HU e-ISSN 1787-2413

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 16 (2015), NUMBER 1

MISKOLC UNIVERSITY PRESS

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VOLUME 16 (2015), NUMBER 1

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Doi: 10.18514/MMN.2015.1221

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 3–15

HU e-ISSN 1787-2413

NOTE ON A q-ANALOGUE OF STANCU-KANTOROVICH OPERATORS

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Received 29 April, 2014

Abstract. The goal of this paper is to introduce new q-Stancu-Kantorovich operators and to study some of their approximation properties. A convergence theorem using the well known Bohman-Korovkin criterion is proven and the rate of convergence involving the modulus of continuity is established. Furthermore, a Voronovskaja type theorem is also proven.

2010 *Mathematics Subject Classification:* 41A10; 41A25; 41A36

Keywords: q-Stancu-Kantorovich operators, modulus of continuity, rate of convergence, Voronovskaja theorem

1. INTRODUCTION

In 1987, A Lupaş [7] introduced and studied q analogue of Bernstein operators, while in 1996 another generalization of these operators based on q-integers was introduced by Philips [13]. Generalizations of the Bernstein operators based on the q-integers attracted a lot of interest and were studied by a number of authors. During the last decade, the applications of q-calculus in the study of linear and positive operators were considered in [5,6,8,10,12,14–16]. The goal of the present paper is to introduce a Kantorovich variant of q-Stancu operators and investigate their approximation properties. Many generalizations and applications of the Stancu operators were considered in the last years ([1,2,11,15]).

Before proceeding further, we mention some basic definitions and notations from q-calculus. Let $q>0$. For each nonnegative integer k, the q-integer $[k]$ and q-factorial $[k]!$ are respectively defined by

$$[k]:=\begin{cases}\dfrac{1-q^k}{1-q}, & q\neq 1,\\ k, & q=1,\end{cases}\quad,\quad [k]!:=\begin{cases}[k][k-1]\cdots[1], & k\geq 1,\\ 1, & k=0.\end{cases}$$

For the integers n,k satisfying $n\geq k\geq 0$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n\\ k\end{array}\right]:=\frac{[n]!}{[k]![n-k]!}.$$

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We denote $(a+b)_q^k = \prod_{j=0}^{k-1}(a+bq^j)$.

The q-Jackson integral on the interval [0,b] is defined as

$$\int_0^b f(t)d_qt = (1-q)b\sum_{j=0}^{\infty} f(q^jb)q^j, 0<q<1,$$

provided that sums converge absolutely. Suppose $0<a<b$. The q-Jackson integral on the interval $[a,b]$ is defined as

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt, 0<q<1.$$

The clasical Bernstein-Kantorovich operators B_n^*, $n=1,2,\dots$ are defined by

$$\begin{aligned} B_n^*(f,x) &:= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{k/n+1}^{k+1/n+1} f(t)dt \\ &= \sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f\left(\frac{k+t}{n+1}\right)dt,\ f:[0,1]\to\mathbb{R}, \end{aligned} \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$.

In [3] the author introduced the q-analogue of the Bernstein-Kantorovich operators

$$B_{n,q}(f;x) = [n+1]\sum_{k=0}^{n} q^{-k}p_{n,k}(q;x)\int_{[k]/[n+1]}^{[k+1]/[n+1]} f(t)d_qt,$$

where $f\in C[0,1]$ and $p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k}$.

In [8], inspired by (1.1), the authors introduced a q-type generalization of Bernstein-Kantorovich operators as follows

$$B_{n,q}^*(f,x) := \sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 f\left(\frac{[k]+q^kt}{[n+1]}\right)d_qt, \tag{1.2}$$

where $f\in C[0,1]$, $0<q<1$.

These operators verify:

Lemma 1 ([8]). *For all $n\in\mathbb{N}$, $x\in[0,1]$ and $0<q<1$ we have*

$$B_{n,q}^*(1,x)=1,\ B_{n,q}^*(t,x) = \frac{2q}{[2]}\frac{[n]}{[n+1]}x + \frac{1}{[2]}\frac{1}{[n+1]},$$

$$B_{n,q}^*(t^2,x) = \frac{q^2(q+2)}{[3]}\frac{[n][n-1]}{[n+1]^2}x^2 + \frac{4q+7q^2+q^3}{[2][3]}\frac{[n]}{[n+1]^2}x + \frac{1}{[3]}\frac{1}{[n+1]^2}.$$

In [15] two kinds of Kantorovich-type q-Bernstein-Stancu operators were introduced. The first version is defined using q-Jackson integral as follows

$$S_{n,q}^{(\alpha,\beta)}(f,x)=([n+1]+\beta)\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_{\frac{[k]+\alpha}{[n+1]+\beta}}^{\frac{[k+1]+\alpha}{[n+1]+\beta}}f(t)d_qt. \tag{1.3}$$

To guarantee the positivity of $S_{n,q}^{(\alpha,\beta)}$ in [15] a Riemann-type q-integral is considered (see [9]), with the following definition:

$$\int_a^b f(t)d_q^Rt=(1-q)(b-a)\sum_{j=0}^{\infty}f\left(a+(b-a)q^j\right)q^j.$$

M.Y. Ren and X.M. Zeng redefine $S_{n,q}^{(\alpha,\beta)}(f;x)$ by putting the Riemann-type q-integral into the operators instead of the q-Jackson integral as

$$\tilde{S}_{n,q}^{(\alpha,\beta)}(f,x)=([n+1]+\beta)\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_{\frac{[k]+\alpha}{[n+1]+\beta}}^{\frac{[k+1]+\alpha}{[n+1]+\beta}}f(t)d_q^Rt. \tag{1.4}$$

The idea of using Riemann-type q-integral in the study of linear and positive operators, in order to guarantee the positivity, was first used by Ö. Dalmanoğlu and O. Doğru [4] for q-Bernstein-Kantorovich operators.

Inspired by (1.2) we introduce a q-type generalization of Stancu-Kantorovich operators as follows

$$S_{n,q}^{*(\alpha,\beta)}(f,x)=\sum_{k=0}^{n}p_{n,k}(q;x)\int_0^1 f\left(\frac{[k]+q^kt+\alpha}{[n+1]+\beta}\right)d_qt, \tag{1.5}$$

where $0\le\alpha\le\beta$ and $f\in C[0,1]$.

2. Uniform approximation by q-Stancu-Kantorovich operators

Using the Bohman-Korovkin criterion we shall prove a convergence theorem for the q-Stancu-Kantorovich operators. First, we need the following two lemmas.

Lemma 2. *For all $n\in\mathbb{N}$, $x\in[0,1]$ and $0<q<1$, we have*

$$S_{n,q}^{*(\alpha,\beta)}(t^m,x)=\frac{[n+1]^m}{([n+1]+\beta)^m}\sum_{\nu=0}^{m}\frac{\alpha^{m-\nu}}{[n+1]^{m-\nu}}\binom{m}{\nu}B_{n,q}^{*}(t^\nu,x). \tag{2.1}$$

Proof. By direct computation we obtain

$$S_{n,q}^{*(\alpha,\beta)}(t^m,x)=\frac{[n+1]^m}{([n+1]+\beta)^m}\sum_{k=0}^{n}p_{n,k}(q;x)\int_0^1\left(\frac{[k]+q^kt+\alpha}{[n+1]}\right)^m d_qt$$

$$=\frac{[n+1]^m}{([n+1]+\beta)^m}\sum_{k=0}^{n}p_{n,k}(q;x)\int_0^1\sum_{\nu=0}^{m}\binom{m}{\nu}\left(\frac{[k]+q^kt}{[n+1]}\right)^\nu\frac{\alpha^{m-\nu}}{[n+1]^{m-\nu}}d_qt$$

$$= \frac{[n+1]^m}{([n+1]+\beta)^m} \sum_{\nu=0}^{m} \frac{\alpha^{m-\nu}}{[n+1]^{m-\nu}} \binom{m}{\nu} \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]+q^k t}{[n+1]} \right)^{\nu} d_q t$$

$$= \frac{[n+1]^m}{([n+1]+\beta)^m} \sum_{\nu=0}^{m} \frac{\alpha^{m-\nu}}{[n+1]^{m-\nu}} \binom{m}{\nu} B^*_{n,q}(t^\nu, x).$$

□

Lemma 3. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < 1$, we have*

$$S^{*(\alpha,\beta)}_{n,q}(1,x) = 1,\ S^{*(\alpha,\beta)}_{n,q}(t,x) = \frac{2q}{[2]} \frac{[n]}{[n+1]+\beta} x + \frac{\alpha}{[n+1]+\beta} + \frac{1}{[2]([n+1]+\beta)},$$

$$S^{*(\alpha,\beta)}_{n,q}(t^2,x) = \frac{1}{([n+1]+\beta)^2} \left\{ \frac{q^2(q+2)}{[3]} [n][n-1]x^2 \right.$$
$$\left. + \frac{q[n]}{[2]} \left(4\alpha + \frac{4+7q+q^2}{[3]} \right) x + \frac{2\alpha}{[2]} + \frac{1}{[3]} + \alpha^2 \right\}.$$

Proof. Taking into account (2.1) and Lemma 1, by direct computation we obtain explicit formulas for $S^{*(\alpha,\beta)}_{n,q}(t^i,x)$, $i = 0,1,2$, as follows

$$S^{*(\alpha,\beta)}_{n,q}(1,x) = B^*_{n,q}(1,x) = 1,$$

$$S^{*(\alpha,\beta)}_{n,q}(t,x) = \frac{[n+1]}{[n+1]+\beta} \left\{ \frac{\alpha}{[n+1]} B^*_{n,q}(1,x) + B^*_{n,q}(t,x) \right\}$$
$$= \frac{2q}{[2]} \frac{[n]}{[n+1]+\beta} x + \frac{\alpha}{[n+1]+\beta} + \frac{1}{[2]([n+1]+\beta)},$$

$$S^{*(\alpha,\beta)}_{n,q}(t^2,x) = \frac{[n+1]^2}{([n+1]+\beta)^2} \left\{ \frac{\alpha^2}{[n+1]^2} B^*_{n,q}(1,x) + \frac{2\alpha}{[n+1]} B^*_{n,q}(t,x) + B^*_{n,q}(t^2,x) \right\}$$
$$= \frac{1}{([n+1]+\beta)^2} \left\{ \frac{q^2(q+2)}{[3]} [n][n-1]x^2 \right.$$
$$\left. + \frac{q[n]}{[2]} \left(4\alpha + \frac{4+7q+q^2}{[3]} \right) x + \frac{2\alpha}{[2]} + \frac{1}{[3]} + \alpha^2 \right\}.$$

□

Lemma 4. *For all $n \in \mathbb{N}$, $x \in [0,1]$ and $0 < q < 1$, we have*

$$S^{*(\alpha,\beta)}_{n,q}\left((t-x)^2,x\right) \le \frac{2[n+1]^2}{([n+1]+\beta)^2} \left\{ \frac{4}{[n]} \left(x(1-x) + \frac{1}{[n]} \right) + \left(\frac{\alpha}{[n+1]} - \frac{\beta}{[n+1]} x \right)^2 \right\},$$

$$S^{*(\alpha,\beta)}_{n,q}\left((t-x)^4,x\right) \le \frac{8[n+1]^2}{([n+1]+\beta)^2} \left\{ \frac{C}{[n]^2} \left(x(1-x) + \frac{1}{[n]^2} \right) + \left(\frac{\alpha}{[n+1]} - \frac{\beta}{[n+1]} x \right)^4 \right\},$$

where C is a positive absolute constant.

Proof. The proof is based on the estimations of the second and fourth order central moments of the q-Bernstein-Kantorovich operators (Lemma 2.4 in [8]):

$$B^*_{n,q}\left((t-x)^2,x\right) \leq \frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right),$$
$$B^*_{n,q}\left((t-x)^4,x\right) \leq \frac{C}{[n]^2}\left(x(1-x)+\frac{1}{[n]^2}\right).$$

We have

$$S^{*(\alpha,\beta)}_{n,q}\left((t-x)^2,x\right) = \sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{[k]+\alpha+q^k t}{[n+1]+\beta}-x\right)^2 d_q t$$
$$= \frac{[n+1]^2}{([n+1]+\beta)^2}\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{[k]+q^k t}{[n+1]}-x+\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^2 d_q t$$
$$\leq \frac{2[n+1]^2}{([n+1]+\beta)^2}\left\{\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{[k]+q^k t}{[n+1]}-x\right)^2 d_q t\right.$$
$$\left.+\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^2 d_q t\right\}$$
$$= \frac{2[n+1]^2}{([n+1]+\beta)^2}\left\{\frac{4}{[n]}\left(x(1-x)+\frac{1}{[n]}\right)+\left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^2\right\}.$$

A similar calculus leads to

$$S^{*(\alpha,\beta)}_{n,q}\left((t-x)^4,x\right) \leq \frac{8[n+1]^2}{([n+1]+\beta)^2}\left\{\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{[k]+q^k t}{[n+1]}-x\right)^4 d_q t\right.$$
$$\left.+\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 \left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^4 d_q t\right\}$$
$$\leq \frac{8[n+1]^2}{([n+1]+\beta)^2}\left\{\frac{C}{[n]^2}\left(x(1-x)+\frac{1}{[n]^2}\right)+\left(\frac{\alpha}{[n+1]}-\frac{\beta}{[n+1]}x\right)^4\right\}.$$

□

Theorem 1. *Let $(q_n)_n$, $0<q_n<1$ be a sequence satisfying the following conditions*

$$\lim_{n\to\infty} q_n = 1,\ \lim_{n\to\infty} q_n^n = a,\ a\in[0,1). \tag{2.2}$$

Then for any $f\in C[0,1]$, the sequence $\{S^{*(\alpha,\beta)}_{n,q_n}(f,x)\}$ converges to f uniformly on $[0,1]$.

Proof. Using the Bohman-Korovkin theorem, it is sufficient to verify

$$\lim_{n\to\infty} S^{*(\alpha,\beta)}_{n,q_n}(e_i,x) = x^i,\ i=0,1,2,\ \text{uniformly on } [0,1]. \tag{2.3}$$

If $q_n \to 1$, then $\lim_{n\to\infty}[n]=\infty$ and $\lim_{n\to\infty}\dfrac{[n]}{[n+1]+\beta}=1$, $\lim_{n\to\infty}\dfrac{[n][n-1]}{([n+1]+\beta)^2}=1$, hence (2.3) follows from Lemma 3.

□

3. The rate of convergence of q-Stancu-Kantorovich operators

In this section we will estimate the rate of convergence in terms of modulus of continuity. Furthermore, we calculate the rate of convergence in terms of the first modulus of continuity of the derivative of the function.

Recall that the first modulus of continuity of $f \in C[0,b]$ for $\delta > 0$ is given by

$$\omega(f,\delta) = \sup_{|y-x|\le\delta} |f(y)-f(x)|,\ x,y\in[0,b].$$

Theorem 2. *Let $(q_n)_n$ be a sequence satisfying conditions (2.2). Then*

$$\left|S^{*(\alpha,\beta)}_{n,q_n}(f,x)-f(x)\right| \le 2\omega(f;\sqrt{\delta_n(x)})$$

for all $f\in C[0,1]$, where

$$\delta_n(x) = S^{*(\alpha,\beta)}_{n,q_n}\left((t-x)^2,x\right)$$

Proof. Since $S^{*(\alpha,\beta)}_{n,q_n}(e_0,\cdot)=e_0$, we have

$$\begin{aligned}\left|S^{*(\alpha,\beta)}_{n,q_n}(f,x)-f(x)\right| &= \left|\sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1\left[f\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}\right)-f(x)\right]d_{q_n}t\right| \\ &\le \sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1\left|f\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}\right)-f(x)\right|d_{q_n}t.\end{aligned}$$

Using the following well known property of modulus of continuity

$$|f(t)-f(x)| \le \omega(f;\delta)\left(\frac{|t-x|}{\delta}+1\right),\ \delta>0 \tag{3.1}$$

we get

$$\left|S^{*(\alpha,\beta)}_{n,q_n}(f,x)-f(x)\right| \le \sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1\left(\frac{1}{\delta}\left|\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right|+1\right)\omega(f,\delta)d_{q_n}t$$

$$= \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1\left|\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right|d_{q_n}t.$$

By the Cauchy-Schwarz inequality it follows

$$\begin{aligned}
&\sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1 \left|\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right| d_{q_n}t \\
&\leq \sum_{k=0}^{n} p_{n,k}(q_n;x)\left\{\int_0^1 \left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{1/2}\left\{\int_0^1 d_{q_n}t\right\}^{1/2} \\
&= \sum_{k=0}^{n} \left\{p_{n,k}(q_n;x)\right\}^{1/2}\left\{p_{n,k}(q_n;x)\int_0^1 \left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{1/2} \\
&\leq \left\{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)\right\}^{1/2}.
\end{aligned}$$

Therefore

$$\left|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)\right| \leq \left(1+\frac{1}{\delta}\sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)}\right)\omega(f,\delta).$$

But, by Lemma 4 and conditions (2.2), $\lim_{n\to\infty} S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)=0$. So, letting $\delta_n(x)=\sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)}$ and taking $\delta=\sqrt{\delta_n(x)}$, we get the result. □

Denote

$$Lip_M(\gamma)=\{f\in C[0,1], |f(t)-f(x)|\leq M|t-x|^{\gamma}\}, 0<\gamma\leq 1.$$

The next result give the rate of convergence of operators $S_{n,q_n}^{*(\alpha,\beta)}$ in terms of the Lipschitz class.

Theorem 3. *Let $(q_n)_n$ be a sequence satisfying conditions (2.2) and $f\in Lip_M(\gamma)$. Then*

$$|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)|\leq M(\delta_n(x))^{\gamma/2},$$

where $\delta_n(x)=S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)$.

Proof. Since $S_{n,q_n}^{*(\alpha,\beta)}(e_0,\cdot)=e_0$ and $f\in Lip_M(\gamma)$, we have

$$\begin{aligned}
|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)| &\leq \sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1 \left|f\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}\right)-f(x)\right| d_{q_n}t \\
&\leq M\sum_{k=0}^{n} p_{n,k}(q_n;x)\int_0^1 \left|\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right|^{\gamma} d_{q_n}t.
\end{aligned}$$

Appling the Hölder's inequality with $p=\dfrac{2}{\gamma}$ and $q=\dfrac{2}{2-\gamma}$, we get

$$\begin{aligned}
&\left|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)\right| \\
&\leq M\sum_{k=0}^{n}p_{n,k}(q_n;x)\left\{\int_0^1\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{\gamma/2}\left\{\int_0^1 d_{q_n}t\right\}^{\frac{2-\gamma}{2}} \\
&= M\sum_{k=0}^{n}p_{n,k}(q_n;x)\left\{\int_0^1\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{\gamma/2} \\
&= M\sum_{k=0}^{n}\left(p_{n,k}(q_n;x)\right)^{\frac{2-\gamma}{2}}\left\{p_{n,k}(q_n;x)\int_0^1\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{\gamma/2} \\
&\leq M\left\{\sum_{k=0}^{n}p_{n,k}(q_n;x)\int_0^1\left(\frac{[k]+q_n^k t+\alpha}{[n+1]+\beta}-x\right)^2 d_{q_n}t\right\}^{\gamma/2} \\
&= M\left\{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)\right\}^{\gamma/2}.
\end{aligned}$$

□

Theorem 4. *Let (q_n), $0<q_n<1$ be a sequence satisfying the conditions (2.2). Then for any $f\in C^1[0,1]$ and $x\in[0,1]$ we have*

$$\left|S_{n,q_n}^{*(\alpha,\beta)}(f;x)-f(x)\right|\leq\lambda_n(x)|f'(x)|+2\sqrt{\delta_n(x)}\omega(f',\sqrt{\delta_n(x)}), \tag{3.2}$$

where $\lambda_n(x)=S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)$ and $\delta_n(x)=S_{n,q_n}^{*(\alpha,\beta)}((t-x)^2,x)$.

Proof. Let $f\in C^1[0,1]$. For any $t\in[0,1]$, $x\in[0,1]$, we have

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t\left(f'(u)-f'(x)\right)du,$$

so we get

$$\begin{aligned}
&S_{n,q_n}^{*(\alpha,\beta)}\left(f(t)-f(x),x\right) \\
&\qquad=f'(x)S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)+S_{n,q_n}^{*(\alpha,\beta)}\left(\int_x^t(f'(u)-f'(x))du,x\right).
\end{aligned}$$

Using the well known property of modulus of continuity (3.1) we have

$$\left|\int_x^t|f'(u)-f'(x)|du\right|\leq\omega(f';\delta)\left[\frac{(t-x)^2}{\delta}+|t-x|\right].$$

Therefore

$$|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)| \le |f'(x)|\cdot|S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)| + \omega(f';\delta)\left\{\frac{1}{\delta}S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)+S_{n,q_n}^{*(\alpha,\beta)}(|t-x|,x)\right\}.$$

Using Cauchy-Schwartz inequality, we obtain

$$S_{n,q_n}^{*(\alpha,\beta)}(|t-x|,x) \le \sqrt{S_{n,q_n}^{*(\alpha,\beta)}(1,x)}\cdot\sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)},$$
$$S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right) \le \sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)}\cdot\sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)},$$

so we have

$$\begin{aligned}|S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)| &\le |f'(x)|\cdot|S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)| \\ &\quad+\omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)}+1\right\}\sqrt{S_{n,q_n}^{*}\left((t-x)^2,x\right)} \\ &= |f'(x)|\lambda_n(x)+\omega(f',\delta)\cdot\left\{\frac{1}{\delta}\sqrt{\delta_n(x)}+1\right\}\sqrt{\delta_n(x)}.\end{aligned}$$

Choosing $\delta=\sqrt{\delta_n(x)}$, we find relation (3.2). □

4. A Voronovskaya theorm for q-Stancu-Kantorovich operators

In this section we shall establish a Voronovskaja type theorem for $S_{n,q}^{*(\alpha,\beta)}$. First, we need the auxiliary result contained in the following lemma.

Lemma 5. *Assume that $0<q_n<1$, $q_n\to 1$ and $q_n^n\to a$, $a\in[0,1)$ as $n\to\infty$. Then we have*

$$\lim_{n\to\infty}[n]S_{n,q_n}^{*(\alpha,\beta)}(t-x;x) = -\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2},$$
$$\lim_{n\to\infty}[n]S_{n,q_n}^{*(\alpha,\beta)}((t-x)^2;x) = -\frac{2a+1}{3}x^2+x.$$

Proof. To prove the lemma we use the formulas for $S_{n,q_n}^{*(\alpha,\beta)}(t^i,x), i=0,1,2$ given in Lemma 3.

$$\begin{aligned}&\lim_{n\to\infty}[n]S_{n,q_n}^{*(\alpha,\beta)}(t-x;x)\\ &=\lim_{n\to\infty}[n]\left\{\left(\frac{2q_n}{[2]}\frac{[n]}{[n+1]+\beta}-1\right)x+\frac{\alpha}{[n+1]+\beta}+\frac{1}{[2]([n+1]+\beta)}\right\}\\ &=\lim_{n\to\infty}\left\{\frac{[n]}{[2]([n+1]+\beta)}(-1-q_n^{n+1}-[2]\beta)x+\frac{\alpha[n]}{[n+1]+\beta}+\frac{[n]}{[2]([n+1]+\beta)}\right\}\\ &=-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}.\end{aligned}$$

$$\lim_{n\to\infty}[n]S_{n,q_n}^{*(\alpha,\beta)}((t-x)^2;x)=\lim_{n\to\infty}[n]\left\{S_{n,q_n}^{*(\alpha,\beta)}(t^2,x)-x^2-2xS_{n,q_n}^{*(\alpha,\beta)}(t-x,x)\right\}$$

$$=\lim_{n\to\infty}[n]\left(\frac{q_n^2(q_n+2)[n][n-1]}{[3]([n+1]+\beta)^2}-1\right)x^2$$

$$+\lim_{n\to\infty}\frac{[n]^2}{[2]([n+1]+\beta)^2}\left(4q_n\alpha+\frac{4q_n+7q_n^2+q_n^3}{[3]}\right)x$$

$$+\lim_{n\to\infty}[n]\left\{\frac{\alpha^2}{([n+1]+\beta)^2}+\frac{2\alpha}{[2]([n+1]+\beta)^2}+\frac{1}{[3]([n+1]+\beta)^2}\right\}$$

$$-\lim_{n\to\infty}2x[n]S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)$$

$$=\lim_{n\to\infty}[n]\left(\frac{q_n(q_n+2)([n]^2-[n])}{[3]([n+1]+\beta)^2}-1\right)x^2+(2\alpha+2)x-2x\left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)$$

$$=\lim_{n\to\infty}\frac{[n]}{[3]([n+1]+\beta)^2}\left\{q_n(q_n+2)[n]^2-q_n(q_n+2)[n]\right.$$

$$\left.-[3](1+q_n[n]+\beta)^2\right\}x^2+x+x^2(1+a+2\beta)$$

$$=\lim_{n\to\infty}\frac{[n]}{[3]([n+1]+\beta)^2}\left\{\left(q_n(q_n+2)-[3]q_n^2\right)[n]^2\right.$$

$$\left.-q_n\left(q_n+2+2[3](1+\beta)\right)[n]-[3](1+\beta)^2\right\}x^2$$

$$+x+x^2(1+a+2\beta)=-\frac{2a+1}{3}x^2+x.$$

□

The main result of this section is the following Voronovskaja type theorem:

Theorem 5. *Let $f''\in C[0,1]$ and $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to a$, $a\in[0,1)$ as $n\to\infty$. Then we have*

$$\lim_{n\to\infty}[n](S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x))=\left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)$$
$$+\frac{1}{2}\left(-\frac{2a+1}{3}x^2+x\right)f''(x).$$

Proof. From the Taylor's theorem, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+\frac{1}{2}(t-x)^2(f''(\xi)-f''(x)),\quad(4.1)$$

where ξ lies between t and x.

Applying $S_{n,q_n}^{*(\alpha,\beta)}$ on both side of (4.1), we obtain

$$[n]\left(S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x)\right)=[n]f'(x)S_{n,q_n}^{*(\alpha,\beta)}(t-x,x)$$

$$+\frac{1}{2}[n]f''(x)S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2,x\right)$$
$$+[n]S_{n,q_n}^{*(\alpha,\beta)}\left(\frac{(t-x)^2}{2}(f''(\xi)-f''(x)),x\right). \quad (4.2)$$

For all $x,y\in[0,1]$, $|f''(\xi)-f''(x)|\leq\omega_{f''}(\delta)\left(1+\frac{(t-x)^2}{\delta^2}\right)$, for any $\delta>0$.

In view of the Lemma 4 and Lemma 5, we have

$$\left|[n]S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2(f''(\xi)-f''(x)),x\right)\right|$$
$$\leq\omega_{f''}(\delta)[n]S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2+\frac{(t-x)^4}{\delta^2},x\right)=\omega_{f''}(\delta)\left(O(1)+\frac{1}{\delta^2}O\left(\frac{1}{[n]}\right)\right).$$

Choosing $\delta=[n]^{-1/2}$, we get

$$\left|[n]S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2(f''(\xi)-f''(x)),x\right)\right|=\omega_{f''}\left([n]^{-1/2}\right)\cdot O(1).$$

Hence

$$\lim_{n\to\infty}\left|[n]S_{n,q_n}^{*(\alpha,\beta)}\left((t-x)^2(f''(\xi)-f''(x)),x\right)\right|=0.$$

In view of Lemma 5, we obtain

$$\lim_{n\to\infty}[n](S_{n,q_n}^{*(\alpha,\beta)}(f,x)-f(x))=\left(-\frac{1+a+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)$$
$$+\frac{1}{2}\left(-\frac{2a+1}{3}x^2+x\right)f''(x).$$

□

5. NUMERICAL EXAMPLE

Let us take $f(x)=x^2$. In Table 5 we compute error estimation for operators (1.4) and (1.5) to the function f.

TABLE 1. Error estimate ($n=30$, $\alpha=2$, $\beta=4$, $q=0.89$)

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\tilde{S}_{n,q}^{(\alpha,\beta)}$	0.0393	0.0631	0.0743	0.0730	0.0590	0.0325	0.0067	0.0584	0.1227	0.1996	0.2890
$S_{n,q}^{*(\alpha,\beta)}$	0.0393	0.0638	0.0763	0.0768	0.0652	0.0417	0.0062	0.0412	0.1007	0.1722	0.2556

For $n=30$, $\alpha=2$, $\beta=4$, $q=0.89$, the convergence of operators (1.4) and (1.5) to function $f(x)=x^2$ is illustrated in Figure 1.

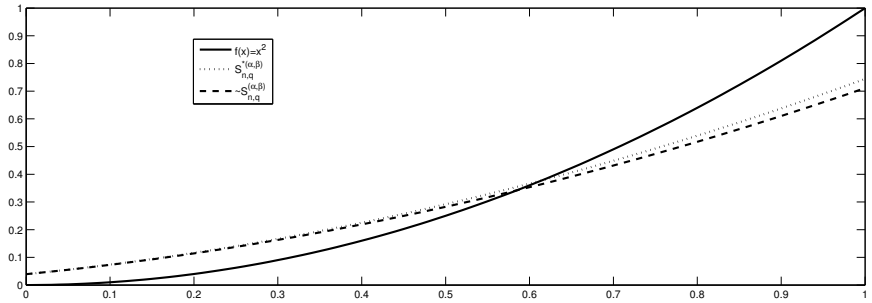

FIGURE 1. Function $f(x) = x^2$ and q-Stancu-Kantorovich operators.

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Doi: 10.18514/MMN.2015.1297

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 17–24

HU e-ISSN 1787-2413

ON STRONG COMMUTATIVITY PRESERVING LIKE MAPS IN RINGS WITH INVOLUTION

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Received 25 June, 2014

Abstract. The main purpose of this paper is to prove the following result: Let R be a prime ring with involution of the second kind and with $char(R) \neq 2$. If R admits a nonzero derivation $d : R \to R$ such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, then R is commutative. We also provide an example which shows that the above result does not holds in case the involution is of the first kind. Moreover, a related result has also been obtained.

2010 *Mathematics Subject Classification:* 16N60; 16W10; 16W25.

Keywords: prime ring, involution, derivation, strong commutativity preserving

1. INTRODUCTION

Throughout this article, R will represent an associative ring with centre $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. However, given two subsets A and B of R, $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[x, y]$ where $x \in A$ and $y \in B$ and $A \circ B$ is defined similarly. Further, $\overline{A}$ will be the subring of R generated by A. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and is called a semiprime ring in case $aRa = (0)$ implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An

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example is the ring of quaternions. A description of such rings can be found in [13], where further references can be found.

An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the last 30 years, several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps on R. The first result in this direction is due to Divinsky[12], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphim. Two years later, Posner [21] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined and extended these results in various directions (viz., [3, 5, 6], where further references can be found).

We say that a map $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [8, 10] for references). Following [7], let S be a subset of R, a map $f : R \to R$ is said to be strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [4], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. Precisely, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [11] proved that if there exist a derivation d of a semiprime ring R and a map $f : I \to R$ defined on a nonzero ideal I of R such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Further, Ali and Huang [2], showed that if R is a 2-torsion free semiprime ring and d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal. Many related generalizations of these results can be found in the literature (see for instance [9, 15–20, 22, 23]).

The main purpose of the present paper is to initiate the study of a more general concept than SCP mappings. More precisely, we consider an additive mapping $f : R \to R$ satisfying $[f(x), f(x^*)] = [x, x^*]$ for all $x \in R$. In fact, we investigate the commutativity of a prime ring with involution, when the mapping f is assumed to be a derivation of R. Moreover, a related result is obtained by replacing the commutator by anti-commutator.

2. Results

We begin with the following lemma, which is essentially proved in [1, Lemma 2.1].

Lemma 1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Theorem 1. *Let R be a prime ring with involution $*$ of the second kind and with $char(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$. Then R is commutative.*

Proof. By the assumption, we have

$$[d(x), d(x^*)] = [x, x^*] \tag{2.1}$$

for all $x \in R$. A lineralization of (2.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = [x, y^*] + [y, x^*] \tag{2.2}$$

for all $x, y \in R$. Replacing y by xx^* in (2.2) and making use of (2.1), we arrive at

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \tag{2.3}$$

for all $x \in R$. Replacing x by $x + h^{'}$, where $h^{'} \in H(R) \cap Z(R)$, we obtain

$$d(h^{'})[d(x), x^*] + [d(x), x]d(h^{'}) + d(h^{'})[x^*, d(x^*)] + [x, d(x^*)]d(h^{'}) = 0.$$

This can be further written as

$$d(h^{'})([d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]) = 0$$

for all $h^{'} \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors we get either $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Suppose

$$d(h^{'}) = 0 \text{ for all } h^{'} \in H(R) \cap Z(R). \tag{2.4}$$

Replacing $h^{'}$ by $(k^{'})^2$ in (2.4), where $k^{'} \in S(R) \cap Z(R)$, we get

$$0 = d(h^{'}) = d((k^{'})^2) = d(k^{'})k^{'} + k^{'}d(k^{'}) = 2d(k^{'})k^{'}.$$

Since $char(R) \neq 2$, we arrive at

$$d(k^{'})k^{'} = 0 \text{ for all } k^{'} \in S(R) \cap Z(R).$$

For each $k^{'} \in S(R) \cap Z(R)$, the last expression yields that either $d(k^{'}) = 0$ or $k^{'} = 0$. Since $k^{'} = 0$ implies $d(k^{'}) = 0$, we may write

$$d(k^{'}) = 0 \text{ for all } k^{'} \in S(R) \cap Z(R). \tag{2.5}$$

Let $x \in Z(R)$. Since $char(R) \neq 2$, every $x \in Z(R)$ can be represented as $2x = h + k$, where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. This implies that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = 0$. Since $char(R) \neq 2$, we get

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.6}$$

Replacing y by $k^{'}y$ in (2.2), where $k^{'} \in S(R) \cap Z(R)$ and using (2.6), we arrive at

$$k^{'}(-[d(x), d(y^*)] + [d(y), d(x^*)] + [x, y^*] - [y, x^*]) = 0$$

for all $k' \in S(R) \cap Z(R)$ and $x, y \in R$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-[d(x), d(y^*)] + [d(y), d(x^*)] = -[x, y^*] + [y, x^*] \tag{2.7}$$

for all $x, y \in R$. On comparing (2.2) and (2.7), we obtain $2[d(x), d(y^*)] = 2[x, y^*]$ for all $x, y \in R$. Replacing y by y^* and using the fact that $char(R) \neq 2$, we conclude that $[d(x), d(y)] = [x, y]$ for all $x, y \in R$. Therefore in view of [4, Theorem 1], R is commutative. Now we consider the case

$$[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we obtain $4[d(k), h)] = 0$. Since $char(R) \neq 2$, we are force to conclude that

$$[d(k), h] = 0 \text{ for all } h \in H(R) \; and \; k \in S(R). \tag{2.8}$$

Replacing h by $k_0 k'$, where $k_0 \in S(R)$ and $k' \in S(R) \cap Z(R)$, we arrive at $([d(k), k_0])k' = 0$. Since R is prime and $S(R) \cap Z(R) \neq (0)$, we get

$$[d(k), k_0] = 0 \; for \; all \; k, k_0 \in S(R). \tag{2.9}$$

Now since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R), k \in S(R)$, so in view of equations (2.8) and (2.9), we are force to conclude that

$$[d(k), x] = 0 \; for \; all \; k \in S(R) \; and \; x \in R. \tag{2.10}$$

for all $k \in S(R)$ and $x \in R$. That is, $d(k) \in Z(R)$ for all $k \in S(R)$. First we assume that $d(S(R)) = (0)$. Then, we have $d(x - x^*) = 0$ for all $x \in R$. That is, $d(x) = d(x^*)$ for all $x \in R$. Thus, we have $0 = [d(x), d(x^*)] = [x, x^*]$ for all $x \in R$. Hence R is normal and we are done by Lemma 1. Now suppose that $d(S(R)) \neq (0)$. For $k_o \in S(R)$ with $d(k_o) \neq 0$ and $k \in [S(R), S(R)]$, we have $d(k_o k k_o) \in Z(R)$. The last expression can be written as $d(k_o)kk_o + k_o k d(k_o) \in Z(R)$, since $d([S(R), S(R)]) = (0)$. Thus $d(k_o)(k_o k + k k_o) \in Z(R)$ and hence $k_o k + k k_o \in Z(R)$ for all $k \in [S(R), S(R)]$. This implies that $d(k_o k + k k_o) \in Z(R)$ and hence $2d(k_o)k \in Z(R)$. Since $char(R) \neq 2$ and R is prime, the above relation yields that $k \in Z(R)$. That is, $[S(R), S(R)] \subseteq Z(R)$. Suppose $[S(R), S(R)] \neq (0)$ and $k, k_o \in S(R)$ such that $[k, k_o] \neq 0$. Since $k k_o k \in S(R)$, we have $[k, k k_o k] = k[k, k_o]k = k^2[k, k_o] \in Z(R)$. This implies that $k^2 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$ as proved earlier. Therefore, R is commutative in view of Lemma 1. Now suppose $[S(R), S(R)] = (0)$. Since $\overline{S(R)^2}$ is both a Lie ideal and a commutative subring of R, by [13, Theorem 2.1.2], $k^2 \in Z(R)$ for all $k \in S(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. Thus, R is normal and hence R is commutative by Lemma 1. This completes the proof of the theorem. □

If we replace commutator by anti-commutator in Theorem 1, the corresponding result also holds.

Theorem 2. *Let R be a prime ring with involution $*$ of the second kind and with $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(x^*) = x \circ x^*$ for all $x \in R$. Then R is commutative.*

Proof. By the given hypothesis, we have $d(x) \circ d(x^*) = x \circ x^*$ for all $x \in R$. This can be further written as

$$d(x)d(x^*) + d(x^*)d(x) = xx^* + x^*x \tag{2.11}$$

for all $x \in R$. A lineralization of (2.11) yields that

$$d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) + d(y^*)d(x) = xy^* + yx^* + x^*y + y^*x \tag{2.12}$$

for all $x, y \in R$. Replacing y by $h^{'}x$, $(h^{'} \in H(R) \cap Z(R))$ in (2.12) and using (2.11), we get

$$d(h^{'})d(x)x^* + d(h^{'})xd(x^*) + d(h^{'})d(x^*)x + d(h^{'})x^*d(x) = 0.$$

That is, $d(h^{'})d(x \circ x^*) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors, we have either $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $d(x \circ x^*) = 0$ for all $x \in R$. Suppose $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$. This further implies that $d(x) = 0$ for all $x \in Z(R)$. Replacing y by $k^{'}y$, $(k^{'} \in S(R) \cap Z(R))$ in (2.12) and using the fact $d(x) = 0$ for all $x \in Z(R)$, we obtain

$$\begin{aligned} k^{'}(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = \\ = k^{'}(-xy^* + yx^* + x^*y - y^*x). \end{aligned}$$

Again using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we arrive at

$$\begin{aligned} -d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x) \\ = -xy^* + yx^* + x^*y - y^*x \end{aligned} \tag{2.13}$$

for all $x, y \in R$. Comparing (2.12) and (2), we get $2(d(x)d(y^*) + d(y^*)d(x)) = 2(xy^* + y^*x)$ for all $x, y \in R$. Since $char(R) \neq 2$ and replace y by y^* to get $d(x) \circ d(y) = x \circ y$ for all $x, y \in R$. Hence, R is commutative in view of [3, Theorem 4.4]. On the other hand, suppose $d(x \circ x^*) = 0$ for all $x \in R$. The above equation can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) = 0 \tag{2.14}$$

for all $x \in R$. Replacing x by $h \in H(R) \cap Z(R)$ in (2.14), and using the fact that $char(R) \neq 2$, we obtain

$$d(h)h = 0 \text{ for all } h \in H(R) \cap Z(R).$$

Now since the centre of a prime ring is free from zero divisors, we get for each $h \in H(R) \cap Z(R)$ either $d(h) = 0$ or $h = 0$. Since $h = 0$ implies $d(h) = 0$, we

may write $d(h)=0$ for all $h\in H(R)\cap Z(R)$ and hence $d(x)=0$ for all $x\in Z(R)$. Linearizing (2.14), we obtain

$$d(x)y^*+d(y)x^*+xd(y^*)+yd(x^*)+d(x^*)y+d(y^*)x+x^*d(y)+y^*d(x)=0 \tag{2.15}$$

for all $x,y\in R$. Replacing y by $y_o\in Z(R)$ in (2.15) and using the fact that $d(x)=0$ for all $x\in Z(R)$, we get

$$d(x)y_o^*+y_od(x^*)+d(x^*)y_o+y_o^*d(x)=0 \tag{2.16}$$

for all $y_o\in Z(R)$ and $x\in R$. In particular, taking $y_o=h_o\in H(R)\cap Z(R)$ in (2.16), we get $2(d(x)h_o+d(x^*)h_o)=0$. Since $char(R)\neq 2$, we obtain $d(x)h_o+d(x^*)h_o=0$. This can be further written as

$$d(x+x^*)h_o=0 \tag{2.17}$$

for all $h_o\in H(R)\cap Z(R)$ and $x\in R$. Using the primeness of R, we get either $d(x+x^*)=0$ or $H(R)\cap Z(R)=(0)$. But $H(R)\cap Z(R)=(0)$ implies that $S(R)\cap Z(R)=(0)$, which gives a contradiction since we have assumed $S(R)\cap Z(R)\neq(0)$. Therefore, we are left with the case $d(x+x^*)=0$ for all $x\in R$. Replacing x by $h+k$ in the above equation, we get $2d(h)=0$. This implies that $d(h)=0$ for all $h\in H(R)$. Further $d(x+x^*)=0$ implies that $d(x)=-d(x^*)$ for all $x\in R$. Replacing x by xh, where $h\in H(R)$ in the last expression we get $d(x)h=-hd(x^*)$, since $d(h)=0$. This further implies that $d(x)h=hd(x)$ for all $x\in R$. Therefore in view of the theorem of [14], we conclude that $h\in Z(R)$ for all $h\in H(R)$. Hence R is commutative in view of Lemma 1. Thereby completing the proof of the theorem. □

At the end, let us write an example which shows that the restriction of second kind involution in Theorem 1 is not superfluous.

Example 1. Let $R=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,\middle|\, a,b,c,d\in Z\right\}$. Of course R with matrix addition and matrix multiplication is a prime ring. Define mappings $d:R\longrightarrow R$, and $*:R\longrightarrow R$ such that $d\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 0 & -b\\ c & 0\end{pmatrix}$, $\begin{pmatrix} a & b\\ c & d\end{pmatrix}^*=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$. Obviously, $Z(R)=\left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\,\middle|\, a\in Z\right\}$. Then $x^*=x$ for all $x\in Z(R)$, and hence $Z(R)\subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d is nonzero and the following condition $[d(x),d(x^*)]=[x,x^*]$ for all $x\in R$, is satisfied. However, R is not commutative. Hence, in Theorem 1, the hypothesis of second kind involution is crucial.

ACKNOWLEDGMENTS

The authors are grateful to the learned referee's for carefully reading the manuscript. The valuable suggestions have simplified and clarified the paper greatly.

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Doi: 10.18514/MMN.2015.1168

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 25–32

HU e-ISSN 1787-2413

ON A GENERAL CLASS OF WEAKLY PICARD OPERATORS

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Received 15 March, 2014

Abstract. In this paper we show that on complete metric spaces the class of weakly Picard operators contains some operators which are more general than the class of almost contractions.

2010 *Mathematics Subject Classification:* 54H25, 47H10

Keywords: Fixed point, Picard operator, weakly Picard operator, almost contraction, almost θ-contraction.

1. Introduction and Preliminaries

One of the most important concepts of fixed point theory is the Picard operator. We know that a self mapping T of a nonempty set X is Picard operator, if T has a unique fixed point and the sequence of successive approximation for any initial point converges to this unique fixed point. Therefore, lot of different types of contractions are Picard operators. (See for more details [1–4]). On the other hand, Rus [6] introduced the concept of weakly Picard operator: a self mapping T of a nonempty set X is weakly Picard operator, if the set of fixed points of T is nonempty and the sequence of successive approximation for any initial point converges to a fixed point of T. There exist many weakly Picard operators in the literature, but one of the most interesting of them is the class of almost contraction, which was introduced by Berinde [2], on complete metric space. Berinde [3] shows that the concept of almost contraction is more general than the lot of different types of contractions in the literature.

For the sake of completeness we recall the concept of almost contraction and some related results.

Let (X,d) be a metric space and $T : X \to X$ be a map. T is said to be an almost contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx) \tag{1.1}$$

for all $x, y \in X$.

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Note that, by the symmetry property of the distance, the almost contractive condition implicitly includes the following dual one

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(x,Ty) \tag{1.2}$$

for all $x, y \in X$. So, in order to check the almost contractiveness of a mapping T, it is necessary to check both (1.1) and (1.2).

Theorem 1 ([3]). *Every almost contraction on complete metric space is a weakly Picard operator.*

On the other hand, a new type of contractive maps has been introduced by Jleli and Samet [5]. Throughout this study, we called it as θ-contraction.

Let Θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is nondecreasing,

(Θ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$,

(Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Definition 1 ([5]). Let (X,d) be a metric space and $T : X \to X$ be a mapping. Given $\theta \in \Theta$, we say that T is θ-contraction if there exists $k \in (0,1)$ such that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k . \tag{1.3}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$.

If we consider the different type of mapping θ in Definition 1, we obtain some of variety of contractions. For example, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then (1.3) turns to

$$d(Tx,Ty) \leq k^2 d(x,y), \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.4}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty)$ $\leq k^2 d(x,y)$ also holds. Therefore T is an ordinary contraction. Similarly, let $\theta : (0,\infty) \to (1,\infty)$ be given by $\theta(t) = e^{\sqrt{te^t}}$. It is clear that $\theta \in \Theta$. Then (1.3) turns to

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq k^2, \text{ for all } x, y \in X, Tx \neq Ty. \tag{1.5}$$

In addition, we have concluded that every θ-contraction T is a contractive mapping, i.e.,

$$d(Tx,Ty) < d(x,y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thus, every θ-contraction is a continuous mapping. On the other side, Example in [5] shows that the mapping T is not ordinary contraction but, but it is a θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$. Thus, the following theorem, which was given as a corollary by Jleli and Samet [5], is a proper generalization of Banach Contraction Principle.

Theorem 2 (Corollary 2.1 of [5])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point in X.*

If we examine the proof of the above theorem, we can say that every θ-contraction on complete metric space is a Picard operator.

In the next section, we introduce a new concept of almost θ-contraction on metric spaces. Then we show that the class of almost θ-contractions are more general than the class of almost contractions and we also show that they are still weakly Picard operators on complete metric spaces.

2. Main result

Definition 2. Let (X,d) be a metric space and $T : X \to X$ be a mapping. Given $\theta \in \Theta$, we say that T is almost θ-contraction if there exist $k \in (0,1)$ and $\lambda \geq 0$ such that, for all $x, y \in X$ with $d(Tx,Ty) > 0$

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y) + \lambda d(y,Tx))]^k . \tag{2.1}$$

Note that, by the symmetry property of the distance, the almost θ-contractive condition implicitly includes the following dual one

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y) + \lambda d(x,Ty))]^k \tag{2.2}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$. So, in order to check the almost θ-contractiveness of a mapping T, it is necessary to check both (2.1) and (2.2).

By the considering $\theta(\alpha) = e^{\sqrt{t}}$, we can say that every almost contraction is also almost θ-contraction.

Our main result is as follows:

Theorem 3. *Every almost θ-contraction on complete metric space is a weakly Picard operator.*

Proof. Let (X,d) be a complete metric space and $T : X \to X$ be an almost θ-contraction. Let $x_0 \in X$ be an arbitrary point in X. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. So we can suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now, from (2.1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \theta(d(x_n,x_{n+1})) &= \theta(d(Tx_{n-1},Tx_n)) \\ &\leq [\theta(d(x_{n-1},x_n) + \lambda d(x_n,Tx_{n-1}))]^k \\ &= [\theta(d(x_{n-1},x_n)]^k . \end{aligned}$$

Therefore we have

$$1 < \theta(d(x_n,x_{n+1})) \leq [\theta(d(x_0,x_1)]^{k^n} \tag{2.3}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.3), we obtain

$$\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1 \tag{2.4}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0^+.$$

Therefore from (Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

As in the proof of Theorem 2.1 in [5], we can obtain

$$\lim_{n\to\infty} n\,[d(x_n, x_{n+1})]^r = 0.$$

Thus, there exits $n_1 \in \mathbb{N}$ such that $n\,[d(x_n, x_{n+1})]^r \le 1$ for all $n \ge n_1$. So, we have, for all $n \ge n_1$

$$d(x_n, x_{n+1}) \le \frac{1}{n^{1/r}}. \tag{2.5}$$

Let $m > n > n_1$, then considering (2.5), we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \le \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ is convergent, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete metric space, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

On the other hand, from (Θ_1) and (2.1), it is easy to conclude that

$$d(Tx, Ty) < d(x,y) + \lambda d(y, Tx)$$

for all $x, y \in X$ with $Tx \ne Ty$. Therefore, for all $x, y \in X$

$$d(Tx, Ty) \le d(x,y) + \lambda d(y, Tx) \tag{2.6}$$

is satisfied. Thus, from (2.6),

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\le d(x_n, z) + \lambda d(z, x_{n+1}). \end{aligned}$$

Letting the $n \to \infty$ in the above inequality, we have $d(z, Tz) = 0$ and so $z = Tz$. Therefore, T is a weakly Picard operator on X. □

In the following example, the mapping T is not almost contraction, but it is almost θ-contraction on a complete metric space and so is weakly Picard operator.

Example 1. Consider the complete metric space (X,d), where $X=\{0,1,2,\cdots\}$ and $d: X\times X\to[0,\infty)$ is given by

$$d(x,y)=\begin{cases} 0 & , \quad x=y \\ \\ x+y & , \quad x\neq y \end{cases}.$$

Let $T: X\to X$ be defined by

$$Tx=\begin{cases} 0 & , \quad x\in\{0,1\} \\ \\ x-1 & , \quad x\geq 2 \end{cases}.$$

Then T is not almost contraction. Indeed, for $y\geq 2$ and $x=y+1$, then we have $d(y,Tx)=0$ and

$$\sup\left\{\frac{d(Tx,Ty)}{d(x,y)}: x,y\in X, x=y+1\right\}=1.$$

Thus, we can not find $\delta\in(0,1)$ and $L\geq 0$ satisfying (1.1). Therefore, using Theorem 1 we can not decide that whether T is weakly Picard operator.

Now, we claim that T is almost θ-contraction with $\theta(t)=e^{\sqrt{te^t}}$, $k=e^{-\frac{1}{2}}$ and $\lambda\geq 0$. To see this we have to show that

$$\theta(d(Tx,Ty))\leq[\theta(d(x,y)+\lambda d(y,Tx))]^k$$

for all $x,y\in X$ with $d(Tx,Ty)>0$. For this, it is sufficient to show

$$\theta(d(Tx,Ty))\leq[\theta(d(x,y))]^k$$

or equivalently

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)}\leq e^{-1}. \tag{2.7}$$

for all $x,y\in X$ with $d(Tx,Ty)>0$. First, observe that

$$d(Tx,Ty)>0\Leftrightarrow \text{the set } \{x,y\}\cap\{0,1\} \text{ is singleton or empty.}$$

Since (2.7) is symmetric with respect to x and y, we may assume $x>y$ in the following cases.

Case 1. Let $\{x,y\}\cap\{0,1\}$ is singleton. Then $d(Tx,Ty)=x-1$ and $d(x,y)\leq\max\{x,x+1\}$, and so we have

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)}\leq\frac{x-1}{x}e^{-1}\leq e^{-1}.$$

Case 2. Let $\{x,y\}\cap\{0,1\}$ is empty. Then $d(Tx,Ty)=x+y-2$ and $d(x,y)=x+y$ and so we have

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)}=\frac{x+y-2}{x+y}e^{-2}\leq e^{-2}\leq e^{-1}.$$

This shows that T is almost θ-contraction with with $\theta(t) = e^{\sqrt{te^t}}$ and so by Theorem 3, T is a weakly Picard operator.

Now we give some examples showing that T is almost θ-contraction but not θ-contraction.

Example 2. Let $X = [0,1] \cup \{2,3\}$ and $d(x,y) = |x-y|$, then (X,d) is complete metric space. Define a map $T : X \to X$,

$$Tx = \begin{cases} \frac{1-x}{2} & , \quad x \in [0,1] \\ \\ x & , \quad x \in \{2,3\} \end{cases}.$$

Since $d(T2,T3) = 1 = d(2,3)$, then for all $\theta \in \Theta$ and $k \in (0,1)$ we have

$$\theta(d(T2,T3)) = \theta(1) > [\theta(1)]^k = [\theta(d(2,3))]^k.$$

Therefore, T is not θ-contraction. That is, Theorem 2 can not be applied to this example.

Now, we claim that T is almost θ-contraction with $\theta(t) = e^{\sqrt{t}}$, $k = \frac{1}{\sqrt{2}}$ and $\lambda = 4$. To see this have to show that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y) + \lambda d(y,Tx))]^k$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$ or equivalently

$$d(Tx,Ty) \leq \frac{1}{2} d(x,y) + 2d(y,Tx) \tag{2.8}$$

for all $x, y \in X$ with $x \neq y$.

Now we consider the following cases:

Case 1. Let $x, y \in [0,1]$, then

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{2}|x-y| = \frac{1}{2} d(x,y) \\ &\leq \frac{1}{2} d(x,y) + 2d(y,Tx), \end{aligned}$$

that is, (2.8) is satisfied.

Case 2. Let $x, y \in \{2,3\}$, then

$$\begin{aligned} d(Tx,Ty) &= |x-y| \leq \frac{5}{2}|x-y| \\ &= \frac{1}{2} d(x,y) + 2d(y,Tx), \end{aligned}$$

that is, (2.8) is satisfied.

Case 3. Let $x \in [0,1]$ and $y \in \{2,3\}$, then

$$d(Tx,Ty) = d(y,Tx)$$

$$\leq \frac{1}{2}d(x,y)+2d(y,Tx),$$

that is, (2.8) is satisfied.

Case 4. Let $x \in \{2,3\}$ and $y \in [0,1]$, then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{2x+y-1}{2}\right| = \frac{2x+y-1}{2} \\ &\leq \frac{5}{2}(x-y) \\ &= \frac{1}{2}d(x,y)+2d(y,Tx), \end{aligned}$$

that is, (2.8) is satisfied.

Therefore T is almost θ-contraction with $\theta(t) = e^{\sqrt{t}}$ and so it is a weakly Picard operator.

Example 3. Consider the complete metric space (X,d), where $X = \{1,3,5,\cdots\} \cup \{2\}$ and $d(x,y) = |x-y|$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 3 & , \quad x \in \{2,3\} \\ 1 & , \quad \text{otherwise} \end{cases}.$$

Then it is easy to see that T is not θ-contraction. But T is almost θ-contraction with $k = \frac{1}{\sqrt{2}}$, $\lambda \geq 3$ and $\theta(t) = e^{\sqrt{t}}$.

In the following theorem, by adding an extra condition on almost θ-contractions on complete metric space, we show that they are also Picard operators.

Theorem 4. *Let (X,d) be a complete metric space and $T : X \to X$ be an almost θ-contraction. Suppose that T also satisfies the following condition: there exist $\theta_1 \in \Theta$ and some $\lambda_1 \geq 0$ and $k_1 \in (0,1)$ such that*

$$\theta_1(d(Tx,Ty)) \leq [\theta_1(d(x,y)+\lambda_1 d(x,Tx))]^{k_1} \tag{2.9}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$. Then T is a Picard operator.

Proof. Suppose that, there are two fixed point z and w of T. If $d(z,w) = 0$, it is clear that $z = w$. Assume that $d(z,w) > 0$. By (2.9) with $x = z$ and $y = w$, we have

$$\begin{aligned} \theta_1(d(z,w)) &= \theta_1(d(Tz,Tw)) \\ &\leq [\theta_1(d(z,w)+\lambda_1 d(z,Tz))]^{k_1} \\ &= [\theta_1(d(z,w))]^{k_1}, \end{aligned}$$

which is a contradiction. Therefore T has a unique fixed point and so it is a Picard operator. □

ACKNOWLEDGEMENT

The authors are grateful to the referees because their suggestions contributed to the improvement of the paper.

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Doi: 10.18514/MMN.2015.1089

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 33–44

HU e-ISSN 1787-2413

ON CERTAIN DIFFERENTIAL IDENTITIES IN PRIME RINGS WITH INVOLUTION

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Received 27 December, 2013

Abstract. In the present paper we investigate commutativity of $*$-prime ring R, which satisfies certain differential identities on $*$-ideals of R. Some results already known for prime rings on ideals have also been deduced. Finally, we provide several examples to justify that various restrictions imposed in the hypotheses of our theorems are not superfluous.

2010 *Mathematics Subject Classification:* 16W25; 16N60; 16U80; 16W10

Keywords: derivation, $*$-prime ring, $*$-ideal and commutativity

1. INTRODUCTION

Throughout the paper, R will represent an associative ring with center Z. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol xoy will stand for the anticommutator $xy + yx$. R is called a prime ring if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xRx = \{0\}$ implies $x = 0$. Given an integer $n > 1$, ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ of R into itself is called an involution on R if it satisfies the conditions: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution '$*$' is called a ring with involution or a $*$-ring. A ring R with involution '$*$' is said to be $*$-prime if $aRb = aRb^* = \{0\}$, where $a, b \in R$ (equivalently $aRb = a^*Rb = \{0\}$) implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution '$*$' is $*$-prime but the converse is not true in general. Of course, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. Let R be a $*$-prime ring. The set of symmetric and skew symmetric elements of R will be denoted by $Sa_*(R)$ i.e.; $Sa_*(R) = \{x \in R \mid x^* = \pm x\}$. An ideal I of R is called a $*$-ideal of R if $I^* = I$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation on R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let R be a $*$-prime ring, $a \in R$ and $aRa = \{0\}$. This implies that $aRaRa^* = \{0\}$ also. Now $*$-primeness of R insures that $a = 0$ or $aRa^* = \{0\}$. $aRa^* = \{0\}$ together with $aRa = \{0\}$ gives us $a = 0$. Thus we conclude that every $*$-prime ring is a semiprime

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ring.

Ashraf, Bell, Herstein and Hongon etc. [1–4] studied the commutativity of prime and semiprime rings, satisfying certain differential identities on some appropriate subsets of R. Hence it is natural to question that what can we say about the commutativity of $*$-prime rings in which derivations satisfy certain identities on $*$-ideals. In this direction Oukhtite , Salhi, Vukman and Kosi-Ulbl etc. [5–11] have already investigated several differential identities on $*$-ideals. In this paper we have obtained the commutativity of $*$-prime rings satisfying any one of the following differential identities on $*$-ideal I under some restrictions: (i) $d(xoy) = d(x)oy$ (ii) $d(x)oy = xoy$, (iii) $d([x,y]) = \pm(xoy)$ (iv) $d(xoy) = \pm[x,y]$, (v) $d(x)oy \in Z$, (vi) $d([x,y]) \pm (xoy) \in Z$, (vii) $d(xoy) \pm [x,y] \in Z$, $(viii)$ $d(x)od(y) = xoy$ and (ix) $(d(x)oy) - (xod(y)) \in Z$ for all $x, y \in I$. We have also shown that there exists no nonzero derivation d satisfying the following differential identities on $*$-ideal I in a $*$-prime ring R under some constraints viz.; (i) $d(xoy) = d(x)oy$ (ii) $d(x)oy = xoy$, (iii) $d(x)oy = xod(y)$, (iv) $d(x)oy = d(x)od(y)$ (v) $xod(y) = d(x)od(y)$ for all $x, y \in I$. Several examples have also been constructed to justify the necessity of $*$-primeness or "characteristic different from 2" of the ring R while dealing with the above differential identities.

2. Preliminary results

We begin with the following lemmas which are essential for developing the proof of our main results. The proofs of Lemma 1-3 can be seen in [6] while that of Lemma 4 can be found in [8].

Lemma 1. *Let R be a $*$-prime ring and I be a nonzero $*$-ideal of R. If $x, y \in R$ satisfy $xIy = xIy^* = \{0\}$, then $x = 0$ or $y = 0$.*

Lemma 2. *Let R be a $*$-prime ring admitting a nonzero derivation d which commutes with '$*$'. If I is a nonzero $*$-ideal of R and $[x,R]Id(x) = \{0\}$ for all $x \in I$, then R is commutative.*

Lemma 3. *Let R be a $*$-prime ring admitting a nonzero derivation d which commutes with '$*$'. If I is a nonzero $*$-ideal of R and $[d(x),x] = 0$ for all $x \in I$, then R is commutative.*

Lemma 4. *Let d be a nonzero derivation of a 2-torsion free $*$-prime ring R and I a nonzero $*$-ideal of R. If $r \in Sa_*(R)$ satisfies $[d(x),r] = 0$ for all $x \in I$, then $r \in Z$. Furthermore, if $d(I) \subseteq Z$, then R is commutative.*

Now we prove the following:

Lemma 5. *If R is a $*$-prime ring admitting a nonzero central $*$-ideal I i.e.; $I \subseteq Z$, then R is commutative.*

Proof. Let $r,s \in R$ and $x \in I$. Using hypothesis we get $rsx = rxs = srx$. This implies that $[r,s]I = \{0\}$ and hence $[r,s]Il = [r,s]Il^* = \{0\}$, where $0 \neq l \in R$. In view of Lemma 1, we get the required result. □

Lemma 6. *Let R be a $*$-prime ring admitting a nonzero derivation d which commutes with '$*$'. If I is a nonzero $*$-ideal of R and $d(x)I[x,R] = \{0\}$ for all $x \in I$, then R is commutative.*

Proof. For the proof, first we show that $d(x)I[x,R] = \{0\}$ for all $x \in I$ if and only if $[x,R]Id(x) = \{0\}$ for all $x \in I$. Suppose that $d(x)I[x,R] = \{0\}$ for all $x \in I$. This implies that $[x,R]^*I^*\{d(x)\}^* = \{0\}$ for all $x \in I$. Since I is a $*$-ideal of R and d commutes with '$*$', we conclude that $[x^*,R]Id(x^*) = \{0\}$ for all $x \in I$. Now replacing x by x^* in the last relation we obtain that $[x,R]Id(x) = \{0\}$ for all $x \in I$. Converse can be proved in similar way. Finally using Lemma 2, we get the required result. □

Lemma 7. *If R is a $*$-prime ring of characteristic different from 2, then R is 2-torsion free.*

Proof. Suppose that $x \in R$ such that $2x = 0$. This implies that $2xrs = 0$ for all $r,s \in R$ i.e.; $xR(2s) = \{0\}$ for all $s \in R$. Since characteristic of R is different from 2 and $R \neq \{0\}$, this provides us a nonzero element $l \in R$ such that $2l \neq 0$. Now we conclude that $xR(2l) = \{0\} = xR(2l)^*$. Finally $*$-primeness of R provides us $x = 0$ and hence R is 2-torsion free. □

3. Main results

There has been a great deal of work concerning commutativity of prime and semiprime rings satisfying certain differential identities on some appropriate subsets of the ring, see [1–4] where further references can be found. Motivated by such work, we explore the commutativity of $*$-prime ring involving derivations satisfying some identities on nonzero $*$-ideal of the ring. We begin with the following differential identity on a nonzero $*$-ideal of a $*$-prime ring, which is of independent interest and insures the commutativity of the ring:

Theorem 1. *Let R be a $*$-prime ring, I be a nonzero $*$-ideal of R and d a nonzero derivation of R such that d commutes with '$*$'. If $d(xoy) = d(x)oy$ for all $x,y \in I$ or $d(x)oy = xoy$ for all $x,y \in I$, then R is commutative.*

Proof. Assume that $d(xoy) = d(x)oy$ for all $x,y \in I$. Now replacing y by xy we arrive at $d(xoxy) = d(x)oxy$ i.e.; $d(x)(xoy) + xd(xoy) = d(x)xy + xyd(x)$. Using hypothesis we obtain that $d(x)(xoy) + x(d(x)oy) = d(x)xy + xyd(x)$. This implies that $d(x)xy + d(x)yx + xd(x)y + xyd(x) = d(x)xy + xyd(x)$ i.e.;

$$d(x)yx = -xd(x)y \tag{3.1}$$

for all $x, y \in I$. Putting yr, where $r \in R$, for y in the relation (3.1) and using it again we conclude that $d(x)yrx = -xd(x)yr = d(x)yxr$ i.e.; $d(x)I[x, R] = \{0\}$ for all $x \in I$ and Lemma 6 forces that R is commutative.
Now suppose that $d(x)oy = xoy$ for all $x, y \in I$. Replacing x by yx, we get $d(yx)oy = yxoy$ i.e.; $d(yx)oy = y(xoy)$ for all $x, y \in I$. Using our hypothesis we obtain that $d(yx)y + yd(yx) = y(d(x)oy)$ i.e.; $d(y)xy + yd(x)y + yd(y)x + y^2d(x) = yd(x)y + y^2d(x)$ and therefore $d(y)xy = -yd(y)x$ for all $x, y \in I$. In view of the latter relation we arrive at $d(x)yx = -xd(x)y$ for all $x, y \in I$. This is identical with the relation (3.1). Arguing as in above we conclude that R is commutative. □

Theorem 2. *Let R be a $*$-prime ring of characteristic different from 2, I be a nonzero $*$-ideal of R and d a derivation of R such that 'd' commutes with '$*$'. If $d(xoy) = d(x)oy$ for all $x, y \in I$ or $d(x)oy = xoy$ for all $x, y \in I$, then $d = 0$*

Proof. Suppose that $d(xoy) = d(x)oy$ for all $x, y \in I$. Then we have to show that $d = 0$. Suppose on contrary that $d \neq 0$. Therefore by Theorem 1 we conclude that R is commutative. By hypothesis given we have $2d(xy) = 2d(x)y$ for all $x, y \in I$ and hence $d(x)y + xd(y) = d(x)y$ for all $x, y \in I$. This yields that $xd(y) = 0$ for all $x, y \in I$ and since R is commutative we arrive at $d(x)y = 0$ for all $x, y \in I$. Replacing y by sy where $s \in R$ in the last relation we obtain that $d(x)sy = 0$ i.e.; $d(x)Ry = \{0\}$ for all $x, y \in I$. Since I is a $*$-ideal of R, we conclude that $d(x)Ry^* = \{0\}$ for all $x, y \in I$ also. $I \neq \{0\}$ and $*$-primeness of R provide us $d(x) = 0$ for all $x \in I$. Now putting xt where $t \in R$ in place of x in the last relation and using the same again we arrive at $xd(t) = 0$ i.e.; $Id(t) = \{0\}$. Using hypothesis this relation provides us $lId(t) = lI\{d(t)\}^* = \{0\}$ where $0 \neq l \in R$. Finally Lemma 1 assures that $d = 0$, leading to a contradiction.
Now assume that $d(x)oy = xoy$ for all $x, y \in I$. Then we have to show that $d = 0$. If $d \neq 0$, then by Theorem 1 we conclude that R is commutative. By hypothesis given we have $2d(x)y = 2xy$ for all $x, y \in I$, then $d(x)y = xy$ for all $x, y \in I$. Replacing x by rx, where $r \in R$ in the last relation and using the same again we infer that $d(rx)y = rxy$ i.e.; $d(r)xy + rd(x)y = rxy$. This implies that $d(r)xy = 0$ for all $r \in R$ and $x, y \in I$. Finally we conclude that $d(r)Iy = \{0\}$ for all $r \in R$ and $y \in I$. In particular we also obtain that $d(r)Iy^* = \{0\}$. Lemma 1 and $I \neq \{0\}$ assure that $d = 0$, leading to a contradiction. □

The following example shows that the existence of "characteristic different from 2" in the hypothesis of the above theorem is not superfluous.

Example 1. Suppose that $R = \mathbb{Z}_2[x] \times \mathbb{Z}_2[x]$, where $\mathbb{Z}_2[x]$ is the polynomial ring over $\mathbb{Z}_2$. Let us consider $D, * : R \longrightarrow R$ such that

$$D(f(x), g(x)) = (d(f(x)), d(g(x))) \text{ and } (f(x), g(x))^* = (g(x), f(x)),$$

where d is the usual differentiation in $\mathbb{Z}_2[x]$. It is easy to check that R is a $*_{ex}$-prime ring since '$*$' is an involution of R, known as exchange involution denoted by $*_{ex}$ and D is a derivation on R. Moreover, it is obvious that R is a ring of characteristic 2 and $D*_{ex} = *_{ex}D$. Further assume that $I = [x^2]$ is the ideal of $\mathbb{Z}_2[x]$ generated by $x^2 \in \mathbb{Z}_2[x]$. Then it can be easily shown that $\mathcal{I} = I \times I$ is a nonzero $*_{ex}$-ideal of R such that $D(xoy) = D(x)oy$ for all $x, y \in \mathcal{I}$ and $D(x)oy = xoy$ for all $x, y \in \mathcal{I}$. However $D \neq 0$.

In the year 2007, Oukhtite and Salhi [9] obtained the commutativity of $*$-prime ring R having "characteristic different from 2" and admitting a nonzero derivation d which commutes with '$*$', such that $d([x, y]) = 0$ for all x, y in a nonzero $*$-ideal of R. We have improved this result and showed that the restriction of "characteristic different from 2" on R used in the above theorem is redundant. In addition we have also investigated similar other differential identities which insure the commutativity of $*$-prime rings. In fact we have obtained the following.

Theorem 3. *Let R be a $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d which commutes with '$*$' and satisfies any one of the following differential identities: (i) $d([x, y]) = 0$ for all $x, y \in I$, (ii) $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, (iii) $d([x, y]) = \pm(xoy)$ for all $x, y \in I$, (iv) $d(xoy) = 0$ for all $x, y \in I$, (v) $d(xoy) = \pm(xoy)$ for all $x, y \in I$ and (vi) $d(xoy) = \pm[x, y]$ for all $x, y \in I$. Then R is commutative.*

Proof. (i) By hypothesis we have $d([x, y]) = 0$, for all $x, y \in I$. Now replacing y by yx and using the hypothesis, we obtain that $[x, y]d(x) = 0$ for all $x, y \in I$ i.e.;

$$xyd(x) = yxd(x) \tag{3.2}$$

for all $x, y \in I$. Replacing y by ry, where $r \in R$ in the relation (3.2) and using it again, we arrive at $[x, r]yd(x) = 0$ for all $x, y \in I$ and for all $r \in R$ i.e.; $[x, R]Id(x) = \{0\}$ for all $x \in I$. Now by Lemma 2, the result follows.

(ii) By hypothesis we have $d([x, y]) = \pm[x, y]$, for all $x, y \in I$. Now replacing y by yx and using the hypothesis, we infer that $[x, y]d(x) = 0$ for all $x, y \in I$ i.e.; $xyd(x) = yxd(x)$ for all $x, y \in I$. This is identical with the relation (3.2). Now arguing in the similar way as above (i), we get our required result.

(iii) Using the same trick as used in (ii), result follows.

(iv) By hypothesis we have $d(xoy) = 0$, for all $x, y \in I$. Now replacing y by yx and using the hypothesis, we obtain that $(xoy)d(x) = 0$ for all $x, y \in I$ i.e.;

$$xyd(x) = -yxd(x) \tag{3.3}$$

for all $x, y \in I$. Replacing y by ry, where $r \in R$ in the relation (3.3) and using it again, we arrive at $[x, r]yd(x) = 0$ for all $x, y \in I$ and for all $r \in R$ i.e.;

$[x, R]Id(x) = \{0\}$ for all $x \in I$. Now by Lemma 2, the result follows.

(v) By hypothesis we have $d(xoy) = \pm(xoy)$, for all $x, y \in I$. Now replacing y by yx and using the hypothesis, we conclude that $(xoy)d(x) = 0$ for all $x, y \in I$ i.e.; $xyd(x) = -yxd(x)$ for all $x, y \in I$. This is identical with the relation (3.3). Now using similar arguments as used in (iv), we get our required result.

(vi) Using the same arguments as used in (v), result follows. □

Corollary 1. *Let R be a prime ring and I a nonzero ideal of R. If R admits a nonzero derivation d satisfying any one of the following differential identities: (i) $d([x, y]) = 0$ for all $x, y \in I$, (ii) $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, (iii) $d([x, y]) = \pm(xoy)$ for all $x, y \in I$, (iv) $d(xoy) = 0$ for all $x, y \in I$, (v) $d(xoy) = \pm(xoy)$ for all $x, y \in I$ and (vi) $d(xoy) = \pm[x, y]$ for all $x, y \in I$. Then R is commutative.*

Proof. Let d be a nonzero derivation of R satisfying any one of above differential identities. Since R is a prime ring, consider $\mathcal{R} = R \times R^o$, which is clearly a $*_{ex}$-prime ring. Set $\mathcal{I} = I \times I^o$ is a nonzero $*_{ex}$-ideal of $\mathcal{R}$. Now define $D : \mathcal{R} \longrightarrow \mathcal{R}$ by $D(x, y) = (d(x), d(y))$. Using hypothesis it can be easily proved that D is a nonzero derivation of $\mathcal{R}$. Moreover $D*_{ex} = *_{ex}D$ and (i) $D([x, y]) = 0$ for all $x, y \in \mathcal{I}$, (ii) $D([x, y]) = \pm[x, y]$ for all $x, y \in \mathcal{I}$ (iii) $D([x, y]) = \pm(xoy)$ for all $x, y \in \mathcal{I}$, (iv) $D(xoy) = 0$ for all $x, y \in \mathcal{I}$, (v) $D(xoy) = \pm(xoy)$ for all $x, y \in \mathcal{I}$ and (vi) $D(xoy) = \pm[x, y]$ for all $x, y \in \mathcal{I}$. Using the Theorem 3, we deduce that $\mathcal{R}$ is commutative and in turn we obtain that R is also commutative. □

Theorem 4. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d such that $d(x)oy \in Z$ for all $x, y \in I$, then R is commutative.*

Proof. Assume that

$$d(x)oy \in Z \tag{3.4}$$

for all $x, y \in I$. The relation (3.4) implies that $d(x)y + yd(x) \in Z$ for all $x, y \in I$. Since I is a nonzero ideal of R, $d(x)y + yd(x) \in I$ for all $x, y \in I$ also. Now we conclude that $d(x)y + yd(x) \in Z \cap I$ for all $x, y \in I$. Now we break the proof in two cases.
Case I: If $Z \cap I = \{0\}$, we obtain that $d(x)y + yd(x) = 0$ i.e.;

$$d(x)y = -yd(x) \tag{3.5}$$

for all $x, y \in I$. Substituting ry, where $r \in R$ for y in the relation (3.5) and using it again we arrive at $d(x)ry = rd(x)y$ i.e.; $[d(x), R]y = \{0\}$. This implies that $[d(x), R]Is = \{0\} = [d(x), R]Is^*$, where $0 \neq s \in R$. Now by Lemma 1, we infer that $[d(x), R] = \{0\}$ for all $x \in I$ i.e; $d(I) \subseteq Z$. Finally, Lemma 4 assures that R is commutative.

Case II: If $Z \cap I \neq \{0\}$, there exists $0 \neq z \in Z \cap I$. By hypothesis we have $d(x)y + yd(x) \in Z$ for all $x, y \in I$. In particular we conclude that $d(x)z + zd(x) \in Z$ i.e.; $2d(x)z \in Z$ for all $x \in I$. Now we have $2d(x)zr = 2rd(x)z$ for all $x \in I$ and $r \in R$. This yields that $[d(x), r]z = 0$ for all $x \in I$ and $r \in R$ i.e.; $[d(x), R]Iz = \{0\}$ for all $x \in I$. We already know that $0 \neq z \in Z \cap I$. Since $I = I^*$ and $Z = Z^*$, the latter relation implies that $0 \neq z^* \in Z \cap I$. Now using z^* in place of z and arguing in the similar way as in just above lines we arrive at $[d(x), R]Iz^* = \{0\}$ for all $x \in I$. Finally we conclude that $[d(x), R]Iz = \{0\} = [d(x), R]Iz^*$ for all $x \in I$, where $0 \neq z$. Using Lemma 1 & Lemma 4, we get the required result for this case. □

The following example demonstrates that the $*$-primeness in the hypothesis of the above theorem can not be omitted.

Example 2. Let $R = \mathbb{R}[x] \times \mathbb{H}$, where $\mathbb{R}[x]$ is the polynomial ring over the ring $\mathbb{R}$ of real numbers and $\mathbb{H}$ is the ring of real quaternions. R is clearly a ring of characteristic different from 2. Define $D : R \longrightarrow R$ as $D(f(x), q) = (d(f(x), 0)$, where d is the usual differentiation of the polynomial ring $\mathbb{R}[x]$. Also define $* : R \longrightarrow R$ as $*(f(x), q) = (f(-x), \bar{q})$, where $f(x) \in \mathbb{R}[x]$ and $\bar{q} = \alpha - \beta i - \gamma j - \delta k$, where $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$. It can be easily shown that D and '$*$' are a nonzero derivation and an involution of R respectively. Suppose that $I = \mathbb{R}[x] \times \{0\}$. It is obvious that I is a $*$-ideal of R. Let $0 \neq u(x) \in \mathbb{R}[x]$ and $0 \neq v \in \mathbb{H}$. Then we have $(u(x), 0)R(0, v) = \{(0, 0)\} = (u(x), 0)R(0, v)^*$, where $(0, 0) \neq (u(x), 0), (0, 0) \neq (0, v) \in R$. This implies that R is not a $*$-prime ring but it is a semiprime ring. It can be easily seen that $D(m)on \in Z$ for all $m, n \in I$ but R is noncommutative.

Theorem 5. *Let R be a $*$-prime ring of characteristic different from 2, I a nonzero $*$-ideal of R. If R admits a nonzero derivation d which commutes with '$*$' such that $d([x, y]) \pm (xoy) \in Z$ for all $x, y \in I$. Then R is commutative.*

Proof. It is clear that $d([x, y]) \pm (xoy) \in I$ for all $x, y \in I$ also. Now in view of our hypothesis we conclude that $d([x, y]) \pm (xoy) \in Z \cap I$ for all $x, y \in I$.
Case I: If $Z \cap I = \{0\}$, then $d([x, y]) \pm (xoy) = 0$ for all $x, y \in I$, using Theorem 3 we get our required result.
Case II: If $Z \cap I \neq \{0\}$, then suppose $0 \neq z \in Z \cap I$. Replacing y by z, we arrive at $d([x, z]) \pm (xoz) \in Z \cap I$ for all $x \in I$ i.e.; $2xz \in Z$ for all $x \in I$ and hence $xz \in Z$ for all $x \in I$ i.e.; $xzr = rxz$ for all $r \in R$. This implies that $[x, R]Rz = \{0\}$ for all $x \in I$. Since $0 \neq z^* \in Z \cap I$, arguing in the similar lines as above we also obtain that $[x, R]Rz^* = \{0\}$ for all $x \in I$. By $*$-primeness of R we conclude that $I \subseteq Z$. Finally by Lemma 5, the result follows. □

Theorem 6. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R. If R admits a nonzero derivation d which commutes with '$*$' such that $d(xoy) \pm [x, y] \in Z$ for all $x, y \in I$. Then R is commutative.*

Proof. It is clear that $d(xoy) \pm [x,y] \in I$ for all $x, y \in I$ also. Now including the hypothesis we conclude that $d(xoy) \pm [x,y] \in Z \cap I$ for all $x, y \in I$.
Case I: If $Z \cap I = \{0\}$, we find that $d(xoy) \pm [x,y] = 0$ for all $x, y \in I$ and hence using Theorem 3 we get our required result.
Case II: Suppose $Z \cap I \neq \{0\}$. Let $0 \neq z \in Z \cap I$. Replacing y by z, we arrive at $d(xoz) \in Z \cap I$ for all $x \in I$ i.e.; $2d(xz) \in Z$ for all $x \in I$ and hence $d(xz) \in Z$ for all $x \in I$ i.e.; $d(x)zx + xd(z)x = xd(x)z + xxd(z)$ for all $x \in I$. Using the fact that $d(Z) \subseteq Z$, we conclude that $[d(x),x]Rz = \{0\}$ for all $x \in I$. since $Z^* = Z$ and $I^* = I$, we obtain that $0 \neq z^* \in Z \cap I$. Now arguing in the similar lines as above we also obtain that $[d(x),x]Rz^* = \{0\}$ for all $x \in I$. By $*$-primeness of R we conclude that $[d(x),x] = 0$ for all $x \in I$. Finally by Lemma 3, the result follows. □

Corollary 2. *Let R be a prime ring of characteristic not 2, I a nonzero ideal and d a nonzero derivation of R satisfying either of the following differential identities (i) $d([x,y]) \pm (xoy) \in Z$ for all $x, y \in I$ or (ii) $d(xoy) \pm [x,y] \in Z$ for all $x, y \in I$. Then R is commutative.*

Proof. Assume that d is a nonzero derivation of R such that (i) $d([x,y]) \pm (xoy) \in Z$ for all $x, y \in I$ or (ii) $d(xoy) \pm [x,y] \in Z$ for all $x, y \in I$. Since R is a prime ring of characteristic not 2, consider $\mathcal{R} = R \times R^o$, which is clearly a $*_{ex}$-prime ring of characteristic different from 2. Set $\mathcal{I} = I \times I^o$ a nonzero $*_{ex}$-ideal of $\mathcal{R}$. Now define $D : \mathcal{R} \longrightarrow \mathcal{R}$ by $D(x,y) = (d(x), d(y))$. Using hypothesis it can be easily proved that D is a nonzero derivation of $\mathcal{R}$. Moreover $D*_{ex} = *_{ex}D$ and (i) $D([x,y]) \pm (xoy) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{I}$ or (ii) $D(xoy) \pm [x,y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{I}$. In view of the Theorem 5 and Theorem 6 we deduce that $\mathcal{R}$ is commutative and in turn we obtain that R is also commutative. □

The following example shows that the $*$-primeness in the hypotheses of the Theorems 5 & 6 can not be omitted.

Example 3. Let $R = \mathbb{R}[x] \times \mathbb{H}$, where $\mathbb{R}[x]$ is the polynomial ring over ring $\mathbb{R}$ of real numbers and $\mathbb{H}$ is the ring of real quaternions. Clearly, R is a ring of characteristic different from 2. Define $D : R \longrightarrow R$ as $D(f(x), q) = (0, d_i(q))$, where d_i is the inner derivation of $\mathbb{H}$, determined by $i \in \mathbb{H}$, i.e.; $d_i(q) = [i,q]$ for all $q \in \mathbb{H}$. Also define $* : R \longrightarrow R$ as $*(f(x), q) = (f(x), \bar{q})$, where $f(x) \in \mathbb{R}[x]$ and $\bar{q} = \alpha - \beta i - \gamma j - \delta k$, where $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$. It can be easily shown that D and $*$ are a nonzero derivation and an involution of R respectively such that $D* = *D$. Suppose that $I = \mathbb{R}[x] \times \{0\}$. It is obvious that I is a $*$-ideal of R. Let $0 \neq u(x) \in \mathbb{R}[x]$ and $0 \neq v \in \mathbb{H}$. Then we have $(u(x),0)R(0,v) = \{(0,0)\} = (u(x),0)R(0,v)^*$, where $(0,0) \neq (u(x),0), (0,0) \neq (0,v) \in R$. This implies that R is not a $*$-prime ring but it is a semiprime ring. Here it is obvious to observe that (i) $D([m,n]) \pm (mon) \in Z$ for all $m, n \in I$ and (ii) $D(mon) \pm [m,n] \in Z$ for all $m, n \in I$, but R is noncommutative.

We now consider differential identities involving anticommutators in the next two results and show that there does not exist nonzero derivation satisfying these differential identities.

Theorem 7. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R such that $Z \cap I \neq \{0\}$. Then there exists no nonzero derivation d such that $d(x)oy = xod(y)$ for all $x, y \in I$.*

Proof. By hypothesis we have $d(x)y + yd(x) - xd(y) - d(y)x = 0$ for all $x, y \in I$. Let $z \in Z \cap I$. Replacing y by z in the hypothesis, we arrive at $d(x)z + zd(x) - xd(z) - d(z)x = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Now since R has characteristic different from 2, and $d(Z) \subseteq Z$, we find that $d(x)z - xd(z) = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Substituting xy, where $y \in I$ for x in the last relation and using the same again we conclude that $d(x)yz = 0$ for all $x, y \in I$ and for all $z \in Z \cap I$. But since $Z^* = Z$ and $I^* = I$, we also have $Z^* \cap I^* = Z \cap I$. These arguments show that $d(x)yz^* = 0$ for all $x, y \in I$ and for all $z \in Z \cap I$. Finally we infer that $d(x)Iz = \{0\} = d(x)Iz^*$ for all $x \in I$ and for all $z \in Z \cap I$. Lemma 1 and the fact that $Z \cap I \neq \{0\}$ insure that $d(x) = 0$ for all $x \in I$. Replacing x by xr, where $r \in R$ in the last relation and using the same again we arrive at $Id(r) = \{0\}$. This implies that $sId(r) = \{0\} = s^*Id(r)$, where $0 \neq s \in R$. Finally by Lemma 1, we obtain that $d = 0$. □

Theorem 8. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R such that $Z \cap I \neq \{0\}$. Then there exists no nonzero derivation d which commutes with '$*$' and satisfies either (i) $d(x)oy = d(x)od(y)$ for all $x, y \in I$ or (ii) $xod(y) = d(x)od(y)$ for all $x, y \in I$.*

Proof. (i) By hypothesis we have $d(x)y + yd(x) - d(x)d(y) - d(y)d(x) = 0$ for all $x, y \in I$. Let $z \in Z \cap I$. Replacing x by z in the hypothesis, we arrive at $d(z)y + yd(z) - d(z)d(y) - d(y)d(z) = 0$ for all $y \in I$ and for all $z \in Z \cap I$. But since R has characteristic different from 2 and $d(Z) \subseteq Z$ we arrive at $d(z)y - d(z)d(y) = 0$ for all $y \in I$ and for all $z \in Z \cap I$. Now we infer that $d(z)I(d(y) - y) = \{0\}$ for all $y \in I$ and for all $z \in Z \cap I$. But it is obvious to see that $Z^* \cap I^* = Z \cap I$. Since $d* = *d$, we also observe that $\{d(z)\}^*I(d(y) - y) = \{0\}$ for all $y \in I$ and for all $z \in Z \cap I$. Using Lemma 1, we obtain that $d(z) = 0$ for all $z \in Z \cap I$ or $d(y) = y$ for all $y \in I$. If first case holds, then hypothesis gives us $d(x)oz = 0$ for all $x \in I$ and for all $z \in Z \cap I$. Since R has characteristic different from 2, Lemma 7 provides us $d(x)z = 0$ for all $x \in I$ and for all $z \in Z \cap I$. This implies that $d(x)Iz = \{0\} = d(x)Iz^*$ for all $x \in I$ and for all $z \in Z \cap I$. Lemma 1 and the fact that $Z \cap I \neq \{0\}$ insure that $d(x) = 0$ for all $x \in I$. Now arguing in the similar way as in the above Theorem 7, we conclude that $d = 0$. If second case holds, then we have $d(y) = y$ for all $y \in I$. Putting yr, where $r \in R$ in the last relation and using the same again we conclude that $yd(r) = 0$ for all $y \in I$ and for all $r \in R$. This shows that $lId(r) = \{0\} = l^*Id(r)$ for all $r \in R$ and $0 \neq l \in R$. Finally Lemma 1

gives our required result.

(ii) Using similar arguments as above, one can obtain the proof. □

The following example justifies that "characteristic different from 2" in the hypothesis of the above Theorems 7 and 8 is not superfluous.

Example 4. Consider R, D, d, $*_{ex}$, I and $\mathcal{I}$ as discussed in the Example 1. It is obvious to observe that $Z \cap \mathcal{I} = \mathcal{I} \neq \{0\}$. It is easy to check that (i) $D(x)oy = xoD(y)$ for all $x, y \in \mathcal{I}$, (ii) $D(x)oy = D(x)oD(y)$ for all $x, y \in \mathcal{I}$ and (iii) $xoD(y) = D(x)oD(y)$ for all $x, y \in \mathcal{I}$. However $D \neq 0$.

Theorem 9. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R such that $Z \cap I \neq \{0\}$. If R admits a nonzero derivation d which commutes with '$*$' and satisfies $d(x)od(y) = xoy$ for all $x, y \in I$, then R is commutative.*

Proof. Given that $d(x)od(y) = xoy$ for all $x, y \in I$. Choose $z \in Z \cap I$. Replacing y by yz in the hypothesis we obtain that $d(x)od(yz) = xoyz$ for all $x \in I$ and for all $z \in Z \cap I$. Now we have

$$\begin{aligned} d(x)od(yz) &= d(x)(d(y)z + yd(z)) + (d(y)z + yd(z))d(x) \\ &= d(x)d(y)z + d(x)yd(z) + d(y)zd(x) + yd(z)d(x) \end{aligned}$$

and on the other hand using the hypothesis we obtain that

$$\begin{aligned} xoyz &= (xoy)z \\ &= (d(x)od(y))z \\ &= d(x)d(y)z + d(y)d(x)z. \end{aligned}$$

Equating the above two expressions and using the fact that $d(Z) \subseteq Z$, we conclude that $(d(x)y + yd(x))d(z) = 0$ i.e.; $(d(x)y + yd(x))Id(z) = \{0\}$ for all $x, y \in I, z \in Z \cap I$. It is obvious that $Z^* \cap I^* = Z \cap I$. Since $d* = *d$, we also infer that $(d(x)y + yd(x))I\{d(z)\}^* = \{0\}$ for all $x, y \in I, z \in Z \cap I$. By Lemma 1 we arrive at $(d(x)y + yd(x)) = 0$ for all $x, y \in I$ or $d(z) = 0$ for all $z \in Z \cap I$. We claim that $d(z) \neq 0$ for all $z \in Z \cap I$. For otherwise hypothesis provides us $xoz = 2xz = d(x)od(z) = 0$ for all $x \in I$ and $z \in Z \cap I$. Since R has characteristic different from 2, Lemma 7 insures that $xz = 0$ i.e.; $xIz = \{0\}$ for all $x \in I, z \in Z \cap I$. Since $Z^* \cap I^* = Z \cap I$. This fact shows that $xIz^* = \{0\}$ for all $x \in I, z \in Z \cap I$. By Lemma 1, we deduce that either $I = \{0\}$ or $Z \cap I = \{0\}$. This leads to a contradiction. Finally we conclude that $(d(x)y + yd(x)) = 0$ for all $x, y \in I$. Replacing y by yr, where $r \in I$ in the last relation and using the same again we obtain that $y[d(x), r] = 0$ for all $x, y \in I, r \in R$. This implies that $lI[d(x), R] = \{0\} = l^*I[d(x), R]$, for all $x \in I$, where $0 \neq l \in R$. By Lemma 1, we find that $d(I) \subseteq Z$. Finally using Lemma 4, we get that R is commutative. □

Theorem 10. *Let R be a $*$-prime ring of characteristic different from 2 and I a nonzero $*$-ideal of R such that $Z \cap I \neq \{0\}$. If R admits a nonzero derivation d which commutes with '$*$' and satisfies $(d(x)oy)-(xod(y)) \in Z$ for all $x, y \in I$, then R is commutative.*

Proof. Replacing y by z, where $z \in Z \cap I$ in the hypothesis and using the fact that $d(Z) \subseteq Z$, we arrive at $2(d(x)z - xd(z)) \in Z$ for all $x \in I$. This implies that $(d(x)z - xd(z)) \in Z$ for all $x \in I$ and for all $z \in Z \cap I$ and hence $(d(x)z - xd(z))x = x(d(x)z - xd(z))$ for all $x \in I$ and for all $z \in Z \cap I$. In turn we conclude that $(d(x)x - xd(x))z = 0$ i.e.; $(d(x)x - xd(x))Iz = \{0\}$ for all $x \in I$ and for all $z \in Z \cap I$. Since $Z^* \cap I^* = Z \cap I$, we obtain that $(d(x)x - xd(x))Iz^* = \{0\}$ for all $x \in I$ and for all $z \in Z \cap I$. Now hypothesis and Lemma 1 provide us that $(d(x)x - xd(x)) = 0$ i.e.; $[d(x), x] = 0$ for all $x \in I$. Finally Lemma 3, completes the proof. □

The following example demonstrates that the $*$-primeness in the hypothesis in the above theorem is necessary.

Example 5. Let $S = M_{2\times 2}(\mathbb{R}[x])$, the ring of all 2×2 matrices over ring $\mathbb{R}[x]$, where $\mathbb{R}[x]$ is the polynomial ring over ring of real numbers. Suppose that $R = S \times S$, which is clearly a ring of characteristic different from 2. Define $D : R \longrightarrow R$ as $D(A, B) = (0, D'(B))$, where

$$D'\begin{pmatrix} f(x) & g(x) \\ h(x) & u(x) \end{pmatrix} = \begin{pmatrix} d(f(x)) & d(g(x)) \\ d(h(x)) & d(u(x)) \end{pmatrix},$$

$B = \begin{pmatrix} f(x) & g(x) \\ h(x) & u(x) \end{pmatrix}$ and d is the usual differentiation of the polynomial ring $\mathbb{R}[x]$. Also define $* : R \longrightarrow R$ as $*(A, B) = (A^t, B^t)$, where A^t and B^t are the transpose of the matrices A and B respectively. It can be easily shown that D and $*$ are a nonzero derivation and an involution of R respectively such that $D* = *D$. Suppose that $I = M_{2\times 2}(\mathbb{R}[x]) \times \{0\}$. It is obvious that I is a $*$-ideal of R and $Z \cap I \neq \{0\}$. Let $0 \neq U, 0 \neq V \in M_{2\times 2}(\mathbb{R}[x])$. Then we have $(U, 0)R(0, V) = \{(0, 0)\} = (U, 0)R(0, V)^*$, where $(0, 0) \neq (U, 0), (0, 0) \neq (0, V) \in R$. This implies that R is not a $*$-prime ring but it is a semiprime ring. Here it is obvious to see that $(D(p)oq) - (poD(q)) \in Z$ for all $p, q \in I$, but R is noncommutative.

ACKNOWLEDGEMENT

The authors are indebted to the referee for his/her useful suggestions and valuable comments.

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Doi: 10.18514/MMN.2015.1266

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 45–55

HU e-ISSN 1787-2413

AN EXTENDED THREE-TERM CONJUGATE GRADIENT METHOD WITH SUFFICIENT DESCENT PROPERTY

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Received 27 May, 2014

Abstract. An extension of the three-term conjugate gradient method proposed by Zhang et al. is suggested. Based on an eigenvalue analysis on the search direction matrix, it is shown that the method possesses the sufficient descent property, no matter whether the line search is exact or not as well as the objective function is convex or not. It is interesting that the method can be considered as a hybridization of the conjugate gradient methods proposed by Zhang et al., and Hestenes and Stiefel. Global convergence of the method is established for uniformly convex objective functions. Comparative numerical results demonstrating efficiency of the proposed method are reported.

2010 *Mathematics Subject Classification:* 65K05; 90C53; 49M37; 15A18

Keywords: unconstrained optimization, large-scale optimization, three-term conjugate gradient method, eigenvalue, sufficient descent condition, global convergence

1. INTRODUCTION

Conjugate gradient (CG) methods comprise a class of unconstrained optimization algorithms with the attractive features of low memory requirement, simple computation and strong global convergence, making them efficient for solving large-scale problems in the form of $\min_{x\in\mathbb{R}^n} f(x)$, with the differentiable nonlinear function $f:\mathbb{R}^n\to\mathbb{R}$. Iterations of the CG methods are given by

$$x_0\in\mathbb{R}^n,\ x_{k+1}=x_k+s_k,\ s_k=\alpha_k d_k,\ k=0,1,..., \tag{1.1}$$

where α_k is a steplength to be computed by a line search procedure and d_k is the search direction defined by

$$d_0=-g_0,\ d_{k+1}=-g_{k+1}+\beta_k d_k,\ k=0,1,...,$$

in which $g_k=\nabla f(x_k)$ and β_k is a scalar called the CG (update) parameter. Different CG methods mainly correspond to different choices for the CG parameter [14]. Although the CG methods are equivalent in the linear case, i.e., when the objective

This research was supported by Research Councils of Semnan University and Ferdowsi University of Mashhad.

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function is strictly convex and the steplength is computed exactly, their numerical behavior for general functions may be quite different [1].

One of the essential CG methods has been proposed by Hestenes and Stiefel [15] (HS), with the following CG parameter:

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k},$$

where $y_k = g_{k+1} - g_k$. Although the HS method is numerically efficient [1] and has the nice property of satisfying the conjugacy condition $d_{k+1}^T y_k = 0$, $\forall k \geq 0$, independent of the line search and the objective function convexity, the method lacks the following descent property:

$$g_k^T d_k < 0, \ \forall k \geq 0. \tag{1.2}$$

This motivated the researchers to deal with modifications of the HS method with descent property (1.2), or its stronger version called the sufficient descent property, i.e.,

$$g_k^T d_k \leq -c||g_k||^2, \ \forall k \geq 0, \tag{1.3}$$

where c is a positive constant and $||.||$ stands for the Euclidean norm. In this context, Hager and Zhang [12] (HZ) proposed the following CG parameter:

$$\beta_k^{HZ} = \beta_k^{HS} - 2\frac{||y_k||^2}{d_k^T y_k}\frac{g_{k+1}^T d_k}{d_k^T y_k}, \tag{1.4}$$

which guarantees (1.3) with $c = 7/8$. Afterwards, the following extension of β_k^{HZ} has been proposed in [14]:

$$\beta_k^{\theta} = \beta_k^{HS} - \theta_k\frac{||y_k||^2}{d_k^T y_k}\frac{g_{k+1}^T d_k}{d_k^T y_k}, \tag{1.5}$$

where θ_k is a nonnegative parameter. It is important that if $\theta_k \geq \bar{\theta} > 1/4$, then the sufficient descent condition (1.3) is achieved with $c = 1 - 1/(4\bar{\theta})$ [2]. Recently, Dai and Kou [8] (DK) proposed the following family of CG parameters:

$$\beta_k^{DK} = \beta_k^{HS} - \left(\tau_k + \frac{||y_k||^2}{s_k^T y_k} - \frac{s_k^T y_k}{||s_k||^2}\right)\frac{g_{k+1}^T s_k}{d_k^T y_k}, \tag{1.6}$$

and established the sufficient descent property (1.3) for four special choices of τ_k which is a nonnegative parameter corresponding to the scaling factor in the scaled memoryless BFGS method [18]. More recently, Babaie-Kafaki and Ghanbari [3] proposed the following family of CG parameters:

$$\beta_k^{p,q} = \beta_k^{HS} - \left(p\frac{||y_k||^2}{s_k^T y_k} - q\frac{s_k^T y_k}{||s_k||^2}\right)\frac{g_{k+1}^T s_k}{d_k^T y_k}, \tag{1.7}$$

with the real parameters p and q, guaranteeing the descent property (1.2) for $p > 1/4$ and $q \leq 1/4$. Note that all the CG parameters β_k^{HZ} defined by (1.4), β_k^{θ} defined by (1.5), β_k^{DK} defined by (1.6), and $\beta_k^{p,q}$ defined by (1.7) can be viewed as adaptive versions of the CG parameter proposed by Dai and Liao [9], i.e.,

$$\beta_k^{DL} = \beta_k^{HS} - t\frac{g_{k+1}^T s_k}{d_k^T y_k},$$

where t is a nonnegative parameter to be chosen appropriately. In another attempt, Zhang et al. [21] (ZZL) proposed a computationally efficient three-term CG method with the following search directions:

$$d_0 = -g_0,\ d_{k+1} = -g_{k+1} + \beta_k^{HS} d_k - \frac{g_{k+1}^T d_k}{d_k^T y_k} y_k,\ \forall k \geq 0, \tag{1.8}$$

which satisfies the sufficient descent condition $g_k^T d_k = -||g_k||^2$, $\forall k \geq 0$. It is interesting that under the exact line search, since we have $g_{k+1}^T d_k = 0$, all the CG methods of HZ, DK, ZZL and the descent CG method with the parameter (1.7) reduce to the HS method; hence, they can be considered as descent extensions of the HS method.

Several other attempts have been made to develop the three-term CG methods with the important sufficient descent property. Examples include the CG methods proposed by Cheng [6], Cheng and Liu [7], Narushima et al. [16], Sugiki et al. [17], Zhang et al. [19, 20], and Babaie-Kafaki and Ghanbari [4]. Here, we suggest an extension of the three-term CG method proposed by Zhang et al. [21]. Our method can be considered as a hybridization of the HS and ZZL methods. The remainder of this work is organized as follows. In Section 2, we propose our method and discuss its global convergence. In Section 3, we numerically compare our method with the CG_Descent [13] and ZZL methods, and report comparative testing results. Finally, we make conclusions in Section 4.

2. A THREE-TERM CONJUGATE GRADIENT METHOD

Here, we deal with an extended form of the search directions (1.8), obtained by embedding a parameter $t_k \in [0, 1]$ on the third term of d_{k+1}, $k \geq 0$. More exactly, the search directions of our method, namely the EZZL method, are given by

$$d_0 = -g_0,\ d_{k+1} = -g_{k+1} + \beta_k^{HS} d_k - t_k \frac{g_{k+1}^T d_k}{d_k^T y_k} y_k,\ \forall k \geq 0. \tag{2.1}$$

Note that if $t_k = 0$, $\forall k \geq 0$, or the exact line search is used, then the method reduces to the HS method. Also, if $t_k = 1$, $\forall k \geq 0$, then the method is precisely the ZZL method. Hence, the three-term CG method EZZL can be considered as a hybridization of the HS and ZZL methods with the hybridization parameter t_k. Next, we compute the

parameter t_k such that the EZZL method satisfy the sufficient descent condition (1.3). In what follows, we assume that $d_k^T y_k \neq 0$, $\forall k \geq 0$, as guaranteed by the popular Wolfe conditions [18], i.e.,

$$f(x_k+\alpha_k d_k)-f(x_k) \leq \delta \alpha_k \nabla f(x_k)^T d_k, \tag{2.2}$$

$$\nabla f(x_k+\alpha_k d_k)^T d_k \geq \sigma \nabla f(x_k)^T d_k, \tag{2.3}$$

where $0<\delta<\sigma<1$.

It is worth noting that from (2.1) the search directions of the EZZL method can be written as:

$$d_{k+1} = -Q_{k+1} g_{k+1},\ k=0,1,..., \tag{2.4}$$

where

$$Q_{k+1} = I - \frac{s_k y_k^T}{s_k^T y_k} + t_k \frac{y_k s_k^T}{s_k^T y_k},\ t_k \in [0,1].$$

Since from (2.4) we can write

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T Q_{k+1}^T g_{k+1} = -g_{k+1}^T \frac{Q_{k+1}^T + Q_{k+1}}{2} g_{k+1}, \tag{2.5}$$

to study the sufficient descent property of the EZZL method with a proper value for t_k, in our analysis we need to find the eigenvalues of the following symmetric matrix:

$$A_{k+1} = \frac{Q_{k+1}^T + Q_{k+1}}{2} = I + \frac{t_k-1}{2}\frac{s_k y_k^T}{s_k^T y_k} + \frac{t_k-1}{2}\frac{y_k s_k^T}{s_k^T y_k}. \tag{2.6}$$

Because $\alpha_k > 0$ and $d_k^T y_k \neq 0$, we have $s_k^T y_k \neq 0$ and consequently, $s_k \neq 0$ and $y_k \neq 0$. So, there exists a set of mutually orthonormal vectors $\{u_k^i\}_{i=1}^{n-2}$ such that

$$s_k^T u_k^i = y_k^T u_k^i = 0,\ i=1,...,n-2,$$

which leads to

$$A_{k+1} u_k^i = u_k^i,\ i=1,...,n-2.$$

That is, the vectors u_k^i, $i=1,...,n-2$, are the eigenvectors of A_{k+1} corresponding to the eigenvalue 1. Next, we find the two remaining eigenvalues of A_{k+1}, namely λ_k^- and λ_k^+.

Since the trace of a square matrix is equal to the sum of its eigenvalues, we have

$$\mathrm{tr}(A_{k+1}) = n + t_k - 1 = \underbrace{1+...+1}_{(n-2)\text{ times}} + \lambda_k^- + \lambda_k^+,$$

which leads to

$$\lambda_k^- + \lambda_k^+ = t_k + 1. \tag{2.7}$$

On the other hand, from the properties of the Frobenius norm, we have

$$||A_{k+1}||_F^2 = \mathrm{tr}(A_{k+1}^T A_{k+1}) = \mathrm{tr}(A_{k+1}^2)$$

$$= n-2+\frac{1}{2}(t_k+1)^2+\frac{1}{2}(t_k-1)^2\frac{||s_k||^2||y_k||^2}{(s_k^T y_k)^2}$$
$$= \underbrace{1+\ldots+1}_{(n-2)\text{ times}} +\lambda_k^{-^2}+\lambda_k^{+^2},$$

which leads to

$$\lambda_k^{-^2}+\lambda_k^{+^2} = \frac{1}{2}(t_k+1)^2+\frac{1}{2}(t_k-1)^2\frac{||s_k||^2||y_k||^2}{(s_k^T y_k)^2}. \tag{2.8}$$

Now, from (2.7) and (2.8), after some algebraic manipulations we get

$$\lambda_k^-\lambda_k^+ = \frac{1}{4}(t_k+1)^2-\frac{1}{4}(t_k-1)^2\frac{||s_k||^2||y_k||^2}{(s_k^T y_k)^2}, \tag{2.9}$$

and consequently, from (2.7) and (2.9), λ_k^- and λ_k^+ can be computed by

$$\lambda_k^{\pm} = \frac{1}{2}(1+t_k)\pm\frac{1}{2}(1-t_k)\frac{||s_k||||y_k||}{s_k^T y_k}.$$

From Cauchy-Schwarz inequality, it can be shown that $\lambda_k^+ \geq 1 \geq \lambda_k^-$. So, in order to guarantee the positive definiteness of the matrix A_{k+1} defined by (2.6), it is necessary to have $\lambda_k^- = \xi \in (0,1]$. Thus, the parameter t_k should be computed by

$$t_k = \frac{(2\xi-1)(s_k^T y_k)+||s_k||||y_k||}{s_k^T y_k+||s_k||||y_k||},\ \xi\in(0,1]. \tag{2.10}$$

It can be seen that for the parameter t_k defined by (2.10) we have $t_k \in (0,1]$. Also, since the matrix A_{k+1} with the parameter t_k given by (2.10) is positive definite and its smallest eigenvalue is equal to ξ, from (2.5) we have

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T A_{k+1} g_{k+1} \leq -\xi||g_{k+1}||^2.$$

So, the following theorem is now immediate.

Theorem 1. *The search directions defined by (2.1) in which the parameter t_k is computed by (2.10) satisfy the sufficient descent condition (1.3) with some positive constant $c \in (0,1]$.*

Now, we discuss global convergence of the EZZL method. In our analysis, we need to make the following basic assumptions on the objective function.

Assumption 1. *The level set $\mathcal{L} = \{x|\ f(x) \leq f(x_0)\}$, with x_0 to be the starting point of the iterative method (1.1), is bounded. Also, in some open convex neighborhood $\mathcal{N}$ of $\mathcal{L}$, f is continuously differentiable and its gradient is Lipschitz continuous; that is, there exists a positive constant L such that*

$$||\nabla f(x)-\nabla f(y)|| \leq L||x-y||,\ \forall x,y\in\mathcal{N}. \tag{2.11}$$

Note that these assumptions imply that there exists a positive constant γ such that

$$||\nabla f(x)|| \leq \gamma, \ \forall x \in \mathcal{L} \text{ (see Proposition 3.1 of [5])}. \tag{2.12}$$

The following important lemma plays an essential role in proving the global convergence theorem of the EZZL method.

Lemma 1 ([17]). *Suppose that Assumption 1 holds. Consider any iterative method in the form of (1.1) for which the sufficient descent condition (1.3) holds and the steplength α_k satisfies the Wolfe conditions (2.2) and (2.3). If*

$$\sum_{k\geq 0} \frac{1}{||d_k||^2} = \infty,$$

then the method converges in the sense that

$$\liminf_{k\to\infty} ||g_k|| = 0. \tag{2.13}$$

Now, we can prove the following global convergence theorem for the EZZL method.

Theorem 2. *Suppose that Assumption 1 holds. Consider the iterative method (1.1) in which the search direction d_k is computed by (2.1) with the parameter t_k given by (2.10), and the steplength α_k is determined such that the Wolfe conditions (2.2) and (2.3) are satisfied. If the objective function f is uniformly convex on $\mathcal{N}$, then the method converges in the sense that (2.13) holds.*

Proof. At first, note that from Theorem 1, because the search directions of the method are descent directions, from (2.2) the sequence $\{x_k\}_{k\geq 0}$ is a subset of the level set $\mathcal{L}$. Also, $d_k \neq 0$, $\forall k \geq 0$, and consequently, using Lemma 1, to complete the proof it is enough to show that $||d_k||$ is bounded above.

The uniform convexity of the differentiable function f ensures that there exists a positive constant μ such that

$$s_k^T y_k \geq \mu ||s_k||^2 \text{ (see Theorem 1.3.16 of [18])}. \tag{2.14}$$

Now, from (2.1), (2.11), (2.12), (2.14), and since for t_k defined by (2.10) we have $t_k \in (0, 1]$, we get

$$\begin{aligned} ||d_{k+1}|| &\leq ||g_{k+1}|| + \frac{||y_k||||g_{k+1}||}{s_k^T y_k}||s_k|| + t_k \frac{||g_{k+1}||||s_k||}{s_k^T y_k}||y_k|| \\ &\leq \gamma + \frac{L||s_k||\gamma}{\mu||s_k||^2}||s_k|| + \frac{\gamma||s_k||}{\mu||s_k||^2} L||s_k|| = \left(1 + 2\frac{L}{\mu}\right)\gamma, \end{aligned}$$

which completes the proof.

□

3. Numerical experiments

Here, we present some numerical results obtained by applying C++ implementations of our extended three-term CG method EZZL with the search directions (2.1) in which the parameter t_k is computed by (2.10), the three-term CG method ZZL proposed by Zhang et al. [21] with the search directions (1.8), and the CG_Descent algorithm [13]. The codes were run on a PC with 3.2 GHz Intel I3 of CPU, 4 GB of RAM and Centos 6.2 server Linux operation system. Furthermore, the experiments were performed on a set of 145 unconstrained optimization test problems of the CUTEr collection [11], with default dimensions as given in Hager's home page: 'http://www.math.ufl.edu/∼hager/'.

In our experiments, we used CG_Descent version 5.1. For the EZZL method, we set $\xi = 0.96$ in (2.10) because of the promising numerical results obtained among the different values $\xi \in \{0.02k\}_{k=1}^{50}$. Similar to the CG_Descent algorithm, in the methods of EZZL and ZZL we used the effective approximate Wolfe conditions proposed by Hager and Zhang [13] in the line search procedure, with the same parameter values as suggested in [13]. Moreover, all attempts to solve the test problems were terminated when $||g_k||_\infty < 10^{-6}(1 + |f(x_k)|)$.

Efficiency comparisons were made using the performance profile of Dolan and Moré [10], on the running time and the total number of function and gradient evaluations being equal to $N_f + 3N_g$ [13], where N_f and N_g respectively denote the number of function and gradient evaluations. Performance profile gives, for every $\omega \geq 1$, the proportion $p(\omega)$ of the test problems that each considered algorithmic variant has a performance within a factor of ω of the best. Figures 1 and 2 show the comparisons results. As the figures demonstrate, the EZZL method is preferable to the ZZL method while the CG_Descent method outperforms the EZZL and ZZL methods, both in the perspectives of the total number of function and gradient evaluations, and the CPU time.

Since the CG methods have been mainly designed for solving large-scale problems, we made furtherer comparisons of the three methods EZZL, ZZL and CG_Descent, on the numerical results of the 64 out of 145 test problems with the minimum dimensions being equal to 1000. Figures 3 and 4 demonstrate the comparisons results. As the figures show, with respect to the total number of function and gradient evaluations, and also with respect to the running time, the EZZL method is preferable to the methods of CG_Descent and ZZL. Thus, the EZZL method seem to be practically promising, especially for solving large-scale problems.

4. Conclusions

An extended three-term conjugate gradient method which can be considered as a hybridization of the three-term conjugate gradient method suggested by Zhang et al. [21] and the conjugate gradient method suggested by Hestenes and Stiefel [15] has

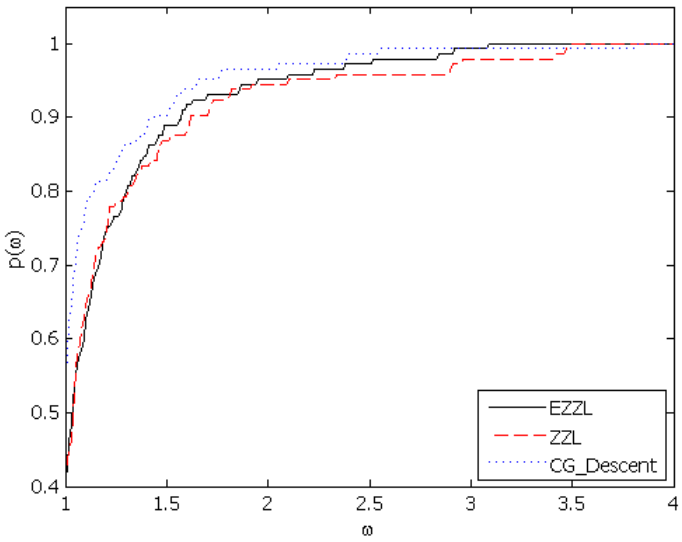

FIGURE 1. Total number of function and gradient evaluations performance profiles for EZZL, ZZL and CG_Descent.

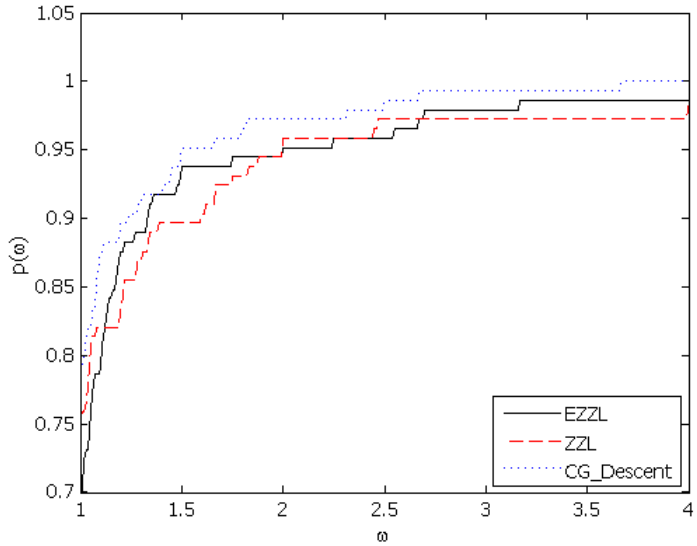

FIGURE 2. CPU time performance profiles for EZZL, ZZL and CG_Descent.

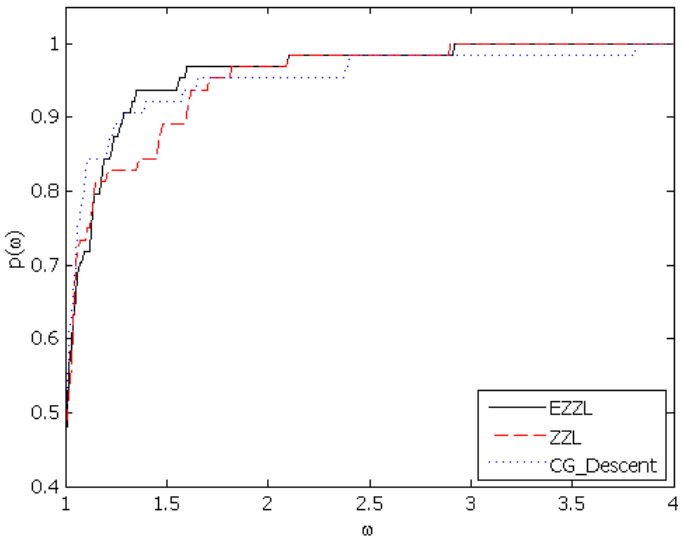

FIGURE 3. Total number of function and gradient evaluations performance profiles for EZZL, ZZL and CG_Descent in large-scale cases.

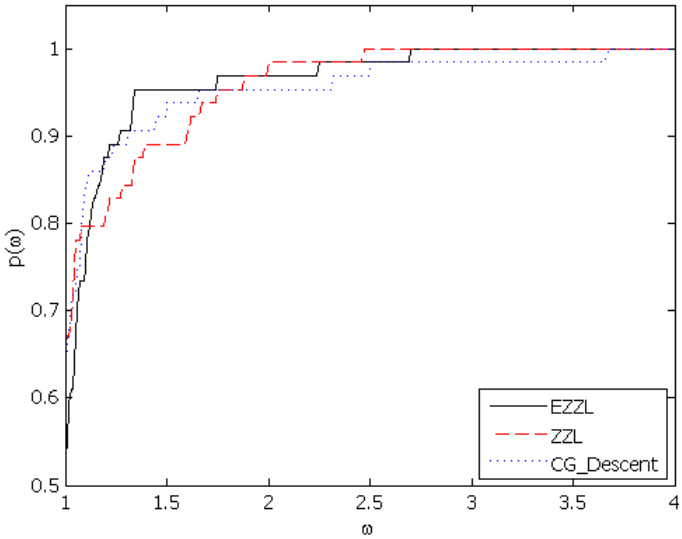

FIGURE 4. CPU time performance profiles for EZZL, ZZL and CG_Descent in large-scale cases.

been proposed. Based on an eigenvalue study, it has been shown that the method possesses the important sufficient descent property, independent of the line search and the objective function convexity. Global convergence of the method has been established for uniformly convex objective functions. Numerical experiments showed the efficiency of the proposed method in the sense of the performance profile introduced by Dolan and Moré [10], particularly for solving large-scale problems.

ACKNOWLEDGEMENT

The authors are grateful to Professor William W. Hager for providing the line search code. They also thank the anonymous referee for his/her useful comments helped to improve the presentation.

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Doi: 10.18514/MMN.2015.1124

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 57–63

HU e-ISSN 1787-2413

A NEW FACTOR THEOREM ON ABSOLUTE RIESZ SUMMABILITY

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Received 06 February, 2014

Abstract. In this paper, a known theorem dealing with an application of quasi power increasing sequences has extended by using a new class of increasing sequences instead of a quasi-σ-power increasing sequence. This theorem also includes some new and known results.

2010 *Mathematics Subject Classification:* 26D15; 40D15; 40F05; 40G99; 46A45

Keywords: Riesz mean, absolute summability, infinite series, sequence spaces, power increasing sequences, Hölder inequality, Minkowski inequality

1. Introduction

A positive sequence $X = (X_n)$ is said to be a quasi-(σ,γ)-power increasing sequence if there exists a constant $K = K(X,(\sigma,\gamma)) \geq 1$ such that $Kn^{\sigma}(\log n)^{\gamma} Xn \geq m^{\sigma}(\log m)^{\gamma} Xm$ for all $n \geq m \geq 1$, where $\gamma \geq 0$ and $0 < \sigma < 1$ (see [14]). If we take γ=0, then we get a quasi-σ-power increasing sequence (see [13]). A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). We denote by t_n^{α} the nth Cesàro mean of order α, with $\alpha > -1$, of the sequences (na_n), that is (see[9])

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \qquad (t_n{}^1 = t_n) \tag{1.1}$$

where

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^{\alpha}), \quad A_{-n}^{\alpha} = 0 \quad for \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $| C,\alpha;\delta |_k$, $k \geq 1$ and $\delta \geq 0$, if (see [11])

$$\sum_{n=1}^{\infty} n^{\delta k-1} | t_n^{\alpha} |^k < \infty. \tag{1.3}$$

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If we set $\delta = 0$, then we get $| C, \alpha |_k$ summablility (see [10]).
Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [12]). The series $\sum a_n$ is said to be summable $| \bar{N}, p_n; \delta |_k$, $k \geq 1$ and $\delta \geq 0$, if (see [4])

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k + k - 1} \mid w_n - w_{n-1} \mid^k < \infty. \tag{1.6}$$

If we set $\delta = 0$, then we obtain $| \bar{N}, p_n |_k$ summability (see [2]). In the special case $p_n = 1$ for all values of n, the $| \bar{N}, p_n; \delta |_k$ summability is the same as $| C, 1; \delta |_k$ summability. Also if we take $\delta = 0$ and $k = 1$, then we get $| \bar{N}, p_n |$ summability.

2. The known result

The following theorem is known dealing with the absolute Riesz summability factors of infinite series.

Theorem 1 ([5]). *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$ and let there be sequences (β_n) and (λ_n) such that*

$$| \Delta \lambda_n | \leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta \beta_n \mid X_n < \infty, \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{2.4}$$

If (p_n) is a sequence such that

$$P_n = O(n p_n), \tag{2.5}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.6}$$

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v} \right)^{\delta k} \frac{| t_v |^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{2.7}$$

$$\sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}=O((\frac{P_v}{p_v})^{\delta k}\frac{1}{P_v})\quad as\quad m\to\infty, \tag{2.8}$$

then the series $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ *is summable* $\mid \bar{N},p_n;\delta\mid_k$, $k\geq 1$ *and* $0\leq\delta<1/k$.

Remark 1. In the hypothesis of Theorem 1, we have added the condition "$(\lambda_n)\in\mathcal{BV}$".

3. The main result

The aim of this paper is to extend Theorem 1 by using a quasi-(σ,γ)-power increasing sequence instead of quasi-σ-power increasing sequence. Now we shall prove the following theorem.

Theorem 2. *Let* $(\lambda_n)\in\mathcal{BV}$ *and let* (X_n) *be a quasi-*(σ,γ)*-power increasing sequence. If all conditions of Theorem 1 are satisfied, then the series* $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ *is summable* $\mid \bar{N},p_n;\delta\mid_k$, $k\geq 1$, *and* $0\leq\delta<1/k$.

Remark 2. If we set γ=0, then we get Theorem 1. If we take $\delta=0$, then we get a new result dealing with $\mid \bar{N},p_n\mid_k$ summability factors. If we take $p_n=1$ for all values of n, then we get a new result dealing with $\mid C,1;\delta\mid_k$ summability factors. Also, if we take γ=0 and δ=0, the we get the known result (see [6]). Finally, if we take (X_n) as an almost increasing sequence, then we get the known result which was published in [7].

We need the following lemmas for the proof of the theorem.

Lemma 1 ([8])**.** *Except for the condition* $(\lambda_n)\in\mathcal{BV}$, *under the conditions on* (X_n), (β_n) *and* (λ_n) *as expressed in the statement of the theorem, we have the following;*

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty, \tag{3.1}$$

$$nX_n\beta_n=O(1). \tag{3.2}$$

Lemma 2 ([3])**.** *If the conditions (2.5) and (2.6) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n^2p_n}\right)=O\left(\frac{1}{n^2}\right). \tag{3.3}$$

4. Proof of Theorem 2

Let (T_n) be the sequence of $(\bar{N},p_n)$ mean of the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$. Then, by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=1}^{n}p_v\sum_{r=1}^{v}\frac{a_rP_r\lambda_r}{rp_r}=\frac{1}{P_n}\sum_{v=1}^{n}(P_n-P_{v-1})\frac{a_vP_v\lambda_v}{vp_v}. \tag{4.1}$$

Then, for $n \geq 1$ we obtain that

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v v \lambda_v}{v^2 p_v}.$$

Using Abel's transformation, we get that

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v (v+1) \frac{t_v}{v^2 p_v} \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta(P_v / v^2 p_v) \\ &\quad + \lambda_n t_n (n+1)/n^2 = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned}$$

To complete the proof of the theorem by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | T_{n,r} |^k < \infty, \quad for \quad r = 1,2,3,4. \tag{4.2}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned} &\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | T_{n,1} |^k \\ &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v | t_v | | \lambda_v | \frac{1}{v} \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v | t_v |^k | \lambda_v |^k \frac{1}{v^k} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v | t_v |^k | \lambda_v |^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k | \lambda_v |^{k-1} | \lambda_v | p_v | t_v |^k \frac{1}{v^k} \frac{1}{P_v} (\frac{P_v}{p_v})^{\delta k} \end{aligned}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \frac{1}{v^k}(\frac{P_v}{p_v})^{\delta k}$$

$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} v^{k-1}\frac{1}{v^k} \mid \lambda_v \mid\mid t_v \mid^k$$

$$= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid (\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r \mid^k}{r} + O(1) \mid \lambda_m \mid \sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m$$

$$= O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and Lemma 1. Now, by using (2.5), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k$$

$$= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\mid t_v \mid\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\mid t_v \mid^k \times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m} P_v\beta_v \mid t_v \mid^k \sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\beta_v \mid t_v \mid^k = O(1)\sum_{v=1}^{m} v\beta_v(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r \mid^k}{r} + O(1)m\beta_m\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m$$

$$= O(1) \quad as \quad m \to \infty,$$

in view of the hypotheses of the theorem and Lemma 1. By using Lemma 2, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | T_{n,3} |^k$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v | \lambda_{v+1} || t_v | \frac{1}{v} \frac{v+1}{v} \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v | \lambda_{v+1} | \frac{1}{v} | t_v | \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} | \lambda_{v+1} |^k | t_v |^k \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} | \lambda_{v+1} |^{k-1} | \lambda_{v+1} || t_v |^k \times \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{k-1} \frac{1}{v^k} | \lambda_{v+1} || t_v |^k (\frac{P_v}{p_v})^{\delta k}$$

$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} v^{k-1} \frac{1}{v^k} | \lambda_{v+1} || t_v |^k$$

$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} | \lambda_{v+1} | \frac{| t_v |^k}{v} = O(1) \quad as \quad m \to \infty.$$

Finally, as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | T_{n,4} |^k = O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\frac{n+1}{n}\right)^k \frac{1}{n^k} | \lambda_n |^k | t_n |^k$$

$$= O(1) \sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} n^{k-1} \frac{1}{n^k} | \lambda_n |^{k-1} | \lambda_n || t_n |^k$$

$$= O(1) \sum_{n=1}^{m} | \lambda_n | (\frac{P_n}{p_n})^{\delta k} \frac{| t_n |^k}{n} = O(1) \quad as \quad m \to \infty.$$

This completes the proof of the theorem.

ACKNOWLEDGEMENT

The author expresses his thanks to the referee for his/her valuable comments and suggestions for the improvement of this paper.

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Doi: 10.18514/MMN.2015.1325

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 65–78

HU e-ISSN 1787-2413

A NOTE ON ANNIHILATORS IN DISTRIBUTIVE NEARLATTICES

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Received 10 September, 2014

Abstract. In this note we propose a definition of relative annihilator in distributive nearlattices with greatest element different from that given in [6] and we present some new characterizations of the distributivity. Later, we study the class of normal and p-linear nearlattices, the lattice of filters and semi-homomorphisms that preserve annihilators.

2010 *Mathematics Subject Classification:* 06A12; 03G10; 06D50

Keywords: distributive nearlattice, relative annihilator, prime ideal, semi-homomorphism

1. Introduction and preliminaries

Abbott in [1] established a correspondence between the class of Tarski algebras, or implication algebras, and join-semilattices in which every principal filter is a Boolean lattice with respect to the induced order. There is an algebraic structure that generalizes the class of Tarski algebras: *nearlattices*. A nearlattice is a join-semilattice in which every principal filter is a lattice. The class of nearlattices forms a variety that has been studied in [9] and [11] by Cornish and Hickman, and in [4], [6] and [7] by Chajda, Kolařík, Halaš and Kühr. In [2] the authors showed that the axiom systems given in [11] and [4] are dependent and that the variety of nearlattices is 2-based. An important class of nearlattices is the class of *distributive nearlattices*. Recently in [3], a full duality between distributive nearlattices with greatest element and certain topological spaces with a distinguished basis was developed.

It is well known that the notion of distributivity in a lattice can be characterized in different ways, for example, a lattice A is distributive if and only if the lattice $\mathrm{Fi}(A)$ of all filters of A is distributive. Another way is through some special subsets, called *annihilators*. In a lattice A, the *annihilator of a relative to b* is defined as the set $\langle a,b\rangle = \{x \in A : x \wedge a \leq b\}$. In [12], Mandelker studied the properties of relative annihilators and characterized the distributivity of a lattice in terms of its relative annihilators. To be more precise, a lattice A is distributive if and only if $\langle a,b\rangle$ is an ideal of A for all $a,b \in A$. These results were generalized by Varlet to the

Both authors were partially supported by PIP 112-201101-00636 CONICET.

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class of distributive semilattices ([14]) and subsequently to the variety of distributive nearlattices by Chajda and Kolařík ([6]).

The main aim of this paper is to propose a definition of relative annihilator in distributive nearlattices with greatest element different from that given in [6]. In Section 2 we introduce the relative annihilators and we develop new characterizations. In Section 3 we present the class of normal and p-linear nearlattices. In Section 4 we study the lattice of filters of a distributive nearlattice. Finally, in Section 5 we characterize the semi-homomorphisms that preserve annihilators.

Given a poset $\langle X, \leq \rangle$, a set $Y \subseteq X$ is called *increasing* if it is closed under $\leq$, i.e., if for every $x \in Y$ and every $y \in X$, if $x \leq y$ then $y \in Y$. Dually, $Y \subseteq X$ is said to be *decreasing* if for every $x \in Y$ and every $y \in X$, if $y \leq x$ then $y \in Y$. The complement of a subset $Y \subseteq X$ will be denoted by $X - Y$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y \ (y \leq x)\}$ ($(Y] = \{x \in X : \exists y \in Y \ (x \leq y)\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

A *join-semilattice with greatest element* is an algebra $\langle A, \vee, 1 \rangle$ of type $(2,0)$ such that the operation $\vee$ is idempotent, commutative, associative and $x \vee 1 = 1$ for all $x \in A$. The binary relation $\leq$ defined by $x \leq y$ if and only if $x \vee y = y$ is a partial order. In what follows, we shall write simply *semilattice*.

A *filter* of a semilattice A is a subset $F \subseteq A$ such that $1 \in F$, F is increasing and if $x, y \in F$ then $x \wedge y \in F$, whenever $x \wedge y$ exists. The set of all filters of A is denoted by $\mathrm{Fi}(A)$. Let X be a non-empty subset of A. The least filter containing X is called the *filter generated by* X and will be denoted by $F(X)$. Note that if $X = \{a\}$ then $F(\{a\}) = [a)$, called the principal filter of a.

A subset I of A is called an *ideal* if I is decreasing and if $x, y \in I$ then $x \vee y \in I$. The least ideal containing X is called the *ideal generated by* X and will be denoted by $I(X)$. We shall say that a non-empty proper ideal P is *prime* if for all $x, y \in A$, if $x \wedge y \in P$, whenever $x \wedge y$ exists, then $x \in P$ or $y \in P$. We will denote by $\mathrm{Id}(A)$ and $X(A)$ the set of all ideals and prime ideals of A, respectively. Finally, we will say that a non-empty ideal I of A is *maximal* if it is proper and for every $J \in \mathrm{Id}(A)$, if $I \subseteq J$ then $J = I$ or $J = A$. We denote by $\mathrm{Idm}(A)$ the set of all maximal ideals of A. It is easy to prove that every maximal ideal is prime.

Definition 1. A *nearlattice* is a semilattice A such that for each $a \in A$ the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded lattice.

The class of nearlattices forms a variety since every nearlattice A can be described as an algebra with one ternary operation: if $x, y, a \in A$, the element $m(x, y, a) = (x \vee a) \wedge_a (y \vee a)$ is correctly defined because $x \vee a, y \vee a \in [a)$ and $[a)$ is a lattice, where $\wedge_a$ denotes the meet in $[a)$.

Proposition 1 ([2])**.** *Let A be a nearlattice. The following identities are satisfied:*

(1) $m(x, y, x) = x$,

(2) $m(m(x,y,z),m(y,m(u,x,z),z),w) = m(w,w,m(y,m(x,u,z),z))$,
(3) $m(x,x,1) = 1$.

Conversely, let $\langle A,m,1\rangle$ be an algebra of type $(3,0)$ satisfying the identities (1)–(3). If we define $x \vee y = m(x,x,y)$, then A is a semilattice and for each $a \in A$, $[a)$ is a bounded lattice, where the meet of $x, y \in [a)$ is $x \wedge_a y = m(x,y,a)$. Hence A is a nearlattice.

Definition 2. Let A be a nearlattice. We call A *distributive* if the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded distributive lattice for each $a \in A$.

Theorem 1 ([7])**.** *Let A be a nearlattice. Then A is distributive if and only if satisfies either of the following identities:*

(1) $m(x,m(y,y,z),w) = m(m(x,y,w),m(x,y,w),m(x,z,w))$,
(2) $m(x,x,m(y,z,w)) = m(m(x,x,y),m(x,x,z),w)$.

Theorem 2 ([10])**.** *Let A be a distributive nearlattice. Let $I \in \mathrm{Id}(A)$ and let $F \in \mathrm{Fi}(A)$ such that $I \cap F = \varnothing$. Then there exists $P \in X(A)$ such that $I \subseteq P$ and $P \cap F = \varnothing$.*

For distributive nearlattices we have the following lemma which characterizes the generated filters and can be deduced from the results given in [9].

Lemma 1. *Let A be a distributive nearlattice. Let $X \subseteq A$ be a non-empty subset. Then*

$$F(X) = \{a \in A \ : \ \exists x_1,...,x_n \in [X) \ \exists x_1 \wedge ... \wedge x_n \ (x_1 \wedge ... \wedge x_n = a)\}.$$

A filter H is said to be *finitely generated* if $H = F(X)$ for some finite non-empty subset X of A. We will denote by $\mathrm{Fi}_f(A)$ the set of all finitely generated filters of A.

Recall that if A is a distributive nearlattice, then $\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice, where the least element is $\{1\}$, the greatest element is A, and for all $G, H \in \mathrm{Fi}(A)$ we have that $G \veebar H = F(G \cup H)$ and $G \barwedge H = G \cap H$.

Theorem 3 ([4, 9])**.** *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *$\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice.*
(3) *$\langle \mathrm{Fi}_f(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice.*

A function $h : A \to B$ between distributive nearlattices is a *semi-homomorphism* if $h(1) = 1$ and $h(a \vee b) = h(a) \vee h(b)$ for all $a, b \in A$. A *homomorphism* is a semi-homomorphism h such that for all $a, b \in A$, if $a \wedge b$ exists, then $h(a \wedge b) = h(a) \wedge h(b)$. In [3] it was shown that there exists a duality between semi-homomorphisms of distributive nearlattices and certain binary relations.

2. Relative annihilators

In this section we will develop new characterizations of the distributivity of a nearlattice through relative annihilators and relative maximal ideals.

Definition 3. Let A be a semilattice. For $a, b \in A$, the *annihilator of a relative to b* is the set

$$a \circ b = \{x \in A \,:\, b \leq x \vee a\}.$$

Let A be a semilattice. Let $a, b \in A, I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. We introduce the following subsets of A:

$$\begin{array}{rcl} I \circ b & = & \{x \in A \,:\, \exists i \in I \ (b \leq x \vee i)\}, \\ a \circ F & = & \{x \in A \,:\, \exists f \in F \ (f \leq x \vee a)\}. \end{array}$$

Let $X, Y \subseteq A$. We denote by $X \circ Y$ the set

$$X \circ Y = \bigcup \{a \circ b \,:\, (a,b) \in X \times Y\}.$$

Remark 1. Note that $a \circ b = (a] \circ b = a \circ [b) = (a] \circ [b)$ for all $a, b \in A$.

The following theorem characterizes distributive nearlattices.

Theorem 4. *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *$a \circ b \in \mathrm{Fi}(A)$ for all $a, b \in A$.*
(3) *$I \circ b \in \mathrm{Fi}(A)$ for all $I \in \mathrm{Id}(A)$ and $b \in A$.*
(4) *$a \circ F \in \mathrm{Fi}(A)$ for all $F \in \mathrm{Fi}(A)$ and $a \in A$.*
(5) *$I \circ F \in \mathrm{Fi}(A)$ for all $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$.*

Proof. (1) $\Rightarrow$ (2) It is obvious that $1 \in a \circ b$. Let $x, y \in A$ such that $x \leq y$ and $x \in a \circ b$. Then, $x \vee a \leq y \vee a$ and $b \leq x \vee a$. So, $b \leq y \vee a$ and $y \in a \circ b$. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Then, $b \leq x \vee a$ and $b \leq y \vee a$, i.e., $x \vee a,\ y \vee a \in [b)$. Since $[b)$ is a bounded distributive lattice, $b \leq (x \vee a) \wedge_b (y \vee a) = (x \wedge y) \vee a$. Then, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$.

(2) $\Rightarrow$ (3) Let $b \in A$ and $I \in \mathrm{Id}(A)$. We note that $i \circ b \subseteq I \circ b$ for all $i \in I$. It is easy to prove that $1 \in I \circ b$ and that $I \circ b$ is increasing. Let $x, y \in I \circ b$ such that $x \wedge y$ exists. Then there exist $i_1, i_2 \in I$ such that $b \leq x \vee i_1$ and $b \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. So, $b \leq x \vee i$ and $b \leq y \vee i$. Since $x, y \in i \circ b$ and $i \circ b \in \mathrm{Fi}(A)$, we have that $x \wedge y \in i \circ b \subseteq I \circ b$. Therefore, $I \circ b \in \mathrm{Fi}(A)$.

(3) $\Rightarrow$ (4) Let $a \in A$ and $F \in \mathrm{Fi}(A)$. It follows easily that $1 \in a \circ F$ and that $a \circ F$ is increasing. Let $x, y \in a \circ F$ such that $x \wedge y$ exists. Then there exist $f_1, f_2 \in F$ such that $f_1 \leq x \vee a$ and $f_2 \leq y \vee a$. So, $x \vee a, y \vee a \in F$. Since $x \vee a, y \vee a \in [a)$, $(x \vee a) \wedge_a (y \vee a)$ exists and $(x \vee a) \wedge_a (y \vee a) \in F$. As $(x \vee a) \wedge_a (y \vee a) \leq x \vee a$ and $(x \vee a) \wedge_a (y \vee a) \leq y \vee a$, we have $x, y \in a \circ ((x \vee a) \wedge_a (y \vee a))$. By Remark 1, $a \circ ((x \vee a) \wedge_a (y \vee a)) = (a] \circ ((x \vee a) \wedge_a (y \vee a))$ and by hypothesis $(a] \circ ((x \vee a) \wedge_a$

$(y \vee a)) \in \mathrm{Fi}(A)$. Then $x \wedge y \in a \circ ((x \vee a) \wedge_a (y \vee a))$, but as $(x \vee a) \wedge_a (y \vee a) \in F$ we have that $x \wedge y \in a \circ F$. So, $a \circ F \in \mathrm{Fi}(A)$.

(4) $\Rightarrow$ (5) Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. It is easy to see that $1 \in I \circ F$ and that $I \circ F$ is increasing. Let $x, y \in I \circ F$ such that $x \wedge y$ exists. Then there exist $(i_1, f_1), (i_2, f_2) \in I \times F$ such that $x \in i_1 \circ f_1$ and $y \in i_2 \circ f_2$, i.e., $f_1 \leq x \vee i_1$ and $f_2 \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. On the other hand, $x \vee f_1, y \vee f_2 \in F$ and $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2)$ exists in $[x \wedge y)$. It follows that $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \in F$. We consider $i \circ F$. We note that $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \leq x \vee i$ and $(x \vee f_1) \wedge_{x \wedge y} (y \vee f_2) \leq y \vee i$. So, $x, y \in i \circ F$. By hypothesis $i \circ F \in \mathrm{Fi}(A)$ and $x \wedge y \in i \circ F$, i.e., there exists $f \in F$ such that $f \leq (x \wedge y) \vee i$. Then $x \wedge y \in i \circ f$ and $x \wedge y \in I \circ F$. Thus, $I \circ F \in \mathrm{Fi}(A)$.

(5) $\Rightarrow$ (1) Let $a \in A$ and $x, y, z \in [a)$. We know that the inequality $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ always holds. We prove the other inequality. As $(x \vee y) \wedge (x \vee z) \leq y \vee x$ and $(x \vee y) \wedge (x \vee z) \leq z \vee x$ then $y, z \in x \circ ((x \vee y) \wedge (x \vee z))$. By Remark 1, $x \circ ((x \vee y) \wedge (x \vee z)) = (x] \circ [(x \vee y) \wedge (x \vee z))$ and by hypothesis $(x] \circ [(x \vee y) \wedge (x \vee z)) \in \mathrm{Fi}(A)$. So, $y \wedge z \in (x] \circ [(x \vee y) \wedge (x \vee z))$, i.e., there exist $i \in (x]$ and $f \in [(x \vee y) \wedge (x \vee z))$ such that $y \wedge z \in i \circ f$. So, $f \leq (y \wedge z) \vee i$. It follows that $(x \vee y) \wedge (x \vee z) \leq x \vee (y \wedge z)$ and $[a)$ is a bounded distributive lattice. □

In lattice theory, a lattice is distributive if and only if every proper ideal is an intersection of prime ideals. Here we present a generalization of this characterization.

Theorem 5. *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *Every proper ideal of A is an intersection of prime ideals.*

Proof. (1) $\Rightarrow$ (2) See Corollary 2.9 of [3].

(2) $\Rightarrow$ (1) Let $a, b \in A$. We prove that $a \circ b \in \mathrm{Fi}(A)$. It is easy to see that $1 \in a \circ b$ and that $a \circ b$ is increasing. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Let $Q = ((x \wedge y) \vee a]$ and suppose that $b \notin Q$. So, Q is a proper ideal and by hypothesis we have that $Q = \bigcap \{P \in X(A) : Q \subseteq P\}$. Then there exists $P \in X(A)$ such that $(x \wedge y) \vee a \in P$ and $b \notin P$. So, $x \wedge y \in P$ and $a \in P$. As P is prime, $x \in P$ or $y \in P$. Suppose that $x \in P$. Then $x \vee a \in P$ and since $x \in a \circ b$, i.e., $b \leq x \vee a$, we have that $b \in P$ which is a contradiction. The same reasoning applies when $y \in P$. Then $b \in Q$ and $b \leq (x \wedge y) \vee a$. Therefore, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$. It follows from Theorem 4 that A is distributive. □

We study a new characterization of distributive nearlattices in terms of the notion of relative maximal ideal with respect to a set.

Definition 4. Let A be a semilattice. Let S be an increasing subset of A. An ideal I of A is called a *relative maximal ideal with respect to S*, when I is maximal among all the ideals which are disjoint to S.

Lemma 2. *Let A be a semilattice. Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$. Then I is a relative maximal ideal with respect to F if and only if $(H \circ F) \cap I \neq \varnothing$ for all $H \in \mathrm{Id}(A)$ such that $H \not\subseteq I$.*

Proof. Suppose that I is a relative maximal ideal with respect to F. Let $H \in \mathrm{Id}(A)$ such that $H \not\subseteq I$. We consider the ideal $I \vee H$. Since I is a relative maximal ideal with respect to F and $I \subseteq I \vee H$ then $(I \vee H) \cap F \neq \varnothing$, i.e., there exist $f \in F, i \in I$ and $h \in H$ such that $f \leq i \vee h$. So, $i \in h \circ f$ and $i \in H \circ F$. Therefore, $(H \circ F) \cap I \neq \varnothing$.

Assume that $(H \circ F) \cap I \neq \varnothing$ for all $H \in \mathrm{Id}(A)$ such that $H \not\subseteq I$. Suppose that I is not a relative maximal ideal with respect to F. Then there exists $J \in \mathrm{Id}(A)$ such that $I \subset J$ and $J \cap F = \varnothing$. Since $J \not\subseteq I$, by hypothesis we get $(J \circ F) \cap I \neq \varnothing$. Then there exist $i \in I$ and $(j, f) \in J \times F$ such that $i \in j \circ f$, i.e., $f \leq i \vee j$. As $i \in I \subset J$ and $i \vee j \in J$, we have that $f \in J$. So, $J \cap F \neq \varnothing$ which is a contradiction. □

Theorem 6. *Let A be a nearlattice. The following conditions are equivalent:*

(1) *A is distributive.*
(2) *Every relative maximal ideal I with respect to $a \circ b$ is prime for all $a, b \in A$.*

Proof. (1) ⇒ (2) Let $a, b \in A$ and $I \in \mathrm{Id}(A)$ such that I is a relative maximal ideal with respect to $a \circ b$. We prove that I is prime. Let $x, y \in A$ such that $x \wedge y$ exists and $x \wedge y \in I$. Suppose that $x \notin I$ and $y \notin I$. Let $I_x = I \vee (x]$ and $I_y = I \vee (y]$. Then $I_x \cap a \circ b \neq \varnothing$ and $I_y \cap a \circ b \neq \varnothing$, i.e., there exist $f_1, f_2 \in a \circ b$ and $i_1, i_2 \in I$ such that $f_1 \leq x \vee i_1$ and $f_2 \leq y \vee i_2$. Let $i = i_1 \vee i_2 \in I$. So, $x \vee i, y \vee i \in a \circ b$ and $(x \vee i) \wedge_i (y \vee i)$ exists in $[i)$. From Theorem 4, it follows that $a \circ b \in \mathrm{Fi}(A)$ and $(x \vee i) \wedge_i (y \vee i) = (x \wedge y) \vee i \in a \circ b$. On the other hand, as I is an ideal, $(x \wedge y) \vee i \in I$. Thus, $I \cap a \circ b \neq \varnothing$ which is a contradiction. Then I is prime.

(2) ⇒ (1) By Theorem 4, it is sufficient to prove that $a \circ b \in \mathrm{Fi}(A)$. It is easy to see that $1 \in a \circ b$ and that $a \circ b$ is increasing. Let $x, y \in a \circ b$ such that $x \wedge y$ exists. Suppose that $x \wedge y \notin a \circ b$. Then $(x \wedge y] \cap a \circ b = \varnothing$. We consider the following family

$$\mathcal{F} = \{I \in \mathrm{Id}(A) \,:\, (x \wedge y] \subseteq I \text{ and } I \cap a \circ b = \varnothing\}.$$

So, $\mathcal{F} \neq \varnothing$. By Zorn's Lemma there exists a maximal element $M \in \mathcal{F}$. It is not difficult to show that M is a relative maximal ideal with respect to $a \circ b$. So, $x \wedge y \in M$ and by hypothesis M is prime. Then $x \in M$ or $y \in M$. Thus, $M \cap a \circ b \neq \varnothing$ which is a contradiction. Therefore, $x \wedge y \in a \circ b$ and $a \circ b \in \mathrm{Fi}(A)$. □

3. Normal and p-linear nearlattices

Let A be a semilattice and $a \in A$. From Definition 3 we have the following relative annihilator

$$a^{\mathsf{T}} = a \circ 1 = \{x \in A \,:\, x \vee a = 1\},$$

called the *annihilator of* a. We have the following result.

Lemma 3. *Let A be a distributive nearlattice. Let $a, b \in A$ and $I \in \mathrm{Id}(A)$. Then*

(1) *$I \cap a \circ b = \varnothing$ if and only if there exists $Q \in X(A)$ such that $I \subseteq Q$, $a \in Q$ and $b \notin Q$.*
(2) *$I \cap a^{\mathsf{T}} = \varnothing$ if and only if there exists $Q \in X(A)$ such that $I \subseteq Q$ and $a \in Q$.*
(3) *$I \cap a^{\mathsf{T}} = \varnothing$ if and only if there exists $U \in \mathrm{Idm}(A)$ such that $I \subseteq U$ and $a \in U$.*
(4) *$U \in \mathrm{Idm}(A)$ if and only if for all $a \in A$, $a \notin U$ if and only if $U \cap a^{\mathsf{T}} \neq \varnothing$.*

Proof. (1) Let $J \in \mathrm{Id}(A)$ such that $J \cap a \circ b = \varnothing$. Let $H = I(J \cup \{a\})$. We prove that $H \cap [b) = \varnothing$. If $x \in H \cap [b)$ then there exists $j \in J$ such that $x \leq j \vee a$ and $b \leq x$. So, $b \leq j \vee a$ and $j \in a \circ b$ which is a contradiction. Then $H \cap [b) = \varnothing$ and by Theorem 2 there exists $Q \in X(A)$ such that $J \subseteq Q$, $a \in Q$ and $b \notin Q$.

The other direction is immediate.

(2) It follows from (1).

(3) If $I \cap a^{\mathsf{T}} = \varnothing$ then there exists $Q \in X(A)$ such that $I \subseteq Q$ and $a \in Q$. We consider the family

$$\mathcal{Z} = \{R \in \mathrm{Id}(A) - \{A\} \ : \ I \subseteq R \text{ and } a \in R\}.$$

So, $\mathcal{Z} \neq \varnothing$ because $Q \in \mathcal{Z}$. Then, by Zorn's Lemma, there exists a maximal element $U \in \mathcal{Z}$. It is clear that U is proper. We prove that U is a maximal ideal. Let $b \in A$ such that $b \notin U$. If $U \cap b^{\mathsf{T}} = \varnothing$ then $H = I(U \cup \{b\})$ is a proper ideal. Otherwise, if $1 \in H$ then there exists $p \in U$ such that $p \vee b = 1$, i.e., $p \in U \cap b^{\mathsf{T}}$ which is a contradiction. So $U \subset H$ and $H \in \mathcal{Z}$, which is a contradiction because U is maximal. Then $U \cap b^{\mathsf{T}} \neq \varnothing$ and there exists $c \in U$ such that $c \vee b = 1$. Therefore, $H = A$ and U is maximal.

Suppose that $I \cap a^{\mathsf{T}} \neq \varnothing$. Then there exists $i \in I$ such that $i \vee a = 1$. So, there exists $U \in \mathrm{Idm}(A)$ such that $I \subseteq Q$ and $a \in Q$. Then $i \vee a = 1 \in U$, which is a contradiction because U is maximal.

(4) Let $U \in \mathrm{Idm}(A)$. Suppose that $a \notin U$. As U is maximal, $I(U \cup \{a\}) = A$. Then $1 \in I(U \cup \{a\})$, i.e., there exists $p \in U$ such that $p \vee a = 1$. So, $p \in a^{\mathsf{T}}$ and $U \cap a^{\mathsf{T}} \neq \varnothing$.

If $U \cap a^{\mathsf{T}} \neq \varnothing$ and $a \in U$ then there exists $p \in U$ such that $p \vee a = 1$. Thus, $1 \in U$ which is a contradiction.

Conversely, let $I \in \mathrm{Id}(A)$ such that $U \subset I$. Then there exists $a \in I$ such that $a \notin U$. So, $U \cap a^{\mathsf{T}} \neq \varnothing$, i.e., there exists $p \in U$ such that $p \vee a = 1$. Since $U \subset I$, we have that $p \in I$ and $a \vee p = 1 \in I$. Therefore, $I = A$ and U is maximal. □

We recall that a bounded distributive lattice is *normal* if each prime ideal contains a unique minimal prime ideal. This concept was introduced by Cornish in [8] and extended to the class of distributive semilattices in [13]. Now, we introduce a generalization of this notion.

Definition 5. Let A be a distributive nearlattice. We say that A is *normal* if each prime ideal is contained in a unique maximal ideal.

We say that A is *p-linear* if the family of prime ideals which contain a prime ideal is a chain.

Remark 2. We note that every normal nearlattice is p-linear.

Remark 3. If A is a bounded distributive lattice then the Definition 5 is equivalent to saying that every prime filter contains a unique minimal filter, which is a concept dual to the definition given by Cornish in [8].

The following results characterize normal nearlattices through annihilators.

Lemma 4. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *For every $P \in X(A)$ and for all $a,b \in A$ with $a \vee b = 1$, $P \cap a^{\mathsf{T}} \neq \varnothing$ or $P \cap b^{\mathsf{T}} \neq \varnothing$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(A)$ and $a,b \in A$ such that $a \vee b = 1$. Suppose that $P \cap a^{\mathsf{T}} = \varnothing$ and $P \cap b^{\mathsf{T}} = \varnothing$. So, by Lemma 3, there exist $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$, $P \subseteq U_2$, $a \in U_1$ and $b \in U_2$. Since A is normal, $U_1 = U_2$. Then $a,b \in U_1$, but $a \vee b = 1 \in U_1$ which is a contradiction. Thus, $P \cap a^{\mathsf{T}} \neq \varnothing$ or $P \cap b^{\mathsf{T}} \neq \varnothing$.
(2) $\Rightarrow$ (1) Let $P \in X(A)$ and $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. If $U_1 \neq U_2$ then there exists $a \in U_1$ such that $a \notin U_2$. As U_2 is maximal, $I(U_2 \cup \{a\}) = A$. Then $1 \in I(U_2 \cup \{a\})$, i.e., there exists $b \in U_2$ such that $a \vee b = 1$. On the other hand, by Lemma 3, $P \cap a^{\mathsf{T}} = \varnothing$ and $P \cap b^{\mathsf{T}} = \varnothing$, which contradicts the assumption. □

Lemma 5. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *For all $a,b \in A$, $(a \vee b)^{\mathsf{T}} = F(a^{\mathsf{T}} \cup b^{\mathsf{T}})$.*
(3) *For all $a,b \in A$ with $a \vee b = 1$, $F(a^{\mathsf{T}} \cup b^{\mathsf{T}}) = A$.*

Proof. (1) $\Rightarrow$ (2) Let $a,b \in A$. Note that the inclusion $F(a^{\mathsf{T}} \cup b^{\mathsf{T}}) \subseteq (a \vee b)^{\mathsf{T}}$ always holds. Let us prove the other inclusion. Suppose that there exists $x \in (a \vee b)^{\mathsf{T}}$ such that $x \notin F(a^{\mathsf{T}} \cup b^{\mathsf{T}})$. So, by Theorem 2, there exists $P \in X(A)$ such that $x \in P$ and $P \cap F(a^{\mathsf{T}} \cup b^{\mathsf{T}}) = \varnothing$. Since $a^{\mathsf{T}}, b^{\mathsf{T}} \subseteq F(a^{\mathsf{T}} \cup b^{\mathsf{T}})$, we have that $P \cap a^{\mathsf{T}} = \varnothing$ and $P \cap b^{\mathsf{T}} = \varnothing$. Then, by Lemma 3, there exist $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$, $P \subseteq U_2$, $a \in U_1$ and $b \in U_2$. As A is normal, $U_1 = U_2$ and $a,b \in U_1$. Also, $x \in U_1$. Then $x \vee (a \vee b) = 1 \in U_1$ which is a contradiction. Therefore, $(a \vee b)^{\mathsf{T}} = F(a^{\mathsf{T}} \cup b^{\mathsf{T}})$.
(2) $\Rightarrow$ (3) It is immediate.
(3) $\Rightarrow$ (1) Let $P \in X(A)$ and $U_1, U_2 \in \mathrm{Idm}(A)$ such that $P \subseteq U_1$ and $P \subseteq U_2$. Suppose that there exists $a \in U_1$ such that $a \notin U_2$. Since U_2 is maximal, there exists $b \in U_2$ such that $a \vee b = 1$. Then $F(a^{\mathsf{T}} \cup b^{\mathsf{T}}) = A$. Let $x \in P$. Hence, $x \in F(a^{\mathsf{T}} \cup b^{\mathsf{T}})$ and there exist $x_1, \ldots, x_n \in a^{\mathsf{T}} \cup b^{\mathsf{T}}$ such that $x_1 \wedge \ldots \wedge x_n$ exists and $x_1 \wedge \ldots \wedge x_n = x$.

So, $x_1 \wedge ... \wedge x_n \in P$ and by the primality of P there exists $x_i \in \{x_1, ..., x_n\}$ such that $x_i \in P$. If $x_i \in a^{\mathsf{T}}$ then $x_i \in U_1$ and $x_i \vee a = 1 \in U_1$, which is a contradiction because U_1 is proper. If $x_i \in b^{\mathsf{T}}$, we arrive at a contradiction. Thus, $U_1 \subseteq U_2$ and consequently $U_1 = U_2$. Therefore, A is normal. □

Corollary 1. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is normal.*
(2) *The application $\rho : A \to \mathrm{Fi}(A)$ defined by $\rho(a) = a^{\mathsf{T}}$ is a homomorphism of distributive nearlattices.*

Proof. It follows from Lemma 5. □

Theorem 7. *Let A be a distributive nearlattice. The following conditions are equivalent:*

(1) *A is p-linear.*
(2) *For every $P \in X(A)$ and for all $a, b \in A$, $P \cap a \circ b \neq \varnothing$ or $P \cap b \circ a \neq \varnothing$.*
(3) *For all $a, b \in A$, $F((a \circ b) \cup (b \circ a)) = A$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(A)$ and $a, b \in A$. Assume that $P \cap a \circ b = \varnothing$ and $P \cap b \circ a = \varnothing$. By Lemma 3 there exist $Q_1, Q_2 \in X(A)$ such that $P \subseteq Q_1$, $a \in Q_1$, $b \notin Q_1$, $P \subseteq Q_2$, $b \in Q_2$, and $a \notin Q_2$. As A is p-linear, $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. If $Q_1 \subseteq Q_2$ then $a \in Q_2$, which is impossible. If $Q_2 \subseteq Q_1$, then $b \in Q_1$ which is a contradiction. Thus, $P \cap a \circ b \neq \varnothing$ or $P \cap b \circ a \neq \varnothing$.

(2) $\Rightarrow$ (3) Suppose that there exist $a, b \in A$ such that $F((a \circ b) \cup (b \circ a)) \neq A$. Then there exists $c \in A$ such that $c \notin F((a \circ b) \cup (b \circ a))$. Since $F((a \circ b) \cup (b \circ a)) \in \mathrm{Fi}(A)$, by Theorem 2 there exists $P \in X(A)$ such that $c \in P$ and $P \cap F((a \circ b) \cup (b \circ a)) = \varnothing$. So, $P \cap a \circ b = \varnothing$ and $P \cap b \circ a = \varnothing$ which is impossible. Thus, $F((a \circ b) \cup (b \circ a)) = A$.

(3) $\Rightarrow$ (1) Let $P, Q_1, Q_2 \in X(A)$ such that $P \subseteq Q_1$ and $P \subseteq Q_2$. If Q_1 and Q_2 are incomparable, then there exist $a, b \in A$ such that $a \in Q_1 - Q_2$ and $b \in Q_2 - Q_1$. Let $x \in P$. Since $A = F((a \circ b) \cup (b \circ a))$, there exist $x_1, ..., x_n \in (a \circ b) \cup (b \circ a)$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. So, $x_1 \wedge ... \wedge x_n \in P$ and by the primality of P there exists $x_i \in \{x_1, ..., x_n\}$ such that $x_i \in P$. If $x_i \in a \circ b$, then $b \leq x_i \vee a$. As $x_i, a \in Q_1$, we have $x_i \vee a \in Q_1$ and $b \in Q_1$ which is a contradiction. Similarly, if $x_i \in b \circ a$ we arrive at a contradiction. Thus, Q_1 and Q_2 are comparable and A is p-linear. □

4. The lattice of filters

In this section we study the structure of the lattice of filters of a distributive nearlattice. Recall that a Heyting algebra is an algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the operation $\Rightarrow$ satisfies

the following condition: $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$ for all $a, b, c \in A$. The *pseudocomplement* of an element $a \in A$ is the element $a^* = a \Rightarrow 0$.

Let A be a distributive nearlattice. For each pair $F, H \in \mathrm{Fi}(A)$ let us define the subset $F \rhd H$ of A as follows:

$$F \rhd H = \{a \in A \,:\, [a) \cap F \subseteq H\}.$$

Theorem 8. *Let A be a distributive nearlattice. Let $F, H \in \mathrm{Fi}(A)$. Then*

(1) $F \rhd H \in \mathrm{Fi}(A)$.
(2) $F \rhd H = \{a \in A : \forall f \in F \; \exists h \in H \; (h \leq a \vee f)\}$.
(3) *The structure $\langle \mathrm{Fi}(A), \veebar, \barwedge, \rhd, \{1\}, A\rangle$ is a Heyting algebra.*

Proof. (1) Let $F, H \in \mathrm{Fi}(A)$. We prove that $F \rhd H \in \mathrm{Fi}(A)$. Since $[1) \cap F = \{1\} \subseteq H$, then $1 \in F \rhd H$. Let $a, b \in A$ such that $a \leq b$ and $a \in F \rhd H$. Thus, $[b) \subseteq [a)$ and $[a) \cap F \subseteq H$. It follows that $[b) \cap F \subseteq H$, i.e., $b \in F \rhd H$. Let $a, b \in F \rhd H$ and suppose that $a \wedge b$ exists. By Theorem 3, the lattice $\mathrm{Fi}(A)$ is distributive and

$$\begin{aligned}[a \wedge b) \cap F &= ([a) \veebar [b)) \cap F \\ &= ([a) \cap F) \veebar ([b) \cap F) \\ &\subseteq H.\end{aligned}$$

Therefore, $a \wedge b \in F \rhd H$ and $F \rhd H \in \mathrm{Fi}(A)$.

(2) Let $F, H \in \mathrm{Fi}(A)$. Let $X = \{a \in A : \forall f \in F \; \exists h \in H \; (h \leq a \vee f)\}$ and $a \in X$. We prove that $a \in F \rhd H$, i.e., $[a) \cap F \subseteq H$. If $x \in [a) \cap F$, then $a \leq x$ and $x \in F$. Since $a \in X$ and $x \in F$, there exists $h \in H$ such that $h \leq a \vee x = x$. As H is a filter, $x \in H$. So, $a \in F \rhd H$. Conversely, let $a \in F \rhd H$ and $f \in F$. Then $a \vee f \in [a) \cap F$ and by hypothesis, $a \vee f \in H$. It follows that $a \in X$. Therefore, $F \rhd H = X$.

(3) By Theorem 3, $\langle \mathrm{Fi}(A), \veebar, \barwedge, \{1\}, A\rangle$ is a bounded distributive lattice. Let $F, G, H \in \mathrm{Fi}(A)$. We prove that $F \cap G \subseteq H$ if and only if $F \subseteq G \rhd H$. Suppose that $F \cap G \subseteq H$ and let $a \in F$. If $x \in [a) \cap G$, then $a \leq x$ and $x \in G$. Therefore $x \in F \cap G$ and by hypothesis $x \in H$, i.e., $F \subseteq G \rhd H$. Conversely, let $x \in F \cap G$. By hypothesis $F \subseteq G \rhd H$, then $x \in G \rhd H$. As $[x) \cap G \subseteq H$, we have that $x \in H$. □

Remark 4. As a particular case, we have $a \circ b = [a) \rhd [b)$. Indeed

$$\begin{array}{lllll} x \in a \circ b & \text{iff} & b \leq x \vee a & \text{iff} & [x \vee a) \subseteq [b) \\ & \text{iff} & [x) \cap [a) \subseteq [b) & \text{iff} & x \in [a) \rhd [b). \end{array}$$

Then we can write the annihilator of a relative to b in terms of the binary operation $\rhd$.

Note that if $F \in \mathrm{Fi}(A)$, then $F^* = F \rhd \{1\} = \{a \in A : [a) \cap F = \{1\}\}$. The following result describes the filter F^* in a different manner.

Proposition 2. *Let A be a distributive nearlattice. Then for every $F \in \mathrm{Fi}(A)$,*

$$F^* = \{a \in A \,:\, \forall f \in F \; (a \vee f = 1)\}.$$

Proof. Let $C = \{a \in A : \forall f \in F\ (a \vee f = 1)\}$ and $a \in C$. We prove that $[a) \cap F = \{1\}$. Let $x \in A$ such that $a \leq x$ and $x \in F$. Then $x = a \vee x = 1$. Thus, $a \in F^*$. Conversely, let $a \in F^*$. Then $[a) \cap F = \{1\}$. Since $a \leq a \vee f$ and $f \leq a \vee f$ for every $f \in F$, and as F is a filter, $a \vee f \in [a) \cap F$. So, $a \vee f = 1$ for each $f \in F$. Therefore, $a \in C$ and $F^* = C$. □

Remark 5. We note that $[a)^* = \{x \in A : x \vee a = 1\}$, i.e., $a^{\intercal} = [a)^*$.

We prove that the pseudocomplement of a subset X is the pseudocomplement of the filter generated by X. This result was proved for Tarski algebras in [5].

Theorem 9. *Let A be a distributive nearlattice. Then for every subset $X \subseteq A$, we have $X^* = F(X)^*$.*

Proof. Since $X \subseteq F(X)$, we have that $F(X)^* \subseteq X^*$. Conversely, let $x \in X^*$. We prove that for every $a \in F(X)$, $x \vee a = 1$. Suppose that there exists $a \in F(X)$ such that $x \vee a \neq 1$. Then there exist $x_1, ..., x_n \in [X)$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = a$. So, there exist $y_1, ..., y_n \in X$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. As $x \in X^*$, $x \vee y_i = 1$ for all $y_i \in \{y_1, ..., y_n\}$. Then $x \vee x_i = 1$ for all $x_i \in \{x_1, ..., x_n\}$. Since $x \vee a \neq 1$, by Theorem 2, there exists $P \in X(A)$ such that $x \vee a \in P$ and $1 \notin P$. Then $x_1 \wedge ... \wedge x_n \in P$ and as P is a prime ideal, there exists $x_i \in \{x_1, ..., x_n\}$ such that $x_i \in P$. On the other hand, $x \vee a \in P$ and $x \in P$. So, $x \vee x_i = 1 \in P$ which is a contradiction. Thus, $x \vee a = 1$ for all $a \in F(X)$ and consequently $x \in F(X)^*$. □

5. Semi-homomorphisms preserving annihilators

Our next aim is to study a particular class of semi-homomorphisms: semi-homomorphisms preserving annihilators. We give some characterizations in terms of prime and maximal ideals.

Definition 6. Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. We say that h is a *semi-homomorphism preserving annihilators*, or $\intercal$*-semi-homomorphism*, if $F(h(a^{\intercal})) = h(a)^{\intercal}$ for all $a \in A$.

Remark 6. We note that $F(h(a^{\intercal})) \subseteq h(a)^{\intercal}$ for all $a \in A$. If $x \in F(h(a^{\intercal}))$ then there exist $x_1, ..., x_n \in [h(a^{\intercal}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. As $x_1, ..., x_n \in [h(a^{\intercal}))$, there exist $y_1, ..., y_n \in h(a^{\intercal})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. So, there exist $t_1, ..., t_n \in a^{\intercal}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Thus, $t_1 \vee a = \cdots = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = \cdots = y_n \vee h(a) = 1$. Then $x \vee h(a) = [(y_1 \vee x_1) \wedge ... \wedge (y_n \vee x_n)] \vee h(a)$. As $[x)$ is a bounded distributive lattice, $x \vee h(a) = (y_1 \vee x_1 \vee h(a)) \wedge ... \wedge (y_n \vee x_n \vee h(a)) = 1$. Therefore, $x \vee h(a) = 1$ and $x \in h(a)^{\intercal}$.

Let $h : A \to B$ be a semi-homomorphism between distributive nearlattices. In general, $h^{-1}(P) \notin X(A)$ for each $P \in X(B)$. Now, we prove that if P is maximal and h is a $\intercal$-semi-homomorphism, then $h^{-1}(P)$ is maximal and therefore prime.

Lemma 6. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a $\top$-semi-homomorphism. Then $h^{-1}(P) \in \mathrm{Idm}(A)$ for every $P \in \mathrm{Idm}(B)$.*

Proof. Let $P \in \mathrm{Idm}(B)$. Since h is a semi-homomorphism, $h^{-1}(P)$ is an ideal. As $h(1) = 1 \notin P$, we have that $h^{-1}(P)$ is proper. Let $a \in A$ such that $a \notin h^{-1}(P)$. Then $h(a) \notin P$ and as P is maximal, by Lemma 3, $P \cap h(a)^{\top} \neq \varnothing$. So, there exists $x \in A$ such that $x \in P \cap F(h(a^{\top})) \neq \varnothing$, i.e., there exist $x_1, ..., x_n \in [h(a^{\top}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. Thus, there exist $y_1, ..., y_n \in h(a^{\top})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in a^{\top}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then $t_1 \vee a = ... = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = ... = y_n \vee h(a) = 1$. As

$$x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$$

and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$, i.e., $t_i \in h^{-1}(P)$ and $h^{-1}(P) \cap a^{\top} \neq \varnothing$. Conversely, it is easy to prove that if $h^{-1}(P) \cap a^{\top} \neq \varnothing$, then $a \notin h^{-1}(P)$. Therefore, by Lemma 3, $h^{-1}(P) \in \mathrm{Idm}(A)$. □

Theorem 10. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. Then the following conditions are equivalent:*

(1) *h is a $\top$-semi-homomorphism.*
(2) *For all $P \in X(B)$ and for every $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$.*
(3) *$\mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)]$ for all $P \in X(B)$.*

Proof. (1) ⇒ (2) Let $P \in X(B)$ and $Q \in X(A)$ such that $h^{-1}(P) \subseteq Q$. Let us consider the ideal $H = I(P \cup h(Q))$. We note that H is a proper ideal. Indeed, if we assume otherwise, there exists $p \in P$ and $q \in Q$ such that $p \vee h(q) = 1$. So, $p \in h(q)^{\top} = F(h(q^{\top}))$. Then, there exist $x_1, ..., x_n \in [h(q^{\top}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = p$. So, there exist $y_1, ..., y_n \in h(q^{\top})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in q^{\top}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then $t_1 \vee q = ... = t_n \vee q = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(q) = ... = y_n \vee h(q) = 1$. As $x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$ and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$, i.e., $t_i \in h^{-1}(P) \subseteq Q$ and since $q \in Q$, $t_i \vee q = 1 \in Q$ which is a contradiction. Therefore, H is a proper ideal and there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$.

(2) ⇒ (3) Let $P \in X(B)$ and $Q \in \mathrm{Idm}(A) \cap [h^{-1}(P))$. Then $Q \in X(A)$ and $h^{-1}(P) \subseteq Q$. By hypothesis, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}(D)$. Since $h^{-1}(D)$ is an ideal and Q is maximal, $Q = h^{-1}(D)$. So, $D \in h^{-1}[X(B) \cap [P)]$ and $\mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)]$.

(3) ⇒ (1) Let $a \in A$. We prove that $h(a)^{\top} \subseteq F(h(a^{\top}))$. Suppose that there exists $x \in h(a)^{\top}$ such that $x \notin F(h(a^{\top}))$. By Theorem 2, there exists $P \in X(A)$ such that

$x \in P$ and $P \cap F(h(a^{\intercal})) = \varnothing$. Then,

$$P \cap h(a)^{\intercal} \neq \varnothing \text{ and } P \cap h(a^{\intercal}) = \varnothing.$$

Thus, $h^{-1}(P) \cap a^{\intercal} = \varnothing$ and $h^{-1}(P) \in \mathrm{Id}(A)$. By Lemma 3, there exists $U \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq U$ and $a \in U$. So, we get that

$$U \in \mathrm{Idm}(A) \cap [h^{-1}(P)) \subseteq h^{-1}[X(B) \cap [P)].$$

Then, there exists $D \in X(B)$ such that $P \subseteq D$ and $U = h^{-1}(D)$. Since $a \in U$, $h(a) \in D$ and $D \cap h(a)^{\intercal} = \varnothing$. On the other hand, $P \cap h(a)^{\intercal} \neq \varnothing$ and $P \subseteq D$. Thus, $D \cap h(a)^{\intercal} \neq \varnothing$ which is a contradiction. Therefore, $h(a)^{\intercal} \subseteq F(h(a^{\intercal}))$ and h is a $^{\intercal}$-semi-homomorphism. □

It is possible to give another characterization of the $^{\intercal}$-semi-homomorphisms in normal nearlattices.

Proposition 3. *Let A, B be two distributive nearlattices and let $h : A \to B$ be a semi-homomorphism. Suppose that B is normal. Then h is a $^{\intercal}$-semi-homomorphism if and only if:*

(1) *For all $P \in X(B)$ and for all $Q_1, Q_2 \in \mathrm{Idm}(A)$, if $h^{-1}(P) \subseteq Q_1 \cap Q_2$ then $Q_1 = Q_2$.*
(2) *$h^{-1}(P) \in \mathrm{Idm}(A)$ for every $P \in \mathrm{Idm}(B)$.*

Proof. $\Rightarrow$) Suppose that h is a $^{\intercal}$-semi-homomorphism. By Lemma 6, we only need to prove (1). Let $P \in X(B)$ and $Q_1, Q_2 \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq Q_1 \cap Q_2$. Suppose that there exists $a \in Q_1$ such that $a \notin Q_2$. Since Q_2 is maximal, there exists $b \in Q_2$ such that $a \vee b = 1$. Then $h(a \vee b) = h(a) \vee h(b) = h(1) = 1$. As B is normal, by Lemma 4, we have that $P \cap h(a)^{\intercal} \neq \varnothing$ or $P \cap h(b)^{\intercal} \neq \varnothing$, and since h is $^{\intercal}$-semi-homomorphism, $P \cap F(h(a^{\intercal})) \neq \varnothing$ or $P \cap F(h(b^{\intercal})) \neq \varnothing$. If $P \cap F(h(a^{\intercal})) \neq \varnothing$, then there exists $x \in P$ such that $x \in F(h(a^{\intercal}))$, i.e., there exist $x_1, ..., x_n \in [h(a^{\intercal}))$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = x$. So, there exist $y_1, ..., y_n \in h(a^{\intercal})$ such that $y_i \leq x_i$ for all $i \in \{1, ..., n\}$. It follows that there exist $t_1, ..., t_n \in a^{\intercal}$ such that $h(t_i) = y_i$ for $1 \leq i \leq n$. Then, $t_1 \vee a = ... = t_n \vee a = 1$ and since h is a semi-homomorphism, we have that $y_1 \vee h(a) = ... = y_n \vee h(a) = 1$. As $x = x_1 \wedge ... \wedge x_n = (x_1 \vee y_1) \wedge ... \wedge (x_n \vee y_n) \in P$ and P is prime, there exists $i \in \{1, ..., n\}$ such that $x_i \vee y_i \in P$. So, $y_i = h(t_i) \in P$ and $t_i \in h^{-1}(P) \subseteq Q_1 \cap Q_2$. Since $a, t_i \in Q_1$, we have that $t_i \vee a = 1 \in Q_1$, which is a contradiction because Q_1 is maximal. If $P \cap F(h(b^{\intercal})) \neq \varnothing$, we get also a contradiction. Therefore, $Q_1 \subseteq Q_2$ and consequently $Q_1 = Q_2$.

$\Leftarrow$) Let $a \in A$. We prove that $h(a)^{\intercal} \subseteq F(h(a^{\intercal}))$. Suppose that there exists $x \in h(a)^{\intercal}$ such that $x \notin F(h(a^{\intercal}))$. Then there exists $P \in X(B)$ such that $x \in P$ and $P \cap F(h(a^{\intercal})) = \varnothing$, i.e., $P \cap h(a)^{\intercal} \neq \varnothing$ and $P \cap F(h(a^{\intercal})) = \varnothing$. Since B is normal, there exists a unique $Q \in \mathrm{Idm}(B)$ such that $P \subseteq Q$. We note that $h(a) \notin Q$. Indeed, if $h(a) \in Q$ then $h(a)^{\intercal} \cap Q \neq \varnothing$ and there exists $x \in Q$ such that $h(a) \vee x = 1 \in Q$,

which is a contradiction. Since $a \notin h^{-1}(Q)$ and by (2) we have $h^{-1}(Q) \in \mathrm{Idm}(A)$, then, by Lemma 3, $h^{-1}(Q) \cap a^{\intercal} \neq \varnothing$, i.e., $Q \cap F(h(a^{\intercal})) \neq \varnothing$. On the other hand, since $P \cap F(h(a^{\intercal})) = \varnothing$ then $h^{-1}(P) \cap a^{\intercal} = \varnothing$ and, by Lemma 3, there exists $U \in \mathrm{Idm}(A)$ such that $h^{-1}(P) \subseteq U$ and $a \in U$. Then $h^{-1}(P) \subseteq h^{-1}(Q) \cap U$ and by (1), we have $h^{-1}(Q) = U$, which is a contradiction because $a \in U$ and $a \notin h^{-1}(Q)$. Therefore, $h(a)^{\intercal} = F(h(a^{\intercal}))$ and h is a $\intercal$-semi-homomorphism. □

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Doi: 10.18514/MMN.2015.1112

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 79–89

HU e-ISSN 1787-2413

MULTIVARIABLE MATRIX GENERALIZATION OF GOULD-HOPPER POLYNOMIALS

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Received 28 January, 2014

Abstract. The main object of this investigation is to define a multivariable matrix generalization of Gould-Hopper polynomials and to reveal some relations such as matrix generating function, matrix recurrence relation, matrix differential equation for them. Furthermore, more general families of bilinear and bilateral matrix generating functions are obtained for these matrix polynomials.

2010 *Mathematics Subject Classification:* 33C25; 15A60

Keywords: Gould-Hopper polynomials, generating matrix function, matrix recurrence relation, matrix differential equation

1. INTRODUCTION

Matrix polynomials and special matrix functions, which have many applications on statistics, group representation theory, scattering theory, interpolation and quadrature, splines and medical imaging, comprise an emerging field of study with important results in literature .

In the recent papers, the matrix generalizations of many polynomials were introduced by many authors and their various properties were given from the scalar case, see for example [1, 3–6, 9–13].

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus in [7], it follows that:

$$f(A)g(A)=g(A)f(A).$$

If D is the complex plane cut along the negative real axis and $log(z)$ denotes the principle logarithm of z, then $z^{1/2}$ represents $\exp((1/2)log(z))$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D$, then $A^{1/2}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

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Let $A \in \mathbb{C}^{N\times N}$ so that $Re(z) > 0,\ \forall z \in \sigma(A)$, then we say that A is a positive stable matrix in $\mathbb{C}^{N\times N}$. Hermite matrix polynomials $H_n(x,A)$ are defined by [9]:

$$H_n(x,A) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k} \quad , \quad n \geq 0$$

and satisfy the three term matrix recurrence relation:

$$cH_n(x,A) = xI\sqrt{2A}H_{n-1}(x,A) - 2(n-1)H_{n-2}(x,A);\ \ n \geq 1$$
$$H_{-1}(x,A) = \theta,\ \ H_0(x,A) = I$$

where I is unit matrix and θ is zero matrix in $\mathbb{C}^{N\times N}$. Also, for these matrix polynomials, it follows

$$\frac{d}{dx}H_n(x,A) = n\sqrt{2A}H_{n-1}(x,A),\ \ n \geq 1$$
$$\exp\left(xt\sqrt{2A} - t^2 I\right) = \sum_{n=0}^{\infty}\frac{1}{n!}H_n(x,A)t^n,\ \ |t| < +\infty.$$

In this paper, we deal with matrix version of the multivariable extension of Gould-Hopper polynomials which are generalization of Hermite polynomials.

We organize the paper as follows:

In Section 2, we construct matrix extension of Gould-Hopper polynomials and give some matrix recurrence relations and matrix differential equation for these polynomials. In Section 3, multivariable generalization of the matrix polynomials presented in Section 2 is defined and their properties are examined. In the last section, bilinear and bilateral generating matrix functions are derived for the multivariable Gould-Hopper matrix polynomials and some applications of our results are presented.

We recall that the Gould–Hopper polynomials $g_n^m(x,y)$ are specified by the generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} g_n^m(x,y)\frac{t^n}{n!} \tag{1.1}$$

where m is positive integer [8] (see also [14]).

In the special case $m = 2$, we have $g_n^2(x,y) = H_n(x,y)$ where $H_n(x,y)$ denotes the two-variable Hermite–Kampé de Fériet polynomials generated by (see [2])

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Furhermore, we note that $H_n(\nu x,-1)=H_{n,m,\nu}(x)$,where $H_{n,m,\nu}(x)$ is the generalized Hermite polynomials defined by Lahiri [15] and they are generated by

$$\exp(\nu xt-t^m)=\sum_{n=0}^{\infty}H_{n,m,\nu}(x)\frac{t^n}{n!}. \tag{1.3}$$

From the equations (1.1), (1.2) and (1.3) we have the special cases $g_n^2(2x,-1)=H_n(x)$, $H_n(2x,-1)=H_n(x)$ and $H_{n,2,2}(x)=H_n(x)$, respectively where $H_n(x)$ denotes Hermite polynomials.

2. Matrix generalization of Gould-Hopper polynomials

We define a matrix version of Gould-Hopper polynomials as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}g_n^m(x,y;A,B)\frac{t^n}{n!}&=G(x,y,t)\\&=\exp(xt\sqrt{2A})\exp(Byt^m)\end{aligned} \tag{2.1}$$

where A,B are matrices in $\mathbb{C}^{N\times N}$ satisfying that A is a positive stable and m is positive integer.

By using (2.1) and Taylor series at $t=0$, we can write

$$\begin{aligned}\sum_{n=0}^{\infty}g_n^m(x,y;A,B)\frac{t^n}{n!}&=\left(\sum_{n=0}^{\infty}\frac{\left(xt\sqrt{2A}\right)^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(Byt^m)^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{\left(\sqrt{2A}\right)^{n-mk}(B)^k}{(n-mk)!\,k!}x^{n-mk}y^kt^n.\end{aligned}$$

Then comparing coefficients of t^n, we have explicit representation for matrix version of Gould-Hopper polynomials as:

$$g_n^m(x,y;A,B)=\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{n!\left(\sqrt{2A}\right)^{n-mk}(B)^k}{(n-mk)!\,k!}x^{n-mk}y^k. \tag{2.2}$$

Now, we consider some special cases as follows.

Remark 1. The case $B=I\in\mathbb{C}^{N\times N}$ in (2.1) gives the matrix polynomials defined by [16].

Remark 2. If we take $m=2$ in (2.1), we have matrix version of the two-variable Hermite–Kampé de Fériet polynomials $H_n(x,y)$ which are generated by (1.2).

Remark 3. Setting νx instead of x and $y=-1$ in (2.1), we get matrix extension of the generalized Hermite polynomials $H_{n,m,\nu}(x)$ specified by (1.3).

Remark 4. If we take $m=2, y=-1$ and $B=I\in\mathbb{C}^{N\times N}$ in (2.1), we have

$$\begin{aligned} g_n^2(x,-1;A,I) &= \sum_{k=0}^{[n/2]} \frac{n!\,(-1)^k \left(x\sqrt{2A}\right)^{n-2k}}{(n-2k)!\,k!} \\ &= H_n(x,A) \end{aligned}$$

where $H_n(x,A)$ is Hermite matrix polynomial given by [9].

Remark 5. For $x\to(-x)$ in (2.2), we get

$$g_n^m(-x,y;A,B)=(-1)^n g_n^m(x,(-1)^m y;A,B).$$

If we write $x=0$ in (2.1), we have

$$\sum_{n=0}^{\infty} g_n^m(0,y;A,B)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{(Byt^m)^n}{n!}. \tag{2.3}$$

On the other hand, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} g_n^m(0,y;A,B)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} g_{nm}^m(0,y;A,B)\frac{t^{nm}}{(nm)!} \\ &\quad+\sum_{n=0}^{\infty} g_{nm+1}^m(0,y;A,B)\frac{t^{nm+1}}{(nm+1)!} \\ &\quad+\ldots+\sum_{n=0}^{\infty} g_{nm+m-1}^m(0,y;A,B)\frac{t^{nm+m-1}}{(nm+m-1)!}. \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), we have

$$\begin{aligned} g_{nm}^m(0,y;A,B) &= \frac{(By)^n\,(mn)!}{n!} \\ g_{nm+k}^m(0,y;A,B) &= \theta;\ \ k=1,2,\ldots,m-1. \end{aligned} \tag{2.5}$$

Remark 6. For $m=2, y=-1$ and $B=I$ in (2.5), we have

$$g_{2n}^2(0,-1;A,I)=H_{2n}(0,A)=\frac{(-1)^n\,(2n)!I}{n!}$$

and

$$g_{2n+1}^2(0,-1;A,I)=H_{2n+1}(0,A)=\theta.$$

If we differentiate (2.1) with respect to x, matrix polynomials $g_n^m(x,y;A,B)$ satisfy recurrence relations as follows:

$$\frac{\partial}{\partial x}g_n^m(x,y;A,B)=n\sqrt{2A}\,g_{n-1}^m(x,y;A,B);\ n\geq 1$$

$$\frac{\partial^k}{\partial x^k} g_n^m(x,y;A,B) = n\,(n-1)\ldots(n-k+1)\left(\sqrt{2A}\right)^k \times\ g_{n-k}^m(x,y;A,B);\ n \geq k. \tag{2.6}$$

Similarly, differentiating (2.1) with respect to y, the following holds for matrix polynomials $g_n^m(x,y;A,B)$

$$\frac{\partial}{\partial y} g_n^m(x,y;A,B) = \frac{n!}{(n-m)!} g_{n-m}^m(x,y;A,B)B\ \ ;\ n \geq m. \tag{2.7}$$

By using the derivative of the generating function (2.1) with respect to t, we can give next relation

$$g_{n+1}^m(x,y;A,B) - \sqrt{2A}x g_n^m(x,y;A,B) = \frac{my\ n!}{(n-m+1)!} g_{n-m+1}^m(x,y;A,B)B;\ \ n \geq m-1.$$

On the other hand, from (2.1) we get

$$\left.\begin{aligned} \frac{\partial}{\partial x} G(x,y,t) &= t\sqrt{2A}G(x,y,t) \\ \frac{\partial}{\partial t} G(x,y,t) &= x\sqrt{2A}G(x,y,t) + G(x,y,t)mBy t^{m-1} \end{aligned}\right\} \tag{2.8}$$

from which, we obtain that

$$t\sqrt{2A}\frac{\partial}{\partial t} G(x,y,t) = x\sqrt{2A}\frac{\partial}{\partial x} G(x,y,t) + \frac{\partial}{\partial x} G(x,y,t)mBy t^{m-1}.$$

In view of (2.1) and the last relation, we derive

$$n\sqrt{2A}g_n^m(x,y;A,B) - x\sqrt{2A}\frac{\partial}{\partial x} g_n^m(x,y;A,B) = my\frac{\partial}{\partial x} g_{n-m+1}^m(x,y;A,B)\frac{n!}{(n-m+1)!}B;\ \ n \geq m-1. \tag{2.9}$$

Now, let's find matrix differential equation for the Gould-Hopper matrix polynomials. We start with writing $n+1$ instead of n in (2.7). Then we have

$$\frac{\partial}{\partial y} g_{n+1}^m(x,y;A,B) = \frac{(n+1)!}{(n+1-m)!} g_{n+1-m}^m(x,y;A,B)B. \tag{2.10}$$

Then differentiating (2.10) with respect to x and then using (2.6), we get

$$(n+1)\sqrt{2A}\frac{\partial}{\partial y} g_n^m(x,y;A,B) = \frac{(n+1)!}{(n+1-m)!}\frac{\partial}{\partial x} g_{n+1-m}^m(x,y;A,B)B. \tag{2.11}$$

If we write (2.11) in (2.9), we arrive at the following matrix differential equation

$$n g_n^m(x,y;A,B) = x\frac{\partial}{\partial x} g_n^m(x,y;A,B) + my\frac{\partial}{\partial y} g_n^m(x,y;A,B).$$

3. Multivariable extension of Gould-Hopper matrix polynomials

The multivariable matrix extension of the Gould-Hopper polynomials given by (2.1) is defined by the following generating function

$$\begin{aligned} & G_n^m(\mathrm{X},\mathrm{Y};\mathrm{T},\mathrm{A},\mathrm{B}) \\ &= G_{n_1}^{m_1}(x_1,y_1;t_1,A_1,B_1)\,G_{n_2}^{m_2}(x_2,y_2;t_2,A_2,B_2)\ldots G_{n_r}^{m_r}(x_r,y_r;t_r,A_r,B_r) \\ &= \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\} \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} \end{aligned} \tag{3.1}$$

where $\mathrm{X}=(x_1,\ldots,x_r)$, $\mathrm{Y}=(y_1,\ldots,y_r)$, $m=(m_1,\ldots,m_r)$, $n=(n_1,\ldots,n_r)$, $|n|=n_1+\ldots+n_r$ $(n_1,\ldots,n_r\in\mathbb{N}_0)$, $m_1,\ldots,m_r\in\mathbb{N}$, $\mathrm{T}=(t_1,\ldots,t_r)$, $\mathrm{A}=(A_1,\ldots,A_r)$, $\mathrm{B}=(B_1,\ldots,B_r)$ and A_i, B_i are matrices in $\mathbb{C}^{N\times N}$ satisfying that A_i is positive stable for $1\le i\le r$.

For the multivariable Gould-Hopper matrix polynomials $g_n^m(\mathrm{X},\mathrm{Y};\mathrm{T},\mathrm{A},\mathrm{B})$, explicit form is

$$\begin{aligned} g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) = & \sum_{k_1,\ldots,k_r=0}^{[n_1/m_1,\ldots,n_r/m_r]} \frac{n_1!(\sqrt{2A_1})^{n_1-m_1k_1}B_1^{k_1}\ldots n_r!(\sqrt{2A_r})^{n_r-m_rk_r}B_r^{k_r}}{(n_1-m_1k_1)!k_1!\ldots(n_r-m_rk_r)!k_r!} \\ & x_1^{n_1-m_1k_1}y_1^{k_1}\ldots x_r^{n_r-m_rk_r}y_r^{k_r}. \end{aligned}$$

Similar to the relations (2.6) and (2.7), if we differentiate (3.1) with respect to the variables x_i and y_i $(i=1,2,\ldots,r)$, we can easily obtain

$$\begin{aligned} & \frac{\partial}{\partial x_i}g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) \\ & \qquad = n_i\sqrt{2A_i}\,g_{n_1,n_2,\ldots n_{i-1},n_i-1,n_{i+1},\ldots,n_r}^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B});\ n_i\ge 1, \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \frac{\partial^k}{\partial x_i^k}g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) \\ & \quad = (n_i-k+1)_k\left(\sqrt{2A_i}\right)^k g_{n_1,n_2,\ldots n_{i-1},n_i-k,n_{i+1},\ldots,n_r}^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B});\ n_i\ge k, \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial y_i}g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) \\ & \qquad = \frac{n_i!}{(n_i-m_i)!}g_{n_1,n_2,\ldots n_{i-1},n_i-m_i,n_{i+1},\ldots,n_r}^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B});\ n_i\ge m_i, \end{aligned} \tag{3.3}$$

$$\frac{\partial^r}{\partial x_1 ... \partial x_r} g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) = n_1 \sqrt{2A_1} ... n_r \sqrt{2A_r}\, g^m_{n_1-1,n_2-1,...,n_r-1} (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}),$$

for all $n_i \geq 1$, $i = 1,2,...,r$ and

$$\frac{\partial^r}{\partial y_1 ... \partial y_r} g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) = \frac{n_1!...n_r!}{(n_1-m_1)!...(n_r-m_r)!} g^m_{n_1-m_1,n_2-m_2,...,n_r-m_r} (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}),$$

for all $n_i \geq m_i$, $i = 1,2,...,r$ where all of the matrices are commutative with each other.

Using the same method in (2.8), we get

$$n_i \sqrt{2A_i}\, g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) - \sqrt{2A_i}\, x_i \frac{\partial}{\partial x_i} g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})$$
$$= m_i\, y_i \frac{n_i!}{(n_i-m_i+1)!} \frac{\partial}{\partial x_i} g^m_{n_1,n_2,...n_{i-1},n_i-m_i+1,n_{i+1},...,n_r} (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})\, B_i \quad (3.4)$$

for $n_i \geq m_i - 1$ where all of the matrices are commutative with each other.

In view of the equations (3.2), (3.3) and (3.4), we obtain the following matrix differential equation for multivariable Gould-Hopper matrix polynomials

$$\sum_{i=1}^{r} n_i\, g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) = \sum_{i=1}^{r} x_i \frac{\partial}{\partial x_i} g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) + \sum_{i=1}^{r} m_i\, y_i \frac{\partial}{\partial y_i} g_n^m (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})$$

where all of the matrices are commutative with each other.

4. Bilinear and bilateral generating matrix functions for multivariable Gould-Hopper matrix polynomials

In order to obtain several families of bilinear and bilateral generating matrix functions for multivariable Gould-Hopper matrix polynomials, we first state our result as the following.

Theorem 1. *For a non-vanishing function $\Omega_\mu(z_1,...,z_s)$ of s complex variables $z_1,...,z_s$ ($s \in \mathbb{N}$) and of complex order μ, let*

$$\Lambda_{\mu,\nu}(z_1,...,z_s;\eta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(z_1,...,z_s)\eta^k \ ; \ (a_k \neq 0,\ \mu,\nu \in \mathbb{C})$$

and

$$\Theta^m_{n,p,\mu,\nu}(\mathrm{X},\mathrm{Y};z_1,...,z_s;\zeta) := \sum_{k=0}^{[n_1/p]} \frac{a_k}{(n_1-pk)!n_2!...n_r!} g^m_{n_1-pk,n_2,...,n_r} (\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B}) \times \Omega_{\mu+\nu k}(z_1,...,z_s)\zeta^k \quad (4.1)$$

where $m=(m_1,...,m_r)$, $n=(n_1,...,n_r)$, $\mathrm{X}=(x_1,...,x_r)$, $\mathrm{Y}=(y_1,...,y_r)$, $p\in\mathbb{N}$ *and (as usual)* $[\alpha]$ *represents the greatest integer in* $\alpha\in\mathbb{R}$. *Then we have*

$$\sum_{n_1,...,n_r=0}^{\infty}\Theta_{n,p,\mu,\nu}^{m}\left(\mathrm{X},\mathrm{Y};z_1,...,z_s;\frac{\eta}{t_1^p}\right)t_1^{n_1}t_2^{n_2}...t_r^{n_r}$$
$$=\prod_{i=1}^{r}\left\{\exp(x_it_i\sqrt{2A_i})\exp(B_iy_it_i^{m_i})\right\}\Lambda_{\mu,\nu}(z_1,...,z_s;\eta)\quad(4.2)$$

provided that each member of (4.2) *exists*.

Proof. For convenience, let S denote the left hand-side of the assertion (4.2) of Theorem 1. Then, upon substituting for the polynomials $\Theta_{n,p,\mu,\nu}^{m}\left(\mathrm{X},\mathrm{Y};z_1,...,z_s;\frac{\eta}{t_1^p}\right)$ from the definition (4.1) into the left-hand side of (4.2), we obtain

$$S=\sum_{n_1,...,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]}a_k\frac{g_{n_1-pk,n_2,...,n_r}^{m}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!...n_r!}$$
$$\cdot\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^kt_1^{n_1-pk}t_2^{n_2}...t_r^{n_r}.\quad(4.3)$$

Upon inverting the order of summation in (4.3), if we replace n_1 by n_1+pk, we can write

$$S=\sum_{n_1,...,n_r=0}^{\infty}\sum_{k=0}^{\infty}a_k\frac{g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{n_1!n_2!...n_r!}\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^kt_1^{n_1}...t_r^{n_r}$$
$$=\left(\sum_{n_1,...,n_r=0}^{\infty}\frac{g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{n_1!n_2!...n_r!}t_1^{n_1}...t_r^{n_r}\right)\left(\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^k\right)$$
$$=\prod_{i=1}^{r}\left\{\exp(x_it_i\sqrt{2A_i})\exp(B_iy_it_i^{m_i})\right\}\Lambda_{\mu,\nu}(z_1,...,z_s;\eta).$$

The proof is completed. □

In order to give some applications of Theorem 1, we consider the expresses of the multivariable function $\Omega_{\mu+\nu k}(y_1,...,y_s)$ $(k\in\mathbb{N}_0,\ s\in\mathbb{N})$ in terms of simpler function of one and more variables.
First of all, let's get $s=2r$ and $\Omega_{\mu+\nu k}(u_1,...,u_r;v_1,...,v_r)=P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})$ in Theorem 1, where $P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})$ denotes the multivariable matrix Humbert polynomials generated by [1]

$$\sum_{k=0}^{\infty}P_k^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})t^k$$

$$= \prod_{i=1}^{r} (D_i - l_i u_i t + v_i t^{l_i})^{-C_i} \left(\left| l_i u_i t - v_i t^{l_i} \right| < |D_i| \ ; \ i = 1,2,...,r \right) \quad (4.4)$$

where $C_i \in \mathbb{C}^{N\times N}$, $\mathrm{U} = (u_1,...,u_r)$, $\mathrm{V} = (v_1,...,v_r)$, $\mathrm{D} = (D_1,...,D_r)$; $D_i \neq 0$, $i = 1,2,...,r$, $\mathrm{L} = (l_1,...,l_r)$ and l_i $(i = 1,2,...,r)$ is positive integer.

Then we obtain the following class of bilateral generating functions for the $g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})$ and $P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})$.

Corollary 1. *If* $\Lambda_{\mu,\nu}^{\mathrm{L}}(\mathrm{U},\mathrm{V};\eta) := \sum_{k=0}^{\infty} a_k P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})\eta^k$ *where* $(a_k \neq 0, \mu,\nu \in \mathbb{N}_0)$; *and*

$$\Theta_{n,p,\mu,\nu}^{m,\mathrm{L}}(\mathrm{X},\mathrm{Y};\mathrm{U},\mathrm{V};\zeta) := \sum_{k=0}^{[n_1/p]} a_k \frac{g_{n_1-pk,n_2,...,n_r}^{m}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!...n_r!} P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})\zeta^k$$

where $n, p \in \mathbb{N}$. *Then we have*

$$\sum_{n_1,...,n_r=0}^{\infty} \Theta_{n,p,\mu,\nu}^{m,\mathrm{L}}\left(\mathrm{X},\mathrm{Y};\mathrm{U},\mathrm{V};\frac{\eta}{t_1^p}\right) t_1^{n_1}...t_r^{n_r}$$

$$= \prod_{i=1}^{r} \left\{ \exp(x_i t_i \sqrt{2A_i}) \exp(B_i y_i t_i^{m_i}) \right\} \Lambda_{\mu,\nu}^{\mathrm{L}}(\mathrm{U},\mathrm{V};\eta) \quad (4.5)$$

provided that each member of (4.5) *exists.*

Remark 7. Using the generating relation (4.4) and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1,...,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \frac{g_{n_1-pk,n_2,...,n_r}^{m}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!...n_r!} P_k^{(C_1,...,C_r)}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})\eta^k t_1^{n_1-pk} t_2^{n_2}...t_r^{n_r}$$

$$= \prod_{i=1}^{r} \left\{ \exp(x_i t_i \sqrt{2A_i}) \exp(B_i y_i t_i^{m_i}) \right\} \prod_{i=1}^{r} \left\{ (D_i - l_i u_i \eta + v_i \eta^{l_i})^{-C_i} \right\}.$$

Now, setting $s = 2$ and $\Omega_{\mu+\nu k}(z_1,z_2) = g_{\mu+\nu k}^{l}(z_1,z_2;C,D)$ in Theorem 1, we obtain the following class of bilinear generating function for the matrix version of the Gould-Hopper polynomials.

Corollary 2. *If* $\Lambda_{\mu,\nu}^{l}(z_1,z_2;\eta) := \sum_{k=0}^{\infty} a_k g_{\mu+\nu k}^{l}(z_1,z_2;C,D)\eta^k$ *where* $(a_k \neq 0, \ \mu,\nu \in \mathbb{N}_0)$; *and*

$$\Theta_{n,p,\mu,\nu}^{m,l}(\mathrm{X},\mathrm{Y},z_1,z_2;\zeta) := \sum_{k=0}^{[n_1/p]} a_k \frac{g_{n_1-pk,n_2,...,n_r}^{m}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!...n_r!} g_{\mu+\nu k}^{l}(z_1,z_2;C,D)\zeta^k$$

where $n_1, p \in \mathbb{N}$. *Then we have*

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \Theta_{n,p,\mu,\nu}^{m,l}\left(\mathrm{X},\mathrm{Y},z_1,z_2;\frac{\eta}{t_1^p}\right) t_1^{n_1}\ldots t_r^{n_r} = \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\} \Lambda_{\mu,\nu}^{l}(z_1,z_2;\eta) \quad (4.6)$$

provided that each member of (4.6) *exists.*

Remark 8. Using Corollary 2 and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1,\ldots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]} \frac{g_{n_1-pk,n_2,\ldots,n_r}^{m}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!\ldots n_r!} g_k^l(z_1,z_2;C,D)\eta^k t_1^{n_1-pk} t_2^{n_2}\ldots t_r^{n_r} = \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\}\exp(z_1\eta\sqrt{2C})\exp(Dz_2\eta^l).$$

We remark that for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(z_1,\ldots,z_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 1 can be applied in order to derive various families of multilinear and multilateral generating functions for the $g_n^m(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})$.

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Doi: 10.18514/MMN.2015.1021

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 91–99

HU e-ISSN 1787-2413

THE MASCHKE-TYPE THEOREM OF SMASH PRODUCTS OF GENERALIZED QUANTUM COMMUTATIVE ALGEBRAS OVER WEAK HOPF ALGEBRAS

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Received 19 October, 2013

Abstract. The paper is concerned with the semisimplicity of smash products of generalized quantum commutative algebras in weak Hopf algebra setting. Let H be a weak Hopf algebra over a field k and A any semisimple and generalized quantum commutative weak Yetter-Drinfeld H-module algebra. It is shown that $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module. Applying results to quasitriangular (weak) Hopf algebras is considered.

2010 *Mathematics Subject Classification:* 16W30

Keywords: weak Hopf algebra, smash product, module algebra

1. INTRODUCTION

In [10], Yang and Wang have proved the following statement:
Suppose that H is a finite dimensional quasitriangular Hopf algebra acting on an algebra A and A is quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module.

The statement above is extended to weak Hopf algebras setting by Zhai and Zhang in [11].

Weak Hopf algebras were introduced by Böhm et al. in [1] as an important generalization of ordinary Hopf algebras and groupoid algebras besides quasi-Hopf algebras, multiplier Hopf algebras, Hopf quasigroups, etc ([5, 6, 9]). The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The initial motivation to study weak Hopf algebras comes from the fact that some classical theory and lots of basic properties of ordinary Hopf algebras have "weak" analogues (see [3, 4, 7, 8]).

The author was supported by the National Natural Science Foundation of China (No.11261063), the Foundation for Excelent Youth Science and Technology Innovation Talents of Xin Jiang Uygur Autonomous Region(No. 2013721043).

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Given a quasitriangular weak Hopf algebra (H,R), a weak H-module algebra A can be endowed with a suitable coaction associated to R to make A be a weak quantum Yetter-Drinfeld H-module algebra which is similar to one studied by Caenepeel et al. in the setting of Hopf algebras [2].

The purpose of this paper is to discuss the validity of the statement such as the start under the more general assumption.

The paper is organized as follows.

In Section 2, we recall basic definitions and give a summary of fundamental properties concerning weak Hopf algebras and quasitriangular weak Hopf algebras and weak Yetter-Drinfeld H-module algebras. In Section 3, as the main content, we discuss the semisimplicity of smash products of generalized quantum commutative algebras in weak Hopf algebra setting. Finally, the application of our results is considered.

2. Preliminaries

Throughout the paper k is a fixed field. Unless otherwise stated, all vector spaces are over k and all maps are k-linear. We will use the Heyneman-Sweedler notation $\Delta(c)=c_{(1)}\otimes c_{(2)}$ for coproduct (summation understood). $\otimes$ mean $\otimes_k$ unless otherwise specified, etc.

2.1. *Weak Hopf algebras*

Recall from Böhm et al. ([1]) that a *weak Hopf algebra* (H,Δ,ε,S) is both an associative algebra and a coalgebra with an antipode $S:H\to H$ satisfying the following conditions (1)-(4):

$$\begin{cases} (1)\quad \Delta(xy)=\Delta(x)\Delta(y) \text{ for all } x,y\in H;\\ (2)\quad \Delta^2(1)=(\Delta(1)\otimes 1)(1\otimes\Delta(1)),\ \ \Delta^2(1)=(1\otimes\Delta(1))(\Delta(1)\otimes 1);\\ (3)\quad \varepsilon(xyz)=\varepsilon(xy_{(1)})\varepsilon(y_{(2)}z),\ \ \varepsilon(xyz)=\varepsilon(xy_{(2)})\varepsilon(y_{(1)}z) \text{ for all } x,y,z\in H;\\ (4)\quad x_{(1)}S(x_{(2)})=\varepsilon(1_{(1)}x)1_{(2)},\ \ S(x_{(1)})x_{(2)}=1_{(1)}\varepsilon(x1_{(2)}),\\ \qquad\qquad\qquad\qquad S(x_{(1)})x_{(2)}S(x_{(3)})=S(x), \text{for all } x\in H. \end{cases}$$

Let H be a weak Hopf algebra. Then we define the maps $\varepsilon_t,\varepsilon_s:H\to H$ by the formulas

$$\varepsilon_t(x)=\varepsilon(1_{(1)}x)1_{(2)},\ \ \varepsilon_s(x)=1_{(1)}\varepsilon(x1_{(2)})$$

and denote by H_t the image $\varepsilon_t(H)$ and denote by H_s the image $\varepsilon_s(H)$.

Let H be a weak Hopf algebra. The following results hold from Böhm et al. ([1]), for all $h,g\in H$,

(W1) H_t and H_s are two sub-algebras of H,

(W2) $\Delta(1)=1_{(1)}\otimes 1_{(2)}\in H_s\otimes H_t,\ \ \varepsilon_t(h)\varepsilon_s(g)=\varepsilon_s(g)\varepsilon_t(h)$,

(W3) $\Delta(\varepsilon_t(h))=1_{(1)}\varepsilon_t(h)\otimes 1_{(2)},\ \ \Delta(\varepsilon_s(g))=1_{(1)}\otimes\varepsilon_s(g)1_{(2)}$,

(W4) $h_{(1)}\otimes\varepsilon_s(h_{(2)})=h1_{(1)}\otimes S(1_{(2)}),\ \ \varepsilon_t(h_{(1)})\otimes h_{(2)}=S(1_{(1)})\otimes 1_{(2)}h$,

(W5) $h_{(1)} \otimes \varepsilon_t(h_{(2)}) = 1_{(1)}h \otimes 1_{(2)},\ \ \varepsilon_s(h_{(1)}) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)}$,
(W6) $\varepsilon_t \circ \varepsilon_t = \varepsilon_t,\ \ \varepsilon_s \circ \varepsilon_s = \varepsilon_s$,
(W7) $\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s,\ \ \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t$,
(W8) $S(hg) = S(g)S(h),\ S(h_{(2)}) \otimes S(h_{(1)}) = S(h)_{(1)} \otimes S(h)_{(2)}$ and
$S(1) = 1,\ \varepsilon \circ S = \varepsilon$,
(W9) $h_1\varepsilon_s(g) \otimes h_2 = h_{(1)} \otimes h_{(2)}S(\varepsilon_s(g)),\ \ h_1 \otimes \varepsilon_t(g)h_2 = S(\varepsilon_t(g))h_{(1)} \otimes h_{(2)}$.

2.2. *Quasitriangular weak Hopf algebras*

Recall from Nikshych et al. in [8] that a *quasitriangular weak Hopf algebra* is a pair (H, R), where H is a weak Hopf algebra and $R = R^1 \otimes R^2 \in \Delta^{op}(1)(H \otimes H)\Delta(1)$ such that
(Q1) There exists $\bar{R} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)$ with $R\bar{R} = \Delta^{op}(1)$ and
$\bar{R}R = \Delta(1)$.
(Q2) For all $h \in H$, we have

$$\begin{cases} \Delta^{op}(h)R = R\Delta(h), \\ (id \otimes \Delta)R = R_{13}R_{12}, \\ (\Delta \otimes id)R = R_{13}R_{23} \end{cases}$$

where Δ^{op} denotes the comultiplication opposite to Δ, $R_{12} = R \otimes 1,\ \ R_{23} = 1 \otimes R$, etc., as usual.

Let (H, R) be a quasitriangular weak Hopf algebra. Then the following six identities hold:

$$\begin{cases} (\varepsilon_s \otimes id)(R) = \Delta(1),\ \ (id \otimes \varepsilon_s)(R) = (S \otimes id)\Delta^{op}(1), \\ (\varepsilon_t \otimes id)(R) = \Delta^{op}(1),\ \ (id \otimes \varepsilon_t)(R) = (S \otimes id)\Delta(1), \\ (S \otimes id)(R) = (id \otimes S^{-1})(R) = \bar{R},\ \ (S \otimes S)(R) = R. \end{cases}$$

2.3. *Weak module algebras*

Let H be a weak Hopf algebra. An algebra A is called a left *weak H-module algebra*, if A is a left H-module via $h \otimes a \mapsto h \cdot a$ such that, for any $a, b \in A$ and $h \in H$,

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \qquad h \cdot 1_A = \varepsilon_t(h) \cdot 1_A.$$

Let H be a weak Hopf algebra and A a weak left H-module algebra. Recall from Nikshych in [8] that a *weak smash product* $A\sharp H$ of A and H is defined on a k-vector space $A \otimes_{H_t} H$, where H is a left H_t -module via its multiplication and A is a right H_t- module via

$$a \cdot x = S^{-1}(x) \cdot a = a(x \cdot 1_A),\ a \in A,\ x \in H_t.$$

Its multiplication is given by the following formula:

$$(a\sharp h)(b\sharp g) = a(h_{(1)} \cdot b)\sharp h_{(2)}g,$$

for any $a, b \in A$ and $g, h \in H$. Then $A\sharp H$ is an associative algebra with unit $1_A\sharp 1_H$.

2.4. *Weak quantum Yetter-Drinfeld H-module algebras*

A k-algebra A is called a *weak quantum Yetter-Drinfeld H-module algebra*, if A satisfies the following conditions:

(WQ1) A is a weak left $H-$module algebra,

(WQ2) A is a right H^{op}-comodule algebra, i.e., the comodule structure map $\rho : A \to A \otimes H$ satisfies

$$a_{[0]} \otimes a_{[1]} = 1_{(1)} \cdot a_{[0]} \otimes 1_{(2)} a_{[1]},\ \rho(ab) = a_{[0]} b_{[0]} \otimes b_{[1]} a_{[1]},\ \rho(1) = (id \otimes \varepsilon_t)\rho(1),$$

where $\rho(a) = a_{[0]} \otimes a_{[1]}$ denotes the coaction.

(WQ3) (WQ1) and (WQ2) satisfy the Yetter-Drinfeld condition

$$(h_{(2)} \cdot a)_{[0]} \otimes (h_{(2)} \cdot a)_{[1]} h_{(1)} = h_{(1)} \cdot a_{[0]} \otimes h_{(2)} a_{[1]},$$

for all $h \in H, a \in A$.

Remark 1. The Yetter-Drinfeld condition is equivanlent to

$$\rho(h \cdot a) = h_{(2)} \cdot a_{[0]} \otimes h_{(3)} a_{[1]} S^{-1}(h_{(1)}). \tag{2.1}$$

Example 1. Let H be a weak Hopf algebra. $\{1_{(1)} h S(1_{(2)}) | h \in H\}$ is a weak quantum Yetter-Drinfeld H-module algebra with the action and coaction given as follows:

$$g \cdot (1_{(1)} h S(1_{(2)})) = g_{(1)} h S(g_{(2)}),\ \rho(1_{(1)} h S(1_{(2)})) = 1_{(1)} h_{(2)} S(1_{(2)}) \otimes S^{-1}(h_{(1)}).$$

Example 2. Let (H, R) be a quasitriangular weak Hopf algebra. Given any left weak H-module algebra A, one can define a right H^{op}-coaction on A as follows:

$$\rho(a) = R^2 \cdot a \otimes R^1.$$

With the above coaction, it is easily checked that A is a weak quantum Yetter-Drinfeld H-module algebra.

3. The main results

In this section, we assume that H is a weak Hopf algebra with bijective antipode S, and A a weak left H -module algebra, and $A\sharp H$ the weak smash product algebra.

Definition 1. Let H be a weak Hopf algebra and A a weak quantum Yetter-Drinfeld H-module algebra. A is called a *generalized quantum commutative algebras*, if A satisfies, for all $a, b \in H$

$$ab = b_{[0]}(b_{[1]} \cdot a).$$

Lemma 1. *For any weak quantum Yetter-Drinfeld H-module algebra A. Then, for all $a \in A$,*

$$1_{[0]} \otimes 1_{[1](1)} \otimes 1_{[1](2)} = 1_{[0]} \otimes 1_{(1)} 1_{[1]} \otimes 1_{(2)} \tag{3.1}$$

$$a_{[0]} \otimes \varepsilon_t(a_{[1]}) = a 1_{[0]} \otimes 1_{[1]}, \tag{3.2}$$

$$a_{[0]} \otimes \varepsilon_s(a_{[1]}) = 1_{[0]}a \otimes S(1_{[1]}). \tag{3.3}$$

Lemma 2. *For any generalized quantum commutative algebra A. Then, for all $a, b \in A$,*

$$ab = (S(a_{[1]}) \cdot b)a_{[0]} \Longleftrightarrow ab = b_{[0]}(b_{[1]} \cdot a). \tag{3.4}$$

Proof. For all $a, b \in A$, we have

$$\begin{aligned}(S(a_{[1]}) \cdot b)a_{[0]} &= a_{[0][0]}(a_{[0][1]}S(a_{[1]}) \cdot b) \\ &= a_{[0]}(\varepsilon_t(a_{[1]}) \cdot b) \\ &= a1_{[0]}(1_{[1]} \cdot b) = ab.\end{aligned}$$

Conversely, For all $a, b \in A$, we have

$$\begin{aligned}b_{[0]}(b_{[1]} \cdot a) &= (S(b_{[0][1]})b_{[1]} \cdot a)b_{[0][0]} \\ &= (S(b_{[1](1)})b_{[1](2)} \cdot a)b_{[0]} \\ &= (S(1_{[1]}) \cdot a)1_{[0]}b = ab.\end{aligned}$$

So we finish the proof. □

Lemma 3. *For any left weak H-module algebra A. Then M is a left $A\sharp H$-module if and only if M is both a left H-module and a left A-module and satisfies the following compatible condition*

$$h \cdot (a \cdot m) = (h_{(1)} \cdot a) \cdot (h_{(2)} \cdot m),$$

for all $h \in H$, $a \in A$ and $m \in M$.

Lemma 4. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. Then, for all $a, b \in A$,*

$$a_{[0]}b_{[0]} \otimes b_{[1]}a_{[1]} = b_{[0]}(b_{[1](2)} \cdot a)_{[0]} \otimes (b_{[1](2)} \cdot a)_{[1]}b_{[1](1)}. \tag{3.5}$$

Proof. For all $a, b \in A$, apply ρ to the identity $ab = b_{[0]}(b_{[1]} \cdot a)$, we have

$$\begin{aligned}\rho(b_{[0]}(b_{[1]} \cdot a)) &= b_{[0][0]}(b_{[1]} \cdot a)_{[0]} \otimes (b_{[1]} \cdot a)_{[1]}b_{[0][1]} \\ &= b_{[0]}(b_{[1](2)} \cdot a)_{[0]} \otimes (b_{[1](2)} \cdot a)_{[1]}b_{[1](1)} \\ &= \rho(ab).\end{aligned}$$

The proof is completed. □

Lemma 5. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. If M is a left $A\sharp H$-module, then M is an A-bimodule with the right module action of A on M as follows*

$$\leftharpoonup : M \otimes A \to M, m \otimes a \mapsto m \leftharpoonup a = a_{[0]} \cdot (a_{[1]} \cdot m),$$

and for all $h \in H$,

$$h \cdot (m \leftharpoonup a) = (h_{(1)} \cdot m) \leftharpoonup (h_{(2)} \cdot a).$$

Proof. First, we shall check that M is a right A-module. In fact, for all $m \in M$ and $a, b \in A$, we have

$$\begin{aligned}(m \leftharpoonup a) \leftharpoonup b &= (a_{[0]} \cdot (a_{[1]} \cdot m)) \leftharpoonup b \\ &= b_{[0]} \cdot (b_{[1]} \cdot (a_{[0]} \cdot (a_{[1]} \cdot m))) \\ &= b_{[0]} \cdot ((b_{[1](1)} \cdot a_{[0]}) \cdot (b_{[1](2)} a_{[1]} \cdot m)) \\ &= (b_{[0]}(b_{[1](2)} \cdot a)_{[0]}) \cdot ((b_{[1](2)} \cdot a)_{[1]} b_{[1](1)} \cdot m) \\ &= a_{[0]} b_{[0]} \cdot (b_{[1]} a_{[1]} \cdot m) \\ &= (ab)_{[0]}((ab)_{[1]} \cdot m) \\ &= m \leftharpoonup ab\end{aligned}$$

$$\begin{aligned}m \leftharpoonup 1 &= 1_{[0]} \cdot (1_{[1]} \cdot m) \\ &= 1_{[0]}(1_{(1)} \cdot 1_A) \cdot (1_{(2)} 1_{[1]} \cdot m) \\ &= 1_{[0]}(S(1_{(1)}) \cdot 1_A) \cdot (1_{(2)} \varepsilon_t(1_{[1]}) \cdot m) \\ &= (1_{[0]}(\varepsilon_t(1_{[1]}) S(1_{(1)}) \cdot 1_A)) \cdot (1_{(2)} \cdot m) \\ &= (1_{[0]}(1_{[1]} S(1_{(1)}) \cdot 1_A)) \cdot (1_{(2)} \cdot m) \\ &= (S(1_{(1)}) \cdot 1_A) \cdot (1_{(2)} \cdot m) \\ &= (1_{(1)} \cdot 1_A) \cdot (1_{(2)} \cdot m) = m.\end{aligned}$$

Now, we shall check M is an A-bimodule, i.e., $(a \cdot m) \leftharpoonup b = a \cdot (m \leftharpoonup b)$. As a matter of fact,

$$\begin{aligned}(a \cdot m) \leftharpoonup b &= b_{[0]} \cdot (b_{[1]} \cdot (a \cdot m)) \\ &= b_{[0]} \cdot ((b_{[1](1)} \cdot a) \cdot (b_{[1](2)} \cdot m)) \\ &= (b_{[0]}(b_{[1](1)} \cdot a)) \cdot (b_{[1](2)} \cdot m) \\ &= (b_{[0][0]}(b_{[0][1]} \cdot a)) \cdot (b_{[1]} \cdot m) \\ &= ab_{[0]} \cdot (b_{[1]} \cdot m) \\ &= a \cdot (m \leftharpoonup b).\end{aligned}$$

Finally, for all $a \in A, m \in M$ and $h \in H$,

$$\begin{aligned}(h_{(1)} \cdot m) \leftharpoonup (h_{(2)} \cdot a) &= (h_{(2)} \cdot a)_{[0]}((h_{(2)} \cdot a)_{[1]} h_{(1)} \cdot m) \\ &= (h_{(1)} \cdot a_{[0]})(h_{(2)} a_{[1]} \cdot m) \\ &= h \cdot (a_{[0]} \cdot (a_{[1]} \cdot m)) = h \cdot (m \leftharpoonup a).\end{aligned}$$

The proof is completed. □

Lemma 6. *For all left $A\sharp H$-module M, we have*

$$_{A\sharp H}Hom(A,M)\cong M^H, F: f\mapsto f(1_A),$$

where $M^H=\{m\in M|h\cdot m=\varepsilon_t(h)\cdot m,\quad \forall h\in H\}$ and A is left $A\sharp H$-module via

$$(a\sharp h)\cdot b=a(h\cdot b).$$

Proof. For given $0\neq m\in M^H$, we define a map via

$$f: a\mapsto (a\sharp 1_H)\cdot m.$$

Throughout standard computation, we can show that $f\in_{A\sharp H}\mathrm{Hom}(A,M)$. Based on this, we can check that F is bijective in a straightforward way. □

Lemma 7. *Let H be a weak Hopf algebra and A a generalized quantum commutative algebra. Then, for all left $A\sharp H$-modules M and N.*

(1) *$Hom_A(M_A,N_A)\in_{A\sharp H}M$,*
(2) *$Hom_A(M_A,N_A)^H={}_{A\sharp H}Hom(M,N)$,*

where $Hom_A(M_A,N_A)$ denotes the space of the right A-module homomorphisms.

Proof. (1) Let $M,N\in_{A\sharp H}M$. Then M,N are both A-bimodules from Lemma 5. For all $a\sharp h\in A\sharp H$, $f\in\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H}M$, define action of $A\sharp H$ on $\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H}M$ by

$$((a\sharp h)\cdot f)(m)=a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot m)),$$

for all $m\in M$. We shall check $(a\sharp h)\cdot f\in\mathrm{Hom}_A(M_A,N_A)$. For all $c\in A$, we have

$$\begin{aligned}((a\sharp h)\cdot f)(m\leftharpoonup c)&=a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot(m\leftharpoonup c)))\\&=a\cdot((h_{(1)}\cdot f(S(h_{(4)})\cdot m))\leftharpoonup h_{(2)}S(h_{(3)})\cdot c)\\&=a\cdot((1_{(1)}h_{(1)}\cdot f(S(h_{(2)})\cdot m))\leftharpoonup 1_{(2)}\cdot c)\\&=a\cdot((h_{(1)}\cdot f(S(h_{(2)})\cdot m))\leftharpoonup c)\\&=(a\cdot(h_{(1)}\cdot f(S(h_{(2)})\cdot m)))\leftharpoonup c\\&=((a\sharp h)\cdot f)(m)\leftharpoonup c.\end{aligned}$$

It is checked directly that $\mathrm{Hom}_A(M_A,N_A)\in_{A\sharp H}M$ is a left $A\sharp H$-module.

(2) For all $f\in_{A\sharp H}\mathrm{Hom}(M,N)$, then f is both a left H-module morphism and a left A-module morphism between M and N. We shall check that $f\in\mathrm{Hom}_A(M_A,N_A)^H$. For all $m\in M$ and $a\in A$, since

$$f(m\leftharpoonup a)=f(a_{[0]}\cdot(a_{[1]}\cdot m))=a_{[0]}\cdot(a_{[1]}\cdot f(m))=f(m)\leftharpoonup m,$$

we conclude that f is a right A-linear. Also, for any $h\in H$ and $m\in M$, we have

$$(h\cdot f)(m)=h_{(1)}\cdot f(S(h_{(2)})\cdot m)=h_{(1)}S(h_{(2)})\cdot f(m)=\varepsilon_t(h)\cdot f(m),$$

i.e., $f\in\mathrm{Hom}_A(M_A,N_A)^H$. Conversely, First, we can define the left H_s-action on M by restricting the $A\sharp H$-action on M : $x\cdot m=m\leftharpoonup(h\cdot 1_A)$, for all $m\in M,h\in H_s$.

Using Remark 1 and Lemma 3, it is easy to see that right A-module homomorphisms are morphisms of left H_s-modules. For all $f \in \mathrm{Hom}_A(M_A, N_A)^H$, we have

$$\begin{aligned} h \cdot f(m) &= h_{(1)}\varepsilon_s(h_{(2)}) \cdot f(m) \\ &= h_{(1)} \cdot f(\varepsilon_s(h_{(2)}) \cdot m) \\ &= (h_{(1)} \cdot f)(h_{(2)} \cdot m) \\ &= (\varepsilon_t(h_{(1)}) \cdot f)(h_{(2)} \cdot m) \\ &= (\varepsilon_t(1_{(1)}) \cdot f)(1_{(2)}h \cdot m) \\ &= (1_{(1)} \cdot f)(1_{(2)}h \cdot m) \\ &= 1_{(1)} \cdot f(S(1_{(2)})1_{(3)}h \cdot m) \\ &= 1_{(1)}\varepsilon_s(1_{(2)}) \cdot f(h \cdot m) = f(h \cdot m), \end{aligned}$$

i.e., f is a left H-module map. Now, we shall check that f is a left A-module map. Indeed, for all $a \in A$ and $m \in M$,

$$\begin{aligned} f(a \cdot m) &= f(a1_{[0]} \cdot (1_{[1]} \cdot m)) \\ &= f(a_{[0]} \cdot (\varepsilon_t(a_{[1]}) \cdot m)) \\ &= f(a_{[0]} \cdot (a_{[1]1}S(a_{[1]2}) \cdot m)) \\ &= f(a_{[0][0]} \cdot (a_{[0][1]}S(a_{[1]}) \cdot m)) \\ &= f((S(a_{[1]}) \cdot m) \leftharpoonup a_{[0]}) \\ &= (S(a_{[1]}) \cdot f(m)) \leftharpoonup a_{[0]} = a \cdot f(m). \end{aligned}$$

So we get that $f \in_{A\sharp H} \mathrm{Hom}(M, N)$. □

Now, we can present the main result in this section.

Theorem 1. *Let H be a weak Hopf algebra and A a generalized quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module algebra.*

Proof. Assume A is a projective left $A\sharp H$-module, then the functor ${}_{A\sharp H}\mathrm{Hom}(A, -)$ is exact. For any left $A\sharp H$-module M, it is viewed as a right A-module via " $\leftharpoonup$ " in Lemma 5. Since A is semisimple, M is projective as a right A-module. Hence the functor $\mathrm{Hom}_A(M, -)$ is exact. Further, the composition functor ${}_{A\sharp H}\mathrm{Hom}(A, \mathrm{Hom}_A(M, -))$ is also exact. From Lemma 6 and 7, we get

$${}_{A\sharp H}\mathrm{Hom}(A, \mathrm{Hom}_A(M, N)) \cong_{A\sharp H} \mathrm{Hom}(M, N),$$

for any left $A\sharp H$-module M and N. Then M is a projective left $A\sharp H$-module, hence $A\sharp H$ is semisimple. The converse is obvious. □

Next, we shall apply Theorem 1 to Example 1. Given a quasitriangular weak Hopf algebra (H, R) and weak H-module algebra A, A is a weak quantum Yetter- Drinfeld

H-module algebra with the coaction defined in Example 1. Then the generalized quantum commutative condition in Definition 1 takes the following form

$$ab = (R^2 \cdot b)(R^1 \cdot a).$$

With the assumption above and by Theorem 1, we have the main result of Zhai and Zhang in [11].

Corollary 1. *Let (H, R) be a quasitriangular weak Hopf algebra and A a quantum commutative. If A is semisimple, then $A\sharp H$ is semisimple if and only if A is a projective left $A\sharp H$-module algebra.*

Remark 2. If $\Delta(1) = 1 \otimes 1$, weak Hopf algebras are just Hopf algebras. Corollary 1 recovers to the results of Yang and Wang in [10].

ACKNOWLEDGEMENT

The author would like to thank the referee for his/her valuable comments on this article.

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Doi: 10.18514/MMN.2015.1237

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 101–114

HU e-ISSN 1787-2413

A ONE-SIDED THEOREM FOR THE PRODUCT OF ABEL AND CESÀRO SUMMABILITY METHODS

YILMAZ ERDEM AND İBRAHİM ÇANAK

Received 09 May, 2014

Abstract. In this paper, a one-sided condition is given to recover (C,α) summability of a sequence from its $(A)(C,\alpha+1)$ summability. Our result extends and generalizes the well known classical Tauberian theorems given for Abel and Cesàro summability methods.

2010 *Mathematics Subject Classification:* 40E05; 40G05; 40G10

Keywords: Abel summability, Cesàro summability, $(A)(C,\alpha)$ summability, (C,α) summability, Tauberian conditions and theorems, product methods

1. INTRODUCTION

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real numbers with partial sums $s_n = \sum_{k=0}^{n} a_k$. For all nonnegative integers m, we define

$$(n\Delta)_m s_n = n\Delta((n\Delta)_{m-1}s_n),$$

where $(n\Delta)_0 s_n = s_n$ and $(n\Delta)_1 s_n = n\Delta s_n$.

The backward difference Δs_n of s_n is defined to be $\Delta s_n = s_n - s_{n-1}$, $n \geq 1$, with $\Delta s_0 = s_0$.

Let A_n^{α} be defined by generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^{\alpha} x^n$ $(|x|<1)$, where

$$A_0^{\alpha} = 0, \quad A_n^{\alpha} = \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$$

for $\alpha > -1$.

A sequence (s_n) is said to be summable by the Cesàro mean of order α, or (C,α) summable to s, where $\alpha > -1$, and we write $s_n \to s$ (C,α) if

$$s_n^{\alpha} = \frac{S_n^{\alpha}}{A_n^{\alpha}} = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k \to s$$

as $n \to \infty$.

We write $\tau_n = n a_n$ and denote the (C,α) mean of (τ_n) by τ_n^{α}. Borwein [4] showed that if a sequence is (C,α) summable to s for any $\alpha > -1$, it is (C,β) summable to

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s for any $\beta > \alpha$. It is also well known that the (C,α) summability method is regular (see [3]). Note that $(C,0)$ summability reduces to the ordinary convergence.

A sequence (s_n) is said to be Abel summable to s, and we write $s_n \to s\ (A)$ if the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for $0 \le x < 1$ and tends to s as $x \to 1^-$. It is well known that if a sequence is (C,α) summable to s for any $\alpha > -1$, then it is Abel summable to s (see [2]).

A sequence (s_n) is said to be $(A)(C,\alpha)$ summable to s, and we write $s_n \to s\ (A)(C,\alpha)$ if the series $\sum_{n=0}^{\infty}(s_n^{\alpha} - s_{n-1}^{\alpha})x^n$, with $s_{-1}^{\alpha} = 0$, is convergent for $0 \le x < 1$ and tends to s as $x \to 1^-$. Note that $(A)(C,\alpha)$ summability reduces to the Abel summability when $\alpha = 0$.

The identity $s_n - s_n^1 = \tau_n^1$ is known as the Kronecker identity and it will be used in the proof of the main result. Throughout this paper we use the symbols $s_n = o(1)$ and $s_n = O(1)$ to mean that $s_n \to 0$ as $n \to \infty$ and (s_n) is bounded for large enough n.

2. Preliminary results

A theorem due to Abel [1] states that if (s_n) converges to s, then it is Abel summable to s. The converse Abel's theorem is not necessarily true. For example the series $\sum_{n=0}^{\infty}(-1)^n$ is not convergent, but it is Abel summable to $1/2$. However, the converse of Abel's theorem may be valid under some condition which we call Tauberian condition. Any theorem stating that convergence follows from a summability method and a Tauberian condition is called a Tauberian theorem.

By imposing some restriction on a_n, Tauber [17] obtained the first partial converses of Abel's theorem.

Theorem 1. *If (s_n) is Abel summable to s and $\tau_n = o(1)$, then (s_n) converges to s.*

Theorem 2. *If (s_n) is Abel summable to s and $\tau_n^1 = o(1)$, then (s_n) converges to s.*

Littlewood [12] replaced the condition $\tau_n = o(1)$ by $\tau_n = O(1)$ and later Hardy and Littlewood [9] obtained the following one-sided Tauberian theorem.

Theorem 3. *If (s_n) is Abel summable to s and $\tau_n \ge -H$ for some nonnegative constant H, then (s_n) converges to s.*

A generalization of Theorem 3 was given by Szász [16].

Theorem 4. *If (s_n) is Abel summable to s and $\tau_n^1 \ge -H$ for some nonnegative constant H, then (s_n) is $(C,1)$ summable to s.*

Pati [14] have recently obtained more general Tauberian theorems generalizing the classical results for the product of the Abel and (C,α) summability methods.

Theorem 5. *If (s_n) is $(A)(C,\alpha)$ summable to s, where $\alpha > 0$, and $\tau_n^\alpha \geq -H$ for some nonnegative constant H, then (s_n) is (C,α) summable to s.*

Theorem 6. *The necessary and sufficient condition that the $(A)(C,\alpha+1)$ summability of (s_n) to s, where $\alpha > -1$, implies the (C,α) summability of (s_n) to s, is that $\tau_n^{\alpha+1} = o(1)$.*

Tauberian theorems in the sense of Pati were generalized by Çanak et al. [6], Çanak and Erdem [5] and Erdem and Çanak [7]. Çanak et al. [6] proved that if (s_n) is $(A)(C,\alpha)$ summable to s and $(n\Delta)_m \tau_n^{\alpha+m} = o(1)$ for $m = 1,2$, then (s_n) is convergent to s. Later, Erdem and Çanak [7] proved the main result in Çanak et al. [6] for all integers $m \geq 1$. Recently, Çanak and Erdem [5] have recovered convergence, (C,α) convergence, and (C,α) slow oscillation of (s_n) depending on the conditions given in terms of $(n\Delta)_m \tau_n^{\alpha+m}$ for some special cases of m.

In this paper, we recover (C,α) convergence of (s_n) from its $(A)(C,\alpha+1)$ summability under the one-sided boundedness of $((n\Delta)_m \tau_n^{\alpha+m})$, where $m \geq 1$ and $\alpha > -1$.

3. Main result

Our result is based on Theorem 1 and Theorem 3.

Theorem 7. *If (s_n) is $(A)(C,\alpha+1)$, where $\alpha > -1$, summable to s, and for some integer $m \geq 0$,*

$$(n\Delta)_m \tau_n^{\alpha+m} \geq -H \tag{3.1}$$

then (s_n) is (C,α) summable to s.

From Theorem 7, we deduce the following corollary:

Corollary 1. *If (s_n) is $(A)(C,\alpha+1)$, where $\alpha > -1$, summable to s, and for some integer $m \geq 0$,*

$$(n\Delta)_m \tau_n^{m} \geq -H \tag{3.2}$$

then (s_n) converges to s.

4. Auxiliary results

We need the following lemmas for the proof of Theorem 7.

Lemma 1 ([10, 11])**.** *For $\alpha > -1$, $\tau_n^\alpha = n\Delta s_n^\alpha = n(s_n^\alpha - s_{n-1}^\alpha)$.*

Lemma 2 ([8, 11])**.** *For $\alpha > -1$, $\tau_n^{\alpha+1} = (\alpha+1)(s_n^\alpha - s_n^{\alpha+1})$.*

Lemma 3 ([6])**.** *For $\alpha > -1$, $n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^\alpha - \tau_n^{\alpha+1})$.*

Lemma 4 ([13])**.** *For $-1 < \alpha < \beta$, $(A)(C,\alpha) \subset (A)(C,\beta)$.*

Lemma 5 ([7]). *Let $\alpha > -1$. For any integer $m \geq 2$,*

$$(n\Delta)_m \tau_n^{\alpha+m} = \sum_{j=1}^{m} (-1)^{j+1} A_m^{(j)}(\alpha) n\Delta \tau_n^{(\alpha+j)},$$

where

$$A_m^{(j)}(\alpha) = a_m^{(j-1)}(\alpha) + a_m^{(j)}(\alpha), \qquad a_m^{(0)}(\alpha) = 0,$$

and

$$a_m^{(j)}(\alpha) = \prod_{k=j+1}^{m} (\alpha + k) \left[\sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq m \\ r<s \Rightarrow t_r \leq t_s}} (\alpha + t_1)(\alpha + t_2) \ldots (\alpha + t_{j-1}) \right]$$

where $j = 1, 2, 3, \ldots, m$.

Lemma 6 ([7]). *Let $\alpha > -1$. For any integer $m \geq 2$,*

$$(\alpha + j) A_m^{(j-1)}(\alpha + 1) + (\alpha + j + 1) A_m^{(j)}(\alpha + 1) = A_{m+1}^{(j)}(\alpha),$$

where $A_m^{(j)}(\alpha)$ is as in Lemma 5.

Lemma 7 ([15]). *For $\alpha > -1$, $\sigma_n(s^\alpha) = \dfrac{1}{\alpha+1} s_n^{\alpha+1} + \left(1 - \dfrac{1}{\alpha+1}\right) \sigma_n(s^{\alpha+1})$.*

Lemma 8. *i) For all $\lambda > 1$ and large enough n, that is, when $[\lambda n] > n$,*

$$s_n^\alpha - s_n^{\alpha+1} = \frac{\alpha}{\alpha+1} \left[\frac{[\lambda n]+1}{[\lambda n]-n} \left(\sigma_{[\lambda n]}(s^{\alpha+1}) - \sigma_n(s^{\alpha+1}) \right) + \left(\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1} \right) \right]$$
$$+ \frac{1}{\alpha} \frac{[\lambda n]+1}{[\lambda n]-n} \left(s_{[\lambda n]}^{\alpha+1} - s_n^{\alpha+1} \right) - \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \left(s_k^\alpha - s_n^\alpha \right) \tag{4.1}$$

ii) For all $0 < \lambda < 1$ and large enough n, that is, when $n > [\lambda n]$,

$$s_n^\alpha - s_n^{\alpha+1} = \frac{\alpha}{\alpha+1} \left[\frac{[\lambda n]+1}{n-[\lambda n]} \left(\sigma_n(s^{\alpha+1}) - \sigma_{[\lambda n]}(s^{\alpha+1}) \right) + \left(\sigma_n(s^{\alpha+1}) - s_n^{\alpha+1} \right) \right]$$
$$+ \frac{1}{\alpha} \frac{[\lambda n]+1}{n-[\lambda n]} \left(s_n^{\alpha+1} - s_{[\lambda n]}^{\alpha+1} \right) - \frac{1}{n-[\lambda n]} \sum_{k=[\lambda n]+1}^{n} \left(s_n^\alpha - s_k^\alpha \right), \tag{4.2}$$

where $[\lambda n]$ denotes the integer part of the product λn.

Proof. Let $\tau_{n,[\lambda n]}^\alpha = \dfrac{1}{[\lambda n]-n} \displaystyle\sum_{k=n+1}^{[\lambda n]} s_k^\alpha$. Then we have,

$$\tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1} = \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} s_k^\alpha - s_n^{\alpha+1}$$

$$= \frac{1}{[\lambda n]-n}\left(\sum_{k=0}^{[\lambda n]} s_k^\alpha - \sum_{k=0}^{n} s_k^\alpha\right) - s_n^{\alpha+1}$$
$$= \frac{[\lambda n]+1}{([\lambda n]-n)([\lambda n]+1)}\sum_{k=0}^{n} s_k^\alpha - \frac{n+1}{([\lambda n]-n)(n+1)}\sum_{k=0}^{n} s_k^\alpha - s_n^{\alpha+1}$$
$$= \frac{1}{[\lambda n]-n}\left(([\lambda n]+1)\sigma_{[\lambda n]}(s^\alpha) - (n+1)\sigma_n(s^\alpha)\right) - s_n^{\alpha+1}$$

By lemma 7, we have

$$\begin{aligned}\tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1} &= \frac{1}{[\lambda n]-n}\left(([\lambda n]+1)\left(\frac{\alpha}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right)\right.\\ &\quad \left.-(n+1)\left(\frac{\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) + \frac{1}{\alpha+1}s_n^{\alpha+1}\right)\right) - s_n^{\alpha+1}\\ &= \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right.\\ &\quad \left.-\frac{(n+1)\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{n+1}{\alpha+1}s_n^{\alpha+1} - ([\lambda n]-n)s_n^{\alpha+1}\right).\end{aligned}$$

or

$$\begin{aligned}s_n^\alpha - s_n^{\alpha+1} &= s_n^\alpha - \tau_{n,[\lambda n]}^\alpha + \tau_{n,[\lambda n]}^\alpha - s_n^{\alpha+1}\\ &= s_n^\alpha - \tau_{n,[\lambda n]}^\alpha + \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right.\\ &\quad \left.-\frac{(n+1)\alpha}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)\\ &= s_n^\alpha - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} s_k^\alpha + \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1})\right.\\ &\quad \left.+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1} - \frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)\end{aligned}$$

We finally have

$$\begin{aligned}s_n^\alpha - s_n^{\alpha+1} &= \frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1}) + \frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}\right.\\ &\quad \left.-\frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1}) - \frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{[\lambda n]-n}\left(\sum_{k=n+1}^{[\lambda n]} s_k^{\alpha}-([\lambda n]-n)s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left(\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_{[\lambda n]}(s^{\alpha+1})-\frac{\alpha(n+1)}{\alpha+1}\sigma_n(s^{\alpha+1})\right.\\
&+\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_n(s^{\alpha+1})-\frac{\alpha([\lambda n]+1)}{\alpha+1}\sigma_n(s^{\alpha+1})\\
&\left.+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}-\frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right)\\
&-\frac{1}{[\lambda n]-n}\left(\sum_{k=n+1}^{[\lambda n]} s_k^{\alpha}-\sum_{k=n+1}^{[\lambda n]} s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left[\frac{\alpha([\lambda n]+1)}{\alpha+1}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&-\frac{\alpha}{\alpha+1}\left((n+1)\sigma_n(s^{\alpha+1})-([\lambda n]+1)\sigma_n(s^{\alpha+1})\right)\\
&\left.+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}-\frac{\alpha[\lambda n]-\alpha n+[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right]\\
&-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right)\\
&=\frac{1}{[\lambda n]-n}\left[\frac{\alpha([\lambda n]+1)}{\alpha+1}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&-\frac{\alpha(n-[\lambda n])}{\alpha+1}\sigma_n(s^{\alpha+1})+\frac{[\lambda n]+1}{\alpha+1}s_{[\lambda n]}^{\alpha+1}+\frac{\alpha(n-[\lambda n])}{\alpha+1}s_n^{\alpha+1}\\
&\left.-\frac{[\lambda n]+1}{\alpha+1}s_n^{\alpha+1}\right]-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right)\\
&=\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\right.\\
&\left.+(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1})\right]+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{[\lambda n]}^{\alpha+1}-s_n^{\alpha+1}\right)\\
&-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\left(s_k^{\alpha}-s_n^{\alpha}\right).
\end{aligned}
$$

This completes the proof. □

The proof for ii) is similar to that of i).

5. Proof of Theorem 7

By hypothesis, $s_n^{\alpha+1} \to s\ (A)$. By Lemma 4, we have $s_n^{\alpha+1} \to s\ (A)$, $s_n^{\alpha+2} \to s\ (A), \ldots, s_n^{\alpha+m} \to s\ (A)$ where m is any positive integer. Hence, by Lemma 2, we get

$$\begin{aligned}
(\alpha+2)(s_n^{\alpha+1} - s_n^{\alpha+2}) &= \tau_n^{\alpha+2} \\
(\alpha+3)(s_n^{\alpha+2} - s_n^{\alpha+3}) &= \tau_n^{\alpha+3} \\
&\vdots \\
(\alpha+m+1)(s_n^{\alpha+m} - s_n^{\alpha+m+1}) &= \tau_n^{\alpha+m+1}.
\end{aligned}$$

Since $s_n^{\alpha+k} \to s\ (A)$ for $k = 1, 2, \ldots, m+1$, we have

$$\begin{aligned}
\tau_n^{\alpha+2} &\to 0\ (A) \\
\tau_n^{\alpha+3} &\to 0\ (A) \\
&\vdots \\
\tau_n^{\alpha+m+1} &\to 0\ (A)
\end{aligned} \tag{5.1}$$

and by Lemma 3,

$$\begin{aligned}
(\alpha+3)(\tau_n^{\alpha+2} - \tau_n^{\alpha+3}) &= n\Delta\tau_n^{\alpha+3} \to 0\ (A) \\
(\alpha+4)(\tau_n^{\alpha+3} - \tau_n^{\alpha+4}) &= n\Delta\tau_n^{\alpha+4} \to 0\ (A) \\
&\vdots \\
(\alpha+m+1)(\tau_n^{\alpha+m} - \tau_n^{\alpha+m+1}) &= n\Delta\tau_n^{\alpha+m+1} \to 0\ (A).
\end{aligned}$$

Since

$$(n\Delta)_m \tau_n^{\alpha+m} \geq -H \tag{5.2}$$

then

$$(n\Delta)_m \tau_n^{\alpha+m+j} \geq -H_1 \tag{5.3}$$

for $j = 1, \ldots, m-1$, by Lemma 5, we have

$$\begin{aligned}
(n\Delta)_{m-1}\tau_n^{\alpha+m+1} &= \sum_{j=1}^{m-1} (-1)^{j+1} A_{m-1}^{j} (\alpha+2) n\Delta\tau_n^{\alpha+2+j} \\
&= A_{m-1}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-1}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4} + \ldots \\
&\quad + (-1)^m A_{m-1}^{m-1}(\alpha+2)n\Delta\tau_n^{\alpha+m+1}.
\end{aligned}$$

For $j = 1, \ldots, m-1$ we have $n\Delta\tau_n^{\alpha+2+j} \to 0\ (A)$. Hence, we get

$$(n\Delta)_{m-1}\tau_n^{\alpha+m+1} \to 0\ (A) \tag{5.4}$$

It follows from (5.3) that

$$(n\Delta)_{m-1}\tau_n^{\alpha+m+1} = o(1) \tag{5.5}$$

by Theorem 3. By Lemma 3, we obtain

$$(\alpha+m+1)((n\Delta)_{m-1}\tau_n^{\alpha+m} - (n\Delta)_{m-1}\tau_n^{\alpha+m+1}) = (n\Delta)_m\tau_n^{\alpha+m+1}. \tag{5.6}$$

Substituting (5.3) and (5.5) into (5.6), we have

$$(n\Delta)_{m-1}\tau_n^{\alpha+m} \geq -H_2. \tag{5.7}$$

Since

$$\begin{aligned}(n\Delta)_{m-2}\tau_n^{\alpha+m} &= \sum_{j=1}^{m-2}(-1)^{j+1}A_{m-2}^{j}(\alpha+2)n\Delta\tau_n^{\alpha+2+j}\\ &= A_{m-2}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-2}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4}+\dots\\ &\quad +(-1)^{m-2}A_{m-2}^{m-2}(\alpha+2)n\Delta\tau_n^{\alpha+m}\end{aligned}$$

by Lemma 5, we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m} \to 0\ (A). \tag{5.8}$$

From (5.7) and (5.8), we obtain, by Theorem 3,

$$(n\Delta)_{m-2}\tau_n^{\alpha+m} = o(1). \tag{5.9}$$

By Lemma 3, we obtain

$$(\alpha+m)((n\Delta)_{m-2}\tau_n^{\alpha+m-1} - (n\Delta)_{m-2}\tau_n^{\alpha+m}) = (n\Delta)_{m-1}\tau_n^{\alpha+m}. \tag{5.10}$$

Substituting (5.7) and (5.9) into (5.10), we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1} \geq -H_3. \tag{5.11}$$

Since

$$\begin{aligned}(n\Delta)_{m-3}\tau_n^{\alpha+m-1} &= \sum_{j=1}^{m-3}(-1)^{j+1}A_{m-3}^{j}(\alpha+2)n\Delta\tau_n^{\alpha+2+j}\\ &= A_{m-3}^{1}(\alpha+2)n\Delta\tau_n^{\alpha+3} - A_{m-3}^{2}(\alpha+2)n\Delta\tau_n^{\alpha+4}+\dots\\ &\quad +(-1)^{m-2}A_{m-3}^{m-3}(\alpha+2)n\Delta\tau_n^{\alpha+m-2}\end{aligned}$$

by Lemma 3, we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-1} \to 0\ (A). \tag{5.12}$$

From (5.11) and (5.12), we have, by Theorem 3,

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-1} = o(1). \tag{5.13}$$

By Lemma 3, we obtain

$$(\alpha+m-1)((n\Delta)_{m-3}\tau_n^{\alpha+m-2} - (n\Delta)_{m-3}\tau_n^{\alpha+m-1}) = (n\Delta)_{m-2}\tau_n^{\alpha+m-1}. \tag{5.14}$$

Substituting (5.11) and (5.13) into (5.14), we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2} \geq -H_4. \tag{5.15}$$

Continuing in this way, we obtain

$$(n\Delta)_2\tau_n^{\alpha+4} \to 0\ (A). \tag{5.16}$$

and

$$(n\Delta)_3\tau_n^{\alpha+4} \geq -H_5. \tag{5.17}$$

From (5.16) and (5.17), we have, by Theorem 3,

$$(n\Delta)_2\tau_n^{\alpha+4} = o(1). \tag{5.18}$$

By Lemma 3, we obtain

$$(\alpha+4)((n\Delta)_2\tau_n^{\alpha+3} - (n\Delta)_2\tau_n^{\alpha+4}) = (n\Delta)_3\tau_n^{\alpha+4}. \tag{5.19}$$

Substituting (5.17) and (5.18) into (5.19), we have

$$(n\Delta)_2\tau_n^{\alpha+3} \geq -H_6. \tag{5.20}$$

From $n\Delta\tau_n^{\alpha+3} \to 0\ (A)$ and (5.20), we have, by Theorem 3,

$$n\Delta\tau_n^{\alpha+3} = o(1). \tag{5.21}$$

By Lemma 3, we obtain

$$(\alpha+3)(n\Delta\tau_n^{\alpha+2} - n\Delta\tau_n^{\alpha+3}) = (n\Delta)_2\tau_n^{\alpha+3}. \tag{5.22}$$

Substituting (5.20) and (5.21) into (5.22), we have

$$n\Delta\tau_n^{\alpha+2} \geq -H_7. \tag{5.23}$$

It follows from (5.1) and (5.23) by Theorem 1, we get

$$\tau_n^{\alpha+2} = o(1). \tag{5.24}$$

Substituting (5.23) and (5.24) into

$$(\alpha+2)(\tau_n^{\alpha+1} - \tau_n^{\alpha+2}) = n\Delta\tau_n^{\alpha+2}$$

we have

$$\tau_n^{\alpha+1} \geq -H_8.$$

Since $s_n^{\alpha+1} \to s\ (A)$ and

$$\tau_n^{\alpha+1} = n\Delta s_n^{\alpha+1} \geq -H_8, \tag{5.25}$$

we obtain, by Theorem 3,

$$s_n^{\alpha+1} \to s. \tag{5.26}$$

Now we need to show that

$$\tau_n^{\alpha} = n\Delta s_n^{\alpha} \geq -C \tag{5.27}$$

for some constant C.

From (5.2), by Lemma 5, we have, for $j = 1,\dots,m-1$,

$$\begin{aligned}(n\Delta)_{m-1}\tau_n^{\alpha+m} &= \sum_{j=1}^{m-1}(-1)^{j+1}A_{m-1}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-1}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2} - A_{m-1}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\\ &\quad +(-1)^m A_{m-1}^{m-1}(\alpha+1)n\Delta\tau_n^{\alpha+m}.\end{aligned}$$

We have $n\Delta\tau_n^{\alpha+1+j} \to 0\ (A)$ for $j = 1,\dots,m-1$. Hence, we get

$$(n\Delta)_{m-1}\tau_n^{\alpha+m} \to 0\ (A). \tag{5.28}$$

It follows from (5.2) that

$$(n\Delta)_{m-1}\tau_n^{\alpha+m} = o(1) \tag{5.29}$$

by Theorem 3. By Lemma 3, we obtain

$$(\alpha+m)((n\Delta)_{m-1}\tau_n^{\alpha+m-1} - (n\Delta)_{m-1}\tau_n^{\alpha+m}) = (n\Delta)_m\tau_n^{\alpha+m}. \tag{5.30}$$

Substituting (5.2) and (5.29) into (5.30), we have

$$(n\Delta)_{m-1}\tau_n^{\alpha+m-1} \geq -H_{10}. \tag{5.31}$$

Since

$$\begin{aligned}(n\Delta)_{m-2}\tau_n^{\alpha+m-1} &= \sum_{j=1}^{m-2}(-1)^{j+1}A_{m-2}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-2}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2} - A_{m-2}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\\ &\quad +(-1)^{m-2}A_{m-2}^{m-2}(\alpha+1)n\Delta\tau_n^{\alpha+m-1}\end{aligned}$$

by Lemma 5, we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1} \to 0\ (A). \tag{5.32}$$

From (5.31) and (5.32), we obtain, by Theorem 3,

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-1} = o(1). \tag{5.33}$$

By Lemma 3, we obtain

$$(\alpha+m-1)((n\Delta)_{m-2}\tau_n^{\alpha+m-2} - (n\Delta)_{m-2}\tau_n^{\alpha+m-1}) = (n\Delta)_{m-1}\tau_n^{\alpha+m-1}. \tag{5.34}$$

Substituting (5.31) and (5.33) into (5.34), we have

$$(n\Delta)_{m-2}\tau_n^{\alpha+m-2} \geq -H_{11}. \tag{5.35}$$

Since

$$\begin{aligned}(n\Delta)_{m-3}\tau_n^{\alpha+m-2} &= \sum_{j=1}^{m-3}(-1)^{j+1}A_{m-3}^{j}(\alpha+1)n\Delta\tau_n^{\alpha+1+j}\\ &= A_{m-3}^{1}(\alpha+1)n\Delta\tau_n^{\alpha+2} - A_{m-3}^{2}(\alpha+1)n\Delta\tau_n^{\alpha+3}+\dots\end{aligned}$$

$$+(-1)^{m-2}A_{m-3}^{m-3}(\alpha+1)n\Delta\tau_n^{\alpha+m-3}$$

by Lemma 3, we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2}\to 0\ (A). \tag{5.36}$$

From (5.35) and (5.36), we have, by Theorem 3,

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-2}=o(1). \tag{5.37}$$

By Lemma 3, we obtain

$$(\alpha+m-2)((n\Delta)_{m-3}\tau_n^{\alpha+m-3}-(n\Delta)_{m-3}\tau_n^{\alpha+m-2})=(n\Delta)_{m-2}\tau_n^{\alpha+m-2}. \tag{5.38}$$

Substituting (5.35) and (5.37) into (5.38), we have

$$(n\Delta)_{m-3}\tau_n^{\alpha+m-3}\geq -H_{12}. \tag{5.39}$$

Continuing in this way, we obtain

$$(n\Delta)_2\tau_n^{\alpha+3}\to 0\ (A). \tag{5.40}$$

and

$$(n\Delta)_3\tau_n^{\alpha+3}\geq -H_{13}. \tag{5.41}$$

From (5.40) and (5.41), we have, by Theorem 3,

$$(n\Delta)_2\tau_n^{\alpha+3}=o(1). \tag{5.42}$$

By Lemma 3, we obtain

$$(\alpha+3)((n\Delta)_2\tau_n^{\alpha+2}-(n\Delta)_2\tau_n^{\alpha+3})=(n\Delta)_3\tau_n^{\alpha+3}. \tag{5.43}$$

Substituting (5.41) and (5.42) into (5.43), we have

$$(n\Delta)_2\tau_n^{\alpha+2}\geq -H_{14}. \tag{5.44}$$

From $n\Delta\tau_n^{\alpha+2}\to 0\ (A)$ and (5.44), we have, by Theorem 3,

$$n\Delta\tau_n^{\alpha+2}=o(1). \tag{5.45}$$

By Lemma 3, we obtain

$$(\alpha+2)(n\Delta\tau_n^{\alpha+1}-n\Delta\tau_n^{\alpha+2})=(n\Delta)_2\tau_n^{\alpha+2}. \tag{5.46}$$

Substituting (5.44) and (5.45) into (5.46), we have

$$n\Delta\tau_n^{\alpha+1}\geq -H_{15}. \tag{5.47}$$

Substituting (5.47), (5.25) and (5.27) into

$$(\alpha+1)(\tau_n^{\alpha}-\tau_n^{\alpha+1})=n\Delta\tau_n^{\alpha+1}$$

we have

$$\tau_n^{\alpha}=n\Delta s_n^{\alpha}\geq -H_{16}. \tag{5.48}$$

By Lemma 8 i) and (5.48), we have

$$s_n^{\alpha}-s_n^{\alpha+1}=\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})\ -s_n^{\alpha+1}\right)\right]$$

$$
\begin{aligned}
&+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)-\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta s_j^{\alpha}\\
&\leq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]\\
&\quad+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{H}{j}\\
&\leq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]\\
&\quad+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)+H\log\frac{[\lambda n]}{n}
\end{aligned}
$$

Taking the lim sup of both sides, we get

$$
\begin{aligned}
\limsup_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\leq\limsup_{n\to\infty}\Bigg\{&\frac{\alpha}{\alpha+1}\Bigg[\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(s^{\alpha+1})-\sigma_n(s^{\alpha+1})\right)\\
&+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\Bigg]+\frac{1}{\alpha}\frac{[\lambda n]+1}{[\lambda n]-n}\left(s_{\lambda n}^{\alpha+1}-s_n^{\alpha+1}\right)\Bigg\}+H\log\lambda,
\end{aligned}
$$

where $H>0$. Since

$$
\frac{[\lambda n]+1}{[\lambda n]-n}\leq\frac{2\lambda}{\lambda-1} \tag{5.49}
$$

for $\lambda>1$ and sufficiently large n and $s_n^{\alpha+1}\to s$, we have

$$
\lim_{\lambda\to1^+}\limsup_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\leq0 \tag{5.50}
$$

By Lemma 8 ii) and (5.48), we have

$$
\begin{aligned}
s_n^{\alpha}-s_n^{\alpha+1}&=\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]\\
&\quad+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)+\frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}\sum_{j=n+1}^{k}\Delta s_j^{\alpha}\\
&\geq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]\\
&\quad+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{\lambda n}^{\alpha+1}\right)+\frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}\sum_{j=n+1}^{k}\frac{H}{j}\\
&\geq\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]
\end{aligned}
$$

$$+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)-H\log\frac{[\lambda n]}{n}$$

Taking the liminf of both sides, we get

$$\liminf_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\geq\liminf_{n\to\infty}\left\{\frac{\alpha}{\alpha+1}\left[\frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n(s^{\alpha+1})-\sigma_{[\lambda n]}(s^{\alpha+1})\right)\right.\right.$$
$$\left.\left.+\left(\sigma_n(s^{\alpha+1})-s_n^{\alpha+1}\right)\right]+\frac{1}{\alpha}\frac{[\lambda n]+1}{n-[\lambda n]}\left(s_n^{\alpha+1}-s_{[\lambda n]}^{\alpha+1}\right)\right\}+H\log\lambda$$

where $H>0$. Since

$$\frac{[\lambda n]+1}{n-[\lambda n]}\leq\frac{2\lambda}{1-\lambda} \tag{5.51}$$

for $0<\lambda<1$ and sufficiently large n and $s_n^{\alpha+1}\to s$, we have

$$\lim_{\lambda\to1^+}\liminf_{n\to\infty}(s_n^{\alpha}-s_n^{\alpha+1})\geq 0. \tag{5.52}$$

Combining (5.50) and (5.52) provides

$$\lim s_n^{\alpha}=\lim s_n^{\alpha+1}.$$

This completes the proof.

We like to note that we used H to denote a constant, possibly different at each occurrence above.

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Doi: 10.18514/MMN.2015.1726

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 115–121

HU e-ISSN 1787-2413

SOME RING THEORY FROM JENŐ SZIGETI

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Received 10 August, 2015

Abstract. A selection of ring theory papers by Jenő Szigeti is reviewed with an emphasis on aspects related to matrix algebras.

The present overview concentrates on three areas of Szigeti's work. "Eulerian polynomial identities" deals essentially with polynomials in several non-commuting indeterminates corresponding to directed Eulerian graphs. "Lie nilpotent determinant theory" adapts to the non-commutative case the classical concepts of determinant, adjoint and characteristic polynomial to yield analogues of well known linear algebra results, especially over Lie nilpotent rings. "Centralizers and zero-level centralizers" is about some non-commutative extensions of theorems on centralizers and double centralizers in matrix algebras, with additional considerations of two-sided annihilators.

We aim at a condensed but self contained presentation of a selection of results.

1. Eulerian polynomial identities

Let Γ be a directed graph with vertex set $V(\Gamma) = \{1, 2, \ldots, k\}$ and edge set $E(\Gamma) = \{x_1, x_2, \ldots, x_N\}$. For each $1 \leq r \leq N$, x_r is an edge from $\sigma(r)$ to $\tau(r)$, both in $V(\Gamma)$. A directed Eulerian path which starts at p and ends at q is viewed as a permutation π of the edges (or rather their indices). The polynomial $P_\Gamma(X)$ in the set $X = E(\Gamma)$ of non-commuting indeterminates, induced by Γ, is defined as follows:

$$P_\Gamma(X) = \sum_{\pi \in \Pi(\Gamma, p, q)} \operatorname{sgn}(\pi) x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(N)},$$

where $\Pi(\Gamma, p, q)$ is the set of all directed Eulerian paths of Γ from p to q. The main result of [16] states that if $N \geq 2kn$, then $P_\Gamma(X) = 0$ is a polynomial identity on the $n \times n$ matrix ring $\mathrm{M}_n(\Omega)$, where Ω is an arbitrary commutative ring with 1. This is a broad generalization of the famous Amitsur-Levitzki theorem. Surprisingly, the above graph based construction led to essentially new identities even for 3×3

Dedicated to the occasion of Jenő Szigeti's 60th birthday.

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matrices. The Capelli polynomials in [4] appear as Eulerian polynomials corresponding to a particular class of graphs, where the authors proved that the inequalities conditioning the corresponding matrix identities are sharp.

Assuming certain lower bounds on out degrees the sign free (permanental) version

$$Q_\Gamma(X) = \sum_{\pi \in \Pi(\Gamma,p,q)} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(N)}$$

of $P_\Gamma(X)$ also yields the identity $Q_\Gamma(X) = 0$ on $\mathrm{M}_n(\Omega)$, where Ω is a commutative ring of characteristic d. The lower bounds assumed depend on both d and n (see [7]). A variation on these assumptions on the graph Γ which ensure the validity of the identity $Q_\Gamma(X) = 0$ on $\mathrm{M}_n(\Omega)$ can be found in [2].

It is worth to note that the proof of the matrix identity $P_\Gamma(X) = 0$ is based on the use of Swan's theorem on the signs of directed Eulerian paths, while the proof of $Q_\Gamma(X) = 0$ is based on a formula (due to Aardenne-Ehrenfest and de Bruijn) counting the number of directed Eulerian paths. A particularly concise form of the mentioned theorem of Swan is formulated in [3] using a special adjacency matrix over the Grassmann algebra.

The form of the polynomial $P_\Gamma(X)$ is enriched in [8] by an involution operator $*$ applied to indeterminates traversed in opposite direction in an undirected Eulerian path, providing identities involving matrix transposition also.

2. Lie nilpotent determinant theory

The following are the outlines of a new determinant theory developed in the series of papers [5, 6, 9–12, 14, 15, 18, 19].

For an $n \times n$ matrix $A = [a_{i,j}]$ over a not necessarily commutative ring (or K-algebra) R with 1, the element

$$\mathrm{sdet}(A) = \sum_{\alpha,\beta \in \mathrm{S}_n} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) a_{\alpha(1),\beta(1)} \cdots a_{\alpha(t),\beta(t)} \cdots a_{\alpha(n),\beta(n)}$$

of R is called the *symmetric determinant* of A. The (r,s) entry of the *symmetric adjoint* $A^* = [a^*_{i,j}]$ of A is the signed symmetric determinant $a^*_{r,s} = (-1)^{r+s}\mathrm{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s-th row and the r-th column of A. If R is commutative, then $\mathrm{sdet}(A) = n!\det(A)$ and $A^* = (n-1)!\mathrm{adj}(A)$, where $\det(A)$ and $\mathrm{adj}(A)$ denote the ordinary determinant and adjoint of A.

The *right adjoint sequence* $(P_k)_{k \geq 1}$ of A is defined by the following recursion: $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ for $k \geq 1$. The *k-th right determinant* of A is the trace

$$\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k)$$

and the *k-th right adjoint* of A is the product

$$\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k.$$

An important observation is that

$$\mathrm{rdet}_{(1)}(A) = \mathrm{tr}(AA^*) = \mathrm{sdet}(A) = \mathrm{tr}(A^*A).$$

For 2×2 and 3×3 matrices we have the following symmetric analogues of the Newton trace formulae : if $A \in \mathrm{M}_2(R)$, then

$$\mathrm{sdet}(A) = \mathrm{tr}^2(A) - \mathrm{tr}(A^2)$$

and if $A \in \mathrm{M}_3(R)$, then

$$\begin{aligned}\mathrm{sdet}(A) = \mathrm{tr}^3(A) &- [\mathrm{tr}(A)\cdot\mathrm{tr}(A^2) + \mathrm{tr}(A\cdot\mathrm{tr}(A)\cdot A) + \mathrm{tr}(A^2)\cdot\mathrm{tr}(A)] \\ &+ [\mathrm{tr}(A^3) + \mathrm{tr}\left((A^\top)^3\right)],\end{aligned}$$

where $A^\top$ denotes the transpose of A. The term preceded by the negative sign can also be expressed more symmetrically, with the notation $\mathrm{tr}(A) = a$, as the trace of $aAA + AaA + AAa$.

The k-th *right characteristic polynomial* of A is the k-th right determinant of the $n\times n$ matrix $xI_n - A$ in $\mathrm{M}_n(R[x])$:

$$p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A)$$

and $p_{A,k}(x)$ is of the form

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k},$$

where $\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ and $\lambda_{n^k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$. Under additional assumptions on A it is proved in a recent work [14] that the coefficients of $p_{A,k}(x)$ are in the fixed ring of a certain group of automorphisms of R.

If R satisfies the commutator identity

$$[[[\ldots[[x_1, x_2], x_3], \ldots], x_k], x_{k+1}] = 0$$

(i.e. if R is Lie nilpotent of index k), then the product $A\cdot\mathrm{radj}_{(k)}(A)$ is a scalar matrix

$$A\cdot\mathrm{radj}_{(k)}(A) = nAP_1\cdots P_k = \mathrm{rdet}_{(k)}(A)\cdot I_n$$

and the Cayley-Hamilton identity with right scalar coefficients

$$(A)p_{A,k} = I_n\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

holds for A. We also have $(A)u = 0$, where $u(x) = p_{A,k}(x)h(x)$ and $h(x) \in R[x]$ is arbitrary. Clearly, these results can be viewed as the index k Lie nilpotent version of classical determinant theory.

At this stage the application of the new theory culminates in the following theorem (see [14]). Let R be a Lie nilpotent algebra (over $\mathbb{Q}$) of index $k \geq 1$. If $\delta : R \longrightarrow R$ is an automorphism with $\delta^n = \mathrm{id}_R$, then the skew polynomial algebra $R[w, \delta]$ is right

integral over $\mathrm{Fix}(\delta)[w^n]$ of degree n^k. In other words, for any $f(w) \in R[w,\delta]$ we have

$$g_0(w^n) + f(w)g_1(w^n) + \cdots + f^{n^k-1}(w)g_{n^k-1}(w^n) + f^{n^k}(w) = 0$$

for some $g_t(w^n) \in \mathrm{Fix}(\delta)[w^n]$, $0 \le t \le n^k - 1$.

In case of an arbitrary R and $A \in \mathrm{M}_n(R)$, we have

$$n(xI_n - A)(xI_n - A)^* = p_{A,1}(x)I_n + C_0 + C_1x + \cdots + C_nx^n,$$

where the matrices $C_i \in \mathrm{M}_n(R)$ are uniquely determined by A, $\mathrm{tr}(C_i) = 0$ and each entry of C_i is in the additive subgroup $[R,R]$ generated by the commutators, i.e. $C_i \in \mathrm{M}_n([R,R])$ for all $0 \le i \le n$. Now the identity

$$(\lambda_0^{(1)}I_n + C_0) + A(\lambda_1^{(1)}I_n + C_1) + \cdots + A^{n-1}(\lambda_{n-1}^{(1)}I_n + C_{n-1}) + A^n(n!I_n + C_n) = 0$$

with right matrix coefficients holds for A. Since commutativity of R would imply $C_0 = C_1 = \cdots = C_{n-1} = C_n = 0$, this is a direct right sided generalization of the classical Cayley-Hamilton theorem.

All the results above have a natural left sided version.

Matrix algebras over the infinitely generated Grassmann algebra, which is Lie nilpotent of index 2, are particularly in the purview of this non-commutative determinant theory. Polynomial identities satisfied by these matrix algebras play a central role in Kemer's classification of T-prime T-ideals.

3. Centralizers and zero level centralizers

For a linear transformation of a finite dimensional vector space over an algebraically closed field the existence of the Jordan normal base is well known in elementary linear algebra. An astonishing fact is that a far reaching generalization of this fundamental result can be formulated for nilpotent complete join homomorphisms of lattices (see [13]). The description of the centralizers and zero level centralizers (two sided annihilators) in Szigeti's papers ([1, 17]) depends on the use of the following remarkable consequence of this lattice theoretical Jordan normal base theorem.

A doubly indexed subset $X = \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \le i \le k_\gamma\}$ of a (unitary) left R-module ${}_RM$ is called a *nilpotent Jordan normal base with respect to* $\varphi \in \mathrm{End}_R(M)$ if each R-submodule $Rx_{\gamma,i} \le M$ is simple,

$$\bigoplus_{\gamma\in\Gamma, 1\le i\le k_\gamma} Rx_{\gamma,i} = M$$

is a direct sum, $\varphi(x_{\gamma,i}) = x_{\gamma,i+1}$, for all $\gamma \in \Gamma$, $1 \le i \le k_\gamma - 1$, $\varphi(x_{\gamma,k_\gamma}) = 0$ and the set $\{k_\gamma \mid \gamma \in \Gamma\}$ of integers is bounded (Γ is called the set of *Jordan blocks* and the *size* of the block $\gamma \in \Gamma$ is the integer k_γ). The existence of such Jordan normal base implies that $\varphi^n = 0 \ne \varphi^{n-1}$, where $n = \max\{k_\gamma \mid \gamma \in \Gamma\}$.

For $\varphi \in \mathrm{End}_R(M)$ the following are equivalent:
(i) ${}_RM$ is semisimple and φ is nilpotent.
(ii) ${}_RM$ has a nilpotent Jordan normal base with respect to φ.

The rest of the section follows literally the exposition in [1] and [17].

If R is a local ring with Jacobson radical J and ${}_RM$ is a finitely generated semisimple left R-module, then the centralizer $\mathrm{Cen}(\varphi)$ of a nilpotent $\varphi \in \mathrm{End}_R(M)$ is isomorphic to the opposite of the factor $\mathcal{N}(X)/\mathcal{J}(X)$ as an algebra over the center $\mathrm{Z}(R)$:

$$\mathrm{Cen}(\varphi) = \{\psi \in \mathrm{End}_R(M) \mid \psi \circ \varphi = \varphi \circ \psi\} \cong (\mathcal{N}(X)/\mathcal{J}(X))^{\mathrm{op}},$$

where

$$\mathcal{N}(X) = \begin{bmatrix} R[z] & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_2}) & R[z] & R[z] & \cdots & R[z] \\ J[z]+(z^{k_1-k_3}) & J[z]+(z^{k_2-k_3}) & R[z] & \cdots & R[z] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1-k_m}) & J[z]+(z^{k_2-k_m}) & J[z]+(z^{k_3-k_m}) & \cdots & R[z] \end{bmatrix}$$

is a subalgebra of $\mathrm{M}_m(R[z])$ and

$$\mathcal{J}(X) = \begin{bmatrix} J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1}) & J[z]+(z^{k_2}) & \cdots & J[z]+(z^{k_m}) \end{bmatrix}$$

is an ideal of $\mathcal{N}(X)$. The number of blocks $m = \dim_R(\ker(\varphi))$, $\Gamma = \{1,2,\ldots,m\}$ and the finite sequence of block sizes $k_1 \geq k_2 \geq \ldots \geq k_m \geq 1$ in a nilpotent Jordan normal base $X = \{x_{\gamma,i} \mid \gamma \in \Gamma, 1 \leq i \leq k_\gamma\}$ with respect to φ are independent of the choice of X and uniquely determined by φ.

For another not necessarily nilpotent endomorphism $\sigma \in \mathrm{End}_R(M)$, the containment $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ holds if and only if there is an R-generating set $\{y_j \in M \mid 1 \leq j \leq d\}$ of ${}_RM$ and there are elements $a_1, a_2, \ldots, a_n \in R$ such that

$$a_1\psi(y_j) + a_2\varphi(\psi(y_j)) + \cdots + a_n\varphi^{n-1}(\psi(y_j)) = \sigma(\psi(y_j))$$

for all $1 \leq j \leq d$ and all $\psi \in \mathrm{Cen}(\varphi)$.

Since $\mathrm{Cen}(\varphi) \subseteq \mathrm{Cen}(\sigma)$ is equivalent to $\sigma \in \mathrm{Cen}(\mathrm{Cen}(\varphi))$, the last statement can be viewed as a non-commutative generalization of Schur's double centralizer theorem in the nilpotent case.

Keeping the above conditions and notations, the zero-level centralizer $\mathrm{Cen}_0(\varphi)$ of the nilpotent φ is isomorphic to the opposite of the factor $\mathcal{N}_0(X)/\mathcal{J}(X)$ as a $\mathrm{Z}(R)$-algebra:

$$\mathrm{Cen}_0(\varphi) = \{\psi \in \mathrm{End}_R(M) \mid \psi \circ \varphi = \varphi \circ \psi = 0\} \cong (\mathcal{N}_0(X)/\mathcal{J}(X))^{\mathrm{op}},$$

where

$$\mathcal{N}_0(X) = \begin{bmatrix} J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \\ J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ J[z]+(z^{k_1-1}) & J[z]+(z^{k_2-1}) & \cdots & J[z]+(z^{k_m-1}) \end{bmatrix}$$

is a subalgebra of $\mathrm{M}_m(R[z])$, still containing $\mathcal{J}(X)$ as an ideal.

Now let $\varphi \in \mathrm{End}_R(M)$ be an arbitrary R-endomorphism of a finitely generated semisimple left R-module ${}_RM$. Then there exist R-submodules W_1, W_2 and V of M such that $W = W_1 \oplus W_2$ and $M = V \oplus W$ are direct sums, $\ker(\varphi) \subseteq W$, $\varphi(W) = W_2$, $\varphi(V) = V$, $\dim_R(W_1) = \dim_R(\ker(\varphi))$, $(\varphi \restriction W) \in \mathrm{End}_R(W)$ is nilpotent and for the zero-level centralizer of φ we have $\mathrm{Cen}_0(\varphi) \cong T$, where

$$T = \{\theta \in \mathrm{End}_R(W) \mid \theta(W_1) \subseteq \ker(\varphi) \text{ and } \theta(W_2) = \{0\}\} = \mathrm{Cen}_0(\varphi \restriction W).$$

The double zero-centralizer theorem states that for another endomorphism $\sigma \in \mathrm{End}_R(M)$ the following conditions are equivalent:
(i) $\sigma \in \mathrm{Cen}_0(\mathrm{Cen}_0(\varphi))$,
(ii) $\mathrm{Cen}_0(\varphi) \subseteq \mathrm{Cen}_0(\sigma)$,
(iii) $\ker(\varphi) \subseteq \ker(\sigma)$ and $\mathrm{im}(\sigma) \subseteq \mathrm{im}(\varphi)$.

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Doi: 10.18514/MMN.2015.691

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 123–131

HU e-ISSN 1787-2413

EXISTENCE AND UNIQUENESS RESULTS FOR BEST PROXIMITY POINTS

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Received 24 September, 2013

Abstract. Let us consider a non-self mapping $T : A \to B$, where A and B are two nonempty subsets of a metric space (X, d). The aim of this paper is to solve the nonlinear programming problem that consists in minimizing the real valued function $x \longmapsto d(x, Tx)$, where T belongs to a new class of non-self mappings. In especial case, existence and uniqueness of fixed point for Kannan type self mappings are also obtained.

2010 *Mathematics Subject Classification:* 47H10; 47H09

Keywords: best proximity point, fixed point, proximal Kannan non-self mapping

1. INTRODUCTION

Let A and B be two nonempty subsets of a metric space X. A non-self mapping $T : A \to B$ is said to be a *contraction* if there exists a constant $r \in [0, 1)$, such that $d(Tx, Ty) \leq rd(x, y)$, for all $x, y \in A$. The well-known Banach contraction principle states that if A is a complete subset of X and T is a contraction self-mapping, then the fixed point equation $Tx = x$ has exactly one solution.

The Banach contraction principle is a very important tools in nonlinear analysis and there are many extensions of this principle; see, e.g., [13] and the references therein.

Let (X, d) be a metric space. A self-mapping $T : X \to X$ is called *Kannan mapping* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$. We know that if X is complete metric space, every Kannan self-mapping defined on X has a unique fixed point ([12]). Note that, the notion of contraction mappings and Kannan mappings are independent. That is, there exists a contraction mapping, which is not Kannan and a Kannan mapping, which is not a contraction. Therefore, we cannot compare these two class of mappings directly.

Recently, Kikkawa and Suzuki in [14], established the following fixed point theorem, which is an extension of Kannan's fixed point theorem.

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Theorem 1 ([14]). *Define a non-increasing function φ from $[0,1)$ into $(\frac{1}{2},1]$ by*

$$\varphi(r)=\begin{cases}1 & \text{if} \quad 0\leq r<\frac{1}{\sqrt{2}},\\ \frac{1}{1+r} & \text{if} \quad \frac{1}{\sqrt{2}}\leq r<1.\end{cases}$$

Let (X,d) be a complete metric space and let T be a self-mapping on X. Let $\alpha\in[0,\frac{1}{2})$ and put $r:=\frac{\alpha}{1-\alpha}\in[0,1)$. Assume that

$$\varphi(r)d(x,Tx)\leq d(x,y) \quad \text{implies} \quad d(Tx,Ty)\leq\alpha[d(x,Tx)+d(y,Ty)],$$

for all $x,y\in X$. Then T has a unique fixed point z and $\lim_n T^n x=z$ holds for every $x\in X$.

It is interesting to note that the function $\varphi(r)$ defined in Theorem 1 is the best constant for every r (see Theorem 2.4 of [14]).

2. Preliminaries

Consider the non-self mapping $T:A\to X$, in which A is a nonempty subset of a metric space (X,d). Clearly, the fixed point equation $Tx=x$ may not have solution. Hence, it is contemplated to find an approximate $x\in A$ such that the error $d(x,Tx)$ is minimum. Indeed, best approximation theory has been derived from this idea. Here, we state the following well-known best approximation theorem due to Kay Fan.

Theorem 2 ([8]). *Let A be a nonempty compact convex subset of a normed linear space X and $T:A\to X$ be a continuous mapping. Then there exists $x\in A$ such that $\|x-Tx\|=dist(Tx,A):=\inf\{\|Tx-a\|:a\in A\}$.*

A point $x\in A$ in the above theorem is called a *best approximant point* of T in A. The notion of best proximity point for non-self mappings is introduced in a similar fashion:

Definition 1. Let A and B be nonempty subsets of a metric space (X,d) and $T:A\to B$ be a non-self mapping. A point $p\in A$ is called a best proximity point of T if

$$d(p,Tp)=dist(A,B):=\{d(x,y):(x,y)\in A\times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem

$$\min_{x\in A} d(x,Tx), \tag{2.1}$$

has at least one solution.

Best proximity point theory is an interesting topic in optimization theory which recently attracted the attention of many authors (see for instance [1–9, 16]).

Let A and B be two nonempty subsets of a metric space (X,d). Let us fix the following notations which will be needed throughout this article:

$$A_0:=\{x\in A:d(x,y)=dist(A,B) \quad \text{for some} \quad y\in B\},$$

$$B_0 := \{y \in B : d(x,y) = dist(A,B) \quad \text{for some} \quad x \in A\}.$$

It is easy to see that if (A,B) is a nonempty and weakly compact pair of subsets of a Banach space X, then (A_0,B_0) is also nonempty pair X.

The notion of *proximal contractions* was defined by Sadiq Basha, as follows.

Definition 2 ([15])**.** Let (A,B) be a pair of nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative real number $\alpha < 1$ such that, for all $u_1,u_2,x_1,x_2 \in A$,

$$\begin{cases} d(u_1,Tx_1) = dist(A,B) \\ d(u_2,Tx_2) = dist(A,B) \end{cases} \Rightarrow d(u_1,u_2) \leq \alpha d(x_1,x_2).$$

Definition 3 ([15])**.** Let A,B be two nonempty subsets of a metric space (X,d). A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y,x_n) \to D(y,A)$ for some $y \in B$ has a convergent subsequence.

Next theorem is the main result of [15].

Theorem 3. *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty and B is approximatively compact with respect to A. Assume that $T : A \to B$ is a proximal contraction such that $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

We mention that in [10], the current author extended Theorem 3 and established a best proximity point theorem under weaker conditions with respect to Theorem 3, due to Sadiq Basha (see Theorem 2.1 and Corollary 2.1 of [10]).

In this article, we introduce a new class of mappings called *weak proximal Kannan non-self mappings* and obtain a similar result of Theorem 1 for this new class of non-self mappings.

3. Main results

To establish our main results, we introduce the following new class of non-self mappings.

Definition 4. Define a strictly decreasing function θ from $[0,\frac{1}{2})$ onto $(\frac{1}{2},1]$ by

$$\theta(r) = 1 - r.$$

Let (A,B) be a nonempty pair of subsets of a metric space (X,d). Let $\alpha \in [0,\frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha}$. Then $T : A \to B$ is said to be a weak proximal Kannan non-self mapping if for all $u,v,x,y \in A$ with

$$d(u,Tx) = dist(A,B) \ \ \& \ \ d(v,Ty) = dist(A,B),$$

we have

$$\theta(r)d^*(x,Tx) \leq d(x,y) \ \text{ implies } \ d(u,v) \leq \alpha[d^*(x,Tx) + d^*(y,Ty)]. \qquad (3.1)$$

The notion of *proximal Kannan non-self mapping* can be defined as below.

Definition 5. Let (A,B) be a nonempty pair of subsets of a metric space (X,d). Then $T : A \to B$ is said to be a proximal Kannan non-self mapping if there exists $\alpha \in [0, \frac{1}{2})$ such that for all $u, v, x, y \in A$ with

$$d(u,Tx) = dist(A,B) \ \ \& \ \ d(v,Ty) = dist(A,B),$$

we have

$$d(u,v) \leq \alpha[d^*(x,Tx) + d^*(y,Ty)].$$

It is clear that the class of weak proximal Kannan non-self mappings contains the class of proximal Kannan non-self mappings as a subclass. Also, the class of proximal Kannan non-self mappings contains the class of Kannan non-self mappings.

We now state our main result of this article.

Theorem 4. *Let (A,B) be a nonempty pair of subsets of a complete metric space (X,d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a weak proximal Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A,B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A,B)$, then $x_n \to x^*$.*

Proof. Assume $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A,B)$. Again, since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A,B)$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A,B), \ \text{ for all } \ n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Thus,

$$d(x_0,Tx_0) \leq d(x_0,x_1) + d(x_1,Tx_0) = d(x_0,x_1) + dist(A,B),$$

and so,

$$\theta(r)d^*(x_0,Tx_0) \leq d^*(x_0,Tx_0) \leq d(x_0,x_1) \quad \& \quad \begin{cases} d(x_1,Tx_0) = dist(A,B), \\ d(x_2,Tx_1) = dist(A,B). \end{cases}$$

Since T is a weak proximal Kannan non-self mapping, we conclude that

$$\begin{aligned} d(x_1,x_2) &\leq \alpha[d^*(x_0,Tx_0) + d^*(x_1,Tx_1)] \\ &\leq \alpha[d(x_0,x_1) + d^*(x_1,Tx_0) + d(x_1,x_2) + d^*(x_2,Tx_1)] \\ &= \alpha[d(x_0,x_1) + d(x_1,x_2)]. \end{aligned}$$

Therefore,

$$d(x_1,x_2) \leq \frac{\alpha}{1-\alpha} d(x_0,x_1) = rd(x_0,x_1).$$

Similarly, we can see that

$$\theta(r)d^*(x_1,Tx_1)\leq d(x_1,x_2) \quad \& \quad \begin{cases} d(x_2,Tx_1)=dist(A,B),\\ d(x_3,Tx_2)=dist(A,B).\end{cases}$$

This implies that

$$\begin{aligned} d(x_2,x_3) &\leq \alpha[d^*(x_1,Tx_1)+d^*(x_2,Tx_2)]\\ &\leq \alpha[d(x_1,x_2)+d^*(x_2,Tx_1)+d(x_2,x_3)+d^*(x_3,Tx_2)]\\ &= \alpha[d(x_1,x_2)+d(x_2,x_3)].\end{aligned}$$

So,

$$d(x_2,x_3)\leq \frac{\alpha}{1-\alpha}d(x_1,x_2)=rd(x_1,x_2)\leq r^2d(x_0,x_1).$$

Hence, by induction, we conclude that

$$d(x_n,x_{n+1})\leq r^n d(x_0,x_1),$$

which implies that

$$\Sigma_{n=1}^{\infty}d(x_n,x_{n+1})\leq \Sigma_{n=1}^{\infty}r^n d(x_0,x_1)<\infty.$$

That is, $\{x_n\}$ is a Cauchy sequence in A_0. Since A_0 is closed and X is complete metric space, we deduce that $\{x_n\}$ is a convergent sequence. Let $x^*\in A_0$ be such that $x_n\to x^*$. We claim that x^* is a unique best proximity point of T. At first, we prove that

$$d^*(x^*,Tx)\leq \alpha d(x^*,x), \quad \forall x\in A_0 \quad \text{with} \quad x\neq x^*. \tag{3.3}$$

Let $x\in A_0$ and $x\neq x^*$. Since $T(A_0)\subseteq B_0$, there exists $y\in A_0$ such that $d(y,Tx)=dist(A,B)$. By the fact that $x_n\to x^*$, there exists $N_1\in\mathbb{N}$ such that

$$d(x_n,x^*)\leq \frac{1}{3}d(x,x^*), \quad \forall n\geq N_1.$$

We now have

$$\begin{aligned} \theta(r)d^*(x_n,Tx_n) &\leq d^*(x_n,Tx_n)=d(x_n,Tx_n)-dist(A,B)\\ &\leq d(x_n,x^*)+d(x^*,x_{n+1})+d(x_{n+1},Tx_n)-dist(A,B)\\ &= d(x_n,x^*)+d(x^*,x_{n+1})\leq \frac{2}{3}d(x,x^*)\\ &= d(x,x^*)-\frac{1}{3}d(x,x^*)\leq d(x,x^*)-d(x_n,x^*)\\ &\leq d(x_n,x).\end{aligned}$$

Thus,

$$\theta(r)d^*(x_n,Tx_n)\leq d(x_n,x) \quad \& \quad \begin{cases} d(x_{n+1},Tx_n)=dist(A,B),\\ d(y,Tx)=dist(A,B).\end{cases}$$

Again, since T is weak proximal Kannan non-self mapping we conclude that

$$d(x_{n+1},y)\leq\alpha[d^*(x_n,Tx_n)+d^*(x,Tx)]\leq\alpha[d(x_n,x_{n+1})+d^*(x,Tx)].$$

Thereby,

$$\begin{aligned}d(x^*,Tx)&=\lim_{n\to\infty}d(x_n,Tx)\\&\leq\lim_{n\to\infty}[d(x_n,x_{n+1})+d(x_{n+1},y)+d(y,Tx)]\\&\leq\lim_{n\to\infty}[(1+\alpha)d(x_n,x_{n+1})+\alpha d^*(x,Tx)+d(y,Tx)]\\&\leq\lim_{n\to\infty}[(1+\alpha)r^nd(x_0,x_1)+\alpha d^*(x,Tx)+dist(A,B)]\\&=\alpha d^*(x,Tx)+dist(A,B).\end{aligned}$$

Then,

$$d^*(x^*,Tx)\leq\alpha d^*(x,Tx),\quad\forall x\in A_0,\quad\text{with}\quad x\neq x^*.$$

That is, (3.3) holds. It now follows from (3.3) that

$$\begin{aligned}d^*(x_n,Tx_n)&\leq d(x_n,x^*)+d^*(x^*,Tx_n)\\&\leq d(x_n,x^*)+\alpha d^*(x_n,Tx_n).\end{aligned}$$

Thus,

$$\theta(r)d^*(x_n,Tx_n)=(1-r)d^*(x_n,Tx_n)\leq(1-\alpha)d^*(x_n,Tx_n)\leq d(x_n,x^*).\quad(3.4)$$

On the other hand, since $x^*\in A_0$ and $T(A_0)\subseteq B_0$, there exists $y^*\in B_0$ such that $d(y^*,Tx^*)=dist(A,B)$. Therefore,

$$\theta(r)d^*(x_n,Tx_n)\leq d(x_n,x^*)\quad\&\quad\begin{cases}d(x_{n+1},Tx_n)=dist(A,B),\\d(y^*,Tx^*)=dist(A,B),\end{cases}$$

which implies that

$$\begin{aligned}d(x_{n+1},y^*)&\leq\alpha[d^*(x_n,Tx_n)+d^*(x^*,Tx^*)]\\&\leq\alpha[d(x_n,x_{n+1})+d^*(x_{n+1},Tx_n)+d^*(x^*,Tx^*)].\end{aligned}$$

If in above relation $n\to\infty$, we obtain

$$\begin{aligned}d(y^*,x^*)&\leq\alpha d^*(x^*,Tx^*)\\&=\alpha[d(x^*,y^*)+d^*(y^*,Tx^*)]=\alpha d(x^*,y^*).\end{aligned}$$

This deduces that $d(x^*,y^*)=0$ or $x^*=y^*$. Hence x^* is a best proximity point of the mapping T. The uniqueness of best proximity point follows from the condition that T is weak proximal Kannan non-self mapping. That is, suppose that x_1^*,x_2^* are two distinct points in A such that $d(x_i^*,Tx_i^*)=dist(A,B)$, for $i=1,2$. So,

$$\theta(r)d^*(x_1^*,Tx_1^*)\leq d(x_1^*,x_2^*)\quad\&\quad\begin{cases}d(x_1^*,Tx_1^*)=dist(A,B),\\d(x_2^*,Tx_2^*)=dist(A,B),\end{cases}$$

Then,

$$0 < d(x_1^*, x_2^*) \leq \alpha[d^*(x_1^*, Tx_1^*) + d^*(x_2^*, Tx_2^*)] = 0,$$

which is a contradiction. Hence, the best proximity point is unique.

□

The following corollaries are obtained from Theorem 4.

Corollary 1. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a proximal Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$ then, $x_n \to x^*$.*

Corollary 2. *Let (A, B) be a nonempty pair of a complete metric space (X, d) such that A_0 is nonempty and closed. Assume that $T : A \to B$ is a Kannan non-self mapping such that $T(A_0) \subseteq B_0$. Then there exists a unique point $x^* \in A$ such that $d(x^*, Tx^*) = dist(A, B)$. Moreover, if $\{x_n\}$ is a sequence in A such that $d(x_{n+1}, Tx_n) = dist(A, B)$ then, $x_n \to x^*$.*

Corollary 3. *Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a self mapping such that*

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in A$, where $\theta(r)$ is defined as in the Definition 4. Then T has a unique fixed point $x^* \in A$. Moreover, if $x_0 \in A$ and we define $x_{n+1} = Tx_n$, then $x_n \to x^*$.

Corollary 4 (Kannan fixed point theorem)**.** *Let A be a nonempty and closed subset of a complete metric space (X, d). Assume that $T : A \to A$ is a Kannan mapping. Then T has a unique fixed point. Moreover, for each $x_0 \in A$ if we define $x_{n+1} = Tx_n$ then the sequence $\{x_n\}$ converges to the fixed point of T.*

Example 1. Suppose that $X = \mathbb{R}$ with the usual metric. Suppose that

$$A := [0, 2] \cup \{5\} \quad \& \quad B := [3, 4].$$

Then A and B are nonempty closed subsets of X and $A_0 = \{2, 5\}$ and $B_0 = \{3, 4\}$. Note that $dist(A, B) = 1$. Let $T : A \to B$ be a mapping defined as

$$T(x) = \begin{cases} \frac{7}{2} & \text{if} \quad x = 0, \\ 4 & \text{if} \quad x \neq 0. \end{cases}$$

It is easy to see that T is weak proximal Kannan non-self mapping for each $\alpha \in [0, \frac{1}{2})$. Indeed, it is sufficient to note that $d(u, Tx) = dist(A, B)$, holds for $u = 5$ and $x \in A - \{0\}$. Therefore, Theorem 4 guaranties the existence and uniqueness of a best proximity point for T and this point is $x^* = 5$.

Example 2. Suppose that $X = \mathbb{R}$ with the usual metric. Suppose that

$$A := [0, \frac{1}{100}] \cup \{1\} \quad \& \quad B := [2, 3].$$

Then A and B are nonempty closed subsets of X and $dist(A, B) = 1$. Define a non-self mapping $T : A \to B$ as follows

$$T(x) = \begin{cases} 2 & \text{if} \quad x \in \mathbb{Q} \cap A, \\ \frac{101}{50} & \text{if} \quad x \in \mathbb{Q}^c \cap A. \end{cases}$$

Note that T is not continuous. We claim that T is Kannan nons-elf mapping with $\alpha = \frac{1}{3}$. For this purpose, it is sufficient to consider two following cases.

Case 1. If $x \in \mathbb{Q} \cap A - \{1\}$ and $y \in \mathbb{Q}^c \cap A$, then

$$\alpha[d^*(x, Tx) + d^*(y, Ty)] = \frac{1}{3}[\frac{101}{50} - (x + y)] \geq \frac{2}{3} > \frac{1}{50} = d(Tx, Ty).$$

Case 2. If $x = 1$ and $y \in \mathbb{Q}^c \cap A$, then

$$\alpha[d^*(x, Tx) + d^*(y, Ty)] = \frac{1}{3}[\frac{51}{50} - y] \geq \frac{1}{3} \times \frac{101}{100} > \frac{1}{50} = d(Tx, Ty).$$

It now follows from Corollary 2 that T has a unique best proximity point and this point is $x^* = 1$.

Remark 1. We mention that in [11] the author studied the existence of best proximity points in metric spaces with a partial order, where weak proximal Kannan non-self mappings are satisfied only for comparable elements.

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Doi: 10.18514/MMN.2015.1031

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 133–144

HU e-ISSN 1787-2413

CYCLIC RELATIVELY NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Received 23 October, 2013

Abstract. A new class of mappings, called generalized orbital cyclic contractions, is introduced and used to study the existence of best proximity points. Existence results of best proximity points for cyclic relatively nonexpansive mappings in the setting of convex metric spaces are also obtained.

2010 *Mathematics Subject Classification:* 47H10; 47H09

Keywords: best proximity point, fixed point, non-diametral point, cyclic relatively nonexpansive mapping

1. INTRODUCTION

Let (X,d) be a metric space and let A and B be nonempty subsets of X. A mapping $T : A\cup B \to A\cup B$ is said to be a *cyclic mapping* provided that $T(A)\subseteq B$ and $T(B)\subseteq A$. A mapping $T : A\cup B \to A\cup B$ is said to be a *cyclic relatively nonexpansive* if T is cyclic and $d(Tx,Ty)\leq d(x,y)$ whenever $x\in A$ and $y\in B$.

If $A\cap B=\varnothing$ then a cyclic mapping cannot have a fixed point, instead it is interesting to study the existence of *best proximity points*, that is, a point $x\in A\cup B$ such that

$$d(x,Tx)=\operatorname{dist}(A,B):=\inf\{d(x,y):(x,y)\in A\times B\}.$$

The relevance of best proximity points is that they provide optimal solutions for the problem of best approximation between two sets.

Eldred, Kirk and Veeramani [11] established the existence of a best proximity point for cyclic relatively nonexpansive mappings by using a geometric notion of *proximal normal structure* in the setting of Banach spaces.

Theorem 1 ([11])**.** *Let (A,B) be a nonempty, weakly compact and convex pair in a Banach space X. Let $T : A\cup B \to A\cup B$ be a cyclic relatively nonexpansive mapping and suppose that (A,B) has the proximal normal structure. Then T has a best proximity point.*

This research was in part supported by a grant from IPM (No. 93470047).

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After that in [27], the authors established the existence of best proximity points for cyclic relatively nonexpansive mappings without invoking proximal normal structure.

In [12], the authors proved the following existence, uniqueness and convergence theorem in uniformly convex Banach spaces.

Theorem 2 ([12]). *Let (A, B) be a nonempty, closed, and convex pair of subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction mapping, that is, T is cyclic and*

$$\|Tx - Ty\| \leq \alpha\|x - y\| + (1-\alpha)\text{dist}(A, B),$$

for every $(x, y) \in A \times B$ and for some $\alpha \in [0, 1)$. Then T has a unique best proximity point $x^* \in A$ and $\{T^{2n}x\}$ converges to x^* for every $x \in A$.

Note that the class of cyclic relatively nonexpansive mappings contains the class of cyclic contractions as a subclass. Suzuki et al. [29] generalized Theorem 2 to metric spaces with the property UC (see also [13, 14]). Existence results of best proximity points is an interesting topic in nonlinear analysis which recently attracted the attention of many authors (see for instance [1–8, 10, 15, 18, 20–26, 28, 31]).

In this article, we introduce a new class of cyclic mappings called *generalized cyclic orbital contractions* which properly contains the class of cyclic contractions and we study sufficient conditions for the existence of a best proximity point for this class of mappings in convex metric spaces. Moreover, we prove the existence of a best proximity point for cyclic relatively nonexpansive mappings in the setting of convex metric spaces by using a geometric notion of *semi-normal structure*.

2. Preliminaries

In [30], Takahashi introduced the notion of convexity in metric spaces as follows.

Definition 1. Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [30].

To describe our results, we need some definitions and notations of [11, 30].

Definition 2. A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Proposition 1. *Let $(X, d, \mathcal{W})$ be a convex metric space and let $B(x; r)$ denote the closed ball centered at $x \in X$ with radius $r \geq 0$. Then $B(x; r)$ is a convex subset of X.*

Proposition 2. *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of X.*

Definition 3. A convex metric space $(X, d, \mathcal{W})$ is said to have property (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

For example every weakly compact convex subset of a Banach space has the property (C).

Let A and B be two nonempty subsets of a convex metric space $(X, d, \mathcal{W})$. We shall say that a pair (A, B) in a convex metric space $(X, d, \mathcal{W})$ satisfies a property if both A and B satisfy that property. For instance, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C$, and $B \subseteq D$. We shall also adopt the following notations.

$$\delta_x(A) := \sup\{d(x, y) : y \in A\} \text{ for all } x \in X,$$
$$\delta(A, B) := \sup\{d(x, y) : x \in A,\ y \in B\},$$
$$\operatorname{diam}(A) := \delta(A, A).$$

The *closed and convex hull* of a set A will be denoted by $\overline{\operatorname{cov}}(A)$ and defined as below.

$$\overline{cov}(A) := \bigcap \{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The pair $(x, y) \in A \times B$ is said to be *proximal* in (A, B) if $d(x, y) = \operatorname{dist}(A, B)$. Moreover, we set

$$A_0 := \{x \in A : d(x, y') = \operatorname{dist}(A, B), \text{ for some } y' \in B\},$$
$$B_0 := \{y \in B : d(x', y) = \operatorname{dist}(A, B), \text{ for some } x' \in A\}.$$

We note that if (A, B) is a nonempty weakly compact and convex pair of subsets of a Banach space X then also is the pair (A_0, B_0) and it is easy to see that $\operatorname{dist}(A, B) = \operatorname{dist}(A_0, B_0)$.

For a cyclic mapping $T : A \cup B \to A \cup B$ and $x \in A \cup B$, we define the orbit setting at x by

$$\mathcal{O}_{T^2} x := \{x, T^2 x, T^4 x, ..., T^{2n} x, ...\},$$

where $T^{2n} x = T(T^{2n-1} x)$ for $n \geq 1$ and $T^0 x = x$. Note that if $(x, y) \in A \times B$ then $\mathcal{O}_{T^2} x \subseteq A$ and $\mathcal{O}_{T^2} y \subseteq B$.

Definition 4. Let A be a nonempty subset in a metric space (X, d). A point p in A is said to be a diametral point if $\delta_p(A) = \operatorname{diam}(A)$.

Definition 5 ([9])**.** A convex metric space $(X, d, \mathcal{W})$ is said to have normal structure if for each bounded, closed and convex subset E of X which contains at least two points, there exists an element $p \in E$ which is a non-diametral point, that is, $\delta_p(E) < \operatorname{diam}(E)$.

Here, we recall the geometric notion of proximal normal structure which was introduced in [11].

Definition 6. A pair (A, B) of subsets of a normed linear space X is said to be a proximal pair if for each $(x, y) \in A \times B$ there exists $(x', y') \in A \times B$ such that

$$\|x - y'\| = \|x' - y\| = \operatorname{dist}(A, B).$$

Definition 7. A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure if for any bounded, closed and convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \operatorname{dist}(H_1, H_2)$, there exits $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta_{x_1}(H_2) < \delta(H_1, H_2), \quad \delta_{x_2}(H_1) < \delta(H_1, H_2).$$

It was announced in [11] that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X has a proximal normal structure (Proposition 2.1 of [11]).

3. Generalized cyclic orbital contractions

Motivated by the concept of cyclic contractions and its applications in Theorem 2, we introduce the notion of generalized cyclic orbital contraction and extend the result on cyclic contraction, from a closed convex pair of a uniformly convex Banach space to a bounded closed and convex pair of a convex metric space.

Definition 8. Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a generalized cyclic orbital contraction map if T is cyclic on $A \cup B$ and there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\{\delta_x(\mathcal{O}_{T^2 y}), \delta_y(\mathcal{O}_{T^2 x})\} + (1 - r)\operatorname{dist}(A, B), \tag{3.1}$$

for all $(x, y) \in A \times B$.

Note that every cyclic contraction mapping is a generalized cyclic orbital contraction map. The following example shows that the reverse implication dose not hold.

Example 1. Consider $X := \mathbb{R}$ with the usual metric. Let $A = B = [0, \frac{1}{2}]$. Define a mapping $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \frac{1}{8}x & \text{if } 0 \leq x \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < x \leq \frac{1}{2}. \end{cases}$$

Then T is generalized cyclic orbital contraction with $r = \frac{7}{8}$.

Proof. In cases $0 \leq x, y \leq \frac{1}{4}$ and $\frac{1}{4} < x, y \leq \frac{1}{2}$, it is easy to verify that (3.1) holds. Let $0 \leq x \leq \frac{1}{4}$ and $\frac{1}{4} < y \leq \frac{1}{2}$. Then $d(Tx, Ty) = \frac{1}{8}x$ and

$$\delta_x(\mathcal{O}_{T^2y}) = \sup_{n\geq 0} |x - T^{2n}y| = \max\{x, y-x\} \quad \& \quad \delta_y(\mathcal{O}_{T^2x}) = \sup_{n\geq 0} |y - T^{2n}x| = y.$$

Thus $\max\{\delta_x(\mathcal{O}_{T^2y}), \delta_y(\mathcal{O}_{T^2x})\} = \max\{x, y-x, x\} = y$. We now have

$$d(Tx, Ty) = \frac{1}{8}x \leq \frac{7}{8}y = r \max\{\delta_x(\mathcal{O}_{T^2y}), \delta_y(\mathcal{O}_{T^2x})\}.$$

that is, T is generalized cyclic orbital contraction. Note that T is not cyclic contraction. Indeed, if $(x, y) = (\frac{1}{4}, \frac{26}{100})$ then $d(Tx, Ty) = \frac{1}{8} \times \frac{1}{4}$. Now, if T is cyclic contraction for some $r \in [0, 1)$, then we must have

$$\frac{1}{32} = d(Tx, Ty) \leq rd(x, y) = r \times \frac{1}{100},$$

which is a contradiction.

□

Let us state our main result of this section.

Theorem 3. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$. Suppose that $T : A \cup B \to A \cup B$ is a generalized cyclic orbital contraction. If X has the property (C) then T has a best proximity point.*

Proof. Let Σ denote the set of all nonempty, bounded, closed and convex pairs (E, F) which are subsets of (A, B) and such that T is cyclic on $E \cup F$. Note that $(A, B) \in \Sigma$. Also, Σ is partially ordered by revers inclusion, that is $(E_1, F_1) \leq (E_2, F_2) \Leftrightarrow (E_2, F_2) \subseteq (E_1, F_1)$. By the fact that X has the property (C), every increasing chain in Σ is bounded above. So, by using Zorn's lemma we obtain a maximal element say $(C, D) \in \Sigma$ which is minimal w.r.t inclusion relation. We note that $(\overline{cov}(T(D)), \overline{cov}(T(C)))$ is a nonempty, bounded, closed and convex pair in X and $(\overline{cov}(T(D)), \overline{cov}(T(C))) \subseteq (C, D)$. Further,

$$T(\overline{cov}(T(D))) \subseteq T(C) \subseteq \overline{cov}(T(C)),$$

and also,

$$T(\overline{cov}(T(C))) \subseteq \overline{cov}(T(D)),$$

that is, T is cyclic on $\overline{cov}(T(D)) \cup \overline{cov}(T(C))$. It now follows from the minimality of (C, D) that

$$\overline{cov}(T(D)) = C \,, \overline{cov}(T(C)) = D.$$

Let $x \in C$, then $D \subseteq B(x; \delta_x(D))$. Now, if $y \in D$ we have

$$\begin{aligned} d(Tx, Ty) &\leq r \max\{\delta_x(\mathcal{O}_{T^2y}), \delta_y(\mathcal{O}_{T^2x})\} + (1-r)\text{dist}(A, B) \\ &\leq r\delta(C, D) + (1-r)\text{dist}(A, B). \end{aligned}$$

Hence, for all $y \in D$ we conclude that

$$Ty \in B(Tx; r\delta(C,D) + (1-r)\text{dist}(A,B)),$$

and so,

$$T(D) \subseteq B(Tx; r\delta(C,D) + (1-r)\text{dist}(A,B)).$$

Therefore,

$$C = \overline{cov}(T(D)) \subseteq B(Tx; r\delta(C,D) + (1-r)\text{dist}(A,B)),$$

which deduces that

$$d(z,Tx) \leq r\delta(C,D) + (1-r)\text{dist}(A,B), \quad \forall z \in C.$$

Thus,

$$\delta_{Tx}(C) \leq r\delta(C,D) + (1-r)\text{dist}(A,B). \tag{3.2}$$

Similarly, if $y \in D$ we conclude that

$$\delta_{Ty}(D) \leq r\delta(C,D) + (1-r)\text{dist}(A,B). \tag{3.3}$$

Set

$$E := \{x \in C : \delta_x(D) \leq r\delta(C,D) + (1-r)\text{dist}(A,B)\},$$
$$F := \{y \in D : \delta_y(C) \leq r\delta(C,D) + (1-r)\text{dist}(A,B)\}.$$

Note that $T(D) \subseteq E$ and $T(C) \subseteq F$. Besides, it is easy to check that

$$E = \bigcap_{y \in D} B(y; r\delta(C,D) + (1-r)\text{dist}(A,B)) \cap C,$$

$$F = \bigcap_{x \in C} B(x; r\delta(C,D) + (1-r)\text{dist}(A,B)) \cap D.$$

Furthermore, if $x \in E$ then by (3.2), $Tx \in F$, i.e. $T(E) \subseteq F$ and also by the relation (3.3) $T(F) \subseteq E$, that is, T is cyclic on $E \cup F$. Again, by the minimality of (C,D) we obtain $E = C$ and $F = D$. So, we conclude that

$$\delta_x(D) \leq r\delta(C,D) + (1-r)\text{dist}(A,B), \quad \forall x \in C.$$

Hence,

$$\delta(C,D) = \text{dist}(A,B).$$

Now, for each pair $(x^*, y^*) \in C \times D$ we must have

$$d(x^*, Tx^*) = d(Ty^*, y^*) = \text{dist}(A,B).$$

This completes the proof. □

The following corollary ensures the existence and uniqueness of a best proximity point in the setting of strictly convex Banach spaces.

Corollary 1. *Let (A,B) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a generalized cyclic orbital contraction. Then T has a unique best proximity point.*

Proof. It now follows from Theorem 3 that there exists a pair $(x^*, y^*) \in A \times B$ such that $\|x^* - Tx^*\| = \|Ty^* - y^*\| = \text{dist}(A, B)$. The uniqueness of the best proximity point of T follows from the strictly convexity of the Banach space. Indeed, if $(u^*, v^*) \in A \times B$ be such that $\|u^* - Tu^*\| = \|Tv^* - v^*\| = \text{dist}(A, B)$ then

$$\begin{aligned}\text{dist}(A, B) &\leq \|\frac{x^* + u^*}{2} - \frac{Tx^* + Tu^*}{2}\| = \|\frac{x^* - Tx^*}{2} + \frac{u^* - Tu^*}{2}\| \\ &< \frac{1}{2}(\|x^* - Tx^*\| + \|u^* - Tu^*\|) = \text{dist}(A, B),\end{aligned}$$

which is a contradiction, that is $x^* = u^*$. Equivalently, we can see that $y^* = v^*$. □

4. Cyclic relatively nonexpansive mappings

In this section we study the existence of a best proximity point for cyclic relatively nonexpansive mappings in the setting of convex metric spaces. For this purpose, we recall a geometric notion of *semi-normal structure* which was introduced by the current author in [16].

Definition 9 ([16])**.** A convex pair (K_1, K_2) in a convex metric space $(X, d, \mathcal{W})$ is said to have semi normal structure if for any bounded, closed and convex pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\delta(H_1, H_2) > dist(H_1, H_2)$, there exits $(x_1, x_2) \in H_1 \times H_2$ such that

$$d(x_1, x_2) = \text{dist}(K_1, K_2) \quad \& \quad \max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} < \delta(H_1, H_2).$$

We note that if in the above definition $K_1 = K_2$, then (K_1, K_2) has semi normal structure if the set K_1 has normal structure in the sense of Brodskil and Milman ([9]).

In [30], Takahashi extended Kirk's fixed point theorem ([19]) to convex metric spaces as follows.

Theorem 4. *Suppose that $(X, d, \mathcal{W})$ is a convex metric space such that X has the property (C). Let K be a nonempty, bounded, closed and convex subset of X with normal structure. If $T : K \to K$ is a nonexpansive mapping, then T has a fixed point.*

Definition 10 ([17])**.** Let (A, B) be a nonempty pair of subsets of a metric space (X, d). We say that the pair (A, B) is proximal compactness provided that every net $(\{x_\alpha\}, \{y_\alpha\})$ of $A \times B$ satisfying the condition that $d(x_\alpha, y_\alpha) \to \text{dist}(A, B)$, has a convergent subnet in $A \times B$.

It is clear that if (A, B) is a compact pair in a metric space (X, d), then (A, B) is proximal compactness. Next, we shall prove our main result of this section.

Theorem 5. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$ such that A_0 is nonempty and (A, B) is proximal compactness. Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If (A, B) has the semi-normal structure and X has the property (C), then T has a best proximity point.*

Proof. Let $\mathcal{F}$ denote the set of all nonempty, closed and convex pairs (E,F) which are subsets of (A,B) and such that T is cyclic on $E\cup F$ and $d(x,y)=\text{dist}(A,B)$ for some $(x,y)\in E\times F$. Note that $(A,B)\in\mathcal{F}$ by the fact that A_0 is nonempty. Also, $\mathcal{F}$ is partially ordered by revers inclusion. Assume that $\{(E_\alpha,F_\alpha)\}_\alpha$ is a descending chain in $\mathcal{F}$. Set $E:=\bigcap E_\alpha$ and $F:=\bigcap F_\alpha$. Since X has the property (C), we conclude that (E,F) is nonempty and also is a closed pair. Also, by Proposition 2, (E,F) is a convex pair. Moreover,

$$T(E)=T(\bigcap E_\alpha)\subseteq\bigcap T(E_\alpha)\subseteq\bigcap F_\alpha=F.$$

Similarly, we can see that $T(F)\subseteq E$, that is, T is cyclic on $E\cup F$. Now, let $(x_\alpha,y_\alpha)\in E_\alpha\times F_\alpha$ be such that $d(x_\alpha,y_\alpha)=\text{dist}(A,B)$. Since (A,B) is proximal compactness, (x_α,y_α) has a convergent subsequence say $(x_{\alpha_i},y_{\alpha_i})$ such that $x_{\alpha_i}\to x\in A$ and $y_{\alpha_i}\to y\in B$. Thus

$$d(x,y)=\lim_i d(x_{\alpha_i},y_{\alpha_i})=\text{dist}(A,B).$$

Therefore, there exists an element $(x,y)\in E\times F$ such that $d(x,y)=\text{dist}(A,B)$. Hence, every increasing chain in $\mathcal{F}$ is bounded above w.r.t revers inclusion relation. Then by using Zorn's Lemma we can get a minimal element say (K_1,K_2). Let $(p,q)\in K_1\times K_2$ be such that $d(p,q)=\text{dist}(A,B)$. If $\delta(K_1,K_2)=\text{dist}(K_1,K_2)$, then for each $(x,y)\in K_1\times K_2$ we have

$$d(x,Tx)=d(Ty,y)=\text{dist}(A,B),$$

and we are finished. Thus we assume that $\delta(K_1,K_2)>\text{dist}(K_1,K_2)$. By a similar argument of Theorem 3, we conclude that $(\overline{cov}(T(K_2)),\overline{cov}(T(K_1)))\subseteq(K_1,K_2)$ is a nonempty, closed and convex pair such that T is cyclic on $\overline{cov}(T(K_2))\cup\overline{cov}(T(K_1))$. On the other hand,

$$\text{dist}(\overline{cov}(T(K_2)),\overline{cov}(T(K_1)))\leq d(Tq,Tp)\leq d(p,q)=\text{dist}(A,B),$$

and so, $(\overline{cov}(T(K_2)),\overline{cov}(T(K_1))\in\mathcal{F}$. It now follows from the minimality of (K_1,K_2) that $\overline{cov}(T(K_2))=K_1$ and $\overline{cov}(T(K_1))=K_2$. Since (A,B) has the semi-normal structure, there exists a pair $(x^*,y^*)\in K_1\times K_2$ such that $d(x^*,y^*)=\text{dist}(K_1,K_2)(=\text{dist}(A,B))$ and if we set

$$r_1:=\delta_{x^*}(K_2),\quad r_2:=\delta_{y^*}(K_1)\quad\text{and}\quad r:=\max\{r_1,r_2\},$$

then $r<\delta(K_1,K_2)$. Suppose that

$$C_r(K_2):=K_1\bigcap(\cap_{x\in K_2}B(x;r))\quad\&\quad C_r(K_1):=K_2\bigcap(\cap_{x\in K_1}B(x;r)).$$

Propositions 1 and 2, imply that $(C_r(K_2),C_r(K_1))$ is a nonempty, closed and convex pair in X. Furthermore, $(x^*,y^*)\in C_r(K_2)\times C_r(K_1)$. We now claim that T is cyclic on $C_r(K_2)\cup C_r(K_1)$. Let $u\in C_r(K_2)$. We must prove that $Tu\in C_r(K_1)$, or $K_1\subseteq B(Tu;r)$. Since T is cyclic relatively nonexpansive,

$$d(Tv,Tu)\leq d(u,v)\leq r,\ \ \forall v\in K_2.$$

So, $Tv \in B(Tu;r)$ for all $v \in K_2$ which implies that

$$K_1 = \overline{cov}(T(K_2)) \subseteq B(Tu;r).$$

This deduces that $Tu \in C_r(K_1)$, that is, $T(C_r(K_2)) \subseteq C_r(K_1)$. Similarly, we can see that $T(C_r(K_1)) \subseteq C_r(K_2)$. Thus T is cyclic on $C_r(K_2) \cup C_r(K_1)$. The minimality of (K_1, K_2) concludes that

$$C_r(K_1) = K_2 \quad \& \quad C_r(K_2) = K_1.$$

Hence, $K_2 \subseteq \bigcap_{x \in K_1} B(x;r)$. Therefore, for each $y \in K_2$ we have $\delta_y(K_1) \leq r$ and so,

$$\delta(K_1, K_2) = \sup_{y \in K_2} \delta_y(K_1) \leq r,$$

which is a contradiction. This completes the proof. □

The following result is another version of Theorem 5 in the setting of reflexive Banach spaces.

Theorem 6. *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If (A, B) has the semi-normal structure, then T has a best proximity point.*

Proof. At first, we note that (A_0, B_0) is a nonempty pair in X. Indeed, if $\{(x_n, y_n)\}$ is a sequence in $A \times B$ such that $\|x_n - y_n\| \to \text{dist}(A, B)$, then by the fact that (A, B) is a bounded, closed and convex pair in a reflexive Banach space X, the sequence $\{(x_n, y_n)\}$ has a weakly convergent subsequence say $\{(x_{n_k}, y_{n_k})\}$ such that $x_{n_k} \rightharpoonup x \in A$ and $y_{n_k} \rightharpoonup y \in B$, where "$\rightharpoonup$" denotes the weakly convergence. But, according to a well-known fact in basic functional analysis, we have

$$\text{dist}(A, B) \leq \|x - y\| \leq \liminf_{k \to \infty} \|x_{n_k} - y_{n_k}\| = \text{dist}(A, B),$$

which implies that (A_0, B_0) is nonempty. Now, let $\mathcal{F}$ be the collection of all nonempty, closed and convex pairs (E, F) which are subsets of (A, B) and such that T is cyclic on $E \cup F$ and $\|x - y\| = \text{dist}(A, B)$ for some $(x, y) \in E \times F$. It is interesting to note that $(A_0, B_0) \in \mathcal{F}$. In fact, it is easy to verify that the pair (A_0, B_0) is closed and convex. Also, if $x \in A_0$, then there exists $y \in B_0$ such that $\|x - y\| = \text{dist}(A, B)$. Since T is cyclic relatively nonexpansive,

$$\text{dist}(A, B) \leq \|Tx - Ty\| \leq \|x - y\| = \text{dist}(A, B),$$

which implies that $T(A_0) \subseteq B_0$. Similarly, we can see that $T(B_0) \subseteq A_0$, that is, T is cyclic on $A_0 \cup B_0$. Now, if $\{(E_\alpha, F_\alpha)\}_\alpha$ is a descending chain in $\mathcal{F}$ and we put $E := \bigcap E_\alpha$ and $F := \bigcap F_\alpha$, then by the reflexivity of the Banach space X, (E, F) is nonempty. Moreover, the pair (E, F) is closed and convex and we can see that T is cyclic on $E \cup F$. Further, if $\{(x_\alpha, y_\alpha)\}$ is a net in $E_\alpha \times F_\alpha$ such that $\|x_\alpha - y_\alpha\| = \text{dist}(A, B)$, then by the fact that X is a reflexive Banach space, there

exists a subnet $\{(x_{\alpha_i}, y_{\alpha_i})\}$ such that $x_{\alpha_i} \rightharpoonup x \in E$ and $y_{\alpha_i} \rightharpoonup y \in F$. Thus we must have

$$\operatorname{dist}(E, F) \leq \|x - y\| = \operatorname{dist}(A, B).$$

Hence, $(E, F) \in \mathcal{F}$. It now follows from the Zorn's Lemma that $\mathcal{F}$ has a minimal element say (K_1, K_2). Similar argument of Theorem 5, deduces that each point $(x^*, y^*) \in K_1 \times K_2$ is a best proximal point of T. □

Corollary 2. *Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If (A, B) has the semi-normal structure, then T has a best proximity point.*

Corollary 3. *Let A be a nonempty, bounded, closed and convex subset of a reflexive Banach space X. Suppose that $T : A \to A$ is a nonexpansive mapping. If X has the normal structure, then T has a fixed point.*

The following best proximity point result is valid in uniformly convex Banach spaces.

Theorem 7 ([11])**.** *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. Then T has a best proximity point.*

Proof. By a similar manner of Theorem 6, there exists a minimal, closed and convex pair $(K_1, K_2) \subseteq (A, B)$ which is T-invariant and such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$. Note that minimality of (K_1, K_2) implies that (K_1, K_2) is a proximal pair. If $\delta(K_1, K_2) = \operatorname{dist}(K_1, K_2)$, then T has a best proximity point and we are finished. So, we assume that $\delta(K_1, K_2) > \operatorname{dist}(K_1, K_2)$. We now consider two following cases:

Case 1. $\min\{diam(K_1), diam(K_2)\} = 0$.
In this case, one of the sets K_1 or K_2 is singleton. Now, by the fact that (K_1, K_2) is proximal and that X is strictly convex, the other set must be singleton which is a contradiction.

Case 2. If $\min\{diam(K_1), diam(K_2)\} > 0$.
Suppose that T does not have a best proximity point. Let $(x^*, y^*) \in K_1 \times K_2$ be such that $\|x^* - y^*\| = \operatorname{dist}(A, B)$. Since T is cyclic relatively nonexpansive, we have

$$\|Tx^* - Ty^*\| \leq \|x^* - y^*\| = \operatorname{dist}(A, B).$$

Thus, we must have $x^* \neq Ty^*$ and $y^* \neq Tx^*$. It now follows from the strictly convexity of X that

$$\left\| \frac{x^* + Ty^*}{2} - \frac{y^* + Tx^*}{2} \right\| = \operatorname{dist}(A, B).$$

Set $R := \delta(K_1, K_2)$ and $r := \min\{\|x^* - Ty^*\|, \|y^* - Tx^*\|\}$. Thus $r \in [0, 2R]$. Now, for all $v \in K_2$ we have

$$\begin{cases} \|x^* - v\| \leq R, \\ \|Ty^* - v\| \leq R, \\ \|x^* - Ty^*\| \geq r. \end{cases}$$

By the fact that X is a uniformly convex Banach space, we conclude that

$$\|\frac{x^* + Ty^*}{2} - v\| \leq (1 - \delta(\frac{r}{R}))R, \quad \forall v \in K_2,$$

where, δ is a modulus of the convexity of X. Hence, $\delta_{\frac{x^*+Ty^*}{2}}(K_2) < R$. Similarly, we can see that $\delta_{\frac{Tx^*+y^*}{2}}(K_1) < R$. Therefore,

$$\max\{\delta_{\frac{x^*+Ty^*}{2}}(K_2), \delta_{\frac{Tx^*+y^*}{2}}(K_1)\} < \delta(K_1, K_2).$$

Now, by an equivalent argument of the proof of Theorem 5, we get a contradiction. □

Remark 1. We mention that the proof of Theorem 3.6 of [16] should be revised by a similar proof of Theorem 7.

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Doi: 10.18514/MMN.2015.1364

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 145–150

HU e-ISSN 1787-2413

ON ALGEBRAIC EXTENSIONS MODULO I

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Received 06 October, 2014

Abstract. Let I be a nonzero ideal of a ring T, let $\varphi : T \to E := T/I$ denote the canonical projection, let D be a ring contained in E, and let $R = \varphi^{-1}(D)$. The main purpose of this paper is to characterize when the ring extension $R \subset T$ is n- (resp., universally) algebraic modulo I in case I is an intersection of finitely many maximal ideals of T.

2010 *Mathematics Subject Classification:* 13A15; 13B25; 13C15

Keywords: integral domain, prime ideal, algebraic extension, algebraic extension modulo I, residually algebraic extension, pullback

1. Introduction

All rings considered below are commutative with identity but *not necessarily integral domains*. All subrings and inclusions of rings are (unital) ring extensions; all ring/algebra homomorphisms are unital. Let A be a ring and $n \geq 1$ be an integer. We denote by $A[n]$ the ring of polynomials in n indeterminates over A (for $n = 1$, $A[1] = A[X]$ is the ring of polynomials in one indeterminate). For convenience, we write $A = A[0]$.

Let I be a nonzero ideal of a ring T, $\varphi : T \to E := T/I$ the natural projection, and D a ring contained in E. Then $R = \varphi^{-1}(D)$ is the ring arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/I = E \end{array}$$

Following [4], we say that R is the ring of the (T, I, D) construction and we set $R := (T, I, D)$. We shall assume that D is properly contained in E (and hence, that R is properly contained in T), and we shall refer to this as a *pullback diagram of type* $(\Box)$. If I is an intersection of finitely many maximal ideals of T, we shall refer to this as a diagram $(\Box_\cap)$. A very good account of pullback constructions has been given in [4, 5] and [6]. It has fashionable in recent years to study rings via pullback diagrams. It is well worth noting that pullback constructions provide a rich source of examples and counterexamples in commutative algebra (see [1–5, 11, 12]). Unless

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otherwise specified, the symbols T, D, I, R have the above meaning throughout the paper.

In [8] the authors introduced the concept of n-algebraic extension modulo I for a diagram $(\Box)$ when T and D are integral domains and $n \geq 0$ is an integer. More precisely, the ring extension $R \subset T$ (of integral domains) is said to be *n-algebraic modulo I* if for every two prime ideals $Q' \subset Q$ of $T[n]$ such that $I[n] \not\subseteq Q'$, $I[n] \subseteq Q$ and $ht(Q \cap R[n]/Q' \cap R[n]) = 1$, then $R[n]/(Q \cap R[n]) \subseteq T[n]/Q$ is algebraic. This concept was first used to characterize when an integral domain R of the form $D + I$, (where I is a nonzero ideal of an integral domain T and D is a subring of T satisfying $D \cap I = (0)$) is a (stably) strong S-domain (cf. [8, Théorème 1.7]). In [2], the authors dealt with a more general situation and used this concept to characterize when a ring R arising from a diagram $(\Box)$ is a (stably) strong S-domain. The main purpose of this paper is to study n-algebraic extensions modulo I for a diagram $(\Box_\cap)$ in order to deepen our knowledge about such extensions. We first extend this notion to arbitrary commutative rings. Our motivation is an example constructed by Fontana et al (see [8, Exemple 1.8]) of a diagram $(\Box_\cap)$ in order to produce a ring extension $R \subset T$ which is 0-algebraic modulo I but not 1-algebraic modulo I. For this reason, M. Fontana et al (see [8]) have introduced the following definition: The ring extension $R \subset T$ is said to be *universally algebraic modulo I*, if $R \subset T$ is *n-algebraic modulo I* for each positive integer n. Our contribution (see Theorem 1) is to prove that for a diagram $(\Box_\cap)$, $R \subset T$ is n-algebraic modulo I if and only if $R \subset T$ is 1-algebraic modulo I if and only if $R \subset T$ is a residually algebraic extension. The key step (Lemma 1) is to show, for any diagram $(\Box)$, that if $R \subset T$ is n-algebraic modulo I (where $n \geq 1$), then $R \subset T$ is $(n-1)$-algebraic modulo I.

Throughout the paper, we use "$\subset$" to denote proper containment and "$\subseteq$" to denote containment. Transcendence degrees paly an important role in our study; if $A \subseteq B$ are two domains, we denote by $tr.deg[B : A]$ the transcendence degree of the quotient field of B over that of A. Any unexplained terminology is standard as in [9, 10]. Relevant terminology and results will be recalled as needed through the paper.

2. Main results

We extend Fontana-Izelgue-Kabbaj's definition, mentioned in the introduction, to arbitrary commutative rings in the following way:

Definition 1. Let $n \geq 0$ be an integer. For a diagram $(\Box)$, the extension $R \subset T$ is said to be *n-algebraic modulo I* if for every two prime ideals $Q' \subset Q$ of $T[n]$ such that $I[n] \not\subseteq Q'$, $I[n] \subseteq Q$ and $ht(Q \cap R[n]/Q' \cap R[n]) = 1$, then $R[n]/(Q \cap R[n]) \subseteq T[n]/Q$ is algebraic.

Definition 2. For a diagram $(\Box)$, the extension $R \subset T$ is said to be *universally algebraic modulo I* if $R \subset T$ is n-algebraic modulo I for each integer $n \geq 0$.

Recall that an extension of rings $A \subseteq B$ is said to be *residually algebraic* if for each prime ideal Q of B, the extension $A/(Q \cap A) \subseteq B/Q$ is algebraic. It is clear that if $R \subset T$ is a residually algebraic extension, then so is $R[n] \subset T[n]$ for any positive integer n (cf. [7, Lemme 1.4]). Hence $R \subset T$ is universally algebraic modulo I.

Recall from [10, Section 1-5] that if p is a prime ideal of a ring A, and Q is a prime ideal of $A[X]$ with $Q \cap A = p$, but with $Q \neq p[X]$, then we call Q *an upper* to p in $A[X]$ (or more simply, an upper to p, or just an upper).

The main result of this paper is the following theorem which identifies n-algebraic extensions modulo I for a diagram $(\square_{\cap})$. We assume that all rings are finite-dimensional.

Theorem 1. *Let $n \geq 1$ be an integer. For a diagram $(\square_{\cap})$, consider the following statements:*

(1) *$R \subset T$ is 1-algebraic modulo I.*
(2) *$tr.deg[T/M : R/(M \cap R)] = 0$ for each maximal ideal M of T containing I.*
(3) *$R \subset T$ is a residually algebraic extension.*
(4) *$R \subset T$ is universally algebraic modulo I.*
(5) *$R \subset T$ is n-algebraic modulo I.*
(6) *$R \subset T$ is 0-algebraic modulo I.*

Then:

(a) *In general,* $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6)$.
(b) *If, in addition, $I \in Max(T)$, then the above statements $(1)-(6)$ are equivalent.*

To prove the implications (5)⇒(1) and (5)⇒(6) in Theorem 1, we need the following lemma.

Lemma 1. *Let $n \geq 1$ be an integer. For a diagram $(\square)$, if $R \subset T$ is n-algebraic modulo I, then $R \subset T$ is $(n-1)$-algebraic modulo I.*

Proof. Let $Q' \subset Q$ be two prime ideals of $T[n-1]$ such that $I[n-1] \not\subseteq Q'$ and $I[n-1] \subseteq Q$. Set $P' = Q' \cap R[n-1]$, $P = Q \cap R[n-1]$ and suppose that $P' \subset P$ are consecutive. Our task is to show that $R[n-1]/P \subseteq T[n-1]/Q$ is an algebraic extension. Let $\mathcal{Q}' = Q' + X_n T[n-1][X_n]$ and $\mathcal{Q} = Q + X_n T[n-1][X_n]$. It is obvious that $\mathcal{Q}'$ respectively $\mathcal{Q}$ are uppers to Q' respectively Q. Set $\mathcal{P}' = \mathcal{Q}' \cap R[n]$ and $\mathcal{P} = \mathcal{Q} \cap R[n]$. One can check easily that $\mathcal{P}' = P' + X_n R[n]$ and $\mathcal{P} = P + X_n R[n]$. As $X_n R[n] \subseteq \mathcal{P}' \subset \mathcal{P}$, then $\mathcal{P}' \subset \mathcal{P}$ are consecutive. On the other hand, since $R \subset T$ is n-algebraic modulo I, then $tr.deg[T[n]/\mathcal{Q} : R[n]/\mathcal{P}] = 0$. As $T[n]/\mathcal{Q} \cong T[n-1]/Q$ and $R[n]/\mathcal{P} \cong R[n-1]/P$, it follows that $tr.deg[T[n-1]/Q : R[n-1]/P] = 0$, as desired. □

Before proceeding to the proof of Theorem 1 it is convenient to recall the following Cahen's lemma [4, Proposition 4]. We shall make use of this result in the proof of

Theorem 1. Note that this lemma holds even for polynomial rings since if $R := (T, I, D)$, then $R[n] := (T[n], I[n], D[n])$.

Lemma 2. *For a diagram* ($\Box$)*, if $P_0 \subset \ldots \subset P_n$ is a chain of primes in R such that P_n is minimal among primes of R containing I and P_{n-1}, then this chain lifts in T.*

We now prove Theorem 1.

Proof of Theorem 1. (a) (1)⇒(2) Let Ω be the finite subset of $Max(T)$ such that $I = \bigcap_{M\in\Omega} M$. We discuss the following two cases.
Case 1. $\mid \Omega \mid \geq 2$. Since $M + \bigcap_{M'\in\Omega\setminus\{M\}} M' = T$, then there exist $u \in \bigcap_{M'\in\Omega\setminus\{M\}} M'$ and $v \in M$ such that $u + v = 1$. Let $P_1' = ((X-u)T[X]) \cap R[X]$ and $P_1 = (M[X] + (X-u)T[X]) \cap R[X]$. The prime ideals $P_1' \subset P_1$ are not necessarily consecutive. Since $T[X]$ is finite-dimensional, there exist two prime ideals P' and P of $T[X]$ such that P' is maximal among the primes such that $P_1' \subseteq P' \subset P_1$ and not containing I, and P is minimal such that $P_1' \subseteq P' \subset P \subseteq P_1$. Therefore P' does not contain I, P contains I and $P' \subset P$ are consecutive. The chain $P_1' \subseteq P' \subset P$ lifts in $T[X]$ as $Q_1' \subseteq Q' \subset Q$. Notice that $Q_1' = (X-u)T[X]$ because P_1' does not contain I and so it lifts uniquely in $T[X]$. Hence Q contains $X-u$ and I. The prime ideal Q cannot contain any prime containing u (if so, it would contain X, thus $X \in P_1$ and hence $u \in M$, which is absurd). Consequently Q is above M. Furthermore Q is an upper to M because $X-u \in Q \setminus M[X]$. The prime ideal P is above $p = M \cap R$. Next, we demonstrate that P is an upper to p. Consider the polynomial $f = (X-u)(X-v) = X^2 - X + uv$. Since $uv \in I$, then clearly f belongs to $P_1' = ((X-u)T[X]) \cap R[X]$. Thus $f \in P$. As $f \notin p[X]$, we deduce that P is an upper to p. As $R \subset T$ is 1-algebraic modulo I, it follows that $T[X]/Q$ is algebraic over $R[X]/P$. Since Q and P are uppers respectively to M and p, we deduce that T/M is algebraic over R/p.
Case 2. $\mid \Omega \mid = 1$. In this case $I = M$, where M is a maximal ideal of T. The proof in this case proceeds along the same lines as in the proof of Case 1 with some modifications. Set $P_1' = ((X-1)T[X]) \cap R[X]$ and $P_1 = (M[X] + (X-1)T[X]) \cap R[X]$. These prime ideals are not necessarily consecutive, so let P' be maximal among the primes such that $P_1' \subseteq P' \subset P_1$ and not containing I, and P be minimal such that $P_1' \subseteq P' \subset P \subseteq P_1$. Therefore P' does not contain I, P contains I, $P' \subset P$ are consecutive and the chain $P_1' \subseteq P' \subset P$ lifts in $T[X]$ as $Q_1' = (X-1)T[X] \subseteq Q' \subset Q$. It is clear that $Q \cap T$ contains I, and as I is a maximal ideal of T, then $Q \cap T = M$. Moreover, since Q contains $X-1$, then Q is an upper to M. The prime ideal P is above $p = M \cap R$. We claim that P is an upper to p. Consider the polynomial $f = (X-1)^2 = X^2 - 2X + 1$. It is obvious that $f \in P_1' = ((X-1)T[X]) \cap R[X]$ and $f \notin p[X]$. Hence $f \in P \setminus p[X]$. Therefore P is an upper to p as claimed. Since $R \subset T$ is 1-algebraic modulo I, it results that $T[X]/Q$ is algebraic over $R[X]/P$. As Q and P are uppers respectively to M and p, it follows that T/M is algebraic

over R/p.
(2)⇒(3) Let $q \in Spec(T)$. Our purpose is to show that $R/(q \cap R) \subseteq T/q$ is an algebraic extension. If $I \not\subseteq q$, then $T_q \simeq R_{q\cap R}$ (see [4, Proposition 0]). So $tr.deg[T/q : R/(q \cap R)] = 0$. If $I \subseteq q$, then $q \in \Omega$. Hence $tr.deg[T/q : R/(q \cap R)] = 0$.
(3)⇒(4)⇒(5) are trivial.
(5)⇒(1) The conclusion is clear if $n = 1$. So assume that $n \geq 2$. The conclusion follows readily from Lemma 1.
(5)⇒(6) Follows readily from Lemma 1.
(b) We now assume that $I \in Max(T)$. We will prove that (6)⇒(2). To this end, we have only to show that $tr.deg[T/I : R/I] = 0$. Let q' be a prime ideal of T such that $q' \subset I$ are consecutive in T (such ideal exists since T is finite-dimensional). Let $p' = q' \cap R$, then $p' \subset I$ are also consecutive in R. Indeed, assume that there exists a prime ideal p of R such that $p' \subset p \subset I$. This chain lifts in T to $q' \subset q \subset I$ (notice that the unique prime ideal of T lying over I is I itself since $I \in Max(T)$). The desired contradiction since $q' \subset I$ are consecutive. As $R \subset T$ is 0-algebraic modulo I, then $tr.deg[T/I : R/I] = 0$, as asserted. □

Remark 1. If we leave out the assumption "$I \in Max(T)$" in the statement of Theorem 1 (b), the conclusion does not hold. More precisely, Fontana et al (see [8, Exemple 1.8]) have already constructed a diagram ($\square_\cap$), where I is an intersection of two maximal ideals of T, such that $R \subset T$ is 0-algebraic modulo I, whereas $R \subset T$ is not 1-algebraic modulo I.

ACKNOWLEDGEMENT

The authors would like to thank the referee for many valuable suggestions.

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Doi: 10.18514/MMN.2015.740

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 151–164

HU e-ISSN 1787-2413

CONTINUITY FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR ON BESOV SPACES

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Received 24 September, 2013

Abstract. In this paper we prove the continuity for the multilinear commutator associated to the Marcinkiewicz operator on Besov spaces.

2010 *Mathematics Subject Classification:* 42B20; 42B25

Keywords: multilinear commutator, Marcinkiewicz operator, Besov space

1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–4, 14]). From [2, 4, 9, 13], we know that the commutators and multilinear operators generated by singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce the multilinear commutator associated to the Marcinkiewicz operator and prove the continuity properties for the multilinear commutator on Besov spaces.

2. Preliminaries and Theorems

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [14, 15])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

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For $\beta \geq 0$, the Besov space $\dot{\wedge}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\wedge}_\beta} = \sup_{\substack{x,h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [12]).

For $b_j \in \dot{\wedge}_\beta(R^n)$ $(j = 1, \cdots, m)$, set

$$||\vec{b}||_{\dot{\wedge}_\beta} = \prod_{j=1}^{m} ||b_j||_{\dot{\wedge}_\beta}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\dot{\wedge}_\beta} = ||b_{\sigma(1)}||_{\dot{\wedge}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\wedge}_\beta}$.

Definition 1. Let $0 < p, q \leq \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,\, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,\, p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,\, p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,\, p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,\, p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,\, p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \leq q < \infty$, $\alpha \in R$. The central Campanato space is defined by (see [17])

$$CL_{\alpha,\, q}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{CL_{\alpha,\, q}} < \infty\},$$

where

$$||f||_{CL_{\alpha,\, q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{1/q}.$$

Definition 3. Let $0<\delta<n$, $0<\gamma\le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$. Assume that $\Omega\in Lip_\gamma(S^{n-1})$, that is there exists a constant $M>0$ such that for any $x,y\in S^{n-1}$, $|\Omega(x)-\Omega(y)|\le M|x-y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu^{\vec{b}}_{\Omega,\delta}(f)(x)=\left(\int_0^\infty |F^{\vec{b}}_{t,\delta}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

where

$$F^{\vec{b}}_{t,\delta}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}\left[\prod_{j=1}^m(b_j(x)-b_j(y))\right]f(y)dy.$$

Set

$$F_{t,\delta}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}f(y)dy,$$

we also define that

$$\mu_{\Omega,\delta}(f)(x)=\left(\int_0^\infty |F_{t,\delta}(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [16]).

Let H be the space $H=\left\{h:||h||=\left(\int_0^\infty|h(t)|^2\frac{dt}{t^3}\right)^{1/2}<\infty\right\}$. Then, it is clear that

$$\mu_{\Omega,\delta}(f)(x)=||F_{t,\delta}(f)(x)|| \text{ and } \mu^{\vec{b}}_{\Omega,\delta}(f)(x)=||F^{\vec{b}}_{t,\delta}(f)(x)||.$$

Note that when $b_1=\cdots=b_m$, $\mu^{\vec{b}}_{\Omega,\delta}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–9, 11–13]). Our main purpose is to study the boundedness properties for the multilinear commutator $\mu^{\vec{b}}_{\Omega,\delta}$ on Besov spaces.

Now we state our theorems as following.

Theorem 1. *Let* $0<\delta<n$, $1<r<n/\delta$, $1/s=1/r-\delta/n$, $0<\beta<\min(1/2m,\gamma/m)$ *for* $0<\gamma\le 1$, *and* $b_j\in\dot{\wedge}_\beta(R^n)$ *for* $j=1,\cdots,m$. *Then* $\mu^{\vec{b}}_{\Omega,\delta}$ *is bounded from* $L^p(R^n)$ *to* $\dot{\wedge}_{\delta+m\beta-n/p}(R^n)$ *for any* $n/(\delta+m\beta)\le p\le n/\delta$.

Theorem 2. *Let* $0<\delta<n$, $0<\beta<\min(1/2m,\gamma/m)$ *for* $0<\gamma\le 1$, $1<q_1<n/(\delta+m\beta)$, $1/q_2=1/q_1-(\delta+m\beta)/n$, $max(-n/q_2-1/2,-n/q_2-\gamma)<\alpha\le -n/q_2$ *and* $b_j\in\dot{\wedge}_\beta(R^n)$ *for* $j=1,\cdots,m$. *Then* $\mu^{\vec{b}}_{\Omega,\delta}$ *is bounded from* $\dot{K}^{\alpha,\infty}_{q_1}(R^n)$ *to* $CL_{-\alpha/n-1/q_2,q_2}(R^n)$.

Remark 1. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1 ([12]). *For $0<\beta<1, 1\le p\le\infty$, we have*

$$\|b\|_{\dot\wedge_\beta}\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-b_Q|dx\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|b(x)-b_Q|^p dx\right)^{1/p}$$
$$\approx\sup_Q\inf_c\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-c|dx\approx\sup_Q\inf_c\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q|b(x)-c|^p dx\right)^{1/p}.$$

Lemma 2 ([10]). *For $\alpha<0, 0<q<\infty$, we have*

$$\|f\|_{\dot K_q^{\alpha,\infty}}\approx\sup_{\mu\in Z}2^{\mu\alpha}\|f\chi_{B_\mu}\|_{L^q}.$$

Lemma 3 ([10]). *Let $0<\eta<n$, $1<p<n/\eta$. Suppose $b\in\dot\wedge_\beta(R^n)$, then*

$$|b_{2^{k+1}B}-b_B|\le C\|b\|_{\dot\wedge_\beta}k|2^{k+1}B|^{\beta/n}\ \text{for } k\ge1.$$

Lemma 4 ([16]). *Let $0<\delta<n$, $1<p<n/\delta$ and $1/q=1/p-\delta/n$. Then $\mu_{\Omega,\delta}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Lemma 5 ([7]). *Let $0\le\eta<n$, $1<r<n/\eta$, $1/r-1/s=\eta/n$ and $b_j\in\dot\wedge_\beta(R^n)$ for $j=1,\cdots,m$. Then $\mu^{\vec b}_{\Omega,\delta}$ is bounded from $L^r(R^n)$ to $L^s(R^n)$.*

Proof of Theorem 1. It is only to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|\mu^{\vec b}_{\Omega,\delta}(f)(x)-C_0|dx\le C\|f\|_{L^p}.$$

Fix a cube Q, $Q=Q(x_0,d)$, we decompose f into $f=f_1+f_2$ with $f_1=f\chi_Q$, $f_2=f\chi_{(R^n\setminus Q)}$.

When $m=1$, for $C_0=\mu_{\Omega,\delta}(((b_1)_Q-b_1)f_2)(x_0)$, we have

$$F^{b_1}_{t,\delta}(f)(x)=(b_1(x)-(b_1)_Q)F_{t,\delta}(f)(x)$$
$$-F_{t,\delta}((b_1-(b_1)_Q)f_1)(x)-F_{t,\delta}((b_1-(b_1)_Q)f_2)(x).$$

Then

$$\begin{aligned}&|\mu^{b_1}_{\Omega,\delta}(f)(x)-\mu_{\Omega,\delta}(((b_1)_Q-b_1)f_2)(x_0)|\\&=\Big|\|F^{b_1}_{t,\delta}(f)(x)\|-\|F_{t,\delta}(((b_1)_Q-b_1)f_2)(x_0)\|\Big|\\&\le\|(b_1(x)-(b_1)_Q)F_{t,\delta}(f)(x)\|+\|F_{t,\delta}((b_1-(b_1)_Q)f_1)(x)\|\\&\quad+\|F_{t,\delta}((b_1-(b_1)_Q)f_2)(x)-F_{t,\delta}((b_1-(b_1)_Q)f_2)(x_0)\|\\&=A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent$1/q + 1/q' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q |A(x)|dx\\
&\le \frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\left(\int_Q |b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\left(\int_Q |\mu_{\Omega,\delta}(f)(x)|^q dx\right)^{1/q}\\
&\le C\frac{|Q|^{\beta/n+1/q'}}{|Q|^{1+(\delta+\beta)/n-1/p}}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\left(\int_Q |f(x)|^p dx\right)^{1/p}\\
&\le C||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}.
\end{aligned}$$

For $B(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Hölder's inequality with exponent $1/t + 1/t' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q |B(x)|dx\\
&\le \frac{1}{|Q|^{(\delta+\beta)/n-1/p}}\left(\frac{1}{|Q|}\int_{R^n} |\mu_{\Omega,\delta}((b_1(x)-(b_1)_Q)f\chi_Q)(x)|^s dx\right)^{1/s}\\
&\le C\frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}}\left(\int_Q |(b_1(x)-(b_1)_Q)f(x)|^r dx\right)^{1/r}\\
&\le C\frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}}\left(\int_Q |b_1(x)-(b_1)_Q|^{rt'}dx\right)^{1/rt'}\left(\int_Q |f(x)|^{rt}dx\right)^{1/rt}\\
&\le C||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}.
\end{aligned}$$

For $C(x)$, note that $|x_0 - y| \approx |x - y|$ for $y \in Q^c$, we have

$$\begin{aligned}
C(x) &= ||F_{t,\delta}((b_1-(b_1)_Q)f_2)(x) - F_{t,\delta}((b_1-(b_1)_Q)f_2)(x_0)||\\
&= \left(\int_0^\infty \left|\int_{|x-y|\le t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1-\delta}}(b_1(y)-(b_1)_Q)dy\right.\right.\\
&\quad \left.\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1-\delta}}(b_1(y)-(b_1)_Q)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\le \left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}|(b_1(y)-(b_1)_Q)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad +\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|b_1(y)-(b_1)_Q\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad +\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right|\right.\right.
\end{aligned}$$

$$\times |b_1(y)-(b_1)_Q||f_2(y)|dy]^2\frac{dt}{t^3})^{1/2}$$
$$\equiv I_1+I_2+I_3.$$

By the Minkowski's inequality and Hölder's inequality and Lemmas 1, 3:

$$\begin{aligned}
I_1 &\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}||x_0-y|^{\delta}\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\
&\quad\times\left[\int_{2^{k+1}Q}(|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}+|(b_1)_{2^{k+1}Q}-(b_1)_Q|^{p'})dy\right]^{1/p'}\\
&\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\
&\quad\times\left[|2^{k+1}Q|^{\beta/n+1/p'}\frac{1}{|2^{k+1}Q|^{\beta/n}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}dy\right)^{1/p'}\right.\\
&\quad\left.+|(b_1)_{2^{k+1}Q}-(b_1)_Q||2^{k+1}Q|^{1/p'}\right]\\
&\le C\sum_{k=0}^{\infty}k2^{k(-1/2+\delta+\beta-n/p)}|Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot\Lambda_\beta}||f||_{L^p}\\
&\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.
\end{aligned}$$

Similarly, we have $I_2\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot\Lambda_\beta}||f||_{L^p}$.
We now estimate I_3. By the following inequality:

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\le C\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\right),$$

we obtain

$$\begin{aligned}
I_3 &\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)||x-x_0|}{|x_0-y|^{n-\delta}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\quad+C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)||x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_Q|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy
\end{aligned}$$

$$\le C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|b_1(y)-(b_1)_Q||f(y)|dy$$
$$\le C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}$$
$$\times\left[|2^{k+1}Q|^{\beta/n+1/p'}\frac{1}{|2^{k+1}Q|^{\beta/n}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}dy\right)^{1/p'}\right.$$
$$\left.+|(b_1)_{2^{k+1}Q}-(b_1)_Q||2^{k+1}Q|^{1/p'}\right]$$
$$\le C\sum_{k=0}^{\infty}k(2^{k(-1+\delta+\beta-n/p)}+2^{k(-\gamma+\delta+\beta-n/p)})|Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot\Lambda_\beta}||f||_{L^p}$$
$$\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.$$

Thus

$$\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|C(x)|dx$$
$$\le C||b_1||_{\dot\Lambda_\beta}||f||_{L^p}\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|Q|^{(\delta+\beta)/n-1/p}dx\le C||b_1||_{\dot\Lambda_\beta}||f||_{L^p}.$$

This completes the proof of case $m=1$.

Now, we consider the case $m\ge 2$. We write, for $\vec{b}=(b_1,...,b_m)$,

$$F^{\vec{b}}_{t,\delta}(f)(x)=\int_{|x-y|\le t}\left[\prod_{j=1}^{m}((b_j(x)-(b_j)_Q)-(b_j(y)-(b_j)_Q))\right]f(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$
$$=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_Q)_\sigma\int_{|x-y|\le t}(b(y)-(b)_Q)_{\sigma^c}f(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$
$$=\prod_{j=1}^{m}(b_j(x)-(b_j)_Q F_{t,\delta}(f)(x)+(-1)^m F_{t,\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)f)(x)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_Q)_\sigma F^{\vec{b}_{\sigma^c}}_{t,\delta}(f)(x),$$

thus, set $C_0=\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)$,

$$|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|$$

$$\begin{aligned}
&\le ||\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)F_{t,\delta}(f)(x)|| + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_Q)_\sigma F_{t,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|| \\
&\quad + ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_1)(x)|| + ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x) \\
&\quad - F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|| \\
&= S_1(x)+S_2(x)+S_3(x)+S_4(x).
\end{aligned}$$

For $S_1(x)$, for $1<p<q<n/\delta$, $1/q=1/p-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent$1/q_1'+\cdots+1/q_m'+1/q=1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |S_1(x)|dx \\
&\le C\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\prod_{j=1}^{m}\left(\int_Q |b_j(x)-(b_j)_Q|^{q_j'}dx\right)^{1/q_j'}\left(\int_Q |\mu_{\Omega,\delta}(f_1)|^q dx\right)^{1/q} \\
&\le C\frac{|Q|^{m\beta/n+1/q_1'+\cdots q_m'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}\prod_{j=1}^{m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q_j'}dx\right)^{1/q_j'} \\
&\quad \times\left(\int_Q |f(x)|^p dx\right)^{1/p} \\
&\le C||\vec{b}||_{\dot{\wedge}_\beta}||f||_{L^p}.
\end{aligned}$$

For $S_2(x)$, denoting $p=rt$, $1<r<s<n/\delta$, $1/s=1/r-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |S_2(x)|dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\left(\int_Q |b(x)-(b_Q)_\sigma|^{s'}dx\right)^{1/s'} \\
&\quad \times\left(\int_Q \mu_{\Omega,\delta}((b-b_Q)_{\sigma^c}f)(x)|^s dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}\frac{1}{|Q|^{|\sigma|\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x)-(b_Q)_\sigma|^{s'}dx\right)^{1/s'}
\end{aligned}$$

$$\times\left(\int_Q |(b(x)-b_Q)_{\sigma^c} f(x)|^r dx\right)^{1/r}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|Q|^{m\beta/n+1/s'+1/rt'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}||b_\sigma||_{\dot\Lambda_\beta}||b_{\sigma^c}||_{\dot\Lambda_\beta}||f||_{L^p}$$

$$\le C||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.$$

For $S_3(x)$, for $1<r<s<n/\delta$, $1/s=1/r-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, taking $p=rt$, $1<r<p<\infty$, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |S_3(x)|dx$$

$$\le \frac{1}{|Q|^{(\delta+m\beta)/n-1/p}}\left(\frac{1}{|Q|}\int_{R^n}|\mu_{\Omega,\delta}(\prod_{j=1}^m((b_j-(b_j)_Q))f\chi_B)(x)|^s dx\right)^{1/s}$$

$$\le C\frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}}\left(\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_Q)f(x)|^r dx\right)^{1/r}$$

$$\le C\frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}}\prod_{j=1}^m\left(\int_Q|b_j(x)-(b_j)_Q|^{rt_j}dx\right)^{1/rt_j}\left(\int_Q|f(x)|^{rt}dx\right)^{1/rt}$$

$$\le C||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.$$

For $S_4(x)$, similar to the proof of $C(x)$ in the case $m=1$, we get

$$S_4(x)\le\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$

$$+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$

$$+(\int_0^\infty[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right|$$

$$\times\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right||f_2(y)|dy]^2\frac{dt}{t^3})^{1/2}$$

$$\equiv V_1+V_2+V_3,$$

Thus, we choose $1<p_j<\infty$, $j=1,\cdots,m$, $1/p_1+\cdots+1/p_m+1/p=1$ and get

$$V_1\le C\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right||x_0-y|^\delta\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy$$

$$\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\prod_{j=1}^{m}\Big[|2^{k+1}Q|^{m\beta/n+1/p'}||b_j||_{\dot\Lambda_\beta}$$
$$+k|2^{k+1}Q|^{m\beta/n+1/p'}||b_j||_{\dot\Lambda_\beta}\Big]||f||_{L^p}$$
$$\le C\sum_{k=0}^{\infty}k2^{k(-1/2+\delta+m\beta-n/p)}|Q|^{m\beta/n+1/p'-1+\delta/n}||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}$$
$$\le C|Q|^{(\delta+m\beta)/n-1/p}||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.$$

Similarly, we have $V_2\le C|Q|^{(\delta+m\beta)/n-1/p}||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}$. For V_3, we obtain

$$V_3\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\le C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right||f(y)|dy$$
$$\le C\sum_{k=0}^{\infty}k(2^{k(-1+\delta+m\beta-n/p)}+2^{k(-\gamma+\delta+m\beta-n/p)})$$
$$|2^{k+1}Q|^{m\beta/n+1/p'-1+\delta/n}||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}$$
$$\le C|Q|^{(\delta+m\beta)/n-1/p}||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.$$

Thus

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|S_4(x)|dx$$
$$\le C||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|Q|^{(\delta+m\beta)/n-1/p}dx\le C||\vec{b}||_{\dot\Lambda_\beta}||f||_{L^p}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Fix a ball $B=B(0,l)$, there exists $\epsilon_0\in\mathbf{Z}$ such that $2^{\epsilon_0-1}\le l<2^{\epsilon_0}$. We choose x_0 such that $2l<|x_0|<3l$. It is only to prove that

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\le C||f||_{\dot K^{\alpha,\infty}_{q_1}}.$$

We write, for $f_1=f\chi_{4B_{\epsilon_0}}$ and $f_2=f\chi_{R^n\setminus 4B_{\epsilon_0}}$, then

$$|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|\le|\mu^{\vec{b}}_{\Omega,\delta}(f_1)(x)|+|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|.$$

So

$$
\begin{aligned}
&2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&\le 2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f_1)(x)|^{q_2}dx\right)^{1/q_2}\\
&\quad+2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&= J_1+J_2.
\end{aligned}
$$

For J_1, by the (L^{q_1},L^{q_2})-boundedness of $\mu^{\vec{b}}_{\Omega,\delta}$ (see Lemma 4) and Lemma 2, we get

$$
\begin{aligned}
J_1 &\le C2^{\epsilon_0(\alpha+n/q_2)}2^{-\epsilon_0 n/q_2}\left(\int_{R^n}|f_1(x)|^{q_1}dx\right)^{1/q_1}\\
&\le C2^{\epsilon_0\alpha}||f\chi_{B_{\epsilon_0}}||_{L^{q_1}}\le C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}
$$

For J_2, similar to the estimates of Theorem 1, we obtain using Hölder's inequality and recalling that $\max(-n/q_2-1/2,-n/q_2-\gamma)<\alpha\le -n/q_2$, $1/q_2=1/q_1-(\delta+m\beta)/n$,

$$
\begin{aligned}
&|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|\\
&\le|\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)(f_2)(x)-\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_B)(f_2)(x_0)|\\
&\quad+|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)||\mu_{\Omega,\delta}(f_2)(x)-\mu_{\Omega,\delta}(f_2)(x_0)|=W_1(x)+W_2(x).
\end{aligned}
$$

For $W_1(x)$, similar to the proof of $S_4(x)$ in Theorem 1, set $1/v_1+\cdots+1/v_m+1/q_1=1$, by the Minkowski's inequality, we have

$$W_1(x)\le V_1+V_2+V_3.$$

For V_1, we have

$$
\begin{aligned}
V_1&\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2-\delta}}dy\\
&\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}}|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)|dy
\end{aligned}
$$

$$\leq C\sum_{k=1}^{\infty}2^{k(\delta+m\beta-n/q_1-\alpha-1/2)}2^{\epsilon_0(\delta+m\beta-n/q_1-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\leq C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

Similarly, we have $V_2 \leq C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$.
For V_3, we obtain

$$V_3 \leq C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}}+\frac{2^{\epsilon_0\gamma}}{2^{(\epsilon_0+k)(n+\gamma-\delta)}}\right)|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}(2^{k(m\beta+n-n/q_1-n+\delta-1)}+2^{k(m\beta+n-n/q_1-n+\delta-\gamma)})$$
$$\times 2^{\epsilon_0(m\beta+n-n/q_1-n+\delta)}||\vec{b}||_{\dot{\wedge}_\beta}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\leq C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

For $W_2(x)$, we have,

$$||F_{t,\delta}(f_2)(x)-F_{t,\delta}(f_2)(x_0)||$$
$$\leq\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\leq t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|\leq t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$=V_1'+V_2'+V_3'.$$

For V_3', similar to the proof of I_3 in Theorem 1, we get

$$V_3' \leq C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\leq C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}}+\frac{2^{\epsilon_0\gamma}}{2^{(\epsilon_0+k)(n+\gamma-\delta)}}\right)2^{(\epsilon_0+k)(n-n/q_1)}$$
$$\left(\int_{B_{\epsilon_0+k}}|f(y)|^{q_1}dy\right)^{1/q_1}$$

$$\le C\sum_{k=1}^{\infty}(2^{k(\delta-n/q_1-\alpha-1)}+2^{k(\delta-n/q_1-\alpha-\gamma)})2^{\epsilon_0(\delta-n/q_1-\alpha)}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\le C2^{\epsilon_0(\delta-\alpha-n/q_1)}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

The proofs of V_1', V_2' are similar to that of V_3', we omit the details. Thus, by Hölder's inequality with $1/v_1+\cdots+1/v_m=1$, we have

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|W_2(x)|^{q_2}dx\right)^{1/q_2}$$
$$\le C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|^{q_2}dx\right)^{1/q_2}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$
$$\le C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}2^{\epsilon_0(m\beta+n/q_2(1/v_1+\cdots+1/v_m))}$$
$$\times\left(\frac{1}{2^{\epsilon_0 n}}\frac{1}{|B|^{m\beta/n}}\prod_{j=1}^{m}\left(\frac{1}{|B|}\int_{|x|<2^{\epsilon_0}}|(b_j(x)-(b_j)_B)|^{q_2v_j}dx\right)^{1/q_2v_j}\right)||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$$
$$\le C||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

Thus

$$J_2\le C||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

This completes the proof of Theorem 2. □

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the referee for his comments and suggestions. The Project supported by Hunan Provincial Natural Science Foundation of China (12JJ6003) and Scientific Research Fund of Hunan Provincial Education Departments (12K017).

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Doi: 10.18514/MMN.2015.766

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 165–180

HU e-ISSN 1787-2413

SOME ASPECTS OF $L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$

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Received 24 September, 2013

Abstract. Let $1 \leq q, p < \infty$ and v, w be Beurling's weight functions on $\mathbb{R}^d$. In this article we deal with harmonic properties of intersection space $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right) = L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$ defined by aid of weighted Lebesgue space $L_v^q\left(\mathbb{R}^d\right)$ and weighted Sobolev space $W_k^{p,w}\left(\mathbb{R}^d\right)$. We research the inclusions and inequalities between the spaces $A_{k,v,w}^{q,p}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ be an open set. Finally, we proved that the spaces $M\left(A_{k,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right)$ can be identified with the weighted spaces of bounded measures $M_w\left(\mathbb{R}^d\right)$.

2010 *Mathematics Subject Classification:* 46E35; 46J10; 46H25; 43A22

Keywords: weighted Sobolev space, Banach module, Banach algebra, approximate identity, continuous embedding, multiplier

1. Introduction and preliminaries

Throughout this work, $\mathbb{R}^d$ denote $d-$dimensional real Euclidean space with Lebesgue measure dx. We use Beurling's weight function, i.e., a measurable, locally bounded function on $\mathbb{R}^d$ satisfying $\omega(x) \geq 1$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^d$ [2]. We denote weighted Lebesgue space $L_\omega^p\left(\mathbb{R}^d\right) = \left\{f \mid f\omega \in L^p\left(\mathbb{R}^d\right)\right\}$ which is a Banach space under the norm

$$\|f\|_{p,\omega} = \int_{\mathbb{R}^d} |f(x)|^p \omega^p(x)\, dx.$$

Some well-known terms such as convolution, translation invariant, continuous embeddings, Banach algebra, Banach module, essential Banach ideal, approximate identity will be used frequently through this paper; their definitions may be found in [6], [13],[15], [22]. It is known that $L_\omega^p\left(\mathbb{R}^d\right)$ is translation invariant and

$$\|L_s f\|_{p,\omega} \leq \omega(s)\|f\|_{p,\omega} \tag{1.1}$$

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for any $f \in L_\omega^p\left(\mathbb{R}^d\right)$. The translation operator L_s ($L_s f(x) = f(x-s)$) is continuous on $L_\omega^p\left(\mathbb{R}^d\right)$. For two weight functions ω_1 and ω_2, we write $\omega_1 \prec \omega_2$ if and only if there exists a constant $c > 0$ such that $\omega_1(x) \leq c\omega_2(x)$ for all $x \in \mathbb{R}^d$. We write $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$. Recall that one has $L_{\omega_1}^p\left(\mathbb{R}^d\right) \subset L_{\omega_2}^p\left(\mathbb{R}^d\right)$ if and only if $\omega_2 \prec \omega_1$. The space $L_\omega^p\left(\mathbb{R}^d\right)$ is a Banach module over $L_\omega^1\left(\mathbb{R}^d\right)$ under the convolution [8].

If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{R}^d$ is an d-tuple of nonnegative integers α_i, then it is written $\alpha \in \mathbb{Z}_+^d$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Similarly if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq d$, then

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_d^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}}$$

denotes a differential operator of order α. For given two locally integrable functions u and v on $\mathbb{R}^d$, we say that v is α^{th}-weak derivative of u, written $D^\alpha u = v$, provided

$$\int_{\mathbb{R}^d} u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x) \varphi(x) dx$$

for all $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$, where $C_c^\infty\left(\mathbb{R}^d\right)$ is the space of all infinitely differentiable functions on $\mathbb{R}^d$, each with compact support. It is known that a weak derivative, if it exists, is uniquely defined up to a set of measure zero and also it is linear [19].

Let w be Beurling's weight function. For any nonnegative integer k and $1 \leq p < \infty$, the weighted Sobolev space $W_k^{p,w}\left(\mathbb{R}^d\right)$ is defined as the space of the functions $u \in L_w^p\left(\mathbb{R}^d\right)$ such that $D^\alpha u$ exists and $D^\alpha u \in L_w^p\left(\mathbb{R}^d\right)$ for all $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \leq k$. $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a Banach space with the norm

$$\|u\|_{W_k^{p,w}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p,w} \text{ [12],[21].}$$

Weighted Sobolev spaces are defined by aid of weighted Lebesgue space $L_w^p\left(\mathbb{R}^d\right)$ by Kufner in 1980s. Clearly, $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a subspace of $L_w^p\left(\mathbb{R}^d\right)$ and also $W_0^{p,w}\left(\mathbb{R}^d\right) = L_w^p\left(\mathbb{R}^d\right)$. For any k, it is obvious the embedding $W_k^{p,w}\left(\mathbb{R}^d\right) \hookrightarrow L_w^p\left(\mathbb{R}^d\right)$. If $\omega = 1$, $W_k^{p,w}\left(\mathbb{R}^d\right) = W_k^p\left(\mathbb{R}^d\right)$. If we take norm $\|.\|_{p,w}$ instead of $\|.\|_p$, we get the following properties by using the method in [10] and [12]. If $w_2 \prec$

w_1 and $k > l$, then $W_k^{p,w_1}(\Omega) \hookrightarrow W_l^{p,w_2}(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^d$. $W_k^{p,w}\left(\mathbb{R}^d\right)$ is translation invariant and

$$\|L_s f\|_{W_k^{p,w}} \leq \omega(s) \|f\|_{W_k^{p,w}} \tag{1.2}$$

for any $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. The translation operator is continuous on $W_k^{p,w}\left(\mathbb{R}^d\right)$. Also $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a Banach module over $L_\omega^1\left(\mathbb{R}^d\right)$ under the convolution.

Sobolev spaces $W_k^p\left(\mathbb{R}^d\right)$ of integer order were introduced by S.L. Sobolev in [17], [18]. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^d$ by using subspaces of Lebesgue spaces. Many generalizations and specializations of these spaces have been constructed and studied in years. In particular, there are extensions that allow arbitrary real values of k, weighted spaces that introduce weight functions into the L^p-norms and other generalizations involve different orders of differentitaion and different L^p-norms in different coordinate directions. Finally, there has been much work on Sobolev spaces and its related areas. To an interested reader, we can suggest our main reference book [1] and the references therein.

Let E and F be two translation invariant Banach spaces. A multiplier on E to F is a bounded linear operator commuting with all translations. We denote by $M(E,F)$ the space of all multipliers on E to F [14].

2. Some results in $L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$

If one looks for Sobolev algebras in literature, one sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to [3], [5], [20]. In [5], it is showed that the space $L_\alpha^p(G) \cap L^\infty(G)$ is an algebra with respect to pointwise multiplication, where G is a connected unimodular Lie group. Also, sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given in [20].

In [3], Chu defined $A_k^p\left(\mathbb{R}^d\right) = L^1\left(\mathbb{R}^d\right) \cap W^{k,p}\left(\mathbb{R}^d\right)$ spaces and showed some algebraic properties of these spaces (Segal algebras). In this section, we will generalize his results to weighted Sobolev algebras.

Let $1 \leq q, p < \infty$, k be a nonnegative integer and v, w be Beurling's weight functions on $\mathbb{R}^d$. We deal with the some harmonic properties of the intersection space $L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$. This space, denoted by $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$, is a normed space

with the norm

$$\|.\|_{k,v,w}^{q,p} = \|.\|_{q,v} + \|.\|_{W_k^{p,w}}.$$

Theorem 1. *$\left(A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right), \|.\|_{k,v,w}^{q,p}\right)$ is a Banach space.*

Proof. Assume that (f_n) be a Cauchy sequence in $\mathrm{A}_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$. Clearly (f_n) is a Cauchy sequence in both $L_v^q\left(\mathbb{R}^d\right)$ and $W_k^{p,w}\left(\mathbb{R}^d\right)$. For this reason, (f_n) converges to $f \in L_v^q\left(\mathbb{R}^d\right)$ and $g \in W_k^{p,w}\left(\mathbb{R}^d\right)$. By using the inequalities $\|.\|_q \leq \|.\|_{q,v}$ and $\|.\|_p \leq \|.\|_{p,w} \leq \|.\|_{W_k^{p,w}}$, we can easily demonstrate that there exist a subsequence $\left(f_{n_k}\right)$ of (f_n) such that $f_{n_k} \to f$ a.e. and a subsequence $\left(f_{n_{k_l}}\right)$ of $\left(f_{n_k}\right)$ such that $f_{n_{k_l}} \to g$ a.e. Therefore, we get $f = g$ a.e. □

Theorem 2. ***(i)*** *$\left(A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right), \|.\|_{k,v,w}^{q,p}\right)$ is translation invariant and*

$$\|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\,\|f\|_{k,v,w}^{q,p}$$

for all $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.

(ii) *The function $s \to L_s f$ is continuous from $\mathbb{R}^d$ to $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ for any $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.*

Proof. **(i)** We know that the spaces $L_v^q\left(\mathbb{R}^d\right)$ and $W_k^{p,w}\left(\mathbb{R}^d\right)$ are translation invariant. Hence $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is translation invariant. We get

$$\begin{aligned}\|L_s f\|_{k,v,w}^{q,p} &= \|L_s f\|_{q,v} + \|L_s f\|_{W_k^{p,w}} \\ &\leq v(s)\,\|f\|_{q,v} + w(s)\,\|f\|_{W_k^{p,w}} \\ &\leq (v+w)(s)\,\|f\|_{k,v,w}^{q,p}.\end{aligned}$$

by (1.1) and (1.2).

(ii) Since $s \to L_s f$ is continuous in $L_v^q\left(\mathbb{R}^d\right)$, for any $\varepsilon > 0$ and $s_0 \in \mathbb{R}^d$ there is a neighbourhood V_1 of s_0 such that

$$\|L_s f - L_{s_0} f\|_{q,v} < \frac{\varepsilon}{2} \tag{2.1}$$

for all $s \in V_1$. There is a neighbourhood V_2 of s_0 such that

$$\|L_s f - L_{s_0} f\|_{W_k^{p,w}} < \frac{\varepsilon}{2} \tag{2.2}$$

for all $s \in V_2$, because the function $s \to L_s f$ is continuous in $W_k^{p,w}\left(\mathbb{R}^d\right)$. Consequently $V = V_1 \cap V_2$ is a neighbourhood of s_0 and we get

$$\|L_s f - L_{s_0} f\|_{k,v,w}^{q,p} < \varepsilon$$

for $s \in V$ by (2.1) and (2.2). □

Theorem 3. *$A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is a BF-space.*

Proof. Let $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ and any compact subset $K \subset \mathbb{R}^d$. Using Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\begin{aligned}\int_K |f(x)|\,dx &= \int_{\mathbb{R}^d} |f(x)|\,\chi_K(x)\,dx \\ &\leq \left(\int_{\mathbb{R}^d} |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} (\chi_K(x))^{p'}\,dx\right)^{\frac{1}{p'}} \\ &\leq \|f\|_{p,w}\,\mu(K)^{\frac{1}{p'}} \leq \|f\|_{W_k^{p,w}}\,\mu(K)^{\frac{1}{p'}}\end{aligned}$$

for any $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. If we write $M_K = \mu(K)^{\frac{1}{p'}}$, there exists $M_K > 0$ such that

$$\int_K |f(x)|\,dx \leq M_K \|f\|_{W_k^{p,w}}. \tag{2.3}$$

Also since $L_v^q\left(\mathbb{R}^d\right)$ is a BF-space, there exists $N_K > 0$ such that

$$\int_K |f(x)|\,dx \leq N_K \|f\|_{q,v}. \tag{2.4}$$

If we write $C_K = \max\{M_K, N_K\}$, we get

$$\int_K |f(x)|\,dx \leq C_K \|f\|_{k,v,w}^{q,p}$$

by (2.3) and (2.4). □

Theorem 4. *If $v \prec w'$ and $w \prec w'$, then $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is Banach module over $L_{w'}^1\left(\mathbb{R}^d\right)$ under the convolution.*

Proof. Assume that $v \prec w'$ and $w \prec w'$. Then we know that $L^1_{w'}\left(\mathbb{R}^d\right) \subset L^1_v\left(\mathbb{R}^d\right)$ and $L^1_{w'}\left(\mathbb{R}^d\right) \subset L^1_w\left(\mathbb{R}^d\right)$. Consequently there exist $c_1, c_2 > 0$ such that $\|g\|_{1,v} \leq c_1 \|g\|_{1,w'}$ and $\|g\|_{1,w} \leq c_2 \|g\|_{1,w'}$ for any $g \in L^1_{w'}\left(\mathbb{R}^d\right)$. Since $L^q_v\left(\mathbb{R}^d\right)$ is a Banach module over $L^1_v\left(\mathbb{R}^d\right)$ and $W^{p,w}_k\left(\mathbb{R}^d\right)$ is a Banach module over $L^1_w\left(\mathbb{R}^d\right)$ under the convolution, we get

$$\begin{aligned}\|f * g\|^{q,p}_{k,v,w} &= \|f * g\|_{q,v} + \|f * g\|_{W^{p,w}_k} \\ &\leq \|f\|_{q,v} \|g\|_{1,v} + \|f\|_{W^{p,w}_k} \|g\|_{1,w} \\ &\leq \|f\|_{q,v} c_1 \|g\|_{1,w'} + \|f\|_{W^{p,w}_k} c_2 \|g\|_{1,w'} \\ &\leq \max\{c_1, c_2\} \|f\|^{q,p}_{k,v,w} \|g\|_{1,w'}\end{aligned}$$

for any $f \in A^{q,p}_{k,v,w}\left(\mathbb{R}^d\right)$ and $g \in L^1_{w'}\left(\mathbb{R}^d\right)$. □

Theorem 5. *If $1 \leq p < \infty$ and $w \prec v$, then $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ is Banach algebra under the convolution.*

Proof. Suppose that $w \prec v$. So, there is a constant $c > 0$ such that $\|f\|_{1,w} \leq c \|f\|_{1,v}$ for any $f \in L^1_v\left(\mathbb{R}^d\right)$. Now we take any $f, g \in A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$. Since $L^1_v\left(\mathbb{R}^d\right)$ is a Beurling algebra and $W^{p,w}_k\left(\mathbb{R}^d\right)$ is $L^1_w\left(\mathbb{R}^d\right)$−module, we find

$$\begin{aligned}\|f * g\|^{1,p}_{k,v,w} &= \|f * g\|_{1,v} + \|f * g\|_{W^{p,w}_k} \leq \|f\|_{1,v} \|g\|_{1,v} + \|f\|_{W^{p,w}_k} \|g\|_{1,w} \\ &\leq \|f\|_{1,v} \|g\|_{1,v} + \|f\|_{W^{p,w}_k} c \|g\|_{1,v} \leq \max\{1, c\} \|f\|^{1,p}_{k,v,w} \|g\|_{1,v} \\ &\leq \max\{1, c\} \|f\|^{1,p}_{k,v,w} \|g\|^{1,p}_{k,v,w}.\end{aligned}$$

If we define a new function on $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ such that $|||.||| = \max\{1, c\} \|.\|^{1,p}_{k,v,w}$, then we can see easily that it is a norm. Moreover, the norms $|||.|||$ and $\|.\|^{1,p}_{k,v,w}$ on $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ are equivalent. Hence we obtain

$$\begin{aligned}|||f * g||| &= \max\{1, c\} \|f * g\|^{1,p}_{k,v,w} \\ &\leq \max\{1, c\} \max\{1, c\} \|f\|^{1,p}_{k,v,w} \|g\|^{1,p}_{k,v,w} \\ &\leq |||f||| \, |||g|||.\end{aligned}$$

□

Definition 1. A sequence of functions φ_n in $C_c^\infty\left(\mathbb{R}^d\right)$ satisfies the following conditions:

(i) $\varphi_n(x) \geq 0$ for all $x \in \mathbb{R}^d$

(ii) $\int\limits_{\mathbb{R}^d} \varphi_n(x)\,dx = 1$

(iii) The support of φ_n is in $[-\varepsilon_n, \varepsilon_n]^d$, $\varepsilon_n > 0$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$ [9].

Theorem 6. *The sequence of functions φ_n is an approximate identity for $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.*

Proof. Since φ_n is an approximate identity, for any $f \in L_v^q\left(\mathbb{R}^d\right)$ and $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\|f * \varphi_n - f\|_{q,v} < \frac{\varepsilon}{2} \tag{2.5}$$

for all $n \geq n_1$. Also we can see that there exists a $n_2 \in \mathbb{N}$ such that

$$\|f * \varphi_n - f\|_{W_{k,w}^p} < \frac{\varepsilon}{2} \tag{2.6}$$

for all $n \geq n_2$ by using the method in [4],[22]. If we set $n_0 = \max\{n_1, n_2\}$, then by (2.5) and (2.6) we obtain

$$\|f * \varphi_n - f\|_{k,v,w}^{q,p} < \varepsilon$$

for all $n \geq n_0$. □

Theorem 7. *For each $f \neq 0$, $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ there exists $c(f) > 0$ such that*

$$c(f)(v+w)(s) \leq \|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\,\|f\|_{k,v,w}^{q,p}.$$

Proof. For given $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$, we write $f \in L_v^q\left(\mathbb{R}^d\right)$ and $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. Let K be any compact subset of $\mathbb{R}^d$. Since $\|L_s f\|_{W_k^{p,w}} \geq \|D^\alpha L_s f\|_{p,w}$ for all $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$, we find

$$\|L_s f\|_{W_k^{p,w}} \geq \|D^\alpha L_s f\|_{p,w} = \left\{\int\limits_{\mathbb{R}^d} |D^\alpha f(x-s)|^p\, w^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{\mathbb{R}^d} |D^\alpha f(u)|^p\, w^p(u+s)\,du\right\}^{\frac{1}{p}} \geq \left\{\int\limits_K |D^\alpha f(u)|^p \frac{w^p(s)}{w^p(-u)}du\right\}^{\frac{1}{p}}$$

$$\geq \left\{\int\limits_K |D^\alpha f(u)|^p \frac{w^p(s)}{\sup\limits_{u\in K} w^p(-u)}du\right\}^{\frac{1}{p}} \geq \frac{w(s)}{\sup\limits_{u\in K} w(-u)}\,\|D^\alpha f \chi_K\|_p.$$

If we set $c_1(f) = \frac{\|D^\alpha f \chi_K\|_p}{\sup\limits_{u\in K} w(-u)}$, then there exists a constant $c_1(f) > 0$ such that

$$\|L_s f\|_{W_k^{p,w}} \geq c_1(f)\, w(s). \tag{2.7}$$

Also we know that there exists a constant $c_2(f) > 0$ such that

$$\|L_s f\|_{q,v} \geq c_2(f)\, v(s) \tag{2.8}$$

for all $f \in L_v^q\left(\mathbb{R}^d\right)$. If we set $c(f) = \min\{c_1(f), c_2(f)\}$, then we get

$$\|L_s f\|_{k,v,w}^{q,p} \geq c(f)(v+w)(s) \tag{2.9}$$

by inequalities (2.7) and (2.8). Also we know that

$$\|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\, \|f\|_{k,v,w}^{q,p} \tag{2.10}$$

by Theorem 2. Hence the proof is completed from (2.9) and (2.10). □

Proposition 1. *Let $1 \leq q_1, q_2, p_1, p_2 < \infty$ and v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$. Then*

$$A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$$

if and only if there is a constant $M > 0$ such that

$$\|f\|_{k,v_2,w_2}^{q_2,p_2} \leq M \|f\|_{k,v_1,w_1}^{q_1,p_1}$$

for every $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$.

Proof. Assume that $A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$. We define the norm

$$\|f\| = \|f\|_{k,v_1,w_1}^{q_1,p_1} + \|f\|_{k,v_2,w_2}^{q_2,p_2}$$

for all $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$. Let (f_n) be a Cauchy sequence in $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|\right)$. Hence (f_n) is a Cauchy sequence in

$$\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|_{k,v_1,w_1}^{q_1,p_1}\right) \text{ and } \left(A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right), \|.\|_{k,v_2,w_2}^{q_2,p_2}\right).$$

Since $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|_{k,v_1,w_1}^{q_1,p_1}\right)$ and $\left(A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right), \|.\|_{k,v_2,w_2}^{q_2,p_2}\right)$ are Banach spaces, there exist $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$ and $g \in A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$ such that

$$\|f_n - f\|_{k,v_1,w_1}^{q_1,p_1} \to 0 \text{ and } \|f_n - g\|_{k,v_2,w_2}^{q_2,p_2} \to 0.$$

If we use the inequalities $\|.\|_{p_1} \leq \|.\|_{k,v_1,w_1}^{q_1,p_1}$ and $\|.\|_{p_2} \leq \|.\|_{k,v_2,w_2}^{q_2,p_2}$, then we find $\|f_n - f\|_{p_1} \to 0$ and $\|f_n - g\|_{p_2} \to 0$. Thus there is a subsequence $\left(f_{n_k}\right)$ of (f_n) such that $f_{n_k} \to f$ a.e. and also there is a subsequence $\left(f_{n_{k_l}}\right)$ of $\left(f_{n_k}\right)$ such that $f_{n_{k_l}} \to g$ a.e. Therefore we find $f = g$ a.e., consequently we get $\|f_n - f\| \to 0$.

Hence $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|\right)$ is a Banach space. We consider the unit function I from $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|\right)$ onto $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|_{k,v_1,w_1}^{q_1,p_1}\right)$. Since $\|I(f)\|_{k,v_1,w_1}^{q_1,p_1} = \|f\|_{k,v_1,w_1}^{q_1,p_1} \leq \|f\|$, the unit function is continuous. Then it is homeomorphism by Banach Theorem. This means that $\|.\|$ and $\|.\|_{k,v_1,w_1}^{q_1,p_1}$ are equivalent, so there is a constant $M > 0$ such that

$$\|f\| \leq M \|f\|_{k,v_1,w_1}^{q_1,p_1} \tag{2.11}$$

for all $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$. If we use the definiton of $\|.\|$ and the inequality (2.11), then we obtain $\|f\|_{k,v_2,w_2}^{q_2,p_2} \leq M \|f\|_{k,v_1,w_1}^{q_1,p_1}$.

Conversely, if $\|f\|_{k,v_2,w_2}^{q_2,p_2} \leq M \|f\|_{k,v_1,w_1}^{q_1,p_1}$ for all $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$, we can easily that the inclusion $A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$ holds. □

It is easy to obtain the following proposition by aid of Proposition 1.

Proposition 2. *Let v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ and $1 \leq q, p < \infty$. If $v_2 \prec v_1$ and $w_2 \prec w_1$, then $A_{k,v_1,w_1}^{q,p}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q,p}\left(\mathbb{R}^d\right)$.*

Theorem 8. *Let $\Omega \subset \mathbb{R}^d$ be an open set and v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$ and $w_2 \prec w_1$. Then*

$$A_{k,v_1,w_1}^{q,p}(\Omega) \hookrightarrow A_{l,v_2,w_2}^{q,p}(\Omega)$$

for all $k, l \in \mathbb{Z}^+$ where $k > l$.

Proof. Let $f \in A_{k,v_1,w_1}^{q,p}(\Omega)$ be given, so we write $f \in L_{v_1}^q(\Omega)$ and $f \in W_k^{p,w_1}(\Omega)$. It is known that $L_{v_1}^q(\Omega) \subset L_{v_2}^q(\Omega)$ where $v_2 \prec v_1$. Also we know that $W_k^{p,w_1}(\Omega) \subset W_l^{p,w_2}(\Omega)$ where $w_2 \prec w_1$ and $k > l$. Therefore we obtain $f \in L_{v_2}^q(\Omega) \cap W_l^{p,w_2}(\Omega) = A_{l,v_2,w_2}^{q,p}(\Omega)$. So we find $A_{k,v_1,w_1}^{q,p}(\Omega) \subset A_{l,v_2,w_2}^{q,p}(\Omega)$.

There exists a constant $c_1 > 0$ such that

$$\|f\|_{q,v_2} \leq c_1 \|f\|_{q,v_1} \tag{2.12}$$

for all $f \in L_{v_1}^q(\Omega)$, because $v_2 \prec v_1$. Moreover, since $W_k^{p,w_1}(\Omega) \hookrightarrow W_l^{p,w_2}(\Omega)$ where $k > l$ and $w_2 \prec w_1$, there exists a constant $c_2 > 0$ such that

$$\|f\|_{W_l^{p,w_2}} \leq c_2 \|f\|_{W_k^{p,w_1}} \tag{2.13}$$

for all $f \in W_k^{p,w_1}(\Omega)$. If we set $c = \max\{c_1, c_2\}$, we get

$$\begin{aligned} \|f\|_{l,v_2,w_2}^{q,p} &\leq c\left(\|f\|_{q,v_1} + \|f\|_{W_k^{p,w_1}}\right) \\ &\leq c\|f\|_{k,v_1,w_1}^{q,p} \end{aligned}$$

from the inequalities (2.12) and (2.13) This completes the proof. □

We prove the following theorem with using method in [23].

Theorem 9. *Let v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then*

$$A^{s,r}_{k,v_1,w_1}(\Omega) \subset A^{q,p}_{l,v_2,w_2}(\Omega)$$

where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

Proof. Assume that $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, so we write $f \in L^s_{v_1}(\Omega)$ and $f \in W^{r,w_1}_k(\Omega)$. If we set $\alpha = \frac{s}{q}$ where $1 \leq q < s < \infty$ and let β be conjugate exponent of α, then we find

$$\|f\|^q_{q,v_1} = \int_\Omega |f(x)|^q v_1^q(x)\,dx \leq \left\{\int_\Omega \left[|f(x)|^q v_1^q(x)\right]^{\frac{s}{q}} dx\right\}^{\frac{q}{s}} \left\{\int_\Omega (\chi_\Omega)^\beta\,dx\right\}^{\frac{1}{\beta}}$$

$$\leq \left(\int_\Omega |f(x)|^s v_1^s(x)\,dx\right)^{\frac{q}{s}} [\mu(\Omega)]^{\frac{1}{\beta}} = \|f\|^q_{s,v_1} [\mu(\Omega)]^{\frac{1}{\beta}} \tag{2.14}$$

by Hölder inequality. Since $\mu(\Omega) < \infty$ and $f \in L^s_{v_1}(\Omega)$, we obtain $f \in L^q_{v_1}(\Omega)$ from (2.14). Hence we have $f \in L^q_{v_2}(\Omega)$, because $v_2 \prec v_1$. Also we can see that $W^{r,w_1}_k(\Omega) \subset W^{p,w_1}_k(\Omega)$ where $1 \leq p < r < \infty$ and $\mu(\Omega) < \infty$ by similar method. Since $w_2 \prec w_1$ and $k > l$, we find $W^{p,w_1}_k(\Omega) \subset W^{p,w_2}_l(\Omega)$. So we get $W^{r,w_1}_k(\Omega) \subset W^{p,w_2}_l(\Omega)$, therefore we write $f \in W^{p,w_2}_l(\Omega)$. Thus we obtain $f \in L^q_{v_2}(\Omega) \cap W^{p,w_2}_l(\Omega) = A^{q,p}_{l,v_2,w_2}(\Omega)$. This completes the proof. □

Theorem 10. *Let v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then there exist $c(f) > 0$ and $c > 0$ such that*

$$c(f)(v_2 + w_2)(s) \leq \|L_s f\|^{q,p}_{l,v_2,w_2} \leq c(v_2 + w_2)(s)\ \|f\|^{s,r}_{k,v_1,w_1}$$

for all $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, $f \neq 0$ where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

Proof. For given $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, there exists a constant $c > 0$ such that

$$c(f)(v_2 + w_2)(s) \leq \|L_s f\|^{q,p}_{l,v_2,w_2} \tag{2.15}$$

by Theorem 7 and Theorem 9. Since $v_2 \prec v_1$, there is a constant $c_1 > 0$ such that

$$\|f\|_{q,v_2} \leq c_1 \|f\|_{q,v_1}. \tag{2.16}$$

Also since $W^{p,w_1}_k(\Omega) \hookrightarrow W^{p,w_2}_l(\Omega)$ where $w_2 \prec w_1$ and $k > l$, there is a constant $c_2 > 0$ such that

$$\|f\|_{W^{p,w_2}_l} \leq c_2 \|f\|_{W^{p,w_1}_k}. \tag{2.17}$$

If we set $m_1 = \max\{c_1, c_2\}$, we obtain

$$\begin{aligned}\|L_s f\|_{l,v_2,w_2}^{q,p} &= \|L_s f\|_{q,v_2} + \|L_s f\|_{W_l^{p,w_2}} \\ &\leq v_2(s)\|f\|_{q,v_2} + w_2(s)\|f\|_{W_l^{p,w_2}} \\ &\leq v_2(s)c_1\|f\|_{q,v_1} + w_2(s)c_2\|f\|_{W_k^{p,w_1}} \\ &\leq m_1(v_2+w_2)(s)\|f\|_{k,v_1,w_1}^{q,p}\end{aligned}$$

by using (2.16) and (2.17). Also we can see that $A_{k,v_1,w_1}^{s,r}(\Omega) \subset A_{k,v_1,w_1}^{q,p}(\Omega)$ by Theorem 9 and so there exists a constant $m_2 > 0$ such that

$$\|f\|_{k,v_1,w_1}^{q,p} \leq m_2\|f\|_{k,v_1,w_1}^{s,r}$$

by Proposition 1. Thus there exists a constant $c > 0$ such that

$$\|L_s f\|_{l,v_2,w_2}^{q,p} \leq c(v_2+w_2)(s)\|f\|_{k,v_1,w_1}^{s,r} \tag{2.18}$$

for all $f \in A_{k,v_1,w_1}^{s,r}(\Omega)$. If we combine (2.15) with (2.18), the proof is completed. □

We prove the following theorem with using method in [1].

Theorem 11. *Let $\Omega \subset \mathbb{R}^d$ be open set, v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\frac{1}{s} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}$, $\frac{1}{r} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}$ for some λ with $0 < \lambda < 1$, then*

$$A_{k,v_1,w_1}^{q_1,p_1}(\Omega) \cap A_{k,v_1,w_1}^{q_2,p_2}(\Omega) \subset A_{l,v_2,w_2}^{s,r}(\Omega)$$

where $1 \leq q_1 < s < q_2 < \infty$ and $1 \leq p_1 < r < p_2 < \infty$.

Proof. Suppose that $f \in A_{k,v_1,w_1}^{q_1,p_1}(\Omega) \cap A_{k,v_1,w_1}^{q_2,p_2}(\Omega)$, so we write $f \in L_{v_1}^{q_1}(\Omega) \cap L_{v_1}^{q_2}(\Omega)$ and $f \in W_k^{p_1,w_1}(\Omega) \cap W_k^{p_2,w_1}(\Omega)$. If we set $t = \frac{q_1}{s\lambda}$, then we see $t' = \frac{q_2}{s(1-\lambda)}$ is conjugate exponent of t. Thus we obtain

$$\begin{aligned}\|f\|_{s,v_1}^s &= \int_\Omega |f(x)|^s v_1^s(x)\,dx \\ &= \int_\Omega |f(x)|^{s\lambda} v_1^{s\lambda}(x)\,|f(x)|^{s(1-\lambda)} v_1^{s(1-\lambda)}(x)\,dx \\ &\leq \left\{\int_\Omega \left[|f(x)|^{s\lambda} v_1^{s\lambda}(x)\right]^{\frac{q_1}{s\lambda}} dx\right\}^{\frac{s\lambda}{q_1}}\end{aligned}$$

$$\left\{\int_{\Omega}\left[\left|f(x)\right|^{s(1-\lambda)}v_1^{s(1-\lambda)}(x)\right]^{\frac{q_2}{s(1-\lambda)}}dx\right\}^{\frac{s(1-\lambda)}{q_2}}$$
$$=\|f\|_{q_1,v_1}^{s\lambda}\|f\|_{q_2,v_1}^{s(1-\lambda)}$$

by Hölder inequality. Since $f\in L_{v_1}^{q_1}(\Omega)\cap L_{v_1}^{q_2}(\Omega)$, we get $f\in L_{v_1}^{s}(\Omega)$. Also we can show that $f\in W_k^{r,w_1}(\Omega)$ by similar method under the hypothesis. Hence we find $f\in L_{v_1}^{s}(\Omega)\cap W_k^{r,w_1}(\Omega)=A_{k,v_1,w_1}^{s,r}(\Omega)$. We know that $A_{k,v_1,w_1}^{s,r}(\Omega)\subset A_{l,v_2,w_2}^{s,r}(\Omega)$ where $v_2\prec v_1$, $w_2\prec w_1$ and $k>l$ by Theorem 8, therefore we get $f\in A_{l,v_2,w_2}^{s,r}(\Omega)$. □

3. Multiplier spaces of $\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right),L_w^1\left(\mathbb{R}^d\right)\right)$

In this section we call the intersection space $L_w^1\left(\mathbb{R}^d\right)\cap W_k^{p,w}\left(\mathbb{R}^d\right)$ as $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and equipped with the sum norm $\|f\|_{k,w}^{1,p}=\|f\|_{1,w}+\|f\|_{W_k^{p,w}}$. We denote the space of multipliers from $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ to $L_w^1\left(\mathbb{R}^d\right)$ by $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right),L_w^1\left(\mathbb{R}^d\right)\right)$. It is known that $L_w^1\left(\mathbb{R}^d\right)$ is a closed ideal in the space $M_w\left(\mathbb{R}^d\right)$ which is defined by

$$M_w\left(\mathbb{R}^d\right)=\left\{\mu:\mu\text{ is a bounded measure and }\|\mu\|_\omega=\int_{\mathbb{R}^d}w\,d\,|\mu|<\infty\right\}.$$

We will show that $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right),L_w^1\left(\mathbb{R}^d\right)\right)\cong M_w\left(\mathbb{R}^d\right)$ by using results in the second section.

Proposition 3. *If $\mu\in M_w\left(\mathbb{R}^d\right)$ and $f\in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$, then $\mu*f\in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\|\mu*f\|_{k,w}^{1,p}\leq\|\mu\|_\omega\|f\|_{k,w}^{1,p}$.*

Proof. Since $s\to L_sf$ is a continuous function from $\mathbb{R}^d$ to $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ for $f\in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and μ is a bounded Borel measure, then $\int_{\mathbb{R}^d}\|L_sf\|_{k,w}^{1,p}\,d\,|\mu|(s)<\infty$. So, the integral $\int_{\mathbb{R}^d}L_sf\,d\mu(s)$ belong to $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ by [16, Proposition 3.2.62]. Therefore we get

$$\|\mu * f\|_{k,w}^{1,p} = \left\| \int_{\mathbb{R}^d} L_s f d\mu(s) \right\|_{k,w}^{1,p} \leq \int_{\mathbb{R}^d} \|L_s f\|_{k,w}^{1,p} d|\mu|(s)$$
$$\leq \int_{\mathbb{R}^d} \|f\|_{k,w}^{1,p} w(s) d|\mu|(s) = \|f\|_{k,w}^{1,p} \|\mu\|_\omega .$$

□

Proposition 4. *$A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is an essential Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$.*

Proof. Let $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $g \in L_w^1\left(\mathbb{R}^d\right)$. By Theorem 4, we can easily see that $f * g \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and we find

$$\|f * g\|_{k,w}^{1,p} = \|f * g\|_{1,w} + \|f * g\|_{W_k^{p,w}}$$
$$\leq \|f\|_{1,w} \|g\|_{1,w} + \|f\|_{W_k^{p,w}} \|g\|_{1,w}$$
$$\leq \|f\|_{k,w}^{1,p} \|g\|_{1,w} .$$

We known that $C_c^\infty\left(\mathbb{R}^d\right)$ is a dense subset of $L_w^1\left(\mathbb{R}^d\right)$ [10] and we can easily see that $C_c^\infty\left(\mathbb{R}^d\right) \subset A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$. Hence we find that $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is a dense subset of $L_w^1\left(\mathbb{R}^d\right)$**.** So we get that $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is a dense Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$. Now let $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\varepsilon > 0$. By Theorem 2, there is a neighbourhood U of the unit element e of $\mathbb{R}^d$ such that

$$\|L_s f - f\|_{k,w}^{1,p} < \varepsilon$$

for all $s \in \mathbb{R}^d$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be as in Definition 1, so there exists $n_0 \in \mathbb{N}$ such that $\text{supp}\varphi_{n_0} \subset U$. Thus

$$\|\varphi_n * f - f\|_{k,w}^{1,p} = \left\| \int_{\mathbb{R}^d} \varphi_n(s)(L_s f - f) ds \right\|_{k,w}^{1,p}$$
$$\leq \|L_s f - f\|_{k,w}^{1,p} \int_{\mathbb{R}^d} \varphi_n(s) ds$$
$$= \|L_s f - f\|_{k,w}^{1,p} < \varepsilon$$

for all $n \geq n_0$. Therefore $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is an essential Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$. □

Theorem 12. *Let $T : A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right) \to L^1_w\left(\mathbb{R}^d\right)$ be a linear transformation, then the following are equivalent.*

i) $T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$.

ii) *There exists a unique measure $\mu \in M_w\left(\mathbb{R}^d\right)$ such that $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$.*

Moreover the correspondence between T and μ defines an isometric algebra isomorphism of $M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$ onto $M_w\left(\mathbb{R}^d\right)$.

Proof. Let $\mu \in M_w\left(\mathbb{R}^d\right)$ and $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$. Then,

$$\begin{aligned}
\|Tf\|_{1,w} = \|\mu * f\|_{1,w} &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-s)\,\mu(s)\,ds \right| w(x)\,dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x-s)|\,|\mu(s)|\,ds \right) w(x)\,dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x)|\,|\mu(s)|\,ds \right) w(x+s)\,dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x)|\,w(x)\,dx \right) w(s)\,|\mu(s)|\,ds \\
&\leq \|f\|_{1,w}\,\|\mu\|_w \leq \|f\|^{1,p}_{k,w}\,\|\mu\|_w .
\end{aligned}$$

Hence we get $T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$ and $\|T\| \leq \|\mu\|_\omega$.

Conversely, suppose that $T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$. Therefore we have

$$\|Tf\|_{1,w} \leq \|T\|\,\|f\|^{1,p}_{k,w} = \|T\|\left(\|f\|_{1,w} + \|f\|_{W^{p,w}_k}\right)$$

for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$. In [7, Lemma 2.1], it is obtained $\lim_{s\to\infty} \|f + L_s f\|_{p,w} = 2^{\frac{1}{p}}\,\|f\|_{p,w}$ for all $f \in L^p_w\left(\mathbb{R}^d\right)$ using the method in [11]. Since the norm $\|.\|_{W^{p,w}_k}$ is a finite sum of L^p_w norms, we find $\lim_{s\to\infty} \|f + L_s f\|_{W^{p,w}_k} = 2^{\frac{1}{p}}\,\|f\|_{W^{p,w}_k}$. So we get

$$2\|Tf\|_{1,w} = \lim_{s\to\infty} \|Tf + TL_s f\|_{1,w} = \lim_{s\to\infty} \|T(f + L_s f)\|_{1,w}$$

$$\leq \lim_{s\to\infty} \|T\| \left(\|f + L_s f\|_{1,w} + \|f + L_s f\|_{W_k^{p,w}} \right)$$
$$\leq \|T\| \left(2\|f\|_{1,w} + 2^{\frac{1}{p}} \|f\|_{W_k^{p,w}} \right).$$

Therefore we have

$$\|Tf\|_{1,w} \leq \|T\| \left(\|f\|_{1,w} + 2^{\frac{1}{p}-1} \|f\|_{W_k^{p,w}} \right) .$$

Repeating this process n times, we see that

$$\|Tf\|_{1,w} \leq \|T\| \left(\|f\|_{1,w} + 2^{n\left(\frac{1}{p}-1\right)} \|f\|_{W_k^{p,w}} \right) .$$

Since $p > 1$ we obtain $\lim_{n\to\infty} 2^{n\left(\frac{1}{p}-1\right)} = 0$ and so we conclude that

$$\|Tf\|_{1,w} \leq \|T\| \|f\|_{1,w} .$$

Hence T is continuous on $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$, considered as a subspace of $L_w^1\left(\mathbb{R}^d\right)$. Thus T defines a continuous linear transformation from $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ as a subspace of $L_w^1\left(\mathbb{R}^d\right)$ to $L_w^1\left(\mathbb{R}^d\right)$ which commutes with translation. Since $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is dense in $L_w^1\left(\mathbb{R}^d\right)$, T determines a unique element T' of $M\left(L_w^1\left(\mathbb{R}^d\right)\right)$ and $\|T'\| \leq \|T\|$. There exists a unique element $\mu \in M_w\left(\mathbb{R}^d\right)$ such that $T'f = \mu * f$ for each $f \in L_w^1\left(\mathbb{R}^d\right)$ and $\|\mu\|_w = \|T'\|$. Consequently $Tf = \mu * f$ for each $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\|\mu\|_w \leq \|T\|$. Hence (i) and (ii) are equivalent. It is evident that the correspondence between T and μ defines isometric algebra isomorhism from $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right)$ onto $M_w\left(\mathbb{R}^d\right)$. □

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Doi: 10.18514/MMN.2015.1135

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 181–194

HU e-ISSN 1787-2413

SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED S-METRIC SPACES

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Received 18 February, 2014

Abstract. In this paper, we prove some fixed point results using mixed weakly monotone property and altering distance function in the setting of S-metric space. Our result generalizes the result of Lakshmikantham and Ciric [V. Lakshmikantham and L. Ciric, Coupled common fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis, 70, (2009), 4341 - 4349].

2010 *Mathematics Subject Classification:* 54H25; 47H10

Keywords: fixed point, S-metric space, partially ordered space, mixed weakly monotone property

1. INTRODUCTION

In functional analysis, the concept of space is very broad and astonishing. A space will be a set of unspecified elements satisfying certain axioms, and by choosing different set of axioms, we shall obtain different type of spaces. In partially ordered set, as the name indicates, ordering and sequencing is defined between the elements of a set. Although there are some elements in the set which are not related. A set in which all elements are related is called totally ordered set. In [18] Marr defines the concept of convergence in partially ordered metric space. Also, he tried to obtain a relation between metric space and partially ordered metric space, and claimed that the fixed point theorems in metric spaces are the particular cases of fixed point results in partially ordered metric spaces.

Definition 1 ([18])**.** A partially ordered space is a set X with a binary relation $\leq$, which satisfy the three conditions:

(1) $x \leq x$ for all $x \in X$;
(2) $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$;
(3) $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

There is vast literature of obtaining fixed points using different contractive conditions. In this direction, Dolbosco [3] took initiation to gave fixed point results using altering distance function or control function. Altering distance function measures

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the distance between two points. Khan et. al. [15] worked on the concept of altering distance between points, for pair of self mappings. Also, Altun [1], Aydi [2], Gupta et al [8,9], Iseki [11] established theorems on sequence of self mappings with altering distance function.

Definition 2 ([15])**.** An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$, which is

(1) monotone increasing and continuous,
(2) $\psi(t) = 0$ iff $t = 0$.

In 1963, Gahler [6] introduced the generalization of metric space and called it 2-metric space. Fixed point theorems in 2-metric space has been proved initially by Iseki [12]. After that several authors proved fixed point results in the setting of 2-metric space. In 1979, Fisher [5] gave common fixed point results using commuting mapping. Jungck [13] and Kubiak [16] also proved some results using commuting and semi-commuting mappings. In 1992, Murthy [19] used compatible type mapping to find fixed point, which is more general than commuting and semi-commuting maps. After that in 1978, Khan [14] proved a result by taking a uniformly convergent sequence of 2-metrics in X. In 1988, Ha et al [10] objected on Gahler theory that 2-metric space is not the generalization of metric space. He claimed that the contraction mappings in these spaces are unrelated. These considerations inspired Dhage [4] to introduce a new metric namely D-metric. After that Sedghi and Shobe [22] gave D^* metric space which is more general than D-metric space. Further the topological properties of D-metric space are objected by Mustafa [20] in 2003, and Mustafa [21] gave a more generalized metric called G-metric. He defined many concepts like convergence, continuity, completeness, compactness, product of spaces in the setting of G-metric space and stated that every G-metric space is topologically equivalent to a metric space.

Definition 3 ([21])**.** Let X be a non-empty set and $G : X \times X \times X \to [0,\infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$,

(1) $G(x,y,z) = 0$ if $x = y = z$;
(2) $0 < G(x,x,y)$ for all $x, y \in X$ with $x \neq y$;
(3) $G(x,x,y) \leq G(x,y,z)$ for all $x, y, z \in X$ with $x \neq y$;
(4) $G(x,y,z) = G(x,z,y) = G(y,z,x)...$;
(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$.

Then the function G is called G-metric on X and the pair (X,G) is called a G-metric space.

In 2011, Sedghi and Shobe [23] introduced a more general space namely S-metric space which is generalization of D^* metric space and G-metric space.

Definition 4 ([23]). Let X be a non-empty set. A generalized metric or S-metric on X is a function, $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$;
(2) $S(x, y, z) = 0$ iff $x = y = z$;
(3) $S(x, y, z) \leq S(a, y, z) + S(a, x, x)$.

Then the pair (X, S) is called a S-metric space.

Example 1 ([23]). If $X = R^n$, define

(1) $S(x, y, z) = \|y + x - 2z\| + \|y - z\|$
(2) $S(x, y, z) = d(x, y) + d(x, z)$, here d is the ordinary metric on X.

Then (X, S) is S-metric space.

Definition 5 ([23]). Let (X, S) be a S-metric space and $A \subset X$,

(1) If for every $x \in A$, there exists $r > 0$ such that $B_S(x, r) \subset A$, then subset A is called open subset of X.
(2) Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x, y, y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x, x) = S(x, x_n, x_n) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that $S(x, x_n, x_n) < \epsilon$ for all $n \geq n_0$.
(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in N$ such that $S(x_n, x_m, x_m) < \epsilon$ for each $n, m \geq n_0$.
The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.
(5) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology induced by the S-metric S on X.

Also, Sedghi et. al. [24] gave some properties of S-metric space and proved fixed point theorems in this space. Nguyen [25] proved coupled fixed point in partially ordered S-metric space with the help of mixed weakly monotone maps.
Lakshmikantham and Ciric [17] introduced the concept of mixed g-monotone mapping and proved coincidence fixed point theorems in partially ordered metric space.

Definition 6 ([17]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has mixed g-monotone property if F is monotone g-non-decreasing in its first argument, and is monotone g-non-increasing in its second argument,
that is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \text{ implies } F(x_1, y) \leq F(x_2, y)$$
$$\text{and } y_1, y_2 \in X, g(y_1) \leq g(y_2) \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Theorem 1 ([17]). *Let $(X,\leq)$ be a partially ordered set and $F : X\times X\to X$ and suppose there is a metric d on X such that (X,d) is a complete metric space. Assume that there is a function $\psi : [0,\infty)\to[0,\infty)$ with $\psi(t) < t$ and*

$$\lim_{r\to t^+}\phi(r) < t \ \ for\ each\ t > 0.$$

Also suppose $F : X\times X\to X$ and $g : X\to X$ are such that F has the mixed g-monotone property satisfying:

$$d(F(x,y),F(u,v)) \leq \phi\left(\frac{d(g(x),g(u)) + d(g(y),g(v))}{2}\right),$$

for all $x,y,u,v\in X$, for which $g(x)\leq g(u)$ and $g(y)\geq g(v)$. Consider $F(X\times X)\subseteq g(X)$, g is continuous and commutes with F and also suppose either

(1) *F is continuous*
or
(2) *X has the following property:*
   (a) *if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leq x$ for all n;*
   (b) *if a non-increasing sequence $\{y_n\}\to y$, then $y\leq y_n$ for all n.*

If there exist $x_0,y_0\in X$ such that $g(x_0)\leq F(x_0,y_0)$ and $g(y_0)\geq F(y_0,x_0)$ then there exist $x,y\in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

In 2012, Gordji et. al. [7] introduced the concept of the mixed weakly inceasing property of mappings and proved a coupled fixed point result.

Definition 7 ([7]). Let $(X,\leq)$ be a partially ordered set and $f,g : X\times X\to X$ be mappings. We say that a pair (f,g) has the mixed weakly monotone property on X, if for any $x,y\in X$

$$x\leq f(x,y), y\geq f(y,x),$$

this implies

$$f(x,y)\leq g(f(x,y),f(y,x)), f(y,x)\geq g(f(y,x),f(x,y))$$

and

$$x\leq g(x,y), y\geq g(y,x),$$

this implies

$$\Rightarrow g(x,y)\leq f(g(x,y),g(y,x)), g(y,x)\geq f(g(y,x),g(x,y)).$$

Theorem 2 ([7]). *Let $(X,\leq,d)$ be a partially ordered complete metric space. Let $f,g : X\times X\to X$ are mappings such that pair (f,g) has the mixed weakly monotone property on X. Suppose that there exist $p,q,r,s\geq 0$ with $p+q+r+2s < 1$ such that*

$$\begin{aligned} d\left(f\left(x,y\right),g\left(u,v\right)\right) \leq & \frac{p}{2}D\left(\left(x,y\right),\left(u,v\right)\right) \\ & + \frac{q}{2}D\left(\left(x,y\right),\left(f\left(x,y\right),f\left(y,x\right)\right)\right) \\ & + \frac{r}{2}D\left(\left(u,v\right),\left(g\left(u,v\right),g\left(v,u\right)\right)\right) \\ & + \frac{s}{2}D\left(\left(x,y\right),\left(g\left(u,v\right),g\left(v,u\right)\right)\right) \\ & + \frac{s}{2}D\left(\left(u,v\right),\left(f\left(x,y\right),f\left(y,x\right)\right)\right), \end{aligned}$$

for all $x,y,u,v \in X$ *with* $x \leq u$ *and* $y \geq v$. *Let* $x_0, y_0 \in X$ *be such that* $x_0 \leq f\left(x_0,y_0\right), y_0 \geq f\left(y_0,x_0\right)$ *or* $x_0 \leq g\left(x_0,y_0\right), y_0 \geq g\left(y_0,x_0\right)$.
If f or g is continuous, then f and g have a coupled common fixed point in X.

This result is generalized by Nguyen [25] to S-metric space as follows:

Theorem 3 ([25])**.** *Let* $(X,\leq,S)$ *be a partially ordered S-metric space and* $f,g : X \times X \to X$ *be two maps such that*

(1) *X is complete;*
(2) *The pair* (f,g) *has the mixed weakly monotone property on X,* $x_0 \leq f\left(x_0,y_0\right)$, $f\left(y_0,x_0\right) \leq y_0$ *or* $x_0 \leq g\left(x_0,y_0\right), g\left(y_0,x_0\right) \leq y_0$ *for some* $x_0, y_0 \in X$*;*
(3) *There exist* $p,q,r,s \geq 0$ *satisfying* $p+q+r+2s<1$ *and*

$$\begin{aligned} d\left(f\left(x,y\right),f\left(x,y\right),g\left(u,v\right)\right) \leq & \frac{p}{2}D\left(\left(x,y\right),\left(x,y\right),\left(u,v\right)\right) \\ & + \frac{q}{2}D\left(\left(x,y\right),\left(x,y\right),\left(f\left(x,y\right),f\left(y,x\right)\right)\right) \\ & + \frac{r}{2}D\left(\left(u,v\right),\left(u,v\right),\left(g\left(u,v\right),g\left(v,u\right)\right)\right) \\ & + \frac{s}{2}D\left(\left(x,y\right),\left(x,y\right),\left(g\left(u,v\right),g\left(v,u\right)\right)\right) \\ & + \frac{s}{2}D\left(\left(u,v\right),\left(u,v\right),\left(f\left(x,y\right),f\left(y,x\right)\right)\right), \end{aligned}$$

for all $x,y,u,v \in X$ *with* $x \leq u$ *and* $y \geq v$,
where $D = \left(\left(x,y\right),\left(u,v\right),\left(z,w\right)\right) = S\left(x,u,z\right) + S\left(y,v,w\right)$;
(4) *f or g is continuous or X has the following property:*
  (a) *If* $\{x_n\}$ *is an increasing sequence with* $x_n \to x$, *then* $x_n \leq x$ *for all* $n \in N$,
  (b) *If* $\{x_n\}$ *is an decreasing sequence with* $x_n \to x$, *then* $x \leq x_n$ *for all* $n \in N$.

Then f and g have a coupled common fixed point in X.

Now our aim is to prove new common coupled fixed points theorems by using altering distance function and mixed weakly monotone maps in partially ordered S-metric space.

2. Main result

Theorem 4. *Let $(X,\leq,S)$ be a partially ordered complete S-metric space and the mappings $f,g:X\times X\to X$ satisfies the mixed weakly monotone property on X; $x_0\leq f(x_0,y_0)$, $f(y_0,x_0)\leq y_0$ or $x_0\leq g(x_0,y_0)$, $g(y_0,x_0)\leq y_0$ for some $x_0,y_0\in X$. Consider a function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(t)<t$ and*

$$\lim_{r\to t^+}\phi(r)<t \quad for\ each\ t>0.$$

such that

$$S(f(x,y),f(x,y),g(u,v))\leq\phi\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right); \tag{2.1}$$

$\forall x,y,u,v\in X$ with $x\leq u$ and $y\geq v$.
Also, assume that either f or g is continuous or X has the following property:

(1) *If $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in N$;*
(2) *If $\{y_n\}$ is an decreasing sequence with $y_n\to y$ then $y\leq y_n$ for all $n\in N$.*

Then f and g have a coupled common fixed point in X.

Proof. Consider $x_0\leq f(x_0,y_0)$ and $y_0\geq f(y_0,x_0)$ and let $f(x_0,y_0)=x_1$ and $f(y_0,x_0)=y_1$.
Now, $x_1=f(x_0,y_0)\leq g(f(x_0,y_0),f(y_0,x_0))=g(x_1,y_1)=x_2$(say) and $y_1=f(y_0,x_0)\leq g(f(y_0,x_0),f(x_0,y_0))=g(y_1,x_1)=y_2$(say).
Continuing in this way, we have $x_{2n+1}=f(x_{2n},y_{2n})$, $y_{2n+1}=f(y_{2n},x_{2n})$ and $x_{2n+2}=g(x_{2n+1},y_{2n+1})$, $y_{2n+2}=g(y_{2n+1},x_{2n+1})$.
Thus, we conclude that $\{x_n\}$ is increasing and $\{y_n\}$ is decreasing sequence.
Similarly, from the condition $x_0\leq g(x_0,y_0)$ and $y_0\geq g(y_0,x_0)$, we can say that the sequences $\{x_n\}$ and $\{y_n\}$ are increasing or decreasing.
Now from (2.1), we obtain

$$S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\leq\phi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right),$$

which implies,

$$S(x_{2n+1},x_{2n+1},x_{2n+2})\leq\phi\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right). \tag{2.2}$$

Again, using the same step, we have

$$S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \leq \phi\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}\right). \quad (2.3)$$

Adding (2.2) and (2.3), we get

$$S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \leq 2\phi\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}\right).$$

Thus,

$$t_{2n} \leq 2\phi\left(\frac{t_{2n-1}}{2}\right), \quad (2.4)$$

where $t_{2n} = S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})$.
Now, using the property that $\phi(t) < t$, $t > 0$ in (2.4), this gives

$$t_{2n} \leq 2\left(\frac{t_{2n-1}}{2}\right) = t_{2n-1}. \quad (2.5)$$

Interchanging the role of mappings f and g, and using (2.1), we have

$$S(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(x_{2n+2}, y_{2n+2})) \leq \phi\left(\frac{S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{2}\right).$$

Proceeding as above one can get, $t_{2n+1} \leq 2\phi\left(\frac{t_{2n}}{2}\right)$.
Again using the fact that $\phi(t) < t$ above inequality gives

$$t_{2n+1} \leq t_{2n}. \quad (2.6)$$

From (2.5) and (2.6), we conclude that $\{t_n\}$ is a decreasing sequence. Therefore there exists some $t \geq 0$ such that

$$\lim_{n\to\infty} t_n = t.$$

Also, using the properties of function ϕ

$$t = \lim_{n\to\infty} t_{2n} \leq 2 \lim_{n\to\infty} \phi\left(\frac{t_{2n-1}}{2}\right) < t,$$

This is a contradiction. Hence $t = 0$.
That is,

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = 0. \quad (2.7)$$

Using property of S-metric space, for $n \leq m$, we have

$$S(x_n, x_n, x_m) + S(y_n, y_n, y_m) \leq 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) + 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_m).$$

on taking limit as $n,m \to \infty$ and using (2.7), we obtain

$$\begin{aligned}
&\lim_{n,m\to\infty}\Big[S(x_n,x_n,x_m)+S(y_n,y_n,y_m)\Big]\\
&\leq \lim_{n,m\to\infty}\Big[S(x_{n+1},x_{n+1},x_m)+S(y_{n+1},y_{n+1},y_m)\Big]\\
&\leq \lim_{n,m\to\infty}\Big[2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m)\\
&\qquad +2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m)\Big]\\
&= \lim_{n,m\to\infty}\Big[S(x_{n+2},x_{n+2},x_m)+S(y_{n+2},y_{n+2},y_m)\Big].
\end{aligned}$$

By repeatedly use of property of S-metric space, we get

$$\begin{aligned}
&\lim_{n,m\to\infty}\Big[S(x_n,x_n,x_m)+S(y_n,y_n,y_m)\Big]\\
&\leq \lim_{n,m\to\infty}\Big[2S(x_{m-1},x_{m-1},x_m)+S(x_m,x_m,x_{m+1})\\
&\qquad +2S(y_{m-1},y_{m-1},y_m)+S(y_m,y_m,y_{m+1})\Big],
\end{aligned}$$

which gives,

$$\lim_{n,m\to\infty}\Big[S(x_n,x_n,x_m)+S(y_n,y_n,y_m)\Big]=0.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X. Since X is complete, $\exists\, x,y\in X$ such that

$$x_n \to x \ \ and \ \ y_n \to y \ \ as \ \ n\to\infty.$$

As f is supposed to be continuous, therefore

$$\begin{aligned}
x &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n},y_{2n})\\
&= f\left(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}\right) = f(x,y)
\end{aligned}$$

and

$$\begin{aligned}
y &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n},x_{2n})\\
&= f\left(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}\right) = f(y,x).
\end{aligned}$$

From (2.1),

$$\begin{aligned}
&S(f(x,y),f(x,y),g(x,y))+S(f(y,x),f(y,x),g(y,x))\\
&\qquad \leq \phi\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)+\phi\left(\frac{S(y,y,y)+S(x,x,x)}{2}\right),
\end{aligned}$$

$S(x,x,g(x,y))+S(y,y,g(y,x))=0.$

Thus we have, $g(x,y)=x$ and $g(y,x)=y$.
Hence (x,y) is coupled common fixed point of f and g.
Similarly, the result follows, when g is assumed to be continuous.

Now, consider the other assumption that for an increasing sequence$\{x_n\}$ with $x_n \to x$ we have $x_n \leq x$ and for decreasing sequence $\{y_n\}$ with $y_n \to y$ we have $y \leq y_n$ for all $n \in N$.
Consider,

$$\begin{aligned} S(x,x,f(x,y)) &\leq 2S(x,x,x_n)+S(f(x,y),f(x,y),x_n) \\ &= 2S(x,x,x_n)+S(f(x,y),f(x,y),g(x_{n-1},y_{n-1})) \\ &\leq 2S(x,x,x_n)+\phi\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} S(y,y,f(y,x)) &\leq 2S(y,y,y_n)+S(f(y,x),f(y,x),y_n) \\ &= 2S(y,y,y_n)+S(f(y,x),f(y,x),g(y_{n-1},x_{n-1})) \\ &\leq 2S(y,y,y_n)+\phi\left(\frac{S(y,y,y_{n-1})+S(x,x,x_{n-1})}{2}\right). \end{aligned}$$

Adding the above inequalities, one can get

$$\begin{aligned} &S(x,x,f(x,y))+S(y,y,f(y,x)) \\ &\quad \leq 2S(x,x,x_n)+2S(y,y,y_n)+2\phi\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right), \end{aligned}$$

on taking $n \to \infty$, we get $S(x,x,f(x,y))+S(y,y,f(y,x)) < 0$, which shows that $f(x,y)=x$ and $f(y,x)=y$. By interchanging the role of functions f and g, we get the same result for g. Thus (x,y) is the common coupled fixed point of f and g. □

Corollary 1. *Let $(X,\leq,S)$ be a partially ordered complete S-metric space and the mappings $f,g : X \times X \to X$ satisfies the mixed weakly monotone property on X. Consider a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and*

$$\lim_{r \to t^+} \phi(r) < t \quad for\ each\ t > 0$$

such that

$$S(f(x,y),f(x,y),g(u,v)) \leq \frac{a}{2}[S(x,x,u)+S(y,y,v)], \tag{2.8}$$

$\forall x,y,u,v \in X$ with $x \leq u$ and $y \geq v$.
Also, assume that either f or g is continuous or X has the following property:

(1) *If $\{x_n\}$ is an increasing sequence with $x_n \to x$ then $x_n \leq x$ for all $n \in N$;*
(2) *If $\{y_n\}$ is an decreasing sequence with $y_n \to y$ then $y \leq y_n$ for all $n \in N$.*

Proof. Taking $\phi(t) = at$ with $a \in (0,1)$ in Theorem 4, we obtain the result. □

Corollary 2. *Let $(X,\leq,S)$ be a partially ordered complete S-metric space and the mapping $f : X\times X\to X$ satisfies the mixed monotone property on X. Consider a function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(t)<t$*

$$\lim_{r\to t^+}\phi(r)<t \quad for\ each\ t>0$$

such that

$$S(f(x,y),f(x,y),f(u,v))\leq\frac{a}{2}[S(x,x,u)+S(y,y,v)], \tag{2.9}$$

$\forall x,y,u,v\in X$ with $x\leq u$ and $y\geq v$.
Also assume that either f is continuous or X has the following property:-

(1) *If $\{x_n\}$ is an increasing sequence with $x_n\to x$ then $x_n\leq x$ for all $n\in N$;*
(2) *If $\{y_n\}$ is an decreasing sequence with $y_n\to y$ then $y\leq y_n$ for all $n\in N$.*

Proof. Taking $\phi(t)=at$ with $a\in(0,1)$ and $f=g$ in Theorem 4, we obtain the corollary or result. □

Corollary 3. *Assume that X is totally ordered set in addition to the hypothesis of Theorem 4. Then f and g have unique common fixed point.*

Proof. From Theorem 4, f and g have a coupled common fixed point (x,y). Let (l,m) be another coupled common fixed point of f and g. Without loss of generality we may assume that $(x,y)\leq(l,m)$. Then from (2.1) we have,

$$S(f(x,y),f(x,y),g(l,m))\leq\phi\left(\frac{S(x,x,l)+S(y,y,m)}{2}\right),$$
$$S(x,x,l)\leq\phi\left(\frac{S(x,x,l)+S(y,y,m)}{2}\right).$$

Similarly,

$$S(y,y,m)\leq\phi\left(\frac{S(y,y,m)+S(x,x,l)}{2}\right).$$

On adding the above inequalities, we have

$$S(x,x,l)+S(y,y,m)\leq 2\phi\left(\frac{S(x,x,l)+S(y,y,m)}{2}\right).$$

Using the definition of function ϕ, we get

$$S(x,x,l)+S(y,y,m)<S(x,x,l)+S(y,y,m),$$

which is a contradiction. Hence $x=l$ and $y=m$. This proves that the coupled common fixed point of f and g is unique.
Again from (2.1), we have

$$S(f(x,y),f(x,y),g(y,x))\leq\phi\left(\frac{S(x,x,y)+S(y,y,x)}{2}\right),$$

$$S(x,x,y) \leq \phi\left(\frac{S(x,x,y)+S(y,y,x)}{2}\right).$$

Similarly,

$$S(y,y,x) \leq \phi\left(\frac{S(y,y,x)+S(x,x,y)}{2}\right).$$

On adding the above inequalities, we have

$$S(x,x,y)+S(y,y,x) \leq 2\phi\left(\frac{S(x,x,l)+S(y,y,m)}{2}\right).$$

Thus we get $x = y$. □

Theorem 5. *Let $(X,\leq,S)$ be a partially ordered complete S-metric space and the mappings $f,g : X \times X \to X$ satisfies the mixed weakly monotone property on X. Consider a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and*

$$\lim_{r\to t^+} \phi(r) < t \ \ for\ each\ t > 0$$

such that

$$\int_0^{S(f(x,y),f(x,y),g(u,v))} \varphi(t)\,dt \leq \phi \int_0^{\frac{S(x,x,u)+S(y,y,v)}{2}} \varphi(t)\,dt, \tag{2.10}$$

$\forall x,y,u,v \in X$ with $x \leq u$ and $y \geq v$. Here $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function as a summable for each compact R^+, non-negative and such that for each $\epsilon > 0$, $\int \varphi(t)\,dt > 0$.
Also, assume that either f or g is continuous or X has the following property:

(1) *If $\{x_n\}$ is an increasing sequence with $x_n \to x$ then $x_n \leq x$ for all $n \in N$;*
(2) *If $\{y_n\}$ is an decreasing sequence with $y_n \to y$ then $y \leq y_n$ for all $n \in N$.*

Then f and g have a coupled common fixed point in X.

Proof. As in Theorem 4, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ and using (2.10), we have

$$\int_0^{S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))} \varphi(t)\,dt \leq \phi \int_0^{\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)} \varphi(t)\,dt,$$

$$\int_0^{S(x_{2n+1},x_{2n+1},x_{2n+2})} \varphi(t)\,dt < \int_0^{\left(\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}\right)} \varphi(t)\,dt,$$

This gives,

$$S\left(x_{2n+1},x_{2n+1},x_{2n+2}\right)<\left(\frac{S\left(x_{2n},x_{2n},x_{2n+1}\right)+S\left(y_{2n},y_{2n},y_{2n+1}\right)}{2}\right). \quad (2.11)$$

In the same way, we get

$$S\left(y_{2n+1},y_{2n+1},y_{2n+2}\right)<\left(\frac{S\left(y_{2n},y_{2n},y_{2n+1}\right)+S\left(x_{2n},x_{2n},x_{2n+1}\right)}{2}\right). \quad (2.12)$$

On adding the above inequalities and using the properties of function ϕ, we get $\{t_n\}$ be a decreasing sequence and $lim_{n\to\infty}t_n=0$.
Again, by using the properties of S-metric space we observe that $\{x_n\}$ and $\{y_n\}$ are the Cauchy sequences in X. By continuity of function f, we have $x=f\left(x,y\right)$ and $y=f\left(y,x\right)$.
Now from (2.10), we get

$$\int_0^{S(f(x,y),f(x,y),g(x,y))}\varphi\left(t\right)dt\leq\phi\int_0^{\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)}\varphi\left(t\right)dt,$$

which implies $S\left(x,x,g\left(x,y\right)\right)=0$ or $g\left(x,y\right)=x$. Similarly $g\left(y,x\right)=y$.
Thus (x,y) is the coupled common fixed point of f and g.

Now, assuming the condition that for increasing sequence $\{x_n\}$ with $x_n\to x$ then $x_n\leq x$ and for decreasing sequence $\{y_n\}$ with $y_n\to y$ then $y_n\geq y$ for all $n\in N$,

$$\begin{aligned}\int_0^{S(x,x,f(x,y))}\varphi\left(t\right)dt&\leq\int_0^{2S(x,x,x_n)}\varphi\left(t\right)dt+\int_0^{S(f(x,y),f(x,y),x_n)}\varphi\left(t\right)dt\\&=\int_0^{2S(x,x,x_n)}\varphi\left(t\right)dt+\int_0^{S(f(x,y),f(x,y),g(x_{n-1},y_{n-1}))}\varphi\left(t\right)dt\\&\leq\int_0^{2S(x,x,x_n)}\varphi\left(t\right)dt+\phi\int_0^{\left(\frac{S(x,x,x_{n-1})+S(y,y,y_{n-1})}{2}\right)}\varphi\left(t\right)dt.\end{aligned} \quad (2.13)$$

Taking limit as $n\to\infty$, we obtain

$$\int_0^{S(x,x,f(x,y))}\varphi\left(t\right)dt\leq\int_0^{2S(x,x,)}\varphi\left(t\right)dt+\phi\int_0^{\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)}\varphi\left(t\right)dt,$$

This shows that f has coupled fixed point. By interchanging the role of mappings f and g, we get the coupled fixed point of g.

Hence, we conclude that in both cases f and g have coupled fixed point.

□

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Doi: 10.18514/MMN.2015.1140

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 195–203

HU e-ISSN 1787-2413

BOUNDS FOR LAPLACIAN–TYPE GRAPH ENERGIES

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Received 24 February, 2014

Abstract. Let G be an undirected simple and connected graph with n vertices ($n \geq 3$) and m edges. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$, respectively, the Laplacian, signless Laplacian, and normalized Laplacian eigenvalues of G. The Laplacian energy, signless Laplacian energy, and normalized Laplacian energy of G are defined as $LE = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, $SLE = \sum_{i=1}^{n} \left|\gamma_i - \frac{2m}{n}\right|$, and $NLE = \sum_{i=1}^{n} |\rho_i - 1|$, respectively. Lower bounds for LE, SLE, and NLE are obtained.

2010 *Mathematics Subject Classification:* 05C50; 05C07

Keywords: energy (of graph), Laplacian energy, signless Laplacian energy, normalized Laplacian energy, Randić energy

1. INTRODUCTION

Let G be an undirected simple and connected graph with n vertices ($n \geq 2$) and m edges, and let $d_1, d_2, \ldots, d_n$ be its vertex degrees.

If the i-th and j-th vertex of the graph G are adjacent, we write $i \sim j$. Then the adjacency matrix $\mathbf{A} = (a_{ij})$ of G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $\mathbf{A}$ form the (ordinary) spectrum of G; for details on the respective spectral theory see [9].

Denote by $\mathbf{D}$ the diagonal matrix of the vertex degrees of G. The Laplacian matrix of G is $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and its eigenvalues are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ (see [3, 16, 25]). In addition, $\mathbf{Q} = \mathbf{D} + \mathbf{A}$ is the signless Laplacian matrix of G and its eigenvalues will be denoted by $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$ [10, 11].

Because the graph G is assumed to be connected, it has no isolated vertices (i.e., $d_i > 0$ for all $1 \leq i \leq n$) and therefore the matrix $\mathbf{D}^{-1/2}$ is well–defined. Then $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ is called the normalized Laplacian matrix of the graph G. Its

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eigenvalues are $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$. For details of the spectral theory of the normalized Laplacian matrix see [8].

It is convenient to write the normalized Laplacian matrix as $\mathbf{I} - \mathbf{R}$, where $\mathbf{R}$ is the so-called Randić matrix [4, 29, 30], whose (i, j)-entry is

$$r_{ij} = \begin{cases} 1/\sqrt{d_i\, d_j} & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

The (ordinary) energy of the graph G is defined as [23]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1.1}$$

Its theory is nowadays well elaborated [23]. Energy–like spectral invariants have been introduced also for other graph matrices [18]. In this paper we are concerned with the Laplacian [21, 23], signless Laplacian [1], and normalized Laplacian (or Randić) energies [5, 20], defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

$$SLE = SLE(G) = \sum_{i=1}^{n} \left| \gamma_i - \frac{2m}{n} \right|$$

$$NLE = NLE(G) = \sum_{i=1}^{n} |\rho_i - 1|$$

respectively. In what follows lower bounds for LE, SLE and NLE are obtained.

Remark 1. In analogy to (1.1), the "Randić energy" is defined as the sum of the absolute values of the eigenvalues of the Randić matrix. It has been shown in [20], that the Randić energy coincides with the normalized signless Laplacian energy.

Remark 2. One could also consider the normalized signless Laplacian matrix, $\mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2}$ and its "energy" (sum of absolute values of eigenvalues). However, the energy of this matrix is exactly the same as the normalized Laplacian energy, NLE [20]. For the general definition of the energy of a matrix see [28].

The Laplacian, signless Laplacian, and normalized (or Randić) Laplacian spreads of a graph G are defined as $LS(G) = \mu_1 - \mu_{n-1}$, $SLS(G) = \gamma_1 - \gamma_n$, and $NLS(G) = \rho_1 - \rho_{n-1}$, respectively (see [5, 13, 15, 24]).

2. Preliminaries

In this section we recall some results from spectral graph theory, and state a few analytical inequalities needed for our work.

Lemma 1 ([3]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

where M_1 is the sum of squares of the vertex degrees, usually referred to as the first Zagreb index (see [2, 7, 19]).

Lemma 2 ([12]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\frac{M_1}{m} \geq 2\sqrt{\frac{M_1}{n}} \geq \frac{4m}{n} \,. \tag{2.1}$$

Lemma 3 ([31]). *Let G an (n,m)-graph, such that $n \geq 3$ and $m \geq 1$. Then*

$$LE(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n} \tag{2.2}$$

with equality if and only if $n = 3$ or for $n \geq 4$ if $\mu_2 = \cdots = \mu_{n-2} = \frac{2m}{n}$.

Lemma 4 ([26]). *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LS(G) = \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2}{n-1}} \sqrt{(n-1)(M_1 + 2m) - 4m^2} \,. \tag{2.3}$$

Equality holds if and only if $G \cong K_n$.

Lemma 5 ([27]). *Let $a_1, a_2, \ldots, a_n$ be real numbers and $p_1, p_2, \ldots, p_n$ non-negative real numbers with the property $p_1 + p_2 + \cdots + p_n = 1$. Then, for each α, $\alpha \leq 0$ and $\alpha \geq 1$,*

$$\sum_{i=1}^{n} p_i \, a_i^{\alpha} \geq \left(\sum_{i=1}^{n} p_i \, a_i \right)^{\alpha} . \tag{2.4}$$

For the case $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality in (2.4) holds if and only if $\alpha = 0$ or $\alpha = 1$ or $a_1 = a_2 = \cdots = a_n$.

Lemma 6 ([6]). *Let $a_1, a_2, \ldots, a_n$ be real numbers, and assume that there are $r, R \in \mathbb{R}$ such that $-\infty < r \leq a_i \leq R < +\infty$, for each $i = 1, 2, \ldots, n$. Then for any non-negative $p_1, p_2, \ldots, p_n$ with the property $p_1 + p_2 + \cdots + p_n = 1$,*

$$0 \leq \sum_{i=1}^{n} p_i \, a_i^2 - \left(\sum_{i=1}^{n} p_i \, a_i \right)^2 \leq \frac{1}{2}(R - r) \sum_{i=1}^{n} p_i \left| a_i - \sum_{i=1}^{n} p_i \, a_i \right| . \tag{2.5}$$

The constant $\frac{1}{2}$ is sharp.

Lemma 7 ([32]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n-1} \rho_i = n \qquad \text{and} \qquad \sum_{i=1}^{n} \rho_i^2 = n + 2R_{-1} \tag{2.6}$$

where $R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j}$; for details on the graph invariant R_{-1} see [4, 22].

Lemma 8 ([17]). *Let G be an undirected simple and connected graph with n, $n \geq 2$, vertices and m edges. Then*

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

where M_1 is the first Zagreb index.

Lemma 9 ([17]). *The signless Laplacian spread has an upper bound*

$$SLS(G) \leq \sqrt{\frac{2[n(M_1 + 2m) - 4m^2]}{n}} \, .$$

Lemma 10 ([14]). *Suppose that G is a graph without isolated vertices. Then*

$$\mu_1 - \mu_{n-1} \geq \frac{2}{n-1} \sqrt{(n-1)(2m + M_1) - 4m^2} \, . \tag{2.7}$$

3. Main results

3.1. *Lower bound for Laplacian energy*

Theorem 1. *Let G be an undirected connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1} \sqrt{(n-1)(2m + M_1) - 4m^2} \, . \tag{3.1}$$

Proof. Inequality (3.1) directly follows from inequalities (2.2) and (2.7). □

Corollary 1. *Let G be an undirected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1} \sqrt{\frac{2m(n(n-1) - 2m)}{n}} \, .$$

Corollary 2. *Let G be an undirected simple and connected k-regular graph with n, $n \geq 3$, vertices and m edges, $1 < k \leq n-1$. Then*

$$LE(G) \geq k + \frac{2}{n-1} \sqrt{nk(n-k-1)} \, .$$

Theorem 2. *Let G be an undirected simple and connected graph with n, $n \geq 3$ vertices and m edges. Then*

$$LE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{(n-1)(M_1+2m)-4m^2}\,. \tag{3.2}$$

Proof. For $n-1$ and $p_i := \frac{1}{n-1}$, $a_i := \mu_i$, $i = 1,2,\ldots n-1$, $r := \mu_{n-1}$ and $R := \mu_1$, the inequality (2.5) transforms into

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\mu_i\right)^2 \leq \frac{\mu_1-\mu_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\mu_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i\right|$$

i.e., based on Lemma 1,

$$(n-1)(M_1+2m)-4m^2 \leq \frac{n-1}{2}(\mu_1-\mu_{n-1})\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right|.$$

Since

$$\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right| \leq \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = LE(G)$$

using inequality (2.3), from the above inequality we obtain (3.2). □

Using Lemma 2, we arrive at the following (n,m)-type lower bound for the Laplacian energy:

Corollary 3. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$LE(G) \geq \sqrt{\frac{4m(n(n-1)-2m)}{n(n-1)}}\,. \tag{3.3}$$

Corollary 4. *Let G be an undirected simple and connected k-regular graph with n, $n \geq 3$, vertices and m edges, $1 < k \leq n-1$. Then*

$$LE(G) > \sqrt{\frac{2nk(n-k-1)}{n-1}}\,.$$

Remark 3. Since for undirected k-regular graphs, $LE = E$, the inequality in Corollary 4 provides a lower bound also for the ordinary energy.

Inequalities (3.1) and (3.2) are incomparable. Thus, for example, if $G \cong K_n$, then inequality (3.1) is stronger than (3.2), but if $G \cong K_{1,n-1}$, $n \geq 8$, then the opposite is valid.

3.2. *Lower bound for signless Laplacian energy*

Theorem 3. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$SLE(G) \geq \sqrt{\frac{2(n(M_1+2m)-4m^2)}{n}} \,. \tag{3.4}$$

Proof. For $p_i := \frac{1}{n}$, $a_i = \gamma_i$, $i = 1,2,\ldots,n$, $r = \gamma_n$ and $R = \gamma_1$, the inequality (2.5) becomes

$$\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n}\gamma_i\right)^2 \leq \frac{\gamma_1-\gamma_n}{2n}\sum_{i=1}^{n}\left|\gamma_i - \frac{2m}{n}\right| .$$

Bearing in mind Lemma 8, the above inequality becomes

$$n(M_1+2m)-4m^2 \leq \frac{n}{2}\,SLS(G)\times SLE(G) \,.$$

By Lemma 9 and the above inequality, we obtain (3.4). □

Bearing in mind Lemma 2 and inequality (3.4), we arrive at a lower bound for $SLE(G)$ depending only on the parameter m.

Corollary 5. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$SLE(G) \geq 2\sqrt{m} \,.$$

Corollary 6. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges, which is k-regular, $1 < k \leq n$. Then*

$$SLE(G) \geq \sqrt{2nk} \,.$$

3.3. *Lower bound for normalized Laplacian energy*

Theorem 4. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Let, as before, $R_{-1} = \sum_{i\sim j}\frac{1}{d_i\,d_j}$. Then*

$$NLS(G) \leq \sqrt{\frac{2}{n-1}}\,\sqrt{2(n-1)R_{-1}-n} \,. \tag{3.5}$$

Equality holds if and only if $G \cong K_n$.

Proof. According to (2.6) we have that

$$\begin{aligned}(n-1)(n+2R_{-1})-n^2 &= (n-1)\sum_{i=1}^{n-1}\rho_i^2 - \left(\sum_{i=1}^{n-1}\rho_i\right)^2 \\ &= \sum_{1\leq i<j\leq n-1}(\rho_i-\rho_j)^2 \,. \end{aligned} \tag{3.6}$$

By Lemma 5, i.e., by inequality (2.4), for $n = 2$ and $\alpha = 2$, we get

$$(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2 \geq \frac{1}{2}(\rho_1 - \rho_{n-1})^2 \tag{3.7}$$

for each $i = 2, 3, \ldots, n-2$. Then,

$$\begin{aligned}\sum_{1 \leq i < j \leq n-1} (\rho_i - \rho_j)^2 &\geq \sum_{i=2}^{n-2} [(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2] + (\rho_1 - \rho_{n-1})^2 \\ &\geq \frac{n-3}{2}(\rho_1 - \rho_{n-1})^2 + (\rho_1 - \rho_{n-1})^2 \\ &= \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2\end{aligned}$$

which combined with (3.6) yields

$$(n-1)(n+2R_{-1}) - n^2 = 2(n-1)R_{-1} - n \geq \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2$$

from which the inequality (3.5) follows.

Equality in (3.7) holds if and only if $\rho_1 = \rho_2 = \cdots = \rho_{n-1}$. Therefore, equality in (3.5) holds if and only if $G \cong K_n$. This completes the proof of Theorem 4. □

Corollary 7. *Let G be an undirected simple nad connected k-regular graph, $1 < k \leq n-1$, with n, $n \geq 3$, vertices and m edges. Then*

$$NLS(G) \leq \sqrt{\frac{2n(n-k-1)}{(n-1)k}}\,.$$

Equality holds if and only if $k = n-1$, i.e., $G \cong K_n$.

We now state a theorem, analogous to Theorem 2, which provides a lower bound for NLE in terms of parameters n and R_{-1}.

Theorem 5. *Let G be an undirected simple and connected graph with n, $n \geq 3$, vertices and m edges. Then*

$$NLE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{2(n-1)R_{-1} - n}\,. \tag{3.8}$$

Proof. For $n := n-1$, $p_i := \frac{1}{n-1}$, $a_i := \rho_i$, $i = 1, 2, \ldots, n-1$, $r = \rho_{n-1}$ and $R = \rho_1$, inequality (2.5) becomes

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\rho_i\right)^2 \leq \frac{\rho_1 - \rho_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\rho_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i\right|.$$

Having in mind Lemma 7, the above inequality transforms into

$$(n-1)(n+2R_{-1}) - n^2 \leq \frac{n-1}{2}NLS(G)\sum_{i=1}^{n-1}\left|\rho_i - \frac{n}{n-1}\right|\,. \tag{3.9}$$

Since

$$\sum_{i=1}^{n-1}\left|\rho_i - \frac{n}{n-1}\right| \le \sum_{i=1}^{n}|\rho_i - 1|$$

according to (3.9) we obtain

$$(n-1)(n+2R_{-1})-n^2 \le \frac{n-1}{2}\,NLS(G)NLE(G)\,. \tag{3.10}$$

Combining (3.5) and (3.10) we arrive at (3.8). □

Remark 4. For a k-regular graph, $R_{-1} = m/k^2 = n/(2k)$. Since for k-regular graphs, $NLE = \frac{1}{k}E = \frac{1}{k}LE$, inequality (3.8) is equivalent to the result proven in Corollary 4.

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Doi: 10.18514/MMN.2015.1369

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 205–211

HU e-ISSN 1787-2413

SIMPLE PROOFS OF SOME THEOREMS IN RESULTANT THEORY

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Received 10 October, 2014

Abstract. In this note, using elementary results from algebraic geometry, we give new and simple proofs of some known theorems in resultant theory.

2010 *Mathematics Subject Classification:* 13P15

Keywords: homogeneous polynomials, multivariate resultants, three ternary quadrics system

1. INTRODUCTION

The multivariate resultant is a fundamental tool of computational algebraic geometry which was introduced by *F. S. Macaulay* [4] in 1902 after earlier work due to Euler, Sylvester and Cayley. The resultant of a polynomial system has many important properties for the geometry of the variety that the system defines. Indeed, the resultant is an algebraic condition in terms of the coefficients of a given system of polynomials which is satisfied if and only if the system has a common solution. More precisely, suppose that K is an algebraically closed field. Let $f = a_0 + \cdots + a_l x^l$ and $g = b_0 + \cdots + b_m x^m$ be two univariate polynomials where a_i's and b_i's belong to K and $a_l, b_m \neq 0$. The *Sylvester* matrix of f, g is defined as

$$S(f,g) = \begin{bmatrix}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & & & b_1 & b_0 & & \\
a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\
\vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\
a_l & \vdots & \ddots & a_1 & b_m & \vdots & \ddots & b_1 \\
 & a_l & & a_2 & & b_m & & b_2 \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & a_l & & & & b_m
\end{bmatrix}.$$

The univariate resultant of f and g is the determinant of $S(f,g)$ and is denoted by $\mathrm{Res}(f,g)$. It is well known that $\mathrm{Res}(f,g) = 0$ if and only if f and g have a

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nontrivial common root (see [1, Chapter 3, Proposition 1.7]). A generalization of this result can be used to decide whether a system of $n+1$ homogeneous equations in $n+1$ variables has a solution or not. Throughout this note we let $R = K[x_0,\dots,x_n]$. To state the next theorem, we need to introduce some notation. Let $F_0,\dots,F_n \in R$ be $n+1$ homogeneous polynomials of degrees $d_0,\dots,d_n$. So, we can write F_i as $\sum_{|\alpha|=d_i} c_{i,\alpha}x^\alpha$ where some of the $c_{i,\alpha}$'s may be zero. For each possible pair of indices i,α, we introduce a new variable $u_{i,\alpha}$. Let us associate a polynomial $P \in \mathbb{Z}[u_{i,\alpha}]$ to the *universal* homogeneous polynomials $\sum_{|\alpha|=d_i} u_{i,\alpha}x^\alpha$ for $i = 0,\dots,n$. Then, when the polynomial P for a particular collection of polynomials $\sum_{|\alpha|=d_i} c_{i,\alpha}x^\alpha$ for $i = 0,\dots,n$ is considered, for each i and α, the variable $u_{i,\alpha}$ is substituted by $c_{i,\alpha}$ into P.

Theorem 1 ([1, Chapter 3, Theorem 2.3])**.** *Let us fix the degrees $d_0,\dots,d_n$. Then there is a unique polynomial* $\mathrm{Res} \in \mathbb{Z}[u_{i,\alpha}]$ *which has the following properties:*

(1) *If $F_0,\dots,F_n \in R$ are homogeneous of degrees, respectively $d_0,\dots,d_n$, then the system $F_0 = \cdots = F_n = 0$ have a nontrivial solution over K if and only if* $\mathrm{Res}(F_0,\dots,F_n) = 0$,
(2) $\mathrm{Res}(x_0^{d_0},\dots,x_n^{d_n}) = 1$,
(3) Res *is irreducible as a polynomial in* $K[u_{i,\alpha}]$.

If we fix the degrees $d_0,\dots,d_n$, then the polynomial $\mathrm{Res} = \mathrm{Res}_{d_0,\dots,d_n} \in \mathbb{Z}[u_{i,\alpha}]$ is called the *resultant* polynomial corresponding to $d_0,\dots,d_n$. Further, the constant $\mathrm{Res}(F_0,\dots,F_n) \in K$ is called the *resultant* of $F_0,\dots,F_n$. For more details, we refer the reader to the books [1,2,5]. The resultant has many algebraic properties that make it a convenient tool in constructive algebra. In this note, we consider the symmetry and multiplicativity property of resultant.

Theorem 2. *Suppose that $F_0,\dots,F_n \in R$ are homogeneous of degrees, respectively $d_0,\dots,d_n$.*

(*a*) *If $i < j$ then*

$$\mathrm{Res}(F_0,\dots,F_i,\dots,F_j,\dots,F_n) = (-1)^{d_0\cdots d_n}\mathrm{Res}(F_0,\dots,F_j,\dots,F_i,\dots,F_n).$$

(*b*) *If $F_j = F_j'F_j''$ is a product of homogeneous polynomials of degrees d_j' and d_j'' then*

$$\mathrm{Res}(F_0,\dots,F_n) = \mathrm{Res}(F_0,\dots,F_j',\dots,F_n)\mathrm{Res}(F_0,\dots,F_j'',\dots,F_n).$$

In 1991, Jouanolou [3, Section 5], proved this theorem, however, in Section 2 we give a new and elementary proof for it. Further, in Section 3, we use some effective elementary algebraic geometry results to prove the necessary and sufficient conditions for the satisfiability of the "three ternary quadrics" system. For the classical proof of this result, we refer the reader to ([6, Art. 90] and [1, page 88]).

2. THE PROOF OF THEOREM 2

In order to prove this theorem, we need some properties of ideals and varieties. Let $R = K[x_0, \ldots, x_n]$ be a polynomial ring over an algebraically closed field K and $I \subset R$ a homogeneous ideal. Further, let $f_1, \ldots, f_k$ be polynomials (not necessarily homogeneous) in R. Then, the *variety* defined by $f_1, \ldots, f_k$ is the set

$$\mathbf{V}(f_1, \ldots, f_k) = \{(a_0, \ldots, a_n) \in K^{n+1} \mid f_i(a_0, \ldots, a_n) = 0 \text{ for all } i\}.$$

A subset $V \subset K^{n+1}$ is called a variety if there exist $f_1, \ldots, f_k \in R$ so that $V = \mathbf{V}(f_1, \ldots, f_k)$. Further, the ideal of a variety V is defined to be
$\mathbf{I}(V) = \{f \in R \mid f(a_0, \ldots, a_n) = 0 \text{ for all } (a_0, \ldots, a_n) \in V\}$.

Theorem 3 (Hilbert's Nullstellensatz)**.** *Suppose that $I = \langle f_1, \ldots, f_k \rangle$ is the ideal generated by the f_i's. Then, $\mathbf{I}(\mathbf{V}(f_1, \ldots, f_k)) = \sqrt{I}$.*

For more details on this topic we refer the reader to [1, 2].

Theorem 4 ([1, Theorem 3.1, page 95])**.** *Let us fix the degrees $d_0, \ldots, d_n$, then the polynomial $\mathrm{Res}_{d_0,\ldots,d_n}$ has the total degree $\sum_{j=0}^{n} d_0 \cdots d_{j-1} d_{j+1} \cdots d_n$.*

We continue with some notation. Let $F \in R$ be a homogeneous polynomial. Then, we denote the polynomial $F(x_0, \ldots, x_{n-1}, 0)$ and $F(x_0, \ldots, x_{n-1}, 1)$ by $\bar{F}(x_0, \ldots, x_{n-1})$ and $f(x_0, \ldots, x_{n-1})$, respectively.

Theorem 5 ([1, Theorem 3.4, page 96])**.** *If $\mathrm{Res}(\bar{F}_0, \ldots, \bar{F}_{n-1}) \neq 0$, then we have $\mathrm{Res}(F_0, \ldots, F_n) = \mathrm{Res}(\bar{F}_0, \ldots, \bar{F}_{n-1})^{d_n} . \det(m_{f_n})$ where m_{f_n} is the linear multiplication map by f_n on $K[x_0, \ldots, x_{n-1}]/\langle f_0, \ldots, f_{n-1} \rangle$.*

Proof. of Theorem 2: (a) Let us denote by $R_{i,j}$ and $R_{j,i}$ the polynomials $\mathrm{Res}(F_0, \ldots, F_n)$ and $\mathrm{Res}(F_0, \ldots, F_j, \ldots, F_i, \ldots, F_n)$, respectively. By Theorem 1, if $R_{j,i}$ vanishes, then

$$F_0 = \cdots = F_j = \cdots = F_i = \cdots = F_n = 0$$

has a nontrivial solution. On the other hand, every nontrivial solution of this system is a nontrivial solution of $F_0 = \cdots = F_n = 0$. Hence $R_{i,j}$ vanishes on the set $\mathbf{V}(R_{j,i})$. This shows that

$$R_{i,j} \in \mathbf{I}(\mathbf{V}(R_{j,i})) = \sqrt{\langle R_{j,i} \rangle}.$$

However, $R_{j,i}$ is irreducible, which implies that $R_{i,j} \in \langle R_{j,i} \rangle$. By Theorem 4, $R_{i,j}$ and $R_{j,i}$ have the same degree. This follows that there exists a constant c so that $R_{i,j} = c R_{j,i}$. We shall prove that $c = (-1)^{d_0 \cdots d_n}$. Since c is constant, it's enough to show that

$$\mathrm{Res}(x_0^{d_0}, \ldots, x_n^{d_n}) = (-1)^{d_0 \cdots d_n} \mathrm{Res}(x_0^{d_0}, \ldots, x_j^{d_j}, \ldots, x_i^{d_i}, \ldots, x_n^{d_n}).$$

In doing so, we will use induction on the number of polynomials. For $n = 2$, the claim is implied by the main properties of determinants. Now assume that the claim holds

for n polynomials of degrees $d_0,\dots,d_{n-1}$. By applying Theorem 5 on $x_0^{d_0},\dots,x_n^{d_n}$ and $x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n}$, it follows respectively that

$$\det(m_{x_n^{d_n}}) = \frac{\mathrm{Res}(x_0^{d_0},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_{n-1}^{d_{n-1}})^{d_n}}$$

$$\det(m_{x_n^{d_n}}) = \frac{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_{n-1}^{d_{n-1}})^{d_n}}.$$

Thus, we can write

$$\frac{\mathrm{Res}(x_0^{d_0},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_{n-1}^{d_{n-1}})^{d_n}} = \frac{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_{n-1}^{d_{n-1}})^{d_n}}$$

and therefore from the induction hypothesis we can conclude that

$$\begin{aligned}
&\mathrm{Res}(x_0^{d_0},\dots,x_n^{d_n})\\
=\ &\frac{\mathrm{Res}(x_0^{d_0},\dots,x_{n-1}^{d_{n-1}})^{d_n}\cdot\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_{n-1}^{d_{n-1}})^{d_n}}\\
=\ &\frac{(-1)^{d_0\cdots d_n}\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_{n-1}^{d_{n-1}})^{d_n}\cdot\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n})}{\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_{n-1}^{d_{n-1}})^{d_n}}\\
=\ &(-1)^{d_0\cdots d_n}\mathrm{Res}(x_0^{d_0},\dots,x_j^{d_j},\dots,x_i^{d_i},\dots,x_n^{d_n})
\end{aligned}$$

which proves the assertion. To prove (b), let us denote by R, R', R'' the polynomials $\mathrm{Res}(F_0,\dots,F_j,\dots,F_n), \mathrm{Res}(F_0,\dots,F_j',\dots,F_n), \mathrm{Res}(F_0,\dots,F_j'',\dots,F_n)$. We shall show that $R = R'R''$. Note that R may be considered as a polynomial in the coefficients of $F_0,\dots,F_j',F_j'',\dots,F_n$. By Theorem 1, if either R' or R'' vanishes, then either the system $F_0 = \cdots = F_j' = \cdots = F_n = 0$ or the system $F_0 = \cdots = F_j'' = \cdots = F_n = 0$ has a nontrivial solution. Every nontrivial solution of these systems is a nontrivial solution of the system $F_0 = \cdots = F_n = 0$. Note that we consider R, R' and R'' as the polynomials in the coefficients of polynomials $F_0,\dots,F_j',F_j'',\dots,F_n,$. Then due to the irreducibility of R' and R'', we have

$$R \in \mathbf{I}(\mathbf{V}(R')) = \sqrt{\langle R'\rangle} = \langle R'\rangle, \qquad R \in \mathbf{I}(\mathbf{V}(R'')) = \sqrt{\langle R''\rangle} = \langle R''\rangle.$$

This proves that $R' \mid R$ and $R'' \mid R$. It should be noted that R' and R have degrees 0 and $d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ in the coefficients of polynomial F_j'', respectively. Also, R' and R have degree $d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ in the coefficients of polynomial F_j'. Thus, there exists a polynomial g such that $R = gR'$ where g has degree 0 and $d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ in the coefficients of F_j' and F_j'', respectively. From $R \in \langle R''\rangle$ it implies that $R'' \mid gR'$. On the other hand, R'' is irreducible, which implies that R'' divides either g or R'. Since R' and R'' have degrees 0 and $d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ in

the coefficients of F_j'', it follows that $R'' \mid g$. Since R'' and g have the same degree in the coefficients of polynomial F_j'' there exists a polynomial c of degree 0 in the coefficients of F_j'' such that $g = cR''$. Thus, we conclude that $R = cR'R''$. Note that R and $R'R''$ have the same total degree. Therefore c is constant. From the equality

$$F_0 = x_0^{d_0}, \ldots, F_j' = x_j^{d_j'}, F_j'' = x_j^{d_j''}, \ldots, F_n = x_n^{d_n},$$

we see that $R = R' = R'' = 1$ and thus $c = 1$, which implies that $R = R'R''$ and this ends the proof. □

3. THREE TERNARY QUADRICS SYSTEM

In this section we consider the classical system of three ternary quadrics. This is the following system

$$\begin{array}{lllllllllllll} F_0 = c_{01}x^2 & + & c_{02}y^2 & + & c_{03}z^2 & + & c_{04}xy & + & c_{05}xz & + & c_{06}yz = 0 \\ F_1 = c_{11}x^2 & + & c_{12}y^2 & + & c_{13}z^2 & + & c_{14}xy & + & c_{15}xz & + & c_{16}yz = 0 \\ F_2 = c_{21}x^2 & + & c_{22}y^2 & + & c_{23}z^2 & + & c_{24}xy & + & c_{25}xz & + & c_{26}yz = 0 \end{array}$$

where the c_{ij}'s are parameters. By Theorem 1, $\mathrm{Res}(F_0, F_1, F_2)$ vanishes exactly when this system has a nontrivial solution in x, y and z. However, $\mathrm{Res}(F_0, f_1, F_2)$ is a large polynomial in 18 variables with 21894 terms (see [1, page 88]). The aim of this section is to provide a new and simple proof for a compact representation of $\mathrm{Res}(F_0, F_1, F_2)$. For the original proof, we refer to [6, Art. 90]. Let us denote by J the Jacobian determinant of F_0, F_1, F_2 w.r.t. the variables x, y and z. Then, the partial derivatives of J are quadratic and hence we can write

$$\begin{array}{lllllllllllll} \frac{\partial J}{\partial x} & = & b_{01}x^2 & + & b_{02}y^2 & + & b_{03}z^2 & + & b_{04}xy & + & b_{05}xz & + & b_{06}yz \\ \frac{\partial J}{\partial y} & = & b_{11}x^2 & + & b_{12}y^2 & + & b_{13}z^2 & + & b_{14}xy & + & b_{15}xz & + & b_{16}yz \\ \frac{\partial J}{\partial z} & = & b_{21}x^2 & + & b_{22}y^2 & + & b_{23}z^2 & + & b_{24}xy & + & b_{25}xz & + & b_{26}yz. \end{array}$$

Proposition 1.

$$\mathrm{Res}(F_0, F_1, F_2) = \frac{-1}{512} \det \begin{bmatrix} c_{01} & c_{02} & c_{03} & c_{04} & c_{05} & c_{06} \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ b_{01} & b_{02} & b_{03} & b_{04} & b_{05} & b_{06} \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}.$$

Proof. We refer to the right hand side matrix as A. If we regard the monomials $x^2, y^2, z^2, xy, xz, yz$ as unknowns, then $F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z}$ are linear and

$$\mathrm{Res}_{1,1,1,1,1,1}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z}) = \det(A).$$

Note that each b_{ij} is a cubic polynomial in the c_{ij}'s. Hence the polynomial $\mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z})$ has total degree 12 in $c_{01}, \ldots, c_{26}$. By Theorem 1, the resultant vanishes iff

$$F_0 = F_1 = F_2 = \frac{\partial J}{\partial x} = \frac{\partial J}{\partial y} = \frac{\partial J}{\partial z} = 0$$

has a nontrivial solution $(x^2, y^2, z^2, xy, xz, yz)$. Also, it can be easily verified that every nontrivial solution of this system is a nontrivial solution of $F_0 = F_1 = F_2 = 0$. Hence (x, y, z) is a nontrivial solution of $F_0 = F_1 = F_2 = 0$. Then $\mathrm{Res}_{2,2,2}(F_0, F_1, F_2) = 0$. Thus $\mathrm{Res}_{2,2,2}(F_0, F_1, F_2)$ vanishes on the set

$$\mathbf{V}(\mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z})).$$

This means that $\mathrm{Res}_{2,2,2}(F_0, F_1, F_2)$ belongs to the ideal of this variety, i.e.

$$\mathrm{Res}_{2,2,2}(F_0, F_1, F_2) \in \sqrt{\langle \mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z}) \rangle}.$$

On the other hand, if we substitute every b_{ij} in the above matrix by the corresponding cubic polynomial in the c_{ij}'s, using the function `irreduc` of MAPLE, we can see easily that $\det(A)$ and therefore

$$\mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z})$$

is irreducible in the c_{ij}'s which yields that

$$\mathrm{Res}_{2,2,2}(F_0, F_1, F_2) \in \langle \mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z}) \rangle.$$

By Theorem 4, $\mathrm{Res}_{2,2,2}(F_0, F_1, F_2)$ has the total degree 12 in $c_{01}, \ldots, c_{26}$. Hence, for a $c \in K$, we have

$$\mathrm{Res}_{2,2,2}(F_0, F_1, F_2) = c \cdot \mathrm{Res}(F_0, F_1, F_2, \frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial z}).$$

But, for the special case; if we set $(F_0, F_1, F_2) = (x^2, y^2, z^2)$, we can conclude that

$$\mathrm{Res}(x^2, y^2, z^2, 8yz, 8xz, 8xy) = \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{bmatrix} = -512.$$

Finally, by Theorem 1, we have $\mathrm{Res}_{2,2,2}(x^2, y^2, z^2) = 1$ and so $c = \frac{-1}{512}$. □

ACKNOWLEDGEMENTS

The research of the first author was supported in part by a grant from IPM (No. 93550420). The authors are grateful to the referee and Prof. David Cox for their useful comments and advices for the improvement of this note.

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Doi: 10.18514/MMN.2015.1050

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 213–217

HU e-ISSN 1787-2413

A NOTE ON A SERIES CONTAINING THE LAGUERRE POLYNOMIALS

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Received 05 November, 2013

Abstract. Expressions for the summation of the series involving the Laguerre polynomials

$$S(\pm\nu,\pm j)\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1\pm\nu\pm j)_n}$$

for any non-negative integer j are obtained in terms of generalized hypergeometric functions. These results provide alternative, and in some cases simpler expressions to those recently obtained in the literature.

2010 *Mathematics Subject Classification:* 33C15; 33C20

Keywords: Laguerre polynomials, generalized hypergeometric functions, generalized Kummer summation theorem

1. Introduction

The generalized Laguerre polynomials $L_n^{(\nu)}(x)$ are encountered in many branches of pure and applied mathematics. They form an orthogonal set on $[0,\infty)$ with the weight function $x^\nu e^{-x}$, with the first three polynomials given by

$$\begin{aligned}L_0^{(\nu)}(x)&=1,\\ L_1^{(\nu)}(x)&=1-x+\nu,\\ L_2^{(\nu)}(x)&=\frac{1}{2}x^2-(\nu+2)x+\frac{1}{2}(\nu+1)(\nu+2).\end{aligned}$$

In general, $L_n^{(\nu)}(x)$ can be represented as a terminating confluent hypergeometric function ${}_1F_1$ in the form

$$L_n^{(\nu)}(x)=\frac{(\nu+1)_n}{n!}\,{}_1F_1(-n;\nu+1;x).$$

Here $(a)_n$ denotes the Pochhammer symbol, or rising factorial, defined by $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$.

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In [2], Kim *et al.* obtained summation formulas for the two series involving the generalized Laguerre polynomial $L_n^{(\nu)}(x)$ given by

$$\sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1 \pm \nu + j)_n}$$

for integer j, where $-5 \le j \le 5$. Recently, Brychkov [1] has extended these results for any integer j. The aim of this note is to derive alternative expressions for the summation of the series

$$S(\pm\nu, \pm j) \equiv e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1 \pm \nu \pm j)_n}$$

for any non-negative integer j. Our results are different from, and in some cases simpler, than those obtained in [1].

2. The series $S(\nu, \pm j)$

We start with the transformation [2, (3.5)]

$$e^{-x} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} (-xy)^n L_n^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} {}_{p+2}F_q \left[\begin{matrix} -n, -n-\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; y \right], \quad (2.1)$$

where p and q are non-negative integers and ${}_pF_q$ denotes the generalized hypergeometric function. In this, if we take $p = 0$, $q = 1$, $b_1 = 1 + \nu + j$ and $y = -1$, then

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu+j)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} {}_2F_1 \left[\begin{matrix} -n, -n-\nu \\ 1+\nu+j \end{matrix} ; -1 \right]. \quad (2.2)$$

The ${}_2F_1$ series on the right-hand side of (2.2) can be evaluated with the help of the generalized Kummer summation theorem [3]

$$ {}_2F_1 \left[\begin{matrix} a, b \\ 1+a-b+j \end{matrix} ; -1 \right] = \frac{2^{-a} \Gamma(\frac{1}{2}) \Gamma(b-j) \Gamma(1+a-b+j)}{\Gamma(b) \Gamma(\frac{1}{2}a - b + \frac{1}{2}j + \frac{1}{2}) \Gamma(\frac{1}{2}a - b + \frac{1}{2}j + 1)} \times \sum_{r=0}^{j} (-1)^r \binom{j}{r} \frac{\Gamma(\frac{1}{2}a - b + \frac{1}{2}j + \frac{1}{2}r + \frac{1}{2})}{\Gamma(\frac{1}{2}a - \frac{1}{2}j + \frac{1}{2}r + \frac{1}{2})} \quad (2.3)$$

for $j = 0, 1, 2, \dots$.

After some straightforward simplification, we obtain

$$S(\nu, j) \equiv e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu+j)_n}$$

$$= \frac{(-1)^j 2^{2\nu+j}\,\Gamma(1+\nu)}{\Gamma(1+2\nu+j)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \left\{ \frac{\Gamma(\nu+\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}j+\frac{1}{2}r)} \right.$$
$$\times {}_4F_5\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}\nu,\ 1+\frac{1}{2}\nu,\ \frac{1}{2}+\nu+\frac{1}{2}j+\frac{1}{2}r,\ \frac{1}{2}+\frac{1}{2}j-\frac{1}{2}r \\ \frac{1}{2},\ \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}j,\ 1+\frac{1}{2}\nu+\frac{1}{2}j,\ \frac{1}{2}+\nu+\frac{1}{2}j,\ 1+\nu+\frac{1}{2}j \end{matrix} ; -x^2 \right]$$
$$- \frac{4x(1+\nu)}{(1+\nu+j)(1+2\nu+j)} \frac{\Gamma(\nu+\frac{1}{2}j+\frac{1}{2}r+1)}{\Gamma(\frac{1}{2}r-\frac{1}{2}j)}$$
$$\left. \times {}_4F_5\left[\begin{matrix} 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu, 1+\nu+\frac{1}{2}j+\frac{1}{2}r, 1+\frac{1}{2}j-\frac{1}{2}r \\ \frac{3}{2}, 1+\frac{1}{2}\nu+\frac{1}{2}j, \frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}j, 1+\nu+\frac{1}{2}j, \frac{3}{2}+\nu+\frac{1}{2}j \end{matrix} ; -x^2 \right] \right\} \quad (2.4)$$

for $j = 0, 1, 2, \ldots$.

Again, in (2.1), if we take $p = 0$, $q = 1$, $b_1 = 1+\nu-j$ and $y = -1$, then

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu-j)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} {}_2F_1\left[\begin{matrix} -n, -n-\nu \\ 1+\nu-j \end{matrix} ; -1 \right]. \quad (2.5)$$

The ${}_2F_1$ series on the right-hand side of (2.5) can be evaluated with the help of the known result [3]

$${}_2F_1\left[\begin{matrix} a, b \\ 1+a-b-j \end{matrix} ; -1 \right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-j)}{\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+1)}$$
$$\times \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})} \quad (2.6)$$

for $j = 0, 1, 2, \ldots$ and, after some simplification, we obtain

$$S(\nu, -j) \equiv e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu-j)_n}$$
$$= \frac{2^{2\nu-j}\,\Gamma(1+\nu-j)}{\Gamma(1+2\nu-j)} \sum_{r=0}^{j} \binom{j}{r} \left\{ \frac{\Gamma(\nu-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}j+\frac{1}{2}r)} \right.$$
$$\times {}_2F_3\left[\begin{matrix} \frac{1}{2}+\nu-\frac{1}{2}j+\frac{1}{2}r, \frac{1}{2}+\frac{1}{2}j-\frac{1}{2}r \\ \frac{1}{2}, \frac{1}{2}+\nu-\frac{1}{2}j, 1+\nu-\frac{1}{2}j \end{matrix} ; -x^2 \right]$$
$$\left. - \frac{4x\Gamma(\nu-\frac{1}{2}j+\frac{1}{2}r+1)}{(1+2\nu-j)\Gamma(\frac{1}{2}r-\frac{1}{2}j)} {}_2F_3\left[\begin{matrix} 1+\nu-\frac{1}{2}j+\frac{1}{2}r, 1+\frac{1}{2}j-\frac{1}{2}r \\ \frac{3}{2}, 1+\nu-\frac{1}{2}j, \frac{3}{2}+\nu-\frac{1}{2}j \end{matrix} ; -x^2 \right] \right\} \quad (2.7)$$

for $j = 0, 1, 2, \ldots$.

3. THE SERIES $S(-\nu,\pm j)$

Further, if we take $p=0$, $q=1$, $b_1=1-\nu+j$ and $y=-1$ in (2.1), we find

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1-\nu+j)_n}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,-n-\nu\\1-\nu+j\end{matrix};-1\right]. \tag{3.1}$$

The ${}_2F_1$ series on the right-hand side of (3.1) can be evaluated by (2.3) to produce the result after some simplification

$$\begin{aligned}S(-\nu,j)&\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1-\nu+j)_n}\\&=\frac{(-2)^j}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\left\{\frac{\Gamma(-\frac12\nu+\frac12 j+\frac12 r+\frac12)}{\Gamma(-\frac12\nu-\frac12 j+\frac12 r+\frac12)}\right.\\&\quad\times{}_3F_4\left[\begin{matrix}1,\frac12+\frac12\nu+\frac12 j-\frac12 r,\frac12-\frac12\nu+\frac12 j+\frac12 r\\\frac12+\frac12 j,1+\frac12 j,\frac12-\frac12\nu+\frac12 j,1-\frac12\nu+\frac12 j\end{matrix};-x^2\right]\\&\quad-\frac{4x}{(j+1)(1-\nu+j)}\,\frac{\Gamma(-\frac12\nu+\frac12 j+\frac12 r+1)}{\Gamma(-\frac12\nu-\frac12 j+\frac12 r)}\\&\quad\left.\times{}_3F_4\left[\begin{matrix}1,1+\frac12\nu+\frac12 j-\frac12 r,1-\frac12\nu+\frac12 j+\frac12 r\\1+\frac12 j,\frac32+\frac12 j,1-\frac12\nu+\frac12 j,\frac32-\frac12\nu+\frac12 j\end{matrix};-x^2\right]\right\}\end{aligned} \tag{3.2}$$

for $j=0,1,2,\ldots$.

Finally, if we take $p=0$, $q=1$, $b_1=1-\nu-j$ and $y=-1$ in (2.1), we find

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1-\nu-j)_n}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,-n-\nu\\1-\nu-j\end{matrix};-1\right]. \tag{3.3}$$

The ${}_2F_1$ series on the right-hand side of (3.3) can be evaluated by (2.6) to produce the result after some simplification

$$\begin{aligned}S(-\nu,-j)&\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1-\nu-j)_n}\\&=2^{-j}\sum_{r=0}^{j}\binom{j}{r}\left\{{}_2F_3\left[\begin{matrix}\frac12-\frac12\nu-\frac12 j+\frac12 r,\frac12+\frac12\nu+\frac12 j-\frac12 r\\\frac12,\frac12-\frac12\nu-\frac12 j,1-\frac12\nu-\frac12 j\end{matrix};-x^2\right]\right.\\&\quad\left.-\frac{2x(\nu+j-r)}{\nu+j-1}\,{}_2F_3\left[\begin{matrix}1-\frac12\nu-\frac12 j+\frac12 r,1+\frac12\nu+\frac12 j-\frac12 r\\\frac32,1-\frac12\nu-\frac12 j,\frac32-\frac12\nu-\frac12 j\end{matrix};-x^2\right]\right\}\end{aligned} \tag{3.4}$$

for $j=0,1,2,\ldots$.

4. Concluding remarks

To conclude we make a brief comparison of the results (2.4), (2.7), (3.2) and (3.4) with those obtained in [1]. The summations $S(\nu, \pm j)$ derived by Brychkov were expressed respectively in terms of finite sums of ${}_2F_3(-x^2)$ functions and Bessel functions of the first kind. The summations $S(-\nu, \pm j)$ were expressed respectively in terms of finite sums of four ${}_4F_3(-x^2)$ functions and four ${}_6F_7(-x^2)$ functions, including the Jacobi polynomials of zero argument. Our expressions in (3.2) and (3.4) involve simpler finite sums of two ${}_3F_4(-x^2)$ and two ${}_2F_3(-x^2)$ functions, respectively.

Finally, we mention that the summations $S(\pm\nu, \pm j)$ have been verified numerically with the help of *Mathematica*.

Acknowledgement

One of the authors (YSK) acknowledges the support of the Wonkwang University Research Fund (2014).

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Doi: 10.18514/MMN.2015.1253

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 219–235

HU e-ISSN 1787-2413

NEW INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE s-LOGARITHMICALLY CONVEX FUNCTIONS WITH APPLICATIONS

M. A. LATIF AND S. S. DRAGOMIR

Received 18 May, 2014

Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type are presented for functions whose nth derivatives in absolute value are s-logarithmically convex. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are s-logarithmically convex functions as special cases. Our results may provide refinements of some results for s-logarithmically convex functions already exist in literature. Finally, applications to special means of the established results are given.

2010 *Mathematics Subject Classification:* 26D15; 26D20; 26E60; 41A55

Keywords: Hermite-Hadamard's inequality, s-logarithmically convex function, Hölder integral inequality

1. INTRODUCTION

A function $f : I \to \mathbb{R}$, $\varnothing \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known as the Hermite-Hadamard inequality (see [8]). The inequalities (1.1) hold in reversed direction if f is concave.

For recent results on Hermite-Hadamard type integral inequalities for convex functions see [5, 7, 10–13, 15, 18, 19] and closely related references therein.

The classical convexity has been generalized in diverse ways such as s-convexity, m-convexity, (α, m)-convexity, h-convexity, logarithmically-convexity, s- logarithmically convexity, (α, m)- logarithmically convexity and h-log-convexity. Many papers have been written by a number of mathematicians concerning Hermite-Hadamard

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type inequalities for these classes of convex functions see for instance the recent papers [1–4, 6, 8, 9, 14, 17, 20–25, 27] and the references therein.

The notion of logarithmically convex functions is defined as follows.

Definition 1 ([1, 25, 26]). If a function $f : I \subseteq \mathbb{R} \to (0,\infty)$ satisfies

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda}, \tag{1.2}$$

for all $x, y \in I$, $\lambda \in [0,1]$, the function f is called logarithmically convex on I. If the inequality (1.2) reverses, the function f is called logarithmically concave on I.

The concept of logarithmically convex functions was further generalized as in the definition below.

Definition 2 ([1, 25, 26]). For some $s \in (0,1]$, a positive function $f : I \subseteq \mathbb{R} \to (0,\infty)$ is said to be s-logarthmically convex on I if and only if

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda^s} [f(y)]^{(1-\lambda)^s}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

It is obvious that when $s = 1$ in Definition 2, the s-logarithmically convexity becomes the usual logarithmically convexity.

Xi et al. [25], obtained the following Hermite-Hadamard type inequalities for s-logarithmically convex functions.

Theorem 1 ([25]). *Let $f : I \subseteq [0,\infty) \to (0,\infty)$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. If $\left|f'\right|^q$ for $q \geq 1$ is s-logarithmically convex on $[a,b]$ for some given $s \in (0,1]$, then*

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4} \left(\frac{1}{2}\right)^{1-1/q} \left\{ 3^{(q-1)/q} [L_1(\mu,q)]^{1/q} + [L_2(\mu,q,b)]^{1/q} \right\}, \tag{1.3}$$

where

$$L_1(\mu,q) \leq \begin{cases} \left|f'(a) f'(b)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f'(a)\right|, \left|f'(b)\right| \leq 1, \\ \left|f'(a) f'(b)\right|^{q/(2s)} F_1(\mu_2), & 1 \leq \left|f'(a)\right|, \left|f'(b)\right|, \\ \left|f'(a) f'(b)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f'(a)\right| \leq 1 < \left|f'(b)\right|, \\ \left|f'(a) f'(b)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f'(b)\right| \leq 1 < \left|f'(a)\right|, \end{cases}$$

$$L_2(\mu,q,u) \le \begin{cases} \left|f'(u)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f'(a)\right|, \left|f'(b)\right| \le 1, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_2), & 1 \le \left|f'(a)\right|, \left|f'(b)\right|, \\ \left|f'(u)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f'(a)\right| \le 1 < \left|f'(b)\right|, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f'(b)\right| \le 1 < \left|f'(a)\right|, \end{cases}$$

$$F_1(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(2\nu - 1 - \frac{\nu-1}{\ln \nu}\right) & \nu \ne 1, \\ \frac{3}{2} & \nu = 1, \end{cases}$$

$$F_2(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(\nu - \frac{\nu-1}{\ln \nu}\right) & \nu \ne 1, \\ \frac{1}{2} & \nu = 1, \end{cases}$$

and

$$\mu_1 = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2}, \mu_2 = \left|\frac{f'(a)}{f'(b)}\right|^{q/(2s)}, \mu_3 = \frac{\left|f'(a)\right|^{sq/2}}{\left|f'(b)\right|^{q/(2s)}}, \mu_4 = \frac{\left|f'(a)\right|^{q/(2s)}}{\left|f'(b)\right|^{qs/2}}.$$

Theorem 2 ([25]). *Under the conditions of Theorem 1, we have*

$$\left|f(b) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{[L_2(\mu,q,a)]^{1/q} + 3^{(q-1)/q}\left[L_1\left(\mu^{-1},q\right)\right]^{1/q}\right\}, \quad (1.4)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ and μ_i for $i = 1, 2, 3, 4$ are defined as in Theorem 1.

Theorem 3 ([25]). *Under the conditions of Theorem 1, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{[L_2(\mu,q,b)]^{1/q} + \left[L_1\left(\mu^{-1},q,a\right)\right]^{1/q}\right\}, \quad (1.5)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ and μ_i for $i = 1, 2, 3, 4$ are defined as in Theorem 1.

Applications to special means of positive numbers of the above results can also be seen in [25].

For further results on Hermite-Hadamard type inequalities for s-logarithmically convex we refer the reader to [1, 9, 26, 27]. The main purpose of the present paper is to establish a new Hermite-Hadamard type integral inequalities in Section 2 by using the notion of s-logarithmically convexity and new identity for n-times differentiable functions from [15]. The applications of our results to special means of positive real numbers are also given in Section 3.

2. Main results

The following Lemmas are essential in establishing our main results in this section.

Lemma 1 ([15]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where $a, b \in I^\circ$ with $a < b$, we have the identity*

$$\begin{aligned}\frac{f(a)+f(b)}{2} &- \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\\ &= \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\ &+ \frac{(-1)^n(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t)\,f^{(n)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt, \quad (2.1)\end{aligned}$$

where an empty sum is understood to be nil.

Lemma 2 ([16]). *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 t^n \mu^t\,dt = \frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}} + n!\mu\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!\,(\ln\mu)^{k+1}} \quad (2.2)$$

for $n \in \mathbb{N}$.

Lemma 3. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 (1-t)^n \mu^t\,dt = \frac{n!\mu}{(\ln\mu)^{n+1}} - n!\sum_{k=0}^{n}\frac{1}{(n-k)!\,(\ln\mu)^{k+1}} \quad (2.3)$$

for $n \in \mathbb{N}$.

Proof. By making the substitution $t = 1-u$ in (2.2), in which μ is replaced by $\frac{1}{\mu}$, we get (2.3). □

Lemma 4 ([3]). *For $\alpha > 0$ and $\mu > 0$, we have*

$$J(\alpha,\mu) := \int_0^1 (1-t)^{\alpha-1}\mu^t\,dt = \sum_{k=1}^{\infty}\frac{(\ln\mu)^{k-1}}{(\alpha)_k} < \infty, \tag{2.4}$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

Theorem 4. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where $a, b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for some $s \in (0,1]$ and $q \in [1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$

$$\le \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{[F_1(\mu,n)]^{\frac{1}{q}} + \left[F_1\left(\mu^{-1},n\right)\right]^{\frac{1}{q}}\right\}, \tag{2.5}$$

where $\mu = \left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$,

$$(\delta,\theta) = \begin{cases} (s/2,s/2), & \text{if } 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \le 1, \\ (1-s/2,1-s/2), & \text{if } 1 \le \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ (s/2,1-s/2), & \text{if } 0 < \left|f^{(n)}(a)\right| \le 1 \le \left|f^{(n)}(b)\right|, \\ (1-s/2,s/2) & \text{if } 0 < \left|f^{(n)}(b)\right| \le 1 \le \left|f^{(n)}(a)\right|, \end{cases}$$

and

$$F_1(v,n) = \begin{cases} \frac{n!v(\ln v-1)}{(\ln v)^{n+1}} + \frac{1}{\ln v} - n!\sum_{k=1}^{n}\frac{\ln v-1}{(n-k)!(\ln v)^{k+1}}, & v \ne 1, \\ \frac{n}{n+1}, & v = 1. \end{cases}$$

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right.$$

$$-\frac{1}{b-a}\int_a^b f(x)\,dx\Bigg| \le \frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right\}. \quad (2.6)$$

Since for $0<\xi\le 1\le\eta$, $0\le\lambda\le 1$ and $0<s\le 1$. Then

$$\xi^{\lambda^s}\le\xi^{s\lambda} \quad \text{and} \quad \eta^{\lambda^s}\le\eta^{\lambda s+1-s}. \quad (2.7)$$

When $0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\le 1$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\le\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(b)\right|^{sq\left(\frac{1+t}{2}\right)}dt$$
$$=\left|f^{(n)}(a)\,f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$=\left|f^{(n)}(a)\,f^{(n)}(b)\right|^{sq/2}F_1(\mu,n). \quad (2.8)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\le\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(a)\right|^{sq\left(\frac{1+t}{2}\right)}dt$$
$$=\left|f^{(n)}(a)\,f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-t}\,dt$$
$$=\left|f^{(n)}(a)\,f^{(n)}(b)\right|^{sq/2}F_1\left(\mu^{-1},n\right). \quad (2.9)$$

When $\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\ge 1$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$

$$\leq \left|f^{(n)}(a)\, f^{(n)}(b)\right|^{q(1-s/2)} \int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\, f^{(n)}(b)\right|^{q(1-s/2)} F_1(\mu,n). \quad (2.10)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s} dt$$
$$\leq \left|f^{(n)}(a)\, f^{(n)}(b)\right|^{q(1-s/2)} \int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-t}\,dt$$
$$= \left|f^{(n)}(a)\, f^{(n)}(b)\right|^{q(1-s/2)} F_1\left(\mu^{-1},n\right). \quad (2.11)$$

When $0 < \left|f^{(n)}(a)\right| \leq 1 \leq \left|f^{(n)}(b)\right|$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s} dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)} \int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-s/2)} F_1(\mu,n). \quad (2.12)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s} dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)} \int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-t}\,dt$$
$$= \left|f^{(n)}(a)\right|^{q/2s}\left|f^{(n)}(b)\right|^{q(1-s/2)} F_1\left(\mu^{-1},n\right). \quad (2.13)$$

When $0 < \left|f^{(n)}(b)\right| \leq 1 \leq \left|f^{(n)}(a)\right|$, by using Lemma 3 and (2.7), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s} dt$$
$$\leq \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2} \int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^t\,dt$$
$$= \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2} F_1(\mu,n). \quad (2.14)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t}dt$$
$$= \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}F_1\left(\mu^{-1},n\right), \quad (2.15)$$

where $\mu = \left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$. A combination of (2.8)-(2.15) into (2.6) gives the desired result. This completes the proof of the Theorem. □

Corollary 1. *Under the assumptions of Theorem 4, if $q=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{F_1(\mu,n)+F_1\left(\mu^{-1},n\right)\right\}, \quad (2.16)$$

where $F_1(\nu,n)$, μ and (δ,θ) are defined as in Theorem 4.

Corollary 2. *Under the assumptions of Theorem 4, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_1(\mu,1)\right]^{\frac{1}{q}}+\left[F_1\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\}, \quad (2.17)$$

where

$$F_1(\nu,1)=\begin{cases}\frac{\nu(\ln\nu-1)+1}{(\ln\nu)^2}, & \nu\neq 1\\ \frac{1}{2}, & \nu=1\end{cases}, \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1,\\ (1-s/2,1-s/2), & \text{if } 1\leq\left|f'(a)\right|,\left|f'(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f'(a)\right|\leq 1\leq\left|f'(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f'(b)\right|\leq 1\leq\left|f'(a)\right|.\end{cases}$$

Corollary 3. *If we take $q = 1$ in Corollary 2, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{b-a}{4}\right)\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{\left[F_1(\mu,1)\right]+\left[F_1\left(\mu^{-1},1\right)\right]\right\}, \quad (2.18)$$

where $F_1(\nu,1)$, μ and (δ,θ) are defined as in Corollary 2.

Corollary 4. *Suppose the assumptions of Theorem 4 are fulfilled and if $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_1(\mu,2)\right]^{\frac{1}{q}}+\left[F_1\left(\mu^{-1},2\right)\right]^{\frac{1}{q}}\right\}, \quad (2.19)$$

where

$$F_1(\nu,2)=\begin{cases}\frac{2\nu(\ln\nu-1)-(\ln\nu)^2+2}{(\ln\nu)^3}, & \nu\neq 1,\\ \frac{2}{3}, & \nu=1,\end{cases} \quad ;\ \mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f''(a)\right|,\left|f''(b)\right|\leq 1,\\ (1-s/2,1-s/2), & \text{if } 1\leq\left|f''(a)\right|,\left|f''(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f''(a)\right|\leq 1\leq\left|f''(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f''(b)\right|\leq 1\leq\left|f''(a)\right|.\end{cases}$$

Corollary 5. *If $q = 1$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{16}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{\left[F_1(\mu,2)\right]+\left[F_1\left(\mu^{-1},2\right)\right]\right\}, \quad (2.20)$$

where $F_1(\nu,2)$, μ and (δ,θ) are defined as in Corollary 4.

Theorem 5. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for some $n \in \mathbb{N}$, where $a, b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for some $s \in (0,1]$ and $q \in (1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx \right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{[F_2(\mu,n)]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},n\right)\right]^{\frac{1}{q}}\right\}, \quad (2.21)$$

where

$$F_2(\nu,n)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{(nq-q+1)_k}<\infty, & \nu\neq 1,\\ \frac{1}{nq-q+1}, & \nu=1,\end{cases}$$
$$(nq-q+1)_k=(nq-q+1)(nq-q+2)\ldots(nq-q+k),$$

μ and (δ,θ) are defined as in Theorem 4.

Proof. Using Lemma 1, the Hölder inequality and the s-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(n-1+t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1(1-t)^{q(n-1)}\left[\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1(1-t)^{q(n-1)}\left[\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right\}. \quad (2.22)$$

The proof follows by using similar arguments as in proving Theorem 4 and using Lemma 4. □

Corollary 6. *Under the assumptions of Theorem 5, if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{[F_2(\mu,1)]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\}, \tag{2.23}$$

where

$$F_2(\nu,1)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{k!}<\infty, & \nu\neq 1,\\ 1, & \nu=1,\end{cases};\ \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and (δ,θ) is defined as in Corollary 2.

Corollary 7. *Under the assumptions of Theorem 5, if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{16}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{[F_2(\mu,2)]^{\frac{1}{q}}+\left[F_2\left(\mu^{-1},2\right)\right]^{\frac{1}{q}}\right\}, \tag{2.24}$$

where

$$F_2(\nu,2)=\begin{cases}\sum_{k=1}^{\infty}\frac{(\ln\nu)^{k-1}}{(q+1)_k}<\infty, & \nu\neq 1,\\ \frac{1}{q+1}, & \nu=1,\end{cases}$$

$$(q+1)_k=(q+1)(q+2)\ldots(q+k),\ \mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2}$$

and (δ,θ) are defined as in Corollary 4.

Theorem 6. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)}\in L([a,b])$ for some $n\in\mathbb{N}$, where $a,\ b\in I^{\circ}$ with $a<b$. If $\left|f^{(n)}\right|^{q}$ is s-logarithmically convex on $[a,b]$ for $q\in(1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right) -\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{n^{n+1-\frac{1}{q}}(b-a)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$

$$\times \left| f^{(n)}(a) \right|^{\delta} \left| f^{(n)}(b) \right|^{\theta} \left\{ [F_3(\mu)]^{\frac{1}{q}} + \left[F_3\left(\mu^{-1}\right) \right]^{\frac{1}{q}} \right\}, \quad (2.25)$$

where

$$F_3(v) = \begin{cases} \frac{v-1}{\ln v}, & v \neq 1, \\ 1, & v = 1, \end{cases} , \mu = \left| \frac{f^{(n)}(b)}{f^{(n)}(a)} \right|^{sq/2},$$

$$B(z;\alpha,\beta) = \int_0^z t^{\alpha-1}(1-t)^{1-\beta}\, dt, 0 \leq z \leq 1, \alpha > 0, \beta > 0$$

is the incomplete Beta function and (δ,θ) *are defined as in Theorem 4.*

Proof. Using Lemma 1, the Hölder inequality and the s-logarithmically convexity of $\left| f^{(n)} \right|^q$ on $[a,b]$, we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right. \\ &\qquad \left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\ &\leq \frac{(b-a)^n}{2^{n+1}n!} \left(\int_0^1 (1-t)^{q(n-1)/(q-1)} (n-1+t)^{q/(q-1)}\, dt \right)^{1-1/q} \\ &\qquad \times \left\{ \left(\int_0^1 \left| f^{(n)}(a) \right|^{q\left(\frac{1-t}{2}\right)^s} \left| f^{(n)}(b) \right|^{q\left(\frac{1+t}{2}\right)^s} dt \right)^{1/q} \right. \\ &\qquad \left. + \left(\int_0^1 \left| f^{(n)}(b) \right|^{q\left(\frac{1-t}{2}\right)^s} \left| f^{(n)}(a) \right|^{q\left(\frac{1+t}{2}\right)^s} dt \right)^{1/q} \right\}. \quad (2.26) \end{aligned}$$

By using (2.7) and the fact that

$$\begin{aligned} &\int_0^1 (1-t)^{q(n-1)/(q-1)} (n-1+t)^{q/(q-1)}\, dt \\ &= n^{\frac{nq+q-1}{q-1}} \int_0^{\frac{1}{n}} t^{\frac{(n-1)q}{q-1}} (1-t)^{\frac{q}{q-1}}\, dt = n^{\frac{nq+q-1}{q-1}} B\left(\frac{1}{n}; \frac{nq-1}{q-1}, \frac{2q-1}{q-1} \right), \end{aligned}$$

we get the required inequality (2.25) from (2.26). □

Corollary 8. *Suppose the assumptions of Theorem 6 are satisfied and if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{[F_3(\mu)]^{\frac{1}{q}}+\left[F_3\left(\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (2.27)$$

where

$$F_3(v)=\begin{cases}\frac{v-1}{\ln v}, & v\neq 1,\\ 1, & v=1,\end{cases};\ \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and (δ,θ) are defined as in Corollary 2.

Corollary 9. *Suppose the assumptions of Theorem 6 are satisfied and if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{2^{1+1/q}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left\{[F_3(\mu)]^{\frac{1}{q}}+\left[F_3\left(\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (2.28)$$

where

$$F_3(v)=\begin{cases}\frac{v-1}{\ln v}, & v\neq 1,\\ 1, & v=1,\end{cases};\mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2},$$

$B(z;\alpha,\beta)$ is the incomplete Beta function as defined in Theorem 6 and (δ,θ) are defined as in Corollary 4.

Remark 1. We can get several interesting inequalities for log-convex functions by setting $s=1$ in the above results. However, the details are left to the interested reader.

3. Applications to special means

For positive numbers $a>0$, $b>0$, define

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a, & a=b\end{cases}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq 0,-1 \text{ and } a\neq b,\\ \frac{b-a}{\ln b-\ln a}, & p=-1 \text{ and } a\neq b,\\ I(a,b), & p=0 \text{ and } a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that A, G, H, $L=L_{-1}$, $I=L_0$ and L_p are called the arithmetic, geometric, harmonic, logarithmic, exponential and generalized logarithmic means of positive numbers a and b.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 7. *Let $0<a<b\leq 1$, $r<0$, $r\neq -1$, $s\in(0,1]$ and $q\geq 1$.*

(1) *If $r\neq -2$, then*

$$\begin{aligned}&\left|A\left(a^{r+1},b^{r+1}\right)-\left[L_{r+1}(a,b)\right]^{r+1}\right|\\ &\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}|r+1|\left[G\left(a^r,b^r\right)\right]^2\\ &\times\left\{a^{-rs}\left[\frac{2\left(1-b^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)\right)}{rqs(\ln b-\ln a)}\right]^{1/q}\right.\\ &\left.+b^{-rs}\left[\frac{2\left(a^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)-1\right)}{rqs(\ln b-\ln a)}\right]^{1/q}\right\}.\end{aligned}$$

(2) *If $r=-2$, then*

$$\begin{aligned}&\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right|\\ &\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left[G\left(a^{-2},b^{-2}\right)\right]^2\\ &\times\left\{a^{2s}\left[\frac{1-b^{qs}L(a^{-qs},b^{-qs})}{qs(\ln a-\ln b)}\right]^{1/q}+b^{2s}\left[\frac{a^{qs}L(a^{-qs},b^{-qs})-1}{qs(\ln a-\ln b)}\right]^{1/q}\right\}.\end{aligned}$$

Proof. Let $f(x)=\frac{x^{r+1}}{r+1}$ for $0<x\leq 1$. Then $|f'(x)|=x^r$ and

$$\ln\left|f'(\lambda x+(1-\lambda)y)\right|^q\leq\lambda^s\ln\left|f'(x)\right|^q+(1-\lambda)^s\left|f'(y)\right|^q$$

for $x,\ y \in (0,1]$, $\lambda \in [0,1]$, $s \in (0,1]$ and $q \geq 1$. This shows that $|f'(x)|^q = x^{rq}$ is s-logarthmically convex function on $(0,1]$. Since$|f'(a)| > |f'(b)| = b^r \geq 1$, hence

$$\mu = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2} = \left(\frac{b}{a}\right)^{rqs/2}, \mu^{-1} = \left(\frac{a}{b}\right)^{rqs/2}$$

and

$$\begin{aligned}
&\left|f'(a)\, f'(b)\right|^{(1-s/2)} \left\{[F_1(\mu,1)]^{\frac{1}{q}} + \left[F_1\left(\mu^{-1},1\right)\right]^{\frac{1}{q}}\right\} \\
&= \left[G\left(a^r,b^r\right)\right]^2 \left\{a^{-rs}\left[\frac{2\left(1-b^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)\right)}{rqs\,(\ln b-\ln a)}\right]^{1/q}\right. \\
&\left. + b^{-rs}\left[\frac{2\left(a^{-rqs/2}L\left(a^{rqs/2},b^{rqs/2}\right)-1\right)}{rqs\,(\ln b-\ln a)}\right]^{1/q}\right\}.
\end{aligned}$$

Substituting the above quantities in Corollary 2, we get the required inequality. □

Remark 2. Many interesting inequalities of means of positive real numbers can be obtained by using the other results, however, the details are left to the interested reader.

ACKNOWLEDGEMENT

The authors would like to say thank to the anonymous reviewer for his/her very useful and constructive comments which have been implemented in the final version of the paper.

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Doi: 10.18514/MMN.2015.704

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 237–248

HU e-ISSN 1787-2413

ON THE ZEROS OF SOLUTIONS AND THEIR DERIVATIVES OF SECOND ORDER NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Received 24 September, 2013

Abstract. This paper is devoted to studying the growth and oscillation of solutions and their derivatives of equations of the type $f'' + A(z) f' + B(z) f = F(z)$, where $A(z), B(z) (\not\equiv 0)$ and $F(z) (\not\equiv 0)$ are meromorphic functions of finite order.

2010 *Mathematics Subject Classification:* 34M10, 30D35

Keywords: Linear differential equations, Meromorphic functions, Exponent of convergence of the sequence of zeros

1. Introduction and main results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [8, 17]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f. In the following, we give the necessary notations and basic definitions.

Definition 1 ([17])**.** Let f be a meromorphic function. Then the hyper-order of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 2 ([8,11])**.** The type of a meromorphic function f of order ρ $(0 < \rho < \infty)$ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho}}.$$

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If f is entire function of order ρ $(0 < \rho < \infty)$, we can define the type by

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^\rho}.$$

Remark 1. We have not always the equality $\tau_M(f) = \tau(f)$, for example $\tau(e^z) = \frac{1}{\pi} < 1 = \tau_M(e^z)$.

Definition 3 ([7,17])**.** Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| \leqslant r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| \leqslant r\}$.

The study of oscillation of solutions of linear differential equations has attracted many interests since the work of Bank and Laine [1,2], for more details, see [9]. The main subject of this research is the zeros distribution of solutions and their derivatives of linear differential equations. In this paper, we first discuss the growth of solutions of second order linear differential equation

$$f'' + A(z) f' + B(z) f = F(z), \tag{1.1}$$

where $A(z)$, $B(z)$ $(\not\equiv 0)$ and $F(z)$ $(\not\equiv 0)$ are meromorphic functions of finite order. Some results on the growth of entire solutions of (1.1) have been obtained by several researchers (see [5, 6, 12, 14]). Li and Wang (see [12]) investigated the non-homogeneous linear differential equation

$$f'' + e^{-z} f' + h(z) e^{bz} f = H(z), \tag{1.2}$$

where $h(z)$ is a transcendental entire function of finite order $\rho(h) < \frac{1}{2}$, and b is a real constant. They proved that all nontrivial solutions of (1.2) are of infinite order, provided that $\rho(H) < 1$. After their, Wang and Laine (see [14]) studied the differential equation

$$f'' + A_1(z) e^{az} f' + A_0(z) e^{bz} f = H(z), \tag{1.3}$$

where $A_0(z)$, $A_1(z)$, $H(z)$ are entire functions of order less than one, and $a, b \in \mathbb{C}$, and obtained.

Theorem 1 ([14]). *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of (1.3) is of infinite order.*

J. Tu and co-authors investigated the hyper-exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$ $(j = 0, 1, 2, \ldots)$, where f is a solution of

$$f'' + A(z) f' + B(z) f = 0 \tag{1.4}$$

and $\varphi(z)$ is an entire function satisfying $\rho(\varphi) < \rho(f)$ or $\rho_2(\varphi) < \rho_2(f)$, and obtained the following result.

Theorem 2 ([13]). *Let $A(z)$ and $B(z)$ be entire functions with finite order. If $\rho(A) < \rho(B) < \infty$ or $0 < \rho(A) = \rho(B) < \infty$ and $\tau_M(A) < \tau_M(B)$, then for every solution $f \not\equiv 0$ of (1.4) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\rho_2(\varphi) < \rho_2(f)$, we have*

$$\overline{\lambda_2}\left(f^{(j)} - \varphi\right) = \rho_2(f) = \rho(B) \quad (j = 0, 1, 2, \ldots).$$

Recently in [15, 16], H. Y. Xu, J. Tu, X. M. Zheng and H. Y. Xu, J. Tu have investigated the relationship between small functions and the derivatives of solutions of higher order linear differential equations with entire and meromorphic functions. It is a natural to ask what about the exponent of convergence of zeros of $f^{(j)}(z)$ $(j = 0, 1, 2, \ldots)$, where f is a solution of (1.1). The main purpose of this paper is to give an answer to this question. The method used in the proofs of our theorems is quite different from the method used in the papers [13, 16]. Before we state our results we need to define the following notations

$$A_j(z) = A_{j-1}(z) - \frac{B'_{j-1}(z)}{B_{j-1}(z)} \quad \text{for} \quad j = 1, 2, 3, \ldots, \tag{1.5}$$

$$B_j(z) = A'_{j-1}(z) - A_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} + B_{j-1}(z) \quad \text{for} \quad j = 1, 2, 3, \ldots \tag{1.6}$$

and

$$F_j(z) = F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} \quad \text{for} \quad j = 1, 2, 3, \ldots, \tag{1.7}$$

where $A_0(z) = A(z)$, $B_0(z) = B(z)$ and $F_0(z) = F(z)$. We obtain the following results.

Theorem 3. *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions with finite order such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3, \ldots)$. If f is a meromorphic solution of (1.1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \quad (j = 0, 1, 2, \ldots)$$

and

$$\overline{\lambda}_2\left(f^{(j)}\right)=\lambda_2\left(f^{(j)}\right)=\rho\ (j=0,1,2,\ldots).$$

Theorem 4. *Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be meromorphic functions with finite order such that $B_j(z)\not\equiv 0$ and $F_j(z)\not\equiv 0$ $(j=1,2,3,\ldots)$. If f is a meromorphic solution of (1.1) with*

$$\rho(f)>\max\{\rho(A),\rho(B),\rho(F)\},$$

then

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho(f)\ (j=0,1,2,\ldots).$$

Remark 2. The conditions $B_j(z)\not\equiv 0$ and $F_j(z)\not\equiv 0$ $(j=1,2,3,\ldots)$ are necessary. For example $f(z)=e^{-z}+1$ satisfies (1.1), where $A(z)=\frac{z}{z+1}$, $B(z)=-\frac{1}{z+1}$ and $F(z)=-\frac{1}{z+1}$. On the other hand

$$A_1=A-\frac{B'}{B}=1,$$

$$B_1=A'-A\frac{B'}{B}+B\equiv 0,\ F_1=F'-F\frac{B'}{B}\equiv 0$$

and

$$\lambda(f)=1>\lambda\left(f^{(j)}\right)=0\ (j=1,2,3,\ldots).$$

Here, we will give some sufficient conditions on the coefficients which guarantee $B_j(z)\not\equiv 0$ and $F_j(z)\not\equiv 0$ $(j=1,2,3,\ldots)$:

Theorem 5. *Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be entire functions with finite order such that $\rho(B)>\max\{\rho(A),\rho(F)\}$. Then all nontrivial solutions of (1.1) satisfy*

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=+\infty\ (j=0,1,2,\ldots)$$

with at most one possible exceptional solution f_0 such that

$$\rho(f_0)=\max\left\{\overline{\lambda}(f_0),\rho(B)\right\}.$$

Remark 3. The condition $\rho(B)>\max\{\rho(A),\rho(F)\}$ does not ensure that all solutions of (1.1) are of infinite order. For example we can see that $f_0(z)=e^{-z^2}$ satisfies the differential equation

$$f''+2zf'+(e^{z^2}+2)f=1,$$

where

$$\overline{\lambda}(f_0)=0<\rho(f_0)=\rho(B)=2.$$

In the next, we note

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log m(r,f)}{\log r}.$$

Theorem 6. *Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be meromorphic functions with finite order such that $\sigma(B)>\max\{\sigma(A),\sigma(F)\}$. If f is a meromorphic solution of (1.1) with $\rho(f)=\infty$ and $\rho_2(f)=\rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=+\infty\ (j=0,1,2,\ldots)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right)=\lambda_2\left(f^{(j)}\right)=\rho\ (j=0,1,2,\ldots).$$

Theorem 7. *Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be entire functions with finite order such that $\rho(B)=\rho(A)>\rho(F)$ and $\tau(B)>k\tau(A)$, $k\geqslant 1$ is an integer. If f is a nontrivial solution of (1.1) with $\rho(f)=\infty$ and $\rho_2(f)=\rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=+\infty\ (j=0,1,\ldots,k)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right)=\lambda_2\left(f^{(j)}\right)=\rho\ (j=0,1,\ldots,k).$$

Corollary 1. *Suppose that $A_0\not\equiv 0$, $A_1\not\equiv 0$, $H\not\equiv 0$ are entire functions of order less than one, and the complex constants a,b satisfy $ab\neq 0$ and $|b|>k|a|$, $k\geqslant 1$ is an integer. Then every nontrivial solution f of (1.3) satisfies*

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=+\infty\ (j=0,1,\ldots,k).$$

2. Preliminary lemmas

Lemma 1 ([8]). *Let f be a meromorphic function and let $k\geqslant 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f),$$

where $S(r,f)=O(\log T(r,f)+\log r)$, possibly outside of an exceptional set $E\subset(0,+\infty)$ of r with finite linear measure. If f is of finite order of growth, then

$$m\left(r,\frac{f^{(k)}}{f}\right)=O(\log r).$$

Lemma 2 ([3,4]). *Let $A_0,A_1,\ldots,A_{k-1},F\not\equiv 0$ be finite order meromorphic functions.*

(i) If f is a meromorphic solution of the equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=F \tag{2.1}$$

with $\rho(f)=+\infty$, *then* f *satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

(*ii*) *If* f *is a meromorphic solution of equation (2.1) with* $\rho(f)=+\infty$ *and* $\rho_2(f)=\rho$, *then*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)=\rho.$$

Lemma 3 ([13]). *Let* $A_0,A_1,\ldots,A_{k-1},F\not\equiv 0$ *be finite order meromorphic functions. If* f *is a meromorphic solution of equation (2.1) with*

$$\max\left\{\rho\left(A_j\right)\ (j=0,1,\ldots,k-1),\rho(F)\right\}<\rho(f)<\infty,$$

then

$$\overline{\lambda}(f)=\lambda(f)=\rho(f).$$

Lemma 4 ([10]). *Let* f *and* g *be meromorphic functions in the complex plane such that* $0<\rho(f),\rho(g)<\infty$ *and* $0<\tau(f),\tau(g)<\infty$. *Then we have*
(*i*) *If* $\rho(f)>\rho(g)$, *then we obtain*

$$\tau(f+g)=\tau(fg)=\tau(f).$$

(*ii*) *If* $\rho(f)=\rho(g)$ *and* $\tau(f)\neq\tau(g)$, *then we get*

$$\rho(f+g)=\rho(fg)=\rho(f)=\rho(g).$$

Lemma 5 ([6]). *Let* $A,B_1,\ldots,B_{k-1},\ F\not\equiv 0$ *be entire functions of finite order, where* $k\geqslant 2$. *Suppose that either* (*i*) *or* (*ii*) *below holds:*
(*i*) $\rho\left(B_j\right)<\rho(A)\ (j=1,\ldots,k-1)$,
(*ii*) $B_1,\ldots,\ B_{k-1}$ *are polynomials and* A *is transcendental. Then we have*
(*a*) *All solutions of the differential equation*

$$f^{(k)}+B_{k-1}f^{(k-1)}+\cdots+B_1f'+Af=F$$

satisfy

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty$$

with at most one possible solution f_0 *of finite order.*
(*b*) *If there exists an exceptional solution* f_0 *in case* (*a*), *then* f_0 *satisfies*

$$\rho(f_0)\leqslant\max\left\{\rho(A),\rho(F),\overline{\lambda}(f_0)\right\}<\infty. \tag{2.2}$$

Furthermore, if $\rho(A)\neq\rho(F)$ *and* $\overline{\lambda}(f_0)<\rho(f_0)$, *then*

$$\rho(f_0)=\max\{\rho(A),\rho(F)\}.$$

3. Proof of the theorems and corollary

Proof of Theorem 3. We prove this theorem by using mathematical induction. Since $B \not\equiv 0$, $F \not\equiv 0$, then by using Lemma 2, we have

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Dividing both sides of (1.1) by B, we obtain

$$\frac{1}{B}f'' + \frac{A}{B}f' + f = \frac{F}{B}. \tag{3.1}$$

Differentiating both sides of equation (3.1), we have

$$\frac{1}{B}f^{(3)} + \left(\left(\frac{1}{B}\right)' + \frac{A}{B}\right)f'' + \left(\left(\frac{A}{B}\right)' + 1\right)f' = \left(\frac{F}{B}\right)'. \tag{3.2}$$

Multiplying now (3.2) by B, we get

$$f^{(3)} + A_1 f'' + B_1 f' = F_1, \tag{3.3}$$

where

$$A_1 = A - \frac{B'}{B},$$

$$B_1 = A' - A\frac{B'}{B} + B$$

and

$$F_1 = F' - F\frac{B'}{B}.$$

Since $B_1 \not\equiv 0$, $F_1 \not\equiv 0$ are meromorphic functions with finite order, then by using Lemma 2, we obtain

$$\overline{\lambda}(f') = \lambda(f') = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f') = \lambda_2(f') = \rho_2(f) = \rho.$$

Dividing now both sides of (3.3) by B_1, we obtain

$$\frac{1}{B_1}f^{(3)} + \frac{A_1}{B_1}f'' + f' = \frac{F_1}{B_1}. \tag{3.4}$$

Differentiating both sides of equation (3.4) and multiplying by B_1, we get

$$f^{(4)} + A_2 f^{(3)} + B_2 f'' = F_2, \tag{3.5}$$

where $A_2, B_2 \not\equiv 0$ and $F_2 \not\equiv 0$ are meromorphic functions defined in (1.5)-(1.7). By using Lemma 2, we obtain

$$\overline{\lambda}(f'') = \lambda(f'') = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2\left(f''\right)=\lambda_2\left(f''\right)=\rho_2\left(f\right)=\rho.$$

We suppose now that

$$\overline{\lambda}\left(f^{(k)}\right)=\lambda\left(f^{(k)}\right)=\rho\left(f\right)=+\infty,\ \overline{\lambda}_2\left(f^{(k)}\right)=\lambda_2\left(f^{(k)}\right)=\rho_2\left(f\right)=\rho \quad (3.6)$$

for all $k=0,1,2,\ldots,j-1$, and we prove that (3.6) is true for $k=j$. By the same method as before, we can obtain

$$f^{(j+2)}+A_j f^{(j+1)}+B_j f^{(j)}=F_j,$$

where $A_j, B_j \not\equiv 0$ and $F_j \not\equiv 0$ are meromorphic functions defined in (1.5) -(1.7). By using Lemma 2, we obtain

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho\left(f\right)=+\infty$$

and

$$\overline{\lambda}_2\left(f^{(j)}\right)=\lambda_2\left(f^{(j)}\right)=\rho_2\left(f\right)=\rho.$$

Thus, the proof of Theorem 3 is completed. □

Proof of Theorem 4. By a similar reasoning as in the proof of Theorem 3, and by using Lemma 3, we obtain

$$\overline{\lambda}\left(f^{(j)}\right)=\lambda\left(f^{(j)}\right)=\rho\left(f\right)\quad (j=0,1,2,\ldots).$$

□

Proof of Theorem 5. By Lemma 5, all nontrivial solutions of (1.1) are of infinite order with at most one exceptional solution f_0 of finite order. By using (1.5) and Lemma 1 we have

$$m\left(r,A_j\right)\leqslant m\left(r,A_{j-1}\right)+O\left(\log r\right)$$

for all $j=1,2,3,\ldots$, which we can rewrite as

$$m\left(r,A_j\right)\leqslant m\left(r,A\right)+O\left(\log r\right)\quad (j=1,2,3,\ldots). \quad (3.7)$$

On the other hand, we have from (1.6)

$$B_j=A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}}-\frac{B'_{j-1}}{B_{j-1}}\right)+B_{j-1}$$

$$=A_{j-1}\left(\frac{A'_{j-1}}{A_{j-1}}-\frac{B'_{j-1}}{B_{j-1}}\right)+A_{j-2}\left(\frac{A'_{j-2}}{A_{j-2}}-\frac{B'_{j-2}}{B_{j-2}}\right)+B_{j-2}$$

$$=\sum_{k=0}^{j-1}A_k\left(\frac{A'_k}{A_k}-\frac{B'_k}{B_k}\right)+B. \quad (3.8)$$

Now we prove that $B_j \not\equiv 0$ for all $j = 1,2,3,\ldots$. Suppose there exists an integer $j = 1,2,3,\ldots$ such that $B_j \equiv 0$. By (3.7) and (3.8)

$$T(r,B) = m(r,B) \leqslant \sum_{k=0}^{j-1} m(r,A_k) + O(\log r)$$
$$\leqslant j\, m(r,A) + O(\log r) = j\, T(r,A) + O(\log r) \tag{3.9}$$

which implies the contradiction $\rho(B) \leqslant \rho(A)$. Hence $B_j \not\equiv 0$ for all $j = 1,2,3,\ldots$. Suppose now there exists an integer $j = 1,2,3,\ldots$ that is the first index for which $F_j \equiv 0$. Then, by (1.7) and $F_{j-1}(z) \not\equiv 0$ we have

$$F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} = 0$$

which implies

$$F_{j-1}(z) = c B_{j-1}(z),$$

where $c \in \mathbb{C}\setminus\{0\}$. By (3.8) we have

$$\frac{1}{c}F_{j-1} = \sum_{k=0}^{j-2} A_k\left(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\right) + B. \tag{3.10}$$

On the other hand, we have from (1.7)

$$m\left(r,F_j\right) \leqslant m(r,F) + O(\log r) \quad (j = 1,2,3,\ldots). \tag{3.11}$$

By (3.10), (3.11) and Lemma 1, we obtain

$$T(r,B) = m(r,B) \leqslant \sum_{k=0}^{j-2} m(r,A_k) + m\left(r,F_{j-1}\right) + O(\log r)$$
$$\leqslant (j-1)\, m(r,A) + m(r,F) + O(\log r)$$
$$= (j-1)\, T(r,A) + T(r,F) + O(\log r) \tag{3.12}$$

which implies the contradiction $\rho(B) \leqslant \max\{\rho(A),\rho(F)\}$. Since $B_j \not\equiv 0$ and $F_j \not\equiv 0$ $(j = 1,2,3,\ldots)$, then by applying Theorem 3 and Lemma 5 we have

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0,1,2,\ldots)$$

with at most one exceptional solution f_0 of finite order. Since $\rho(B) > \max\{\rho(A),\rho(F)\}$, then by (2.2) we obtain

$$\rho(f_0) \leqslant \max\left\{\rho(B),\overline{\lambda}(f_0)\right\}. \tag{3.13}$$

On the other hand by (1.1), we can write

$$B = \frac{F}{f_0} - \left(\frac{f''_0}{f_0} + A\frac{f'_0}{f_0}\right).$$

It follows that

$$T(r,B) = m(r,B) \leqslant m\left(r,\frac{F}{f_0}\right) + m(r,A) + O(\log r)$$
$$\leqslant T(r,f_0) + T(r,F) + T(r,A) + O(\log r)$$

which implies

$$\rho(B) \leqslant \max\{\rho(f_0),\rho(A),\rho(F)\} = \rho(f_0). \tag{3.14}$$

Since $\overline{\lambda}(f_0) \leqslant \rho(f_0)$, then by using (3.13) and (3.14) we obtain

$$\rho(f_0) = \max\left\{\rho(B),\overline{\lambda}(f_0)\right\}.$$

This completes the proof of Theorem 5. □

Proof of Theorem 6. By using the same reasoning as in the proof of Theorem 5, we can prove Theorem 6. □

Proof of Theorem 7. First, we prove that $B_j \not\equiv 0$ for all $j = 1,2,\ldots,k$. Suppose there exists an integer s, $1 \leqslant s \leqslant k$ such that $B_s \equiv 0$. By (3.7) and (3.8), we have

$$T(r,B) = m(r,B) \leqslant \sum_{k=0}^{s-1} m(r,A_k) + O(\log r)$$
$$\leqslant sm(r,A) + O(\log r) = sT(r,A) + O(\log r) \tag{3.15}$$

which implies the contradiction $k\tau(A) < \tau(B) \leqslant s\tau(A)$. Hence $B_j \not\equiv 0$ for all $j = 1,2,\ldots,k$. Now, we prove that $F_j \not\equiv 0$ for all $j = 1,2,\ldots,k$. Suppose there exists an integer s, $1 \leqslant s \leqslant k$ such that $F_s \equiv 0$. From (3.12), we have

$$T(r,B) \leqslant (s-1)m(r,A) + m(r,F) + O(\log r)$$
$$= (s-1)T(r,A) + T(r,F) + O(\log r) \tag{3.16}$$

which implies the contradiction $k\tau(A) < \tau(B) \leqslant (s-1)\tau(A)$. Hence $F_j \not\equiv 0$ for all $j = 1,2,\ldots,k$. Since $B_j \not\equiv 0$ and $F_j \not\equiv 0$ $(j = 1,\ldots,k)$, then by Theorem 3 we have

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0,1,\ldots,k)$$

and

$$\overline{\lambda_2}\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \ (j = 0,1,\ldots,k).$$

□

Proof of Corollary 1. Since $ab \neq 0$, $|b| > k|a|$, then by Theorem 1, every non-trivial solution f of (1.3) is of infinite order. By using Lemma 4 we have

$$\tau\left(A_0 e^{bz}\right) = \tau\left(e^{bz}\right) = \frac{|b|}{\pi} > k\frac{|a|}{\pi} = k\tau\left(A_1 e^{az}\right).$$

Then by Theorem 7, we obtain

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = \rho(f) = +\infty \ (j = 0, 1, \ldots, k).$$

□

4. Open problem

It's interesting to study whether the condition $|b| > k|a|$, where $k \geqslant 1$ is an integer is necessary in Corollary 1. For that, we pose the following problem.

Conjecture 1. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, $H \not\equiv 0$ are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $|b| > |a|$. Then every nontrivial solution f of (1.3) satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \ (j = 0, 1, \ldots).$$

Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions to improve the paper.

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Doi: 10.18514/MMN.2015.1131

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 249–256

HU e-ISSN 1787-2413

OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MT-CONVEX FUNCTIONS

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Received 12 February, 2014

Abstract. Some inequalities of Ostrowski type for MT-convex functions via fractional integrals are obtained. These results not only generalize those of [25], but also provide new estimates on these types of Ostrowski inequalities for fractional integrals.

2010 *Mathematics Subject Classification:* 26A33; 26A51; 26D07; 26D10; 26D15

Keywords: Ostrowski type inequality, MT-convex function, fractional integral

1. INTRODUCTION

The following result is known in the literature as the Ostrowski inequality (see [17, page 468] or [18]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a,b]$.

Theorem 1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a,b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall\, x \in [a,b]. \tag{1.1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [1,4,5,8,10–15,20,24] and the references cited therein). In [23] (see also [25, 26]), Tunç and Yidirim defined the following so-called MT-convex function:

Definition 1. A function: $I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of MT(I), if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx+(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

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In [25], Tunç derived some inequalities of Ostrowski type for MT-convex functions.

Theorem 2. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|$ is MT-convex on $[a,b]$ and $|f'(x)|\le M$, $x\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{M\pi\left[(x-a)^2+(b-x)^2\right]}{4(b-a)} \tag{1.3}$$

for each $x\in[a,b]$.

Theorem 3. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)|\le M$, $x\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{1.4}$$

for each $x\in[a,b]$.

Theorem 4. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q\ge1$ and $|f'(x)|\le M$, $x\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\pi^{\frac{1}{q}}\frac{(x-a)^2+(b-x)^2}{(b-a)} \tag{1.5}$$

for each $x\in[a,b]$.

Fractional calculus [7, 16, 19] was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 2. Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha>0$ with $a\ge0$ are defined by

$$J^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b,$$

respectively, where $\Gamma(\alpha)=\int_0^{\infty}e^{-u}u^{\alpha-1}du$. Here, $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$. In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional integral inequalities, we refer the reader to the papers [2,3,6,9,21] and the reference cited therein.

Motivated by these results, in the present paper, we establish some Ostrowski type inequalities for MT-convex functions via Riemann-Liouville fractional integrals. So, new estimates on these types of Ostrowski inequalities via fractional integrals are provided and the results of [25] are generalized.

2. Ostrowski type fractional integral inequalities for MT-convex functions

In this section, we apply the following fractional integral identity from Set [22] to derive some new Ostrowski type fractional integral inequalities for MT-convex functions.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L_1[a,b]$, then for all $x\in[a,b]$ and $\alpha>0$, one has*

$$\begin{aligned}&\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\\&=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)\,dt-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)\,dt.\end{aligned}\tag{2.1}$$

Using this lemma, we can obtain the following Ostrowski type fractional integral inequalities for MT-convex functions.

Theorem 5. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|$ is MT-convex on $[a,b]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$\begin{aligned}&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|\\&\le M\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}.\end{aligned}\tag{2.2}$$

Proof. From (2.1) and since $|f'|$ is MT-convex, we have

$$\begin{aligned}&\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|\\&\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt\\&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|\right]dt\\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(b)\right|\right]dt\\
&\leq \frac{M\,(x-a)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt\\
&\quad+\frac{M\,(b-x)^{\alpha+1}}{b-a}\int_0^1\left[t^\alpha\frac{\sqrt{t}}{2\sqrt{1-t}}+t^\alpha\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt\\
&=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\int_0^1\left[t^{\alpha+\frac{1}{2}}(1-t)^{-1/2}+t^{\alpha-\frac{1}{2}}(1-t)^{1/2}\right]\mathrm{d}t\\
&=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\left[\beta\left(\alpha+\frac{3}{2},\frac{1}{2}\right)+\beta\left(\alpha+\frac{1}{2},\frac{3}{2}\right)\right]\\
&=\frac{M[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}]}{2(b-a)}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha+1)},
\end{aligned}
$$

where we have used the Beta function of Euler type, which is defined as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},\quad \forall\ x,y>0.$$

The proof is completed. □

Remark 1. In Theorem 5, if we choose $\alpha=1$, we get the inequality in Theorem 2.

Theorem 6. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$
\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(b)\right]\right|\\
&\leq\frac{M}{(1+p\alpha)^{1/p}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}.
\end{aligned}
\tag{2.3}
$$

Proof. From Lemma 1 and using the well-known Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^\alpha_{x-}f(a)+J^\alpha_{x+}f(b)\right]\right|\\
&\leq\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha\left|f'(tx+(1-t)a)\right|dt
\end{aligned}
$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is MT-convex and $|f'(x)|\leq M$, we get

$$\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt \leq \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q+\frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|^q\right]dt$$
$$\leq M^q\int_0^1\left[\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right]dt=\frac{\pi}{2}M^q$$

and similarly

$$\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \leq \frac{\pi}{2}M^q.$$

By simple computation, we have

$$\int_0^1 t^{p\alpha}dt=\frac{1}{p\alpha+1}.$$

Using these results, we complete the proof of (2.3). □

Remark 2. In Theorem 6, if we choose $\alpha=1$, we get the inequality in Theorem 3.

Theorem 7. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L_1[a,b]$. If $|f'|^q$ is MT-convex on $[a,b]$, $q\geq 1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequalities for fractional integrals with $\alpha>0$ and $x\in[a,b]$ hold:*

$$\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|$$
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right)^{\frac{1}{q}}\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}. \tag{2.4}$$

Proof. From Lemma 1 and using the well-known power mean inequality, we have

$$\left|\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)\right]\right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt$$

$$
\begin{aligned}
&+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right| dt \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^q$ is MT-convex on $[a,b]$ and $|f'(x)| \le M$, we get

$$
\begin{aligned}
&\int_0^1 t^\alpha \left|f'(tx+(1-t)a)\right|^q dt \\
&\le \int_0^1 \left[t^\alpha \frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q + t^\alpha \frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|^q\right] dt \\
&\le M^q \int_0^1 \left[t^\alpha \frac{\sqrt{t}}{2\sqrt{1-t}} + t^\alpha \frac{\sqrt{1-t}}{2\sqrt{t}}\right] dt = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)} M^q
\end{aligned}
$$

and similarly

$$
\int_0^1 t^\alpha \left|f'(tx+(1-t)b)\right|^q dt \le \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)} M^q.
$$

Using these inequalities, we complete the proof of (2.4). □

Remark 3. In Theorem 7, if we choose $\alpha = 1$, we get the inequality in Theorem 4.

ACKNOWLEDGEMENTS

This work was partly supported by the National Natural Science Foundation of China (Grant No. 11301277), the Qing Lan Project of Jiangsu Province, and the Training Abroad Project of Outstanding Young and Middle-Aged University Teachers and Presidents.

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Doi: 10.18514/MMN.2015.1223

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 257–275

HU e-ISSN 1787-2413

NECESSARY AND SUFFICIENT CONDITIONS FOR THE SOLVABILITY OF INVERSE PROBLEM FOR A CLASS OF DIRAC OPERATORS

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Received 01 May, 2014

Abstract. In this paper we consider a problem for the first order Dirac differential equations system with spectral parameter dependent on boundary condition. The asymptotic behaviors of eigenvalues, eigenfunctions and normalizing numbers of this system are investigated. The expansion formula with respect to eigenfunctions is obtained and Parseval equality is given. The main theorem on necessary and sufficient conditions for the solvability of inverse problem is proved and the algorithm of reconstruction of potential from spectral data (the sets of eigenvalues and normalizing numbers) is given.

2010 *Mathematics Subject Classification:* 34A55; 34L40

Keywords: Dirac operator, eigenvalues and normalizing numbers, inverse problem, necessary and sufficient conditions

1. INTRODUCTION

We consider the boundary value problem generated by one dimensional Dirac differential equations system

$$By' + \Omega(x)y = \lambda y, \quad 0 < x < \pi \tag{1.1}$$

with boundary condition

$$\begin{aligned} U(y) &:= y_1(0) = 0, \\ V(y) &:= (\lambda + h_1)\, y_1(\pi) + h_2 y_2(\pi) = 0, \end{aligned} \tag{1.2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real valued functions in $L_2(0,\pi)$, λ is a spectral parameter, h_1 and h_2 are real numbers, $h_2 > 0$.

In the finite interval, the inverse problems for Dirac differential equations system by different spectral characteristics (e.g., one spectrum and norming constants or two spectra or spectral function) are solved in [2, 6, 10, 11, 18, 20, 21]. When spectral

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parameter contained in boundary condition, reconstruction of potential from spectral function is carried out in [18] and the uniqueness of the inverse problem for Dirac operator according to Weyl function is worked in [3]. Inverse spectral problem for Dirac operator with potentials belong entrywise to $L_p(0,1)$; for some $p \in [1,\infty)$, was studied in [2] and in this work, not only Gelfand-Levitan-Marchenko method but also Krein method [12] was used. Using Weyl-Titschmarsh function, direct and inverse problems for Dirac type-system were investigated in [8, 22]. For weighted Dirac systems, inverse spectral problems was examined in [25].

On a positive half line, inverse scattering problem for a system of Dirac equations of order $2n$ is completely solved in [9] and when boundary condition involving spectral parameter, for Dirac operator, inverse scattering problem worked in [4, 17]. The applications of Dirac differential equations system has been widespread in various areas of physics, such as [5, 23, 24] ever since Dirac equation was discovered to be associated with nonlinear wave equation in [1].

This paper is organized as follows: In Section 2, the asymptotic formulas of eigenvalues, eigenfunctions and normalizing numbers of the boundary value problem (1.1), (1.2) are investigated. By the eigenfunctions, completeness theorem is proved and expansion formula is obtained. Parseval equality is given. In Section 3, the main equation namely Gelfand-Levitan-Marchenko type equation is derived. In Section 4, we show that the boundary value problem (1.1),(1.2) can be uniquely determined from its eigenvalues and normalizing numbers. Finally, in Section 5, the solution of the inverse problem is obtained. Let's express this more clearly. We can state the inverse problem for a system of Dirac equations in the following way: let λ_n and α_n, $(n \in \mathbb{Z})$ are respectively eigenvalues and normalizing numbers of boundary value problem (1.1), (1.2) and the quantities $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ are called spectral data. Knowing the spectral data $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ to indicate a method of determining the potential $\Omega(x)$ and to find necessary and sufficient conditions for λ_n and α_n, $(n \in \mathbb{Z})$ to be the spectral data of a problem (1.1),(1.2), for this, we derive differential equation, Parseval equality and boundary conditions. The main theorem on the necessary and sufficient conditions for the solvability of inverse problem is proved and then algorithm of the construction of the function $\Omega(x)$ by spectral data is given. Note that throughout this paper, we use the following notation: $\tilde{\phi}$ denotes the transposed matrix of ϕ.

2. Preliminaries

The inner product in Hilbert space $H = L_2(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}$ is defined by

$$\langle Y, Z \rangle := \int_0^\pi [y_1(x)\overline{z}_1(x) + y_2(x)\overline{z}_2(x)]\,dx + \frac{1}{h_2} y_3 \overline{z}_3,$$

where

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \end{pmatrix} \in H, \qquad Z = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \end{pmatrix} \in H.$$

Let us define

$$L(Y) := \begin{pmatrix} l(y) \\ -h_1 y_1(\pi) - h_2 y_2(\pi) \end{pmatrix}$$

with

$$D(L) = \left\{ \begin{array}{c} Y \mid Y = (y_1(x), y_2(x), y_3)^T \in H,\ y_1(x),\ y_2(x) \in AC[0,\pi], \\ y_3 = y_1(\pi),\ y_1(0) = 0,\ l(y) \in L_2(0,\pi;\mathbb{C}^2), \end{array} \right\}$$

where

$$l(y) = \begin{pmatrix} y_2' + p(x)y_1 + q(x)y_2 \\ -y_1' + q(x)y_1 - p(x)y_2 \end{pmatrix}.$$

The boundary value problem (1.1), (1.2) is equivalent to the equation $LY = \lambda Y$.

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions of the system (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \psi(\pi,\lambda) = \begin{pmatrix} h_2 \\ -\lambda - h_1 \end{pmatrix}. \tag{2.1}$$

The solution $\varphi(x,\lambda)$ has the following representation (see [3])

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x K(x,t)\varphi_0(t,\lambda)dt, \tag{2.2}$$

where $\varphi_0(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix}$, $K_{ij}(x,.) \in L_2(0,\pi)$, $i,j = 1,2$ for fixed $x \in [0,\pi]$ and $K(x,t)$ is the solution of the problem

$$\begin{aligned} BK_x'(x,t) + K_t'(x,t)B &= -\Omega(x)K(x,t), \\ \Omega(x) &= K(x,x)B - BK(x,x), \\ K_{11}(x,0) &= K_{21}(x,0) = 0. \end{aligned} \tag{2.3}$$

Formula (2.3) gives the relation between the kernel $K(x,t)$ and the coefficient of $\Omega(x)$ of equation (1.1).

The characteristic function $\Delta(\lambda)$ of the problem L is

$$\Delta(\lambda) := W[\varphi(x,\lambda), \psi(x,\lambda)] = \varphi_2(x,\lambda)\psi_1(x,\lambda) - \varphi_1(x,\lambda)\psi_2(x,\lambda), \tag{2.4}$$

where $W[\varphi(x,\lambda), \psi(x,\lambda)]$ is the Wronskian of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ and independent of $x \in [0,\pi]$. The zeros of $\Delta(\lambda)$ coincide with the eigenvalues λ_n of problem L. The functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are eigenfunctions and there exists a sequence β_n such that

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{2.5}$$

Normalizing numbers are

$$\alpha_n := \int_0^\pi \left(|\varphi_1(x,\lambda_n)|^2 + |\varphi_2(x,\lambda_n)|^2\right) dx + \frac{1}{h_2}|\varphi_1(\pi,\lambda_n)|^2.$$

The following relation holds (see [16]):

$$\dot{\Delta}(\lambda_n) = \beta_n \alpha_n, \tag{2.6}$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

Theorem 1. *i) The eigenvalues λ_n, $(n \in \mathbb{Z})$ of boundary value problem (1.1), (1.2) are*

$$\lambda_n = \lambda_n^0 + \epsilon_n, \quad \{\epsilon_n\} \in l_2, \tag{2.7}$$

where $\lambda_n^0 = n$ are zeros of function $\lambda \sin \lambda\pi$. For large n, the eigenvalues are simple.

ii) The eigenfunctions of the boundary value problem can be represented in the form

$$\varphi(x,\lambda_n) = \begin{pmatrix} \sin nx \\ -\cos nx \end{pmatrix} + \begin{pmatrix} \zeta_n^{(1)}(x) \\ \zeta_n^{(2)}(x) \end{pmatrix}, \tag{2.8}$$

where $\sum_{n=-\infty}^{\infty} \left\{ \left|\zeta_n^{(1)}(x)\right|^2 + \left|\zeta_n^{(2)}(x)\right|^2 \right\} \le C$, here C is a positive number.

iii) Normalizing numbers of the problem (1.1), (1.2) are as follows

$$\alpha_n = \pi + \tau_n, \quad \{\tau_n\} \in l_2. \tag{2.9}$$

Proof. The proof of this theorem is given in [15]. □

We note that using (2.2), as $|\lambda| \to \infty$ uniformly in $x \in [0,\pi]$ the following asymptotic formulas are obtain:

$$\varphi_1(x,\lambda) = \sin \lambda x + O\left(\frac{1}{|\lambda|} e^{|Im\lambda| x}\right),$$
$$\varphi_2(x,\lambda) = -\cos \lambda x + O\left(\frac{1}{|\lambda|} e^{|Im\lambda| x}\right). \tag{2.10}$$

Substituting the asymptotic formulas (2.10) into

$$\Delta(\lambda) = (\lambda + h_1)\varphi_1(\pi,\lambda) + h_2\varphi_2(\pi,\lambda),$$

we get as $|\lambda| \to \infty$

$$\Delta(\lambda) = \lambda \sin \lambda\pi + O\left(e^{|Im\lambda|\pi}\right). \tag{2.11}$$

Proposition 1. *The specification of the eigenvalues λ_n $(n \in \mathbb{Z})$ uniquely determines the characteristic function $\Delta(\lambda)$ by the formula*

$$\Delta(\lambda) = -\pi(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{(\lambda_n^2 - \lambda^2)}{n^2}. \tag{2.12}$$

Proof. Since the function $\Delta(\lambda)$ is entire function of order 1, from Hadamard's theorem (see [13]) using (2.11) we obtain (2.12). □

Theorem 2. *a) The system of eigenfunctions* $\{\varphi(x,\lambda_n)\}$, $(n \in \mathbb{Z})$ *of boundary value problem (1.1), (1.2) is complete in space* $L_2(0,\pi;\mathbb{C}^2)$.

b) Let $f(x)$, $x \in [0,\pi]$ *be an absolutely continuous function. Then the expansion formula*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \varphi(x,\lambda_n), \tag{2.13}$$

$$a_n = \frac{1}{\alpha_n} \langle f(x), \varphi(x,\lambda_n) \rangle$$

is valid and the series converges uniformly with respect to $x \in [0,\pi]$. *For* $f(x) \in L_2(0,\pi;\mathbb{C}^2)$ *series (2.13) converges in* $L_2(0,\pi;\mathbb{C}^2)$, *moreover, Parseval equality holds*

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} \alpha_n |a_n|^2. \tag{2.14}$$

Proof. The proof of this theorem is obtained in [15]. □

3. Main equation

Theorem 3. *For each fixed* $x \in (0,\pi]$ *the kernel* $K(x,t)$ *satisfies the following equation:*

$$F(x,t) + K(x,t) + \int_0^x K(x,\xi) F(\xi,t) d\xi = 0, \quad 0 < t < x, \tag{3.1}$$

where

$$F(x,t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\alpha_n} \varphi_0(x,\lambda_n) \tilde{\varphi}_0(t,\lambda_n) - \frac{1}{\pi} \varphi_0(x,\lambda_n^0) \tilde{\varphi}_0(t,\lambda_n^0) \right]. \tag{3.2}$$

Proof. Using the transformation operators (see [14]), according to (2.2) we have

$$\varphi_0(x,\lambda) = \varphi(x,\lambda) + \int_0^x H(x,t) \varphi(t,\lambda) dt. \tag{3.3}$$

It follows from (2.2) and (3.3) that

$$\sum_{n=-N}^{N} \frac{1}{\alpha_n} \varphi(x,\lambda_n) \tilde{\varphi}_0(t,\lambda_n)$$

$$= \sum_{n=-N}^{N} \frac{1}{\alpha_n} \varphi_0(x,\lambda_n) \tilde{\varphi}_0(t,\lambda_n) + \int_0^x K(x,\xi) \sum_{n=-N}^{N} \frac{1}{\alpha_n} \varphi_0(\xi,\lambda_n) \tilde{\varphi}_0(t,\lambda_n) d\xi$$

and

$$\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)$$
$$=\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\tilde{\varphi}(t,\lambda_n)+\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^t\tilde{\varphi}(\xi,\lambda_n)\tilde{H}(t,\xi)d\xi.$$

Using the last two equalities, we obtain

$$\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi(x,\lambda_n)\tilde{\varphi}(t,\lambda_n)-\frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]$$
$$=\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(x,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)-\frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]$$
$$+\int_0^x K(x,\xi)\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(\xi,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)-\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]d\xi$$
$$+\int_0^x K(x,\xi)\sum_{n=-N}^{N}\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)d\xi$$
$$-\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^t\tilde{\varphi}(\xi,\lambda_n)\tilde{H}(t,\xi)d\xi$$

or

$$\Phi_N(x,t)=I_{N1}(x,t)+I_{N2}(x,t)+I_{N3}(x,t)+I_{N4}(x,t), \tag{3.4}$$

where

$$\Phi_N(x,t):=\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi(x,\lambda_n)\tilde{\varphi}(t,\lambda_n)-\frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right],$$
$$I_{N1}(x,t):=\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(x,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)-\frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right],$$
$$I_{N2}(x,t):=\int_0^x K(x,\xi)\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(\xi,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)-\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]d\xi,$$
$$I_{N3}(x,t):=\int_0^x K(x,\xi)\sum_{n=-N}^{N}\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)d\xi,$$

$$I_{N4}(x,t) := \sum_{n=-N}^{N} \frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^t \tilde{\varphi}(\xi,\lambda_n)\tilde{H}(t,\xi)d\xi.$$

Let $f(x)$ be an absolutely continuous function. Then using expansion formula (2.13),

$$\lim_{N\to\infty}\int_0^\pi \Phi_N(x,t)f(t)dt = \lim_{N\to\infty}\int_0^\pi \sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\tilde{\varphi}(t,\lambda_n)f(t)dt$$

$$-\lim_{N\to\infty}\int_0^\pi \sum_{n=-N}^{N}\frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)f(t)dt = 0, \tag{3.5}$$

$$\lim_{N\to\infty}\int_0^\pi I_{N1}(x,t)f(t)dt$$

$$= \lim_{N\to\infty}\int_0^\pi \sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(x,\lambda_n)\tilde{\varphi}_0(t,\lambda_n) - \frac{1}{\pi}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]f(t)dt$$

$$= \int_0^\pi F(x,t)f(t)dt, \tag{3.6}$$

$$\lim_{N\to\infty}\int_0^\pi I_{N2}(x,t)f(t)dt = \lim_{N\to\infty}\int_0^\pi\int_0^x K(x,\xi)\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\varphi_0(\xi,\lambda_n)\tilde{\varphi}_0(t,\lambda_n)\right.$$

$$\left.-\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]f(t)d\xi dt = \int_0^\pi\left(\int_0^x K(x,\xi)F(\xi,t)d\xi\right)f(t)dt, \tag{3.7}$$

$$\lim_{N\to\infty}\int_0^\pi I_{N3}(x,t)f(t)dt$$

$$= \lim_{N\to\infty}\int_0^\pi\int_0^x K(x,\xi)\sum_{n=-N}^{N}\frac{1}{\pi}\varphi_0(\xi,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)f(t)d\xi dt$$

$$= \int_0^x K(x,\xi)f(\xi)d\xi, \tag{3.8}$$

$$\lim_{N\to\infty}\int_0^\pi I_{N4}(x,t)f(t)dt$$

$$= \lim_{N\to\infty}\int_0^\pi\left(\sum_{n=-N}^{N}\frac{1}{\alpha_n}\varphi(x,\lambda_n)\int_0^t\tilde{\varphi}(\xi,\lambda_n)\tilde{H}(t,\xi)d\xi\right)f(t)dt$$

$$= \int_x^\pi \tilde{H}(t,x) f(t) dt. \tag{3.9}$$

We put $K(x,t) = H(x,t) = 0$ for $x < t$. Using (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) we obtain

$$\int_0^\pi F(x,t) f(t) dt + \int_0^\pi \left(\int_0^x K(x,\xi) F(\xi,t) d\xi \right) f(t) dt + \int_0^x K(x,\xi) f(\xi) d\xi - \int_x^\pi \tilde{H}(t,x) f(t) dt = 0.$$

Since $f(x)$ can be chosen arbitrarily,

$$F(x,t) + K(x,t) + \int_0^x K(x,\xi) F(\xi,t) d\xi - \tilde{H}(t,x) = 0.$$

When $t < x$, this equality implies (3.1). □

4. Theorem for the solution of the inverse problem

Lemma 1. *For each fixed $x \in (0,\pi]$ the main equation (3.1) has a unique solution $K(x,t) \in L_2(0,x)$.*

Proof. It suffices to prove that homogeneous equation

$$g(t) + \int_0^x g(s) F(s,t) ds = 0$$

has only trivial solution $g(t) = 0$. Let $g(t) = (g_1(t),\ g_2(t)) \in L_2(0,x)$ be a solution of the above equation and $g(t) = 0$ for $t \in (x,\pi)$. Then

$$(g(t),\ g(t)) + \left(\int_0^x g(s) F(s,t) ds,\ g(t) \right) = 0.$$

Using the expression (3.2), we get

$$\begin{aligned}
&\int_0^x \left(g_1^2(t) + g_2^2(t) \right) dt \\
&+ \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \int_0^x \left[\int_0^x \left(g_1(s) \sin \lambda_n s - g_2(s) \cos \lambda_n s \right) ds \right] g_1(t) \sin \lambda_n t \, dt \\
&- \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \int_0^x \left[\int_0^x \left(g_1(s) \sin \lambda_n s - g_2(s) \cos \lambda_n s \right) ds \right] g_2(t) \cos \lambda_n t \, dt \\
&- \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_0^x \left[\int_0^x \left(g_1(s) \sin \lambda_n^0 s - g_2(s) \cos \lambda_n^0 s \right) ds \right] g_1(t) \sin \lambda_n^0 t \, dt \\
&+ \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_0^x \left[\int_0^x \left(g_1(s) \sin \lambda_n^0 s - g_2(s) \cos \lambda_n^0 s \right) ds \right] g_2(t) \cos \lambda_n^0 t \, dt
\end{aligned}$$

$$= \int_0^x \left(g_1^2(t)+g_2^2(t)\right)dt + \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^x \left[g_1(t)\sin\lambda_n t - g_2(t)\cos\lambda_n t\right]dt\right)^2$$
$$- \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left(\int_0^x \left[g_1(t)\sin\lambda_n^0 t - g_2(t)\cos\lambda_n^0 t\right]dt\right)^2 = 0.$$

Thus, it follows from the last relation that

$$\int_0^x \left(g_1^2(t)+g_2^2(t)\right)dt + \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^x g(t)\varphi_0(t,\lambda_n)dt\right)^2$$
$$- \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left(\int_0^x g(t)\varphi_0(t,\lambda_n^0)dt\right)^2 = 0.$$

Using Parseval equality,

$$\|g\|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left(\int_0^x g(t)\varphi_0(t,\lambda_n^0)dt\right)^2$$

we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^x g(t)\varphi_0(t,\lambda_n)dt\right)^2 = 0.$$

Since $\alpha_n > 0$ and from statement a) in Theorem 2, the system $\{\varphi_0(t,\lambda_n)\}$, $(n \in \mathbb{Z})$ is complete in $L_2(0,\pi;\mathbb{C}^2)$, we obtain $g(t) = 0$. □

Theorem 4. *Let $L(\Omega(x),h_1,h_2)$ and $\hat{L}(\hat{\Omega}(x),\hat{h}_1,\hat{h}_2)$ be two boundary value problems and*

$$\lambda_n = \hat{\lambda}_n, \quad \alpha_n = \hat{\alpha}_n, \quad (n \in \mathbb{Z}).$$

Then

$$\Omega(x) = \hat{\Omega}(x) \text{ a.e. on } (0,\pi), \quad h_1 = \hat{h}_1, \quad h_2 = \hat{h}_2.$$

Proof. According to (3.2), $F(x,t) = \hat{F}(x,t)$. Then, from the main equation (3.1), we have $K(x,t) = \hat{K}(x,t)$. It follows from (2.3) that $\Omega(x) = \hat{\Omega}(x)$ a.e. on $(0,\pi)$. Taking into account (2.2), we find $\varphi(x,\lambda_n) = \hat{\varphi}(x,\lambda_n)$. In consideration of (2.11), we get $\dot{\Delta}(\lambda_n) \equiv \dot{\hat{\Delta}}(\lambda_n)$ and from (2.6), $\beta_n = \hat{\beta}_n$. Thus, using (2.1) and (2.5), we obtain $h_1 = \hat{h}_1$, $h_2 = \hat{h}_2$. □

5. Solution of the inverse problem

Let the real numbers $\{\lambda_n,\alpha_n\}$, $(n \in \mathbb{Z})$ of the form (2.7) and (2.9) be given. Using these numbers, we construct the function $F(x,t)$ by the formulas (3.2) and determine $K(x,t)$ from the main equation (3.1).

Now, let us construct the functions $\varphi(x,\lambda)$ by the formula (2.2), $\Omega(x)$ by the formula (2.3), $\Delta(\lambda)$ by the formula (2.12) and β_n by the formula (2.6) respectively, i.e.,

$$\varphi(x,\lambda) := \varphi_0(x,\lambda) + \int_0^x K(x,t)\varphi(t,\lambda)dt,$$
$$\Omega(x) := K(x,x)B - BK(x,x),$$
$$\Delta(\lambda) := -\pi(\lambda_0^2-\lambda^2)\prod_{n=1}^{\infty}\frac{(\lambda_n^2-\lambda^2)}{n^2},$$
$$\beta_n := \frac{\dot{\Delta}(\lambda_n)}{\alpha_n} \neq 0.$$

The function $F(x,t)$ can be rewritten as follows:

$$F(x,t) = \frac{1}{2}[a(x-t) + a(x+t)T],$$

where

$$a(x) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{\alpha_n}\begin{pmatrix}\cos\lambda_n x & -\sin\lambda_n x\\ \sin\lambda_n x & \cos\lambda_n x\end{pmatrix} - \frac{1}{\pi}\begin{pmatrix}\cos nx & -\sin nx\\ \sin nx & \cos nx\end{pmatrix}\right]$$

and $T = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. It is shown analogously in ([7], Lemma 1.5.4) that the function $a(x) \in W_2^1[0,2\pi]$.

5.1. *Derivation of the differential equation*

Theorem 5. *The following relations hold:*

$$B\varphi'(x,\lambda) + \Omega(x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda), \tag{5.1}$$
$$\varphi_1(0,\lambda) = 0, \quad \varphi_2(0,\lambda) = -1. \tag{5.2}$$

Proof. Differentiating on x and t the main equation (3.1) respectively, we get

$$F_x'(x,t) + K_x'(x,t) + K(x,x)F(x,t) + \int_0^x K_x'(x,\xi)F(\xi,t)d\xi = 0, \tag{5.3}$$

$$F_t'(x,t) + K_t'(x,t) + \int_0^x K(x,\xi)F_t'(\xi,t)d\xi = 0. \tag{5.4}$$

It follows from (3.2) that

$$F_t'(x,t)B + BF_x'(x,t) = 0. \tag{5.5}$$

Since $F(x,0)BS = 0$, where $S = \begin{pmatrix}0\\ -1\end{pmatrix}$, using the main equation (3.1), we obtain

$$K(x,0)BS = 0, \tag{5.6}$$

or

$$K_{11}(x,0) = K_{21}(x,0) = 0.$$

Multiplying the equation (5.3) on the left by B, we get

$$BF'_x(x,t) + BK'_x(x,t) + BK(x,x)F(x,t) + \int_0^x BK'_x(x,\xi)F(\xi,t)d\xi = 0,$$

and multiplying the equation (5.4) on the right by B, we find

$$F'_t(x,t)B + K'_t(x,t)B + \int_0^x K(x,\xi)F'_t(\xi,t)Bd\xi = 0.$$

Adding the last two equalities and using (5.5)

$$BK'_x(x,t) + BK(x,x)F(x,t) + \int_0^x BK'_x(x,\xi)F(\xi,t)d\xi$$
$$= -K'_t(x,t)B + \int_0^x K(x,\xi)BF'_\xi(\xi,t)d\xi \equiv I(x,t). \quad (5.7)$$

Integrating by parts in (5.7) and from (5.6),

$$I(x,t) = -K'_t(x,t)B + K(x,x)BF(x,t) - \int_0^x K'_\xi(x,\xi)BF(\xi,t)d\xi \quad (5.8)$$

is obtained. Substituting (5.8) into (5.7), we have

$$BK'_x(x,t) + BK(x,x)F(x,t) + K'_t(x,t)B - K(x,x)BF(x,t)$$
$$+ \int_0^x \left[BK'_x(x,\xi) + K'_\xi(x,\xi)B\right]F(\xi,t)d\xi = 0. \quad (5.9)$$

Multiplying (3.1) on the left by $\Omega(x)$ in form of (2.3) and adding to (5.9),

$$BK'_x(x,t) + K'_t(x,t)B + \Omega(x)K(x,t)$$
$$+ \int_0^x \left[BK'_x(x,\xi) + K'_\xi(x,\xi)B + \Omega(x)K(x,\xi)\right]F(\xi,t)d\xi = 0 \quad (5.10)$$

is obtained. Setting

$$J(x,t) := BK'_x(x,t) + K'_t(x,t)B + \Omega(x)K(x,t),$$

we can rewrite equation (5.10) as follows

$$J(x,t) + \int_0^x J(x,\xi)F(\xi,t)d\xi = 0. \quad (5.11)$$

According to Lemma 1, homogeneous equation (5.11) has only trivial solution, i.e.,

$$BK'_x(x,t) + K'_t(x,t)B + \Omega(x)K(x,t) = 0, \quad 0 < t < x. \quad (5.12)$$

Now, differentiating (2.2) and multiplying on the left by B, we have

$$B\varphi'_x(x,\lambda) = \lambda B \begin{pmatrix} \cos\lambda x \\ \sin\lambda x \end{pmatrix} + BK(x,x) \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix}$$

$$+\int_0^x BK_x'(x,t)\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt. \tag{5.13}$$

On the other hand, multiplying (2.2) on the left by λ and integrating by parts and using (5.6), we find

$$\lambda\varphi(x,\lambda)=\lambda B\begin{pmatrix}\cos\lambda x\\ \sin\lambda x\end{pmatrix}+K(x,x)B\begin{pmatrix}\sin\lambda x\\ -\cos\lambda x\end{pmatrix}$$
$$-\int_0^x K_t'(x,t)B\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\lambda\varphi(x,\lambda)=B\varphi_x'(x,\lambda)+[K(x,x)B-BK(x,x)]\begin{pmatrix}\sin\lambda x\\ -\cos\lambda x\end{pmatrix}$$
$$-\int_0^x\left[K_t'(x,t)B+BK_x'(x,t)\right]\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt.$$

Taking into account (2.3) and (5.12),

$$B\varphi_x'(x,\lambda)+\Omega(x)\varphi(x,\lambda)=\lambda\varphi(x,\lambda)$$

is obtained. For $x=0$, from (2.2) we get (5.2). □

5.2. *Derivation of Parseval equality*

Theorem 6. *For any $g(x)\in L_2(0,\pi;\mathbb{C}^2)$, the following relation holds:*

$$\|g\|_{L_2}^2=\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi\tilde\varphi(t,\lambda_n)g(t)dt\right)^2. \tag{5.15}$$

Proof. Denote

$$Q(\lambda):=\int_0^\pi\tilde\varphi(t,\lambda)g(t)dt.$$

It follows from (2.2) that

$$Q(\lambda):=\int_0^\pi\tilde\varphi_0(t,\lambda)h(t)dt,$$

where

$$h(t)=g(t)+\int_t^\pi\tilde K(s,t)g(s)ds. \tag{5.16}$$

Similarly, in view of (3.3),

$$g(t)=h(t)+\int_t^\pi\tilde H(s,t)h(s)ds, \tag{5.17}$$

where for the kernel $H(x,t)$ we have the identity

$$\tilde H(x,t)=F(t,x)+\int_0^t K(t,\xi)F(\xi,x)d\xi. \tag{5.18}$$

Using (5.16), we get

$$\begin{aligned}\int_0^\pi F(x,t)h(t)dt &= \int_0^\pi F(x,t)\left[g(t)+\int_t^\pi \tilde{K}(s,t)g(s)ds\right]dt\\ &= \int_0^\pi \left[F(x,t)+\int_0^t F(x,s)\tilde{K}(t,s)ds\right]g(t)dt\\ &= \int_0^x \left[F(x,t)+\int_0^t F(x,s)\tilde{K}(t,s)ds\right]g(t)dt\\ &\quad+\int_x^\pi \left[F(x,t)+\int_0^t F(x,s)\tilde{K}(t,s)ds\right]g(t)dt.\end{aligned}$$

It follows from (3.1) and (5.18) that

$$\int_0^\pi F(x,t)h(t)dt = \int_0^x H(x,t)g(t)dt - \int_x^\pi \tilde{K}(t,x)g(t)dt. \tag{5.19}$$

From (3.2) and Parseval equality, we obtain

$$\begin{aligned}&\int_0^\pi \left[h_1^2(t)+h_2^2(t)\right]dt+\int_0^\pi\int_0^\pi \tilde{h}(x)F(x,t)h(t)dtdx\\ &=\int_0^\pi \left[h_1^2(t)+h_2^2(t)\right]dt+\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n)h(t)dt\right)^2\\ &\quad-\sum_{n=-\infty}^{\infty}\frac{1}{\pi}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n^0)h(t)dt\right)^2\\ &=\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n)h(t)dt\right)^2\\ &=\sum_{n=-\infty}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}.\end{aligned}$$

Taking into account (5.19), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi \left[h_1^2(t)+h_2^2(t)\right]dt+\int_0^\pi \tilde{h}(x)\left(\int_0^x H(x,t)g(t)dt\right)dx\\ &\quad-\int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{K}(t,x)g(t)dt\right)dx\\ &=\int_0^\pi \left[h_1^2(t)+h_2^2(t)\right]dt+\int_0^\pi \left(\int_t^\pi \tilde{h}(x)H(x,t)dx\right)g(t)dt\\ &\quad-\int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{K}(t,x)g(t)dt\right)dx,\end{aligned}$$

whence by formulas (5.16) and (5.17),

$$\sum_{n=-\infty}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n} = \int_0^{\pi} \left[h_1^2(t)+h_2^2(t)\right] dt + \int_0^{\pi} \left(\tilde{g}(t)-\tilde{h}(t)\right) g(t) dt$$
$$- \int_0^{\pi} \tilde{h}(x)\left(h(x)-g(x)\right) dx$$
$$= \int_0^{\pi} \left(g_1^2(t)+g_2^2(t)\right) dt$$

is obtained, i.e., the relation (5.15) is valid. □

Corollary 1. *For any function $f(x)$ and $g(x) \in L_2(0,\pi;\mathbb{C}^2)$, the following relation holds:*

$$\int_0^{\pi} \tilde{g}(x) f(x) dx = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \left(\int_0^{\pi} \tilde{g}(t) \varphi(t,\lambda_n) dt \right) \left(\int_0^{\pi} \tilde{\varphi}(t,\lambda_n) f(t) dt \right). \tag{5.20}$$

Lemma 2. *For any $f(x) \in W_2^1[0,\pi]$, the expansion formula*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x,\lambda_n) \tag{5.21}$$

is valid, where

$$c_n = \frac{1}{\alpha_n} \int_0^{\pi} \tilde{\varphi}(x,\lambda_n) f(x) dx.$$

Proof. Consider the series

$$f^*(x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x,\lambda_n), \tag{5.22}$$

where

$$c_n := \frac{1}{\alpha_n} \int_0^{\pi} \tilde{\varphi}(x,\lambda_n) f(x) dx. \tag{5.23}$$

Using Theorem 2.5 and integrating by parts , we get

$$c_n = \frac{1}{\alpha_n \lambda_n} \int_0^{\pi} \left[-\frac{\partial}{\partial x} \tilde{\varphi}(x,\lambda_n) B + \tilde{\varphi}(x,\lambda_n) \Omega(x) \right] f(x) dx$$
$$= -\frac{1}{\alpha_n \lambda_n} \left[\tilde{\varphi}(\pi,\lambda_n) B f(\pi) - \tilde{\varphi}(0,\lambda_n) B f(0) \right]$$
$$+ \frac{1}{\alpha_n \lambda_n} \int_0^{\pi} \tilde{\varphi}(x,\lambda_n) \left[B f'(x) + \Omega(x) f(x) \right] dx.$$

Applying the asymptotic formulas in Theorem 1.1, we find $\{c_n\} \in l_2$. Consequently the series (5.22) converges absolutely and uniformly on $[0,\pi]$. According to (5.20) and (5.23), we have

$$\begin{aligned}\int_0^\pi \tilde{g}(x) f(x) dx &= \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi \tilde{g}(t) \varphi(t,\lambda_n) dt \right) \left(\int_0^\pi \tilde{\varphi}(t,\lambda_n) f(t) dt \right) \\ &= \sum_{n=-\infty}^{\infty} c_n \left(\int_0^\pi \tilde{g}(t) \varphi(t,\lambda_n) dt \right) = \int_0^\pi \tilde{g}(t) \left(\sum_{n=-\infty}^{\infty} c_n \varphi(t,\lambda_n) \right) \\ &= \int_0^\pi \tilde{g}(t) f^*(t) dt.\end{aligned}$$

Since $g(x)$ is arbitrary, $f(x) = f^*(x)$ is obtained, i.e., the expansion formula (5.21) is proved. □

5.3. *Derivation of boundary condition*

Lemma 3. *The following equality holds:*

$$\sum_{n=-\infty}^{\infty} \frac{\varphi(x,\lambda_n)}{\alpha_n \beta_n} = 0. \tag{5.24}$$

Proof. Using residue theorem, we get

$$\sum_{n=-\infty}^{\infty} \frac{\varphi(x,\lambda_n)}{\alpha_n \beta_n} = \sum_{n=-\infty}^{\infty} \frac{\varphi(x,\lambda_n)}{\dot{\Delta}(\lambda_n)} = \sum_{n=-\infty}^{\infty} \operatorname*{Res}_{\lambda=\lambda_n} \frac{\varphi(x,\lambda)}{\Delta(\lambda)} = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{\varphi(x,\lambda)}{\Delta(\lambda)} d\lambda, \tag{5.25}$$

where $\Gamma_N = \left\{\lambda : |\lambda| = N + \frac{1}{2}\right\}$. From (2.12) and ([19], Lemma 3.4.2),

$$\Delta(\lambda) = \lambda \sin \lambda\pi + O(e^{|Im\lambda|\pi}). \tag{5.26}$$

We denote $G_\delta = \{\lambda : |\lambda - n| \geq \delta,\ \ n = 0, \pm 1, \pm 2 ...\}$ for some small fixed $\delta > 0$ and $|\sin \lambda\pi| \geq C_\delta e^{|Im\lambda|\pi}$, $\lambda \in G_\delta$, where C_δ positive number. Therefore, we have

$$|\Delta(\lambda)| \geq C_\delta |\lambda| e^{|Im\lambda|\pi}, \ \ \lambda \in G_\delta.$$

Using this inequality and (2.10), we obtain (5.24). □

Theorem 7. *The following relation is valid:*

$$(\lambda_n + h_1)\varphi_1(\pi,\lambda_n) + h_2\varphi_2(\pi,\lambda_n) = 0. \tag{5.27}$$

Proof. From (5.24), we can write for any $n_0 \in \mathbb{Z}$

$$\frac{\varphi(x,\lambda_{n_0})}{\alpha_{n_0}} = -\sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} \frac{\beta_{n_0} \varphi(x,\lambda_n)}{\alpha_n \beta_n} \tag{5.28}$$

Let $k \neq n_0$ be any fixed number and $f(x) = \varphi(x, \lambda_k)$. Then substituting (5.28) in (5.21)

$$\varphi(x, \lambda_k) = \sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} c_{nk} \varphi(x, \lambda_n),$$

where

$$c_{nk} = \frac{1}{\alpha_n} \int_0^{\pi} \left[\tilde{\varphi}(t, \lambda_n) - \frac{\beta_{n_0}}{\beta_n} \tilde{\varphi}(t, \lambda_{n_0}) \right] \varphi(t, \lambda_k) dt.$$

The system of functions $\{\varphi_0(x, \lambda_n)\}$, $(n \in \mathbb{Z})$ is orthogonal in $L_2(0, \pi; \mathbb{C}^2)$. Then by (2.2), the system of functions $\{\varphi(x, \lambda_n)\}$, $(n \in \mathbb{Z})$ is orthogonal in $L_2(0, \pi; \mathbb{C}^2)$ as well. Therefore, $c_{nk} = \delta_{nk}$, where δ_{nk} is Kronecker delta. Let us define

$$a_{nk} := \int_0^{\pi} \tilde{\varphi}(t, \lambda_n) \varphi(t, \lambda_k) dt. \tag{5.29}$$

Using this expression, we have for $n \neq k$

$$a_{kk} - \frac{\beta_n}{\beta_k} a_{nk} = \alpha_k. \tag{5.30}$$

It follows from (5.29) that $a_{nk} = a_{kn}$. Taking into account this equality and (5.30),

$$\beta_k^2 (\alpha_k - a_{kk}) = \beta_n^2 (\alpha_n - a_{nn}) = H, \quad k \neq n,$$

where H is a constant. Then, we have

$$\int_0^{\pi} \tilde{\varphi}(t, \lambda_n) \varphi(t, \lambda_n) dt = \alpha_n - \frac{H}{\beta_n^2}$$

and

$$\int_0^{\pi} \tilde{\varphi}(t, \lambda_k) \varphi(t, \lambda_n) dt = -\frac{H}{\beta_k \beta_n}, \quad k \neq n.$$

It is easily obtained that for $k \neq n$,

$$\int_0^{\pi} [\varphi_1(x, \lambda_k) \varphi_1(x, \lambda_n) + \varphi_2(x, \lambda_k) \varphi_2(x, \lambda_n)] dx$$
$$= \frac{1}{(\lambda_k - \lambda_n)} [\varphi_2(\pi, \lambda_k) \varphi_1(\pi, \lambda_n) - \varphi_1(\pi, \lambda_k) \varphi_2(\pi, \lambda_n)] = -\frac{H}{\beta_k \beta_n}.$$

According to the last equation, for $n \neq k$,

$$\beta_k \varphi_2(\pi, \lambda_k) \beta_n \varphi_1(\pi, \lambda_n) - \beta_k \varphi_1(\pi, \lambda_k) \beta_n \varphi_2(\pi, \lambda_n) = -H(\lambda_k - \lambda_n). \tag{5.31}$$

We denote

$$D_n := \beta_n \varphi_1(\pi, \lambda_n), \quad E_n := \beta_n \varphi_2(\pi, \lambda_n). \tag{5.32}$$

Then, we can rewrite equation (5.31) as follows

$$D_k E_n - E_k D_n = H(\lambda_k - \lambda_n), \quad n \neq k. \tag{5.33}$$

Let $i,\ j,\ k,\ n$ be pairwise distinct integers, then we get

$$D_k E_n - E_k D_n = H(\lambda_k - \lambda_n),$$
$$D_n E_i - E_n D_i = H(\lambda_n - \lambda_i),$$
$$D_i E_k - E_i D_k = H(\lambda_i - \lambda_k).$$

Adding them together, we find

$$D_n(E_i - E_k) + E_n(D_k - D_i) = E_i D_k - D_i E_k.$$

In this equation, replacing n by j, we get another equation

$$D_j(E_i - E_k) + E_j(D_k - D_i) = E_i D_k - D_i E_k.$$

Subtracting the last two equation,

$$(D_n - D_j)(E_i - E_k) = (D_i - D_k)(E_n - E_j).$$

In the case of $E_n = E_j$, for some $n,\ j \in \mathbb{Z}$, then E_n =const. From (5.33), $D_n = \mu_1\lambda_n + \mu_2$. In the case of $E_n \neq E_j$, then we obtain $D_n = \mu_1\lambda_n + \mu_2$ and $E_n = \mu_3\lambda_n + \mu_4$, where in both cases $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ are constant. Therefore, using these relation in (5.32), we find

$$\beta_n\varphi_1(\pi,\lambda_n) = \mu_1\lambda_n + \mu_2, \quad \beta_n\varphi_2(\pi,\lambda_n) = \mu_3\lambda_n + \mu_4.$$

Using

$$\varphi_1(\pi,\lambda_n) = O\left(\frac{1}{n}\right), \quad \varphi_2(\pi,\lambda_n) = (-1)^{n+1} + O\left(\frac{1}{n}\right),$$

$\lambda_n = n + O\left(\frac{1}{n}\right)$ and $\beta_n = n(-1)^n + O(1)$ derived from (2.6) and (5.26), we obtain $\mu_1 = 0$, $\mu_3 = -1$. Denoting $h_2 := \mu_2$ and $h_1 := -\mu_4$,

$$h_2\varphi_2(\pi,\lambda_n) = -(\lambda_n + h_1)\varphi_1(\pi,\lambda_n), \ \ n \in \mathbb{Z}$$

is obtained and it follows from (5.33) that $H = h_2$. □

Thus, we have proved the following theorem:

Theorem 8. *For the sequences $\{\lambda_n,\alpha_n\}$, $(n \in \mathbb{Z})$, to be the spectral data for a certain boundary value problem $L(\Omega(x),h_1,h_2)$ of the form (1.1), (1.2) with $\Omega(x) \in L_2(0,\pi)$ it is necessary and sufficient that the relations (2.7) and (2.9) hold.*

Algorithm for construction of the function $\Omega(x)$ by spectral data $\{\lambda_n,\alpha_n\}$, $(n \in \mathbb{Z})$ follows from the proof of the theorem:
1) By the given numbers $\{\lambda_n,\alpha_n\}$, $(n \in \mathbb{Z})$ the function $F(x,t)$ is constructed by formula (3.2),
2) The function $K(x,t)$ is found from equation (3.1),
3) $\Omega(x)$ is calculated by the formula (2.3).

ACKNOWLEDGEMENT

This work is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK).

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Doi: 10.18514/MMN.2015.1256

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 277–281

HU e-ISSN 1787-2413

A DOUBLE INEQUALITY INVOLVING ERDŐS-BORWEIN CONSTANTS

MIRCEA MERCA

Received 20 May, 2014

Abstract. In this note, the author proves a double inequality by introducing an approximation of Erdős-Borwein constants.

2010 *Mathematics Subject Classification:* 26D15; 26D07; 30B10

Keywords: inequalities, number of divisors, series

1. Introduction

Recently, after an experiment, we stumbled upon the following double inequality

$$\left(x-\frac{1}{x+1}\right)^{-1}<\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n}<\left(x-\frac{1}{x}\right)^{-1},\quad x>1. \tag{1.1}$$

In this paper, we present a proof of this inequality using several tools from number theory.

Recall that a Mersenne number is a positive integer that is one less than a power of two:

$$2^n-1.$$

The sum of the reciprocals of all Mersenne numbers, namely

$$E=\sum_{n=1}^{\infty}\frac{1}{2^n-1}=1.606695152415291763 3833\ldots$$

is known as the Erdős-Borwein constant. This mysterious number is known to be irrational, as shown by Erdős [4] in 1948.

More recently, using Padé approximant techniques, P. Borwein [2] established the irrationality of more general numbers

$$EB(x)=\sum_{n=1}^{\infty}\frac{1}{x^n-1}$$

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for x a positive integer, $x>1$. We remark that the numbers $EB(x)$ can be written as

$$EB(x)=\frac{1}{x-1}+\frac{1}{x-1}\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n},\qquad x>1.$$

We have the following approximation of these irrational numbers.

Theorem 1. *Let x be a real number such that $x>1$. Then*

$$\frac{1}{x}+\frac{1}{x^4}\cdot\frac{x^2+1}{x+1}<\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n}<\frac{x}{x^2-1}-\frac{x}{x^5-1}+\frac{x}{x^6-1}-\frac{x}{x^7-1}.$$

It is clear that this series has an asymptotic behaviour, i.e.,

$$\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n}\sim\frac{1}{x}\qquad(x\to\infty).$$

The case $x=2$ of this theorem can be written as

$$\frac{29}{48}<E-1<\frac{153320}{248031}.$$

In other words, we have

$$1.60416\ldots<E<1.61814\ldots$$

2. Proof of Theorem 1

Let n be a positive integer. The divisors function $\tau(n)$ is defined as the number of divisors of n, unity and n itself included, i.e.,

$$\tau(n)=\sum_{d\mid n}1.$$

In number theory, the inequality

$$\tau(n)\leqslant\lfloor 2\sqrt{n}\rfloor$$

is well-known. It is an easy exercise to show that

$$\lfloor 2\sqrt{n}\rfloor<1+\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor-\left\lfloor\frac{n}{7}\right\rfloor,\qquad n\geqslant 36$$

and

$$\tau(n)\leqslant 1+\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor-\left\lfloor\frac{n}{7}\right\rfloor,\qquad n<36.$$

Clearly,

$$\tau(n)\leqslant 1+\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor-\left\lfloor\frac{n}{7}\right\rfloor,$$

for any positive integer n.

To prove the inequality we consider: the generating function of $\tau(n)$ [1, s. 24.3.3],

$$\sum_{n=1}^{\infty}\tau(n)q^n=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n},\qquad |q|<1,$$

the generating function of 1,

$$\sum_{n=0}^{\infty}q^n=\frac{1}{1-q},\qquad |q|<1,$$

and the generating function of $\lfloor n/k\rfloor$,

$$\sum_{n=0}^{\infty}\left\lfloor\frac{n}{k}\right\rfloor q^n=\frac{q^k}{(1-q)(1-q^k)},\qquad |q|<1,$$

where k is a positive integer.

For $0<q<1$, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}&<\sum_{n=1}^{\infty}\left(1+\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor-\left\lfloor\frac{n}{7}\right\rfloor\right)q^n\\&=\frac{q}{1-q}+\frac{q^2}{(1-q)(1-q^2)}-\frac{q^5}{(1-q)(1-q^5)}\\&\quad+\frac{q^6}{(1-q)(1-q^6)}-\frac{q^7}{(1-q)(1-q^7)}.\end{aligned}$$

Thus we deduce that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{q^n}{1-q^n}&<\frac{q^2}{(1-q)(1-q^2)}-\frac{q^5}{(1-q)(1-q^5)}\\&\quad+\frac{q^6}{(1-q)(1-q^6)}-\frac{q^7}{(1-q)(1-q^7)},\qquad 0<q<1\end{aligned}$$

or

$$\sum_{n=1}^{\infty}\frac{q^n}{1+q+q^2+\cdots+q^n}<\frac{q}{1-q^2}-\frac{q^4}{1-q^5}+\frac{q^5}{1-q^6}-\frac{q^6}{1-q^7},\qquad 0<q<1.$$

Replacing q by $1/x$ in this inequality, we get

$$\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n}<\frac{x}{x^2-1}-\frac{x}{x^5-1}+\frac{x}{x^6-1}-\frac{x}{x^7-1},\qquad x>1.$$

On the other hand, due to Clausen's [3], we have the following identity:

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}=\sum_{n=1}^{\infty}q^{n^2}\frac{1+q^n}{1-q^n},\qquad |q|<1.$$

For $x>1$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{1+x+x^2+\cdots+x^n}&=(x-1)EB(x)-1\\&=(x-1)\sum_{n=1}^{\infty}x^{-n^2}\frac{x^n+1}{x^n-1}-1\\&>(x-1)\left(\frac{1}{x}\cdot\frac{x+1}{x-1}+\frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1}\right)-1\\&=\frac{1}{x}+\frac{1}{x^4}\cdot\frac{x^2+1}{x+1}\end{aligned}$$

and Theorem 1 is proved.

3. Concluding remarks

Lower and upper bounds of Erdős-Borwein constants has been introduced in the paper. For $x>1$, it is an easy exercise to prove that

$$\frac{x+1}{x^2+x-1}<\frac{1}{x}+\frac{1}{x^4}\cdot\frac{x^2+1}{x+1}$$

and

$$\frac{x}{x^2-1}-\frac{x}{x^5-1}+\frac{x}{x^6-1}-\frac{x}{x^7-1}<\frac{x}{x^2-1}.$$

So by Theorem 1, we derive the double inequality (1.1) that can be written in terms of Erdős-Borwein constants as follows

Corollary 1. *For $x>1$,*

$$\left(x-\frac{1}{x+1}\right)^{-1}<(x-1)EB(x)-1<\left(x-\frac{1}{x}\right)^{-1}.$$

Acknowledgement

Special thanks go to the anonymous referees for their helpful assistance. The author expresses his gratitude to Dr. Oana Merca for the careful reading of the manuscript and helpful remarks.

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Doi: 10.18514/MMN.2015.754

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 283–304

HU e-ISSN 1787-2413

SOME SPACES OF DIFFERENCE SEQUENCES AND LACUNARY STATISTICAL CONVERGENCE IN n-NORMED SPACE DEFINED BY SEQUENCE OF ORLICZ FUNCTIONS

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Received 24 September, 2013

Abstract. In the present paper we introduce some new sequence spaces in the n-normed space. We define here some new spaces of double sequences by using the notions of lacunary sequences, difference sequences, almost convergence, sequence of Orlicz functions and statistical convergence. We examine some topological properties of these spaces of double sequences, inclusion relations between these newly defined sequence spaces and establish relation with lacunary statistical convergence. The study of these new sequence spaces provides a tool to deal with convergence problems of double sequences.

2010 *Mathematics Subject Classification:* 40A05; 46A45; 46B70

Keywords: difference sequences, double sequence spaces, Orlicz function, n-normed space, statistical convergence, lacunary statistical convergence

1. Introduction and preliminaries

The initial work on double sequences is found in Bromwich [5]. Later on, it was studied by Hardy [14], Moricz [21], Moricz and Rhoades [22], Mursaleen [24], [23], Basarir and Sonalcan [4], Altay and Basar [2], Basar and Sever [3], Mursaleen and Mohiuddine [29, 30], Alotaibi et al [1] and many others. Mursaleen and Edely [27] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [24] and Mursaleen and Edely [28] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [2] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined

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the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Basar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known classical sequence space ℓ_q and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [36]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.
An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [15] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

It is shown in [15] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. The notion of difference sequence spaces was introduced by Kızmaz [17], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.
Let n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in Z\},$$

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_k)$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

Taking $n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Et and Çolak [7].
The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan ([11, 12]) and Gunawan and Mashadi [13] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where

$d \geq n \geq 2$. A real valued function $||\cdot,\cdots,\cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha|\ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot,\cdots,\cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot,\cdots,\cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot,\cdots,\cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} ||x_k - x_i, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

A double sequence $x = (x_{k,l})$ of real numbers is called almost convergent to a limit L if

$$P - \lim_{p,q\to\infty} \sup_{m,n\geq 0} \left|\frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} x_{k,l} - L\right| = 0$$

i.e. the average value of $(x_{k,l})$ taken over any rectangle $\{(k,l) : m \leq k \leq m+p-1,\ n \leq l \leq n+q-1\}$ tends to L as both p and q tends to ∞, and this P-convergence is uniform in m and n.

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty\ (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be

denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [10] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \text{ and } l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$.

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz function and $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers. Then, we define the following sequence spaces:

$$\Big[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] = \Big\{x = (x_{k,l}) \in w(X) :$$
$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\Big\}$$

$$\Big[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0 = \Big\{x = (x_{k,l}) \in w(X) :$$
$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we have

$$\Big[AC_{\theta_{r,s}}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] = \Big\{x = (x_{k,l}) \in w(X) :$$
$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$
$$= 0, \text{ uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho > 0\Big\}$$

$$\Big[AC_{\theta_{r,s}}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0 = \Big\{x = (x_{k,l}) \in w(X) :$$

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_{k,l}}$$
$$=0,\ \text{uniformly in } m \text{ and } n \text{ for some } \rho>0\Big\}.$$

If we take $p=(p_{k,l})=1$, we have

$$\left[AC_{\theta_{r,s}},\mathcal{M},\Delta^v,||\cdot,\cdots,\cdot||\right]=\Big\{x=(x_{k,l})\in w(X):$$
$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\right)\right]$$
$$=0,\ \text{uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho>0\Big\}$$

$$\left[AC_{\theta_{r,s}},\mathcal{M},\Delta^v,||\cdot,\cdots,\cdot||\right]_0=\Big\{x=(x_{k,l})\in w(X):$$
$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)\right]$$
$$=0,\ \text{uniformly in } m \text{ and } n \text{ for some } \rho>0\Big\}.$$

If we take $p=(p_{k,l})=1$ and $\mathcal{M}(x)=x$, we have

$$\left[AC_{\theta_{r,s}},\Delta^v,||\cdot,\cdots,\cdot||\right]=\Big\{x=(x_{k,l})\in w(X):$$
$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\right)$$
$$=0,\ \text{uniformly in } m \text{ and } n \text{ for some } L \text{ and } \rho>0\Big\}$$

$$\left[AC_{\theta_{r,s}},\Delta^v,||\cdot,\cdots,\cdot||\right]_0=\Big\{x=(x_{k,l})\in w(X):$$
$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)$$
$$=0,\ \text{uniformly in } m \text{ and } n \text{ for some } \rho>0\Big\}.$$

The following inequality will be used throughout the paper. If $0\le p_{k,l}\le \sup p_{k,l}=H$, $D=\max(1,2^{H-1})$ then

$$|a_{k,l}+b_{k,l}|^{p_{k,l}}\le D\{|a_{k,l}|^{p_{k,l}}+|b_{k,l}|^{p_{k,l}}\} \tag{1.1}$$

for all k,l and $a_{k,l},b_{k,l}\in\mathbb{C}$. Also $|a|^{p_{k,l}}\le\max(1,|a|^H)$ for all $a\in\mathbb{C}$.

The main aim of this paper is to study some topological properties and inclusion relation between the spaces $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]$ and $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]_0$ in the second section of the paper. In the third section of this paper we defined sequence spaces $\left[\mathcal{F}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]$ and $\left[\mathcal{F}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]$ and proved very interesting inclusion relations between the spaces defined in this section and spaces $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]$ and $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]_0$. We also make an effort to study statistical convergence in the fourth section of this paper.

2. Some topological properties

Theorem 1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be a factorable double sequence of positive real numbers, then the spaces $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]$ and $\left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]_0$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_{k,l})$, $y = (y_{k,l}) \in \left[AC_{\theta_{r,s}}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\right]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} = 0,$$

uniformly in m and n for some $\rho_1 > 0$,

and

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} = 0,$$

uniformly in m and n for some $\rho_2 > 0$.

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing convex function, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left[M_{k,l}\left(||\frac{\Delta^v(\alpha x_{k+m,l+n} + \beta y_{k+m,l+n})}{\rho_3}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\leq D\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(||\frac{\Delta^v(x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\quad + D\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}}
\end{aligned}$$

$$\leq D\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v(x_{k+m,l+n})}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$+D\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$\longrightarrow 0$ as $r\longrightarrow\infty$, uniformly in m and n.

Thus, we have $\alpha x+\beta y\in\Big[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\Big]_0$. Hence $\Big[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\Big]_0$ is a linear space. Similarly, we can prove that $\Big[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\Big]$ is a linear space. □

Theorem 2. *For any sequence of Orlicz functions $\mathcal{M}=(M_{k,l})$ and $p=(p_{k,l})$ be a factorable double sequence of positive real numbers, the space $\Big[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\Big]_0$ is a topological linear space paranormed by $g(x)=$*

$$\inf\Big\{\rho^{\frac{p_{r,s}}{K}}:\Big(\frac{1}{h_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}\leq 1,r,s\in\mathbb{N}\Big\},$$

where $K=\max(1,\sup_{k,l}p_{k,l}<\infty)$.

Proof. Clearly $g(x)\geq 0$ for $x=(x_{k,l})\in\Big[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\Big]_0$. Since $M_{k,l}(0)=0$, we get $g(0)=0$. Again, if $g(x)=0$, then

$$\inf\Big\{\rho^{\frac{p_{r,s}}{K}}:\Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}\leq 1,r,s\in\mathbb{N}\Big\}$$

$=0$. This implies that for a given $\epsilon>0$, there exists some $\rho_\epsilon(0<\rho_\epsilon<\epsilon)$ such that

$$\Big(\frac{1}{h_{r,s}}\sum_{k,l\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}\leq 1.$$

Thus

$$\Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}$$

$$\leq\Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_\epsilon},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}\leq 1,$$

for each r,s,m and n. Suppose that $x_{k,l}\neq 0$ for each $k,l\in N$. This implies that $\Delta^v x_{k+m,l+n}\neq 0$, for each $k,l,m,n\in N$.

Let $\epsilon \to 0$, then $\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big) \to \infty$. It follows that

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \to \infty,$$

which is a contradiction. Therefore, $\Delta^v x_{k+m,l+n} = 0$ for each k, l, m and n and thus $x_{k,l} = 0$ for each $k, l \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \leq 1$$

and

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \leq 1$$

for each r, s, m and n. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n} + y_{k+m,l+n})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \\
&\leq \Big(\sum_{(k,l)\in I_{r,s}} \Big[\frac{\rho_1}{\rho_1+\rho_2} M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\
&\quad + \frac{\rho_2}{\rho_1+\rho_2} M_{k,l}\Big(||\frac{\Delta^v (y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \\
&\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v (y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \\
&\leq 1.
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_{r,s}}{K}} : \\
&\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n} + y_{k+m,l+n})}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \\
&\qquad \leq 1, r, s \in \mathbb{N}\Big\}, \\
&\leq \inf\Big\{\rho_1^{\frac{p_{r,s}}{K}} : \Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v (x_{k+m,l+n})}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}
\end{aligned}$$

$$\leq 1, r, s \in \mathbb{N}\Big\}$$

$$+\inf\Big\{\rho_2^{\frac{p_{r,s}}{K}} : \Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v(y_{k+m,l+n})}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}}$$

$$\leq 1, r, s \in \mathbb{N}\Big\}.$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_{r,s}}{K}} :$$

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v \lambda x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\}.$$

Then

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{p_{r,s}}{K}} :$$

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_{r,s}} \leq \max(1, |\lambda|^{\sup p_{r,s}})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_{r,s}}) \inf\Big\{t^{\frac{p_r}{K}} :$$

$$\Big(\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{K}} \leq 1, r, s \in \mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

To prove the next theorem we need the following lemma.

Lemma 1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $M(x) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 3. *For a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ which satisfies Δ_2-condition, we have $[AC_{\theta_{r,s}}, \Delta^v, ||\cdot, \cdots, \cdot||] \subseteq [AC_{\theta_{r,s}}\mathcal{M}, \Delta^v, ||\cdot, \cdots, \cdot||]$.*

Proof. Let $x = (x_{k,l}) \in [AC_{\theta_{r,s}}, \Delta^v||\cdot, \cdots, \cdot||]$ so that for each m and n, we have

$$D_{r,s} = \Big\{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}|| = 0,$$

uniformly in m and n for some $L\Big\}$.

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for every t with $0 \le t \le \delta$. Now, we have

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} ||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||$$

$$= \lim_{r,s} \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}|| \le \delta}} M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)$$

$$+ \lim_{r,s} \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}|| > \delta}} M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)$$

$$\le \frac{1}{h_{r,s}}(h_{r,s}\epsilon) + \lim_{r,s} \frac{1}{h_{r,s}} \sum_{\substack{(k,l)\in I_{r,s} \\ ||\frac{||\Delta^v x_{k+m,l+n}-L}{\rho}, z_1, \cdots, z_{n-1}|| > \delta}} M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)$$

$$< \frac{1}{h_{r,s}}(h_{r,s}\epsilon) + \frac{1}{h_{r,s}} K\delta^{-1} M_{k,l}(2) h_{r,s} D_{r,s}.$$

Therefore by Lemma 1 as r and s goes to infinity in the Pringsheim sense, for each m and n, we have $x = (x_{k,l}) \in [AC_{\theta_{r,s}}\mathcal{M}, \Delta^v||\cdot, \cdots, \cdot||]$. This completes the proof of the theorem. □

Theorem 4. *Let* $0 < \inf p_{k,l} = h \le p_{k,l} \le \sup p_{k,l} = H < \infty$ *and* $\mathcal{M} = (M_{k,l})$, $\mathcal{M}' = (M'_{k,l})$ *be two sequences of Orlicz functions which satisfying* Δ_2*-condition, we have*

(i) $\Big[AC_{\theta_{r,s}}, \mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big] \subset \Big[AC_{\theta_{r,s}}, \mathcal{M} \circ \mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big]$ *and*

(ii) $\Big[AC_{\theta_{r,s}}, \mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big]_0 \subset \Big[AC_{\theta_{r,s}}, \mathcal{M} \circ \mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big]_0$.

Proof. Let $x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big]$. Then we have

$$\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M'_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0,$$

uniformly in m and n for some L and $\rho > 0$.

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for $0 \le t \le \delta$. Let

$$y_{k,l} = M'_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big) \text{ for all } k, l \in \mathbb{N}.$$

We can write

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [M_{k,l}(y_{k,l})]^{p_{k,l}} = \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}} + \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}>\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}}.$$

Since $\mathcal{M} = (M_{k,l})$ satisfies Δ_2-condition, we have

$$\begin{aligned}\frac{1}{h_{r,s}} &\sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}} \\ &\le [M_{k,l}(1)]^H \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}} \\ &\le [M_{k,l}(2)]^H \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}\le\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}} \qquad (2.1)\end{aligned}$$

For $y_{k,l} > \delta$

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.$$

Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, it follows that

$$M_{k,l}(y_{k,l}) < M_{k,l}\Big(1 + \frac{y_{k,l}}{\delta}\Big) < \frac{1}{2}M_{k,l}(2) + \frac{1}{2}M_{k,l}\Big(\frac{2y_{k,l}}{\delta}\Big).$$

Also $(M_{k,l})$ satisfies Δ_2-condition, we can write

$$M_{k,l}(y_{k,l}) < \frac{1}{2}T\frac{y_{k,l}}{\delta}M_{k,l}(2) + \frac{1}{2}T\frac{y_{k,l}}{\delta}M_{k,l}(2) = T\frac{y_{k,l}}{\delta}M_{k,l}(2).$$

Hence,

$$\begin{aligned}\frac{1}{h_{r,s}} &\sum_{(k,l)\in I_{r,s}, y_{k,l}>\delta} [M_{k,l}(y_{k,l})]^{p_{k,l}} \\ &\le \max\Big(1, \Big(\frac{TM_{k,l}(2)}{\delta}\Big)^H\Big)\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, y_{k,l}>\delta} [(y_{k,l})]^{p_{k,l}} \qquad (2.2)\end{aligned}$$

From equations (2.1) and (2.2), we have

$$x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', p, \Delta^v||\cdot, \cdots, \cdot||\Big].$$

This completes the proof of (i). Similarly, we can prove that $\Big[AC_{\theta_{r,s}}, \mathcal{M}', p, \Delta^v||\cdot,\cdots,\cdot||\Big]_0 \subset \Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', p, \Delta^v||\cdot,\cdots,\cdot||\Big]_0$. □

Theorem 5. *If $0 < p_{k,l} \leq q_{k,l} < \infty$ for all $k,l \in \mathbb{N}$ and $\left(\frac{q_{k,l}}{p_{k,l}}\right)$ be bounded, then* $\Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', q, \Delta^v||\cdot,\cdots,\cdot||\Big] \subset \Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', p, \Delta^v||\cdot,\cdots,\cdot||\Big]$.

Proof. Let $x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', q, \Delta^v||\cdot,\cdots,\cdot||\Big]$. Write

$$t_{k,l} = \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_{k,l}}$$

and $\mu_{k,l} = \frac{p_{k,l}}{q_{k,l}}$ for all $k,l \in \mathbb{N}$. Then $0 < \mu_{k,l} \leq 1$ for $k,l \in \mathbb{N}$. Take $0 < \mu < \mu_{k,l}$ for $k,l \in \mathbb{N}$. Define the sequences $(u_{k,l})$ and $(v_{k,l})$ as follows: For $t_{k,l} \geq 1$, let $u_{k,l} = t_{k,l}$ and $v_{k,l} = 0$ and for $t_{k,l} < 1$, let $u_{k,l} = 0$ and $v_{k,l} = t_{k,l}$. Then clearly for all $k,l \in \mathbb{N}$, we have

$$t_{k,l} = u_{k,l} + v_{k,l}, \qquad t_{k,l}^{\mu_{k,l}} = u_{k,l}^{\mu_{k,l}} + v_{k,l}^{\mu_{k,l}}$$

Now it follows that $u_{k,l}^{\mu_{k,l}} \leq u_{k,l} \leq t_{k,l}$ and $v_{k,l}^{\mu_{k,l}} \leq v_{k,l}^{\mu}$. Therefore,

$$\begin{aligned}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} t_{k,l}^{\mu_{k,l}} &= \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} (u_{k,l}^{\mu_{k,l}} + v_{k,l}^{\mu_{k,l}}) \\ &\leq \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} t_{k,l} + \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} v_{k,l}^{\mu}.\end{aligned}$$

Now for each k,l we have

$$\begin{aligned}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} v_{k,l}^{\mu} &= \sum_{(k,l)\in I_{r,s}} \Big(\frac{1}{h_{r,s}} v_{k,l}\Big)^{\mu}\Big(\frac{1}{h_{r,s}}\Big)^{1-\mu} \\ &\leq \Big(\sum_{(k,l)\in I_{r,s}} \Big[\Big(\frac{1}{h_{r,s}} v_{k,l}\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{(k,l)\in I_{r,s}} \Big[\Big(\frac{1}{h_{r,s}}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\ &= \Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} v_{k,l}\Big)^{\mu}\end{aligned}$$

and so

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} t_{k,l}^{\mu_{k,l}} \leq \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} t_{k,l} + \Big(\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} v_{k,l}\Big)^{\mu}.$$

Hence $x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}\circ\mathcal{M}', p, \Delta^v||\cdot,\cdots,\cdot||\Big]$. □

3. INCLUSION RELATIONS

Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [19], it was shown that

$$f = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

In [19] Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers. Then we define the following sequence spaces:

$$\Big[\mathcal{F}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] = \Big\{x = (x_{k,l}) \in w(X) :$$

$$P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho > 0\Big\}$$

and

$$\Big[\mathcal{F}, \mathcal{M}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0 = \Big\{x = (x_{k,l}) \in w(X) :$$

$$P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho > 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we have

$$\Big[\mathcal{F}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big] = \Big\{x = (x_{k,l}) \in w(X) :$$

$$P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \Big(||\frac{\Delta^v x_{k+m,l+n} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_{k,l}} = 0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho > 0\Big\}$$

and

$$\Big[\mathcal{F}, p, \Delta^v, ||\cdot, \cdots, \cdot||\Big]_0 = \Big\{x = (x_{k,l}) \in w(X) :$$

$$P-\lim_{p,q}\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)^{p_{k,l}}=0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho>0\Big\}.$$

If we take $p=(p_{k,l})=1$, we get

$$\left[\mathcal{F},\mathcal{M},\Delta^v,||\cdot,\cdots,\cdot||\right]=\Big\{x=(x_{k,l})\in w(X):$$

$$P-\lim_{p,q}\frac{1}{pq}\sum_{k,l=1,1}^{p,q}M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\right)=0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho>0\Big\}$$

and

$$\left[\mathcal{F},\mathcal{M},\Delta^v,||\cdot,\cdots,\cdot||\right]_0=\Big\{x=(x_{k,l})\in w(X):$$

$$P-\lim_{p,q}\frac{1}{pq}\sum_{k,l=1,1}^{p,q}M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)=0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho>0\Big\}.$$

If we take $\mathcal{M}(x)=x$ and $p=(p_{k,l})=1$, we get

$$\left[\mathcal{F},\Delta^v,||\cdot,\cdots,\cdot||\right]=\Big\{x=(x_{k,l})\in w(X):$$

$$P-\lim_{p,q}\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left(||\frac{\Delta^v x_{k+m,l+n}-L}{\rho},z_1,\cdots,z_{n-1}||\right)=0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } L \text{ and } \rho>0\Big\}$$

and

$$\left[\mathcal{F},\Delta^v,||\cdot,\cdots,\cdot||\right]_0=\Big\{x=(x_{k,l})\in w(X):$$

$$P-\lim_{p,q}\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho},z_1,\cdots,z_{n-1}||\right)=0,$$

$$\text{uniformly in } m \text{ and } n, \text{ for some } \rho>0\Big\}.$$

In this section of the paper we study inclusion relations between the spaces $\left[AC_{\theta_{r,s}},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\right]_0$, $\left[\mathcal{F},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\right]$ and $\left[\mathcal{F},\mathcal{M},p,\Delta^v,||\cdot,\cdots,\cdot||\right]$.

Theorem 6. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers, then the spaces $\left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]$ and $\left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. It is easy to prove. □

Theorem 7. *For any sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ and $p = (p_{k,l})$ be a factorable double sequence of positive real numbers, the space $\left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]$ is a topological linear space paranormed by*

$$g(x) = \inf\left\{\rho^{\frac{p_{p,q}}{K}} : \left(\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}}\right)^{\frac{1}{K}} \leq 1\right\},$$

where $K = \max(1, \sup_{k,l} p_{k,l} < \infty)$.

Proof. It is easy to prove in view of Theorem 2, so we omit the details. □

Theorem 8. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$, we have*

(i) $\left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0 \subset \left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0$,

(ii) $\left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right] \subset \left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]$.

Proof. Let $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$, then there exists $\delta > 0$ such that $q_r > 1 + \delta$ and $\bar{q}_s > 1 + \delta$. This implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Then for $x = (x_{k,l}) \in \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0$, we can write for each m and n

$$\begin{aligned}
B_{r,s} &= \frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&= \frac{1}{h_{r,s}}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\quad - \frac{1}{h_{r,s}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\quad - \frac{1}{h_{r,s}}\sum_{k=k_{r-1}+1}^{k_r}\sum_{l=1}^{l_{s-1}}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\quad - \frac{1}{h_{r,s}}\sum_{k=1}^{k_{r-1}}\sum_{l=l_{s-1}+1}^{l_s}\left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}}
\end{aligned}$$

$$= \frac{k_r k_s}{h_{rs}}\Big(\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)$$

$$- \frac{k_{r-1} l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)$$

$$- \frac{1}{h_r}\sum_{k=k_{r-1}+1}^{k_r}\frac{l_{s-1}}{h_s}\frac{1}{l_{s-1}}\sum_{l=1}^{l_{s-1}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$- \frac{1}{h_s}\sum_{l=l_{s-1}+1}^{l_s}\frac{k_{r-1}}{h_r}\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}.$$

Since $x = (x_{k,l}) \in \Big[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\Big]$ the last two terms tends to zero uniformly in m, n in the Pringsheim sense, thus for each m and n, we have

$$B_{r,s} = \frac{k_r k_s}{h_{rs}}\Big(\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{k_s}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big)$$

$$- \frac{k_{r-1} l_{s-1}}{h_{rs}}\Big(\frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}\Big) + o(1).$$

Since $h_{rs} = k_r l_s - k_{r-1} l_{s-1}$ we are granted for each m and n the following:

$$\frac{k_r l_s}{h_{rs}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{rs}} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

and

$$\frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}}\Big[M_{k,l}\Big(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

are both Pringsheim null sequences for all m and n. Thus $B_{r,s}$ is a Pringsheim null sequence for each m and n. Therefore $x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\Big]_0$. This completes the proof of (i). Similarly, we can prove (ii). □

Theorem 9. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$, we have*

(i) $\left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0 \subset \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0$,

(ii) $\left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right] \subset \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]$.

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s. Let $x = (x_{k,l}) \in \left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\right]_0$ and $\epsilon > 0$. Also there exist $r_0 > 0$ and $s_0 > 0$ such that for every $i \geq r_0$ and $j \geq s_0$ and all m and n, we have

$$A'_{i,j} = \frac{1}{h_{ij}} \sum_{(k,l)\in I_{i,j}} \left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} < \epsilon.$$

Let $N = \max\{A'_{ij} : 1 \leq i \leq r_0 \text{ and } 1 \leq j \leq s_0\}$, p and q be such that $k_{r-1} < p \leq k_r$ and $l_{s-1} < q \leq l_s$. Thus we obtain the following:

$$\begin{aligned}
&\frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left[M_{k,l}\left(||\frac{\Delta^v x_{k_m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{k,l=1,1}^{k_r l_s} \left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}} \\
&\leq \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r,s} \left(\sum_{k,l\in I_{t,u}} \left[M_{k,l}\left(||\frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right]^{p_{k,l}}\right) \\
&= \frac{1}{k_{r-1}l_{s-1}} \sum_{t,u1,1}^{r_0,s_0} h_{t,u}A'_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} h_{t,u}A'_{t,u} \\
&\leq \frac{N}{k_{r-1}l_{s-1}} \sum_{t,u=1,1}^{r_0,s_0} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} h_{t,u}A'_{t,u} \\
&\leq \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} h_{t,u}A'_{t,u} \\
&\leq \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \left(\sup_{t\geq r_0 \cup u\geq s_0} A'_{t,u} \frac{1}{k_{r-1}l_{s-1}} \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} h_{t,u}\right. \\
&\leq \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\epsilon \sum_{(r_0<t\leq r)\cup(s_0<u\leq s)} h_{t,u}
\end{aligned}$$

$$\leq \frac{N k_{r_0} l_{s_0} r_0 s_0}{k_{r-1} l_{s-1}} + \epsilon H^2.$$

Since k_r and l_s both approaches infinity as both p and q approaches infinity, it follows that

$$\frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left[M_{k,l}\left(|| \frac{\Delta^v x_{k+m,l+n}}{\rho}, z_1, \cdots, z_{n-1} || \right) \right]^{p_{k,l}} \to 0, \text{ uniformly in } m \text{ and } n.$$

Therefore $x = (x_{k,l}) \in \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot|| \right]$. This completes the proof of (i). Similarly, we can prove (ii). □

Theorem 10. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{r,s} \leq \limsup_{r,s} q_{r,s} < \infty$, we have*

(i) $\left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot|| \right]_0 = \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot|| \right]_0$

(ii) $\left[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot|| \right] = \left[\mathcal{F}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot|| \right]$.

Proof. The proof directly follows from Theorem 8 and Theorem 9. □

4. Statistical convergence

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [34] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [9], Connor [6], Salat [32], Mursaleen et al [25], Mursaleen and Edely [27], Mursaleen and Mohiuddine [31], Isık [16], Savas [33], Kolk [18], Maddox [19], Mursaleen et al [26] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.
A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists: $\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.
A sequence $x = (x_{k,l})$ is said to be lacunary Δ^v-statistically convergent to L, if for every $\epsilon > 0$

$$\lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : ||(\Delta^v x_{k,l} - L), z_1, \cdots, z_{n-1}|| \geq \epsilon \right\} \right| = 0.$$

In this case we write $x_{k,l} \to L\Big(S_{\theta_{r,s}}(\Delta^v)\Big)$. The set of all lacunary Δ^v-statistically convergent sequences is denoted by $S_{\theta_{r,s}}(\Delta^v)$.

Theorem 11. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$. Then $\Big[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\Big] \subset S_{\theta_{r,s}}(\Delta^v)$.*

Proof. Let $x = (x_{k,l}) \in \Big[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\Big]$ and $\epsilon > 0$ be given. Then

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$\geq \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, \left(||\frac{\Delta^v x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}||\right)\geq\epsilon} \Big[M_{k,l}\Big(||\frac{(\Delta^v x_{k,l} - L)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$\geq \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, \left(||\frac{\Delta^v x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}||\right)\geq\epsilon} \big[M_{k,l}(\epsilon)\big]^{p_{k,l}}$$

$$\geq \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, \left(||\frac{\Delta^v x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}||\right)\geq\epsilon} \min\Big(\big[M_{k,l}(\epsilon)\big]^h, \big[M_{k,l}(\epsilon)\big]^H\Big)$$

$$\geq \frac{1}{h_{r,s}} \Big|\{(k,l) \in I_{r,s} : \big(||(\Delta^v x_{k,l} - L), z_1, \cdots, z_{n-1}||\big) \geq \epsilon\}\Big|$$

$$\min\Big(\big[M_{k,l}(\epsilon)\big]^h, \big[M_{k,l}(\epsilon)\big]^H\Big).$$

Hence $x = (x_{k,l}) \in S_{\theta_{r,s}}(\Delta^v)$. □

Theorem 12. *Let $\mathcal{M} = (M_{k,l})$ be a bounded sequence of Orlicz functions and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$. Then $S_{\theta_{r,s}}(\Delta^v) \subset \Big[AC_{\theta_{r,s}}, \mathcal{M}, \Delta^v, p, ||\cdot, \cdots, \cdot||\Big]$.*

Proof. Since $M = (M_{k,l})$ is bounded, so there exists an integer K such that $M_{k,l}(t) < K$, for all $t \geq 0$. Then

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}, \left(||\frac{\Delta^v x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}||\right)\geq\epsilon} \Big[M_{k,l}\Big(||\frac{\Delta^v x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_{k,l}}$$

$$+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s},\ \left(||\frac{\Delta^v x_{k,l}-L}{\rho},z_1,\cdots,z_{n-1}||\right)<\epsilon}\left[M_{k,l}\left(||\frac{\Delta^v x_{k,l}-L}{\rho},z_1,\cdots,z_{n-1}||\right)\right]^{p_{k,l}}$$

$$\leq\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s},\left(||\frac{\Delta^v x_{k,l}-L}{\rho},z_1,\cdots,z_{n-1}||\right)\geq\epsilon}\max(K^h,K^H)$$

$$+\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s},\ \left(||\frac{\Delta^v x_{k,l}-L}{\rho},z_1,\cdots,z_{n-1}||\right)<\epsilon}[M_{k,l}(\epsilon)]^{p_{k,l}}$$

$$\leq\max(K^h,K^H)\frac{1}{h_{r,s}}\left|\{(k,l)\in I_{r,s}:\left(||\frac{\Delta^v x_{k,l}-L}{\rho},z_1,\cdots,z_{n-1}||\right)\geq\epsilon\}\right|$$

$$+\max\left(\left[M_{k,l}(\epsilon)\right]^h,\left[M_{k,l}(\epsilon)\right]^H\right).$$

Hence $x=(x_{k,l})\in\left[AC_{\theta_{r,s}},\mathcal{M},\Delta^v,p,||\cdot,\cdots,\cdot||\right]$. □

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Doi: 10.18514/MMN.2015.1215

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 305–319

HU e-ISSN 1787-2413

BI-STARLIKE FUNCTION OF COMPLEX ORDER ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Received 22 April, 2014

Abstract. In the present paper, two new subclasses of bi-starlike and bi-convex function of complex order associated with hypergeometric functions are introduced and coefficient estimates for functions in these classes are obtained. Several new (or known) consequences of the results are also pointed out.

2010 *Mathematics Subject Classification:* 30C45

Keywords: analytic functions, univalent functions, bi-univalent functions, bi-starlike and bi-convex functions of complex order, coefficient bounds, hypergeometric functions

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in Δ. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in Δ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$in Δ. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

$$\text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in Δ if both $f(z)$ and $f^{-1}(z)$ are univalent in Δ.Let Σ denote the class of bi-univalent functions in Δ given by (1.1). Earlier, Brannan and Taha [3]introduced certain subclasses of bi-univalent function

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class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex function of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively.

A function $f(z) \in \mathcal{A}$ is in the class of strongly bi-starlike functions $\mathcal{S}^*_\Sigma[\alpha]$ ([3, 19])of order α $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$\left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \text{ and } \left| arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}$$

and strongly bi-convex functions $\mathcal{K}^*_\Sigma[\alpha]$ ([3, 19])of order α $(0 < \alpha \leq 1)$

$$\left| arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2}, \text{ and } \left| arg\left(1+\frac{wg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2}$$

where g is given by (1.2). For each of the function classes $\mathcal{S}^*_\Sigma[\alpha]$ and $\mathcal{K}_\Sigma[\alpha]$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were found [3, 19]. But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem(see[2,3,8,12,19]). Many researchers (see[16,17,20,21]) have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{z\ f'(z)}{f(z)}$ or $1 + \frac{z\ f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\Delta, \phi(0) = 1, \phi'(0) > 0$, and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\ f'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z\ f''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two

functions $f,h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f*h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta \tag{1.3}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

For complex parameters $\alpha_1,\ldots,\alpha_l$ and $\beta_1,\ldots,\beta_m$ $(\beta_j \neq 0,-1,\ldots; j = 1,2,\ldots,m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1,\ldots\alpha_l;\beta_1,\ldots,\beta_m;z) &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\, l,m \in \mathbb{N}_0 &:= \mathbb{N} \cup \{0\}; z \in \Delta) \end{aligned} \tag{1.4}$$

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.5}$$

For positive real values of $\alpha_1,\ldots,\alpha_l$ and $\beta_1,\ldots,\beta_m$ $(\beta_j \neq 0,-1,\ldots; j = 1,2,\ldots,m)$, let

$$\mathcal{H}(\alpha_1,\ldots\alpha_l;\beta_1,\ldots,\beta_m) : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$(\mathcal{H}(\alpha_1,\ldots\alpha_l;\beta_1,\ldots,\beta_m))(f)(z) = z\, {}_lF_m(\alpha_1,\alpha_2,\ldots\alpha_l;\beta_1,\beta_2\ldots,\beta_m;z) * f(z)$$

$$(\mathcal{H}_m^l)(f)(z) = z + \sum_{n=2}^{\infty} \Gamma_n\, a_n\, z^n \tag{1.6}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{1}{(n-1)!} \tag{1.7}$$

$\alpha_i > 0, (i = 1,2,...l), \beta_j > 0, (j = 1,2,...m), l \leq m+1;\ l,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For notational simplicity, we use a shorter notation

$$\mathcal{H}_m^l[\alpha_1,\beta_1]f(z) = \mathcal{H}_m^l f(z)$$

for

$$\mathcal{H}(\alpha_1,\ldots\alpha_l;\beta_1,\ldots,\beta_m) = \mathcal{H}_m^l[\alpha_1,\beta_1]$$

in the sequel. It follows from (1.6) that

$$\mathcal{H}_1^2[1,1]f(z) = f(z), \mathcal{H}_1^2[2,1]f(z) = z f'(z).$$

The linear operator $\mathcal{H}_m^l[\alpha_1,\beta_1]$ is called Dziok-Srivastava operator (see [6]). Further by using the Gaussian hypergeometric function given by (1.6), Hohlov [7] introduced a generalized convolution operator $H_{a,b,c}$ as $H_{a,b,c}f(z) = z\, {}_2F_1(a,b,c;z) *$

$f(z)$, contains as special cases most of the known linear integral or differential operators. For the suitable choices of l,m in turn the operator $\mathcal{H}_m^l[\alpha_1,\beta_1]$ includes various operators as remarked below:

Remark 1. For $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(a,1;c)f(z) = \mathcal{L}(a,c)f(z)$$
$$= \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n\right) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$$

was considered by Carlson and Shaffer [4].

Remark 2. For $f \in \mathcal{A}$, and

$$\mathcal{H}_1^2(\delta+1,1;1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^{\delta} f(z), (\delta > -1)$$

given by $\mathcal{D}^{\delta} f(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \delta+n-1 \\ n-1 \end{pmatrix} a_n z^n$, was introduced by Ruscheweyh [15].

Remark 3. For $f \in \mathcal{A}$, and $\mathcal{H}_1^2(c+1,1;c+2)f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt = \mathcal{J}_c f(z)$ where $c > -1$. The operator $\mathcal{J}_c$ was introduced by Bernardi [1]. In particular, the operator $\mathcal{J}_1$ was studied earlier by Libera [9] and Livingston[10].

Remark 4. For $f \in \mathcal{A}, \mathcal{H}_1^2(2,1;2-\lambda)f(z) = \Gamma(2-\lambda)z^{\lambda}\mathcal{D}_z^{\lambda} f(z) = \Omega^{\lambda} f(z),\ \lambda \notin \mathbb{N}\setminus\{1\}$ called Owa-Srivastava operator[18] and Ω^{λ} is also called Srivastava-Owa fractional derivative operator, where $\mathcal{D}_z^{\lambda} f(z)$ denotes the fractional derivative of $f(z)$ of order λ, studied by Owa [13].

Motivated by the earlier work of Deniz[5], in the present paper we introduce two new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}\setminus\{0\}$,involving Dziok-Srivastava operator $\mathcal{H}_m^l$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ. Several related classes are also considered , and connection to earlier known results are made.

1.1. *The function class of bi-univalent functions*

In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.8}$$

Further we assume $\alpha_i > 0, (i = 1,2,...l), \beta_j > 0, (j = 1,2,...m), l \le m+1;\ l,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in our present study unless otherwise stated.

Definition 1. A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{H}^l_m f(z))'}{\mathcal{H}^l_m f(z)}+\frac{\lambda z^2(\mathcal{H}^l_m f(z))''}{\mathcal{H}^l_m f(z)}-1\right)\prec\phi(z) \tag{1.9}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{H}^l_m g(w))'}{\mathcal{H}^l_m g(w)}+\frac{\lambda w^2(\mathcal{H}^l_m g(w))''}{\mathcal{H}^l_m g(w)}-1\right)\prec\phi(w) \tag{1.10}$$

where $\gamma\in\mathbb{C}\backslash\{0\}$, $\lambda\geq 0, z, w\in\Delta$ and the function g is given by (1.2).

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{[z(\mathcal{H}^l_m f(z))'+\lambda z^2(\mathcal{H}^l_m f(z))'']'}{(\mathcal{H}^l_m f(z))'}-1\right)\prec\phi(z) \tag{1.11}$$

and

$$1+\frac{1}{\gamma}\left(\frac{[w(\mathcal{H}^l_m g(w))'+\lambda w^2(\mathcal{H}^l_m g(w))'']'}{(\mathcal{H}^l_m g(w))'}-1\right)\prec\phi(w) \tag{1.12}$$

where $\gamma\in\mathbb{C}\backslash\{0\}$, $\lambda\geq 0, z, w\in\Delta$ and the function g is given by (1.2).

Remark 5. A function $f(z)\in\Sigma$ given by (1.1) and for $\lambda=0$, we note that

$$\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)\equiv\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma)$$

satisfies the following conditions :

$$\left[1+\frac{1}{\gamma}\left(\frac{z(\mathcal{H}^l_m f(z))'}{\mathcal{H}^l_m f(z)}-1\right)\right]\prec\phi(z) \text{ and } \left[1+\frac{1}{\gamma}\left(\frac{w(\mathcal{H}^l_m g(w))'}{\mathcal{H}^l_m g(w)}-1\right)\right]\prec\phi(w)$$

and

$$\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)\equiv\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$$

satisfies the following conditions :

$$\left[1+\frac{1}{\gamma}\left(\frac{z(\mathcal{H}^l_m f(z))''}{(\mathcal{H}^l_m f(z))'}\right)\right]\prec\phi(z) \text{ and } \left[1+\frac{1}{\gamma}\left(\frac{w(\mathcal{H}^l_m g(w))''}{(\mathcal{H}^l_m g(w))'}\right)\right]\prec\phi(w)$$

where $\gamma\in\mathbb{C}\backslash\{0\}$ $z, w\in\Delta$ and the function g is given by (1.2).

Remark 6. A function $f(z)\in\Sigma$ given by (1.1) and for $\gamma=1$, we note that $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)\equiv\mathcal{S}^{l,m}_{\Sigma,\phi}(\lambda)$ and $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)\equiv\mathcal{K}^{l,m}_{\Sigma,\phi}(\lambda)$ satisfies the following conditions respectively:

$$\left(\frac{z(\mathcal{H}^l_m f(z))'}{\mathcal{H}^l_m f(z)}+\frac{\lambda z^2(\mathcal{H}^l_m f(z))''}{\mathcal{H}^l_m f(z)}\right)\prec\phi(z),$$

$$\left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \frac{\lambda w^2(\mathcal{H}_m^l g(w))''}{\mathcal{H}_m^l g(w)}\right) \prec \phi(w),$$

$$\left(\frac{[z(\mathcal{H}_m^l f(z))' + \lambda z^2(\mathcal{H}_m^l f(z))'']'}{(\mathcal{H}_m^l f(z))'}\right) \prec \phi(z),$$

and

$$\left(\frac{[w(\mathcal{H}_m^l g(w))' + \lambda w^2(\mathcal{H}_m^l g(w))'']'}{(\mathcal{H}_m^l g(w))'}\right) \prec \phi(w)$$

where $\lambda \geq 0, z, w \in \Delta$ and the function g is given by (1.2).

2. Coefficient estimates for the function class $\mathcal{S}_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$ and $\mathcal{K}_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$

For deriving our main results, we need the following lemma.

Lemma 1 ([14]). *If $h \in \mathcal{P}$, then*

$$|c_k| \leq 2$$

for each k, where $\mathcal{P}$ is the family of all functions h analytic in Δ for which $\Re(h(z)) > 0$ and

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \text{ for } z \in \Delta.$$

Lemma 2 (see [14]). *Let the function $\varphi(z)$ given by*

$$\varphi(z) = \sum_{n=1}^{\infty} C_n z^n \qquad (z \in \Delta)$$

be convex in $\mathbb{U}$. Suppose also that the function $\mathfrak{h}(z)$ given by

$$\mathfrak{h}(z) = \sum_{n=1}^{\infty} \mathfrak{h}_n z^n$$

is holomorphic in Δ. If

$$\mathfrak{h}(z) \prec \varphi(z) \qquad (z \in \Delta),$$

then

$$|\mathfrak{h}_n| \leqq |C_1| \qquad (n \in \mathbb{N}).$$

Theorem 1. *Let $f(z)$ is given by (1.1) be in the class $\mathcal{S}_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$, $\gamma \in \mathbb{C}\backslash\{0\}$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{\left|\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\right|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{2(1+3\lambda)\Gamma_3}. \tag{2.2}$$

Proof. It follows from (1.9) and (1.10) that

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)} + \frac{\lambda z^2 (\mathcal{H}_m^l f(z))''}{\mathcal{H}_m^l f(z)} - 1\right) = \phi(u(z)) \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)} + \frac{\lambda w^2 (\mathcal{H}_m^l g(w))''}{\mathcal{H}_m^l g(w)} - 1\right) = \phi(v(w)). \tag{2.4}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots,$$

or equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in Δ with $p(0) = 1 = q(0)$. Since $u, v : \Delta \to \Delta$, the functions $p(z)$ and $q(z)$ have a positive real part in Δ, and for each i,

$$|p_i| \leq 2 \quad \text{and} \quad |q_i| \leq 2.$$

Since $p(z)$ and $q(w)$ in $\mathcal{P}$, we have the following forms:

$$\begin{aligned}\phi(u(z)) &= \phi\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]\right) \\ &= \frac{1}{2} B_1 p_1 z + \left[\frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2\right] z^2 \end{aligned} \tag{2.5}$$

and

$$\phi(v(w)) = \phi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \cdots\right]\right)$$

$$= \frac{1}{2}B_1q_1w + \left[\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2\right]w^2 \tag{2.6}$$

It follows from (2.3) -(2) that

$$\frac{1}{\gamma}(1+2\lambda)\Gamma_2 a_2 = \frac{1}{2}B_1p_1, \tag{2.7}$$

$$\frac{1}{\gamma}\left[2(1+3\lambda)\Gamma_3 a_3 - (1+2\lambda)\Gamma_2^2 a_2^2\right] = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2p_1^2, \tag{2.8}$$

$$-\frac{1}{\gamma}(1+2\lambda)\Gamma_2 a_2 = \frac{1}{2}B_1q_1, \tag{2.9}$$

and

$$\frac{1}{\gamma}\left([4(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]a_2^2 - 2(1+3\lambda)\Gamma_3 a_3\right) = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \tag{2.11}$$

and

$$8(1+2\lambda)^2\Gamma_2^2 a_2^2 = \gamma^2 B_1^2(p_1^2 + q_1^2). \tag{2.12}$$

Upon adding (2.8)and (2.10),if we make use of (2.11) and (2.12), we obtain

$$\left(4\{\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\}\right)a_2^2$$
$$= \gamma^2 B_1^3(p_2 + q_2).$$

which yields

$$a_2^2 = \frac{\gamma^2 B_1^3(p_2+q_2)}{4\{\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\}}$$

Since, by definition, $p(z), q(w) \subset h(\Delta)$, by applying Lemma (1) for the coefficients p_2 q_2 and $B_1 > 0$, we have

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.8) from (2.10) and using (2.11), we get

$$\frac{4}{\gamma}(1+3\lambda)\Gamma_3(a_3 - a_2^2) = \frac{B_1}{2}(p_2 - q_2).$$

Upon substituting the value of a_2^2 from (2.12), we get

$$a_3 = \frac{\gamma^2 B_1^2(p_1^2+q_1^2)}{8(1+2\lambda)^2\Gamma^2} + \frac{\gamma B_1(p_2-q_2)}{8(1+3\lambda)\Gamma_3}.$$

Applying Lemma (1) once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{2(1+3\lambda)\Gamma_3}.$$

□

Theorem 2. *Let $f(z)$ given by (1.1) be in the class $K_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$, $\gamma \in \mathbb{C}\backslash\{0\}$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{2\{\left|\gamma[3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2]B_1^2 + 2(1+2\lambda)^2(B_1 - B_2)\Gamma_2^2\right|\}}} \tag{2.13}$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{4(1+2\lambda)^2 \Gamma_2^2} + \frac{|\gamma| B_1}{6(1+3\lambda)\Gamma_3}. \tag{2.14}$$

Proof. We can write the argument inequalities in (1.11) and (1.12) equivalently as follows:

$$1 + \frac{1}{\gamma}\left(\frac{[z(\mathcal{H}_m^l f(z))' + \lambda z^2 (\mathcal{H}_m^l f(z))'']'}{(\mathcal{H}_m^l f(z))'} - 1\right) = \phi(u(z)) \tag{2.15}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{[w(\mathcal{H}_m^l g(w))' + \lambda w^2 (\mathcal{H}_m^l g(w))'']'}{(\mathcal{H}_m^l g(w))'} - 1\right) = \phi(v(w)), \tag{2.16}$$

and proceeding as in the proof of Theorem 1, we can arrive the following relations from (2.15) and (2.16)

$$\frac{2}{\gamma}(1+2\lambda)\Gamma_2 a_2 = \frac{1}{2} B_1 p_1, \tag{2.17}$$

$$\frac{1}{\gamma}[6(1+3\lambda)\Gamma_3 a_3 - 4(1+2\lambda)\Gamma_2^2 a_2^2] = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2, \tag{2.18}$$

and

$$-\frac{2}{\gamma}(1+2\lambda)\Gamma_2 a_2 = \frac{1}{2} B_1 q_1, \tag{2.19}$$

$$\frac{1}{\gamma}[6(1+3\lambda)(2a_2^2 - a_3)\Gamma_3 - 4(1+2\lambda)\Gamma_2^2 a_2^2] = \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2. \tag{2.20}$$

From (2.17) and (2.19), we get

$$p_1 = -q_1 \tag{2.21}$$

and

$$32(1+2\lambda)^2 \Gamma_2^2 a_2^2 = \gamma^2 B_1^2 (p_1^2 + q_1^2). \tag{2.22}$$

Now from (2.18), (2.20) and (2.22), we obtain

$$a_2^2 = \frac{\gamma^2 B_1^3 (p_2 + q_2)}{8[\gamma[3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2]B_1^2 + 2(1+2\lambda)^2(B_1 - B_2)\Gamma_2^2]}. \tag{2.23}$$

Applying Lemma (1) for the coefficients p_2 and q_2, we have the desired inequality given in (2.13).

Next, in order to find the bound on $|a_3|$, by subtracting (2.18) from (2.20), and using (2.21), we get

$$\frac{12}{\gamma}(1+3\lambda)(a_3 - a_2^2)\Gamma_3 = \frac{B_1}{2}(p_2 - q_2).$$

Upon substituting the value of a_2^2 given (2.22),the above equation lead to

$$a_3 = \frac{\gamma B_1 (p_2 - q_2)}{24(1+3\lambda)\Gamma_3} + \frac{\gamma^2 B_1^2 (p_1^2 + q_1^2)}{32(1+2\lambda)^2 \Gamma_2^2}.$$

Applying the Lemma (1) once again for the coefficients p_1, p_2, q_1 and q_2, we get the desired coefficient given in (2.14). □

Putting $\lambda = 0$, in Theorems (1) and (2) , we can state the coefficient estimates for the functions in the subclasses $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma)$ and $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$ defined in Remark (5).

Corollary 1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma)$, then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(2\Gamma_3 - \Gamma_2^2)B_1^2 + (B_1 - B_2)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\gamma|^2 B_1^2}{\Gamma_2^2} + \frac{|\gamma| B_1}{2\Gamma_3}.$$

Corollary 2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$,then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{2|\gamma(3\Gamma_3 - 2\Gamma_2^2)B_1^2 + 4(B_1 - B_2)\Gamma_2^2|}} \quad \text{and} \quad |a_3| \leq \frac{|\gamma|^2 B_1^2}{4\Gamma_2^2} + \frac{|\gamma| B_1}{6\Gamma_3}.$$

Taking $\gamma = 1$, in Theorems (1) and (2) , we can state the coefficient estimates for the functions in the subclasses $\mathcal{S}^{l,m}_{\Sigma,\phi}(\lambda)$ and $\mathcal{K}^{l,m}_{\Sigma,\phi}(\lambda)$ defined in Remark (6).

Corollary 3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}^{l,m}_{\Sigma,\phi}(\lambda)$, then*

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|[2(1+3\lambda)\Gamma_3 - (1+2\lambda)\Gamma_2^2]B_1^2 + (1+2\lambda)^2(B_1 - B_2)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{B_1^2}{(1+2\lambda)^2 \Gamma_2^2} + \frac{B_1}{2(1+3\lambda)\Gamma_3}.$$

Corollary 4. *Let $f(z)$ given by (1.1) be in the class $\mathcal{K}^{l,m}_{\Sigma,\phi}(\lambda)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|\{[3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2]B_1^2 + 2(1+2\lambda)^2(B_1-B_2)\Gamma_2^2\}|}}$$

and

$$|a_3| \leq \frac{B_1^2}{4(1+2\lambda)^2\Gamma_2^2} + \frac{B_1}{6(1+3\lambda)\Gamma_3}.$$

3. Corollaries and its consequences

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \tag{3.1}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

Corollary 5. *By choosing $\phi(z)$ of the form (3.1), we state the following results*

(1) for functions $f \in \mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$, by Theorem1,

$$|a_2| \leq \frac{2|\gamma|\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2|\gamma|\alpha(1+2\lambda)]\Gamma_2^2 + 4|\gamma|\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2\alpha^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{|\gamma|\alpha}{(1+3\lambda)\Gamma_3}.$$

(2) for functions $f \in \mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$, by Theorem2,

$$|a_2| \leq \frac{|\gamma|\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2|\gamma|\alpha(1+2\lambda)]\Gamma_2^2 + 3|\gamma|\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2\alpha^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{|\gamma|\alpha}{3(1+3\lambda)\Gamma_3}$$

On the other hand if we take

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1), \tag{3.2}$$

then we have $B_1 = B_2 = 2(1-\beta)$.

Corollary 6. *By choosing $\phi(z)$ of the form (3.2), we state the following results*
(1) for functions $f \in \mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$, by Theorem1,

$$|a_2| \leq \frac{|\gamma|\sqrt{2(1-\beta)}}{\sqrt{|\gamma[2(1+3\lambda)\Gamma_3-(1+2\lambda)\Gamma_2^2]|}}$$

and

$$|a_3| \leq \frac{4|\gamma|^2(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2}+\frac{|\gamma|(1-\beta)}{(1+3\lambda)\Gamma_3}.$$

(2) for functions $f \in \mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma,\lambda)$, by Theorem2,

$$|a_2| \leq \frac{|\gamma|\sqrt{(1-\beta)}}{\sqrt{|\gamma[3(1+3\lambda)\Gamma_3-2(1+2\lambda)\Gamma_2^2]|}}$$

and

$$|a_3| \leq \frac{|\gamma|^2(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2}+\frac{|\gamma|(1-\beta)}{3(1+3\lambda)\Gamma_3}$$

Corollary 7. *Let $f(z)$ given by (1.1) be in the $\mathcal{S}^{l,m}_{\Sigma,\phi}(\lambda)$ and $\phi(z)$ is of the form (3.1), then from Theorem (1),we have*

$$|a_2| \leq \frac{2\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2-2\alpha(1+2\lambda)]\Gamma_2^2+4\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+2\lambda)^2\Gamma_2^2}+\frac{\alpha}{(1+3\lambda)\Gamma_3}.$$

Remark 7. From Corollary (7),taking $\lambda = 0$, we obtain the following results

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1-3\alpha)\Gamma_2^2+4\alpha\Gamma_3}} \quad \text{and} \quad |a_3| \leq \frac{4\alpha^2}{\Gamma_2^2}+\frac{\alpha}{\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we obtain the results of Srivastava et al [17].

Corollary 8. *Let $f(z)$ given by (1.1) be in the $\mathcal{S}^{l,m}_{\Sigma,\phi}(\lambda)$ and $\phi(z)$ is of the form (3.2), then from Theorem (1), we have*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2(1+3\lambda)\Gamma_3-(1+2\lambda)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{|(1+2\lambda)^2\Gamma_2^2}+\frac{(1-\beta)}{(1+3\lambda)\Gamma_3|}.$$

Remark 8. From Corollary (8),taking $\lambda = 0$, we get

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{|2\Gamma_3 - \Gamma_2^2|}} \text{ and } |a_3| \leq \frac{4(1-\beta)^2}{\Gamma_2^2} + \frac{(1-\beta)}{\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \sqrt{2(1-\beta)} \text{ and } |a_3| \leq 4(1-\beta)^2 + (1-\beta).$$

Corollary 9. *Let $f(z)$ given by (1.1) be in the $\mathcal{K}_{\Sigma,\phi}^{l,m}(\lambda)$ and $\phi(z)$ as defined in (3.1)then from Theorem (2), we have*

$$|a_2| \leq \frac{\alpha}{\sqrt{|[(1-\alpha)(1+2\lambda)^2 - 2\alpha(1+2\lambda)]\Gamma_2^2 + 3\alpha(1+3\lambda)\Gamma_3|}}$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{\alpha}{3(1+3\lambda)\Gamma_3}.$$

Remark 9. From Corollary (9),taking $\lambda = 0$, we get

$$|a_2| \leq \frac{\alpha}{\sqrt{|3\alpha\Gamma_3 + (1-3\alpha)\Gamma_2^2|}} \text{ and } |a_3| \leq \frac{\alpha^2}{\Gamma_2^2} + \frac{\alpha}{3\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \alpha \text{ and } |a_3| \leq \alpha^2 + \frac{\alpha}{3}.$$

Corollary 10. *Let $f(z)$ given by (1.1) be in the $\mathcal{K}_{\Sigma,\phi}^{l,m}(\lambda)$ and $\phi(z)$ is of the form (3.2),then from Theorem (2), we have*

$$|a_2| \leq \sqrt{\frac{1-\beta}{|3(1+3\lambda)\Gamma_3 - 2(1+2\lambda)\Gamma_2^2|}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1+2\lambda)^2\Gamma_2^2} + \frac{(1-\beta)}{3(1+3\lambda)\Gamma_3}.$$

Remark 10. From Corollary (10), by taking $\lambda = 0$, we get

$$|a_2| \leq \sqrt{\frac{1-\beta}{|3\Gamma_3 - 2\Gamma_2^2|}} \text{ and } |a_3| \leq \frac{(1-\beta)^2}{\Gamma_2^2} + \frac{(1-\beta)}{3\Gamma_3}.$$

Further by taking $l = 2, m = 1$, we have

$$|a_2| \leq \sqrt{(1-\beta)} \text{ and } |a_3| \leq (1-\beta)^2 + \frac{(1-\beta)}{3}.$$

Concluding remarks: By further specializing the parameters l, m, one can define various other interesting subclasses of Σ based on the differential operators stated in the Remarks 1 to 4, and we can easily derived analogous results (as in Theorems 1 and 2) and the corresponding corollaries as mentioned above. The details involved may be left as an exercise for the interested reader.

ACKNOWLEDGEMENT

The authors record their sincere thanks to the referees for their valuable suggestions to improve the results and the present form of the paper.

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Doi: 10.18514/MMN.2015.1133

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 321–352

HU e-ISSN 1787-2413

COMMON FIXED POINT THEOREMS SATISFYING INTEGRAL TYPE RATIONAL CONTRACTIVE CONDITIONS AND APPLICATIONS

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Received 15 February, 2014

Abstract. Inspired by the concept of integral type (Branciari type) contractive condition and respective fixed point results, we derive some new common fixed point theorems for quadruple self mappings satisfying a generalized $\mathcal{W}$-contractive condition or a generalized ϕ-contractive condition of integral type, also using the notions of compatibility and subsequential continuity (alternately subcompatibility and reciprocal continuity) in the frame of metric spaces. We furnish some examples to illustrate the useability of our main results. Several results existing in the literature are obtained as consequences of the main results. At the end of this article the existence and uniqueness of solutions for certain systems of integral equations and functional equations arising in dynamic programming are demonstrated with the help of the obtained results.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: metric space, compatible mappings, reciprocal continuity, subcompatible mappings, subsequential continuity, fixed point

1. INTRODUCTION

The Banach Contraction Principle is a very popular tools which is used to solve existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions—modifying the basic contractive condition or changing the ambiental space. This celebrated theorem can be stated as follows.

Theorem 1 ([12])**.** *Let (X,d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx,Ty) \leq kd(x,y),\ \forall\, x,y \in X,$$

where k is a constant in $(0,1)$. Then, T has a unique fixed point $x^* \in X$.

Banach Contraction Principle provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Later, many authors have extended, generalized and improved Banach fixed point theorem in different ways (see [22–26, 40, 47]).

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During the late 20th century, metrical common fixed point theory saw a trend of investigation which moved around commuting nature of two maps. Several conditions were introduced, including weak commutativity (Sessa [52]), compatibility (Jungck [33]), weak compatibility (Jungck and Rhoades [34]) and many others, and a lot of respective common fixed point results were obtained. A survey of these notions and relationship among them can be seen in [35].

We recall that two mappings $A, S : X \to X$ are called weakly compatible if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$.

In the study of common fixed points of compatible-type mappings we often require assumption of completeness of the space or continuity of mappings involved besides some contractive condition, but the study of common fixed points of non-compatible mappings can be extended to the class of non-expansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or the completeness of the space. Aamri and El Moutawakil [1] generalized the concept of non-compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive conditions. Although (E.A) property is a generalization of the concept of non-compatible maps, yet it requires either completeness of the whole space or some of the range spaces or continuity of maps. After that, Liu et al. [37] defined the notion of common property (E.A) for a hybrid pair of single- and multi-valued mappings and proved some fixed point results under hybrid contractive conditions.

Further, Al-Thagafi and Shahzad [3] introduced the notion of occasionally weakly compatible mappings which is the most general among all the commutativity concepts. However, in an interesting note, Djorić et al. [27] observed that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weak compatibility. In 2009 Bouhadjera and Godet-Thobie [17] further enlarged the class of compatible (reciprocally continuous) pairs by introducing the concept of subcompatible (subsequential continuous) pair which is substantially weaker than using compatibility (reciprocal continuity). Since then, Imdad et al. [31], as well as Chauhan et al. [21] improved the results of [17] and showed that these results can easily be recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocal continuity.

On the other hand, Branciari [19] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle with an integral type inequality. This celebrated theorem can be stated as follow.

Theorem 2. *Let (X, d) be a complete metric space, $k \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \leq k \int_0^{d(x,y)} \varphi(t)\,dt$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a Lebesgue-integrable function which is summable (i.e., with finite integral) on each compact subset of* $[0, +\infty)$*, non-negative and such that* $\int_0^\epsilon \varphi(t)\,dt > 0$*, for each* $\epsilon > 0$*. Then* f *has a unique fixed point* $a \in X$ *such that for each* $x \in X$*,* $\lim_{n\to\infty} f^n x = a$*.*

Later, some interesting and rich fixed point theorems involving more general contractive conditions of integral type were obtained in [2,5–11,28,48,50,51,53–55,57]. Recently, Chauhan et al. [20] studied integral type fixed point results for two pairs of mappings which are non-commutative and non-continuous, and gave applications to dynamic systems.

Inspired by the concept of integral type (Branciari type) conditions and notions of compatibility and subsequential continuity (alternately subcompatibility and reciprocal continuity), an attempt has been made in this article to prove fixed point theorems for two pairs of self mappings under a new type of contractive conditions, namely generalized $\mathcal{W}$-contractive condition and generalized ϕ-contractive condition of integral type in the setting of metric spaces. Some examples are furnished to show validity of hypothesis of results in dissimilar circumstances. Several results existing in the literature are obtained as consequences of the main results. Our results improve and generalize several comparable results in the existing literature. We wrap up the paper with applications of our results to an existence result for certain system of integral equations and solvability of certain system of functional equations arising in dynamic programming.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ is the set of positive integers. To start with, we state the following definitions.

Definition 1. Let $A, S : X \to X$ be two self mappings of a metric space (X, d). The mappings A and S are said to be

(1) commuting if $ASx = SAx$, for all $x \in X$;
(2) weakly commuting [52] if $d(ASx, SAx) \le d(Ax, Sx)$, for all $x \in X$;
(3) compatible [33] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$;
(4) non-compatible [43] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent;
(5) weakly compatible [34] if they commute at their coincidence points, i.e., $ASu = SAu$ whenever $Au = Su$, for some $u \in X$;
(6) occasionally weakly compatible [3] if and only if there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute.

Definition 2 ([41]). A pair (A, S) of self mappings of a metric space (X, d) is called reciprocally continuous if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$, whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

It is easy to see that if two self mappings are continuous, then they are obviously reciprocally continuous but the converse is not true [42, Example 2]. Moreover, in the setting of common fixed point theorems for compatible pairs of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity but not conversely (see [41]).

Definition 3 ([1]). A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy the property (E.A) if there exist a sequence $\{x_n\}$ in X and some $z \in X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \in X.$$

It can be noticed that any non-compatible self mappings of a metric space (X, d) satisfy the property (E.A) but two mappings satisfying the property (E.A) need not be non-compatible (see [29, Example 1]). Also, weak compatibility and the property (E.A) are independent to each other (see [44, Examples 2.1-2.2]).

Definition 4 ([4]). Two pairs (A, S) and (B, T) of self mappings of a metric space (X, d) are said to satisfy the common property (E.A), if there exist two sequences $\{x_n\}, \{y_n\}$ in X for some z in X and such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z.$$

Definition 5 ([17]). A pair (A, S) of self mappings of a metric space (X, d) is said to be subcompatible if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$.

A pair of non-compatible or subcompatible mapping satisfies the property (E.A). Obviously, compatible mappings which satisfy the property (E.A) are subcompatible but the converse statement does not hold in general (see [49, Example 2.3]). Two occasionally weakly compatible mappings are subcompatible, however the converse is not true in general (see [18, Example 1.2]).

Definition 6 ([17]). A pair (A, S) of self mappings of a metric space (X, d) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$.

One can easily check that if two self mappings are continuous or reciprocally continuous, then they are naturally subsequentially continuous. However, there exist subsequentially continuous pairs of mappings which are neither continuous nor reciprocally continuous (see [18, Example 1.4]).

3. Common fixed point results under $\mathcal{W}$-weakly contractive condition

The attempted improvements in this section are five-fold.

(i) The condition on containment of ranges amongst the involved mappings is relaxed.

(ii) Continuity requirements of all the involved mappings are completely relaxed.

(iii) The (E.A) property is replaced by compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) property which are the most general among all existing weak commutativity concepts.

(iv) The condition on completeness of the whole space is relaxed.

(v) A generalized $\mathcal{W}$-weakly contractiveness is taken into consideration to derive our results.

In what follows, we denote by Ψ the collection of all functions $\mathcal{W} : [0,\infty) \to [0,\infty)$ which are continuous and satisfy $\mathcal{W}(t) < t$, for all $t > 0$.

Also, we will denote by Φ the set of functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

(φ_1) φ is a Lebesgue integrable function on each compact subset of $[0,+\infty)$.

(φ_2) $\displaystyle\int_0^{\varepsilon} \varphi(s)\,ds > 0$ for all $\varepsilon > 0$.

In this section we state and prove the following theorem.

Theorem 3. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X,d). If the pairs (A,S) and (B,T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A,S) has a coincidence point,*

(2) *the pair (B,T) has a coincidence point.*

For all $x, y \in X$, we define

$$\Theta(x,y) = \max\left\{ \begin{array}{c} d(Ax,Sx), d(By,Ty), d(Ty,Sx), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)], \\ \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{array} \right\}.$$

We assume that, for all $x, y \in X$,

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \le \int_0^{\Theta(x,y)} \varphi(t)dt - \mathcal{W}\left(\int_0^{\Theta(x,y))} \varphi(t)\,dt\right), \tag{3.1}$$

where $\mathcal{W} \in \Psi$ and $\varphi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

Proof. The proof of the theorem is divided into the following two parts:

Part I: Since the pair (A,S) (as well as (B,T)) is subsequentially continuous and compatible, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \zeta,$$

for some $\zeta \in X$, and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = d(A\zeta, S\zeta) = 0,$$

with $A\zeta = S\zeta$, whereas in respect to the pair (B,T), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \xi,$$

for some $\xi \in X$, and

$$\lim_{n\to\infty} d(BTy_n, TBy_n) = d(B\xi, T\xi) = 0,$$

with $B\xi = T\xi$. Hence ζ is a coincidence point of the pair (A,S), and ξ is a coincidence point of the pair (B,T).

Now we prove that $\zeta = \xi$. By putting $x = x_n$ and $y = y_n$ in inequality (3.1), we have

$$\int_0^{d(Ax_n,By_n)} \varphi(t)\,dt \le \int_0^{\Theta(x_n,y_n)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x_n,y_n)} \varphi(t)\,dt\right) \tag{3.2}$$

where

$$\Theta(x_n,y_n) = \max \left\{ \begin{array}{c} d(Ax_n,Sx_n), d(By_n,Ty_n), d(Ty_n,Sx_n), \frac{1}{2}[d(Ax_n,Ty_n)+d(By_n,Sx_n)], \\ \frac{d(Ax_n,Sx_n)d(By_n,Ty_n)}{1+d(Ty_n,Sx_n)}, \frac{d(Ax_n,Ty_n)d(By_n,Sx_n)}{1+d(Ty_n,Sx_n)}, \frac{d(Ax_n,Ty_n)d(By_n,Sx_n)}{1+d(Ax_n,By_n)} \end{array} \right\}.$$

Passing to the limit as $n \to \infty$ in inequality (3.2), we have

$$\int_0^{d(\zeta,\xi)} \varphi(t)\,dt \le \int_0^{\Theta(\zeta,\xi)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(\zeta,\xi)} \varphi(t)\,dt\right),$$

where

$$\Theta(\zeta,\xi) = \max\left\{d(\zeta,\xi), d(\zeta,\xi), d(\zeta,\xi), \frac{d^2(\zeta,\xi)}{1+d(\zeta,\xi)}, \frac{d^2(\zeta,\xi)}{1+d(\zeta,\xi)}\right\} = d(\zeta,\xi).$$

Therefore

$$\int_0^{d(\zeta,\xi)} \varphi(t)\,dt \le \int_0^{d(\zeta,\xi)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{d(\zeta,\xi)} \varphi(t)\,dt\right) < \int_0^{d(\zeta,\xi)} \varphi(t)\,dt$$

which is a contradiction. Thus, we have $\zeta = \xi$. Now we assert that $A\zeta = \zeta$. Putting $x = \zeta$ and $y = y_n$ in inequality (3.1), we get

$$\int_0^{d(A\zeta,By_n)} \varphi(t)\,dt \le \int_0^{\Theta(\zeta,y_n)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(\zeta,y_n)} \varphi(t)\,dt\right) \tag{3.3}$$

where

$$\Theta(\zeta, y_n) = \max\left\{\begin{array}{c} d(A\zeta,S\zeta), d(By_n,Ty_n), d(Ty_n,S\zeta), \frac{1}{2}[d(A\zeta,Ty_n)+d(By_n,S\zeta)], \\ \frac{d(A\zeta,S\zeta)d(By_n,Ty_n)}{1+d(Ty_n,S\zeta)}, \frac{d(A\zeta,Ty_n)d(By_n,S\zeta)}{1+d(Ty_n,S\zeta)}, \frac{d(A\zeta,Ty_n)d(By_n,S\zeta)}{1+d(A\zeta,By_n)} \end{array}\right\}.$$

Passing to the limit as $n \to \infty$ in condition (3.3), we obtain

$$\int_0^{d(A\zeta,\xi)} \varphi(t)\,dt \le \int_0^{\Theta(\zeta,\xi)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(\zeta,\xi)} \varphi(t)\,dt\right),$$

where

$$\Theta(\zeta,\xi) = \max\left\{d(\xi,A\zeta), d(A\zeta,\xi), \frac{d^2(A\zeta,\xi)}{1+d(\xi,A\zeta)}, \frac{d^2(A\zeta,\xi)}{1+d(A\zeta,\xi)}\right\} = d(A\zeta,\xi).$$

Therefore we get

$$\int_0^{d(A\zeta,\zeta)} \varphi(t)\,dt \le \int_0^{d(Az,z)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{d(Az,z)} \varphi(t)\,dt\right) < \int_0^{d(A\zeta,\zeta)} \varphi(t)\,dt,$$

which is a contradiction. Thus, we have $A\zeta = \zeta$. Therefore, $A\zeta = S\zeta = \zeta$.

Now we show that $B\zeta = \zeta$. Putting $x = x_n$ and $y = \zeta$ in inequality (3.1), we have

$$\int_0^{d(Ax_n,B\zeta)} \varphi(t)\,dt \le \int_0^{\Theta(x_n,\zeta)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x_n,\zeta)} \varphi(t)\,dt\right) \tag{3.4}$$

where

$$\Theta(x_n,\zeta) = \max\left\{\begin{array}{c} d(Ax_n,Sx_n), d(B\zeta,T\zeta), d(T\zeta,Sx_n), \frac{1}{2}[d(Ax_n,T\zeta)+d(B\zeta,Sx_n)], \\ \frac{d(Ax_n,Sx_n)d(B\zeta,T\zeta)}{1+d(T\zeta,Sx_n)}, \frac{d(Ax_n,T\zeta)d(B\zeta,Sx_n)}{1+d(T\zeta,Sx_n)}, \frac{d(Ax_n,T\zeta)d(B\zeta,Sx_n)}{1+d(Ax_n,B\zeta)}. \end{array}\right\}.$$

Passing to the limit as $n \to \infty$ in inequality (3.4), we have

$$\int_0^{d(\zeta,B\zeta)} \varphi(t)\,dt \le \int_0^{\Theta(\zeta,\zeta)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(\zeta,\zeta)} \varphi(t)\,dt\right)$$

where

$$\Theta(\zeta,\zeta) = \max\left\{d(B\zeta,\zeta), d(\zeta,B\zeta), \frac{d^2(\zeta,T\zeta)}{1+d(B\zeta,\zeta)}, \frac{d^2(\zeta,T\zeta)}{1+d(\zeta,B\zeta)}\right\} = d(\zeta,B\zeta).$$

Therefore we get

$$\int_0^{d(\zeta,B\zeta)} \varphi(t)\,dt \le \int_0^{d(\zeta,B\zeta)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{d(\zeta,B\zeta)} \varphi(t)\,dt\right) < \int_0^{d(\zeta,B\zeta)} \varphi(t)\,dt,$$

which is a contradiction. Thus, we get $B\zeta = S\zeta = \zeta$. Therefore in all, $\zeta = A\zeta = S\zeta = B\zeta = T\zeta$, i.e., ζ is a common fixed point of A, B, S and T.

Finally, to prove the uniqueness of common fixed point, let ζ' be another common fixed point of A, B, S and T.

Putting $x = \zeta$ and $y = \zeta'$ in inequality (3.1), we have

$$\int_0^{d(\zeta,\zeta')} \varphi(t)\,dt = \int_0^{d(A\zeta,B\zeta')} \varphi(t)\,dt \le \int_0^{\Theta(\zeta,\zeta')} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(\zeta,\zeta'))} \varphi(t)\,dt\right),$$

where

$$\begin{aligned}
&\Theta(\zeta,\zeta') \\
&= \max\left\{ \begin{array}{c} d(A\zeta,S\zeta), d(B\zeta',T\zeta'), d(T\zeta',S\zeta), \frac{1}{2}[d(A\zeta,T\zeta') + d(B\zeta',S\zeta)], \\ \frac{d(A\zeta,S\zeta)d(B\zeta',T\zeta')}{1+d(T\zeta',S\zeta)}, \frac{d(A\zeta,T\zeta')d(B\zeta',S\zeta)}{1+d(T\zeta',S\zeta)}, \frac{d(A\zeta,T\zeta')d(B\zeta',S\zeta)}{1+d(A\zeta,B\zeta')} \end{array} \right\} \\
&= \max\left\{ d(\zeta',\zeta), d(\zeta,\zeta'), \frac{d^2(\zeta,\zeta')}{1+d(\zeta',\zeta)}, \frac{d^2(\zeta,\zeta')}{1+d(\zeta,\zeta')} \right\} \\
&= d(\zeta',\zeta).
\end{aligned}$$

Therefore

$$\int_0^{d(\zeta,\zeta')} \varphi(t)\,dt \le \int_0^{d(\zeta,\zeta')} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{d(\zeta,\zeta'))} \varphi(t)\,dt\right) < \int_0^{d(\zeta,\zeta')} \varphi(t)\,dt$$

which is a contradiction. Hence $\zeta = \zeta'$.jn

Part II: Since the pair (A, S) (as well as (B, T)) is subcompatible and reciprocally continuous, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \zeta,$$

for some $\zeta \in X$, and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = d(A\zeta, S\zeta) = 0,$$

whereas in respect to the pair (B, T), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \xi,$$

for some $\xi \in X$, and

$$\lim_{n\to\infty} d(BTy_n, TBy_n) = d(B\xi, T\xi) = 0.$$

Therefore, $A\zeta = S\zeta$ and $B\xi = T\xi$, i.e., ζ is a coincidence point of the pair (A,S) whereas ξ is a coincidence point of the pair (B,T). The rest of the proof can be completed on the lines of **Part I**. □

3.1. *Consequences*

Some integral versions of well known results in the literature can be obtained as particular cases of our Theorem 3. For example, we state the following corollaries.

Corollary 1. *Let $A,B,S,T : X \to X$ be four self mappings of a metric space (X,d). If the pairs (A,S) and (B,T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A,S) has a coincidence point,*
(2) *the pair (B,T) has a coincidence point.*

We assume that, for all $x, y \in X$,

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \le \int_0^{\max\{d(Ax,Sx),d(By,Ty),d(Ty,Sx),\frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\max\{d(Ax,Sx),d(By,Ty),d(Ty,Sx),\frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}} \varphi(t)\,dt\right),$$

where $\mathcal{W} \in \Psi$ and $\varphi \in \Phi$. Then A,B,S and T have a unique common fixed point in X.

Corollary 2. *Let $A,B,S,T : X \to X$ be four self mappings of a metric space (X,d). If the pairs (A,S) and (B,T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A,S) has a coincidence point,*
(2) *the pair (B,T) has a coincidence point.*

We assume that, for all $x, y \in X$,

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \le \int_0^{d(Ty,Sx)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{d(Ty,Sx)} \varphi(t)\,dt\right),$$

where $\mathcal{W} \in \Psi$ and $\varphi \in \Phi$. Then A,B,S and T have a unique common fixed point in X.

If $\mathcal{W}(t) = (1-k)t$ for $k \in (0,1)$ in Theorem 3, we have following result:

Corollary 3. *Let $A,B,S,T : X \to X$ be four self mappings of a metric space (X,d). If the pairs (A,S) and (B,T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A,S) has a coincidence point,*
(2) *the pair (B,T) has a coincidence point.*

For all $x, y \in X$, we define

$$\Theta(x,y) = \max\left\{ \begin{array}{c} d(Ax,Sx), d(By,Ty), d(Ty,Sx), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)], \\ \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{array} \right\}.$$

We assume that, for all $x, y \in X$,

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \leq k \int_0^{\Theta(x,y)} \varphi(t)\,dt$$

where $k \in (0,1)$ and $\varphi \in \Phi$. Then A, B, S and T have a unique common fixed point in X.

If we take $\varphi(t) = 1$ for all t in Theorem 3, we get the following consequences:

Corollary 4. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

For all $x, y \in X$, we define

$$\Theta(x,y) = \max\left\{ \begin{array}{c} d(Ax,Sx), d(By,Ty), d(Ty,Sx), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)], \\ \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{array} \right\}.$$

We assume that, for all $x, y \in X$,

$$d(Ax,By) \leq \Theta(x,y) - \mathcal{W}(\Theta(x,y))$$

where $\mathcal{W} \in \Psi$. Then A, B, S and T have a unique common fixed point in X.

Corollary 5. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

We assume that, for all $x, y \in X$,

$$\begin{aligned} d(Ax,By) &\leq \max\left\{ d(Ax,Sx), d(By,Ty), d(Ty,Sx), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \right\} \\ &\quad - \mathcal{W}\left(\max\left\{ d(Ax,Sx), d(By,Ty), d(Ty,Sx), \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \right\} \right) \end{aligned}$$

where $\mathcal{W} \in \Psi$. Then A, B, S and T have a unique common fixed point in X.

Corollary 6. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

We assume that, for all $x, y \in X$,

$$d(Ax, By) \leq d(Ty, Sx) - \mathcal{W}(d(Ty, Sx))$$

where $\mathcal{W} \in \Psi$. Then A, B, S and T have a unique common fixed point in X.

If $\mathcal{W}(t) = (1-k)t$ for $k \in (0,1)$ in Corollary 4, we have the following result:

Corollary 7. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

We assume that, for all $x, y \in X$,

$$d(Ax, By) \leq k \max \left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), d(Ty, Sx), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)], \\ \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{array} \right\}$$

where $k \in (0,1)$. Then A, B, S and T have a unique common fixed point in X.

3.2. *Illustrative examples*

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over some recently established results.

In the examples, we show the importance of each condition or property of Theorem 3 required for getting coincidence (common fixed) point of the mappings.

Example 1. Let $X = [0, \infty)$ and d be the usual metric on X. Set $A = B$ and $S = T$. Define self mappings A and S by

$$Ax = \begin{cases} x^2, & \text{if } x \in [0,1); \\ 2x-1, & \text{if } x \in [1,\infty); \end{cases} \quad Sx = \begin{cases} 3x-2, & \text{if } x \in [0,1); \\ x+3, & \text{if } x \in [1,\infty). \end{cases}$$

Consider the sequence $\{x_n\} = \left\{1 - \frac{1}{n}\right\}_{n \in \mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^2 = 1 = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} 3\left(1 - \frac{1}{n}\right) - 2$$

Also,

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left(1 - \frac{3}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^2 = 1 = A(0),$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S\left(1-\frac{1}{n}\right)^2 = \lim_{n\to\infty} 3\left(1-\frac{1}{n}\right)^2 - 2 = 1 \neq S(1),$$

but

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0.$$

Thus, the pair (A,S) is compatible but neither subsequentially continuous nor reciprocally continuous. By routine calculation we can check that the condition (3.1) is not satisfied. Therefore Theorem 3 cannot be used to prove the existence of coincidence point of A and S. Note that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both or by involving conditions on completeness (or closedness) of underlying space (or subspaces). Also, in this example neither X is complete nor any range-space is closed, since $A(X) = [0,1)\cup[1,\infty)$ and $S(X) = [-2,1)\cup[4,\infty)$.

Example 2. Let $X = [0,\infty)$ and d be the usual metric on X. Set $A = B$ and $S = T$. Define the self mappings A and S by

$$Ax = \begin{cases} 2, & \text{if } x \in [0,3); \\ x, & \text{if } x \in [3,\infty); \end{cases} \quad Sx = \begin{cases} 2x-4, & \text{if } x \in [0,3]; \\ 3, & \text{if } x \in (3,\infty). \end{cases}$$

Consider the sequence $\{x_n\} = \left\{3+\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty}\left(3+\frac{1}{n}\right) = 3 = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} 3.$$

Also,

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A3 = 3 = A(3), \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S\left(3+\frac{1}{n}\right) = 3 \neq S(3),$$

but

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0.$$

Thus, the pair (A,S) is compatible but not reciprocally continuous.

Consider another sequence $\{x_n\} = \left\{3-\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty}\left(3-\frac{1}{n}\right) = 2 \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} 2\left(3-\frac{1}{n}\right) - 4 = 2.$$

Next,

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left(3-\frac{1}{n}\right) = 2 = A(2),$$
$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S2 = 0 \neq S(2),$$

and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) \neq 0.$$

Thus, the pair (A,S) is subsequentially continuous but not compatible. By routine calculation we can check that the condition (3.1) is not satisfied. Therefore Theorem

3 do not imply the existence of coincidence point of A and S. It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both or by involving conditions on completeness (or closedness) of underlying space (or subspaces). Also, in this example neither X is complete nor any range-space are closed, since $A(X) = \{2\} \cup [3, \infty)$ and $S(X) = [-4, 2] \cup \{3\}$.

Example 3. Let $X = [0, \infty)$ and d be the usual metric on X. Set $A = B$ and $S = T$. Define self mappings A and S by

$$Ax = \begin{cases} \frac{x}{6}, & \text{if } x \in [0,2]; \\ 2x-2, & \text{if } x \in (2,\infty); \end{cases} \quad Sx = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,2]; \\ 3x-4, & \text{if } x \in (2,\infty). \end{cases}$$

Consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \left(\frac{1}{6n}\right) = 0, \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} \left(\frac{1}{4n}\right) = 0.$$

Also,

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left(\frac{1}{4n}\right) = \lim_{n\to\infty} \left(\frac{1}{24n}\right) = 0 = A(0),$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S\left(\frac{1}{6n}\right) = \lim_{n\to\infty} \left(\frac{1}{24n}\right) = 0 = S(0),$$

and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0.$$

Consider another sequence $\{x_n\} = \left\{2 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} 2\left(2+\frac{1}{n}\right) - 2 = 2 = \lim_{n\to\infty} \left(4 + \frac{4}{n} - 2\right) = 2 = \lim_{n\to\infty} Sx_n.$$

Also,

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left(2+\frac{3}{n}\right) = \lim_{n\to\infty} 2\left(2+\frac{3}{n}-2\right) = 2 \neq A(2),$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S\left(2+\frac{2}{n}\right) = \lim_{n\to\infty} \left(6+\frac{6}{n}-4\right) = 2 \neq S(2),$$

but $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$. Thus, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous. By Theorem 3, A and S have a coincidence point; here $u = 0$ is a coincidence point of A and S. It is noted by routine calculation that condition (3.1) is satisfied for $\mathcal{W}(t) = \frac{2t}{3}$ and $\varphi(s) = 1$. Thus, all the conditions of Theorem 3 (also of Corollary 4) are satisfied and 0 is a unique common fixed point of the pairs (A, S) and (B, T).

Now, we support the uniqueness of the common fixed point of the pairs (A, S) and (B, T) of Theorem 3 by the following example. Note that inequality (3.1) is necessary to prove the existence and uniqueness of the pairs (A, S) and (B, T).

Example 4. Let $X = \{0, 1, 2, 3, \ldots\}$ and consider the metric $d: X \times X \to X$ given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \max\{x, y\}, & \text{if } x \neq y. \end{cases}$$

Define $A, B, S, T: X \to X$ by

$$Ax = \begin{cases} 0, & \text{if } x = 0; \\ 2x + 1, & \text{if } x \neq 0; \end{cases} \qquad Bx = \begin{cases} 0, & \text{if } x = 0; \\ 2x + 2, & \text{if } x \neq 0; \end{cases}$$

and

$$Sx = \begin{cases} 0, & \text{if } x = 0; \\ 4x + 4, & \text{if } x \neq 0; \end{cases} \qquad Tx = \begin{cases} 0, & \text{if } x = 0; \\ 4x + 2, & \text{if } x \neq 0; \end{cases}$$

Also, define $\varphi, \mathcal{W} : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = 2t$ and $\mathcal{W}(t) = \sqrt{t}$. Then:

(1) The pairs (A, S) and (B, T) are compatible and subsequentially continuous.
(2) φ is a Lebesgue-integrable function which is summable, nonnegative and such that $\int_0^{\epsilon} \varphi(t)\, dt > 0$ for all $\epsilon > 0$.
(3) $\mathcal{W} \in \Psi$.
(4) For any $x, y \in X$, we have

$$\int_0^{d(Ax,By)} \varphi(t)\, dt \leq \int_0^{\Theta(x,y)} \varphi(t)\, dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} \varphi(t)\, dt\right),$$

where

$$\Theta(x, y) = \max\left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), d(Ty, Sx), \frac{1}{2}[d(Ax, Ty) + d(By, Sx)], \\ \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ty,Sx)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{array} \right\}.$$

Proof. To prove (1), take $x_n = 0$ for all but finitely many n. Then $Ax_n = 0$ and $Sx_n = 0$ for all but finitely many n. Hence, $d(Ax_n, 0) \to 0$ and $d(Sx_n, 0) \to 0$ as $n \to +\infty$; that is, $\lim_{n\to+\infty} Ax_n = 0$ and $\lim_{n\to+\infty} Sx_n = 0$. Also $\lim_{n\to+\infty} SAx_n = 0 = S0$ and $\lim_{n\to+\infty} ASx_n = 0 = A0$.

The pair (A, S) is compatible and subsequentially continuous. Similarly, we can show that the pair (B, T) is compatible and subsequentially continuous. The proof of (2) and (3) is clear. To prove (4), given $x, y \in X$, we divide the proof into the following cases:

Case 1: Let $x = y = 0$. Here $Ax = By = Sx = Ty = 0$. Thus condition (4) holds trivially.

Case 2. Let $x = 0$ and $y \neq 0$. Here $Ax = 0, By = 2y + 2, Sx = 0, Ty = 4y + 2$. Thus $d(Ax, By) = 2y + 2$ and $d(Sx, Ty) = 4y + 2$. Therefore

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{2y+2} (2t)\,dt = (2y+2)^2 \leq (4y+2)^2 - (4y+2) \\ &= d^2(Sx,Ty) - d(Sx,Ty) \leq [\Theta(x,y)]^2 - \Theta(x,y) \\ &= \int_0^{\Theta(x,y)} 2t\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} 2t\,dt\right) \\ &= \int_0^{\Theta(x,y)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} \varphi(t)\,dt\right).\end{aligned}$$

Case 3. Let $x \neq 0$ and $y = 0$. Here $Ax = 2x + 1, By = 0, Sx = 4x + 4, Ty = 0$. Thus $d(Ax, By) = 2x + 1$ and $d(Sx, Ty) = 4x + 4$. Therefore

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{2x+1} (2t)\,dt = (2x+1)^2 \leq (4x+4)^2 - (4x+4) \\ &= d^2(Sx,Ty) - d(Sx,Ty) \leq [\Theta(x,y)]^2 - \Theta(x,y) \\ &= \int_0^{\Theta(x,y)} 2t\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} 2t\,dt\right) \\ &= \int_0^{\Theta(x,y)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} \varphi(t)\,dt\right).\end{aligned}$$

Case 4. Let $x > y > 0$.

• If $x = 2y + 1$, that is $y = \frac{x-1}{2}$, then $Ax = 2x + 1, By = x + 1, Sx = 4x + 4$, and $Ty = 2x$. Thus $d(Ax, By) = 2x + 1$ and $d(Sx, Ty) = 4x + 4$. Therefore

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{2x+1} (2t)\,dt = (2x+1)^2 \leq (4x+4)^2 - (4x+4) \\ &= d^2(Sx,Ty) - d(Sx,Ty) \leq [\Theta(x,y)]^2 - \Theta(x,y) \\ &= \int_0^{\Theta(x,y)} 2t\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} 2t\,dt\right) \\ &= \int_0^{\Theta(x,y)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} \varphi(t)\,dt\right).\end{aligned}$$

• If $x > 2y + 1$, then $Ax = 2x + 1, By = x + 1, Sx = 4x + 4$, and $Ty = 2x$. Thus $d(Ax, By) = 2x + 1$ and $d(Sx, Ty) = 4x + 4$. Therefore as before we get the conclusion.

Case 5. Let $y > x > 0$. If $y = 2x + 2$, that is $x = \frac{y-2}{2}$, then $Ax = y - 1, By = y + 1, Sx = 2y$, and $Ty = 4y + 2$. Thus $d(Ax, By) = y + 1$ and $d(Sx, Ty) = 4x + 2$.

Therefore

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{y+1} (2t)\,dt = (y+1)^2 \le (4x+2)^2 - (4x+2)\\ &= d^2(Sx,Ty) - d(Sx,Ty) \le [\Theta(x,y)]^2 - \Theta(x,y)\\ &= \int_0^{\Theta(x,y)} 2t\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} 2t\,dt\right)\\ &= \int_0^{\Theta(x,y)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(x,y)} \varphi(t)\,dt\right).\end{aligned}$$

Thus in all cases contractive condition (3.1) is satisfied. Hence, all the hypotheses of Theorem 3 are satisfied and 0 is a unique common fixed point of the pairs (A,S) and (B,T). Note that all the involved mappings are discontinuous at their unique common fixed point. □

4. Common fixed point results under generalized ϕ-contractive conditions

In this section we prove our results by taking into consideration different contractive condition, namely the one known as generalized ϕ-contractive condition.

In what follows, we denote

$$\Phi_1 = \left\{ \begin{array}{c} \phi : (\mathbb{R}^+)^9 \to \mathbb{R}^+ : \phi \text{ is upper semicontinuous and nondecreasing in} \\ \text{each coordinate variable and } \phi_1(t) = \max\{\phi(t,t,t,t,t,t,t,t,0,), \\ \phi(0,0,t,0,t,t,t,0,t)\} < t \end{array} \right\}$$

and

$$\Phi_2 = \left\{ \begin{array}{c} \phi : (\mathbb{R}^+)^7 \to \mathbb{R}^+ : \phi \text{ is upper semicontinuous and nondecreasing} \\ \text{in each coordinate variable and } \phi_2(t) = \phi(t,t,t,t,t,t,t) < t \end{array} \right\}.$$

Theorem 4. *Let $A,B,S,T : X \to X$ be four self mappings of a metric space (X,d). If the pairs (A,S) and (B,T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A,S) has a coincidence point,*
(2) *the pair (B,T) has a coincidence point.*

If there exists $\phi \in \Phi_1$ such that

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \tag{4.1}$$

$$\leq \phi \left(\begin{array}{c} \int_0^{d(Ax,Sx)} \varphi(t)\,dt, \int_0^{d(By,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(Ax,Sx)+d(By,Ty)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)]} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(By,Ty)+d(Sx,Ty)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax,Ty)+d(By,Sx)]} \varphi(t)\,dt, \\ \int_0^{\frac{d(Ax,Sx)d(By,Ty)}{1+d(Ax,By)}} \varphi(t)\,dt, \int_0^{\frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)}} \varphi(t)\,dt \end{array} \right)$$

for all $x, y \in X$ and some $\varphi \in \Phi$, then A, B, S and T have a unique common fixed point in X.

Proof. Suppose the pair (A, S) (as well as (B, T)) is subsequentially continuous and compatible. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u,$$

for some $u \in X$, and

$$\lim_{n\to\infty} d(Ax_n, Sy_n) = d(Au, Su) = 0,$$

with $Au = Su$, whereas in respect to the pair (B, T), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w,$$

for some $w \in X$, and

$$\lim_{n\to\infty} d(By_n, Ty_n) = d(Bw, Tw) = 0,$$

with $Bw = Tw$. Hence, u is a coincidence point of the pair (A, S), whereas w is a coincidence point of the pair (B, T).

Now we assert that $u = w$. To accomplish this, using inequality (4.1) with $x = x_n$, $y = y_n$, we have

$$\int_0^{d(Ax_n,By_n)} \varphi(t)\,dt \tag{4.2}$$

$$\leq \phi \left(\begin{array}{c} \int_0^{d(Ax_n,Sx_n)} \varphi(t)\,dt, \int_0^{d(By_n,Ty_n)} \varphi(t)\,dt, \int_0^{d(Sx_n,Ty_n)} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(Ax_n,Sx_n)+d(By_n,Ty_n)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax_n,Sx_n)+d(Sx_n,Ty_n)]} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(By_n,Ty_n)+d(Sx_n,Ty_n)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax_n,Ty_n)+d(By_n,Sx_n)]} \varphi(t)\,dt, \\ \int_0^{\frac{d(Ax_n,Sx_n)d(By_n,Ty_n)}{1+d(Ax_n,By_n)}} \varphi(t)\,dt, \int_0^{\frac{d(Ax_n,Ty_n)d(By_n,Sx_n)}{1+d(Ax_n,By_n)}} \varphi(t)\,dt \end{array} \right).$$

Passing to the limit as $n \to \infty$ in inequality (4.2), we get

$$\int_0^{d(u,w)} \varphi(t)\,dt \leq \phi \left(\begin{array}{c} 0, 0, \int_0^{d(u,w)} \varphi(t)\,dt, 0, \int_0^{\frac{1}{2}d(u,w)} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}d(u,w)} \varphi(t)\,dt, \int_0^{d(u,w)} \varphi(t)\,dt, 0, \int_0^{\frac{d^2(u,w)}{1+d(u,w)}} \varphi(t)\,dt \end{array} \right)$$

$$\leq \phi \left(\begin{array}{c} 0, 0, \int_0^{d(u,w)} \varphi(t)\,dt, 0, \int_0^{d(u,w)} \varphi(t)\,dt, \\ \int_0^{d(u,w)]} \varphi(t)\,dt, \int_0^{d(u,w)} \varphi(t)\,dt, 0, \int_0^{d(u,w)} \varphi(t)\,dt \end{array} \right)$$

$$= \phi_1\left(\int_0^{d(u,w)} \varphi(t)\,dt\right) < \int_0^{d(u,w)} \varphi(t)\,dt$$

a contradiction, so we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = 0$$

Hence, $u = w$. Now we prove that $Au = u$. Putting $x = u$ and $y = y_n$ in inequality (4.1), we obtain

(4.3)

$$\begin{aligned}
&\int_0^{d(Au,By_n)} \varphi(t)\,dt \\
&\leq \phi\begin{pmatrix}
\int_0^{d(Au,Su)} \varphi(t)\,dt, \int_0^{d(By_n,Ty_n)} \varphi(t)\,dt, \int_0^{d(Su,Ty_n)} \varphi(t)\,dt, \\
\int_0^{\frac{1}{2}[d(Au,Su)+d(By_n,Ty_n)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Au,Su)+d(Su,Ty_n)]} \varphi(t)\,dt, \\
\int_0^{\frac{1}{2}[d(By_n,Ty_n)+d(Su,Ty_n)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Au,Ty_n)+d(By_n,Su)]} \varphi(t)\,dt, \\
\int_0^{\frac{d(Au,Su)d(By_n,Ty_n)}{1+d(Au,By_n)}} \varphi(t)\,dt, \int_0^{\frac{d(Au,Ty_n)d(By_n,Su)}{1+d(Au,By_n)}} \varphi(t)\,dt
\end{pmatrix}.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in inequality (4.3), we get

$$\begin{aligned}
&\int_0^{d(Au,u)} \varphi(t)\,dt \\
&\leq \phi\begin{pmatrix}
0, 0, \int_0^{d(Au,z)} \varphi(t)\,dt, 0, \int_0^{\frac{1}{2}d(Au,u)} \varphi(t)\,dt, \int_0^{\frac{1}{2}d(Au,u)} \varphi(t)\,dt, \\
\int_0^{d(Au,u)} \varphi(t)\,dt, 0, \int_0^{\frac{d^2(Au,u)}{1+d(Au,u)}} \varphi(t)\,dt
\end{pmatrix} \\
&\leq \phi\begin{pmatrix}
0, 0, \int_0^{d(Au,u)} \varphi(t)\,dt, 0, \int_0^{d(Au,u)} \varphi(t)\,dt, \int_0^{d(Au,u)} \varphi(t)\,dt, \\
\int_0^{d(Au,u)} \varphi(t)\,dt, 0, \int_0^{d(Au,u)} \varphi(t)\,dt
\end{pmatrix} \\
&= \phi_1\left(\int_0^{d(Au,u)} \varphi(t)\,dt\right) < \int_0^{d(Au,u)} \varphi(t)\,dt
\end{aligned}$$

a contradiction, so we have

$$\int_0^{d(Au,u)} \varphi(t)\,dt = 0$$

and so $Au = u$. Therefore, $Au = Su = u$. Now we assert that $Bu = u$. To accomplish this, on using inequality (4.1) with $x = x_n$, $y = u$, we have

$$\int_0^{d(Ax_n,Bu)} \varphi(t)\,dt \tag{4.4}$$

$$\leq \phi \begin{pmatrix} \int_0^{d(Ax_n,Sx_n)} \varphi(t)\,dt, \int_0^{d(Bu,Tu)} \varphi(t)\,dt, \int_0^{d(Sx_n,Tu)} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(Ax_n,Sx_n)+d(Bu,Tu)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax_n,Sx_n)+d(Sx_n,Tu)]} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(Bu,Tu)+d(Sx_n,Tu)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax_n,Tu)+d(Bu,Sx_n)]} \varphi(t)\,dt, \\ \int_0^{\frac{d(Ax_n,Sx_n)d(Bu,Tu)}{1+d(Ax_n,Bu)}} \varphi(t)\,dt, \int_0^{\frac{d(Ax_n,Tu)d(Bu,Sx_n)}{1+d(Ax_n,Bu)}} \varphi(t)\,dt \end{pmatrix}.$$

Passing to the limit as $n \to \infty$ in inequality (4.4), we get

$$\int_0^{d(u,Bu)} \varphi(t)\,dt$$
$$\leq \phi \begin{pmatrix} 0, 0, \int_0^{d(u,Bu)} \varphi(t)\,dt, 0, \int_0^{\frac{1}{2}d(u,Bu)} \varphi(t)\,dt, \int_0^{\frac{1}{2}d(u,Bu)} \varphi(t)\,dt, \\ \int_0^{d(Bu,u)} \varphi(t)\,dt, 0, \int_0^{\frac{d^2(Bu,u)}{1+d(Bu,u)}} \varphi(t)\,dt \end{pmatrix}$$
$$\leq \phi \begin{pmatrix} 0, 0, \int_0^{d(u,Bu)} \varphi(t)\,dt, 0, \int_0^{d(u,Bu)} \varphi(t)\,dt, \int_0^{d(u,Bu)} \varphi(t)\,dt, \\ \int_0^{d(Bu,u)} \varphi(t)\,dt, 0, \int_0^{d(Bu,u)} \varphi(t)\,dt \end{pmatrix}$$
$$\leq \phi_1 \left(\int_0^{d(Bu,u)} \varphi(t)\,dt \right) < \int_0^{d(Bu,u)} \varphi(t)\,dt$$

which is a contradiction, so we have

$$\int_0^{d(z,Bz)} \varphi(t)\,dt = 0.$$

Thus $Bz = Sz = z$. Therefore in all, $z = Az = Sz = Bz = Tz$, i.e., z is a common fixed point of A, B, S and T. The uniqueness of common fixed point is an easy consequence of inequality (4.1).

The proof is similar if we assume that (A, S) (as well as (B, T)) is subcompatible and reciprocally continuous. This completes the proof of the theorem. □

Now we state the following theorem whose proof is similar to the one of Theorem 4.

Theorem 5. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

If there exists $\phi \in \Phi_2$ such that

$$\int_0^{d(Ax,By)} \varphi(t)\,dt \tag{4.5}$$

$$\leq \phi \begin{pmatrix} \int_0^{d(Ax,Sx)} \varphi(t)\,dt, \int_0^{d(By,Ty)} \varphi(t)\,dt, \int_0^{d(Sx,Ty)} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(Ax,Sx)+d(By,Ty)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)]} \varphi(t)\,dt, \\ \int_0^{\frac{1}{2}[d(By,Ty)+d(Sx,Ty)]} \varphi(t)\,dt, \int_0^{\frac{1}{2}[d(Ax,Ty)+d(By,Sx)]} \varphi(t)\,dt \end{pmatrix}$$

for all $x, y \in X$ and some $\varphi \in \Phi$, then A, B, S and T have a unique common fixed point in X.

4.1. *Consequences*

In this section we derive some consequences of Theorem 4 and Theorem 5.
If $\varphi(t) = 1$ for all t in Theorem 4 then we get the following natural result.

Corollary 8. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

If there exists $\phi \in \Phi_1$ such that

$$d(Ax, By) \leq \phi \begin{pmatrix} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}[d(Ax,Sx)+d(By,Ty)], \\ \frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)], \frac{1}{2}[d(By,Ty)+d(Sx,Ty)], \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)], \frac{d(Ax,Sx)d(By,Ty)}{1+d(Ax,By)}, \frac{d(Ax,Ty)d(By,Sx)}{1+d(Ax,By)} \end{pmatrix}$$

for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.

If we consider $\varphi(t) = 1$ for all t in Theorem 5, we obtain the following corollary:

Corollary 9. *Let $A, B, S, T : X \to X$ be four self mappings of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

(1) *the pair (A, S) has a coincidence point,*
(2) *the pair (B, T) has a coincidence point.*

If there exists $\phi \in \Phi_2$ such that

$$d(Ax, By) \leq \phi \begin{pmatrix} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}[d(Ax,Sx)+d(By,Ty)], \\ \frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)], \frac{1}{2}[d(By,Ty)+d(Sx,Ty)], \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \end{pmatrix}$$

for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.

Remark 1. Theorems 3–5 extend the results of Altun and Türkoğlu [7,8], Aliouche [6], Rhoades [48], Li et al. [36] for single-valued mappings, whereas the conditions

on completeness of the underlying subspaces and containment of ranges amongst involved mappings are relaxed.

On choosing A, B, S and T suitably in Theorems 3–5 and Corollaries 4–9, we can deduce several corollaries for three and two self mappings.

4.2. *Illustrative examples*

In what follows, we support the result of Theorem 5 by an example, inspired by [56].

Example 5. Let $X = \{0, 1, 2, 3, \ldots\}$ and consider the metric $d: X \times X \to X$ given by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ x + y, & \text{if } x \neq y. \end{cases}$$

Define $A, B, S, T: X \to X$ by

$$Ax = \begin{cases} 0, & \text{if } x = 0; \\ x + 1, & \text{if } x \neq 0; \end{cases} \qquad Bx = \begin{cases} 0, & \text{if } x = 0; \\ x + 2, & \text{if } x \neq 0; \end{cases}$$

and

$$Sx = \begin{cases} 0, & \text{if } x = 0; \\ 2x + 2, & \text{if } x \neq 0; \end{cases} \qquad Tx = \begin{cases} 0, & \text{if } x = 0; \\ 2x + 1, & \text{if } x \neq 0; \end{cases}$$

Also, define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = te^t + t$. Then:

(1) The pairs (A, S) and (B, T) are compatible and subsequentially continuous.
(2) φ is a Lebesgue-integrable function which is summable, nonnegative and such that $\int_0^{\epsilon} \varphi(t)\, dt > 0$ for all $\epsilon > 0$.
(3) For any $x, y \in X$, we have

$$\int_0^{d(Ax,By)} \varphi(t)\, dt \leq e^{-1} \int_0^{\Delta(x,y)} \varphi(t)\, dt,$$

where

$$\Delta(x, y) = \max \left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}[d(Ax, Sx) + d(By, Ty)], \\ \frac{1}{2}[d(Ax, Sx) + d(Sx, Ty)], \frac{1}{2}[d(By, Ty) + d(Sx, Ty)], \\ \frac{1}{2}[d(Ax, Ty) + d(By, Sx)] \end{array} \right\}.$$

Proof. We can prove (1) as in Example 4. The proof of (2) is clear. To prove (3), given $x, y \in X$, with $x \geq y$ and divide the proof into the following cases:

Case 1: $x = y = 0$. Here $Ax = By = Sx = Ty = 0$. Thus

$$\int_0^{d(Ax,Ay)} \varphi(t)\, dt \leq e^{-1} \int_0^{\Delta(x,y)} \varphi(t)\, dt.$$

Case 2. $y = 0$ and $x > 0$. Then we have $Ax = x + 1$, $By = 0$, $Sx = 2x + 2$ and $Ty = 0$. Consequently, we obtain

$$d(Ax, By) = d(x + 1, 0) = x + 1$$

and

$$\begin{aligned}
&\Delta(x, y)\\
&= \max \left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}[d(Ax, Sx) + d(By, Ty)], \\ \frac{1}{2}[d(Ax, Sx) + d(Sx, Ty)], \frac{1}{2}[d(By, Ty) + d(Sx, Ty)], \\ \frac{1}{2}[d(Ax, Ty) + d(By, Sx)] \end{array} \right\}\\
&= \max\\
&\left\{ \begin{array}{c} d(x + 1, 2x + 2), d(B0, T0), d(2x + 2, 0), \frac{1}{2}[d(x + 1, 2x + 2) + d(0, 0)], \\ \frac{1}{2}[d(x + 1, 2x + 2) + d(2x + 2, 0)], \frac{1}{2}[d(0, 0) + d(x + 1, 0)], \\ \frac{1}{2}[d(x + 1, 0) + d(0, 2x + 2)] \end{array} \right\}\\
&= 3x + 3.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^{d(Ax,By)} \varphi(t)\, dt = \int_0^{x+1} (te^t + t)\, dt &= \int_0^{x+1} (te^t)\, dt = (x + 1)e^{x+1}\\
&\leq e^{-1}(3x + 3)e^{3x+3} = e^{-1}\Delta(x, y) \times e^{\Delta(x,y)}\\
&= e^{-1} \int_0^{\Delta(x,y)} \varphi(t)\, dt.
\end{aligned}$$

Case 3. Suppose $x > y > 0$. We need to consider the following subcases:

• If $x = y + 1$, or equivalently $y = x - 1$, then we have $d(Ax, By) = d(Ax, B(x - 1)) = d(x + 1, x + 1) = 0$.

• If $x > y + 1$, then we have $Ax = x + 1$, $By = y + 2$, $Sx = 2x + 2$ and $Ty = 2y + 1$. Now, if $x = 2y$, then $d(Ax, By) = d(A(2y), By) = d(2y + 1, y + 2) = 3y + 3$ and

$$\begin{aligned}
&\Delta(x, y)\\
&= \max \left\{ \begin{array}{c} d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{1}{2}[d(Ax, Sx) + d(By, Ty)], \\ \frac{1}{2}[d(Ax, Sx) + d(Sx, Ty)], \frac{1}{2}[d(By, Ty) + d(Sx, Ty)], \\ \frac{1}{2}[d(Ax, Ty) + d(By, Sx)] \end{array} \right\}\\
&= \max \left\{ \begin{array}{c} d(2y + 1, 4y + 2), d(y + 2, 2y + 1), d(4y + 2, 2y + 1), \\ \frac{1}{2}[d(2y + 1, 4y + 2) + d(y + 2 + 2y + 1)], \\ \frac{1}{2}[d(2y + 1, 4y + 2) + d(4y + 2, 2y + 1)], \\ \frac{1}{2}[d(y + 2 + 2y + 1) + d(4y + 2, 2y + 1)], \\ \frac{1}{2}[d(2y + 1, 2y + 1) + d(y + 2, 4y + 2)] \end{array} \right\}\\
&= 6y + 3.
\end{aligned}$$

Thus we get

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{3y+3} (te^t + t)\,dt = \int_0^{3y+3} (te^t)\,dt = (3x+3)e^{3y+3} \\ &\le e^{-1}(6y+3)e^{6y+3} = e^{-1}\Delta(x,y)\times e^{\Delta(x,y)} \\ &= e^{-1}\int_0^{\Delta(x,y)} \varphi(t)\,dt.\end{aligned}$$

• On the other hand, if $x < 2y$ then $d(Ax,By) = d(x+1,y+2) = x+y+3$ and

$$\begin{aligned}&\Delta(x,y) \\ &= \max\left\{\begin{array}{c} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}[d(Ax,Sx)+d(By,Ty)], \\ \frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)], \frac{1}{2}[d(By,Ty)+d(Sx,Ty)], \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\end{array}\right\} \\ &= \max\left\{\begin{array}{c} d(x+1,2x+2), d(y+2,2y+1), d(2x+2,2y+1), \\ \frac{1}{2}[d(x+1,2x+2)+d(y+2,2y+1)], \\ \frac{1}{2}[d(x+1,2x+2)+d(2x+2,2y+1)], \\ \frac{1}{2}[d(y+2,2y+1)+d(2x+2,2y+1)], \\ \frac{1}{2}[d(x+1,2y+1)+d(y+2,2x+2)]\end{array}\right\} \\ &= 2x+2y+3.\end{aligned}$$

Thus we get

$$\begin{aligned}\int_0^{d(Ax,By)} \varphi(t)\,dt &= \int_0^{x+y+3} (te^t + t)\,dt = \int_0^{x+y+3} (te^t)\,dt = (3x+3)e^{x+y+3} \\ &\le e^{-1}(2x+2y+3)e^{2x+2y+3} = e^{-1}\Delta(x,y)\times e^{\Delta(x,y)} \\ &= e^{-1}\int_0^{\Delta(x,y)} \varphi(t)\,dt.\end{aligned}$$

• Finally, if $x > 2y$, then $d(Ax,By) = d(x+1,y+2) = x+y+3$ and

$$\begin{aligned}&\Delta(x,y) \\ &= \max\left\{\begin{array}{c} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}[d(Ax,Sx)+d(By,Ty)], \\ \frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)], \frac{1}{2}[d(By,Ty)+d(Sx,Ty)], \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\end{array}\right\} \\ &= \max\left\{\begin{array}{c} d(x+1,2x+2), d(y+2,2y+1), d(2x+2,2y+1), \\ \frac{1}{2}[d(x+1,2x+2)+d(y+2,2y+1)], \\ \frac{1}{2}[d(x+1,2x+2)+d(2x+2,2y+1)], \\ \frac{1}{2}[d(y+2,2y+1)+d(2x+2,2y+1)], \\ \frac{1}{2}[d(x+1,2y+1)+d(y+2,2x+2)]\end{array}\right\} \\ &= 3x+3.\end{aligned}$$

Therefore, we have

$$\int_0^{d(Ax,By)} \varphi(t)\,dt = \int_0^{x+y+3} (te^t + t)\,dt = \int_0^{x+y+3} (te^t)\,dt = (x+y+3)e^{x+y+3}$$
$$\leq e^{-1}(3x+3)e^{3x+3} = e^{-1}\Delta(x,y) \times e^{\Delta(x,y)}$$
$$= e^{-1}\int_0^{\Delta(x,y)\}} \varphi(t)\,dt.$$

Case 4. Assume $x = y > 0$. Consequently, we get $d(Ax, Bx) = d(x+1, x+2) = 2x+3$ and

$$\Delta(x,y)$$
$$= \max\left\{\begin{array}{c} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}[d(Ax,Sx)+d(By,Ty)], \\ \frac{1}{2}[d(Ax,Sx)+d(Sx,Ty)], \frac{1}{2}[d(By,Ty)+d(Sx,Ty)], \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} d(x+1,2x+2), d(x+2,2x+1), d(2x+2,2x+1), \\ \frac{1}{2}[d(x+1,2x+2)+d(x+2,2x+1)], \\ \frac{1}{2}[d(x+1,2x+2)+d(2x+2,2x+1)] \\ \frac{1}{2}[d(x+2,2x+1)+d(2x+2,2x+1)], \\ \frac{1}{2}[d(x+1,2x+1)+d(x+2,2x+2)]\} \end{array}\right\}$$
$$= 4x+3.$$

Therefore, we have

$$\int_0^{d(Ax,By)} \varphi(t)\,dt = \int_0^{2x+3} (te^t + t)\,dt = \int_0^{2x+3} (te^t)\,dt = (2x+3)e^{2x+3}$$
$$\leq e^{-1}(4x+3)e^{4x+3} = e^{-1}\Delta(x,y) \times e^{\Delta(x,y)}$$
$$= e^{-1}\int_0^{\Delta(x,y)\}} \varphi(t)\,dt.$$

If we define a function $\phi_2 : [0,+\infty) \to [0,+\infty)$ by $\phi_2(t) = e^{-1}t$, for all $t \geq 0$, then clearly $\phi_2 \in \Phi$. Thus, in all possible cases, the contractive condition (4.5) is satisfied. Hence, all the hypotheses of Theorem 5 are satisfied in this example and 0 is a unique common fixed point of the pairs (A,S) and (B,T). Note that all the involved mappings are discontinuous at their unique common fixed point. □

Remark 2. Example 5 can also be used to illustrate Theorem 4.

5. An application to systems of integral equations

Consider the following system of integral equations:

$$u(t) = \int_0^T K_1(t,s,u(s))\,ds + g(t),$$

$$u(t) = \int_0^T K_2(t,s,u(s))\,ds + g(t), \tag{5.1}$$
$$u(t) = \int_0^T K_3(t,s,u(s))\,ds + g(t),$$

$t \in I = [0,T]$, where $T > 0$. The purpose of this section is to give an existence theorem for a solution of the system (5.1) using Corollary 9.

Consider the set

$$C(I) := \{u : I \to \mathbb{R} \mid u \text{ is continuous on } I\},$$

and define $d : C(I) \times C(I) \to \mathbb{R}$ by:

$$d(u,v) = \max_{t\in I} |u(t) - v(t)|, \quad \forall\, u,v \in C(I).$$

Then $(C(I),d)$ is a complete metric space. Define, further, mappings $T_i : C(I) \to C(I)$ by

$$T_i x(t) = \int_0^T K_i(t,s,x(s))\,ds + g(t), \quad t \in I, \quad i \in \{1,2,3\}. \tag{5.2}$$

Consider the following conditions:

(i) $K_1, K_2, K_3 : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous,

(ii) there exists a continuous function $G : I \times I \to \mathbb{R}_+$ such that

$$|K_1(t,s,u(t)) - K_2(t,s,v(t))| \le G(t,s)$$
$$\phi\begin{pmatrix} |T_1u(t) - T_3u(t)|, |T_2v(t) - T_3v(t)|, |T_3u(t) - T_3v(t)|, \\ \frac{1}{2}[|T_1u(t) - T_3u(t)| + |T_2v(t) - T_3v(t)|], \\ \frac{1}{2}[|T_1u(t) - T_3u(t)| + |T_3u(t) - T_3v(t)|], \\ \frac{1}{2}[|T_2v(t) - T_3v(t)| + |T_3u(t) - T_3v(t)|], \\ \frac{1}{2}[|T_1u(t) - T_3v(t)| + |T_2v(t) - T_3u(t)|] \end{pmatrix}$$

for all $u,v \in C(I)$, and all $s,t \in I$,

(iii) $\max\limits_{t\in I} \int_0^T G(t,s)\,ds < 1$.

(iv) there exists $\{h_n\} \in C(I)$ such that $\lim_{n\to\infty} A_1h_n = \lim_{n\to\infty} A_2h_n = h^* \in C(I)$ and

$$\lim_{n\to\infty} \sup_{x\in I} |A_1A_2h_n - A_2A_1h_n| = 0,$$

(v) there exists $\{k_n\} \in C(I)$ such that $\lim_{n\to\infty} T_1k_n = \lim_{n\to\infty} T_2k_n = k^* \in C(I)$ and

$$\lim_{n\to\infty} \sup_{x\in I} |T_1T_2h_n - T_2T_1k_n| = 0.$$

We will prove the following result.

Theorem 6. *Suppose that the hypotheses **(i)–(v)** hold. Then the system 5.1 has a unique solution $x^* \in C(I)$.*

Proof. Since K_i and g, for $i = 1,2,3$ are continuous, then also the operators T_i from equation (5.2) are continuous and so reciprocally continuous. By hypothesis **(iv)** and **(v)** the pairs (T_1, T_3) and (T_2, T_3) are subcompatible with respect to T_3. Now, for all $u, v \in C(I)$, by **(ii)** and **(iii)**, we have:

$$\begin{aligned}
&|T_1u(t) - T_2v(t)| \\
&\leq \int_0^T |K_1(t,s,u(s)) - K_2(t,s,v(s))|\, ds \\
&\leq \int_0^T G(t,s)ds \\
&\times\phi\begin{pmatrix} |T_1u(t) - T_3u(t)|, |T_2v(t) - T_3v(t)|, |T_3u(t) - T_3v(t)|, \\ \frac{1}{2}[|T_1u(t) - T_3u(t)| + |T_2v(t) - T_3v(t)|], \\ \frac{1}{2}[|T_1u(t) - T_3u(t)| + |T_3u(t) - T_3v(t)|], \\ \frac{1}{2}[|T_2v(t) - T_3v(t)| + |T_3u(t) - T_3v(t)|], \\ \frac{1}{2}[|T_1u(t) - T_3v(t)| + |T_2v(t) - T_3u(t)|] \end{pmatrix}.
\end{aligned}$$

On routine calculations, we get

$$\begin{aligned}
&d(T_1u, T_2v) \\
&\leq \phi\begin{pmatrix} d(T_1u, T_3u), d(T_2v, T_3v), d(T_3u, T_3v), \frac{1}{2}[d(T_1u, T_3u) + d(T_2v, T_3v)], \\ \frac{1}{2}[d(T_1u, T_3u) + d(T_3u, T_3v)], \\ \frac{1}{2}[d(T_2v, T_3v) + d(T_3u, T_3v)], \frac{1}{2}[d(T_1u, T_3v) + d(T_2v, T_3u)] \end{pmatrix}.
\end{aligned}$$

Then, putting $A = T_1$, $B = T_2$ and $S = T = T_3$, Corollary 9 is applicable in the case $X = C(I)$, and so T_1, T_2 and T_3 have a unique common fixed point. Then, there exists a unique $x^* \in C(I)$, a common fixed point of T and S, that is x^* is a unique solution to (5.1). □

6. Applications to existence theorems for functional equations arising in dynamic programming

The existence, uniqueness, and iterative approximations of solutions for several classes of functional equations arising in dynamic programming were studied by a lot of researchers. Bellman [14] first studied the existence of solutions for some classes of functional equations arising in dynamic programming. Bellman and Lee [15] pointed out that the basic form of the functional equations in dynamic programming is as follows:

$$q(x) = \sup_{y \in D}\{G(x, y, q(\tau(x, y)))\}, \quad x \in W,$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U as well as V are Banach spaces.

In 1984, Bhakta and Mitra [16] obtained some existence theorem for the following functional equation which arises in multistage decision process related to dynamic programming

$$q(x) = \sup_{y\in D}\{g(x,y) + G(x,y,q(\tau(x,y)))\}, \quad x \in W,$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U as well as V are Banach spaces.

Thereafter a lot of work has been done in this direction and existence and uniqueness results were obtained for solutions and common solutions of some functional equations and systems of functional equations in dynamic programming with the use of fixed point results. For details see [13,32,38,39,45,46] and the references therein.

Let U and V be Banach spaces, $W \subseteq U$ be a state space and $D \subseteq V$ be a decision space. Now, by using the fixed point results obtained in the previous section, we study the solvability of the following system of functional equations arising in dynamic programming:

$$\begin{cases} F_i(x) = \sup_{y\in D}\{g(x,y) + H_i(x,y,F_i(\tau(x,y)))\}, x \in W,\ i = 1,2, \\ G_i(x) = \sup_{y\in D}\{g(x,y) + K_i(x,y,G_i(\tau(x,y)))\}, x \in W,\ i = 1,2, \end{cases} \tag{6.1}$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $H_i, K_i : W \times D \times \mathbb{R} \to \mathbb{R}$.

Let $C(W)$ denote the space of all continuous real-valued functions on W. Clearly, this space endowed with the metric given by

$$d(h,k) = \sup_{x\in W} |h(x) - k(x)|, \text{ for all } h,k \in C(W)$$

is a complete metric space.

We will prove the following result.

Theorem 7. *Let $H_i, K_i : W \times D \times \mathbb{R} \to \mathbb{R}$, for $i = 1,2$, be four bounded functions and let $A_i, T_i : C(W) \to C(W)$, for $i = 1,2$, be four operators defined by*

$$A_i h(x) = \sup_{y\in D}\{g(x,y) + H_i(x,y,h(\tau(x,y)))\},\ i = 1,2, \tag{6.2}$$

$$T_i k(x) = \sup_{y\in D}\{g(x,y) + K_i(x,y,k(\tau(x,y)))\}, i = 1,2, \tag{6.3}$$

for all $h,k \in C(W)$ and $x \in W$. Assume that the following conditions hold:

(i) *there exists $\{h_n\} \in C(W)$ such that $\lim_{n\to\infty} A_1 h_n = \lim_{n\to\infty} A_2 h_n = h^* \in C(W)$ and $\lim_{n\to\infty} \sup_{x\in W} |A_1 A_2 h_n - A_2 A_1 h_n| = 0$,*

(ii) *there exists $\{k_n\} \in C(W)$ such that $\lim_{n\to\infty} T_1 k_n = \lim_{n\to\infty} T_2 k_n = k^* \in C(W)$ and $\lim_{n\to\infty} \sup_{x\in W} |T_1 T_2 h_n - T_2 T_1 k_n| = 0$,*

(iii)

$$\int_0^{|H_1(x,y,h(\tau(x,y)))-K_1(x,y,k(\tau(x,y)))|} \varphi(s)\,ds \leq \int_0^{\Theta(h,k)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(h,k)} \varphi(t)\,dt\right), \tag{6.4}$$

where

$$\Theta(h,k) = \max \left\{ \begin{array}{c} |A_1h - A_2h|, |T_1k - T_2k|, |T_2k - A_2k|, \frac{1}{2}[|A_1h - T_2k| + |T_1k - A_2k|], \\ \\ \frac{|A_1k-A_2k||T_1k-T_2k|}{1+|T_2k-A_2h|}, \frac{|A_1h-T_2k||T_1k-A_2h|}{1+|T_2k-A_2h|}, \frac{|A_1h-T_2k||T_1k-A_2h|}{1+|A_1h-T_1k|} \end{array} \right\}$$

where $h,k \in C(W)$, $x \in W$, $y \in D$, $\mathcal{W} \in \Psi$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a non-negative summable Lebesgue integrable function such that*

$$\int_0^{\epsilon} \varphi(s)\,ds > 0,$$

for each $\epsilon > 0$. *Then the system of functional equations (6.1) has a unique bounded solution.*

Proof. Notice that the system of functional equations (6.1) has a unique bounded solution if and only if the operators (6.2) and (6.3) have a unique common fixed point. Now, since H_1, H_2, K_1 and K_2 are bounded, there exists a positive number Λ such that

$$\sup\{|H_i(x,y,z)|, |K_i(x,y,z)| : (x,y,z) \in W \times D \times \mathbb{R}, i = 1,2\} \leq \Lambda.$$

By using a property from the integration theory [30, Theorem 12.34], and the properties of φ, we conclude that for each positive number ε, there exists a positive number $\delta(\varepsilon)$ such that

$$\int_{\Gamma} \varphi(s)\,ds \leq \varepsilon, \tag{6.5}$$

for all $\Gamma \subseteq [0, 2\Lambda]$ with $m(\Gamma) \leq \delta(\varepsilon)$, where $m(\Gamma)$ is the Lebesgue measure of Γ.

Let $x \in W$ and $h_1, h_2 \in C(W)$, then there exist $y_1, y_2 \in D$ such that

$$A_1h_1(x) < g(x,y_1) + H_1(x,y_1,h_1(\tau(x,y_1))) + \delta(\varepsilon), \tag{6.6}$$

$$T_1h_2(x) < g(x,y_2) + K_1(x,y_2,h_2(\tau(x,y_2))) + \delta(\varepsilon), \tag{6.7}$$

$$A_1h_1(x) \geq g(x,y_2) + H_1(x,y_2,h_1(\tau(x,y_2))), \tag{6.8}$$

$$T_1h_2(x) \geq g(x,y_1) + K_1(x,y_1,h_2(\tau(x,y_1))). \tag{6.9}$$

Then from (6.6) and (6.9), it follows easily that

$$A_1h_1(x) - T_1h_2(x) < H_1(x,y_1,h_1(\tau(x,y_1))) - K_1(x,y_1,h_2(\tau(x,y_1))) + \delta(\varepsilon)$$

$$\leq |H_1(x,y_1,h_1(\tau(x,y_1))) - K_1(x,y_1,h_2(\tau(x,y_1)))| + \delta(\varepsilon).$$

Hence we get

$$\begin{aligned} &A_1h_1(x) - T_1h_2(x) \\ &< |H_1(x,y_1,h_1(\tau(x,y_1))) - K_1(x,y_1,h_2(\tau(x,y_1)))| + \delta(\varepsilon). \end{aligned} \tag{6.10}$$

Similarly, from (6.7) and (6.8) we obtain

$$\begin{aligned} &T_1h_2(x) - A_1h_1(x) \\ &< |K_1(x,y_2,h_2(\tau(x,y_2))) - H_1(x,y_2,h_1(\tau(x,y_2)))| + \delta(\varepsilon). \end{aligned} \tag{6.11}$$

Therefore, from (6.10) and (6.11) we have

$$\begin{aligned} &|A_1h_1(x) - T_1h_2(x)| \\ &< |H_1(x,y_1,h_1(\tau(x,y_1))) - K_1(x,y_1,h_2(\tau(x,y_1)))| + \delta(\varepsilon). \end{aligned} \tag{6.12}$$

In view of (7), (6.5) and (6.12), it follows easily that

$$\int_0^{|A_1h_1(x)-T_1h_2(x)|} \varphi(s)\,ds \leq \int_0^{\Theta(h_1,h_2)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta(h_1,h_2)} \varphi(t)\,dt\right) + \varepsilon$$

where

$$\Theta(h_1,h_2) = \max\left\{ \begin{array}{c} |A_1h_1 - A_2h_1|, |T_1h_2 - T_2h_2|, |T_2h_2 - A_2h_2|, \\ \frac{1}{2}[|A_1h_1 - T_2h_2| + |T_1h_2 - A_2h_2|], \frac{|A_1h_2-A_2h_2||T_1h_2-T_2h_2|}{1+|T_2h_2-A_2h_1|}, \\ \frac{|A_1h_1-T_2h_2||T_1h_2-A_2h_1|}{1+|T_2h_2-A_2h_1|}, \frac{|A_1h_1-T_2h_2||T_1h_2-A_2h_1|}{1+|A_1h_1-T_1h_2|} \end{array} \right\}.$$

Since the above inequality is true for any $x \in W$ and $\varepsilon > 0$ is taken arbitrary, then we deduce easily that

$$\int_0^{d(A_1h_1,T_1h_2)} \varphi(s)\,ds \leq \int_0^{\Theta_d(h_1,h_2)} \varphi(t)\,dt - \mathcal{W}\left(\int_0^{\Theta_d(h_1,h_2)} \varphi(t)\,dt\right),$$

where

$$\Theta_d(h_1,h_2) = \max\left\{ \begin{array}{c} d(A_1h_1,A_2h_1), d(T_1h_2,T_2h_2), d(T_2h_2,A_2h_2), \\ \frac{1}{2}[d(A_1h_1,T_2h_2) + d(T_1h_2,A_2h_2)], \frac{d(A_1h_2,A_2h_2)d(T_1h_2,T_2h_2)}{1+d(T_2h_2,A_2h_1)}, \\ \frac{d(A_1h_1,T_2h_2))d(T_1h_2,A_2h_1)}{1+d(T_2h_2,A_2h_1)}, \frac{d(A_1h_1,T_2h_2)d(T_1h_2,A_2h_1)}{1+d(A_1h_1,T_1h_2)} \end{array} \right\}.$$

Next, since H_i and K_i, for $i = 1,2$, are continuous, the operators (6.2) and (6.3) are also continuous and so reciprocally continuous. Also, by conditions (i) and (ii), the pairs (A_1,A_2) and (T_1,T_2) are subcompatible. Thus all the hypotheses of Theorem 3 are satisfied. Consequently, the operators (6.2) and (6.3) have a unique common fixed point, that is, the system of functional equations (6.1) has a unique bounded solution. □

ACKNOWLEDGEMENT

The author would like to thank the editor and anonymous reviewers for their constructive comments, which contributed to improve the presentation of the paper.

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Doi: 10.18514/MMN.2015.783

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 353–360

HU e-ISSN 1787-2413

TREES ON HYPERBOLIC LATTICES

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Received 24 September, 2013

Abstract. In the hyperbolic plane there are infinite regular lattices. From a fix vertex of a lattice tree graphs can be constructed recursively to the next layers with edges of the lattice. In this article we examine the properties of the growing of trees and the probabilities of length of trees considering the vertices on level i.

2010 *Mathematics Subject Classification:* 05C05; 52C20

Keywords: hyperbolic mosaics, regular mosaics, honeycombs, graph, tree

1. INTRODUCTION

In a regular mosaic we can define belts of cells around a fix vertex of the mosaic. Belt 0 is the fix vertex. The first belt consists of the cells of the mosaic having common (finite) points with the fix vertex. If belt i is known, let belt $(i+1)$ consist of the cells that have a common (finite) point (not necessarily a common vertex) with the belt i, but have not with the belt $(i-1)$. Figure 1 shows the first three belts in mosaic $\{4,5\}$. Earlier studies have dealt with the problem of the growing of belts. Let v_i be the number of the cells in the belt i. The crystal-growing ratio, $\lim_{i\to\infty}(v_{i+1}/v_i)$, is known for all 2-dimensional and some 3- and 4-dimensional regular mosaics in hyperbolic spaces ([4–9]).

In this article we consider a regular planar mosaic with Schläfli's symbol $\{p,q\}$ ([2]) as a lattice and construct tree graphs along the edges . The number of the trees grows from belt to belt and we examine the intensity of this growth. If $(p-2)(q-2)=4$, then the lattice is Euclidean, while for $(p-2)(q-2)>4$ the lattice is hyperbolic. There are only three regular lattices ($\{3,6\}$, $\{4,4\}$, $\{6,3\}$) on the Euclidean plane but there are infinite ones on the hyperbolic plane. (Some papers have studied percolation problems on hyperbolic lattices, where mosaics are considered to be lattices [1,3].)

2. TREES

Let us fix a B_0 vertex of the lattice as a main root (label it layer 0 or level 0). Let the outer boundary of belt i be layer i or level i. Now we connect the vertices from

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layer 0 to layer 1 along the edges of the lattice. We build the trees from level $(i-1)$ to level i using the maximum number of edges between level $(i-1)$ and level i. (All vertices on level i are connected to only one vertex of the previous level. We do not let leaves on level $(i-1)$.) We never connect edges on the same layer. The rest vertices on layer i will be also roots of new trees. In this recursive way, we obtain infinitely long trees. Let B denote the roots and A the other vertices. In Figure 1 and 2 the thick edges show the trees from level 0 to level 4. (The dual problem is the case when we get trees by connecting the centres of the cells of the mosaic (Figure 3).)

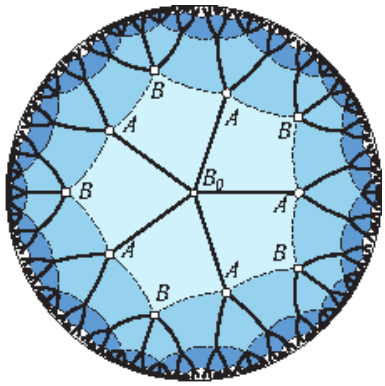

FIGURE 1. *Trees of the mosaic* $\{p,q\} = \{4,5\}$.

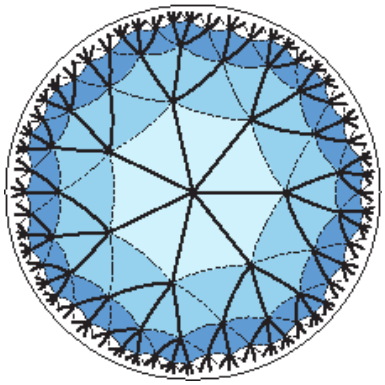

FIGURE 2. *Trees of the mosaic* $\{p,q\} = \{3,7\}$.

In case of $q = 3$ there is not any tree with this definition, because only one edge is not enough to connect the layers. If $p = 3$ the algorithm does not give roots except the main one (Figure 2). Let a_i and b_i be the numbers of the vertices A and B on level i, respectively.

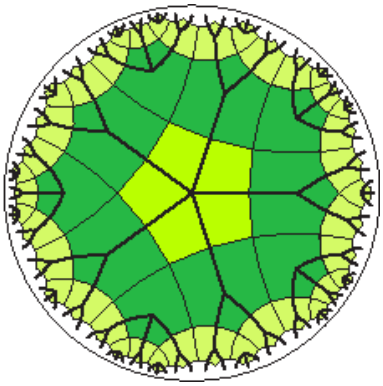

FIGURE 3. *Trees of the mosaic* $\{5,4\}$*, dual of mosaic* $\{4,5\}$.

In the following we shall give some properties of the sequences a_i and b_i for all hyperbolic planar lattices $\{p,q\}$ except the case $p=3$ or $q=3$.

In case of all $\{p,q\}$ lattices $a_0=0$, $b_0=1$ and $a_1=q$, $b_1=q(p-3)$.

Lemma 1. *If* $p>3$, $q>3$ *and* $i\geq 1$, *then* $a_{i+1}=(q-3)a_i+(q-2)b_i$ *and* $b_{i+1}=((q-3)(p-3)-1)a_i+((q-2)(p-3)-1)b_i$.

Proof. The degrees of all vertices are q. On level i a vertex has an edge from level $(i-1)$ and two on level i. So, this vertex is connected to the next level with $q-3$ edges to $q-3$ different A (in Figure 4 we can see a part of level i and $(i+1)$, where a circle denotes a vertex A, a square a vertex B and a triangle a vertex whose type is unknown). Similarly, a root has two edges on level i, so it is connected to $q-2$ new vertices A on the next level.

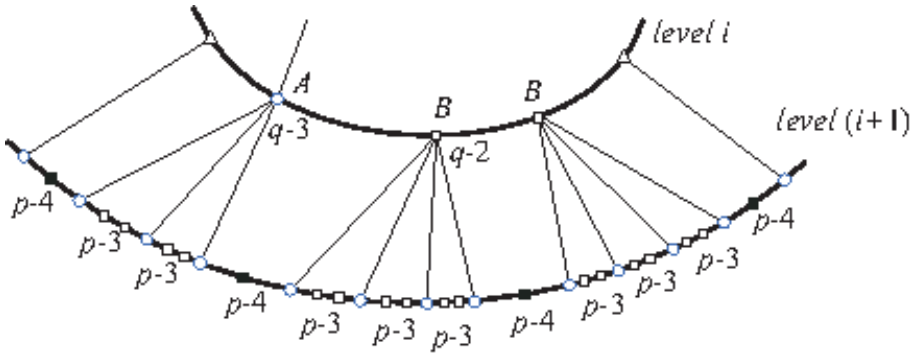

FIGURE 4. *Vertices and edges from level* i *to level* $(i+1)$.

All vertices A are rounded by q pieces of p-gons. Two of them are in the belt i, the others are in the belt $(i+1)$. Among them there are $q-4$ pieces of p-gons that

have $p-3$ vertices which are not connected to the tree of A but they are on level $(i+1)$. So they are roots in the next level. Two further p-gons can have vertices as roots, but they are connected not only to A but also to other vertices on level i (filled squares in Figure 4). Due to the multiplicity we calculate only half of them to the vertex A. Then the number of roots on level $(i+1)$ calculated from a vertex A is $(q-4)(p-3)+2(p-4)/2=(q-3)(p-3)-1$.

We can determine the number of new roots in a similar way in case of a vertex B. Now there are $q-3$ pieces of p-gons and all have $p-3$ vertices which are new roots. Similarly to case A, there are two polygons around B having an other vertex on level i (filled squares in Figure 4). Due to the multiplicity we have to divide their number by two to obtain the correct number of new roots. So the new roots are $(q-3)(p-3)+2(p-4)/2=(q-2)(p-3)-1$ altogether. □

From Lemma 1 we derive a recursive equation system ($p>3, q>3, i\geq 1$)

$$a_{i+1} = (q-3)a_i+(q-2)b_i \tag{2.1}$$

$$b_{i+1} = \big((q-3)(p-3)-1\big)a_i+\big((q-2)(p-3)-1\big)b_i, \tag{2.2}$$

that can also be written in a matrix form

$$\mathbf{w}_{i+1}=\mathbf{M}\mathbf{w}_i, \tag{2.3}$$

where $\mathbf{w}_i=[a_i \;\; b_i]^T$, $\mathbf{M}=\begin{pmatrix} q-3 & q-2 \\ (q-3)(p-3)-1 & (q-2)(p-3)-1\end{pmatrix}$.
All $\{r_i\}_{i=1}^{\infty}$ recursive sequences are defined by

$$r_i = \alpha^T\mathbf{w}_i, \tag{2.4}$$

and can be determined explicitly as ([6,7])

$$r_i=g_{r1}z_1^i+g_{r2}z_2^i, \tag{2.5}$$

where α is a real vector and if $c=(p-2)(q-2)-2>2$ then $z_1=\frac{c+\sqrt{c^2-4}}{2}$, $z_2=\frac{c-\sqrt{c^2-4}}{2}$. The quantities z_1, z_2 are the eigenvalues of matrix $\mathbf{M}$ and $z_1=|z_1|>|z_2|\neq 0$ with

$$g_{r1}=\frac{r_2-z_2r_1}{z_1(z_1-z_2)}\neq 0,\quad g_{r2}=\frac{z_1r_1-r_2}{z_2(z_1-z_2)}.$$

Theorem 1. *The growing ratios of the vertices, of the roots and in addition, of all the vertices are equal to* z_1, $\lim\limits_{i\to\infty}\frac{a_{i+1}}{a_i}=\lim\limits_{i\to\infty}\frac{b_{i+1}}{b_i}=\lim\limits_{i\to\infty}\frac{a_{i+1}+b_{i+1}}{a_i+b_i}=z_1$ $(i\geq 1)$.

Proof. Let sequences r_i be equal to a_i, b_i or a_i+b_i. Then $\alpha^T=[0\;1]^T\mathbf{M}^{-1}$, $\alpha^T=[1\;0]^T\mathbf{M}^{-1}$ or $\alpha^T=[1\;1]^T\mathbf{M}^{-1}$, respectively, and from [6] we obtain that the limits are equal to the largest eigenvalue z_1 of matrix $\mathbf{M}$. □

With the help of equation (2.5), we can write $a_i=g_{a1}z_1^i+g_{a2}z_2^i$, $b_i=g_{b1}z_1^i+g_{b2}z_2^i$, $a_i+b_i=g_{ab1}z_1^i+g_{ab2}z_2^i$. Let $L=\frac{g_{b1}}{g_{a1}}$ and $K=\frac{L}{1+L}$.

Theorem 2. $\lim\limits_{i\to\infty}\frac{b_i}{\sum_{j=0}^{i}b_j}=\frac{z_1-1}{z_1}$ $(i\geq 1)$. $\lim\limits_{i\to\infty}\frac{b_i}{a_i}=L$, $\lim\limits_{i\to\infty}\frac{b_i}{a_i+b_i}=K$ $(i\geq 1)$.

Proof. The first limit for $r_i=b_i$ comes from [6].

As $\lim\limits_{i\to\infty}\left(\frac{z_2}{z_1}\right)^i=0$, then

$$\lim_{i\to\infty}\frac{b_i}{a_i}=\lim_{i\to\infty}\frac{g_{b1}z_1^i+g_{b2}z_2^i}{g_{a1}z_1^i+g_{a2}z_2^i}=\lim_{i\to\infty}\frac{g_{b1}+g_{b2}\left(\frac{z_2}{z_1}\right)^i}{g_{a1}+g_{a2}\left(\frac{z_2}{z_1}\right)^i}=\frac{g_{b1}}{g_{a1}}=L. \tag{2.6}$$

$$\lim_{i\to\infty}\frac{b_i}{a_i+b_i}=\lim_{i\to\infty}\frac{\frac{b_i}{a_i}}{1+\frac{b_i}{a_i}}=K. \tag{2.7}$$

□

3. Probability

In what follows let us suppose that i is large enough. We consider a vertex V from level i. Let $p_{i,j}$ $(0\leq j\leq i)$ be the probability that the root of the vertex V on level i is on level j and let $p_{i,-j}$ be the probability that the root of the vertex V on level i is on level j or on a level below. Let $M=\frac{hL}{1+hL}$, where $h=\frac{q-2}{q-3}$ and $q>3$.

Theorem 3. *If* $0<j<i$, *then* $p_{i,-j}=(1-K)(1-M)^{i-j-1}$ *and* $p_{i,j}=(1-K)M(1-M)^{i-j-1}$. *Moreover* $p_{i,i}=K$ *and* $p_{i,0}=p_{i,-0}=(1-K)(1-M)^{i-1}$.

Proof. The ratio of the roots and all vertices on level i is $\frac{b_i}{a_i+b_i}$. If i is large enough, then $\frac{b_i}{a_i+b_i}\approx K$ (in case of mosaic $\{4,5\}$ if $i=7$ then the difference between the two values is less than 10^{-6}). Similarly, $\frac{b_{i-1}}{a_{i-1}+b_{i-1}}\approx K$. Let V be a vertex on level i. Thus the probability that the vertex V is a root is K and that V is not a root is $1-K$. So $p_{i,i}=K$ and $p_{i,-(i-1)}=1-K$.

On level $(i-1)$ the ratio of the numbers of roots and other vertices is $\frac{b_{i-1}}{a_{i-1}}\approx L$. Equation (2.1) implies that $\frac{q-2}{q-3}L$ gives the ratio between the numbers of vertices with roots on level $(i-1)$ and with roots below. If k_1 is the number of the vertices which have roots on level $(i-1)$, then $\frac{k_1}{a_1}\approx M$. Thus the probability that the root of V is neither on level i, nor on level $(i-1)$ is $(1-K)(1-M)$, therefore $p_{i,-(i-2)}=(1-K)(1-M)$.

Similarly, if k_2 is the number of the vertices on level i whose roots are on level $(i-2)$ then $\frac{k_2}{a_1-k_1}\approx M$. Thus $p_{i,-(i-3)}=(1-K)(1-M)^2$.

Generally, we obtain $p_{i,-j}=(1-K)(1-M)^{i-(j+1)}$, here $0\leq j<i$. And in case of $0<j<i$, $p_{i,j}=(1-K)(1-M)^{i-(j+1)}-(1-K)(1-M)^{i-j}=(1-K)(1-M)^{i-j-1}\big(1-(1-M)\big)=(1-K)M(1-M)^{i-j-1}$. □

The probabilities in Theorem 3 are more precise the higher the i is and the closer the j is to i.

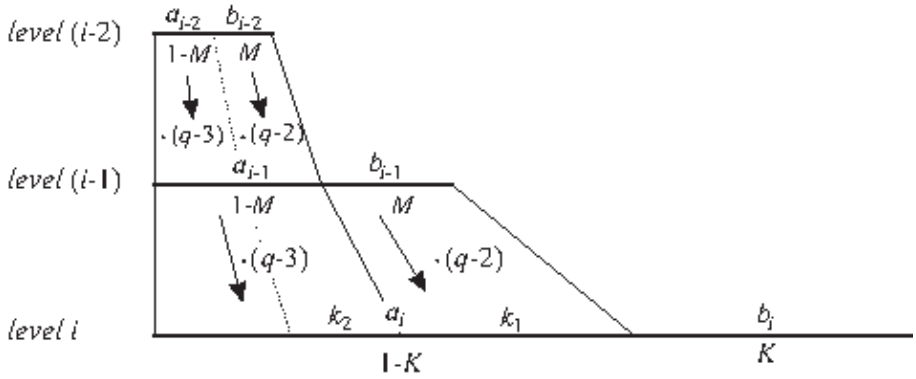

FIGURE 5. *Probability of vertices on level i.*

Examples for lattice $\{4,5\}$.

For lattice $\{4,5\}$ (Figure 1) we give all the results discussed above. The numbers of vertices and roots on level i are in Table 1 ($0 \le i \le 10$).

TABLE 1. Numbers of vertices and roots on level i

i	0	1	2	3	4	5	6	7	8	9	10
a_i	0	5	25	95	335	1325	4945	18455	68875	257045	959305
b_i	1	5	15	55	205	765	2855	10655	39765	148405	553855
$a_i + b_i$	1	10	40	150	560	2090	7800	29110	108640	405450	1513160

The recursion matrix of growing is $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and the crystal-growing ratio is $z_1 = 2 + \sqrt{3} \approx 3.732051$ (while the other eigenvalue is $z_2 = 2 - \sqrt{3}$). In case of $i = 10$ the difference between the corresponding ratios and their limits are less then 10^{-9} (in case of $i = 100$ the difference is less then 10^{-113}).

The values of g_{r1} for the recursive sequences a_i, b_i, $a_i + b_i$ are $g_{a1} = -\frac{5}{2} + \frac{5}{2}\sqrt{3} \approx 1.830127$, $g_{b1} = \frac{5}{2} - \frac{5}{6}\sqrt{3} \approx 1.056624$, $g_{ab1} = \frac{5}{3}\sqrt{3} \approx 2.886751$. Other important values are $L = \frac{1}{3}\sqrt{3} \approx 0.577350$, $K = \frac{1}{2}(\sqrt{3} - 1) \approx 0.366025$ and $M = -3 + 2\sqrt{3} \approx 0.464102$. The difference between the value of K ($p_{i,i} \approx K$) and the exact probability $p_{10,10} = \frac{b_{10}}{a_{10}+b_{10}}$ is less then 10^{-10}.

For the probabilities from Theorem 3 we obtain the values in Table 2.

TABLE 2. Probabilities $p_{i,j}$

j	$i=7$	$i=10$
10	-	0.366025
9	-	0.294228
8	-	0.157677
7	0.366025	0.084499
6	0.294229	0.045283
5	0.157677	0.024267
4	0.084499	0.013005
3	0.045283	0.006969
2	0.024267	0.003735
1	0.013005	0.002001
0	0.015016	0.002311

We can give the number of the vertices on level i which have the common main root given by the expression $s_i = q(q-3)^{(i-1)}$. Then the exact probability is $p_{i,0} = \frac{s_i}{a_i+b_i}$. If $i=7$ or $i=10$, then $p_{7,0} = \frac{320}{29110} \approx 0.010993$ or $p_{10,0} = \frac{2560}{1513160} \approx 0.001692$ and we obtain that $O(p_{7,0}) \approx 10^{-3}$ or $O(p_{10,0}) \approx 10^{-4}$ (in case of $i = 100$, $O(p_{100,0}) \approx 10^{-28}$).

We can conclude that the probabilities from Theorem 3 are exact enough even in case of $i=7$. The worst result is in case of $j=0$, but as j is getting closer to i the result is getting more precise.

Remarks

We can join the main trees in their common main root creating an infinite main tree without root and we can connect the other roots to the main tree by edges, this way we obtain a spanning tree of the vertices of the lattice. In case of $p=3$ the definition in the introduction always gives a spanning tree and we can go back to the main root from all vertices on level i.

There is only one regular Euclidean planar lattice when $p>3$ and $q>3$. Now we examine that $\{4,4\}$ lattice (Figure 6). The $a_0=0$, $b_0=1$, $a_i=8i-4$ and $b_i=4$ is constant. As $z_1=z_2=1$, the growing of the vertices from level to level is slow, if i is large enough, the growth is almost constant. The probability that the root of a vertex V on level i is on level j $(0<j<i)$ is $p_{i,-j} = \frac{8j+4}{8i} = \frac{j+\frac{1}{2}}{i}$ and $\lim_{i\to\infty} p_{i,-j} = 0$.

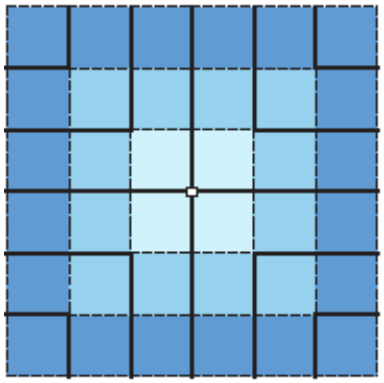

FIGURE 6. *Trees of the mosaic* $\{p,q\} = \{4,4\}$.

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Doi: 10.18514/MMN.2015.1260

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 361–368

HU e-ISSN 1787-2413

APPROXIMATELY ALGEBRAIC TENSOR PRODUCTS

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Received 23 May, 2014

Abstract. Let X and Y be normed spaces over a complete field $\mathbb{F}$ with dual spaces X' and Y' respectively. Under certain hypotheses, for given $x \in X$, $y \in Y$ and a mapping u from $X' \times Y'$ to $\mathbb{F}$, we apply Hyers–Ulam approach to find a unique bounded bilinear mapping v near to u such that $||v|| = ||x \otimes y||$.

2010 *Mathematics Subject Classification:* 46B10; 47L50; 39B72

Keywords: algebraic tensor product, dual space, stability

1. Introduction

Let X, Y, and Z be normed linear spaces over the same field $\mathbb{F}$. A mapping $\phi : X \times Y \longrightarrow Z$ is said to be bilinear if the mappings $x \longmapsto \phi(x, y)$ and $y \longmapsto \phi(x, y)$ are linear. A bilinear mapping $\phi : X \times Y \longrightarrow Z$ is said to be bounded if there exists $M > 0$ such that $||\phi(x, y)|| \leq M||x||||y||$ for all $x \in X$ and $y \in Y$. The norm of ϕ is then defined by

$$||\phi|| := \sup\{||\phi(x, y)|| : (x, y) \in \mathcal{B}_X \times \mathcal{B}_Y\},$$

where $\mathcal{B}_X := \{x \in X : ||x|| \leq 1\}$. The set of all bounded bilinear mappings from $X \times Y$ to Z is denoted by $\mathcal{BL}(X \times Y, Z)$. Let X' and Y' be dual spaces of X and Y respectively. For given $x \in X$ and $y \in Y$, $x \otimes y$ is an element of $\mathcal{BL}(X' \times Y', \mathbb{F})$ defined by $x \otimes y(f, g) := f(x)g(y)$ for all $f \in X'$ and $g \in Y'$. The algebraic tensor product of X and Y, $X \otimes Y$, is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $\mathcal{BL}(X' \times Y', \mathbb{F})$ (see [3]).

A classical question in the theory of functional equations is the following (see [4], [6], [7], [9], [10], [8], [12], [14], [15], [20], [19], [17], [18], [21], [13], [22]): *When is it true that a function which approximately satisfies a functional equation ζ must be close to an exact solution of ζ?*
If the problem accepts a solution, we say that the equation ζ is stable. There are cases in which each approximate solution is actually a true solution. In such cases, we call the equation ζ superstable.

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The first stability problem concerning group homomorphisms was raised by Ulam [22] during his talk before a Mathematical Colloquium at the University of Wisconsin in 1940. Ulam's problem was partially solved by Hyers [7] for mappings between Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [16] has provided a lot of influence in the development of what is called the generalized Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Gavruta [5] in 1994 by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. Badora [2] proved the generalized Hyers-Ulam stability of ring homomorphisms, which generalizes the result of D. G. Bourgin. Miura [11] proved the generalized Hyers-Ulam stability of Jordan homomorphisms.

In this paper, under certain hypotheses and using Hyers–Ulam approach, we find a unique bounded bilinear mapping v near to a given mapping $u : X' \times Y' \longrightarrow \mathbb{F}$ such that $||v|| = ||x \otimes y||$ for $x \in X$, $y \in Y$. Throughout this paper, it is assumed that X and Y are normed spaces over a complete field $\mathbb{F}$ with dual spaces X' and Y' respectively.

2. Results

Theorem 1. *Let $u : X' \times Y' \to \mathbb{F}$ be a mapping for which there exist positive real valued functions φ_1, φ_2, and φ on $X' \times X' \times Y'$, $X' \times Y' \times Y'$, and $X' \times Y'$, respectively such that*

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi_1(2^i f, 2^i f, g) < \infty, \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi_1(2^n f_1, 2^n f_2, g) = \lim_{n\to\infty} \frac{1}{2^n} \varphi_2(2^n f, g_1, g_2) = \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n f, g) = 0, \tag{2.2}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \le \varphi_1(f_1, f_2, g), \tag{2.3}$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \le \varphi_2(f, g_1, g_2) \tag{2.4}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$. Then, there exists a unique bilinear mapping v from $X' \times Y'$ to $\mathbb{F}$ such that

$$|u(f,g) - v(f,g)| \le \tilde{\varphi}(f,g) \quad (f \in X', g \in Y'). \tag{2.5}$$

Moreover, if the mapping u satisfies

$$||u(f,g)| - |f(x)g(y)|| \le \varphi(f,g) \tag{2.6}$$

for some fixed $x \in X$ and $y \in Y$, then $||v|| = ||x \otimes y||$ and so in particular v is bounded.

Proof. Putting $c = 1$ and replacing f_1 and f_2 in (2.3) by f and dividing both sides by 2, we get

$$|\frac{1}{2}u(2f,g)-u(f,g)| \leq \frac{1}{2}\varphi_1(f,f,g) \tag{2.7}$$

for all $f \in X'$ and $g \in Y'$. Replacing f by $2f$ in (2.7) and dividing both sides by 2, we find that

$$|\frac{1}{2^2}u(2^2f,g)-\frac{1}{2}u(2f,g)| \leq \frac{1}{2^2}\varphi_1(2f,2f,g) \tag{2.8}$$

for all $f \in X'$ and $g \in Y'$. Combining (2.7) with (2.8), we obtain

$$|\frac{1}{2^2}u(2^2f,g)-u(f,g)| \leq \frac{1}{2}\varphi_1(f,f,g)+\frac{1}{2^2}\varphi_1(2f,2f,g)$$

for all $f \in X'$ and $g \in Y'$. By induction on n, we conclude that

$$|\frac{1}{2^n}u(2^nf,g)-u(f,g)| \leq \sum_{i=0}^{n-1}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g) \tag{2.9}$$

for all $f \in X'$ and $g \in Y'$. We now turn to use the Cauchy convergence criterion. Replace f by 2^kf in (2.9) and divide both sides by 2^k, where k is an arbitrary positive integer, to get

$$|\frac{1}{2^{n+k}}u(2^{n+k}f,g)-\frac{1}{2^k}u(2^kf,g)| \leq \sum_{i=k}^{n+k-1}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g)$$

for all $f \in X'$, $g \in Y'$, and all positive integers $n \geq k$. It follows from the last inequality and (2.1) that the sequence $\{\frac{1}{2^n}u(2^nf,g)\}$ is a Cauchy sequence for all $f \in X'$ and $g \in Y'$. Since $\mathbb{F}$ is a complete field, this sequence converges. Define $v(f,g) := \lim_{n\to\infty}\frac{1}{2^n}u(2^nf,g)$. Taking the limit as $n \to \infty$ in (2.9), we find that the inequality (2.5) holds for all $f \in X'$ and $g \in Y'$. Replace f_1 and f_2 in (2.3) by 2^nf_1 and 2^nf_2 respectively and divide both sides by 2^n and take the limit as $n \to \infty$ and apply then (2.2) to get the mapping $f \longmapsto v(f,g)$ is linear. By a similar way one can replace f in (2.4) by 2^nf and divide both sides by 2^n to deduce that the mapping $g \longmapsto v(f,g)$ is linear. Consequently, the mapping v is bilinear. Our next claim is to prove that v is unique. Let v' be another mapping satisfying (2.5). Hence,

$$\begin{aligned}|v(f,g)-v'(f,g)| &= \frac{1}{2^k}|v(2^kf,g)-v'(2^kf,g)|\\ &\leq \frac{2}{2^k}\tilde{\varphi}(2^kf,g)\\ &= 2\sum_{i=k}^{\infty}\frac{1}{2^{i+1}}\varphi_1(2^if,2^if,g)\end{aligned}$$

for all $f \in X'$ and $g \in Y'$. Passing to the limit as $k \to \infty$, we conclude that v is unique. Replace f by $2^n f$ in (2.6) and divide both sides by 2^n, to arrive at

$$\left| \frac{1}{2^n} |u(2^n f, g)| - |f(x)g(y)| \right| \leq \frac{1}{2^n} \varphi(2^n f, g) \tag{2.10}$$

for all $f \in X'$ and $g \in Y'$. Taking the limit as $n \to \infty$ in (2.10) and applying the definition of the norm, we conclude that $||v|| = ||x \otimes y||$ and so v is bounded. □

Remark 1. Under the same hypotheses of Theorem 1, with (2.1) and (2.2) replaced by

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi_2(f, 2^i g, 2^i g) < \infty, \tag{2.11}$$

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi_1(f_1, f_2, 2^n g) = \lim_{n\to\infty} \frac{1}{2^n} \varphi_2(f, 2^n g_1, 2^n g_2) = \lim_{n\to\infty} \frac{1}{2^n} \varphi(f, 2^n g) = 0, \tag{2.12}$$

there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying (2.5). Note that by using (2.4) and the same method as in the proof of Theorem 1, we can define $v(f,g) := \lim_{n\to\infty} \frac{1}{2^n} u(f, 2^n g)$.

In the following corollaries, as a consequence of Theorem 1, we show the Rassias stability of algebraic tensor products.

Corollary 1. *Let $x \in X$, $y \in Y$, and $u : X' \times Y' \to \mathbb{F}$ be a mapping such that*

$$||u(f,g)| - |f(x)g(y)|| \leq \alpha + \beta(||f||^p + ||g||^p) + \gamma ||f||^p ||g||^p, \tag{2.13}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \alpha + \beta(||f_1||^q + ||f_2||^q + ||g||^q) + \gamma ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q,$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \leq \alpha + \beta(||f||^r + ||g_1||^r + ||g_2||^r) + \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$, where p, q, r, α, β, and γ are constants with $0 \leq p, q, r < 1$, $\alpha > 0$, and $\beta, \gamma \geq 0$. Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \leq \alpha + \beta(2k||f||^q + ||g||^q) + \gamma k |f||^q ||g||^q \tag{2.14}$$

for all $f \in X'$ and $g \in Y'$, where $k = \frac{1}{2-2^q}$.

Remark 2. Under the hypotheses of Corollary 1 and using Remark 1, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \leq \alpha + \beta(||f||^r + 2k||g||^r) + \gamma k |f||^r ||g||^r$$

for all $f \in X'$ and $g \in Y'$, where $k = \frac{1}{2-2^r}$.

Theorem 2. *Let $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ be linearly independent sets in X and Y respectively and u be a mapping from $X'\times Y'$ to $\mathbb{F}$ for which there exist mappings $\varphi_1 : X'\times X'\times Y'\longrightarrow \mathbb{R}^+$, $\varphi_2 : X'\times Y'\times Y'\longrightarrow \mathbb{R}^+$, and $\varphi : X'\times Y'\longrightarrow \mathbb{R}^+$ satisfying (2.1), (2.2), (2.3), (2.4) and*

$$\left||u(f,g)|-\sum_{i=1}^{m}|f(x_i)g(y_i)|\right|\leq\varphi(f,g) \tag{2.15}$$

for all $f\in X'$, $g\in Y'$. Then, there exists a unique mapping $v\in\mathcal{BL}(X'\times Y',\mathbb{F})$ such that

$$|u(f,g)-v(f,g)|\leq\tilde{\varphi}(f,g)\ \ (f\in X',g\in Y'),\ \ ||v||\leq\sum_{i=1}^{m}||x_i\otimes y_i||. \tag{2.16}$$

In the following our interest is to provide a dual for Theorem 1.

Theorem 3. *Let $x\in X$, $y\in Y$, and let $u : X'\times Y'\to\mathbb{F}$ be a mapping for which there exist mappings $\varphi_1 : X'\times X'\times Y'\longrightarrow \mathbb{R}^+$, $\varphi_2 : X'\times Y'\times Y'\longrightarrow \mathbb{R}^+$, and $\varphi : X'\times Y'\longrightarrow \mathbb{R}^+$ satisfying (2.3), (2.4), (2.6), and*

$$\tilde{\varphi}(f,g):=\sum_{i=0}^{\infty}2^i\varphi_1(\frac{f}{2^{i+1}},\frac{f}{2^{i+1}},g)<\infty, \tag{2.17}$$

$$\lim_{n\to\infty}2^n\varphi_1(\frac{f_1}{2^n},\frac{f_2}{2^n},g)=\lim_{n\to\infty}2^n\varphi_2(\frac{f}{2^n},g_1,g_2)=\lim_{n\to\infty}2^n\varphi(\frac{f}{2^n},g)=0 \tag{2.18}$$

for all $f,f_1,f_2\in X'$, $g,g_1,g_2\in Y'$. Then, there exists a unique mapping $v\in\mathcal{BL}(X'\times Y',\mathbb{F})$ satisfying (2.5).

Proof. By induction on n, we conclude that

$$|u(f,g)-2^nu(\frac{f}{2^n},g)|\leq\sum_{i=0}^{n-1}2^i\varphi_1(\frac{f}{2^{i+1}},\frac{f}{2^{i+1}},g) \tag{2.19}$$

for all $f\in X'$ and $g\in Y'$. Replace f by $\frac{f}{2^k}$ in (2.19) and multiply both sides by 2^k, where k is an arbitrary positive integer, to get

$$|2^ku(\frac{f}{2^k},g)-2^{n+k}u(\frac{f}{2^{n+k}},g)|\leq\sum_{i=k}^{n+k-1}2^i\varphi_1(\frac{f}{2^{i+1}},\frac{f}{2^{i+1}},g)$$

for all $f\in X'$, $g\in Y'$, and all positive integers $n\geq k$. In order to use the Cauchy convergence criterion, the last inequality and (2.17) imply the sequence $\{2^nu(\frac{f}{2^n},g)\}$ is a Cauchy sequence for all $f\in X'$ and $g\in Y'$. Due to completeness of $\mathbb{F}$, this sequence converges. Define $v(f,g):=\lim_{n\to\infty}2^nu(\frac{f}{2^n},g)$. Taking the limit as $n\to\infty$ in (2.19), we deduce that the inequality (2.5) holds for all $f\in X'$ and $g\in Y'$. The rest of the proof is similar to that of Theorem 1. □

Remark 3. Under the same hypotheses of Theorem 3, with (2.17) and (2.18) replaced by

$$\tilde{\varphi}(f,g) := \sum_{i=0}^{\infty} 2^i \varphi_2(f, \frac{g}{2^{i+1}}, \frac{g}{2^{i+1}}) < \infty, \tag{2.20}$$

$$\lim_{n\to\infty} 2^n \varphi_1(f_1, f_2, \frac{g}{2^n}) = \lim_{n\to\infty} 2^n \varphi_2(f, \frac{g_1}{2^n}, \frac{g_2}{2^n}) = \lim_{n\to\infty} 2^n \varphi(f, \frac{g}{2^n}) = 0, \tag{2.21}$$

there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying (2.5). We remark that by using (2.4) and the same method as in the proof of Theorem 3, one can define $v(f,g) := \lim_{n\to\infty} 2^n u(f, \frac{g}{2^n})$.

Corollary 2. *Let $x \in X$, $y \in Y$, and $u : X' \times Y' \to \mathbb{F}$ be a mapping such that*

$$||u(f,g)| - |f(x)g(y)|| \leq \alpha ||f||^p ||g||^p, \tag{2.22}$$

$$|u(cf_1 + f_2, g) - cu(f_1, g) - u(f_2, g)| \leq \beta ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q,$$

$$|u(f, cg_1 + g_2) - cu(f, g_1) - u(f, g_2)| \leq \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$$

for all $f, f_1, f_2 \in X'$, $g, g_1, g_2 \in Y'$, and $c \in \mathbb{F}$, where $p, q, r > 1$, and $\alpha, \beta, \gamma > 0$. Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \leq \frac{\beta}{2^q - 2} ||f||^q ||g||^q \quad (f \in X', g \in Y').$$

Proof. It is enough to define $\varphi(f,g) := \alpha ||f||^p ||g||^p$, $\varphi_1(f_1, f_2, g) := \beta ||f_1||^{\frac{q}{2}} ||f_2||^{\frac{q}{2}} ||g||^q$, and $\varphi_2(f, g_1, g_2) := \gamma ||f||^r ||g_1||^{\frac{r}{2}} ||g_2||^{\frac{r}{2}}$ for all $f, f_1, f_2 \in X'$ and $g, g_1, g_2 \in Y'$ and then apply Theorem 3. □

Remark 4. Under the hypotheses of Corollary 2 and using Remark 3, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ such that $||v|| = ||x \otimes y||$ and

$$|u(f,g) - v(f,g)| \leq \frac{\gamma}{2^r - 2} ||f||^r ||g||^r \quad (f \in X', g \in Y').$$

Theorem 4. *Let $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ be linearly independent sets in X and Y respectively and u be a mapping from $X' \times Y'$ to $\mathbb{F}$ for which there exist mappings $\varphi_1 : X' \times X' \times Y' \longrightarrow \mathbb{R}^+$, $\varphi_2 : X' \times Y' \times Y' \longrightarrow \mathbb{R}^+$, and $\varphi : X' \times Y' \longrightarrow \mathbb{R}^+$ satisfying (2.17), (2.18), (2.15), (2.3), (2.4). Then, there exists a unique mapping $v \in \mathcal{BL}(X' \times Y', \mathbb{F})$ satisfying (2.16).*

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referee for his/her valuable comments concerning the statement of Theorem 1.

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Doi: 10.18514/MMN.2015.710

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 369–383

HU e-ISSN 1787-2413

FIXED POINTS AND COMPLETENESS ON PARTIAL METRIC SPACES

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Received 24 September, 2013

Abstract. Recently, Suzuki [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008), 1861-1869] proved a fixed point theorem that is a generalization of the Banach contraction principle and characterizes the metric completeness. Paesano and Vetro [D. Paesano and P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology Appl., **159** (2012), 911-920] proved an analogous fixed point result for a self-mapping on a partial metric space that characterizes the partial metric 0-completeness. In this paper we prove a fixed point result for a new class of contractions of Berinde-Suzuki type on a partial metric space. Moreover, using our results, as application we obtain a new characterization of partial metric 0-completeness. Finally, we give a typical application of fixed point methods to integral equation, by using our results.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: fixed and common fixed points, Suzuki fixed point theorem, partial metric spaces, ordered partial metric spaces, partial metric 0-completeness

1. INTRODUCTION

In 2008, in order to characterize the completeness of underlying metric spaces, Suzuki [14] introduced a weaker notion of contraction and proved the following theorem.

Theorem 1. *Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r)=\begin{cases}1 & \text{if } 0\le r\le(\sqrt{5}-1)/2\\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2<r<2^{-1/2}\\ (1+r)^{-1} & \text{if } 2^{-1/2}\le r<1.\end{cases}$$

Then for a metric space (X,d), the following are equivalent:

(i) *X is complete.*

The second author is supported by Università degli Studi di Palermo (Local University Project ex 60%).

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(ii) *There exists* $r \in (0,1)$ *such that every mapping* T *on* X *satisfying the following has a fixed point:*

$$\theta(r)d(x,Tx) \le d(x,y) \quad implies \quad d(Tx,Ty) \le rd(x,y) \text{ for all } x,y \in X.$$

In 1994 Matthews [9] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification.

Recently, Romaguera [13] proved that a partial metric space is 0-complete if and only if every Caristi type mapping on X has a fixed point. The result of Romaguera extended Kirk's [7] characterization of metric completeness to a kind of complete partial metric spaces.

In [11], Paesano and Vetro proved the following result.

Theorem 2. *Let* (X,p) *be a 0-complete partial metric space and* T *be a self-mapping on* X. *Define a nonincreasing function* $\theta : [0,1) \to (1/2,1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2 \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 < r < 2^{-1/2} \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ *such that*

$$\theta(r)p(x,Tx) \le p(x,y) \quad implies \quad p(Tx,Ty) \le rp(x,y) \tag{1.1}$$

for all $x,y \in X$. *Then there exists a unique fixed point* z *of* T. *Moreover* $\lim_{n\to+\infty} T^n x = z$ *for all* $x \in X$.

In this paper, in the framework of partial metric spaces, we prove fixed point results for a class of weaker contractive self-mappings of Berinde-Suzuki type (see [2,3,14]). We deduce, also, common fixed point results for two self-mappings. Moreover, using our results, as application we obtain a new characterization of partial metric 0-completeness in terms of fixed point. This characterization extends Suzuki's characterization of metric completeness. Finally, we give a typical application of fixed point methods to integral equation, by using our results.

2. Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [4, 6, 8–11, 13, 15] and references therein. A partial metric on a nonempty set X is a function $p : X \times X \to [0,+\infty)$ such that for all $x,y,z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \le p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic examples of partial metric spaces is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$. Notice that the function $p^s : X \times X \to [0, +\infty)$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

is a metric on X, induced by the partial metric p.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\},$$

for all $x \in X$, $\epsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$. Also, a sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s). A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$. Moreover, (X, p) is complete if and only if the metric space (X, p^s) is complete.

A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 1 (Lemma 2 in [11]). *Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to x \in X$ and $p(x, x) = 0$, then $\lim_{n \to +\infty} p(x_n, z) = p(x, z)$ for all $z \in X$.*

Lemma 2 (Lemma 1 in [11]). *Let (X, p) be a partial metric space and $\{x_n\} \subset X$ a 0-Cauchy sequence. The sequence $\{p(x, x_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$.*

Let X be a non-empty set and $T, f : X \to X$. The mappings T and f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. We denote with $F(T) := \{y \in X : Ty = y\}$ the set of fixed points of T, and with $PC(T, f) := \{y \in X : Tx = fx = y$ for some $x \in X\}$ the set of points of coincidence of T and f. The following lemma is obvious.

Lemma 3. *Let X be a non-empty set and let $T, f : X \to X$ be weakly compatible. If T and f have a unique point of coincidence v in X, then v is a unique common fixed point of T and f.*

Let X be a nonempty set together with a partial order, that is, a binary relation $\preceq$ which is reflexive, antisymmetric and transitive. Then $(X, \preceq)$ is said to be a partially ordered set and $x, y \in X$ are called comparable if one of the elements precedes the other ($x \preceq y$ or $y \preceq x$). Also, if (X, p) is a partial metric space then $(X, p, \preceq)$ is called an ordered partial metric space. Let $f, T : X \to X$ be two mappings, T is said to be f-increasing if $fx \prec fy$ implies $Tx \prec Ty$ for all $x, y \in X$. If f is the identity mapping on X, then T is increasing. If $Tx \preceq Ty$ whenever $x, y \in X$ and $x \preceq y$, then T is said to be nondecreasing.

Lemma 4. *Let (X, p) be a 0-complete partial metric space and let $\{x_n\} \subset X$. Assume that there exists $r \in [0, 1)$ such that*

$$p(x_n, x_{n+1}) \leq r p(x_{n-1}, x_n) \tag{2.1}$$

for all $n \in \mathbb{N}$. Then there exists $z \in X$ such that $\lim_{n\to+\infty} p(x_n, z) = p(z, z) = 0$. Further, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$p(z, x_{n(k)+1}) \leq p(z, x_{n(k)}) \qquad for\ all \qquad k \in \mathbb{N}. \tag{2.2}$$

Proof. From (2.1), we deduce that

$$p(x_{j-1+n}, x_{j+n}) \leq r^n p(x_{j-1}, x_j) \tag{2.3}$$

for all $n, j \in \mathbb{N}$. This implies that $\{x_n\}$ is a 0-Cauchy sequence. As (X, p) is 0-complete, there exists $z \in X$ such that $\lim_{n\to+\infty} p(x_n, z) = p(z, z) = 0$.

Now, we show that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that (2.2) holds. Arguing by contradiction, we assume that there exists $\nu \in \mathbb{N}$ such that

$$p(z, x_j) > p(z, x_{j-1}) \qquad \text{for all} \qquad j \geq \nu. \tag{2.4}$$

By induction on n from (2.4), we deduce that

$$p(z, x_{j-1+n}) > p(z, x_{j-1}) \qquad \text{for all} \qquad j, n \in \mathbb{N},\ \ j \geq \nu. \tag{2.5}$$

Now by (2.3), we get that

$$\begin{aligned} p(x_{j-1+n}, x_{j-1}) &\leq p(x_{j-1}, x_j) + \cdots + p(x_{j-2+n}, x_{j-1+n}) - \sum_{k=j}^{j+n-2} p(x_k, x_k) \\ &\leq (1 + \cdots + r^{n-1})\, p(x_{j-1}, x_j) \\ &\leq \frac{1}{1-r}\, p(x_{j-1}, x_j). \end{aligned}$$

As $n \to +\infty$, by Lemma 1, we obtain that $p(z, x_{j-1}) \leq \frac{1}{1-r}\, p(x_{j-1}, x_j)$ for all $j \in \mathbb{N}$. This implies that

$$p(z, x_{j-1+n}) \leq \frac{1}{1-r}\, p(x_{j-1+n}, x_{j+n}) \leq \frac{r^n}{1-r}\, p(x_{j-1}, x_j) \tag{2.6}$$

for all $n, j \in \mathbb{N}$. By (2.5) and (2.6) we get

$$p(z, x_{j-1}) < \frac{r^n}{1-r} p(x_{j-1}, x_j) \quad \text{for all} \quad n \in \mathbb{N},\ j \geq \nu.$$

From this relation, as $n \to +\infty$, we obtain that $p(z, x_{j-1}) \leq 0$, that is $z = x_{j-1}$ for all $j \geq \nu$, which is in contradiction with (2.4). Hence, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that (2.2) holds. □

3. Fixed point theorem in partial metric spaces

We start this section by presenting a fixed point theorem.

Theorem 3. *Let (X, p) be a 0-complete partial metric space and T be a mapping on X. Assume that there exist $r \in [0, 1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y, x) \text{ implies } p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y)) \quad (3.1)$$

for all $x, y \in X$. Then T has a fixed point in X. Moreover, if $r + L < 1$ there exists a unique fixed point z of T.

Proof. By (p_2), we have $p(Tx, Tx) \leq p(Tx, x)$ for every $x \in X$. By (3.1), we get that

$$p(Tx, T^2x) \leq rp(x, Tx) + L(p(Tx, Tx) - p(Tx, Tx)) = rp(x, Tx), \quad (3.2)$$

for every $x \in X$. Now, we fix $x_0 \in X$ and define a sequence $\{x_n\} \subset X$ by $x_n = T^n x_0 = Tx_{n-1}$, for all $n \in \mathbb{N}$. From (3.2), we deduce that

$$p(x_n, x_{n+1}) \leq rp(x_{n-1}, x_n),$$

that is (2.1) holds for all $n \in \mathbb{N}$. By Lemma 4, there exists $z \in X$ such that $\lim_{n \to +\infty} p(x_n, z) = p(z, z) = 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$p(z, x_{n(k)+1}) \leq p(z, x_{n(k)}), \quad \text{for all} \quad k \in \mathbb{N}.$$

Now, using (3.1) and $p(z, z) = 0$, we obtain that

$$p(Tx_{n(k)}, Tz) \leq rp(x_{n(k)}, z) + Lp(z, x_{n(k)+1}) \quad \text{for all} \quad k \in \mathbb{N}.$$

As $k \to +\infty$, by Lemma 1, we get

$$p(z, Tz) \leq (r + L)\, p(z, z) = 0$$

and hence $Tz = z$, that is z is a fixed point of T. Now, we assume that $r + L < 1$ and let $y \in X$ be another fixed point of T with $y \neq z$. Then, we have

$$p(z, y) = p(z, Ty) \leq p(z, y)$$

and, by (3.1), we get

$$p(y, z) = p(Ty, Tz) \leq rp(y, z) + Lp(z, Ty) - Lp(z, z) \leq (r + L)\, p(y, z).$$

This is possible only if $p(z, y) = 0$, which is a contradiction. Therefore z is a unique fixed point of T. □

Example 1. Let $X = \{0,1\} \cup [r^{-1},4]$, where $r \in (3/4,1)$, be endowed with the partial metric $p(x,y) = \max\{x,y\}$ and $T : X \to X$ be defined by

$$Tx = \begin{cases} 0 & \text{if } x \in \{0,1\}, \\ 1 & \text{if } x \in [r^{-1},4]. \end{cases}$$

For every $x, y \in X$ we have:

$$p(y,Tx) = \begin{cases} y & \text{if } x \in \{0,1\}, \\ y & \text{if } y \in [r^{-1},4], \\ \max\{1,y\} & \text{if } x \in [r^{-1},4] \end{cases}$$

and hence $p(y,Tx) \leq p(y,x)$ and $p(Tx,Ty) \leq rp(x,y)$ for all $x, y \in X$. So T satisfies the condition (3.1) in Theorem 3 with $L = 0$.

Since (X,p) is a 0-complete partial metric space, using Theorem 3, we deduce that T has a unique fixed point.

Example 2. Let $X = [0,1] \cup [r^{-1},4]$, where $r \in (3/4,1)$, be endowed with the partial metric

$$p(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1] \\ \max\{x,y\} & \text{if } \{x,y\} \cap [r^{-1},4] \neq \varnothing \end{cases}$$

and $T : X \to X$ be defined by

$$Tx = \begin{cases} x & \text{if } x \in [0,1] \\ 1 & \text{if } x \in [r^{-1},4], \end{cases}$$

For every $x, y \in X$, we deduce easily that

$$p(y,Tx) \leq p(y,x) \quad \text{and} \quad p(Tx,Ty) \leq rp(x,y) + L(p(y,Tx) - p(y,y)),$$

where $L \geq 1-r$. So T satisfy the condition (3.1) in Theorem 3 with $L \geq 1-r$.

Since (X,p) is a 0-complete partial metric space, using Theorem 3, we deduce that T has a fixed point.

Now, we use the following lemma, that is a consequence of the axiom of choice, to obtain common fixed point results for two self-mappings defined on a partial metric space.

Lemma 5 (Lemma 2.1 in [5])**.** *Let X be a non-empty set and $f : X \to X$ a mapping. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one.*

Theorem 4. *Let (X,p) be a partial metric space and let T, f be self-mappings on X, such that $TX \subset fX$. Assume that there exists $r \in [0,1)$ and $L \geq 0$ such that*

$$p(fy,Tx) \leq p(fy,fx) \quad implies$$
$$p(Tx,Ty) \leq rp(fx,fy) + L(p(fy,Tx) - p(fy,fy)) \tag{3.3}$$

for all $x, y \in X$. Then T and f have a point of coincidence in X. Moreover, if $r + L < 1$, T and f are weakly compatible and fX is 0-complete, then T and f have a unique common fixed point.

Proof. By Lemma 5, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by} \quad Sfx = Tx \quad \text{for all} \quad fx \in fE.$$

Since f is one-to-one on E and $TX \subset fX$, S is well defined. By condition (3.3), for all $fx, fy \in fE$, we have

$$p(fy, Sfx) \leq p(fy, fx)$$

implies

$$p(Sfx, Sfy) \leq rp(fx, fy) + L(p(fy, Sfx) - p(fy, fy)).$$

Then, since fE is 0-complete, by Theorem 3, we have that S has a fixed point on fE, say fz. This implies that $u = fz = Tz$ is a point of coincidence of T and f. Now, we assume that $r + L < 1$ and we prove that T and f have a unique point of coincidence. Let fw be another point of coincidence of T and f with $fw \neq fz$. Since $p(fw, Tz) \leq p(fw, fz)$, using (3.3) we get that

$$p(Tz, Tw) \leq rp(fz, fw) + L(p(fw, Tz) - p(fw, fw)) \leq (r + L)p(Tz, Tw)$$

which is a contradiction and hence T and f have a unique point of coincidence. In conclusion, since T and f are weakly compatible, by Lemma 3, we have that fz is the unique common fixed point of T and f. □

4. Fixed point theorems on ordered partial metric spaces.

In this section we prove fixed point results on the framework of ordered partial metric spaces. For some interesting papers in this direction on metric spaces, we refer to [1, 11, 12].

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that the partial metric space (X, p) is 0-complete. Let $T : X \to X$ be an increasing mapping with respect to $\preceq$. Assume that there exists $r \in [0, 1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y, x) \text{ implies } p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y)) \quad (4.1)$$

for all comparable $x, y \in X$ with $x \neq y$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$, we have $x_n \prec x$, for all $n \in \mathbb{N}$;*

then T has a fixed point in X. Moreover, if

(iii) *for all $x, y \in F(T)$ there exists $v \in X$ such that x, v and y, v are comparable;*

(iv) $r + L < 1$,

then T has a unique fixed point.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $x_0 = Tx_0$, then the result is proved. Hence, we suppose $x_0 \prec Tx_0$. Define a sequence $\{x_n\} \subset X$ such that $x_{n+1} = T^{n+1}x_0 = Tx_n$, for all $n \in \mathbb{N}$. Since T is increasing and $x_0 \prec Tx_0$, we have

$$x_0 \prec Tx_0 = x_1 \prec T^2x_0 = x_2 \prec \cdots \prec x_n \prec \cdots$$

and so $\{x_n\}$ is an increasing sequence.

Note that $p(x_n, Tx_{n-1}) \leq p(x_n, x_{n-1})$ holds for all $n \in \mathbb{N}$. Since x_n and x_{n-1} are comparable for all $n \in \mathbb{N}$, from (4.1) with $x = x_{n-1}$ and $y = x_n$, we deduce that

$$p(x_n, x_{n+1}) \leq rp(x_{n-1}, x_n) + L(p(x_n, x_n) - p(x_n, x_n)) = rp(x_{n-1}, x_n).$$

This implies that (2.1) holds for all $n \in \mathbb{N}$. By Lemma 4, we deduce that there exist $z \in X$ with $p(z,z) = 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\lim_{n\to+\infty} p(x_n, z) = p(z,z)$ and

$$p(z, x_{n(k)+1}) \leq p(z, x_{n(k)}) \qquad \text{for all } k \in \mathbb{N}.$$

By (ii), we have that z is comparable with $x_{n(k)} \neq z$ for all k and, by condition (4.1) and $p(z,z) = 0$, we obtain

$$p(x_{n(k)+1}, Tz) \leq rp(x_{n(k)}, z) + Lp(z, x_{n(k)+1}) \qquad \text{for all } k \in \mathbb{N}.$$

Taking the limit as $k \to +\infty$, we deduce that $p(z, Tz) \leq (r + L)p(z,z) = 0$ and so $Tz = z$. Hence z is a fixed point of T.

Now, we assume that hypotheses (iii) and (iv) hold and we prove the uniqueness of the fixed point. Suppose that there exists $y \in X$ such that $Ty = y$, with $y \neq z$. We have two possible cases:

Case 1. y and z are comparable. Then, since

$$p(z, y) = p(z, Ty) \leq p(z, y)$$

by (4.1), it follows that

$$p(y, z) = p(Ty, Tz) \leq rp(y, z) + L(p(z, Ty) - p(z, z)) \leq (r + L)p(y, z),$$

which is possible only if $p(y, z) = 0$, which leads to contradiction.

Case 2. y and z are not comparable. In this case, there exists $v \in X \setminus \{y, z\}$, comparable with y and with z. Since T is increasing with respect to $\preceq$, it follows that $T^{n-1}y$, $T^{n-1}v$ are comparable and $T^{n-1}y \neq T^{n-1}v$ for all $n \in \mathbb{N}$. Now, since

$$p(T^{n-1}v, T^n y) \leq p(T^{n-1}v, T^{n-1}y) \qquad \text{for all } n \in \mathbb{N},$$

by (4.1), we obtain that

$$\begin{aligned} p(T^n y, T^n v) &\leq rp(T^{n-1}y, T^{n-1}v) + L(p(T^{n-1}v, T^n y) - p(T^{n-1}v, T^{n-1}v)) \\ &\leq (r + L)p(T^{n-1}y, T^{n-1}v) \end{aligned}$$

for all $n \in \mathbb{N}$. Iterating this process, it follows that

$$p(T^n y, T^n v) \leq (r+L)\, p(T^{n-1} y, T^{n-1} v) \leq \cdots \leq (r+L)^n\, p(y,v),$$

that is $p(T^n y, T^n v) \to 0$ as $n \to +\infty$. With similar arguments, we deduce that $p(T^n z, T^n v) \to 0$ as $n \to +\infty$. Hence,

$$0 < p(y,z) = p(T^n y, T^n z) \leq p(T^n y, T^n v) + p(T^n v, T^n z) \to 0$$

as $n \to +\infty$, which is a contradiction.

In both cases, we deduce that T has a unique fixed point. □

From the above theorem, we deduce the following consequence.

Corollary 1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that the partial metric space (X, p) is 0-complete. Let $T : X \to X$ be an increasing mapping with respect to $\preceq$. Assume that there exist $r \in [0,1)$ and $L \geq 0$ such that*

$$p(y, Tx) \leq p(y,x) \text{ implies } p(Tx, Ty) \leq r\, p(x,y) + L(p(y,Tx) - p(y,y)) \quad (4.2)$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq T x_0$;*
(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$, we have $x_n \prec x$, for all $n \in \mathbb{N}$;*

then T has a fixed point in X. Moreover, if

(iii) *for all $x, y \in F(T)$ there exists $v \in X$ such that x, v and y, v are comparable;*
(iv) $r + L < 1$,

then T has a unique fixed point.

Example 3. Let $X = [0,1] \cup [r^{-1}, 4]$, where $r \in (3/4, 1)$, be endowed with the partial metric

$$p(x,y) = \begin{cases} |x-y| & \text{if } x, y \in [0,1], \\ \max\{x,y\} & \text{if } \{x,y\} \cap [r^{-1}, 4] \neq \varnothing, \end{cases}$$

and a partial order $\preceq$ on X defined by $x \preceq y$ if:

$$x = y,\ x \leq y \text{ and } x, y \in [0,1] \cap \mathbb{Q} \text{ or } x \in [0,1] \text{ and } y \in [r^{-1}, 4].$$

Then $(X, p, \preceq)$ is a 0-complete ordered partial metric space. The mapping $T : X \to X$ defined by

$$Tx = \begin{cases} x/2 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ \frac{x}{1+x} & \text{if } x \in [0,1] \setminus \mathbb{Q}, \\ 1 & \text{if } x \in [r^{-1}, 4], \end{cases}$$

is increasing with respect to $\preceq$. In order to show that T satisfies the condition (4.2) in Corollary 1, we consider the following cases:

Case 1. $x \leq y$ and $x, y \in [0,1] \cap \mathbb{Q}$, then $p(Tx, Ty) = \frac{1}{2} p(x,y)$.

Case 2. $x \in [0,1]$ and $y \in [r^{-1},4]$, then

$$p(Tx,Ty) = 1 - Tx \leq 1 = rr^{-1} \leq ry = r\,p(x,y).$$

So T satisfies the condition (4.2) in Corollary 1 with $L = 0$.

Hence Corollary 1 applies and T has a unique fixed point.

Note that the mapping T does not satisfy the condition (3.1) of Theorem 3, in fact for $x \in [0,1] \setminus \mathbb{Q}$ and $y = Tx$, we have $p(y,Tx) = 0 < p(x,y)$. If there exist $r \in [0,1]$ and $L \geq 0$ such that (4.1) holds, then

$$\begin{aligned} p(Tx,Ty) &= \frac{x-y}{(1+x)(1+y)} \\ &\leq r(x-y) + L(p(y,Tx) - p(y,y)) \\ &= r(x-y). \end{aligned}$$

This implies that

$$\frac{1}{(1+x)(1+y)} \leq r$$

and letting $x \to 0$ we get that $1 \leq r$, which is a contradiction.

Theorem 6. *Let $(X,\preceq)$ be a partially ordered set and let T, f be self-mappings on X, such that T is f-increasing and $TX \subset fX$. Suppose that there exists a partial metric p on X such that fX is a 0-complete subset of X. Assume that there exists $r \in [0,1)$ and $L \geq 0$ such that*

$$\begin{aligned} p(fy,Tx) \leq p(fy,fx) \quad implies \\ p(Tx,Ty) \leq rp(fx,fy) + L(p(fy,Tx) - p(fy,fy)) \end{aligned} \tag{4.3}$$

for all comparable $fx, fy \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*
(ii) *for every increasing sequence $\{x_n\} \subset X$ converging to $x \in X$ we have $x_n \preceq x$, for all $n \in \mathbb{N}$;*
(iii) *for all $x, y \in PC(T,f)$ there exists $v \in fX$ such that x, v and y, v are comparable and T and f are weakly compatible;*
(iv) $r + L < 1$,

then T and f have a unique common fixed point.

Proof. By Lemma 5, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by} \quad Sfx = Tx \qquad \text{for all } fx \in fE.$$

Since f is one-to-one on E and since $TX \subset fX$, S is well defined. By hypothesis (4.3), for all comparable $fx, fy \in fE$, we have

$$p(fy,Sfx) \leq p(fy,fx)$$

implies

$$p(Sfx, Sfy) \leq rp(fx, fy) + L(p(fy, Sfx) - p(fy, fy)).$$

Furthermore, since T is f-increasing, we obtain that for all $x, y \in E$,

$$fx \prec fy \quad \text{implies} \quad Sfx = Tx \prec Ty = Sfy,$$

that is S is increasing on fE. Now, we note that if fz is a fixed point of S, then $fz \in PC(T, f)$. So, condition (iii) of Theorem 6 ensures that condition (iii) of Corollary 1 holds. Then, since fE is 0-complete, by Corollary 1, we have that S has a unique fixed point on fE, say fz, if $r + L < 1$.

Now, we assume that $r + L < 1$ and we prove that T and f have a unique point of coincidence. Let $fy \in PC(T, f)$ with $fy \neq fz$. Proceeding as in Corollary 1, Case 1 if fy and fz are comparable or Case 2 if fy and fz are not comparable, we deduce that $fy = fz$. In conclusion, since T and f are weakly compatible, by Lemma 3, we have that fz is the unique common fixed point of T and f. □

5. Completeness in partial metric spaces and fixed points

In this section we characterize those partial metric spaces for which every self-mapping of Berinde-Suzuki type has a fixed point in the style of Suzuki's characterization of metric completeness. This will be done by means of the notion of 0-complete partial metric space which was introduced by Romaguera in [13].

Theorem 7. *For a partial metric space (X, p) the following are equivalent:*

(i) *(X, p) is 0-complete;*

(ii) *There exsist $r \in (0, 1)$ and $L \geq 0$ such that every mapping T on X satisfying the following has a fixed point:*

$$p(y, Tx) \leq p(y, x) \quad \textit{implies} \quad p(Tx, Ty) \leq rp(x, y) + L(p(y, Tx) - p(y, y))$$

for all $x, y \in X$.

Proof. By Theorem 3, (i) implies (ii). Let us prove that (ii) implies (i). We assume (ii). Arguing by contradiction, we also assume that (X, p) is not 0-complete, that is, there exists a 0-Cauchy sequence $\{y_n\}$ which does not converge.

Define a function from X into $[0, +\infty)$ by $fx = \lim_{n \to +\infty} p(x, y_n)$ for $x \in X$. Since, by Lemma 2, the sequence $\{p(x, y_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$, then the function f is well defined. The following are obvious:

P1) $fx > 0$ for all $x \in X$.
In fact, if $fx = \lim_{n \to +\infty} p(x, y_n) = 0$ then $p(x, x) = 0$ and $\lim_{n \to +\infty} y_n = x$, which is a contradiction.

P2) $\lim_{n \to +\infty} fy_n = \lim_{n, m \to +\infty} p(y_n, y_m) = 0$.

P3) $fx - fy \leq p(x, y) \leq fx + fy \quad$ for all $x, y \in X$.
It follows from

- $p(x,y) \le p(x,y_n) + p(y_n,y) - p(y_n,y_n)$;
- $p(x,y_n) \le p(x,y) + p(y,y_n) - p(y,y)$.

Now, by $P1$) and $P2$), there exists $\nu : X \to \mathbb{N}$ such that $fy_{\nu(x)} \le \dfrac{r}{3+2r} fx$ for each $x \in X$. Define $T : X \to X$, by $Tx = y_{\nu(x)}$. Then it is obvious that

$$fTx \le \frac{r}{3+2r} fx \quad \text{and} \quad Tx \in \{y_n : n \in \mathbb{N}\},$$

for all $x \in X$. T does not have a fixed point, since $Tx \ne x$ for all $x \in X$ because $fTx < fx$. Fix $x, y \in X$ with $p(y,Tx) \le p(x,y)$. In the case where $fx > 2fy$, by $P3$), we have

$$\begin{aligned} p(Tx,Ty) \le fTx + fTy &< \frac{r}{3}(fx+fy) \\ &< \frac{r}{3}(fx+fy) + \frac{2r}{3}(fx-2fy) \\ &= r(fx-fy) \le rp(x,y). \end{aligned}$$

In the other case, where $fx \le 2fy$, we have

$$\begin{aligned} p(x,y) \ge p(y,Tx) &\ge fy - fTx \\ &\ge fy - \frac{r}{3+2r} fx \ge \left(1 - \frac{2r}{3+2r}\right) fy \\ &= \frac{3+2r-2r}{3+2r} fy = \frac{3}{3+2r} fy \end{aligned}$$

and hence

$$\begin{aligned} p(Tx,Ty) \le fTx + fTy &\le \frac{r}{3+2r}(fx+fy) \\ &\le \frac{3r}{3+2r} fy \le rp(x,y). \end{aligned}$$

By (ii), T has a fixed point which leads to contradiction. Hence we obtain that X is 0-complete. This completes the proof. □

6. Application to integral equation

In this section we give a typical application of fixed point methods to integral equation, by using our theorems in ordered partial metric spaces. Here, we consider the following integral equation:

$$x(t) = g(t) + \int_0^t K(t,s,x(s))ds, \qquad \text{for all } t \in [0,I], \tag{6.1}$$

where $I > 0$. Let $X = C([0,I],\mathbb{R})$ be the space of all continuous functions defined on $[0,I]$. Notice that $C([0,I])$ endowed with the metric

$$d(x,y) = \|x-y\|_\infty = \max_{t\in[0,I]} |x(t)-y(t)|$$

is a complete metric space. In the sequel, we consider X equipped with the partial order $\preceq$ given by

$$x, y \in X, \qquad x \preceq y \iff (x(t) \leq y(t) \text{ and } \|x\|_\infty, \|y\|_\infty \leq 1) \text{ or } (x(t) = y(t)),$$

for all $t \in [0, I]$.

Now, we state and prove our theorem:

Theorem 8. *Let $T : C([0,I]) \to C([0,I])$ be the operator defined by*

$$T(x)(t) = g(t) + \int_0^t K(t,s,x(s))ds, \qquad \text{for all } t \in [0,I].$$

Suppose that the following conditions hold:

(i) $K \in C([0,I] \times [0,I] \times \mathbb{R}, \mathbb{R})$ *and* $g \in X$;
(ii) $K(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing, for all* $t, s \in [0,I]$;
(iii) *there exists* $x_0 \in X$ *such that*

$$x_0(t) \leq g(t) + \int_0^t K(t,s,x_0(s))ds, \qquad \text{for all } t \in [0,I];$$

(iv) *there exist* $r \in [0,1)$, $L > 0$ *and a continuous function* $G : [0,I] \times [0,I] \to [0,+\infty)$ *such that*

$$p(y, T(x)) \leq p(y,x)$$

implies

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq rI^{-1}|x(s) - y(s)| + G(t,s)|y(s) - T(x)(s)|,$$

for all $t, s \in [0,I]$ *and* $x, y \in \mathbb{R}$ *with* $x \preceq y$;
(v) $\max_{t \in [0,I]} \int_0^t G(t,s)ds = L$;
(vi) $\|T(x)\|_\infty \leq 1$ *whenever* $\|x\|_\infty \leq 1$ *for* $x \in X$.

Then (6.1) has at least one solution $x^* \in X$. *Moreover, if* $r + L < 1$, *then* T *has a unique fixed point.*

Proof. First, condition (iii) implies that $x_0 \preceq T(x_0)$, that is condition (i) of Theorem 5 holds true. Clearly, by condition (ii) the operator T is increasing. Also, in [6], the authors showed that $(X, \preceq)$ satisfies condition (ii) of Theorem 5. Now, let X be endowed with the partial metric defined by

$$p(x,y) = \begin{cases} d(x,y) & \text{if } \|x\|_\infty, \|y\|_\infty \leq 1, \\ d(x,y) + L & \text{otherwise.} \end{cases}$$

Notice that (X, p) is 0-complete but is not complete. Indeed we have

$$p^s(x,y) = \begin{cases} 2d(x,y) & \text{if } (\|x\|_\infty, \|y\|_\infty \leq 1) \text{ or } (\|x\|_\infty, \|y\|_\infty > 1), \\ 2d(x,y) + L & \text{otherwise,} \end{cases}$$

and so (X, p^s) is not complete.
By definition of T it is clear that $x^* \in X$ is a solution of (6.1) if and only if x^* is a fixed point of T. Then, let $x, y \in X$ such that $x \preceq y$. From condition (iv), we deduce that

$$\begin{aligned} |T(x)(t) - T(y)(t)| &= \left| \int_0^t K(t,s,x(s)) - K(t,s,y(s))ds \right| \\ &\leq \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|ds \\ &\leq \int_0^t [rI^{-1}|x(s) - y(s)| + G(t,s)|y(s) - T(x)(s)|]ds. \end{aligned}$$

On routine calculations, for all $x, y \in X$ with $x \prec y$, since $\|x\|_\infty, \|y\|_\infty \leq 1$, we have

$$p(T(x), T(y)) \leq rp(x,y) + Lp(y, T(x)).$$

Therefore, without loss of generality, we write

$$p(y, T(x)) \leq p(y,x) \Longrightarrow p(T(x), T(y)) \leq rp(x,y) + Lp(y, T(x)),$$

for all comparable $x, y \in X$ with $x \neq y$. Therefore, (4.1) holds true for all comparable $x, y \in X$ with $x \neq y$, since $p(y,y) = d(x,y) = 0$. Thus, Theorem 5 applies to this case and so we deduce that the operator T has a fixed point $x^* \in X$, which is a solution of the integral equation (6.1). Now, if $r + L < 1$, since x and $\max\{x, y\}$ as well as y and $\max\{x, y\}$ are comparable for all $x, y \in X$, we get that the operator T has a unique fixed point $x^* \in X$, which is a unique solution of the integral equation (6.1). □

ACKNOWLEDGEMENT

The authors would like to thank the Editor and anonymous reviewer(s) for their constructive comments, which contributed to improve the final version of the paper.

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Doi: 10.18514/MMN.2015.1012

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 385–390

HU e-ISSN 1787-2413

THE NON-EXTENSIBILITY OF $D(-2k+1)$-TRIPLES $\{1, k^2, k^2+2k-1\}$

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Abstract. In this paper we prove that for an integer k such that $|k| \geq 2$, the $D(-2k+1)$-triple $\{1, k^2, k^2+2k-1\}$ cannot be extended to a $D(-2k+1)$-quadruple.

2010 *Mathematics Subject Classification:* 11D45; 11D99

Keywords: Diophantine m-tuples, Pell equations

1. INTRODUCTION

Let n be an integer. A set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ is said to have the property $D(n)$ if $a_i \cdot a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Such a set is called a Diophantine m-tuple with the property $D(n)$ (or $D(n)$-m-tuple, or P_n-set of size m). Fermat found the first example of a $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$. Baker and Davenport [1] proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella and Pethő [8] proved that even the $D(1)$-pair $\{1, 3\}$ cannot be extended to a $D(1)$-quintuple. Dujella [6] proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. Brown [2], Gupta and Singh [10] and Mohanty and Ramasamy [12] proved independently that if n is an integer such that $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple. On the other hand, Dujella [4] proved that if an integer n satisfies $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exist at least one $D(n)$-quadruple.

Many authors considered the problems of (non)extensibility of particular $D(n)$-triples, but also for parametric families of $D(n)$-triples (see references given in [3]). Let $\{a, b, c\}$ be a $D(n)$-triple. The problem of its extensibility to a $D(n)$-quadruple leads to a system of Diophantine equations. These equations are Pellian equations, unless one of the numbers (of polynomials in the parametric case) ab, ac and bc is a perfect square. One such case was considered by Fujita and Togbé [9]. Namely, they considered $D(-k^2)$-triple $\{1, k^2, k^2+1\}$ and they proved that if $\{1, k^2, k^2+1, d\}$ is

This research is supported by TUBITAK (The Scientific and Technological Research Council of Turkey) and Necmettin Erbakan University Scientific Research Project Coordinatorship (BAP). A. D. has been supported by Croatian Science Foundation under the project no. 6422.

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a $D(-k^2)$-quadruple, then $d = 4k^2+1$, and in this case, $3k^2+1$ has to be a perfect square.

In the present paper, we will consider the $D(-2k+1)$-triple $\{1, k^2, k^2+2k-1\}$. This is indeed a $D(-2k+1)$-triple, since $1\cdot k^2+(-2k+1) = (k-1)^2$, $1\cdot(k^2+2k-1)+(-2k+1) = k^2$ and $k^2\cdot(k^2+2k-1)+(-2k+1) = (k^2+k-1)^2$. Note that for an integer k such that $k \neq 0, \pm 1$, the elements of $\{1, k^2, k^2+2k-1\}$ are nonzero and distinct integers.

2. Results

Theorem 1. *Let k be an integer such that $|k| \geq 2$. Then the $D(-2k+1)$-triple $\{1, k^2, k^2+2k-1\}$ cannot be extended to a $D(-2k+1)$-quadruple.*

Note that the statement of Theorem 1 is very simple for $k \equiv 2 \pmod 4$. Indeed, in this case we have the $D(n)$ triple $\{a,b,c\}$ with $n \equiv 5 \pmod 8$, $a \equiv 1 \pmod 8$, $b \equiv 4 \pmod 8$, $c \equiv 7 \pmod 8$. Assume that $\{a,b,c,d\}$ is a $D(n)$-quadruple. If d is odd, then $d \equiv a \pmod 4$ or $d \equiv c \pmod 4$, so one of the numbers $ad+n$, $cd+n$ is $\equiv 2 \pmod 4$, and thus it is not a perfect square. If d is even, then $bd+n \equiv 5 \pmod 8$ is not a perfect square.

In the proof of Theorem 1 we will use the fact that the product of first two elements of the triple $\{1, k^2, k^2+2k-1\}$ is a perfect square. This will allow us to get simple upper bound for the solutions of the corresponding Diophantine equation. The other equation in the system will be a Pellian equation. We will determine the sequences to its solution by using tools of the Diophantine approximations (continued fractions). The comparison with previously obtained upper bounds for the solutions will leave only a few initial elements in the sequences to be checked whether they satisfy the other equation of the system. Technical details (in particular, continued fraction expansion of the corresponding quadratic irrationals) differ slightly depending on whether the parameter k is positive or negative. We will handle the positive case in Section 3 and the negative case in Section 4.

3. Proof of Theorem 1 for positive k

Assume that there is a positive integer d such that the set $\{1, k^2, k^2+2k-1, d\}$ is a $D(-2k+1)$-quadruple. Then there exist nonnegative integers x, y, z satisfying

$$d-2k+1 = x^2, \tag{3.1}$$

$$dk^2-2k+1 = y^2, \tag{3.2}$$

$$d(k^2+2k-1)-2k+1 = z^2. \tag{3.3}$$

By eliminating d from the above equations, we obtain the system of equations

$$y^2-k^2x^2 = (k^2-1)(2k-1), \tag{3.4}$$

$$k^2z^2-(k^2+2k-1)y^2 = (2k-1)^2. \tag{3.5}$$

From (3.4), we have $y^2-k^2x^2>0$, which implies $0\le kx\le y-1$. Thus, we have

$$y^2\ge 2k^3-k^2-2k+1>k^3 \tag{3.6}$$

and

$$y^2\le (y-1)^2+2k^3-k^2-2k+1,$$

which implies

$$y\le (2k^3-k^2-2k+2)/2<k^3. \tag{3.7}$$

Therefore,

$$k^{3/2}<y<k^3. \tag{3.8}$$

Next consider the equation (3.5). We have

$$0<kz-\sqrt{k^2+2k-1}\,y=\frac{(2k-1)^2}{kz+\sqrt{k^2+2k-1}\,y}<\frac{4k^2}{2ky}=\frac{2k}{y}.$$

Thus, we get

$$\left|\frac{z}{y}-\frac{\sqrt{k^2+2k-1}}{k}\right|<\frac{2}{y^2},$$

so z/y is "almost" a convergent of continued fraction of $\frac{\sqrt{k^2+2k-1}}{k}$. If $\gcd(z,y)=g>1$, $z=gz'$, $y=gy'$, then

$$\left|\frac{z'}{y'}-\frac{\sqrt{k^2+2k-1}}{k}\right|<\frac{1}{2y'^2},$$

and z'/y' is a convergent.

By applying the algorithm for continued fraction expansion of quadratic irrationals (see e.g. [13, Section 7.7]) to $\alpha=\frac{\sqrt{k^2+2k-1}}{k}=\frac{\sqrt{(k^2+2k-1)k^2}}{k^2}$, we get $s_0=0$, $t_0=k^2$, $a_0=1$, $s_1=k^2$, $t_1=2k-1$, $a_1=k+1$, $s_2=k^2+k-1$, $t_2=1$, $a_2=2k^2+2k-2$, $s_3=k^2+k-1$, $t_3=2k-1$, $a_3=k+1$, $s_4=k^2$, $t_4=k^2$, $a_4=2$, $(s_5,t_5)=(s_1,t_1)$. Thus, we have the following continued fraction expansion

$$\frac{\sqrt{k^2+2k-1}}{k}=[1,\overline{k+1,2k^2+2k-2,k+1,2}].$$

By [7, Lemma 1], we get

$$k^2z'^2-(k^2+2k-1)y'^2=1,-2k+1 \text{ or } k^2.$$

Thus

$$(2k-1)^2=g^2,g^2(-2k+1) \text{ or } g^2k^2,$$

so the only possibility is $g=2k-1$. Then we get the sequence of possible y' satisfying the equation

$$k^2z'^2-(k^2+2k-1)y'^2=1.$$

Its initial values are $y'_1=k+1$, $y'_2=4k^5+20k^4+32k^3+16k^2-k-1,\ldots$. Since $y=gy'$, only the first value gives y satisfying the inequalities (3.8). Thus, the only

candidate for a solution is $y = 2k^2 + k - 1$. By (3.4), it gives $x^2 = 4k^2 + 2k - 2$, which is not possible (only solutions are $k = \pm 1$).

It remains to consider the case $g = 1$. Now candidates for solutions of (3.5) are not necessarily convergents, but also small linear combinations of successive convergents. More precisely, by Worley's theorem [5, 14], we have $z = rp_i \pm sp_{i-1}$, $y = rq_i \pm sq_{i-1}$, where $0 \leq rs < 4$. By using [7, Lemma 1] again, we find that all solutions of (3.5) with $g = 1$ are given by $z = p_{4j+1} - 2p_{4j}$, $y = q_{4j+1} - 2q_{4j}$ and $z = 2p_{4j+2} + p_{4j+1}$, $y = 2q_{4j+2} + q_{4j+1}$. So, the sequence of solutions in y starts as

$$y_1 = k - 1,\ y_2 = 4k^3 + 8k^2 + k - 1,\ y_3 = 4k^5 + 12k^4 + 4k^3 - 8k^2 - k + 1,\ \ldots.$$

Thus, clearly there is no solution satisfying the inequality (3.8).

4. Proof of Theorem 1 for negative k

The proof of Theorem 1 for negative k's follows the same lines as the proof for positive k's given in the previous section. So, we will give only the sketch of the proof here. We write $k = -K$, with positive integer $K \geq 2$. For $K = 2$, we have the $D(5)$-triple $\{1, 4, -1\}$ which clearly cannot be extended to a $D(5)$-quadruple because it contains elements with mixed signs. For $K = 3$, we have the $D(7)$-triple $\{1, 9, 2\}$ and its non-extensibility has been proved by Kaygisiz and Senay [11]. Thus, we may assume that $K \geq 4$.

Assume that there is a positive integer d such that the set $\{1, K^2, K^2 - 2K - 1, d\}$ is a $D(2K+1)$-quadruple. Then there exist nonnegative integers x, y, z satisfying

$$d + 2K + 1 = x^2, \tag{4.1}$$

$$dK^2 + 2K + 1 = y^2, \tag{4.2}$$

$$d(K^2 - 2K - 1) + 2K + 1 = z^2. \tag{4.3}$$

Now, the system (3.4) and (3.5) becomes

$$y^2 - K^2x^2 = -(K^2 - 1)(2K + 1), \tag{4.4}$$

$$K^2z^2 - (K^2 - 2K - 1)y^2 = (2K + 1)^2. \tag{4.5}$$

In this case, from (4.4), we first get $y \leq Kx - 1$, then $K^2x^2 \leq (Kx - 1)^2 + 2K^3 + K^2 - 2K - 1$, which implies $x < 2K^2$, and finally

$$y < 2K^3. \tag{4.6}$$

Similarly as before, from the equation (4.5), we obtain

$$0 < Kz - \sqrt{K^2 - 2K - 1}\,y = \frac{(2K+1)^2}{Kz + \sqrt{K^2 - 2K - 1}\,y} < \frac{4K}{y},$$

and thus

$$\left|\frac{z}{y}-\frac{\sqrt{K^2-2K-1}}{K}\right|<\frac{4}{y^2}.$$

If $\gcd(z,y)=g>1$, then from (4.2) and (4.3) we see that y and z cannot be both even, hence $g\geq 3$. Thus, by putting $z=gz'$, $y=gy'$, we obtain

$$\left|\frac{z'}{y'}-\frac{\sqrt{K^2-2K-1}}{K}\right|<\frac{1}{2y'^2},$$

and z'/y' is a convergent of the continued fraction expansion of $\alpha'=\frac{\sqrt{K^2-2K-1}}{K}$. We have the following continued fraction expansion

$$\frac{\sqrt{K^2-2K-1}}{K}=[0,1,\overline{K-3,1,2K^2-2K-4,1,K-3,2}].$$

By [7, Lemma 1], we get

$$K^2z'^2-(K^2-2K-1)y'^2=1,2K+1,K^2,-K^2+2K+1 \text{ or } -2K^2+4K+4.$$

Thus we have only two possibilities: the first is $g=2K+1$, and the second can appear only if $2K+1$ is a perfect square, say $2K+1=G^2$, in which case we may have $g=G$. The first possibility leads to the equation

$$K^2z'^2-(K^2-2K-1)y'^2=1,$$

for which the only solution satisfying the inequality (4.6) is $(y',z')=(K-1,K-2)$. This leads to $y=2K^2-K-1$ and $x^2=4K^2-2K-2$, which has no solutions with $K\neq\pm 1$. For the second possibility, the only solution of the corresponding Pellian equation

$$K^2z'^2-(K^2-2K-1)y'^2=2K+1,$$

satisfying the inequality (4.6) is $(y',z')=(1,1)$. Hence $y^2=2K+1$, and (4.2) implies that $d=0$, which, by definition, is not considered as a proper extension to a quadruple.

It remains the case $g=1$. Similarly as above, by applying Worley's theorem, we get that the only solutions of (4.5) with $g=1$ in the range given by (4.6) is $(y,z)=(K+1,K)$. This, together with (4.2), gives $d=1$, which is again not considered as a proper extension to a quadruple, since the starting triple already contains an element 1.

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Doi: 10.18514/MMN.2015.1195

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 391–406

HU e-ISSN 1787-2413

FIXED POINT THEOREMS IN VECTOR METRIC SPACES FOR MULTIVALUED OPERATORS

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Received 06 April, 2014

Abstract. In this work we prove some extensions of the multivalued contraction principle and of Reich's fixed point theorem for multivalued operators in vector metric spaces. We also present a theory for the multivalued contraction principle in vector metric spaces in the spirit of the approach given by I. A. Rus and A. Petruşel. Some examples and open questions are pointed out.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: multivalued contraction principle, multivalued Reich contraction, contractive mapping, data dependence, fixed point, strict fixed point, generalized contraction, Hausdorff convergence, limit shadowing property, Picard operator, Riesz space, multivalued operator, vector metric space, weakly Picard operator, well-posedness of fixed point problem

1. PRELIMINARIES

We present first some auxiliary concepts and results, see the monograph of Aliprantis and Border [1] and the manuscripts [4, 7, 18].

A partially ordered set $(X, \geq)$ is a *lattice* if each pair of elements $x, y \in X$ has a supremum and an infimum.

A real vector space E with an order relation $\geq$ on E that is compatible with the algebraic structure of E in the sense that satisfies properties:

(1) $x \geq y$ implies $x + z \geq y + z$ for each $z \in E, x, y \in E$ and

(2) $x \geq y$ implies $tx \geq ty$ for each $t > 0, x, y \in E$

is called an *ordered vector space*.

An ordered vector space that is also a lattice is called a *Riesz space* or *vector lattice*. Many familiar spaces are Riesz spaces, as is shown by the following standard examples (see [1, 4]):

(1) The space $\mathbb{R}^n$ with the Euclidean norm and with the componentwise ordering relation is a Riesz space.

(2) Given a compact Hausdorff space T, the vector space $C(T)$ of all continuous real functions and the vector space $C_b(T)$ of all bounded continuous real functions with the supremum norm and with the pointwise ordering relation

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are Riesz spaces. In particular, the space c of convergent real sequences and the space c_0 of real null-sequences are Riesz spaces.

(3) Given a measure space (X, A, μ) and $1 \leq p \leq \infty$, the Lebesgue space $L_p(\mu)$ is a Riesz space with the standard $\|\cdot\|_p$ norm and the almost everywhere pointwise ordering relation. In particular, the ℓ_p spaces of real sequences are also Riesz spaces.

The symbolistic $x_n \downarrow x$ means that x_n is a decreasing sequence and $\inf\{x_n\} = x$. Some basic properties of decreasing sequences are:

- $x_n \downarrow x$ and $y_m \downarrow y$ implies $x_n + y_m \downarrow x + y$;
- $x_n \downarrow x$ implies $\lambda x_n \downarrow \lambda x$, for $\lambda > 0$ and $\lambda x_n \uparrow \lambda x$, for $\lambda < 0$;
- $x_n \downarrow x$ and $y_m \downarrow y$ implies $x_n \vee y_m \downarrow x \vee y$ and $x_n \wedge y_m \downarrow x \wedge y$.

The meaning of $x_n \uparrow x$ and some basic properties of increasing sequences are similar.

A Riesz space E is *Archimedean* if $\frac{1}{n}x \downarrow 0$ holds for every $x \in E^+$, where

$$E^+ = \{x \in E : x \geq 0\} \text{ is the positive cone of } E.$$

A Riesz space E is *order complete* or *Dedekind complete* if every non-empty subset of E which is bounded above has a supremum. Any order complete Riesz space is Archimedean. The converse is false.

Let E be a Riesz space. A sequence (b_n) in E is called *order-convergent* (or *o-convergent*) to b, if there exists a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ holds for all n, written $b_n \xrightarrow{o} b$, where

$$|x| = x \vee (-x) \text{ for any } x \in E.$$

A sequence (b_n) in E is called *order-Cauchy* (or *o-Cauchy*), if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $\left|b_n - b_{n+p}\right| \leq a_n$ holds for all n and p. A Riesz space E is called *o-Cauchy complete* if every o-Cauchy sequence is o-convergent.

Let E, F be two Riesz spaces and $f : E \to F$. The function f is *order continuous* (or *o-continuous*) if $b_n \xrightarrow{o} b$ in E implies $f(b_n) \xrightarrow{o} f(b)$ in F.

For notations and other facts regarding lattice and order properties, order convergence and order continuity in Riesz spaces we refer to [1].

We present now the concept of vector metric space.

Definition 1. Let X be a non-empty set and E be a Riesz space. The function $d : X \times X \to E$ is said to be a *vector metric* (or *E-metric*) if it satisfies the following properties:

$$(a)\ d(x, y) = 0 \text{ if and only if } x = y,$$
$$(b)\ d(x, y) \leq d(x, z) + d(y, z), \text{ for all } x, y, z \in X.$$

Also, the triple (X, d, E) is said to be a *vector metric space*.

Example 1. A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by

$$d(x, y) = |x - y|.$$

This vector metric is called the *absolute valued metric* on E.

Now we give some useful definitions for our main results.

Let (X, d, E) be a vector metric space. A sequence (x_n) in X is called *vectorially converging* (or E-*converges*) to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n. A sequence (x_n) in X is called *vectorially Cauchy* (or E-*Cauchy*), if there is a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p. A vector metric space X is called E-*complete* if each E-Cauchy sequence in X E-converges to a limit in X. If $E = \mathbb{R}$, the concepts of E-convergence and metric convergence are the same, respectively the concepts of E-Cauchy sequence and Cauchy sequence are the same. If $X = E$ and d is the absolute valued vector metric on X, then the concepts of vectorially convergence and convergence in order are the same. Notice than, in particular, if $X = E$ and d is an absolute valued vector metric on E, then we obtain fixed point theorems of Riesz space E.

The subset $\varnothing \neq A \subseteq X$ defined by

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}$$

is called the E-*diameter* of X if $\sup\{d(x, y) : x, y \in A\}$ in E. Furthermore, if there exists an $a > 0$ in E such that $d(x, y) \leq a$, for $x, y \in A$, then A is called an E-*bounded set*.

Let X, Y be two vector metric spaces and $f : X \to Y$. The function f is *vectorially continuous* (or E-*continuous*) if $x_n \xrightarrow{d,E} x$ in X implies $f(x_n) \xrightarrow{d,E} f(x)$ in Y.

If X is a nonempty set and $T : X \to P(X)$ is a multivalued operator, we denote by $F_T := \{x \in X \mid x \in T(x)\}$, where

$$\mathcal{P}(X) := \{Y \mid Y \subseteq X\};$$
$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \varnothing\}$$

and in the context of a vector metric space (X, d, E), we denote by

$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is } E\text{-closed}\};$$
$$P_b(X) := \{Y \in P(X) \mid Y \text{ is } E\text{-bounded}\};$$
$$\text{Graph}(T) := \{(x, y) \in X \mid y \in T(x)\}.$$

2. Fixed point theorems for contractions and generalized contractions in vector metric spaces

In this paragraph, we prove the Contraction Principle and Reich fixed point theorem for multivalued operators in vector metric spaces. Also, examples are given to obtain the existence of the fixed point using the new results.

Remark that there exists on the market some powerful background for our study, the papers of Berinde [2], Daffer & Kaneko [5], Kaneko [8, 9], Mizoguchi & Takahashi [10], and more others obtain in their results same conclusions, but not in a vector metric space under an Archimedean assumption. Notice that in the setting of a vector metric space can appear some difficulties, especially when we want to give an example or an application of our result to a differential inclusion. For example, in the case when the vector metric space is $\mathbb{R}^n$, we need to introduce an absolute value property for a matrix which converges to zero to can give an application to a differential inclusions system, see I.-R. Petre [12] (Theorem 2.2), in the case when the vector metric space is a Riesz space E, for a working application we need to assume that E is order complete which guarantees that a certain infimum exists in E, see I.-R. Petre [11] (Theorem 2.28), and also for an application in generalized b-metric spaces, see I.-R. Petre, M. Bota [13] (Theorem 3.13).

Definition 2. Let (X,d,E) be a vector metric space. The operator $T : X \to P_{cl}(X)$ is said to be a multivalued k-contraction, if and only if $k \in [0,1)$ and for any $x, y \in X$ and any $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u,v) \leq k\, d(x,y).$$

Definition 3. Let (X,d,E) be a vector metric space and $T : X \to P(X)$ be a multivalued operator. The sequence $(x_n)_{n\in\mathbb{N}} \subset X$, recursively defined by

$$\begin{cases} x_0 = x,\ x_1 = y; \\ x_{n+1} \in T(x_n), \text{ for all } n \in \mathbb{N}, \end{cases}$$

is called the sequence of successive approximations of T starting from $(x,y) \in \operatorname{Graph}(T)$.

Theorem 1. *Let (X,d,E) be an E-complete vector metric space with E Archimedean and let $T : X \to P_{cl}(X)$ be a multivalued k-contraction. Then T has a fixed point in X and for any $x \in X$, there exists a sequence of successive approximations of T starting from $(x,y) \in \operatorname{Graph}(T)$ which E-converges in (X,d,E) to the fixed point of T.*

Proof. Let $x_0 \in X$ such that $x_1 \in Tx_0$. Then there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) \leq k\, d(x_0,x_1)$. Thus, define the sequence $(x_n) \in X$ by $x_{n+1} \in Tx_n$ and $d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n)$, for any $n \in \mathbb{N}^*$. Inductively, we obtain

$$d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n) \leq \ldots \leq k^n d(x_0,x_1), \text{ for any } n \in \mathbb{N}^*.$$

We have

$$\begin{aligned} d\left(x_n, x_{n+p}\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + \ldots + d\left(x_{n+p-1}, x_{n+p}\right) \\ &\leq \left(k^n + k^{n+1} + \ldots + k^{n+p-1}\right) d\left(x_0, x_1\right) = k^n \frac{1-k^p}{1-k} d\left(x_0, x_1\right) \\ &\leq \frac{k^n}{1-k} d\left(x_0, x_1\right) = a_n \cdot a := b_n, \text{ for any } n \in \mathbb{N}^* \text{ and } p \in \mathbb{N}^*, \end{aligned}$$

where $a_n = \frac{k^n}{1-k} \downarrow 0$ and $a = d\left(x_0, x_1\right) \in E^+$.
On the other hand, by E Archimedean hypothesis, we have $b_n \downarrow 0$. So, the sequence (x_n) is E-Cauchy în X. By the E-completness of X, there is $z \in X$ such that $x_n \xrightarrow{d,E} z$. Hence, there exists (a_n) in E such that $a_n \downarrow 0$ and $d\left(x_n, z\right) \leq a_n$.

We know $x_{n+1} \in T x_n$ for any $n \in \mathbb{N}$ and by the multivalued k-contraction condition it follows that there exists $u \in Tz$ such that

$$d\left(x_{n+1}, u\right) \leq k d\left(x_n, z\right), \text{ for any } n \in \mathbb{N}.$$

Then the following estimation holds:

$$\begin{aligned} d\left(z, u\right) &\leq d\left(x_{n+1}, u\right) + d\left(x_{n+1}, z\right) \\ &\leq k d\left(x_n, z\right) + a_{n+1} \leq (k+1) a_n \downarrow 0. \end{aligned}$$

Thus, we have there exists $z = u \in Tz$, i.e., T has a fixed point in X. □

We can give an example to show that Theorem 1 generalize Nadler's classical metric fixed point theorem.

Example 2. Let $E = \mathbb{R}^2$ with componentwise ordering and let

$$X = \left\{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\} \cup \left\{(0, x) \in \mathbb{R}^2 : 0 \leq x \leq 1\right\}.$$

The mapping $d : X \times X \to E$ is defined by

$$\begin{aligned} d\left((x,0),(y,0)\right) &= \left(\frac{4}{3}|x-y|, |x-y|\right), \\ d\left((0,x),(0,y)\right) &= \left(|x-y|, \frac{2}{3}|x-y|\right), \\ d\left((x,0),(0,y)\right) &= \left(\frac{4}{3}x + y, x + \frac{2}{3}y\right). \end{aligned}$$

Then X is an E-complete vector metric space. Let $T : X \to P_{cl}(X)$ with $T\left(x_1, x_2\right) = \left\{u\left(x_1, x_2\right), v\left(x_1, x_2\right)\right\}$, where $u, v : X \to X$ are defined by

$$\begin{aligned} u\left((x,0)\right) &= (0,x) \text{ and } u\left((0,x)\right) = \left(\frac{x}{2}, 0\right); \\ v\left((x,0)\right) &= (0,x) \text{ and } v\left((0,x)\right) = \left(\frac{x}{3}, 0\right). \end{aligned}$$

We have the following possibilities

Case 1: for any $(x,0),(y,0)\in X$ and any $(0,x)\in T(x,0)$, there exists $(0,y)\in T(y,0)$

Case 2: for any $(x,0),(0,y)\in X$ and any $(0,x)\in T(x,0)$, there exists $\left(\frac{y}{2},0\right)\in T(0,y)$ or $\left(\frac{y}{3},0\right)\in T(0,y)$

Case 3: for any $(0,x),(y,0)\in X$ and any $\left(\frac{x}{2},0\right)\in T(0,x)$ or any $\left(\frac{x}{3},0\right)\in T(0,x)$, there exists $(0,y)\in T(y,0)$

Case 4: for any $(0,x),(0,y)\in X$ and any $\left(\frac{x}{2},0\right)\in T(0,x)$, there exists $\left(\frac{y}{2},0\right)\in T(0,y)$, respectively for any $(0,x),(0,y)\in X$ and any $\left(\frac{x}{3},0\right)\in T(0,x)$, there exists $\left(\frac{y}{3},0\right)\in T(0,y)$

such that for all of these cases condition (2) holds for $k=\frac{3}{4}$.

From Theorem 1 it follows that T has a fixed point in X, but T is not a contraction mapping (for example, using the Chebyshev metric) on $X\subseteq\mathbb{R}^2$, thus we can not apply the classical Nadler's fixed point theorem.

Definition 4. Let (X,d,E) be a vector metric space. The operator $T:X\to P_{cl}(X)$ is said to be a multivalued (a,b,c)-contraction, if and only if $a,b,c\in\mathbb{R}_+$ with $a+b+c<1$ and for any $x,y\in X$ and any $u\in T(x)$, there exists $v\in T(y)$ such that

$$d(u,v)\leq ad(x,y)+bd(x,u)+cd(y,v).$$

Theorem 2. *Let (X,d,E) be an E-complete vector metric space with E Archimedean and let $T:X\to P_{cl}(X)$ be a multivalued (a,b,c)-contraction. Then T has a fixed point in X and for any $x\in X$, there exists a sequence of successive approximations of T starting from $(x,y)\in\mathrm{Graph}(T)$ which E-converges in (X,d,E) to the fixed point of T.*

Proof. Let $x_0\in X$ such that $x_1\in Tx_0$. Then there exists $x_2\in Tx_1$ such that $d(x_1,x_2)\leq ad(x_0,x_1)+bd(x_0,x_1)+cd(x_1,x_2)$, where $a,b,c\in\mathbb{R}_+$ with $a+b+c<1$. Thus, define the sequence $(x_n)\in X$ by $x_{n+1}\in Tx_n$ and $d(x_n,x_{n+1})\leq(a+b)d(x_{n-1},x_n)+cd(x_n,x_{n+1})$, for any $n\in\mathbb{N}^*$. Inductively, we obtain

$$d(x_n,x_{n+1})\leq\frac{a+b}{1-c}d(x_{n-1},x_n)\leq\ldots\leq\left(\frac{a+b}{1-c}\right)^n d(x_0,x_1),\text{ for any }n\in\mathbb{N}^*.$$

Since $a,b,c\in\mathbb{R}_+$ with $a+b+c<1$, we have $0\leq a+b<1-c$. We denote $k:=\frac{a+b}{1-c}\in[0,1)$.

We have

$$\begin{aligned}d(x_n,x_{n+p})&\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{n+p-1},x_{n+p})\\&\leq\left(k^n+k^{n+1}+\ldots+k^{n+p-1}\right)d(x_0,x_1)=k^n\frac{1-k^p}{1-k}d(x_0,x_1)\\&\leq\frac{k^n}{1-k}d(x_0,x_1)=a_n\cdot a:=b_n,\text{ for any }n\in\mathbb{N}^*\text{ and }p\in\mathbb{N}^*,\end{aligned}$$

where $a_n = \frac{k^n}{1-k} \downarrow 0$ and $a = d(x_0, x_1) \in E^+$.
On the other hand, by E Archimedean hypothesis, we have $b_n \downarrow 0$. So, the sequence (x_n) is E-Cauchy în X. By the E-completness of X, there is $z \in X$ such that $x_n \xrightarrow{d,E} z$. Hence, there exists (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, z) \leq a_n$.

We know $x_{n+1} \in Tx_n$ for any $n \in \mathbb{N}$ and by the multivalued (a,b,c)-contraction condition it follows that there exists $u \in Tz$ such that

$$d(x_{n+1}, u) \leq ad(x_n, z) + bd(x_n, x_{n+1}) + cd(z, u), \text{ for any } n \in \mathbb{N}.$$

Since

$$\begin{aligned} d(z,u) &\leq d(x_{n+1}, u) + d(x_{n+1}, z) \\ &\leq ad(x_n, z) + bd(x_n, x_{n+1}) + cd(z,u) + a_{n+1} \\ &\leq (a+1)a_n + bd(x_n, x_{n+1}) + cd(z,u), \text{ hence} \\ d(z,u) &\leq \frac{a+1}{1-c} a_n + \frac{b}{1-c} d(x_n, x_{n+1}) \downarrow 0. \end{aligned}$$

Thus, we have there exists $z = u \in Tz$, i.e., T has a fixed point in X. □

Example 3. Let $E = \mathbb{R}^2$ and consider the set X, the mapping $d : X \times X \to E$ and the multivalued operator $T : X \to P_{cl}(X)$ be defined as in Example 2. It is easily to seen that T is a multivalued $\left(\frac{1}{7}, \frac{1}{7}, \frac{2}{3}\right)$ contraction. From Theorem 2 it follows that that T has a fixed point in X, but T is not a $\left(\frac{1}{7}, \frac{1}{7}, \frac{2}{3}\right)$ contraction mapping with respect to the Chebyshev metric on $X \subseteq \mathbb{R}^2$. Thus we can not apply the classical Reich's fixed point theorem.

Remark 1. Theorem 1 and Theorem 2 generalize several known results in the theory of fixed points (see Nadler [17], Covitz-Nadler [14], Reich [15], Bucur, Guran, Petruşel [3]), etc. Notice also that here we do not need a closed graph condition on T, as in [3], for example.

3. The theory of the contraction principle in vector metric spaces

Our aim is to construct a theory of the fixed point theorems for multivalued operators in vector metric spaces. The novelity is given by a study of several properties of fixed points such as: the existence of fixed points and strict fixed points, data dependence of fixed points, convergence of the fixed point sets for a sequence of multivalued operators, Ulam stability of the inclusion $x \in T(x)$, well-posedness property of the fixed point problem, limit shadowing property of the multivalued operator and others, using the approach introduced in [16] and [14]. This theory can be similarly considered for fixed point theorems for classes of multivalued generalized contractions in vector metric spaces.

Definition 5. Let X, Y be two nonempty subsets and $T : X \to P(Y)$ be a multivalued operator. Then a singlevalued operator $t : X \to Y$ is called a selection of T if and only if $t(x) \in T(x)$, for any $x \in X$.

Definition 6. Let (X, d, E) be a vector metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

$i)$ $x_0 = x, x_1 = y$;

$ii)$ $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

$iii)$ the sequence $(x_n)_{n\in\mathbb{N}}$ is E-convergent and its limit is a fixed point of T.

Definition 7. Let (X, d, E) be a vector metric space and $T : X \to P(X)$ be a MWP operator. Then we define the multivalued operator $T^\infty : \mathrm{Graph}(T) \to P(F_T)$ by the formula $\{T^\infty(x, y) = z \in F_T$: there exists a sequence of successive approximations of T starting from (x, y) that E-converges to $z\}$.

Definition 8. Let (X, d, E) be a vector metric space and $T : X \to P(X)$ a MWP operator. Then T is said to be a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \leq c\, d(x, y)$, for all $(x, y) \in \mathrm{Graph}(T)$.

Definition 9. Let (X, d, E) be a vector metric space and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of ordered sets in $P_{cl}(X)$. Then T_n is called Hausdorff E-convergent to an ordered and closed set T of X, denoted by $T_n \overset{H,d,E}{\longrightarrow} T$ as $n \to \infty$ if and only if there exists a sequence $(a_n) \subset E$ such that $a_n \downarrow 0$ as $n \to \infty$ and for any $u_n \in T_n(x)$, there exists $v \in T(x)$ (respectively for any $v \in T(x)$, there exists $u_n \in T_n(x)$) such that

$$d(u_n, v) \leq a_n, \text{ for any } n \in \mathbb{N}.$$

Definition 10. Let (X, d, E) be an E-complete vector metric space and $T : X \to P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if:

$i)$ $(SF)_T = F_T = \{x^*\}$;

$ii)$ $T^n(x) \overset{H,d,E}{\longrightarrow} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Lemma 1 (Extended Cauchy Lemma)**.** *Let E be an order complete Riesz space. Let $a_n \in \mathbb{R}_+, b_n \in E^+, n \in \mathbb{N}^*$ such that $\sum_{i=0}^{\infty} |a_i| < +\infty$ and $b_n \overset{o}{\longrightarrow} 0$ as $n \to \infty$. Then*

$$\sum_{i=0}^{n} a_{n-i} b_i \overset{o}{\longrightarrow} 0 \text{ as } n \to \infty.$$

Proof. We split the above sum in two parts:

$$\sum_{i=0}^{n} a_{n-i} b_i = \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{n-i} b_i + \sum_{i=\left[\frac{n}{2}\right]+1}^{n} a_{n-i} b_i .$$

From $b_i \xrightarrow{o} 0$ as $i \to \infty$ it follows that there exists a sequence $c_i \downarrow 0$ in E such that $|b_i| \le c_i$ holds for all i.

For the first part of the sum we have:

$$\left| \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{n-i} b_i \right| \le \sum_{i=0}^{\left[\frac{n}{2}\right]} |a_{n-i}|\,|b_i| \le \sum_{i=0}^{\left[\frac{n}{2}\right]} |a_{n-i}|\, c_i$$

$$= \sum_{i=0}^{\left[\frac{n}{2}\right]} |a_{n-i}|\, c_0 \le c_0 \sum_{i=\left[\frac{n}{2}\right]+1}^{\infty} |a_i| := x_n \downarrow 0 \text{ as } n \to \infty,$$

since E is Archimedean.

Since $c_i \downarrow 0$ as $i \to \infty$, we have there exists $m_n := \sup\left\{c_i : i \ge \left[\frac{n}{2}\right] + 1\right\} \downarrow 0$ as $n \to \infty$, because the space E is order complete.

For the second part of the sum we have:

$$\left| \sum_{i=\left[\frac{n}{2}\right]+1}^{n} a_{n-i} b_i \right| \le \sum_{i=\left[\frac{n}{2}\right]+1}^{n} |a_{n-i}|\,|b_i|$$

$$\le \sum_{i=\left[\frac{n}{2}\right]+1}^{n} |a_{n-i}|\, c_i \le m_n \sum_{i=0}^{\infty} |a_i| := y_n \downarrow 0 \text{ as } n \to \infty.$$

Since

$$\left| \sum_{i=0}^{n} a_{n-i} b_i \right| \le \left| \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{n-i} b_i \right| + \left| \sum_{i=\left[\frac{n}{2}\right]+1}^{n} a_{n-i} b_i \right| \le x_n + y_n = z_n \downarrow 0 \text{ as } n \to \infty,$$

thus

$$\sum_{i=0}^{n} a_{n-i} b_i \xrightarrow{o} 0 \text{ as } n \to \infty.$$

□

Theorem 3. *Let (X,d,E) be an E-complete vector metric space with E Archimedean and let $T:X\to P_{cl}(X)$ a multivalued k-contraction. Then the following statements hold:*

$i)$ $F_T\neq\varnothing$;

$ii)$ T is a $\frac{1}{1-k}$-multivalued weakly Picard operator;

$iii)$ Let $S:X\to P_{cl}(X)$ be a multivalued k-contraction and $\eta\in E^+$ such that for any $u\in S(x)$, there exists $v\in T(x)$ such that $d(u,v)\leq\eta$ (respectively for any $u\in T(x)$, there exists $v\in S(x)$ such that $d(u,v)\leq\eta$). Then for any $p\in F_S$, there exists $q\in F_T$ such that $d(p,q)\leq\frac{1}{1-k}\eta$ (respectively for any $p\in F_T$, there exists $q\in F_S$ such that $d(p,q)\leq\frac{1}{1-k}\eta$).

$iv)$ Let $T_n:X\to P_{cl}(X),n\in\mathbb{N}$ be a sequence of multivalued k-contractions such that $T_n(x)\overset{H,d,E}{\longrightarrow}T(x)$ as $n\to\infty$, uniformly with respect to $x\in X$. Then, $F_{T_n}\overset{H,d,E}{\longrightarrow}F_T$ as $n\to\infty$.

$v)$ (Ulam-Hyers stability of the inclusion $x\in T(x)$) Let $\epsilon\in E^+$ be such that there exists $y\in T(x):d(x,y)\leq\epsilon$. Then, there exists $x^*\in F_T$ such that $d(x,x^*)\leq\frac{1}{1-k}\epsilon$.

Proof. $i)$ It follows from Theorem 1.

$ii)$ Let $x_0=x,x_1=y\in X$ and define the sequence $x_{n+1}\in T(x_n)$, for all $n\in\mathbb{N}$, which from Theorem 1 E-converges in (X,d,E) to x^*, for all $x\in X$ and $F_T\neq\varnothing$. We know that $d\left(x_n,x_{n+p}\right)\leq k^n\frac{1-k^p}{1-k}d(x_0,x_1)$, for any $n\in\mathbb{N}^*$ and $p\in\mathbb{N}^*$ and letting $p\to\infty$, we get that

$$d\left(x_n,x^*\right)\leq\frac{k^n}{1-k}d(x_0,x_1),\text{ for any }n\in\mathbb{N}^*.$$

For $n=1$, we get

$$d\left(x_1,x^*\right)\leq\frac{k}{1-k}d(x_0,x_1).$$

Then

$$d\left(x_0,x^*\right)\leq d(x_0,x_1)+d\left(x_1,x^*\right)\leq\frac{1}{1-k}d(x_0,x_1).$$

Hence, T is a $\frac{1}{1-k}$-multivalued weakly Picard operator.

$iii)$ Let $x_0\in F_S$ be arbitrary chosen. Then, by $ii)$, we have that there exists $t^\infty(x_0,x_1)\in F_T$ such that

$$d(x_0,t^\infty(x_0,x_1))\leq\frac{1}{1-k}d(x_0,x_1),\text{ for any }x_1\in T(x_0).$$

Since there exists $x_1\in T(x_0)$ such that $d(x_0,x_1)\leq\eta$, we have that

$$d(x_0,t^\infty(x_0,x_1))\leq\frac{1}{1-k}\eta.$$

By a similar procedure we can prove that, for any $y_0 \in F_T$ there exists $s^\infty(y_0, y_1) \in F_S$ such that

$$d(y_0, s^\infty(y_0, y_1)) \leq \frac{1}{1-k}\eta.$$

iv) Since $T_n(x) \overset{H,d,E}{\longrightarrow} T(x)$ as $n \to \infty$, thus there exists (η_n) in E such that $\eta_n \downarrow 0$ and for any $u_n \in T_n(x)$, there exists $v \in T(x)$ (respectively for any $v \in T(x)$, there exists $u_n \in T_n(x)$) such that

$$d(u_n, v) \leq \eta_n, \text{ for any } n \in \mathbb{N}. \tag{3.1}$$

By *iii*), follows immediately that for any $x_{0_n} \in F_{T_n}$ (respectively for any $y_0 \in F_T$) we have that there exists $t^\infty(x_{0_n}, x_1) \in F_T$ (respectively, there exists $s^\infty(y_0, y_{1_n}) \in F_{T_n}$) such that

$$d(x_{0_n}, t^\infty(x_{0_n}, x_1)) \leq \frac{1}{1-k}\eta_n, \text{ for any } n \in \mathbb{N}, \text{ respectively}$$
$$d(y_0, s^\infty(y_0, y_{1_n})) \leq \frac{1}{1-k}\eta_n, \text{ for any } n \in \mathbb{N}.$$

Letting in (3.1) $n \to \infty$, we obtain that there exists $a_n := \frac{1}{1-k}\eta_n \subset E$ such that $a_n \downarrow 0$ as $n \to \infty$ and for any $p_n := x_{0_n} \in F_{T_n}$, there exists $q := t^\infty(x_{0_n}, x_1) \in F_T$ (respectively for any $q := y_0 \in F_T$, there exists $p_n := s^\infty(y_0, y_{1_n}) \in F_{T_n}$) such that

$$d(p_n, q) \leq a_n, \text{ for any } n \in \mathbb{N}.$$

Hence, $F_{T_n} \overset{H,d,E}{\longrightarrow} F_T$ as $n \to \infty$.

v) Let $\epsilon \in E^+$ and $x \in X$. By hypothesis, there exists $y \in T(x)$ such that $d(x, y) \leq \epsilon$. By the proof of *ii*), we have that there exists $x^* := t^\infty(x, y) \in F_T$ such that

$$d(x, x^*) \leq \frac{1}{1-k}d(x, y) \leq \frac{1}{1-k}\epsilon.$$

□

A second result for multivalued contractions in vector metric spaces is as follows.

Theorem 4. *Let (X, d, E) be an E-complete vector metric space with E Archimedean and let $T : X \to P_{cl}(X)$ a multivalued k-contraction with $(SF)_T \neq \varnothing$. Then the following statements hold:*

i) $F_T = (SF)_T = \{x^*\}$;

ii) $F_{T^n} = (SF)_{T^n} = \{x^*\}$ *for* $n \in \mathbb{N}^*$;

iii) $T^n(x) \overset{H,d,E}{\longrightarrow} \{x^*\}$ *as* $n \to \infty$, *for each* $x \in X$;

iv) *Let $S : X \to P_{cl}(X)$ a multivalued operator and $\eta \in E^+$ such that $F_S \neq \varnothing$ and for any $u \in S(x)$, there exists $v \in T(x)$ such that $d(u, v) \leq \eta$ (respectively for any $u \in T(x)$, there exists $v \in S(x)$ such that $d(u, v) \leq \eta$). Then for any $p \in F_S$,*

there exists $q \in F_T$ *such that* $d(p,q) \leq \frac{1}{1-k}\eta$ *(respectively for any* $p \in F_T$*, there exists* $q \in F_S$ *such that* $d(p,q) \leq \frac{1}{1-k}\eta$*);*

v) *Let* $T_n : X \to P_{cl}(X), n \in \mathbb{N}$ *be a sequence of multivalued operators such that* $F_{T_n} \neq \varnothing$ *for each* $n \in \mathbb{N}$ *and* $T_n(x) \overset{H,d,E}{\longrightarrow} T(x)$ *as* $n \to \infty$*, uniformly with respect to* $x \in X$*. Then,* $F_{T_n} \overset{H,d,E}{\longrightarrow} F_T$ *as* $n \to \infty$*.*

vi) *(Well-posedness property of the fixed point problem) If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that there exists* $y_n \in T(x_n), n \in \mathbb{N}$ *with the property*

$$d(x_n, y_n) \overset{o}{\longrightarrow} 0, \text{ as } n \to \infty,$$

then $x_n \overset{d,E}{\longrightarrow} x^*$*, as* $n \to \infty$*.*

vii) *(Limit shadowing property of the multivalued operator) If* $(y_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that there exists* $u_n \in T(y_n), n \in \mathbb{N}$ *with the property*

$$d(y_{n+1}, u_n) \overset{o}{\longrightarrow} 0, \text{ as } n \to \infty,$$

then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ *of successive approximations for* T*, such that*

$$d(x_n, y_n) \overset{o}{\longrightarrow} 0, \text{ as } n \to \infty.$$

Proof. i) Let $x^* \in (SF)_T$ and we prove that x^* is the unique strict fixed point of T.

We suppose there exists $y \in (SF)_T, y \neq x^*$, then $T(y) = \{y\}$.

From

$$d(x^*, y) = d(T(x^*), T(y)) \leq k d(x^*, y),$$

we obtain the contradiction $k \geq 1$. Thus, $(SF)_T = \{x^*\}$.

We know that $\{x^*\} \subset F_T$ and prove that $F_T \subset \{x^*\}$.

Let $y \in F_T, y \neq x^*$. Since $x^*, y \in X$ and for any $u := y \in T(y)$, there exists $v \in T(x^*) = \{x^*\}$ such that

$$d(y, x^*) \leq k d(x^*, y),$$

we obtain the contradiction $k \geq 1$. Thus, $F_T \subset \{x^*\}$.
Hence, $F_T = \{x^*\}$.

ii) Let $x^* \in (SF)_{T^n}$, for $n \in \mathbb{N}^*$ and we prove that x^* is the unique strict fixed point of T^n.
We suppose there exists $y \in (SF)_{T^n}, y \neq x^*$, then $T^n(y) = \{y\}$, for $n \in \mathbb{N}^*$.
Let $x^*, y \in X$ and for any $u_1 \in T(y)$, there exists $v_1 \in T(x^*)$ such that

$$d(u_1, v_1) \leq k d(x^*, y).$$

Let $u_1, v_1 \in X$ and for any $u_2 \in T(u_1) \subset T(T(y)) = T^2(y)$, there exists $v_2 \in T(v_1) \subset T(T(x^*)) = T^2(x^*)$ such that

$$d(u_2, v_2) \leq k d(u_1, v_1) \leq k^2 d(x^*, y).$$

Inductively, let $u_{n-1}, v_{n-1} \in X, n \geq 2$ and for any $u_n \in T(u_{n-1}) \subset T^n(y) = \{y\}$, there exists $v_n \in T(v_{n-1}) \subset T^n(x^*) = \{x^*\}$ such that

$$d(y, x^*) \leq k^n d(x^*, y), \text{ for any } n \in \mathbb{N}^*.$$

Thus, $y = x^*$ and $(SF)_{T^n} = \{x^*\}$.
We know that $\{x^*\} \subset F_{T^n}$ and prove that $F_{T^n} \subset \{x^*\}$.
Let $y \in F_{T^n}, y \neq x^*$. Similarly, we have that for any $x^*, y \in X$ and for any $u_n := y \in T^n(y)$, there exists $v_n \in T^n(x^*) = \{x^*\}$ such that

$$d(y, x^*) \leq k^n d(x^*, y), \text{ for any } n \in \mathbb{N}^*.$$

Thus, $y = x^*$ and hence $F_{T^n} = (SF)_{T^n} = \{x^*\}$.

iii) Let $x, x^* \in X$ and for any $u_1 := x \in T(x)$, there exists $v_1 \in T(x^*)$ such that

$$d(u_1, v_1) \leq k d(x, x^*).$$

Let $u_1, v_1 \in X$ and for any $u_2 \in T(u_1) \subset T(T(x)) = T^2(x)$, there exists $v_2 \in T(v_1) \subset T(T(x^*)) = T^2(x^*)$ such that

$$d(u_2, v_2) \leq k d(u_1, v_1) \leq k^2 d(x, x^*).$$

Inductively, let $u_{n-1}, v_{n-1} \in X, n \geq 2$ and for any $u_n \in T(u_{n-1}) \subset T^n(x)$, there exists $v_n \in T(v_{n-1}) \subset T^n(x^*) = \{x^*\}$ such that

$$d(u_n, v_n) \leq k^n d(x, x^*), \text{ for any } n \in \mathbb{N}^*. \tag{3.2}$$

By a similar procedure, we can obtain that for any $u_{n-1}, v_{n-1} \in X, n \geq 2$ and for any $u_n \in T(v_{n-1}) \subset T^n(x^*) = \{x^*\}$, there exists $v_n \in T(v_{u-1}) \subset T^n(x)$ such that

$$d(u_n, v_n) \leq k^n d(x^*, x), \text{ for any } n \in \mathbb{N}^*. \tag{3.3}$$

Letting in (3.2) and (3.3) $n \to \infty$, we obtain that there exists $a_n := k^n d(x^*, x) \subset E$ such that $a_n \downarrow 0$ as $n \to \infty$ and hence $T^n(x) \overset{H,d,E}{\longrightarrow} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

iv) Let $y \in F_S$. Then, there exists $x^* \in T(y)$ such that

$$d(y, x^*) \leq d(y, x) + d(x, x^*), \text{ for any } x \in T(y).$$

Since, for any $x \in T(y)$, there exists $x^* \in T(x^*) = \{x^*\}$ such that $d(x, x^*) \leq k d(y, x^*)$, we obtain

$$d(y, x^*) \leq \eta + k d(y, x^*)$$

and thus,

$$d(y, x^*) \leq \frac{1}{1-k}\eta.$$

Hence, for any $y \in F_S$, there exists $x^* \in F_T$ such that

$$d(y, x^*) \leq \frac{1}{1-k}\eta.$$

By a similar procedure, we obtain that for any $y \in F_T$, there exists $x^* \in F_S$ such that

$$d\left(y, x^*\right) \leq \frac{1}{1-k}\eta.$$

v) It follows from iv) (see also, Theorem 3 iv)).

vi) Let $x_n, x^* \in X, n \in \mathbb{N}$. Then, for any $y_n \in T\left(x_n\right)$, there exists $u \in T\left(x^*\right) = \{x^*\}$ such that

$$d\left(y_n, x^*\right) \leq k d\left(x_n, x^*\right).$$

Since

$$d\left(x_n, x^*\right) \leq d\left(x_n, y_n\right) + d\left(y_n, x^*\right) \leq d\left(x_n, y_n\right) + k d\left(x_n, x^*\right),$$

thus

$$d\left(x_n, x^*\right) \leq \frac{1}{1-k} d\left(x_n, y_n\right), \text{ for any } n \in \mathbb{N}. \tag{3.4}$$

By $d\left(x_n, y_n\right) \xrightarrow{o} 0$ as $n \to \infty$, we obtain that there exists (a_n) in E such that $a_n \downarrow 0$ as $n \to \infty$ and

$$\left|d\left(x_n, y_n\right)\right| \leq a_n, \text{ for any } n \in \mathbb{N}. \tag{3.5}$$

Letting in (3.4) $n \to \infty$ and using (3.5), we have

$$d\left(x_n, x^*\right) \leq \frac{1}{1-k} a_n \downarrow 0, \text{ as } n \to \infty.$$

Hence, $x_n \xrightarrow{d,E} x^*$ as $n \to \infty$.

vii) Let $y_n, x^* \in X, n \in \mathbb{N}$. Then, for any $u_n \in T\left(y_n\right)$, there exists $v_n \in T\left(x^*\right) = \{x^*\}$ such that

$$d\left(u_n, x^*\right) \leq k d\left(y_n, x^*\right).$$

Then:

$$\begin{aligned} d\left(y_{n+1}, x^*\right) &\leq d\left(y_{n+1}, u_n\right) + d\left(u_n, x^*\right) \leq d\left(y_{n+1}, u_n\right) + k d\left(y_n, x^*\right) \\ &\leq d\left(y_{n+1}, u_n\right) + k\left[d\left(y_n, u_{n-1}\right) + k d\left(y_{n-1}, x^*\right)\right] \\ &= d\left(y_{n+1}, u_n\right) + k d\left(y_n, u_{n-1}\right) + k^2 d\left(y_{n-1}, x^*\right) \end{aligned}$$

and thus, inductively we get that:

$$\begin{aligned} d\left(y_{n+1}, x^*\right) &\leq d\left(y_{n+1}, u_n\right) + k d\left(y_n, u_{n-1}\right) + \ldots + k^n d\left(y_1, u_0\right) + k^{n+1} d\left(y_0, x^*\right) \\ &= \sum_{i=0}^{n} k^{n-i} d\left(y_{i+1}, u_i\right) + k^{n+1} d\left(y_0, x^*\right) \xrightarrow{o} 0 \text{ as } n \to \infty, \end{aligned}$$

which follows from Lemma 1.

By o-convergence definition to 0, we have there exists (a_n) in E such that $a_n \downarrow 0$ as $n \to \infty$ and $\left|d\left(y_{n+1}, x^*\right)\right| \leq a_{n+1}$, for any $n \in \mathbb{N}$.

On the other hand, by the proof of Theorem 3 i)-ii), we know that there exists a sequence $\left(x_n\right)_{n \in \mathbb{N}}$ of successive approximations for T starting from arbitrary $(x, y) \in \operatorname{Graph}(T)$ which E-converge to a fixed point $x^* \in X$ of the operator T. Since, the

fixed point is unique, we get that $d(x_n, x^*) \xrightarrow{o} 0$ as $n \to \infty$ (by definition, there exists (b_n) in E such that $b_n \downarrow 0$ as $n \to \infty$ and $|d(x_n, x^*)| \leq b_n$, for any $n \in \mathbb{N}$). Thus, for such a sequence $(x_n)_{n\in\mathbb{N}}$, we have

$$|d(x_n, y_n)| \leq \left|d\left(x_n, x^*\right)\right| + \left|d\left(x^*, y_n\right)\right| \leq (b_n + a_n) \downarrow 0 \text{ as } n \to \infty, \text{ for each } x \in X.$$

Hence,

$$d(x_n, y_n) \xrightarrow{o} 0, \text{ as } n \to \infty.$$

□

A third result for multivalued k-contractions in vector metric spaces is the following.

Theorem 5. *Let (X, d, E) be an E-complete vector metric space with E Archimedean and let $T : X \to P_{cp}(X)$ be a multivalued k-contraction such that $T(F_T) = F_T$. Then the following statements hold:*

i) $T(x) = F_T$, for each $x \in F_T$;

ii) If $(x_n)_{n\in\mathbb{N}} \subset X$ is a sequence such that $x_n \xrightarrow{d,E} x^* \in F_T$ as $n \to \infty$, then $T(x_n) \xrightarrow{H,d,E} F_T$ as $n \to \infty$.

Proof. i) Let $x \in F_T$ be arbitrary chosen. Then, $x \in T(x)$ and thus $F_T \subset T(x)$. On the other hand $T(x) \subset T(F_T) \subset F_T$.
Thus, $T(x) = F_T$, for each $x \in F_T$.

ii) Since $x_n \xrightarrow{d,E} x^* \in F_T$ as $n \to \infty$, it follows that there exists a sequence $(a_n) \subset E$ such that $a_n \downarrow 0$ as $n \to \infty$ and

$$d\left(x_n, x^*\right) \leq a_n, \text{ for any } n \in \mathbb{N}.$$

Let $x_n \in X, n \in \mathbb{N}, x^* \in X$. Then, for any $u_n \in T(x_n)$, there exists $u \in T(x^*) = F_T$ (respectively, for any $u \in T(x^*) = F_T$, there exists $u_n \in T(x_n)$) such that

$$d(u_n, u) \leq k d\left(x_n, x^*\right) \leq k a_n \downarrow 0, \text{ as } n \to \infty,$$

since E is Archimedean. Hence, $T(x_n) \xrightarrow{H,d,E} F_T$ as $n \to \infty$. □

Open questions.

The above considerations give rise to the following tasks:

(a) To construct a similar theory for some multivalued generalized contraction in vector metric spaces.

(b) To give fixed point theorems for nonlinear contractions in vector metric spaces (see [2, 5, 6, 10, 18, 19]).

ACKNOWLEDGEMENT

The author wishes to thank the referee for the valuable suggestions leading to an improved version of the manuscript.

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Doi: 10.18514/MMN.2015.828

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 407–421

HU e-ISSN 1787-2413

A GENERALIZATION OF HYERS-ULAM STABILITY ON m-SEMIGROUPS

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Received 24 September, 2013

Abstract. In this paper we investigate the generalized Hyers-Ulam stability of mappings of $m-$semigroups $m \in \mathbb{N}$; $m \geq 2$ into Banach spaces. For $m = 3$ the results can be found in Amyari and Moslehian [*Approximate homomorphisms of ternary semigroups*, Lett. Math. Phys. 77 (2006), 1-9] with the mention that they are true in the class of normal $m-$semigroups which is larger than the class of commutative $m-$semigroups. For $m = 2$ we find certain results of Hyers [*On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA, 27 (1941), 222-224], Aoki [*On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64-66], Rassias, Th. M. [*On the stability of the linear mapping in Banach space*, Proc. Amer. Math, Soc. 72 (1978), 297-300] and Rassias, J. M. [*Solution of a Problem of Ulam,* J. Approx. Theory Math. 57 (1989), 268-273]. In addition, we establish the superstability of $m-$ary homomorphims into Banach algebras endowed with multiplicative norms, generalizing the results of Szekelyhidi [*On a theorem of Baker, Lawrence and Zorzitto,* Proc. Amer. Math. Soc., 84 (1982), 95-96] and Amyari and Moslehian (2006) .

2010 *Mathematics Subject Classification:* 39B52; 39B82; 46B99; 20N15

Keywords: m-semigroup, stability of mappings, superstability, approximately mappings, Banach algebras

1. Introduction

In 1940 S.M. Ulam [22] proposed the famous problem "Give conditions in order for a linear mapping near an approximately linear mapping to exist". More precisely, given a group $(G,\cdot)$, a metric group $(G',\cdot,d)$ and a positive number $\varepsilon > 0$, does there exist $\delta > 0$ such that if a function $f : G \to G'$ satisfies the inequality $d(f(x \cdot y),\ f(x) \cdot f(y)) < \delta$ for all $x, y \in G$, then there is a homomorphism $T : G \to G'$ such that $d(f(x),\ T(x)) < \varepsilon$ for all $x \in G$? When this problem has a solution, we say that the homomorphisms from G to G' are stable.

In 1941 D.H. Hyers [13] solved the Ulam problem for the case of approximate additive mappings under the assumption that G and G' are Banach spaces. This result was generalized by T. Aoki [3] in 1950 and improved by Th. M. Rassias [19] in 1978. After 1982 J. M. Rassias [17, 18] , Th. M. Rassias [20] and L. Szekelyhidi [21] established the stability for Ulam problem for different mappings. The aspect

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of stability that was introduced in these papers is called Hyers-Ulam (shortly H-U) stability. During past few years several mathematicians, like P. Găvruţa [12], Z. Gajda [10] have published on various generalizations and applications of H-U stability to a number of functional equations .

In 2006 Amyari and M. S. Moslehian [1] generalized the H-U stability of mappings on commutative ternary semigroup homomorphisms into Banach spaces. Also, the superstability ("stronger" than the concept of stability) of ternary semigroup homomorphisms into Banach algebras endowed with multiplicative norms has been investigated.

In 2012 M. Dehghanian and M. S. Modarres [6] investigated the generalized H-U stability of ternary $\gamma-$homomorphisms on ternary semigroups. The stability of ternary algebras homomorphisms has been studied by I. S. An and C. Park [2] , M. S. Moslehian and L. Szekelyhidi [15], also.

The object of the present article is thus to formulate and solve some stability problems for m-ary case, in such a way to retrieve some known results from the binary or ternary case. $m-$Ary algebraic structures have many applications in different branches. For example, in the theory of automata [11], some $m-$groupoids are applied in the theory of quantum groups, ternary structures [14] and $m-$Lie algebras are used in physics. The first paper focused on $m-$groups was written by W. Dörnte [7] in 1928. In 1940 E. Post [16] published an extensive study of $m-$groups. The other mathematicians like W. Dudek [8], K. Glazek [9] investigated different $m-$ary structures.

In this paper, using a sequence of Hyers type, we generalize the H-U stability of mappings on $m-$semigroups, $m \geq 2$ into Banach spaces. As a particular case, for $m = 2$ we obtain the results of P. Găvruţă [12], D. H. Hyers [13], T. Aoki [3], Th. M. Rassias [19, 20], J. M. Rassias [18], and for $m = 3$ we find those of M. Amyari and M. S. Moslehian [1] with the mention that they are true in the class of normal $m-$semigroups which is larger than the class of commutative $m-$semigroups. In addition, we establish the superstability of $m-$ary homomorphims into Banach algebras endowed with multiplicative norms, generalizing the results of L. Szekelyhidi [21] and Amyari and Moslehian [1].

2. Preliminaries

First, we recall notations, definitions and properties related to $m-$semigroups which are used throughout this paper.

Let $(S, ()_\circ)$ be an $m-$semigroup, $m \in \mathbb{N}$; $m \geq 2$, i.e. a set S with associative operation $()_\circ : S^m \to S$. A sequence of simbols $x_1 x_2 \dots x_i$ whether they are sets or individual elements, will be abbreviated to x_1^i. When $x_1 = x_2 = \dots = x_i = x$, then we will write $x_1 x_2 \dots x_i = \overset{(i)}{x}$. With this convention, the associative law is written as

$$((x_1^m)_\circ\, x_{m+1}^{2m-1})_\circ = (x_1^i\, (x_{i+1}^{i+m})_\circ\, x_{i+m+1}^{2m-1})_\circ$$

for all $x_1,\dots,x_{2m-1}\in S$ and $i=1,2,\dots,m-1$.

Recursively, one may also define $x^{[0]}=x$, $x^{[1]}=(\overset{(m)}{x})_\circ$, $x^{[k+1]}=(x^{[k]}\overset{(m-1)}{x})_\circ$ for every natural number k. Therefore $x^{[k]}=(\overset{(k(m-1)+1)}{x})_\circ$ which is the so-called long product.

The following laws are true:

$$(x^{[k_1]},\dots,x^{[k_m]})_\circ=x^{[k_1+\dots+k_m+1]},\ \text{for all } k_1,\dots,k_m\in\mathbb{N}$$

$$(x^{[k]})^{[s]}=x^{[ks(m-1)+k+s]},\ \forall\ k,s\in\mathbb{N}$$

We remark that

$$(x^{[1]})^{[1]}=x^{[m+1]}=(\overset{(m^2)}{x})_\circ;\left(x^{[m+1]}\right)^{[1]}=(\overset{(m^3)}{x})_\circ=x^{[m^2+m+1]}$$

and

$$(\overset{(m^k)}{x})_\circ=x^{[m^{k-1}+m^{k-2}+\dots+m+1]},\ \forall\ k\in\mathbb{N}^*.$$

An $m-$semigroup $(S,()_\circ)$ will be called:

- commutative, iff $(x_1^m)_\circ=(x_{\sigma(1)}^{\sigma(m)})_\circ$ for all $x_1\dots,x_m\in S$ and for each permutation σ of $1,2,\dots,m$;

- semicommutative, iff $(x_1^m)_\circ=(x_mx_2^{m-1}x_1)_\circ$ for all $x_1,x_2,\dots,x_m\in S$;

- entropic, iff $((x_{11}^{1m})_\circ(x_{21}^{2m})_\circ\dots(x_{m1}^{mm})_\circ)_\circ=((x_{11}^{m1})_\circ,(x_{12}^{m2})_\circ,\dots,(x_{1m}^{mm})_\circ)_\circ$ for any m by m matrix $(x_{ij})_{i,j=\overline{1,m}}$ of elements from S;

- normal (or exponential), iff $(x_1^m)_\circ^{[n]}=(x_1^{[n]},\dots,x_m^{[n]})_\circ$ for all $x_1,\dots,x_m\in S$ and for all $n\in\mathbb{N}$.

We remark that commutativity implies semicommutativity, semicommutativity implies entropy and entropy implies the normality of $m-$semigroup S.

An m-semigroup $(S,()_\circ)$ is called an m-group if for all $a_0,a_1,\dots,a_m\in S$ and fixed $i\in\{1,...,m\}$ there exists an unique element $x\in S$ such that $(a_1^{i-1},x,a_{i+1}^m)_\circ=a_0$. In some m-groups there is an element $e\in A$ (called an identity or a neutral element) such that $(\overset{(i-1)}{e},x,\overset{(m-i)}{e})_\circ=x$ holds for all $x\in S$ and for all $i=1,\dots,m$. It is interesting that there are m-groups with two or more neutral elements or which do not contain such elements [7, 16].

We mention that if $(S,*)$ is a semigroup (group) and we define the m- ary operation $()_\circ:S^m\to S$, $(x_i^m)_\circ=x_1*x_2\cdots*x_m$, then $(S,()_\circ)$ is an m-semigroup ($m-$group), so called the m-semigroup (m-group) derived from the binary semigroup (group) $(S,*)$. In addition, there are m-semigroups that are not derived from a binary semigroup.

Definition 1. *A normed $m-$semigroup is a triplet $(S,()_\circ,\|\ \|)$ such that $(S,()_\circ)$ is an $m-$semigroup and $\|\ \|:S\to[0,\infty)$, $\left\|(x_1^m)_\circ\right\|\le\|x_1\|+\dots+\|x_m\|$ for all $x_1,...,x_m\in S$.*

Definition 2. *Let $(X,*,\|\ \|)$ be a Banach space over the field of real numbers or complex numbers. If we define an m-ary operation $()_\circ : X^m \to X$; $(x_1^m)_\circ = x_1 * x_2 \cdots * x_m$ then $(X,()_\circ)$ is an commutative $m-$group with neutral element. We say that $(X,()_\circ,\|\ \|)$ is an $m-$ary Banach space derived from $(X,*,\|\ \|)$.*

Extending the notion of ternary Banach algebra given in [5],[23] we have

Definition 3. *An m-ary Banach algebra is a normed Banach space $(X,\|\ \|)$ equipped with an m-ary operation $()_\circ : X^m \to X$ which is associative and satisfies $\|(x_1^m)_\circ\| \leq \|x_1\| \ldots \|x_m\|$ for all $x_1,...,x_m \in S$. If $\|(x_1^m)_\circ\| = \|x_1\| \ldots \|x_m\|$ then we say that the norm is multiplicative.*

3. The generalized Hyers-Ulam stability

P. Găvruţă [12] obtained a generalization of Th. M. Rassias's theorem by replacing the Cauchy differences by a control mapping φ satisfying a condition of convergence. In the same manner, using the Hyers "direct method" we can prove the following theorem, which generalizes Theorem 2.1 of Amyari and Moslehian [1] for ternary semigroups.

Theorem 1. *Let $(S,()_\circ)$ be an $m-$semigroup, X a Banach space and $\varphi : S^m \to [0,\infty)$ be a mapping such that*

$$\tilde{\varphi}(x_1,\ldots,x_m) := \frac{1}{m}\sum_{n=0}^{\infty} m^{-n}\varphi(({}^{(m^n)}_{x_1})_\circ,\ldots,({}^{(m^n)}_{x_m})_\circ) < \infty \tag{3.1}$$

Suppose that the map $f : S \to X$ satisfies the condition

$$\left\| f((x_1,\ldots,x_m)_\circ) - \sum_{i=1}^{m} f(x_i) \right\| \leq \varphi(x_1,\ldots,x_m). \tag{3.2}$$

Then there exists an unique mapping $T : S \to X$ such that

$$\|f(x) - T(x)\| \leq \tilde{\varphi}(x,x,\ldots,x) \tag{3.3}$$

and

$$T(x^{[1]}) = mT(x)\,, \text{for all } x \in S. \tag{3.4}$$

Moreover, if S is a normal $m-$semigroup then T is a homomorphism of S on $(X,\sum)$, the $m-$semigroup derived from $(X,+)$.

Proof. For $x_1 = \ldots = x_m = x$ in (3.2) we get

$$\left\| f(x^{[1]}) - mf(x) \right\| \leq \varphi(x,\ldots,x)$$

or

$$\left\| \frac{1}{m} f(x^{[1]}) - f(x) \right\| \leq \frac{1}{m}\varphi(x,\ldots,x).$$

Because

$$({}^{(m^{k+1})}_{x})_{\circ} = x^{[m^k+m^{k-1}+\ldots+m+1]} = (x^{[m^{k-1}+m^{k-2}+\ldots+1]})^{[1]} = (({}^{(m^k)}_{x})_{\circ})^{[1]}$$

we also have

$$\left\| f(({}^{(m^{k+1})}_{x})_{\circ}) - mf({}^{(m^k)}_{x})_{\circ} \right\| \leq \varphi(({}^{(m^k)}_{x})_{\circ}, \ldots, ({}^{(m^k)}_{x})_{\circ}).$$

Then, recursively, we obtain

$$\begin{aligned}
\left\| m^{-n} f(({}^{(m^n)}_{x})_{\circ}) - f(x) \right\| &= \left\| \sum_{k=0}^{n-1} \left(m^{-(k+1)} f(({}^{(m^{k+1})}_{x})_{\circ}) - m^{-k} f(({}^{(m^k)}_{x})_{\circ}) \right) \right\| \\
&= \left\| \sum_{k=0}^{n-1} m^{-k} \left(\frac{1}{m} f(({}^{(m^{k+1})}_{x})_{\circ}) - f(({}^{(m^k)}_{x})_{\circ}) \right) \right\| \\
&\leq \sum_{k=0}^{n-1} m^{-k} \left\| \frac{1}{m} f(({}^{(m^{k+1})}_{x})_{\circ}) - f(({}^{(m^k)}_{x})_{\circ}) \right\| \\
&\leq \frac{1}{m} \sum_{k=0}^{n-1} m^{-k} \varphi(x^{[\sum_{i=0}^{k-1} m^i]}, \ldots, x^{[\sum_{i=0}^{k-1} m^i]}).
\end{aligned}$$

Hence

$$\left\| m^{-n} f(x^{[\sum_{i=0}^{n-1} m^i]}) - f(x) \right\| \leq \frac{1}{m} \sum_{k=0}^{n-1} m^{-k} \varphi(x^{[\sum_{i=0}^{k-1} m^i]}, \ldots, x^{[\sum_{i=0}^{k-1} m^i]}). \tag{3.5}$$

Similarly, for all $x \in S$, $n, r \in \mathbb{N}$; $r < n$ we can show that

$$\left\| m^{-n} f(x^{[\sum_{i=0}^{n-1} m^i]}) - m^{-r} f(x^{[\sum_{i=0}^{r-1} m^i]}) \right\|$$

$$\leq \tfrac{1}{m} \textstyle\sum_{k=r}^{n-1} m^{-k} \varphi(x^{[\sum_{i=0}^{k-1} m^i]}, \ldots, x^{[\sum_{i=0}^{k-1} m^i]}).$$

From this inequality and condition (3.1) we conclude that the sequence

$$a_n(x) = m^{-n} f(x^{[\sum_{i=0}^{n-1} m^i]})$$

(a so called generalized Hyers-Ulam sequence) is a Cauchy sequence in X. In view of above conclusion, corroborated with the completeness of X, it follows that $\{a_n(x)\}_{n\in\mathbb{N}}$ is convergent. Let $T : S \to X$ be their limit

$$T(x) = \lim_{n\to\infty} m^{-n} f(x^{[\sum_{i=0}^{n-1} m^i]}). \tag{3.6}$$

Because $(x^{[1]})^{[\sum_{i=0}^{n-1} m^i]} = x^{[\sum_{i=0}^{n} m^i]}$, for all $x \in S$, we obtain

$$T(x^{[1]}) = \lim_{n\to\infty} m^{-n} f((x^{[1]})^{[\sum_{i=0}^{n-1} m^i]}) =$$
$$= m \lim_{n\to\infty} m^{-(n+1)} \underset{n\to\infty}{f} (x^{[\sum_{i=0}^{n} m^i]}) = mT(x).$$

By (3.1) and (3.6) and by letting $n \to \infty$ in inequality (3.5), we have

$$\|T(x) - f(x)\| \le \tilde{\varphi}(x, \dots, x)$$

for all $x \in S$.

We intend now to prove that T is uniquely defined. Let $T' : S \to X$ be another mapping with these properties. Then by (3.4) we conclude

$$\begin{aligned}
\left\|T(x) - T'(x)\right\| &= m^{-n} \left\|m^n T(x) - m^n T'(x)\right\| \\
&= m^{-n} \left\| T(x^{[\sum_{i=0}^{n-1} m^i]}) - T'(x^{[\sum_{i=0}^{n-1} m^i]}) \right\| \\
&\le m^{-n} \left\| T(x^{[\sum_{i=0}^{n-1} m^i]}) - f(x^{[\sum_{i=0}^{n-1} m^i]}) \right\| + \\
&+ m^{-n} \left\| f(x^{[\sum_{i=0}^{n-1} m^i]}) - T'(x^{[\sum_{i=0}^{n-1} m^i]}) \right\|
\end{aligned}$$

and by (3.3) we have

$$\left\|T(x) - T'(x)\right\| \le 2m^{-n} \tilde{\varphi}(x^{[\sum_{i=0}^{n-1} m^i]}, \dots, x^{[\sum_{i=0}^{n-1} m^i]}).$$

Taking the limit in this inequality as $n \to \infty$, we obtain $T(x) = T'(x)$ for all $x \in S$ and so T is unique.

In the sequel, we suppose that $(S, ()_\circ)$ is a normal $m-$semigroup, i.e.

$$(x_1, \dots, x_m)_\circ^{[k]} = \left(x_1^{[k]}, \dots, x_m^{[k]}\right)_\circ \tag{3.7}$$

for all $x_1, \dots, x_m \in S$ and all $k \in \mathbb{N}$. Replacing x_j by $x_j^{[\sum_{i=0}^{n-1} m^i]}$, $j = 1, \dots, m$ in inequality (3.2) we have

$$\left\| f\left([x_1^{[\sum_{i=0}^{n-1} m^i]}, \dots, x_m^{[\sum_{i=0}^{n-1} m^i]}]_\circ\right) - \sum_{j=1}^{m} f\left(x_j^{[\sum_{i=0}^{n-1} m^i]}\right) \right\|$$
$$\le \varphi(x_1^{[\sum_{i=0}^{n-1} m^i]}, \dots, x_m^{[\sum_{i=0}^{n-1} m^i]}).$$

Using the equality (3.7), if we divide both sides by m^n we obtain

$$\left\| m^{-n} f\left((x_1, \dots, x_m)_\circ^{[\sum_{i=0}^{n-1} m^i]}\right) - \sum_{j=1}^{m} m^{-n} f\left(x_j^{[\sum_{i=0}^{n-1} m^i]}\right) \right\|$$
$$\le m^{-n} \varphi(x_1^{[\sum_{i=0}^{n-1} m^i]}, \dots, x_m^{[\sum_{i=0}^{n-1} m^i]}).$$

In the last inequality, by letting $n \to \infty$, and by (3.1) and (3.6), we get

$$T((x_1,\dots,x_m)_\circ)=\sum_{j=1}^{m}T(x_j)$$

i.e. T is an $m-$ semigroup homomorphism. □

Therefore, we have proven that under suitable conditions on the function involved, "near" any solution of inequality (3.2) there exists a solution of equation $f((x_1^m)_\circ)=\sum_{j=1}^{m}f(x_j)$. The word "near" means that the distance of the solution of the equation from the solution of the inequality is explicitly evaluated through the function $\varphi: S^m\to[0,\infty)$.

As a consequence, if $\varphi(x_1,\dots,x_m)=$ constant, Theorem 1 gives a generalization of the well known Hyers result [13].

Corollary 1. *Let $(S,()_\circ)$ be an $m-$semigroup, $m\geq 2$, X be a Banach space and $\varepsilon>0$. Suppose that $f:S\to X$ is a mapping satisfying*

$$\left\|f((x_1,\dots,x_m)_\circ)-\sum_{j=1}^{m}f(x_j)\right\|<\varepsilon$$

for all $x_1,\dots,x_m\in S$.

Then there exists an unique mapping $T:S\to X$ such that

$$\|f(x)-T(x)\|<\frac{\varepsilon}{m-1}$$

and

$$T(x^{[1]})=mT(x),\forall\, x\in S.$$

Moreover, if S is a normal $m-$semigroup, then T is an $m-$semigroup homomorphism.

Proof. Because $m\geq 2$, the series $\sum_{n=0}^{\infty}\frac{\varepsilon}{m^{n+1}}$ is convergent with sum $\frac{\varepsilon}{m-1}$. Applying Theorem 1 there exists an unique mapping $T:S\to X$ such that

$$T(x)=\lim_{n\to\infty}m^{-n}f(x^{[m^{n-1}+\dots+m+1]})$$

with

$$T(x^{[1]})=mT(x);\ \forall\, x\in S$$

and

$$\|f(x)-T(x)\|<\frac{\varepsilon}{m-1}.$$

□

For various forms of the function $\varphi(x_1,\dots,x_m)$ we obtain other generalizations of some results on stability published in the last years.

First, we suppose that $f:S\to X$ is a mapping satisfying a condition weaker than Hyers's condition on approximate additive mappings in terms of a product of powers of norms.

Corollary 2. *Let $(S,()_\circ,\|\ \|_1)$ be a normed $m-$semigroup, $m\in\mathbb{N}, m\geq 2$, $(X,\|\ \|_2)$ be a Banach space and $\varepsilon>0$. Suppose that $k_1,\ldots,k_m\in\mathbb{R}$; $0\leq p=k_1+\ldots+k_m<1$ and $f:S\to X$ is a mapping satisfying the condition*

$$\left\| f((x_1,\ldots,x_m)_\circ)-\sum_{j=1}^{m} f(x_j)\right\|_2 < \varepsilon\|x_1\|_1^{k_1}\|x_2\|_1^{k_2}\ldots\|x_m\|_1^{k_m},$$
$$\forall x_1,\ldots,x_m\in S.$$

Then there is an unique mapping $T:S\to X$ such that

$$\|f(x)-T(x)\|_2<\frac{\varepsilon\|x\|_1^p}{m-m^p}$$

and

$$T(x^{[1]})=mT(x),\ \forall x\in S.$$

Moreover, if S is a normal $m-$semigroup, then T is an $m-$ary homomorphism.

Proof. Because $(S,()_\circ,\|\ \|_1)$ is a normed $m-$semigroup, we have for all $x\in S$,

$$\left\|x^{[1]}\right\|_1=||(\overset{(m)}{x})_\circ||_1\leq m||x||_1.$$

By induction, we have $||x^{[k]}||_1\leq((m-1)k+1)||x||_1$ for all $k\in\mathbb{N}$.
Hence

$$||x^{[m^{n-1}+\ldots+m+1]}||_1\leq m^n||x||_1,$$

for all $n\in\mathbb{N}$ and all $x\in S$.

Let $\varphi:S^m\to[0,\infty)$ be defined by $\varphi(x_1,\ldots,x_m)=\varepsilon||x_1||_1^{k_1}\ldots||x_m||_1^{k_m}$ where $0\leq p=k_1+\ldots+k_m<1$.

Because

$$\begin{aligned}\varphi((\overset{(m^n)}{x_1})_\circ,\ldots,(\overset{(m^n)}{x_m})_\circ)&=\varepsilon||(\overset{(m^n)}{x_1})_\circ||_1^{k_1}\ldots||(\overset{(m^n)}{x_m})_\circ||_1^{k_m}\\&\leq\varepsilon(m^n||x_1||_1)^{k_1}\ldots(m^n||x_m||_1)^{k_m}\\&=\varepsilon(m^n)^{k_1+\ldots+k_m}||x_1||_1^{k_1}\ldots||x_m||_1^{k_m}\\&=\varepsilon m^{np}||x_1||_1^{k_1}\ldots||x_m||_1^{k_m},\end{aligned}$$

the series

$$\frac{1}{m}\sum_{n=0}^{\infty}m^{-n}\varphi((\overset{(m^n)}{x_1})_\circ\ldots(\overset{(m^n)}{x_m})_\circ)\leq\frac{1}{m}\sum_{n=0}^{\infty}\varepsilon(m^{p-1})^n||x_1||_1^{k_1}\ldots||x_m||_1^{k_m}<\infty$$

is convergent with the sum

$$\widetilde{\varphi}(x_1,\ldots,x_m)\leq\frac{\varepsilon}{m-m^p}||x_1||_1^{k_1}\ldots||x_m||_1^{k_m}$$

and

$$\widetilde{\varphi}(x,\ldots,x) \leq \frac{\varepsilon||x||_1^p}{m-m^p}.$$

According to Theorem 1 there is an unique mapping $T : S \to X$ such that

$$\|f(x)-T(x)\|_2 \leq \frac{\varepsilon||x||_1^p}{m-m^p}$$

and

$$T(x^{[1]}) = mT(x), \text{ for all } x \in S.$$

If S is a normal normed $m-$semigroup, then T is an $m-$ary homomorphism. □

We remark that, in the special case where $m = 2$, Corollary 2 is Theorem 1.1 from J. M. Rassias [18].

Corollary 3. *Let $(S, ()_\circ, \| \|_1)$ be a normed $m-$semigroup $m \in \mathbb{N}$, $m \geq 2$, $(X, \| \|_2)$ be an $m-$ary Banach space and $\varepsilon > 0$. Suppose that $f : S \to X$ is a mapping satisfying the inequality*

$$\left\| f(x_1,x_2,\ldots,x_m)_\circ - \sum_{j=1}^m f(x_j) \right\|_2 < \varepsilon(||x_1||_1^p + \ldots + ||x_m||_1^p)$$

for $x_1,\ldots,x_m \in S$, where $0 \leq p < 1$.

Then there is an unique mapping $T : S \to X$ such that

$$\|f(x)-T(x)\|_2 \leq \frac{\varepsilon m||x||_1^p}{m-m^p}$$

and

$$T(x^{[1]}) = mT(x), \ \forall \ x \in S.$$

Moreover, if S is a normal $m-$semigroup, then T is an $m-$ary homomorphism.

Proof. We apply Theorem 1 to the mapping $\varphi : S^m \to X$,

$$\varphi(x_1,\ldots,x_m) = \varepsilon(||x_1||_1^p + \ldots + ||x_m||_1^p).$$

Because for $0 \leq p < 1$, the series $\sum_{n=0}^{\infty} m^{n(p-1)}$ converges to $\dfrac{m}{m-m^p}$ and so we obtain the result above. □

In particular, the case $m = 2$ gives the results of T. Aoki [3] and Th. M. Rassias [19].

4. SUPERSTABILITY

Let $(S,()_\circ)$ be an $m-$semigroup, X a Banach space and $f:S\to X$ be a mapping. Investigations concerning stability properties of the functional equation

$$f((x_1^m)_\circ)=\sum_{j=1}^{m}f(x_j)$$

proved that any "approximate" solution of this equation can be approximated by an exact solution, i.e., by an $m-$ary homomorphism if we regard $(X,\sum)$ as an $m-$semigroup derived from $(X,+)$. The equation $f((x_1^m)_\circ)=\prod\limits_{j=1}^{m}f(x_j)$ possesses a more surprising stability property, namely, any unbounded approximate solution must be an $m-$ary homomorphism. Such a phenomenon is called superstability (see L. Szekelyhidi [21], for $m=2$ and M. Amyari and M.S. Moslehian [1] for $m=3$). More exactly, suppose that we are given a functional equation $E(f)=0$, such that the notion of boundedness of f and $E(f)$ makes sense and furthermore, we assume that $E(f)$ is bounded whenever f is bounded. The functional equation $E(f)=0$ is stable if any function g approximately satisfying this equation is near to a true solution of this equation. This functional equation is superstable if the boundedness of $E(f)$ implies that either f is bounded or $E(f)=0$. Therefore the notion of superstability is "stronger" than the notion of stability.

In this section we will study the superstability of $m-$ary homomorphism, generalizing the results of Amyari and Moslehian. [1].

Theorem 2. *Let $(S,()_\circ)$ be an $m-$semigroup, A a normed algebra whose norm is multiplicative and $\varepsilon\geq 0$. If $f:S\to A$ satisfies the inequality*

$$\left\|f((x_1,\dots,x_m)_\circ)-\prod\nolimits_{j=1}^{m}f(x_j)\right\|\leq\varepsilon \tag{4.1}$$

for all $x_1,\dots,x_m\in S$, then there is $\delta_\varepsilon>1$ such that either

$$\|f(x)\|\leq\delta_\varepsilon, \text{ for all } x\in S, \tag{4.2}$$

or

$$f((x_1,\dots,x_m)_\circ)=\prod\nolimits_{j=1}^{m}f(x_j),\ \forall x_1,\dots,x_m\in S. \tag{4.3}$$

Proof. By inequality (4.1) we have $\left\|f(x^{[1]})-(f(x))^m\right\|\leq\varepsilon$.

We remark that the equation $\delta^m-\delta=\varepsilon$ has a unique solution $\delta_\varepsilon>1$. Suppose that there exists $a\in S$ such that $\|f(a)\|>\delta_\varepsilon$ i.e. $\|f(a)\|=\delta_\varepsilon+p$ for some $p>0$. Then

$$\begin{aligned}\left\|f(a^{[1]})\right\|&=\left\|(f(a))^m-((f(a))^m-f(a^{[1]}))\right\|\\&\geq\|f(a)\|^m-\left\|(f(a))^m-f(a^{[1]})\right\|\end{aligned}$$

$$\begin{aligned}&\geq (\delta_\varepsilon + p)^m - \varepsilon\\ &= \delta_\varepsilon^m + m\delta_\varepsilon^{m-1} p + \ldots + p^m - (\delta_\varepsilon^m - \delta_\varepsilon)\\ &> \delta_\varepsilon + mp.\end{aligned}$$

Assume by induction that

$$\left\| f(a^{[m^{n-1}+\ldots+m+1]}) \right\| > \delta_\varepsilon + m^n p.$$

Then

$$\begin{aligned}\left\| f(a^{[m^n+\ldots+m+1]}) \right\| &= \left\| f((a^{[m^{n-1}+\ldots+m+1]})^{[1]}) \right\|\\ &\geq \left\| f(a^{[m^{n-1}+\ldots+m+1]}) \right\|^m -\\ &- \left\| (f(a^{[m^{n-1}+\ldots+m+1]}))^m - f((a^{[m^{n-1}+\ldots+m+1]})^{[1]}) \right\|\\ &\geq (\delta_\varepsilon + m^n p)^m - \varepsilon > \delta_\varepsilon + m^{n+1} p.\end{aligned}$$

Therefore

$$\left\| f(a^{[m^{n-1}+\ldots+m+1]}) \right\| > \delta_\varepsilon + m^n p \tag{4.4}$$

holds for all positive integers n.

Because for every $x_1, x_2, \ldots, x_{2m-1} \in S$ according to the inequality (4.1) we have

$$\left\| f((x_1^m)_\circ x_{m+1}^{2m-1})_\circ) - f((x_1^m)_\circ) f(x_{m+1}) \ldots f(x_{2m-1}) \right\| \leq \varepsilon$$

and

$$\left\| f((x_1^{m-1}(x_m^{2m-1})_\circ)_\circ) - f(x_1) \ldots f(x_{m-1}) f((x_m^{2m-1})_\circ)_\circ \right\| \leq \varepsilon.$$

By associativity of $m-$ary operation $()_\circ$, we obtain

$$\left\| f((x_1^m)_\circ\ f(x_{m+1}) \ldots f(x_{2m-1}) - f(x_1) \ldots f(x_{m-1})_\circ f((x_m^{2m-1})_\circ) \right\| \leq 2\varepsilon.$$

Because the norm $\|\ \|$ is multiplicative, we have

$$\left\| f((x_1^m)_\circ)\ f(x_{m+1}) \ldots f(x_{2m-1}) - \prod_{j=1}^{2m-1} f(x_j) \right\|$$

$$\leq \left\| f((x_1^m)_\circ)\ f(x_{m+1}) \ldots f(x_{2m-1}) - f(x_1) \ldots f(x_{m-1}) f((x_m^{2m-1})_\circ) \right\| +$$

$$+ \left\| f(x_1) \ldots f(x_{m-1}) (f((x_m^{2m-1})_\circ) - f(x_m) \ldots f(x_{2m-1})) \right\|$$

$$\leq 2\varepsilon + \|f(x_1)\| \ldots \|f(x_{m-1})\| \varepsilon.$$

In particular, for $x_{m+1} = x_{m+2} = \ldots = x_{2m-1} = a^{[m^{n-1}+\ldots+m+1]}$ we obtain

$$\left\| f((x_1^m)_\circ) - \prod_{j=1}^m f(x_j) \right\| \cdot \left\| f(a^{[m^{n-1}+\ldots m+1]}) \right\|^{m-1}$$

$$\leq 2\varepsilon + \|f(x_1)\| \ldots \|f(x_{m-1})\| \varepsilon.$$

By (4.4)

$$\|f((x_1^m)_\circ) - f(x_1)\dots f(x_m)\| \le \frac{2\varepsilon + \|f(x_1)\|\dots\|f(x_{m-1})\|\varepsilon}{(\delta_\varepsilon + m^n p)^{m-1}}.$$

Letting $n \to \infty$ then

$$f(x_1^m)_\circ = f(x_1)\dots f(x_m)$$

for all $x_1, \dots, x_m \in S$. □

The Baker's example [4] extended to $m-$ary product shows that the crucial step in the proof above is the fact that the norm is multiplicative.
Given $\varepsilon > 0$ there is an unique solution δ_ε greater than 1 of the equation $|\delta^m - \delta| = \varepsilon$. If $(S = \mathbb{R}, (\,)_\circ)$ is the $m-$semigroup with $(x_1^m)_\circ = x_1 + \ldots + x_m$, $M_2(\mathbb{R})$ is the algebra of 2x2 matrices with real elements and $f : \mathbb{R} \to M_2(\mathbb{R})$, $f(x) = \begin{pmatrix} e^x & 0 \\ 0 & \delta_\varepsilon \end{pmatrix}$ then for the usual matrix norm we have

$$\|f(x_1 + \ldots + x_m) - f(x_1)\dots f(x_m)\| =$$
$$= \left\| \begin{pmatrix} 0 & 0 \\ 0 & \delta - \delta^m \end{pmatrix} \right\| = |\delta - \delta^m| = \varepsilon,$$

for all $x_1, \dots, x_n \in \mathbb{R}$. Hence $f(x_1 + \ldots + x_m) \neq f(x_1)...f(x_m)$ while f is unbounded.

Szekelyhidi [21] proved that the invariance of the vector space where the functions are defined is enough to ensure superstability. The notions and results of Theorem 2 can be generalized in the following:

Definition 4. *Let $(S, (\,)_\circ)$ be an $m-$semigroup and let V be the linear space of complex $\mathbb{C}-$valued function on S. The space V is called right (left) invariant if $\varphi : S \to \mathbb{C}$ belongs to V implies that the right (left) translation*

$$\varphi_{y_1^{m-1}} : S \to \mathbb{C};\ \varphi_{y_1^{m-1}}(x) = \varphi((x, y_1^{m-1})_\circ)$$

$$({}_{y_1^{m-1}}\varphi : S \to \mathbb{C};\ {}_{y_1^{m-1}}\varphi(x) = \varphi((y_1^{m-1}, x)_\circ))$$

belongs to V for every $y_1, y_2, \dots, y_{m-1} \in S$.

Theorem 3. *Let $(S, (\,)_\circ)$ be an $m-$semigroup, V a right invariant linear space of complex valued functions on S and $\varphi,\ f : S \to \mathbb{C}$ be nonzero functions for which the function $\psi_{y_1^{m-1}} : S \to \mathbb{C}$;*

$$\psi_{y_1^{m-1}}(x) = \varphi((x, y_1^{m-1})_\circ) - \varphi(x) f(y_1) \cdot \ldots \cdot f(y_{m-1}),$$

belongs to V for every $y_1, \dots, y_{m-1} \in S$. Then either $\varphi \in V$, or f is an $m-$ary homomorphism of $m-$semigroup $(S, (\,)_\circ)$ into $(\mathbb{C}, \Pi)-$the derived $m-$semigroup from the semigroup $(\mathbb{C}, \cdot)$.

Proof. Suppose that f is not an $m-$ary homomorphism of S into $\mathbb{C}$. Hence there are elements $x_2, x_3, \ldots, x_{m+1} \in S$ such that

$$f((x_2^{m+1})_\circ) \neq f(x_2) f(x_3) \ldots f(x_{m+1})$$

and therefore there is

$$[f((x_2^{m+1})_\circ) - f(x_2) \ldots f(x_{m+1})]^{-1} \overset{\text{denoted}}{=} a^{-1}.$$

Assume that $x_i \in S$ such that $b_i = f(x_i) \neq 0$; $i \in \{m+2, ..., 2m-1\}$.

Then, by associativity law of $m-$ary operation we have

$$\varphi(((x_1^m)_\circ x_{m+1}^{2m-1})_\circ) - \varphi((x_1^m)_\circ) f(x_{m+1}) \ldots f(x_{2m-1})$$

$$= [\varphi((x_1 (x_2^{m+1})_\circ x_{m+2}^{2m-1})_\circ) - \varphi(x_1) f((x_2^{m+1})_\circ) f(x_{m+2}) \ldots f(x_{2m-1})]$$

$$-[\varphi((x_1^m)_\circ) - \varphi(x_1) f(x_2) \ldots f(x_m)] f(x_{m+1}) \ldots f(x_{2m-1})$$

$$+\varphi(x_1)[f((x_2^{m+1})_\circ) - f(x_2) \ldots f(x_{m+1})] f(x_{m+2}) \ldots f(x_{2m-1})$$

and hence

$$\varphi(x_1) = \{[\varphi(((x_1^m)_\circ x_{m+1}^{2m-1})_\circ) - \varphi((x_1^m)_\circ) f(x_{m+1}) \ldots f(x_{2m-1})]$$

$$-[\varphi((x_1 (x_2^{m+1})_\circ x_{m+2}^{2m-1})_\circ) - \varphi(x_1) f((x_2^{m+1})_\circ) f(x_{m+2}) \ldots f(x_{2m-1})]$$

$$+[\varphi((x_1^m)_\circ) - \varphi(x_1) f(x_2) \ldots f(x_m)] f(x_{m+1}) \ldots f(x_{2m-1})\}$$

$$\cdot [f(x_2^{m+1}) - f(x_2) \ldots f(x_{m+1})]^{-1} [f(x_{m+2})]^{-1} \ldots [f(x_{2m-1})]^{-1}.$$

Therefore

$$\varphi(x_1) = [\psi_{x_{m+1}^{2m-1}}((x_1^m)_\circ) - \psi_{(x_2^{m+1})_\circ x_{m+2}^{2m-1}}(x_1)$$

$$+\psi_{x_2^m}(x_1) f(x_{m+1}) \ldots f(x_{2m-1})] a^{-1} b_{m+2}^{-1} b_{m+3}^{-1} \ldots b_{2m-1}^{-1}.$$

Since V is right-invariant linear space we conclude that the right-hand side of above equality as a function of x_1 belongs to V, then hence so does φ. □

As a consequence we obtain a natural generalization of Theorem 3.4 of Amyari and Moslehian [1]:

Corollary 4. *Let $(S, ()_\circ)$ be an $m-$semigroup and $\varphi, f : S \to \mathbb{C}$ be nonzero functions for which there exists a bounded function $\alpha : S^{m-1} \to [0, \infty)$ such that*

$$|\varphi((x_1^m)_\circ) - \varphi(x_1) f(x_2) \ldots f(x_m)| \leq \alpha(x_2, x_3, \ldots, x_m)$$

for all $x_1, x_2, \ldots, x_m \in S$.

Then either φ *is bounded or* f *is an* $m-$*ary homomorphism of* $(S, ()_\circ)$ *into* $(\mathbb{C}, \Pi)$.

Proof. Let V be the linear space of all bounded $\mathbb{C}-$valued function on S. The function $\psi_{x_2^m} : S \to \mathbb{C}$

$$\psi_{x_2^m}(x_1) = \varphi((x_1^m)_\circ) - \varphi(x_1) f(x_2) \ldots f(x_m)$$

belongs to V for each $x_2, x_3, \ldots, x_m$. Then we can apply Theorem 3 which implies that either φ is bounded or f is an $m-$ary homomorphism. □

In the special case where $f = \varphi : S \to \mathbb{C}$ and mapping $\alpha = \varepsilon$, $\varepsilon > 0$ we obtain a generalization of Corollary 3.5 from [1]:

Corollary 5. *Let* $(S, ()_\circ)$ *be an* $m-$*semigroup,* $\varepsilon > 0$ *and* $f : S \to \mathbb{C}$ *be a nonzero function such that*

$$\left| f((x_1^m)_\circ) - \prod_{j=1}^{m} f(x_j) \right| \leq \varepsilon$$

for all $x_1, x_2, \ldots, x_m \in S$.

Then either f *is bounded or* f *is an* $m-$*ary homomorphism.*

ACKNOWLEDGEMENT

We are thankful to the anonymous referees for theirs valuable suggestions that led to an improvement in the presentation of the paper.

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Doi: 10.18514/MMN.2015.1412

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 423–438

HU e-ISSN 1787-2413

REPRESENTATION OF SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH DELAY AND WITH LINEAR PARTS DEFINED BY PAIRWISE PERMUTABLE MATRICES

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Received 05 November, 2014

Abstract. This paper is devoted to the study of linear systems of neutral differential equations with delay. Assuming the linear parts to be given by pairwise permutable matrices, representation of a solution of a nonhomogeneous initial value problem using matrix polynomial of a degree depending on time is derived. Examples illustrating the obtained results are given.

2010 *Mathematics Subject Classification:* 34K40; 34K06; 39B12

Keywords: neutral differential equations, delayed equation, matrix polynomial

1. INTRODUCTION AND PRELIMINARIES

In [8], the authors considered the initial value problem for the system of delay differential equations of the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bx(t-\tau) + f(t), \quad t \geq 0, \\ &x(t) = \varphi(t), \quad -\tau \leq t \leq 0 \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, A, B are $n \times n$ matrices, $\tau > 0$ is a delay, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$ is a given initial function, and $f \in C([0, \infty), \mathbb{R}^n)$ is a given function. They proved the following result.

Theorem 1. *Let A, B be $n \times n$ permutable matrices, i.e. $AB = BA$, $f \in C([0, \infty), \mathbb{R}^n)$ and $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$ be given functions. Any solution of the initial value problem (1.1) has the form*

$$\begin{aligned} x(t) = \mathrm{e}^{A(t+\tau)} \mathrm{e}_\tau^{B_1 t} \varphi(-\tau) + \int_{-\tau}^{0} \mathrm{e}^{A(t-s)} \mathrm{e}_\tau^{B_1 (t-\tau-s)} (\varphi'(s) - A\varphi(s)) \mathrm{d}s \\ + \int_0^t \mathrm{e}^{A(t-s)} \mathrm{e}_\tau^{B_1 (t-\tau-s)} f(s) \mathrm{d}s, \end{aligned}$$

M. Pospíšil was supported by the Grant VEGA-SAV 2/0029/13.

L. Škripková was supported by the project CZ.1.07/2.3.00/30.0039 Excellent young researcher at Brno University of Technology.

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where

$$\mathrm{e}_\tau^{Bt} = \begin{cases} \Theta, & t < -\tau, \\ E, & -\tau \le t < 0, \\ E + Bt + B^2 \frac{(t-\tau)^2}{2} + \cdots + B^k \frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \le t < k\tau, k \in \mathbb{N}, \end{cases} \tag{1.2}$$

Θ and E are the zero and the identity matrix, respectively, and $B_1 = \mathrm{e}^{-A\tau}B$.

Here e_τ^{Bt} is called delayed matrix exponential. Note that it is a matrix polynomial of time-dependent degree.

Theorem 1 led to stability and controllability results for differential equations with one or multiple delays, existence results for boundary value problems, representation of solutions of oscillating systems, as well as results in the theory of difference, partial differential, and functional differential equations (see e.g. [3–5, 10, 13–15] and references therein).

Note that in [14], a result, analogical to Theorem 1 with $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, was derived for differential equations with variable delay. Here we recall the result for a constant delay.

Theorem 2. *Let A, B be $n \times n$ permutable matrices, i.e., $AB = BA$, $f \in C([0,\infty), \mathbb{R}^n)$ and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$ be given functions. Any solution of the initial value problem (1.1) has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \le t < 0, \\ \mathrm{e}^{At}\mathrm{e}_\tau^{B_1(t-\tau)}\varphi(0) + \displaystyle\int_{-\tau}^{0} \mathrm{e}^{A(t-s)}\mathrm{e}_\tau^{B_1(t-2\tau-s)} B_1\varphi(s)\mathrm{d}s & \\ \quad + \displaystyle\int_0^t \mathrm{e}^{A(t-s)}\mathrm{e}_\tau^{B_1(t-\tau-s)} f(s)\mathrm{d}s, & 0 \le t. \end{cases} \tag{1.3}$$

In the present paper, we derive representation of solutions of the initial value problem for neutral differential equation with a delay

$$\dot{x}(t) - C\dot{x}(t-\tau) = Ax(t) + Bx(t-\tau) + f(t), \quad t \ge 0, \tag{1.4}$$

$$x(t) = \varphi(t), \quad -\tau \le t \le 0 \tag{1.5}$$

for $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, on suppose that the matrices A, B, C are pairwise permutable, i.e., $AB = BA$, $BC = CB$, $CA = AC$. So at the end of Section 2, we obtain a result similar to Theorem 2 (see Theorem 6 below), which can be used when investigating stability, controllability, observability, boundary value problems, etc., for neutral differential equations. When investigating these problems one may find the papers [2,3,6,9–14] useful, where analogical problems for differential equations with delay are discussed.

In Section 3, we provide examples of systems covered by the discussed theory. We note that a family of permutable matrices can be constructed using e.g. [1].

Throughout the paper, we consider $\tau > 0$, $\mathbb{N}_0$ denotes the set of all nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and θ denotes the n-dimensional vector of zeros. Moreover, we assume the property of empty sum and empty product, i.e.,

$$\sum_{i\in\varnothing} f(i) = 0, \qquad \sum_{i\in\varnothing} F(i) = \Theta, \qquad \prod_{i\in\varnothing} f(i) = 1, \qquad \prod_{i\in\varnothing} F(i) = E$$

for any function f and matrix function F, whether they are defined or not for indicated argument. Finally, $\lfloor\cdot\rfloor$ denotes the floor function.

2. Main results

In this section, we derive a representation of solutions of the initial value problem (1.4), (1.5) using piecewise defined matrix polynomial.

First, we consider the matrix equation of the form

$$\dot{X}(t) - C\dot{X}(t-\tau) = BX(t-\tau) \tag{2.1}$$

for $t \geq 0$ with initial condition

$$X(t) = \begin{cases} \Theta, & t < 0, \\ E, & t = 0. \end{cases} \tag{2.2}$$

The solution of (2.1), (2.2) is called the fundamental matrix solution. In fact, this solution solves equation (2.1) on

$$M := \bigcup_{k\in\mathbb{N}_0} (k\tau, (k+1)\tau),$$

and the discontinuity of the initial condition is transfered to each $k\tau$, $k \in \mathbb{N}$ (see [7]). So, at the points $k\tau$, $k \in \mathbb{N}$ we consider right-sided derivatives. Equivalently to (2.1), X satisfies the corresponding integral equation

$$X(t) = E + CX(t-\tau) + B\int_0^t X(s-\tau)\mathrm{d}s, \quad t \geq 0 \tag{2.3}$$

with jumps at $k\tau$, $k \in \mathbb{N}$. In the following theorem, we state the solution $X(t)$.

Theorem 3. *Let B, C be $n \times n$ permutable matrices. Then the fundamental solution $X(t)$ of (2.1), (2.2) has the form*

$$X(t) = \begin{cases} \Theta, & t < 0, \\ \displaystyle\sum_{j=0}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!}, & t \in [k\tau, (k+1)\tau), k \in \mathbb{N}_0. \end{cases} \tag{2.4}$$

Proof. Let $t \in [0,\tau)$. Then by (2.4), $X(t) = E$, $X(t-\tau) = \dot{X}(t) = \dot{X}(t-\tau) = \Theta$. So X solves (2.1) on $[0,\tau)$.

Now, let $t \in [k\tau,(k+1)\tau)$ for some $k \in \mathbb{N}$. Then

$$X(t-\tau) = \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^j}{j!},$$

and differentiating (2.4), we get

$$\begin{aligned} \dot{X}(t) &= \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}, \\ \dot{X}(t-\tau) &= \sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^{j-1}}{(j-1)!}. \end{aligned} \tag{2.5}$$

Hence,

$$\begin{aligned} &BX(t-\tau) + C\dot{X}(t-\tau) \\ &= \underbrace{\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} C^i B^{j+1} \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^j}{j!}}_{=:S_1} \\ &\quad + \underbrace{\sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} C^{i+1} B^j \binom{i+j}{i} \frac{[t-(i+j+1)\tau]^{j-1}}{(j-1)!}}_{=:S_2}. \end{aligned}$$

Changing the variable $j+1 \to j$ in S_1, and $i+1 \to i$ in S_2 results in

$$\begin{aligned} &BX(t-\tau) + C\dot{X}(t-\tau) \\ &= \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!} \\ &\quad + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} C^i B^j \binom{i+j-1}{i-1} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}. \end{aligned} \tag{2.6}$$

Now, we write the first term as

$$\sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

$$= \sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} C^i B^j \binom{i+j-1}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

using the empty sum property. So we can join the last sum in the above equality and the last sum in (2.6), and apply the combinatorial identity

$$\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$$

for any $a, b \in \mathbb{N}$, $a \geq b$, to obtain

$$BX(t-\tau) + C\dot{X}(t-\tau)$$
$$= \sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} + \sum_{j=1}^{k-1}\sum_{i=1}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}. \tag{2.7}$$

Note that the first term on the right-hand side of the latter equality can be rewritten as

$$\sum_{j=1}^{k} B^j \frac{[t-j\tau]^{j-1}}{(j-1)!} = \sum_{j=1}^{k} C^0 B^j \binom{0+j}{0} \frac{[t-(0+j)\tau]^{j-1}}{(j-1)!}.$$

Consequently, the terms on the right-hand side of (2.7) can be summed (the last sum is extended to $j = 1, 2, \ldots k$ by the empty sum property), which results in

$$BX(t-\tau) + C\dot{X}(t-\tau) = \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}$$

where the right-hand side is exactly $\dot{X}(t)$ of (2.5). The statement is proved. □

Remark 1. Formula (2.4) was obtained using step-by-step integration method with the aid of integral equation (2.3).

Some properties of the fundamental solution at the points $k\tau$, $k \in \mathbb{N}_0$ are concluded in the next lemma.

Lemma 1. *Let X be defined by (2.4). For any $k \in \mathbb{N}_0$ the following holds true:*

(1) $\lim_{t \to k\tau^+} X(t) = \lim_{t \to k\tau^-} X(t) + C^k$,

(2) $\lim_{t \to k\tau^+} \dot{X}(t) = \lim_{t \to k\tau^-} \dot{X}(t) + kC^{k-1}B$.

Proof. If $k = 0$, the statements can be proved easily. For better clarity, we shall use the lower index $k \in \mathbb{N}_0$ to denote

$$X_k(t) := \sum_{j=0}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!}, \quad t \geq k\tau. \tag{2.8}$$

Note that using the empty sum property, this notation can be extended to negative integers as $X_k(t) = \Theta$ for $k < 0$, $t \in \mathbb{R}$.

Let $k \in \mathbb{N}$ be arbitrary and fixed. Then

$$\begin{aligned}\lim_{t\to k\tau^+} X(t) &= \lim_{t\to k\tau^+} X_k(t) = \lim_{t\to k\tau^+} \sum_{j=0}^{k-1}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^j}{j!} \\ &= \sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[k\tau-(i+j)\tau]^j}{j!} \\ &\quad + \lim_{t\to k\tau^+} \sum_{j=0}^{k-1} C^{k-j} B^j \binom{k}{k-j} \frac{[t-k\tau]^j}{j!} \\ &= X_{k-1}(k\tau) + C^k = \lim_{t\to k\tau^-} X(t) + C^k.\end{aligned}$$

To prove the second statement, we differentiate (2.8) to get

$$\dot{X}_k(t) = \sum_{j=1}^{k}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!}, \quad t \geq k\tau.$$

Consequently,

$$\begin{aligned}\lim_{t\to k\tau^+} \dot{X}(t) &= \lim_{t\to k\tau^+} \dot{X}_k(t) = \lim_{t\to k\tau^+} \sum_{j=1}^{k-1}\sum_{i=0}^{k-j} C^i B^j \binom{i+j}{i} \frac{[t-(i+j)\tau]^{j-1}}{(j-1)!} \\ &= \sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} C^i B^j \binom{i+j}{i} \frac{[k\tau-(i+j)\tau]^{j-1}}{(j-1)!} \\ &\quad + \lim_{t\to k\tau^+} \sum_{j=1}^{k-1} C^{k-j} B^j \binom{k}{k-j} \frac{[t-k\tau]^{j-1}}{(j-1)!} \\ &= \dot{X}_{k-1}(k\tau) + kC^{k-1}B = \lim_{t\to k\tau^-} \dot{X}(t) + kC^{k-1}B.\end{aligned}$$

The proof is complete. □

Now we find a solution of the homogeneous initial value problem.

Theorem 4. *Let B, C be $n \times n$ permutable matrices, and $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$ be a given function satisfying*

$$\dot{\varphi}(0) = B\varphi(-\tau) + C\dot{\varphi}(-\tau). \tag{2.9}$$

Any solution of the initial value problem consisting of the equation

$$\dot{x}(t) - C\dot{x}(t-\tau) = Bx(t-\tau), \quad t \geq 0 \tag{2.10}$$

and initial condition (1.5)*, has the form*

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ X(t)(\varphi(0)-C\varphi(-\tau))+B\displaystyle\int_{-\tau}^{0}X(t-\tau-s)\varphi(s)\mathrm{d}s & \\ \quad+C\displaystyle\int_{-\tau}^{0}\dot{X}(t-\tau-s)\varphi(s)\mathrm{d}s+\psi(t), & 0\le t\end{cases}\tag{2.11}$$

where $X(t)$ *is a fundamental matrix solution given by* (2.4)*,*

$$\psi(t):=C^{\lfloor\frac{t}{\tau}\rfloor+1}\varphi\left(t-\left(\left\lfloor\frac{t}{\tau}\right\rfloor+1\right)\tau\right).\tag{2.12}$$

Proof. Let $t\in(0,\tau)$. Then by (2.11),

$$x(t)=\varphi(0)-C\varphi(-\tau)+B\int_{-\tau}^{t-\tau}\varphi(s)\mathrm{d}s+C\varphi(t-\tau).$$

Hence, $\dot{x}(t)=B\varphi(t-\tau)+C\dot{\varphi}(t-\tau)$, $x(t-\tau)=\varphi(t-\tau)$ and $\dot{x}(t-\tau)=\dot{\varphi}(t-\tau)$. So it is easy to see that, x solves (2.10) on $(0,\tau)$. Moreover,

$$\lim_{t\to0^+}x(t)=\varphi(0)-C\varphi(-\tau)+C\varphi(-\tau)=\varphi(0)=\lim_{t\to0^-}x(t)$$

and

$$\lim_{t\to0^+}\dot{x}(t)=B\varphi(-\tau)+C\dot{\varphi}(-\tau)=\dot{\varphi}(0)=\lim_{t\to0^-}\dot{x}(t)$$

by (2.9). Thus x and $\dot{x}$ are both continuous at 0.

Now, let $t\in(k\tau,(k+1)\tau)$ for some $k\in\mathbb{N}$. Using the notation (2.8) and definition (2.12) of function ψ, we can write

$$\begin{aligned}x(t)&=X_k(t)(\varphi(0)-C\varphi(-\tau))\\&\quad+\int_{-\tau}^{t-(k+1)\tau}\left(BX_k(t-\tau-s)+C\dot{X}_k(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\&\quad+\int_{t-(k+1)\tau}^{0}\left(BX_{k-1}(t-\tau-s)+C\dot{X}_{k-1}(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\&\quad+C^{k+1}\varphi(t-(k+1)\tau).\end{aligned}\tag{2.13}$$

Hence,

$$\begin{aligned}x(t-\tau)&=X_{k-1}(t-\tau)(\varphi(0)-C\varphi(-\tau))\\&\quad+\int_{-\tau}^{t-(k+1)\tau}\left(BX_{k-1}(t-2\tau-s)+C\dot{X}_{k-1}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\&\quad+\int_{t-(k+1)\tau}^{0}\left(BX_{k-2}(t-2\tau-s)+C\dot{X}_{k-2}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\&\quad+C^{k}\varphi(t-(k+1)\tau).\end{aligned}\tag{2.14}$$

Differentiating (2.13) and applying Lemma 1 leads to

$$\begin{aligned}\dot{x}(t) &= \dot{X}_k(t)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{t-(k+1)\tau}\left(B\dot{X}_k(t-\tau-s)+C\ddot{X}_k(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+\int_{t-(k+1)\tau}^{0}\left(B\dot{X}_{k-1}(t-\tau-s)+C\ddot{X}_{k-1}(t-\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+(k+1)C^kB\varphi(t-(k+1)\tau)+C^{k+1}\dot{\varphi}(t-(k+1)\tau).\end{aligned}\tag{2.15}$$

On the other side, differentiating (2.14) and applying Lemma 1 gives

$$\begin{aligned}\dot{x}(t-\tau) &= \dot{X}_{k-1}(t-\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{t-(k+1)\tau}\left(B\dot{X}_{k-1}(t-2\tau-s)+C\ddot{X}_{k-1}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+\int_{t-(k+1)\tau}^{0}\left(B\dot{X}_{k-2}(t-2\tau-s)+C\ddot{X}_{k-2}(t-2\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+kC^{k-1}B\varphi(t-(k+1)\tau)+C^k\dot{\varphi}(t-(k+1)\tau).\end{aligned}\tag{2.16}$$

Note that by Theorem 3,

$$\dot{X}_k(t)=BX_{k-1}(t-\tau)+C\dot{X}_{k-1}(t-\tau),\quad t\in[k\tau,(k+1)\tau)$$

for any $k\in\mathbb{N}_0$. Clearly, it is also valid for $k=-1$ provided that $X_{-1}(t)=\Theta=X_{-2}(t)$ for any $t\in\mathbb{R}$.

Now we can use formulas (2.14), (2.15) and (2.16) to directly derive equation (2.10) on $(k\tau,(k+1)\tau)$.

Using (2.13) and Lemma 1 we get

$$\begin{aligned}\lim_{t\to k\tau^+}x(t) &= X_k(k\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{0}\left(BX_{k-1}((k-1)\tau-s)+C\dot{X}_{k-1}((k-1)\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+C^{k+1}\varphi(-\tau)=X_{k-1}(k\tau)(\varphi(0)-C\varphi(-\tau))\\ &+\int_{-\tau}^{0}\left(BX_{k-1}((k-1)\tau-s)+C\dot{X}_{k-1}((k-1)\tau-s)\right)\varphi(s)\mathrm{d}s\\ &+C^k\varphi(0)=\lim_{t\to k\tau^-}x(t)\end{aligned}$$

where the last equality follows again, by (2.13) with $k-1$ instead of k. Finally, the continuity of $\dot{x}$ at the point $k\tau$, $k\in\mathbb{N}$ is a consequence of continuity of x and $\dot{x}$ at $(k-1)\tau$ due to the form of (2.10), e.g. for $k=1$ it holds

$$\lim_{t\to\tau^+}\dot{x}(t)=\lim_{t\to\tau^+}Bx(t-\tau)+C\dot{x}(t-\tau)=\lim_{t\to 0^+}Bx(t)+C\dot{x}(t)$$

$$= \lim_{t\to 0^-} Bx(t) + C\dot{x}(t) = \lim_{t\to\tau^-} \dot{x}(t).$$

This completes the proof. □

Remark 2. Condition (2.9) is necessary and sufficient for the solution to have continuous derivative at the points $k\tau, k \in \mathbb{N}_0$ (cf. [7]). Later, we will have analogical conditions (see (2.17), (2.22)) for the case of nonhomogeneous equation without and with a linear non-delayed term.

In the rest of the paper, we denote $x_h(t)$ solution $x(t)$ of (2.10), (1.5), given by (2.11) to emphasize the fact that, it is a solution of a homogeneous equation.

The following theorem provides a result on a solution of a nonhomogeneous neutral differential equation with a given initial condition.

Theorem 5. *Let B, C be $n\times n$ permutable matrices, $f \in C([0,\infty),\mathbb{R}^n)$ and $\varphi \in C^1([-\tau,0],\mathbb{R}^n)$ be given functions satisfying*

$$\dot{\varphi}(0) = B\varphi(-\tau) + C\dot{\varphi}(-\tau) + f(0). \tag{2.17}$$

Any solution of the initial value problem consisting of the equation

$$\dot{x}(t) - C\dot{x}(t-\tau) = Bx(t-\tau) + f(t), \quad t \geq 0 \tag{2.18}$$

and initial condition (1.5), has the form

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ X(t)(\varphi(0) - C\varphi(-\tau)) + B\displaystyle\int_{-\tau}^{0} X(t-\tau-s)\varphi(s)\mathrm{d}s \\ + C\displaystyle\int_{-\tau}^{0} \dot{X}(t-\tau-s)\varphi(s)\mathrm{d}s + \psi(t) + \int_0^t X(t-s)f(s)\mathrm{d}s, \\ & 0 \leq t \end{cases} \tag{2.19}$$

where $X(t)$ is a fundamental matrix solution given by (2.4), and $\psi(t)$ is given by (2.12).

Proof. Note that $x(t) = x_h(t) + \int_0^t X(t-s)f(s)\mathrm{d}s$ where $x_h(t)$ is a solution of homogeneous equation (2.10). Let $t \in (0,\tau)$. Then

$$x(t) = \varphi(0) - C\varphi(-\tau) + B\int_{-\tau}^{t-\tau} \varphi(s)\mathrm{d}s + C\varphi(t-\tau) + \int_0^t f(s)\mathrm{d}s,$$

$x(t-\tau) = \varphi(t-\tau)$, $\dot{x}(t) = B\varphi(t-\tau) + C\dot{\varphi}(t-\tau) + f(t)$. Hence (2.18) is verified. Moreover,

$$\lim_{t\to 0^+} x(t) = \lim_{t\to 0^+} x_h(t) = \varphi(0) = \lim_{t\to 0^-} x(t)$$

and

$$\lim_{t\to 0^+} \dot{x}(t) = B\varphi(-\tau) + C\dot{\varphi}(-\tau) + f(0) = \dot{\varphi}(0) = \lim_{t\to 0^-} \dot{x}(t)$$

by (2.17).

Next, let $t \in (k\tau,(k+1)\tau)$ for some $k \in \mathbb{N}$. Since $X(t)$ of (2.4) is piecewise defined, we have to split the integral $\int_0^t X(t-s)f(s)\mathrm{d}s$ to be able to differentiate it later. So we write

$$x(t) = x_h(t) + \int_0^{t-k\tau} X_k(t-s)f(s)\mathrm{d}s + \sum_{j=0}^{k-1}\int_{t-(j+1)\tau}^{t-j\tau} X_j(t-s)f(s)\mathrm{d}s.$$

Similarly, we have

$$\begin{aligned} x(t-\tau) = x_h(t-\tau) &+ \int_0^{t-k\tau} X_{k-1}(t-\tau-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-2}\int_{t-(j+2)\tau}^{t-(j+1)\tau} X_j(t-\tau-s)f(s)\mathrm{d}s. \end{aligned}$$

When computing the derivatives, we apply Lemma 1 to obtain

$$\begin{aligned} \dot{x}(t) &= \dot{x}_h(t) + \int_0^{t-k\tau} \dot{X}_k(t-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-1}\int_{t-(j+1)\tau}^{t-j\tau} \dot{X}_j(t-s)f(s)\mathrm{d}s + X_k(k\tau)f(t-k\tau) \\ &+ \sum_{j=0}^{k-1}\Big(X_j(j\tau)f(t-j\tau) - X_j((j+1)\tau)f(t-(j+1)\tau)\Big) \\ &= \dot{x}_h(t) + \int_0^{t-k\tau} \dot{X}_k(t-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-1}\int_{t-(j+1)\tau}^{t-j\tau} \dot{X}_j(t-s)f(s)\mathrm{d}s + \sum_{j=1}^{k} C^j f(t-j\tau) + f(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x}(t-\tau) &= \dot{x}_h(t-\tau) + \int_0^{t-k\tau} \dot{X}_{k-1}(t-\tau-s)f(s)\mathrm{d}s \\ &+ \sum_{j=0}^{k-2}\int_{t-(j+2)\tau}^{t-(j+1)\tau} \dot{X}_j(t-\tau-s)f(s)\mathrm{d}s + X_{k-1}((k-1)\tau)f(t-k\tau) \\ &+ \sum_{j=0}^{k-2}\Big(X_j(j\tau)f(t-(j+1)\tau) - X_j((j+1)\tau)f(t-(j+2)\tau)\Big) \\ &= \dot{x}_h(t-\tau) + \int_0^{t-k\tau} \dot{X}_{k-1}(t-\tau-s)f(s)\mathrm{d}s \end{aligned}$$

$$+\sum_{j=0}^{k-2}\int_{t-(j+2)\tau}^{t-(j+1)\tau}\dot{X}_j(t-\tau-s)f(s)\mathrm{d}s+\sum_{j=1}^{k-1}C^j f(t-(j+1)\tau)+f(t-\tau).$$

Using the above formulas, one can easily verify equation (2.10) on $(k\tau,(k+1)\tau)$.

To prove the continuity at $k\tau$, we use the continuity of $x_h(t)$,

$$\lim_{t\to k\tau^+}x(t)=\lim_{t\to k\tau^+}x_h(t)+\sum_{j=0}^{k-1}\int_{(k-j-1)\tau}^{(k-j)\tau}X_j(k\tau-s)f(s)\mathrm{d}s$$

$$=\lim_{t\to k\tau^-}x_h(t)+\sum_{j=0}^{k-1}\int_{(k-j-1)\tau}^{(k-j)\tau}X_j(k\tau-s)f(s)\mathrm{d}s=\lim_{t\to k\tau^-}x(t).$$

Continuity of $\dot{x}$ at $k\tau$, $k\in\mathbb{N}$ can be proved inductively as in the proof of Theorem 4. □

Remark 3. Putting $C=\Theta$ in the latter theorem states that,

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ \mathrm{e}_\tau^{B(t-\tau)}\varphi(0)+B\displaystyle\int_{-\tau}^{0}\mathrm{e}_\tau^{B(t-2\tau-s)}\varphi(s)\mathrm{d}s & \\ \quad+\displaystyle\int_0^t\mathrm{e}_\tau^{B(t-\tau-s)}f(s)\mathrm{d}s, & 0\le t\end{cases}$$

for e_τ^{Bt} defined by (1.2), is a solution of equation

$$\dot{x}(t)=Bx(t-\tau)+f(t),\quad t\ge 0 \tag{2.20}$$

along with initial condition (1.5) assuming that

$$\dot{\varphi}(0)=B\varphi(-\tau)+f(0). \tag{2.21}$$

As $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$, one can make per partes leading to

$$x(t)=\mathrm{e}_\tau^{Bt}\varphi(-\tau)+\int_{-\tau}^{0}\mathrm{e}_\tau^{B(t-\tau-s)}\dot{\varphi}(s)\mathrm{d}s+\int_0^t\mathrm{e}_\tau^{B(t-\tau-s)}f(s)\mathrm{d}s$$

for any $t\ge 0$. Note that for $t\in[-\tau,0)$ this formula becomes $x(t)=\varphi(t)$. This corresponds to a solution of (2.20), (1.5) from [9, Theorems 1, 2]. The assumption (2.21) means the continuity of $\dot{x}(t)$ at 0.

Finally, we give a result on a representation of a solution of initial value problem (1.4), (1.5).

Theorem 6. *Let A, B, C be $n\times n$ pairwise permutable matrices, $f\in C([0,\infty),\mathbb{R}^n)$ and $\varphi\in C^1([-\tau,0],\mathbb{R}^n)$ be given functions satisfying*

$$\dot{\varphi}(0)=A\varphi(0)+B\varphi(-\tau)+C\dot{\varphi}(-\tau)+f(0). \tag{2.22}$$

Any solution of the initial value problem (1.4), (1.5) *has the form*

$$x(t)=\begin{cases}\varphi(t), & -\tau\le t<0,\\ \mathrm{e}^{At}Y(t)(\varphi(0)-C\varphi(-\tau))+\widetilde{B}\displaystyle\int_{-\tau}^{0}\mathrm{e}^{A(t-s)}Y(t-\tau-s)\varphi(s)\mathrm{d}s & \\ +\widetilde{C}\displaystyle\int_{-\tau}^{0}\mathrm{e}^{A(t-s)}\dot{Y}(t-\tau-s)\varphi(s)\mathrm{d}s+\psi(t) & \\ +\displaystyle\int_{0}^{t}\mathrm{e}^{A(t-s)}Y(t-s)f(s)\mathrm{d}s, & 0\le t\end{cases}\tag{2.23}$$

where $Y(t)$ *is given by*

$$Y(t)=\begin{cases}\Theta, & t<0,\\ \displaystyle\sum_{j=0}^{k}\sum_{i=0}^{k-j}\widetilde{C}^{i}\widetilde{B}^{j}\binom{i+j}{i}\frac{[t-(i+j)\tau]^{j}}{j!}, & t\in[k\tau,(k+1)\tau),k\in\mathbb{N}_0,\end{cases}\tag{2.24}$$

$\widetilde{B}=(B+AC)\mathrm{e}^{-A\tau}$, $\widetilde{C}=C\mathrm{e}^{-A\tau}$, *and* ψ *is given by* (2.12).

Proof. Let us introduce the substitution $x(t)=\mathrm{e}^{At}y(t)$. Then $y(t)$ fulfills

$$\dot{y}(t)-\widetilde{C}\dot{y}(t-\tau)=\widetilde{B}y(t-\tau)+\widetilde{f}(t),\quad t\ge 0,$$
$$y(t)=\widetilde{\varphi}(t),\quad -\tau\le t\le 0$$

with $\widetilde{f}(t)=\mathrm{e}^{-At}f(t)$, $\widetilde{\varphi}(t)=\mathrm{e}^{-At}\varphi(t)$. By application of Theorem 5, we obtain the solution

$$y(t)=\begin{cases}\widetilde{\varphi}(t), & -\tau\le t<0,\\ Y(t)(\widetilde{\varphi}(0)-\widetilde{C}\widetilde{\varphi}(-\tau))+\widetilde{B}\displaystyle\int_{-\tau}^{0}Y(t-\tau-s)\widetilde{\varphi}(s)\mathrm{d}s & \\ +\widetilde{C}\displaystyle\int_{-\tau}^{0}\dot{Y}(t-\tau-s)\widetilde{\varphi}(s)\mathrm{d}s+\widetilde{\psi}(t) & \\ +\displaystyle\int_{0}^{t}Y(t-s)\widetilde{f}(s)\mathrm{d}s, & 0\le t\end{cases}$$

where

$$\widetilde{\psi}(t):=\widetilde{C}^{\lfloor\frac{t}{\tau}\rfloor+1}\widetilde{\varphi}\left(t-\left(\left\lfloor\frac{t}{\tau}\right\rfloor+1\right)\tau\right).$$

Note that to get a C^1–solution we need to change condition (2.17) to (2.22). It is easy to see that $\widetilde{\varphi}(0)=\varphi(0)$, $\widetilde{C}\widetilde{\varphi}(-\tau)=C\varphi(-\tau)$ and $\widetilde{\psi}(t)=\mathrm{e}^{-At}\psi(t)$. Therefore, $\mathrm{e}^{At}y(t)$ gives the solution $x(t)$ of (2.23). □

Remark 4. Let us put $C = \Theta$ in Theorem 6. Then $\widetilde{C} = \Theta$, $\widetilde{B} = B_1 = \mathrm{e}^{-A\tau}B$ and

$$Y(t) = \begin{cases} \Theta, & t < 0, \\ \displaystyle\sum_{j=0}^{k} B_1^j \frac{(t-j\tau)^j}{j!}, & t \in [k\tau,(k+1)\tau), k \in \mathbb{N}_0. \end{cases}$$

Thus it is easy to see that $Y(t) = \mathrm{e}_\tau^{B_1(t-\tau)}$ for e_τ^{Bt} defined by (1.2). Moreover, $\psi(t) = \theta$ and the solution $x(t)$ given by (2.23) coincides with the one stated by (1.3) in Theorem 2, or, after per partes, with the one from Theorem 1. Note that condition (2.22), in this case

$$\dot{\varphi}(0) = A\varphi(0) + B\varphi(-\tau) + f(0),$$

implies the continuity of $\dot{x}(t)$ at 0, and the solution gets smoother at $k\tau$ for each $k \in \mathbb{N}$ as was proved in [14].

3. Examples

In this section, we provide two examples illustrating the proved results. On the first one, the behavior of solutions of a scalar equation can be seen from a numerically computed graph. The second one provides an example of permutable matrices of a higher dimension.

Example 1. Let us consider the scalar neutral differential equation with a delay

$$\begin{aligned} \dot{x}(t) - \dot{x}(t-1) &= ax(t) + 2x(t-1) + 2\cos t, \quad t \geq 0 \\ x(t) &= t, \quad -1 \leq t \leq 0 \end{aligned} \tag{3.1}$$

for $a \in \mathbb{R}$.

One does not have to verify the condition of permutability of matrices, since $n = 1$. Moreover, condition (2.22) is satisfied for any $a \in \mathbb{R}$. So the solution can be immediately obtained by Theorem 6. Figure 1 depicts different solutions corresponding to particular values of parameter a.

Example 2. Let us consider the system

$$\begin{aligned} \dot{x}(t) - \dot{x}(t-\pi) + 0.2\dot{y}(t-\pi) &= -0.4x(t) - 0.02y(t) - 0.02z(t) \\ &\quad - 0.5x(t-\pi) + 0.1y(t-\pi) \\ &\quad + 0.1z(t-\pi) + f_1(t) \\ \dot{y}(t) - \dot{y}(t-\pi) &= -0.4y(t) - 0.5y(t-\pi) + f_2(t) \\ \dot{z}(t) - \dot{z}(t-\pi) &= -0.02y(t) - 0.5z(t) + 0.1y(t-\pi) + f_3(t) \end{aligned}$$

for $t \geq 0$ along with initial condition

$$(x(t), y(t), z(t)) = (\sin(t), \sin(2t), \cos(t)), \quad -\pi \leq t \leq 0,$$

where $f \in C([0,\infty), \mathbb{R}^3)$ is a given function such that $f(0) = (2.52, 0, 0.5)^T$.

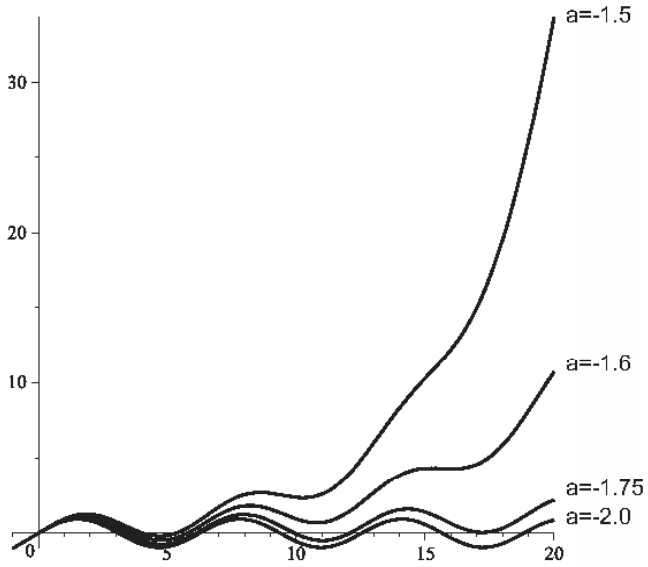

FIGURE 1. Solutions of (3.1) corresponding to different values of parameter a.

The latter example can be considered as a system modeling the sizes of changes in three given populations. So, $x > 0$ ($x < 0$) means that the first population is growing (dying). Here it is assumed by the matrix

$$C = \begin{pmatrix} 1 & -0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

that the speed of changes in each population is similar to the one before time π. The number -0.2 above the diagonal means that the speed of growth of the second population affects negatively the speed of growth of the first one. Matrices

$$A = \begin{pmatrix} -0.4 & -0.02 & -0.02 \\ 0 & -0.4 & 0 \\ 0 & -0.02 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 & 0.1 & 0.1 \\ 0 & -0.5 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}$$

reflect the effort of populations before the time π and now to stabilize the growth or diminution of the populations, i.e., to prevent hardly predictable changes in the number of members of each population; but also influences coming from changes in other populations.

Now, it is easy to verify that all the assumptions of Theorem 6 are satisfied, and to obtain the solution.

Remark 5. The space of matrices commuting with the matrix C of (3.2) has dimension 5, and these have the form

$$\Xi = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix}$$

with real parameters a, b, c, d, e, as can be derived from [1].

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Doi: 10.18514/MMN.2015.1725

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 439–441

HU e-ISSN 1787-2413

THE SIXTH INTERNATIONAL WORKSHOP – 2015 "CONSTRUCTIVE METHODS FOR NON-LINEAR BOUNDARY VALUE PROBLEMS"

MIKLÓS RONTÓ

Received 27 July, 2015

In the period from 9th July to 12th July, 2015, the Sixth International Workshop – 2015 "Constructive Methods for Non-Linear Boundary-Value Problems" had taken place in Miskolc, Hungary. The list of participants and the program of the meeting can be found on the workshop web site:

http://mat76.mat.uni-miskolc.hu/bvpworkshop/about

The workshop was organized by the Institute of Mathematics of the University of Miskolc and Institute of Mathematics of the Academy of Sciences of Czech Republic in cooperation with the Regional Committee of the Hungarian Academy of Sciences in Miskolc and Junior Mathematical Society in Miskolc.

The scope of the conference covered various topics related to the modern theory of boundary value problems in a broad sense, including initial value and periodic problems for functional differential equations, differential variational inequalities, certain singular problems, the method of lower and upper functions, stability analysis, limit cycles, problems with state-dependent impulses, sign-constant and oscillatory solutions of boundary value problems, certain topics of theory of integral, viability and invariance in set-valued dynamics, differential-algebraic systems, vector valued metric approach in the theory of differential equations and inclusions, semilinear hyperbolic functional differential equations, boundary value problems with nonlocal conditions, asymptotic decompositions, multivalued operators and related fixed point theorems, numerical-analytic methods for boundary value problems, discontinuous differential equations with delayed or advanced arguments, equations with p-Laplacian, and Navier–Stokes equations.

The workshop was attended by more than 50 researchers from China, Chile, Czech Republic, Hungary, Latvia, Poland, Romania, Russia, Slovak Republic, and Ukraine, who held 20 and 30 minute lectures.

The Organizing and Program Committee of the workshop consisting of V. Berinde, A. Cabada, J. Diblík, A. Domoshnitsky, J. Džurina, M. Fečkan, R. Hakl, P. Körtesi, V. Marinec, I. Piller, M. Perestyuk, A. Petruşel, I. Rachůnková, A. Rontó, M. Růžičková, A. Samoilenko, F. Sadyrbaev, S. Staněk, J. Szigeti, M. Tvrdý and the author of

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this note (chairman) gratefully acknowledges the financial support provided by the following funds and insitutions:

- Institute of Mathematics of the University of Miskolc
- Faculty of Mechanical Engineering and Informatics of the University of Miskolc
- Tudomány Támogatásáért Észak-Magyarországon Alapítvány.

This meeting was a continuation of the five previous workshops held in Miskolc in 2000 and 2003, in Sárospatak in 2006 [2], in Eger in 2009 [3], and in Tokaj in 2012 [1].

We hope that the fruitful tradition of this kind of "compact" international meetings oriented mostly at the exchange of ideas of researchers working in closely related fields will continue in the future.

The participants of the workshop in front of the university main building

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Doi: 10.18514/MMN.2015.1280

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 443–452

HU e-ISSN 1787-2413

TORSION UNITS OF INTEGRAL GROUP RING OF THE SIMPLE GROUP $S_4(4)$

ANTONIO L. ROSA

Received 12 June, 2014

Abstract. We consider the Zassenhaus conjecture for the normalized unit group of the integral group ring of the sympletic simple group $S_4(4)$. As a consequence, we confirm for this group the prime graph conjecture.

2010 *Mathematics Subject Classification:* 16S34; 20C05; 20D08

Keywords: Zassenhaus conjecture, torsion unit, partial augmentation, integral group ring

1. Introduction, conjectures and main results

In the integral group ring $\mathbb{Z}G$, where G is a finite group, we consider the group of normalized units

$$V(\mathbb{Z}G) = \left\{u = \sum_{g\in G} \alpha_g g \in U(\mathbb{Z}G) \mid \varepsilon(u) = \sum_{g\in G} \alpha_g = 1\right\}.$$

A conjecture of Zassenhaus **(ZC)** states that:

> **(ZC)** Every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate in the rational group algebra $\mathbb{Q}G$ to an element $g \in G$. That is, for every $u \in V(\mathbb{Z}G)$, there exists $v \in \mathbb{Q}G$ such that $vuv^{-1} = g \in G$.

The main tool in the investigation of the Zassenhaus conjecture **(ZC)** for finite groups with small numbers of normal subgroups is the Luthar-Passi method, which appeared in [24] to show **(ZC)** in the case where $G = A_5$. M. Hertweck (see [19,20]) applied the theory of Brauer characters to the Luthar-Passi method and used it to study the conjecture of Zassenhaus **(ZC)** for $G = PSL(2, p^n)$.

Now the Luthar-Passi method is very useful to study of **(ZC)** for arbitrary groups. As recent results, we cite [4, 18, 21–23, 25–27]. Now, we need to introduce some notation. For $P_d(H)$ denote the set of all prime divisors of the orders of elements of torsion part $t(H)$ of a group H. The prime graph (or Gruenberg-Kegel graph) of H, denoted by $\pi(H)$, is the graph with vertices as the primes in $P_d(H)$ and we connect of p to q if there is an element of order pq in the torsion part $t(H)$. In [23] was proposed the following conjecture,

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(PGC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

Clearly, this conjecture is a weaker version of the conjecture of Zassenhaus **(ZC)**. In [23], W. Kimerle showed that **(PGC)** holds for finite Frobenius groups and solvable groups. V. Bovdi and M. Hertweck in [3] completed the p-version of the Zassenhaus conjecture for Frobenius groups. In [2,5–15] the conjecture **(PGC)** was studied for several simple sporadic groups.

Here, we are investigating the conjecture **(ZC)** for the symplectic simple group $S_4(4)$. Using the Luthar-Passi method, we can not prove the rational conjugacy **(ZC)** for all torsion units in $V(\mathbb{Z}S_4(4))$, but for units of order $\{3,5,17\}$ we are able to prove **(ZC)**. Also our main result gives a lot of information about the partial augmentations of some units. Finally, as a consequence we obtain that **(PGC)** is valid for this group.

Let $G = S_4(4)$. It is well known (see [17]) that $|G| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ and $\exp(G) = 2^2 \cdot 3 \cdot 5 \cdot 17$. Let

$$\begin{aligned}\mathcal{C} = \{&C_1, C_{2a}, C_{2b}, C_{2c}, C_{3a}, C_{3b}, C_{4a}, C_{4b}, C_{5a}, C_{5b}, C_{5c}, C_{5d}, C_{5e}, C_{6a}, C_{6b},\\ &C_{10a}, C_{10b}, C_{10c}, C_{10d}, C_{15a}, C_{15b}, C_{15c}, C_{15d}, C_{17a}, C_{17b}, C_{17c}, C_{17d}\}\end{aligned}$$

be the collection of all conjugacy classes of $S_4(4)$, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by

$$\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$$

the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [1]) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1.1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following

Theorem 1. *Let G denote the symplectic group $S_4(4)$. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the tuple*

$$\begin{aligned}(&\nu_{2a}, \nu_{2b}, \nu_{2c}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d}, \nu_{5e}, \nu_{6a}, \nu_{6b}, \nu_{10a},\\ &\nu_{10b}, \nu_{10c}, \nu_{10d}, \nu_{15a}, \nu_{15b}, \nu_{15c}, \nu_{15d}, \nu_{17a}, \nu_{17b}, \nu_{17c}, \nu_{17d}) \in \mathbb{Z}^{26}\end{aligned}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. The following properties hold.

(i) *There is no elements of orders 34, 51, 85 in $V(\mathbb{Z}G)$.*

(ii) *If $|u| \in \{3,5,17\}$, then u is rationally conjugate to some $g \in G$.*

(iii) *If $|u| = 2$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{2b} and ν_{2c}, and the tuple $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ is one of*

$$\{(1,-3,3), (2,-2,1), (1,-2,2), (0,-2,3), (3,-1,-1), (2,-1,0),$$

$$(1,-1,1),(0,-1,2),(-1,-1,3),(2,0,-1),(1,0,0),(0,0,1),$$
$$(-1,0,2),(-2,0,3),(1,1,-1),(0,1,0),(-1,1,1),(-2,1,2),$$
$$(-3,1,3),(0,2,-1),(-1,2,0),(-2,2,1),(-1,3,-1)\}.$$

As an immediate consequence of part (i) of the Theorem we obtain

Corollary 1. *If $G = S_4(4)$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. Preliminaries

The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1 (see [24]). *Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

This fact shows that several partial augmentations are zero.

Proposition 2 (see [19], Proposition 3.1; [20], Proposition 2.2). *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3 (see [20,24]). *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G, then for every integer l the number*

$$\mu_l(u,\chi,p) = \frac{1}{k}\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\} \tag{2.1}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u,\chi,*)$ for $\mu_l(u,\chi,0)$.
Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4 (see [16]). *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

3. Proof of the Theorem

Throughout this section we denote $S_4(4)$ by G. The character table of G, as well as the p-Brauer character tables, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2,3,5,17\}$, can be found using the computational algebra system GAP [17]. For the

characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

First of all we start to investigate units of orders 2, 3, 5 and 17, since the group G possesses elements of these orders. After this, by Proposition 4, the order of each torsion unit divides the exponent of G, so to prove the Kimmerle's conjecture, it remains to consider units of orders 34, 51 and 85. We prove that no units of all these orders do appear in $V(\mathbb{Z}G)$.

Now we consider each case of possible orders separately.

Let u be an involution. By (1.1) and Proposition 2 we get

$$\nu_{2a} + \nu_{2b} + \nu_{2c} = 1.$$

Put $t_1 = 3\nu_{2a} + 3\nu_{2b} - \nu_{2c}$ and $t_2 = 5\nu_{2a} - 3\nu_{2b} + \nu_{2c}$. By (2.1) we obtain the system of inequalities

$$\mu_1(u,\chi_2,*) = \tfrac{1}{2}(2t_1+18) \geq 0; \qquad \mu_0(u,\chi_2,*) = \tfrac{1}{2}(-2t_1+18) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{2}(2t_2+34) \geq 0; \qquad \mu_1(u,\chi_3,*) = \tfrac{1}{2}(-2t_2+34) \geq 0,$$

from which $-9 \leq t_1 \leq 9$ and $-17 \leq t_2 \leq 17$. Furthermore, from the system of linear equations

$$\nu_{2a} + \nu_{2b} + \nu_{2c} = 1, \quad 3\nu_{2a} + 3\nu_{2b} - \nu_{2c} = t_1, \quad 5\nu_{2a} - 3\nu_{2b} + \nu_{2c} = t_2,$$

we select only integer solutions, and using the condition that all $\mu_i(u,\chi_j,*)$ are non-negative integers, we obtain twenty three tuples $(\nu_{2a},\nu_{2b},\nu_{2c})$.

Let u be a unit of order 3. By (1.1) and Proposition 2 we get

$$\nu_{3a} + \nu_{3b} = 1.$$

Put $t_1 = \nu_{3a} + 4\nu_{3b}$. Using $\mathfrak{BCT}(2)$ and by (2.1) we obtain the system of inequalities

$$\mu_0(u,\chi_7,2) = \tfrac{1}{3}(2t_1+16) \geq 0, \qquad \mu_1(u,\chi_7,2) = \tfrac{1}{3}(-t_1+16) \geq 0,$$

from which $t_1 \in \{1+3k \mid -3 \leq k \leq 5\}$. Using inequalities

$$\mu_0(u,\chi_2,2) = \tfrac{1}{3}(-4\nu_{3a}+2\nu_{3b}+4) \geq 0, \qquad \mu_0(u,\chi_4,2) = \tfrac{1}{3}(2\nu_{3a}-4\nu_{3b}+4) \geq 0,$$

we obtain only two integral solutions $(\nu_{3a},\nu_{3b}) \in \{(0,1),(1,0)\}$.

Let u be a unit of order 5. By (1.1) and Proposition 2 we get

$$\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} + \nu_{5e} = 1.$$

Put

$$\alpha = 3\nu_{5a} + 3\nu_{5b} + 3\nu_{5c} + 3\nu_{5d} - 2\nu_{5e} \text{ and } \beta = 4\nu_{5a} + 4\nu_{5b} - \nu_{5c} - \nu_{5d} - \nu_{5e}.$$

Using $\mathfrak{BCT}(2)$ and by (2.1) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{5}(4\alpha+18) \geq 0; \qquad \mu_1(u,\chi_2,*) = \tfrac{1}{5}(-\alpha+18) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{5}(4\beta+34) \geq 0; \qquad \mu_0(u,\chi_6,2) = \tfrac{1}{5}(-4\beta+16) \geq 0,$$

so $\alpha \in \{-2,3,8,13,18\}$ and $\beta \in \{-6,-1,4\}$ and we obtain only five integral solutions

$$(\nu_{5a},\nu_{5b}) \in \{(1,0,0,0,0),(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(0,1,0,0,0)\}.$$

Let u be a unit of order 17. By (1.1) and Proposition 2 we get

$$\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1 = 13\nu_{17a}-4\nu_{17b}-4\nu_{17c}-4\nu_{17d}, \quad t_2 = 4\nu_{17a}-13\nu_{17b}+4\nu_{17c}+4\nu_{17d},$$
$$\text{and} \quad t_3 = 4\nu_{17a}+4\nu_{17b}-13\nu_{17c}+4\nu_{17d}.$$

By (2.1) we have

$$\mu_1(u,\chi_2,2)=\tfrac{1}{17}(t_1+4)\geq 0; \qquad \mu_3(u,\chi_{12},2)=\tfrac{1}{17}(-t_1+64)\geq 0;$$
$$\mu_6(u,\chi_{12},2)=\tfrac{1}{17}(t_2+64)\geq 0; \qquad \mu_2(u,\chi_2,2)=\tfrac{1}{17}(-t_2+4)\geq 0;$$
$$\mu_2(u,\chi_{12},2)=\tfrac{1}{17}(t_3+64)\geq 0; \qquad \mu_3(u,\chi_2,2)=\tfrac{1}{17}(-t_3+4)\geq 0.$$

This yields

$$t_1 \in \{-4,13,30,47,64\}, \quad t_2 \in \{-64,-47,-30,-13,4\},$$
$$\text{and} \quad t_3 \in \{-64,-47,-30,-13,4\}.$$

Using inequality

$$\mu_6(u,\chi_2,2)=\tfrac{1}{17}(-4\nu_{17a}-4\nu_{17b}-4\nu_{17c}+13\nu_{17d}+4)\geq 0$$

we have the four trivial solutions.

Let u be a unit of order 34. By (1.1) and Proposition 2 we have

$$\nu_{2a}+\nu_{2b}+\nu_{2c}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1 = 6\nu_{2a}+6\nu_{2b}-2\nu_{2c}-\nu_{17a}-\nu_{17b}-\nu_{17c}-\nu_{17d},$$
$$t_2 = 5\nu_{2a}-3\nu_{2b}+\nu_{2c}, \quad t_3 = 3\nu_{2a}-5\nu_{2b}-\nu_{2c},$$
$$t_4 = 15\nu_{2a}+15\nu_{2b}-\nu_{2c}-4\nu_{17a}+13\nu_{17b}-4\nu_{17c}-4\nu_{17d}$$
$$t_5 = 15\nu_{2a}+15\nu_{2b}-\nu_{2c}+13\nu_{17a}-4\nu_{17b}-4\nu_{17c}-4\nu_{17d}$$
$$t_6 = 15\nu_{2a}+15\nu_{2b}-\nu_{2c}-4\nu_{17a}-4\nu_{17b}-4\nu_{17c}+13\nu_{17d}$$

Since $|u^{17}|=2$, for any character χ of G we need to consider twenty three cases, defined by part (ii) of the Theorem:

(1) Let

$$\chi(u^{17}) \in \{\chi(2a),\chi(2a)-2\chi(2b)+2\chi(2c),2\chi(2a)-\chi(2b),-\chi(2b)+2\chi(2c),$$
$$\chi(2b),-\chi(2a)+2\chi(2c),-2\chi(2a)+\chi(2b)+2\chi(2c),-\chi(2a)+2\chi(2b)\}.$$

Then by (2.1) we obtain two incompatible inequalities

$$\mu_{17}(u,\chi_2,*)=\tfrac{1}{34}(16t_1+\alpha)\geq 0, \qquad \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+\beta)\geq 0,$$

where $(\alpha,\beta)\in\{(40,28),(24,44)\}$.

(2) Let

$$\chi(u^{17})\in\{\chi(2c),2\chi(2a)-2\chi(2b)+\chi(2c),\chi(2a)-\chi(2b)+\chi(2c),\\ -\chi(2a)+\chi(2b)+\chi(2c),-2\chi(2a)+2\chi(2b)+\chi(2c)\}.$$

Using (2.1), we obtain the incompatible system of inequalities

$$\mu_{17}(u,\chi_2,*)=\tfrac{1}{34}(16t_1+32)\geq 0;\\ \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+36)\geq 0;\\ \mu_1(u,\chi_2,*)=\tfrac{1}{34}(-t_1+15)\geq 0.$$

(3) Let

$$\chi(u^{17})\in\{-2\chi(2b)+3\chi(2c),-2\chi(2a)+3\chi(2c),\\ 2\chi(2a)-\chi(2c),2\chi(2b)-\chi(2c)\}.$$

Using (2.1), we obtain the system of inequalities

$$\mu_0(u,\chi_3,*)=\tfrac{1}{34}(32t_2+\alpha)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2+\beta)\geq 0,$$

where $(\alpha,\beta)\in\{(52,16),(20,48)\}$, which has no integral solution.

(4) Let $\chi(u^{17})=\chi(2a)-3\chi(2b)+3\chi(2c)$. Using (2.1), we calculate the following system of inequalities

$$\mu_{17}(u,\chi_2,*)=\tfrac{1}{34}(16t_1+16)\geq 0;\qquad \mu_1(u,\chi_2,*)=\tfrac{1}{34}(-t_1-1)\geq 0;\\ \mu_1(u,\chi_3,*)=\tfrac{1}{34}(2t_2)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2)\geq 0;\\ \mu_2(u,\chi_4,*)=\tfrac{1}{34}(2t_3+4)\geq 0;\qquad \mu_0(u,\chi_4,*)=\tfrac{1}{34}(-32t_3+4)\geq 0;\\ \mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+271)\geq 0;\qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+205)\geq 0;\\ \mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+254)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+188)\geq 0;\\ \mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+254)\geq 0;\qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+188)\geq 0.$$

It follows that $t_1=-1$, $t_2=0$, $t_3=-2$,

$$t_4\in\{1+34k\mid -8\leq k\leq 6\},\ t_5,t_6\in\{18+34k\mid -8\leq k\leq 5\},$$

and we have no solutions.

(5) Let $\chi(u^{17})=3\chi(2a)-\chi(2b)-\chi(2c)$. Using (2.1), we obtain the system of inequalities

$$\mu_2(u,\chi_2,*)=\tfrac{1}{34}(t_1+3)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+20)\geq 0;\\ \mu_1(u,\chi_3,*)=\tfrac{1}{34}(2t_2)\geq 0;\qquad \mu_{17}(u,\chi_3,*)=\tfrac{1}{34}(-32t_2)\geq 0;\\ \mu_2(u,\chi_4,*)=\tfrac{1}{34}(2t_3+4)\geq 0;\qquad \mu_0(u,\chi_4,*)=\tfrac{1}{34}(-32t_3+4)\geq 0;\\ \mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+190)\geq 0;\qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+252)\geq 0;\\ \mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+190)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+252)\geq 0;$$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+207) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+269) \geq 0.$

It follows that $t_1 = -3$, $t_2 = 0$, $t_3 = -2$,

$$t_4, t_5 \in \{14+34k \mid -6 \leq k \leq 7\},\ t_6 \in \{31+34k \mid -7 \leq k \leq 7\},$$

and we have no solutions again.

(6) Let $\chi(u^{17}) = -\chi(2a) - \chi(2b) + 3\chi(2c)$. Again, using (2.1), we obtain the system of inequalities

$\mu_{17}(u,\chi_2,*) = \frac{1}{34}(16t_1+16) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{34}(-t_1-1) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+36) \geq 0; \quad \mu_{17}(u,\chi_3,*) = \frac{1}{34}(-32t_2+32) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3+32) \geq 0; \quad \mu_0(u,\chi_4,*) = \frac{1}{34}(-32t_3+36) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+254) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+188) \geq 0;$

$\mu_2(u,\chi_{17},*) = \frac{1}{34}(t_5+254) \geq 0; \quad \mu_9(u,\chi_{17},*) = \frac{1}{34}(-t_5+188) \geq 0;$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+271) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+205) \geq 0.$

It follows that $t_1 = -1$, $t_2 = 1$, $t_3 = -1$,

$$t_4, t_5 \in \{18+34k \mid -8 \leq k \leq 5\},\ t_6 \in \{1+34k \mid -8 \leq k \leq 6\},$$

and we have no solutions again.

(7) Let $\chi(u^{17}) = \chi(2a) + \chi(2b) - \chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$\mu_2(u,\chi_2,*) = \frac{1}{34}(t_1+3) \geq 0; \quad \mu_0(u,\chi_2,*) = \frac{1}{34}(-16t_1+20) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+36) \geq 0; \quad \mu_{17}(u,\chi_3,*) = \frac{1}{34}(-32t_2+32) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3+32) \geq 0; \quad \mu_0(u,\chi_4,*) = \frac{1}{34}(-32t_3+36) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+190) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+252) \geq 0;$

$\mu_2(u,\chi_{17},*) = \frac{1}{34}(t_5+190) \geq 0; \quad \mu_9(u,\chi_{17},*) = \frac{1}{34}(-t_5+252) \geq 0;$

$\mu_{12}(u,\chi_{17},*) = \frac{1}{34}(t_6+207) \geq 0; \quad \mu_3(u,\chi_{17},*) = \frac{1}{34}(-t_6+269) \geq 0.$

It follows that $t_1 = -3$, $t_2 = 1$, $t_3 = -1$,

$$t_4, t_5 \in \{14+34k \mid -6 \leq k \leq 7\},\ t_6 \in \{31+34k \mid -7 \leq k \leq 7\},$$

and we have no solutions.

(8) Let $\chi(u^{17}) = -3\chi(2a) + \chi(2b) + 3\chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$\mu_{17}(u,\chi_2,*) = \frac{1}{34}(16t_1+16) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{34}(-t_1-1) \geq 0;$

$\mu_0(u,\chi_3,*) = \frac{1}{34}(32t_2+4) \geq 0; \quad \mu_2(u,\chi_3,*) = \frac{1}{34}(-2t_2+4) \geq 0;$

$\mu_{17}(u,\chi_4,*) = \frac{1}{34}(32t_3) \geq 0; \quad \mu_1(u,\chi_4,*) = \frac{1}{34}(-2t_3) \geq 0;$

$\mu_4(u,\chi_{17},*) = \frac{1}{34}(t_4+254) \geq 0; \quad \mu_1(u,\chi_{17},*) = \frac{1}{34}(-t_4+188) \geq 0;$

$$\mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+271)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+205)\geq 0;$$
$$\mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+254)\geq 0;\qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+188)\geq 0.$$

It follows that $t_1=-1$, $t_2=2$, $t_3=0$,

$$t_4,t_6\in\{18+34k\mid -8\leq k\leq 5\},\ t_5\in\{1+34k\mid -8\leq k\leq 6\},$$

and we have no solutions.

(9) Let $\chi(u^{17})=-\chi(2a)+3\chi(2b)-\chi(2c)$. Then, by (2.1), we obtain the system of inequalities

$$\mu_2(u,\chi_2,*)=\tfrac{1}{34}(t_1+3)\geq 0;\qquad \mu_0(u,\chi_2,*)=\tfrac{1}{34}(-16t_1+20)\geq 0;$$
$$\mu_0(u,\chi_3,*)=\tfrac{1}{34}(32t_2+4)\geq 0;\qquad \mu_2(u,\chi_3,*)=\tfrac{1}{34}(-2t_2+4)\geq 0;$$
$$\mu_{17}(u,\chi_4,*)=\tfrac{1}{34}(32t_3)\geq 0;\qquad \mu_1(u,\chi_4,*)=\tfrac{1}{34}(-2t_3)\geq 0;$$
$$\mu_4(u,\chi_{17},*)=\tfrac{1}{34}(t_4+190)\geq 0;\qquad \mu_1(u,\chi_{17},*)=\tfrac{1}{34}(-t_4+252)\geq 0;$$
$$\mu_2(u,\chi_{17},*)=\tfrac{1}{34}(t_5+207)\geq 0;\qquad \mu_9(u,\chi_{17},*)=\tfrac{1}{34}(-t_5+269)\geq 0;$$
$$\mu_{12}(u,\chi_{17},*)=\tfrac{1}{34}(t_6+190)\geq 0;\qquad \mu_3(u,\chi_{17},*)=\tfrac{1}{34}(-t_6+252)\geq 0.$$

It follows that $t_1=-3$, $t_2=2$, $t_3=0$,

$$t_4,t_6\in\{14+34k\mid -6\leq k\leq 7\},\ t_5\in\{31+34k\mid -7\leq k\leq 7\},$$

and we have no solutions.

Let u be a unit of order 51. By (1.1) and Proposition 2 we have

$$\nu_{3a}+\nu_{3b}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1=\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}.$$

Then using (2.1) we obtain the non-compatible system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{51}(32t_1+34)\geq 0,\qquad \mu_{17}(u,\chi_2,*)=\tfrac{1}{51}(-16t_1+34)\geq 0.$$

Let u be a unit of order 85. By (1.1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{5c}+\nu_{5d}+\nu_{5e}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}=1.$$

Put

$$t_1=3\nu_{5a}+3\nu_{5b}+3\nu_{5c}+3\nu_{5d}-2\nu_{5e}+\nu_{17a}+\nu_{17b}+\nu_{17c}+\nu_{17d}.$$

Since $|u^{17}|=5$, for any character χ of G we need to consider five cases, defined by part (iii) of the Theorem:

(1) Let $\chi(u^{17})\in\{\chi(5a),\chi(5b),\chi(5c),\chi(5d)\}$. Using (2.1), we obtain the system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{85}(64t_1+46)\geq 0,\qquad \mu_{17}(u,\chi_2,*)=\tfrac{1}{85}(-16t_1+31)\geq 0,$$

which has no integral solution.

(2) Let $\chi(u^{17}) = \chi(5e)$. Again, using (2.1), we obtain the system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{85}(64t_1 + 26) \geq 0, \qquad \mu_5(u, \chi_2, *) = \tfrac{1}{85}(-4t_1 + 9) \geq 0,$$

which has no integral solution.

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Doi: 10.18514/MMN.2015.1108

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 453–461

HU e-ISSN 1787-2413

KINDS OF DERIVATIONS ON HILBERT C^*-MODULES AND THEIR OPERATOR ALGEBRAS

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Received 24 January, 2014

Abstract. Let $\mathcal{M}$ be a Hilbert C^*-module. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a derivation if $d(< x, y > z) =< dx, y > z+ < x, dy > z+ < x, y > dz$ for all $x, y, z \in \mathcal{M}$. We give some results for derivations and automatic continuity of them on $\mathcal{M}$. Also, we will characterize generalized derivations and strong higher derivations on the algebra of compact operators and adjointable operators of Hilbert C^*-modules, respectively.

2010 *Mathematics Subject Classification:* 46L08; 16W25

Keywords: derivation, higher derivations, Hilbert C^*-module

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a C^*-algebra. A $pre-Hilbert \quad \mathcal{A}-module \quad \mathcal{M}$ is a left $\mathcal{A}$-module equipped with a sesquilinear form $< \cdot, \cdot >: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}$ which satisfies the following axioms for all $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$:
(1) $< x, x > \geq 0$;
(2) $< x, x >= 0 \Longleftrightarrow x = 0$;
(3) $< x, y >^* =< y, x >$;
(4) $< ax, y >= a < x, y >$.
For every $x \in \mathcal{M}$, set $\| x \| = \| < x, x > \|^{1/2}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to this norm is called a $Hilbert \quad \mathcal{A}-module$. For example, a complex Hilbert space H is a Hilbert C^*-module over the C^*-algebra of complex numbers or a C^*-algebra $\mathcal{A}$ is a Hilbert C^*-module over $\mathcal{A}$ by $< a, b >= ab^*$, for all $a, b \in \mathcal{A}$. A linear mapping $T : \mathcal{M} \to \mathcal{M}$ is called an $operator$ if T is continuous and $\mathcal{A}$-linear (i.e. $T(ax) = aT(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$). By $End(\mathcal{M})$, we denote the set of all operators on $\mathcal{M}$. A mapping $T : \mathcal{M} \to \mathcal{M}$ is called $adjointable$ if there exists a mapping $T^* : \mathcal{M} \to \mathcal{M}$ such that $< Tx, y >=< x, T^* y >$ for all $x, y \in \mathcal{M}$. As a well-known result, every adjointable mapping $T : \mathcal{M} \to \mathcal{M}$ is an operator. The set of all adjointable mappings on $\mathcal{M}$ is denoted by $End^*(\mathcal{M})$ which is a C^*-algebra under the usual operator norm. For $x, y \in \mathcal{M}$, define $\theta_{x,y} : \mathcal{M} \to \mathcal{M}$ by $\theta_{x,y}(z) =< z, y > x$, for all $z \in \mathcal{M}$. Clearly, $\theta_{x,y} \in End^*(\mathcal{M})$ with $\theta^*_{x,y} = \theta_{y,x}$.

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Note that $\theta_{x,y}$ is quite different from rank one projections in Hilbert spaces. For example we can not infer $x = 0$ or $y = 0$ from $\theta_{x,y} = 0$. We denote by $\mathcal{K}(\mathcal{M})$ the closed linear span of $\{\theta_{x,y} : x, y \in \mathcal{M}\}$. The elements of $\mathcal{K}(\mathcal{M})$ are called $compact \quad operators$. This concept of compact operators is different from compact operators in the usual sense. However, this concept coincides with the concept of usual compact operators when we choose a Hilbert space as a Hilbert C^*-module. Set $I = \text{span}\{< x, y >: x, y \in \mathcal{M}\}$. It is easy to see that I is a $*$-bi-ideal of $\mathcal{A}$. An important class of Hilbert C^*-modules are $full$ modules. A Hilbert C^*-module $\mathcal{M}$ is called full if $\overline{I} = \mathcal{A}$, where $\overline{I}$ is the norm closure of I in $\mathcal{A}$. For example, $\mathcal{A}$ is a full $\mathcal{A}$-module. It is well-known that the derivations on Banach algebras are the generators of certain dynamical systems. A linear mapping $\phi : \mathcal{M} \to \mathcal{M}$ is called a $homomorphism$ if $\phi(< x, y > z) = < \phi x, \phi y > \phi z$ for all $x, y, z \in \mathcal{M}$. A dynamical system on $\mathcal{M}$ is a strongly continuous one-parameter family $(u_t)_{t\in\mathbb{R}}$ of homomorphisms. A linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a $derivation$ if $d(< x, y > z) = < dx, y > z + < x, dy > z + < x, y > dz$ for all $x, y, z \in \mathcal{M}$, see [1] and [2]. In [1], Abbaspour and Skeide proved that a C_0-group $u = (u_t)_{t\in\mathbb{R}}$ is a dynamical system if and only if its generator is a derivation and every derivation on full Hilbert C^*-module $\mathcal{M}$ is a generalized derivation i.e. there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ax) = \delta(a)x + ad(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{M}$. Also, they proved that every derivation on full Hilbert C^*-modules extends as a $*$-derivation to the linking algebra. In this paper, we consider derivations on Hilbert C^*-modules and give some results about adjointable derivations and automatic continuity of them.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A σ-$derivation$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in \mathcal{A}$. If $\sigma = I$, where I is the identity operator on $\mathcal{A}$, then d is a derivation. A generalized derivation on $\mathcal{A}$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. In [7], P. Li, D. Han and W. S. Tang proved that every derivation on $End^*(\mathcal{M})$ is inner if $\mathcal{A}$ is commutative and unital. In section 3, we will characterize generalized derivations on $\mathcal{K}(\mathcal{M})$ without commutativity condition. Suppose that $\{d_n\}_{n=0}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. It$'$s called a $higher \quad derivation$ if $d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$ for all $a, b \in \mathcal{A}$ and all $n \geqslant 0$. If $d_0 = I$, $\{d_n\}_{n=0}^{\infty}$ is called a $strong \quad higher \quad derivation$. Let δ be a derivation on $\mathcal{A}$ and define the sequence $\{d_n\}_{n=0}^{\infty}$ on $\mathcal{A}$ by $d_0 = I$ and $d_n = \dfrac{\delta^n}{n!}$ for every $n \geqslant 1$. By Leibnitz rule, $\{d_n\}_{n=0}^{\infty}$ is a higher derivation on $\mathcal{A}$. Higher derivations were introduced by Hasse and Schmidt [4] and algebraists sometimes call them Hasse-Schmidt derivations. For a higher derivation obviously, d_0 is a homomorphism and d_1 is a d_0-derivation in the sense of [11]. Therefore, higher derivations are the generalizations of homomorphisms and derivations. In [12], higher derivations

are applied to study generic solving of higher differential equations. For more information about higher derivations and its applications see [5], [6], [9] and [10]. The last author in [10], characterized the strong higher derivations in terms of derivations. In section 4 we give a characterization of higher derivation on $End^*(\mathcal{M})$ with use of elements whose product is in $\mathcal{K}(M)$.

2. Derivations on Hilbert C^*-modules

Let $\mathcal{M}$ be a Hilbert C^*-module. Recall that a linear mapping $d : \mathcal{M} \to \mathcal{M}$ is called a $derivation$ if

$$d(< x, y > z) =< dx, y > z+ < x, dy > z+ < x, y > dz$$

for all $x, y, z \in \mathcal{M}$. Note that if $d : \mathcal{M} \to \mathcal{M}$ is an adjointable map with $d^* = -d$, then d is a derivation. But the converse is not true. For example suppose that H is a Hilbert space. Set $u_0 \in B(H)$ such that $u^* = -u$ and u is not in the center of $B(H)$. Define $d : B(H) \to B(H)$ by $d(v) = u_0 v - v u_0$ for every $v \in B(H)$. It is easy to see that d is a derivation on $B(H)$ as a $B(H)$-module but d is not adjointable. Otherwise, d is $\mathcal{A}$-linear and Therefore,

$$u_0 vv - vvu_0 = d(vv) = vd(v) = vu_0 v - vvu_0$$

for every $v \in B(H)$. This implies that u_0 is in the center of $B(H)$, which is a contradiction. Let M be a full Hilbert C^*-module. Note that if there exists $a \in \mathcal{A}$ such that $ax = o$ for every $x \in \mathcal{M}$, then $a = o$. Therefore, we have the following theorem:

Theorem 1. *Let $\mathcal{M}$ be a full Hilbert C^*-module. Then $d \in End^*(\mathcal{M})$ is a derivation if and only if $d^* = -d$.*

Proof. Suppose that $d \in End^*(\mathcal{M})$ is a derivation. Then $(< dx, y > + < x, dy >)z = 0$ for all $x, y, z \in \mathcal{M}$. Hence $d^* = -d$. The converse is trivial. □

A set of non-zero elements $\{x_i\}_{i \in I} \subseteq \mathcal{M}$ is called a standard basis for $\mathcal{M}$ if the reconstruction formula $x = \sum_{i \in I} < x, x_i > x_i$ holds for every $x \in \mathcal{M}$. Let

$$L_n(\mathcal{A}) = \{(a_1, a_2, \cdots, a_n) : a_i \in \mathcal{A}, 1 \leq i \leq n\}.$$

Then $L_n(\mathcal{A})$ a Hilbert C^*-module over C^*-algebra $\mathcal{A}$ with module product $a(a_1, a_2, \cdots, a_n) = (aa_1, aa_2, \cdots, aa_n)$ and inner product

$$< (a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) >= a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^*$$

for every $a \in \mathcal{A}$ and $(a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) \in L_n(\mathcal{A})$, see [8]. If $\mathcal{A}$ is unital then $L_n(\mathcal{A})$ has standard basis $\{e_i\}_{i=1}^n$ such that
$e_i = (0, 0, \cdots, 1_{i \cdot th}, \cdots, 0)$ and $1 \leq i \leq n$. In [3], D. Bakic and proved that every Hilbert C^*-module over the C^*-algebra of the compact operators possesses a standard basis.

Theorem 2. *Let $\mathcal{M}$ have a standard basis and $d \in End^*(\mathcal{M})$. Then d is a derivation if and only if $d^* = -d$.*

Proof. Let $\{x_i\}_{i\in I}$ be a standard basis for $\mathcal{M}$ and $d \in End^*(\mathcal{M})$ be a derivation. Then $d(x) = \sum_{i\in I} < x, x_i > dx_i$. On the other hand,

$$\begin{aligned} dx &= \sum_{i\in I} < dx, x_i > x_i + \sum_{i\in I} < x, dx_i > x_i + \sum_{i\in I} < x, x_i > dx_i \\ &= dx + \sum_{i\in I} < d^*x, x_i > x_i + \sum_{i\in I} < x, x_i > dx_i \\ &= dx + d^*x + dx. \end{aligned}$$

So, $d^* = -d$. □

Lemma 1. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then there exist $x_1, \cdots, x_n \in \mathcal{M}$ such that $\sum_{i=1}^n < x_i, x_i >= 1$.*

Proof. See [8]. □

A Hilbert C^*-module $\mathcal{M}$ over C^*-algebra $\mathcal{A}$ is called simple if the only closed submodules of $\mathcal{M}$ over A are $\{0\}$ and $\mathcal{M}$. For example, let H be a Hilbert space and $\mathcal{K}(H)$ denotes the the algebra of compact operator on H. Then $\mathcal{K}(H)$ is a simple Hilbert C^*-module over itself.

Theorem 3. *Let $\mathcal{M}$ be a full and simple Hilbert C^*-module over the unital C^*-algebra $\mathcal{A}$ and d be a derivation on $\mathcal{M}$ with closed range. Then d is continuous or surjective.*

Proof. Define the separating space S(d)=$\{y \in \mathcal{M} : \exists\{x_n\} \to 0$ in $\mathcal{M}$ such that $dx_n \to y\}$. As a well-known result $S(d)$ is a closed subspace of $\mathcal{M}$. By lemma 1, there exist $x_1, \cdots, x_m$ such that $\sum_{i=1}^m < x_i, x_i >= 1$. Therefore, $a = \sum_{i=1}^m < ax_i, x_i >$ for all $a \in \mathcal{A}$. For $z \in S(d)$ there exists a sequence $z_n \to 0$ such that that $dz_n \to z$. Hence

$$d(az_n) = \sum_{i=1}^m < adx_i, x_i > z_n + \sum_{i=1}^m < ax_i, dx_i > z_n + \sum_{i=1}^m a < x_i, x_i > dz_n \to az. \tag{2.1}$$

This implies that $S(d)$ is a submodule of $\mathcal{M}$. Since $\mathcal{M}$ is simple, $S(d) = \{0\}$ or $S(d) = \mathcal{M}$. If $S(d) = \{0\}$, by closed graph theorem, d is continuous. If $S(d) = \mathcal{M}$ by (2.1), $A\mathcal{M} \subseteq \overline{Im(d)}$. Since $\mathcal{A}$ is unital $\mathcal{A}\mathcal{M} = \mathcal{M}$. Therefore, $\overline{Im(d)} = Im(d) = \mathcal{M}$ and T is surjective. □

Lemma 2. *Let $\mathcal{M}$ be a Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then $I\mathcal{M} = \mathcal{M}$.*

Proof. Clearly, $I\mathcal{M} \subseteq \mathcal{M}$. let $z \in \mathcal{M}$ and set

$$x = \lim_{n\to\infty} (\frac{1}{n} + < z, z >^{1/3})^{-1} z.$$

One can see that $z =< x, x > x$ and therefore, $I\mathcal{M} = \mathcal{M}$. For more detail see [8]. □

Theorem 4. *Let $\mathcal{M}$ be a Hilbert C^*-module over unital C^* algebra $\mathcal{A}$. Suppose that $\mathcal{M}$ is a simple I-module and d be a derivation on $\mathcal{M}$ with closed range. Then d is continuous or surjective.*

Proof. Let $a \in I$ and $z \in S(d)$. So there exist a sequence

$$z_n \to 0, \quad x_1, x_2, \cdots, x_m, \quad y_1, y_2, \cdots, y_m \in \mathcal{M}$$

for some $m \in \mathbb{N}$ such that $dz_n \to z$ and $a = \sum_{i=1}^{m} < x_i, y_i >$. But

$$d(az_n) = \sum_{i=1}^{m} < dx_i, y_i > z_n + \sum_{i=1}^{m} < x_i, dy_i > z_n + \sum_{i=1}^{m} a dz_n \to az. \tag{2.2}$$

This implies that $S(d)$ is a submodule of $\mathcal{M}$. Therefore, $S(d) = \{0\}$ or $S(d) = \mathcal{M}$. If $S(d) = \{0\}$, by closed graph theorem, d is continuous. If $S(d) = \mathcal{M}$, by (2.2), $I\mathcal{M} \subseteq \overline{Im(d)}$. Therefore, by lemma 2, $Im(d) = \mathcal{M}$ and T is surjective. □

3. Characterization of generalized derivations on the algebra of compact operators

Let $\mathcal{A}$ be an algebra. Recall that a derivation on $\mathcal{A}$ is a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. A generalized derivation on $\mathcal{A}$ is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that there exists a derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. Recall that a linear mapping $\Pi : \mathcal{A} \to \mathcal{A}$ is called a left multiplier if $\Pi(ab) = \Pi(a)b$ for all $a, b \in \mathcal{A}$. For a generalized derivation d, set $\Pi = d - \delta$. One can easily see that Π is a left multiplier. let $d : \mathcal{A} \to \mathcal{A}$ de a linear mapping. As a well-known result d is a generalized derivation if and only if there exist a derivation $\delta : \mathcal{A} \to \mathcal{A}$ and left multiplier $\Pi : \mathcal{A} \to A$ such that $d = \delta + \Pi$.

Theorem 5. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then linear mapping $\Pi : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ is a left multiplier if and only if there exists $T \in End(\mathcal{M})$ such that $\Pi(A) = TA$ for all $A \in \mathcal{K}(\mathcal{M})$.*

Proof. Let $T \in End(\mathcal{M})$. Define $\Pi : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ by $\Pi(A) = TA$, for every $A \in \mathcal{K}(\mathcal{M})$. Clearly Π is a left multiplier. Conversely, since $\mathcal{M}$ is full, by lemma 1, there exist $x_1, \cdots, x_n$ such that $\sum_{i=1}^{i=n} < x_i, x_i >= 1$. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T(x) = \sum_{i=1}^{n} \Pi(\theta_{x,x_i})x_i,$$

for every $x \in \mathcal{M}$. For every $A \in \mathcal{K}(\mathcal{M})$ we have

$$TA(x) = \sum_{i=1}^{n} \Pi(\theta_{Ax,x_i})x_i = \sum_{i=1}^{n} \Pi(A\theta_{x,x_i})x_i = \sum_{i=1}^{n} \Pi(A)(\theta_{x,x_i})x_i$$

$$= \sum_{i=1}^{n} < x_i, x_i > \Pi(A)x = \Pi(A)x.$$

So $\Pi(A) = TA$. T is obviously a continuous linear mapping. To show that $T \in End(\mathcal{M})$ it's remain to show that T is $\mathcal{A}$-linear. Now suppose that $a \in \mathcal{A}$, $x \in \mathcal{M}$ and $A \in \mathcal{K}(\mathcal{M})$ We have,

$$\Pi(A)(ax) = TA(ax) = T(aA(x))$$

On the other hand

$$\Pi(A)(ax) = a\Pi(A)(a) = aTA(x)$$

and so $T(aA(x)) = aTA(x)$ for every $a \in \mathcal{A}$, $x \in \mathcal{M}$ and $A \in \mathcal{K}(\mathcal{M})$. Now lemma 2 implies that T is $\mathcal{A}$-linear. □

Definition 1. By $\mathcal{L}_0(\mathcal{M})$ we denote the set of all linear mapping A on $\mathcal{M}$ such that $AB - CA \in \mathcal{K}(\mathcal{M})$ for all $B, C \in \mathcal{K}(\mathcal{M})$.
Clearly $End^*(\mathcal{M}) \subset \mathcal{L}_0(\mathcal{M})$.

Theorem 6. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$. Then linear mapping $\delta : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ is a derivation if and only if there exists $T \in \mathcal{L}_0(\mathcal{M})$ such that $\delta(A) = TA - AT$ for all $A \in \mathcal{K}(\mathcal{M})$.*

Proof. Let $T \in \mathcal{L}_0(\mathcal{M})$. Define $\delta : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ by $\delta(A) = TA - AT$, for every $A \in \mathcal{K}(\mathcal{M})$. Clearly, δ is a derivation. Conversely, since $\mathcal{M}$ is full by lemma 2 there exist $x_1, \cdots, x_n$ such that $\sum_{i=1}^{i=n} < x_i, x_i >= 1$. Define $T : \mathcal{M} \to \mathcal{M}$ by

$$T(x) = \sum_{i=1}^{i=n} \delta(\theta_{x,x_i})x_i,$$

for every $x \in \mathcal{M}$. For every $A \in \mathcal{K}(\mathcal{M})$ we have

$$\begin{aligned} TA(x) &= \sum_{i=1}^{n} \delta(\theta_{Ax,x_i})x_i \\ &= \sum_{i=1}^{n} \delta(A\theta_{x,x_i})x_i \\ &= \sum_{i=1}^{n} \delta(A)(\theta_{x,x_i})x_i + \sum_{i=1}^{n} A\delta(\theta_{x,x_i})x_i \\ &= \delta(A)x + AT(x). \end{aligned}$$

□

Theorem 7. *Let $\mathcal{M}$ be a full Hilbert C^*-module over unital C^*-algebra $\mathcal{A}$ and $d : \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$ be a linear mapping. Then d is a generalized derivation if and only if there exist $T_1 \in \mathcal{L}_0(\mathcal{M})$ and $T_2 \in End(\mathcal{M})$ such that $d(A) = T_1A - AT_1 + T_2A$ for every $A \in \mathcal{K}(\mathcal{M})$.*

4. Characterization of higher derivation on the algebra of adjointable operators

Let $\mathcal{A}$ be an algebra and suppose that $\{d_n\}_{n=0}^{\infty}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{A}$. It's called a higher derivation if

$$d_n(ab) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b) \tag{4.1}$$

for all $a, b \in \mathcal{A}$ and all $n \geqslant 0$. If $d_0 = I$, $\{d_n\}_{n=0}^{\infty}$ is called a strong higher derivation. If (4.1) holds for all $x, y \in \mathcal{A}$ and $n = 0, 1, 2, \ldots, m$, it is called a higher derivation of rank m. Now we are going to give a characterization of strong higher derivations in terms of operators whose product is compact.

Theorem 8. *Let $\mathcal{M}$ be a full Hilbert C^*-module over the unital C^*-algebra $\mathcal{A}$. Let $\{d_n : End^*(\mathcal{M}) \to End^*(\mathcal{M})\}_{n=o}^{\infty}$ be a sequence of linear mappings such that $d_0 = I$. Then $\{d_n\}_{n=o}^{\infty}$ is a strong higher derivation if and only if $d_n(AB) = \sum_{i=0}^{n} d_i(A)d_{n-i}(B)$ for all $A, B \in End^*(\mathcal{M})$ such that $AB \in \mathcal{K}(\mathcal{M})$ and all $n \geq 1$.*

Proof. By lemma 1, there exist $x_1, \cdots, x_n$ such that $\sum_1^n < x_i, x_i >= 1$. Let x_i for some $1 \leq i \leq n$, $x \in \mathcal{M}$, $A, B \in End^*(\mathcal{M})$, and $m \geq 1$ be arbitrary elements. Since $\mathcal{K}(\mathcal{M})$ is a two sided ideal in $End^*(\mathcal{M})$,

$$d_m(A\theta_{x,x_i}) = \sum_{i=0}^{m} d_i(A)d_{m-i}(\theta_{x,x_i})$$

and

$$d_m(AB\theta_{x,x_i}) = d_m(AB)\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(AB)d_{m-i}(\theta_{x,x_i}).$$

On the other hand,

$$d_m(AB\theta_{x,x_i}) = d_m(A)B\theta_{x,x_i} + \sum_{i=0}^{m-1} d_i(A)d_{m-i}(B\theta_{x,x_i}). \tag{4.2}$$

Take $m = 1$. By comparing these equalities, we obtain

$$d_1(AB)\theta_{x,x_i} = d_1(A)B\theta_{x,x_i} + Ad_1(B)\theta_{x,x_i}.$$

So

$$\begin{aligned} d_1(AB)x &= \sum_{i=1}^{n} d_1(AB) < x_i, x_i > x \\ &= \sum_{i=1}^{n} d_1(A)B < x_i, x_i > x + \sum_{i=1}^{n} Ad_1(B) < x_i, x_i > x \\ &= d_1(A)Bx + Ad_1(B)x. \end{aligned}$$

This implies that d_1 is a derivation. As an induction suppose that $\{d_0, d_1, \cdots, d_m\}$ is a higher derivation of rank m. By induction, we get

$$\begin{aligned} d_{m+1}(AB\theta_{x,x_i}) &= d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m} d_i(AB)d_{m+1-i}(\theta_{x,x_i}) \\ &= d_{m+1}(AB)\theta_{x,x_i} + \sum_{i=0}^{m}\sum_{j=0}^{i} d_j(A)d_{i-j}(B)d_{m+1-i}(\theta_{x,x_i}) \end{aligned}$$

and by (4.2),

$$\begin{aligned} d_{m+1}(AB\theta_{x,x_i}) &= d_{m+1}(A)B\theta_{x,x_i} + \sum_{i=0}^{m} d_i(A)d_{m+1-i}(B\theta_{x,x_i}) \\ &= d_m(A)B\theta_{x,x_i} + \sum_{i=0}^{m}\sum_{j=0}^{m-i} d_i(A)d_j(B)d_{m+1+i-j}(\theta_{x,x_i}). \end{aligned}$$

One can see that

$$\sum_{i=0}^{m}\sum_{j=0}^{i} d_j(A)d_{i-j}(B)d_{m+1-i}(\theta_{x_0,x_i}) = \sum_{i=0}^{m}\sum_{j=0}^{m-i} d_i(A)d_j(B)d_{m+1+i-j}(\theta_{x,x_i}).$$

Therefore,

$$d_{m+1}(AB)\theta_{x,x_i} = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)\theta_{x,x_i}.$$

So

$$d_{m+1}(AB)x = \sum_{i=0}^{m+1} d_i(A)d_{m+1-i}(B)x.$$

will imply that $\{d_0, d_1, \cdots, d_{m+1}\}$ is a higher derivation of rank $m+1$. □

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Doi: 10.18514/MMN.2015.739

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 463–473

HU e-ISSN 1787-2413

CONVERGENCE TO COMMON FIXED POINT FOR A PAIR OF QUASI-NONEXPANSIVE MAPPING AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPING IN THE INTERMEDIATE SENSE

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Received 24 September, 2013

Abstract. The purpose of this paper is to study modified two-step iteration process for a pair of quasi-nonexpansive mapping and asymptotically quasi-nonexpansive mapping in the intermediate sense. We also establish some strong convergence theorems and a weak convergence theorem for this iteration scheme and mappings in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize some previous results from the existing literature (see for example [1, 3, 5–7, 9–13]).

2010 *Mathematics Subject Classification:* 47H09; 47H10; 47J25

Keywords: asymptotically quasi-nonexpansive mapping in the intermediate sense, quasi-nonexpansive mapping, modified two-step iteration process, common fixed point, strong convergence, weak convergence, Banach space

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T: K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T: K \to K$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$.

(2) quasi-nonexpansive if $F(T) \neq \varnothing$ and

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$.

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(3) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(5) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

(6) uniformly quasi-Lipschitzian if there exists $L \in [1, +\infty)$ such that

$$\|T^n x - p\| \leq L\|x - p\|$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

Remark 1. It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive is uniformly Lipschitzian. Also, if $F(T) \neq \varnothing$, then a nonexpansive mapping is a quasi-nonexpansive mapping, an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive, a uniformly L-Lipschitzian mapping must be uniformly quasi-Lipschitzian and an asymptotically quasi-nonexpansive mapping must be uniformly quasi-Lipschitzian mapping but the converse is not true in general.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. Recall also that a mapping $T: K \to K$ is said to be asymptotically quasi-nonexpansive in the intermediate sense [17] provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x\in K,\ p\in F(T)} \Big(\|T^n x - p\| - \|x - p\|\Big) \leq 0.$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example:

Example 1. Let $X = \mathbb{R}$ be a normed linear space and $K = [0,1]$. For each $x \in K$, we define

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n |x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\limsup_{n\to\infty}\{|T^n x - T^n y| - |x - y|\} = \limsup_{n\to\infty}\{k^n |x - y| - |x - y|\} \leq 0$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$, for all $x, y \in K$, $n \in \mathbb{N}$ and T is continuous. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 2. Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in K$, define

$$T(x) = \begin{cases} k\, x\, sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not asymptotically nonexpansive mapping.

Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see,for example, [4, 6, 7, 9–12, 16] and references therein).

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1 \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Later on, many authors, for example Khan and Takahashi [6], Shahzad and Udomene [13] and Takahashi and Tamura [14] have studied the two mappings case of iterative schemes for different types of mappings.

Recently, Khan et al. [5] studied the modified two-step iteration process for two mappings as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \geq 1 \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Inspired and motivated by Agarwal et al. [1], Khan et al. [5] and many others, in this paper we introduce the following iteration scheme for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. The proposed iteration scheme is as follows:

Definition 1. Let $S: K \to K$ be a quasi-nonexpansive mapping and $T: K \to K$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense on a closed convex subset K of a real Banach space E with $K + K \subset K$. Let $\{x_n\}$ be the sequence defined as:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + v_n,\ n \geq 1 \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{u_n\}$ and $\{v_n\}$ are two sequences in K. The iteration scheme (1.3) is called modified two-step iteration process with errors for a pair of mappings.

The aim of this paper is to establish some strong convergence theorems and a weak convergence theorem for newly proposed iteration scheme (1.3) in the framework of real Banach spaces. The results presented in this paper extend, improve and generalize some previous known corresponding results from the literature (e.g. [1, 3, 5–7, 9–13] and many others).

2. Preliminaries

For the sake of convenience, we restate the following concepts.

A mapping $T: K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition $\{x_n\}$ converges weakly to $x \in K$ and $\{Tx_n\}$ converges strongly to 0 imply $Tx = 0$.

A mapping $T: K \to K$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

We say that a Banach space E satisfies the *Opial's condition* [8] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [8].

Now, we state the following useful lemma to prove our main results:

Lemma 1 (see [15]). *Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of nonnegative numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. If one of the following conditions is satisfied:*

(i) $\alpha_{n+1} \leq \alpha_n + \beta_n$, $n \geq 1$,

(ii) $\alpha_{n+1} \leq (1 + \beta_n)\alpha_n$, $n \geq 1$,

then the limit $\lim_{n\to\infty} \alpha_n$ *exists.*

3. Main results

In this section, we prove some strong convergence theorems and a weak convergence theorem of the iteration scheme (1.3) for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense in the framework of real Banach spaces.

Theorem 1. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $S: K \to K$ be a quasi-nonexpansive mapping and $T: K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$. Put*

$$G_n = \max\left\{ \sup_{x\in K,\ q\in F} \left(\|T^n x - q\| - \|x - q\| \right) \vee 0,\ \ \forall\, n \geq 1 \right\} \tag{3.1}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Then $\{x_n\}$ converges to a common fixed point of the mappings S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf_{p\in F} d(x, p)$.

Proof. The necessity is obvious. Thus we only prove the sufficiency. Let $q \in F$. Then from (1.3) and (3.1), we have

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n + v_n - q\| \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n \|T^n x_n - q\| + \|v_n\| \\ &\leq (1-\beta_n)\|x_n - q\| + \beta_n [\|x_n - q\| + G_n] + \|v_n\| \\ &= (1-\beta_n)\|x_n - q\| + \beta_n \|x_n - q\| + \beta_n G_n + \|v_n\| \\ &\leq \|x_n - q\| + G_n + \|v_n\|. \end{aligned} \tag{3.2}$$

Again using (1.3), (3.1) and (3.2), we obtain

$$\begin{aligned}\|x_{n+1}-q\| &= \|(1-\alpha_n)T^n x_n+\alpha_n Sy_n+u_n-q\|\\ &\le (1-\alpha_n)\|T^n x_n-q\|+\alpha_n\|Sy_n-q\|+\|u_n\|\\ &\le (1-\alpha_n)[\|x_n-q\|+G_n]+\alpha_n\|y_n-q\|+\|u_n\|\\ &\le (1-\alpha_n)\|x_n-q\|+\alpha_n[\|x_n-q\|+G_n+\|v_n\|]\\ &\quad +(1-\alpha_n)G_n+\|u_n\|\\ &\le \|x_n-q\|+G_n+\|u_n\|+\alpha_n\|v_n\|\\ &= \|x_n-q\|+\theta_n \end{aligned} \tag{3.3}$$

where $\theta_n = G_n+\|u_n\|+\alpha_n\|v_n\|$. Since by assumption of the theorem $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\sum_{n=1}^{\infty}\|v_n\| < \infty$, it follows that $\sum_{n=1}^{\infty}\theta_n < \infty$. Thus by Lemma 1 we know that the limit $\lim_{n\to\infty}\|x_n-q\|$ exists. Also from (3.3), we obtain

$$d(x_{n+1},F) \le d(x_n,F)+\theta_n, \tag{3.4}$$

for all $n \ge 1$. From Lemma 1 and (3.4), we know that $\lim_{n\to\infty} d(x_n,F)$ exists. Since $\liminf_{n\to\infty} d(x_n,F) = 0$, we have that $\lim_{n\to\infty} d(x_n,F) = 0$.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Therefore, for any $m,n \ge 1$ and for given $p \in F$, from (3.3), we have

$$\begin{aligned}\|x_{n+m}-p\| &\le \|x_{n+m-1}-p\|+\theta_{n+m-1}\\ &\le \|x_{n+m-2}-p\|+\theta_{n+m-2}+\theta_{n+m-1}\\ &\le \dots\\ &\le \|x_n-p\|+\sum_{j=n}^{n+m-1}\theta_j. \end{aligned} \tag{3.5}$$

Since

$$\lim_{n\to\infty} d(x_n,F) = 0, \qquad \sum_{n=1}^{\infty}\theta_n < \infty \tag{3.6}$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n,F) < \frac{\varepsilon}{8}, \qquad \sum_{j=n}^{\infty}\theta_j < \frac{\varepsilon}{4} \quad \forall\, n \ge n_1. \tag{3.7}$$

Hence, there exists $q \in F$ such that

$$\|x_n-q\| < \frac{\varepsilon}{4} \quad \forall\, n \ge n_1. \tag{3.8}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (3.5), we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq \|x_n - q\| + \sum_{j=n}^{n+m-1} \theta_j + \|x_n - q\| \\ &\leq 2\|x_n - q\| + \sum_{j=n}^{n+m-1} \theta_j \\ &< 2 \times \frac{\varepsilon}{4} + 2 \times \frac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{3.9}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that

$$\|x_n - q^*\| < \frac{\varepsilon_1}{2(L+1)}, \ \forall\, n \geq n_2. \tag{3.10}$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, there must exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$, we have

$$d(x_n, F) < \frac{\varepsilon_1}{3(L+1)}, \tag{3.11}$$

and in particular, we have

$$d(x_{n_3}, F) < \frac{\varepsilon_1}{3(L+1)}. \tag{3.12}$$

Therefore, there exists $z^* \in F$ such that

$$\|x_{n_3\|-z^*} < \frac{\varepsilon_1}{2(L+1)}. \tag{3.13}$$

Consequently, we have

$$\begin{aligned}\|Tq^* - q^*\| &= \|Tq^* - z^* + z^* - x_{n_3} + x_{n_3} - q^*\| \\ &\leq \|Tq^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\ &\leq L\|q^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\ &\leq L\|q^* - x_{n_3} + x_{n_3} - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\ &\leq L[\|q^* - x_{n_3}\| + \|x_{n_3} - z^*\|] + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\ &\leq (L+1)\|q^* - x_{n_3}\| + (L+1)\|z^* - x_{n_3}\| \\ &< (L+1) \times \frac{\varepsilon_1}{2(L+1)} + (L+1) \times \frac{\varepsilon_1}{2(L+1)} < \varepsilon_1. \end{aligned} \tag{3.14}$$

This implies that $q^* \in F(T)$. Similarly, we can show that $q^* \in F(S)$. Since S is

quasi-nonexpansive, so it is uniformly quasi-1 Lipschitzian (by Remark 1), so here taking $L = 1$, we have

$$\begin{aligned}
\|Sq^* - q^*\| &= \|Sq^* - z^* + z^* - x_{n_3} + x_{n_3} - q^*\| \\
&\le \|Sq^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\le \|q^* - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\le \|q^* - x_{n_3} + x_{n_3} - z^*\| + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\le [\|q^* - x_{n_3}\| + \|x_{n_3} - z^*\|] + \|z^* - x_{n_3}\| + \|x_{n_3} - q^*\| \\
&\le 2\|q^* - x_{n_3}\| + 2\|z^* - x_{n_3}\| \\
&< 2 \times \frac{\varepsilon_1}{4} + 2 \times \frac{\varepsilon_1}{4} \\
&< \varepsilon_1. \qquad (3.15)
\end{aligned}$$

This shows that $q^* \in F(S)$. Thus $q^* \in F$, that is, q^* is a common fixed point of the mappings S and T. This completes the proof. □

Theorem 2. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K + K \subset K$. Let $S: K \to K$ be a quasi-nonexpansive mapping and $T: K \to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F = F(S) \cap F(T) \neq \varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$. Put*

$$G_n = \max\left\{ \sup_{x \in K,\ q \in F} \left(\|T^n x - q\| - \|x - q\| \right) \vee 0,\ \ \forall\, n \ge 1 \right\}$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Suppose that the mappings S and T satisfy the following conditions:

(C_1) $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$,

(C_2) *there exists a constant $A > 0$ such that*

$$\left\{ \|x_n - Sx_n\| + \|x_n - Tx_n\| \right\} \ge A d(x_n, F)$$

for all $n \ge 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T.

Proof. From conditions (C_1) and (C_2), we have $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 1, that $\{x_n\}$ must converges strongly to a common fixed point of the mappings S and T. This completes the proof. □

Theorem 3. *Let E be a real Banach space and K be a nonempty closed convex subset of E with $K+K\subset K$. Let $S:K\to K$ be a quasi-nonexpansive mapping and $T:K\to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F=F(S)\cap F(T)\neq\varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta,1-\delta]$ for some $\delta\in(0,1)$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}\|u_n\|<\infty$ and $\sum_{n=1}^{\infty}\|v_n\|<\infty$. Put*

$$G_n=\max\left\{\sup_{x\in K,\ q\in F}\left(\|T^nx-q\|-\|x-q\|\right)\vee 0,\ \ \forall n\geq 1\right\}$$

such that $\sum_{n=1}^{\infty}G_n<\infty$. Suppose $\lim_{n\to\infty}\|x_n-Sx_n\|=0$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. If at least one of the mappings S and T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. Without loss of generality, we may assume that T is semi-compact. By Theorem 1, $\{x_n\}$ is bounded and by assumption of the theorem $\lim_{n\to\infty}\|x_n-Sx_n\|=0$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. This means that there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\to x^*\in K$ as $n_k\to\infty$. Now again by the hypothesis of the theorem, we find

$$\|x^*-Tx^*\|=\lim_{n_k\to\infty}\|x_{n_k}-Tx_{n_k}\|=0$$

and

$$\|x^*-Sx^*\|=\lim_{n_k\to\infty}\|x_{n_k}-Sx_{n_k}\|=0.$$

This shows that $x^*\in F$. According to Theorem 1, the limit $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Then

$$\lim_{n\to\infty}\|x_n-x^*\|=\lim_{n_k\to\infty}\|x_{n_k}-x^*\|=0,$$

which means that $\{x_n\}$ converges to $x^*\in F$. Thus the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings S and T. This completes the proof. □

Theorem 4. *Let E be a real Banach space satisfying Opial's condition and K be a weakly compact subset of E with $K+K\subset K$. Let $S:K\to K$ be a quasi-nonexpansive mapping and $T:K\to K$ be uniformly L-Lipschitzian asymptotically quasi-nonexpansive mapping in the intermediate sense such that $F=F(S)\cap F(T)\neq\varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\delta,1-\delta]$ for some $\delta\in(0,1)$. Let $\{x_n\}$ be the sequence defined by (1.3) with the restrictions $\sum_{n=1}^{\infty}\|u_n\|<\infty$ and $\sum_{n=1}^{\infty}\|v_n\|<\infty$. Put*

$$G_n=\max\left\{\sup_{x\in K,\ q\in F}\left(\|T^nx-q\|-\|x-q\|\right)\vee 0,\ \ \forall n\geq 1\right\}$$

such that $\sum_{n=1}^{\infty}G_n<\infty$. Suppose that S and T have a hybrid fixed point, $I-S$ and $I-T$ are demiclosed at zero and $\{x_n\}$ is an approximating fixed point sequence

for S and T, that is, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of S and T.*

Proof. First, we show that $\omega_w(x_n) \subset F$. Let $x_{n_k} \to x$ weakly. By assumption, we have $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since $I - S$ and $I - T$ are demiclosed at zero, $x \in F$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a common fixed point of S and T. This completes the proof. □

Remark 2. Our results extend, improve and generalize many known results from the existing literature to the case of more general class of mappings and modified two-step iteration scheme with errors considered in this paper.

4. Conclusion

We proved several results on the iteration scheme (1.3) for a pair of quasi-nonexpansive and asymptotically quasi-nonexpansive mapping in the intermediate sense. This class is more general than the classes of nonexpansive, quasi-nonexpansive, asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings. Thus our results provide improvements and extensions of the results of the existing literature.

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Doi: 10.18514/MMN.2015.1261

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 475–482

HU e-ISSN 1787-2413

ON AN INEQUALITY OF REDHEFFER

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Received 23 May, 2014

Abstract. We offer two new proofs of famous Redheffer's inequality, as well establish two converse inequalities for it. Also a hyperbolic analogue is pointed out.

2010 *Mathematics Subject Classification:* 26D05; 26D07; 26D99

Keywords: trigonometric and hyperbolic functions, Redheffer's inequality, Cusa-Huygens inequality

1. INTRODUCTION

In 1969, Redheffer [8] proposed the following inequality

$$\frac{\pi^2-x^2}{\pi^2+x^2} \leq \frac{\sin x}{x}, \quad x \in \mathbb{R}, \tag{1.1}$$

which was proved by Williams [12] in the same year. In literature, this inequality is known as *Redheffer's inequality*. By using the infinite product and induction method, Williams verified this inequality also in 1969, [11]. Motivated by his work, many developments such as generalizations, refinements and applications took place, e.g., see [4].

Thereafter some Redheffer's-type inequalities for other trigonometric, hyperbolic and Bessels function were established, e.g., see [2, 3, 6, 10, 13] and the references therein.

Recently a new proof of (1.1) has appeared in [5], where authors are using the Lagrange mean value theorem, combined with induction , which is very complicated for the reader.

The inequality (1.1) is valid for all $x \in \mathbb{R}$. It is immediate that we may assume $x > 0$ and $x \in (0, \pi)$ as for $x > \pi$, we may let $x = \pi + t$ for $t > 0$, then inequality (1.1) becomes

$$\frac{\sin t}{t} < \frac{2\pi^2+3\pi t+t^2}{2\pi^2+2\pi t+t^2}.$$

This is obvious, as right side is greater than one, and left side less than one. Thus, we may consider $x \in (0, \pi)$.

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So far, all the authors have given the proof of (1.1) by using the induction method. From our proof, it is obvious that the induction method is not needed. In this paper we give new interesting proofs of (1.1), which are based on the elementary calculus. The authors think that this proof could be one from the "Book" (See [1]).

2. New proof of inequality and its converse

Lemma 1. *For all $x \in (0,\pi)$ one has*

$$x+\sin(x) > x^2\frac{\cos(x/2)}{\sin(x/2)}.$$

Proof. Let $h_1(x) = x+\sin(x)-x^2\cos(x/2)/\sin(x/2)$. Then

$$h_1'(x) = 1+\cos(x)-\frac{2x\sin(x)-x^2}{2\sin(x/2)^2} = \frac{\sin(x)^2+x^2-2x\sin(x)}{2\sin(x/2)^2} > 0,$$

where we have used $1+\cos(x) = 2\cos(x/2)^2$ and $2\sin(x/2)\cos(x/2) = \sin(x)$. As $h_1(x) > h_1(0) = 0$, the result follows. □

Theorem 1. *The following inequalities*

$$\frac{\pi^2-x^2}{\pi^2+x^2} \le \frac{\sin(x)}{x} < \frac{12-x^2}{12+x^2} \tag{2.1}$$

hold for $x \in (0,\pi]$.

Proof. Let

$$f_1(x) = \frac{x^2(x+\sin(x))}{x-\sin(x)},$$

for $x \in (0,\pi]$. After some elementary computations, one has

$$\frac{(x-\sin(x))^2}{2x} f_1'(x) = x^2(1+\cos(x)) - \sin(x)(x+\sin(x)) = g_1(x).$$

As $1+\cos(x) = 2\cos(x/2)^2$, and $\sin(x) = 2\sin(x/2)\cos(x/2)$, we get

$$\begin{aligned} g_1'(x) &= 2\cos\left(\frac{x}{2}\right)\left[x^2\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)(x+\sin(x))\right] \\ &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\left[x^2\frac{\cos(x/2)}{\sin(x/2)} - (x+\sin(x))\right] < 0 \end{aligned}$$

by Lemma 1. Thus $f_1'(x) < 0$, and f_1 is strictly decreasing in $x \in (0,\pi]$. We get $f_1(x) > \lim_{x\to\pi} f_1(x) = \pi^2$, which is equivalent to

$$\frac{x+\sin(x)}{x-\sin(x)} > \frac{\pi^2}{x^2},$$

thus the first inequality in (2.1) follows. The second inequality in (2.1) follows similarly from $f_1(x) < \lim_{x\to 0} f_1(x) = 12$. □

The right side of (2.1) improves the known inequality:

$$\frac{\sin(x)}{x} < 1 - \frac{x^2}{\pi^2}, \quad 0 < x < \frac{\pi}{2}.$$

Indeed, this follows by $x^2 + 12 < 2\pi^2$, which is true, as $\pi^2/4 + 12 < 2\pi^2$ becomes $48 < 7\pi^2$.

Proposition 1. *The second inequality in* (2.1) *refines the relation*

$$\frac{\sin(x)}{x} < \frac{\cos(x)+2}{3}, \quad 0 < x < \frac{\pi}{2},$$

so-called Cusa-Huygens inequality [7, 9].

Proof. It is equivalent to prove that,

$$(12 + x^2)\cos(x) + 5x^2 - 12 = s(x) > 0.$$

One has successively:

$$s'(x) = 2x(5 + \cos(x)) - (12 + x^2)\sin(x) - 12,$$

$$s''(x) = -4x\sin(x) - (10 + x^2)\cos(x) + 10,$$

$$s'''(x) = x^2\sin(x) + 6(\sin(x) - x\cos(x)) > 0,$$

as $\sin(x) > x\cos(x)$ (i.e. $\tan(x) > x$). Thus we get

$$s''(x) > s''(0) = 0,\ s'(x) > s'(0) = 0,$$

and finally $s(x) > s(0) = 0$. □

3. An other proof of (1.1)

Lemma 2. *Let $f(x)$ be two times differentiable function on $(0, \pi)$. Define $g(x) = f(x)/\sin(x)$, $h(x) = \sin(x)^2 g'(x)$ and $F(x) = f(x) + f''(x)$. Then the sign of $h'(x)$ depends on the sign of $F(x)$.*

Proof. One has

$$\sin(x)^2 g'(x) = f'(x)\sin(x) - f(x)\cos(x) = h(x),$$

and

$$h'(x) = (f(x) + f''(x))\sin(x) = F(x)\sin(x).$$

As $\sin(x) > 0$ for all $x \in (0, \pi)$, the result follows. □

Theorem 2. *For $x \in (0, \pi)$, we have*

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} < \frac{\sin(x)}{x} < c_1 \frac{\pi^2 - x^2}{\pi^2 + x^2},$$

where $c_1 = 1.07514$.

Proof. Let $g(x) = f(x)/\sin(x)$ for $x \in (0,\pi)$, where

$$f(x) = \frac{x(\pi^2 - x^2)}{\pi^2 + x^2}.$$

One has

$$f'(x) = \frac{\pi^4 - 4\pi^2 x^2 - x^4}{(\pi^2 + x^2)^2},$$

so we get

$$F(x) = f(x) + f''(x) = \frac{xp(x)}{(\pi^2 + x^2)^3},$$

i.e, by Lemma 2 the sign of $h'(x)$ depends on the sign of $p(x)$. Here

$$\begin{aligned} h(x) &= f'(x)\sin(x) - f(x)\cos(x) \\ &= \frac{x\left(\pi^4 - x^4\right)\cos(x) + \left(-x^4 - 4\pi^2 x^2 + \pi^4\right)\sin(x)}{\left(x^2 + \pi^2\right)^2} \\ &= \frac{k(x)}{\left(x^2 + \pi^2\right)^2} \end{aligned}$$

and

$$p(x) = -x^6 - \pi^2 x^4 + \pi^2.x^2(\pi^2 + 4) - 12\pi^4 + \pi^6.$$

Here $h(0) = h(\pi) = 0$ and $h(x_0) = 0$.

Elementary computation gives $k(2\pi/3) < 0$, while with the use of a computer we get $k(17\pi/24) > 0$. So $k(x)$ has a root x_0 between $2\pi/3 \approx 2.0944$ and $17\pi/24 \approx 2.22529$.

Now, by letting $y = x^2$, we get

$$p(x) = q(y) = -y^3 - \pi^2 y^2 + \pi^2 y(\pi^2 + 4) + \pi^6 - 12\pi^4.$$

Since

$$q'(y) = -3y^2 - 2\pi^2 y + \pi^2(\pi^2 + 4),$$

and $y > 0$, the only root of $q'(y) = 0$ is $y^* = (2\pi\sqrt{3 + \pi^2 + 3} - \pi^2)/3$, which lies between π and π^2. One can verify that $q(0) < 0$ and $q(\pi) > 0$. As $q'(y) > 0$ for $y \in (0, y^*)$ and < 0 in $y \in (y^*, \pi^2)$, and as $q(\pi^2) < 0$, we get the following: q is increasing from $q(0)$ to $q(y^*)$ and decreasing from $q(y^*)$ to $q(\pi^2)$. Thus there exist only two roots in $(0, \pi^2)$ to $q(y)$, let them z_1 and z_2. Clearly, z_1 is in $(0,\pi)$ and y_2 in (y^*, π^2). More precisely, as $q(6) < 0$ and $q(7) > 0$, one finds that $y_2 > 6$. These imply that $q(y) < 0$ in $(0, z_1)$, > 0 in (z_1, z_2) and $<$ in (z_2, π^2). In terminology of $P(x)$, we get that $p(x) < 0$ in $(0, \sqrt{z_1})$; > 0 in $(\sqrt{z_1}, \sqrt{z_2})$, and < 0 in $(\sqrt{z_2}, \pi)$. In $(0, \sqrt{z_1})$ clearly $h'(x) < 0$, so $h(x) < h(0) = 0$; similarly in (z_2, π) one has $h'(x) < 0$, so $h(x) > h(\pi) = 0$. Remains the interval $(\sqrt{z_1}, \sqrt{z_2})$. As $z_1 < \pi$ and $z_2 > 6$, we get that

$$\sqrt{z_1} < \sqrt{\pi} < 2 < x_0 < \sqrt{6} \approx 2.44949 < \sqrt{z_2},$$

so we find that x_0 lies between $\sqrt{z_1}$ and $\sqrt{z_2}$. Then clearly $h(x) < h(x_0) = 0$ in $(\sqrt{z_1}, x_0)$, and $h(x) > h(x_0) = 0$ in $(x_0, \sqrt{z_2})$, so all is done.

Thus the minimum point of g is at x_0. As $g(x)$ tends to 1 when x tends to 0 or π, thus the Redheffer's inequality follows. On the other hand, we get also $g(x) \geq g(x_0)$, i.e. the best possible converse to Redheffer's inequality. Now, with the aid of a computer one can find the more precise approximation $x_0 \approx 2.12266$, giving $g(x_0) \approx 0.93012 = 1/c_1$ so the converse to the Redheffer's inequality holds true. □

Lemma 3. *For $a = 2.175$, the function*

$$Q(x) = -3ax^{2a+2} - 2a\pi^a x^{a+2}$$
$$+2a(a-1)(2a-1)\pi^a x^a + a\pi^{2a}x^a - 2a(a+1)(a-2)\pi^{2a}$$

has exactly two roots y_1 and y_2 in $(0, \pi)$.

Proof. We have $Q(1/2) < 0$, $Q(\pi/2) > 0$, and $Q(\pi) < 0$, so Q has two roots y_1 in $(1/2, \pi/2)$, resp. y_2 in $(\pi/2, \pi)$. To show that Q has no other zeros, we have to consider $Q'(x) = xR(x)$, where

$$R(x) = -3a(2a+2)x^{2a} - 2a(a+2)\pi^a x^a - 2a^2(a-a)(2a-1)\pi^a x^{a-2} + 2a\pi^{2a}.$$

One has further

$$R'(x) = 2a^2 x^{a-3} T(x),$$

here

$$T(x) = -3(2a+2)x^{a+2} - (a+2)\pi^a x^2 - 2a^2(a-1)(a-2)(2a-1)\pi^a.$$

Since $a > 2$, we get $T(x) < 0$, so $R'(x) < 0$. One has $Q'(x) = xR(x)$, where $R'(x) < 0$. Since $R(0) > 0$ and $R(\pi) < 0$, and $R(x)$ is strictly decreasing, $R(x) = 0$ can have exactly one root r in $(0, \pi)$. Therefore, $Q(x)$ has exactly one extremal point. Since $Q(0) < 0$, $Q(\pi) < 0$ and Q takes also positive values, clearly $Q(r)$ will be a maximum of $Q(x)$. This shows that Q has exactly two roots in $(0, \pi)$: one in $(0, r)$ and the other one in (r, π).

□

Theorem 3. *For $x \in (0, \pi)$, the following inequality holds*

$$\frac{\sin(x)}{x} < \frac{\pi^a - x^a}{\pi^a + x^a},$$

where $a = 87/40 = 2.175$.

Proof. Inequality can be written as $g(x) = f(x)/\sin(x) > 1$, where

$$f(x) = \frac{x(\pi^a - x^a)}{\pi^a + x^a}.$$

First of all, similarly to the proof of Theorem 2, one has

$$f'(x) = \frac{\pi^{2a} - 2a\pi^a x^a - x^{2a}}{(\pi^a + x^a)^2},$$

so we get

$$h(x) = \frac{K(x)}{(\pi^a + x^a)^2},$$

where

$$K(x) = (\pi^2 - 2a\pi^a x^a - x^{2a})\sin(x) - x(\pi^{2a} - x^{2a})\cos(x).$$

One finds

$$F(x) = \frac{xP(x)}{(\pi^a + x^a)^3},$$

where

$$P(x) = -x^{3a} - \pi^a x^{2a} + 2a(a-1)\pi^a x^{2a-2} + \pi^{2a} x^a - 2a(a+1)\pi^{2a} x^{a-2} + \pi^{3a}.$$

Now the proof of Theorem runs as follows: Since $K(1/2) > 0$ and $K(\pi/4) < 0$, $K(\pi/2) > 0$ and $K(2\pi/3) < 0$, we get x_1 in $(1/2, \pi/4)$ such that $K(x_1) = 0$ and x_2 in $(\pi/2, 2\pi/3)$ such that $K(x_2) = 0$. As $P(0) > 0$ and $P(\pi/4) < 0$. One has

$$P'(x) = xQ(x),$$

where $Q(x)$ is as in Lemma 3. It follows from Lemma 3 that that, $Q(x) < 0$ for x in $(0, y_1)$ and (y_2, π), and $Q(x) > 0$ for x in (y_1, y_2). This shows that $P(x)$ is strictly decreasing in $(0, y_1)$ and (y_2, π) and strictly increasing in (p_1, p_2). This implies that $P(x)$ has a unique root p_1 in $(0, y_1)$, as well as a unique p_2 in (y_1, y_2) and p_3 in (y_2, π). By approximate computation we can see that $p_1 < 2/3$, $p_2 > 1$ and $p_3 > 2$, and $p_1 < p_2 < p_3$. This shows that for the roots x_1 and x_2 of function $k(x)$ one has that x_1 is in (p_1, p_2) and x_2 in (p_2, p_3). As $h(0) = h(x_1) = h(x_2) = h(\pi) = 0$, we get the following;

(1) for $x \in (0, p_1)$ one has $h(x) > h(0) = 0$,
(2) for $x \in (p_1, x_1)$ one has $h(x) > h(x_1) = 0$,
(3) for $x \in (x_1, p_2)$ one has $h(x) < h(x_1) = 0$,
(4) for $x \in (p_2, x_2)$ one has $h(x) < h(x_2) = 0$,
(5) for $x \in (x_2, p_3)$ one has $h(x) > h(x_2) = 0$,
(6) for $x \in (p_3, \pi)$ one has $h(x) > h(\pi) = 0$.

From the above it follows that x_1 is a local minimum point , while x_2 a local maximum point of $g(x)$. Clearly, $g(x_1) > \lim g(x)$, when x tends to zero, $= 1$ and $\lim g(x)$ at $x = \pi$ is $a/2 > 1$. This completes the proof.

□

4. A HYPERBOLIC ANALOGUE

Lemma 4. *For $x \in (0, \infty)$,*

$$x + \sinh(x) > x^2 \coth(x/2).$$

Let $h_2(x) = x + \sinh(x) - x^2 \coth(x/2)$.

Proof. One has

$$h_2'(x) = \frac{(\sinh(x)-x)^2}{2\sinh(x/2)^2} > 0,$$

and $h_2(x) > \lim_{x\to 0} h_2(x) = 0$. Thus, inequality holds. □

Theorem 4. *For $x \in (0,\infty)$, we have*

$$\frac{\sinh(x)+x}{\sinh(x)-x} > \frac{12}{x^2}. \tag{4.1}$$

Proof. Let

$$f_2(x) = \frac{x^2(\sinh(x)+x)}{\sinh(x)-x},$$

for $x \in (0,\infty)$. We get

$$\frac{(\sinh(x)-x)^2}{2x} f_2'(x) = (\sinh(x/2))((x+\sinh(x)) - x^2\cosh(x/2)^2),$$

which is positive by Lemma 4. Thus, f_2 is strictly increasing in $x \in (0,\infty)$, and the inequality (4.1) follows from $f_2(x) > \lim_{x\to 0} f_2(x) = 12$. Therefore, we have an analogue of the second inequality in the circular case, and this is (4.1). When $x^2 < 12$, then (4.1) becomes

$$\frac{\sinh(x)}{x} < \frac{12+x^2}{12-x^2}.$$

□

It is interesting to observe that

$$\frac{12+x^2}{12-x^2} < \frac{\pi^2+x^2}{\pi^2-x^2},$$

if $x < \pi$. Indeed this becomes: $\pi^2 < 12$.

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Doi: 10.18514/MMN.2015.1214

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 483–490

HU e-ISSN 1787-2413

POWER SUBGROUPS OF THE EXTENDED HECKE GROUPS

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Received 22 April, 2014

Abstract. We consider the extended Hecke groups $\overline{H}(\lambda_q)$ generated by $T(z) = -1 / z$, $S(z) = -1/(z+\lambda_q)$ and $R(z) = 1/\overline{z}$ with $\lambda_q = 2\cos(\pi/q)$ for $q \geq 3$ integer. In this article, we study the abstract group structures of the power subgroups $\overline{H}^m(\lambda_q)$ of $\overline{H}(\lambda_q)$ for each positive integer m. Then, we give the relations between commutator subgroups and power subgroups.

2010 *Mathematics Subject Classification:* 20H10; 11F06

Keywords: extended Hecke groups, power subgroups

1. INTRODUCTION

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \text{ and } S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number.

E. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \geq 3$ integer, or $\lambda \geq 2$. We will focus on the discrete case with $\lambda < 2$, i.e., those with $\lambda = \lambda_q$, q an integer ≥ 3. These groups have come to be known as the *Hecke Groups*, and we will denote them $H(\lambda_q)$ for $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q and it has a presentation

$$H(\lambda_q) = < T, S \mid T^2 = S^q = I > \cong C_2 * C_q.$$

Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is, all the groups $H(\lambda_q)$ are triangle groups. The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2,\mathbb{Z}[\lambda_q])$, for $q \geq 4$. The groups $H(\sqrt{2})$ and $H(\sqrt{3})$ are of particular interest, since they are the only Hecke groups, aside from the modular group, whose elements are completely known.

The extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$, for $q \geq 3$ integer,

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in [9, 10] and [6]. Thus, the extended Hecke group $\overline{H}(\lambda_q)$ has a presentation,

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = I, RT = TR, RS = S^{q-1}R >\cong D_2 *_{C_2} D_q. \tag{1.1}$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

Now we give some information about the power subgroups of $\overline{H}(\lambda_q)$.

Let m be a positive integer. Let us define $\overline{H}^m(\lambda_q)$ to be the subgroup generated by the m^{th} powers of all elements of $\overline{H}(\lambda_q)$. The subgroup $\overline{H}^m(\lambda_q)$ is called the $m-th$ power subgroup of $\overline{H}(\lambda_q)$. As fully invariant subgroups, they are normal in $\overline{H}(\lambda_q)$.

From the definition one can easily deduce that

$$\overline{H}^{mk}(\lambda_q) \leq \overline{H}^m(\lambda_q)$$

and

$$\overline{H}^{mk}(\lambda_q) \leq \left(\overline{H}^m(\lambda_q)\right)^k.$$

Using the last two inequalities imply that $\overline{H}^m(\lambda_q).\overline{H}^k(\lambda_q) = \overline{H}^{(m,k)}(\lambda_q)$ where (m,k) denotes the greatest common diviser of m and k.

The power subgroups of the Hecke groups $H(\lambda_q)$ have been studied and classified in [2,3] and [5]. For $q \geq 3$ prime, the power subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ were studied by Sahin, Ikikardes and Koruoğlu in [11, 12] and [13].

The aim of this paper is to study the power subgroups $\overline{H}^m(\lambda_q)$ of the extended Hecke groups $\overline{H}(\lambda_q)$, $q \geq 3$ integer. For each positive integer m, we determine the abstract group structures and generators of $\overline{H}^m(\lambda_q)$. Also, we give the signatures of $\overline{H}^m(\lambda_q)$. To get all these results, we use the techniques of combinatorial group theory (Reidemeister-Schreier method, permutation method and Riemann-Hurwitz formula). Finally, we give the relations between commutator subgroups and power subgroups.

2. The group structure of power subgroups of $\overline{H}(\lambda_q)$

Now we consider the presentation of the extended Hecke group $\overline{H}(\lambda_q)$ given in (1.1):

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = I, TR = RT, RS = S^{-1}R >$$

Firstly, we find a presentation for the quotient $\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q)$ by adding the relation $X^m = I$ for all $X \in \overline{H}(\lambda_q)$ to the presentation of $\overline{H}(\lambda_q)$. The order of $\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q)$ gives us the index. We have,

$$\begin{aligned}\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) <T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = T^m = S^m \\ = R^m = (TS)^m = (RS)^m = (TR)^m = ... = I >.\end{aligned} \tag{2.1}$$

Thus we use the Reidemeister-Schreier process to find the generators and the presentations of the power subgroups $\overline{H}^m(\lambda_q)$, $q \geq 3$ integer (for the method, please see [2] and [5]).

Firstly, we now discuss the group theoretical structure of these subgroups for $q \geq 3$ odd integer. We start with the case $m = 2$.

Theorem 1. 1) *If $q \geq 3$ is an odd integer, then $\overline{H}^2(\lambda_q)$ is the free product of two finite cyclic groups of order q, i.e.,*

$$\overline{H}^2(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

2) *If $q > 3$ is an even integer, then $\overline{H}^2(\lambda_q)$ is the free product of the infinite cyclic group and two finite cyclic groups of order $q/2$, i.e.,*

$$\begin{aligned}\overline{H}^2(\lambda_q) &= \left\langle S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I \right\rangle, \\ &\cong C_{q/2} * C_{q/2} * \mathbb{Z}.\end{aligned}$$

Proof. 1) By (2.1), we have

$$\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q) \cong \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong C_2 \times C_2,$$

since $S^2 = S^q = I$ and $(2,q) = 1$. Now we can choose $\{I, T, R, TR\}$ as a Schreier transversal for $\overline{H}^2(\lambda_q)$. According to the Reidemeister-Schreier method (see [8]), we get the generators of $\overline{H}^2(\lambda_q)$ as the followings :

$I.T.(T)^{-1} = I$,	$I.S.(I)^{-1} = S$,	$I.R.(R)^{-1} = I$,
$T.T.(I)^{-1} = I$,	$T.S.(T)^{-1} = TST^{-1}$,	$T.R.(TR)^{-1} = I$,
$R.T.(TR)^{-1} = RTRT$,	$R.S.(R)^{-1} = RSR^{-1}$,	$R.R.(I)^{-1} = I$,
$TR.T.(R)^{-1} = TRTR$,	$TR.S.(TR)^{-1} = TRSR^{-1}T^{-1}$,	$TR.R.(T)^{-1} = I$.

Since $TRTR = RTRT = I$, $RSR^{-1} = S^{-1}$ and $TRSR^{-1}T^{-1} = (TST)^{-1}$, the generators of $\overline{H}^2(\lambda_q)$ are S and TST. Thus $\overline{H}^2(\lambda_q)$ has a presentation

$$\overline{H}^2(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_q * C_q.$$

Also, using the permutation method (see [14]) and the Riemann-Hurwitz formula, we get the signature of $\overline{H}^2(\lambda_q)$ as $(0; q, q, \infty) = (0; q^{(2)}, \infty)$.

2) By 2.1, the quotient group $\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q)$ is

$$\begin{aligned}\overline{H}(\lambda_q)/\overline{H}^2(\lambda_q) &\cong \langle T, S, R \mid T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = (TS)^2 = I \rangle \\ &\cong C_2 \times C_2 \times C_2,\end{aligned}$$

since $S^2 = S^q = I$. Now we can choose Schreier transversal as $\{I, T, S, R, TS, TR, SR, TSR\}$. According to the Reidemeister-Schreier method, all possible products are

$$\begin{array}{ll}
I.T.(T)^{-1} = I, & TS.T.(S)^{-1} = TSTS^{-1}, \\
T.T.(I)^{-1} = I, & TR.T.(R)^{-1} = I, \\
S.T.(TS)^{-1} = STS^{-1}T, & SR.T.(TSR)^{-1} = SRTRS^{-1}T, \\
R.T.(TR)^{-1} = I, & TSR.T.(SR)^{-1} = TSRTRS^{-1}, \\
I.S.(S)^{-1} = I, & TS.S.(T)^{-1} = TS^2T, \\
T.S.(TS)^{-1} = I, & TR.S.(TSR)^{-1} = I, \\
S.S.(I)^{-1} = S^2, & SR.S.(R)^{-1} = I, \\
R.S.(SR)^{-1} = RSRS^{-1}, & TSR.S.(TR)^{-1} = TSRSRT, \\
I.R.(R)^{-1} = I, & TS.R.(TSR)^{-1} = I, \\
T.R.(TR)^{-1} = I, & TR.R.(T)^{-1} = I, \\
S.R.(SR)^{-1} = I, & SR.R.(S)^{-1} = I, \\
R.R.(I)^{-1} = I, & TSR.R.(TS)^{-1} = I,
\end{array}$$

Since $SRTRS^{-1}T = STS^{-1}T$, $TSRTRS^{-1} = TSTS^{-1}$, $TSTS^{-1} = (STS^{-1}T)^{-1}$, $RSRS^{-1} = S^{-2}$ and $TSRSRT = I$, the generators of $\overline{H}^2(\lambda_q)$ are S^2,TS^2T and $TSTS^{-1}$. Thus $\overline{H}^2(\lambda_q)$ has a presentation

$$\overline{H}^2(\lambda_q) = \left\langle S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I \right\rangle.$$

Therefore $\overline{H}^2(\lambda_q)$ has the signature $\left(0; (q/2)^{(2)}, \infty^{(2)}\right)$. □

Corollary 1. 1) *If $q \geq 3$ is an odd integer and if m is a positive even integer such that $(m,q) = 1$, then $\overline{H}^m(\lambda_q) = \overline{H}^2(\lambda_q)$.*

2) *If $q \geq 3$ is an integer and if m is a positive odd integer, then $\overline{H}^m(\lambda_q) = \overline{H}(\lambda_q)$.*

Proof. 1) If $q \geq 3$ is an odd integer and if m is a positive even integer such that $(m,q) = 1$, then by (2.1), we get

$$\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) \cong < T, R \mid T^2 = R^2 = (TR)^2 = I > \cong D_2,$$

from the relations

$$R^2 = R^m = I,\ S^q = S^m = I \text{ and } T^2 = T^m = I.$$

Since $\overline{H}^2(\lambda_q)$ is the only normal subgroup of index 4, we have $\overline{H}^m(\lambda_q) = \overline{H}^2(\lambda_q)$.

2) If $q \geq 3$ is an integer and if m is a positive odd integer, then by (2.1), we obtain

$$\overline{H}^m(\lambda_q) = \overline{H}(\lambda_q),$$

from the relations

$$R^2 = R^m = I,\ T^2 = T^m = I \text{ and } (RS)^2 = (RS)^m = I.$$

□

Theorem 2. *Let $q > 3$ be an even integer and let m be a positive even integer such that $(m,q) = 2$. The normal subgroup $\overline{H}^m(\lambda_q)$ is the free product of finite cyclic groups m of order $q/2$ and the infinite cyclic group $\mathbb{Z}$, i.e.,*

$$\overline{H}^m(\lambda_q) = < \underbrace{(TS)(TS)...(TS)}_{(m-1)\ times} TS^{-1} > * < S^2 > * < TS^2T > *$$
$$< TSTS^2TS^{-1}T > *...* < \underbrace{(TS)(TS)...(TS)}_{(m-2)\ times} TS^2T \underbrace{(S^{-1}T)(S^{-1}T)...(S^{-1}T)}_{(m-2)\ times} > .$$

Proof. By (2.1), we have

$$\overline{H}(\lambda_q)/\overline{H}^m(\lambda_q) \cong \left\langle T,S,R \mid T^2 = S^2 = R^2 = (TR)^2 = (RS)^2 = (TS)^m = I \right\rangle,$$
$$\cong C_2 \times D_m,$$

since $S^q = S^m = I$. Now we can choose $\{I,\ T,\ S,\ TS,\ TST,\ TSTS,\ ...,\ \underbrace{(TS)(TS)...(TS)}_{(m-1)\ \text{times}},\ R,\ TR,\ SR,\ TSR,\ TSTR,\ TSTSR,\ ...,\ \underbrace{(TS)(TS)...(TS)}_{(m-1)\ \text{times}}R\}$ as a Schreier transversal for $\overline{H}^m(\lambda_q)$. According to the Reidemeister-Schreier method, we find the generators generators of $\overline{H}^m(\lambda_q)$ as

$$a_1 = \underbrace{(TS)(TS)...(TS)}_{(m-1)\ \text{times}} TS^{-1},\ a_2 = S^2,\ a_3 = TS^2T,\ a_4 = TSTS^2TS^{-1}T, ...,$$
$$a_{m+1} = \underbrace{(TS)(TS)...(TS)}_{(m-2)\ \text{times}} TS^2T \underbrace{(S^{-1}T)(S^{-1}T)...(S^{-1}T)}_{(m-2)\ \text{times}}.$$

Thus $\overline{H}^m(\lambda_q)$ has a presentation $\overline{H}^m(\lambda_q) =$

$$\left\langle a_1,a_2,a_3,a_4,...,a_{m+1} \mid (a_2)^{q/2} = (a_3)^{q/2} = (a_4)^{q/2} = \cdots = (a_{m+1})^{q/2} = I \right\rangle.$$

Also the signature of $\overline{H}^m(\lambda_q)$ is $\left(0; (q/2)^{(m)}, \infty^{(2)}\right)$. □

We are only left to consider the case where $(m,2) = 2$ and $(m,q) = d > 2$. In this case, the above techniques do not say much about $\overline{H}^m(\lambda_q)$. But, we can say something about $\overline{H}^m(\lambda_q)$ some special cases of q. To do these, we need the following results about the commutator subgroups of $\overline{H}(\lambda_q)$ in [9] and [10] .

Lemma 1. 1) *For an odd number $q \geq 3$:*

$$\overline{H}'(\lambda_q) = \left\langle S, TST \mid S^q = (TST)^q = I \right\rangle \cong C_q * C_q.$$

2) $\overline{H}'(\lambda_q)/\ \overline{H}''(\lambda_q) =< S, TST \mid S^q = (TST)^q = (S.TST)^q = I > \cong C_q \times C_q.$
3) *For an even integer $q > 3$:*

$$\overline{H}'(\lambda_q) =< S^2, TS^2T, TSTS^{-1} \mid (S^2)^{q/2} = \left(TS^2T\right)^{q/2} = (TSTS^{-1})^{\infty} = I >$$
$$\cong C_{q/2} * C_{q/2} * \mathbb{Z}.$$

By using Lemma 1 and Theorem 1 we get the following

Corollary 2. *Let $q \geq 3$ be an integer. Then $\overline{H}'(\lambda_q) \cong \overline{H}^2(\lambda_q)$.*

Theorem 3. *Let $q \geq 3$ be an odd integer and let m be a positive integer. The groups $\overline{H}^{2q}(\lambda_q)$ are the subgroups of the second commutator subgroup $\overline{H}''(\lambda_q)$.*

Proof. Since $\overline{H}'(\lambda_q) = \overline{H}^2(\lambda_q)$, we get $\left(\overline{H}^2(\lambda_q)\right)^q \leq \overline{H}^2(\lambda_q)$ and $\left(\overline{H}'(\lambda_q)\right)^q \leq \overline{H}'(\lambda_q)$. Also we know that

$$\overline{H}'(\lambda_q)/\left(\overline{H}'(\lambda_q)\right)^q = < S, TST \mid S^q = (TST)^q = (S.TST)^q = \cdots = I > .$$

Therefore the index $\left|\overline{H}'(\lambda_q) : \left(\overline{H}'(\lambda_q)\right)^q\right|$ is greater than or equal to the index $\left|\overline{H}'(\lambda_q) : \overline{H}''(\lambda_q)\right| = q^2$. Thus we have

$$\overline{H}^{2q}(\lambda_q) \subset \overline{H}''(\lambda_q).$$

□

By means of this results, we are going to be able to investigate the subgroups $\overline{H}^{2qm}(\lambda_q)$. We have by Schreier's theorem the following theorem:

Theorem 4. *Let $q \geq 3$ be an odd integer. The groups $\overline{H}^{2qm}(\lambda_q)$ are free.*

Finally, we can only say something the case $q = 4$. This Hecke group is very important and studied by many authors, see [1] and [7].

Theorem 5. *i)* $\left|\overline{H}^2(\lambda_4) : (\overline{H}^2)^2(\lambda_4)\right| = 8$.

ii) The group $(\overline{H}^2)^2(\lambda_4)$ is a free group of rank 9.

Proof. i) If we take $k_1 = S^2$, $k_2 = TS^2T$ and $k_3 = TSTS^3$, then the quotient group $\overline{H}^2(\lambda_4)/(\overline{H}^2)^2(\lambda_4)$ is the group obtained by adding the relation $k_i^2 = I$ to the relations of $(\overline{H}^2)^2(\lambda_4)$, for $i \in \{1,2,3\}$. Thus we have

$$\overline{H}^2(\lambda_4)/(\overline{H}^2)^2(\lambda_4) \cong C_2 \times C_2 \times C_2.$$

Therefore, we obtain $\left|\overline{H}^2(\lambda_4) : (\overline{H}^2)^2(\lambda_4)\right| = 8$.

ii) Let $\Sigma = \{I, k_1, k_2, k_3, k_1k_2, k_1k_3, k_2k_3, k_1k_2k_3\}$ be a Schreier transversal for $(\overline{H}^2)^2(\lambda_4)$. Using the Reidemeister-Schreier method, we obtain the generators of $(\overline{H}^2)^2(\lambda_4)$ as follows:

$$
\begin{aligned}
&I.k_1.(k_1)^{-1} = I, && I.k_2.(k_2)^{-1} = I,\\
&k_1.k_1.(I)^{-1} = I, && k_1.k_2.(k_1k_2)^{-1} = I,\\
&k_2.k_1.(k_1k_2)^{-1} = k_2k_1k_2k_1, && k_2.k_2.(I)^{-1} = I,\\
&k_3.k_1.(k_1k_3)^{-1} = k_3k_1k_3^{-1}k_1, && k_3.k_2.(k_2k_3)^{-1} = k_3k_2k_3^{-1}k_2,\\
&k_1k_2.k_1.(k_2)^{-1} = k_1k_2k_1k_2, && k_1k_2.k_2.(k_1)^{-1} = I,\\
&k_1k_3.k_1.(k_3)^{-1} = k_1k_3k_1k_3^{-1}, && k_1k_3.k_2.(k_1k_2k_3)^{-1} = k_1k_3k_2k_3^{-1}k_2k_1,\\
&k_2k_3.k_1.(k_1k_2k_3)^{-1} = k_2k_3k_1k_3^{-1}k_2k_1, && k_2k_3.k_2.(k_3)^{-1} = k_2k_3k_2k_3^{-1},\\
&k_1k_2k_3.k_1.(k_2k_3)^{-1} = k_1k_2k_3k_1k_3^{-1}k_2, && k_1k_2k_3.k_2.(k_1k_3)^{-1} = k_1k_2k_3k_2k_3^{-1}k_1,
\end{aligned}
$$

$$
\begin{aligned}
&I.k_3.(k_3)^{-1} = I,\\
&k_1.k_3.(k_1k_3)^{-1} = I,\\
&k_2.k_3.(k_2k_3)^{-1} = I,\\
&k_3.k_3.(I)^{-1} = k_3^2,\\
&k_1k_2.k_3.(k_1k_2k_3)^{-1} = I,\\
&k_1k_3.k_3.(k_1)^{-1} = k_1k_3^2k_1,\\
&k_2k_3.k_3.(k_2)^{-1} = k_2k_3^2k_2,\\
&k_1k_2k_3.k_3.(k_1k_2)^{-1} = k_1k_2k_3^2k_2k_1.
\end{aligned}
$$

After some calculations, we get the generators of $(\overline{H}^2)^2(\lambda_4)$ *as*

$$
\begin{array}{lll}
k_1k_2k_3k_2k_3^{-1}k_1, & k_1k_2k_1k_2, & k_3^2,\\
k_1k_2k_3k_1k_3^{-1}k_2, & k_1k_3k_1k_3^{-1}, & k_1k_3^2k_1,\\
k_1k_2k_3^2k_2k_1, & k_2k_3k_2k_3^{-1}, & k_2k_3^2k_2.
\end{array}
$$

Also, we find the signature of $(\overline{H}^2)^2(\lambda_4)$ *as* $(1; \underbrace{\infty, \infty, \cdots, \infty}_{8 \text{ times}}) = (1; \infty^{(8)})$. □

Notice that the group $(\overline{H}^2)^2(\lambda_4) = (H^2)^2(\lambda_4)$ is the principal congruence subgroup $H_4(\lambda_4)$ of $H(\lambda_4)$.

Since $\overline{H}^{4k}(\lambda_4) \leq \overline{H}^4(\lambda_4) \leq (\overline{H}^2)^2(\lambda_4)$, we are going to be able to investigate the subgroups $\overline{H}^{4k}(\lambda_4)$. We have by Schreier's theorem the following theorem:

Corollary 3. *The groups* $\overline{H}^{4k}(\lambda_4)$ *are free.*

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Doi: 10.18514/MMN.2015.1099

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 491–501

HU e-ISSN 1787-2413

SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR s-CONVEX FUNCTIONS

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Received 15 January, 2014

Abstract. Some new results related of the left-hand side of the Hermite-Hadamard type inequalities for the class of mappings whose second derivatives at certain powers are $s-$convex in the second sense are established. Also, some applications to special means of real numbers are provided.

2010 *Mathematics Subject Classification:* 26A51; 26D07; 26D10; 26D15

Keywords: Hermite-Hadamard type inequality, $s-$convex function, Hölder inequality

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard inequality [6]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if f is concave. For recent results, generalizations and new inequalities related to the Hermite-Hadamard inequality see [3,4,9,11,12,14,17].

The classical Hermite- Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$.

Definition 1. Let I be on interval in $\mathbb{R}$. Then $f : I \to R$ is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The class of functions which are s-convex in the second sense has been stated as the following (see [7]).

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Definition 2. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex (in the second sense), if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$. Some interesting and important inequalities for s-convex (in the second sense) functions can be found in [1,2,5,8,10,15,16,18,19]. It can be easily seen that convexity means just s-convexity when $s = 1$.

In [13], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $f'' \in L_1[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt,$$

where

$$m(t) := \begin{cases} t^2 & ,t \in [0,\frac{1}{2}) \\ (1-t)^2 & ,t \in [\frac{1}{2},1]. \end{cases}$$

Also, the main inequalities in [13], pointed out as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)| + |f''(b)|}{2}\right]. \tag{1.1}$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a,b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8(2p+1)^{1/p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{1/q} \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose second derivatives at certain powers are s-convex functions in the second sense.

2. MAIN RESULTS

In order to prove our main results we need the following lemma:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$, then the following equaliy holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)$$

$$= \frac{(b-a)^2}{16}\left\{\int_0^1 t^2 f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt + \int_0^1 t^2 f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}. \quad (2.1)$$

Proof. By integration by parts, we have the following identity

$$\begin{aligned}
&\int_0^1 t^2 f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt + \int_0^1 t^2 f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&= \frac{4}{b-a}\left\{\int_0^1 t f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt - \int_0^1 t f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\} \\
&= \frac{4}{b-a}\left\{t\frac{2}{a-b}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\Big|_0^1 + \frac{2}{b-a}\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right. \\
&\quad \left. + t\frac{2}{b-a}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\Big|_0^1 + \frac{2}{b-a}\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\} \\
&= -\frac{16}{(b-a)^2}f\left(\frac{a+b}{2}\right) + \frac{8}{(b-a)^2}\int_0^1 f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
&\quad + \frac{8}{(b-a)^2}\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.
\end{aligned}$$

Using the change of the variable in last integrals, we get the required identity (2.1). □

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be twice differantiable mapping on I°, $a,b \in I$ with $a < b$. If $|f''|$ is s- convex in the second sense on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequaliy holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2\left(2^{s+4}-(4s+12)\right)}{2^{s+4}(s+1)(s+2)(s+3)}\left[|f''(a)|+|f''(b)|\right]. \quad (2.2)$$

Proof. Using Lemma 2 and the s- convexity in the second sense of $|f''|$, we find

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt \\
&\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[\left(\frac{t}{2}\right)^s\left|f''(a)\right|+\left(\frac{2-t}{2}\right)^s\left|f''(b)\right|\right. \\
&\quad \left.+\left(\frac{2-t}{2}\right)^s\left|f''(a)\right|+\left(\frac{t}{2}\right)^s\left|f''(b)\right|\right]dt \\
&= \frac{(b-a)^2\left(\left|f''(a)\right|+\left|f''(b)\right|\right)}{2^{s+4}}\left[\int_0^1\left[t^{s+2}+t^2(2-t)^s\right]dt\right] \\
&= \frac{(b-a)^2\left(2^{s+4}-(4s+12)\right)}{2^{s+4}(s+1)(s+2)(s+3)}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]
\end{aligned}
$$

where we have used the fact that

$$
\int_0^1 t^2(2-t)^s\,dt=\frac{2^{s+4}-\left(s^2+7s+14\right)}{(s+1)(s+2)(s+3)},\quad\text{and}\quad\int_0^1 t^{s+2}dt=\frac{1}{s+3}.
$$

The proof is completed. □

Remark 1. If we take $s=1$ in Theorem 3, then inequality (2.2) becomes inequality (1.1).

The next theorem gives a new upper bound of the left-hand side Hermite-Hadamard inequality for s-convex functions:

Theorem 4. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be twice differantiable mapping on I°, $a,b\in I$ with $a<b$. If $|f''|^q$ is s- convex in the second sense on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequaliy holds:*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right] \qquad (2.3)\\
&\leq \frac{(b-a)^2}{8}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\frac{1}{(s+1)^{\frac{1}{q}}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using Hölder's inequality and the s-convexity in the second sense of $|f''|^q$, we find

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{16}\left(\int_0^1 |t|^{2p}\,dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left[\left(\frac{t}{2}\right)^s|f''(a)|^q + \left(\frac{2-t}{2}\right)^s|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\left(\frac{2-t}{2}\right)^s|f''(a)|^q + \left(\frac{t}{2}\right)^s|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{|f''(a)|^q+\left(2^{s+1}-1\right)|f''(b)|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left(2^{s+1}-1\right)|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

Let $a_1 = |f''(a)|^q$, $b_1 = \left(2^{s+1}-1\right)|f''(b)|^q$, $a_2 = \left(2^{s+1}-1\right)|f''(a)|^q$, $b_2 = |f''(b)|^q$. Here , $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that

$$\sum_{k=1}^n (a_k+b_k)^s \le \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s.$$

for $(0 \le s < 1)$, $a_1, a_2, ..., a_n \ge 0$, $b_1, b_2, ..., b_n \ge 0$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left[\left(|f''(a)|+\left(2^{s+1}-1\right)^{\frac{1}{q}}|f''(b)|\right)+\left(\left(2^{s+1}-1\right)^{\frac{1}{q}}|f''(a)|+|f''(b)|\right)\right]$$
$$= \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(1+\left(2^{s+1}-1\right)^{\frac{1}{q}}\right)\left(|f''(a)|+|f''(b)|\right)\right]$$
$$\le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(2^{s+1}\right)\left(|f''(a)|+|f''(b)|\right).$$

This completes the proof. □

Corollary 1. *Under assumption Theorem 4, if we take $s=1$, then the inequality (2.3) becomes the following inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{16\,(2p+1)^{\frac{1}{p}}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^2}{2^{\frac{2}{q}+2}\,(2p+1)^{\frac{1}{p}}}\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

Theorem 5. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be twice differantiable mapping on I°, $a,b\in I$ with $a<b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$, for some fixed $s\in(0,1]$ and $q\geq 1$, then the following inequaliy holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \qquad (2.4)\\
&\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^s(s+3)}|f''(a)|^q+\frac{2^{s+4}-(s^2+7s+14)}{2^s(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2^{s+4}-(s^2+7s+14)}{2^s(s+1)(s+2)(s+3)}|f''(a)|^q+\frac{1}{2^s(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 2, using the well known power mean inequality for $q\geq 1$ and the s- convexity in the second sense of $|f''|^q$, we find

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{16}\left\{\left(\int_0^1 t^2\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^2\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right. \\
&\left.\left.+\left(\int_0^1 t^2\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]\right\} \\
&\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^2\left[\left(\frac{t}{2}\right)^s|f''(a)|^q+\left(\frac{2-t}{2}\right)^s|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^1 t^2\left[\left(\frac{2-t}{2}\right)^s\left|f''(a)\right|^q+\left(\frac{t}{2}\right)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{2^s(s+3)}\left|f''(a)\right|^q+\frac{2^{s+4}-\left(s^2+7s+14\right)}{2^s(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{2^{s+4}-\left(s^2+7s+14\right)}{2^s(s+1)(s+2)(s+3)}\left|f''(a)\right|^q+\frac{1}{2^s(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

which completes the proof of Theorem 5. □

Corollary 2. *Under assumption Theorem 5, if we take $s=1$, then inequality (2.4) becomes the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^2}{48}\left[\left(\frac{3\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{5\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right].$$

3. Applications to special means

In [1], the following result is given.

Let $g: I\to I_1\subseteq[0,\infty)$ be a non-negative convex functions on I. Then $g^s(x)$ is $s-$convex on I, $0<s<1$.

For arbitrary positive real numbers a,b $(a\neq b)$, we shall consider the following special means:

(a) The arithmetic mean: $A=A(a,b):=\dfrac{a+b}{2}$, $a,b>0$,

(b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b>0,$$

(c) The logarithmic mean:

$$L=L(a,b):=\begin{cases}a & \text{if } a=b\\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b\end{cases},\ a,b>0,$$

(d) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \; p \in \mathbb{R} \setminus \{-1,0\}; \; a,b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

(1) Let $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^{s+1}$, $s \in (0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)dx = L_{s+1}^{s+1}(a,b),$$
$$\frac{f(a)+f(b)}{2} = A(a^{s+1},b^{s+1}),$$
$$f\left(\frac{a+b}{2}\right) = A^{s+1}(a,b).$$

(a) From Theorem 3, we obtain

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{(b-a)^2 s(2^{s+4}-(4s+12))}{2^{s+3}(s+2)(s+3)} A(a^{s-1},b^{s-1}).$$

For instance, if $s = 1$ then we get

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{1}{12}(b-a)^2.$$

(b) From Theorem 4, we have

$$\begin{aligned} &\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \\ &\leq \frac{s(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{(s-1)q} + \left(2^{s+1}-1\right)b^{(s-1)q}\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\left(2^{s+1}-1\right)a^{(s-1)q} + b^{(s-1)q}\right)^{\frac{1}{q}}\right] \\ &\leq \frac{s(b-a)^2}{4}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}-1}} A(a^{s-1},b^{s-1}), \end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4}\left(\frac{1}{4p+2}\right)^{\frac{1}{p}}, \; p > 1.$$

(c) From Theorem 5, we get

$$\left|L_{s+1}^{s+1}(a,b)-A^{s+1}(a,b)\right| \leq \frac{(b-a)^2}{2^{s+4}(s+2)(s+3)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}$$
$$\left[\left(s(s+1)(s+2)a^{(s-1)q}+\left(2^{s+4}s-s^3-7s^2-14s\right)b^{(s-1)q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(2^{s+4}s-s^3-7s^2-14s\right)a^{(s-1)q}+s(s+1)(s+2)b^{(s-1)q}\right)^{\frac{1}{q}}\right]$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\left|L_2^2(a,b)-A^2(a,b)\right| \leq \frac{(b-a)^2(48)^{\frac{1}{q}}}{576}, \quad q>1.$$

(2) Let $f:[a,b]\subseteq[0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\dfrac{1}{x^s}\in K_s^2$, $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)dx = L_{-s}^{-s}(a,b),$$
$$\frac{f(a)+f(b)}{2} = A(a^{-s},b^{-s}),$$
$$f\left(\frac{a+b}{2}\right) = A^{-s}(a,b).$$

(a) From Theorem 3, we obtain

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right| \leq \frac{(b-a)^2\left(2^{s+4}-(4s+12)\right)}{2^{s+4}(s+1)(s+2)(s+3)}\left[a^{(-s-2)}+b^{(-s-2)}\right].$$

For instance, if $s=1$ then we get

$$\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right| \leq \frac{(b-a)^2}{24}A\left(a^{-3},b^{-3}\right).$$

(b) From Theorem 4, we have

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right|$$
$$\leq \frac{s(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{(-s-2)q}+\left(2^{s+1}-1\right)b^{(-s-2)q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(2^{s+1}-1\right)a^{(-s-2)q}+b^{(-s-2)q}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{8}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}-1}}\left[a^{(-s-2)}+b^{(-s-2)}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\begin{aligned}&\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right|\\&\leq\frac{(b-a)^2}{32}\left(\frac{2}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{-3q}+3b^{-3q}\right)^{\frac{1}{q}}+\left(3a^{-3q}+b^{-3q}\right)^{\frac{1}{q}}\right]\\&\leq\frac{(b-a)^2}{8}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}A\left(a^{-3},b^{-3}\right)\end{aligned}$$

where $q>1$.

(c) From Theorem 5, we get

$$\begin{aligned}&|L_{-s}^{-s}(a,b)-A^{-s}(a,b)|\leq\frac{(b-a)^2}{2^{s+4}(s+2)(s+3)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\\&\left[\left(s(s+1)(s+2)a^{(-s-2)q}+\left(2^{s+4}s-s^3-7s^2-14s\right)b^{(-s-2)q}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\left(2^{s+4}s-s^3-7s^2-14s\right)a^{(-s-2)q}+s(s+1)(s+2)b^{(-s-2)q}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right|\leq\frac{(b-a)^2}{384}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(6a^{-3q}+10b^{-3q}\right)^{\frac{1}{q}}+\left(10a^{-3q}+6b^{-3q}\right)^{\frac{1}{q}}\right],$$

where $q>1$.

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Doi: 10.18514/MMN.2015.1180

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 503–506

HU e-ISSN 1787-2413

REMARKS ON YU MIAO AND SHOUFANG XU'S PAPER "ALMOST SURE CONVERGENCE OF WEIGHTED SUMS"

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Received 26 March, 2014

Abstract. It is pointed out that a general theorem presented in Yu Miao and Shoufang Xu's paper has a serious gap. In this small note, a counterexample for this theorem is provided exhibiting a triangular array of real numbers and a sequence of independent identically distributed random variables for which its conclusion fails. An reformulation of Yu Miao and Shoufang Xu's general theorem is performed making this statement true.

2010 *Mathematics Subject Classification:* 60F15

Keywords: almost sure convergence, triangular arrays, weighted sums

1. INTRODUCTION

In paper [11], the authors announced a general theorem (see Theorem 6 of [11], page 175) which supports important improvements in several well-known results about the almost sure convergence of the weighted sum

$$\sum_{k=1}^{n} a_{n,k} X_k \tag{1.1}$$

where $\{X_n, n \geqslant 1\}$ is a sequence of random variables and $\{a_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ is a triangular array of constants.

Over the past years, many authors have studied the almost sure convergence of (1.1) (see, for instance, [2–4, 6, 7, 10, 13–18]). In 1995, Li, Rao and Wang proved a result in [10] which seems to say that for almost all choices of weights, the law of the Iterated Logarithm fails, establishing what they call the law of Logarithm. Notwithstanding, a specific example with a concrete array of real numbers and sequence of independent identically distributed random variables for which the law of the Iterated Logarithm fails was not included in the quoted paper. Only one year later, Li and Rao presented the desired example (see [9]) which can be used to claim

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project PEst-OE/MAT/UI0297/2014 (Centro de Matemática e Aplicações).

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that Corollary 3 of [11] is false. Indeed, a simple adaptation of Li and Rao's example can be made to prove that Theorem 6 of [11] is also false. Contrary to what Miao and Xu declared in page 176 of [11], we cannot rearrange the random sequence $\{a_{n,k}X_k,\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ when the order of $\{a_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ does not satisfy $a_{n,1} \geqslant a_{n,2} \geqslant \ldots \geqslant a_{n,n}$.

2. Counterexample

Consider a sequence $\{X_n, n \geqslant 1\}$ of independent identically distributed random variables each with normal distribution with zero mean and unit variance. Let $\mathbf{H}_n$ be a sequence of Hadamard matrices of order n (see [1] or [8]) and suppose the $\infty \times \infty$ matrix $\left(h_{n,k}\right)_{n,k\geqslant 1}$ given by

$$\begin{pmatrix} \mathbf{H}_1 & \mathbf{O} & \mathbf{O} & \ldots \\ \mathbf{O} & \mathbf{H}_2 & \mathbf{O} & \ldots \\ \mathbf{O} & \mathbf{O} & \mathbf{H}_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\{a_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ be the triangular array of constants defined by

$$a_{2n,k} = \frac{h_{n,k}}{\log\log\log(n+15)\,\sqrt{n\log\log(n+2)}},$$
$$a_{2n-1,k} = \frac{1}{\log\log\log(n+15)\,\sqrt{n\log\log(n+2)}}.$$

Since $\left|h_{n,k}\right| \leqslant 1$ for each n,k we have

$$\max_{1\leqslant k\leqslant n}\left|a_{n,k}\right| \leqslant \frac{1}{\log\log\log(n+15)\,\sqrt{n\log\log(n+2)}}$$

and the sequence b_n in Theorem 6 of Yu Miao and Shoufang Xu's paper is

$$b_n = \log\log\log(n+15)\,\sqrt{n\log\log(n+2)}.$$

From Hartman-Wintner law of the iterated logarithm (see [12], page 248) we get

$$\frac{X_1+\ldots+X_n}{b_n} \xrightarrow{\text{a.s.}} 0.$$

On the other hand,

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n} a_{n,k}X_k\right| \geqslant \limsup_{n\to\infty}\frac{1}{b_n}\left|\sum_{k=1}^{2n} h_{n,k}X_k\right| \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty}\frac{1}{\sqrt{4n\log(2n)}}\left|\sum_{k=1}^{2n} h_{n,k}X_k\right| \geqslant \frac{1}{\sqrt{2}} \quad \text{a.s.}$$

(see [9]) which implies

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n} a_{n,k}X_k\right| = \infty \quad \text{a.s.}$$

Therefore, subsequent corollaries in [11] which use Theorem 6 cannot be validated until a correct proof be given.

3. General theorem

We shall announce the following definition.

Definition 1. Let $\{a_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of constants. We say that $\{a_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ is *rowwise increasing* (resp. *rowwise decreasing*) if $a_{n,k+1} \geqslant a_{n,k}$, for all $1 \leqslant k \leqslant n-1, n \geqslant 2$ (resp. $a_{n,k+1} \leqslant a_{n,k}$, for all $1 \leqslant k \leqslant n-1$, $n \geqslant 2$). An array of constants $\{a_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ is said to be *rowwise monotone* if it is rowwise increasing or rowwise decreasing.

The result below will permit us to obtain the desired almost sure convergence of (1.1) in all corollaries of [11] assuming additionally the rowwise monotonicity of the triangular array of constants. The proof uses Abel transformation (see [5], page 114) and is similar to Cèsaro lemma (see [19], page 116) being, therefore, omitted.

Theorem 1. *Let $\{x_n\}$ be a sequence of constants and $\{a_{n,k}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ an rowwise monotone array of constants satisfying*

$$\max_{1\leqslant k\leqslant n}\left|a_{n,k}\right| \leqslant b_n^{-1}$$

where $0 < b_n \uparrow \infty$. If

$$\frac{1}{b_n}\sum_{k=1}^{n} x_k \to 0$$

then

$$\sum_{k=1}^{n} a_{n,k}x_k \to 0.$$

Remark 1. Let us observe that the assumption "$\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed random variables" in Theorem 6 of [11] is totally superfluous.

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Doi: 10.18514/MMN.2015.1029

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 507–516

HU e-ISSN 1787-2413

ON A CO-UNIVERSAL ARROW IN THE CONSTRUCT OF n-ARY HYPERALGEBRAS

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Received 23 October, 2013

Abstract. We introduce and study a new operation of product of n-ary hyperalgebras which lies, with respect to set inclusion, between their cartesian product and the cartesian product of their idempotent hulls. For every fixed n-ary hyperalgebra, the product introduced gives an endofunctor of the construct of n-ary hyperalgebras. We define a power of n-ary hyperalgebras and specify a class of n-ary hyperalgebras such that, with respect to the endofunctor, the power together with the evaluation map constitute a co-universal arrow for each hyperalgebra of the class.

2010 *Mathematics Subject Classification:* 08B25; 08C05

Keywords: hyperalgebra, diagonality, mediality, combined product, co-universal arrow with respect to a functor

1. INTRODUCTION

Hyperalgebras proved to be useful generalizations of algebras having many application in various branches of mathematics and computer science (automata theory). This lead to a rapid development of the theory of hyperalgebras since the beginning of 90's of the last century - see e.g. [3]. The aim of this paper is to contribute to the development.

In his pioneering paper [2], G. Birkhoff introduced the cardinal (i.e., direct) arithmetic of partially ordered sets and showed that it behaves analogously to the arithmetic of natural numbers. Several authors then extended the cardinal arithmetic to relational systems - see e.g. [7, 8] and [10, 11]. Conversely, the cardinal arithmetic has been restricted from relational systems to n-ary and universal algebras in [13] and [12], respectively. Finally, in [14], the cardinal arithmetic was extended from universal algebras to universal hyperalgebras.

In [14], an operation of a power of hyperalgebras (of the same type) was introduced and studied. The results of [14] determine a class of hyperalgebras such that, with respect to the endofunctor of the construct of hyperalgebras (of the same type) given by

This work was supported by the project NETME+ (LO1202) cofunded by the Ministry of Education of the Czech Republic under the National Sustainability Programme I.

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the cartesian product and a fixed idempotent hyperalgebra, the power together with the evaluation map constitute a co-universal arrow for each hyperalgebra of the class. In the present note, we restrict our considerations to n-ary hyperalgebras, i.e., hyperalgebras with just one n-ary hyperoperation. For n-ary hyperalgebras, we introduce an operation of product which is obtained as the restriction of the product of relational systems introduced and studied in [8]. The operation lies, with respect to set inclusion, between the cartesian product of n-ary hyperalgebras and the cartesian product of their idempotent hulls. For idempotent hyperalgebras, the cartesian product and the introduced one coincide. We will show that the above mentioned class has the property that, with respect to the endofunctor of the construct of n-ary hyperalgebras given by the new product and arbitrary (not only idempotent) but fixed n-ary hyperalgebra, the power together with the evaluation map constitute a co-universal arrow for each hyperalgebra of the class. We then get the first exponential law with respect to the new product which, unlike the first exponential law with respect to the cartesian product, is valid for arbitrary n-ary algebras in the exponents, not only for the idempotent ones.

2. Combined product of hyperalgebras

For the basic concepts used concerning n-ary hyperalgebras we refer to [2] and for those concerning category theory we refer to [1]. Let n be a nonnegative integer. By an n-ary *hyperalgebra* we understand a pair $\mathbf{G} = (G, p)$ where G is a set, the so-called *carrier* of $\mathbf{G}$, and $p : G^n \to \exp G \setminus \{\varnothing\}$ is a map, the so-called *n-ary hyperoperation* of $\mathbf{G}$. To avoid some nonwanted singularities, only the n-ary hyperalgebras with $n > 0$ are considered in this note. An n-ary hyperalgebra $\mathbf{G} = (G, p)$ with the property $\operatorname{card} p(x_1, \ldots, x_n) = 1$ for every $x_1, \ldots, x_n \in G$ is called an n-ary *algebra*. Binary hyperalgebras are usually called *hypergroupiods*.

Let $\mathbf{H} = (H, q), \mathbf{G} = (G, p)$ be a pair of n-ary hyperalgebras. A map $f : H \to G$ is called a *homomorphism* of $\mathbf{H}$ into $\mathbf{G}$ if $f(q(x_1, \ldots, x_n)) \subseteq p(f(x_1), \ldots, f(x_n))$. The set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$ will be denoted by $Hom(\mathbf{H}, \mathbf{G})$. We denote by HYP_n the construct of n-ary hyperalgebras with homomorphisms as morphisms. Subobjects in HYP_n are called *subhyperalgebras*. Clearly, $\mathbf{G}$ is a subhyperalgebra of $\mathbf{H}$ if $G \subseteq H$ and $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$, whenever $x_1, \ldots x_n \in H$. We write $\mathbf{G} \cong \mathbf{H}$ if $\mathbf{G}$ and $\mathbf{H}$ are isomorphic in HYP_n and we write $\mathbf{G} \preceq \mathbf{H}$ if $\mathbf{G}$ may be embedded into $\mathbf{H}$ in HYP_n, i.e., if there exists a subalgebra $\mathbf{H}'$ of $\mathbf{H}$ such that $\mathbf{G} \cong \mathbf{H}'$.

Given n-ary hyperalgebras $\mathbf{G} = (G, p)$ and $\mathbf{H} = (G, q)$, we put $\mathbf{G} \leq \mathbf{H}$ if $p(x_1, \ldots, x_n) \subseteq q(x_i, \ldots, x_n)$ for every $x_1, \ldots, x_n \in G$.

If p is an n-ary hyperoperation on a set G and $X_1, \ldots, X_n$ are nonempty subsets of G, we put $p(X_1, \ldots, X_n) = \bigcup \{p(x_1, \ldots, x_n); x_1 \in X_1, \ldots, x_n \in X_n\}$.

Let $\mathbf{G}_i = (G_i, p_i), i \in I$, be a family of n-ary hyperalgebras. We denote by $\prod_{i \in I} \mathbf{G}_i$ the cartesian product of the family in HYP_n. Thus, $\prod_{i \in I} \mathbf{G}_i$

$= (\prod_{i\in I} G_i, p)$ is the n-ary hyperalgebra where $p(f_1,...,f_n) = \prod_{i\in I} p_i(f_1(i),...,f_n(i))$ whenever $f_1,...,f_n \in \prod_{i\in I} G_i$. As usual, if the set I is finite, say $I = \{1,\dots,m\}$, then we write $\mathbf{G}_1 \times \dots \times \mathbf{G}_m$ instead of $\prod_{i\in I} \mathbf{G}_i$. Further, we define the *direct sum* of a family $\mathbf{G}_i = (G, p_i), i \in I$, of n-ary hyperalgebras to be the n-ary hyperalgebra $\sum_{i\in I} \mathbf{G}_i = (G,q)$ where, for every $x_1,\dots,x_n \in G, q(x_1,\dots,x_n) = \bigcup_{i\in I} p_i(x_1,\dots,x_n)$. If the set I is finite, say $I = \{1,\dots,m\}$, then we write $\mathbf{G}_1 \uplus \dots \uplus \mathbf{G}_m$ instead of $\sum_{i\in I} \mathbf{G}_i$.

An n-ary hyperalgebra $\mathbf{G} = (G, p)$ is said to be *idempotent* if, for every $x \in G$, $x \in p(x_1,\dots,x_n)$ whenever $x = x_1 = \dots = x_n$. Let $\mathbf{G} = (G, p)$ be an n-ary hyperalgebra. For every $x_1,\dots,x_n \in G$, we put

$$\bar{p}(x_1,\dots,x_n) = \begin{cases} p(x_1,\dots,x_n) \cup \{x_1\} & \text{if } x_1 = \dots = x_n, \\ p(x_1,\dots,x_n) & \text{if } x_i \neq x_j \text{ for some } i,j \in \{1,\dots,n\}. \end{cases}$$

The n-ary hyperalgebra $(G, \bar{p})$ is called the *idempotent hull* of $\mathbf{G}$ and is denoted by $\bar{\mathbf{G}}$.

Let $\mathbf{G}_i = (G_i, p_i), i \in I$, be a non-empty family of n-ary hyperalgebras. The *combined product* of the family is the n-ary hyperalgebra $\bigotimes_{i\in I} \mathbf{G}_i = \sum_{i\in I} \prod_{j\in I} \mathbf{G}_{ij}$ where

$$\mathbf{G}_{ij} = \begin{cases} \bar{\mathbf{G}}_j & \text{if } i = j, \\ \mathbf{G}_j & \text{if } i \neq j. \end{cases}$$

Thus, $\bigotimes_{i\in I} \mathbf{G}_i = (\prod_{i\in I} G_i, r)$ where, for any $f, f_1,...,f_n \in \prod_{i\in I} G_i$, $f \in r(f_1,...,f_n)$ if and only if there exists a subset $J \subseteq I$, card $J \leq 1$, such that $f(i) \in p_i(f_1(i),...,f_n(i))$ for every $i \in I \setminus J$ and $f(i) = f_1(i) = \dots = f_n(i)$ for every $i \in J$.

If the set I is finite, say $I = \{1,\dots,m\}$, we write $\mathbf{G}_1 \otimes \dots \otimes \mathbf{G}_m$ instead of $\bigotimes_{i\in I} G_i$. We then clearly have $\mathbf{G}_1 \otimes \dots \otimes \mathbf{G}_m = (\mathbf{G}_{11} \times \mathbf{G}_{12} \times \dots \times \mathbf{G}_{1m}) \uplus (\mathbf{G}_{21} \times \mathbf{G}_{22} \times \dots \times \mathbf{G}_{2m}) \uplus \dots \uplus (\mathbf{G}_{m1} \times \mathbf{G}_{m2} \dots, \times \mathbf{G}_{mm}) = (\bar{\mathbf{G}}_1 \times \mathbf{G}_2 \times \dots \times \mathbf{G}_m) \uplus (\mathbf{G}_1 \times \bar{\mathbf{G}}_2 \times \dots \times \mathbf{G}_m) \uplus \dots \uplus (\mathbf{G}_1 \times \mathbf{G}_2 \times \dots \times \bar{\mathbf{G}}_m)$.

In particular, if $I = \{1,2\}$, then, for any $(x_1,y_1),\dots,(x_n,y_n) \in \mathbf{G}_1 \times \mathbf{G}_2$, $(x,y) \in r((x_1,y_1),\dots,(x_n,y_n))$ if and only if one of the following three conditions is satisfied:

(i) $x \in p_1(x_1,\dots,x_n)$ and $y \in p_2(y_1,\dots,y_n)$,
(ii) $x = x_1 = x_2 = \dots = x_n$ and $y \in p_2(y_1,\dots,y_n)$,
(iii) $x \in p_1(x_1,\dots,x_n)$ and $y = y_1 = y_2 = \dots = y_n$.

If card $I = 1$, say $I = \{i\}$, then $\bigotimes_{i\in I} \mathbf{G}_i = \bar{\mathbf{G}}_i$.

For every n-ary hyperalgebra $\mathbf{H} = (H,q)$, the operation of combined product defines an endofunctor $\mathbf{H} \otimes - : HYP_n \to HYP_n$ which assigns to every morphism $f : \mathbf{G}_1 \to \mathbf{G}_2$ in HYP_n the morphism $id_H \times f : \mathbf{H} \otimes \mathbf{G}_1 \to \mathbf{H} \otimes \mathbf{G}_2$. This endofunctor will be studied in the next section.

Remark 1. Let $\mathbf{G}_i, i \in I$, be a non-empty family of n-ary hyperalgebras. We clearly have $\prod_{i\in I} \mathbf{G}_i \leq \bigotimes_{i\in I} \mathbf{G}_i \leq \prod_{i\in I} \bar{\mathbf{G}}_i = \bigotimes_{i\in I} \bar{\mathbf{G}}_i$ and $\overline{\bigotimes_{i\in I} \mathbf{G}_i} \leq \bigotimes_{i\in I} \bar{\mathbf{G}}_i$.

If card $I \leq 2$, then $\overline{\bigotimes_{i\in I}\mathbf{G}_i} = \prod_{i\in I}\bar{\mathbf{G}}_i$. If $\mathbf{G}_i$ is idempotent for every $i \in I$, then all the previous inequalities become equalities. If at most one of the n-ary hyperalgebras $\mathbf{G}_i, i \in I$, is not idempotent, then $\bigotimes_{i\in I}\mathbf{G}_i$ is idempotent.

Theorem 1. *Let* $\mathbf{G}_i = (G_i, p_i), i \in I \neq \emptyset$, *and* $\mathbf{H} = (H, q)$ *be* n*-ary hyperalgebras. Then* $\sum_{i\in I}\mathbf{H}\otimes\mathbf{G}_i = \mathbf{H}\otimes\sum_{i\in I}\mathbf{G}_i$.

Proof. Let $\sum_{i\in I}\mathbf{G}_i = (G, s), \mathbf{H}\otimes\mathbf{G}_i = (H\times G, r_i)$ for each $i \in I$, $\mathbf{H}\otimes\sum_{i\in I}\mathbf{G}_i = (H\times G, u)$ and $\sum_{i\in I}\mathbf{H}\otimes\mathbf{G}_i = (H\times G, v)$. We will show that $u((x_1,y_1),\ldots,(x_n,y_n)) = v((x_1,y_1),\ldots,(x_n,y_n))$ for every $(x_1,y_1),\ldots,(x_n,y_n) \in H\times G$. It is easy to see that the following five conditions are equivalent:

(a) $(x,y) \in u((x_1,y_1),\ldots,(x_n,y_n))$;
(b) one of the following three cases occurs:
  (i) $x \in q(x_1,\ldots,x_n)$ and $y \in s(y_1,\ldots,y_n)$,
  (ii) $x = x_1 = x_2 = \ldots = x_n$ and $y \in s(y_1,\ldots,y_n)$,
  (iii) $x \in q(x_1,\ldots,x_n)$ and $y = y_1 = y_2 = \ldots = y_n$;
(c) one of the following three cases occurs:
  (i) $x \in q(x_1,\ldots,x_n)$ and $y \in p_i(y_1,\ldots,y_n)$ for some $i \in I$,
  (ii) $x = x_1 = x_2 = \ldots = x_n$ and $y \in p_i(y_1,\ldots,y_n)$ for some $i \in I$,
  (iii) $x \in q(x_1,\ldots,x_n)$ and $y = y_1 = y_2 = \ldots = y_n$;
(d) $(x,y) \in r_i((x_1,y_1),\ldots,(x_n,y_n))$ for some $i \in I$;
(e) $(x,y) \in v((x_1,y_1),\ldots,(x_n,y_n))$.

This proves the statement. □

Definition 1. An n-ary hyperalgebra (G,p) is called *medial* if, for every $n\times n$-matrix (a_{ij}) over G, from $x_i \in p(a_{i1},\ldots,a_{in})$ for each $i = 1,\ldots,n$ and $y_j \in p(a_{1j},\ldots,a_{nj})$ for each $j = 1,\ldots,n$ it follows that $p(x_1,\ldots,x_n) = p(y_1,\ldots,y_n)$.

The medial n-ary algebras are studied in [13]. The mediality may be extended, in a natural way, from n-ary algebras onto universal algebras. The medial universal algebras are often called commutative and they were dealt with e.g. in [5,6] and [12]. The medial groupoids are studied in [4].

Example 1. (1) Let $(X,\leq)$ be a partially ordered set with a least element 0 and let A be the set of all atoms of $(X,\leq)$. Put $0' = \{0\}$ for any $x \in X$ with $x \neq 0$ put $x' = \{y \in X; y < x \text{ and } y \in A\cup\{0\}\}$. Further, for any pair $x,y \in X$ put $x * y = x'\cap y'$. Then $(X,*)$ is a medial hypergroupoid. Indeed, it can easily be seen that for any $a,b,c,d \in X$ we have $x * y = f * g$ whenever $x \in a * b$, $y \in c * d$, $f \in a * c$ and $g \in b * d$.

(2) Every unary algebra $\mathbf{G}$ consisting of two-element cycles is medial and, moreover, the unary hyperalgebra $\bar{\mathbf{G}}$ is medial, too.

It may easily be seen that the cartesian product of a family of medial n-ary hyperalgebras is a medial n-ary hyperalgebra. But this is not generally true for the combined

product of n-ary hyperalgebras. With respect to the combined product, we have the following result:

Proposition 1. *Let* $\mathbf{G},\mathbf{H}$ *be* n*-ary hyperalgebras. If* $\bar{\mathbf{G}},\bar{\mathbf{H}}$ *are medial, then so is* $\overline{\mathbf{G}\otimes\mathbf{H}}$.

Proof. Since the cartesian product of a family of medial n-ary hyperalgebras is a medial n-ary hyperalgebra, the statement follows from the obvious equality $\overline{\mathbf{G}\otimes\mathbf{H}} = \bar{\mathbf{G}}\times\bar{\mathbf{H}}$ (see Remark 1). □

3. Power of hyperalgebras

Lemma 1. *Let* $\mathbf{H} = (H,q)$ *and* $\mathbf{G} = (G,p)$ *be* n*-ary hyperalgebras and let* $f_1,\dots,f_n \in Hom(\mathbf{H},\mathbf{G})$. *Let* $f : H \to G$ *be a map such that* $f(x) \in p(f_1(x),\dots,f_n(x))$ *for every* $x \in H$. *If* $\mathbf{G}$ *is medial, then* f *is a homomorphism from* $\mathbf{H}$ *into* $\mathbf{G}$.

Proof. Let $x_1,\dots,x_n \in H$. We will show that $f(q(x_1,\dots,x_n)) \subseteq p(f(x_1),\dots,f(x_n))$. Let $y \in f(q(x_1,\dots,x_n))$. Then there exists $x \in q(x_1,\dots,x_n)$ such that $y = f(x)$, so $f(x) \in f(q(x_1,\dots,x_n))$. Since $f_1,\dots,f_n \in Hom(\mathbf{H},\mathbf{G})$ and $x \in q(x_1,\dots,x_n)$, we have $f_j(x) \in f_j(q(x_1,\dots,x_n)) \subseteq p(f_j(x_1),\dots,f_j(x_n))$ for each $j = 1,\dots,n$. By the assumption, we have $f(x_i) \in p(f_1(x_i),\dots,f_n(x_i))$ for each $i = 1,\dots,n$ and $f(x) \in p(f_1(x),\dots,f_n(x))$. Since $\mathbf{G}$ is medial, $p(f_1(x),\dots,f_n(x)) = p(f(x_1),\dots,f(x_n))$. Thus, $y = f(x) \in p(f(x_1),\dots,f(x_n))$. □

Given sets G,H, we denote by G^H the set of all maps of H into G. The bijection $\varphi : (G^H)^K \to G^{H\times K}$ (where G,H,K are sets and $\times$ denotes the cartesian product) given by $\varphi(h)(x,y) = h(y)(x)$ whenever $h \in (G^H)^K, x \in H$ and $y \in K$ will be called *canonical*.

Definition 2. Let $\mathbf{H} = (H,q)$ and $\mathbf{G} = (G,p)$ be n-ary hyperalgebras and let $\mathbf{G}$ be medial. The *power* of $\mathbf{G}$ and $\mathbf{H}$ is the n-ary hyperalgebra $\mathbf{G}^{\mathbf{H}} = (Hom(\mathbf{H},\mathbf{G}),r)$ where, for any $f_1,\dots,f_n \in Hom(\mathbf{H},\mathbf{G})$, $r(f_1,\dots,f_n) = \{f \in G^H; f(x) \in p(f_1(x),\dots,f_n(x))$ for each $x \in H\}$.

It is easy to see that the power $\mathbf{G}^{\mathbf{H}}$ of n-ary hyperalgebras is idempotent whenever $\mathbf{G}$ is idempotent (and medial).

Proposition 2. *Let* $\mathbf{G}_i$, $i \in I$, *be a nonempty family of medial* n*-ary hyperalgebras and* $\mathbf{H}$ *be an* n*-ary hyperalgebra. If* $\bigotimes_{i\in I}\mathbf{G}_i$ *is medial, then*

$$\bigotimes_{i\in I}\mathbf{G}_\mathbf{i}^{\mathbf{H}} \preceq (\bigotimes_{\mathbf{i}\in\mathbf{I}}\mathbf{G}_\mathbf{i})^{\mathbf{H}}.$$

Proof. Let $\mathbf{G}_i = (G_i,p^i)$ for every $i \in I$, $\mathbf{H} = (H,q)$, $\bigotimes_{i\in I}\mathbf{G}_\mathbf{i} = (\prod_{i\in I}G_i,r)$, $\mathbf{G}_\mathbf{i}^{\mathbf{H}} = (Hom(\mathbf{H},\mathbf{G}_\mathbf{i}),u^i)$ for every $i \in I$, $\bigotimes_{i\in I}\mathbf{G}_\mathbf{i}^{\mathbf{H}} = (\prod_{i\in I}Hom(\mathbf{H},\mathbf{G}_\mathbf{i}),s)$ and

$(\bigotimes_{i\in I}\mathbf{G_i})^{\mathbf{H}} = (Hom(\mathbf{H},\bigotimes_{i\in I}\mathbf{G_i}),t)$. We define a map $\alpha : \prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i}) \to (\prod_{i\in I} G_i)^H$ by $\alpha(f^i;\ i\in I)(z) = (f^i(z);\ i\in I)$ for each $z\in H$.

Let $(f^i;\ i\in I)\in\prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i})$. Let $h\in\alpha(f^i;\ i\in I)(q(x_1,\dots,x_n))$. Then $h=\alpha(f^i;\ i\in I)(x)$ where $x\in q(x_1,\dots,x_n)$. Thus, $h=\alpha(f^i;i\in I)(x) = (f^i(x);i\in I)$. Since $x\in q(x_1,\dots,x_n)$ and $f^i\in Hom(\mathbf{H},\mathbf{G_i})$ for every $i\in I$, we have $f^i(x)\in p^i(f^i(x_1),\dots,f^i(x_n))$ for every $i\in I$. Thus, $h=(f^i(x);i\in I)\in r((f^i(x_1);i\in I),\dots,(f^i(x_n);i\in I)) = r(\alpha(f^i;\ i\in I)(x_1),\dots,\alpha(f^i;\ i\in I)(x_n))$. Consequently, $\alpha(f^i;\ i\in I)(q(x_1,\dots,x_n))\subseteq r(\alpha(f^i;\ i\in I)(x_1),\dots,\alpha(f^i;\ i\in I)(x_n))$. Therefore, $\alpha(f^i;\ i\in I)\in Hom(\mathbf{H},\bigotimes_{i\in I}\mathbf{G_i})$. We have shown that α maps $\prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i})$ into $Hom(\mathbf{H},\bigotimes_{i\in I}\mathbf{G_i})$.

Suppose that $\alpha(f^i;\ i\in I)=\alpha(g^i;\ i\in I)$ where $(f^i;\ i\in I),(g^i;\ i\in I)\in\prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i})$. Then $(f^i(x);\ i\in I)=\alpha(f^i;\ i\in I)(x)=\alpha(g^i;\ i\in I)(x) = (g^i(x);\ i\in I)$ for every $x\in H$. Therefore, $f^i(x)=g^i(x)$ for every $i\in I$ and for every $x\in H$. Hence, $f^i=g^i$ for every $i\in I$. Thus, α: $\prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i})\to Hom(\mathbf{H},\bigotimes_{i\in I}\mathbf{G_i})$ is an injection.

Finally, let $(f_1^i;\ i\in I),\dots,(f_n^i;\ i\in I)\in\prod_{i\in I} Hom(\mathbf{H},\mathbf{G_i})$. We are to show that $\alpha(s((f_1^i;\ i\in I),\dots,(f_n^i;\ i\in I))) = t(\alpha(f_1^i;\ i\in I),\dots,\alpha(f_n^i;\ i\in I))$. It is easy to see that the following seven conditions are equivalent:

(a) $f\in\alpha(s_\lambda((f_1^i;\ i\in I),\dots,(f_n^i;\ i\in I)))$;

(b) $f=\alpha(f^i;i\in I)$ where $(f^i;i\in I)\in s((f_1^i;i\in I),\dots,(f_n^i;i\in I))$;

(c) $f=\alpha(f^i;i\in I)$ and there exists $J\subseteq I$, card $J\leq 1$, such that $f^i\in u^i(f_1^i,\dots,f_n^i)$ for every $i\in I\smallsetminus J$ and $f^i=f_1^i=f_2^i=\dots=f_n^i$ for every $i\in J$;

(d) $f(x)=(f^i(x);i\in I)$ for every $x\in H$ and there exists $J\subseteq I$, card $J\leq 1$, such that $f^i(x)\in p_\lambda^i(f_1^i(x),\dots,f_n^i(x))$ for every $i\in I\smallsetminus J$ and every $x\in H$ and $f^i(x)=f_1^i(x)=f_2^i(x)=\dots=f_n^i(x)$ for every $i\in J$ and every $x\in H$;

(e) $f(x)=(f^i(x);i\in I)\in r((f_1^i(x);\ i\in I),\dots,(f_n^i(x);\ i\in I))$ for every $x\in H$;

(f) $f(x)\in r(\alpha(f_1^i;\ i\in I)(x),\dots,\alpha(f_n^i;\ i\in I)(x))$ for every $x\in H$;

(g) $f\in t(\alpha(f_1^i;\ i\in I),\dots,\alpha(f_n^i;\ i\in I))$.

Consequently, $\alpha(s((f_1^i;\ i\in I),\dots,(f_n^i;\ i\in I))) = t(\alpha(f_1^i;\ i\in I),\dots,\alpha(f_n^i;\ i\in I))$, which yields $\bigotimes_{i\in I}\mathbf{G_i^H}\preceq(\bigotimes_{\mathbf{i\in I}}\mathbf{G_i})^{\mathbf{H}}$. □

Remark 2. It may easily be shown that, in the previous Proposition, we may write $\cong$ instead of $\preceq$ provided that $\mathbf{G_i}$ is idempotent for every $i\in I$. We then obtain the so-called second exponential law for the cartesian product, i.e., the law $\prod_{i\in I}\mathbf{G_i^H}\cong(\prod_{\mathbf{i\in I}}\mathbf{G_i})^{\mathbf{H}}$.

Theorem 2. *Let* $\mathbf{G},\mathbf{H}$ *be n-ary hyperalgebras. If* $\mathbf{G}$ *is medial, then so is* $\mathbf{G^H}$.

Proof. Let $\mathbf{H}=(H,q)$, $\mathbf{G}=(G,p)$, $\mathbf{G^H}=(Hom(\mathbf{H},\mathbf{G}),r)$ and let (f_{ij}) be an $n\times n$-matrix over $Hom(\mathbf{H},\mathbf{G})$. Suppose that $h_i\in r(f_{i1},\dots,f_{in})$ for all $i=1,\dots,n$

and $g_j \in r(f_{1j},\dots,f_{nj})$ for all $j = 1,\dots,n$. For every $x \in H$, we have $h_i(x) \in p(f_{i1}(x),\dots,f_{in}(x))$ for all $i = 1,\dots,n$ and $g_j(x) \in p(f_{1j}(x),\dots,f_{nj}(x))$ for all $j = 1,\dots,n$. Since $\mathbf{G}$ is medial, we have $p(h_1(x),\dots,h_n(x)) = p(g_1(x),\dots,g_n(x))$ for every $x \in H$. Therefore, $r(h_1,\dots,h_n) = r(g_1,\dots,g_n)$, so that $\mathbf{G}^{\mathbf{H}}$ is medial. □

Definition 3. An n-ary hyperalgebra (G,p) is called *diagonal* if, for every $n \times n$-matrix (a_{ij}) over G, we have $p(p(a_{11},\dots,a_{1n}),\dots,p(a_{n1},\dots,a_{nn})) \cap p(p(a_{11},\dots,a_{n1}),\dots,p(a_{1n},\dots,a_{nn})) \subseteq p(a_{11},\dots,a_{nn})$.

For idempotent algebras, the diagonality introduced coincides with the diagonality studied in [9].

Example 2. It may easily be seen that the hypergroupoid from Example 1(1) is diagonal.

The combined product of diagonal n-ary hyperalgebras is not be diagonal in general. But we have:

Proposition 3. *Let* $\mathbf{G},\mathbf{H}$ *be* n*-ary hyperalgebras. If* $\bar{\mathbf{G}},\bar{\mathbf{H}}$ *are diagonal, then so is* $\overline{\mathbf{G}\otimes\mathbf{H}}$.

Proof. The proof is analogous to that of Proposition 1. □

Theorem 3. *Let* $\mathbf{G},\mathbf{H}$ *be* n*-ary hyperalgebras. If* $\mathbf{G}$ *is medial and diagonal, then* $\mathbf{G}^{\mathbf{H}}$ *is diagonal.*

Proof. Let $\mathbf{H} = (H,q)$, $\mathbf{G} = (G,p)$, $\mathbf{G}^{\mathbf{H}} = (Hom(\mathbf{H},\mathbf{G}),r)$ and let (f_{ij}) be an $n \times n$-matrix over $Hom(\mathbf{H},\mathbf{G})$. Suppose that $f \in r(r(f_{11},\dots,f_{1n}),\dots,r(f_{n1},\dots,f_{nn})) \cap r(r(f_{11},\dots,f_{n1}),\dots,r(f_{1n},\dots,f_{nn}))$. Then $f(x) \in p(p(f_{11}(x),\dots,f_{1n}(x)),\dots,r(f_{n1}(x),\dots,f_{nn}(x))) \cap p(p(f_{11}(x),\dots,f_{n1}(x)),\dots,p(f_{1n}(x),\dots,f_{nn}(x)))$ for every $x \in H$. Since $\mathbf{G}$ is diagonal, we have $f(x) \in p(f_{11}(x),\dots,f_{nn}(x))$ for every $x \in H$. Thus, $f \in r(f_{11},\dots,f_{nn})$. Hence, $\mathbf{G}^{\mathbf{H}}$ is diagonal. □

We denote by $MDHYP_n$ the full subconstruct of HYP_n whose objects are the n-ary hyperalgebras that are medial and diagonal.

Lemma 2. *Let* $\mathbf{G} \in MDHYP_n$ *and* $\mathbf{H},\mathbf{K} \in HYP_n$. *Then the canonical bijection* $\varphi : (G^H)^K \to G^{H\times K}$ *restricted to* $Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ *is a bijection of* $Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ *onto* $Hom(\mathbf{H}\otimes\mathbf{K},\mathbf{G})$.

Proof. Let $\mathbf{H} = (H,q)$, $\mathbf{G} = (G,p)$, $\mathbf{K} = (K,s)$, $\mathbf{G}^{\mathbf{H}} = (Hom(\mathbf{H},\mathbf{G}),r)$, $\mathbf{H}\otimes\mathbf{K} = (H\times K,v)$ and let $\mathbf{G}$ be diagonal and medial. Let $h \in Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ and let $(y_1,z_1),\dots,(y_n,z_n) \in H\times K$. To show that $\varphi(h)(v((y_1,z_1),\dots,(y_n,z_n))) \subseteq p(\varphi(h)(y_1,z_1),\dots,\varphi(h)(y_n,z_n))$, let $x \in \varphi(h)(v((y_1,z_1),\dots,(y_n,z_n)))$. Then there exists $(y,z) \in v((y_1,z_1),\dots,(y_n,z_n))$ such that $x = \varphi(h)(y,z)$. Thus, one of the following three conditions is satisfied:

(i) $y \in q(y_1,\dots,y_n)$ and $z \in s(z_1,\dots,z_n)$,

(ii) $y = y_1 = y_2 = \ldots = y_n$ and $z \in s(z_1,\ldots,z_n)$,
(iii) $y \in q(y_1,\ldots,y_n)$ and $z = z_1 = z_2 = \ldots = z_n$.

If $z \in s(z_1,\ldots,z_n)$, then $h(z) \in h(s(z_1,\ldots,z_n)) \subseteq r(h(z_1),\ldots,h(z_n))$, hence $\varphi(h)(y,z) = h(z)(y) \in p((h(z_1)(y),\ldots,h(z_n)(y)) = p(\varphi(h)(y,z_1),\ldots,\varphi(h)(y,z_n)$ for each $y \in H$. Thus, $\varphi(h)(y,z) \in p(\varphi(h)(y_1,z_1),\ldots,\varphi(h)(y_n,z_n))$ provided that (ii) is satisfied.

If $y \in q(y_1,\ldots,y_n)$, then $\varphi(h)(y,z) = h(z)(y) \in h(z)(q(y_1,\ldots,y_n)) \subseteq p(h(z)(y_1),\ldots,h(z)(y_n)) = p(\varphi(h)(y_1,z),\ldots,\varphi(h)(y_n,z))$ for every $z \in K$. Thus, $\varphi(h)(y,z) \in p(\varphi(h)(y_1,z_1),\ldots,\varphi(h)(y_n,z_n))$ provided that (iii) is satisfied.

Suppose that (i) is satisfied. Since $z \in s(z_1,\ldots,z_n)$, by the above considerations we get $\varphi(h)(y_i,z) \in p(\varphi(h)(y_i,z_1),\ldots,\varphi(h)(y_i,z_n))$ for every $i = 1,\ldots,n$ and $\varphi(h)(y,z) \in p(\varphi(h)(y,z_1),\ldots,\varphi(h)(y,z_n))$. As $y \in q(y_1,\ldots,y_n)$, by the above considerations we obtain $\varphi(h)(y,z_j) \in p(\varphi(h)(y_1,z_j),\ldots,\varphi(h)(y_n,z_j))$ for every $j = 1,\ldots,n$ and $\varphi(h)(y,z) \in p(\varphi(h)(y_1,z),\ldots,\varphi(h)(y_n,z))$. The diagonality of $\mathbf{G}$ implies $\varphi(h)(y,z) \in p(\varphi(h)(y_1,z_1),\ldots,\varphi(h)(y_n,z_n))$. We have shown that $\varphi(h) \in Hom(\mathbf{H}\otimes\mathbf{K},\mathbf{G})$.

Let $g \in Hom(\mathbf{H}\otimes\mathbf{K},\mathbf{G})$ and $z_1,\ldots,z_n \in K$. To show that $\varphi^{-1}(g)(s(z_1,\ldots,z_n)) \subseteq r\ (\varphi^{-1}(g)(z_1)\ ,\ldots,\varphi^{-1}(g)(z_n))$, let $x \in \varphi^{-1}(g)(s(z_1,\ldots,z_n))$. Then there exists $z \in s(z_1,\ldots,z_n)$ such that $x \in \varphi^{-1}(g)(z)$. Let $y \in H$. Then $(y,z) \in v((y,z_1),\ldots,(y,z_n))$. Since $g \in Hom(\mathbf{H}\otimes\mathbf{K},\mathbf{G})$, we have $g(y,z) \in g(v((y,z_1)\ ,\ldots,(y,z_n)) \subseteq p(g(y,z_1),\ldots,g(y,z_n))$. It follows that $\varphi^{-1}(g)(z)(y) \in p(\varphi^{-1}(g)(z_1)(y)\ ,\ldots,\varphi^{-1}(g)(z_n)(y))$, hence $\varphi^{-1}(g)(z) \in p(\varphi^{-1}(g)(z_1),\ldots,\varphi^{-1}(g)(z_n))$. Therefore, $\varphi^{-1}(g)(s(z_1,\ldots,z_n)) \subseteq r(\varphi^{-1}(g)(z_1),\ldots,\varphi^{-1}(g)(z_n))$. Consequently, $\varphi^{-1}(g) \in Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$. □

Given $\mathbf{G} \in MDHYP_n$ and $\mathbf{H} = (H,q) \in HYP_n$, the *evaluation map* is the map $e : \mathbf{H}\otimes\mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ defined by $e(y,f) = f(y)$ whenever $y \in H$ and $f \in Hom(\mathbf{H},\mathbf{G})$.

Theorem 4. *Let $\mathbf{G} \in MDHYP_n$ and $\mathbf{H} \in HYP_n$. Then the pair $(\mathbf{G}^{\mathbf{H}},e)$, where $e : \mathbf{H}\otimes\mathbf{G}^{\mathbf{H}}$ is the evaluation map, is a co-universal arrow for $\mathbf{G}$ with respect to the functor $\mathbf{H}\otimes - : HYP_n \to HYP_n$.*

Proof. Let $\mathbf{G} = (G,p)$, $\mathbf{H} = (H,q)$ and $\mathbf{H}\otimes\mathbf{G}^{\mathbf{H}} = (H\times Hom(\mathbf{H},\mathbf{G}),w)$. For every $\mathbf{K} = (K,r) \in HYP_n$ and every morphism $f : \mathbf{H}\otimes\mathbf{K} \to \mathbf{G}$, the map $f^* : \mathbf{K} \to \mathbf{G}^{\mathbf{H}}$ given by $f^*(z)(y) = f(y,z)$ is a morphism in HYP_n by the previous Lemma. We will show that the evaluation map $e : \mathbf{H}\otimes\mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ is morphism in HYP_n, too. To this end, let $x \in e(w((y_1,h_1),\ldots,(y_n,h_n)))$. Then $x = e(y,h) = h(y)$ where $(y,h) \in w((y_1,h_1),\ldots,(y_n,h_n))$. Thus, we have

(i) $y \in q(y_1,\ldots,y_n)$ and $h \in r(h_1,\ldots,h_n)$ or
(ii) $y = y_1 = y_2 = \ldots = y_n$ and $h \in r(h_1,\ldots,h_n)$ or
(iii) $y \in q(y_1,\ldots,y_n)$ and $h = h_1 = h_2 = \ldots = h_n$.

Suppose that (i) is satisfied. Then $h(y) \in p(h(y_1),\ldots,h(y_n))$ and $h(y_i) \in p(h_1(y_i),\ldots,h_n(y_i))$ for every $i = 1,\ldots,n$. Thus, we have $x = h(y) \in$

$p(p(h_1(y_1),\ldots,h_n(y_1)),\ldots,p(h_1(y_n),\ldots,h_n(y_n)))$. Further, we have $h(y) \in p(h_1(y),\ldots,h_n(y))$ and $h_j(y) \in p(h_j(y_1),\ldots,h_j(y_n))$ for every $j = 1,\ldots,n$. Hence, $h(y) \in p(p(h_1(y_1),\ldots,h_1(y_n)),\ldots,p(h_n(y_1),\ldots,h_n(y_n)))$. Since $\mathbf{G}$ is diagonal, we have $x \in p(h_1(y_1),\ldots,h_n(y_n)) = p(e(y_1,h_1),\ldots,e(y_n,h_n))$. Therefore, $e : \mathbf{H} \otimes \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ is a homomorphism.

Suppose that (ii) is true. Then $h(y) = h(y_1) = \ldots = h(y_n)$ and $h(y_i) \in p(h_1(y_i),\ldots,h_n(y_i))$ for every $i = 1,\ldots,n$. Thus, we have $x = h(y) \in p(h_1(y),\ldots,h_n(y)) = p(e(y_1,h_1),\ldots,e(y_n,h_n))$. Therefore, $e : \mathbf{H} \otimes \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ is a homomorphism.

Finally, let (iii) be true. Then $(h) \in p(h(y_1),\ldots,h(y_n))$ and $h(y_i) = h_1(y_i) = \ldots = h_n(y_i)$ for every $i = 1,\ldots,n$. Thus, we have $x = h(y) \in p(h(y_1),\ldots,h(y_n)) = p(e(y_1,h_1),\ldots,e(y_n,h_n))$. Therefore, $e : \mathbf{H} \otimes \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ is a homomorphism. □

Remark 3. The full subconstruct $MDHYP_n$ of HYP_n is closed under powers but it is not closed under combined products. By the previous Theorem, for every full subconstruct $\mathcal{A}$ of $MDHYP_n$ closed under both the powers and combined products, and every $\mathcal{A}$-object $\mathbf{H}$, the functor $\mathbf{H} \otimes - : \mathcal{A} \to \mathcal{A}$ is a co-adjoint with the corresponding adjoint being the functor $-^{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$. But it follows from [14] that the functor $-^{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$ is an adjoint with the corresponding co-adjoint being the functor $\mathbf{H} \times - : \mathcal{A} \to \mathcal{A}$. Consequently, the functors $\mathbf{H} \otimes - : \mathcal{A} \to \mathcal{A}$ and $\mathbf{H} \times - : \mathcal{A} \to \mathcal{A}$ are naturally isomorphic and $\mathcal{A}$ is cartesian closed. Therefore, all objects of $\mathcal{A}$ have to be idempotent. Such a subconstruct $\mathcal{A}$ may be obtained by putting $\mathcal{A} = IMDHYP_n$ where $IMDHYP_n$ is the full subconstruct of $MDHYP_n$ given by its idempotent objects. It follows from [14] that $IMDHYP_n$ is cartesian closed.

Proposition 4. *Let* $\mathbf{G},\mathbf{H},\mathbf{K}$ *be n-ary hyperalgebras. If* $\mathbf{G}$ *is medial and diagonal, then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\otimes\mathbf{K}}.$$

Proof. Let $\mathbf{H} = (H,q), \mathbf{G} = (G,p), \mathbf{K} = (K,s), \mathbf{G}^{\mathbf{H}} = (Hom(\mathbf{H},\mathbf{G}),r), \mathbf{H}\otimes\mathbf{K} = (H\times K,v), (\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} = (Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}}),t), \mathbf{G}^{\mathbf{H}\otimes\mathbf{K}} = (Hom(\mathbf{H}\otimes\mathbf{K},\mathbf{G}),u)$ and let $\mathbf{G}$ be diagonal and medial.

Let $h_1,\ldots,h_n \in Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ and $x \in \varphi(t(h_1,\ldots,h_n))$. Then there exists $h \in t(h_1,\ldots,h_n)$ such that $x = \varphi(h)$. Since $h \in t(h_1,\ldots,h_n)$, we have $h(z) \in r(h_1(z),\ldots,h_n(z))$ for every $z \in K$. Therefore, $h(z)(y) \in p(h_1(z)(y),\ldots,h_n(z)(y))$ for eacu $y \in H$. Thus, $\varphi(h)(y,z) \in p(\varphi(h_1)(y,z),\ldots,\varphi(h_n)(y,z))$ for every $(y,z) \in H\times K$. Hence, $\varphi(h) \in u(\varphi(h_1),\ldots,\varphi(h_n))$. Consequently, $\varphi(t(h_1,\ldots,h_n)) \subseteq u(\varphi(h_1),\ldots,\varphi(h_n))$.

Conversely, let $x \in u(\varphi(h_1),\ldots,\varphi(h_n))$. By Lemma 2, there exists $h \in Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ with $x = \varphi(h)$, so that $\varphi(h) \in u(\varphi(h_1),\ldots,\varphi(h_n))$. Then $\varphi(h)(y,z) \in p(\varphi(h_1)(y,z)\ ,\ldots,\varphi(h_n)(y,z))$ for every $(y,z) \in H\times K$, which implies $h(z)(y) \in p(h_1(z)(y)\ ,\ldots,\ h_n(z)(y))$ for every $y \in H$ and $z \in K$. It follows that $h(z) \in$

$r(h_1(z),\dots,h_n(z))$ for every $z \in K$. Therefore, $h \in t(h_1,\dots,h_n)$, thus $\varphi(h) \in \varphi(t(h_1,\dots,h_n))$. Hence, $u(\varphi(h_1),\dots,\varphi(h_n)) \subseteq \varphi(t(h_1,\dots,h_n))$.

We have shown that $\varphi(t(h_1,\dots,h_n)) = u(\varphi(h_1),\dots,\varphi(h_n))$ whenever $h_1, \dots, h_n \in Hom(\mathbf{K},\mathbf{G}^{\mathbf{H}})$. Thus, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ is isomorphic to $\mathbf{G}^{\mathbf{H}\otimes\mathbf{K}}$ by Lemma 2. □

Remark 4. It follows from Proposition 4 that the exponential law $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ is valid whenever $\mathbf{G},\mathbf{H},\mathbf{K}$ are n-ary hyperalgebras, $\mathbf{G}$ medial and diagonal and $\mathbf{H},\mathbf{K}$ idempotent. This result follows from [4]. But Proposition 4 says that, when replacing the cartesian product by the combined one, the above exponential law is satisfied for any n-ary hyperalgebras $\mathbf{H},\mathbf{K}$, not only for the idempotent ones.

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Doi: 10.18514/MMN.2015.770

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 517–525

HU e-ISSN 1787-2413

EXISTENCE RESULTS FOR A NON-RESONANT NONLOCAL BOUNDARY VALUE PROBLEMS

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Received 24 September, 2013

Abstract. In this paper we consider the following nonlocal boundary value problems

$$(p(t)x')' = f(t,x,x'), \quad x'(0) = 0, \quad x(1) = \int_0^1 x'(s)dg(s)$$

and

$$(p(t)x')' = f(t,x,x'), \quad x'(0) = 0, \quad x(1) = \int_0^1 x(s)dg(s),$$

where $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ and the integrals are meant in the sense of Riemann-Stieltjes. Under a sign condition on the function f, we prove the existence of solutions.

2010 *Mathematics Subject Classification:* 34B10; 34B15

Keywords: nonlocal boundary value problem, non-resonant boundary value problem, Leray-Schauder degree theory

1. Introduction

In this paper the following ordinary differential equations

$$(p(t)x')' = f(t,x,x'), \quad t \in [0,1] \tag{1.1}$$

with the initial condition

$$x'(0) = 0 \tag{1.2}$$

and the non-local boundary condition

$$x(1) = \int_0^1 x'(s)dg(s) \tag{1.3}$$

or

$$x(1) = \int_0^1 x(s)dg(s), \tag{1.4}$$

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where $f = (f_1, \dots, f_k) : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous, p is of class C^1, $p(t) > 0$ and $p'(t) \geq 0$, $t \in [0,1]$, $g = (g_1, \dots, g_k) : [0,1] \to \mathbb{R}^k$ has bounded variation and $\int_0^1 x'(s)dg(s) = [\int_0^1 x_1'(s)dg_1(s), \dots, \int_0^1 x_k'(s)dg_k(s)]$, $\int_0^1 x(s)dg(s) = [\int_0^1 x_1(s)dg_1(s), \dots, \int_0^1 x_k(s)dg_k(s)]$, is studied.

In other words, for instance, (1.1), (1.2) and (1.3) is the system of k BVPs

$$\begin{cases} (p(t)x_i'(t))' = f_i(t, x(t), x'(t)), \\ x_i'(0) = 0, \\ x_i(1) = \int_0^1 x_i'(s)dg_i(s), \end{cases}$$

where $t \in [0,1]$, $i = 1, \dots, k$ and the integrals $\int_0^1 x_i'(s)dg_i(s)$ are meant in the sense of Riemann-Stieltjes.

Nonlocal boundary value problems (BVPs) arise in different areas of applied mathematics and physics. Such problems, inter alia, have applications in chemical engineering, thermo-elasticity, underground water flow and population dynamics (see for instance [1, 19] and the references therein).

The study of nonlocal BVPs in the case of linear second-order ordinary differential equations was initiated by Bitsadze and Samarski [2] and later continued by Il'in and Moiseev [7, 8]. Since then, many authors began to consider second-order differential equations with various multi-point boundary conditions (compare for instance [3, 5, 13]).

BVPs with Riemann-Stieltjes integral boundary conditions include as special cases multi-point BVPs. According to the best of our knowledge, [15] is one of the first article in which the Authors consider the Riemann-Stieltjes integral boundary conditions for ordinary differential equations. Nowadays, the problem of the existence of solutions for various types of such nonlocal BVPs is the subject of many papers (we refer the reader to [10–12, 14, 18] and the references therein).

BVPs of the type: $x'' = f$, $x'(0) = 0$, wherein in the second boundary condition $x(1)$ depends on the derivative of the function x at some of its points, were considered for instance in [4, 6]. As far as we are aware, the BVP (1.1), (1.2), (1.3) has not been studied in this generality so far.

Nonlocal BVPs with conditions of the type $x'(0) = 0$, $x(1) = \int_0^1 x(s)dg(s)$ one can find for instance in [5], [18] or [17]. To our best knowledge, problem (1.1), (1.2), (1.4) was considered for the first time in [9] and [16] - in [9] as a multi-point boundary value problem, in [16] as a nonlocal boundary value problem with the Riemann-Stieltjes boundary condition.

In all papers mentioned above the Authors deal with scalar problems.

The purpose of this paper is to obtain conditions for the existence of a solution for the BVPs (1.1), (1.2), (1.3) and (1.1), (1.2), (1.4), using the properties of the Leray-Schauder topological degree. Our results extend results in the references.

2. Main result

First, we shall introduce notation and terminology. Denote by $C^1([0,1],\mathbb{R}^k)$ the Banach space of all continuous functions $x:[0,1]\to\mathbb{R}^k$ which have continuous first derivatives x' with the norm

$$\|x\| = \max\left\{|x(1)|,\ \sup_{t\in[0,1]}\left|x'(t)\right|\right\}, \tag{2.1}$$

where $|\cdot|$ means the Euclidean norm in $\mathbb{R}^k$.

We shall also need the following result (which is a straight consequence of the classical Arzelà−Ascoli theorem):

Lemma 1. *For a set $Z\subset \mathrm{C}^1\left([0,1],\mathbb{R}^k\right)$ to be relatively compact, it is necessary and sufficient that:*
(1) there exists $M>0$ such that for any $x\in Z$ and $t\in[0,1]$ we have $|x(1)|\le M$ and $|x'(t)|\le M$;
(2) for every $t_0\in[0,1]$ the family $Z':=\{x'\mid x\in Z\}$ is equicontinuous at t_0.

First let us consider BVP (1.1), (1.2), (1.3). Since the problem is always non-resonant (the homogeneous linear problem has only trivial solution), we will find an equivalent integral equation.

Let us consider the equation (1.1) and integrate it from 0 to t, we get

$$x'(t) = \frac{1}{p(t)}\int_0^t f(s,x(s),x'(s))ds + \frac{x'(0)}{p(t)}.$$

It then follows from (1.2) that

$$x'(t) = \frac{1}{p(t)}\int_0^t f(s,x(s),x'(s))ds. \tag{2.2}$$

Now, integrating (2.2) from t to 1, we have

$$x(t) = -\int_t^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))duds + x(1). \tag{2.3}$$

By (1.3) and (2.3), we obtain

$$x(1) = \int_0^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))dudg(s).$$

As an immediate corollary we have the following:

Lemma 2. *A function $x\in C^1([0,1],\mathbb{R}^k)$ is a solution of the problem (1.1), (1.2), (1.3) only and only if x satisfies the following integral equation*

$$x(t) = -\int_t^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))duds+$$

$$+\int_0^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))dudg(s).$$

Let $A:\mathrm{C}^1\left([0,1],\mathbb{R}^k\right)\to\mathrm{C}^1\left([0,1],\mathbb{R}^k\right)$ be such that

$$(Ax)(t)=-\int_t^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))duds+$$
$$+\int_0^1 \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))dudg(s).$$

We thus conclude that

$$(Ax)'(t)=\frac{1}{p(t)}\int_0^t f(s,x(s),x'(s))ds.$$

Functions $Ax,(Ax)':[0,1]\to\mathbb{R}^k$ are continuous. It follows that the operator A is well-defined.

Let $x\in\mathrm{C}^1\left([0,1],\mathbb{R}^k\right)$ and $\|x\|=M$. It is to be noted that

$$|x(t)|\le(1-t)\sup_{t\in[0,1]}\left|x'(t)\right|+|x(1)|\le 2M. \tag{2.4}$$

Now, since f is continuous g has bounded variation, by (2.4) and Lemma 1, it is clear that

Lemma 3. *The operator A is completely continuous.*

The existence result is given in the following theorem:

Theorem 1. *Assume that the following conditions hold*

- (i) $f=(f_1,\dots,f_k):[0,1]\times\mathbb{R}^k\times\mathbb{R}^k\to\mathbb{R}^k$ *is a continuous function;*
- (ii) *there exists $M>0$ such that $(y|f(t,x,y))<0$ for $t\in[0,1]$, $x\in\mathbb{R}^k$ and $|y|\ge M$, where $(\cdot\,|\,\cdot)$ means the scalar product in $\mathbb{R}^k$ corresponding to the Euclidean norm;*
- (iii) $p\in C^1([0,1],\mathbb{R})$ *are such that $p(t)>0$ and $p'(t)\ge 0$;*
- (iv) $g=(g_1,\dots,g_k):[0,1]\to\mathbb{R}^k$ *and Var*(g)<1, *where Var*(g) *means the variation of g on the interval* $[0,1]$;

Then BVP (1.1), (1.2) *and* (1.3) *has at least one solution.*

Proof. Let us consider the continuous family of BVPs:

$$(p(t)\varphi')'=\lambda f(t,\varphi,\varphi'),\quad \varphi'(0)=0,\quad \varphi(1)=\int_0^1\varphi'(s)dg(s), \tag{2.5}$$

depending on a parameter $\lambda\in[0,1]$. Notice that the problem (2.5) is equivalent to the following integral equation $\varphi(t)=\lambda A\varphi(t)$. It follows from Lemma 3 that the operator λA is completely continuous.

Consider the homotopy $H : [0,1] \times C^1\left([0,1], \mathbb{R}^k\right) \to C^1\left([0,1], \mathbb{R}^k\right)$ given by

$$H(\lambda, \varphi) = \varphi - \lambda A\varphi$$

in $\Omega = B(0, M)$, where M is the positive constant from the assumption (ii).

We shall show that $H(\lambda, \varphi) = 0$ has no solution for $\lambda \in [0,1]$ and φ belonging to $\partial\Omega$.

Conversely, suppose that $H(\lambda, \varphi) = 0$ for $\lambda \in [0,1]$ and $\varphi \in \partial\Omega$.

If $H(\lambda, \varphi) = 0$ for $\lambda = 0$ and $\varphi \in \partial\Omega$, then BVP (2.5) has only a trivial solution, which contradicts the fact that $\varphi \in \partial\Omega$.

Now, let $H(\lambda, \varphi) = 0$ with $\lambda \in (0,1]$ and $\varphi \in \partial\Omega$. By (2.1),

$$M = \|\varphi\| = \max\left\{|\varphi(1)|, \sup_{t\in[0,1]} |\varphi'(t)|\right\}$$

.

First, assume that $|\varphi(1)| = M$. In this case, by (1.3) and (iv), we get

$$M = |\varphi(1)| = \left|\int_0^1 \varphi'(s)dg(s)\right| \leq M \operatorname{Var}(g) < M,$$

a contradiction.

Now, consider the case when $\sup_{t\in[0,1]} |\varphi'(t)| = M$. Let $\psi(t) = |\varphi'(t)|^2$. Notice that $\varphi'(0) = 0$. Hence ψ has a maximum equal to M^2 for some $t_0 \in (0,1]$.

If $t_0 \in (0,1]$, then, by assumptions (ii), (iii) and (2.5), since $|\varphi'(t_0)| = M$, we have

$$\begin{aligned} 0 \leq \psi'(t_0) &= 2\left(\varphi'(t_0) \mid \varphi''(t_0)\right) \\ &= 2\left(\varphi'(t_0) \mid \frac{\lambda f(t_0, \varphi(t_0), \varphi'(t_0))}{p(t_0)} - \frac{p'(t_0)}{p(t_0)}\varphi'(t_0)\right) \\ &= 2\frac{\lambda}{p(t_0)}\left(\varphi'(t_0) \mid f\left(t_0, \varphi(t_0), \varphi'(t_0)\right)\right) - 2\frac{p'(t_0)}{p(t_0)}\left|\varphi'(t_0)\right|^2 < 0. \end{aligned}$$

Hence, we get a contradiction.

Hence homotopy H does not vanish on the boundary of Ω for $\lambda > 0$. We thus conclude that $H(\lambda, \varphi) \neq 0$ for $\lambda \in [0,1]$ and $\varphi \in \partial\Omega$.

Now, by the properties of the Leray-Schauder topological degree, we have

$$\begin{aligned} &\deg_{LS}(I - A, \Omega) = \deg_{LS}(H(1, \cdot), \Omega) \\ &= \deg_{LS}(H(0, \cdot), \Omega) = \deg_{LS}(I, \Omega) = 1 \neq 0. \end{aligned}$$

Hence A has a fixed point in Ω, i.e. BVP (1.1), (1.2) and (1.3) has a solution in Ω. □

Now, let us consider BVP (1.1), (1.2), (1.4). For this problem we will assume that Var(g) $< 1/2$. If so, then $\int_0^1 dg_i(s) \neq 1$ and the problem is non-resonant. Proceeding similarly as in the case of the first problem, we get

Lemma 4. *A function $x \in C^1([0,1],\mathbb{R}^k)$ is a solution of the problem (1.1), (1.2), (1.4) only, and only if x satisfies the following integral equation*

$$x_i(t) = -\int_t^1 \frac{1}{p(s)} \int_0^s f_i(u,x(u),x'(u))duds + \\ +\alpha_i \left[\int_0^1 \int_s^1 \frac{1}{p(u)} \int_0^u f_i(z,x(z),x'(z))dzdudg_i(s) \right],$$

for every $i = 1,\ldots,k$, where $\alpha_i := \left(\int_0^1 dg_i(s) - 1\right)^{-1}$.

Let $B : \mathrm{C}^1\left([0,1],\mathbb{R}^k\right) \to \mathrm{C}^1\left([0,1],\mathbb{R}^k\right)$ is given by

$$(Bx)(t) = -\int_t^1 \frac{1}{p(s)} \int_0^s f(u,x(u),x'(u))duds + \\ +\alpha \left[\int_0^1 \int_s^1 \frac{1}{p(u)} \int_0^u f(z,x(z),x'(z))dzdudg(s) \right].$$

It is clear that $Bx, (Bx)' : [0,1] \to \mathbb{R}^k$ are continuous. It follows that B is well-defined. Moreover, since f is continuous g has bounded variation, by (2.4) and Lemma 1, it is clear that

Lemma 5. *The operator B is completely continuous.*

Theorem 2. *Assume that the following conditions hold*

- (i) $f = (f_1,\ldots,f_k) : [0,1]\times\mathbb{R}^k\times\mathbb{R}^k \to \mathbb{R}^k$ *is a continuous function;*
- (ii) *there exists $M > 0$ such that $(y|f(t,x,y)) < 0$ for $t \in [0,1]$, $x \in \mathbb{R}^k$ and $|y| \geq M$;*
- (iii) $p \in C^1([0,1],\mathbb{R})$ *are such that $p(t) > 0$ and $p'(t) \geq 0$;*
- (iv) $g = (g_1,\ldots,g_k) : [0,1] \to \mathbb{R}^k$ *and Var(g) $< 1/2$.*

Then BVP (1.1), (1.2) and (1.4) has at least one solution.

Proof. Let us consider the continuous family of BVPs:

$$(p(t)\varphi')' = \lambda f(t,\varphi,\varphi'), \quad \varphi'(0) = 0, \quad \varphi(1) = \int_0^1 \varphi(s)dg(s), \tag{2.6}$$

depending on a parameter $\lambda \in [0,1]$. Notice that the problem (2.6) is equivalent to the following integral equation $\varphi(t) = \lambda B\varphi(t)$. It follows from Lemma 5 that the operator λB is completely continuous.

Consider the homotopy $H : [0,1]\times\mathrm{C}^1\left([0,1],\mathbb{R}^k\right) \to \mathrm{C}^1\left([0,1],\mathbb{R}^k\right)$ given by

$$H(\lambda,\varphi) = \varphi - \lambda B\varphi$$

in $\Omega = B(0,M)$, where M is the positive constant from the assumption (ii).

Notice that $H(\lambda,\varphi) = 0$ has no solution for $\lambda \in [0,1]$ and φ belonging to the boundary of the ball Ω.

Indeed, $H(0,\varphi) = 0$ has only a trivial solution, so $\lambda \neq 0$.

Now, let $H(\lambda,\varphi) = 0$ with $\lambda \in (0,1]$ and $\varphi \in \partial\Omega$. By (2.1),

$$M = \|\varphi\| = \max\left\{|\varphi(1)|, \sup_{t\in[0,1]} |\varphi'(t)|\right\}$$

.

First, assume that $|\varphi(1)| = M$. In this case, by (1.4) and (iv), we get

$$M = |\varphi(1)| = \left|\int_0^1 \varphi(s)dg(s)\right| \leq 2M \operatorname{Var}(g) < M,$$

a contradiction.

The proof of the fact that $\sup_{t\in[0,1]} |\varphi'(t)| \neq M$ is similar to the proof of Theorem 1.

So, $H(\lambda,\varphi) = 0$ has no solution for $\lambda \in [0,1]$ and $\varphi \in \partial\Omega$. Hence A has a fixed point in Ω, i.e. BVP (1.1), (1.2) and (1.4) has a solution in Ω. □

3. Application

Let us consider the following BVP

$$(p(t)x')' = f(t,x,x'), \quad x'(0) = 0, \quad x(1) = \int_0^1 x'(s)dg(s), \tag{3.1}$$

where $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y))$,

$$\begin{aligned} f_1(t,x,y) &= -(t+1)\left(\exp(-x_1^2)+1\right)y_1 - (t+1)\left(\exp(-x_2^2)+1\right)y_2, \\ f_2(t,x,y) &= (t+1)\left(\exp(-x_2^2)+1\right)y_1 - (t+1)\left(\exp(-x_1^2)+1\right)y_2 + \\ &\quad -2(t+1)\left(\exp(-x_1^2)+1\right) \end{aligned}$$

and p, g are arbitrary functions satisfying the conditions (iii) and (iv).

Observe that f satisfies the assumptions (i) and (ii). Indeed, for any $M > 2$ and $|y| \geq M$, $x \in \mathbb{R}^2$ and $t \in [0,1]$, we have

$$\begin{aligned} (y \mid f(t,x,y)) &= y_1\left(-(t+1)\left(\exp(-x_1^2)+1\right)y_1 - (t+1)\left(\exp(-x_2^2)+1\right)y_2\right) + \\ &\quad + y_2\left((t+1)\left(\exp(-x_2^2)+1\right)y_1 - (t+1)\left(\exp(-x_1^2)+1\right)y_2\right) + \\ &\quad -2y_2(t+1)\left(\exp(-x_1^2)+1\right) = \\ &\quad = -(t+1)\left(\exp(-x_1^2)+1\right)\left(y_1^2 + y_2^2 + 2y_2\right) < 0. \end{aligned}$$

Indeed, notice that $y_1^2 + y_2^2 + 2y_2 > 0$, if $y_2 \in (-\infty,-2] \cup [0,\infty)$. If $y_2 \in (-2,0)$, we obtain

$$y_1^2 + y_2^2 + y_2 \geq M^2 + y_2 > 2 + y_2 > 0.$$

Hence, there exists at least one nontrivial solution of (3.1) what ends the example and completes the paper.

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Doi: 10.18514/MMN.2015.738

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 527–541

HU e-ISSN 1787-2413

COUPLED COINCIDENCE POINT THEOREMS FOR COMPATIBLE MAPPINGS IN ORDERED UNIFORM SPACE

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Received 24 September, 2013

Abstract. In this paper, we use the order relation on uniform spaces defined by [5] to introduce the notion of compatibility of mappings in an ordered uniform space and use this notion to establish coupled coincidence point theorems to ordered uniform space. An example is also given.

2010 *Mathematics Subject Classification:* 54H25; 47H10

Keywords: coupled fixed point, non-linear contractions, ordered uniform space

1. INTRODUCTION

There exists considerable literature of fixed point theory dealing with results on fixed or common fixed points in uniform space (e.g. [1–3, 5, 16, 21]). But the majority of these results are proved for contractive or contractive type mapping (notice from the cited references). Recently, Aamri and El Moutawakil [1] have introduced the concept of E-distance function on uniform spaces and utilize it to improve some well known results of the existing literature involving both E-contractive or E- expansive mappings. Lately, I. Altun and M. Imdad [5] have introduced a partial ordering on uniform spaces utilizing E- distance function and have used the same to prove a fixed point theorem for single-valued non-decreasing mappings on ordered uniform spaces. The Banach contraction principle is the most celebrated fixed point theorem. Boyd and Wong [7] extended the Banach contraction principle to the case of non-linear contraction mappings. Afterward Ciric and Lakshmikantham [9, 13] obtained important fixed point theorems. Recently Bhaskar and Lakshmikantham [6], B. S. Choudhury and A. Kundu [8], Nieto and Lopez [14, 15], Ran and Reurings [20] and Agarwal, El-Gebeily and O'Regan [4] presented some new results for contractions in partially ordered metric spaces. Common fixed point results for commuting mappings in metric spaces were first deduced by Jungck [10]. The concept of commuting has been weakened in various directions and in several ways over the years. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck [11]. Lately, Petre and Páles [18, 19] gived fixed point theorems in E-Banach

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spaces and E-metric spaces. In common fixed point problems, this concept and its generalizations have been used extensively [12, 17].

In this paper, we use the partial ordering on uniform spaces which is defined by [5], so we proved some coupled coincidence and coupled common fixed point theorems for a pair of mappings. We also discuss an example.

Now, we mention some relevant definitions and properties from the foundation of uniform spaces. We call a pair (X,ϑ) to be a uniform space which consists of a non-empty set X together with a uniformity ϑ wherein the latter begins with a special kind of filter on $X\times X$ whose all elements contain the diagonal $\Delta=\{(x,x):x\in X\}$. If $V\in\vartheta$ and $(x,y)\in V$, $(y,x)\in V$ then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V\in\vartheta$, there exists $N\geq 1$ such that x_n and x_m are V-close for $m,n\geq N$. A uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x\in X$ are the sets $V(x)=\{y\in X:(x,y)\in V\}$ when V runs over ϑ.

A uniform space (X,ϑ) is said to be Hausdorff if and only if the intersection of all the $V\in\vartheta$ reduces to diagonal Δ of X i.e. $(x,y)\in V$ for $V\in\vartheta$ implies $x=y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V=V^{-1}=\{(y,x):(x,y)\in V\}$. Since each $V\in\vartheta$ contains a symmetrical $W\in\vartheta$ and if $(x,y)\in W$ then x and y are both W and V-close and then one may assume that each $V\in\vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X,ϑ), they are naturally interpreted with respect to the topological space $(X,\tau(\vartheta))$.

2. Preliminaries

We shall require the following definitions and lemmas in the sequel.

Definition 1 ([1]). Let (X,ϑ) be a uniform space. A function $p:X\times X\to\mathbb{R}^+$ is said to be an E-distance implies

(p_1) For any $V\in\vartheta$ there exists $\delta>0$ such that $p(z,x)\leq\delta$ and $p(z,y)\leq\delta$ for some $z\in X$, imply $(x,y)\in V$,

(p_2) $p(x,y)\leq p(x,z)+p(z,y)$, for any $x,y,z\in X$.

The following lemma embodies some useful properties of E-distance.

Lemma 1 ([1, 2]). *Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x,y,z\in X$, the following holds:*

(a) If $p(x_n,y)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $y=z$. In particular, if $p(x,y)=0$ and $p(x,z)=0$, then $y=z$.

(b) If $p(x_n,y_n)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a $p-$Cauchy sequence in (X, ϑ).

Let (X, ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2 ([1,2]). Let (X, ϑ) be a uniform space and p be an E-distance on X. Then

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$,

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(fx_n, fx) = 0$,

(iv) $f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark 1 ([1]). Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.

Lemma 2 ([5]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$. Define the relation " $\preceq$ " on X as follows;*

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x, y) \leq \varphi(x) - \varphi(y).$$

Then " $\preceq$ " is a (partial) order on X induced by φ.

Definition 3 ([6]). We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping T if $T(x, y) = x$, $T(y, x) = y$.

Definition 4 ([13]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $T : X \times X \to X$ and $g : X \to X$ implies
$T(x, y) = g(x)$, $T(y, x) = g(y)$.

Definition 5 ([13]). Let X be a non-empty set and $T : X \times X \to X$ and $g : X \to X$. We say T and g are commutative implies
$g(T(x, y)) = T(g(x), g(y))$
for any $x, y \in X$.

Definition 6 ([8]). The mappings T and g where $T : X \times X \to X$ and $g : X \to X$, are said to be compatible implies
$\lim_{n\to\infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) = 0$

and
$\lim_{n\to\infty} p\left(g\left(T\left(y_n, x_n\right)\right), T\left(g\left(y_n\right), g\left(x_n\right)\right)\right) = 0$
whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that
$\lim_{n\to\infty} T\left(x_n, y_n\right) = \lim_{n\to\infty} g\left(x_n\right) = x$ and $\lim_{n\to\infty} T\left(y_n, x_n\right) = \lim_{n\to\infty} g\left(y_n\right) = y$,
for any $x, y \in X$ are satisfied.

3. Main results

Definition 7. Let (X, ϑ) be a uniform space and let " $\preceq$ " be an order relation on X and let $T : X \times X \to X$ be an operator. We say that T has the mixed monotone property if $T(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow T\left(x_1, y\right) \preceq T\left(x_2, y\right) \tag{3.1}$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow T\left(x, y_1\right) \succeq T\left(x, y_2\right). \tag{3.2}$$

Definition 8. Let (X, ϑ) be a uniform space and let " $\preceq$ " be an order relation on X and let $T : X \times X \to X$, $g : X \to X$ be operators. We say T has the mixed $g-$monotone property if T is monotone $g-$non-decreasing in its first argument and is monotone $g-$non-increasing in its second argument, that is, for any $x, y \in X$,
$x_1, x_2 \in X$, $g\left(x_1\right) \preceq g\left(x_2\right)$ implies

$$T\left(x_1, y\right) \preceq T\left(x_2, y\right) \tag{3.3}$$

and
$y_1, y_2 \in X$, $g\left(y_1\right) \preceq g\left(y_2\right)$ implies

$$T\left(x, y_1\right) \succeq T\left(x, y_2\right) \tag{3.4}$$

If g is the identity mapping, then Definition 8 reduces to Definition 7.

Theorem 1. *Let (X, ϑ) be a uniform space, " $\preceq$ " is an order on X and suppose there is an $E-$distance p on X such that (X, p) is a $p-$Cauchy complete uniform space. Let $T : X \times X \to X$ be a $\tau(\vartheta)$ continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*
$p\left(T\left(x, y\right), T\left(u, v\right)\right) \leq \frac{k}{2}\left[p\left(x, u\right) + p\left(y, v\right)\right]$
for all comparable x, u and all comparable y, v. If there exist $x_0, y_0 \in X$ such that
$x_0 \preceq T\left(x_0, y_0\right)$ *and* $y_0 \succeq T\left(y_0, x_0\right)$,
then there exist $x, y \in X$ such that $x = T(x, y)$ and $y = T(x, y)$.

Proof. Since $x_0 \preceq T\left(x_0, y_0\right) = x_1$ and $y_0 \succeq T\left(x_0, y_0\right) = y_1$
letting $x_2 = T\left(x_1, y_1\right)$ and $y_2 = T\left(y_1, x_1\right)$, we denote

$T^2(x_0,y_0)=T(T(x_0,y_0),T(y_0,x_0))=T(x_1,y_1)=x_2.$
$T^2(y_0,x_0)=T(T(y_0,x_0),T(x_0,y_0))=T(y_1,x_1)=y_2.$
With this notation, we now have, due to the mixed monotone property of T,
$x_2=T^2(x_0,y_0)=T(x_1,y_1)\succeq T(x_0,y_0)=x_1$ and
$y_2=T^2(y_0,x_0)=T(y_1,x_1)\preceq T(y_0,x_0)=y_1.$
Further, for $n=1,2,...$ we let
$x_{n+1}=T^{n+1}(x_0,y_0)=T(T^n(x_0,y_0),T^n(y_0,x_0))$
and
$y_{n+1}=T^{n+1}(y_0,x_0)=T(T^n(y_0,x_0),T^n(x_0,y_0)).$
We can easily verify that
$x_0\preceq T(x_0,y_0)=x_1\preceq T^2(x_0,y_0)=x_2\preceq ...\preceq T^{n+1}(x_0,y_0)\preceq ...$
and
$y_0\succeq T(y_0,x_0)=y_1\succeq T^2(y_0,x_0)=y_2\succeq ...\succeq T^{n+1}(y_0,x_0)\succeq ....$
Now, we claim that, for $n\in\mathbb{N}$,
$p\left(T^{n+1}(x_0,y_0),T^n(x_0,y_0)\right)\leq\frac{k^n}{2}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)]$ ...(i)
$p\left(T^{n+1}(y_0,x_0),T^n(y_0,x_0)\right)\leq\frac{k^n}{2}[p(T(y_0,x_0),y_0)+p(T(x_0,y_0),x_0)]$ ...(ii)
Indeed, for $n=1$, using $T(x_0,y_0)\succeq x_0$ and $T(y_0,x_0)\preceq y_0$, we get
$$\begin{aligned}p\left(T^2(x_0,y_0),T(x_0,y_0)\right)&=p(T(T(x_0,y_0),T(y_0,x_0)),T(x_0,y_0))\\&\leq\tfrac{k}{2}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)].\end{aligned}$$
Similarly,
$$\begin{aligned}p\left(T^2(y_0,x_0),T(y_0,x_0)\right)&=p(T(T(y_0,x_0),T(x_0,y_0)),T(y_0,x_0))\\&=p(T(y_0,x_0),T(T(y_0,x_0),T(x_0,y_0)))\\&\leq\tfrac{k}{2}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)].\end{aligned}$$
Now, assume that (i) and (ii) hold. Using
$T^{n+1}(x_0,y_0)\succeq T^n(x_0,y_0)$ and $T^{n+1}(y_0,x_0)\preceq T^n(y_0,x_0)$ we get
$$\begin{aligned}p\left(T^{n+2}(x_0,y_0),T^{n+1}(x_0,y_0)\right)&=p(T\left(T^{n+1}(x_0,y_0),T^{n+1}(y_0,x_0)\right)\\&\quad,T(T^n(x_0,y_0),T^n(y_0,x_0)))\\&\leq\tfrac{k}{2}[p\left(T^{n+1}(x_0,y_0),T^n(x_0,y_0)\right)+p\left(T^{n+1}(y_0,x_0),T^n(y_0,x_0)\right)]\\&\leq\tfrac{k^{n+1}}{2}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)].\end{aligned}$$
Similarly, one can show that
$p\left(T^{n+2}(y_0,x_0),T^{n+1}(y_0,x_0)\right)\leq\frac{k^{n+1}}{2}[p(T(y_0,x_0),y_0)+p(T(x_0,y_0),x_0)].$
This implies that $\{T^n(x_0,y_0)\}$ and $\{T^n(y_0,x_0)\}$ are $p-$Cauchy sequences in X.
Indeed, let $m>n$, then
$$\begin{aligned}p(T^m(x_0,y_0),T^n(x_0,y_0))&\leq p(T^m(x_0,y_0),T^{m-1}(x_0,y_0))+\\&\quad...+p(T^{n+1}(x_0,y_0),T^n(x_0,y_0))\\&\leq\tfrac{k^{m-1}+...+k^n}{2}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)]\\&\leq\tfrac{k^n}{2(1-k)}[p(T(x_0,y_0),x_0)+p(T(y_0,x_0),y_0)].\end{aligned}$$
Similarly, we can verify that $\{T^n(y_0,x_0)\}$ is also a $p-$Cauhy sequence.
Since X is a $p-Cauchy$ complete uniform space, there exist $x,y\in X$ such that
$\lim_{n\to\infty}T^n(x_0,y_0)=x$ and $\lim_{m\to\infty}T^m(y_0,x_0)=y.$

Finally, we claim $T(x,y)=x$ and $T(y,x)=y$.

Let $\varepsilon>0$. Since T is $\tau(\vartheta)-$continuous at (x,y), for a given $\frac{\varepsilon}{2}>0$, there exists a $\delta>0$ such that

$p(x,u)+p(y,v)<\delta$ implies $p(T(x,y),T(u,v))<\frac{\varepsilon}{2}$.

Since $\{T^n(x_0,y_0)\}\to x$ and $\{T^n(y_0,x_0)\}\to y$, for $\eta=\min\left(\frac{\varepsilon}{2},\frac{\delta}{2}\right)>0$, there exist n_0,m_0 such that, for $n\geq n_0$, $m\geq m_0$,

$p(T^n(x_0,y_0),x)<\eta$ and $p(T^m(y_0,x_0),y)<\eta$.

Now, for $n\in\mathbb{N}$, $n\geq\max\{n_0,m_0\}$,

$$\begin{aligned}p(T(x,y),x)&\leq p\left(T(x,y),T^{n+1}(x_0,y_0)\right)+p\left(T^{n+1}(x_0,y_0),x\right)\\&=p(T(x,y),T(T^n(x_0,y_0),T^n(y_0,x_0)))+p(T^{n+1}(x_0,y_0),x)\\&<\tfrac{\varepsilon}{2}+\eta\leq\varepsilon.\end{aligned}$$

This implies that $T(x,y)=x$. Similarly, we can show that $T(y,x)=y$. □

Theorem 2. *Let (X,ϑ) be a uniform space, "$\preceq$" is an order on X and suppose there is an $E-$distance p on X such that (X,p) is a $p-$Cauchy complete uniform space. Assume there is a function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(t)<t$ and $\lim\limits_{r\to t+}\phi(r)<t$ for each $t>0$ and also suppose $T:X\times X\to X$ and $g:X\to X$ are such that T has the mixed $g-$monotone property and*

$$p(T(x,y),T(u,v))\leq\phi\left(\frac{p(g(x),g(u))+p(g(y),g(v))}{2}\right)\tag{3.5}$$

for all $x,y,u,v\in X$ for which $g(x)$, $g(u)$ are comparable and $g(y)$, $g(v)$ are comparable. Suppose

$T(X\times X)\subseteq g(X)$, g is $\tau(\vartheta)-$ continuous and monotone increasing and T and g be compatible mappings. Also suppose

(a) *T is $\tau(\vartheta)-$ continuous or*

(b) *X has the following property :*

(i) *if a non-decreasing sequence $\{x_n\}\to x$, then*

$$x_n\preceq x\tag{3.6}$$

for all n,

(ii) *if a non-increasing sequence $\{y_n\}\to y$, then*

$$y\preceq y_n\tag{3.7}$$

for all n.

If there exist $x_0,y_0\in X$ such that
$g(x_0)\preceq T(x_0,y_0)$ and $g(y_0)\succeq T(y_0,x_0)$,
then there exist $x,y\in X$ such that
$g(x)=T(x,y)$ and $g(y)=T(y,x)$,
that is, T and g have a coupled coincidence point in X.

Proof. Let $x_0,\ y_0 \in X$, be such that
$g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$. Since
$T(X \times X) \subseteq g(X)$, we can define $x_1, y_1 \in X$ such that
$g(x_1) = T(x_0, y_0)$ and $g(y_1) = T(y_0, x_0)$.
In the same way we construct,
$g(x_2) = T(x_1, y_1)$ and $g(y_2) = T(y_1, x_1)$.
Continuing in this way we construct two sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that,
$g(x_{n+1}) = T(x_n, y_n)$ and

$$g(y_{n+1}) = T(y_n, x_n) \tag{3.8}$$

for all $n \geq 0$.
Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{3.9}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{3.10}$$

Since $g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$, in view of $g(x_1) = T(x_0, y_0)$ and $g(y_1) = T(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (3.9) and (3.10) hold for $n = 0$.
We presume that (3.9) and (3.10) hold for some $n > 0$. As T has the mixed $g-$monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (8), we get
$g(x_{n+1}) = T(x_n, y_n) \preceq T(x_{n+1}, y_n)$ and

$$T(y_{n+1}, x_n) \preceq T(y_n, x_n) = g(y_{n+1}). \tag{3.11}$$

Also for the same reason we have $g(x_{n+2}) = T(x_{n+1}, y_{n+1}) \succeq T(x_{n+1}, y_n)$ and

$$T(y_{n+1}, x_n) \succeq T(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{3.12}$$

Then from (3.11) and (3.12)
$g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$.
Then, by mathematical induction it follows that (3.9) and (3.10) hold for all $n \geq 0$.
Let, $\delta_n = p(g(x_n), g(x_{n+1})) + p(g(y_n), g(y_{n+1}))$ and $\delta_n^{|} = p(g(x_{n+1}), g(x_n)) + p(g(y_{n+1}), g(y_n))$.
Next we prove that
$\delta_n \leq 2\phi\left(\frac{\delta_{n-1}}{2}\right)$ and

$$\delta_n^{|} \leq 2\phi\left(\frac{\delta_{n-1}^{|}}{2}\right). \tag{3.13}$$

Since for all $n \geq 0$, $g(x_{n-1}) \preceq g(x_n)$ and $g(y_{n-1}) \succeq g(y_n)$, we have from (3.5) and (3.8),
$p(g(x_n), g(x_{n+1})) = p(T(x_{n-1}, y_{n-1}), T(x_n, y_n))$

$$\leq \phi\left(\frac{p\left(g\left(x_{n-1}\right),g\left(x_n\right)\right)+p\left(g\left(y_{n-1}\right),g\left(y_n\right)\right)}{2}\right)=\phi\left(\frac{\delta_{n-1}}{2}\right) \tag{3.14}$$

and

$p\left(g\left(x_{n+1}\right),g\left(x_n\right)\right)=p\left(T\left(x_n,y_n\right),T\left(x_{n-1},y_{n-1}\right)\right)$

$\leq \phi\left(\frac{p(g(x_n),g(x_{n-1}))+p(g(y_n),g(y_{n-1}))}{2}\right)=\phi\left(\frac{\delta'_{n-1}}{2}\right)$.

Similarly from (3.5) and (3.8), we have for all $n \geq 0$,

$p\left(g\left(y_n\right),g\left(y_{n+1}\right)\right)=p\left(T\left(y_{n-1},x_{n-1}\right),T\left(y_n,x_n\right)\right)$

$$\leq \phi\left(\frac{p\left(g\left(y_{n-1}\right),g\left(y_n\right)\right)+p\left(g\left(x_{n-1}\right),g\left(x_n\right)\right)}{2}\right)=\phi\left(\frac{\delta_{n-1}}{2}\right) \tag{3.15}$$

and

$p\left(g\left(y_{n+1}\right),g\left(y_n\right)\right)=p\left(T\left(y_n,x_n\right),T\left(y_{n-1},x_{n-1}\right)\right)$

$\leq \phi\left(\frac{p(g(y_n),g(y_{n-1}))+p(g(x_n),g(x_{n-1}))}{2}\right)=\phi\left(\frac{\delta'_{n-1}}{2}\right)$.

Combining (3.14) and (3.15) we obtain(3.13).

Since $\phi(t)<t$ for $t>0$, it follows from (3.13) that the sequences $\{\delta_n\}$ and $\{\delta'_n\}$ are monotone decreasing sequence of non-negative real numbers. Hence there exist $\delta \geq 0$ and $\delta' \geq 0$ such that $\lim_{n\to\infty}\delta_n=\delta$ and $\lim_{n\to\infty}\delta'_n=\delta'$. If possible, let $\delta>0$. Taking the limit as $n\to\infty$ in (3.13) and using $\lim_{r\to t^+}\phi(r)<t$ for all $t>0$, we obtain

$$\delta=\lim_{n\to\infty}\delta_n\leq 2\lim_{n\to\infty}\phi\left(\frac{\delta_{n-1}}{2}\right)=2\lim_{\delta_{n-1}\to\delta^+}\phi\left(\frac{\delta_{n-1}}{2}\right)<2\frac{\delta}{2}=\delta,$$

which is a contradiction. Thus $\delta=0$. Hence we have

$$\lim_{n\to\infty}[p(g\left(x_n\right),g\left(x_{n+1}\right))+p(g\left(y_n\right),g\left(y_{n+1}\right))]=\lim_{n\to\infty}\delta_n=0 \tag{3.16}$$

and similarly $\delta'=0$ that is $\lim_{n\to\infty}[p(g\left(x_{n+1}\right),g\left(x_n\right))+p(g\left(y_{n+1}\right),g\left(y_n\right))]=\lim_{n\to\infty}\delta'_n=0$.

Next we show that $\{g(x_n)\}$ and $\{g\left(y_n\right)\}$ are p-Cauchy sequences. Let at least one of $\{g\left(x_n\right)\}$ and $\{g\left(y_n\right)\}$ be not a $p-Cauchy$ sequence. Then there exists $\varepsilon>0$ and sequences of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every natural number k

$m(k)>l(k)\geq k$ and

$$p_k=p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)}\right)\right)\geq\varepsilon. \tag{3.17}$$

Now corresponding to $l(k)$ we can choose $m(k)$ to be the smallest positive integer for which (3.17) holds. Then,

$$p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)-1}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)-1}\right)\right)<\varepsilon. \tag{3.18}$$

Further from (3.17) and (3.18), for all $k\geq 0$, we have,

$\varepsilon\leq p_k\leq p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)-1}\right)\right)+p\left(g\left(x_{m(k)-1}\right),g\left(x_{m(k)}\right)\right)$

$$+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)-1}\right)\right)+p\left(g\left(y_{m(k)-1}\right),g\left(y_{m(k)}\right)\right)$$
$$=p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)-1}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)-1}\right)\right)+\delta_{m(k)-1}<\varepsilon+\delta_{m(k)-1}.$$

Taking the limit as $k\to\infty$, we have by (3.16),

$$\lim_{k\to\infty}p_k=\varepsilon. \tag{3.19}$$

Again, for all $k\geq 0$, we have,

$$\begin{aligned}p_k&=p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)}\right)\right)\\&\leq p\left(g\left(x_{l(k)}\right),g\left(x_{l(k)+1}\right)\right)\\&+p\left(g\left(x_{l(k)+1}\right),g\left(x_{m(k)+1}\right)\right)+p\left(g\left(x_{m(k)+1}\right),g\left(x_{m(k)}\right)\right)\\&+p\left(g\left(y_{l(k)}\right),g\left(y_{l(k)+1}\right)\right)\\&+p\left(g\left(y_{l(k)+1}\right),g\left(y_{m(k)+1}\right)\right)+p\left(g(y_{m(k)+1}),g\left(y_{m(k)}\right)\right)\\&=p\left(g\left(x_{l(k)}\right),g\left(x_{l(k)+1}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{l(k)+1}\right)\right)\\&+p\left(g\left(x_{l(k)+1}\right),g\left(x_{m(k)+1}\right)\right)\\&+p(g\left(y_{l(k)+1}\right),g\left(y_{m(k)+1}\right)+p\left(g\left(x_{m(k)+1}\right),g\left(x_{m(k)}\right)\right)\\&+p\left(g\left(y_{m(k)+1}\right),g\left(y_{m(k)}\right)\right).\end{aligned}$$

Hence, for all $k\geq 0$

$$p_k\leq\delta_{l(k)}+\delta^{\mathrm{l}}_{m(k)}+p\left(g\left(x_{l(k)+1}\right),g\left(x_{m(k)+1}\right)\right)+p\left(g\left(y_{l(k)+1}\right),g\left(y_{m(k)+1}\right)\right). \tag{3.20}$$

From (3.5),(3.8),(3.9),(3.10) and (3.17), for all $k\geq 0$, we obtain

$$\begin{aligned}&p\left(g\left(x_{l(k)+1}\right),g\left(x_{m(k)+1}\right)\right)=p\left(T\left(x_{l(k)},y_{l(k)}\right),T\left(x_{m(k)},y_{m(k)}\right)\right)\\&\leq\phi\left(\frac{p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)}\right)\right)}{2}\right)=\phi\left(\frac{p_k}{2}\right).\end{aligned} \tag{3.21}$$

Also by (3.5),(3.8),(3.9),(3.10) and (3.17), for all $k\geq 0$, we have,

$$\begin{aligned}&p\left(g\left(y_{l(k)+1}\right),g\left(y_{m(k)+1}\right)\right)=p\left(T\left(y_{l(k)},x_{l(k)}\right),T\left(y_{m(k)},x_{m(k)}\right)\right)\\&\leq\phi\left(\frac{p\left(g\left(x_{l(k)}\right),g\left(x_{m(k)}\right)\right)+p\left(g\left(y_{l(k)}\right),g\left(y_{m(k)}\right)\right)}{2}\right)=\phi\left(\frac{p_k}{2}\right).\end{aligned} \tag{3.22}$$

Putting (3.21) and (3.22) in (3.20) for all $k\geq 0$, we obtain, $p_k\leq\delta_{l(k)}+\delta^{\mathrm{l}}_{m(k)}+2\phi\left(\frac{p_k}{2}\right)$.

Letting $n\to\infty$ in the above inequality and using (3.16),(3.17) and (3.19) we obtain,

$$\varepsilon\leq 2\lim_{k\to\infty}\phi\left(\frac{p_k}{2}\right)=2\lim_{p_k\to\varepsilon^+}\phi\left(\frac{p_k}{2}\right)<2\frac{\varepsilon}{2}=\varepsilon, \tag{3.23}$$

which is a contradiction. Therefore, $\{g(x_n)\}$ and $\{g(y_n)\}$ are p-Cauchy sequences in X and hence they are convergent in the p-Cauchy complete uniform space (X,ϑ). Let

$$\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \text{ and}$$
$$\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y. \tag{3.24}$$

Since T and g are compatible mappings, we have by (3.24),

$$\lim_{n\to\infty} p(g(T(x_n, y_n)), T(g(x_n), g(y_n))) = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} p(g(T(y_n, x_n)), T(g(y_n), g(x_n))) = 0. \tag{3.26}$$

Next we prove that $g(x) = T(x, y)$ and $g(y) = T(y, x)$.

Let (a) hold.

For all $n \geq 0$, we have,

$$p(g(x_n), T(g(x_n), g(y_n)))$$
$$\leq p(g(x_n), g(T(x_n, y_n))) + p(g(T(x_n, y_n)), T(g(x_n), g(y_n))).$$

Taking the limit as $n \to \infty$, using (3.8), (3.24),(3.25) and the fact that T and g are continuous, we have $p(g(x_n), T(x, y)) = 0$.

Similarly, from (3.8), (3.24),(3.26) and the continuities of T and g, we have $p(g(y_n), T(y, x)) = 0$.

Combining the above two results we get $g(x) = T(x, y)$ and $g(y) = T(y, x)$.

Next we suppose that (b) holds.

By (3.9), (3.10) and (3.24) we have $\{g(x_n)\}\}$ is non-decreasing sequence, $g(x_n) \to x$ and $\{g(y_n)\}$ is non-increasing sequence,

$g(y_n) \to y$ as $n \to \infty$. Then by (3.6) and (3.7) we have for all $n \geq 0$,

$$g(x_n) \preceq x \text{ and } g(y_n) \succeq y. \tag{3.27}$$

Since, T and g are compatible mappings and g is continuous, by (3.25) and (3.26) we have,

$$\lim_{n\to\infty} g(g(x_n)) = g(x) = \lim_{n\to\infty} g(T(x_n, y_n)) = \lim_{n\to\infty} T(g(x_n), g(y_n)) \tag{3.28}$$

and

$$\lim_{n\to\infty} g(g(y_n)) = g(y) = \lim_{n\to\infty} g(T(y_n, x_n)) = \lim_{n\to\infty} T(g(y_n), g(x_n)). \tag{3.29}$$

Now we have $p(g(x), T(x, y)) \leq p(g(x), g(gx_{n+1})) + p(g(g(x_{n+1})), T(x, y))$.

Taking the limit as $n \to \infty$ in the above inequality, using (3.8) and (3.28) we have,

$$p(g(x), T(x, y))$$
$$\leq \lim_{n\to\infty} p(g(x), g(g(x_{n+1}))) + \lim_{n\to\infty} p(g(T(x_n, y_n)), T(x, y))$$
$$\leq \lim_{n\to\infty} p(T(g(x_n), g(y_n)), T(x, y)).$$

Since the mapping g is monotone increasing, by (3.5), (3.27) and the above inequality, we have for all $n \geq 0$,

$p(g(x), T(x,y)) \leq \lim_{n\to\infty} \phi\left(\frac{p(g(g(x_n)),g(x))+p(g(g(y_n)),g(y))}{2}\right)$.
Using (3.24) and the property of $\phi-$ function we obtain, $p(g(x), T(x,y)) \leq 0$.
That is
$g(x) = T(x,y)$
and similarly, by virtue of (3.8), (3.24) and (3.29) we obtain
$g(y) = T(y,x)$.
Thus we have proved that T and g have coupled coincidence point in X.
This completes the proof of the Theorem 2. □

Remark 2. If we assume $g(x) = x$ for all $x \in X$ and $\phi(t) = kt$ for $0 \leq k < 1$ then we obtain the results of Theorem 1.

Corollary 1. *Let (X, ϑ) be a uniform space, " $\preceq$ " is an order on X and suppose there is an $E-$distance p on X such that (X, p) is a $p-$Cauchy complete uniform space. Assume there is a function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$ and $\lim_{r\to t+} \phi(r) < t$ for each $t > 0$ and also suppose $T : X \times X \to X$ and $g : X \to X$ are such that T has the mixed $g-$monotone property and*

$$p(T(x,y), T(u,v)) \leq \phi\left(\frac{p(g(x), g(u)) + p(g(y), g(v))}{2}\right) \tag{3.30}$$

for all $x, y, u, v \in X$ for which comparable $g(x)$, $g(u)$ and comparable $g(y)$, $g(v)$. Suppose

$T(X \times X) \subseteq g(X)$, g is $\tau(\vartheta)-$ continuous and commutes with T and also suppose either

(a) *T is $\tau(\vartheta)-$ continuous or*
(b) *X has the following property :*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then*

$$x_n \preceq x \tag{3.31}$$

  for all n,
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then*

$$y \preceq y_n \tag{3.32}$$

  for all n.

If there exist $x_0, y_0 \in X$ such that
$g(x_0) \preceq T(x_0, y_0)$ and $g(y_0) \succeq T(y_0, x_0)$,
then there exist $x, y \in X$ such that
$g(x) = T(x,y)$ and $g(y) = T(y,x)$,
that is, T and g have a coupled coincidence.

Example 1. Let $X = [0,1] \times [0,1]$, $p(x,y) = | x_1 - x_2 | + | y_1 - y_2 |$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and " $\preceq$ " is a partially ordered with the natural ordering of

real numbers. Then $(X,\preceq)$ be an ordered uniform space and (X,p) is a $p-Cauchy$ complete uniform space. Let $g:X\to X$ be defined as

$g(x)=x^2=\left(x_1^2,x_2^2\right)$ for all $x=(x_1,x_2)\in X$.

Let $T:X\times X\to X$ be defined as;

$$T(x,y)=\begin{cases}\left(\frac{(x_1-y_1)^2}{3},\frac{(x_2-y_2)^2}{3}\right) & x,y\in X,\ x\succeq y\\ (0,0), & x\prec y\end{cases}$$

T corresponds the mixed $g-$monotone property.

Let $\phi:[0,\infty)\to[0,\infty)$ be defined as

$\phi(t)=\frac{2}{3}t,\ for\ t\in[0,\infty)$.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that,

$\lim_{n\to\infty}T(x_n,y_n)=a,\ \lim_{n\to\infty}g(x_n)=a,\ a=(a_1,a_2)$

$\lim_{n\to\infty}T(y_n,x_n)=b,\ \lim_{n\to\infty}g(y_n)=b,\ b=(b_1,b_2)$.

Then obviously, $a=(0,0)$ and $b=(0,0)$.

Now, for all $n\geq 0$;

$g(x_n)=x_n^2=(x_{n(1)}^2,x_{n(2)}^2),\ x_n=(x_{n(1)},x_{n(2)})\in X$

$g(y_n)=y_n^2=(y_{n(1)}^2,y_{n(2)}^2),\ y_n=(y_{n(1)},y_{n(2)})\in X$

$$T(x_n,y_n)=\begin{cases}\left(\frac{x_{n(1)}^2-y_{n(1)}^2}{3},\frac{x_{n(2)}^2-y_{n(2)}^2}{3}\right), & if\ x_n\succeq y_n\\ (0,0), & x_n\prec y_n\end{cases}$$

$$T(y_n,x_n)=\begin{cases}\left(\frac{y_{n(1)}^2-x_{n(1)}^2}{3},\frac{y_{n(2)}^2-x_{n(2)}^2}{3}\right), & if\ y_n\succeq x_n\\ (0,0), & y_n\prec x_n\end{cases}$$

Then, it follows that;

$\lim_{n\to\infty}p(g(T(x_n,y_n)),T(g(x_n),g(y_n)))\to 0$ as $n\to\infty$

and

$\lim_{n\to\infty}p(g(T(y_n,x_n)),T(g(y_n),g(x_n)))\to 0$ as $n\to\infty$.

Hence, the mappings T and g are compatible in X.

Also, $x_0=0=(0,0)$ and $y_0=c=(c_1,c_2)$ are two points in X such that

$g(x_0)=g((0,0))=(0,0)=T((0,0),(c_1,c_2))=T(x_0,y_0)$

and

$$g(y_0)=g((c_1,c_2))=\left(c_1^2,c_2^2\right)\succeq\left(\frac{c_1^2}{3},\frac{c_2^2}{3}\right)=T((c_1,c_2),(0,0))=T(y_0,x_0).$$

We next verify inequality (5) of Theorem 2. We take $x,y,u,v\in X$, such that $g(x)=g((x_1,x_2))\preceq g((u_1,u_2))=g(u)$ and $g(y)\succeq g(v)$, that is, $x^2\preceq u^2\Rightarrow x_1^2\preceq u_1^2$ and $x_2^2\preceq u_2^2$ and $y^2\succeq v^2\Rightarrow y_1^2\succeq v_1^2$ and $y_2^2\succeq v_2^2$.

We consider the following cases:

Case 1: $x\succeq y$ and $u\succeq v$.

Then $p(T(x,y),T(u,v))=p\left(\left(\frac{x_1^2-y_1^2}{3},\frac{x_2^2-y_2^2}{3}\right),\left(\frac{u_1^2-v_1^2}{3},\frac{u_2^2-v_2^2}{3}\right)\right)$

$$= \left|\frac{x_1^2-y_1^2}{3} - \frac{u_1^2-v_1^2}{3}\right| + \left|\frac{x_2^2-y_2^2}{3} - \frac{u_2^2-v_2^2}{3}\right|$$
$$= \left|\frac{x_1^2-u_1^2}{3} - \frac{y_1^2-v_1^2}{3}\right| + \left|\frac{x_2^2-u_2^2}{3} - \frac{y_2^2-v_2^2}{3}\right|$$
$$\preceq \left(\left|\frac{x_1^2-u_1^2}{3}\right| + \left|\frac{x_2^2-u_2^2}{3}\right|\right) + \left(\left|\frac{y_1^2-v_1^2}{3}\right| + \left|\frac{y_2^2-v_2^2}{3}\right|\right)$$
$$= \frac{2}{3}\left\{\frac{|x_1^2-u_1^2|+|x_2^2-u_2^2|}{2} + \frac{|y_1^2-v_1^2|+|y_2^2-v_2^2|}{2}\right\}$$
$$= \phi\left(\frac{p((g(x),g(u)))+p((g(y),g(v)))}{2}\right).$$

Case 2: $x \succeq y$ and $u \prec v$.

Then $p(T(x,y),T(u,v)) = p\left(\left(\frac{x_1^2-y_1^2}{3}, \frac{x_2^2-y_2^2}{3}\right),(0,0)\right)$
$$= \left|\frac{x_1^2-y_1^2}{3}\right| + \left|\frac{x_2^2-y_2^2}{3}\right|$$
$$= \left|\frac{u_1^2+x_1^2-y_1^2-u_1^2}{3}\right| + \left|\frac{u_2^2+x_2^2-y_2^2-u_2^2}{3}\right|$$
$$= \left|\frac{(u_1^2-y_1^2)-(u_1^2-x_1^2)}{3}\right| + \left|\frac{(u_2^2-y_2^2)-(u_2^2-x_2^2)}{3}\right|$$
$$\preceq \left|\frac{(v_1^2-y_1^2)+(u_1^2-x_1^2)}{3}\right| + \left|\frac{(v_2^2-y_2^2)+(u_2^2-x_2^2)}{3}\right| \text{ (since } v \succ u)$$
$$\preceq \left(\left|\frac{u_1^2-x_1^2}{3}\right| + \left|\frac{u_2^2-x_2^2}{3}\right|\right) + \left(\left|\frac{v_1^2-y_1^2}{3}\right| + \left|\frac{v_2^2-y_2^2}{3}\right|\right)$$
$$= \frac{2}{3}\left\{\frac{|x_1^2-u_1^2|+|x_2^2-u_2^2|}{2} + \frac{|y_1^2-v_1^2|+|y_2^2-v_2^2|}{2}\right\}$$
$$= \phi\left(\frac{p((g(x),g(u)))+p((g(y),g(v)))}{2}\right).$$

Case 3: $x \prec y$ and $u \succeq v$.

Then $p(T(x,y),T(u,v)) = p\left((0,0),\left(\frac{u_1^2-v_1^2}{3}, \frac{u_2^2-v_2^2}{3}\right)\right)$
$$= \left|\frac{u_1^2-v_1^2}{3}\right| + \left|\frac{u_2^2-v_2^2}{3}\right|$$
$$= \left|\frac{u_1^2+x_1^2-v_1^2-x_1^2}{3}\right| + \left|\frac{u_2^2+x_2^2-v_2^2-x_2^2}{3}\right|$$
$$= \left|\frac{(x_1^2-v_1^2)-(u_1^2-x_1^2)}{3}\right| + \left|\frac{(x_2^2-v_2^2)-(u_2^2-x_2^2)}{3}\right|$$
$$\preceq \left|\frac{(y_1^2-v_1^2)+(u_1^2-x_1^2)}{3}\right| + \left|\frac{(y_2^2-v_2^2)+(u_2^2-x_2^2)}{3}\right| \text{ (since } y \succ x)$$
$$\preceq \left(\left|\frac{u_1^2-x_1^2}{3}\right| + \left|\frac{u_2^2-x_2^2}{3}\right|\right) + \left(\left|\frac{y_1^2-v_1^2}{3}\right| + \left|\frac{y_2^2-v_2^2}{3}\right|\right)$$

$$= \frac{2}{3}\left\{\frac{|x_1^2-u_1^2|+|x_2^2-u_2^2|}{2}+\frac{|y_1^2-v_1^2|+|y_2^2-v_2^2|}{2}\right\}$$
$$= \phi\left(\frac{p((g(x),g(u)))+p((g(y),g(v)))}{2}\right).$$

Case 4: $x \prec y$ and $u \prec v$ with $x^2 \preceq u^2$ and $y^2 \succeq v^2$.

Then $T(x,y)=0$ and $T(u,v)=0$, that is $p(T(x,y),T(u,v))=0$. Obviously (5) is satisfied.

Thus it is verified that the functions T, g, ϕ satisfy all the conditions of Theorem 2. Here $(0,0)$ is the coupled coincidence point of T and g in X.

Remark 3. In this example T and g are not commuting maps. Thus the result of [13] which is noted here in Corollary 1 is not applicable to this example.

ACKNOWLEDGEMENT

We would like to thank the referees for their suggestions that improved the presentation of the paper.

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Doi: 10.18514/MMN.2015.1283

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 543–551

HU e-ISSN 1787-2413

MODULES THAT HAVE A SUPPLEMENT IN EVERY COATOMIC EXTENSION

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Received 13 June, 2014

Abstract. Let R be a ring and M be an R-module. M is said to be *an E^*-module (respectively, an EE^*-module)* if M has a supplement (respectively, ample supplements) in every coatomic extension N, i.e. $\frac{N}{M}$ is coatomic. We prove that if a module M is an EE^*-module, every submodule of M is an E^*-module, and then we show that a ring R is left perfect iff every left R-module is an E^*-module iff every left R-module is an EE^*-module. We also prove that the class of E^*-modules is closed under extension. In addition, we give a new characterization of left V-rings by cofinitely injective modules.

2010 *Mathematics Subject Classification:* 16D50; 16L30

Keywords: supplement, coatomic extension, E^*-module, EE^*-module, (semi)perfect ring

1. INTRODUCTION

Throughout this paper, we assume that all rings are associative with identity and all modules are unital left modules. Let M be such a module. As usual, the notation $K \subseteq M$ means that K is a submodule of M. A submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap U \neq 0$ for every non-zero submodule $U \subseteq M$. Dually, a proper submodule K of M is said to be *small* in M and denoted by $K << M$, if $M \neq K + T$ for every proper submodule T of M. By radical of M, denoted by $Rad(M)$, we will indicate the sum of all small submodules, or, equivalently intersection of all maximal submodules of M (see [8]). If $M = Rad(M)$, i.e., M has no maximal submodules, M is called *radical*.

Let M be a module. M is said to be *coatomic* if $Rad(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$ for some submodule K of M in [9]. M is coatomic if and only if every proper submodule of M is contained in a maximal submodule of M. Semisimple modules are coatomic. In addition, every factor module of a coatomic module is again coatomic.

Let $U, V \subseteq M$ be modules. V is called *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V << V$. A submodule S of M has *ample supplements* in M if, whenever $M = S + L$, L contains a supplement K of S in M [8].

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Let $0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$ be a short exact sequence of modules. Then, N is called *an extension* of M by K. To simplify the notation, we think of M as a submodule N. In [3], a module N is said to be a *cofinite extension* of M provided $M \subseteq N$ and $\frac{N}{M}$ is finitely generated. In light of this fact that finitely generated modules are coatomic, we call a module N *coatomic extension* of M if the factor module $\frac{N}{M}$ of N is coatomic.

It is well known that a module M is injective if and only if it is a direct summand of every extension N of M. Since every direct summand is a supplement, Zöschinger studied in [10] modules that have a supplement in every extension and termed these modules the property (E) as a generalization of injective modules. In particular, he proved in [10, Lemma 1.2] that every submodule of a module M has the property (E) if and only if M has ample supplements in every extension, namely the property (EE). It is obvious that the class of modules with the property (EE) contains properly artinian modules.

In [3], a module M is said to have *the property* (CE) (respectively, *the property* (CEE)) if M has a supplement (respectively, ample supplements) in every cofinite extension. It is shown in [3, Theorem 2.12] that R is semiperfect if and only if every left R-module has the property (CE).

Let M be a module. We call M *an* E^**-module* if M has a supplement in every coatomic extension, and M *an* EE^**-module* if it has ample supplements in every coatomic extension. The notation of E^*-modules lies between modules with (E) and modules with the property (CE). Some examples are given to show that these inclusions are proper.

In this paper, we study some basic properties of E^*-modules and EE^*-modules. We show that the class of E^*-modules closed under finite direct sums, extensions and direct summands. We prove that, over a left hereditary ring, every factor module of a coatomic E^*-module is an E^*-module. In Proposition 2, we show that if M is an EE^*-module, then every submodule of M is an E^*-module. This gives us that over semilocal rings every EE^*-module is strongly radical supplemented. Also, we prove in Theorem 2 that a ring R is left perfect if and only if every left R-module is an E^*-module if and only if every left R-module is an EE^*-module. In addition, we show that every simple left R-module is cofinitely injective if and only if, for every finitely generated left R-module M, $Rad(M) = 0$ if and only if R is a left V-ring (i.e., rings whose simple left modules are injective). Finally, we prove in Proposition 5 that every E^*-module over a left V-ring is injective.

2. E^*-Modules and EE^*-Modules

It is clear that every module with the property (E) is an E^*-module, but the following example shows that an E^*-module doesn't have the property (E), in general.

Recall from [1] that a module M is called *strongly radical supplemented* if every submodule N of M containing $Rad(M)$ has a supplement in M. It is proven in [1,

Corollary 2.1] that finite sums of strongly radical supplemented modules are strongly radical supplemented. Note that every radical module is strongly radical supplemented.

Example 1. For a non-complete local dedekind domain R, let M be the direct sum of left R-modules R^*, $K^{(I)}$ and R, where R^* is the completion of R, K is the quotient field of R and I is an index set, respectively. Since injective modules over a dedekind domain are strongly radical supplemented, it follows from [10, Lemma 3.3] that M is an E^*-module. On the other hand, M doesn't have the property (E) by [10, Theorem 3.5].

It is shown in [10, Lemma 1.3 (a)] that direct summands of modules with the property (E) have the property (E). Now we give an analogue of this fact for E^*-modules.

Proposition 1. *Every direct summand of an E^*-module is an E^*-module.*

Proof. Let M be an E^*-module and U be a direct summand of M. Then, we can write $M = U \oplus V$ for some submodule V of M. For any coatomic extension T of U, we consider the external direct product of these modules T and V. Put $W = T \oplus V$. Now we take the monomorphism $\Phi : M \longrightarrow W$ by $\Phi(m) = \Phi(u+v) = (u,v)$ for all $m = u + v \in U \oplus V = M$. It can be seen that $\Phi(M)$ is an E^*-module. Now

$$\frac{W}{\Phi(M)} = \frac{T \oplus V}{\Phi(M)} \cong \frac{T}{U}$$

is coatomic. It follows that $\Phi(M)$ has a supplement, say $U^{'}$, in W. Therefore, $T = U + \Psi(U^{'})$, where $\Psi : W \longrightarrow T$ is the projection. Since $ker(\Psi) \subseteq \Phi(M)$, we have $U \cap \Psi(U^{'}) << \Psi(U^{'})$ by [8, 19.3]. Hence, $\Psi(U^{'})$ is a supplement of U in T. □

A submodule of an E^*-module need not be an E^*-module, in general. To see this actuality, we shall consider the left $\mathbb{Z}$-modules $\mathbb{Z} \subseteq \mathbb{Q}$. But we have:

Proposition 2. *If M is an EE^*-module, then every submodule U of M is an E^*-module.*

Proof. Let N be a coatomic extension of U. We shall show that U has a supplement in N. By W, we denote the external direct product of M and N. Put $F = \frac{W}{W'}$, where the submodule $W^{'} = \{(u,-u) \in W \,|\, u \in U\} \subseteq W$. For these inclusion homomorphism $\iota_1 : U \longrightarrow N$ and $\iota_2 : U \longrightarrow M$, we can draw the pushout in the following:

$$\begin{array}{ccc} U & \xrightarrow{\iota_1} & N \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \phi} \\ M & \xrightarrow{\xi} & F \end{array}$$

where ξ and ϕ are monomorphisms. Then $F = Im(\xi) + Im(\phi)$. Therefore $\frac{N}{U} \cong \frac{F}{Im(\xi)}$ is coatomic. Since $Im(\xi) \cong M$ is an EE^*-module, there exists a submodule L of $Im(\phi)$ such that L is a supplement of $Im(\xi)$ in F. Now

$$N = \phi^{-1}(F) = \phi^{-1}(Im(\xi)) + \phi^{-1}(L) = U + \phi^{-1}(L)$$

and

$$U \cap \phi^{-1}(L) << \phi^{-1}(L).$$

This means that $\phi^{-1}(L)$ is a supplement of U in N. □

A ring R is called *semilocal* if $\frac{R}{Rad(R)}$ is a semisimple artinian ring ([8]). The following corollary is an immediate consequence of Proposition 2.

Corollary 1. *Let M be an EE^*-module over a semilocal ring R. Then, M is strongly radical supplemented.*

Proof. Let $Rad(M) \subseteq U \subseteq M$. Then $\frac{M}{U}$ is a factor module of $\frac{M}{Rad(M)}$. Since R is a semilocal ring, $\frac{M}{Rad(M)}$ is semisimple as a $\frac{R}{Rad(R)}$-module. Therefore, $\frac{M}{U}$ is a coatomic R-module. By the hypothesis, U has a supplement in M. This means that M is strongly radical supplemented. □

Let Γ be a class of modules. Then, Γ is called *closed under extension* if $M, \frac{N}{M} \in \Gamma$ implies $N \in \Gamma$. The following crucial lemma is used to show that the class of E^*-modules is closed under extensions.

Lemma 1. *Let M be a module and K be a small submodule of M. Then, M is coatomic if and only if the factor module $\frac{M}{K}$ is coatomic.*

Proof. ($\Longrightarrow$) It is clear.
($\Longleftarrow$) Let U be a proper submodule of M. Since $K \ll M$, then $\frac{U+K}{K}$ is a proper submodule of $\frac{M}{K}$. Since $\frac{M}{K}$ is coatomic, $\frac{U+K}{K}$ is contained in a maximal submodule of $\frac{M}{K}$, say $\frac{V}{K}$. Therefore, V is a maximal submodule of M. Hence, M is coatomic as required. □

Recall from [9, Lemma 1.5 (a)] that the class of coatomic modules is closed under extensions.

Theorem 1. *Let $M \subseteq N$ be modules. If M and $\frac{N}{M}$ are E^*-modules, then N is an E^*-module.*

Proof. Let K be a coatomic extension of N. For $M \subseteq N \subseteq K$,

$$\frac{K}{N} \cong \frac{\frac{K}{M}}{\frac{N}{M}}$$

is coatomic, and thus $\frac{K}{M}$ is a coatomic extension of $\frac{N}{M}$. By the hypothesis, the submodule $\frac{N}{M}$ has a supplement, say $\frac{L}{M}$, in $\frac{K}{M}$. So we can write $\frac{N}{M}+\frac{L}{M}=\frac{K}{M}$ and $\frac{N}{M}\cap\frac{L}{M}=\frac{N\cap L}{M}\ll\frac{L}{M}$. Therefore, $K=N+L$. Now

$$\frac{\frac{L}{M}}{\frac{N\cap L}{M}}\cong\frac{L}{N\cap L}\cong\frac{N+L}{N}=\frac{K}{N}$$

is coatomic. Applying Lemma 1, we obtain that $\frac{L}{M}$ is coatomic. Since M is an E^*-module, there exists a submodule $M^{'}$ of L such that $M+M^{'}=L$ and $M\cap M^{'}\ll M^{'}$. Then, $K=N+L=N+(M+M^{'})=N+M^{'}$. Assume that $N+M^{''}=K$ for some submodule $M^{''}\subseteq M^{'}$. Then, $M+M^{''}\subseteq L$. Since $\frac{L}{M}$ is a supplement of $\frac{N}{M}$ in $\frac{K}{M}$, it follows that $L=M+M^{''}$. By the minimality of $M^{'}$, we have $M^{''}=M^{'}$. Therefore M is an E^*-module. □

Note that, by Theorem 1, a finitely generated semisimple module is an E^*-module.

Corollary 2. *Let M be a module and K be a maximal submodule of M. If K is an E^*-module, then M is an E^*-module. In particular, modules containing a simple maximal submodule are E^*-modules.*

Proof. Let K be an E^*-module. Since simple modules are E^*-modules, the factor module $\frac{M}{K}$ is an E^*-module. Applying Theorem 1, we get M is an E^*-module. □

Now we can prove every finite direct sum of E^*-modules in the following Proposition.

Proposition 3. *Let M_i $(i\in I)$ be any finite collection of E^*-modules and $M=M_1\oplus M_2\oplus\ldots\oplus M_n$. Then, M is an E^*-module.*

Proof. In order to show that M is an E^*-module, we use induction on n. Suppose that $n=2$. Other case can prove by a similar way. Let $M=M_1\oplus M_2$. Then, $M_2\cong\frac{M}{M_1}$. By the hypothesis and Theorem 1, we obtain that M is an E^*-module. □

Recall that over a left hereditary ring every factor module of an injective module is injective. In the following, we show that every factor module of a coatomic E^*-module over a left hereditary ring is an E^*-module.

Proposition 4. *Let R be a left hereditary ring and M be a coatomic E^*-module. Then every factor module of M is an E^*-module.*

Proof. For any submodule U of M, let N be a coatomic extension of $\frac{M}{U}$. Then, N is coatomic. By $E(M)$, we denote the injective hull of M. Since R is left hereditary, $\frac{E(M)}{U}$ is injective, and so there exists a commutative diagram with exact rows in the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\iota_1} & M & \xrightarrow{\pi} & \frac{M}{U} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \iota_2} & & \\ 0 & \longrightarrow & U & \xrightarrow{f} & K & \xrightarrow{\Phi} & N & \longrightarrow & 0 \end{array}$$

i.e., $f\,id = \Psi\iota_1$ and $\Phi\Psi = \iota_2\pi$, where $\Psi : M \longrightarrow K$ is a monomorphism by [7, Lemma 2.16] . It follows that $N \cong \frac{K}{\Psi(M)}$. Since M is an E^*-module, $\Psi(M)$ has a supplement, say V, in K. By the last part proof of Proposition 2, we obtain that $\Phi(V)$ is a supplement of $\frac{M}{U}$ in N. Hence, $\frac{M}{U}$ is an E^*-module. □

Recall from [6] that an epimorphism $f : P \longrightarrow M$ is called *a small cover* if $Ker(f) << P$, and a projective module P together with a small cover $f : P \longrightarrow M$ is called *a projective cover* of M. A ring R is called *semiperfect* if every finitely generated left (or right) R-module has a projective cover, and it is called *left perfect* if every left R-module has a projective cover.

It is known that a ring R is semiperfect if and only if R is semilocal and idempotents can be lifted modulo $Rad(R)$, and it is left perfect if and only if R is semilocal and $Rad(R)$ is a left t-nilpotent ideal. Local rings are semiperfect ([6]).

Now we give a characterization of left perfect rings via E^*-modules. Firstly, we have the following lemma.

Lemma 2. *The following statements are equivalent over an arbitrary ring.*

(1) *Every left module is an E^*-module.*
(2) *Every left module is an EE^*-module.*

Proof. Suppose that every left module is an E^*-module. Let M be any module. For a coatomic extension N of M, let $N = M + S$ for some submodule S of N. Then $\frac{N}{M} \cong \frac{S}{M\cap S}$ is coatomic. By the hypothesis, $M \cap S$ has a supplement in S, say W. So we can write $S = (M \cap S) + W$ and $(M \cap S) \cap W = M \cap W \ll W$. Then we have $N = M + W$ and $N \cap W \ll W$. Therefore, M is an EE^*-module. The converse is clear by definitions. □

Theorem 2. *Let R be a ring. The following three statements are equivalent.*

(1) *R is left perfect.*
(2) *Every left R-module is an E^*-module.*
(3) *Every left R-module is an EE^*-module.*

Proof. (1) ⇒ (2) By [4, 39.9], over a left perfect ring every left module has the property (E). This completes the proof of (2).

(2) ⇒ (3) It follows from Lemma 2.

(3) ⇒ (1) Since EE^*-modules have the property (CEE), by the hypothesis, every R-module has the property (CEE). It follows from [3, Theorem 2.12] that R is semiperfect. Therefore, R is semilocal.

Now it is enough to show that every left R-module is strongly radical supplemented by [2, Theorem 1]. Let M be any left R-module. By the hypothesis, M is an EE^*-module. Applying Corollary 1, we deduce that M is strongly radical supplemented. □

Now we give an example of a module, which has the property (CE), but not an E^*-module.

Example 2. Let p be a prime integer in $\mathbb{Z}$. Consider the local dedekind domain $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a,b \in \mathbb{Z} \text{ and } p \nmid b\}$. Let N be the left R-module $R^{(\mathbb{N})}$. Put $M = Rad(N)$. Since R is a local ring, the factor module $\frac{N}{M}$ is semisimple as a $\frac{R}{Rad(R)}$−module. Therefore, $\frac{N}{M}$ is a semisimple R-module and so N is a coatomic extension of M. It follows from [3, Theorem 2.12] that M has the property (CE). On the other hand, M is not an E^*-module by [2, Theorem 1].

In [5], a ring R is said to be *a left V-ring* if every simple left R-module is injective. It is well known that a ring R is a left V-ring if and only if $Rad(M) = 0$ for every left R-module M. Recall from [3] that a module M is called *cofinitely injective* if M is a direct summand of every cofinite extension N of M. Clearly, injective modules are cofinitely injective, and a cofinitely injective module has the property (CE). Now we have the next result:

Proposition 5. *Let R be a left V-ring and M be an E^*-module over the ring. Then, M is injective.*

Proof. Let N be an extension of M. Since every module over a left V-ring is coatomic, $\frac{N}{M}$ is coatomic. Therefore, N is a coatomic extension of M. By assumption, we can write $N = M + K$ and $M \cap K << K$ for some submodule $K \subseteq N$. Since R is a left V-ring, we obtain that $M \cap K \subseteq Rad(K) \subseteq Rad(N) = 0$. This means that M is a direct summand of N. Hence, M is injective. □

Theorem 3. *The following statements are equivalent for a ring R.*

(1) *Every simple left R-module is cofinitely injective.*
(2) *If M is a finitely generated left R-module, $Rad(M) = 0$.*
(3) *Every proper left ideal I of R is an intersection of maximal left ideals.*

Proof. (1) $\Longrightarrow$ (2) Let M be an arbitrary finitely generated left R-module and let $m \in Rad(M)$. We claim that $m = 0$. Put $K = Rm$. Then, K has a maximal submodule L. Therefore, $\frac{K}{L}$ is a simple left R-module and by assumption $\frac{K}{L}$ is cofinitely injective. Now, for $L \subseteq K \subseteq M$ modules,

$$\frac{\frac{M}{L}}{\frac{K}{L}} \cong \frac{M}{K}$$

is coatomic since M is finitely generated. So there exists the decomposition $\frac{M}{L} = \frac{K}{L} \oplus \frac{T}{L}$ for some submodule $\frac{T}{L} \subseteq \frac{M}{L}$. Note that

$$\frac{K}{L} \cong \frac{\frac{M}{L}}{\frac{T}{L}} \cong \frac{M}{T}$$

is simple. Thus, T is a maximal submodule of M. Therefore, $m \in K \cap T \subseteq L$. It follows that $m = 0$.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) follow from [5, Theorem 6.1]. □

As a consequence of the above, we have the following.

Corollary 3. *Let R be a ring. Then, R is a left V-ring if and only if every simple left R-module is cofinitely injective.*

Proof. The proof follows from Theorem 3 and [5, Theorem 6.1]. □

ACKNOWLEDGEMENT

I would like to thank the referee for the valuable suggestions and comments which improved the revision of the paper.

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Doi: 10.18514/MMN.2015.1203

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 553–563

HU e-ISSN 1787-2413

PSEUDO ASYMPTOTIC BEHAVIOR OF MILD SOLUTION FOR SEMILINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Received 14 April, 2014

Abstract. In this paper, by the weighted ergodic function based on the measure theory, we study the pseudo asymptotic behavior of mild solution for semilinear fractional integro-differential equations. The existence, unique of μ-pseudo anti-periodic (μ-pseudo periodic, μ-pseudo almost periodic, μ-pseudo almost automorphic) solution are investigated. Moreover, an application to fractional partial differential equations is given.

2010 *Mathematics Subject Classification:* 47G05; 35B40

Keywords: semilinear fractional integro-differential equation, pseudo asymptotic, sectorial operator, measure theory

1. INTRODUCTION

The study of asymptotic behavior of solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Particularly, pseudo asymptotic behavior of solutions, such as pseudo anti-periodicity; pseudo periodicity; pseudo almost periodicity; pseudo almost automorphy have received much attention in recent years. Many authors have made important contributions to this theory. With the help of weighted measure [10], the notion of weighted class of functions, such as weighted pseudo anti-periodic [2]; weighted pseudo periodic [2]; weighted pseudo almost periodic [10]; weighted pseudo almost automorphic [6] are introduced and applications to differential equations are investigated.

Recently, Blot, Cieutat and Ezzinbi [4, 5] use the results of the measure theory to establish μ-ergodic and introduce the new concepts of μ-pseudo almost periodic function, μ-pseudo almost automorphic function, which are more general than weighted pseudo almost automorphic function, weighted pseudo almost automorphic

This research is supported by the National Natural Science Foundation of China (Grant No.11426201) and the Natural Science Foundation of Zhejiang Province (Grant No. LQ13A010015).

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function, respectively. They developed some results like completeness and composition theorems to investigate evolution equations, partial functional differential equations in Banach space.

In this paper, by the method of measure theory [4, 5], we introduce the notions of μ-pseudo anti-periodic function, μ-pseudo periodic function, explore fundamental properties and establish composition theorem. Then we conduct further studies on $\mathcal{P}$-class of functions (i.e., μ-pseudo anti-periodic, μ-pseudo periodic, μ-pseudo almost periodic, μ-pseudo almost automorphic) and its applications in semilinear fractional integro-differential equations in Banach space.

The paper is organized as follows. In Section 2, we recall some fundamental results about the notion of $\mathcal{P}$-class of functions including composition theorem. Sections 3 is devoted to the existence and uniqueness of $\mathcal{P}$ mild solutions to semilinear fractional integro-differential equations. In Section 4, an application to fractional partial differential equations is given.

2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. $C(\mathbb{R}, X)$ denotes the set of continuous functions from $\mathbb{R}$ to X. $BC(\mathbb{R}, X)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to X with the supremum norm.

2.1. *Sectorial operators and Riemann-Liouville fractional derivative*

Definition 1 ([14])**.** A closed and densely defined linear operator A is said to be sectorial of type $\widetilde{\omega}$ if there exist $0 < \theta < \pi/2$, $M > 0$, and $\widetilde{\omega} \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\widetilde{\omega} + S_\theta := \{\widetilde{\omega} + \lambda : \lambda \in \mathbb{C}, |arg(-\lambda)| < \theta\},$$
$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M}{|\lambda - \widetilde{\omega}|}, \quad \lambda \notin \widetilde{\omega} + S_\theta.$$

The sectorial operators are well studied in the literature, we refer to [14] for more details.

Definition 2 ([3])**.** Let $1 < \alpha < 2$ be given. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A is the generator of a solution operator if there exist $\widetilde{\omega} \in \mathbb{R}$ and a strong continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \widetilde{\omega}\} \subset \rho(A)$ and $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt$, $Re\lambda > \widetilde{\omega}$, $x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Note that if A is sectorial of type $\widetilde{\omega}$ with $0<\theta<\pi(1-\alpha/2)$, then A is the generator of a solution operator given by

$$S_\alpha(t):=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}d\lambda,$$

where γ is a suitable path lying outside the sector $\widetilde{\omega}+S_\theta$ ([8]). Recently, Cuesta ([8]) has proved that if A is a sectorial operator of type $\widetilde{\omega}<0$ for some $0<\theta<\pi(1-\alpha/2)$ $(1<\alpha<2), M>0$, there exists $C>0$ such that

$$\|S_\alpha(t)\|\leq\frac{CM}{1+|\widetilde{\omega}|t^\alpha},\quad t\geq 0. \tag{2.1}$$

Note that

$$\int_0^\infty\frac{1}{1+|\widetilde{\omega}|t^\alpha}dt=\frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}$$

for $1<\alpha<2$, therefore $S_\alpha(t)$ is integrable on $(0,\infty)$.

Definition 3 ([16])**.** The fractional order integral of order $\alpha>0$ with the low limit $t_0>0$ for a function f is defined as

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^t(t-s)^{\alpha-1}f(s)ds,\quad t>t_0,\ \alpha>0,$$

provided the right-hand side is pointwise defined on $[t_0,\infty)$, where Γ is the Gamma function.

Definition 4 ([16])**.** Riemann-Liouville derivative of order $\alpha>0$ with the low limit $t_0>0$ for a function $f:[t_0,\infty)\to\mathbb{R}$ can be written as

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^t(t-s)^{n-\alpha-1}f(s)ds,\quad t>t_0,\ n-1<\alpha<n.$$

2.2. *μ-ergodic and functions by measure theory*

$\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ stands for the set of all positive measure μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R})=\infty$ and $\mu([a,b])<\infty$ for all $a,b\in\mathbb{R}$ $(a\leq b)$. We formulate the following hypothesis:

(H_0) For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that $\mu(\{a+\tau,a\in A\})\leq\beta\mu(A)$ if $A\in\mathcal{B}$ satisfies $A\cap I=\varnothing$.

Definition 5. Let $\mu\in\mathcal{M}$. A function $f\in BC(\mathbb{R},X)$ is said to be μ-ergodic if

$$\lim_{T\to+\infty}\frac{1}{\mu([-T,T])}\int_{[-T,T]}\|f(t)\|d\mu(t)=0.$$

Denote by $\mathcal{E}(\mathbb{R},X,\mu)$ the set of such functions.

Lemma 1. *Let $\mu\in\mathcal{M}$ and satisfies (H_0), then $\mathcal{E}(\mathbb{R},X,\mu)$ is translation invariant.*

Definition 6. A function $f \in C(\mathbb{R}, X)$ is said to be anti-periodic if there exists a $\omega \in \mathbb{R}\backslash\{0\}$ with the property that $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. The collection of those functions is denoted by $P_{\omega ap}(\mathbb{R}, X)$.

Definition 7. A function $f \in C(\mathbb{R}, X)$ is said to be periodic if there exists a $\omega \in \mathbb{R}\backslash\{0\}$ with the property that $f(t+\omega) = f(t)$ for all $t \in \mathbb{R}$. The collection of those ω-periodic functions is denoted by $P_{\omega}(\mathbb{R}, X)$.

Definition 8 ([11])**.** A function $f \in \mathrm{C}(\mathbb{R}, X)$ is said to be almost periodic if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$, such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t+\tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $AP(\mathbb{R}, X)$ the set of such functions.

Definition 9 ([7])**.** A function $f \in \mathrm{C}(\mathbb{R}, X)$ is said to be almost automorphic if for every sequence of real numbers $(s_n^{'})_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ is well defined for each $t \in \mathbb{R}$, and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R}, X)$ the set of such functions.

Next, we recall the functions by the measure theory.

Definition 10. Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be μ-pseudo anti-periodic if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PP_{\omega ap}(\mathbb{R}, X, \mu)$ the collection of such functions.

Definition 11. Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be a μ-pseudo periodic if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega}(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PP_{\omega}(\mathbb{R}, X, \mu)$ the collection of such functions.

Definition 12 ([5])**.** Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be μ-pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PAP(\mathbb{R}, X, \mu)$ the collection of such functions.

Definition 13 ([4])**.** Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be μ-pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PAA(\mathbb{R}, X, \mu)$ the collection of such functions.

Remark 1. (*i*) If the measure μ is the Lebesgue measure, then the spaces $PP_{\omega ap}(\mathbb{R}, X, \mu)$, $PP_{\omega}(\mathbb{R}, X, \mu)$, $PAP(\mathbb{R}, X, \mu)$, $PAA(\mathbb{R}, X, \mu)$ are the functions: pseudo anti-periodic ($PP_{\omega ap}(\mathbb{R}, X)$ [17]), pseudo periodic ($PP_{\omega}(\mathbb{R}, X)$ [13]), pseudo almost periodic ($PAP(\mathbb{R}, X)$ [18]), pseudo almost automorphic ($PAA(\mathbb{R}, X)$ [15]), respectively. One can see [1, 4, 5] for more details.

(*ii*) Let $\rho(t) > 0$ a.e. on $\mathbb{R}$ for the Lebesgue measure. μ denotes the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for } A \in \mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$, then $PP_{\omega ap}(\mathbb{R},X,\mu)$, $PP_{\omega}(\mathbb{R},X,\mu)$, $PAP(\mathbb{R},X,\mu)$, $PAA(\mathbb{R},X,\mu)$ are the weighted class of functions: weighted pseudo anti-periodic ($WPP_{\omega ap}(\mathbb{R},X)$ [2]), weighted pseudo periodic ($WPP_{\omega}(\mathbb{R},X)$ [2]), weighted pseudo almost periodic ($WPAP(\mathbb{R},X)$ [10]), weighted pseudo almost automorphic ($WPAA(\mathbb{R},X)$ [6]), respectively.

Let

$$\mathcal{A}(\mathbb{R},X)=\{P_{\omega ap}(\mathbb{R},X),P_{\omega}(\mathbb{R},X),AP(\mathbb{R},X),AA(\mathbb{R},X)\}$$

$$\mathcal{P}(\mathbb{R},X,\mu)=\{PP_{\omega ap}(\mathbb{R},X,\mu),PP_{\omega}(\mathbb{R},X,\mu),PAP(\mathbb{R},X,\mu),PAA(\mathbb{R},X,\mu)\}$$

It is not difficult to see that $f\in\mathcal{P}(\mathbb{R},X,\mu)$ if and only if it can be decomposed as $f=g+\varphi$, where $g\in\mathcal{A}(\mathbb{R},X)$ and $\varphi\in\mathcal{E}(\mathbb{R},X,\mu)$.

Definition 14. Let $\mu_1,\mu_2\in\mathcal{M}$, μ_1 is said to be equivalent to μ_2 ($\mu_1\sim\mu_2$) if there exist constants $\alpha,\beta>0$ and a bounded interval I (eventually $I=\varnothing$) such that $\alpha\mu_1(A)\le\mu_2(A)\le\beta\mu_1(A)$ for $A\in\mathcal{B}$ satisfies $A\cap I=\varnothing$.

Similarly as the proof of [4, 5], we have the following results for the class of functions $\mathcal{P}(\mathbb{R},X,\mu)$.

Lemma 2. *Let $\mu\in\mathcal{M}$, then the following properties hold:*

(*i*) $f\pm g\in\mathcal{P}(\mathbb{R},X,\mu)$ *if* $f,g\in\mathcal{P}(\mathbb{R},X,\mu)$.

(*ii*) $\lambda f\in\mathcal{P}(\mathbb{R},X,\mu)$ *if* $\lambda\in\mathbb{R}$, $f\in\mathcal{P}(\mathbb{R},X,\mu)$.

(*iii*) $\mathcal{P}(\mathbb{R},X,\mu)$ *is a Banach space with the supremum norm* $\|\cdot\|$.

(*iv*)

$$PP_{\omega ap}(\mathbb{R},X,\mu)\subset PP_{\omega}(\mathbb{R},X,\mu)\subset PAP(\mathbb{R},X,\mu)\subset PAA(\mathbb{R},X,\mu)\subset BC(\mathbb{R},X)$$

.

Lemma 3. *Let $\mu_1,\mu_1\in\mathcal{M}$. If $\mu_1\sim\mu_2$, then*

$$\mathcal{E}(\mathbb{R},X,\mu_1)=\mathcal{E}(\mathbb{R},X,\mu_2),\quad \mathcal{P}(\mathbb{R},X,\mu_1)=\mathcal{P}(\mathbb{R},X,\mu_2).$$

Theorem 1. *Let $\mu\in\mathcal{M}$, $f\in\mathcal{P}(\mathbb{R}\times X\times X,X,\mu)$, and satisfies*

(*i*) *f is uniformly continuous on each compact set $K_1\times K_2$ in $X\times X$ with respect to the second, third variable u,v.*

(*ii*) *For all bounded subset E_1,E_2 of X, f is bounded on $\mathbb{R}\times E_1\times E_2$.*

then $f(\cdot,h_1(\cdot),h_2(\cdot))\in\mathcal{P}(\mathbb{R},X,\mu)$ if $h_1(\cdot),h_2(\cdot)\in\mathcal{P}(\mathbb{R},X,\mu)$.

Corollary 1. *Let $\mu\in\mathcal{M}$, $f\in\mathcal{P}(\mathbb{R}\times X\times X,X,\mu)$, and there exists a constant $L_f>0$ such that*

$$\|f(t,u_1,u_2)-f(t,v_1,v_2)\|\le L_f(\|u_1-v_1\|+\|u_2-v_2\|),$$
$$t\in\mathbb{R},\ u_1,u_2,v_1,v_2\in X,$$

then $f(\cdot,h_1(\cdot),h_2(\cdot))\in\mathcal{P}(\mathbb{R},X,\mu)$ if $h_1(\cdot),h_2(\cdot)\in\mathcal{P}(\mathbb{R},X,\mu)$.

Lemma 4. *Let $\{S(t)\}_{t\geq 0} \subset L(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t), t \in \mathbb{R}^+$, where $\phi \in L^1(\mathbb{R}^+)$. If $f \in \mathcal{P}(\mathbb{R}, X, \mu)$, $\mu \in \mathcal{M}$ and satisfies (H_0), then*

$$(\Lambda f)(t) := \int_{-\infty}^{t} S(t-s) f(s) ds \in \mathcal{P}(\mathbb{R}, X, \mu).$$

Proof. Let $f(t) = f_1(t) + f_2(t)$, where $f_1 \in \mathcal{A}(\mathbb{R}, X)$, $f_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$, then

$$(\Lambda f)(t) = \int_{-\infty}^{t} S(t-s) f_1(s) ds + \int_{-\infty}^{t} S(t-s) f_2(s) ds := (\Lambda_1 f)(t) + (\Lambda_2 f)(t),$$

where

$$(\Lambda_1 f)(t) = \int_{-\infty}^{t} S(t-s) f_1(s) ds, \quad (\Lambda_2 f)(t) = \int_{-\infty}^{t} S(t-s) f_2(s) ds, \ \ t \in \mathbb{R}.$$

By $f_1 \in \mathcal{A}(\mathbb{R}, X)$, it is not difficult to see that $\Lambda_1 f \in \mathcal{A}(\mathbb{R}, X)$. Next, we show that $\Lambda_2 f \in \mathcal{E}(\mathbb{R}, X, \mu)$, that is,

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|(\Lambda_2 f)(t)\| \, d\mu(t) = 0.$$

In fact, for $T > 0$, by using Fubini's theorem, one has

$$\begin{aligned}
&\frac{1}{\mu([-T,T])} \int_{[-T,T]} \|(\Lambda_2 f)(t)\| \, d\mu(t) \\
&= \frac{1}{\mu([-T,T])} \int_{[-T,T]} \left\| \int_{-\infty}^{t} S(t-s) f_2(s) ds \right\| d\mu(t) \\
&\leq \frac{1}{\mu([-T,T])} \int_{[-T,T]} \int_{-\infty}^{t} \|\phi(t-s) f_2(s)\| \, ds d\mu(t) \\
&= \frac{1}{\mu([-T,T])} \int_{[-T,T]} \int_{0}^{\infty} \|\phi(s) f_2(t-s)\| \, ds d\mu(t) \\
&\leq \int_{0}^{\infty} \phi(s) \Phi_T(s) ds,
\end{aligned}$$

where

$$\Phi_T(s) = \frac{1}{\mu([-T,T])} \int_{[-T,T]} \| f_2(t-s)\| d\mu(t).$$

Since $\mu \in \mathcal{M}$ and satisfies (H_0), by Lemma 1, it follows that $f_2(\cdot - s) \in \mathcal{E}(\mathbb{R}, X, \mu)$ for each $s \in \mathbb{R}$, hence $\lim_{T\to\infty} \Phi_T(s) = 0$ for all $s \in \mathbb{R}$. Since $\|\Phi_T\| < \infty$ and $\phi \in L^1(\mathbb{R}^+)$, then $\Lambda_2 f \in \mathcal{E}(\mathbb{R}, X, \mu)$ by using the Lebesgue dominated convergence theorem, so $\Lambda f \in \mathcal{P}(\mathbb{R}, X, \mu)$. □

3. Fractional integro-differential equations

This section is devoted to the existence, uniqueness of $\mathcal{P}(\mathbb{R},X,\mu)$ solution to the following fractional integro-differential equations

$$\begin{aligned} D_t^{\alpha}u(t) &= Au(t)+D_t^{\alpha-1}f(t,u(t),Ku(t)), \quad t\in\mathbb{R},\ 1<\alpha<2,\\ Ku(t) &= \int_{-\infty}^{t} k(t-s)g(s,u(s))ds. \end{aligned} \tag{3.1}$$

where $A:D(A)\subset X\to X$ is a linear densely operator of sectorial type on a complex Banach space $(X,\|\cdot\|)$, f,g,k are continuous functions. The fractional derivative D_t^{α} is to be understood in Riemann-Liouville sense.

Definition 15 ([9, 12])**.** Assume that A generates an integral solution operator $S_{\alpha}(t)$. A continuous function $u:\mathbb{R}\to X$ satisfying the integral equation

$$u(t)=\int_{-\infty}^{t} S_{\alpha}(t-s)f(s,u(s),Ku(s))ds, \quad t\in\mathbb{R},$$

is called a mild solution on $\mathbb{R}$ to (3.1).

To study (3.1), we introduce the following assumptions:

(H_1) A is a sectorial operator of type $\widetilde{\omega}<0$ with $0<\theta<\pi(1-\alpha/2)$.

(H_2) $k(t)\in L^1(\mathbb{R}^+)$.

(H_3) $g\in\mathcal{P}(\mathbb{R}\times X,X,\mu)$, and there exists a constant $L_g>0$ such that

$$\|g(t,u)-g(t,v)\|\le L_g\|u-v\|, \quad t\in\mathbb{R},\ u,v\in X.$$

(H_4) $f\in\mathcal{P}(\mathbb{R}\times X\times X,X,\mu)$, and there exists a constant $L_f>0$ such that

$$\begin{gathered}\|f(t,u_1,u_2)-f(t,v_1,v_2)\|\le L_f(\|u_1-v_1\|+\|u_2-v_2\|),\\ t\in\mathbb{R},\ u_1,u_2,v_1,v_2\in X.\end{gathered}$$

(H_5) $\mu\in\mathcal{M}$ and satisfies (H_0).

Lemma 5. *If $u\in\mathcal{P}(\mathbb{R},X,\mu)$, and assume that (H_1), (H_2), (H_3) and (H_5) hold, then $Ku\in\mathcal{P}(\mathbb{R},X,\mu)$.*

Proof. Since $g\in\mathcal{P}(\mathbb{R}\times X,X,\mu)$, by (H_3) and Theorem 1, it is clear that $g(\cdot,u(\cdot))\in\mathcal{P}(\mathbb{R},X,\mu)$. Similarly as the proof of Lemma 4, $Ku\in\mathcal{P}(\mathbb{R},X,\mu)$. □

Theorem 2. *Assume that (H_1)-(H_5) hold, if*

$$CML_f\left(1+L_g\|k\|_{L^1}\right)|\widetilde{\omega}|^{-1/\alpha}\pi<\alpha\sin(\pi/\alpha),$$

then (3.1) has a unique mild solution $u(t)\in\mathcal{P}(\mathbb{R},X,\mu)$.

Proof. Define the operator $\mathcal{F} : \mathcal{P}(\mathbb{R}, X, \mu) \to \mathcal{P}(\mathbb{R}, X, \mu)$ by

$$(\mathcal{F}u)(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s, u(s), Ku(s)) ds, \quad t \in \mathbb{R}. \tag{3.2}$$

First, we show that $\mathcal{F}$ is well defined. In fact, if $u \in \mathcal{P}(\mathbb{R}, X, \mu)$, by Lemma 5, $Ku \in \mathcal{P}(\mathbb{R}, X, \mu)$, then $f(\cdot, u(\cdot), Ku(\cdot)) \in \mathcal{P}(\mathbb{R}, X, \mu)$ by Corollary 1. On the other hand, by (2.1), we have

$$\|S_\alpha(t)\| \le \frac{CM}{1 + |\widetilde{\omega}| t^\alpha}, \quad t \ge 0.$$

Since $1 < \alpha < 2$ and $CM/(1 + |\widetilde{\omega}| t^\alpha) \in L^1(\mathbb{R}^+)$, $\mathcal{F}$ maps $\mathcal{P}(\mathbb{R}, X, \mu)$ into itself by Lemma 4.

For any $u, v \in \mathcal{P}(\mathbb{R}, X, \mu)$,

$$\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\le \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|f(s, u(s), Ku(s)) - f(s, v(s), Kv(s))\| ds \\
&\le L_f \int_{-\infty}^{t} \|S_\alpha(t-s)\| (\|u(s) - v(s)\| + \|Ku(s) - Kv(s)\|) ds \\
&\le L_f \left(1 + L_g \|k\|_{L^1}\right) \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|u(s) - v(s)\| ds \\
&\le L_f \left(1 + L_g \|k\|_{L^1}\right) \|u - v\| \int_0^{\infty} \frac{CM}{1 + |\widetilde{\omega}| s^\alpha} ds \\
&\le \frac{CM L_f \left(1 + L_g \|k\|_{L^1}\right) |\widetilde{\omega}|^{-1/\alpha} \pi}{\alpha \sin(\pi/\alpha)} \|u - v\|.
\end{aligned}$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{P}(\mathbb{R}, X, \mu)$, which is the unique $\mathcal{P}(\mathbb{R}, X, \mu)$ mild solution to (3.1). □

Next, we formulate the following hypotheses:

(H_6) $f \in \mathcal{P}(\mathbb{R} \times X \times X, X, \mu)$ and satisfies

$$\|f(t, u_1, u_2) - f(t, v_1, v_2)\| \le L_f(t)(\|u_1 - v_1\| + \|u_2 - v_2\|),$$
$$t \in \mathbb{R}, \ u_1, u_2, v_1, v_2 \in X.$$

where $L_f \in L^1(\mathbb{R}, \mathbb{R}^+)$.

(H_7) For all bounded sets E_1, E_2 of X, f is bounded on $\mathbb{R} \times E_1 \times E_2$.

Theorem 3. *Assume that* (H_1)-(H_3), (H_5)-(H_7) *hold, then* (3.1) *has a unique mild solution* $u(t) \in \mathcal{P}(\mathbb{R}, X, \mu)$.

Proof. Define the operator $\mathcal{F}$ as in (3.2). Similarly as the proof Theorem 2, $\mathcal{F}$ is well defined. Lemma 4, Lemma 5 and Theorem 1 imply that $\mathcal{F}$ maps $\mathcal{P}(\mathbb{R}, X, \mu)$ into itself.

Let $u, v \in \mathcal{P}(\mathbb{R}, X, \mu))$ and define $C_\alpha = \sup_{t\in\mathbb{R}^+} \|S_\alpha(t)\|$, one has

$$\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\leq \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))\| ds \\
&\leq \int_{-\infty}^{t} L_f(s)\|S_\alpha(t-s)\|(\|u(s)-v(s)\| + \|Ku(s)-Kv(s)\|) ds \\
&\leq \left(1 + L_g\|k\|_{L^1}\right) \int_{-\infty}^{t} L_f(s)\|S_\alpha(t-s)\| \|u(s)-v(s)\| ds \\
&\leq C_\alpha \left(1 + L_g\|k\|_{L^1}\right) \|u-v\| \int_{-\infty}^{t} L_f(s) ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\|\mathcal{F}^2 u - \mathcal{F}^2 v\right\| \\
&\leq C_\alpha \left(1 + L_g\|k\|_{L^1}\right) \int_{-\infty}^{t} L_f(s)\|(\mathcal{F}u)(s) - (\mathcal{F}v)(s)\| ds \\
&\leq \left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right)\right)^2 \|u-v\| \int_{-\infty}^{t} L_f(s) \int_{-\infty}^{s} L_f(\sigma) \mathrm{d}\sigma ds \\
&= \left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right)\right)^2 \|u-v\| \int_{-\infty}^{t} \left(\int_{-\infty}^{s} L_f(\sigma) d\sigma\right) d\left(\int_{-\infty}^{s} L_f(\sigma) d\sigma\right) \\
&\leq \frac{\left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right)\right)^2}{2!} \|u-v\| \left(\int_{-\infty}^{t} L_f(s) ds\right)^2.
\end{aligned}$$

By the method of mathematical induction, one has

$$\|\mathcal{F}^n u - \mathcal{F}^n v\| \leq \frac{\left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right)\right)^n}{n!} \|u-v\| \left(\int_{-\infty}^{t} L_f(s) ds\right)^n.$$

Moreover, since $L_f \in L^1(\mathbb{R}, \mathbb{R}^+)$,

$$\|\mathcal{F}^n u - \mathcal{F}^n v\| \leq \frac{\left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right) \|L_f\|_{L^1}\right)^n}{n!} \|u-v\|.$$

For sufficiently large n, we have $\left(C_\alpha \left(1 + L_g\|k\|_{L^1}\right) \|L\|_{L^1}\right)^n / n! < 1$, by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{P}(\mathbb{R}, X, \mu)$, which is the unique $\mathcal{P}(\mathbb{R}, X, \mu)$ mild solution to (3.1). The proof is complete. □

4. EXAMPLE

Consider the fractional partial differential equation

$$\begin{aligned} &D_t^{\alpha} w(t,x) = \frac{\partial^2 w(t,x)}{\partial x^2} - a w(t,x) + D_t^{\alpha-1} f(t,x,w(t,x),Kw(t,x)), \\ &\quad t \in \mathbb{R}, x \in (0,1), \\ &Kw(t,x) = \int_{-\infty}^{t} k(t-s) g(s,x,w(s,x)) ds, \\ &w(t,0) = w(t,1) = 0, \end{aligned} \tag{4.1}$$

where $1 < \alpha < 2$, $a > 0$ and $k(\cdot)$ is a real valued continuous nonincreasing function such that $|k(t)| \leq C_k e^{-bt}$ for $t \geq 0$ and C_k, b are positive constants.

Let $X = L^2[0,1]$ and define the operator A by

$$\begin{aligned} &D(A) := \{w \in L^2[0,1] : w'' \in L^2[0,1],\ w(0) = w(1) = 0\}, \\ &Aw = w'' - aw, \qquad \text{for all} \quad w \in D(A). \end{aligned}$$

It is well know that A is the infinitisimal generator of an analytic semigroup on $L^2[0,1]$ ([14]), thus A is of sectorial type $\widetilde{\omega} = -a < 0$.

Let $u(t) = w(t,x), t \in \mathbb{R}, x \in [0,1]$, then (4.1) can be rewritten as the abstract form (3.1). By Theorem 2,

Theorem 4. *Under assumptions (H_3), (H_4) and (H_5), then (4.1) has a unique $\mathcal{P}(\mathbb{R}, X, \mu)$ mild solution if $CL_f \left(1 + L_g \|k\|_{L^1}\right) |a|^{-1/\alpha} \pi < \alpha \sin(\pi/\alpha)$.*

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Doi: 10.18514/MMN.2015.540

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 565–573

HU e-ISSN 1787-2413

ON A NOTE OF CONVERGENCE THEOREMS FOR ZEROS OF GENERALIZED LIPSCHITZ Φ-QUASI-ACCRETIVE OPERATORS

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Received 17 May, 2012

Abstract. In this paper, the convergence of Mann iterative process with errors for generalized Lipschitz Φ-quasi-accretive operators is proved in uniformly smooth Banach spaces. Our results improve the corresponding results of Chidume et al.[2].

2010 *Mathematics Subject Classification:* 47H10; 47H09

Keywords: generalized Lipschitz, Φ-quasi-accretive operator, Mann iterative process with errors, fixed point, uniformly smooth Banach space

1. Introduction

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* :< x, f >= \|x\|^2 = \|f\|^2\},$$

for all $x \in E$, where $< \cdot, \cdot >$ denotes the generalized duality pairing. It is well known that

(1) If E is a smooth Banach space, then the mapping J is single-valued.
(2) $J(\alpha x) = \alpha J(x)$ for all $x \in E$ and $\alpha \in \Re$.
(3) If E is a uniformly smooth Banach space, then the mapping J is uniformly continuous on any bounded subset of E(see [1] and [3]).

In the sequel, we denote the single-valued normalized duality mapping by j.

Definition 1 ([2]). Let $T : E \to E$ be an operator. T is said to be strongly accretive if there is a positive constant $k \in (0,1)$ such that for every $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$< Tx - Ty, j(x-y) >\geq k\|x-y\|^2. \tag{1.1}$$

Let τ denote the class of all strictly increasing continuous function $f : [0,+\infty) \to [0,+\infty)$ with $f(0) = 0$. Given $\phi \in \tau$, we say that T is ϕ-strongly accretive if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$< Tx - Ty, j(x-y) >\geq \phi(\|x-y\|)\|x-y\|. \tag{1.2}$$

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Further, let $\Phi \in \tau$, T is called Φ-accretive if there exists $j(x-y) \in J(x-y)$, such that the inequality

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|) \tag{1.3}$$

holds for all $x, y \in E$.

It is shown that the class of Φ-accretive operators not only properly includes the class of ϕ-strongly accretive operators, but also that of strongly accretive operators. Let $N(T) = \{x \in E : Tx = 0\} \neq \varnothing$. If the above inequalities (1.1), (1.2) and (1.3) hold for any $x \in E$ and $y \in N(T)$, then the corresponding operator T is called strongly quasi-accretive, ϕ-strongly quasi-accretive and Φ-quasi-accretive, respectively. Closely related to the class of accretive operators is that of pseudocontractive types.

Let $F(T) = \{x \in E : Tx = x\} \neq \varnothing$. A mapping $T : E \to E$ is called strongly hemi-contractive, ϕ-strongly hemi-contractive and Φ-hemi-contractive if and only if $I - T$ is strongly quasi-accretive, ϕ-strongly quasi-accretive and Φ-quasi-accretive, respectively. Here I denotes the identity mapping of E.

Definition 2 (see [7])**.** For arbitrary given $x_0 \in E$, Mann iterative process with errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n T x_n + c_n u_n, n \geq 0, \tag{1.4}$$

where $\{u_n\}$ is any bounded sequence in E; $\{a_n\}$ and $\{c_n\}$ are two real sequences in $[0,1]$ satisfying $a_n + c_n \leq 1$ for any $n \geq 0$.

Definition 3 (see [8,9])**.** A mapping $T : E \to E$ is called generalized Lipschitz if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(1 + \|x - y\|)$$

for all $x, y \in E$.

Recently, C. E. Chidume and C. O. Chidume [2] established an approximation theorem for the zeros of generalized Lipschitz Φ-quasi-accretive operators. Their result is as follows.

Chidume's Theorem. Let E be a uniformly smooth real Banach space and $A : E \to E$ be a mapping with $N(A) \neq \varnothing$. Suppose A is a generalized Lipschitz generalized Φ-quasi-accretive mapping. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$; (ii) $\Sigma_{n=0}^{\infty}(b_n + c_n) = \infty$; (iii) $\Sigma_{n=0}^{\infty} c_n < \infty$; (iv) $\lim_{n\to\infty} b_n = 0$. Let $\{x_n\}$ be generated iteratively from arbitrary $x_0 \in E$ by,

$$x_{n+1} = a_n x_n + b_n S x_n + c_n u_n, n \geq 0,$$

where $S : E \to E$ is defined by $Sx := f + x - Ax, \forall x \in E$ and $\{u_n\}$ is an arbitrary bounded sequence in E. Then, there exists $\gamma_0 \in \Re$ such that if $b_n + c_n \leq \gamma_0, \forall n \geq 0$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.

This result improves a lot of recent contributions in the area. However, there exists a gap in its provided proof. Precisely, $c_n = min\{\frac{\epsilon}{4\beta}, \frac{1}{2\sigma}\Phi(\frac{\epsilon}{2})\alpha_n\}$ does not holds in line 14 of Claim 2 of page 248, i.e., $c_n \leq \frac{1}{2\sigma}\Phi(\frac{\epsilon}{2})\alpha_n$ is a wrong case, and it was applied to the formula of line 3rd of page 249.

Example 1. Setting the iteration parameters: $a_n = 1 - b_n - c_n$, where
$\{b_n\}_{n=0}^{\infty} : b_0 = b_1 = 0, b_n = \frac{1}{n}, n \geq 2.$
$\{c_n\}_{n=0}^{\infty} : 0, \frac{1}{\sqrt{1^2}}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{\sqrt{4^2}}, \frac{1}{5^2}, \cdots, \frac{1}{8^2}, \frac{1}{\sqrt{9^2}}, \frac{1}{10^2}, \cdots, \frac{1}{15^2}, \frac{1}{\sqrt{16^2}}, \frac{1}{17^2}, \cdots.$
Then $\Sigma_{n=0}^{\infty} c_n < +\infty$, but $c_n \neq o(b_n + c_n)$. Therefore, the proof of Theorem 3.1 of [2] is incorrect.

The aim of this paper is to establish a convergence result relative to the Mann iteration with errors for generalized Lipschitz Φ-quasi-accretive operators in uniformly smooth real Banach spaces. The following auxiliary facts will be needed.

Lemma 1 (see [3]). *Let E be a uniformly smooth real Banach space and let $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, J(x + y) >$$

for all $x, y \in E$.

Lemma 2 (see [6]). *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative real numbers sequence satisfying the condition*

$$\rho_{n+1} \leq (1 - \theta_n)\rho_n + o(\theta_n), n \geq 0,$$

where $\theta_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\theta_n = \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Results

Theorem 1. *Let E be an arbitrary uniformly smooth real Banach space and $T : E \to E$ be a generalized Lipschitz Φ-quasi-accretive operator with $N(T) \neq \varnothing$. Let $\{a_n\}, \{c_n\}$ be two real numbers sequences in $[0, 1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$; (ii) $a_n, c_n \to 0$ as $n \to \infty$ and $c_n = o(a_n)$; (iii) $\Sigma_{n=0}^{\infty} a_n = \infty$. For some $x_0 \in E$, let $\{u_n\}$ be any bounded sequence and $\{x_n\}$ be Mann iterative sequence with errors defined by*

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n S x_n + c_n u_n, n \geq 0, \tag{2.1}$$

where $S : E \to E$ is defined by $Sx = x - Tx$ for any $x \in E$. Then sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$.

Proof. Let $q \in N(T)$, that is $q \in F(S)$. Since $T : E \to E$ is a generalized Lipschitz Φ-quasi-accretive operator, then S is a generalized Lipschitz Φ-hemi-contractive, i.e., there exists $\Phi \in \tau$ such that

$$< Sx - Sq, J(x - q) > \leq \|x - q\|^2 - \Phi(\|x - q\|),$$

and

$$\|Sx - Sy\| \leq L(1 + \|x - y\|)$$

for any $x, y \in E$.

Step 1: There exists $x_0 \in D$ and $x_0 \neq Sx_0$ such that

$$r_0 = \|x_0 - Sx_0\| \cdot \|x_0 - q\| \in R(\Phi).$$

Indeed, if $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$; if $\sup\{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$, then for $q \in E$, there exists a sequence $\{w_n\}$ in E such that $w_n \to q$ as $n \to \infty$ with $w_n \neq q$. Furthermore, we obtain that $\{w_n - Sw_n\}$ is bounded. Hence there exists a natural number n_0 such that $\|w_n - Sw_n\| \cdot \|w_n - q\| < \frac{r_1}{2}$ for $n \geq n_0$, then we renew define $x_0 = w_{n_0}$ and $\|x_0 - Sx_0\| \cdot \|x_0 - q\| \in R(\Phi)$.

Step 2: For any $n \geq 0$, $\{x_n\}$ is bounded.

Setting $R = \Phi^{-1}(r_0)$, then from Definition 2, we obtain that $\|x_0 - q\| \leq R$. Denote

$$B_1 = \{x \in D : \|x - q\| \leq R\}, B_2 = \{x \in D : \|x - q\| \leq 2R\}.$$

Since S is generalized Lipschitz, so S is bounded. Let

$$M = \sup_{x \in B_2} \{\|Sx - q\| + 1\} + \sup_n \{\|u_n - q\|\}.$$

Next, we want to prove that $x_n \in B_1$. If $n = 0$, then $x_0 \in B_1$. Now assume that it holds for some n, i.e., $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose it is not the case, then $\|x_{n+1} - q\| > R$. Since J is uniformly continuous on bounded subset of E, then for $\epsilon_0 = \frac{\Phi(\frac{R}{2})}{4[L+(1+L)R]}$, there exists $\delta > 0$ such that $\|Jx - Jy\| < \epsilon$ when $\|x - y\| < \delta, \forall x, y \in B_2$. Now denote

$$\tau_0 = \min\{\frac{R}{2M}, \frac{\Phi(\frac{R}{2})}{8R(M + 2R)}, \frac{\delta}{2(M + 2R)}, \frac{R + L(1 + R)}{2(M + R)}\}.$$

Since $a_n, c_n \to 0$ as $n \to \infty$, without loss of generality, we assume that $0 \leq a_n, c_n \leq \tau_0$ for any $n \geq 0$. Since $c_n = o(a_n)$, denote $c_n < a_n \tau_0$. So we have

$$\|u_n - x_n\| \leq \|x_n - q\| + \|u_n - q\| \leq R + M, \tag{2.2}$$

$$\|x_n - Sx_n\| \leq L + (1 + L)\|x_n - q\| \leq L + (1 + L)R, \tag{2.3}$$

$$\|x_n - q\| \geq \|x_{n+1} - q\| - a_n\|Sx_n - q\| - c_n\|u_n - q\| > \frac{R}{2}, \tag{2.4}$$

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n - c_n)\|x_n - q\| + a_n\|Sx_n - q\| + c_n\|u_n - q\| \\ &\leq R + \tau_0 M \leq 2R, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\|(x_{n+1}-q)-(x_n-q)\| &\leq a_n\|Sx_n-x_n\|+c_n\|u_n-x_n\| \\ &\leq a_n(\|Sx_n-q\|+\|x_n-q\|) \\ &+c_n(\|u_n-q\|+\|x_n-q\|) \\ &\leq \tau_0(M+2R) \\ &<\delta.\end{aligned} \tag{2.6}$$

So

$$\|J(x_{n+1}-q)-J(x_n-q)\|<\epsilon_0.$$

Using Lemma 1 and above formulas, we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &\leq \|x_n-q\|^2+2a_n<Sx_n-x_n,J(x_{n+1}-q)> \\ &+2c_n<u_n-x_n,J(x_{n+1}-q)> \\ &\leq \|x_n-q\|^2-2a_n\Phi(\|x_n-q\|)+2a_n\|x_n-Sx_n\| \\ &\cdot\|J(x_{n+1}-q)-J(x_n-q)\| \\ &+2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\| \\ &\leq R^2-2a_n\Phi(\frac{R}{2})+2a_n(L+(1+L)R)\epsilon_0 \\ &+4a_n\tau_0(M+R)R \\ &\leq R^2,\end{aligned} \tag{2.7}$$

which is a contradiction. Hence $x_{n+1}\in B_1$, i.e., $\{x_n\}$ is a bounded sequence.

Step 3: We want to prove that $\|x_n-q\|\to 0$ as $n\to\infty$.

Setting

$$M_1=\sup_n\|x_n-q\|+\sup_n\|u_n-q\|.$$

Since $a_n,c_n\to 0$ as $n\to\infty$, then

$$\|(x_{n+1}-q)-(x_n-q)\|\to 0.$$

Hence

$$\|J(x_{n+1}-q)-J(x_n-q)\|\to 0$$

as $n \to \infty$. From (2.7), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \|x_n-q\|^2 - 2a_n\Phi(\|x_n-q\|) \\
&\quad + 2a_n\|x_n - Sx_n\|\cdot\|J(x_{n+1}-q)-J(x_n-q)\| \\
&\quad + 2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\| \\
&\le \|x_n-q\|^2 - 2a_n\Phi(\|x_n-q\|) \\
&\quad + 2a_n[(1+L)\|x_n-q\|+L]A_n \\
&\quad + 2c_n\|u_n-x_n\|\cdot\|x_{n+1}-q\| \\
&\le \|x_n-q\|^2 - 2a_n\Phi(\|x_n-q\|) \\
&\quad + a_nA_n(1+L)\|x_n-q\|^2 + a_nA_n(1+L) \\
&\quad + 2a_nLA_n + 4c_nM_1^2 \\
&\le \|x_n-q\|^2 - 2a_n\Phi(\|x_n-q\|) \\
&\quad + a_nA_n(1+L))M_1^2 + a_nA_n(1+3L) + 4c_nM_1^2 \\
&= \|x_n-q\|^2 - 2a_n\Phi(\|x_n-q\|) + C_n \\
&= \|x_n-q\|^2 + 2a_n[B_n - \Phi(\|x_n-q\|)],
\end{aligned} \tag{2.8}$$

where

$$A_n = \|J(x_{n+1}-q)-J(x_n-q)\|,\ \ B_n = \frac{C_n}{2a_n},$$

$$C_n = a_nA_n(1+L))M_1^2 + a_nA_n(1+3L) + 4c_nM_1^2.$$

Letting $\inf_{n\ge 0}\frac{\Phi(\|x_n-q\|)}{1+\|x_{n+1}-q\|^2} = \lambda$, then $\lambda = 0$. If it is not the case, we assume that $\lambda > 0$. Let $0 < \gamma < \min\{1,\lambda\}$, then $\frac{\Phi(\|x_n-q\|)}{1+\|x_{n+1}-q\|^2} \ge \gamma$, i.e.,

$$\Phi(\|x_n-q\|) \ge \gamma + \gamma\|x_{n+1}-q\|^2 \ge \gamma\|x_{n+1}-q\|^2.$$

Thus, from (2.8) that

$$\|x_{n+1}-q\|^2 \le \|x_n-q\|^2 + 2a_n(B_n - \gamma\|x_{n+1}-q\|^2), \tag{2.9}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \tfrac{1}{1+2a_n\gamma}\|x_n-q\|^2 + \tfrac{2a_nB_n}{1+2a_n\gamma} \\
&= (1-\tfrac{2a_n\gamma}{1+2a_n\gamma})\|x_n-q\|^2 + \tfrac{2a_nB_n}{1+2a_n\gamma}.
\end{aligned} \tag{2.10}$$

Let $\rho_n = \|x_n-q\|^2, \lambda_n = \frac{2a_n\gamma}{1+2a_n\gamma}, \sigma_n = \frac{2a_nB_n}{1+2a_n\gamma}$. Then we get that

$$\rho_{n+1} \le (1-\lambda_n)\rho_n + \sigma_n.$$

Applying Lemma 2, then $\rho_n \to 0$ as $n \to \infty$. Thus $\lambda = 0$, it is a contradiction. Therefore, there exists an infinite subsequence such that $\frac{\Phi(\|x_{n_i}-q\|)}{1+\|x_{n_i+1}-q\|^2} \to 0$ as $i \to$

∞. Since $0 \leq \frac{\Phi(\|x_{n_i}-q\|)}{1+M_1^2} \leq \frac{\Phi(\|x_{n_i}-q\|)}{1+\|x_{n_i+1}-q\|^2}$, then $\Phi(\|x_{n_i}-q\|) \to 0$ as $i \to \infty$. By the strictly increasing and continuity of Φ, we get $\|x_{n_i}-q\| \to 0$ as $i \to \infty$. Next we prove $\|x_n - q\| \to 0$ as $n \to \infty$. Let $\forall \varepsilon \in (0,1)$, there exists n_{i_0} such that $\|x_{n_i}-q\| < \epsilon, a_n, c_n < \frac{\epsilon}{8M_1}, B_n < \frac{1}{2}\Phi(\frac{\epsilon}{2})$, for any $n_i, n \geq n_{i_0}$. First, we prove $\|x_{n_i+1}-q\| < \epsilon$. Suppose that it is not this case, then $\|x_{n_i+1}-q\| \geq \epsilon$. By Definition 2, we estimate the following formula:

$$\begin{aligned}
\|x_{n_i}-q\| &\geq \|x_{n_i+1}-q\| - a_{n_i}\|Sx_{n_i}-x_{n_i}\| - c_{n_i}\|u_{n_i}-x_{n_i}\| \\
&> \epsilon - a_{n_i}[\|Sx_{n_i}-q\| + \|x_{n_i}-q\|] - c_{n_i}[\|u_{n_i}-q\| + \|x_{n_i}-q\|] \\
&\geq \epsilon - (b_{n_i}+c_{n_i})2M_1 \\
&> \frac{\epsilon}{2}.
\end{aligned} \tag{2.11}$$

Since Φ is strictly increasing, (2.11) leads to

$$\Phi(\|x_{n_i}-q\|) \geq \Phi(\frac{\epsilon}{2}).$$

From (2.8), we have

$$\begin{aligned}
\|x_{n_i+1}-q\|^2 &\leq \|x_{n_i}-q\|^2 + 2a_{n_i}[B_{n_i} - \Phi(\|x_{n_i}-q\|)] \\
&< \epsilon^2 + 2a_{n_i}[\frac{1}{2}\Phi(\frac{\epsilon}{2}) - \Phi(\frac{\epsilon}{2})] \\
&\leq \epsilon^2,
\end{aligned} \tag{2.12}$$

which is a contradiction. Thus $\|x_{n_i+1}-q\| < \epsilon$. Suppose that $\|x_{n_i+m}-q\| < \epsilon$ holds. Repeating the above course, we can easily show that $\|x_{n_i+m+1}-q\| < \epsilon$ holds. Therefore, we obtain that $\|x_{n_i+m}-q\| < \epsilon$ for any positive integer m, which means $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Remark 1. In above theorem we assume a condition to have $N(T) \neq \varnothing$ for generalized Lipschitz Φ-quasi-accretive operator. However before presenting the main results, we note that this assumption is not necessary in [4] and [5]. The reason is that for Lipschitz or continuous Φ-accretive operators we have $N(T) \neq \varnothing$, but for a generalized Lipschitz mapping it can not be assumed that it must be Lipschitz or continuous, which means that $N(T)$ may be empty. For this, we add a sufficient condition. Therefore our Theorem 1 includes the past results of [4] and [5] which are known as the existence theorems for Lipschitz or continuous Φ-accretive operators and which are special cases of our Theorem 1.

Theorem 2. *Let D be a nonempty closed convex subset of uniformly smooth real Banach space E, and $T : D \to D$ a generalized Lipschitz Φ-hemi-contractive mapping with $q \in F(T) \neq \varnothing$. Let $\{a_n\}, \{c_n\}$ be real sequences in $[0,1]$ and satisfy the conditions (i) $a_n + c_n \leq 1$; (ii) $a_n \to 0$ as $n \to \infty$; (iii) $\Sigma_{n=0}^{\infty} a_n = \infty$; (iv) $c_n = o(a_n)$.*

Let $\{u_n\}$ be any bounded sequence in D. For some $x_0 \in D$, let $\{x_n\}$ be Mann iterative scheme with errors defined by (1.4). Then $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof. Since $T : D \to D$ is a generalized Lipschitz Φ-hemi-contractive mapping, then

$$Tx - Tq, J(x-q) >\leq \|x-q\|^2 - \Phi(\|x-q\|)$$

and

$$\|Tx - Ty\| \leq L(1 + \|x-y\|)$$

hold for any $x, y \in E, q \in F(T)$. The rest follows as in Theorem 1. □

Remark 2. In Theorem 1 and Theorem 2, the condition $\Sigma_{n=0}^{\infty} c_n < \infty$ of the iteration parameter $\{c_n\}$ is replaced by $c_n = o(a_n)$, and these two conditions are not included each other (See above Counterexample). Up to now, it is unknown whether the results of Chidume et al. [2] hold for the condition $\Sigma_{n=0}^{\infty} c_n < \infty$. Hence our Theorem 1 and Theorem 2 improve Theorem 3.1 and Theorem 3.2 of [2], respectively.

ACKNOWLEDGEMENT

The authors are grateful to the anonymous referees for their appreciation, comments and suggestions which helped the improvement of the paper.

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Doi: 10.18514/MMN.2015.1258

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 575–586

HU e-ISSN 1787-2413

MONOTONE ITERATIVE TECHNIQUE BY UPPER AND LOWER SOLUTIONS WITH INITIAL TIME DIFFERENCE

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Received 21 May, 2014

Abstract. In this work, the monotone iterative technique have been investigated by choosing upper and lower solutions with initial time difference that start at different initial times for the initial value problem. This method offers a way of proving existence of maximal and minimal solutions in addition to obtaining solutions in closed sectors.

2010 *Mathematics Subject Classification:* 34A99; 34C60; 34A12

Keywords: initial time difference, monotone iterative technique, existence theorems, comparison results

1. INTRODUCTION

The original method of monotone iterative technique provides an explicit analytic representation for the solution of nonlinear differential equations which yields pointwise upper and lower estimates for the solution of problem whenever the functions involved are monotone nondecreasing and nonincreasing [1–3, 8, 10–13]. As a result, the method has been popular in applied areas [1, 2],[4–6],[9, 12, 13]. The monotone iterative technique [2, 6–8], uncoupled with the method of upper and lower solutions, offers monotone sequences that converge uniformly and monotonically to the extremal solutions of the given nonlinear problem. Since each member of such a sequence is the solution of a certain (ODEs) which can be explicitly computed, the advantage and the importance of this technique needs no special emphasis. Moreover, this method can successfully be employed to generate two sided pointwise bounds on solutions of initial value problems of ODEs from which qualitative and quantitative behavior can be investigated. In this paper especially we employed monotone iterative technique for ODEs with initial time difference.

2. PRELIMINARIES

In this section we will give some basic definition and theorems by [4] which are very useful to use in our future references.

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We consider the following initial value problem

$$x'(t) = f(t, x(t)), x(t_0) = x_0 \text{ for } t \geq t_0, t_0 \in R_+ \tag{2.1}$$

where $f \in C[R_+ \times R, R]$ and $t \in [t_0, t_0 + T]$.

Definition 1. (i) Let $r(t)$ be a solution of the (2.1) on $t \in [t_0, t_0 + T]$. Then $r(t)$ is said to be a maximal solution of (2.1) if, for every solution $x(t)$ of (2.1) existing on $[t_0, t_0 + T]$ the inequality

$$x(t) \leq r(t), t \in [t_0, t_0 + T] \tag{2.2}$$

holds.

(ii) Let $\rho(t)$ be a solution of the (2.1) on $t \in [t_0, t_0 + T]$. Then $\rho(t)$ is said to be a minimal solution of (2.1) if, for every solution $x(t)$ of (2.1) existing on $[t_0, t_0 + T]$ the inequality

$$x(t) \geq \rho(t), t \in [t_0, t_0 + T] \tag{2.3}$$

holds.

Definition 2. (i) A function $\beta \in C^1[[t_0, t_0 + T], R]$ is said to be an upper solution of (2.1) if

$$\beta' \geq f(t, \beta), \beta(t_0) \geq x_0, t \in [t_0, t_0 + T] \tag{2.4}$$

(ii) A function $\alpha \in C^1[[t_0, t_0 + T], R]$ is said to be a lower solution of (2.1) if

$$\alpha' \leq f(t, \alpha), \alpha(t_0) \leq x_0, t \in [t_0, t_0 + T] \tag{2.5}$$

Theorem 1 ([4]). *Let $\alpha, \beta \in C^1[[t_0, t_0 + T], R]$ be lower and upper solutions of (2.1) respectively. Suppose that $x \geq y$, f satisfies the inequality*

$$f(t, x) - f(t, y) \leq M(x - y) \tag{2.6}$$

where M is a positive constant .Then $\alpha(t_0) \leq \beta(t_0)$ implies that $\alpha(t) \leq \beta(t), t \in [t_0, t_0 + T]$.

Remark 1 ([4]). Let the assumptions of Theorem 1 hold. Then every solution $x(t)$ of (2.1) such that $\alpha(t_0) \leq x(t_0) \leq \beta(t_0)$ satisfies the estimate

$$\alpha(t) \leq x(t) \leq \beta(t), t \in [t_0, t_0 + T]. \tag{2.7}$$

Theorem 2 ([4]). *Let $f \in C[[t_0, t_0 + T] \times R, R]$ and $|f(t, x)| \leq L$. Then there exist a solution of the IVP (2.1) on $[t_0, t_0 + T]$.*

3. Comparison Theorems and Existence Results relative to Initial Time Difference

In this section, we will give some basic comparison theorems and existence results relative to initial time difference.

Theorem 3 ([5]). *Assume that* $f \in C[R_+ \times R, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0 + T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0 + T], R], \eta_0 > 0$, *and*

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0 + T] \tag{3.1}$$

$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0 + T] \tag{3.2}$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $f(t,x) - f(t,y) \leq M(x-y), x \geq y, M > 0$;
(iii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x.
Then (A) $\alpha(t) \leq \beta(t + (\sigma + \xi))$ *for* $t \geq \tau_0$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.
(B) $\alpha(t - \sigma) \leq \beta(t + \xi)$ *for* $t \geq t_0$,*where* $\sigma = t_0 - \tau_0$ *and* $\xi = \eta_0 - t_0$.
(C) $\alpha(t - (\sigma + \xi) \leq \beta(t), t \geq \eta_0$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.

Proof of Theorem 3. Please see [5] for the details of the proof by simple modification of (A), (B) and (C). □

The following theorem is the existence result in the closed sectors.

Theorem 4. *Assume that* $f \in C[R_+ \times \Omega, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0 + T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0 + T], R], \eta_0 > 0$

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0 + T]$$

$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0 + T]$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x;
(iii) $\alpha(t - \sigma) \leq \beta(t + \xi)$ *for* $t_0 \leq t \leq t_0 + T$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.
Then there exist a solution of initial value problem (2.1) *satisfying*

$$\alpha(t - \sigma) \leq x(t) \leq \beta(t + \xi)$$

for $t_0 \leq t \leq t_0 + T$.

Proof of Theorem 4. Let $\beta_0(t) = \beta(t + \xi)$ and $\alpha_0(t) = \alpha(t - \sigma)$ for $t \in [t_0, t_0 + T] = I$. Then we get $\beta_0(t_0) = \beta(\eta_0) \geq \alpha_0(t_0) = \alpha(\tau_0)$ and

$$\beta_0'(t) \geq f(t + \xi, \beta_0(t)), \alpha_0'(t) \leq f(t - \sigma, \alpha_0(t)).$$

Assume that $\alpha_0(t_0) \leq x_0 \leq \beta_0(t_0)$ and $p : [t_0, t_0 + T] \times R \to R$ such that

$$p(t,x) = \max[\alpha_0(t), \min[x, \beta_0(t)]]$$

Then $f(t, p(t,x))$ defines a continuous extension of f to $[t_0, t_0 + T] \times R$ which is also bounded since f is bounded on Ω, where

$$\Omega = \{(t,x) \in R_+ \times R : \alpha(t - \sigma) \leq x \leq \beta(t + \xi) \text{ for } t_0 \leq t \leq t_0 + T\}. \tag{3.3}$$

Therefore, the initial value problem $x' = f(t, p(t,x)), x(t_0) = x_0$ according to Theorem 2 has a solution on I. For sufficiently small $\varepsilon > 0$, consider

$$\alpha_{0_\varepsilon}(t) = \alpha_0(t) - \varepsilon(1 + t) \tag{3.4}$$

$$\beta_{0_\varepsilon}(t) = \beta_0(t) + \varepsilon(1+t). \tag{3.5}$$

Clearly

$$\alpha_{0_\varepsilon}(t_0) < \alpha_0(t_0) \leq x_0 \leq \beta_0(t_0) < \beta_{0_\varepsilon}(t_0)$$

and hence $\alpha_{0_\varepsilon}(t_0) < x_0 < \beta_{0_\varepsilon}(t_0)$. We wish to show that

$$\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t) \text{ on } I$$

Suppose that it is not true, then there exists a $t_1 \in (t_0, t_0+T]$ such that

$$\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t) \text{ on } [t_0, t_1) \text{ and } \beta_{0_\varepsilon}(t_1) = x(t_1).$$

Then $x(t_1) > \beta_0(t_1)$ and so

$$p(t_1, x(t_1)) = \beta_0(t_1)$$

Also $\alpha_0(t_1) \leq p(t_1, x(t_1)) \leq \beta_0(t_1)$. Hence

$$\beta_0'(t_1) \geq f(t_1, \beta_0(t_1)) = f(t_1, p(t_1, x(t_1))) = x'(t_1)$$

Since $\beta_{0_\varepsilon}'(t_1) > \beta_0'(t_1) \geq x'(t_1)$ we have $\beta_{0_\varepsilon}'(t_1) > x'(t_1)$. This contradicts $x(t) < \beta_{0_\varepsilon}(t)$ for $t \in [t_0, t_1)$. The other case can be proved similarly. Consequently, we obtain $\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t)$ on I. Letting $\varepsilon \to 0$ we get

$$\alpha_0(t) \leq x(t) \leq \beta_0(t) \text{ on } I.$$

Therefore the proof is completed. □

Remark 2. Assume that $f \in C[R_+ \times R, R]$, assumptions (i), (ii) of Theorem 4 hold and

$(iii)^*$ $\alpha(t) \leq \beta(t + (\sigma + \xi))$ for $\tau_0 \leq t \leq \tau_0 + T$ where $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$. Then there exist a solution satisfying

$$\alpha(t) \leq x(t+\sigma) \leq \beta(t + (\sigma + \xi))$$

for $\tau_0 \leq t \leq \tau_0 + T$.

Remark 3. Assume that $f \in C[R_+ \times R, R]$, assumptions (i), (ii) of Theorem 4 hold and

$(iii)^{**}$ $\alpha(t - (\sigma + \xi)) \leq \beta(t)$ for $\eta_0 \leq t \leq \eta_0 + T$ where $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$. Then there exist a solution satisfying

$$\alpha(t - (\sigma + \xi)) \leq x(t - \xi) \leq \beta(t)$$

for $\eta_0 \leq t \leq \eta_0 + T$.

4. Monotone Iterative Technique with the different initial data

In this section we have applied the monotone iterative technique for the nonlinear initial value problem of (2.1) by choosing lower and upper solutions with known at the different initial data.

Theorem 5. *Assume that* $f \in C[R_+ \times R, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0 + T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0 + T], R], \eta_0 > 0$

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0 + T]$$
$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0 + T]$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x;
(iii) $\alpha(t-\sigma) \leq \beta(t+\xi)$ *for* $t_0 \leq t \leq t_0 + T, \sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$;
(iv) $f(t,x) - f(t,y) \geq -M(x-y)$ *where* $M > 0$ *and* $\alpha(t-\sigma) \leq y \leq x \leq \beta(t+\xi)$ *for* $t \in [t_0, t_0 + T]$.
Then there exist monotone sequences $\left\{\widetilde{\alpha}_n\right\}$ *and* $\left\{\widetilde{\beta}_n\right\}$ *which converge uniformly and monotonically on* $[t_0, t_0 + T]$ *such that* $\widetilde{\alpha}_n \to \rho$ *and* $\widetilde{\beta}_n \to r$ *as* $n \to \infty$. *Moreover,* ρ *and* r *are minimal and maximal solutions such that* ρ *is the minimal solution of the initial value problem of* $x' = f(t,x), x(\tau_0) = x_0$ *on* $[\tau_0, \tau_0 + T]$ *and* r *is the maximal solution of the initial value problem of* $x' = f(t,x), x(\eta_0) = x_0$ *on* $[\eta_0, \eta_0 + T]$ *respectively where* $\widetilde{\beta}_0(t) = \beta(t+\xi), \widetilde{\alpha}_0(t) = \alpha(t-\sigma)$.

Proof of Theorem 5. Since $\widetilde{\beta}_0(t) = \beta(t+\xi), \widetilde{\beta}_0(t_0) = \beta(t_0+\xi) = \beta(\eta_0) \geq \alpha_0(\tau_0)$ and $\widetilde{\beta}_0'(t) \geq f(t+\xi, \widetilde{\beta}_0(t)), \widetilde{\alpha}_0'(t) \leq f(t-\sigma, \widetilde{\alpha}_0(t)), t \in [t_0, t_0 + T]$. Consider the following linear initial value problems

$$\widetilde{\alpha}_{n+1}'(t) = f(t-\sigma, \widetilde{\alpha}_n(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)), \widetilde{\alpha}_{n+1}(t_0) = x_0 \tag{4.1}$$
$$\widetilde{\beta}_{n+1}'(t) = f(t+\xi, \widetilde{\beta}_n(t)) - M(\widetilde{\beta}_{n+1}(t) - \widetilde{\beta}_n(t)), \widetilde{\beta}_{n+1}(t_0) = x_0 \tag{4.2}$$

Setting $p(t) = \widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)$ where $p(t_0) \leq 0$ for $t \in I$.

$$\begin{aligned} p'(t) &= \widetilde{\beta}_1'(t) - \widetilde{\beta}_0'(t) \\ &= f(t+\xi, \widetilde{\beta}_0(t)) - M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) - \widetilde{\beta}_0'(t) \\ &\leq f(t+\xi, \widetilde{\beta}_0(t)) - M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) - f(t+\xi, \widetilde{\beta}_0(t)) \\ p'(t) &\leq -Mp(t) \end{aligned}$$

This shows that $p(t) \leq p(t_0)e^{-Mt} \leq 0$ since we have $p(t_0) \leq 0$. Hence $\widetilde{\beta}_1(t) \leq \widetilde{\beta}_0(t)$ on I. Similarly we can show that $\widetilde{\alpha}_0(t) \leq \widetilde{\alpha}_1(t)$ on I. Setting $p(t) = \widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)$ and $p(t_0) \geq 0$ for $t \in [t_0, t_0 + T]$.

$$\begin{aligned} p'(t) &= \widetilde{\alpha}'_1(t) - \widetilde{\alpha}'_0(t) \\ &= f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha_1}(t) - \widetilde{\alpha}_0(t)) - \widetilde{\alpha}'_0(t) \\ &\geq f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t-\sigma, \widetilde{\alpha}_0(t)) \\ p'(t) &\geq -Mp(t) \end{aligned}$$

This shows that $p(t) \geq p(t_0)e^{-Mt} \geq 0$. Hence $\widetilde{\alpha}_0(t) \leq \widetilde{\alpha}_1(t)$ on I. Now we can show $\widetilde{\alpha}_1(t) \leq \widetilde{\beta}_1(t)$. Setting $p(t) = \widetilde{\alpha}_1(t) - \widetilde{\beta}_1(t)$ where $p(t_0) \leq 0$ for $t \in [t_0, t_0 + T]$.

$$\begin{aligned} p'(t) &= \widetilde{\alpha}'_1(t) - \widetilde{\beta}'_1(t) \\ &= f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t+\xi, \widetilde{\beta}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\ &\leq f(t, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t, \widetilde{\beta}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\ &= f(t, \widetilde{\alpha}_0(t)) - f(t, \widetilde{\beta}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\ &\leq -\left[-M(\widetilde{\beta}_0(t) - \widetilde{\alpha}_0(t))\right] - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\ &= M(\widetilde{\beta}_1(t) - \widetilde{\alpha}_1(t)) \\ p'(t) &\leq -Mp(t) \end{aligned}$$

This shows that $p(t) \leq p(t_0)e^{-Mt} \leq 0$ on I. Hence $\widetilde{\alpha}_1(t) \leq \widetilde{\beta}_1(t)$ on $[t_0, t_0 + T]$. Consequently, we have

$$\widetilde{\alpha}_0(t) \leq \widetilde{\alpha}_1(t) \leq \widetilde{\beta}_1(t) \leq \widetilde{\beta}_0(t) \text{ on } I.$$

To employ the method of mathematical induction, assume that for some $k > 1$

$$\widetilde{\alpha}_{k-1}(t) \leq \widetilde{\alpha}_k(t) \leq \widetilde{\beta}_k(t) \leq \widetilde{\beta}_{k-1}(t) \text{ on } I$$

we then show that

$$\widetilde{\alpha}_k(t) \leq \widetilde{\alpha}_{k+1}(t) \leq \widetilde{\beta}_{k+1}(t) \leq \widetilde{\beta}_k(t) \text{ on } I.$$

where $\widetilde{\alpha}_{k+1}(t)$ and $\widetilde{\beta}_{k+1}(t)$ are the solutions of the linear IVPs

$$\begin{aligned} \widetilde{\alpha}'_{k+1}(t) &= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)),\ \widetilde{\alpha}_{k+1}(t_0) = x_0 \\ \widetilde{\beta}'_{k+1}(t) &= f(t+\xi, \widetilde{\beta}_k(t)) - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)),\ \widetilde{\beta}_{k+1}(t_0) = x_0. \end{aligned}$$

As we have done before, we set $p(t) = \widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)$ where $p(t_0) \geq 0$.

$$\begin{aligned}
p'(t) &= \widetilde{\alpha}'_{k+1}(t) - \widetilde{\alpha}'_k(t) \\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) \\
&\quad - \widetilde{\alpha}_k(t)) - f(t-\sigma, \widetilde{\alpha}_{k-1}(t)) + M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t)) \\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - f(t-\sigma, \widetilde{\alpha}_{k-1}(t)) \\
&\quad - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) + M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t)) \\
&\geq -M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t)) \\
&= -M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) \\
p'(t) &\geq -Mp(t)
\end{aligned}$$

This shows that $p(t) \geq p(t_0)e^{-Mt} \geq 0$ on I. Hence $\widetilde{\alpha}_k(t) \leq \widetilde{\alpha}_{k+1}(t)$ on I. Setting $p(t) = \widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)$ where $p(t_0) \leq 0$.

$$\begin{aligned}
p'(t) &= \widetilde{\beta}'_{k+1}(t) - \widetilde{\beta}'_k(t) \\
&= f(t+\xi, \widetilde{\beta}_k(t)) - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) \\
&\quad - f(t+\xi, \widetilde{\beta}_{k-1}(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t)) \\
&= f(t+\xi, \widetilde{\beta}_k(t)) - f(t+\xi, \widetilde{\beta}_{k-1}(t)) \\
&\quad - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t)) \\
&\leq -\left[-M(\widetilde{\beta}_{k-1}(t) - \widetilde{\beta}_k(t))\right] - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t)) \\
p'(t) &\leq -Mp(t)
\end{aligned}$$

This shows that $p(t) \leq p(t_0)e^{-Mt} \leq 0$ on I. Hence $\widetilde{\beta}_{k+1}(t) \leq \widetilde{\beta}_k(t)$ on I. Now setting $p(t) = \widetilde{\alpha}_{k+1}(t) - \widetilde{\beta}_{k+1}(t)$ where $p(t_0) \leq 0$.

$$\begin{aligned}
p'(t) &= \widetilde{\alpha}'_{k+1}(t) - \widetilde{\beta}'_{k+1}(t) \\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) \\
&\quad - f(t+\xi, \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) \\
&\leq f(t, \widetilde{\alpha}_k(t)) - f(t, \widetilde{\beta}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) + M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t))
\end{aligned}$$

$$\leq -\left[-M(\widetilde{\beta}_k(t)-\widetilde{\alpha}_k(t))\right]-M(\widetilde{\alpha}_{k+1}(t)-\widetilde{\alpha}_k(t))+M(\widetilde{\beta}_{k+1}(t)-\widetilde{\beta}_k(t))$$

$$p'(t) \leq -Mp(t)$$

This shows that $p(t) \leq p(t_0)e^{-Mt} \leq 0$ on I. Hence $\widetilde{\alpha}_{k+1}(t) \leq \widetilde{\beta}_{k+1}(t)$ on I. Consequently, for all $k \in N$ and for $t \in I$. We get

$$\widetilde{\alpha}_k(t) \leq \widetilde{\alpha}_{k+1}(t) \leq \widetilde{\beta}_{k+1}(t) \leq \widetilde{\beta}_k(t) \text{ on } I.$$

Hence it follows that for all $n \in N$ and $t \in I$, we have

$$\widetilde{\alpha}_0(t) \leq \widetilde{\alpha}_1(t) \leq ... \leq \widetilde{\alpha}_n(t) \leq \widetilde{\beta}_n(t) \leq ... \leq \widetilde{\beta}_1(t) \leq \widetilde{\beta}_0(t) \text{ on } I. \tag{4.3}$$

It is clear that the sequences $\left\{\widetilde{\alpha}_n\right\}$ and $\left\{\widetilde{\beta}_n\right\}$ are uniformly bounded and equicontinuous sequence of functions on $[t_0, t_0+T]$ and consequently by Ascoli-Arzela's theorem there exist subsequences $\left\{\widetilde{\alpha}_{n_k}\right\}$ and $\left\{\widetilde{\beta}_{n_k}\right\}$ that converge uniformly on $[t_0, t_0+T]$. In view of (4.3) it follows that the entire sequences $\left\{\widetilde{\alpha}_n\right\}$ and $\left\{\widetilde{\beta}_n\right\}$ converge uniformly and monotonically to $\widetilde{\rho}$ and $\widetilde{r}$,respectively, as $n \to \infty$. We have obtained the following corresponding Volterra integral equation for (4.1) and (4.2)

$$\widetilde{\alpha}_{n+1}(t) = x_0 + \int_{t_0}^{t} (f(s-\sigma, \widetilde{\alpha}_n(s)) - M(\widetilde{\alpha}_{n+1}(s)-\widetilde{\alpha}_n(s))ds$$

$$\widetilde{\beta}_{n+1}(t) = x_0 + \int_{t_0}^{t} (f(s+\xi, \widetilde{\beta}_n(s)) - M(\widetilde{\beta}_{n+1}(s)-\widetilde{\beta}_n(s))ds.$$

Therefore, we get

$$\widetilde{\rho}'(t) = f(t-\sigma, \widetilde{\rho}(t)), \widetilde{\rho}(t_0) = x_0 \tag{4.4}$$

$$\widetilde{r}'(t) = f(t+\xi, \widetilde{r}(t)), \widetilde{r}(t_0) = x_0 \tag{4.5}$$

as $n \to \infty$ where $\widetilde{\rho}(t) = \rho(t-\sigma)$ and $\widetilde{r}(t) = r(t+\xi)$, respectively. Finally, we must show that $\widetilde{\rho}$ and $\widetilde{r}$ are the minimal and maximal solutions of the IVP (4.4) and (4.5), respectively. Let $x(t)$ be any solution of (2.1) such that

$$\widetilde{\alpha}_0(t) \leq x(t) \leq \widetilde{\beta}_0(t) \text{ on } [t_0, t_0+T].$$

Then we need to prove

$$\widetilde{\alpha}_0(t) \leq \widetilde{\rho} \leq x(t) \leq \widetilde{r} \leq \widetilde{\beta}_0(t) \text{ on } [t_0, t_0+T].$$

Suppose that for some n,

$$\widetilde{\alpha}_n(t) \le x(t) \le \widetilde{\beta}_n(t).$$

Then, we set $p(t) = \widetilde{\alpha}_{n+1}(t) - x(t)$ where $p(t_0) = 0$. Thus

$$\begin{aligned} p'(t) &= \widetilde{\alpha}'_{n+1}(t) - x'(t) \\ &= f(t-\sigma, \widetilde{\alpha}_n(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) - f(t,x) \\ &\le f(t, \widetilde{\alpha}_n(t)) - f(t,x) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) \\ &\le -M(\widetilde{\alpha}_n(t) - x(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) \\ &= -M(\widetilde{\alpha}_{n+1}(t) - x(t)) \\ &= -Mp(t). \end{aligned}$$

This shows that $p(t) \le p(t_0)e^{-Mt} \le 0$ on I since we have $p(t_0) \le 0$. Hence $\widetilde{\alpha}_{n+1}(t) \le x(t)$ on I. In a similar manner, we can show that $x(t) \le \widetilde{\beta}_n(t)$ on $[t_0, t_0+T]$. This proves by induction that $\widetilde{\alpha}_n(t) \le x(t) \le \widetilde{\beta}_n(t)$ for all n taking limit as $n \to \infty$ we arrive at $\widetilde{\rho} \le x(t) \le \widetilde{r}$ on $[t_0, t_0+T]$. Therefore the proof is completed. □

Corollary 1. *If in addition to the assumption of Theorem 5, we assume*

$$f(t,x) - f(t,y) \le M(x-y), \alpha(t-\sigma) \le y \le x \le \beta(t+\xi), M > 0$$

then we have unique solution of (2.1) such that $\widetilde{\rho} = x = \widetilde{r}$.

Proof. If we set $p = r - \rho$ then $p' = r' - \rho' = f(t,r) - f(t,\rho) \le M(r-\rho)$, which gives $p' \le Mp$ and $p(t_0) = 0$.Hence we get $p(t) \le 0$ on $[t_0, t_0+T]$ which implies $r \le \rho$. Also , utilizing the fact that $\rho \le r$, we have $\rho = x = r$ is the unique solution of (2.1). □

Corollary 2. *If in addition to the assumption* $(i), (ii)$ *of Theorem 5, we assume*
(iii) $\alpha(t) \le \beta(t + (\sigma + \xi))$ *for* $\tau_0 \le t \le \tau_0 + T$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$;
(iv) $f(t,x) - f(t,y) \ge -M(x-y)$ *where* $M > 0$ *and* $\alpha(t) \le y \le x \le \beta(t + (\sigma + \xi))$ *for* $t \in [\tau_0, \tau_0 + T]$.

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\left\{\widetilde{\beta}_n\right\}$ *which converge uniformly and monotonically on* $[\tau_0, \tau_0 + T]$ *such that* $\alpha_n \to \rho$ *and* $\widetilde{\beta}_n \to \widetilde{r}$ *as* $n \to \infty$. *Moreover,* ρ *and* r *are minimal and maximal solutions such that* ρ *is the minimal solution of the initial value problem of* $x' = f(t,x), x(\tau_0) = x_0$ *on* $[\tau_0, \tau_0 + T]$ *and* r *is the maximal solution of the initial value problem of*

$$\widetilde{x}'(t) = f(t+\sigma, \widetilde{x}(t)), \widetilde{x}(\tau_0) = x_0, t \in [\tau_0, \tau_0 + T] \tag{4.6}$$

respectively where $\widetilde{\beta}_0(t)=\beta(t+(\sigma+\xi)),\widetilde{\alpha}_0(t)=\alpha_0(t)$.

Corollary 3. *If in addition to the assumption* $(i),(ii)$ *of Theorem 5, we assume*
(iii) $\alpha(t-(\sigma+\xi))\leq\beta(t)$ *for* $\eta_0\leq t\leq\eta_0+T$ *where* $\sigma=t_0-\tau_0,\xi=\eta_0-t_0$;
(iv) $f(t,x)-f(t,y)\geq -M(x-y)$ *where* $M>0$ *and* $\alpha(t-(\sigma+\xi))\leq y\leq x\leq\beta(t)$ *for* $t\in[\eta_0,\eta_0+T]$.

Then there exist monotone sequences $\left\{\widetilde{\alpha}_n\right\}$ *and* $\{\beta_n\}$ *which converge uniformly and monotonically on* $[\eta_0,\eta_0+T]$ *such that* $\widetilde{\alpha}_n\to\widetilde{\rho}$ *and* $\beta_n\to r$ *as* $n\to\infty$. *Moreover,* $\widetilde{\rho}$ *and* r *are minimal and maximal solutions such that* $\widetilde{\rho}$ *is the minimal solution of the initial value problem of*

$$\widetilde{x}'(t)=f(t-\xi,\widetilde{x}(t)),\widetilde{x}(\eta_0)=x_0,t\in[\eta_0,\eta_0+T] \tag{4.7}$$

and r *is the maximal solution of the initial value problem of* $x'=f(t,x),x(\eta_0)=x_0$ *on* $[\eta_0,\eta_0+T]$ *where* $\widetilde{\beta}_0(t)=\beta_0(t),\widetilde{\alpha}_0(t)=\alpha_0(t-(\sigma+\xi))$.

5. Examples

Example 1. Consider the nonlinear initial value problem

$$x'(t)=e^t x^2,\ x(1)=-1 \text{ for } t\geq 1 \tag{5.1}$$

where $f(t,x)=e^t x^2\in C\left[R_+\times R,R\right]$ and $t\in[1,4]$.

(E_1) $\alpha(t)=-\frac{2}{e^t}$, $\alpha(0)=-2$, $\alpha(t)\in C^1\left[[0,3],\ R\right]$ and $\beta(t)=-\frac{1}{2e^t}$, $\beta(2)=-\frac{1}{2e^2}$, $\beta(t)\in C^1\left[[2,5],\ R\right]$, then we get for $T=3$

$$\alpha'(t)=\frac{2}{e^t} \text{ and } f(t,\alpha)=\frac{4}{e^t} \text{ then } \alpha'(t)\leq f(t,\alpha) \text{ for } t\in[0,3]$$
$$\beta'(t)=\frac{1}{2e^t} \text{ and } f(t,\beta)=\frac{1}{4e^t} \text{ then } \beta'(t)\geq f(t,\alpha) \text{ for } t\in[2,5].$$

Therefore, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions, respectively and

$$\alpha(\tau_0)=\alpha(0)=-2<x(t_0)=x(1)=-1<\beta(\eta_0)=\beta(2)=-\frac{1}{2e^2}.$$

(E_2) $0<1<2$ and $f(t,x)$ is nondecreasing in t for each x and $\alpha(1)=\frac{-2}{e}\leq\beta(1)=-\frac{1}{2e}$.

(E_3) $f(t,x)-f(t,y)\geq -M(x-y)$ where $M=4e^4>0$ is the Lipschitz constant for $\alpha(1)\leq y\leq x\leq\beta(1)$, $t\in[0,5]$. Also $\widetilde{\alpha}_0(t)=\alpha(t-1)$ and $\widetilde{\beta}_0(t)=\beta(t+1)$ for $t\in[1,4]$.

Therefore, $\widetilde{\alpha}_{n+1}$ is a lower solution and $\widetilde{\beta}_{n+1}$is an upper solution of (5.1) for $t\in[1,3]$. Thus $\widetilde{\alpha}_{n+1}(t)\leq\widetilde{\beta}_{n+1}(t)$ for $t\in[1,4]$.

Consequently, we have for all n,

$$\widetilde{\alpha}_0(t) \leq \widetilde{\alpha}_1(t) \leq \cdots \leq \widetilde{\alpha}_n(t) \leq \widetilde{\beta}_n(t) \leq \cdots \leq \widetilde{\beta}_1(t) \leq \widetilde{\beta}_0(t) \text{ for } t \in [1,4]$$

Employing the standard monotone technique it can be shown that the monotone sequence $\left\{\widetilde{\alpha}_n(t)\right\}$ converges to $\widetilde{\rho}$ which is the minimal solution of (5.1) as $n \to \infty$ and monotone sequence $\left\{\widetilde{\beta}_n(t)\right\}$ converges to $\widetilde{r}$ which is the maximal solution of (5.1) as $n \to \infty$. We arrive at $\widetilde{\rho} \leq x(t) \leq \widetilde{r}$ on $[1,3]$. In this example,since the solution $x(t) = -\frac{1}{e^t+1-e}$ of (5.1) is unique, $\widetilde{\rho} = x(t) = \widetilde{r}$.

ACKNOWLEDGEMENT

The authors would like to thank the referees for their insightful comments and detailed suggestions which improved the quality of the paper.

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Doi: 10.18514/MMN.2015.1033

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 587–605

HU e-ISSN 1787-2413

THE HARMONIC INDEX FOR UNICYCLIC AND BICYCLIC GRAPHS WITH GIVEN MATCHING NUMBER

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Received 24 October, 2013

Abstract. The harmonic index of a graph G is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. In this paper, we present the minimum harmonic indices for unicyclic and bicyclic graphs with n vertices and matching number m ($2 \leq m \leq \lfloor \frac{n}{2} \rfloor$), respectively. The corresponding extremal graphs are also characterized.

2000 *Mathematics Subject Classification:* 05C07; 05C70; 05C35; 92E10

Keywords: harmonic index, unicyclic graph, bicyclic graph, matching number

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index $R(G)$, proposed by Randić [20] in 1975, is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}},$$

where $d(u)$ denotes the degree of a vertex u of G. The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationship studies. Mathematical properties of this descriptor have been studied extensively (see [9, 10, 14, 15, 19] and the references cited therein).

In this paper, we consider a closely related variant of the Randić index, named the harmonic index. For a graph G, the harmonic index $H(G)$ is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}.$$

This index first appeared in [6], and it can also be viewed as a particular case of the general sum-connectivity index proposed by Zhou and Trinajstić in [32].

This work was supported by the Fundamental Research Funds for the Central Universities (No. NS2015078).

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Favaron, Mahéo and Saclé [7] considered the relation between the harmonic index and the eigenvalues of graphs. Zhong [28, 29], Zhong and Xu [30] determined the minimum and maximum harmonic indices for simple connected graphs, trees, unicyclic and bicyclic graphs, and characterized the corresponding extremal graphs. Wu, Tang and Deng [23] found the minimum harmonic index for graphs (triangle-free graphs, respectively) with minimum degree at least 2, and characterized the corresponding extremal graphs. Deng, Balachandran, Ayyaswamy and Venkatakrishnan [2] considered the relation between the harmonic index and the chromatic number of a graph by using the effect of removal of a minimum degree vertex on the harmonic index. Liu [17] proposed a conjecture concerning the relation between the harmonic index and the diameter of a connected graph, and showed that the conjecture is true for trees. Ilić [12], Xu [25], Zhong and Xu [31] established some relationships between the harmonic index and several other topological indices. The chemical applicability of the harmonic index was also recently investigated [8, 11]. See [3, 18, 24, 26] for more information of this index.

In this paper, we determine the minimum harmonic indices for unicyclic and bicyclic graphs with n vertices and matching number m ($2 \leq m \leq \lfloor \frac{n}{2} \rfloor$), respectively. The corresponding extremal graphs are also characterized. The related problems have been well-studied for several other topological indices, such as the Randić index [16, 33], the modified Randić index [13] and the sum-connectivity index [4, 5, 21, 22].

2. Preliminaries

Let G be a graph. For any vertex $v \in V(G)$, we use $N_G(v)$ (or $N(v)$ if there is no ambiguity) to denote the set of neighbors of v in G. A pendent vertex is a vertex of degree 1. For two distinct vertices u and v of G, the distance $d(u, v)$ between u and v is the number of edges in a shortest path joining u and v in G. A unicyclic graph is a connected graph with n vertices and n edges, and a bicyclic graph is a connected graph with n vertices and $n + 1$ edges. We use C_n to denote the cycle on n vertices.

A matching M in a graph G is a subset of $E(G)$ such that no two edges in M share a common vertex. A matching M in G is said to be maximum, if for any other matching M' in G, $|M'| \leq |M|$. The matching number of G is the number of edges in a maximum matching of G. If M is a matching in G and the vertex $v \in V(G)$ is incident with an edge of M, then v is said to be M-saturated, and if every vertex in G is M-saturated, then M is a perfect matching.

For any vertex $v \in V(G)$, we use $G - v$ to denote the graph resulting from G by deleting the vertex v and its incident edges. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G.

We now establish some lemmas which will be used frequently in later proofs.

Lemma 1. *Let G be a connected graph on $n \geq 4$ vertices with a pendent vertex u. Let v be the unique neighbor of u with $d(v) = s$, and let w be a neighbor of v different from u with $d(w) = t$.*

(i) If $s = 2$ and w is adjacent to at most one pendent vertex in G, then

$$H(G) \geq H(G-u-v) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3}$$

with equality if and only if one neighbor of w has degree 1 and the other neighbors of w have degree 2.

(ii) If v is adjacent to at most k pendent vertices in G, then

$$H(G) \geq H(G-u) + \frac{2(s-k)}{s+2} + \frac{2(2k-s)}{s+1} - \frac{2(k-1)}{s}$$

with equality if and only if k neighbors of v have degree 1 and the other neighbors of v have degree 2.

Proof. (i) Let $N(w) = \{w_0 = v, w_1, \ldots, w_{t-1}\}$. Since w is adjacent to at most one pendent vertex in G, we may assume that $d(w_1) \geq 1$, and $d(w_i) \geq 2$ for each $2 \leq i \leq t-1$ (if $t \geq 3$). Note that $\frac{2}{t+x} - \frac{2}{t-1+x}$ is increasing for $x \geq 1$, we have

$$\begin{aligned} H(G) &= H(G-u-v) + \sum_{i=1}^{t-1} \left(\frac{2}{t+d(w_i)} - \frac{2}{t-1+d(w_i)} \right) + \frac{2}{t+2} + \frac{2}{3} \\ &\geq H(G-u-v) + \left(\frac{2}{t+1} - \frac{2}{t} \right) + (t-2) \left(\frac{2}{t+2} - \frac{2}{t+1} \right) + \frac{2}{t+2} + \frac{2}{3} \\ &= H(G-u-v) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \end{aligned}$$

with equality if and only if $d(w_1) = 1$ and $d(w_i) = 2$ for each $2 \leq i \leq t-1$ (if $t \geq 3$). This proves (i).

(ii) Let r $(1 \leq r \leq k)$ be the number of pendent neighbors of v in G, and let $N(v) = \{v_0 = u, v_1, \ldots, v_{s-1}\}$. Without loss of generality, we may assume that $d(v_i) = 1$ for each $1 \leq i \leq r-1$ (if $r \geq 2$), and $d(v_i) \geq 2$ for each $r \leq i \leq s-1$ (if $s \geq r+1$). Note that $\frac{2}{s+x} - \frac{2}{s-1+x}$ is increasing for $x \geq 1$ and $\frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} < 0$, we have

$$\begin{aligned} H(G) &= H(G-u) + (r-1) \left(\frac{2}{s+1} - \frac{2}{s} \right) \\ &\quad + \sum_{i=r}^{s-1} \left(\frac{2}{s+d(v_i)} - \frac{2}{s-1+d(v_i)} \right) + \frac{2}{s+1} \\ &\geq H(G-u) + (r-1) \left(\frac{2}{s+1} - \frac{2}{s} \right) + (s-r) \left(\frac{2}{s+2} - \frac{2}{s+1} \right) + \frac{2}{s+1} \\ &= H(G-u) + r \left(\frac{4}{s+1} - \frac{2}{s+2} - \frac{2}{s} \right) + \frac{2s}{s+2} - \frac{2s}{s+1} + \frac{2}{s} \end{aligned}$$

$$\geq H(G-u)+k\left(\frac{4}{s+1}-\frac{2}{s+2}-\frac{2}{s}\right)+\frac{2s}{s+2}-\frac{2s}{s+1}+\frac{2}{s}$$
$$=H(G-u)+\frac{2(s-k)}{s+2}+\frac{2(2k-s)}{s+1}-\frac{2(k-1)}{s}$$

with equalities if and only if $r=k$ and $d(v_i)=2$ for each $k\leq i\leq s-1$ (if $s\geq k+1$). This completes the proof of the lemma. □

Lemma 2. (*i*) *The function* $\frac{2(x-1)}{x+2}-\frac{2(x-3)}{x+1}-\frac{2}{x}$ *is decreasing for* $x\geq 2$.

(*ii*) *For* $k\geq 1$, *the function* $\frac{2(x-k)}{x+2}+\frac{2(2k-x)}{x+1}-\frac{2(k-1)}{x}$ *is decreasing for* $x\geq k+1$.

Proof. (i) Let $f(x)=\frac{2(x-1)}{x+2}-\frac{2(x-3)}{x+1}-\frac{2}{x}=\frac{8}{x+1}-\frac{6}{x+2}-\frac{2}{x}$. For $x\geq 2$, we have

$$f'(x)=-\frac{8}{(x+1)^2}+\frac{6}{(x+2)^2}+\frac{2}{x^2}=\frac{-8x^3+24x+8}{x^2(x+1)^2(x+2)^2}$$
$$=\frac{-8x(x^2-4)-8(x-1)}{x^2(x+1)^2(x+2)^2}<0,$$

and hence (i) holds.

(ii) Let $g(x)=\frac{2(x-k)}{x+2}+\frac{2(2k-x)}{x+1}-\frac{2(k-1)}{x}$ and $g_1(x)=\frac{2(k-1)}{x}+\frac{2(x-1-k)}{x+1}$. Then $g(x)=g_1(x+1)-g_1(x)$. For $x\geq k+1\geq 2$, we have

$$g_1''(x)=\frac{4(k-1)}{x^3}-\frac{4(k+2)}{(x+1)^3}=\frac{-12x^3+12(k-1)x^2+12(k-1)x+4(k-1)}{x^3(x+1)^3}$$
$$=\frac{-12x^2(x-k)-12x(x-k)-4(3x-k+1)}{x^3(x+1)^3}<0,$$

and $g'(x)=g_1'(x+1)-g_1'(x)<0$. So the assertion of the lemma holds. □

Lemma 3. *Let* G *be a connected graph, and let* u *be a vertex of degree* 2 *in* G *with two neighbors* v *and* w *such that* $d(v)\geq 2$ *and* $vw\notin E(G)$. *Let* $G'=G-uw+vw$, *then* $H(G)>H(G')$.

Proof. Let $d(v)=p\geq 2$ and let $N(v)=\{v_0=u,v_1,\ldots,v_{p-1}\}$. Then

$$H(G)-H(G')$$
$$=\left(\sum_{i=1}^{p-1}\frac{2}{p+d(v_i)}+\frac{2}{2+d(w)}\right)-\left(\sum_{i=1}^{p-1}\frac{2}{p+1+d(v_i)}+\frac{2}{p+1+d(w)}\right)$$
$$=\sum_{i=1}^{p-1}\left(\frac{2}{p+d(v_i)}-\frac{2}{p+1+d(v_i)}\right)+\left(\frac{2}{2+d(w)}-\frac{2}{p+1+d(w)}\right)>0.$$

This proves the lemma. □

3. MINIMUM HARMONIC INDEX FOR UNICYCLIC GRAPHS WITH GIVEN MATCHING NUMBER

Let $\mathscr{U}_n$ be the set of unicyclic graphs with $n \geq 3$ vertices, and let $\mathscr{U}_{n,m}$ be the set of unicyclic graphs with n vertices and matching number m, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In this section, we determine the minimum harmonic index for graphs in $\mathscr{U}_{n,m}$, and characterize the corresponding extremal graphs.

For a unicyclic graph G with the cycle C_p, the forest obtained from G by deleting the edges in C_p consists of p vertex-disjoint trees, each containing a vertex of C_p, which is called the root of this tree in G. These trees are called the branches of G. Chang and Tian [1] showed the following lemma.

Lemma 4. *Let $G \in \mathscr{U}_{2m,m}$ $(m \geq 3)$, and let T be a branch of G with root r. If $u \in V(T)$ is a pendent vertex which is furthest from the root r with $d(u,r) \geq 2$, then u is adjacent to a vertex of degree 2.*

The second lemma was proved by Yu and Tian [27].

Lemma 5. *Let $G \in \mathscr{U}_{n,m}$ $(n > 2m)$ and $G \ncong C_n$. Then there exists a maximum matching M and a pendant vertex u in G such that u is not M-saturated.*

Zhong [29] proved the following result.

Lemma 6. *Let $G \in \mathscr{U}_n$ with $n \geq 3$. Then $H(G) \leq \frac{n}{2}$ with equality if and only if $G \cong C_n$.*

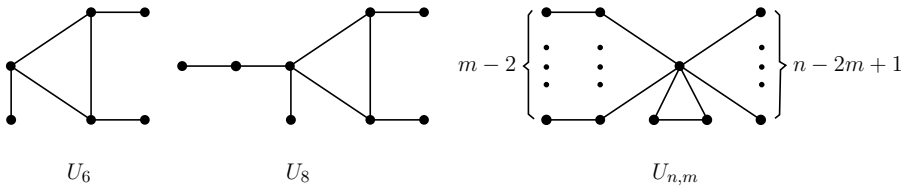


FIGURE 1. The graphs U_6, U_8 and $U_{n,m}$.

Let U_6 be the unicyclic graph on 6 vertices obtained by attaching a pendent vertex to every vertex of a triangle, and let U_8 be the unicyclic graph on 8 vertices obtained by attaching a path on two vertices to one vertex of degree 3 of U_6. For $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, we use $U_{n,m}$ to denote the unicyclic graph on n vertices obtained by attaching $n - 2m + 1$ pendent vertices and $m-2$ paths on two vertices to one vertex of a triangle. See Figure 1 for an illustration.

Theorem 1. *Let $G \in \mathscr{U}_{2m,m} \setminus \{U_6, U_8\}$, where $m \geq 2$. Then*

$$H(G) \geq \frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equality if and only if $G \cong U_{2m,m}$.

Proof. We prove the theorem by induction on m. If $m = 2$, then either $G \cong C_4$ or $G \cong U_{4,2}$. Since $H(C_4) = 2 > \frac{9}{5} = H(U_{4,2})$, we see that the assertion of the theorem holds. So we may assume that $m \geq 3$ and the result holds for graphs in $\mathscr{U}_{2(m-1),m-1} \setminus \{U_6, U_8\}$. By Lemma 6, since C_{2m} is the unique unicyclic graph on $2m$ vertices with the maximum harmonic index, we may further assume that $G \ncong C_{2m}$. Let M be a maximum matching in G, then $|M| = m$. By Lemma 4, we need only consider the following two cases.

Case 1. There exists a pendent vertex u in G which is adjacent to a vertex v of degree 2.

Let w be the neighbor of v different from u with $d(w) = t \geq 2$, and let $G' = G - u - v$. Then $uv \in M$ and $G' \in \mathscr{U}_{2(m-1),m-1}$. Since M contains exactly one edge incident with w and there are m edges of G outside M, we have $t \leq m + 1$. Note that w is adjacent to at most one pendent vertex in G.

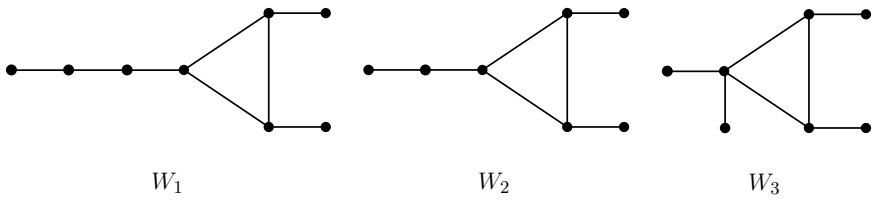


FIGURE 2. The graphs W_1, W_2 and W_3.

If $G' \cong U_6$, then we have $G \cong W_1$ (since we assume $G \ncong U_8$), see Figure 2. Since $H(W_1) = \frac{107}{30} > \frac{139}{42} = H(U_{8,4})$, we know that the result holds.

If $G' \cong U_8$, then $t \leq 5$. By Lemma 1(i) and Lemma 2(i), we have

$$\begin{aligned} H(G) &\geq H(U_8) + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \\ &\geq \frac{347}{105} + \frac{2\cdot(5-1)}{5+2} - \frac{2\cdot(5-3)}{5+1} - \frac{2}{5} + \frac{2}{3} = \frac{85}{21} > \frac{113}{28} = H(U_{10,5}), \end{aligned}$$

and hence the assertion of the theorem holds.

Now suppose that $G' \ncong U_6, U_8$. Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

$$\begin{aligned} H(G) &\geq H(G') + \frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t} + \frac{2}{3} \\ &\geq \left(\frac{2(m-1)}{(m-1)+3} + \frac{2}{(m-1)+2} + \frac{2[(m-1)-2]}{3} + \frac{1}{2} \right) \\ &\quad + \frac{2[(m+1)-1]}{(m+1)+2} - \frac{2[(m+1)-3]}{(m+1)+1} - \frac{2}{m+1} + \frac{2}{3} \end{aligned}$$

$$= \frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equalities if and only if $G' \cong U_{2(m-1),m-1}$ and $t = m+1$, i.e., $G \cong U_{2m,m}$. This proves Case 1.

Case 2. G is a unicyclic graph with maximum degree 3 obtained by attaching $2m-p$ pendent vertices to some vertices of a cycle C_p ($m \le p \le 2m-1$).

If $m = 3$, then G is either the unicyclic graph obtained by attaching a pendent vertex to one vertex of C_5 or the unicyclic graph obtained by attaching a pendent vertex to two adjacent vertices of C_4 (since we assume $G \ncong U_6$). Then we have $H(G) \ge \frac{79}{30} > \frac{77}{30} = H(U_{6,3})$, and the theorem holds. So we may assume that $m \ge 4$. We consider two subcases according to the value of p.

Subcase 2.1. $p = m$.

Then every vertex of C_p is attached by a pendent vertex and $H(G) = \frac{5m}{6}$. Let $f(x) = \frac{5x}{6} - \left(\frac{2x}{x+3} + \frac{2}{x+2} + \frac{2(x-2)}{3} + \frac{1}{2}\right) = \frac{x}{6} + \frac{6}{x+3} - \frac{2}{x+2} - \frac{7}{6}$. For $x \ge 4$, we have

$$f'(x) = \frac{1}{6} - \frac{6}{(x+3)^2} + \frac{2}{(x+2)^2} \ge \frac{1}{6} - \frac{6}{(4+3)^2} + \frac{2}{(x+2)^2} > 0.$$

This implies that $f(x)$ is increasing for $x \ge 4$, and thus $f(m) \ge f(4) = \frac{1}{42} > 0$, i.e., $H(G) > H(U_{2m,m})$.

Subcase 2.2. $m+1 \le p \le 2m-1$.

In this subcase, there exists at least one edge, say xy, on C_p such that $xy \in M$. Then $d(x) = d(y) = 2$; for otherwise, the pendent vertex adjacent to x or y can not be M-saturated. Let z be the neighbor of x different from y in G, and let $G'' = G - xz + yz$. Then $G'' \in \mathscr{U}_{2m,m} \setminus \{U_8\}$. By Lemma 3, we have $H(G) > H(G'')$. Comparing with the graph G, we see that the length of the unique cycle in G'' decreases by 1. Repeating this operation from G to G'', we eventually obtain the unicyclic graph described in Subcase 2.1 and the result holds. This finishes the proof of the theorem. □

Since $H(U_{6,3}) = \frac{77}{30} > \frac{5}{2} = H(U_6)$ and $H(U_{8,4}) = \frac{139}{42} > \frac{347}{105} = H(U_8)$, by Theorem 1, we immediately obtain the following two results.

Corollary 1. *Let $G \in \mathscr{U}_{6,3}$, then $H(G) \ge \frac{5}{2}$ with equality if and only if $G \cong U_6$.*

Corollary 2. *Let $G \in \mathscr{U}_{8,4}$, then $H(G) \ge \frac{347}{105}$ with equality if and only if $G \cong U_8$.*

We now prove the main result of this section.

Theorem 2. *Let $G \in \mathscr{U}_{n,m} \setminus \{U_6, U_8\}$, where $2 \le m \le \lfloor \frac{n}{2} \rfloor$. Then*

$$H(G) \ge \frac{2m}{n-m+3} + \frac{2(n-2m+1)}{n-m+2} + \frac{2(m-2)}{3} + \frac{1}{2}$$

with equality if and only if $G \cong U_{n,m}$.

Proof. We prove Theorem 2 by induction on n. If $n = 2m$, then by Theorem 1, the assertion of the theorem holds. So we may assume that $n > 2m$ and the result holds for graphs in $\mathscr{U}_{n-1,m} \setminus \{U_6, U_8\}$. By Lemma 6, since C_n is the unique unicyclic graph on n vertices with the maximum harmonic index, we may also assume that $G \ncong C_n$. Then by Lemma 5, there exists a maximum matching M and a pendant vertex u in G such that u is not M-saturated. Let v be the unique neighbor of u with $d(v) = s \geq 2$, and let $G' = G - u$. Then $G' \in \mathscr{U}_{n-1,m}$. Since M contains exactly one edge incident with v and there are $n-m$ edges of G outside M, we have $s \leq n-m+1$. Let r be the number of pendant neighbors of v in G, where $1 \leq r \leq s-1$. Note that at least $r-1$ pendant neighbors of v are not M-saturated, and there are $n-2m$ vertices are not M-saturated in G. Then $r \leq n-2m+1$.

If $G' \cong U_6$, then $n = 7$, $m = 3$ and either $G \cong W_2$ or $G \cong W_3$ (see Figure 2). Since $H(W_2) = \frac{46}{15} > H(W_3) = \frac{284}{105} > \frac{113}{42} = H(U_{7,3})$, we see that the result holds.

If $G' \cong U_8$, then $n = 9$, $m = 4$ and $s \leq 5$. By Lemma 1(ii) (with $k = n-2m+1 = 2$) and Lemma 2(ii), we have

$$\begin{aligned} H(G) &\geq H(U_8) + \frac{2(s-2)}{s+2} + \frac{2(4-s)}{s+1} - \frac{2}{s} \\ &\geq \frac{347}{105} + \frac{2\cdot(5-2)}{5+2} + \frac{2\cdot(4-5)}{5+1} - \frac{2}{5} = \frac{24}{7} > \frac{143}{42} = H(U_{9,4}), \end{aligned}$$

and thus the assertion of the theorem holds.

Therefore we may assume that $G' \ncong U_6, U_8$. Then by Lemma 1(ii) (with $k = n-2m+1$), Lemma 2(ii) and the induction hypothesis, we conclude that

$$\begin{aligned} H(G) &\geq H(G') + \frac{2[s-(n-2m+1)]}{s+2} + \frac{2[2(n-2m+1)-s]}{s+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{s} \\ &\geq \left(\frac{2m}{(n-1)-m+3} + \frac{2[(n-1)-2m+1]}{(n-1)-m+2} + \frac{2(m-2)}{3} + \frac{1}{2} \right) \\ &\quad + \frac{2[(n-m+1)-(n-2m+1)]}{(n-m+1)+2} + \frac{2[2(n-2m+1)-(n-m+1)]}{(n-m+1)+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{n-m+1} \\ &= \frac{2m}{n-m+3} + \frac{2(n-2m+1)}{n-m+2} + \frac{2(m-2)}{3} + \frac{1}{2} \end{aligned}$$

with equalities if and only if $G' \cong U_{n-1,m}$, $s = n-m+1$ and $r = n-2m+1$, i.e., $G \cong U_{n,m}$. This completes the proof of the theorem. □

By applying Theorem 2, we can also obtain the minimum harmonic index for graphs in $\mathscr{U}_n$ ($n \geq 4$). This is one of the main results in [29].

Corollary 3. *Let $G \in \mathscr{U}_n$ with $n \geq 4$. Then*

$$H(G) \geq \frac{4}{n+1} + \frac{2(n-3)}{n} + \frac{1}{2}$$

with equality if and only if $G \cong U_{n,2}$.

Proof. Let M be a maximum matching in G, then $2 \leq |M| = m \leq \lfloor \frac{n}{2} \rfloor$ (since $n \geq 4$). If $m = 2$, then by Theorem 2, we have

$$\begin{aligned} H(G) &\geq \frac{2\cdot 2}{n-2+3} + \frac{2(n-2\cdot 2+1)}{n-2+2} + \frac{2\cdot(2-2)}{3} + \frac{1}{2} \\ &= \frac{4}{n+1} + \frac{2(n-3)}{n} + \frac{1}{2} \end{aligned}$$

with equality if and only if $G \cong U_{n,2}$. So we may assume that $m \geq 3$.

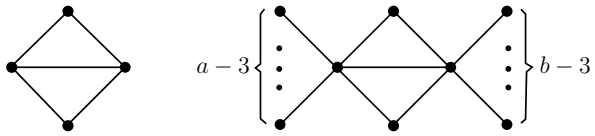
If $G \cong U_6$, then $H(G) = \frac{5}{2} > \frac{29}{14} = H(U_{6,2})$, we see that the result holds. If $G \cong U_8$, then $H(G) = \frac{347}{105} > \frac{79}{36} = H(U_{8,2})$, and the result also holds. Now suppose that $G \ncong U_6, U_8$. Then by Theorem 2 and Lemma 3, we see that $H(G) \geq H(U_{n,m}) > H(U_{n,m-1}) > \cdots > H(U_{n,2})$. So the assertion of the corollary holds. □

4. Minimum harmonic index for bicyclic graphs with given matching number

Let $\mathscr{B}_n$ be the set of bicyclic graphs with $n \geq 4$ vertices, and let $\mathscr{B}_{n,m}$ be the set of bicyclic graphs with n vertices and matching number m, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In this section, we present the minimum harmonic index for graphs in $\mathscr{B}_{n,m}$, and characterize the corresponding extremal graphs.

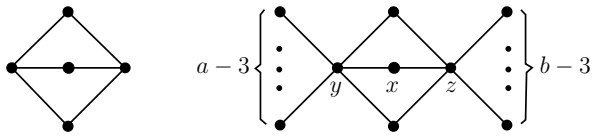
We denote by $\tilde{\mathscr{B}}_n$ the set of bicyclic graphs with $n \geq 4$ vertices containing no pendent vertices. Let $\mathscr{B}_n^1$ be the set of bicyclic graphs on $n \geq 6$ vertices obtained by connecting two vertex-disjoint cycles by a new edge, and let $\mathscr{B}_n^2$ be the set of bicyclic graphs on $n \geq 7$ vertices obtained by connecting two vertex-disjoint cycles by a path of length at least two. Let $\mathscr{B}_n^3$ be the set of bicyclic graphs on $n \geq 5$ vertices obtained by identifying a vertex of a cycle and a vertex of the other cycle. Let $\mathscr{B}_n^4$ be the set of bicyclic graphs on $n \geq 4$ obtained from C_n by adding a new edge, and let $\mathscr{B}_n^5$ be the set of bicyclic graphs on $n \geq 5$ obtained by connecting two non-adjacent vertices by a path of length at least two. Clearly, $\tilde{\mathscr{B}}_n = \bigcup_{i=1}^5 \mathscr{B}_n^i$.

For $i = 4, 5$, we use B_i to denote the unique bicyclic graph on i vertices in $\mathscr{B}_n^i$. Let $B_{n,a,b}$ be the bicyclic graph on n vertices obtained by attaching $a-3$ and $b-3$ pendent vertices to the two vertices of degree 3 of B_4, respectively, where $a \geq b \geq 3$ and $a+b = n+2$. Let $B'_{n,a,b}$ be the bicyclic graph on n vertices obtained by attaching $a-3$ and $b-3$ pendent vertices to the two vertices of degree 3 of B_5, respectively, where $a \geq b \geq 3$ and $a+b = n+1$. Then $B_4 \cong B_{4,3,3}$ and $B_5 \cong B'_{5,3,3}$. See Figure 3 and Figure 4 for an illustration. We first determine the minimum harmonic index for graphs in $\mathscr{B}_n$ with matching number 2.



B_4 $B_{n,a,b}$

FIGURE 3. The graphs B_4 and $B_{n,a,b}$.



B_5 $B'_{n,a,b}$

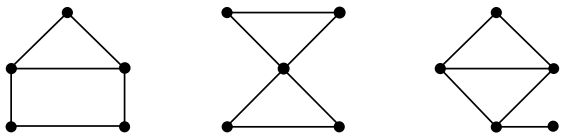
FIGURE 4. The graphs B_5 and $B'_{n,a,b}$.

Theorem 3. *Let $G \in \mathscr{B}_{n,2}$ with $n \geq 4$. Then*

$$H(G) \geq \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}$$

with equality if and only if $G \cong B_{n,n-1,3}$.

Proof. Since B_4 is the unique bicyclic graph on 4 vertices in $\mathscr{B}_{4,2}$, we see that the result holds for $n = 4$. If $n = 5$, then $G \in \{F_i | 1 \leq i \leq 3\} \cup B_5 \cup B_{5,4,3}$, where F_i $(1 \leq i \leq 3)$ are shown in Figure 5. It is easy to calculate that $H(F_1) = \frac{73}{30} > H(B_5) = \frac{12}{5} > H(F_2) = \frac{7}{3} > H(F_3) = \frac{23}{10} > \frac{226}{105} = H(B_{5,4,3})$, and hence the assertion of the theorem holds. So we may assume that $n \geq 6$. We consider three cases according to the structure of G.



F_1 F_2 F_3

FIGURE 5. The graphs F_1, F_2 and F_3.

Case 1. $G \cong B_{n,a,b}$, where $a \geq b \geq 3$ and $a+b = n+2$.

Let $f(x) = \frac{4}{x+1} - \frac{8}{x}$. For $x \geq 3$, we have

$$f''(x) = \frac{8}{(x+1)^3} - \frac{16}{x^3} = \frac{-8(x^3+6x^2+6x+2)}{x^3(x+1)^3} < 0.$$

This implies that $f(x+1) - f(x)$ is decreasing for $x \geq 3$. Suppose $a \geq b \geq 4$. Then

$$\begin{aligned}
&H(B_{n,a+1,b-1}) - H(B_{n,a,b}) \\
&= \left(\frac{4}{(a+1)+2} + \frac{2[(a+1)-3]}{(a+1)+1} + \frac{4}{(b-1)+2} + \frac{2[(b-1)-3]}{(b-1)+1}\right. \\
&\quad \left. + \frac{2}{(a+1)+(b-1)}\right) - \left(\frac{4}{a+2} + \frac{2(a-3)}{a+1} + \frac{4}{b+2} + \frac{2(b-3)}{b+1} + \frac{2}{a+b}\right) \\
&= \left(\frac{4}{a+3} - \frac{12}{a+2} + \frac{8}{a+1}\right) - \left(\frac{4}{b+2} - \frac{12}{b+1} + \frac{8}{b}\right) \\
&= [f(a+2) - f(a+1)] - [f(b+1) - f(b)] < 0,
\end{aligned}$$

i.e., $H(B_{n,a,b}) > H(B_{n,a+1,b-1})$ for $a \geq b \geq 4$. So we conclude that $H(B_{n,a,b}) \geq H(B_{n,n-1,3})$ with equality if and only if $a = n-1$ and $b = 3$.

Case 2. G is the bicyclic graph obtained by attaching $n-4$ pendent vertices to one vertex of degree 2 of B_4.

Then

$$\begin{aligned}
&H(G) - H(B_{n,n-1,3}) \\
&= \left(\frac{4}{n+1} + \frac{2(n-4)}{n-1} + \frac{4}{5} + \frac{1}{3}\right) - \left(\frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}\right) \\
&= \frac{8}{n} - \frac{2}{n+2} - \frac{6}{n-1} + \frac{1}{3} = \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{n(n-1)} + \frac{1}{3} \\
&\geq \left(\frac{2}{n} - \frac{2}{n+2}\right) - \frac{6}{6\cdot(6-1)} + \frac{1}{3} > 0.
\end{aligned}$$

So Case 2 holds.

Case 3. $G \cong B'_{n,a,b}$, where $a \geq b \geq 3$ and $a+b = n+1$.

Let x be one vertex of degree 2, and let y, z be the two vertices of degree at least 3 in G, see Figure 4. Let $G' = G - xz + yz$, then $G' \cong B_{n,a+1,b}$. By Lemma 3, we have $H(G) > H(G')$. Hence by the argument in Case 1, we deduce that $H(G) > H(B_{n,n-1,3})$. This completes the proof of the theorem. □

The following lemma was proved by Zhu, Liu and Wang [33], which will be used in the following argument.

Lemma 7. *Let $G \in \mathscr{B}_{n,m}$ $(n > 2m \geq 6)$ and G contains at least one pendent vertex. Then there exists a maximum matching M and a pendent vertex u in G such that u is not M-saturated.*

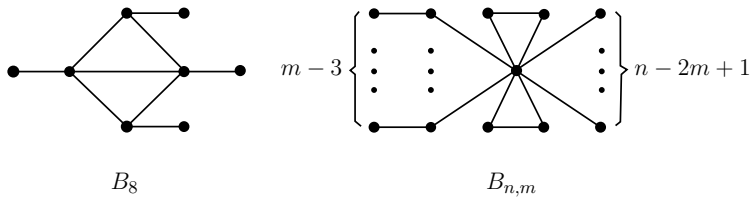


FIGURE 6. The graphs B_8 and $B_{n,m}$.

Let B_8 be the bicyclic graph on 8 vertices obtained by attaching a pendent vertex to every vertex of B_4. For $3 \le m \le \lfloor \frac{n}{2} \rfloor$, we use $B_{n,m}$ to denote the bicyclic graph on n vertices obtained by attaching $n-2m+1$ pendent vertices and $m-3$ paths on two vertices to the vertex of degree 4 of F_2, see Figure 6.

Lemma 8. *Let $G \in \mathscr{B}_{2m,m} \setminus \{B_8\}$ $(m \ge 3)$ and no pendent vertex has neighbor of degree 2. Then*

$$H(G) \ge \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1$$

with equality if and only if $G \cong B_{6,3}$.

Proof. Let M be a maximum matching in G, then $|M| = m$ and every vertex in G is adjacent to at most one pendent vertex. Since $G \in \mathscr{B}_{2m,m} \setminus \{B_8\}$ and no pendent vertex has neighbor of degree 2, we see that G can be obtained by attaching some pendent vertices to a bicyclic graph $\tilde{G} \in \tilde{\mathscr{B}}_k$ $(m \le k \le 2m)$. We consider two cases according to G contains vertices of degree 2 or not.

Case 1. There is no vertex of degree 2 in G.

Then either $k = m$ or $k = m+1$. If $k = m$, then G can be obtained by attaching a pendent vertex to every vertex of a bicyclic graph $\tilde{G} \in \tilde{\mathscr{B}}_m$. If $k = m+1$, then G can be obtained by attaching a pendent vertex to every vertex of degree 2 of a bicyclic graph $\tilde{G} \in \mathscr{B}^1_{m+1} \cup \mathscr{B}^4_{m+1}$.

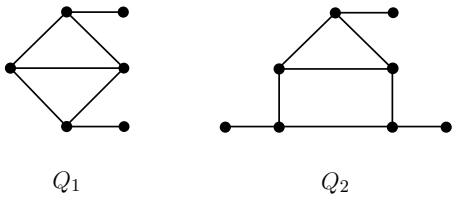


FIGURE 7. The graphs Q_1 and Q_2.

If $m = 3$, then $\tilde{G} \cong B_4$ and $G \cong Q_1$ (see Figure 7). Since $H(Q_1) = \frac{8}{3} > \frac{52}{21} = \frac{2\cdot(3+1)}{3+4} + \frac{2}{3+3} + \frac{2\cdot(3-3)}{3} + 1$, we know that the lemma holds.

If $m = 4$, since we assume $G \ncong B_8$, we have $\tilde{G} \cong F_1$ and $G \cong Q_2$ (see Figure 7). So the assertion of the lemma holds because $H(Q_2) = \frac{7}{2} > \frac{269}{84} = \frac{2\cdot(4+1)}{4+4} + \frac{2}{4+3} + \frac{2\cdot(4-3)}{3} + 1$.

Now assume that $m \geq 5$. Then

$$H(G) = \begin{cases} \frac{5m}{6} - \frac{59}{420}, & \text{if } \tilde{G} \in \mathscr{B}_m^1 \cup \mathscr{B}_m^4, \\ \frac{5m}{6} - \frac{16}{105}, & \text{if } \tilde{G} \in \mathscr{B}_m^2 \cup \mathscr{B}_m^5, \\ \frac{5m}{6} - \frac{1}{6}, & \text{if } \tilde{G} \in \mathscr{B}_m^3, \\ \frac{5m}{6} + \frac{1}{6}, & \text{if } \tilde{G} \in \mathscr{B}_{m+1}^1 \cup \mathscr{B}_{m+1}^4. \end{cases}$$

Let $f(x) = \left(\frac{5x}{6} - \frac{1}{6}\right) - \left(\frac{2(x+1)}{x+4} + \frac{2}{x+3} + \frac{2(x-3)}{3} + 1\right) = \frac{x}{6} + \frac{6}{x+4} - \frac{2}{x+3} - \frac{7}{6}$. For $x \geq 5$, we have

$$f'(x) = \frac{1}{6} - \frac{6}{(x+4)^2} + \frac{2}{(x+3)^2} \geq \frac{1}{6} - \frac{6}{(5+4)^2} + \frac{2}{(x+3)^2} > 0.$$

This implies that $f(x)$ is increasing for $x \geq 5$, and thus $f(m) \geq f(5) = \frac{1}{12} > 0$, i.e., $H(G) > \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1$.

Case 2. There exists a vertex, say u, of degree 2 in G.

Let v and w be the two neighbors of u in G such that $d(v) = s \geq 2$ and $d(w) = t \geq 2$. By the symmetry between v and w, we may assume that $uv \in M$.

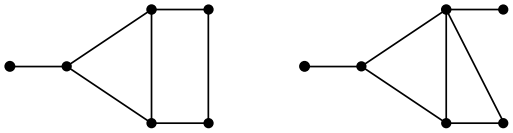
Suppose that no vertex of degree 2 is contained in the cycles of G. Since no pendent vertex has neighbor of degree 2 in G, we conclude that $\tilde{G} \in \mathscr{B}_k^2$ and u lies on the path connecting two vertex-disjoint cycles of G. Hence $vw \notin E(G)$. Let $G' = G - uw + vw$, then $G' \in \mathscr{B}_{2m,m} \setminus \{B_8\}$. By Lemma 3, we have $H(G) > H(G')$. Comparing with the graph G, we see that the number of vertices of degree 2 in G' decreases by 1. Repeating this operation from G to G', we finally obtain a bicyclic graph described in Case 1, and hence the result holds.

So we may choose u such that u lies on some cycle of G. Let $N(w) = \{w_0 = u, w_1, \ldots, w_{t-1}\}$, and let $G'' = G - uw$. Then G'' is a unicyclic graph on $2m$ vertices with a perfect matching M, i.e., $G'' \in \mathscr{U}_{2m,m}$. Note that $2 \leq s, t \leq 5$ and w is adjacent to at most one pendent vertex. Since $\frac{2}{s+2} - \frac{2}{s+1}$ is increasing for $s \geq 2$, $\frac{2}{t+x} - \frac{2}{t-1+x}$ is increasing for $x \geq 1$ and by Lemma 2(i), we have

$$\begin{aligned} H(G) &= H(G'') + \sum_{i=1}^{t-1} \left(\frac{2}{t + d(w_i)} - \frac{2}{t - 1 + d(w_i)}\right) + \frac{2}{t+2} + \left(\frac{2}{s+2} - \frac{2}{s+1}\right) \\ &\geq H(G'') + \left(\frac{2}{t+1} - \frac{2}{t}\right) + (t-2)\left(\frac{2}{t+2} - \frac{2}{t+1}\right) + \frac{2}{t+2} \\ &\quad + \left(\frac{2}{2+2} - \frac{2}{2+1}\right) \\ &= H(G'') + \left(\frac{2(t-1)}{t+2} - \frac{2(t-3)}{t+1} - \frac{2}{t}\right) - \frac{1}{6} \end{aligned}$$

$$\geq H(G'') + \left(\frac{2\cdot(5-1)}{5+2} - \frac{2\cdot(5-3)}{5+1} - \frac{2}{5}\right) - \frac{1}{6}$$
$$= H(G'') - \frac{19}{210} \qquad (*)$$

with equalities if and only if $s = 2$, $t = 5$, one neighbor of w has degree 1 and the other neighbors of w have degree 2.



R_1 R_2

FIGURE 8. The graphs R_1 and R_2.

If $G'' \cong U_6$, then either $G'' \cong R_1$ or $G'' \cong R_2$ (see Figure 8). Since $H(R_1) = \frac{14}{5} > H(R_2) = \frac{533}{210} > \frac{52}{21} = \frac{2\cdot(3+1)}{3+4} + \frac{2}{3+3} + \frac{2\cdot(3-3)}{3} + 1$, the assertion of the lemma holds. If $G'' \cong U_8$, then by $(*)$, we have

$$H(G) \geq H(U_8) - \frac{19}{210} = \frac{347}{105} - \frac{19}{210} = \frac{45}{14}$$
$$> \frac{269}{84} = \frac{2\cdot(4+1)}{4+4} + \frac{2}{4+3} + \frac{2\cdot(4-3)}{3} + 1,$$

and the result holds. So suppose that $G'' \ncong U_6, U_8$. It follows from Lemma 2(i) that

$$\frac{2[(m+2)-1]}{(m+2)+2} - \frac{2[(m+2)-3]}{(m+2)+1} - \frac{2}{m+2}$$
$$\leq \frac{2\cdot[(3+2)-1]}{(3+2)+2} - \frac{2\cdot[(3+2)-3]}{(3+2)+1} - \frac{2}{3+2} = \frac{8}{105}$$

since $m \geq 3$. Then by $(*)$ and Theorem 1, we have

$$H(G) \geq H(G'') - \frac{19}{210}$$
$$\geq \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-2)}{3} + \frac{1}{2}\right) - \frac{19}{210}$$
$$= \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-3)}{3} + 1\right) + \frac{8}{105}$$
$$\geq \left(\frac{2m}{m+3} + \frac{2}{m+2} + \frac{2(m-3)}{3} + 1\right)$$

$$+\left(\frac{2[(m+2)-1]}{(m+2)+2}-\frac{2[(m+2)-3]}{(m+2)+1}-\frac{2}{m+2}\right)$$
$$=\frac{2(m+1)}{m+4}+\frac{2}{m+3}+\frac{2(m-3)}{3}+1$$

with equalities if and only if $s=2$, $t=5$, $G''\cong U_{2m,m}$ and $m=3$, i.e., $G\cong B_{6,3}$. This finishes the proof of the lemma. □

Theorem 4. *Let $G\in\mathscr{B}_{2m,m}\setminus\{B_8\}$, where $m\geq 3$. Then*

$$H(G)\geq\frac{2(m+1)}{m+4}+\frac{2}{m+3}+\frac{2(m-3)}{3}+1$$

with equality if and only if $G\cong B_{2m,m}$.

Proof. We prove Theorem 4 by induction on m. If $m=3$, then by Lemma 7, we may assume that there exists a pendent vertex in G whose neighbor is a vertex of degree 2. Hence G is the bicyclic graph obtained from B_4 by attaching a path on two vertices to either one vertex of degree 3 or one vertex of degree 2. Then we have $H(G)\geq\frac{289}{105}>\frac{52}{21}=H(B_{6,3})$, and the assertion of the theorem holds. So we may assume that $m\geq 4$ and the result holds for graphs in $\mathscr{B}_{2(m-1),m-1}\setminus\{B_8\}$. Let M be a maximum matching in G, then $|M|=m$. If no pendent vertex has neighbor of degree 2 in G, then by Lemma 7, we see that the result holds.

Now suppose that there exists a pendent vertex u in G whose neighbor v is a vertex of degree 2. Let w be the neighbor of v different from u with $d(w)=t\geq 2$, and let $G'=G-u-v$. Then $uv\in M$ and $G'\in\mathscr{B}_{2(m-1),m-1}$. Since M contains exactly one edge incident with w and there are $m+1$ edges of G outside M, we have $t\leq m+2$. Note that w is adjacent to at most one pendent vertex in G.

If $G'\cong B_8$, then $t\leq 5$. By Lemma 1(i) and Lemma 2(i), we have

$$H(G)\geq H(B_8)+\frac{2(t-1)}{t+2}-\frac{2(t-3)}{t+1}-\frac{2}{t}+\frac{2}{3}$$
$$\geq\frac{447}{140}+\frac{2\cdot(5-1)}{5+2}-\frac{2\cdot(5-3)}{5+1}-\frac{2}{5}+\frac{2}{3}=\frac{551}{140}>\frac{47}{12}=H(U_{10,5}),$$

and hence the assertion of the theorem holds.

So we may further assume that $G'\ncong B_8$. Then by Lemma 1(i), Lemma 2(i) and the induction hypothesis, we conclude that

$$H(G)\geq H(G')+\frac{2(t-1)}{t+2}-\frac{2(t-3)}{t+1}-\frac{2}{t}+\frac{2}{3}$$
$$\geq\left(\frac{2[(m-1)+1]}{(m-1)+4}+\frac{2}{(m-1)+3}+\frac{2[(m-1)-3]}{3}+1\right)$$
$$+\frac{2[(m+2)-1]}{(m+2)+2}-\frac{2[(m+2)-3]}{(m+2)+1}-\frac{2}{m+2}+\frac{2}{3}$$

$$= \frac{2(m+1)}{m+4} + \frac{2}{m+3} + \frac{2(m-3)}{3} + 1$$

with equalities if and only if $G' \cong B_{2(m-1),m-1}$ and $t = m+2$, i.e., $G \cong B_{2m,m}$. So Theorem 4 holds. □

Since $H(B_{8,4}) = \frac{269}{84} > \frac{447}{140} = H(B_8)$, by Theorem 4, we immediately obtain the following result.

Corollary 4. *Let $G \in \mathscr{B}_{8,4}$, then $H(G) \geq \frac{447}{140}$ with equality if and only if $G \cong B_8$.*

We now present the minimum harmonic index for graphs in $\mathscr{B}_{n,m} \setminus \{B_8\}$, where $3 \leq m \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 5. *Let $G \in \mathscr{B}_{n,m} \setminus \{B_8\}$, where $3 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$H(G) \geq \frac{2(m+1)}{n-m+4} + \frac{2(n-2m+1)}{n-m+3} + \frac{2(m-3)}{3} + 1$$

with equality if and only if $G \cong B_{n,m}$.

Proof. We prove the theorem by induction on n. If $n = 2m$, then by Theorem 4, the assertion of the theorem holds. So we may assume that $n > 2m$ and the result holds for graphs in $\mathscr{B}_{n-1,m} \setminus \{B_8\}$. If there is no pendent vertex in G, then $G \in \tilde{\mathscr{B}}_n$ and $n = 2m+1$. It is easy to check that

$$H(G) = \begin{cases} m + \frac{13}{30}, & \text{if } G \in \mathscr{B}^1_{2m+1} \cup \mathscr{B}^4_{2m+1}, \\ m + \frac{2}{5}, & \text{if } G \in \mathscr{B}^2_{2m+1} \cup \mathscr{B}^5_{2m+1}, \\ m + \frac{1}{3}, & \text{if } G \in \mathscr{B}^3_{2m+1}. \end{cases}$$

This implies that

$$\begin{aligned} &H(G) - H(B_{2m+1,m}) \\ &\geq \left(m + \frac{1}{3}\right) - \left(\frac{2(m+1)}{(2m+1)-m+4} + \frac{2[(2m+1)-2m+1]}{(2m+1)-m+3} + \frac{2(m-3)}{3} + 1\right) \\ &= \frac{m}{3} + \frac{8}{m+5} - \frac{4}{m+4} - \frac{2}{3} = \frac{m-2}{3} + \frac{4(m+3)}{(m+4)(m+5)} > 0, \end{aligned}$$

i.e., $H(G) > H(B_{2m+1,m})$.

So we may assume that G contains at least one pendent vertex. Then by Lemma 7, there exists a maximum matching M and a pendent vertex u in G such that u is not M-saturated. Let v be the unique neighbor of u with $d(v) = s \geq 2$, and let $G' = G - u$. Then $G' \in \mathscr{B}_{n-1,m}$. Since M contains exactly one edge incident with v and there are $n+1-m$ edges of G outside M, we have $s \leq n-m+2$. Let r be the number of pendant neighbors of v in G, where $1 \leq r \leq s-1$. Note that at least $r-1$ pendant neighbors of v are not M-saturated, and there are $n-2m$ vertices are not M-saturated in G. Then $r \leq n-2m+1$.

If $G' \cong B_8$, then $n = 9$, $m = 4$ and $s \le 5$. By Lemma 1(ii) (with $k = n-2m+1 = 2$) and Lemma 2(ii), we deduce that

$$\begin{aligned} H(G) &\ge H(B_8) + \frac{2(s-2)}{s+2} + \frac{2(4-s)}{s+1} - \frac{2}{s} \\ &\ge \frac{447}{140} + \frac{2\cdot(5-2)}{5+2} + \frac{2\cdot(4-5)}{5+1} - \frac{2}{5} = \frac{1393}{420} > \frac{59}{18} = H(B_{9,4}), \end{aligned}$$

and hence the assertion of the theorem holds.

Therefore we may assume that $G' \ncong B_8$. Then by Lemma 1(ii) (with $k = n - 2m + 1$), Lemma 2(ii) and the induction hypothesis, we have

$$\begin{aligned} H(G) &\ge H(G') + \frac{2[s-(n-2m+1)]}{s+2} + \frac{2[2(n-2m+1)-s]}{s+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{s} \\ &\ge \left(\frac{2(m+1)}{(n-1)-m+4} + \frac{2[(n-1)-2m+1]}{(n-1)-m+3} + \frac{2(m-3)}{3} + 1 \right) \\ &\quad + \frac{2[(n-m+2)-(n-2m+1)]}{(n-m+2)+2} + \frac{2[2(n-2m+1)-(n-m+2)]}{(n-m+2)+1} \\ &\quad - \frac{2[(n-2m+1)-1]}{n-m+2} \\ &= \frac{2(m+1)}{n-m+4} + \frac{2(n-2m+1)}{n-m+3} + \frac{2(m-3)}{3} + 1 \end{aligned}$$

with equalities if and only if $G' \cong B_{n-1,m}$, $s = n-m+2$ and $r = n-2m+1$, i.e., $G \cong B_{n,m}$. This completes the proof of the theorem. □

We can also determine the minimum harmonic index for graphs in $\mathscr{B}_n$ (see also in [31]) by using Theorem 3 and Theorem 5.

Corollary 5. *Let $G \in \mathscr{B}_n$ with $n \ge 4$. Then*

$$H(G) \ge \frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}$$

with equality if and only if $G \cong B_{n,n-1,3}$.

Proof. Let M be a maximum matching in G, then $2 \le |M| = m \le \lfloor \frac{n}{2} \rfloor$ (since $n \ge 4$). If $m = 2$, then the result follows immediately from Theorem 3.

If $m = 3$, then by Theorem 5, we have

$$\begin{aligned} H(G) &\ge \frac{2\cdot(3+1)}{n-3+4} + \frac{2(n-2\cdot 3+1)}{n-3+3} + \frac{2\cdot(3-3)}{3} + 1 \\ &= \frac{8}{n+1} + \frac{2(n-5)}{n} + 1 \end{aligned}$$

with equality if and only if $G \cong B_{n,3}$. Note that in this case $n \geq 6$. Since

$$\begin{aligned}
&H(B_{n,3}) - H(B_{n,n-1,3}) \\
&= \left(\frac{8}{n+1} + \frac{2(n-5)}{n} + 1\right) - \left(\frac{2}{n+2} + \frac{4}{n+1} + \frac{2(n-4)}{n} + \frac{4}{5}\right) \\
&= \left(\frac{4}{n+1} - \frac{2}{n+2} - \frac{2}{n}\right) + \frac{1}{5} = \frac{-4}{n(n+1)(n+2)} + \frac{1}{5} \\
&\geq \frac{-4}{6\cdot(6+1)\cdot(6+2)} + \frac{1}{5} = \frac{79}{420} > 0,
\end{aligned}$$

we know that the assertion of the corollary holds.

So we may assume that $m \geq 4$. If $G \cong B_8$, then $H(G) = \frac{447}{140} > \frac{22}{9} = H(B_{8,7,3})$, we see that Corollary 5 holds. Now suppose that $G \ncong B_8$. Then by Theorem 5 and Lemma 3, we see that $H(G) \geq H(B_{n,m}) > H(B_{n,m-1}) > \cdots > H(B_{n,3}) > H(B_{n,n-1,3})$. This finishes the proof of the corollary. □

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Doi: 10.18514/MMN.2015.1138

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 607–623

HU e-ISSN 1787-2413

NEW MODULAR FORM IDENTITIES ASSOCIATED TO GENERALIZED VERTEX OPERATOR ALGEBRAS

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Received 21 February, 2014

Abstract. New identities appearing from consideration of higher genus characters for generalized vertex operator algebras with a formal VOSA parameter associated to a local coordinate on a self-sewn Riemann surface are considered. Genus two version of twisted product Jacobi identity is reviewed. Further generalization of classical number theory identities for modular forms are proposed.

2010 *Mathematics Subject Classification:* 17B69; 30F10; 32A25; 11F03

Keywords: vertex operator superalgebras, intertwining operators, Riemann surfaces, automorphic forms, theta-functions, Jacobi product identities

1. INTRODUCTION

This is the a continuation of the papers [42,43] describing modular form identities obtained as a result of computation of characters for vertex operator algebra modules on genus two (and higher) Riemann surfaces [33–38]. In thise part we announce new modular form identities proved (in collaboration with M.P. Tuite) in [37] and obtained in the consideration of genus one and genus two characters for generalized vertex operator algebras with a formal VOSA parameter associated to a local coordinate on a self-sewn Riemann surface.

In [28, 34, 37] we have established modular properties of genus one and genus two partition and n-point functions for free fermionic vertex operator superalgebras. Modular properties of correlations functions for classes of vertex algebras on the torus were proven in [7, 29, 41], [21–28], [33–38] and others. In [34] we established that the genus two partition function for the rank two free fermion VOSA is modular invariant with respect to the group $G = (SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})) \rtimes \mathbb{Z}_2$, i.e.,

$$Z_{V_{\mathbb{Z}}}^{(2)}\left[\begin{array}{c} f^{(2)} \\ g^{(2)} \end{array}\right]\Bigg|\gamma = e_{\gamma}^{(2)}\left[\begin{array}{c} f^{(2)} \\ g^{(2)} \end{array}\right] Z_{V_{\mathbb{Z}}}^{(2)}\left[\begin{array}{c} f^{(2)} \\ g^{(2)} \end{array}\right],$$

where $|\gamma$ denotes G-action on the partition function, and $e_{\gamma}^{(2)}\left[\begin{array}{c} f^{(2)} \\ g^{(2)} \end{array}\right]$ is a multiplier system. Note that modular invariance of the genus two partition function can also

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inferred from using modular properties of the Riemann theta function and the Heisenberg genus two partition function described in [21]. Similarly, we can also obtain [34, 37] modular invariance for the two rank two free fermion n-point generating differentials $\mathcal{G}^{(2)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix}(x_1,y_1,\dots,x_n,y_n)$, (5.4) as well as for all n-point function differentials $\mathcal{F}^{(2)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix}((v_1,z_1),\dots,(v_n,z_n))$ (5.3) (see subsection 5.2 below) and extend this result to higher genus [34]. In more complicated cases at genus g [38] we can explicitly prove automorphic properties of generating functions by using their explicit representations following from vertex operator algebra computations.

In these notes we mention new identities for modular forms which are right hand sides of normalized (with respect to Heisenberg vertex operator algebra partition functions) partition and generating functions in the case of self-sewing of the torus to forma a genus two Riemann surface as well as some higher genus cases. For main definition related to vertex operator algebras and computation of their correlation functions see [3, 4, 6, 16, 26, 28].

2. Genus one identities

2.1. *Self-sewing of the torus*

Let M be the Heisenberg vertex operator algebra [16], $M\otimes e^{\alpha}$ be its irreducible M-module for some $\alpha\in\mathbb{C}$. The torus partition function is given by

$$Z^{(1)}_{\alpha}(q)=\mathrm{Tr}_{M\otimes e^{\alpha}}\left(q^{L(0)-1/24}\right)=\frac{q^{\frac{1}{2}\alpha^2}}{\eta(q)},$$

where $\eta(q)=q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta for modular parameter q. Thus

$$\frac{Z^{(1)}_{\alpha}(q)}{Z^{(1)}_{M}(\tau)}=q^{\frac{1}{2}\alpha^2},$$

where $Z^{(1)}_{M}(\tau)=1/\eta(q)$ is the genus one Heisenberg rank one partition function. Twisted torus n-point function is defined in [37] by

$$Z^{(1)}_{\alpha}\left(u_1\otimes e^{\beta_1},z_1;\dots;u_n\otimes e^{\beta_n},z_n;q\right) \tag{2.1}$$
$$=\mathrm{Tr}_{M\otimes e^{\alpha}}\left(\mathcal{Y}\left(q_1^{L(0)}(u_1\otimes e^{\beta_1}),q_1\right)\dots\mathcal{Y}\left(q_n^{L(0)}(u_n\otimes e^{\beta_n}),q_n\right)q^{L(0)-1/24}\right),$$

for formal $q_i=e^{z_i}$ with $i=1,\dots,n$. Since $e^{\beta\widehat{q}}M\otimes e^{\alpha}=M\otimes e^{\alpha+\beta}$ it follows that the n–point function vanishes when $\sum_{i=1}\beta_i\neq 0$.

We next describe a natural generalization of previous results in [21, 28]. Firstly, consider the n–point functions for n highest weight vectors $\mathbf{1}\otimes e^{\beta_i}$, which we abbreviate below to e^{β_i}, for $i=1,\dots,n$.

Proposition 1. *For* $\sum_{i=1}^{n}\beta_i = 0$ *then*

$$\frac{Z_\alpha^{(1)}\left(e^{\beta_1},z_1;\dots;e^{\beta_n},z_n;q\right)}{Z_M^{(1)}(\tau)} = q^{\frac{1}{2}\alpha^2}\exp\left(\alpha\sum_{i=1}^{n}\beta_i z_i\right)\prod_{1\le r<s\le n}K(z_{rs},\tau)^{\beta_r\beta_s}, \tag{2.2}$$

where $z_{rs} = z_r - z_s$ *and* $K(z,\tau)$ *is the genus one prime form.*

Similarly to [21], we may obtain a closed form for the general n–point function (2.1). In particular, we apply standard genus one Zhu recursion theory [34] to reduce (2.1) to an explicit multiple of (2.2) to find

Proposition 2. *For* $\sum_{i=1}^{n}\beta_i = 0$ *then*

$$\frac{Z_\alpha^{(1)}\left(u_1\otimes e^{\beta_1},z_1;\dots;u_n\otimes e^{\beta_n},z_n;q\right)}{Z_\alpha^{(1)}\left(e^{\beta_1},z_1;\dots;e^{\beta_n},z_n;q\right)} = Q_\alpha^{\beta_1,\dots,\beta_n}(u_1,z_1;\dots u_n,z_n;q),$$

where $Q_\alpha^{\beta_1,\dots,\beta_n}(u_1,z_1;\dots u_n,z_n;q)$ *is an explicit sum of elliptic and quasi-modular forms (see [21] for details).*

In Propositions 14 and 15 of [28] all torus orbifold n–point functions for vectors in $V_{\mathbb{Z}}$ are computed by means of a generating function. In that analysis we made use of an explicit Zhu reduction formula developed for a VOSA with real grading. However, in the present case we do not have an intertwining Zhu reduction formula because of the absence of a commutator formula for interwiners [6]. Instead, here we adopt an alternative approach for computing the intertwining n–point functions by exploiting the bosonized formalism. Thus for example, we find

$$\begin{aligned}\frac{Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}(e^{\kappa},w;e^{-\kappa},0;q)}{Z_M^{(1)}(\tau)} &= \sum_{\mu\in\mathbb{Z}+\alpha_1} e^{2\pi i\mu\beta_1}\frac{Z_\mu^{(1)}(e^{\kappa},w;e^{-\kappa},0;q)}{Z_M^{(1)}(\tau)}\\ &= \frac{\vartheta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)}{K(w,\tau)^{\kappa^2}},\end{aligned} \tag{2.3}$$

for genus one theta series. More generally, we define a generating function for all $n+2$–point functions by the following formal differential form

$$\mathcal{G}_n^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}(x_1,y_1,\dots,x_n,y_n) \tag{2.4}$$

$$\equiv Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}\left(\psi^+,x_1;\,\psi^-,y_1;\,\dots;\psi^+,x_n;\,\psi^-,y_n;e^{\kappa},w;\,e^{-\kappa},0;q\right)\cdot\prod_{i=1}^{n}dx_i^{\frac{1}{2}}dy_i^{\frac{1}{2}},$$

for $V_{\mathbb{Z}}$ generators $\psi^{\pm} = e^{\pm 1}$ alternatively inserted at x_i, y_i for $i = 1,\dots,n$.

Proposition 3. *The generating form* (2.1) *is given by*

$$\frac{\mathcal{G}_n^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(x_1,y_1,\dots,x_n,y_n)}{Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa},w;e^{-\kappa},0;q)} = \det S_{\kappa}, \tag{2.5}$$

where S_κ denotes the $n\times n$ matrix with components $S_\kappa\left(x_i,y_j\right)$ for $i,j=1,\dots,n$ for Szegő kernel.

Finally, we obtain the following generalization of Proposition 15 of [28] concerning the generating properties of (2.1).

Proposition 4. *For a pair of square bracket mode twisted Fock vectors*

$$\Psi_{\kappa}[\boldsymbol{k}_1,\boldsymbol{l}_2] = e^{\kappa\widehat{q}}\psi^{+}[-k_{11}]\dots\psi^{+}[-k_{1s_1}]\psi^{-}[-l_{21}]\dots\psi^{-}[-l_{2t_2}]\mathbf{1},$$
$$\Psi_{-\kappa}[\boldsymbol{k}_2,\boldsymbol{l}_1] = e^{-\kappa\widehat{q}}\psi^{+}[-k_{21}]\dots\psi^{+}[-k_{2s_2}]\psi^{-}[-l_{11}]\dots\psi^{-}[-l_{1t_1}]\mathbf{1},$$

for $p = s_1+s_2 = t_1+t_2 > 0$, the generating function $\mathcal{G}_m^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(x_1,y_1,\dots,x_m,y_m)$ for all torus orbifold intertwining $n+2$–point functions is given by

$$\frac{Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(\Psi_{\kappa}[\boldsymbol{k}_1,\boldsymbol{l}_2],w;\Psi_{-\kappa}[\boldsymbol{k}_2,\boldsymbol{l}_1],0;q)}{Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa},w;\ e^{-\kappa},0;q)} = \epsilon\ \det C_{ab}(\boldsymbol{k}_a,\boldsymbol{l}_b), \tag{2.6}$$

where

$$\epsilon = (-1)^{(t_1+s_2)t_2+\lfloor\frac{1}{2}p\rfloor}e^{i\pi B\kappa(s_2-t_1)},$$

for some odd integer B and

$$C_{ab}(\boldsymbol{k}_a,\boldsymbol{l}_b) = \begin{bmatrix} C_{11}(\boldsymbol{k}_1,\boldsymbol{l}_1) & C_{12}(\boldsymbol{k}_1,\boldsymbol{l}_2) \\ C_{21}(\boldsymbol{k}_2,\boldsymbol{l}_1) & C_{22}(\boldsymbol{k}_2,\boldsymbol{l}_2) \end{bmatrix},$$

is a $p\times p$ block matrix with components $C_{ab}(k_{ai_a},l_{bj_b})$ for $i_a = 1,\dots,s_a$ and $j_b = 1,\dots,t_b$ for $a,b=1,2$ for S_κ expansion coefficients [37].

3. Genus two identities

In [34, 43] we have proposed a way to derive higher genus generalizations of the classical identities for modular forms. In particular, in [34] we gave a genus two example of the Jacobi product identity.

3.1. *The self-sewing partition function*

The partition or 0–point function is expressible in terms of the basic genus one twisted 2-point function (2.3) and the infinite Szegő moment matrix $T = T(\tau, w, \rho|\theta_1, \theta_2, \phi_1, \kappa)$ of [37] as follows:

Theorem 1. *The genus two continuous orbifold partition function for $V_{\mathbb{Z}}$ on a Riemann surface in the ρ-sewing formalism is convergent on $\mathcal{D}^{\rho}$ and is given by*

$$\frac{Z^{(2)}_{V_{\mathbb{Z}}}\begin{bmatrix} f \\ g \end{bmatrix}(\tau, w, \rho)}{Z^{(1)}_{V_{\mathbb{Z}}}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}(e^{\kappa}, w;\, e^{-\kappa}, 0;\, q)} = e^{2i\pi\beta_2\kappa}(e^{i\pi B}\rho)^{\frac{1}{2}\kappa^2}\det(I - T). \tag{3.1}$$

3.2. *An identity for the Szegő kernel*

The "source" genus two Szegő kernel on the torus is defined by [35]

$$S_{\kappa}(x, y) = \left(\frac{\vartheta_1(x-w,\tau)\vartheta_1(y,\tau)}{\vartheta_1(x,\tau)\vartheta_1(y-w,\tau)}\right)^{\kappa} \frac{\vartheta\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}(x - y + \kappa w, \tau)}{\vartheta\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}(\kappa w, \tau)\, K(x - y, \tau)} dx^{\frac{1}{2}} dy^{\frac{1}{2}}, \tag{3.2}$$

for $\kappa \neq -\frac{1}{2}$ (with a different expression when $\kappa = -\frac{1}{2}$ given in [35]). Recall also the definition of the Szegő kernel moments [35, 37],

$$G_{ab}(k, l) = \rho^{\frac{1}{2}(k_a + l_b - 1)} C_{ab}(k, l), \tag{3.3}$$

$$h_a(x, k) = \rho^{\frac{1}{2}(k_a - \frac{1}{2})} d_a(x, k), \tag{3.4}$$

$$\overline{h}_a(y, k) = \rho^{\frac{1}{2}(k_a - \frac{1}{2})} \overline{d}_a(y, k), \tag{3.5}$$

with associated infinite matrix $G = (G_{ab}(k, l))$ and row vectors $h(x) = (h_a(x, k))$ and $\overline{h}(y) = (\overline{h}_a(y, k))$. Here $d_a(y, k)$ are half-order differentials

$$d_a(x, k) = \frac{1}{2\pi i} \oint_{\mathcal{C}_a(y_a)} y_a^{-k_a} S_{\kappa}(x, y_a) dy_a^{\frac{1}{2}}, \tag{3.6}$$

$$\overline{d}_a(y, k) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\overline{a}}(x_{\overline{a}})} x_{\overline{a}}^{-k_a} S_{\kappa}(x_{\overline{a}}, y) dx_{\overline{a}}^{\frac{1}{2}}, \tag{3.7}$$

and $C_{ab}(k, l)$ is the infinite block moment matrix

$$C_{ab}(k, l) = \frac{1}{(2\pi i)^2} \oint_{\mathcal{C}_{\overline{a}}(x_{\overline{a}})} \oint_{\mathcal{C}_b(y_b)} (x_{\overline{a}})^{-k_a} (y_b)^{-l_b} S_{\kappa}(x_{\overline{a}}, y_b)\, dx_{\overline{a}}^{\frac{1}{2}}\, dy_b^{\frac{1}{2}}. \tag{3.8}$$

Thus the genus two Szegő kernel on the self-sewn torus is given by

$$S^{(2)}(x, y) = S_{\kappa}(x, y) + \xi h(x) D^{\theta} (I - T)^{-1} \overline{h}^{T}(y), \tag{3.9}$$

where $\overline{h}^T(y)$ denotes the transpose of the infinite row vector $\overline{h}(y)$.

$$T = \xi G D^{\theta_2}, \tag{3.10}$$

$$D^{\theta_2}(k,l) = \begin{bmatrix} \theta_2^{-1} & 0 \\ 0 & -\theta_2 \end{bmatrix} \delta(k,l). \tag{3.11}$$

Using (3.9) we define matrices

$$S^{(2)} = \left(S^{(2)}(x_i, y_j)\right), \quad S_\kappa = \left(S_\kappa(x_i, y_j)\right), \tag{3.12}$$

$$H = \left((h(x_i))(k,a)\right), \quad \overline{H}^T = \left(\left(\overline{h}(y_i)\right)(l,b)\right)^T.$$

$S^{(2)}$ and S_κ are finite matrices indexed by $i, j = 1, \ldots, n$; H is semi-infinite with n rows indexed by i and columns indexed by $k \geq 1$ and $a = 1, 2$ and $\overline{H}^T$ is semi-infinite with rows indexed by $l \geq 1$ and $b = 1, 2$ and with n columns indexed by j. Similarly to Proposition 3 in [35] we have

Proposition 5.

$$\det \begin{bmatrix} S_\kappa & -\xi H D^{\theta_2} \\ \overline{H}^T & I - T \end{bmatrix} = \det S^{(2)} \det(I - T), \tag{3.13}$$

with T, D^{θ_2} of (3.10) *and* (3.11).

3.3. *The genus two n-point generating form*

Similarly to the genus one situation (2.1), we define a genus two continuous orbifold generating differential form by

$$\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \ldots, x_n, y_n) \tag{3.14}$$

$$= Z_{V_{\mathbb{Z}}}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\psi^+, x_1; \psi^-, y_1; \ldots, \psi^+, x_n; \psi^-, y_n; \tau, w, \rho) \prod_{i=1}^{n} dx_i^{\frac{1}{2}} dy_i^{\frac{1}{2}}.$$

Then we have

Theorem 2. *The generating form* (3.3) *is given by*

$$\frac{\mathcal{G}_n^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (x_1, y_1, \ldots, x_n, y_n)}{Z_{V_{\mathbb{Z}}}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho)} = \det S^{(2)}, \tag{3.15}$$

for $n \times n$ matrix $S^{(2)}$ of (3.12) *and $Z_{V_{\mathbb{Z}}}^{(2)} \begin{bmatrix} f \\ g \end{bmatrix} (\tau, w, \rho)$ is the genus two twisted partition function* (3.1).

Normalized relative to the genus two partition function, the 2-point function for ψ^+ and ψ^- is thus given by the Szegő kernel and more generally, the generating function is a Szegő kernel determinant. This agrees with the assumed form in [30] or as found by string theory methods using a Schottky parametrization in [5] and in the genus two ϵ-sewing scheme [35].

3.4. *Genus two Heisenberg partition function*

We may compute the genus two twisted partition and generating functions in an alternative way by use of a bosonic basis $u \otimes e^{\nu} \in V_{\mathbb{Z}+\kappa}$ for $u \in M$ and $\nu \in \mathbb{Z}+\kappa$. In particular, we can immediately exploit and extend the results for lattice VOAs in [24].

Let us firstly recall from [22, 24] the following definitions:

$$\begin{aligned} P_2(\tau,z) &= \wp(\tau,z) + E_2(\tau) \\ &= \frac{1}{z^2} + \sum_{k=2}^{\infty} (k-1) E_k(\tau) z^{k-2}, \end{aligned} \tag{3.16}$$

for Weierstrass function $\wp(\tau,z)$ and Eisenstein series $E_k(\tau)$. We define

$$P_{k+1}(\tau,z) = -\frac{1}{k}\partial_z P_k(\tau,z),$$

for $k \geq 2$ and for $k,l \geq 1$

$$\begin{aligned} C(k,l,\tau) &= (-1)^{k+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} E_{k+l}(\tau), \\ D(k,l,\tau,z) &= (-1)^{k+1} \frac{(k+l-1)!}{(k-1)!(l-1)!} P_{k+l}(\tau,z), \\ R_{ab}(k,l) &= -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \begin{bmatrix} D(k,l,\tau,w) & C(k,l,\tau) \\ C(k,l,\tau) & D(l,k,\tau,w) \end{bmatrix}. \end{aligned}$$

Theorems 5.1 and 5.6 of [24] tell us that:

Theorem 3. *The genus two normalized partition function for the Heisenberg VOA M in the self-sewing scheme is*

$$\frac{Z_M^{(2)}(\tau,w,\rho)}{Z_M^{(1)}(\tau)} = (\det(1-R))^{-1/2}, \tag{3.17}$$

where $Z_M^{(2)}(\tau,w,\rho)$ is holomorphic on $\mathcal{D}^{\rho}$.

3.5. *Lattice VOA*

Theorem 6.1 of [24] concerns the genus two partition function for a lattice VOA. We obtain a natural generalization for a twisted genus two partition function defined by

$$Z^{(2)}_{\mu,\nu}(\tau,w,\rho)=\sum_{\Psi_\nu\in M\otimes e^\nu} Z^{(1)}_{\mu}\left(\Psi_\nu,w;\overline{\Psi}_\nu,0;q\right), \tag{3.18}$$

for $\mu,\nu\in\mathbb{C}$ where the sum is taken over a $M\otimes e^\nu$ basis, $\overline{\Psi}_\nu$ is the square bracket dual (with respect to a non-generate form, see [34, 37]) and the summand is a genus one 2-point function (2.1) for $\mathcal{M}$. We find that the normalized partition function is given by the theta-function.

Theorem 4.

$$\frac{Z^{(2)}_{\mu,\nu}(\tau,w,\rho)}{Z^{(2)}_{M}(\tau,w,\rho)}=e^{i\pi\left(\mu^2\Omega^{(2)}_{11}+2\mu\nu\Omega^{(2)}_{12}+\nu^2\Omega^{(2)}_{22}\right)}, \tag{3.19}$$

where $\Omega^{(2)}$ is the genus two period matrix.

We are now able to compute the genus two twisted partition function (3.1) by use of a bosonic basis to obtain

Theorem 5. *The genus two normalized twisted partition function for $V_{\mathbb{Z}}$ on a Riemann surface in the self-sewing formalism is given by*

$$\frac{Z^{(2)}_{V_{\mathbb{Z}}}\begin{bmatrix}f\\ g\end{bmatrix}(\tau,w,\rho)}{Z^{(2)}_{M}(\tau,w,\rho)}=\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\Omega^{(2)}\right), \tag{3.20}$$

for genus two Riemann theta function with some characteristics α_1, $\alpha_2=\kappa,\beta_1,\beta_2$ and where $Z^{(2)}_{M}(\tau,w,\rho)$ is the genus two Heisenberg partition function.

3.6. *Generalizations of classical identities for modular forms*

In [34] we proved a generalization of the classical Jacobi identity suitable for genus two. By comparison of the direct computation via algebraic technique [21]–[26], [34]–[38] of the partition and n-point functions with the bosonization technique [21], we obtain [34, 37] the genus two version of the Jacobi product and Frobenius–Fay trisecant identities. Comparing with the fermionic expression (3.1) and noting that

$$\left(e^{i\pi B}\rho\right)^{\frac{1}{2}\kappa^2}\frac{Z^{(1)}_{V_{\mathbb{Z}}}\begin{bmatrix}f_1\\ g_1\end{bmatrix}(e^{\kappa},w;\,e^{-\kappa},0;\,q)}{Z^{(1)}_{M}(\tau)}=\left(\frac{e^{i\pi B}\rho}{K(w,\tau)^2}\right)^{\frac{1}{2}\kappa^2}\vartheta^{(1)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau), \tag{3.21}$$

we obtain a genus two analogue of the classical Jacobi triple product identity for the elliptic theta function (separate to a similar identity shown in [34]) as follows:

Theorem 6. *The ratio of genus two and genus one Riemann theta functions on* $\mathcal{D}^\rho$ *is given by*

$$\frac{\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\Omega^{(2)}\right)}{\vartheta^{(1)}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)} = e^{2i\pi\beta_2\kappa}\left(\frac{e^{i\pi B}\rho}{K(w,\tau)^2}\right)^{\frac{1}{2}\kappa^2}\det(I-T)\det(I-R)^{\frac{1}{2}}.$$

As suggested in [30], we also remark that Fay's Trisecant Identity at genus two follows by comparing the generating function $\mathcal{G}_n^{(2)}\begin{bmatrix}f\\ g\end{bmatrix}(x_1,y_1,\ldots,x_n,y_n)$ of (3.3) to its form in terms of a bosonic basis.

Thus we see that even at the partition function level, automorphic objects (in particular, ratios of θ-functions) can be related to certain functions expressed via determinants [23], [28], [34], [36] containing data (coefficients in expansions of regular parts of ordinary or half-order (e.g., Szegő kernel) differentials) coming from sewing of lower genus Riemann surfaces. These identities reflect automorphic–differential outcome of the boson–fermion correspondence.

4. Higher genus identities

4.1. *The partition functions for Heisenberg VOA*

Let $Z_M^{(2)}\left(\Omega^{(2)}\right)$ be the genus two partition function for the rank one free Heisenberg VOA M [23]. It has been computed in [23, 25]

$$\frac{Z_M^{(2)}\left(\Omega^{(2)}\right)}{Z_M^{(1)}(\tau_1)\,Z_M^{(1)}(\tau_2)} = \det\left(I-A_1^{(1)}A_2^{(1)}\right)^{-1/2}. \tag{4.1}$$

Here $A_a^{(1)}$ for $a=1,2$ are infinite matrices with components indexed by $k,l\geq 1$ [23, 24],

$$A_a(k,l,\tau_a,\epsilon) = \epsilon^{(k+l)/2}\frac{(-1)^{k+1}(k+l-1)!}{\sqrt{kl}\,(k-1)!(l-1)!}\,E_{k+l}(\tau_a), \tag{4.2}$$

where ϵ is Riemann surface sewing scheme parameter [24, 40], and $E_n(\tau)$ are the Eisenstein series [31]. Note that $E_n(\tau)$ appear explicitly as a result of vertex operator algebra computation.

At genus g, in two curves sewing formalism, we obtain [38] similar formula for the rank two Heisenberg VOA module

$$\frac{Z_{M_2}^{(g)}}{Z_{M_2}^{(g_1)}Z_{M_2}^{(g_2)}} = \det\left(I-A^{(g_1,g_2)}\right)^{-1}, \tag{4.3}$$

where $Z_{M_2}^{(g)}$ is a non-vanishing holomorphic function on the sewing domain, and is automorphic with respect to $SL(2,\mathbb{Z})\times\ldots\times SL(2,\mathbb{Z})\subset Sp(2g,\mathbb{Z})$ with automorphy factor $\det(C\Omega^{(g)}+D)^{-1}$, and a multiplier system. Here $A^{(g_1,g_2)}$ is an infinite moment matrix containing genus g sewing data, and $\Omega^{(g)}$ is the genus g period matrix. Automorphic properties of the above mentioned genus two partition functions follow [34, 37] from the structure of the determinant and lower genus partition functions in (4.3). An alternative construction of the genus g partition function for the Heisenberg vertex operator algebra is performed in [33] via the Schottky parameterization.

4.2. *The free fermion partition functions*

The genus one free fermion twisted partition function is given by [16, 28]

$$Z_{V_Z}^{(1)}\left[\begin{array}{c} f \\ g \end{array}\right](\tau)=q^{\alpha^2/2-1/24}\prod_{l\geq 1}\left(1-\theta^{-1}q^{l-\frac{1}{2}+\alpha}\right)\left(1-\theta q^{l-\frac{1}{2}-\alpha}\right), \tag{4.4}$$

which represents a form of the genus one Jacobi triple identity [16]. It is clearly a modular invariant object. This partition function can be also obtained [35, 37] in the form analogous to (4.3) as a determinant of an infinite matrix containing moments of corresponding genus zero differentials when considering a self-sewing of the sphere.

Genus g formula for the partition function can be also conjectured [38]:

$$\frac{Z_{V_{\mathbb{Z}}}^{(g)}\left[\begin{array}{c} f^{(g)} \\ g^{(g)} \end{array}\right]\left(\Omega^{(g)}\right)}{Z_{V_{\mathbb{Z}}}^{(g_1)}\left[\begin{array}{c} f^{(g_1)} \\ g^{(g_1)} \end{array}\right]\left(\Omega^{(g_1)}\right)Z_{V_{\mathbb{Z}}}^{(g_2)}\left[\begin{array}{c} f^{(g_2)} \\ g^{(g_2)} \end{array}\right]\left(\Omega^{(g_2)}\right)}=\det\left(I-Q^{(g_1,g_2)}\right), \tag{4.5}$$

where $g=g_1+g_2$, $Q^{(g_1,g_2)}$ is an infinite matrix, and $\Omega^{(g)}$, $\Omega^{(g_1)}$, $\Omega^{(g_2)}$ are period matrices for a sewn genus g curve and lower genus curves of genuses g_1, g_2. In particular in [34] we proved that the genus two twisted partition function for the rank two fermion VOSA is given by

$$\frac{Z_{V_{\mathbb{Z}}}^{(2)}\left[\begin{array}{c} f^{(2)} \\ g^{(2)} \end{array}\right](\tau_1,\tau_2,\epsilon)}{Z_{V_{\mathbb{Z}}}^{(1)}\left[\begin{array}{c} f_1 \\ g_1 \end{array}\right](\tau_1)\ Z_{V_{\mathbb{Z}}}^{(1)}\left[\begin{array}{c} f_2 \\ g_2 \end{array}\right](\tau_2)}=\det\left(I-Q^{(1,1)}\right). \tag{4.6}$$

5. Differential side: Ward identities

Another group of identities are used to prove an autmorphic–differential correspondence [45].

Let X be a compact connected simply connected algebraic curve. As it is explained in [1, 3, 13], twisted D-modules on the moduli stack Bun_G of G-bundles over X arise in conformal field theories as sheaves of conformal blocks on the moduli space $\mathcal{M}_{g,n}$ of pointed complex curves of genus g. These $\mathcal{D}$-modules encode chiral correlation functions of a model. For CFT's with Lie algebraic symmetries

[13], correlation functions satisfy Ward identities which involve [2, 11] projectively flat connections on a bundle of conformal blocks. For sheaves of conformal blocks on Bun_G, one obtains a projective connection in the bundle or the structure of a twisted $\mathcal{D}$-module on the sheaf of conformal blocks.

Based on algebraic properties of VOA modules, we show [36] that certain normalized (i.e., divided by appropriate Heisenberg VOA partition function, e.g., (5.1), (5.2) and (5.5), see subsections 5.1 and 5.2) generating differentials for n-point functions can be represented as differential operators acting on functions with nice automorphic properties (such as θ-functions). Then such differential operators turn out to be connections in bundles over Riemann surfaces. This construction can be extended to differentials associated to arbitrary n-point functions (e.g., (5.3), see subsection 5.2) even on higher genus Riemann surfaces. This construction represents another side of the geometric correspondence for vertex algebras.

5.1. *The partition functions for free fermionic VOSA*

Let $V_{\mathbb{Z}}$ be a free fermionic vertex operator superalgebra V module [28]. The normalized form of the genus one twisted partition function (see (4.4) in subsection 4.2) for the rank two free fermionic vertex operator superalgebra (see, e.g., [16], [28]) can be also expressed as

$$\frac{Z^{(1)}_{V_{\mathbb{Z}}}\begin{bmatrix} f \\ g \end{bmatrix}(\tau)}{Z^{(1)}_{M}(\tau)} = e^{-2\pi i\alpha\beta}\cdot\vartheta^{(1)}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau), \tag{5.1}$$

for the torus theta function with characteristics α, β. The genus two normalized twisted partition function (4.6) can be computed in the bosonization formalism to obtain [21, 34]

$$\frac{Z^{(2)}_{V_{\mathbb{Z}}}\begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix}\left(\Omega^{(2)}\right)}{Z^{(2)}_{M}\left(\Omega^{(2)}\right)} = e^{-2\pi i\alpha^{(2)}\cdot\beta^{(2)}}\cdot\vartheta^{(2)}\begin{bmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{bmatrix}\left(\Omega^{(2)}\right), \tag{5.2}$$

for the genus two Riemann theta function with characteristics $\alpha^{(2)} = (\alpha_1, \alpha_2)$, $\beta^{(2)} = (\beta_1, \beta_2)$ where $Z^{(2)}_{M}\left(\Omega^{(2)}\right)$ is the genus two Heisenberg partition function. Natural formulas relating the free fermion normalized twisted partition functions and corresponding theta-functions are also available at higher genus [38].

5.2. *Free fermion genus two twisted generating n-point functions*

For $v_i \in V$, and points z_i, $i = 1, \ldots n$, on a genus g curve, let us introduce [34] the following differentials:

$$\mathcal{F}^{(g)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(g)} \\ g^{(g)} \end{bmatrix}((v_1, z_1), \ldots, (v_n, z_n))$$

$$\equiv F^{(g)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(g)} \\ g^{(g)} \end{bmatrix}((v_1,z_1),\dots,(v_n,z_n))\prod_{i=1}^{n} dz_i^{wt(v_i)}, \tag{5.3}$$

where $wt(v_i)$ are weights of the states v_i, [16]. In [34] we proved that for $g=2$ the generating differential for all rank two free fermion VOSA n-point functions is given by the differential

$$\mathcal{G}^{(2)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix}(x_1,y_1,\dots,x_n,y_n) \tag{5.4}$$

$$\equiv F^{(2)}_{V_{\mathbb{Z}},n}\begin{bmatrix} f^{(2)} \\ g^{(2)} \end{bmatrix}((\psi^+,x_1),(\psi^-,y_1)\dots,(\psi^+,x_n),(\psi^-,y_n))\cdot\prod_{i,j=1}^{n} dx_i^{1/2}dy_j^{-1/2},$$

where $F^{(2)}_{V_{\mathbb{Z}},n}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]((\psi^+,x_1),(\psi^-,y_1)\dots,(\psi^+,x_n),(\psi^-,y_n))$ is the genus two $2n$-point function in coordinates x_i, y_i, $i=1,\dots,n$; $\psi^{\pm}$ are generating states, and $f^{(2)}$, $g^{(2)}$ being vectors containing pairs of V-twisting automorphisms. Then its normalized form is given by

$$\frac{\mathcal{G}^{(2)}_{V_{\mathbb{Z}},n}\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right](x_1,y_1,\dots,x_n,y_n)}{Z^{(2)}_{M}} = e^{-2\pi i\alpha\cdot\beta}\det S^{(2)}\cdot\vartheta^{(2)}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\Omega^{(2)}\right), \tag{5.5}$$

where elements of the matrix $S^{(2)}=\left[S^{(2)}\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](x_i,y_j)\right]$, $i,\ j=1,\dots,n$ are genus two Szegő kernels, i.e.,

$$S^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x_i,y_j) = K^{(2)}\begin{bmatrix}\theta^{(2)}\\ \phi^{(2)}\end{bmatrix}(x_i,y_j)\,dx_i^{1/2}\,dy_j^{1/2}, \tag{5.6}$$

and $K^{(2)}\left[\begin{smallmatrix}\theta\\ \phi\end{smallmatrix}\right](z_1,z_2)$ is the functional part of the genus two prime form [9].

5.3. *Genus two one-point Virasoro vector function*

In [34] (see also [37]) we gave an example of a genus two generating function relation containing a flat connection over a Riemann surface. In [34] we computed the one-point function for the Virasoro vector $\widetilde{\omega}=\frac{1}{2}a[-1]a$, where a is the Heisenberg element in the bosonized version of the rank two free fermion VOSA on a genus two Riemann surface obtained in the sewing procedure of two tori.

The Virasoro vector can be written as $\widetilde{\omega}=\frac{1}{2}(\psi^+[-2]\psi^- + \psi^-[-2]\psi^+)$ for the rank two free fermion generators $\psi^{\pm}$ in the Zhu equivalent VOSA formulation. Let w, z be two distinct points on a genus two Riemann surface. To compute a one point

function we use the normalized generating form

$$\frac{\mathcal{G}^{(2)}_{V_{\mathbb{Z}},1}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](w,z)}{Z^{(2)}_{V_{\mathbb{Z}}}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]} = S^{(2)}\left[\begin{smallmatrix} \theta^{(2)} \\ \phi^{(2)} \end{smallmatrix}\right](w,z).$$

Then we find [36]

$$\frac{\partial_w F^{(2)}_{V_{\mathbb{Z}},2}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](\psi^+,w;\psi^-,z)}{Z^{(2)}_{V_{\mathbb{Z}}}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]} = \frac{\partial_w F^{(2)}_{V_{\mathbb{Z}},1}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](Y[\psi^+,w-z]\psi^-,z)}{Z^{(2)}_{V_{\mathbb{Z}}}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]}$$
$$= -\frac{1}{(w-z)^2} + \frac{F^{(2)}_{V_{\mathbb{Z}},1}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](\psi^+[-2]\psi^-,z)}{Z^{(2)}_{V_{\mathbb{Z}}}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]} + \dots,$$

and similarly for ∂_z. Due to (5.6) it follows that the Virasoro vector one-point differential form is

$$\frac{\mathcal{F}^{(2)}_{V_{\mathbb{Z}},1}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](\widetilde{\omega},z)}{Z^{(2)}_{V_{\mathbb{Z}}}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right]} = dz^2 \lim_{w\to z}\left(\frac{1}{(w-z)^2} + \frac{1}{2}(\partial_w - \partial_z)\,K^{(2)}(w,z)\right). \tag{5.7}$$

An alternative expression [34] for this is given below in (5.8).

5.4. *Flat connections*

Introduce the differential operator [39], [21], [34]

$$\mathcal{D}^{(g)} = \frac{1}{2\pi i}\sum_{1\le i\le j\le g} \nu_i^{(g)}(x)\,\nu_j^{(g)}(x)\frac{\partial}{\partial\Omega^{(g)}_{ij}}.$$

It includes holomorphic 1-forms $\nu_i^{(g)}$ as well as derivative with respect to the genus g Riemann surface period matrix $\Omega^{(g)}$. The genus g projective connection $s^{(g)}$ is defined by [15]

$$s^{(g)}(x) = 6\lim_{x\to y}\left(\omega^{(g)}(x,y) - \frac{dxdy}{(x-y)^2}\right),$$

where $\omega^{(g)}$ is the meromorphic differential of the second kind on a Riemann surface. The projective connection $s^{(g)}(x)$ is not a global 2-form but rather transforms under a general conformal transformation $x\to\phi(x)$ as follows $s^{(g)}(\phi(x)) = s^{(g)}(x) - \{\phi;x\}dx^2$, where $\{\phi;x\} = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$ is the Schwarzian derivative.

5.5. *Genus two Ward identity*

It turns out that the Virasoro one-point differential form on genus two Riemann surface can be represented by means of the Ward identity as a flat connection in certain bundle. Let $Z_M^{(2)}$ be the genus two partition function for the rank one Heisenberg VOA M (see explicit formula (4.1)). Using results of [25] we proved in [34] that the Virasoro one-point normalized differential form for the rank two fermion VOSA satisfies the genus two Ward identity

$$\frac{\mathcal{F}^{(2)}_{V_{\mathbb{Z}},1}\left[\begin{smallmatrix} f^{(2)} \\ g^{(2)} \end{smallmatrix}\right](\tilde{\omega}, z)}{Z_M^{(2)}} = e^{-2\pi i \alpha^{(2)}\cdot\beta^{(2)}}\left[\mathcal{D}^{(2)} + \frac{1}{12}s^{(2)}(z)\right]\cdot\vartheta^{(2)}\begin{bmatrix}\alpha^{(2)} \\ \beta^{(2)}\end{bmatrix}\left(\Omega^{(2)}\right). \tag{5.8}$$

Here the expression for the normalized genus two one-point differential form is represented as the action of a differential operator on an automorphic function. One can generalize [38] (5.8) to genus g formula. Note that the Ward identities coherent with the above formula appeared previously in [8, 17]. In contrast to the pure algebraic–geometry [11, 17, 32, 39] and physics approaches [2, 11], the Ward identity shows up from algebraic properties of corresponding vertex algebra. The Ward identities were also used in [12] to establish a version of the geometric Langlands correspondence for Kac–Moody Lie algebras at critical level.

One can also represent the genus two free fermion generating differentials (5.4) (see subsection 5.2) using projective connections [34]. In particular, due to definitions of the genus g Szegő kernel and prime form [9], it is possible to rewrite more general formula for the normalized twisted generating differential (5.5) for all n-point free fermion functions as a quite complicated operator acting on a theta function in a form of (5.8). Similar Ward identities are also available at higher genus cases [14, 38]. Another interesting way to represent the formula (5.5) (and thus (5.8), (5.7), see subsection 5.3) is by using an alternative formulation of the prime form given in [10]. This point will be discussed in [45].

Computations involving boson–fermion correspondence compared to direct computations of generating functions give [28] us various higher genus generalizations of the Frobenius–Fay identities [9]. Using the Ward identity form (e.g., (5.8)) for generating differentials, one is also able to deduce projective connection form for those identities by comparison with pure vertex operator algebra computations.

In examples given above we see that for fermionic VOSAs there is some sense to write expressions for the partition and generating functions in normalized form sinse then differential (projective connection structure) is clear. On the other hand, unnormalized form reflects the automorphic side of the correspondence.

Another interesting direction in generating identities for modular form is related to Jacobi forms via vertex operator algebras [18–20]. We find in this way analogues of the Frobenius–Fay identities [44] for Jacobi forms.

ACKNOWLEDGEMENT

We would like to express our gratitude to the organizers of the Conference Representation Theory XIII, Dubrovnik, Croatia, 2013. This text is based on the author's talk on the Conference.

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Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*, and *Реферативный журнал „Математика"*

Secretariat of the Vice-Rector for Research and International Relations, University of Miskolc
Responsible for publication: Prof. Dr. Tamás Kékesi
Published by the Miskolc University Press under the leadership of Attila Szendi
Responsible for duplication: Works manager Erzsébet Pásztor
Number of copies printed: 70
Put to the Press on August 12, 2015
Number of permission: TNRT 2015-260-ME **HU e-ISSN 1787-2413**