligned}
\sum_{uv\in E(G)} \frac{\delta}{\sqrt{d_u d_v}} &\leq \sum_{uv\in E(G)} \frac{\sqrt{d_u d_v}}{\frac{1}{2}(d_u + d_v)}, \\
\delta R(G) &\leq GA_1(G).
\end{aligned}$$

If the graph is regular, then $R(G) = \frac{m}{\delta}$, the lower and upper bound are the same, and they are equal to $GA_1(G) = m$.
If the equality in the lower bound is attained, then

$$\delta = \frac{2 d_u d_v}{d_u + d_v}$$

for every $uv \in E(G)$. By Lemma 3, the graph $G$ is regular.

If the equality in the upper bound is attained, then

$$\Delta = \frac{2d_u d_v}{d_u + d_v}$$

for every $uv \in E(G)$, and $G$ is regular by Lemma 3. □

Note that Theorem 6 improves the upper bound $GA_1(G) \leq \Delta R(G)$: since $\frac{m}{\Delta} \leq R(G)$, we have

$$\sqrt{\Delta m R(G)} \leq \Delta R(G).$$

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# INVERSION FORMULAS FOR GRAPHS

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*Abstract.* In this note we derive some combinatorial identities from inversion formulas in the completion of the dual of graph Hopf algebra. As a consequence some identities involving Stirling and Bell numbers are obtained.

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*Keywords:* graph, Hopf algebra, Stirling and Bell numbers

## 1. Introduction

The graph Hopf algebra $\mathcal{G}$, introduced by Schmitt [3] as the incidence Hopf algebra of graphs, is a well known example of a combinatorial Hopf algebra. It is also called chromatic Hopf algebra in [1], because of its relation with chromatic polynomials of graphs.

The graded Hopf algebra dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$ is isomorphic to the polynomial Hopf algebra $k[\mathbb{G}_{\text{conn}}]$ generated by connected simple graphs. This determines the completion $\widehat{\mathcal{G}}^*$ of the graded dual of the graph Hopf algebra as the Hopf algebra of formal power series $k[[\mathbb{G}_{\text{conn}}]]$ in variables corresponding to connected graphs. For a class of graphs $\mathcal{C}$ is defined the characteristic element $c(\mathcal{C}) \in \widehat{\mathcal{G}}^*$. We define elements which count the numbers of ordered and unordered decompositions of graphs onto subgraphs from the class $\mathcal{C}$. By using inversion formulas in the algebra $\widehat{\mathcal{G}}^*$ we obtain some combinatorial identities for graphs which are satisfied by these numbers. As a consequence, some numerical identities involving the Stirling numbers of the second kind and the ordered Bell numbers are obtained.

## 2. Hopf algebra of graphs $\mathcal{G}$

For a graph $\Gamma$ denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges. All graphs considered are simple, i.e. without multiple edges and loops. By $|\Gamma|$ we denote the number of vertices of a graph $\Gamma$. Let $\Gamma|_I$ be the induced subgraph of a graph $\Gamma$ on the set of vertices $I \subset V(\Gamma)$.

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For graphs $\Gamma$ and $\Gamma_1,\dots,\Gamma_k$, define $\binom{\Gamma}{\Gamma_1,\dots,\Gamma_k}$ to be the number of ordered set partitions $I : I_1 \sqcup \dots \sqcup I_k = V(\Gamma)$ such that $\Gamma|_{I_j}$ is isomorphic to $\Gamma_j$ for all $j = 1,\dots,k$.

2.1. *Hopf algebras*

For a detailed exposition of Hopf algebras see [5]. Fix a field $k$. A bialgebra $\mathcal{H}$ is a vector space over $k$ equipped with linear maps

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \text{ and } \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H},$$

respectively the multiplication and comultiplication, such that the following properties are satisfied:

(1) $(\mathcal{H}, m, u)$ is an associative algebra with the unit $u : k \to \mathcal{H}$

(2) $(\mathcal{H}, \Delta, \epsilon)$ is a coassociative coalgebra with the counit $\epsilon : \mathcal{H} \to k$

(3) $\Delta$ and $\epsilon$ are multiplicative morphisms (equivalently, $m$ and $u$ are comultiplicative morphisms). If there exists a bialgebra automorphism $S : \mathcal{H} \to \mathcal{H}$ such that

$$m \circ (S \otimes I) \circ \Delta = m \circ (I \otimes S) \circ \Delta = u \circ \epsilon,$$

where $I : \mathcal{H} \to \mathcal{H}$ is the identity map, then $\mathcal{H}$ is a Hopf algebra and $S$ is its antipode. A Hopf algebra $\mathcal{H}$ is graded if $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ and the multiplication and comultiplication respect this decomposition

$$m(\mathcal{H}_i \otimes \mathcal{H}_j) \subset \mathcal{H}_{i+j},\ \Delta(\mathcal{H}_k) \subset \sum_{i+j=k} \mathcal{H}_i \otimes \mathcal{H}_j.$$

$\mathcal{H}$ is connected if $dim(\mathcal{H}_0) = 1$. A graded connected bialgebra $\mathcal{H}$ posses the antipode $S$ determined recursively as follows: $S(h) = h$ for $h \in \mathcal{H}_0$, and $(m \circ (S \otimes I) \circ \Delta)(h) = 0$ for $h \in \mathcal{H}_i, i > 0$.

*Example* 1. Let $X = \{x_1, x_2, \dots\}$ be a countable set with the rank function $rk : X \to \mathbb{N}$ and $k[X]$ be the polynomial algebra over a field $k$ generated by $X$. Define the comultiplication $\Delta : k[X] \to k[X] \otimes k[X]$ on variables by $\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1, n \geq 1$ and extend it algebraically to $k[X]$. It turns $k[X]$ into a graded, connected Hopf algebra over $k$. The antipode $S : k[X] \to k[X]$ is uniquely determined by $S(x_n) = -x_n$ for all $n$. The Hopf algebra $k[X]$ is called the *polynomial Hopf algebra* generated by the set $X$.

Recall the definition of the graph Hopf algebra $\mathcal{G}$, introduced in [3]. It is linearly spanned by all isomorphism classes of finite simple graphs. A graduation $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_n$ is given by the number of vertices. The space $\mathcal{G}$ is a Hopf algebra with the multiplication defined by disjoint union of graphs $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$ and the comultiplication

$$\Delta(\Gamma) = \sum_{I \subset V(\Gamma)} \Gamma|_I \otimes \Gamma|_{V(\Gamma) \setminus I}. \tag{2.1}$$

The Hopf algebra $\mathcal{G}$ is graded, connected, commutative and cocommutative. The antipode $S : \mathcal{G} \to \mathcal{G}$ is determined by Takeuchi's general formula

$$S(\Gamma) = \sum_{k\geq 1}(-1)^k \sum_{J_1\sqcup\ldots\sqcup J_k = V(\Gamma)} \prod_{j=1}^{k} \Gamma|_{J_j},$$

where the inner sum is over all ordered set partitions $(J_1, \ldots, J_k)$ of the set of vertices $V(\Gamma)$. A more combinatorial formula involving acyclic orientations on graphs is obtained recently in [2].

$\mathcal{G}$ is algebraically isomorphic to the polynomial algebra $k[\mathbb{G}_{\text{conn}}]$ generated by the family of all isomorphism classes of connected simple graphs.

Let $\mathcal{G}^* = \bigoplus_{n\geq 0} \mathcal{G}_n^*$ be the graded dual of the Hopf algebra of graphs $\mathcal{G}$. If we denote the value of $\delta \in \mathcal{G}^*$ on $\Gamma \in \mathcal{G}$ by $\langle \delta, \Gamma\rangle$, the multiplication and comultiplication on $\mathcal{G}^*$ are determined by the identities

$$\langle \delta\eta, \Gamma\rangle = m \circ (\delta \otimes \eta) \circ \Delta(\Gamma),$$
$$\langle \Delta(\delta), \Gamma_1 \otimes \Gamma_2\rangle = \delta(\Gamma_1\Gamma_2).$$

The Hopf algebra $\mathcal{G}^*$ is commutative and cocommutative with the set of linear generators formed by all $\delta_\Gamma$, where

$$\langle \delta_\Gamma, \Gamma'\rangle = \begin{cases} 1, & \Gamma = \Gamma' \\ 0, & \text{otherwise} \end{cases}.$$

The following product formula holds

$$\delta_{\Gamma_1}\cdots\delta_{\Gamma_k} = \sum_{|\Gamma|=n} \binom{\Gamma}{\Gamma_1, \ldots, \Gamma_k} \delta_\Gamma, \tag{2.2}$$

where $n = |\Gamma_1| + \cdots + |\Gamma_k|$. Note that taking duals is not an algebra morphism since $\delta_{\Gamma_1}\delta_{\Gamma_2} \neq \delta_{\Gamma_1\Gamma_2}$, but the restriction to connected graphs generates the algebra morphism $\Phi : k[\mathbb{G}_{\text{conn}}] \to \mathcal{G}^*$. It is actually an isomorphism of Hopf algebras which is a consequence of Schmitt's work on Whitney systems [4].

**Theorem 1.** *The graded dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$ is isomorphic to the polynomial Hopf algebra $k[\mathbb{G}_{\text{conn}}]$ generated by connected simple graphs.*

## 3. Inversion formulas

Let $\widehat{\mathcal{G}}^*$ be the completion of the graded dual $\mathcal{G}^*$ of the graph Hopf algebra $\mathcal{G}$. Theorem 1 implies that $\widehat{\mathcal{G}}^*$ is isomorphic to the Hopf algebra of formal power series $k[[\mathbb{G}_{\text{conn}}]]$. Given an element $x \in \widehat{\mathcal{G}}^*$, by $[n]x$ we denote its $n$-th homogeneous summand.

For a class of graphs $\mathcal{C}$ let $c(\mathcal{C}) = \sum_{\Gamma\in\mathcal{C}} \delta_\Gamma$ be its characteristic element

$$c(\mathcal{C})(\Gamma) = \begin{cases} 1, & \Gamma \in \mathcal{C} \\ 0, & \text{otherwise} \end{cases}.$$

Define

$$u(\mathcal{C}) = \sum_{k\geq 0} c(\mathcal{C})^k = \frac{\epsilon}{\epsilon - c(\mathcal{C})} \quad \text{and} \quad \exp(c(\mathcal{C})) = \sum_{k\geq 0} \frac{1}{k!} c(\mathcal{C})^k.$$

These elements have the following combinatorial meanings.

**Lemma 1.** *The values $u(\mathcal{C})(\Gamma)$ and $\exp(c(\mathcal{C}))(\Gamma)$ are the numbers of all ordered and unordered decompositions of a graph $\Gamma$ onto subgraphs from the class $\mathcal{C}$.*

*Proof.* The $n$-th homogeneous summand of the $k$-power of the characteristic element $c(\mathcal{C})$ is given by

$$[n]c(\mathcal{C})^k = \sum_{\substack{\Gamma_1,\dots,\Gamma_k \in \mathcal{C} \\ |\Gamma_1|+\cdots+|\Gamma_k| = n}} \delta_{\Gamma_1}\cdots\delta_{\Gamma_k}.$$

Therefore, for a graph $\Gamma$ on $n$ vertices we have by the product formula (2.2)

$$c(\mathcal{C})^k(\Gamma) = \sum_{\substack{\Gamma_1,\dots,\Gamma_k \in \mathcal{C} \\ |\Gamma_1|+\cdots+|\Gamma_k| = n}} \binom{\Gamma}{\Gamma_1\cdots\Gamma_k}, \tag{3.1}$$

which is precisely the number of all ordered decompositions of the graph $\Gamma$ onto $k$ subgraphs from the class $\mathcal{C}$. The lemma follows from definitions of $u(\mathcal{C})$ and $\exp(c(\mathcal{C}))$. □

**Theorem 2.** *Given a class $\mathcal{C}$ of simple graphs and a graph $\Gamma$ on the vertex set $V \neq \varnothing$, then*

$$c(\mathcal{C})(\Gamma) = \sum_{I_1\sqcup\ldots\sqcup I_k = V} (-1)^{k-1} \prod_{j=1}^{k} u(\mathcal{C})(\Gamma|_{I_j}), \tag{3.2}$$

$$c(\mathcal{C})(\Gamma) = \sum_{I_1\sqcup\ldots\sqcup I_k = V} \frac{(-1)^{k-1}}{k} \prod_{j=1}^{k} \exp(c(\mathcal{C}))(\Gamma|_{I_j}), \tag{3.3}$$

*where the sums are over all ordered decompositions of $V$.*

*Proof.* The statement follows from the inversion formulas

$$c(\mathcal{C}) = \frac{u(\mathcal{C})-\epsilon}{u(\mathcal{C})} = \sum_{k>0} (-1)^{k-1}(u(\mathcal{C})-\epsilon)^k,$$

$$c(\mathcal{C}) = \log(\epsilon + (\exp(c(\mathcal{C}))-\epsilon)) = \sum_{k>0} \frac{(-1)^{k-1}}{k}(\exp(c(\mathcal{C}))-\epsilon)^k,$$

by calculating at the graph $\Gamma$. □

For an integer $n$ let $\alpha = (i_1, \dots, i_k) \models n$ be a composition, i.e. $i_j, j = 1, \dots, k$ are positive integers and $i_1 + \dots + i_k = n$. Denote by $\alpha(j) = i_j$ the $j$-th part of $\alpha$ and by $k(\alpha) = k$ the number of its parts. Let $\binom{n}{\alpha} = \frac{n!}{i_1! i_2! \cdots i_k!}$ be the multinomial coefficient. It counts the number of all ordered set partitions $(I_1, \dots, I_k)$ of the finite set $V$ with prescribed sizes of parts $|I_1| = i_1, \dots, |I_k| = i_k$. Recall that a *proper coloring* of a graph $\Gamma$ with at most $k$ colors is the map $f : V(\Gamma) \to [k]$ such that there are no monochromatic edges, i.e. $f(v_1) \neq f(v_2)$ if $\{v_1, v_2\}$ is an edge of $\Gamma$. The chromatic polynomial $\chi(\Gamma, k)$ counts the numbers of proper colorings of a graph $\Gamma$.

**Theorem 3.**

$$\sum_{I_1 \sqcup \dots \sqcup I_k = V(\Gamma)} (-1)^{k-1} \prod_{j=1}^{k} \chi(\Gamma|_{I_j}, |I_j|) = \begin{cases} 1, & \Gamma \text{ is discrete} \\ 0, & \text{otherwise} \end{cases}, \tag{3.4}$$

*where the sum is over all ordered set partitions $(I_1, \dots, I_k)$ of the vertex set $V(\Gamma)$.*

*Proof.* Let $\mathcal{C} = \{D_n\}_{n>0}$ be the class of all discrete graphs. Note that $\binom{\Gamma}{D_{i_1} \cdots D_{i_k}}$ is the number of all proper colorings of $\Gamma$ with exactly $k$ colors such that a color $j$ is taken $i_j$ times, for all $j = 1, \dots, k$. Therefore, from (3.1), we have that $c(\mathcal{C})^k(\Gamma)$ counts the number of all proper colorings with exactly $k$ colors and $(u(\mathcal{C}) - \epsilon)(\Gamma) = \chi(\Gamma, |\Gamma|)$. We apply the inversion formula (3.2) to obtain (3.4). □

Recall that the Stirling number of the second kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{\alpha \models n : k(\alpha) = k} \binom{n}{\alpha}$$

is the number of all set partitions of an $n$-element set into $k$ parts. The ordered Bell number

$$F(n) = \sum_{\alpha \models n} \binom{n}{\alpha}$$

counts the total number of ordered set partitions on an $n$-element set. The following corollaries illustrate how Theorems 2 and 3 lead to some identities involving Stirling and Bell numbers.

**Corollary 1.** *The following identity holds for an integer $n$*

$$\sum_{\alpha \models n} \frac{(-1)^{k(\alpha)-1}}{k(\alpha)} \binom{n}{\alpha} = 0. \tag{3.5}$$

*Proof.* Let $\mathcal{C} = \{D_1\}$, where $D_1$ is the graph on the single vertex. Then $c(\mathcal{C}) = \delta_{D_1}$ and $\exp(\delta_{D_1})(\Gamma) = 1$ for any graph $\Gamma$. The identity follows from the inversion formula (3.3) applied on an arbitrary graph $\Gamma$ on $n \neq 1$ vertices. □

Note that the identity (3.5) may be rewritten as

$$\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0.$$

**Corollary 2.** *Given an integer $n$, the following identity holds*

$$\sum_{\alpha\models n}(-1)^{k(\alpha)-1}\binom{n}{\alpha}\prod_{j=1}^{k(\alpha)}F(\alpha(j))) = 1.$$

*Proof.* Set $\Gamma = D_n$ into the formula (3.4) and note that $\chi(D_j, j) = F(j)$. □

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# BALANCING, PELL AND SQUARE TRIANGULAR FUNCTIONS

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*Abstract.* In this work, we derive some functions on balancing, cobalancing, Lucas–balancing, Lucas–cobalancing, Pell, Pell–Lucas and square triangular numbers. At the end of this article we investigated common values of combinatorial numbers and Lucas–balancing numbers.

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## 1. Introduction

A positive integer $n$ is called a balancing number (see [1] and [3]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer $r$ which is called cobalancing number (or balancer). For example $6, 35, 204, 1189$ and $6930$ are balancing numbers with balancers $2, 14, 84$, $492$ and $2870$, respectively. If $n$ is a balancing number with balancer $r$, then from (1.1) one has $\frac{(n-1)n}{2}=rn+\frac{r(r+1)}{2}$, and so

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2} \quad \text{and} \quad n=\frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{1.2}$$

Let $B_n$ denote the $n^{\text{th}}$ balancing number, and let $b_n$ denote the $n^{\text{th}}$ cobalancing number. Then they satisfy $B_1=1, B_2=6, B_{n+1}=6B_n-B_{n-1}$ and $b_1=0, b_2=2, b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$. The zeros of the characteristic equation $x^2-6x+1=0$ for balancing numbers are $\alpha_1=3+\sqrt{8}$ and $\beta_1=3-\sqrt{8}$. Ray derived some nice results on balancing and cobalancing numbers in his Phd thesis [14, p.19]. Since $x$ is a balancing number if and only if $8x^2+1$ is a perfect square, he set $y^2-8x^2=1$ for some $y\neq 0$, which is a Pell equation. The fundamental solution is $(y_1,x_1)=(3,1)$. So $y_n+x_n\sqrt{8}=(3+\sqrt{8})^n$ and $y_n-x_n\sqrt{8}=(3-\sqrt{8})^n$ for $n\geq 1$.

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Thus $x_n = \frac{(3+\sqrt{8})^n-(3-\sqrt{8})^n}{2\sqrt{8}}$ which is the Binet formula for balancing numbers and is denoted by $B_n$.

Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ be the roots of the characteristic equation for Pell and Pell–Lucas numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$, respectively. Notice that $\alpha^2 = 3+\sqrt{8}$ and $\beta^2 = 3-\sqrt{8}$. So the Binet formula for balancing numbers is $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$. Thus there is a connection between balancing and Pell numbers by $B_n = \frac{P_{2n}}{2}$. Similarly, the Binet formula for cobalancing numbers is $b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$, and so $b_n = \frac{P_{2n-1}-1}{2}$.

We note that $B_n$ is a balancing number if and only if $8B_n^2+1$ is a perfect square, and $b_n$ is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus (1.2) implies that

$$C_n = \sqrt{8B_n^2+1} \text{ and } c_n = \sqrt{8b_n^2+8b_n+1} \tag{1.3}$$

are integers called the $n^{\text{th}}$ Lucas–balancing and $n^{\text{th}}$ Lucas–cobalancing number, respectively. Their Binet formulas are $C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$, respectively (for further details see [9, 11–13]).

Balancing numbers and their generalizations have been investigated by several authors, from many aspects. In [5, Theorem 4], Liptai proved that there is no Fibonacci balancing number except 1, and in [6, Theorem 4] he proved that there is no Lucas balancing number. In [15], Szalay considered the same problem and obtained some nice results by a different method. In [4], Kovács, Liptai and Olajos extended the concept of balancing numbers to the $(a,b)$-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a+b)+\cdots+(a(n-1)+b) = (a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some positive integers $n$ and $r$, then $an+b$ is an $(a,b)$-balancing number. The sequence of $(a,b)$-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$.

In [7], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y,k,l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer $x$ with $x \leq y-2$ is called a $(k,l)$-power numerical center for $y$ if

$$1^k+\cdots+(x-1)^k = (x+1)^l+\cdots+(y-1)^l.$$

They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for $(k,l)$-power numerical centers.

For positive integers $k,x$, let

$$\Pi_k(x) = x(x+1)\ldots(x+k-1). \tag{1.4}$$

It was proved in [4, Theorem 3 and Theorem 4] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2,3,4\}$ all solutions were determined. In [17, Theorem 1] Tengely, considered the case $k = 5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$, and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$.

Recently in [2], Dash, Ota and Dash considered the so-called $t$-balancing numbers for an integer $t \geq 2$. A positive integer $n$ is a $t$-balancing number if

$$1+2+\cdots+n = (n+1+t)+(n+2+t)+\cdots+(n+r+t)$$

holds for some positive integer $r$ which is called $t$-cobalancing number. The $t$-balancing numbers are denoted by $B_n^t$ and $t$-cobalancing numbers by $b_n^t$. In [16], Tekcan, Tayat and Özbek derived the general terms of $t$-balancing numbers by solving the Diophantine equation $8x^2 - y^2 + 8x(1+t) + (2t+1)^2 = 0$.

Balancing functions are also interesting in the theory of balancing numbers. There were first considered by Behera, Panda and Ray in [1, 14], who derived the following results.

**Lemma 1** ([1, Theorem 2.1]). *For any balancing number $x$, the functions*

$$F(x) = 2x\sqrt{8x^2+1},\ \ G(x) = 3x + \sqrt{8x^2+1} \text{ and } H(x) = 17x + 6\sqrt{8x^2+1}$$

*are also balancing numbers.*

**Lemma 2** ([1, Theorem 2.2]). *If $x$ any balancing number, then*

$$K(x) = 6x\sqrt{8x^2+1} + 16x^2 + 1$$

*is an odd balancing number.*

**Lemma 3** ([1, Theorem 4.1]). *If $x$ and $y$ are balancing numbers, then*

$$f(x,y) = x\sqrt{8y^2+1} + y\sqrt{8x^2+1}$$

*is also a balancing number.*

**Lemma 4** ([14, 2.4.12 Theorem]). *If $x, y$ and $z$ are balancing numbers, then*

$$f(x,y,z) = x\sqrt{8y^2+1}\sqrt{8z^2+1} + y\sqrt{8x^2+1}\sqrt{8z^2+1} + z\sqrt{8x^2+1}\sqrt{8y^2+1} + 8xyz$$

*is also a balancing number.*

Similarly to balancing numbers, Panda and Ray [12] defined the following functions for any two cobalancing numbers $x$ and $y$:

$$f(x) = 3x + \sqrt{8x^2+8x+1} + 1,$$

$$g(x) = 17x + 6\sqrt{8x^2+8x+1} + 8,$$

$$h(x) = 8x^2 + 8x + 1 + (2x+1)\sqrt{8x^2+8x+1},$$
$$t(x,y) = \frac{1}{2}\Big[2(2x+1)(2y+1) + (2x+1)\sqrt{8y^2+8y+1} + (2y+1)\sqrt{8x^2+8x+1} + \sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1} - 1\Big].$$

They proved the following two lemmas.

**Lemma 5** ([12, Theorem 2.1])**.** *For any two cobalancing numbers $x$ and $y$, $f(x)$, $g(x)$, $h(x)$ and $t(x,y)$ are all cobalancing numbers.*

**Lemma 6** ([12, Theorem 2.2])**.** *If $x$ any cobalancing number, then the cobalancing number next to $x$ is $f(x) = 3x + \sqrt{8x^2+8x+1} + 1$, and consequently, the previous one is $\widetilde{f}(x) = 3x - \sqrt{8x^2+8x+1} + 1$.*

Here we note, [12] originally contained a type at $h(x)$, we always use the correct version.

## 2. Main results

In this section, we deal with balancing, Pell, Pell–Lucas and square triangular functions. Before considering balancing functions, we need the following theorem.

**Theorem 1.** *For any integers $n, k, l \geq 0$, we get*

$$B_{n+k+l} = B_k C_l C_n + C_k B_l C_n + C_k C_l B_n + 8 B_k B_l B_n. \tag{2.1}$$

*Proof.* It is known that $B_{n+k} = C_k B_n + B_k C_n$ and $C_{k+l} = C_k C_l + 8 B_k B_l$ for integers $n, l, k \geq 0$. So we easily get

$$\begin{aligned} &B_k C_l C_n + C_k B_l C_n + C_k C_l B_n + 8 B_k B_l B_n \\ &= C_n (B_k C_l + C_k B_l) + B_n (C_k C_l + 8 B_k B_l) \\ &= C_n B_{k+l} + B_n C_{k+l} \\ &= B_{n+k+l} \end{aligned}$$

as we wanted. □

In Lemmas 1–5, Behara, Panda and Ray did not determine precisely value of the functions $F(x)$, $G(x)$, $H(x)$, $K(x)$, $f(x,y)$, $f(x,y,z)$ and $f(x)$, $g(x)$, $h(x)$, $t(x,y)$. In the following theorem, we are able to do that.

**Theorem 2.** (1) *For the balancing functions $F(x), G(x), H(x), K(x), f(x,y)$, $f(x,y,z)$, we have*

$$F(B_n) = B_{2n},\ G(B_n) = B_{n+1},\ H(B_n) = B_{n+2},\ K(B_n) = B_{2n+1},$$
$$f(B_n, B_k) = B_{n+k} \text{ and } f(B_n, B_k, B_l) = B_{n+k+l}.$$

(2) *For the cobalancing functions* $f(x), g(x), h(x), t(x,y)$, *we have*

$$f(b_n) = b_{n+1},\ g(b_n) = b_{n+2},\ h(b_n) = b_{2n} \text{ and } t(b_n, b_m) = b_{n+m}.$$

*Proof.* (1) Since $\sqrt{8B_n^2+1} = C_n$, we easily get

$$F(B_n) = 2\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right) = \frac{\alpha^{4n}-\beta^{4n}}{4\sqrt{2}} = B_{2n}.$$

Taking $k = 1$ and $l = 0$ in (2.1), one obtains

$$B_{n+1} = 3B_n + C_n = 3B_n + \sqrt{8B_n^2+1} = G(B_n).$$

If $k = 2$ and $l = 0$, then it leads to

$$B_{n+2} = 17B_n + 6C_n = 17B_n + 6\sqrt{8B_n^2+1} = H(B_n)$$

and taking $k = 1$ and $l = n$, we get

$$B_{2n+1} = 6B_nC_n + 16B_n^2 + 1 = 6B_n\sqrt{8B_n^2+1} + 16B_n^2 + 1 = K(B_n).$$

Finally, if $l = 0$, we have

$$B_{n+k} = B_nC_k + B_kC_n = B_n\sqrt{8B_k^2+1} + B_k\sqrt{8B_n^2+1} = f(B_n, B_k).$$

The last assertions is obvious from (2.1) since

$$\begin{aligned} B_{n+k+l} &= B_nC_kC_l + B_kC_nC_l + B_lC_nC_k + 8B_nB_kB_l \\ &= B_n\sqrt{8B_k^2+1}\sqrt{8B_l^2+1} + B_k\sqrt{8B_n^2+1}\sqrt{8B_l^2+1} \\ &\quad + B_l\sqrt{8B_n^2+1}\sqrt{8B_k^2+1} + 8B_nB_kB_l \\ &= f(B_n, B_k, B_l). \end{aligned}$$

(2) It can be proved similarly. □

Apart from above theorem, we can give the following theorem.

**Theorem 3.** (1) *If $x$ is the $n^{th}$ balancing number and $y$ is the $n^{th}$ cobalancing number, then the function*

$$B_1(x,y) = \frac{-8xy + \sqrt{8x^2+1}\sqrt{8y^2+8y+1} + 1}{4}$$

*is the $n^{th}$ balancing number, and the function*

$$B_2(x,y) = x + 2y + \sqrt{8x^2+1} + \sqrt{8y^2+8y+1} + 1$$

*is the $(n+1)^{st}$ balancing number.*

(2) *(The generalized type of Lemma 2) Let $B_k$ denote the $k^{th}$ balancing number, let $b_k$ denote the $k^{tk}$ cobalancing number and let $C_k$ denote the $k^{tk}$ Lucas–balancing number for an integer $k \geq 0$. Then for any balancing number $x$, the function*

$$B_k^1(x) = B_k x\sqrt{8x^2+1} + C_k x^2 + 1 + 2b_{2n+k} - \sum_{i=0}^{n-2} B_{2i+k+1}$$

*is a balancing number. In fact if $x$ is the $n^{th}$ balancing number, then $B_k^1(x)$ is the $(2n+k)^{th}$ balancing number. So $B_k^1(x)$ is odd balancing number for odd $k$ and even balancing number for even $k$.*

(3) *Let $B_k$ denote the $k^{th}$ balancing number, let $b_k$ denote the $k^{tk}$ cobalancing number and let $C_k$ denote the $k^{th}$ Lucas–balancing number for an integer $k \geq 0$. Then for any cobalancing number $x$, the function*

$$b_k^1(x) = C_k x + B_k\left(\sqrt{8x^2+8x+1}+1\right) + b_k$$

*is a cobalancing number. In fact, if $x$ is the $n^{th}$ cobalancing number, then $b_k(x)$ is the $(n+k)^{th}$ cobalancing number (Note that $b_1^1(x) = f(x)$ and $b_2^1(x) = g(x)$).*

(4) *If $x$ is the $n^{th}$ balancing number, $y$ is the $m^{th}$ cobalancing number and $z$ is the $n^{th}$ Lucas–balancing number, then the function*

$$b_1(x) = \frac{2x + \sqrt{8x^2+1} - 1}{2}$$

*is the $(n+1)^{th}$ cobalancing number and the function*

$$b_2(x,y,z) = \frac{2x\sqrt{8y^2+8y+1} + 2y\sqrt{8x^2+1} + z - 1}{2}$$

*is the $(n+m)^{th}$ cobalancing number.*

*Proof.* (1) Notice that $B_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}, b_n = \frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, C_n = \frac{\alpha^{2n}+\beta^{2n}}{2}$ and $c_n = \frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$. So for any balancing number $x$ and cobalancing number $y$, we have

$$\begin{aligned} B_1(x,y) &= \frac{-8xy + \sqrt{8x^2+1}\sqrt{8y^2+8y+1} + 1}{4} \\ &= \frac{-8\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}\right) + \left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right)\left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}\right) + 1}{4} \\ &= \frac{\left\{\begin{array}{c} \frac{-\alpha^{4n-1}+\alpha^{2n}\beta^{2n-1}+\beta^{2n}\alpha^{2n-1}-\beta^{4n-1}}{4} + \frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}} \\ + \frac{\alpha^{4n-1}+\alpha^{2n}\beta^{2n-1}+\beta^{2n}\alpha^{2n-1}+\beta^{4n-1}}{4} + 1 \end{array}\right\}}{4} \end{aligned}$$

$$= \frac{\frac{(\alpha\beta)^{2n}(\alpha^{-1}+\beta^{-1})}{2} + \frac{\alpha^{2n}-\beta^{2n}}{\sqrt{2}} + 1}{4}$$
$$= \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}$$
$$= B_n$$

since $\alpha\beta = -1$ and $\alpha^{-1} + \beta^{-1} = -2$. The others can be proved similarly. □

As in Theorem 3, there are infinitely many Lucas–balancing and Lucas–cobalancing functions given below which can be proved similarly.

**Theorem 4.** *Let $B_k$ denote the $k^{th}$ balancing number, $C_k$ denote the $k^{th}$ Lucas–balancing number and let $c_k$ denote the $k^{tk}$ Lucas–cobalancing number. Then for any balancing number $x$ and any cobalancing number $y$ with the same order, say $n$,*

(1) *the function*

$$C_k(x,y) = C_{k-1}\sqrt{8x^2+1} + c_{k-1}\sqrt{8y^2+8y+1} + c_{n+k-1}$$

*is the $(n+k-1)^{st}$ Lucas–balancing number, that is, $C_k(x,y) = C_{n+k-1}$ for $k \geq 2$.*

(2) *the function*

$$c_k(x,y) = C_{k-1}\sqrt{8y^2+8y+1} + c_{k-1}\sqrt{8x^2+1} + C_{n-k+1}$$

*is the $(n+k-1)^{st}$ Lucas–cobalancing number, that is, $c_k(x,y) = c_{n+k-1}$ for $n \geq k-1 \geq 1$, and the function*

$$c_k(y) = 4B_k(2y+1) + C_k\sqrt{8y^2+8y+1}$$

*is the $(n+k)^{th}$ Lucas–cobalancing number, that is, $c_k(y) = c_{n+k}$.*

For Pell and Pell–Lucas numbers, we can give the following theorem.

**Theorem 5.** *Let $B_k$ denote the $k^{th}$ balancing number, let $b_k$ denote the $k^{th}$ cobalancing number, let $C_k$ denote the $k^{tk}$ Lucas–balancing number, let $c_k$ denote the $k^{tk}$ Lucas–cobalancing number, let $P_k$ denote the $k^{th}$ Pell number and let $Q_k$ denote the $k^{th}$ Pell–Lucas number for an integer $k \geq 0$. Then for any balancing number $x$, for any cobalancing number $y$, for any Lucas–balancing number $z$ and for any Lucas–cobalancing number $w$ with the same order, say $n$,*

(1) *the function*

$$P_k^1(x,y) = \frac{8B_k(x+y) + C_k(\sqrt{8x^2+1} + \sqrt{8y^2+8y+1})}{2} + P_{2k}$$

*is the* $(2n+2k)^{th}$ *Pell number, that is,* $P_k^1(x,y) = P_{2n+2k}$, *the function*

$$P_k^2(x,y) = \frac{C_k\sqrt{8x^2+1} + c_k\sqrt{8y^2+8y+1}}{2}$$

*is the* $(2n+2k-1)^{st}$ *Pell number, that is,* $P_k^2(x,y) = P_{2n+2k-1}$, *and the function*

$$P_1(x,y,z,w) = \frac{z\sqrt{8x^2+1} + w\sqrt{8y^2+8y+1}}{2}$$

*is the* $(4n-1)^{st}$ *Pell number, that is,* $P_1(x,y,z,w) = P_{4n-1}$.

(2) *the function*

$$Q_k(x) = 32C_k x^2 + 32B_k x\sqrt{8x^2+1} + Q_{2k}$$

*is the* $(4n+2k)^{th}$ *Pell–Lucas number, that is,* $Q_k(x) = Q_{4n+2k}$.

(3) *We set*

$$G_k(x,y) = B_k\sqrt{8x^2+1} + b_k\sqrt{8y^2+8y+1}.$$

(a) *If* $k=1$, *then the function* $G_1(x,y)$ *is the* $n^{th}$ *Lucas–balancing number, that is,* $G_1(x,y) = C_n$.

(b) *If* $k=2$, *then the function* $G_2(x,y)$ *is the four times of the* $(2n+1)^{st}$ *Pell number, that is,* $G_2(x,y) = 4P_{2n+1}$.

(c) *The function* $G_k(x,y)$ *is the sum of Lucas–balancing numbers from* $n$ *to* $(n+k-1)$, *that is,*

$$G_k(x,y) = \sum_{i=n}^{n+k-1} C_i$$

*for every* $k \geq 1$, *or is the sum of Pell numbers, that is,*

$$G_k(x,y) = \begin{cases} 4\sum\limits_{i=1}^{\frac{k}{2}} P_{2n+4i-3} & \text{for even } k \geq 2 \\ P_{2n} + P_{2n-1} + 4\sum\limits_{i=0}^{\frac{k-3}{2}} P_{2n+4i+3} & \text{for odd } k \geq 3. \end{cases}$$

*Proof.* (1) Notice that $C_n + c_n = 2P_{2n}$, $B_n + b_n = \frac{C_n-1}{2}$ and $P_{2n} = 2B_n$. So

$$\begin{aligned} P_k^1(x,y) &= \frac{8B_k(x+y) + C_k(\sqrt{8x^2+1} + \sqrt{8y^2+8y+1})}{2} + P_{2k} \\ &= \frac{8B_k(B_n+b_n) + C_k(C_n+c_n)}{2} + 2B_k \\ &= \frac{8B_k(\frac{C_n-1}{2}) + 2C_k P_{2n}}{2} + 2B_k \end{aligned}$$

$$
\begin{aligned}
&= 2B_kC_n + C_kP_{2n} \\
&= 2\left(\frac{\alpha^{2k}-\beta^{2k}}{4\sqrt{2}}\right)\left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right) + \left(\frac{\alpha^{2k}+\beta^{2k}}{2}\right)\left(\frac{\alpha^{2n}-\beta^{2n}}{2\sqrt{2}}\right) \\
&= \frac{\left\{\begin{array}{c}\alpha^{2(k+n)}+\alpha^{2k}\beta^{2n}-\beta^{2k}\alpha^{2n}-\beta^{2(n+k)} \\ +\alpha^{2(k+n)}-\alpha^{2k}\beta^{2n}+\beta^{2k}\alpha^{2n}-\beta^{2(k+n)}\end{array}\right\}}{4\sqrt{2}} \\
&= \frac{\alpha^{2(n+k)}-\beta^{2(n+k)}}{2\sqrt{2}} \\
&= P_{2n+2k}.
\end{aligned}
$$

The others can be proved similarly. □

There is a connection between balancing numbers and triangular numbers denoted by $T_n$ which are the numbers of the form $T_n = \frac{n(n+1)}{2}$ for $n \geq 0$. There are infinitely many triangular numbers that are also square numbers which are called square triangular numbers and is denoted by $S_n$. For the $n^{\text{th}}$ square triangular number $S_n$, we can write $S_n = s_n^2 = \frac{t_n(t_n+1)}{2}$, where $s_n$ and $t_n$ are the sides of the corresponding square and triangle. Their Binet formulas are

$$S_n = \left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)^2,\ s_n = \frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}} \text{ and } t_n = \frac{\alpha^{2n}+\beta^{2n}-2}{4}.$$

In [10], Özkoç, Tekcan and Gözeri derived some nice results on triangular, square triangular and balancing numbers.

For the square triangular numbers, squares and triangles, we can give the following theorem.

**Theorem 6.** *For any balancing number $x$ and any cobalancing number $y$ with the same order, say $n$,*

(1) *the function*

$$S(x,y) = \frac{4x^2+4y(y+1)+\sqrt{8x^2+1}\sqrt{8y^2+8y+1}+1}{8}$$

*is the $n^{th}$ square triangular number, that is, $S(x,y) = S_n$.*

(2) *the function*

$$s(x,y) = \frac{6y-2\sqrt{8x^2+1}+3\sqrt{8y^2+8y+1}+3}{2}$$

*is the $(n-1)^{st}$ square, that is, $s(x,y) = s_{n-1}$.*

(3) *the function*

$$t(x,y) = \frac{2(x-y-1)+\sqrt{8x^2+1}-\sqrt{8y^2+8y+1}}{2}$$

*is the $n^{th}$ triangle, that is, $t(x,y) = t_n$.*

*Proof.* (1) Since $1+\alpha^{-2}+2\alpha^{-1} = 1+\beta^{-2}+2\beta^{-1} = 2$ and $\alpha\beta = -1$, we get

$$\begin{aligned}
S(x,y) &= \frac{4x^2+4y(y+1)+\sqrt{8x^2+1}\sqrt{8y^2+8y+1}+1}{8} \\
&= \frac{\left\{\begin{array}{c} 4\left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)^2+4\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}\right)\left[\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}+1\right] \\ +\left(\frac{\alpha^{2n}+\beta^{2n}}{2}\right)\left(\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}\right)+1 \end{array}\right\}}{8} \\
&= \frac{\alpha^{4n}(1+\alpha^{-2}+2\alpha^{-1})+\beta^{4n}(1+\beta^{-2}+2\beta^{-1})-4}{64} \\
&= \frac{\alpha^{4n}-2(\alpha\beta)^{2n}+\beta^{4n}}{32} \\
&= \left(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}\right)^2 \\
&= S_n.
\end{aligned}$$

The others can be proved similarly. □

## 3. Numerical results

In [8], the authors dealt with the common values of balancing–type sequences. At the end of this paper we investigate this problem between Lucas–balancing and combinatorial numbers.

Let us consider the equation

$$C_n = f(x), \tag{3.1}$$

where $f(x)$ is a polynomial with integer coefficients and $C_n$ is the $n^{th}$ Lucas–balancing number.

In this paper we study (3.1) when $f(x)$ is one of the polynomials $\binom{x}{k}$, and

$$S_k(x) = 1^k + 2^k + \cdots + (x-1)^k,$$
$$\Pi_k(x) = x(x+1)\ldots(x+k-1)$$

for fixed integer $k \geq 2$.

We have to mention that we can deduce the following property of Lucas–balancing from equation (1.3):

**Lemma 7.** *If $x = \pm C_n$ and $y = \pm B_n$, then the equation*

$$x^2 - 8y^2 = 1$$

*is valid.*

Our numerical result is the following.

**Theorem 7.** *Let $2 \le k \le 4$ and $f(x)$ be one of the polynomials $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$. Then the solutions $(C_n, x)$ of equation (3.1) are in the table*

| $C_m$ | $B_m$ | $f(x)$ | $x$ | $k$ |
|---|---|---|---|---|
| 3 | 1 | $\binom{x}{k}$ | 3 | 2 |
| 3 | 1 | $S_{k-1}(x)$ | 3 | 2 |

*Proof.* Consider the equation (3.1) when $f(x)$ one of polynomials $\binom{x}{2}$, $\binom{x}{3}$ and $\binom{x}{4}$. Using the multiplications by suitable powers of 2 and transformations $X = x$, $X = 2(x-1)^2+1$, $X = x^2-3x+1$ respectively, for the polynomials above. Then by Lemma 7 we obtain

$$(2^3 y)^2 = 2X^4 - 4X^3 + 2X^2 - 8,$$
$$\left(2^4 3y\right)^2 = X^3 - 7X^2 + 15X - 297,$$
$$\left(2^5 3y\right)^2 = 2X^4 - 4X^2 - 1150,$$

where $y$ is a balancing number. These types of equations are solvable by MAGMA (`IntegralQuarticPoints` and `IntegralPoints`), so after testing them we get the solutions above.

The suitable command at the first case is `IntegralQuarticPoints([2,-4,2,0,-8],[3,8])`, because the point $[3,8]$ is on the curve. We get the solutions `(2,8),(2,0),(1,0),(3,8)` for $(X, 2^3 y)$. Using these results we realize that there is only one solution for $B_n$, $C_n$ and $x$, where $(B_n, C_n, x) = (1,3,3)$.

Let us consider the second equation. In this case there is no solution, since from the property (1.3) follows that all Lucas–balancing numbers are odd, while $\binom{x}{3}$ is even.

We use in the third case the command `IntegralQuarticPoints([2,0,-4,0,-1150])` and we found no solution.

*Remark* 1. We mention that we can use `IntegralQuarticPoints([])` in the case also when the constant term is not square, but the odd powers are missing from the right hand side.

Now let $f(x)$ be equal to $S_{k-1}(x)$. If $k = 2$, then $S_1(x) = \binom{x}{2}$, and its unique solutions has already been determined.

Applying the transformations $X = 2(2x-1)^2$, $X = \binom{x}{2}$, respectively, to the equation (3.1) when $f(x) = S_2(x)$ and $f(x) = S_3(x)$ we get

$$\left(2^6 3y\right)^2 = X^3 - 4X^2 + 4X - 4608,$$
$$(2^2 y)^2 = 2X^4 - 2.$$

Using the commands `IntegralPoints(EllipticCurve([0,-4,4,0,-4608]))` and `IntegralQuarticPoints([2,0,0,0,-2])`, it provides only the solution $(1,0)$, in the second case, that is there is no solution for Lucas–balancing number.

Finally let $f(x) = \Pi_2(x)$, $\Pi_3(x)$, $\Pi_4(x)$ and by using the transformations $X = 2x+1$, $X = 2(x+1)^2+1$ and $X = x^2+3x$ we arrive that

$$(2^4 y)^2 = 2X^4 - 4X^2 - 30,$$
$$(2^3 y)^2 = X^3 - 7X^2 + 15X - 17,$$
$$y^2 = 2X^4 + 8X^3 + 8X^2 - 2.$$

By MAGMA, there exist no solution in the first and the second cases. In the third case we can easily conclude the same as follows. Since $X$ is always even, therefore $y$ is also even and we have

$$y^2 \equiv 0 \pmod 4 \text{ and } 2X^4 + 8X^3 + 8X^2 - 2 \equiv 2 \pmod 4,$$

which is impossible. □

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# ON THE PRIME SPECTRUM OF MODULES

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*Abstract.* Let $R$ be a commutative ring and let $M$ be an $R$-module. Let us denote the set of all prime submodules of $M$ by $Spec(M)$. In this article, we explore more properties of strongly top modules and investigate some conditions under which $Spec(M)$ is a spectral space.

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*Keywords:* prime spectrum, Zariski topology, spectral space

## 1. Introduction, etc

Throughout this article, all rings are commutative with identity elements, and all modules are unital left modules. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote respectively the natural numbers, the ring of integers and the field of quotients of $\mathbb{Z}$. If $N$ is a subset of an $R$-module $M$, then $N \leq M$ denotes $N$ is an submodule of $M$.

Let $M$ be an $R$-module. For any submodule $N$ of $M$, we denote the annihilator of $M/N$ by $(N : M)$, i.e. $(N : M) = \{r \in R | rM \subseteq N\}$. A submodule $P$ of $M$ is called *prime* if $P \neq M$ and whenever $r \in R$ and $e \in M$ satisfy $re \in P$, then $r \in (P : M)$ or $e \in P$.

The set of all prime submodule of $M$ is denoted by $Spec(M)$ (or $X$). For any ideal $I$ of $R$ containing $Ann(M)$, $\overline{I}$ and $\overline{R}$ will denote $I/Ann(M)$ and $R/Ann(M)$, respectively. Also the map $\psi : Spec(M) \to Spec(\overline{R})$ given by $P \mapsto \overline{(P : M)}$ is called the *natural map* of $X$. $M$ is called *primeful* (resp. $X$*-injective*) if either $M = \mathbf{0}$ or $M \neq \mathbf{0}$ and the natural map $\psi$ is surjective (resp. if either $X = \varnothing$ or $X \neq \varnothing$ and natural map $\psi$ is injective). (See [3, 11] and [13].)

The *Zariski topology* on $X$ is the topology $\tau$ described by taking the set $\Omega = \{V(N) | N$ is a submodule of $M\}$ as the set of closed sets of $X$, where $V(N) = \{P \in X | (P : M) \supseteq (N : M)\}$ [11].

The *quasi-Zariski topology* on $X$ is described as follows: put $V^*(N) = \{P \in X | \, P \supseteq N\}$ and $\Omega^* = \{V^*(N) | N$ is a submodule of $M\}$. Then there exists a topology $\tau^*$ on $X$ having $\Omega^*$ as the set of it's closed subsets if and only if $\Omega^*$ is closed under the finite union. When this is the case, $\tau^*$ is called a quasi-Zariski topology on $X$ and $M$ is called a *top $R$-module* [15].

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Let $Y$ be a topological space. $Y$ is irreducible if $Y \neq \varnothing$ and for every decomposition $Y = A_1 \cup A_2$ with closed subsets $A_i \subseteq Y, i = 1, 2$, we have $A_1 = Y$ or $A_2 = Y$. A subset $T$ of $Y$ is irreducible if $T$ is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets $F$ , $G$ which are closed in $Y$ and satisfy $T \subseteq F \cup G$, $T \subseteq F$ or $T \subseteq G$. Let $F$ be a closed subset of $Y$. An element $y \in Y$ is called a generic point of $Y$ if $Y = cl(\{y\})$ (here for a subset $Z$ of $Y$, $cl(Z)$ denotes the topological closure of $Z$).

A topological space $X$ is a *spectral space* if $X$ is homeomorphic to $Spec(S)$ with the Zariski topology for some ring $S$. This concept plays an important role in studying of algebraic properties of an $R$-module $M$ when we a have a related topology. For an example, when $Spec(M)$ is homeomorphic to $Spec(S)$, where $S$ is a commutative ring, we can transfer some of known topological properties of $Spec(S)$ to $Spec(M)$ and then by using these properties explore some of algebraic properties of $M$.

Spectral spaces have been characterized by M. Hochster as quasi-compact $T_0$-spaces $X$ having a quasi-compact open base closed under finite intersection and each irreducible closed subset of $X$ has a generic point [9, p. 52, Proposition 4].

The concept of strongly top modules was introduced in [2] and some of its properties have been studied. In this article, we get more information about this class of modules and explore some conditions under which $Spec(M)$ is a spectral space for its Zariski or quasi-Zariski topology.

In the rest of this article, $X$ will denote $Spec(M)$. Also the set of all maximal submodules of $M$ is denoted by $Max(M)$.

## 2. Main results

**Definition 1** (Definition 3.1 in [1])**.** Let $M$ be an $R$-module. $M$ is called a *strongly top* module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $V^*(N) = V^*(IM)$.

**Definition 2** (Definition 3.1 in [2])**.** Let $M$ be an $R$-module. $M$ is called a *strongly top* module if $M$ is a top module and $\tau^* = \tau$.

*Remark* 1. Definition 1 and Definition 2 are equivalent. This follows from the fact that if $N$ is a submodule of $M$, then by [11, Result 3], we have

$$V(N) = V((N : M)M) = V^*((N : M)M).$$

*Remark* 2 (Theorem 6.1 in [11]). Let $M$ be an $R$-module. Then the following are equivalent:

(a) $(X, \tau)$ is a $T_0$ space;
(b) The natural map of $X$ is injective;
(c) $V(P) = V(Q)$, that is, $(P : M) = (Q : M)$ implies that $P = Q$ for any $P, Q \in X$;

(d) $|Spec_p(M)| \leq 1$ for every $p \in Spec(R)$.

*Remark* 3. (a) Let $M$ be an $R$-module and $p \in Spec(R)$. The saturation of a submodule $N$ with respect to $p$ is the contraction of $N_p$ in $M$ and denoted by $S_p(N)$. It is known that

$$S_p(N) = N^{ec} = \{x \in M \,|\, tx \in N \text{ for some } t \in R \setminus p\}.$$

(b) Let $M$ be an $R$-module and $N \leq M$. The radical of $N$, denoted by $rad(N)$, is the intersection of all prime submodules of $M$ containing $N$; that is, $rad(N) = \bigcap_{P \in V^*(N)} P$ ([14]).

(c) A topological space $X$ is Noetherian provided that the open (respectively, closed) subsets of $X$ satisfy the ascending (respectively, descending) chain condition ([4, p. 79, Exercises 5-12]).

**Proposition 1.** *Let $M$ be an strongly top module and $\psi$ be the natural map of $X$. Then*

(a) $(X, \tau) = (X, \tau^*) \cong Im\psi$.

(b) *If $X$ is Noetherian, then $X$ is a spectral space.*

*Proof.* (a) By [15, Theorem 3.5] and Remark 2, $\psi|_{Im\psi}$ is bijective. Also we have

$$\psi(V(N)) = \{\overline{(P : M)} \,|\, P \in X, (P : M) \supseteq (N : M)\}.$$

Now by [11, Proposition 3.1] and the above arguments, $\psi$ is continuous and a closed map. Consequently we have $(X, \tau) = (X, \tau^*) \cong Im\psi$.

(b) Let $Y = V^*(N)$ be an irreducible closed subset of $X$. Now by [6, Theorem 3.4], we have

$$V^*(N) = V^*(rad(N)) = cl(\{rad(N)\}).$$

Hence $Y$ has a generic point. Also $X$ is Noetherian and it is a $T_0$-space by [6, Proposition 3.8 (i)]. Hence it is a spectral space by [9, Pages 57 and 58]. □

An $R$- module $M$ is said to be a *weak multiplication module* if either $X = Spec(M) = \varnothing$ or $X \neq \varnothing$ and for every prime submodule $P$ of $M$, we have $P = IM$ for some ideal $I$ of $R$ (see [5]).

The following theorem extends [1, Proposition 3.5], [1, Corollary 3.6], [1, Theorem 3.9 (1)], and [1, Theorem 3.9 (7)]. In fact, in part (a) of this theorem, we withdraw the restrictions of finiteness and Noetherian property from [1, Proposition 3.5] and [1, Corollary 3.6], respectively. In part (b), we remove the conditions " $M$ is primeful " and " $R$ is a Noetherian ring " in [1, Theorem 3.9 (1)] and instead of them, we put the weaker conditions " $Im(\psi)$ is closed in $Spec(\overline{R})$ " and " $Spec(\overline{R})$ is a Noetherian space ". In part (c), we withdraw the condition " $R$ has Noetherian spectrum " from [1, Theorem 3.9 (7)] and put the weaker condition " the intersection of every infinite family of maximal ideals of $R$ is zero ".

**Theorem 1.** *Let $M$ be an $R$-module. Then we have the following.*

(a) *Let $(M_i)_{i\in I}$ be a family of $R$-modules and let $M = \oplus_{i\in I} M_i$. If $M$ is an strongly top $R$-module, then each $M_i$ is an strongly top $R$-module.*
(b) *If $M$ be an strongly top $R$-module and $\psi$ be the natural map of $X$, then we have*
  (i) *If $Im(\psi)$ is closed in $Spec(\overline{R})$, then $(X,\tau) = (X,\tau^*)$ is a spectral space.*
  (ii) *If $Spec(\overline{R})$ is Noetherian, then $(X,\tau) = (X,\tau^*)$ is a spectral space.*
(c) *Suppose $R$ is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If $M$ is a weak multiplication $R$-module, then $M$ is a top module.*

*Proof.* (a) Each $M_i$ is a homomorphic image of $M$, hence it is strongly top by [1, Proposition 3.3].
(b) (i) By Proposition 1, we have $(X,\tau) = (X,\tau^*) \cong Im(\psi)$. Now the claim follows by [11, Theorem 6.7].
  (ii) As $Spec(\overline{R})$ is Noetherian, $Im(\psi)$ is also Noetherian. Now the claim follows from Proposition 1.
(c) Use the technique of [3, Theorem 3.18].

□

The following theorem extends [1, Theorem 3.9(3) ].

**Theorem 2.** *Suppose $R$ is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If $M$ is $X$-injective with $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$, then $M$ is a top module.*

*Proof.* If $S_0(\mathbf{0}) = M$, then $X = \varnothing$ and there is nothing to prove. Otherwise, by [12, Corollary 3.7], $S_0(\mathbf{0})$ is a prime submodule so that $S_0(\mathbf{0}) = rad(\mathbf{0})$. Hence the natural map $f : Spec(M/S_0(\mathbf{0})) \to Spec(M)$ is a homeomorphism by [7, Proposition 1.4]. But by [3, Theorem 3.7 (a)] and [3, Theorem 3.15 (e)], $M/S_0(\mathbf{0})$ is a weak multiplication module. Now the result follows because by Theorem 1 (c), $M/S_0(\mathbf{0})$ is a top module. □

Let $M$ be an $R$-module. Then $M$ is called a *content* module if for every $x \in M$, $x \in c(x)M$, where $c(x) = \bigcap\{I \mid I$ is an ideal of $R$ such that $x \in IM\}$ (see [13, p. 140]).

In below we generalize [1, Theorem 3.9(4)].

**Theorem 3.** *Suppose $R$ is a one dimensional integral domain and let $M$ be a content $R$-module. Then we have the following.*

(a) *If $M$ is $X$-injective , then $M$ is a top module.*
(b) *If $M$ is $X$-injective and $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$, then $M$ is an strongly top module. Furthermore, if $Spec(\overline{R})$ is Noetherian, then $(X,\tau^*)$ is spectral.*

*Proof.* (a) By [3, Theorem 3.21], we have

$$Spec(M) = \{S_p(pM) | p \in V(Ann(M)), S_p(pM) \neq M\} = \{S_0(\mathbf{0})\} \cup Max(M),$$

where

$$Max(M) = \{pM | p \in Max(R), pM \neq M\}.$$

Let $N \leq M$ and let $N \nsubseteq S_0(\mathbf{0})$. Then

$$rad(N) = \bigcap_{N \subseteq P \in Spec(M)} P = \bigcap_{N \subseteq P \in Max(M)} P.$$

So by the above arguments, there is an index set $I$ such that $rad(N) = \bigcap_{i \in I}(p_i M)$. Since $M$ is content module,

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i \in I}(p_i M)) = V((\bigcap_{i \in I} p_i M).$$

Now if $N \subseteq S_0(\mathbf{0})$, then by [10, Lemma 2],

$$\begin{aligned} V^*(N) = V^*(rad(N)) &= V^*(S_0(\mathbf{0}) \cap (\bigcap_{i \in I}(p_i M))) \\ &= V^*(S_0(\mathbf{0}) \cap ((\bigcap_{i \in I} p_i)M)) \\ &= V^*(S_0(\mathbf{0})) \cup V^*((\bigcap_{i \in I} p_i)M) \\ &= V^*(S_0(\mathbf{0})) \cup V((\bigcap_{i \in I} p_i)M). \end{aligned}$$

By the above arguments, it follows that $M$ is a top module.

(b) By [3, Theorem 3.21],

$$Spec(M) = \{S_0(\mathbf{0})\} \cup Max(M) \; and \; Max(M) = \{pM | p \in Max(R), pM \neq M\}.$$

Let $N \leq M$. If $N \subseteq S_0(\mathbf{0})$, then $V^*(N) = V^*(\mathbf{0}) = X$. Otherwise, we have $rad(N) = \bigcap_{i \in I}(p_i M)$ by [3, Theorem 3.21]. Since $M$ is content, by [11, Result 3] we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i \in I}(p_i M)) = V((\bigcap_{i \in I} p_i)M).$$

Hence $M$ is an strongly top module. The second assertion follows from Theorem 1 (b). □

**Theorem 4.** *If $M$ is content weak multiplication, then $M$ is an strongly top module. Moreover, if $Spec(R)$ is Noetherian, then $(X, \tau^*)$ is a spectral space.*

*Proof.* Let $N \leq M$. Then we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{N \leq P} P).$$

Since $M$ is a weak multiplication module, for each prime submodule $P$ of $M$ containing $N$, there exists an ideal $I_P$ of $R$ such that $P = I_P M$. Hence since $M$ is a content module,

$$V^*(N) = V^*(\bigcap_{N \leq P} (I_P M)) = V^*((\bigcap_{N \leq P} I_P)M).$$

This implies that $M$ is an strongly top module. Since $Spec(R)$ is Noetherian, so is $Spec(\overline{R})$. Hence by Theorem 1 (b), $(X, \tau^*)$ is a spectral space. □

**Theorem 5.** *Let $R$ be a one-dimensional integral domain and let $M$ be an $X$-injective $R$-module such that $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$. If the intersection of every infinite number of maximal submodules of $M$ is zero, then $M$ is strongly top and $(X, \tau^*)$ is a spectral space.*

*Proof.* If $S_0(\mathbf{0}) = M$, then $X = \varnothing$ and there is nothing to prove. Otherwise, by [3, Theorem 3.21], we have $Spec(M) = \{S_0(\mathbf{0})\} \cup Max(M)$ and $Max(M) = \{pM \mid p \in Max(R), pM \neq M\}$. Now let $N \leq M$. If $N = \mathbf{0}$, then claim clear because $V^*(N) = V^*(\mathbf{0}) = V^*(0M) = Spec(M)$. So we assume that $N \neq \mathbf{0}$. We consider two cases.

(1) $N \subseteq S_0(\mathbf{0})$. In this case, we have $V^*(N) = V^*(\mathbf{0}) = V^*(0M) = Spec(M)$.

(2) $N \nsubseteq S_0(\mathbf{0})$. Then since $N \neq 0$ and the intersection of every infinite number of maximal submodules of $M$ is zero, $rad(N) = \bigcap_{i=1}^n (p_i M)$, where $p_i M \in Max(M)$ for each $i$ $(1 \leq i \leq n)$. Hence we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i=1}^n (p_i M)).$$

Now we show that $V^*(\bigcap_{i=1}^n (p_i M)) = V^*((\bigcap_{i=1}^n p_i)M)$. Clearly, $V^*(\bigcap_{i=1}^n (p_i M)) \subseteq V^*((\bigcap_{i=1}^n p_i)M)$. Too see this reverse inclusion, let $P \in V^*((\bigcap_{i=1}^n p_i)M)$. If $P = S_0(\mathbf{0})$, then $(\bigcap_{i=1}^n p_i)M \subseteq S_0(\mathbf{0})$ implies that $\bigcap_{i=1}^n p_i \subseteq ((\bigcap_{i=1}^n p_i)M : M) \subseteq (S_0(\mathbf{0}) : M) = 0$. Thus, there exists $j$ $(1 \leq j \leq n)$ such that $p_j = 0$, a contradiction. Hence we must have $P = qM$, where $q \in Max(R)$. Then, similar the above arguments, there exists $j$ $(1 \leq j \leq n)$ such that $q = p_j$. Therefore, $P = qM = p_j M \in V^*(\bigcap_{i=1}^n (p_i M))$. So we have

$$V^*(N) = V^*(\bigcap_{i=1}^n (p_i M)) = V^*((\bigcap_{i=1}^n p_i)M).$$

Hence $M$ is strongly top so that $\tau = \tau^*$. On the other hand, $\tau = \tau^*$ is a subset of a finite complement topology. This implies that $(X, \tau^*)$ is Noetherian. Now by Proposition 1, $(X, \tau^*) = (X, \tau)$ is spectral. □

**Theorem 6.** *If for each submodule $N$ of $M$, $rad(N) = \sqrt{(N:M)}M$, then $M$ is an strongly top module. Moreover, if $Spec(R)$ is Noetherian , then $(X,\tau^*)$ is spectral.*

*Proof.* Let $N \leq M$. Then we have

$$V^*(N) = V^*(rad(N)) = V^*(\sqrt{(N:M)}M)$$
$$= V(\sqrt{(N:M)}M) = V(rad(N)) = V(N).$$

Hence $M$ is an strongly top module. Now the result follows by using similar arguments as in the proof of Theorem 4. □

*Remark* 4. Theorems 4, 5, and 6 improve respectively [1, Theorem 3.9(5)], [1, Theorem 3.9(8)], and [1, Theorem 3.9(6)]. They show that the notion of " top modules " can be replaced by " strongly top modules " and the proofs can be shortened considerably.

In below we generalize [1, Theorem 3.36].

**Theorem 7.** *Let $M$ be a primeful $R$-module. Then we have the following.*

(a) *If $(X,\tau)$ is discrete, then $Spec(M) = Max(M)$.*
(b) *If $R$ is Noetherian and $Spec(M) = Max(M)$, then $(X,\tau)$ is a finite discrete space.*

*Proof.* (a) Since $(X,\tau)$ is discrete, it is a $T_1$-space. Now by [3, Theorem 4.3], we have $Spec(M) = Max(M)$.

(b) By [3, Theorem 4.3], $Spec(\overline{R}) = Max(\overline{R})$. Hence $\overline{R}$ is Artinian. Now by [3, Theorem 4.3], $(X,\tau)$ is a $T_0$-space. Thus by Remark 2, $M$ is $X$-injective. But $M$ is a cyclic $\overline{R}$-module and hence a cyclic $R$-module by [3, Remark 3.13] and [3, Theorem 3.15]. Also $(Spec(M),\tau)$ is homoeomorphic to $Spec(\overline{R})$ by [11, Theorem 6.5(5)]. Hence $X$ is a finite discrete space by [4, Chapter 8, Exe 2].

□

It is well known that if $R$ is a PID and $Max(R)$ is not finite, then the intersection every infinite number of maximal ideals of $R$ is zero. Now it is natural to ask the following question: Is the same true when $R$ is a one dimensional integral domain with infinite maximal ideals? In below, we show that this true when $Spec(R)$ is a Noetherian space. Although this is not a simple fact, it used by some authors without giving any proof.

**Theorem 8.** (a) *Let $I$ be an ideal of $R$ and let $k,n \in \mathbb{N}$. Then $(\sqrt{I}:a^k) = (\sqrt{I}:a^n)$.*

(b) *Let $I$ be an ideal of $R$ and let $a \in R$, $n \in \mathbb{N}$. Then $\sqrt{I} = \sqrt{(\sqrt{I}:a^n)} \cap \sqrt{\langle \sqrt{I}, a^n \rangle}$.*

(c) *Suppose $Spec(R)$ is a Noetherian topological space. Then for every ideal $I$ of $R$, $\sqrt{I}$ has a primary decomposition.*

(d) *Suppose $R$ is a one dimensional integral domain and $Spec(R)$ is a Noetherian topological space. Then the intersection of every infinite number of maximal ideals is zero.*

*Proof.* (a) It is clear.

(b) Let $f \in \sqrt{(\sqrt{I} : a^n)} \cap \sqrt{\langle \sqrt{I}, a^n \rangle}$. Then there is $m \in \mathbb{N}$ such that $f^m \in (\sqrt{I} : a^n) \cap \langle \sqrt{I}, a^n \rangle$. It follows that $f^m = g + xa^n$ for some $g \in \sqrt{I}$ and $x \in R$ and we also get $a^n f^m \in \sqrt{I}$. Hence $a^n f^m = a^n g + xa^{2n}$. This implies that $xa^{2n} \in \sqrt{I}$ and so $x \in (\sqrt{I} : a^n)$ by part (a). Thus $xa^n \in \sqrt{I}$. It follows that $f \in \sqrt{I}$. The reverse inclusion is clear.

(c) Set $\Sigma =$

$$\{\sqrt{I} \mid I \textit{ is a proper ideals of } R \textit{ and } \sqrt{I} \textit{ doesn't have any primary decomposition}\}.$$

Since $Spec(R)$ is Noetherian, the radicals of ideals satisfy the a.c.c. condition. So $\Sigma$ has a maximal member, $\sqrt{I_0}$ say. Thus $\sqrt{I_0} \notin Spec(R)$. In other words,

$$\exists a, b \in R \textit{ s.t. } ab \in \sqrt{I_0} \textit{ and } a \notin \sqrt{I_0} \textit{ and } b \notin \sqrt{I_0}.$$

By part (b) we have $\sqrt{I_0} = \sqrt{(\sqrt{I_0} : b)} \cap \sqrt{\langle \sqrt{I_0}, b \rangle}$. Further, $\sqrt{I_0} \subsetneq \sqrt{(\sqrt{I_0} : b)}$ and $\sqrt{I_0} \subsetneq \sqrt{\langle \sqrt{I_0}, b \rangle}$. Since $\sqrt{(\sqrt{I_0} : b)}$ and $\sqrt{\langle \sqrt{I_0}, b \rangle}$ have primary decompositions by hypothesis, $\sqrt{I_0}$ has a primary decomposition, a contradiction.

(d) Since $R$ is one dimensional integral domain, $Spec(R) = \{0\} \cup Max(R)$. Suppose $\{m_i\}_{i \in I}$ is an infinite family of maximal ideals of $R$ such that $\bigcap_{i \in I} m_i \neq 0$. By part (c), $\sqrt{\bigcap_{i \in I} m_i}$ has a primary decomposition. Hence

$$\sqrt{\bigcap_{i \in I} m_i} = \bigcap_{j=1}^{n} m'_j, \quad m'_j \in Max(R).$$

This implies that $\{m_i\}_{i \in I}$ is a finite family, a contradiction. So the proof is compeleted. □

*Example* 1. We show that $\mathbb{Z}[i\sqrt{5}]$ is a one dimensional Noetherian integral domain which has infinite number of maximal ideals and it is not a PID. To see this, let $\phi : \mathbb{Z}[X] \to \mathbb{Z}[i\sqrt{5}]$ be the natural epimorphism given by $p(x) \mapsto p(i\sqrt{5})$. by using [8] or [16], one can see that

$$Spec(\mathbb{Z}[X]) = \{\langle p \rangle, \langle f \rangle, \langle q, g \rangle \mid p \textit{ and } q \textit{ are prime numbers, } f \textit{ is a primary irreducible polynomial in } \mathbb{Q}[X], \textit{ and } g \textit{ is an irreducible polynomial in } \mathbb{Z}_q[X]\}.$$

Now we have $ker\phi = \langle X^2 + 5 \rangle$. A simple verification shows that

$$Spec(\mathbb{Z}[i\sqrt{5}]) = \{0\} \cup Max(\mathbb{Z}[i\sqrt{5}])$$

$$= \{0\} \cup \{\langle q, g(\sqrt{-5})\rangle \mid \langle q, g\rangle \in Spec(\mathbb{Z}[X]) \text{ and } X^2 + 5 \in \langle q, g\rangle\}.$$

Further $\mathbb{Z}[i\sqrt{5}]$ contains a finite number elements which are invertible by [17, p. 38]. So $\mathbb{Z}[i\sqrt{5}]$ is a Noetherian one dimensional integral domain with infinite number of maximal ideals. Hence the intersection of every infinite number of maximal ideals of $\mathbb{Z}[i\sqrt{5}]$ is zero by Theorem 8 (c). Note that $\mathbb{Z}[i\sqrt{5}]$ is not a PID by [17, p. 38].

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# ON TAUBERIAN REMAINDER THEOREMS FOR CESÀRO SUMMABILITY METHOD OF NONINTEGER ORDER

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*Abstract.* In this paper, we prove some Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$.

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*Keywords:* Tauberian remainder theorem, $\lambda$-bounded series, $(C,\alpha)$ summability, Cesàro means

## 1. Introduction

Let $A_n^\alpha$ be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$, $(|x|<1)$, where $\alpha > -1$. For a real sequence $u = (u_n)$, the Cesàro means of the sequence $(u_n)$ of noninteger order $\alpha$ are defined by

$$\sigma_n^{(\alpha)}(u) = \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^{\alpha-1} u_j .$$

We say that a sequence $(u_n)$ is $(C,\alpha)$ summable to a finite number $s$, where $\alpha > -1$ if

$$\lim_{n\to\infty} \sigma_n^{(\alpha)}(u) = s, \tag{1.1}$$

and we write $u_n \to s$ $(C,\alpha)$. We denote the backward difference of $(u_n)$, by $\Delta u_n = u_n - u_{n-1}$, with $\Delta u_0 = u_0$. We define $\tau_n(u) = n\Delta u_n$ $(n = 0,1,2,..)$ and indicate $\tau_n^{(\alpha)}(u)$ as $(C,\alpha)$ mean of $(\tau_n(u))$.

Note that if taking $\alpha = k$ where $k$ is a nonnegative integer, then we obtain the $(C,k)$ summability method and for $\alpha = 0$, the $(C,0)$ summability is ordinary convergence.

The $(C,\alpha)$ summability method is regular, more generally, if a sequence $(u_n)$ is $(C,\alpha)$ summable to $s$, where $\alpha > -1$ and $\beta \geq \alpha$ for $\alpha,\beta$, then $(u_n)$ is also $(C,\beta)$ summable to $s$. However, the converse is not always true. The converse of this statement is valid under some conditions called Tauberian conditions. Any theorem which states that convergence of a sequence follows from a summability method and some Tauberian condition(s) is said to be a Tauberian theorem. Recently, a number

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of authors such as Estrada and Vindas [4, 5], Natarajan [15], Çanak et al. [2], Erdem and Çanak [3], Çanak and Erdem [1] have investigated Tauberian theorems for several summability methods.

For a sequence $(u_n)$ and for each integer $m \geq 1$,

$$(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n), \tag{1.2}$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

For $\alpha > -1$, the identity

$$\tau_n^{(\alpha)}(u) = n\Delta\sigma_n^{(\alpha)}(u) \tag{1.3}$$

was proved by Kogbetliantz [9]. Note that $\tau_n^{(0)}(u) = \tau_n(u)$.

The identity

$$\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u) = \frac{1}{\alpha+1}\tau_n^{(\alpha+1)}(u) \tag{1.4}$$

is used in the various steps of proofs (see [10]).

Çanak et al. [2] represent the identity

$$n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1}), \tag{1.5}$$

for $\alpha > -1$.

Erdem and Çanak [3] prove that for $\alpha > -1$ and any integer $k \geq 1$

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k}(-1)^{j+1}A_k^{(j)}(\alpha)n\Delta\tau_n^{(\alpha+j)}(u), \tag{1.6}$$

where $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $\quad a_k^{(0)}(\alpha) = 0$ and

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r < s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

## 2. Tauberian remainder theorems

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence $(u_n)$ is called bounded with the rapidity $(\lambda_n)$ (in short $\lambda$-bounded) if

$$\lambda_n(u_n - s) = O(1),$$

with $\lim_{n\to\infty} u_n = s$. Let

$$m^{\lambda} = \{u = (u_n) | \lim_{n\to\infty} u_n = s \quad \text{and} \quad \lambda_n(u_n - s) = O(1)\}. \tag{2.1}$$

A sequence $(u_n)$ is called $\lambda$-bounded by the $(C,\alpha)$ method of summability if

$$\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1), \tag{2.2}$$

with $\lim_{n\to\infty} \sigma_n^{(\alpha)}(u) = s$. Shortly, we write $u \in ((C,\alpha), m^{\lambda})$.

G. Kangro [7] introduced the concepts of Tauberian remainder theorems using summability with given rapidity $\lambda$. G. Kangro [8] and Tammeraid [16, 17] proved some Tauberian remainder theorems for several summability method, such as Riesz, Cesàro, Hölder and Euler-Knopp methods. Recently, various authors have represented some Tauberian remainder theorems (see [12, 13]). In [18], Tammeraid proved some Tauberian remainder theorems in which the $(C,\alpha)$ summability method is used. Tauberian remainder theorems have also been studied by many authors via the Fourier integral method. [6, 11]

Meronen and Tammeraid [14] proved the following Tauberian remainder theorems:

**Theorem 1.** *Let the condition*

$$\lambda_n \tau_n^{(1)}(u) = O(1)$$

*be satisfied. If* $u \in ((C,1), m^\lambda)$, *then* $u \in m^\lambda$.

**Theorem 2.** *Let the conditions*

$$\lambda_n \tau_n(u) = O(1),$$
$$\lambda_n n \Delta \tau_n^{(1)}(u) = O(1)$$

*be satisfied. If* $u \in ((C,1), m^\lambda)$, *then* $u \in m^\lambda$.

The main purpose of this paper is to prove several Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$. Our main theorems improve Theorem 1 and Theorem 2 given by Meronen and Tammeraid [14].

## 3. A LEMMA

We require the following lemma to be used in the proofs of main theorems.

**Lemma 1.** *Let* $\alpha > -1$. *For any integer* $k \geq 2$,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\tau_n^{(\alpha)}(u) - B_{1,1}\sigma_n^{(\alpha)}(u) + B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \left( B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u) \right),$$

*where* $B_{m,l} = (\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha)$ *and* $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $a_k^{(0)}(\alpha) = 0$ *and*

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r < s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

*Proof.* From identity (1.6), we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k} (-1)^{j+1} A_k^{(j)}(\alpha) n\Delta \tau_n^{(\alpha+j)}(u)$$
$$= A_k^{(1)}(\alpha) n\Delta \tau_n^{(\alpha+1)}(u) + \sum_{j=2}^{k} (-1)^{j+1} A_k^{(j)}(\alpha) n\Delta \tau_n^{(\alpha+j)}(u).$$

It follows from identity (1.5) that

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$+ \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

By identity (1.4), we can write the above equation as

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha) \tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha) \left(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)\right)$$
$$+ \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha) \Big((\alpha+j-1)(\sigma_n^{(\alpha+j-2)}(u) - \sigma_n^{(\alpha+j-1)}(u))$$
$$-(\alpha+j)(\sigma_n^{(\alpha+j-1)}(u) - \sigma_n^{(\alpha+j)}(u))\Big).$$

Therefore,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1) A_k^{(1)}(\alpha) \tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha) \sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha) \sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \Big((\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j-2)}(u)$$
$$- (\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j-1)}(u)$$
$$- (\alpha+j)^2 (-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j-1)}(u) + (\alpha+j)^2 (-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j)}(u)\Big).$$

Hence, we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1) A_k^{(1)}(\alpha) \tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha) \sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha) \sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \Big((\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j-2)}(u)$$
$$- (\alpha+j)(2\alpha+2j-1)(-1)^{j+1} A_k^{(j)}(\alpha) \sigma_n^{(\alpha+j-1)}(u)$$

$$+ (\alpha+j)^2(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u)\Big).$$

Taking $(\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha) = B_{m,l}$, we obtain

$$(n\Delta)_k\tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\tau_n^{(\alpha)}(u) - B_{1,1}\sigma_n^{(\alpha)}(u) + B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+\sum_{j=2}^{k}\Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u)\Big).$$

Thus, we conclude that Lemma 1 is true for each integer $k \geq 2$. □

## 4. Main results

In the main theorems, we prove some Tauberian remainder theorems to recover $\lambda$-bounded by the $(C,\alpha)$ summability of a sequence out of $\lambda$-bounded by the $(C,\alpha+j)$ summability for $j = 1,2$ and any integer $j = k$, and some suitable conditions. In special cases of main theorems, we obtain some classical type Tauberian remainder theorems for the $(C,1)$ summability method.

**Theorem 3.** *Let the conditions*

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = O(1), \tag{4.1}$$

*and*

$$\lambda_n\tau_n^{(\alpha)}(u) = O(1) \tag{4.2}$$

*be satisfied for* $\alpha > -1$. *If* $u \in ((C,\alpha+1),m^\lambda)$, *then* $u \in ((C,\alpha),m^\lambda)$.

*Proof.* From identity (1.5), we have

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = \lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$= \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).$$

From identity (1.4), we obtain

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)).$$

Rewritten the above equation, we have

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - s) = \lambda_n(\alpha+1)^2(\sigma_n^{(\alpha+1)}(u) - s)$$
$$+\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n n\Delta\tau_n^{(\alpha+1)}(u).$$

Using (4.1) and (4.2), we get

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1)$. That means $u \in ((C,\alpha),m^\lambda)$. □

Notice that taking $\alpha = 0$, we obtain Theorem 2.

**Proposition 1.** *Let the conditions*

$$\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) = O(1), \tag{4.3}$$

$$\lambda_n \tau_n^{(\alpha)}(u) = O(1), \tag{4.4}$$

*and*

$$\lambda_n (\sigma_n^{(\alpha+2)}(u) - s) = O(1) \tag{4.5}$$

*be satisfied for* $\alpha > -1$. *If* $u \in ((C,\alpha+1), m^\lambda)$, *then* $u \in ((C,\alpha), m^\lambda)$.

*Proof.* Taking $k = 2$ in Lemma 1, we have

$$\begin{aligned}
\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) &= \lambda_n (\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u)\right) \\
&\quad - \lambda_n (\alpha+2)^2 \left(\tau_n^{(\alpha+1)}(u) - \tau_n^{(\alpha+2)}(u)\right) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha+1)}(u) \\
&\quad - \lambda_n (\alpha+2)^2 \tau_n^{(\alpha+1)}(u) + \lambda_n (\alpha+2)^2 \tau_n^{(\alpha+2)}(u).
\end{aligned}$$

From identity (1.4), we get

$$\begin{aligned}
&\lambda_n (n\Delta)_2 \tau_n^{(\alpha+2)}(u) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+2)(\alpha+1)\left((\alpha+1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u))\right) \\
&\quad - \lambda_n (\alpha+2)^2 \left((\alpha+1)(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u))\right) \\
&\quad + \lambda_n (\alpha+2)^2 \left((\alpha+2)(\sigma_n^{(\alpha+1)}(u) - \sigma_n^{(\alpha+2)}(u))\right) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha)}(u) \\
&\quad + \lambda_n (\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha+1)}(u) - \lambda_n (\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha)}(u) \\
&\quad + \lambda_n (\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha+1)}(u) + \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+1)}(u) - \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+2)}(u) \\
&= \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)\alpha_n^{(\alpha)}(u) \\
&\quad + \lambda_n ((\alpha+2)^3 + (\alpha+2)^2(\alpha+1) + (\alpha+1)^2(\alpha+2))\sigma_n^{(\alpha+1)}(u) \\
&\quad - \lambda_n (\alpha+2)^3\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Rewritten the above equation, we have

$$\begin{aligned}
&\lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u) - s) \\
&= -\lambda_n n\Delta\tau_n^{(\alpha+2)}(u) + \lambda_n (\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u) + \lambda_n ((\alpha+2)^3 + (\alpha+2)^2(\alpha+1) \\
&\quad + (\alpha+1)^2(\alpha+2) - s)\sigma_n^{(\alpha+1)}(u) - \lambda_n ((\alpha+2)^3 - s)\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Using (4.3), (4.4) and (4.5), we get

$$\lambda_n (\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s)=O(1)$. That means $u\in((C,\alpha),m^\lambda)$. □

Now, we represent a Tauberian remainder theorem which generalizes Theorem 3 and Proposition 1.

**Theorem 4.** *Let the conditions*

$$\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u)=O(1), \tag{4.6}$$

$$\lambda_n\tau_n^{(\alpha)}(u)=O(1), \tag{4.7}$$

*and*

$$\lambda_n(\sigma_n^{(\alpha+j)}(u)-s)=O(1)\quad \textit{for}\quad 2\le j\le k \tag{4.8}$$

*be satisfied for* $\alpha>-1$. *If* $u\in((C,\alpha+1),m^\lambda)$, *then* $u\in((C,\alpha),m^\lambda)$.

*Proof.* From Lemma 1 we have

$$\begin{aligned}\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u)&=B_{1,1-\alpha}\lambda_n\tau_n^{(\alpha)}(u)-B_{1,1}\lambda_n\sigma_n^{(\alpha)}(u)+B_{1,1}\lambda_n\sigma_n^{(\alpha+1)}(u)\\&\quad+\lambda_n\sum_{j=2}^{k}\Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u)-2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u)+B_{j,j}\sigma_n^{(\alpha+j)}(u)\Big),\end{aligned}$$

Rewritten the above equation, we have

$$\begin{aligned}&B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s)\\&=B_{1,1-\alpha}\lambda_n\tau_n^{(\alpha)}(u)-\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u)+B_{1,1}\lambda_n(\sigma_n^{(\alpha+1)}(u)-s)\\&\quad+\lambda_n\sum_{j=2}^{k}B_{j,j-1}(\sigma_n^{(\alpha+j-2)}(u)-s)-\lambda_n\sum_{j=2}^{k}2B_{j,j-\frac{1}{2}}(\sigma_n^{(\alpha+j-1)}(u)-s)\\&\quad+\lambda_n\sum_{j=2}^{k}B_{j,j}(\sigma_n^{(\alpha+j)}(u)-s).\end{aligned}$$

Using (4.6), (4.7) and (4.8), we get

$$B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s)=O(1)+O(1)+O(1)+O(1)+O(1)+O(1)=O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s)=O(1)$. That means $u\in((C,\alpha),m^\lambda)$. □

**Theorem 5.** *Let the condition*

$$\lambda_n\tau_n^{(\alpha+j+1)}(u)=O(1)\quad \textit{for}\quad 0\le j\le k-1, \tag{4.9}$$

*be satisfied for* $\alpha>-1$. *If* $u\in((C,\alpha+k),m^\lambda)$, *then* $u\in((C,\alpha),m^\lambda)$.

*Proof.* Suppose that $u\in((C,\alpha+k),m^\lambda)$. Taking $j=k-1$ in (4.9), it follows from the idendity

$$\tau_n^{(\alpha+k)}(u)=(\alpha+k)(\sigma_n^{(\alpha+k-1)}(u)-\sigma_n^{(\alpha+k)}(u))$$

that we obtain

$$\begin{aligned}\lambda_n(\alpha+k)(\sigma_n^{(\alpha+k-1)}(u)-s) &= \lambda_n\tau_n^{(\alpha+k)}(u)+\lambda_n(\alpha+k)(\sigma_n^{\alpha+k}(u)-s)\\ &= O(1)+O(1)=O(1)\end{aligned}$$

then we obtain $\lambda_n(\sigma_n^{(\alpha+k-1)}-s)=O(1)$. Hence, that means

$$u\in((C,\alpha+k-1),m^\lambda).$$

From identity (1.4), we have

$$\tau_n^{(\alpha+k-1)}(u)=(\alpha+k-1)(\sigma_n^{(\alpha+k-2)}(u)-\sigma_n^{(\alpha+k-1)}(u)).$$

Taking $j=k-2$ in (4.9), we obtain

$$\begin{aligned}&\lambda_n(\alpha+k-1)(\alpha_n^{(\alpha+k-2)}(u)-s)\\ &=\lambda_n\tau_n^{(\alpha+k-1)}(u)+\lambda_n(\alpha+k-1)(\sigma_n^{(\alpha+k-1)}(u)-s)=O(1)+O(1)=O(1)\end{aligned}$$

Therefore we have

$$u\in((C,\alpha+k-2),m^\lambda).$$

Continuing in this way, we obtain that

$$u\in((C,\alpha+1),m^\lambda).$$

Taking $j=0$ in (4.9), we obtain $\lambda_n\tau_n^{(\alpha+1)}=O(1)$. From identity (1.4), we have

$$\begin{aligned}\lambda_n(\alpha+1)(\sigma_n^{(\alpha)}(u)-s) &= \lambda_n\tau_n^{(\alpha+1)}(u)+\lambda_n(\alpha+1)(\sigma_n^{(\alpha+1)}(u)-s)\\ &= O(1)+O(1)=O(1)\end{aligned}$$

This completes the proof. □

**Theorem 6.** *Let the condition*

$$\lambda_n(n\Delta)_j\tau_n^{(\alpha+j)}(u)=O(1)\quad \textit{for}\quad 0\le j\le k, \tag{4.10}$$

*be satisfied for* $\alpha>-1$. *If* $u\in((C,\alpha+k),m^\lambda)$, *then* $u\in((C,\alpha),m^\lambda)$.

*Proof.* By identity (1.6) for $k=1$, it follows

$$\begin{aligned}\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) &= \lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u))\\ &= \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).\end{aligned}$$

Taking $j=0$ and $j=1$ in (4.10), we obtain

$$\lambda_n\tau_n^{(\alpha+1)}(u)=O(1)$$

From identity (1.6) for $k=2$, we get

$$\begin{aligned}\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u) &= \lambda_n(\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u)\right)\\ &\quad-\lambda_n(\alpha+2)^2\left(\tau_n^{(\alpha+1)}(u)-\tau_n^{(\alpha+2)}(u)\right).\end{aligned}$$

Taking $j = 0$ and $j = 2$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+2)}(u) = O(1)$$

Continuing in this way, by Lemma 1, we obtain

$$\lambda_n (n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha) \lambda_n (\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$+ \lambda_n \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

Taking $j = 0$ and $j = k$ in (4.10), we obtain

$$\lambda_n \tau_n^{(\alpha+k)}(u) = O(1).$$

The conditions in Theorem 5 hold, the proof is completed. □

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# ON THE LOCAL KERNEL BASED APPROXIMATION OF HIGHLY OSCILLATORY INTEGRALS

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*Abstract.* A local kernel based interpolation scheme for computing oscillatory integrals is presented. David levin [17] transformed the integration problem into a systems of ODEs/PDEs and suggested collocation methods for solving the resultant ODEs/PDEs numerically. We extended the approach of David Levin and propose a local kernel based interpolation scheme for solving the resultant ODEs/PDEs. The proposed numerical scheme is capable of computing the oscillatory integrals for large data points in the domain of integrations.

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*Keywords:* local interpolation schemes, radial kernels basis, oscillatory integrals

## 1. Introduction

In this section, we describe a local meshless method for integrals of the form

$$I = \int_D f(\mathbf{x}) e^{iq(\mathbf{x})} d\mathbf{x}, \tag{1.1}$$

where $f$ is non oscillatory, $D$ is a finite interval $[a,b]$ in one-dimensional case $\mathbf{x} = x$ and $|q'(\mathbf{x})| \gg (b-a)^{-1}$, while a rectangle $[a,b]\times[c,d]$ in two-dimensional case $\mathbf{x} = (x,y)$ and $q_x(x,y) \gg (b-a)^{-1}$ and $q_y(x,y) \gg (d-c)^{-1}$. The function $q(x)$ is called the oscillator of (1.1). If we write $q(x) = \omega g(x)$ intuitively, the value $I$ of the integral will decreases with the increment in frequency parameter $\omega$, because increasing integral oscillations cancel out.

The standard integration technique are unable to compute highly oscillatory integrals, when the frequency is significantly larger than the number of quadrature points. Many integration techniques have been developed for computing highly oscillatory integrals. [4, 7, 11–14, 17, 18, 20, 21]. Highly oscillatory integral (1.1) occurs in a wide range of practical problems and applications ranging from celestial mechanics to nonlinear optics, plasma transport, fluid dynamics, Schrodinger spectra, computerized tomography [13]. The methods developed in [2, 7, 18] for evaluating rapidly oscillatory integrals are applicable to any integral of the form (1.1) and performed very well when $q' = W$ a constant frequency. For varying frequency $q'$ David levin

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[17] has shown that if the function $f$ is of the form

$$\mathcal{L}p(x) \equiv p'(x) + iq'(x)p(x) = f(x),\ a \le x \le b, \tag{1.2}$$

then the integral could be evaluated directly as

$$I = \int_a^b \frac{d}{dx}(p(x)e^{iq(x)})dx = p(b)e^{iq(b)} - p(a)e^{iq(a)}, \tag{1.3}$$

and for the two-dimensional case the formulations in equations (1.2)-(1.3) are given as

$$\mathcal{L}p(x,y) \equiv p_{xy} + iq_y p_x + iq_x p_y + (iq_{xy} - q_x q_y)p = f(x,y), \tag{1.4}$$

and the integral could readily be obtained as

$$\begin{aligned} I &= \int_a^b \int_c^d \frac{\partial^2}{\partial x \partial y}(pe^{iq})dxdy \\ &= p(b,d)e^{iq(b,d)} - p(a,d)e^{iq(a,d)} - p(b,c)e^{iq(b,c)} + p(a,c)e^{iq(a,c)}. \end{aligned} \tag{1.5}$$

The general solution of this equation is so that in general is as oscillatory as the integrand in (1.1). If the functions $f$ and $q'$ are slowly oscillatory, then there also exists a slowly oscillatory solution $p_0$ of (1.2) [17]. By looking for this $p_0$ we avoid all the difficulties caused by the rapid oscillations. Writing the general solution of (1.1) in the form

$$p(x) = p_0(x) + ce^{-iq(x)}, \tag{1.6}$$

and using (1.3) we get

$$I = p_0(b)e^{iq(b)} - p_0(a)e^{iq(a)}. \tag{1.7}$$

Therefore, $p_0(a)$ and $p_0(b)$ are actually all we need. A problem arises since we do not have any initial condition for $p_0$. Furthermore, even if $p_0(a)$ were known, any forward integration scheme for (1.2) will develop high oscillations due to round-off errors. However, the rapidly oscillatory homogeneous solution $ce^{-iq(x)}$ can be excluded automatically by solving (1.2) by collocation using slowly oscillatory basis functions. By this method $p_0$ is singled out without specifying any boundary conditions.

**Theorem 1.** *If $f$ and $q$ are slowly oscillatory and $|q'(x)| \gg 1$ then the differential equation $\mathcal{L}p(x) \equiv p'(x) + iq'(x)p(x) = f(x)$, has a particular solution $p_0$ which is slowly oscillatory in comparison with the rapidly oscillatory homogeneous solution $e^{iq}$.*

The resultant system of ODEs/PDEs recovered in the Levin method can be approximated by using suitable interpolation technique with carefully selected basis functions. To avoid the difficulties of large linear systems, we could divide the domain of integration, or we could use some compactly supported basis in the interpolation technique. An other way to use some localized interpolation technique. In the

present study we construct a local kernel based interpolation scheme for approximating highly oscillatory integrals.

## 2. Collocation with kernel basis function

A collocation with radial kernels (RBFs) of multivariate data points $x_k, k = 1, ..., N \in \Omega$ takes the form

$$p(x) = \sum_{k=1}^{n} \lambda_k \phi(\|x - x_k\|). \tag{2.1}$$

Here $\|\cdot\|$ denotes the standard Euclidean vector norm, $\phi_\varepsilon(r)$ is some radial kernel, where the entries of the system matrix $\mathbf{A}$ are $\mathbf{A}_{i,j} = \phi(\|x_i - x_j\|), i = 1, ..., N, j = 1, ..., N$. Similarly the kernel basis interpolant of the function $f(x)$ in the David Levin method is defined as

$$\mathcal{L}p(x) = \sum_{j=1}^{N} \lambda_j \phi'(\|x - x_j\|) + i\omega q'(x) \sum_{j=1}^{N} \lambda_j \phi(\|x - x_j\|),\ j = 1, ..., N. \tag{2.2}$$

The coefficients $\lambda_k$ are determined in such a way that $\mathcal{L}p(x_j) = f(x_j)$ at interpolation nodes $x_1, ..., x_N \in \Omega$, we have

$$\mathbf{B}^{\smile} = \mathbf{f}. \tag{2.3}$$

Computing the above linear system at the evaluations points $x_1, ..., x_M \in \Omega$, the solution $p(x)$ is given as

$$\mathbf{p} = \mathbf{M}\mathbf{B}^{-1}\mathbf{f} = \mathbf{W}\mathbf{f}. \tag{2.4}$$

The approximation to the integrals (1.3) and (1.5) are

$$I_N = p(b)e^{iq(b)} - p(a)e^{iq(a)}, \tag{2.5}$$

and

$$I_N = p(b,d)e^{iq(b,d)} - p(a,d)e^{iq(a,d)} - p(b,c)e^{iq(b,c)} + p(a,c)e^{iq(a,c)}, \tag{2.6}$$

respectively. In this Kansa's global approach [15] the differentiation matrix $\mathbf{W}$ is full dense, and is very hard to solve the above linear system, when the data points increases, especially in multi-dimensions.

## 3. Local kernel basis function interpolation

Here we construct a local variant of Kansa's method for solving the ODEs/PDEs in David Levin method. The present approximation scheme is based on the local methods in [31]. For a given set of interpolation nodes $x_1, ..., x_N, \in \Omega$, these nodes can be chosen any where in the domain. In the present interpolation scheme at a given node $x_i \in \Omega$, the local interpolant takes the form

$$p(x_i) = \sum_{j=1}^{n} \lambda_j \phi(\|x_i - x_j\|),\ x_j \in \Omega_i \subset \Omega, \tag{3.1}$$

where $\Omega_i$ is a small local domain contains the $n$ neighboring nodes corresponding to the nodes $x_i$. Hence to each local domain $\Omega_i, i = 1, ..., N \in \Omega$, the $N$ number of $n \times n$ linear systems,

$$\mathbf{p}^i = \mathbf{A}^{i \backsim i}, \quad i = 1, ..., N. \tag{3.2}$$

The entries of the matrix $\mathbf{A}^i$ are $a^i_{kj} = \phi(\|x_k - x_j\|)$, $x_k, x_j \in \Omega_i$, $i = 1, ..., N$. The matrix $\mathbf{A}^i$ is called local interpolation matrix, and each system have to be solved for the expansion coefficients.

Now to approximate $\mathcal{L}p(x)$, where $\mathcal{L}$ is a linear differential operator, we have

$$\mathcal{L}p(x_i) = \sum_{j=1}^{n} \lambda_j \mathcal{L}\phi(\|x_i - x_j\|), \; x_j \in \Omega_i \subset \Omega. \tag{3.3}$$

We write equation (3.3) as a dot product of two vectors, as

$$\mathcal{L}p(x_i) = \mathbf{v}^{\mathbf{i}} \cdot \backsim^{\mathbf{i}}, \quad \mathbf{v}^i = \mathcal{L}\phi(\|x_i - x_j\|), \quad x_j \in \Omega_i. \tag{3.4}$$

The expansion coefficients from (3.3) is substituted in (3.4), we get

$$\mathcal{L}p(x_i) = \mathbf{v}^i (\mathbf{A}^i)^{-1} \mathbf{p}^i = \mathbf{w}^i \mathbf{p}^i, \tag{3.5}$$

where,

$$\mathbf{w}^i = \mathbf{v}^i (\mathbf{A}^i)^{-1}, \tag{3.6}$$

is the weight corresponding to center $x_i$. Hence for all centers locations, we have

$$\mathcal{L}p = \mathbf{W}\mathbf{p}, \tag{3.7}$$

consequently, the local interplant of the function $f$ is

$$\mathcal{L}\mathbf{p} = \mathbf{W}\mathbf{p} = \mathbf{f}. \tag{3.8}$$

The solution $\mathbf{p}$

$$\mathbf{p} = \mathbf{W}^{-1}\mathbf{f}. \tag{3.9}$$

Here the differentiation matrix $\mathbf{W}$ is sparse, each row of the matrix $\mathbf{W}$ contains $n$ non-zeros elements and N-n zeros elements, where $n$ is the number of elements in the local domain $\Omega_i$.

### 3.1. *Choosing a good value of shape parameter*

In our local scheme, we used the multiquadrics, $\phi_\varepsilon(r) = \sqrt{1 + (\varepsilon r)^2}$, as well as the Gaussian kernel $\phi_\varepsilon(r) = e^{-\varepsilon^2 r^2}$, as the basis functions in our local interpolant. As usual these kernel functions contain a parameter $\varepsilon$ and the solution accuracy greatly depends on this parameter. In the literature a variety of algorithm are available for choosing good value of shape parameters [6, 9, 10, 23, 27, 29, 30]. In our computation we used the following algorithm based on the uncertainty principle [24] related to the kernel based interpolation theory for choosing good value of parameter.

**Algorithm**
$\kappa = 1$
$10^{13} < \kappa < 10^{15}$
while $\kappa < \kappa_{min}$ and $\kappa > \kappa_{max}$
$\mathbf{L}, \mathbf{S}, \mathbf{U} = svd(\mathbf{A}^i)$
$\kappa = \max(D)/\min(D)$
if $\kappa < \kappa_{min}$, $\varepsilon = \varepsilon - \delta\varepsilon$
if $\kappa > \kappa_{max}$, $\varepsilon = \varepsilon + \delta\varepsilon$
$\varepsilon$ (good) $= \varepsilon$.
Here svd is the singular value decomposition of the interpolation matrix $\mathbf{A}^i$. $\mathbf{L}$, $\mathbf{U}$ are $n \times n$ orthogonal matrices and $\mathbf{D}$ is $n \times n$ diagonal matrix contains the $n$ singular values of $\mathbf{A}^i$, and $\kappa = \max(D)/\min(D)$ is the condition number of the matrix $\mathbf{A}^i$. The good value of shape parameter is obtained from the above algorithm, then the svd is used to compute $(\mathbf{A}^i)^{-1} = (\mathbf{LSU^T})^{-1} = US^{-1}L^T$ (see [5]).

**Theorem 2.** *If $\Phi(r) = \phi(\sqrt{r})$ is completely monotone but not constant on $[0,\infty)$, then for any points $x_k$ in $R^d$, the system matrix $\mathbf{A}$ in the kernel interpolation is positive definite.*

**Definition 1.** A function $\Phi(r)$ is completely monotone on $[0,\infty)$, if

(1) $\Phi(r) \in C[0,\infty)$.

(2) $\Phi(r) \in C^\infty[0,\infty)$.

(3) $(-1)^k \frac{d^k}{dr^k}\Phi(r) \geq 0$, for $r > 0$ and $k = 0,1,2,\ldots$.

**Theorem 3.** *If $\phi(r) \in C[0,\infty)$ with $\phi(0) > 0$ and $\phi(\rho) < 0$ for some $\rho > 0$, then there is an upper limit on the dimension $d$ for which the interpolation problem is nonsingular for all points distributions.*

Various choices of radial kernels have been used in the literature. The existence and uniqueness of the interpolants $p(x)$ have been discussed in the literature [1, 19, 22] for the radial kernels listed in Table 1.

**Lemma 1** ([32]). *Suppose $q(x)$ is real-valued and smooth in $(a,b)$, and that $q^k(x) \geq 1$ for all $x \in (a,b)$ for a fixed value of $k$. Then*

$$\left| \int_a^b e^{i\omega q(x)} \right| \leq c(k)/\omega^{1/k}, \tag{3.10}$$

*holds when (i) $k \geq 2$, or (ii) $k = 1$ and $q'(x)$ is monotonic. The bound $c(k)$ is independent of $q(x)$ and $\omega$, $c(k) = 5 \cdot 2^{k-1} - 2$.*

TABLE 1. Various types of finitely and infinitely smooth radial basis functions.

| | |
|---|---|
| Duchon spline | $\phi(r)=r^{2k}\log r, k\in\mathbb{N},$ |
| Wendland | $\phi(r)=(1-r)_+^k p(r),$ p is a polynomial, $k\in\mathbb{N}$ |
| Matern | $\phi(r)=\frac{2^{1-\nu}}{\Gamma(\nu)}r^\nu K_\nu(r), \nu>0$ |
| Gaussian (GA) | $\phi(r)=e^{-(\varepsilon r)^2}$ |
| Multiquadric (MQ) | $\phi(r)=(1+(\varepsilon r)^2)^{1/2}$ |
| Generalized multiquadric (GMQ) | $\phi(r)=(1+(\varepsilon r)^2)^{\nu/2},\ \nu\neq 0$ |

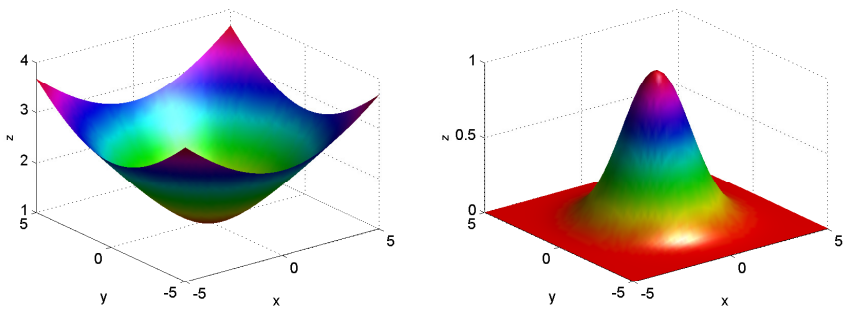


FIGURE 1. Plots of MQ and GA radial basis functions in the domain $[-5,5]$, with shape parameter $\varepsilon=0.5$

**Lemma 2** ([32]). *Under the assumption on $q(x)$ in Lemma 3.3, then*

$$\left|\int_a^b e^{i\omega q(x)}g(x)dx\right|\leq c(k)/\omega^{1/k}\left(|g(b)|+\int_a^b|g'(x)|dx\right). \tag{3.11}$$

**Theorem 4.** *Suppose that $f(x)$ and $q(x)$ are suitably smooth and $q'(x)\neq 0$ for all $x$ in $[a,b]$. Then the quadrature (1.3) at arbitrary interpolation nodes $x_1,\ldots,x_N$ satisfies*

$$E=|I-I_N|\leq\frac{3(N+1)\|F^N(x)\|_\infty(b-a)^N}{\omega N!}, \tag{3.12}$$

*where $F(x)=f(x)-\mathcal{L}p(x)$.*

Proof. Let $\mathcal{L}p(x)$ be the radial kernel interpolant of the function $f(x)$ defined by

$$\mathcal{L}p(x)=\sum_{j=1}^N\lambda_j\phi'(\|x-x_j\|)+i\omega q'(x)\sum_{j=1}^N\lambda_j\phi(\|x-x_j\|),\ j=1,\ldots,N, \tag{3.13}$$

where the interpolation nodes $x_1,\ldots,x_N\in[a,b]$, then

$$E = |I - I_N| = \left| \int_a^b (f(x) - \mathcal{L}p(x)) e^{i\omega q(x)} dx \right| = \left| \int_a^b F(x) e^{i\omega q(x)} dx \right|, \quad (3.14)$$

where, $F(x) = f(x) - \mathcal{L}p(x)$ satisfy the interpolation condition $F(x_j) = 0, j = 1, ..., N$. So there exist $x_j \in (x_j, x_{j+1})$, such that $F'(x_j) = 0,\ j = 1, ..., N-1$. Then $F(x)$ and $F'(x)$ are represented by Lagrange interpolating polynomials

$$F(x) = \frac{F^N(\zeta_1)}{N!} \prod_{j=1}^{N} (x - x_j) \quad (3.15)$$

and

$$F'(x) = \frac{F^N(\zeta_2)}{(N-1)!} \prod_{j=1}^{N-1} (x - x_j) \quad (3.16)$$

for some $\zeta_1, \zeta_2 \in [a, b]$, then for $k = 1$, $c(k) = 3$

$$\begin{aligned} E = |I - I_N| = \left| \int_a^b (f(x) - \mathcal{L}p(x)) e^{i\omega q(x)} dx \right| &= \left| \int_a^b F(x) e^{i\omega q(x)} dx \right| \\ &\leq c(k)/\omega^{1/k} \left( |F(b)| + \int_a^b |F'(x)| dx \right) \\ &\leq \frac{3(N+1)(b-a)^N \|F^N(x)\|_\infty}{\omega N!}. \quad (3.17) \end{aligned}$$

Various types of radial kernels contain a shape parameter $\varepsilon$ in the form $\phi_\varepsilon = \phi(\varepsilon r)$. This types of basis function $\phi_\varepsilon$ becomes increasingly flat when $\varepsilon$ is very small. In such a case we have good approximation properties for both interpolation problems and solving elliptic partial differential equations (see [8, 16, 25] ). However the condition number of the system matrix **A** is, quite large when $\varepsilon$ is very small (see [26]). This conflict situation is known as *uncertainty principle* [24]. This uncertainty principle holds only if the direct approach uses the system matrix **A** to compute the expansion coefficients ˘. It was investigated in [3] that, for infinitely smooth basic functions $\phi(\|.\|)$, such as multiquadrics the limit as $\varepsilon \to 0$, limiting interpolants often exist and take the form of polynomials. In the 1-D case, it is proved in [3] that with simple conditions on the basis function, the radial kernels interpolants converge to the Lagrange interpolating polynomial. In 2-D case, the limiting behavior of the kernel based interpolants as $\varepsilon \to 0$ to be a multivariate polynomial [28].

Under the assumption that when $\varepsilon \to 0$, the kernel based interpolant $\mathcal{L}p(x))$ converges to the Lagrange interpolating polynomial. In other words $(\mathcal{L}p(x))^N \equiv 0$,

which imply $\|F^N(x)\|_\infty = \|f^N(x)\|_\infty$, and from (3.17) we have,

$$E = |I - I_N| \leq \frac{3(N+1)(b-a)^N \|f^N(x)\|_\infty}{\omega N!}. \tag{3.18}$$

## 4. Application of the method and numerical experiments

In this section, the proposed method is applied to check the accuracy and validity of the present method for computing the oscillatory integrals.

**Problem 1**: Here we consider the integrals of the form

$$I = \int_0^1 \sin x \; e^{i\omega(x+Cx^2)} dx, \text{ and } J = \int_{100}^{200} (1+\ln x)\; e^{i(x\ln x)} dx. \tag{4.1}$$

We compute the integrals $I$ and $J$ by the present method using two types of radial kernels, the multiquadric and the Gaussian. The results are compared with Romberg integration technique and with the Levin method [17] (see Tables 2-4). The good accuracy over the Romberg integration technique is achieved. The results of the present method is also well agreed with method of Levin. The error bound developed for the present scheme is tested, and the good agreement between the actual error and the error bound is observed.

TABLE 2. Exact value of the real part of the integral I=$\int_0^1 \sin x \; e^{i\omega(x+Cx^2)} dx$= 4.59859e-004, when $\omega = 500$, $C = 1$, Problem 4.1.

| N | | L method (MQ) | L method (GA) | RI | Levin method [17] |
|---|---|---|---|---|---|
| 5 | $I$ | 4.5793e-004 | 8.1457e-004 | 3.04379076e-002 | 4.60098e-004 |
| 10 | $I$ | 5.8445e-005 | 1.5156e-004 | 2.81033592e-003 | 4.59863e-004 |

TABLE 3. Absolute error and the error bound (3.17), I=$\int_0^1 \sin x \; e^{i\omega(x+Cx^2)} dx$, when $C = 1$, Problem 4.1.

| $\omega$ | 10 | $10^2$ | $10^3$ | $10^4$ |
|---|---|---|---|---|
| $\lvert I - I_N \rvert$ | 8.1902e-002 | 5.9239e-003 | 5.1740e-004 | 9.8644e-005 |
| Error bound(N=5) | 1.5000e-002 | 1.5000e-003 | 1.5000e-004 | 1.5000e-005 |
| $\varepsilon$(MQ) | 0.0300 | 0.0300 | 0.0300 | 0.0300 |
| $\kappa$ | 3.2626e+016 | 3.2626e+016 | 3.2626e+016 | 3.2626e+016 |

**Problem 2**: Here we consider the integral of the form

$$I = \int_0^1 \int_0^1 \cos(x+y)\; e^{i\omega(x+y+C(x^2+y^2))} dx, \tag{4.2}$$

TABLE 4. Exact value of the real part of the integral J=$\int_{100}^{200}(1+\ln x)\, e^{i(x\ln x)}dx$= -1.77429e+00, when $\omega = 1$, Problem 4.1.

| N,n | | L method (MQ) | L method (GA) |
|---|---|---|---|
| 500,70 | $I$ | -1.5377e+000 | -1.3643e+000 |
| | $\tau$ | 2.006854 | 2.081579 |

We compute the integral for $N = 25$ equally spaced nodes in the domain of integration $[0,1]\times[0,1]$, and $n = 8$ nodes in each local domain $\Omega_i$, for the choice $\omega = 100$ and $C = 1$. We used two radial kernels the multiquadric and the Gaussian. The results of the present method are well agreed with the results of the Levin method [17].

TABLE 5. Exact value of the real part of the integral I= -6.92072e-009 $[0,1]^2$, $\omega = 10^4$, $C = 1$, corresponding to Problem 4.2.

| N,n | | L method (MQ) | L method (GA) | Levin method [17] |
|---|---|---|---|---|
| 25,8 | $I$ | -7.2313e-008 | -7.5778e-009 | -6.9208e-009 |

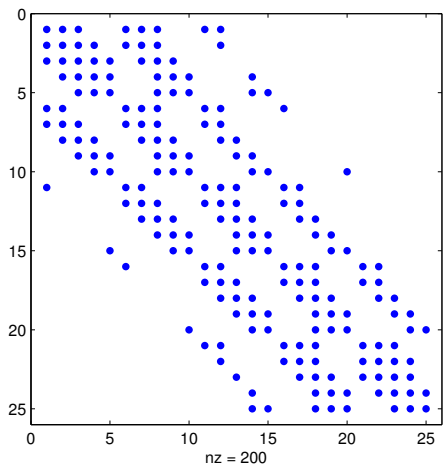


FIGURE 2. Sparsity of the differentiation matrix $W$ in Problem 4.2, where out of 625 entries only 200 are non zeros and the rest are zeros.

## 5. Conclusions

In this work the local kernel based interpolation scheme in constructed for evaluating the oscillatory integrals. The error bound of the present scheme is also developed. The proposed scheme is based on the work of [17, 31]. The differentiation matrices in the proposed kernel based method are sparse in nature. Most of the interpolation schemes including Levin method [17] faces problems for large number of nodes in the domain of integration due to the dense interpolation matrices. The current numerical scheme is capable of handling thousand of nodes in the domain of integration. In fact we are solving small size interpolation problems of size $n \leq N$ corresponding to each domain $\Omega_i$.

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