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RIESZ BASES OF EXPONENTIALS AND SINE-TYPE FUNCTIONS

S. A. AVDONIN (Leningrad) and I. JOÓ (Budapest)

The notion of sine-type functions has been introduced by B. Ja. Levin [1] in connection with the description of bases from exponentials $e^{i\lambda_n x}$, $\lambda_n \in \mathbf{C}$, $n \in \mathbf{Z}$, into the space $L^2(0, a)$ ($0<a<\infty$ everywhere in this paper). This notion appears in almost every work devoted to this problem. In a sense, the nearness of the λ_n's to the zeros of a sine-type function is a known ([2], [3]) sufficient condition for $\{e^{i\lambda_n x}\}$ to have the Riesz property in $L^2(0, a)$.

In the present paper we prove that in some sense the nearness of $\{\lambda_n\}$ to the zeros of a sine-type function is also necessary for the Riesz basis property. Our investigations are based on the necessary and sufficient conditions of the Riesz basis property of $\{e^{i\lambda_n x}\}$ given by B. S. Pavlov [4] and on the equivalent form of these conditions obtained by S. V. Hruščev [5].

1. The problem of description of the Riesz bases from exponentials has a long history. The first result of this type seems to be obtained by R. Paley and N. Wiener [6]. They proved that $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0, 2\pi)$ if $\sup\limits_{n\in\mathbf{Z}} |\lambda_n - n| =: d < \pi^{-2}$. R. J. Duffin and J. J. Eachus [7] proved the same statement for $d<\pi^{-1}\log 2$. In these terms the problem was solved by M. I. Kadeč [8]. Namely, he obtained the result for $d<1/4$. The examples of A. E. Ingham [9] and N. Levinson [10] show that Kadeč's result cannot be improved.

All the investigations mentioned above are based on the fact that $\{e^{i\lambda_n x}\}$ is close to the orthonormal basis $\{e^{inx}\}$. A different approach to the description of the Riesz bases from exponentials was developed by B. Ja. Levin [1]. In this approach the main role is played by the "generator function" of $\{e^{i\lambda_n x}\}$. This function was also used earlier by R. E. A. C. Paley and N. Wiener [6].

DEFINITION 1. Let $\Lambda := \{\lambda_n\} \subset \mathbf{C}$, $a>0$. The generator function of (Λ, a) (if it exists) is defined as an entire function F of exponential type with conjugate diagram $[0, ia]$ and with nullset Λ [11].

We shall restrict ourselves to a special class of Λ's for which $\{e^{i\lambda_n x}\}$ is a Riesz basis. In this case we have a simple formula for F in terms of Λ and we do not need the notion of the "conjugate diagramm". Given $f, g\colon X \to R_+$ we write $f(x) \asymp g(x)$ $(x\in X)$ if there exist constants $c_1, c_2>0$ such that $c_1 \leqq f(x)/g(x) \leqq c_2$ $(x\in X)$.

DEFINITION 2 [1]. An entire function F of exponential type is called to be of sine type if

1) the zeros of F lie in $\{z\in\mathbf{C}\colon |y|\leqq h,\ z=x+iy\}$ for some $h>0$,
2) there is $y_0\in\mathbf{R}$ such that $|F(x+iy_0)| \asymp 1$ $(x\in R)$ holds.

Note that 1) and 2) imply 2) for any y_0, $|y_0|>h$.

B. Ja. Levin [1] and V. D. Golovin [12] obtained the first results for the basis problem of exponentials in terms of the generator function.

THEOREM A [1], [12]. *$\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0,a)$ if*

1) *the generator function of $(\{\lambda_n\}, a)$ is of sine type and*
2) *$\{\lambda_n\}$ is separate, i.e.* $\inf_{n\neq m} |\lambda_n-\lambda_m|>0$.

V. E. Katznelson [2] generalized Kadec's theorem for zeros of sine type functions.

THEOREM B [2]. *Suppose the generator function of $(\{\lambda_n\}, a)$ is of sine type and $\{\delta_n\}$ is a bounded sequence of complex numbers with*

$$|\mathrm{Re}\,\delta_n| \leqq d \inf_{m\in\mathbf{Z},\, m\neq n} |\mathrm{Re}\,(\lambda_m-\lambda_n)|, \quad d<1/4. \tag{1}$$

If the sequence $\{\lambda_n+\delta_n\}$ is separate, then $\{e^{i(\lambda_n+\delta_n)x}\}$ is a Riesz basis in $L^2(0,a)$.

This theorem has been strengthened in [3]. Namely condition (1) for $\{\delta_n\}$ can be replaced by its averaged analogue. To formulate this result we need the following

DEFINITION 3 [3]. Let $\Lambda=\{\lambda_n\}$ be a separate set with $|\mathrm{Im}\,\lambda_n|\leqq h<\infty$. A partition $\Lambda=\bigcup_{j\in\mathbf{Z}}\Lambda_j$ is called to be a J-partition, if there exists an increasing sequence $\{\alpha_j\}$, $j\in\mathbf{Z}$, such that $l_j:=\alpha_{j+1}-\alpha_j\leqq\mathrm{const}$ and $\Lambda_j=\{\lambda_n\in\Lambda\colon\ \alpha_j\leqq\mathrm{Re}\,\lambda_n<\alpha_{j+1}\}$.

THEOREM C [3]. *If we replace condition* (1) *in Theorem* B *by*

$$\Big|\sum_{n:\lambda_n\in\Lambda_j}\mathrm{Re}\,\delta_n\Big| \leqq dl_j, \quad d<1/4,\ j\in\mathbf{Z}, \tag{2}$$

for some J-partition of Λ, then the statement of Theorem B *remains valid.*

We shall prove below that, roughly speaking, the hypotheses of Theorem C are also necessary for the Riesz bases property of exponentials. [3] also contains the following generalization of Theorem A.

THEOREM D [3]. *Let $\Lambda=\{\lambda_n\}$ be a separate set with $|\mathrm{Im}\,\lambda_n|\leqq h<\infty$. Suppose the generator function F_Λ of (Λ, a) satisfies $|F_\Lambda(x+iy_0)|\asymp|x|^\alpha$ $(x\in\mathbf{R})$ for some $\alpha\in(-1/2, 1/2)$ and for some y_0, $|y_0|>h$. Then $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0,a)$.*

In [3] Theorem D is also proved for more general functions than $|x|^\alpha$ and these results show that $|F_\Lambda|$ can oscillate.

In all the papers mentioned which follow Levin's method, the main role is played by properties of the generator function F_Λ. The main technical tool is an estimation of infinite products. B. S. Pavlov suggested a "geometrical approach" (in the sense of the geometry of Hilbert spaces) for the investigation of the basis problem for exponentials. First he proved Theorem A [13] with the help of his method and then he obtained a complete description of Riesz bases of the form $\{e^{i\lambda_n x}\}$ in $L^2(0,a)$ [4] in terms of the generator function F_Λ. The core of Pavlov's method is the following. He considers the functions $\{e^{i\lambda_n x}\}$ in $L^2(0,\infty)$ (since the operator of multiplication by $e^{-(h+\delta)x}$ is bounded and invertible in $L^2(0,a)$ we may assume without loss of generality that $0<\delta\leqq\mathrm{Im}\,\lambda_n\leqq 2h+\delta$). Denote by E_Λ the closure of the linear hull of $\{e^{i\lambda_n x}\}$ in $L^2(0, \infty)$. It is well known [20] that $\{e^{i\lambda_n x}\}$ is an unconditional basis (not necessarily normed Riesz basis) in E_Λ if and only if the Carleson condition [14]

$$\text{(C)} \qquad \inf_{n\in\mathbf{Z}} \prod_{k\neq n} \left| \frac{\lambda_k-\lambda_n}{\lambda_k-\bar{\lambda}_n} \right| > 0$$

holds. Furthermore, if $0<\delta\leqq \operatorname{Im}\lambda_n \leqq 2h+\delta$ then (C) is equivalent to $\inf\limits_{n\neq m} |\lambda_n-\lambda_m|>0$ (cf. e.g. [15]). Consider now the orthogonal projection of $L^2(0, \infty)$ onto $L^2(0, a)$ defined as multiplication by the characteristic function $\chi[0, a]$ of $[0, a]$. If the restriction of this projection to E_Λ is an isomorphism, then it transforms the Riesz basis $\{e^{i\lambda_n x}\}$ of E_Λ onto the Riesz basis $\{e^{i\lambda_n x}\chi_{[0,a]}\}$ of $L^2(0, a)$. The condition of the existence of the inverse of the projection just mentioned can be expressed in terms of the generator function F_Λ of $(\{\lambda_n\}, a)$.

THEOREM E [4]. *$\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0, a)$ if and only if* 1) *the set $\{\lambda_n\}$ is separate and $|\operatorname{Im}\lambda_n|\leqq h<\infty$, $n\in\mathbf{Z}$;* 1) *for some y_0, $|y_0|>h$, the function $w(x):=|F_\Lambda(x+iy_0)|^2$ statisfies the Muckenhoupt condition*

$$(\mathrm{A}_2) \qquad \sup_{I\in\mathscr{I}} \frac{1}{|I|}\int_I w(x)\,dx \cdot \frac{1}{|I|}\int_I [w(x)]^{-1}\,dx < \infty,$$

where $\mathscr{I}$ denotes the set all finite intervals on $\mathbf{R}$ *and F_Λ is the generator function of* $(\{\lambda_n\}, a)$.

It is well known ([16], [17]) that the condition (A_2) is equivalent to that of H. Helson and G. Szegö [16]:

DEFINITION 4 [16]. A non-negative function w satisfies the Helson—Szegö (shortly (HS)) condition, if there exist functions $u, v\in L^\infty(\mathbf{R})$ such that $\|v\|_{L^\infty(\mathbf{R})}<\pi/2$ and

$$\text{(HS)} \qquad w(x) = \exp\{u(x)+\tilde{v}(x)\},$$

where

$$(3) \qquad \tilde{v}(x) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbf{R}} \left[\frac{1}{x-t}+\frac{t}{1+t^2}\right] v(t)\,dt$$

is the conform-invariant form of the Hilbert transform of v. Hence, one can replace condition 2) in Theorem E by (HS) for the function $|F_\Lambda(x+iy_0)|^2$. S. V. Hruščev [5] has proved, that the latter form of the condition of the Riesz basis property for $\{e^{i\lambda_n x}\}$ is very convenient from the point of view of verification. He has derived in this way all the known sufficient conditions for the basis property of $\{e^{i\lambda_n x}\}$.

In the present paper we shall also use the "HS-form" of the conditions of the Riesz basis property.

Finally, we mention the fundamental work [18] which, besides the results mentioned above, also contains some new forms of the conditions of the basis property of exponentials. Connections of this problem with the theory of Hankel operators, the functional model of Nagy—Foiaş, as well as applications to the theory of interpolation and to the spectral theory of operators can also be found in [18].

2. The aim of the present paper is to describe Riesz bases of the form $\{e^{i\lambda_n x}\}$ in $L^2(0, a)$ in term of the nearness of λ_n's to the zeros of sine-type functions.

Now we give some simple geometrical properties of $\{\lambda_n\}$ which are necessary for the Riesz basis property of $\{e^{i\lambda_n x}\}$. The proofs of these statements are given in [4], [5], [18]. In what follows, suppose that $\operatorname{Re}\lambda_n \leqq \operatorname{Re}\lambda_m$ if and only if $n \leqq m$. Since the norms of the elements of any Riesz basis are bounded from below and above it follows that for any Riesz basis $\{e^{i\lambda_n x}\}$ we have $\sup_{n\in\mathbf{Z}} |\operatorname{Im}\lambda_n| < \infty$. Furthermore

1) $\lim\limits_{n\to\pm\infty} \dfrac{\lambda_n}{n} = 2\pi/a,$

2) $\exists \lim\limits_{R\to\infty} \sum\limits_{n:|\lambda_n|\leqq R} \dfrac{1}{\lambda_n} < \infty.$

Consequently, the generator function F_Λ belongs to the Cartwright class [11] and has the form

$$F_\Lambda(z) = e^{i\frac{a}{2}z}\,\text{p.v.}\prod_n \left(1-\frac{z}{\lambda_n}\right) := e^{i\frac{a}{2}z} \lim_{R\to\infty} \prod_{n:|\lambda_n|\leqq R} \left(1-\frac{z}{\lambda_n}\right). \tag{4}$$

Here without loss of generality we may suppose that $0 \notin \{\lambda_n\}$ (otherwise one can shift a finite number of λ_n's without affecting the Riesz basis property) and $F(0)=1$.

The sequence $\{\lambda_n\}$ is called *standard* in [5] if it is separate, satisfies 1), 2) and $\sup_{n\in\mathbf{Z}} |\operatorname{Im}\lambda_n| \leqq h < \infty$. In what follows we assume that $\{\lambda_n\}$ is standard.

It follows from (4) (cf. e.g. [3]) that for any y_1, y_2 with $|y_1| > h$, $|y_2| > h$ we have

$$|F_\Lambda(x+iy_1)| \asymp F_\Lambda(x+iy_2)| \quad (x\in\mathbf{R}). \tag{5}$$

Hence, if the (HS) condition

$$|F_\Lambda(x+iy)|^2 = \exp\{u(x)+\tilde{v}(x)\} \quad (x\in\mathbf{R}), \quad \|u\|_{L^\infty(\mathbf{R})} < \infty, \ \|v\|_{L^\infty(\mathbf{R})} < \pi/2, \tag{6}$$

for $|F_\Lambda|^2$ holds for some y_1, $|y_1| > h$, then it also holds for any y_2, $|y_2| > h$.

For our further considerations it is convenient to define the set $\{\hat{\lambda}_n := \lambda_n - i(\delta+h)\}$, $\delta > 0$. The transformation of $\{\lambda_n\}$ to $\{\hat{\lambda}_n\}$ corresponds to the operator of multiplication by $e^{(h+\delta)x}\colon L^2(0, a) \to L^2(0, a)$.

This operator is a bounded operator with bounded inverse. Consequently $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0, a)$ if and only if $\{e^{i\hat{\lambda}_n x}\}$ is a Riesz basis. For brevity we use in our work the notation $\{\lambda_n\}$ instead of $\{\hat{\lambda}_n\}$ and assume that

$$-2h-\delta \leqq \operatorname{Im}\lambda_n \leqq -\delta, \quad n\in\mathbf{Z}. \tag{7}$$

In view of (5) one can consider F_Λ only on the real line, i.e. consider condition (HS) in the form

$$|F_\Lambda(x)|^2 = \exp\{u(x)+\tilde{v}(x)\} \quad (x\in\mathbf{R}), \quad \|u\|_{L^\infty(\mathbf{R})} < \infty, \ \|v\|_{L^\infty(\mathbf{R})} < \pi/2. \tag{8}$$

It is known [1, 18/III] that zero set $\{\mu_n\}$ of any sine-type function satisfies $\sup_{n\in\mathbf{Z}} |\mu_{n+1}-\mu_n| < \infty$. All known examples of the Riesz bases $\{e^{i\lambda_n x}\}$ have analogous property and we include it into the conditions of the following

THEOREM. *Consider the following statements:*

1) $\{e^{i\lambda_n x}\}$ *is a Riesz basis in* $L^2(0, a)$;

2) *there exists a sine-type function with zero set* $\{\mu_n\}\subset\mathbf{R}$ *such that for some* $d\in(0, 1/4)$

$$d\,\mathrm{Re}\,(\lambda_{n-1}-\lambda_n) \leqq \mathrm{Re}\,(\mu_n-\lambda_n) \leqq d\,\mathrm{Re}\,(\lambda_{n+1}-\lambda_n) \quad (n\in\mathbf{Z}). \tag{9}$$

Then

a) 2) *implies* 1);

b) 1) *implies* 2) *if* $\sup_{n\in\mathbf{Z}} |\lambda_{n+1}-\lambda_n|<\infty$.

PROOF. a) Suppose there exists a sine-type function having zeros μ_n satisfying (9). Let $\Lambda=\bigcup_{j\in\mathbf{Z}}\Lambda_j$ be some J-partition of Λ. Summing up the inequalities (9) for every n for which $\lambda_n\in\Lambda_j$, we obtain for some $0<d<1/4$ the inequalities

$$d\,[\max(\mathrm{Re}\,\Lambda_{j-1})-\max(\mathrm{Re}\,\Lambda_j)] \leqq \sum_{n:\lambda_n\in\Lambda_j} \mathrm{Re}\,\delta_n \leqq \tag{10}$$

$$\leqq d\,[\min(\mathrm{Re}\,\Lambda_{j+1})-\min(\mathrm{Re}\,\Lambda_j)],$$

where $\delta_n:=\mu_n-\lambda_n$.

Since $\{\mu_n\}$ are the roots of a sine-type function, we have $\sup_{n\in\mathbf{Z}}|\mu_{n+1}-\mu_n|<\infty$, and consequently, taking into account (9), we get

$$\sup_{n\in\mathbf{Z}}|\lambda_n-\mu_n|<\infty, \quad \sup_{n\in\mathbf{Z}}|\lambda_{n+1}-\lambda_n|<\infty.$$

Taking the union of some Λ_j if necessary, we arrive to a J-partition of Λ for which we obtain by (10) the following inequalities with some $d_1\in(0, 1/4)$:

$$(d_1(\mu^{j-1}_{\max}-\mu^j_{\max}) \leqq \sum_{n:\lambda_n\in\Lambda_j} \mathrm{Re}\,\delta_n \leqq d_1(\mu^j_{\max}-\mu^{j-1}_{\max})$$

where $\mu^j_{\max}:=\max\{\mu_n\colon \lambda_n\in\Lambda_j\}$. Applying Theorem C we obtain that $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0, a)$.

To prove the statement of Theorem b) we need some lemmas.

We know ([21]) that 1) implies $\sup_{n\in\mathbf{Z}}|\lambda_{n+1}-\lambda_n|<\infty$.

LEMMA 1. *Condition* (HS) *holds for* $|F_\Lambda(x)|^2$ *if and only if there exist* $u, v\in L^\infty(\mathbf{R})$ *with differentiable* v *and such that* $v'\in L^\infty(\mathbf{R})$, $\|v\|_{L^\infty(\mathbf{R})}<\pi/2$,

$$|F_\Lambda(x)|^2 = \exp\left\{u(x)+\frac{1}{\pi}\int_{K(x)}\left(\frac{1}{x-t}+\frac{1}{t}\right)v(t)\,dt\right\} \quad (x\in\mathbf{R}), \tag{11}$$

where

$$K(x) = \{t\in\mathbf{R}\colon |t|\geqq 1,\ |x-t|\geqq 1\}. \tag{12}$$

PROOF. Suppose (8) is fulfilled. Since $u, v \in \mathrm{BMO}$, they can be extended by Poisson's formula for $y>0$ as harmonic functions [19]:

$$u(x, y) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{y}{(x-t)^2+y^2} u(t)\, dt \tag{13}$$

and similar holds for $\tilde{v}$. The resulting function $\exp\{u(x, y)+\tilde{v}(x, y)\}$ coincides with $|F_\Lambda(x+iy)|^2$ on the halfplane $y>0$ (cf. 18/III, the proof of the Theorem 8). Since the harmonic continuation and the operation $\tilde{}$ defined in (3) commute, we have for any fixed $H>0$

$$F_\Lambda(x+iH)|^2 = \exp\{u(x, H)+\widetilde{[v(x, H)]}\} \quad (x\in\mathbf{R}).$$

$v(x)$ can not be smooth ($v\in L^\infty(\mathbf{R})$ only), but $v(x, H)$ is smooth, namely it follows from (13) that $v(x, H)$ is differentiable and for any $x\in\mathbf{R}$

$$|v(x, H)| \leqq \|v(\cdot)\|_{L^\infty(\mathbf{R})}, \quad |v'(x H)| \leqq 2\|v(\cdot)\|_{L^\infty(\mathbf{R})}.$$

It is obvious that for any $f(t)$ with $|f'(t)|\leqq M$ $(t\in\mathbf{R})$ we have

$$F(x) := \text{p.v.} \int_{x-1}^{x+1} \frac{f(t)}{x-t}\, dt \in L^\infty(\mathbf{R}).$$

Indeed, by p.v. $\int_{-1}^{1} \frac{dt}{t} = 1$ we obtain

$$|F(x)| = \left|\text{p.v.} \int_{-1}^{1} \frac{f(x-t)}{t}\, dt\right| = \left|\text{p.v.} \int_{-1}^{1} \frac{f(x-t)-f(x)}{t}\, dt\right| \leqq \int_{-1}^{1} M\, dt = 2M.$$

Apply this to $f(t)=v(t, H)$, then we obtain from the (HS) condition for $|F_\Lambda(x)|^2$ the following expression for $|F_\Lambda(x+iH)|^2$:

$$|F(x+iH)|^2 = \exp\left\{u(x)+\int_{K(x)} \left(\frac{1}{x-t}+\frac{1}{t}\right) v(t)\right\} \quad (x\in\mathbf{R})$$

where u and v satisfy the conditions of Lemma 1. In order to complete the proof of (HS)$\Rightarrow$(11) it is enough to use (5). The converse, i.e. (HS)$\Leftarrow$(11) follows from the boundedness of p.v. $\int_{x-1}^{x+1} v(t)(x-t)^{-1}\, dt$ for v with the properties in Lemma 1. Lemma 1 is proved.

Before formulating Lemma 2 we need some notations. Suppose

$$0 < \inf_{n\in\mathbf{Z}} (\lambda_{n+1}-\lambda_n) \leqq \sup_{n\in\mathbf{Z}} (\lambda_{n+1}-\lambda_n) < \infty, \quad \{\lambda_n\} \subset \mathbf{R}. \tag{$*$}$$

Define

$$\begin{aligned}
\lambda(t) &:= \lambda_n+(\lambda_{n+1}-\lambda_n)(t-n), \quad n \leqq t < n+1,\\
D(x) &:= \{n\in\mathbf{Z}\colon |n| \geqq 1, \quad |x-n| \geqq 1\},\\
D_\Lambda(x) &:= \{n\in\mathbf{Z}\colon |\lambda_n| \geqq 1, \quad |x-\lambda_n| \geqq 1\},\\
K_\Lambda(x) &:= \{t\in\mathbf{R}\colon |\lambda(t)| \geqq 1, \quad |x-\lambda(t)| \geqq 1\},
\end{aligned}$$

and for $v_n, \delta_n, v(t), \delta(t)\in\mathbf{R}$, $d>0$ $(n\in\mathbf{Z},\ t\in\mathbf{R})$

$$\sigma(x, \{v_n\}) := \sum_{n\in D(x)} v_n\left(\frac{1}{x-n}+\frac{1}{n}\right), \quad |v_n|\leqq d; \tag{14}$$

$$\sigma_\Lambda(x, \{\delta_n\}) := \sum_{n\in D_\Lambda(x)} \delta_n\left(\frac{1}{x-\lambda_n}+\frac{1}{\lambda_n}\right), \quad \left|\frac{\delta_n}{\lambda_{n+1}-\lambda_n}\right|\leqq d; \tag{15}$$

$$I(x, v) := \int_{K(x)} v(t)\left(\frac{1}{x-t}+\frac{1}{t}\right)dt, \quad |v(t)|\leqq d; \tag{16}$$

$$I_\Lambda(x, \delta) := \int_{K_\Lambda(x)} \delta(t)\left[\frac{1}{x-\lambda(t)}+\frac{1}{\lambda(t)}\right]dt, \quad \left|\frac{\delta(t)}{\lambda'(t)}\right|\leqq d \quad \text{a.e. on } \mathbf{R}. \tag{17}$$

LEMMA 2. *For any function φ of the form $(13+i)$, $1\leqq i\leqq 4$, and for any $j\in\{1, 2, 3, 4\}$ there exists a function ψ of the form $(13+j)$ such that* $\exp\varphi(x)\asymp\exp\psi(x)$ $(x\in\mathbf{R})$.

PROOF. It is enough to prove the cases $(i, j)=(1, 3), (3, 1), (2, 4), (4, 2), (3, 4), (4, 3)$.

1) *The cases* $(1, 3)$ *and* $(3, 1)$. In these cases for a given $\{v_n\}$ define $v(t):=v_n$, $n\leqq t<n+1$, and for a given $v(t)$ consider $v_n:=\int_n^{n+1} v(t)\,dt$, resp. Obviously

$$\left|\int_n^{n+1}\frac{v(t)}{t}\,dt-\frac{v_n}{n}\right| = \left|\int_n^{n+1}\left\{\frac{v(t)}{t}-\frac{v(t)}{n}+\frac{v(t)}{n}-\frac{v_n}{n}\right\}dt\right| =$$

$$= \left|\int_n^{n+1} v(t)\frac{n-t}{nt}\right| \leqq d\max\left\{\frac{1}{n^2}, \frac{1}{(n+1)^2}\right\},$$

and

$$\left|\int_n^{n+1}\frac{v(t)}{x-t}\,dt-\frac{v_n}{n}\right| \leqq d\max\left\{\frac{1}{|x-n|^2}, \frac{1}{|x-n-1|^2}\right\}.$$

After this it is enough to take into account

$$\sup_{x\in\mathbf{R}}\sum_{n\in D(x)}\frac{1}{|x-n|^2}<\infty.$$

2) *The cases* $(2, 4)$ *and* $(4, 2)$. In these cases define $\delta(t):=\delta_n$, $n\leqq t<n+1$ and $\delta_n:=\int_n^{n+1}\delta(t)\,dt$, resp. The necessary estimates follow analogously as in 1) using $(*)$.

3) *The cases* $(3, 4)$ *and* $(4, 3)$. In these cases we use the change of variable $t=\lambda(\tau)$ and $\tau=\lambda(t)$ and use the functions $\delta(\tau):=\lambda'(\tau)v(\lambda(\tau))$ and $v(\tau)=\delta(\lambda^{-1}(\tau))/\lambda'(\lambda^{-1}(\tau))$, resp. Lemma 2 is proved.

We also need the following three lemmas from [3] for standard sequences $\{\lambda_n\}$.

LEMMA 3. *Let F and G be generator functions with zeros λ_n and* $\operatorname{Re}\lambda_n$ *respectively. Then for any $H\in\mathbf{R}$ with $|H|>\sup_{n\in\mathbf{Z}}|\operatorname{Im}\lambda_n|$ the relation*

$$|F(x+iH)|\asymp|G(x+iH)|\quad(x\in\mathbf{R})\tag{18}$$

holds.

LEMMA 4. *Let F and G be generator functions with zeros λ_n and $\lambda_n+\delta_n$ respectively, where $\{\delta_n\}$ is a bounded sequence. Then for any $H\in\mathbf{R}$ with $|H|>\sup_{n\in\mathbf{Z}}(|\operatorname{Im}\lambda_n|+|\operatorname{Im}\delta_n|)$ the estimate*

$$\left|\frac{F(x+iH)}{G(x+iH)}\right|\asymp\exp\operatorname{Re}\left\{\sum_{n\in\mathbf{Z}}\delta_n\left(\frac{1}{x+iH-\lambda_n}+\frac{1}{\lambda_n}\right)\right\}\quad(x\in\mathbf{R})\tag{19}$$

holds.

LEMMA 5. *Suppose the conditions of Lemma 4 are fulfilled and $\sum_{n:\lambda_n\in\Lambda_j}\operatorname{Re}\delta_n=0$ $(j\in\mathbf{Z})$ holds for some J-partition of $\{\lambda_n\}$. Then we have*

$$|F(x+iH)|\asymp|G(x+iH)|\quad(x\in\mathbf{R})\tag{20}$$

for any $H\in\mathbf{R}$, $|H|>\sup_{n\in\mathbf{Z}}(|\operatorname{Im}\lambda_n|+|\operatorname{Im}\delta_n|)$.

Now we can prove statement b) of the Theorem. Suppose that $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0,a)$ and denote by F_Λ the generator function of $(\{\lambda_n\}a)$. Then $|F_\Lambda(x)|^2$ satisfies (HS) and hence, according to Lemma 1, it also satisfies (11). First suppose that $\{\operatorname{Re}\lambda_n\}$ is separate. Then by (5) using Lemmas 2, 3 we obtain

$$|F_\Lambda(x)|\asymp\exp\left\{\sum_{n\in D_{\Lambda^0}(x)}\left(\frac{1}{x-\lambda_n^0}+\frac{1}{\lambda_n^0}\right)'\delta_n^0\right\}\quad(x\in\mathbf{R}),$$

where $\Lambda^0:=\{\lambda_n^0\}:=\{\operatorname{Re}\lambda_n\}$, $\{\delta_n^0\}\subset\mathbf{R}$ and for some $d_1\in(0,1/4)$

$$\left|\frac{\delta_n^0}{\lambda_{n+1}^0-\lambda_n^0}\right|<d_1\quad(n\in\mathbf{Z}).$$

According to Lemma 4 the generator function G of $(\{\lambda_n^0+\delta_n^0\},a)$ is a sine-type function.

Now we show that there exists $\{\delta_n\}\subset\mathbf{R}$ such that for some $d\in(0,1/4)$

$$d(\lambda_{n-1}^0-\lambda_n^0)\leqq\delta_n\leqq d(\lambda_{n+1}^0-\lambda_n^0)\quad(n\in\mathbf{Z})\tag{21}$$

holds, further for some J-partition $\Lambda=\bigcup_{j\in\mathbf{Z}}\Lambda_j$ of Λ

$$\sum_{n:\lambda_n\in\Lambda_j}(\delta_n-\delta_n^0)=0\quad(j\in\mathbf{Z})\tag{22}$$

is fulfilled.

Let $\Lambda=\bigcup_{j\in\mathbf{Z}}\Lambda_j$ be an arbitrary J-partition of Λ

$$\gamma_n(t):=\frac{1}{4}[(\lambda_{n-1}^0-\lambda_n^0)+t(\lambda_{n+1}^0-\lambda_{n-1}^0)],\quad t\in[0,1],$$

$$S_j(t):=\sum_{n:\lambda_n\in\Lambda_j}[\gamma_n(t)-\delta_n^0]\quad (j\in\mathbf{Z}).$$

Estimate $S_j(0)$ and $S_j(1)$. Obviously

$$S_j(0)=\sum_{n:\lambda_n\in\Lambda_j}\left[\frac{1}{4}(\lambda_{n-1}^0-\lambda_n^0)-\delta_n^0\right]=\frac{1}{4}\max(\Lambda_{j-1}^0-\max\Lambda_j^0)-\sum_{n:\lambda_n\in\Lambda_j}\delta_n^0\leqq$$

$$\leqq\frac{1}{4}(\max\Lambda_{j-1}^0-\max\Lambda_j^0)+d_1\sum_{n:\lambda_n\in\Lambda_j}(\lambda_{n+1}^0-\lambda_n^0)=$$

$$=\frac{1}{4}(\max\Lambda_{j-1}^0-\max\Lambda_j^0)+d_1(\min\Lambda_{j+1}^0-\min\Lambda_j^0).$$

Noting that $d_1<1/4$ and taking the union of some Λ_j if necessary, we obtain

$$S_j(0)\leqq\left(d_1-\frac{1}{4}\right)\inf_{j\in\mathbf{Z}}(\max\Lambda_j^0-\max\Lambda_{j-1}^0)+2d_1\sup_{n\in\mathbf{Z}}(\lambda_{n+1}^0-\lambda_n^0)<0.$$

On the other hand

$$S_j(1)=\sum_{n:\lambda_n\in\Lambda_j}\left[(\lambda_{n+1}^0-\lambda_n^0)\frac{1}{4}-\delta_n^0\right]=\frac{1}{4}(\min\Lambda_{j+1}^0-\min\Lambda_j^0)-\sum_{n:\lambda_n\in\Lambda_j}\delta_n^0\geqq$$

$$\geqq\frac{1}{4}(\min\Lambda_{j+1}^0-\min\Lambda_j^0)-d_1(\min\Lambda_{j+1}^0-\min\Lambda_j^0)\geqq$$

$$\geqq\left(\frac{1}{4}-d_1\right)\inf_{n\in\mathbf{Z}}(\lambda_{n+1}^0-\lambda_n^0)>0\quad(j\in\mathbf{Z})$$

because $\{\lambda_n^0\}$ is separate. Consequently, for every $j\in\mathbf{Z}$ there exists $t_j\in[\alpha,\beta]\subset(0,1)$ such that $S_j(t_j)=0$. (We have used $S_j(0)=O(1)$ and $S_j(1)=O(1)$ which follow from the definition of the J-partition.) Now let $\delta_n:=\gamma_n(t_j)$ for those n for which $\lambda_n\in\Lambda_j$, $j\in\mathbf{Z}$. It is easy to see that (21) and (22) hold. Now from Lemma 5 we obtain statement b) of the Theorem (with $\mu_n:=\operatorname{Re}\lambda_n+\delta_n$) for the case when the set $\{\operatorname{Re}\lambda_n\}$ is separate.

Consider the case when $\{\operatorname{Re}\lambda_n\}$ is not separate.

1) Construct a J-partition $\Lambda=\bigcup_{j\in\mathbf{Z}}\tilde\Lambda_j$ such that $\inf_{j\in\mathbf{Z}}(\min\operatorname{Re}\tilde\Lambda_{j+1}-\max\operatorname{Re}\tilde\Lambda_j)>$ >0. Such a J-partition exists since $\{\lambda_n\}$ is separate and $\sup_{n\in\mathbf{Z}}|\operatorname{Im}\lambda_n|<\infty$.

2) Construct a J-partition $\Lambda=\bigcup_{j\in\mathbf{Z}}\Lambda_j$ where

$$\Lambda_j=\tilde\Lambda_{3j-1}\cup\tilde\Lambda_{3j}\cup\tilde\Lambda_{3j+1},\quad j\in\mathbf{Z}.$$

We have $\sup_{j\in\mathbf{Z}} \operatorname{card} \Lambda_j < \infty$ and

$$\inf_{j\in\mathbf{Z}} (\max \operatorname{Re} \Lambda_j - \min \operatorname{Re} \Lambda_j) > 0. \tag{23}$$

3) Construct $\nu = \{\nu_n\}_{n\in\mathbf{Z}} \subset \mathbf{R}$ and a J-partition $\nu = \bigcup_{j\in\mathbf{Z}} \nu_j$ such that

$$\begin{cases} \text{a) } \operatorname{card} \nu_j = \operatorname{card} \Lambda_j, \ \min \nu_j = \min \operatorname{Re} \Lambda_j, \ \max \nu_j = \max \operatorname{Re} \Lambda_j, \\ \quad \nu_j \text{ is separate (consequently } \nu \text{ is separate too),} \\ \text{b) } \sum_{n:\lambda_n\in\Lambda_j} (\operatorname{Re} \lambda_n - \nu_n) = 0 \quad (j\in\mathbf{Z}). \end{cases} \tag{24}$$

According to Lemma 5

$$|F_\nu(x+i)| \asymp |F_\Lambda(x+i)| \quad (x\in\mathbf{R})$$

where F_ν and F_Λ are generator functions of (ν, a) and (Λ, a) respectively. Hence $\{e^{i\nu_n x}\}$ is a Riesz basis in $L^2(0, a)$.

4) According to the statement of the separate case (which is proved) there exists a sine-type function with zero set $\{\xi_n\} \subset \mathbf{R}$ such that

$$d(\nu_{n-1} - \nu_n) \leqq \xi_n - \nu_n \leqq d(\nu_{n+1} - \nu_n) \quad (n\in\mathbf{Z}) \tag{25}$$

for some $d\in(0, 1/4)$.

5) From (25) it follows that

$$d(\max \nu_{j-1} - \max \nu_j) \leqq \sum_{n:\nu_n\in\nu_j} (\xi_n - \nu_n) \leqq d(\min \nu_{j+1} - \min \nu_j) \quad (j\in\mathbf{Z}).$$

Taking into account (24) we have

$$d(\max \operatorname{Re} \Lambda_{j-1} - \max \operatorname{Re} \Lambda_j) \leqq \sum_{n:\lambda_n\in\Lambda_j} (\xi_n - \operatorname{Re} \lambda_n) \leqq d(\min \operatorname{Re} \Lambda_{j+1} - \min \operatorname{Re} \Lambda_j).$$

6) Now we show that there exists $\{\mu_n\} \subset \mathbf{R}$ such that for some $d_1\in(0, 1/4)$

$$d_1 \operatorname{Re}(\lambda_{n-1} - \lambda_n) \leqq \operatorname{Re}(\mu_n - \lambda_n) \leqq d_1 \operatorname{Re}(\lambda_{n+1} - \lambda_n) \quad (n\in\mathbf{Z}) \tag{26}$$

and for some J-partition $\Lambda = \bigcup_{j\in\mathbf{Z}} \Lambda_j$

$$\sum_{n:\lambda_n\in\Lambda_j} (\mu_n - \xi_n) = 0 \quad (j\in\mathbf{Z}). \tag{27}$$

From (27) and Lemma 5 it follows that $\{\mu_n\}$ is the zero set of a sine-type function.

The proof of (26), (27) is analogous to that of (21), (22). Consider the J-partition $\Lambda = \bigcup_{j\in\mathbf{Z}} \Lambda_j$ constructed in 2) and let

$$\gamma_n(t) := \lambda_n^0 + \frac{1}{4}[(\lambda_{n-1}^0 - \lambda_n^0) + t(\lambda_{n+1}^0 - \lambda_{n-1}^0)], \quad t\in[0, 1],$$

$$S_j(t) := \sum_{n:\lambda_n\in\Lambda_j} [\xi_n - \gamma_n(t)] \quad (j\in\mathbf{Z}),$$

where

$$\lambda_n^0 := \operatorname{Re}\lambda_n, \quad \Lambda_j^0 := \operatorname{Re}\Lambda_j.$$

Estimate $S_j(0)$ and $S_j(1)$:

$$S_j(0) = \sum_{n:\lambda_n\in\Lambda_j}\left[\xi_n-\lambda_n^0+\frac{1}{4}(\lambda_n^0-\lambda_{n-1}^0)\right] \geqq$$

$$\geqq d(\max\Lambda_{j-1}^0-\max\Lambda_j^0)+\frac{1}{4}(\max\Lambda_j^0-\max\Lambda_{j-1}^0).$$

Noting that $d<1/4$ we obtain $\inf_{j\in\mathbf{Z}} S_j(0)>0$. On the other hand

$$S_j(1) = \sum_{n:\lambda_n\in\Lambda_j}\left[\xi_n-\lambda_n^0-\frac{1}{4}(\lambda_{n+1}^0-\lambda_n^0)\right] \leqq$$

$$\leqq d(\min\Lambda_{j+1}^0-\min\Lambda_j^0)-\frac{1}{4}(\min\Lambda_{j+1}^0-\min\Lambda_j^0).$$

Hence $\sup_{j\in\mathbf{Z}} S_j(1)<0$. Consequently for every $j\in\mathbf{Z}$ there exists $t_j\in[\alpha,\beta]\subset(0,1)$ such that $S_j(t_j)=0$.

Now let $\mu_n:=\gamma_n(t_j)$ for those n for which $\lambda_n\in\Lambda_j$, $j\in\mathbf{Z}$. Conditions (26), (27) hold. The Theorem is proved.

REMARKS. 1. According to Lemma 3 we can prescribe $\{\operatorname{Im}\mu_n\}$ (with $\sup_{n\in\mathbf{Z}}|\operatorname{Im}\mu_n|<\infty$) arbitrarily.

2. Inequalities (9) in the conditions of the Theorem can be replaced by

$$d(\mu_{n-1}-\mu_n) \leqq \operatorname{Re}(\lambda_n-\mu_n) \leqq d(\mu_{n+1}-\mu_n) \quad (n\in\mathbf{Z}).$$

3. In the proof of the Theorem (see Lemmas 1, 2) we have also proved the following statement which is interesting in itself. $\{e^{i\lambda_n x}\}$ is a Riesz basis in $L^2(0,a)$ if and only if there exists $\{\delta_n\}\subset\mathbf{R}$, $\sup_{n\in\mathbf{Z}}|\delta_n|<1/4$ such that

$$|F_\Lambda(x)| \asymp \exp\left\{\sum_{n\in D(x)}\delta_n\left(\frac{1}{x-n}+\frac{1}{n}\right)\right\}.$$

Here F_Λ is the generator function of $(\{\lambda_n\}, a)$, and we do not suppose that $\sup_{n\in\mathbf{Z}}|\lambda_{n+1}-\lambda_n|<\infty$.

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LENINGRAD STATE UNIVERSITY
DEP. OF APPL. MATH.
BIBLIOTECHNAYA PL. 2
198904 LENINGRAD, USSR

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF MATHEMATICS
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

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ON LINKING JORDAN CURVES

M. BOGNÁR (Budapest)

The aim of this paper is to define the notion of the linking Jordan curve and to show that under certain conditions we can find such a curve.

The subject is based on the axiomatization of the linking theory which is described in [4], [5] and [6]. In a certain sense this paper is a continuation of the previous one [6].

1. Introduction

1.1. First recall some notions and notations from [4] and [5].

By a *homology theory* we always mean a partially exact homology theory defined on the category of compact pairs. For a homology theory H and a compact space X we denote by $\tilde{H}_t(X)$ the t-dimensional homology group $H_t(X)$ for $t>0$ and the reduced zero-dimensional homology group for $t=0$. If $f\colon X\to Y$ is a continuous mapping then $\tilde{H}_t(f)$ or also $\tilde{H}(f)$ will denote the map of $\tilde{H}_t(X)$ into $\tilde{H}_t(Y)$ defined by the induced map $f_*\colon H_t(X)\to H_t(Y)$.

1.2. A mapping $\mathfrak{v}\colon A\times B\to C$ where A, B and C are abelian groups is said to be a *bihomomorphism* if the following condition is satisfied:

$$\mathfrak{v}(a+a', b+b') = \mathfrak{v}(a, b)+\mathfrak{v}(a', b)+\mathfrak{v}(a, b')+\mathfrak{v}(a', b').$$

We say that a bihomomorphism $\mathfrak{v}\colon A\times B\to C$ is *trivial* if $\mathfrak{v}(A\times B)=0$.

1.3. Now suppose given two homology theories H and H', an abelian group C, the n-dimensional euclidean space R^n and integers t, t' satisfying $t+t'=n-1$.

DEFINITION. The map $\mathfrak{B}=\mathfrak{B}_{H,H',C,t,t'}$ which makes each ordered pair (N, N') of disjoint compact subsets of R^n correspond to a bihomomorphism $\mathfrak{v}_{N,N'}\colon \tilde{H}_t(N)\times\times\tilde{H}'_{t'}(N')\to C$ will be called a *theory of linking of compacts in R^n for the homology theories H and H'* if for any compacts M, M', N and N' in R^n satisfying $M\subset N$, $M'\subset N'$ and of course $N\cap N'=\emptyset$, the condition

$$\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{N,N'}\big(\tilde{H}_t(i)(u), \tilde{H}'_{t'}(i')(u')\big)$$

is satisfied for every $u\in\tilde{H}_t(M)$ and $u'\in\tilde{H}'_{t'}(M')$ where $i\colon M\subset N$ and $i'\colon M'\subset N'$ are the inclusion maps. The group C is said to be the *range* of the theory $\mathfrak{B}$.

Notice as a direct consequence of this definition that if $\mathfrak{B}=\mathfrak{B}_{H,H',C,t,t'}$ is a theory of linking of compacts in R^n and M, N, N' are compacts in R^n such that

$M \subset N \subset R^n \setminus N'$ then

$$\mathfrak{v}_{M,N'}(u, u') = \mathfrak{v}_{N,N'}(\tilde{H}_t(i)(u), u')$$

holds for every $u \in H_t(M)$ and $u' \in H'_{t'}(N')$ where $i \colon M \subset N$ is the inclusion map.

Likewise the relation

$$\mathfrak{v}_{N,M'}(u, u') = \mathfrak{v}_{N,N'}(u, \tilde{H}'_{t'}(i')(u'))$$

holds for every $u \in \tilde{H}_t(N)$ and $u' \in \tilde{H}'_{t'}(M')$ where $i' \colon M' \subset N'$ is the inclusion map and $N' \subset R^n \setminus N$.

1.4. We shall say that the theory of linking $\mathfrak{V} = \mathfrak{V}_{H,H',C,t,t'}$ of compacts in R^n is *degenerate* if for every pair of nonintersecting compact subsets M, M' of R^n $\mathfrak{v}_{M\,M'}$ is a trivial bihomomorphism.

2. Linking Jordan curves

2.1. Let p be a prime and G' an elementary cyclic p-group, i.e. G' is isomorphic to the group Z_p of integers $\bmod\, p$. Let H' be a continuous homology theory defined on the category $\mathcal{A}_c$ of all compact pairs and based on the coefficient group G'. Thus H' is isomorphic on $\mathcal{A}_c$ to the Čech homology theory over G'. We shall keep it fixed in the sequel.

Let R^n be the n-euclidean space where $n \geqq 2$, and let P be a compact subspace of R^n.

Let H be a homology theory and let u be an element of $\tilde{H}_{n-2}(P)$. Let C be an abelian group and let $\mathfrak{V} = \mathfrak{V}_{H,H',C,n-2,1}$ be a theory of linking of compacts in R^n.

DEFINITION. A Jordan curve J of $R^n \setminus P$ is said to be a *linking Jordan curve of* u (with respect to $\mathfrak{V} = \mathfrak{V}_{H,H',C,n-2,1}$) if there is a $u' \in H'_1(J)$ such that $\mathfrak{v}_{P,J}(u, u') \neq 0$.

2.2. REMARK. Let J be a linking Jordan curve of u. Then for each nonzero element u'_1 of $H'_1(J)$ we have $\mathfrak{v}_{P,J}(u, u'_1) \neq 0$.

In fact, select $u' \in H'_1(J)$ such that $\mathfrak{v}_{P,J}(u, u') \neq 0$. Since $H'_1(J)$ is isomorphic to Z_p it follows the existence of an integer m such that $u' = m u'_1$. Consequently

$$0 \neq \mathfrak{v}_{P,J}(u, u') = m \mathfrak{v}_{P,J}(u, u'_1)$$

implies $\mathfrak{v}_{P,J}(u, u'_1) \neq 0$ indeed.

Now we can state the following theorem.

2.3. THEOREM. *Let p and H' be the same as in* 2.1. *Suppose that G is an elementary cyclic p-group and H is a continuous homology theory defined on the category of all compact pairs and based on the coefficient group G. (We shall keep it fixed in the sequel, too.) Let C be an abelian group. Suppose that $\mathfrak{V} = \mathfrak{V}_{H,H'C,n-2,1}$ is a nondegenerate theory of linking for compacts in R^n where $n \geqq 2$. Let P be a compact set in R^n and u_0 a nonzero element of $\tilde{H}_{n-2}(P)$. Then there exists a linking Jordan curve of u_0 with respect to $\mathfrak{V}$.*

PROOF. Let S and S' be spheres of dimensions $n-2$ and 1, respectively, contained in R^n such that they are mutually linked, i.e.

(a) the center of S belongs to S' and the center of S' belongs to S,

(b) the planes R and R' supporting S and S', respectively, intersect in a line,

(c) R and R' are perpendicular in the natural sense that any vectors a in R and a' in R' which are perpendicular to the line $R\cap R'$ are mutually perpendicular.

Observe that if (N, N') is an ordered pair of disjoint compact sets of R^n then $\mathfrak{v}_{N,N'}$ will denote in the sequel the value of the theory $\mathfrak{V}_{H,H',C,n-2,1}$ on the pair (N, N').

The uniqueness and existence theorem of the theory of linking (see [5] and [6]) shows that $\mathfrak{v}_{S,S'}\colon \tilde{H}_{n-2}(S)\times H_1'(S')\to C$ is a nontrivial bihomomorphism (see [5] Corollary 2). Thus we can select $u_1\in\tilde{H}_{n-2}(S)$ and $u_1'\in H_1'(S')$ such that $\mathfrak{v}_{S,S'}(u_1, u_1')\neq\emptyset$. Hence $u_1\neq 0$, $u_1'\neq 0$ and since the groups $\tilde{H}_{n-2}(S)$ and $H_1'(S')$ are isomorphic to G and G', respectively, and thus they are isomorphic to Z_p it follows that u_1 and u_1' are generators of $\tilde{H}_{n-2}(S)$ and $H_1'(S')$, respectively.

Let $b=\mathfrak{v}_{S,S'}(u_1, u_1')$ and let B be the subgroup of C generated by b. Since

$$pb=\mathfrak{v}_{S,S'}(pu_1, u_1')=\mathfrak{v}_{S,S'}(0, u_1')=0$$

and $b\neq 0$ it follows that B is isomorphic to Z_p.

Observe that for every pair (N, N') of disjoint compacts in R^n we have

$$\mathfrak{v}_{N,N'}\big(\tilde{H}_{n-2}(N)\times H_1'(N')\big)\subset B$$

(see Corollary 1 of [5]).

Now let K be the continuous group of rotations of a circle which we will consider as an additive group of real numbers defined up to an additive integer and let $\psi\colon B\to K$ be a nonzero homomorphism. There exists obviously such a homomorphism and this homomorphism is a monomorphic mapping. Further if for any $v\in\tilde{H}_{n-2}(S)$ and $v'\in H_1'(S')$ we introduce the product

$$vv'=\psi\big(\mathfrak{v}_{S,S'}(v, v')\big)$$

then the compact discrete groups $\tilde{H}_{n-2}(S)$ and $H_1'(S')$ form a couple (see [8] Definition 3) and the groups $\tilde{H}_{n-2}(S)$ and $H_1'(S')$ are orthogonal in the sense that to each nonzero element v of $\tilde{H}_{n-2}(S)$ one can find a $v'\in H_1'(S')$ with $vv'\neq 0$ and to each nonzero element v' of $H_1'(S')$ there is a $v\in\tilde{H}_{n-2}(S)$ such that $vv'\neq 0$.

In the paper [6], starting from a bihomomorphism $\mathfrak{v}_0\colon \tilde{H}_{n-2}(S)\times H_1'(S')\to C$, we have constructed a bihomomorphism $\mathfrak{v}_1\colon G\times G'\to C$ (see [6] 19—22). In the same way starting from $\mathfrak{v}_{S,S'}$ we get a bihomomorphism $\mathfrak{v}_2\colon G\times G'\to B$, where for $g\in G$ and $g'\in G'$

$$\mathfrak{v}_2(g, g')=v_{S,S'}\big(\lambda(g), \lambda'(g')\big)$$

and

$$\lambda\colon G\to\tilde{H}_{n-2}(S),\quad \lambda'\colon G'\to H_1'(S')$$

are isomorphisms.

These isomorphisms λ and λ' can be constructed in the following way.

Let q and q' be the centers and R and R' the supporting planes of S and S', respectively. Let σ and σ' be simplexes of dimensions $n-1$ and 2, respectively, lying in R^n such that

(a') The supporting plane of σ is R and that of σ' is R'.

(b′) Let $\dot\sigma$ and $\dot\sigma'$ be the boundaries of σ and σ' in R and R', respectively. Then

$$\dot\sigma \cap \dot\sigma' = \emptyset, \quad \sigma \cap \dot\sigma' = \{q\}, \quad \dot\sigma \cap \sigma' = \{q'\}.$$

(c′) q is the interior point of a 1-face σ_1' of σ' (with respect to the supporting line of σ_1'). There exist obviously such simplexes σ and σ'.

Now let f_1; $\dot\sigma \to S$ and f_1': $\dot\sigma' \to S'$ be projections with centers q and q', respectively, i.e. for $x \in \dot\sigma$ and $x' \in \dot\sigma'$ x and $f_1(x)$ lie on the same ray issuing from q while x' and $f_1'(x')$ lie on the same ray issuing from q'.

Let us take an ordering $A^0 < A^1 < \ldots < A^{n-1}$ of the vertices of σ and an ordering $B^0 < B^1$ of the vertices of σ_1'. Let us take an ordering $D^0 < D^1 < D^2$ of the vertices of σ' such that $D^1 = B^0$ and $D^2 = B^1$. Thus we get ordered simplexes s, s_1' and s' (see [7] p. 55). The spaces $|s|$, $|s_1'|$ and $|s'|$ of s, s_1' and s' are the simplexes σ, σ_1' and σ', respectively (see [7] p. 55). Let t, t_1' and t' be the orientations of σ, σ_1' and σ' taking the value $+1$ on the sequences $|A^0, \ldots, A^{n-1}|$, $|B^0, B^1|$, $|D^0, D^1, D^2|$, respectively (see [2] p. 4).

Select the orientation ξ of R^n such that the intersection number $t \times t_1'$ of t and t_1' should be $+1$ (see [2] p. 10).

Let the mappings φ: $G \to H_{n-1}(\sigma, \dot\sigma)$ and φ': $G' \to H_2(\sigma', \dot\sigma')$ be defined by the relations

$$\varphi(g) = gs \quad \text{and} \quad \varphi'(g') = g's' \quad (g \in G,\ g' \in G')$$

(see [7] p. 80).

Let

$$\partial\colon H_{n-1}(\sigma, \dot\sigma) \to \tilde H_{n-2}(\dot\sigma) \quad \text{and} \quad \partial'\colon H_2(\sigma', \dot\sigma') \to H_1(\dot\sigma')$$

be the boundary operators of the compact pairs $(\sigma, \dot\sigma)$ and $(\sigma', \dot\sigma')$, respectively. Then

$$\lambda = \tilde H_{n-2}(f_1)\partial\varphi, \quad \lambda' = f_{1*}'\partial'\varphi' = \tilde H_1'(f_1')\partial'\varphi'.$$

Since all the mappings φ, φ', ∂, ∂', $\tilde H_{n-2}(f_1)$ and f_{1*}' are clearly isomorphisms it follows that λ and λ' are isomorphisms.

Now introducing the product $gg' = \psi(\mathfrak{v}_2(g, g'))$ for $g \in G$ and $g' \in G'$ it follows that G and G' form a couple and the groups G and G' are orthogonal with respect to this multiplication.

Now let Q be a compact metric space, Γ an abelian group and m a nonnegative integer. In [6] 24, there was defined the isomorphism

$$\eta_{m,Q,\Gamma}\colon \tilde\Delta^m(Q, \Gamma) \to \tilde H_m^1(Q, \Gamma)$$

where $\tilde H_m^1(Q, \Gamma)$ is the m-dimensional reduced Čech homology group of Q over the group Γ and $\tilde\Delta^m(Q, \Gamma)$ is the m-th reduced homology group of Q over Γ, i.e. the group of the homology classes of the normal proper cycles (in [9] true cycles) of Q. (The terminology is taken from [2].)

Moreover in [6], Section 29 we have defined the isomorphisms

$$h_0\colon G = H_0(P_0) \to H_0^1(P_0, G) \quad \text{and} \quad h_0'\colon G' = H_0'(P_0) \to H_0^1(P_0, G')$$

where $P_0 = \{\emptyset\}$ is the fixed one point space and for any abelian group Γ, $H_0^1(P_0, \Gamma)$ is the 0-dimensional Čech homology group of P_0 over Γ. (Observe that for any homology theory H'' the group $H_0''(P_0)$ is the coefficient group of H''.)

Let $h\colon \boldsymbol{H}\to\boldsymbol{H}^1(.,G)$ and $h'\colon \boldsymbol{H}'\to\boldsymbol{H}^1(.,G')$ be the extensions of these isomorphisms (see [7] p. 287) where $\boldsymbol{H}^1(.,G)$ and $\boldsymbol{H}^1(.,G')$ are the Čech homology theories over the groups G and G', respectively. Since by hypothesis the homology theories $\boldsymbol{H}$ and $\boldsymbol{H}'$ are continuous it follows that h and h' are isomorphisms (see [7] p. 288).

We also remark that for any compact T_2-space Q and for any nonnegative integer m we have obviously

$$h(m,Q,\emptyset)\big(\tilde{H}_m(Q)\big)=\tilde{H}^1_m(Q,G). \tag{1}$$

Now the uniqueness theorem and the proof of the existence theorem of the theory of linking (see [6]) show that for each compact subset N' of $R^n\setminus P$ and for every $u'\in H_1'(N')$ we have

$$\mathfrak{v}_{P,N}(u_0,u')=\mathfrak{v}\big(\eta^{-1}_{n-2,P,G}h(n-2,P,\emptyset)(u_0),\ \eta^{-1}_{1,N',G'}h'(1,N',\emptyset)(u')\big) \tag{2}$$

(see Sections 27 and 30 in [6]) where for $w\in\tilde{\Delta}^{n-2}(P,G)$ and $w'\in\Delta^1(N',G')$ starting from the bihomomorphism $\mathfrak{v}_2$ the linking coefficient $\mathfrak{v}(w,w')$ is defined in the usual way (see [3]). In details this means the following:

Let $x^q=\sum_{i=1}^{y}g_it_i^q$ and $x'^{q'}=\sum_{j=1}^{y'}g_j't_j'^{q'}$ be two chains in R^n in relative general position (see [3] p. 73) where $q+q'=n$ and $g_i\in G$ and $g_j'\in G'$ hold for $i=1,\dots,y$ and $j=1,\dots,y'$. Then the intersection number $x^q\times x'^{q'}$ of the chains x^q and $x'^{q'}$ is defined by the relation

$$x^q\times x'^{q'}=\sum_{i=1}^{y}\sum_{j=1}^{y'}t_i^q\times t_j'^{q'}\mathfrak{v}_2(g_i,g_j').$$

Chapter XV of [3] deals with the particular case $G=G'$ where G is a ring with unity and $\mathfrak{v}_2(a,b)=ab$ for $a,b\in G$. However the main results of this chapter remain true also after our modification. In particular we can define the linking number $\mathfrak{v}(z_1^q,z_2^{q'-1})$ of two nonintersecting cycles z_1^q and $z_2^{q'-1}$ in R^n with $q+q'=n$ and with z_1^q a normal cycle if $q=0$ so as it is done in [3] p. 80. Likewise we can define the linking number $\mathfrak{v}(\mathfrak{z}^q,z^{q'-1})$ of the normal proper cycle $\mathfrak{z}^q$ of the compact subset Q of R^n and of the cycle $z^{q'-1}$ of the open set $\Omega=R^n\setminus Q$. Furthermore we can define the linking number $\mathfrak{v}(\mathfrak{z}^q,\mathfrak{z}'^{q'-1})$ of the normal proper cycles $\mathfrak{z}^q$ and $\mathfrak{z}'^{q'-1}$ of the disjoint compact subsets Q and Q' of R^n so as it is done in [3] p. 86. Now let $\mathfrak{z}^{n-2}\in w$ and $\mathfrak{z}'^1\in w'$. Then by definition we have

$$\mathfrak{v}(w,w')=\mathfrak{v}(\mathfrak{z}^{n-2},\mathfrak{z}'^1). \tag{3}$$

Let us compare these considerations with the notions of [9]. In [9] one starts from the bihomomorphism $\psi\mathfrak{v}_2\colon G\times G'\to K$ where G and G' form an orthogonal couple with respect to the multiplication

$$gg'=\psi\mathfrak{v}_2(g,g')\quad(g\in G,\ g'\in G').$$

Thus we get the index of intersection $X(x^q,x'^{q'})$ with respect to [9] of the chains

$$x^q=\sum_{i=1}^{y}g_it_i^q\quad\text{and}\quad x'^{q'}=\sum_{j=1}^{y'}g_j't_j'^{q'}$$

lying in relative general position in R^n and satisfying the condition $q+q'=n$ as follows:

$$X(x^q, x'^{q'}) = \sum_{i=1}^{y} \sum_{j=1}^{y'} g_i g'_j (t_i^q \times t_j'^{q'}).$$

Consequently

$$X(x^q, x'^{q'}) = \sum_{i=1}^{y} \sum_{j=1}^{y'} \psi\big(\mathfrak{v}_2(g_i, g'_j)\big) t_i^q \times t_j'^{q'} =$$

$$= \psi\left(\sum_{i=1}^{y} \sum_{j=1}^{y'} t_i^q \times t_j'^{q'} \mathfrak{v}_2(g_i, g'_j)\right) = \psi(x^q \times x'^{q'}).$$

Hence denoting the coefficient of linking defined in [9] by $\mathfrak{v}'$ and taking also the disagreement between the Definition XV. 1.42 of [3] and Definition 2 of [9] into account we get the relation

$$\mathfrak{v}'(z_1^q, z_2^{q'-1}) = (-1)^{q+1} \psi\big(\mathfrak{v}(z_1^q, z_2^{q'-1})\big)$$

for any two nonintersecting cycles z_1^q and $z_2^{q'-1}$ of R^n. Also, it is supposed that $q+q'=n$, z_1^q is a cycle over the group G and it is normal if $q=0$. Moreover $z_2^{q'-1}$ is a cycle over the group G' and it is normal if $q'-1=0$.

Consequently if $\mathfrak{z}^q$ is a proper cycle of a compact set Q in R^n over the group G and it is normal in the case $q=0$ and $z'^{q'-1}$ is a cycle in $R^n \setminus Q$ over the group G' and it is normal in the case $q'-1=0$ and $q+q'=n$ we then have

$$\mathfrak{v}'(\mathfrak{z}^q, z'^{q'-1}) = (-1)^{q+1} \psi\big(\mathfrak{v}(\mathfrak{z}^q, z'^{q'-1})\big).$$

Now since $h(n-2, P, \emptyset)\colon H_{n-2}(P) \to H^1_{n-2}(P, G)$ is an isomorphism and $u_0 \neq 0$ by hypothesis, taking also (1) into account it follows that

$$0 \neq h(n-2, P, \emptyset)(u_0) \in \tilde{H}^1_{n-2}(P, G)$$

and thus

$$w_0 = \eta^{-1}_{n-2, P, G} h(n-2, P, \emptyset)(u_0) \neq 0. \tag{4}$$

Since G is a compact T_0-topological group it follows that the group $\tilde{\Delta}^{n-2}(P, G)$ can be considered as a compact T_0-topological group satisfying the second axiom of countability (see [9]).

Let $\mathfrak{z}_0^{n-2}$ be a proper cycle of the homology class $w_0 \in \tilde{\Delta}^{n-2}(P, G)$. Then applying the theorem of duality of Pontrjagin (see [9]) taking also $w_0 \neq 0$ into account we can state that there exists a cycle z'^1 of $R^n \setminus P$ over the group G' such that

$$\mathfrak{v}'(\mathfrak{z}_0^{n-2}, z'^1) = (-1)^{n-1} \psi\big(\mathfrak{v}(\mathfrak{z}_0^{n-2}, z'^1)\big) \neq 0$$

and thus $\mathfrak{v}(\mathfrak{z}_0^{n-2}, z'^1) \neq 0$ which yields $z'^1 \neq 0$.

The proof of the theorem of duality of Pontrjagin also shows that z'^1 can be selected such that it should be a cycle of a finite 1-dimensional euclidean complex K' lying in $R^n \setminus P$.

We now interrupt the proof of the theorem and formulate a lemma which we shall use in the sequel.

To prepare it first observe that every finite 1-dimensional euclidean complex K'' (in terms of [1] a triangulation of dimension 1) lying in R^n can be considered as a finite graph. The edges of this graph are the 1-simplexes of K'' and the vertices are the 0-simplexes. The proper faces of a 1-simplex are the endpoints of this edge.

Let Γ be an abelian group. For any 1-chain x^1 of K'' over the group Γ we use the notation $|x^1|$ for the subcomplex of K'' consisting of all edges of K'' on which x^1 does not vanish and of all endpoints of such edges, while the body $\overline{x^1}$ of x^1 is the body of the complex $|x^1|$ i.e. the subspace of R^n consisting of all points of all simplexes of $|x^1|$ (cf. [1] p. 136).

We now turn to the formulation of the lemma.

LEMMA. *Let K'' be a finite 1-dimensional euclidean complex lying in R^n. Let Γ be an abelian group and let z' be a nonzero 1-cycle of K'' over Γ. Then z' can be represented in the form*

$$z' = z_1' + \ldots + z_s'$$

such that for $i=1, \ldots, s$ z_i' is a nonzero cycle of K'' and $|z_i'|$ is a circuit of the graph K''.

The proof is quite simple, thus it will be omitted.

Continuing the proof of the theorem, consider now a representation

$$z'^1 = z_1'^1 + \ldots + z_s'^1$$

of z'^1 as it is required in the preceding lemma. We then have

$$0 \neq \mathfrak{v}(\mathfrak{z}_0^{n-2}, z'^1) = \mathfrak{v}(\mathfrak{z}_0^{n-2}, z_1'^1 + \ldots + z_s'^1) = \sum_{i=1}^{s} \mathfrak{v}(\mathfrak{z}_0^{n-2}, z_i'^1).$$

Consequently there exists an i such that $\mathfrak{v}(\mathfrak{z}_0^{n-2}, z_i'^1) \neq 0$. Since $|z_i'^1|$ is a circuit of K' it follows that the body of $z_i'^1$ is a Jordan curve J contained in $R^n \setminus P$.

Now for any nonnegative integer y let $z_{i,y}'^1$ be the barycentric subdivision of order y of the cycle $z_i'^1$. Then

$$\mathfrak{z}_0'^1 = (z_i'^1 = z_{i,0}'^1, \ldots, z_{i,y}'^1, \ldots)$$

is a proper cycle of J over the group G'. Moreover we have

$$0 \neq \mathfrak{v}(\mathfrak{z}_0^{n-2}, z_i'^1) = \mathfrak{v}(\mathfrak{z}_0^{n-2}, \mathfrak{z}_0'^1)$$

(see also [3] p. 86 and [6] 42). Let w_0' be the homology class of $\mathfrak{z}_0'^1$ in the group $\Delta^1(J, G')$. Then (3) shows that $\mathfrak{v}(w_0, w_0') = \mathfrak{v}(\mathfrak{z}_0^{n-2}, \mathfrak{z}_0'^1)$ and thus $\mathfrak{v}(w_0, w_0') \neq 0$.

Let

$$u_0' = h'(1, J, \emptyset)^{-1} \eta_{1,J,G'}(w_0').$$

Then $u_0' \in H_1'(J)$ and by (2) and (4) we have $\mathfrak{v}_{P,J}(u_0, u_0') = \mathfrak{v}(w_0, w_0')$. Consequently $\mathfrak{v}_{P,J}(u_0, u_0') \neq 0$. J is a linking Jordan curve of u_0 with respect to $\mathfrak{B}$ indeed.

The proof of the Theorem is complete.

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EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

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PROXIMITIES, SCREENS, MEROTOPIES, UNIFORMITIES. III

Á. CSÁSZÁR (Budapest), member of the Academy

9. Semi-merotopies. The category **Scr** can be brought into connection with another important topological category introduced by M. Katětov ([14], [15]), namely that of merotopies. For our purposes, it will be useful to study a somewhat more general concept.

If X is a given set, we say that a subset $\mathfrak{M}\neq\emptyset$ of $\exp\exp X$ is a *semi-merotopy* on X iff

M1. $\mathfrak{m}\neq\emptyset$ for $\mathfrak{m}\in\mathfrak{M}$,

M2. For $x\in X$, there is $\mathfrak{m}\in\mathfrak{M}$ such that $\mathfrak{m}$ is fixed at x,

M3. If $\mathfrak{m}_1\in\mathfrak{M}$, $\mathfrak{m}_2\in\exp\exp X$, $\mathfrak{m}_1<\mathfrak{m}_2$, then $\mathfrak{m}_2\in\mathfrak{M}$.

$\mathfrak{M}$ is said to be a *merotopy* on X iff, moreover,

M4. If $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2\in\mathfrak{M}$ then either $\mathfrak{m}_1\in\mathfrak{M}$ or $\mathfrak{m}_2\in\mathfrak{M}$.

If $\mathfrak{M}$ and $\mathfrak{M}'$ are semi-merotopies on X and Y, respectively, and $f\colon X\to Y$, then f will be said to be $(\mathfrak{M}, \mathfrak{M}')$*-continuous* iff $\mathfrak{m}\in\mathfrak{M}$ implies $f(\mathfrak{m})\in\mathfrak{M}'$. Then the semi-merotopies and $(\mathfrak{M}, \mathfrak{M}')$-continuous maps constitute a concrete category **Smer**. The full subcategory of **Smer** the objects of which are the merotopies will be denoted by **Mer** (by [11], p. 75, **Smer** is isomorphic to **P-Near**).

We show that **Smer** is a strongly topological category. Observe first that, if $\mathfrak{M}$, $\mathfrak{M}'$ are semi-merotopies on X, then $\{\emptyset\}\in\mathfrak{M}$, $\{\{x\}\}\in\mathfrak{M}$ for $x\in X$, and $\mathfrak{M}$ is coarser than $\mathfrak{M}'$ iff $\mathfrak{M}\supset\mathfrak{M}'$.

(9.1) LEMMA. *Let $\mathfrak{M}_0$ be a semi-merotopy on X, $g\colon Z\to X$, and define $g^{-1}(\mathfrak{M}_0)$ to be the collection of all elements $\mathfrak{m}\in\exp\exp X$ such that $g(\mathfrak{m})\in\mathfrak{M}_0$. Then $g^{-1}(\mathfrak{M}_0)$ is a semi-merotopy on Z such that g is $(\mathfrak{M}, \mathfrak{M}_0)$-continuous iff $\mathfrak{M}$ is finer than $g^{-1}(\mathfrak{M}_0)$.*

PROOF. $g(\{\emptyset\})=\{\emptyset\}$, $g(\{\{z\}\})=\{\{g(z)\}\}$ for $z\in Z$, $g(\emptyset)=\emptyset$, $\mathfrak{m}_1<\mathfrak{m}_2$ implies $g(\mathfrak{m}_1)<g(\mathfrak{m}_2)$. g is $(\mathfrak{M}, \mathfrak{M}_0)$-continuous iff $\mathfrak{M}\subset g^{-1}(\mathfrak{M}_0)$. $\square$

(9.2) LEMMA. *If $g\colon Z\to X$, $f\colon X\to Y$, and $\mathfrak{M}_0$ is a semi-merotopy on Y, then $(f\circ g)^{-1}(\mathfrak{M}_0)=g^{-1}(f^{-1}(\mathfrak{M}_0))$.* $\square$

(9.3) LEMMA. *Let $\mathfrak{M}_i$ be a semi-merotopy on X for $i\in I$. If $I=\emptyset$, let $\mathfrak{M}=\sup\{\mathfrak{M}_i\colon i\in I\}$ denote the set of all non-empty elements of* $\exp\exp X$. *If $I\neq\emptyset$, then $\mathfrak{M}=\bigcap_{i\in I}\mathfrak{M}_i$ is a semi-merotopy on X, and a semi-merotopy $\mathfrak{M}'$ on X is finer than every $\mathfrak{M}_i$ iff $\mathfrak{M}'$ is finer than $\mathfrak{M}$, so that we can define $\mathfrak{M}=\sup\{\mathfrak{M}_i\colon i\in I\}$.*

PROOF. The case $I=\emptyset$ is obvious. If $I\neq\emptyset$, then $\emptyset\notin\mathfrak{M}$, $\{\emptyset\}\in\mathfrak{M}$, $\{\{x\}\}\in\mathfrak{M}$ for $x\in X$, and $\mathfrak{m}_1<\mathfrak{m}_2\in\exp\exp X$, $\mathfrak{m}_1\in\mathfrak{M}$ implies $\mathfrak{m}_2\in\mathfrak{M}$, so that $\mathfrak{M}$ is a semi-merotopy on X. $\square$

(9.4) LEMMA. *If $\mathfrak{M}_i$ is a semi-merotopy on X for $i\in I$, $g\colon Z\to X$, then*

$$\sup\{g^{-1}(\mathfrak{M}_i)\colon i\in I\} = g^{-1}\big(\sup\{\mathfrak{M}_i\colon i\in I\}\big). \quad \square$$

(9.5) THEOREM (cf. [11], 6.4). *The category* **Smer** *is strongly topological.*

PROOF. (0.1). $\square$

A set $\mathfrak{B}\subset\mathfrak{M}$ will be called a *base* for the semi-merotopy $\mathfrak{M}$ iff, for $\mathfrak{m}\in\mathfrak{M}$, there is $\mathfrak{b}\in\mathfrak{B}$ such that $\mathfrak{b}<\mathfrak{m}$.

(9.6) LEMMA. *If $\mathfrak{B}$ is a base for the semi-merotopy $\mathfrak{M}$ on X then*

B1. $\mathfrak{b}\neq\emptyset$ *for* $\mathfrak{b}\in\mathfrak{B}$,
B2. *For $x\in X$, there is a $\mathfrak{b}\in\mathfrak{B}$ fixed at x.*

Conversely if $\mathfrak{B}$ is a subset of $\exp\exp X$ *satisfying* B1 *and* B2, *then $\mathfrak{B}$ is a base for one and only one semi-merotopy $\mathfrak{M}$, composed of all $\mathfrak{m}\in\exp\exp X$ such that $\mathfrak{b}<\mathfrak{m}$ for some $\mathfrak{b}\in\mathfrak{B}$.* $\square$

(9.7) LEMMA. *If $\mathfrak{S}$ is a screen on X, then it is a base for a merotopy on X.*

PROOF. $\mathfrak{S}$ satisfies B1 and B2. If $\mathfrak{s}\in\mathfrak{S}$, $\mathfrak{s}<\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2\in\exp\exp X$, and, indirectly, we assume that neither $\mathfrak{m}_1$ nor $\mathfrak{m}_2$ is finer than $\mathfrak{s}$, then there are $S_i\in\mathfrak{s}$ such that S_i does not contain any subset belonging to $\mathfrak{m}_i$ $(i=1,2)$. Then $S=S_1\cap\cap S_2\in\mathfrak{s}$ does not contain any subset belonging to $\mathfrak{m}$; a contradiction. $\square$

(9.8) LEMMA. *If $\mathfrak{B}$ is a base for a semi-merotopy $\mathfrak{M}$, then $\mathfrak{M}$ is the smallest semi-merotopy containing $\mathfrak{B}$. Conversely if $\mathfrak{M}$ is a semi-merotopy on X, $\mathfrak{F}\subset\mathfrak{M}$, and $\mathfrak{M}$ is the smallest semi-merotopy containing $\mathfrak{F}$, then $\mathfrak{B}=\mathfrak{F}\cup\{\dot{x}\colon x\in X\}$ (where $\dot{x}=\{A\in\exp X\colon x\in A\}$) is a base for $\mathfrak{M}$. If the elements of $\mathfrak{F}$ are filters in X, and $\mathfrak{M}$ is the smallest merotopy containing $\mathfrak{F}$, then $\mathfrak{B}$ is a base for $\mathfrak{M}$ again.*

PROOF. If $\mathfrak{M}'\supset\mathfrak{B}$ is a semi-merotopy, and $\mathfrak{B}$ is a base for $\mathfrak{M}$, then by (9.6) $\mathfrak{M}'\supset\mathfrak{M}$. If $\mathfrak{M}$ is the smallest semi-merotopy containing $\mathfrak{F}$, then by (9.6) $\mathfrak{B}$ is a base for a semi-merotopy $\mathfrak{M}'$. Since $x\in X$ implies $\{\{x\}\}\in\mathfrak{M}$ and $\{\{x\}\}<\dot{x}$, $\mathfrak{B}\subset\mathfrak{M}$ and $\mathfrak{M}'\supset\mathfrak{M}$ (because $\mathfrak{M}'\supset\mathfrak{F}$), while $\mathfrak{M}'\subset\mathfrak{M}$ by (9.6). Hence $\mathfrak{B}$ is a base for $\mathfrak{M}$. If $\mathfrak{F}$ is composed of filters and $\mathfrak{M}$ is the smallest merotopy containing $\mathfrak{F}$, then $\mathfrak{B}$ is a screen, hence $\mathfrak{M}'$ is a merotopy by (9.7), and $\mathfrak{M}'\subset\mathfrak{M}\subset\mathfrak{M}'$ again. $\square$

A merotopy $\mathfrak{M}$ is said to be a *filter-merotopy* ([14], 2.1) iff there is a set $\mathfrak{F}$ of filters such that $\mathfrak{M}$ is the smallest merotopy containing $\mathfrak{F}$.

(9.9) LEMMA. *A semi-merotopy is a filter-merotopy iff there is a base for $\mathfrak{M}$ composed of filters.*

PROOF. (9.7), (9.8). $\square$

The full subcategory of **Smer** the objects of which are the filter-merotopies will be denoted by **Fmer**. We now examine the subcategories **Mer** and **Fmer** of **Smer**.

(9.10) LEMMA. *Let* $\mathfrak{M}$ *be a semi-merotopy on* X. *The union* $\mathfrak{M}^q$ *of all merotopies contained in* $\mathfrak{M}$ *is the coarsest merotopy finer than* $\mathfrak{M}$. $\mathfrak{M}^q$ *is composed of all systems* $\mathfrak{m}\in\mathfrak{M}$ *such that, whenever* $\mathfrak{m}=\bigcup_1^n \mathfrak{m}_i$, $n\in\mathbf{N}$, *there is an i satisfying* $\mathfrak{m}_i\in\mathfrak{M}$.

PROOF. If $\mathfrak{M}'\subset\mathfrak{M}$ is a merotopy and $\mathfrak{m}\in\mathfrak{M}'$, $\mathfrak{m}=\bigcup_1^n \mathfrak{m}_i$, then there is an i such that $\mathfrak{m}_i\in\mathfrak{M}'$; this can be shown by an easy induction on n. Therefore it suffices to check that the systems $\mathfrak{m}\in\mathfrak{M}$ described in the last sentence of the statement constitute a merotopy $\mathfrak{M}_0$.

Now $\{\{x\}\}\in\mathfrak{M}_0$ for $x\in X$, and $\mathfrak{m}\in\mathfrak{M}_0$, $\mathfrak{m}<\mathfrak{m}'$ implies $\mathfrak{m}'\in\mathfrak{M}_0$; in fact, if $\mathfrak{m}'=\bigcup_1^n \mathfrak{m}_i'$, we can write $\mathfrak{m}=\bigcup_1^n \mathfrak{m}_i$ in such a manner that $\mathfrak{m}_i<\mathfrak{m}_i'$ ($M\in\mathfrak{m}$ belongs to $\mathfrak{m}_i$ iff it has a subset belonging to $\mathfrak{m}_i'$). Now $\mathfrak{m}_i\in\mathfrak{M}$ for at least one i, and $\mathfrak{m}_i'\in\mathfrak{M}$ for the same i. Finally if $\mathfrak{m}\in\mathfrak{M}_0$, $\mathfrak{m}=\mathfrak{m}'\cup\mathfrak{m}''$, then either $\mathfrak{m}'\in\mathfrak{M}_0$ or $\mathfrak{m}''\in\mathfrak{M}_0$; otherwise we would have $\mathfrak{m}'=\bigcup_1^m \mathfrak{m}_i'$, $\mathfrak{m}''=\bigcup_1^n \mathfrak{m}_j''$, $\mathfrak{m}_i'\notin\mathfrak{M}$, $\mathfrak{m}_j''\notin\mathfrak{M}$ for every i and j, in contradiction with $\mathfrak{m}\in\mathfrak{M}_0$. $\square$

(9.11) LEMMA. *Let* $\mathfrak{M}$ *be a semi-merotopy on* X. *The filters belonging to* $\mathfrak{M}$ *constitute a screen, hence a base for a filter-merotopy* $\mathfrak{M}^f$, *which is the coarsest filter-merotopy finer than* $\mathfrak{M}$.

PROOF. By $\dot{x}\in\mathfrak{M}$ for $x\in X$ and (9.7), $\mathfrak{M}^f$ is a filter-merotopy contained in $\mathfrak{M}$. If $\mathfrak{M}'\subset\mathfrak{M}$ is a filter-merotopy, then there is a base $\mathfrak{B}'\subset\mathfrak{M}'$ for $\mathfrak{M}'$ composed of filters, so that $\mathfrak{M}'\subset\mathfrak{M}^f$. $\square$

(9.12) LEMMA. *If* $\mathfrak{M}$ *is a semi-merotopy on* X, *and* $g\colon Z\to X$, *then*

$$g^{-1}(\mathfrak{M}^q)=g^{-1}(\mathfrak{M})^q,\quad g^{-1}(\mathfrak{M}^f)=g^{-1}(\mathfrak{M})^f.$$

PROOF. If $\mathfrak{M}$ is a merotopy, then clearly $g^{-1}(\mathfrak{M})$ is a merotopy as well. Hence $g^{-1}(\mathfrak{M}^q)$ is a merotopy contained in $g^{-1}(\mathfrak{M})$, and $g^{-1}(\mathfrak{M})^q\supset g^{-1}(\mathfrak{M}^q)$.

On the other hand, let $\mathfrak{M}'\subset g^{-1}(\mathfrak{M})$ be a merotopy, and define

$$\mathfrak{B}=\{g(\mathfrak{m}')\colon \mathfrak{m}'\in\mathfrak{M}'\}\cup\{\dot{x}\colon x\in X\}.$$

Then $\mathfrak{B}\subset\mathfrak{M}$ is a base for a semi-merotopy $\mathfrak{M}_0\subset\mathfrak{M}$. $\mathfrak{M}_0$ is a merotopy; in fact $g(\mathfrak{m}')<\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$, $\mathfrak{m}'\in\mathfrak{M}'$ implies $\mathfrak{m}'=\mathfrak{m}_1'\cup\mathfrak{m}_2'$ where $\mathfrak{m}_i'$ is composed of those $M'\in\mathfrak{m}'$ for which $g(\mathfrak{M}')$ contains a subset belonging to $\mathfrak{m}_i$. Then $\mathfrak{m}_i'\in\mathfrak{M}'$ for some i, and $g(\mathfrak{m}_i')<\mathfrak{m}_i\in\mathfrak{M}_0$. If $\dot{x}<\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$, then either $\{x\}$ or $\emptyset$ belongs to $\mathfrak{m}$, hence to $\mathfrak{m}_i$ for some i, and $\dot{x}<\mathfrak{m}_i\in\mathfrak{M}_0$. Now we have $\mathfrak{B}\subset\mathfrak{M}_0\subset\mathfrak{M}^q$, and clearly $\mathfrak{M}'\subset g^{-1}(\mathfrak{M}^q)$, so that $g^{-1}(\mathfrak{M})^q\subset g^{-1}(\mathfrak{M}^q)$.

Now let $\mathfrak{M}$ be a filter-merotopy with a base $\mathfrak{B}$ composed of filters. Then the filters in Z generated by the filter bases of the form $g^{-1}(\mathfrak{b})$, $\mathfrak{b}\in\mathfrak{B}$, constitute a base $\mathfrak{B}'$ for $g^{-1}(\mathfrak{M})$; in fact, $\mathfrak{b}<g(g^{-1}(\mathfrak{b}))$ shows $g^{-1}(\mathfrak{b})\in g^{-1}(\mathfrak{M})$ for $\mathfrak{b}\in\mathfrak{B}$, and $g^{-1}(\mathfrak{b})<\mathfrak{b}'\in g^{-1}(\mathfrak{M})$ for the filter $\mathfrak{b}'$ generated by $g^{-1}(\mathfrak{b})$. Hence $\mathfrak{B}'\subset g^{-1}(\mathfrak{M})$. On the other hand, if $\mathfrak{m}'\in g^{-1}(\mathfrak{M})$, then $\mathfrak{b}<g(\mathfrak{m}')$ for some $\mathfrak{b}\in\mathfrak{B}$, hence $g^{-1}(\mathfrak{b})<$ $<\mathfrak{m}'$, and $\mathfrak{b}'<g^{-1}(\mathfrak{b})<\mathfrak{m}'$, $\mathfrak{b}'\in\mathfrak{B}'$ for $\mathfrak{b}'$ as above (except when $\emptyset\in g^{-1}(\mathfrak{b})$; in this case $\emptyset\in\mathfrak{m}'$, $\mathfrak{b}'<\mathfrak{m}'$ for an arbitrary $\mathfrak{b}'$ obtained from a $\mathfrak{b}\in\mathfrak{B}$ such that $\mathfrak{b}<\dot{x}$, $x=g(z)$, $z\in Z$). Therefore $g^{-1}(\mathfrak{M})$ is a filter-merotopy provided so is $\mathfrak{M}$.

Now, as in the first part of the proof, we get

$$g^{-1}(\mathfrak{M}^f) \subset g^{-1}(\mathfrak{M})^f.$$

If $\mathfrak{s}' \in g^{-1}(\mathfrak{M})$ is a filter, then $g(\mathfrak{s}') \in \mathfrak{M}$ is a filter base that generates in X a filter $\mathfrak{s}$ such that $g(\mathfrak{s}') < \mathfrak{s} \in \mathfrak{M}$, and $\mathfrak{s} \in \mathfrak{M}^f$, $\mathfrak{s} < g(\mathfrak{s}') \in \mathfrak{M}^f$. Thus $\mathfrak{s}' \in g^{-1}(\mathfrak{M}^f)$ for every filter $\mathfrak{s}'$ from $g^{-1}(\mathfrak{M})$, and $g^{-1}(\mathfrak{M})^f \subset g^{-1}(\mathfrak{M}^f)$. □

Since obviously $\mathfrak{M} = \mathfrak{M}^q$ for a merotopy, $\mathfrak{M} = \mathfrak{M}^f$ for a filter-merotopy, we obtain from (0.3):

(9.13) THEOREM. *The subcategories* **Mer** *and* **Fmer** *are bicoreflective in* **Smer** *and in every full subcategory of* **Smer** *containing them, the coreflections of* $\mathfrak{M}$ *are* $\mathfrak{M}^q$ *and* $\mathfrak{M}^f$, *respectively. Both* **Mer** *and* **Fmer** *are strongly topological with the operations*

$$g^{-1}_{\mathbf{Mer}}(\mathfrak{M}) = g^{-1}_{\mathbf{Fmer}}(\mathfrak{M}) = g^{-1}_{\mathbf{Smer}}(\mathfrak{M}),$$

$$\sup\nolimits_{\mathbf{Mer}}\{\mathfrak{M}_i \colon i \in I\} = \Big(\bigcap_{i \in I} \mathfrak{M}_i\Big)^q \quad (I \neq \emptyset),$$

$$\sup\nolimits_{\mathbf{Fmer}}\{\mathfrak{M}_i \colon i \in I\} = \Big(\bigcap_{i \in I} \mathfrak{M}_i\Big)^f \quad (I \neq \emptyset),$$

$$\sup\nolimits_{\mathbf{Mer}}\{\mathfrak{M}_i \colon i \in I\} = \sup\nolimits_{\mathbf{Fmer}}\{\mathfrak{M}_i \colon i \in I\} = \sup\nolimits_{\mathbf{Smer}}\{\mathfrak{M}_i \colon i \in I\}$$

for $I = \emptyset$. □

(For **Mer**, see [11], 7.2.)

The relationship between filter-merotopies and screens can be expressed in a more precise way:

(9.14) LEMMA. *If* $\mathfrak{M}$ *is a filter-merotopy on* X, *the filters that belong to* $\mathfrak{M}$ *constitute an ascending screen* $\mathfrak{S}(\mathfrak{M})$ *on* X. *If* $\mathfrak{S}$ *is an ascending screen on* X, *it is a base for a filter-merotopy* $\mathfrak{M}(\mathfrak{S})$ *on* X. *We have*

$$\mathfrak{M}(\mathfrak{S}(\mathfrak{M})) = \mathfrak{M}, \quad \mathfrak{S}(\mathfrak{M}(\mathfrak{S})) = \mathfrak{S}$$

for every filter-merotopy $\mathfrak{M}$ *and every ascending screen* $\mathfrak{S}$.

PROOF. $\mathfrak{S}(\mathfrak{M})$ is a screen since $\dot{x} \in \mathfrak{M}$ for $x \in X$, and it is obviously ascending. The second statement is contained in (9.7). The equalities are straightforward. □

(9.15) LEMMA. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be filter-merotopies on* X *and* Y, *respectively. A map* $f \colon X \to Y$ *is* $(\mathfrak{M}, \mathfrak{M}')$-*continuous iff it is* $(\mathfrak{S}(\mathfrak{M}), \mathfrak{S}(\mathfrak{M}'))$-*continuous.*

PROOF. If f is $(\mathfrak{M}, \mathfrak{M}')$-continuous, $\mathfrak{s} \in \mathfrak{S}(\mathfrak{M})$, then $f(\mathfrak{s}) \in \mathfrak{M}'$, hence $\mathfrak{s}' < f(\mathfrak{s})$ for some $\mathfrak{s}' \in \mathfrak{S}(\mathfrak{M}')$, and f is $(\mathfrak{S}(\mathfrak{M}), \mathfrak{S}(\mathfrak{M}'))$-continuous.

Conversely, if f is $(\mathfrak{S}(\mathfrak{M}), \mathfrak{S}(\mathfrak{M}'))$-continuous, $\mathfrak{m} \in \mathfrak{M}$, then $\mathfrak{s} < \mathfrak{m}$ for some $\mathfrak{s} \in \mathfrak{S}(\mathfrak{M})$, hence $\mathfrak{s}' < f(\mathfrak{s})$ for some $\mathfrak{s}' \in \mathfrak{S}(\mathfrak{M}')$, and $\mathfrak{s}' < f(\mathfrak{s}) < f(\mathfrak{m})$ implies $f(\mathfrak{m}) \in \mathfrak{M}'$ so that f is $(\mathfrak{M}, \mathfrak{M}')$-continuous. □

(9.16) THEOREM. *We induce an isomorphism* F: **Fmer** $\to$ **Ascr** *by defining* $F(\mathfrak{M}) = \mathfrak{S}(\mathfrak{M})$. □

More generally, the second part of (9.12) can be completed in the following manner:

(9.17) LEMMA. *Let the screen $\mathfrak{S}$ be a base for the filter-merotopy $\mathfrak{M}$ on X, $g\colon Z\to X$; then $g^{-1}_{\mathrm{Scr}}(\mathfrak{S})$ is a base for $g^{-1}_{\mathrm{Fmer}}(\mathfrak{M})$.*

PROOF. If $\mathfrak{s}\in\mathfrak{S}$ and $\mathfrak{s}'$ denotes the filter generated in Z by $g^{-1}(\mathfrak{s})$, then $\mathfrak{s}\in\mathfrak{M}$ implies $g(g^{-1}(\mathfrak{s}))\in\mathfrak{M}$ by $\mathfrak{s}<g(g^{-1}(\mathfrak{s}))$, hence $g^{-1}(\mathfrak{s})\in g^{-1}_{\mathrm{Smer}}(\mathfrak{M})$, and $\mathfrak{s}'\in g^{-1}_{\mathrm{Smer}}(\mathfrak{M})=$ $=g^{-1}_{\mathrm{Fmer}}(\mathfrak{M})$ by (9.13) and $g^{-1}(\mathfrak{s})<\mathfrak{s}'$.

Conversely, if $\mathfrak{m}'\in g^{-1}_{\mathrm{Fmer}}(\mathfrak{M})$, then $g(\mathfrak{m}')\in\mathfrak{M}$ and there is an $\mathfrak{s}\in\mathfrak{S}$ such that $\mathfrak{s}<g(\mathfrak{m}')$. Then $g^{-1}(\mathfrak{s})<g^{-1}(g(\mathfrak{m}'))<\mathfrak{m}'$ and the filter $\mathfrak{s}'$ generated by $g^{-1}(\mathfrak{s})$ satisfies $\mathfrak{s}'\in g^{-1}_{\mathrm{Scr}}(\mathfrak{S})$, $\mathfrak{s}'<\mathfrak{m}'$ (if $\emptyset\in\mathfrak{m}'$, the same is true for a filter $\mathfrak{s}\in\mathfrak{S}$ fixed at x for some $x\in g(Z)$). $\square$

A similar statement holds for the operation sup:

(9.18) LEMMA. *Let $\mathfrak{S}_i$ ($i\in I$) be a screen on X and a base for a filter-merotopy $\mathfrak{M}_i$. Then $\sup_{\mathrm{Scr}}\{\mathfrak{S}_i\colon i\in I\}$ is a base for $\sup_{\mathrm{Fmer}}\{\mathfrak{M}_i\colon i\in I\}$.*

PROOF. The case $I=\emptyset$ is obvious. Assume $I\neq\emptyset$.

If the filter $\mathfrak{s}$ is generated by the finite intersections of $\bigcup_{i\in I}\mathfrak{s}_i$ where $\mathfrak{s}_i\in\mathfrak{S}_i$, then $\mathfrak{s}_i\subset\mathfrak{s}$ for each i, hence $\mathfrak{s}$ belongs to $\bigcap_{i\in I}\mathfrak{M}_i$ and also to $(\bigcap_{i\in I}\mathfrak{M}_i)^f=$ $=\sup_{\mathrm{Fmer}}\{\mathfrak{M}_i\colon i\in I\}$ (by (9.13)).

Conversely, if $\mathfrak{m}\in(\bigcap_{i\in I}\mathfrak{M}_i)^f$, then there is a filter $\mathfrak{s}\in\bigcap_{i\in I}\mathfrak{M}_i$ satisfying $\mathfrak{s}<\mathfrak{m}$. For every i, there is $\mathfrak{s}_i\in\mathfrak{S}_i$ such that $\mathfrak{s}_i<\mathfrak{s}$. Then the finite intersections of $\bigcup_{i\in I}\mathfrak{s}_i$ belong to $\mathfrak{s}$, and the filter generated by them is an element of $\sup_{\mathrm{Scr}}\{\mathfrak{S}_i\colon i\in I\}$ coarser than $\mathfrak{s}$ and $\mathfrak{m}$. $\square$

10. Semi-merotopies and semi-proximities. We know that proximities can be induced by screens; now we show that semi-proximities are in a similar relationship with semi-merotopies.

(10.1) LEMMA. *Let $\mathfrak{M}$ be a semi-merotopy on X, and define, for $P, Q, A, B\subset X$,*

$$P\delta Q \textit{ iff } P,\ Q\in\sec\mathfrak{m} \textit{ for some } \mathfrak{m}\in\mathfrak{M},$$

$$A<B \textit{ iff } \mathfrak{m}\in\mathfrak{M},\ A\in\sec\mathfrak{m} \textit{ implies } \{B\}<\mathfrak{m}.$$

Then δ is a semi-proximity on X, and $<$ is the symmetric semi-topogenous order on X associated with δ.

PROOF. P1, P2, P4 are obvious, $x\in P\cap Q$ implies $P, Q\in\sec\dot{x}$, $\dot{x}\in\mathfrak{M}$. Clearly $A<B$ iff $A\bar{\delta}X-B$. $\square$

We say that $\mathfrak{M}$ *induces* δ and $<$, and we write $\delta=\delta(\mathfrak{M})$, $<\,=\,<(\mathfrak{M})$. We also say that $\mathfrak{M}$ is *compatible* with δ and $<$. If $\mathfrak{M}$ is a filter-merotopy and $\mathfrak{S}=\mathfrak{S}(\mathfrak{M})$, then clearly $\delta(\mathfrak{M})=\delta(\mathfrak{S})$; in this case, $\delta(\mathfrak{M})$ is a proximity. We have, more generally:

(10.2) LEMMA. *If $\mathfrak{M}$ is a merotopy on X, then $\delta(\mathfrak{M})$ is a proximity.*

PROOF. Assume $P\bar{\delta}Q$, $P\bar{\delta}R$ for $\delta=\delta(\mathfrak{M})$, but $P\delta Q\cup R$. Then there is $\mathfrak{m}\in\mathfrak{M}$ such that P, $Q\cup R\in\sec\mathfrak{m}$. Now $Q\cup R\in\sec\mathfrak{m}$ implies, for every $M\in\mathfrak{m}$, either $Q\cap M\neq\emptyset$ or $R\cap M\neq\emptyset$; denote by $\mathfrak{m}_1$ the collection of the $M\in\mathfrak{m}$ of the first type, by $\mathfrak{m}_2$ that of the $M\in\mathfrak{m}$ of the second type. Hence $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$, and $\mathfrak{m}_i\in\mathfrak{M}$ for some i. Thus either $P, Q\in\sec\mathfrak{m}_1$ or $P, R\in\sec\mathfrak{m}_2$, a contradiction. □

The question arises whether every semi-proximity admits compatible semi-merotopies. The answer to this question (similar to the one answered in (6.11)) is quite elementary.

(10.3) LEMMA. *If $\mathfrak{M}$ is a semi-merotopy then every $\mathfrak{m}\in\mathfrak{M}$ is $\delta(\mathfrak{M})$-compressed.* □

(10.4) LEMMA. *If δ is a semi-proximity on X, and $P\delta Q$, then there exists a δ-compressed $\emptyset\neq\mathfrak{m}\in\exp\exp X$ such that $P, Q\in\sec\mathfrak{m}$.*

PROOF. Let $\mathfrak{m}$ denote the system of all subsets $M\subset X$ such that $M\cap P\neq\emptyset\neq M\cap Q$. Then $P, Q\in\sec\mathfrak{m}$, and $X\in\mathfrak{m}$ because $P\neq\emptyset\neq Q$. We show that $\mathfrak{m}$ is δ-compressed.

In fact, suppose $A\in\sec\mathfrak{m}$. Then either $A\supset P$ or $A\supset Q$; indeed, $P-A\neq\emptyset\neq Q-A$ would imply $\{p, q\}\in\mathfrak{m}$ for some $p\in P-A$, $q\in Q-A$, but $\{p, q\}\cap A=\emptyset$. Similarly, $B\in\sec\mathfrak{m}$ implies either $B\supset P$ or $B\supset Q$. Now if $A, B\in\sec\mathfrak{m}$, then $A\supset P, B\supset P$ or $A\supset Q, B\supset Q$ implies $A\delta B$ since $P\neq\emptyset$, $Q\neq\emptyset$, while $A\supset P$, $B\supset Q$ or $A\supset Q, B\supset P$ implies $A\delta B$ according to $P\delta Q$. □

(10.5) THEOREM. *If δ is an arbitrary semi-proximity on X, then the collection of all δ-compressed systems $\emptyset\neq\mathfrak{m}\in\exp\exp X$ is a semi-merotopy $\mathfrak{M}$ such that $\delta=\delta(\mathfrak{M})$.*

PROOF. $\{\{x\}\}$ is δ-compressed for $x\in X$, hence $\mathfrak{M}$ is a semi-merotopy. By (10.4) $P\delta Q$ implies $P, Q\in\sec\mathfrak{m}$ for some $\mathfrak{m}\in\mathfrak{M}$, and conversely, $P, Q\in\sec\mathfrak{m}$, $\mathfrak{m}\in\mathfrak{M}$ implies $P\delta Q$ since $\mathfrak{m}$ is δ-compressed. □

Let us write $\mathfrak{M}(\delta)=\mathfrak{M}$ for this $\mathfrak{M}$. We say that a semi-merotopy $\mathfrak{M}$ on X is *saturated* iff it is of the form $\mathfrak{M}(\delta)$, i.e. iff it is composed of all $\delta(\mathfrak{M})$-compressed, non-empty systems of subsets of X.

We introduce a functor F: **Smer**$\to$**Sprox** by $F(\mathfrak{M})=\delta(\mathfrak{M})$:

(10.6) LEMMA. *If $f\colon X\to Y$ is $(\mathfrak{M}, \mathfrak{M}')$-continuous, where $\mathfrak{M}$ and $\mathfrak{M}'$ are semi-merotopies on X and Y, respectively, then f is $(\delta(\mathfrak{M}), \delta(\mathfrak{M}'))$-continuous. The converse is true provided $\mathfrak{M}'$ is saturated.*

PROOF. If $P, Q\in\sec\mathfrak{m}$, $\mathfrak{m}\in\mathfrak{M}$, then $f(P), f(Q)\in\sec f(\mathfrak{m})$, $f(\mathfrak{m})\in\mathfrak{M}'$ provided f is $(\mathfrak{M}, \mathfrak{M}')$-continuous. If f is $(\delta(\mathfrak{M}), \delta(\mathfrak{M}'))$-continuous and $\mathfrak{m}\in\mathfrak{M}$, then $\mathfrak{m}$ is $\delta(\mathfrak{M})$-compressed by (10.3), hence $f(\mathfrak{m})$ is $\delta(\mathfrak{M}')$-compressed, and $f(\mathfrak{m})\in\mathfrak{M}'$. □

(10.7) THEOREM. *$F(\mathfrak{M})=\delta(\mathfrak{M})$ induces an isomorphism onto* **Sprox** *from the full subcategory of* **Smer** *the objects of which are the saturated semi-merotopies.* □

Let us examine the behaviour of the functor F with respect to the usual operations.

(10.8) LEMMA. *If* $\mathfrak{M}$ *is a semi-merotopy on* X, $g\colon Z\to X$, *then* $\delta(g^{-1}(\mathfrak{M}))=g^{-1}(\delta(\mathfrak{M}))$.

PROOF. Let us write $\delta=\delta(\mathfrak{M})$, $g^{-1}(\mathfrak{M})=\mathfrak{M}'$. If $P\delta(\mathfrak{M}')Q$, then there is $\mathfrak{m}'\in\mathfrak{M}'$ such that $P, Q\in\sec\mathfrak{m}'$, hence $g(\mathfrak{m}')\in\mathfrak{M}$, $g(P), g(Q)\in\sec g(\mathfrak{m}')$, and $g(P)\delta g(Q)$, $Pg^{-1}(\delta)Q$.

Conversely if $Pg^{-1}(\delta)Q$, then $g(P)\delta g(Q)$, hence $g(P), g(Q)\in\sec\mathfrak{m}$ for some $\mathfrak{m}\in\mathfrak{M}$, $P, Q\in\sec g^{-1}(\mathfrak{m})$, and $\mathfrak{m}<g(g^{-1}(\mathfrak{m}))\in\mathfrak{M}$, $g^{-1}(\mathfrak{m})\in\mathfrak{M}'$, so that $P\delta(\mathfrak{M}')Q$. $\square$

In contrast to the situation concerning screens and proximities (cf. (8.5)), we have:

(10.9) LEMMA. *For semi-merotopies* $\mathfrak{M}_i$ *on* X, $\delta_i=\delta(\mathfrak{M}_i)$, $\mathfrak{M}=\sup\{\mathfrak{M}_i\colon i\in I\}$, $\delta=\delta(\mathfrak{M})$, *the equality*

$$\delta = \sup_{\mathrm{Sprox}}\{\delta_i\colon i\in I\}$$

is valid.

PROOF. The case $I=\emptyset$ is obvious. Assume $I\neq\emptyset$. Let us denote by δ' the right hand side. By (10.6) each δ_i is coarser than δ, hence so is δ'. On the other hand, if $A\delta' B$, then $A\delta_i B$ for every i, hence there is $\mathfrak{m}_i\in\mathfrak{M}_i$ satisfying $A, B\in\sec\mathfrak{m}_i$. Define $\mathfrak{m}=\bigcup_{i\in I}\mathfrak{m}_i$. Clearly $\mathfrak{m}_i<\mathfrak{m}$, hence $\mathfrak{m}\in\mathfrak{M}_i$ for every i, and $\mathfrak{m}\in\mathfrak{M}=\bigcap_{i\in I}\mathfrak{M}_i$, further $A, B\in\sec\mathfrak{m}$, so that $A\delta B$. $\square$

Unfortunately, the behaviour of $\delta(\mathfrak{M})$ with respect to $\mathfrak{M}^q$ and $\mathfrak{M}^f$ is less advantageous.

(10.10) EXAMPLE. Let $X=Y=\mathbf{R}$, $Z=X\times Y$, and let p_1 and p_2 be the projections from Z onto X and Y, respectively. Let $\mathfrak{M}$ be composed of all systems $\mathfrak{m}\in\exp\exp Z$ such that, for every $\varepsilon>0$, there are $\mathfrak{M}_k\in\mathfrak{m}$ having projections $p_k(M_k)$ with diameters less than ε $(k=1, 2)$. Clearly $\mathfrak{M}$ is a semi-merotopy on Z. Set $\delta=\delta(\mathfrak{M})$; we show $\delta^q\neq\delta(\mathfrak{M}^q)=\delta'$.

For this purpose, consider

$$A = \{(x, x)\colon x\in\mathbf{R}\},\quad B = \{(x, y)\colon |x-y|\geqq 1\}.$$

As in (8.5), we can show that, whenever

$$A = \bigcup_1^m A_i,\quad B = \bigcup_1^n B_j,$$

there are i, j and $x_1, x_2\in\mathbf{R}$ such that (x_1, x_1), $(x_2, x_2)\in A_i$, $(x_1, x_2)\in B_j$. Then

$$\{\{(x_1, x_1), (x_1, x_2)\},\ \{(x_2, x_2), (x_1, x_2)\}\}\in\mathfrak{M},$$

hence $A_i\delta B_j$. Therefore $A\delta^q B$.

On the other hand, suppose $A, B\in\sec\mathfrak{m}$, $\mathfrak{m}\in\mathfrak{M}^q$. Then $M\in\mathfrak{m}$ implies that at least one of the projections $p_1(M)$ and $p_2(M)$ has a diameter $>\frac{1}{4}$. Consequently we can set $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$ such that $p_k(M)$ has a diameter $>\frac{1}{4}$ if $M\in\mathfrak{m}_k$. Thus $\mathfrak{m}_k\notin\mathfrak{M}$ for $k=1, 2$, and, *a fortiori*, $\mathfrak{m}_k\notin\mathfrak{M}^q$: a contradiction. Therefore $A\bar{\delta}'B$. □

(10.11) Example. Let $X=\mathbf{R}$. If $\alpha=(a_n)$ and $\beta=(b_n)$ are two sequences in $\mathbf{R}$ such that $|a_n|\to\infty$, $|b_n|\to\infty$, set

$$\mathfrak{b}(\alpha, \beta) = \big\{\{a_n, b_n\}\colon n\in\mathbf{N}\big\}.$$

Then all systems $\mathfrak{b}(\alpha, \beta)$, together with the systems $\big\{\{x\}\big\}$ for $x\in\mathbf{R}$, constitute a base for a semi-merotopy $\mathfrak{M}$ on X. In fact, $\mathfrak{M}$ is a merotopy. If $\big\{\{x\}\big\}<\mathfrak{m}=\mathfrak{m}_1\cup\cup\mathfrak{m}_2$, then clearly either $\big\{\{x\}\big\}<\mathfrak{m}_1$ or $\big\{\{x\}\big\}<\mathfrak{m}_2$. If $\mathfrak{b}(\alpha, \beta)<\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$, then we can set $\mathbf{N}=N_1\cup N_2$ such that $\big\{\{a_n, b_n\}\colon n\in N_i\big\}<\mathfrak{m}_i$. Now either N_1 or N_2 is infinite, say, N_1. Then the subsequences α' and β' of α and β composed of the a_n and b_n, respectively, satisfying $n\in N_1$, yield a system $\mathfrak{b}(\alpha', \beta')$ such that $\mathfrak{b}(\alpha', \beta')<\mathfrak{m}_1$, hence $\mathfrak{m}_1\in\mathfrak{M}$.

Thus $\delta=\delta(\mathfrak{M})$ is a proximity; it can be easily described: $A\delta B$ iff either $A\cap B\neq\emptyset$ or A and B are unbounded. In fact, $A\cap B=\emptyset$, $A\delta B$ implies $A, B\in\sec\mathfrak{m}$ for some $\mathfrak{m}\in\mathfrak{M}$ satisfying $\mathfrak{b}(\alpha, \beta)<\mathfrak{m}$ for suitable sequences α, β. Then $A, B\in\sec\mathfrak{b}(\alpha, \beta)$, and A, B are unbounded. Conversely, if A and B are unbounded, A contains all elements of a sequence α, B does the same for a sequence β, and $A, B\in\sec\mathfrak{b}(\alpha, \beta)$.

Since a system $\mathfrak{b}(\alpha, \beta)$ contains disjoint elements, no filter $\mathfrak{s}$ can fulfil $\mathfrak{b}(\alpha, \beta)<\mathfrak{s}$, and $\mathfrak{M}$ does not contain any other filters than $\dot{x}$ $(x\in X)$. Therefore $A\delta(\mathfrak{M}^f)B$ iff $A\cap B\neq\emptyset$, and $\delta(\mathfrak{M})^q=\delta(\mathfrak{M})\neq\delta(\mathfrak{M}^f)$. □

A positive result can be obtained for saturated semi-merotopies.

(10.12) Lemma. *Let δ be a semi-proximity on X. A filter in X is δ-compressed iff it is δ^q-compressed.*

Proof. A δ^q-compressed system is δ-compressed, of course. Conversely, if a filter $\mathfrak{s}$ in X is δ-compressed, $A\overline{\delta^q}B$, $A\in\sec\mathfrak{s}$, then

$$A=\bigcup_1^m A_i,\quad B=\bigcup_1^n B_j,\quad A_i\bar{\delta}B_j \text{ for every } i \text{ and } j.$$

Now $A_i\in\sec\mathfrak{s}$ for at least one i because otherwise $X-A_i\in\mathfrak{s}$ would follow for each i, hence $X-A=\bigcap_1^m (X-A_i)\in\mathfrak{s}$: a contradiction. For this i, $X-B_j\in\mathfrak{s}$ for every j, and $X-B=\bigcap_1^n (X-B_j)\in\mathfrak{s}$, so that $\mathfrak{s}$ is δ^q-compressed. □

(10.13) Theorem. *Let δ be a semi-proximity on X, $\mathfrak{M}=\mathfrak{M}(\delta)$. Then $\delta^q=\delta(\mathfrak{M}^q)=\delta(\mathfrak{M}^f)$.*

Proof. By (10.12) the δ^q-compressed filters are precisely the filters contained in $\mathfrak{M}$, i.e. they are by (9.11) the elements of a screen $\mathfrak{S}$, which is a base for $\mathfrak{M}^f$. By (6.11), $\delta^q=\delta(\mathfrak{S})$ and clearly $\delta(\mathfrak{S})=\delta(\mathfrak{M}^f)$, hence $\delta^q=\delta(\mathfrak{M}^f)$. Now $\delta(\mathfrak{M}^q)$ is a proximity by (10.2), finer than $\delta(\mathfrak{M})=\delta$ by (10.6), hence finer than δ^q. At the same time $\delta(\mathfrak{M}^q)$ is coarser than $\delta(\mathfrak{M}^f)$, i.e. than δ^q, and $\delta(\mathfrak{M}^q)=\delta^q$. □

The good behaviour of the operation sup with respect to the categories **Smer** and **Sprox** is no more valid for **Mer** and **Prox**.

(10.14) EXAMPLE. Let $X=Y=\mathbf{R}$, $Z=X\times Y$, and let p_1 and p_2 be the projections (as in (10.10)). Let $\mathfrak{M}_i$ be the collection of all systems $\mathfrak{m}\in\exp\exp Z$ such that, for $\varepsilon>0$, there is an $M\in\mathfrak{m}$ such that the diameter of $p_i(M)$ is less than ε $(i=1,2)$. Clearly $\mathfrak{M}_i$ is a merotopy on Z for $i=1,2$, and $\mathfrak{M}_1\cap\mathfrak{M}_2=\mathfrak{M}$ where $\mathfrak{M}$ is the same as in (10.10). Hence, for $\delta_i=\delta(\mathfrak{M}_i)$, we have

$$\sup_{\mathbf{Sprox}}\{\delta_1,\delta_2\}=\delta(\mathfrak{M})=\delta$$

by (10.9). Now we know from (10.10) that

$$\sup_{\mathbf{Prox}}\{\delta_1,\delta_2\}=\delta^q\neq\delta(\mathfrak{M}^q)$$

where $\mathfrak{M}^q=\sup_{\mathbf{Mer}}\{\mathfrak{M}_1,\mathfrak{M}_2\}$. □

A similar negative result is valid for **Fmer** and **Prox**, namely that, if $\mathfrak{M}_i$ is a filter-merotopy on X for $i\in I$, $\delta_i=\delta(\mathfrak{M}_i)$, $\mathfrak{M}=\sup_{\mathbf{Fmer}}\{\mathfrak{M}_i\colon i\in I\}$, then, in general,

$$\sup_{\mathbf{Prox}}\{\delta_i\colon i\in I\}\neq\delta(\mathfrak{M}).$$

According to (9.16) this is shown by (8.5); for the sake of completeness, let us formulate it again in the language of filter-merotopies:

(10.15) EXAMPLE. For X, Y, Z, p_1, p_2 as in (8.5), let $\mathfrak{S}_i$ denote the ascending screen composed of all filters $\mathfrak{s}$ in Z such that $p_i(\mathfrak{s})$ converges with respect to the usual topology c of $\mathbf{R}$ $(i=1,2)$. $\mathfrak{S}_i$ is a base for a filter-merotopy $\mathfrak{M}_i$ on Z by (9.7). Clearly $\delta(\mathfrak{M}_i)=\delta(\mathfrak{S}_i)=\delta_i$, where $P\delta_i Q$ iff $c(p_i(P))\cap c(p_i(Q))\neq\emptyset$. By (8.5), $A\delta' B$ for the sets A and B in (8.5) and $\delta'=\sup_{\mathbf{Prox}}\{\delta_1,\delta_2\}$. On the other hand, a base for

$$\sup_{\mathbf{Fmer}}\{\mathfrak{M}_1,\mathfrak{M}_2\}=(\mathfrak{M}_1\cap\mathfrak{M}_2)^f=\mathfrak{M}$$

is composed of the filters $\mathfrak{s}$ in Z such that both $p_1(\mathfrak{s})$ and $p_2(\mathfrak{s})$ converge; therefore $A\bar{\delta}B$ for $\delta=\delta(\mathfrak{M})$. □

If δ is a semi-proximity, then $\mathfrak{M}(\delta)$ is a semi-merotopy without being a merotopy in general ((10.5), (10.2)). The situation is still the same if δ is a proximity:

(10.16) EXAMPLE. Let $X=\{a,b,c\}$, $|X|=3$, and let δ be the discrete proximity on X. Then $\mathfrak{m}=\{\{a,b\},\{a,c\},\{b,c\}\}$ is δ-compressed. In fact, if $P\cap Q=\emptyset$, then one of the sets P and Q has cardinality $\leqq 1$, say, $P\subset\{a\}$; therefore $P\cap\{b,c\}=\emptyset$ and, *a fortiori*, $P\notin\sec\mathfrak{m}$.

Now $\{\{a,b\}\}$ is not δ-compressed because $\{a\}\bar{\delta}\{b\}$. Similarly $\{\{a,c\}\}$ and $\{\{b,c\}\}$ are not δ-compressed, so that $\mathfrak{m}\in\mathfrak{M}(\delta)$ is the union of three subsets not belonging to $\mathfrak{M}(\delta)$. □

Hence, for a proximity δ, we have in general

$$\mathfrak{M}(\delta)\neq\mathfrak{M}(\delta)^q.$$

The latter is, of course, the coarsest merotopy compatible with δ ((10.5), (10.13), (10.3), (9.10)).

Let us now brifly examine the connection of semi-merotopies and semi-closures. If $\mathfrak{M}$ is a semi-merotopy on X, $\delta=\delta(M)$, let us write $c(\mathfrak{M})=c_\delta$.

In contrast to the situation illustrated by (10.10), we can now state:

(10.17) LEMMA. *If $\mathfrak{M}$ is a semi-merotopy on X, $c=c(\mathfrak{M})$, then $c^q=c(\mathfrak{M}^q)$.*

PROOF. Set $c'=c(\mathfrak{M}^q)$; then c' is a closure by (10.2) and (3.10), finer than c by (10.6) and (3.10), hence finer than c^q. Conversely c^q is finer than c'. Assume, in fact, $x\in c^q(A)$, and define

$$\mathfrak{m}=\{\{x,a\}:\ a\in A\}.$$

Then $\mathfrak{m}\in\mathfrak{M}$ because $x\in c(A)$, hence there is $\mathfrak{m}_0\in\mathfrak{M}$ such that $\mathfrak{m}_0$ is fixed at x and $A\in\sec\mathfrak{m}_0$; clearly $\mathfrak{m}_0<\mathfrak{m}$. Moreover, $\mathfrak{m}\in\mathfrak{M}^q$; indeed, if $\mathfrak{m}=\bigcup\limits_1^n\mathfrak{m}_i$, then $A=\bigcup\limits_1^n A_i$,

$$\mathfrak{m}_i=\{\{x,a\}:\ a\in A_i\},$$

and $x\in c(A_i)$ for some i. Choose $\mathfrak{m}'\in\mathfrak{M}$ fixed at x and such that $A_i\in\sec\mathfrak{m}'$, then obviously $\mathfrak{m}'<\mathfrak{m}_i$, $\mathfrak{m}_i\in\mathfrak{M}$. By (9.10) $\mathfrak{m}\in\mathfrak{M}^q$, $\mathfrak{m}$ is fixed at x, $A\in\sec\mathfrak{m}$, so that $x\in c'(A)$. □

An example based on the same idea as (10.11) (but slightly more complicated) shows that the behaviour of $c(\mathfrak{M})$ with respect to $\mathfrak{M}^f$ is still bad:

(10.18) EXAMPLE. Let $X=\mathbf{R}$, and denote α and β two sequences $\alpha=(a_n)$ and $\beta=(b_n)$ in $\mathbf{R}$ with the same limit with respect to the usual topology c of $\mathbf{R}$: $a_n\to x$, $b_n\to x$. Denote $\mathfrak{b}(\alpha,\beta)=\{\{a_n,b_n\}:\ n\in\mathbf{N}\}$, and let $\mathfrak{B}$ be the collection of all systems $\mathfrak{b}(\alpha,\beta)$. Then $\mathfrak{B}$ is a base for a semi-merotopy $\mathfrak{M}$ on X (observe that $\{\{x\}\}=\mathfrak{b}(\alpha,\beta)$ for $a_n=b_n=x$). Similarly to the argument in (10.11), it turns out that $\mathfrak{M}$ is a merotopy. For $\delta=\delta(\mathfrak{M})$, clearly $A\delta B$ iff $c(A)\cap c(B)\neq\emptyset$, hence $c_\delta=c$. On the other hand, if $\mathfrak{b}(\alpha,\beta)<\mathfrak{s}$ for a filter $\mathfrak{s}$, then $\mathfrak{b}(\alpha,\beta)$ cannot contain two disjoint elements, so that, for every n, either $a_n=x$ or $b_n=x$ (if $a_n\to x$, $b_n\to x$), and necessarily $\mathfrak{s}=\dot{x}$. Hence $\mathfrak{M}^f$ induces the proximity $\delta'=\delta(\mathfrak{M}^f)$ for which $A\delta'B$ iff $A\cap B\neq\emptyset$, and $c_{\delta'}$ is the discrete topology on $\mathbf{R}$. □

If $\mathfrak{M}$ is a semi-merotopy on X and $g\colon Z\to X$, then $g^{-1}(c(\mathfrak{M}))=c(g^{-1}(\mathfrak{M}))$ by (10.8) and (3.9). If $\mathfrak{M}_i$ is a semi-merotopy on X for $i\in I$, then

$$c(\sup_{\mathrm{Smer}}\{\mathfrak{M}_i:\ i\in I\})=\sup_{\mathrm{Scl}}\{c(\mathfrak{M}_i):\ i\in I\}$$

by (10.9) and (3.9); if every $\mathfrak{M}_i$ is a merotopy, then

$$c(\sup_{\mathrm{Mer}}\{\mathfrak{M}_i:\ i\in I\})=\sup_{\mathrm{Cl}}\{c(\mathfrak{M}_i):\ i\in I\}$$

by the previous equality and (10.17). However, if every $\mathfrak{M}_i$ is a filter-merotopy, then, in general,

$$c(\sup_{\mathrm{Fmer}}\{\mathfrak{M}_i:\ i\in I\})\neq\sup_{\mathrm{Cl}}\{c(\mathfrak{M}_i):\ i\in I\};$$

this fact is shown by (8.6), if we define $\mathfrak{M}_i$ to be the filter-merotopy for which $\mathfrak{S}_i$ is a base $(i=1,2)$. In fact, obviously $c_{\delta_i}=c(\mathfrak{M}_i)$ for $\delta_i=\delta(\mathfrak{S}_i)$, and sup $\{\mathfrak{S}_1,\mathfrak{S}_2\}$

is a base for

$$(\mathfrak{M}_1 \cap \mathfrak{M}_2)^f = \sup_{\mathrm{Fmer}}\{\mathfrak{M}_1, \mathfrak{M}_2\}$$

(see (9.18)).

Part IV will deal with the relation of merotopies to uniformities.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF MATHEMATICS
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

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ALMOST SURE LIMIT POINTS OF SAMPLE EXTREMES OF AN I.I.D. SEQUENCE

S. S. NAYAK (Mysore)

1. Introduction

Let $\{X_n,\ n\geqq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s.) with a common distribution function (d.f.) F. Define $Y_n=\max(X_1, X_2, \dots, X_n)$. Laurens De Haan and Arie Hordijk [1] have studied the behaviour of the sequence $\{Y_n\}$ under the following assumptions:

(a) $F(x)$ has a positive derivative $f(x)$ for all real x and $\lim\limits_{x\to\infty} g(x)/x=c$ $(0\leqq c<\infty)$.

(b) $F(x)$ has a positive derivative $f(x)$ for all real x and $F(x)$ is twice differentiable with $\lim\limits_{x\to\infty} g'(x)=0$, where

$$g(x)=\frac{\{1-F(x)\}\log\log\{1/(1-F(x))\}}{f(x)}.$$

Let $\{b_n\}$ and $\{c_n\}$ be two sequences of real numbers defined by $1-F(b_n)=1/n$ and $c_n=g(b_n)$. The above authors have established that under (a),

$$\limsup_{n\to\infty} Y_n/b_n=e^c \quad \text{a.s.,} \qquad \liminf_{n\to\infty} Y_n/b_n=1 \quad \text{a.s.,} \tag{1.1}$$

$$1-F(b_n x)=\frac{(\log n)^{r_n(x)}}{n} \quad \text{for all} \quad x>0, \tag{1.2}$$

where for all $x>0$, $\lim\limits_{n\to\infty} r_n(x)=-(\log x)/c$ and under (b),

$$\limsup_{n\to\infty}(Y_n-b_n)/c_n=1 \quad \text{a.s.,} \qquad \liminf_{n\to\infty}(Y_n-b_n)/c_n=0 \quad \text{a.s.,} \tag{1.3}$$

$$1-F(b_n+c_n x)=\frac{(\log n)^{r_n^*(x)}}{n} \quad \text{for all real} \quad x \tag{1.4}$$

where $\lim\limits_{n\to\infty} r_n^*(x)=-x$ for all real x.

Wichura [8] has proved the above results in the functional form. In this paper, the almost sure limit sets of the random vector consisting of the first two largest values (properly normalised) in a random sample are derived under (a) and (b). This problem has been considered by Vishnu Hebbar [6] for Gaussian sequences under certain conditions on the covariance function. He has also studied some related problems for Gaussian sequences (see [5], [7]). The almost sure limit sets of the random vector containing $p(\geqq 2)$ independent copies of Y_n are studied by S. S. Nayak [4] under the assumptions (a) and (b). For a good review of multivariate extreme values see [2].

In this paper, we assume that $0<c<\infty$. A generic constant is denoted by the letter d with a suffix. Some preliminary lemmas are given in the next section and the main problem is considered in the last section.

2. Preliminary lemmas

LEMMA 2.1. *Let $\{A_n\}$ be a sequence of events in a probability space. If*

(i) $\sum_{n=1}^{\infty} P(A_n)=\infty$ *and*

(ii) $\liminf_{n\to\infty} \sum\sum_{1\le j<k\le n} \left(P(A_j\cap A_k)-P(A_j)P(A_k)\right)/\left(\sum_{j=1}^{n} P(A_j)\right)^2 \leqq 0,$

then $P(A_n \text{ i.o.})=1$.

PROOF. See Ortega and Wschebor [3], Lemma 1, p. 86.

LEMMA 2.2. *Let S_n be the second largest among $X_1, X_2, \ldots, X_n$. Then under* (a), *for all $x_1>x_2>1$ we have*

$$P(Y_n/b_n>x_1,\ S_n/b_n>x_2) \sim (\log n)^{r_n(x_1)+r_n(x_2)} \quad as \quad n\to\infty.$$

PROOF. The joint probability density (p.d.f.) of Y_n and S_n is $n(n-1)F^{n-2}(y)\cdot$ $\cdot f(x)f(y)$ for $x>y$ and zero otherwise. Assume that n is sufficiently large so that $b_n>0$. Hence we have

$$(2.1)\qquad P(Y_n/b_n>x_1,\ S_n/b_n>x_2)=1-F^n(b_nx_1)-n\left(1-F(b_nx_1)\right)F^{n-1}(b_nx_2).$$

Write $u_n=1-F(b_nx_1)$ and $v_n=1-F(b_nx_2)$. By (1.2), observe that for all $x_1>1$ and $x_2>1$, $nu_n\to 0$ and $nv_n\to 0$ as $n\to\infty$. The right side of (2.1) can be written as

$$nu_n-\frac{(n-1)}{2n}(nu_n)^2+\frac{(n-1)(n-2)}{6n^2}\left(1+o(1)\right)(nu_n)^3-$$

$$-\frac{nu_n}{1-v_n}\left(1-nv_n+\frac{(n-1)}{2n}\left(1+o(1)\right)(nv_n)^2\right)\sim n^2u_nv_n \quad \text{as} \quad n\to\infty,$$

since, for all $x_1>x_2>0$, $u_n/v_n\to 0$ as $n\to\infty$ which is a consequence of (1.2). The proof is complete.

LEMMA 2.3. *Let $\{n_k\}$ be a monotonically increasing subsequence of $\{n\}$. Then under* (a), *for all $x>1$ and positive integers s and t such that $\lim_{s,t\to\infty} n_s/n_t=\alpha(<\infty)$ we have*

$$1-F^{n_s}(b_{n_t}x)\sim (n_s/n_t)(\log n_t)^{r_{n_t}(x)} \quad as \quad s,t\to\infty.$$

PROOF. Let $x_t=1-F(b_{n_t}x)$. Since $\lim_{s,t\to\infty} n_s/n_t=\alpha(<\infty)$, by (1.2) it follows that for all $x>1$, $n_s x_t \to 0$ as $s,t\to\infty$. Hence,

$$F^{n_s}(b_{n_t}x)=(1-x_t)^{n_s}=1-n_s x_t(1+o(1)).$$

This along with (1.2) implies the lemma.

The proofs of the following two lemmas, being similar to those of the above two lemmas, are omitted.

LEMMA 2.4. *Under* (b), *we have for all* $x_1>x_2>0$

$$P(Y_n>b_n+c_nx_1,\ S_n>b_n+c_nx_2)\sim(\log n)^{r_n^*(x_1)+r_n^*(x_2)}\quad as\quad n\to\infty.$$

LEMMA 2.5. *Let* $\{n_k\}$ *be a monotonically increasing subsequence of* $\{n\}$. *Then under* (b), *for all* $x>0$ *and positive integers s and t such that* $\lim_{s,t\to\infty} n_s/n_t=\beta(<\infty)$ *we have*

$$1-F^{n_s}(b_{n_t}+c_{n_t}x)\sim(n_s/n_t)(\log n_t)^{r_{n_t}^*(x)}\quad as\quad s,t\to\infty.$$

3. Almost sure limit points

THEOREM 3.1. *Let* Y_n *and* S_n *be the first two largest values among* $X_1, X_2, \dots, X_n$. *Then under* (a), *the set of all almost sure limit points of* $(Y_n/b_n, S_n/b_n)$ *is*

$$S=\{(x_1,x_2):\ 1\le x_2\le x_1\le e^c,\ x_1x_2\le e^c\}.$$

PROOF. Let $\varepsilon>0$ be arbitrarily fixed. For all x_1 and x_2 satisfying $x_1>x_2\ge 1$ and $1<x_1x_2<e^c$ define a positive valued function $h(x_1,x_2)=h$ by:

$$\max\big(1,\ c/\log x_1(x_2+\varepsilon),\ c/\log x_2(x_1+\varepsilon)\big)<h(x_1,x_2)<c/\log x_1x_2.$$

Let $n_k=[\exp k^h]$ where $[u]$ is the greatest integer less than or equal to u. The theorem will be established through the following lemmas.

LEMMA 3.1. *For* $\varepsilon>0$ *and all* x_1 *and* x_2 *such that* $x_1>x_2>1$ *and* $x_1x_2<e^c$ *we have*

(i) $P(Y_{n_k}/b_{n_k}>x_1+\varepsilon,\ S_{n_k}/b_{n_k}>x_2 \text{ i.o.})=0$ *and*

(ii) $P(Y_{n_k}/b_{n_k}>x_1,\ S_{n_k}/b_{n_k}>x_2+\varepsilon \text{ i.o.})=0.$

PROOF. By Lemma 2.2 and (1.2) we have

$$P(Y_{n_k}/b_{n_k}>x_1+\varepsilon,\ S_{n_k}/b_{n_k}>x_2)<d_1k^{h(\varepsilon_1-(1/c)\log x_2(x_1+\varepsilon))}$$

for $\varepsilon_1>0$ and all $k\ge k_1$. By choosing ε_1 such that $0<\varepsilon_1<(1/c)\log x_2(x_1+\varepsilon)-1/h$, we notice that $\sum P(Y_{n_k}/b_{n_k}>x_1+\varepsilon,\ S_{n_k}/b_{n_k}>x_2)<\infty$ and hence by the Borel—Cantelli lemma we get (i).

If $x_1>x_2+\varepsilon$ then the proof of (ii) is similar to that of (i). Let $1<x_2<x_1\le$ $\le x_2+\varepsilon$ and $x_1x_2<e^c$. Then

$$P(Y_{n_k}/b_{n_k}>x_1,\ S_{n_k}/b_{n_k}>x_2+\varepsilon)=P(S_{n_k}/b_{n_k}>x_2+\varepsilon)=$$
$$=1-n_kF^{n_k-1}(b_{n_k}(x_2+\varepsilon))+(n_k-1)F^{n_k}(b_{n_k}(x_2+\varepsilon)).$$

Let $x_k^* = 1 - F(b_{n_k}(x_2+\varepsilon))$. Since $x_2 > 1$, by (1.2) it follows that $n_k x_k^* \to 0$ as $k \to \infty$. In terms of x_k^* the above probability can be written as

$$1 - n_k(1-x_k^*)^{n_k-1} + (n_k-1)(1-x_k^*)^{n_k} =$$

$$= 1-(1-x_k^*)^{n_k} - n_k x_k^*(1-x_k^*)^{n_k}/(1-x_k^*) \sim (n_k x_k^*)^2/2 \quad \text{as} \quad k \to \infty.$$

But, by (1.2) we have

$$(n_k x_k^*)^2 < d_2 k^{2h(\varepsilon_2 - (1/c)\log(x_2+\varepsilon))}$$

for $\varepsilon_2 > 0$ and all $k \geqq k_2$. Since $x_1 \leqq x_2 + \varepsilon$, we can choose $\varepsilon_2 > 0$ such that $\Sigma (n_k x_k^*)^2 < \infty$. Now an application of the Borel—Cantelli lemma completes the proof.

LEMMA 3.2. *For all* $x_1 > x_2 > 1$ *with* $x_1 x_2 < e^c$ *we have*

$$P(Y_{n_k}/b_{n_k} > x_1,\, S_{n_k}/b_{n_k} > x_2 \text{ i.o.}) = 1.$$

PROOF. Define

$$E_k = \{Y_{n_k}/b_{n_k} > x_1,\, S_{n_k}/b_{n_k} > x_2\}.$$

By (1.2) and Lemma 2.2 we have

$$P(E_k) > d_3 k^{-h(\varepsilon_3 + (\log x_1 x_2)/c)}$$

for $\varepsilon_3 > 0$ and all $k \geqq k_3$ and all x_1 and x_2 satisfying $x_1 > x_2 > 1$, $x_1 x_2 < e^c$. We can choose $\varepsilon_3 > 0$ such that

$$\sum P(E_k) = \infty \quad \text{for all} \quad x_1 > x_2 > 1 \quad \text{with} \quad x_1 x_2 < e^c. \tag{3.1}$$

Let s and t be two positive integers with $s < t$. Denote the first maximum and the second maximum among $X_{n_s+1}, X_{n_s+2}, \ldots, X_{n_t}$ by $Y'_{n_t-n_s}$ and $S'_{n_t-n_s}$ respectively. Let s and t be sufficiently large so that $b_{n_s} > 0$ and $b_{n_t} > 0$. Hence for s and t large,

$$P(E_s \cap E_t) = P(Y_{n_s} > b_{n_s}x_1,\; S_{n_s} > b_{n_s}x_2,\; Y_{n_t} > b_{n_t}x_1,\; S_{n_t} > b_{n_t}x_2).$$

Let

$$P(A_1) = P(Y_{n_s} > b_{n_s}x_1,\, S_{n_s} > b_{n_s}x_2,\, \max(Y_{n_s}, Y'_{n_t-n_s}) > b_{n_t}x_1,$$
$$S_{n_t} > b_{n_t}x_2,\, Y_{n_s} < Y'_{n_t-n_s})$$

and

$$P(A_2) = P(Y_{n_s} > b_{n_s}x_1,\, S_{n_s} > b_{n_s}x_2,\, \max(Y_{n_s}, Y'_{n_t-n_s}) > b_{n_t}x_1,$$
$$S_{n_t} > b_{n_t}x_2,\, Y_{n_s} \geqq Y'_{n_t-n_s}).$$

Then we have

$$P(E_s \cap E_t) = P(A_1) + P(A_2). \tag{3.2}$$

We evaluate $P(A_1)$ and $P(A_2)$ as follows: Note that $\max(Y_{n_s}, Y'_{n_t-n_s}) = Y'_{n_t-n_s}$ if and only if $\max(Y_{n_s}, S'_{n_t-n_s}) = S_{n_t}$. Hence

$$P(A_1) = P\big(Y_{n_s} > b_{n_s}x_1, S_{n_s} > b_{n_s}x_2, Y'_{n_t-n_s} > b_{n_t}x_1,$$
$$\max(Y_{n_s}, S'_{n_t-n_s}) > b_{n_t}x_2, Y_{n_s} < Y'_{n_t-n_s}\big) =$$
$$= P\big(Y_{n_s} > \max(b_{n_s}x_1, b_{n_t}x_2), S_{n_s} > b_{n_s}x_2, Y'_{n_t-n_s} > b_{n_t}x_1, Y_{n_s} < Y'_{n_t-n_s}\big) +$$
$$+ P\big(b_{n_s}x_1 < Y_{n_s} \leqq b_{n_t}x_2, S_{n_s} > b_{n_s}x_2, Y'_{n_t-n_s} > b_{n_t}x_1, S'_{n_t-n_s} > b_{n_t}x_2, Y_{n_s} < Y'_{n_t-n_s}\big).$$

Observe that Y_{n_s} and $Y'_{n_t-n_s}$ are independent and so also are S_{n_s} and $S'_{n_t-n_s}$. By using the independence and the fact that $x_1 > x_2$ we obtain

$$P(A_1) = P\big(Y_{n_s} > \max(b_{n_s}x_1, b_{n_t}x_2), S_{n_s} > b_{n_s}x_2, Y'_{n_t-n_s} > b_{n_t}x_1, Y_{n_s} < Y'_{n_t-n_s}\big) +$$
$$+ P\big(b_{n_s}x_1 < Y_{n_s} \leqq b_{n_t}x_2, S_{n_s} > b_{n_s}x_2\big) P\big(Y'_{n_t-n_s} > b_{n_t}x_1, S'_{n_t-n_s} > b_{n_t}x_2\big) =$$
$$= P(A_2) + P(A_3)\,P(A_4), \text{ say.}$$

Let $g_n(x, y)$ be the joint p.d.f. of Y_n and S_n. Note that $g_n(x, y) = n(n-1)F^{n-2}(y) \cdot \cdot f(x) f(y)$ for $x > y$ and zero otherwise. Put $v_n = \max(b_{n_s}x_1, b_{n_t}x_2)$. Then we get

$$p(A_2) = \int_{b_{n_t}x_1}^{\infty} \int_{v}^{u} \int_{b_{n_s}x_2}^{x} g_{n_s}(x, y)\,dy\,dx\,dP\,(Y'_{n_t-n_s} \leqq u) =$$
$$= \frac{n_t - n_s}{n_t}\big(1 - F^{n_t}(b_{n_t}x_1)\big) - \frac{n_s(n_t - n_s)}{n_t - n_s + 1} F^{n_s-1}(b_{n_s}x_2)\big(1 - F^{n_t-n_s+1}(b_{n_t}x_1)\big) +$$
$$+ \big(1 - F^{n_t-n_s}(b_{n_t}x_1)\big)\big(n_s F(v) F^{n_s-1}(b_{n_s}x_2) - F^{n_s}(v)\big).$$

$$P(A_3) = \int_{b_{n_s}x_1}^{b_{n_t}x_2} \int_{b_{n_s}x_2}^{x} g_{n_s}(x, y)\,dy\,dx =$$
$$= F^{n_s}(b_{n_t}x_2) - F^{n_s}(b_{n_s}x_1) - n_s F^{n_s-1}(b_{n_s}x_2)\big(F(b_{n_t}x_2) - F(b_{n_s}x_1)\big)$$

and

$$P(A_4) = \int_{b_{n_t}x_1}^{\infty} \int_{b_{n_t}x_2}^{x} g_{n_t-n_s}(x, y)\,dy\,dx =$$
$$= 1 - F^{n_t-n_s}(b_{n_t}x_1) - (n_t - n_s) F^{n_t-n_s-1}(b_{n_t}x_2)\big(1 - F(b_{n_t}x_1)\big).$$

Observing that $\max(Y_{n_s}, Y'_{n_t-n_s}) = Y_{n_s}$ if and only if $\max(S_{n_s}, Y'_{n_t-n_s}) = S_{n_t}$, we have

$$P(B_1) = P(Y_{n_s} > b_{n_t}x_1, S_{n_s} > b_{n_s}x_2, \max(S_{n_s}, Y'_{n_t-n_s}) > b_{n_t}x_2, Y_{n_s} \geqq Y'_{n_t-n_s}).$$

Since $x_1 > x_2$ and $b_{n_s} < b_{n_t}$, after some simplification we get

$$P(B_1) = P(Y_{n_s} > b_{n_t}x_1, S_{n_s} > b_{n_t}x_2, Y_{n_s} \geqq Y'_{n_t-n_s}) +$$
$$+ P(Y_{n_s} > b_{n_t}x_1, b_{n_s}x_2 < S_{n_s} \leqq b_{n_t}x_2, Y'_{n_t-n_s} > b_{n_t}x_2, Y_{n_s} \geqq Y'_{n_t-n_s}) =$$
$$= P(B_2) + P(B_3), \quad \text{say.}$$

Consider

$$P(B_2)=\int_{-\infty}^{\infty}P(Y_{n_s}>\max(x,b_{n_t}x_1),\,S_{n_s}>b_{n_t}x_2)\,dP(Y'_{n_t-n_s}\leqq x)=$$

$$=P(Y'_{n_t-n_s}\leqq b_{n_t}x_1)\int_{b_{n_t}x_1}^{\infty}\int_{b_{n_t}x_2}^{\infty}g_{n_s}(u,v)\,dv\,du+$$

$$+\int_{b_{n_t}x_1}^{\infty}\int_{x}^{\infty}\int_{b_{n_t}x_2}^{u}g_{n_s}(u,v)\,dv\,du\,dP(Y'_{n_t-n_s}\leqq x)=$$

$$=\bigl(1-F^{n_s}(b_{n_t}x_1)-n_s(1-F(b_{n_t}x_1))\,F^{n_s-1}(b_{n_t}x_2)\bigr)F^{n_t-n_s}(b_{n_t}x_1)+$$

$$+\bigl(1-F^{n_t-n_s}(b_{n_t}x_1)\bigr)\bigl(1-n_sF^{n_s-1}(b_{n_t}x_2)\bigr)+$$

$$+\frac{n_s(n_t-n_s)}{n_t-n_s+1}(1-F^{n_t-n_s+1}(b_{n_t}x_1))F^{n_s-1}(b_{n_t}x_2)-\frac{n_t-n_s}{n_t}(1-F^{n_t}(b_{n_t}x_1)).$$

Let

$$I_1=P(Y_{n_s}>b_{n_t}x_1,\,b_{n_s}x_2<S_{n_s}\leqq b_{n_t}x_2)\int_{b_{n_t}x_2}^{b_{n_t}x_1}dP(Y'_{n_t-n_s}\leqq x)$$

and

$$I_2=\int_{b_{n_t}x_1}^{\infty}P(Y_{n_s}>x,\,b_{n_s}x_2<S_{n_s}\leqq b_{n_t}x_2)\,dP(Y'_{n_t-n_s}\leqq x).$$

Then $P(B_3)=I_1+I_2$. Since $x_1>x_2$, for all $x\geqq b_{n_t}x_1$ we have

$$P(Y_{n_s}>x,\,b_{n_s}x_2<S_{n_s}\leqq b_{n_t}x_2)=\int_{x}^{\infty}\int_{b_{n_s}x_2}^{b_{n_t}x_2}g_{n_s}(u,v)\,dv\,du=$$

$$=n_s\bigl(F^{n_s-1}(b_tx_2)-F^{n_s-1}(b_{n_s}x_2)\bigr)\bigl(1-F(x)\bigr).$$

Hence, I_1 and I_2 will simplify to

$$I_1=n_s\bigl(1-F(b_{n_t}x_1)\bigr)\bigl(F^{n_s-1}(b_{n_t}x_2)-F^{n_s-1}(b_{n_s}x_2)\bigr)\bigl(F^{n_t-n_s}(b_{n_t}x_1)-F^{n_t-n_s}(b_{n_t}x_2)\bigr)$$

and

$$I_2=n_s(n_t-n_s)\bigl(F^{n_s-1}(b_{n_t}x_2)-F^{n_s-1}(b_{n_s}x_2)\bigr)\times$$

$$\times\left(\frac{1-F^{n_t-n_s}(b_{n_t}x_1)}{n_t-n_s}-\frac{1-F^{n_t-n_s+1}(b_{n_t}x_1)}{n_t-n_s+1}\right).$$

Substituting these probabilities in (3.2) we get that for sufficiently large s and t

$(s<t)$ and for all $x_1>x_2>1$ with $x_1x_2<e^c$,

$$\begin{aligned}P(E_s\cap E_t) &= \big(1-F^{n_t-n_s}(b_{n_t}x_1)\big)\big(1-F^{n_s}(v)-n_s(1-F(v))F^{n_s-1}(b_{n_s}x_2)\big)+\\ &+\big(1-F^{n_t-n_s}(b_{n_t}x_1)-(n_t-n_s)(1-F(b_{n_t}x_1))F^{n_t-n_s-1}(b_{n_t}x_2)\times\\ &\times\big(F^{n_s}(b_{n_t}x_2)-F^{n_s}(b_{n_s}x_1)-n_s\big(F(b_{n_t}x_2)-F(b_{n_s}x_1)\big)F^{n_s-1}(b_{n_s}x_2)\big)+\\ &+\big(1-F^{n_s}(b_{n_t}x_1)-n_s(1-F(b_{n_t}x_1))F^{n_s-1}(b_{n_t}x_2)\big)F^{n_t-n_s}(b_{n_t}x_1)+\\ &+n_s\big(1-F(b_{n_t}x_1)\big)\big(F^{n_s-1}(b_{n_t}x_2)-F^{n_s-1}(b_{n_s}x_2)\big)\big(F^{n_t-n_s}(b_{n_t}x_1)-F^{n_t-n_s}(b_{n_t}x_2)\big)=\\ &= C_1C_2+C_3C_4+C_5+n_s\big(1-F(b_{n_t}x_1)\big)C_6C_7, \quad \text{say.}\end{aligned}$$

By noting that $v\geqq b_{n_t}x_2$ and using (1.2) we majorize the above terms and write

$$P(E_s\cap E_t)\leqq C_1\big(1-F^{n_s}(b_{n_t}x_2)\big)+C_3C_4+C_5+C_7(n_s/n_t)(\log n_t)^{r_{n_t}(x_1)}. \tag{3.3}$$

Note that $P(E_s)$ is obtained from (2.1) by replacing n by n_s. From the proof of Lemma 2.2 we notice that

$$P(E_s)\sim(\log n_s)^{r_{n_s}(x_1)+r_{n_s}(x_2)} \quad \text{as} \quad s\to\infty.$$

We now find the estimates for C_1, C_3, C_4, C_5 and C_7. We illustrate the method by estimating C_5. By Lemma 2.3 we have

$$1-F^{n_t-n_s}(b_{n_t}x_1)\sim\frac{n_t-n_s}{n_t}(\log n_t)^{r_{n_t}(x_1)} \quad \text{as} \quad s,t\to\infty.$$

In applying Lemma 2.3, we have used the fact that $n_s/n_t\to 0$. This is implied by the observation $\lim_{s\to\infty} n_s/n_{s+1}=0$ which is a consequence of the fact that $h>1$. Also note that $(\log n_t)^{r_{n_t}(x_1)}\to 0$ as $t\to\infty$ because $x_1>1$. Thus $F^{n_t-n_s}(b_{n_t}x_1)\to 1$ as $s,t\to\infty$. Hence as $s,t\to\infty$ we have

$$C_5\sim 1-F^{n_s}(b_{n_t}x_1)-n_s\big(1-F(b_{n_t}x_1)\big)F^{n_s-1}(b_{n_t}x_2).$$

Let

$$a_t=\frac{(\log n_t)^{r_{n_t}(x_1)}}{n_t} \quad \text{and} \quad b_t=\frac{(\log n_t)^{r_{n_t}(x_2)}}{n_t}.$$

Since $x_1>x_2>1$ and $n_s/n_t\to 0$ as $s,t\to\infty$, by (1.2) it follows that $n_sa_t\to 0$, $(n_s-1)b_t\to 0$ as $s,t\to\infty$ and $a_t/b_t\to 0$ as $t\to\infty$. By (1.2) we have

$$\begin{aligned}1-F^{n_s}(b_{n_t}x_1)-n_s\big(1-F(b_{n_t}x_1)\big)F^{n_s-1}(b_{n_t}x_2)&=\\ =1-(1-a_t)^{n_s}-n_sa_t(1-b_t)^{n_s-1}&=\end{aligned}$$

$$=-\frac{n_s(n_s-1)}{2}a_t^2\big(1+o(1)\big)+n_s(n_s-1)a_tb_t\big(1+o(1)\big)\sim n_s^2a_tb_t \quad \text{as} \quad s,t\to\infty.$$

Thus

$$C_5\sim(n_s/n_t)^2(\log n_t)^{r_{n_t}(x_1)+r_{n_t}(x_2)}\sim(n_s/n_t)^2P(E_t) \quad \text{as} \quad s,t\to\infty.$$

Similarly we obtain as $s, t\to\infty$,

$$C_1\big(1-F^{n_s}(b_{n_t}x_2)\big)\sim(n_s/n_t)P(E_t),\ C_3\sim P(E_t),\ C_4\sim P(E_s)\quad\text{and}\quad C_7\sim(\log n_t)^{r_{n_t}(x_2)}.$$

Using these estimates in (3.3), its right side becomes $P(E_s)P(E_t)\big(1+(2+n_s/n_t)\cdot\cdot(n_s/n_t P(E_s))\big)$. We now show that $n_s/n_t P(E_s)\to 0$ as $s, t\to\infty$. Note that

$$n_s/n_t P(E_s)\leqq n_s/n_{s+1}P(E_s)\sim n_s/n_{s+1}(\log n_s)^{r_{n_s}(x_1)+r_{n_s}(x_2)}\text{ as } s\to\infty.$$

By (1.2) we now get

$$n_s/n_t P(E_s)<d_4 m!/\big((s+1)^h-s^h\big)^m s^{-h(\varepsilon_4+(\log x_1x_2)/c)}$$

for all $s\geqq s_1$, $\varepsilon_4>0$ and any integer $m\geqq 1$. Since $(1+x)^h-1\sim hx$ as $x\to 0$, the above expression is less than

$$d_5 m!/h^m s^{(h-1)m-h(\varepsilon_4+(\log x_1x_2)/c)}$$

for all $s\geqq s_2$. This tends to zero by choosing m such that

$$m>h\big(\varepsilon_4+(\log x_1x_2)/c\big)/(h-1).$$

Thus we have shown that for any $\varepsilon_5>0$, there corresponds a positive integer s_3 such that for all $s\geqq s_3$ and all $x_1>x_2>1$ with $x_1x_2<e^c$ we have

$$P(E_s\cap E_t)<(1+\varepsilon_5)P(E_s)P(E_t).$$

This along with (3.1) implies that

$$\text{(3.4)}\qquad \liminf_{n\to\infty}\sum_{1<s<t<h}\sum\big(P(E_s\cap E_t)-P(E_s)P(E_t)\big)/\Big(\sum_{s=1}^{n}P(E_s)\Big)^2\leqq\varepsilon_5.$$

Since $\varepsilon_5>0$ is arbitrary, by (3.1), (3.4) and Lemma 2.1 we conclude that $P(E_k \text{ i.o.})=1$ for all x_1 and x_2 such that $x_1>x_2>1$ and $x_1x_2<e^c$. The proof of the lemma is complete.

LEMMA 3.3. *For all $x_1>x_2\geqq 1$ with $x_1x_2\geqq e^c$ and $\varepsilon>0$ we have*

$$P\big(Y_n>b_n(x_1+\varepsilon),\ S_b>b_n(x_2+\varepsilon)\quad\text{i.o.}\big)=0.$$

PROOF. Define $m_k=[\exp k]$. Note that for any x with $x+\varepsilon>0$, $(x+\varepsilon)b_r$ is monotonically increasing in r. Hence

$$P\big(Y_n>(x_1+\varepsilon)b_n,\ S_n>(x_2+\varepsilon)b_n\quad\text{for infinitely many } n\big)\leqq$$
$$\leqq P\big(Y_{m_{k+1}}>(x_1+\varepsilon)b_{m_k},\ S_{m_{k+1}}>(x_2+\varepsilon)b_{m_k}\quad\text{for infinitely many } k\big).$$

By proceeding as in the proof of Lemma 2.2 we get that

$$P\big(Y_{m_{k+1}}>(x_1+\varepsilon)b_{m_k},\ S_{m_{k+1}}>(x_2+\varepsilon)b_{m_k}\big)\sim$$
$$\sim(m_{k+1}/m_k)^2(\log m_k)^{r_{m_k}(x_1+\varepsilon)+r_{m_k}(x_2+\varepsilon)}\text{ as } k\to\infty.$$
$$<d_6k^{\varepsilon_6-(1/c)\log(x_1+\varepsilon)(x_2+\varepsilon)}\quad\text{for}\quad\varepsilon_6>0\quad\text{and}\quad\text{all}\quad k\geqq k_4.$$

The last inequality follows from (1.2). Choose $\varepsilon_6>0$ such that $0<\varepsilon_6<(1/c)\cdot\log(x_1+\varepsilon)(x_2+\varepsilon)-1$. This is possible since $x_1x_2\geqq e^c$. Now an appeal to the Borel—Cantelli lemma completes the proof.

LEMMA 3.4. *For* $\varepsilon>0$, $P(S_n<(1-\varepsilon)b_n \text{ i.o.})=0$.

PROOF. It is enough to prove the lemma when $0<\varepsilon<1$. Define $m_k=[e^k]$. Let $a_k(x)=(\log m_{k+1})^{r_{m_{k+1}}(x)}$. In view of (1.2), observe that for all x such that $0<x<1$, $a_k(x)\to\infty$ and $a_k(x)/m_{k+1}\to 0$ as $k\to\infty$. We have for all real x

$$P(S_n\leqq x)=F^n(x)+n\big(1-F(x)\big)F^{n-1}(x).$$

Hence, for all x such that $0<x<1$, by (1.2) we now get that

$$P(S_{m_k}\leqq b_{m_{k+1}}x)=\big(1-a_k(x)/m_{k+1}\big)^{m_k}+(m_k/m_{k+1})\big(1-a_k(x)/m_{k+1}\big)^{m_k-1}a_k(x)\sim$$
$$\sim(m_k/m_{k+1})a_k(x)\big(1-a_k(x)/m_{k+1}\big)^{m_k}\sim(m_k/m_{k+1})a_k(x)e^{-(m_k/m_{k+1})a_k(x)}\quad\text{as}\quad k\to\infty,$$

because $m_ka_k^2(x)/m_{k+1}^2\to 0$ as $k\to\infty$. Consider

$$\sum P(S_{m_k}<(1-\varepsilon)b_{m_{k+1}})<d_7+d_8\sum_{k_5}^{\infty}a_k(1-\varepsilon)e^{-(m_k/m_{k+1})a_k(1-\varepsilon)}<$$
$$<d_7+d_8\sum_{k_5}^{\infty}v!(m_{k+1}/m_k)^v a_k^{-(v-1)}(1-\varepsilon)<d_9+d_{10}\sum_{k_6}^{\infty}k^{-(v-1)r_{m_{k+1}}(1-\varepsilon)}.$$

Here v is any integer greater than 1. Note that

$$r_{m_{k+1}}(1-\varepsilon)\to-\frac{\log(1-\varepsilon)}{c}>0\quad\text{as}\quad k\to\infty.$$

Let ε_7 be such that $0<\varepsilon_7<-\dfrac{\log(1-\varepsilon)}{c}$. Choose the integer v such that $v>1-1\Big/\left(\varepsilon_7+\dfrac{\log(1-\varepsilon)}{c}\right)$. Then there corresponds a positive integer k_7 such that for all $k\geqq k_7$ we have $(v-1)r_{m_{k+1}}(1-\varepsilon)>1$. This implies that $\sum P\big(S_k<(1-\varepsilon)b_{m_{k+1}}\big)<\infty$ for all ε such that $0<\varepsilon<1$. Since

$P\big(S_n<(1-\varepsilon)b_n$ for infinitely many $n\big)\leqq P\big(S_{m_k}<(1-\varepsilon)b_{m_{k+1}}$ for infinitely many $k\big)$,

an application of the Borel—Cantelli lemma completes the proof.

LEMMA 3.5. *No point of the set* $\{(x_1,x_2)\colon 1\leqq x_1<x_2\leqq e^c\}$ *is a limit point of* $(Y_n/b_n,\ S_n/b_n)$.

PROOF. If possible, let (x_1,x_2) with $x_1<x_2$ be a limit point of $(Y_n/b_n,\ S_n/b_n)$. Then there is a subsequence $\{\omega_k\}$ such that

$$\lim_{k\to\infty}Y_{\omega_k}/b_{\omega_k}=x_1\quad\text{and}\quad\lim_{k\to\infty}S_{\omega_k}/b_{\omega_k}=x_2.$$

Since $Y_{\omega_k}\geqq S_{\omega_k}$ a.s., we should have $x_1\geqq x_2$ which is a contradiction. The proof of the lemma is complete.

Let us now complete the proof of the theorem. Lemmas 3.3, 3.4 and 3.5 along with (1.1) imply that the limit set is contained in S. By Lemmas 3.1 and 3.2 we get that every point in the set $S_1=\{(x_1, x_2):\ x_1>x_2>1,\ x_1x_2<e^c\}$ is a limit point of $(Y_n/b_n, S_n/b_n)$. By continuity considerations it now follows that S is the required limit set.

COROLLARY 3.1. *Every point in the interval* $[1, e^{c/2}]$ *is a limit point of* S_n/b_n.

THEOREM 3.2. *Let* Y_n *and* S_n *be the first two largest values among* $X_1, X_2, \dots, X_n$. *Then under* (b), *the set of all almost sure limit points of* $((Y_n-b_n)/c_n, (S_n-b_n)/c_n)$ *is*

$$S^* = \{(x_1, x_2):\ 0 \leqq x_2 \leqq x_1 \leqq 1,\quad x_1+x_2 \leqq 1\}.$$

PROOF. Let $\varepsilon>0$ be arbitrarily fixed. For all x_1 and x_2 satisfying $1\geqq x_1>x_2\geqq 0$, $0<x_1+x_2<1$, define a positive valued function $h(x_1, x_2)=h$ by

$$\max\left(1, (x_1+x_2+\varepsilon)^{-1}\right) < h(x_1, x_2) < (x_1+x_2)^{-1}.$$

Let $n_k=[\exp k^h]$. The rest of the proof is similar to that of the previous theorem with the change that (1.3), (1.4) and Lemmas 2.4 and 2.5 are to be used in place of (1.1), (1.2) and Lemmas 2.2 and 2.3 respectively.

COROLLARY 3.2. *Every point of the interval* $[0, 1/2]$ *is a limit point of* $(S_n-b_n)/c_n$.

REMARK 3.1. The method in the proofs of the above two theorems leads to many complicated terms when we consider three or more largest values in a sample. We strongly believe that the limit sets in the case of $p(\geqq 2)$ largest values $Y_n^{(i)}$, $i=1, 2, \dots, p$ are as follows:

Under (a), the almost sure limit set of $(Y_n^{(i)}/b_n,\ i=1, 2, \dots, p)$ is

$$S(p) = \left\{(x_1, x_2, \dots, x_p):\ e^c \geqq x_1 \geqq x_2 \geqq \dots \geqq x_p \geqq 1,\ \prod_{i=1}^{p} x_i \leqq e^c\right\},$$

and under (b), the set of all almost sure limit points of $((Y_n^{(i)}-b_n)/c_n,\ i=1, 2, \dots, p)$ is

$$S^*(p) = \left\{(x_1, x_2, \dots, x_p):\ 1 \geqq x_1 \geqq x_2 \geqq \dots \geqq x_p \geqq 0,\ \sum_{i=1}^{p} x_i \leqq 1\right\}.$$

The author has verified the truth of these statements for $p=3$. For obtaining $S(3)$, the function $h(x_1, x_2, x_3)=h$ should satisfy

$$\max\Big(1,\ \max_{1\leqq i\leqq 3} c/(\log x_i+\varepsilon+\sum_{j=1,\, j\neq i}^{3} \log(x_j+\varepsilon))\Big) < h < c\Big/\sum_{i=1}^{3} \log x_i$$

where $e^c\geqq x_1>x_2>x_3\geqq 1$, $x_1x_2x_3<e^c$ and $\varepsilon>0$.

For obtaining $S^*(3)$, the function $h(x_1, x_2, x_3)=h$ is chosen such that

$$\max\left(1,(\varepsilon+x_1+x_2+x_3)^{-1}\right)<h<(x_1+x_2+x_3)^{-1},$$

where $1\geqq x_1>x_2>x_3\geqq 0$, $0<x_1+x_2+x_3<1$ and $\varepsilon>0$. In either case $n_k=[\exp k^h]$.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF MYSORE
MYSORE—570006, INDIA

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A NOTE ON INVERSE-CLOSED SUBALGEBRAS OF $C(X)$

J. L. BLASCO (Burjasot)

Introduction

All topological spaces are assumed to be Tychonoff. As usual, $C(X)$ will denote the ring of all continuous real-valued functions on a topological space X. An algebra on X is a subring of $C(X)$ which contains the constants, separate points and closed sets, and is closed under uniform convergence and inversion in $C(X)$. In [5], Hager and Johnson have proved that if the Hewitt realcompactification vX is Lindelöf, then each algebra on X is isomorphic to $C(Y)$ for any space Y and they ask whether the converse of this assertion holds. From ([3], Theorem 10) the answer is yes if X is paracompact. In this paper we provide another partial answer to the above question by showing the equivalence when X is a Hausdorff topological group.

It is known that an algebra A on X is a $C(Y)$ if and only if the collection $Z(A)$ of all zero-sets of functions in A is a complete base on X ([2], Theorem 5). We shall prove our result by using a method provided in ([3], Theorem 2) for constructing noncomplete bases.

Preliminaries

The set of points of X where a member f of $C(X)$ is equal to zero is called the zero-set of f and will be denoted by $Z(f)$. We write $Z(X)$ for the family of all zero-sets in X. Let M be a nonempty subset of X. The Q-closure of M in X is the set $Q(M, X)$ of all points $p\in X$ for which each G_δ-set about p meets M. We write βX for the Stone—Čech compactification of X. It is known that vX is the Q-closure of X in βX.

A subset B of X is said to be z-separated from M if there is a zero-set Z in X such that $B\subset Z$ and $Z\cap M=\emptyset$[1]. The following result is needed:

(F1) [7] *A space X is Lindelöf if and only if there is a Hausdorff compactification K of X such that every compact subset of $K\sim X$ is z-separated from X.*

We will write $v(X, \mathscr{D})$ for the Wallman realcompactification associated with a given base $\mathscr{D}$ on X. For definitions and basic results the reader is referred to [1], [2] and [9]. A base $\mathscr{D}$ on X is called complete if it coincides with the trace on X of all zero-sets in $v(X, \mathscr{D})$. For later use, the following fundamental result is needed:

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[1] This notion is due to E. F. Steiner [8] who uses the term separating nest generated intersection ring. An equivalent concept is the strong delta normal base dueto R. A. Alò and H. L. Shapiro [1].

(F2) ([3], Theorem 2) *Let K be a Hausdorff compactification of X such that there exist two compact subsets C_1 and C_2 of $K\sim Q(X, K)$ which are disjoint and homeomorphic. If C_2 is not z-separated from X, then the family $\{Z\cap X\colon Z\in Z(K)\}$ contains a noncomplete base on X.*

The results

If there exists a locally finite family of open sets of a space X which is not countable, then by ([10], Theorem 2.6) there is an uncountable discrete family of open sets of X. On the other hand, in the proof of Corollary 4 of [3] we see that there is a noncomplete base on X if X has an uncountable C-embedded closed discrete subset. Consequently, we have the following

PROPOSITION 1. *If each base on X is complete, then each locally finite family of open sets of X is countable.*

We leave the verification of the following simple lemma to the reader.

LEMMA 2. *Let X be a space and let φ be an automorphism of βX such that $\varphi(X)=X$. If E is a subset of X, then*

$$\varphi(\mathrm{cl}_{vX} E) = \mathrm{cl}_{vX}\varphi(E) \quad \text{and} \quad \varphi(\mathrm{cl}_{\beta X} E \sim \mathrm{cl}_{vX} E) = \mathrm{cl}_{\beta X}\varphi(E) \sim \mathrm{cl}_{vX}\varphi(E).$$

LEMMA 3. *In a topological group G, the family of all symmetric neighborhoods of the identity e which are zero-sets is a base of neighborhoods of e.*

PROOF. Let W be a neighborhood of e. Since G is completely regular there exists $f\in C(G)$ such that $f(e)=0$ and $f(G\sim W)=\{1\}$. Put $F=\{x\in G\colon f(x)\leqq 1/2\}$. Then F is a neighborhood of e and a zero-set of G. Since the map $x\to x^{-1}$ is an automorphism of G, the function $g(x)=f(x^{-1})$ is continuous on G. Therefore the set $\{x\in G\colon g(x)\leqq 1/2\}=F^{-1}$ is a zero-set of G and a neighborhood of e. Then the set $F\cap F^{-1}$ is a zero-set of G and a symmetric neighborhood of e which is contained in W. □

Let G be a topological group and let $a\in G$. Consider the map $L_a\colon G\to G$ given by $L_a(x)=ax$. This function is an automorphism of G and from the properties of the Stone—Čech compactification, L_a can be extended to an automorphism L_a^β of βG.

The following theorem is the main result.

THEOREM 4. *For every topological group G the following statements are equivalent:*

(1) *vG is Lindelöf.*
(2) *Each algebra on G is a $C(Y)$.*
(3) *Each base on G is complete.*

PROOF. (1)⇒(2) follows from ([5], 4.4).
(2)⇒(3) is an inmediate consequence of ([2], Theorem 5).
(3)⇒(1). Suppose that vG is not Lindelöf and let W be a neighborhood of e such that $G\sim W\neq\emptyset$. By Lemma 3 there exists a symmetric neighborhood V of e such that $V\in Z(G)$ and $V^2\subset W$. The family $\mathscr{V}=\{xV\colon x\in G\}$ is a uniform cover of G and therefore is a normal cover. According to ([6], Theorem 1.2) it has a locally

finite open refinement $\mathcal{U}$. From (3) and Proposition 1 it follows that $\mathcal{U}$ is countable, therefore there is a sequence $\{x_n: n\in N\}$, $x_n\in G$, such that $G=\bigcup\{x_nV: b\in N\}$. Since $vG=Q(G, \beta G)$ we have $vG=\bigcup\{\mathrm{cl}_{vG}(x_nV): n\in N\}$. By assumption vG is not Lindelöf, therefore there exists $m\in N$ such that $\mathrm{cl}_{vG}(x_mV)$ is not Lindelöf. By Lemma 2

$$L^{\beta}_{x_m}(\mathrm{cl}_{vG}V) = \mathrm{cl}_{vG}(x_mV)$$

hence $\mathrm{cl}_{vG}V$ is not Lindelöf. Then by F1, there exists a compact subset K of $\mathrm{cl}_{\beta G}V\sim\sim\mathrm{cl}_{vG}V$ which is not z-separated from $\mathrm{cl}_{vG}V$.

Pick $y\in G\sim W$. Then $V\cap yV=\emptyset$, and since V, yV are zero-sets of G it follows that $\mathrm{cl}_{\beta G}V\cap\mathrm{cl}_{\beta G}(yV)\neq\emptyset$. Moreover, by Lemma 2 $L^{\beta}_{y}(K)$ is a compact subset of $\mathrm{cl}_{\beta G}(yV)\sim\mathrm{cl}_{vG}(yV)$. Hence K and $L^{\beta}_{y}(K)$ are disjoint, homeomorphic, compact subsets of $\beta G\sim vG$. Finally, since K is not z-separated from $\mathrm{cl}_{vG}V$ and $vG=Q(G, \beta G)$, it follows that K is not z-separated from G. The proof is concluded by applying F2 to βG. □

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CATEDRA DE MATEMATICAS II
FACULTAD DE CIENCIAS MATEMATICAS
BURJASOT, VALENCIA
SPAIN

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ON PARA-UNIFORM NEARNESS SPACES AND *D*-COMPLETE REGULARITY

H. BRANDENBURG (Berlin)

1. Introduction

A topological space X is called *D-completely regular* if it has an F_σ-base, i.e. a base $\mathscr{B}$ for the open sets such that for each $B\in\mathscr{B}$ there exists a countable subcollection $\mathscr{A}$ of $\mathscr{B}$ satisfying $B=\bigcup\{X\setminus A|A\in\mathscr{A}\}$. Clearly, every perfect space (i.e. closed sets are G_δ's) is D-completely regular, hence every semi-stratifiable space, every semi-metrizable space, and every σ-space. D-completely regular spaces were introduced in [1], where it was shown that they are precisely those topological spaces which can be embedded into products of developable spaces (see also [5, Theorem 3.5] and [2, Theorem 3]). In particular, every completely regular space is D-completely regular. However, there exist regular T_1-spaces which are not D-completely regular (e.g. see [6, Example 7.7]).

Since completely regular spaces are uniformizable, it is tempting to ask whether D-complete regularity can be characterized in a similar way by means of some kind of generalized uniform structure. Our main result yields an affirmative answer to this question: As a suitable generalization of uniform spaces we introduce para-uniform nearness spaces. We prove that a nearness space [7] is para-uniform if and only if it can be embedded into a product of nearness spaces which have a countable base (Theorem 1). From [3, Theorem 1] it follows that a topological nearness space is para-uniform if and only if its induced topology is D-paracompact in the sense of C. M. Pareek [8] (Theorem 2). Finally we show that a topological space X is D-completely regular if and only if it is para-uniformizable, i.e. iff there exists a para-uniform nearness structure on X which induces the topology of X (Theorem 3).

2. Results

We recall some definitions from the theory of nearness spaces. Let X be a set and consider a nonempty collection μ of covers of X. For each subset A of X the interior of A with respect to μ is defined as

$$\operatorname{int}_\mu A = \{x\in A| \text{ there exists a } \mathscr{U}\in\mu \text{ such that } \operatorname{St}(x,\mathscr{U})\subset A\},$$

where $\operatorname{St}(x,\mathscr{U})=\bigcup\{U\in\mathscr{U}|x\in U\}$. Following H. Herrlich [7] the pair (X,μ) is called a *nearness space* (and μ is a *nearness structure* on X), if the following conditions are satisfied:

(N.1) if $\mathscr{U}\in\mu$ and $\mathscr{U}$ refines $\mathscr{V}$, then $\mathscr{V}\in\mu$;
(N.2) if $\mathscr{U},\mathscr{V}\in\mu$, then $\mathscr{U}\wedge\mathscr{V}=\{U\cap V|U\in\mathscr{U},\ V\in\mathscr{V}\}\in\mu$;
(N.3) if $\mathscr{U}\in\mu$, then $\operatorname{int}_\mu\mathscr{U}=\{\operatorname{int}_\mu U|U\in\mathscr{U}\}\in\mu$.

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Any subcollection β of μ with the property that for each $\mathscr{V}\in\mu$ there exists a $\mathscr{U}\in\beta$ which refines $\mathscr{V}$ is called a *base* of μ. It is a *subbase* of μ if the collection of all covers of the form $\mathscr{U}_1\wedge\ldots\wedge\mathscr{U}_n$, $n\in\mathbf{N}$, $\mathscr{U}_i\in\beta$, forms a base of μ. The following simple observation will be useful later:

PROPOSITION 1. *Let β be a nonempty collection of covers of a set X. If β satisfies condition $(*)$, then β is a subbase of a nearness structure on X:*

$(*)$ *For each $\mathscr{V}\in\beta$ there exists a $\mathscr{U}\in\beta$ which refines* $\operatorname{int}_\beta\mathscr{V}=\{\operatorname{int}_\beta V|V\in\mathscr{V}\}$.

PROOF. Clearly, the collection $\mu(\beta)$ consisting of all covers of X which are refined by a cover of the form $\mathscr{U}_1\wedge\ldots\wedge\mathscr{U}_n$, $n\in\mathbf{N}$, $\mathscr{U}_i\in\beta$, satisfies the axioms (N.1) and (N.2) for a nearness structure. In order to show that it also satisfies (N.3) consider an arbitrary $\mathscr{V}\in\mu(\beta)$. There exist $\mathscr{U}_1, \ldots, \mathscr{U}_n\in\beta$ such that $\mathscr{U}_1\wedge\ldots\wedge\mathscr{U}_n$ refines $\mathscr{V}$. Using $(*)$ we can find $\mathscr{W}_1, \ldots, \mathscr{W}_n\in\beta$ such that $\mathscr{W}_i$ refines $\operatorname{int}_\beta\mathscr{U}_i$ for each $i\in\{1, \ldots, n\}$. Since it is easily seen that $\mathscr{W}_1\wedge\ldots\wedge\mathscr{W}_n$ refines $\operatorname{int}_{\mu(\beta)}\mathscr{V}=\{\operatorname{int}_{\mu(\beta)}V|V\in\mathscr{V}\}$, we conclude that $\operatorname{int}_{\mu(\beta)}\mathscr{V}\in\mu(\beta)$, i.e. that $\mu(\beta)$ is a nearness structure for which β is a subbase. $\square$

We call a collection β of covers of a set *kernel-normal* if it is nonempty and satisfies $(*)$ in the preceding Proposition. Note, that if $X=\{a, b, c, d\}$, $\mathscr{U}=\{\{a, d\}, \{b, d\}, \{b, c\}\}$, and $\mathscr{V}=\{\{a, b\}, \{a, c\}, \{c, d\}\}$, then $\beta=\{\mathscr{U}, \mathscr{V}\}$ is a subbase of a nearness structure on X (namely the discrete nearness structure) which is not kernel-normal. A cover $\mathscr{U}$ of a nearness space (X, μ) is called *kernel-normal* if there exists a countable kernel-normal subcollection of μ containing $\mathscr{U}$. Finally we say that a nearness space (X, μ) is *para-uniform,* if each $\mathscr{U}\in\mu$ is kernel-normal. Clearly, every uniform (nearness) space [7] is para-uniform, for if $\mathscr{U}\in\mu$ is a normal cover in the sense of J. W. Tukey [9], then it is easily seen that $\mathscr{U}$ is kernel-normal. If μ is the collection of all covers of the set $\mathbf{N}$ of natural numbers which are refined by some cover of $\mathbf{N}$ which is open with respect to the co-finite topology on $\mathbf{N}$, then $(\mathbf{N}, \mu)$ is a para-uniform nearness space which is not uniform.

PROPOSITION 2. (i) *Every subspace of a para-uniform nearness space is para-uniform.*

(ii) *Every product of para-uniform nearness spaces is para-uniform.*

(iii) *Every sum of para-uniform nearness spaces is para-uniform.*

PROOF. All assertions follow directly from the definitions. As a sample we prove (ii): Let $((X_i, \mu_i))_{i\in I}$ be a family of para-uniform nearness spaces. Moreover let μ be the product nearness structure [7] on $\prod_{i\in I} X_i$ with respect to this family and consider an arbitrary $\mathscr{V}\in\mu$. Then there exist $i_1, \ldots, i_n\in I$ and covers $\mathscr{U}_{i_1}\in\mu_{i_1}, \ldots, \mathscr{U}_{i_n}\in\mu_{i_n}$ such that $p_{i_1}^{-1}\mathscr{U}_{i_1}\wedge\ldots\wedge p_{i_n}^{-1}\mathscr{U}_{i_n}$ refines $\mathscr{V}$, where for each $k\in\{1, \ldots, n\}$, $p_{i_k}\colon \prod_{i\in I} X_i\to X_{i_k}$ is the projection and $p_{i_k}^{-1}\mathscr{U}_{i_k}=\{p_{i_k}^{-1}[U]|U\in\mathscr{U}_{i_k}\}$. Every $\mathscr{U}_{i_k}$ belongs to some countable kernel-normal subcollection β_{i_k} of μ_{i_k}.

If

$$\beta=\{\mathscr{V}\}\cup\{p_{i_1}^{-1}\mathscr{W}_{i_1}\wedge\ldots\wedge p_{i_n}^{-1}\mathscr{W}_{i_1}|\mathscr{W}_{i_1}\in\beta_{i_1}, \ldots, \mathscr{W}_{i_n}\in\beta_{i_n}\},$$

then β is a countable subcollection of μ containing $\mathscr{V}$, which is easily seen to be kernel-normal. Hence the product $(\prod_{i\in I} X_i, \mu)$ is para-uniform. $\square$

Recall that a mapping f from a nearness space (X, μ) into a nearness space (Y, η) is called *uniformly continuous* if $f^{-1}\mathcal{U}=\{f^{-1}[U] | U\in\mathcal{U}\}\in\mu$ for each $\mathcal{U}\in\eta$. (X, μ) and (Y, η) are *isomorphic* if there exists a bijective mapping f from X onto Y such that both f and f^{-1} are uniformly continuous with respect to μ and η. Our first theorem shows that the class of para-uniform nearness spaces is precisely the epireflective hull in the category of nearness spaces of the class of nearness spaces having a countable base.

THEOREM 1. *The following conditions are equivalent for a nearness space* (X, μ):

(i) (X, μ) *is para-uniform.*

(ii) (X, μ) *is isomorphic to a subspace of a product of nearness spaces which have a countable base.*

(iii) *For every* $\mathcal{V}\in\mu$ *there exists a countable subcollection* ξ *of* μ *containing a* $\mathcal{U}$ *which refines* $\operatorname{int}_\xi \mathcal{V}=\{\operatorname{int}_\xi V | V\in\mathcal{V}\}$.

PROOF. (i) implies (ii): Assumint (i) we can find a countable kernel-normal subcollection $\beta(\mathcal{U})$ of μ containing $\mathcal{U}$ for each $\mathcal{U}\in\mu$. By virtue of Proposition 1 every $\beta(\mathcal{U})$ is a subbase of a nearness structure $\mu(\mathcal{U})$ on X which has a countable base. Since the identity id: $(X, \mu)\to(X, \mu(\mathcal{U}))$ is uniformly continuous for each $\mathcal{U}\in\mu$, the mapping f from (X, μ) into $\prod_{\mathcal{U}\in\mu}(X, \mu(\mathcal{U}))$, defined by $x\mapsto(x_{\mathcal{U}})_{\mathcal{U}\in\mu}$, where $x_{\mathcal{U}}=x$ for each $\mathcal{U}\in\mu$, is uniformly continuous. More precisely, it can be easily shown that (X, μ) is isomorphic to the subspace $f[X]$ of $\prod_{\mathcal{U}\in\mu}(X, \mu(\mathcal{U}))$, which proves the implication.

That (ii) implies (i) follows from Proposition 2 and the observation that every nearness space with a countable base is para-uniform. Clearly, condition (iii) is formally weaker than (i). Hence it only remains to verify that (iii) implies (i). To this end consider an arbitrary $\mathcal{U}\in\mu$. For technical reasons we define $\mathcal{U}(0, n)=\{X\}$ for each $n\in\mathbf{N}$. Using complete induction and (iii) it can be shown for each $k\in\mathbf{N}$ there exists a $\mathcal{U}(k, k)\in\mu$ and a sequence $\beta(k)=(\mathcal{U}(k, n))_{n>k}$ in μ such that

(a) $\mathcal{U}=\mathcal{U}(1, 1)$ and $\mathcal{U}(k, k)=\mathcal{U}(k-1, k)$ for each $k>1$;

(b) $\mathcal{U}(k, k+1)$ refines $\operatorname{int}_{\beta(k)}\mathcal{U}(k, k)$ for each $k\in\mathbf{N}$;

(c) $\mathcal{U}(k, n)$ refines $\mathcal{U}(k-1, n)$ for each $k\in\mathbf{N}$ and for each $n\geqq k$.

We define $\beta=\{\mathcal{U}(k, k) | k\in\mathbf{N}\}$ and claim that β is kernel-normal. To prove this assertion we note that $\operatorname{int}_{\beta(k)} A\subset\operatorname{int}_\beta A$ for every subset A of X and for each $k\in\mathbf{N}$. For if $x\in\operatorname{int}_{\beta(k)} A$, there exists an $n>k$ such that $\operatorname{St}(x, \mathcal{U}(k, n))\subset A$. Condition (c) implies that

$$\operatorname{St}(x, \mathcal{U}(n, n))\subset\operatorname{St}(x, \mathcal{U}(n-1, n))\subset\ldots\subset\operatorname{St}(x, \mathcal{U}(k, n)).$$

Therefore $x\in\operatorname{int}_\beta A$. In particular it follows that $\operatorname{int}_{\beta(k)}\mathcal{U}(k, k)$ refines $\operatorname{int}_\beta\mathcal{U}(k, k)$ for each $k\in\mathbf{N}$. By virtue of (a) and (b) $\mathcal{U}(k+1, k+1)=\mathcal{U}(k, k+1)$ refines $\operatorname{int}_{\beta(k)}\mathcal{U}(k, k)$. Hence $\mathcal{U}(k+1, k+1)$ refines $\operatorname{int}_\beta\mathcal{U}(k, k)$ for each $k\in\mathbf{N}$, which proves that β is kernel-normal. Since $\mathcal{U}\in\beta$, the proof is complete. □

Every nearness structure μ on a set X induces a topology τ_μ on X which is given by $\tau_\mu=\{A\subset X | \operatorname{int}_\mu A=A\}$. If every τ_μ-open cover of X belongs to μ, then (X, μ)

is called a *topological nearness space*. It is wellknown that a topological nearness space is uniform if and only if its induced topology is fully normal in the sense of J. W. Tukey [9]. Fully normal spaces are precisely those topological spaces which admit an $\mathscr{A}$-mapping onto a metrizable space for every open cover $\mathscr{A}$, where a continuous mapping f from a topological space X onto a topological space Y is called an $\mathscr{A}$-mapping if there exists an open cover $\mathscr{B}$ of Y such that $\{f^{-1}[B] \| B \in \mathscr{B}\}$ refines the open cover $\mathscr{A}$ of X. In [8] C. M. Pareek has initiated the study of those topological spaces which admit an $\mathscr{A}$-mapping onto a developable T_1-space for every open cover $\mathscr{A}$, which he called *D-paracompact*. Every D-paracompact space is subparacompact, but there exist subparacompact spaces which are not D-paracompact (see [3] and [4] for recent results on D-paracompactness). Using the main result of [3] we can now prove:

THEOREM 2. *A topological nearness space is para-uniform if and only if its induced topology is D-paracompact.*

PROOF. Let (X, μ) be a topological nearness space and consider an arbitrary τ_μ-open cover $\mathscr{A}$ of X. If (X, μ) is para-uniform, there exists a countable kernel-normal subcollection β of μ containing $\mathscr{A}$. Since $\{\mathrm{int}_\mu \mathscr{U} | \mathscr{U} \in \mu\}$ is a countable collection of τ_μ-open covers containing $\mathscr{A}$ which is also kernel-normal, it follows from [3, Theorem 1] that (X, τ_μ) is D-paracompact. Conversely, if (X, τ_μ) is supposed to be D-paracompact, then, by virtue of [3, Theorem 1], there exists a countable kernel-normal collection $\eta(\mathscr{U})$ of τ_μ-open covers of X containing $\mathrm{int}_\mu \mathscr{U}$ for each $\mathscr{U} \in \mu$. Since (X, μ) is topological, the countable kernel-normal collection $\{\mathscr{U}\} \cup \eta(\mathscr{U})$ belongs to μ for each $\mathscr{U} \in \mu$, hence (X, μ) is para-uniform. □

Our main result shows that para-uniform nearness structures are well-chosen in order to prove an external characterization of D-completely regular spaces similar to the classical characterization of complete regularity by means of uniformities:

THEOREM 3. *A topological space is D-completely regular if and only if it is para-uniformizable, i.e. if and only if its topology is induced by a para-uniform nearness structure.*

PROOF. Let (X, τ) be a topological space. If there exists a para-uniform nearness structure μ on X such that $\tau = \tau_\mu$, then for every $\mathscr{V} \in \mu$ we can find a countable kernel-normal subcollection $\xi(\mathscr{V})$ of μ containing $\mathscr{V}$. By virtue of Proposition 1, every $\xi(\mathscr{V})$ is a subbase of a nearness structure $\mu(\mathscr{V})$ on X which has a countable base and is contained in μ. Therefore, if $\tau_{\mu(\mathscr{V})}$ denotes the topology on X induced by $\mu(\mathscr{V})$, then $\tau_{\mu(\mathscr{V})} \subset \tau$ for each $\mathscr{V} \in \mu$. We claim that $\mathscr{B} = \bigcup \{\tau_{\mu(\mathscr{V})} | \mathscr{V} \in \mu\}$ is an F_σ-base of τ. In fact, from $\tau = \tau_\mu$ it follows that if $A \in \tau$ and $x \in A$, then there exists a $\mathscr{V} \in \mu$ such that $\mathrm{St}(x, \mathscr{V}) \subset A$. Consequently $x \in \mathrm{int}_{\mu(\mathscr{V})} V \subset A$ for some $V \in \mathscr{V}$, which shows that $\mathscr{B}$ is a base of τ. In order to verify that $\mathscr{B}$ is an F_σ-base consider an arbitrary $B \in \mathscr{B}$. There exists a $\mathscr{V} \in \mu$ such that $B \in \tau_{\mu(\mathscr{V})}$. Let $\beta(\mathscr{V})$ be a countable base of $\mu(\mathscr{V})$. For each $\mathscr{W} \in \beta(\mathscr{V})$ define

$$B(\mathscr{W}) = \bigcup \{\mathrm{int}_{\mu(\mathscr{V})} W | W \in \mathscr{W},\ \mathrm{int}_{\mu(\mathscr{V})} W \cap (X \setminus B) \neq \emptyset\}.$$

Then $\{B(\mathscr{W}) | \mathscr{W} \in \beta(\mathscr{V})\}$ is a countable subcollection of $\mathscr{B}$ satisfying $B = \bigcup \{X \setminus B(\mathscr{W}) | \mathscr{W} \in \beta(\mathscr{V})\}$. Hence $\mathscr{B}$ is an F_σ-base, i.e. (X, τ) is D-completely regular.

To prove the converse implication assume now that (X, τ) is D-completely regular. Let $\mathcal{B}$ be an F_σ-base of (X, τ). We may assume that $\mathcal{B}$ is closed with respect to the formation of finite intersections. If β is the collection of all finite covers of X consisting of sets from $\mathcal{B}$, then it is easily seen that $\mathcal{U}=\{\operatorname{int}_\beta U \mid U\in\mathcal{U}\}$ for each $\mathcal{U}\in\beta$. Consequently β is kernel-normal and hence a subbase of a nearness structure $\mu(\beta)$ on X (Proposition 1). In fact, because of our assumption on $\mathcal{B}$ it is already a base of $\mu(\beta)$. We claim that $\tau=\tau_{\mu(\beta)}$. Clearly, $\tau_{\mu(\beta)}\subset\tau$. On the other hand, if $U\in\tau$ and $x\in U$, then there exists a $B\in\mathcal{B}$ such that $x\in B\subset U$. Moreover, since $\mathcal{B}$ is an F_σ-base, $x\in X\setminus A\subset\mathcal{B}$ for some $A\in\mathcal{B}$. In particular, $\mathcal{U}=\{A, B\}$ belongs to β. Now $\operatorname{St}(x, \mathcal{U})\subset U$, which shows that $x\in\operatorname{int}_{\mu(\beta)} U$. This implies $\tau=\tau_{\mu(\beta)}$, which proves our claim.

We complete the proof by showing that $(X, \mu(\beta))$ is para-uniform. To this end consider an arbitrary $\mathcal{V}\in\mu$. Since β is a base of $\mu(\beta)$, there exists a $\mathcal{U}\in\beta$ which refines $\mathcal{V}$. For each $U\in\mathcal{U}$ we can find a countable subcollection $\mathcal{B}_U$ of $\mathcal{B}$ such that $U=\bigcup\{X\setminus B \mid B\in\mathcal{B}_U\}$. If ξ is the countable subcollection of $\mu(\beta)$ consisting of $\mathcal{V}$ and of all covers of the form $\{U, B\}$, $U\in\mathcal{U}$ and $B\in\mathcal{B}_U$, then $\mathcal{V}=\{\operatorname{int}_\xi V \mid V\in\mathcal{V}\}$. Therefore we infer from Theorem 1 (iii) that $(X, \mu(\beta))$ is para-uniform. $\square$

We conclude this note by mentioning an open problem which arises naturally in connection with Proposition 2:

PROBLEM. Does there exist a para-uniform nearness space the completion of which (see 7, §5) is not para-uniform?

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FREIE UNIVERSITÄT BERLIN
INSTITUT FÜR MATHEMATIK I
D—1000 BERLIN 33

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ON RINGS WITH UNIQUE MINIMAL SUBRINGS

Y. HIRANO (Okayama)

§1.

In this note, we consider which rings have a unique minimal subring. This problem was raised by Szász [5, Problem 80]. In §2, we consider this problem under some additional assumptions on rings. Kruse and Price [3] determined the structure of nilpotent p-rings with unique minimal subring. Using their result, we shall completely determine the structure of right (or left) artinian rings with unique minimal subring. In §3, we shall consider the problem under general situation.

Throughout this note, all rings are associative. For any subset A of a ring R, $[A]$ (resp (A)) denotes the subring (resp. two-sided ideal) of R generated by A. We denote by $T(R)$ the torsion ideal of a ring R; $T(R)=\{a\in R|na=0$ for some non-zero integer $n\}$. We say that R is torsion-free if $T(R)=0$. For any prime p, we set $R_p=$ $=\{a\in R|p^n a=0$ for some $n\}$. A ring R is called a p-ring if $R=R_p$. R^+ denotes the additive group of R. Given a, b in R, we write $a-ab$ (resp. $a-ba$) formally as $a(1-b)$ (resp. $(1-b)a$).

§2.

We begin with the following lemma.

LEMMA 1. *If S is a ring having no non-trivial subring, then S is either a field of order p or a null ring of order p for some p.*

PROOF. By hypothesis, $S=[x]$ for any non-zero $x\in S$. If $x^2\neq 0$, then $x\in[x^2]$, and so we can write $x=xy$ for some $y\in[x]$. In this case, y is an idempotent, and $S=\mathbf{Z}y\cong\mathbf{Z}/(p)$ for some prime p. If $x^2=0$, then $S^2=0$, and so S^+ has no non-trivial subgroup. In this case, S is a null ring of order p for some prime p.

First we shall consider the case that the unique minimal subring is a field. We consider the following condition:

$(*)$ *If a, b are non-zero elements of a ring R such that $ab=0$, then either $[a]$ or $[b]$ is infinite.*

PROPOSITION 1. *The following statements are equivalent:*

1) *R has a unique minimal subring S, and S is a field.*

2) *R has a non-zero finite subring, and R satisfies the condition $(*)$.*

PROOF. 1)$\Rightarrow$2). Let a, b be non-zero elements of R such that $[a]$ and $[b]$ are finite. Then the field S is contained in both of $[a]$ and $[b]$, and so $S\subseteq[a][b]$. Hence, we have $ab\neq 0$. This shows that R satisfies $(*)$.

2)$\Rightarrow$1). The condition $(*)$ implies that any finite subring of R is a field. Therefore, by our hypothesis, there exists at least one minimal subring S. We shall show that S is the unique minimal subring of R. Let e be the identity element of S and let f be any idempotent of R such that $[f]$ is a minimal subring. Then $g=e+ef-efe$ is an idempotent, and $eg=g$, $ge=e$. Hence $[e, g]$ is a finite field. Thus we see that $g=e$, and so $ef=efe$. Similarly, we have $fe=efe$, and hence $ef=fe$. Therefore, $[e, f]$ is finite, and so we conclude that $e=f$.

A ring R is called π-regular if for every $a\in R$ there exists a positive integer n (depending on a) and an element $x\in R$ such that $a^n xa^n=a^n$. For example, right (or left) artinian rings are π-regular (see e.g. [2]).

COROLLARY 1. *Let R be a π-regular ring. Then the following conditions are equivalent:*

1) *R has a unique minimal subring S, and $S\cong GF(p)$.*

2) *R is a direct sum of a division ring of characteristic p and a torsion-free ring.*

PROOF. In view of Lemma 1, it suffices to prove that 1) implies 2). Let e be the identity of S. Since $eR(1-e)$ is square zero and $peR(1-e)=0$, we have that $eR(1-e)=0$. Similarly, we have $(1-e)Re=0$, and so e is a central idempotent. By Proposition 1, R satisfies $(*)$ and any non-zero subring of $(1-e)R$ is infinite. To show that $(1-e)R$ is torsion-free, let a be an element of $T((1-e)R)$. Since $(1-e)R$ is also π-regular, there exists a positive integer n and an element $x\in(1-e)R$ such that $a^n xa^n=a^n$. If $a^n\neq 0$, then $[a^n x]$ is a non-zero finite subring of $(1-e)R$. This contradiction shows that $a^n=0$. Thus, $[a]$ is finite, and hence $a=0$. Therefore, $(1-e)R$ is torsion-free. Since $pe=0$, eR does not have non-zero nilpotent elements. Thus eR is a π-regular ring without non-zero nilpotent elements. Similarly, we see that eR has no non-trivial idempotents. Hence eR is a division ring.

Kruse and Price [3] determined the structure of nilpotent p-rings with unique minimal subring. The result is the following.

PROPOSITION 2. *A nilpotent p-ring N contains only one subring S of order p if and only if N and S satisfy one of the following conditions:*

(1) *N^+ is cyclic or quasi-cyclic.*

(2) *Let $U=\{a\in N \mid pa=0\}$. Then U^+ has rank 2 or 3, $U^2=S$, and $b\in U^2$, $b^2=0$ implies $b\in S$. There is, moreover, an ideal C of N such that $N=C+U$, $C\cap U=S$, and C^+ is cyclic or quasi-cyclic.*

Given a commutative ring F and an F-algebra A, we denote by $F\propto A$ the ring whose additive group is the direct sum of F and A with multiplication given by

$$(f, a)(f', a') = (ff', fa'+f'a+aa').$$

A ring R is called a *local ring* if R/J is a division ring, where J denotes the Jacobson radical of R.

PROPOSITION 3. *Let R be a local ring of characteristic p^n $(n\geqq 2)$ with nilpotent Jacobson radical J. Then R has a unique minimal subring S if and only if R and S satisfy one of the following conditions:*

(3) *$R\cong \mathbf{Z}/(p^n)$.*

(4) *Let $U=\{a\in R \mid pa=0\}$. Then U^+ has rank 2 or 3, $U^2=S$, and $b\in U$, $b^2=0$ implies $b\in S$. Moreover, R is isomorphic to the ring $(\mathbf{Z}/(p^n)\propto U)/((p^{n-1}\cdot 1, s))$, where s is a non-zero element of S.*

PROOF. It suffices to prove the only if part. Clearly, S is a null ring of order p. If a is an invertible element of R, then $p^{n-1}a \neq 0$. Conversely, let a be an element with $p^{n-1}a \neq 0$. Then $[p^{n-1}a]=S=[p^{n-1}\cdot 1]$, and hence $p^{n-1}(a-r\cdot 1)=0$ for some $0<r<p$. Since the (additive) order of the invertible elements of R is p^n, we conclude that $a-r\cdot 1 \in J$, and so a is invertible. Thus we have shown that $R/J \cong GF(p)$ and the characteristic of J is p^{n-1}. By our hypothesis, J and S satisfy either (1) or (2) in Proposition 2. First, we consider the case that they satisfy (1). In this case, J^+ is cyclic and generated by $p\cdot 1$, because the characteristic of J equals the order of $p\cdot 1$. Then R^+ is also a cyclic group generated by 1, and so R is isomorphic to $\mathbf{Z}/(p^n)$. Next, assume that (2) is satisfied. Then $U=\{a\in R \mid pa=0\}=\{a\in J \mid pa=0\}$ satisfies the same condition as in (2). Moreover, we have that $R=[1]+U$ and $[1]\cap U=S$. Since $[1]\cong \mathbf{Z}/(p^n)$, we have the natural epimorphism $\psi\colon \mathbf{Z}/(p^n)\propto U \to (1)+U=R$. Since $[1]\cap U=[p^{n-1}\cdot 1]=S$, $\operatorname{Ker}(\psi) \ni (p^{n-1}\cdot 1, s)$ for some non-zero $s\in S$. Comparing the orders of R and $(\mathbf{Z}/(p^n)\propto U)/((p^{n-1}\cdot 1, s))$, we have the desired isomorphism. Hence, in this case, R satisfies (4).

We can now prove the main theorem.

THEOREM 1. *A right artinian ring R has a unique minimal subring if and only if R satisfies one of the following conditions:*

(i) *R is a direct sum of a division ring of characteristic $p(\neq 0)$ and a torsion-free ring with right identity.*

(ii) *For some prime p, R_p is non-zero and satisfies one of* (1)—(4), *and R is a direct sum of R_p and a torsion-free ring with right identity.*

PROOF. It suffices to prove the only if part. By hypothesis, there exists a prime p such that $R_p \neq 0$ and R/R_p is torsion-free. By [4, Satz 4] or [6, Theorem 5], R is a direct sum of R_p and a torsion-free ring with right identity. Since R_p is artinian, either R_p has a non-zero idempotent or R_p is nilpotent. If R_p is nilpotent, then R_p satisfies (1) or (2). If R_p has a non-zero idempotent e, then e must be the identity of R_p, because if e is not an identity of R_p both of eR_p and $(1-e)R_p$ contain minimal subrings. Since R_p is an artinian ring without non-trivial idempotents, R_p is a local ring of characteristic p^n for some $n \geqq 1$. If $n=1$, then R_p has the minimal subring $S\cong GF(p)$ generated by the identity of R_p. In this case, R satisfies (i) by Corollary 1. If $n\geqq 2$, then, by Proposition 3, R_p satisfies (3) or (4).

COROLLARY 2. *A finite ring R has a unique minimal subring if and only if R satisfies one of the following conditions:*

(a) *R is a finite field.*

(b) *R is a nilpotent p-ring with cyclic additive group.*

(c) *R is a nilpotent p-ring, and satisfies* (2).

(d) *$R\cong \mathbf{Z}/(p^n)$ for some prime p and $n\geqq 2$.*

(e) *$R\cong(\mathbf{Z}/(p^n)\propto U)/((p^{n-1}\cdot 1, s))$, where s is a non-zero element of S and U is the same as in* (4).

§3.

Throughout this section, R denotes a ring with unique minimal subring S. We consider the torsion ideal $T=T(R)$. We can easily see that $T=\bigoplus_q R_q$, where q runs over all primes. Thus there exists a prime p such that $S\subseteq R_p$. If $q\neq p$, then any non-zero element of R_q generates an infinite multiplicative subsemigroup. For example, consider the polynomial ring $K[x]$ over the field $K=GF(q)$. Then the ring $xK[x]$ has characteristic q, and any non-zero element of $xK[x]$ generates an infinite subsemigroup.

By Lemma 1, the unique minimal subring S is either a field or a null ring of order p.

Case 1: S is isomorphic to $GF(p)$. In this case, R_p has no non-zero nilpotent elements, because if R_p contains a non-zero nilpotent element, then R_p contains a null subring of order p, which contradicts the hypothesis. Let e be the identity of S. Then e is a central idempotent of R, and any non-zero subring of $(1-e)R$ is infinite. By Proposition 1, any finite subring of eR is a field containing S. However, eR is not necessarily a domain. As an example, consider the ring $GF(p)[x, y]/(xy)$.

Case 2: S is a null ring of order p. In this case, S is an ideal of R. To see this, let $S=[s]$ and let $a\in R$. If $sas\neq 0$, then $[sas]=[s]$, and so $isas=s$ for some $0<i<p$. Then $e=isa$ is a nonzero idempotent, and $[e]\cong GF(p)$, which is a contradicition. Hence, sa and as are nilpotent and hence $sa,\ as\in S$. Therefore, S is a two-sided ideal of R. We denote by P the prime radical of R_p. Since P is locally nilpotent (see e.g., [1, p. 51]), any finitely generated subring is a nilpotent p-ring with unique minimal subring, and hence satisfies (1) or (2) in Proposition 2. Since P is the direct limit of its finitely generated subrings, P also satisfies (1) or (2).

Summarizing the above results, we obtain the following theorem.

THEOREM 2. *Let R be a ring with unique minimal subring S. If $S\cong GF(p)$, then R decomposes as follows: $R=R'\oplus D$, where R' is a ring all of whose non-zero subrings are infinite, D has characteristic p and any finite subring of D is a field containg S. If S is a null ring of order p, then S is an ideal of R, $T(R)/R_p$ has no non-zero nilpotent elements and the prime radical of R_p is a nil p-ring satisfying* (1) *or* (2).

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA 700, JAPAN

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JORDAN DERIVATIONS AND JORDAN HOMOMORPHISMS ON PRIME RINGS OF CHARACTERISTIC 2

R. AWTAR (Ile-Ife)

Throughout this paper we assume that φ is an additive mapping from a ring R onto a non-commutative prime ring S of characteristic 2 and φ is a monomorphism satisfying the following conditions:

(i) $\varphi(xy+yx)=\varphi(x)\varphi(y)+\varphi(y)\varphi(x)$, i.e., $\varphi[x, y]=[\varphi(x), \varphi(y)]$, where $[x, y]$ denotes the commutator $xy-yx$, for all $x, y\in R$.

(ii) $\varphi(x^3)=\varphi(x)^3$ for all $x\in R$.

We denote the center of a ring R by $Z(R)$. For $x, y\in R$, let $\delta(x, y)=\varphi(xyx)+ +\varphi(x)\varphi(y)\varphi(x)$. For any subset A of R, we denote $R-A=\{x\in R|x\notin A\}$.
We begin with the following lemma.

LEMMA 1. *For all* $x, y\in R$, $\varphi(x^2)+\varphi(x)^2\in Z(S)$.

PROOF. See [1, page 851, lines 17—19].

LEMMA 2. *For all* $x, y\in R$,

$$\varphi(xyx+yxy) = \varphi(x)\varphi(y)\varphi(x)+\varphi(y)\varphi(x)\varphi(y),$$

i.e., $\delta(x, y)=\delta(y, x)$.

PROOF. See [1, page 851 lines 13—24]. By linearizing Lemma 2 on x, we get

LEMMA 3. *For all* $x, y, z\in R$,

$$\varphi(xyz+zyx) = \varphi(x)\varphi(y)\varphi(z)+\varphi(z)\varphi(y)\varphi(x).$$

LEMMA 4. *For all* $x, y, z\in R$, $[\delta(x, y), \varphi(z)]=\delta(x, [y, z])$.

PROOF. See [1, page 850 lines 4—13].

LEMMA 5. *For* $x, y\in R$, $\delta(x, y)$ *commutes with* $\varphi(x)$ *and* $\varphi(y)$.

PROOF. Let $z=y$ in Lemma 4. Then $[\delta(x, y), \varphi(y)]=0$. After interchanging x and y, we get $[\delta(y, x), \varphi(x)]=0$ and in view of Lemma 2, we get the desired conclusion that $[\delta(x, y), \varphi(x)]=0$.

LEMMA 6. *For all* $x\in R$, $\varphi(x^2)=\varphi(x)^2$.

PROOF. From Lemmas 5, 4 and 3, for all $x, y \in R$, we get

$$0 = [\delta(x, y), \varphi(x)] = \delta(x, [y, x]) = \varphi(x[y, x]x) + \varphi(x)\varphi[y, x]\varphi(x) =$$
$$= \varphi(x^2yx + xyx^2) + \varphi(x)[\varphi(y), \varphi(x)]\varphi(x) = \varphi(x^2)\varphi(y)\varphi(x) + \varphi(x)\varphi(y)\varphi(x^2) +$$
$$+ \varphi(x)\varphi(y)\varphi(x)^2 + \varphi(x)^2\varphi(y)\varphi(x) = \{\varphi(x^2) + \varphi(x)^2\}[\varphi(x), \varphi(y)],$$

by Lemma 1. Put $\lambda(x) = \varphi(x^2) + \varphi(x)^2$, so λ is an additive mapping from R into S. If $\lambda(x) \neq 0$, since S is prime and $\lambda(x) \in Z(S)$, then $[\varphi(x), \varphi(y)] = 0$ and so $x \in Z(R)$. Hence $R - Z(R) \subseteqq \operatorname{Ker} \lambda$. Since $R - Z$ generates R under addition, so $R \subseteqq \operatorname{Ker} \lambda$. Hence $\lambda(x) = \varphi(x^2) + \varphi(x)^2 = 0$ for all $x \in R$.

LEMMA 7. *For $x, y \in R$, if $xy = yx$ then $\varphi(xy) = \varphi(x)\varphi(y)$.*

PROOF. Since φ is a monomorphism and by Lemma 6 $\varphi(x^2) = \varphi(x)^2$, so the proof follows from Lemma 5 of [1].

LEMMA 8. *For fixed $x \in R$, if $\delta(x, y)$ and $\delta(x^2, y)$ are in $Z(S)$ for all $y \in R$ then $\delta(x, y) = 0$ for all $y \in R$.*

PROOF. See [1, page 850, lines 31—38 and page 851, line 1].

Now we are in a position to prove the main lemma of this paper.

LEMMA 9. *For all $x, y \in R$, $\delta(x, y) = \varphi(xyx) + \varphi(x)\varphi(y)\varphi(x) = 0$.*

PROOF Since the charactristic of S is 2, so $[y, z] = [z, y]$ in S which together with Lemmas 2 and 4 shows that $\delta(x, [y, z])$ is symmetrical in x, y and z. So, for all $x, y, z \in R$

$$[\delta(x^2, y), \varphi(z)] = [\delta(y, z), \varphi(x^2)] = [\delta(y, z), \varphi(x)^2] =$$
$$= [[\delta(y, z), \varphi(x)], \varphi(x)] = [\delta(x, [y, z]), \varphi(x)] = 0,$$

by Lemmas 1 and 5. Thus $\delta(x^2, y) \in Z(S)$ for all $x, y \in R$. Linearizing it on x, yields $\delta([x, y], z) \in Z(S)$ for all $x, y, z \in R$. From the above, we have $\delta([x, y]^2, z) \in Z(S)$ for all $x, y, z \in R$. Hence by Lemma 8, $\delta([x, y], z) = 0$ and so $[\delta(x, y), \varphi(z)] = 0$ by Lemma 4. Thus $\delta(x, y) \in Z(S)$ for all $x, y \in R$. Again in view of Lemma 8, we get the desired conclusion that $\delta(x, y) = 0$ for all $x, y \in R$.

From this point the proof given by Simley [3] can be used to prove the following result which extends some due to Herstein and Kleinfeld [1, Theorem 2].

THEOREM 1. *If φ is an additive mapping from a ring R onto a non-commutative prime ring S of characteristic 2; moreover, φ is a monomorphism satisfying $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ and $\varphi(x^3) = \varphi(x)^3$ for all $x, y \in R$, then φ is either an isomorphism or an anti-isomorphism.*

The next lemma is an analogoue of Lemma 9 for Jordan (Lie) derivations. The proof is almost identical in pattern with Lemma 9 and so will be omitted.

LEMMA 10. *Let R be a non-commutative prime ring of characteristic 2 and let, $'$ be an additive mapping from R into itself such that $(xy + yx)' = x'y + xy' + y'x + yx'$ and $(x^3)' = x^2x' + xx'x + x'x^2$ for all $x, y \in R$. Then $(xyx)' = x'yx + xy'x + xyx'$ for all $x, y \in R$.*

Now, the proof given by Herstein [2, Theorem 4.1] can be used to show the following theorem which generalizes one due to Herstein [2, Theorem 4.1].

THEOREM 2. *Let R be a non-commutative prime ring of characteristic* 2, *and let,* $'$ *be an additive mapping from* R *into itself satisfying* $(xy+yx)'=x'y+xy'+y'x+yx'$ *and* $(x^3)'=x^2x'+xx'x+x'x^2$ *for all* $x, y \in R$. *Then* $'$ *is a derivation of* R.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF IFE
ILE—IFE
NIGERIA

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A NOTE ON RINGS WITH CHAIN CONDITIONS

DINH VAN HUYNH (Hanoi)

1. Introduction

In this note we consider associative rings (not necessary with identity) and study the question: When does the chain condition on the one side imply the same chain condition on the other side?

We will give a new solution for the following problem raised by A. Kertész [10, Problem 95]: Which right artinian rings are left artinian? This problem was solved by Widiger [11, Theorem 1]. He explicitly gave there three classes of rings such that a right artinian ring A is left artinian if and only if A has no homomorphic images contained in one of these three classes. We will prove that a right artinian ring A is left artinian if and only if $J/(J^2+D)$ is a finitely generated left A-module, where J and D are the Jacobson radical and the maximal divisible torsion ideal of A, respectively (Theorem 1). (If, for example, the left (or right) annihilator of A is zero, then $D=(0)$.) Comparing with Widiger's solution our criterion seems to be easier to test because for verifying whether A is left artinian or not we only need to consider the left A-module $M=J/(J^2+D)$ which is a direct sum of simple left A-modules M_i with $AM_i=M_i$ and of a finite trivial left A-module E: $M=\sum_{i\in I}^{\oplus} M_i \oplus E$. Then A is left artinian if and only if the index set I is finite.

An interesting result of Lenagan [9, Proposition] stated that if I is an ideal of a right noetherian ring A (with identity) such that ${}_AI$ is artinian, then I_A is also artinian. We will prove that a similar result holds also for almost right noetherian rings under an additional condition $(*)$ (Theorem 3). We can do that because fortunately every semiprime almost right noetherian ring is right Goldie ([8, Lemma 3]). We guess moreover that every semiprime almost right noetherian ring is right noetherian, but we are unable to show that.

2. A condition for right artinian rings to be left artinian

It is known that there are right artinian rings which are not left artinian. For example the matrix ring

$$A=\begin{bmatrix} Q & 0 \\ Q & 0 \end{bmatrix}$$

is right but *not left* artinian, where Q is the field of rational numbers. It is also known that if a right artinian ring A is left artinian then A/D is left (and right) noetherian. Hence $J/(J^2+D)$ is a finitely generated left A-module. The following theorem shows that this condition is even sufficient.

THEOREM 1. *Let A be a right artinian ring. Then A is left artinian if and only if $J/(J^2+D)$ is a finitely generated left A-module, where J and D are the Jacobson radical and the maximal divisible torsion ideal of A, respectively.*

As is well-known, for a right artinian ring A, if $D=(0)$ then A is right noetherian (Fuchs and Szele). A ring A is called restricted right noetherian if every factor ring of A by a nonzero ideal is right noetherian. Such a ring is denoted by RN-ring for short. A PRN-ring is an RN-ring which is not right noetherian (properly restricted right noetherian).

For the proof of Theorem 1 we shall prove the following Lemma which is also of its own interest.

LEMMA 2. *The prime radical of a* PRN-*ring is a zero ring.*

PROOF. Let A be a PRN-ring and N be the prime radical of A. (If A/N is right artinian then the statement is proved by Widiger [12, Satz 22].) Let B and C be two non-zero ideals of A. Then by assumption, A/B and A/C are right noetherian. Hence we must have

$$B\cap C\neq(0). \tag{1}$$

In case $N\neq(0)$, i.e. A is non-prime, there are non-zero ideals B and C of A with $BC=(0)$. By (1), $D=B\cap C\neq(0)$ but $D^2=(0)$. From this it is easy to verify that N is nilpotent. There exists a positive integer m such that $N^m\neq(0)$, $N^{m+1}=(0)$. We assume in contrary that $m\geqq 2$.

As is well-known, we can embedd A into a ring $A^*=A\times Z$ with identity such that every right (left) ideal of A is a right (left) ideal of A^*, where Z is the ring of all integers. Further, since $A^*/A\cong Z$, for every ideal $I\neq(0)$ of A, A^*/I is right noetherian. Considering the factor rings A^*/N^2 and A^*/N^m we get

$$N = x_1A^*+\ldots+x_kA^*+N^2 \quad (x_i\in N,\quad i = 1, 2, \ldots, k) \tag{2}$$

and

$$N^{m-1} = y_1A^*+\ldots+y_hA^*+N^m \quad y_j\in N^{m-1},\quad j = 1, \ldots, h) \tag{3}$$

respectively. Now, $N^m=N\cdot N^{m-1}$. For $x\in N^m$ we have $x=ab$ with $a\in N$, $b\in N^{m-1}$. By (2) and (3)

$$a = x_1a_1+\ldots+x_ka_k+a' \quad (a_i\in A^*,\ a'\in N^2),$$

$$b = y_1b_1+\ldots+y_hb_h+b' \quad (b_j\in A^*,\ b'\in N^m).$$

Then

$$ab = \big(x_1(a_1y_1b_1)+\ldots+x_1(a_1y_hb_h)\big)+\ldots+\big(x_k(a_ky_1b_1)+\ldots+x_k(a_ky_hb_h)\big). \tag{4}$$

Since every $a_iy_jb_j$ is contained in N^{m-1} we get by (3)

$$a_iy_jb_j = y_1c_{ij}^{(1)}+\ldots+y_hc_{ij}^{(h)}+c_i \quad (c_{ij}^{(t)}\in A^*,\ c'_{ij}\in N^m). \tag{5}$$

Putting (5) in (4) we get

$$x = x_1y_1d_{11}+\ldots+x_1y_hd_{1h}+x_2y_1d_{21}+\ldots+x_2y_hd_{2h}+\ldots+x_ky_1d_{k1}+\ldots+x_ky_hd_{kh}$$

with $d_{ij}=c_{i1}^{(j)}+\ldots+c_{ih}^{(j)}\in A^*$. Hence N^m is a finitely generated right A-module, namely it is generated by $x_iy_j\in N^m$ $(i=1, \ldots, k;\ j=1, \ldots, h)$. Since $N^m\cdot N=(0)$, N^m is a finitely generated right module over the right noetherian ring A^*/N. Hence N^m is a noetherian right A-module, implying that A^* is right noetherian, a contradiction.

REMARKS. A ring A is called a PK-ring if (i) every factor ring of A by a non-zero ideal of A has right Krull dimension and (ii) A does not have right Krull dimension.

QUESTION. Is the prime radical of a PK-ring a zero ring? (For the modules and rings with Krull dimension we refer to Gordon and Robson [6].)

Using the third Theorem of Goldie (cf. Herstein [7, Theorem 4.8]), Lemma 2 and a method of Widiger [13, Satz 5], one can describe the structure of all non-prime PRN-rings with identity whose left ideals are principal.

PROOF OF THEOREM 1. Let A be a right artinian ring, J and D be the Jacobson radical and the maximal divisible torsion ideal of A, respectively. If A is left artinian, then A/D is left noetherian. Hence $(J/D)/\big((J^2+D)/D\big)$ is a finitely generated left A/D-module, therefore $J/(J^2+D)$ is a finitely generated left A-module.

Conversely, assume that $J/(J^2+D)$ is a finitely generated left A-module. If A/D is left artinian, then A is also left artinian. Hence we can assume without loss of generality that $D=(0)$. Hence A is right noetherian. If A is not left noetherian, there exists an ideal B of A which is maximal with respect to the condition that $A'=A/B$ is not left noetherian. Clearly, A' is then a PRN-ring. By Lemma 2, $J'^2=(0)$, where J' is the Jacobson radical of A'. By the Wedderburn—Artin Theorem we have

$$A/J(B) = B/J(B)\oplus C/J(B),$$

where $J(B)=B\cap J$, the Jacobson radical of B and C is an ideal of A containing $J(B)$. Evidently, it holds $C/J(B)\cong A'$. Hence $J^2\subseteqq J(B)$. The left A-module $J/J(B)$ is not finitely generated because otherwise $C/J(B)$ is left noetherian. Hence J/J^2 is as a left A-module not finitely generated. This contradicts the assumption. Thus A is left noetherian which implies that A is left artinian.

The proof of Theorem 1 is complete.

3. Almost artinian ideals in almost noetherian rings

In this section all rings are without identity, unless the result coincides with those of Lenagan [9, Proposition], because an almost right noetherian ring with two sided identity is right noetherian.

Following Cater [1], we say that a right A-module M is almost artinian (resp. almost noetherian) if for each infinite descending (resp. ascending) chain $M_1\supseteqq M_2\supseteqq\ldots$ (resp. $M_1\subseteqq M_2\subseteqq\ldots$) of submodules M_i of M there exist positive integers m, q such that $M_mA^q\subseteqq M_i$ (resp. $M_iA^q\subseteqq M_m$) for all i, or equivalently, there exists a positive integer p such that $M_pA^p\subseteqq M_i$ (resp. $M_iA^p\subseteqq M_p$) for all i. If A_A is almost artinian (resp. almost noetherian) we say that A is an almost right artinian (resp. almost right noetherian) ring. Almost artinian (resp. almost noetherian) rings and modules have been studied in [1], [8], [3], [4].

THEOREM 3. *Let A be an almost right noetherian ring and I be an ideal of A such that ${}_AI$ is almost artinian. If for any infinite descending chain of ideals I_i of A contained in I,*

$$(*) \qquad I_1 \supseteqq I_1 \supseteqq \dots$$

there exists a positive integer p such that $I_pA^p \subseteqq I_i$ for all i, then I_A is almost artinian.

PROOF. Let $I_0=\{x;\, x\in I,\ xA=(0)\}$ and $I_i=\{x;\, x\in I,\ xA\subseteqq I_{i-1}\}$, $i=1, 2, \dots$. Then each I_i is an ideal of A in I with

$$(1) \qquad I_0 \subsetneqq I_1 \subsetneqq \dots$$

and $I_iA \nsubseteqq I_{i-2}$ for $i=2, 3, \dots$. Since A is almost right noetherian, (1) consists of only finitely many numbers. Let I_{p_0} be the greatest number in (1) and put $A'=A/I_{p_0}$. Then A' is almost right noetherian and the image I' of I in A' has the same properties as I which are stated in Theorem 3. We have moreover

$$(2) \qquad x\in I', \quad xA' = (0) \quad \text{implies} \quad x=0.$$

Hence for every non-zero ideal N' of A' contained in I', $N'A'\neq(0)$. By $(*)$ we can use [8, Proposition 1 (1)] to get a minimal ideal N' of A' in I'. ${}_{A'}N$ is almost artinian. We first prove that $N'_{A'}$ is almost artinian, too.

If $A'N'=(0)$, N' is a minimal right ideal of A', proving the statement. If $A'N'\neq(0)$, for every non-zero submodule N_i' of ${}_{A'}N'$, $A'N_i'\neq(0)$. Hence by [8, Proposition 1 (1)], ${}_{A'}N'$ contains a minimal submodule N_1' with $A'N_1'=N_1'$. From this the socle $\mathrm{Soc}\,({}_{A'}N')$ is a non-zero ideal of A' in N', therefore $\mathrm{Soc}\,({}_{A'}N')=N'$, then

$$(3) \qquad {}_{A'}N' = N_1'\oplus \dots \oplus N_t',$$

where each N_i' is a simple left A'-module with $N_i'=A'N_i'$. Now, let us consider the factor ring $S=A'/r(N')$, where $r(N')=\{x;\, x\in A',\ N'x=(0)\}$. Then S is a prime almost right noetherian ring. By [8, Lemma 3], S is a prime right Goldie ring which has a simple artinian quotient ring $Q(S)$. From this, (3) and (2) we can follow the proof of Lenagan in [9] for obtaining that $N'_{A'}$ is artinian.

Now, let N_1 be the complete inverse image of N' in A. Then by an easy induction on p_0 (cf. (1)) we get that N_1 is an almost artinian right A-module. If $N_1\neq I$, we find an ascending chain

$$(4) \qquad I_0^{(1)} \subsetneqq I_1^{(1)} \subsetneqq \dots$$

of ideals $I_i^{(1)}$ of A in I with $I_0^{(1)}=\{x;\, x\in I,\ xA\subseteqq N_1\}$ and $I_i^{(1)}=\{x;\, x\in I,\ xA\subseteqq I_{i-1}^{(1)}\}$, $i=1, 2, \dots$. Since $I_i^{(1)}A\nsubseteqq I_{i-2}^{(1)}$ $(i=2, 3, \dots)$, (4) must stop at some p_1. If $I_{p_1}^{(1)}\neq I$, by the previous way we find an ideal N_2 of A in I such that $N_2\supset I_{p_1}^{(1)}$ and $N_2/I_{p_1}^{(1)}$ is minimal in $A/I_{p_1}^{(1)}$, $N_2/I_{p_1}^{(1)}\,.\, A/I_{p_1}^{(1)}\neq(0)$ and N_2/N_1 is an almost artinian right A-module. This implies that N_2 is an almost artinian right A-module by [1, Proposition 7]. So we finally find an ascending chain

$$(5) \qquad N_1 \subsetneqq N_2 \subsetneqq \dots$$

with $N_iA\nsubseteqq N_{i-1}$ for all $i=2, 3, \dots$. Since A is almost right noetherian, (5) must stop at some t. Then N_t is an almost right A-module, and either $N_t=I$, or there is a positive integer p_t with

$$N_t \subsetneqq I_0^{(t)} \subsetneqq \dots \subsetneqq N_{p_t}^{(t)} = I,$$

where $I_0^{(t)}=\{x;\ x\in I,\ xA\subseteq N_t\}$; $I_i^{(t)}=\{x;\ x\in I,\ xA\subseteq I_{i-1}^{(t)}\}$ $i=1, 2, \dots, p_t$. From this it is easy to see that I_A is almost right artinian.

The proof of Theorem is complete.

REMARKS. The following example shows that without (*) the statement of Theorem 3 is not true in general.

EXAMPLE 1. Let Z be the ring of all integers. Then the matrix ring

$$A = \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix}$$

is right noetherian. The ideal $I=\begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}$ of A is an almost artinian left A-module, but I_A is not almost artinian. The reason is that I does not satisfy the condition (*).

By the result of Lenagan [9] and the proof given there one sees that if I is an ideal of a (right and left) noetherian ring A such that I_A (and then implying $_AI$) is artinian then $A/r(I)$ and $A/l(I)$ are artinian rings. In our case the same result does not hold in general.

EXAMPLE 2. As is well-known, there exist simple (right and left) noetherian rings with identity which are not (right and left) artinian (see for example [5, Theorem 7.45, p. 362]). Let A be such a ring. Then A contains a maximal right ideal R. Then $M=A/R$ is a unital simple right A-module and so the matrix ring

$$\tilde{A} = \begin{bmatrix} A & 0 \\ M & 0 \end{bmatrix}$$

is right noetherian and almost left noetherian. For the ideal $I=\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}$ of $\tilde{A}$, $_{\tilde{A}}I$ is almost artinian, and $I_{\tilde{A}}$ is simple. Clearly $\tilde{A}/r(I)\cong A$ so that $\tilde{A}/r(I)$ is not (almost) right (or left) artinian.

By Theorem 3, it is not difficult to show that an almost left artinian ring is almost right artinian if and only if it is almost right noetherian. We point out that this is also an easy consequence of [8, Theorem 4, 1)⇔3)].

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INSTITUTE OF MATHEMATICS
P.O. BOX 631 BO HO
HANOI
VIETNAM

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SUMMABILITY AND AMARTS OF FINITE ORDER IN FRÉCHET SPACES

DINH QUANG LUU (Hanoi)

§1. Introduction

The extension of Bochner integrals in Banach spaces to locally convex spaces (l.c.s's) is well-known (see, [1, 4, 6]). Thus it is natural to study vector-valued asymptotic martingales (amarts) in such spaces, see [6, 7, 8, 11, 12] etc. The class of vector-valued amarts of finite order [9] is here extended to Fréchet spaces and will be more precisely characterized. Theorem 2.1.3 in [13] and Theorem 1 in [14, I.6] will be applied to obtain some other characterizations of absolutely summing operators in Fréchet spaces. In particular, some special properties of amarts of finite order in nuclear Fréchet spaces are given.

§2. Preliminaries

Let E be a Fréchet space with the 0-neighborhood base $U(E)$. For each $U\in U(E)$, let U^0 and p_U denote the polar and the continuous seminorm, associated with U, resp. Given a vector measure $\mu\colon \mathcal{A}\to E$, where $(\Omega, \mathcal{A}, P)$ is a fixed probability space, the total U-variation $V_U(\mu)$ and the U-semi-variation $S_U(\mu)$ of μ are defined by

$$V_U(\mu) = \sup\left\{\sum_{j=1}^{k} p_U(\mu(A_j))\,\middle|\,\langle A_j\rangle_{j=1}^{k}\in\Pi(\mathcal{A}, \Omega)\right\}$$

and

$$S_U(\mu) = \sup\{|\langle\mu, e\rangle|(\Omega)\,|\,e\in U^0\},$$

resp. where $\Pi(\mathcal{A}, \Omega)$ is the collection of all finite $\mathcal{A}$-measurable partitions of Ω. By $V(\mathcal{A}, E)$ or $S(\mathcal{A}, E)$ we denote the space of all V-equivalence or S-equivalence classes of E-valued V-bounded or S-bounded measures, resp. Thus $V(\mathcal{A}, E)$ or $S(\mathcal{A}, E)$ is topologized by the family of seminorms $(V_U|U\in U(E))$ or $(S_U|U\in U(E))$, resp. Moreover, using the same arguments given in [13] for the spaces $l^1_N(E)$ and $l^1_N\{E\}$, we can establish easily the following result:

LEMMA 2.1. *Both $(V(\mathcal{A}, E)$, V-topology$)$ and $(S(\mathcal{A}, E)$, S-topology$)$ are Fréchet spaces.*

For definition of Bochner integrable functions $f\colon \Omega\to E$, we refer to [3, 4]. Let $L_1(\mathcal{A}, E)$ be the space of such functions. It is known that every $f\in L_1(\mathcal{A}, E)$ is Pettis integrable. Define

$$B_U(f) = \int_\Omega p_U(f)\,dP$$

and

$$P_U(f) = \sup\left\{\int_\Omega |\langle e, f\rangle|\, dP,\ e\in U^0\right\} \quad (U\in U(E), f\in L_1(\mathcal{A}, E)).$$

Thus *the Bochner topology or the Pettis topology of $L_1(\mathcal{A}, E)$ are given by the family of seminorms $(B_U|U\in U(E))$ or $(P_U|U\in U(E))$*, resp. Moreover, with the identification

$$f\mapsto \mu_f\colon \mathcal{A}\to E\colon \mu_f(A) = \int_A f\,dP \quad (A\in\mathcal{A})$$

one can regard ($L_1(\mathcal{A}, E)$, Bochner topology) *as a closed linear subspace of* ($V(\mathcal{A}, E)$, V-topology) *and*

$$B_U(f) = V_U(\mu_f) \quad (U\in U(E),\quad f\in L_1(\mathcal{A}, E)).$$

Finally, as in the Banach space case (see, e.g. [2]) we get the following result:

LEMMA 2.2. *Let $\mu\in S(\mathcal{A}, E)$, $U\in U(E)$ and $f\in L_1(\mathcal{B}, E)$, where $\mathcal{B}$ is a sub σ-field of $\mathcal{A}$. Then*

(i) $S_U(\mu)\leqq V_U(\mu)$,

(ii) $q_U(\mu)\leqq S_U(\mu)\leqq 4q_U(\mu)$,

where $q_U(\mu) = \sup\{p_U(\mu(A))|A\in\mathcal{A}\}$,

(iii) $q_U(f) \overset{\mathrm{df}}{=} q_U(\mu_f)\leqq P_U(f)\leqq 4q_U^{\mathcal{B}}(\mu_f)$,

where $q_U^{\mathcal{B}}(\mu) = \sup\{p_U(\mu(A))|A\in\mathcal{B}\}$.

§3. Summability and amarts of finite order

Let X, Y be linear topological spaces. By $T\in\mathcal{L}(X, Y)$ we mean a linear continuous operator from X into Y. In particular, if X and Y are locally convex spaces whose topologies are generated by the family of seminorms $\{p\}$ and $\{q\}$, resp. then by Theorem 1 in [14, I.6], *$T\in\mathcal{L}(X, Y)$ if and only if for every $q\in\{q\}$ there is some $p\in\{p\}$ and $\beta(q, p)>0$ such that $q(Tx)\leqq\beta(q, p)p(x)$ $(x\in X)$.* Let ($l_N^1(X)$, ε-topology) or ($l_N^1\{Y\}$, Π-topology) be the space of all summable or absolutely summable sequences in X or in Y, resp. (see [13]). Then $T\in\mathcal{L}(X, Y)$ is said to be absolutely summing if it maps $l_N^1(X)$ into $l_N^1\{Y\}$.

THEOREM 3.1. *Let E, F be Fréchet spaces and $T\in\mathcal{L}(E, F)$. Then the following conditions are equivalent:*

(i) *T is absolutely summing.*

(ii) *$T^0\in\mathcal{L}((S(\mathcal{A}, E)$, S-topology), ($V(\mathcal{A}, F)$, V-topology)), where $(T^0\mu)(A)= =T(\mu(A))$ $(\mu\in S(\mathcal{A}, E),\ A\in\mathcal{A})$.*

(iii) *$T^1\in\mathcal{L}((L_1(\mathcal{A}, E)$, Pettis topology), ($L_1(\mathcal{A}, F)$, Bochner topology)), where $(T^1f)(\omega)=T(f(\omega))$ $(f\in L_1(\mathcal{A}, E),\ \omega\in\Omega)$.*

(iv) *T^1 satisfies* (iii) *for the special probability space $(N, \mathcal{P}(N), \gamma)$, where $N= =\{1, 2, \dots\}$, $\mathcal{P}(N)$ the σ-field of all subsets of N and $\gamma\colon\mathcal{P}(N)\to[0, 1]$, given by $\gamma(Z)=\sum_{n\in Z}2^{-n}(Z\in\mathcal{P}(N))$.*

PROOF. Let $T\in\mathscr{L}(E,F)$ be absolutely summing. Thus by Theorem 2.1.3 in [13], T "lifts" to an operator $T_N\in\mathscr{L}(l_N^1(E),\ l_N^1\{F\})$ given by

$$T_N(\langle x_n\rangle)=\langle Tx_n\rangle\quad (\langle x_n\rangle\in l_N^1(E)).$$

Consequently, by Theorem 1 in [14, I.6], for each $C\in U(F)$ one can choose some $U\in U(E)$ and $\beta(C,U)>0$ such that

$$\Pi_C(\langle Tx_n\rangle)\leqq\beta(C,U)\,\varepsilon_U(\langle x_n\rangle)\quad (\langle x_n\rangle\in l_N^1(E)).$$

Equivalently,

$$\sum_N p_C(Tx_n)\leqq\beta(C,U)\sup\{\sum_N|\langle x_n,e\rangle|\,|e\in U^0\}\quad (\langle x_n\rangle\in l_N^1(E)).$$

Let $\langle x_j\rangle_{j=1}^k$ be a finite sequence of E. Then the element $\langle x_1,x_2,\dots,x_k,0,\dots,0\dots\rangle\in\in l_N^1(E)$. Hence, the last inequality yields

$$\sum_{j=1}^k p_C(Tx_j)\leqq\beta(C,U)\sup\{\sum_{j=1}^k|\langle x_j,e\rangle|\,|e\in U^0\}. \tag{3.1}$$

Now let $\mu\in S(\mathscr{A},E)$ and $\langle A_j\rangle_{j=1}^k\in\Pi(\mathscr{A},\Omega)$. Applying (3.1) to the finite sequence $\langle\mu(A_j)\rangle_{j=1}^k$ of E, we get

$$\sum_{j=1}^k p_C((T^0\mu)(A))=\sum_{j=1}^k p_C(T(\mu(A_j)))\leqq\beta(C,U)\sup\{\sum_{j=1}^k|\langle\mu(A_j),e\rangle|\,|e\in U^0\leqq$$
$$\leqq\beta(C,U)\sup\{|\langle\mu,e\rangle|(\Omega)\,|e\in U^0\}=\beta(C,U)\,S_U(\mu).$$

This implies that

$$V_C(T^0\mu)=\sup\{\sum_{j=1}^k p_C((T^0\mu)(A_j))\,|\langle A_j\rangle_{j=1}^k\in\Pi(\mathscr{A},\Omega)\}\leqq$$
$$\leqq\beta(C,U)\,S_U(\mu)\quad (\mu\in S(\mathscr{A},E)).$$

Consequently, by Theorem 1 in [14, I.6], T^0 satisfies (ii), noting that the linearity of T^0 is automatically satisfied.

As the implications (ii)→(iii)→(iv) are easy, it remains to prove (iv)→(i). Suppose that T fails to be absolutely summing. Then by definition there is some $\langle x_n\rangle\in l_N^1(E)$ such that $\langle Tx_n\rangle\notin l_N^1\{F\}$. Equivalently, there is some $C\in U(F)$ such that $\sum_N p_C(Tx_n)=\infty$. Therefore, one can choose a strictly increasing sequence $\langle n_k\rangle$ of N such that

$$\sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j)\geqq k\quad (k\in N).$$

Now define $f_k\colon\Omega\to E$ by

$$f_k=\sum_{j=n_k+1}^{n_{k+1}} 2^j x_j 1_{\{j\}}\quad (k\in N)$$

where 1_A is the characteristic function of $A\in\mathscr{A}$. Obviously, each $f_k\in L_1(\mathscr{P}(N),E)$ and, by [13, 1.2.6] $\langle f_k\rangle$ is convergent to 0 in the Pettis topology. On the other hand, since

$$\int_N p_C(T^1 f_k)\,d\gamma = \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geqq k \quad (k\in N),$$

the sequence $\langle T^1 f_k\rangle$ fails to be convergent in the Bochner topology of $L_1(\mathscr{P}(N), F)$. This contradicts (4) and therefore completes the proof.

REMARK. *If E, F are Banach spaces*, the theorem has been proved recently by Bru—Heinrich [2], using Proposition 2.2.1 in [13] *which cannot be applied to Fréchet spaces.*

Combining Theorem 4.2.5 in [13] and the theorem, one can prove easily the following corollary in which the equivalence (i)$\leftrightarrow$(iii) has been recently proved by Egghe [7] *who used directly Proposition* 4.1.5 *in* [13].

COROLLARY 3.2. *For a Fréchet space E, the following properties are equivalent:*
(i) *E is nuclear.*
(ii) $(V(\mathscr{A}, E)$, *V-topology*$)\equiv(S(\mathscr{A}, E)$, *S-topology*$)$.
(iii) $(L_1(\mathscr{A}, E)$, *Pettis topology*$)\equiv(L_1(\mathscr{A}, E)$, *Bochner topology*$)$.
(iv) $(L_1(\mathscr{P}(N), E)$, *Pettis topology*$)\equiv(L_1(\mathscr{P}(N), E)$, *Bochner topology*$)$.

Hereafter, let $\langle\mathscr{A}_n\rangle$ be a sequence of sub-σ-fields of $\mathscr{A}$ with $\Sigma=\bigcup_N \mathscr{A}_n$ and $\mathscr{A}=\sigma(\Sigma)$. A sequence $\langle f_n\rangle$ in $L_1(\mathscr{A}, E)$ is said to be *adapted to* $\langle\mathscr{A}_n\rangle$, if each $f_n\in L_1(\mathscr{A}_n, E)$ and it has a property $(*)$ if so has the sequence $\langle\mu_n\rangle$ of measures, associated with $\langle\mathscr{A}_n\rangle$, given by

$$\mu_n\colon \mathscr{A}_n \to E\colon \mu_n(A) = \int_A f_n\,dP \quad (n\in N, A\in\mathscr{A}_n).$$

A sequence $\langle\mu_n\rangle$ in $S(\mathscr{A}, E)$ is said to be adapted to $\langle\mathscr{A}_n\rangle$, if each $\mu_n\in S^n(E)=$ $=S(\mathscr{A}_n, E)$. We shall study only such sequences of measures. Let T^∞ be the set of all bounded stopping times. Given $\langle\mu_n\rangle$, $\langle f_n\rangle$ and $\tau\in T^\infty$, we define $\mathscr{A}_\tau$, μ_τ and f_τ as in [2].

DEFINITION 3.3 (see [11]). A sequence $\langle\mu_n\rangle$ in $S(\mathscr{A}, E)$ is said to be a martingale, if

$$\mu_{m,n} = \mu_m|_{\mathscr{A}_n} = \mu_n \quad (m \geqq n\in N).$$

DEFINITION 3.4. A sequence $\langle\mu_n\rangle$ in $S(\mathscr{A}, E)$ is said to be an *amart of finite order*, if for every $d\in N$ the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^d}$ converges in E, where T^d is the subset of all bounded stopping times each of which takes essentially at most d values. Moreover, if the net is convergent for $d=\infty$, then as in [2, 11], $\langle\mu_n\rangle$ is called an *amart*.

Obviously, by definition, *every amart is that of finite order*. The remark at the end shows that *there is a real-valued amart of finite order which fails to be an amart.*

LEMMA 3.5. *Let $\langle\mu_n\rangle$ be a sequence in $S(\mathscr{A}, E)$. Then the following assertions are equivalent:*
(i) *$\langle\mu_n\rangle$ is an amart of finite order.*
(ii) $\lim_{m\in N} \sup_{m\geqq n} S_U^n(\mu_{m,n}-\mu_n) = 0 \quad (U\in U(E))$.

(iii) $\langle\mu_n\rangle$ *has a (weak) Riesz decomposition:* $\mu_n=\alpha_n+\beta_n$ $(n\in N)$, *where* $\langle\alpha_n\rangle$ *is a martingale and* $\langle\beta_n\rangle$ *a Pettis potential, i.e.*

$$\lim_{n\in N} S_U^n(\beta_n)=0 \quad (U\in U(E)).$$

(iv) *There is a finitely additive measure* $\mu_\infty\colon \Sigma\to E$ *such that each* $\mu_{\infty,n}=$ $=\mu_\infty|_{\mathcal{A}_n}\in S(\mathcal{A}_n, E)$ *and*

$$\lim_{n\in N} S_U^n(\mu_n-\mu_{\infty,n})=0 \quad (U\in U(E)).$$

PROOF. (i)$\to$(ii). Let $\langle\mu_n\rangle$ be as in (i). Then by definition, the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^2}$ converges in E. Thus, for each $U\in U(E)$ and $\varepsilon>0$, one can choose some $\tau(\varepsilon)\in T^2$ such that if $\sigma,\tau\in T^2$ with $\sigma,\tau\geqq\tau(\varepsilon)$ then

$$p_U(\mu_\sigma(\Omega)-\mu_\tau(\Omega))\leqq 4^{-1}\varepsilon. \tag{3.2}$$

Now let $m,n\in N$ with $m\geqq n\geqq\tau(\varepsilon)$ and $A\in\mathcal{A}_n$. Define

$$\sigma=m1_\Omega \quad\text{and}\quad \tau=n1_A+m1_{\Omega\setminus A}.$$

Obviously $\sigma,\tau\in T^2$, $\sigma\geqq\tau\geqq\tau(\varepsilon)$. Hence by (3.2)

$$p_U(\mu_m(A)-\mu_n(A))\leqq p_U(\mu_\sigma(\Omega)-\mu_\tau(\Omega))\leqq 4^{-1}\varepsilon.$$

Therefore by Lemma 2.2, it follows that

$$S_U^n(\mu_{m,n}-\mu_n)\leqq 4q_U^{\mathcal{A}_n}(\mu_{m,n}-\mu_n)=4\sup\{p_U(\mu_m(A)-\mu_n(A))\,|\,A\in\mathcal{A}_n\}\leqq\varepsilon.$$

This proves (ii).

Since the next implications (ii)$\to$(iii)$\to$(iv) can be established easily, using the arguments, similar to those given in the proof of Theorem 2.2 in [11], it remains to prove (iv)$\to$(i). Let $\langle\mu_n\rangle$ satisfy (iv). We shall show that $\langle\mu_n\rangle$ is an amart of finite order. Indeed, let $d\in N$ be any but fixed. For each $U\in U(E)$ and $\varepsilon>0$, choose $n(\varepsilon)\in N$ such that

$$\sup_{n\geqq n(\varepsilon)} S_U^n(\mu_n-\mu_{\infty,n})\leqq d^{-1}\varepsilon.$$

Let $\tau\in T^d$ with $\tau\geqq n(\varepsilon)$. Obviously,

$$p_U(\mu_\tau(\Omega)-\mu_\infty(\Omega))\leqq \sum_{n\geqq n(\varepsilon)} p_U(\mu_n(\{\tau=n\})-\mu_\infty(\{\tau=n\}))\leqq$$

$$\leqq d\sup_{n\geqq n(\varepsilon)} q_U^{\mathcal{A}_n}(\mu_n-\mu_{\infty,n})\leqq d\sup_{n\geqq n(\varepsilon)} S_U^n(\mu_n-\mu_{\infty,n})\leqq\varepsilon.$$

This means that the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^d}$ converges in E (exactly to $\mu_\infty(\Omega)$). This proves (i) and the lemma.

REMARK. The inspection of the proof shows that a sequence $\langle\mu_n\rangle$ in $S(\mathcal{A},E)$ is an amart of finite order if and only if for some $d\in\{2,3,\dots\}$ the net $\langle\mu_\tau(\Omega)\rangle_{\tau\in T^d}$ converges in E. Therefore the notion of amarts of order d introduced in [9] must be omitted.

DEFINITION 3.6. Following [10], call a sequence $\langle\mu_n\rangle$ in $V(\mathcal{A},E)$ an *L^1-amart*, if

$$\lim_{n\in N}\sup_{m\geqq n} V_U^n(\mu_{m,n}-\mu_n)=0 \quad (U\in U(E)).$$

Clearly, by Lemmas 3.5 and 2.2, *every L^1-amart is an amart of finite order*. We shall see later that *the converse is true if and only if the Fréchet space E is nuclear*. More generally using Lemma 3.5, Definition 3.6, and Theorem 3.1 we can prove easily the following theorem which gives an answer to the question what operators map amarts of finite order into L^1-amarts.

THEOREM 3.7. *Let E, F be Fréchet spaces and $T\in\mathscr{L}(E, F)$. Then the following conditions are equivalent:*

(i) *T is absolutely summing.*

(ii) *T maps E-valued amarts of finite order in $S(\mathscr{A}, E)$ into F-valued L^1-amarts in $V(\mathscr{A}, F)$.*

(iii) *T maps E-valued amarts of finite order in $L_1(\mathscr{A}, E)$ into F-valued L^1-amarts in $L_1(\mathscr{A}, F)$.*

COROLLARY 3.8. *For a Fréchet Banach space E, the following conditions are equivalent:*

(i) *E is nuclear.*

(ii) *Every amart of finite order in $S(\mathscr{A}, E)$ is an L^1-amart in $V(\mathscr{A}, E)$.*

(iii) *Every amart of finite order $\langle f_n\rangle$ in $L_1(\mathscr{A}, E)$ has a (strong) Riesz decomposition $f_n=g_n+h_n$ $(n\in N)$, where $\langle g_n\rangle$ is a martingale and $\langle h_n\rangle$ is an L^1-potential, i.e.*

$$\lim_N \int_\Omega p_U(f_n)\,dP = 0 \quad (U\in U(E)).$$

(iv) *Every Pettis uniformly integrable amart of finite order in $L_1(\mathscr{A}, E)$ is convergent in the Bochner topology.*

PROOF. (i)$\Leftrightarrow$(ii) follows from Theorem 4.2.5 in [13] and Theorem 3.7.

(i)$\to$(iii). Let $\langle f_n\rangle$ be an amart of finite order in $L_1(\mathscr{A}, E)$. Suppose that E is nuclear. Then by (i)$\Leftrightarrow$(ii), $\langle f_n\rangle$ must be L^1-amart, i.e. if $\langle \mu_n\rangle$ is the sequence of measures associated with $\langle f_n\rangle$ then

$$\lim_{n\in N}\sup_{m\geqq n} V_U^n(\mu_{m,n}-\mu_n) = 0 \quad (U\in U(E)). \tag{3.3}$$

Now remark that E is a nuclear Fréchet space, then by [3, 4], E has the Radon—Nikodym property. Therefore, one can define the conditional expectation operator on $L_1(\mathscr{A}, E)$. Let $E^{\mathscr{A}_n}(f_m)=f_{m,n}$ be the $\mathscr{A}_n$-conditional expectation of f_m $(m\geqq n\in N)$. It is clear that (3.3) is equivalent to

$$\lim_{n\in N}\sup_{m\geqq n} B_U(f_{m,n}-f_n) = 0 \quad (n\in N).$$

Therefore, as for L^1-amarts with values in Banach spaces (see [10]), one can show that for each $n\in N$, the sequence $\langle f_{m,n}\rangle_{m=n}^\infty$ is Cauchy in the Bochner topology. But as it has been noted in [3, 4], the class $L_1(\mathscr{A}_n, E)$ with the Bochner topology is a Fréchet space, every sequence $\langle f_{m,n}\rangle_{m=n}^\infty$ must be convergent to some $g_n\in L_1(\mathscr{A}_n, E)$ in the Bochner topology. It is not hard to show that the sequence $\langle g_n\rangle$ must be a martingale. Moreover, if we put $h_n=f_n-g_n$ $(n\in N)$, then $\langle h_n\rangle$ must be an L^1-potential. This proves (i)$\to$(iii).

(i)$\to$(iv). Let $\langle f_n\rangle$ be a Pettis uniformly integrable amart of finite order. Then by (i)$\to$(iii), $\langle f_n\rangle$ must be written in a form $f_n=g_n+h_n$ $(n\in N)$, where $\langle g_n\rangle$ is a martingale and $\langle h_n \rangle$ an L^1-potential. On the other hand, by (1) and Corollary 3.2 $\langle f_n \rangle$ must be uniformly integrable in the Bochner topology, i.e. every sequence $\langle p_U(f_n) \rangle$ is uniformly integrable. Hence so is the martingale $\langle g_n \rangle$. To finish the proof we note that by [3], every nuclear Fréchet space has the Radon—Nikodym property. Therefore, by Theorem 2.1 [12], $\langle g_n \rangle$ must be convergent to some $f \in L_1(\mathcal{A}, E)$ in the Bochner topology, hence so does the amart of finite order $\langle f_n \rangle$ which completes the proof of (i)$\to$(iv).

Finally, taking $E=F$, and $T\equiv$the identical operator, the example given in the proof of Theorem 3.1 proves (iii)$\to$(i), (iv)$\to$(i) and the corollary.

REMARK. The corollary seems to be new even for the real-valued case. Further, since every real-valued L^1-amart is an amart of finite order then by [10], there is a nonnegative real-valued amart of finite order which fails to be an amart. Finally, many results and references, related to amarts in Banach spaces can be found in [5].

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INSTITUTE OF MATHEMATICS
P.O. BOX 631, BO HO
HANOI
VIETNAM

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COGENERATORS OF RADICALS

W. G. LEAVITT (Lincoln)

1. Radical and semisimple classes

All rings considered will be contained in some universal class W of not necessarily associative rings. For a class M of rings we write, as usual, $UM=\{R\in W|$ all $0\neq R/I\notin M\}$ and $SM=\{R\in W|$ if $0\neq I\lhd R$ then $I\notin M\}$. It is well-known that every class M generates a radical class LM (the "lower radical") which is the smallest radical class containing M. It can be obtained by simply intersecting all the radical classes containing M (see [4]) or can be constructed by the Kurosh (see [1]) or some other construction (such as in [7]). There are a number of equivalent definitions of a radical class (such as that it is closed under homomorphisms, homomorphic extensions, and unions of ascending chains of ideals). An easy criterion is:

PROPOSITION 1. *A class P is a radical class if and only if $P=USP$.*

PROOF. This can be checked directly or one can note that for P homomorphically closed USP is the first Kurosh step. Thus from $P=USP$ it follows that P is homomorphically closed and all Kurosh steps are equal, that is $P=LP$ is radical. Conversely $P=LP$ implies $P=USP$.

The concept dual to that of a radical is that of a semisimple class (sometimes called "Coradical"; see [6; p. 781]. Note that in [6] notations M^* and M_* are used for SM and UM, and $\overline{M}$ is used for LM.). There are also various characterizations of a semisimple class, one of which is:

PROPOSITION 2. *M is a semisimple class if and only if $M=SUM$.*

PROOF. Conditions such as those of [6; p. 781] or [3; p. 312] can be checked, or one can note the equivalence to a characterization such as that of [8; p. 21].

It is, of course, well-known that if P is a radical class then SP is semisimple (sometimes called the "P-free" class), and when M is semisimple then UM is radical.

As we noted, every class is contained in a smallest radical and dually we would like to have, for any class M, the largest radical relative to which every $R\in M$ is semisimple, or equivalently the smallest semisimple class containing M. In a universal class in which semisimple classes are hereditary (such as the class of all associative or all alternative rings) this is simply $SU\overline{M}$ where $\overline{M}$ is the hereditary closure of M. However such a smallest semisimple class does not in general exist:

EXAMPLE 1. Let a ring R be generated over Z_2 by symbols x, y, z where $x^2=x$, $xy=yx=xz=zx=y^2=y$, $yz=zy=z$, and $z^2=0$. It is clear that R has only one proper ideal $I=\{y, z, y+z, 0\}$ and I only the ideal $J=\{z, 0\}$. Also $I/J\cong R/I\cong$ $\cong Z_2\not\cong J$, since $J\cong Z_2^0$ (the zero ring on Z_2). Let $M=\{R\}$ and $\overline{M}=\{R, I, J\}$, $M^1=\{R, I/J\}$. We have $M\subset\overline{M}\subset SU\overline{M}$ and $M\subset M^1\subset SUM^1$, so $SU\overline{M}$ and SUM^1

are semisimple classes containing M. They are incomparable since $Z_2^0 \in S U \overline{M}$ and $Z_2^0 \notin SUM^1$, while $Z_2 \in SUM^1$ and $Z_2 \notin SU\overline{M}$. Now suppose there is a radical P such that $M \subseteq SP$ with $SP \subseteq SU\overline{M}$ and $SP \subseteq SUM^1$. Then $R \in SP$ would imply $I \notin P$ so either $I \in SP$ from which $J \notin P$ so $J \in SP$ contradicting $SP \subseteq \subseteq SUM^1$, or else $I/J \in SP$ contradicting $SP \subseteq SU\overline{M}$. Thus M is not contained in a smallest semisimple class.

A class M is called "regular" (see 8; p. 21) if $M \subseteq SUM$. It is well-known that:

PROPOSITION 3. *Every regular class M is contained in a smallest semisimple class SUM where UM is radical.*

When UM is radical it is called the "upper radical" defined by M. More generally, we have

PROPOSITION 4 [2; p. 219]. *A class M defines an upper radical UM if and only if every $0 \neq R \in M$ has an image $0 \neq R/I \in SUM$.*

Remark that while every class is contained in a smallest radical, not every class contains a largest radical (for an example see [5; p. 684]). It is also true that not every class contains a largest semisimple class (an example will be given later).

2. Cogenerating classes

We will call a class M coregular if $SUM \subseteq M$. Thus a class is semisimple if and only if it is both regular and coregular.

LEMMA 1. *For M any class, SUM is coregular.*

PROOF. If $A \in SUSUM$ then if $0 \neq I \lhd A$ we have some $0 \neq I/J \in SUM$. But then I/J has an image in M, and so $A \in SUM$.

PROPOSITION 5. *Every coregular class M contains a largest semisimple class $SLUM$.*

PROOF. For an arbitrary class M we have $UM \subseteq LUM$ so that $SLUM \subseteq SUM$, and if M is coregular $SLUM \subseteq M$. Now if $SP \subseteq M$ for a radical P then $UM \subseteq P$ so $LUM \subseteq P$. Then $SP \subseteq SLUM$ so $SLUM$ is the largest semisimple class contained in M.

The following example shows that not every class contains a largest semisimple class. Note that the construction can take place in any universal class W (including the class of all associative rings).

EXAMPLE 2. Let V and T be any two radicals with $V \cap T = 0$ (such as LA, LB with A and B hereditary and $A \cap B = 0$, say $A = \{Z_2\}$ and $B = \{Z_3\}$). Let M be any class with $SV \cup ST \subseteq M \neq W$. Then $U(SV \cup ST) \subseteq USV = V$ and $U(SV \cup ST) \subseteq T$, so $U(SV \cup ST) = 0$. Thus if $SV \cup ST \subseteq Q \subseteq M$ with Q semisimple then $UQ \subseteq U(SV \cup ST) = 0$ so the contradiction $W = SUQ = Q \subseteq M$.

A class M will be said to "cogenerate" a radical $\underline{M}$ (see [6; p. 780]) if $\underline{M}$ is the smallest radical class such that $S\underline{M} \subseteq M$, or equivalently if $S\underline{M}$ is the largest semisimple class contained in M. Thus Example 2 shows that not every class is a cogenerator. However it is immediate from Proposition 5 that:

PROPOSITION 6. *Every coregular class M cogenerates a radical $\underline{M}=LUM$.*

COROLLARY 1. *For M an arbitrary class SUM is a cogenerator.*

We also have

PROPOSITION 7 [6; 0.16*, p. 783]. *If M cogenerates a radical $\underline{M}$ then $LUM\subseteq\underline{M}$, and $LUM=\underline{M}$ if and only if $SLUM\subseteq M$.*

PROOF. Since $S\underline{M}\subseteq M$ we have $UM\subseteq\underline{M}$ so $LUM\subseteq\underline{M}$. Thus $S\underline{M}\subseteq SLUM$ and if $SLUM\subseteq M$ then $SLUM=S\underline{M}$, so $LUM=\underline{M}$.

LEMMA 2. *The union of any class of regular classes is regular.*

PROOF. Let $Q=\cup Q_i$ where each Q_i is regular. If $R\in Q$ then $R\in Q_i$ for some i. Then $R\in Q_i\subseteq SUQ_i\subseteq SUQ$. Thus $Q\subseteq SUQ$.

We can thus generalize Proposition 6 to:

PROPOSITION 8. *If $SUN\subseteq M$ for every regular class $N\subseteq M$ then M is a cogenerator.*

PROOF. Let $\{Q_i\}$ be the class of all semisimple subclasses of M. By Lemma 2 the union $Q=\cup Q_i$ is regular, so the semisimple class $SUQ\subseteq M$. Since every $Q_i\subseteq Q\subseteq\subseteq SUQ$ we have $UQ=\underline{M}$.

A class N will be called "M-regular" if $N\subseteq SUN\subseteq M$. Then the criterion for a cogenerator is:

THEOREM 1. *A class M is a cogenerator if and only if every union of M-regular classes is M-regular.*

PROOF. Suppose first that M cogenerates $\underline{M}$. Let $\{N_i\}$ be any class of M-regular classes then $N=\cup N_i$ is regular by Lemma 2. Now $S\underline{M}$ contains all semisimple subclasses of M so all $SUN_i\subseteq S\underline{M}$. Thus we have $N=\cup N_i\subseteq\cup SUN_i\subseteq S\underline{M}$, so that $N\subseteq SUN\subseteq SUS\underline{M}\subseteq S\underline{M}\subseteq M$, that is N is M-regular.

On the other hand, suppose every union of M-regular classes is M-regular. Let $Q=\cup Q_i$ where $\{Q_i\}$ is the class of all semisimple subclasses of M. Since every Q_i is M-regular it follows that Q is M-regular. Thus $Q\subseteq SUQ\subseteq M$ so SUQ is one of the $\{Q_i\}$, that is $SUQ\subseteq Q$. Therefore $Q=SUQ$ is the largest semisimple class contained in M, that is $\underline{M}=UQ$.

COROLLARY 2. *M is a cogenerator if and only if $SUQ\subseteq M$ where Q is the union of all the semisimple subclasses of M.*

COROLLARY 3. *M is a cogenerator if and only if the union of all the semisimple subclasses of M is semisimple.*

Remark that while every coregular class is a cogenerator, the converse is not true:

EXAMPLE 3. Take any radical, say the Jacobson radical J, and let $M=SJ\cup\{B\}$ where B is some simple radical ring. Then SJ is the largest semisimple class contained in M but rings like $B\oplus B$ are in SUM but not in M.

LEMMA 3. *If M is homomorphically closed then SM is coregular.*

PROOF. Let $A \in SUSM$. If $A \notin SM$ then there is some $0 \neq I \lhd A$ with $I \in M$. But every non-zero ideal of A has a non-zero image in SM, contradicting $I \in M$.

PROPOSITION 9. *If M is homomorphically closed then SM is a cogenerator with $\underline{SM} = LUSM$.*

PROOF. This follows from Lemma 3 and Proposition 6.

PROPOSITION 10 [6; 0.16, p. 783]. *If SM is a cogenerator then $\underline{SM} \subseteq \underline{LM}$.*

PROOF. Since $M \subseteq LM$ we have $SLM \subseteq SM$ and since SLM is semisimple $SLM \subseteq S\underline{SM}$. Thus $\underline{SM} \subseteq LM$.

Note that not all classes of form SM are cogenerators:

EXAMPLE 4. Again take $V = L\{Z_2\}$, $T = L\{Z_3\}$ and let $M = \{Z_2 \oplus Z_3\}$. If $A \in SV$ then $A \in SM$ since otherwise $Z_2 \oplus Z_3 \lhd A$ giving the contradiction $Z_2 \lhd A$. Thus $SV \subseteq SM$ and similarly $ST \subseteq SM$. Therefore $SV \cup ST \subseteq SM$ so, as in Example 2, SM is not a cogenerator.

We can improve Proposition 9 to:

PROPOSITION 11 [6; 0.16, p. 783]. *If M is homomorphically closed then SM is a cogenerator with $\underline{SM} = LM$.*

PROOF. It follows from Propositions 9 and 10 that SM is a cogenerator with $\underline{SM} = LUSM \subseteq LM$. But USM is the first Kurosh step so if M is homomorphically closed $M \subseteq USM$. Thus $LM \subseteq LUSM$ so $\underline{SM} = LM$.

Note that SM can be a cogenerator even if M is not homomorphically closed:

EXAMPLE 5. Let V be a vector space of dimension $\geqq \aleph_1$ and let A be the ring of all linear transformations of rank $\leqq \aleph_1$. Then we have $C \lhd B \lhd A$ where B is the set of all linear formations of V of rank $\leqq \aleph_0$ and C those of finite rank. It is known that C, B/C, and A/B are simple non-isomorphic rings. Let $M = \{A, A/B, B/C\}$ so that M is not homomorphically closed. It is easily checked that $M \subseteq USM$ so $SUSM \subseteq SM$, that is SM is coregular hence a cogenerator. Note that in this case $LUSM = LM$. However SM can be a cogenerator even if $LUSM \neq LM$:

EXAMPLE 6. Let $M = \{Z_2, Z_2 \oplus Z_3\}$ so if $A \notin SM$ then $0 \neq I \lhd A$ with either $I \cong Z_2$ or $I \cong Z_2 \oplus Z_3$. But in the latter case $Z_2 \cong J \lhd A$. Thus in either case A has an ideal in USM so $A \notin SUSM$. Therefore $SUSM \subseteq SM$ so SM is a cogenerator with $\underline{SM} = LUSM \subset LM$. The last inclusion is proper since $Z_3 \in LM$ but $Z_3 \notin LUSM$.

On [6; p. 781] two properties were considered, namely;

CI. M is closed under subdirect products.

CII. If $J \lhd I \lhd A$ with $A/I \in M$ and $I/J \in M$ then there exists some $C \lhd A$ with $C \subseteq J$ such that $A/C \in M$.

PROPOSITION 12. *If M satisfies* CI *and* CII *then M is coregular.*

PROOF. Let $A \in SUM$ and suppose $0 \neq I \lhd A$ with I minimal relative to $A/I \in M$. Then from $A \in SUM$ there is some $0 \neq I/J \in M$, so by CII the minimality of I would be contradicted. Thus $I = 0$, so $A \in M$ and hence $SUM \subseteq M$.

COROLLARY 4 [6; 0.16*, p. 783]. *If M satisfies* CI *and* CII *then M cogenerates $M=LUM$.*

LEMMA 4. *For M an arbitrary class, $M\cup SUM$ is a coregular class.*

PROOF. If $A\in SU(M\cup SUM)$ then for every $0\neq I\lhd A$ either I has an image in M or an image in SUM. But every ring in SUM has an image in M so, in any case, I has an image in M, that is $A\in SUM$. Thus $SU(M\cup SUM)\subseteq SUM\subseteq M\cup SUM$.

Remark that a coregular class need not satisfy CII:

EXAMPLE 7. Let A, B, C be as in Example 5. Let $N=\{A/C, C\}$ and $M=N\cup SUN$. Then M is coregular by Lemma 4. But $A\notin N$ and since both $B\notin N$ and $B/C\notin N$ we have $A\notin SUN$. Thus $A\notin M$.

It is also true that a coregular class need not satisfy CI:

EXAMPLE 8. Let $A=Z_2[x, e, f]$ where $x^2=x, xe=ex=e, xf=fx=f$, and $e^2=f^2=ef=fe=0$. Set $I_1=\{0, e\}$ and $I_2=\{0, f\}$ so $A/I_1\cong A/I_2$. Let $N=\{A/I_1\}$ and $M=N\cup SUN$ is coregular. Since $0=I_1\cap I_2=\bigcap\{I\lhd A|A/I\in M\}$ it follows that A is a subdirect sum of members of M. However $A\notin N$ and $A\notin SUN$ since $I_1\lhd A$ has no image in N. Thus $A\notin M$.

We conclude with a smallest theorem:

THEOREM 2. *Every class M is contained in a smallest coregular class $M\cup SUM$.*

PROOF. By Lemma 4 $M\cup SUM$ is coregular and suppose $M\subseteq N$ where $SUN\subseteq N$. Then $SUM\subseteq SUN\subseteq N$ so $M\cup SUM\subseteq N$.

Remark that it is an open question as to whether or not every class is contained in a smallest cogenerator. However Example 2 shows that there are classes which do not contain a largest cogenerator, or even a largest coregular class.

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UNIVERSITY OF NEBRASKA—LINCOLN
DEPARTMENT OF MATHEMATICS AND STATISTICS
LINCOLN, NEBRASKA 68588—0323
U.S.A.

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A LIMIT THEOREM FOR PROBABILITIES RELATED TO THE RANDOM BOMBARDMENT OF A SQUARE

R. ISAAC (New York)

§0. Introduction and notation

Consider the unit square S bombarded by darts (i.e. points) thrown independently and uniformly onto S. A u-square (US) is a square with dimensions $u\times u$, $0<u<1$, lying within S and with sides parallel to those of S. Say that an n-success occurs if there exists at least one US which contains none of the first n darts in its interior. Let $p_n(u)=P(n\text{-success})$; in this paper we shall prove that for any u, $0<u<1$

(0.1) $$\lim_{n\to\infty} \frac{p_n(u)}{n^2(1-u^2)^n} = L(u), \quad 0 < L(u) < \infty$$

exists (Theorem 5.1). This improves the result of [2]:

(0.2) If $u=\frac{1}{2}$ there exist constants $A>0$ and $B<\infty$ such that for all $n>1$

$$A < \frac{p_n\left(\frac{1}{2}\right)}{n^2\left(\frac{3}{4}\right)^n} < B.$$

This problem is in the class of problems in geometric probability (discussed, for example, in [3] and [4]) dealing with prescribed geometric conditions required to hold after random bombardment of a geometric figure. This will be described further in Section 6.

The author is indebted to Dr. Leopold Flatto for suggesting the problem. Dr. Flatto obtained the upper bound B in (0.2) using the clever idea of using a grid to discretize the problem. This basic idea is at the heart of this paper as well as of [2].

The outline of the paper is as follows: in Section 1 we prove the extension of (0.2) for general u; this result will be required for the proof of (0.1). Section 2 discusses refinements of a grid. The important point here is that a sequence of refinements will yield a monotone sequence of probabilities and will allow us in Section 3 to approximate $p_n(u)$ for each fixed n in a useful way. In Section 4 a ratio inequality is proved setting the stage for the proof of (0.1) in Section 5. It is in Section 5 that (1.1), the extension of (0.2), is crucially used. Section 6 is devoted to a discussion of related problems and applications.

We remark that the precise values of $L(u)$ remain undetermined, although by using the idea in section one for calculating the upper and lower bounds we can get rough bounds on $L(u)$.

Throughout this paper we will suppress the dependence upon u in the notation (writing "p_n" for "$p_n(u)$", for example). This simplifies the exposition considerably and causes no confusion, since we may consider u chosen in the unit interval and fixed at the outset.

A summary of necessary definitions and notations follows.

S: the unit square, taken as target for a sequence of n independent and uniformly distributed points (referred to as "darts"). It will be convenient to speak in terms of the pattern of darts "at time n", that is, the pattern after the n-th dart is thrown.

u-square (US): a square with dimensions $u\times u$, $0<u<1$, lying within S and with sides parallel to those of S.

N: any integer used to define a grid on S.

grid (or *N-grid*): the sides of S are subdivided into N equal parts by lattice points denoted by the Cartesian pairs (i, j), $0\leqq i$, $j\leqq N$ (the origin (0,0) is thus the lower left vertex of S). Drawing vertical and horizontal lines through opposite lattice points we obtain the grid.

tile: each of the squares of dimension $\frac{1}{N}\times\frac{1}{N}$ composing an N-grid.

interior canonical square (ICS): this depends on given N and u. We must have $Nu\geqq 2$ and then, given an N-grid, an ICS is any square with vertices on lattice points and length of edge$=[Nu-1]\cdot\frac{1}{N}$. ($[x]$ is the greatest integer $\leqq x$.) An ICS contains exactly $[Nu-1]^2$ tiles.

exterior canonical square (ECS): analogous to ICS, except that length of edge$=$ $=[Nu+1]\cdot\frac{1}{N}$. An ECS contains exactly $[Nu+1]^2$ tiles.

intact: refers to an area within S that has not been hit by any darts (at time n).

c: used throughout for the ratio $\frac{n}{N}$.

p_n: the probability of an n-success, that is, that there exists at least one intact US at time n.

$r_{N,n}$: the probability of at least one intact ICS formed from an N-grid at time n.

$q_{N,n}$: the probability of at least one intact ECS formed from an N-grid at time n.

llv: abbreviation for the lower left vertex of a US, ICS, or ECS.

N_I (respectively, N_E): the total number of ICS's (respectively, ECS's)$=(N+1-$ $-[Nu-1])^2$ (respectively, $(N+1-[Nu+1])^2$) formed from an N-grid.

frame: Given an ICS I and any US Q with $Q\supset I$, a frame of I is the region $Q\sim I$. If the grid size$=N$, the area of a frame $\leqq\frac{4u}{N}$.

(0.3) Let the collection of $N-$ICS's be counted off in any way, that is, assign to each N-ICS a unique integer i, $1\leqq i\leqq N_I$. The collection of N-ICS's can therefore be represented as

$$\{I_N^i,\ 1\leqq i\leqq N_I\}.$$

(0.4) Let R be any region of S. Define the events $IR_n=R$ is intact at time n, $HR_n=R$ is hit by time n, i.e. R is not intact at time n.

Thus we can write $II_{N,n}^i$, $HI_{N,n}^i$, etc. Moreover, by putting $R_n=I_{N,n}$ or $R_n=\mathrm{US}_n$ we mean that at least one N-ICS or at least one US (respectively) is intact or not intact after the n-th dart is thrown.

§1. Bounds for p_n

In this section we shall prove the following extension of (0.2):

THEOREM 1.1. *Given u fixed,* $0<u<1$, *there exist constants* $A>0$ *and* $B<\infty$ *(depending upon u) such that for all* $n>1$

$$A < \frac{p_n}{n^2(1-u^2)^n} < B. \tag{1.1}$$

If $u=\frac{1}{2}$, (0.2) is obtained. The proof to follow is essentially the argument of [2] changed in obvious ways; for further details the reader can consult [2].

The event that there is an n-success is the union of an uncountable number of events of the form "U_1 is intact" where U_1 is a specific US. This uncountable union turns out to be measurable so that p_n is indeed defined; this will follow from Proposition 3.1. (1.1) will be shown by finding bounds on $r_{N,n}$ and $q_{N,n}$. The reason this can be done is given in the following lemma.

LEMMA 1.1. *Let* U_1 *be any* US. *Then for any N-grid there always exists an* ICS *I with* $I\subset U_1$. *If E is any* ECS *determined from an N-grid, there always exists a* US U_2 *with* $E\supset U_2$. *Thus, for all n and all* $N\geqq\frac{2}{u}$ *we have*

$$q_{N,n} < p_n < r_{N,n}. \tag{1.2}$$

PROOF. If (a, b) are the coordinates of the llv of U_1 according to the grid cartesian system (see Section 0), there are numbers k and l, $0\leqq k, l<1$ such that $(a+k, b+l)$ is a lattice point. A simple computation verifies that the ICS I with llv at this lattice point is entirely contained within U_1. Similarly, let E have llv (a, b). Then the US U_2 with llv also at (a, b) is certainly contained within E. (1.2) is a consequence of these relations.

PROPOSITION 2.1. *There exists a constant* $B<\infty$ *such that*

$$p_n < Bn^2(1-u^2)^n, \quad n \geqq 1. \tag{1.3}$$

PROOF. For a given N-grid let $\{I_N^i,\ 1\leqq i\leqq N_I\}$ be the set of all ICS's. Letting i range over all $i\in N_I$ we obtain

$$p_n < r_{N,n} \leqq \sum_i P(II^i_{N,n}) \leqq \sum_i \left(1-\left([Nu-1]\frac{1}{N}\right)^2\right)^n \leqq N_I\left(1-\left(u-\frac{2}{N}\right)^2\right)^n \leqq \tag{1.4}$$

$$\leqq N_I(1-u^2)^n\left[1+\frac{4}{N(1-u)}-\frac{4}{N^2(1-u)}\right]^n \leqq (N(1-u)+3)^2(1-u^2)^n \exp\left(\frac{4n}{N(1-u)}\right).$$

If N and $n\to\infty$ such that $cN=n$ with c constant, then there is certainly a constant B such that for $n\geqq 1$ the right side of (1.4) $\leqq Bn^2(1-u^2)^n$.

PROPOSITION 2.2. *There exists a constant* $A>0$ *such that*

$$p_n > An^2(1-u^2)^n, \quad n \geqq 1. \tag{1.5}$$

PROOF. References to "intact" below always refer to a fixed number of darts n. The notation E_i refers to an ECS relative to a fixed N-grid.

Let i_0 be a fixed integer, $1 \leqq i_0 \leqq N_E$, and consider the ECS E_{i_0}. Let t be another integer and put

$$p_{i_0,t} = P(E_{i_0} \text{ and } E_t \text{ are both intact}) = (1-\text{area of } E_{i_0} \cup E_t)^n.$$

First we find an upper bound for

$$p_{i_0} = \sum_{t \neq i_0} p_{i_0,t}.$$

The computation is broken up into two cases, an "essential" part in which the two ECS's are close together, and an "inessential" part in which the two are "far" apart. Let us suppose that the llv of E_{i_0} is the origin (0, 0) and the llv of the fixed ECS E_{t_0} has coordinates (j, k).

Case 1: both j and k are $\leqq u\sqrt{N}$. The length of edge of an ECS satisfies the inequalities

$$u < [Nu+1]\frac{1}{N} = \text{length} \leqq u + \frac{1}{N}.$$

Therefore for fixed N and u the length can be represented as $u+\dfrac{\beta}{N}$ for fixed β, $0 < \beta \leqq 1$.

The part of E_{t_0} not overlapping with E_{i_0} consists of two rectangles with areas $\dfrac{ju}{N}+o\left(\dfrac{1}{N}\right)$ and $\dfrac{ku}{N}+o\left(\dfrac{1}{N}\right)$, and these rectangles overlap in a rectangle with area $\dfrac{jk}{N^2}$. It follows that

$$p_{i_0,t_0} \leqq \left(1-\left(u^2+\frac{2u\beta}{N}+\frac{(j+k)u}{N}+o\left(\frac{1}{N}\right)-\frac{jk}{N^2}\right)\right)^n \leqq$$

$$\leqq (1-u^2)^n \exp\left(-\left(2du\beta+du(j+k)-\frac{djk}{N}\right)\frac{n}{N}\right)$$

where we have set $d=(1-u^2)^{-1}$ and used the relation $(1-q)^n \leqq e^{-qn}$ valid for $0 \leqq q \leqq 1$ (which will be true for N large enough). Since we are in Case 1, $jk \leqq Nu^2$ so the above inequality becomes upon placing $c=\dfrac{n}{N}$

$$p_{i_0,t_0} \leqq (1-u^2)^n \exp(-2du\beta c)\exp(du^2 c)\exp(-du(j+k)c). \tag{1.6}$$

To find an upper bound for

$$\sum_{t \in T} p_{i_0,t}$$

where T is the set of indices t such that the pair E_{i_0} and E_{t_0} fall into Case 1 $(t_0 \neq i_0)$, we shall sum the terms $\exp(-du(j+k)c)$ appearing in (1.6) over all possible pairs (j, k) for $0 \leqq j$, $k < \infty$. The matrix of (j, k) pairs that arises is $m_{ij}=(i, j)$, $i \geqq 0$ and $j \geqq 0$, with the single term (0, 0) missing. If we add row s of this matrix, $s \geqq 1$, and assume c is so large that we get geometric series of ratio less than 1, we have the sum

$$\exp(-du(sc))(1-\exp(-duc))^{-1} \tag{1.7}$$

and the sum for $s=0$ is the same as that for $s=1$. Since the llv of E_{i_0} is (0, 0), E_t is necessarily positioned "northeast" of E_{i_0}. For E_{i_0} in a general position, however, E_t could have one of four directions relative to E_{i_0}. Therefore we throw in the factor 4 and add over each row sum to obtain from (1.6) and (1.7)

$$\begin{aligned}\sum_{t\in T} p_{i_0,t} &\leqq 4(1-u^2)^n \exp(-2du\beta c)\exp(du^2c)\times \\ &\times\{\exp(-duc)[(1-\exp(duc))^{-2}+(1-\exp(-duc))^{-1}]\}\leqq \\ &\leqq 8(1-u^2)^n\exp(-2du\beta c)\exp(-d(u-u^2)c)(1-\exp(-duc))^{-2}.\end{aligned} \tag{1.8}$$

Case 2: at least one of $j, k>u\sqrt{N}$. In this case the two ECS's are far apart: the area of $E_{i_0}\cup E_{t_0}$ is larger than

$$u^2+\frac{2u\beta}{N}+u\max\left(\frac{j}{N},\frac{k}{N}\right)\geqq u^2+\frac{2u\beta}{N}+\frac{u^2\sqrt{N}}{N}$$

where we recall that the length of edge of E_{i_0} is $u+\frac{\beta}{N}$. As in (1.6) we may now obtain for N large enough

$$p_{i_0,t_0}\leqq\left(1-\left(u^2+\frac{2u\beta}{N}+\frac{u^2\sqrt{N}}{N}\right)\right)^n\leqq(1-u^2)^n\exp(-2du\beta c)\exp(-du^2c\sqrt{N}).$$

Let T' be the set of indices t such that the pair E_{i_0} and E_{t_0} fall into Case 2. From the above we have

$$\begin{aligned}\sum_{t\in T'} p_{i_0,t} &\leqq N_E(1-u^2)^n\exp(-2du\beta c)\exp(-du^2c\sqrt{N})\leqq \\ &\leqq KN^2(1-u^2)^n\exp(-2du\beta c)\exp(-du^2c\sqrt{N})\end{aligned} \tag{1.9}$$

where the constant K can be taken as $(2-u)^2$.

Estimating p_n. Let n and N be considered fixed but large with $c=\frac{n}{N}$ chosen so large that the coefficients of $(1-u^2)^n\exp(-2du\beta c)$ in (1.8) and (1.9) are each less than $\frac{\varepsilon}{4}$. Therefore we obtain

$$p_{i_0}=\sum_{t\neq i_0}p_{i_0,t}=\sum_{t\in T}p_{i_0,t}+\sum_{t\in T'}p_{i_0,t}<\frac{\varepsilon}{2}(1-u^2)^n\exp(-du\beta c). \tag{1.10}$$

Moreover,

$$P(E_{i_0}\text{ is intact at time }n)=\left(1-\left(u+\frac{\beta}{N}\right)^2\right)^n=$$

$$=(1-u^2)^n\left(1-\left(\frac{2du\beta}{N}+\frac{\beta^2d}{N^2}\right)\right)^n\geqq(1-u^2)^n\exp(-2du\beta c)(1-\lambda_n)$$

where $\lambda_n \to 0$ as $n \to \infty$. By a well-known inequality of Bonferroni ([1], p. 100)

$$(1.11) \qquad p_n > q_{N,n} \geqq \sum_{i \leqq N_E} P(E_i \text{ is intact at time } n) - \sum_{i \leqq N_E} p_i >$$

$$> N_E(1-u^2)^n \exp(-2du\beta c)\left(1-\lambda_n-\frac{\varepsilon}{2}\right) \geqq (1-\varepsilon)\frac{(1-u)^2}{c^2}\exp(-2du\beta c)n^2(1-u^2)^n$$

for n large enough. Now let n and N increase so that c remains fixed. Since N is changing, β will vary but always $\beta \leqq 1$ (see Case 1) so that the minimum value for the coefficient of $n^2(1-u^2)^n$ in (1.11) is

$$A_1 = (1-\varepsilon)\frac{(1-u)^2}{c^2}\exp(-2duc)$$

for all n and N sufficiently large. Since $p_n \neq 0$ for each n, the constant A_1 can be replaced by a constant A so that (1.11) becomes

$$p_n > An^2(1-u^2)^n, \quad n \geqq 1$$

and the proof of (1.5) is finished.

§2. The refinement lemma

The lemma of this section will be useful in the sequel. Let N_2 and N_1 be two integers such that each tile of the N_1-grid is a union of tiles of the N_2-grid. We say that the N_2-grid is a *refinement* of the N_1-grid.

LEMMA 2.1. *Let ε be given, $0<\varepsilon<1$. There exist sequences of numbers $\{\delta_n\}$ and $\{\omega_n\}$ satisfying:*

(a) $\lim \delta_n = \varepsilon$.

(b) *ω_n is a sequence of non-decreasing integers with $\omega_n \uparrow \infty$.*

(c) $n^{1+\delta_n} = 2^{\omega_n}, \quad n \geqq 1.$

Moreover, consider the grid numbers $N(n) = n^{1+\delta_n} = 2^{\omega_n} \geqq \frac{2}{u}$. If $s \geqq t$ the $N(s)$ grid is a refinement of the $N(t)$ grid and

(d) $r_{N(s),n} \leqq r_{N(t),n}$ *for each fixed n.*

PROOF. Let $k = \frac{\ln 2}{1+\varepsilon}$ and choose $\omega_n = \left[\frac{\ln n}{k}\right]$ where the brackets, as usual, denote the "greatest integer" function. Taking logarithms of both sides of (c) shows that

$$1+\delta_n = \left[\frac{\ln n}{k}\right]\frac{\ln 2}{\ln n}, \quad n \geqq 2$$

with δ_n, $n \geqq 2$ defined by the equation. This implies

$$\frac{\ln 2}{k} - \frac{\ln 2}{\ln n} \leqq 1+\delta_n \leqq \frac{\ln 2}{k} = 1+\varepsilon$$

proving (a). Letting $\delta_1=0$ we observe (c) remains true for $n=1$. (b) is immediate from the definition of ω_n. If $s \geqq t$

$$N(s) = s^{1+\delta_s} = 2^{\omega_s} = 2^f \cdot 2^{\omega_t} = 2^f(t^{1+\delta_t}) = 2^f N(t)$$

where $f \geqq 0$. If $f=0$, $N(s)=N(t)$ and there is nothing to prove. If $f>0$, then the $N(s)$ grid is obtained from the $N(t)$ grid by halving each tile edge in the $N(t)$ grid f times. Thus the $N(s)$ grid is a refinement of the $N(t)$ grid and a computation shows each $N(s)$ ICS contains an $N(t)$ ICS, yielding (d).

In the applications of Lemma 2.1 the particular value of ε will be unimportant. When the lemma is invoked we will assume that some ε has been chosen to define appropriate sequences $\{\delta_n\}$ and $\{\omega_n\}$. Whatever ε is chosen we have for all sufficiently large n

$$\frac{1}{n^{1+\delta_n}} \leqq \frac{1}{n^{1+\varepsilon/2}}. \tag{2.1}$$

§3. Approximation of p_n

The main result of this section is Proposition 3.2 which shows, loosely speaking, that one can uniformly approximate the terms p_n by the terms $r_{N_n,n}$ where N_n is a sequence of grid numbers to be determined.

LEMMA 3.1. *Let A and B be disjoint regions of S,* area $(A) \leqq a$, area $(B) \leqq b < 1$, $a+b \leqq 1$. *Then*

$$P(IA_n | IB_n) \geqq \left(1-\frac{a}{1-b}\right)^n.$$

PROOF. Let area $(A)=x$, area $(B)=y$. We have

$$P(IA_n | IB_n) = \frac{(1-(y+x))^n}{(1-y)^n} = \left(1-\frac{x}{1-y}\right)^n \geqq \left(1-\frac{a}{1-b}\right)^n.$$

PROPOSITION 3.1. *For each fixed n, the sequence* $N(k)=k^{1+\delta_k}$, *where δ_k is as defined in Lemma* 2.1, *satisfies: as $k\to\infty$ we have*

$$r_{N(k),n} \downarrow p_n. \tag{3.1}$$

PROOF. Fix n. The events $II_{N(k),n}$ form a decreasing sequence by Lemma 2.1 and

$$\bigcap_k II_{N(k),n} = V_n \supseteqq \mathrm{IUS}_n$$

and this implies

$$r_{N(k),n} \downarrow P(V_n) \geqq p_n. \tag{3.2}$$

If V_n occurs there is an intact $N(k)$-ICS for each k. Let us consider the set of all possible sequences of ICS's where the k^{th} term of the sequence is an intact $N(k)$-ICS. We call such sequences *configurations*. There are a countable number of possible configurations C_i. If a given configuration C_i occurs, there will be an intact US if, in addition to C_i, a frame around at least one of the ICS's composing C_i, is intact. This frame has area disjoint from C_i at most $\frac{4u}{N(k)}$ for an $N(k)$-ICS$\subset C_i$. We may assume the union of the areas of all ICS's in the configuration is $b<1$, for $b=1$ implies all darts have fallen into an area of probability 0. Apply Lemma 3.1 to obtain

$$(3.3)\quad P(\mathrm{IUS}_n|\,C_i) \geqq P\left(\text{an intact frame of area} \leqq \frac{4u}{N(k)}\text{disjoint from } C_i|\,C_i\right) \geqq$$

$$\geqq \left(1-\frac{4u}{N(k)}\frac{1}{1-b}\right)^n.$$

(3.3) is valid for all $N(k)\uparrow\infty$ since each C_i contains an $N(k)$-ICS. Thus

$$(3.4)\qquad P(\mathrm{IUS}_n|\,C_i) = 1.$$

Now since there are a countable number of configurations we have, using (3.4)

$$(3.5)\qquad P(\mathrm{HUS}_n|\,V_n) \leqq \sum_T P(\mathrm{HUS}_n|\,C_i)\frac{P(C_i)}{P(V_n)} = 0$$

where T is the countable set of indices corresponding to all possible configurations. (3.5) is equivalent to

$$(3.6)\qquad P(\mathrm{IUS}_n|V_n) = 1$$

proving

$$p_n = P\,(\mathrm{IUS}_n) \geqq P(V_n)$$

which, in conjunction with (3.2), proves (3.1).

PROPOSITION 3.2. *Given* $\varepsilon>0$ *there exists a subsequence* i_n *such that* $N(i_n)$ *satisfies*

$$(3.7)\qquad \frac{p_n}{r_{N(i_n),n}} > 1-\varepsilon \quad \textit{for all } n.$$

PROOF. For each fixed n, $\varepsilon p_n>0$. Proposition 3.1 implies the existence of an integer $k_0(n)$ such that for all $k\geqq k_0(n)$

$$r_{N(k),\,n}-p_n < \varepsilon p_n.$$

Therefore, if $k\geqq k_0(n)$

$$\frac{p_n}{r_{N(k),n}} > \frac{1}{1+\varepsilon} > 1-\varepsilon.$$

Choose the sequence $\{i_n\}$ inductively as follows: set

$$\text{(a)}\qquad i_1 = k_0(1)$$

and

$$\text{(b)}\qquad \text{if}\quad n>1 \quad \text{let}\quad i_n = \max\left(i_{n-1}, k_0(n)\right)+1.$$

This sequence clearly yields (3.7).

§4. A ratio inequality

The principal result to be proved now (Corollary 4.1) gives an important upper bound for the ratio of terms $r_{M,m}$ and $r_{N,n}$ for $m>n$ and M and N certain values that depend upon m and n. The upper bound is independent of M and N. Below we use the notation of (0.3).

PROPOSITION 4.1. *Let* $N(n)=n^{1+\delta_n}$ *where* δ_n *is a sequence defined in Lemma* 2.1. *Let* $n_2 \geqq n_1 \geqq n$. *Then*

$$\frac{r_{N(n_2),n+1}}{r_{N(n_1),n}} \leqq (1-u^2)\left(\frac{n+1}{n}\right)^2 \tag{4.1}$$

for all sufficiently large n.

PROOF. Let $N(n_1)=N$ and fix an index i. Say that dart $n+1$ *bingos* if all I_N^j for $j<i$ are hit by time $n+1$ but not all are hit by time n, that is, the event

$$HI^j_{N,n+1} \quad \text{for all} \quad j<i \sim HI^j_{N,n} \quad \text{for all} \quad j<i \tag{4.2}$$

occurs.

We will say that an ICS is *hit uniquely* (by time $n+1$) if exactly one of the darts s, $1 \leqq s \leqq n+1$ hits the ICS.

Consider the set of ICS's I_N^j for $j<i$ that are hit uniquely by a given dart x, $1 \leqq x \leqq n+1$. An ICS I_N^k is a *leader* (or *leader for dart x*) if I_N^k is hit uniquely by dart x but I_N^j for $j<k$ is not hit uniquely by dart x. We observe that if I_N^k is a leader for dart x then:

(a) some I_N^j for $j<k$ may be hit by dart x, but not uniquely.

(b) some I_N^j for $j<k$ may be leaders, but not for dart x.

Define the event

$$ {}_sLI_N^k = I_N^k \quad \text{is the } s^{\text{th}} \text{ leader}, \quad 1 \leqq s \leqq n+1. \tag{4.3}$$

Notice that if $1 \leqq x \leqq n+1$

$$P(\text{dart } x \text{ hits } I_N^k \mid II^i_{N,n+1}, HI^j_{N,n+1} \quad \text{for} \quad j<i, {}_sLI_N^k) = \frac{1}{n+1}. \tag{4.4}$$

This follows because, first, the left side of (4.4) has the same value by symmetry for any x, $1 \leqq x \leqq n+1$, and, second, the events "dart x hits I_N^k" for $1 \leqq x \leqq n+1$ define a conditional partition of the universe given ${}_sLI_N^k$. If dart $n+1$ bingos then dart $n+1$ hits some ICS uniquely and this event can be decomposed in terms of s^{th} leaders to yield from (4.4)

$$\begin{gathered} P(\text{dart } n+1 \text{ bingos at leader } s \mid II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j<i) = \\ = \sum_{k<i} P({}_sLI_N^k \mid II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j<i) \times \\ \times P(\text{dart } n+1 \text{ hits } I_N^k \mid II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j<i, {}_sLI_N^k) \leqq \frac{1}{n+1} \times \text{conditional} \\ \text{probability of at least } s \text{ leaders.} \end{gathered} \tag{4.5}$$

Put, for a moment, $I^i_{N,n+1}=W$. Using a variation of the argument leading to (1.8) we have

$$P(\text{at least } s \text{ leaders} \mid IW) \leqq$$

$$\leqq P(W \text{ and at least one other ICS are avoided by } n \text{ darts} \mid IW) <$$

$$< \exp(-kn/N) < (n+1)^{-1}$$

provided N, n get large at an appropriate rate, e.g. $N=n^{1/2}$. Thus

$$P(n+1 \text{ bingos} \mid IW) < (n+1)^{-1}$$

for large n when this rate holds. But it is seen that for fixed n the inequality remains valid when N is taken larger (because, roughly, it becomes harder to bingo when sets get larger). At the rate given above, one also has

$$P(HI^j_{N,n+1} \text{ for } j < i \mid IW) \to 1$$

implying

$$P(n+1 \text{ bingos} \mid IW, HI^j_{N,n+1} \text{ for } j < i) < (n+1)^{-1}$$

at this rate, and again the inequality still holds for larger N when n is fixed.

It follows that

(4.6)

$$\frac{P(II^i_{N,n+1}, HI^j_{N,n} \text{ for } j < i)}{P(II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j < i)} = P(HI^j_{N,n} \text{ for } j < i \mid II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j < i) =$$

$$= 1 - P(\text{dart } n+1 \text{ bingos} \mid II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j < i) \geqq 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Moreover,

$$\frac{P(II^i_{N,n+1}, HI^j_{N,n} \text{ for } j < i)}{P(II^i_{N,n}, HI^j_{N,n} \text{ for } j < i)} = P(II^i_{N,n+1} \mid II^i_{N,n}, HI^j_{N,n} \text{ for } j < i) = \tag{4.7}$$

$$= P(\text{dart } n+1 \text{ avoids } I^i_N) = 1 - \left(u - \frac{\lambda}{N}\right)^2 \leqq$$

$$\leqq (1-u^2) + \frac{2u\lambda}{N} \leqq 1 - u^2 + \frac{4u}{N} = (1-u^2)\left(1 + \frac{4du}{N}\right)$$

where the length of edge of an $\text{ICS} = u - \frac{\lambda}{N}$ is such that $1 \leqq \lambda \leqq 2$ by virtue of the inequalities

$$u - \frac{2}{N} \leqq [Nu-1]\frac{1}{N} \leqq u - \frac{1}{N}$$

and we have set $(1-u^2)^{-1}=d$ as in Section 1. Now write

$$r_{N,n+1} = \sum_i P(II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j < i) = \tag{4.8}$$

$$= \sum_i \frac{P(II^i_{N,n+1}, HI^j_{N,n} \text{ for } j < i)}{P(II^i_{N,n}, HI^j_{N,n} \text{ for } j < i)} \frac{P(II^i_{N,n+1}, HI^j_{N,n+1} \text{ for } j < i)}{P(II^i_{N,n+1}, HI^j_{N,n} \text{ for } j < i)} \times$$

$$\times P(II^i_{N,n}, HI^j_{N,n} \text{ for } j < i) \leqq$$

$$\leqq \sum_i (1-u^2)\left(1+\frac{4du}{N}\right)\left(\frac{n+1}{n}\right) P(II^i_{N,n}, HI^j_{N,n} \text{ for } j < i) =$$

$$= (1-u^2)\left(1+\frac{4du}{N}\right)\left(\frac{n+1}{n}\right) r_{N,n}$$

where we have used (4.6) and (4.7). Since $N=N(n_1)\geqq N(n)=n^{1+\delta_n}$, by (2.1)

$$1+\frac{4du}{N} \leqq 1+\frac{1}{n} = \frac{n+1}{n}$$

for n sufficiently large, and so for such n (4.8) implies

$$\frac{r_{N(n_1),n+1}}{r_{N(n_1),n}} \leqq (1-u^2)\left(\frac{n+1}{n}\right)^2. \tag{4.9}$$

According to Lemma 2.1 we have

$$r_{N(n_2),n+1} \leqq r_{N(n_1),n+1} \tag{4.10}$$

and replacing in (4.9) gives (4.1).

COROLLARY 4.1. *Let* $n_2>n_1$ *and* $m>n$ *with* $n_1\geqq n$ *and* $n_2\geqq m$. *Then*

$$\frac{r_{N(n_2),m}}{r_{N(n_1),n}} \leqq (1-u^2)^{m-n}\left(\frac{m}{n}\right)^2 \tag{4.11}$$

for all sufficiently large m *and* n.

Case 1: $m>n_1$. Let $n_1>n$. For $0\leqq k\leqq n_1-n-1$

$$\frac{r_{N(n_1),n+k+1}}{r_{N(n_1),n+k}} \leqq (1-u^2)\left(\frac{n+k+1}{n+k}\right)^2 \tag{4.12}$$

by (4.8) and the relation $N(n_1)\geqq N(n+k)$. Similarly, for $n_1\leqq j\leqq m-1$

$$\frac{r_{N(j+1),j+1}}{r_{N(j),j}} \leqq (1-u^2)\left(\frac{j+1}{j}\right)^2 \tag{4.13}$$

where we again use (4.10). Multiply all the terms appearing in (4.12) and (4.13) to obtain a collapsing product, yielding

$$\frac{r_{N(m),m}}{r_{N(n_1),n}} \leqq (1-u^2)^{m-n}\left(\frac{m}{n}\right)^2$$

and then (4.10) allows us to replace $N(m)$ by $N(n_2)$ to get (4.11). If $n_1=n$ we obtain the same result by multiplying the terms of the form (4.13) together.

Case 2: $m\leqq n_1$. For $0\leqq k\leqq m-n-1$ we only have terms of the form (4.12), the product giving

$$\frac{r_{N(n_1),m}}{r_{N(n_1),n}}\leqq(1-u^2)^{m-n}\left(\frac{m}{n}\right)^2.$$

Again (4.10) allows us to replace $N(n_1)$ by $N(n_2)$ to get (4.11).

§5. Proof of main theorem

Let us set

$$\bar{p}_n=\frac{p_n}{n^2(1-u^2)^n}.$$

THEOREM 5.1. *For each u there exists a number L, $0<L<\infty$ (L depending upon u) such that*

$$\lim_{n\to\infty}\bar{p}_n=L. \tag{5.1}$$

PROOF. If the limit fails to exist two subsequences $\{n_i\}$ and $\{m_i\}$ may be determined so that we have by (1.1)

$$\bar{p}_{n_i}\to L_1,\quad \bar{p}_{m_i}\to L_2,\quad 0<L_1<L_2<\infty.$$

Let ε be chosen so small that $(1-\varepsilon)^{-1}<\dfrac{L_2}{L_1}$. By Proposition (3.2) there is a subsequence i_{n_i} with

$$\frac{p_{n_i}}{r_{N(i_{n_i}),n_i}}>1-\varepsilon \text{ for all } n_i. \tag{5.2}$$

Without loss of generality it may be assumed that $m_i>n_i$ for each i. Thus $i_{m_i}>i_{n_i}$, $i_{n_i}\geqq n_i$, and $i_{m_i}\geqq m_i$, so that Corollary 4.1 applies. Using (4.11) and (5.2) we have

$$\frac{p_{m_i}}{(1-\varepsilon)^{-1}p_{n_i}}\leqq\frac{r_{N(i_{m_i}),m_i}}{r_{N(i_{n_i}),n_i}}\leqq(1-u^2)^{m_i-n_i}\left(\frac{m_i}{n_i}\right)^2$$

which implies

$$\frac{\bar{p}_{m_i}}{\bar{p}_{n_i}}\leqq(1-\varepsilon)^{-1}.$$

Let $i\to\infty$. We obtain

$$\frac{L_2}{L_1}\leqq(1-\varepsilon)^{-1}<\frac{L_2}{L_1},$$

a contradiction. This is true regardless of the fixed u chosen, $0<u<1$, and the proof is complete.

§6. Related problems and applications

The random bombardment problem of this paper is one of a large class of bombardment and related covering problems well known in geometric probability; see [3] and [4] for discussions and further references. The one-dimensional analog of the square bombardment is treated in [3], p. 31, where the probability that n random points on an interval fall such that the largest interval between two adjacent points exceeds a fixed value u is calculated explicitly. More generally ([3], p. 28) the joint distribution of the length of intervals between adjacent points is obtained. From formula (2.38) of [3] it follows that the one-dimensional analog of (0.1) is

$$\lim_{n\to\infty} \frac{p_n}{n(1-u)^n} = 1.$$

In this case, of course, we have an explicit expression for the error term.

The dual nature of bombardment and covering problems is striking in the one-dimensional case if we consider random arcs of length u on a unit circumference and ask for the probability that n such arcs completely cover the circumference. If each arc is determined by its most clockwise endpoint, these points determine n intervals whose joint distribution is that of the n intervals formed by $n-1$ random points in an interval of unit length ([3], p. 33). In the case of a square this duality cannot be fully realized. If Q is the $u\times u$ square centered on the origin of the unit square S and $X_1, X_2, ..., X_n$ are the positions of the n random darts, then if the union of the squares $Q+X_i$ does not cover S, each point y in this uncovered region is such that $Q+y$ is not hit by any of the darts. The only problem is that $Q+y$ will not be entirely within S if y is too close to the edge of S. We obtain the duality when we consider a torus rather than a square as in the one-dimensional case where we consider a unit circumference rather than a unit interval. A solution to the problem when the duality is complete is given in [4].

Another related problem concerns n cylinders of a fixed length that strike a sphere at random and stand up perpendicularly from the surface. The cylinder throws a "shadow" on the sphere in the form of a circular cap. One is required to find the probability that the entire surface of the sphere is shaded. This problem has been used to model the attack of a virus by antibodies (see [3], p. 111 and [5], Chap. 4).

The problem treated in this paper is clearly applicable to a variety of detection problems. Here is a possible medical application. Suppose that cancer cells are distributed at random throughout a region of tissue where the cells play the role of the darts and the region is S. A biopsy is performed, that is, a small sample of tissue is taken from S for examination. The biopsy may be negative because the sample (or US) does not contain any cancer cells. p_n is then a measure of the sensitivity of the biopsy as a test for detecting disease (suggested by Professor Valerie Miké).

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HERBERT H. LEHMAN COLLEGE
CUNY
U.S.A.

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GENERATING COTORSION THEORIES AND INJECTIVE CLASSES

H. P. GOETERS (Middletown)*

Several authors have studied injective classes of abelian groups, a problem proposed in [4] (Problem 46). Here we study ways of generating injective classes, and in the process, give characterizations of the cotorsion theories in the category TF.

1. TF will denote the category of all torsion-free abelian groups of finite rank and homomorphisms. The term group will mean a member of obj TF (unless specified otherwise). In particular, an exact sequence $0 \to A \to B \to C \to 0$ of groups, will mean $A, B, C \in \operatorname{obj} TF$. The symbols π, Z, Z_p, Q and $\hat{Z}_p$ will denote the set of primes, integers, integers localized at p, rationals and p-adic integers, respectively. If h, h': $\pi \to \{0, 1, 2, \ldots\} \cup \{\infty\}$, we declare them equivalent provided $\Sigma_p |h(p) - h'(p)| < \infty$. An equivalence class is called a type. Let $\bar{0}$ denote the type determined by the zero map, and $\bar{\infty}$ the type containing h where $h(p) = \infty$ for every p. An idempotent type is the type of a function h where $h(p) = 0$ or ∞. The support of a group G is defined as $\operatorname{supp} G = \{p \mid pG \neq G\}$. Generally, the notation will be that of [4] and [5] or [1].

2. Cotorsion theories in TF. Cotorsion theories in $\mathcal{Ab}$ (the category of all abelian groups) were studied in [7]. Here, we describe the cotorsion theories in TF.

DEFINITION. An ordered pair $(\mathcal{F}, \mathcal{C})$ of classes of groups (in TF) is called a cotorsion theory if:

i) $\operatorname{Ext}(A, X) = 0$ for all $A \in \mathcal{F}$ and $X \in \mathcal{C}$,
ii) $\operatorname{Ext}(Y, X) = 0$ for all $X \in \mathcal{C}$ implies $Y \in \mathcal{F}$,
iii) $\operatorname{Ext}(A, Y) = 0$ for all $A \in \mathcal{F}$ implies $Y \in \mathcal{C}$.

Given a class H of groups, we define the following classes. Let $P(H)$ (dually, $I(H)$) be the class of all those groups Y satisfying $\operatorname{Ext}(Y, X) = 0$ (resp. $\operatorname{Ext}(X, Y) = 0$) for all $X \in H$.

Clearly, $(P(H), IP(H))$ and $(PI(H), I(H))$ are cotorsion theories. The former is called the cotorsion theory generated by H, and the latter, the cotorsion theory cogenerated by H. Moreover, by observing that both I and P reverse containment, if $(\mathcal{F}, \mathcal{C})$ is any cotorsion theory then $(\mathcal{F}, \mathcal{C}) = (P(\mathcal{C}), IP(\mathcal{C})) = (PI(\mathcal{F}), I(\mathcal{F}))$. (Generally, we define a partial order "$\leqq$" by $(\mathcal{F}, \mathcal{C}) \leqq (\mathcal{F}', \mathcal{C}')$ iff $\mathcal{F}' \subseteq \mathcal{F}$, or equivalently, iff $\mathcal{C} \subseteq \mathcal{C}'$. Under this relation, $(\mathcal{F}, \mathcal{C}) = (\mathcal{F}', \mathcal{C}')$ iff $\mathcal{C} = \mathcal{C}'$, or, iff

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* This note is part of my thesis, written while attending the University of Connecticut. I would like to extend my gratitude to my thesis advisor, William J. Wickless, for his help and guidance then and now.

$\mathcal{F}=\mathcal{F}'$.) Hence, we need only consider cotorsion theories (co) generated by classes of groups.

The next result is captured from the proof of Theorem 12 in [11]. Let $S\subset\pi$, and let t be a type. We will write $t\sim_S\bar{0}$ if, for every $h\in t$, $\sum_{p\in S} h(p)<\infty$.

LEMMA 1. *Let $A, Y\in TF$. Then* $\operatorname{Ext}(Y, A)=0$ *iff $OT(Y)\sim_S\bar{0}$ for $S=\operatorname{supp} A$.*

The proof is immediate from the proof of the above theorem so it is omitted. The next result is the dual to a result in [11].

THEOREM 2. *The cotorsion-free class $P(H)$ (generated by H) is the class of those groups Y for which $OT(Y)\sim_{\operatorname{supp} A}\bar{0}$ for every $A\in H$. The cotorsion closure of H, $IP(H)$, is the class of those groups X satisfying: $X\in IP(H)$ iff*

(∗) *if t is a type with $t\sim_{\operatorname{supp} A}\bar{0}$ for every $A\in H$ then $t\sim_{\operatorname{supp} X}\bar{0}$.*

PROOF. Since $P(H)=\bigcap_{A\in H} P(\{A\})$, the first statement follows directly from Lemma 1.

Assume (∗) holds for a group X and let $Y\in P(H)$. By Lemma 1, $OT(Y)\sim_{\operatorname{supp} A}\bar{0}$ for every $A\in H$. Thus, by (∗), $OT(Y)\sim_{\operatorname{supp} X}\bar{0}$, i.e. $\operatorname{Ext}(Y, X)=0$. Since Y was arbitrary, $X\in IP(H)$.

Conversely, let $X\in IP(H)$ and let t be a type with $t\sim_{\operatorname{supp} A}\bar{0}$ for every $A\in H$. If W is a rank 1 group of type t, then by Lemma 1, $W\in P(H)$. Consequently, $\operatorname{Ext}(W, X)=0$ and again invoking Lemma 1, we have that $OT(W)=t\sim_{\operatorname{supp} X}\bar{0}$. □

Given an ordered pair $(\mathcal{F}, \mathcal{C})$ of classes of groups, we can say that $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory iff

i) If $Y\in\mathcal{F}$ then $X\in\mathcal{C}$ iff $OT(Y)\sim_{\operatorname{supp} X}\bar{0}$ and

ii) If $X\in\mathcal{C}$ then $Y\in\mathcal{F}$ iff $OT(Y)\sim_{\operatorname{supp} X}\bar{0}$.

EXAMPLE 3. $P(\{Z\})$ is the class of all (finite rank) free groups since $\operatorname{supp} Z=\pi$ and therefore $OT(A)\sim_{\operatorname{supp} Z}\bar{0}$ iff $OT(A)=\bar{0}=IT(A)$ ([1] p. 15).

We can show that if H is a finite set then $Y\in IP(H)$ iff $\operatorname{supp} Y\subseteqq\bigcup_{X\in H}\operatorname{supp} X$. This is not true if the set of isomorphism classes on H is infinite, however, for let $H=\{Z_p|p\in\pi\}$. Then $\pi\subset\bigcup_p \operatorname{supp} Z_p$, but $Z\notin IP(H)$.

THEOREM 4. *The class $IP(H)$ is the class of those groups Y satisfying:*

i) $\operatorname{supp} Y\subseteqq\bigcup_{x\in H}\operatorname{supp} X$ *and*

ii) *if C is an infinite subset of* $\operatorname{supp} Y$, *then there exists an $X\in H$ with* $\operatorname{supp} X\cap C$ *infinite.*

PROOF. Suppose $\operatorname{supp} Y$ satisfies conditions i) and ii), and let r be a type satisfying $t\sim_{\operatorname{supp} X}\bar{0}$ for all $X\in H$. For $h\in t$, let $C=\{p|p\in\operatorname{supp} Y$ and $h(p)\neq 0\}$. Since $t\sim_{\operatorname{supp} X}\bar{0}$, $C\cap\operatorname{supp} X$ is finite for all $X\in H$. Thus C is finite. If $p\in C$, then $p\in\operatorname{supp} X$ for some $X\in H$. Therefore, $h(p)<\infty$, and consequently, $t\sim_{\operatorname{supp} Y}\bar{0}$.

Conversely, if $Y\in IP(H)$ and C is an infinite subset of $\operatorname{supp} Y$, let $h(p)=\begin{cases}1 & p\in C\\ 0 & p\notin C\end{cases}$. Then $[h]\nsim_{\operatorname{supp} Y}\bar{0}$ which implies $[h]\nsim_{\operatorname{supp} Y}\bar{0}$ for some $X\in H$. For this X, $\operatorname{supp} X \cap C$ is infinite. If $p \in \operatorname{supp} Y$, let t be the idempotent type with support$=\pi \setminus \{p\}$ $(t \sim_{\pi \setminus \{p\}} \bar{0}$ and $t(p)=\infty)$. Then $t \not\sim_{\operatorname{supp} Y} \bar{0}$ so that $t \not\sim_{\operatorname{supp} X} \bar{0}$ for some $X \in H_1$. I.e. $p \in \operatorname{supp} X$. $\square$

If H is a class of groups, let $K=\{S \subseteq \pi | X \in H$ with $\operatorname{supp} X=S\}$. For each $S \in K$, let X_s be any subgroup of Q with $\operatorname{supp} X_s=S$. Then $IP(H)=IP(\{X_s | S \in K\})$ using Theorem 4. Hence, every cotorsion theory is generated by a set of rank 1 groups. This is analogous to the result proved in [7] about cotorsion theories in $\mathcal{Ab}$ (the category of abelian groups).

In that paper, L. Salce asks for a characterization of all cotorsion theories in $\mathcal{Ab}$ as well as answers several other unsolved problems. The questions have easy answers in TF. For instance: We say that a cotorsion theory $(\mathcal{F}, \mathcal{C})$ has enough injectives (projectives) if for any group G, there is an exact sequence $0 \to G \to C \to X \to 0$ $(\to C \to \to X \to b \to 0)$ where $C \in \mathcal{C}$ and $X \in \mathcal{F}$. He asks if all cotorsion theories in $\mathcal{Ab}$ have enough injectives.

Corollary 5. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence. Then* $B \in IP(H)$ $(B \in P(H))$ *iff* $A, C \in IP(H)$ $(A, C \in P(H))$.

The proof follows from Theorems 2 and 4 and properties of Ext, after noting that $OT(A) \leqq OT(B)$ and $\operatorname{supp} C \subseteq \operatorname{supp} B$.

As a consequence, we can see that no cotorsion theory in TF save $(\operatorname{obj} T\mathcal{F}, \mathcal{D})$, ($\mathcal{D}$=class of divisible groups) has enough projectives. Also, no cotorsion theory except $(\mathcal{F}, \operatorname{obj} TF)$, ($\mathcal{F}$=class of free groups) has enough injectives.

However, questions about cotorsion theories in $\mathcal{Ab}$ are hard to answer if one renders judgement from the status of Whitehead's conjecture (i.e. in $\mathcal{Ab}$, $(P(\{Z\})$ is the class of all free groups). This is known to be independent of the ZFC axioms of set theory.

Let X and C be groups. It is well known that X is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ iff $\operatorname{Ext}(C, X)=0$. Hence, if $\mathcal{E}$ is the class of all exact sequences $0 \to A \to B \to C' \to 0$ where $C' \in P(H)$, then the class of all groups injective with respect to each $E \in \mathcal{E}$ is just $IP(H)$, for any class of groups H. In this way, H generates the injective clas $IP(H)$.

3. Some injective classes. We generate an injective class $I'C(H)$ which, generally, is unequal to $IP(H)$. Let $C(H)$ be the class of these groups A such that for any pure embedding $\alpha\colon A \to B$, B any group, each $X \in H$ is injective with respect to $0 \to A \xrightarrow{\alpha} B$. For $\mathcal{E}$ equal to the class of all exact sequences $0 \to A \to B \to C \to 0$ with $A \in C(H)$, let $I'C(H)$ denote the class of all groups injective with respect to each $E \in \mathcal{E}$.

For each $X \in H$, write $X=D_x \oplus R_x$ where D_x is divisible and R_x reduced. For any exact sequence E $0 \to A \to B \to C \to 0$, X is injective with respect to E iff R_x is. Hence if H_R=class of all R_x, $X \in H$ and $H_R \neq \{0\}$ then $C(H_R)=C(H)$. Since if H contains only divisible groups, $C(H)=\operatorname{obj} TF$ and $I'C(H)$ is the class of all divisible groups, we will assume, without loss of generality, that H contains only reduced groups.

The following appears in the literature.

Definition. For a class of groups H, let $H^{\perp}$ be the class of groups G satisfying $\operatorname{Hom}(X, G)=0$ for all $X \in H$, and dually, ${}^{\perp}H$ the class of groups G satisfying

$\mathrm{Hom}(G, X)=0$ for all $X \in H$. We will write $X^{\perp}$ and ${}^{\perp}X$ for $\{X\}^{\perp}$ and ${}^{\perp}\{X\}$ respectively.

We will show that ${}^{\perp}H=C(H)$ and (modulo divisible summands) $({}^{\perp}H)^{\perp}= =I'C(H)$. We need some results which were offered by W. Wickless.

PROPOSITION 6. *Let R be a reduced group and let s and t be types with $s<t$. There are at most countable many rank 2 subgroups U of R, such that $A \oplus \oplus B \leqq U$, with A and B both rank 1 pure subgroups of R of type S and U cohomogeneous of type t (i.e. $0 \neq f$ $\mathrm{Hom}(U, Q) \Rightarrow \mathrm{type}\ f(U)=t$).*

Before proving this we will recall the characterization of groups put forth in [3]. For the moment, let $A=\langle a\rangle_*$ and $B=\langle b\rangle_*$ and regard U as a subgroup of $QU= =Qa \oplus Qb$. Let $A_0=\{x \in Qa \mid y \in Qb$ with $x+y \in U\}$ and $B_0=\{y \in Qb \mid x \in Qa$ with $x+y \in U\}$. The following hold:

i) $U/A \cong B_0$ under the isomorphism $(x+y)+A \to Y$

ii) $U/B \cong A_0$ under the isomorphism $(x+y)+A \to X$ and

iii) $U/A \oplus B \cong A_0/A$ under the isomorphism $(x+y)+A \oplus B \to x+A$.

Conversely, given a rational vector space $Qa \oplus Qb$ and subgroups $0 \neq A \leqq \leqq A_0 \leqq Qa$, $0 \neq B \leqq B_0 \leqq Qb$ with $A_0/A \cong^{\theta} B_0/A$, then $U=\{(x, y) \mid \theta(x+A)=y+B\}$ is a rank 2 group; $A, B \lhd U$ and i)—iii) hold. Moreover, U is uniquely determined by the quintuple (A_0, A, B_0, B, θ).

Suppose $t=\mathrm{type}\ A_0$ and $s=\mathrm{type}\ A$ with $t>s$. Let h_0 and h be the height vector of a in A_0 and A respectively. Define $t-s=[h_0-h]$. We have that $A_0/A \cong \cong \bigoplus_p Z(p^{h_0(p)-h(p)})$, i.e. for $k=h_0-h \in t-s$, $A_0/A \cong \bigoplus_p Z(p^{k(p)})$.

Furthermore, $\mathrm{End}\left(\bigoplus_p Z(p^{k(p)})\right)$ is an uncountable group since $t-s \neq \bar{0}$. This implies that there are uncountably many rank 2 groups U homogeneous of type s, cohomogeneous of type t (one fore each quintuple (A_0, A, B_0, B, θ), $\theta \in \mathrm{Aut}(A_0/A)$).

PROOF OF PROPOSITION 6. Let A and B be pure rank 1 subgroups of type s and let $h \in t-s$ with $U/A \oplus B \cong \bigoplus_p Z(p^{h(p)})$. We note that $U/A \oplus B \leqq \langle A \oplus B\rangle_*/A \oplus B$, the latter being a fixed torsion group of p-rank 1 for every p. Let $f\colon \langle A \oplus B\rangle_*/A \oplus B \to \to Q/Z$ be a fixed embedding. Then $f(U/A \oplus B)=\bigoplus_p Z(p^{h(p)})$. Moreover, the totality of groups U' (as in the statement) is the set of all $f^{-1}\left(\bigoplus_p Z(p^{h'(p)})\right)$ where $h' \in t-s$. Since $t-s$ is at most countable (any $h' \in t-s$ is a rational multiple of h) and there are at most countably many pairs A and B with $A, B \lhd R$, the proposition follows. □

We make one further observation: it is a consequence that there are at most countably subgroups U of R, cohomogeneous of type t containing $A \oplus B$ with type $A=s=\mathrm{type}\ B$ and A and B almost pure in R (i.e. $\langle A\rangle_*/A$ and $\langle B\rangle_*/B$ are finite).

The next lemma is due to W. Wickless and is almost the dual to a result in [9].

LEMMA 7. *Let Y be a reduced group. There is a rank 2 group A, homogeneous to type $\bar{0}$, with $\mathrm{Hom}(A, Y)=0$.*

PROOF. Let V be a rational vector space with basis $\{a, b\}$. We construct A by defining its localizations A_p with $Z_pa\oplus Z_pb\leqq A_p\leqq Qa\oplus Qb$. Let $S=\operatorname{supp} Y$. By the hypothesis, $S\neq\emptyset$. If $q\notin S$ let $A_q=Z_pa\oplus Z_qb$. Let $p\in S$. There are uncountable many groups U with $Z_pa, Z_pb<U$, $Z_pa\oplus Z_pb\leqq U$, and U being cohomogeneous of type ∞ (see the discussion following Proposition 6). Let U be one such group. Write $Y_p=D_p\oplus R_p$ where D_p is divisible and R_p is reduced. If $0\neq f\in\operatorname{Hom}(U, R)_p$, then f is an embedding (since every factor of U is divisible). Also, since R_p is reduced, $f(Z_pa)$ and $f(Z_pb)$ are almost pure in R_p (if $f(Z_pa)$ were not almost pure, it would be divisible). By the remark after Proposition 6, R_p contains at most countably many groups $f(U)$. Since there are uncountable many choices for U, there is a group U_0 with Z_pa, $Z_pb\leqq U_0$, $Z_pa\oplus Z_pb\leqq U_0\leqq Qa\oplus Qb$, U_0 cohomogeneous of type $\overline{\infty}$ and U_0 is not isomorphic to any subgroup of R_p. Let $A_p=U_0$.

Take $A=\bigcap_p A_p$. We will first show that $Za, Zb\lhd A$. If $my=na$, write $m=$ $=P_i^{e_i}\dots P_s^{e_s}$. For each i, let $y_1=(P_1^{e_1}\dots \hat{P_i^{e_i}}\dots P_s^{e_s})$ (where ^ means that the ith term is deleted). Since $Z_{p_i}a\lhd A_{p_i}$ and $P_i^{e_i}y_i=na$, $y_i=(n/p_i^{e_i})a\in Z_{p_i}a$. But this says that $p_i^{e_i}$ divides n. Since i was arbitrary, $m|n$ and therefore $y=(n/m)a\in Za$. Similarly $Zb\lhd A$, so that A is homogeneous of type $\bar{0}$.

Let $f\in\operatorname{Hom}(A, Y)$. Consider $f_p\in\operatorname{Hom}(A_p, Y)$. By construction, $f_p(A_p)\subseteqq D_p$ for every p. It follows that $f(A)=\bigcap_p f_p(A_p)\subseteqq\bigcap_p D_p=0$. □

If $R\leqq Q$ containing Z consider $A\otimes R$ (A as above). Since $A\cong A\times Z$ is a full subgroup of $A\otimes R$ and $\operatorname{Hom}(A\otimes Z, Y)=0$ we must have $\operatorname{Hom}(A\otimes R, Y)=0$. Hence, given any type t, there is a homogeneous, rank 2 group B, of type t with $\operatorname{Hom}(B, Y)=0$.

We use this to get the following:

LEMMA 8. *Let Y and A be groups, $0\neq f\in\operatorname{Hom}(A, Y)$ and assume that Y is reduced. There exists a group C and a pure embedding $i\colon A\to C$ so that $f\in i^*(\operatorname{Hom}(C, Y))$ (where $i^*\colon\operatorname{Hom}(C, Y)\to\operatorname{Hom}(A, Y)$ by $i^*(g)=gi$).*

PROOF. Pick $a\in A\setminus\operatorname{Ker} f$ and let $t=\operatorname{type}\langle a\rangle_*$. Let B be a group which is homogeneous of type t and satisfies $\operatorname{Hom}(B, Y)=0$. In particular, the diagram:

(1)
$$\begin{array}{ccc} 0\to\langle a\rangle_* & \xrightarrow{k} & B \\ \downarrow{\scriptstyle f} & & \\ Y & & \end{array}$$

cannot be completed, where k is any pure embedding. Let C be the pushout of

$$\begin{array}{ccc} 0\to\langle a\rangle_* & \xrightarrow{k} & B, \\ {\scriptstyle i}\downarrow & & \\ A & & \end{array}$$

where i is the inclusion map. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \langle a\rangle_* & \xrightarrow{k} & B & \to & B/\langle a\rangle_* & \to & 0 \\ & & {\scriptstyle i}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \xrightarrow[g]{} & C & \to & C/gA & \to & 0. \end{array}$$

Since $B/\langle a\rangle \cong C/gA$, g is a pure embedding. Also, any completion of

$$\begin{array}{ccc} 0 \to A & \xrightarrow{g} & C, \\ {\scriptstyle t}\downarrow & & \\ Y & & \end{array}$$

would yield a completion of (1), which is impossible. □

We now invoke the assumption that H contains only reduced groups.

THEOREM 9. ${}^{\perp}H=C(H)$.

PROOF. "$\supseteq$" If $\operatorname{Hom}(A, Y)=0$ for every $Y\in H$, then clearly Y is injective with respect to any embedding $\alpha\colon A\to B$, for any $Y\in H$.

"$\subseteq$" If $A\in C(H)$ and $Y\in H$ is such that $\operatorname{Hom}(A, Y)\neq 0$, then let $0\neq f\in \operatorname{Hom}(A, Y)$. Let C and i be as in Lemma 8. Then, the diagram

$$\begin{array}{ccc} 0 \to A & \xrightarrow{i} & C \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array}$$

cannot be completed, a contradiction. □

For instance, $C(\{Z\})={}^{\perp}Z$, the class of groups which have no free summands. While $C(\{Q\})=\operatorname{obj} TF$, and ${}^{\perp}Q=0$.

THEOREM 10. *If Y is a reduced group then $Y\in I'C(H)$ iff $Y\in({}^{\perp}H)^{\perp}$.*

The proof is analogous to the one above. If Y is reduced and $X\leqq Y$ then $\operatorname{Hom}(A, X)\leqq \operatorname{Hom}(A, Y)$. Consequently, if $Y\in I'C(H)$, $\operatorname{Hom}(A, X)=0$ for every $A\in C(H)$, which implies that $X\in I'C(H)$.

Generally, if Y is not divisible, write $Y=D\oplus R$ where R is reduced and nonzero. If $Y\in I'C(H)$ then $R\in({}^{\perp}H)^{\perp}$. Since $({}^{\perp}H)^{\perp}\cap {}^{\perp}H=0$, $R\notin {}^{\perp}H$. Hence, there is an $X\in H$ with $\operatorname{Hom}(R, X)\neq 0$, and consequently, $\operatorname{Hom}(Y, X)\neq 0$. In particular, if H is a set of rank 1 groups, then there is a pure subgroup K of Y of corank 1 and $X\in H$ with type $Y/K\leqq$ type X.

COROLLARY 11. *If H is a set of rank 1 groups and Y is a group with $OT(Y)\leqq$ type X for some $X\in H$, then $Y\in I'C(H)$.*

PROOF. If $\{y_1 \dots y_n\}$ is a maximally linearly independent subset of Y, let $Y_i=\langle y_1 \dots \hat{y}_i \dots y_n\rangle_*$ and $\bar{Y}_i=Y/Y_i$. Then the map $f\colon Y\to\oplus\bar{Y}_i$; defined by $f(a)=(a+Y_1)\oplus\dots\oplus(a+Y_n)$ is a monomorphism. Since type $\bar{Y}_i\leqq$ type X, we can embed, $\oplus\bar{Y}_i$ into X^n. Since Y is isomorphic to a subgroup of X^n, $Y\in I'C(H)$. □

The converse is not true as Example 13 shows.

COROLLARY 12. *Let H be a set of rank 1 groups and Y a reduced group in $I'C(H)$. If Y is a Butler group then $OT(Y)\leqq$ type $X_1\vee\dots\vee$ type X_n for some $X_1, \dots, X_n\in H$.*

PROOF. Since Y is a reduced Butler group there are subgroups $C_1, \dots, C_n$ of Q and a surjection $f\colon C_1\oplus\dots\oplus C_n\to Y$. Since each subgroup of Y is in $I'C(H)$,

$f(C_i)\in I'C(H)$. Hence, by Corollary 11, there exists $X_i\in H$ with type $X_i \geqq$ type $f(C_i)$ for $i=1, 2, \dots, n$. That means $OT(Y) \leqq OT(C_1\oplus\dots\oplus C_n)=\bigvee_{i=1}^{n}$ type $C_1 \leqq$ $\leqq \bigvee_{i=1}^{n}$ type $f(C_i) \leqq \bigvee_{i=1}^{n}$ type X_i. □

EXAMPLE 13. Let $X\leqq Q$ containing Z and having type t, where t is any type such that there is a type s with $\bar{0}<s\leqq t$, supp $s=\pi$ and $r=t+s>t$. For example, if t is a nonzero type with supp $t=\pi$, take $s=t$. □

Let B be a rank 1 group of type s and $A\geqq B$, a rank 1 group of type t with $A/B \overset{\theta}{\cong} X/Z$ (because of the conditions on s and t, such a pair exists).

If Y is any rank 2 group determined by the quintuple (A, B, X, Z, θ) then $Y\in({}^{\perp}X)^{\perp}$. To see this we note that $C\in{}^{\perp}X$ implies $C/K\in{}^{\perp}X$ for any $K\lhd C$ (generally, ${}^{\perp}H$ is closed under torsion free factors). We show that Y has no subgroups other than 0 in ${}^{\perp}X$ which will imply that Hom $(C, Y)=0$ for every $C\in{}^{\perp}X$.

If $Y\in R$, R a rank 1 subgroup of Y, write $y=(a, b)\in A\oplus X$. If $b\neq 0$, define $f\colon RX$ by $f(z, x)=x$. Since $f(y)=b\neq 0$, Hom $(R, X)\neq 0$. If $b=0$ then type $\langle y\rangle_*=$ $=s=$ type $B'\leqq t$, hence R can be imbedded in X.

If $C\leqq Y$, rank $C=2$ let $f\colon C\to(C+A)/A$ be the natural map. Let $g\colon Y/B\cong X$ and consider $fg\in$ Hom (C, X). Since C is rank 2, $fg\neq 0$.

Thus, no subgroup of Y is in ${}^{\perp}X$, and consequently. $Y\in({}^{\perp}X)^{\perp}$. But $OT(Y)=$ $=r=s+t>t$. □

A type t satisfies the hypothesis of Example 13 iff t is not idempotent. If t is idempotent, then an analogous example cannot be found.

PROPOSITION 14. *Let X be a subring of Q. If G is a reduced rank 2 group then $G\in({}^{\perp}X)^{\perp}$ iff $OT(G)\leqq$ type X.*

PROOF. Assume $G\in({}^{\perp}X)^{\perp}$. Since $G\notin{}^{\perp}X$, there is a pure subgroup B of G with G/B isomorphic to a subgroup of X.

Let $a\in G\setminus B$ and $A=\langle a\rangle_*$. Let A' and B' be rank 1 groups with $A'/A\cong B'/B$. (See the remarks following Proposition 6.) It follows that type $A'-$ type $A=$ type $B'-$ type B.

Since $G/B\cong A'$ and $A, B\in({}^{\perp}X)^{\perp}$, type A', type A, type $B\leqq$ type X. If $S=$ $=$ supp X then since type X is idempotent, type $A'-$type $A\sim_s\bar{0}$ and type $B\sim_s\bar{0}$. Consequently, type $B'\sim_s\bar{0}$ which implies that type $B'\leqq$ type X. Thus, $OT(G)=$ $=$ type $B'\vee$ type $A'\leqq$ type X.

The other implication is covered by Corollary 11. □

Let H be any class of reduced groups. For any groups A and B, A^n is a full subgroup of $A\otimes B$ for $n=$ rank B. Consequently, if $A\in{}^{\perp}H$, then $A\otimes B\in{}^{\perp}H$ for any B. In fact, every $C\in{}^{\perp}H$ is $A\otimes B$ for some A and B.

There is a natural isomorphism between Hom $(A\otimes B, Y)$ and Hom $(A$, Hom $(B, Y))$ (see [6] p. 37). If $X\in I'C(H)$, $A\in{}^{\perp}H$ and B is any group, then $0=$ $=$ Hom $(A\otimes B, X)\cong$ Hom $(A$, Hom $(B, X))$. Consequently, Hom $(B, X)\in I'C(H)$ for any $X\in I'C(H)$ and any B. But Hom (B, X) does not necessarily fill out $I'C(H)$ as B and X vary.

PROPOSITION 15. *Let X be a rank 1 group. There exists an $M\in({}^{\perp}X)$ with $M\not\cong\mathrm{Hom}(B, X)$ for any group B.*

We refer to P. Warfield's paper [8], where the following is found.

COROLLARY. *If M is any group and X is rank 1, then there is a group B with $M\cong\mathrm{Hom}(B, X)$ iff M is locally free over $R(M)$, $R(M)=R(X)$ and $OT(M)\leqq\leqq\mathrm{type}\, X$ (here, $R(M)$ is the subring of Q generated by 1 and $1/p$ for every $p\notin\mathrm{supp}\, M$).*

PROOF. *Case* 1: $\pi\setminus\mathrm{supp}\, X=\emptyset$. From Example 13, we know there is a group G with $G\in({}^{\perp}X)^{\perp}$ but $OT(G)>\mathrm{type}\, X$. Take this G as M.

Case 2: $\pi\setminus\mathrm{supp}\, X\neq\emptyset$. Let $p\in\pi\setminus\mathrm{supp}\, X$. There is no B with $\mathrm{Hom}(B, X)\cong Z$ since $R(Z)=Z\subseteqq Z_p\subseteqq R(X)$. But Z is a subgroup of X so that $Z\in({}^{\perp}X)^{\perp}$. □

Assume each group in H has no free summands. If one were so inclined, one could show (in an analogous manner) that the class $H^{\perp}$ is the class of groups U such that each $X\in H$ is projective with respect to any surjection $V\xrightarrow{\alpha}U\to 0$. Also, that for $\mathscr{E}$, the class of all exact sequences $0\to W\to V\to U\to 0$ with $U\in H^{\perp}$, ${}^{\perp}(H^{\perp})$ is the class of all groups projective with respect to each $E\in\mathscr{E}$ (modulo free summands). The following replaces Lemma 7 in the analogue. It is found in the paper [9].

LEMMA. *Let Y be a group and U and V rank 1 groups with* type $U<$type V. *There is a rank 2 group and a surjection $H\colon G\to V$ such that*

i) *G is homogeneous of type*$=$type U,
ii) *G is cohomogeneous of type*$=$type V,
iii) *given any map $f\colon Y\to V$, $hf(Y)$ is not quasi-equal to V.*

As this is not central to our theme, generating injective classes, we will refrain from further discussion of this.

4. Relations and applications. The following is a consequence of the results presented.

LEMMA 16. ${}^{\perp}H\cap P(H)=0$.

PROOF. If $A\in P(H)$ then $OT(A)\sim_{\mathrm{sup}\, X}\bar{0}$ for every $X\in H$. This implies that $OT(A)\leqq IT(X)$ for every $X\in H$. Consequently, $\mathrm{rank}\,(\mathrm{Hom}(A, X))=(\mathrm{rank}\, A)\cdot\cdot(\mathrm{rank}\, X)$ (see [8]), and thus $\mathrm{Hom}(A, X)\neq 0$ for every $X\in H$. □

Hence $\mathrm{Hom}(C, X)=0=\mathrm{Ext}(C, X)$ implies $C=0$ for any group X. (See [4] p. 225.) The dual statement $H^{\perp}\cap I(H)=0$ also holds.

By construction $H\subseteqq I'C(H)\cap IP(H)$. Moreover, let $\mathscr{E}(H)$ be the class of all exact sequences E in TF relative to which each $X\in H$ is injective. Let $\mathscr{I}(\mathscr{E}(H))$ be the class of all groups Y injective with respect to each $E\in\mathscr{E}(H)$. $\mathscr{I}(\mathscr{E}(H))$ is called the injective class generated by H. (If $\mathscr{I}(\mathscr{E}(H))$ can be determined for an arbitrary class H, Problem 46 in [4] would be answered in TF). Since $I'C(H)$ and $IP(H)$ are both injective classes (generated by H) of smaller classes of exact sequence, a direct verification shows that $\mathscr{I}(\mathscr{E}(H))\subseteqq I'C(H)\cap IP(H)$.

PROPOSITION 17. *If X is a subring of Q then $\mathscr{I}(\mathscr{E}(\{X\}))$ is the class of all $R\oplus D$ where $R\cong X^m$ and $D\cong Q^n$ for some integers m and n.*

PROOF. Let $R\in I'C(\{X\})\cap IP(\{X\})$ be a reduced group. We will show, by induction on $n=\operatorname{rank} R$, that R is quasi-isomorphic to X^n.

If $n=1$, then $\operatorname{type} R\leqq \operatorname{type} X$ by the remark preceding Corollary 11, and $\operatorname{supp} R\subseteq \operatorname{supp} X$ by Theorem 4. But $\operatorname{type} R\leqq \operatorname{type} X$ implies that $\operatorname{supp} R\geqq \operatorname{supp} X$. Hence $\operatorname{supp} R=\operatorname{supp} X$ and since X has indempotent type, $\operatorname{type} R=\operatorname{type} X$.

Both classes, $I'C(H)$ and $IP(H)$ are closed under quasi-isomorphisms. If Y is quasi-isomorphic to $C\oplus B$ then $Q\operatorname{Hom}(A, Y)\cong Q\operatorname{Hom}(A, C)\oplus Q\operatorname{Hom}(A, B)$. Also, $\operatorname{supp} Y=\operatorname{supp} C\cup \operatorname{supp} B$. Hence $Y\in I'C(H)$ $(Y\in IP(H))$ iff $C, B\in I'C(H)$ $(C, B\in IP(H))$.

Inductively, since $RI'C(\{X\})$ there is a $B\lhd R$, of corank 1 with $\operatorname{type} R/B\leqq$ $\leqq \operatorname{type} X$.

Choose $a\in R\setminus B$. Since $R\in IP(\{X\})$, so is $\langle a\rangle_*$, and therefore $\langle a\rangle_*\in I'C(\{X\})\cap$ $\cap IP(\{X\})$. By the above, $\operatorname{type}\langle a\rangle_*=\operatorname{type} X$ and therefore $\operatorname{type} X=\operatorname{type}\langle a\rangle_*=$ $=\operatorname{type} R/B$. The last equality guarantees a quasi-splitting of $0\to B\to R\to R/B\to 0$. Since $B\in I'C(\{X\})\cap IP(\{X\})$, by the inductive hypothesis R is quasi-isomorphic to X^n.

Since the inner type and outer type are quasi-isomorphism invariant $R\cong X^n$. $\square$

This proposition was originally proved in [10] using different techniques. Unfortunately, $\mathcal{I}(\mathcal{E}(\{X\}))$ is properly contained in $I'C(\{X\})\cap IP(\{X\})$ if X is a rank 1 group without idempotent type. As an example, let X be the subgroup of Q generated by $1/p$ for every $p\in\pi$. Then $Z\leqq X$ so that $Z\in I'C(\{X\})$. Also $\operatorname{supp} Z=\operatorname{supp} X$ so that $Z\in IP(\{X\})$. But let G be a rank 2 group, homogeneous of type $\bar{0}$, and cohomogeneous of type=type X. Since $OT(G)=\operatorname{type} X$, $E\colon 0\to Z\to G\to X\to 0$ is in $\mathcal{E}(\{X\})$ (Proposition 1. 7, [10]). But Z is not injective with respect to E since Z is not a summand of G.

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WESLEYAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
MIDDLETOWN, CT 06457
U.S.A.

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THE ORDER OF APPROXIMATION IN THE CENTRAL LIMIT THEOREM FOR RANDOM SUMMATION

A. KRAJKA and Z. RYCHLIK (Lublin)

1. Introduction. Let $\{X_n, n\geqq 1\}$ be a sequence of independent random variables such that $EX_n=a_n$ and $E(X_n-a_n)^2=\sigma_n^2<\infty$, $n\geqq 1$. Let us put

$$S_n=\sum_{k=1}^{n} X_k, \quad A_n=\sum_{k=1}^{n} a_k, \quad s_n^2=\sum_{k=1}^{n}\sigma_k^2, \quad n\geqq 1.$$

In various fields of applications, e.g., in random walk problems, in sequential analysis, in the theory of Markov chains, in connection with Monte Carlo Methods and in the theory of queues (cf. [6]) one has to deal with a random summation index N_n, $n\geqq 1$, instead of the constant summation index n.

Several authors (cf. e.g., [1], [12], [10], [3], [2], [5]) have investigated the asymptotic distribution S_{N_n} for $n\to\infty$. For example, from the results of [2] or [5] we can derive the following

THEOREM 1. *Let $\{X_n, n\geqq 1\}$ be a sequence of independent random variables such that $EX_n=a_n$, $\sigma^2X_n=\sigma_n^2<\infty$, $n\geqq 1$, and let $\{N_n, n\geqq 1\}$ be a sequence of positive integer-valued random variables. If there exists a sequence $\{k_n, n\geqq 1\}$ of positive integers such that $k_n\to\infty$ as $n\to\infty$, and $s_{N_n}^2/s_{k_n}^2\xrightarrow{P}\lambda$ as $n\to\infty$, for some positive random variable λ, then $(S_n-A_n)/s_n\xrightarrow{D}N(0,1)$ implies*

$$(S_{N_n}-A_{N_n})/s_{N_n}\xrightarrow{D}N(0,1) \tag{1}$$

and

$$(S_{N_n}-A_{N_n})/\lambda^{1/2}s_{k_n}\xrightarrow{D}N(0,1), \tag{2}$$

where $N(0,1)$ denotes a standard normal random variable with the distribution $\varphi(x)$.

Now the question arises whether convergence orders are also available in (1) and (2). For the case of a fixed summation index n this problem was solved by the theorem of Berry—Esséen and its generalizations. In the case when $\{X_n, n\geqq 1\}$ is a sequence of independent and identically distributed random variables the rate of convergence in (1) and (2) have been studied in [9], [8] and [4].

The purpose of this paper is to give the rate of convergence in (1) and (2) in the case when $\{X_n, n\geqq 1\}$ is a sequence of independent random variables not necessarily identically distributed. This case has been considered in [14] and [13] but under much stronger assumptions on the random variables $\{X_n, n\geqq 1\}$ as well as on the random variables $\{N_n, n\geqq 1\}$.

2. Rate of convergence. Let

$$T_n = \max\{k:\ s_k^2 \leqq \lambda s_n^2\},\quad s_0^2 = 0, \tag{3}$$

and

$$\Delta_n(h) = \sum_{k=1}^{\infty} (h_k)^{-1} I(s_{k-1}^2 < \lambda s_n^2 \leqq s_k^2), \tag{4}$$

where $h_k = L_k + B_k$ and

$$L_k = s_k^{-2} \sum_{i=1}^{k} E(X_i - a_i)^2 I(|X_i - a_i| \geqq s_k),$$

$$B_k = s_k^{-3} \sum_{i=1}^{k} E|X_i - a_i|^3 I(|X_i - a_i| < s_k).$$

THEOREM 2. *Let $\{X_n, n\geqq 1\}$ be a sequence of independent random variables such that $EX_n = a_n$, $E(X_n - a_n)^2 = \sigma_n^2 < \infty$, $n \geqq 1$, and let $\{N_n, n\geqq 1\}$ be a sequence of positive integer-valued random variables. Assume $\{\varepsilon_n, n\geqq 1\}$ is a sequence with*

$$h_n^2 \leqq \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

If there exist constants C_1 and C_2 such that

$$P(|s_{N_n}^2 / s_{T_n}^2 - 1| > C_1 \varepsilon_n) = O(\varepsilon_n^{1/2}) \tag{6}$$

and

$$P\big(\Delta_n(h) < C_2/\varepsilon_n^{1/2}\big) = O(\varepsilon_n^{1/2}), \tag{7}$$

then

$$\sup_x |P(S_{N_n} - A_{N_n} < x s_{T_n}) - \varphi(x)| = O(\varepsilon_n^{1/2}) \tag{8}$$

and

$$\sup_x |P(S_{N_n} - A_{N_n} < x s_{N_n}) - \varphi(x)| = O(\varepsilon_n^{1/2}) \tag{9}$$

provided the positive random variable λ in (3) *is independent of $\{X_n, n\geqq 1\}$.*

If, in addition, there exists a constant C_0 such that $\sigma_{k+1} s_k^{-1} h_{k+1}^{-1} \leqq C_0$, for every $k \geqq 1$, then

$$\sup_x |P(S_{N_n} - A_{N_n} < x s_n \lambda^{1/2}) - \varphi(x)| = O(\varepsilon_n^{1/2}).$$

We remark that in accordance with the result presented in [8] the assumption "λ is independent of $\{X_n, n\geqq 1\}$" can not, in general, be omitted.

Let G denote the class of functions $g(x)$ satisfying the conditions:

(a) $g(x)$ is nonnegative, even and non-descreasing on $(0, \infty)$ with $\lim_{x\to\infty} g(x) = \infty$.

(b) The function $x/g(x)$ does not decrease on $(0, \infty)$.

Let us observe that for every function $g \in G$, by (a),

$$L_k \leqq \sum_{i=1}^{k} E(X_i - a_i)^2 g(X_i - a_i) I(|X_i - a_i| \geqq s_k) / s_k^2 g(s_k),$$

and, by (b),

$$B_k \leqq \sum_{i=1}^{k} E(X_i-a_i)^2 g(X_i-a_i) I(|X_i-a_i| < s_k)/s_k^2 g(s_k).$$

Thus, for every $g \in G$, we get

$$h_k = L_k + B_k \leqq h_k^* = \sum_{i=1}^{k} E(X_i-a_i)^2 g(X_i-a_i)/s_k^2 g(s_k)$$

and $\Delta_n(h) \geqq \Delta_n(h^*)$, so that from Theorem 2 we immediately get

THEOREM 3. *Let $\{X_n, n \geqq 1\}$ be a sequence of independent random variables such that $EX_k=0$, $\sigma^2 X_k = \sigma_k^2$ and $EX_k^2 g(X_k) < \infty$, $k \geqq 1$, for some function $g \in G$, and let $\{N_n, n \geqq 1\}$ be a sequence of positive integer-valued random variables. Assume $\{\varepsilon_n^*, n \geqq 1\}$ is a sequence with*

$$h_n^* = \sum_{k=1}^{n} EX_k^2 g(X_k)/s_n^2 g(s_n) \leqq \varepsilon_n^{*1/2}, \quad n \geqq 1,$$

and $\varepsilon_n^* \to 0$ as $n \to \infty$. If (6) and (7) hold with $\{\varepsilon_n^*, n \geqq 1\}$ and $\{h_n^*, n \geqq 1\}$ instead of $\{\varepsilon_n, n \geqq 1\}$ and $\{h_n, n \geqq 1\}$, respectively, then (8) and (9) hold with $\{\varepsilon_n^*, n \geqq 1\}$.

Let us observe that if $\{X_n, n \geqq 1\}$ is a sequence of independent and identically distributed random variables such that $EX_1=0$, $EX_1^2=1$ and $E|X_1|^{2+\delta} < \infty$, for some $0 < \delta \leqq 1$, then Theorem 3 gives the results presented in [9], [8] (one dimensional case) and [4]. In this case, in Theorem 3, $g(x)=|x|^\delta$, $T_n=[n\lambda]$, $s_{N_n}^2 = N_n$,

$$h_n^* = \sum_{k=1}^{n} E|X_k|^{2+\delta}/n^{1+\delta/2} = E|X_1|^{2+\delta}/n^{\delta/2},$$

so that the assumptions of Theorem 3 reduce to those given in [9], [8] and [4].

We would also like to mention that, in general, the bounds $O(\varepsilon_n^{1/2})$ and $O(\varepsilon_n^{*1/2})$ in Theorems 2 and 3, respectively, cannot be improved neither by assuming that $P(|s_{N_n}^2/s_{T_n}^2 - 1| > C_1 \varepsilon_n) = 0$ nor by imposing that $P(|s_{N_n}^2/s_{T_n}^2 - 1| > 0) = O(\varepsilon_n^{1/2})$. This fact can be proved similarly as in [9].

Let us observe that condition (7) is always fulfilled if λ is bounded away from zero and the sequence $\{\varepsilon_n, n \geqq 1\}$ is nonincreasing and satisfies (5). Namely, if $C_2 < 1$ is a constant such that $P(\lambda^{3/2} < C_2) = 0$, then by our assumptions

$$P(\Delta_n(h) < C_2/\varepsilon_n^{1/2}) \leqq P(\lambda^{3/2} < C_2).$$

This inequality follows from the following relations. We have

$$P = P(\Delta_n(h) < C_2 \varepsilon_n^{-1/2}) = \int_0^\infty I\Big(\sum_{k=1}^{\infty} h_k^{-1} I(s_{k-1}^2 < x s_n^2 \leqq s_k^2) < C_2 \varepsilon_n^{-1/2}\Big)\, dF_\lambda(x),$$

where F_λ is the distribution function of the random variable λ. But the sequence $\{s_n^2, n \geqq 1\}$ is nondecreasing, so that for every $x > 0$ there exists exactly one $k = k(x)$

such that $s^2_{k(x)-1} < xs^2_n \leqq s^2_{k(x)}$. Hence

$$P = \int_0^\infty I(h^{-1}_{k(x)} < C_2 \varepsilon_n^{-1/2})\, dF_\lambda(x) = \int_0^\infty I(h^{-1}_{k(x)} < C_2 \varepsilon_n^{-1/2};\ s_{k(x)} \geqq s_n)\, dF_\lambda(x) +$$

$$+ \int_0^\infty I(h^{-1}_{k(x)} < C_2 \varepsilon_n^{-1/2};\ s_{k(x)} < s_n)\, dF_\lambda(x) = P_1 + P_2.$$

On the other hand $s_{k(x)} \geqq s_n$ implies $k(x) \geqq n$ and, in consequence, $\varepsilon_{k(x)} \leqq \varepsilon_n$. Thus, by (5),

$$I(\varepsilon_n^{1/2} < C_2 h_{k(x)};\ s_{k(x)} \geqq s_n) \leqq I(\varepsilon_n^{1/2} < h_{k(x)};\ s_{k(x)} \geqq s_n) \leqq$$

$$\leqq I(\varepsilon_n < \varepsilon_{k(x)};\ s_{k(x)} \geqq s_n) \leqq I(\varepsilon_n < \varepsilon_{k(x)};\ \varepsilon_{k(x)} \leqq \varepsilon_n) = 0,$$

so that $P_1 = 0$. Furthermore for the sake of simplicity we may and do assume $EX_k = 0$, $k \geqq 1$. Then, taking into account (5), we get

$$P_2 = \int_0^\infty I(\varepsilon_n^{1/2} < C_2 h_{k(x)};\ s_{k(x)} < s_n)\, dF_\lambda(x) \leqq$$

$$\leqq \int_0^\infty I\Big(\varepsilon_n^{1/2} < C_2 \Big\{ \sum_{i=1}^{k(x)} EX_i^2 I(|X_i| \geqq s_n)/s^2_{k(x)} + \Big[B_{k(x)} + \sum_{i=1}^{k(x)} EX_i^2 \cdot$$

$$\cdot I(s_{k(x)} \leqq |X_i| < s_n)/s^2_{k(x)}\Big]\Big\},\ s_{k(x)} < s_n\Big)\, dF_\lambda(x) \leqq$$

$$\leqq \int_0^\infty I\Big(\varepsilon_n^{1/2} < C_2 \Big\{L_n x^{-1} + \Big[B_{k(x)} + \sum_{i=1}^{k(x)} E|X_i|^3 I(s_{k(x)} \leqq$$

$$\leqq |X_i| < s_n)/s^3_{k(x)}\Big]\Big\},\ s_{k(x)} < s_n\Big)\, dF_\lambda(x) \leqq \int_0^\infty I\big(\varepsilon_n^{1/2} < C_2\{L_n x^{-1} + B_n x^{-3/2}\}\big)\, dF_\lambda(x) \leqq$$

$$\leqq \int_0^1 I(\varepsilon_n^{1/2} < C_2 h_n x^{-3/2})\, dF_\lambda(x) + \int_1^\infty I(\varepsilon_n^{1/2} < C_2 h_n x^{-1})\, dF_\lambda(x) \leqq$$

$$\leqq \int_0^1 I(x^{3/2} < C_2)\, dF_\lambda(x) + \int_1^\infty I(x < C_2)\, dF_\lambda(x) = P(\lambda^{3/2} < C_2) + 0.$$

3. Proof of Theorem 2. Without loss of generality we may and do assume that $EX_k = 0$, $k \geqq 1$. Let $S_{T_n} = X_1 + \ldots + X_{T_n}$, where $X_0 = 0$. Then using the fact that λ is independent of $\{X_n, n \geqq 1\}$ we get

$$|P(S_{T_n} < x s_{T_n}) - \varphi(x)| = \Big| \sum_{k=1}^\infty P(T_n = k)\big(P(S_k < x s_k) - \varphi(x)\big)\Big| \leqq \tag{10}$$

$$\leqq P\big(\Delta_n(h) < C_2/\varepsilon_n^{1/2}\big) + \sum_{k \in D_n} P(T_n = k)|P(S_k < x s_k) - \varphi(x)|.$$

Here and in what follows,

$$D_n = \{k:\ h_k \leqq \varepsilon_n^{1/2}/C_2\}.$$

But by Theorem 8 [11, p. 148], there exists an absolute constant C such that for every $k \geqq 1$

(11) $$\sup_x |P(S_k < xs_k) - \varphi(x)| \leqq Ch_k,$$

so that by (10), (7) and (11)

(12) $$\sup_x |P(S_{T_n} < xs_{T_n}) - \varphi(x)| = O(\varepsilon_n^{1/2}).$$

Let

$$I_n = \{k \geqq 1 \colon |s_k^2 - s_{T_n}^2| \leqq C_1 \varepsilon_n s_{T_n}^2\}.$$

Then, according to (6),

(13) $$P(N_n \notin I_n) = O(\varepsilon_n^{1/2}).$$

Let us put

$$A_n(x) = [\max_{k \in I_n} S_k < xs_{T_n}],\ B_n(x) = [\min_{k \in I_n} S_k < xs_{T_n}].$$

Then, by (13), we get

(14) $$P(A_n(x)) - O(\varepsilon_n^{1/2}) \leqq P(S_{N_n} < xs_{T_n}) \leqq P(B_n(x)) + O(\varepsilon_n^{1/2}).$$

On the other hand

(15) $$P(A_n(x)) \leqq P(S_{T_n} < xs_{T_n}) \leqq P(B_n(x)),$$

so that (12), (14) and (15) yield

(16) $$\sup_x |P(S_{N_n} < xs_{T_n}) - \varphi(x)| \leqq \sup_x [P(B_n(x)) - P(A_n(x))] + O(\varepsilon_n^{1/2}).$$

Thus (16) yields (8) if it is shown that

(17) $$\sup_x (P(B_n(x)) - P(A_n(x))) = O(\varepsilon_n^{1/2}).$$

But (17) is bounded from above by

(18) $$P(\Delta_n(h) < C_2/\varepsilon_n^{1/2}) + \sup_x \sum_{k \in D_n} P(T_n = k)\big(P(\min_{i \in I(k,n)} S_i < xs_k) -$$
$$- P(\max_{i \in I(k,n)} S_i < xs_k)\big),$$

where $I(k, n) = \{i \colon s_k^2(1 - C_1\varepsilon_n) \leqq s_i^2 \leqq s_k^2(1 + C_1\varepsilon_n)\}$. Furthermore, for every $p \in I(k, n)$ we have

(19) $$P(\min_{i \in I(k,n)} S_i < xs_k) - P(\max_{i \in I(k,n)} S_i < xs_k) =$$
$$= P(\min_{i \in I(k,n)} S_i < xs_k \leqq S_p) + P(S_p < xs_k \leqq \max_{i \in I(k,n)} S_i).$$

At first we prove that

(20) $$\sup_x P(S_p < xs_k \leqq \max_{i \in I(k,n)} S_i) = O(\varepsilon_n^{1/2}).$$

Let us put $p(k,n)=p=\min\{i\colon i\in I(k,n)\}$ and $q(k,n)=q=\max\{i\colon i\in I(k,n)\}$, and let P_n denote the distribution of X_n. Then using Fubini's theorem and Theorem 8 [11, p. 148] we get

$$P(S_p<xs_k\leqq\max_{p\leqq i\leqq q}S_i)=P_1*\ldots*P_q\Big\{(z_1,\ldots,z_q)\colon\ \sum_{i=1}^{p}z_i<xs_k\leqq\max_{p\leqq i\leqq q}\sum_{j=1}^{i}z_i\Big\}=$$

$$=\int P_1*\ldots*P_p\Big\{(z_1,\ldots,z_p)\colon xs_k/s_p-\max_{p\leqq j\leqq q}\sum_{i=p+1}^{j}z_i/s_p\leqq$$

$$\leqq\sum_{i=1}^{p}z_i/s_p<xs_k/s_p\Big\}\,dP_{p+1}*\ldots*P_q(z_{p+1},\ldots,z_q)\leqq$$

$$\leqq Ch_p+\int\Big|\varphi(xs_k/s_p)-\varphi\Big(xs_k/s_p-\max_{p\leqq j\leqq q}\sum_{i=p+1}^{j}z_i/s_p\Big)\Big|dP_{p+1}*\ldots*P_q(z_{p+1},\ldots,z_q)\leqq$$

$$\leqq Ch_p+E(\max_{p\leqq j\leqq q}|S_j-S_p|)/s_p.$$

Furthermore, $h_p=L_p+B_p$,

$$B_p\leqq B_k(s_k/s_p)^3\leqq B_k(1-C_1\varepsilon_n)^{-3/2},$$

and

$$L_p=s_p^{-2}\sum_{i=1}^{p}EX_i^2I(|X_i|\geqq s_k)+s_p^{-2}\sum_{i=1}^{p}EX_i^2I(s_p\leqq|X_i|<s_k)\leqq$$

$$\leqq L_k(s_k/s_p)^2+B_k(s_k/s_p)^3\leqq L_k(1-C_1\varepsilon_n)^{-1}+B_k(1-C_1\varepsilon_n)^{-3/2},$$

so that

$$h_p\leqq 2h_k(1-C_1\varepsilon_n)^{-3/2}.$$

On the other hand, by Hölder's inequality, and then by Doob's one [7 p. 15]

$$E(\max_{p\leqq j\leqq q}|S_j-S_p|)/s_p\leqq\big(E(S_q-S_p)^2\big)^{1/2}/2s_p\leqq(2C_1\varepsilon_n)^{1/2}/2(1-C_1\varepsilon_n)^{1/2}=O(\varepsilon_n^{1/2}).$$

Thus (20) is proved. Similarly we prove that $P(\min_{i\in I(k,n)}S_i<xs_k\leqq S_p)=O(\varepsilon_n^{1/2})$. In fact it is enough to replace in (20) X_i by $-X_i$. Hence (18), (19) and (20) give (17) what proves (8). On the other hand, by (6),

$$P\big(|s_{N_n}/s_{T_n}-1|>(C_1\varepsilon_n)^{1/2}\big)\leqq P(|s_{N_n}^2/s_{T_n}^2-1|>C_1\varepsilon_n)=O(\varepsilon_n^{1/2}),$$

so that (9) follows from (8) and Lemma 10 [9]. The last assertion of Theorem 2 is also a consequence of Lemma 10 [9], (7) and (8). Namely, if $\sigma_{k+1}s_k^{-1}h_{k+1}^{-1}\leqq C_0$, $k\geqq 1$, then

$$P\big(|(\lambda s_n^2/s_{T_n}^2)^{1/2}-1|>C_0C_2^{-1}\varepsilon_n^{1/2}\big)\leqq P(|\lambda s_n^2/s_{T_n}^2-1|\geqq C_0^2C_2^{-2}\varepsilon_n)=$$

$$=P(\lambda s_n^2-s_{T_n}^2\geqq C_0^2C_2^{-2}\varepsilon_n s_{T_n}^2)=$$

$$=\sum_{k=1}^{\infty}P(\lambda s_n^2\geqq s_k^2+C_0^2C_2^{-2}\varepsilon_n s_k^2,\ s_k^2\leqq\lambda s_n^2<s_{k+1}^2)\leqq$$

$$\leqq\sum_{k=1}^{\infty}P(\lambda s_n^2\geqq s_k^2+\sigma_{k+1}^2h_{k+1}^{-2}C_2^{-2}\varepsilon_n,\ s_k^2\leqq\lambda s_n^2<s_{k+1}^2)\leqq P\big(\Delta_n(h)<C_2/\varepsilon_n^{1/2}\big)=O(\varepsilon_n^{1/2}).$$

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INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA UNIVERSIT
20-031 LUBLIN, UL. NOWOTKI 10
POLAND

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К ТЕОРИИ ИНТЕРПОЛЯЦИИ С ГРАНИЧНЫМИ УСЛОВИЯМИ

Д. Л. БЕРМАН (Ленинград)

1. Обозначим через C множество всех функций, непрерывных в $[-1, 1]$. Пусть задана матрица чисел

(M) $\quad \{x_k^{(n)}\}, \quad k = 1, 2, \ldots, n, \ n = 1, 2, \ldots, \ -1 < x_n^{(n)} < x_{n-1}^{(n)} < \ldots < x_1^{(n)} < 1$

и пусть $H_n(f, x)$ — полином степени $2n-1$, однозначно определяющийся из условий

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H_n'(f, x_k^{(n)}) = 0, \quad k = 1, 2, \ldots, n. \tag{1}$$

Классическая теорема Л. Фейера [1] утверждает, что если n-я строчка матрицы (M) состоит из чисел

$$x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n, \tag{2}$$

то для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$H_n(f, x) \to f(x), \quad n \to \infty. \tag{3}$$

Рассмотрим полином $F_n(f, x)$ степени $2n+3$, который однозначно определяется из условий

$$F_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad F_n'(f, x_k^{(n)}) = 0, \quad k = 1, 2, \ldots, n$$

и из граничных условий $F_n(f, \pm 1) = f(\pm 1)$, $F_n'(f, \pm 1) = 0$. Оказалось, что процесс $\{F_n(f, x)\}$, построенный для $f(x) = |x|$ при узлах (2), расходится в точке $x = 0$, [2—3]. В [4] было доказано, что он расходится всюду в $(-1, 1)$. Такое же утверждение имеет место для $f(x) = x^2$ и для $f(x) = x$ при $x \neq 0$ [5—6]. Все эти результаты являются неожиданными, если учесть результат Л. Фейера (3).

Рассмотрим процесс $\{Q_n(f, x)\}$, где полином $Q_n(f, x)$ степени $2n+1$, однозначно определяется из «внутренних» условий

$$Q_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad Q_n'(f, x_k^{(n)}) = 0$$

и из граничных условий $Q_n(f, 1) = f(1)$, $Q_n'(f, 1) = 0$. Известно [7], что при узлах (2) процесс $\{Q_n(f, x)\}$, построенный для $f(x) = x^2$, расходится всюду в $[-1, 1)$. В связи с указанными фактами естественно поставить следующую задачу. Пусть $P_{n,q}(f, x)$ алгебраический полином степени $2n+q$, который однозначно определяется из «внутренних» условий

$$P_{n,q}(f, x_k^{(n)}) = f(x_k^{(n)}), \quad P_{n,q}'(f, x_n^{(k)}) = 0, \quad k = 1, 2, \ldots, n,$$

и из граничных условий $P_{n,q}(f,1)=f(1)$, $P^{(s)}_{n,q}(f,1)=0$, $s=1,2,\ldots,q$, где $f\in C$. Требуется определить для каких функций сходится интерполяционный процесс $\{P_{n,q}(f,x)\}$? Аналогичные задачи могут быть поставлены для левого конца отрезка $[-1,1]$ и для обоих концов отрезка $[-1,1]$. В данной статье будет рассмотрен только тот случай, когда граничные условия относятся к правому концу отрезка $[-1,1]$. Для процесса $\{Q_n(f,x)\}$ эта задача изучалась в [15] и [16], то есть она изучалась для $q=1$. Здесь будет рассмотрен случай произвольного натурального q.

2. Пусть $P_{n,i}(f,x)$ обозначает полином степени $2n+i$, однозначно определяющийся из условий

$$P_{n,i}(f,x_k^{(n)})=f(x_k^{(n)});\ P'_{n,i}(f,x_k^{(n)})=0,\quad k=1,\ldots,n; \tag{4}$$
$$P_{n,i}(f,1)=f(1);\quad P^{(j)}_{n,i}(f,1)=0,\quad j=1,2,\ldots,i.$$

Лемма 1. *Пусть полиномы $P_{n,i}$ и $P_{n,i+1}$ построены для n-ой строчки матрицы* (M). *Тогда*

(5)
$$P_{n,i+1}(f,x)-P_{n,i}(f,x)=-\frac{\omega_n^2(x)}{\omega_n^2(1)}\frac{(x-1)^{i+1}}{(i+1)!}P^{(i+1)}_{n,i}(f,1),\ \omega_n(x)=\prod_{k=1}^{n}(x-x_k^{(n)}).$$

Доказательство. Из определения полиномов $P_{n,i}$ и $P_{n,i+1}$ следует, что

$$P_{n,i+1}(f,x)-P_{n,i}(f,x)=A_1\omega_n^2(x)(x-1)^{i+1}, \tag{6}$$

где постоянное A_1 определяется из равенства

$$-P^{(i+1)}_{n,i}(f,1)=A_1\left[\omega_n^2(x)(x-1)^{i+1}\right]^{(i+1)}_{x=1}. \tag{7}$$

Очевидно, что из (6) и (7) следует (5).

Теорема 1. *Для любой $f\in C$ выполняется равенство*

(8)
$$P^{(i+1)}_{n,i}(f,1)=(i+1)!\left(\sum_{k=1}^{n}\frac{f(x_k^{(n)})-f(1)}{(x_k^{(n)}-1)^{i+1}}h_k^{(n)}(1)+(i+1)\sum_{k=1}^{n}\frac{f(x_k^{(n)})-f(1)}{(x_k^{(n)}-1)^{i+1}}[l_k^{(n)}(1)]^2\right),$$

где

$$l_k^{(n)}(x)=\frac{\omega_n(x)}{(x-x_k^{(n)})\omega'_n(x_k^{(n)})},\ h_k^{(n)}(x)=v_k^{(n)}(x)[l_k^{(n)}(x)]^2,\ \omega_n(x)=\prod_{k=1}^{n}(x-x_k^{(n)}),$$

$$v_k^{(n)}(x)=(x_k^{(n)}-x)\omega''_n(x_k^{(n)})\left(\omega'_n(x_k^{(n)})\right)^{-1}+1. \tag{9}$$

Доказательство. Найдем полином $P_{n,i}(f,x)$ по интерполяционной формуле Эрмита [8]. По этой формуле, исходя из условий (4), получим, что

$$P_{n,i}(f,x)=\sum_{k=1}^{n}\frac{A(x)}{(x-x_k)^2}f(x_k)\left\{\frac{(x-x_k)^2}{A(x)}\right\}^{(1)}_{x_k}+\omega_n^2(x)f(1)\left\{\frac{1}{\omega_n^2(x)}\right\}^{(i)}_{1}, \tag{10}$$

где $A(x)=\omega_n^2(x)(x-1)^{i+1}$ и $\{\varphi(x)\}^{(s)}_{x_i}$ есть частная сумма разложения функции $\varphi(x)$ в ряд Тейлора около точки x_i до степени $(x-x_i)^s$ включительно. После простых вычислений получим, что

$$\left\{\frac{(x-x_k)^2}{A(x)}\right\}^{(1)}_{x_k} = \frac{1}{(x_k-1)^2(\omega_n'(x_k))^2}\left(1-\frac{\omega_n''(x_k)}{\omega_n'(x_k)}(x-x_k)-\frac{i+1}{x_k-1}(x-x_k)\right).$$

Поэтому из (10) выводим, что

$$(11)\quad P_{n,i}(f,x) = \sum_{k=1}^{n}\left(\frac{x-1}{x_k-1}\right)^{i+1} f(x_k)h_k(x)+(i+1)\sum_{k=1}^{n}\left(\frac{x-1}{x_k-1}\right)^{i+1} f(x_k)\,l_k^2(x)+$$

$$+f(1)\,\omega_n^2(x)\left\{\frac{1}{\omega_n^2(x)}\right\}^{(i)}_1.$$

Продифференцируем (11) $(i+1)$ раз и положим $x=1$, тогда имеем

$$(12)\quad P_{n,i}^{(i+1)}(f,1) = (i+1)!\left(\sum_{k=1}^{n}\frac{f(x_k)}{(x_k-1)^{i+1}}h_k(1)+(i+1)\sum_{k=1}^{n}\frac{f(x_k)}{(x_k-1)^{i+1}}l_k^2(1)\right)+$$

$$+f(1)\left[\omega_n^2(x)\left\{\frac{1}{\omega_n^2(x)}\right\}^{(i)}_1\right]^{(i+1)}_{x=1}.$$

Положим в (12) $f(x)\equiv 1$. В силу единственности полинома $P_{n,i}$ имеем, что в этом случае $P_{n,i}(f,x)\equiv 1$. Поэтому $P_{n,i}^{(i+1)}(f,x)\equiv 0$. Отсюда и из (12) вытекает, что

$$(13)\quad \left[\omega_n^2(x)\left\{\frac{1}{\omega_n^2(x)}\right\}^{(i)}_1\right]^{(i+1)}_{x=1} = -(i+1)!\left(\sum_{k=1}^{n}\frac{h_k(1)}{(x_k-1)^{i+1}}+(i+1)\sum_{k=1}^{n}\frac{l_k^2(1)}{(x_k-1)^{i+1}}\right).$$

Из (12) и (13) следует (8).

Будем говорить, что матрица узлов (M) обладает свойством (F), если выполняются условия:

1) $h_k^{(n)}(1)\geqq 0,\ k=1,\dots,n,\ n=1,2,\dots;$ 2) $\lim\limits_{n\to\infty}\sum_{k=1}^{n}x_k^{(n)}h_k^{(n)}(1)=1;$

3) $\sum_{k=1}^{n}[l_k^{(n)}(1)]^2\leqq C,\ n=1,2,\dots,$

где постоянное $C>0$ не зависит от n.

Лемма 2. *Пусть матрица узлов* (M) *обладает свойством* (F) *и пусть* $f^{(i+1)}(x)$ *непрерывна в* $[-1,1]$. *Тогда, если* $f'(1)=f''(1)=\dots=f^{(i+1)}(1)=0$, *то*

$$\lim_{n\to\infty}P_{n,i}^{(i+1)}(f,1)=0,\quad i=0,1,\dots.$$

Доказательство. Согласно теореме 1 достаточно доказать, что правая часть из равенства (8) стремится к нулю, когда $n\to\infty$. Рассмотрим только вторую сумму из правой части равенства (8), ибо первая сумма из (8) рассматривается аналогично. Так как $f'(1)=f''(1)=\dots=f^{(i)}(1)=0$, то

$$f(x_k)=f(1)+\frac{f^{(i+1)}(c_k)}{(i+1)!}(x_k-1)^{i+1},\quad x_k<c_k<1.$$

Поэтому

$$q_n \equiv \sum_{k=1}^{n} \frac{f(x_k)-f(1)}{(x_k-1)^{i+1}} l_k^2(1) = \frac{1}{(i+1)!} \sum_{k=1}^{n} f^{(i+1)}(c_k) l_k^2(1).$$

По условию $f^{(i+1)}(1)=0$. Следовательно,

$$q_n = \frac{1}{(i+1)!} \sum_{k=1}^{n} \left(f^{(i+1)}(c_k) - f^{(i+1)}(1)\right) l_k^2(1).$$

Так как $f^{(i+1)}(x)$ непрерывна в $[-1. 1]$, то по $\varepsilon>0$ найдется $\delta>0$ такое, что $|f^{(i+1)}(1)-f^{(i+1)}(c_k)|<\varepsilon$, если $1-c_k<\delta$. Поэтому

$$|q_n| \leqq \frac{\varepsilon}{(i+1)!} \sum_{k=1}^{n} l_k^2(1) + \frac{2\|f^{(i+1)}\|}{(i+1)!} \sum_{1-x_k>\delta} l_k^2(1),$$

где $\|f^{(i+1)}\| = \max\limits_{-1\leqq x\leqq 1} |f^{(i+1)}(x)|$. При этом учтено, что из $x_k<c_k<1$ следует, что $1-c_k<1-x_k$. По условию теоремы $\sum\limits_{k=1}^{n} [l_k^{(n)}(1)]^2 \leqq C$, $n=1, 2, \ldots$. Поэтому выводим, что

$$|q_n| \leqq \frac{C\varepsilon}{(i+1)!} + \frac{2\|f^{(i+1)}\|}{(i+1)!} \sum_{1-x_k>\delta} l_k^2(1). \tag{14}$$

Из тождества

$$x = \sum_{k=1}^{n} x_k h_k(x) + \sum_{k=1}^{n} (x-x_k) l_k^2(x)$$

следует, что

$$\sum_{k=1}^{n} (1-x_k) l_k^2(1) = 1 - \sum_{k=1}^{n} x_k h_k(1). \tag{15}$$

Из условия 2) матрицы узлов, обладающей свойством (F) и из (15) получим, что

$$\lim_{n\to\infty} \sum_{k=1}^{n} (1-x_k^{(n)})[l_k^{(n)}(1)]^2 = 0. \tag{16}$$

Так как

$$\sum_{1-x_k>\delta} [l_k^{(n)}(1)]^2 \leqq \frac{1}{\delta} \sum_{k=1}^{n} (1-x_k^{(n)})[l_k^{(n)}(1)]^2,$$

то из (16) следует, что

$$\lim_{n\to\infty} \sum_{1-x_k^{(n)}>\delta} [l_k^{(n)}(1)]^2 = 0.$$

Отсюда и из (14) выводим, что $q_n\to 0$, $n\to\infty$.

Теперь можно доказать следуюшую теорему

Теорема 2. *Пусть матрица узлов* (M) *удовлетворяет условиям:*

1) $h_k^{(n)}(x) \geqq 0, |x| \leqq 1, \quad k = 1, \dots, n, \quad n = 1, 2, \cdots;$

2) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (x_k^{(n)})^i h_k^{(n)}(x) = x^i, \quad i = 1, 2.$

3) $\sum\limits_{k=1}^{n} [l_k^{(n)}(1)]^2 \leqq C,$

где постоянное $C > 0$ *не зависит от* n.

4) *выполняется неравенство* $|\omega_n(x)| \leqq C|\omega_n(1)|$, $|x| \leqq 1$, *где* $C > 0$ — *константа и* $\omega_n(x) = \prod\limits_{k=1}^{n} (x - x_k^{(n)})$.

Пусть $f^{(i)}(x)$ *непрерывна в* $[-1, 1]$. *Тогда, если* $f'(1) = f''(1) = \dots = f^{(i)}(1) = 0$, *то равномерно в* $[-1, 1]$ *выполняется соотношение*

$$P_{n,i}(f, x) \to f(x), \quad n \to \infty, \quad i = 0, 1, \dots.$$

Доказательство. Доказательство будем вести по индукции. Установим сперва справедливость теоремы при $i=0$ и $i=1$. Известно [9], что полином $H_n(f, x)$, который однозначно определяется из условий (1) может быть представлен в виде

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k^{(n)}) h_k^{(n)}(x),$$

где $h_k^{(n)}(x)$ определяется согласно (9). Поэтому из условия 1) теоремы 2 следует, что оператор $H_n(f, x)$ — положительный. Стало быть, в силу условия 2) теоремы 2 и равенства $\sum\limits_{k=1}^{n} h_k^{(n)}(x) \equiv 1$, согласно теореме П. П. Коровкина [10], заключаем, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$H_n(f, x) \to f(x), \quad n \to \infty. \tag{17}$$

В соответствии с принятыми нами обозначениями полином $P_{n,0}(f, x)$ степени $2n$ и однозначно определяется из условий $P_{n,0}(f, x_k) = f(x_k)$, $P'_{n,0}(f, x_k) = 0$, $k = 1, \dots, n$, $P_{n,0}(f, 1) = f(1)$. Поэтому

$$P_{n,0}(f, x) - H_n(f, x) = \frac{\omega_n^2(x)}{\omega_n^2(1)} \big(f(1) - H_n(f, 1)\big).$$

Отсюда и из условия 4) теоремы 2 выводим, что

$$|P_{n,0}(f, x) - H_n(f, x)| \leqq C|f(1) - H_n(f, 1)|, \quad |x| \leqq 1.$$

Следовательно, из (17) вытекает, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$P_{n,0}(f, x) \to f(x), \quad n \to \infty.$$

При $i=0$ равенство (5) принимает вид:

$$P_{n,1}(f,x)-P_{n,0}(f,x)=-\frac{\omega_n^2(x)}{\omega_n^2(1)}(x-1)P'_{n,0}(f,1). \tag{18}$$

Очевидно, что матрица узлов из теоремы 2 обладает свойством (F). Стало быть, применима лемма 2. То есть $\lim\limits_{k\to\infty} P'_{n,0}(f,1)=0$. Поэтому из условия 4) теоремы 2 и из (18) следует, что для любой $f\in C$ выполняется равномерно в $[-1,1]$ соотношение $P_{n,1}(f,x)\to f(x)$, $n\to\infty$. Итак, теорема 2 доказана для $i=0$ и для $i=1$.

Пусть теорема 2 справедлива при $i=m$. Докажем, что она справедлива и при $i=m+1$. Согласно лемме 1

$$P_{n,m+1}(f,x)-P_{n,m}(f,x)=-\frac{\omega_n^2(x)}{\omega_n^2(1)}\frac{(x-1)^{m+1}}{(m+1)!}P_{n,m}^{(m+1)}(f,1).$$

Применим к (5) условие 4) из теоремы 2, тогда получим, что

$$|P_{n,m+1}(f,x)-P_{n,m}(f,x)|\leqq\frac{2^{m+1}C}{(m+1)!}|P_{n,m}^{(m+1)}(f,1)|,\quad |x|\leqq 1. \tag{19}$$

По условию $f'(1)=f''(1)=\ldots=f^{(m+1)}(1)=0$. Поэтому согласно лемме 2 $\lim\limits_{n\to\infty} P_{n,m}^{(m+1)}(f,1)=0$. Отсюда и из (19) заключаем, что из равномерной сходимости в $[-1,1]$ процесса $\{P_{n,m}(f,x)\}$ следует равномерная сходимость в $[-1,1]$ процесса $\{P_{n,m+1}(f,x)\}$.

3. Пусть n-я строчка матрицы (M) состоит из корней полинома $\omega_n(x)=\prod\limits_{k=1}^{n}(x-x_k^{(n)})$. Согласно Л. Фейеру [9] матрица узлов (M) называется ϱ-нормальной, если существует такое число $\varrho>0$, что всюду в $[-1,1]$ выполняется неравенство $v_k^{(n)}(x)>\varrho>0$, $k=1,\ldots,n$, $n=1,2,\ldots$, где $v_k^{(n)}(x)$ определяется согласно (9). Л. Фейер [9] доказал, что, если матрица (M) состоит из корней полиномов Якоби $J_n^{(\alpha_n,\beta_n)}(x)$, где $-1<\alpha_n,\beta_n<-\gamma<0$, $n=1,2,\ldots$, а γ — сколь угодно малое фиксированное число, то она ϱ-нормальная.

Г. Грюнвальд [11] доказал, что при ϱ-нормальной матрице узлов (M) для любой $f\in C$ выполняется в $[-1,1]$ равномерно соотношение $H_n(f,x)\to f(x)$, $n\to\infty$. Поэтому из теоремы 2 вытекает

Теорема 3. *Пусть интерполяционный процесс* $\{P_{n,i}(f,x)\}$ *построен для ϱ-нормальной матрицы узлов* (M), *удовлетворяющей условию* 4) *из теоремы* 2. *Пусть* $f^{(i)}(x)$ *непрерывна в* $[-1,1]$ *и* $f'(1)=f''(1)=\ldots=f^{(i)}(1)=0$. *Тогда процесс* $\{P_{n,i}(f,x)\}$ *удовлетворяет равномерно в* $[-1,1]$ *соотношению*

$$P_{n,i}(f,x)\to f(x),\quad n\to\infty,\quad i=0,1,2,\ldots.$$

Теорема 3 позволяет изучить поведение интерполяционного процесса $\{P_{n,i}(f,x)\}$, построенного при ультрасферических узлах. Имеет место

Теорема 4. *Пусть n-я строчка матрицы* (M) *состоит из корней ультрасферических полиномов* $J_n^{(\alpha)}(x)$, *где*

$$-\frac{1}{2} \leqq \alpha < 0. \tag{20}$$

Пусть $f^{(i)}(x)$ *непрерывна в* $[-1, 1]$ *и* $f'(1)=f''(1)=\ldots=f^{(i)}(1)=0$. *Тогда процесс* $\{P_{n,i}(f, x)\}$ *удовлетворяет равномерно в* $[-1, 1]$ *соотношению* $P_{n,i}(f, x)\to f(x)$, $n\to\infty$, $i=0, 1, \ldots$.

Доказательство. Теорема 4 следует из теоремы 3, ибо при выполнении (20) матрица узлов (M) ϱ-нормальная (Л. Фейер [9]). Выполняется условие 3) из теоремы 2, ибо согласно [9] при (20) $-1/2<\alpha<0$, $\sum_{k=1}^{n} l_k^2(1) \leqq -\frac{1}{\alpha}$. Условие 4) из теоремы 2 также выполняется, ибо известно [12], что при $\alpha \geqq -\frac{1}{2}$,

$$|J_n^{(\alpha)}(x)| \leqq |J_n^{(\alpha)}(1)|, \quad |x| \leqq 1, \quad n = 1, 2, \ldots .$$

В связи с этой теоремой возникает вопрос о поведении процесса $\{P_{n,i}(f, x)\}$, когда неравенство (20) заменяется условием $\alpha \in (-1, \infty) \setminus \left[-\frac{1}{2}, 0\right)$. Возможно, что для решения этого вопроса будут полезны исследования J. Szabados [13] и P. Vértesi [15, 17]. В заключение выражаю благодарность рецензенту за внимание к моей работе.

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HYPERSTONEAN COVER AND SECOND DUAL EXTENSION

V. K. ZAHAROV (Leningrad)

Introduction

It is known in functional analysis that the most remarkable extension of the family $C(S)$ of all continuous functions on a compact space S is *the second dual* $C(S)''$. The second dual was extensively investigated as a vector-lattice extension of $C(S)$ ([1]—[13]) and as a ring extension of $C(S)$ allotted by Arens product ([8]—[10], [13], [14]—[20]). However the natural question, what properties distinguish the second dual extension $C(S)''$ among all other extensions of $C(S)$, had no answer.

In this connection the following problems arise: first to select some categories connected with the second dual extension, secondly to select characteristic properties of this extension within the scope of these categories and thirdly to prove that the selected properties characterize this extension in these categories.

The paper is devoted to a solution of the mentioned problems. Two new categories of extensions of the family $C(S)$ are introduced: the category of *vector-lattice extensions inheriting Lebesgue decomposition* and the category of *C-ring extensions inheriting Lebesgue decomposition.* A series of properties is introduced for objects of the given categories.

In order to prove that the chosen properties characterize the second dual extension it was required to characterize the hyperstonean cover of S, which is defined in the following way: the lattice algebra $C(S)''$ is isomorphic to a lattice algebra $C(R)$ for some compact R and the natural imbedding $u\colon C(S)\to C(S)''$ generates a surjective continuous mapping $\varrho\colon R\to S$. The space R with this mapping is called *the hyperstonean cover of S.* This cover was considered in the papers [16], [18], [4], [5], [21], [8]—[12], [22]—[29]. As far as the hyperstonean cover is defined in non-topological terms and at the same time is a topological invariant, at the IVth Prague Conference in 1976 the following problem was set up by J. Flachsmeyer ([21]): "Is there a nice topological descriptive characterization of the hyperstonean cover induced by the second functional dual of $C(S)$? This question is far from being answered." The second section of the paper is devoted to a solution of this problem. For this purpose a new topological category of *perfect preimages lifting Kelley covering* and some new topological properties of objects of this category are introduced. And within the scope of this category some descriptive topological characterization of the hyperstonean cover is given (see also [29]).

Finally with the help of this result some vector-lattice and C-ring characterizations of the second dual extension are given (see also [30]). Note that these characterizations are formulated in terms of the space S only without using measures on S and real numbers, although it has been noted in ([10], p. 234) that there is no hope for such description of $C(S)''$.

In the given paper a detailed account of the mentioned results is represented

(Theorems 1—3). The account is led at once for a general case of a completely regular space. As far as in this case the notion of the second dual and the definition of the hyperstonean cover require additional more precise definitions, they are transferred to the introductions to each section.

We shall adhere the terminology accepted in the books [31]—[37].

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§1. Basic categories and properties

1.1. Kelley ideals

Let T be a completely regular space and $\mathscr{P}(T)$ the field of all subsets of T.

1.1.1. A subset E of T will be called a *K_σ-set* if $E=\bigcup F_k$ for some sequence of compact subsets F_k. The set of all the K_σ-subsets of T will be denoted by $\mathscr{K}_\sigma(T)$.

A σ-ideal N in $\mathscr{P}(T)$ will be called *regular* (or more exactly *compactly regular*) if the following conditions are fulfilled:

a) for any $P\in N$ there exists a sequence of open sets G_k such that $P\subset\bigcap G_k\in N$;

b) for any open set G in T there exists a K_σ-set $E\subset G$ such that $G\setminus E\in N$;

c) for any K_σ-set E there exists a disjoint K_σ-set E' such that $T\setminus(E\cup E')\in N$.

1.1.2. Let $\mathscr{B}$ be some subset of $\mathscr{P}(T)$ and $\{B_p|p\leqq m\}$ a finite sequence of elements of $\mathscr{B}$. The number

$$i_N\{B_p\}\equiv\max\left\{\frac{l}{m}\,|\exists\, 1\leqq p_1<\ldots<p_l\leqq m(B_{p_1}\cap\ldots\cap B_{p_l}\notin N)\right\}$$

will be called *the intersection number of the sequence $\{B_p\}$ with respect to the ideal N.* We shall say that $\mathscr{B}$ has a *nonzero intersection number with respect to N* if $i_N\{B_p\}\geqq\frac{1}{r}$ for some natural number r and any finite sequence $\{B_p\}$ in $\mathscr{B}$.

A regular σ-ideal N will be called *Kelley ideal* if $\mathscr{K}_\sigma(T)\setminus N$ is the union of a sequence of subsets $\mathscr{K}_k$, which satisfy the following conditions:

a) if $E\in\mathscr{K}_k$ and $E'\in\mathscr{K}_\sigma(T)$ is equivalent to E with respect to N, then $E'\in\mathscr{K}_k$;

b) every $\mathscr{K}_k$ has a nonzero intersection number with respect to N;

c) if $\{E_k|k<+\infty\}$ is an increasing sequence in $\mathscr{K}_\sigma(T)\setminus N$ and $\bigcup E_k\in\mathscr{K}_m$ then $E_{k_0}\in\mathscr{K}_m$ for some k_0.

1.1.3. Let $\mathscr{B}(T)$ be the σ-field of all Borel subsets of T. Let ν be a Radon measure on T, i.e. a bounded countably additive real-valued function ν on the field $\mathscr{B}(T)$ such that $\nu B=\sup\{\nu K|K\subset B\ \&\ K$ is compact$\}$ for any Borel set B. The set of all Radon measures on T will be denoted by $M(T)$. Let $n\equiv\{\mu\in M(T)|\mu\ll\nu\ll\nu\}$ be the class of all measures coabsolutely continuous with measure ν and let $\mathscr{M}(T)$ be the set of all such classes.

LEMMA. *For any Kelley ideal N there exists a measure $\nu\in M(T)$ such that $N=\{P\in\mathscr{P}(T)|\exists B\in\mathscr{B}(T)(P\subset B\ \&\ \nu B=0)\}$. The mapping $\zeta\colon N\mapsto n$ is a bijection between $\mathscr{N}(T)$ and $\mathscr{M}(T)$.*

PROOF. Let ν be a measure, $N_0\equiv\{B\in\mathscr{B}(T)|\nu B=0\}$ the corresponding ideal and $\bar\nu$ a strictly positive σ-additive measure on the Boolean algebra $\mathscr{B}_N(T)\equiv\mathscr{B}(T)/N_0$, resp. According to Kelley's criterion (see [38] or [37], §. 42), $\mathscr{B}_N(T)\setminus\{0\}$ is a union of some sequence of subsets $\mathscr{E}_k$ satisfying the Kelley conditions:

a) any $\mathscr{E}_k$ has a nonzero intersection number, i.e.

$$i\{\bar B_p\}\equiv\max\left\{\frac{l}{m}|\exists 1\leqq p_1<\ldots<p_l\leqq m(\bar B_{p_1}\wedge\ldots\wedge\bar B_{p_l}\neq 0)\right\}\geqq\frac{1}{r}$$

for some natural number $r=r(k)$ and any finite sequence $\{\bar B_p|p\leqq m\}$ in $\mathscr{E}_k$;

b) if $\{\bar B_p|p<+\infty\}$ is an increasing sequence in $\mathscr{B}_N(T)\setminus\{0\}$ and $\sup\bar B_p\in\mathscr{E}_k$ then $\bar B_{p_0}\in\mathscr{E}_k$ for some p_0.

Consider the σ-ideal $N\equiv\{P\in\mathscr{P}(T)|\exists B\in N_0\ (P\subset B)\}$. It is clear that N is regular. Let $\mathscr{K}_k\equiv\{E\in\mathscr{K}_\sigma(T)|\bar E\in\mathscr{E}_k\}$. Then $\mathscr{K}_\sigma(T)\setminus N=\bigcup\mathscr{K}_k$ and every $\mathscr{K}_k$ satisfies the conditions from the definition of Kelley ideals. Hence N is a Kelley ideal.

Conversely, let N be a Kelley ideal. Then $\mathscr{K}_\sigma(T)\setminus N=\bigcup\mathscr{K}_k$ and $\mathscr{K}_k$ satisfy the conditions from the definition of Kelley ideals. Let $\mathscr{B}_k\equiv\{B\in\mathscr{B}(T)|\exists E\in\mathscr{K}_k\ (E\subset B\ \&\ B\setminus E\in N)\}$. Consider the set $\mathscr{M}(T,N)\equiv\{P\cup E|P\in N\ \&\ E\in\mathscr{K}_\sigma(T)\}$. Then $\mathscr{M}(T,N)$ is a σ-field. In fact, for E there is a disjoint K_σ-set E' such that $P'\equiv T\setminus(E\cup E')\in N$. Let $E'=\bigcup F_k$ and $P\subset\bigcap G_k\in N$. Consider the sets $F_{ki}\equiv F_k\setminus G_i$, $Q\equiv(P'\setminus P)\cup\cup((\bigcap G_k\setminus P)\cap E)$ and $B'\equiv\bigcup_k\bigcup_i F_{ki}\cup Q$. Then $B'\cup B=T$ and $B'\cap B=\emptyset$, i.e. B' is the complement to the set $B\equiv E\cup P$. Besides $\mathscr{M}(T,N)$ is closed under countable unions. Since $\mathscr{M}(T,N)$ contains all open sets and is a σ-field we conclude that $\mathscr{B}(T)\subset\mathscr{M}(T,N)$. Therefore for any $B\in\mathscr{B}(T)$ there exists a K_σ-set $E\subset B$ such that $B\setminus E\in N$. Consequently $\bigcup\mathscr{B}_k=\mathscr{B}(T)\setminus N$ and $\mathscr{B}_k$ contains together with each of its elements all of its classes of N-equivalence. Let $N_0\equiv N\cap\mathscr{B}(T)$, $\mathscr{B}_N(T)\equiv\equiv\mathscr{B}(T)\setminus N_0$ and $\mathscr{E}_k\equiv\{\bar B|B\in\mathscr{B}_k\}=\{\bar E|E\in\mathscr{K}_k\}$. Then $\mathscr{B}_N(T)\setminus\{0\}=\bigcup\mathscr{E}_k$. Let $\{\bar E_p\}$ be a finite sequence in $\mathscr{E}_k$. Then $\{E_p\}\subset\mathscr{K}_k$ and $i\{\bar E_p\}=i_N\{E_p\}\geqq\frac{1}{r}$. Consequently, $\mathscr{E}_k$ has a nonzero intersection number. Let $\{\bar E_p\}$ be an increasing sequence in $\mathscr{B}_N(T)\setminus N$ and $\sup\bar E_p\in\mathscr{E}_k$. Then we can choose E_p such that they are increasing. As $\overline{\bigcup E_p}\in\mathscr{E}_k$ then $\bigcup E_p\in\mathscr{K}_k$. Hence $E_{p_0}\in\mathscr{K}_k$ and therefore $\bar E_{p_0}\in\mathscr{E}_k$. Thus $\mathscr{E}_k$ satisfy the above mentioned Kelley conditions a) and b). By virtue of Kelley's criterion there exists a finite strictly positive σ-additive measure $\bar\nu$ on the Boolean algebra $\mathscr{B}_N(T)$. Carry over this measure on $\mathscr{B}(T)$ by setting $\nu B\equiv\overline{\nu B}$. Then $N_0=\{B\in\mathscr{B}(T)|\nu B=0\}$; consequently N has the form specified in the lemma. As for any $B\in\mathscr{B}(T)$ there exists a K_σ-set $E=\bigcup F_k\subset B$ such that $B\setminus E\in N_0$, then $\nu B=\nu\bigcup F_k=\sup\nu F_k$, i.e. the measure ν is compactly regular. The lemma is proved.

1.2. Category of perfect preimages lifting Kelley covering

We shall suppose that all spaces considered are completely regular and all mappings considered are perfect.

Let T and R be completely regular spaces and $\varrho: R \to T$ a surjective perfect mapping.

1.2.1. Let T_N denote the support of the Kelley ideal N, i.e. the complement to the union of all open elements from N. The covering $\{T_N | N \in \mathcal{N}(T)\}$ will be called *the Kelley covering of the space T.*

The preimage R will be called *lifting Kelley covering* if R has a family of closed subsets $\{R_N | N \in \mathcal{N}(T)\}$ such that $\bigcup R_N$ is dense in R, $\varrho R_N = T_N$ and $N_1 \subset N_2$ implies $R_{N_2} \subset R_{N_1}$. The mapping $T_N \mapsto R_N$ will be called *the lifting of the Kelley covering* and the defined preimage will be denoted by $\{R, \varrho: R \to T, T_N \mapsto R_N\}$.

A perfect mapping $\gamma: R \to \mathring{R}$ such that $\varrho = \mathring{\varrho} \circ \gamma$ and $\gamma R_N \subset \mathring{R}_N$ will be called *a mapping* (or *a morphism*) of the preimage $\{R, \varrho: R \to T, T_N \mapsto R_N\}$ into the preimage $\{\mathring{R}, \mathring{\varrho}: \mathring{R} \to T, T_N \mapsto \mathring{R}_N\}$ and will be denoted by $\{\gamma\}: \{R, \varrho: R \to T, T_N \mapsto R_N\} \to \{\mathring{R}, \mathring{\varrho}: \mathring{R} \to T, T_N \mapsto \mathring{R}_N\}$.

The preimage $\{R, \varrho: R \to T, T_N \mapsto R_N\}$ will be called *larger* than the preimage $\{\mathring{R}, \mathring{\varrho}: \mathring{R} \to T, T_N \mapsto \mathring{R}_N\}$ if there exists a mapping $\{\gamma\}$ of the first preimage into the second one such that the mapping $\gamma: R \to \mathring{R}$ is surjective and $\gamma R_N = \mathring{R}_N$.

1.2.2. Let $\{R, \varrho: R \to T, T_N \mapsto R_N\}$ be a perfect preimage of T lifting Kelley covering.

The preimage R will be called *saturated* if for any R_N and any open set G intersecting R_N there exists an R_M such that $\emptyset \neq R_M \subset R_N \cap G$ and $M \supset N$.

The preimage R will be called *σ-filled* if $\bigcup R_{N_k}$ is dense in R_N for any sequence of ideals N_k such that $\bigcap N_k = N$. Any saturated preimage is σ-filled.

The preimage R will be called *disjoined* if $G \cap R_N = \emptyset$ implies $(\mathrm{cl}\, G) \cap R_N = \emptyset$ for an open set G in R.

1.3. Category of vector-lattice extensions inheriting Lebesgue decomposition

We shall suppose that all vector lattices considered are Archimedian, have fixed strong units and are uniformly complete with respect to their units and that all vector-lattice homomorphisms considered preserve these units.

Let T be a completely regular space and $C^*(T)$ the vector lattice of all bounded continuous functions on T. Let X be a vector lattice and $u: C^*(T) \to X$ an injective vector-lattice homomorphism. We shall say that X is *an extension of* $C^*(T)$ and shall identify $C^*(T)$ with its image in X.

1.3.1. The extension X is called Dedekind complete if any subset of X, which is bounded above, has a supremum in X. An ideal Y is called a component of X if $y_\xi \in Y$, $x \in X$ and $x = \sup y_\xi$ imply $x \in Y$.

1.3.2. For any Kelley ideal $N \in \mathcal{N}(T)$ consider the ideal $C_N^*(T) \equiv \equiv \{f \in C^*(T) | f(T_N) = 0\}$ in $C^*(T)$. The family $\{C_N^*(T) | N \in \mathcal{N}(T)\}$ will be called *the Lebesgue decomposition of the vector lattice* $C^*(T)$.

The extension X of $C^*(T)$ will be called *inheriting Lebesgue decomposition* if X has a family of uniformly closed ideals $\{X_N | X\in\mathcal{N}(T)\}$ such that $\bigcap X_N=\{0\}$, $uf\in X_N$ iff $f\in C^*_N(T)$ and $N_1\subset N_2$ implies $X_{N_1}\subset X_{N_2}$. The mapping $C^*_N(T)\mapsto X_N$ will be called *the inheritance of Lebesgue decomposition* and the defined extension will be denoted by $\{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}$.

A vector-lattice homomorphism $v\colon X\to\mathring{X}$ such that $v\circ u=\mathring{u}$ and $vX_N\subset\mathring{X}_N$ will be called *a morphism of the extension* $\{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}$ *into the extension* $\{\mathring{X}, \mathring{u}\colon C^*(T)\to\mathring{X},\ C^*_N(T)\mapsto\mathring{X}_N\}$ and will be denoted by $\{v\}\colon \{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}\to\{\mathring{X}, u\colon C^*(T)\to\mathring{X},\ C^*_N(T)\mapsto\mathring{X}_N\}$.

The extension $\{\mathring{X}, \cdot\cdot\colon C^*(T)\to\mathring{X},\ C^*_N(T)\mapsto\mathring{X}_N\}$ will be called *larger* than the extension $\{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}$ if there exists a morphism $\{v\}$ from the second extension into the first one such that the homomorphism $v\colon X\to\mathring{X}$ is injective and $vx\in\mathring{X}_N$ iff $x\in X_N$.

1.3.3. Let $\{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}$ be an extension of $C^*(T)$ inheriting Lebesgue decomposition.

The extension X will be called *saturated* if for any X_N and any proper component Y such that $Y^d\equiv\{x\in X | \forall y\in Y(|x|\wedge|y|=0)\}\not\subset X_N$ there exists an X_M such that $X_N\cup\cup Y\subset X_M$ and $M\supset N$.

The extension X will be called *σ-filled* if $\bigcap X_{N_k}=X_N$ for any sequence of ideals N_k such that $\bigcap N_k=N$.

The extension X will be called *component* if every ideal X_N is a component of X.

LEMMA. *Any saturated extension X is σ-filled.*

PROOF. On the strength of Yosida's theorem ([36], § 45) there is a compact R such that the vector lattice X is isomorphic to the vector lattice $C(R)$. Consider the non-empty closed subsets $R_N\equiv\{s\in R | \forall x\in X_N(x(s)=0)\}$. Let $\bigcap N_k=N$. Then $\bigcup R_{N_k}$ is dense in R_N. In fact, assume that there exists an open set G such that $G\cap(R_N\setminus\bigcup R_{N_k})\neq\emptyset$. Take a regular closed set $F\subset G$ such that $R_N\cap\operatorname{int}F\neq\emptyset$. Consider the proper component $Y\equiv\{y\in X | y(F)=0\}$. Then there exists an $M\supset N$ such that $X_N\cup Y\subset X_M\neq X$. Therefore $R_M\subset G$.

Assume that for any k there exists a set $P_k\in M$ such that $T\setminus P_k\in N_k$. Then $P\equiv\bigcup P_k\in M$ and $T\setminus P\in N$ imply $\mathbf{1}\in C^*_M(T)$. But this is impossible because X_M is a proper ideal. Therefore there exists a number k such that $T\setminus P\notin N_k$ for any $P\in M$.

Consider the bijection $\zeta\colon N\mapsto n$ from Section 1.3. Let $n_k\equiv\zeta N_k$ and $m\equiv\zeta M$. Then $m_k\equiv n_k\wedge m\neq 0$. Take the proper ideal $M_k\equiv\zeta^{-1}m_k$. Then $\emptyset\neq R_{M_k}\subset R_{N_k}\cap R_M=\emptyset$. From this contradiction we conclude that such a set G does not exist.

Now take a $0\leqq x\in\bigcap X_{N_k}$. Then $x(R_N)=0$. Consider the functions $x_p\equiv\equiv\left(x-\frac{1}{p}\mathbf{1}\right)\vee 0$. From the property $R_N\cap\operatorname{cl}\operatorname{coz}x_p=\emptyset$ we conclude that $x_p\in X_N$. As this ideal is uniformly closed we get $x\in X_N$. The lemma is proved.

1.4. Category of C-ring extensions inheriting Lebesgue decomposition

We shall assume that all rings considered are commutative with a unit and all ring homomorphisms are unitary.

1.4.1. A ring X will be called *C-ring* if X has the following properties ([39]):
a) for any x, y there exists z such that $x^2+y^2=z^2$;
b) for any x there exist y and z such that $x=y^2-z^2$ and $yz=0$;
c) if for x and for any $n \geqq 1$ there exists $y=y(n)$ such that $n(x^2+y^2)=\mathbf{1}$ then $x=0$;
d) for any x there exists $(\mathbf{1}+x^2)^{-1}$;
e) for any x there exist y and $n \in \mathbf{N}$ such that $x^2+y^2=n\mathbf{1}$;
f) if $\{x_n\}$ is a sequence such that for any $k \geqq 1$ there exists $n_0=n_0(k)$ such that that $m, n \geqq n_0$ implies $k((x_m-x_n)^2+y^2)=\mathbf{1}$ then there exists x such that for any $k \geqq 1$ there exists $m=m(k)$ such that $k((x-x_m)^2+z^2)=\mathbf{1}$.

A ring ideal Y of the C-ring X will be called a *C-ideal* if for any sequence $\{y_n\}$ and any x such that for any $k \geqq 1$ there exists $m=m(k)$ such that $k((x-y_m)^2+z^2)=\mathbf{1}$, the condition $\{y_n\} \subset Y$ implies $x \in Y$.

The importance of the class of C-rings follows from the following.

Delfosse's theorem ([39]). *A commutative ring X with a unit is a C-ring iff X is isomorphic to a ring $C(K)$ of all continuous functions on some compact space K.*

Corollary. *With respect to an order defined by the cone $P \equiv \{x \in X \mid \exists y \in X (x=y^2)\}$ the C-ring X is a lattice ring and the ring isomorphism $X \cong C(K)$ is a lattice-ring isomorphism.*

1.4.2. Let T be a completely regular space and $C^*(T)$ the C-ring of all bounded continuous functions on T. Let X be a C-ring and $u\colon C^*(T) \to X$ an injective ring homomorphism. We shall say that X is *a C-ring extension of $C^*(T)$* and shall identify $C^*(T)$ with its image in X.

1.4.3. If Y and Z are modules over the C-ring X then the set of all module homomorphisms from Y into Z is denoted by $\mathrm{Hom}_X(Y, Z)$. Let Y and Z be ring ideals in the C-ring X. A homomorphism $g \in \mathrm{Hom}_X(Y, Z)$ will be called *bounded* if there is a natural number n such that $|gy| \leqq n|y|$ for any $y \in Y$. The subset of $\mathrm{Hom}_X(Y, Z)$ consisting of all the bounded homomorphisms will be denoted by $\mathrm{Hom}_X^*(Y, Z)$.

The annihilator and the second annihilator of a subset Y of X will be denoted as usual by Y^* and Y^{**}.

The extension X will be called *continuing* if for any ring ideal Y of the ring X and for any homomorphism $g \in \mathrm{Hom}_X^*(Y, Y^{**})$ there exists a homomorphism $h \in \mathrm{Hom}_X^*(X, Y^{**})$ extending g.

A ring ideal Z in X will be called *a segment of X* if for any ring ideal Y of X and for any pair of homomorphisms $g \in \mathrm{Hom}_X^*(Y, Y^{**})$ and $h \in \mathrm{Hom}_X^*(X, Y^{**})$ such that h extends g, the condition $gY \subset Z$ implies $hX \subset Z$.

1.4.4. For any Kelley ideal $N \in \mathcal{N}(T)$ consider the C-ideal $C_N^*(T) \equiv \{f \in C^*(T) \setminus f(T_N)=0\}$ in the C-ring $C^*(T)$. The family $\{C_N^*(T) \mid N \in \mathcal{N}(T)\}$ will be called *the Lebesgue decomposition of the C-ring $C^*(T)$*.

The extension X of $C^*(T)$ will be called *inheriting Lebesgue decomposition* if X has a family of C-ideals $\{X_N | N \in \mathcal{N}(T)\}$ such that $\bigcap X_N = \{0\}$, $uf \in X_N$ iff $f \in C_N^*(T)$ and $N_1 \subset N_2$ implies $X_{N_1} \subset X_{N_2}$. The mapping $C_N^*(T) \mapsto X_N$ will be called *the inheritance of Lebesgue decomposition* and the defined extension will be denoted by $\{X, u\colon C^*(T) \to X, C_N^*(T) \mapsto X_N\}$.

A ring homomorphism $v\colon X \to \mathring{X}$ such that $v \circ u = \mathring{u}$ and $vX_N \subset \mathring{X}_N$ will be called *a morphism of the extension* $\{X, u\colon C^*(T) \to X, C_N^*(T) \mapsto X_N\}$ *into the extension* $\{\mathring{X}, \mathring{u}\colon C^*(T) \to \mathring{X}, C_N^*(T) \mapsto \mathring{X}_N\}$ and will be denoted by $\{v\}\colon \{X, u\colon C^*(T) \to X, C_N^*(T) \mapsto X_N\} \to \{\mathring{X}, \mathring{u}\colon C^*(T) \to \mathring{X}, C_N^*(T) \mapsto \mathring{X}_N\}$.

The partial preorder on C-ring extensions of $C^*(T)$ inheriting Lebesgue decomposition is defined as in Section 1.3.2.

1.4.5. Let $\{X, u\colon C^*(T) \to X, C_N^*(T) \mapsto X_N\}$ be an extension of $C^*(T)$ inheriting Lebesgue decomposition.

A ring ideal Y in X is called an annihilator ideal if $Y = Y^{**}$.

The extension X will be called *saturated* if for any X_N and any proper annihilator ideal Y such that $Y^* \not\subset X_N$ there exists an X_M such that $X_N \cup Y \subset X_M$ and $M \supset N$.

The extension X will be called *σ-filled* if $\bigcap X_{N_k} = X_N$ for any sequence of ideals N_k such that $\bigcap N_k = N$. Any saturated extension is σ-filled.

The extension X will be called *segment* if any X_N is a segment of X.

§2. Hyperstonean cover

Let T be a completely regular space and $C^*(T)$ the vector lattice of all bounded continuous functions on T. An order bounded functional φ on $C^*(T)$ is called *tight* (or else with Prohorov property) if for any $\varepsilon > 0$ there is a compact set $K = K(\varepsilon)$ such that $|\varphi f| < \varepsilon$ for any $f \in C^*(T)$ such that $|f| \leq \chi(T \setminus K)$. Let $C^*(T)^t$ denote the vector lattice of all order bounded tight functionals on $C^*(T)$. This vector lattice is remarkable by the fact that according to Riesz—Prohorov's theorem ([34], IX, § 5, n.2) it coincides with the vector lattice $M(T)$ of all bounded Radon measures on T. Further let $C^*(T)^{t'}$ denote the vector lattice of all order bounded functionals on $C^*(T)^t$. In the case of a compact space S this coincides with the usual second dual $C(S)''$. The vector lattice $C^*(T)^{t'}$ will be called *the second tight dual of* $C^*(T)$.

The vector lattice $C^*(T)^{t'}$ is isomoprhic by some isomorphism $v\colon C^*(T)^{t'} \to C(R_0)$ to a vector lattice $C(R_0)$ for some compact space R_0 (see [15], [4]), and the canonical injection $u\colon C^*(T) \to C^*(T)^{t'}$ generates a surjective continuous mapping $\varrho_0\colon R_0 \to \beta T$ such that $vuf = f' \circ \varrho_0$, where $f' \in C(\beta T)$ is the extension of the function $f \in C^*(T)$ (see [4]). Consider the subspace $R \equiv \varrho_0^{-1} T$ and the mapping $\varrho \equiv \varrho_0 | R$. The space R with the mapping $\varrho\colon R \to T$ will be called *the hyperstonean cover of T.*

2.1. Construction and properties of hyperstonean cover

Since the hyperstonean cover R is defined in non-topological terms it is a natural question to find a purely topological construction for R. Kelley ideals give us such a possibility.

Lemma 1. *If N_1 and N_2 are Kelley ideals then $N_1 \cap N_2$ is a Kelley ideal too.*

PROOF. It is clear that $N\equiv N_1\cap N_2$ is a regular σ-ideal. Further for N_1 there exists a sequence $\mathcal{K}_k^1$ and for N_2 there exists a sequence $\mathcal{K}_k^2$ satisfying the conditions a)—c) from the definition of Kelley ideals. Denote $\mathcal{K}_k^1\cup\mathcal{K}_k^2$ by $\mathcal{K}_k$. Let $\{B_p|p\leqq m\}\subset\mathcal{K}_k$. Then $\{B_p|p\leqq m\}$ can be represented as a union of two disjoint sequences $\{B_u\in\mathcal{K}_k^1|u\leqq m_1\}$ and $\{B_v\in\mathcal{K}_k^2|v\leqq m_2\}$. Since $i_{N_1}\{B_u\}\geqq\frac{1}{r_1}$ and $i_{N_2}\{B_v\}\geqq\frac{1}{r_2}$, there exist sequences $1\leqq u_1<\ldots<u_{l_1}\leqq m_1$ and $1\leqq v_1<\ldots<v_{l_2}\leqq m_2$ such that $\frac{l_1}{m_1}\geqq\frac{1}{r_1}$, $\frac{l_2}{m_2}\geqq\frac{1}{r_2}$, $B_{u_1}\cap\ldots\cap B_{u_{l_1}}\notin N_1$ and $B_{v_1}\cap\ldots\cap B_{v_{l_2}}\notin N_2$. Then these intersections do not belong to N. Therefore

$$i_N\{B_p\}\geqq\frac{l_1\vee l_2}{m}\geqq\frac{(r_1\wedge r_2)(m_1\vee m_2)}{mr_1r_2}\geqq\frac{1}{2r_1r_2}.$$

Hence $\mathcal{K}_k$ has a non-zero intersection number with respect to N, i.e. condition b) is fulfilled. The validity of conditions a) and c) is evident. The lemma is proved.

Consider the set R_0' of all the ultrafilters from $\mathcal{N}(T)$. Associate the set R_{0_N} of all the ultrafilters not containing N with the ideal N. Consider the topology in R_0' with the open base $\{R_{0_N}|N\in\mathcal{N}(T)\}$. Denote by $\Delta_c(R_0')$ the set of all compact open subsets of R_0'.

LEMMA 2. *The space R_0' is locally compact and extremally disconnected. The mapping $N\mapsto R_{0N}$ is a bijection between $\mathcal{N}(T)$ and $\Delta_c(R_0')$, $R_{0N_1}\cup R_{0N_2}=R_{0N_1\cap N_2}$ for any N_1 and N_2 and $R_{0N_1}\cap R_{0N_2}=\emptyset$ iff there exists a $P\in N_1$ such that $T\setminus P\in N_2$.*

PROOF. The direct proof of this lemma is highly laborious. Therefore we shall adduce a roundabout proof with the help of the lemma from Section 1.1.3. As $N_2\subset N_1$ is equivalent to $n_1\leqq n_2$ then the bijection ζ is anti-isotone. Therefore $\zeta(N_1\cap N_2)=n_1\vee n_2$. Let P_0' denote the set of all maximal ideals in the lattice $\mathcal{M}(T)$. Consider in P_0' the topology with the open base $\{P_{0n}|n\in\mathcal{M}(T)\}$ where P_{0n} consists of all the maximal ideals not containing n. As the lattice $\mathcal{M}(T)$ has a least element and relative complements, by virtue of Stone's theorem (see [36], § 6—8) the mapping $n\mapsto P_{0n}$ is an isomorphism between the lattices $\mathcal{M}(T)$ and $\Delta_c(P_0')$. Therefore the space P_0' is locally compact and zerodimensional. As the lattice $\mathcal{M}(T)$ is complete the space P_0' is extremally disconnected. The bijection ζ generates a homeomorphism $\pi\colon P_0'\to R_0'$ such that $\pi P_{0n}=R_{0N}$. Consequently the space R_0' possesses the mentioned properties too. Further

$$R_{0N_1}\cup R_{0N_2}=\pi(P_{0n_1}\cup P_{0n_2})=\pi(P_{0n_1\vee n_2})=R_{0N_1\cap N_2}.$$

Let there exist a $P\in N_1$ such that $T\setminus P\in N_2$. Then $n_1\wedge n_2=0$. Therefore $R_{0N_1}\cap R_{0N_2}=\pi(P_{0n_1}\cap P_{0n_2})=\pi(P_{0n_1\wedge n_2})=\emptyset$. The lemma is proved.

Let $s\in R_0'$ and let s correspond to an ultrafilter θ_s. Consider the set $P_s\equiv\equiv\cap\{T_N|N\notin\theta_s\}$ and the subspace $R'\equiv\{s\in R_0'|P_s\neq\emptyset\}$. Let $R_N\equiv R_{0N}\cap R'$.

LEMMA 3. *For any $s\in R'$ the set P_s consists of one point only. The mapping $\varrho'\colon R'\to T$ such that $\varrho's\equiv P_s$ is surjective and continuous and $\varrho'R_N=T_N$. Moreover, R_N is dense in R_{0N}.*

PROOF. Let $s\in R_N$ and let $t_1\neq t_2$ belong to P_s. Since $N\notin\theta_s$, thus $P_s\subset T_N$.

Consider an open set G_1 such that $t_1 \in G_1$ and $t_2 \notin F_1 \equiv \mathrm{cl}\, G_1$. Then $t_2 \in G_2$ and $t_1 \notin F_2 \equiv \mathrm{cl}\, G_2$. Consider the Kelley ideals $N_1 \equiv \{P \in \mathscr{P}(T) | P \cap F_1 \in N\}$ and $N_2 \equiv \equiv \{P \in \mathscr{P}(T) | P \cap G_2 \in N\}$. As $t_1 \in G_1 \cap T_N$, thus $t_1 \notin T_{N_1}$. $T_{N_1} \subset F$ implies $t_2 \notin T_{N_1}$; similarly $t_2 \in T_{N_2}$ and $t_1 \notin T_{N_2}$. Further $T_N = T_{N_1 \cap N_2} = T_{N_1} \cup T_{N_2}$. It follows from $N \notin \theta_s$ that either $N_1 \notin \theta_s$ or $N_2 \notin \theta_s$ because of $N = N_1 \cap N_2$. But this means that the points t_1 and t_2 belong either to T_{N_1} or to T_{N_2}. From this contradiction it follows that P_s consists of one point only. Therefore the mapping ϱ' is defined correctly. Moreover, $\varrho' R_N \subset T_N$.

Check that $\varrho' R_N = T_N$. Let $t \in T_N$; consider an open set G containing t and the Kelley ideal $N_G \equiv \{P \in \mathscr{P}(T) | P \cap G \in N\}$. If $t \in G_1 \cap G_2$ then $N_{G_1 \cap G_2} \supset N_{G_1} \cup N_{G_2}$ implies $R_{{}^0N_{G_1}} \cap R_{{}^0N_{G_2}} \supset R_{{}^0N_{G_1 \cap G_2}} \neq \emptyset$. The same is valid for finite intersections. Besides $R_{{}^0N_G} \subset R_{{}^0N}$. By virtue of compactness of $R_{{}^0N}$ there exists a point $s \in \in \bigcap \{R_{{}^0N_G} | t \in G\} \subset R_{{}^0N}$. Consider the ultrafilter $\theta_s \equiv \{M \in \mathscr{N}(T) | R_{{}^0M} \notin s\}$. Assume that $M \notin \theta_s$ and $t \notin T_M$. Consider the set $G_0 \equiv T \setminus T_M$. According to the previous lemma $R_{{}^0M} \cap R_{{}^0N_{G_0}} = \emptyset$. Hence $t \in T_M$ because this intersection contains the point s. Consequently $P_s \equiv \bigcap \{T_M | M \notin \theta_s\} \ni t$. Therefore $s \in R_{{}^0N} \cap R' = R_N$, and $\varrho' s = t$.

As $\bigcup T_N = T$, the mapping ϱ' is surjective. Check now its continuity. Let $s \in R_N$ and V be a neighbourhood of the point $t \equiv \varrho' s$. Consider an open set G such that $F \equiv \mathrm{cl}\, G \subset V$ and $t \in G$. Consider the Kelley ideals $N_1 \equiv \{P \in \mathscr{P}(T) | P \cap F \in N\}$ and $N_2 \equiv \{P \in \mathscr{P}(T) | P \cap (T \setminus F) \in N\}$. Then $N = N_1 \cap N_2$ implies $R_N = R_{N_1} \cup R_{N_2}$. Let U be an arbitrary neighbourhood of t. By $t \in U \cap G \cap T_N$, $t \in T_{N_1}$. Assume that $s \notin R_{N_1}$; then $s \in R_{N_2}$ implies $t \in T_{N_2}$. But $T_{N_2} \subset \mathrm{cl}\,(T \setminus F) \not\ni t$, whence $s \in R_{N_1}$ and $\varrho' R_{N_1} = T_{N_1} \subset F \subset V$.

Check now the density of R_N in $R_{{}^0N}$. Take an arbitrary basic open set $R_{{}^0M}$ from $R_{{}^0N}$ and let $t \in T_M$. According to the property proved above there exists a point $s \in R_M$ such that $\varrho' s = t$. Therefore $R_{{}^0M} \cap R_N = R_M \neq \emptyset$. From here the required density follows. The lemma is proved.

COROLLARY 1. *The space R' is extremally disconnected.*

COROLLARY 2. *The mapping $N \mapsto R_N$ is injective, $R_{N_1} \cup R_{N_2} = R_{N_1 \cap N_2}$ for any N_1, N_2 and $R_{N_1} \cap R_{N_2} = \emptyset$ iff there exists a $P \in N_1$ such that $T \setminus P \in N_2$.*

Let $s \in R_0'$ and let s correspond to an ultrafilter θ_s from $\mathscr{N}(T)$ and consider the set $P_s' \equiv \bigcap \{\mathrm{cl}_{\beta T} T_N | N \notin \theta_s\}$.

LEMMA. 4. *For any $s \in R_0'$ the set P_s' consists of one point. The mapping ϱ_0': $R_0' \to \beta T$ such that $\varrho_0' s \equiv P_s'$ is surjective and continuous, $\varrho' = \varrho_0' | R'$, $\varrho_0' R_{{}^0N} = \mathrm{cl}_{\beta T} T_N$ and $R_N = R_{{}^0N} \cap (\varrho_0')^{-1} T_N$.*

PROOF. Let $s \in R_0'$ and assume that N_1, $N_2 \notin \theta_s$ and $\mathrm{cl}\, T_{N_1} \cap \mathrm{cl}\, T_{N_2} = \emptyset$. Then $N \equiv N_1 \cap N_2 \notin \theta_s$ implies $s \in R_{{}^0N}$. By Lemma 2, $R_{{}^0N} = R_{{}^0N_1} \cup R_{{}^0N_2}$ and $R_{{}^0N_1} \cap R_{{}^0N_2} \neq \emptyset$. Assumping $s \in R_{{}^0N_1}$ we get $N_1 \in \theta_s$ but this is false. Hence $P_s' \neq \emptyset$. In just the same way as in the previous lemma it is checked that $\varrho_0' R_{{}^0N} = \mathrm{cl}\, T_N$, consequently the mapping ϱ_0' is surjective.

Let $s \in R'$, then $\varrho_0' s = \varrho' s$. Consequently $\varrho' = \varrho_0' | R'$ and $\varrho_0' R_N = \varrho' R_N = T_N$. Let $s \in R_{{}^0N}$ and $\varrho_0' s \in T_N$. Then we have $P_s = \bigcap \{T \cap \mathrm{cl}\, T_M | M \notin \theta_s\} = T \cap \varrho_0' s = \varrho_0' s$.

Hence $s\in R'\cap R_{0N}=R_N$ and $R_N=R_{0N}\cap(\varrho_0')^{-1}T_N$. In just the same way as in the proof of the previous lemma it is checked that the mapping ϱ_0' is continuous.

Let $R_0\equiv\beta R_0'=\beta R'$ and $\varrho_0\colon R_0\to\beta T$ be the extension of the mappings ϱ_0' and ϱ'. Consider the subspace $R\equiv\varrho_0^{-1}T$ and the surjective perfect mapping $\varrho\colon R\to T$ such that $\varrho\equiv\varrho_0|R$.

LEMMA 5. *The space R is extremally disconnected, for any N the set R_N is open-closed in R, $\bigcup\{R_N|N\in\mathcal{N}(T)\}$ is dense in R and $\varrho R_N=T_N$. For any set R_N and any open-closed set U in R there exists an ideal $M\supset N$ such that $R_M=U\cap R_N$.*

PROOF. As R_0' is locally compact, R_0' is open in R_0. Therefore R_{0N} is open and compact in R_0. From the previous lemma we have

$$R_N=R_{0N}\cap(\varrho_0')^{-1}T_N=R_{0N}\cap\varrho_0^{-1}T_N=R_{0N}\cap\varrho_0^{-1}T=R_{0N}\cap R.$$

Hence R_N is open-closed in R.

Let U be an open-closed set in R. Then $V\equiv\mathrm{cl}_{R_0}U$ is open-closed in R_0. It follows from Lemma 2 that $V\cap R_{0N}=R_{0M}$ for some ideal $M\supset N$. From here $R_M=R\cap R_{0M}=R\cap V\cap R_N=U\cap R_N$. The lemma is proved.

For any Borel set B in T and any Kelley ideal N consider the ideal $N_B\equiv\{P\in\mathcal{P}(T)|P\cap B\in N\}$.

LEMMA 6. *The ideal N_B is a Kelley ideal.*

PROOF. At first we verify the regularity. Let $P\in N_B$. Then there exists a sequence of open sets G_k such that $P\cap B\subset\bigcap G_k\in N$. As it has been established in the proof of the lemma from Section 1.1.3, for B there exists a K_σ-set $\bigcup F_j\subset B$ such that $B\setminus\bigcup F_j\in N$. Then $P\subset\bigcap\bigcap(G_k\cup(P\setminus F_j))\equiv H$ and $H\cap B\subset(\bigcap G_k)\cup(P\cap B\setminus\bigcup F_j)\in N$ implies $H\in N_B$. Let G be an open set in T. As above for the Borel set $G\cap B$ there exists a K_σ-set $E\subset G\cap B$ such that $G\cap B\setminus E\in N$. Then $G\setminus E\in N_B$. Consequently conditions a) and b) from the definition of the regularity are fulfilled. The third condition is evident.

Further as N is a Kelley ideal, $\mathcal{K}_\sigma(T)\setminus N=\bigcup\mathcal{K}_k$ and each set $\mathcal{K}_k$ has a non-zero intersection number with respect to N. Consider the sets

$$\mathcal{E}_k\equiv\{E\in\mathcal{K}_\sigma(T)|E\cap\bigcup F_j\in\mathcal{K}_k\}.$$

Let $E\in\mathcal{E}_k$ and $E'\triangle E\in N_B$, then $(E'\cap\bigcup F_j)\triangle(E\cap\bigcup F_j)\in N$ implies $E'\in\mathcal{E}_k$. Further let $\{E_p\}\subset\mathcal{E}_k$, then $i_{N_B}\{E_p\}=i_N\{E_p\cap\bigcup F_j\}$ means that $\mathcal{E}_k$ has a non-zero intersection number with respect to N_B. As the third condition from the definition of Kelley ideals is obvious the lemma is proved.

Denote by $\Delta(R_N)$ the set of all open-closed subsets from the space R_N. Denote by $\mathcal{B}_N(T)$ the set of all classes of N-equivalence $\bar{B}$ of elements $B\in\mathcal{B}(T)$. Let $B_1\in\bar{B}$. Then $N_{B_1}=N_B$. Therefore we can define correctly the mapping $i_N\colon\mathcal{B}_N(T)\to\Delta(R_N)$ by setting $i_N\bar{B}\equiv R_{N_B}$.

LEMMA 7. *The mapping $i_N\colon\mathcal{B}_N(T)\to\Delta(R_N)$ is an isomorphism of Boolean algebras.*

PROOF. Let $\bar{B}\neq\bar{C}$; for example $C\setminus B\notin N$. Then $C\setminus B\in N_B$ and $C\setminus B\notin N_C$. By Lemma 2, $R_{0N_B}\neq R_{0N_C}$. By Lemma 3, $i_N\bar{B}\neq i_N\bar{C}$, thus i_N is injective.

Let U be an open-closed subset in R_N and let V be its relative complement. Then by Lemma 5 $U=R_M$, $V=R_L$ and $M\cap L\supset N$. By Corollary 2 to Lemma 3 and the regularity of the ideals, there exists a Borel set B such that $B\in L$ and $C\equiv T\setminus B\in M$. Then $R_M\subset R_{N_B}\subset R_C$ and $R_L\subset R_{N_C}\subset R_N$. Since $C\in N_B$ and $T\setminus C\in N_C$, we have $R_{N_B}\cap R_{N_C}=\emptyset$. Hence $U=R_M=R_{N_B}$, and i_N is surjective.

If $\bar{B}\leqq\bar{C}$ then $N_B\supset N_C$ implies $i_N\bar{B}\subset i_N\bar{C}$. Conversely, if $i_N\bar{B}\subset i_N\bar{C}$ then $R_{N_C}=R_{N_C}\cup R_{N_B}=R_{N_C\cap N_B}$ implies $N_C\subset N_B$. Consider the set $P\equiv B\setminus C$. Then $P\cap B\in N$ because of $P\in N_C$. Hence $\bar{B}\leqq\bar{C}$. Thus the given mapping is isotonic. From here and the bijectivity established above it follows that this mapping is an isomorphism. The lemma is proved.

COROLLARY. *The space R_N satisfies the Souslin condition, i.e. there exists in R_N at most a countable set of mutually disjoint open subsets.*

LEMMA 8. *In the space R_N, any meager set is nowhere dense.*

PROOF. Let F_k be a sequence of closed nowhere dense subsets in R_N. Assume that $\mathrm{cl}\bigcup F_k$ is not nowhere dense, i.e. there is an open-closed set U in R_N such that $U\subset\mathrm{cl}\bigcup F_k$. We may assume that $F_k\subset U$ and therefore $U=\mathrm{cl}\bigcup F_k$. According to the previous corollary, for every k there exists a sequence $\{V_{kj}\}$ of decreasing open-closed subsets in U with nowhere dense intersection, containing the set F_k. Let $V_{kj}=$ $=i_N\bar{B}_{kj}$ and $U=i_N\bar{B}$. We may assume that $B\supset B_{kj}\supset B_{kj+1}$. Take $v\in\zeta N$, then $\inf vB_{kj}=v(\bigcap B_{kj})=0$ for every k. Take some j_k such that $vB_{kj}<vB/2^{k+1}$, and consider $B_0\equiv\bigcup B_{kj}\subset B$. We have $vB_0\leqq\sum\limits_k vB_{kj_k}<vB$. Denote $B_1\equiv B\setminus B_0$, then $V\equiv i_N\bar{B}_1\neq\emptyset$. So we get $\bigcup F_k\cap V\subset\bigcup(V_{kj_k}\cap V)=\emptyset$ and $V\subset U$ but this is impossible. The lemma is proved.

COROLLARY. *The space R_N is Baire.*

Now we need a classification of Borel sets. The classification of Young [40], used usually in mathematical literature, is not suitable for us since it is valid only for such spaces whose open sets have the type F_σ. It can be checked that the following classification used for the first time in [41] is valid: $\mathscr{B}(T)=\bigcup\{\mathscr{B}_\alpha(T)|\alpha<\omega_1\}$ where $\mathscr{B}_0(T)$ consists of all open sets and

$$\mathscr{B}_\alpha(T)\equiv\{\bigcup_k(B_k\cup(T\setminus C_k))|\exists\beta_k<\alpha\exists\gamma_k<\alpha(B_k\in\mathscr{B}_{\beta_k}(T),C_k\in\mathscr{B}_{\gamma_k}(T))\}.$$

Denote by ϱ_N the restriction of ϱ on R_N.

LEMMA 9. *The set $i_N\bar{B}\Delta\varrho_N^{-1}B$ is nowhere dense in the space R_N for any Borel set B.*

PROOF. Let G be an open set in T. Check that $i_N\bar{G}=\mathrm{cl}\,\varrho_N^{-1}G$. Let $s\in\varrho_N^{-1}G\cap$ $\cap i_N(\overline{T\setminus G})$. Then $\varrho_N s\in T_{N_{T\setminus G}}\subset T\setminus G$ but this is impossible. On the other hand assume that there exists $\emptyset\neq U\equiv i_N\bar{B}$ such that $U\subset i_N\bar{G}\setminus\mathrm{cl}\,\varrho_N^{-1}G$. We can suppose that $B\subset G$. Since $B\notin N$, there exists a compact set $F\subset B$ such that $F\notin N$. Let $V\equiv i_N\bar{F}$. Then $\emptyset\neq\varrho_N V=T_{N_F}\subset F\subset G$. But $V\subset U\subset\varrho_N^{-1}(T\setminus G)$ implies $\varrho_N V\subset$ $\subset T\setminus G$. From this contradiction we conclude that the assertion of the lemma is valid for sets of class zero. By induction, using Lemma 8 it is proved that the same is valid for all other classes. The lemma is proved.

COROLLARY 1. *Let B be a Borel set. Then $B\in N$ iff $\varrho_N^{-1}B$ is nowhere dense in the space R_N.*

COROLLARY 2. *Let B be a Borel set. Then $i_N B\Delta\varrho_N^{-1}B\not\supset R_M$ for any $M\supset N$.*

LEMMA 10. *There exists a unique surjective homomorphism of the Boolean algebras $j_N\colon \mathscr{B}(R)\to\Delta(R_N)$ such that $j_N B\Delta(B\cap R_N)$ is a nowhere-dense set in the space R_N. Moreover $j_N B=\mathrm{cl\,int}\,(B\cap R_N)=\mathrm{int\,cl}\,(B\cap R_N)$.*

PROOF. Let G be an open set in R. Then $j_N G=\mathrm{cl}\,(G\cap R_N)$. By induction and using Lemma 8, j_N is defined for all other classes. The uniqueness of j_N follows from the corollary to Lemma 8.

From here and Lemma 8 it follows that $B'\equiv B\cap R_N=U\Delta F$ for the set $U\equiv j_N B$ and some nowhere dense set F. Since $U\setminus\mathrm{cl}\,F\subset\mathrm{int}\,B'\subset B'\subset\mathrm{cl}\,B'\subset U\cup\mathrm{cl}\,F$, we get $\mathrm{cl\,int}\,B'\sim\mathrm{int}\,B'\sim B'\sim U$ and $\mathrm{int\,cl}\,B'\sim\mathrm{cl}\,B'\sim B'\sim U$ relatively to the first category. Consequently $U=\mathrm{cl\,int}\,B'=\mathrm{int\,cl}\,B'$. The lemma is proved.

COROLLARY. *If $B\in\mathscr{B}(T)$ then $j_N(\varrho^{-1}B)=i_N\bar{B}$.*

Consider the ideals $N'\equiv\{P\in\mathscr{P}(R)\mid \mathrm{int\,cl}\,(P\cap R_N)=\emptyset\}$ on R for all $N\in\mathscr{N}(T)$.

LEMMA 11. *The ideal N' is a Kelley ideal.*

PROOF. Let $P\in N'$, i.e. $P\cap R_N\subset F$ for some closed nowhere dense set F from R_N. By the corollary to Lemma 7 there exists a sequence V_k of decreasing open-closed sets in R_N with a nowhere dense intersection, containing the set F. Let $G_k\equiv(R\setminus R_N)\cup \cup V_k$. Then $P\subset\bigcap G_k\in N'$. Let G be an open set in R. By what has been said above there exists in $G'\equiv G\cap R_N$ a sequence U_k of open-closed sets such that $G'\setminus\bigcup U_k$ is nowhere dense. But $U_k=i_N\bar{B}_k$ where it can be supposed that B_k is a K_σ-set. As the mapping ϱ is perfect, $E_k\equiv\varrho^{-1}B_k\cap U_k$ is a K_σ-set in R. Let $E\equiv\bigcup E_k$, then $E\subset G$ and $G\setminus E\in N'$. At last let E be a K_σ-set in R. Then there exists a sequence of open-closed sets W_k such that $\bigcup W_k$ is dense in $G\equiv R_N\setminus\mathrm{cl}\,(E\cap R_N)$. But $W_k= =i_N\bar{B}_k$ where B_k can be supposed to be a K_σ-set. Consider the K_σ-set $E'\equiv\bigcup(\varrho^{-1}B_k\cap \cap W_k)$. Then $G\setminus E'$ is a meager set. By Lemma 10 the set $\mathrm{cl}\,(E\cap R_N)\setminus(E\cap R_N)$ is the same. Therefore $R\setminus(E\cup E')\in N'$. Consequently N' is a regular ideal.

Let $\mathscr{K}_\sigma(T)\setminus N=\bigcup\mathscr{K}_k$ and consider the sets $\mathscr{E}_k\equiv(E\in\mathscr{K}_\sigma(R)\mid\exists F\in\mathscr{K}_k(F\in i_N^{-1}j_N E)\}$. Then $\mathscr{K}_\sigma(R)\setminus N'=\bigcup\mathscr{E}_k$. Let $E\Delta E'\in N'$ and $E\in\mathscr{E}_k$. Then $j_N E=j_N E'$ implies $E'\in\mathscr{E}_k$. Let $\{E_p\}$ be a finite sequence from $\mathscr{E}_k$ and $F_p\in i_N^{-1}j_N E_p$. Then the intersection number of $\{E_p\}$ with respect to N' is equal to that of $\{E_p\}$ with respect to N. Consequently each $\mathscr{E}_k$ has a non-zero intersection number with respect to N'. Further let $\{E_p\}$ be an increasing sequence from $\mathscr{K}_\sigma(R)\setminus N'$ and $\bigcup E_p\in\mathscr{E}_k$. Then the corresponding sequence $\{F_p\}$ can be chosen to be increasing. As $i_N\overline{\bigcup F_p}=\sup j_N E_p= =j_N\bigcup E_p$, $\bigcup F_p\in\mathscr{K}_k$. Therefore $F_{p_0}\in\mathscr{K}_k$ implies $E_{p_0}\in\mathscr{E}_k$. The lemma is proved.

Finally, having all the necessary apparatus we can prove that the preimage R constructed above is the hyperstonean cover of T. For that we need in addition a concrete expression of the essential upper integral by means of the corresponding measure on $\mathscr{B}(T)$. According to ([34], IX, § 3, N° 2) for a given measure $v\in M(T)$ the essential upper integral $\dot{v}$ is constructed on the set $B^*(T)$ of all bounded Borel functions on T. Also for v the abstract Lebesgue integral $\int x\,dv$ on $B^*(T)$ can be constructed ([42]) by the formula

$$\int x\,d\nu \equiv \sup\left\{\sum \inf\{x(t)\,|\,t\in B_k\}\nu B_k\,|\,B_k\in\mathscr{B}(T),\ B_k\cap B_j=\emptyset,\ \cup\{B_k|k\leqq m\}=T\right\}.$$

It can be checked that $\dot{\nu}x=\int x\,d\nu$.

PROPOSITION 1. *The space R with the mapping $\varrho\colon R\to T$ is the hyperstonean cover of T.*

PROOF. By the Riesz—Prohorov theorem, for any functional $\varphi\in C^*(T)^t$ there is a unique measure $\nu\in M(T)$ such that $\varphi f=\int f\,d\nu$ for any function $f\in C^*(T)$, and the mapping $\varphi\mapsto\nu$ is a vector-lattice isomorphism between $C^*(T)^t$ and $M(T)$ (see [34], IX, § 5, N^0 3).

Define for ν the corresponding measure ν' on R. We shall apply a method used for the first time in the paper [43]. For every $B\in\mathscr{B}(R)$ we put $\nu'B\equiv\nu(i_N^{-1}j_N B)$ where N is a Kelley ideal such that $\nu\in\zeta N$. The function ν' is a bounded Borel measure. It is clear that $\nu'\in\zeta N'$. Let $B\in\mathscr{B}(R)$. Then it follows from the proof of the lemma in Section 1.1.3 and from Lemma 11 that there exists a K_σ-set $E\subset B$ such that $B\setminus E\in N'$. Consequently $\nu'B=\nu'E$. But $E=\cup F_k$ for some increasing sequence of compact sets, therefore $\nu'B=\sup\nu'F_k$.

Let $\nu\ll\mu$ and $\mu'B=0$. Then the set $B\cap R_M$ is nowhere dense. Therefore so is the set $B\cap R_N$ because R_N is an open-closed subset in R_M. Hence $\nu'B=0$, and $\nu'\ll\mu'$.

The mapping $\nu\mapsto\nu'$ from $M(T)$ into $M(R)$ is linear and monotone. In fact let A be a Borel set on R. For μ and ν consider the measure $\lambda\equiv|\mu|+|\nu|$. Since μ, ν and $\mu+\nu$ are absolutely continuous with respect to λ, just the same is valid for their images in $M(R)$. For A and λ there exists a Borel set B on T such that $A\sim\varrho^{-1}B$ with respect to λ'. Therefore $A\sim\varrho^{-1}B$ also with respect to μ', ν' and $(\mu+\nu)'$. From here we have $(\mu+\nu)'A=(\mu+\nu)'(\varrho^{-1}B)=(\mu+\nu)B=\mu'(\varrho^{-1}B)+\nu'(\varrho^{-1}B)=\mu'A+\nu'A$. The preservation of the order is checked similarly.

The functional $\varphi'\in C^*(R)'$ corresponds to the measure ν' on R such that $\varphi'f=\int f\,d\nu'$ for any $f\in C^*(R)$. Let $f\in C^*(R)$. Consider the mapping $\varkappa\colon C^*(T)^t\to\mathbf{R}$ such that $\varkappa\varphi\equiv\varphi'f$. It follows from what has been said above that $\varkappa$ is an order bounded linear functional. Therefore we can define correctly the linear monotone mapping $w\colon C^*(R)\to C^*(T)^{t'}$ by setting $wf\equiv\varkappa$. The mapping w is isotone. In fact let $wf\leqq wg$. Then $\varphi'f\leqq\varphi'g$ for any positive tight functional φ. Assume that $f(t)>g(t)$ for all t from some open-closed set U. It follows from Lemma 5 that $\emptyset\neq R_{N_0}\subset U$ for some measure $\nu_0\in\zeta N_0$. Then $\varphi_0'(f-g)>0$ but this is impossible. Hence $f\leqq g$. Further if $wf=0$ then $\varphi'f=0$ for any φ. From the density of the union of the supports of all measures ν' (Lemmas 5 and 11) it follows that $f\triangleq 0$. Consequently the mapping w is injective.

Now we check surjectivity. Let $\varkappa\in C^*(T)^{t'}=M(T)'$. We shall use the reasoning from the proof of Proposition 27.2.2 from [35]. Let $\mu\in M(T)$ and $M_\mu\equiv\{\nu\in M(T)|\nu\ll\ll\mu\}$. Let B_μ^1 denote the vector lattice of classes of μ-equivalence $\bar{x}^\mu$ of all Borel functions x on T integrable with respect to μ, and let B_μ^* denote the vector sublattice of all bounded classes. By virtue of Radon—Nikodým theorem the mapping u_μ: $B_\mu^1\to M_\mu$ such that $(u_\mu\bar{y})B\equiv\int y\chi(B)\,d\mu$ is a vector-lattice isomoprhism. Let w_μ be an isomorphism between B_μ^* and the vector lattice of all order bounded functionals on B_μ^1 such that $(w_\mu\bar{x})\bar{y}=\int yx\,d\mu$. Consider the elements $\bar{x}_\mu\equiv w_\mu^{-1}(\varkappa\circ u_\mu)\in B_\mu^*$. Let

$\mu \ll \nu$. Then $x_\mu \sim x_\nu$ with respect to μ. In fact, take an arbitrary $\lambda \ll \mu$. Then $\lambda = u_\mu \bar{y}$ and $\lambda = u_\nu \bar{z}$. By Radon—Nikodým theorem $\int g\,d\lambda = \int gy\,d\mu$ and $\int g\,d\lambda = \int g\,z d\nu$ for any $g \in B^1_\lambda$. Therefore $\varkappa\lambda = (\varkappa \circ u_\mu)\bar{y} = (w_\mu \bar{x}_\mu)\bar{y} = \int y x_\mu\,d\mu$. On the other hand $\varkappa\lambda = (\varkappa \circ u_\nu)\bar{z} = (w_\nu \bar{x}_\nu)\bar{z} = \int z x_\nu\,d\nu = \int x_\nu\,d\lambda = \int x_\nu y\,d\mu$. Hence $\int y x_\mu\,d\mu = \int y x_\nu\,d\mu$. As we took λ arbitrarily, $\bar{y}$ runs through all B^1_μ. Therefore $x_\mu \sim x_\nu$ with respect to μ.

However for x_ν there exists a sequence of step Borel functions x_k such that $|x_\nu - x_k| \leqq \frac{1}{k}\mathbf{1}$. Let $x_k = \sum a_p \chi(B_p)$ for some finite Borel partition $\{B_p\}$, and consider the sets $U_p \equiv i_N \bar{B}_p$ and the function $f_k \equiv \sum a_p \chi(U_p) \in C^*(R)$. As $U_p \sim \varrho^{-1} B_p$, $f_k \sim x_k \circ \varrho$ with respect to ν'. The inequality $|x_k \circ \varrho - x_j \circ \varrho| \leqq \frac{2}{k}\mathbf{1}$ for all $j \geqq k$ implies $|f_k(s) - f_j(s)| \leqq \frac{2}{k}$ for any $s \in R_N$. Therefore there exists a function $f_\nu \in C^*(R)$ such that $|f_\nu - f_k| \leqq \frac{4}{k}\mathbf{1}$. It is clear that $f_\nu \sim x_\nu \circ \varrho$ with respect to ν'. Let $M \supset N$. Then $x_\nu \circ \varrho_M \sim x_\mu \sim \varrho_N \sim f_\mu | R_M$ and $x_\nu \circ \varrho_M \sim f_\nu | R_N$ with respect to the first category in R_M. Therefore $f_\mu | R_M = f_\nu | R_M$. Let M and N be arbitrary Kelley ideals. Then by Lemma 5 $R_M \cap R_N = R_L$ for some ideal $L \supset N \cup M$. Consequently $f_\mu | R_M \cap R_N = f_l | R_L = f_\nu | R_M \cap R_N$. Thus we can define correctly the function $f \in C^*(R)$ by setting $f | R_N \equiv f_\nu | R_N$. It is clear that $f \sim x_\nu \circ \varrho$ with respect to ν'.

Hence $(wf)\varphi = \int (x_\nu \circ \varrho)\,d\nu'$. But $\int (x \circ \varrho)\,d\nu' = \int x\,d\nu$ for any Borel function x. In fact for x there exists a sequence x_k of Borel step functions such that $|x - x_k| \leqq \frac{1}{k}\mathbf{1}$. By Levi's theorem ([42], 12.22) $\int x\,d\nu = \lim \int x_k\,d\nu$. But $x_k = \sum a_p \chi(B_p)$ for some finite Borel partition $\{B_p\}$. As $|x \circ \varrho - x_k \circ \varrho| \leqq \frac{1}{k}\mathbf{1}$, by Levi's theorem

$$\int (x \circ \varrho)\,d\nu' = \lim \int (x_k \circ \varrho)\,d\nu' = \lim \int \sum a_p \chi(\varrho^{-1} B_p)\,d\nu' =$$
$$= \lim \sum a_p \nu'(\varrho^{-1} B_p) = \lim \sum a_p \nu B_p = \lim \int x_k\,d\nu = \int x\,d\nu.$$

Consequently $(wf)\varphi = \int x_\nu\,d\nu = \varkappa\nu$, i.e. $wf = \varkappa$. Hence the mapping w is surjective.

Hence and from the isotony of w it follows that w is a vector lattice isomorphism. Since $R_0 = \beta R$, there is an isomorphism $\beta\colon C^*(R) \to C(R_0)$. Consider the isomorphism $v\colon C^*(T)^{t'} \to C(R_0)$ by setting $v \equiv \beta \circ w^{-1}$.

Let $f \in C^*(T)$, then $\big(w(f \circ \varrho)\big)\varrho = \int (f \circ \varrho)\,d\nu' = \int f\,d\nu = \varphi f = (uf)\varrho$, i.e. $w(f \circ \varrho) = uf$. Hence $vuf = f' \circ \varrho_0$, where f' denotes the extension of f on βT.

Thus the space R_0, the mapping $\varrho_0\colon R_0 \to \beta T$, the space $R \equiv \varrho_0^{-1} T$ and the mapping $\varrho \equiv \varrho_0 | R$ satisfy the initial definition of the hyperstonean cover. The proposition is proved.

2.2. Uniqueness of the definition of hyperstonean cover

At first sight there are several possibilities to extend the notion of hyperstonean cover from compact spaces to completely regular spaces. Now we shall show that the definition of the hyperstonean cover of a completely regular space accepted above is not only natural but also the only possible one.

Similarly to the hyperstonean cover $hS\equiv R$ of a compact space S, it would be possible to look for the "hyperstonean cover" of a completely regular T as for a continuous preimage hT of T relative to a mapping $\varrho\colon hT\to T$ such that $C^*(hT)\cong\cong C^*(T)''$. However, no such preimage exists. In fact, consider the mapping ϱ_0: $\beta hT\to\beta T$ extending the mapping ϱ. As $C(\beta hT)\cong C^*(hT)\cong C^*(T)''\cong C(\beta T)''$, $R_0\equiv\beta hT=h\beta T$. Let $s\in\beta T\setminus T$, and consider the Kelley ideal $N\equiv\{P\in\mathscr{P}(\beta T)|s\notin P\}$. Then according to Lemma 5, there exists an open-closed set R_{0N} in R_0 such that $\varrho_0R_{0N}=s$. Consequently $R_{0N}\cap hT=\emptyset$. But this contradicts the equality $R_0\equiv\beta hT$. Hence in fact a continuous preimage of T realizing the second dual $C^*(T)''$ does not exist.

Also analogously to the hyperstonean cover of a compact space, it would be possible to look for the "hyperstonean cover" of a completely regular space T as for a continuous preimage hT relative to a mapping $\varrho\colon hT\to T$ such that every order bounded functional φ on $C^*(T)$ can be extended to an order continuous functional φ' on $C^*(hT)$. However, such preimage does not exist either. In fact, consider ϱ_0: $\beta hT\to\beta T$. As $\varphi f'=\varphi f=\varphi'(f\circ\varrho)=\varphi'(f'\circ\varrho_0)$, where f' denotes the extension of the function $f\in C^*(T)$ on βT, every order bounded functional on $C(\beta T)$ extends to an order continuous functional on $C(\beta hT)$. Consider a point $s\in\beta T\setminus T$ and the point functional φ_s on $C(\beta T)$ corresponding to this point. Since the functional φ_s' is order continuous on $C(\beta hT)$, $\operatorname{supp}\varphi_s'\equiv\beta hT\setminus\bigcup\{\operatorname{coz}g|g\geqq 0,\ \varphi_s'g=0\}$ has a non-empty interior in βhT. Further for any $t\neq s$ there exists a function $g\in C(\beta T)$ such that $s\notin\operatorname{coz}g\ni t$. Then $\varphi_s'(g\circ\varrho_0)=\varphi_sg=0$ implies $\operatorname{coz}(g\circ\varrho_0)\cap\operatorname{supp}\varphi_s'=\emptyset$. Hence $\operatorname{supp}\varphi_s'\subset\varrho_0^{-1}s$. Therefore $\operatorname{supp}\varphi_s'\cap hT=\emptyset$ but this is impossible. Consequently a continuous preimage of T improving all the functionals at the same time in fact does not exist.

2.3. Characterization of hyperstonean cover

It follows from Section 2.1 that the hyperstonean cover is a perfect saturated extremally disconnected disjoined preimage of T lifting Kelley covering.

The following lemma shows an intrinsic structure of saturated preimages.

LEMMA 12. *Let R be a saturated preimage of T lifting Kelley covering. Then for any open set G from R and any Kelley ideal N there exists a K_σ-set $E=E(G,N)$ from T such that $G\Delta\varrho^{-1}E\not\supset R_M$ for any $M\supset N$.*

PROOF. We can assume $R_N\cap G\neq\emptyset$. Consider the nonempty set $\Gamma\equiv\{M\in\in\mathscr{N}(T)|R_M\subset G\cap R_N,\ M\supset N\}$ and the bijection from Section 1.1.3. Let $\zeta M=m$ and $\zeta N=n$. As $m\leqq n$ and the lattice $\mathscr{M}(T)$ is complete, there exists $n_0\equiv\equiv\sup\{m|M\in\Gamma\}$. Denote by n_1 the complement of n_0 to n and consider the ideals $N_0\equiv\zeta^{-1}n_0$ and $N_1\equiv\zeta^{-1}n_1$. There exists a Borel set $B\in N_1$ such that $T\setminus B\in N_0$. It is clear that we can suppose $B\subset T_{N_0}$. For B there exists a K_σ-set $E\subset B$ such that $B\setminus E\in N_0$. Then $T\setminus E\in N_0$ and $E\in N_1$.

Assume that there exists an $M\supset N$ such that $R_M\subset G\Delta\varrho^{-1}E$ and $R_M\cap(G\setminus\setminus\varrho^{-1}E)\neq\emptyset$. Then there exists an $M_1\supset M$ such that $R_{M_1}\subset G\cap R_M$. Therefore $M_1\in\Gamma$. As $R_{M_1}\cap\varrho^{-1}E=\emptyset$, $E\in M_1$ and $T\setminus E\in N_0$ means that $m_1\wedge n_0=0$. But this contradicts the inequality $m_1\leqq n_0$. Consequently $R_M\subset\varrho^{-1}E\setminus G$. In this case $T_M \in N_1$ and $T \setminus T_M \in M$ means that $n_1 \wedge m = 0$. But then $n = n_0 \vee n_1$ and $m \leqq n$ imply $m \leqq n_0$. Therefore there exists an $M_1 \in \Gamma$ such that $m_2 = m \wedge m_1 \neq 0$. Consequently $\emptyset \neq R_{M_2} \subset R_M \cap R_{M_1} \subset (\varrho^{-1}E \setminus G) \cap G = \emptyset$. It follows from the contradiction thus obtained that such an ideal M does not exist. The lemma is proved.

Uniqueness is understood up to isomorphism in the category of the perfect preimages of T lifting Kelley covering.

THEOREM 1. *Let R be the hyperstonean cover of T. Then*

1) *R is the unique largest among all perfect saturated preimages of T lifting Kelley covering;*

2) *R is the unique smallest among all perfect σ-filled extremally disconnected disjoined preimages of T lifting Kelley covering; moreover R is the unique universal among all such preimages;*

3) *R is the unique perfect saturated extremally disconnected disjoined preimage of T lifting Kelley covering.*

PROOF. Let $\{R, \varrho\colon R \to T,\ T_N \mapsto R_N\}$ be a preimage of T having the properties from 1).

A family $\theta \equiv \{B_N \in \mathscr{B}(T) | N \in \mathscr{N}(T)\}$ will be called *a Borel pseudoset* if $N_1 \subset N_2$ implies $B_{N_1} \Delta B_{N_2} \in N_2$. The pseudosets $\theta \equiv \{B_N\}$ and $\theta' \equiv \{B'_N\}$ will be called *equivalent* if $B_N \Delta B'_N \in N$ for any N. Define $\theta \vee \theta'$ as $\{B_N \cup B'_N\}$. The Boolean algebra of classes of equivalences $\bar\theta$ of all Borel pseudosets θ will be denoted by Ψ.

Let G be an open set from R. Then by the previous lemma for any N there exists a K_σ-set E_N such that $G \Delta \varrho^{-1} E_N \not\supset R_M$ for any $M \supset N$. Let $N_1 \subset N_2$. Assume that $E \equiv E_{N_i} \setminus E_{N_j} \notin N_2$. Then there exists a closed set $F \subset E$ also not belonging to this ideal. Consider the ideal $L \equiv \{P \in \mathscr{P}(T) | P \cap F \in N_2\} \supset N_2$. Then $R_L \subset \varrho^{-1} E_{N_i} \setminus \varrho^{-1} E_{N_j}$. If $R_L \cap G \neq \emptyset$ we shall take an $R_M \subset R_L \cap G$. Then $R_M \subset G \setminus \varrho^{-1} E_{N_j}$ but this is impossible. If $R_L \cap G = \emptyset$ then $R_L \subset \varrho^{-1} E_{N_i} \setminus G$ but this is impossible, too. Hence $E \in N_2$ and $\theta \equiv \{E_N\}$ is a Borel pseudoset.

Let for G there exist a Borel set B such that $G \Delta \varrho^{-1} B \not\supset R_M$ for all $M \supset N$. Then by analogous arguments $E_N \Delta B \in N$. Therefore we can define correctly the mapping $k\colon \mathscr{G}(R) \to \Psi$ by setting $kG \equiv \bar\theta$ where $\mathscr{G}(R)$ denotes the lattice of all open sets in R.

Now we verify that k is a lattice homomorphism. Let $kG_1 = \bar\theta_1$, $kG_2 = \bar\theta_2$, $\theta_1 = \{E^1_N\}$ and $\theta_2 = \{E^2_N\}$. Assume that $R_L \subset (G_1 \cup G_2) \Delta \varrho^{-1}(E^1_N \cup E^2_N)$ for some $L \supset N$. Further temporarily we shall omit the index N. If $R_L \cap ((G_1 \cup G_2) \setminus \varrho^{-1}(E^1 \cup E^2)) \neq \emptyset$ then there exists an $R_{L_1} \subset R_L \cap (G_1 \cup G_2)$. Let $R_{L_1} \cap G_1 \neq \emptyset$, then there exists an $R_{L_2} \subset R_{L_1} \cap G_1$. Therefore $R_{L_2} \subset G_1 \setminus \varrho^{-1} E^1$ but this is impossible. Hence $R_L \subset \subset \varrho^{-1}(E^1 \cup E^2) \setminus (G_1 \cup G_2)$. Then for example $E^1 \notin L$. Therefore there exists an $M \supset L$ such that $T_M \subset E^1$. Hence $R_M \subset \varrho^{-1} E^1 \vee G_1$ but this is impossible. Thus $k(G_1 \cup G_2) = kG_1 \vee kG_2$. Now check the preservation of intersection. Assume that $R_L \subset (G_1 \cap G_2) \Delta \varrho^{-1}(E^1_N \cap E^2_N)$. Further again we shall omit the index N temporarily. If $R_L \cap ((G_1 \cap G_2) \setminus \varrho^{-1}(E^1 \cap E^2)) \neq \emptyset$ then there exists an $R_{L_1} \subset R_L \cap (G_1 \cap G_2)$. Hence it follows that $R_{L_1} \cap \varrho^{-1} E^1 \neq \emptyset$. Assume that $E^1 \in L_1$, then there exists an ideal $L_2 \supset L_1$ such that $T_{L_2} \subset T \setminus E^1$. In this case $R_{L_2} \subset G_1 \setminus \varrho^{-1} E^1$ but this is impossible. Hence $E^1 \notin L_1$. But then similarly $T_{L_2} \subset E^1$ for some $L_2 \supset L_1$. In this case $R_{L_2} \subset \varrho^{-1} E^1 \cap R_{L_1}$. As $R_{L_1} \cap \varrho^{-1}(E^1 \cap E^2) \neq \emptyset$, we have $R_{L_2} \subset G_2 \setminus \varrho^{-1} E^2$ but this is impossible. Thus $R_L \subset \varrho^{-1}(E^1 \cap E^2) \setminus (G_1 \cap G_2)$. Hence it follows that

$R_L \cap G_1 \neq \emptyset$. Therefore $R_{L_1} \subset R_L \cap G_1$ for some $L_1 \supset L$. In this case $R_{L_1} \subset \varrho^{-1}E^2 \setminus G_2$ but this is impossible. It follows from the contradiction thus obtained that $k(G_1 \cap G_2) = kG_1 \wedge kG_2$. It is clear that k preserves the unit.

Let $G \neq \emptyset$, then $G \cap R_N \neq \emptyset$ for some ideal. Therefore there exists an $R_M \subset G \cap R_N$. In this case $R_M \cap \varrho^{-1}E_M \neq \emptyset$. Assume that $E_M \in M$, then there exists an ideal $L \supset M$ such that $T_L \subset T \setminus E_M$. Therefore $R_L \subset G \setminus \varrho^{-1}E_M$ but this is impossible. Hence $E_M \notin M$ and $kG \neq 0$. Conversely let $kG \neq 0$, then $E_N \notin N$ for some ideal. Therefore there exists an ideal $L \supset N$ such that $T_L \subset E_N$. In this case $G \cap R_N \neq \emptyset$, i.e. $G \neq \emptyset$.

Now let $\{R, \varrho\colon R \to T, T_N \mapsto R_N\}$ be a preimage of T with the properties from 2). Denote $\varrho | R_N$ by ϱ_N. Let B be a Borel set not belonging to some ideal N. Then there exist K_σ-sets $\emptyset \neq E_1 \equiv \cup F_j \subset B$ and $E_2 \equiv \cup F_k \subset T \setminus B$ such that $T \setminus (E_1 \cup E_2) \in N$. Consider the open-closed sets $U_j \equiv \operatorname{int} \varrho_N^{-1} F_j$ and $U \equiv \operatorname{cl} \cup U_j \neq \emptyset$ from R_N. Consider the ideals $M_j \equiv \{P \in \mathcal{P}(T) | P \cap F_j \in N\}$ and $M_k \equiv \{P \in \mathcal{P}(T) | P \cap F_k \in N\}$. Then it follows from the property of disjointedness that $R_j \equiv R_{M_j} \subset U_j$ and $R_k \equiv R_{M_k} \subset \subset R_N \setminus U_j$. Therefore $\cup R_j \subset U \subset R_N \setminus \cup R_k$. Assume that for B there exist other K_σ-sets $E_1' \equiv \cup F_p \subset B$ and $E_2' \equiv \cup F_q \subset T \setminus B$ such that $T \setminus (E_1' \cup E_2') \in N$. Let U', M_p, M_q, R_p and R_q be the corresponding sets. Consider the ideals $M_{jp} \equiv \{P \in \mathcal{P}(T) | P \cap (F_j \cap F_p) \in N\}$ and $M_{kq} \equiv \{P \in \mathcal{P}(T) | P \cap (F_k \cap F_q) \in N\}$. Denote the sets $R_{M_{rs}}$ by R_{rs}. Then $R_{rs} \subset R_r \cap R_s$. Therefore $R_{jp} \subset U \subset R \setminus \cup R_{kq}$ and similarly $R_{jp} \subset U' \subset \subset R \setminus \cup R_{kq}$. Hence $U \Delta U' \subset R_N \setminus (\cup R_{jp}) \cup (\cup R_{kq})$. As $(\cap M_{jp}) \cap (\cap M_{kq}) = N$ and R is σ-filled, $(\cup R_{jp}) \cup (\cup R_{kq})$ is dense in R_N. Therefore $U = U'$. Thus we get the mapping $B \mapsto U$. Let $B \to U$, $B' \mapsto U'$ and $B \Delta B' \in N$. Consider the K_σ-sets $E_1 \equiv \equiv \cup F_j \subset B \cap B'$ and $E_2 \equiv \cup F_k \subset T \setminus B \cup B'$ such that $T \setminus (E_1 \cup E_2) \in N$. Then $\cup R_j \subset U \subset R_N \setminus \cup R_k$ and $\cup R_j \subset U' \subset R_N \setminus \cup R_k$ means that $U = U'$. Thus we get the mapping $\bar{B} \mapsto U$ from $\mathcal{B}_N(T)$ into $\Delta(R_N)$. This mapping will be denoted by i_N. If $i_N \bar{B} = \emptyset$ then $\cup R_j = \emptyset$ implies $F_j \in N$. Consequently $\bar{B} = 0$.

We now check that this mapping is a Boolean homomorphism. Let $i_N \bar{B}_1 = U_1$, $i_N \bar{B}_2 = U_2$ and $i_N(\bar{B}_1 \vee \bar{B}_2) = U$. Take corresponding K_σ-sets such that $\cup F_j \subset B_1 \subset \subset T \setminus \cup F_k$ and $\cup F_p \subset B_2 \subset T \setminus \cup F_q$, and consider the sets $F_{jp} \equiv F_j \cap F_p$. Analogously, define the sets F_{jq}, F_{kp} and F_{kq}. Consider the ideals $M_{rs} \equiv \{P \in \mathcal{P}(T) | P \cap F_{rs} \in \in N\}$ for $r, s \in \{j, k, p, q\}$. Denote $R_{M_{rs}}$ by R_{rs} and $(\cup R_{jp}) \cup (\cup R_{jq}) \cup (\cup R_{kp}) \cup \cup (\cup R_{kq})$ by Q. Then by the definition of i_N we have $(\cup R_{jp}) \cup (\cup R_{jq}) \subset U_1 \subset R_N \setminus \setminus (\cup R_{kp}) \cup (\cup R_{kq})$, $(\cup R_{jp}) \cup (\cup R_{kp}) \subset U_2 \subset R_N \setminus (\cup R_{jq}) \cup (\cup R_{kq})$ and $(\cup R_{jp}) \cup \cup (\cup R_{jq}) \cup (\cup R_{kp}) \subset U \subset R_N \setminus \cup R_{kq}$. Hence we obtain $P \equiv U \Delta (U_1 \cup U_2) \subset R_N \setminus Q$. As $(\cap M_{jp}) \cap (\cap M_{jq}) \cap (\cap M_{kp}) \cap (\cap M_{kq}) = N$ then Q is dense in R_N. Therefore $P = \emptyset$. Consequently i_N preserves supremum. Evidently this mapping preserves unit and complement as well.

Let $\bar{\theta} \in \Psi$ and $\theta \equiv \{B_N\}$ and consider the open-closed sets $U_N \equiv i_N \bar{B}_N$ and $U \equiv \operatorname{cl} \cup U_N$. Define the mapping $i\colon \Psi \to \Delta(R)$ setting $i\bar{\theta} \equiv U$. Let $N \supset M$ and take K_σ-sets $E_1 \equiv \cup F_j \subset B_M$ and $E_2 \equiv \cup F_k \subset T \setminus B_M$ such that $T \setminus (E_1 \cup E_2) \in M$. Consider the ideals $M_j' \equiv \{P \in \mathcal{P}(T) | P \cap F_j \in M\}$ and $M_j'' \equiv \{P \in \mathcal{P}(T) | P \cap F_j \in N\}$. Similarly define the ideals M_k' and M_k''. Let R_j', R_k', R_j'' and R_k'' be the corresponding sets. Then $\cup R_j' \subset U_M \subset R_M \setminus \cup R_k'$. By what has been proved above $\cup R_j'' \subset U_N \subset R_N \setminus \setminus \cup R_k''$. As $R_j'' \subset R_j' \cap R_N$ and $R_k'' \subset R_k' \subset R_N$, $\cup R_j'' \subset U_M \cap R_N \subset R_N \setminus \cup R_k''$. Therefore $U_M \cap R_N = U_N$.

From this property we obtain $U_N \subset (\cup U_M) \cap R_N \subset (\cup U_{M \cap N}) \cap R_N = U_N$. Therefore $U \cap R_N = \operatorname{cl}((\cup U_M) \cap R_N) = U_N$. Hence the injectivity of i follows. Moreover, from this property and the homomorphy of i_N it follows that i is a Boolean homomorphism.

Now let $\{\mathring{R}, \mathring{\varrho}\colon \mathring{R}\to T,\ T_N\mapsto \mathring{R}_N\}$ and $\{R, \varrho\colon R\to T,\ T_N\mapsto R_N\}$ be preimages with the properties from 1) and 2), resp. Consider the mappings $k\colon \mathscr{G}(\mathring{R})\to\Psi$ and $i\colon \Psi\to\Delta(R)$ defined above. Consider the unit preserving lattice homomorphism $\alpha\colon \mathscr{G}(\mathring{R})\to\Delta(R)$ such that $\alpha\equiv i\circ k$. Let $t\in R$ and consider the sets $\Gamma\equiv\{G\in\mathscr{G}(\mathring{R})\mid t\in\in\alpha G\}$ and $P\equiv\mathring{\varrho}^{-1}\varrho t$. Assume that $P\cap \operatorname{cl} G=\emptyset$ for some $G\in\Gamma$. Then $\varrho t\in C\equiv\equiv T\setminus \operatorname{cl}\mathring{\varrho}G$ implies $t\in\varrho^{-1}C\subset\operatorname{cl}\varrho^{-1}C=iC$. Denote $\mathring{\varrho}^{-1}C$ by G_1, then $kG_1=C$. Therefore $C\wedge kG=k(G_1\cap G)=0$ and $iC\cap\alpha G=\emptyset$, but this is false. Hence $P\cap\operatorname{cl} G\neq\neq\emptyset$ for any $G\in\Gamma$. Let $G_1, G_2\in\Gamma$. Then $G_1\cap G_2\in\Gamma$ implies $P\cap\operatorname{cl} G_1\cap\operatorname{cl} G_2\neq\emptyset$. Therefore $P_t\equiv\bigcap\{P\cap\operatorname{cl} G\mid G\in\Gamma\}\neq\emptyset$ by virtue of compactness of P. Assume that there exist $s_1, s_2\in P_t$. Then there exist open sets G_1 and G_2 such that $s_1\in G_1$, $s_2\in G_2$, $s_1\notin\operatorname{cl} G_2$, $s_2\notin\operatorname{cl} G_1$ and $G_1\cup G_2=\mathring{R}$. Hence $R=\alpha G_1\cup\alpha G_2$. Assume that $t\in\alpha G_1$, then $s_2\in\operatorname{cl} G_1$ but this is false. Hence P_t consists of one point only. Therefore we can define correctly the mapping $\gamma\colon R\to\mathring{R}$ by setting $\gamma t\equiv P_t$. This mapping is continuous. In fact, let C be a neighbourhood of a point $s\equiv\gamma t$. Consider an open set G such that $s\in G\subset\operatorname{cl} G\subset C$. Then there exists an open set G_1 such that $s_1\notin\operatorname{cl} G_1$ and $G\cup G_1=\mathring{R}$. Hence $R=\alpha G\cup\alpha G_1$. If $t\in\alpha G_1$ then $s\in\operatorname{cl} G_1$ but this is false. Hence $t\in\alpha G$. Let $t_1\in\alpha G$, then $\gamma t_1\in\operatorname{cl} G\subset C$. This gives the continuity of γ because the set αG is open.

This mapping is surjective. In fact, consider a point $s\in\mathring{R}$, the set $\Gamma\equiv\equiv\{G\in\mathscr{G}(\mathring{R})\mid s\in G\}$ and the set $P\equiv\varrho^{-1}\mathring{\varrho}s$. Assume that $\alpha G\cap P=\emptyset$ for some $G\in\Gamma$. Then $\mathring{\varrho}s\in C\equiv T\setminus\varrho\alpha G$. Therefore $s\in\mathring{\varrho}^{-1}C\equiv G_1$. On the other hand $\varrho^{-1}C\cap\cap\alpha G=\emptyset$ implies $\alpha(G_1\cap G)=ikG_1\cap\alpha G=iC\cap\alpha C=\operatorname{cl}\varrho^{-1}C\cap\alpha G=\emptyset$. Hence we obtain $G_1\cap G_2=\emptyset$, but this is false. We conclude from this contradiction that there exists a point $t\in\bigcap\{\alpha G\cap P\mid G\in\Gamma\}$. Consequently $\gamma t\in\{\operatorname{cl} G\cap G\in\Gamma\}=s$.

The property $\mathring{\varrho}\circ\gamma=\varrho$ follows from the definition of γ. In turn this property implies the perfectness of γ ([31], VI, § 2, 56).

Prove that $\gamma R_N=\mathring{R}_N$. Assume that there exists an open set G such that $G\cap\mathring{R}_N\neq\emptyset$ and $\operatorname{cl} G\cap\gamma R_N=\emptyset$, and a point $t\in\alpha G\cap R_N$. Then $\gamma t\in\operatorname{cl} G\cap\gamma R_N=\emptyset$, but this is impossible. Therefore $\alpha G\cap R_N=\emptyset$. Consider an $\{E_N\}\in kG$. Then $i_N\bar{E}_N==ikG\cap R_N=\emptyset$ implies $E_N\in N$. Further there exists an ideal $M\supset N$ such that $\mathring{R}_M\subset G\cap\mathring{R}_N$. Then $E_N\in M$ means that there exists an ideal $M_1\supset M$ such that $T_{M_1}\subset T\setminus E_N$. As a result we obtain $\mathring{R}_{M_1}\subset G\setminus\mathring{\varrho}^{-1}E_N$, but this is impossible. From this contradiction we conclude that $\mathring{R}_N\subset\gamma R_N$.

Conversely, assume that there exists an open set G such that $G\cap\gamma R_N\neq\emptyset$ and $G\cap\mathring{R}_N=\emptyset$. Consider an $\{E_N\}\in kG$. Assume that $E_N\notin N$. Then there exists an ideal $M\supset N$ such that $T_M\subset E_N$. This implies $\mathring{R}_M\subset\mathring{\varrho}^{-1}E_N\setminus G$ but this is impossible. Therefore $E_N\in N$ and $ikG\cap R_N=i_N\bar{E}_N=\emptyset$. Let $t\in\gamma^{-1}G$, then there exists an open set G_1 such that $\gamma t\notin\operatorname{cl} G_1$ and $G\cup G_1=\mathring{R}$. Hence $\alpha G\cup\alpha G_1=R$. This shows that $t\in\alpha G$. Therefore $\gamma^{-1}G\subset\alpha G$. So we have obtained $G\cap\gamma R_N=\emptyset$ but this contradicts our assumption.

Thus R is larger than $\mathring{R}$. Now let R be the hyperstonean cover of T. As R has the properties from 1) and 2) simultaneously, we get that the hyperstonean cover is the largest of all preimages with the properties from 1) and the smallest of all preimages with the properties from 2). Let $\mathring{R}$ be some other largest preimage of T. Then there are mappings $\gamma: R\to\mathring{R}$ and $\delta: \mathring{R}\to R$ such that R is larger than $\mathring{R}$ relative to γ and $\mathring{R}$ is larger than R relative to δ. Let $t\in R_N$, then $t=\bigcap\{R_M\}$. This implies $\delta\gamma t\in\bigcap\{R_M\}=t$. As $\bigcup R_N$ is dense in R we conclude that $\delta\circ\gamma=\mathrm{id}$. This means that γ and δ are mutually inverse homeomorphisms and therefore the preimages R and $\mathring{R}$ are isomorphic.

The uniqueness of the smallest preimage and assertion 3) are checked in a similar manner. The theorem is proved.

Note that if a perfect preimage of T lifting Kelley covering is smaller than the hyperstonean cover then it is saturated. This assertion is a converse to the first assertion of the theorem.

Further the theorem will be used essentially for a characterization of the second dual extension $C^*(T)\subset C^*(T)^{t'}$.

§3. Second dual extension as a vector lattice

Let T be a completely regular space and $C^*(T)^{t'}$ the second tight dual to $C^*(T)$ described in the introduction to § 2. The second dual can be defined in a different way. Let β denote the strict topology on $C^*(T)$ introduced by Giles (see [44], II, 1). Let $C^*(T)^\beta$ denote the vector lattice of order bounded functionals on $C^*(T)$ continuous with respect to the topology β. Then by the Buck—Giles theorem (see [36], II, 3.3) the vector lattice $C^*(T)^\beta$ coincides with the vector lattice $M(T)$ of all bounded Radon measures on T, consequently by the Riesz—Prohorov theorem (see [34], IX, § 5, 2) it coincides with the vector lattice $C^*(T)^t$. The vector lattice $C^*(T)^{\beta'}$ of all order bounded functionals on $C^*(T)^\beta$ will be called *the second strict dual to* $C^*(T)$. From the coincidence of $C^*(T)^t$ and $C^*(T)^\beta$ it follows that the second tight dual and the second strict dual coincide. Therefore we shall not distinguish them further and shall call simply *the second dual to* $C^*(T)$.

Let $u: C^*(T)\to C^*(T)^{t'}$ be the canonical imbedding, and $\zeta: N\mapsto n$ the bijection from Section 1.1.3. Let $\nu\in n=\zeta N$ and φ_ν be the functional on $C^*(T)$ corresponding to ν. Consider the vector-lattice ideal $C^*(T)^{t'}_N\equiv\{\varkappa\in C^*(T)^{t'}|\forall\nu\in n(|\varkappa||\varphi_\nu|=0)\}$ in $C^*(T)^{t'}$. Then $\{C^*(T)^{t'},\ u: C^*(T)\to C^*(T)^{t'},\ C^*_N(T)\mapsto C^*(T)^{t'}_N\}$ is an extension of $C^*(T)$ inheriting Lebesgue decomposition. This extension will be called *the second dual extension of* $C^*(T)$.

3.1. Functional description of the second dual extension by functions on hyperstonean cover

Let R be the hyperstonean cover of T and $\varrho: R\to T$ the canonical mapping. Consider the injective vector-lattice homomorphism $\varphi: C^*(T)\to C^*(R)$ such that $\varphi f\equiv f\circ\varrho$. For a Kelley ideal N consider the ideal $C^*_N(R)\equiv\{f\in C^*(R)|f(R_N)=0\}$. Then $\{C^*(R),\ \varphi: C^*(T)\to C^*(R),\ C^*_N(T)\mapsto C^*_N(R)\}$ is a vector-lattice extension of $C^*(T)$ inheriting Lebesgue decomposition.

PROPOSITION 2. *The extensions $C^*(T)^{t'}$ and $C^*(R)$ are isomorphic saturated Dedekind complete component extensions of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Denote $C^*(T)^{t'}$ by X, $C^*(T)^{t'}_N$ by X_N, $C^*(R)$ by Φ and $C^*_N(R)$ by Φ_N. In the proof of Proposition 1 it has been established that there exists a vector-lattice isomorphism $w\colon \Phi\to X$ such that $\mathring{\varphi}_\nu f=\varphi'_\nu(wf)$ and $w\circ\varphi=u$, where $\mathring{\varphi}_\nu$ and φ'_ν are the functionals on Φ and X, respectively, extending the functional φ_ν. Let $f\in\Phi_N$, then $R_N=\operatorname{supp}\nu'$ for any $\nu\in n=\zeta N$, where ν' denotes the measure on R defined in the proof of Proposition 1. Therefore $(wf)(\varphi_\nu)=\varphi'_\nu(wf)=\mathring{\varphi}_\nu f=\int f\,d\nu'=0$ for any $\nu\in n$. Hence $wf\in X_N$. Conversely, consider the vector-lattice isomorphism $v\equiv w^{-1}$. Let $0<\varkappa\in X_N$, then $\int v\varkappa\,d\nu'=\mathring{\varphi}_\nu(v\varkappa)=\varphi'_\nu(wv\varkappa)=\varkappa\varphi_\nu=0$ for $\nu'>0$. As $\operatorname{supp}\nu'=R_N$, $(v\varkappa)(R_N)=0$ and $v\varkappa\in X_N$ for any $\varkappa>0$ and hence for any $\varkappa$. Therefore the given extensions are isomorphic.

Let Y be a proper component of Φ such that $Y^d\not\subset\Phi_N$. Consider the non-empty set $P\equiv\{s\in R\mid\forall g\in Y(g(s)=0)\}$. Then $Y=\{f\in\Phi\mid f(P)=0\}$. By assumption $P\cap R_N=\emptyset$ from the inclusion $\bigcup\{\operatorname{coz} g\mid g\in Y^d\}\subset P$ we obtain $Y^d\subset\Phi_N$ but this is false. Consequently $P\cap R_N\neq\emptyset$. As P is a regular closed set, $\operatorname{int}P\cap R_N\neq\emptyset$. Therefore there exists a proper ideal $M\supset N$ such that $R_M\subset\operatorname{int}P\cap R_N$. Hence we obtain $\Phi_N\cup Y\subset\Phi_M$. This means that Φ is saturated.

Since the space R is extremally disconnected and the sets R_N are open-closed, Φ is Dedekind complete and component. The proposition is proved.

3.2. Characterization of the second dual extension as a vector lattice

In what follows uniqueness is understood up to isomorphism in the category of the vector-lattice extensions of $C^*(T)$ inheriting Lebesgue decomposition.

THEOREM 2. 1) *$C^*(T)^{t'}$ is the unique largest of all saturated extensions of $C^*(T)$ inheriting Lebesgue decomposition;*

2) *$C^*(T)^{t''}$ is the unique smallest of all σ-filled Dedekind complete component extensions of $C^*(T)$ inheriting Lebesgue decomposition; moreover $C^*(T)^{t''}$ is the unique universal among all such extensions;*

3) *$C^*(T)^{t''}$ is the unique saturated Dedekind complete component extension of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Let $\{X, u\colon C^*(T)\to X,\ C^*_N(T)\mapsto X_N\}$ be an extension having the properties from 1). By Yosida's theorem ([36], § 45) there is a unique compact R_0 such that the vector lattice X is isomorphic to the vector lattice $C(R_0)$ relative to an isomorphism r_0. Then the mapping u generates a unique surjective continuous mapping $\varrho_0\colon R_0\to\beta T$ such that $r_0uf=f'\circ\varrho_0$ where f' denotes the extension of a function $f\in C^*(T)$ on βT.

Consider the space $R\equiv\varrho_0^{-1}T$ and the perfect mapping $\varrho\colon R\to T$ which is the restriction of ϱ_0. Consider the vector lattice Φ consisting of the restrictions on R of all functions from $C(R_0)$, the homomorphism $r\colon X\to\Phi$ such that $rx\equiv r_0x|R$, and the homomorphism $\varphi\colon C^*(T)\to\Phi$ such that $\varphi f\equiv f\circ\varrho$.

For a Kelley ideal N consider the ideals $\Phi_{0N}\equiv r_0X_N$ and $\Phi_N=rX_N$ and the closed subsets $R_{0N}\equiv\{s\in R_0\mid\forall f\in\Phi_{0N}(f(s)=0)\}\neq\emptyset$ and $R_N\equiv R_{0N}\cap R$. Then $\bigcup R_{0N}$ is dense in R_0 and $\varrho_0R_{0N}=\operatorname{cl}T_N$. It is clear that $N_1\subset N_2$ implies $R_{N_1}\supset R_{N_2}$. Take for N a proper ideal $M \supset N$ such that T_M is a compact set. Then $R_M = R_{0M}$. It follows from this fact that $R_N \neq \emptyset$ for any N.

Let $0 \leqq f \in C(R_0)$ and $f(R_{0N}) = 0$. Consider the functions $f_k \equiv \left(f - \frac{1}{k}\mathbf{1}\right) \vee 0$. From the property $R_{0N} \cap \operatorname{cl} \operatorname{coz} f_k = \emptyset$ we conclude that $f_k \in \Phi_{0N}$. This implies that f belongs to this ideal also. Therefore $\Phi_{0N} = \{f \in C(R_0) | f(R_{0N}) = 0\}$.

Let C be the cozero-set of a function $f \in C(R_0)$ such that $C \cap R_{0N} \neq \emptyset$. Take a sequence of compact sets F_k such that $F_k \notin N$ and $T \setminus \bigcup F_k \in N$. Consider the proper ideals $N_k \equiv \{P \in \mathscr{P}(T) | P \cap F_k \in N\}$. Then $\bigcap N_k = N$. As X is σ-filled we have $f \notin \Phi_{0N_k}$ for some k. Therefore $C \cap R_N \supset C \cap R_{0N_k} \neq \emptyset$. This means that R_N is dense in R_{0N}. As a consequence we get $\Phi_N = \{f \in \Phi | f(R_N) = 0\}$ and $\varrho R_N = T_N$.

Moreover, we have established that R is dense in R_0. Hence the triplet $\{\Phi, \varphi: C^*(T) \to \Phi, C_N^*(T) \mapsto \Phi_N\}$ is an extension isomorphic to the initial one.

In addition we get that $\bigcup R_N$ is dense in R. Consequently R is a preimage of T lifting Kelley covering.

Let G be an open set in R and $G \cap R_N \neq \emptyset$. Take a proper regular closed set $F \subset G$ such that $\operatorname{int} F \cap R_N \neq \emptyset$. Consider the proper component $Y \equiv \{f \in \Phi \setminus f(F) = 0\}$. As $Y^d \not\subset \Phi_N$ we get by virtue of the saturatedness of Φ that there exists an ideal Φ_M containing the set $\Phi_N \cup Y$. This means that $R_M \subset R_N \cap G$. Thus R is a saturated preimage.

Now let $\{\mathring{X}, \mathring{u}: C^*(T) \to \mathring{X}, C_N^*(T) \mapsto \mathring{X}_N\}$ be an extension having the properties from 2). Consider, as above, the isomorphic extension $\{\mathring{\Phi}, \mathring{\varphi}: C^*(T) \to \mathring{\Phi}, C_N^*(T) \mapsto \mathring{\Phi}_N\}$ for the corresponding preimage $\{\mathring{R}, \mathring{\varrho}: \mathring{R} \to T, T_N \mapsto \mathring{R}_N\}$.

Let $\bigcap N_k = N$, then $\mathring{\Phi}_N = \bigcap \mathring{\Phi}_{N_k}$ implies that $\bigcup \mathring{R}_{N_k}$ is dense in $\mathring{R}_N$. This means that the preimage $\mathring{R}$ is σ-filled. As Φ is Dedekind complete, $\mathring{R}$ is extremally disconnected. As Φ_N is a component, $\mathring{R}_N$ is open-closed. Hence the preimage $\mathring{R}$ is disjoined.

By Theorem 1 there exists a mapping $\gamma: \mathring{R} \to R$ such that $\mathring{R}$ is larger than R relative to γ.

Let $f \in \Phi$. As $\mathring{R}_0$ is an extremally disconnected compact, $\mathring{R}_0 = \beta \mathring{R}$ (see [31], VI, § 5, 173). Therefore $\mathring{\Phi} = C^*(\mathring{R})$. Consequently $f \circ \gamma \in \mathring{\Phi}$. This means that we can define correctly the injective vector-lattice homomorphism $v: \Phi \to \mathring{\Phi}$ by setting $vf \equiv f \circ \gamma$. Then $\mathring{\varphi} = v \circ \varrho$. Let $f \in \Phi_N$, then $(vf)(\mathring{R}_N) = 0$ implies $vf \in \mathring{\Phi}_N$. Thus the extension $\mathring{\Phi}$ is larger than the extension Φ. This fact is valid for the initial extensions $\mathring{X}$ and X, too.

Now let Φ be the extension from Proposition 2 isomorphic to the second dual extension $C^*(T)''$. As Φ has the properties from 1) and 2) simultaneously we get that Φ is the largest of all extensions with the properties from 1) and the smallest of all extensions with the properties from 2).

Let $\mathring{X}$ be some other largest extension of $C^*(T)$. Consider, as above, the isomorphic extension $\{\mathring{\Phi}, \mathring{\varphi}: C^*(T) \to \mathring{\Phi}, C_N^*(T) \mapsto \Phi_N\}$ for the preimage $\{\mathring{R}, \mathring{\varrho}: \mathring{R} \to T, T_N \mapsto \mathring{R}_N\}$. Take some mapping $w: \Phi \to \mathring{\Phi}$ such that $\mathring{\Phi}$ is larger than Φ relative to w. Define the surjective perfect mapping $\delta: \mathring{R} \to R$ by setting $\delta s \equiv \bigcap \{\operatorname{cl} \operatorname{coz} f \cap \varrho^{-1} \mathring{\varrho} s | s \in \operatorname{coz} wf\}$. Then $\varrho \circ \delta = \mathring{\varrho}$. We check that $wf = f \circ \delta$ for any $0 \leqq f \in \Phi$. Assume that there exists a point s such that $(wf)(s) \neq (f \circ \delta)(s)$. If $(wf)(s) > (f \circ \delta)(s)$ then we shall consider the function $g\equiv f$; otherwise $g\equiv -f$. Denote the number $((wg)(s)+(g\circ\delta)(s))/2$ by a. Consider the function $h\equiv(g-a\mathbf{1})\vee 0$. Take a neighbourhood G of s such that $(wg)(t)>a$ for any $t\in G$. Also take a neighbourhood U of the point δs such that $g(r)<a$ for any $r\in U$. Then $U\subset R\setminus \operatorname{coz} h$ and $G\cap\delta^{-1}U\subset \subset \operatorname{coz} wh$. Therefore $\delta s\notin \operatorname{cl}\operatorname{coz} h$ and $\delta s\in\operatorname{cl}\operatorname{coz} h$ but this is impossible. From this contradiction we conclude that such point does not exist.

We now check that $\delta\mathring{R}_N\subset R_N$. Assume that there exists a point $s\in\delta\mathring{R}_N\setminus R_N$. Take a function $f\in\Phi_N$ such that $s\in\operatorname{coz} f$, then for some point $t\in\mathring{R}_N$ such that $s=\delta t$ we get $(wf)(t)\neq 0$. On the other hand, $wf\in\mathring{\Phi}_N$ implies $(wf)(t)=0$. It follows from this contradiction that this inclusion is valid.

Now take the mapping $v\colon \mathring{\Phi}\to\Phi$ defined above. Let $s\in R_N$. By virtue of the saturatedness of the hyperstonean cover we have $s=\bigcap\{R_M\}$. Then $\delta\gamma s\in\bigcap\{R_M\}=s$. From this fact we conclude that $\delta\gamma s=s$ for any point $s\in R$, i.e. $(wvf)(s)=f(s)$. Thus v and w are mutually inverse isomorphisms of the vector lattices, and the extensions Φ and $\mathring{\Phi}$ are isomorphic.

The uniqueness of the smallest extension and assertion 3) are checked in a similar way. The theorem is proved.

Note that the class of saturated extensions of $C^*(T)$ inheriting Lebesgue decomposition contains extensions not isomorphic to $C^*(T)^{t\prime}$. The most important such extensions are the vector lattices $B^*(T)$ of all bounded Borel functions and the vector lattice $L^*(T)$ of all bounded universally measurable (see [34], V, § 3, 4) functions on T.

§4. Second dual extension as a C-ring

Let T be a completely regular space and $C^*(T)^{t\prime}$ the second tight dual to $C^*(T)$ described in the introduction to § 2. Provide it by the Arens product in the following way. For $f,g\in C^*(T)$ and $\xi\in C^*(T)^t$ define $\xi\cdot f\in C^*(T)^t$ by setting $(\xi\cdot f)g\equiv \equiv\xi(fg)$. For $\psi\in C^*(T)^{t\prime}$ define $\psi\cdot\xi\in C^*(T)^t$ by setting $(\psi\cdot\xi)f\equiv\psi(\xi\cdot f)$. For $\varphi\in C^*(T)^{t\prime}$ define $\varphi\psi\in C^*(T)^{t\prime}$ by setting $(\varphi\psi)\xi\equiv\varphi(\psi\cdot\xi)$. Then $C^*(T)^{t\prime}$ is a C-ring, containing $C^*(T)$ as a C-subring. In the case of a compact space S the possibility of converting $C^*(S)''$ into a ring extending the ring $C(S)$ has been proved in different ways by Vulih [14], Kakutani [15], Arens [16] and Grothendieck [17]. However it follows from Vulih's theorem (see [14] or [45], V. 8.2) that all these ring structures on $C(S)''$ coincide.

Let $u\colon C^*(T)\to C^*(T)^{t\prime}$ be the canonical imbedding. Consider in $C^*(T)^{t\prime}$ the C-ideals $C^*(T)^{t\prime}_N$ introduced in the previous section. Then $\{C^*(T)^{t\prime},\ u\colon C^*(T)\to \to C^*(T)^{t\prime},\ C^*_N(T)\mapsto C^*(T)^{t\prime}_N\}$ is a C-ring extension of $C^*(T)$ inheriting Lebesgue decomposition. This extension will be called *the second dual extension of* $C^*(T)$.

4.1. Functional description of the second dual extension by functions on hyperstonean cover

Let R be the hyperstonean cover of T and $\varrho\colon R\to T$ the canonical mapping. Consider the injective ring homomorphism $\varphi\colon C^*(T)\to C^*(R)$ such that $\varphi f\equiv f\circ\varrho$. For a Kelley ideal N consider the C-ideal $C^*_N(R)\equiv\{f\in C^*(R)\mid f(R_N)=0\}$. Then

$\{C^*(R),\ \varphi\colon C^*(T)\to C^*(R),\ C_N^*(T)\mapsto C_N^*(R)\}$ is a C-ring extension of $C^*(T)$ inheriting Lebesgue decomposition.

PROPOSITION 3. *The extensions $C^*(T)^{tt'}$ and $C^*(R)$ are isomorphic saturated continuing segment extensions of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Denote $C^*(R)$ by Φ and $C_N^*(R)$ by Φ_N. In just the same way as in the proof of Proposition 2 it is established that the given extensions are isomorphic and Φ is a saturated extension.

Let Y be a ring ideal in the ring Φ and $g\in \mathrm{Hom}_\Phi^*(Y, Y^{**})$. Let $y_1, y_2\in Y$ and $t\in \mathrm{coz}\, y_1\cap \mathrm{coz}\, y_2$, then $(gy_1)(s)/y_1(s)=(gy_2)(s)/y_2(s)$. Consequently we can define correctly the function z by setting $z(s)\equiv(gy)(s)/y(s)$ for any $y\in Y$ and for any $t\in \mathrm{coz}\, y$ and $z(t)\equiv 0$ for any $t\notin U\equiv \mathrm{cl}\,\{\mathrm{coz}\, y|y\in Y\}$. As the function z is bounded and continuous on a dense subset of R and the space R is extremally disconnected then z can be extended to a continuous function on the whole space R (see [31], VI, § 5, 158 and 173).

As $z\in Y^{**}$ we can define correctly the homomorphism $h\in \mathrm{Hom}_\Phi^*(\Phi, Y^{**})$ by setting $hf\equiv fz$. Let $y\in Y$ and $s\in \mathrm{coz}\, y_1$ for some $y_1\in Y$. Then $y_1(s)(hy)(s)= =y(s)(gy_1)(s)=y_1(s)(gy)(s)$ implies $(hy)(s)=(gy)(s)$. As hy and gy belong to Y^{**} we have $(hy)(s)=0=(gy)(s)$ for all $s\notin U$. This means that $hy=gy$. Thus Φ is continuing.

Now let g and h be the homomorphisms from the definition of the segment and $gY\subset \Phi_N$. Let $f\in\Phi$ and $s\in R_N\cap \mathrm{coz}\, y$ for some $y\in Y$. Then $y(s)(hf)(s)= =f(s)(gy)(s)=0$ and consequently $(hf)(s)=0$. If $s\in R_N\setminus U$ then $(hf)(s)=0$ for $hf\in Y^{**}$. Hence $hf\in\Phi_N$. This means that Φ_N is a segment of Φ. The proposition is proved.

4.2. Characterization of the second dual extension as a C-ring

In what follows uniqueness is understood up to isomorphism in the category of the C-ring extensions of $C^*(T)$ inheriting Lebesgue decomposition.

THEOREM 3. 1) *$C^*(T)^{tt'}$ is the unique largest of all saturated extensions of $C^*(T)$ inheriting Lebesgue decomposition;*

2) *$C^*(T)^{tt'}$ is the unique smallest of all σ-filled continuing segment extensions of $C^*(T)$ inheriting Lebesgue decomposition; moreover $C^*(T)^{tt'}$ is the unique universal among all such extensions;*

3) *$C^*(T)^{tt'}$ is the unique saturated continuing segment extension of $C^*(T)$ inheriting Lebesgue decomposition.*

PROOF. Let $\{X,\ u\colon C^*(T)\to X,\ C_N^*(T)\mapsto X_N\}$ be an extension having the properties from 1). By Delfosse's theorem ([39]) there is a compact R_0 such that the lattice ring X is isomorphic to the lattice ring $C(R_0)$ relative to an isomorphism r_0. The lattice-ring homomorphism u generates a unique surjective continuous mapping $\varrho_0\colon R_0\to\beta T$ such that $r_0uf=f'\circ\varrho_0$ where f' denotes the extension of a function $f\in C^*(T)$ on βT. Further by completely the same arguments as in the proof of Theorem 2 we obtain the preimage $\{R,\ \varrho\colon R\to T,\ T_N\mapsto R_N\}$ of T and the corresponding extension $\{\Phi, \varphi\colon C^*(T)\to\Phi,\ C_N^*(T)\mapsto\Phi_N\}$ isomorphic to the initial one.

In just the same way as in the proof of Theorem 2 it is established that the preimage R is saturated.

Now let $\{\mathring{X}, \mathring{u}\colon C^*(T)\to\mathring{X},\ C_N^*(T)\mapsto\mathring{X}_N\}$ be an extension having the properties from 2). Consider the isomorphic extension $\{\mathring{\Phi}, \mathring{\varphi}\colon C^*(T)]\to\mathring{\Phi}, C_N^*(T)\mapsto\mathring{\Phi}_N\}$ for the corresponding preimage $\{\mathring{R}, \mathring{\varrho}\colon \mathring{R}\to T,\ T_N\mapsto\mathring{R}_N\}$. Then the preimage $\mathring{R}$ is σ-filled.

Let G be an open set from $\mathring{R}$. Consider the ring ideal $Y\equiv\{y\in\mathring{\Phi}|\mathrm{coz}\, y\subset G\}$ of the ring Φ, and define the homomorphism $g\in\mathrm{Hom}^*_{\mathring{\Phi}}(Y, Y^{**})$ by setting $gy\equiv y$. Then there exists a bounded $\mathring{\Phi}$-module homomorphism $h\colon\mathring{\Phi}\to Y^{**}$ extending g. Consider the function $u\equiv h\mathbf{1}\in\mathring{\Phi}$ and the set $U\equiv\mathrm{cl}\, G$. It is clear that $u(\mathring{R}\setminus U)=0$. Let $s\in G$, then $s\in\mathrm{coz}\, y$ for some $y\in Y$. Therefore $y(s)u(s)=(gy)(s)=y(s)$ implies $u(s)=1$. Since the function u is continuous we conclude that $u=\chi(U)$ and $U\in\Delta(\dot{R})$. This means that the preimage $\mathring{R}$ is extremally disconnected.

Let $G\cap\mathring{R}_N=\emptyset$. Then $gY\subset\mathring{\Phi}_N$ implies $u\in\mathring{\Phi}_N$ and $U\cap\mathring{R}_N=\emptyset$. Hence the preimage $\mathring{R}$ is disjoined.

The rest of the proof is exactly the same as that of Theorem 2.

Note that the class of saturated extension of $C^*(T)$ inheriting Lebesgue decomposition contains extensions not isomorphic to $C^*(T)''$. The most important such extensions are the C-ring $B^*(T)$ of all bounded Borel functions and the C-ring $L^*(T)$ of all bounded universally measurable (see [34], V, § 3. 4) functions on T.

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INSTITUTE OF TEXTILE
AND LIGHT INDUSTRY
HERZENA 18
191065 LENINGRAD
USSR

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PROXIMITIES, SCREENS, MEROTOPIES, UNIFORMITIES. IV

Á. CSÁSZÁR (Budapest), member of the Academy

11. Semi-uniformities. It is well-known that uniformities can be defined in two completely equivalent manners, introduced in [24] and [23], respectively. In order to recall and generalize these definitions, let X be a set and $\mathfrak{a}, \mathfrak{b} \in \exp X$; we say that $\mathfrak{a}$ *refines* $\mathfrak{b}$, or that $\mathfrak{a}$ is a *refinement* of $\mathfrak{b}$, denoted by $\mathfrak{a} \ll \mathfrak{b}$, iff $A \in \mathfrak{a}$ implies the existence of $B \in \mathfrak{b}$ such that $A \subset B$. $\mathfrak{a} \neq \emptyset$ is a *cover* of X iff $\bigcup \mathfrak{a} = X$.

Now, according to [23], $\mathfrak{C}$ is a *uniformity* on X iff $\mathfrak{C} \neq \emptyset$ is a set whose elements are covers of X, and the following conditions U1—U3 are fulfilled:

U1. $\mathfrak{c} \in \mathfrak{C}$, $\mathfrak{c} \ll \mathfrak{c}'$ implies $\mathfrak{c}' \in \mathfrak{C}$,
U2. $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathfrak{C}$ implies the existence of $\mathfrak{c} \in \mathfrak{C}$ such that $\mathfrak{c} \ll \mathfrak{c}_i$ for $i=1, 2$,
U3. $\mathfrak{c} \in \mathfrak{C}$ implies the existence of $\mathfrak{c}' \in \mathfrak{c}$ such that $\mathfrak{c}^* \ll \mathfrak{c}$, where $\mathfrak{c}^* = \{C^* : C \in \mathfrak{c}'\}$, $C^* = \bigcup \{C' \in \mathfrak{c}' : C' \cap C \neq \emptyset\}$.

On the other hand, according to [24], $\mathcal{U}$ is a *W-uniformity* on $X \neq \emptyset$ iff $\mathcal{U}$ is a filter in $X \times X$ satisfying

W1. $\Delta_X = \{(x, x) : x \in X\} \subset U$ for $U \in \mathcal{U}$,
W2. $U \in \mathcal{U}$ implies $U \cap U^{-1} \in \mathcal{U}$ for $U^{-1} = \{(y, x) : (x, y) \in U\}$,
W3. $U \in \mathcal{U}$ implies the existence of $U' \in \mathcal{U}$ such that $U' \circ U' \subset U$ where $U' \circ U' = \{(x, z)$: there is $y \in X$ such that $(x, y) \in U'$, $(y, z) \in U'\}$.

For convenience, if $X = \emptyset$, the set $\{\emptyset\}$ is considered to be a W-uniformity on X.

The equivalence of these two definitions (for $X \neq \emptyset$) is established if we assign, to a uniformity $\mathfrak{C}$ on X, the W-uniformity $\mathcal{U}$ on X composed of those sets $U \subset X \times X$ that contain a subset of the form $\bigcup \{C \times C : C \in \mathfrak{c}\}$ for some $\mathfrak{c} \in \mathfrak{C}$, and conversely, to a W-uniformity $\mathcal{U}$, we assign the uniformity $\mathfrak{C}$ composed of the covers $\mathfrak{c}$ that possess a refinement of the form $\mathfrak{c}' = \{C \subset X : C \times C \subset U\}$ for some $U \in \mathcal{U}$. These operations are inverse to each other; we say that the corresponding $\mathfrak{C}$ and $\mathcal{U}$ are *associated* with each other.

Let $\mathfrak{c} \in \exp \exp X$, $U \subset X \times X$, and $\mathfrak{a} \in \exp \exp X$. We say that $\mathfrak{a}$ is $\mathfrak{c}$-*Cauchy* iff there exist $C \in \mathfrak{c}$ and $A \in \mathfrak{a}$ such that $A \subset C$, and that $\mathfrak{a}$ is U-*Cauchy* iff there is $A \in \mathfrak{a}$ such that $A \times A \subset U$. For a uniformity $\mathfrak{C}$ on X, we say that $\mathfrak{a}$ is $\mathfrak{C}$-*Cauchy* iff it is $\mathfrak{c}$-Cauchy for every $\mathfrak{c} \in \mathfrak{C}$. Similarly, for a W-uniformity $\mathcal{U}$, we say that $\mathfrak{a}$ is $\mathcal{U}$-*Cauchy* iff it is U-Cauchy for every $U \in \mathcal{U}$. It is easily seen that, if $\mathfrak{C}$ and $\mathcal{U}$ are associated with each other, then $\mathfrak{C}$-Cauchy means the same as $\mathcal{U}$-Cauchy.

A set $\mathfrak{C} \neq \emptyset$ of covers of X will be said to be a *semi-uniformity* iff it fulfils U1 and a *pseudo-uniformity* iff it satisfies U1 and U2 ("quasi-uniformity" in [13]). Similarly, we say that an ascending system $\mathcal{U} \neq \emptyset$ in $X \times X$ is a *W-semi-uniformity* iff it fulfils W1 and W2, and that a filter $\mathcal{U}$ in $X\times X$ satisfying W1 and W2 is a *W-pseudo-uniformity* ("semi-uniformity" in [1]).

It is well-known (cf. e.g. [16]) that pseudo-uniformities are closely related to merotopies. We shall show that the same holds for semi-uniformities and semi-merotopies.

If $\mathfrak{C}$ and $\mathfrak{C}'$ are semi-uniformities on X and Y, respectively, and $f\colon X\to Y$, then f is said to be $(\mathfrak{C}, \mathfrak{C}')$-*continuous* iff $\mathfrak{c}'\in\mathfrak{C}'$ implies the existence of $\mathfrak{c}\in\mathfrak{C}$ such that $f(\mathfrak{c})\ll\mathfrak{c}'$.

(11.1) Lemma. *The semi-uniformities and the* $(\mathfrak{C}, \mathfrak{C}')$-*continuous maps constitute a concrete category* **Sunif.** □

The definition of a $\mathfrak{C}$-Cauchy system can be obviously generalized for the case when $\mathfrak{C}$ is a semi-uniformity.

(11.2) Lemma. *Let $\mathfrak{C}$ be a semi-uniformity on X. Then the $\mathfrak{C}$-Cauchy systems constitute a semi-merotopy on X denoted by $\mathfrak{M}(\mathfrak{C})$.*

Proof. $\{\emptyset\}$ is $\mathfrak{C}$-Cauchy, and the same holds for $\{\{x\}\}$ if $x\in X$. If $\mathfrak{m}$ is $\mathfrak{C}$-Cauchy, then $\mathfrak{m}\neq\emptyset$ because $\mathfrak{C}\neq\emptyset$. If $\mathfrak{m}$ is $\mathfrak{C}$-Cauchy and $\mathfrak{m}<\mathfrak{m}'$, then $\mathfrak{m}'$ is $\mathfrak{C}$-Cauchy. □

(11.3) Lemma. *If $\mathfrak{C}$ is a pseudo-uniformity, then $\mathfrak{M}(\mathfrak{C})$ is a merotopy.*

Proof. Let $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$ be $\mathfrak{C}$-Cauchy, and assume that neither $\mathfrak{m}_1$ nor $\mathfrak{m}_2$ is $\mathfrak{C}$-Cauchy. Then there are $\mathfrak{c}_1, \mathfrak{c}_2\in\mathfrak{C}$ such that no $C\in\mathfrak{c}_i$ contains any subset $M\in\mathfrak{m}_i$. Choose $\mathfrak{c}\in\mathfrak{C}$, $\mathfrak{c}\ll\mathfrak{c}_i$ for $i=1, 2$. Then no $C\in\mathfrak{c}$ contains any subset $M\in\mathfrak{m}$: a contradiction. □

(11.4) Lemma. *Let $\mathfrak{M}$ be a semi-merotopy on X. Then those $\mathfrak{c}\in\exp\exp X$ for which every $\mathfrak{m}\in\mathfrak{M}$ is $\mathfrak{c}$-Cauchy constitute a semi-uniformity $\mathfrak{C}(\mathfrak{M})$ on X.*

Proof. $\{X\}\in\mathfrak{C}(\mathfrak{M})$ since $\mathfrak{m}\neq\emptyset$ for $\mathfrak{m}\in\mathfrak{M}$. $\mathfrak{M}\neq\emptyset$ and $\{\{x\}\}\in\mathfrak{M}$ for $x\in X$ imply that every $\mathfrak{c}\in\mathfrak{C}(\mathfrak{M})$ is a cover of X. If $\mathfrak{c}\ll\mathfrak{c}'$ and $\mathfrak{m}$ is $\mathfrak{c}$-Cauchy then $\mathfrak{m}$ is $\mathfrak{c}'$-Cauchy as well. □

(11.5) Lemma. *If $\mathfrak{M}$ is a merotopy, then $\mathfrak{C}(\mathfrak{M})$ is a pseudo-uniformity.*

Proof. Let $\mathfrak{c}_1, \mathfrak{c}_2\in\mathfrak{C}(\mathfrak{M})$ and define

$$\mathfrak{c} = \{C_1\cap C_2\colon\ C_i\in\mathfrak{c}_i\quad (i=1, 2)\}.$$

Then $\mathfrak{c}$ is a cover of X, $\mathfrak{c}\ll\mathfrak{c}_i$ $(i=1, 2)$. Assume $\mathfrak{c}\notin\mathfrak{C}(\mathfrak{M})$. Then there is $\mathfrak{m}\in\mathfrak{M}$ such that no set $C\in\mathfrak{c}$ contains a subset $M\in\mathfrak{m}$. Now for $M\in\mathfrak{m}$, either $M\subset C_1\in\mathfrak{c}_1$ or $M\subset C_2\in\mathfrak{m}_2$ is impossible (otherwise $M\subset C_1\cap C_2\in\mathfrak{c}$ would hold). Thus $\mathfrak{m}=$ $=\mathfrak{m}_1\cup\mathfrak{m}_2$ and, for $M\in\mathfrak{m}_i$, $M\subset C_i\in\mathfrak{c}_i$ is impossible, so that $\mathfrak{m}_i$ is not $\mathfrak{c}_i$-Cauchy; this contradicts the fact that either $\mathfrak{m}_1$ or $\mathfrak{m}_2$ belongs to $\mathfrak{M}$. □

(11.6) Lemma. *If $\mathfrak{C}$ is a semi-uniformity, $\mathfrak{M}$ a semi-merotopy on X, then*

$$\mathfrak{C}(\mathfrak{M}(\mathfrak{C})) = \mathfrak{C},\quad \mathfrak{M}(\mathfrak{C}(\mathfrak{M})) = \mathfrak{M}.$$

Proof. By definition, $\mathfrak{C}\subset\mathfrak{C}(\mathfrak{M}(\mathfrak{C}))$. If $\mathfrak{c}'\in\exp\exp X$, $\mathfrak{c}'\notin\mathfrak{C}$, then $\mathfrak{c}\in\mathfrak{C}$, $\mathfrak{c}\ll\mathfrak{c}'$ is impossible, hence every $\mathfrak{c}\in\mathfrak{C}$ contains an element $C\in\mathfrak{c}$ that is not a subset of any $C'\in\mathfrak{c}'$. Let $\mathfrak{m}$ denote the collection of these sets C. Then $\mathfrak{m}\in\mathfrak{M}(\mathfrak{C})$ but $\mathfrak{m}$ is not $\mathfrak{c}'$-Cauchy, so that $\mathfrak{c}'\notin\mathfrak{C}(\mathfrak{M}(\mathfrak{C}))$.

Similarly, $\mathfrak{M}\subset\mathfrak{M}(\mathfrak{C}(\mathfrak{M}))$ by definition. If $\mathfrak{m}'\in\exp\exp X$, $\mathfrak{m}'\notin\mathfrak{M}$, then $\mathfrak{m}<\mathfrak{m}'$ is impossible for $\mathfrak{m}\in\mathfrak{M}$, so that every $\mathfrak{m}\in\mathfrak{M}$ contains an element $M\in\mathfrak{m}$ that does not contain any subset $M'\in\mathfrak{m}'$. The collection $\mathfrak{c}$ of these sets M belongs to $\mathfrak{C}(\mathfrak{M})$, but $\mathfrak{m}'$ is not $\mathfrak{c}$-Cauchy, $\mathfrak{m}'\notin\mathfrak{M}(\mathfrak{C}(\mathfrak{M}))$. □

(11.7) LEMMA. *Let $\mathfrak{C}$ and $\mathfrak{C}'$ be semi-uniformities on X and Y, respectively. A map $f\colon X\to Y$ is $(\mathfrak{C},\mathfrak{C}')$-continuous iff it is $(\mathfrak{M}(\mathfrak{C}),\mathfrak{M}(\mathfrak{C}'))$-continuous.*

PROOF. If $\mathfrak{c}\in\mathfrak{C}$, $\mathfrak{c}'\in\mathfrak{C}'$, $f(\mathfrak{c})\ll\mathfrak{c}'$, and $\mathfrak{m}$ is $\mathfrak{c}$-Cauchy, then clearly $f(\mathfrak{m})$ is $f(\mathfrak{c})$-Cauchy and $\mathfrak{c}'$-Cauchy. Hence $(\mathfrak{C},\mathfrak{C}')$-continuity implies $(\mathfrak{M}(\mathfrak{C}),\mathfrak{M}(\mathfrak{C}'))$-continuity.

Conversely, assume that, for a $\mathfrak{c}'\in\mathfrak{C}'$, there is no $\mathfrak{c}\in\mathfrak{C}$ satisfying $f(\mathfrak{c})\ll\mathfrak{c}'$. Then we can choose from every $\mathfrak{c}\in\mathfrak{C}$ a set $C\in\mathfrak{c}$ such that $f(C)$ is not a subset of any $C'\in\mathfrak{c}'$. The collection $\mathfrak{m}$ of these sets C belongs to $\mathfrak{M}(\mathfrak{C})$, and $f(\mathfrak{m})$ is not $\mathfrak{c}'$-Cauchy. Therefore $(\mathfrak{M}(\mathfrak{C}),\mathfrak{M}(\mathfrak{C}'))$-continuity implies $(\mathfrak{C},\mathfrak{C}')$-continuity. □

(11.8) THEOREM ([11], 3.3—3.6). *The functor F defined by $F(\mathfrak{C})=\mathfrak{M}(\mathfrak{C})$ induces an isomorphism from* **Sunif** *onto* **Smer** *that carries onto* **Mer** *the full subcategory* **Psunif** *of* **Sunif** *the objects of which are the pseudo-uniformities. Hence* **Sunif** *is a strongly topological category and* **Psunif** *is a strongly topological bicoreflective subcategory of* **Sunif**. □

The part concerning **Psunif** and **Mer** is well-known as it was mentioned above; it motivates the fact that sometimes (e.g. in [12]) pseudo-uniformities are called merotopies.

Since $\mathfrak{M}$ is a filter-merotopy iff $\mathfrak{M}=\mathfrak{M}^f$, we obtain:

(11.9) COROLLARY. *With the notations of* (11.8), *a semi-uniformity $\mathfrak{C}$ belongs to F^{-1}*(**Fmer**) *iff every $\mathfrak{C}$-Cauchy system is finer than some $\mathfrak{C}$-Cauchy filter.* □

On the other hand, [20] contains a characterization of those merotopies that belong to F(**Unif**) (using W-uniformities instead of uniformities).

In order to examine the category **Unif** as a subcategory of **Psunif**, let us first describe the fundamental operations in **Sunif**.

(11.10) LEMMA. *If $\mathfrak{C}$ is a semi-uniformity on X, $g\colon Z\to X$, then $\overset{-1}{g}_{\mathbf{Sunif}}(\mathfrak{C})$ is composed of those $\mathfrak{c}'\in\exp\exp Z$ for which there is a $\mathfrak{c}\in\mathfrak{C}$ such that $g^{-1}(\mathfrak{c})\ll\mathfrak{c}'$.*

PROOF. By (11.8) and (11.6)

$$\overset{-1}{g}_{\mathbf{Sunif}}(\mathfrak{C})=\mathfrak{C}\big(\overset{-1}{g}_{\mathbf{Smer}}(\mathfrak{M}(\mathfrak{C}))\big).$$

By (9.1) $\mathfrak{c}'\in\overset{-1}{g}_{\mathbf{Sunif}}(\mathfrak{C})$ iff $g(\mathfrak{m}')\in\mathfrak{M}(\mathfrak{C})$ implies that $\mathfrak{m}'$ is $\mathfrak{c}'$-Cauchy. If $\mathfrak{c}\in\mathfrak{C}$, $g^{-1}(\mathfrak{c})\ll\mathfrak{c}'$, and $g(\mathfrak{m}')$ is $\mathfrak{C}$-Cauchy, then there are $M'\in\mathfrak{m}'$ and $C\in\mathfrak{c}$ such that $g(M')\subset C$, i.e. $M'\subset g^{-1}(C)$, and then $M'\subset g^{-1}(C)\subset C'$ for some $C'\in\mathfrak{c}'$. On the other hand, if there does not exist a $\mathfrak{c}\in\mathfrak{C}$ satisfying $g^{-1}(\mathfrak{c})\ll\mathfrak{c}'$, then there is, for every $\mathfrak{c}\in\mathfrak{C}$, a set $C\in\mathfrak{c}$ such that $g^{-1}(C)$ is not a subset of any $C'\in\mathfrak{c}'$. These sets $g^{-1}(C)$ constitute a system $\mathfrak{m}'$ satisfying $g(\mathfrak{m}')\in\mathfrak{M}(\mathfrak{C})$ that is not $\mathfrak{c}'$-Cauchy. □

(11.11) LEMMA. *If $\mathfrak{C}_i$ is a semi-uniformity on X for $i\in I$, then* $\sup_{\mathbf{Sunif}}\{\mathfrak{C}_i$:

$i\in I\}=\bigcup_{i\in I}\mathfrak{C}_i$ *if* $I\neq\emptyset$, *and it is composed of all* $\mathfrak{c}\in\exp\exp X$ *such that* $X\in\mathfrak{c}$ *if* $I=\emptyset$.

PROOF. By (11.8) and (11.6) again,

$$\sup_{\mathbf{Sunif}}\{\mathfrak{C}_i:\ i\in I\}=\mathfrak{C}\big(\sup_{\mathbf{Smer}}\{\mathfrak{M}(\mathfrak{C}_i):\ i\in I\}\big).$$

If $I=\emptyset$, then $\{\{X\}\}$ belongs to the sup at the right-hand side, and this system is $\mathfrak{c}$-Cauchy iff $X\in\mathfrak{c}$; for such a $\mathfrak{c}$, every non-empty element of $\exp\exp X$ is $\mathfrak{c}$-Cauchy as well.

If $I\neq\emptyset$, then the sup at the right-hand side is $\bigcap_{i\in I}\mathfrak{M}(\mathfrak{C}_i)$. If $\mathfrak{c}'\in\bigcup_{i\in I}\mathfrak{C}_i$ then every $\mathfrak{m}$ that is $\mathfrak{C}_i$-Cauchy for each $i\in I$ is $\mathfrak{c}'$-Cauchy, too. If $\mathfrak{c}'\notin\bigcup_{i\in I}\mathfrak{C}_i$, then $\mathfrak{c}\ll\mathfrak{c}'$, $\mathfrak{c}\in\mathfrak{C}_i$ is impossible, hence, for each $\mathfrak{c}\in\mathfrak{C}_i$, $i\in I$, there is $C\in\mathfrak{c}$ that is not a subset of any $C'\in\mathfrak{c}'$. These sets C constitute a system $\mathfrak{m}\in\bigcap_{i\in I}\mathfrak{M}(\mathfrak{C}_i)$ that is not $\mathfrak{c}'$-Cauchy. □

(11.12) LEMMA. *The* **Psunif**-*coreflection of a semi-uniformity* $\mathfrak{C}$ *on* X *is the collection* $\mathfrak{C}^q$ *of all* $\mathfrak{c}\in\exp\exp X$ *such that there are* $\mathfrak{c}_i\in\mathfrak{C}$ $(i=1,\dots,n;\ n\in\mathbf{N})$ *satisfying*

$$\Big\{\bigcap_1^n C_i:\ C_i\in\mathfrak{c}_i\quad(i=1,\dots,n)\Big\}\ll\mathfrak{c}.$$

PROOF. By (11.8) and (11.6) the coreflection in question is $\mathfrak{C}\big(\mathfrak{M}(\mathfrak{C})^q\big)$. Choose $\mathfrak{c}_i\in\mathfrak{C}$ for $i=1,\dots,n$ and set

$$\mathfrak{c}'=\Big\{\bigcap_1^n C_i:\ C_i\in\mathfrak{c}_i\quad(i=1,\dots,n)\Big\}.\tag{11.2.1}$$

If $\mathfrak{c}'\ll\mathfrak{c}$, and $\mathfrak{m}\in\mathfrak{M}(\mathfrak{C})^q$, then $\mathfrak{m}$ is $\mathfrak{c}$-Cauchy; in fact $\mathfrak{m}$ is $\mathfrak{c}'$-Cauchy, otherwise there would be, for every $M\in\mathfrak{m}$, at least one i such that $M\subset C\in\mathfrak{c}_i$ is impossible. Denote by $\mathfrak{m}_i$ the collection of all $M\in\mathfrak{m}$ for which this holds for i. Then $\mathfrak{m}=\bigcup_1^n\mathfrak{m}_i$ so that, by (9.10), $\mathfrak{m}_i\in\mathfrak{M}(\mathfrak{C})$ for at least one i. For this i, $\mathfrak{m}$ is not $\mathfrak{c}_i$-Cauchy: a contradiction.

On the other hand, assume that $\mathfrak{c}\in\exp\exp X$ has no refinement of the form (11.12.1). Then, for every finite subset $\{\mathfrak{c}_i:\ i=1,\dots,n\}$ of $\mathfrak{C}$, there are $C_i\in\mathfrak{c}_i$ such that $\bigcap_1^n C_i$ is not a subset of any $C\in\mathfrak{c}$. Let $\mathfrak{m}$ denote the system composed of all these intersections $\bigcap_1^n C_i$. Then $\mathfrak{m}\in\mathfrak{M}(\mathfrak{C})^q$. In fact, if $\mathfrak{m}=\bigcup_1^n\mathfrak{m}_i$ and no $\mathfrak{m}_i$ belonged to $\mathfrak{M}(\mathfrak{C})$, then, for each i, there would be a $\mathfrak{c}_i\in\mathfrak{C}$ such that $M\in\mathfrak{m}_i$, $C\in\mathfrak{c}_i$, $M\subset C$ is impossible. Now the intersection $\bigcap_1^n C_i$ corresponding to this choice of the covers $\mathfrak{c}_i$ cannot belong to $\mathfrak{m}$: a contradiction. Thus we obtained a system $\mathfrak{m}\in\mathfrak{M}(\mathfrak{C})^q$ that is not $\mathfrak{c}$-Cauchy. □

Of course, (11.10)—(11.13) could be checked directly, without referring to (11.8).

(11.13) LEMMA. *If $\mathfrak{C}$ is a uniformity on X, $g\colon Z\to X$, then $\overset{-1}{g}_{\mathrm{Sunif}}(\mathfrak{C})=\overset{-1}{g}_{\mathrm{Psunif}}(\mathfrak{C})$ is a uniformity.*

PROOF. The equality results from (11.8) and (9.13). For $\mathfrak{c}''\in g^{-1}(\mathfrak{C})=\overset{-1}{g}_{\mathrm{Sunif}}(\mathfrak{C})$, choose a $\mathfrak{c}\in\mathfrak{C}$ such that $g^{-1}(\mathfrak{c})\ll\mathfrak{c}''$ (see (11.10)), then select $\mathfrak{c}'\in\mathfrak{C}$ satisfying the condition in U3. Clearly $g^{-1}(\mathfrak{c}')\in g^{-1}(\mathfrak{C})$ fulfils the condition in U3 for $g^{-1}(\mathfrak{c})$ and, *a fortiori*, for $\mathfrak{c}''$. $\square$

(11.14) LEMMA. *Let $\mathfrak{C}_i$ be a uniformity on X for $i\in I$. Then* $\sup_{\mathrm{Psunif}}\{\mathfrak{C}_i\colon i\in I\}=(\sup_{\mathrm{Sunif}}\{\mathfrak{C}_i\colon i\in I\})^q$ *is a uniformity.*

PROOF. By (11.8), (9.13) and (11.12), then equality is valid. The case $I=\emptyset$ is obvious. Assume $I\neq\emptyset$.

Denoting the common value of both sides by $\mathfrak{C}$, for $\mathfrak{c}\in\mathfrak{C}$, choose $i_k\in I$ $(k=1, \dots, n)$ and $\mathfrak{c}_{i_k}\in\mathfrak{C}_{i_k}$ such that

$$\mathfrak{c}'=\Big\{\bigcap_{k=1}^{n} C_k\colon\ C_k\in\mathfrak{c}_{i_k}\quad (k=1, \dots, n)\Big\}\ll\mathfrak{c}.$$

For each $\mathfrak{c}_{i_k}$, select $\mathfrak{c}'_{i_k}\in\mathfrak{C}_{i_k}$ satisfying U3. Then it is easily seen that

$$\mathfrak{c}''=\Big\{\bigcap_{k=1}^{n} C'_k\colon\ C'_k\in\mathfrak{c}'_{i_k}\quad (k=1, \dots, n)\Big\}\in\mathfrak{C}$$

fulfils U3 for the given $\mathfrak{c}$. $\square$

(11.15) THEOREM ([12]). *The category* **Unif** *is a bireflective subcategory of every full subcategory of* **Psunif** *in which it is contained.* **Unif** *is strongly topological with the operations*

$$\overset{-1}{g}_{\mathrm{Unif}}(\mathfrak{C})=\overset{-1}{g}_{\mathrm{Psunif}}(\mathfrak{C})=\overset{-1}{g}_{\mathrm{Sunif}}(\mathfrak{C}),$$

$$\sup_{\mathrm{Unif}}\{\mathfrak{C}_i\colon i\in I\}=\sup_{\mathrm{Psunif}}\{\mathfrak{C}_i\colon i\in I\}=(\sup_{\mathrm{Sunif}}\{\mathfrak{C}_i\colon i\in I\})^q.$$

PROOF. (0.2), (11.13), (11.14). $\square$

12. W-semi-uniformities. In contrast to the equivalence of uniformities and W-uniformities, we shall see that semi-uniformities and W-semi-uniformities (and similarly, pseudo-uniformities and W-pseudo-uniformities) are essentially different concepts.

If $f\colon X\to Y$ and $\mathcal{U}, \mathcal{V}$ are W-semi-uniformities on X and Y, respectively, we say that f is *$(\mathcal{U}, \mathcal{V})$-continuous* iff, for $V\in\mathcal{V}$, there is $U\in\mathcal{U}$ such that $(x, y)\in U$ implies $(f(x), f(y))\in V$. This can be easier formulated if we denote, for $f\colon X\to Y$, by f_+ the map from $X\times X$ into $Y\times Y$ defined by $f_+(x, y)=(f(x), f(y))$; then the $(\mathcal{U}, \mathcal{V})$-continuity of f means that $V\in\mathcal{V}$ implies the existence of $U\in\mathcal{U}$ such that $f_+(U)\subset V$, or that $f_+^{-1}(V)\in\mathcal{U}$ whenever $V\in\mathcal{V}$ (because $\mathcal{U}$ is ascending).

(12.1) LEMMA. *The W-semi-uniformities and the $(\mathcal{U}, \mathcal{V})$-continuous maps constitute a concrete category* **Wsunif**. $\square$

(12.2) LEMMA. *If $\mathcal{U}$ is a W-semi-uniformity on X, $g\colon Z\to X$, then all sets $U'\subset Z\times Z$ such that $U'\supset g_+^{-1}(U)$ for some $U\in\mathcal{U}$ constitute a W-semi-uniformity*

$g^{-1}(\mathcal{U})$ on Z that is the coarsest W-semi-uniformity $\mathcal{U}'$ on Z such that g is $(\mathcal{U}', \mathcal{U})$-continuous. □

(12.3) LEMMA. *If $\mathcal{U}_i$ is a W-semi-uniformity on X for $i\in I$, then $\sup\{\mathcal{U}_i\colon i\in I\}=\bigcup_{i\in I}\mathcal{U}_i$ is the coarsest W-semi-uniformity on X finer than each $\mathcal{U}_i$, provided $I\neq\emptyset$; for $I=\emptyset$, let $\sup\{\mathcal{U}_i\colon i\in I\}$ denote $\{X\times X\}$, i. e. the coarsest W-semi-uniformity on X.* □

(12.4) LEMMA. *If $g\colon Z\to X$, $f\colon X\to Y$, and $\mathcal{V}$ is a W-semi-uniformity on Y, then $(f\circ g)^{-1}(\mathcal{V})=g^{-1}(f^{-1}(\mathcal{V}))$.* □

(12.5) LEMMA. *If $g\colon Z\to X$, and $\mathcal{U}_i$ is a W-semi-uniformity on X for $i\in I$, then*

$$g^{-1}(\sup\{\mathcal{U}_i\colon i\in I\}) = \sup\{g^{-1}(\mathcal{U}_i)\colon i\in I\}.\quad\square$$

(12.6) THEOREM. *The category* **Wsunif** *is strongly topological.* □

Let **Wpsunif** and **Wunif** denote the full subcategories of **Wsunif** the objects of which are the W-pesudo-uniformities and W-uniformities, respectively.

(12.7) LEMMA. *If $\mathcal{U}$ is a W-semi-uniformity on X, then the collection of all intersections $\bigcap_1^n U_i$, where $U_i\in\mathcal{U}$, $n\in\mathbf{N}$, is the coarsest W-pseudo-uniformity $\mathcal{U}^q$ finer than $\mathcal{U}$.* □

(12.8) LEMMA. *$\mathcal{U}^q=\mathcal{U}$ for every W-pseudo-uniformity $\mathcal{U}$. If $\mathcal{U}$ is a W-semi-uniformity on X, $g\colon Z\to X$, then*

$$g^{-1}(\mathcal{U}^q) = g^{-1}(\mathcal{U})^q.\quad\square$$

(12.9) THEOREM. **Wpsunif** *is a bicoreflective subcategory of* **Wsunif** *and strongly topological with the operations*

$$g^{-1}_{\mathbf{Wpsunif}}(\mathcal{U}) = g^{-1}_{\mathbf{Wsunif}}(\mathcal{U}),$$

$$\sup\nolimits_{\mathbf{Wpsunif}}\{\mathcal{U}_i\colon i\in I\} = (\sup\nolimits_{\mathbf{Wsunif}}\{\mathcal{U}_i\colon i\in I\})^q.$$

The **Wpsunif**-*coreflection of a W-semi-uniformity $\mathcal{U}$ is $\mathcal{U}^q$.*

PROOF. (0.3), (12.7), (12.8). □

(12.10) LEMMA. *If $\mathcal{U}$ is a W-uniformity on X, $g\colon Z\to X$, then $g^{-1}(\mathcal{U})$ is a W-uniformity on Z.*

PROOF. $U_1\circ U_1\subset U$, U_1, $U\subset X\times X$ implies

$$g_+^{-1}(U_1)\circ g_+^{-1}(U_1)\subset g_+^{-1}(U).\quad\square$$

(12.11) LEMMA. *If $\mathcal{U}_i$ is a W-uniformity on X for $i\in I$, then $\sup_{\mathbf{Wpsunif}}\{\mathcal{U}_i\colon i\in I\}$ is a W-uniformity as well.*

PROOF. $U_i'\circ U_i'\subset U_i$, U_i', $U_i\subset X\times X$ $(i=1, \dots, n)$ implies

$$\Big(\bigcap_1^n U_i'\Big)\circ\Big(\bigcap_1^n U_i'\Big)\subset\bigcap_1^n U_i.\quad\square$$

(12.12) THEOREM. **Wunif** *is a bireflective subcategory of every full subcategory of* **Wpsunif** *in which it is contained, and it is strongly topological with the operations*

$$g^{-1}_{\mathbf{Wunif}}(\mathscr{U}) = g^{-1}_{\mathbf{Wpsunif}}(\mathscr{U}) = g^{-1}_{\mathbf{Wsunif}}(\mathscr{U}),$$

$$\sup_{\mathbf{Wunif}}\{\mathscr{U}_i:\ i\in I\} = \sup_{\mathbf{Wpsunif}}\{\mathscr{U}_i:\ i\in I\} = \left(\sup_{\mathbf{Wsunif}}\{\mathscr{U}_i:\ i\in I\}\right)^q.$$

PROOF. (0.2), (12.10), (12.11). □

Now we investigate the connection of W-semi-uniformities and semi-merotopies.

(12.13) LEMMA. *If $\mathscr{U}$ is a W-semi-uniformity on X, then the $\mathscr{U}$-Cauchy systems constitute a semi-merotopy $\mathfrak{M}(\mathscr{U})$ on X.*

PROOF. $\mathscr{U}\neq\emptyset$ implies $\mathfrak{m}\neq\emptyset$ for a $\mathscr{U}$-Cauchy system $\mathfrak{m}$. If $\mathfrak{m}$ is $\mathscr{U}$-Cauchy and $\mathfrak{m}<\mathfrak{m}'$, then $\mathfrak{m}'$ is $\mathscr{U}$-Cauchy. $\{\{x\}\}$ is $\mathscr{U}$-Cauchy for $x\in X$ because $\Delta_X\subset U$ for $U\in\mathscr{U}$. □

(12.14) LEMMA. *Let $\mathfrak{M}$ be a semi-merotopy on $X\neq\emptyset$. Then the collection of all sets $U\subset X\times X$ for which every $\mathfrak{m}\in\mathfrak{M}$ is U-Cauchy is a W-semi-uniformity $\mathscr{U}(\mathfrak{M})$ on X.*

PROOF. Clearly $X\times X\in\mathscr{U}$, and $U\in\mathscr{U}$, $U\subset U'\subset X\times X$ implies $U'\in\mathscr{U}$. If $x\in X$, then $\{\{x\}\}\in\mathfrak{M}$, hence $\Delta_X\subset U$ for $U\in\mathscr{U}$. If $\mathfrak{m}$ is U-Cauchy, then it is $U\cap U^{-1}$-Cauchy as well. □

(12.15) LEMMA. *For a W-semi-uniformity $\mathscr{U}$ on X, we have $\mathscr{U}(\mathfrak{M}(\mathscr{U}))=\mathscr{U}$.*

PROOF. Obviously $\mathscr{U}\subset\mathscr{U}(\mathfrak{M}(\mathscr{U}))$. Conversely, if $U'\notin\mathscr{U}$, $U'\subset X\times X$, then $U-U'\neq\emptyset$ for $U\in\mathscr{U}$ (because $\mathscr{U}$ is ascending). Choose $(x,y)\in(U\cap U^{-1})-U'$ for $U\in\mathscr{U}$. The collection of the sets $\{x,y\}$ is $\mathscr{U}$-Cauchy without being U'-Cauchy, hence $U'\notin\mathscr{U}(\mathfrak{M}(\mathscr{U}))$. □

On the other hand, we can only state:

(12.16) LEMMA. *For a semi-merotopy $\mathfrak{M}$ on X, we have $\mathfrak{M}\subset\mathfrak{M}(\mathscr{U}(\mathfrak{M}))$.* □

The sign $\subset$ cannot be replaced by $=$ in general.

(12.17) EXAMPLE. Let X be an infinite set and $\mathfrak{S}$ the collection of all filters $\dot{A}=\{S:\ A\subset S\subset X\}$, where A is a subset of X containing two elements at most. Clearly $\mathfrak{S}$ is a screen on X and it is a base for a filter-merotopy $\mathfrak{M}$ on X. Now $U\in\mathscr{U}(\mathfrak{M})$ necessarily contains every $(x,y)\in X\times X$, hence $\mathscr{U}(\mathfrak{M})=\{X\times X\}$, and every $\mathfrak{m}\in\exp\exp X$, $\mathfrak{m}\neq\emptyset$ is $\mathscr{U}(\mathfrak{M})$-Cauchy. For $\mathfrak{m}=\{X\}$, $\mathfrak{m}$ is not finer than any element of $\mathfrak{S}$. □

(12.18) LEMMA. *For a semi-merotopy $\mathfrak{M}$ on X, there exists a W-semi-uniformity $\mathscr{U}$ on X such that $\mathfrak{M}=\mathfrak{M}(\mathscr{U})$ iff $\mathfrak{M}=\mathfrak{M}(\mathscr{U}(\mathfrak{M}))$, and then $\mathscr{U}(\mathfrak{M})$ is the unique W-semi-uniformity looked for.*

PROOF. By (12.15), $\mathfrak{M}=\mathfrak{M}(\mathscr{U})$ implies $\mathscr{U}=\mathscr{U}(\mathfrak{M})$, hence the condition is necessary. It is sufficient by (12.14). □

(12.19) *Let $\mathcal{U}$ and $\mathcal{V}$ be W-semi-uniformities on X and Y, respectively, and $f\colon X\to Y$. f is $(\mathcal{U},\mathcal{V})$-continuous iff it is $(\mathfrak{M}(\mathcal{U}),\mathfrak{M}(\mathcal{V}))$-continuous. More generally, if $\mathfrak{M}$ and $\mathfrak{M}'$ are semi-merotopies on X and Y, respectively, then the $(\mathfrak{M},\mathfrak{M}')$-continuity of f implies its $(\mathcal{U}(\mathfrak{M}),\mathcal{U}(\mathfrak{M}'))$-continuity.*

Proof. For $V\in\mathcal{V}$, $\mathfrak{m}\in\exp\exp X$ is $f_+^{-1}(V)$-Cauchy iff $f(\mathfrak{m})$ is V-Cauchy. Hence the $(\mathcal{U},\mathcal{V})$-continuity of f implies its $(\mathfrak{M}(\mathcal{U}),\mathfrak{M}(\mathcal{V}))$-continuity. If f is $(\mathfrak{M},\mathfrak{M}')$-continuous and $V\in\mathcal{U}(\mathfrak{M}')$, then $\mathfrak{m}\in\mathfrak{M}$ implies $f(\mathfrak{m})\in\mathfrak{M}'$, hence $f(\mathfrak{m})$ is V-Cauchy and $\mathfrak{m}$ is $f_+^{-1}(V)$-Cauchy so that $f_+^{-1}(V)\in\mathcal{U}(\mathfrak{M})$. If f is $(\mathfrak{M}(\mathcal{U}),\mathfrak{M}(\mathcal{V}))$-continuous, then it is, by the above result, $(\mathcal{U}(\mathfrak{M}(\mathcal{U})),\mathcal{U}(\mathfrak{M}(\mathcal{V})))$-continuous, i.e. $(\mathcal{U},\mathcal{V})$-continuous by (12.15). □

(12.20) Theorem. *The functor F defined by $F(\mathcal{U})=\mathfrak{M}(\mathcal{U})$ is an isomorphism of* **Wsunif** *onto the full subcategory of* **Smer** *the objects of which are the semi-merotopies $\mathfrak{M}$ satisfying $\mathfrak{M}=\mathfrak{M}(\mathcal{U}(\mathfrak{M}))$.*

Proof. (12.13), (12.18), (12.19). □

(12.21) Lemma. *If $\mathcal{U}$ is a W-pseudo-uniformity on X, then $\mathfrak{M}(\mathcal{U})$ is a merotopy.*

Proof. If $\mathcal{U}$ is a W-pseudo-uniformity, $\mathfrak{m}\in\mathfrak{M}(\mathcal{U})$, and $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$, assume that $m_i\notin\mathfrak{M}(\mathcal{U})$ for $i=1,2$. Then there are $U_i\in\mathcal{U}$ such that U_i does not contain any subset $M\times M$ where $M\in\mathfrak{m}_i$. Now $U_1\cap U_2\in\mathcal{U}$ does not contain any $M\times M$ for $M\in\mathfrak{m}$: a contradiction. □

(12.22) Lemma. *If $\mathfrak{M}$ is a merotopy on X, then $\mathcal{U}(\mathfrak{M})$ is a W-pseudo-uniformity.*

Proof. Let $U_1, U_2\in\mathcal{U}(\mathfrak{M})$, and assume $U_1\cap U_2\notin\mathcal{U}(\mathfrak{M})$. Then there is $\mathfrak{m}\in\mathfrak{M}$ such that $M\times M\subset U_1\cap U_2$ is impossible for $M\in\mathfrak{m}$. Hence $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2$ where $\mathfrak{m}_i$ is the set of those $M\in\mathfrak{m}$ for which $M\times M$ is not a subset of U_i. Thus $\mathfrak{m}_i\in\mathfrak{M}$ for at least one i, in contradiction with the fact that $\mathfrak{m}_i$ is not U_i-Cauchy. □

(12.23) Lemma. *Let $\mathcal{U}$ be a W-semi-uniformity on X. $\mathcal{U}$ is a W-pseudo-uniformity iff $\mathfrak{M}(\mathcal{U})$ is a merotopy.*

Proof. (12.21), (12.15), (12.22). □

(12.24) Theorem. *The functor F defined in* (12.20) *induces an isomorphism from* **Wpsunif** *onto the full subcategory of* **Mer** *the objects of which are the merotopies $\mathfrak{M}$ satisfying $\mathfrak{M}=\mathfrak{M}(\mathcal{U}(\mathfrak{M}))$.*

Proof. (12.20), (12.23). □

Let us denote by **Wsmer** and **Wmer** the subcategories F(**Wsunif**) and F(**Wpsunif**) of **Smer** and **Mer**, respectively. They are proper subcategories by (12.17).

(12.25) Lemma. *Let $\mathcal{U}$ be a W-semi-uniformity on X, $g\colon Z\to X$. Then*

$$g^{-1}_{\mathbf{Smer}}(\mathfrak{M}(\mathcal{U}))=\mathfrak{M}(g^{-1}_{\mathbf{Wsunif}}(\mathcal{U})).$$

Proof. Just as in (12.19), $\mathfrak{m}\in\exp\exp Z$ is $g^{-1}(\mathcal{U})$-Cauchy iff $g(\mathfrak{m})$ is $\mathcal{U}$-Cauchy. □

(12.26) LEMMA. *Let $\mathcal{U}_i$ be a W-semi-uniformity on X for $i\in I$. Then*

$$\sup_{\mathbf{Smer}}\{\mathfrak{M}(\mathcal{U}_i)\colon i\in I\} = \mathfrak{M}\big(\sup_{\mathbf{Wsunif}}\{\mathcal{U}_i\colon i\in I\}\big).$$

PROOF. (12.3), (9.3). □

(12.27) THEOREM. **Wsmer** *is a bireflective subcategory in every full subcategory of* **Smer** *in which it is contained. The reflection in* **Wsmer** *of a semi-merotopy $\mathfrak{M}$ is $\mathfrak{M}(\mathcal{U}(\mathfrak{M}))$.*

PROOF. (0.2), (12.25), (12.26), (12.16) can be completed by the observation that, if $\mathcal{U}$ is a W-semi-uniformity, $\mathfrak{M}$ is a semi-merotopy on X, and $\mathfrak{M}(\mathcal{U})$ is coarser than $\mathfrak{M}$, then $\mathfrak{M}(\mathcal{U})=\mathfrak{M}\big(\mathcal{U}(\mathfrak{M}(\mathcal{U}))\big)\supset\mathfrak{M}\big(\mathcal{U}(\mathfrak{M})\big)$ by (12.15) and (12.19). □

(12.28) LEMMA. *If $\mathcal{U}$ is a W-semi-uniformity on X, then $\mathfrak{M}(\mathcal{U}^q)=\mathfrak{M}(\mathcal{U})^q$.*

PROOF. By (12.7) and (12.23) $\mathfrak{M}(\mathcal{U}^q)$ is a merotopy on X, finer than $\mathfrak{M}(\mathcal{U})$ by (12.19), hence $\mathfrak{M}(\mathcal{U}^q)\subset\mathfrak{M}(\mathcal{U})$ and $\mathfrak{M}(\mathcal{U}^q)\subset\mathfrak{M}(\mathcal{U})^q$ by (9.10). Conversely, for $\mathfrak{m}\in\mathfrak{M}(\mathcal{U})^q$, assume $\mathfrak{m}\notin\mathfrak{M}(\mathcal{U}^q)$. Then there are $U_i\in\mathcal{U}$ $(i=1, \dots, n)$ such that $M\times M\subset\bigcap_1^n U_i$ is impossible for $M\in\mathfrak{m}$. Denote by $\mathfrak{m}_i$ the set of those $M\in\mathfrak{m}$ for which $M\times M\subset U_i$ does not hold. Then $\mathfrak{m}=\bigcup_1^n \mathfrak{m}_i$ and $\mathfrak{m}_i\in\mathfrak{M}(\mathcal{U})$ for at least one i; this is in contradiction with the fact that $\mathfrak{m}_i$ is not U_i-Cauchy. □

(12.29) LEMMA. *If $\mathcal{U}$ is a W-pseudo-uniformity on X, $g\colon Z\to X$, then*

$$g^{-1}_{\mathbf{Mer}}(\mathfrak{M}(\mathcal{U})) = g^{-1}_{\mathbf{Smer}}(\mathfrak{M}(\mathcal{U})) = \mathfrak{M}\big(g^{-1}_{\mathbf{Wsunif}}(\mathcal{U})\big) = \mathfrak{M}\big(g^{-1}_{\mathbf{Wpsunif}}(\mathcal{U})\big).$$

PROOF. (9.13) and (12.25) show the first two equalities; the third one results from (12.9). □

(12.30) LEMMA. *If $\mathcal{U}_i$ is a W-pseudo-uniformity on X for $i\in I$, then*

$$\sup_{\mathbf{Mer}}\{\mathfrak{M}(\mathcal{U}_i)\colon i\in I\} = \big(\sup_{\mathbf{Smer}}\{\mathfrak{M}(\mathcal{U}_i)\colon i\in I\}\big)^q = \mathfrak{M}\big(\sup_{\mathbf{Wsunif}}\{\mathcal{U}_i\colon i\in I\}\big)^q =$$
$$= \mathfrak{M}\big((\sup_{\mathbf{Wsunif}}\{\mathcal{U}_i\colon i\in I\})^q\big) = \mathfrak{M}\big(\sup_{\mathbf{Wpsunif}}\{\mathcal{U}_i\colon i\in I\}\big).$$

PROOF. (9.13), (9.3), (12.26), (12.28), (12.9). □

(12.31) THEOREM. **Wmer** *is a bireflective subcategory of every full subcategory of* **Mer** *in which it is contained. The reflection in* **Wmer** *of a merotopy $\mathfrak{M}$ is $\mathfrak{M}(\mathcal{U}(\mathfrak{M}))$.*

PROOF. (0.2), (12.29), (12.30), (12.27), (12.22), (12.21). □

(12.32) THEOREM. **Wmer** *is a bicoreflective subcategory of every full subcategory of* **Wsmer** *in which it is contained. The* **Wmer**-*coreflection of $\mathfrak{M}=\mathfrak{M}(\mathcal{U})$, where $\mathcal{U}$ is a W-semi-uniformity, is $\mathfrak{M}^q=\mathfrak{M}(\mathcal{U}^q)$.*

PROOF. (12.20), (12.24), (12.9), (12.28). □

13. Characterization of Wsmer. Our next purpose is to characterize, based on an idea of [20], those semi-merotopies $\mathfrak{M}$ that can be written in the form $\mathfrak{M}(\mathcal{U})$ for some W-semi-uniformity $\mathcal{U}$.

For a given set X, let us denote by $\mathfrak{B}_X$ the collection of all $\mathfrak{m}\in\exp\exp X$ such that each $M\in\mathfrak{m}$ has two elements at most. For $\mathfrak{m}\in\mathfrak{B}_X$, let $B(\mathfrak{m})\subset X\times X$ be defined by

$$B(\mathfrak{m}) = \{(x, y)\colon \{x, y\}\in\mathfrak{m}\} = B(\mathfrak{m})^{-1}.$$

For a semi-merotopy $\mathfrak{M}$, we define

$$\mathscr{B}(\mathfrak{m}) = \{B(\mathfrak{m})\colon \mathfrak{m}\in\mathfrak{M}\cap\mathfrak{B}_X\}.$$

(13.1) LEMMA. *Let $\mathfrak{M}$ be a semi-merotopy on X and $U=U^{-1}\subset X\times X$. Then $U\in\mathscr{U}(\mathfrak{M})$ iff $U\in\sec\mathscr{B}(\mathfrak{M})$.*

PROOF. If $U\in\mathscr{U}(\mathfrak{M})$, $\mathfrak{m}\in\mathfrak{M}\cap\mathfrak{B}_X$, $B=B(\mathfrak{m})$, there is $\{x, y\}\in\mathfrak{m}$ such that $\{x, y\}\times\{x, y\}\subset U$ so that $(x, y)\in U\cap B$. Conversely assume $U\in\sec\mathscr{B}(\mathfrak{M})$, $\mathfrak{m}\in\mathfrak{M}$, but $(M\times M)-U\neq\emptyset$ for every $M\in\mathfrak{m}$. Then, for $M\in\mathfrak{m}$, there is $(x, y)\in\in(M\times M)-U$; the sets $\{x, y\}$ obtained from these pairs (x, y) constitute a system $\mathfrak{m}^*$ finer than $\mathfrak{m}$, hence belonging to $\mathfrak{M}\cap\mathfrak{B}_X$. Clearly $U\cap B(\mathfrak{m}^*)=\emptyset$: a contradiction. □

(13.2) LEMMA. *Let $\mathscr{U}$ be a W-semi-uniformity on X, $\mathfrak{m}\in\mathfrak{B}_X$. Then $\mathfrak{m}\in\mathfrak{M}(\mathscr{U})$ iff $B(\mathfrak{m})\in\sec\mathscr{U}$.*

PROOF. $B(\mathfrak{m})\in\sec\mathscr{U}$ iff $B(\mathfrak{m})\cap U\neq\emptyset$ for every $U\in\mathscr{U}$ such that $U=U^{-1}$. For such a set U, $(x, y)\in B(\mathfrak{m})\cap U$ holds iff $\{x, y\}\times\{x, y\}\subset U$. □

(13.3) LEMMA. *Let $\mathfrak{M}$ be a semi-merotopy on X. If $\mathfrak{m}'\in\mathfrak{B}_X\cap\mathfrak{M}(\mathscr{U}(\mathfrak{M}))$, then $\mathfrak{m}'\in\mathfrak{M}$.*

PROOF. By (13.2), $B(\mathfrak{m}')\in\sec\mathscr{U}(\mathfrak{M})$. Then there is $\mathfrak{m}\in\mathfrak{M}\cap\mathfrak{B}_X$ such that $B(\mathfrak{m}')\supset B(\mathfrak{m})$. In fact, in the opposite case there would be, for every $\mathfrak{m}\in\mathfrak{M}\cap\mathfrak{B}_X$, a pair $(x, y)\in B(\mathfrak{m})-B(\mathfrak{m}')$, and then these pairs (x, y) and the corresponding pairs (y, x) would constitute a set U such that $U=U^{-1}$ and $U\in\sec\mathscr{B}(\mathfrak{M})$, whence $U\in\mathscr{U}(\mathfrak{M})$, $U\cap B(\mathfrak{m}')=\emptyset$: a contradiction. Now $B(\mathfrak{m})\subset B(\mathfrak{m}')$ implies $\mathfrak{m}\subset\mathfrak{m}'$ so that $\mathfrak{m}'\in\mathfrak{M}$. □

(13.4) LEMMA. *If $\mathscr{U}$ is a W-semi-uniformity on X and $\mathfrak{M}=\mathfrak{M}(\mathscr{U})$, then $\mathfrak{M}$ fulfils the following condition:*

WM. $\mathfrak{m}\in\exp\exp X$, $\mathfrak{m}\notin\mathfrak{M}$ *implies the existence of* $\mathfrak{m}'\in\mathfrak{B}_X$ *such that* $\mathfrak{m}<\mathfrak{m}'$, $\mathfrak{m}'\notin\mathfrak{M}$.

PROOF. If $\mathfrak{m}$ is not $\mathscr{U}$-Cauchy, then there is $U\in\mathscr{U}$ such that $M\in\mathfrak{m}$, $M\times M\subset U$ is impossible. Select, for $M\in\mathfrak{m}$, $x, y\in M$ such that $(x, y)\notin U$. The corresponding sets $\{x, y\}$ constitute a system $\mathfrak{m}'\in\mathfrak{B}_X$ such that $\mathfrak{m}<\mathfrak{m}'$ and $\mathfrak{m}'$ is not U-Cauchy. □

(13.5) THEOREM. *A semi-merotopy $\mathfrak{M}$ on X can be written in the form $\mathfrak{M}=\mathfrak{M}(\mathscr{U})$ for some W-semi-uniformity $\mathscr{U}$ iff it satisfies* WM.

PROOF. By (13.4) WM is necessary. Conversely, by (13.3) every element of $\mathfrak{B}_X\cap\mathfrak{M}(\mathscr{U}(\mathfrak{M}))$ belongs to $\mathfrak{M}$, so that, if $\mathfrak{M}$ satisfies WM, then $\mathfrak{M}(\mathscr{U}(\mathfrak{M}))\subset\mathfrak{M}$. By (12.16) $\mathfrak{M}=\mathfrak{M}(\mathscr{U}(\mathfrak{M}))$. □

(13.6) COROLLARY. *For a merotopy $\mathfrak{M}$ on X, there is a W-pseudo-uniformity $\mathcal{U}$ such that $\mathfrak{M}=\mathfrak{M}(\mathcal{U})$ iff $\mathfrak{M}$ satisfies* WM.

PROOF. (13.5), (12.18), (12.22). $\square$

14. W-semi-uniformities and semi-proximities. Now let $\mathcal{U}$ be a W-semi-uniformity on X, $\mathfrak{M}=\mathfrak{M}(\mathcal{U})$. We denote by $\delta(\mathcal{U})$ the semi-proximity $\delta(\mathfrak{M})$.

(14.1) LEMMA. *For a W-semi-uniformity $\mathcal{U}$ on X, $A\delta(\mathcal{U})B$ iff $A\times B\in\sec\mathcal{U}$.*

PROOF. If $A\delta(U)B$, then there is $\mathfrak{m}\in\mathfrak{M}(\mathcal{U})$ such that $A, B\in\sec\mathfrak{m}$ and, for $U\in\mathcal{U}$, there exists $M\in\mathfrak{m}$ satisfying $M\times M\subset U$. Then clearly $(A\times B)\cap U\neq\emptyset$.
Conversely, if $A\times B\in\sec\mathcal{U}$, select, for $U\in\mathcal{U}$, $U=U^{-1}$ a pair $(x, y)\in\in(A\times B)\cap U$. Then the corresponding sets $\{x, y\}$ constitute a system $\mathfrak{m}\in\mathfrak{M}(\mathcal{U})$ such that $A, B\in\sec\mathfrak{m}$, so that $A\delta(\mathcal{U})B$. $\square$

(14.2) LEMMA. *If $\mathfrak{M}$ is a semi-merotopy on X, $\mathcal{U}=\mathcal{U}(\mathfrak{M})$, then $\delta(\mathcal{U})=\delta(\mathfrak{M})$.*

PROOF. By (12.16) $\mathfrak{M}(\mathcal{U})$ is coarser than $\mathfrak{M}$, hence $\delta(\mathcal{U})$ is coarser than $\delta(\mathfrak{M})$. Conversely suppose $A\overline{\delta(\mathfrak{M})}B$. Then

$$V=(X\times X)-(A\times B)\in\mathcal{U}.$$

In fact,

$$V=((X-A)\times X)\cup(X\times(X-B)),$$

and if $\mathfrak{m}\in\mathfrak{M}$ satisfies $A\in\sec\mathfrak{m}$, then $B\notin\sec\mathfrak{m}$, hence $M\subset X-B$, $M\times M\subset\subset X\times(X-B)$ for some $M\in\mathfrak{m}$; similarly $A\notin\sec\mathfrak{m}$ implies $M\subset X-A$, $M\times M\subset\subset(X-A)\times X$ for a suitable $M\in\mathfrak{m}$.
Now $(A\times B)\cap V=\emptyset$ implies $A\overline{\delta(\mathcal{U})}B$ by (14.1). $\square$

(14.3) LEMMA. *Let δ be a semi-proximity on X. Then $\mathfrak{M}=\mathfrak{M}(\delta)=\mathfrak{M}(\mathcal{U})$ for $\mathcal{U}=\mathcal{U}(\mathfrak{M})$, and $\mathcal{U}$ is the coarsest W-semi-uniformity such that $\delta(U)=\delta$.*

PROOF. By (10.5) and (10.3) $\mathfrak{M}$ is the coarsest semi-merotopy satisfying $\delta(\mathfrak{M})=\delta$. By (14.2) $\delta(\mathcal{U})=\delta(\mathfrak{M}(\mathcal{U}))=\delta$, and by (12.16) $\mathfrak{M}(\mathcal{U})$ is coarser than $\mathfrak{M}$. Hence $\mathfrak{M}=\mathfrak{M}(\mathcal{U})$. If $\mathcal{U}'$ is a W-semi-uniformity such that $\delta(\mathcal{U}')=\delta$, then $\mathfrak{M}(\mathcal{U}')$ is finer than $\mathfrak{M}$, hence $\mathcal{U}'=\mathcal{U}(\mathfrak{M}(\mathcal{U}'))$ is finer than $\mathcal{U}(\mathfrak{M})=\mathcal{U}$ (cf. (12.15)). $\square$

(14.4) COROLLARY. *For every semi-proximity δ, there are W-semi-uniformities $\mathcal{U}$ such that $\delta(\mathcal{U})=\delta$.* $\square$

(14.5) LEMMA. *Let δ be a proximity on X. Then $\mathfrak{M}=\mathfrak{M}(\delta)^q$ is the coarsest merotopy such that $\delta(\mathfrak{M})=\delta$, and $\mathcal{U}=\mathcal{U}(\mathfrak{M})$ is the coarsest W-pseudo-uniformity satisfying $\delta(\mathcal{U})=\delta$. We have also $\mathfrak{M}(\mathcal{U})=\mathfrak{M}$.*

PROOF. By (10.5) $\delta(\mathfrak{M}(\delta))=\delta$ and by (10.13) $\delta(\mathfrak{M})=\delta(\mathfrak{M}(\delta)^q)=\delta^q=\delta$. $\mathfrak{M}$ is a merotopy and if a merotopy $\mathfrak{M}'$ satisfies $\delta(\mathfrak{M}')=\delta$, then, by (10.3), $\mathfrak{M}'$ is finer than $\mathfrak{M}(\delta)$, hence finer than $\mathfrak{M}(\delta)^q=\mathfrak{M}$.
$\mathcal{U}$ is a W-pseudo-uniformity by (12.22), and by (14.2) $\delta(\mathcal{U})=\delta(\mathfrak{M})=\delta$. $\mathfrak{M}(\mathcal{U})$ is a merotopy by (12.21), satisfies $\delta(\mathfrak{M}(\mathcal{U}))=\delta(\mathcal{U}(\mathfrak{M}(\mathcal{U})))=\delta(\mathcal{U})=\delta$ by (14.2) and (12.15), hence $\mathfrak{M}(\mathcal{U})=\mathfrak{M}$. If a W-pseudo-uniformity $\mathcal{U}'$ satisfies $\delta(\mathcal{U}')=\delta$,

then the merotopy $\mathfrak{M}(\mathcal{U}')$ (see (12.23)) is, by a similar argument, finer than $\mathcal{U}=\mathcal{U}(\mathfrak{M})$. $\square$

(14.6) Corollary. *For every proximity δ on X, there exist W-pseudo-uniformities $\mathcal{U}$ such that $\delta(\mathcal{U})=\delta$.* $\square$

Another proof of (14.6) is obtained from the following:

(14.7) Lemma. *Let δ be a proximity on X, $\mathfrak{S}$ the screen composed of all δ-compressed filters, $\mathfrak{M}$ the filter-merotopy for which $\mathfrak{S}$ is a base, $\mathcal{U}=\mathcal{U}(\mathfrak{M})$. Then $\delta(\mathcal{U})=\delta$, $\mathfrak{M}$ is the coarsest filter-merotopy compatible with δ, and $\mathfrak{S}$ is composed of all $\mathcal{U}$-Cauchy filters.*

Proof. $U\in\mathcal{U}$ iff every $\mathfrak{s}\in\mathfrak{S}$ is U-Cauchy, and $\delta=\delta(\mathfrak{S})$ by (6.11). Clearly $\delta(\mathfrak{S})=\delta(\mathfrak{M})$, hence by (14.2) $\delta(\mathcal{U})=\delta$.

Every $\mathfrak{s}\in\mathfrak{S}$ is obviously $\mathcal{U}$-Cauchy. Conversely, a $\mathcal{U}$-Cauchy filter $\mathfrak{s}$ is δ-compressed by $\delta(\mathcal{U})=\delta(\mathfrak{M}(\mathcal{U}))=\delta$ and (10.3). If a filter-merotopy $\mathfrak{M}'$ with a base $\mathfrak{S}'$ composed of filters satisfies $\delta(\mathfrak{M}')=\delta$, then every $\mathfrak{s}\in\mathfrak{S}'$ is δ-compressed, $\mathfrak{S}'\subset\mathfrak{S}$, $\mathfrak{M}'\subset\mathfrak{M}$. $\square$

The merotopy $\mathfrak{M}$ in (14.7) need not coincide with $\mathfrak{M}(\delta)$; in fact, by (10.16), the latter is not a merotopy in general. The author does not know whether the merotopy $\mathfrak{M}$ in (14.7) is always equal to $\mathfrak{M}(\delta)^q$, or whether at least $\mathfrak{M}=\mathfrak{M}(\mathcal{U}(\mathfrak{M}))$; by (12.17), a filter-merotopy $\mathfrak{M}$ need not fulfil the last equality.

If $\mathcal{U}$ is a W-semi-uniformity on X, let us denote by $\mathfrak{S}(\mathcal{U})$ the screen composed of all $\mathcal{U}$-Cauchy filters. For the W-pseudo-uniformity $\mathcal{U}$ in (14.7), we had $\delta(\mathfrak{S}(\mathcal{U}))=$ $=\delta(\mathcal{U})$. This equality need not hold in general.

(14.8) Example. Let $\mathcal{U}$ denote the Euclidean W-uniformity on $X=\mathbf{R}$, then $\delta(U)$ is the Euclidean proximity by (14.1), while $A\delta(\mathfrak{S}(\mathcal{U}))B$ iff there is a convergent filter $\mathfrak{s}$ such that $A, B\in\sec\mathfrak{s}$, i.e. iff $c(A)\cap c(B)\neq\emptyset$ for the usual topology c of $\mathbf{R}$. $\square$

In this example, $\delta(\mathcal{U})$ and $\delta(\mathfrak{S}(\mathcal{U}))$ are distinct but they induce the same topology. This is always true if $\mathcal{U}$ is a W-uniformity: the above argument shows that $A\delta(\mathfrak{S}(\mathcal{U}))B$ iff $c'(A)\cap c'(B)\neq\emptyset$ where c' is the topology of the completion $\mathcal{U}'$ of $\mathcal{U}$, and it is well-known that c' is an extension of c_δ for $\delta=\delta(\mathcal{U})$.

However, for W-pseudo-uniformities $\mathcal{U}$, $c_\delta\neq c_{\delta'}$, in general, where $\delta=\delta(\mathcal{U})$, $\delta'=\delta(\mathfrak{S}(\mathcal{U}))$.

(14.9) Example. For $X=\mathbf{R}$, define on $X\times X$ a topology for which (x, y) is isolated if $x\neq y$ and a neighbourhood base of (x, x) is composed of the sets $V_\varepsilon(x, x)$ $(\varepsilon>0)$, where

$$V_\varepsilon(x,x)=\{(t,x)\colon x\leqq t<x+\varepsilon\}\cup\{(x,t)\colon x\leqq t<x+\varepsilon\}.$$

Since the symmetry map $\varphi(x,y)=(y,x)$ is a homeomorphism for this topology, it is easily seen that the neighbourhoods of the diagonal Δ_X constitute a W-pseudo-uniformity $\mathcal{U}$ on X. If

$$U=\bigcup_{x\in X}V_{\varepsilon(x)}(x,x),\quad \varepsilon(x)>0 \quad\text{for}\quad x\in X,$$

then $U(x)=[x, x+\varepsilon(x))$ so that, for $\delta=\delta(\mathcal{U})$, c_δ is the Sorgenfrey topology on X. On the other hand, $\mathcal{U}$ is obviously finer than the Euclidean W-uniformity of $\mathbf{R}$, hence a $\mathcal{U}$-Cauchy filter $\mathfrak{s}$ is convergent with respect to the Euclidean topology of $\mathbf{R}$; suppose $\mathfrak{s}\to x$. Now

$$U = V_1(x,x)\cup \bigcup_{y\neq x} V_{\left|\frac{y-x}{2}\right|}(y,y)\in\mathcal{U}$$

and $S\in\mathfrak{s}$, $S\times S\subset U$ implies $S=\{x\}$ so that $\mathfrak{s}=\dot{x}$, and $\delta'=\delta(\mathfrak{S}(\mathcal{U}))$ is the discrete proximity, $c_{\delta'}\neq c_\delta$ is the discrete topology. □

Another case when $\delta(\mathcal{U})=\delta(\mathfrak{S}(\mathcal{U}))$ is furnished by the following:

(14.10) LEMMA. *Let c be an S_1-topology on X, and let $\mathcal{U}$ be composed of the neighbourhoods of Δ_X with respect to the product topology. Then $\mathcal{U}$ is a W-pseudo-uniformity on X, and $\delta(U)=\delta(\mathfrak{S}(\mathcal{U}))=\delta_c$.*

PROOF. Clearly $\mathcal{U}$ is a W-pseudo-uniformity on X. If $A\delta_c B$, i.e. if $c(A)\cap c(B)\neq\emptyset$, then the c-neighbourhood filter $\mathfrak{s}$ of some $x\in c(A)\cap c(B)$ is clearly $\mathcal{U}$-Cauchy and satisfies $A, B\in\sec\mathfrak{s}$; hence δ_c is finer than $\delta(\mathfrak{S}(\mathcal{U}))$. The proximity $\delta(\mathfrak{S}(\mathcal{U}))$ is obviously finer than $\delta(\mathcal{U})$, and the latter is finer than δ_c because if $c(A)\cap c(B)=\emptyset$, we can find for $x\in X$ a c-neighbourhood V_x such that either $V_x\cap A=\emptyset$ or $V_x\cap B=\emptyset$, so that $U=\bigcup_{x\in X}(V_x\times V_x)\in\mathcal{U}$ fulfils $(A\times B)\cap U=\emptyset$. □

It is not surprising that, in (14.9), a W-pseudo-uniformity on X was defined as the collection of all neighbourhoods of Δ_X with respect to some topology on $X\times X$:

(14.11) LEMMA. *If c_+ is a topology on $X\times X$ such that the map $\varphi\colon X\times X\to X\times X$ defined by $\varphi(x,y)=(y,x)$ is (c_+, c_+)-continuous, then the c_+-neighbourhoods of Δ_X constitute a W-pseudo-uniformity on X.* □

(14.12) LEMMA. *If $\mathcal{U}$ is a W-pseudo-uniformity on X, then there exists a T_0-topology on $X\times X$ such that φ in* (14.11) *is (c_+, c_+)-continuous and $\mathcal{U}$ is the collection of all c_+-neighbourhoods of Δ_X.*

PROOF. Let the points of $(X\times X)-\Delta_X$ be isolated for c_+, and, for $x\in X$, let the system

$$\{U-F\colon\ U\in\mathcal{U},\quad F\subset\Delta_X \ \text{ is finite},\quad (x,x)\notin F\}$$

be a c_+-neighbourhood base of (x,x). Clearly c_+ is a T_0-topology on $X\times X$ and φ is (c_+, c_+)-continuous. Every $U\in\mathcal{U}$ is a c_+-neighbourhood of Δ_X and, if V is a c_+-neighbourhood of Δ_X, then, for $x\in X$, there are $U\in\mathcal{U}$ and a finite set $F\subset\Delta_X$ such that $(x,x)\in U-F\subset V$; now $U\subset V$ because $U-\Delta_X\subset V$, $\Delta_X\subset V$. □

(14.13) COROLLARY. *If $\mathcal{U}$ is a separated W-pseudo-uniformity on X* (i.e. a W-pseudo-uniformity such that $\bigcap\mathcal{U}=\Delta_X$), *then the topology c_+ constructed in* (14.12) *is T_1.* □

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EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
BUDAPEST, MÚZEUM KRT. 6—8
H—1088

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SOLVING THE FARTHEST POINT PROBLEM IN FINITE CODIMENSIONAL SUBSPACE OF $C(\Omega)$

Á. P. BOSZNAY (Budapest)

Introduction

Let $(X, \|\cdot\|)$ be a real normed linear space, $\emptyset \neq K \subset X$ a bounded set. Let us define the mapping $Q_K\colon X \to 2^K$ by

$$Q_K(x) = \{y \in K;\ \|x-y\| = \sup_{k\in K} \|x-k\|\}.$$

We call K a uniquely remotal set, if for all $x \in X$, $Q_K(x)$ consists of exactly one element. In this case we denote by $Q_K(x)$ this element of K.

The following problem is called the farthest point problem. The problem — as far as we know — is open.

Let K be a uniquely remotal set in $(X, \|\cdot\|)$. Is K a singleton?

There are many special cases in which the problem is solved affirmatively.

This is the case for finite dimensional X [1], for norm-compact K [1], for norm-continuous Q_K [2], in the Banach spaces c_0, c [7]; with a suitable renorming in all normed linear spaces [5].

In [4] we have given a positive answer in many special finite codimensional subspaces of $C[0, 1]$, including C. In this note we give a general solution for all finite codimensional subspaces of $C(\Omega)$, where Ω is a compact metric space.

The result

THEOREM. *Let $(X, \|\cdot\|)$ be a finite codimensional subspace of $C(\Omega)$, where Ω is a compact metric space. Let $K \subset X$ be a uniquely remotal set. Then K is a singleton.*

PROOF. We need several lemmas.

LEMMA 1 [6, Lemma 1.2]. *With the assumptions of the Theorem, for all $\varepsilon > 0$ and $V \subset \Omega$ infinite and open, there exists an $f \in X$ with the properties $\|f\| = 1$, $f(\Omega \setminus V) = 0$, $f \geqq -\varepsilon$.*

In fact, this is a special case of Lemma 1.2 in [6].

LEMMA 2. *With the assumptions of the Theorem, let us assume that the set $Q_K(X) = \{Q_K(x);\ x \in X\}$ is countable. Then K is a singleton.*

PROOF OF LEMMA 2. Let us assume that $Q_K(X) = \{k_1, k_2, k_3, \ldots\}$. It is easy to show that the sets $Q_K^{-1}(k_n)$ are disjoint and closed, and also $\bigcup_{n=1} Q_K^{-1}(k_n) = X$. Using a lemma of Asplund [7], all but one of the sets $Q_K^{-1}(k_n)$ are void. This implies the assertion of the Lemma.

Lemma 3. *Let $V_1, V_2 \subset \Omega$ be open infinite sets such that $V_1 \cap V_2 = \emptyset$, and let $f_1, f_2 \in X$ be such that $\|f_1\| = \|f_2\| = 1$,*

$$f_1(\Omega \setminus V_1) = 0, \quad f_2(\Omega \setminus V_2) = 0, \quad f_1, f_2 \geqq -\frac{1}{3}.$$

Then there exist $c_1, c_2 \in \mathbf{R}^+$ such that

$$\|c_1 f_1 - Q_K(c_1 f_1)\| = c_1 f_1(\omega_1) - Q_K(c_1 f_1)(\omega_1)$$

for some $\omega_1 \in V_1$, and

$$\|c_2 f_2 - Q_K(c_2 f_2)\| = c_2 f_2(\omega_2) - Q_K(c_2 f_2)(\omega_2)$$

for some $\omega_2 \in V_2$, and $Q_K(c_1 f_1) = Q_K(c_2 f_2)$.

Proof of Lemma 3. First, we shall verify that if $c_1, c_2 > 3 \sup_{k \in K} \|k\|$ then

$$\|c_1 f_1 - Q_K(c_1 f_1)\| = c_1 f_1(\omega_1) - Q_K(c_1 f_1)(\omega_1) \tag{1}$$

for some $\omega_1 \in V_1$, and

$$\|c_2 f_2 - Q_K(c_2 f_2)\| = c_2 f_2(\omega_2) - Q_K(c_2 f_2)(\omega_2) \tag{2}$$

for some $\omega_2 \in V_2$.

By

$$|Q_K(c_1 f_1)(\omega)| \leqq \sup_{k \in K} \|k\| \quad (\omega \in \Omega),$$

we have

$$|c_1 f_1(\omega) - Q_K(c_1 f_1)(\omega)| \leqq \sup_{k \in K} \|k\| \quad \text{for} \quad \omega \in \Omega \setminus V_1. \tag{3}$$

Also $c_1 f_1(\omega') = c_1$ for some $\omega' \in V_1$. Using this,

$$c_1 f_1(\omega') - Q_K(c_1 f_1)(\omega') \geqq c_1 - \sup_{k \in K} \|k\|. \tag{4}$$

On the other hand $c_1 f_1(\omega'') \geqq -\frac{c_1}{3}$ for arbitrary $\omega'' \in \Omega$, so,

$$Q_K(c_1 f_1)(\omega'') - c_1 f_1(\omega'') \leqq \sup_{k \in K} \|k\| + \frac{c_1}{3}. \tag{5}$$

Using $c_1 > 3 \sup_{k \in K} \|k\|$, (3), (4) and (5) imply (1). The proof of (2) is similar.

Because of continuity we can fix c_1, c_2 so that

$$\|c_1 f_1 - Q_K(c_1 f_1)\| = \|c_2 f_2 - Q_K(c_2 f_2)\|.$$

We now show that $Q_K(c_1 f_1) = Q_K(c_2 f_2)$. Introducing the element $f = c_1 f_1 + c_2 f_2 \in X$, $c_1, c_2 > 3 \sup_{k \in K} \|k\|$ and $V_1 \cap V_2 = \emptyset$ implies that

$$\|f - Q_K(f)\| = c_1 f_1(\omega) + c_2 f_2(\omega) - Q_K(f)(\omega) \tag{6}$$

for some $\omega\in V_1\cup V_2$. Assuming that $\omega\in V_1$, $c_1f_1(\omega)+c_2f_2(\omega)=c_1f_1(\omega)$, so

$$c_1f_1(\omega)+c_2f_2(\omega)-Q_K(f)(\omega) = c_1f_1(\omega)-Q_K(f)(\omega) \leqq \tag{7}$$
$$\leqq \|c_1f_1-Q_K(f)\| \leqq \|c_1f_1-Q_K(c_1f_1)\| = c_1f_1(\omega_1)-Q_K(c_1f_1)(\omega_1).$$

Thus $Q_K(f)=Q_K(c_1f_1)$. For $\omega_2\in V_2$ we have

$$c_1f_1(\omega_2)+c_2f_2(\omega_2)-Q_K(c_2f_2)(\omega_2) = c_2f_2(\omega_2)-Q_K(c_2f_2)(\omega_2) =$$
$$= c_1f_1(\omega_1)-Q_K(c_1f_1)(\omega_1).$$

By (2), (6) and (7), the latter implies

$$\|c_1f_1+c_2f_2-Q_K(c_2f_2)\| = \|c_1f_1+c_2f_2-Q_K(c_1f_1+c_2f_2)\|,$$

and finally we get $Q_K(f)=Q_K(c_2f_2)$. Lemma 3 is proved.

LEMMA 4. *Let $f\in X$ be arbitrary,*

$$\|f-Q_K(f)\| = f(\omega)-Q_K(f)(\omega) \quad \textit{for some} \quad \omega\in\Omega.$$

Then $\min\limits_{k\in K} k(\omega)=Q_K(f)(\omega)$.

The proof of Lemma 4 is left to the reader.

LEMMA 5. *Let us introduce the set $\Omega_d\subset\Omega$ in the following way: $\Omega_d=\{\omega\in\Omega$, ω is not isolated, $\exists x\in X$ such that $x(\omega)-Q_K(x)(\omega)=\|x-Q_K(x)\|\}$. Then there exists an element $k_d\in K$ such that*

$$\inf_{k\in K} k(\omega) = k_d(\omega)$$

for all $\omega\in\Omega_d$.

PROOF OF LEMMA 5. Let $d_1, d_2\in\Omega_d$ be fixed, let $\varepsilon>0$ be such that dist $(d_1, d_2)>2\varepsilon$, V_1, V_2 are open infinite nonempty sets with the properties $\overline{V}_1\cap\overline{V}_2=\emptyset$, diam $(V_1)<\varepsilon$, diam $(V_2)<\varepsilon$, $d_1\in V_1$, $d_2\in V_2$. Let $d\notin\overline{V}_1$, $d\in\Omega_d$. Then there exists an open infinite nonempty set U such that $d\in U$, $U\cap\overline{V}_1=\emptyset$. We can choose the diameter of U arbitrarily small. Lemma 3 implies that there exist $d_1'\in V_1$, $d'\in U$, $g_1, g\in X$ such that

$$\|g_1-Q_K(g_1)\| = g_1(d_1')-Q_K(g_1)(d_1'),$$
$$\|g-Q_K(g)\| = g(d')-Q_K(g)(d'), \quad Q_K(g_1) = Q_K(g).$$

Using Lemma 4,

$$\min_{k\in K} k(d') = Q_K(g)(d') = Q_K(g)(d') = Q_K(g_1)(d'), \tag{8}$$

$$\min_{k\in K} k(d_1') = Q_K(g)(d_1') = Q_K(g_1)(d_1'). \tag{9}$$

From the uniqueness of $Q_K(g)$ and $Q_K(g_1)$ it follows that no other element of K but $Q_K(g)=Q_K(g_1)$ can stay on the right side of (8) and (9).

Let the diameter of U tend to 0. Elementary continuity reasoning shows that

$$\min_{k\in K} k(d) = Q_K(g_1)(d) = Q_K(g)(d).$$

So,

$$\min_{k\in K} k(d) = Q_K(g_1)(d), \tag{10}$$

when $d\in\Omega_d\setminus\bar{V}_1$.

Now let $d\notin\bar{V}_2$, $d\in\Omega_d$. Repeating the same arguments as above, we have a $g_2\in X$, such that

$$\min_{k\in K} k(d) = Q_K(g_2)(d), \tag{11}$$

when $d\in\Omega_d\setminus\bar{V}_2$. Now let f_1, f_2 fulfill the requirements of Lemma 3, with open infinite sets V_1, V_2. Using (10) and (11), Lemma 3 implies

$$Q_K(c_1f_1) = Q_K(g_2),\quad Q_K(c_2f_2) = Q_K(g_1)\quad \text{and}\quad Q_K(c_1f_1) = Q_K(c_2f_2).$$

Finally, we have $Q_K(g_1)=Q_K(g_2)$. Lemma 5 is proved.

LEMMA 6. *Let us introduce the set $\Omega_U\subset\Omega$ in the following way: $\Omega_U=\{\omega\in\Omega$, ω is not isolated, $\exists x\in X$ such that $Q_K(x)(\omega)-x(\omega)=\|x-Q_K(x)\|\}$. Then there exists an element $k_u\in K$ such that*

$$\sup_{k\in K} k(\omega) = k_u(\omega)$$

for all $\omega\in\Omega_U$.

PROOF OF LEMMA 6. Similar to the proof of Lemma 5.

Finally, we shall prove the Theorem. Let $x\in X$ be such that

$$\|x-Q_K(x)\| = x(\omega)-Q_K(x)(\omega)$$

for some $\omega\in\Omega$. We consider now two cases.

Case I: $\omega\in\Omega_d$. Here using Lemma 5 one can prove elementarily that $Q_K(x)=k_d$.

Case II: ω is an isolated point. Using the fact that the set of isolated points in Ω is at most countable, one can easily prove that $Q_K(x)$ is an element of a fixed countable set (i.e., the set $\{\tilde{k}\in K;\ \tilde{k}(\omega)=\inf_{k\in K} k(\omega)$ and ω is isolated$\}$). For an $x\in X$ with property

$$\|x-Q_K(x)\| = Q_K(x)(\omega)-x(\omega)$$

we can proceed on the same way. Summing up, $Q_K(X)$ is countable. Applying Lemma 2, the Theorem is proved.

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BUDAPEST UNIVERSITY OF TECHNOLOGY
FACULTY OF MECHANICAL ENGINEERING
DEPARTMENT OF MATHEMATICS
H—1521. BUDAPEST, MŰEGYETEM RKP. 3

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VIBRATING STRINGS WITH FREE ENDS

M. HORVÁTH (Budapest)

In this paper we shall consider the equation

$$\varrho(x)\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y(x,t)}{\partial x}\right) + \delta(x-a)f(t), \tag{1}$$

$$0 < x < 1, \quad 0 < t < T$$

which describes the forced motion of a string with density $\varrho(x)$ and modulus of elasticity $p(x)$. At time t the external force $f(t)$ is given at the point $0<a<1$ of the string. This equation was investigated by many authors. In the case $p=\varrho\equiv 1$, A. G. Butkowski [1] considered the control problem of the string. Roughly speaking, the string is called controllable at time $T>0$ if for each initial conditions

$$y(\cdot, 0) = y_0, \quad y_t(\cdot, 0) = y_1$$

given in some function spaces $\left(\text{here } y_t \text{ means } \dfrac{\partial y}{\partial t}\right)$, we can find a control $f(t)$ such that

$$y(\cdot, T) = y_t(\cdot, T) \equiv 0.$$

Butkowskii proved for the case of fixed endpoints

$$y(0, \cdot) = y(1, \cdot) \equiv 0$$

that if we allow distributions $f(t)$ as controls then the string will be controllable in finite time for exactly those points $0<a<1$ which can not be well approximated by rational numbers. D. L. Russell [2] obtained another controllability result in the case of Sturm—Liouville boundary conditions. He supposed also that the spatial distribution of external forces is a function $g(x)\in L^2(0,1)$ (that is, in [2] the right hand side of (1) is $g(x)f(t)$).

I. Joó [3], [4] investigated the structure of the set of reachable movement states

$$\mathcal{D}_a(T) := \{(y(\cdot, T), y_t(\cdot, T)) : f(t) \in L^2(0, T)\}$$

where the boundary is fixed:

$$y(0, \cdot) = y(1, \cdot) \equiv 0$$

and at $t=0$ the string is relaxed:

$$y(\cdot, 0) = y_t(\cdot, 0) \equiv 0.$$

The aim of the present paper is to generalize the results of I. Joó for the case of strongly regular boundary conditions (cf. [5]). The Schrödinger equations with strongly regular boundary conditions have discrete spectrum and the asymptotic properties of the eigenvalues and eigenfunctions are well-known, cf. [5].

Introduce the following notations: $y_i:=y(i)$, $y_i':=y'(i)$, $i=0, 1$. The strongly regular boundary conditions are equivalent to one of the following three cases:

$$\text{(I)} \qquad U_1(y) = y_0 = 0, \quad U_2(y) = y_1 = 0.$$

$$\text{(II)} \qquad \begin{cases} U_1(y) = a_1 y_0' + b_1 y_1' + a_0 y_0 + b_0 y_1 = 0, \\ U_2(y) = c_0 y_0 + d_0 y_1 = 0 \end{cases}$$

if we have $b_1 c_0 + a_1 d_0 \neq 0$, $a_1 \neq \pm b_1$; $c_0 \neq \pm d_0$.

$$\text{(III)} \qquad \begin{cases} U_1(y) = y_0' + \alpha_{11} y_0 + \alpha_{12} y_1 = 0, \\ U_2(y) = y_1' + \alpha_{21} y_0 + \alpha_{22} y_1 = 0. \end{cases}$$

Remark that the boundary conditions adjoint to U_1 and U_2 with respect to some Schrödinger operator

$$Ly = y'' + qy$$

have the form

$$\text{(I*)} \qquad y_0 = 0, \quad y_1 = 0,$$

$$\text{(II*)} \qquad d_0 y_0' + c_0 y_1' + c_1 y_0 + d_1 y_1 = 0, \quad b_1 y_0 + a_1 y_1 = 0,$$

$$\text{(III*)} \qquad y_0' + \beta_{11} y_0 + \beta_{12} y_1 = 0, \quad y_1' + \beta_{21} y_0 + \beta_{22} y_1 = 0.$$

Consider the equation (1) with strongly regular boundary conditions

$$U_1(y(\cdot, t)) = 0, \quad U_2(y(\cdot, t)) = 0, \quad 0 < t < T.$$

We suppose $p, \varrho \in C^2[0, 1]$, $f(t) \in L^2(0, T)$, $p > 0$, $\varrho > 0$, $p(0) = p(1)$, $\varrho(0) = \varrho(1)$. The initial conditions are given by

$$y(\cdot, 0) = y_0 \in H, \quad y_t(\cdot, 0) \in L^2(0, 1)$$

where the space H consists of the functions $h \in H^1(0, 1)$ for which $U_1(h) = U_2(h) = 0$ in case of (I), $U_2(h) = 0$ in case of (II) and $H = H^1(0, 1)$ in case of (III). First apply the transform

$$u(x^*, t) = y(\varphi(x^*), t) \sqrt[4]{(p \circ \varphi)(\varrho \circ \varphi)(x^*)}$$

where

$$\varphi := r^{-1} \quad \text{and} \quad r(x) := \int_0^x \sqrt{\frac{\varrho(s)}{p(s)}}\, ds.$$

For convenience, write x instead of x^*; then $u(x, t)$ satisfies the equation

$$\text{(1')} \qquad u_{tt} - u_{xx} - q(x) u = \frac{\delta(x - a')}{\alpha(a)} f, \quad a' := \int_0^a \sqrt{\frac{\varrho}{p}}$$

in $(x, t)\in(0, l)\times(0, T)$, where

$$q\in C[0, l],\quad l:=\int_0^1\sqrt{\frac{\varrho}{p}},\quad \alpha(a):=\sqrt[4]{\frac{\varrho^3(a)}{p(a)}}.$$

It is easy to see that $u(x, t)$ satisfies some strongly regular boundary conditions

$$(2)\qquad V_1(u(\cdot, t)) = 0,\quad V_2(u(\cdot, t)) = 0,\quad 0 < t < T$$

and some initial conditions

$$(3)\qquad u(\cdot, 0) = u_0\in H_l,\qquad u_t(\cdot, 0) = u_1\in L^1(0, l);$$

here $H_l\subset H^1(0, l)$ is defined with V_1 and V_2 just as $H\subseteq H^1(0, 1)$. It will be convenient to use the following definition which is obtained from (1′) by twofold integration by parts:

DEFINITION ([3]). The solution of (1′), (2), (3) is a function

$$u(x, t)\in L^2((0, l)\times(0, T))$$

satisfying the equality

$$(4)\quad \int_0^l\int_0^T u(z_{tt}-z_{xx}-qz)\,dt\,dx = \int_0^l [u_1 z(\cdot, 0)-u_0 z_t(\cdot, 0)]\,dx+\int_0^T \frac{z(a', \cdot)}{\alpha(a)} f\,dt$$

for all $z\in C^2([0, l]\times[0, T])$ with the properties

$$z(\cdot, T) = z_t(\cdot, T) \equiv 0,\quad W_1(z(\cdot, t)) = W_2(z(\cdot, t)) = 0,\quad 0 < t < T$$

where W_1 and W_2 are the (strongly regular) boundary conditions adjoint to V_1 and V_2.

Consider the Schrödinger operator $Ly:=y''+qy$ and the boundary conditions $V_1(y)=V_2(y)=0$. An eigenfunction (of order 0) with the eigenvalue λ is a function $y\in C^2[0, l]$ such that

$$y''+qy+\lambda y = 0,\quad V_1(y) = V_2(y) = 0.$$

An eigenfunction of order $i\geqq 1$ is a function $y\in C^2[0, l]$ satisfying $y''+qy+\lambda y=y^*$, $V_1(y)=V_2(y)=0$, where y^* is an eigenfunction of order $i-1$. Mihajlov and Kesselmann ([6], [7]) proved that if V_1 and V_2 are strongly regular then the eigenfunctions (v_n) of order $\geqq 0$ of the operator L form a Riesz basis in $L^2(0, l)$ and its biorthogonal system is the system (w_n) of the eigenfunctions of order $\geqq 0$ of L with the conditions $W_1=W_2=0$. This means that

$$\langle v_n, w_k\rangle = \delta_{n,k},\quad n, k = 1, 2, \ldots.$$

In order to study the equation (4) we need some preliminary investigations. Using the asymptotic formulae given in [5, Ch. II, 4.9] and writing the eigenfunctions in the form

$$y = y_1\, U_1(y_2)-y_2\, U_1(y_1)\quad\text{or}\quad y = y_1\, U_2(y_2)-y_2\, U_2(y_1)$$

where y_1, y_2 are the basic solutions of $Ly+\lambda y=0$ given in [5, Ch. II, 4.5] we have the following estimates: in case (I)

(5) $$\sqrt{\lambda_n} = \frac{n\pi}{l}+O\left(\frac{1}{n}\right),$$

$$v_n(x) = \sin\frac{n\pi}{l}x+O\left(\frac{1}{n}\right), \quad \frac{v_n'(x)}{\sqrt{\lambda_n}} = \cos\frac{n\pi}{l}x+O\left(\frac{1}{n}\right);$$

in case (II)

(6) $$\sqrt{\lambda_n} = \alpha_n+O\left(\frac{1}{n}\right),$$

$$v_n(x) = c_0\sin\alpha_n x+d_0\sin\alpha_n(x-l)+O\left(\frac{1}{n}\right),$$

$$\frac{v_n'(x)}{\sqrt{\lambda_n}} = c_0\cos\alpha_n x+d_0\cos\alpha_n(x-l)+O\left(\frac{1}{n}\right)$$

where (for $n=2m, 2m+1$)

$$\alpha_{2m} = \frac{1}{l}\left(2m\pi+\frac{\ln s_1}{i}\right), \quad \alpha_{2m+1} = \frac{1}{l}\left(2m\pi+\frac{\ln s_2}{i}\right)$$

and s_1, s_2 are the roots of the equation

$$(b_1c_0+a_1d_0)\left(s+\frac{1}{s}\right)+2(a_1c_0+b_1d_0) = 0$$

(it follows from (II) that $s_1\neq s_2$, $s_1, s_2\neq 1$); or equivalently

(6′) $$v_n(x) = a_1\cos\alpha_n x+b_1\cos\alpha_n(x-l)+O\left(\frac{1}{n}\right),$$

$$-\frac{v_n'(x)}{\sqrt{\lambda_n}} = a_1\sin\alpha_n x+b_1\sin\alpha_n(x-l)+O\left(\frac{1}{n}\right);$$

in case (III)

(7) $$\sqrt{\lambda_n} = \frac{n\pi}{l}+O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos\frac{n\pi}{l}x+O\left(\frac{1}{n}\right), \quad -\frac{v_n'(x)}{\sqrt{\lambda_n}} = \sin\frac{n\pi}{l}x+O\left(\frac{1}{n}\right).$$

LEMMA 1. *For large N the system* $\left(\frac{v_n'}{\sqrt{\lambda_n}}\right)_{n=N}^{\infty}$ *is a Riesz basis in* $L^2(0, l)$ *(in its spanned subspace).*

PROOF. In case (I) and (III) we have only to refer to (5), (7) and the following well-known theorem of Bari: If $\varphi_1, \varphi_2, \ldots$ forms a Riesz basis in some Banach space in its closed linear hull $V(\varphi_n\colon n\geqq 1)$ and if $\sum_{n=1}^{\infty}\|\varphi_n-\psi_n\|^2<\infty$ then for large N the system $\psi_N, \psi_{N+1}, \dots$ is also a Riesz basis in $V(\psi_n\colon n\geqq N)$. In case (II) consider the boundary conditions

$$\tilde{V}_1 := c_0 y_0' + d_0 y_1' = 0, \quad \tilde{V}_2 := a_1 y_0 + b_1 y_1 = 0.$$

This is strongly regular since so are V_1 and V_2. Using (6), (6′) and the above mentioned results of Mihajlov and Kesselmann on the eigenfunctions in the boundary problem $\tilde{V}_1=\tilde{V}_2=0$ we see that the Bari theorem can be applied again. Lemma 1 is proved.

LEMMA 2. *The following assertions are equivalent for a function* $\sum c_n v_n \in H^1(0,l)$:

(i) $\sum c_n v_n \in H_l$,

(ii) $\sum c_n v_n' = (\sum c_n v_n)'$,

(iii) $\sum |\lambda_n|\cdot|c_n|^2<\infty$.

PROOF. (ii)⇔(iii) is an easy consequence of Lemma 1. Indeed, (ii) holds if and only if $\sum c_n v_n'$ converges in norm and this is equivalent to (iii) by Lemma 1.

(i)⇔(iii). Consider the system $w_1, w_2, \dots$ of all eigenfunctions (of order $\geqq 0$) of the boundary value problem

$$Ly = y'' + qy, \quad W_1(y) = W_2(y) = 0$$

where W_1, W_2 are adjoint to V_1, V_2. Suppose

$$w_n'' + q w_n + \bar{\lambda}_n w_n = \theta_{n-1} w_{n-1}, \quad n = 1, 2, \dots \tag{8}$$

where $w_0 \equiv 0$ and $\theta_{n-1}=0$ or 1. Let $\varphi\in H^1(0,l)$, $\varphi=\sum c_n v_n$, then for large n

$$\sqrt{\bar{\lambda}_n}\, c_n = \sqrt{\bar{\lambda}_n}\langle \varphi, w_n\rangle = \frac{1}{\sqrt{\bar{\lambda}_n}}\langle \varphi, \theta_{n-1}w_{n-1} - q w_n\rangle + \left\langle \varphi', \frac{w_n'}{\sqrt{\bar{\lambda}_n}}\right\rangle - \left[\varphi \frac{w_n'}{\sqrt{\bar{\lambda}_n}}\right]_0^l$$

holds. Here the first and second member on the right hand side belong to l_2, hence (iii) holds if and only if

$$\left[\varphi, \frac{w_n'}{\sqrt{\bar{\lambda}_n}}\right]_0^l \in l_2. \tag{9}$$

In case (I), (9) means by (5) that

$$(\varphi(l) + (-1)^n \varphi(0)) \in l_2 \quad \text{i.e.} \quad \varphi(0) = \varphi(l) = 0.$$

In case (II) observe first that W_1, W_2 have the form

$$W_1(y) = d_0 y_0' + c_0 y_1' + \beta_0 y_0 + \beta_1 y_1 = 0, \quad W_2(y) = b_1 y_0 + a_1 y_1 = 0.$$

Hence, supposing $c_0\neq 0$, say,

$$\frac{w_n'(l)}{\sqrt{\bar{\lambda}_n}} = -\frac{d_0}{c_0}\frac{w_n'(0)}{\sqrt{\bar{\lambda}_n}} + O\left(\frac{1}{n}\right),$$

so

$$c_0\left[\varphi, \frac{w_n'}{\sqrt{\lambda_n}}\right]_0^l = -\frac{w_n'(0)}{\sqrt{\lambda_n}}(d_0\varphi(l)+c_0\varphi(0))+O\left(\frac{1}{n}\right) =$$

$$= (-1)^n c_0\left(\sin\frac{\ln s}{i}\right)(d_0\varphi(l)+c_0\varphi(0))+O\left(\frac{1}{n}\right)$$

by (6). Since $s\neq\pm 1$ is a consequence of the conditions given in (II) we see that $(9)\Leftrightarrow\varphi\in H_l$. In case (III) the same assertion easily follows from (7). Lemma 2 is proved.

THEOREM 1. *In case* (I) *and* (II) *the system* $\left(\frac{v_n'}{1+|\sqrt{\lambda_n}|}\right)_{n=1}^{\infty}$ *forms a Riesz basis in* $L^2(0, l)$ *in* $V\left(\frac{v_n'}{1+\sqrt{|\lambda_n|}}: n\geqq 1\right)$.

PROOF. From Lemma 1 it follows that we have only to verify the linear independence of all $\frac{v_n'}{1+\sqrt{|\lambda_n|}}$, i.e. the following property:

$$\sum_{n=1}^{\infty}|c_n|^2<\infty \quad\text{and}\quad \sum_{n=1}^{\infty}c_n\frac{v_n'}{1+|\sqrt{\lambda_n}|}=0$$

imply $c_n=0$ for all n. But $\sum c_n\frac{v_n'}{1+|\sqrt{\lambda_n}|}=0$ implies $\sum\frac{c_n}{1+|\sqrt{\lambda_n}|}v_n=\text{const}$.

Lemma 2 states that $\text{const}\in H_l$ and this is possible only when $\text{const}=0$. Hence the Riesz basis property of $(v_n)_{n=1}^{\infty}$ implies $c_n=0$ for all n. Theorem 1 is proved.

THEOREM 2. *The equation* (4) *has a unique solution* $u\in L^2((0, l)\times(0, T))$, *which has the expansion*

$$u(x, t) = \sum c_n(t)\,v_n(x) \tag{10}$$

converging in L^2, *for each initial condition* $u_0\in H_l$, $u_1\in L^2(0, l)$ *and for each control* $f\in L^2(0, T)$. *The coefficients* $c_n(t)$ *have the form* (15), (16), (17). *Further the series* (10) *can be differentiated term by term, i.e.*

$$u\in H^1((0, l)\times(0, T))$$

and

$$u_t(x, t) = \sum c_n'(t)\,v_n(x), \tag{11}$$

$$u_x(x, t) = \sum c_n(t)\,v_n'(x). \tag{12}$$

PROOF. Define

$$z(x, t):= \overline{w_n(x)}\,b(t)$$

where (w_n) is the system described in (8) and $b\in C^2[0, T]$, $b(T)=b'(T)=0$. Suppose

$$u_0(x) =: \sum c_{n,0}v_n(x), \quad u_1(x) =: \sum c_{n,0}'v_n(x). \tag{13}$$

Then (4) gives

$$\int_0^T [c_n(t)b''(t)+\lambda_n c_n(t)b(t)-\theta_{n-1}c_{n-1}(t)b(t)]\,dt =$$
$$= c'_{n,0}b(0)-c_{n,0}b'(0)+\frac{\overline{w_n(a')}}{\alpha(a)}\int_0^T f(t)b(t)\,dt$$

i.e. the boundary value problem

$$(14)\qquad c''_n+\lambda_n c_n-\theta_{n-1}c_{n-1} = \frac{\overline{w_n(a')}}{\alpha(a)}f,\quad c_n(0) = c_{n,0},\quad c'_n(0) = c'_{n,0}.$$

Its solution can be given as follows. If $\lambda_1=0$ and the eigenvectors $w_1, \dots, w_{N_0}$ form a chain (i.e. $\theta_1=\dots=\theta_{N_0-1}=1$, $\theta_{N_0}=0$) then for $1\leqq n\leqq N_0$ we have

$$(15)\qquad c_n(t) = c_{n,0}+tc'_{n,0}+\dots+c_{1,0}\frac{t^{2n-2}}{(2n-2)!}+c'_{1,0}\frac{t^{2n-1}}{(2n-1)!}+$$
$$+\int_0^t f(\xi)\left[\beta_n(t-\xi)+\beta_{n-1}\frac{(t-\xi)^3}{3!}+\dots+\beta_1\frac{(t-\xi)^{2n-1}}{(2n-1)!}\right]d\xi\quad \left(\beta_n:=\frac{\overline{w_n(a')}}{\alpha(a)}\right).$$

Suppose that for $n\geqq N_0+1$, $\lambda_n\neq 0$ and for $n\geqq N_1+1$ all eigenvalues λ_n are simple. Then using the symbols

$$c:=\begin{pmatrix} c_{N_1}\\ \vdots \\ c_{N_0+1}\end{pmatrix},\quad A:=\begin{pmatrix} \lambda_{N_1} & \theta_{N_1-1} & & & \\ & \lambda_{N_1-1} & \theta_{N_1-2} & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \theta_{N_0+1} \\ & & & & \lambda_{N_0+1}\end{pmatrix},\quad \beta:=\begin{pmatrix} \beta_{N_1}\\ \vdots \\ \beta_{N_0+1}\end{pmatrix},$$

$$c_0:=\begin{pmatrix} c_{N_1,0}\\ \vdots \\ c_{N_0+1,0}\end{pmatrix},\quad c'_0:=\begin{pmatrix} c'_{N_1,0}\\ \vdots \\ c'_{N_0+1,0}\end{pmatrix}$$

we can write

$$c''+Ac = \beta f,\quad c(0) = c_0,\quad c'(0) = c'_0.$$

Since $\det A\neq 0$, there is a matrix B such that $A=B^2$. Define $b_1:=c'+iBc$, $b_2:=c'-iBc$, then

$$b'_{1,2}\mp iBb_{1,2} = \beta f \quad\text{and}\quad b_{1,2}(0) = c'_0\pm iBc_0.$$

Hence

$$(16)\qquad b_{1,2}(t) = e^{\pm iBt}b_{1,2}(0)+\int_0^t f(\tau)e^{\pm iB(t-\tau)}\beta\,d\tau,$$

$$c = B^{-1}\frac{b_1-b_2}{2i},\quad c' = \frac{b_1+b_2}{2}.$$

The same arguments show that for $n>N_1$

$$c_n'(t)\pm i\sqrt{\lambda_n}c_n(t) = e^{\pm i\sqrt{\lambda_n}t}\left[c_{n,0}'\pm i\sqrt{\lambda_n}c_{n,0}+\frac{\overline{w_n(a')}}{\alpha(a)}\int_0^t f(\tau)e^{\mp i\sqrt{\lambda_n}\tau}d\tau\right]. \tag{17}$$

From $w_n(a')=O(1)$ it follows that

$$\sum |c_n(t)|^2<\infty, \qquad \int_0^T \sum |c_n(t)|^2\,dt<\infty.$$

Now define the function $u(x,t)$ by (10), (15), (16) and (17). Then $u\in L^2((0,l)\times(0,T))$ and it satisfies (4) for $z(x,t)=\overline{w_n(x)}b(t)$, $n\in\mathbf{N}$. Moreover it turned out that if a solution of (1′), (2), (3) exists then it is uniquely determined by its coefficients. We shall prove (4) for an arbitrary $z(x,t)$ with the properties indicated there. Expand $z(x,t)$ with respect to the system $(\bar{w}_n)$:

$$z(x,t)=:\sum d_n(t)\overline{w_n(x)}.$$

It is easy to see that $d_n\in C^2[0,T]$, $d_n(T)=d_n'(T)=0$ further

$$z_{tt}(x,t)=\sum d_n''(t)\overline{w_n(x)}, \quad z(x,t)=\sum d_n(t)\overline{w_n(x)}$$

are convergent series in $L^2((0,l)\times(0,T))$. The convergence of

$$z(x,0)=\sum d_n(0)\overline{w_n(x)}, \quad z_t(x,0)=\sum d_n'(0)w_n(x)$$

in $L^2(0,l)$ are obvious and since $\langle v_n, \overline{z_{xx}(\cdot,t)}\rangle=\langle v_n'', \overline{z(\cdot,t)}\rangle$ implies

$$\int_0^T \sum |\lambda_n|^2|d_n(t)|^2\,dt<\infty,$$

the series

$$z_{xx}(x,t)=\sum d_n(t)\overline{w_n''(x)}$$

is also convergent in $L^2((0,l)\times(0,T))$. These considerations imply that (4) is satisfied for all $z(x,t)$. The existence of a solution of (4) is proved.

Finally the relations (11) and (12) are consequences of the estimates

$$\sum \left(|\lambda_n|\cdot|c_n(t)|^2+|c_n'(t)|^2\right)<\infty,$$

$$\int_0^T \sum\left(|\lambda_n||c_n(t)|^2+|c_n'(t)|^2\right)dt<\infty.$$

which follow from the formulae (5), (6), (7) and from the Bari theorem mentioned in proving Lemma 1. Theorem 2 is proved.

LEMMA 3. *If* $\delta>\sup\left|\operatorname{Im}\sqrt{\lambda_n}\right|$, *then the map*

$$L\colon H_l\to l_2, \quad \sum_{n=1}^{\infty} c_n v_n \mapsto \left((i\delta+\sqrt{\lambda_n})c_n\right)_{n=1}^{\infty}$$

is an isomorphism (onto l_2*).*

PROOF. L is linear, one-to-one and onto by Lemma 2. It follows from the open mapping principle that it is enough to prove the continuity of L^{-1}. But

$$\left\|\sum c_n v_n\right\|^2_{L^2(0,l)} \leqq \text{const} \sum |c_n|^2$$

since (v_n) is a Riesz basis and

$$\left\|\sum c_n v_n'\right\|^2_{L^2(0,l)} \leqq \text{const} \sum (1+|\lambda_n|)|c_n|^2$$

since $\left(\frac{v_n'}{\sqrt{\lambda_n}}\right)_{n=N}$ is a Riesz basis in its span by Lemma 1. These inequalities imply the continuity of L^{-1} and the proof is complete.

Now define the reachability set ([3], [4])

$$\mathcal{D}_a(T) := \{(u(\cdot, T), u_t(\cdot, T)) \in H_l \oplus L^2(0, l): f \in L^2(0, T)\}.$$

Define further

$$\mathcal{B}_a(T) := \{(\gamma_n)_{n=2}^{\infty} \in l_2: \gamma_{2n} := c_n'(T) + i\sqrt{\lambda_n}\, c_n(T),$$
$$\gamma_{2n+1} := c_n'(T) - i\sqrt{\lambda_n}\, c_n(T);\ f \in L^2(0, T)\}.$$

We see from Lemma 3 that if $\lambda_n \neq 0$ then

$$I: H_l \oplus L^2(0, l) \to l_2,$$

$$\left(\sum_{n=1}^{\infty} c_n v_n, \sum_{n=1}^{\infty} c_n' v_n\right) \mapsto \left(c'_{\left[\frac{k}{2}\right]} + (-1)^k i \sqrt{\lambda_{\left[\frac{k}{2}\right]}}\, c_{\left[\frac{k}{2}\right]}\right)_{k=2}^{\infty}$$

is an (onto) isomorphism and for all a and T, $I(\mathcal{D}_a(T)) = \mathcal{B}_a(T)$.

THEOREM 3. *Suppose that* $\lambda_n \neq 0$ *and* $u_0 = u_1 = 0$. *Then for a.e.* $a \in (0, 1)$ *the following assertions are valid:*

(i) $\mathcal{D}_a(T)$ *increases strongly with* T *for* $T < 2l$,

(ii) $\mathcal{D}_a(T)$ *is closed in* $H_l \oplus L^2(0, l)$ *if* $T < 2l$.

PROOF. We shall prove the same properties for $\mathcal{B}_a(T)$. From (17) we know that for large n

$$\gamma_{2n} = \frac{\overline{w_n(a')}}{\alpha(a)} \int_0^T f(\tau) e^{i\sqrt{\lambda_n}(T-\tau)}\, d\tau,$$

$$\gamma_{2n+1} = \frac{\overline{w_n(a')}}{\alpha(a)} \int_0^T f(\tau) e^{-i\sqrt{\lambda_n}(T-\tau)}\, d\tau.$$

It is known from the theory of exponential bases (cf. for example [8]) that for large N

$$(e^{\pm i\sqrt{\lambda_n} x})_{n=N}^{\infty}$$

is a Riesz basis in $L^2(0, 2l)$ (in its span which has a finite codimension). For $n = 1, \ldots, 2N$ we have also

$$\gamma_n = \frac{\overline{w_n(a')}}{\alpha(a)} \int_0^T f(\tau) h_n(\tau)\, d\tau$$

with some $h_n \in L^2(0, 2l)$.

The rest of the proof is similar to that of Theorem 1 in [4]. For every $0<T<2l$ we have an $\varepsilon=\varepsilon(T)>0$ such that there exists a sequence

$$(n_k)\subset\left\{n\geqq N:\ \frac{|w_n(a')|}{\alpha(a)}\geqq\varepsilon\right\}$$

such that $(e^{\pm i\sqrt{\lambda_{n_k}}x})$ is a (complete) Riesz basis in $L^2(0, T)$. Now the properties (18) and (19) can be easily verified using only the coordinates n_k. Theorem 3 is proved.

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EÖTVÖS LORÁND UNIVERSITY
MATHEMATICAL INSTITUTE
BUDAPEST, MÚZEUM KRT. 6—8.
H—1088

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ON A CONJECTURE ABOUT FUNCTIONS NOT BELONGING TO L^2

Z. MAGYAR (Budapest)

I, Joó has called the author's attention to the following problem (cf. [2]). Let $\mathcal{F}$ denote the set of those continuous real functions f on $\mathbf{R}_+$ which are bounded, strictly positive and satisfy

$$\int_0^{+\infty} f^2(x)\,dx = +\infty.$$

If $c>0$ and $f\in\mathcal{F}$ then let $x_n(c)$ be defined by

$$\int_0^{x_n(c)} f(t)\,dt = nc \quad (n = 1, 2, 3, \dots).$$

Question. Is it true for any $f\in\mathcal{F}$ that $h(c):=\sum_{n=1}^{\infty} f(x_n(c))=+\infty$ for almost every c?

The answer is no. Namely, there is an $f\in\mathcal{F}$ satisfying $h(c)<+\infty$ for almost every c.

Remarks. 1. M. Horváth, independently of the author, got the same result.

2. Our construction yields as a secondary product an example of a set of infinite measure in $\mathbf{R}_+$ which intersects the series nc only in finite set for almost every c (cf. [3]).

Proof. We give a concrete $f\in\mathcal{F}$ of this property.

If $r, s>0$, $r\geqq s$ then let $g_{r,s}$ be "the thorn-function of height 1, width $2s$ and center r", that is

$$g_{r,s}(x) = \max(0, 1-s^{-1}\cdot|x-r|). \tag{1}$$

Then it is clear that $\int_0^{+\infty} g^2_{r,s}(x)\,dx=2s/3$. Therefore if we have a family $\{(r_i, s_i);\ i\in I\}$ where I is an index set satisfying

$$\sum_{i\in I} s_i = +\infty \tag{2}$$

and

$$|r_i-r_{i'}| \geqq s_i+s_{i'}, \quad \text{if} \quad i \neq i' \tag{3}$$

then the function f defined by

$$f(x) = (x+1)^{-1} + \sum_{i\in I} g_{r_i, s_i}(x) \quad (x\in\mathbf{R}_+) \tag{4}$$

belongs to $\mathcal{F}$.

We construct a family satisfying (2) and (3) in the following way. Let $N(k):=$ $:=[\exp(2^k)]$ for $k=1, 2, 3, \ldots$ and let $I:=\{(k, m);\ k=1, 2, \ldots,\ m=1, \ldots, N(k)\}$. We define an ordering on I by setting

$$(k, m) < (k', m') \quad \text{if} \quad k < k' \quad \text{or} \quad [k = k' \quad \text{and} \quad m > m']. \tag{5}$$

Now let

$$s_{k,m} = (m\cdot 2^k)^{-1} \tag{6}$$

and

$$r_i = \exp\left(\frac{N(k)!}{m} - \frac{s_i}{2} + \sum_{j<i} s_j\right) - 1, \quad \text{where} \quad i = (k, m). \tag{7}$$

Then

$$\sum_{m=1}^{N(k)} s_{k,m} = 2^{-k}\big(1+\ldots+1/N(k)\big) \geqq 2^{-k}\cdot\log\big(N(k)+1\big) \geqq 1,$$

and hence we get (2). We see also that

$$\sum_{m=1}^{N(k)} s_{k,m} < 2^{-k}\big(1+\log N(k)\big) < 2.$$

This implies that $r_i > s_i$ for all $i\in I$, and $r_i - r_j > s_i + s_j$ if $j<i$. Thus (3) holds, and setting $F(x):=\int_0^x f(t)\,dt$ (where f is defined by (4)) we infer that

$$F(r_{k,m}) = m^{-1}\cdot\big(N(k)!\big) \tag{8}$$

and

$$\log(x+1) \leqq F(x) \leqq 2\cdot\log(x+1) \quad \text{for all } x\in\mathbf{R}_+. \tag{9}$$

We want to prove that $h(c)=\sum_{n=1}^{\infty} f\big(F^{-1}(nc)\big)$ is finite for almost every $c>0$. Denote the interval $\big(F(r_{k,m}-s_{k,m}), F(r_{k,m}+s_{k,m})\big)$ by $B_{k,m}$. Then we get from (8) that

$$B_{k,m}\subset\big(m^{-1}(N(k)!-2^{-k}), m^{-1}(N(k)!+2^{-k})\big). \tag{10}$$

On the other hand, writing $B=\bigcup_{(k,m)\in I} B_{k,m}$, clearly $f\big(F^{-1}(x)\big)=\big(F^{-1}(x)+1\big)^{-1}$ whenever $x\notin B$, and hence we infer by (9) that

$$f\big(F^{-1}(x)\big) \leqq \exp(-x/2) \quad \text{if} \quad x\notin B. \tag{11}$$

This implies that $h(c)$ is finite if c satisfies the following condition:

(12) there is an $n_0(c)$ such that $nc\notin B$ if $n>n_0(c)$.

We assert that (12) holds for almost every c. It is enough to prove this assertion on the interval $[2^M, 2^{M+1}]$, where M is an arbitrary integer. Fix an M, and define a probability measure P on the Borel subsets of $[2^M, 2^{M+1}]=\Omega$ by setting $P(A):=$ $:=2^{-M}\cdot\lambda(A)$, where λ is the Lebesgue measure. Let

$$A_k:=\{c\in\Omega;\ \exists n:\ nc\in \bigcup_{m=1}^{N(k)} B_{k,m}\}. \tag{13}$$

Then we see from (10) that in case $c\in A_k$ there are n, m such that $|nc-m^{-1}N(k)!|<$ $<m^{-1}\cdot 2^{-k}$, and hence there is another positive integer n satisfying

$$|nc-N(k)!| < 2^{-k}. \tag{14}$$

Let

$$A_{k,n}=\left(n^{-1}(N(k)!-2^{-k}), n^{-1}(N(k)!+2^{-k})\right)\cap\Omega.$$

It follows from (14) that $P(A_k)\leqq \sum_{n=1}^{\infty} P(A_{k,n})$. On the other hand, if $A_{k,n}\neq\emptyset$ then

$$2^{-M-1}(N(k)!-1) < n < 2^{-M}(N(k)!+1). \tag{15}$$

Since $P(A_{k,n})\leqq n^{-1}\cdot 2^{1-k}\cdot 2^{-M}$, we have by (15)

$$\sum_{n-1}^{\infty} P(A_{k,n}) \leqq \{2^{-M}(N(k)!+1)-$$

$$2^{-M-1}(N(k)!-1)+1\}\cdot 2^{-M}\cdot 2^{M+1}2^{1-k}(N(k)!-1)^{-1}2^{1-k} \leqq C_M\cdot 2^{-k}$$

where C_M depends only on M. Thus $P(A_k)\leqq C_M\cdot 2^{-k}$, and hence $\sum_{k=1}^{\infty} P(A_k)<+\infty$. This implies $P(H)=0$, where $H=\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} A_k$. Observe that each element of $\Omega\setminus H$ satisfies (12). Thus our proof is complete.

REMARK. If $0\leqq a<b$ are arbitrary then $\int_a^b h(c)\,dc=\sum_{n=1}^{\infty} n^{-1}\cdot \int_{F^{-1}(na)}^{F^{-1}(nb)} f^2(t)\,dt$ for any $f\in\mathcal{F}$. It is not very hard to show that the right side of this equality is

$$\geqq \int_{F^{-1}(a)}^{+\infty} f^2(t)\left(1-a/b-a/F(t)\right)dt=+\infty \quad\text{if}\quad f\in\mathcal{F}.$$

Thus our construction yields a Borel measurable function h whose integral on each interval equals $+\infty$, although h is finite a.e.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF APPLIED ANALYSIS
1088 BUDAPEST, MÚZEUM KRT. 6—8
HUNGARY

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ON FUNCTIONS ADDITIVE WITH RESPECT TO INTERVAL FILLING SEQUENCES

Z. DARÓCZY (Debrecen), corresponding member of the Academy and
I. KÁTAI (Budapest), member of the Academy

1. Introduction

The notion of functions additive with respect to interval filling sequences has been introduced in [4]; earlier we have investigated important special cases ([1], [2], [3]). In this paper we consider the following problem: What can be said about the structure of an additive and continuous function which is positive for positive values of the variable? Our main result will say that in case of *smooth* interval filling sequences these functions are linear. With the help of this we are going to prove that if a function additive and continuous with respect to a smooth interval filling sequence is *differentiable on a set of positive measure,* then it is linear. This result is important because earlier we have shown ([3]) that there exists a function additive and continuous with respect to a smooth interval filling sequence, which is *nowhere differentiable.*

Many steps of the method of proof here employed are effective also for non-smooth interval filling sequences. These partial results will be formulated also without the condition of smoothness. Nevertheless, up to now we have not yet succeeded in eliminating smoothness from our main result. Since no counterexample is known, it can be guessed that our main results will turn out to be valid also for non-smooth interval filling sequences.

2. Interval filling sequences

Let Λ denote the set of those real sequences $\{\lambda_n\}$, for which the conditions $\lambda_n > \lambda_{n+1} > 0$ $(n\in\mathbf{N})$ and $L := \sum_{n=1}^{\infty} \lambda_n < \infty$ are satisfied.

DEFINITION 2.1. The sequence $\{\lambda_n\}\in\Lambda$ is said to be *interval filling,* if for any $x\in[0, L]$ there exists a sequence $\varepsilon_n\in\{0, 1\}$ $(n\in\mathbf{N})$, such that $x=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n$.

We have ([1], [3]) the following

THEOREM 2.2. *The sequence* $\{\lambda_n\}\in\Lambda$ *is interval filling if and only if*

$$\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i \tag{2.1}$$

holds for any $n\in\mathbf{N}$.

Let $\{\lambda_n\}\in\Lambda$ be interval filling and $x\in[0, L]$. Let moreover be, by induction on n,

$$\varepsilon_n(x):=\begin{cases}1 & \text{for} \quad \sum_{i=1}^{n-1}\varepsilon_i(x)\lambda_i+\lambda_n\leqq x\\ 0 & \text{for} \quad \sum_{i=1}^{n-1}\varepsilon_i(x)\lambda_i+\lambda_n> x\end{cases} \tag{2.2}$$

and

$$\varepsilon_n^*(x):=\begin{cases}1 & \text{for} \quad \sum_{i=1}^{n-1}\varepsilon_i^*(x)\lambda_i+\lambda_n< x\\ 0 & \text{for} \quad \sum_{i=1}^{n-1}\varepsilon_i^*(x)\lambda_i+\lambda_n\geqq x.\end{cases} \tag{2.3}$$

Then ([1], [3], [4])

$$x=\sum_{n=1}^{\infty}\varepsilon_n(x)\lambda_n=\sum_{n=1}^{\infty}\varepsilon_n^*(x)\lambda_n. \tag{2.4}$$

The first of the representations (2.4) will be called the *regular*, and the second one the *quasiregular* expansion of x ([1], [4]).

Definition 2.3. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call the number $x\in[0, L]$ *finite* if there exists N such that $\varepsilon_n(x)=0$ for $n>N$. If x is finite and $\varepsilon_m(x)=1$, moreover $\varepsilon_n(x)=0$ for $n>m$, then we say that x has *length* m, and write $h(x)=m$. We define $h(0)=0$.

Let

$$V_N:=\{x\mid x\in[0, L],\quad h(x)\leqq N\} \tag{2.5}$$

and

$$U_N:=\{x\mid x\in[0, L],\quad h(x)=N\}. \tag{2.6}$$

Definition 2.4. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $x\in]0, L]$. The number

$$b_N(x):=\max\{t\mid t\in V_N,\quad t<x\} \tag{2.7}$$

will be called the *left hand neighbor* of x in V_N ([4]).

Lemma 2.5. *If* $\{\lambda_n\}\in\Lambda$ *is interval filling, then*

$$\operatorname{card}(U_N)<\frac{L}{\lambda_N}\quad (N\in\mathbf{N}) \tag{2.8}$$

where card (A) *denotes the cardinality of the finite set* A.

Proof. If $x, y\in U_N$ and $x<y$ then $x\leqq y-\lambda_N$. Indeed, the left hand neighbor of y in V_N is $y-\lambda_N$ and $x\in V_N$, hence $x\leqq y-\lambda_N$. This implies that if we put $I_x:=$ $:=]x-\lambda_N, x[$ $(x\in U_N)$ then $I_x\cap I_y=\emptyset$ for $x, y\in U_N$ and $x\neq y$. Thus

$$\operatorname{meas}\Big(\bigcup_{x\in U_N}I_x\Big)=\operatorname{card}(U_N)\,\lambda_N<L$$

whence (2.8) follows. □

LEMMA 2.6. *If* $\{\lambda_n\}\in\Lambda$ *is interval filling, then*

$$\text{card}(V_N) < 1+L\sum_{i=1}^{N}\frac{1}{\lambda_i} \quad (N\in\mathbf{N}). \tag{2.9}$$

PROOF. In view of

$$V_N = \{0\}\cup U_1\cup U_2\cup\ldots\cup U_N$$

where the right hand side is a disjoint union, Lemma 2.5 implies

$$\text{card}(V_N) = 1+\sum_{i=1}^{N}\text{card}(U_i) < 1+L\sum_{i=1}^{N}\frac{1}{\lambda_i}. \quad \square$$

LEMMA 2.7. *Let* $\{\lambda_n\}\in\Lambda$ *be an interval filling sequence and* $L_{N+1}:=\sum_{i=N+1}^{\infty}\lambda_i$ $(N\in\mathbf{N})$. *Then*

$$\frac{L}{L_{N+1}} \leqq \text{card}(V_N) \quad (N\in\mathbf{N}). \tag{2.10}$$

PROOF. If $x\in[0, L]$ then

$$x = \sum_{i=1}^{N}\varepsilon_i(x)\lambda_i+\eta_{N+1}(x)$$

where

$$\eta_{N+1}(x) = \sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i \leqq L_{N+1}$$

and

$$S_N(x):= \sum_{i=1}^{N}\varepsilon_i(x)\lambda_i\in V_N.$$

Thus $\bigcup_{t\in V_N}[t, t+L_{N+1}]\supset[0, L]$, and from this $L\leqq\text{card}(V_N)L_{N+1}$ follows, i.e. (2.10) holds. $\square$

3. Smooth sequences

DEFINITION 3.1. We call the sequence $\{\lambda_n\}\in\Lambda$ *smooth,* if there exists $K>0$ such that

$$\sum_{i=n+1}^{\infty}\lambda_i < K\lambda_n \tag{3.1}$$

for any $n\in\mathbf{N}$ ([5]).

LEMMA 3.1. *The sequence* $\{\lambda_n\}\in\Lambda$ *is smooth if and only if there exists* $T\in\mathbf{N}$ *such that*

$$\lambda_{n+T} < \frac{1}{2}\lambda_n \tag{3.2}$$

for any $n\in\mathbf{N}$.

PROOF. (i) Let $\{\lambda_n\}\in\Lambda$ be a smooth sequence. Then for any $n, t\in\mathbf{N}$ we have

$$t\lambda_{n+t} \leqq \lambda_{n+1}+\lambda_{n+2}+\ldots+\lambda_{n+t} < K\lambda_n.$$

Now there exists $T\in\mathbf{N}$ such that $\frac{K}{T}<\frac{1}{2}$, hence

$$\lambda_{n+T} < \frac{K}{T}\lambda_n < \frac{1}{2}\lambda_n$$

for any $n\in\mathbf{N}$, i.e. (3.2) holds.

(ii) Let $\{\lambda_n\}\in\Lambda$ satisfy (3.2) for some $T\in\mathbf{N}$. Then

$$\lambda_{n+kT} < \frac{1}{2^k}\lambda_n$$

for any k and n. Hence

$$\lambda_{n+1}+\lambda_{n+2}+\ldots+\lambda_{n+T} < T\lambda_n,$$

$$\lambda_{n+T+1}+\lambda_{n+T+2}+\ldots+\lambda_{n+2T} < \frac{T}{2}\lambda_n,$$

$$\lambda_{n+2T+1}+\lambda_{n+2T+2}+\ldots+\lambda_{n+3T} < \frac{T}{2^2}\lambda_n$$

$$\ldots$$

These inequalities imply

$$\sum_{i=n+1}^{\infty}\lambda_i < T\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\right)\lambda_n = 2T\lambda_n,$$

i.e. (3.1) holds for $K:=2T$. $\square$

LEMMA 3.2. *If the sequence $\{\lambda_n\}\in\Lambda$ is smooth, then there exist numbers H, $s\in\mathbf{N}$, such that for any $N\in\mathbf{N}$ one can find among the numbers N, $N+1, \ldots, N+H$ a value n satisfying*

$$\lambda_{n+s}+\lambda_{n+1} < \lambda_n. \tag{3.3}$$

PROOF. By Lemma 3.1 there exists $T\in\mathbf{N}$ such that $\lambda_{N+2T}<\frac{1}{4}\lambda_N$ for any $N\in\mathbf{N}$. Let $H:=2T$. Then

$$\frac{1}{4} > \frac{\lambda_{N+2T}}{\lambda_N} = \frac{\lambda_{N+2T}}{\lambda_{N+2T-1}}\,\frac{\lambda_{N+2T-1}}{\lambda_{N+2T-2}}\cdots\frac{\lambda_{N+1}}{\lambda_N}.$$

In the product on the right hand side let $\alpha:=\frac{\lambda_{n+1}}{\lambda_n}$ $(N\leqq n\leqq N+2T=N+H)$ be the smallest factor. Then $\alpha^{2T}<\frac{1}{4}$, i.e.

$$\frac{\lambda_{n+1}}{\lambda_n}=\alpha<\left(\frac{1}{2}\right)^{\frac{1}{T}}. \tag{3.4}$$

On the other hand, for any $l\in\mathbf{N}$ we have by (3.2)

$$\lambda_{n+1}+\lambda_{n+lT}<\left(\frac{1}{2}\right)^{\frac{1}{T}}\lambda_n+\frac{1}{2^l}\lambda_n,$$

and this implies the existence of an $l_0\in\mathbf{N}$ such that $\left(\frac{1}{2}\right)^{1/T}+\frac{1}{2^{l_0}}<1$. Hence, putting $s:=l_0T$ we get from (3.4)

$$\lambda_{n+1}+\lambda_{n+s}=\lambda_{n+1}+\lambda_{n+l_0T}<\left[\left(\frac{1}{2}\right)^{\frac{1}{T}}+\frac{1}{2^{l_0}}\right]\lambda_n<\lambda_n,$$

i.e. (3.3) holds for some index $n\in\{N, N+1, \dots, N+H\}$. □

LEMMA 3.3. *If the sequence* $\{\lambda_n\}\in\Lambda$ *is smooth, then there exists a constant* $c>1$ *such that*

$$\sum_{i=1}^{N}\frac{1}{\lambda_i}<\frac{c}{\lambda_N} \tag{3.5}$$

for any $N\in\mathbf{N}$.

PROOF. By Lemma 3.1 there exists $T\in\mathbf{N}$, such that

$$\lambda_{n+T}<\frac{1}{2}\lambda_n \quad (n\in\mathbf{N}). \tag{3.6}$$

If $N\leqq T$ then

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\dots+\frac{1}{\lambda_N}\leqq\frac{T}{\lambda_N}<\frac{2T}{\lambda_N}.$$

If $N>T$ and $N=kT+r$ $(0<r\leqq T)$ then (3.6) implies

$$\frac{1}{\lambda_{N-T}}<\frac{1}{2\lambda_N}$$

whence by iteration

$$\frac{1}{\lambda_N}+\frac{1}{\lambda_{N-T}}+\frac{1}{\lambda_{N-2T}}+\dots+\frac{1}{\lambda_{N-kT}}\leqq\frac{1}{\lambda_N}\left(1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^k}\right)<\frac{2}{\lambda_N}.$$

Let us now write this same inequality replacing N by each of the numbers $N-1$, $N-2, \dots, N-T+1$ in turn, and let us add the inequalities thus obtained. We obtain

$$\sum_{i=1}^{N}\frac{1}{\lambda_i}<\frac{2}{\lambda_N}+\frac{2}{\lambda_{N-1}}+\dots+\frac{2}{\lambda_{N-T+1}}<\frac{2T}{\lambda_N},$$

i.e. (3.5) holds for $c:=2T(>1)$. □

THEOREM 3.4. *If* $\{\lambda_n\}\in\Lambda$ *is a smooth interval filling sequence, then there exist constants* $0<C_1<C_2$ *such that*

$$\frac{C_1}{\lambda_N}<\operatorname{card}(V_N)<\frac{C_2}{\lambda_N} \tag{3.7}$$

for any $N\in\mathbf{N}$.

PROOF. By Lemma 2.7 smoothness implies

$$\operatorname{card}(V_N)\geqq\frac{L}{L_{N+1}}>\frac{L}{K\lambda_N},$$

i.e. the left hand side of (3.7) is valid for $C_1:=\frac{L}{K}>0$. Now Lemmas 2.6 and 3.3 together imply

$$\operatorname{card}(V_N)<1+L\sum_{i=1}^{N}\frac{1}{\lambda_i}<1+\frac{LC}{\lambda_N}<\frac{C_2}{\lambda_N},$$

where $C_2>0$ is some upper bound of $\{\lambda_N+LC\}$ $(N\in\mathbf{N})$. This completes the proof of the theorem. □

4. Additive and continuous functions

We have introduced the notion of additive function in [4].

DEFINITION 4.1. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call the function $F\colon[0,L]\to\mathbf{R}$ *additive* (with respect to the sequence $\{\lambda_n\}$), if for any $x\in[0,L]$ the equality

$$F(x)=F\Big(\sum_{n=1}^{\infty}\varepsilon_n(x)\lambda_n\Big)=\sum_{n=1}^{\infty}\varepsilon_n(x)F(\lambda_n) \tag{4.1}$$

holds, where $\sum_{n=1}^{\infty}|F(\lambda_n)|<\infty$. In (4.1) $\varepsilon_n(x)$ denotes the digits 0, 1 defined by the *regular algorithm* (2.2).

REMARK. If $\{a_n\}$ is a real sequence satisfying $\sum_{n=1}^{\infty}|a_n|<\infty$, then putting $F(\lambda_n):=$ $:=a_n$ $(n\in\mathbf{N})$ we get an additive function

$$F(x):=\sum_{n=1}^{\infty}\varepsilon_n(x)a_n\quad(x\in[0,L])$$

and conversely, if $F\colon[0,L]\to\mathbf{R}$ is additive, then for $a_n:=F(\lambda_n)$ $(n\in\mathbf{N})$ we have $\sum_{n=1}^{\infty}|a_n|<\infty$. Thus, in case of a fixed interval filling sequence any additive function is uniquely determined by a real sequence $\{a_n\}$, such that $\sum_{n=1}^{\infty}|a_n|<\infty$.

In the sequel we shall need a condition, necessary and sufficient in order that an additive function $F\colon [0, L]\to\mathbf{R}$ be continuous in $[0, L]$ ([4]).

DEFINITION 4.2. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call the function $F\colon [0, L]\to\mathbf{R}$ *quasiadditive* (with respect to the sequence $\{\lambda_n\}$), if for any $x\in\in[0, L]$ the equality

$$F(x)=F\Big(\sum_{n=1}^{\infty}\varepsilon_n^*(x)\lambda_n\Big)=\sum_{n=1}^{\infty}\varepsilon_n^*(x)F(\lambda_n) \tag{4.2}$$

holds, where $\sum_{n=1}^{\infty}|F(\lambda_n)|<\infty$. In (4.2) $\varepsilon_n^*(x)$ denotes the digits 0, 1 defined by the *quasiregular algorithm* (2.3).

Our following result will be of fundamental importance ([4]).

THEOREM 4.3. *Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $F\colon [0, L]\to\mathbf{R}$ an additive function. Then F is continuous in $[0, L]$ if and only if it is quasiadditive.*

REMARK. The statement formulated in Theorem 4.3 is a consequence of the following ones ([4]):

(1) If F is additive, then F is continuous at any *nonfinite* point x.

(2) If F is additive, then F is continuous from the right at any *finite* point x.

(3) If F is additive and continuous from the left at any finite point $x>0$, then putting $a_n:=F(\lambda_n)$ $(n\in\mathbf{N})$ we have

$$a_n:=\sum_{i=n+1}^{\infty}\varepsilon_i^*(\lambda_n)a_i \tag{4.3}$$

for any $n\in\mathbf{N}$, and conversely, if (4.3) is satisfied for any $n\in\mathbf{N}$, then F is continuous from the left at any finite point $x>0$.

(4) If F is additive, then (4.3) is valid for any $n\in\mathbf{N}$ if and only if F is quasiadditive.

5. On positive and continuous additive functions

Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $F\colon [0, L]\to\mathbf{R}$ an additive and continuous function, such that $a_n:=F(\lambda_n)>0$ $(n\in\mathbf{N})$. Then it is clear that $F(x)>0$ for any $x\in]0, L]$. Let $\mathcal{P}[\{\lambda_n\}]$ denote the set of all *positive and continuous additive* functions.

THEOREM 5.1. *If $F\in\mathcal{P}[\{\lambda_n\}]$, then F is strictly monotone increasing in* $[0, L]$.

PROOF. Let $0\leqq\alpha<\beta\leqq L$ be arbitrary and by continuity

$$\min_{x\in[\alpha,\beta]}F(x)=F(\xi),$$

where $\alpha\leqq\xi\leqq\beta$. If $\alpha<\xi$, then let

$$\xi=\sum_{n=1}^{\infty}\varepsilon_n^*(\xi)\lambda_n.$$

Since on the right hand $\varepsilon_n^*(\xi)=1$ for infinitely many values of n, there exists $N_0\in\mathbf{N}$ such that for

$$\xi_{N_0}:=\sum_{n=1}^{N_0}\varepsilon_n^*(\xi)\lambda_n$$

one has $\alpha<\xi_{N_0}<\xi$. Then by Theorem 4.3 and in view of $F(\lambda_n)>0$ one gets

$$F(\xi_{N_0})=\sum_{n=1}^{N_0}\varepsilon_n^*(\xi)F(\lambda_n)<\sum_{n=1}^{\infty}\varepsilon_n^*(\xi)F(\lambda_n)=F(\xi),$$

a contradiction. Hence $\alpha=\xi$, i.e. $F(\alpha)\leqq F(\beta)$, so that F is monotone increasing. If $F(\alpha)=F(\beta)$ were true, then by monotonicity $F(x)=F(\alpha)$ would hold for any $x\in[\alpha,\beta]$. Then

$$\beta=\sum_{n=1}^{\infty}\varepsilon_n^*(\beta)\lambda_n$$

and there exists N_0 such that

$$\alpha<\sum_{n=1}^{N_0}\varepsilon_n^*(\beta)\lambda_n=\beta_{N_0}<\beta.$$

From this

$$F(\beta_{N_0})=\sum_{n=1}^{N_0}\varepsilon_n^*(\beta)F(\lambda_n)<\sum_{n=1}^{\infty}\varepsilon_n^*(\beta)F(\lambda_n)=F(\beta)$$

and this contradicts the equality $F(\beta_{N_0})=F(\alpha)=F(\beta)$. Thus $F(\alpha)<F(\beta)$, i.e. F is strictly monotone increasing. □

REMARKS. (i) If $F\colon[0,L]\to\mathbf{R}$ is an additive and continuous function such that $F(x)\geqq 0$ for any $x\in[0,L]$, i.e. $a_n:=F(\lambda_n)\geqq 0$ $(n\in\mathbf{N})$ then considerations similar to the proof of Theorem 5.1 show that F is *monotone increasing*.

(ii) It is important to know that the monotone increasing property of an additive function does not imply continuity. Indeed, if with the notation $a_n:=F(\lambda_n)$ $(n\in\mathbf{N})$ we have $a_1:=1$ and $a_k:=0$ $(k\geqq 2)$, then the additive function F determined by the sequence $\{a_n\}$ is the following:

$$F(x)=\begin{cases}0 & \text{for}\quad 0\leqq x<\lambda_1\\ 1 & \text{for}\quad \lambda_1\leqq x\leqq L.\end{cases}$$

Clearly, F is monotone increasing but fails to be continuous for $x=\lambda_1$.

LEMMA 5.2. *If $F\in\mathcal{P}[\{\lambda_n\}]$ then the sequence $\{a_n:=F(\lambda_n)\}\in\Lambda$ is interval filling.*

PROOF. By Theorem 5.1 F is strictly monotone increasing, hence $\{a_n\}\in\Lambda$. On the other hand, by continuity

$$a_n=\sum_{i=n+1}^{\infty}\varepsilon_i^*(\lambda_n)a_i\leqq\sum_{i=n+1}^{\infty}a_i$$

for any $n\in\mathbf{N}$, i.e. by Theorem 2.2 the sequence $\{a_n\}\in\Lambda$ is interval filling. □

DEFINITION 5.3. Let $\{\lambda_n\}\in\Lambda$ be interval filling and

$$\mathscr{R}[\{\lambda_n\}] := \{\varepsilon(x) := (\varepsilon_1(x), \varepsilon_2(x), \ldots) \mid x\in[0, L]\}.$$

Clearly, $\mathscr{R}[\{\lambda_n\}]\subset\{0, 1\}^{\mathbf{N}}$. If $\{a_n\}\in\Lambda$ is interval filling and $\mathscr{R}[\{\lambda_n\}]=\mathscr{R}[\{a_n\}]$ then we say that the two interval filling sequences are *isomorphic;* in notation $\{\lambda_n\}\cong\{a_n\}$.

THEOREM 5.4. *If* $F\in\mathscr{P}[\{\lambda_n\}]$ *then the interval filling sequences* $\{a_n := F(\lambda_n)\}$ *and* $\{\lambda_n\}$ *are isomorphic* $(\{a_n\}\cong\{\lambda_n\})$.

PROOF. (i) Let $\varepsilon\in\mathscr{R}[\{\lambda_n\}]$. Then there exists $x\in[0, L]$ such that $x=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n$ with $\varepsilon_n=\varepsilon_n(x)$ for any $n\in\mathbf{N}$. Let

$$y := \sum_{n=1}^{\infty}\varepsilon_n(x)\,a_n\in[0, A] \quad\text{with}\quad A := \sum_{n=1}^{\infty}a_n < \infty.$$

Let us show that if

$$y = \sum_{n=1}^{\infty}\delta_n(y)\,a_n$$

is a regular expansion (Lemma 5.2) of the number $y\in[0, A]$ with respect to the interval filling sequence $\{a_n\}\in\Lambda$, then $\delta_n(y)=\varepsilon_n(x)$ for any $n\in\mathbf{N}$, i.e. $\varepsilon\in\mathscr{R}[\{a_n\}]$. Suppose the contrary, i.e. let $\varepsilon(x)\neq\delta(y)$. Then there exists $N\in\mathbf{N}$, such that $\varepsilon_i(x)=\delta_i(y)$ for $i=1, 2, \ldots, N-1$ and $\varepsilon_N(x)\neq\delta_N(y)$. Then by the greedy property of the regular expansion, $\varepsilon_N(x)=0$ and $\delta_N(y)=1$. Hence

$$x = \sum_{i=1}^{N-1}\varepsilon_i(x)\lambda_i+\eta_N(x),$$

where

$$\eta_N(x) = \sum_{i=N}^{\infty}\varepsilon_i(x)\lambda_i = \sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i$$

and $\eta_N(x)<\lambda_N$. On the other hand

$$y \geqq \sum_{i=1}^{N-1}\delta_i(y)a_i+a_N. \tag{5.1}$$

By the strictly monotone increasing property of F we have $F(\eta_N(x))<F(\lambda_N)$; moreover, the right hand side of the equality $\eta_N(x)=\sum_{i=N+1}^{\infty}\varepsilon_i(x)\lambda_i$ is the regular expansion of $\eta_N(x)$, hence by the additivity of F

$$F(\eta_N(x)) = \sum_{i=N+1}^{\infty}\varepsilon_i(x)a_i.$$

Thus

$$y = F(x) = \sum_{i=1}^{N-1}\varepsilon_i(x)\,a_i+\sum_{i=N+1}^{\infty}\varepsilon_i(x)\,a_i = \sum_{i=1}^{N-1}\delta_i(y)\,a_i+F(\eta_N(x)) < \sum_{i=1}^{N-1}\delta_i(y)\,a_i+a_N$$

and this contradicts (5.1). Therefore $\mathscr{R}[\{\lambda_n\}]\subset\mathscr{R}[\{a_n\}]$.

(ii) If $\delta\in\mathcal{R}[\{a_n\}]$, then there exists $y\in[0, A]$ such that $y=\sum_{n=1}^{\infty}\delta_n(y)a_n$, where $\delta_n=\delta_n(y)$ $(n\in\mathbf{N})$. (Here $\delta_n(y)$ denotes now the digits of the regular expansion of the number y with respect to $\{a_n\}$.) Since F is strictly monotone increasing, there exists $x\in[0, L]$ such that $x=F^{-1}(y)$. Let $x=\sum_{n=1}^{\infty}\varepsilon_n(x)\lambda_n$. Then by additivity

$$F(x) = y = \sum_{n=1}^{\infty}\varepsilon_n(x)a_n.$$

We now show that $\varepsilon_n(x)=\delta_n(y)$ $(n\in\mathbf{N})$. Suppose the contrary, i.e. let $\varepsilon(x)\neq\delta(y)$. Then there exists $N\in\mathbf{N}$ such that $\varepsilon_i(x)=\delta_i(y)$ for $i=1, 2, \ldots, N-1$, and $\varepsilon_N(x)=0$, $\delta_N(y)=1$. Hence

$$x = \sum_{i=1}^{N-1}\varepsilon_i(x)\lambda_i+\eta_N(x),$$

where $\eta_N(x)<\lambda_N$. On the other hand

$$y \geqq \sum_{i=1}^{N-1}\delta_i(y)a_i+a_N.$$

F being strictly increasing we have $F(\eta_N(x))<F(\lambda_N)$; furthermore $F(\eta_N(x))= =\sum_{i=N}^{\infty}\varepsilon_i(x)a_i$, hence

$$y = F(x) = \sum_{i=1}^{N-1}\varepsilon_i(x)a_i+\sum_{i=N}^{\infty}\varepsilon_i(x)a_i = \sum_{i=1}^{N-1}\delta_i(y)a_i+F(\eta_N(x)) < \sum_{i=1}^{N-1}\delta_i(y)a_i+a_N$$

which is a contradiction. Therefore $\mathbf{R}[\{a_n\}]\subset\mathbf{R}[\{\lambda_n\}]$. □

Corollary. *If $\{\lambda_n\}\cong\{a_n\}$, then let us denote the set of numbers having length N, and the set of numbers having length not greater than N by U_N and V_N, respectively, for the sequence $\{\lambda_n\}$, and by U_N^* and V_N^*, respectively, for the sequence $\{a_n\}$. Then*

$$\operatorname{card}(U_N) = \operatorname{card}(U_N^*)$$

and

$$\operatorname{card}(V_N) = \operatorname{card}(V_N^*).$$

6. The main result

The aim of our investigations so far conducted was to determine the structure of a positive, additive and continuous function. In the case of an arbitrary interval filling sequence we are not yet able to answer this question. However, for the case when the interval filling sequence $\{\lambda_n\}\in\Lambda$ is *smooth*, a complete solution of the problem will be given in what follows.

Our result generalizes the results of [2], and also, in a certain sense, that of [6].

The following result will be needed in the sequel:

LEMMA 6.1. *Let* $\{\lambda_n\}\in\Lambda$ *be a smooth interval filling sequence. If* $F\in\mathcal{P}[\{\lambda_n\}]$ *then the sequence* $\{a_n:=F(\lambda_n)\}\in\Lambda$ *is interval filling and smooth.*

PROOF. By Lemma 5.2 $\{a_n\}\in\Lambda$ is interval filling, hence it suffices to prove the smoothness of $\{a_n\}$. By Lemma 3.2 there exist $H, S\in\mathbf{N}$, such that for any $N\in\mathbf{N}$ there exists a number $n\in\{N, N+1, N+2, ..., N+H\}$ satisfying $\lambda_{n+1}+\lambda_{n+s}<\lambda_n$. Then the regular expansion of the number $x:=\lambda_{n+1}+\lambda_{n+s}$ is itself, i.e. by the additivity and the strict increasingness (Theorem 5.4) of F we have

$$2a_{N+H+S} = 2F(\lambda_{N+H+S}) \leqq 2F(\lambda_{n+s}) \leqq F(\lambda_{n+s})+F(\lambda_{n+1}) =$$
$$= F(\lambda_{n+s}+\lambda_{n+1}) < F(\lambda_n) \leqq F(\lambda_N) = a_N,$$

i.e. putting $T:=H+S$ we get

$$a_{N+T} < \frac{1}{2}a_N$$

for any $N\in\mathbf{N}$. Thus by Lemma 3.1 $\{a_n\}\in\Lambda$ is a smooth sequence. □

We are now able to formulate our main result in the form of the following

THEOREM 6.2. *If* $\{\lambda_n\}\in\Lambda$ *is a smooth interval filling sequence and* F: $[0, L]\to\mathbf{R}$ *is an additive and continuous function satisfying* $F(x)>0$ *for* $x>0$ (i.e. $F\in\mathcal{P}[\{\lambda_n\}]$). *then there exists a constant* $\gamma>0$, *such that* $F(x)=\gamma x$ *for any* $x\in[0, L]$.

PROOF. Let $a_n:=F(\lambda_n)>0$ $(n\in\mathbf{N})$. By Theorem 3.4 there exist constants $0<c_1<c_2$ such that

$$\frac{c_1}{\lambda_N} < \text{card}\,(V_N) < \frac{c_2}{\lambda_N} \quad (N\in\mathbf{N}). \tag{6.1}$$

Lemma 6.1 and Theorem 3.4 together imply the existence of constants $0<c_1^*<c_2^*$, such that

$$\frac{c_1^*}{a_N} < \text{card}\,(V_N^*) < \frac{c_2^*}{a_N} \quad (N\in\mathbf{N}) \tag{6.2}$$

where V_N^* denotes the set of numbers having length not greater than N with respect to the interval filling sequence $\{a_n\}\in\Lambda$. By Theorem 5.4 $\{a_n\}\cong\{\lambda_n\}$, hence card $(V_N)=$ $=$card (V_N^*) $(N\in\mathbf{N})$. The inequalities (6.1) and (6.2) now yield

$$\frac{c_1^*}{a_N} < \frac{c_2}{\lambda_N} \quad (N\in\mathbf{N}),$$

i.e. putting $K_1:=\dfrac{c_1^*}{c_2}>0$ we get

$$K_1 < \frac{a_N}{\lambda_N} \quad (N\in\mathbf{N}). \tag{6.3}$$

From the inequality (6.3) and from additivity we infer that

$$0 < K_1 < \frac{F(x)}{x} \quad (x\in]0, L]). \tag{6.4}$$

Let $\gamma := \inf\limits_{x>0} \frac{F(x)}{x}$, where $0<K_1 \leqq \gamma$. Then we have the following two possibilities:

(i) There exists $x_0 \in]0, L]$ such that $\frac{F(x_0)}{x_0} = \gamma$;

(ii) $\frac{F(x)}{x} > \gamma$ for any $x \in]0, L]$.

ad (i). Then $0 \leqq F(x) - \gamma x =: G(x)$, and $G(x_0)=0$ $(x_0>0)$. Since $G\colon [0, L] \to \mathbf{R}$ is additive and continuous, G is also monotone increasing. Hence $G(x)=0$ for $x \in [0, x_0]$. Now there exists N_0, such that $G(\lambda_n)=0$ for $n>N_0$. On the other hand, the continuity of G implies

$$G(\lambda_{N_0}) = \sum_{i=N_0+1}^{\infty} \varepsilon_i^*(\lambda_{N_0}) G(\lambda_i) = 0,$$

whence it follows $G(\lambda_N)=0$ for any $N \in \mathbf{N}$. This implies $G(x)=0$ for any x, i.e. $F(x)=\gamma x$ $(x \in [0, L])$.

ad (ii). Then $0<F(x)-\gamma x =: G(x)$ for any $x>0$. Since G is additive and continuous, by (6.4) there exists $0<K_2$, such that $K_2 < \frac{G(x)}{x}$ for any $x>0$. This yields

$$0 < K_2 \leqq \inf_{x>0} \frac{G(x)}{x} = \inf_{x>0} \frac{F(x)-\gamma x}{x} = \inf_{x>0} \frac{F(x)}{x} - \gamma = 0,$$

and this contradiction shows the impossibility of case (ii). □

7. Additive and continuous functions differentiable on a set of positive measure

The results obtained so far enable us to investigate further classes of continuous additive functions. Since in [3] we have shown by an example that there exists a continuous and additive function which is *nowhere differentiable,* it makes sense to ask, what can be said in case a continuous and additive function is differentiable on some nonvoid set $E \subset [0, L]$.

LEMMA 7.1. *Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence. If $x \in [0, L]$ is a non-finite number, then there exists $N_0 = N_0(x)$, such that $\lambda_{N_0} < x$. For $N \geqq N_0$, let $k := k(N, x)$ be the greatest among the numbers $\{1, 2, \dots, N\}$ for which $\varepsilon_k(x) = 1$. Then for $N \geqq N_0$ there exists $\beta_N(x) \in U_N$, so that*

$$0 < x - \beta_N(x) < \big(N - k(N, x) + 1\big) L_{N+1}. \tag{7.1}$$

PROOF. If $N \geqq N_0$, then

$$S_N(x) := \sum_{n=1}^{N} \varepsilon_n(x) \lambda_n = \sum_{n=1}^{k(N,x)-1} \varepsilon_n(x) \lambda_n + \lambda_{k(N,x)},$$

where $k(N, x)\in\{1, 2, \dots, N\}$. It is clear that

$$0 < x - S_N(x) = \sum_{n=N+1}^{\infty} \varepsilon_n(x)\lambda_n < L_{N+1}.$$

Let us define the number $\beta_N(x)\in U_N$ as follows:

1) For $k(N, x)=N$ let $\beta_N(x):=S_N(x)$.
2) For $1\leqq k(N, x)<N$ let

$$x_1 := \sum_{n=1}^{k(N,x)-1} \varepsilon_n(x)\lambda_n + 0 + \sum_{n=k(N,x)+1}^{N} \varepsilon_n^*(\lambda_{k(N,x)})\lambda_n. \tag{7.2}$$

Then the right hand side of (7.2) is the regular expansion of x_1, and $\lambda_{k+1}<\lambda_k$ implies $\varepsilon^*_{k(N,x)+1}(\lambda_{k(N,x)})=1$, i.e. in the regular expansion of x_1 the maximal index l_1 for which $\varepsilon_{l_1}(x_1)=1$, is necessarily greater than $k(N, x)$. If $l_1=N$, then let $\beta_N(x):=$ $:=x_1\in U_N$ so that

$$0 < S_N(x) - x_1 = \sum_{n=N+1}^{\infty} \varepsilon_n^*(\lambda_{k(N,x)})\lambda_n < L_{N+1}$$

holds. If $l_1<N$, then we continue the procedure in the previous manner. In finitely many steps (if r is the number of steps, then $r\leqq N-k(N, x)$) we reach an x_r satisfying $x_r\in U_N$. As

$$0 < x_{r-1} - x_r = \sum_{n=N+1}^{\infty} \varepsilon_n^*(\lambda_{l_{r-1}})\lambda_n < L_{N+1}$$

where $l_{r-1}<N$ denotes the maximal index occurring in the regular expansion of x_{r-1} for which $\varepsilon_{l_{r-1}}(x_{r-1})=1$, we obtain

$$0 < x - x_r = x - \beta_N(x) < (r+1)L_{N+1} \leqq (N-k(N, x)+1)L_{N+1}$$

i.e. (7.1) holds. □

LEMMA 7.2. *If $\{\lambda_n\}\in\Lambda$ is an interval filling and smooth sequence, then the sequence $\beta_N(x)\in U_N$ $(N\geqq N_0=N_0(x))$ constructed in Lemma 7.1 satisfies*

$$0 < x - \beta_N(x) < KN\lambda_N$$

where $\lim_{N\to\infty} N\lambda_N=0$, i.e. $\beta_N(x)$ converges uniformly to x on the set of nonfinite numbers.

PROOF. Owing to smoothness $L_{N+1}\leqq K\lambda_N$, whence

$$\sum_{N=1}^{\infty} N\lambda_N = L + \sum_{N=1}^{\infty} L_{N+1} \leqq L + KL < \infty$$

i.e. $\lim_{N\to\infty} N\lambda_N=0$, therefore the lemma is true. □

THEOREM 7.3. *Let $\{\lambda_n\}\in\Lambda$ be an interval filling and smooth sequence. If F: $[0, L]\to\mathbf{R}$ is an additive function, differentiable at a nonfinite point $x>0$, then* *for* $N \geqq N_0 = N_0(x)$ *(see Lemma* 7.1*) we have*

$$\left|\frac{F(\lambda_N)}{\lambda_N}\right| \leqq c_1 + c_2\big(N - k(N, x)\big) \tag{7.3}$$

where $k(N, x)$ *is the quantity defined in Lemma* 7.1, *and* $c_1 > 0$, $c_2 > 0$ *are absolute constants depending on* x.

Proof. There exists a function $E_x\colon\,]0, L[\to \mathbf{R}$ such that $\lim\limits_{y\to x} E_x(y) = 0$ and

$$F(x) - F(y) = F'(x)(x-y) + E_x(y)(x-y) \tag{7.4}$$

for any $y \in]0, L[$. Let $\beta_N(x) \in U_N$ $(N \geqq N_0 = N_0(x))$ be the sequence constructed in Lemma 7.1 and let $\gamma_N(x) := \beta_N(x) - \lambda_N$. Now (7.4) implies

$$F(x) - F(\beta_N(x)) = F'(x)(x - \beta_N(x)) + E_x(\beta_N(x))(x - \beta_N(x))$$

and

$$F(x) - F(\gamma_N(x)) = F'(x)(x - \gamma_N(x)) + E_x(\gamma_N(x))(x - \gamma_N(x)).$$

From these equations we get

$$F(\lambda_N) = F(\beta_N(x)) - F(\gamma_N(x)) =$$

$$= F'(x)\lambda_N - E_x(\beta_N(x))(x - \beta_N(x)) + E_x(\gamma_N(x))(x - \gamma_N(x))$$

for $N \geqq N_0 = N_0(x)$. Owing to Lemmas 7.1 and 7.2 $0 < x - \beta_N(x) < KN\lambda_N$ and $0 < x - \gamma_N(x) < KN\lambda_N + \lambda_N$, moreover $N\lambda_N \to 0$ $(N \to \infty)$, hence by smoothness

$$\left|\frac{F(\lambda_N)}{\lambda_N}\right| \leqq |F'(x)| + |E_x(\beta_N(x))| \frac{(N - k(N, x) + 1)L_{N+1}}{\lambda_N} +$$

$$+ |E_x(\gamma_N(x))| \left[\frac{(N - k(N, x) + 1)L_{N+1}}{\lambda_N} + 1\right] \leqq c_1 + c_2\big(N - k(N, x)\big)$$

for $N \geqq N_0$, where $c_1 > 0$ and $c_2 > 0$ are constants depending on x. □

Theorem 7.4. *Let* $\{\lambda_n\} \in \Lambda$ *be an interval filling and smooth sequence. If* $F\colon [0, L] \to \mathbf{R}$ *is additive (with respect to* $\{\lambda_n\}$*) and differentiable on a set of positive measure, then* $\left\{\dfrac{F(\lambda_n)}{\lambda_n}\right\}$ *is a bounded sequence.*

Proof. Suppose the contrary, i.e. let

$$\limsup_{n\to\infty} \left|\frac{F(\lambda_N)}{\lambda_N}\right| = \infty.$$

Then there exists a sequence $M_1 < M_2 < M_3 < \dots$ of indices, such that

$$\left|\frac{F(\lambda_{M_i})}{\lambda_{M_i}}\right| > i^2 \quad (i \in \mathbf{N}). \tag{7.4}$$

Let

$$\mathscr{F}_j := \{x \mid x \in [0, L],\, \varepsilon_n(x) = 0 \quad \text{for} \quad n = M_j - j, M_j - j + 1, \dots, M_j\}.$$

If $x \in \mathcal{F}_j$, then

$$x = \sum_{n=1}^{M_j-j-1} \varepsilon_n(x)\lambda_n + \sum_{n=M_j+1}^{\infty} \varepsilon_n(x)\lambda_n,$$

hence by the smoothness of $\{\lambda_n\}$ and by the inequalities (3.7) and (3.2) we get

$$\text{meas}\,(\mathcal{F}_j) < \text{card}\,(V_{M_j-j})L_{M_j+1} < \frac{cK\lambda_{M_j}}{\lambda_{M_j-j}} < C_0\left(\frac{1}{2}\right)^{j/t_0}.$$

This yields

$$\sum_{j=1}^{\infty} \text{meas}(\mathcal{F}_j) < \infty. \tag{7.5}$$

By the supposition of the theorem there exists a measurable set $E \subset [0, L]$, such that meas $(E) = \delta > 0$ and F is differentiable in the points of E. By (7.5) there exists $T \in \mathbf{N}$ such that

$$\sum_{j=T}^{\infty} \text{meas}\,(\mathcal{F}_j) < \delta.$$

Hence there exists $x \in E$ with the property that for any $j \geqq T$ the set

$$\{\varepsilon_n(x) | n = M_j - j, M_j - j + 1, \ldots, M_j\}$$

contains the value 1. This means that for $j \geqq T$ we have

$$M_j - k(M_j, x) \leqq j+1.$$

As F is differentiable in x, for $j \geqq T$ Theorem 7.3 yields

$$\left|\frac{F(\lambda_{M_j})}{\lambda_{M_j}}\right| < c_1 + c_2(j+1)$$

and this contradicts (7.4). □

THEOREM 7.5. *Let $\{\lambda_n\} \in \Lambda$ be a smooth interval filling sequence. If the additive and continuous function $F\colon [0, L] \to \mathbf{R}$ is differentiable on a set of positive measure, then there exists $\alpha \in \mathbf{R}$, such that $F(x) = \alpha x$ for any $x \in [0, L]$.*

PROOF. By Theorem 7.4 there exists $c > 0$, such that

$$\frac{F(\lambda_n)}{\lambda_n} < c \quad (n \in \mathbf{N}).$$

From this we infer by additivity that $F(x) < cx$ $(x \in]0, L])$. Let $G(x) := cx - F(x)$ $(>0$ for $x \in]0, L])$. Then G is an additive, continuous and positive function, hence by Theorem 6.2 $G(x) = \gamma x$, with $\gamma > 0$ a constant. From this $F(x) = cx - G(x) = (c-\gamma)x$ follows, i.e. putting $\alpha := c - \gamma$ we see that the theorem is valid. □

REMARK. The question whether the differentiability condition occurring in Theorem 7.5 can be weakened to differentiability at a single point remains unanswered. E.g. for the smooth interval filling sequence $\left\{\lambda_n := \frac{1}{q^n}\right\}$ $(1 < q < 2)$ we have been able to give a positive answer ([2]).

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DEPARTMENT OF MATHEMATICS
L. KOSSUTH UNIVERSITY DEBRECEN
DEBRECEN 10, HUNGARY

DEPARTMENT OF MATHEMATICS
L. EÖTVÖS UNIVERSITY BUDAPEST
BUDAPEST, MÚZEUM KRT. 6—8
HUNGARY

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A REMARK ON THE NORM OF THE BANACH SPACE OF UNIFORMLY STRONG CONVERGENT TRIGONOMETRIC SERIES

J. DOBI (Szeged)

N. Tanovic-Miller introduced the set S of functions f whose Fourier series converge strongly uniformly on $[0, 2\pi]$ and showed that $U \supset S \supset A$, where U and A denote the classes of functions f whose Fourier series converge uniformly and absolutely on $[0, 2\pi]$, resp. ([1], Theorem 4.)

It is known that the set C of continuous functions and the set U are Banach spaces with the norms

$$\|f\|_C = \sup_{0 \leqq t \leqq 2\pi} |f(t)|, \quad \|f\|_U = \sup_N \|S_N(f)\|_C$$

where $S_N(f)$ denotes the N-th partial sum of the Fourier series of f. I. Szalay showed that the set S is a Banach space with the norm

$$\text{(1)} \qquad \|f\|_S = \sup_M \frac{1}{M+1} \Big\| \sum_{N=0}^{M} |(N+1)S_N(f) - NS_{N-1}(f)| \Big\|_C \quad (S_{-1}(f) = 0)$$

([2], Theorem 1, [3], Theorem 1).

H. G. Feichtinger raised the problem whether S is a Banach space with the simpler norm

$$\text{(2)} \qquad \|f\| = \sup_N \|S_N(f)\|_C + \sup_M \frac{1}{M+1} \Big\| \sum_{N=1}^{M} N|S_N(f) - S_{N-1}(f)| \Big\|_C \equiv$$

$$\equiv \|f\|_U + \sup_M \frac{1}{M+1} \Big\| \sum_{N=1}^{M} N|A_N(f)| \Big\|_C.$$

We answer this question in the following

THEOREM. *S is Banach space with the norm defined under* (2).

PROOF. Considering that

$$\frac{1}{M+1} \Big\| \sum_{N=0}^{M} |(N+1)S_N(f) - NS_{N-1}(f)| \Big\|_C = \frac{1}{M+1} \Big\| \sum_{N=0}^{M} |NA_N(f) + S_N(f)| \Big\|_C \leqq$$

$$\leqq \frac{1}{M+1} \Big\| \sum_{N=0}^{M} N|A_N(f)| \Big\|_C + \frac{1}{M+1} \Big\| \sum_{N=0}^{M} |S_N(f)| \Big\|_C,$$

it is obvious that

$$\text{(3)} \qquad \|f\|_S \leqq \|f\|$$

because for any non-negative integer M

$$(4)\quad \frac{1}{M+1}\Big\|\sum_{N=0}^{M}|S_N(f)|\Big\|_C \leqq \frac{1}{M+1}\sum_{N=0}^{M}\|S_N(f)\|_C \leqq \frac{1}{M+1}\sum_{N=0}^{M}\|f\|_U = \|f\|_U.$$

On the other hand it is known ([2], Theorem 1, [3], Theorem 1), that

$$(5)\qquad \|f\|_U \leqq \|f\|_S$$

and we show that

$$(6)\qquad \sup_M \frac{1}{M+1}\Big\|\sum_{N=1}^{M} N|A_N(f)|\Big\|_C \leqq 2\|f\|_S.$$

Considering the estimation

$$\frac{1}{M+1}\Big\|\sum_{N=0}^{M} N|A_N(f)|\Big\|_C = \frac{1}{M+1}\Big\|\sum_{N=0}^{M}|(N+1)S_N(f)-NS_{N-1}(f)-S_N(f)|\Big\|_C \leqq$$

$$\leqq \frac{1}{M+1}\Big\|\sum_{N=0}^{M}|(N+1)S_N(f)-N\cdot S_{N-1}(f)|\Big\|_C + \frac{1}{M+1}\Big\|\sum_{N=0}^{M}|S_N(f)|\Big\|_C,$$

and using (1), (4) and (5) we obtain (6). On the basis of (2), (3), (5) and (6)

$$(7)\qquad \|f\|_S \leqq \|f\| \leqq 3\|f\|_S.$$

Using the fact that S is a Banach space with the norm (1), (7) implies that S is a Banach space with the norm (2), too.

We observe that for the function

$$f(t) = \sum_{n=1}^{\infty}\frac{\cos nt}{n(n+1)}$$

we have $\|f\|_s \neq \|f\|$. Indeed,

$$\|f\|_U = \sup_N \Big\|\Big|\sum_{n=1}^{N}\frac{\cos nt}{n(n+1)}\Big|\Big\|_C = \sup_N\left|\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right| = \sup_N\left|1-\frac{1}{N+1}\right| = 1,$$

$$\|f\|_S = \sup_M \frac{1}{M+1}\Big\|\sum_{N=0}^{M}\big|N(S_N(f)-S_{N-1}(f))+S_N(f)\big|\Big\|_C =$$

$$= \sup_M \frac{1}{M+1}\left\|\sum_{N=0}^{M}\left|\frac{\cos Nt}{N+1}+\sum_{n=1}^{N}\frac{\cos nt}{n(n+1)}\right|\right\|_C = \sup_M \frac{1}{M+1}\sum_{N=0}^{M}\left|\frac{1}{N+1}+\right.$$

$$\left.+\left(1-\frac{1}{N+1}\right)\right| = \sup_M \frac{1}{M+1}\Big|\sum_{N=0}^{M} 1 = \sup_M \frac{M+1}{M+1} = 1,$$

and

$$\|f\| = 1+\sup_M \frac{1}{M+1}\left\|\sum_{N=1}^{M} N\left|\frac{\cos Nt}{N(N+1)}\right|\right\|_C = 1+\sup_M \frac{1}{M+1}\left|\sum_{N=1}^{M}\frac{1}{N+1}\right| = 1+\frac{5}{18}.$$

This example shows that the norm $\|f\|_S$ in certain sense is better than the norm $\|f\|$, because assuming $f\in A$ by the estimation

$$\frac{1}{M+1}\sum_{N=0}^{M}|(N+1)S_N(f)-NS_{N-1}(f)| \leqq \frac{1}{M+1}\sum_{N=0}^{M}|S_N(f)|+\frac{1}{M+1}\sum_{N=0}^{M}N|A_N(f)| \leqq$$

$$\leqq \frac{1}{M+1}\sum_{N=0}^{M}(M+1-N)|A_N(f)|+\frac{1}{M+1}\sum_{N=0}^{M}N|A_N(f)| = \sum_{N=0}^{M}|A_N(f)|$$

we have

$$\|f\|_S \leqq \|f\|_A, \tag{8}$$

which does not hold for $\|f\|$. Finally we mention that (8) is sharper than the inequality (1.9) in [2].

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RÓZSA FERENC SZAKKÖZÉPISKOLA
SZEGED, ODESSZAI KRT. 38
H—6725, HUNGARY

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ON A PEXIDER-TYPE FUNCTIONAL EQUATION FOR QUASIDEVIATION MEANS

ZS. PÁLES (Debrecen—Karlsruhe)

1. Introduction

It is well-known that the Shannon entropy $S(P)=S(p_1, \dots, p_n)=\sum_{i=1}^{n} -p_i \ln p_i$, where $P=(p_1, \dots, p_n)$ is a finite probability distribution, satisfies the functional equation

$$S(P*Q) = S(P)+S(Q) \tag{1}$$

for each probability distributions $P=(p_1, \dots, p_n)$ and $Q=(q_1, \dots, q_m)$ with

$$P*Q = (p_1 q_1, \dots, p_n q_1, \dots, p_1 q_m, \dots, p_n q_m).$$

If we denote

$$M(P) = \exp\big(-S(P)\big) = \prod_{i=1}^{n} p_i^{p_i}$$

then (1) turns into

$$M(P*Q) = M(P)\,M(Q). \tag{2}$$

On the other hand, it is easy to check that $M(p_1, \dots, p_n)$ is a mean value of the variables $p_1, \dots, p_n$. These properties of M led Rényi [15], [16], Aczél and Daróczy [3] and Daróczy [7] to investigate (2) in several classes of means. A summary of these results can be found in the book of Aczél and Daróczy [5].

In 1980 Daróczy and the author [11] investigated the multiplicativity equation

$$M(x*y) = M(x)\,M(y) \tag{3}$$

where $x=(x_1, \dots, x_n)\in \mathbf{R}_+^n$, $y=(y_1, \dots, y_m)\in \mathbf{R}_+^m$, $x*y=(x_1y_1, \dots, x_ny_1, \dots, x_1y_m, \dots \dots, x_ny_m)\in \mathbf{R}_+^{nm}$, $n, m\in \mathbf{N}$, further M is assumed to be a deviation mean (see [8]). (Here x and y are not supposed to be finite probability distributions.) Our result reads as follows (see [10, Theorem 9]):

THEOREM 1. *Let M be a deviation mean on $\mathbf{R}_+$. Then M is multiplicative on $\mathbf{R}_+$ (i.e. satisfies* (3) *for all indicated x and y) if and only if there exist a multiplicative function m: $\mathbf{R}_+ \to \mathbf{R}_+$ (i.e. $m(xy)=m(x)m(y)$ for all $x, y\in \mathbf{R}_+$) and a real constant c such that $M(x)=M_{c,m}(x)$ for all $x\in \mathbf{R}_+^n$ and $n\in \mathbf{N}$ where*

$$M_{c,m}(x_1, \dots, x_n) = \begin{cases} \left(\sum_{i=1}^{n} m(x_i)x_i^c \Big/ \sum_{i=1}^{n} m(x_i)\right)^{1/c}, & \text{if } c \neq 0, \\ \exp\left(\sum_{i=1}^{n} m(x_i)\ln x_i \Big/ \sum_{i=1}^{n} m(x_i)\right), & \text{if } c = 0. \end{cases}$$

In the present note we study the more general Pexider-type functional equation

$$K(x*y) = M(x)\,N(y) \tag{4}$$

where M, N, K are quasideviation means on the given open intervals $I, J, H \subseteqq \mathbf{R}_+$, respectively and where $x \in I^n$, $y \in J^m$ with $x*y \in H^{nm}$ and $n, m \in \mathbf{N}$ are arbitrary values.

If $I=J=H=\mathbf{R}_+$ then substituting $m=1, y=1$ and $n=1$, $x=1$ we obtain that $K=M$ and $K=N$, respectively. Therefore, in this case, (4) is equivalent to (3). But the general case is much more complicated. To overcome the difficulties, we shall improve the method of [10].

If $IJ \cap H = \emptyset$ then (4) does not say anything for the means M, N and K, that is, any means M, N and K satisfy (4). Therefore it is reasonable to assume that $IJ \cap H \neq \emptyset$ throughout this paper.

2. Basic notations and concepts

Let $I \subseteqq \mathbf{R}$ be an interval. The function $E\colon I^2 \to \mathbf{R}$ is called a *quasideviation* on I if

E1) $\operatorname{sign} E(x, y) = \operatorname{sign}(x-y)$ for all $x, y \in I$;

E2) $y \to E(x, y)$ is a continuous function on I for each fixed $x \in I$;

E3) $y \to E(x, y)/E(x', y)$ is a strictly decreasing function on $]x, x'[$ for each fixed $x < x'$ in I.

If $x_1, \ldots, x_n$ are arbitrary values in I then (as it has been proved in [13, Theorem 2.1]) the equation

$$E(x_1, y) + \ldots + E(x_n, y) = 0$$

has a unique solution $y=y_0$ in I. This solution y_0 is called the *E-quasideviation mean* of $x_1, \ldots, x_n$ and is denoted by $\mathfrak{M}_E(x_1, \ldots, x_n)$.

The following result summarizes the properties of quasideviation means.

THEOREM 2. *Let $I \subseteqq \mathbf{R}$ be an open interval and let $M\colon \bigcup_{n=1}^{\infty} I^n \to \mathbf{R}$ be an arbitrary function. Then M is a quasideviation mean (that is there exists a quasideviation E on I such that $M = \mathfrak{M}_E$) if and only if the following properties are satisfied:*

(i) *M is reflexive, that is, for $x \in I$, $M(x) = x$;*

(ii) *M is symmetric, that is*

$$M(x_1, \ldots, x_n) = M(x_{p(1)}, \ldots, x_{p(n)})$$

for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ and permutation p of the set $\{1, \ldots, n\}$;

(iii) *For any $x < u < v < y$ in I there exist natural numbers n, m such that*

$$u < M(\underbrace{x, \ldots, x}_{n}, \underbrace{y, \ldots, y}_{m}) < v;$$

(iv) *M is strongly internal in the following sense:*

$$\min\big(M(x_1, \ldots, x_n), M(y_1, \ldots, y_m)\big) < M(x_1, \ldots, x_n, y_1, \ldots, y_m) < \tag{5}$$
$$< \max\big(M(x_1, \ldots, x_n), M(y_1, \ldots, y_m)\big)$$

for all $n, m \in \mathbf{N}$, $x_i, y_j \in I$ *unless* $M(x_1, \ldots, x_n) = M(y_1, \ldots, y_m)$. *In this latter case* (5) *holds with equality at both places.*

For the proof of this result see [14, Theorem 4] and for the original version of this characterization theorem see [13, Characterization Theorem 2].

REMARK. The most important examples for quasideviation means are the quasiarithmetic means [12], the quasiarithmetic means with weight function [6], and the deviation means [8]. We mention only this latter concept: $E\colon I^2 \to \mathbf{R}$ is called a *deviation* if E1) and E2) are satisfied, further,

E3)* $y \to E(x, y)$ is strictly decreasing on I for each fixed $x \in I$.

It is easy to check that E3)* implies E3) but the converse statement is not true. Therefore the class of means generated by quasideviations is wider than the one generated by deviations. The use of quasideviation means is convenient since a characterization theorem is known for them, but for deviation means a similar result has not been obtained yet.

3. Basic functional equations

Let I, J and H be fixed open subintervals of $\mathbf{R}_+$. For the sake of convenience, we shall use the following notations:

$$I_0 = I \cap (H/J), \quad J_0 = J \cap (H/I), \quad H_0 = H \cap (IJ)$$

and

$$I^* = I_0/I_0, \quad J^* = J_0/J_0, \quad H^* = H_0/H_0.$$

Then it is obvious that I_0, J_0, H_0, I^*, J^* and H^* are open intervals, further $I_0 \subseteqq H_0/J_0$, $J_0 \subseteqq H_0/I_0$, $H_0 \subseteqq I_0 J_0$ and $I^* \subseteqq H^*/J^*$, $J^* \subseteqq H^*/I^*$, $H^* \subseteqq I^* J^*$.

THEOREM 3. *Let E, F and G be quasideviations on I, J and H, respectively. Then*

$$\mathfrak{M}_G(x*y) = \mathfrak{M}_E(x) \cdot \mathfrak{M}_F(y) \tag{6}$$

is satisfied for all $n, m \in \mathbf{N}$, $x \in I^n$, $y \in J^m$ *with* $x*y \in H^{nm}$ *if and only if*

$$\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} E(x_{3-i}, x_0) F(y_{3-j}, y_0) G(x_i y_j, x_0 y_0) = 0 \tag{7}$$

holds for $x_i \in I_0$, $y_j \in J_0$ *with* $x_i y_j \in H_0$ $(i, j = 0, 1, 2)$.

PROOF. First we prove that (7) is a necessary condition. If $IJ \cap H = \emptyset$ then $I_0 = J_0 = H_0 = \emptyset$ therefore (7) is obvious. If $IJ \cap H \neq \emptyset$ then I_0, J_0, H_0 are nonempty open subintervals of $\mathbf{R}_+$.

Let first $x_1, x_2 \in I_0$, $y_1, y_2 \in J_0$ be fixed values with $x_1 < x_2$, $y_1 < y_2$, $x_1 y_1$, $x_2 y_2 \in H_0$. We shall prove that then (7) is satisfied for all $x_1 < x_0 < x_2$, $y_1 < y_0 < y_2$. Put

$$x = (\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2}), \quad y = (\underbrace{y_1, \ldots, y_1}_{m_1}, \underbrace{y_2, \ldots, y_2}_{m_2}) \tag{8}$$

into (6). Then we obtain

$$(9)\quad \mathfrak{M}_G(\underbrace{x_1y_1,\dots,x_1y_1}_{n_1m_1},\ \underbrace{x_1y_2,\dots,x_1y_2}_{n_1m_2},\ \underbrace{x_2y_1,\dots,x_2y_1}_{n_2m_1}),\ \underbrace{x_2y_2,\dots,x_2y_2}_{n_2m_2}) = x_0y_0,$$

where

$$(10)\qquad x_0 = x_0(n_1,n_2) = \mathfrak{M}_E(\underbrace{x_1,\dots,x_1}_{n_1},\underbrace{x_2,\dots,x_2}_{n_2}),$$

$$(11)\qquad y_0 = y_0(m_1,m_2) = \mathfrak{M}_F(\underbrace{y_1,\dots,y_1}_{m_1},\underbrace{y_2,\dots,y_2}_{m_2}).$$

By the definition of quasideviation means, it follows from (9), (10) and (11) that

$$(12)\qquad \sum_{i=1}^{2}\sum_{j=1}^{2} n_i m_i G(x_iy_j, x_0y_0) = 0,$$

$$(13)\qquad n_1E(x_1,x_0)+n_2E(x_2,x_0) = 0$$

and

$$(14)\qquad m_1F(y_1,y_0)+m_2F(y_2,y_0) = 0.$$

Applying (13) and (14), we can see that

$$n_i = (-1)^iE(x_{3-i},x_0)A \quad (i=1,2)$$

and

$$m_j = (-1)^jF(y_{3-j},y_0)B \quad (j=1,2)$$

with some nonzero values A and B independent of i and j, respectively. Substituting these values into (12) and dividing the resulting equation by AB we obtain that (7) is valid for $x_0=x_0(n_1,n_2)$ and $y_0=y_0(m_1,m_2)$ where n_1,n_2,m_1,m_2 are arbitrary natural numbers. By property (iii) of quasideviation means in Theorem 2, the values $x_0=x_0(n_1,n_2)$, $y_0=y_0(m_1,m_2)$ form a dense subset of $]x_1,x_2[$ and $]y_1,y_2[$, respectively. Now, applying that quasideviations are continuous in the second variable, we can see that (7) is valid for all $x_1<x_0<x_2$, $y_1<y_0<y_2$.

To prove (7) on the wider domain described in the Theorem, fix $x_0\in I_0$, $y_0\in J_0$ with $x_0y_0\in H_0$ and rewrite (7) into the form

$$(15)\qquad A(x_1,y_1)-A(x_2,y_1)-A(x_1,y_2)+A(x_2,y_2) = 0,$$

where A is defined by

$$A(x,y) = \frac{G(xy,x_0y_0)}{E(x,x_0)F(y,y_0)}$$

for $x\in I_0\setminus\{x_0\}$, $y\in J_0\setminus\{y_0\}$ with $xy\in H_0$.

First let $y_1<y_0<y_2$ be fixed values in J_0 and let $x_1,x_2\in I_0\setminus\{x_0\}$ with $x_iy_j\in H_0$. If $x_1<x_0<x_2$ or $x_2<x_0<x_1$ then (15) is valid (as we have proved it). If $x_1,x_2<x_0$ then choose $x_3\in I_0$ so that $x_0<x_3$ and $x_3y_j\in H_0$. (If x_3 is close enough to x_0 then these conditions are satisfied.) Now, applying (15) for $x_1<x_0<x_3$, $y_1<y_0<y_2$ and for $x_2<x_0<x_3$, $y_1<y_0<y_2$ we obtain

$$A(x_1,y_1)-A(x_3,y_1)-A(x_1,y_2)+A(x_3,y_2) = 0$$

and

$$A(x_2, y_1) - A(x_3, y_1) - A(x_2, y_2) + A(x_3, y_2) = 0.$$

Subtracting the second equation from the first one we get that (15) is valid for $x_1, x_2 < x_0$. If $x_0 < x_1, x_2$ then choosing $x_3 < x_0$ in I_0 with $x_3 y_j \in H_0$, the same argument gives (15).

Thus we have seen that (15) is valid if $x_1, x_2 \in I_0 \setminus \{x_0\}$ and $y_1, y_2 \in J_0 \setminus \{y_0\}$ with $y_1 < y_0 < y_2$, $x_i y_j \in H_0$. Now fixing x_1 and x_2, a similar argument shows that the restriction $y_1 < y_0 < y_2$ can also be omitted.

Rearranging (15) to its original form and using the continuity of quasideviations in the second variable, we get that (7) is satisfied on the domain indicated in the theorem. Thus the necessity of (7) is proved.

To see the sufficiency, let $x_i \in I$, $y_j \in J$ with $x_i y_j \in H$ $(1 \leqq i \leqq n,\ 1 \leqq j \leqq m)$. Then it is obvious that $x_i \in I_0$, $y_j \in J_0$, $x_i y_j \in H_0$ is also valid. Denote by x_0 and y_0 the mean values $\mathfrak{M}_E(x_1, \ldots, x_n)$ and $\mathfrak{M}_F(y_1, \ldots, y_m)$, respectively. Choose $x^* \in I_0 \setminus \{x_0\}$, $y^* \in J_0 \setminus \{y_0\}$ with $x^* y^* \in H_0$ so that $x^* y_j$, $x_i y^* \in H_0$ be satisfied. (If x^* and y^* are close to x_0 and y_0, respectively, then these relations hold since $x_0 y_j$, $x_i y_0 \in H_0$.)

Putting $x_1 = x$, $x_2 = x^*$, $y_1 = y$, $y_2 = y^*$ into (7) and rearranging the inequality we get

$$G(xy, x_0 y_0) = P(y) E(x, x_0) + Q(x) F(y, y_0) + RE(x, x_0) F(y, y_0), \tag{16}$$

where

$$P(y) = \frac{G(x^* y, x_0 y_0)}{E(x^*, x_0)}, \quad Q(x) = \frac{G(x y^*, x_0 y_0)}{F(y^*, y_0)}$$

and

$$R = -\frac{G(x^* y^*, x_0 y_0)}{E(x^*, x_0) F(y^*, y_0)}.$$

Substitute $x = x_i$, $y = y_j$ into (16) and add the inequalities obtained. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} G(x_i y_j, x_0 y_0) = \sum_{j=1}^{m} P(y_j) \sum_{i=1}^{n} E(x_i, x_0) + \sum_{i=1}^{n} Q(x_i) \sum_{j=1}^{m} F(y_j, y_0) + \tag{17}$$

$$+ R \sum_{i=1}^{n} E(x_i, x_0) \sum_{j=1}^{m} F(y_j, y_0).$$

By the definition of x_0 and y_0 we know that

$$\sum_{i=1}^{n} E(x_i, x_0) = 0 \quad \text{and} \quad \sum_{j=1}^{m} F(y_j, y_0) = 0.$$

Therefore it follows from (17) that

$$\sum_{i=1}^{n} \sum_{j=1}^{\hat{m}} G(x_i y_j, x_0 y_0) = 0,$$

i.e.

$$\mathfrak{M}_G(x_1 y_1, \ldots, x_1 y_m, \ldots, x_n y_1, \ldots, x_n y_m) = x_0 y_0,$$

which was to be proved.

In the next step we give necessary and sufficient conditions in order that (7) be valid.

THEOREM 4. *Let E, F and G be quasideviations on I, J and H, respectively. Then E, F and G satisfy the functional equation* (7) *for all* $x_i \in I_0$, $y_j \in J_0$ *with* $x_i y_j \in H_0$ $(i, j=0, 1, 2)$ *if and only if*

(i) *for all* $x_i \in I_0$, $y_0 \in J_0$ *with* $x_i y_0 \in H_0$ $(i=0, 1, 2)$,

$$E(x_1, x_0) G(x_2 y_0, x_0 y_0) = E(x_2, x_0) G(x_1 y_0, x_0 y_0); \tag{18}$$

(ii) *for all* $x_0 \in I_0$, $y_j \in J_0$ *with* $x_0 y_j \in H_0$ $(j=0, 1, 2)$,

$$F(y_1, y_0) G(x_0 y_2, x_0 y_0) = F(y_2, y_0) G(x_0 y_1, x_0 y_0); \tag{19}$$

(iii) *for each fixed* $z_0 \in H_0$ *there exists a function*

$$a\colon [(I^* \cup J^*) \cap (H_0/z_0)] \setminus \{1\} \to \mathbf{R}$$

such that

$$\frac{G(stz_0, z_0)}{G(sz_0, z_0) G(tz_0, z_0)} = a(s) + a(t) \tag{20}$$

if $s \in I^* \setminus \{1\}$, $t \in J^* \setminus \{1\}$, $s, t, st \in H_0/z_0$.

PROOF. Assume first that (7) is satisfied. Then putting $y_1 \neq y_0 = y_2$ and $x_1 \neq \neq x_0 = x_2$ into (7), we easily obtain (18) and (19), respectively.

Denoting the value of the right hand side of (18) and (19) by c and d, respectively, we have that

$$E(x_{3-i}, x_0) = c/G(x_i y_0, x_0 y_0)$$

and

$$F(y_{3-j}, y_0) = d/G(x_0 y_j, x_0 y_0)$$

for $i, j=1, 2$. Applying these equations, (7) turns into the form

$$\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} \frac{G(x_i y_j, x_0 y_0)}{G(x_i y_0, x_0 y_0) G(x_0 y_j, x_0 y_0)} = 0 \tag{21}$$

where $x_i \in I_0$, $y_j \in J_0$, $x_i y_j \in H_0$ $(i, j=0, 1, 2)$, $x_1 \neq x_0 \neq x_2$, $y_1 \neq y_0 \neq y_2$. Now let $z_0 \in H_0$ be fixed and define the function A by

$$A(s, t) = \frac{G(stz_0, z_0)}{G(sz_0, z_0) G(tz_0, z_0)}$$

where $s, t, st \in H_0/z_0$.

We are going to show that A satifies the equation

$$\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} A(s_i, t_j) = 0 \tag{22}$$

if $s_i \in I^* \setminus \{1\}$, $t_j \in J^* \setminus \{1\}$ and $s_i, t_j, s_i t_j \in H_0/z_0$ $(i, j=1, 2)$. First assume, in addition, that $s_1/s_2 \in I^*$ and $t_1/t_2 \in J^*$ is also satisfied and let $s_0 = 1 = t_0$. Then, by our assumptions,

$$(I_0/s_i) \cap (I_0/s_j) \neq \emptyset$$

and

$$(t_i z_0/J_0)\cap(t_j z_0/J_0) \neq \emptyset$$

for $i, j=0, 1, 2$.

On the other hand, since $s_i t_j z_0 \in H_0 \subset I_0 J_0$, hence

$$(I_0/s_i)\cap(t_j z_0/J_0) \neq 0$$

for $i, j=0, 1, 2$. Thus we have proved that the family of intervals

$$\{I_0,\ I_0/s_1,\ I_0/s_2,\ z_0/J_0,\ z_0 t_1/J_0,\ z_0 t_2/J_0\}$$

satisfies the 2-intersection property, that is, each two elements of this family have a nonempty intersection. Then, by the well-known Helley's theorem (see [11]) there exists a common element of these intervals. Denote it by x_0 and let $y_0=z_0/x_0$. Then

$$x_i := s_i x_0 \in I_0, \quad y := t_j y_0 \in J_0$$

and

$$x_i y_j = s_i t_j x_0 y_0 = s_i t_j z_0 \in H_0.$$

Applying (21) for these values, we obtain (22) in this case.

If $s_1/s_2 \notin I^*$ is not satisfied but $t_1/t_2 \in J^*$ is, then choose $s^* \in I^* \setminus \{1\}$ so that $s^*, s^* t_j \in H_0/z_0$, $s_i/s^* \in I^*$ $(i, j=1, 2)$ be satisfied. (If s^* is close enough to 1 then this is possible.) Now apply (22) for the values s_1, s^*, t_1, t_2 and for s^*, s_2, t_1, t_2. Then

$$A(s_1, t_1)-A(s^*, t_1)-A(s_1, t_2)+A(s^*, t_2) = 0$$

and

$$A(s^*, t_1)-A(s_2, t_1)-A(s^*, t_2)+A(s_2, t_2) = 0.$$

Adding the first equation to the second, we obtain (22). A similar argument shows that the restriction $t_1/t_2 \in J^*$ can also be omitted. Thus we have proved (22) on the domain indicated.

Now define the function a. For $r \in [(I^* \cup J^*)\cap(H_0/z_0)]\setminus\{1\}$ let

$$a(r) = A(r, r_0)-(1/2)A(r_0, r_0), \tag{23}$$

where $1 \neq r_0 \in I^* \cap J^*$ with $rr_0, r_0^2 \in H_0/z_0$.

We have to show that the definition of a is correct, that is, $a(r)$ does not depend on the choice of r_0.

If $r \in I^* \cap (H_0/z_0)$ and r^* satisfies the same conditions as r_0, then, applying (23) and the symmetry of A, we have that

$$A(r, r_0)-A(r, r^*)-A(r_0, r_0)+A(r_0, r^*) = 0$$

and

$$(1/2)A(r^*, r^*)-A(r_0, r^*)+(1/2)A(r_0, r_0) = 0.$$

Adding these equations we obtain

$$A(r, r_0)-(1/2)A(r_0, r_0)-A(r, r^*)+(1/2)A(r^*, r^*) = 0.$$

If $r \in J^* \cap (H_0/z_0)$ then a similar argument shows that the definition of a is correct.

At last we show that

$$A(s, t) = a(s)+a(t)$$

if $s\in I^*\setminus\{1\}$, $t\in J^*\setminus\{1\}$, $s, t, st\in H_0/z_0$.

Choosing the same r_0 in the definition of $a(s)$ and $a(t)$ and applying (22), we have

$$a(s)+a(t) = A(s, r_0)+A(r_0, t)-A(r_0, r_0) = A(s, t).$$

Thus the necessity of conditions (i), (ii) and (iii) of the Theorem is proved.

The proof of the sufficiency is far easier. If (20) is satisfied on the indicated domain, then

$$\frac{G(xy, x_0y_0)}{G(xy_0, x_0y_0)G(x_0y, x_0y_0)} = a(x/x_0)+a(y/y_0)$$

if $x\in I_0\setminus\{x_0\}$, $y\in J_0\setminus\{y_0\}$, $x_0y, xy_0, xy\in H_0$.

Applying this formula, it is easy to check that (21) is valid if $x_i\in I_0$, $y_j\in J_0$, $x_iy_j\in H_0$ $(i, j=0, 1, 2)$, $x_1\neq x_0\neq x_2$, $y_1\neq y_0\neq y_2$. Using equations (18) and (19), it follows from (21) that (7) is also satisfied. Thus the proof is complete.

For the sake of brevity introduce the notations

(24) $$U:=(I^*\setminus\{1\})\cap(H_0/z_0),\quad V:=(J^*\setminus\{1\})\cap(H_0/z_0),\quad W = H_0/z_0$$

(24) $$b(r) = G(rz_0, z_0)\quad (r\in W),$$

(where $z_0\in H_0$ is fixed). Then it is obvious that U, V, W are open sets, $U\subseteq W$, $V\subseteq W$, $W\subseteq UV$, further, by property E1) of quasideviations,

(25) $$\operatorname{sign} b(r) = \operatorname{sign}(r-1)$$

for $r\in W$.

Now condition (iii) of Theorem 4 means that there exists a function $a\colon U\cup V\to\mathbf{R}$ such that

(26) $$\frac{b(st)}{b(s)b(t)} = a(s)+a(t)$$

if $s\in U$, $t\in V$ and $st\in W$.

In what follows we derive a functional equation for the unknown function a.

THEOREM 5. *Let* $a\colon U\cup V\to\mathbf{R}$, $b\colon W\to\mathbf{R}$ *and assume that* (25) *and* (26) *is satisfied for* $r\in W$ *and for* $s\in U$, $t\in V$ *with* $st\in W$, *respectively. Then there exists a constant* $c\in\mathbf{R}$ *such that*

(i) *if* $s\in U$, $t\in V$, $st\in W$ *then*

(28) $$a(st)\big(a(s)+a(t)\big)-a(s)a(t) = c,$$

further

(ii) *for* $r^2\in U\cap V$ $(r>0)$,

(29) $$\operatorname{sign} a(r) = \operatorname{sign}(r-1);$$

(iii)

$$a(s)+a(t)=0 \tag{30}$$

is satisfied for $s\in U$, $t\in V$ *with* $st\in U\cup V$ *if and only if* $st=1$.

PROOF. Denote by $B(s,t)$ the left hand side of (28) on the open domain

$$D=\{(s,t)|s\in U,\ t\in V,\ st\in U\cup V\}.$$

Let $(s_0,t_0)\in D$ be fixed and define D_{s_0,t_0} by

$$D_{s_0,t_0}=\{(s,t)\mid s,t,st\in U\cap V,\ s_0t\in U,\ s_0st,\ s_0t_0t\in W\}.$$

We are going to show that

$$B(s_0,t_0)=B(s,t) \tag{31}$$

for $(s,t)\in D_{s_0,t_0}$.

Consider the identities

$$\frac{b(s_0t_0t)}{b(s_0t_0)b(t)}\cdot\frac{b(s_0t_0)}{b(s_0)b(t_0)}=\frac{b(s_0tt_0)}{b(s_0t)b(t_0)}\,\frac{b(s_0t)}{b(s_0)b(t)} \tag{32}$$

and

$$\frac{b(s_0st)}{b(s_0t)b(s)}\cdot\frac{b(s_0t)}{b(s_0)b(t)}=\frac{b(s_0st)}{b(s_0)b(st)}\,\frac{b(st)}{b(s)b(t)}. \tag{33}$$

If $(s,t)\in D_{s_0,t_0}$, then either $(s_0,t_0,t)\in D$ or $(t,s_0t_0)\in D$, further (s_0,t_0), (s_0t,t_0), (s_0,t), (s_0t,s), (s_0,t), (s_0,st) and (s,t) are also in D. Therefore, by (26), it follows from (32) and (33) that

$$[a(s_0t_0)+a(t)][a(s_0)+a(t_0)]=[a(s_0t)+a(t_0)][a(s_0)+a(t)]$$

and

$$[a(s_0t)-a(s)][a(s_0)+a(t)]=[a(s_0)+a(st)][a(s)+a(t)],$$

respectively, whence

$$B(s_0,t_0)=B(s_0,t)=B(s,t),$$

i.e. (31) holds.

Now we prove that B is a constant. Let $(s_1,t_1)\in D$ be arbitrary. Then it is easy to check that there exists $(s,t)\in D_{s_0,t_0}\cap D_{s_1,t_1}$. (If s and t are close enough to 1 then this condition is satisfied.) This implies that

$$B(s_0,t_0)=B(s,t)=B(s_1,t_1).$$

Denoting by c the constant value of $B(s,t)$, we get (28).

To prove (ii), substitute $s=t=r$ into (26) and use (25). Property (iii) of the function a is also a simple consequence of (25) and (26).

4. Solution of the basic functional equations

First we solve equation (28).

THEOREM 6. *Let* $c\in\mathbf{R}$ *be an arbitrary constant. Then the function* $a\colon U\cup V\to\mathbf{R}$ *satisfies properties* (i), (ii) *and* (iii) *of Theorem* 5 *if and only if there exists a positive constant* p *such that*

$$a(r)=\begin{cases}\sqrt{c}\,\coth(\sqrt{c}\,\ln r^p), & \text{if } c\neq 0,\\ 1/\ln r^p & \text{if } c=0\end{cases}\tag{34}$$

for $r\in U\cup V$.

PROOF. First we are going to prove the existence of $1<r_0$ such that if $r\in]1/r_0, r_0[\setminus\{1\}$, then (34) holds with a suitable constant p.

Choose $1<r_0$ so that r_0^2, $1/r_0^2\in U\cap V$. If $r\in]1/r_0, r_0[\setminus\{1\}$, then we have that (29) is satisfied. Now we distinguish three cases.

Case I: $c<0$. Define the function $q\colon]1, r_0[\to\mathbf{R}$ by

$$q(r)=\operatorname{arcotg}\left(a(r)/\sqrt{-c}\right).$$

Then it is obvious that $0<q(r)<\pi/2$ since $0<a(r)$. Let $s, t, st\in]1, r_0[$. Using the functional equation (28), we have

$$\cot q(st)\left(\cot q(s)+\cot q(t)\right)=\cot q(s)\cdot\cot q(t)-1.\tag{35}$$

On the other hand, since $q(s), q(t), q(s)+q(t)\in]0,\pi[$, we can apply the well-known addition formula of the cotangent function. Thus we get

$$\cot q(s)\cot q(t)-1=\cot\left(q(s)+q(t)\right)\left(\cot q(s)+\cot q(t)\right).\tag{36}$$

Since $\cot q(s)<0$, $\cot q(t)>0$, hence it follows from (35) and (36) that

$$\cot q(st)=\cot\left(q(s)+q(t)\right).\tag{37}$$

However $q(st)$, $q(s)+q(t)\in]0,\pi[$, therefore (37) implies that

$$q(st)=q(s)+q(t).\tag{38}$$

Let $A(x)=q(e^x)$ if $x\in]0,\ln r_0[$. By (38)

$$A(x+y)=A(x)+A(y)$$

if $x, y, x+y\in]0,\ln r_0[$. Using the well-known extension theorem of Aczél—Erdős [2] and Daróczy—Losonczi [9], we can state the existence of an additive function $\bar{A}\colon\mathbf{R}\to\mathbf{R}$ which is the extension of A. If $x\in]0,\ln r_0[$ then $\bar{A}(x)=A(x)=q(e^x)>0$, therefore there exists a positive constant $k>0$ such that $\bar{A}(x)=kx$ for $x\in\mathbf{R}$ (see Aczél [1]). Let $p=k/\sqrt{-c}$. Then

$$q(r)=A(\ln r)=\bar{A}(\ln r)=k\ln r=p\sqrt{-c}\ln r,$$

and thus, for $r\in]1, r_0[$,

$$a(r)=\sqrt{-c}\cot q(r)=\sqrt{-c}\cot(p\sqrt{-c}\ln r)=\sqrt{c}\coth(\sqrt{c}\ln r^p).$$

If $r \in]1/r_0, 1[$, then using property (iii) of a, we get

$$a(r) = -a(1/r) = -\sqrt{c}\coth(\sqrt{c}\ln 1/r^p) = \sqrt{c}\coth(\sqrt{c}\ln r^p).$$

Thus we have proved that (34) is valid for $r \in]1/r_0, r_0[\setminus\{1\}$ in the case $c<0$.

Case II: $c=0$. Let $q(r)=1/a(r)$ for $r \in]1, r_0[$, Then, applying equation (28), we can see that (38) is satisfied for all $s, t, st \in]1, r_0[$.

Using the same argument as in Case I, we obtain that there exists a positive constant p such that

$$q(r) = p\ln r = \ln r^p.$$

Therefore $a(r)=1/\ln r^p$ for $r \in]1, r_0[$.

If $r \in]1/r_0, r_0[$, then, by (iii),

$$a(r) = -a(1/r) = -1/\ln(1/r^p) = 1/\ln r^p.$$

This completes the proof of (34) in this case.

Case III: $c>0$. First we shall show that

$$a(r) > \sqrt{c} \tag{39}$$

if $r \in]1, r_0[$.

Put $s=t=\sqrt{r}$ into (28). Then apply the arithmetic-geometric mean inequality to obtain

$$a(r) = \frac{c+a^2(\sqrt{r})}{2a(\sqrt{r})} = \frac{1}{2}\left(\frac{c}{a(\sqrt{r})}+a(\sqrt{r})\right) \geqq \sqrt{c}.$$

If $a(r)=\sqrt{c}$ then, by the above inequality, $a(\sqrt{r})=\sqrt{c}$ is also satisfied. On the other hand, by (iii),

$$a(1/r) = -a(r) = -\sqrt{c}.$$

Thus

$$a(1/r)+a(\sqrt{r}) = -\sqrt{c}+\sqrt{c} = 0. \tag{40}$$

Using (iii) again, (40) implies that $(1/r)\sqrt{r}=1$, which is a contradiction. Therefore (39) is valid.

For $r \in]1, r_0[$ let

$$q(r) = \operatorname{arcoth}(a(r)/\sqrt{c}).$$

(The definition of q is correct since $a(r)>\sqrt{c}$.) Then the functional equation (28) turns into the form

$$\coth q(st)(\coth q(s)+\coth q(t)) = \coth q(s)\cdot\coth q(t)+1 =$$
$$= \coth(q(s)+q(t))(\coth q(s)+\coth q(t)).$$

Since $\coth q(s), \coth q(t)>0$, hence

$$\coth q(st) = \coth(q(s)+q(t))$$

and therefore q satisfies (38) if $s, t, st\in]1, r_0[$. Applying the same argument again, we obtain that $q(r)=k\ln r$ with a suitable positive constant k. Let $p=k/\sqrt{c}$. Then, for $]1, r_0[$

$$a(r) = \sqrt{c}\coth q(r) = \sqrt{c}\coth(\sqrt{c}\, p\ln r) = \sqrt{c}\coth(\sqrt{c}\ln r^p).$$

Using property (iii), we can see that this equation remains valid if $r\in]1/r_0, 1[$.

Thus we have proved (34) in all the possible cases provided that $r\in]1/r_0, r_0[\setminus\{1\}$. To verify (34) in the general case we shall use induction.

Let $r^*\in U\cup V$ be arbitrary. Then $\sqrt[n]{r^*}\in]1/r_0, r_0[\setminus\{1\}$ if n is large enough. Let $r_i=(\sqrt[n]{r^*})^i$. Then we know that (34) is satisfied for $r=r_1$. Assume that we have proved (34) for $r=r_i$. Then either $s=r_i\in U$, $t=r_1\in V$ or $s=r_1\in U$, $t=r_i\in V$, therefore we can apply equation (28) for these values:

$$a(r_{i+1})\big(a(r_i)-a(r_1)\big)-a(r_i)a(r_1)=c,$$

i.e.

$$a(r_{i+1}) = \frac{c+a(r_i)a(r_1)}{a(r_i)+a(r_1)}. \tag{41}$$

Using (34) for $r=r_1$ and for $r=r_i$, it follows from (41) that (34) is also valid for $r=r_{i+1}$. Thus (34) is proved for all $r\in U\cup V$. The proof is complete.

THEOREM 7. *Let $b\colon W\to\mathbf{R}$ and assume that* (25) *is satisfied for all $r\in W$. Then there exists $a\colon U\cup V\to\mathbf{R}$ such that* (26) *is valid for $s\in U$, $t\in V$ and $st\in W$ if and only if there can be found a multiplicative function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ (i.e. $m(xy)=m(x)m(y)$, $x, y>0$) a real constant k and a positive real constant p such that*

$$b(r) = pm(r)S(\ln r, \sqrt{k}) \tag{42}$$

for $r\in W$ and where S is defined by

$$S(x, z) = \begin{cases} \sinh zx/z & \text{if } z\in\mathbf{C}\setminus\{0\}, x\in\mathbf{R},\\ x & \text{if } z=0, x\in\mathbf{R}.\end{cases}$$

PROOF. If there exists $a\colon U\cup V\to\mathbf{R}$ such that (26) holds, then by Theorems 5 and 6, we know that a is of the form (34) where c is a real and p is a positive constant.

Assume first that $c\neq 0$. Then, applying (34) and (26), we have, for $s\in U$, $t\in V$ with $st\in W$, that

$$\frac{b(st)}{b(s)b(t)} = \sqrt{c}\coth(\sqrt{c}\ln s^p)+\sqrt{c}\coth(\sqrt{c}\ln t^p) = \tag{43}$$

$$=\sqrt{c}\,\frac{\sinh(\sqrt{c}\ln s^p+\sqrt{c}\ln t^p)}{\sinh(\sqrt{c}\ln s^p)\sinh(\sqrt{c}\ln t^p)} = \frac{S(\ln(st)^p, \sqrt{c})}{S(\ln s^p, \sqrt{c})S(\ln t^p, \sqrt{c})}.$$

It can be seen very easily that this latter formula remains valid also in the case $c=0$.

Let

$$\overline{m}(r) = \begin{cases} S(\ln r^p, \sqrt{c})/b(r) & \text{if} \quad r \in W \setminus \{1\}, \\ 1 & \text{if} \quad r = 1. \end{cases}$$

Then it follows from (43) that $\overline{m}(st) = \overline{m}(s)\overline{m}(t)$ if $s \in U \cup \{1\}$, $t \in V \cup \{1\}$, $st \in W$.

Since $W \subset UV$ and $\overline{m}$ is not identically zero, hence $\overline{m}$ is strictly positive on W. Let $A(x) = \ln \overline{m}(e^x)$ for $x \in \ln W$. Then $A(x+y) = A(x) + A(y)$ for all $x \in \ln (U \cup \{1\})$, $y \in \ln (V \cup \{1\})$ with $x+y \in \ln W$. By the extension theorem of Daróczy and Losonczi [9], there exists an additive function $\overline{A}$: $\mathbf{R} \to \mathbf{R}$ such that $\overline{A}$ is the extension of A. Then

$$\overline{m}(r) = \exp(A(\ln r)) = \exp(\overline{A}(\ln r))$$

if $r \in W$. Let

$$m(r) = \exp(-\overline{A}(\ln r))$$

for $r \in \mathbf{R}_+$. Then clearly m is multiplicative function and

$$m(r) = 1/\overline{m}(r) = b(r)/S(\ln r^p, \sqrt{c})$$

if $r \in W$. It follows from this formula that

$$b(r) = m(r) S(\ln r^p, \sqrt{c}) = pm(r) S(\ln r, p\sqrt{c}).$$

Defining k as cp^2 we obtain (42), which proves the necessity of the above representation.

The proof of the sufficiency is far easier. It is easy to see that (26) is satisfied with the function b given in (42) if

$$a(r) = \begin{cases} (\sqrt{k}/p) \coth(\sqrt{k} \ln r) & (k \neq 0), \\ 1/\ln r^p & (k = 0). \end{cases}$$

In the next theorem we solve the functional equations (18), (19) and (20).

THEOREM 8. *Let E, F and G be quasideviations on I, J and H respectively. Then E, F and G satisfy conditions* (i), (ii) *and* (iii) *of Theorem* 4 *if and only if there exist a multiplicative function m: $\mathbf{R}_+ \to \mathbf{R}_+$, three positive continuous functions e: $I_0 \to \mathbf{R}_+$, f: $J_0 \to \mathbf{R}_+$ and g: $H_0 \to \mathbf{R}_+$ further a real constant k such that*

(i)

(44) $$E(x, x_0) = e(x_0) m(x) S(\ln(x/x_0), \sqrt{k})$$

if $x, x_0 \in I_0$, $x/x_0 \in H^*$;

(ii)

(45) $$F(y, y_0) = f(y_0) m(y) S(\ln(y/y_0), \sqrt{k})$$

if $y, y_0 \in J_0$, $y/y_0 \in H^*$;

(iii)

(46) $$G(z, z_0) = g(z_0) m(z) S(\ln(z/z_0), \sqrt{k})$$

if $z, z_0 \in H_0$.

PROOF. First we prove the necessity of the above representation. If condition (iii) of Theorem 4 is satisfied then for each fixed $z_0 \in H_0$ the function b defined by (24) satisfies the condition of Theorem 7. Therefore, for each fixed z_0, there exists a multiplicative function $m = m_{z_0}$, a real constant $k = k_{z_0}$ and a positive constant $p = p_{z_0}$ such that (42) is valid. Therefore

$$(47) \quad G(z, z_0) = p_{z_0} m_{z_0}(z/z_0) S(\ln(z/z_0), \sqrt{k_{z_0}}) = (p_{z_0}/m_{z_0}(z_0)) m_{z_0}(z) S(\ln(z/z_0), \sqrt{k_{z_0}}).$$

Define $g: H_0 \to \mathbf{R}_+$ by

$$g(z_0) = p_{z_0}/m_{z_0}(z_0), \quad z_0 \in H_0.$$

Then (47) can be rewritten as

$$(48) \qquad G(z, z_0) = g(z_0) m_{z_0}(z) S(\ln(z/z_0), \sqrt{k_{z_0}}).$$

We are going to show that m_{z_0} and k_{z_0} do not depend on the choice of z_0.

Let $z_0 \in H_0$ be fixed and let $z \in H_0 \cap (z_0 J^*)$ be arbitrary. Let $s_1, s_2 \in I^* \setminus \{1\}$ with $s_1/s_2 \in I^*$, $s_i z_0$, $s_i z \in H_0$ $(i = 1, 2)$ and let for the brevity $s_0 = 1$. Then, by our assumptions

$$(I/s_i) \cap (I/s_j) \neq \emptyset, \quad (z_0/J_0) \cap (z/J_0) \neq \emptyset$$

and

$$(I_0/s_i) \cap (z_0/J_0) \neq \emptyset, \quad (I_0/s_i) \cap (z/J_0) \neq \emptyset$$

for all $i, j = 0, 1, 2$. Therefore the system of intervals

$$I_0 = I_0/s_0, \quad I_0/s_1, \quad I_0/s_2, \quad z_0/J_0, \quad z/J_0$$

satisfies the 2-intersection property. Thus there exists a common element, say x_0. Then

$$x_0, s_1 x_0, s_2 x_0 \in I_0; \quad y_0 = z_0/x_0, \ y = z/x_0 \in J_0,$$

further $x_i y_0 = s_i z_0$ and $x_i y = s_i z$ are in H_0. Now applying (18) for the values x_0, $x_1 = s_1 x_0$, $x_2 = s_2 x_0$, y_0 and using (48), we obtain

$$(49) \qquad \frac{E(s_1 x_0, x_0)}{E(s_2 x_0, x_0)} = \frac{G(s_1 z_0, z_0)}{G(s_2 z_0, z_0)} = \frac{m_{z_0}(s_1) S(\ln s_1, \sqrt{k_{z_0}})}{m_{z_0}(s_2) S(\ln s_2, \sqrt{k_{z_0}})}.$$

Applying (18) again for the values $x_0, x_1 = s_1 x_0$, $x_2 = s_2 x_0$, y we get

$$(50) \qquad \frac{E(s_1 x_0, x_0)}{E(s_2 x_0, x_0)} = \frac{G(s_1 z, z)}{G(s_2 z, z)} = \frac{m_z(s_1) S(\ln s_1, \sqrt{k_z})}{m_z(s_2) S(\ln s_2, \sqrt{k_z})}.$$

It follows from (50) and (49) that

$$\frac{m_{z_0}(s_1) S(\ln s_1, \sqrt{k_{z_0}})}{m_z(s_1) S(\ln s_1, \sqrt{k_z})} = \frac{m_{z_0}(s_2) S(\ln s_2, \sqrt{k_{z_0}})}{m_z(s_2) S(\ln s_2, \sqrt{k_z})}$$

if $s_1, s_2 \in (I^* \setminus \{1\}) \cap (H_0/z_0) \cap (H_0/z)$, $s_1/s_2 \in I^*$. This equation shows that the function

$$(51) \qquad s \to q(s) = q_{z_0, z}(s) = \frac{m_{z_0}(s) S(\ln s, \sqrt{k_{z_0}})}{m_z(s) S(\ln s, \sqrt{k_z})}$$

is constant on the set $(I^*\setminus\{1\})\cap(H_0/z_0)\cap(H_0/z)$ (that is in a neighbourhood of $s=1$). Denote this constant value by $q=q_{z_0,z}$.

Let s be positive and $s^2\in(I^*\setminus\{1\})\cap(H_0/z_0)\cap(H_0/z)$. Then in the case $k_z\neq 0\neq k_{z_0}$ a simple calculation gives

$$\frac{1}{q}=\frac{S(\ln s^2,\sqrt{k_{z_0}})}{S^2(\ln s,\sqrt{k_{z_0}})}\cdot\frac{S^2(\ln s,\sqrt{k_z})}{S(\ln s^2,\sqrt{k_z})}=\frac{\sqrt{k_{z_0}}\coth(\sqrt{k_{z_0}}\ln s)}{\sqrt{k_z}\coth(\sqrt{k_z}\ln s)}.$$

If $s\to 1$ then the right hand side of this equation tends to 1 therefore $q=1$ and we have

$$\sqrt{k_{z_0}}\coth(\sqrt{k_{z_0}}x)=\sqrt{k_z}\coth(\sqrt{k_z}x) \tag{52}$$

if $x\in(1/2)\ln[(I^*\setminus\{1\})\cap(H_0/z_0)\cap(H_0/z)]$. Differentiating by x, (52) turns into

$$(1/k_{z_0})\sinh^2(\sqrt{k_{z_0}}x)=(1/k_z)\sinh^2(\sqrt{k_z}x),$$

whence we get (even for $x=0$)

$$(1/\sqrt{k_{z_0}})\sinh\sqrt{k_{z_0}}x=(1/\sqrt{k_z})\sinh\sqrt{k_z}x.$$

Expanding in Maclaurin series we can easily see that $k_{z_0}=k_z$. (In the case $k_zk_{z_0}=0$ the proof is similar.) But then, since $q=1$, it follows from (51) that $m_{z_0}(s)=m_z(s)$ in a neighbourhood of $s=1$. Therefore $m_{z_0}\equiv m_z$.

Thus we have proved that if $z\in H_0\cap(z_0J^*)$ then $k_{z_0}=k_z$ and $m_{z_0}=m_z$, i.e. for each fixed $z_0\in H_0$ there exists a neighbourhood of z_0 (namely $H_0\cap(z_0J^*)$) where k_z and m_z do not depend on z. This proves that k_z and m_z is independent of z on H_0, and consequently, (46) holds.

We shall verify (44) only since the proof of (45) is similar.

If $x_0,x_1,x_2\in I_0$, $x_i/x_j\in H^*$, then $(H_0/x_0)\cap(H_0/x_1)\cap(H_0/x_2)\cap J_0$ is not empty. This means that there exists an element $y_0\in J_0$ so that $x_iy_0\in H_0$ $(i=0,1,2)$. Therefore (18) can be applied for these values. Thus we obtain

$$\frac{E(x_1,x_0)}{m(x_1)S(\ln(x_1/x_0),\sqrt{k})}=\frac{E(x_2,x_0)}{m(x_2)S(\ln(x_2/x_0),\sqrt{k})}.$$

It follows from this equation that the function

$$x\to E(x,x_0)/(m(x)S(\ln(x/x_0),\sqrt{k})) \tag{53}$$

is a positive constant on the set $I_0\cap(x_0H^*)$ for each fixed $x_0\in I_0$. Denote this constant by $e(x_0)$. Thus we have defined the function $e\colon I_0\to\mathbf{R}_+$ and, by this definition, (44) is satisfied.

The continuity of e, f and g is an easy consequence of property E2) of quasi-deviations.

The proof of the sufficiency is a simple calculation therefore we omit it.

5. Main results

We begin with the following

LEMMA. *Let $k\in\mathbf{R}$ and let $m\colon \mathbf{R}_+\to\mathbf{R}_+$ be a multiplicative function. Then the function*

$$D(x, y) = D_{k,m}(x, y) = m(x)\,S(\ln(x/y), \sqrt{k}) \tag{54}$$

is a quasideviation on the interval $]u, v[\subseteq\mathbf{R}_+$ if and only if

$$0 \leqq k+\pi^2/(\ln(v/u))^2. \tag{55}$$

PROOF. If $k\geqq 0$ then (55) does not mean any restriction therefore we have to show that D is a quasideviation on $\mathbf{R}_+$.

Let $k>0$. Then

$$D(x, y) = m(x)\big((x/y)^{\sqrt{k}}-(x/y)^{-\sqrt{k}}\big)/(2\sqrt{k}).$$

It is obviuos from this formula that D has both property E1) and property E2) of quasideviations. To check E3), let $0<x<x'$. Then

$$\frac{D(x, y)}{D(x', y)} = \frac{m(x)}{m(x')}\cdot\frac{(x')^{\sqrt{k}}}{x^{\sqrt{k}}}\cdot\frac{x^{2\sqrt{k}}-y^{2\sqrt{k}}}{(x')^{2\sqrt{k}}-y^{2\sqrt{k}}},$$

whence easily follows that $y\to D(x, y)/D(x', y)$ is strictly decreasing on $]x, x'[$.

In the case $k=0$ a similar argument shows that D is a quasideviation on $\mathbf{R}_+$.

If $k<0$ then

$$D(x, y) = m(x)\sin\big(\sqrt{-k}\ln(x/y)\big)/\sqrt{-k}.$$

Therefore E1) is satisfied if and only if

$$\operatorname{sign}\sin\sqrt{-k}\,t = \operatorname{sign} t$$

for $t\in]\ln u/v, \ln v/u[$. This means that $t\sqrt{-k}$ must be in the interval $]-\pi, \pi[$, whence we get $\ln v/u\leqq\pi/\sqrt{-k}$. Rearranging this inequality, we obtain that (55) is a necessary condition and it is also sufficient for E1) to hold. Property E2) is trivially satisfied. To see E3), let $u<x<x'<v$. Then

$$\frac{D(x, y)}{D(x', y)} = \frac{m(x)}{m(x')}\cdot\frac{\sin\big(\sqrt{-k}\ln(x/y)\big)}{\sin\big(\sqrt{-k}\ln(x'/y)\big)}.$$

Derivating with respect to y, we have

$$\frac{d}{dy}\left(\frac{D(x, y)}{D(x', y)}\right) = \frac{m(x)}{m(x')}\cdot\frac{\sqrt{-k}}{y}\cdot\frac{\sin\big(\sqrt{-k}\ln(x/x')\big)}{\sin^2\big(\sqrt{-k}\ln(x'/y)\big)} < 0,$$

since, by property E1),

$$\operatorname{sign}\big(\sin\big(\sqrt{-k}\ln x/x'\big)\big) = \operatorname{sign}(x-x') = -1.$$

Thus the lemma is proved.

Now let $x_1, \ldots, x_n \in \mathbf{R}_+$ and assume that

$$0 < k+\pi^2/(\ln(x_i/x_j))^2 \tag{56}$$

for $i, j=1, \ldots, n$ with $x_i \neq x_j$. We shall denote the set of all such vectors $x=(x_1, \ldots, x_n)$ by Δ_k. Then there exist $0<u<v<\infty$ such that $x_i \in]u, v[$ and (55) is satisfied. By the Lemma we know that $D=D_{k,m}$ is a quasideviation on $]u, v[$, therefore the equation

$$D(x_1, y)+\ldots+D(x_n, y) = 0 \tag{57}$$

has a unique solution $y=y_0$ between $\min(x_1, \ldots, x_n)$ and $\max(x_1, \ldots, x_n)$. We shall denote this solution y_0 by $\mathfrak{N}_{k,m}(x_1, \ldots, x_n)$.

If $k>0$ then multiplying (57) by $y^{\sqrt{k}}$ we have

$$m(x_1)(x_1^{\sqrt{k}}-x_1^{-\sqrt{k}}y^{2\sqrt{k}})+\ldots+m(x_n)(x_n^{\sqrt{k}}-x_n^{-\sqrt{k}}y^{2\sqrt{k}}) = 0,$$

whence

$$\mathfrak{N}_{k,m}(x_1, \ldots, x_n) = y_0 = \Big(\sum_{i=1}^{n} m(x_i)x_i^{\sqrt{k}} \Big/ \sum_{i=1}^{n} m(x_i)x^{-\sqrt{k}}\Big)^{1/(2\sqrt{k})}. \tag{58}$$

In the case $k=0$ we find that

$$\mathfrak{N}_{0,m}(x_1, \ldots, x_n) = y_0 = \exp\Big(\sum_{i=1}^{n} m(x_i)\ln x_i \Big/ \sum_{i=1}^{n} m(x_i)\Big). \tag{59}$$

If $k<0$ then (57) turns into

$$m(x_1)\sin(\sqrt{-k}\ln(x_1/y))+\ldots+m(x_n)\sin(\sqrt{-k}\ln(x_n/y)) = 0. \tag{60}$$

Let $w=\sqrt{uv}$. Then $\sqrt{u/v}<y/w<\sqrt{v/u}$ and therefore $\cos(\sqrt{-k}\,\ln(y/w)) \neq$ $\neq 0$. Dividing (60) by this term and applying the addition formula for the sine function, we obtain

$$m(x_1)\sin(\ln(x_1/w)^{\sqrt{-k}})+\ldots+m(x_n)\sin(\ln(x_n/w)^{\sqrt{-k}}) =$$
$$= \tan(\ln(y/w)^{\sqrt{-k}})\big[m(x_1)\cos(\ln(x_1/w)^{-\sqrt{k}})+\ldots+m(x_n)\cos(\ln(x_n/w)^{\sqrt{-k}})\big],$$

whence

$$\mathfrak{N}_{k,m}(x_1, \ldots, x_n) = y_0 = w\left[\exp\operatorname{arctg}\left(\frac{\sum_{i=1}^{n} m(x_i)\sin(\ln(x_i/w)^{\sqrt{-k}})}{\sum_{i=1}^{n} m(x_i)\cos(\ln(x_i/w)^{\sqrt{-k}})}\right)\right]^{1/\sqrt{-k}}. \tag{61}$$

Of course, we can see from equation (60) that the above value y_0 does not depend on the choice of u and v.

The most important property of the means introduced in formulas (58), (59) and (61) is that they are multiplicative, i.e. if $x, y, x*y \in \Delta_k$ then

$$\mathfrak{N}_{k,m}(x*y) = \mathfrak{N}_{k,m}(x)\,\mathfrak{N}_{k,m}(y). \tag{62}$$

The proof of this identity is a simple calculation therefore we omit it. ((62) also follows from Theorem 3, since the deviations $E=F=G=D=D_{k,m}$ satisfy the functional equation (7).)

We remark that if $k \geqq 0$ then the above class of means is equivalent to the class $M_{c,m}$ introduced in Theorem 1. (Namely, $M_{c,m^*}=\mathfrak{N}_{k,m}$ if $|c|=2\sqrt{k}$ and $m^*(x) \cdot x^{c/2}=m(x)$.) Thus the "new" multiplicative means obtained in this paper are the $\mathfrak{N}_{k,m}$ means for negative k. The multiplicativity property of this class of means was first discovered by Aczél—Daróczy [4].

Now we can state our main result.

MAIN THEOREM. *Let M, N and K be quasideviation means on the interval I, J and H, respectively. Then*

$$K(x*y) = M(x)N(y) \tag{4}$$

is satisfied for all $n, m \in \mathbf{N}$, $x \in I^n$, $y \in J^m$ with $x*y \in H^{nm}$ if and only if there exist a real constant k and a multiplicative function $m\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$M(x) = \mathfrak{N}_{k,m}(x) \tag{63}$$

for all $x=(x_1, \ldots, x_n) \in I_0^n$, $n \in \mathbf{N}$ with $x_i/x_j \in H^*$ $(i, j=1, \ldots, n)$,

$$N(y) = \mathfrak{N}_{k,m}(y) \tag{64}$$

for all $y=(y_1, \ldots, y_n) \in J_0^n$, $n \in \mathbf{N}$ with $y_i/y_j \in H^*$ $(i, j=1, \ldots, n)$,

$$K(z) = \mathfrak{N}_{k,m}(z) \tag{65}$$

for all $z=(z_1, \ldots, z_n) \in H_0^n$, $n \in \mathbf{N}$.

PROOF. First we prove the necessity. Since M, N and K are quasideviation means, hence there are quasideviations E, F and G such that $M=\mathfrak{M}_E$, $N=\mathfrak{M}_F$ and $K=\mathfrak{M}_G$. Then the assumption implies that (6) holds on the domain given in Theorem 3. Applying Theorems 3, 4 and 8, we can see that there exist k and m such that conditions (i), (ii) and (iii) of Theorem 8 are satisfied.

To prove (63) let $x=(x_1, \ldots, x_n) \in I_0^n$ with $x_i/x_j \in H^*$. If $\min(x_1, \ldots, x_n) < x_0 < \max(x_1, \ldots, x_n)$ then $x_i/x_0 \in H^*$ also holds, therefore, by (i), we have

$$E(x_i, x_0) = e(x_0) D_{k,m}(x_i, x_0).$$

This means that the mean value $M(x)=\mathfrak{M}_E(x)$ is the solution x_0 of the equation

$$\sum_{i=1}^{n} E(x_i, x_0) = e(x_0) \sum_{i=1}^{n} D_{k,m}(x_i, x_0) = 0$$

i.e., (63) is valid.

Equations (64) and (65) can analogously be proved.

To prove the sufficiency, let $x \in I^n$, $y \in J^m$ with $x*y \in H^{nm}$. Then it is obvious that $x_i \in I_0$ (since $x_i \in H/y_j \subset H/J$), $y_j \in J_0$ (since $y_j \in H/x_i \subset H/I$) and $x_i y_j \in H_0$ (since $x_i y_j \in IJ$), further $x_i/x_j = x_i y_1/x_j y_1 \in H_0/H_0 = H^*$ and $y_i/y_j = x_1 y_i/x_1 y_j \in H_0/H_0 = H^*$. Therefore for these x, y and $z=x*y$ we have (63), (64) and (65). But then, by equation (62), we get (4) at once.

The proof is complete.

The most interesting special case of the Main Theorem is when $I=J=H=\,]u, v[$ and $K=M=N$. In this case we obtain the following

COROLLARY. *Let M be a quasideviation mean on the interval $]u, v[\subset \mathbf{R}_+$ $(0\leqq u<v\leqq\infty)$. Then M is multiplicative on $]u, v[$ that is*

$$M(x*y) = M(x)M(y) \tag{3}$$

for all $x\in\,]u, v[^n$, $y\in\,]u, v[^m$, $x*y\in\,]u, v[^{nm}$, $n, m\in\mathbf{N}$ if and only if there exist $k\in\mathbf{R}$ and a multiplicative function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ such that (63) *holds if*

$$x\in\,]\max(u, u/v),\ \min(v, v/u)[^n,\quad n\in\mathbf{N} \tag{66}$$

and if

$$x\in\,]\max(u, u^2),\ \min(v, v^2)[^n,\quad n\in\mathbf{N}. \tag{67}$$

PROOF. Apply the Main Theorem. Let $I=J=H=\,]u, v[$ and $M=N=K$. Now

$$I_0 = J_0 = \,]u, v[\cap]u/v, v/u[\, = \,]\max(u, u/v),\ \min(v, v/u)[,$$

$$H_0 = \,]u, v[\cap]u^2, v^2[\, = \,]\max(u, u^2),\ \min(v, v^2)[,$$

and further

$$H^* = H_0/H_0 = \,]\max(u/v, u^2/v, u/v^2),\ \min(v/u, v/u^2, v^2/u)[\, = I_0/I_0 = J_0/J_0.$$

Therefore the domain of equations (63) and (64) is given by (66) and the domain of (65) is (67). This proves the Corollary.

In a certain special case the above Corollary turns into a

CHARACTERIZATION THEOREM. *Let $0\leqq u<v\leqq\infty$ and assume that either $u\leqq 1\leqq v$ or $u/v=0$. Then the function $M\colon \bigcup_{n=1}^{\infty}\,]u, v[^n\to\mathbf{R}$ can be represented in the form $M=\mathfrak{N}_{k,m}$ with a suitable $k\in\mathbf{R}$ and a multiplicative function $m\colon \mathbf{R}_+\to\mathbf{R}_+$ if and only if it satisfies conditions* (i)—(iv) *of Theorem* 2 *and is multiplicative on $]u, v[$.*

PROOF. If $M=\mathfrak{N}_{k,m}$ then it is a quasideviation mean, therefore conditions (i)—(iv) of Theorem 2 are valid. The multiplicativity of M is obvious.

If M fulfils conditions (i)—(iv) then it is a quasideviation mean. On the other hand, by the Corollary, the multiplicativity of M implies the existence of k and m so that $M=\mathfrak{N}_{k,m}$ on the interval $]\max(u, u/v), \min(v, v/u)[$, which is, by our assumption, equal to the interval $]u, v[$.

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DEPARTMENT OF MATHEMATICS
KOSSUTH L. UNIVERSITY
H 4010 DEBRECEN, PF. 12
HUNGARY

MATHEMATISCHES INSTITUT I
UNIVERSITÄT KARLSRUHE
D—7500 KARLSRUHE
WEST GERMANY

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ON THE SPECTRAL SINGULARITIES OF THE TRUNCATED SHIFT

B. NAGY (Budapest)

1. Introduction

The spectral singularities of certain linear operators were first studied by J. Schwartz [17], Naimark [13], Ljance [9], [10], B. Pavlov [15], [16], and also by M. Krein and Langer [8]. A general notion of the sets of the spectral singularities of a closed linear operator in a Banach space has been given by the author in [12]. Deep structure theorems, including generalized spectral decompositions, for the truncated shift have been proved by Vasyunin [19] (cf. also [14]).

The purpose of this paper is to study the sets of the spectral singularities of the truncated shift operators (hence of contractions of class C_{00} with one-dimensional defect spaces, see [18]). In Theorem 1 we establish the connection between S-spectrality in the sense of [1], [11] and a kind of spectrality in the sense of [19] for the truncated shift. This will enable us to apply several results of Vasyunin [19] to our problems: these will in general be given as propositions. Theorem 2 will extend a result of Foias [5] and show that the support of the singular continuous part of the representing measure of the characteristic function F is contained in the set $S(T_F)$ of the spectral singularities of the corresponding truncated shift. Theorems 3 and 4 will show that there are truncated shifts with arbitrary closed sets on the unit circle as the sets of the spectral singularities and with prescribed types of characteristic functions. This contrasts with the fact that these operators are all decomposable in the sense of Foias (cf. [4]): thus they have good but not excellent spectral decomposition properties.

The results in the last section deal with the set of the spectral singularities in the strict sense, i.e. with the set $\hat{S}(T_F)$. Lemma 2 shows that the support of the representing singular measure (of the singular factor) of the characteristic function F is contained in $\hat{S}(T_F)$. Proposition 7 gives necessary and sufficient conditions for an operator T_F to be S-scalar. Finally, it is shown that even if F is a Blaschke product, the sets $S(T_F)$ and $\hat{S}(T_F)$ can be equal to and, in another case, can be very far from each other.

Summarizing, the results will show that the uniform spectral behavior of this class of operators (decomposability) gives place to a large variety when finer concepts of spectral decomposition are considered.

2. Preliminaries and notations

Throughout the paper N will denote the set of the positive integers, $\mathbf{C}$ the complex plane, D the open unit disc and C the unit circle. If X is a Hilbert space and U is a bounded linear operator in X (in notation: $U \in B(X)$), the Hilbert space and

Banach space adjoints of U will be denoted by U^* and U', respectively. So $U^*=JU'J^{-1}$, where J denotes the canonical isometry of the Banach space dual X' onto X. T will always denote the shift acting in H^2 of the disc as $(Tf)(z)=zf(z)$ $(z\in D)$. If F is an inner function (cf. [6]) in H^2 with representing measure m^F (see [14; p. 44]), we denote by $F(e)$ the inner function represented by m_e^F: here $m_e^F(b)=m^F(e\cap b)$, e and b are Borel sets in $\mathbf{C}$. $F(e;z)$ will denote the value of $F(e)$ at $z\in D$, and we define

$$K=K_F=H^2\ominus FH^2,\quad T_F=P_K T|K_F,\quad T_F^*=T^*|K_F,\quad K(e)=K_{F(e)},$$

where $|$ denotes restriction, and P_K the orthogonal projection operator of H^2 onto K. $\vee M_a$ will stand for the closed linear span of the subspaces M_a, and sometimes $m(p)$ will stand for $m(\{p\})$ if m is a measure and $\{p\}$ is a singleton. For $e\subset\mathbf{C}$, e^* will denote the set of the complex conjugates of the elements of e.

Let L, M be closed subspaces of a Hilbert space such that $L\cap M=\{0\}$. On the linear manifold $L+M$ the projection $P(L, M)$ onto L parallel to M is then determined. This projection is bounded if and only if the manifold $L+M$ is closed. The sine of the angle between L and M is defined as in [14; pp. 253—254] by

$$\sin(L, M)=\inf_{x\in L}\frac{|(I-P_M)x|}{|x|}=|P(L, M)|^{-1}$$

where P_M denotes the orthogonal projection of $L+M=\overline{L+M}$ (the closure) onto M. Applying the notation of the preceding paragraph, we shall write $P(e)=P(K(e), K(e^c))$ for (a fixed inner function F and) a Borel set e if and only if the latter projection is bounded (here and in the following e^c means $\mathbf{C}\setminus e$).

We shall need from Nagy [12] the following notations and facts. Let A be a Boolean algebra of certain Borel subsets of $\mathbf{C}$ with unit $\mathbf{C}$, and let X be a complex Banach space. A Boolean algebra homomorphism E of A onto a Boolean algebra of projections in X such that $E(\mathbf{C})=I$ (the identity) and E is countably additive on A in the strong operator topology of X is called an A-spectral measure. The bounded operator U in X is called A-spectral if there is an A-spectral measure E such that (σ will denote the spectrum)

$$E(a)\,U=UE(a),\quad \sigma(U|E(a)X)\subset\bar{a}\quad (a\in A).$$

In this case E is said to be an A-resolution of the identity for U. Let $S=\bar{S}\subset\sigma(U)$, and let A be one of the following algebras:

$$B_S=\{b \text{ Borel set in } \mathbf{C}\colon\ b\supset S \text{ or } b\cap S=\emptyset\},$$

$$A(S)=\{b \text{ Borel set in } \mathbf{C}\colon\ h(b\cap\sigma(U))\cap S=\emptyset\},$$

where h denotes the boundary (of a subset of $\sigma(U)$) in the relative topology of $\sigma(U)$. Instead of B_S-spectral and B_S-resolution of the identity, we shall simply say *S-spectral* and *S-resolution of the identity,* respectively.

It is proved in [12] that an operator U is $A(S)$-spectral if and only if U is S_1-spectral for every closed neighborhood S_1 (in the relative topology of $\sigma(U)$) of S. Further, for any operator U there is a smallest one, $S(U)$, among the sets $S=\bar{S}\subset\sigma(U)$ for which U is $A(S)$-spectral: by definition, $S(U)$ is called *the set of the spectral singularities* of the operator U.

It is proved in [12] that the S-resolution or the $A(S)$-resolution of the identity of an operator U, if it exists, is unique in its class. Let E denote the $A(S)$-resolution of the identity for U, and assume that $U|E(b)X$ is a spectral operator of scalar type in the sense of Dunford ([3; XV. 4.1]) for every $b\in A(S)$ such that $\bar{b}\cap S=\emptyset$. In this case we say that U is $A(S)$-scalar. By [12; Theorem 6], there is a smallest one, $\hat{S}(U)$, among the sets $S=\bar{S}\subset\sigma(U)$ for which U is $A(S)$-scalar: by definition, $\hat{S}(U)$ is called *the set of the spectral singularities in the strict sense* for U. If D denotes the S-resolution of the identity for an operator V, and $V|D(S^c)X$ is a spectral operator of scalar type, we say that V is *S-scalar.*

As a standard reference on the truncated shift we shall use the monograph by N. K. Nikolskiy [14].

3. On the set $S(T_F)$

LEMMA 1. *Let F be an inner function, let $K=K_F=H^2\ominus FH^2$, and let A be a Boolean algebra of Borel sets in* $\mathbf{C}$ *such that $e\in A$ implies that $P(e)$ exists. Then P is a Boolean algebra homomorphism of A into a Boolean algebra of projections in $B(K)$.*

PROOF. Let $e,f\in A$. Then $K(e\cap f)=K(e)\cap K(f)$ (see, e.g., [14, pp. 28, 44, 272]) and, similarly, $K(e^c\cap f)=K(e^c)\cap K(f)$. Further, $K(f)=K(f\cap e)\vee K(f\cap e^c)$. Hence for every $k_f\in K(f)$ there is a sequence $\{k^n_{fe}+k^n_{fe^c}\}$ converging to k_f and such that $k^n_{fe}\in K(f\cap e)$, $k^n_{fe^c}\in K(f\cap e^c)$. Therefore,

$$P(e)k_f=\lim_n k^n_{fe}\in K(f\cap e),\quad P(e^c)k_f=\lim_n k^n_{fe^c}\in K(f\cap e^c),$$

thus we obtain that ($\dot{\oplus}$ will denote topological [not necessarily orthogonal] direct sum)

$$K(f)=K(f\cap e)\dot{\oplus}K(f\cap e^c)=K(f)\cap K(e)\dot{\oplus}K(f)\cap K(e^c).$$

Putting f^c in place of f, we see (cf. [3; VI. 9. 24]) that $P(e)$ and $P(f)$ commute, thus $P(e)P(f)$ is a projection. Its range is $K(e)\cap K(f)=K(e\cap f)$, whereas its kernel is $K(e^c)\vee K(f^c)=K(e^c\cup f^c)=K((e\cap f)^c)$. Hence $P(e)P(f)=P(e\cap f)$.

It is clear from the definition that $P(e^c)=I-P(e)$, therefore

$$P(e\cup f)=I-P(e^c\cap f^c)=I+(P(e)-I)P(f^c)=P(e)P(f^c)+P(f)=$$
$$=P(e)+P(f)-P(e)P(f).$$

Thus P is indeed an algebra homomorphism.

THEOREM 1. *Let $K=K_F=H^2\ominus FH^2$, $T_F=P_KT|K_F$ and let $S=\bar{S}\subset\sigma(T_F)$. The following statements are equivalent:*

(i) $\sup_{e\in B_s}|P(e)|<\infty$,

(ii) *T^*_F is S^*-spectral,*

(iii) T'_F is *S-spectral.*

Further, if (i) *holds, then the S^*-resolution of the identity R for T_F^* is given by*

$$R(e) = P(e^*) \quad (e \in B_{S^*}), \tag{1}$$

and the S-resolution of the identity E' for T'_F is defined by

$$E'(e) = J^{-1}P(e)J \quad (e \in B_S). \tag{2}$$

PROOF. Since the equivalence of (ii) and (iii) can be proved in a straightforward manner, we shall show here that (i) and (ii) are equivalent. Note that $T_F^* = T^*|K_F$.

Assume first that (i) holds. By Lemma 1, P is then a (uniformly bounded) Boolean algebra homomorphism of B_S into a Boolean algebra of projections in $B(K)$, which obviously satisfies $P(\mathbf{C})K = K(\mathbf{C}) = K$, i.e. $P(\mathbf{C}) = I$. The Boolean algebra of projections $\{P(e)\colon e \in B_S\}$ is σ-complete in the sense of [3; XVII. 3.1]. In fact, it is easily seen that

$$\sup\{P(e_n)\colon n \in N\} = P(\cup e_n), \quad \inf\{P(e_n)\colon n \in N\} = P(\cap e_n),$$

$$P(\cup e_n)K = K(\cup e_n) = \vee\{K(e_n)\colon n \in N\},\ P(\cap e_n)K = \cap\{K(e_n)\colon\ n \in N\}.$$

By [3; XVIII. 3.4], the B_S-spectral measure P is (strongly) countably additive. For any $e \in B_S$ we have $K = P(e)K \oplus P(e^c)K$, and the operator T_F^* leaves each subspace $P(e)K = K(e)$ invariant. Hence T_F^* commutes with each $P(e)$. By the Livsic—Möller theorem (cf. [14; p. 81]),

$$\sigma(T_F^*|P(e)K) = \sigma(T_{F(e)})^* = \sigma(F(e))^* \subset \bar{e}^* = \overline{e^*},$$

thus T_F^* is S^*-spectral with resolution of the identity R defined by $R(e^*) = P(e)$ $(e^* \in B_{S^*})$.

Conversely, assume that T_F^* is S^*-spectral with a certain resolution of the identity R, and that Q is defined on B_S by $Q(e) = R(e^*)$. Let $e \in B_S$, $e = \bar{e}$. Then we have

$$Q(e)K = R(e^*)K = \{k \in K\colon \sigma_{T^*}(k) \subset e^*\} \tag{3}$$

where σ_{T^*} denotes the local spectrum of the element (cf., e.g., [11]) with respect to the operator T_F^*. This can be proved for the case $e \supset S$ as in Nagy [11; Lemma 4]. For the case $e \cap S = \emptyset$ we note that T_F^* is decomposable in the sense of Foias (cf. [14; p. 111]), hence has the single-valued extension property (this is also seen from the fact that the interior of $\sigma(T_F^*)$ is void). Thus the proof of [11; Lemma 4] can be easily modified to yield (3) in this case.

Since $\sigma(T_F^*|K(e)) \subset e^*$, we have $P(e)K = K(e) \subset Q(e)K$, and the last set is, by (3) and [2; Proposition 1.3.8], a spectral maximal subspace for T_F^*. By [14; p. 111], we see that

$$Q(e)K = K(f) \quad \text{where} \quad f^* = \sigma(T_F^*|Q(e)K) \subset e^*.$$

Hence $F(f)$ divides $F(e)$, therefore

$$Q(e)K = K(f) = K(f) \cap K(e) = Q(e)K \cap P(e)K = P(e)K. \tag{4}$$

The σ-algebra B_S is the σ-algebra of all Borel sets in the locally compact metric space X whose points are the points of $\mathbf{C} \setminus S$ plus the point S (with the induced metric). Since X is separable, every compact subset of X is a G_δ set. The σ-algebra B_S is clearly generated by the class of these compact sets, whose complements are, consequently, F_σ sets. Let f be the union of an increasing sequence $\{e_n\}$ of closed sets in B_S. Since the Boolean algebra of projections $\{Q(e)\colon e\in B_S\}$ is, by [3; XVII. 3.10], σ-complete, we have

$$Q(f)K = \bigvee_{n=1}^{\infty} Q(e_n)K = \bigvee_{n=1}^{\infty} P(e_n)K = K\Big(\bigcup_{n=1}^{\infty} e_n\Big) = P(f)K. \tag{5}$$

From (4) and (5), for every compact e in B_S we have $Q(e)=P(e)$. For every sequence $\{b_n\}\subset B_S$, the σ-completeness of $\{Q(e)\colon e\in B_S\}$ implies

$$Q\Big(\bigcup_{n=1}^{\infty} b_n\Big)K = \bigvee_{n=1}^{\infty} Q(b_n)K, \quad Q\Big(\bigcap_{n=1}^{\infty} b_n\Big)K = \bigcap_{n=1}^{\infty} Q(b_n)K.$$

By definition and by [14; p. 28], the same holds if we replace Q everywhere by P. If, in addition, $Q(b_n)=P(b_n)$ for every n, then this implies $Q(b)=P(b)$ for both $b=\bigcup_{n=1}^{\infty} b_n$ and $b=\bigcap_{n=1}^{\infty} b_n$. Hence $P(e)=Q(e)$ for every $e\in B_S$. Since a B_{S^*}-resolution of the identity is uniformly bounded, we obtain (i).

Now (2) follows in this case immediately.

Proposition 1. *With the notations of Theorem* 1, *the operator T_F^* is S^*-spectral if and only if T_F is S-spectral. If, in this case, R_0 denotes the S-resolution of the identity for T_F, then $R_0(e)=P(e)^*$; hence $R_0(e)K=K\ominus K(e^c)$ for every e in B_S.*

Proof. Since $K=K_F$ is reflexive, [3; XVI. 4.6] yields the statement of the first sentence (cf. [3; IV. 10.1]). Assume now that R_0 is the S-resolution of the identity for T_F, and $e\in B_S$. Then, by the proof of [3; XVI. 4.6] and by Theorem 1, $P(e)= =R_0(e)^*$. On the other hand, $R_0(e)^*K=K\ominus R_0(e^c)K$. Hence $R_0(e^c)K=K\ominus P(e)K= =K\ominus K(e)$ for every e in B_S.

Proposition 2. *With the notations of Theorem* 1, *the following are equivalent:*

(i) *T_F^* is S^*-spectral;*

(ii) $\inf\limits_{e\in B_S\cap S^c} \sin\big(K(e), K(e^c)\big) > 0$;

(iii) $\inf\limits_{e\in B_S\cap S^c} \inf\limits_{z\in D} \big(|F(e;\, z)|+|F(e^c;\, z)|\big) > 0.$

Proof. By Theorem 1, (i) is equivalent to

$$\sup_{e\in B_S\cap S^c} |P(e)| < \infty.$$

The rest follows from the definition of the sine and from Vasyunin's theorem on the angles (cf. [14; pp. 274 and 277]).

Proposition 3. *With the notations of Theorem* 1, *assume that there are singletons $p_i\subset S^c$ $(i=1, 2, \ldots)$ such that, setting $p_0=S$, we have $F=\prod_{n=0}^{\infty} F(p_n)$. Let*

$f_n=F(p_n)$, $F_n=F/f_n=\prod_{\substack{k=0\\k\neq n}}^{\infty} f_k$. *The operator* T_F^* *is* S^**-spectral if and only if*

$$\inf_{n=0,1,\dots} \inf_{z\in D} (|f_n(z)|+|F_n(z)|) > 0. \tag{6}$$

PROOF. By Proposition 2, T_F^* is S^*-spectral if and only if (iii) holds. By the main result of Vasyunin [19; p. 21], under our conditions this is equivalent to (6).

REMARK. The structure of the spectrum of any truncated shift T_F shows that the set of the spectral singularities $S(T_F)$ is contained in the unit circle C. In the following we shall obtain more information about the set $S(T_F)$.

Let F be an inner function with canonical factorization $F=BZ$ where B is a Blaschke product and Z is a singular function. Let m^Z be the representing singular measure (on the unit circle C) of Z with canonical decomposition $m^Z=m_a+m$ into purely atomic and continuous parts. The following theorem is, in a sense, an extension of a result of Foias [5; p. 1193] (cf. also [19; pp. 45—46]).

THEOREM 2. *Let F and m be as above,* $K=K_F=H^2\ominus FH^2$, $T_F=P_K T|K_F$, *let M denote the support of m* (supp m), *and $S(T_F)$ the set of the spectral singularities of the operator T_F. Then*

$$M\subset S(T_F)=S(T_F^*)^*.$$

PROOF. Assume that $M\setminus S(T_F)$ is nonvoid. Then there is a closed neighborhood S of $S(T_F)$ such that T_F is S-spectral and $M\setminus S$ is nonvoid (cf. Section 2). By Propositions 1 and 2, this implies

$$\inf_{e\in B_S\cap S^c} \inf_{z\in D} (|F(e;\ z)|+|F(e^c;\ z)|) > 0.$$

Since $F=BZ_aZ_c$ where Z_a and Z_c are the singular inner functions represented by the measures m_a and m, respectively, and every inner function is bounded (in modulus) in D by 1, we obtain

$$\inf_{e\in B_S\cap(C\setminus S)} \inf_{z\in D} (|Z_c(e;\ z)|+|Z_c(C\setminus e;\ z)|) > 0. \tag{7}$$

On the other hand, $M\setminus S\neq\emptyset$ implies that $m(C\setminus S)>0$. Since m is continuous, there are pairwise disjoint subintervals $\tilde d_n$ of $C\setminus S$ with $m(\tilde d_n)>0$ for every $n=1, 2, \dots$. Since m is singular with respect to the (normalized) Lebesgue measure λ on C, there are intervals $d_n\subset\tilde d_n$ such that $m(d_n)\geqq\lambda(d_n)n$. The continuity of m implies that each d_n can be partitioned into two intervals d_n^1 and d_n^2 such that $m(d_n^1)=m(d_n^2)=\frac{1}{2}m(d_n)$. The arguments and estimations in [19; pp. 45—46] (cf. also [14; pp. 299—300]) then yield that

$$\inf_n \inf_{z\in D} (|Z_c(d_n^1;\ z)|+|Z_c(d_n^2;\ z)|) = 0.$$

Since the left-hand side of (7) is not greater than

$$\inf_n \inf_{z\in D} (|Z_c(d_n^1;\ z)|+|Z_c(C\setminus d_n^1;\ z)|),$$

and

$$|Z_c(C\setminus d_n^1;\ z)| = \exp\left\{-\int_{C\setminus d_n^1} \frac{1-|z|^2}{|v-z|^2}\, m(dv)\right\} \leqq |Z_c(d_n^2;\ z)|,$$

we obtain that the left-hand side of (7) is zero: a contradiction. Thus $M\subset S(T_F)$.

PROPOSITION 4. *Let F be a singular inner function with a purely atomic representing measure m,* $K=K_F=H^2\ominus FH^2$, $T_F=P_K T|K_F$, *and let* $S=\bar{S}\subset\sigma(T_F)$.

(i) *If*

$$\sup_{p\in S^c} \sum_{r\in C\setminus\{p\}} \frac{m(p)m(r)}{|p-r|^2} < \infty, \quad \sum_{\substack{p\in S^c\\ r\in S}} \frac{m(p)m(r)}{|p-r|^2} < \infty,$$

then T_F *is S-spectral.*

(ii) *If* T_F *is S-spectral, then*

$$\sup_{\substack{p,r\in S^c\\ p\neq r}} \frac{m(p)m(r)}{|p-r|^2} < \infty.$$

PROOF. By Propositions 1 and 3, T_F is S-spectral if and only if (6) holds. By the theorem on the angles (cf. [14; p. 227]), this is equivalent to

$$\inf_{p\in S^c} \sin\left(K(\{p\}), K(\{p\}^c)\right) > 0;\ \sin\left(K(S), K(S^c)\right) > 0. \tag{8}$$

By the estimation of the angles (cf. [14; pp. 304—305]), for any Borel set $e\subset S^c\cap C$ we have

$$\sin\left(K(e), K(e^c)\right) \geqq \exp\left\{-2\int_e\int_{e^c} \frac{m(dp)m(dr)}{|p-r|^2}\right\}. \tag{9}$$

If the condition of (i) holds then, by (9), we obtain (8), so T_F is S-spectral. On the other hand, if T_F is S-spectral then, by Propositions 1 and 3,

$$0 < \inf_{p\in S^c} \inf_{z\in D} \left(|F(\{p\};\ z)| + |F(\{p\}^c;\ z)|\right) \leqq \inf_{p\in S^c} \inf_{r\in\{p\}^c} \inf_{z\in D} \left(|F(\{p\};\ z)| + |F(\{r\};\ z)|\right).$$

By the theorem on the angles, then

$$0 < \inf_{p\in S^c} \inf_{r\in\{p\}^c} \sin\left(K(\{p\}), K(\{r\})\right).$$

By Vasyunin [19; Lemma 4.6, p. 27], this implies

$$\sup_{p\in S^c} \sup_{r\in\{p\}^c} \frac{m(p)m(r)}{|p-r|^2} < \infty.$$

Hence the assertion of (ii) follows.

THEOREM 3. *Let S be any closed subset of the unit circle C. There is a singular inner function F with purely atomic representing measure m such that the set* $S(T_F)$ *of the spectral singularities of* T_F *is S.*

PROOF. Let S be void. The example of [14; pp. 320—321] yields a spectral operator T_F (even with $\sigma(T_F)=C$). Therefore we shall assume that $S\neq\emptyset$, and choose a countable dense subset S_0 of S. The set S_0 is the disjoint union of the set I of all isolated and of the set A_0 of all nonisolated points of S_0. For each $s^k \in S_0$ choose a sequence $\{s_n^k : n \in N\}$ of distinct points of C with the following properties: $s_1^k = s^k$, $d(s_n^k, s^k) < n^{-3/2}$ (d is the Euclidean distance in $\mathbf{C}$), further

(i) if $s^k \in A_0$, then $s_n^k \in S_0$ for every $n \in N$,

(ii) if $s^k \in I$, then $d(s_n^k, s^k) < \frac{1}{3} d(S \setminus \{s^k\}, s^k)$.

Now define (with the notation $m(p)$ again for $m(\{p\})$)

$$\tilde{m}(s_n^k) = (kn)^{-2}, \quad m(s_n^k) = \sum_{\substack{j,r \\ s_r^j = s_n^k}} \tilde{m}(s_r^j).$$

Then $m(s_n^k) \geqq (kn)^{-2}$, and m (as defined by these masses at these points) is a purely atomic singular finite measure:

$$\sum_{\{s_n^k\}} m(s_n^k) = \sum_{j,r} \tilde{m}(s_r^j) < \infty.$$

Let F denote the singular inner function represented by m. Further define the sets S_r^k $(k, r \in N)$ by

$$S_r^k = \begin{cases} \{s_n^k : n \geqq r\} & \text{if } s^k \in I, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then we have (cf. [14; p. 81])

$$\sigma(T_F) = \sigma(F) = \operatorname{supp} m = \bar{A}_0 \cup \overline{\bigcup_{k=1}^{\infty} S_1^k}.$$

Define the sets Z_r $(r \in N)$ by

$$Z_r = \bar{A}_0 \cup \overline{\bigcup_{\substack{k=1 \\ k \neq r}}^{\infty} S_1^k} \cup \{s^r\}.$$

They are closed subsets of $\sigma(T_F)$, and $\sigma(T_F) \setminus (Z_r \cup S_q^r)$ is a finite set for each pair of positive integers q, r. Hence T_F is $(Z_r \cup S_q^r)$-spectral (take the Riesz-projections), and we clearly have

$$\bigcap_{q,r=1}^{\infty} (Z_r \cup S_q^r) = \bar{A}_0 \cup \bar{I} = \bar{S}_0 = S.$$

By Nagy [12; Theorem 5] (cf. Section 2), T_F is $A(S)$-spectral.

Assume now that T_F is $A(Q)$-spectral, where Q is a proper closed subset of S. For every closed neighborhood Q_1 of Q the operator T_F is then Q_1-spectral (cf. Section 2). We choose such a Q_1 with the additional property that there is an $s^k \in S_0 \setminus Q_1$. By Proposition 4 and by the construction above, there is an $n_0 > 0$ such that

$$\infty > \sup_{n > n_0} \frac{m(s^k) m(s_n^k)}{|s^k - s_n^k|^2} \geqq \sup_{n > n_0} \frac{n^3}{k^2 k^2 n^2} = \infty,$$

a contradiction. Hence $S(T_F) = S$.

Let $k\colon D\to N\cup\{0\}$ be a map such that $\sum_{v\in D} k(v)(1-|v|)<\infty$. Let

$$b_0(z)=z,\quad b_v(z)=\frac{|v|(v-z)}{v(1-\bar{v}z)}\quad (v\in D\setminus\{0\}),\quad B=B(k,\,\cdot\,)=\prod_{v\in D} b_v^{k(v)}$$

be the corresponding Blaschke product, and let $B_v=Bb_v^{-k(v)}$. The spectrum of T_B is (cf. [14; p. 81])

$$\sigma(T_B)=\sigma(B)=\overline{B^{-1}(\{0\})}.$$

Proposition 5. *Let $S=\bar{S}\subset\sigma(T_B)$. The operator T_B is S-spectral if and only if there is $p>0$ such that for every $z\in D$*

$$|B(z)|\geqq p\min\{|\prod_{v\in S} b_v(z)^{k(v)}|;\ \inf_{v\in S^c}|b_v(z)^{k(v)}|\}.$$

Proof. Proposition 3 and Vasyunin [19; pp. 20—21].

In what follows k_S is the product of k and of the characteristic function of S, and $B(S;\,\cdot\,)$ is the Blaschke product determined by k_S: $B(S;\,\cdot\,)=B(k_S;\,\cdot\,)$).

Proposition 6. *Let $S=\bar{S}\subset\sigma(T_B)$.*

(i) *If T_B is S-spectral, then*

$$\inf_{v\in S^c}|B_v(v)|>0,\quad \inf_{\substack{v,u\in S^c\\ v\neq u}}|b_v(u)|^{k(v)k(u)}>0,\quad \inf_{z\in D}(|B(S;z)|+|B(S^c;z)|)>0.$$

(ii) *If*

$$\inf_{v\in S^c}|B_v(v)|^{k(v)}>0,\quad \prod_{z\in D}|B(S;z)|^{k_{S^c}(z)}>0,$$

then the operator T_B is S-spectral.

Proof. (i) Assume T_B is S-spectral. By Proposition 2 and by [14; p. 314], this implies

$$0<\inf_{v\in S^c}\sin\big(K(\{v\}),K(\{v\}^c)\big)=\inf_{v\in S^c}|B_v(v)|. \tag{10}$$

From (10) it follows that

$$\inf_{\substack{v,u\in S^c\\ v\neq u}}\sin\big(K(\{v\}),K(\{u\})\big)>0.$$

Making use of the fact that the function $x\mapsto x^{-1}\log^2 x$ decreases on the interval $(0,1)$, Corollary 1 in [14; p. 312] implies that

$$\inf_{\substack{v,u\in S^c\\ v\neq u}}|b_v(u)|^{k(v)k(u)}>0.$$

The last assertion is contained in Proposition 2.

(ii) If the conditions hold, then [14; p. 304] shows that

$$\inf_e \sin\big(K(e),K(e^c)\big)>0$$

where e runs over S and any $\{v\}$, $v\in S^c$. By the theorem on the angles ([14; p. 277]) and by Proposition 3, this implies that T_B is S-spectral.

THEOREM 4. *Let* $S=\bar{S}\subset C$. *There is a Blaschke product B such that the set* $S(T_B)$ *of the spectral singularities of the operator* T_B *is S.*

PROOF. If $S=\emptyset$, then B can be e.g. any finite Blaschke product. Now let $S\neq\emptyset$. There is a countable dense subset S_0 (with elements s_k) of S. Let

$$p_{kn}=(kn)^{-2},\quad v_{kn}=s_k(1-p_{kn})\quad (k=1,2,\ldots;\ n=2,3,\ldots).$$

Then $\lim\limits_n v_{kn}=s_k$ (along the radius), and $\sum\limits_{k,n}(1-|v_{kn}|)=\sum\limits_{k,n}p_{kn}<\infty$. Therefore the Blaschke product $B=\prod\limits_{k,n}b_{v_{kn}}$ converges uniformly on compacts of D. The structure of $\sigma(T_B)=\sigma(B)$ shows that if S_1 is a closed neighborhood (in **C**) of S, then T_B is S_1-spectral. By [12; Theorem 5], T_B is therefore $A(S)$-spectral.

Assume now that T_B is $A(Q)$-spectral where Q is a proper closed subset of S. Then there is a closed neighborhood Q_1 of Q such that T_B is Q_1-spectral, and $s_k\in S_0\setminus Q_1$ for some $k\in N$. By Proposition 6(i), we have then $\inf\limits_{n>n_0}|b_{v_{kn}}(v_{k,n+1})|>0$ for some $n_0\in N$.

On the other hand, from the definition of the v_{kn}'s we obtain

$$|b_{v_{kn}}(v_{k,n+1})|=\frac{|v_{kn}-v_{k,n+1}|}{|1-\bar{v}_{kn}v_{k,n+1}|}=\frac{|p_{kn}-p_{k,n+1}|}{|1-(1-p_{kn})(1-p_{k,n+1})|}=\frac{k^2(2n+1)}{k^2(n+1)^2+k^2n^2-1},$$

a contradiction. Therefore $S(T_B)=S$.

4. On the set $\hat{S}(T_F)$

LEMMA 2. *Let F be an inner function with singular factor Z and with representing singular measure* $m=m^Z$. *Assume that* $S=\bar{S}\subset\sigma(T_F)$ *and that* T_F *is S-scalar. Then*

$$\text{supp}\, m\subset S. \tag{11}$$

PROOF. The support of the continuous part of m is contained in S, by Theorem 2. Thus we shall prove (11) for the case when m is purely atomic. Assume that, on the contrary, there is a $z\in\text{supp}\, m\cap S^c\cap C$. Since S^c is open, we may and will assume that z, in addition, satisfies $m(\{z\})>0$. This implies that the singular function $F(\{z\},\cdot)$ defined by

$$F(\{z\},w)=\exp\left\{\frac{w+z}{w-z}\,m(\{z\})\right\}$$

is not identically constant, so $FH^2\neq F(\{z\}^c)H^2$ (cf. [14; p. 35]).

Let E denote the S-resolution of the identity for T_F. By assumption, $T_F|E(S^c)K_F$ is scalar. Hence (cf. [3; XV. 8. 3]) $E(\{z\})K_F=\{0\}$, for z is not in the point spectrum of T_F. On the other hand, by Proposition 1, we have (cf. [14; p. 104])

$$E(\{z\})K_F=K_F\ominus K_F(\{z\}^c)=F(\{z\}^c)H^2\ominus FH^2\neq\{0\},$$

a contradiction.

COROLLARY. *Let F and m be as above. Then* $\text{supp}\, m\subset\hat{S}(T_F)$.

For the following lemma we do not claim novelty: its assertion is usually attributed to Kacnelson [7]. However, for lack of an explicit statement and proof, we show how it can be proved by using results contained in [14].

LEMMA 3. *The operator T_F is spectral of scalar type in the sense of Dunford (i.e. $\emptyset$-scalar) if and only if the inner function F is a Blaschke product with simple zeros:*

$$F = B = \prod_{v\in D} b_v^{k(v)}, \quad k(v) \leqq 1, \tag{12}$$

furthermore Carleson's following condition holds:

$$\inf \{|B_v(v)|: v\in\sigma(T_F)\cap D\} > 0. \tag{13}$$

PROOF. If (12) and (13) hold then, by [14; pp. 173—175], the family $Y=\{y_v: v\in D\cap\sigma(T_F)\}$ defined by

$$y_v(z) = (1-|v|^2)^{1/2} B_v(v)^{-1} B_v(z)(1-\bar{v}z)^{-1} \quad (z\in D)$$

is a Riesz basis in the space $K_B=H^2\ominus BH^2$. By [14; pp. 176—178], $T_F=T_B$ is then similar to a normal operator, hence is scalar as stated.

Conversely, assume that T_F is spectral of scalar type. By Lemma 2, F is then a Blaschke product $B=\prod_{v\in D} b_v^{k(v)}$. Thus [14; p. 106] shows that T_F is a complete operator. Since $T_F=T_B$ is scalar, the kernel of T_B-v is, according to [3; XV. 8.2], equal to the kernel of $(T_B-v)^n$ for every positive integer n. By [14; pp. 104—105], $k(v)\leqq 1$ for $v\in D$, i.e. (12) holds. Since T_B is similar to a normal operator, [14; pp. 176—178] yield that the family Y is a Riesz basis for the space K_B. A theorem of Shapiro and Shields (cf. [14; p. 175]) shows that then (13) holds.

PROPOSITION 7. *Let F be an inner function, and let $S=\bar{S}\subset\sigma(T_F)$. The operator T_F is S-scalar if and only if the inner function $F(S^c)$ is a Blaschke product B with simple zeros, Carleson's condition* (13) *holds for the operator T_B, and the projection $P(S)$ (cf. Section 2) exists.*

PROOF. If T_F is S-scalar (hence S-spectral) with S-resolution of the identity E, then (cf. Proposition 1 and Theorem 1)

$$E(e) = P(e)^*, \quad E(e)K_F = K_F\ominus K_{F(e^c)} \quad (e\in B_S).$$

By [14; p. 104], we have

$$E(e)K_F = K_F\ominus K_{F(e^c)} = F(e^c)H^2\ominus FH^2.$$

Since $F=F(e^c)F(e)$, [14; p. 105] shows that $T_F|E(e)K_F$ is unitarily equivalent to $T_{F(e)}=T_{F/F(e^c)}$. Let $e=S^c$. By assumption, $T_F|E(S^c)K_F$ is spectral of scalar type, hence so is $T_{F(S^c)}$. By Lemma 3, the statements about the inner function $F(S^c)$ are valid. Finally, $P(S)=E(S)^*$ clearly exists.

The assumptions of the converse imply, by Lemma 3, that $T_{F(S^c)}$ is spectral of scalar type. The projection $P(S)$ commutes with T_F^* (cf. the proof of Theorem 1), hence the projection $E(S)=P(S)^*$ commutes with T_F. As in the preceding paragraph, we see that $T_F|E(e)K_F$ is unitarily equivalent to $T_{F(e)}$ for $e=S$ or $e=S^c$

(where $E(S^c)$ is defined as $I-E(S)$). Hence

$$\sigma(T_F|E(S)K_F) = \sigma(T_{F(S)}) = \sigma(F(S)) \subset S$$

(cf. [14; p. 81]), and the operator $T_F|E(S^c)K_F$ is spectral of scalar type. Since

$$T_F = T_F|E(S^c)K_F \oplus T_F|E(S)K_F,$$

the operator T_F is S-scalar (cf. [1; p. 377]).

REMARK. Let $F=B\cdot Z$ be the canonical factorization of F, and let the Blaschke product B have only simple zeros. Since Carleson's condition (13) locally holds for T_B (in a neighborhood of any z in $\sigma(T_B)\cap D$), we obtain that the set of the spectral singularities in the strict sense is (for any such T_F) a subset of C. Further, Theorem 4 shows that for any $S=\bar{S}\subset C$ there is a Blaschke product B such that $\hat{S}(T_B)=$ $=S(T_B)=S$.

Our final result will show that these two sets of the spectral singularities can, in general, be very different from each other.

THEOREM 5. *Let $S=\bar{S}\subset C$. There is a Blaschke product B such that $S(T_B)=\emptyset$, $\hat{S}(T_B)=\sigma(T_B)$, $\sigma(T_B)\cap C=S$.*

PROOF. Let S_0 be a countable dense subset of S. Then S_0 contains all the isolated points of S, and any $s\in S_0$ is isolated in S_0 if and only if in S. Let $\{s_n\}$ be a sequence containing each nonisolated point of S_0 exactly once and each isolated point of S_0 $\aleph_0$ times as elements. Further let

$$v_n = s_n(1-2^{-n}), \quad B = \prod_{n=1}^{\infty} b_{v_n}^2.$$

The set $V=\{v_n\}$ satisfies the Newman condition (cf. [14; p. 203]):

$$\sup\left\{\frac{1-|v|}{1-|z|}:\ v\in V\setminus\{z\},\ |v|\geqq|z|,\ z\in V\right\} \leqq 2^{-1} < 1.$$

Since

$$|B_v(v)| = \Big|\prod_{z\in V\setminus\{v\}} b_z(v)^2\Big|,$$

[14; p. 206] shows that Newman's condition implies Carleson's:

$$\inf\{|B_v(v)|:\ v\in V\} > 0.$$

Since in the Blaschke product B we have $k(v_n)=2$ for every n, [19; p. 28] shows that the family $\{K(\{v\}):\ v\in V\}$ is a Riesz basis for K_B. Hence (cf. [19; pp. 20—21]) $S(T_B)=\emptyset$. Since B has no simple zeros, by Proposition 7 we have $\hat{S}(T_B)=\sigma(T_B)$. Finally, we have clearly

$$\sigma(T_B)\cap C = \bar{V}\cap C = S.$$

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DEPARTMENT OF MATHEMATICS
FACULTY OF CHEMICAL ENGINEERING
TECHNICAL UNIVERSITY OF BUDAPEST
BUDAPEST, HUNGARY

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A GRÜNWALD—MARCINKIEWICZ TYPE THEOREM FOR LAGRANGE INTERPOLATION BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

F. PINTÉR and P. VÉRTESI (Budapest)

1. Introduction. Definitions. Preliminary results

1.1. The classical Grünwald—Marcinkiewicz result can be stated as follows.

THEOREM 1.1. *There exists a continuous on* $[-1, 1]$ *funtion* f *such that*

$$\varlimsup_{n\to\infty} |L_n(f, T, x)| = \infty \quad \text{for any} \quad x\in[-1, 1]. \tag{1.1}$$

Here, as usual, $T=\left\{\cos\frac{2k-1}{2n}\pi\right\}$, $k=1, 2, \dots, n$, $n\in\mathbf{N}$,[1] is the Chebyshev matrix, L_n is the Lagrange interpolatory polynomial of degree $\leq n-1$ based on n-th row of T. For further (classical and recent) results and references, see Vértesi [1], where the corresponding trigonometric and complex cases can be found, too.

1.2. On the other hand, problems of Lagrange interpolation by entire functions of exponential type approximating functions $f\in\mathrm{UCB}(\mathbf{R})$ (see Definition 1.4) have received little attention.

Our main goal is here to prove a theorem analogous to Theorem 1.1 using the function class B_σ. For this aim let us consider some definitions (cf. [2], [3] and [4]).

DEFINITION 1.2. Let $0\leq\sigma<\infty$. If for the entire function $g\in E$

$$|g(z)| = O(e^{(\sigma+\varepsilon)|z|}), \quad z\in\mathbf{C} \tag{1.2}$$

for any fixed $\varepsilon>0$ then g is called an *entire function of exponential type* σ.

DEFINITION 1.3. $g\in B_\sigma$ iff $g\in E_\sigma$ and g is bounded on the real line.

DEFINITION 1.4. The function f: $\mathbf{R}\to\mathbf{R}$ is from the function class $\mathrm{UCB}(\mathbf{R})$ iff f is uniformly continuous and bounded.

If for any fixed $\sigma>0$ and $\alpha_\sigma\in I_\sigma:=\left[-\frac{\pi}{2\sigma}, \frac{\pi}{2\sigma}\right)$

$$x_{k\sigma}:= \alpha_\sigma+\frac{k\pi}{\sigma}, \quad k\in\mathbf{Z}, \tag{1.3}$$

then $X:=\{x_{k\sigma}\}$, $k\in\mathbf{Z}$, $\sigma>0$, is an *equidistant interpolatory matrix* (or system of nodes) on $\mathbf{R}$. The corresponding *fundamental functions of Lagrange interpolation* can

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[1] **N**, **R**, **C** and **Z** stand for the set of the natural numbers, real numbers, complex numbers and integers, respectively.

be defined as follows:

$$(1.4)\qquad g_{k\sigma}(X,z):=\begin{cases}\dfrac{\sin\sigma(z-x_{k\sigma})}{\sigma(z-x_{k\sigma})}+\dfrac{1}{k\pi}\sin\sigma(z-x_{k\sigma}), & z\in\mathbf{C},\quad\text{if}\quad k\neq 0,\\[2mm] \dfrac{\sin\sigma(z-\alpha_\sigma)}{\sigma(z-\alpha_\sigma)}, & z\in\mathbf{C},\quad\text{if}\quad k=0.\end{cases}$$

One can see that for $\sigma>0$, $k,j\in\mathbf{Z}$, $g_{k\sigma}\in B_\sigma$, $g_{k\sigma}(x_{j\sigma})=\delta_{kj}$ and $g'_{k\sigma}(\alpha_\sigma)=0$; moreover they are the only functions in E_σ having these properties ([4; 4.3.1 and 4.3.11]). Further, using a remark of the second author (cf. [3; 2.2] and (3.41)), let

$$(1.5)\qquad L_\sigma(f,X,z):=\sum_{k=-\infty}^{\infty} f(x_{k\sigma})g_{k\sigma}(z),\quad z\in\mathbf{C}$$

whenever $f\in\mathrm{UCB}(\mathbf{R})$. It is easy to see that $L_\sigma(f,x_{k\sigma})=f(x_{k\sigma})$, $k\in\mathbf{Z}$ and $L_\sigma(f)\in E_\sigma$ (cf. [4; 4.3.11]). $L_\sigma(f,X)$ is the *Lagrange interpolatory operator;* the expression

$$(1.6)\qquad \lambda_\sigma(X,x):=\sum_{k=-\infty}^{\infty}|g_{k\sigma}(X,x)|,\quad x\in\mathbf{R}$$

is the *σ-th Lebesgue function.*

1.3. In their paper [2], R. Gervais, Q. I. Rahman and G. Schmeisser obtained a Faber-type theorem for a process similar to (1.5) with $\alpha_\sigma=0$. Using essentially their argument one can prove that *for any fixed interval $[a,b]\supset[-2\pi,2\pi]$ and equidistant matrix X, there exists a function $f\in\mathrm{UCB}(\mathbf{R})$ for which*

$$\sup_{\sigma>0}\max_{a\le x\le b}|L_\sigma(f,X,x)|=\infty.$$

On the other hand, if $f\in\mathrm{UCB}(\mathbf{R})$ then $L_\sigma(f,X,x)$ *uniformly tends to f on any fixed interval $[a,b]$ whenever* $\lim_{\sigma\to\infty}\omega\left(f,\dfrac{1}{\sigma}\right)\log\sigma=0$. Here X is an equidistant matrix;

$$\omega(f,\delta):=\max_{\substack{x,\,x+t\in\mathbf{R}\\ |t|\le\delta}}|f(x+t)-f(x)|$$

is the usual modulus of continuity of f (cf. [2; Theorem 1] and [3; Theorem 2.2]).

2. The result

2.1. Let us consider a special equidistant matrix X, namely the one with $\alpha_\sigma=-\dfrac{\pi}{2\sigma}$. Then we get the system of nodes

$$(2.1)\qquad C:=\left\{c_{k\sigma}:=\frac{2k-1}{2\sigma}\pi,\quad k\in\mathbf{Z},\quad \sigma>0\right\}$$

(cf. (1.3)), corresponding to T. We prove

THEOREM 2.1. *One can find a function* $f \in \mathrm{UCB}(\mathbf{R})$ *such that*

$$\sup_{\sigma>0} |L_\sigma(f, C, x)| = \infty \tag{2.2}$$

for arbitrary $x \in \mathbf{R} \setminus \{0\}$.

On the other hand, for any $f \in \mathrm{UCB}(\mathbf{R})$

$$L_\sigma(f, C, 0) \to f(0) \quad \textit{when} \quad \sigma \to \infty. \tag{2.3}$$

2.2. Using the same argument, the above theorem can be proved for some other equidistant matrices X (cf. 3.7). On the other hand, if X is an *arbitrary* equidistant matrix, proving the analogous statement is an open question.

3. Proof

3.1. The main ideas are analogous to those used by G. Grünwald and J. Marcinkiewicz in the classical case properly applied to our situation.

First let $\sigma > 0$ be fixed and

$$S_\sigma := \left\{\frac{k\pi}{\sigma};\ k \in \mathbf{Z}\right\}, \quad C_\sigma := \left\{\frac{2k-1}{2\sigma}\pi;\ k \in \mathbf{Z}\right\}, \quad \sigma > 0.$$

Then if $N > 0$ is a fixed integer, we get that

$$S_\sigma \cup C_\sigma \cup C_{2\sigma} \cup \ldots \cup C_{2^N\sigma} = S_{2^{N+1}\sigma}. \tag{3.1}$$

3.2. LEMMA 3.1. *For any fixed* $\varepsilon, \sigma > 0$ *and* $\varphi \in \mathrm{UCB}(\mathbf{R})$ *one can find an entire function* R *of exponential type such that*

$$R\left(\frac{k\pi}{\sigma}\right) = \varphi\left(\frac{k\pi}{\sigma}\right), \quad \textit{if} \quad k \in \mathbf{Z} \tag{3.2}$$

and

$$|R(x) - \varphi(x)| \leqq \varepsilon, \quad \textit{if} \quad x \in \mathbf{R}. \tag{3.3}$$

PROOF. Using the method of [2; Part 4] it is easy to get that the operator

$$H_\sigma(\varphi, z) = H_\sigma(\varphi, X, z) := \sum_{k=-\infty}^{\infty} \varphi(x_{k\sigma}) \left(\frac{\sin \sigma(z - x_{k\sigma})}{\sigma(z - x_{k\sigma})}\right)^2, \tag{3.4}$$

$$z \in \mathbf{C}, \quad \sigma > 0 \quad \varphi \in \mathrm{UCB}(\mathbf{R}),$$

of Hermite—Fejér type has the following properties:

(i) $H_\sigma(\varphi, z) \in B_{2\sigma}$,

(ii) $H_\sigma(\varphi, x_{k\sigma}) = \varphi(x_{k\sigma})$ if $k \in \mathbf{Z}$,

(iii) $H'_\sigma(\varphi, x_{k\sigma}) = 0$ if $k \in \mathbf{Z}$,

(iv) $\sup\limits_{-\infty<x<\infty} |\varphi(x) - H_\sigma(\varphi, x)| \to 0$ as $\sigma \to \infty$.

I.e., with a proper $m>0$

$$|H_{m\sigma}(\varphi, x)-\varphi(x)| \leqq \frac{\varepsilon}{2} \quad \text{if} \quad x\in\mathbf{R}.$$

Now let

$$R(z):=-H_\sigma(H_{m\sigma}(\varphi)-\varphi, z)+H_{m\sigma}(\varphi, z) \quad \text{if} \quad z\in\mathbf{C}. \tag{3.5}$$

Then, if $\alpha_\sigma=0$ at (3.4),

$$R\left(\frac{k\pi}{\sigma}\right)=-H_\sigma\left(H_{m\sigma}(\varphi), \frac{k\pi}{\sigma}\right)+H_\sigma\left(\varphi, \frac{k\pi}{\sigma}\right)+H_{m\sigma}\left(\varphi, \frac{k\pi}{\sigma}\right)= \tag{3.6}$$

$$=-H_{m\sigma}\left(\varphi, \frac{k\pi}{\sigma}\right)+\varphi\left(\frac{k\pi}{\sigma}\right)+H_{m\sigma}\left(\varphi, \frac{k\pi}{\sigma}\right)=\varphi\left(\frac{k\pi}{\sigma}\right) \quad \text{if} \quad k\in\mathbf{Z};$$

further one can write

$$|R(x)-\varphi(x)| \leqq |H_\sigma(H_{m\sigma}(\varphi)-\varphi, x)|+|H_{m\sigma}(\varphi, x)-\varphi(x)| \leqq \tag{3.7}$$

$$\leqq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \text{if} \quad x\in\mathbf{R},$$

as it was stated.

3.3. Next we prove our main

LEMMA 3.2. *For an arbitrary integer $p\geqq 2$ one can find a function R of exponential type with*

$$|R(x)| \leqq 2 \quad \textit{when} \quad x\in\mathbf{R} \tag{3.8}$$

further if $\tilde{x}\in\left[\frac{\pi}{p}, \pi p\right]$, then there exists a $\tau=\tau(\tilde{x})\geqq p$ for which

$$|L_\tau(R, C, \tilde{x})| > p. \tag{3.9}$$

PROOF. Let $m:=2p\exp(4\pi p^3)$, say, be fixed. Then we define a function φ as follows. If $\sigma>0$, let

$$\varphi\left(\frac{k\pi}{\sigma}\right):=0 \quad \text{for} \quad k\in\mathbf{Z}. \tag{3.10}$$

Now we define φ on $C_\sigma, C_{2\sigma}, \ldots, C_{2^{2mp+1}\sigma}$ i.e. on $S_{2^{2mp+2}\sigma}$ (see (3.1)) as follows. If $1\leqq i\leqq mp+1$ and $i, k\in\mathbf{Z}$, then

$$\varphi\left(\frac{2k-1}{2^{2i-1}\sigma}\pi\right):=\begin{cases}0 & \text{if} \quad \dfrac{2k-1}{2^{2i-1}\sigma}\pi<\dfrac{i\pi}{m}, \\ (-1)^k & \text{if} \quad \dfrac{2k-1}{2^{2i-1}\sigma}\pi\geqq\dfrac{i\pi}{m},\end{cases} \tag{3.11}$$

$$\varphi\left(\frac{2k-1}{2^{2i}\sigma}\pi\right):=\begin{cases}0 & \text{if} \quad \dfrac{2k-1}{2^{2i}\sigma}\pi<\dfrac{i\pi}{m}, \\ (-1)^k & \text{if} \quad \dfrac{2k-1}{2^{2i}\sigma}\pi\geqq\dfrac{i\pi}{m}.\end{cases} \tag{3.12}$$

Finally let φ be linear between the neighbouring nodes. Then, obviously $\varphi \in \mathrm{UCB}(\mathbf{R})$ i.e., by Lemma 3.1, with $\varepsilon=1$ and $2^{2mp+2}\sigma$ (instead of σ) we get the corresponding function R. Using (3.3), $|R(x)| \leqq 2$ for $x \in \mathbf{R}$.

Let $j \in \mathbf{Z}$ be that index for which

$$\frac{j-1}{m}\pi \leqq \tilde{x} < \frac{j}{m}\pi. \tag{3.13}$$

By $\tilde{x} \in \left[\frac{\pi}{p}, \pi p\right]$ and $m>p$, $1 \leqq j \leqq mp+1$. If we choose the index r such that

$$c_{r-1, 2^{2j-2}\sigma} < \frac{j\pi}{m} \leqq c_{r, 2^{2j-2}\sigma},$$

then by (3.13)

$$\tilde{x} < \frac{j\pi}{m} \leqq c_{r, 2^{2j-2}\sigma}. \tag{3.14}$$

Now let $\tau_1(\tilde{x}) := 2^{2j-2}\sigma$, using (1.4), with $\alpha_\sigma = -\frac{\pi}{2\sigma}$ and the definition of R, we get

$$|L_{\tau_1}(R, C, \tilde{x})| = \Big|\sum_{k=-\infty}^{\infty} R(c_{k\tau_1}) g_{k\tau_1}(C, \tilde{x})\Big| = \tag{3.15}$$

$$= \Big|\sum_{k=r}^{\infty} (-1)^k g_{k\tau_1}(C, \tilde{x})\Big| = \left|\sum_{k=r}^{\infty} (-1)^k \frac{\cos(\tau_1 \tilde{x})(-1)^k \left(\tilde{x} + \frac{\pi}{2\tau_1}\right)}{k\pi(\tilde{x} - c_{k\tau_1})}\right| =$$

$$= \frac{|\cos(\tau_1 \tilde{x})| \left(\tilde{x} + \frac{\pi}{2\tau_1}\right)}{\pi} \sum_{k=r}^{\infty} \frac{1}{k(c_{k\tau_1} - \tilde{x})} =: A_1.$$

To estimate A_1, which actually equals $\sum_{k=r}^{\infty} |g_{k\tau_1}(C, \tilde{x})|$ (compare (1.6)), we estimate the sum as follows:

$$\sum_{k=r}^{\infty} \frac{1}{k(c_{k\tau_1} - \tilde{x})} = \sum_{k=r}^{\infty} \frac{1}{\left(\frac{2k-1}{2^{2j-1}\sigma}\pi - \tilde{x}\right)k} \geqq \tag{3.16}$$

$$\geqq \sum_{k=r}^{\infty} \frac{1}{\left(\frac{2k-1}{2^{2j-1}\sigma}\pi - \frac{j-1}{m}\pi\right)k} =: A_2.$$

Choose $\sigma = p$. By $c_{r-1, 2^{2j-2}\sigma} < \frac{j\pi}{m}$ we get $r \leqq \left[\frac{2^{2j-2}\sigma j}{m} + \frac{3}{2}\right] =: r^*$. If $a :=$ $:= \frac{1}{2^{2j-2}p}$ and $b := \frac{-1}{2^{2j-1}p} - \frac{j-1}{m}$, then let $g(x) := -\frac{1}{b}\log\left(\frac{ax+b}{x}\right)$ for $x \geqq$ $\geqq r > 0$.

By (3.14) $ar+b>0$, i.e. this definition has meaning, moreover $g'(x)=\dfrac{1}{x(ax+b)}$ which means that g' is strictly decreasing on $[r,\infty)$. So

$$A_2=\sum_{k=r}^{\infty}\frac{1}{(ak+b)k\pi}\geqq\frac{1}{\pi}\sum_{k=r^*}^{\infty}\frac{1}{(ak+b)k}\geqq\frac{1}{\pi}\int_{r^*}^{\infty}\frac{dx}{x(ax+b)}=$$

$$=\frac{1}{\pi}[g]_{r^*}^{\infty}=\frac{1}{\pi b}\log\left(\frac{ar^*+b}{r^*}\right)-\frac{1}{\pi b}\log a=:A_3.$$

By definition

$$\log a=-\log 2^{2j-2}-\log p. \tag{3.17}$$

Clearly $r^*\geqq\dfrac{2^{2j-2}pj}{m}+\dfrac{1}{2}$, from where

$$\log r^*\geqq\log 2^{2j-2}+\log p+\log j-\log m. \tag{3.18}$$

Further

$$ar^*+b\leqq\frac{1}{2^{2j-2}p}\left(\frac{2^{2j-2}pj}{m}+\frac{3}{2}\right)-\frac{1}{2^{2j-1}p}-\frac{j-1}{m}=\frac{1}{2^{2j-2}p}+\frac{1}{m},$$

which means

$$\log(ar^*+b)\leqq\log\left(\frac{1}{2^{2j-2}p}+\frac{1}{m}\right). \tag{3.19}$$

Finally we remark that

$$-b=\frac{1}{2^{2j-1}p}+\frac{j-1}{m}\leqq 1+p\leqq 2p. \tag{3.20}$$

Now by (3.17)—(3.20) we get

$$A_3=-\frac{1}{\pi b}\left(\log a+\log r^*-\log(ar^*+b)\right)\geqq\frac{1}{2\pi p}\log\frac{2^{2j-2}pj}{2^{2j-2}p+m}.$$

By $\dfrac{\pi}{p}\leqq\tilde{x}<\dfrac{j\pi}{m}$, $m<jp$, from where $2^{2j-2}p+m<2^{2j-2}p\left(1+\dfrac{j}{2^{2j-2}}\right)\leqq 2^{2j-1}p$, i.e. by $jp>m=2p\exp(4\pi p^3)$

$$A_3\geqq\frac{1}{2\pi p}\log\frac{j}{2}>2p^2. \tag{3.21}$$

Now using (3.15), (3.16), (3.21) and $\left|\tilde{x}+\dfrac{\pi}{2\tau_1}\right|>\tilde{x}\geqq\dfrac{\pi}{p}$ we get (using $A_2\geqq A_3$, too)

$$|L_{\tau_1}(R,C,\tilde{x})|>2p|\cos\tau_1\tilde{x}|. \tag{3.22}$$

Similarly, with $\tau_2(\tilde{x})=2\tau_1(\tilde{x})$ we get

$$|L_{\tau_2}(R, C, \tilde{x})| > 2p\,|\cos\tau_2\tilde{x}|. \tag{3.23}$$

Now using the estimation

$$\max(|\cos\vartheta|, |\cos 2\vartheta|) \geqq \frac{1}{2} \quad \text{for any} \quad \vartheta\in\mathbf{R} \tag{3.24}$$

$\Big($which can be proved as follows: if $|\cos\vartheta|\geqq\frac{1}{2}$, we are ready; if it is not the case, (3.24) comes from $|\cos 2\vartheta|=|1-2\cos^2\vartheta|>1-2\cdot\frac{1}{4}=\frac{1}{2}\Big)$, we get that $\max(|\cos\tau_1\tilde{x}|, |\cos\tau_2\tilde{x}|)\geqq\frac{1}{2}$, i.e. by (3.22) and (3.23) we get (3.9).

3.4. Similarly to Lemma 3.2, we can prove

LEMMA 3.3. *For an arbitrary integer $p\geqq 2$ one can find a function R of exponential type with*

$$|R(x)| \leqq 2 \quad \textit{when} \quad x\in\mathbf{R}; \tag{3.25}$$

further, if $\tilde{x}\in\Delta(p):=\left[-\pi p, -\frac{\pi}{p}\right]\cup\left[\frac{\pi}{p}, \pi p\right]$, then there exists a $\tau=\tau(\tilde{x})\geqq p$ for which

$$|L_\tau(R, C, \tilde{x})| > p. \tag{3.26}$$

PROOF. The essential difference is the definition of φ. Instead of (3.10)—(3.12), on $S_{2^{2mp+2}\sigma}$ let

$$\varphi\left(\frac{k\pi}{\sigma}\right) := 0 \quad \text{if} \quad k\in\mathbf{Z}, \tag{3.27}$$

$$\varphi\left(\frac{2k-1}{2^{2i-1}\sigma}\pi\right) := \begin{cases} 0 & \text{if} \quad \dfrac{|2k-1|}{2^{2i-1}\sigma}\pi < \dfrac{i\pi}{m}, \\ (-1)^k & \text{if} \quad \dfrac{|2k-1|}{2^{2i-1}\sigma}\pi \geqq \dfrac{i\pi}{m}, \end{cases} \tag{3.28}$$

$$\varphi\left(\frac{2k-1}{2^{2i}\sigma}\right) := \begin{cases} 0 & \text{if} \quad \dfrac{|2k-1|}{2^{2i}\sigma}\pi < \dfrac{i\pi}{m}, \\ (-1)^k & \text{if} \quad \dfrac{|2k-1|}{2^{2i}\sigma}\pi \geqq \dfrac{i\pi}{m}. \end{cases} \tag{3.29}$$

We omit the further details.

3.5. By [4; 4.8(5)] we state

LEMMA 3.4. *If $f\in B_\sigma$ $(\sigma\geqq 0)$ then*

$$\sup_{x\in\mathbf{R}}|f'(x)| \leqq \sigma \sup_{x\in\mathbf{R}}|f(x)|. \tag{3.30}$$

Another inequality estimates the Lebesgue function (see [2; Lemma 5] and [3; 1.3]).

LEMMA 3.5. *With a proper constant* $c_0>0$ *we have*

$$\lambda_\sigma(X,x) \leqq c_0 \log\big((\sigma+2)(|x|+2)\big) \tag{3.31}$$

for any $x\in\mathbf{R}$ *and* $\sigma>0$.

3.6. Now the construction of $F(x)$ can be as follows. If $p\geqq 2$, consider the exponential function R_p of type $t(p)$ ensured by Lemma 3.3. Then if $\tilde{x}\in\Delta(p)$, we have

$$|L_\tau(R_p, C, \tilde{x})| > p \tag{3.32}$$

with a proper $\tau=\tau(\tilde{x},p)\geqq p$, where

$$p \leqq \tau(\tilde{x},p) \leqq 2^{2mp+1}p =: N(p). \tag{3.33}$$

Let us construct a sequence $\{p_k\}\subset\mathbf{N}$ such that

$$2 \leqq p_1 < p_2 < p_3 < \dots, \tag{3.34}$$

$$p_{k+1} > \max\big(t(p_1), t(p_2), \dots, t(p_k)\big) \quad \text{for any} \quad k\in\mathbf{N}, \tag{3.35}$$

$$p_{k+1} > p_k^2 \quad \text{if} \quad k\in\mathbf{N}, \tag{3.36}$$

$$p_{k+1} > \log^2 N(p_k) \quad \text{if} \quad k\in\mathbf{N}. \tag{3.37}$$

Now let

$$F(x) := \sum_{k=1}^{\infty} \frac{R_{p_k}(x)}{\sqrt{p_k}}. \tag{3.38}$$

By (3.34) and (3.36) F is continuous and bounded on $\mathbf{R}$. To get that F is *uniformly* continuous, we use the inequality $|F(x)-F(y)|\leqq|F(x)-F_n(x)|+|F_n(x)-F_n(y)|+|F_n(y)-F(y)|$ and the uniform continuity of $F_n(x):=\sum_{k=1}^{n} R_{p_k}(x)/\sqrt{p_k}$ (see Lemma 3.4). In this way we get the relation $F\in\mathrm{UCB}(\mathbf{R})$.

Now let $\tilde{x}\in\mathbf{R}\setminus\{0\}$. Then, with a proper $n\in\mathbf{N}$, $\tilde{x}\in\Delta(p_n)$. We write

$$F(x) = S_1(x) + \frac{R_{p_n}(x)}{\sqrt{p_n}} + S_2(x), \quad x\in\mathbf{R}, \tag{3.39}$$

where

$$S_1(x) := \sum_{k=1}^{n-1} \frac{R_{p_k}(x)}{\sqrt{p_k}}, \qquad S_2(x) := \sum_{k=n+1}^{\infty} \frac{R_{p_k}(x)}{\sqrt{p_k}}, \quad x\in\mathbf{R}. \tag{3.40}$$

Here $S_1(x)$ is an entire function of exponential type, its type is $\leqq\max\big(t(p_1), t(p_2), \dots, t(p_{n-1})\big) < p_n \leqq \tau(\tilde{x},p_n) =: \tau_n$, S_1 is bounded on $\mathbf{R}$, i.e. $S_1\in B_{\tau_n}$.

Using the representation

$$G(x) = L_\sigma(G,X,x) + \frac{\sin\sigma(x-\alpha_\sigma)}{\sigma} G'(\alpha_\sigma) \quad \text{if} \quad G\in B_\sigma, \tag{3.41}$$

(see [4; 4.3 (13)]), we get

$$L_{\tau_n}(S_1, C, \tilde{x}) = S_1(\tilde{x}) - \frac{1}{\tau_n} \sin \tau_n \left(\tilde{x} + \frac{\pi}{2\tau_n}\right) S_1'\left(-\frac{\pi}{2\tau_n}\right), \tag{3.42}$$

from where by $K_1 := \sum_{k=1}^{n} 2/\sqrt{p_k}$ and Lemma 3.4

$$|L_{\tau_n}(S_1, C, \tilde{x})| \leqq K_1 + \frac{1}{\tau_n} K_1 \tau_n = 2K_1. \tag{3.43}$$

Further, if $K_2 := \sum_{k=n+1}^{\infty} 2/\sqrt{p_k}$, by Lemma 3.5 and

$$K_2 < \frac{1}{\sqrt{p_{n+1}}} \left(2 + \frac{2}{\sqrt{p_{n+1}}} + \frac{2}{\sqrt{p_{n+2}}} + \dots\right) \leqq \frac{2+K_1}{\sqrt{p_{n+1}}}$$

(see (3.36)) we obtain

$$\begin{aligned} |L_{\tau_n}(S_2, C, \tilde{x})| &\leqq K_2 \lambda_{\tau_n}(C, \tilde{x}) \leqq K_2 c_0 \log\left((\tau_n+2)(|\tilde{x}|+2)\right) \leqq \\ &\leqq (2+K_1)\, c_0 \frac{\log\left((\tau_n+2)(|\tilde{x}|+2)\right)}{\sqrt{p_{n+1}}} =: T. \end{aligned} \tag{3.44}$$

Here, by (3.33), $|\tilde{x}| \leqq \pi p_n$, (3.34) and (3.37)

$$T \leqq (2+K_1)\, c_0 \left(\frac{\log(N(p_n)+2)}{\sqrt{p_{n+1}}} + \frac{\log(\pi p_n + 2)}{\sqrt{p_{n+1}}}\right) \leqq K_3,$$

where this K_3 does not depend on $\tilde{x}$ and n.

To estimate the middle term of (3.39), we use (3.32)

$$\left|L_{\tau_n}\left(\frac{R_{p_n}}{\sqrt{p_n}}, C, \tilde{x}\right)\right| < \frac{1}{\sqrt{p_n}} p_n = \sqrt{p_n}, \tag{3.45}$$

i.e., by (3.43)—(3.45) we get

$$|L_{\tau_n}(f, C, \tilde{x})| \geqq \sqrt{p_n} - 2K_1 - K_3 \tag{3.46}$$

which gives (2.2) when we replace n by $n+1$, $n+2$, e.t.c.

To verify (2.3), let $f \in \mathrm{UCB}(\mathbf{R})$. If $\alpha_\sigma = -\frac{\pi}{4\sigma}$, $\sigma > 0$, by the properties of $H_\sigma(f, X, z)$ (see 3.2) we can write

$$\begin{aligned} |f(0) - L_{2\sigma}(f, C, 0)| &\leqq |f(0) - H_\sigma(f, 0)| + |H_\sigma(f, 0) - L_{2\sigma}(f, C, 0)| \leqq \\ &\leqq \varepsilon + \left|L_{2\sigma}(H_\sigma(f), C, 0) - L_{2\sigma}(f, C, 0)\right| = \varepsilon + \left|L_{2\sigma}(H_\sigma(f) - f, C, 0)\right| \leqq \\ &\leqq \varepsilon\left(1 + \lambda_{2\sigma}(C, 0)\right), \end{aligned}$$

if we apply (3.41) for H_σ and the relation $H'_\sigma\left(f, \frac{-\pi}{4\sigma}\right)=0$. Here

$$\lambda_\sigma(C,0) = \sum_{k=-\infty}^{\infty} |g_{k\sigma}(C,0)| = \frac{1}{\pi}\sum_{k\neq 0}\frac{1}{|k(2k-1)|}+\frac{2}{\pi}<\infty, \tag{3.47}$$

whence, being $\varepsilon>0$ arbitrarily small, we get the relation (2.3).

3.7. If for an equidistant matrix X we can prove a relation corresponding to (3.24), in many cases the previous argument (more exactly a very similar one) can be repeated, i.e. Theorem 2.1 holds true for that X, too. Here is a simple example.

An equidistant matrix X is of *Chebyshev type* iff for any $\sigma>0$ the nodes $\{x_{k\sigma}\}$, $k\in\mathbf{N}$, and $\{x_{j,2\sigma}\}$, $j\in\mathbf{N}$, denoted by $\circ$ and $\times$, respectively, are situated as follows:

$$-\circ\underbrace{\text{——}}_{\frac{\pi}{4\sigma}}\times\underbrace{\text{————}}_{\frac{\pi}{2\sigma}}\times\underbrace{\text{——}}_{\frac{\pi}{4\sigma}}\circ\text{——}\times\text{————}\times\text{——}\circ\text{——}\times\text{————}\times\text{——}\circ-$$

Here we obviously have the relation $|\alpha_\sigma-\sigma_{2\sigma}|=\frac{\pi}{4\sigma}$. The corresponding factors in $g_{k\sigma}$ and $g_{k,2\sigma}$ are $\sin(\sigma(x-\alpha_\sigma)-k\pi)=\pm\sin\sigma(x-\alpha_\sigma)$ and $\sin\big(2\sigma(x-\alpha_{2\sigma})-k\pi\big)=\pm\sin 2\sigma\left(x-\alpha_\sigma\pm\frac{\pi}{4\sigma}\right)=\pm\cos 2\sigma(x-\alpha_\sigma)$, respectively (cf. (1.4), (3.15) and (3.23)). On the other hand, $\max(|\sin\alpha|, |\cos 2\alpha|)\geqq\frac{1}{2}$, by $\cos 2\alpha=1-2\sin^2\alpha$. We omit the further details.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
1053, BUDAPEST, REÁLTANODA U. 13—15
HUNGARY

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GROUP ALGEBRAS WITH EVERY PROPER QUOTIENT FINITE DIMENSIONAL

J. B. SRIVASTAVA and S. K. SHAH (New Delhi)

Throughout the paper $K[G]$ will denote the group algebra of the group G over the field K. Our object is to study the following problem.

PROBLEM. What are all infinite groups G such that $K[G]/I$ is finite dimensional over K for every nonzero ideal I of $K[G]$?

A group G will be called a solution to the problem if G is infinite and $K[G]/I$ is finite dimensional for every nonzero ideal I of $K[G]$.

The problem in this generality appears to be difficult. However, if the group G posseses a nontrivial normal Abelian subgroup or the FC-subgroup $\Delta(G)$ of G is nontrivial, we have obtained a complete solution to the problem. It, then, follows that an infinite nilpotent group is a solution to the problem if and only if it is infinite cyclic and an infinite solvable non-nilpotent group is a solution to the problem if and only if it is infinite dihedral. Further, it has been shown that the families of algebraically closed groups and of universal groups are solutions to the problem. These groups are infinite simple and their FC-subgroup is trivial. Another example of an infinite simple group is given which is not a solution to the problem. Some more elementary facts are also obtained.

1. Preliminaries. For a subgroup H of G, $\omega(H)$ will denote the right ideal of the group algebra $K[G]$ generated by the subset $\{1-h|h\in H\}$. If H is a normal subgroup of G, then $\omega(H)$ is a two-sided ideal of $K[G]$ and $K[G/H]\cong K[G]/\omega(H)$. Thus $K[G]/\omega(G)\cong K$. For details, see Connell ([2], Proposition 1). We remark that in the terminology of [3], $\omega(H)=\omega(K[H])K[G]$ and $\omega(G)=\omega(K[G])$ is the augmentation ideal.

For an arbitrary group G, the FC-subgroup $\Delta(G)$ of G consists of all those elements of G which have only a finite number of conjugates in G. Thus $\Delta(G)=\{x\in G\,\big|\,|G:C_G(x)|<\infty\}$; for more details see ([3], Lemma 4.1.6, p. 117). By ([2], Theorem 8; or [3], Theorem 4.2.10, p. 129), the group algebra $K[G]$ is prime if and only if G has no non-identity finite normal subgroup and this happens if and only if $\Delta(G)$ is torsion-free Abelian. Finally, if $K[G]$ is a prime group algebra and H is a normal subgroup of finite index in G, then $I\cap K[H]\neq 0$ for every nonzero ideal I of $K[G]$. This last result is a consequence of a beautiful interesection theorem due to Zalesskii (see [3], Corollary 9.1.9, p. 359).

2. The main result. Let H be a normal subgroup of G, written as $H\trianglelefteq G$. An ideal L of $K[H]$ is G-invariant if $g^{-1}Lg\subseteq L$ for every $g\in G$, in fact $g^{-1}Lg=L$ for every $g\in G$. If L is a G-invariant ideal of $K[H]$, then $I=LK[G]=K[G]L$ is an ideal of $K[G]$. We start with a simple lemma.

LEMMA 1. *Let $H \trianglelefteq G$ with $|G:H| < \infty$, and let L be a G-invariant ideal of $K[H]$ with $I = LK[G]$. Then $K[G]/I$ is finite dimensional over K if and only if $K[H]/L$ is finite dimensional over K.*

PROOF. Let $|G:H| = n$, and let $\{g_1 = 1, g_2, \dots, g_n\}$ be a transversal of H in G, then $K[G]$ is a free $K[H]$-module having $\{g_1 = 1, g_2, \dots, g_n\}$ as a free $K[H]$-basis. We have

$$K[G] = K[H] \oplus K[H]g_2 \oplus \dots \oplus K[H]g_n$$

and

$$I = LK[G] = L \oplus Lg_2 \oplus \dots \oplus Lg_n.$$

Now it is clear that

$$K[G]/I \cong K[H]/L \dotplus K[H]g_2/Lg_2 \dotplus \dots \dotplus K[H]g_n/Lg_n$$

where the right hand side is an external direct-sum of K-spaces. Thus $K[G]/I$ is finite dimensional over K if and only if $K[H]/L$ is finite dimensional.

LEMMA 2. *Let $K[G]$ be a prime group ring, and let $H \trianglelefteq G$ with $|G:H| < \infty$. Then $K[H]$ is prime and every nonzero ideal of $K[H]$ contains a nonzero G-invariant ideal of $K[H]$.*

PROOF. It is easy to see that $\Delta(H) \subseteq \Delta(G)$. Now $\Delta(G)$ is torsion-free Abelian as $K[G]$ is prime, hence $\Delta(H)$ is torsion-free Abelian and $K[H]$ is prime.

Now let L be a non-zero ideal of $K[H]$. Also let $|G:H| = n$ and $\{g_1 = 1, g_2, \dots, g_n\}$ a transversal of H in G. Define $L_0 = \bigcap_{g \in G} g^{-1}Lg$, then it is easy to see that $L_0 = \bigcap_{i=1}^{n} g_i^{-1}Lg_i$. Further since $K[H]$ is prime, the finite product $\prod_{i=1}^{n} g_i^{-1}Lg_i$ of non-zero ideals of $K[H]$ is non-zero and clearly it is contained in L_0. Thus L_0 is a non-zero G-invariant ideal of $K[H]$ contained in L.

LEMMA 3. *Let $K[G]$ be a prime group ring, and let $H \trianglelefteq G$ with $|G:H| < \infty$. If $K[H]/L$ is finite dimensional over K for every nonzero ideal L of $K[H]$, then $K[G]/I$ is also finite dimensional over K for every nonzero ideal I of $K[G]$.*

PROOF. Let I be any nonzero ideal of $K[G]$, and let $L = I \cap K[H]$. Since $K[G]$ is prime and $H \trianglelefteq G$ with G/H a finite group, so $L \neq 0$ (see [3], Corollary 9.1.9, p. 359). Also L is G-invariant and it is given that $K[H]/L$ is finite dimensional. By Lemma 1, $K[G]/J$ is finite dimensional where $J = LK[G] = (I \cap K[H])K[G] \subseteq I$. This gives that $K[G]/I$ is finite dimensional, as desired.

Now we proceed towards the proof of our main result of this section. The case $\Delta(G) \neq 1$ is completely settled.

THEOREM 4. *Let K be a field and let G be an infinite group such that either $\Delta(G) \neq 1$ or G possesses a nontrivial normal Abelian subgroup. Then $K[G]/I$ is finite dimensional over the field K for every nonzero ideal I of $K[G]$ if and only if either G is infinite cyclic or G is infinite dihedral.*

PROOF. We divide the proof into four steps. First three steps give the proof one way and the fourth step gives the proof of the converse.

First suppose that $K[G]/I$ is finite dimensional for every nonzero ideal I of $K[G]$.

Step I. If $H \trianglelefteq G$, $H \neq 1$, then $|G:H| < \infty$ and H is infinite. Thus $K[G]$ is prime, and $K[H]/L$ is finite dimensional for every nonzero ideal L of $K[H]$.

Since $H \trianglelefteq G$, $H \neq 1$, so $\omega(H)$ is a nonzero ideal of $K[G]$ and therefore $K[G]/\omega(H)$ is finite dimensional. But $K[G]/\omega(H) \cong K[G/H]$, hence G/H is a finite group and $|G:H| < \infty$. Also G is infinite, so H is infinite. Thus G does not contain any nontrivial finite normal subgroup. This gives that $K[G]$ is prime and $\Delta(G)$ is torsion-free Abelian. To see the last part, let L be a nonzero ideal of $K[H]$. By Lemma 2 there exists a nonzero G-invariant ideal L_0 of $K[H]$ with $L_0 \subseteq L$. By the given hypothesis $K[G]/I$ is finite dimensional, where $I = L_0K[G]$. By Lemma 1, $K[H]/L_0$ is finite dimensional. But $L_0 \subseteq L$, so $K[H]/L$ is finite dimensional.

Step II. $\Delta(G)$ is infinite cyclic and $C_G(\Delta(G)) = \Delta(G)$.

Suppose G possesses a nontrivial normal Abelian subgroup N. By Step I $|G:N| < \infty$ and so $N \subseteq \Delta(G)$, since Abelian subgroups of finite index are contained in $\Delta(G)$. Thus $\Delta(G) \neq 1$. Again by Step I $|G:\Delta(G)| < \infty$, $\Delta(G)$ is torsion-free Abelian, and $K[\Delta(G)]/L$ is finite dimensional for every nonzero ideal L of $K[\Delta(G)]$. Now Step I applied to $K[\Delta(G)]$ and the fact that $\Delta(G)$ is nontrivial torsion-free Abelian imply that every nontrivial subgroup of $\Delta(G)$ is of finite index. By a result due to Fedorov ([4], 15.1.20) $\Delta(G)$ is infinite cyclic. Let $\Delta(G) = \langle x \rangle$.

Clearly, $\Delta(G) \subseteq C_G(\Delta(G))$, because $\Delta(G)$ is Abelian. On the other hand if $z \in C_G(\Delta(G))$, then $\Delta(G) \subseteq C_G(z)$ and $|G:C_G(z)| < \infty$. Thus $z \in \Delta(G)$ and $C_G(\Delta(G)) = \Delta(G)$.

Step III. $G = \langle x \rangle$ is infinite cyclic or $G = \langle x, y \mid y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$ is infinite dihedral.

If $G = \Delta(G)$, then by Step II $G = \langle x \rangle$ is infinite cyclic. So assume that $G \neq \Delta(G)$. We claim that $|G:\Delta(G)| = 2$. Now $\Delta(G) = \langle x \rangle$ is infinite cyclic, so it has only one nontrivial automorphism sending x to x^{-1}. Thus for any $g \notin C_G(\Delta(G)) = \Delta(G)$, $g^{-1}xg = x^{-1}$. Obviously then for any pair of elements g_1, g_2 not in $\Delta(G)$, we have $g_1^{-1}xg_1 = x^{-1} = g_2^{-1}xg_2$. Hence $g_1g_2^{-1} \in C_G(x) = C_G(\Delta(G)) = \Delta(G)$ and $\Delta(G)g_1 = \Delta(G)g_2$. We get $|G:\Delta(G)| = 2$. Let $y \in G$ such that $y \notin \Delta(G) = \langle x \rangle$, then $y^{-1}xy = x^{-1}$, and $y^2 \in Z(G) \cap \langle x \rangle = 1$, where $Z(G)$ denotes the centre of G. Thus we have $G = \langle x, y \mid y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$, as desired.

Step IV. If G is infinite cyclic or infinite dihedral, then $K[G]/I$ is finite dimensional for every nonzero ideal I of $K[G]$.

If G is infinite cyclic, then the conclusion is well known as well as easy to prove.

Next let $G = \langle x, y \mid y^2 = 1,\ y^{-1}xy = x^{-1} \rangle$ be an infinite dihedral group. Take $H = \langle x \rangle$, then H is infinite cyclic, $|G:H| = 2$, and $\Delta(G) = H$. Thus $K[G]$ is prime. Also, from above $K[H]/L$ is finite dimensional for every nonzero ideal L of $K[H]$. By Lemma 3, then, $K[G]/I$ is finite dimensional for every nonzero ideal I of $K[G]$.

An immediate consequence of this theorem is the following corollary.

COROLLARY 5. *An infinite nilpotent group is a solution to the problem if and only if it is infinite cyclic. Also an infinite solvable non-nilpotent group is a solution to the problem if and only if it is infinite dihedral.*

PROOF. It is enough to observe that if G is nilpotent then $Z(G) \neq 1$ and if G is solvable then it possesses a nontrivial normal Abelian subgroup. Rest follows from the proof of Step III above.

3. Examples. In what follows, for definitions and other details, we refer to ([1], [3], Chapt. 9, Section 4). If G is an algebraically closed group or a universal group then it has been proved in [1], (also see [3], Corollaries 9.4.6 and 9.4.10), that the augmentation ideal $\omega(G)=\omega(K[G])$ is the unique non-zero proper ideal of $K[G]$. Also $K[G]/\omega(G)\cong K$. Therefore the families of algebraically closed groups and universal groups are solutions to the problem. If G is an algebraically closed or universal group, then G is infinite simple and $\Delta(G)=1$. By a result due to Scott ([3], Theorem 9.4.4) every group G can be embedded in an algebraically closed group, and further by a similar result due to P. Hall [(3], Theorem 9.4.8) every locally finite group G can be embedded in a universal group. All this shows that there is a rich supply of infinite simple groups G with $\Delta(G)=1$, which are also solutions to the problem.

In Theorem 4 we observed that if a group G is a solution to the problem, then every nontrivial normal subgroup of G is of finite index in G. The converse of this is not true. The following example is really interesting in many ways.

Let X be an infinite set, and let S_X denote the group of all restricted permutations on X moving only finitely many points of X. Then the alternating subgroup A_X is the only nontrivial normal subgroup of S_X and $|S_X:A_X|=2$. Also A_X is an infinite simple group. We claim that both S_X and A_X are not solutions to the problem. Clearly $K[S_X]$ is a prime group ring.

Let $M=\left\{\sum_{x\in X} a_x x \mid a_x\in K,\ x\in X,\ a_x=0 \text{ except for finitely many } x\right\}$ be the permutation module of $K[S_X]$, where S_X acts on the right by permuting the basis X. Let $J=\mathrm{Ann}_{K[S_X]}(M)$; we claim that J is a nonzero ideal of $K[S_X]$ and $K[S_X]/J$ is not finite dimensional over K.

First observe that if x_1, x_2, x_3, x_4 are four distinct elements of X and $\sigma=(x_1, x_2)$, $\varrho=(x_3, x_4)$ are two disjoint transpositions in S_X, then $(1-\sigma)(1-\varrho)$ is a nonzero element of $K[S_X]$ and it is easy to verify that $x(1-\sigma)(1-\varrho)=0$ for every $x\in X$. This shows that $(1-\sigma)(1-\varrho)$ annihilates M, therefore, it belongs to J and $J\neq 0$, as claimed. Further $K[S_X]/J$ is infinite dimensional over K, as the transpositions $(1, i)$, $i=2, 3, \ldots$ are clearly independent $\bmod J$. Also by our Lemma 3, $K[A_X]/L$ cannot be finite dimensional for every nonzero ideal L of $K[A_X]$. Thus A_X is infinite simple, but is not a solution to the problem.

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DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
HAUZ KHAS: NEW DELHI-11 0016
INDIA

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ESTIMATE FOR THE DIFFERENCE OF THE FIRST DERIVATIVES OF PARTIAL SUMS ARISING BY NON-SELFADJOINT STURM—LIOUVILLE OPERATORS

N. H. LOI (Budapest)

In this paper we prove an exact estimate for the difference of the first derivatives of the partial sums arising by non-selfadjoint Sturm—Liouville operators. Our result extends to that of [3] for the case when the eigenvalues may be arbitrary complex numbers. The present proof is based on an efficient method due to V. A. Il'in [6], [7].

1. Formulation of the main result

Let $G=(a, b)$ be a finite interval on the real line. We consider the non-selfadjoint operators

$$Lu := -u'' + q(x)u(x), \qquad \hat{L}u := -u'' + \hat{q}(x)u(x),$$

where

$$q(x) \in L^p(G), \quad \hat{q}(x) \in L^{\hat{p}}(G); \quad p, \hat{p} \in (1, \infty). \tag{1}$$

Denote $\{u_k\}$ and $\{\hat{u}_k\}$ the complete orthogonal systems from the eigenfunctions of the corresponding operators; further let $\{\lambda_k\}$ and $\{\hat{\lambda}_k\}$ be the eigenvalues λ_k, $\hat{\lambda}_k \in \mathbf{C}$.

Let $f(x)$ be an absolutely continuous function on the closed interval $[a, b]$; further let μ be an arbitrary non-negative number. Consider the partial sums

$$\sigma_\mu(f, x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, u_k \rangle u_k(x), \quad \hat{\sigma}_\mu(f, x) := \sum_{|\mathrm{Re}\sqrt{\hat{\lambda}_k}|<\mu} \langle f, \hat{u}_k \rangle \hat{u}_k(x).$$

The aim of this paper is to prove the following result:

THEOREM. *Suppose K is an arbitrary compact subset of G, and let the potentials $q(x)$, $\hat{q}(x)$ fulfill the condition* (1). *For any $f \in W_1^1(G)$ the estimates*

$$|\sigma'_\mu(f, x) - \hat{\sigma}'_\mu(f, x)| \leqq C(K) \|f\|_{W_1^1(G)}$$

holds uniformly in x on the compact K. The constant $C(K)$ depends only on K.

REMARK 1. Our estimate is exact in the order, i.e. we cannot change the constant $C(K)$ to $o(1)$.

2. Preliminary results

In this section we recall some well-known results which are necessary for our proof:

$$(2)\qquad \left|\frac{2}{\pi}\int_0^R \sin \mu t \cos \varrho t \frac{\operatorname{ch} \nu t - 1}{t}\, dt\right| \leqq C_1(R)\frac{\operatorname{ch} \nu R}{2+\left|\mu - |\varrho|\right|},$$

$$(3)\qquad \left|\int_0^R \frac{\sin \mu t \sin \lambda(t-|x-\xi|)}{\pi t\lambda}\, dt\, d\xi\right| \leqq C_2(R)\frac{\operatorname{ch} \operatorname{Im} \lambda R}{(1+|\lambda|)^{3/4}},$$

$$(R>0;\ \mu, \varrho, \nu\in\mathbf{R};\ \lambda\in\mathbf{C})$$

(cf. [2]).

$$(4)\qquad \|u_k\|_{L^\infty(G)} \leqq C_1\left(1+\left|\operatorname{Im}\sqrt{\lambda_k}\right|\right)^{1/p}\|u_k\|_{L^p(G)}\quad (1\leqq p\leqq\infty),$$

$$(5)\qquad \|u_k'\|_{L^\infty(G)} \leqq C_2\left(1+\left|\sqrt{\lambda_k}\right|\right)\|u_k\|_{L^\infty(G)}$$

(cf. [1]).

$$(6)\qquad C_3\|u_k\|_{L^q(K)} \leqq \exp\left\{\left|\operatorname{Im}\sqrt{\lambda_k}\right|R\right\}\left[1+\left|\operatorname{Im}\sqrt{\lambda_k}\right|\right]^{1/s-1/q}\|u_k\|_{L^s(K_{-R})} \leqq C_4\|u_k\|_{L^q(K)}$$

where $1\leqq q\leqq s\leqq\infty$, $K_{-R}\subset K\subset G$ and $R:=\operatorname{dist}(K_{-R}, \partial K)\leqq\frac{1}{2}\operatorname{mes} K$. K can coincide with G (cf. [4]).

Given any compact interval $K:=[c, d]\subset G$, there exists an $R>0$ such that $K_R:=[c-R, d+R]\subset G$ and

$$(7)\qquad \sup_{\mu>0}\sum_{\left|\mu-|\operatorname{Re}\sqrt{\lambda_k}|\right|\leqq 1}\left(\|u_k\|_{L^\infty(K_R)}\operatorname{ch}\operatorname{Im}\sqrt{\lambda_k}R\right)^2 < U < \infty$$

(cf. [2]).

In what follows we shall need the estimate

$$(8)\qquad \left|\int_{y_1}^{y_2} u_k(y)\, dy\right| \leqq C_5\frac{\|u_k\|_{L^\infty(G)}}{1+\left|\sqrt{\lambda_k}\right|}\quad (q(x)\in L^p(G), p>1;\ y_1, y_2\in G).$$

This follows directly from the definition of the eigenfunctions.

We shall systematically use the following inequalities:

$$(9)\qquad |\sin z| \leqq 2|z| \operatorname{ch}\operatorname{Im} z,\quad z\in\mathbf{C},$$

$$(10)\qquad |\sin z| \leqq \operatorname{ch}\operatorname{Im} z\quad z\in\mathbf{C}.$$

In the above estimates the constants $C_1, C_2, \ldots, C_5$ do not depend on the eigenvalue λ_k.

3. Estimation of the spectral function

Let K be an arbitrary compact subset of G and fix a number R_0 such that $0<R_0\leqq\frac{1}{4}\operatorname{dist}(K, \partial G)$; further suppose $R_0\leqq R\leqq 2R_0$. Denote S_{R_0} the average operator introduced by V. A. Il'in [5]:

$$S_{R_0}[f]:=\frac{1}{R_0}\int_{R_0}^{2R_0} f(R)\, dR.$$

Fixing $\mu>0$ and $x\in K$ arbitrarily, let us introduce the function $W: C\to \mathbf{R}$ by

$$W(x, y, \mu) := \begin{cases} \dfrac{1}{\pi}\dfrac{\sin \mu(y-x)}{y-x} & \text{if } |y-x| \leqq R, \\ 0 & \text{if } |y-x| > R. \end{cases}$$

Denote $\theta(x, y, \mu)$ the spectral function of the operator, i.e.

$$\theta(x, y, \mu) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} u_k(x)\overline{u_k(y)}.$$

The purpose of this section is to prove the following

Proposition. *Suppose* $q(x)\in L^p(G)$, $p>1$. *Then the estimate*

$$\left|\int_{y_1}^{y_2} \frac{\partial}{\partial x}[S_{R_0}W(x, y, \mu)-\theta(x, y, \mu)]\right| dy \leqq M(K) \tag{11}$$

is uniform in x *on the compact* K *and in* $y_1, y_2\in[a, b]$. *The constant* $M(K)$ *depends only on the compact* K *and the potential* $q(x)$.

For the proof the following lemmas are necessary. For brevity we denote by μ_k an arbitrary square root of λ_k and put $\varrho_k := \mathrm{Re}\,\mu_k$, $\nu_k := \mathrm{Im}\,\mu_k$.

Lemma 1. *Given any compact* $K\subset G$, *there exists a constant* $C_3(R_0)$ *such that for all* $k\in\mathbf{N}$, $\mu>0$, *and* $R_0\leqq R\leqq 2R_0$

$$\left|S_{R_0}\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\,\frac{\mathrm{ch}\,\nu_k t-1}{t}\,dt\right| \leqq \frac{C_3(R_0)\,\mathrm{ch}^2\,\nu_k 2R_0}{1+|\mu-|\varrho_k||^2}.$$

Proof. For the sake of simplicity, setting $\varrho_k := \varrho$, $\nu_k := \nu$ in case $\mu\neq\varrho_k$, we can write

$$S_{R_0}\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\,\frac{\mathrm{ch}\,\nu_k t-1}{t}\,dt =$$

$$= S_{R_0}\left\{\frac{1}{\pi}\left[-\frac{\cos(\mu+\varrho)t}{\mu+\varrho}-\frac{\cos(\mu-\varrho)t}{\mu-\varrho}\right]\frac{\mathrm{ch}\,\nu t-1}{t}\Bigg|_0^R+\right.$$

$$\left.+\frac{1}{\pi}\int_0^R\left[\frac{\cos(\mu+\varrho)t}{\mu+\varrho}+\frac{\cos(\mu-\varrho)t}{\mu-\varrho}\right]\left(\frac{\mathrm{ch}\,\nu t-1}{t}\right)'dt\right\} =$$

$$= S_{R_0}\left\{-\frac{1}{\pi}\left[\frac{\cos(\mu+\varrho)R}{\mu+\varrho}+\frac{\cos(\mu-\varrho)R}{\mu-\varrho}\right]\frac{\mathrm{ch}\,\nu R-1}{R}\right\}+$$

$$+S_{R_0}\left\{\frac{1}{\pi}\left[\frac{\sin(\mu+\varrho)t}{(\mu+\varrho)^2}+\frac{\sin(\mu-\varrho)t}{(\mu-\varrho)^2}\right]\left(\frac{\mathrm{ch}\,\nu t-1}{t}\right)'\Bigg|_0^R-\right.$$

$$\left.-\frac{1}{\pi}\int_0^R\left[\frac{\sin(\mu+\varrho)t}{(\mu+\varrho)^2}+\frac{\sin(\mu-\varrho)t}{(\mu-\varrho)^2}\right]\left(\frac{\mathrm{ch}\,\nu t-1}{t}\right)''dt\right\} = S_{R_0}(J_1)+S_{R_0}(J_2).$$

Here

$$S_{R_0}(J_1) := S_{R_0}\left\{-\frac{1}{\pi}\left[\frac{\cos(\mu+\varrho)R}{\mu+\varrho}+\frac{\cos(\mu-\varrho)R}{\mu-\varrho}\right]\frac{\operatorname{ch}\nu R-1}{R}\right\} =$$

$$= -\frac{1}{\pi R_0}\left[\frac{\sin(\mu+\varrho)R}{(\mu+\varrho)^2}+\frac{\sin(\mu-\varrho)R}{(\mu-\varrho)^2}\right]\frac{\operatorname{ch}\nu R-1}{R}\Bigg|_{R_0}^{2R_0}+$$

$$+\frac{1}{\pi R_0}\int_{R_0}^{2R_0}\left[\frac{\sin(\mu+\varrho)R}{(\mu+\varrho)^2}+\frac{\sin(\mu-\varrho)R}{(\mu-\varrho)^2}\right]\left(\frac{\operatorname{ch}\nu R-1}{R}\right)' dR.$$

Considering that the function $\left(\frac{\operatorname{ch}\nu R-1}{R}\right)'$ does not change sign, hence we obtain

$$|S_{R_0}(J_1)| \leqq \frac{12}{\pi R_0^2|\mu-|\varrho||^2}(\operatorname{ch}\nu 2R_0-1). \tag{12}$$

Now we estimate $S_{R_0}(J_2)$. Considering that the function $\left(\frac{\operatorname{ch}\nu t-1}{t}\right)''$ does not change sign, too, hence we have

$$|J_2| \leqq \frac{2}{\pi}\frac{1}{|\mu-|\varrho||^2}\left|\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'\Bigg|_0^R\right|+\left|\int_0^R\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'' dt\right| \leqq \tag{13}$$

$$\leqq C_2(R_0)\frac{1}{|\mu-|\varrho||^2}\operatorname{ch}^2\nu 2R_0,$$

From (13) we have

$$|S_{R_0}(J_2)| \leqq \frac{C_2(R_0)}{|\mu-|\varrho||^2}\operatorname{ch}^2\nu 2R_0. \tag{14}$$

On the other hand, we have obviously

$$\left|S_{R_0}\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k t-1}{t}dt\right| \leqq \frac{2}{\pi R_0}(\operatorname{ch}\nu 2R_0-1). \tag{15}$$

Lemma 1 follows from (12), (14) and (15).

LEMMA 2. *Given any compact $K\subset G$, there exists a constant $C_4(R_0)$ such that for all $k\in\mathbf{N}$, $\mu>0$, and $R_0\leqq R\leqq 2R_0$*

$$\left|S_{R_0}\frac{2i}{\pi}\int_0^R \sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}dt\right| \leqq \frac{C_4(R_0)\operatorname{ch}^2\nu_k 2R_0}{1+|\mu-|\varrho_k||^2}.$$

PROOF. This proof can be made in a completely similar way applied for that of Lemma 1, taking into account that the functions $\left(\frac{\operatorname{sh}\nu t}{t}\right)'$, $\left(\frac{\operatorname{sh}\nu t}{t}\right)''$ do not change sign with $t\geqq 0$.

LEMMA 3. *Given any compact* $K \subset G$, *there exists a constant* $C_5(R_0)$ *such that for all* $k \in \mathbf{N}$, $\mu > 0$, $R_0 \leqq R \leqq 2R_0$

$$\left|\frac{1}{\pi}\int_r^R \frac{\sin \mu t \cos \mu_k(t-r)}{t}\,dt\right| \leqq C_5(R_0)\left\{\frac{1}{r^\delta}+\frac{\operatorname{ch} \nu_k R}{1+\big|\mu-|\varrho_k|\big|}\right\}$$

$$(0<\delta<1),$$

$$\left|\frac{1}{\pi}\int_r^R \frac{\sin \mu t \cos \mu_k(t-r)}{t}\,dt\right| \leqq C_5(R_0)\left\{\frac{1}{r^\delta\big|\mu-|\varrho_k|\big|^{\delta/2}}+\frac{\operatorname{ch} \nu_k R}{1+\big|\mu-|\varrho_k|\big|}\right\}$$

$$\left(0<\delta<1,\ \big|\mu-|\varrho_k|\big| \geqq 1\right).$$

PROOF. Consider the integral

$$I=\frac{1}{\pi}\int_r^R \frac{\sin \mu t \cos \mu_k(t-r)}{t}\,dt=$$

$$=\frac{1}{\pi}\int_r^R \frac{\sin \mu t \cos \varrho(t-r)}{t}\,dt+\frac{1}{\pi}\int_r^R \sin \mu t \cos \varrho(t-r)\frac{\operatorname{ch} \nu(t-r)-1}{t}\,dt-$$

$$-\frac{i}{\pi}\int_r^R \sin \mu t \sin \varrho(t-r)\frac{\operatorname{sh} \nu(t-r)}{t}\,dt=I_1+I_2+I_3.$$

The following estimates are true:

$$(16)\qquad |I_1|=\frac{1}{\pi}\left|\int_r^R \frac{\sin \mu t \cos \varrho(t-r)}{t}\,dt\right| \leqq \frac{D_1}{r^\delta}\quad (0<\delta<1),$$

$$(17)\qquad |I_1|=\frac{1}{\pi}\left|\int_r^R \frac{\sin \mu t \cos \varrho(t-r)}{t}\,dt\right| \leqq \frac{D_2}{r^\delta\big|\mu-|\varrho|\big|^{\delta/2}}\quad \left(0<\delta<1,\ \big|\mu-|\varrho|\big| \geqq 1\right)$$

(cf. [6]).

Now we estimate the quantities I_2 and I_3. Considering that

$$\frac{\operatorname{ch} \nu(t-r)-1}{t},\quad \frac{\operatorname{sh} \nu(t-r)}{t},\quad \left(\frac{\operatorname{ch} \nu(t-r)-1}{t}\right)',\quad \left(\frac{\operatorname{sh} \nu(t-r)}{t}\right)'$$

do not change sign, we obtain easily

$$(18)\qquad |I_2| \leqq \frac{1}{\pi}\left|\int_r^R \frac{\operatorname{ch} \nu(t-r)-1}{t}\,dt\right|=$$

$$=\frac{1}{\pi}\left\{\left|[\operatorname{ch} \nu(t-r)-1]_r^R\right|+\left|\int_r^R t\left[\frac{\operatorname{ch} \nu(t-r)-1}{t}\right]' dt\right|\right. \leqq \frac{2}{\pi}\operatorname{ch} \nu R,$$

$$(19)\qquad |I_3| \leqq \frac{1}{\pi}\left|\int_r^R \frac{\operatorname{sh} \nu(t-r)}{t}\,dt\right| \leqq$$

$$\leqq \frac{1}{\pi}\left|\operatorname{sh} \nu(t-r)|_r^R\right|+\frac{1}{\pi}\left|\int_r^R t\left(\frac{\operatorname{sh} \nu(t-r)}{t}\right)' dt\right| \leqq \frac{2}{\pi}\operatorname{ch} \nu R.$$

In case $\mu \neq \varrho$ by the method applied in Lemma 1 we have

$$|I_2| \leqq \frac{2}{\pi}\frac{1}{\big|\mu-|\varrho|\big|}\left\{\frac{\operatorname{ch}\nu R}{R}+\left|\int_r^R\left(\frac{\operatorname{ch}\nu(t-r)-1}{t}\right)' dt\right|\right\} \leqq \frac{4}{\pi}\frac{1}{\big|\mu-|\varrho|\big|}\frac{\operatorname{ch}\nu R}{R}, \tag{20}$$

$$|I_3| \leqq \frac{2}{\pi}\frac{1}{\big|\mu-|\varrho|\big|}\left\{\frac{|\operatorname{sh}\nu(R-r)|}{R}+\left|\int_r^R\left(\frac{\operatorname{sh}\nu(t-r)}{t}\right)' dt\right|\right\} \leqq \tag{21}$$

$$\leqq \frac{4}{\pi}\frac{1}{\big|\mu-|\varrho|\big|}\frac{|\operatorname{sh}\nu(R-r)|}{R}.$$

Lemma 3 follows from the relations (16)—(21).

LEMMA 4. *Given any compact $K \subset G$, there exists a constant $C_6(K)$ such that for $q(x) \in L^1(G)$*

$$\sum_{|\mu-|\varrho_k||\leqq 1}\|u_k'\|^2_{L^\infty(K_R)} \leqq C_6(K)\mu^2, \quad \mu \geqq 1.$$

PROOF. Using (5) and (7) we get

$$\sum_{|\mu-|\varrho_k||\leqq 1}\|u_k'\|^2_{L^\infty(K_R)} \leqq C_2^2 \sum_{|\mu-|\varrho_k||\leqq 1}(1+|\mu_k|)^2\|u_k\|^2_{L^\infty(K_R)} \leqq$$

$$\leqq C\mu^2 \sum_{|\mu-|\varrho_k||\leqq 1}(1+|\nu_k|)^2\|u_k\|^2_{L^\infty(K_R)} \leqq$$

$$\leqq C\mu^2 \sum_{|\mu-|\varrho_k||\leqq 1}(\|u_k\|_{L^\infty(K_R)}\operatorname{ch}\nu_k R)^2 \leqq CU\mu^2 = C_6(K)\mu^2, \quad \mu \geqq 1.$$

LEMMA 5. *Given any compact $K \subset G$, there exists a constant $C_7(K)$ such that for $0<R\leqq 2R_0$; $q(x)\in L^p(G)$, $p>1$; $y_1, y_2 \in [a, b]$*

$$\sum_{|\mu-|\varrho_k||\leqq 1}\left|\int_{y_1}^{y_2} u_k(y)\,dy \operatorname{ch}\nu_k R\right|^2 \leqq \frac{C_7(K)}{\mu^2}, \quad \mu \geqq 1.$$

PROOF. Using (6) and (8) we have for $q(x)\in L^p(G)$, $p>1$; $y_1, y_2 \in [a, b]$

$$\left|\int_{y_1}^{y_2} u_k(y)\,dy \operatorname{ch}\nu_k R\right| \leqq C_5\frac{\|u_k\|_{L^\infty(G)}}{1+|\mu_k|}\operatorname{ch}\nu_k R \leqq C_5 C\frac{\|u_k\|_{L^\infty(K)}\exp|\nu_k|R}{1+|\mu_k|}\operatorname{ch}\nu_k R.$$

Therefore using (7) we get easily

$$\sum_{|\mu-|\varrho_k||\leqq 1}\left|\int_{y_1}^{y_2} u_k(y)\,dy \operatorname{ch}\nu_k R\right|^2 \leqq C_5^2 C^2 U\frac{1}{\mu^2} = \frac{C_7(K)}{\mu^2}, \quad \mu \geqq 1.$$

PROOF OF THE PROPOSITION. We determine the Fourier coefficients of the function $W(x, y, \mu)$ according to the system $\{u_k\}$:

$$\langle u_k, W\rangle = \int_{x-R}^{x+R} u_k(y)\frac{1}{\pi}\frac{\sin\mu(y-x)}{y-x}\,dy = \int_0^R \frac{1}{\pi}\frac{\sin\mu t}{t}[u_k(x+t)+u_k(x-t)]\,dt.$$

Applying the Titchmarsch formula [12]:

$$u_k(x+t)+u_k(x-t) = 2u_k(x)\cos\mu_k t+\int_{x-t}^{x+t} q(\xi)u_k(\xi)\frac{\sin\mu_k(t-|x-\xi|)}{\mu_k}\,d\xi$$

we obtain for the Fourier coefficients of W the expansions

$$\langle u_k, W\rangle = u_k(x)\frac{2}{\pi}\int_0^R \frac{\sin\mu t\cos\varrho_k t}{t}\,dt-u_k(x)\frac{2}{\pi}\int_R^\infty \frac{\sin\mu t\cos\varrho_k t}{t}\,dt+ \tag{22}$$

$$+u_k(x)\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k t-1}{t}\,dt-u_k(x)\frac{2i}{\pi}\int_0^R \sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt+$$

$$+\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^R \frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi.$$

Similarly, we introduce the function

$$\omega(x,y,\mu) := \begin{cases} \dfrac{1}{\pi}\dfrac{\sin\mu R}{R} & \text{if } |x-y|\leqq R,\\ 0 & \text{if } |x-y|>R.\end{cases}$$

We obtain

$$\langle u_k, \omega\rangle = u_k(x)\frac{2}{\pi}\frac{\sin\mu R\sin\mu_k R}{R\mu_k}+ \tag{23}$$

$$+\frac{\sin\mu R}{R}\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^R \frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi.$$

It is well-known (cf. [2]) that

$$\frac{2}{\pi}\int_0^R \frac{\sin\mu t\cos\varrho_k t}{t}\,dt = \delta(\mu, |\varrho_k|), \tag{24}$$

where

$$\delta(\eta, |\varrho_k|) := \begin{cases} 1 & \text{if } \mu>|\varrho_k|,\\ \dfrac{1}{2} & \text{if } \mu=|\varrho_k|,\\ 0 & \text{if } \mu<|\varrho_k|.\end{cases}$$

Let

$$I^\mu_{|\varrho_k|}(R) := \frac{2}{\pi}\int_R^\infty \frac{\sin\mu t\cos\varrho_k t}{t}\,dt = \frac{2}{\pi}\frac{\sin\mu R\sin|\varrho_k|R}{|\varrho_k|R}+\frac{\mu}{|\varrho_k|}\cdot K^\mu_{|\varrho_k|}(R), \tag{25}$$

where

$$K^{\mu}_{|\varrho_k|}(R) := -\frac{2}{\pi\mu}\int\limits_R^{\infty}\left(\frac{\sin\mu t}{t}\right)'\sin|\varrho_k|t\,dt.$$

On the other hand we have

$$\frac{2}{\pi}\frac{\sin\mu R\sin\mu_k R}{R\mu_k} = \frac{2}{\pi}\frac{\sin\mu R\sin|\varrho_k|R}{R|\varrho_k|}+h(R,\mu_k,\mu), \tag{26}$$

where

$$h(R,\mu_k,\mu) := \frac{\sin\mu R\sin\varrho_k R(\operatorname{ch}\nu_k R-1)}{R\mu_k}+$$

$$+i\frac{\sin\mu R\cos\varrho_k R\operatorname{sh}\nu_k R-\nu_k\sin\mu R\sin\varrho_k R}{R\mu_k}.$$

From (22)—(26) in case $|\varrho_k|>1$ we have

$$\langle u_k, W\rangle-\langle u_k,\omega\rangle = u_k(x)\,\delta(\mu,|\varrho_k|)-u_k(x)\cdot\frac{\mu}{|\varrho_k|}\cdot K^{\mu}_{|\varrho_k|}(R)+ \tag{27}$$

$$+u_k(x)\frac{2}{\pi}\int\limits_0^R\sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k t-1}{t}\,dt-u_k(x)\frac{2i}{\pi}\int\limits_0^R\sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt+$$

$$+\int\limits_{x-R}^{x+R}q(\xi)u_k(\xi)\int\limits_{|x-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi-$$

$$-\frac{\sin\mu R}{R}\int\limits_{x-R}^{x+R}q(\xi)u_k(\xi)\int\limits_{|x-\xi|}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi-u_k(x)h(R,\mu_k,\mu).$$

Using (2), (3), (7), (9), (10) and the trivial inequalities

$$|K^{\mu}_{|\varrho_k|}(R)| \leqq C_1(R_0)\quad(\forall\varrho_k,\forall\mu),$$

$$|\sin\mu R\sin\varrho_k R(\operatorname{ch}\nu_k R-1)| \leqq \operatorname{ch}\nu_k R,$$

$$|\sin\mu R\cos\varrho_k R\operatorname{sh}\nu_k R| \leqq \operatorname{ch}\nu_k R,$$

$$|\nu_k\sin\mu R\sin\varrho_k R| \leqq C_3(R)\operatorname{ch}\nu_k R$$

one can prove easily that the following series are convergent:

$$\sum_{|\varrho_k|>1}\left|u_k(x)\frac{\mu}{|\varrho_k|}K^{\mu}_{|\varrho_k|}R\right|^2,$$

$$\sum_{k=1}^{\infty}\left|u_k(x)\frac{2}{\pi}\int_0^R \sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k-1}{t}\,dt\right|^2,$$

$$\sum_{k=1}^{\infty}\left|u_k(x)\frac{2i}{\pi}\int_0^R \sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt\right|^2,$$

$$\sum_{k=1}^{\infty}\left|\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi\right|^2,$$

$$\sum_{k=1}^{\infty}\left|\int_{x-R}^{x+R} q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi\right|^2,$$

$$\sum_{|\varrho_k|>1}|u_k(x)h(R,\mu_k,\mu)|^2.$$

Multiplying (27) by $\overline{u_k(y)}$, summing up for all $k\in\mathbf{N}$, applying the average operator to both sides (taking into account that any series from the corresponding series on the right hand side of (27) still are convergent in the metrics of the space $L^2(G)$) and integrating in y from y_1 to y_2 $(a\leqq y_1\leqq y_2\leqq b)$ we have the following relation:

$$\int_{y_1}^{y_2}[S_{R_0}W(x,y,\mu)-\theta(x,y,\mu)]\,dy=\int_{y_1}^{y_2}S_{R_0}\omega(x,y,\mu)\,dy+ \tag{28}$$

$$+\frac{1}{2}\sum_{\varrho_k=\mu}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k(x)-\frac{2}{\pi}\sum_{0\leqq|\varrho_k|\leqq1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy u_k(x)S_{R_0}\left\{\frac{\sin\mu R\sin\mu_k R}{R\mu_k}\right\}-$$

$$-\sum_{0\leqq|\varrho_k|\leqq1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy u_k(x)S_{R_0}I^{\mu}_{|\varrho_k|}(R)-\sum_{|\varrho_k|>1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy u_k(x)\frac{\mu}{|\varrho_k|}S_{R_0}K^{\mu}_{|\varrho_k|}(R)-$$

$$-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy u_k(x)S_{R_0}\left\{\frac{2}{\pi}\int_0^R\sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k t-1}{t}\,dt\right\}-$$

$$-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy u_k(x)S_{R_0}\left\{\frac{2i}{\pi}\int_0^R\sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt\right\}+$$

$$+\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\int_{x-R}^{x+R}q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi-$$

$$-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\frac{\sin\mu R}{R}\int_{x-R}^{x+R}q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu_k(t-|x-\xi|)}{\pi\mu_k}\,dt\,d\xi-$$

$$-\sum_{|\varrho_k|>1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k(x)\,S_{R_0}\{h(R,\mu_k,\mu)\}.$$

REMARK 2. In the special case $\nu_k=0$, i.e. $\mu_k=\varrho_k$, in (28) the sixth, seventh and tenth terms vanish and we have (42) in [3] in case of the eigenvalues being non-negative numbers.

Denote $g(x)$ $(h(x))$ the left (right) hand side of (28). Our aim is to prove that the derivatives $g'(x), h'(x)$ exist and are bounded. With the above notations the following relation is true:

$$g'(x)=\int_{y_1}^{y_2}\frac{\partial}{\partial x}[S_{R_0}W(x,y,\mu)-\theta(x,y,\mu)]\,dy. \tag{29}$$

Denote by $h_i(x)$ $(i=1,2,\dots,10)$ the i^{th} member on the right hand side of (28). It is easy to see that on the compact K the derivatives $h_i'(x)$ $(i=1,2,4)$ exist. The existence of the derivative $h_5'(x)$ can be proved similarly by repeating word for word the analogous proof applied in [3] for the proof of (45).

Here we shall prove that the derivatives $h_i'(x)$ $(i=3,6,\dots,10)$ exist and are bounded. One can assume that $\mu\geqq 1$.

Consider the function

$$h_3(x):=\frac{2}{\pi}\sum_{0\leqq|\varrho_k|\leqq 1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k(x)\,S_{R_0}\left\{\frac{\sin\mu R\sin\mu_k R}{R\mu_k}\right\}.$$

Using Lemmas 4, 5 and (9) we have

$$S_3(x)=\frac{2}{\pi}\sum_{0\leqq|\varrho_k|\leqq 1}\left|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\right||u_k'(x)|\left|S_{R_0}\left\{\frac{\sin\mu R\sin\mu_k R}{R\mu_k}\right\}\right|\leqq \tag{30}$$

$$\leqq\frac{2}{\pi}\Big\{\sum_{0\leqq|\varrho_k|\leqq 1}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,\mathrm{ch}\,\nu_k 2R_0\Big|^2\Big\}^{1/2}\Big\{\sum_{0\leqq|\varrho_k|\leqq 1}|u_k'(x)|^2\Big\}^{1/2}\leqq C_6^{1/2}(K)\,C_7^{1/2}(K)<\infty.$$

Hence the derivative $h_3'(x)$ exists and

$$h_3'(x)=\frac{2}{\pi}\sum_{0\leqq|\varrho_k|\leqq 1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)\,S_{R_0}\left\{\frac{\sin\mu R\sin\mu_k R}{R\mu_k}\right\}. \tag{31}$$

The proof of the existence of the derivatives $h_6'(x)$ and $h_7'(x)$ can be proved similarly. Using Lemmas 1, 2 and (5)—(8) we get

$$h_6'(x)=\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)\,S_{R_0}\frac{2}{\pi}\int_0^R\sin\mu t\cos\varrho_k t\,\frac{\mathrm{ch}\,\nu_k t-1}{t}\,dt \tag{32}$$

and

$$h_7'(x)=\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)\,S_{R_0}\frac{2i}{\pi}\int_0^R\sin\mu t\sin\varrho_k t\,\frac{\mathrm{sh}\,\nu_k t}{t}\,dt. \tag{33}$$

Now consider the function

$$h_8(x)=\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\int_{x-R}^{x+R}q(\xi)u_k(\xi)\int_{|x-\xi|}^{R}\frac{\sin\mu t\sin\mu_k(t-|x-\xi|)}{\pi t\mu_k}\,dt\,d\xi.$$

We show that on the compact K the following relation is true:

(34)

$$h_8'(x)=\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\Big\{-\frac{1}{\pi}\int_{x-R}^{R}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\,dt\,d\xi+$$

$$+\frac{1}{\pi}\int_{x}^{x+R}q(\xi)u_k(\xi)\int_{\xi-x}^{R}\frac{\sin\mu t\cos\mu_k(t+x-\xi)}{t}\,dt\,d\xi\Big\}=S_{81}+S_{82},$$

where

$$S_{81}:=\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\Big\{-\frac{1}{\pi}\int_{x-R}^{x}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\,dt\,d\xi\Big\}.$$

In what follows we shall need the estimate

$$\int_{x-R}^{x}\frac{q(\xi)}{|x-\xi|^{\delta}}\,d\xi\le C(R_0)\quad (q(x)\in L^p(G),\ p>1) \tag{35}$$

(cf. [8]).

Using Lemma 3 and (35) we obtain that

$$|S_{81}|\le C_5(R_0)C(R_0)\Big\{\sum_{0\le|\varrho_k|\le1}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\,\|u_k\|_{L^\infty(K_{2R_0})}+ \tag{36}$$

$$+\sum_{1<|\varrho_k|\le\mu/2}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\frac{\|u_k\|_{L^\infty(K_{2R_0})}}{\big|\mu-|\varrho_k|\big|^{\delta/2}}+\sum_{\mu/2<|\varrho_k|<\mu-1}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\frac{\|u_k\|_{L^\infty(K_{2R})}}{\big|\mu-|\varrho_k|\big|^{\delta/2}}+$$

$$+\sum_{|\mu-|\varrho_k||\le1}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\,\|u_k\|_{L^\infty(K_{2R_0})}+\sum_{|\varrho_k|>\mu+1}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\frac{\|u_k\|_{L^\infty(K_{2R_0})}}{\big|\mu-|\varrho_k|\big|^{\delta/2}}\Big\}+$$

$$+C_5(R)\sum_{k=1}^{\infty}\Big|\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\Big|\frac{\|u_k\|_{L^\infty(K_{2R_0})}\operatorname{ch}\nu_k 2R_0}{1+\big|\mu-|\varrho_k|\big|}.$$

1. Using Lemma 5 and (8) we have

$$\sum_{0\le|\varrho_k|\le1}\le C_7^{1/2}(K)U<\infty. \tag{37}$$

2. If $1\le|\varrho_k|\le\frac{\mu}{2}$, the relation $\big|\mu-|\varrho_k|\big|\ge|\varrho_k|$ is true, too. Then using (4), (6) and (8) we get

$$\sum_{1<|\varrho_k|\le\mu/2}\le CC_1C_5C_6(R_0)\sum_{i=1}^{\infty}\frac{1}{i^{1+\delta/2}}<\infty. \tag{38}$$

Here we used that $\|u_k\|^2_{L^2(G)}=1$ and $\dfrac{1+|\nu_k|}{\exp|\nu_k|2R_0}\le C_6(R_0)$, $\forall\nu_k\in\mathbf{R}$.

3. If $\frac{\mu}{2}<|\varrho_k|<\mu-1$, the relation $|\varrho_k|>\big|\mu-|\varrho_k|\big|$ is also true. Then using also (4), (6) and (8) we have

$$\sum_{\mu/2<|\varrho_k|<\mu-1} \leqq CC_1C_5C_6(R_0)\sum_{i=1}^{[\mu/2]}\frac{1}{i^{\delta/2}(\mu-i)} \leqq D(K,\delta)<\infty. \tag{39}$$

4. Using the Cauchy—Schwarz inequality, Lemma 5 and (7) we obtain

$$\sum_{|\mu-|\varrho_k||\leqq 1} \leqq C_7^{1/2}(K)U^{1/2}<\infty. \tag{40}$$

5. Using (4), (6) and (8) we have

$$\sum_{|\varrho_k|>\mu+1} \leqq CC_1C_5C_6(R_0)\sum_{i=1}^{\infty}\frac{1}{i^{1+\delta/2}}<\infty. \tag{41}$$

6. Applying (6)—(8) we get

$$\sum_{k=1}^{\infty} \leqq 4CC_5U<\infty. \tag{42}$$

Substituting (37)—(42) into (36) we obtain the estimate for the quantity S_{81}:

$$|S_{81}|<\infty.$$

We have a similar estimate for S_{82}, too. Hence the derivative $h_8'(x)$ exists and (34) follows.

One can prove the existence and the boundedness of the derivatives $h_9'(x)$ and $h_{10}'(x)$ similarly.

Summarising the above argument we obtain from (26), (31)—(34)

$$\begin{aligned}
&\int_{y_1}^{y_2}\frac{\partial}{\partial x}[S_{R_0}W(x,y,\mu)-\theta(x,y,\mu)]\,dy= \qquad (43)\\
&=\frac{d}{dx}\int_{y_1}^{y_2}S_{R_0}\omega(x,y,\mu)\,dy+\frac{1}{2}\sum_{|\varrho_k|=\mu}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)-\\
&-\frac{2}{\pi}\sum_{0<|\varrho_k|\leqq 1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)S_{R_0}\left\{\frac{\sin\mu R\sin\mu_k R}{R\mu_k}\right\}-\\
&-\sum_{0\leqq|\varrho_k|\leqq 1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)S_{R_0}I^{\mu}_{|\varrho_k|}(R)-\\
&-\sum_{|\varrho_k|\geqq 1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)\frac{\mu}{|\varrho_k|}S_{R_0}K^{\mu}_{|\varrho_k|}(R)-\\
&-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)S_{R_0}\left\{\frac{2}{\pi}\int_0^R\sin\mu t\cos\varrho_k t\frac{\operatorname{ch}\nu_k t-1}{t}\,dt\right\}-\\
&-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)S_{R_0}\left\{\frac{2i}{\pi}\int_0^R\sin\mu t\sin\varrho_k t\frac{\operatorname{sh}\nu_k t}{t}\,dt\right\}+
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\left\{-\frac{1}{\pi}\int_{x-R}^{x}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\frac{\sin\mu t\cos\mu_k(t-x+\xi)}{t}\,td\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\int_{x}^{x+R}q(\xi)u_k(\xi)\int_{\xi-x}^{R}\frac{\sin\mu t\cos\mu_k(t+x-\xi)}{t}\,dt\,d\xi\right\}-$$

$$-\sum_{k=1}^{\infty}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,S_{R_0}\left\{\frac{1}{\pi}\frac{\sin\mu R}{R}\int_{x-R}^{x}q(\xi)u_k(\xi)\int_{x-\xi}^{R}\cos\mu_k(t-x+\xi)\,dt\,d\xi+\right.$$

$$\left.+\frac{1}{\pi}\frac{\sin\mu R}{R}\int_{x}^{x+R}q(\xi)u_k(\xi)\int_{\xi-x}^{R}\cos\mu_k(t+x-\xi)\,dt\,d\xi\right\}-$$

$$-\sum_{|\varrho_k|>1}\int_{y_1}^{y_2}\overline{u_k(y)}\,dy\,u_k'(x)\,S_{R_0}\{h(R,\mu_k,\mu)\}.$$

Above we proved that all the series on the right hand side of (43) are convergent uniformly in x on the compact K and are bounded. Therefore

$$\left|\int_{y_1}^{y_2}\frac{\partial}{\partial x}[S_{R_0}W(x,y,\mu)-\theta(x,y,\mu)]\,dy\right|\leqq M(K). \tag{44}$$

The proposition is proved.

4. Proof of the theorem

The idea of the proof is the following. Introduce the notation

$$S_\mu(f,x):=\int_a^b\frac{\partial}{\partial x}S_{R_0}W(y)\cdot f(y)\,dy. \tag{45}$$

Using (44), by the method applied in [3] one can prove that

$$|\sigma_\mu'(f,x)-S_\mu(f,x)|\leqq M_1(K)\|f\|_{W_1^1(G)}.$$

An analogous estimate holds also for the quantity $\frac{d}{dx}\hat{\sigma}_\mu(f,x)$. Therefore applying the triangle inequality we have the desired estimate

$$|\sigma'(f,x)-\hat{\sigma}_\mu'(f,x)\leqq C(K)\|f\|_{W_1^1(G)}.$$

The theorem is proved.

5. Proof of Remark 1

Consider the operators $Lu:=-u''$ on the interval $(0,\pi)$ defined by the boundary conditions $u(0)=u(\pi)=0$; and $\hat{L}u:=-u''$ on the mentioned interval defined by the boundary conditions $u'(0)=u'(\pi)=0$.

Choosing $f(x)\equiv 1$ we get

$$\sigma_{2n-1}'(f,x)-\hat{\sigma}_{2n-1}'(f,x)=\frac{2}{\pi}\frac{\sin nx}{\sin x}.$$

Let $K\subset G$ be an arbitrary compact which does not contain the point $x_0=\frac{\pi}{2}$. Then we have for any $x\in K$

$$\overline{\lim_{n\to\infty}}\,[\sigma'_{2n-1}(f,x)-\hat{\sigma}'_{2n-1}(f,x)]>0.$$

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MATHEMATICAL INSTITUTE
EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
MÚZEUM KRT. 6—8.
1088 BUDAPEST, HUNGARY

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A PROOF OF MIKUSIŃSKI'S THEOREM ON BOUNDED MOMENTS WITH BERNSTEIN POLYNOMIALS

U. ABEL (Giessen)

The purpose of this paper is to give a new proof of Mikusiński's theorem on bounded moments [8], which may be formulated in the following form:

THEOREM A. *Let* $(\lambda_n)_{n=1}^{\infty}$ *be a sequence of positive numbers satisfying*

$$(1)\qquad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty \quad \textit{and} \quad \lambda_{n+1}-\lambda_n \geqq c > 0 \quad (n = 1, 2, \ldots),$$

and let α *be a function of bounded variation on* $[0, \infty)$, *continuous on the left on* $(0, \infty)$ *such that*

$$(2)\qquad \int_0^{\infty} e^{-t\lambda_n}\, d\alpha(t) = O(e^{-q\lambda_n}) \quad (n \to \infty),$$

then $\alpha(t) \equiv \alpha(0)$ *for all* $t \in [0, q]$.

Integration by parts in (2) gives (assuming $\alpha(0)=0$)

$$\int_0^{\infty} e^{-t\lambda_n}\, d\alpha(t) = \lambda_n \int_0^{\infty} e^{-t\lambda_n} \alpha(t)\, dt$$

and Theorem A follows from

THEOREM B. *Let* $(\lambda_n)_{n=1}^{\infty}$ *be a sequence of positive numbers satisfying* (1), *and let* f *be a bounded measurable function on* $[0, \infty)$. *Then*

$$(3)\qquad \int_0^{\infty} e^{-t\lambda_n} f(t)\, dt = O(e^{-q\lambda_n}) \quad (n \to \infty)$$

implies $f(t)=0$ *a.e. on* $[0, q]$.

Mikusiński's proof [8] (see [10], [6], [7] for earlier attempts to the matter) is based on a certain discontinuity factor, the idea of which goes back to Phragmén. In [3], R. P. Boas Jr. applies deep properties of analytic functions. The proof in [2] uses the Poisson integral and works only in the case $\lambda_n = n$ $(n=1, 2, \ldots)$. The proof presented here is based on the generalized Bernstein polynomials

$$B_n(t) = \sum_{j=0}^{n} a_j e^{-t\lambda_n},$$

$$a_j = \sum_{v=0}^{j} \frac{f(-\log b_{nv})(-1)^{n-v} \prod_{k=v+1}^{n} \lambda_k}{(\lambda_j-\lambda_v)(\lambda_j-\lambda_{v+1})\dots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\dots(\lambda_j-\lambda_n)},$$

$$b_{nv} = \prod_{k=v+1}^{n}\left(1-\frac{\lambda_1}{\lambda_k}\right)^{1/\lambda_1} \quad (0 \leqq v \leqq n-1), \quad b_{nn} = 1,$$

which converge uniformly on $[0, \infty)$ to f, if the sequence $0=\lambda_0<\lambda_1<\lambda_2<\dots$ satisfies (1), f is continuous on $[0, \infty)$ and $\lim_{t\to+\infty} f(t)$ exists (see [5], p. 44—46, and [4]).

The crux in the proof of Theorem B is contained in the following lemma on the growth of the coefficients a_j (see [1], p. 4f).

LEMMA. *Suppose $f(t)=0$ $(t\geqq t_0)$. Then $a_j=0$ if $-\log b_{nj}\geqq t_0$ and*

$$\log |a_j| \leqq \log \sup_{t\geqq 0} |f(t)|+K\{\lambda_j\delta|\log\delta|+|\log\lambda_1|+|\log\lambda_j|+|\log\delta|\}+\lambda_j K(\delta)t_0$$

for all $\delta\in(0,1)$, $K(\delta)=\max\left\{|\log\delta|;\frac{1}{\delta}\right\}$, *if* $-\log b_{nj}<t_0$.

PROOF OF THEOREM B. Let $r\in(0, q)$ be a Lebesgue point of f and let $\varepsilon>0$ be given. For every $h>0$ we define the function

$$\omega_r(t; h) = \begin{cases} h^{-1} \ (r \leqq t \leqq r+h), \\ \text{linear on } [r-h^2, r] \text{ and } [r+h, r+h+h^2], \\ 0 \quad \text{elsewhere} \end{cases}$$

such that ω_r is continuous for all real t. Then for sufficiently small h we have $0<r-h^2<r+h+h^2<q$, and thus

$$\int_0^\infty \omega_r(t; h)f(t)\,dt = \frac{1}{h}\int_r^{r+h} f(t)\,dt+\int_{r-h^2}^{r}\omega_r(t; h)f(t)\,dt+\int_{r+h}^{r+h+h^2}\omega_r(t; h)f(t)\,dt.$$

Since the first integral on the right hand side tends to $f(r)$ as $h\to 0+$ and each of the last two integrals can be estimated by $h\|f\|_\infty$ we can choose $h_0>0$ such that

$$\left|f(r)-\int_0^\infty \omega_r(t; h_0)f(t)\,dt\right| < \varepsilon. \tag{4}$$

Now we approximate $\omega_r(t;h_0)$ by a Bernstein polynomial $B(t_n)=\sum_{j=0}^{n} a_j e^{-t\lambda_j}$ such that

$$|\omega_r(t; h_0)-B_n(t)| < \varepsilon \quad (t \geqq 0).$$

Then (4) yields

$$(5)\qquad \left|f(r)-\int_0^\infty B(t)f(t)\,dt\right| < \varepsilon+\int_0^\infty |\omega_r(t;\,h_0)-B_n(t)||f(t)|\,dt \leqq \varepsilon+\varepsilon\|f\|_1.$$

The Lemma gives for the coefficients of B_n $a_j=0$ $(0\leqq j\leqq p-1)$ and

$$|a_j| \leqq h_0^{-1}\exp\{K(|\log\lambda_1|+|\log\delta|+|\log\lambda_j|)\}\times$$
$$\times\exp\{[K\delta\,|\log\delta|+K(\delta)(r+h_0+h_0^2)]\lambda_j\}\quad (p\leqq j\leqq n)$$

where p is the integer with

$$-\log b_{np} < r+h_0+h_0^2 \leqq -\log b_{np-1}$$

and $\delta\in(0,1)$.

Without loss of generality we may assume $\lambda_j\geqq 1$. Furthermore we choose δ so close to 1 that $K\delta|\log\delta|+K(\delta)(r+h_0+h_0^2)\leqq q-\beta<q$. Then we get

$$(6)\qquad |a_j| \leqq K_1\lambda_j^K e^{(q-\beta)\lambda_j}\quad (p\leqq j\leqq n).$$

By (3), $\lambda_{j+1}-\lambda_j\geqq c>0$ $(j=0,2,\dots)$ and (6), it follows that, for some constant $M<+\infty$,

$$\left|\int_0^\infty B_n(t)f(t)\,dt\right| \leqq \sum_{j=p}^n |a_j|\left|\int_0^\infty e^{-t\lambda_j}f(t)\,dt\right| \leqq$$

$$\leqq M\sum_{j=p}^n |a_j|e^{-q\lambda_j} \leqq MK_1\sum_{j=p}^\infty \lambda_j^K e^{-\beta\lambda_j} \leqq K_2\sum_{j=p}^\infty e^{-(1/2)\beta\lambda_j} \leqq$$

$$\leqq K_2\sum_{j=p}^\infty e^{-(1/2)\beta cj} = O(e^{-(1/2)\beta cp})\quad (p\to\infty).$$

Since $\sum_{j=1}^\infty \lambda_j^{-1}=+\infty$, we have $p\to\infty$ as $n\to\infty$. Thus we can choose n so large that

$$\left|\int_0^\infty B_n(t)f(t)\,dt\right| < \varepsilon,$$

and, by (5), it holds $|f(r)|<2\varepsilon+\varepsilon\|f\|_1$. Hence $f(t)=0$ for every Lebesgue point t in $(0,q)$, i.e., $f\equiv 0$ a.e. in $(0,q)$.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT
ARNDTSTRASSE 2
D—6300 GIESSEN

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BOEHMIANS AND GENERALIZED FUNCTIONS

P. MIKUSIŃSKI (Santa Barbara)

Introduction. Since S. Sobolev in 1936 [19] and L. Schwartz in 1945 [17] introduced the notion of distributions, there arose a number of theories of generalized functions; see e.g. [1—3], [5—11], [14], [16], [21]. These theories differ from one another by generality, by applications or by language which is used to build them. One of the youngest generalizations of functions is Boehmians.

The idea of the construction of Boehmians was initiated by the concept of regular operators introduced by T. K. Boehme in [4]. Regular operators form a subalgebra of the field of Mikusinski operators and hence they include only such functions whose support is bounded from the left. Attempts were made to generalize the notion of regular operators in order to embrace all continuous functions. A general construction of Boehmians presented in [12] suits this aim. In a concrete case the space of Boehmians contains all regular operators, all distributions and some objects which are not operators nor distributions. An example of such a space is given in [12]. A concept of convergence of Boehmians was introduced and discussed in [13]. In the same paper the concrete space of Boehmians mentioned above is discussed in more detail. The space furnished with the introduced convergence appears to be a complete quasi-normed space.

The present note completes the previous notes devoted to Boehmians (i.e. [12] and [13]) with some remarks and some new results. In the first section, we recall the general definition of Boehmians. Then we describe Boehmians as multipliers on suitable spaces. The third section is devoted to the convergence of Boehmians. We list basic properties of the convergence and prove some new facts. In Section 4, we present a few examples of spaces of Boehmians. In the last section, we discuss connections between Boehmians and other generalized functions.

1. General construction of Boehmians. For every ring without divisors of zero, there exists the corresponding field of quotients. The space C^+ of all continuous functions on the real line R with supports bounded from the left forms a ring without divisors of zero with respect to the convolution; (the definition of the convolution can be found in Section 2). The field of quotients for this space is known as the field of Mikusiński operators (see [11]). When replacing C^+ by the space C of all continuous functions, the construction of the field of quotients is impossible because there are divisors of zero in C. The construction of Boehmians is similar to the construction of the field of quotients and in some cases it gives just the field of quotients (see [12]). On the other hand the construction is possible when there are divisors of zero, for example in the case of space C (with the operations of pointwise addition and convolution).

Let G be a linear space and let S be a subspace of G. We assume that to each pair of elements $f\in G$ and $\varphi\in S$ there is assigned the product $f*\varphi$ so that the following conditions are satisfied:

(1) If $\varphi, \psi\in S$, then $\varphi*\psi\in S$ and $\varphi*\psi=\psi*\varphi$,

(2) if $f\in G$ and $\varphi, \psi\in S$, then $(f*\varphi)*\psi=f*(\varphi*\psi)$,

(3) if $f, g\in G$, $\varphi\in S$ and $\lambda\in R$, then $(f+g)*\varphi=f*\varphi+g*\varphi$ and $\lambda(f*\varphi)=$ $=(\lambda f)*\varphi$.

Let Δ be a family of sequences of elements from S such that

(4) if $f, g\in G$, $(\delta_n)\in\Delta$ and $f*\delta_n=g*\delta_n$ for $n=1, 2, \ldots$, then $f=g$,

(5) if (φ_n), $(\psi_n)\in\Delta$, then $(\varphi_n*\psi_n)\in\Delta$.

Elements of Δ will be called *delta sequences*.

Let $\mathscr{A}(G, \Delta)$ be the family of all pairs of sequences $(f_n)\in G^N$ and $(\varphi_n)\in\Delta$ such that $f_i*\varphi_j=f_j*\varphi_i$ for all $i, j\in N$. Such a pair will be denoted, for short, by f_n/φ_n. It can be easily verified that the relation defined as follows

$$f_n/\varphi_n \sim g_n/\psi_n \quad \text{if} \quad f_i*\psi_j = g_j*\varphi_i \quad \text{for all} \quad i, j\in N$$

is an equivalence in $\mathscr{A}(G, \Delta)$. We put $\mathscr{B}(G, \Delta)=\mathscr{A}(G, \Delta)/\sim$. Elements of $\mathscr{B}(G, \Delta)$ will be called *Boehmians* and denoted by small letters like $f, g, \ldots$ or by $[f_n/\varphi_n]$, $[g_n/\psi_n], \ldots$.

The sum of two Boehmians and multiplication by scalar can be defined in a natural way:

$$[f_n/\varphi_n]+[g_n/\psi_n] = [(f_n*\psi_n+g_n*\varphi_n)/\varphi_n*\psi_n] \quad \text{and} \quad \lambda[f_n/\varphi_n] = [\lambda f_n/\varphi_n].$$

Then $\mathscr{B}(G, \Delta)$ becomes a linear space and we have

LEMMA 1.1. *Let $(\delta_n)\in\Delta$. The mapping u given by the formula*

(6) $u(f)=[f*\delta_n/\delta_n]$

yields an algebraic isomorphism of G into $\mathscr{B}(G, \Delta)$. The mapping u does not depend on the choice of (δ_n).

For convenience, G will be considered as a subset of $\mathscr{B}(G, \Delta)$.

The multiplication defined on $G\times S$ can be extended onto $\mathscr{B}(G, \Delta)\times(\mathscr{B}(S, \Delta)$, where $\mathscr{B}(S, \Delta)$ denotes the set of all elements of $\mathscr{B}(G, \Delta)$ which can be written in the form $[g_n/\psi_n]$ with $g_n\in S$ for all $n=1, 2, \ldots$ If $f=[f_n/\varphi_n]\in\mathscr{B}(G, \Delta)$ and $g=[g_n/\psi_n]\in$ $\in\mathscr{B}(S, \Delta)$ then we put $f*g=[f_n*g_n/\varphi_n*\psi_n]$.

LEMMA 1.2. *If $f, g\in\mathscr{B}(G, \Delta)$, $(\delta_n)\in\Delta$ and $f*\delta_n=g*\delta_n$ for all $n\in\mathbf{N}$, then $f=g$.*

LEMMA 1.3. *If $[f_n/\varphi_n]\in\mathscr{B}(G, \Delta)$, then $[f_n/\varphi_n]*\varphi_i=f_i$ for each $i\in\mathbf{N}$.*

LEMMA 1.4. *If, for $f\in\mathscr{B}(G, \Delta)$, there exists $(\delta_n)\in\Delta$ such that $f*\delta_n\in G$ for each $n\in\mathbf{N}$, then $f=[f*\delta_n/\delta_n]$.*

Proofs of all the above lemmas can be found in [13].

2. Multipliers. As usual by $\mathscr{D}(R^m)$ we denote the space of all infinitely differentiable functions with compact support. For $f\in C(R^m)$ (where $C(R^m)$ is the space of all continuous functions) and $\varphi\in\mathscr{D}(R^m)$, by $f*\varphi$ we mean the convolution of f and φ, i.e.

$$(f*\varphi)(x) = \int_{R^m} f(y)\varphi(x-y)\,dy.$$

The space $\mathscr{D}(R^m)$ with the convolution forms a ring.

R. A. Struble in [20] has proved the following

THEOREM 2.1. *The space of all mappings* $M\colon \mathscr{D}(R^m)\to C(R^m)$ *satisfying the condition*

$$M(\varphi*\psi) = M(\varphi)*\psi \quad \textit{for each} \quad \varphi,\psi\in\mathscr{D}(R^m)$$

is isomorphic to the space $\mathscr{D}'(R^m)$ *of all distributions.*

The theorem shows that $\mathscr{D}'$ can be described by a purely algebraic condition concerning the convolution (note that neither continuity nor linearity of M is assumed). A similar description is possible for Boehmians. However, the characterization is not so ismple. In this section we present the characterization of Boehmians as multipliers. The results will be used in Section 5.

From (1) and (2) it follows that S is a multiplicative semigroup. Let Q be a subsemigroup of S. A mapping $M\colon Q\to G$ is called a multiplier on Q if

$$M(\varphi*\psi) = M(\varphi)*\psi \quad \text{for each} \quad \varphi,\psi\in Q. \tag{7}$$

The family of all multipliers on Q will be denoted by $\mathscr{M}(Q, G)$. Clearly, $\mathscr{M}(Q, G)$ is a linear space. Put $\mathscr{B}_Q(G,\Delta)=\{f\in\mathscr{B}(G,\Delta)\colon f*\varphi\in G$ for each $\varphi\in Q\}$. If $f\in\mathscr{B}_Q(G,\Delta)$, then the mapping

$$M_f(\varphi) = f*\varphi \tag{8}$$

is a multiplier on Q. It is easy to prove that $\mathscr{B}_Q(G,\Delta)$ is isomorphic to a subset of $\mathscr{M}(Q, G)$. On the other hand we have the following

THEOREM 2.2. *If* $\Delta\cap Q^N\neq\emptyset$ (*i.e. there is a delta sequence* (δ_n) *in* Δ *elements of which are in* Q), *then for each multiplier* $M\in\mathscr{M}(Q,G)$ *there is* $f\in\mathscr{B}_Q(G,\Delta)$ *such that* $M(\varphi)=f*\varphi$ *for each* $\varphi\in Q$. *Hence* $\mathscr{M}(Q,G)$ *and* $\mathscr{B}_Q(G,\Delta)$ *are isomorphic.*

PROOF. Let $(\delta_n)\in\Delta\cap Q^N$ and let M be a multiplier on Q. Consider the Boehmian $f=[M(\delta_n)/\delta_n]$; ($M(\delta_n)/\delta_n$ belongs to $\mathscr{A}(G,\Delta)$ because of (7)). If $\varphi\in Q$, then $f*\varphi=[M(\delta_n)*\varphi/\delta_n]=[M(\varphi)*\delta_n/\delta_n]=M(\varphi)\in G$. Hence $f\in\mathscr{B}_Q(G,\Delta)$ and $M(\varphi)=f*\varphi$ for all $\varphi\in Q$.

REMARK. From the above facts it follows that any space of Boehmians can be embedded into a suitable *multiplier extension of admissible vector module* introduced by Á. Száz in [22]. More precisely, $\mathscr{B}(G,\Delta)$ can be identified with the collection of all quotients multipliers from $\mathscr{M}(S, G)$ whose domains contain delta sequences from Δ.

3. Convergence of Boehmians. In this section we shall discuss a particular case of Boehmians for which G is a quasi-normed space and the family Δ is defined in a special way.

Let G be a linear space and let $\|\,.\,\|$ be a quasi-norm on G (by a quasi-norm we mean a real function on G such that $\|f\|=0$ implies $f=0$, $\|f+g\|\leqq\|f\|+\|g\|$, $\|-f\|=\|f\|$ and $\lim\|\lambda_n f_n-\lambda f\|=0$ if $\lambda_n, \lambda\in R$, $\lim\lambda_n=\lambda$ and $\lim\|f_n-f\|=0$). Moreover, we assume that

$$\text{(9)} \qquad \text{if } \|f_n\|\to 0, \quad \text{then} \quad \|f_n*\varphi\|\to 0 \quad \text{for each} \quad \varphi\in S.$$

Let S_0 be a subsemigroup of S (with respect to the product) and let s be a positive functional on S_0 satisfying

$$\text{(10)} \qquad s(\varphi*\psi)\leqq s(\varphi)+s(\psi) \quad \text{for each} \quad \varphi,\psi\in S_0,$$

$$\text{(11)} \qquad \text{if } \|f_n-f\|\to 0 \quad \text{and} \quad s(\varphi_n)\to 0, \quad \text{then} \quad \|f_n*\varphi_n-f\|\to 0.$$

It is easily verified that the family

$$\Delta=\{(\delta_n)\in S_0^N:\ s(\delta_n)\to 0\}$$

satisfies all the conditions imposed on the family of delta sequences Δ (see Section 1)˙

We say that a sequence of Boehmians $f_n\in\mathscr{B}(G,\Delta)$ is Δ-convergent to a Boehmian f, and we write Δ-$\lim f_n=f$, if there is $(\varphi_n)\in\Delta$ such that $(f_n-f)*\varphi_n\in G$ for all $n\in\mathbf{N}$ and $\|(f_n-f)*\varphi_n\|\to 0$.

Now we list some basic properties of Δ-convergence. (Proofs of the following theorems can be found in [13].)

THEOREM 3.1. *The mapping u defined by* (6) *is continuous.*

THEOREM 3.2. G is dense in $\mathscr{B}(G,\Delta)$ (with respect to Δ-convergence).

THEOREM 3.3. *Let $g=[g_n/\delta_n]\in\mathscr{B}(G,\Delta)$ and $g_n\in S$ for each $n\in\mathbf{N}$. If Δ-$\lim f_n=f$, then Δ-$\lim f_n*g=f*g$.*

THEOREM 3.4. *If G is complete and the following conditions are satisfied*

(12) *if $\varphi_n\in S_0$, $s(\varphi_n)\leqq c$ (for all $n\in\mathbf{N}$) and $\|\varphi_n-\varphi\|\to 0$, then $\varphi\in S_0$ and $s(\varphi)\leqq c$;*

(13) *if $\|f_n\|\to 0$ and $s(\varphi_n)\leqq c$ for all $n\in\mathbf{N}$, then there exists $c'\in\mathbf{R}$ such that $\|f_n\varphi_n\|\leqq$ $\leqq c'\|f_n\|$ for all $n\in\mathbf{N}$,*

then the space $\mathscr{B}(G,\Delta)$ is a complete quasi-normed space (with respect to Δ-convergence).

Let Q, as in Section 2, be a subsemigroup of S. We are going to describe the set $\mathscr{B}_Q(G,\Delta)$ in terms of convergence. Let $f_n, f\in\mathscr{B}_Q(G,\Delta)$ $(n=1,2,\ldots)$. We say that (f_n) is *Q-convergent* to f if $\|(f_n-f)*\varphi\|\to 0$ for each $\varphi\in Q$. If $Q^N\cap\Delta\neq\emptyset$, then Q-convergence is a Hausdorff convergence. A sequence (f_n) of elements from $\mathscr{B}_Q(G,\Delta)$ is called a *Q-Cauchy sequence* if for each pair of increasing sequences of positive integers p_n and q_n we have $\|(f_{p_n}-f_{q_n})*\varphi\|\to 0$ for each $\varphi\in Q$.

Up to the end of this section we assume that G is complete, $Q^N\cap\Delta\neq\emptyset$ and that conditions (12) and (13) are satisfied.

LEMMA 3.5. *Let $f_n \in \mathscr{B}_Q(G, \Delta)$ for $n=1, 2, \ldots$. If (f_n) is a Q-Cauchy sequence, then (f_n) is Δ-convergent (to an element of $\mathscr{B}(G, \Delta)$).*

PROOF. It suffices to note that Q-convergence implies Δ-convergence.

THEOREM 3.6. *$\mathscr{B}_Q(G, \Delta)$ is Q-complete (i.e. each Q-Cauchy sequence in $\mathscr{B}_Q(G, \Delta)$ is Q-convergent).*

PROOF. Let (f_n) be a Q-Cauchy sequence $(f_n \in \mathscr{B}_Q(G, \Delta))$. By Lemma 3.5, the sequence (f_n) is Δ-convergent to some element $f \in \mathscr{B}(G, \Delta)$. We have to show that $f \in \mathscr{B}_Q(G, \Delta)$ and that (f_n) is Q-convergent to f. In fact, for each $\varphi \in Q$, the sequence $(f_n * \varphi)$ is a Cauchy sequence in G. By the completeness of G, $(f_n * \varphi)$ is convergent to an element $f_\varphi \in G$. Hence, by Theorem 3.1, we have $\Delta\text{-}\lim (f_n * \varphi) = f_\varphi$. On the other hand, in view of Theorem 3.3, we have $\Delta\text{-}\lim (f_n * \varphi) = (\Delta\text{-}\lim f_n) * \varphi = f * \varphi$, whence $f * \varphi = f_\varphi \in G$ and $\|(f_n - f) * \varphi\| \to 0$. The proof is complete.

THEOREM 3.7. *G is dense in $\mathscr{B}_Q(G, \Delta)$ (with respect to Q-convergence).*

PROOF. Let $f = [f_n/\varphi_n] \in \mathscr{B}_Q(G, \Delta)$ with $(\varphi_n) \in Q^N \cap \Delta$. For each $\varphi \in Q$ we have $\|(f_n - f) * \varphi\| = \|(f * \varphi_n - f) * \varphi\| = \|(f * \varphi) * \varphi_n - (f * \varphi)\| \to 0$, by Lemma 1.3 and (11). Hence (f_n) is Q-convergent to f, which proves the theorem.

COROLLARY 3.8. *The space $\mathscr{B}_Q(G, \Delta)$ is identical with the completion of G with respect to Q-convergence.*

4. Examples. We shall use the following notation:
C = the space of all continuous functions on R^m,
L_1 = the space of all integrable functions on R^m,
L_1^0 = the space of all functions from L_1 with bounded support,
$\mathscr{D}$ = the space of all infinitely differentiable functions on R^m with compact support,
$\mathscr{D}'$ = the space of all distributions on R^m,

$$|x| = (x_1^2 + \ldots + x_m^2)^{1/2} \quad \text{for} \quad x = (x_1, \ldots, x_m) \in R^m,$$

$$K_\varepsilon = \{x \in R^m \colon |x| < \varepsilon\}.$$

From now on, by the product on $G \times S$ we mean the convolution

$$(f * \varphi)(x) = \int_{R^m} f(y)\, \varphi(x-y)\, dy.$$

$\mathscr{B}(C, \Delta_c^+)$. In this case S consists of all continuous functions with compact support. Let Δ_c^+ be the family of delta sequences defined as in Section 3 with

$$S_0 = \left\{\varphi \in S \colon \varphi \geqq 0 \quad \text{and} \quad \int \varphi = 1\right\} \quad \text{and} \quad s(\varphi) = \inf\{\varepsilon > 0 \colon \operatorname{supp} \varphi \subset K_\varepsilon\}.$$

Let $\|\cdot\|$ be a quasi-norm on C which generates uniform convergence on compact subsets of R^m. It is easily verified that in this case all the conditions (9)—(13) are satisfied. Hence the space $\mathscr{B}(C, \Delta_c^+)$ endowed with the Δ-convergence is a complete quasi-normed space.

THEOREM 4.1. *Let $(\delta_n) \in \Delta_c^+ \cap \mathscr{D}^N$. The mapping $u \colon \mathscr{D}' \to \mathscr{B}(C, \Delta_c^+)$ defined by the formula $u(f) = [f * \delta_n/\delta_n]$ is a continuous embedding of $\mathscr{D}'$ into $\mathscr{B}(C, \Delta_c^+)$.*

PROOF. Clearly, u is an injection of $\mathscr{D}'$ into $\mathscr{B}(C, \Delta_c^+)$. We shall show that it is continuous. Let $f_n \to f$ in $\mathscr{D}'$ (weakly). Then $\|(f_n - f) * \varphi\| \to 0$ for each $\varphi \in \mathscr{D}$ (see [18], p. 53). Since $(\delta_n) \in \Delta_c^+ \cap \mathscr{D}^N$, $\|(f_n - f) * \delta_k\| \to 0$ for each $k \in \mathbf{N}$. Applying the diagonal method we can find a subsequence (f_{p_n}) of (f_n) such that $\|(f_{p_n} - f) * \delta_n\| \to 0$, which means that (f_{p_n}) is Δ-convergent to f. Since Δ-convergence is of quasi-norm type, the convergence of the sequence (f_n) follows.

The mapping u^{-1} is not continuous. A suitable example is presented in [13]. It is easily seen that the mapping u does not depend on the choice of the delta sequence (δ_n).

In view of Theorem 4.1, we may identify each distribution with its respective Boehmian. Theorem 2.1 gives us the following simple characterization of elements of $\mathscr{B}(C, \Delta_c^+)$ (and also $\mathscr{B}(C, \Delta_c)$; see the next example) which may be identified with distributions.

THEOREM 4.2. *A Boehmian $f \in \mathscr{B}(C, \Delta_c^+)$ is an element of $u(\mathscr{D}')$ iff $f * \varphi \in C$ for every $\varphi \in \mathscr{D}$.*

Let us observe that each Boehmian from $\mathscr{B}(C, \Delta_c^+)$ has all derivatives which are again elements of $\mathscr{B}(C, \Delta_c^+)$. In fact, for each $f \in \mathscr{B}(C, \Delta_c^+)$, there exists a representative $f = [f_n/\varphi_n]$ such that all functions f_n are infinitely differentiable. Then we can adopt the following definition:

$$D^\alpha [f_n/\varphi_n] = [D^\alpha f_n/\varphi_n].$$

It is easy to check that this definition coincides with the usual definition of derivatives when f is a differentiable function.

Let $(\delta_n) \in \Delta_c^+ \cap \mathscr{D}$ and let $\delta = [\delta_n/\delta_n]$. We have the following simple

LEMMA 4.3. *$D^\alpha f = f * (D^\alpha \delta)$ for every $f \in \mathscr{B}(C, \Delta_c^+)$.*

From the above lemma and Theorem 3.3 we obtain at once

THEOREM 4.4. *The operation D^α is continuous (with respect to Δ-convergence).*

As in the case of derivatives, the notion of support of a function can be extended to each Boehmian from $\mathscr{B}(C, \Delta_c^+)$. Let $f \in \mathscr{B}(C, \Delta_c^+)$ and let Ω be an open set in R^m. We say that f is equal to zero on Ω if for each compact subset K of Ω there exists $(\delta_n) \in \Delta_c^+$ such that $f * \delta_n \in C$ and $f * \delta_n = 0$ on K for each $n \in \mathbf{N}$. If f is equal to zero on Ω_ν for each $\nu \in I$, then f is equal to zero on $\bigcup_{\nu \in I} \Omega_\nu$. Therefore, the support of f can be defined as the complement of the largest open set on which f is equal to zero.

$\mathscr{B}(C, \Delta_c)$. In this example we enlarge the family of delta sequences letting

$$S_0 = \left\{\varphi \in S \colon \int \varphi = 1\right\} \quad (S = C \cap L_1^0) \quad \text{and} \quad s(\varphi) = \inf\{\varepsilon > 0 \colon \operatorname{supp} \varphi \subset K_\varepsilon\} + \\ + \ln \int |\varphi|.$$

Denote the corresponding family of delta sequences by Δ_c. Since in this case, as in the first example, all the conditions (9)—(13) are satisfied, the space $\mathscr{B}(C, \Delta_c)$ endowed with the Δ-convergence is a complete quasi-normed space. Clearly we have $\Delta_c^+ \subset \Delta_c$. Thus, we have also $\mathscr{B}(C, \Delta_c^+) \subset \mathscr{B}(C, \Delta_c)$, when $[f_n/\varphi_n] \in \mathscr{B}(C, \Delta_c^+)$ is identified with

$[f_n/\varphi_n]\in\mathscr{B}(C, \Delta_c)$. Moreover, each Δ-convergent sequence in $\mathscr{B}(C, \Delta_c^+)$ is Δ-convergent in $\mathscr{B}(C, \Delta_c)$ (to the same limit). We shall prove that $\mathscr{B}(C, \Delta_c^+)$ is a proper subspace of $\mathscr{B}(C, \Delta_c)$. Since both spaces are complete quasi-normed spaces, it is enough to show (by the open mapping theorem) that there is a Δ-convergent sequence in $\mathscr{B}(C, \Delta_c)$ which does not converge in $\mathscr{B}(C, \Delta_c^+)$. In the following example we put $C=C(R)$ and $\mathscr{D}=\mathscr{D}(R)$.

Let $(\varphi_n)\in\Delta_c^+\cap\mathscr{D}$. Put $\psi_n(t)=\varphi_n(t)-(1/\alpha_n)\varphi_n'(t)$, where $\alpha_n=\int|\varphi_n'|$ $(n=1, 2, \ldots)$. Since $\alpha_n\geqq\max\varphi_n$, the sequence (α_n) is unbounded. It is easy to check that for each $g\in L_1$ the sequence (ψ_n*g) converges to g in L_1. Hence for each fixed $k\in\mathbf{N}$ we have $\int|\psi_n*\varphi_k|-\int|\varphi_k|\to 0$. Since $\int|\varphi_k|=1$, we have $\int|\psi_n*\varphi_k|\to 1$ for each $k\in\mathbf{N}$ (as $n\to\infty$). Thus we can find an increasing sequence of positive integers i_n such that $\psi_{i_n}*\varphi_n\to 1$. Therefore $(\psi_{i_n}*\varphi_n)\in\Delta_c$. Let $f_n(t)=e^{\alpha_n t}$ for $n=1, 2, \ldots$. Since, for each $n\in\mathbf{N}$, $f_n*\psi_{i_n}=0$, we have $f_n*(\psi_{i_n}*\varphi_n)=0$. Thus, we have proved that the sequence of functions f_n is Δ-convergent to 0 in $\mathscr{B}(C, \Delta_c)$. On the other hand, for any $(\delta_n)\in\Delta_c^+$, the sequence $(f_n*\delta_n)(t)$ is unbounded for every $t>0$. Therefore, (f_n) does not converge in $\mathscr{B}(C, \Delta_c^+)$.

Note that the definition of derivatives and the definition of the support given for $\mathscr{B}(C, \Delta_c^+)$ make sense in $\mathscr{B}(C, \Delta_c)$, and Theorem 4.4. remains true.

$\mathscr{B}(L_1, \Delta_c)$. In this case we have $G=S=L_1$. The family of delta sequences is the same as is the preceding example. Endow L_1 with the convergence generated by the norm $\int|f|$. Then $\mathscr{B}(L_1, \Delta_c)$ is a complete quasi-normed space (with respect to the Δ-convergence).

THEOREM 4.5. *Let $(\delta_n)\in\Delta_c$. The mapping $u\colon\mathscr{B}(L_1, \Delta_c)\to\mathscr{B}(C, \Delta_c)$ given by the formula*

$$u([f_n/\varphi_n]) = [f_n*\delta_n/\varphi_n*\delta_n]$$

is a continuous injection.

PROOF. Clearly u is an injection. To prove that it is continuous assume that (f_n) is a sequence of Boehmians Δ-convergent to 0 in $\mathscr{B}(L_1, \Delta_c)$. This means that there is a delta sequence $(\varphi_n)\in\Delta_c$ such that $f_n*\varphi_n\in L_1$ $(n=1, 2, \ldots)$ and $\int|f_n*\varphi_n|\to 0$. Hence, for a suitable delta sequence $(\psi_n)\in\Delta_c$, $g_n=f_n*\varphi_n*\psi_n$ is a sequence of continuous functions convergent uniformly to 0 and so is the sequence of convolutions $u(f_n)*(\varphi_n*\psi_n)=g_n*\delta_n$, which completes the proof.

It is easy to see that u^{-1} is not continuous.

Note that the following formula

$$[f_n/\varphi_n]*[g_n/\psi_n] = [f_n*g_n/\varphi_n*\psi_n]$$

extends the notion of convolution to all pairs of elements from $\mathscr{B}(L_1, \Delta_c)$. Similarly, the integral can be defined for all elements of $\mathscr{B}(L_1, \Delta_c)$.

THEOREM 4.6. *The mapping $\int\colon\mathscr{B}(L_1, \Delta_c)\to R$ defined by the formula*

$$\int[f_n/\varphi_n] = \int f_1(x)\,dx$$

is a continuous linear functional on $\mathscr{B}(L_1, \Delta_c)$. The testriction of the functional $\int$ to L_1 gives the integral, i.e. $\int f=\int f(x)\,dx$ for $f\in L_1$.

PROOF. First, we have to prove that the definition of the functional is correct. Let $f \in \mathscr{B}(L_1, \Delta_c)$ and let $f=[f_n/\varphi_n]=[g_n/\psi_n]$. We have to show that $\int f_1(x)\,dx = \int g_1(x)\,dx$. In fact, we have $f_1 * \psi_1 = g_1 * \varphi_1$. Since $\int (f_1 * \psi_1)(x)\,dx = \int f_1(x)\,dx$ and $\int (g_1 * \varphi_1)(x)\,dx = \int g_1(x)\,dx$, the equality follows.

If $f \in L_1$, then $\int [f * \varphi_n/\varphi_n] = \int (f * \varphi_1)(x)\,dx = \int f(x)\,dx$.

The linearity of $\int$ is obvious. To prove the continuity assume that (f_n) is a sequence of Boehmians convergent to 0 in $\mathscr{B}(L_1, \Delta_c)$. Note that, if $f * \varphi \in L_1$ $\left(f \in \mathscr{B}(L_1, \Delta_c),\ \varphi \in L_1 \text{ and } \int \varphi = 1\right)$, then $\int f = \int (f * \varphi)(x)\,dx$. There is $(\varphi_n) \in \Delta_c$ such that $\int (f_n * \varphi_n)(x)dx \to 0$. Thus $\int f_n = \int (f_n * \varphi_n)(x)\,dx \to 0$, which completes the proof.

The Fourier transform can be also easily extended onto the whole space $\mathscr{B}(L_1, \Delta_c)$, which follows from the following simple .

LEMMA 4.7. *If* $[f_n/\varphi_n] \in \mathscr{B}(L_1, \Delta_c)$, *then the sequence* $\hat{f}_n$ *converges uniformly on compact subsets of* $\mathbf{R}^n$ *(by* $\hat{f}$ *we denote the Fourier transform of* f*).*

PROOF. Let K be a compact set. Since, for each $(\varphi_n) \in \Delta_c$, the sequence $(\hat{\varphi}_n)$ converges almost uniformly to the constant function 1, there is $k \in \mathbf{N}$ such that $\hat{\varphi}_k > 0$ on K. Then we have

$$\hat{f}_n = \hat{f}_n \frac{\hat{\varphi}_k}{\hat{\varphi}_k} = \frac{\widehat{f_n * \varphi_k}}{\hat{\varphi}_k} = \frac{\widehat{f_k * \varphi_n}}{\hat{\varphi}_k} = \frac{\hat{f}_k}{\hat{\varphi}_k} \cdot \hat{\varphi}_n.$$

Thus $(\hat{f}_n)$ converges to $\dfrac{\hat{f}_k}{\hat{\varphi}_k}$ on K.

In view of the above lemma, we can define the Fourier transform of $[f_n/\varphi_n]$ as the limit of the sequence $(\hat{f}_n)$. Thus, the Fourier transform of a Boehmian from $\mathscr{B}(L_1, \Delta_c)$ is a continuous function.

The above definition of Fourier transform was given by J. Burzyk (oral communication). Another definition of the Fourier transform was given by D. Nemzer in his doctoral thesis:

Let $f=[f_n/\varphi_n] \in \mathscr{B}(L_1, \Delta_c)$. Then for $z \in \mathbf{C}$ we put $\hat{f}(z) = \dfrac{\hat{f}_{n_0}(z)}{\hat{\varphi}_{n_0}(z)}$, where n_0 is the least positive integer such that $\hat{\varphi}_{n_0}(z) \neq 0$.

It can be easily proved that the definition is correct and is equivalent to Burzyk's definition.

5. Boehmians and other generalized functions. In Section 4 it was shown that the space of distributions can be identified with a subspace of $\mathscr{B}(C, \Delta_c^+)$. In this section we are going to discuss connections between Boehmians and other types of generalized functions. The following theorem gives conditions for a subspace of $\mathscr{D}$, under which its dual can be embedded into $\mathscr{B}(C, \Delta)$ (where Δ is any family of delta sequences).

THEOREM 5.1. *Let $\mathcal{F} \subset \mathcal{D}$ be a locally convex space such that*

(14) *if* $\varphi, \psi \in \mathcal{F}$, *then* $\varphi * \psi \in \mathcal{F}$ *and* $\tau_x \varphi \in \mathcal{F}$ *for each* $x \in R^m$ $((\tau_x \varphi)(y) = \varphi(y-x))$;

(15) $T_\varphi(x) = \tau_x \varphi$ *is a continuous mapping from* R^m *to* $\mathcal{F}$ *for each* $\varphi \in \mathcal{F}$;

(16) $\mathcal{F}$ *is a countable inductive limit of Fréchet spaces.*

If $\mathcal{F}^N \cap \Delta \neq \emptyset$ *for some family of delta sequences* Δ, *then the space* $\mathcal{F}'$ *of all continuous linear functionals on* $\mathcal{F}$ *is isomorphic to a subspace of* $\mathcal{B}(C, \Delta)$.

PROOF. Let f be a continuous linear functional on $\mathcal{F}$. For any $\varphi \in \mathcal{F}$, by $f * \varphi$ we denote the function defined by the equation $(f*\varphi)(x) = f(\tau_{-x}\varphi)$. From (15) it follows that $f * \varphi$ is a continuous function $(f * \varphi \in C)$.

Let $(\delta_n) \in \mathcal{F}^N \cap \Delta$. Then the mapping

$$I(f) = [f * \delta_n / \delta_n] \tag{17}$$

defines an isomorphism of $\mathcal{F}'$ into $\mathcal{B}(C, \Delta)$. To prove that I is an isomorphism we have to show that $[f * \delta_n / \delta_n]$ represents a Boehmian (for each $f \in \mathcal{F}'$) and that I is linear and injective.

To prove that $[f * \delta_n / \delta_n]$ represents a Boehmian it is enough to show that

$$f * (\varphi * \psi) = (f * \varphi) * \psi \quad \text{for each } f \in \mathcal{F}' \text{ and each } \varphi, \psi \in \mathcal{F}. \tag{18}$$

By the equation $v(x) = \psi(x) \tau_x \varphi$ we define a continuous mapping from R^m to $\mathcal{F}$ (continuity of v follows from (15)) which is zero outside a compact subset of R^m (because the support of ψ is compact). Moreover, the convex hull of $v(R^m)$ has compact closure in $\mathcal{F}$. Hence, the equality

$$(\varphi * \psi)(x) = \int \psi(y) (\tau_y \varphi)(x) \, dy$$

may be written as an $\mathcal{F}$-valued integral

$$\varphi * \psi = \int_K \psi(y) (\tau_y \varphi) \, dy \quad (\text{where } K = \operatorname{supp} \psi)$$

and by Theorem 3.27 [15, p. 74] we have

$$(f * (\varphi * \psi))(0) = f(\varphi * \psi) = \int_K \psi(y) f(\tau_y \varphi) \, dy = \int_K \psi(y) (f * \varphi)(-y) \, dy =$$

$$= ((f * \varphi) * \psi)(0).$$

Since $(f * \varphi)(x) = (f * \tau_{-x} \varphi)(0)$, equality (18) follows.

Linearity of I is obvious. To prove that I is injective it suffices to note that $f * \delta_n = g * \delta_n$ implies $f * \delta_n * \varphi = g * \delta_n * \varphi$, and hence $(f * \varphi)(0) = (g * \varphi)(0)$ for each $\varphi \in \mathcal{F}$. Therefore $f = g$, which completes the proof.

REMARK. In the above theorem no connection between the topology of $\mathcal{F}$ and the topology of $\mathcal{D}$ is assumed.

In particular, from Theorem 5.1 it follows that every space of Roumieu ultradistributions (and hence also equivalent ultradisributions like Beurling ultradistributions or ω-ultradistributions; see [5]) can be embedded into $\mathcal{B}(C, \Delta_c^+)$.

Theorem 5.1 gives conditions under which functionals defined on a space of test-functions can be identified with Boehmians. On the other hand, we can prove that each Boehmian from $\mathscr{B}(C, \Delta_c)$ can be obtained as a functional on a suitable space of test-functions. Namely, we have the following

THEOREM 5.2. *For each $f\in\mathscr{B}(C, \Delta_c)$ there is a space of test-functions $\mathscr{D}_f$ such that $\mathscr{D}'_f$ is isomorphic to a subspace of $\mathscr{B}(C, \Delta_c)$ containing f.*

PROOF. Let $f\in\mathscr{B}(C, \Delta_c)$ and

$$\mathscr{D}_f = \{\varphi\in\mathscr{D}\colon\ f*\varphi\in C\}.$$

Clearly $\mathscr{D}_f$ is a linear space. We define a locally convex topology on $\mathscr{D}_f$ by the family of seminorms consisting of all seminorms which define the usual topology of $\mathscr{D}$ and the following sequence of seminorms

$$\|\varphi\|_n = \sup\,\{|(f*\varphi)(x)|\colon\ x\in K_n\}.$$

We shall prove that conditions (14)—(16) are satisfied for $\mathscr{F}=\mathscr{D}_f$. The first part of (14) is obvious. The second part follows from the equality $f*\tau_x\varphi=\tau_x(f*\varphi)$, which holds for each $\varphi\in\mathscr{D}_f$.

To prove (15) assume that $x_n\to x$ in R^m. If $\varphi\in\mathscr{D}_f$, then the sequence $(\tau_{x_n}\varphi)$ converges to $\tau_x\varphi$ in $\mathscr{D}$. Moreover, $f*\tau_{x_n}\varphi=\tau_{x_n}(f*\varphi)\xrightarrow{c}\tau_x(f*\varphi)=f*\tau_x\varphi$.

Since $\mathscr{D}_f$ is a countable strict inductive limit of metric spaces, to prove completeness of $\mathscr{D}_f$ it suffices to show that each Cauchy sequence is convergent. Thus, assume that (φ_n) is a Cauchy sequence in $\mathscr{D}_f$. Then, in particular, (φ_n) is a Cauchy sequence in $\mathscr{D}$ and hence converges to some $\varphi\in\mathscr{D}$. If $[f_n/\delta_n]$ is a representative of f, then

$$(f*\varphi_n)*\delta_k = f_k*\varphi_n \xrightarrow{c} f_k*\varphi = (f*\varphi)*\delta_k, \quad \text{for each } k\in\mathbf{N}.$$

Therefore, there is an increasing sequence of indices k_n such that $\Delta\text{-}\lim f*\varphi_{k_n}= =f*\varphi$. On the other hand, the sequence $(f*\varphi_n)$ is a Cauchy sequence in C and hence it converges to some $g\in C$. Since the convergence in C is stronger than Δ-convergence, we have also $\Delta\text{-}\lim f*\varphi_n=g$. Hence $f*\varphi=g\in C$, which proves (16).

Since $\mathscr{D}_f^N\cap\Delta_c\neq\emptyset$, form Theorem 5.1 it follows that $\mathscr{D}'_f$ is isomorphic to a subspace of $\mathscr{B}(C, \Delta_c)$. The isomorphism is defined by (17). To complete the proof we have to show that $f\in I(\mathscr{D}'_f)$. In fact, the mapping $f^*(\varphi)=(f*\varphi)(0)$ is a continuous linear functional on $\mathscr{D}_f$. Moreover,

$$(f^* * \delta_n)(x) = f^*(\tau_{-x}\delta_n) = (f*\tau_{-x}\delta_n)(0) = (f*\delta_n)(x)$$

and hence

$$I(f^*) = [f^* * \delta_n/\delta_n] = [f*\delta_n/\delta_n] = f.$$

The proof is complete.

Let us recall a known theorem on distributions. We use the notation from Section 3.

The space $\mathscr{D}'$ is isomorphic to the completion of C with respect to $\mathscr{D}$-convergence.

From Corollary 3.8. we get a general

THEOREM 5.3. *Let $\mathscr{F}$ be a subalgebra of the convolution algebra $\mathscr{D}$. If $\mathscr{F}^N\cap \cap\Delta_c\neq\emptyset$ then the completion of C with respect to $\mathscr{F}$-convergence is isomorphic to $\mathscr{B}_{\mathscr{F}}(C, \Delta_c)$.*

Finally, note that the space $\mathscr{B}(C, \Delta_c)$ is isomorphic to a subspace of the space $\mathscr{M}$ introduced by Á. Száz in [20]. Roughly speaking $\mathscr{M}$ consists of all multipliers on subspaces $\mathscr{D}$ which are not divisors of zero in C. Hence all elements of $\mathscr{B}(C, \Delta_c)$ as multipliers on subsets of $\mathscr{D}$ are included in $\mathscr{M}$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106
U.S.A.

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INTERPOLATION BY ALMOST QUARTIC SPLINES

ANJULA SAXENA (Lucknow)

In this paper we define a new class of quartic splines of class C^2 and solve the problem of (0, 2) interpolation by the elements of this class. We also study the possibility of the solution of the problems of (0, 1) and (1, 2) interpolations by the elements of the same class. Error estimates are obtained in each case.

1. Introduction

Let

$$\Delta: \; 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$$

be a partition of the interval $I=[0, 1]$ with $x_{k+1}-x_k=h_k$, $k=0, \ldots, n-1$. Denote by $S^{(3)}_{n,5}$ the class of quintic splines $s(x)$ satisfying the condition that $s(x)\in C^3(I)$ and is quintic in each of the sub-intervals of I. It is known from Meir and Sharma [2] that if $h_k=h=1/n$, and n is odd, then there exists a unique $s(x)\in S^{(3)}_{n,5}$, which is the solution of the (0, 2) interpolation problem:

$$(1.1) \qquad s(x_k) = y_k, \quad s''(x_k) = y''_k, \quad k = 0, \ldots, n, \quad s'(x_0) = x'_0, \quad s'(x_n) = y'_n;$$

where y_k, y''_k, $k=0, \ldots, n$ and y'_0, y'_n are given real numbers.

It is also known (Sallam [3]) that for non-uniform partitions of I, this problem is uniquely solvable by the elements of $S^{(3)}_{n,5}$, provided $h_{k+1}>h_k$.

Our aim here is to seek the solution of the problem (1.1) in some other class of spline functions of order less than the previous one and which yields better error bounds. With this aim, we define a class of almost quartic splines $S^{(2)*}_{n,4}$ and consider the possibility of solving three interpolation problems, namely the (0, 2), (1, 2) and (0, 1) interpolation problems by the elements of $S^{(2)*}_{n,4}$.

In Section 2, we define the class $S^{(2)*}_{n,4}$ and prove three theorems on the existence and uniqueness of the elements of $S^{(2)*}_{n,4}$ which are solutions of these problems. We find that they do not require any mesh restrictions. In Section 3, we obtain the error bounds in each case when $f\in C^4(I)$ and $r=0, \ldots, 3$. We find that the order of approximation is almost the same as obtained by the elements of the class $S^{(3)}_{n,5}$. However, the order is better in the end intervals and the constants are small. We finish the paper by making concluding remarks in Section 4.

2. Existence and uniqueness

DEFINITION. $S^*_{n,4}$ is the class of spline functions $s(x)$ such that

(1) $$s(x)\in C^2(I),$$

(2) $s(x)$ is a quintic in one of the end intervals say in $[x_0, x_1]$ and is a quartic in each $[x_k, x_{k+1}]$, $k=1, \dots, n-1$.

We shall prove the following theorems:

THEOREM 1. *Given Δ and the numbers y_k, y_k'', $k=0, \dots, n$; y_k', $k=0, n$, there exists a unique $s_\Delta(x)\in S^{(2)*}_{n,4}$ such that*

(2.1) $$s_\Delta(x_k) = y_k,\quad s_\Delta''(x_k) = y_k'',\quad k = 0, \dots, n,\quad s_\Delta'(x_k) = y_k',\quad k = 0, n.$$

THEOREM 2. *Given Δ and the numbers y_k', y_k'', $k=0, \dots, n$; y_k, $k=0, n$, there exists a unique $\bar{s}_\Delta(x)\in S^{(2)*}_{n,4}$ such that*

(2.2) $$\bar{s}_\Delta'(x_k) = y_k',\quad \bar{s}_\Delta''(x_k) = y_k'',\quad k = 0, \dots, n,\quad \bar{s}_\Delta(x_k) = y_k,\quad k = 0, n.$$

THEOREM 3. *Given Δ and the numbers y_k, y_k', $k=0, \dots, n$; y_k'', $k=0, n$, there exists a unique $\bar{s}_\Delta(x)\in S^{(2)*}_{n,4}$ such that*

(2.3) $$\hat{s}_\Delta(x_k) = y_k,\quad \hat{s}_\Delta'(x_k) = y_k',\quad k = 0, \dots, n,\quad \hat{s}_\Delta''(x_k) = y_k'',\quad k = 0, n.$$

To prove these theorems we follow the idea of Győrvári [1].

PROOF OF THEOREM 1. If we set

(2.4) $$s_\Delta(x) = \begin{cases} s_0(x) & \text{when } x_0 \leqq x \leqq x_1 \\ s_k(x) & \text{when } x_k \leqq x \leqq x_{k+1},\quad k = 1, \dots, n-2 \\ s_{n-1}(x) & \text{when } x_{n-1} \leqq x \leqq x_n, \end{cases}$$

then

$$s_0(x) = y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}a_{0,3}+\frac{(x-x_0)^4}{4!}a_{0,4}+\frac{(x-x_0)^5}{5!}a_{0,5},$$

(2.5) $$s_k(x) = y_k+(x-x_k)a_{k,1}+\frac{(x-x_k)^2}{2!}y_k''+\frac{(x-x_k)^3}{3!}a_{k,3}+\frac{(x-x_k)^4}{4!}a_{k,4},$$

$$s_{n-1}(x) = y_{n-1}+(x-x_{n-1})a_{n-1,1}+\frac{(x-x_{n-1})^2}{2!}y_{n-1}''+\frac{(x-x_{n-1})^3}{3!}a_{n-1,3}+ \\ +\frac{(x-x_{n-1})^4}{4!}a_{n-1,4};$$

where the coefficients are determined by the interpolatory conditions (2.1) and the continuity requirement that $s_\Delta(x)\in C^2(I)$. If we apply these conditions, we have the following three sets of equations:

$$(2.6)\qquad \begin{cases} y_1 = y_0+h_0y_0'+\dfrac{h_0^2}{2!}y_0''+\dfrac{h_0^3}{3!}a_{0,3}+\dfrac{h_0^4}{4!}a_{0,4}+\dfrac{h_0^5}{5!}a_{0,5}, \\ y_1'' = y_0''+h_0a_{0,3}+\dfrac{h_0^2}{2!}a_{0,4}+\dfrac{h_0^3}{3!}a_{0,5}, \\ a_{1,1} = y_0'+h_0y_0''+\dfrac{h_0^2}{2!}a_{0,3}+\dfrac{h_0^3}{3!}a_{0,4}+\dfrac{h_0^4}{4!}a_{0,5}, \end{cases}$$

$$(2.7)\qquad \begin{cases} y_{k+1} = y_k+h_ka_{k,1}+\dfrac{h_k^2}{2!}y_k''+\dfrac{h_k^3}{3!}a_{k,3}+\dfrac{h_k^4}{4!}a_{k,4}, \\ y_{k+1}'' = y_k''+h_ka_{k,3}+\dfrac{h_k^2}{2!}a_{k,4}, \\ a_{k+1,1} = a_{k,1}+h_ky_k''+\dfrac{h_k^2}{2!}a_{k,3}+\dfrac{h_k^3}{3!}a_{k,4}, \end{cases}$$

$$(2.8)\qquad \begin{cases} y_n = y_{n-1}+h_{n-1}a_{n-1,1}+\dfrac{h_{n-1}^2}{2!}y_{n-1}''+\dfrac{h_{n-1}^3}{3!}a_{n-1,3}+\dfrac{h_{n-1}^4}{4!}a_{n-1,4}, \\ y_n' = a_{n-1,1}+h_{n-1}y_{n-1}''+\dfrac{h_{n-1}^2}{2!}a_{n-1,3}+\dfrac{h_{n-1}^3}{3!}a_{n-1,4}, \\ y_n'' = y_{n-1}''+h_{n-1}a_{n-1,3}+\dfrac{h_{n-1}^2}{2!}a_{n-1,4}. \end{cases}$$

Solving (2.8) we get

$$(2.9)\quad a_{n-1,1} = \frac{2}{h_{n-1}}\left(y_n-y_{n-1}-\frac{h_{n-1}^2}{2}y_{n-1}''\right)-(y_n'-h_{n-1}y_{n-1}'')+\frac{h_{n-1}}{6}(y_n''-y_{n-1}''),$$

(2.10)

$$a_{n-1,3} = \frac{-12}{h_{n-1}^3}\left(y_n-y_{n-1}-\frac{h_{n-1}^2}{2}y_{n-1}''\right)+\frac{12}{h_{n-1}^2}(y_n'-h_{n-1}y_{n-1}'')-\frac{3}{h_{n-1}}(y_n''-y_{n-1}''),$$

(2.11)

$$a_{n-1,4} = \frac{24}{h_{n-1}^4}\left(y_n-y_{n-1}-\frac{h_{n-1}^2}{2}y_{n-1}''\right)-\frac{24}{h_{n-1}^3}(y_n'-h_{n-1}y_{n-1}'')+\frac{8}{h_{n-1}^2}(y_n''-y_{n-1}'').$$

From (2.7) we have

$$(2.12)\qquad a_{k,1}+a_{k+1,1} = \frac{2}{h_k}(y_{k+1}-y_k)+\frac{h_k}{6}(y_{k+1}''-y_k''),$$

$$(2.13)\qquad a_{k,3} = \frac{12}{h_k^3}\left(y_{k+1}-y_k-\frac{h_k^2}{2}y_k''\right)-\frac{1}{h_k}(y_{k+1}''-y_k'')-\frac{12}{h_k^2}a_{k,1},$$

$$(2.14)\qquad a_{k,4} = -\frac{24}{h_k^4}\left(y_{k+1}-y_k-\frac{h_k^2}{2}y_k''\right)+\frac{4}{h_k^2}(y_{k+1}''-y_k'')+\frac{24}{h_k^3}a_{k,1},$$

and from (2.6)

(2.15)

$$a_{0,3} = -\frac{24}{h_0^2}a_{1,1}+\frac{60}{h_0^3}\left(y_1-y_0-h_0y_0'-\frac{h_0^2}{2}y_0''\right)+\frac{24}{h_0^2}(y_0'+h_0y_0'')+\frac{3}{h_0}(y_1''-y_0''),$$

(2.16)

$$a_{0,4} = \frac{168}{h_0^3}a_{1,1}-\frac{360}{h_3^4}\left(y_1-y_0-h_0y_0'-\frac{h_0^2}{2}y_0''\right)-\frac{168}{h_0^3}(y_0'+h_0y_0'')-\frac{24}{h_0^2}(y_1''-y_0''),$$

(2.17)

$$a_{0,5} = -\frac{360}{h_0^4}a_{1,1}+\frac{720}{h_0^5}\left(y_1-y_0-h_0y_0'-\frac{h_0^2}{2}y_0''\right)+\frac{360}{h_0^4}(y_0'+h_0y_0'')+\frac{60}{h_0^3}(y_1''-y_0'').$$

The coefficient matrix of the system of equations (2.9) and (2.12) in unknowns $a_{k,1}$, $k=1, \dots, n-1$ is a non-singular matrix and hence they are uniquely determined and so are, therefore, the coefficients $a_{k,3}$ and $a_{k,4}$.

Since $a_{1,1}$ is already determined, the coefficients $a_{0,k}$, $k=3, 4, 5$ are also uniquely determined. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Here again we express $\bar{s}_\Delta(x)$ in the form (2.4) where we write

$$\bar{s}_0(x) = y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}-y_0''+\frac{(x-x_0)^3}{3!}b_{0,3}+$$

$$+\frac{(x-x_0)^4}{4!}b_{0,4}+\frac{(x-x_0)^5}{5!}b_{0,5},$$

$$\bar{s}_k(x) = b_{k,0}+(x-x_k)y_k'+\frac{(x-x_k)^2}{2!}y_k''+\frac{(x-x_k)^3}{3!}b_{k,3}+\frac{(x-x_k)^4}{4!}b_{k,4},$$

$$\bar{s}_{n-1}(x) = b_{n-1,0}+(x-x_{n-1})y_{n-1}'+\frac{(x-x_{n-1})^2}{2!}y_{n-1}''+$$

$$+\frac{(x-x_{n-1})^3}{3!}b_{n-1,3}+\frac{(x-x_{n-1})^4}{4!}b_{n-4,4}.$$

If we apply the interpolatory conditions (2.2) and the continuity requirement we get a system of equations involving the coefficients to be determined. On solving these equations we obtain

(2.18)

$$b_{n-1,0}=\left(y_n-h_{n-1}y'_{n-1}-\frac{h_{n-1}^2}{2}y''_{n-1}\right)-\frac{h_{n-1}}{2}(y'_n-y'_{n-1}-h_{n-1}y''_{n-1})+\frac{h_{n-1}^2}{12}(y''_n-y''_{n-1}),$$

$$b_{n-1,3}=\frac{6}{h_{n-1}^2}(y'_n-y'_{n-1}-h_{n-1}y''_{n-1})-\frac{2}{h_{n-1}}(y''_n-y''_{n-1}), \tag{2.19}$$

$$b_{n-1,4}=-\frac{12}{h_{n-1}^3}(y'_n-y'_{n-1}-h_{n-1}y''_{n-1})+\frac{6}{h_{n-1}^2}(y''_n-y''_{n-1}), \tag{2.20}$$

$$b_{k,3}=\frac{6}{h_k^2}(y'_{k+1}-y'_k-h_k y''_k)-\frac{2}{h_k}(y''_{k+1}-y''_k), \tag{2.21}$$

$$b_{k,4}=-\frac{12}{h_k^3}(y'_{k+1}-y'_k-h_k y''_k)+\frac{6}{h_k^2}(y''_{k+1}-y''_k), \tag{2.22}$$

$$b_{k+1,0}-b_{k,0}=h_k y'_k+\frac{h_k^2}{2}y''_k+\frac{h_k}{2}(y'_{k+1}-y'_k-h_k y''_k)-\frac{h_k^2}{12}(y''_{k+1}-y''_k), \tag{2.23}$$

$$b_{0,3}=\frac{60}{h_0^3}b_{1,0}-\frac{60}{h_0^3}\left(y_0+h_0y'_0+\frac{h_0^2}{2}y''_0\right)-\frac{24}{h_0^2}(y'_1-y'_0-h_0y''_0)+\frac{3}{h_0}(y''_1-y''_0), \tag{2.24}$$

$$b_{0,4}=-\frac{360}{h_0^4}b_{1,0}+\frac{360}{h_0^4}\left(y_0+h_0y'_0+\frac{h_0^2}{2}y''_0\right)+ \tag{2.25}$$

$$+\frac{168}{h_0^3}(y'_1-y'_0-h_0y''_0)-\frac{24}{h_0^2}(y''_1-y''_0),$$

$$b_{0,5}=\frac{720}{h_0^5}b_{1,0}-\frac{720}{h_0^5}\left(y_0+h_0y'_0+\frac{h_0^2}{2}y''_0\right)+ \tag{2.26}$$

$$+\frac{360}{h_0^4}(y'_1-y'_0-h_0y''_0)+\frac{60}{h_0^3}(y''_1-y''_0).$$

By the arguments of Theorem 1, we see that these coefficients are uniquely determined. So, Theorem 2 is proved.

PROOF OF THEOREM 3. We again express $\hat{s}_\Delta(x)$ in the form (2.4) and write

$$\hat{s}_0(x)=y_0+(x-x_0)y'_0+\frac{(x-x_0)^2}{2!}y''_0+\frac{(x-x_0)^3}{3!}C_{0,3}+\frac{(x-x_0)^4}{4!}C_{0,4}+\frac{(x-x_0)^5}{5!}C_{0,5},$$

$$\hat{s}_k(x)=y_k+(x-x_k)y'_k+\frac{(x-x_k)^2}{2!}C_{k,2}+\frac{(x-x_k)^3}{3!}C_{k,3}+\frac{(x-x_k)^4}{4!}C_{k,4},$$

$$\hat{s}_{n-1}(x)=y_{n-1}+(x-x_{n-1})y'_{n-1}+\frac{(x-x_{n-1})^2}{2!}C_{n-1,2}+$$

$$+\frac{(x-x_{n-1})^3}{3!}C_{n-1,3}+\frac{(x-x_{n-1})^4}{4!}C_{n-1,4}.$$

Similarly to the procedures in Theorems 1 and 2, we obtain:

$$(2.27)\qquad C_{n-1,2} = y_n'' + \frac{12}{h_{n-1}^2}(y_n - y_{n-1} - h_{n-1}y_{n-1}') - \frac{6}{h_{n-1}}(y_n' - y_{n-1}'),$$

$$(2.28)\qquad C_{n-1,3} = -\frac{6}{h_{n-1}}y_n'' - \frac{48}{h_{n-1}^3}(y_n - y_{n-1} - h_{n-1}y_{n-1}') + \frac{30}{h_{n-1}^2}(y_n' - y_{n-1}'),$$

$$(2.29)\qquad C_{n-1,4} = \frac{12}{h_{n-1}^2}y_n'' + \frac{72}{h_{n-1}^4}(y_n - y_{n-1} - h_{n-1}y_{n-1}') - \frac{48}{h_{n-1}^3}(y_n' - y_{n-1}');$$

$$(2.30)\qquad C_{k,3} = -\frac{6}{h_k}C_{k,2} + \frac{24}{h_k^3}(y_{k1+} - y_k - h_k y_k') - \frac{6}{h_k^2}(y_{k+1}' - y_k'),$$

$$(2.31)\qquad C_{k,4} = \frac{12}{h_k^2}C_{k,2} - \frac{72}{h_k^4}(y_{k+1} - y_k - h_k y_k') + \frac{24}{h_k^3}(y_{k+1}'y - y_k'),$$

$$(2.32)\qquad C_{k+1,2} - C_{k,2} = -\frac{12}{h_k^2}(y_{k+1} - y_k - h_k y_k') + \frac{6}{h_k}(y_{k+1}' - y_k');$$

$$(2.33)\qquad C_{0,3} = \frac{3}{h_0}(C_{1,2} - y_0'') + \frac{60}{h_0^3}\left(y_1 - y_0 - h_0 y_0' - \frac{h_0^2}{2}y_0''\right) - \frac{24}{h_0^2}(y_1' - y_0' - h_0 y_0''),$$

$$(2.34)\qquad C_{0,4} = -\frac{24}{h_0^2}(C_{1,2} - y_0'') - \frac{360}{h_0^4}\left(y_1 - y_0 - h_0 y_0' - \frac{h_0^2}{2}y_0''\right) + \frac{168}{h_0^3}(y_1' - y_0' - h_0 y_0''),$$

(2.35)

$$C_{0,5} = \frac{60}{h_0^3}(C_{1,2} - y_0'') + \frac{720}{h_0^5}\left(y_1 - y_0 - h_0 y_0' - \frac{h_0^2}{2}y_0''\right) - \frac{360}{h_0^4}(y_1' - y_0' - h_0 y_0'');$$

from which we see that the coefficients are determined uniquely.

3. Error bounds

In this section we shall obtain error bounds of the spline functions $s_\Delta(x)$, $\bar{s}_\Delta(x)$ and $\hat{s}_\Delta(x)$ considered in Section 2 when they are interpolant of a certain function $f(x)$.

We shall first prove

THEOREM 4. *Let Δ be the uniform partition of I with $h_k = h = 1/n$ and $f \in C^4(I)$. Then for the unique spline $s_\Delta(x)$ of Theorem 1 with $y_k = f(x_k)$, $y_k'' = f''(x_k)$, $k = 0, \ldots, n$; $y_0' = f(x_0)$, $y_n' = f'(x_n)$, we have*

$$(3.1)\qquad |s_\Delta^{(r)}(x) - f^{(r)}(x)| \leqq K_1 h^{3-r}\omega_4(h), \quad r = 0, \ldots, 3,$$

where

$$K_1 = \begin{cases} 62 & \textit{when} \quad x \in [x_0, x_1] \\ 6 & \textit{when} \quad x \in [x_k, x_{k+1}] \\ 7h & \textit{when} \quad x \in [x_{n-1}, x_n]; \end{cases}$$

and $\omega_4(h)$ is the modulus of continuity of $f^{(4)}$. *

For the proof of this theorem, we shall need the following

LEMMA 1. *Let $f \in C^4(I)$. Then*

$$e_{k,1} < \frac{(n-k)}{6} h^3 \omega_4(h), \quad k = 1, \dots, n-1,$$

where

$$e_{k,1} = a_{k,1} - y'_k. \tag{3.2}$$

PROOF. If $f \in C^4(I)$, then from (2.12) on using Taylor's formula, we have

$$e_{k,1} + e_{k+1,1} = (a_{k,1} - y'_k) + (a_{k+1,1} - y'_{k+1}) = \frac{h^3}{12} [f^{(4)}(\xi_k) - 2f^{(4)}(\eta_k) + f^{(4)}(\zeta_k)],$$

$$x_k < \xi_k, \eta_k, \zeta_k < x_{k+1}, \quad k = 1, \dots, n-2;$$

and from (2.9)

$$a_{n-1,1} - y'_{n-1} = \frac{h^3}{12} [f^{(4)}(\xi_{n-1}) - 2f^{(4)}(\eta_{n-1}) + f^{(4)}(\zeta_{n-1})], \tag{3.3}$$

$$x_{n-1} < \xi_{n-1}, \quad \eta_{n-1}, \quad \zeta_{n-1} < x_n.$$

We easily see that the system of equations (3.2) and (3.3) in the unknowns $e_{k,1}$, $k=1, \dots, n-1$ has a unique solution:

$$e_{k,1} = d_k - d_{k+1} + d_{k+2} - \dots + (-1)^{n-1-k} d_{n-1},$$

where

$$d_k = \frac{h^3}{12} [f^{(4)}(\xi_k) - 2f^{(4)}(\eta_k) + f^{(4)}(\zeta_k)].$$

It is clear that $d_k \leqq \frac{h^3}{6} \omega_4(h)$. Hence,

$$e_{k,1} \leqq \frac{(n-k)}{6} h^3 \omega_4(h), \quad k = 1, \dots, n-1.$$

PROOF OF THEOREM 4. Let $x \in [x_k, x_{k+1}]$, $k=1, \dots, n-2$. From (2.5) we have

$$s'''_k(x) = a_{k,3} + (x - x_k) a_{k,4}$$

and

$$f'''(x) = y''''_k + (x - x_k) f^{(4)}(\alpha_k), \quad x_k < \alpha_k < x.$$

* From now on Δ denotes the uniform partition of I with $h_k = h = 1/n$, and $y_k = f(x_k)$, $y'_k = f'(x_k)$ etc., $k=1, \dots, n$.

Then,

$$|s_k'''(x)-f'''(x)| \leqq |a_{k,3}-y_k'''|+(x-x_k)|a_{k,4}-f^{(4)}(\alpha_k)|.$$

From (2.13) and (2.14) by Taylor's expansion we have

$$a_{k,3}-y_k''' = \frac{h}{2}[f^{(4)}(\xi_k)-f^{(4)}(\zeta_k)]-\frac{12}{h^2}e_{k,1},$$

and

$$a_{k,4}-f^{(4)}(\alpha_k) = \frac{24}{h^3}e_{k,1}+[2f^{(4)}(\xi_k)-f^{(4)}(\zeta_k)-f^{(4)}(\alpha_k)]$$

from which owing to Lemma 1, we get

$$|a_{k,3}-y_k'''| \leqq \frac{h}{2}[4(n-k)+1]\omega_4(h) \leqq 2\omega_4(h)$$

and

$$(x-x_k)|a_{k,4}-f^{(4)}(\alpha_k)| \leqq h[4(n-k)+2]\omega_4(h) \leqq 4\omega_4(h).$$

Thus

$$|s_k'''(x)-f'''(x)| \leqq 6\omega_4(h).$$

Set

$$g(x) := s_k''(x)-f''(x).$$

Then by (2.1), $g(x_k)=g(x_{k+1})=0$ and so by Rolle's theorem, there exists $\beta_k\,(x_k<\beta_k<x_{k+1})$ such that

$$g'(\beta_k) = s_k'''(\beta_k)-f'''(\beta_k) = 0,$$

from which we obtain

$$|s_k''(x)-f''(x)| = \left|\int_{\beta_k}^{x}\{s_k'''(t)-f'''(t)\}\,dt\right| \leqq \int_{\beta_k}^{x}|s_k'''(t)-f'''(t)|\,dt \leqq 6h\omega_4(h).$$

Carrying on similar arguments we easily see that

$$|s_k^{(r)}(x)-f^{(r)}(x)| \leqq 6h^{3-r}\omega_4(h), \quad r = 0, \dots, 3.$$

For $x_0\leqq x\leqq x_1$ and $x_{n-1}\leqq x\leqq x_n$, we have from (2.5)

$$|s_0'''(x)-f'''(x)| \leqq |a_{0,3}-y_0'''|+h|a_{0,4}-f^{(4)}(r_0)|+\frac{h^2}{2}|a_{0,5}|,$$

and

$$|s_{n-1}'''(x)-f'''(x)| \leqq |a_{n-1,3}-y_{n-1}'''|+h|a_{n-1,4}-f^{(4)}(r_{n-1})|,$$

where $x_0<r_0<x$, $x_{n-1}<r_{n-1}<x_r$.

From (2.10) and (2.11) we get

$$|a_{n-1,3}-y_{n-1}'''| \leqq 2h\omega_4(h) \quad \text{and} \quad |a_{n-1,4}-f^{(4)}(r_{n-1})| \leqq 5\omega_4(h)$$

so that

$$|s_{n-1}'''(x)-f'''(x)| \leqq 7h\omega_4(h).$$

Similarly by (2.15), (2.16), (2.17) and Lemma 1, we have

$$|a_{0,3}-y_0'''| \leqq 4h\omega_4(h)+\frac{24}{h^2}e_{1,1} \leqq 4\omega_4(h),$$

$$|a_{0,4}-f^{(4)}(r_0)| \leqq 28\omega_4(h)+\frac{168}{h^3}e_{1,1} \leqq 28\omega_4(h)$$

and

$$|a_{0,5}| \leqq \frac{60}{h}\omega_4(h)+\frac{360}{h^4}e_{1,1} \leqq \frac{60}{h}\omega_4(h).$$

Hence

$$|s_0'''(x)-f'''(x)| \leqq 62\omega_4(h).$$

By the method of successive integration we obtain

$$|s_0^{(r)}(x)-f^{(r)}(x)| \leqq 62h^{3-r}\omega_4(h)$$

and

$$|s_{n-1}^{(r)}(x)-f^{(r)}(x)| \leqq 7h^{4-r}\omega_4(h),$$

valid for $r=0, \ldots, 3$.

Theorems 5 and 6 below give the error bounds for the spline functions $\bar{s}_\Delta(x)$ and $\hat{s}_\Delta(x)$ of Theorems 2 and 3 respectively. We omit the proofs as the same can be carried out on the pattern of Theorem 4.

THEOREM 5. *Let $f\in C^4(I)$ Then for the unique spline $\bar{s}_\Delta(x)$ of Theorem* 2, *we have*

$$|\bar{s}_\Delta^{(r)}(x)-f^{(r)}(x)| \leqq K_2h^{3-r}\omega_4(h), \quad r=0, \ldots, 3, \tag{3.2}$$

where

$$K_2 = \begin{cases} 65 & \text{when} \quad x\in[x_0, x_1] \\ 4h & \text{when} \quad x\in[x_k, x_{k+1}], \quad k=1, \ldots, n-1. \end{cases}$$

THEOREM 6. *Let $f\in C^4(I)$. Then for the unique spline $\hat{s}_\Delta(x)$ of Theorem* 3, *we have*

$$|\hat{s}_\Delta^{(r)}(x)-f^{(r)}(x)| \leqq K_3h^{3-r}\omega_4(h), \quad r=0, \ldots, 3, \tag{3.3}$$

where

$$K_3 = \begin{cases} 62 & \text{when} \quad x\in[x_0, x_1] \\ 18 & \text{when} \quad x\in[x_k, x_{k+1}], \quad k=1, \ldots, n-2 \\ 14h & \text{when} \quad x\in[x_{n-1}, x_n]. \end{cases}$$

4. Conclusion

We find from our study that the spline interpolants s_Δ and $\hat{s}_\Delta$ give the same order of approximation to any function $f\in C^4(I)$. The order is better in the end interval $[x_{n-1}, x_n]$. Even the interpolant $\bar{s}_\Delta$, where the value of the function is not prescribed at the knots, yields the same order of approximation which is better in the whole interval I except in $[x_0, x_1]$. Thus $\bar{s}_\Delta$ may be preferred to s_Δ and $\hat{s}_\Delta$ in problems where the function values are not known.

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DEPARTMENT OF MATHEMATICS AND ASTRONOMY
LUCKNOW UNIVERSITY, LUCKNOW—226 007,
INDIA

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FOURIER TRANSFORMATIONS OF FUNCTIONS WITH SYMMETRICAL DIFFERENCES

M. S. YOUNIS (Irbid)

I. Introduction

E. C. Titchmarsh [1, Theorem 84, p. 115] proved that if $f(x)$ belongs to $L^p(\mathbf{R})$ such that

$$\|f(x+h)-f(x)\|_p = O(h^\alpha) \quad \text{as} \quad h \to 0, \quad 1 < p \leqq 2, \quad 0 < \alpha \leqq 1,$$

then its Fourier transform $\hat{f}(u)$ belongs to $L^\beta(\mathbf{R})$ if

$$\frac{p}{p+\alpha p-1} < \beta \leqq p' = \frac{p}{p-1}.$$

The present author see [2] and [3] has extended Ticthmarsh's theorems to higher differences of functions in one and several variables.

In [4], R. G. Mamedov introduces the concept of symmetrical differences defined as

$$\varphi_m(f, x, h) = \Delta_h^m f(x) + \Delta_{-h}^m f(x)$$

where $\Delta_h^m f(x)$ denotes the m^{th} difference of $f(x)$ with step h with respect to x.

Our aim in this paper is to show that the conclusions of Titchmarsh's theorems are still the same if we employ the difference $\varphi_m(f, x, h)$ instead of $\Delta_h^m f$ or $\Delta_{-h}^m f$.

2. Definitions and notations

Here we adhere to the definitions and notations commonly used in the liter ature.

The Fourier transform of $f(x) \in L^p(\mathbf{R})$, $1 \leqq p \leqq 2$ is defined as

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-iux} f(x)\, dx \quad (u \in \mathbf{R}).$$

The Lipschitz class in $L^p(\mathbf{R})$ is the collection of functions $f(x)$ for which

$$\|f(x+h)-f(x)\|_p = O(h^\alpha)$$

$0 < \alpha \leqq 1$, as $h \to 0$ holds, where $\|\,.\,\|_p$ is the usual L^p norm.

The m^{th} difference of $f(x)$ with step h is denoted by

$$\Delta_h^m f(x) = \sum_{i=0}^{m} (-1)^{m-1} \binom{m}{i} f(x+ih),$$

$\Delta_{-h}^{m} f(x)$ with step $(-h)$ is defined similarly. For functions of several variables, we write $f(x)=f(x_1, x_2, ..., x_n)$ $(x\in\mathbf{R}^n)$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(u_1, u_2, ..., u_n) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbf{R}^n} e^{-i(u,x)} f(x)\,dx \quad (u = (u_1, ..., u_n)\in\mathbf{R}^n)$$

where $(u, x)=u_1x_1+u_2x_2+, ..., +u_nx_n$, $\mathbf{R}^n$ is the n-dimensional Euclidean space.

The corresponding definitions and notations for Fourier series can be introduced in a similar manner.

3. Main results

In [2] we proved the following

THEOREM 3.1. *Let* $f(x)\in L^p(\mathbf{R})$, $1<p\leqq 2$, *and let*

$$\|\Delta_h^m f(x)\|_p = O(h^\alpha), \quad 0<\alpha\leqq 1 \quad as \quad h\to 0.$$

Then $\hat{f}(u)$ *belongs to* $L^\beta(\mathbf{R})$ *if*

$$\frac{p}{p+\alpha p-1} < \beta \leqq p' = \frac{p}{p-1}. \tag{2}$$

We now generalize Theorem 3.1 as follows.

THEOREM 3.2. *Let* $f(x)\in L^p(\mathbf{R})$, $1<p\leqq 2$ *such that*

$$\|\varphi_m f\|_p = O(h^\alpha), \quad 0<\alpha\leqq 1, \quad as \quad h\to 0. \tag{3}$$

Then conclusion (2) *holds.*

PROOF. Following the notations in Section 2 we see that the Fourier transform of $\Delta_h^m f$ is equal to $(e^{ihu}-1)^m \hat{f}(u)$ $(u\in\mathbf{R})$, and similarly, the Fourier transform of $\Delta_{-h}^m f$ is equal to $(e^{-ihu}-1)^m \hat{f}(u)$. It is easily seen that

$$(e^{ihu}-1)^m = e^{ihum/2}(-2i)^m \left(\sin\frac{uh}{2}\right)^m$$

and that

$$(e^{-ihu}-1)^m = e^{-ihum/2}(-2i)^m \left(\sin\left(\frac{-uh}{2}\right)\right)^m.$$

This shows that the Fourier transform $\hat{\varphi_m f}$ of $\varphi_m f$ is equal to

$$(-2i)\left[e^{ihum/2}\left(\sin\frac{uh}{2}\right)^m + e^{-ihum/2}\left(\sin\left(\frac{-uh}{2}\right)\right)^m\right]\hat{f}(u) \quad (u\in\mathbf{R}).$$

Applying the Hausdorff—Young theorem to (3) we obtain

$$\left\|(-2i)^m\left[e^{ihum/2}\left(\sin\frac{uh}{2}\right)^m + e^{-ihum/2}\left[\sin\left(\frac{-uh}{2}\right)\right]^m\right]\hat{f}(u)\right\|_{p'} \leqq \tag{4}$$

$$\leqq M\|\varphi_m f\|_p = O(h^\alpha).$$

Since the exponentials are majorized by their modulus which is 1, the left hand side of (4) is of the order of $\left\|\left(\sin\frac{uh}{2}\right)^m \hat{f}(u)\right\|_{p'}$.

Thus we can write

$$\left\|\left(\sin\frac{uh}{2}\right)^m \hat{f}(u)\right\|_{p'} \leqq M\|\varphi_m f\|_p = O(h^\alpha).$$

This gives

$$\int\left|\left(\sin\frac{uh}{2}\right)^m \hat{f}(u)\right|^{p'} du = O(h^{\alpha p})^{p'/p} = O(h^{\alpha p'})$$

and this gives

$$\int_0^{2/h} |u^m \hat{f}|^{p'}\, du = O[h^{(\alpha-m)p'}].$$

The rest of the proof follows exactly that of Theorem 84 [[1] p. 115] and (2) holds.

REMARK 3.3. As we have pointed out in [2] and [3] the validity of (2) does not depend on the order m of the difference $\Delta_h^m f$. This suggests a further generalization of Theorem 3.2 where $\varphi_m f$ is replaced by a combination of differences of orders $m_1, m_2, \ldots, m_k$ respectively. Thus we can take

$$\varphi_{m_1, m_2, \ldots, m_k}(f, x, h) = (\Delta_{\pm h}^{m_1} \pm \Delta_{\pm h}^{m_2} +, \ldots, \pm \Delta_{\pm h}^{m_k}) f(x). \tag{5}$$

By taking the Fourier transform of the last quantity and applying the Hausdorff—Young theorem, the left hand side of (5) will be replaced in this case by an expression of the form

$$\left\|\left\{\left(\sin\frac{uh}{2}\right)^{m_1} + \left(\sin\frac{uh}{2}\right)^{m_2} + \ldots + \left(\sin\frac{uh}{2}\right)^{m_k}\right\}\hat{f}(u)\right\|_{p'} \leqq \tag{5'}$$

$$\leqq M\|\varphi_{m_1, m_2, \ldots, m_k}(f, \cdot, h)\|_p = O(h^\alpha).$$

Since the orders $m_1, m_2, \ldots, m_k$ do not affect the course of the argument, the hand side of (5)′ can be replaced by

$$\left\|\left(\sin\frac{uh}{2}\right)^{m_j} \hat{f}\right\|_{p'} = O(h^\alpha).$$

Thus

$$\int_0^{2/h} |(uh)^{m_j} \hat{f}|^{p'}\, du = O(h^{\alpha p'}), \quad \int_0^{2/h} |u^{m_j} \hat{f}|^{p'}\, du = O(h^{(\alpha-m)p'}),$$

and the same conclusion (2) holds.

In the one dimensional case, still a further extension is feasible. The differences can be taken with different steps $h_1, h_2, \ldots, h_k$. Thus we take

$$\varphi_{m_1, m_2 \ldots m_k}(f, x, h_1, \ldots, h_k) = [\Delta_{\pm h_1}^{m_1} \pm \Delta_{\pm h_2}^{m_2}, \ldots, \pm \Delta_{\pm h_k}^{m_k}] f(x),$$

and consequently

$$\|\varphi_{m_1, \ldots m_k}(f, \cdot, h_1, \ldots, h_k)\|_p = O(h_1^{\alpha_1} + h_2^{\alpha_2} +, \ldots + h_k^{\alpha_k})$$

where $0 < \alpha_1, \alpha_2, \ldots, \alpha_k \leqq 1$.

Here again we have proved in [2] that the final result does not depend on the h's since they all go to zero, but it certainly depends on the α's, and in this case the modified conclusion (2) states that $\hat{f}$ belongs to the intersection of the spaces L^{β_i} where

$$\frac{p}{p+\alpha_i p-1} < \beta_i \leqq p' = \frac{p}{p-1}$$

or in other words, $\hat{f}$ belongs to L^β for $\beta = \max(\beta_1, \ldots, \beta_k)$ which means that

$$\frac{p}{p+\alpha p-1} < \beta \leqq p'$$

for $\alpha = \min(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

4. The special case $p=2$ and $0<\alpha<1$

In this section we try to examine the conclusion of Titchmarsh's Theorem 85 [[1], p. 117]] which was also generalized to higher differences in the following manner.

THEOREM 4.1. *Let $f(x) \in L^2(\mathbf{R})$. Then the conditions*

$$\|\Delta_h^m f\|_2 = O(h^\alpha), \quad 0 < \alpha < 1 \quad as \quad h \to 0, \tag{6}$$

$$\Big[\int_{\infty}^{-X} + \int_X^{\infty} |\hat{f}(u)|^2 \, du\Big] = O(X^{-2\alpha}) \quad as \quad X \to \infty \tag{7}$$

are equivalent.

Here again the order m does not play any crucial role in the proof and in the conclusion.

If we replace $\Delta_h^m f$ in (6) by the quantities

$$\varphi_m f = (\Delta_h^m + \Delta_{-h}^m) f,$$

$$\varphi_{m_1, m_2, \ldots, m_k}(f, x, h) = [\Delta_{\pm h}^{m_1} + \Delta_{\pm h}^{m_2} \pm \ldots \pm \Delta_{\pm h}^{m_k}] f(x)$$

or even by the more general condition

$$\|[\Delta_{\pm h_1}^{m_1} \pm \Delta_{\pm h_2}^{m_2} \pm \ldots \pm \Delta_{\pm h_k}^{m_k}] f\|_2 = O[h_1^{\alpha_1} + h_2^{\alpha_2} + \ldots + h_k^{\alpha_k}] \tag{8}$$

then we can arrive at the first part of the result i.e (8) implies (7) or it implies that

$$\Big[\int_{\infty}^{-X} + \int_X^{\infty}\Big] |\hat{f}|^2 \, du = O[X^{-2\alpha}]$$

as $X\to\infty$ where $\alpha=\min(\alpha_1, \alpha_2, \dots, \alpha_k)$. But the last few steps in the proof of the second part become artificial in the sense that we cannot conclude from the fact

$$\int_{-\infty}^{\infty} |\bar{s}\text{in } uh)^m \hat{f}(u)|\, du = O(h^{2\alpha}) \tag{9}$$

that α was a minimum of $\alpha_1, \alpha_2, \dots, \alpha_k$ or even that (9) leads to

$$\int ||\Delta_h^m \pm \Delta_{-h}^m| f|^2\, dx = O(h^{2\alpha})$$

or whether m was an arbitrary choice from among $m_1, m_2, \dots, m_k$ and so forth.

5. Functions on other groups

In the first place, the situation of periodic functions on T leads exactly to the same results with the necessary modifications for the Fourier series. The main point now is that under conditions similar to those in the previous theorems, the sequence of Fourier coefficients $\hat{f}(n)$ belongs to the sequence space l^β or to the intersection $\bigcap_{j=1}^{k} l^{\beta_j}$ where

$$\frac{p}{p+\alpha_j p-1} < \beta_j \leqq p'$$

as the case might be.

For functions on $\mathbf{R}$ and T^n we would like to confine ourselves to functions of two variables in $\mathbf{R}^2$ for the sake of brevity. Along the lines of Theorem 3.2 we state

THEOREM 5.1. *Let* $f(x, y)\in L^p(\mathbf{R}^2)$, $1<p\leqq 2$, *and let*

$$\|(\Delta_{-h}^{m_1}+\Delta_h^{m_2})(\Delta_{-k}^{n_1}\pm\Delta_k^{n_2})f\|_p = O(h^{\alpha_1}k^{\alpha_2}).$$

Then $\hat{f}(u, v)$ *belongs to* $L^{\beta_1}\cap L^{\beta_2}$ *where*

$$\frac{p}{p+\alpha_i p-1} < \beta_i \leqq p', \quad i = 1, 2.$$

The proof, as in the previous cases, yields

$$\int\int_{\mathbf{R}^2} |(\sin uh)^{m_1}\pm(\sin uh)^{m_2}](\sin uk)^{n_1}\pm(\sin uk)^{n_2}]\hat{f}(u, v)|^{p'}\, du\, dv =$$
$$= O[h^{\alpha_1 p'} k^{\alpha_2 p'}].$$

For $0<u\leqq 1/h$, $0<v\leqq 1/k$, and working with either m_1 or m_2 for the differences in x_1 and with n_1 or n_2 with respect to y we arrive at the estimates

$$\int_0^{1/h}\int_0^{1/k} |(uh)^m (vk)^n \hat{f}|^{p'}\, du\, dv = O(h^{\alpha_1 p'} k^{\alpha_2 p'}),$$

$$\int_0^{1/h}\int_0^{1/k} |u^m v^n \hat{f}|^{p'}\, du\, dv = O[h^{(\alpha_1-m)p'} k^{(\alpha_2-n)p'}]$$

and the proof gives the desired result.

REMARK 5.2. In view of the analysis of 3.3 we can formulate Theorem 5.1 in a more general setting by supposing that

$$\|(\sum_{j=1}^{L} \Delta_{\pm h_j}^{m_j})(\sum_{i=1}^{n} \Delta_{\pm k_i}^{m_i})f\|_p = O[(\sum_1^L h_j^{\alpha_j})(\sum_1^n k_i^{\gamma_i})], \quad 0<\alpha_j \leqq 1, \quad 0<\gamma_i \leqq 1,$$

and the conclusion is that the Fourier transform $\hat{f}(u, v)$ belongs to $L^{\beta}(\mathbf{R}^2)$ where $L^{\beta}=[\bigcap_j L^{\beta_j}]\cap[\bigcap_i L^{\beta_i}]$.

REMARK 5.3. The analysis in Theorems 5.1 and 5.2 shows us what kind of generalizations one would expect along these lines when dealing with functions on $\mathbf{R}^n$ and on T^n so there is no need to state the theorems in that direction; it would be rather complicated to do so.

In the particular situation when $p=2$ and $\alpha_1, \ldots, \alpha_k$ are less than 1, we can prove the first assertion of the two equivalent conditions for functions in $\mathbf{R}^n$ and T^n by using these generalized forms of differences. As we pointed out earlier, however, to prove the converse would be rather artificial. It is worthy of mentioning at this point that in the simplest situation of $f\in L^2(\mathbf{R}^2)$ or f is in $L^2(T^2)$ we do not have two conditions only, rather, we have to prove the equivalence of four conditions each containing an estimate of the Fourier transform (or the Fourier coefficient in the case of T^2) with respect to certain weights along four different regions of the plane such as

$$\int_X^{\infty}\int_Y^{\infty} |\hat{f}(u, v)|^2\, du\, dv = O[X^{-2\alpha_1} Y^{-2\alpha_2}],$$

$$\int_0^{X}\int_Y^{\infty} |u\cdot\hat{f}(u, v)|^2\, du\, dv = O[X^{-2(1+\alpha_1)} Y^{-2\alpha_2}],$$

$$\int_0^{Y}\int_X^{\infty} |v\cdot\hat{f}(u, v)|^2\, du\, dv = O[X^{-2\alpha} Y^{-2(1+\alpha_2)}],$$

and

$$\int_0^{X}\int_0^{Y} |uv\hat{f}(u, v)|^2 = O[X^{-2(\alpha_1+1)} Y^{-2(\alpha_2+1)}]$$

as X, $Y\to\infty$ with similar estimates for the other parts of the plane.

It is clear that the mere statement of theorems along these lines for functions of three or more variables in $\mathbf{R}^n$ and T^n becomes very exhaustive and complicated, but its general flavour is apperent.

REMARK 5.4. In the foregoing analysis we adhered to the important case $0<\alpha\leqq 1$ for the Lipschitz functions. If, however, $\alpha>0$ then let $\alpha=r+\alpha'$ where r is a positive integer and $0<\alpha'<1$. In this case it is all the same whether we make the assumption that $\|\Delta_h^r f\|_p=O[h^{\alpha}]$ or $\|\Delta_h f^{(r)}\|_p=O[h^{\alpha'}]$.

If we employ these stronger conditions in the previous theorems we will arrive at conclusions that reflect the entry of the orders of differences and hence the orders of the derivatives of $f(x)$ in their influence on the exponent β of the dual space L^{β} which contains the Fourier transform of f. The conclusion reads in this case

$$\frac{p}{p+(r+\alpha')p-1}<\beta\leqq p'.$$

REMARK 5.5. In view of Walker [5] and [6] and in view of the work done by Younis ([3], chapter3) on extending Titchmarsh's theorems to compact metric groups we can assert that by using the general forms of differences in the spirit of Mamedov we are able to extend the previous theorems to Lipschitz functions on compact zero dimensional groups. We refer to Walker [5] for definitions and notations, and we shall merely state some theorems in this directions. Thus we have

THEOREM 5.6. *Let G be a compact Abelian metric zero dimensional group with Γ as its dual group, let*

$$\|\varphi_m f\|_p = O(h^\alpha), \quad 0<\alpha\leqq 1, \quad h\to 0$$

where $\varphi_n f=(\Delta_h^m+\Delta_{-h}^m)f(x)$.

Then the Fourier transform $\hat{f}$ belongs to $L^\beta(\Gamma)$ for

$$\frac{p}{p+\alpha p-1}<\beta\leqq p'.$$

The proof is just an adoptation of the corresponding original case, we can even replace $\varphi_m f(x,h)$ with the more general form $\sum_{j=1}^{k}\Delta_{h_j}^{m_j}f$.

We conclude this paper by hinting that the case $\alpha<1$ and $p=2$ can be settled in one direction only, and that there is a strong possibility that the same analysis can be done for functions defined on compact finite dimensional groups. However this shall be dealt with a forthcoming paper.

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DEPARTMENT OF MATHEMATICS
YARMOUK UNIVERSITY
IRBID
JORDAN

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THE COMPLETELY PRIME RADICAL IN NEAR-RINGS

N. J. GROENEWALD (Port Elizabeth)

§ 1. Introduction

Completely prime ideals have been studied for associative rings by Andrunakievic and Rjabuhin in [2] and also by McCoy in [5]. The purpose of this note is to extend these results to near-rings. We define a completely prime radical and show that it coincides with the upper radical determined by the class of all non-zero near-rings without divisors of zero. We also give an element wise characterization of this radical.

§ 2. Definitions and preliminary results

Throughout N stands for a zero-symmetric right near-ring. For terminology and notation we shall refer to [6]. The ideal generated by $a\in N$, is denoted by $\langle a\rangle$.

An ideal I of N is said to be

(i) Completely prime if $a, b\in N$, $ab\in I$ implies $a\in I$ or $b\in I$.

(ii) Completely semiprime if $a\in N$, $a^2\in I$, implies $a\in I$. It is easy to show that I is completely semiprime if $a^n\in I$, n a positive integer, implies $a\in I$.

Another way of stating the above is that $I\in N$ is completely prime if the factor near-ring N/I has no nonzero divisors of zero. An ideal I is completely prime if N/I has no nonzero nilpotent elements. We call N a completely prime (completely semiprime) near-ring if (0) is a completely prime (completely semiprime) ideal.

LEMMA 2.1. *Let I be a completely semiprime ideal in the near-ring N and let a, b be elements of N. Then each of the following is true.*

(i) *If $ab\in I$ then $ba\in I$.*
(ii) *If $ab\in I$ and $x\in N$ then $axb\in I$.*
(iii) *If $a^nb\in I$ then $ab\in I$.*

PROOF. Similar to that for rings (cf. [6], Lemma 1).

COROLLARY 2.1 (cf. [7], Proposition 9.37). *If N is completely prime then $ab=0\Rightarrow$ $\Rightarrow axb=0$ for every $x\in N$.*

The following lemma is an extension of a result proved by Bell [3] and Plasser [8].

LEMMA 2.2. *The following assertions are equivalent for the ideal I of N.*
(a) *If $a, b, n\in N$ then $ab\in I\Rightarrow anb\in I$.*
(b) *For every $n\in N$ then $(I, n)=\{x\in N: xn\in I\}\lhd N$.*
(c) *For every $S\subseteq N$ then $(I, S)=\{x\in N: xS\subseteq I\}\lhd N$.*

PROOF. The proof is obvious.

Recall that an ideal $I \lhd N$ is prime if for $a, b \in N$ $\langle a \rangle \langle b \rangle \subseteq I$ implies $a \in I$ or $b \in I$.

COROLLARY 2.2. *If a prime ideal P is completely semiprime, it is completely prime.*

PROOF. Let $a, b \in N$ and suppose $ab \in P$. Let $a' \in \langle a \rangle$ and $b' \in \langle b \rangle$ be arbitrary. Since P is completely semiprime (P, b) is an ideal and because $\langle a \rangle \subseteq (P, b)$ it follows that $a'b \in P$. But from Lemma 2.1(i) we have $ba' \in P$, i.e. $b \in (P, a')$. Hence, as above, $b'a' \in P$ from which it follows that $\langle a \rangle \langle b \rangle \subseteq P$. Since P is a prime ideal, $a \in P$ or $b \in P$. Consequently P is a completely prime ideal.

§ 3. The completely prime radical

A subset M of N is called an m-system if for all $a, b \in M$ there exist $a_1 \in \langle a \rangle$ and $b_1 \in \langle b \rangle$ such that $a_1 b_1 \in M$.

LEMMA 3.1. *Let N be a near-ring. If M is an m-system and I a completely semiprime ideal such that $M \cap I = \emptyset$ then there exists an ideal P which is maximal in the set of completely semiprime ideals which contain I and do not intersect M. P is completely prime.*

PROOF. The existence of such an ideal P is an immediate consequence of Zorn's lemma. We proceed to show that P is completely prime. Consider $a \in N$ with $a \notin P$. (P, a) is completely semiprime: Let $t^n \in P, a)$, n a positive integer, i.e. $t^n a \in P$. From Lemma 2.1(iii) we have $t \in (P, a)$. We show $P = (P, a)$ for each $a \notin P$.

(i) Let $a \notin P$, $a \in M$ and suppose $(P, a) \cap M \neq \emptyset$ with, say $b \in (P, a) \cap M$. Since $ba \in P$ it follows from Corollary 2.2 that $\langle b \rangle \langle a \rangle \subseteq P$. Since M is an m-system $\langle b \rangle \langle a \rangle \cap \cap M \neq \emptyset$ and consequently $P \cap M \neq \emptyset$. Contradiction, hence $(P, a) \cap M = \emptyset$, and by the the fact that P is maximal with this property, we have $P = (P, a)$.

(ii) Suppose $a \notin P$, $a \notin M$ and $(P, a) \cap M \neq \emptyset$. If $c \in (P, a) \cap M$ then $ca \in P$ with $c \notin P$ since $M \cap P = \emptyset$. From (i) and Lemma 2.1(i) we have $a \in (P, c) = P$. This contradiction implies $(P, a) = P$.

Finally, we show that P is completely prime: Suppose $pq \in P$ with $q \notin P$. From the above $(P, q) = P$ and $p \in (P, q) = P$. This completes the proof of the lemma.

DEFINITION. If $A \lhd N$, we define the completely prime radical $\mathscr{C}(A)$, of A as the intersection of all the completely prime ideals in N containing A.

THEOREM 3.1. *Let N be a near-ring and A an ideal of N. A is completely semiprime if and only if $A = \mathscr{C}(A)$.*

PROOF. Clearly, if $A = \mathscr{C}(A)$ then A is completely semiprime since the intersection of any number of completely prime ideals is completely semiprime.

Suppose A is completely semiprime and let $\mathscr{S} = \{P_\alpha : A \subseteq P_\alpha;\, P_\alpha$ completely prime ideal in $N\}$. $\mathscr{S} \neq \emptyset$ since $N \in \mathscr{S}$. Clearly $A \subseteq \cap P_\alpha$. For the inclusion in the other direction, let $a \notin A$. Let $M = \{a^n : n$ a positive integer$\}$. Since A is completely semiprime, M is an m-system such that $M \cap A = \emptyset$. From Lemma 3.1 there exists a completely prime ideal P which contains A and do not intersect M. Since $a \notin P$ we have $a \notin \cap P_\alpha$ and therefore $\cap P_\alpha \subseteq P$, completing the proof.

THEOREM 3.2 (cf. Pilz [7], Theorem 9.36). *A near-ring N without nilpotent elements is isomorphic to a subdirect sum of near-rings without proper divisors of zero.*

PROOF. If N has no nonzero nilpotent elements, then (0) is completely semi-prime. The rest follows from Theorem 3.1.

An ideal P is called a minimal completely prime ideal of I if P is minimal in the set of completely prime ideals containing I. In [9] Van der Walt proved that the prime radical of an ideal I coincides with the intersection of all the minimal prime ideals of I. We have the following result for $\mathscr{C}(I)$.

PROPOSITION 3.3. *If I is a completely semi-prime ideal of N, then I is the intersection of all the minimal completely prime ideals of I.*

PROOF. The proposition will follow from Theorem 3.1 as soon as we have proved that any completely prime ideal containing I contains a minimal completely prime ideal of I. This follows from Zorn's lemma applied to $\{P \lhd N | P \supseteq I$ and P completely prime, $\supseteq\}$ and the fact that the intersection of a linearly ordered set of completely prime ideals is completely prime.

From [9] we know that the prime radical $\mathscr{P}(N)$ of the near-ring N consists of those elements $n \in N$ with the property that every m-system which contains n, contains 0 also. We now give a similar characterization of the completely prime radical $\mathscr{C}(N)$. Obviously, if I is an ideal of N, then I is a completely prime ideal if and only if $N-I$ is a multiplicative system, i.e. if $a, b \in N-I$ then $ab \in N-I$. The subset $\{x \in N|$ whenever $x \in G$, G a multiplicative system, then $0 \in G\}$ is contained in $\mathscr{C}(N)$. This inclusion is, in general, strict, for if $\mathscr{C}(N)$ is equal to this set, $\mathscr{C}(N)$ is a nilideal. Indeed, if $x \in \mathscr{C}(N)$, let $G=\{x, x^2, x^3, x^4, \ldots\}$. Then G is a multiplicative system. By the assumed equality, $0 \in G$ follows which implies x is nilpotent. Hence $\mathscr{C}(N)$ is a nilideal. However, by [4] we know that, in general, $\mathscr{C}(N) \neq \mathscr{N}(N)$ the nilradical of the near-ring N. Motivated by [10] and [11], we make the following

DEFINITION. An *mc*-system in N is a pair (G, P) where P is an ideal in N and G is a subset of N such that $G \cap P$ contains no nonzero elements of N and for every $a \in G$, $ab \in G$ for all $b \notin P$.

Clearly, an ideal I of N is completely prime if and only if $(N-I, I)$ is an *mc*-system.

We can now prove

PROPOSITION 3.4. *For any near-ring N, $\mathscr{C}(N)=\{x \in N|$ whenever $x \in G$, (G, P) an mc-system for some ideal P, then $0 \in G\}$.*

PROOF. Let $x \in \mathscr{C}(N)$ and suppose $x \in G$, where (G, P) is an *mc*-system. If $0 \notin G$, then $G \cap P = \emptyset$. By Zorn's lemma, choose an ideal Q of N with $P \subseteq Q$ and Q maximal with respect to $G \cap Q = \emptyset$. Then $x \notin Q$. Hence $x \notin P$. Q is completely prime: Suppose $ab \in Q$ and $a \notin Q$. We show $b \in Q$. Suppose $b \notin Q$. The maximal property of Q implies that $Q + \langle b \rangle$ contains an element $g \in G$. Since $a, b \notin Q$ and $P \subseteq Q$, we have $a, b \notin P$. From the fact that G is an *mc*-system it follows that $ga \in G$. Again, since $b \notin P$ and $ga \in G$ we have $gab \in G$. Hence $gab \in G \cap Q$. But this contradicts $G \cap Q = \emptyset$. Hence $b \in Q$ must hold. Thus Q a completely prime ideal follows.

The converse inclusion is obvious.

§ 4. The generalized nil radical

For the notations about general radical theory in near-rings we refer to [12]. Let $\mathcal{M}$ be a class of completely prime near-rings. Clearly $\mathcal{M}$ is hereditary, i.e. if $A\in\mathcal{M}$ and $I\lhd A$ then $I\in\mathcal{M}$. Let $\mathcal{U}\mathcal{M}$ be the upper radical class determined by $\mathcal{M}$.

$\mathcal{U}\mathcal{M}=\{A: A$ has no nonzero homomorphic image in $\mathcal{M}\}$.
$=\{A$: every nonzero homomorphic image of A has a nonzero divisor of zero$\}$.

As in the case of rings [4] and [2] we call $\mathcal{U}\mathcal{M}$ the generalized nil radical and denote it by $\mathcal{N}_g$.

THEOREM 4.1. $\mathcal{U}\mathcal{M}=\mathcal{M}'=\{A$: *A a near-ring such that* $A=\mathcal{C}(A)\}$.

PROOF. $N\in\mathcal{U}\mathcal{M}\Rightarrow N$ has no nonzero homomorphic image in $\mathcal{M}$ $\Rightarrow N$ has no nonzero completely prime ideals $\Rightarrow\mathcal{C}(N)=N$. Hence $N\in\mathcal{M}'$ from which it follows that $\mathcal{U}\mathcal{M}\subseteq\mathcal{M}'$.

For the inclusion in the other direction, we have $N\in\mathcal{M}'\Rightarrow N=\mathcal{C}(N)\Rightarrow N$ has no nonzero completely prime ideals $\Rightarrow N$ has no nonzero homomorphic image which is completely prime. Hence $N\in\mathcal{U}\mathcal{M}$ and this completes the proof.

Let $\mathcal{D}$ denote the class of all distributilvey generated near-rings. If $\mathcal{M}$ is a class of prime near-rings, Kaarli [5] defined the calss $\mathcal{M}$ to be $\mathcal{D}$-special if $\mathcal{M}\cap\mathcal{D}$ is hereditary with respect to ideals and if $U\lhd S\lhd N\in\mathcal{D}$ and $S/U\in\mathcal{M}$ imply $U\lhd N$ and $N/(U: S)\in\mathcal{M}$, where $(U: S)=\{n\in N: nS\subseteq U\}$. Kaarli proved that if $\mathcal{M}$ is the class of all prime near-rings, then $\mathcal{M}$ is $\mathcal{D}$-special with corresponding radical the lower nil radical.

In what follows, let $\mathcal{M}$ be the class of all completely prime nearrings.

LEMMA 4.2. *If* $A\lhd\cdot N$ *and* $A\in\mathcal{M}$, *then* $N\in\mathcal{M}$, *i.e.* $\mathcal{M}$ *is essentially closed.*

PROOF. It is easy to check that $l(A)=\{n\in N: nA=0\}$ is a two-sided ideal of N. But $(l(A)\cap A)^2\subseteq l(A)\cdot A=0$. Since $l(A)\cap A\lhd A$ and $A\in\mathcal{M}$, we have $l(A)\cap A==(0)$. Since $A\lhd\cdot N$, it follows that $l(A)=0$. Let $x, y\in N$ such that $x, y\neq 0$. Since $l(A)=(0)$, we can find $a, b\in A$ such that xa and $yb\neq 0$. Furthermore, we have $ax\neq 0$ for if $ax=0$, then $xaxa=0$ and since $A\in\mathcal{M}$, it follows that $xa=0$. Now we have $ax\cdot yb\neq 0$. Hence $xy\neq 0$. Thus $N\in\mathcal{M}$.

THEOREM 4.3. *The class* $\mathcal{M}$ *of all completely prime near-rings is* $\mathcal{D}$*-special.*

PROOF. $\mathcal{M}\cap\mathcal{D}$ is clearly heriditary. Furthermore, since the class of all prime near-rings is $\mathcal{D}$-special and since a completely prime near-ring is also a prime near-ring, we have that $U\lhd S\lhd N\in\mathcal{D}$ and $S/U\in\mathcal{M}$ implies $U\lhd N$. We only have to show that $(U: S)$ is a completely prime ideal in N. Let $x\in N$ such that $x^2\in(U: S)$. Suppose $x\notin((U: S)$, i.e. there exists $b\in S$ such that $xb\notin U$. We also have $bx\notin U$, for if $bx\in U$ then $xbxb\in U$ and since $S/U\in\mathcal{M}$, i.e. U completely prime ideal of S, it follows that $xb\in U$.

Let $a\in S$ be arbitrary. Since $x^2\in(U: S)$ we have $x^2a\in U$. Therefore $bx^2a\in U$.

From this and the fact that $bx\notin U$, we have $xa\in U$. Consequently, $xS\subseteq U$, i.e. $x\in(U: S)$. Hence $(U: S)$ is a completely semi-prime ideal. But from Kaarli [5] $(U: S)$ is a prime ideal and therefore Corollary 2.2 implies $(U: S)$ is a completely prime ideal. Hence $N/(U: S)\in\mathcal{M}$.

THEOREM 4.4. *If $\mathcal{M}$ is the class of all completely prime near-rings, then for every $N\in\mathcal{D}$ and $I\lhd N$ we have $\mathcal{N}_g(N)\cap I=\mathcal{N}_g(N)$ and $\mathcal{C}(N)=\mathcal{N}_g(N)$.*

PROOF. We show that the conditions of [1], Theorem 1 are satisfied. This follows from the fact that $\mathcal{M}$ is $\mathcal{D}$-special and Lemma 4.2. From [1], Theorem 1 we have $\mathcal{N}_\varrho$ is hereditary and from [1], Proposition 3 we have $\mathcal{S}\mathcal{U}\mathcal{M}$ hereditary. Hence for every $N\in\mathcal{D}$ and every ideal I of N we have $\mathcal{N}_g(I)=N\cap\mathcal{N}_g(N)$. Let $N\in\mathcal{D}$, we show that $\mathcal{C}(N)=\mathcal{N}_g(N)$. Let $\{T_\alpha\}$ be the set of all ideals T of N for which $N/T_\alpha\in\mathcal{M}$. Then we have $\mathcal{N}_g(N)\subseteq B=\cap T_\alpha$. Let us suppose $\mathcal{N}_g(N)\neq B$. In this case we have $\mathcal{N}_g(B)=$ $=B\cap\mathcal{N}_g(N)=\mathcal{N}_g(N)\neq B$. Hence there exists an ideal $C\lhd B$ with $0\neq B/C\in\mathcal{M}$. From Theorem 4.3 we have $N/(C:B)\in\mathcal{M}$. Consequently $(C:B)\in\{T_\alpha\}$. $(C:B)\cap$ $\cap B=C$ for if $x\in(C:B)\cap B$, then $x\in B$ and $x^2\in C$. Now, since $B/C\in\mathcal{M}$, we have $x\in C$. Hence $(C:B)\cap B\subseteq C$. Inclusion in the other direction is clear. Thus $B=$ $=\cap T_\alpha=B\cap(C:B)=C\neq B$ a contradiction. Hence $B=\mathcal{N}_g(N)=C(N)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PORT ELIZABETH
PORT ELIZABETH
SOUTH AFRICA

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О СРЕДНИХ ДЛЯ ЦЕНТРИРОВАННЫХ СИСТЕМ ФУНКЦИЙ

А. В. БАХШЕЦЯН (Ереван)

1. В работе [1] К. Тандори рассматривал средние для систем стохастически независимых функций. В данной статье переносятся на центрированные системы функций результаты, полученные в [1]. Причём метод доказательств нами также заимствован из работ [1] и [2].

Напомним, что система на [0, 1] интегрируемых по Лебегу функций $\Phi = \{\varphi_k\}$ (конечная или бесконечная) называется центрированной, если для любого n и любого $\Lambda \in \mathscr{F}_n(\Phi)$ ($\mathscr{F}_n(\Phi)$ — минимальная σ-алгебра, по которой измеримы функции $\varphi_1, \varphi_2, \ldots, \varphi_n$) имеем[1]

$$\int_{\Lambda} \varphi_{n+1}(x)\,dx = 0. \tag{1}$$

Кроме того, мы будем предпологать, что

$$\int_0^1 \varphi_1(x)\,dx = 0 \tag{2}$$

и

$$\int_0^1 \varphi_n^2(x)\,dx = 1, \quad n = 1, 2, \ldots. \tag{3}$$

Нетрудно проверить, что при этих условиях система Φ будет ортонормированной.

Пусть $1 \leqq K < \infty$. Через $\Omega = \Omega(\infty)$ обозначим класс всех центрированных систем, удовлетворяющих условиям (2) и (3), а через $\Omega(K)$ — класс центрированных систем, которые кроме (2) и (3) удовлетворяют также условию:[2]

$$|\varphi_n(x)| \leqq K \quad (x \in [0, 1];\ n = 1, 2, \ldots).$$

Очевидно, для любых K_1 и K_2 $(1 < K_1 < K_2 < \infty)$

$$\Omega(1) \subset \Omega(K_1) \subset \Omega(K_2) \subset \Omega. \tag{4}$$

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[1] Такие системы иначе называются мартингальными разностями, а частные суммы рядов по этим системам образуют т.н. мартингальные последовательности.

[2] Из доказательств утверждений будет видно, что ограничение (2) не является существенным: мы принимаем его для удобства выкладок.

Далее, зафиксируем положительную неубываюшую последовательность $\lambda=\{\lambda_n\}$ $(0<\lambda_1\leqq\lambda_2\leqq\ldots)$ с $\lim_{n\to\infty}\lambda_n=\infty$, и для бесконечной системы $\Phi=\{\varphi_k\}_{k=1}^{\infty}$ и числовой последовательности $a=\{a_k\}_{k=1}^{\infty}$ будем рассматривать средние

$$\sigma_n(x, a, \Phi) := \frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k(x) \quad (n = 1, 2, \ldots).$$

Через $M(K)$ $(1\leqq K\leqq\infty)$ обозначим класс всех числовых последовательностей $a=\{a_k\}$, для которых

$$\lim_{n\to\infty}\sigma_n(x, a, \Phi) = 0 \tag{5}$$

п. в. на [0, 1] для любой бесконечной системы $\Phi\in\Omega(K)$. Очевидно, в силу (4), имеем при $1<K_1<K_2<\infty$

$$M(1)\supset M(K_1)\supset M(K_2)\supset M(\infty).$$

Оказывается (как и в случае ортонормированных систем стохастически независимых функций; см. [1]), справедлива следующая

Теорема 1. *Для* $1<K<\infty$

$$M(K) = M(1).$$

Замечание 1. Очевидно, что простейшая из центрированных систем — система Радемахера $r=\{r_k\}_{k=1}^{\infty}$ $(r_k(x):=\operatorname{sign}\sin 2^k\pi x,\ \ k=1, 2, \ldots)$, которой мы часто будем пользоваться, принадлежит $\Omega(1)$. С другой стороны, не трудно убедиться, что система Радемахера по существу (с точнотью до эндоморфизма отрезка [0, 1]) единственный представитель (из числа бесконечных систем) класса $\Omega(1)$, в том смысле, что если некоторая $\Phi=\{\varphi_k\}_{k=1}^{\infty}\in\Omega(1)$, то существует сохраняющее меру отображение T: $[0, 1]\to[0, 1]$ такое, что $\varphi_k(x)=r_k(Tx)$ для любого k (см. напр. [5]).

Учитывая это замечание и в силу закона нуля и единицы, из теоремы 1 получаем следующее

Следствие 1. *Если для некоторого* $K<\infty$ *и для некоторой системы* $\Phi\in\Omega(\in)$ *на множестве положительной меры не выполняется* (5), *то п.в.*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\tau_k(x)\neq 0.$$

Замечание 2. Пусть $\Omega^*(K)$, $M^*(K)(1\leqq K\leqq\infty)$ определяются аналогично соответствующим классам $\Omega(K)$, $M(K)$ с заменой условия центрированности на стохастическую независимость системы $\Phi=\{\varphi_k\}_{k=1}^{\infty}$, а условия (2) — на

$$\int_0^1 \varphi_k(x)\,dx = 0, \quad k = 1, 2, \ldots. \tag{2'}$$

Очевидно, при наличии (2′), имеем: $\Omega^*(K)\subset\Omega(K)$ и, следовательно, $M^*(K)\supset\supset M(K)$. Более того, в силу замечания 1, $M^*(1)=M(1)$ и, следовательно, $M^*(K)=M(K)$.

В работе [1] доказано, что для $1<K<\infty$ $M^*(K)=M^*(1)\neq M^*(\infty)$. Отсюда, учитывая замечание 2, получаем.

Следствие 2. Если при некотором $1\leqq K<\infty$ для некоторой системы $\Psi\in\Omega^*(K)$ $\sigma_n(x, a, \Psi)\to 0$ на множестве положительной меры (и, следовательно, в силу закона нуля и единицы, п.в.), то для любой системы $\Phi\in\bigcup\limits_{K<\infty}\Omega(K)$ п.в. выполняется (5).

Замечание 3. Из известной теоремы о сходимости мартингалов (см. [3], стр. 278) и, учитывая замечание 11 работы [1], получаем, что $a\in M(\infty)$ тогда и только тогда, когда

$$\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2}<\infty, \tag{6}$$

или, что то же самое, $M(\infty)=M^*(\infty)$.[3]

Отсюда в частности следует, что $M(1)\neq M(\infty)$.

Для последовательности $a=\{a_k\}$, для любых K $(1\leqq K\leqq\infty)$ и целых M и N $(1\leqq M\leqq N<\infty)$ определим

$$\|a, K, M, N\| := \sup_{\Phi\in\Omega(K)}\Big\{\int_0^1 \sup_{M\leqq n\leqq N}\sigma_n^2(x, a, \Phi)\,dx\Big\}^{1/2};$$

$$\|a, K, M\| := \lim_{N\to\infty}\|a, K, M, N\| \quad (\leqq\infty);$$

$$\|a, K\| := \|a, K, 1\| \quad (\leqq\infty).$$

Мы докажем также следующее утверждение (ср. [1]; теоремы IV, V, VI).

Теорема 2. *Следующие три условия эквиваленты:*

(I) $a\in M(\infty)$;

(II) $\|a, \infty\|<\infty$;

(III) $\lim\limits_{N\to\infty}\|a, \infty, N\|=0$.

Замечание 4. Теорема 2 не переносится на случай $1\leqq K<\infty$. Точнее, при $1\leqq K<\infty$ из $\lim\limits_{N\to\infty}\|a, K, N\|=0$ следует, что $a\in M(K)$, но из $\|a, K\|<\infty$ не следует, что $a\in M(K)$. Доказательство то же самое, что и в случае стохастически независимых функций (см. [1]; теорема V и замечание III).

Наконец, верна

Теорема 3. *Пусть* $1\leqq K<\infty$ *и* $\lim\limits_{N\to\infty}\|a, K, N\|=\varrho$ $(0<\varrho<\infty)$. *Тогда* $a\notin M(K)$.

Вопрос о том, верно ли утверждение теоремы 3 при $\varrho=\infty$, как и в случае стохастически независимых функций, остается открытым.

[3] Итак, мы имеем, что для любого K $(1\leqq K\leqq\infty)$ $M(K)=M^*(K)$, хотя при $K>1$ класс $\Omega(K)$ существенно шире класса $\Omega^*(K)$.

2. Прежде чем перейти к доказательствам теорем, введём некоторые вспомогательные обозначения и докажем несколько лемм.

Пусть $1 \leqq K < \infty$. Через $\bar{\Omega}(\infty)$ обозначим множество тех систем из $\Omega(\infty)$, которые состоят из функций, принимающих не более, чем счётное число значений,[4] а через $\bar{\Omega}(K)$ — множество систем из $\Omega(K)$, состоящих из простых функций, т.е. функций, принимающих конечное число значений.

Для системы $\Psi = \{\psi_n\}_{n=1}^{\infty}$ и для $\varepsilon > 0$ через $\mathscr{F}_k^{\varepsilon}(\Psi)$ $(k = 1, 2, \dots)$ обозначим минимальную σ-алгебру, содержашую множества:

$$\psi_n^{-1}([m\varepsilon, (m+1)\varepsilon)), \quad m = 0, \pm 1, \pm 2, \dots; \quad n = 1, 2, \dots, k.$$

Далее, пусть f — интегрируемая на $[0, 1]$ функция, $\mathscr{F}$ — некоторая σ-алгебра подмножеств, измеримых по Лебегу. Через $E(f|\mathscr{F})$, как обычно, обозначиваем функцию, интегрируемую на $[0, 1]$, измеримую по $\mathscr{F}$ и такую, что для любого $\Lambda \in \mathscr{F}$

$$\int_{\Lambda} E(f|\mathscr{F}) = \int_{\Lambda} f.$$

И наконец, пусть e — измеримое подмножество $[0, 1]$, $0 < t < 1$. Через $e(t)$ и $\tilde{e}(t)$ обозначим множества:

$$e(t) := e \cap [0, \alpha); \quad \tilde{e}(t) := e \cap [\alpha, 1],$$

где α такое, что $\operatorname{mes} e \cap [0, \alpha) = t \operatorname{mes} e$.

Замечание 5. Очевидно, что если $\Psi \in \bar{\Omega}(K)$ $(1 \leqq K < \infty)$, то для любого $\varepsilon > 0$ и k σ-алгебра $\mathscr{F}_k^{\varepsilon}(\Psi)$ содержит конечное число различных атомов[5]: $e_1, e_2, \dots, \dots, e_n$, причём $\sum_{i=1}^{n} \operatorname{mes} e_i = 1$. Если же $\Psi \in \Omega(\infty)$, то число различных атомов может быть и счётным, но опять же $\sum_{i} \operatorname{mes} e_i = 1$.

Лемма 1. *Пусть* $1 < K \leqq \infty$ *и* $\Psi = \{\psi_k\}_{k=1}^{\infty} \in \Omega(K)$. *Тогда для любой последовательности* $a = \{a_k\}_{k=1}^{\infty}$ *и для любых* $q \in (0, 1)$ *и* $\varepsilon > 0$ *существует система* $\Phi = \{\varphi_k\}_{k=1}^{\infty} \in \bar{\Omega}(K)$ *такая, что для любых натуральных чисел* $L \leqq M \leqq N$ *и для любого* y

$$\operatorname{mes}\left\{x\colon \max_{M \leqq n \leqq N} \frac{1}{\lambda_n} \Big|\sum_{k=L}^{n} a_k \varphi_k(x)\Big| > y\right\} \geqq \tag{7}$$

$$\geqq q \operatorname{mes}\left\{x\colon \max_{M \leqq n \leqq N} \frac{1}{\lambda_n} \Big|\sum_{k=L}^{n} a_k \psi_k(x)\Big| > y + \varepsilon\right\}.$$

Доказательство леммы 1. Основная идея доказательства заключается в приближении функций системы Ψ ступенчатыми функциями.

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[4] Или эквивалентные таким функциям. (И ниже всюду мы это будем предпологать.)

[5] Напомним, что атомом σ-алгебры $\mathscr{F}$ называется всякий элемент $e \in \mathscr{F}$ с $\operatorname{mes} e > 0$ такой, что для любого $\Lambda \in \mathscr{F}$ $\operatorname{mes}(e \cap \Lambda) \cdot \operatorname{mes}(e \triangle \Lambda) = 0$. Атомы e и e_1 считаем различными, если $\operatorname{mes}(e \cap e_1) = 0$.

Предположим сначала, что $1<K<\infty$. Выберем $\delta\in(0,1-q)$ так, чтобы выполнялось неравенство

$$\frac{1-(1-\delta)(1-\delta^2)^2}{\delta} \leqq K^2. \tag{8}$$

Пусть, далее, целочисленная последовательность $\{m_k\}_{k=1}^{\infty}$ такая, что

$$\varepsilon_k := \frac{1}{2^{m_k}} \leqq \min\left\{\frac{\varepsilon\lambda_1}{2^k|a_k|},\ \delta^2\right\}.^{6} \tag{9}$$

Рассмотрим систему простых функций (см. замечание 5)

$$\bar{\Phi} = \{\bar{\varphi}_k := E(\psi_k|\mathscr{F}_k^{\varepsilon_k}(\Psi)\}_{k=1}^{\infty}. \tag{10}$$

Очевидно, в силу определения, система $\bar{\Phi}$ также центрирована и $|\bar{\varphi}_k(x)|\leqq K$ при $x\in[0,1]$, $k=1,2,\dots$.

Имеем для любого $k\geqq 1$[7]

$$\|\bar{\varphi}_k\|_2 \geqq \|\psi_k\|_2-\|\bar{\varphi}_k-\psi_k\|_2 \geqq 1-\varepsilon_k \geqq 1-\delta^2. \tag{11}$$

Далее, определим систему $\Phi=\{\varphi_k\}_{k=1}^{\infty}$:

$$\varphi_k(x) := \begin{cases} \bar{\varphi}_k\left(\dfrac{x}{1-\delta}\right) & \text{при} \quad x\in[0,1-\delta); \\ \beta_k r_k\left(\dfrac{x-1+\delta}{\delta}\right) & \text{при} \quad x\in[1-\delta,1], \quad k=1,2,\dots, \end{cases}$$

где числа β_k определяются из условия: $\|\varphi_k\|_2=1$. Следовательно,

$$1=\int_0^{1-\delta}\bar{\varphi}_k^2\left(\frac{x}{1-\delta}\right)dx+\beta_k^2\delta=(1-\delta)\int_0^1\bar{\varphi}_k^2(x)\,dx+\beta_k^2\delta.$$

Отсюда, в силу (11) и (8)

$$\beta_k^2=\frac{1}{\delta}[1-(1-\delta)\|\bar{\varphi}_k\|_2^2] \leqq \frac{1-(1-\delta)(1-\delta^2)^2}{\delta} \leqq K^2.$$

Получили, что система простых функций Φ равномерно ограничена $\big(|\varphi_k(x)|\leqq K$ для любых $x\in[0,1]$ и $k\geqq 1\big)$. Центрированность системы Φ вытекает из центрированности систем $\bar{\Phi}$ и $\{r_k\}$. Итак, $\Phi\in\bar{\Omega}(K)$.

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[6] В случае $a_k=0$ вместо (9) потребуем лишь: $\varepsilon_k\leqq\delta^2$.

[7] Через $\|\cdot\|_2$, как обычно, обозначиваем норму в $L^2[0,1]$.

Пусть $L \leqq M \leqq N$. Из определения системы Φ получаем

$$\operatorname{mes}\left\{x:\ \max_{M \leqq n \leqq N} \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \varphi_k(x)\right| > y\right\} \geqq$$

$$\geqq \operatorname{mes}\left\{x:\quad 0 \leqq x < 1-\delta,\ \max_{M \leqq n \leqq N} \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \varphi_k(x)\right| > y\right\} =$$

$$= (1-\delta) \operatorname{mes}\left\{x:\ \max_{M \leqq n \leqq N} \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \bar{\varphi}_k(x)\right| > y\right\}.$$

С другой стороны, в силу (9), для любого $n = M, M+1, \ldots, N$

$$\frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \bar{\varphi}_k(x)\right| \geqq \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \psi_k(x)\right| - \frac{1}{\lambda_n} \sum_{k=L}^{N} |a_k|\,|\bar{\varphi}_k(x) - \psi_k(x)| \geqq$$

$$\geqq \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \psi_k(x)\right| - \frac{1}{\lambda_1} \sum_{k=1}^{\infty} |a_k|\,\varepsilon_k \geqq \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \psi_k(x)\right| - \varepsilon.$$

Следовательно,

$$\operatorname{mes}\left\{x:\ \max_{M \leqq n \leqq N} \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \varphi_k(x)\right| > y\right\} \geqq$$

$$\geqq (1-\delta) \operatorname{mes}\left\{x:\ \max_{M \leqq n \leqq N} \frac{1}{\lambda_n}\left|\sum_{k=L}^{n} a_k \psi_k(x)\right| > y+\varepsilon\right\}.$$

Отсюда, учитывая, что $1-\delta > q$, получаем (7).

В случае $K=\infty$ доказательство то же самое, даже более упрощается, так как не нужно следить за равномерной ограниченностью систем $\bar{\Phi}$ и Φ. Лемма 1 полностью доказана.

При помощи этой леммы докажем следующее утверждение.

Лемма 2. *Пусть* $1 \leqq K \leqq \infty$ *и* $1 \leqq M < N < \infty$. *Тогда*

$$\sup_{\Phi \in \bar{\Omega}(K)} \left\{\int_0^1 \max_{M \leqq n \leqq N} \sigma_n^2(x, a, \Phi)\, dx\right\}^{1/2} = \|a, K, M, N\|.$$

Доказательство леммы 2. Для $K=1$ лемма тривиальна, так как $\bar{\Omega}(1) = \Omega(1)$. Пусть $1 < K \leqq \infty$ и q — произвольное число, меньше единицы. По определению $\|a, K, M, N\|$ найдётся система $\Psi \in \Omega(K)$ такая, что

$$\int_0^1 \max_{M \leqq n \leqq N} \sigma_n^2(x, a, \psi)\, dx \geqq q\,\|a, K, M, N\|^2. \tag{12}$$

Применив лемму 1 к системе Ψ при

$$\varepsilon \leqq \min\left(\sqrt{\frac{1-q}{2}},\ \frac{1-q}{4}\left(\int_0^1 \max_{M \leqq n \leqq N} |\sigma_n(x, a, \Psi)|\, dx\right)^{-1}\right) \tag{13}$$

получим систему $\Phi\in\bar\Omega(K)$ такую, что при любом y

(14) $$\operatorname{mes}\{x\colon \max_{M\leq n\leq N}|\sigma_n(x, a, \Phi)|>y\}\geqq q\operatorname{mes}\{x\colon \max_{M\leq n\leq N}|\sigma_n(x, a, \Psi)|>y+\varepsilon\}.$$

Левую часть этого неравенства обозначим через $F_\Phi(y)$, а правую — $qF_\Psi(y+\varepsilon)$. Известно, что

$$\int_0^1 \max_{M\leq n\leq N}\sigma_n^2(x, a, \Phi)\,dx = 2\int_0^\infty yF_\Phi(y)\,dy.$$

Следовательно, в силу (12), (13) и (14)

$$\int_0^1 \max_{M\leq n\leq N}\sigma_n^2(x, a, \Phi)\,dx \geqq 2q\int_0^\infty yF_\Psi(y+\varepsilon)\,dy =$$

$$= 2q\int_\varepsilon^\infty (y-\varepsilon)F_\Psi(y)\,dy = 2q\int_0^\infty yF_\Psi(y)\,dy-2q\int_0^\varepsilon yF_\Psi(y)\,dy-2q\varepsilon\int_\varepsilon^\infty F_\Psi(y)\,dy\geqq$$

$$\geqq q\int_0^1 \max_{M\leq n\leq N}\sigma_n^2(x, a, \Psi)\,dx-2q\frac{\varepsilon^2}{2}-2q\varepsilon\int_0^1\max_{M\leq n\leq N}|\sigma_n(x, a, \Psi)|\,dx\geqq$$

$$\geqq q^2\|a, K, M, N\|^2-(1-q)q.$$

Отсюда, в силу произвольности q, получаем утверждение леммы 2. Лемма 2 доказана.

Лемма 3. *Пусть* $\Phi\in\bar\Omega(\infty)$. *Тогда для любой последовательности* $\{c_k\}_{k=1}^\infty$ *и для любого* N

$$\int_0^1 \max_{n\leq N}\Big(\sum_{k=1}^n c_k\varphi_k(x)\Big)^2 dx \leqq A\sum_{k=1}^N c_k^2$$

где A *— абсолютная постоянная.*

Доказательство этой леммы мы здесь не приводим, так как оно по существу не отличается от доказательства теоремы 1 из работы [4].

Лемма 4. *Пусть* $1<K<\infty$ *и* $\Phi=\{\varphi_k\}_{k=1}^\infty\in\bar\Omega(K)$. *Тогда существует система* $\{\pi_k(x)\}_{k=M}^N$, *ортогональная на любом элементе σ-алгебры* $\mathscr{F}(\Phi)$,[8] *такая, что*

(15) $$|\pi_k(x)|\leqq 2 \quad \textit{для любого} \quad x\in[0, 1] \quad (k=M, M+1, \ldots, N)$$

и

$$\Big\{\frac{\varphi_k(x)}{K}+\pi_k(x)\Big\}_{k=M}^N\in\bar\Omega(1).$$

[8] $\mathscr{F}(\Phi)$ — минимальная σ-алгебра, по которой измеримы все функции системы Φ.

Доказательство леммы 4. Пусть $\{e_i\}_{i=1}^n$ — множество различных атомов σ-алгебры $\mathscr{F}(\Phi)$:

$$\text{(16)}\qquad \operatorname{mes} e_i > 0, \quad i = 1, 2, \dots, n; \quad \sum_{i=1}^{n} \operatorname{mes} e_i = 1.$$

Значение функции $\dfrac{1}{K}\varphi_k$ на множестве e_i обозначим через $\omega_{k,i}(k=M, M+ +1, \dots, N;\ i=1, 2, \dots, n)$. Имеем

$$\text{(17)}\qquad |\omega_{k,i}| \leqq 1, \quad k = M, M+1, \dots, N; \quad i = 1, 2, \dots, n.$$

Далее, положим $t_{k,i}=\dfrac{1}{2}(1+\omega_{k,i})$.

Для любого i $(1\leqq i\leqq n)$ определим индуктивно множества $E_{k,i}$ $(k= =M, M+1, \dots, N)$: $E_{M,i}$ — одноэлементное множество $\{e_i\}$; для $k>M$ $E_{k,i}:=\{e(t_{k-1,i}),\ \tilde{e}(t_{k-1,i}):\ e\in E_{k-1,i}\}$.

Очевидно, для любых k и i $(M\leqq k\leqq N,\ 1\leqq i\leqq н)$

$$\text{(18)}\qquad \bigcup_{e\in E_{k,i}} e = e_i.$$

Положим для $k=M, M+1, \dots, N$

$$\pi_k(x) := \sum_{i=1}^{n} \sum_{e\in E_{k,i}} \left[(1-\omega_{k,i})\chi\big(x, e(t_{k,i})\big)-(1+\omega_{k,i})\chi\big(x, \tilde{e}(t_{k,i})\big)\right]$$

$\big(\chi(x, P)$ — характеристическая функция множества $P\big)$.

Из определения $\pi_k(x)$ следует, что в силу (17) имеют место неравенства (15). Далее, для любых i и k $(1\leqq i\leqq n,\ M\leqq k\leqq N)$, учитывая (18), имеем

$$\int_{e_i} \pi_k(x)\,dx = \sum_{e\in E_{k,i}} [(1-\omega_{k,i}) \operatorname{mes} e(t_{k,i})-(1+\omega_{k,i}) \operatorname{mes} \tilde{e}(t_{k,i})] =$$

$$= \sum_{e\in E_{k,i}} [(1-\omega_{k,i})\, t_{k,i} \operatorname{mes} e-(1+\omega_{k,i})(1-t_{k,i}) \operatorname{mes} e] = 0.$$

Если же $M\leqq k<m\leqq N$, то

$$\int_{e_i} \pi_k(x)\pi_m(x)\,dx = \sum_{e\in E_{k,i}} \Big[(1-\omega_{k,i}) \int_{e_i} \pi_m(x)\chi\big(x, e(t_{k,i})\big)\,dx-$$

$$-(1+\omega_{k,i})\int_{e_i} \pi_m(x)\,\chi\big(x, \tilde{e}(t_{k,i})\big)\,dx\Big].$$

Но, так как $e(t_{k,i})\in E_{k+1,i}$ при $e\in E_{k,i}$, то

$$\int_{e_i} \pi_m(x)\chi\big(x, e(t_{k,i})\big)\,dx = \int_{e(t_{k,i})} \pi_m(x)\,dx = \sum_{\substack{e'\in E_{m,i},\\ e'\subset e(t_{k,i})}} \Big[(1-\omega_{m,i}) \int_{e(t_{k,i})} \chi\big(x, e'(t_{m,i})\big)\,dx-$$

$$-(1+\omega_{m,i}) \int_{e(t_{k,i})} \chi\big(x, \tilde{e}'(t_{m,i})\big)\,dx\Big] = \sum_{\substack{e'\in E_{m,i},\\ e'\subset e(t_{k,i})}} [(1-\omega_{m,i})\,t_{m,i} \operatorname{mes} e' -$$

$$-(1+\omega_{m,i})(1-t_{m,i}) \operatorname{mes} e'] = 0.$$

Аналогично, для любого $e \in E_{k,i}$

$$\int_{e_i} \pi_m(x)\chi\big(x, \tilde{e}(t_{k,i})\big)\,dx = 0.$$

Итак, система $\{\pi_k(x)\}$ ортогональна на любом e_i $(i=1, 2, \dots, n)$.

Из определения π_k сразу следует, что $\left|\frac{1}{K}\varphi_k(x)+\pi_k(x)\right|=1$ п.в. $(k=M, M+1, \dots, N)$. Остаётся проверить центрированность системы $\left\{\frac{1}{K}\varphi_k(x)+\pi_k(x)\right\}_{k=M}^{N}$. Для этого достаточно показать, что для любого m $(M\leqq m\leqq N)$ и для любой системы знаков $\varepsilon_k=\pm 1$ $(k=M,\ M+1, \dots, m)$ имеем

$$\text{(19)}\qquad \operatorname{mes}\left\{x\colon \frac{1}{K}\varphi_k(x)+\pi_k(x) = \varepsilon_k;\quad k = M, M+1, \dots, m\right\} = 2^{-m+M-1}.$$

Множество в левой части (19) обозначим через E. Тогда

$$\operatorname{mes} E = \operatorname{mes} \bigcup_{i=1}^{n} (E\cap e_i) = \sum_{i=1}^{n} \operatorname{mes} (E\cap e_i).$$

Но, легко видеть, что для любого i $(1\leqq i\leqq n)$

$$\operatorname{mes} (E\cap e_i) = \operatorname{mes} e_i\cdot \frac{1+\varepsilon_M\omega_{M,i}}{2}\cdot\frac{1+\varepsilon_{M+1}\omega_{M+1,i}}{2}\cdot\ldots\cdot\frac{1+\varepsilon_m\omega_{m,i}}{2}.$$

И, следовательно,

$$\operatorname{mes} E = 2^{-m+M-1}\sum_{i=1}^{n}(1+\varepsilon_M\omega_{M,i})(1+\varepsilon_{M+1}\omega_{M+1,i})\cdot\ldots\cdot(1+\varepsilon_m\omega_{m,i}) \operatorname{mes} e_i =$$

$$= 2^{-m+M-1}\int_0^1\left(1+\frac{\varepsilon_M}{K}\varphi_M(x)\right)\left(1+\frac{\varepsilon_{M+1}}{K}\varphi_{M+1}(x)\right)\ldots\left(1+\frac{\varepsilon_m}{K}\varphi_m(x)\right)dx = 2^{-m+M-1},$$

в силу центрированности системы Φ.

Лемма 4 полностью доказана.

Нам также понадобится следующая лемма, принадлежащая К. Тандори (см. [2], лемма 11).

Лемма 5. *Если для некоторой последовательности* $a=\{a_k\}$

$$\varlimsup_{n\to\infty}\frac{1}{\lambda_n^2}\sum_{k=1}^{n}a_k^2>0,$$

то п.в. на $[0, 1]$

$$\varlimsup_{n\to\infty}\frac{1}{\lambda_n}\left|\sum_{k=1}^{n}a_k r_k(x)\right| > 0.$$

3. Перейдём к доказательствам теорем.

Доказательство теоремы 1. Очевидно при $1<K<\infty$ $M(1)\supset M(K)$. Нам остаётся доказать, что если $a\notin M(K)$, то $a\notin M(1)$.

Пусть $a\notin M(K)$. Если $\overline{\lim\limits_{n\to\infty}}\frac{1}{\lambda_n^2}\sum\limits_{k=1}^{n}a_k^2>0$, то на основании леммы 5

$$\lim_{n\to\infty}\sigma_n(x,a,r)\neq 0$$

п.в. на [0, 1] и, следовательно, $a\notin M(1)$.

Рассмотрим случай, когда

$$\lim_{n\to\infty}\frac{1}{\lambda_n^2}\sum_{k=1}^{n}a_k^2=0. \tag{20}$$

В силу определения $M(K)$, существует система $\Psi=\{\psi_k\}_{k=1}^{\infty}\in\Omega(K)$ и множество P положительной меры ($\operatorname{mes}P=\alpha>0$) такие, что

$$\lim_{N\to\infty}\sup_{n\geqq N}|\sigma_n(x,a,\Psi)|\geqq\varrho>0\quad\text{при}\quad x\in P. \tag{21}$$

Не трудно убедиться, что, в силу (20) и (21), для $\omega<\varrho$ можно индуктивно построить целочисленные последовательности $\{N_m\}_{m=1}^{\infty}$ и $\{M_m\}_{m=1}^{\infty}$ ($N_m<M_m<N_{m+1}$, $m=1,2,\ldots;\ N_1=0$) такие, что для $m=1,2,\ldots$

$$\max_{M_m\leqq n}\frac{1}{\lambda_n}\sum_{k=1}^{N_m}|a_k|\leqq\frac{\omega}{8K}; \tag{22}$$

$$\max_{M_m\leqq n}\frac{1}{\lambda_n^2}\sum_{k=1}^{n}a_k^2\leqq\frac{\omega^2}{128K^2}; \tag{23}$$

$$\max_{M_m\leqq n\leqq N_{m+1}}\frac{1}{\lambda_n}\Big|\sum_{k=N_m+1}^{n}a_k\psi_k(x)\Big|>\omega\quad\text{при}\quad x\in P_m, \tag{24}$$

где $P_m\subset P$ и $\operatorname{mes}P_m\geqq\frac{\alpha}{2}$ $(m=1,2,\ldots)$.

Применяя лемму 1 к системе $\Psi\left(\text{при } q=\frac{1}{2},\ \varepsilon=\frac{\omega}{2}\right)$, получим систему простых функций $\bar\Phi=\{\bar\varphi_k\}_{k=1}^{\infty}\in\bar\Omega(K)$ такую, что для любого $m=1,2,\ldots$

$$\max_{M_m\leqq n\leqq N_{m+1}}\frac{1}{\lambda_n}\Big|\sum_{k=N_m+1}^{n}a_k\bar\varphi_k(x)\Big|>\frac{\omega}{2}\quad\text{при}\quad x\in\tilde P_m,$$

где, в силу (24), $\operatorname{mes}\tilde P_m\geqq\frac{1}{2}\operatorname{mes}P_m\geqq\frac{\alpha}{4}$ $(m=1,2,\ldots)$.

Применим лемму 4 к системам $\{\bar\varphi_k(x)\}_{k=N_m+1}^{N_{m+1}}$ $(m=1,2,\ldots)$. Получим системы $\left\{\tilde\varphi_k(x):=\frac{1}{K}\bar\varphi_k(x)+\pi_k(x)\right\}_{k=N_m+1}^{N_{m+1}}\in\bar\Omega(1)$ $(m=1,2,\ldots)$. Пусть для $m\geqq1$ $\{e_i^m\}_{i=1}^{n_m}$

— различные атомы σ-алгебры $\mathscr{F}_{N_{m+1}}(\bar{\Phi})\left(\sum_{i=1}^{n_m} \operatorname{mes} e_i^m=1\right)$ и пусть Z_m — множество тех индексов i, для которых $e_i^m \cap \tilde{P}_m \neq \emptyset$. Очевидно,

$$\sum_{i\in Z_m} \operatorname{mes} e_i^m \geqq \frac{\alpha}{4}.$$

Пусть, далее, $n(i, m)(\geqq M_m)$ такое, что

$$\max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_m+1}^{n} a_k\bar{\varphi}_k(x)\right| = \frac{1}{\lambda_{n(i,m)}}\left|\sum_{k=N_m+1}^{n(i,m)} a_k\bar{\varphi}_k(x)\right|$$

при $x\in e_i^m$, $i\in Z_m$.

Для $i\in Z_m$, используя неравенство Чебышева и учитывая (23), имеем

$$\operatorname{mes}\left\{x\in e_i^m\colon \max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_m+1}^{n} a_k\tilde{\varphi}_k(x)\right| > \frac{\omega}{4K}\right\} \geqq$$

$$\geqq \operatorname{mes}\left\{x\in e_i^m\colon \frac{1}{\lambda_{n(i,m)}}\left|\sum_{k=N_m+1}^{n(i,m)} a_k\tilde{\varphi}_k(x)\right| > \frac{\omega}{4K}\right\} \geqq$$

$$\geqq \operatorname{mes}\left\{x\in e_i^m\colon \frac{1}{\lambda_{n(i,m)}}\left|\sum_{k=N_m+1}^{n(i,m)} a_k\cdot\frac{1}{K}\tilde{\varphi}_k(x)\right| > \frac{\omega}{2K}\right\}-$$

$$-\operatorname{mes}\left\{x\in e_i^m\colon \frac{1}{\lambda_{n(i,m)}}\left|\sum_{k=N_m+1}^{n(i,m)} a_k\pi_k(x)\right| \geqq \frac{\omega}{4K}\right\} \geqq$$

$$\geqq \operatorname{mes} e_i^m - \frac{16K^2}{\omega^2}\cdot\frac{1}{\lambda^2_{h(i,m)}} \sum_{k=N_m+1}^{n(i,m)} a_k^2 \int_{e_i^m} \pi_k^2(x)\,dx \geqq$$

$$\geqq \operatorname{mes} e_i^m\left(1-\frac{64K^2}{\omega^2}\max_{M_m\leqq n}\frac{1}{\lambda_h^2}\sum_{k=1}^{n} a_k^2\right) \geqq \frac{1}{2}\operatorname{mes} e_i^m.$$

Отсюда получаем, что

$$\operatorname{mes}\left\{x\colon \max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_m+1}^{n} a_k\tilde{\varphi}_k(x)\right| > \frac{\omega}{4K}\right\} \geqq$$

$$\geqq \sum_{i\in Z_m} \operatorname{mes}\left\{x\in e_i^m\colon \max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_m+1}^{n} a_k\tilde{\varphi}_k(x)\right| > \frac{\omega}{4K}\right\} \geqq$$

$$\geqq \frac{1}{2}\sum_{i\in Z_m} \operatorname{mes} e_i^m \geqq \frac{\alpha}{8}.$$

Теперь уже легко видеть, что при помощи систем $\{\tilde{\varphi}_k(x)\}_{k=N_m+1}^{N_{m+1}}$, $m=1, 2, \dots$ можно индуктивно построить систему $\Phi=\{\varphi_k\}_{k=1}^{\infty}\in\Omega(1)$ такую, что для любого $m\geqq 1$

$$\operatorname{mes}\left\{x\colon \max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\left|\sum_{k=N_m+1}^{n} a_k\varphi_k(x)\right| > \frac{\omega}{4K}\right\} \geqq \frac{\alpha}{8}.$$

Для этого нужно положить $\varphi_k=\tilde{\varphi}_k$ при $N_1<k\leqq N_2$ и если φ_k, $k\leqq N_m$ уже построенны, то в качестве φ_k, $N_m<k\leqq N_{m+1}$ взять функции системы $\{\tilde{\varphi}_k\}_{k=N_m+1}^{N_{m+1}}$, «сжатые» на каждый из атомов σ-алгебры $\mathscr{F}_{N_m}(\Phi)$.

Если обозначить $H_m:=\left\{x\colon \max\limits_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\Big|\sum\limits_{k=N_m+1}^{n} a_k\varphi_k(x)\Big|>\frac{\omega}{4K}\right\}$, то из (25) имеем, что

$$\operatorname{mes} \varlimsup_{m\to\infty} H_m>0.$$

Отсюда, учитывая, что, в силу (25) и (22)

$$\max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\Big|\sum_{k=1}^{n} a_k\varphi_k(x)\Big| \geqq \max_{M_m\leqq n\leqq N_{m+1}} \frac{1}{\lambda_n}\Big|\sum_{k=N_m+1}^{n} a_k\varphi_k(x)\Big|-$$

$$-\frac{1}{\lambda_n}\sum_{k=1}^{N_m}|a_k|>\frac{\omega}{4K}-\frac{\omega}{8K}=\frac{\omega}{8K}$$

при $x\in H_m$, получаем, что

$$\varlimsup_{n\to\infty} \frac{1}{\lambda_n}\Big|\sum_{k=1}^{n} a_k\varphi_k(x)\Big|>0$$

на множестве положительной меры, а в силу замечания 1, и п.в.

Итак, $a\notin M(1)$. Теорема 1 полностью доказана.

Доказательство теоремы 2. В силу замечания 3, импликация (II)$\Rightarrow$(I) следует из соответствующей теоремы для стохастически независимых функций (см. [1]: теорема IV). Импликация (III)$\Rightarrow$(II) очевидна. Остаётся доказать, что из (I) следует (III).

Пусть $a\in M(\infty)$. Тогда, в силу замечания 3

$$\sum_{k=1}^{\infty}\frac{a_k^2}{\lambda_k^2}<\infty.$$

Возьмём произвольное $\varepsilon>0$ и пусть N_0 такое, что

$$\sum_{k=N_0}^{\infty}\frac{a_k^2}{\lambda_k^2}<\varepsilon.$$

Для любого $\Phi=\{\varphi_k\}_{k=1}^{\infty}\in\bar{\Omega}(\infty)$ имеем при $N\geqq N_0$

$$\int_0^1 \sup_{n\geqq N}\sigma_n^2(x,a,\Phi)\,dx \leqq \frac{2}{\lambda_N^2}\int_0^1 \Big(\sum_{k=1}^{N_0-1}|a_k|\,|\varphi_k(x)|\Big)^2dx+ \tag{26}$$

$$+2\int_0^1 \sup_{n\geqq N}\left(\frac{1}{\lambda_n}\sum_{k=N_2}^{n}a_k\varphi_k(x)\right)^2dx \leqq \frac{2(N_0-1)}{\lambda_N^2}\sum_{k=1}^{N_0-1}a_k^2+$$

$$+2\int_0^1 \sup_{n\geqq N}\left(\frac{1}{\lambda_n}\sum_{k=N_0}^{n}a_k\varphi_k(x)\right)^2dx.$$

Применив преобразование Абеля, получим

$$\frac{1}{\lambda_n}\sum_{k=N_0}^{n} a_k\varphi_k(x) = \frac{1}{\lambda_n}\sum_{k=N_0}^{n}\lambda_k\frac{a_k}{\lambda_k}\varphi_k(x) =$$

$$= \frac{1}{\lambda_n}\sum_{k=N_0}^{n-1}(\lambda_k-\lambda_{k+1})\sum_{i=N_0}^{k}\frac{a_i}{\lambda_i}\varphi_i(x)+\sum_{i=N_0}^{n}\frac{a_i}{\lambda_i}\varphi_i(x).$$

Следовательно,

$$\sup_{n\geqq N}\frac{1}{\lambda_n}\Big|\sum_{k=N_0}^{n}a_k\varphi_k(x)\Big| \leqq 2\sup_{n\geqq N_0}\Big|\sum_{k=N_0}^{n}\frac{a_k}{\lambda_k}\varphi_k(x)\Big|.$$

Отсюда и из (26), применяя лемму 3, получаем

$$\int_0^1 \sup_{n\geqq N}\sigma_n^2(x,a,\Phi)\,dx \leqq \frac{2(N_0-1)}{\lambda_N^2}\sum_{k=N}^{N_0+1}a_k^2+8\int_0^1\sup_{n\geqq N_0}\left(\sum_{k=N_0}^{n}\frac{a_k}{\lambda_k}\varphi_k(x)\right)^2dx = \tag{27}$$

$$= \frac{2(N_0-1)}{\lambda_N^2}\sum_{k=1}^{N_0-1}a_k^2+8A\sum_{k=N_0}^{\infty}\frac{a_k^2}{\lambda_k^2}\leqq\frac{2(N_0-1)}{\lambda_N^2}\sum_{k=1}^{N_0-1}a_k^2+8A\varepsilon\quad (N=N_0,\,N_0+1,\,\ldots).$$

Но с другой стороны, для достаточно больших $N(>N_1)$ первое слагаемое правой части (27) будет меньше ε. Итак, при $N>N_1$ для любого $\Phi\in\bar\Omega(\infty)$ имеем

$$\int_0^1\sup_{n\geqq N}\sigma_n^2(x,a,\Phi)\,dx \leqq (8A+1)\varepsilon.$$

Учитывая лемму 2, получим, что при $N>N_1$

$$\|a,\infty,N\| = \lim_{M\to\infty}\|a,\infty,N,M\| = \lim_{M\to\infty}\sup_{\Phi\in\bar\Omega(\infty)}\left\{\int_0^1\max_{N\leqq n\leqq M}\sigma_n^2(x,a,\Phi)\,dx\right\}^{1/2}\leqq$$

$$\leqq \sup_{\Phi\in\bar\Omega(\infty)}\left\{\int_0^1\sup_{n\geqq N}\sigma_n^2(x,a,\Phi)\,dx\right\}^{1/2}\leqq\sqrt{(8A+1)\varepsilon},$$

т. е. $\lim_{N\to\infty}\|a,\infty,N\|=0$.

Теорема 2 полностью доказана.

Доказательство теоремы 3. Пусть $1\leqq K<\infty$ и последовательность a такая, что $\|a,K,N\|\downarrow\varrho$ $(\varrho>0)$ при $N\to\infty$. Тогда, очевидно

$$\|a,K\|<\infty. \tag{28}$$

Учитывая, что для любого M

$$\|a,K,M,N\|\to\|a,K,M\|\quad(>\varrho)\quad\text{при}\quad N\to\infty,$$

можно найти целочисленные последовательности $\{N_m\}_{m=1}^{\infty}$ $(N_1=0)$ и $\{M_m\}_{m=1}^{\infty}$ такие, чтобы имели место:

1) $N_m < M_m < N_{m+1} \quad (m = 1, 2, \ldots);$

2) $\dfrac{1}{\lambda_n} \sum\limits_{k=1}^{N_m} |a_k| < \dfrac{\varrho}{8K}$ при $n = M_m, M_m+1, \ldots \quad (m = 2, 3, \ldots);$

3) $\|a, K, M_m, N_{m+1}\| > \dfrac{\varrho}{2} \quad (m = 1, 2, \ldots).$

В силу 3) и леммы 2, существует система $\Phi^{(1)}=\{\varphi_k^{(1)}\}_{k=1}^{\infty}\in\bar{\Omega}(K)$ такая, что

$$\Big\{\int_0^1 \max_{M_1\leqq n\leqq N_2} \sigma_n^2(x, a, \Phi^{(1)})\, dx\Big\}^{1/2} > \frac{\varrho}{2}.$$

Далее, в силу той же леммы, можно построить систему $\Phi^{(2)}=\{\Phi_k^{(2)}\}\in\bar{\Omega}(K)$ такую, что

$$\Big\{\int_0^1 \max_{M_2\leqq n\leqq N_3} \sigma_n^2(x, a, \Phi^{(2)})\, dx\Big\}^{1/2} > \frac{\varrho}{2},$$

причём будем считать, что система $\{\varphi_1^{(1)}, \varphi_2^{(1)}, \ldots, \varphi_{N_2}^{(1)}, \varphi_1^{(2)}, \varphi_2^{(2)}, \ldots\}$ образует центрированную систему. (В противном случае можно систему $\Phi^{(2)}$ «сжать» на каждый атом σ-алгебры $\mathscr{F}_{N_2}(\Phi^{(1)})$). Продолжая таким образом, мы получим последовательность систем $\{\Phi^{(m)}\}$ такую, что для любого $m\geqq 1$

$$\Big\{\int_0^1 \max_{M_m\leqq n\leqq N_{m+1}} \sigma_n^2(x, a, \Phi^{(m)})\, dx\Big\}^{1/2} > \frac{\varrho}{2},$$

причем система $\Phi=\{\varphi_k\}_{k=1}^{\infty}$, определённая следующим образом:

$$\varphi_k(x) = \varphi_k^{(m)}(x), \quad k = N_m+1, N_m+2, \ldots, N_{m+1}; \quad m = 1, 2, \ldots$$

принадлежит $\bar{\Omega}(K)$.

Для системы Φ имеем

$$\Big\{\int_0^1 \sup_{M_m\leqq n} \sigma_n^2(x, a, \Phi)\, dx\Big\}^{1/2} \geqq \Big\{\int_0^1 \sup_{M_m\leqq n\leqq N_{m+1}} \sigma_n^2(x, a, \Phi)\, dx\Big\}^{1/2} \geqq$$

$$\geqq \Big\{\int_0^1 \sup_{M_m\leqq n\leqq N_{m+1}} \Big(\frac{1}{\lambda_n}\sum_{k=N_m+1}^{n} a_k\varphi_k(x)\Big)^2 dx\Big\}^{1/2} - \Big\{\int_0^1 \sup_{M_m\leqq n} \Big(\frac{1}{\lambda_n}\sum_{k=1}^{N_m} a_k\varphi_k(x)\Big)^2 dx\Big\}^{1/2} \geqq$$

$$\geqq \Big\{\int_0^1 \sup_{M_m\leqq n\leqq N_{m+1}} \Big(\frac{1}{\lambda_n}\sum_{k=1}^{n} a_k\varphi_k^{(m)}(x)\Big)^2 dx\Big\}^{1/2} - \Big\{\int_0^1 \sup_{M_m\leqq n} \Big(\frac{1}{\lambda_n}\sum_{k=1}^{N_m} a_k\varphi_k^{(m)}(x)\Big)^2 dx\Big\}^{1/2} -$$

$$-\frac{K}{\lambda_{M_m}}\sum_{k=1}^{N_m}|a_k| \geqq \frac{\varrho}{2}-\frac{\varrho}{8}-\frac{\varrho}{8} = \frac{\varrho}{4}.$$

Пусть $F_N(x)=\sup\limits_{n\geqq N}\sigma_n^2(x, a, \Phi)$. Так как, для любого N $F_N(x)\geqq F_{N+1}(x)$ и, в силу (28) $\int\limits_0^1 F_1(x)dx\leqq\|a, K\|^2<\infty$, то на основании леммы Фату, имеем

$$\int\limits_0^1 \varlimsup_{n\to\infty} \sigma_n^2(x, a, \Phi)\,dx = \int\limits_0^1 \lim_{N\to\infty} F_N(x)\,dx \geqq \lim_{N\to\infty} \int\limits_0^1 F_N(x)\,dx \geqq \frac{\varrho^2}{16}.$$

Отсюда получаем, что $\varlimsup\limits_{n\to\infty} \sigma_n^2(x, a, \Phi)>0$ на множестве положительной меры, т. е. $a\notin M(K)$.

Теорема 3 доказана.

В заключение автор благодарит профессора К. Тандори за постановку задач.

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ЕРЕВАНСКИЙ ГОС. УНИВЕРСИТЕТ
МЕХАНИКО-МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ
УЛ. МРАВЯНА. I
375049. ЕРЕВАН, СССР

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ON EMPTY TRIANGLES DETERMINED BY POINTS IN THE PLANE

M. KATCHALSKI and A. MEIR (Edmonton)

1. In 1980, modifying an early problem of E. Klein [1], P. Erdös [2] asked the following question: For $n \geqq 3$ find the smallest integer $g(n)$ such that from any set S of $g(n)$ points in the plane, no three collinear, one can choose a convex subset of n points whose interior contains no point of S. The value of $g(4)$ was found by E. Klein and $g(5)$ was determined in [3]. Horton [4] showed that $g(7)$ does not exist. We consider here two related problems. We call a set S of n $(n \geqq 3)$ points in the plane *line-free* if S contains no three collinear points. We call a triangle determined by three different points of S *empty* if the triangle contains no point of S in its interior and denote the number of empty triangles in S by $\Delta(S)$. We say that a set T *destroys* a family F of triangles if every triangle in F contains a point of T in its interior.

Our first problem is to determine for $n \geqq 3$ the smallest integer $k(n)$ such that for every line-free set S of n points in the plane there exists a set T of $k(n)$ points which destroys all empty triangles in S. We show that $k(n)=2n-5$ for $n \geqq 3$. Our second problem is as follows. Given a line-free set S of n points in the plane, what can one say about the values of $\Delta(S)$? If S consists of the vertices of a convex n-gon, then clearly $\Delta(S)=\binom{n}{3}$; so the interesting question is to find a lower bound for $\Delta(S)$. As the main result of this paper we show that for

$$d(n) = \inf \{\Delta(S) \colon |S| = n, \quad S \text{ line-free}\},$$

the following inequality holds: There exists a constant $K>0$ such that for all $n \geqq 3$,

$$(n-1)(n-2)/2 \leqq d(n) \leqq Kn^2.$$

These problems, of course, can be generalized, in an obvious way, to higher dimensions. For $\mathbf{R}^3$ we have the following partial results. Let S be any plane-free set (i.e. no four points on a plane) of n points in $\mathbf{R}^3$. Then

(a) the number of *empty* simplexes in S is at least $(n-1)(n-2)(n-3)/6$.
(b) There exists a set T with $|T|=(n-3)^2$, which destroys *all* simplexes in S.

These partial results can be extended to $\mathbf{R}^k$ with $k>3$. We conjecture that for any set S of n points in general position in $\mathbf{R}^k$, $k \geqq 3$, there exists a destroying set with $c_k n$ points, where c_k depends only on k.

2. THEOREM 1. *Let S be a line-free set of n $(n \geqq 3)$ points in the plane whose convex hull is an m-gon. Then*

(a) *There exists a set T with $|T|=2n-m-2$ which destroys all triangles in S;*
(b) *If a set R destroys all empty triangles in S, then $|R| \geqq 2n-m-2$.*

THEOREM 2. (a) *If S is any line-free set of n points in the plane, then* $\Delta(S) \geqq (n-1)(n-2)/2$.

(b) *There exists an absolute constant K with the following property: For every $n \geqq 3$ there exists a line-free set $S=S(n)$ of n points in the plane so that $\Delta(S) \leqq Kn^2$.*

COROLLARY. *For* $n \geqq 3$, $(n-1)(n-2)/2 \leqq d(n) \leqq Kn^2$, *where K is independent of n.*

3. PROOF OF THEOREM 1. (a) Let $S=\{p_1, p_2, \ldots, p_n\}$, where $p_i=(x_i, y_i)$ for $1 \leqq i \leqq n$. We may assume that no two x_i are equal (otherwise we rotate the x, y axes slightly), and also that $x_1 < x_2 < \ldots < x_n$. Since S is line-free, the distance from any point p_i to any segment $[p_j, p_k]$ is positive. Let 2ε be the minimum of all such distances. For $1 \leqq i \leqq n$ we let

$$p_i^+ = (x_i, y_i+\varepsilon), \quad p_i^- = (x_i, y_i-\varepsilon) \quad \text{and} \quad T_1 = \bigcup_{i=1}^{n} \{p_i^+, p_i^-\}.$$

The set T required by (a) is defined as those points of T_1 which also belong to the convex hull of S. Now, the convex hull of S has m vertices including p_1 and p_n. If p_j is a vertex of the convex hull and $1<j<n$, then exactly one of p_j^+ or p_j^- is exterior to the convex hull and the points $p_1^+, p_1^-, p_n^+, p_n^-$ are all exterior. Hence $|T|=2n-m-2$. We still have to show that T destroys all triangles in S. Let $p_ip_jp_k$ be any triangle in S. We may assume that $x_i<x_j<x_k$. Then $1<j<n$ and it is easy to see that either p_j^+ or p_j^- belongs to the interior of the triangle. The same point then, a fortiori, belongs to the convex hull of S and so it is also in T. This proves that T destroys every triangle.

In order to prove (b), we recall that there exists a triangulation of the convex hull of S, whose vertices are the points in S, consisting of $2n-m-2$ triangles. There are therefore, $2n-m-2$ empty triangles in S with pairwise disjoint interior. This implies (b).

The Corollary of Theorem 1 is an immediate consequence of (a) and (b) since always $m \geqq 3$, but $m=3$ is possible.

4. PROOF OF THEOREM 2. (a) Let p_n be a vertex of the convex hull of S. It is easy to see that we may label the remaining points of S so that the angles $\sphericalangle p_1p_np_i$ increase counterclockwise with i for $2 \leqq i \leqq n-1$ and that p_1 may be chosen so that all angles $\sphericalangle p_ip_np_{i+1}$ are less than π. The triangles $p_ip_np_{i+1}$ are then empty in S for $1 \leqq i \leqq n-2$, hence we have at least $n-2$ empty triangles in S with vertex p_n. We remove now p_n from S; we do not create any new empty triangles thereby, since p_n could not be an interior point in any triangle of S. Hence $\Delta(S) \geqq n-2+\Delta(S-\{p_n\})$. By induction if follows that $\Delta(S) \geqq (n-1)(n-2)/2$, as required.

In order to prove (b) we shall need several lemmas.

LEMMA 1. *Let $A=\{a_1, \ldots, a_m\}$ be a set of m points equidistributed on a circle $\mathscr{C}$ with radius ϱ. Let q be a point outside $\mathscr{C}$ such that* dist $(q, \mathscr{C})>2\varrho$ *and p be in or on $\mathscr{C}$. Suppose that the set $A_1=A \cup \{p, q\}$ is line-free. Then the number of empty triangles of the form qpa_i is at most* 6.

PROOF. We shall say that a point a on the circle $\mathscr{C}$ *faces* q if the segment $[a, q]$ does not contain an interior point of $\mathscr{C}$. Now denote the points of intersection of the line l passing through q and p with the circle $\mathscr{C}$ by b and d (see Fig 1). We shall show that there exist at most two a_i facing q and at most four a_i not facing q so that the triangle qpa_i is empty in A_1. Suppose there are three a_i facing q with this property. Then two of these, say a_1 and a_2 are on the same side of the line l as in Fig. 1. But then, clearly, the triangle qpa_2 is not empty, since a_1 belongs to its interior.

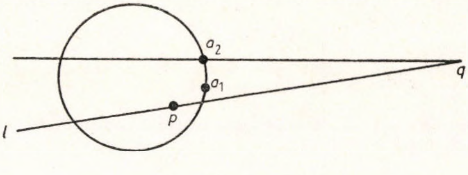

Fig. 1

Now, suppose that there are five a_i not facing q so that the triangles qpa_i are empty in A_1. Then three a_i, say a_1, a_2, a_3 are on the same side of l as in Fig. 2. Let the intersection point of $[a_3, q]$ with $\mathscr{C}$ be denoted by e and set $\sphericalangle\, edp=\alpha$, $\sphericalangle\, a_3ed=\beta$, $\sphericalangle\, eqp=\gamma$. Then $\beta=\alpha+\gamma$ and $\gamma\leqq\alpha$ since $\operatorname{dist}(e, q)\geqq 2\varrho\geqq\operatorname{dist}(e, d)$. Hence $\beta\leqq 2\alpha$. Since the a_i are equidistributed on $\mathscr{C}$, $\beta>2(\pi/n)$ and so $\alpha>\pi/n$. But then the arc $\widehat{eb}$ contains a point a_i in its interior, so the triangle qpa_3 is not empty. This contradiction proves the lemma.

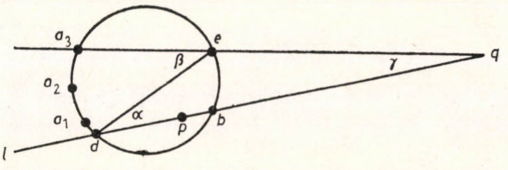

Fig. 2

LEMMA 2. *Let* $A=\{a_1, \dots, a_m\}$ *be equidistributed on a circle* $\mathscr{C}$ *with radius* ϱ. *Let* q *be outside* $\mathscr{C}$ *such that* $\operatorname{dist}(q, \mathscr{C})>2\varrho$. *Suppose* $A_1=A\cup\{q\}$ *is line-free. Then there exists* $r_0>0$ *so that the following holds: If* $\operatorname{dist}(a_1', a_i)<r_0$ *for* $1\leqq i\leqq m$, *and* $\operatorname{dist}(q', q)<r_0$, *then the set* $A_1'=\{a_1', \dots, a_m', q')$ *is line-free and the number of empty triangles of the form* $q'a_j'a_i'$ *is at most* $3m$.

PROOF. Let h_0 be the minimum of all heights of all triangles in A_1. Since A_1 is line-free, $h_0>0$. We choose $r_0=h_0/4$. It is easy to show that if the triangle qa_ja_i is non-empty in A_1, then $q'a_j'a_i'$ is non-empty in A_1'. Now, from Lemma 1 with $p=a_j$, j fixed, the number of empty triangles of the form qa_ja_i is at most 6, so the total number of empty triangles qa_ja_i in A_1 is at most $3m$. Therefore the number of empty triangles $q'a_j'a_i'$ in A_1' is at most $3m$ as well.

LEMMA 3. *Let* $\mathscr{B}$ *and* $\mathscr{B}^*$ *be concentric closed disks with center c and radii* ϱ *and* ϱ^* $(\varrho^*\geqq 2\varrho)$ *respectively. Let* $b_1, b_2, \dots, b_{4k}$ *be* $4k$ *equidistributed points on* $\mathscr{C}^*=\partial\mathscr{B}^*$. *Suppose a segment* $[p_1, p_2]$ *is exterior to* $\mathscr{B}^*$, q *a point in* $\mathscr{B}$ *and* $\sphericalangle\, p_1qp_2\geqq\pi/k$. *Then the triangle* p_1qp_2 *contains a point* b_i *in its interior.*

PROOF. Let e_1 and e_2 be the intersection points of $[p_1, q]$ and $[p_2, q]$ with $\mathscr{C}^*$, respectively (see Fig. 3). Denote the intersection point of the bisector of $\sphericalangle\, e_1ce_2$ with $\mathscr{C}=\partial\mathscr{B}$ by d. Then $\sphericalangle\, e_1de_2 \geqq \sphericalangle\, e_1qe_2 \geqq \pi/k$. It is easy to show that $\sphericalangle\, e_1ce_2 > \frac{1}{2} \sphericalangle\, e_1de_2$, because of the assumption concerning the radii. It follows therefore that $\sphericalangle\, e_1ce_2 > \pi/2k$ and thus the arc $\widehat{e_1e_2}$ of $\mathscr{C}^*$ must contain a point b_i in its interior.

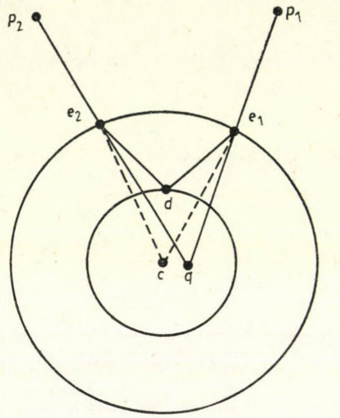

Fig. 3

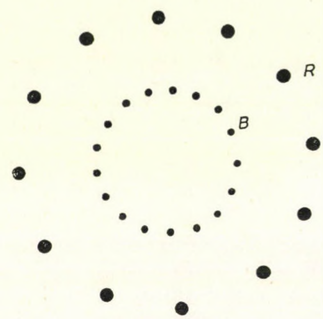

Fig. 4 (r-group with $m=17$)

By a modification of the preceding argument in Lemma 3, we obtain the statement of Lemma 4.

LEMMA 4. *Let $\mathscr{B}_1^*$, $\mathscr{B}_2^*$ and $\mathscr{B}_3^*$ be closed disjoint disks with centers $\gamma_1, \gamma_2, \gamma_3$ respectively, each with radius* 3. *Let $\mathscr{B}_3$ be concentric with $\mathscr{B}_3^*$ having radius ϱ, $\varrho<3/2$. Suppose the points $b_1, b_2, \dots, b_{6m}$ are equidistributed on $\mathscr{C}_3^*=\partial\mathscr{B}_3^*$, $\sphericalangle\, \gamma_1\gamma_3\gamma_2 \geqq \pi/m$ and* $\operatorname{dist}(\gamma_1, \gamma_3) \geqq 9m$, $\operatorname{dist}(\gamma_2, \gamma_3) \geqq 9m$. *Then every triangle $p_1p_2p_3$ with $p_1 \in \mathscr{B}_1^*$, $p_2 \in \mathscr{B}_2^*$ and $p_3 \in \mathscr{B}_3$ contains a point b_i in its interior.*

Now we are ready to prove Theorem 2 (b) constructing a set $S=S(n)$ as required. In fact we shall construct $S(n)$ only for values of n when $n=m^3+18m^2$ $(m \geqq 4)$; it is easy to see that this is no restriction of the generality. Let m be any fixed integer and r any positive number. We shall call a set consisting of m ("black") points equidistributed on a circle (with center c) and radius r and of 12 ("red") points equidistributed on a concentric circle with radius $2r$, an *r-group* (see Fig. 4).

We shall call a set Γ' consisting of m ("white") points $c_1, \dots, c_m$ equidistributed on a unit circle (with center γ) and of $6m$ ("green") points equidistributed on a concentric cirlce of radius 3, a *pre-cluster*.

When the white points $c_1, c_2, \dots, c_m$ in a pre-cluster Γ' are replaced by m r-groups centered at $c_1, c_2, \dots, c_m$ we call the resulting set Γ a *cluster*. We position now m pre-clusters so that their centers $\gamma_1, \gamma_2, \dots, \gamma_m$ are equidistributed on a circle of radius $R=3m^2$. We rotate, if necessary, any of the sets of white and green points that lie on circles so that the set S_1 so obtained is line-free. Let h be the minimum of all heights of all triangles in S_1. We choose r, $0<r<h/8$ and replace every white point in S_1 by an r-group. By rotating, if necessary, any of the sets of black and red points that lie on circles, we can achieve that the new set S is line-free. This set S consists of m clusters and contains a total of m^3 black, $12m^2$ red and $6m^2$ green points.

We wish to show that $\Delta(S)=O(m^6)$. The vertices of any triangle in S may belong to one, two or three clusters; one, two or three groups and can be of different colors (B=black, R=red, G=green), so we shall have to distinguish among several cases.

(i) *Vertices in* 3 *clusters*. Using Lemma 4, we can easily see that all empty triangles must be of the colors GGG. The total number of such triangles is $O(m^6)$.

(ii) *Vertices in* 2 *clusters* Γ_i, Γ_j, $i\neq j$. The total number of triangles with only one black vertex is $O(m^6)$. The remaining cases follow.

BBB in two groups: $O(m^6)$, using Lemma 1.

BBB in three groups: None, each such triangle contains a red point due to Lemma 3, with $k=3$.

BB$\in\Gamma_i$, R$\in\Gamma_j$: $O(m^6)$, using Lemma 2.

BB$\in\Gamma_i$, G$\in\Gamma_j$: $O(m^6)$, using Lemma 2.

B$\in\Gamma_i$, BG$\in\Gamma_j$: $O(m^6)$, using Lemma 1.

B$\in\Gamma_i$, BR$\in\Gamma_j$ (BR in two groups): $O(m^6)$, using Lemma 2.

B$\in\Gamma_i$, BR$\in\Gamma_j$ (BR in one group): $O(m^6)$=total number of such triangles.

(iii) *Vertices in* 1 *cluster*. With the exception of the case BBB in three groups, the total number of all other triangles is $O(m^6)$.

BBB in three groups: None, each such triangle contains a red point, due to Lemma 3 with $k=3$.

In conclusion, $\Delta(S)=O(m^6)=O(n^2)$, since $n\sim m^3$. This completes the proof of Theorem 2.

Addendum (October 10, 1985). Since the submission of this paper for publication, the following information and further results came to our attention.

1) G. B. Purdy has informed us that he had announced the result of our Theorem 2(b) in *AMS Abstracts,* **3** (1982), 318, but he has published no proofs as yet.

2) It was pointed out by B. Grünbaum and J. Zaks that our conjecture, preceding Theorem 1, is false. Utilizing Schlegel diagrams of neighbourly polytopes in 4 dimensions, they show that there exists $c>0$ such that for every n there exists in $\mathbf{R}^3$ a set of cn^2 $(c>0)$ simplexes with pairwise disjoint interiors, whose vertex set has cardinality $\leqq n$.

3) In connection with 2) the following problem may be raised. If A is a set of points in general position in $\mathbf{R}^d$, let $s=s(A)$ denote the largest integer so that there exists s non-degenerate simplexes with pairwise disjoint interiors, whose vertices are in A. Let $t=t(A)$ be the smallest integer such that there exists a set of points with cardinality t, which destroys all simplexes in A. Clearly we must have $s(A)\leqq t(A)$. In $\mathbf{R}^2$, it follows from our result, that $a(A)=t(A)$. Does equality hold in higher dimensions?

4) We were able to extend our earlier results as follows. If A denotes a set of points in general position in $\mathbf{R}^2$, let $Q(A)$ denote the number of empty quadrilaterals

in A and, as before, $\Delta(A)$ the number of empty triangles in A. We have proved that for every A, $Q(A) \geqq (1/8)\Delta(A)$. We also proved that there exists $c>0$ such that for every n there exists $A \in \mathbf{R}^2$ with $|A|=n$ such that $Q(A)<cn^2$. Does there exist for every n a set A in $\mathbf{R}^2$ such that $|A|=n$, $\Delta(A)=O(n^2)$ but $Q(A)>n^{2+\varepsilon}$, $\varepsilon>0$?

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THE UNIVERSITY OF ALBERTA
DEPARTMENT OF MATHEMATICS
EDMONTON, ALBERTA
CANADA

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ON A PROBLEM OF ERDŐS, HERZOG AND PIRANIAN

G. WAGNER (Stuttgart)

1. Preliminaries. We consider polynomials $f(z)=\prod_{\mu=1}^{n}(z-z_\mu)$ of the complex variable z with all (not necessarily distinct) zeros z_μ on the unit circle $C=\{|z|=1\}$. Denote by $S(f)$ the set of points z where the inequality $|f(z)|\leqq 1$ is satisfied. The authors in [1] show that the linear Lebesgue measure of the set $S(f)\cap C$ (denoted by $m(S(f)\cap C)$) can be made arbitrarily small by suitable choice of the polynomial $f(z)$. Moreover, they conjecture that $m\bigl(S(f)\cap C\bigr)>n^{-c}$ ($n=$ degree of the polynomial $f(z)$) holds with some positive constant $c>0$ ([1], pp. 132 and 134). We shall prove this conjecture.

THEOREM 1. *For each polynomial* $f(z)=\prod_{\mu=1}^{n}(z-z_\mu)$, $|z_\mu|=1$, *we have* $m\bigl(S(f)\cap C\bigr)>1/4\sqrt{n}$. *On the other hand, for each* $n\geqq 2$ *there exists a polynomial* $f(z)=\sum_{\mu=1}^{n}(z-z_\mu)$, $|z_\mu|=1$ *such that* $m\bigl(S(f)\cap C\bigr)<16\,(\log n/n)^{1/3}$ *holds.*

Similar results can be obtained if we replace the unit circle C by some analytic curve of logarithmic capacity 1. For example, in the case of the real interval $I=[-2, 2]$ we get (with unspecified constants, using the Vinogradov notation).

THEOREM 1′. *For each polynomial* $f(x)=\prod_{\mu=1}^{n}(x-x_\mu)$, $-2\leqq x_\mu\leqq 2$, *we have* $m\bigl(S(f)\cap I\bigr)\gg 1/n$. *On the other hand, for each* $n\geqq 2$ *there exists a polynomial* $f(x)=\prod_{\mu=1}^{n}(x-x_\mu)$ *with* $m\bigl(S(f)\cap I\bigr)\ll\log^{1/3} n/n^{2/3}$.

In the case of the interval I it seems more natural to ask for the measure $\gamma\bigl(S(f)\cap I\bigr)$, where γ is the equilibrium distribution of I with respect to the logarithmic potential. The corresponding results are $\gamma\bigl(S(f)\cap I\bigr)\gg n^{-1/2}$ and $\gamma\bigl(S(f)\cap I\bigr)\ll\log^{1/3} n/n^{1/3}$.

2. An energy inequality. Let $z_1, z_2, \dots, z_l$ be pairwise different points on the unit circle. Furthermore, let $\alpha_1, \alpha_2, \dots, \alpha_l$ be arbitrary natural numbers $\geqq 1$ ("multiplicities"). Put $\alpha_1+\alpha_2+\dots+\alpha_l=n$. The number $E=\sum_{\substack{1\leqq\mu,\nu\leqq l\\ \mu\neq\nu}}\alpha_\mu\alpha_\nu\log|z_\mu-z_\nu|$ measures the total (logarithmic) energy of a discrete charge distribution with charges α_μ placed at the points z_μ. In the case when all the α_μ's are equal to 1 ($l=n$) it is known that E satisfies the relation $E\leqq n\cdot\log n$. Equality holds if and only if the points $z_1, z_2, \dots, z_n$ are the vertices of a regular n-gon (cf. [2], [3]). The following generalization seems to be of independent interest.

THEOREM 2. *Let $z_1, z_2, \dots, z_l$ be pairwise different points on C with multiplicities $\alpha_1, \alpha_2, \dots, \alpha_l$; $\alpha_1+\alpha_2+\dots+\alpha_l=n$. Then the following inequality holds:*

$$E = \sum_{\mu\neq\nu} \alpha_\mu\alpha_\nu \log|z_\mu - z_\nu| \leqq \sum_{j=1}^{l} \alpha_j^2 \log\frac{ne^2}{\alpha_j}.$$

PROOF. The product $\prod_{1\leqq\mu<\nu\leqq l} |z_\mu - z_\nu|^{\alpha_\mu\alpha_\nu}$ is equal to the absolute value of a Vandermonde-type determinant Δ of the following form (we use the abbreviation $!r!=0!\cdot 1!\cdot\ldots\cdot r!$, $r\geqq 0$):

$$(1)\qquad \Delta = \prod_{\mu=1}^{l} !(\alpha_\mu-1)!^{-1} \begin{vmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{n-1} \\ \frac{d}{dz_1}1 & \frac{d}{dz_1}z_1 & \frac{d}{dz_1}z_1^2 & \dots & \frac{d}{dz_1}z_1^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{d^{\alpha_1-1}}{dz_1^{\alpha_1-1}}1 & \frac{d^{\alpha_1-1}}{dz_1^{\alpha_1-1}}z_1 & & \dots & \frac{d^{\alpha_1-1}}{dz_1^{\alpha_1-1}}z_1^{n-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{d^{\alpha_l-1}}{dz_l^{\alpha_l-1}}1 & \frac{d^{\alpha_l-1}}{dz_l^{\alpha_l-1}}z_l & & \dots & \frac{d^{\alpha_l-1}}{dz_l^{\alpha_l-1}}z_l^{n-1} \end{vmatrix}.$$

Although formula (1) should certainly be known, we could not find it in the literature. The proof, however, is elementary and we omit it. The Hadamard inequality says that the absolute value of a determinant is not greater than the product of the Euclidean lengths if the row vectors. So we have

$$|\Delta|^2 \leqq \prod_{\mu=1}^{l} !(\alpha_\mu-1)!^{-2} \prod_{\nu=1}^{l}\prod_{s=0}^{\alpha_\nu-1} (0^{2s}+1^{2s}+\dots+(n-1)^{2s}) \quad (0^0=1).$$

The inequality

$$0^{2s}+1^{2s}+\dots+(n-1)^{2s} \leqq \int_0^n x^{2s}\,dx = \frac{n^{2s+1}}{2s+1} \leqq n^{2s+1}$$

holds, hence

$$|\Delta|^2 \leqq \prod_{\mu=1}^{l} !(\alpha_\mu-1)!^{-2} \prod_{\nu=1}^{l} n^{\alpha_\nu^2}.$$

Taking logarithms we get

$$E = \log|\Delta|^2 \leqq \sum_{j=1}^{l} \alpha_j^2 \log n - 2\sum_{j=1}^{l} \log !(\alpha_j-1)!.$$

By induction we can prove that

$$\log!(r-1)! \geqq \frac{1}{2} r^2 \log \frac{r}{e^2} \quad (r = 1, 2, \ldots)$$

holds. We obtain

$$E \leqq \sum \alpha_j^2 \log n - \sum \alpha_j^2 \log \frac{\alpha_j}{e^2} = \sum_{j=1}^{l} \alpha_j^2 \log \frac{ne^2}{\alpha_j}.$$

This proves the assertion.

REMARK. With some more effort we can prove a similar result for the interval $I=[-2, 2]$: Let $x_1, x_2, \ldots, x_l$ be pairwise distinct points on I with multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_l \geqq 1$; $\sum_{\mu=1}^{l} \alpha_\mu = n$. Then the following inequality holds:

$$E = \sum_{\mu \neq \nu} \alpha_\mu \alpha_\nu \log |x_\mu - x_\nu| \leqq 2 \sum_{j=1}^{l} \alpha_j^2 \log \frac{ne^2}{\alpha_j}.$$

We do not make use of this result and shall not prove it here.

3. Proof of the lower estimates. We begin with a reduction of the problem. Let $f(z) = \prod_{\mu=1}^{n} (z - z_\mu)$, the zeros $z_\mu \in C$ being not necessarily distinct. The set $S(f) \cap C$ consists of a finite number of pairwise disjoint closed arcs $A_1, A_2, \ldots, A_l$ $(1 \leqq l \leqq n)$, called *"components"* of $S(f)$. Each component contains at least one zero z_μ. Let $z_1, z_2, \ldots, z_{\alpha_1}$ be the zeros in A_1. Choose a branch of arg z which is continuous on A_1. Replace the zeros $z_1, \ldots, z_{\alpha_1}$ by a single zero $z_1^* \in C$ with multiplicity α_1 and $\arg z_1^* = \frac{1}{\alpha_1} \sum_{\mu=1}^{\alpha_1} \arg z_\mu$. Due to the convexity of the function $\log |z - z_\mu|$ we have

$$\sum_{\mu=1}^{\alpha_1} \log |z - z_\mu| \leqq \alpha_1 \log |z - z_1^*|$$

for all $z \in C$ outside the smallest arc containing $z_1, z_2, \ldots, z_{\alpha_1}$. Hence the polynomial $f^*(z) = (z - z_1^*)^{\alpha_1} \sum_{\mu=\alpha_1+1}^{n} (z - z_\mu)$ has components $A_1^*, A_2^*, \ldots, A_k^*$ such that $A_1^* \cup \ldots \cup A_k^* \subset A_1 \cup \ldots \cup A_l$ holds. So, in order to prove the lower estimate for $m(S(f) \cap C)$, we may assume that $S(f)$ has l components $A_1, A_2, \ldots, A_l$ with exactly one zero z_μ (with multiplicity α_μ, $\sum_{\mu=1}^{l} \alpha_\mu = n$) in each A_μ. Denote by z_1', z_1'' the boundary points of the component A_1. Let A_1', A_1'' be the arcs connecting z_1 with z_1', z_1'', respectively. We have

$$0 = \log f(z_1') = \sum_{\mu=1}^{l} \alpha_\mu \log |z_1' - z_\mu| \quad \text{and} \quad 0 = \log f(z_1'') = \sum_{\mu=1}^{l} \alpha_\mu \log |z_1'' - z_\mu|.$$

Noting that $\log|z_1'-z_1|=\log 2\sin\frac{m(A_1')}{2}$ and $\log|z_1''-z_1|=\log 2\sin\frac{m(A_1'')}{2}$, and using convexity of the function $\sum_{\mu=2}^{l}\alpha_\mu\log|z-z_\mu|$ on A_1, we get:

$$(2)\qquad \alpha_1\sum_{\mu=2}^{l}\alpha_\mu\log|z_1-z_\mu| \geqq (m(A_1'')/m(A_1))\cdot\alpha_1\sum_{\mu=2}^{l}\alpha_\mu\log|z_1'-z_\mu|+$$

$$+(m(A_1')/m(A_1))\cdot\alpha_1\sum_{\mu=2}^{l}\alpha_\mu\log|z_1''-z_\mu| = -\alpha_1^2((m(A_1'')/m(A_1))\log|z_1'-z_1|+$$

$$+(m(A_1')/m(A_1))\log|z_1''-z_1|) \geqq -\frac{1}{2}\alpha_1^2(\log|z_1'-z_1|+\log|z_1''-z_1|) =$$

$$= -\frac{1}{2}\alpha_1^2\log 4\sin\frac{m(A_1')}{2}\sin\frac{m(A_1'')}{2} \geqq -\frac{1}{2}\alpha_1^2\log 4\sin^2\frac{m(A_1)}{4} =$$

$$= \alpha_1^2\log\frac{1}{2\sin(m(A_1)/4)} \geqq \alpha_1^2\log\frac{2}{m(A_1)}.$$

Repeating the argument for the components $A_2, A_3, \ldots, A_l$ and adding the inequalities which correspond to (2) we obtain

$$\sum_{\mu\neq\nu}\alpha_\mu\alpha_\nu\log|z_\mu-z_\nu| \geqq \sum_{j=1}^{l}\alpha_j^2\log\frac{2}{m(A_j)}.$$

Comparing with Theorem 2 we get the final inequality

$$(3)\qquad \sum_{j=1}^{l}\alpha_j^2\log\frac{2}{m(A_j)} \leqq \sum_{j=1}^{l}\alpha_j^2\log\frac{ne^2}{\alpha_j}.$$

We will show that the basic inequality (3) has implications for the sum $\sum_{j=1}^{l}m(A_j)= =m(S(f)\cap C)$. Using again convexity of the log function, we have

$$0\leqq\sum_{j=1}^{l}\alpha_j^2\log\frac{ne^2m(A_j)}{2\alpha_j} \leqq \log\frac{\sum\frac{ne^2}{2}\alpha_j m(A_j)}{\sum\alpha_j^2},$$

hence

$$\sum_j\alpha_j m(A_j) \geqq \frac{2}{ne^2}\sum_j\alpha_j^2 > \frac{1}{4n}\sum_j\alpha_j^2.$$

Furthermore,

$$\sum_{j=1}^{l} m(A_j) > \frac{1}{4n}\frac{\sum \alpha_j^2}{\max \alpha_j} \geqq \frac{1}{4n}\left(\max \alpha_j + \frac{n-\max \alpha_j}{\max \alpha_j}\right) \geqq \frac{1}{4n}(2\sqrt{n}-1) \geqq \frac{1}{4\sqrt{n}}.$$

This proves the lower estimate for the unit circle.

We shall prove the lower estimate for the interval $I=[-2, 2]$. Let $f(x)= = \prod_{\mu=1}^{n}(x-x_\mu)$, $x_\mu \in I$, be a polynomial of degree n. Each $x, x_\mu \in I$ can be uniquely represented in the form $x=2\cos\varphi$, $x_\mu=2\cos\varphi_\mu$ $(0\leqq\varphi, \varphi_\mu\leqq\pi)$. Put $z=e^{i\varphi}$, $z_\mu=e^{i\varphi_\mu}$. Then

$$|f(x)| = \prod_{\mu=1}^{n}|2\cos\varphi-2\cos\varphi_\mu| = \prod_{\mu=1}^{n}|z-z_\mu||z-\bar{z}_\mu| = |g(z)|$$

holds with $g(z)=\prod_{\mu=1}^{n}(z-z_\mu)(z-\bar{z}_\mu)$. Since the zeros of $g(z)$ lie symmetric with respect to the real axis, the set $S(g)\cap C$ also has this property. The complex function $\tau(z)=z+z^{-1}$ maps the upper half C^+ of the unit circle onto the interval I, the set $S(g)\cap C^+$ onto $S(f)\cap I$, and a Borel set $B\cap C^+$ onto a set $\tau(B)$ with $\gamma(\tau(B))= =\frac{1}{\pi}m(B)$, $m(\tau(B))\gg(m(B))^2$. Here γ denotes the equilibrium distribution of the interval I. Applying Theorem 1 to the polynomial $g(z)$ we obtain the lower estimates $\gamma(S(f)\cap I)\gg n^{-1/2}$ and $m(S(f)\cap I)\gg n^{-1}$.

Remark. More generally one could ask for the distribution of the values of $|f(z)|$. For example, if we define the set $S_\lambda(f)\cap C=\{z\in C \,|\, |f(z)|\leqq\lambda\}$ for each $\lambda\in(0, 1]$, and if we denote by A_j the components of $S_\lambda(f)\cap C$, we get the corresponding inequality

$$\sum \alpha_j \log\lambda + \sum \alpha_j^2 \log\frac{2}{m(A_j)} \leqq \sum \alpha_j^2 \log\frac{ne^2}{\alpha_j}. \tag{4}$$

Since $\lambda\leqq 1$, we have $\log\lambda\leqq 0$, hence (4) implies

$$\sum \alpha_j^2 \log\lambda + \sum \alpha_j^2 \log\frac{2}{m(A_j)} \leqq \sum \alpha_j^2 \log\frac{ne^2}{\alpha_j}.$$

The same reasoning as in the proof before now yields $m(S_\lambda(f)\cap C)\geqq\frac{\lambda}{4\sqrt{n}}$.

4. Proof of the upper estimates. The following example is taken from [1], p. 134. Let $f(z)=z^n-1$. For a given number $h\in\mathbf{N}$ $(2h+1\leqq n)$ take away the zeros $e^{2\pi ik/n}$ for $-h\leqq k\leqq h$ and replace them by a single $(2h+1)$-fold zero at the point 1. For this polynomial

$$f^*(z) = (z-1)^{2h+1}\prod_{k=h+1}^{n-h-1}(z-e^{2\pi ik/n})$$

we shall estimate the measure $m(S(f^*)\cap C)$. For $\zeta=e^{2\pi i\alpha/n}$, $\frac{n}{2}\geqq|\alpha|>h$, we have

$$\log|f^*(\zeta)|-\log|f(\zeta)| =$$

$$=2\log\left|2\sin\frac{\pi\alpha}{n}\right|-\log\left|2\sin\frac{\pi(\alpha-1)}{n}\right|-\log\left|2\sin\frac{\pi(\alpha+1)}{n}\right|+$$

$$+2\log\left|2\sin\frac{\pi\alpha}{n}\right|-\log\left|2\sin\frac{\pi(\alpha-2)}{n}\right|-\log\left|2\sin\frac{\pi(\alpha+2)}{n}\right|+$$

$$+\ldots+$$

$$+2\log\left|2\sin\frac{\pi\alpha}{n}\right|-\log\left|2\sin\frac{\pi(\alpha-h)}{n}\right|-\log\left|2\sin\frac{\pi(\alpha+h)}{n}\right|.$$

We use convexity of the function $\log|2\sin\pi x|$ $(0<x<1)$ and the relation $\left|\frac{d^2}{dx^2}\log|2\sin\pi x|\right|\geqq\pi^2$. For $\varrho\in\{1, 2, \ldots, h\}$ we get

$$(5)\qquad 2\log\left|2\sin\frac{\pi\alpha}{n}\right|-\log\left|2\sin\frac{\pi(\alpha-\varrho)}{n}\right|-\log\left|2\sin\frac{\pi(\alpha+\varrho)}{n}\right|\geqq\pi^2\frac{\varrho^2}{n^2},$$

hence

$$\log|f^*(\zeta)|-\log|f(\zeta)|\geqq\pi^2\frac{h^3}{3n^2}>\frac{3h^3}{n^2}.$$

Inequality (5) already shows that all the components of $f^*(z)$ that belong to unit roots $e^{2\pi ik/n}$ $\left(\frac{n}{2}\geqq|k|>h\right)$ have smaller lengths than the corresponding components of $f(z)$. In order to get a quantitative result note that $\left(\text{for } 0\leqq t\leqq\frac{\pi}{n}\right)$ we have

$$\log|f(e^{\pm it+2\pi ik/n})|=\log\left(2\sin\frac{nt}{2}\right)\geqq\log\frac{2nt}{\pi},$$

hence by (5):

$$\log|f^*(e^{\pm it+2\pi ik/n})|\geqq\log\left(\frac{2nt}{\pi}e^{3h^3/n^2}\right).$$

For $t\leqq\frac{\pi}{2n}e^{-3h^3/n^2}$ we have $|f^*(e^{\pm it+2\pi ik/n})|\geqq 1$. Thus the length of each component of $f^*(z)$ belonging to $e^{2\pi ik/n}$ $\left(h<|k|\leqq\frac{n}{2}\right)$ is not greater than $2\cdot\frac{\pi}{2n}e^{-3h^3/n^2}$. The length of the component belonging to the zero 1 does not exceed $2\pi\cdot\frac{2h}{n}=$ $=\frac{4\pi h}{n}$. Adding these estimates we get the following inequality for $m(S(f^*)\cap C)$:

$$m(S(f^*)\cap C)\leqq\frac{4\pi h}{n}+n\cdot\frac{\pi}{n}e^{-3h^3/n^2}.$$

Choosing

$$h=[n^{2/3}\log^{1/3}n]\quad \left(\frac{1}{2}n^{2/3}\log^{1/3}n\leqq h\leqq\frac{n-1}{2}\quad\text{for } n\geqq 32\right)$$

we get

$$m\big(S(f^*)\cap C\big)\leqq\frac{4\pi\log^{1/3}n}{n^{1/3}}+\pi\cdot e^{-(3/8)\log n}<16\left(\frac{\log n}{n}\right)^{1/3}. \tag{6}$$

For $2\leqq n<32$ the right hand side of (6) is greater than 2π and the inequality holds trivially. This proves the upper bound in Theorem 1. If we project the zeros of $f^*(z)$ onto the interval I via the mapping $\tau(z)=z+z^{-1}$, we obtain a polynomial $g^*(x)$ for which we can prove the inequalities

$$m\big(S(g^*)\cap I\big)\ll\frac{\log^{1/3}n}{n^{2/3}}\quad\text{and}\quad \gamma\big(S(g^*)\cap I\big)\ll\frac{\log^{1/3}n}{n^{1/3}}$$

in a similar way. This completes the proof of Theorem 1 and Theorem 1'.

5. Conclusion. As a natural counterpart to Theorem 1 one would expect that the inequality $2\pi-m\big(S(f)\cap C\big)\gg n^{-c}$ holds for some positive constant $c>0$. However, we have not been able to prove this result.

The following generalization of the problem considered seems to be interesting. Let $u(x)$ be an integrable real function on the unit interval $[0, 1)$, periodically continued over the real axis, satisfying $\int_0^1 u(x)\,dx=0$. For an n-tuple of points $a_1, a_2, \ldots, \ldots, a_n$ in $[0, 1)$ consider the set $S_n(u)=\{x\in[0, 1)\,|\,\sum_{\mu=1}^{n}u(x-a_\mu)\leqq 0\}$. Does there exist a universal exponent $c>0$ such that

$$m\big(S_n(u)\big)\geqq K(u)\cdot n^{-c} \tag{7}$$

holds for every such function u, each n-tuple $a_1, a_2, \ldots, a_n$ and some constant K depending on u? We proved (7) for $u(x)=\log|2\sin\pi x|$ with $c=\frac{1}{2}$ (Theorem 1), and the same result is true for the function $u(x)=x-\frac{1}{2}$. A general answer to this problem seems to be difficult.

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MATHEMATISCHES INSTITUT A
PFAFFENWALDRING 57
D—7000 STUTTGART 80
FEDERAL REPUBLIC OF GERMANY

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u-ISOMORPHIC SEMIGROUPS OF CONTINUOUS FUNCTIONS IN LOCALLY COMPACT SPACES

Á. CSÁSZÁR (Budapest), member of the Academy

0. Let X be a topological space, $C(X)$ the set of all continuous real-valued functions on X, equipped with the semigroup operation of pointwise multiplication, $C^*(X)$, $C_\infty(X)$, and $C_c(X)$ the subsemigroups of $C(X)$ composed of bounded functions, of functions vanishing at infinity in the case of a locally compact X, and of functions with compact support, respectively. A classical theorem [4] states that, if X_1 and X_2 are compact Hausdorff spaces, and $C(X_1)$ and $C(X_2)$ are isomorphic, then X_1 and X_2 are homeomorphic. It is easy to deduce from this theorem that, if X_1 and X_2 are Tikhonov spaces, and $C^*(X_1)$ and $C^*(X_2)$ are isomorphic, then the Čech—Stone compactifications βX_1 and βX_2 are homeomorphic.

In a recent paper [2], the author has proved a generalization in two directions of this statement. Firstly, the role of $C^*(X_i)$ is given to semigroups composed of continuous functions of X_i into suitable topological semigroups, on the other hand, semigroup isomorphy is replaced by a weaker condition, i.e. u-isomorphy.

The purpose of the present paper is to establish a similar generalization of the following statements of [5]: if X_1 and X_2 are locally compact Hausdorff spaces, and $C_c(X_1)$ and $C_c(X_2)$ (or $C_\infty(X_1)$ and $C_\infty(X_2)$) are isomorphic, then X_1 and X_2 are homeomorphic.

1. Let us recall some definitions from [2]. A topological semigroup $\mathbf{S}$ is said to be *segment-like* iff

(a) it contains $[0, 1]$ as a topological subsemigroup ($[0, 1]$ is always equipped with the multiplication of real numbers and with the topology inherited from the Euclidean topology of $\mathbf{R}$),

(b) 0 is a zero element and 1 is a unity element in $\mathbf{S}$,

(c) there is a continuous homomorphism $x \to |x|$ from $\mathbf{S}$ into $\mathbf{R}$ (equipped with the multiplication of real numbers and Euclidean topology),

(d) $|x|=x$ for $x\in[0, 1]$,

(e) $|x|=0$, $x\in\mathbf{S}$ implies $x=0$,

(f) $a, b\in\mathbf{S}$, $ab=a$ implies either $a=0$ or $b=1$.

In [2], a series of examples of segment-like semigroups is given.

If S is a semigroup, $f, g\in S$, let us write $f>_u g$ iff $fg=f$. A bijection φ from a semigroup S_1 onto a semigroup S_2 is said to be a *u-isomorphism* iff

$$f >_u g \Leftrightarrow \varphi(f) >_u \varphi(g).$$

A semigroup isomorphism is, of course, a u-isomorphism but the converse is false (in a group, $f>_u g$ iff g is the unity element).

If X is a topological space and $\mathbf{S}$ is a segment-like semigroup, let us denote by $S(X)$ the set of all continuous functions from X into $\mathbf{S}$, equipped with pointwise multiplication, by $S_0(X)$, $S_\infty(X)$, and $S_c(X)$ the subsemigroups of $S(X)$ composed of functions with range contained in $[0, 1]$, of those vanishing at infinity in the case if X is locally compact (i.e. having an extension f^* to the one-point compactification $X\cup\{\infty\}$ of X such that $f^*(\infty)=0$), and of those with compact support, respectively, where the support of $f\in S(X)$ is the closure in X of the set $Z^c(f)$, and

$$Z^c(f) = X-Z(f), \quad Z(f) = \{x\in X\colon\ f(x) = 0\},$$
$$E(f) = \{x\in X\colon\ f(x) = 1\}.$$

2. The proof in [2] was based on the concept of u-ideals. Now we shall use a dual concept: in a semigroup S, $U\subset S$ is said to be a *u'-ideal* iff

(a) $\emptyset\neq U\neq S$,

(b) $f\in U, g\in S, f>_u g$ implies $g\in U$,

(c) $f, g\in U$ implies the existence of $h\in U$ such that

$$h >_u f, \quad h >_u g.$$

3. In the following, we assume that X is a locally compact Hausdorff space, $\mathbf{S}$ is a segment-like semigroup, and S is a subsemigroup of $S(X)$ satisfying

$$S_0(X)\cap S_c(X) \subset S \subset S_\infty(X).$$

Observe that, for $f, g\in S$, by Section 1, (f), $f>_u g$ iff $Z(f)\cup E(g)=X$. Also, if $f\in S$, then the closure $\overline{E(f)}$ is compact.

LEMMA 1. *If $K\subset X$ is compact, $F\subset X$ is closed, $K\cap F=\emptyset$, then there is an $h\in S_0(X)\cap S_c(X)$ such that $K\subset \operatorname{int} E(h)$, $F\subset Z(h)$.*

PROOF. By the local compactness of X, there is, for $x\in K$, a compact neighbourhood V_x of x such that $V_x\cap F=\emptyset$. There are $x_1, \dots, x_n\in K$ satisfying $K\subset \subset\bigcup_1^n \operatorname{int} V_{x_i}$. Then $K'=\bigcup_1^n V_{x_i}$ is compact and $K\subset\operatorname{int} K'$, $K'\cap F=\emptyset$. By [1], 5.3.g.9, there exists a $k\in S_0(X)$ such that $K\subset E(k)$, $X-\operatorname{int} K'\subset Z(k)$. Then $h= =\min(2k, 1)$ satisfies $h\in S_0(X)$, $K\subset\operatorname{int} E(h)$, $F\subset Z(h)$, $Z^c(h)\subset K'$, hence $h\in S_c(X)$. □

LEMMA 2. *For a compact set $\emptyset\neq K\subset X$,*

$$U_K = \{f\in S\colon\ K\subset \operatorname{int} E(f)\} \tag{2.1}$$

is a u'-ideal in S.

PROOF. $U_K\neq S$ because the constant 0 belongs to S but not to U_K. $U_K\neq\emptyset$ because, by Lemma 1 (applied for $F=\emptyset$), there exists an $h\in S_0(X)\cap S_c(X)\subset S$ such that $K\subset\operatorname{int} E(h)$. If $f\in U_K$, $g\in S$, $f>_u g$, then $Z(f)\cup E(g)=X$, hence

$$K\subset \operatorname{int} E(f) \subset Z^c(f) \subset E(g),$$

and $g\in U_K$. If f, $g\in U_K$ and, according to Lemma 1, $h\in S_0(X)\cap S_c(X)$ satisfies $K\subset\operatorname{int} E(h)$,

$$X-(\operatorname{int} E(f)\cap \operatorname{int} E(g)) \subset Z(h),$$

then we have $h \in S$, $K \subset \operatorname{int} E(h)$, hence $h \in U_K$, and $Z(h) \cup E(f) = Z(h) \cup E(g) = X$, so that $h >_u f$, $h >_u g$. □

LEMMA 3. *If* $\emptyset \neq K_1 \subset X$, $\emptyset \neq K_2 \subset X$ *are compact sets,* $K_1 \neq K_2$, *then* $U_{K_1} \neq U_{K_2}$.

PROOF. Assume, say, $x \in K_2 - K_1$. Then, by Lemma 1, there is an $h \in S_0(X) \cap \cap S_c(X) \subset S$ such that $K_1 \subset \operatorname{int} E(h)$, $x \in Z(h)$. Clearly $h \in U_{K_1}$, $h \notin U_{K_2}$. □

LEMMA 4. *If U is a u'-ideal in S, then $U = U_K$ for the compact set*

$$K = \bigcap_{f \in U} \overline{E(f)} \neq \emptyset. \tag{4.1}$$

PROOF. $\overline{E(f)}$ is compact for $f \in S$, and it is non-empty for $f \in U$; in fact, by Section 2, (c), there is an $h \in S$ such that $h >_u f$, $Z(h) \cup E(f) = X$, and $E(f) = \emptyset$ would imply $Z(h) = X$, $Z(h) \cup E(g) = X$ for every $g \in S$, whence $h >_u g$, $g \in U$ for $g \in S$ by Section 2, (b), $U = S$, in contradiction with Section 2, (a).

Now, $f, g, h \in U$, $h >_u f$, $h >_u g$ imply

$$Z(h) \cup E(f) = Z(h) \cup E(g) = X, \quad E(h) \subset Z^c(h) \subset E(f) \cap E(g).$$

Therefore $\{\overline{E(f)} : f \in U\}$ is a filter base composed of compact sets and it has a non-empty compact intersection K.

For $f \in U$, there is by Section 2, (c) an $h \in U$ such that $h >_u f$, so that

$$K \subset \overline{E(h)} \subset \{x \in X : |h(x)| = 1\} \subset Z^c(h) \subset E(f),$$

hence $K \subset \operatorname{int} E(f)$ ($Z^c(h)$ is open by Section 1, (c), (d), (e)), finally $f \in U_K$.

On the other hand, $f \in U_K$ impies $K \subset \operatorname{int} E(f)$ so that there is $g \in U$ satisfying $\overline{E(g)} \subset \operatorname{int} E(f)$ (otherwise the compact sets $\overline{E(g)} - \operatorname{int} E(f)$, $g \in U$ would constitute a filter base whose elements would contain a point not belonging to K). Choose $h \in U$, $h >_u g$; then

$$Z(h) \cup E(g) = Z(h) \cup E(f) = X,$$

$h >_u f$, and $f \in U$ by Section 2, (b). □

COROLLARY 5. *The mapping* $K \mapsto U_K$ *establishes a bijection from the set of all compact, non-empty subsets of X onto the set of all u'-ideals in S.* □

LEMMA 6. *For compact, non-empty subsets* K_1, $K_2 \subset X$, *we have*

$$K_1 \subset K_2 \quad \text{iff} \quad U_{K_1} \supset U_{K_2}.$$

PROOF. (2.1) and (4.1). □

LEMMA 7. *The mapping* $\varphi(x) = U_{\{x\}}$ *is a bijection from X onto the set M of all maximal u'-ideals in S, and the image of a compact* $\emptyset \neq K \subset X$ *is composed of all maximal u'-ideals that contain* U_K. □

COROLLARY 8. *The mapping φ defined in Lemma 7 is a homeomorphism from X onto M provided a subset F is defined to be closed in M iff* $F \cap \varphi(K) = \varphi(K')$ *for some compact* $K' \subset X$ *whenever* $K \subset X$ *is compact,* $K \neq \emptyset$.

PROOF. A subset of X is closed iff it intersects every compact, non-empty subset in a compact set. □

THEOREM 9. *Let* X_1 *and* X_2 *be locally compact Hausdorff spaces,* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *segment-like semigroups,* $S_i(X_i)$ *the semigroup of all continuous mappings from* X_i *into* $\mathbf{S}_i$ (equipped with pointwise multiplication), $S_{i0}(X_i)$, $S_{ic}(X_i)$, $S_{i\infty}(X_i)$ *the subsemigroups composed of mappings with range in* $[0, 1]$, *with compact support, and vanishing at infinity, respectively, finally* S_i *a subsemigroup of* $S_i(X_i)$ *satisfying*

$$S_{i0}(X_i) \cap S_{ic}(X_i) \subset S_i \subset S_{i\infty}(X_i).$$

If S_1 *and* S_2 *are u-isomorphic, then* X_1 *and* X_2 *are homeomorphic.* □

A similar theorem can be found in [3].

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6—8.
1088

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ON THE NUMBER OF POLYNOMIALS AND INTEGRAL ELEMENTS OF GIVEN DISCRIMINANT

J. H. EVERTSE* (Amsterdam) and K. GYŐRY* (Debrecen)

§ 1. Introduction

Let K be a field of characteristic 0, let R be a subring of K which has K as its quotient field, let G be a finite, normal extension of K and let R' be an integral extension ring of R in G. We shall suppose that either R is finitely generated over $\mathbf{Z}$ (we shall refer to this as the *absolute case*) or R is finitely generated over a field $\mathbf{k}$ of characteristic 0 which is algebraically closed in K (this will be called the *relative case*). Let $n \geqq 2$ be an integer. By $\Phi(n, R, R')$ we shall denote the set of all polynomials $f(X) \in R[X]$ of degree n which are monic and all of whose zeros are simple and belong to R'. By $\Phi(R, R')$ we denote the set $\bigcup_{n \geqq 2} \Phi(n, R, R')$. Let β be a fixed, non-zero element of R. We shall study the sets of polynomials $f(X) \in \Phi(R, R')$ satisfying

$$D(f) = \beta \tag{1}$$

or more generally

$$D(f) \in \beta R^*.[1] \tag{2}$$

Here $D(f)$ denotes the discriminant of f, i.e. if $f(X) = (X-\alpha_1)\dots(X-\alpha_n)$, then

$$D(f) = \prod_{1 \leqq i < j \leqq n} (\alpha_i - \alpha_j)^2.$$

We call two polynomials $f(X), g(X) \in R[X]$ *R-equivalent* if $g(X) = f(X+a)$ for some $a \in R$ and *weakly R-equivalent* if $g(X) = u^{\deg f} f(X/u + a)$ for some $u \in R^*$ and $a \in R$. The corresponding equivalence classes will be called R-equivalence classes and weak R-equivalence classes, respectively. If two polynomials f, g are R-equivalent then $D(f) = D(g)$ whereas if f, g are weakly R-equivalent then $D(f) = \varepsilon D(g)$ with some $\varepsilon \in R^*$.

In the absolute case Győry [6], [7] proved that if R is integrally closed in K then the polynomials $f(X) \in \Phi(R, R')$ which satisfy (1) belong to at most finitely many R-equivalence classes and the polynomials $f(X) \in \Phi(R, R')$ satisfying (2) belong to at most finitely many weak R-equivalence classes. Further, in [8] he showed that these equivalence classes can be determined effectively provided that R, K, G, R' and β are given explicitly in a certain well-defined sense (cf. [8], § 2.1). As consequences, in [8] (cf. also [9]) he obtained effective finiteness theorems for integral elements with

* The research was done at the University of Leiden in the academic year 1983/1984.

[1,2] If R is a ring, then R^* denotes its group of units and R^+ its additive group.

given discriminant (or which is the same, for irreducible polynomials with given discriminant) and for power bases over R. In [8], he also established effective results in the relative case by giving an effective bound for the Degree (cf. [8], § 2.1) of an appropriate representative of an arbitrary equivalence class. However, these assertions do not lead to finiteness results. For other historical remarks on (1), (2) and for further references, we refer to [4] and [9].

If R is integrally closed in K then $R' \cap K = R$. In the present paper our results will be established in the more general case when R^{+} [2] is a subgroup of finite index in $(R' \cap K)^{+}$. We shall derive both in the absolute and in the relative case explicit upper bounds for the number of R-equivalence classes of polynomials $f \in \Phi(R, R')$ satisfying (1) and for the number of weak R-equivalence classes of polynomials $f \in \Phi(R, R')$ satisfying (2). However, in the relative case we have to restrict ourselves to non-special polynomials (cf. §§ 3, 5). In both cases, we have attempted to give bounds which depend minimally on K, R, G, R' and β. For example, if in particular K is an algebraic number field with degree d and R is its ring of integers then our bounds depend only on d, $[G: K]$ and the number of distinct prime ideal divisors of β.

Our results concerning polynomials will be formulated in § 3. In § 4 we shall deduce similar quantitative finiteness results on integral elements over R with given discriminant and shall point out that our finiteness assertions do not remain valid if the factor group $(R' \cap K)^{+}/R^{+}$ is infinite. As a consequence, we shall give there among other things a generalisation of a result obtained on power bases in [3], which states that for every algebraic number field K of degree d the maximal number of pairwise weakly $\mathbf{Z}$-inequivalent algebraic integers $\alpha \in K$ for which $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is an integral basis of K is bounded above by a constant depending on d only. Here $\alpha, \beta \in K$ are called weakly $\mathbf{Z}$-equivalent if $\beta = \pm\alpha + a$ with some $a \in \mathbf{Z}$.

Our theorems will be proved in §§ 5 to 9. The proofs are based on some recent quantitative finiteness results on unit equations, due to Evertse [2] and Evertse and Győry [3].

§ 2. Preliminaries and notations

Let R_0 be either $\mathbf{Z}$ (the absolute case) or a field $\mathbf{k}$ of characteristic 0 (the relative case) and let K_0 denote the quotient field of R_0. (Thus $K_0 = \mathbf{Q}$ if $R_0 = \mathbf{Z}$ and $K_0 = \mathbf{k}$ if $R_0 = \mathbf{k}$). Let K be a finitely generated extension field of K_0. In case $R_0 = \mathbf{k}$ we suppose that $\mathbf{k}$ is algebraically closed in K. The field K has a finite transcendence basis over K_0, $\{z_1, \ldots, z_q\}$ say, where $q \geqq 0$. Put $K_1 = K_0(z_1, \ldots, z_q)$ and $R_1 = R_0[z_1, \ldots, z_q]$. Then K is a finite extension of K_1. Put $d = [K: K_1]$. We have the following diagram:

$$\begin{array}{ccccc} & & & & K \\ & & & & \cup \\ R_1 = R_0[z_1, \ldots, z_q] & \subset & & & K_1 = K_0(z_1, \ldots, z_q) \\ \cup & & & & \cup \\ R_0 & & \subset & & K_0 \end{array}$$

We note that R_1 is a unique factorisation domain with unit group $R_0^* = \{1, -1\}$ if $R_0 = \mathbf{Z}$ and $R_0^* = \mathbf{k}^*$ if $R_0 = \mathbf{k}$. Let I denote a maximal set of pairwise non-associated irreducible elements of R_1. To every $\pi\in I$ there corresponds a valuation[3] v_π on K_1 which is defined by $v_\pi(\pi)=1$ and $v_\pi(a/b)=0$ for any $a, b\in R_1$ not divisible by π. Note that for every $\alpha\in K_1^*$ there are at most finitely many $\pi\in I$ with $v_\pi(\alpha)\neq 0$. Every valuation v_π with $\pi\in I$ can be extended in at most d pairwise inequivalent ways to K. By replacing these extensions by equivalent valuations if necessary we obtain a set of valuations m_K on K with the following properties:

(3) every $V\in m_K$ has value group $\mathbf{Z}$;

(4) if $\alpha\in K^*$ then $V(\alpha)=0$ for all but finitely many $V\in m_K$;

(5) if $\alpha\in R_1$ then $V(\alpha)\geqq 0$ for all $V\in m_K$;

(6) if $\alpha\in R_0^*$ then $V(\alpha)=0$ for all $V\in m_K$.

In the sequel we shall use the following notations. If T is a subset of m_K, then we denote by $\mathcal{O}_T$ the ring $\{\alpha\in K: V(\alpha)\geqq 0$ for all $V\in m_K\setminus T\}$. Note that $\mathcal{O}_T^*= =\{\alpha\in K: V(\alpha)=0$ for all $V\in m_K\setminus T\}$.

If L/K is a finite extension, of degree p say, then one can construct in a similar way as above a set of valuations m_L on L with value group $\mathbf{Z}$. If we choose the same transcendence basis $\{z_1, \ldots, z_q\}$ for L, these valuations are, up to equivalence, just the extensions of the valuations in m_K to L. If $V\in m_K$, $W\in m_L$ and if W is equivalent to an extension of V to L then we say that W lies above V. For every $V\in m_K$ there are at most p valuations $W\in m_L$ lying above V.

The elements of the abelian group generated by m_K will be called *divisors*. Thus every divisor $\mathfrak{h}$ can be expressed as

$$\mathfrak{h} = \sum_{V\in m_K} V(\mathfrak{h})V,$$

where the $V(\mathfrak{h})$ are integers of which at most finitely many are non-zero. If $\alpha\in K^*$ then the divisor (α) is defined by $(\alpha)=\sum_{V\in m_K} V(\alpha)V$. If K is an algebraic number field then there exists an isomorphism $\mathfrak{C}_K$ of the additive group of divisors of K onto the multiplicative group of fractional ideals in K which is defined by $\mathfrak{C}_K(\mathfrak{h})=\{\alpha\in K: V(\alpha)\geqq V(\mathfrak{h})$ for all $V\in m_K\}$. $\mathfrak{C}_K$ maps m_K onto the set of prime ideals in K.

Let L/K be a finite extension of degree p in a fixed, finite, normal extension G of K. Let $\sigma_1, \ldots, \sigma_p$ denote the distinct K-isomorphisms of L in G and if $\alpha\in L$ put $\sigma_i(\alpha)=\alpha^{(i)}$. If $\mathbf{x}=(x_1, \ldots, x_p)\in L^p$ then

$$D(\mathbf{x}) = [\det(x_j^{(i)})_{\substack{i=1,\ldots,p\\ j=1,\ldots,p}}]^2$$

denotes the discriminant of $\mathbf{x}$ with respect to L/K. It is known that $D(\mathbf{x})\neq 0$ if and only if $x_1, \ldots, x_p$ are linearly independent over K. If $\mathbf{x}=(1, \alpha, \ldots, \alpha^{p-1})$ for some $\alpha\in L$ then we put $D_{L/K}(\alpha)=D(\mathbf{x})$. Then we have

$$D_{L/K}(\alpha) = \prod_{1\leqq i<j\leqq p} (\alpha^{(i)}-\alpha^{(j)})^2. \tag{7}$$

[3] By a valuation we shall always mean an additive, non-trivial, discrete valuation. By an absolute value we shall mean a non-trivial multiplicative valuation.

Finally, if $\mathbf{x}=(x_1, \ldots, x_p)$, $\mathbf{y}=(y_1, \ldots, y_p)\in L^p$ are vectors such that $y_i=\sum_{j=1}^{p} \xi_{ij} x_j$ for certain $\xi_{ij}\in K$, then

(8) $$D(\mathbf{y}) = [\det (\xi_{ij})_{\substack{i=1, \ldots, p \\ j=1, \ldots, p}}]^2 D(\mathbf{x}).$$

Let R' be a subring of L having L as its quotient field. We define the *discriminant divisor* $\mathfrak{D}_K(R')$ of R' over K by

$$V(\mathfrak{D}_K(R')) = \max \{0, \min_{\mathbf{x}\in R'^p} V(D(\mathbf{x}))\} \quad \text{for all} \quad V\in m_K.$$

By (4) this is indeed a divisor. If K is an algebraic number field and if R' is the ring of integers of L then the ideal $\mathfrak{C}_K(\mathfrak{D}_K(R'))$ is just the discriminant of L over K.

Let R be a subring of K and suppose that R' is an integral extension ring of R in L and that R' is a free R-module with basis $\mathbf{w}=(\omega_1, \ldots, \omega_p)$ say. Let T be a subset of m_K such that $R\subset \mathcal{O}_T$. If $\mathbf{w}'$ is an arbitrary vector in R'^p then, by (8),

(9) $$D(\mathbf{w}')\in D(\mathbf{w})R.$$

Hence

(10) $$V(\mathfrak{D}_K(R')) = V(D(\mathbf{w})) \quad \text{for all} \quad V\in m_K\setminus T.$$

§ 3. On polynomials with given discriminant

Let K, R_0, K_0, $\{z_1, \ldots, z_q\}$, R_1, K_1, d, m_K have the same meaning as in § 2. Thus R_0 is either $\mathbf{Z}$ (the absolute case) or a field $\mathbf{k}$ of characteristic 0 which is algebraically closed in K (the relative case). Let G/K be a finite, normal extension of degree g. Let $\overline{K}_0=K_0(=\mathbf{Q})$ if $R_0=\mathbf{Z}$ and let $\overline{K}_0$ be the algebraic closure of $K_0(=\mathbf{k})$ in G in the relative case. Let R be a subring of K which is finitely generated over R_0 and which has K as its quotient field. Further, let R' be an integral extension ring of R in G such that

(11) $$\mathcal{I}:= (R'\cap K^+) : R^+] < \infty.$$

We note that if R is integrally closed in K then $\mathcal{I}=1$. Further, in the relative case, (11) implies that $\mathcal{I}=1$, i.e. $R'\cap K=R$. Indeed, if (in the relative case) $R'\cap K\neq R$ and $a\in(R'\cap K)\setminus R$ then the elements in $a\mathbf{k}$ are contained in distinct cosets of $(R'\cap K)^+/R^+$. Hence $\mathcal{I}=\infty$.

Let β be a fixed, non-zero element of R and let T, T' be the smallest subsets of m_K such that $R\subset\mathcal{O}_T$, $R[\beta^{-1}]\subset\mathcal{O}_{T'}$. Then, by (4), T, T' have finite cardinalities, t, t' respectively, say.

Before stating our results we have to introduce the notion of *special* polynomials. In the absolute case, every polynomial $f(X)\in R[X]$ is called non-special. In the relative case, a polynomial $f(X)$ is called *special* in $R[X]$ if $f(X)\in R[X]$ and if

(12) $$f(X) = \mu^r h((X+a)^{n_0}/\mu)(X+a)^\delta,$$

where r, n_0, δ are integers with $r>0$, $n_0>0$, $\delta\in\{0, 1\}$, $rn_0+\delta\geqq 3$ and $\delta=0$ if $n_0=1$, where $a\in R$, where $\mu\in K^*$ is integral over R and where $h(X)\in\mathbf{k}[X]$ is a monic polynomial of degree r with non-zero discriminant[4] which has its zeros in $\bar{K}_0$ and $h(0)\neq 0$ if $n_0>1$. The polynomial $f\in R[X]$ is called non-special if it is not of the type (12). We notice that all polynomials which are weakly R-equivalent to a special polynomial in $R[X]$ must be special in $R[X]$ themselves.

As in § 1, $\Phi(n, R, R')(n\geqq 2)$ denotes the set of all monic polynomials of degree n with coefficients in R and with only simple zeros belonging to R'. Further, we put $\Phi(R, R')=\bigcup_{n\geqq 2}\Phi(n, R, R')$. By $N_1(R, R', \beta)$, $N_1(n, R, R', \beta)$ we shall denote the number of R-equivalence classes of *non-special* polynomials $f\in\Phi(R, R')$ and $f\in\Phi(n, R, R')$ respectively, which satisfy

$$D(f)=\beta, \tag{1}$$

whereas by $N_2(R, R', \beta)$, $N_2(n, R, R', \beta)$ we shall denote the number of weak R-equivalence classes of *non-special* polynomials $f\in\Phi(R, R')$ and $f\in\Phi(n, R, R')$ respectively, which satisfy

$$D(f)\in\beta R^*. \tag{2}$$

THEOREM 1. *Let n be an integer with $n\geqq 2$. Both in the absolute and in the relative case we have*

$$N_1(n, R, R', \beta)\leqq n(n-1)\frac{(4\cdot 7^{g(3d+2t')})^{n-2}}{(n-2)!}\mathcal{I},$$

$$N_2(n, R, R', \beta)\leqq \{n(n-1)\}^{[\bar{K}_0:\, K_0](d+t)}\frac{(4\cdot 7^{g(3d+2t')})^{n-2}}{(n-2)!}\mathcal{I}.$$

Let $\mathcal{W}_1$ be the set of special polynomials in $\Phi(n, R, R')$ satisfying (1) and let $\mathcal{W}_2$ be the set of special polynomials in $\Phi(n, R, R')$ satisfying (2) ($n\geqq 3$). We shall prove in § 5 that in the relative case $\mathcal{W}_2$ contains infinitely many weak R-equivalence classes, provided that $R'\supset\bar{K}_0$ and that $\mathcal{W}_2$ contains a special polynomial with $r\geqq 2$. We shall also show that $\mathcal{W}_1$ contains infinitely many R-equivalence classes in case $\mathbf{k}$ is algebraically closed and $\mathcal{W}_1$ contains a special polynomial with $r\geqq 2$.

We shall now present some consequences of Theorem 1.

COROLLARY 1. *Both in the absolute and in the relative case we have*

$$N_1(R, R', \beta)\leqq\mathcal{I}\exp\{8\cdot 7^{g(3d+2t')}\},$$

$$N_2(R, R', \beta)\leqq\mathcal{I}\exp\{8[\bar{K}_0:K_0](d+t)\cdot 7^{g(3d+2t')}\}.$$

PROOF. For $A=4\cdot 7^{g(3d+2t')}$ and for $p\in\mathbf{Z}$, $p\geqq 1$, we have, since $\{(k+2)(k+1)\}^p\leqq 2(p+1)^{2p+k-2}$ for $k\geqq 0$,

$$\sum_{k=0}^{\infty}\{(k+2)(k+1)\}^p\frac{A^k}{k!}\mathcal{I}\leqq 2(p+1)^{2p-2}\mathcal{I}\sum_{k=0}^{\infty}\frac{\{(p+1)A\}^k}{k!}=$$

$$=2(p+1)^{2p-2}\mathcal{I}e^{pA}\leqq\mathcal{I}e^{2pA}.$$

Hence our assertion follows from Theorem 1.

[4] For a linear polynomial $h(X)$, we put $D(h)=1$.

COROLLARY 2. *Let* $\gamma\in R$. *Then both in the absolute and in the relative case*

(i) *for every* $n\geqq 2$ *the number of non-special polynomials* $f\in\Phi(n, R, R')$ *which satisfy* (1) *and* $f(0)=\gamma$ *is at most*

$$n^2(n-1)\frac{(4\cdot 7^{g(3d+2t')})^{n-2}}{(n-2)!},$$

(ii) *the number of non-special polynomials* $f\in\Phi(R, R')$ *which satisfy* (1) *and* $f(0)=\gamma$ *is at most*

$$\exp\{8\cdot 7^{g(3d+2t')}\}.$$

PROOF. The ring $\tilde{R}=R'\cap K$ is finitely generated over R_0 (cf. [11], [12]). In the relative case (11) implies $\tilde{R}=R$. Further, both in the absolute and the relative case $\tilde{R}\subset\mathcal{O}_T$, $\tilde{R}[\beta^{-1}]\subset\mathcal{O}_{T'}$. Since $\Phi(n, R, R')\subset\Phi(n, \tilde{R}, R')$ and $\Phi(R, R')\subset\Phi(\tilde{R}, R')$, it suffices to prove our assertion with $\tilde{R}$ instead of R. The first part of Corollary 2 follows now immediately from Theorem 1, on noting that all polynomials in a fixed $\tilde{R}$-equivalence class are of the type $f(X)=f_0(X+a)$, where $a\in\tilde{R}$ and f_0 is a fixed representative of this class, and that there are at most n values of a for which $f_0(a)=\gamma$. The second part of Corollary 2 follows at once from the first part, on noting that for $A=4\cdot 7^{g(3d+2t')}$,

$$\sum_{k=0}^{\infty}(k+2)^2(k+1)\frac{A^k}{k!}=(A^3+8A^2+14A+4)e^A\leqq e^{2A}.$$

Corollary 1 already shows that a polynomial $f\in\Phi(R, R')$ which is non-special and which satisfies (2) must have bounded degree. More explicitly we have

THEOREM 2. *Both in the absolute and the relative case, every non-special polynomial* $f\in\Phi(R, R')$ *which satisfies* (2) *has degree at most*

$$2+4\cdot 7^{g(3d+2t')}.$$

In the absolute case, the finiteness assertions of Theorems 1, 2 and their corollaries above were earlier proved by Győry [6] (cf. also Győry [7]) under the restriction that R is integrally closed in K. Effective versions of these results were later obtained by Győry [8]. Further, he established in [8] certain effective analogues also in the relative case.

We shall now specialise our results above to the case of algebraic number fields. Let K be an algebraic number field of degree d with ring of integers $\mathcal{O}_K$ and let G/K be a normal extension of degree g. Let $\mathcal{O}_G$ be the ring of integers of G. Let $\beta\in\mathcal{O}_K\setminus\{0\}$ and let $S=\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be a (possibly empty) set of prime ideals in K. Let t' denote the number of prime ideals which belong to S or divide $\langle\beta\rangle$.[5] We call two polynomials $f(X), g(X)\in\mathcal{O}_K[X]$ *weakly S-equivalent* if there are $a, b, c\in\mathcal{O}_K$ such that $\langle b\rangle$, $\langle c\rangle$ are solely composed of prime ideals from S (b, c are units if $t=0$) and such that

$$g(X)=\left(\frac{b}{c}\right)^{\deg f}f\left(\frac{cX+a}{b}\right).$$

[5] $\langle\alpha\rangle$ denotes the ideal in $\mathcal{O}_K$ generated by α.

COROLLARY 3. *Let n be an integer with $n \geqq 2$. Then the polynomials $f(X) \in \in \Phi(n, \mathcal{O}_K, \mathcal{O}_G)$ with the property*

$$\langle D(f)\rangle = \langle\beta\rangle \mathfrak{p}_1^{k_1} \dots \mathfrak{p}_t^{k_t} \tag{13}$$

for certain rational integers $k_1, \dots, k_t$ belong to at most

$$\{n(n-1)\}^{d+t} \frac{(4 \cdot 7^{g(3d+2t')})^{n-2}}{(n-2)!}$$

weak S-equivalence classes.

For an effective finiteness result concerning the polynomials $f \in \Phi(n, \mathcal{O}_K, \mathcal{O}_G)$ which satisfy (13), see Győry [5].

PROOF OF COROLLARY 3. Let $\mathfrak{C}_K$ be the isomorphism of the group of divisors of K onto the group of fractional ideals in K (cf. § 2) and let $T = \mathfrak{C}_K^{-1}(S)$. Now Corollary 3 follows at once from Theorem 1 on noting that every polynomial $f(X) \in \in \Phi(n, \mathcal{O}_K, \mathcal{O}_G)$ which satisfies (13) also satisfies $D(f) \in \beta\mathcal{O}_T^*$ and that two polynomials $f(X), g(X) \in \Phi(n, \mathcal{O}_K, \mathcal{O}_G)$ are weakly S-equivalent if and only if they are weakly $\mathcal{O}_T$-equivalent.

§ 4. On integral elements with given discriminant

Let K, R_0, K_0, $\{z_1, \dots, z_q\}$, R_1, K_1, d, m_K have the same meaning as in § 2. Let L/K be a finite extension of degree $m \geqq 2$ and let G denote the normal closure of L over K. Put $[G: K] = g$. In the relative case (when $R_0 = \mathbf{k}$) we assume something stronger than in § 2, namely that $\mathbf{k}$ is algebraically closed in G. Let $\sigma_1, \dots, \sigma_m$ denote the distinct K-isomorphisms of L in G. If $\alpha \in L$ then we put $\alpha^{(i)} = \sigma_i(\alpha)$, $i = 1, \dots, m$. Let R be a subring of K which is finitely generated over R_0 and let $R' \subset L$ be an integral extension ring of R with quotient field L such that

$$\mathcal{I} = [(R' \cap K)^+ : R^+] < \infty. \tag{11}$$

If $\alpha \in R'$, then by (7) the discriminant $D_{L/K}(\alpha)$ of α is equal to $\prod_{1 \leqq i < j \leqq d} (\alpha^{(i)} - \alpha^{(j)})^2$. Hence if $L = K(\alpha)$ then $D_{L/K}(\alpha)$ is equal to the discriminant of the minimal polynomial of α over K. For that reason we call two elements $\alpha_1, \alpha_2 \in R'$ *R-equivalent* if $\alpha_2 = \alpha_1 + a$ for some $a \in R$ and *weakly R-equivalent* if $\alpha_2 = u\alpha_1 + a$ for some $a \in R$, $u \in R^*$. As usual, the corresponding equivalence classes will be called R-equivalence classes and weak R-equivalence classes, respectively. If $\alpha_1, \alpha_2 \in R'$ are R-equivalent then $D_{L/K}(\alpha_1) = D_{L/K}(\alpha_2)$ while if $\alpha_1, \alpha_2 \in R'$ are weakly R-equivalent then $D_{L/K}(\alpha_1) = \varepsilon D_{L/K}(\alpha_2)$ with some $\varepsilon \in R^*$.

Let T be the smallest subset of m_K such that $R \subset \mathcal{O}_T$. Let $\mathfrak{D}_K(R')$ be the discriminant divisor of R' over K and let β be a fixed element of K^*. Let T'' be the smallest subset of m_K such that $R \subset \mathcal{O}_{T''}$ and $V(\beta) = V(\mathfrak{D}_K(R'))$ for all $V \in m_K \setminus T''$. The sets T, T'' have finite cardinalities t, t'' respectively, say. Let $M_1(R, R', \beta)$ denote the number of R-equivalence classes of $\alpha \in R'$ satisfying

$$D_{L/K}(\alpha) = \beta \tag{14}$$

and let $M_2(R, R', \beta)$ denote the number of weak R-equivalence classes of $\alpha \in R'$ satisfying

(15) $$D_{L/K}(\alpha) \in \beta R^*.$$

THEOREM 3. *Both in the absolute and the relative case we have*

$$M_1(R, R', \beta) \leqq m(m-1)(4 \cdot 7^{g(3d+2t'')})^{m-2} \cdot \mathscr{I},$$

$$M_2(R, R', \beta) \leqq \{m(m(-1)\}^{d+t}(4 \cdot 7^{g(3d+2t'')})^{m-2} \cdot \mathscr{I}.$$

We note that $g \leqq m!$. Notice that we have also a finiteness result (without exclusion of "special" integral elements) in the relative case. It is not clear whether such a finiteness result holds if $\mathbf{k}$ is not algebraically closed in G. Finally, we remark that if $\mathscr{I}=\infty$ and if there is an $\alpha \in R'$ satisfying (14) (resp. (15)) then $M_1(R, R', \beta)$ (resp. $M_2(R, R', \beta)$) is infinite. Indeed, in this case the (weak) $(R' \cap K)$-equivalence class of α in question splits into infinitely many (weak) R-equivalence classes.

Let $N_{L/K}$ denote the norm with respect to L/K. Then every $(R' \cap K)$-equivalence class of elements of R' contains at most m elements α for which $N_{L/K}(\alpha)$ assumes some fixed value. Thus, applying Theorem 3 to $M_1(R' \cap K, R', \beta)$ we have

COROLLARY 4. *Let $\gamma \in K$. Then the number of $\alpha \in R'$ with $D_{L/K}(\alpha)=\beta$ and $N_{L/K}(\alpha)=\gamma$ is at most*

$$m^2(m-1)(4 \cdot 7^{g(3d+2t'')})^{m-2}.$$

The above argument shows that Corollary 4 is true without assuming $\mathscr{I}<\infty$.

Let $\alpha \in R'$. We call $\{1, \alpha, \ldots, \alpha^{m-1}\}$ a *power basis* if $\{1, \alpha, \ldots, \alpha^{m-1}\}$ is a basis of R' as a free R-module. If this is the case and if $\alpha' \in R'$ is weakly R-equivalent to α then $\{1, \alpha', \ldots, \alpha'^{m-1}\}$ is also an R-basis of R'. From Theorem 3 it follows

COROLLARY 5. *Those $\alpha \in R'$ for which $\{1, \alpha, \ldots, \alpha^{m-1}\}$ is an R-basis of R' belong to at most*

$$\{m(m-1)\}^{d+t}(4 \cdot 7^{g(3d+2t)})^{m-2} \cdot \mathscr{I}$$

weak R-equivalence classes.

In [3] (cf. Theorem 11) we derived the bound $(4 \cdot 7^{g(3d+2t)})^{m-2}$ in case $R_0=\mathbf{Z}$ and R is integrally closed in K. If $R_0=\mathbf{k}$ and R is integrally closed in K then it is also possible to get rid of the factor $\{m(m-1)\}^{d+t}$ but we shall not work this out here.

In the absolute case, Győry [6] (cf. also Győry [7]) proved earlier the finiteness assertions of Theorem 3 and its corollaries above under the assumption that R is integrally closed in K. Later he obtained [8], [9] effective versions of these results. In [8], certain effective analogues have been established also in the relative case.

PROOF OF COROLLARY 5. Suppose that R' has an R-basis of the form $\{1, \alpha_0, \ldots, \alpha_0^{m-1}\}$. This is clearly no restriction. In view of (9), $\{1, \alpha, \ldots, \alpha^{m-1}\}$ is an R-basis of R' only if

(16) $$D_{L/K}(\alpha) \in D_{L/K}(\alpha_0) R^*.$$

By (10), $V(\mathfrak{D}_K(R'))=V(D_{L/K}(\alpha_0))$ for all $V \in m_K \setminus T$. Now Corollary 5 follows immediately from (16) and Theorem 3 with $\beta=D_{L/K}(\alpha_0)$.

Let K, L be algebraic number fields with rings of integers $\mathcal{O}_K$, $\mathcal{O}_L$ respectively, where $K\subset L$, $[K:\mathbf{Q}]=d$ and $[L:K]=m$. Let G denote the normal closure of L over K and put $g=[G:K]$. Let $\mathfrak{D}_{L/K}$ denote the discriminant of L over K. For every $\alpha\in\mathcal{O}_L$ with $D_{L/K}(\alpha)\neq 0$ the ideal $\langle D_{L/K}(\alpha)\rangle\mathfrak{D}_{L/K}^{-1}$ is the square of an integral ideal, $\mathfrak{J}(\alpha)$ say, which is called the *index* of α with respect to L/K. Let $\mathfrak{a}$ be a fixed ideal in $\mathcal{O}_K$ and let $S=\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ be a finite (possibly empty) set of prime ideals in $\mathcal{O}_K$. We shall now deal with the set of $\alpha\in\mathcal{O}_L$ satisfying

$$\mathfrak{J}(\alpha)=\mathfrak{a}\mathfrak{p}_1^{k_1}\dots\mathfrak{p}_t^{k_t} \quad \text{for certain} \quad k_1, \dots, k_t\in\mathbf{Z}. \tag{17}$$

We call $\alpha_1, \alpha_2\in\mathcal{O}_L$ weakly S-equivalent if there are $a, b, c\in\mathcal{O}_K$ with $\langle b\rangle$, $\langle c\rangle$ solely composed of prime ideals from S, such that

$$\alpha_2=\frac{b\alpha_1+a}{c}.$$

If α satisfies (17) then all elements of $\mathcal{O}_L$ which are S-equivalent to α also satisfy (17). Let t'' denote the number of prime ideals which divide $\mathfrak{a}$ or belong to S. Then we have

COROLLARY 6. *The numbers* $\alpha\in\mathcal{O}_L$ *which satisfy* (17) *belong to at most*

$$\{m(m-1)\}^{d+t}(4\cdot 7^{g(3d+2t'')})^{m-2}$$

weak S-equivalence classes.

An effective finiteness result concerning the elements $\alpha\in\mathcal{O}_L$ satisfying (17) can be found in Győry [5].

PROOF OF COROLLARY 6. Let $T=\mathfrak{C}_K^{-1}(S)$ (cf. § 2 and the proof of Corollary 3 in § 3). Suppose that (17) is solvable. Let α_0 be a solution of (17) and put $D_{L/K}(\alpha_0)=\beta$. Then every solution $\alpha\in\mathcal{O}_L$ of (17) satisfies $D_{L/K}(\alpha)\in\beta\mathcal{O}_T^*$ and two elements $\alpha_1, \alpha_2\in\mathcal{O}_L$ are S-equivalent if and only if they are $\mathcal{O}_T$-equivalent. Now Corollary 6 follows easily from Theorem 3.

§ 5. On special polynomials

Let $\mathbf{k}$ be a field of characteristic 0, let K be a field which is finitely generated over $\mathbf{k}$ and let G/K be a finite, normal extension. As in § 2, we suppose that $\mathbf{k}$ is algebraically closed in K. The algebraic closure of $\mathbf{k}$ in G is denoted by $\bar{K}_0$. Let R be a subring of K which has K as its quotient field (and which is now not necessarily finitely generated over $\mathbf{k}$). We extend the concept of special polynomials defined in § 3 by calling a polynomial $f(X)$ *special* in $R[X]$ if $f(X)\in R[X]$ and if

$$f(X)=\mu^r h\big((X+a)^{n_0}/\mu\big)(X+a)^\delta, \tag{12}$$

where r, n_0, δ are integers with $r>0$, $n_0>0$, $\delta\in\{0, 1\}$, $rn_0+\delta\geqq 3$ and $\delta=0$ if $n_0=1$, where $a\in R$, where $\mu\in K^*$ is integral over R and where $h(X)$ is a monic polynomial of degree r with coefficients in $\mathbf{k}$ and zeros in $\overline{K}_0$ such that $D(h)\neq 0$ and $h(0)\neq 0$ if $n_0>1$. If f satisfies (12) then $\deg f=rn_0+\delta\geqq 3$ and

$$D(f)=(-1)^{rn_0(n_0-1)/2}n_0^{rn_0}\mu^{r(rn_0-1+2\delta)}h(0)^{n_0-1+2\delta}D(h)^{n_0}\neq 0 \tag{18}$$

(with the convention that $h(0)^{n_0-1+2\delta}=1$ if $n_0=1$ and $h(0)=0$).

LEMMA 1. *Let $n\geqq 3$ be an integer and let $f(X)\in R[X]$ be a polynomial of degree n with zeros $\alpha_1, \dots, \alpha_n \in G$. Then the following statements are equivalent:*

(i) *f is special in $R[X]$;*

(ii) *there are $a\in R$, $\lambda\in G^*$ and $c_1, \dots, c_n\in\overline{K}_0$ such that $\alpha_i=c_i\lambda-a$ $(i=1, \dots, n)$;*

(iii) *there are integers $i, j\in\{1, \dots, n\}$ with $i\neq j$ such that for all $k\in\{1, \dots, n\}$ we have $(\alpha_i-\alpha_k)/(\alpha_i-\alpha_j)\in\overline{K}_0$.*

PROOF. (i)⇒(ii). Suppose that f satisfies (12). Let $\Theta_1, \dots, \Theta_r$ be the zeros of $h(X)$ in $\overline{K}_0$ and suppose that $\Theta_1\neq 0$. Then f can be written as

$$f(X)=\prod_{i=1}^{r}\{(X+a)^{n_0}-\Theta_i\mu\}(X+a)^{\delta}.$$

Choose $\lambda\in G^*$ such that $\lambda^{n_0}=\Theta_1\mu$. Then there are $c_1, \dots, c_n\in\overline{K}_0$ such that

$$f(X)=\prod_{i=1}^{n}(X+a-c_i\lambda).$$

This clearly proves (ii).

(ii)⇒(iii). If $\alpha_i=c_i\lambda-a$ for $i=1, \dots, n$, where $a\in R$, $\lambda\in G^*$ and $c_1, \dots, c_n\in\overline{K}_0$, then we have for all triples (i, j, k) with $1\leqq i, j, k\leqq n$ and $i\neq j$ that

$$\frac{\alpha_i-\alpha_k}{\alpha_i-\alpha_j}=\frac{c_i-c_k}{c_i-c_j}\in\overline{K}_0.$$

(iii)⇒(ii). Put $\lambda=\alpha_i-\alpha_j$. Then we have for $k, l\in\{1, \dots, n\}$

$$\frac{\alpha_k-\alpha_l}{\alpha_i-\alpha_j}=\frac{\alpha_i-\alpha_l}{\alpha_i-\alpha_j}-\frac{\alpha_i-\alpha_k}{\alpha_i-\alpha_j}\in\overline{K}_0,$$

hence

$$\alpha_k-\alpha_l=c_{kl}\lambda \tag{19}$$

for some $c_{kl}\in\overline{K}_0$. Put $a=-(\alpha_1+\dots+\alpha_n)/n$ and $c_k=(c_{k1}+\dots+c_{kn})/n$. Then $c_k\in\overline{K}_0$ and $a\in R$, in view of the facts that $f(X)\in R[X]$ and $n^{-1}\in\mathbf{k}\subset R$. Therefore, by (19), on taking the sum over all l, we have

$$\alpha_k=c_k\lambda-a \quad \text{for} \quad k=1, \dots, n.$$

This proves (ii).

(ii)⇒(i). Let $g(X)=f(X-a)=\prod_{i=1}^{n}(X-c_i\lambda)$. Then $g(X)\in R[X]$. Let A be the set of rational integers m such that $\lambda^m=c\zeta$ for some $c\in\overline{K}_0$ and $\zeta\in K$. It is easy to show that A is an ideal in $\mathbf{Z}$. Since at least one coefficient of g is non-zero, A contains nonzero integers. Let n_0 be a positive integer which generates A. Let r, δ be integers with $n=rn_0+\delta$ and $0\leqq\delta<n_0$. Then $g(X)$ can be written as

$$g(X) = X^n+d_1X^{n-n_0}\lambda^{n_0}+\ldots+d_rX^\delta\lambda^{rn_0}, \tag{20}$$

where $d_1, \ldots, d_r\in\bar{K}_0$. Note that $D(g)=D(f)\neq 0$, whence $\delta\in\{0, 1\}$. Choose $c\in\bar{K}_0$ such that $\lambda^{n_0}=c\mu$ where $\mu\in K$. Then μ is integral over R. Put $h_i=d_ic^i$ $(i=1, \ldots, r)$, $h(X)=X^r+h_1X^{r-1}+\ldots+h_r$. Since $d_i\lambda^{in_0}=h_i\mu^i$ for $i=1, \ldots, r$ and $g(X)\in R[X]$ we have $h(X)\in\mathbf{k}[X]$. By (20) we obtain

$$g(X) = \mu^r h(X^{n_0}/\mu)X^\delta \quad (r>0,\ n_0>0,\ \delta\in\{0, 1\},\ rn_0+\delta = n). \tag{21}$$

The zeros of h obviously belong to $\bar{K}_0$. It is also clear, by our choice of r, δ, that $\delta=0$ if $n_0=1$ and $h(0)\neq 0$ if $n_0>1$. Now (i) follows immediately from (21) and $f(X)=g(X+a)$.

Let R be a finitely generated subring of K over $\mathbf{k}$ which has K as its quotient field, and let R' be an integral extension ring of R in G such that $R'\cap K=R$. In the lemma below we shall state some results about the sets of polynomials

$$\mathscr{V}_1 = \{f(X)\in\Phi(n, R, R')\colon f \text{ is special in } R[X] \text{ with } r\geqq 2 \text{ and } D(f)=\beta\},$$

$$\mathscr{V}_2 = \{f(X)\in\Phi(n, R, R')\colon f \text{ is special in } R[X] \text{ with } r\geqq 2 \text{ and } D(f)\in\beta R^*\},$$

where β is an element of $R\setminus\{0\}$ and $n\geqq 3$ is an integer.

LEMMA 2. (i) *Suppose that* $\bar{K}_0\subset R'$. *If* $\mathscr{V}_2$ *is non-empty then it contains infinitely many weak R-equivalence classes of polynomials.*

(ii) *Suppose that* $\mathbf{k}$ *is algebraically closed. If* $\mathscr{V}_1$ *is non-empty then it contains infinitely many R-equivalence classes of polynomials.*

PROOF. If $\bar{K}_0\subset R'$ (which is also the case if $\mathbf{k}$ is algebraically closed) then for every polynomial $f(X)\in\Phi(n, R, R')$ satisfying (12) we have $\mu\in R$. Indeed, there exists a $c\in\bar{K}_0^*$ such that $c\mu$ is the product of certain zeros of f. Therefore $c\mu\in R'$ and hence $\mu\in R'\cap K=R$. Let n_0, r, δ be integers with $n=rn_0+\delta$, $r>0$, $n_0>0$, $\delta\in\{0, 1\}$, $\delta=0$ if $n_0=1$. Let $\mu\in R\setminus\{0\}$. Put $h_m(X)=(X-1)(X-2)(X-6m)\times\times(X-8m)\ldots(X-2rm)$ if $r\geqq 3$ and $h_m(X)=(X-1)(X-m)$ if $r=2$ $(m=1, 2, \ldots)$. Let

$$\mathscr{S} = \mathscr{S}(n_0, r, \delta, \mu) = \{\mu^r h_m(X^{n_0}/\mu)X^\delta\colon m = 1, 2, \ldots\}.$$

We shall show that the polynomials in $\mathscr{S}$ are pairwise R-inequivalent. Let $f_p(X)=\mu^r h_p(X^{n_0}/\mu)X^\delta$, $f_q(X)=\mu^r h_q(X^{n_0}/\mu)X^\delta$ be polynomials in $\mathscr{S}$ which are weakly R-equivalent. Then there are $a\in R$ and $u\in R^*$ such that

$$\mu^r h_q(X^{n_0}/\mu)X^\delta = \mu^r u^n h_p\left(\left(\frac{X+a}{u}\right)^{n_0}\Big/\mu\right)\left(\frac{X+a}{u}\right)^\delta = \tag{22}$$

$$= (\mu u^{n_0})^r h_p\left(\frac{(X+a)^{n_0}}{\mu u^{n_0}}\right)(X+a)^\delta.$$

First suppose that $n_0>1$. Then the left-hand side of (22) can be written as $X^n+\gamma_1X^{n-n_0}+\ldots$, whereas the right-hand side of (22) can be written in the form

$(X+a)^n+\varrho_1(X+a)^{n-n_0}+\ldots=X^n+naX^{n-1}+\ldots$ with some $\gamma_1, \delta_1 \in K$. Hence $a=0$. Therefore, by (22) we have

$$\mu^r h_q(X^{n_0}/\mu)X^\delta = (\mu u^{n_0})^r h_p(X^{n_0}/\mu u^{n_0})X^\delta$$

which implies that $h_q(X)=u^{n_0 r}h_p(X/u^{n_0})$. Thus the zeros of $h_q(X)$ are just equal to the zeros of $h_p(X)$ multiplied by u^{n_0}. But then $u^{n_0}=1$, $p=q$. Hence $f_p(X)=f_q(X)$.

Now suppose that $n_0=1$. Then $\delta=0$ and $r=n\geqq 3$. Hence, by (22),

$$\mu^n h_q(X/\mu) = (\mu u)^n h_p\left(\frac{X+a}{\mu u}\right).$$

This in turn implies that

$$h_q(X) = u^r h_p\left(\frac{X}{u}+\frac{a}{\mu u}\right). \tag{23}$$

Let $\alpha_1, \ldots, \alpha_r$ be the zeros of $h_p(X)$. By (23) there is an $\alpha \in K$ such that $u\alpha_i+\alpha$ $(i=1, \ldots, r)$ are just the zeros of $h_q(X)$. But since $r\geqq 3$, it follows that $u=1, \alpha=0$. Hence $p=q$.

Suppose that $\mathscr{V}_2$ is non-empty and let $f(X)=\mu^r h((X+a)^{n_0}/\mu)X^\delta$ $(rn_0+\delta=n$ and μ, a, h are as in (12)) be an element of $\mathscr{V}_2$. Note that $\mu \in R\setminus\{0\}$. By (18), $\mu^{r(rn_0-1+2\delta)} \in \beta R^*$. By (18) we have also $\mathscr{S}=\mathscr{S}(n_0, r, \delta, \mu) \subseteqq \mathscr{V}_2$. But $\mathscr{S}$ contains infinitely many polynomials which are pairwise weakly R-inequivalent. This proves (i).

Suppose that $\mathscr{V}_1$ is non-empty and let $f(X)=\mu^r h((X+a)^{n_0}/\mu)X^\delta \in \mathscr{V}_2$ (r, n_0, δ, μ, h have the same meaning as in the proof of (i)). Then (18) implies that

$$c\mu^{r(rn_0-1+2\delta)}(-1)^{rn_0(n_0-1)/2}n_0^{rn_0} = \beta, \quad \text{where} \quad c = h(0)^{n_0-1+2\delta}D(h)^{n_0} \neq 0.$$

Put

$$\alpha = \alpha(H) = \left[\frac{c}{H(0)^{n_0-1+2\delta}D(H)^{n_0}}\right]^{1/(r(rn_0+2\delta-1))}, \quad H^*(X) = \alpha^r H(X/\alpha)$$

for every monic polynomial $H(X)\in \mathbf{k}[X]$ of degree r with $D(H)\neq 0$ and $H(0)\neq 0$. Since $\mathbf{k}$ is algebraically closed, $H^*(X)$ is also a monic polynomial of degree r with coefficients in $\mathbf{k}$. Further, $H^*(0)^{n_0-1+2\delta}D(H^*)^{n_0}=c$. Hence the set

$$\mathscr{S}^* = \{\mu^r h_m^*(X^{n_0}/\mu)X^\delta\colon m = 1, 2, \ldots\}$$

is contained in $\mathscr{V}_1$. But it is easy to check that all these polynomials are pairwise R-inequivalent. This proves (ii).

Remark. The question whether the set $\mathscr{V}_1$ contains infinitely many R-equivalence classes of polynomials in case $\mathbf{k}$ is not algebraically closed seems to be far more difficult to answer. Moreover, if (1) (resp. (2)) can only be satisfied by special polynomials with $r=1$ then it is possible that there are only finitely many (weak) R-equivalence classes of special polynomials satisfying (1) (resp. (2)).

§ 6. On units and unit equations

Let K, R_0, K_0, $\{z_1, \dots, z_q\}$, R_1, K_1, d, m_K have the same meaning as in § 2. Let T be a finite subset of m_K of cardinality $t \geqq 0$. In this section we shall state some properties of the group $\mathcal{O}_T^* = \{\alpha \in K : V(\alpha) = 0 \text{ for all } V \in m_K \setminus T\}$.

LEMMA 3. (i) *If* $R_0 = \mathbf{Z}$ *then* $\mathcal{O}_T^* \cong W \times \mathbf{Z}^p$, *where* W *is the finite group of roots of unity in* K *and* $0 \leqq p \leqq d + t - 1$.

(ii) *If* $R_0 = \mathbf{k}$ *and* $\mathbf{k}$ *is algebraically closed in* K *then* $\mathcal{O}_T^*/\mathbf{k}^* \cong \mathbf{Z}^p$ *where* $0 \leqq p \leqq d + t - 1$.

PROOF. First of all we shall prove (ii). There exists a set of pairwise inequivalent absolute values $\{|\,.\,|_v\}_{v \in M_K}$ on K with the following properties (cf. [2], § 3.):

(24) If $\alpha \in K^*$ then $|\alpha|_v = 1$ for all but finitely many $v \in M_K$ and $\prod_{v \in M_K} |\alpha|_v = 1$.

$$M_K = I_K \cup P_K, \quad \text{where} \quad I_K \cap P_K = \emptyset, \tag{25}$$

where the valuations in the set $\{-\log |\,.\,|_v : v \in P_K\}$ are, up to equivalence, equal to the valuations in m_K and where the valuations in the set $\{-\log |\,.\,|_v : v \in I_K\}$ are, up to equivalence, equal to the extensions of the valuation V_∞ on $K_1 = \mathbf{k}(z_1, \dots, z_q)$. Here V_∞ is defined by $V_\infty(F/G) = b - a$ for all polynomials $F, G \in R_1 \setminus \{0\}$ of total degrees a, b respectively.

$$\{\alpha \in K : |\alpha|_v = 1 \quad \text{for all} \quad v \in M_K\} = \mathbf{k}^*. \tag{26}$$

Let $S \subset M_K$ be the set containing the $v \in I_K$ and the $v \in P_K$ for which $-\log |\,.\,|_v$ is equivalent to a valuation in T. Let $S = \{v_1, v_2, \dots, v_s\}$. Since I_K has cardinality $\leqq d$, we have $s \leqq d + t$. Let $\mathfrak{h}$ be the homomorphism from $\mathcal{O}_T^*$ to $\mathbf{R}^s$ defined by

$$\mathfrak{h}(\alpha) = (\log |\alpha|_{v_1}, \dots, \log |\alpha|_{v_s}).$$

The elements α of $\mathcal{O}_T^*$ satisfy $|\alpha|_v = 1$ for $v \in M_K \setminus S$ and $\sum_{i=1}^{s} \log |\alpha|_{v_i} = 0$ (cf. (24)). Hence $\ker \mathfrak{h} = \mathbf{k}^*$ and the image of $\mathfrak{h}$ is a discrete group of rank $\leqq s - 1$. Thus $\mathcal{O}_T^*/\mathbf{k}^* \cong \mathbf{Z}^p$ for some integer p with $0 \leqq p \leqq d + t - 1$.

We now prove (i). Let $\mathbf{k}_0$ denote the algebraic closure of $\mathbf{Q}$ in K. Put $d_1 = [\mathbf{k}_0 : \mathbf{Q}]$, $d_2 = [K : \mathbf{k}_0(z_1, \dots, z_q)]$. Then $d_1 d_2 = d$. Let $m_K^{(1)}$ be the set of valuations in m_K whose restriction to $\mathbf{k}_0$ is non-trivial and let $m_K^{(2)} = m_K \setminus m_K^{(1)}$. Let $T_i = T \cap m_K^{(i)}$ $(i = 1, 2)$ and let t_i denote the cardinality of T_i $(i = 1, 2)$. There exists a one-to-one correspondence between the valuations in $m_K^{(1)}$ and the prime ideals in $\mathbf{k}_0$ (cf. § 2). Let $\mathfrak{p}_1, \dots, \mathfrak{p}_{t_1}$ be the prime ideals corresponding to the valuations in T_1. Then $\mathcal{O}_T^* \cap \mathbf{k}_0^* = \{\alpha \in \mathbf{k}_0^* : \langle \alpha \rangle = \mathfrak{p}_1^{k_1} \dots \mathfrak{p}_{t_1}^{k_{t_1}} \text{ for certain } k_1, \dots, k_{t_1} \in \mathbf{Z}\}$. By Lang [10], Ch. 5, $\mathcal{O}_T^* \cap \mathbf{k}_0^* \cong W \times \mathbf{Z}^{r + t_1}$, where W is the group of roots of unity in $\mathbf{k}_0$ and r is the rank of the group of units in the ring of integers of $\mathbf{k}_0$. The valuations in $m_K^{(2)}$ lie above the valuations on $\mathbf{k}_0(z_1, \dots, z_q)$ which correspond to irreducible polynomials of degree $\geqq 1$ in $\mathbf{k}_0[z_1, \dots, z_q]$. Hence there exists a set of absolute values $\{|\,.\,|_v\}_{v \in M_K}$ satisfying the properties (24) to (26) with $\mathbf{k}_0$, $m_K^{(2)}$ instead of $\mathbf{k}$, m_K, respectively. Hence by (ii), $\mathcal{O}_T^*/\mathcal{O}_T^*\cap\mathbf{k}_0^*\cong\mathbf{Z}^{p_2}$ where p_2 is an integer with $0\leqq p_2\leqq d_2+t_2-1$. This is true since $\mathcal{O}_T^*/\mathcal{O}_T^*\cap\mathbf{k}_0^*\subset\mathcal{O}_{T_2}^*/\mathbf{k}_0^*$. But this shows that

$$\mathcal{O}_T^*\cong W\times\mathbf{Z}^{r+t_1+p_2}=W\times\mathbf{Z}^p$$

say, where $0\leqq p\leqq d_1+t_1-1+d_2+t_2-1\leqq d+t-1$.

Let λ, $\mu\in K^*$. We shall now deal with the equation

$$\lambda x+\mu y=1 \quad \text{in} \quad x, y\in\mathcal{O}_T^*. \tag{27}$$

LEMMA 4. (i) *In the absolute case* (27) *has at most* $4\cdot 7^{3d+2t}$ *solutions.*
(ii) *In the relative case* (27) *has at most* $2\cdot 7^{2d+2t}$ *solutions with* $\lambda x\notin\mathbf{k}$, $\mu y\notin\mathbf{k}$.

PROOF. (i) is exactly Theorem 1 of [3]. In the proof of (ii) we shall use the set of absolute values $\{|\,.\,|_v\}_{v\in M_K}$ with properties (24) to (26). Let $S\subset M_K$ be the set of $v\in M_K$ for which either $v\in I_K$ or $v\in P_K$ and $-\log|\,.\,|_v$ is equivalent to a valuation in T. Let s denote the cardinality of S. Note that $|\alpha|_v=1$ for all $\alpha\in\mathcal{O}_T^*$ and $v\in M_K\setminus S$. By Theorem 2 of [2], (27) has at most $2\cdot 7^{2s}$ solutions with $\lambda x/\mu y\notin\mathbf{k}$. Since $s\leqq d+t$, this proves (ii).

§ 7. Preliminaries to the proofs of Theorem 1, 2, 3

Let K, R_0, K_0, $\{z_1, \dots, z_q\}$, d, m_K have the same meaning as in § 2. Let G/K be a finite, normal extension of degree g. Let $\overline{K}_0=K_0=\mathbf{Q}$ if $R_0=\mathbf{Z}$ and let $\overline{K}_0$ be the algebraic closure of K_0 in G if $R_0=\mathbf{k}$. Let R be a subring of K which has K as its quotient field and which is finitely generated over R_0. Let $R_1, \dots, R_n$ $(n\geqq 2)$ be integral extensions of R in G and let $\tilde{R}=R_1\cap R_2\cap\dots\cap R_n\cap K$. In this section we shall deal with the set $\mathcal{C}$ of tuples $\boldsymbol{\alpha}=(\alpha_1, \dots, \alpha_n)$ with the following properties:

$$\alpha_i\in R_i \quad\text{for}\quad i=1, \dots, n;\ f(\boldsymbol{\alpha}; X):=\prod_{i=1}^{n}(X-\alpha_i)\in K[X];\ \alpha_i\neq\alpha_j \quad\text{for}\quad 1\leqq i<j\leqq n.$$

We shall call the tuples $\boldsymbol{\alpha}'=(\alpha_1', \dots, \alpha_n')$, $\boldsymbol{\alpha}''=(\alpha_1'', \dots, \alpha_n'')\in\mathcal{C}$ *R-equivalent* if $\alpha_i''=\alpha_i'+a$ for some $a\in R$ $(i=1, \dots, n)$ and *weakly R-equivalent* if $\alpha_i''=u\alpha_i'+a$ for some $a\in R$, $u\in R^*$. The corresponding equivalente classes will be called R-equivalence classes and weak R-equivalence classes, respectively. In the absolute case, every $\boldsymbol{\alpha}\in\mathcal{C}$ will be called *non-special*. In the relative case, $\boldsymbol{\alpha}\in\mathcal{C}$ will be called *special* if $f(\boldsymbol{\alpha}; X)$ is special in $K[X]$ (in the general sense defined in § 5) and *non-special* otherwise. If in the relative case $\boldsymbol{\alpha}=(\alpha_1, \dots, \alpha_n)$ is non-special with $n\geqq 3$, then by Lemma 1 we may suppose that

$$\frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_2}\notin\overline{K}_0 \quad\text{for some}\quad i\in\{3, \dots, n\}. \tag{28}$$

Lemmas 5 and 6 below will be used in the proofs of Theorems 1 and 3.

LEMMA 5. *Let* $U\geqq 1$ *and let* $n\geqq 2$ *be an integer. Let* $\mathscr{C}_1\subset\mathscr{C}$ *be a set of non-special tuples* $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)$ *such that for all triples of integers* (i, j, k) *with* $1\leqq i, j, k\leqq n$, $i\neq k$, *the set*

$$\left\{\frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_k}: \boldsymbol{\alpha}\in\mathscr{C}_1, \ \text{if} \ R_0=\mathbf{k} \ \text{then} \ \frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_k}\notin\bar{K}_0\right\}$$

has cardinality at most U. Then the set of tuples

$$\left\{\left(\frac{\alpha_i-\alpha_j}{\alpha_1-\alpha_2}\right)_{1\leqq i, j\leqq n}: \boldsymbol{\alpha}\in\mathscr{C}_1\right\}$$

has cardinality at most U^{n-2} *if* $R_0=\mathbf{Z}$ *and at most* $\max(1, 2^{n-2}-1)U^{n-2}$ *if* $R_0=\mathbf{k}$.

PROOF. Lemma 5 is obvious if $n=2$, so we shall assume that $n\geqq 3$. We notice that $\alpha_i-\alpha_j=(\alpha_1-\alpha_j)-(\alpha_1-\alpha_i)$, whence the tuple $[(\alpha_i-\alpha_j)/(\alpha_1-\alpha_2)]_{1\leqq i, j\leqq n}$ is completely determined by the numbers $(\alpha_1-\alpha_k)/(\alpha_1-\alpha_2)$ $(k=3, \ldots, n)$. This proves Lemma 5 in the case $R_0=\mathbf{Z}$.

Now suppose that $R_0=\mathbf{k}$. Let $\mathscr{S}$ be a non-empty subset of $\{3, \ldots, n\}$ and let l denote the smallest element of $\mathscr{S}$. Let $\mathscr{C}_1(\mathscr{S})$ denote the set of tuples $(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_1$ such that $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2)\notin\bar{K}_0$ if and only if $i\in\mathscr{S}$. By (28), $\mathscr{C}_1$ is the union of all sets $\mathscr{C}_1(\mathscr{S})$, with $\mathscr{S}$ being a non-empty subset of $\{3, \ldots, n\}$. For all $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_1(\mathscr{S})$ we thus have that $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2)\notin\bar{K}_0$ for $i\in\mathscr{S}$ and $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_l)\notin\bar{K}_0$ for $i\in\{3, \ldots, n\}\setminus\mathscr{S}$. Since $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2)=[(\alpha_1-\alpha_i)/(\alpha_1-\alpha_l)][(\alpha_1-\alpha_l)/(\alpha_1-\alpha_2)]$, each tuple $((\alpha_i-\alpha_j)/(\alpha_1-\alpha_2))_{1\leqq i, j\leqq n}$ is completely determined by the numbers $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2)$ $(i\in\mathscr{S})$, $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_l)$ $(i\in\{3, \ldots, n\}\setminus\mathscr{S})$. This shows that the set of tuples

$$\left\{\left(\frac{\alpha_i-\alpha_j}{\alpha_1-\alpha_2}\right)_{1\leqq i, j\leqq n}: (\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_1(\mathscr{S})\right\}$$

has cardinality at most U^{n-2}. But since $\{3, \ldots, n\}$ has only $2^{n-2}-1$ non-empty subsets, this proves Lemma 5 also in the relative case.

Let $\beta\in K^*$ and let γ_{ij} $(1\leqq i, j\leqq n)$ be elements of G. We shall consider the sets

$$\mathscr{C}_2=\left\{\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}: \frac{\alpha_i-\alpha_j}{\alpha_1-\alpha_2}=\gamma_{ij} \ \text{for} \ 1\leqq i<j\leqq n, \prod_{1\leqq i<j\leqq n}(\alpha_i-\alpha_j)^2=\beta\right\},$$

and

$$\mathscr{C}_3=\left\{\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}: \frac{\alpha_i-\alpha_j}{\alpha_1-\alpha_2}=\gamma_{ij} \ \text{for} \ 1\leqq i<j\leqq n, \prod_{1\leqq i<j\leqq n}(\alpha_i-\alpha_j)^2\in\beta R^*\right\}.$$

Let T be the smallest subset of m_K such that $R\subset\mathcal{O}_T$, and let t denote the cardinality of T.

LEMMA 6. *If* $\mathscr{I}:=[\tilde{R}^+: R^+]<\infty$ *then both in the absolute and the relative case* (i) $\mathscr{C}_2$ *is contained in at most* $n(n-1)\mathscr{I}$ *R-equivalence classes and* (ii) $\mathscr{C}_3$ *is contained in at most* $\{n(n-1)\}^{[K_0:K_0](d+t)}\cdot\mathscr{I}$ *weak R-equivalence classes.*

PROOF. We shall call two tuples $\boldsymbol{\alpha}'=(\alpha_1', \ldots, \alpha_n')$, $\boldsymbol{\alpha}''=(\alpha_1'', \ldots, \alpha_n'')\in\mathscr{C}$ $\tilde{R}$-equivalent if $\alpha_i''=\alpha_i'+a$ for some $a\in\tilde{R}$ $(i=1, \ldots, n)$ and weakly $(R, \tilde{R})$-equivalent if $\alpha_i''=u\alpha_i'+a$ for some $u\in R^*$ and $a\in\tilde{R}$ $(i=1, \ldots, n)$. The corresponding equivalence classes will becalled $\tilde{R}$-equivalence classes and weak $(R, \tilde{R})$-equivalence classes, respectively. It is easy to check that every $\tilde{R}$-equivalence class is contained in at most $\mathscr{I}$ R-equivalence classes, and every weak $(T, \tilde{R})$-equivalence class is contained in at most $\mathscr{I}$ weak R-equivalence classes. Therefore it suffices to show the following:

(29) $\mathscr{C}_2$ is contained in at most $n(n-1)$ $\tilde{R}$-equivalence classes,

(30) $\mathscr{C}_3$ is contained in at most $\{n(n-1)\}^{[K_0:K_0](d+t)}$ weak $(R, \tilde{R})$-equivalence classes.

For every $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_3$, put $\psi(\boldsymbol{\alpha})=\alpha_1-\alpha_2$, $S(\boldsymbol{\alpha})=(\alpha_1+\ldots+\alpha_n)/n$. Then $\psi(\boldsymbol{\alpha})\in G^*$, $S(\boldsymbol{\alpha})\in K$. Further, put $\beta_0:=\beta/(\prod_{1\le i<j\le n}\gamma_{ij}^2)$. Let $\boldsymbol{\alpha}'=(\alpha_1', \ldots, \alpha_n')\in\mathscr{C}_3$, $\boldsymbol{\alpha}''=(\alpha_1'', \ldots, \alpha_n'')\in\mathscr{C}_3$. Then

$$\frac{\psi(\boldsymbol{\alpha}')}{\psi(\boldsymbol{\alpha}'')}=\frac{\alpha_i'-\alpha_j'}{\alpha_i''-\alpha_j''}\quad\text{for}\quad 1\le i<j\le n. \tag{31}$$

Hence

$$\frac{\psi(\boldsymbol{\alpha}')}{\psi(\boldsymbol{\alpha}'')}=\frac{\alpha_i'-S(\boldsymbol{\alpha}')}{\alpha_i''-S(\boldsymbol{\alpha}'')}\quad\text{for}\quad i=1, \ldots, n. \tag{32}$$

By (32), $\alpha_i'-\{\psi(\boldsymbol{\alpha}')/(\psi(\boldsymbol{\alpha}'')\}\alpha_i''$ does not depend on i. Since $\tilde{R}=R_1\cap\ldots\cap R_n\cap K\supset R$, we infer that $\psi(\boldsymbol{\alpha}')/\psi(\boldsymbol{\alpha}'')\in R^*$ if and only if $\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}''$ are weakly $(R, \tilde{R})$-equivalent and $\alpha_i''=u\alpha_i'+a$ for some $u\in R^*$, $a\in\tilde{R}$ with $u=\psi(\boldsymbol{\alpha}')/\psi(\boldsymbol{\alpha}'')$. Thus we have the following equivalences

(33) $\psi(\boldsymbol{\alpha}')=\psi(\boldsymbol{\alpha}'')\Leftrightarrow\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ are $\tilde{R}$-equivalent;

(34) $\psi(\boldsymbol{\alpha}')/\psi(\boldsymbol{\alpha}'')\in R^*\Leftrightarrow\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ are weakly $(R, \tilde{R})$-equivalent.

(29) is an immediate consequence of (33), on noting that for every $\boldsymbol{\alpha}\in\mathscr{C}_2$ we have $\psi(\boldsymbol{\alpha})^{n(n-1)}=\beta_0$, whence $\psi(\boldsymbol{\alpha})$ can assume at most $n(n-1)$ values.

In the proof of (30) we shall need some further notations. In the absolute case we put $\bar{K}=K$, $\bar{K}_1=K_1$, $\bar{R}=R$. In the relative case, choose $\zeta\in G$ such that $\bar{K}_0=K_0(\zeta)=\mathbf{k}(\zeta)$ and put $\bar{K}=K(\zeta)$, $\bar{K}_1=K_1(\zeta)$, $\bar{R}=R[\zeta]$. Then $\bar{R}\cap K=R$. Let $\Delta_0=\{1\}$ if $R_0=\mathbf{Z}$ and $\Delta_0=\bar{K}_0^*$ if $R_0=\mathbf{k}$. Both in the absolute and in the relative case, let $\Gamma=\{u\in G^*: u^{n(n-1)}\in\bar{R}^*\}$ and let $\bar{T}$ be the set of valuations in $m_{\bar{K}}$ lying above the valuations in T. Then $\bar{R}^*\subset\Gamma\subset\mathcal{O}_{\bar{T}}^*=\{\Theta\in\bar{K}: V(\Theta)=0 \text{ for all } V\in m_{\bar{K}}\setminus\bar{T}\}$. Put $p=[\bar{K}_0:K_0]$. Then $[\bar{K}:K]=p$. Hence $\bar{T}$ has cardinality at most pt. Together with $[\bar{K}:\bar{K}_1]\le d$ and Lemma 3, this shows that Γ/Δ_0 is the direct product of at most $d+pt$ multiplicative cyclic groups, at most one of which is finite. Using also that $\Delta_0\subset\bar{R}^*\subset\Gamma$ and $(\Gamma/\Delta_0)^{n(n-1)}\subset\bar{R}^*/\Delta_0\subset\Gamma/\Delta_0$, we obtain

$$[\Gamma:\bar{R}^*]=[\Gamma/\Delta_0:\bar{R}^*/\Delta_0]\le[\Gamma/\Delta_0:(\Gamma/\Delta_0)^{n(n-1)}]\le\{n(n-1)\}^{d+pt}. \tag{35}$$

We notice that $\bar{K}/K$ is a normal extension of degree p. Let $\sigma_1, \ldots, \sigma_p$ denote the distinct K-automorphisms of $\bar{K}$, where σ_1 is the identity. For every $\Theta\in G$, $\mathrm{Tr}(\Theta)=\mathrm{Tr}_{G/\bar{K}}\setminus\bar{T}(\Theta)$ denotes the trace of Θ over $\bar{K}$ and for every $\Theta\in G^*$, $\bar{\Theta}$ denotes the coset of Θ in the factor group $G^*/\bar{R}^*$.

We define the mapping $\mathfrak{h}: \mathscr{C}_3 \to G^*/\bar{R}^* \times \{1, ..., n\}^p$ by

$$\mathfrak{h}(\boldsymbol{\alpha}) = (\overline{\psi(\boldsymbol{\alpha})}, i_1, ..., i_p),$$

where $\boldsymbol{\alpha}=(\alpha_1, ..., \alpha_n)\in\mathscr{C}_3$ and where i_j is the smallest integer $k_j\in\{1, ..., n\}$ such that $\sigma_j(\mathrm{Tr}(\alpha_1))=\mathrm{Tr}(\alpha_{k_j})$ for $j=1, ..., p$. (It is easily seen that such integers k_j exist). If $\boldsymbol{\tau}\in\mathscr{C}_3$ then $\overline{\psi(\boldsymbol{\tau})}^{n(n-1)}=\bar{\beta}_0$. Further, the number of cosets $\bar{\varrho}\in G^*/\bar{R}^*$ with $\bar{\varrho}^{n(n-1)}=\bar{\beta}_0$ is at most $[\Gamma: \bar{R}^*]$. Together with (35) and the fact that $i_1=1$ for every $\boldsymbol{\tau}\in\mathscr{C}_3$, this shows that the range of $\mathfrak{h}$ has cardinality at most

$$n^{p-1}\{n(n-1)\}^{d+pt} \leqq \{n(n-1)\}^{p(d+t)}. \tag{36}$$

We shall now show that for $\boldsymbol{\alpha}', \boldsymbol{\alpha}''\in\mathscr{C}_3$ with $\mathfrak{h}(\boldsymbol{\alpha}')=\mathfrak{h}(\boldsymbol{\alpha}'')$ we have $\psi(\boldsymbol{\alpha}')/\psi(\boldsymbol{\alpha}'')\in R^*$. Together with (34) and (36) this proves (30). Let $\boldsymbol{\alpha}'=(\alpha_1', ..., \alpha_n')$, $\boldsymbol{\alpha}''=(\alpha_1'', ..., \alpha_n'')\in\mathscr{C}_3$ with $\mathfrak{h}(\boldsymbol{\alpha}')=\mathfrak{h}(\boldsymbol{\alpha}'')$. Put $u=\psi(\boldsymbol{\alpha}')/\psi(\boldsymbol{\alpha}'')$. Then $u\in\bar{R}^*$. Moreover, by (32)

$$u = \frac{\mathrm{Tr}(\alpha_k')-gS(\boldsymbol{\alpha}')/p}{\mathrm{Tr}(\alpha_k'')-gS(\boldsymbol{\alpha}'')/p} \quad \text{for} \quad k=1, ..., n. \tag{37}$$

Let $\sigma\in\{\sigma_1, ..., \sigma_p\}$ and let k denote the smallest integer in $\{1, ..., n\}$ such that $\sigma(\mathrm{Tr}(\alpha_1'))=\mathrm{Tr}(\alpha_k')$, $\sigma(\mathrm{Tr}(\alpha_1''))=\mathrm{Tr}(\alpha_k'')$. Then (37) implies that $\sigma(u)=u$. From this it follows that $u\in\bar{R}^*\cap K=R^*$.

§ 8. Proofs of Theorems 1 and 2

Let K, R_0, K_0, $\{z_1, ..., z_q\}$, d, m_K be the same as in § 2. Let G/K be a normal extension of finite degree g. Let R be a subring of K which is finitely generated over R_0 and which has K as its quotient field and let R' be an integral extension ring of R in G such that $\mathscr{I}=[(R'\cap K)^+: R^+]<\infty$. Let $\beta\in R\setminus\{0\}$ and let T, T' be the smallest subsets of m_K such that $R\subset\mathcal{O}_T$, $R[\beta^{-1}]\subset\mathcal{O}_{T'}$, respectively. Let t, t' denote the cardinalities of T, T', respectively. Let $\bar{T}'$ be the set of valuations in m_G lying above the valuations in T'. Let $\bar{K}_0=K_0=\mathbf{Q}$ if $R_0=\mathbf{Z}$ and let $\bar{K}_0$ denote the algebraic closure of $\mathbf{k}$ in G if $R_0=\mathbf{k}$. We shall use frequently that [6]

$$[G: \bar{K}_0(z_1, ..., z_q)] \leqq gd, \quad \#(\bar{T}') \leqq gt. \tag{38}$$

We shall now apply the results of § 7 with $R_1=...=R_n=R'$, where $n\geqq 2$. Define the sets

$$\mathscr{C}_4 = \{\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)\in\mathscr{C}: f(\boldsymbol{\alpha}; X)\in\Phi(n, R, R'),\ f(\boldsymbol{\alpha}; X) \text{ is non-special in } K[X],$$
$$D(f(\boldsymbol{\alpha}; X)) = \beta\},$$

$$\mathscr{C}_5 = \{\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)\in\mathscr{C}: f(\boldsymbol{\alpha}; X)\in\Phi(n, R, R'), f(\boldsymbol{\alpha}; X) \text{ is non-special in } K[X],$$
$$D(f(\boldsymbol{\alpha}; X))\in\beta R^*\},$$

[6] For any finite set H, $\#(H)$ will denote the number of elements of H.

where $\mathscr{C}$ has the same meaning as in § 7, but with $R_1=\ldots=R_n=R'$. We note that if $\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}''$ are (weakly) R-equivalent tuples in $\mathscr{C}_5$ then $f(\boldsymbol{\alpha}'; X)$, $f(\boldsymbol{\alpha}''; X)$ are (weakly) R-equivalent polynomials in $\Phi(n, R, R')$. Let N_1 denote the number of R-equivalence classes of tuples in $\mathscr{C}_4$, while N_2 denotes the number of weak R-equivalence classes of tuples in $\mathscr{C}_5$. Let $N_1(n, R, R', \beta)$, $N_2(n, R, R', \beta)$ be the same as in Theorem 1. Then

$$N_1(n, R, R', \beta) \leqq \frac{N_1}{(n-2)!}, \quad N_2(n, R, R', \beta) \leqq \frac{N_2}{(n-2)!}. \tag{39}$$

For $n=2$ this is obvious. If $n\geqq 3$, then (39) follows immediately from the fact that for every polynomial $f(X)\in\Phi(n, R, R')$ there are at least $(n-2)!$ pairwise weakly R-inequalent $\boldsymbol{\alpha}\in\mathscr{C}$ with $f(X)=f(\boldsymbol{\alpha}; X)$. Indeed, let $\alpha_1, \ldots, \alpha_n$ be the zeros of f in R'. Let σ, τ be two distinct permutations of $(3, \ldots, n)$ and let $\boldsymbol{\alpha}'=(\alpha_1, \alpha_2, \alpha_{\sigma(3)}, \ldots, \alpha_{\sigma(n)})$, $\boldsymbol{\alpha}''=(\alpha_1, \alpha_2, \alpha_{\tau(3)}, \ldots, \alpha_{\tau(n)})$. Then the tuples $\big((\alpha_1-\alpha_{\sigma(i)})/(\alpha_1-\alpha_2)\big)_{i=3,\ldots,n}$, $\big((\alpha_1-\alpha_{\tau(i)})/(\alpha_1-\alpha_2)\big)_{i=3,\ldots,n}$ are distinct which easily implies that $\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}''$ are not weakly R-equivalent.

In view of (39), Theorem 1 is an immediate consequence of the following proposition.

PROPOSITION 1. *We have*

$$N_1\leqq n(n-1)(4\cdot 7^{g(3d+2t')})^{n-2}\cdot\mathscr{I} \quad \text{and} \quad N_2\leqq \big(n(n-1)\big)^{[K_0:K_0](d+t)}(4\cdot 7^{g(3d+2t')})^{n-2}\cdot\mathscr{I}.$$

PROOF. Since R' is an integral extension of R, all tuples $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_5$ have the property that $\alpha_i-\alpha_j\in\mathcal{O}_{T'}^*=\{\alpha\in G: V(\alpha)=0 \text{ for all } V\in m_G\setminus\overline{T}'\}$ for all $i, j\in\{1, \ldots, n\}$ with $i\neq j$. Together with (38), Lemma 4 and the relations

$$\frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_k}+\frac{\alpha_j-\alpha_k}{\alpha_i-\alpha_k}=1,$$

this shows that for each triple (i, j, k) with $1\leqq i, j, k\leqq n$ and $i\neq k$, the set

$$\left\{\frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_k}\colon (\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_5,\ \frac{\alpha_i-\alpha_j}{\alpha_i-\alpha_k}\notin\overline{K}_0 \quad \text{if} \quad R_0=\mathbf{k}\right\}$$

has cardinality most A if $R_0=\mathbf{Z}$ and at most $A/2$ if $R_0=\mathbf{k}$, where $A=4\cdot 7^{g(3d+2t')}$. But this in turn implies, together with Lemma 5, that both in the absolute and the relative case the set

$$\left\{\left(\frac{\alpha_i-\alpha_j}{\alpha_1-\alpha_2}\right)_{1\leqq i, j\leqq n}\colon (\alpha_1, \ldots, \alpha_n)\in\mathscr{C}_5\right\}$$

has cardinality at most A^{n-2}. Now Proposition 1 follows immediately from Lemma 6.

PROOF OF THEOREM 2. Let $f(X)\in\Phi(R, R')$ be a non-special polynomial in $R[X]$ which satisfies (2). Suppose that f has degree $n\geqq 3$ and zeros $\alpha_1, \ldots, \alpha_n\in R'$. We shall use that

$$\alpha_i-\alpha_j\in\mathcal{O}_{T'}^* \quad \text{for} \quad i, j\in\{1, \ldots, n\} \quad \text{with} \quad i\neq j. \tag{40}$$

First of all suppose that $R_0=\mathbf{Z}$. Note that

$$\frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_2}+\frac{\alpha_i-\alpha_2}{\alpha_1-\alpha_2}=1 \quad \text{for} \quad i=3, \ldots, n,$$

and that the numbers $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2)$ $(i=3, \ldots, n)$ are pairwise distinct. Hence by Lemma 4, (38) and (40) we have

$$n-2 \leqq 4 \cdot 7^{g(3d+2t')}.$$

Now suppose that $R_0=\mathbf{k}$. Further, we assume that $(\alpha_1-\alpha_3)/(\alpha_1-\alpha_2) \notin \overline{K}_0$ (where $\overline{K}_0$ is the algebraic closure of $\mathbf{k}$ in G), which is by Lemma 1 no restriction. Let $\mathscr{S}$ be the subset of $\{3, \ldots, n\}$ consisting of those i for which $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_2) \notin \overline{K}_0$. By (38), (40), (41) and Lemma 4 we have

$$\#(\mathscr{S}) \leqq 2 \cdot 7^{g(3d+2t')}.$$

If $i \in \{3, \ldots, n\} \setminus \mathscr{S}$, then $(\alpha_1-\alpha_i)/(\alpha_1-\alpha_3) \notin \overline{K}_0$. Hence by (40), the identities

$$\frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_3}+\frac{\alpha_i-\alpha_3}{\alpha_1-\alpha_3}=1 \quad (i \in \{3, \ldots, n\} \setminus \mathscr{S}),$$

(38) and Lemma 4, we have also

$$\#(\{3, \ldots, n\} \setminus \mathscr{S}) \leqq 2 \cdot 7^{g(3d+2t')}.$$

Together with (42) this shows that also in the relative case

$$n-2 \leqq 4 \cdot 7^{g(3d+2t')}.$$

§ 9. Proof of of Theorem 3

Suppose that K, R_0, K_0, $\{z_1, \ldots, z_q\}$, R_1, K_1, d, m_K have the same meaning as in § 2. Let L be a finite extension of K of degree $m \geqq 2$ and let G denote the normal closure of L over K. Put $g=[G: K]$. In the relative case we assume that $\mathbf{k}$ is algebraically closed in G. Let R be a subring of K which is finitely generated over R_0 and which has K as its quotient field. Let $R' \subset L$ be an integral extension of R having L as its quotient field and suppose that $\mathscr{I}=[(R' \cap K)^+: R^+] < \infty$. Let $\sigma_1, \ldots, \sigma_m$ be the K-isomorphisms of L in G. For $\alpha \in L$, put $\alpha^{(i)}=\sigma_i(\alpha)$ $(i=1, \ldots, m)$. Let $\mathfrak{D}_K(R')$ be the discriminant divisor of R' over K. Let T be the smallest subset of m_K such that $R \subset \mathcal{O}_T$ and let t denote the cardinality of T. Let $\beta \in K^*$ and let T'' be the smallest subset of m_K such that $T \subset T''$ and $V(\beta)=V(\mathfrak{D}_K(R'))$ for all $V \in m_K \setminus T''$. Let t'' be the cardinality of T''. Further, let $\overline{T}''$ be the set of valuations in m_G lying above the valuations in T''. We shall use frequently that

$$(43) \qquad [G: K_1] \leqq gd, \quad \#(\overline{T}'') \leqq gt''.$$

If $\alpha \in L$, $\boldsymbol{\alpha}$ will denote the tuple $(\alpha^{(1)}, \ldots, \alpha^{(n)})$. We shall use the same notations as in § 7, however with $n=m$, $R_i=\sigma_i(R')$ for $i=1, \ldots, m$ and $\tilde{R}=R' \cap K$. We shall deal with the sets of tuples

$$\mathscr{C}_6=\{\boldsymbol{\alpha}\colon \alpha \in R', D_{L/K}(\alpha)=\beta\}, \quad \mathscr{C}_7=\{\boldsymbol{\alpha}\colon \alpha \in R', D_{L/K}(\alpha) \in \beta R^*\}.$$

We assert that if $\mathscr{C}_7$ is non-empty then $V(\beta)\geqq V(\mathfrak{D}_K(R'))$ for every $V\in m_K\setminus T$. Indeed, let $\alpha\in R'$ such that $\boldsymbol{\alpha}\in\mathscr{C}_7$. Since $D_{L/K}(\alpha)$ is integral over R, hence $V(\beta)=V(D_{L/K}(\alpha))\geqq 0$ for all $V\in m_K\setminus T$. Together with (7) and the definition of $\mathfrak{D}_K(R')$ this proves our assertion.

LEMMA 7. *Let $\alpha_1, \alpha_2\in R'$ such that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\in\mathscr{C}_7$. Then for $i\neq j$ with $1\leqq i, j\leqq m$*

$$\frac{\alpha_1^{(i)}-\alpha_1^{(j)}}{\alpha_2^{(i)}-\alpha_2^{(j)}}\in\mathcal{O}_{T''}^* = \{\alpha\in G^*\colon V(\alpha)=0 \quad \textit{for all} \quad V\in m_G\setminus\overline{T}''\}.$$

PROOF. Let V be a fixed valuation in $m_G\setminus\overline{T}''$ and let $\alpha_1, \alpha_2\in R'$ such that $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\in\mathscr{C}_7$. Then $D_{L/K}(\alpha_1)\neq 0$, hence $\{1, \alpha, \dots, \alpha^{m-1}\}$ is a K-basis of L. We infer that there are $\xi_1, \dots, \xi_m\in K$ such that $\alpha_2=\sum_{j=1}^m \xi_j\alpha_1^{j-1}$. For $i\in\{1, \dots, m\}$, let $\mathbf{y}_i=(1, \alpha_1, \dots, \alpha_1^{i-1}, \alpha_2, \alpha_1^{i+1}, \dots, \alpha_1^{m-1})$. Then we have by (8) that

$$(44)\qquad D(\mathbf{y}_i)=\det{}^2\begin{pmatrix}1 & & & & 0\\ & \ddots & & & \\ \xi_1 & & \xi_i & & \xi_m\\ & & & \ddots & \\ 0 & & & & 1\end{pmatrix} D_{L/K}(\alpha_1)=\xi_i^2 D_{L/K}(\alpha_1) \quad \text{for} \quad i=1, \dots, m.$$

But by the definition of T'' we have $W(D_{L/K}(\alpha_1))=W(\beta)=W(\mathfrak{D}_K(R'))$ for all $W\in m_K\setminus T''$ and by the definition of $\mathfrak{D}_K(R')$ we have $W(D(\mathbf{y}_i))\geqq W(\mathfrak{D}_K(R'))$ for all $W\in m_K\setminus T''$. Together with (44) this shows that $V(\xi_i)\geqq 0$ for $i=1, \dots, m$. But then we have, since $V(\alpha_1^{(i)})\geqq 0$ for $i=1, \dots, m$,

$$V\left(\frac{\alpha_2^{(i)}-\alpha_2^{(j)}}{\alpha_1^{(i)}-\alpha_1^{(j)}}\right)=V\left(\sum_{k=2}^m \xi_k\frac{(\alpha_1^{(i)})^{k-1}-(\alpha_1^{(j)})^{k-1}}{\alpha_1^{(i)}-\alpha_1^{(j)}}\right)=V\left(\sum_{k=2}^m\sum_{l=0}^{k-2}\xi_k(\alpha_1^{(i)})^{k-2-l}(\alpha_1^{(j)})^l\right)\geqq 0.$$

We can show in a similar way, by interchanging α_1, α_2, that $V((\alpha_1^{(i)}-\alpha_1^{(j)})/(\alpha_2^{(i)}-\alpha_2^{(j)}))\geqq 0$. Hence $V((\alpha_1^{(i)}-\alpha_1^{(j)})/(\alpha_2^{(i)}-\alpha_2^{(j)}))=0$. This proves Lemma 7.

We shall now prove Theorem 3. We remark that two numbers $\alpha_1, \alpha_2\in R'$ are (weakly) R-equivalent if and only if the tuples $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ are (weakly) R-equivalent. Hence in view of Lemma 6 it suffices to prove the following proposition:

PROPOSITION 2. *The set of tuples $\mathscr{V}=\left\{\left(\frac{\alpha^{(i)}-\alpha^{(j)}}{\alpha^{(1)}-\alpha^{(2)}}\right)_{1\leqq i, j\leqq n}\colon \boldsymbol{\alpha}\in\mathscr{C}_7\right\}$ has cardinality at most*

$$(4\cdot 7^{g(3d+2t'')})^{m-2}.$$

PROOF. For convenience we put $B=4\cdot 7^{g(3d+2t'')}$. Let $\boldsymbol{\alpha}_0$ be a fixed element of $\mathscr{C}_7$. We put $\lambda_{ij}=\alpha_0^{(i)}-\alpha_0^{(j)}$ for $1\leqq i, j\leqq m$ with $i\neq j$. Further, for every $\alpha\in R'$ we put $X_{ij}(\alpha)=(\alpha^{(i)}-\alpha^{(j)})/\lambda_{ij}$ for $1\leqq i, j\leqq m$ with $i\neq j$. Then for every $\boldsymbol{\alpha}\in\mathscr{C}_7$ we have by Lemma 7 that $X_{ij}(\alpha)\in\mathcal{O}_{T''}^*$. By Lemma 4, (43) and the relations

$$\frac{\lambda_{ij}}{\lambda_{ik}}\cdot\frac{X_{ij}(\alpha)}{X_{ik}(\alpha)}+\frac{\lambda_{jk}}{\lambda_{ik}}\cdot\frac{X_{jk}(\alpha)}{X_{ik}(\alpha)}=1 \quad (i, j, k\in\{1, \dots, m\},\ i\neq k),$$

we have that for each triple (i, j, k) with $1 \leqq i, j, k \leqq m$, $i \neq k$, the set

$$\left\{\frac{\alpha^{(i)}-\alpha^{(j)}}{\alpha^{(i)}-\alpha^{(k)}}:\ \boldsymbol{\alpha}\in\mathscr{C}_7,\ \frac{\alpha^{(i)}-\alpha^{(j)}}{\alpha^{(i)}-\alpha^{(k)}}\notin\mathbf{k}\ \text{ if }\ R_0=\mathbf{k}\right\}$$

has cardinality at most B if $R_0=\mathbf{Z}$ and at most $\frac{1}{2}B$ if $R_0=\mathbf{k}$. In the absolute case, Proposition 2 is an immediate consequence of Lemma 5. In the relative case we infer that $\mathscr{V}$ contains at most $\max(1, 2^{m-2}-1)(B/2)^{m-2}$ tuples for which $\boldsymbol{\alpha}$ is non-special (i.e. $f(\boldsymbol{\alpha}; X)$ is non-special in $K[X]$). We shall now estimate the number of tuples in $\mathscr{V}$ for which $\boldsymbol{\alpha}$ is special.

Let $\boldsymbol{\alpha}\in\mathscr{C}_7$ such that $\boldsymbol{\alpha}$ is special or, which is the same, the minimal polynomial $f(X)$ of α is special in $K[X]$. Then $m\geqq 3$. Further, there are integers r, n_0, δ with $r>0$, $n_0>0$, $\delta\in\{0,1\}$, $rn_0+\delta=m$ and $\delta=0$ if $n_0=1$, and there are $a\in K$, $\mu\in K^*$ and a monic polynomial $h(X)\in\mathbf{k}[X]$ of degree r with $D(h)\neq 0$ such that

$$f(X) = \mu^r h\big((X+a)^{n_0}/\mu\big)(X+a)^{\delta}.$$

But since f is irreducible we have that $\delta=0$ and h is irreducible. Furthermore, h has its zeros in G and $\mathbf{k}$ is algebraically closed in G. Hence $r=1$. Therefore there exists a $\mu'\in K^*$ such that

$$f(X) = (X+a)^m-\mu'.$$

Let ϱ be a fixed, primitive m-th root of unity and let Θ be a fixed m-th root of μ'. Then $\alpha^{(i)}=\varrho^{k_i}\theta-a$ for $i=1, \ldots, m$, where $(k_1, \ldots, k_m)$ is a permutation of $(1, \ldots, m)$. Hence the tuple

$$\left(\frac{\alpha^{(i)}-\alpha^{(j)}}{\alpha^{(1)}-\alpha^{(2)}}\right)_{1\leqq i, j\leqq m} = \left(\frac{\varrho^{k_i}-\varrho^{k_j}}{\varrho^{k_1}-\varrho^{k_2}}\right)_{1\leqq i, j\leqq m}$$

belongs to a set of cardinality at most $m!$. But this shows that the number of tuples in $\mathscr{V}$ for which $\boldsymbol{\alpha}$ is special is, in view of $m\leqq g$, at most

$$m! \leqq 2\cdot 7^{3m(m-2)} \leqq (B/2)^{m-2}.$$

Therefore, the total number of tuples in $\mathscr{V}$ is also in the relative case at most B^{m-2}.

REMARK. We notice that a weaker version of Theorem 3 can be deduced also from Theorem 1.

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CENTRE FOR MATHEMATICS AND COMPUTER SCIENCE
KRUISLAAN 413
P. O. BOX 4079
1009 AB AMSTERDAM
THE NETHERLANDS

MATHEMATICAL INSTITUTE
KOSSUTH LAJOS UNIVERSITY
4010 DEBRECEN
HUNGARY

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ON TOTALLY SEPARABLE PACKINGS OF EQUAL BALLS

G. KERTÉSZ (Budapest)

In the n-dimensional Euclidean space a set of bodies $\{b_i\}$ is said to be totally separable if to any pair of bodies b_i and b_j there is an $(n-1)$-dimensional plane p such that b_i and b_j lie on opposite sides of p, and for each index k, $p\cap b_k=\emptyset$. It is known [4] that for $n=2$ the density of a densest totally separable packing of congruent copies of a centro-symmetric convex disc is equal to the area of the disc divided by the area of the smallest parallelogram containing the disc. Further results about totally separable circles are contained in [2] and [3].

In the present paper we shall show that in 3-space the density of a densest totally separable packing of equal (open) balls is $\pi/6$. This constant equals the volume of a ball divided by the volume of the circumscribed cube. We shall prove the following sharper

THEOREM. *If a cube of volume V contains a totally separable set of N balls of radius r then $V\geqq 8Nr^3$.*

PROOF. We consider a plane separating any two of the balls without intersecting the others. This plane divides the cube q into two pieces. If any of these contains two balls we divide it similarly as we did with q. Continuing this process we finally obtain a decomposition of q into convex polyhedra each containing exactly one sphere. Let s be the surface area, and V the volume of one of these polyhedra. Again let l be the length of an edge of this polyhedron, and α the angle between the outer normals of the faces meeting at this edge. Then, by a well known inequality of Minkowski (see e.g. [5] p. 287),

$$s^2 \geqq 3v\Sigma l\cdot \tan\frac{\alpha}{2}$$

where the sum extends over all edges of the polyhedron. Since $v\geqq(r/3)s$, we have

$$(*)\qquad v \geqq \frac{1}{3}r^2\Sigma l\cdot\cot\frac{\beta}{2}$$

where $\beta=\pi-\alpha$ is the dihedral angle at the respective edge. Let k be the number of polyhedra meeting along a common segment e of an edge. Let $\beta_1, \ldots, \beta_k$ be the respective dihedral angles. According as e lies

(i) on an edge of q,
(ii) in the interior of a face either of q or of another polyhedron,
(iii) in the interior of q but not in the interior of a face of another polyhedron,

we have

(i) $k \geqq 1,\ \beta_1+\ldots+\beta_k=\pi/2,$
(ii) $k \geqq 2,\ \beta_1+\ldots+\beta_k=\pi,$
(iii) $k \geqq 4,\ \beta_1+\ldots+\beta_k=2\pi.$

Since for $0<x\leqq\pi/2$, $\cot x$ is a convex, decreasing function, we have in each case

$$\cot\frac{\beta_1}{2}+\ldots+\frac{\beta_k}{2}\geqq k.$$

Therefore, denoting the total edge-length of the polyhedra by L, and summing up the inequalities $(*)$ for all N polyhedra, we have

$$V\geqq\frac{r^2}{3}\Sigma L.$$

But a theorem of Besicovith and Eggleston [1] claims that the total edge-length of a convex polyhedron containing a sphere attains its minimum for the circumscribed cube. Therefore $L\leqq 24r$ and conseqently $V\geqq 8Nr^3$ as claimed.

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15.
H—1053

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ON THE ORDER OF APPROXIMATION BY FEJÉR MEANS OF HERMITE—FOURIER AND LAGUERRE—FOURIER SERIES

I. JOÓ (Budapest)

To the memory of Professor Géza Freud

Introduction

It is well known that the trigonometric conjugate functions play an important role in harmonic analysis and in approximation theory. There are many results in the formulation of which the conjugate function appears. A classical result of this type was obtained by G. Alexits [1, 2, 3];

THEOREM A. *For the Fejér sums $\sigma_n(f)$ of the trigonometric Fourier series of the 2π-periodic function f the estimate $\|\sigma_n(f)-f\|_p=O\left(\frac{1}{n}\right)$ holds if and only if the conjugate function $\tilde{f}$ of f belongs to the class* Lip $(1, p)$ $(1\leqq p\leqq\infty)$.

For the proof G. Alexits proved

LEMMA B. *Let (a_n) be any sequence in a Banach space B. Consider the Fejér-sums σ_n^1 and σ_n^2 of the series Σa_n and Σna_n respectively. Then $\|\sigma_n^2\|_B=O(1)$ if and only if there exists $\sigma\in B$ such that $\|\sigma_n^1-\sigma\|_B=O\left(\frac{1}{n}\right)$.*

Lemma B was used by many authors for solving problems in approximation theory and harmonic analysis (see e.g. [5, 7, 11]). In [5] D. Králik generalized (among others) Lemma B proving

LEMMA C. *Let (a_n) be any sequence in a Banach space B. Consider the Fejér-sums σ_n^1 and σ_n^2 of the series Σa_n and $\Sigma n^\alpha a_n$ respectively, where $0<\alpha<1$. If $\|\sigma_n^2\|_B=$ $=O(1)$ then there exists $\sigma\in B$ such that*

$$\|\sigma_n^1-\sigma\|_B=O\left(\frac{1}{n^\alpha}\right)\quad(n\geqq 1).$$

Lemma C was applied in [11] for the investigation of the derivative of fractional order.

1. The aim of the present note is to give another application of Lemma C concerning the Hermite—Fourier and Laguerre—Fourier expansions. We need some notations and definitions.

Denote $L_u^p(a, b)$ the Banach space of functions defined on (a, b) with norm

$$\|f\|_{p,u}:=\Big(\int_a^b |f|^p\cdot u\Big)^{1/p}.$$

For $u=1$ we write simply $L^p(a, b)$. In what follows we suppose $1<p<\infty$. Let $f\in L^p_{e^{-x^2}}(-\infty, \infty)$, $g\in L^p_{x^\alpha e^{-x}}(0, \infty)$. Denote $\tilde{f}$ and $\bar{g}$ the Hermite-conjugate of f and the Laguerre-conjugate of g respectively. This notion was introduced by B. Muckenhoupt [8, 9], further he proved that $\tilde{f}\in L^p_{e^{-x^2}}(-\infty, \infty)$, $\bar{g}\in L^p_{x^\alpha e^{-x}}(0, \infty)$ for $1<p<\infty$. Now we give a simpler and more general definition for Hermite and Laguerre conjugate functions. For $1<p<\infty$ and $f\in L^p_{e^{-x^2}}(-\infty, \infty)$, $g\in L_{x^\alpha e^{-x}}(0, \infty)$ our definition coincides with that of B. Muckenhoupt (this follows from [8, 9]), but we shall not use this fact.

Denote $H=\{h_n\}$, $L^\alpha=\{l_n^\alpha\}$, $p^\lambda=\{p_n^\lambda(u_\lambda^2, x)\}$ the normed Hermite, Laguerre polynomials further the system of orthonormal polynomials on $(-\infty, \infty)$ with respect to the weight function $u_\lambda^2(x)=|x|^\lambda e^{-x^2}$ $(\lambda\geqq 0)$, resp.

For any orthonormal system φ (with respect to some weight) denote $\mathscr{F}(\varphi)$ the set of all functions f which have Fourier series (i.e. coefficients) with respect to the system φ.

DEFINITION. Let $f\in\mathscr{F}(H)$, $g\in\mathscr{F}(L^\alpha)$, $\alpha>-1$,

$$f\sim\sum_{k=0}^{\infty} a_k(f)h_k, \quad g\sim\sum_{k=0}^{\infty} b_k(g)l_k^\alpha. \tag{1}$$

If there exists $\tilde{f}\in\mathscr{F}(H)$ and $y^{-1/2}\bar{g}(y)\in\mathscr{F}(L^{\alpha+1})$ such that

$$\tilde{f}\sim\sum_{k=1}^{\infty} a_k h_{k-1}, \tag{2}$$

$$y^{-1/2}\bar{g}(y)\sim\sum_{k=1}^{\infty} -k^{-1/2} b_k l_{k-1}^{\alpha+1}(y), \tag{3}$$

then $\tilde{f}$ is called to be the Hermite-conjugate function of f and $\bar{g}$ the Laguerre-conjugate of g.

For $f\in\mathscr{F}(H)$ denote $\sigma_n(f)$ the n-th Fejér mean of the Hermite—Fourier series (1) of f. If $g\in\mathscr{F}(L^\alpha)$ $(\alpha>-1)$, denote $h(x):=g(x^2)$, $k(x):=\bar{g}(x^2)$ ($\bar{g}$ denotes the Laguerre-conjugate of g if it exists). We prove two theorems.

THEOREM I. *Let* $1<p<\infty$, $f\in\mathscr{F}(H)$. *Suppose* $\tilde{f}$ *exists, is absolutely continuous on every finite interval and* $e^{-x^2/2}\tilde{f}'(x)$, $(e^{-x^2}\tilde{f}(x))'e^{x^2/2}\in L^p(-\infty, \infty)$. *Then* $e^{-x^2/2}f(x)\in L^p(-\infty, \infty)$ *and*

$$\|e^{-x^2/2}[\sigma_n(f, x)-f(x)]\|_{L^p(-\infty,\infty)} = O\left(\frac{1}{\sqrt{n}}\right) \quad (n\geqq 1). \tag{4}$$

THEOREM II. *Let* $1<p<\infty$, $\alpha\geqq -1/2$, $g\in\mathscr{F}(L^\alpha)$. *Suppose there exists* $\bar{g}$ *further* $k(x)$ *is absolutely continuous on every interval of the form* (ε, δ), $0<\varepsilon<\delta<\infty$ *and* $(u_{\alpha+1/2}^2(x)k(x))'u_{\alpha+1/2}^{-1}(x)\in L^p(-\infty, \infty)$. *Then* $u_{\alpha+1/2}(x)k(x)\in L^p(-\infty, \infty)$ *and*

$$\|u_{\alpha+1/2}(x)[\sigma_n(k, x)-k(x)]\|_{L^p(-\infty,\infty)} = O\left(\frac{1}{\sqrt{n}}\right) \quad (n\geqq 1), \tag{5}$$

where $\sigma_n(k, x)$ denotes the n-th Fejér mean of the Fourier series of h with respect to the system $p^{\alpha+1/2}$.

2. For the proof of the theorems we need some lemmas.

LEMMA 1. *We have for* $n=1, 2, \ldots$

$$h_n'(x) = \sqrt{2n}\, h_{n-1}(x), \tag{6}$$

$$(p_{2n}^{\lambda})'(x) = \sqrt{n}\, p_{2n-1}^{\lambda}(x) \quad (\lambda \geqq 0). \tag{7}$$

PROOF. See [10] and [6] respectively. □

LEMMA 2. *If* $e^{x^2/2}[e^{-x^2}\tilde{f}(x)]' \in L(-\infty, \infty)$, $1<p<\infty$, *then for every algebraic polynomial* $p(x)$

$$\lim_{|x|\to\infty} e^{-x^2} p(x) \tilde{f}(x) = 0. \tag{8}$$

PROOF. We can suppose $\tilde{f}(0)=0$. In this case the Hölder inequality implies $\left(\frac{1}{p}+\frac{1}{q}=1\right)$:

$$|e^{-x^2}p(x)\tilde{f}(x)| = \left|e^{-x^2}p(x)\int_0^x \tilde{f}'(t)\,dt\right| = |e^{-x^2/2}p(x)|\left|e^{-x^2/2}\int_0^x \tilde{f}(t)\,dt\right| =$$

$$\leqq |e^{-x^2/2}p(x)|\left|\int_0^x e^{-t^2/2}|\tilde{f}'(t)|\,dt\right| \leqq$$

$$\leqq |e^{-x^2/2}p(x)| \cdot x^{1/q} \cdot \|e^{-t^2/2}\tilde{f}'(t)\|_{L^p(-\infty,\infty)} \to 0 \quad (|x|\to\infty).$$

LEMMA 3. *Let $\Sigma a_k h_k$ be any formal Hermite-series, denote σ_n its n-th Fejér-mean. Suppose $\|e^{-x^2/2}\sigma_n(x)\|_p = O(1)$, $n \geqq 1$, $1<p<\infty$. Then there exists $f \in L^p(-\infty, \infty)$ such that $\Sigma a_k h_k$ is the Hermite—Fourier series of f.*

PROOF. Define q by $\frac{1}{p}+\frac{1}{q}=1$ and consider the functionals

$$T_n\colon L^q(-\infty, \infty) \to \mathbf{R},$$

$$T_n h := \int_{-\infty}^{\infty} \sigma_n(x) h(x) e^{-x^2/2}\,dx \quad (h \in L^q(-\infty, \infty)).$$

By Hölder inequality we get

$$|T_n h| \leqq \|\sigma_n(x)e^{-x^2/2}\|_{L^p(-\infty,\infty)} \cdot \|h(x)\|_{L^q(-\infty,\infty)} \leqq C\|h\|_q.$$

On the other hand, for any function of the form $e^{-x^2/2}h_m(x)$

$$T_n[e^{-x^2/2}h_m(x)] = \int_{-\infty}^{\infty} \sigma_n(x) h_m(x) e^{-x^2}\,dx = \tag{9}$$

$$= \int_{-\infty}^{\infty} \sum_{k=0}^{n-1} a_k \left(1-\frac{k}{n}\right) h_k(x) h_m(x) e^{-x^2}\,dx = \left(1-\frac{m}{n}\right) a_m \to a_m \quad (n\to\infty).$$

Hence, the sequence $\{T_n\}$ converges for every function of the form $e^{-x^2/2}p(x)$, where $p(x)$ is an arbitrary algebraic polynomial. These functions form a dense set in $L^q(-\infty, \infty)$, so taking into account the Banach—Steinhaus theorem and also the Riesz representation theorem we obtain: there exists a function $f(x)$ on $(-\infty, \infty)$ such that $f(x)e^{-x^2/2}\in L^p(-\infty, \infty)$ and for every $h\in L^q(-\infty, \infty)$, $T_n h \to \int_{-\infty}^{\infty} f(x)h(x)e^{-x^2}dx$ $(n\to\infty)$ holds. Hence, by (9), $a_m = \int_{-\infty}^{\infty} f(x)h_m(x)e^{-x^2}dx$. □

3. Proof of Theorem I. Let

$$f \sim \Sigma a_k h_k, \tag{10}$$

$$\tilde{f} \sim \Sigma a_k h_{k-1}, \tag{11}$$

then, according to the assumption $\varphi(x) := [e^{-x^2}\tilde{f}(x)]' e^{-x^2/2} \in L^p(-\infty, \infty)$ we have $\varphi\in\mathcal{F}(H)$. Denote

$$\varphi(x) \sim \sum_{k=0}^{\infty} b_k h_k(x) e^{-x^2/2}.$$

Integrating by parts and using (6), (8) we have

$$b_k = \int_{-\infty}^{\infty} \varphi(x) h_k(x) e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} [e^{-x^2}\hat{f}(x)]' h_k(x)\,dx =$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\hat{f}(x) h_k'(x)\,dx = \sqrt{2k}\int_{-\infty}^{\infty} \hat{f}(x) h_{k-1}(x) e^{-x^2}\,dx =$$

$$= \sqrt{2k}\, a_{k-1}(\hat{f}) = \sqrt{2k}\, a_k(f), \quad b_0 = 0,$$

i.e.

$$\varphi(x) \sim \sum_{k=1}^{\infty} \sqrt{2k}\, a_k h_k(x) e^{-x^2/2}. \tag{12}$$

We assumed $\varphi(x)\in L^p(-\infty, \infty)$, hence G. Freud's result [4] gives

$$\|\sigma_n(\varphi)\|_{L^p(-\infty,\infty)} = O(1) \quad (n \geqq 1) \tag{13}$$

and hence by Lemma B we get: there exists $\sigma\in L^p(-\infty, \infty)$ such that

$$\|e^{-x^2/2}[\sigma_n(f, x) - \sigma(x)]\|_{L^p(-\infty,\infty)} = O\left(\frac{1}{\sqrt{n}}\right) \quad (n \geqq 1),$$

consequently

$$\|e^{-x^2/2}\sigma_n(f, x)\|_{L^p(-\infty,\infty)} = O(1), \tag{14}$$

and using Lemma 3 we obtain: $e^{-x^2/2}f(x)\in L^p(-\infty, \infty)$. Applying G. Freud's result [4] again, it follows

$$\|e^{-x^2/2}[\sigma_n(f, x) - f(x)]\|_{L^p(-\infty,\infty)} \to 0 \quad (n \to \infty),$$

i.e. $f(x) = \sigma(x)$ and (4) is fulfilled. □

4. PROOF OF THEOREM II. We only sketch the proof, because it is similar to that of Theorem I. Let

$$g(x) \sim \sum_{n=0}^{\infty} a_n l_n^{\alpha}(x),$$

then by definition

$$x^{-1/2}\tilde{g}(x) \sim \sum_{n=1}^{\infty} -n^{-1/2} a_n l_{n-1}^{(\alpha+1)}(x).$$

Applying the following formula of N. X. Ky ([6] formula (3))

$$p_{2n}(W_\beta^2, x) = p_n(u_{\beta-1/2}, x^2), \quad p_{2n+1}(W_\beta^2, x) = xp_n(u_{\beta+1/2}, x^2)$$

$$(\beta \geqq 0,\ u_\beta(x) = x^\beta e^{-x},\ W_\beta(x) = |x|^\beta e^{-x^2/2})$$

we get

$$h(x) \sim \sum_{n=0}^{\infty} a_n p_{2n}(W_{\alpha+1/2}^2, x),$$

$$k(x) \sim \sum_{n=1}^{\infty} -n^{-1/2} a_n p_{2n-1}(W_{\alpha+1/2}^2, x).$$

Now let $\psi(x) := [u_{\alpha+1/2}^2(x) k(x)]' u_{\alpha+1/2}(x)$. Apply (7) and use the method of proof of Theorem I. It follows

$$\psi(x) \sim \sum_{n=1}^{\infty} -\sqrt{n}\, a_n p_n(W_{\alpha+1/2}^2, x) \cdot u_{\alpha+1/2}(x).$$

Because $\psi \in L^p(-\infty, \infty)$, Lemma 2.3 of [6] gives

$$\|\sigma_n(\psi)\|_{L^p(-\infty,\infty)} = O(1),$$

hence, a similar argument as in Section 3 yields

$$\|u_{\alpha+1/2}(x)[\sigma_n(h, x) - h(x)]\|_{L^p(-\infty,\infty)} = O\left(\frac{1}{\sqrt{n}}\right) \quad (n \geqq 1). \quad \square$$

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LORÁND EÖTVÖS UNIVERSITY
DEPARTMENT OF ANALYSIS
1445 BUDAPEST 8, PF. 323
HUNGARY

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ON EMBEDDING OF LOCALLY COMPACT ABELIAN TOPOLOGICAL GROUPS IN EUCLIDEAN SPACES. I

M. BOGNÁR (Budapest)*

All the groups considered in this paper are locally compact abelian T_0-topological groups with countable bases. They are referred to as LCA-groups. For each positive integer m, R^m denotes the m-dimensional euclidean space.

Introduction

The main target of this paper is to prove the following theorem stated in [2].

THEOREM A. *The space of an n-dimensional connected LCA-group can be embedded in R^{n+1} only if it is the direct sum of k-dimensional toroidal and an $(n-k)$-dimensional vector group* $(0 \leqq k \leqq n)$.

This theorem is a generalization of a theorem of K. Kodaira and M. Abe [9] which says:

An n-dimensional compact connected LCA-group can be embedded in R^{n+1} only if it is a toroidal group.

Also, we shall prove some further embedding theorems about LCA-groups. Namely, the following ones.

THEOREM B. *Each n-dimensional LCA-group with locally connected components can be embedded in R^{n+1}.*

THEOREM C. *An n-dimensional* $(n>0)$ *LCA-group can be embedded in R^n if and only if it is locally connected and its components are non-compact sets.*

It is important that in [3] we have proved the following theorem (cf. also [4]).

THEOREM D. *Each n-dimensional LCA-group can be embedded in R^{n+2}.*

Hence if we consider the integers m for which a given LCA-group G can be embedded in R^m then taking into account also the coincidence of the dimension of G and of the dimension of the components of G, Theorems A, B, C, D determine the minimum of these integers m.

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It is to be noted that there are some misprints in [4]. We shall correct them at the end of Section 1.

The paper is divided into two parts. Each of them is equipped with a separate list of references.

Now we are going to prove Theorems B and C.

1. Proof of Theorems B and C

1.1. LEMMA. *Let Γ be an n-dimensional* $(0 \leqq n < \infty)$ *LCA-group. Then Γ has a subspace homeomorphic to R^n.*

PROOF. Let Γ_1 be the component of the zero element of Γ. Let Γ_2 be a subgroup of Γ containing Γ_1 such that Γ/Γ_2 is discrete and Γ_2/Γ_1 is compact (see [10] p. 161). Since Γ has a countable base and Γ/Γ_2 is discrete, it follows that Γ is a countable union of closed subspaces each being homeomorphic to Γ_2. Thus Γ_2 is of dimension n, since otherwise by the sum theorem for dimension (see [8] p. 30) Γ would have the dimension $\leqq n-1$.

Since Γ_2/Γ_1 is compact, it follows that Γ_2 decomposes into the direct sum of a compact subgroup Γ_3 and a vector subgroup V of Γ_2 (see [10] p. 160). Let k be the dimension of Γ_3. Then $k \leqq n$ and by the product theorem for dimension (see [8] p. 33), V is of dimension $\geqq n-k$. Thus V admits a subspace homeomorphic to R^{n-k}. On the other hand, Γ_3 has a subspace homomorphic to R^k (see [10] p. 213). Consequently, Γ_2 and so also Γ has a subspace homeomorphic to $R^k \times R^{n-k}$, i.e. homeomorphic to R^n as required. □

1.2. COROLLARY. *If Γ_1 and Γ_2 are LCA-groups of finite dimensions then the dimension of their direct sum is the sum of the dimensions of Γ_1 and Γ_2.*

1.3. COROLLARY. *Let Γ be an n-dimensional* $(0 \leqq n < \infty)$ *LCA-group. Then each subgroup of Γ containing the component of the zero element of Γ is of dimension n.*

1.4. REMARK. Let Γ_1 be a subgroup of the LCA-group Γ such that Γ/Γ_1 is discrete. Then if Γ_1 can be embedded in an R^m $(m>0)$ then Γ can be embedded in the same R^m, too.

In fact, since Γ has a countable base, it follows that Γ/Γ_1 is countable and this proves the statement.

1.5. REMARK. The topological product of the unit interval **I** of the space of the reals and an n-dimensional toroidal group $(0 \leqq n < \infty)$ can clearly be embedded in R^{n+1}.

1.6. COROLLARY. *The direct sum of an n-dimensional toroidal group and of a* 0*-dimensional LCA-group can be embedded in R^{n+1}.*

1.7. COROLLARY. *The direct sum of an n-dimensional toroidal group and a* 1*-dimensional vector group can be embedded in R^{n+1}.*

1.8. NOTATIONS. We denote by Ξ the factor group of the additive group of the real numbers by the subgroup of integers. Ξ is clearly a compact LCA-group. If x

is a real number then the coset of the group Ξ which contains x will be denoted by $\bar{x}$. Moreover, if r is a positive integer then U_r denotes the set

$$U_r = \left\{\bar{x};\ \frac{-1}{3^r} < x < \frac{1}{3^r}\right\}.$$

$\{U_1, \dots, U_r, \dots\}$ is clearly a base about the point $\bar{0}$ in Ξ.

1.9. Definition. Let G be an LCA-group. Any homomorphism γ of the topological group G into the topological group Ξ is called a *character* of the topological group G. We define on the set Γ of characters a topological group in the following manner: The sum of two characters γ and γ^1 is that character $\gamma+\gamma^1$ for which

$$\gamma+\gamma^1(g) = \gamma(g)+\gamma^1(g)$$

holds for each $g\in G$.

Neighbourhoods V of the zero element of Γ are defined in terms of the neighbourhoods U of zero in Ξ and compact subsets Φ of the space G by putting $\gamma\in V[\Phi, U]$ if $\gamma(\Phi)\subset U$.

In such a way we obtain an LCA-group Γ (see [10] p. 128). This is the *character group of* G.

1.10. Definition. Let Γ be the character group of the LCA-group G. Let G_1 be a (closed) subgroup of G. Let us denote by (Γ, G_1) the set of all elements $\gamma\in\Gamma$ for which $\gamma(g)=\bar{0}$ for every $g\in G_1$. The set (Γ, G_1) is called the *annihilator of the group* G_1 *in the group* Γ and it is a (closed) subgroup of Γ.

Let Γ_1 be a (closed) subgroup of the group Γ and denote by (G, Γ_1) the set of all elements $g\in G$ for which $\gamma(g)=\bar{0}$ for every $\gamma\in\Gamma_1$. The set (G, Γ_1) is called the *annihilator of the group* Γ_1 *in the group* G and is a (closed) subgroup of the group G.

1.11. Lemma. *Let Γ be an n-dimensional* ($0\leqq n<\infty$) *compact LCA-group with locally connected components. Then Γ is the direct sum of an n-dimensional toroidal and* 0*-dimensional compact subgroup.*

Proof. Let G be the character group of Γ. G is a discrete LCA-group (cf. [10] p. 133).

Let $\Gamma(G)$ be the character group of G. Since the topological group $\Gamma(G)$ is isomorphic to Γ (see [10] p. 146), it follows that G is of rank n (see [10]) p. 148). Let Γ_1 be the component of the zero element of Γ and let G_1 be the annihilator of the topological group Γ_1 in the group G. G_1 is conposed of all the elements of G having a finite order (see [10] p. 149) and thus G/G_1 has no element of finite order and its rank is n, too.

Since the topological group Γ_1 is isomorphic to the character group of G/G_1 (see [10] p. 136) and since Γ_1 is locally connected, by hypothesis it follows that G/G_1 has a finite system of linearly independent generators (see [10] p. 168). Consequently, G/G_1 is a free abelian group and thus G is the direct sum of G_1 and of a subgroup G_2 isomorphic to G/G_1. The character group of the free abelian group G_2 with rank n is an n-dimensional toroidal group (see [10] p. 142 and p. 148), while the character group of G_1 is a compact 0-dimensional group (see [10] p. 133 and p. 148). Consequently, the character group $\Gamma(G)$ of G and thus Γ itself is isomorphic to the direct sum of the character groups of G_1 and G_2 (see [10] p. 141) and thus it is the direct sum of an n-dimensional toroidal subgroup and of a 0-dimensional compact subgroup indeed. □

1.12. REMARK. Let Γ be a locally connected LCA-group and let Γ_1 be the component of the zero element of Γ. Then Γ_1 is open in Γ. Consequently Γ_1 is locally connected as well and Γ/Γ_1 is a discrete LCA-group.

1.13. PROOF OF THEOREM B. Let Γ be an n-dimensional $(n \geqq 0)$ LCA-group with locally connected components. Let Γ_1 be the component of the zero element of Γ. Let Γ_2 be a subgroup of Γ containing Γ_1 such that Γ/Γ_2 is discrete and Γ_2/Γ_1 is compact (see [10] p. 161).

Γ_2 is an n-dimensional LCA-group (see 1.3) with locally connected components. Since Γ_2/Γ_1 is compact, it follows that Γ_2 decomposes into the direct sum of a k-dimensional compact subgroup $\Gamma_3 (0 \leqq k \leqq n)$ and an $(n-k)$-dimensional vector subgroup V (see [10] p. 160 and 1.2).

Γ_3 has clearly locally connected components, too. Hence by 1.11, Γ_3 is the direct sum of a k-dimensional toroidal and a 0-dimensional compact subgroup. Thus by 1.6, Γ_3 can be embedded in R^{k+1}. V can be clearly embedded in R^{n-k}. Hence the direct sum Γ_2 of Γ_3 and V can be embedded in $R^{k+1} \times R^{n-k}$ and thus in R^{n+1} as well. Since Γ/Γ_2 is discrete, it follows by 1.4 that Γ can be embedded in R^{n+1} as required.

The proof of Theorem B is complete. □

1.14. PROOF OF THEOREM C. Let Γ be an n-dimensional locally connected LCA-group with non-compact components. Let Γ_1 be the component of the zero element of Γ. Then according to 1.3 and 1.12, Γ_1 is an n-dimensional locally connected open subgroup of Γ, while Γ/Γ_1 is a disrete group.

Since Γ_1 is a connected locally connected n-dimensional LCA-group, it follows that Γ_1 decomposes into the direct sum of a k-dimensional $0 \leqq k \leqq n$ toroidal subgroup Γ_2 and an $(n-k)$-dimensional vector subgroup V (see 1.2 and [10] p. 170). Since Γ_1 is non compact, it follows $n-k \geqq 1$ and thus V decomposes into the direct sum of a 1-dimensional V_1 and an $(n-k-1)$-dimensional V_2 vector subgroup.

According to 1.7, the sum $\Gamma_2 + V_1$ of the groups Γ_2 and V_1 can be embedded in R^{k+1}. On the other hand, V_2 can be embedded into R^{n-k-1}. However, since Γ_1 is the direct sum of $\Gamma_2 + V_1$ and V_2, it follows that the space of Γ_1 can be embedded in $R^{k+1} \times R^{n-k-1}$ and thus in R^n as well. Finally, since Γ/Γ_1 is discrete, it follows by 1.4 that the space of Γ can be embedded in R^n.

Now suppose that the n-dimensional $(0 < n < \infty)$ LCA-group Γ can be embedded in R^n. Let $\varphi : \Gamma \to R^n$ be such an embedding, i.e. φ is a topological map of Γ onto the subspace $\varphi(\Gamma)$ of R^n.

Since the space of Γ is homogeneous, it follows by 1.1 that to each $\gamma \in \Gamma$ we can find a subspace $R^n(\gamma)$ of Γ homomorphic to R^n and containing γ. However, by the theorem of invariance of domain (Gebietsinvarianz), $\varphi(R^n(\gamma))$ is open in R^n and thus $\varphi(\Gamma) = \bigcup_{\gamma \in \Gamma} \varphi(R^n(\gamma))$ is an open subset of R^n as well. Consequently, $\varphi(\Gamma)$ is locally connected and the components of $\varphi(\Gamma)$ are open subsets of $\varphi(\Gamma)$. Hence the components of $\varphi(\Gamma)$ are non-compact sets. These statements imply that the space of Γ is locally connected and the components of Γ are non-compact sets.

The proof of Theorem C is complete. □

We now turn to the correction of the misprints in [4].

	Misprint	*Correct*
p. 345 row 10:	$n-2$	$n+2$
p. 347 row 8:	$s(\xi_1, \xi_2, \ldots, \xi_i)$	$s(\xi_1, \xi_2, \ldots, \xi_i, \ldots)$
p. 347 row 17:	l_{g_l}	lg_l
p. 348 row 6 from below:	a factor-	a vector-
p. 348 last row:	therefore can	therefore Γ can

Now we give a preliminary sketch of the proof of our main theorem A.

2. Sketch of the proof of the main theorem

2.1. DEFINITION. Let X be a topological space and $g: X\to X$ an autohomeomorphism of X. A finite sequence $Z=(L_1, \ldots, L_m)$ of nonempty pairwise disjoint closed subsets of X is said to be an *(X, g)-cycle of order m* if Z is a covering of X and $g(L_i)=L_{i+1}$ whenever $i=1, \ldots, m-1$ and $g(L_m)=L_1$.

The sets L_i are called the *members* of the (X, g)-cycle Z.

The autohomeomorphism $g: X\to X$ is said to be an *absolutely cyclic map* if for each open covering Ω of X there is an (X, g)-cycle Z which is a refinement of Ω.

2.2. DEFINITION. Let D be a space homeomorphic to the Cantor discontinuum and let the autohomeomorphism $g: D\to D$ be an absolutely cyclic map. For each $q\in D$ let us identify the point $(1, q)$ of $\mathbf{I}\times D$ with $\big(0, g(q)\big)\in\mathbf{I}\times D$ where $\mathbf{I}=[0, 1]$ is the same topological space as in 1.5. The quotient space of $\mathbf{I}\times D$ thus obtained is called a *solenoid*.

2.3. REMARK. Since each connected and locally connected n-dimensional LCA-group decomposes into the direct sum of a k-dimensional $(0\leqq k\leqq n)$ toroidal group and an $(n-k)$-dimensional vector group (see 1.2 and [10] p. 170), to prove Theorem A we need only to show that *the space of a connected but non-locally connected n-dimensional LCA-group cannot be embedded in R^{n+1}.*

2.4. We can show that the space of each n-dimensional connected but non-locally connected LCA-group has a subspace homeomorphic to the topological product of a solenoid and an $(n-1)$-dimensional cube.

Thus we need only to show that the topological product $S\times C^{n-1}$ of a solenoid S and an $(n-1)$-dimensional cube C^{n-1} (if $n=1$ then $C^0=C^{1-1}$ is a singleton) cannot be embedded in R^{n+1}.

2.5. Let D be a space homeomorphic to the Cantor discontinuum and $g: D\to D$ an absolutely cyclic map of D. Let S be the solenoid obtained from this map. Let $\eta: \mathbf{I}\times D\to S$ be the natural map of $\mathbf{I}\times D$ to the quotient space S, i.e. $\eta(t, q)=\eta(t', q')$ iff $(t, q)=(t', q')$ or $t=0$, $t'=1$ and $q=g(q')$ or $t=1$, $t'=0$ and $q'=g(q)$. Let $n\geqq 1$ and let C^{n-1} be an $(n-1)$-cube. Consider the subspaces

$$Y=\eta\left(\left[\frac{1}{6}, \frac{5}{6}\right]\times D\right)\times C^{n-1}$$

and

$$Y' = \eta\left(\left(\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]\right) \times D\right) \times C^{n-1}$$

of $S \times C^{n-1}$. Each of these two subspaces is a system of n-bricks. In particular, for $q \in D$ let

$$B_q = \eta\left(\left[\frac{1}{6}, \frac{5}{6}\right] \times \{q\}\right) \times C^{n-1}$$

and

$$B'_q = \eta\left(\left(\left[0, \frac{1}{3}\right] \times \{q\}\right) \cup \left(\left[\frac{2}{3}, 1\right] \times \{g^{-1}(q)\}\right)\right) \times C^{n-1}.$$

Observe that for $q \in D$ the brick B_q of the first system meets two bricks of the second system, namely the bricks B'_q and $B'_{g(q)}$.

Suppose now that $S \times C^{n-1}$ can be embedded in R^{n+1} and let $h: S \times C^{n-1} \to R^{n+1}$ be an injective continuous map. h is a topological map of $S \times C^{n-1}$ onto the subspace $h(S \times C^{n-1})$ of R^{n+1}.

The bricks $h(B_q)$ of the set $h(Y)$ determine in a certain sense a local ordering of D (see also [5]). By joining the bricks B_q and $B'_{q'}$ described above we can prove that g preserves this local ordering. On the other hand, we can show that each absolutely cyclic map of the Cantor discontinuum D disturbs every local ordering on D. Thus we come to a contradiction and this proves the nonexistence of h, i.e. $S \times C^{n-1}$ cannot be embedded in R^{n+1}.

Now we are going to prepare the proof of Theorem A in details.

3. Absolutely cyclic maps

Let X be a T_1-space and $g: X \to X$ an autohomeomorphism of X. We keep them fixed in this section.

The section deals with the fundamental properties of (X, g)-cycles and of absolutely cyclic maps.

Also we shall prove a theorem stated in [5], namely that the only infinite T_1-space which possesses an absolutely cyclic map is the Cantor discontinuum.

3.1. First we introduce a useful notation. Let m be a positive integer and let $i, j \in \{1, 2, \ldots, m\}$. Let

$$i +_m j = \begin{cases} i+j & \text{if } \ i+j \leqq m \\ i+j-m & \text{if } \ m < i+j, \end{cases}$$

$$i -_m j = \begin{cases} i-j & \text{if } \ 0 < i-j \\ i-j+m & \text{if } \ i-j \leqq 0. \end{cases}$$

Using these notations we can simplify the definition 2.1 as follows.

A finite sequence $Z = (L_1, \ldots, L_m)$ of nonempty pairwise disjoint closed subsets of X is said to be an *(X, g)-cycle of order m* if Z is a covering of X and $g(L_i) = L_{i +_m 1}$ for $i = 1, \ldots, m$.

3.2. DEFINITION. Let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle and let $k\in\{1, \dots, m\}$. For $i=1, \dots, m$ let $L_i'=L_{i+_m k}$. Then $Z'=(L_1', \dots, L_m')$ is clearly an (X, g)-cycle of order m as well. It is called the *image of Z by rotation* and we write $Z'\sim Z$. "$\sim$" is obviously a relation of equivalence on the set of (X, g)-cycles.

3.3. REMARK. Let Z and Z' be (X, g)-cycles. Then for the (X, g)-cycles Z and Z', Z' is clearly the image of Z by rotation iff Z and Z' have at least one common member.

3.4. DEFINITION. The (X, g)-cycle $Z=(X)$ of order 1 is called the *trivial* (X, g)-*cycle*. If X is connected then there is no other (X, g)-cycle than $Z=(X)$.

3.5. REMARK. Let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle. Let $i\in\{1, \dots, m\}$ and $q\in L_i$. Then for any positive integer k the point $g^k(q)$ belongs to L_i iff m is a divisor of k, i.e. $m|k$. $g^k=\overset{1}{g}\dots\overset{k}{g}\colon X\to X$ is clearly an autohomeomorphism of X, too.

3.6. REMARK. Let the (X, g)-cycle $Z'=(L_1', \dots, L_s')$ be a refinement of the $(X, g$-) cycle $Z=(L_1, \dots, L_m)$. Then $m\leqq s$ and for each $j\in\{1, \dots, s\}$ there exists a unique $i(j)\in\{1, \dots, m\}$ such that $L_j'\subset L_{i(j)}$.

The function $i(j)$ obviously satisfies the relation

$$i(j+_s 1) = i(j)+_m 1 \tag{1}$$

and thus

$$i(j+_s m) = i(j)+_m m = i(j). \tag{2}$$

Now let $j, j'\in\{1, \dots, s\}$. Then (1) and (2) imply

$$i(j)=i(j')\Leftrightarrow m|j'-j. \tag{3}$$

On the other hand 3.5 shows that m is a divisor of s: $m|s$.

Finally (3) implies that $Z''=(L_m', L_{2m}', \dots, L_s')$ is an $(L_{i(m)}, g^m|_{L_{i(m)}})$ cycle of order s/m and thus

$$L_{i(m)} = L_m'\cup L_{2m}'\cup\dots\cup L_s'.$$

3.7. REMARK. Let $Z=(L_1, \dots, L_m)$ and $Z'=(L_1', \dots, L_m')$ be (X, g)-cycles of the same order m. Suppose that there exists an (X, g)-cycle $Z''=(L_1'', \dots, L_s'')$ such that Z'' is a refinement of both (X, g)-cycles Z and Z'. Then Z' is the image of Z by rotation.

In fact, 3.6 shows that

$$L_{i(m)} = L_m''\cup L_{2m}''\cup\dots\cup L_s'' = L_{i'(m)}'$$

where $L_{i(m)}$ and $L_{i'(m)}'$ are determined by the relations

$$L_m''\subset L_{i(m)},\quad L_m''\subset L_{i'(m)}'.$$

Consequently, by 3.3 we have $Z'\sim Z$ indeed.

3.8. LEMMA. *Suppose that g is an absolutely cyclic map of X* (see 2.1). *Let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle. Then $g^m|_{L_1}\colon L_1\to L_1$ is an absolutely cyclic map of the subspace L_1 of X.*

PROOF. First observe that by 3.5, $g^m(L_i)\subset L_i$ holds for $i=1, ..., m$. However, g^m is an autohomeomorphism of X and the members of Z are pairwise disjoint sets. Consequently for $i=1, ..., m$ we have $g^m(L_i)=L_i$. Hence $g^m|_{L_1}$ is an autohomeomorphism of the subspace L_1 of X.

Let Ω' be an open covering of L_1 and let

$$\Omega = \Omega' \cup \{L_2, ..., L_m\}.$$

Ω is an open covering of the space X and g is an absolutely cyclic map of X. Consequently, there exists an (X, g)-cycle $Z'=(L_1', ..., L_s')$ which is a refinement of Ω. However, Ω is a refinement of Z by construction and thus Z' is a refinement of Z. This implies $s\geqq m$ and that m is a divisor of s (see 3.6). Without loss of generality we can suppose that $L_m'\subset L_1$, since otherwise we can replace Z' by an appropriate image of it by rotation. Hence taking also 3.6 into account, $Z''=(L_m', L_{2m}', ..., L_s')$ is an $(L_1, g^m|_{L_1})$-cycle and since Z' is a refinement of Ω, it clearly follows that Z'' is a refinement of Ω'. $g^m|_{L_1}$ is an absolutely ciclic map of L_1 indeed. □

3.9. LEMMA. *Let k be a positive integer and suppose that the cardinality of X is higher than k. Suppose that $g: X\to X$ is an absolutely cyclic map. Then there exists an (X, g)-cycle of order $\geqq k+1$ and thus for $q\in X$ the points $q, g(q), ..., g^k(q)$ are pairwise distinct* (cf. 3.5).

PROOF. Let $p_1, ..., p_{k+1}$ be pairwise distinct points of X. Let $N=\{p_1, ..., p_{k+1}\}$ and for $i=1, ..., k+1$ let $U_i=(X\setminus N)\cup\{p_i\}$. Let $\Omega=\{U_1, ..., U_{k+1}\}$. Ω is clearly an open covering of X. Let $Z=(L_1, ..., L_m)$ be an (X, g)-cycle which is a refinement of Ω. Since $g: X\to X$ is an absolutely cyclic map, it follows the existence of such a cycle Z.

Now for $i=1, ..., k+1$, let L^i be the member of Z which contains the point p_i. Since for $j\neq i$ we have $p_i\notin U_j$, it follows

$$L^i \not\subset U_j \quad (j\neq i) \tag{4}$$

and thus

$$L^i \subset U_i \quad (i = 1, ..., k+1). \tag{5}$$

(4) and (5) imply that $L^1, L^2, ..., L^{k+1}$ are pairwise distinct members of Z. Consequently, the order m of Z is at least $k+1$ as required. □

3.10. COROLLARY. *Suppose that X is an infinite space and that the autohomeomorphism g: $X\to X$ is absolutely cyclic. Then for every $q\in X$ the points $q, g(q), ..., ..., g^n(q), ...$ are pairwise distinct.*

3.11. LEMMA. *Suppose that $g: X\to X$ is an absolutely cyclic map and let $q\in X$. Then the set $N=\{q, g(q), ..., g^k(q), ...\}$ is dense in X.*

PROOF. Let $p\in X$ and let U be a neighbourhood of p. Let $\Omega=\{U, X\setminus\{p\}\}$. Let $Z=(L_1, ..., L_m)$ be an (X, g)-cycle which is a refinement of the open covering Ω of X. We may suppose $q\in L_1$, since otherwise we can replace Z by an appropriate image by its rotation. Let L_k be the member of Z containing p. Then $L_k\not\subset(X\setminus\{p\})$ and thus $L_k\subset U$. However, $g^{k+m-1}(q)\in L_k$ and thus $U\cap N\neq\emptyset$. N is everywhere dense in X indeed. □

Now we turn to the theorem stated at the very beginning of this section.

3.12. THEOREM. *The only infinite T_1-space which possesses an absolutely cyclic map is the Cantor discontinuum.*

PROOF. Let X be an infinite T_1-space and $g: X \to X$ an absolutely cyclic map.

We first show that X is a Hausdorff space and the components of X are singletons.

In fact, let p_1 and p_2 be distinct points of X and let

$$\Omega = \{X \setminus \{p_1\},\ X \setminus \{p_2\}\}.$$

Let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle which is a refinement of Ω. Let $p_1 \in L_i$. We then have obviously $L_i \not\subset (X \setminus \{p_1\})$ and thus $L_i \subset (X \setminus \{p_2\})$; consequently L_i and $X \setminus L_i$ are disjoint open neighbourhoods of p_1 and p_2. X is a Hausdorff space indeed.

On the other hand, $\{L_i, X \setminus L_i\}$ is a decomposition of X into disjoint open sets and thus for the component K of the point p_1 in X we have $K \subset L_i$ and thus $p_2 \notin K$. Since p_2 may be considered as an arbitrary point of $X \setminus \{p_1\}$ we get $K \subset \{p_1\}$ and this implies $K = \{p_1\}$ as required.

Next we show that X is a compact space.

In fact, let Ω be an open covering of X and let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle which is a refinement of Ω. Select for each L_i a member G_i of Ω containing L_i. Then $\Omega' = \{G_1, \dots, G_m\}$ is a finite subsystem of Ω such that $\bigcup \Omega' = X$. X is compact indeed.

X is a compact Hausdorff space and thus it is a T_4-space.

Consider now the system Σ consisting of every member of all (X, g)-cycles. Σ is a system of open subsets of X. We show that Σ is a base of X.

In fact, let G be an open subset of X and $q \in G$. Consider the open covering $\Omega = \{G, X \setminus \{q\}\}$ of X and let $Z=(L_1, \dots, L_m)$ be an (X, g)-cycle which is a refinement of Ω. Let $q \in L_i$. Then $L_i \not\subset X \setminus \{q\}$ and thus $L_i \subset G$. Hence for $L_i \in \Sigma$ we have $q \in L_i \subset G$. Ω is a base of X indeed.

This base Σ is countable. First observe that since X is an infinite T_2-space, Σ cannot be finite. To prove that Σ is countable we only need to show that for any two (X, g)-cycles Z and Z' of the same order we have $Z \sim Z'$, i.e. Z' is the image of Z by rotation.

Now let $Z=(L_1, \dots, L_m)$ and $Z'=(L_1', \dots, L_m')$ be (X, g)-cycles of the same order m. Let

$$\Omega = \{L_i \cap L_j';\ i = 1, \dots, m,\ j = 1, \dots, m\}$$

and let Z'' be an (X, g)-cycle which is a refinement of Ω. Then Z'' is an (X, g)-cycle which is a refinement of both cycles Z and Z' and thus our statement follows from 3.7.

We have seen that X is a T_4-space with a countable base, consequently X is metrizable. Thus X is a metrizable compact space that fails to contain any proper continuum. Hence X is a discontinuum.

We finally show that X is a perfect space, i.e. each $q \in X$ is a limit point of X.

In fact, let $q \in X$ and $N = \{g(q), g^2(q), \ldots\}$. Then by 3.10 we have $q \notin N$. On the other hand, 3.11 shows that q belongs to the closure of N. Hence q is a limit point of N and thus it is a limit point of X indeed.

We have proved that X is a perfect discontinuum. Consequently, it is homeomorphic to the standard Cantor discontinuum (see [1] p. 121).

This completes the proof of the theorem. □

4. On the structure of non-locally connected topological groups

This section is devoted to the proof of a theorem stated in 2.4 which is as follows.

4.1. THEOREM. *The space of each n-dimensional connected but non-locally connected LCA-group has a subspace homeomorphic to the topological product of a solenoid and an $(n-1)$-dimensional cube.*

The proof of this theorem proceeds in several steps.

4.2. DEFINITION. Let $f: X \to Y$ and $g: X \to Z$ be surjective maps where X, Y and Z are arbitrary sets. Then g is said to be *compatible with* f if $f(x) = f(x')$ implies $g(x) = g(x')$.

f and g are said to be *mutually compatible maps* if f is compatible with g and and g is compatible with f.

4.3. NOTATION. Suppose that the surjective map $g: X \to Z$ is compatible with the surjective map $f: X \to Y$. We then write gf^{-1} for the map $gf^{-1}: Y \to Z$ defined by the relation

$$\{gf^{-1}(y)\} = gf^{-1}(\{y\}) \quad (y \in Y).$$

Since $gf^{-1}(\{y\})$ is a singleton, it follows that the map $gf^{-1}: Y \to Z$ is well defined.

Notice that for any subset $Y' \subset Y$ we have obviously $(gf^{-1})(Y') = g(f^{-1}(Y'))$ and for $Z' \subset Z$ we clearly have

$$(gf^{-1})^{-1}(Z') = f(g^{-1}(Z')).$$

If f and g are mutually compatible maps then the map $gf^{-1}: Y \to Z$ is bijective and one has $(gf^{-1})^{-1} = fg^{-1}$.

4.4. REMARK. Let X, Y, Z be compact T_2-spaces and suppose that the surjective continuous map $g: X \to Z$ is compatible with the surjective continuous map $f: X \to Y$. Then the map $gf^{-1}: Y \to Z$ is clearly continuous.

On the other hand if f and g are mutually compatible surjective continuous maps then $gf^{-1}: Y \to Z$ is a homeomorphism.

4.5. REMARK. Let X be a compact T_2-space and P an equivalence relation on X. Let $\eta: X \to X/P$ be the natural mapping of X onto the quotient space X/P and suppose that η is a closed map. Then X/P is a compact T_2-space as well (see [6] p. 122 and [6] p. 57).

In particular, let F_1 and F_2 be homeomorphic disjoint closed subsets of X and let $g: F_1 \to F_2$ be a topological map. For $q, q' \in X$ let $(q, q') \in P$ if $q = q'$ or $q \in F_1$

and $q'=g(q)$ or $q'\in F_1$ and $q=g(q')$. The relation P on X is then an equivalences. Considering the natural mapping $\eta: X\to X/P$ for any closed subset M of X we have

$$\eta^{-1}(\eta(M)) = M\cup g(M\cap F_1)\cup g^{-1}(M\cap F_2)$$

and thus $\eta^{-1}(\eta(M))$ is a closed subset of X. Consequently, $\eta(M)$ is closed in the space X/P. Hence $\eta: X\to X/P$ is a closed map and thus X/P is a compact T_2-space.

Now it will be convenient to generalize the concept of the solenoid.

4.6. Definition. Let D be a space homeomorphic to the Cantor discontinuum and let T^k $(k\geqq 1)$ be the brick

$$T^k = \left\{(x_1, \ldots, x_k)\in R^k;\ 0\leqq x_1\leqq 1,\ 0\leqq x_i\leqq \frac{2}{3} \text{ for } i=2, \ldots, k\right\}$$

of R^k. Let

$$T_0^{k-1} = \left\{(x_1, \ldots, x_k)\in R^k;\ x_1=0,\ 0\leqq x_i\leqq \frac{2}{3} \text{ for } i=2, \ldots, k\right\},$$

$$T_1^{k-1} = \left\{(x_1, \ldots, x_k)\in R^k;\ x_1=1,\ 0\leqq x_i\leqq \frac{2}{3} \text{ for } i=2, \ldots, k\right\}.$$

$T_0^{k-1}\times D$ and $T_1^{k-1}\times D$ are disjoint closed subspaces of the compact T_2-space $T^k\times D$.

Let $g: D\to D$ be an absolutely cyclic map (see 2.1). Let the map

$$\bar{g}\colon T_1^{k-1}\times D \to T_0^{k-1}\times D$$

be defined by

$$\bar{g}((1, x_2, \ldots, x_k), q) = ((0, x_2, \ldots, x_k), g(q)).$$

$\bar{g}: T_1^{k-1}\times D\to T_0^{k-1}\times D$ is clearly a homeomorphism.

Now for $y, y'\in T^k\times D$ let $(y, y')\in P$ if $y=y'$ or $y\in T_1^{k-1}\times D$ and $y'=\bar{g}(y)$ or $y'\in T_1^{k-1}\times D$ and $y=\bar{g}(y')$. The relation P on $T^k\times D$ is then an equivalence and $(T^k\times D)/P$ is a compact T_2-space (cf. 4.5). This space is called a *k-dimensional solenoid* and we denote it by S_k. S_k is clearly *determined by the map g*. In this context the solenoid defined in 2.2 is the 1-dimensional solenoid.

The concept of the solenoid was introduced by Vietoris [11]. The terminology "solenoid" comes from van Dantzig [7].

4.7. Remark. Observe that the topological product of a k-dimensional solenoid and an m-dimensional cube is homeomorphic to a $(k+m)$-dimensional solenoid.

In fact, the assertion is obviously true in the case $m=0$.

Now suppose that $m\geqq 1$. Let D, T^k, g and S_k be the same as in 4.6. Let

$$C^m = \left\{(z_1, \ldots, z_m)\in R^m;\ 0\leqq z_i\leqq \frac{2}{3} \text{ for } i=1, \ldots, m\right\}.$$

C^m is an m-cube in R^m.

Let S_{k+m} be the $(k+m)$-dimensional soleonid determined by the same map g: $D \to D$ as S_k.

Let $\eta: T^k \times D \to S_k$ and $\eta': T^{k+m} \times D \to S_{k+m}$ be the natural mapping of the space $T^k \times D$ and $T^{k+m} \times D$ onto the quotient space S_k and S_{k+m}, respectively.

Now for

$$y = (((x_1, \ldots, x_k), q), (z_1, \ldots, z_m)) \in (T^k \times D) \times C^m$$

let

$$f(y) = (\eta((x_1, \ldots, x_k), q), (z_1, \ldots, z_m)) \in S_k \times C^m$$

and

$$f'(y) = \eta'((x_1, \ldots, x_k, z_1, \ldots, z_m), q) \in S_{k+m}.$$

$f: (T^k \times D) \times C^m \to S_k \times C^m$ and $f': (T^k \times D) \times C^m \to S_{k+m}$ are continuous surjective mappings where the spaces $(T^k \times D) \times C^m$, $S_k \times C^m$ and S_{k+m} are compact T_2-spaces. On the other hand, the mappings f and f' are clearly mutually compatible. Consequently, 4.4 shows that

$$f' f^{-1}: S_k \times C^m \to S_{k+m}$$

is a homeomorphism. $S_k \times C^m$ and S_{k+m} are homeomorphic spaces indeed.

And now we go back to Theorem 4.1.

4.8. First observe that to prove 4.1 taking also 4.7 into account, clearly we need only to verify the following theorem.

4.9. THEOREM. *The space of each n-dimensional connected but non-locally connected LCA-group has a subspace homeomorphic to an n-dimensional solenoid.*

4.10. It is sufficient to prove this latter Theorem 4.9 only for compact groups.

In fact, let Γ be an n-dimensional connected but non-locally connected LCA-group. Then Γ is the direct sum of a k-dimensional $(0 \leqq k \leqq n)$ compact subgroup Γ_1 and an $(n-k)$-dimensional vector subgroup V (see [10] p. 160 and 1.2).

Since Γ is connected, so is Γ_1 and since Γ is non-locally connected and V is a locally connected group, it follows that Γ_1 is connected and non-locally connected as well. Consequently, $k \geqq 1$ and assuming that our theorem is true for compact groups, it follows that Γ_1 has a subspace homeomorphic to a k-dimensional solenoid S_k. V has obviously a subspace homeomorphic to an $(n-k)$-cube C^{n-k} and thus Γ has a subspace homeomorphic to $S_k \times C^{n-k}$. Consequently, by 4.7 Γ has a subspace homeomorphic to an n-dimensional solenoid as required.

It is sufficient to prove Theorem 4.9 for the character groups of torsion free non-finitely generated abelian groups of finite rank.

In fact, let Γ_1 be a compact connected but non-locally connected k-dimensional LCA-group. Let G be the character group of Γ_1. Then the topological group Γ_1 is isomorphic to the character group of G (see [10] p. 134). G is a discrete (see [10] p. 133) torsion free group of rank k (see [10] p. 148) and it has no finite system of linearly independent generators (see [10] p. 168). Hence if the space of the character group of each torsion free discrete non-finitely generated abelian group of rank k has a subspace homeomorphic to a k-dimensional solenoid S_k, then we may conclude the same statement for Γ_1.

Thus to prove 4.9 we need to verify only the following assertion.

4.11. *The space of the character group of any torsion free discrete non-finitely generated abelian group of finite rank k $(1 \leqq k)$ has a subspace homeomorphic to a k-dimensional solenoid.*

The remainder of this section deals with the proof of this latter assertion 4.11.

4.12. LEMMA. *Let Γ be a compact LCA-topological group and let Ω be an open covering of Γ. Then there is a neighbourhood V of the zero element of Γ such that the covering*

$$\Phi = \{q+V;\ q\in\Gamma\}$$

of Γ is a refinement of Ω.

PROOF. Select a finite subcovering

$$\Omega' = \{U_1, \ldots, U_k\}$$

of Ω. The space of Γ is T_0 and regular, hence it is T_2 and by the compactness of Γ it is T_4. Thus the space of Γ is normal. Consequently, there is a closed covering $\Psi = \{F_1, \ldots, F_k\}$ of Γ such that $F_i \subset V_i$ holds for $i=1, \ldots, k$. Since for $i=1, \ldots, k$, F_i is compact it follows the existence of a neighbourhood V_i of the zero element of Γ such that for each $q\in F_i$ $q+V_i\subset U_i$. Let $V = V_1 \cap \ldots \cap V_k$. Then

$$\Phi = \{q+V;\ q\in\Gamma\}$$

is clearly a refinement of Ω. □

4.13. NOTATION. Let G be a torsion free abelian group and let S be a subset of G. We then denote by $\langle S\rangle_*$ the subgroup of G generated by S and by $\langle S\rangle_p$ the pure subgroup of G generated by S. $\langle S\rangle_*$ is the minimal subgroup of G and $\langle S\rangle_p$ is the minimal pure subgroup of G containing S.

4.14. LEMMA. *Let G be a torsion free non-finitely generated (discrete) abelian group of finite rank k $(k\geqq 1)$. Then G admits a (pure) subgroup G_0 such that G/G_0 is a torsion free non finitely generated group of rank* 1.

PROOF. We proceed by induction with respect to the rank k.

If $k=1$ then the assertion is obviously true.

Suppose now that $k>1$ and that the statement is true if one replaces k by $k-1$. Let $g_1, g_2, \ldots, g_k$ be linearly independent elements of G and let

$$G' = \langle\{g_2, \ldots, g_k\}\rangle_p.$$

Then G' is a torsion free group of rank $(k-1)$ and G/G' is a torsion free group of rank 1. Both of the groups G' and G/G' cannot be finitely generated since otherwise G itself would be finitely generated. If G/G' is not finitely generated then we may select $G_0 = G'$. Conversely, if G/G' is finitely generated then G' is not finitely generated and thus by the induction hypothesis there is a subgroup G_0' of G' for which G'/G_0' is a torsion free non-finitely generated group of rank 1. On the other hand, since G/G' is a torsion free finitely generated group of rank 1, it follows that G/G' is an infinite cyclic group and thus G is the direct sum of G' and a subgroup G_1 isomorphic to G/G'. Let $G_0=G_0'+G_1$. Then G/G_0 is clearly isomorphic to G'/G_0' and thus it is of the required type indeed. $\square$

4.15. Let G be a torsion free non-finitely generated (discrete) abelian group of finite rank k $(k\geqq 1)$. G is clearly an LCA-group. Let G_0 be a pure subgroup of G of rank $k-1$ such that G/G_0 is a torsion free non finitely generated abelian group (of rank 1) (see 4.14). Let $g_1\in G\setminus G_0$ and in the case $k\geqq 2$ let $g_2, \dots, g_k$ be linearly independent elements of G_0.

In the remainder of this section we keep fixed the groups G and G_0 and the elements $g_1, \dots, g_k$. $g_1, \dots, g_k$ are clearly linearly independent elements of G.

Now we are going to construction the k-dimensional solenoid contained in the space of the character group of G.

4.16. Let $g\in G$. Then there are rational integers $m, m_1, \dots, m_k$ such that $m\neq 0$ and

$$mg = m_1 g_1+\dots+m_k g_k$$

and the rationals $m_1/m, \dots, m_k/m$ are uniquely determined by g. We denote them by $q_1(g), \dots, q_k(g)$ and write

$$g = q_1(g)g_1+\dots+q_k(g)g_k.$$

We have

$$q_i(g_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases} \tag{6}$$

Moreover, for $g, g'\in G$ we have

$$q_i(g+g') = q_i(g)+q_i(g') \quad (i=1, \dots, k). \tag{7}$$

Finally $g\in G_0$ implies

$$q_1(g) = 0. \tag{8}$$

4.17. Let $G_1'=\langle\{g_1\}\rangle_*$ and $G_1=\langle G_0\cup G_1'\rangle_*$ (see 4.13). We then have

$$G_1 = \langle G_0\cup\{g_1\}\rangle_*. \tag{9}$$

G/G_1 is clearly a non finite torsion group.

4.18. Let Γ be the character group of G (see 1.9) and consider the annihilators $\Gamma_0=(\Gamma, G_0)$ and $\Gamma_1=(\Gamma, G_1)$ (see 1.10). Γ_1 is isomorphic to the character group of G/G_1 (see [10] p. 136). Since G/G_1 is discrete, it follows that Γ_1 is a compact LCA-group (see [10] p. 133). On the other hand, since G/G_1 is non-compact and it is isomorphic to the character group of Γ_1 (see [10] p. 134), it follows that the T_0-group Γ_1 fails to be discrete. Consequently, Γ_1 is an infinite compact LCA-group.

4.19. For every $t\in\mathbf{R}$ where $\mathbf{R}$ is the set of real numbers let $\gamma^t: G\to\Xi$ (see 1.8) be the mapping defined by the relation $\gamma^t(g)=\overline{tq_1(g)}$ (see 4.16 and 1.8). γ^t is a character of G.

We have $\gamma^0=0$. Moreover, for $t, t'\in\mathbf{R}$ one has

$$\gamma^{t+t'} = \gamma^t+\gamma^{t'}. \tag{10}$$

Notice also that for $g \in G_0$ we have $q_1(g)=0$ (see 4.16 (8)). Consequently

(11) $$\gamma^t \in \Gamma_0$$

holds for every $t \in \mathbf{C}$. On the other hand, let i be a rational integer ($i \in \mathbf{Z}$). Then taking also 4.16 (6) into account, we get

$$\gamma^i(g_1) = \overline{iq_1(g_1)} = \bar{i} = \bar{0}$$

and thus by (11) and 4.17 we have

(12) $$\gamma^i \in \Gamma_1 \quad (i \in \mathbf{Z}).$$

4.20. LEMMA. *Let $g^1, \ldots, g^s$ be elements of G and let*

$$G' = \langle G_1 \cup \{g^1, \ldots, g^s\} \rangle_* .$$

Then there is a $g' \in G$ such that $q_1(g')=1/m$ where m is a positive integer and

$$G' = \langle G_0 \cup \{g'\} \rangle_* .$$

Moreover, for the annihilator $\Gamma'=(\Gamma, G')$ of G' in Γ, Γ' is an open subgroup of Γ_1, Γ_1/Γ' consists of m elements and each of these cosets is of the form $\Gamma'+\gamma^i$ where $1 \leqq i \leqq m$. Finally we also have $\gamma^m \in \Gamma'$.

PROOF. First observe that by 4.17 (9) we have

(13) $$G' = \langle G_0 \cup \{g_1, g^1, \ldots, g^s\} \rangle_* .$$

Now let $\eta_0 : G \to G/G_0$ be the natural mapping of G onto G/G_0. Since $g_1 \in G \setminus G_0$, it follows

(14) $$\eta_0(g_1) \neq 0.$$

Hence $\eta_0(G')$ is a nonzero finitely generated subgroup of the torsion free group G/G_0 of rank 1. Consequently $\eta_0(G')$ is a cyclic group. Select $g' \in G'$ such that G'/G_0 is generated by $\eta_0(g')$. Then

(15) $$\eta_0(g_1) = m\eta_0(g')$$

where m is a nonzero integer. We may suppose that $m>0$, since otherwise g' can be replaced by $-g'$. Moreover, we clearly have

(16) $$G' = \langle G_0 \cup \{g'\} \rangle_* .$$

Now by (15) one has

(17) $$g_1 = mg' + g_0$$

where $g_0 \in G_0$. Consequently, taking also 4.16 (6), 4.16 (7) and 4.16 (8) into account, we get

$$1 = q_1(g_1) = mq_1(g') + q_1(g_0) = mq_1(g')$$

and thus

(18) $$q_1(g') = 1/m.$$

Now consider a coset of Γ_1 modulo Γ'. We show that this coset is of the form $\Gamma'+\gamma^i$ where $i\in\{1, \dots, m\}$.

In fact, let γ be an arbitrary element of this coset. Then by (17) one has

$$\gamma(g_1) = m\gamma(g')+\gamma(g_0)$$

and since $g_1\in G_1$, $g'\in G'$, $g_0\in G_0\subset G_1$ and $\gamma\in\Gamma_1=(\Gamma, G_1)$, it follows $\bar{0}=m\gamma(g')+\bar{0}$. Consequently

$$\gamma(g') = \overline{i/m} \tag{19}$$

where the integer i can be selected such that $1\leqq i\leqq m$. However, (18) shows that $\gamma^i(g')=\overline{i/m}$ (see also 4.19) and thus $(\gamma-\gamma^i)(g')=0$. Since $\gamma, \gamma^i\in\Gamma_1\subset\Gamma_0=(\Gamma, G_0)$ (see also 4.19 (12)), taking also (16) into account, we get

$$\gamma-\gamma^i\in(\Gamma, G') = \Gamma'$$

and thus $\gamma\in\Gamma'+\gamma^i$. The coset in question is of the form $\Gamma'+\gamma^i$ indeed.

However, for $1\leqq i<j\leqq m$ and for arbitrary $\gamma'\gamma''\in\Gamma'$ we have $(\gamma'+\gamma^i)(g')=$ $=\gamma^i(g')=\overline{i/m}\neq\overline{j/m}=(\gamma''+\gamma^j)(g')$ and thus $\Gamma'+\gamma^i\neq\Gamma'+\gamma^j$. Consequently, Γ_1/Γ' has exactly m members and since Γ_1/Γ' is finite and each member of it is closed in Γ_1, it follows that each member of Γ_1/Γ', and thus Γ' itself, is open in Γ_1.

Finally, by (18) we have

$$\gamma^m(g') = \overline{m\cdot 1/m} = \bar{1} = \bar{0}$$

and since $\gamma^m\in\Gamma_0=(\Gamma, G_0)$ (see 4.19 (11)) it follows by (16) that $\gamma^m(G')=0$ and thus $\gamma^m\in(\Gamma, G')=\Gamma'$ as required.

The proof of the Lemma is complete. □

4.21. Lemma. *For* $\gamma\in\Gamma_1$ *let* $\psi(\gamma)=\gamma+\gamma^1\in\Gamma_1$ (cf. also 4.19 (12)). *Then the autohomeomorphism* $\psi:\Gamma_1\to\Gamma_1$ *of* Γ_1 *is an absolutely cyclic map.*

Proof. Let Ω be an open covering of Γ_1 and let V_1 be a neighbourhood of the zero element of Γ_1 such that $\{\gamma+V_1; \gamma\in\Gamma_1\}$ is a refinement of Ω. By 4.12, such a V_1 exists. Let Φ be a compact subset of G and U a neighbourhood of zero in Ξ such that

$$(V[\Phi, U]\cap\Gamma_1)\subset V_1 \quad \text{(see 1.9)},$$

i.e. $\gamma'\in\Gamma_1$ and $\gamma'(\Phi)\subset U$ implies $\gamma'\in V_1$.

G is a discrete group and thus Φ is a finite subset of G, $\Phi=\{g^1, \dots, g^s\}$. Let $G' = \langle G_1\cup\{g^1, \dots, g^s\}\rangle_*$ and let $\Gamma'=(\Gamma, G')$. Then for each $\gamma'\in\Gamma'$ we have $\gamma'\in\Gamma_1=(\Gamma, G_1)$ and $\gamma'(\Phi)=\bar{0}\in U$. Thus $\gamma'\in V[\Phi, U]\cap\Gamma_1$ and this yields

$$\Gamma'\subset V_1. \tag{20}$$

By 4.20, Γ_1/Γ' is a finite group and if its order is m then each member of the group is of the form $\Gamma'+\gamma^i$ where $1\leqq i\leqq m$.

Now for $i=1, \dots, m$ let $L_i=\Gamma'+\gamma^i$.

We first show that $Z=(L_1, \dots, L_m)$ is a (Γ_1, ψ)-cycle.

In fact, by 4.20 $L_1, \ldots, L_m$ are pairwise disjoint closed subsets of Γ_1 and Z is a covering of Γ_1. Moreover, taking also 4.19 (10) into account, for $i=1, \ldots, m-1$ we have

$$\psi(L_i) = \Gamma' + \gamma^i + \gamma^1 = \Gamma' + \gamma^{i+1} = L_{i+1}$$

and since $\gamma^m \in \Gamma'$ (see 4.20) we also have

$$\psi(L_m) = \Gamma' + \gamma^m + \gamma^1 = \Gamma' + \gamma^1 = L_1.$$

Z is a (Γ_1, ψ)-cycle indeed.

Next, we show that the covering Z is a refinement of Ω.

In fact, let $i \in \{1, \ldots, m\}$ and let $\gamma' \gamma'' \in L_i = \Gamma' + \gamma^i$. Then $\gamma'' - \gamma' \in \Gamma'$ and thus $\gamma'' \in \gamma' + \Gamma'$. Consequently by (20) we have $L_i \subset \gamma' + V_1$. However, $\{\gamma + V_1; \gamma \in \Gamma_1\}$ is a refinement of Ω and thus there is a member W of Ω such that

$$L_i \subset \gamma' + V_1 \subset W.$$

Z is a refinement of Ω indeed and thus ψ is an absolutely cyclic map of Γ_1 as required. □

Observe that since Γ_1 is an infinite LCA-group (see 4.18), it follows by 3.12 that *the space of Γ_1 is homeomorphic to the Cantor discontinuum.*

4.22. We now define the characters of the form $\gamma^{t_1, \ldots, t_k}$ of the group G where $t_1, \ldots, t_k$ are arbitrary reals.

In fact, for $g \in G$ let

$$\gamma^{t_1, \ldots, t_k}(g) = \overline{\sum_{i=1}^{k} t_i q_i(g)}$$

(see 4.16 and 1.8). $\gamma^{t_1, \ldots, t_k}\colon G \to \Xi$ is obviously a character of G and we clearly have $\gamma^{t_1, \ldots, t_k} - \gamma^{t'_1, \ldots, t'_k} = \gamma^{t_1 - t'_1, \ldots, t_k - t'_k}$,

$$\gamma^{t_1, 0, \ldots, 0} = \gamma^{t_1}. \tag{21}$$

4.23. Now let T^k be the same as in 4.6 and consider the space $T^k \times \Gamma_1$. Let for $((t_1, \ldots, t_k), \gamma) \in T^k \times \Gamma_1$

$$f((t_1, \ldots, t_k), \gamma) = \gamma + \gamma^{t_1, \ldots, t_k} \in \Gamma.$$

First we show that the mapping $f\colon T^k \times \Gamma_1 \to \Gamma$ is continuous.

In fact, let $y = ((t_1, \ldots, t_k), \gamma) \in T^k \times \Gamma_1$ and let W be a neighbourhood of $f(y)$ in Γ. Since Γ is the character group of G, it follows that there exists a compact subset Φ of G and a positive integer r such that

$$f(y) + V[\Phi, U_r] \subset W$$

(cf. 1.9 and 1.8). Since Φ is compact and G is discrete, it follows that Φ is a finite subset of G, $\Phi = \{g^1, \ldots, g^s\}$. Let $G' = \langle G_1 \cup \Phi \rangle_*$ (cf. 4.13) and $\Gamma' = (\Gamma, G')$ (cf. 1.10). According to 4.20, Γ' is an open subgroup of Γ_1. Now let

$$M = 1 + \sum_{i=1}^{k} \sum_{j=1}^{s} |q_i(g^j)| \quad \text{(cf. 4.16)}$$

and consider the neighbourhood

$$V=\left\{y'=((t_1', \ldots, t_k'), \gamma');\ \gamma'-\gamma\in\Gamma' \quad\text{and}\quad |t_i'-t_i|<\frac{1}{3^r M} \text{ for } i=1, \ldots, k\right\}$$

of y in $T^k\times\Gamma_1$. We then have

$$f(V)\subset f(y)+V[\Phi, U_r]\subset W,$$

which proves the continuity of the mapping $f\colon T_k\times\Gamma_1\to\Gamma$.

Next let S_k be the k-dimensional solenoid determined by the map $\psi\colon\Gamma_1\to\Gamma_1$ and let $\eta\colon T^k\times\Gamma_1\to S_k$ be the natural mapping of $T^k\times\Gamma_1$ onto the quotient space S_k.

We now show that $f\colon T^k\times\Gamma_1\to f(T^k\times\Gamma_1)\subset\Gamma$ and $\eta\colon T^k\times\Gamma_1\to S_k$ are mutually compatible maps (cf. 4.2) where $f(T^k\times\Gamma_1)$ is considered as a subspace of Γ.

In fact, let

$$y=((t_1, \ldots, t_k), \gamma)\in T^k\times\Gamma_1, \quad y'=((t_1', \ldots, t_k'), \gamma')\in T^k\times\Gamma_1$$

and suppose that $t_1\leqq t_1'$, $y\neq y'$ and $\eta(y)=\eta(y')$. Then $t_i=t_i'$ for $i=2, \ldots, k$. Moreover $t_1=0$, $t_1'=1$ and $\gamma=\psi(\gamma')=\gamma'+\gamma^1$ (see 4.6) and thus

$$f(y)=\gamma+\gamma^{0,t_2,\ldots,t_k}=\gamma'+\gamma^1+\gamma^{0,t_2,\ldots,t_k}=$$

$$=\gamma'+\gamma^{1,0,\ldots,0}+\gamma^{0,t_2,\ldots,t_k}=\gamma'+\gamma^{1,t_2,\ldots,t_k}=f(y').$$

On the other hand suppose that $t_1\leqq t_1'$ $y\neq y'$ and $f(y)=f(y')$. That means

$$\gamma+\gamma^{t_1,\ldots,t_k}=\gamma'+\gamma^{t_1',\ldots,t_k'}$$

and thus

$$\gamma^{t_1'-t_1,\ldots,t_k'-t_k}=\gamma-\gamma'\in\Gamma_1. \tag{22}$$

Consequently taking also 4.22 and 4.16 (6) into account we get

$$\overline{t_i'-t_i}=\overline{(t_i'-t_i)q_i(g_i)}=\overline{\sum_{j=1}^{k}(t_j'-t_j)q_j(g_i)}=\gamma^{t_1'-t_1,\ldots,t_k'-t_k}(g_i)=(\gamma-\gamma')(g_i).$$

However $g_1, \ldots, g_k\in G_1$ and $\gamma, \gamma'\in\Gamma_1=(\Gamma, G_1)$ and these yields

$$\overline{t_i'-t_i}=(\gamma-\gamma')(g_i)=\bar{0}. \tag{23}$$

For $i=2, \ldots, k$ we have $0\leqq t_i$, $t_i'\leqq\frac{2}{3}$ and thus by (23) one has

$$t_i=t_i', \quad i=2, \ldots, k. \tag{24}$$

Moreover we have $t_1 \neq t_1'$ since otherwise (24) and (22) would imply $\gamma=\gamma'$ and thus $y=y'$ contradicting the assumption $y \neq y'$. Hence $t_1<t_1'$ and since $0 \leqq t_1 < t_1' \leqq 1$, (23) implies $t_1=0$ and $t_1'=1$. Thus taking also (22) and 4.22 (21) into account we get

$$\gamma-\gamma' = \gamma^1, \quad \gamma = \gamma'+\gamma^1 = \psi(\gamma').$$

That is to say

$$y = \big((0, t_2, \ldots, t_k), \psi(\gamma')\big) \quad \text{and} \quad y' = \big((1, t_2, \ldots, t_k), \gamma'\big)$$

and this implies $\eta(y)=\eta(y')$.

The surjective maps

$$\eta\colon T^k \times \Gamma_1 \to S_k \quad \text{and} \quad f\colon T^k \times \Gamma_1 \to f(T^k \times \Gamma_1) \subset \Gamma$$

of the compact space $T^k \times \Gamma_1$ onto the T_2-spaces S_k and $f(T^k \times \Gamma_1)$ (cf. also 4.6) are mutually compatible indeed.

Now according to 4.4

$$f\eta^{-1}\colon S_k \to f(T^k \times \Gamma_1) \subset \Gamma$$

is a homeomorphism. The space of Γ has a subspace namely $f(T^k \times \Gamma_1)$ homeomorphic to a k-dimensional solenoid indeed.

The proof of 4.11 is complete and so the proof of Theorems 4.9 and 4.1 is complete as well.

5. The structure of S_n

This section deals with systems of n-bricks. We also introduce the notion of joined systems of n-bricks with respect to a joining function. Finally, we show that the n-dimensional solenoid can be considered as the union of two joined systems of n-bricks with respect to two joining functions. One of them is the identical map of the Cantor discontinuum and the second is an absolutely cyclic map of the same discontinuum.

5.1. Definition. A triple (Y, D, f) is called a *system of n-bricks* if

(a) Y and D are topological spaces and $f\colon Y \to D$ is a surjective continuous mapping,

(b) there exists an n-brick B^n in R^n and a homeomorphism $\varphi\colon B^n \times D \to Y$ such that for each $(x, q) \in B^n \times D$ we have

$$f(\varphi(x, q)) = q \tag{25}$$

and thus for each $q \in D$ one has

$$f^{-1}(\{q\}) = \varphi(B^n \times \{q\}). \tag{26}$$

Y is said to be the *body* and D the *base* of the system. f is called the *projection* of (Y, D, f).

Observe that f is clearly an open map.

For any pair (B^n, φ) satisfaying (25), B^n is said to be the *fibre* and φ the *coordinate function* of (B^n, φ). The pair (B^n, φ) itself is said to be a *coordinate pair* of the given system.

5.2. DEFINITION. The n-bricks B^n and B'^n lying in R^n are said to be *properly joined* if they have a common interior point, i.e. if $\operatorname{int} B^n \cap \operatorname{int} B'^n \neq \emptyset$.

5.3. DEFINITION. Let (Y, D, f) and (Y', D', f') be systems of n-bricks where Y and Y' are subspaces of the same topological space and the bases D and D' are homeomorphic. Let $\psi: D \to D'$ be a homeomorphism. We say that (Y, D, f) is *joined to* (Y', D', f') *with respect to the joining function* ψ if there are properly joined n-bricks B^n and B'^n in R^n and for each $q \in D$ there is an open neighbourhood V_q of q in D and a homeomorphism

$$\varphi_q\colon ((B^n \cup B'^n) \times V_q) \to (f^{-1}(V_q) \cup f'^{-1}(\psi(V_q)))$$

such that for each $q' \in V_q$

$$\varphi_q(B^n \times \{q'\}) = f^{-1}(\{q'\}) \tag{27}$$

and

$$\varphi_q(B'^n \times \{q'\}) = f'^{-1}(\{\psi(q')\}). \tag{28}$$

5.4. REMARK. Let the triple (Y, D, f) be a system of n-bricks and let $h: Y \to Y_1$ be a topological mapping. Let

$$f_1 = fh^{-1}\colon Y_1 \to D.$$

Then (Y_1, D, f_1) is clearly a system of n-bricks as well.

5.5. REMARK. Let the system of n-bricks (Y, D, f) be joined to that of (Y', D', f') with respect to a joining function $\psi: D \to D'$. (It is clearly supposed that Y and Y' are subspaces of the same topological space and D and D' are homeomorphic.) Let $h: Y \cup Y' \to Z$ be a topological mapping. Let $Y_1 = h(Y)$, $Y_1' = h(Y')$, $h_1 = h|_Y\colon Y \to Y_1$, $h_1' = h|_{Y'}\colon Y' \to Y_1'$, $f_1 = fh_1^{-1}\colon Y_1 \to D$ and $f_1' = f' h_1'^{-1}\colon Y_1' \to D'$. Then the system of n-bricks (Y_1, D, f_1) is clearly joined to (Y_1', D', f_1') with respect to the same joining function ψ.

5.6. THEOREM. *Let D be a space homeomorphic to the Cantor discontinuum. Let $\psi: D \to D$ be an absolutely cyclic map and let S_n be the n-dimensional solenoid determined by the map ψ* (cf. 4.6). *Then there exist systems of n-bricks (Y, D, f) and (Y', D, f') such that Y and Y' are subspaces of S_n and (Y, D, f) is joined to (Y', D, f') with respect to two distinct joining functions. The first of them is the identical map of D and the second is ψ.*

PROOF. Consider first the following bricks in R^n:

$$T^n = \left\{(x_1, \dots, x_n);\ 0 \leqq x_1 \leqq 1,\ 0 \leqq x_i \leqq \frac{2}{3} \text{ for } i = 2, \dots, n\right\},$$

$$T_0^{n-1} = \{x = (x_1, \dots, x_n);\ x \in T^n \text{ and } x_1 = 0\},$$

$$T_1^{n-1} = \{x = (x_1, \dots, x_n);\ x \in T^n \text{ and } x_1 = 1\},$$

$$(29)\qquad B^n = \left\{x = (x_1, \dots, x_n);\ x \in T^n \text{ and } \frac{1}{6} \leqq x_1 \leqq \frac{5}{6}\right\},$$

$$(30)\qquad B_0^n = \left\{x = (x_1, \dots, x_n);\ x \in T^n \text{ and } 0 \leqq x_1 \leqq \frac{1}{3}\right\},$$

$$(31)\qquad B_1^n = \left\{x = (x_1, \dots, x_n);\ x \in T^n \text{ and } \frac{2}{3} \leqq x_1 \leqq 1\right\},$$

$$(32)\qquad B'^n = \left\{(x_1, \dots, x_n);\ \frac{2}{3} \leqq x_1 \leqq \frac{4}{3} \text{ and } 0 \leqq x_i \leqq \frac{2}{3} \text{ for } i = 2, \dots, n\right\},$$

$$(33)\qquad B''^n = \left\{(x_1, \dots, x_n);\ -\frac{1}{3} \leqq x_1 \leqq \frac{1}{3} \text{ and } 0 \leqq x_i \leqq \frac{2}{3} \text{ for } i = 2, \dots, n\right\}.$$

Observe that B^n and B'^n are properly joined n-bricks in R^n. Likewise B^n and B''^n are properly joined n-bricks in R^n.

Recall from 4.6 that S_n is defined as follows. For $((1, x_2, \dots, x_n), q) \in T_1^{n-1} \times D$ let $\bar{\psi}((1, x_2, \dots, x_n), q) = ((0, x_2, \dots, x_n), \psi(q)) \in T_0^{n-1} \times D$ and for $z, z' \in T^n \times D$ let $(z, z') \in P$ if $z = z'$ or $z \in T_1^{n-1} \times D$ and $z' = \bar{\psi}(z)$ or $z' \in T_1^{n-1} \times D$ and $z = \bar{\psi}(z')$. Now S_n is the quotient space $(T^n \times D)/P$. It is a compact T_2-space.

Let $\eta: T^n \times D \to S_n$ be the natural mapping.

Now let $Y = \eta(B^n \times D)$ (see (29)). $\eta|_{B^n \times D}: B^n \times D \to Y$ is clearly a homeomorphism.

Next for $q' \in D$ and $y \in \eta(B^n \times \{q'\}) \subset Y$ let $f(y) = q'$. $f: Y \to D$ is clearly a surjective continuous map and (Y, D, f) is obviously a system of n-bricks where Y is a subspace of S_n. Moreover for $q' \in D$ we clearly have

$$(34)\qquad f^{-1}(\{q'\}) = \eta(B^n \times \{q'\}).$$

Now let

$$Y' = \eta((B_0^n \cup B_1^n) \times D) \quad \text{(see (30) and (31))}$$

and

$$\eta' = \eta|_{(B_0^n \cup B_1^n) \times D}: (B_0^n \cup B_1^n) \times D \to Y'.$$

Moreover for $z \in (B_0^n \cup B_1^n) \times D$ let

$$(35)\qquad \tilde{f}'(z) = \begin{cases} q' & \text{if } z \in B_0^n \times \{q'\} \\ \psi(q') & \text{if } z \in B_1^n \times \{q'\}. \end{cases}$$

$\tilde{f}'\colon (B_0^n \cup B_1^n)\times D \to D$ is clearly a continuous surjective map and it is compatible with the map $\eta'\colon (B_0^n \cup B_1^n)\times D \to Y' \subset S_n$. Hence according to 4.4

$$f' = \tilde{f}'\eta'^{-1}\colon Y' \to D \tag{36}$$

is a continuous surjective map, and for $q' \in D$ we clearly have

$$f'^{-1}(\{q'\}) = \eta\tilde{f}'^{-1}(\{q'\}) = \eta\big((B_0^n\times\{q'\})\cup(B_1^n\times\{\psi^{-1}(q')\})\big), \tag{37}$$

$$f'^{-1}(\{\psi(q')\}) = \eta\big((B_0^n\times\{\psi(q')\})\cup(B_1^n\times\{q'\})\big). \tag{38}$$

(Y', D, f') is a system of n-bricks as well.

In fact, for $((x_1, \dots, x_n), q') \in (B_0^n \cup B_1^n)\times D$ let

$$\zeta((x_1, \dots, x_n), q') = \begin{cases} ((x_1, \dots, x_n), q') & \text{if } x_1 \leqq \dfrac{1}{3} \\ ((x_1-1, x_2, \dots, x_n), \psi(q')) & \text{if } x_1 \geqq \dfrac{2}{3}. \end{cases} \tag{39}$$

$\zeta\colon (B_0^n \cup B_1^n)\times D \to B''^n \times D$ (cf. (33)) is a continuous surjective map and the maps ζ and η' are clearly mutually compatible. Hence

$$\varphi = \eta'\zeta^{-1}\colon\ B''^n\times D \to Y' \tag{40}$$

is a homeomorphism and for $q' \in D$ we clearly have

$$\begin{aligned} f'^{-1}(\{q'\}) &= \eta\big((B_0^n\times\{q'\})\cup(B_1^n\times\{\psi^{-1}(q')\})\big) = \\ &= \eta'\zeta^{-1}(B''^n\times\{q'\}) = \varphi(B''^n\times\{q'\}) \end{aligned} \tag{41}$$

(see (37) and (39)). (Y', D', f') is a system of n-bricks indeed.

Now let ψ_0 be the identical map of D. We now show that (Y, D, f) is joined to (Y', D, f') with respect to the joining functions ψ_0 and ψ.

In fact, let $Z = (L_1, \dots, L_m)$ be a (D, ψ)-cycle where $m \geqq 2$. Since ψ is an absolutely cyclic map of the infinite T_1-space D, such a cycle Z obviously exists.

Let $q \in D$ and select $L_i \in Z$ such that $q \in L_i$. L_i is an open neighbourhood of q in D. Moreover, L_i and $L_{i-_m1} = \psi^{-1}(L_i)$ (cf. 3.1) are disjoint subsets of D. Likewise L_i and $L_{i+_m1} = \psi(L_i)$ are disjoint as well. Also, by (34), (37) and (38) we clearly have

$$f^{-1}(L_i) = \eta(B^n\times L_i), \tag{42}$$

$$f'^{-1}(\psi_0(L_i)) = f'^{-1}(L_i) = \eta\big((B_0^n\times L_i)\cup(B_1^n\times L_{i-_m1})\big), \tag{43}$$

$$f'^{-1}(\psi(L_i)) = f'^{-1}(L_{i+_m1}) = \eta\big((B_0^n\times L_{i+_m1})\cup(B_1^n\times L_i)\big). \tag{44}$$

Now let

$$C_i = \big((B^n\cup B_1^n)\times L_i\big)\cup(B_0^n\times L_{i+_m1}) \tag{45}$$

and

$$C_i' = \big((B^n\cup B_0^n)\times L_i\big)\cup(B_1^n\times L_{i-_m1}). \tag{46}$$

By (42), (43) and (44) we then have

$$\eta(C_i) = f^{-1}(L_i) \cup f'^{-1}(\psi(L_i)), \tag{47}$$

$$\eta(C_i') = f^{-1}(L_i) \cup f'^{-1}(\psi_0(L_i)). \tag{48}$$

Next, let

$$\eta_i = \eta|_{C_i}\colon\ C_i \to f^{-1}(L_i) \cup f'^{-1}(\psi(L_i)) \tag{49}$$

and

$$\eta_i' = \eta|_{C_i'}\colon\ C_i' \to f^{-1}(L_i) \cup f'^{-1}(\psi_0(L_i)). \tag{50}$$

On the other hand, for $((x_1, \dots, x_n), q') \in C_i$ (see (45)) let

$$\zeta_i((x_1, \dots, x_n), q') = \begin{cases} ((x_1, x_2, \dots, x_n), q') & \text{if } \ q' \in L_i \\ ((x_1+1, x_2, \dots, x_n), \psi^{-1}(q')) & \text{if } \ q' \in L_{i+_m 1} \end{cases} \tag{51}$$

and for $((x_1, \dots, x_n), q') \in C_i'$ (see (46)) let

$$\zeta_i'((x_1, \dots, x_n), q') = \begin{cases} ((x_1, x_2, \dots, x_n), q') & \text{if } \ q' \in L_i \\ ((x_1-1, x_2, \dots, x_n), \psi(q')) & \text{if } \ q' \in L_{i-_m 1}. \end{cases} \tag{52}$$

Hence

$$\zeta_i\colon\ C_i \to (B^n \cup B'^n) \times L_i \tag{53}$$

and

$$\zeta_i'\colon\ C_i' \to (B^n \cup B''^n) \times L_i \tag{54}$$

(see also (29), (32) and (33)) are surjective continuous maps and for $q' \in L_i$ we have

$$\zeta_i^{-1}(B^n \times \{q'\}) = B^n \times \{q'\}, \tag{55}$$

$$\zeta_i^{-1}(B'^n \times \{q'\}) = (B_1^n \times \{q'\}) \cup (B_0^n \times \{\psi(q')\}), \tag{56}$$

$$\zeta_i'^{-1}(B^n \times \{q'\}) = B^n \times \{q'\}, \tag{57}$$

$$\zeta_i'^{-1}(B''^n \times \{q'\}) = (B_0^n \times \{q'\}) \cup (B_1^n \times \{\psi^{-1}(q')\}) \tag{58}$$

(see also (45), (46), (51) and (52)). Moreover, ζ_i and η_i are mutually compatible maps (see (45), (49), (51) and (53)) and thus

$$\varphi_q = \eta_i \zeta_i^{-1}\colon\ (B^n \cup B'^n) \times L_i \to f^{-1}(L_i) \cup f'^{-1}(\psi(L_i))$$

is a homeomorphism (see 4.4). Further for $q' \in L_i$ taking also (55), (34), (56) and (38) into account we get

$$\varphi_q(B^n \times \{q'\}) = \eta_i \zeta_i^{-1}(B^n \times \{q'\}) = \eta(B^n \times \{q'\}) = f^{-1}(\{q'\})$$

and

$$\varphi_q(B'^n \times \{q'\}) = \eta_i \zeta_i^{-1}(B'^n \times \{q'\}) =$$
$$= \eta((B_1^n \times \{q'\}) \cup (B_0^n \times \{\psi(q')\})) = f'^{-1}(\{\psi(q')\}).$$

Hence (Y, D, f) is joined to (Y', D, f') with respect to the joining function ψ (see also 5.3 (27) and (28)).

On the other hand, by (46), (50), (52) and (54), ζ_i' and η_i' are mutually compatible maps as well and thus

$$\varphi_q' = \eta_i' \zeta_i'^{-1}: (B^n \cup B''^n) \times L_i \to f^{-1}(L_i) \cup f'^{-1}(\psi_0(L_i)) = f^{-1}(L_i) \cup f'^{-1}(L_i)$$

is a homeomorphism. Further by (57), (34), (58) and (37) we have for $q' \in L_i$

$$\varphi_q'(B^n \times \{q'\}) = \eta_i' \zeta_i'^{-1}(B^n \times \{q'\}) = \eta(B^n \times \{q'\}) = f^{-1}(\{q'\})$$

and

$$\varphi_q'(B''^n \times \{q'\}) = \eta_i' \zeta_i'^{-1}(B''^n \times \{q'\}) =$$

$$= \eta(B_0^n \times \{q'\}) \cup (B_1^n \times \{\psi^{-1}(q')\}) = f'^{-1}(\{q'\}) = f'^{-1}(\{\psi_0(q')\}).$$

(Y, D, f) is joined to (Y', D, f') with respect to the joining function $\psi_0 = \mathrm{id}_D$ indeed.

The proof of the theorem is complete. □

6. Local ordering

This section discusses some fundamental properties of ordered sets. Also we recall the notion of local ordering introduced in [5] and we prove a theorem stated in [5], namely that each absolutely cyclic map of the Cantor discontinuum D disturbs each local ordering on D.

6.1. Definition. Let M be a set and R a triadic relation on M. The pair (M, R) is called an *ordered set* if

(i) $(a, b, c) \in R$ implies that a, b, c are distinct elements of M,

(ii) $(a, b, c) \in R$ implies $(c, b, a) \in R$,

(iii) for any three distinct elements a, b, c of M from the relations $(a, b, c) \in R$, $(b, c, a) \in R$, $(c, a, b) \in R$ one and only one holds,

(iv) for any four distinct elements a, b, c, d of M $(a, b, c) \in R$ and $(a, b, d) \notin R$ imply $(c, b, d) \in R$.

6.2. Remark. Let (M, R) be an ordered set and let a, b, c, d be distinct elements of M. Then $(a, b, c) \in R$ and $(a, b, d) \in R$ imply $(c, b, d) \notin R$.

In fact, suppose that $(a, b, c) \in R$, $(a, b, d) \in R$ and $(c, b, d) \in R$. Then by 6.1 (iii) we have $(b, c, a) \notin R$, $(c, a, b) \notin R$, $(b, d, a) \notin R$, $(d, a, b) \notin R$, $(b, d, c) \notin R$ and $(d, c, b) \notin$ $\notin R$ and thus by 6.1 (ii) and 6.1 (iv) we get $(a, c, d) \notin R$, $(d, a, c) \notin R$ and $(c, d, a) \notin R$ but this is implossible by 6.1 (iii).

6.3. Notations. Let (M, R) be an ordered set and $M' \subset M$. Then the restriction $R|_{M' \times M' \times M'}$ is denoted by $R|_{M'}$ and instead of $(M', R|_{M'})$ we also write $(M, R)|_{M'}$.

Notice that if $M'' \subset M' \subset M$ then we have obviously $(R|_{M'})|_{M''} = R|_{M''}$.

6.4. Definition. Let (M, R) be an ordered set and $q \in M$. Then taking also 6.2 into account, $M \setminus \{q\}$ decomposes uniquely in two disjoint sets M_1 and M_2 such that for any $b, c \in M \setminus \{q\}$, b and c are contained in different M_i-s iff $(b, q, c) \in R$. We say that M_1 and M_2 are the *sides of q in (M, R)*.

If $q' \in M_i$ then instead of M_i we also write $S_{R,q}(q')$.

It may happen that one of the sides is the empty set or even more that both sides are empty.

6.5. Remark. Let (M, R) be an ordered set and let $M'\subset M$. Let $q\in M'$. Let M_1 and M_2 be the sides of q in (M, R). Then the sides of q in $(M, R)|_{M'}$ are clearly $M_1\cap M'$ and $M_2\cap M'$.

6.6. Remark. Let $g\colon M\to N$ be a bijective map of the set M onto the set N and let $M'\subset M$. Let (M', R) be an ordered set. Denote then by $g(R)$ the relation

$$g(R) = \{(g(a), g(b), g(c)); (a, b, c)\in R\}.$$

$(g(M'), g(R))$ is then clearly an ordered set.

Observe that $g^{-1}(g(R))=R$. Moreover if $M''\subset M'$ then we have

$$g(R|_{M''}) = g(R)|_{g(M'')}.$$

Finally, if in addition $h\colon N\to P$ is a bijective map then one has $(hg)(R)=h(g(R))$.

Observe that if $N=M$ then instead of gg we also write g^2 and instead of g^2g we write g^3 etc.

6.7. Lemma. *Let (M, R) be an ordered set and let $g\colon M\to M$ be a bijective map for which $g(R)=R$. Let $c\in M$ and suppose that c, $g(c)$ and $g^2(c)$ are pairwise distinct. Then for $k=1, 2, \dots$ we have*

$$g^k(c)\notin S_{R, g(c)}(c)\cap S_{R, g^2(c)}(c) \quad \text{(cf. 6.4).}$$

Proof. The assertion is obviously true for $k=1$ and $k=2$.

According to 6.1 (iii), we have to consider three cases. In all of these cases we proceed by induction.

(a) $(g(c), c, g^2(c))\in R$. We then have $(g^2(c), g(c), g^3(c))\in R$ and by 6.1 (iii) $(c, g(c), g^2(c))\notin R$. Thus by 6.1 (iv) we get

$$(c, g(c), g^3(c))\in R. \tag{59}$$

For proving the assertion in this case we need only to show that for each even k where $k\geqq 4$ we have

$$(c, g^2(c), g(c))\in R$$

and for each odd k' where $k'\geqq 3$ one has

$$(c, g(c), g^{k'}(c))\in R.$$

(59) shows that this latter relation is true for $k'=3$. Suppose that for any odd $k'(k'\geqq 3)$ we have $(c, g(c), g^{k'}(c))\in R$. We then have $(g(c), g^2(c), g^{k'+1}(c))\in R$ and by the assumption (a) and 6.1 (iii) we get $(c, g^2(c), g(c))\notin R$. Thus 6.1 (iv) implies

$$(c, g^2(c), g^{k'+1}(c))\in R.$$

Hence the statement is true for the even number $k'+1$.

On the other hand, suppose that for any even k $(k\geqq 4)$ we have $(c, g^2(c), g^k(c))\in R$. Then one has $(g(c), g^3(c), g^{k+1}(c))\in R$ and thus $(g^3(c), g(c), g^{k+1}(c))\notin R$. Taking also (59) into account we get the required relation $(c, g(c), g^{k+1}(c))\in R$.

(b) Suppose now

$$(g^2(c),\ g(c),\ c)\in R. \tag{60}$$

For proving the assertion in this second case we only need to show that for $k=2, 3, \ldots$ we have $(c, g(c), g^k(c))\in R$.

(60) and 6.1 (ii) show that this relation is true for $k=2$. Suppose that for any k $(k\geqq 2)$ we have $(c, g(c), g^k(c))\in R$. Then $(g(c), g^2(c), g^{k+1}(c))\in R$ and thus $(g^2(c), g(c), g^{k+1}(c))\notin R$. Taking also (60) into account by 6.1 (iv) we get the required relation $(c, g(c), g^{k+1}(c))\in R$.

(c) Suppose finally that

$$(c,\ g^2(c),\ g(c))\in R. \tag{61}$$

This implies $(g(c), g^3(c), g^2(c))\in R$ and $(g^2(c), g^4(c), g^3(c))\in R$ and thus

$$(g(c),\ g^2(c),\ g^3(c))\notin R \tag{62}$$

and $(g^3(c), g^2(c), g^4(c))\notin R$.

Hence we also have

$$(g(c),\ g^2(c),\ g^4(c))\notin R. \tag{63}$$

Taking also (61) into account (62) and (63) imply

$$(c,\ g^2(c),\ g^3(c))\in R \tag{64}$$

and

$$(c,\ g^2(c),\ g^4(c))\in R. \tag{65}$$

To prove the assertion in this third case we only need to show that

$$(c,\ g^2(c),\ g^k(c))\in R$$

holds for $k=3, 4, \ldots$.

In fact, by (64) and (65) this relation is true for $k=3$ and $k=4$. Suppose now that $k\geqq 3$ and that $(c, g^2(c), g^k(c))\in R$. We then have $(g^2(c), g^4(c), g^{k+2}(c))\in R$ and thus $(g^4(c), g^2(c), g^{k+2}(c))\notin R$. Hence taking also (65) into account we get the required relation

$$(c,\ g^2(c),\ g^{k+2}(c))\in R.$$

The proof of the Lemma is complete. □

6.8. DEFINITION. Let X be a topological space. The family $\Theta=\{(M_\alpha, R_\alpha);\ \alpha\in A\}$ of ordered sets is said to be a *local ordering of X* if it satisfies the following conditions:

(i) The system $\{M_\alpha;\ \alpha\in A\}$ is an open base for X.

(ii) For any $q\in X$ and $(M_{\alpha'}, R_{\alpha'})$, $(M_{\alpha''}, R_{\alpha''})\in\Theta$ with $q\in M_{\alpha'}\cap M_{\alpha''}$ there is an (M_α, R_α) in Θ such that $q\in M_\alpha\subset(M_{\alpha'}\cap M_{\alpha''})$ and

$$R_{\alpha'}|_{M_\alpha} = R_{\alpha''}|_{M_\alpha} = R_\alpha.$$

(iii) For any $(M_\alpha, R_\alpha)\in\Theta$ and $q\in M_\alpha$ the sides of q in (M_α, R_α) (cf. 6.4) are open sets in X.

6.9. Definition. Two local orderings Θ and Θ' of the space X are said to be *equivalent* — and we write $\Theta\sim\Theta'$ — if $\Theta\cup\Theta'$ is also a local ordering of X.

$\sim$ is clearly a reflexive and symmetric relation on the set of local orderings of X.

6.10. Remark. Let Θ and Θ' be equivalent local orderings of the space X and let $q\in X$. Select $(M_\alpha, R_\alpha)\in\Theta$ and $(M'_{\alpha'}, R'_{\alpha'})\in\Theta'$ such that $q\in M_\alpha\cap M'_{\alpha'}$. Then there exists an $(M_\beta, R_\beta)\in\Theta$ such that $q\in M_\beta\subset(M_\alpha\cap M'_{\alpha'})$ and

$$R_\beta = R_\alpha|_{M_\beta} = R'_{\alpha'}|_{M_\beta}.$$

In fact, since $\Theta\cup\Theta'$ is a local ordering of X, there exists either an $(M_\beta, R_\beta)\in\Theta$ of the required type or an $(M'_{\beta'}, R'_{\beta'})\in\Theta'$ such that

$$q\in M'_{\beta'}\subset(M_\alpha\cap M'_{\alpha'}) \quad\text{and}\quad R'_{\beta'} = R_\alpha|_{M'_{\beta'}} = R'_{\alpha'}|_{M'_{\beta'}}.$$

We have to consider only the second case. However, by 6.8 (i) there is an $(M_\gamma, R_\gamma)\in\Theta$ such that $q\in M_\gamma\subset M'_{\beta'}$ and by 6.8 (ii) there is an $(M_\beta, R_\beta)\in\Theta$ such that

$$q\in M_\beta\subset(M_\gamma\cap M_\alpha) = M_\gamma \quad\text{and}\quad R_\beta = R_\gamma|_{M_\beta} = R_\alpha|_{M_\beta}.$$

Hence $q\in M_\beta\subset(M_\alpha\cap M'_{\alpha'})$ and

$$R_\beta = R_\alpha|_{M_\beta} = (R_\alpha|_{M'_{\beta'}})|_{M_\beta} = (R'_{\alpha'}|_{M'_{\beta'}})|_{M_\beta} = R'_{\alpha'}|_{M_\beta}$$

as required.

6.11. Remark. The relation $\sim$ is an equivalence on the set of local orderings of the space X.

In fact, we have to show only the transitivity of $\sim$.

To this end consider local orderings Θ, Θ' and Θ'' of X such that $\Theta\sim\Theta'$ and $\Theta'\sim\Theta''$. Let $q\in X$. Select $(M_\alpha, R_\alpha)\in\Theta$ and $(M''_{\alpha''}, R''_{\alpha''})\in\Theta''$ such that $q\in M_\alpha\cap M''_{\alpha''}$. In order to prove $\Theta\sim\Theta''$ we clearly have to show the existence of an $(M_\beta, R_\beta)\in\Theta$ such that $q\in M_\beta\subset(M_\alpha\cap M''_{\alpha''})$ and

$$R_\beta = R_\alpha|_{M_\beta} = R''_{\alpha''}|_{M_\beta}.$$

Now by 6.8. (i) there is an $(M'_{\alpha'}, R'_{\alpha'})\in\Theta'$ such that $q\in M'_{\alpha'}$ and thus by $\Theta'\sim\Theta''$ and 6.10 there is a member $(M'_{\beta'}, R'_{\beta'})$ of Θ' such that

$$q\in M'_{\beta'}\subset(M''_{\alpha''}\cap M'_{\alpha'}) \quad\text{and}\quad R'_{\beta'} = R''_{\alpha''}|_{M'_{\beta'}}.$$

On the other hand, since $\Theta\sim\Theta'$ it follows by 6.10 the existence of an $(M_\beta, R_\beta)\in\Theta$ such that $q\in M_\beta\subset(M_\alpha\cap M'_{\beta'})$ and $R_\beta=R_\alpha|_{M_\beta}=R'_{\beta'}|_{M_\beta}$.

Thus we clearly have $q\in M_\beta\subset(M_\alpha\cap M''_{\alpha''})$ and

$$R_\beta = R_\alpha|_{M_\beta} = R'_{\beta'}|_{M_\beta} = (R''_{\alpha''}|_{M'_{\beta'}})|_{M_\beta} = R''_{\alpha''}|_{M_\beta}$$

as required.

6.12. NOTATION. Let $g: X \to X$ be an autohomeomorphism of the topological space X and let Θ be a local ordering of X. Let

$$g(\Theta) = \{(g(M_\alpha), g(R_\alpha)); (M_\alpha, R_\alpha) \in \Theta\}.$$

$g(\Theta)$ is clearly a local ordering of X as well.

6.13. DEFINITION. Let $g: X \to X$ be an autohomeomorphism of the topological space X and let Θ be a local ordering of X. We say that g *disturbs* Θ if $g(\Theta) \nsim \Theta$.

And now we have the following theorem.

6.14. THEOREM. *Each absolutely cyclic map of a space D homeomorphic to the Cantor discontinuum disturbs every local ordering of D* (cf. 2.1).

PROOF. Let $g: D \to D$ be an absolutely cyclic map of D and let $\Theta = \{(M_\alpha, R_\alpha); \alpha \in A\}$ be a local ordering of D. We have to show only that $g(\Theta) \nsim \Theta$.

We argue by contradiction by supposing $g(\Theta) \sim \Theta$. Let $\Omega = \{M_\alpha; \alpha \in A\}$. Ω is an open covering of D, consequently there is a (D, g)-cycle $Z = (L_1, \ldots, L_m)$ which is a refinement of Ω. Select for each $i \in \{1, \ldots, m\}$ an $\alpha[i] \in A$ such that $L_i \subset M_{\alpha[i]}$.

Let $q \in D$. Then there is a unique $i(q) \in \{1, \ldots, m\}$ such that $q \in L_{i(q)}$. Now select $\alpha(q) \in A$ such that

$$M_{\alpha(q)} \subset L_{i(q)}, \tag{66}$$

$$R_{\alpha(q)} = R_{\alpha[i(q)]}|_{M_{\alpha(q)}} \tag{67}$$

and

$$g(R_{\alpha(q)}) = R_{\alpha[i(q) +_m 1]}|_{g(M_{\alpha(q)})}. \tag{68}$$

(see also 6.6 and 3.1). By $g(\Theta) \sim \Theta$ taking also 6.10 into account, there exists such an $\alpha(q)$.

Let $\Omega' = \{M_{\alpha(q)}; q \in D\}$ and let the (D, g)-cycle $Z' = (L'_1, \ldots, L'_{m'})$ be a refinement of Ω'. Z' is clearly a refinement of Z. Now for $j = 1, \ldots, m'$ let

$$R'_j = R_{\alpha[i(j)]}|_{L'_j}. \tag{69}$$

Recall that $i(j)$ is defined by the relation $L'_j \subset L_{i(j)}$ (cf. 3.6).

We now show that for $j = 1, \ldots, m'$ we have

$$g(R'_j) = R'_{j +_{m'} 1}. \tag{70}$$

In fact, select $q \in D$ such that $L'_j \subset M_{\alpha(q)}$. Then (66) shows that $i(j) = i(q)$ and thus by (67) we have

$$R'_j = R_{\alpha[i(j)]}|_{L'_j} = R_{\alpha[i(q)]}|_{L'_j} = (R_{\alpha[i(q)]}|_{M_{\alpha(q)}})|_{L'_j} = R_{\alpha(q)}|_{L'_j}.$$

Consequently, by (68) and 3.6. (1) we clearly obtain

$$g(R'_j) = g(R_{\alpha(q)}|_{L'_j}) = g(R_{\alpha(q)})|_{L'_{j +_{m'} 1}} = (R_{\alpha[i(q) +_m 1]}|_{g(M_{\alpha(q)})})|_{L'_{j +_{m'} 1}} =$$

$$= R_{\alpha[i(j) +_m 1]}|_{L'_{j +_{m'} 1}} = R_{\alpha[i(j +_{m'} 1)]}|_{L'_{j +_{m'} 1}} = R'_{j +_{m'} 1}$$

as required.

However, (70) implies that $g^{m'}(R_1')=R_1'$ and thus

$$g^{m'}|_{L_1'}(R_1') = R_1'. \tag{71}$$

Let $h=g^{m'}|_{L_1'}: L_1' \to L_1'$. By 3.8 $h: L_1' \to L_1'$ is an absolutely cyclic map of the infinite subspace L_1' of D.

Let $q' \in L_1'$. Then 3.10 shows that q', $h(q')$ and $h^2(q')$ are pairwise distinct. Thus by $q' \in S_{R_1', h(q')}(q')$ and $q' \in S_{R_1', h^2(q')}(q')$ (cf. 6.4)

$$V = S_{R_1', h(q')}(q') \cap S_{R_1', h^2(q')}(q')$$

is a nonempty subset of L_1'. However, by 6.5 and (69)

$$S_{R_1', h(q')}(q') = S_{R_{\alpha[i(1)]}, h(q')}(q') \cap L_1'$$

and

$$S_{R_1', h^2(q')}(q') = S_{R_{\alpha[i(1)]}, h^2(q')}(q') \cap L_1'$$

and thus taking also 6.8 (iii) into account, V is an open nonempty subset of L_1'.

Consider now the subset $N=\{h(q'), h^2(q')=h(h(q')), \ldots, h^k(q)'=h^{k-1}(h(q')), \ldots\}$ of L_1'. By (71) and 6.7 we have $V \cap N = \emptyset$. On the other hand, by 3.11, N is a dense subset of L_1' and thus $V \cap N \neq \emptyset$ in contradiction with the preceding statement.

The assumption $g(\Theta) \sim \Theta$ was false and thus $g(\Theta) \nsim \Theta$ indeed.

The proof of the theorem is complete. $\square$

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6—8.
H—1088

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UNIFORM DISTRIBUTION OF SEQUENCES CONNECTED WITH ARITMETICAL FUNCTIONS

I. KÁTAI (Budapest), member of the Academy

1. Notations and definitions. $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{N}_0$ denote the set of complex, real, rational numbers, integers, positive integers, nonnegative integers, respectively. $e(x)$ stands for the abbreaviation of $e^{2\pi i x}$. $\mathcal{A}$ denotes the class of the real valued additive functions, $\mathcal{M}$ denotes the class of the complex valued multiplicative functions. For an integer $q\geqq 2$ let $\mathcal{A}_q$ be the set of real-valued q additive, and $\mathcal{M}_q$ be the set of complex-valued q-multiplicative functions. Let $E_q=\{0, 1, \dots, q-1\}$. Every $n\in\mathbf{N}_0$ can be written uniquely in the form

$$n=\sum_{j=0}^{\infty} a_j(n)q^j, \quad a_j(n)\in E_q.$$

A function f is called q-additive if

$$f(n)=\sum_{j=0}^{\infty} f\big(a_j(n)q^j\big), \quad f(0)=0$$

and g is called q-multiplicative if

$$g(n)=\prod_{j=0}^{\infty} g\big(a_j(n)q^j\big), \quad g(0)=1.$$

For $x\in\mathbf{R}$ let $\|x\|$ denote the distance of x to the nearest integer.

$a|b$ denotes that a divides b.

U denotes the set of sequences uniformly distributed mod 1 (U.D. mod 1). For the definition of well-distributed sequences mod 1 and for some wellknown results in the theory of U.D. mod 1 sequences see the excellent book of L. Kuipers and H. Niederreiter [7].

2. H. Daboussi [1] proved that

$$\text{(2.1)} \qquad \frac{1}{x}\sum_{n\leqq x} f(n)e(n\alpha)\to 0$$

if $\alpha\in\mathbf{R}\setminus\mathbf{Q}$ and $f\in\mathcal{M}$ of modulus $\leqq 1$. (2.1) was proved by H. Delange [2] under the condition

$$\text{(2.2)} \qquad \sum_{n\leqq x} |f(n)|^2 = O(x),$$

and for the much wider class of uniformly summable multiplicative functions by K.-H. Indlekofer [3]. The speed of the convergence is treated by H. L. Montgomery and R. C. Vaughan in [4].

In [5] we considered the sum

$$M(f, t, x) = \sum_{n \leqq x} f(n) e(t(n)) \tag{2.3}$$

with a general $t: \mathbf{N} \to \mathbf{R}$. The main result is cited now as

LEMMA 1. *Assume that for every $K > 0$ there exists a sequence of mutually coprime integers n_j $(j=1, \ldots, s)$ such that*

$$(K <) n_1 < n_2 < \ldots < n_s; \quad \sum_{j=1}^{s} n_j^{-1} > K, \tag{2.4}$$

and

$$\frac{1}{x} \sum_{m \leqq x} e(t(n_u m) - t(n_v m)) \to 0$$

if $u \neq v$. Then

$$M(f, t; x) = o(x) \quad as \quad x \to \infty \tag{2.5}$$

or every $f \in \mathcal{M}$ of modulus $\leqq 1$, uniformly in f.

It is known that the conditions are satisfied if t is a polynomial such that at least one coefficient of $t(x) - t(0)$ is irrational.

Let $\mathcal{T}$ denote the set of those functions $t: \mathbf{N} \to \mathbf{R}$ for which the sequence

$$\eta_n := F(n) + t(n)$$

belongs to U for each $F \in \mathcal{A}$.

By using the Weyl-criterion Daboussi's theorem implies that $t(n) = \alpha n$ belongs to $\mathcal{T}$ if $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, while Lemma 1 can be formulated in the following way.

LEMMA 2. *If for every K there exists a sequence n_j $(j=1, \ldots, s)$ of mutually coprime integers satisfying* (2.4), *and the sequence*

$$\xi_m := t(n_u m) - t(n_v m)$$

belongs to U for every $u \neq v$, then $t \in \mathcal{T}$.

Since the zero function $\in \mathcal{A}$, therefore $\mathcal{T} \subseteq U$. We are unable to give a complete characterization of $\mathcal{T}$. It is obvious that $\mathcal{T} \cap \mathcal{A} = \emptyset$, since if $t \in \mathcal{A}$, then $F = -t \in \mathcal{A}$ and $F(n) + t(n) = 0$ $(\forall n)$, so $F + t \notin U$.

Let now t be a constant multiple of a $g \in \mathcal{M}$, and let $t(n) = \alpha g(n)$. If $g(n) = n^k$, $k \in \mathbf{N}$, then $t \in \mathcal{T}$ for every $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. We shall prove in Theorem 2 that it is not typical in the class of multiplicative functions taking on integer values.

THEOREM 1. *Let g be any integer-valued arithmetical function. Assume that there exists a positive constant ε, an infinite sequence of positive integers $d_1 < d_2 < \ldots$, so that $d_n | d_{n+1}$ $(n = 1, 2, \ldots)$, and a sequence $x_1 < x_2 < \ldots$ of positive reals tending to infinity such that*

$$\# \{n \leqq x_\nu | g(n) \equiv 0 \pmod{d_\nu}\} \geqq \varepsilon x_\nu. \tag{2.6}$$

Then there exists an irrational α for which $\|\alpha g(n)\|\to 0$ on a set of integers having positive upper density, consequently $\alpha g(n)$ is not UD mod 1.

PROOF. Let λ_n be a sequence of positive real numbers tending to zero monotonically.

Let the sequence $e_1, e_2, \ldots$ be a subsequence of $d_1, d_2, \ldots$ defined as follows.

Let $e_1:=d_1$, $Y_1:=x_1$,

$$A_1 = \max_{\substack{n\leqq Y_1\\ e_1|g(n)}} \frac{g(n)}{e_1}, \quad a_1 = \min_{\substack{n\leqq Y_1\\ e_1|g(n)}} \frac{g(n)}{e_1}.$$

Let e_2 be the smallest member of the sequence d_ν $(\nu=1, 2, \ldots)$ for which the inequality

$$e_2 > \frac{2}{\lambda_1} e_1 \max(|a_1|, |A_1|, 1)$$

holds. Assume that $e_2=d_\nu$. Then $Y_2:=x_\nu$.

Assume that $e_1, e_2, \ldots, e_n$, $Y_1, Y_2, \ldots, Y_n$ are defined,

$$A_n = \max_{\substack{m\leqq Y_n\\ e_n|g(m)}} \frac{g(m)}{e_n}, \quad a_n = \min_{\substack{m\leqq Y_n\\ e_n|g(m)}} \frac{g(m)}{e_n}. \tag{2.7}$$

Then e_{n+1} is the smallest member of the sequence d_ν $(\nu=1, 2, \ldots)$ for which

$$e_{n+1} > \frac{2}{\lambda_n} e_n \max(|a_n|, |A_n|, 1) \tag{2.8}$$

holds. If $e_{n+1}=d\mu$, then $Y_{n+1}:=x_\mu$.

Let now

$$\alpha = \frac{1}{e_1}+\frac{1}{e_2}+\ldots. \tag{2.9}$$

By the notation $q_1=e_1$, $q_n=e_n/e_{n-1}$ we can write α in the form

$$\alpha = \frac{1}{q_1}+\frac{1}{q_1q_2}+\frac{1}{q_1q_2q_3}+\ldots.$$

From (2.8) we get that $q_n\to\infty$. But such type of expansions represent irrational numbers. For the proof see the book Galambos [6], Chapter II, Corollary 2.6.

Let us consider those integers $m\leqq Y_n$ for which $g(m)\equiv 0 (\text{mod } e_n)$. From (2.6) we get that the cardinality of this set is at least εY_n. Let $g(m)=e_nu$, $m\leqq Y_n$. By the notation (2.7) we have $a_n\leqq u\leqq A_n$. Then

$$\alpha g(m) = \left(\frac{1}{e_1}+\ldots+\frac{1}{e_n}\right)e_n u+\left(\frac{1}{e_{n+1}}+\ldots\right)e_n u.$$

The first summand on the right hand side is an integer, the absolute value of the second one is less than

$$\frac{2}{e_{n+1}} e_n|u| < \lambda_n.$$

Consequently

$$\#\{m \leqq Y_n | \, \|\alpha g(m)\| < \lambda_n\} > \varepsilon Y_n,$$

and this completes the proof.

COROLLARY 1. *Let g be an integer valued multiplicative function, and for each prime q let*

$$a(q) := \sum_{q|g(p)} \frac{1}{p},$$

where p runs over the set indicated. Assume that $a(q)=\infty$ *for at least one q. Then there exists at least one irrational* α *for which* $\alpha g(n)$ *is not UD* mod 1.

PROOF. Let q be a fixed prime such that $a(q)=\infty$ and $\mathcal{P}$ the set of the primes p satisfying $g(p)\equiv 0 \pmod q$.

Let $S\in\mathcal{A}$ be defined on the set of prime powers p^α by the relation

$$S(p^\alpha) = \begin{cases} 1 & \text{if } \alpha = 1 \text{ and } p\in\mathcal{P} \\ 0 & \text{otherwise.} \end{cases}$$

Then, from the Turán—Kubilius inequality,

$$\sum_{n\leqq x} (S(n)-A_x)^2 \ll xA_x, \quad A_x = \sum_{\substack{p\in\mathcal{P} \\ p\leqq x}} p^{-1},$$

whence

$$x^{-1}\#\left\{n \leqq x | S(n) > \frac{1}{2}A_x\right\} \to 1 \quad (x\to\infty).$$

Since $S(n)>\frac{1}{2}A_x$ implies that $q^{\beta_x}|g(n)$, $\beta_x=\left[\frac{A_x}{2}\right]$, the conditions of Theorem 1 hold with $x_n=n$, $d_n=q^{\beta_n}$. □

THEOREM 2. *Let K be a polynomial with integer coefficients, g an integer valued multiplicative function such that* $g(p)=K(p)$ *for each prime p. Then* $g\in\mathcal{T}$ *if and only if* $K(x)=\pm x^k$ $(k\geqq 0)$.

PROOF. If $K(x)=ax^k$, $a\neq 0$, ± 1, then our theorem follows from Corollary 1. Assume that $K(x)\neq ax^k$. Let $K(x)=K_1(x)\dots K_r(x)$ where $K_i(x)\in Z[x]$ are irreducible over $\mathbf{Q}$. We may assume that $K_1(0)\neq 0$. Then there exists a prime q and an $l\in\mathbf{Z}$, $(l,q)=1$ such that $K_1(l)\equiv 0 \pmod q$. Furthermore, $q|K_1(p)$, $K_1(p)|K(p)$ if $p\equiv l \pmod q$. Since the sum of $1/p$ for $p\equiv l \pmod q$ is divergent, the conditions of Corollary 1 hold; consequently $g\notin\mathcal{T}$.

Let now $K(x)=\varepsilon x^k$, $\varepsilon=1$ or -1. Every $n\in\mathbf{N}$ can be written as $n=Lm$, where L is the square-full part and m is the squarefree part of n, $(L,m)=1$. We have

$$g(n) = g(L)\varepsilon^{\omega(m)}m^k.$$

Let $\alpha\in\mathbf{R}\setminus\mathbf{Q}$, $F\in\mathcal{A}$, $l\in\mathbf{Z}$, $l\neq 0$. We shall prove that

$$x^{-1}\sum_{n\leqq x} e(\alpha l(F(n)+g(n))) \to 0. \tag{2.10}$$

Let $e_L(m)=1$ if $(m, L)=1$ and $=0$ if $(m, L)>0$. Then $e_L \in \mathcal{M}$. Observe that

$$e((-1)^{\omega(m)}\beta m^k) = \frac{1}{2}(1+(-1)^{\omega(m)})e(\beta m^k)+\frac{1}{2}(1-(-1)^{\omega(m)})e(-\beta m^k). \tag{2.11}$$

Let

$$s(\beta, m) = e(\beta \cdot \varepsilon^{\omega(m)} m^k). \tag{2.12}$$

Let now L run over the squarefull numbers. The left hand side of (2.10) can be written as

$$x^{-1} \sum_L e(lF(L)) \cdot \Sigma_L,$$

where

$$\Sigma_L = \sum_{m \leqq x/L} |\mu(m)| e_L(m) e(lF(m)) s(l\alpha g(L), m). \tag{2.13}$$

Let L be fixed. If $\varepsilon=1$, then we may use Lemma 1 with $t(m)=l\alpha g(L)m^k$, and get

$$\Sigma_L = o(x/L) \quad (x \to \infty). \tag{2.14}$$

If $\varepsilon=-1$, then by using (2.11), we can write (2.14) as the sum of four sums of type (1.4) with $t(m)=\pm l\alpha g(L)m^k$, and deduce (2.14). Furthermore, it is obvious that

$$|\Sigma_L| \leqq x/L, \quad \sum_L 1/L < \infty.$$

Consequently (2.10), and so the theorem is true. □

3. Now we consider functions in $\mathcal{A}_q$, $\mathcal{M}_q$. Let $f \in \mathcal{A}_q$,

$$A_N := \frac{1}{q} \sum_{j=0}^{N-1} \sum_{a \in E_q} f(aq^j), \quad B_N := \frac{1}{q} \sum_{j=0}^{N-1} \sum_{a \in E_q} f^2(aq^j).$$

Then

$$\sum_{0 \leqq n < q^N} (f(n)-A_N)^2 \leqq q^N \cdot B_N, \tag{3.1}$$

that follows immediately from the elements of probability theory.

Let $g \in \mathcal{M}_q$ with modulus $\leqq 1$, t be a function, $t: \mathbf{N}_0 \to \mathbf{R}$, $s(n):=e(t(n))$,

$$A(x, y) = \sum_{x \leqq n < x+y} g(n)s(n). \tag{3.2}$$

We shall give upper estimates for (3.2) under some conditions stated for $s(n)$.

Let $I=\{i_1<i_2<\ldots<i_r\}$, $i_\nu \in \mathbf{N}_0$, Δ be the whole set $d \in \mathbf{N}_0$ for which

$$d = \sum_{j=0}^{r-1} \delta_j q^{i_j}, \quad \delta_j \in E_q \quad (j = 0, \ldots, r)$$

holds. Let $\bar{\Delta}$ be the set of those $m \in \mathbf{N}_0$ for which $a_j(m)=0$ $\forall j \in I$. Then each $n \in \mathbf{N}_0$ can be written uniquely as $n=d+m$, $d \in \Delta$, $m \in \bar{\Delta}$, furthermore $g(n)=g(d)g(m)$.

Let now $f\in\mathcal{A}_q$ be defined by

$$f(aq^j)=\begin{cases}1 & \text{if } a=1 \text{ and } j\in I,\\ 0 & \text{otherwise,}\end{cases}$$

and consider the sum

$$E(y):=\sum_{x\le n<x+y}(f(n)-A)^2,\quad A=\frac{r}{q}. \tag{3.3}$$

Assume that $y>T=q^{i_r+1}$, since $f\in\mathcal{A}_q$, therefore

$$E(y)\le\sum_{l=0}^{T-1}(f(l)-A)^2\left(\frac{y}{T}+1\right)\le\frac{2y}{T}\sum_{l=0}^{T-1}(f(l)-A)^2,$$

and by (3.1) we get

$$E(y)\le 2Ay. \tag{3.4}$$

Let

$$B(x,y)=\sum_{x\le n<x+y}g(n)s(n)f(n). \tag{3.5}$$

From (3.4), by using Cauchy's inequality we get

$$|B(x,y)-AA(x,y)|\le\sqrt{2A}\,y,\quad A=\frac{r}{q}. \tag{3.6}$$

To estimate (3.5) we put $n=d+m$, $d\in\Delta$, $m\in\bar{\Delta}$, and we have

$$B(x,y)=\sum_{m\in\bar{\Delta}}g(m)\sum_{d\in\Delta}g(d)\,s(d+m).$$

Here $m\in[x-T,x+y]$ and in the inner sum $d\in[x-m,x+y-m)$. From the Cauchy's inequality we get

$$|B(x,y)|^2\le\Sigma_1\sum_{d_1,d_2\in\Delta}g(d_1)\bar{g}(d_2)f(d_1)f(d_2)\cdot S(d_1,d_2), \tag{3.7}$$

where

$$S(d_1,d_2)=\Sigma s(d_1+m)\bar{s}(d_2+m), \tag{3.8}$$

and the summation is extended over those $m\in\bar{\Delta}$ for which d_1+m, $d_2+m\in[x,x+y]$, furthermore Σ_1 is the number of m in the interval $[x-T,x+y]$. Consequently

$$\Sigma_1\ll\frac{y}{q^r}. \tag{3.9}$$

Similarly,

$$S(d,d)\le\Sigma_1\ll\frac{y}{q^r}.$$

Let

$$H:=\max_{\substack{d_1\ne d_2\\ d_1,d_2\in\Delta}}|S(d_1,d_2)|. \tag{3.10}$$

From (3.7) we get

$$|B(x,y)|^2 \ll \left(\frac{y}{q^r}\right)^2 \sum_{d\in\Delta} f^2(d) + \frac{y}{q^r} H \sum_{\substack{d_1,d_2\in\Delta\\ d_1\ne d_2}} f(d_1)f(d_2).$$

The number of solutions of the equation $f(d)=k$, $d\in\Delta$ is $\binom{r}{k}(q-1)^{r-k}$, therefore

$$\sum_{d\in\Delta} f^2(d) = \sum_{k=0}^{r} \binom{r}{k}(q-1)^{r-k}k^2 < r^2 q^r$$

and similarly,

$$\sum_{\substack{d_1,d_2\in\Delta\\ d_1\ne d_2}} f(d_1)f(d_2) \le \Big(\sum_{d\in\Delta} f(d)\Big)^2 < r^2 q^{2r}.$$

So we have

$$|B(x,y)|^2 \ll \frac{y^2 r^2}{q^r} + yHr^2 q^r,$$

where the constant implied by $\ll$ may depend only on q. Taking into consideration (3.6) we get

$$|A(x,y)|^2 \ll \left|A(x,y)-\frac{1}{A}B(x,y)\right|^2 + \frac{1}{A^2}|B(x,y)|^2 \ll \frac{1}{r}v^2 + \frac{y^2}{q^r} + yHq^r. \tag{3.11}$$

Hence we get immediately the following theorems.

THEOREM 3. *Let t be such a function for which*

$$\eta(n;u,v,M) := t(u+q^M n) - t(v+q^M n) \quad (n=0,1,2,\ldots)$$

is UD mod 1 *for every $M\ge 1$, and every $0\le u\ne v\le q^M-1$. Let $F\in\mathcal{A}_q$. Then the sequence $F(n)+t(n)$ $(n=0,1,2,\ldots)$ is UD* mod 1.

PROOF. Let $l\in\mathbf{Z}$, $l\ne 0$, $f(n)=e(lF(n))$, and put $lt(n)$ instead of $t(n)$. Take $I=\{1,2,\ldots r\}$, $x=1$, and consider $S(d_1,d_2)$ as the sum of some finite subsums of type

$$\sum e^{2\pi i[t(u+q^{r+1}n)-t(v+q^{r+1}n)]}.$$

From the Weyl criterion we know that each of them is $o(y)$ as $y\to\infty$. From (3.11) we get

$$\limsup \left|\frac{A(1,y)}{y}\right| < \frac{c}{\sqrt{r}}$$

with a suitable constant $c>0$. Since r is arbitrary, therefore $A(1,y)=o(y)$ $(y\to\infty)$. $\square$

THEOREM 4. *If $\eta(n;u,v,M)$ are welldistributed* mod 1 *in Theorem* 3, *then so is $F(n)+t(n)$.*

PROOF. The proof is almost the same as that of Theorem 3. We have to use the fact that $H=o(y)$ uniformly in $1\leqq x<\infty$, and use (3.11).

CROOLLARY 2. *Let* $P(x)=\alpha_r x^r+\ldots+\alpha_1 x+\alpha_0$ *be a polynomial such that at least one of* $\alpha_2, \ldots, \alpha_r$ *is an irrational number. Then for each* $F\in\mathcal{A}_q$ *the sequence* $F(n)+$ $+P(n)$ *is well distributed* mod 1.

For the proof see [7].

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INSTITUTE OF MATHEMATICS
L. EÖTVÖS UNIVERSITY
BUDAPEST 1088
HUNGARY

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GENERALIZED NUMBER SYSTEMS IN THE COMPLEX PLANE

Z. DARÓCZY (Debrecen), corresponding member of the Academy
and I. KÁTAI (Budapest), member of the Academy

1. Introduction

Let $\Lambda:=\{\lambda_1>\lambda_2>\dots\}$ be an infinite sequence of positive numbers such that $L:=\Sigma\lambda_n<\infty$. Let $H(\Lambda)$ denote the set of those x which can be written in the form

$$x=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n,\quad \varepsilon_n\in\{0,1\}.$$

It is obvious that $H(\Lambda)\subseteq[0,L]$. In our paper [1] written jointly with A. Járai we called Λ to be interval filling if $H(\Lambda)=[0,L]$. Furthermore we observed that Λ is interval filling if and only if the relation

$$\lambda_N\leqq\lambda_{N+1}+\lambda_{N+2}+\dots \tag{1.1}$$

holds for every $N\geqq 1$. Consequently, if (1.1) holds for every large N, i.e. for each $N\geqq N_0$ then $H(\Lambda)$ contains an interval of positive length.

We were unable to give necessary and sufficient conditions for $H(\Lambda)$ to be totally disconnected.

Let now $\Lambda=\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\Sigma|\lambda_n|<\infty$ and let $H(\Lambda)$ be the set of those complex numbers z which can be written in the form

$$z=\sum_{n=1}^{\infty}\varepsilon_n\lambda_n,\quad \varepsilon_n\in\{0,1\}.$$

The problem of determining those sequences Λ for which 0 is in the interior of $H(\Lambda)$, seems to be quite hopeless in this general formulation.

Let now Λ be the quite regular sequence $\lambda_n=\theta^n$, where $\theta\in\mathbf{C}$, $|\theta|<1$. We shall use the notation $|q|=1/\theta$, $q=Q$, $\arg q=-\psi$, i.e. $q=Q\cdot e^{-i\psi}$, $\theta=\dfrac{1}{Q}e^{i\psi}$.

Let

$$H_k=\Big\{z\,|\,z=\sum_{j=k}^{\infty}\varepsilon_j\theta^j;\ \varepsilon_j\in\{0,1\}\Big\}$$

for $k=0,\pm1,\pm2,\dots$ and

$$E=\bigcup_{k=-\infty}^{\infty}H_k.$$

It is obvious that E is the set of those complex numbers z which can be written in the form

$$z = \delta_M q^M + \ldots + \delta_0 + \varepsilon_1 \theta + \varepsilon_2 \theta^2 + \ldots, \quad \varepsilon_j, \delta_l \in \{0, 1\}.$$

We are interested in determining those numbers θ for which $E = E_\theta$ contains all complex numbers.

If θ is a real number then E is a subset of the real numbers.

The following assertion is almost trivial.

THEOREM 1. *If* $E_\theta = \mathbf{C}$, *then* $|\theta| \geqq 1/\sqrt{2}$.

PROOF. First we observe that

$$H_k = H_{k+1} \cup \{\theta^k + H_{k+1}\}, \quad H_k = \theta^k H_0. \tag{1.2}$$

Furthermore, H_0 and so H_k is a closed set. Let $g_n \in H_0$ $(n = 1, 2, \ldots)$, $g_n \to g$, $g \in \mathbf{C}$. Let

$$g_n = \sum_{m=k}^{\infty} \varepsilon_m^{(n)} \theta^m.$$

Let the sequence δ_m, $\delta_m \in \{0, 1\}$ be chosen as follows. Let $\delta_1 = 0$ if $\varepsilon_1^{(n)} = 0$ for an infinite sequence of integers n, and let $\delta_1 = 1$ otherwise. Let $\mathcal{M}_1 = \{n/\varepsilon_1^{(n)} = \delta_1\}$. Assume that $\delta_1, \ldots, \delta_R$, $\mathcal{M}_R \subseteqq \mathcal{M}_{R-1} \subseteqq \ldots \subseteqq \mathcal{M}_1$ are fixed, $\mathcal{M}_R$ contains the infinite set of the integers n for which

$$\varepsilon_1^{(n)} = \delta_1, \ldots, \varepsilon_R^{(n)} = \delta_R.$$

Then let $\delta_{R+1} = 0$ if $\varepsilon_{R+1}^{(n)} = 0$ holds for infinitely many $n \in \mathcal{M}_R$, and $\delta_{R+1} = 1$ otherwise. Let $\mathcal{M}_{R+1} = \{n | n \in \mathcal{M}_R, \ \varepsilon_{R+1}^{(n)} = \delta_{R+1}\}$.

Let $h_R = \delta_1 \theta + \ldots + \delta_R \theta^R$. Since for $n \in \mathcal{M}_R$

$$|g_n - h_R| \leqq |\theta|^{R+1} + |\theta|^{R+2} + \ldots \leqq \frac{|\theta|^{R+1}}{1 - |\theta|}, \quad g_n \to g,$$

therefore $|g - h_R| \to 0$ as $R \to \infty$. Consequently $g = \sum_{\nu=1}^{\infty} \delta_\nu \theta^\nu$, and so $g \in H_0$.

Let μ denote the Lebesgue measure. Since H_k is closed, therefore it is measurable. From (1.2) we get

$$\mu(H_k) \leqq 2\mu(H_{k+1}), \quad \mu(H_{k+1}) = |\theta|^2 \mu(H_k),$$

and so

$$\mu(H_k) \leqq 2|\theta|^2 \mu(H_k).$$

Then $2|\theta|^2 < 1$ implies $\mu(H_k) = 0$, and so $\mu(E) = 0$. $\square$

2. Let F denote the set of those θ for which $E_\theta = \mathbf{C}$.

THEOREM 2. *Let* $\psi \not\equiv 0, \pi \pmod{2\pi}$. *Then there exists a positive constant* $\varepsilon = \varepsilon(\psi)$ *such that all* θ *with* $1 < \frac{1}{|\theta|} < 1 + \varepsilon(\psi)$, $\arg \theta = \psi$ *belongs to* F.

PROOF. Let $\omega=e^{i\psi}$.

(1) Assume that $\frac{\psi}{2\pi}=$irrational. Then the sequence $\omega, \omega^2, \ldots$ is everywhere dense in the unit circle, consequently there is a suitable K such that

$$\max_{\varphi\in[0,2\pi]} \min_{j=1,\ldots,K} |\omega^j - e^{i\varphi}| < 0.1. \tag{2.1}$$

Let $\varepsilon=\varepsilon(\psi)$ be chosen so that

$$0.9 > (1+\varepsilon)^K - 1. \tag{2.2}$$

Let $1<Q<\varepsilon$. We shall prove that each z in the closed unit disc belongs to H_1. Let $|z|\leqq 1$. Then with a nonnegative integer a, we can write

$$z = \theta^a W, \quad \frac{1}{Q} \leqq |W| \leqq 1.$$

Let $W=re^{i\varphi}$, $\frac{1}{Q}\leqq r\leqq 1$. Let $j\in[1, K]$ be chosen so that $|\omega^j-e^{i\varphi}|<0.1$. Let

$$W= \theta^j+(z-\theta^j) = \theta^j+\theta^j W_1, \quad W_1 = q^j W-1.$$

Since

$$|W-\theta^j| = \left|re^{i\varphi}-\frac{1}{Q^j}\omega^j\right| \leqq \left(r-\frac{1}{Q^j}\right)+\frac{1}{Q^j}|e^{i\varphi}-\omega^j| \leqq r-\frac{1}{Q^j}+\frac{0.1}{Q^j} = r-\frac{0.9}{Q^j},$$

therefore, by (2.2)

$$|W_1| = Q^j r - 0.9 < r = |W|,$$

so we have proved that each z in $|z|\leqq 1$ can be written in the form

$$z = \theta^{s_1}+\theta^{s_1}z_1,$$

where $s_1>0$ is a suitable integer, $|z_1|\leqq 1$. Continuing this procedure,

$$z_j = \theta^{s_{j+1}}+\theta^{s_{j+1}}z_{j+1}, \quad |z_{j+1}| \leqq 1,$$

we get that

$$z = \theta^{s_1}+\theta^{s_1+s_2}+\ldots+\theta^{s_1+\ldots+s_n}+\ldots$$

which is a desired representation of z.

From (1.2) it is obvious that $E=\mathbf{C}$.

(2) Let now $\psi=\frac{2\pi A}{B}$, and let A, B be coprime integers, $B\neq 1, 2$. Then $\omega^B=1$, $\theta^B=\frac{1}{Q^B}$. Let ε be so small that

$$\frac{1}{(1+\varepsilon)^B} > \frac{1}{2}, \tag{2.3}$$

and let $1<Q<1+\varepsilon$. Then the sequence $\lambda_n=\theta^{Bn}$ is interval filling, consequently all nonnegative numbers in the interval $[0, S]$, $S=\frac{\theta^B}{1-\theta^B}$ can be represented in the form

$$\sum_{n=1}^{\infty} \delta_n \theta^{Bn}, \quad \delta_n \in \{0, 1\}.$$

Let now $1 \leqq t < B$ be an integer. Then all the numbers in the segment with endpoints $0, \theta^t \frac{1}{1-\theta^B}$ can be written in the form

$$\sum_{n=0}^{\infty} \delta_n \theta^{t+Bn}.$$

Let now

$$M = \left\{ z \,|\, z = x_1 \frac{\theta}{1-\theta^B} + \frac{x_2 \theta^2}{1-\theta^B} + \dots + \frac{x_B \theta^B}{1-\theta^B};\ 0 \leqq x_j \leqq 1 \right\}.$$

Then $M \subseteqq H_1$. Since $B>2$, therefore the angles between the consecutive ω^j $(j=1, \dots, B)$ are smaller than π. Consequently 0 is in the interior of M. Hence it follows that $E_\theta = \mathbf{C}$. □

3. Let now $\mathcal{A}=\{a_0=0, a_1, \dots, a_{K-1}\}$ be an arbitrary finite set of complex numbers. Let S_m be the set of those complex numbers z which can be written as

$$z = \sum_{\nu=m}^{\infty} b_\nu \theta^\nu, \quad b_\nu \in \mathcal{A} \quad (\nu = m, m+1, \dots). \tag{3.1}$$

Let

$$(F_\theta =) F = \bigcup_{m=-\infty}^{\infty} S_m. \tag{3.2}$$

Since

$$S_{m+1} = \theta S_m, \quad S_m = \bigcup_{j=0}^{K-1} \{a_j \theta^m + S_{m+1}\},$$

therefore

$$\mu(S_{m+1}) = |\theta|^2 \mu(S_m), \quad \mu(S_m) \leqq K \mu(S_{m+1}),$$

and so $F_\theta = \mathbf{C}$ implies that $|\theta|^2 K \geqq 1$.

THEOREM 3. *Let* $\mathcal{A}=\{a_0=0, a_1, \dots, a_{K-1}\}$, $\theta=\frac{1}{Q} e^{i\psi}$. *Let us assume that* $d>\frac{1}{2}$, *and that*

$$\min_{\substack{z \\ |z|=1}} \max_{j=0, \dots, K-1} \cos(\arg a_j - \arg z) \geqq d. \tag{3.3}$$

If

$$1 < Q^2 < d + \frac{1}{2} + \sqrt{d(1+d)}, \tag{3.4}$$

then $E_\theta = \mathbf{C}$.

REMARK. The condition (3.3) does not depend on $\arg \theta$.

PROOF. It is enough to prove that S_1 contains an open disc around the zero. Furthermore, if $E_\theta=\mathbf{C}$ with the coefficients $\mathcal{A}=\{a_0=0, a_1, \ldots, a_{K-1}\}$, then the same is true with $t\mathcal{A}=\{0, ta_1, \ldots, ta_{K-1}\}$ whenever $0<t<\infty$. We may assume that $a_i\neq a_j$ if $i\neq j$, and that $|a_i|\geqq 1$ $(i=1, \ldots, K-1)$.

Let $B:=\{W\,|\,|W|<Q\}$. Assume that (3.3), (3.4) hold. We shall prove that $B\subseteq S_1$. For this we need to prove only the following assertion:

If $w\in B$, then there exists an $a\in\mathcal{A}$ such that

$$W = a\theta+\theta W_1, \quad W_1\in B. \tag{3.5}$$

Iterating (3.5) we shall get the desired expansion of W.

If $|W|<1$ then we may take $a=0$. Then $|W_1|=|qW|<Q$, so $W_1\in B$.

Let now $|W|=r\geqq 1$. Let a_j $(j\in\{1, \ldots, K-1\})$ be so chosen that $\cos(\arg a_j-\arg We^{-i\psi})\geqq d$. Let $a=a_j$. Then $W_1=q(W-a\theta)$, $|W_1|^2=Q^2|W-a\theta|^2$, furthermore $|W-a\theta|^2=|W|^2+|\theta|^2-a\theta\overline{W}-\overline{a\theta}W=r^2+\frac{1}{Q^2}-2\frac{r}{Q}|a|\cos\left(\arg(a\theta)-\arg W\right)\leqq$ $\leqq r^2+\frac{1}{Q^2}-2\frac{r}{Q}d$, and so

$$|W_1|^2 \leqq Q^2r^2+1-2rQd.$$

The maximum of the right hand side in $r\in[1, Q]$ is at $r=Q$. By solving a second order inequality we get that

$$\max_{1\leqq r\leqq Q}(Q^2r^2+1-2rQd) \leqq Q^2$$

if (3.4) holds. So $W_1\in B$. We proved that $B\subseteq H_1$, which gives the theorem immediately. □

REMARK. This argumentation does not work if $d=1/2$, and so in the case $\mathcal{A}=\{0, \varrho, \varrho^2, \varrho^3=1\}$, $\varrho=e^{2\pi i/3}$.

THEOREM 4. *Let* $\mathcal{A}=\{0, \varrho, \varrho^2, \varrho^3=1\}$. *Then there exists a positive constant* c, *such that* $F_\theta=\mathbf{C}$ *for every* θ *satisfying* $\frac{1}{1+c}<|\theta|<1$.

PROOF. First we observe that $F_\theta=\mathbf{C}$ implies that $F_{\theta\rho}=F_{\theta\rho^2}=\mathbf{C}$. So we may assume that $|\arg\theta|\leqq\frac{\pi}{3}$. Observing that the set of the complex conjugates of $\mathcal{A}$ is $\mathcal{A}$, we get that $W\in F_\theta$ implies $\overline{W}\in F_{\overline{\theta}}$. So we may assume that $0\leqq\arg\theta<\frac{\pi}{3}$.

Let now $B=\{W\,\big|\,|W|<\Omega\}$, $W=re^{i\lambda}$. Let K be a positive integer and try to write W as

$$W= a\theta+\theta^{m m}W_1, \quad a\in\mathcal{A}, \quad W_1\in B, \quad m\in[1, K]. \tag{3.6}$$

If $r<1$ then we may take $a=0$, $m=1$. Let $r\geqq 1$. An elementary discussion of the cosine function leads to the following lemma, that we state without proof.

LEMMA. *Let* $0<\varepsilon<0.1$. *Let* T_ε *be the set of those* $\eta\in[0, 2\pi]$ *for which*

$$\max_{k=0,1,2}\cos\left(\eta-\frac{2\pi k}{3}\right) < \frac{1}{2}+\varepsilon. \tag{3.7}$$

Then T_ε is covered by the union of the intervals $|\eta - s| < c\varepsilon$, where $s \in \left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$ $(=: U)$, c is an absolute positive constant.

Assuming $a = e^{2\pi i k/3}$, from (3.6) we get

$$|W_1|^2 = |q^m W - a|^2 = Q^{2m} r^2 + 1 - 2\,\mathrm{Re}(q^m) W \bar{a} = Q^{2m} r^2 + 1 - 2Q^m r \cos\left(\lambda - m\psi - \frac{2\pi k}{3}\right).$$

Let ε be a small positive number. For a fixed $m \in [1, K]$ consider those $W \in B'$ $|W| \geqq 1$ which satisfy the inequality

$$\max_{k=0,1,2} \cos\left(\lambda - m\psi - \frac{2\pi k}{3}\right) \geqq \frac{1}{2} + \varepsilon. \tag{3.8}$$

Let

$$f_m(r) := Q^{2m} r^2 + 1 - 2Q^m r \left(\frac{1}{2} + \varepsilon\right) - Q^2.$$

Then $f_m(1) = Q^{2m} + 1 - (1 + 2\varepsilon) Q^m - Q^2$. Let $Q_0 = Q_0(\varepsilon) > 1$ be such that $f_m(1) < 0$ whenever $1 \leqq Q < Q_0$ for every $m \in [1, \ldots, K]$. Since $f_m(r)$ is continuous therefore there exists a $Q_m(\varepsilon) > 1$ such that

$$\max_{1 \leqq r < Q} f_m(r) < 0$$

whenever $1 < Q < Q_m(\varepsilon)$.

Let now $\tilde{Q} = \min\{Q_0, Q_1, \ldots, Q_K\} > 1$, $1 < Q < \tilde{Q}$. Then W can be written in the form (3.6) if the inequality (3.8) is satisfied for a suitable $m \in [1, K]$.

Let now assume that λ, ψ are such that (3.8) does not hold. Then, by using our Lemma with $\eta = \lambda - m\psi$, we get $\eta \in T_\varepsilon$. So we have

$$\lambda = s_m + m\psi + \xi_m \pmod{2\pi} \quad (m = 1, \ldots, K) \tag{3.9}$$

where

$$|\xi_m| < c\varepsilon, \quad s_m \in U.$$

By taking $m = 1$ and 2, we have $\psi = s_1 - s_2 + \xi_1 - \xi_2$.

Since $0 \leqq \psi < \frac{\pi}{3}$, therefore $\psi = \varkappa + \zeta$, where $|\zeta| < 2c\varepsilon$, $\varkappa = 0$ or $\frac{\pi}{3}$. Let us consider the case $\varkappa = \frac{\pi}{3}$. Then (3.9) with $m = 6$ gives that $\lambda = s_6 + 6\zeta + \xi_6$, whence $|\lambda - s_6| < 13c\varepsilon$. Then, from (3.9) with $m = 1$, we get that $\lambda = s_1 + \frac{\pi}{3} + \zeta + \xi_1$, whence $\left|s_1 + \frac{\pi}{3} - s_6\right| < 16c\varepsilon$. But this cannot occur if $16c\varepsilon < \frac{\pi}{3}$. Let us assume now that $\varepsilon < \frac{\pi}{3 \cdot 16c}$, $x = \frac{\pi}{3}$.

Let $K = 6$. Then there exists at least one representation (3.6) for all $W \in B$.

It has remained to consider the case $\varkappa = 0$, i.e. if $0 \leqq \psi < 2c\varepsilon$. From (3.9) with $m = 1$ we get

$$|\lambda - s_1| < 3c\varepsilon, \quad s_1 \in U. \tag{3.10}$$

Now we shall prove that for these values λ,

$$W = a\theta + b\theta^2 + \theta^2 W_1 \tag{3.11}$$

with suitable chosen $a, b \in \mathcal{A}$.

Let us consider the case $s_1 = \pi$. Then we take $a = \varrho$, $b = \bar{\varrho}$, $\varrho = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. From (3.11) we get $W_1 = q^2 W - \varrho q - \bar{\varrho}$. Since $|W - re^{i\pi}| \leqq 3c\varepsilon$, therefore, estimating roughly, we get

$$|W_1| \leqq 3c\varepsilon Q^2 + |q^2 re^{i\pi} - \varrho q - \bar{\varrho}| \leqq 3c\varepsilon Q^2 + |q^2 r - 1| + |q-1| + |e^{i\pi} - \varrho - \bar{\varrho}|.$$

We have $e^{i\pi} - \varrho - \bar{\varrho} = 0$,

$$|q-1| = |Qe^{i\psi} - 1| \leqq Q - 1 + \psi, \quad |q^2 r - 1| \leqq Q^2 r - 1 + 2\psi.$$

Consequently, by $0 \leqq \psi < 2c\varepsilon$,

$$|W_1| \leqq 3c\varepsilon Q^2 + Q^3 - 1 + (Q-1) + 6c\varepsilon (=: g(Q)).$$

Let us assume that $9c\varepsilon < 1$. Then there exists a constant $Q^* > 1$, such that $g(Q^*) < 1$, and so from the monotonicity of $g(Q)$ in $[1, Q^*]$ we have $g(Q) < 1$, i.e. $|W_1| < 1$.

The cases $s_1 = \frac{\pi}{3}, \frac{5\pi}{3}$ can be reduced to the earlier case by representing first ϱW or $\bar{\varrho} W$ instead of W in the form (3.11) and multiplying the equation by $\bar{\varrho}$ or ϱ, respectively.

So we have proved that $B \subset S_1$ if $1 < Q < \min(\tilde{Q}, Q^*)$, and this completes the proof of the theorem.

4. Theorem 5. *Let us assume that θ is not a real number, $0 < |\theta| < 1$. Then there exists a $k \in \mathbf{N}$ such that $F_\theta = \mathbf{C}$ if $\mathcal{A} = \{0, 1, \ldots, k\}$.*

Proof. If the assertion is true for θ^h where $h \in \mathbf{N}$, then it is true for θ as well.

Let us consider the case when there exists an $h \in \mathbf{N}$ such that $\frac{\pi}{2} < \arg \theta^h < \pi$. Let h be so chosen and let us write θ instead of θ^h. Then the angles determined by the vectorials (θ, θ^2), (θ^2, θ^3), (θ^3, θ) are less than π. Consequently every complex number W can be written uniquely as

$$W = \alpha\theta + \beta\theta^2 + \gamma\theta^3, \quad \alpha, \beta, \gamma \geqq 0, \quad \alpha\beta\gamma = 0. \tag{4.1}$$

Let S denote the disc $S = \{W \mid |W| \leqq Q^2 + Q\}$, and let $t > 0$ be such a number that every $W \in S$ can be written in the form (4.1) with $\alpha, \beta, \gamma \in [0, t]$. The existence of such a t is obvious.

Let now $k = [t] + 1$. Let $W \in S$,

$$W = \alpha\theta + \beta\theta^2 + \gamma\theta^3 = [\alpha]\theta + [\beta]\theta^2 + [\gamma]\theta^3 + \theta^3 W_1.$$

Then

$$\theta^3 W_1 = \{\alpha\}\theta + \{\beta\}\theta^2 + \{\gamma\}\theta^3, \quad W_1 = \{\alpha\} q^2 + \{\beta\} q + \{\gamma\}, \quad \alpha\beta\gamma = 0.$$

Since at least one of α, β, γ is zero, therefore $|W_1| \leqq Q^2+Q$, $W_1 \in S$. Henc we get that $F_\theta = \mathbf{C}$.

It has remained to see the case when there is no $h \in \mathbf{N}$ such that $\frac{\pi}{2} < \arg \theta^h < \pi$. This means that $h\psi \notin \left(\frac{\pi}{2}, \pi\right) \pmod{2\pi}$ $(h=1, 2, \ldots)$. So $\tau = \frac{\psi}{2\pi}$ is a rational number, $\tau = \frac{A}{B}$, $(A, B)=1$, and $\frac{hA}{B} \notin \left(\frac{1}{4}, \frac{1}{2}\right) \pmod 1$ $(h=1, 2, \ldots)$, and so $B = 1, 2, 3, 4$. The cases $B=1, 2$ lead to real θ.

If $B=4$, then $\theta = \pm i\frac{1}{Q}$. But this is a simple case, since the real and the imaginary part can be separated easily, assuming that the coefficient set is real. To prove the theorem it is enough to see that each real number x can be written in the form

$$x = \sum_{\nu=-h}^{\infty} (-1)^\nu a_{2\nu} Q^{-2\nu}, \quad a_\nu \in \mathscr{A} = \{0, 1, \ldots, k\}. \tag{4.2}$$

This is true if $k=[Q^2]$. Let us consider the set

$$S = \left\{x \mid x = \sum_{\nu=1}^{\infty} (-1)^\nu b_\nu Q^{-2\nu},\ b_\nu \in \mathscr{A}\right\}.$$

The maximal element of S is $\frac{[Q^2]}{Q^4-1} = \eta$, the minimal is $\xi = -\frac{[Q]^2 Q^2}{Q^4-1}$. It is enough to prove that $S = [\xi, \eta]$. Let $x \in [\xi, \eta]$. We define x_1 by the relation

$$x = -bQ^{-2} + (-1)Q^{-2}x_1, \quad b \in \mathscr{A}.$$

Then $x_1 = -b - xQ^2$. We have

$$-xQ^2 \in \left[\frac{-[Q^2]Q^2}{Q^4-1}, \frac{[Q^2]Q^4}{Q^4-1}\right].$$

Observing that $\eta - \xi > 1$ we can choose an integer b such that $-b - xQ_s \in S$. But then this belongs to the set $\mathscr{A}$. Since the transformation $x \to x_1$ maps $[\xi, \eta]$ into $[\xi, \eta]$, therefore $[\xi, \eta] = S$. Since $0 \in (\xi, \eta)$, each real number x can be written in the form (4.2). □

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DEPARTMENT OF MATHEMATICS
L. KOSSUTH UNIVERSITY DEBRECEN
DEBRECEN 10, HUNGARY

DEPARTMENT OF MATHEMATICS
L. EÖTVÖS UNIVERSITY
H—1088 BUDAPEST,
HUNGARY

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ON THE REPRESENTATION OF THE DUAL OF A REAL C^*-ALGEBRA

Z. MAGYAR (Budapest)

Let $\mathbf{F}$ denote either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. A is called a C^*-algebra over $\mathbf{F}$, if it is isometrically *-isomorphic to a norm-closed *-subalgebra of the algebra $B(\mathscr{H})$ of all bounded linear operators on some Hilbert space $\mathscr{H}$ over $\mathbf{F}$. If A is a C^*-algebra, we use the notations $A_h=\{a\in A; a=a^*\}$, $A_+=\{a^*a; a\in A\}$, $A'_h=\{f\in A'; f=f^*\}$ (where A' is the Banach space of bounded $\mathbf{F}$-linear functionals on A, and $f^*(a):=\overline{f(a^*)}$), $A'_j=\{f\in A'; f=-f^*\}$, $A'_+=\{f\in A'_h; f(a)\geqq 0\ \forall a\in A_+\}$ and $S=\{f\in A'_+; \|f\|=1\}$.

In case $\mathbf{F}=\mathbf{C}$ it is known that the direct sum π of the so called GNS-representations of A with respect to the elements of S gives us each $g\in A'$ in the form $g(\,.\,)=\sum_{i=1}^{n}\langle\pi(\,.\,)x_i, y_i\rangle$. The aim of this paper is to prove the corresponding statement in case $\mathbf{F}=\mathbf{R}$; more precisely, we will prove the following theorem.

THEOREM 1. *Suppose that $\pi: A\to B(\mathscr{H})$ is a *-homomorphism (where $\mathscr{H}$ is a Hilbert space over $\mathbf{F}$) satisfying the following condition: for all $f\in S$ there is an $x\in\mathscr{H}$ such that $f(a^*a)\leqq\|\pi(a)x\|^2$ $\forall a\in A$. Then for any $g\in A'_h\cup A'_j$ we find $x, y\in\mathscr{H}$ such that $g(a)=\langle\pi(a)x, y\rangle$ $\forall a\in A$.*

First we list some well-known results we need as lemma (with a sketch of their proofs).

LEMMA 1 (Jordan decomposition). *Let A be a C^*-algebra over $\mathbf{F}$ and $f\in A'_h$. Then there are $f_1, f_2\in A'_+$ satisfying $f=f_1-f_2$ and $\|f\|=\|f_1\|+\|f_2\|$; and this decomposition is unique.*

PROOF. Observe that the mapping $g\to g|_{A_h}$ is norm-preserving on A'_h (if $a\in A$ is such that $\|a\|\leqq 1$ and $|g(a)|$ is near $\|g\|$, then $g(b)$ is near $\|g\|$ if $b=\lambda a$ with suitable $\lambda\in\mathbf{F}$, $|\lambda|=1$; hence $g((b+b^*)/2)$ is near $\|g\|$ for $g\in A'_h$). It is known that for any $g\in A_h$ there is a $g\in S$ such that $|g(h)|=\|h\|$ (this can be proved by using the Hahn—Banach theorem and the facts that A_+ is a cone, $A_+=\{a\in A_h; \operatorname{Sp}(a)\subset\mathbf{R}_+\}$ and the norm equals the spectral radius on A_h). In other words, $\|h\|=\max\{g(h); g\in S\cup -S\}$ for all $h\in A_h$.

Therefore if $\varphi(h)>\max\{g(h); g\in S\cup -S\}$ for an $h\in A_h$ then $\|\varphi\|>1$. Let $L=\{sg; s\in[0,1],\ g\in S\}=\{\varphi\in A'_+; \|\varphi\|\leqq 1\}$ and let $K=\operatorname{co}(L\cup -L)$. Then L is compact in the weak*-topology of the dual space X of the real Banach space A_h, and L is convex. Hence K is a compact convex set, and therefore $\varphi\notin K$ implies the existence of an $h\in A_h$ such that $\varphi(h)>\sup\{g(h); g\in K\}$. Thus $\|\varphi\|>1$, that is K

is the unit ball of X. Since $K=\{g_1-g_2; g_1, g_2\in A'_+, \|g_1\|+\|g_2\|\leqq 1\}$ and by the isometry of X and A'_h we get the existence part of the lemma.

Now we prove the unicity. Let $\varepsilon>0$ and $a\in A_h$ be such that $\|a\|\leqq 1$ and $f(a)>\|f\|-\varepsilon$. Let p and n be the positive and negative parts of a (by the commutative Gelfand—Naimark theorem) and let $f=f_1-f_2$ be a decomposition satisfying $f_1, f_2\in A'_+$, $\|f_1\|+\|f_2\|=\|f\|$. Then

$$\|f_1\|-\varepsilon+\|f_2\| = \|f\|-\varepsilon < f_1(p-n)-f_2(p-n) = f_1(p)+f_2(n)-\big(f_1(n)+f_2(p)\big) \leqq$$

$$\leqq \begin{cases} f_1(p)+\|f_2\| \\ \|f_1\|-f_2(p)+\|f_2\|, \end{cases} \quad f_1(p)\leqq\|f_1\|, \quad f_2(n)\leqq\|f_2\|,$$

Hence $f_1(p)>\|f_1\|-\varepsilon$, $f_2(p)<\varepsilon$. Let $q=\sqrt{p}$. We assert that

$$(1) \qquad |f_1(c)-f(qc)| \leqq \sqrt{\varepsilon}\cdot\|c\|\sqrt{2}\sqrt{\|f\|} \quad \forall c\in A.$$

This follows from the Schwarz inequality $|g(b^*c)|^2\leqq g(b^*b)\cdot g(c^*c)$ $\forall b, c\in A$ for positive g, and the fact that $(1-q)^2\leqq 1-q\leqq 1-p$ because

$$|f_1(c)-f(qc)| \leqq \big|f_1((1-q)c)\big|+|f_2(qc)| \leqq f_1^{1/2}\big((1-q)^2\big)f_1^{1/2}(c^*c)+$$

$$+f_2^{1/2}(q^2)f_2^{1/2}(c^*c) \leqq f_1^{1/2}(1-p)\|f_1\|^{1/2}\|c\|+f_2^{1/2}(p)\|f_2\|^{1/2}\|c\| \leqq$$

$$\leqq \|c\|\big\{(\|f_1\|-f_1(p))^{1/2}\|f_1\|^{1/2}+\varepsilon^{1/2}\|f_2\|^{1/2}\big\} \leqq$$

$$\leqq \varepsilon^{1/2}\|c\|(\|f_1\|^{1/2}+\|f_2\|^{1/2}) \leqq \varepsilon^{1/2}\|c\|\,2\left(\frac{\|f_1\|+\|f_2\|}{2}\right)^{1/2}$$

(we can extend f_1 to the unitization of A, in case $1\notin A$, by setting $\tilde{f}_1(1)=\|f_1\|$, and f_1 remains positive). Since (1) holds for any decomposition satisfying $\|f_1\|+\|f_2\|=\|f\|$ and for any ε, we get the unicity: assuming $f=f_1'-f_2'$ and $\|f_1'\|+\|f_2'\|=\|f\|$ we arrive at (1) for f_1' too so that $|f_1(c)-f_1'(c)|\leqq|f_1(c)-f(qc)|+|f_1'(c)-f(qc)|\leqq 2\sqrt{\varepsilon}\,\|c\|\sqrt{2}\cdot\sqrt{\|f\|}$ and thus $f_1=f_1'$, $f_2=f_2'$.

DEFINITION 1. Let A be a C^*-algebra over $\mathbf{F}$ and let $f\in A'_+$. We call a *-homomorphism $p: A\mapsto B(H)$ f-representation (of A over H) if there is an $x\in H$ satisfying:

(r1) $f(a)=\langle p(a)x, x\rangle$ $\forall a\in A$,

(r2) $\{p(a)x; a\in A\}$ is dense in H.

We shall say that p is an f-representation *with* x, if x satisfies (r1) and (r2).

LEMMA 2. *Let p be an f-representation with x of the C^*-algebra A over the Hilbert space H. Writing $p(A)^c:=\{L\in B(H): Lp(a)=p(a)L\ \forall a\in A\}$ we define a function $T\to g_T$ from $p(A)^c_+$ into A'_+ by setting $g_T(\,.\,)=\langle p(\,.\,)x, Tx\rangle$. Then this mapping is an additive, positive homogeneous bijection from $p(A)^c_+$ onto $\{g\in A'_+; \exists c\geqq 0: g\leqq cf\}$.*

PROOF. If $g\in A'_+$ and $g\leqq c\cdot f$ with $c\in\mathbf{R}_+$, then $g(a^*a)\leqq c\cdot f(a^*a)=c\cdot\|p(a)x\|^2$ for all $a\in A$, and therefore we can define a bounded sesquilinear and positive form

$(.,.)$ on H by setting $(p(a)x, p(b)x) := g(b^*a)$. Hence there is a unique $T \in B(H)_+$ satisfying

$$\langle Tp(a)x,\ p(b)x\rangle = g(b^*a) \text{ for all } a, b, \in A.$$

Then

$$\langle Tp(c)p(a)x, p(b)x\rangle = g(b^*ca) = \langle Tp(a)x, p(c^*b)x\rangle =$$
$$= \langle p(c)Tp(a)x,\ p(b)x\rangle \quad \text{for all} \quad a, b, c \in A.$$

Hence we get $T \in p(A)^c$. Thus we have $g(b^*a) = \langle p(b^*a)x, Tx\rangle$, and since A_h is in the linear hull of elements of the form a^*a in the C^*-algebra A, thus $g(\,.\,) = = \langle p(\,.\,)x, Tx\rangle$, for both sides are from A_h'.

Since for any $T \in p(A)^c_+$ clearly $g_T \in A'_+$, $g_T(a^*a) = \|\sqrt{T}\, p(a)x\|^2 \leq \|T\| \cdot f(a^*a)$ and $g_T(b^*a) = \langle p(b^*a)x, Tx\rangle = \langle Tp(a)x, p(b)x\rangle$, thus the proof is complete.

NOTATIONS. If A is a C^*-algebra over $\mathbf{R}$, then we denote by $A_{\mathbf{C}}$ the complexification of A (see e.g. [2], §13) which is known to be a C^*-algebra over $\mathbf{C}$ (consider the complexification of the corresponding Hilbert space). We identify A with its canonical range in $A_{\mathbf{C}}$. We denote the conjugation of $A_{\mathbf{C}}$ by $c \to \bar{c}$ ($\bar{c} = a - ib$ if $c = a + ib$, $a, b \in A$). We write $\bar{f}(c) := \overline{f(\bar{c})}$ for $f \in A_{\mathbf{C}}$, and $f_{\mathbf{C}}(a+ib) := f(a) + i \cdot f(b)$ for $f \in A'$. Of course, $\{f_{\mathbf{C}}; f \in A'\} = \{f \in A'_{\mathbf{C}}; f = \bar{f}\}$.

LEMMA 3. *Let A be a C^*-algebra over $\mathbf{R}$, $\varphi \in A'_+$ and let p be a φ-representation of A over H with suitable $x \in H$. Let q be the corresponding representation of $A_{\mathbf{C}}$ over $H_{\mathbf{C}}$, where $H_{\mathbf{C}}$ is the complexification of H. Denote by J the conjugation of $H_{\mathbf{C}}$ and let $f = \varphi_{\mathbf{C}}$. Then q is an f-representation of $A_{\mathbf{C}}$ over $H_{\mathbf{C}}$ with x, and $g_{JTJ} = \bar{g}_T$ for all $T \in q(A_{\mathbf{C}})^c_+$ (we used the notation of Lemma 2).*

PROOF. Straightforward.

PROOF OF THEOREM 1. First we show that our condition implies that for any $f \in A'_+$ we can find an $x \in \mathscr{K}$ so that $a \to \pi(a)|_H$ is an f-representation with x, where H is the closure of $\pi(A)x$. We write $f \sim x$ in this situation. Consider first the case $f(\,.\,) = \langle \pi(\,.\,)y, y\rangle$ for some $y \in \mathscr{K}$. Let P be the orthogonal projection of $\mathscr{K}$ onto the closure H of $\pi(A)y$. Then H is an invariant subspace for $\pi(A)$ and so is $H^\perp$, for $\pi(A)$ is a *-algebra. Thus $P \in \pi(A)^c$, and $a \to \pi(a)|_H$ is an f-representation with $x = Py$.

Now let $f \in A'_+$ be arbitrary, and let $z \in \mathscr{K}$ be such that $f(a^*a) \leq \|\pi(a)z\|^2\ \forall a \in A$. Then let y be such that $\langle \pi(\,.\,)z, z\rangle \sim y$, and let T be the corresponding operator on $\overline{\pi(A)y}$ satisfying $f(\,.\,) = \langle \pi(\,.\,)y, Ty\rangle$ by Lemma 2. Then $f \sim x = \sqrt{T}y$ $\big(x \in \overline{\pi(A)x}$ follows from $y \in \overline{\pi(A)y}\big)$.

If $f \in A_h'$ then let f_1, f_2 be its Jordan decomposition and let $f_1 + f_2 \sim x$. We see from Lemma 2 that there are operators T_1, T_2 on $H = \pi(A)x$ satisfying $f_j(\,.\,) = = \langle \pi(\,.\,)x, T_jx\rangle$ $(j = 1, 2)$; hence $f(\,.\,) = \langle \pi(\,.\,)x, (T_1 - T_2)x\rangle$.

If $f \in A_j'$ and $\mathbf{F} = \mathbf{C}$ then $i \cdot f \in A_h'$ and hence $f(\,.\,) = \langle \pi(\,.\,)x, y\rangle$ with suitable $x, y \in \mathscr{K}$.

If $\varphi \in A_j'$ and $\mathbf{F} = \mathbf{R}$ then let $f = i \cdot \varphi_{\mathbf{C}}$. It is easy to see that $f \in A'_{\mathbf{C}_h}$, and clearly $\bar{f} = -f$. Let $f = f_1 - f_2$ be the Jordan decomposition of f. Then $\bar{f}_1 - \bar{f}_2 = \bar{f} = = -f = f_2 - f_1$, and $\|\bar{f}_1\| + \|\bar{f}_2\| = \|\bar{f}_1\| + \|f_2\| = \|f\|$ (for the $^-$ is norm-preserving) and hence $f_2=\bar{f}_1$ by the unicity of the Jordan decomposition (we have used that the $^-$ preserves positivity). Thus $f_1+f_2=h_{\mathbf{C}}$ for some $h\in A'_+$. Let $h\sim x$, $p(\,.\,)=\pi(\,.\,)|_{\overline{\pi(A)x}}$ and let q be the corresponding f_1+f_2 representation by Lemma 3. Also let T_j be the operator to f_j $(j=1,2)$ by Lemma 2. Now we see from Lemmas 2 and 3 that $T_2= =JT_1J$ (for $f_2=\bar{f}_1$). Thus $T_2x=JT_1Jx=JT_1x$, and therefore $y:=i(T_1x-T_2x)\in\mathcal{K}$. Then we have for all $a\in A$ $\langle\pi(a)x,y\rangle=-i(\langle q(a)x,T_1x\rangle-\langle q(a)x,T_2x\rangle)=-i\cdot f(a)= =\varphi(a)$. Thus our theorem is proved.

REMARK. The above proof with small supplements allows us to state that the conclusion of Theorem 1 is valid for any $g\in A'$.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF APPLIED ANALYSIS
1088 BUDAPEST VIII., MÚZEUM KRT. 6—8
HUNGARY

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MARKOV TYPE ESTIMATES FOR DERIVATIVES OF POLYNOMIALS OF SPECIAL TYPE

T. ERDÉLYI (Budapest)

In this paper $c_1, c_2, \dots$ and $c_m^{(1)}, c_m^{(2)}, \dots$ will always denote absolute positive constants and positive constants depending only on m, respectively. Let $K := \mathbf{R}$ (mod 2π) and denote by T_n the set of all real trigonometric polynomials of order at most n, by $T_n^k(\omega)$ $(0 \leqq k \leqq 2n, 0 < \omega \leqq \pi)$ the set of those trigonometric polynomials from T_n which have all but at most k roots in $K \setminus (-\omega, \omega)$, by $S_n^k(\omega, \varepsilon)$ $(0 \leqq k \leqq 2n, 0 < \omega \leqq \pi, 0 < \varepsilon \leqq 1)$ the set of those trigonometric polynomials from T_n which have all but at most k roots outside the ellipse $\left\{ z := x + iy \in \mathbf{C} \,|\, x^2 + \frac{y^2}{\varepsilon^2} < \omega^2 \right\}$. In the definition of $T_n^k(\omega)$ and $S_n^k(\omega, \varepsilon)$ the remaining (at most k) roots can be arbitrary complex numbers. Obviously $T_n^k(\omega) \subset S_n^k(\omega, \varepsilon)$.

Bernstein's inequality asserts that

$$\|t'\|_K \leqq n \|t\|_K$$

for any $t \in T_n$, where and in what follows $\|\cdot\|_A$ denotes the supremum norm in A. V. S. Videnskiĭ [1] proved the following Markov type estimate on an interval shorter than the period:

$$\|t'\|_{[-\omega, \omega]} \leqq 2n^2 \cot \frac{\omega}{2} \|t\|_{[-\omega, \omega]} \tag{1}$$

if $t \in T_n$, $0 < \omega < \pi$ and $n \geqq \frac{1}{2} \sqrt{3 \tan^2 \frac{\omega}{2} + 1}$. The trigonometric polynomial

$$\cos \left(2n \arccos \frac{\sin \frac{x}{2}}{\sin \frac{\omega}{2}} \right)$$

shows that this inequality is sharp.

For trigonometric polynomials from $T_n^0(\omega)$ J. Szabados [2] gave a better Markov type estimate than (1), namely he proved

$$\|t'\|_{[-\omega, \omega]} \leqq \frac{c_1 n}{\omega} \|t\|_{[-\omega, \omega]} \tag{2}$$

if $t \in T_n^0(\omega)$. When $0 < \omega \leqq c_2 \frac{\pi}{2}$ with $c_2 < 1$, the trigonometric polynomial $\sin^{2n} \frac{x + \omega}{2}$ shows that up to the constant c_1 this inequality cannot be improved.

Our main purpose is to give a common generalization of Videnskiĭ's and Szabados' result for trigonometric polynomials from $S_n^k(\omega, \varepsilon)$. (Of course $k=2n$ means there is no restriction for the roots.) A very essential tool to prove Markov type estimates for derivatives of trigonometric polynomials from $S_n^k(\omega, \varepsilon)$ is the idea of the so-called trigonometric Lorentz polynomials which was introduced and examined thoroughly in [3], and which can be defined as follows. Denote by $\mathcal{T}_n(\omega)$ the set of trigonometric polynomials of the form

$$t(x) = \sum_{i=1}^{2n} d_i \sin^i \frac{\omega-x}{2} \sin^{2n-i} \frac{x+\omega}{2}$$

with all $d_i \geqq 0$ or all $d_i \leqq 0$ $(i=0, 1, \ldots, 2n)$. Now we shall prove the main result of this paper from which Markov type inequality for derivatives of polynomials from $S_n^k(\omega, \varepsilon)$ will be straightforward by using a theorem from [3].

Theorem 1. *Let* $0<\omega\leqq\pi$, $t_n\in\mathcal{T}_n(\omega)$, $q_k\in T_k$, $r=t_nq_k$, $n\geqq 0$, $k\geqq 0$, $m\geqq 1$. *Then*

$$\|r^{(m)}\|_{[-\omega,\omega]} \leqq c_m^{(1)} \left(\frac{(n+k)(k+1)}{\omega}\right)^m \|r\|_{[-\omega,\omega]}.$$

For the proof we need some lemmas.

Lemma 1. *Let* $0<\varphi\leqq\frac{\pi}{16}$, $q_l\in T_l$, $p(x)=\sin^{2n}\frac{x+\varphi}{2} q_l(x)$, $n\geqq 0$, $l\geqq 0$. *Then*

$$|p'(\varphi)| \leqq c_2 \frac{(n+l)(l+1)}{\varphi} \|p\|_{[-\varphi,\varphi]}.$$

Proof of Lemma 1. It may be supposed that $n, l\geqq 1$, otherwise by (1) the statement is trivial. Denote by $R_n^l(\varphi)$ the set of all p defined in Lemma 1 when n, l and φ are fixed. It is obvious that there exists a $p_n^*\in R_n^l(\varphi)$ for which

$$\frac{|p_n^{*\prime}(\varphi)|}{\|p_n^*\|_{[-\varphi,\varphi]}} = \sup_{p\in R_n^l(\varphi)} \frac{|p'(\varphi)|}{\|p\|_{[-\varphi,\varphi]}}.$$

First we shall verify that there exist $2l$ points $-\varphi<a_1<a_2<\ldots<a_{2l}<\varphi$ for which

$$|p_n^*(a_i)| = \|p_n^*\|_{[-\varphi,\varphi]} \quad (i=1,2,\ldots,2l). \tag{3}$$

Suppose indirectly that there are only at most $2l-1$ points $-\varphi<\gamma_i<\varphi$ $(i=1, 2, \ldots, h\leqq 2l-1)$ for which

$$|p_n^*(\gamma_i)| = \|p_n^*\|_{[-\varphi,\varphi]}.$$

Then there exists a $v\in T_{n+l}$ for which $v(\gamma_i)=\operatorname{sgn} p_n^*(\gamma_i)$ $(i=1, 2, \ldots, h)$, $v(\varphi)=\operatorname{sgn} p_n^*(\varphi)$, $v'(\varphi)=0$ and v has $2n$ repeated roots at $-\varphi$. But then for a sufficiently small $\varepsilon>0$ $p_n^*-\varepsilon v\in R_n^l(\varphi)$ contradicts the maximality of p_*^n. So there exist $-\varphi<a_1<a_2<\ldots<a_{2l}<\varphi$ for which (3) holds, indeed.

Now we distinguish two cases.

Case I: $p_n^{*\prime}(t)\neq 0$ for $t\in K\setminus[-\varphi,\varphi]$. Then $\|p_n^*\|_{[-\varphi,\varphi]}=\|p_n^*\|_K$, so according to Bernstein's inequality

$$|p_n^{*\prime}(\varphi)| \leqq \|p_n^{*\prime}\|_K \leqq n\|p_n^*\|_K = n\|p_n^*\|_{[-\varphi,\varphi]} \leqq \frac{(n+l)l}{\varphi}\|p_n^*\|_{[-\varphi,\varphi]},$$

therefore recalling the maximality of p_n^* we get the statement of Lemma 1 in this case. The other case is much more difficult.

Case II: there exists a $\delta\in K\setminus[-\varphi,\varphi]$ for which $p_n^{*\prime}(\delta)=0$. It may be supposed that $\varphi<\delta<2\pi-\varphi$. As $p_n^{*\prime}\in T_{n+l}$, $p_n^{*\prime}$ has exactly $2n-1$ repeated roots at $-\varphi$, and simple root at each a_i $(i=1,2,\dots,2l)$ and δ. From this it is clear that there exists no $-\varphi<x<\varphi$ differring from each a_i $(i=1,2,\dots,2l)$ for which $|p_n^*(x)|=\|p_n^*\|_{[-\varphi,\varphi]}$. Furthermore from Rolle's Theorem we can deduce that

$$p_n^*(a_i) = -p_n^*(a_{i+1}) \quad (1 \leqq i \leqq 2l-1).$$

It may be supposed that $p_n^*(a_{2l})<0$. We distinguish two cases.

Case II/1: there exists no $a\in K\setminus(-\varphi,\varphi)$ for which $p_n^*(a)=\|p_n^*\|_{[-\varphi,\varphi]}$. Then $\|p_n^*\|_{[-\varphi,\varphi]}=\|p_n^*\|_K$ and we get the desired result similarly to Case I.

Case II/2: there exists a smallest a such that

$$p_n^*(a) = \|p_n^*\|_{[-\varphi,\varphi]}, \quad \varphi \leqq a < 2\pi-\varphi. \tag{4}$$

(There exist at most two values of a for which (4) holds true.) Now p_n^* has $2n$ repeated roots at $-\varphi$ and by using the notation $a_{2l+1}:=a$, p_n^* has a root on each of the intervals (a_i, a_{i+1}) $(i=1,2,\dots,2l)$. Let $b_i\in(a_i,a_{i+1})$ $(i=1,2,\dots,2l)$ be that point for which $p_n^*(b_i)=0$. So we have found each of the $2(n+l)$ zeros of p_n^*. Now we distinguish two cases again.

Case II/2a: $a>2\varphi$. Observe that $p_n^*\in T_{n+l}$ has all but at most one root in $K\setminus(\varphi,2\varphi)$. We need the following

LEMMA 1.1. *Let $p\in T_{N+1}\setminus T_N$ $(N\geqq 1)$ be such that p has all but at most one root in $K\setminus(\varphi,2\varphi)$, $\varphi\leqq\frac{\pi}{2}$. Then*

$$|p'(\varphi)| \leqq \frac{c_3N}{\varphi}\|p\|_{[\varphi,2\varphi]}.$$

PROOF. From the conditions prescribed for p we can deduce (see [3]) that p is of the form

$$p(x) = \left(\sum_{i=0}^{2N} A_i \sin^i\frac{x-\varphi}{2}\sin^{2N-i}\frac{2\varphi-x}{2}\right)t(x)$$

where each $A_i\geqq 0$ or each $A_i\leqq 0$ $(i=0,1,\dots,2n)$ and $t(x)\in T_1$. It may be supposed that each $A_i \geqq 0$ $(i=0, 1, \ldots, 2N)$. Then

$$|p'(\varphi)| \leqq \left|A_0 \sin^{2N}\frac{\varphi}{2}\, t'(\varphi)\right| + \left|NA_0 \sin^{2N-1}\frac{\varphi}{2}\cos\frac{\varphi}{2}\, t(\varphi)\right| + \tag{5}$$

$$+\left|\frac{1}{2} A_1 \sin^{2N-1}\frac{\varphi}{2}\, t(\varphi)\right|.$$

Let $I:=\left[\left(1+\frac{1}{4N}\right)\varphi,\ \left(1+\frac{1}{2N}\right)\varphi\right]$. Choose a $\xi\in I$ for which $|t(\xi)|=\|t\|_I$. Then by a simple calculation we have

$$|t'(\varphi)| \leqq \frac{c_4 N}{\varphi}|t(\xi)| \tag{6}$$

and

$$|t(\varphi)| \leqq c_5 |t(\xi)|. \tag{7}$$

Further fi om $\xi\in I$ it follows that

$$\sin^{2N}\frac{\varphi}{2} \leqq c_6 \sin^{2N}\frac{2\varphi-\xi}{2}. \tag{8}$$

From (6) nd (8) we get

$$A_0 \sin^{2N}\frac{\varphi}{2}\, t'(\varphi)\Big| \leqq A_0 c_6 \sin^{2N}\frac{2\varphi-\xi}{2}\,\frac{c_4 N}{\varphi}|t(\xi)| \leqq c_6 c_4 \frac{N}{\varphi}|p(\xi)|. \tag{9}$$

From (7) and (8) it is clear that

(10)

$$\left|NA_0 \sin^{2N-1}\frac{\varphi}{2}\cos\frac{\varphi}{2}\, t(\varphi)\right| \leqq NA_0 \frac{\pi c_6}{\varphi} \sin^{2N}\frac{2\varphi-\xi}{2}\, c_5|t(\xi)| \leqq \pi c_6 c_5 \frac{N}{\varphi}|p(\xi)|.$$

Finally by (7), (8) and $\xi\in I$ we have

$$\left|\frac{1}{2} A_1 \sin^{2N-1}\frac{\varphi}{2}\, t(\varphi)\right| \leqq \frac{1}{2} A_1 \sin^{2N-1}\frac{\varphi}{2}\, c_5|t(\xi)| \leqq \tag{11}$$

$$\leqq \frac{c_5}{2} A_1 c_6 \sin^{2N-1}\frac{2\varphi-\xi}{2}\sin\frac{\xi-\varphi}{2}\,\frac{1}{\sin\frac{\xi-\varphi}{2}}|t(\xi)| \leqq$$

$$\leqq \frac{c_5 c_6}{2}\,\frac{\pi}{2}\,\frac{2}{\xi-\varphi}|p(\xi)| \leqq 2c_5 c_6 \pi \frac{N}{\varphi}|p(\xi)|.$$

Now by (5), (9), (10) and (11) we get the desired result. By this the proof of Lemma 1.1 is complete.

Using the just proved Lemma 1.1 we can easily finish the proof of Lemma 1 in Case II/2a. Indeed, let $N:=n+l-1$, then Lemma 1.1 can be applied to p_n^*, so using the fact that $a>2\varphi$ implies $\|p_n^*\|_{[\varphi,2\varphi]}\leqq\|p_n^*\|_{[-\varphi,\varphi]}$, we get

$$|p_n^*(\varphi)|\leqq\frac{c_3(n+l)}{\varphi}\|p_n^*\|_{[\varphi,2\varphi]}\leqq\frac{c_3(n+l)l}{\varphi}\|p_n^*\|_{[\varphi,2\varphi]}\leqq\frac{c_3(n+l)l}{\varphi}\|p_n^*\|_{[-\varphi,\varphi]}.$$

By this, because of the maximality of p_n^*, the proof of Lemma 1 is complete in Case II/2a.

Case II/2b: $\varphi\leqq a\leqq 2\varphi$. Observe that

$$0=\frac{p_n^{*\prime}(\delta)}{p_n^*(\delta)}=n\cot\frac{\delta+\varphi}{2}+\frac{1}{2}\sum_{i=1}^{2l}\cot\frac{\delta-b_i}{2}. \tag{12}$$

Using this, we shall verify that $\delta>\frac{\pi}{2}$. Suppose indirectly that $\delta\leqq\frac{\pi}{2}$. As $-\varphi<$ $<b_i<a_{2l+1}=a\leqq 2\varphi\leqq\frac{\pi}{8}$ for each $i=1,2,\dots,2l$, further $\varphi<\delta<2\pi-\varphi$ and $\delta\leqq$ $\leqq\frac{\pi}{2}$ imply that $0<\frac{\delta-b_i}{2}$, $\frac{\delta+\varphi}{2}<\frac{\pi}{2}$ and this contradicts (12). Thus $\frac{\pi}{2}<\delta<$ $<2\pi-\varphi$, indeed. As

$$p_n^{*\prime}(x)=C\sin^{2n-1}\frac{x+\varphi}{2}\prod_{i=1}^{2l}\sin\frac{x-a_i}{2}\sin\frac{x-\delta}{2}$$

where C is independent of x, and $\frac{\pi}{2}<\delta<2\pi-\varphi$, a simple calculation shows that $p_n^{*\prime}$ is monotonically increasing on the interval $\left[a_{2l},\frac{\delta-\varphi}{2}\right]\supset\left[\varphi,\frac{7\pi}{32}\right]$. Since $a_{2l}<\varphi\leqq a\leqq 2\varphi\leqq\frac{\pi}{8}<\frac{7\pi}{32}$, this means $|p_n^{*\prime}(\varphi)|=p_n^{*\prime}(\varphi)\leqq p_n^{*\prime}(a)=|p_n^{*\prime}(a)|$. Therefore it is sufficient to prove that

$$p_n^{*\prime}(a)\leqq\frac{c_2(n+l)l}{\varphi}\|p_n^*\|_{[-\varphi,a]}\left(=\frac{c_2(n+l)l}{\varphi}\|p_n^*\|_{[-\varphi,\varphi]}\right). \tag{13}$$

In order to prove (13) we need the following lemma.

LEMMA 1.2. *Let* $0<\alpha\leqq\frac{\pi}{16}$ *and* $t_0\in K$ *be fixed. Suppose that* $p_n^*\in T_{n+l}$ *has exactly* $2n$ *repeated roots at* $t_0-\alpha$, *simple roots at* $(t_0-\alpha<)b_1<b_2<\dots<b_{2l}(<t_0+\alpha)$; p_n^* *achieves (with alternating signs) the values* $\pm\|p_n^*\|_{[t_0-\alpha,t_0+\alpha]}$ *on each of the intervals* $(t_0-\alpha,b_1),(b_1,b_2),\dots,(b_{2l-1},b_{2l}),(b_{2l},t_0+\alpha)$ *and* $p_n^*(t_0+\alpha)=\|p_n^*\|_{[t_0-\alpha,t_0+\alpha]}$. *Then*

$$|p_n^{*\prime}(t_0+\alpha)|\leqq c_7\frac{(n+l)l}{\alpha}\|p_n^*\|_{[t_0-\alpha,t_0+\alpha]}.$$

If $\alpha=\frac{a+\varphi}{2}\left(\leqq\frac{3\pi}{2}<\frac{\pi}{8}\right)$ and $t_0=\frac{a-\varphi}{2}$ this lemma gives (13).

PROOF OF LEMMA 1.2. Because of the periodicity it may be supposed that $t_0=0$. Then

$$\frac{|p_n^{*\prime}(\alpha)|}{\|p_n^*\|_{[-\alpha,\alpha]}}=\frac{p_n^{*\prime}(\alpha)}{p_n^*(\alpha)}=n\cot\alpha+\frac{1}{2}\sum_{i=1}^{2l}\cot\frac{\alpha-b_i}{2}. \tag{14}$$

Let $\beta:=\frac{5n}{5n+l}\alpha$. An easy calculation shows that the function

$$f(x)=\frac{\sin^{2n}\frac{x+\alpha}{2}}{\sin^{2l}\frac{x-\beta}{2}}\quad (n,l\geqq 1)$$

is monotonically decreasing in $(\beta,\alpha]$. Now let $\gamma:=\frac{\beta+\alpha}{2}$ and

$$s_{2l}(x):=\cos\left(4l\arccos\frac{\sin\frac{x-\gamma}{2}}{\sin\frac{\alpha-\beta}{4}}\right).$$

It is easy to see that s_{2l} is a trigonometric polynomial of order exactly $2l$ which equioscillates $4l+1$ times in $[\beta,\alpha]$ and has $2l$ simple roots in (β,γ) and $2l$ simple roots in (γ,α). Let $\nu_1>\nu_2>\ldots>\nu_{2l}$ and $\eta_1<\eta_2<\ldots<\eta_{2l}$ be the roots of s_{2l} lying in (β,γ) and (γ,α), respectively. Consider the trigonometric polynomial $r_l(x):=\prod_{i=1}^{2l}\sin\frac{x-\eta_i}{2}$, then

$$r_l(x)\sin^{2l}\frac{x-\beta}{2}=Cs_{2l}(x)\prod_{i=1}^{2l}\frac{\sin\frac{x-\beta}{2}}{\sin\frac{x-\nu_i}{2}}$$

where C is independent of x. As $0<\alpha\leqq\frac{\pi}{16}$ and $0<\beta<\nu_i<\gamma$, the function

$$\frac{\sin\frac{x-\beta}{2}}{\sin\frac{x-\nu_i}{2}}$$

is monotonically decreasing in $[\gamma, \alpha]$ for each $i=1, 2, \ldots, 2l$, so their product is monotonically decreasing in $[\gamma, \alpha]$, too. But then so is the function

$$\frac{\sin^{2n}\frac{x+\alpha}{2}}{\sin^{2l}\frac{x-\beta}{2}} \prod_{i=1}^{2l} \frac{\sin\frac{x-\beta}{2}}{\sin\frac{x-v_i}{2}}.$$

Now let

$$g(x) := r_l(x)\sin^{2n}\frac{x+\alpha}{2} = s_{2l}(x)\frac{\sin^{2n}\frac{x+\alpha}{2}}{\sin^{2l}\frac{x-\beta}{2}} \prod_{i=1}^{2l} \frac{\sin\frac{x-\beta}{2}}{\sin\frac{x-v_i}{2}}.$$

Then $g\in T_{n+l}$ and using the notations $\eta_0:=-\alpha$, $\eta_{2l+1}:=\alpha$, from the equioscillation of s_{2l} we can easily deduce that

$$\|g\|_{[\eta_{i-1}, \eta_i]} \geqq \|g\|_{[\eta_i, \eta_{i+1}]} \quad (i = 1, 2, \ldots, 2l).$$

From this property, using the notations of Lemma 1.2 by the method of P. Borwein [4], who examined the algebraic case, we get

$$(15) \qquad b_i \leqq \eta_i \quad (i = 1, 2, \ldots, 2l).$$

From (14), (15), $0<\alpha\leqq\frac{\pi}{16}$ and $n, l\geqq 1$, by a simple calculation we have

$$\frac{|p_n^{*\prime}(\alpha)|}{\|p_n^*\|_{[-\alpha,\alpha]}} = n\cot\alpha+\frac{1}{2}\sum_{i=1}^{2l}\cot\frac{\alpha-b_i}{2} \leqq n\cot\alpha+\frac{1}{2}\sum_{i=1}^{2l}\cot\frac{\alpha-\eta_i}{2} \leqq$$

$$\leqq n\cot\alpha+\frac{1}{2}\sum_{i=1}^{2l}\left(\cot\frac{\alpha-n_i}{2}+\cot\frac{\alpha-v_i}{2}\right) = n\cot\alpha+\frac{s_{2l}'(\alpha)}{s_{2l}(\alpha)} =$$

$$= n\cot\alpha+2\cdot(2l)^2\cot\frac{\alpha-\beta}{4} \leqq \frac{n}{\alpha}+8l^2\frac{4}{\alpha-\beta} \leqq \frac{n}{\alpha}+32l^2\frac{5n+l}{l\alpha} \leqq c_7\frac{(n+l)l}{\alpha}$$

which gives the statement of Lemma 1.2.

As it was already mentioned, Lemma 1.2 implies (13), therefore

$$|p_n^{*\prime}(\varphi)| \leqq |p_n^{*\prime}(a)| \leqq \frac{c_2(n+l)l}{\varphi}\|p_n^*\|_{[-\varphi,\varphi]}.$$

By this, because of the maximality of p_n^*, the proof of Lemma 1 is complete.

LEMMA 2. *Let* $0<\varphi\leqq\frac{\pi}{16}$, $t_n\in\mathcal{T}_n(\varphi)$, $q_k\in T_k$, $r=t_nq_k$, $n\geqq 0$, $k\geqq 0$, $m\geqq 1$. *Then*

$$|r^{(m)}(\varphi)| \leqq c_m^{(2)}\left(\frac{(n+k)(k+1)}{\varphi}\right)^m \|r\|_{[-\varphi,\varphi]}.$$

PROOF. First we show that if p is a polynomial defined in Lemma 1, then

$$|p'(y)| \leqq c_8 \frac{(n+l)(l+1)}{\varphi} \|p\|_{[-\varphi,\varphi]} \quad (|y| \leqq \varphi). \tag{16}$$

If $y \in [0, \varphi]$ then from Lemma 1 we can deduce

$$|p'(y)| \leqq c_2 \frac{(n+l)(l+1)}{(y+\varphi)/2} \|p\|_{[-\varphi, y]} \leqq 2c_2 \frac{(n+l)(l+1)}{\varphi} \|p\|_{[-\varphi,\varphi]}$$

which gives (16). Now suppose that $y \in [-\varphi, 0)$. By a simple result from [3] we have the following representation:

$$p(x) = q_l(x) \sin^{2n} \frac{x+\varphi}{2} = q_l(x) \sum_{i=0}^{2n} K_i \sin^i \frac{\varphi - x}{2} \sin^{2n-i} \frac{x-y}{2}$$

with all $K_i \geqq 0$, so by Lemma 1 it is easy to see that

$$|p'(y)| \leqq \sum_{i=2n-1}^{2n} \left| \left(K_i \sin^i \frac{\varphi-x}{2} \sin^{2n-i} \frac{x-y}{2} q_l(x) \right)' (y) \right| \leqq$$

$$\leqq 2c_2 \frac{(n+l+1)(l+2)}{(\varphi-y)/2} \|p\|_{[y,\varphi]} \leqq c_9 \frac{(n+l)(l+1)}{\varphi} \|p\|_{[-\varphi,\varphi]},$$

thus (16) is proved. From (16) by induction on m we get

$$|p^{(m)}(\varphi)| \leqq \|p^{(m)}\|_{[-\varphi,\varphi]} \leqq c_m^{(3)} \left(\frac{(n+l)(l+1)}{\varphi} \right)^m \|p\|_{[-\varphi,\varphi]}. \tag{17}$$

Now let

$$t_n(x) = \sum_{i=0}^{2n} L_i \sin^i \frac{\varphi-x}{2} \sin^{2n-i} \frac{x+\varphi}{2}$$

where $L_i \geqq 0$ $(i=0, 1, \dots, 2n)$. Using (17), we have

$$|r^{(m)}(\varphi)| \leqq \sum_{i=0}^{2n} L_i \left| \left(\sin^i \frac{\varphi-x}{2} \sin^{2n-i} \frac{x+\varphi}{2} q_k(x) \right)^{(m)} (\varphi) \right| =$$

$$= \sum_{j=0}^{\min\left(\left[\frac{m}{2}\right], n\right)} L_{2j} \left| \left(\sin^{2n-2j} \frac{x+\varphi}{2} \left(\sin^{2j} \frac{\varphi-x}{2} q_k(x) \right) \right)^{(m)} (\varphi) \right| +$$

$$+ \sum_{j=0}^{\min\left(\left[\frac{m-1}{2}\right], n-1\right)} L_{2j+1} \left| \left(\sin^{2n-2j-2} \frac{x+\varphi}{2} \left(\sin \frac{x+\varphi}{2} \sin^{2j+1} \frac{\varphi-x}{2} q_k(x) \right) \right)^{(m)} (\varphi) \right| \leqq$$

$$\leqq (m+1) c_m^{(3)} \left(\frac{(n+(m+k))(m+k+1)}{\varphi} \right)^{(m)} \|r\|_{[-\varphi,\varphi]} \leqq$$

$$\leqq c_m^{(2)} \left(\frac{(n+k)(k+1)}{\varphi} \right)^m \|r\|_{[-\varphi,\varphi]}. \quad \text{Q.E.D.}$$

Now we need a definition. In case of $-\pi \leqq \varphi_1 < \varphi_2 \leqq \pi$ denote by $\mathcal{T}_n(\varphi_1, \varphi_2)$ the set of trigonometric polynomials of the form

$$t(x) = \sum_{i=0}^{2n} d_i \sin^i \frac{\varphi_2 - x}{2} \sin^{2n-i} \frac{x - \varphi_1}{2}$$

where each $d_i \geqq 0$ or each $d_i \leqq 0$ $(i=0, 1, \ldots, 2n)$. Because of the periodicity the result of Lemma 2 can be written as follows.

Corollary 1. *Let* $\varphi_2 - \varphi_1 \leqq \frac{\pi}{8}$, $t_n \in \mathcal{T}_n(\varphi_1, \varphi_2)$, $q_k \in T_k$, $r = t_n q_k$, $n \geqq 0$, $k \geqq 0$, $m \geqq 0$. *Then*

$$|r^{(m)}(\varphi_2)| \leqq c_m^{(4)} \left(\frac{(n+k)(k+1)}{\varphi_2 - \varphi_1} \right)^m \|r\|_{[\varphi_1, \varphi_2]}.$$

The proof of Theorem 1 is now straightforward. Let $y \in [-\varphi, \varphi]$, $\omega \leqq \frac{\pi}{16}$. Without loss of generality we may suppose that $y \in [-\omega, 0]$. Observe that $t_n \in \mathcal{T}_n(\omega) = \mathcal{T}_n(-\omega, \omega)$ implies that $t_n \in \mathcal{T}_n(y, \omega)$. (This simple observation can be found in [3]). Now applying the result of Corollary 1 to the interval $[y, \omega]$ and the trigonometric polynomial $r = t_n q_k$, we have

$$|r^{(m)}(y)| \leqq c_m^{(4)} \left(\frac{(n+k)(k+1)}{\omega - y} \right)^m \|r\|_{[y, \omega]} \leqq 2^m c_m^{(4)} \left(\frac{(n+k)(k+1)}{\omega} \right)^m \|r\|_{[-\omega, \omega]},$$

hence

$$\|r^{(m)}\|_{[-\varphi, \varphi]} \leqq 2^m c_m^{(4)} \left(\frac{(n+k)(k+1)}{\omega} \right)^m \|r\|_{[-\omega, \omega]}$$

if $\omega \leqq \frac{\pi}{16}$. Finally, if $\omega > \frac{\pi}{16}$, then the interval $[-\omega, \omega]$ can be divided into subintervals of length not greater than $\frac{\pi}{8}$ and repeated application of the just proved part of the theorem gives the desired result. By this the proof of Theorem 1 is complete. □

According to Theorem 2 of [3], $t_n \in S_n^0(\omega, \varepsilon)$ $\left(\omega \leqq \frac{\pi}{4},\ 0 < \varepsilon \leqq 1\right)$ implies that $t_n \in \mathcal{T}_l(-\omega, \omega)$ if $l \geqq c_{10} \frac{n}{\varepsilon^2}$. From this and Theorem 1 we get

Corollary 2. *Let* $0 < \omega \leqq \pi$, $0 < \varepsilon \leqq 1$, $0 \leqq k \leqq n$ *and* $r \in S_n^{2k}(\omega, \varepsilon)$. *Then*

$$\|r^{(m)}\|_{[-\omega, \omega]} \leqq c_m^{(5)} \left(\left(\frac{n-k}{\varepsilon^2} + k \right)(k+1) \right)^m \|r\|_{[-\omega, \omega]} \leqq c_m^{(5)} \left(\frac{n(k+1)}{\varepsilon^2} \right)^m \|r\|_{[-\omega, \omega]}.$$

First we get this when $\omega \leqq \frac{\pi}{4}$. In case of $\frac{\pi}{4} < \omega \leqq \pi$ the interval $[-\omega, \omega]$ can be divided into subintervals of length not greater than $\frac{\pi}{2}$ and repeated application of the result for $\omega\leqq\frac{\pi}{4}$ gives the desired result.

Now we examine how sharp the estimates of Theorem 1 and Corollary 2 are.

THEOREM 2. *Let* $0\leqq k\leqq n$, $0<\omega<c_{11}\frac{\pi}{2}$, $c_{11}<1$ *and* $1\leqq m$. *Then*

$$\sup_{r\in T_n^{2k}}\frac{\|r^{(m)}\|_{[-\omega,\omega]}}{\|r\|_{[-\omega,\omega]}}\geqq c_m^{(6)}\left(\frac{n(k+1)}{\omega}\right)^m \quad (n\geqq c_m^{(7)}).$$

PROOF. We distinguish two cases.

Case 1: $k>2\pi(m+1)$. Let

$$D_k(x)=\frac{1}{2^{k-1}}\cos\,(k\arccos x)=\prod_{i=1}^{k}(x-v_i)\quad(-1<v_1<v_2<\ldots<v_k<1)$$

and

$$p(x)=\left(x+\frac{n-2k}{2k}\right)^{n-k}D_k(x).$$

We have

$$|D_k(x)|\leqq\|D_k\|_{[-1,1]}=\frac{1}{2^{k-1}}\leqq 2(1-x)^k\quad\left(-1\leqq x\leqq\frac{1}{2}\right),$$

further obviously

$$|D_k(x)|=\Big|\prod_{i=k}^{k}(x-v_i)\Big|\leqq(1-x)^k\quad(x<-1).$$

Observe that $\left(\left(x+\frac{n-2k}{2k}\right)^{n-k}(x-1)^k\right)'$ vanishes only at $x=\frac{1}{2}$ in $\left(\frac{2k-n}{2k},1\right)$, hence for $\frac{2k-n}{2k}\leqq x\leqq\frac{1}{2}$ we have

$$|p(x)|\leqq 2\left|\left(x+\frac{n-2k}{2k}\right)^{n-k}(x-1)^k\right|\leqq 2\left(\frac{n-k}{2k}\right)^{n-k}\left(\frac{1}{2}\right)^k$$

and for $\frac{1}{2}<x\leqq 1$ it is apparent that

$$|p(x)|\leqq 2\left(\frac{n}{2k}\right)^{n-k}\left(\frac{1}{2}\right)^k=p(1),$$

thus we get

$$\|p\|_{\left[\frac{2k-n}{2k},1\right]}=p(1). \tag{18}$$

Now let $\eta_i=\cos\dfrac{k-i}{k}\pi\ (i=0, 1, \dots, k)$. As $\cos x\geqq 1-x^2/2$ and $k>2\pi(m+1)$, we have

$$\eta_i \geqq 1-\frac{c_m^{(8)}}{k^2} \geqq \frac{7}{8} > \frac{2k-n}{2k} \quad (k-m-1 \leqq i \leqq k) \tag{19}$$

with $c_m^{(8)}=\dfrac{1}{2}(m+1)^2\pi^2$. From (19) and (18) we get

$$|p(\eta_i)| = \left|\left(\eta_i+\frac{n-2k}{2k}\right)^{n-k} D_k(\eta_i)\right| \geqq \tag{20}$$

$$\geqq \left(\frac{n}{2k}-\frac{c_m^{(8)}}{k^2}\right)^{n-k}\frac{1}{2^{k-1}} = \left(\frac{n}{2k}\right)^{n-k}\frac{1}{2^{k-1}}\left(1-\frac{2c_m^{(8)}}{nk}\right)^{n-k} =$$

$$= p(1)\left(1-\frac{2c_m^{(8)}}{nk}\right)^n \geqq c_m^{(9)}\|p\|_{\left[\frac{2k-n}{2k},1\right]} \quad (k-m-1 \leqq i \leqq k).$$

It is obvious that

$$\operatorname{sgn} p(\eta_i) = -\operatorname{sgn} p(\eta_{i+1}) \quad (k-m-1 \leqq i \leqq k-1). \tag{21}$$

Now let

$$r(y) = p\left(\frac{n}{4k}\frac{\sin y}{\sin\omega}+\frac{4k-n}{4k}\right) \tag{22}$$

and define $\tilde\eta_i\in[-\omega,\omega]$ by

$$\frac{n}{4k}\frac{\sin\tilde\eta_i}{\sin\omega}+\frac{4k-n}{4k} = \eta_i \quad (k-m-1 \leqq i \leqq k). \tag{23}$$

From (20), (21), (22) and (23) we can easily see that

$$r\in T_n^{2k}(\omega), \tag{24}$$

$$\|r\|_{[-\omega,\omega]} = \|p\|_{\left[\frac{2k-n}{2k},1\right]}, \tag{25}$$

$$|r(\tilde\eta_i)| \geqq c_m^{(9)}\|r\|_{[-\omega,\omega]} \quad (k-m-1 \leqq i \leqq k), \tag{26}$$

$$\operatorname{sgn} r(\tilde\eta_i) = -\operatorname{sgn} r(\tilde\eta_{i+1}) \quad (k-m-1 \leqq i \leqq k-1), \tag{27}$$

further from (23) and (19)

$$(\omega-\tilde\eta_i)\cos\omega \leqq \sin\omega-\sin\tilde\eta_i = \frac{4k}{n}\sin\omega\,(1-\eta_i) \leqq \omega\frac{4k}{n}\frac{c_m^{(8)}}{k^2} = \omega\frac{4c_m^{(8)}}{nk}$$

$$(k-m-1 \leqq i \leqq k)$$

and by this, using the condition $0<\omega\leqq c_{11}\frac{\pi}{2}$, $c_{11}<1$, we have

$$(28)\qquad \left(1-\frac{c_m^{(10)}}{nk}\right)\omega \leqq \tilde{\eta}_i \leqq \omega \quad (k-m-1 \leqq i \leqq k).$$

Now let $\Omega(x)=\prod_{i=k-m-1}^{k-1}(x-\tilde{\eta}_j)$. By a wellknown theorem for the m^{th} order divided differences there exists a $z\in[\tilde{\eta}_{k-m-1}, \tilde{\eta}_{k-1}]\subset[-\omega, \omega]$ for which

$$(29)\qquad r^{(m)}(z) = m!\, r(\tilde{\eta}_{k-1}, \tilde{\eta}_{k-2}, \dots, \tilde{\eta}_{k-m-1}) = m! \sum_{j=k-m-1}^{k-1} \frac{r(\tilde{\eta}_j)}{\Omega'(\tilde{\eta}_j)}.$$

As

$$\Omega'(\tilde{\eta}_j) = \prod_{\substack{i=k-m-1\\ i-j}}^{k-1} (x-\tilde{\eta}_i) \quad (k-m-1 \leqq j \leqq k-1)$$

and $\operatorname{sgn} \Omega'(\tilde{\eta}_j) = -\operatorname{sgn} \Omega'(\tilde{\eta}_{j+1})$ $(k-m-1\leqq j\leqq k-2)$, from (26), (27), (28) and (29)

$$|r^{(m)}(z)| \geqq m!(m+1)c_m^{(9)}\|r\|_{[-\omega,\omega]}\left(\frac{nk}{c_m^{(10)}\,\omega}\right)^m = c_m^{(11)}\left(\frac{nk}{\omega}\right)^m \|r\|_{[-\omega,\omega]}$$

which gives the desired result in Case 1.

Case 2: $0\leqq k\leqq 2\pi(m+1)$ and $n\geqq c_m^{(7)}$. Now consider the trigonometric polynomial

$$r_n(x)\in \sin^{2n}\frac{x+\omega}{2}\in T_n^0(\omega)\subset T_n^{2k}(\omega).$$

A simple calculation shows that

$$\|r_n^{(m)}\|_{[-\omega,\omega]} \geqq |r_n^{(m)}(\omega)| \geqq c_m^{(12)}\left(\frac{n}{\omega}\right)^m \|r_n\|_{[-\omega,\omega]} \geqq$$

$$\geqq c_m^{(13)}\left(\frac{n(k+1)}{\omega}\right)^m \|r_n\|_{[-\omega,\omega]} \quad (n\geqq c_m^{(7)})$$

and by this the proof of Theorem 2 is complete. □

According to Theorem 2, up to the constant $c_m^{(1)}$ Theorem 1 is sharp when $0<\omega\leqq c_{11}\frac{\pi}{2}$, $c_{11}<1$. But Theorem 1 does not give a sharp result for arbitrary $0<\omega\leqq\pi$. For example it can be proved that

$$\|t'\|_{[-\omega,\omega]} \leqq c_{12}\sqrt{n}\,\|t\|_{[-\omega,\omega]} \quad \left(t\in\mathscr{T}_n(\omega), \frac{\pi}{2}\leqq\omega\leqq\pi\right)$$

and this result cannot be improved. (See [8], Theorem 1.)

From Theorem 1 we can get Markov type estimates for algebraic polynomials of special type. First we need some definitions. Denote by Π_n the set of all real algebraic polynomials of degree at most n, by P_n^k $(0 \leqq k \leqq n)$ the set of those polynomials from Π_n which have all but at most k roots in $\mathbf{R} \setminus (-1, 1)$, by $S_n^k(\varepsilon)$ $(0 \leqq k \leqq n, 0 < \varepsilon \leqq 1)$ the set of those polynomials from Π_n which have all but at most k roots outside the ellipse

$$\left\{z := x + iy \in \mathbf{C} \,|\, x^2 + \frac{y^2}{\varepsilon^2} < 1\right\}.$$

Obviously $P_n^k \subset S_n^k(\varepsilon)$. Further let $\mathscr{P}_n(a, b)$ $(a < b \in \mathbf{R})$ the set of polynomials of the form

$$p(x) = \sum_{i=0}^{n} d_i (b-x)^i (x-a)^{n-i}$$

with all $d_i \geqq 0$ or all $d_i \leqq 0$. By a simple observation of G. G. Lorentz $S_n^0(1) \subset \mathscr{P}_n(-1, 1)$. Finally denote by H_{n+k}^k the set of polynomials of the form $r = t_n q_k$ where $t_n \in \mathscr{P}_n(1-1, 1)$ and $q_k \in \Pi_k$. From $S_n^0(1) \subset \mathscr{P}_n(-1, 1)$ it is easy to see that

$$P_n^k \subset S_n^k(1) \subset H_n^k. \tag{30}$$

It was proved in [4] that

$$S_n^0(\varepsilon) \subset \mathscr{P}_l(-1, 1) \quad \left(l \geqq \frac{3n}{\varepsilon^2}\right), \tag{31}$$

and from this it is obvious that

$$S_n^k(\varepsilon) \subset H_{l+k}^k \quad \left(l \geqq \frac{3(n-k)}{\varepsilon^2}\right). \tag{32}$$

In [5] P. Borwein proved that

$$\|r'\|_{[-1,1]} \leqq c_{13} n(k+1) \|r\|_{[-1,1]} \tag{33}$$

if $p \in P_n^k$. In [6] (33) was extended to the class $S_n^k(1)$. Now we prove (33) for polynomials from $H_n^k (\supset S_n^k(1))$. This can be obtained from Theorem 1 as a simple consequence, which demonstrates the power of Theorem 1.

THEOREM 3. *If* $r \in H_n^k$ $(0 \leqq k \leqq n)$, *then* (33) *holds true.*

PROOF. As $r \in H_n^k$, we have $r = t_{n-k} q_k$, where $t_{n-k} \in \mathscr{P}_{n-k}(-1, 1)$ and $q_k \in \Pi_k$. Let $\omega = \frac{\pi}{4}$ and consider the trigonometric polynomial $\tilde{r}(x) := r\left(\frac{\sin x}{\sin \omega}\right)$. Then we have $\tilde{r}(x) = \tilde{t}_{n-k}(x) \tilde{q}_k(x)$ where $\tilde{t}_{n-k}(x) := t_{n-k}\left(\frac{\sin x}{\sin \omega}\right)$ and $\tilde{q}_k(x) := q_k\left(\frac{\sin x}{\sin \omega}\right)$. A routine calculation shows that $t_{n-k} \in \mathscr{P}_n(-1, 1)$ and $q_k \in \Pi_k$ imply $\tilde{t}_{n-k} \in \mathscr{T}_n(\omega)$ and $\tilde{q}_k \in T_k$, therefore we can apply Theorem 1 to $\tilde{r}$ (in case of $m = 1$) and we get

$$\|\tilde{r}'\|_{[-\omega, \omega]} \leqq \frac{c_{14} n(k+1)}{\omega} \|\tilde{r}\|_{[-\omega, \omega]}$$

and from this, recalling that $\omega=\frac{\pi}{4}$, we deduce

$$\|r'\|_{[-1,1]} \leqq \left\| r'\left(\frac{\sin x}{\sin \omega}\right)\frac{\cos x}{\sin \omega}\right\|_{[-\omega,\omega]} = \|\tilde{r}'\|_{[-\omega,\omega]} \leqq$$

$$\leqq c_{14}\frac{4}{\pi}n(k+1)\|\tilde{r}\|_{[-\omega,\omega]} = c_{15}n(k+1)\|r\|_{[-1,1]}$$

which gives the desired result. □

The properties possessed by polynomials from H_n^k are, in general, not inherited by the derivatives of these polynomials. Therefore to prove a result for higher derivatives of polynomials from H_n^k needs new ideas.

THEOREM 4. *If* $r\in H_n^k$ $(0\leqq k\leqq n)$ *then*

$$\|r^{(m)}\|_{[-1,1]} \leqq c_m^{(14)}(n(k+1))^m\|r\|_{[-1,1]}.$$

PROOF. Observe that $f\in P_n^k$ implies $f'\in P_{n-1}^{k+1}$ (let $P_n^l := \Pi_n$ if $l\geqq n$). Using this observation, from Theorem 3 by induction on m we get

$$\|f^{(m)}\|_{[-1,1]} \leqq c_m^{(15)}(n(k+1))^m\|f\|_{[-1,1]} \quad (f\in P_n^k). \tag{34}$$

Now let $r\in H_n^k$, so $r=t_{n-k}q_k$, where $q_k\in\Pi_k$ and

$$t_{n-k}(x) = \sum_{i=0}^{n-k} d_i(1-x)^i(x+1)^{n-k-i}$$

say with all $d_i\geqq 0$. Let $s:=\min(n-k, m)$. Applying (34) to $(1-x)^i(x+1)^{n-k-i}\times \times q_k(x)\in P_n^k$ $(i=0, 1, \dots, s)$, we obtain

$$|r^{(m)}(1)| \leqq \left|\sum_{i=0}^{s} d_i\{(1-x)^i(x+1)^{n-k-i}q_k(x)\}_{x=1}^{(m)}\right| \leqq \tag{35}$$

$$\leqq c_m^{(15)}(n(k+1))^m \sum_{i=0}^{s} d_i\|(1-x)^i(x+1)^{n-k-i}q_k(x)\|_{[-1,1]} \leqq$$

$$\leqq c_m^{(15)}(m+1)(n(k+1))^m\|t_{n-k}(x)q_k(x)\|_{[-1,1]} = c_m^{(16)}(n(k+1))^m\|r\|_{[-1,1]}.$$

Now let $y\in[-1, 1]$. It may be supposed that $y\in[0, 1]$, the case of $y\in[-1, 0]$ is similar. It is easy to verify that $t_{n-k}\in\mathcal{P}_{n-k}(-1, 1)$ implies $t_{n-k}\in\mathcal{P}_n(-1, y)$. To see this it is sufficient to note that

$$1-x = \frac{1-y}{1+y}(x+1)+\frac{2}{1+y}(y-x)$$

where $\frac{1-y}{1+y}$ and $\frac{2}{1+y}$ are non-negative. Therefore from (35) by a simple linear transformation we deduce

$$|r^{(m)}(y)| \leqq c_m^{(16)}\left(\frac{2}{1+y}\right)^m(n(k+1))^m\|r\|_{[-1,y]} \leqq c_m^{(14)}(n(k+1))^m\|r\|_{[-1,1]}. \quad □$$

From Theorem 4 and (32) we obtain

COROLLARY 3. *We have*

$$\|r^{(m)}\|_{[-1,1]} \leqq c_m^{(17)} \left(\left(\frac{n-k}{\varepsilon^2}+k\right)(k+1)\right)^m \|r\|_{[-1,1]} \leqq \tag{36}$$

$$\leqq c_m^{(17)} \left(\frac{n(k+1)}{\varepsilon^2}\right)^m \|r\|_{[-1,1]} \quad (r\in S_n^k(\varepsilon)).$$

In [6], Corollary 3 was proved for $m=\varepsilon=1$. There we used the result of Borwein for P_n^k (see [5]) and the relation

$$\sup_{p\in P_n^k} \frac{|p'(y)|}{\|p\|_{[-1,1]}} = \sup_{p\in S_n^k(1)} \frac{|p'(y)|}{\|p\|_{[-1,1]}} \quad (|y|<1). \tag{37}$$

But we do not know the answer to the following

PROBLEM 1. Is it true that

$$\sup_{p\in P_n^k} \frac{|p^{(m)}(y)|}{\|p\|_{[-1,1]}} = \sup_{p\in S_n^k(1)} \frac{|p^{(m)}(y)|}{\|p\|_{[-1,1]}} \quad (|y|<1) \tag{38}$$

for all $m=1, 2, \ldots$?

The estimate of Theorem 4 was proved by P. Borwein for polynomials from P_n^k. By induction on m it was a straightforward consequence of (33). As induction on m does not work even in case of $S_n^k(1)$, for a long time I tried to extend Borwein's result for $S_n^k(1)$ by proving (38). Meanwhile I managed to prove Theorem 4, but Problem 1 is still open for $m\geqq 2$.

Up to the constant depending only on m, Theorem 4 cannot be improved. According to Theorem 4 of [7], if $1\leqq m\leqq n$, then there exist polynomials $r_{n,k}\in P_n^k(\subset H_n^k)$ and a constant $c_m^{(18)}>0$ for which

$$\|r_{n,k}^{(m)}\|_{[-1,1]} \geqq c_m^{(18)}\left(n(k+1)\right)^m \|r_{n,k}\|_{[-1,1]}.$$

Inequality (1) does not contain the classical Bernstein's inequality. Our purpose is to give an estimate of type (1) from which the Bernstein's inequality follows.

THEOREM 5. *Let* $t\in T_n$, $n\geqq 1$ *and* $0<\omega\leqq\pi$. *Then*

$$\|t'\|_{[-\omega,\omega]} \leqq \left(n+c_{17}\frac{\pi-\omega}{\omega}n^2\right)\|t\|_{[-\omega,\omega]}.$$

PROOF. According to (1), in case of $n\geqq\frac{1}{2}\sqrt{3\tan^2\frac{\omega}{2}+1}$

$$\|t'\|_{[-\omega,\omega]} \leqq 2n^2\cot\frac{\omega}{2}\|t\|_{[-\omega,\omega]} \leqq \pi\frac{\pi-\omega}{\omega}n^2\|t\|_{[-\omega,\omega]}. \tag{39}$$

This gives the desired result if $\pi-\omega>\frac{1}{2n}$, namely then $\tan\frac{\omega}{2}<4n$, so $\frac{1}{2}\sqrt{3\tan^2\frac{\omega}{2}+1}<4n$, therefore the constant $c_{17}=16\pi$ is suitable. Now let $\pi-\omega\leqq\leqq\frac{1}{2n}$. Bernstein's inequality asserts that

$$\|t'\|_{[-\pi,\pi]}\leqq n\|t\|_{[-\pi,\pi]}\quad (t\in T_n). \tag{40}$$

Using the Mean Value Theorem, from (40) we get

$$\|t\|_{[-\pi,\pi]}\leqq\|t\|_{[-\omega,\omega]}+(\pi-\omega)n\|t\|_{[-\pi,\pi]},$$

so

$$\|t\|_{[-\omega,\omega]}\geqq\bigl(1-n(\pi-\omega)\bigr)\|t\|_{[-\pi,\pi]}.$$

Therefore using $\pi-\omega\leqq\frac{1}{2n}$, we have

$$\|t'\|_{[-\omega,\omega]}\leqq\|t'\|_{[-\pi,\pi]}\leqq n\|t\|_{[-\pi,\pi]}\leqq$$

$$\leqq\frac{n}{1-n(\pi-\omega)}\|t\|_{[-\omega,\omega]}=\left(n+\frac{n^2(\pi-\omega)}{1-n(\pi-\omega)}\right)\|t\|_{[-\omega,\omega]}\leqq$$

$$\leqq\bigl(n+2(\pi-\omega)n^2\bigr)\|t\|_{[-\omega,\omega]}.\quad\square$$

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OF THE HUNGARIAN ACADEMY OF SCIENCES
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