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ON AXIOMATIZATION OF THE THEORY OF LINKING

M. BOGNÁR (Budapest)

In an earlier paper [4], [5], I gave an axiomatically constructed unique theory of linking of homology theories and in connection with this I used to state a theorem of uniqueness and existence. Also, some corollaries of this theorem were introduced. However the proof of the theorem was omitted in this paper. For that very reason the purpose of the present article is to fill in this long standing gap.

First it should be remarked that the terminology and notation are mainly taken from [6]. So in general we do not refer to this book in details.

Next, recall the definitions and the theorem of [5].

By a homology theory we always mean a partially exact homology theory defined on the category of compact pairs. For a homology theory H and a compact space X we denote by $\tilde{H}_t(X)$ the t-dimensional homology group $H_t(X)$ for $t>0$ and the reduced zero-dimensional homology group for $t=0$. In both cases $\tilde{H}_t(X)$ will be called the reduced t-dimensional homology group of X. If $f\colon X\to Y$ is a continuous mapping of the compact space X into the compact space Y, then $\tilde{H}_t(f)$ or also $\tilde{H}(f)$ will denote the map of $\tilde{H}_t(X)$ into $\tilde{H}_t(Y)$ defined by the induced map $f_*\colon H_t(X)\to H_t(Y)$.

A map $\mathfrak{v}\colon A\times B\to C$, where A, B and C are abelian groups, is said to be a *bihomomorphism* if the following condition is satisfied:

$$\mathfrak{v}(a+a', b+b') = \mathfrak{v}(a, b)+\mathfrak{v}(a', b)+\mathfrak{v}(a, b')+\mathfrak{v}(a', b').$$

Now suppose given two homology theories H and H', an abelian group C, the n-dimensional euclidean space R^n and nonnegative integers r, r' satisfying $r+r'=n-1$ (the pairs H, H' and r, r' are ordered).

MAIN DEFINITION. The map $\mathfrak{B}=\mathfrak{B}_{H,H',C,r,r'}$ which makes correspond to each ordered pair (N, N') of disjoint compact subsets of R^n a bihomorphism $\mathfrak{B}_{N,N'}\colon \tilde{H}_r(N)\times\tilde{H}'_{r'}(N')\to C$ will be called a *theory of linking for the compacts in* R^n (given the homology theories H and H' the group C and the numbers r and r'), if for any compacts M, M', N and N' satisfying $M\subset N$, $M'\subset N'$ and of course $N\cap N'=\emptyset$ the condition

$$\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{N,N'}(\tilde{H}_r(i)(u), \tilde{H}'_{r'}(i')(u'))$$

is satisfied for every $u\in\tilde{H}_r(M)$ and $u'\in\tilde{H}'_{r'}(M')$ where $i\colon M\subset N$ and $i'\colon M'\subset N'$ are the inclusion maps.

DEFINITION. We shall say that the spheres S and S' of dimension r and r' respectively contained in R^n are (mutually) linked if the following conditions are satisfied:

(a) $r+r'=n-1$,

(b) the centre of S belongs to S' and the centre of S'' belongs to S,

(c) the planes R and R' supporting S and S' respectively intersect in a line,

(d) R and R' are perpendicular in the natural sense that any vectors a in R and a' in R' which are perpendicular to the line $R \cap R'$ are mutually perpendicular.

THEOREM. *Let H and H' be any homology theories, let C be an arbitrary abelian group and let r and r' be nonnegative integers satisfying $r+r'=n-1$. For every bihomomorphism*

$$\mathfrak{v}_0 = \mathfrak{v}_0 \colon \tilde{H}_r(S) \times \tilde{H}'_{r'}(S') \to C$$

where S and S' are linked spheres in R^n of dimension r and r' respectively there exists one and only one theory of linking $\mathfrak{B} = \mathfrak{B}_{H,H',C,r,r'}$ for the compacts in R^n which takes the value $\mathfrak{v}_0$ on the pair S, S'.

First we prove the uniqueness of $\mathfrak{B}$.

The sketch of the proof can be drawn up as follows.

(A) First we use the fact that for any two transversally intersecting simplexes of appropriate dimensions there exists an isotopy of R^n which maps the given standard spheres onto the boundaries of these simplexes. This allows us to prove that the linking number for boundaries of simplexes is uniquely defined.

(B) In the second part of the proof we approximate the disjoint compacts by their disjoint simplicial neighbourhoods. In these neighbourhoods any homology class of the compacts (in any homology theory) can be represented by union of boundaries of simplexes. This completes the proof.

Now we are going to prove the uniqueness of $\mathfrak{B}$ in details.

First we make some preliminary remarks.

Throughout this paper let n be a positiv integer and R^n the n-dimensional euclidean space.

1. DEFINITION. Let X, Y and W be subsets of R^n such that $X \cup Y \subset W$ and let $f\colon X \to Y$ be a continuous mapping. f is said to be a *deformation in W* if $i_2 f$ and i_1 are homotopic maps of X into W, where $i_1 \colon X \subset W$ and $i_2 \colon Y \subset W$ are the inclusion maps.

In particular let X and Y be subsets of R^n, $f\colon X \to Y$ a surjective continuous mapping and let

$$W = \bigcup_{p \in X} [p, f(p)]$$

where for $p \in X$, $[p, f(p)]$ is the segment joining p and $f(p)$. Then f is clearly a deformation in W.

Observe that if in addition X is compact, then so is the set W, too. In this particular case we shall denote the set $W = \bigcup_{p \in X} [p, f(p)]$ by W_f.

2. Let C_1 and C_2 be compact convex sets in R^n with the same supporting plane R such that the interiors (in the plane R) of the sets C_1 and C_2 intersect. Let X and Y

be the boundaries of C_1 and C_2 respectively (in R) and let $0\in(C_1\setminus X)\cap(C_2\setminus Y)$. For any $t\in X$ let $f(t)$ be the common point of the ray $f_{0,t}$ and Y ($f_{0,t}$ is the ray with the initial point 0 and going through t).

X and Y are compact sets and $f\colon X\to Y$ is a surjective continuous map. We shall denote it by f_{0,C_1,C_2}.

We then obviously have $0\notin W_{f_{0,C_1,C_2}}$.

3. DEFINITION. A topological map $g\colon R^n\to R^n$ is said to be an *isotopic deformation of* R^n if there is a continuous map $h\colon R^n\times I\to R^n$, where $I=[0,1]$ is the unit interval of the space of the reals, such that for each $t\in I$ the map $h_t\colon R^n\to R^n$ defined by the relation $h_t(p)=h(p,t)$ is a homeomorphism of R^n onto itself and $h_0=g$, $h_1=\mathrm{id}_{R^n}$, where id_{R^n} is the identical mapping of R^n.

Observe that an isotopic deformation of R^n is evidently a deformation in R^n.

Observe also, that each orientation preserving affine transformation of R^n is clearly an isotopic deformation of it ("orientation preserving" means that the determinant of its matrix is positive and this fact does not depend on the special choice of the coordinate system).

4. PROPOSITION. *Let $g\colon R^n\to R^n$ be an isotopic deformation of R^n and let M and N be disjoint compact sets of R^n. Then there are a positive integer k and compact sets $M_1,\dots,M_{2k+1}, N_1,\dots,N_{2k+1}$ in R^n and continuous mappings*

$$g_1\colon M_1\to M_3,\dots,g_k\colon M_{2k-1}\to M_{2k+1},\quad g^1\colon N_1\to N_3,\dots,g^k\colon N_{2k-1}\to N_{2k+1}$$

such that

$$M=M_1,\ N=N_1,\ g(M)=M_{2k+1},\ g(N)=N_{2k+1},$$

$$g|M=g_k\dots g_2g_1,\quad g|N=g^k\dots g^2g^1$$

where $g|M$ and $g|N$ are restrictions of g to M and N respectively, and such that for $i=1,\dots,k$ $M_{2i-1}\cup M_{2i+1}\subset M_{2i}$, $N_{2i-1}\cup N_{2i+1}\subset N_{2i}$, $M_{2i}\cap N_{2i}=\emptyset$ moreover $g_i\colon M_{2i-1}\to M_{2i+1}$ is a deformation in M_{2i} and $g^i\colon N_{2i-1}\to N_{2i+1}$ is a deformation in N_{2i}.

PROOF. Let $h\colon R^n\times I\to R^n$ be a continuous map such that for $t\in I$ the map $h_t\colon R^n\to R^n$ defined by $h_t(p)=h(p,t)$ is a homeomorphism of R^n onto itself and $h_0=g$, $h_1=\mathrm{id}_{R^n}$. g is an isotopic deformation of R^n, hence such a map h exists.

For $t\in I$ let U_t and V_t be disjoint open neighbourhoods of $h_t(M)$ and $h_t(N)$ respectively. Since $h_t(M)=h(M\times\{t\})$ and $h_t(N)=h(N\times\{t\})$ are disjoint compact sets in R^n it follows that such neighbourhoods U_t and V_t exist.

Hence $h^{-1}(U_t)$ and $h^{-1}(V_t)$ are neighbourhoods of $M\times\{t\}$ and $N\times\{t\}$ respectively. Let W_t be an open neighbourhood of t in I such that $M\times W_t\subset h^{-1}(U_t)$ and $N\times W_t\subset h^{-1}(V_t)$. By the compactness of M and N, such a W_t clearly exists. Now for $t\in I$ we have

$$h(M\times W_t)\subset U_t,\quad h(N\times W_t)\subset V_t. \tag{1}$$

Let us denote the open covering $\{W_t;\ t\in I\}$ of I by $\mathfrak{W}$. Let $0=t_k<t_{k-1}<\dots<t_0=1$ be a subdivision of the interval $[0,1]=I$ such that for each i ($1\leqq i\leqq k$) the interval $[t_i,t_{i-1}]=T_i$ is contained in some $W_t\in\mathfrak{W}$. Since for each open covering of a compact metric space there is a positive real ε such that every subset of the space

of diameter $<\varepsilon$ is wholly contained in at least one member of the covering (see e.g. [1] p. 35) it follows that such a subdivision exists.

Now for $i=0, \ldots, k$ let

$$M_{2i+1} = h(M\times\{t_i\}) = h_{t_i}(M), \tag{2}$$

$$N_{2i+1} = h(N\times\{t_i\}) = h_{t_i}(N). \tag{3}$$

Moreover for $i=1, \ldots, k$ let

$$M_{2i} = h(M\times T_i), \quad N_{2i} = h(N\times T_i), \tag{4}$$

$$g_i = (h_{t_i}h_{t_{i-1}}^{-1})|M_{2i-1}\colon M_{2i-1} \to M_{2i+1}, \tag{5}$$

$$g^i = (h_{t_i}h_{t_{i-1}}^{-1})|N_{2i-1}\colon N_{2i-1} \to N_{2i+1}. \tag{6}$$

Thus we have

$$M_1 = h_{t_0}(M) = h_1(M) = M, \quad N_1 = h_{t_0}(N) = h_1(N) = N,$$

$$M_{2k+1} = h_{t_k}(M) = h_0(M) = g(M) = (g|M)(M),$$

$$N_{2k+1} = h_{t_k}(N) = h_0(N) = g(N) = (g|N)(N).$$

(1), (2), (3) and (4) show that for $j=1, \ldots, 2k+1$ M_j and N_j are disjoint compact sets of R^n. Furthermore for $i=1, \ldots, k$, M_{2i-1} and M_{2i+1} are subsets of M_{2i} and N_{2i-1} and N_{2i+1} are subsets of N_{2i}.

On the other hand we find by (5) and (6) that

$$g_k \cdots g_2 g_1 = (h_{t_k}h_{t_0}^{-1})|M_1 = (h_0 h_1^{-1})|M = g|M$$

and

$$g^k \ldots g^2 g^1 = (h_{t_k}h_{t_0}^{-1})|N_1 = (h_0 h_1^{-1})|N = g|N.$$

Thus we have only to show that for $i=1, \ldots, k$ g_i is a deformation in M_{2i} and g_i is a deformation in N_{2i}.

Clearly, it suffices to prove the first assertion.

Now for $x\in I$ and for $i=1, \ldots, k$ let

$$f_i(x) = (1-x)t_{i-1}+xt_i \in T_i, \tag{7}$$

moreover for $(p, x)\in M_{2i-1}\times I$ let

$$h^i(p, x) = h\big(h_{t_{i-1}}^{-1}(p), f_i(x)\big). \tag{8}$$

Then

$$h^i(p, x)\in h(M\times T_i) = M_{2i}$$

and the map $h^i\colon M_{2i-1}\times I \to M_{2i}$ is continuous.

Since by (7) $f_i(0)=t_{i-1}$ it follows by (8) that for $p\in M_{2i-1}$

$$h^i(p, 0) = h\big(h_{t_{i-1}}^{-1}(p), t_{i-1}\big) = h_{t_{i-1}}\big(h_{t_{i-1}}^{-1}(p)\big) = p.$$

On the other hand from $f_i(1)=t_i$ (see (7)) it follows by (8) and (5) that

$$h^i(p, 1) = h\big(h_{t_{i-1}}^{-1}(p), t_i\big) = h_{t_i}\big(h_{t_{i-1}}^{-1}(p)\big) = \big((h_{t_i}h_{t_{i-1}}^{-1})|M_{2i-1}\big)(p) = g_i(p).$$

g_i is indeed a deformation in M_{2i}. The proof of the proposition is complete. □

5. PROPOSITION. *Let* $\mathfrak{B}=\mathfrak{B}_{H,H',C,r,r'}$ *be a theory of linking for the compacts in* R^n, *where* H *and* H' *are homology theories,* C *and abelian group and* r, r' *nonnegative integers satisfying the condition* $r+r'=n-1$. *Let* X, Y, W, X', Y', W' *be compact sets in* R^n *such that* $X\cup Y\subset W$, $X'\cup Y'\subset W'$ *and* $W\cap W'=\emptyset$. *Further let* $f\colon X\to Y$ *be a deformation in* W *and* $f'\colon X'\to Y'$ *a deformation in* W' ($f\colon X\to Y$ and $f'\colon X'\to Y'$ are clearly continuous mappings). *Then*

$$\mathfrak{v}_{X,X'}(u, u') = \mathfrak{v}_{Y,Y'}(\tilde{H}_r(f)(u), \tilde{H}'_{r'}(f')(u'))$$

holds for every $u\in\tilde{H}_r(X)$ *and* $u'\in\tilde{H}'_{r'}(X')$.

PROOF. Let $i_1\colon X\subset W$, $i_2\colon Y\subset W$, $i'_1\colon X'\subset W'$ and $i'_2\colon Y'\subset W'$ be inclusion maps. By assumption $i_2 f$ is homotopic to i_1 and $i'_2 f'$ is homotopic to i'_1. Hence $\tilde{H}_r(i_2)\tilde{H}_r(f)=\tilde{H}_r(i_2 f)=\tilde{H}_r(i_1)$ and $\tilde{H}'_{r'}(i'_2)\tilde{H}'_{r'}(f')=\tilde{H}'_{r'}(i'_2 f')=\tilde{H}'_{r'}(i'_1)$. Thus according to the main definition we have for every $u\in\tilde{H}_r(X)$ and $u'\in\tilde{H}'_{r'}(X')$

$$\mathfrak{v}_{X,X'}(u, u') = \mathfrak{v}_{W,W'}(\tilde{H}_r(i_1)(u), \tilde{H}'_{r'}(i'_1)(u')) =$$

$$= \mathfrak{v}_{W,W'}(\tilde{H}_r(i_2)\tilde{H}_r(f)(u), \tilde{H}'_{r'}(i'_2)\tilde{H}'_{r'}(f')(u')) = \mathfrak{v}_{Y,Y'}(\tilde{H}_r(f)(u), \tilde{H}'_{r'}(f')(u'))$$

as required. □

6. PROPOSITION. *Let* H, H', C, r, r' *and* $\mathfrak{B}=\mathfrak{B}_{H,H',C,r,r'}$ *be the same as in Proposition* 5. *Let* $g\colon R^n\to R^n$ *be an isotopic deformation of* R^n *and let* M *and* N *be disjoint compact subsets of* R^n. *Then the relation*

$$\mathfrak{v}_{M,N}(u, u') = \mathfrak{v}_{g(M),g(N)}(\tilde{H}_r(g|M)(u), \tilde{H}'_{r'}(g|N)(u'))$$

holds for each $u\in\tilde{H}_r(M)$ *and* $u'\in\tilde{H}'_{r'}(N)$.

PROOF. Select the positive integer k, the compact sets $M_1, \ldots, M_{2k+1}, N_1, \ldots, N_{2k+1}$ and the continuous mappings $g_1\colon M_1\to M_3, \ldots, g_k\colon M_{2k-1}\to M_{2k+1}$, $g^1\colon N_1\to N_3, \ldots, g^k\colon N_{2k-1}\to N_{2k+1}$ such that the conditions of Proposition 4 should be fulfilled. Let $u\in\tilde{H}_r(M)$ and $u'\in\tilde{H}'_{r'}(N)$. Then by Proposition 5 we have

$$\mathfrak{v}_{M,N}(u, u') = \mathfrak{v}_{M_1,N_1}(u, u') = \mathfrak{v}_{M_3,N_3}(\tilde{H}_r(g_1)(u), \tilde{H}'_{r'}(g^1)(u')) =$$

$$= \ldots = \mathfrak{v}_{M_{2k+1},N_{2k+1}}(\tilde{H}_r(g_k)\ldots\tilde{H}_r(g_1)(u), \tilde{H}'_{r'}(g^k)\ldots\tilde{H}'_{r'}(g^1)(u')) =$$

$$= \mathfrak{v}_{g(M),g(N)}(\tilde{H}_r(g|M)(u), \tilde{H}'_{r'}(g|N)(u'))$$

as required. □

7. PROPOSITION. *Suppose that* $n=1$. *Let* S *and* S' *be mutually linked spheres in* R^1. *They clearly are of dimension* 0. *Let* q *be the centre of* S' *and let* g *be the reflection of* R^1 *about* q. *Let* $S_1=g(S)$ *and let* $\mathfrak{B}=\mathfrak{B}_{H,H',C,0,0}$ *be a theory of linking for the compacts in* R^1 *where* H *and* H' *are homology theories and* C *is an abelian group. Then*

$$\mathfrak{v}_{S,S'}(u, u') = -\mathfrak{v}_{S_1,S'}(\tilde{H}_0(g|S)(u), \tilde{H}'_0(g|S')(u'))$$

holds for each $u\in\tilde{H}_0(S)$ *and* $u'\in\tilde{H}'_0(S')$.

PROOF. Let $u\in\tilde{H}_0(S)$ and $u'\in\tilde{H}'_0(S')$. We then obviously have

$$\tilde{H}'_0(g|S')(u') = -u'. \tag{9}$$

Notice that S is a two point space, i.e. $S=\{q, t\}$ where q is the centre of S'. Moreover S_1 can be represented in the form $S_1=\{q, t_1\}$ where $t_1=g(t)$. Let $S_2=\{t, t_1\}$. Then the segment K' joining the two points of S' is disjoint to S_2. Let $Y=\{q, t, t_1\}$ and let i: $S\subset Y$, i_1: $S_1\subset Y$, i_2: $S_2\subset Y$ and i': $S'\subset K'$ be inclusions. Observe that Y and S' are disjoint sets.

Now a slight computation shows that there is a $u_2\in\tilde{H}_0(S_2)$ such that

$$\tilde{H}_0(i)(u)-\tilde{H}_0(i_1)\tilde{H}_0(g|S)(u)=\tilde{H}_0(i_2)(u_2)$$

and since $\tilde{H}_0(K')=0$, taking also (9) into account we get

$$0=\mathfrak{v}_{S_2,K'}(u_2, \tilde{H}_0'(i')(u'))=\mathfrak{v}_{S_2,S'}(u_2, u')=\mathfrak{v}_{Y,S'}(\tilde{H}_0(i_2)(u_2), u')=\mathfrak{v}_{Y,S'}(\tilde{H}_0(i)(u), u')-$$

$$-\mathfrak{v}_{Y,S'}(\tilde{H}_0(i_1)\tilde{H}_0(g|S)(u), u')=\mathfrak{v}_{S,S'}(u, u')-\mathfrak{v}_{S_1,S'}(\tilde{H}_0(g|S)(u), u')=\mathfrak{v}_{S,S'}(u, u')+$$

$$+\mathfrak{v}_{S_1,S'}(\tilde{H}_0(g|S)(u), \tilde{H}_0'(g|S')(u'))$$

and thus

$$\mathfrak{v}_{S,S'}(u, u')=-\mathfrak{v}_{S_1,S'}(\tilde{H}_0(g|S)(u), \tilde{H}_0'(g|S')(u')$$

indeed. □

8. DEFINITION. Non orientation preserving affine transformations of R^n are said to be *orientation reversing transformations.*

Observe that if g_1 is a reflection of R^n about a hyperplane R^{n-1} of it then each orientation reversing affine transformation g of R^n can be represented as the composition of g_1 and of an orientation preserving affine transformation g_2 of R^n, i.e. $g=g_2g_1$.

Now for any affine transformation g of R^n let

$$\varepsilon(g)=\begin{cases}+1, & \text{if } g \text{ is orientation preserving}\\ -1, & \text{if } g \text{ is orientation reversing.}\end{cases}$$

We then obviously have $\varepsilon(\mathrm{id}_{R^n})=+1$. Moreover $\varepsilon(g_2g_1)=\varepsilon(g_2)\varepsilon(g_1)$ holds for any affine transformations g_1 and g_2 of R^n.

9. PROPOSITION. *Let g be an arbitrary affine transformation of R^n. Let H, H', C, r, r' and $\mathfrak{B}=\mathfrak{B}_{H,H',C,r,r'}$ be the same as in Proposition 5. Let S and S' be mutually linked spheres in R^n of dimensions r and r' respectively. Then the relation*

$$\varepsilon(g)\mathfrak{v}_{S,S'}(u, u')=\mathfrak{v}_{g(S),g(S')}(\tilde{H}_r(g|S)(u), \tilde{H}_{r'}'(g|S')(u'))$$

holds for each $u\in\tilde{H}_r(S)$ and $u'\in\tilde{H}_{r'}'(S')$.

PROOF. Since each orientation preserving affine transformation of R^n is an isotopic deformation of R^n, it follows by Proposition 6 that the statement is true if $\varepsilon(g)=1$. According to Proposition 7 and Section 8 the statement is true for $\varepsilon(g)=-1$ and $n=1$.

Now let $n\geqq 2$ and suppose that $\varepsilon(g)=-1$. Since $r+r'=n-1$ we have either $r>0$ or $r'>0$. Without loss of generality we can suppose $r>0$. Let $z=r+1$ and $z'=r'+1$ and let R^z and $R'^{z'}$ be the planes supporting S and S' respectively. Let $R^1=R^z\cap R'^{z'}$. The centres of the spheres S and S' belong clearly to R^1. Let v be a

nonzero vector parallel to R^z and perpendicular to R^1 and let R^{n-1} be the hyperplane of R^n containing R^1 and being perpendicular to v. Let $R^r=R^z\cap R^{n-1}$.

We then obviously have $R'^{z'}\subset R^{n-1}$, $R^1\subset R^r$. Moreover the reflection g_1 of R^n about R^{n-1} takes R^z into itself and $g_1|R^z$ is a reflection of the supporting plane R^z of S about R^r where the centre of S belongs to R^r. Hence $g_1(S)=S$ and for any $u\in\tilde{H}_r(S)$ we obviously have

$$\tilde{H}_r(g_1|S)(u)=\tilde{H}_r\big((g_1|R^z)|S\big)(u)=-u.$$

On the other hand by $R'^{z'}\subset R^{n-1}$ we have $g_1|S'=\mathrm{id}_{S'}$ and thus for any $u'\in\tilde{H}'_{r'}(S')$ we find $\tilde{H}'_{r'}(g_1|S')(u')=u'$. Consequently

$$\varepsilon(g)\mathfrak{v}_{S,S'}(u,u')=\mathfrak{v}_{S,S'}(-u,u')=\mathfrak{v}_{S,S'}\big(\tilde{H}_r(g_1|S)(u),\tilde{H}'_{r'}(g_1|S')(u')\big). \tag{10}$$

Now let $g=g_2g_1$ where g_2 is an orientation preserving affine transformation of R^n (see Section 8) and thus an isotopic deformation of R^n. Hence by Proposition 6 we have

$$\mathfrak{v}_{S,S'}\big(\tilde{H}_r(g_1|S)(u),\tilde{H}'_{r'}(g_1|S')(u')\big)=$$

$$=\mathfrak{v}_{g_2(S),g_2(S')}\big(\tilde{H}_r(g_2|S)\tilde{H}_r(g_1|S)(u),\tilde{H}'_{r'}(g_2|S)\tilde{H}'_{r'}(g_1|S')(u')\big)$$

and thus by $g_2(S)=g(S)$, $g_2(S')=g(S')$ and (10) we get

$$\varepsilon(g)\mathfrak{v}_{S,S'}(u,u')=\mathfrak{v}_{g(S),g(S')}\big(\tilde{H}_r(g|S)(u),\tilde{H}'_{r'}(g|S')(u')\big)$$

as required. □

10. For any simplex σ in R^n let $\dot{\sigma}$ denote the boundary of σ in the supporting plane of σ.

DEFINITION. We shall say that simplexes σ and σ' of dimension z and z' contained in R^n are *cross cutting simplexes* if the following conditions are satisfied:

(i) $z+z'=n+1$,
(ii) $\dot{\sigma}\cap\dot{\sigma}'=\emptyset$,
(iii) $\dot{\sigma}\cap\sigma'\neq\emptyset$ and $\sigma\cap\dot{\sigma}'\neq\emptyset$.

Notice that if σ and σ' are cross cutting simplexes of dimension z and z' respectively lying in R^n and R^z and $R'^{z'}$ are the planes of R^n supporting σ and σ' respectively then

(a) $z>0$ and $z'>0$,
(b) $R^z\cap R'^{z'}$ is a line R^1,
(c) $\dot{\sigma}\cap\sigma'$ and $\sigma\cap\dot{\sigma}'$ are singletons.

11. PROPOSITION. *Let σ and σ' be cross cutting simplexes of dimension z and z' respectively lying in R^n. Let $r=z-1$ and $r'=z'-1$. Let S and S' be linked spheres in R^n of dimension r and r' respectively. Let H and H' be any homology theories and let C be an arbitrary abelian group. Let $\mathfrak{V}^1=\mathfrak{V}^1_{H,H',C,r,r'}$ and $\mathfrak{V}^2=\mathfrak{V}^2_{H,H',C,r,r'}$ be theories of linking for the compacts in R^n. Then $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ implies $\mathfrak{v}^1_{\dot{\sigma},\dot{\sigma}'}=\mathfrak{v}^2_{\dot{\sigma},\dot{\sigma}'}$.*

PROOF. Let p and p' be the only points of the singletons $\sigma\cap\dot{\sigma}'$ and $\dot{\sigma}\cap\sigma'$ respectively. Let R^z and $R'^{z'}$ be the planes supporting σ and σ' respectively. Let S_1 and S'_1 be the spheres of R^n with the radius $\varrho(\{p\},\{p'\})$ of dimension r and r' with the

centres p and p' and with the supporting planes R^z and $R'^{z'}$ respectively, where $\varrho(\{p\}, \{p'\})$ is the distance of the compact sets $\{p\}$ and $\{p'\}$. Let g be an affine transformation of R^n taking S into S_1 and S' into S_1'. There exists obviously such an affine transformation. Let Q and Q' be the balls in R^n with the same centre, radius and supporting plane as that of S_1 and S_1' respectively.

Then σ and Q are compact convex sets with the same supporting plane R^z and with the common interior point p with respect to R^z. Moreover S_1 is the boundary of Q and $\dot{\sigma}$ the boundary of σ (in R^z). Let

$$f=f_{p,Q,\sigma}\colon S_1\to\dot{\sigma} \text{ (see Section 2) and } W=W_f \text{ (see Section 1).}$$

Likewise we can define the homeomorphism

$$f'=f_{p',Q',\sigma'}\colon S_1'\to\dot{\sigma}'$$

and the compact set $W'=W_{f'}$. f and f' are deformations in W and W' respectively (see Definition 1) and by construction we have obviously $W\cap W'=\emptyset$.

Now let $v\in\tilde{H}_r(\dot{\sigma})$ and $v'\in\tilde{H}_{r'}'(\dot{\sigma}')$. Since all the mappings $f\colon S_1\to\dot{\sigma}$, $f'\colon S_1'\to\dot{\sigma}'$, $g|S\colon S\to S_1$ and $g|S'\colon S'\to S_1'$ are homeomorphisms it follows that $\tilde{H}_r(g|S)$, $\tilde{H}_{r'}'(g|S')$, $\tilde{H}_r(f)$ and $\tilde{H}_{r'}'(f')$ are isomorphisms. Consequently there exist a $u\in\tilde{H}_r(S)$ and a $u'\in\tilde{H}_{r'}'(S')$ such that $v=\tilde{H}_r(f)\tilde{H}_r(g|S)(u)$, $v'=\tilde{H}_{r'}'(f')\tilde{H}_{r'}'(g|S')(u')$. Hence assuming $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ taking also Propositions 5 and 9 into account we get

$$\mathfrak{v}^1_{\dot{\sigma},\dot{\sigma}'}(v,v')=\mathfrak{v}^1_{\dot{\sigma},\dot{\sigma}'}\big(\tilde{H}_r(f)\tilde{H}_r(g|S)(u),\tilde{H}_{r'}'(f')\tilde{H}_{r'}'(g|S')(u')\big)=$$

$$=\mathfrak{v}^1_{S_1,S_1'}\big(\tilde{H}_r(g|S)(u),\tilde{H}_{r'}'(g|S')(u')\big)=\varepsilon(g)\mathfrak{v}^1_{S,S'}(u,u')=\varepsilon(g)\mathfrak{v}^2_{S,S'}(u,u')=$$

$$=\mathfrak{v}^2_{S_1,S_1'}\big(\tilde{H}_r(g|S)(u),\tilde{H}_{r'}'(g|S')(u')\big)=$$

$$=\mathfrak{v}^2_{\dot{\sigma},\dot{\sigma}'}\big(\tilde{H}_r(f)\tilde{H}_r(g|S)(u),\tilde{H}_{r'}'(f')\tilde{H}_{r'}'(g|S')(u')\big)=\mathfrak{v}^2_{\dot{\sigma},\dot{\sigma}'}(v,v'). \quad \square$$

12. PROPOSITION. *Let σ and σ' be simplexes of dimension z and $n+1-z$ respectively contained in R^n such that either $\sigma\cap\dot{\sigma}'=\emptyset$ or $\dot{\sigma}\cap\sigma'=\emptyset$. Let $r=z-1$ and $r'=n-z$ and let $\mathfrak{B}=\mathfrak{B}_{H,H',C,r,r'}$ be a theory of linking for the compacts in R^n where H and H' are homology theories and C an abelian group. Then for any $v\in\tilde{H}_r(\dot{\sigma})$ and $v'\in\tilde{H}_{r'}'(\dot{\sigma}')$ we have $\mathfrak{v}_{\dot{\sigma},\dot{\sigma}'}(v,v')=0$.*

PROOF. Without loss of generality we can suppose that $\sigma\cap\dot{\sigma}'=\emptyset$. Let $v\in\tilde{H}_r(\dot{\sigma})$ and $v'\in\tilde{H}_{r'}'(\dot{\sigma}')$. Let $i\colon\dot{\sigma}\subset\sigma$ be the inclusion map and $i'\colon\dot{\sigma}'\subset\dot{\sigma}'$ the identity map. Since $\tilde{H}_r(\sigma)=0$ it follows $\tilde{H}_r(i)(v)=0$ and thus

$$\mathfrak{v}_{\dot{\sigma},\dot{\sigma}'}(v,v')=\mathfrak{v}_{\sigma,\dot{\sigma}'}\big(\tilde{H}_r(i)(v),\tilde{H}_{r'}'(i')(v')\big)=\mathfrak{v}_{\sigma,\dot{\sigma}'}(0,v')=0$$

as required. $\square$

13. *Let σ and σ' be simplexes of dimension z and z' lying in R^n such that $z+z'=n+1$ and $\dot{\sigma}\cap\dot{\sigma}'=\emptyset$. Let $r=z-1$ and $r'=z'-1$. Let $S, S', H, H', C, \mathfrak{B}^1$ and $\mathfrak{B}^2$ be the same as in Proposition* 11. *Then Propositions* 11 *and* 12 *show that $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ implies $\mathfrak{v}^1_{\dot{\sigma},\dot{\sigma}'}=\mathfrak{v}^2_{\dot{\sigma},\dot{\sigma}'}$.*

14. For any euclidean (simplicial) complex K lying in R^n let $|K|$ denote the space of K and for any rational integer q let K^q denote the q-skeleton of K i.e. the closed subcomplex of K consisting of all simplexes of K of dimension $\leqq q$.

PROPOSITION. *Let K be a euclidean complex lying in R^n and let H be a homology theory. Let t be a nonnegative integer and suppose that $\tilde{H}_t(|K|)=0$. The simplexes of dimension $t+1$ of K should be in turn $\sigma_1, \ldots, \sigma_m$ and for $z=1, \ldots, m$ let $i_z: \dot{\sigma}_z \subset |K^t|$ be the respective inclusion map. Then each $u \in \tilde{H}_t(|K^t|)$ can be represented in the form*

$$u = \sum_{z=1}^{m} \tilde{H}_t(i_z)(u_z)$$

where $u_z \in \tilde{H}_t(\dot{\sigma}_z)$.

PROOF. Let $u \in \tilde{H}_t(|K^t|)$. Let

$$\partial_{t+1}: H_{t+1}(|K^{t+1}|, |K^t|) \to H_t(|K^t|, |K^{t-1}|)$$

denote the boundary operator of the triple $(|K^{t+1}|, |K^t|, |K^{t-1}|)$. Explicitly ∂_{t+1} is the composition of the homomorphisms

$$H_{t+1}(|K^{t+1}|, |K^t|) \xrightarrow{\partial} H_t(|K^t|) \xrightarrow{k_*} H_t(|K^t|, |K^{t-1}|)$$

where ∂ is the boondary homomorphism of the pair $(|K^{t+1}|, |K^t|)$ and the homomorphism k_* is induced by the inclusion k. Now in the diagram

$$H_t(|K|) \xleftarrow{j_*} H_t(|K^t|) \xrightarrow{k_*} H_t(|K^t|, |K^{t-1}|)$$

where $j: |K^t| \subset |K|$ is also an inclusion map we have

$$\ker j_* = k_*^{-1}(\operatorname{im} \partial_{t+1}) = k_*^{-1}(k_*(\operatorname{im} \partial))$$

(see [6] p. 93) and since $\ker k_* = 0$ (see [6] p. 93) we get $\ker j_* = \operatorname{im} \partial$. However by assumption $\tilde{H}_t(|K|)=0$ and thus $u \in \ker j_*$. Consequently there exists a $v \in H_{t+1}(|K^{t+1}|, |K^t|)$ such that $u = \partial v$.

Now for $z=1, \ldots, m$ let $f_z: (\sigma_z, \dot{\sigma}_z) \subset (|K^{t+1}|, |K^t|)$ be the inclusion map and $\partial^z: H_{t+1}(\sigma_z, \dot{\sigma}_z) \to H_t(\dot{\sigma}_z)$ the respective boundary homomorphism. Then v can be represented in the form

$$v = \sum_{z=1}^{m} f_{z*}(v_z)$$

where $v_z \in H_{t+1}(\sigma_z, \dot{\sigma}_z)$ (see [6] p. 85). Also, for $z=1, \ldots, m$ the diagram

$$\begin{array}{ccc} H_{t+1}(\sigma_z, \dot{\sigma}_z) & \xrightarrow{f_{z*}} & H_{t+1}(|K^{t+1}|, |K^t|) \\ \partial^z \downarrow & & \downarrow \partial \\ H_t(\dot{\sigma}_z) & \xrightarrow{i_{z*}} & H_t(|K^t|) \end{array}$$

is commutative, hence

$$u = \partial v = \sum_{z=1}^{m} i_{z*} \partial^z(v_z)$$

where $\partial^z(v_z)\in\tilde{H}_t(\dot{\sigma}_z)$ and thus introducing the notation u_z for $\partial^z(v_z)$ we get

$$u=\sum_{z=1}^{m}\tilde{H}_t(i_z)(u_z)$$

as required. □

Observe that the case $m=0$ is not excluded. However if $u\neq 0$ then $v\neq 0$ and thus we get in this case $m\geqq 1$.

15. DEFINITION. We shall say that the euclidean complexes K and K' contained in R^n are *in relative quasigeneral position* if for any integers q, q' satisfying the condition $q+q'<n$ we have $|K^q|\cap|K'^{q'}|=\emptyset$.

Assuming that K and K' are in relative quasigeneral position observe that for any t-simplex σ of K and t'-simplex σ' of K' the inequlaity $t+t'\leqq n+1$ implies $\dot{\sigma}\cap\dot{\sigma}'=\emptyset$.

16. REMARK. Let K be a euclidean complex in R^n. Then for each $\varepsilon>0$ there exists a vector v of R^n of length $<\varepsilon$ such that K and $K'=K+v=\{\sigma+v;\ \sigma\in K\}$ are in relative quasigeneral position.

17. PROPOSITION. *Let K and K' be euclidean complexes in relative quasigeneral position contained in R^n. Let r and r' be nonnegative integers satisfying the condition $r+r'=n-1$. Let S, S', H, H', C, $\mathfrak{V}^1$ and $\mathfrak{V}^2$ be the same as in Proposition* 11 *and suppose that $\tilde{H}_r(|K|)=0=\tilde{H}'_{r'}(|K'|)$. Then $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ implies*

$$\mathfrak{v}^1_{|K^r|,|K'^{r'}|}=\mathfrak{v}^2_{|K^r|,|K'^{r'}|}.$$

PROOF. Let $u\in\tilde{H}_r(|K^r|)$ and $u'\in\tilde{H}'_{r'}(|K'^{r'}|)$. If $u=0$ or $u'=0$ then obviously

$$\mathfrak{v}^1_{|K^r|,|K'^{r'}|}(u,u')=0=\mathfrak{v}^2_{|K^r|,|K'^{r'}|}(u,u').$$

Thus we can suppose that $u\neq 0$ and $u'\neq 0$.

The $(r+1)$-simplexes of K and the $(r'+1)$-simplexes of K' should be $\sigma_1,\ldots,\sigma_m$ and $\sigma'_1,\ldots,\sigma'_{m'}$ respectively, and for $s=1,\ldots,m$ and $s'=1,\ldots,m'$ let $i_s\colon\dot{\sigma}_s\subset|K^r|$ and $i'_{s'}\colon\dot{\sigma}'_{s'}\subset|K'^{r'}|$ be the respective inclusion map. Then by Proposition 14, $m>0$, $m'>0$ and we have the representations

$$u=\sum_{s=1}^{m}\tilde{H}_r(i_s)(u_s),\quad u'=\sum_{s'=1}^{m'}\tilde{H}'_{r'}(i'_{s'})(u'_{s'})$$

where $u_s\in\tilde{H}_r(\dot{\sigma}_s)$ and $u'_{s'}\in\tilde{H}'_{r'}(\dot{\sigma}'_{s'})$. Taking also Section 13 into account, $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ clearly implies

$$\mathfrak{v}^1_{|K^r|,|K'^{r'}|}(u,u')=\sum_{s=1}^{m}\sum_{s'=1}^{m'}\mathfrak{v}^1_{|K^r|,|K'^{r'}|}\big(\tilde{H}_r(i_s)(u_s),\tilde{H}'_{r'}(i'_{s'})(u'_{s'})\big)=$$

$$=\sum_{s=1}^{m}\sum_{s'=1}^{m'}\mathfrak{v}^1_{\dot{\sigma}_s,\dot{\sigma}'_{s'}}(u_s,u'_{s'})=\sum_{s=1}^{m}\sum_{s'=1}^{m'}\mathfrak{v}^2_{\dot{\sigma}_s,\dot{\sigma}'_{s'}}(u_s,u'_{s'})=$$

$$=\sum_{s=1}^{m}\sum_{s'=1}^{m'}\mathfrak{v}^2_{|K^r|,|K'^{r'}|}\big(\tilde{H}_r(i_s)(u_s),\tilde{H}'_{r'}(i'_{s'})(u'_{s'})\big)=\mathfrak{v}^2_{|K^r|,|K'^{r'}|}(u,u')$$

as required. □

18. THEOREM. (Uniqueness theorem). *Let H and H' be homology theories. Let C be an arbitrary abelian group and let r and r' be nonnegative integers satisfying $r+r'=n-1$. Let S and S' be linked spheres of dimension r and r' respectively contained in R^n and let $\mathfrak{V}^1=\mathfrak{V}^1_{H,H',C,r,r'}$ and $\mathfrak{V}^2=\mathfrak{V}^2_{H,H',C,r,r'}$ be theories of linking for the compacts in R^n. Then $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ implies $\mathfrak{V}^1=\mathfrak{V}^2$.*

PROOF. Let N, N' be an ordered pair of disjoint compact subsets of R^n and let $u\in\tilde{H}_r(N)$ and $u'\in\tilde{H}'_{r'}(N')$. If $u=0$ or $u'=0$ then evidently $\mathfrak{v}^1_{N,N'}(u,u')=0=\mathfrak{v}^2_{N,N'}(u,u')$.

Assume that $u\neq 0$ and $u'\neq 0$. Then $N\neq\emptyset$ and $N'\neq\emptyset$.

Let σ_0 be an n-euclidean simplex in R^n containing the set $N\cup N'$ in its interior. Let $\varepsilon=\varrho(N\cup N', \dot{\sigma}_0)$. Then $\varepsilon>0$ and for any vector v of R^n of length $<\varepsilon$ $N\cup N'$ is contained in the interior of σ_0+v. Let $\delta=\varrho(N,N')$. We have $\delta>0$. Let M be the euclidean complex consisting of σ_0 and all the faces of σ_0. Then evidently $|M|=\sigma_0$. Let K be an iterated barycentric subdivision of M such that mesh $K<\frac{\delta}{2}$ and let v be a vector of R^n of length $<\varepsilon$ such that K and $K'=K+v=\{\sigma+v;\sigma\in K\}$ are in relative quasigeneral position. Then evidently mesh $K'=$mesh $K<\frac{\delta}{2}$, $|K|=\sigma_0\supset N$, $|K'|=\sigma_0+v\supset N'$ and since $\tilde{H}_r(\sigma_0)=0=\tilde{H}'_{r'}(\sigma_0)=\tilde{H}'_{r'}(\sigma_0+v)$ it follows $\tilde{H}_r(|K|)=0=\tilde{H}'_{r'}(|K'|)$.

Let K_1 be the subcomplex of K consisting of all simplexes of K meeting N and all the faces of such simplexes. Likewise let K'_1 be the subcomplex of K' consisting of all simplexes of K' meeting N' and all the faces of such simplexes. Then evidently

$$N\subset|K_1|\subset|K|,\quad N'\subset|K'_1|\subset|K'|,\quad |K_1^r|\subset|K_1|,\quad |K_1^r|\subset|K^r|,$$

$$|K_1'^{r'}|\subset|K'_1|,\quad |K_1'^{r'}|\subset|K'^{r'}|\quad\text{and}\quad |K_1|\cap|K'_1|=\emptyset.$$

Let $i_1\colon N\subset|K_1|$, $i_2\colon |K_1^r|\subset|K_1|$, $i_3\colon |K_1^r|\subset|K^r|$, $i'_1\colon N'\subset|K'_1|$, $i'_2\colon |K_1'^{r'}|\subset|K'_1|$ and $i'_3\colon |K_1'^{r'}|\subset|K'^{r'}|$ be inclusion maps. Since $i_{2*}\colon H_r(|K_1^r|)\to H_r(|K_1|)$ is an onto map (see [6] p. 93) it follows that $\tilde{H}_r(i_2)$ is an epimorphism. Likewise $\tilde{H}'_{r'}(i'_2)$ is an epimorphism as well. Consequently there exist a $u_1\in\tilde{H}_r(|K_1^r|)$ and a $u'_1\in\tilde{H}'_{r'}(|K_1'^{r'}|)$ such that

$$\tilde{H}_r(i_2)(u_1)=\tilde{H}_r(i_1)(u)\quad\text{and}\quad\tilde{H}'_{r'}(i'_2)(u'_1)=\tilde{H}'_{r'}(i'_1)(u').$$

Thus assuming $\mathfrak{v}^1_{S,S'}=\mathfrak{v}^2_{S,S'}$ Proposition 17 shows that

$$\mathfrak{v}^1_{N,N'}(u,u')=\mathfrak{v}^1_{|K_1|,|K'_1|}(\tilde{H}_r(i_1)(u),\tilde{H}'_{r'}(i'_1)(u'))=\mathfrak{v}^1_{|K_1|,|K'_1|}(\tilde{H}_r(i_2)(u_1),\tilde{H}'_{r'}(i'_2)(u'_1))=$$

$$=\mathfrak{v}^1_{|K_1^r|,|K_1'^{r'}|}(u_1,u'_1)=\mathfrak{v}^1_{|K^r|,|K'^{r'}|}(\tilde{H}_r(i_3)(u_1),\tilde{H}'_{r'}(i'_3)(u'_1))=$$

$$=\mathfrak{v}^2_{|K^r|,|K'^{r'}|}(\tilde{H}_r(i_3)(u_1),\tilde{H}'_{r'}(i'_3)(u'_1))=\mathfrak{v}^2_{|K_1^r|,|K_1'^{r'}|}(u_1,u'_1)=$$

$$=\mathfrak{v}^2_{|K_1|,|K'_1|}(\tilde{H}_r(i_2)(u_1),\tilde{H}'_{r'}(i'_2)(u'_1))=$$

$$=\mathfrak{v}^2_{|K_1|,|K'_1|}(\tilde{H}_r(i_1)(u),\tilde{H}'_{r'}(i'_1)(u'))=\mathfrak{v}^2_{N,N'}(u,u').$$

The proof of the uniqueness theorem is complete. □

Now we are going to prove the existence of the required theory of linking $\mathfrak{B}$.

The proof proceeds in the following way.

(A) First startig from the given map $\mathfrak{v}_0\colon \tilde{H}_r(S)\times\tilde{H}'_{r'}(S')\to C$ we construct a bihomorphism defined on the Cartesian product of the coefficient groups of the respective homology theories.

(B) In the second part of the proof for each pair of disjoint compact sets in R^n and for each pair of elements of their respective homology groups we define the linking coefficient of these elements. This will be done by simplicial approximation. However by technical reasons we consider nerves of open coverings of the compacts in R^n instead of simplicial neighbourhoods.

Thus we get a theory of linking $\mathfrak{B}$.

(C) In the third part we show that the value of $\mathfrak{B}$ on the given standard spheres coincides with $\mathfrak{v}_0$. This completes the proof.

Now we are going to prove the existence in details.

19. Suppose we are given two homology theories H and H' an abelian group C, nonnegative integers r, r' satisfying $r+r'=n-1$, linked spheres S and S' of dimension r and r' respectively contained in R^n, and a bihomorphism

$$\mathfrak{v}_0 = \mathfrak{v}_0\colon \tilde{H}_r(S)\times\tilde{H}'_{r'}(S') \to C.$$

We shall keep fixed them in the sequel.

First we are going to construct a bihomorphism $\mathfrak{v}_1\colon H_0(P_0)\times H'_0(P_0)\to C$, where $P_0=\{\emptyset\}$ is a fixed one point space. Also, we denote $H_0(P_0)$ by G and $H'_0(P_0)$ by G'. G and G' are the coefficient groups of the homology theories H and H' respectively.

20. Let $z=r+1$ and $z'=r'+1$. Let R^z and $R'^{z'}$ be the supporting planes and let p and p' be the centres of S and S' respectively. Let Q and Q' be the balls in R^n admitting the same centre, radius and supporting plane as S and S' respectively.

Let σ and σ' be cross cutting simplexes of dimension z and z' respectively contained in R^n such that

(a) The supporting plane of σ is R^z and that of σ' is $R'^{z'}$,

(b) $\{p\}=\sigma\cap\dot{\sigma}'$, $\{p'\}=\dot{\sigma}\cap\sigma'$,

(c) p is the interior point of an r'-face σ'_1 of σ'.

Such simplexes σ and σ' obviously exist.

Observe that the set of all vertices of σ and σ'_1 is in general position (see [1] p. 204.)

Let (see Section 2)

$$f_1 = f_{p,\sigma,Q}\colon \dot{\sigma} \to S \quad \text{and} \quad f'_1 = f_{p',\sigma',Q'}\colon \dot{\sigma}' \to S'.$$

Observe that f_1 and f'_1 are deformations in W_{f_1} and $W_{f'_1}$ respectively where W_{f_1} and $W_{f'_1}$ are disjoint compact subsets of R^n.

Now for $u\in\tilde{H}_r(\dot{\sigma})$ and $u'\in\tilde{H}'_{r'}(\dot{\sigma}')$ let

$$\mathfrak{v}'(u, u') = \mathfrak{v}_0(\tilde{H}_r(f_1)(u), \tilde{H}'_{r'}(f'_1)(u')).$$

The mapping

$$\mathfrak{v}'\colon \tilde{H}_r(\dot{\sigma})\times\tilde{H}'_{r'}(\dot{\sigma}') \to C$$

is obviously a bihomomorphism.

21. Let τ' be the union of all r'-faces of σ' excepting σ_1'. τ' is obviously a non-empty homologically trivial compact set. Moreover, the pair $(\dot\sigma', \tau')$ can be triangulated. Hence the homomorphism

$$\tilde m_*\colon \tilde H_{r'}'(\dot\sigma') \to H_{r'}'(\dot\sigma', \tau')$$

induced by the inclusion $m\colon (\dot\sigma', \emptyset) \subset (\dot\sigma', \tau')$ is an isomorphism.

On the other hand we have $\dot\sigma' = \sigma_1' \cup \tau'$ and $\dot\sigma_1' = \sigma_1' \cap \tau'$. Thus the inclusion $k\colon (\sigma_1', \dot\sigma_1') \subset (\dot\sigma', \tau')$ induces an isomorphism

$$k_*\colon H_{r'}'(\sigma_1', \dot\sigma_1') \to H_{r'}'(\dot\sigma', \tau').$$

Finally, since σ is homologically trivial and the pair $(\sigma, \dot\sigma)$ can be triangulated and $\dot\sigma \neq \emptyset$, it follows that the boundary homomorphism

$$\partial = \partial_{z,\sigma,\dot\sigma}\colon H_z(\sigma, \dot\sigma) \to \tilde H_r(\dot\sigma)$$

of $(\sigma, \dot\sigma)$ — where $z = r+1$ — is an isomorphism as well. Likewise the homomorphism

$$\partial' = \partial_{z',\sigma',\dot\sigma'}'\colon H_{z'}'(\sigma', \dot\sigma') \to \tilde H_{r'}'(\dot\sigma')$$

(where $z' = r'+1$) is an isomorphism, too.

Now for $v \in H_z(\sigma, \dot\sigma)$ and $v' \in H_{r'}'(\sigma_1', \dot\sigma_1')$ let

$$\mathfrak{v}^*(v, v') = \mathfrak{v}'\big(\partial(v), \tilde m_*^{-1} k_*(v')\big).$$

The mapping

$$\mathfrak{v}^*\colon H_z(\sigma, \dot\sigma) \times H_{r'}'(\sigma_1', \dot\sigma_1') \to C$$

is clearly a bihomomorphism.

22. Select an orientation ξ of R^n (see [2] p. 2). We shall keep it fixed in the sequel. Thus we can regard R^n as an oriented n-euclidean space.

Now let us take an ordering $A^0 < \ldots < A^z$ of the vertices of σ and an ordering $B^0 < \ldots < B^{r'}$ of the vertices of σ_1'. Thus we get ordered simplexes s and s_1'. The spaces $|s|$ and $|s_1'|$ of s and s_1' are the simplexes σ and σ_1' respectively. Let t^z and $t_1'^{r'}$ be the orientations of σ and σ_1' taking the value $+1$ on the sequence $|A^0, \ldots, A^z|$ and $|B^0, \ldots, B^{r'}|$ respectively (see [2] p. 4).

Let the isomorphisms $\varphi\colon G \to H_z(\sigma, \dot\sigma)$ and $\varphi'\colon G' \to H_{r'}'(\sigma_1', \dot\sigma_1')$ be defined by the relations

$$\varphi(g) = gs \quad \text{and} \quad \varphi'(g') = g's_1'$$

(see [6] p. 80) where g runs over G and g' over G'. $\varphi(g)$ and $\varphi'(g')$ depend only on the orientations t^z and $t_1'^{r'}$ of σ and σ_1' respectively.

Now define the mapping $\mathfrak{v}_1\colon G \times G' \to C$ by the relation

$$\mathfrak{v}_1(g, g') = t^z \times t_1'^{r'}\, \mathfrak{v}^*\big(\varphi(g), \varphi'(g')\big)$$

(see also [2] p. 10). $\mathfrak{v}_1$ is obviously a bihomomorphism and this mapping does not depend on the special choice of the orientations t^z and $t_1'^{r'}$ of the simplexes σ and σ_1'.

In the remainder we shall keep the notations of this and the previous three sections.

23. DEFINITION. Let

$$x^q = \sum_{i=1}^{y} g_i t_i^q \quad \text{and} \quad x'^{q'} = \sum_{j=1}^{y'} g'_j t'^{q'}_j$$

be two chains in R^n in relative general position (see [3] p. 73) where $q+q'=n$ and for $i=1, \ldots, y$ and $j=1, \ldots, y'$ the relations $g_i \in G$ and $g'_j \in G'$ hold. Then the *intersection number* $x^q \times x'^{q'}$ of the chains x^q and $x'^{q'}$ is by definition

$$x^q \times x'^{q'} = \sum_{i=1}^{y} \sum_{j=1}^{y'} t_i^q \times t'^{q'}_j \mathfrak{v}_1(g_i, g'_j).$$

Chapter XV of the book [3] regards the case $G=G'$ and $\mathfrak{v}_1(g, g')=gg'$ for $g, g' \in G$ where G is a ring with unity. However, the main results of this chapter remain true also after our modifications.

In particular, we can define the linking number of two nonintersecting cycles z_1^q and $z_2^{q'-1}$ in R^n with $q+q'=n$ and with z_1^q a normal cycle if $q=0$ so as it is done in [3] p. 80 and formulas 1.51 of [3] p. 82 remain true.

Likewise we can define the linking number for proper cycles in the same manner as in [3] p. 86 and Theorem 2.11 of [3] p. 86 remains true as well.

Thus for any two disjoint compact sets N, N' of R^n we can define the linking number $\mathfrak{v}(\alpha, \alpha')$ where α belongs to the reduced r-th homology group $\tilde{\Delta}^r(N, G)$ of N over G and α' to the reduced r'-th homology group $\tilde{\Delta}^{r'}(N', G')$ of N' over G'.

Actually $\mathfrak{v}(\alpha, \alpha')$ is defined by the relation

$$\mathfrak{v}(\alpha, \alpha') = \mathfrak{v}(\mathfrak{z}^r, \mathfrak{z}'^{r'})$$

where $\mathfrak{z}^r \in \alpha$ and $\mathfrak{z}'^{r'} \in \alpha'$. This linking number $\mathfrak{v}(\alpha, \alpha')$ is by 2.11 of [3] p. 86 independent of the special choice of the proper cycles $\mathfrak{z}^r$ and $\mathfrak{z}'^{r'}$. Moreover for

$$\alpha_1, \alpha_2 \in \tilde{\Delta}^r(N, G) \quad \text{and} \quad \alpha'_1, \alpha'_2 \in \tilde{\Delta}^{r'}(N', G')$$

the equalities 1.51 of [3] p. 82 show that

$$\mathfrak{v}(\alpha_1+\alpha_2, \alpha'_1+\alpha'_2) = \mathfrak{v}(\alpha_1, \alpha'_1)+\mathfrak{v}(\alpha_1, \alpha'_2)+\mathfrak{v}(\alpha_2, \alpha'_1)+\mathfrak{v}(\alpha_2, \alpha'_2).$$

24. Given a compact metric space M, an abelian group Γ and a nonnegative integer q introduce a homomorphism

$$\eta_{q,M,\Gamma}\colon \tilde{\Delta}^q(M, \Gamma) \to \tilde{H}_q^1(M, \Gamma)$$

of the q-th reduced homology group of M over the group Γ into the q-dimensional reduced Čech homology group $\tilde{H}_q^1(M, \Gamma)$ over the same group as follows.

Let $\omega=\{U_1, \ldots, U_k\}$ be a finite open covering of M. Let $\omega'=\{F_1, \ldots, F_k\}$ be a closed covering of M such that for $i=1, \ldots, k$ $F_i \subset U_i$. Such a closed covering obviously exists. Let δ be the Lebesgue number of ω' (see [1] p. 35). Let $\psi\colon M \to \omega$ be a mapping such that $\psi(x)=U_i$ implies $x \in F_i$. Hence ψ is a simplicial mapping of the complex $K(M, \delta)$ (see [2] p. 148) into the nerve N_ω of ω.

Now let $\alpha \in \tilde{\Delta}^q(M, \Gamma)$ and let $\mathfrak{z}=(z_1^q, \ldots, z_y^q, \ldots)$ be a proper cycle belonging to α. Then there exists a natural number y_0 such that for $y \geqq y_0$ every z_y^q is a δ-cycle and all these δ-cycles are δ-homologous to each other in M. Thus for $y \geqq y_0$ the cycles $\psi(z_y^q)$ of N_ω are well defined and they all belong to the same homology class of

N_ω i.e. to the same element of $\tilde{H}_q(N_\omega, \Gamma)$. This element of $\tilde{H}_q(N_\omega, \Gamma)$ is clearly independent of the special choice of ω', ψ and $\mathfrak{z}$, it dependes only on α. We shall denote it by $\eta_{q,M,\Gamma,\omega}(\alpha)$. Thus we obtain a mapping

$$\eta_{q,M,\Gamma,\omega}\colon \tilde{\Delta}^q(M,\Gamma) \to \tilde{H}_q(N_\omega, \Gamma)$$

which is clearly a homomorphism and for any refinement ω_1 of ω we obviously have

$$\eta_{q,M,\Gamma,\omega} = \pi_\omega^{\omega_1}\eta_{q,M,\Gamma,\omega_1}$$

where

$$\pi_\omega^{\omega_1}\colon \tilde{H}_q(N_{\omega_1}, \Gamma) \to \tilde{H}_q(N_\omega, \Gamma)$$

is the homomorphism induced by any projection $N_{\omega_1} \to N_\omega$ (see [6] p. 234 and p. 237). Consequently the system $\{\eta_{q,M,\Gamma,\omega};\ \omega \in \mathrm{Cov}^f M\}$ — where $\mathrm{Cov}^f M$ is the family of all finite open coverings of M — forms a map of the group $\tilde{\Delta}^q(M,\Gamma)$ into the q-th reduced Čech homology system $\{\tilde{H}_q(N_\omega, \Gamma); \pi_\omega^{\omega_1}\}$ of M and the inverse limit of this map is the required homomorphism

$$\eta_{q,M,\Gamma}\colon \tilde{\Delta}^q(M,\Gamma) \to \tilde{H}_q^1(M,\Gamma).$$

It is easy to see that this homomorphism is an isomorphic mapping of $\tilde{\Delta}^q(M,\Gamma)$ onto $\tilde{H}_q^1(M,\Gamma)$.

25. Let M and N be compact metric spaces and let $f\colon M \to N$ be a continuous mapping. Let q be a nonnegative integer and Γ an abelian group. Then f induces a homomorphism

$$\tilde{f}^*\colon \tilde{\Delta}^q(M,\Gamma) \to \tilde{\Delta}^q(N,\Gamma)$$

(see [2] p. 164) and also a homomorphism

$$\tilde{f}_*\colon \tilde{H}_q^1(M,\Gamma) \to \tilde{H}_q^1(N,\Gamma)$$

(see [6] p. 239) and the commutativity evidently holds in the diagram

$$\begin{array}{ccc} \tilde{\Delta}^q(M,\Gamma) & \xrightarrow{\tilde{f}^*} & \tilde{\Delta}^q(N,\Gamma) \\ {\scriptstyle \eta_{q,M,\Gamma}}\downarrow & & \downarrow{\scriptstyle \eta_{q,N,\Gamma}} \\ \tilde{H}_q^1(M,\Gamma) & \xrightarrow{\tilde{f}_*} & \tilde{H}_q^1(N,\Gamma). \end{array}$$

Observe that if M is a closed metric subspace of N and f is the inclusion map of M into N then each proper cycle of M may be considered as a proper cycle of N. In this particular case for any $\alpha \in \tilde{\Delta}^q(M,\Gamma)$ the image $\tilde{f}^*(\alpha) \in \tilde{\Delta}^q(N,\Gamma)$ of α can be obtained as the homology class of a proper cycle $\mathfrak{z}^q$ in $\tilde{\Delta}^q(N,\Gamma)$ where $\mathfrak{z}^q \in \alpha$.

26. PROPOSITION. *Let N, N' be an ordered pair of disjoint compact subsets of R^n. Let M and M' be closed subsets of N and N' respectively. Let $f\colon M \subset N$ and $f'\colon M' \subset N'$ be the inclusion maps. Then for the induced homomorphisms*

$$\tilde{f}^*\colon \tilde{\Delta}^r(M,G) \to \tilde{\Delta}^r(N,G), \quad \tilde{f}'^*\colon \tilde{\Delta}^{r'}(M',G') \to \tilde{\Delta}^{r'}(N',G')$$

and for any $\alpha \in \tilde{\Delta}^r(M,G)$ and $\alpha' \in \tilde{\Delta}^{r'}(M',G')$ we have

$$\mathfrak{v}(\alpha,\alpha') = \mathfrak{v}\big(\tilde{f}^*(\alpha), \tilde{f}'^*(\alpha')\big).$$

PROOF. Let $\mathfrak{z}^r \in \alpha$ and $\mathfrak{z}'^{r'} \in \alpha'$. Then according to the last remark of the preceding section we have $\mathfrak{z}^r \in \tilde{f}^*(\alpha)$, $\mathfrak{z}'^{r'} \in \tilde{f}'^*(\alpha')$ and thus we get

$$\mathfrak{v}(\tilde{f}^*(\alpha), \tilde{f}'^*(\alpha')) = \mathfrak{v}(\mathfrak{z}^r, \mathfrak{z}'^{r'}) = \mathfrak{v}(\alpha, \alpha')$$

as required. □

27. For any two disjoint compact subsets M and M' of R^n and for $\beta \in \tilde{H}_r^1(M, G)$ and $\beta' \in \tilde{H}_{r'}^1(M', G')$ we can define the linking number $\mathfrak{v}(\beta, \beta')$ by the relation

$$\mathfrak{v}(\beta, \beta') = \mathfrak{v}(\eta_{r,M,G}^{-1}(\beta), \eta_{r',M',G'}^{-1}(\beta')).$$

The mapping

$$\mathfrak{v}: \tilde{H}_r^1(M, G) \times \tilde{H}_{r'}^1(M', G') \to C$$

thus obtained is clearly a bihomomorphism.

Moreover, if N and N' are disjoint compact sets in R^n such that $M \subset N$ and $M' \subset N'$ and $f: M \subset N$ and $f': M' \subset N'$ are inclusions then Proposition 26 and the commutativity of the diagram in Section 25 show that

$$\mathfrak{v}(\beta, \beta') = \mathfrak{v}(\tilde{f}_*(\beta), \tilde{f}'_*(\beta'))$$

holds for every $\beta \in \tilde{H}_r^1(M, G)$ and $\beta' \in \tilde{H}_{r'}^1(M', G')$.

28. Consider the fixed one point space $P_0 = \{\emptyset\}$. This space has only one open covering $\omega_0 = \{P_0\}$. Denote the nerve of this covering by N_0. Let t_0^* be the orientation of the only 0-simplex $\{P_0\}$ of N_0 taking the value $+1$ on the sequence $|P_0|$.

Let Γ be an abelian group and $\gamma \in \Gamma$. Then γt_0^* is a cycle of N_0 over Γ and it appears as the only element of its homology class $\overline{\gamma t_0^*}$ in $H_0(N_0, \Gamma)$, i.e. $\overline{\gamma t_0^*} = \{\gamma t_0^*\}$.

Denote by $\pi_{\omega_0,\Gamma}$ the projection

$$\pi_{\omega_0,\Gamma}: H_0^1(P_0, \Gamma) \to H_0(N_0, \Gamma)$$

where $H_0^1(P_0, \Gamma)$ is the 0-th Čech homology group of P_0 over Γ. $\pi_{\omega_0,\Gamma}$ is clearly an isomorphism.

Denote by γ' the element $\pi_{\omega_0,\Gamma}^{-1}(\overline{\gamma t_0^*})$ of $H_0^1(P_0, \Gamma)$. Hence

$$\pi_{\omega_0,\Gamma}(\gamma^*) = \{\gamma t_0^*\}.$$

29. Consider now the groups $G = H_0(P_0)$ and $G' = H_0'(P_0)$ (see Section 19). For $g \in G$ and $g' \in G'$ let

$$h_0(g) = g^* \in H_0^1(P_0, G)$$

(see also Section 28) and

$$h_0'(g') = g'^* \in H_0^1(P_0, G').$$

Thus we obtain the isomorphisms

$$h_0: H_0(P_0) \to H_0^1(P_0, G) \quad \text{and} \quad h_0': H_0'(P_0) \to H_0'(P_0, G').$$

These isomorphisms can be extended in just one way to homomorphisms

$$h: H \to H^1(., G) \quad \text{and} \quad h': H' \to H^1(., G')$$

where $H^1(.,G)$ and $H^1(.,G')$ are the Čech homology theories over the groups G and G' respectively defined on the category of compact pairs. These homomorphisms h and h' are isomorphisms on the category of triangulable pairs. Moreover, for any compact T_2-space X we have

$$h(0,X,\emptyset)(\tilde{H}_0(X)) \subset \tilde{H}_0^1(X,G) \quad \text{and} \quad h'(0,X,\emptyset)(\tilde{H}_0'(X)) \subset \tilde{H}_0^1(X,G').$$

30. Now let N and N' be disjoint compact subsets of R^n and for $u \in \tilde{H}_r(N)$ and $u' \in \tilde{H}_{r'}'(N')$ define the linking number $\mathfrak{v}_{N,N'}(u,u')$ by the relation

$$\mathfrak{v}_{N,N'}(u,u') = \mathfrak{v}(h(r,N,\emptyset)(u), h'(r',N',\emptyset)(u')).$$

The mapping

$$\mathfrak{v}_{N,N'}\colon \tilde{H}_r(N) \times \tilde{H}_{r'}'(N') \to C$$

thus obtained is clearly a bihomomorphism and according to Section 27 and by the commutativity of the first diagram of [6] p. 289 the correspondence $(N,N') \to \mathfrak{v}_{N,N'}$ where (N,N') runs over all ordered pairs of disjoint compact subsets of R^n is a theory of linking for the compacts in R^n.

Thus we need only to show that $\mathfrak{v}_{S,S'} = \mathfrak{v}_0$.

31. Observe that the mappings f_1 and f_1' are deformations in W_{f_1} and $W_{f_1'}$ respectively where W_{f_1} and $W_{f_1'}$ are disjoint compact subsets of R^n (see Section 20). Thus according to Proposition 5 the relation

$$\mathfrak{v}_{\dot{\sigma},\dot{\sigma}'}(u,u') = \mathfrak{v}_{S,S'}(\tilde{H}_r(f_1)(u), \tilde{H}_{r'}'(f_1')(u')) \tag{11}$$

holds for every $u \in \tilde{H}_r(\dot{\sigma})$ and $u' \in \tilde{H}_{r'}'(\dot{\sigma}')$.

However $f_1\colon \dot{\sigma} \to S$ and $f_1'\colon \dot{\sigma}' \to S'$ are homeomorphisms, consequently the mappings

$$\tilde{H}_r(f_1)\colon \tilde{H}_r(\dot{\sigma}) \to \tilde{H}_r(S) \quad \text{and} \quad \tilde{H}_{r'}'(f_1')\colon \tilde{H}_{r'}'(\dot{\sigma}') \to \tilde{H}_{r'}'(S')$$

are isomorphisms and thus comparing (11) with the definition of $\mathfrak{v}'$ in Section 20 we get that $\mathfrak{v}_0 = \mathfrak{a}_{S,S'}$ is an immediate consequence of $\mathfrak{v}' = \mathfrak{v}_{\dot{\sigma},\dot{\sigma}'}$.

Thus we only need to prove this latter relation.

32. First we introduce some notations.

Let $\omega = \{U_1, \ldots, U_t\}$ be a finite open covering of the space X and let $Y \subset X$. Denote then by $N_{Y,\omega}$ the subcomplex of the nerve N_ω of ω consisting of all simplexes $\{U_{i_1}, \ldots, U_{i_m}\}$ of N_ω with $U_{i_1} \cap \ldots \cap U_{i_m} \cap Y \neq \emptyset$.

Now let K be a euclidean complex contained in R^n and let L be a closed subcomplex of K. For each $A \in K^0$ — where K^0 is the 0-skeleton of K (see Section 14) — consider the open star $\mathrm{st}_K(A)$ of A in K (see [6] p. 57). Let

$$\omega(K) = \{\mathrm{st}_K(A);\ A \in K^0\}.$$

$\omega(K)$ is a finite open covering of the polyhedron $|K|$. Let us take the pair $(N_{\omega(K)}, N_{|L|,\omega(K)})$ where $N_{\omega(K)}$ is the nerve of $\omega(K)$. Then the correspondence

$$A \mapsto \mathrm{st}_K(A) \quad (A \in K^0)$$

determines a simplicial isomorphism

$$v(K, L)\colon (K, L) \to (N_{\omega(K)}, N_{|L|, \omega(K)})$$

of the pair (K, L) onto $(N_{\omega(K)}, N_{|L|, \omega(K)})$.

Let Γ be an abelian group and t a nonnegative integer. Denote by $v(K, L, \Gamma, t)$ the isomorphism

$$v(K, L, \Gamma, t)\colon H_t(K, L, \Gamma) \to H_t(N_{\omega(K)}, N_{|L|, \omega(K)}, \Gamma)$$

induced by $v(K, L)$ and by $\pi(K, L, \Gamma, t)$ the respective projection of the Čech homology group $H_t^1(|K|, |L|, \Gamma)$ into $H_t(N_{\omega(K)}, N_{|L|, \omega(K)}, \Gamma)$ i.e.

$$\pi(K, L, \Gamma, t)\colon H_t^1(|K|, |L|, \Gamma) \to H_t(N_{\omega(K)}, N_{|L|, \omega(K)}, \Gamma)$$

This projection is an isomorphism as well.

Let σ^y be a euclidean y-simplex contained in R^n where y is a nonnegative integer less than $n+1$. Let $K(\sigma^y)$ be the euclidean complex consisting of all faces of σ^y included σ^y itself and let $L(\sigma^y)=K(\sigma^y)\setminus\{\sigma^y\}$. Then instead of $N_{\omega(K(\sigma^y))}$ we write N_{σ^y} and instead of $N_{|L(\sigma^y)|, \omega(K(\sigma^y))}$ we write $N^{\cdot}_{\sigma^y}$. Moreover, instead of $v(K(\sigma^y), L(\sigma^y))$, $v(K(\sigma^y), L(\sigma^y), \Gamma, y)$, $v(L(\sigma^y), \emptyset, \Gamma, y-1)$, $\pi(K(\sigma^y), L(\sigma^y), \Gamma, y)$ and $\pi(L(\sigma^y), \emptyset, \Gamma\ y-1)$ we write v_{σ^y}, $v_{\sigma^y, \Gamma}$, $v^{\cdot}_{\sigma^y, \Gamma}$, $\pi_{\sigma^y, \Gamma}$ and $\pi^{\cdot}_{\sigma^y, \Gamma}$, respectively.

33. Let K be either a euclidean complex contained in R^n or the nerve of an open covering of a space. Let L be a closed subcomplex of K and Γ an abelian group. Let x^t be a chain of K over Γ such that its boundary Δx^t (cf. [2] p. 25) should be a chain of L, i.e. x^t is a t-cycle of K relative to L. Denote then by $\overline{x^t}$ the homology class of x^t. Hence $\overline{x^t} \in H_t(K, L, \Gamma)$.

If $\overline{x^t}$ has only one element, namely the relative cycle x^t then instead of $\overline{x^t}$ we also use the symbol $\{x^t\}$.

Now we can state the following lemma.

34. LEMMA. *Let q be a nonnegative integer less than $n+1$ and let σ^q be a euclidean q-simplex contained in R^n. Let $D^0 < \ldots < D^q$ be an ordering of the vertices of σ^q. Thus we get an ordered simplex s^q, where $|s^q| = \sigma^q$. Let t^q be the orientation of σ^q taking the value $+1$ on the sequence $|D^0, \ldots, D^q|$. Let Γ be an abelian group and let $\gamma \in \Gamma$. Then for*

$$\gamma^{\cdot} s^q \in H_q^1(\sigma^q, \dot{\sigma}^q, \Gamma) = H_q^1(|K(\sigma^q)|, |L(\sigma^q)|, \Gamma)$$

(see Section 28 and [6] *p. 80) and for*

$$\{\gamma t^q\} \in H_q(K(\sigma^q), L(\sigma^q), \Gamma)$$

we have

$$\pi_{\sigma^q, \Gamma}(\gamma^{\cdot} s^q) = v_{\sigma^q, \Gamma}(\{\gamma t^q\}).$$

PROOF. We proceed by induction.

First let $q=0$. Then $\omega(K(\sigma^0))=\{\sigma^0\}$. Denote this open covering by ω_1 and its nerve by N_1. Hence $N_1 = N_{\sigma^0}$. Since $L(\sigma^0)=\emptyset$ and thus $|L(\sigma^0)|=\emptyset$ and $N^{\cdot}_{\sigma^0}=\emptyset$ we get

$$H_0(N_{\sigma^0}, N^{\cdot}_{\sigma^0}, \Gamma) = H_0(N_1, \Gamma).$$

Let t_1^* be the orientation of the only 0-simplex $\{\{\sigma^0\}\}$ of N_1 taking the value $+1$ on the sequence $|\{\sigma^0\}|$.

Let D^0 be the only point of σ^0 and let s^0 be the only ordered simplex of σ^0. Moreover let t^0 be the orientation of σ^0 taking the value $+1$ on the sequence $|D^0|$. We then have evidently

$$\nu_{\sigma^0,\Gamma}(\{\gamma t^0\}) = \{\gamma t_1^*\}. \tag{12}$$

Let $f\colon \sigma^0 \to P_0$ be the only map of σ^0 into P_0. Since $f^{-1}(\omega_0)=\omega_1$ (see also Section 28) it follows that f induces the map $f_{\omega_0}\colon N_1 \to N_0$. Now by the commutativity of the diagram

$$\begin{array}{ccc} H_0^1(P_0,\Gamma) & \xleftarrow{f_*} & H_0^1(\sigma^0,\Gamma) \\ \downarrow{\scriptstyle \pi_{\omega_0,\Gamma}} & & \downarrow{\scriptstyle \pi_{\sigma^0,\Gamma}} \\ H_0(N_0,\Gamma) & \xleftarrow[f_{\omega_0 *}]{} & H_0(N_1,\Gamma) \end{array}$$

one has

$$f_{\omega_0 *}\pi_{\sigma^0,\Gamma}(\gamma^* s^0) = \pi_{\omega_0,\Gamma}(f_*(\gamma^* s^0)).$$

However $f_*(\gamma^* s^0)=\gamma^*$ (see [6] p. 79) and $\pi_{\omega_0,\Gamma}(\gamma^*)=\overline{\gamma t_0^*}$ (see Section 28). Consequently

$$f_{\omega_0 *}\pi_{\sigma^0,\Gamma}(\gamma^* s^0) = \overline{\gamma t_0^*}.$$

On the other hand we have evidently

$$f_{\omega_0 *}(\{\gamma t_1^*\}) = \overline{\gamma t_0^*}$$

and since $f_{\omega_0 *}$ is clearly an isomorphism it follows

$$\pi_{\sigma^0,\Gamma}(\gamma^* s^0) = \{\gamma t_1^*\}.$$

Comparing this relation with (12) we get the required equality

$$\pi_{\sigma^0,\Gamma}(\gamma^* s^0) = \nu_{\sigma^0,\Gamma}(\gamma t^0\}).$$

Now suppose that $q>0$ and that the assertion is true if we replace q by $q-1$.

Let σ^{q-1} be the $(q-1)$-face of σ^q spanned by the vertices $D^1<\ldots<D^q$ and let s^{q-1} be the ordered simplex thus obtained. Let t^{q-1} be the orientation of σ^{q-1} taking the value $+1$ on the sequence $|D^1, \ldots, D^q|$. Let

$$M(\sigma^q) = L(\sigma^q)\setminus\{\sigma^{q-1}\} = K(\sigma^q)\setminus\{\sigma^q, \sigma^{q-1}\}.$$

We then have

$$L(\sigma^q) = K(\sigma^{q-1})\cup M(\sigma^q), \quad L(\sigma^{q-1}) = K(\sigma^{q-1})\cap M(\sigma^q).$$

Let ∂_Γ^1 be the boundary homomorphism

$$\partial_\Gamma^1\colon H_q^1(\sigma^q, \dot\sigma^q, \Gamma) \to H_{q-1}^1(\dot\sigma^q, \Gamma)$$

in the Čech homology theory over Γ. Moreover let ∂_Γ^2 and ∂_Γ^3 be the boundary homomorphisms

$$\partial_\Gamma^2\colon H_q(N_{\sigma^q}, N_{\dot\sigma^q}, \Gamma) \to H_{q-1}(N_{\dot\sigma^q}^*, \Gamma),$$

$$\partial_\Gamma^3\colon H_q(K(\sigma^q), L(\sigma^q), \Gamma) \to H_{q-1}(L(\sigma^q), \Gamma)$$

corresponding to the couples

$$\varkappa, \lambda(N^*_{\sigma^q}, \emptyset) \to (N_{\sigma^q}, \emptyset) \to (N_{\sigma^q}, N^*_{\sigma^q})$$

and

$$\varkappa', \lambda'\big(L(\sigma^q), \emptyset\big) \to \big(K(\sigma^q), \emptyset\big) \to \big(K(\sigma^q), L(\sigma^q)\big)$$

respectively where $\varkappa$, λ, $\varkappa'$ and λ' are inclusions.

Let us introduce the following notations.

$$N_{\sigma^q} = N_1, \quad N^*_{\sigma^q} = N_2, \quad N_{\omega(L(\sigma^q))} = N_3, \quad N_{|M(\sigma^q)|, \omega(L(\sigma^q))} = N_4,$$

$$N_{\sigma^{q-1}} = N_5, \quad N^*_{\sigma^{q-1}} = N_6,$$

$$H^1_q(\sigma^q, \dot\sigma^q, \Gamma) = H_1, \quad H^1_{q-1}(\dot\sigma^q, \Gamma) = H_2, \quad H^1_{q-1}\big(\dot\sigma^q, |M(\sigma^q)|, \Gamma\big) = H_3,$$

$$H^1_{q-1}(\sigma^{q-1}, \dot\sigma^{q-1}, \Gamma) = H_4, \quad H_q(N_1, N_2, \Gamma) = H_5, \quad H_{q-1}(N_2, \Gamma) = H_6,$$

$$H_{q-1}(N_3, \Gamma) = H_7, \quad H_{q-1}(N_3, N_4, \Gamma) = H_8, \quad H_{q-1}(N_5, N_6, \Gamma) = H_9,$$

$$H_q\big(K(\sigma^q), L(\sigma^q), \Gamma\big) = H_{10}, \quad H_{q-1}\big(L(\sigma^q), \Gamma\big) = H_{11},$$

$$H_{q-1}\big(L(\sigma^q), M(\sigma^q), \Gamma\big) = H_{12}, \quad H_{q-1}\big(K(\sigma^{q-1}), L(\sigma^{q-1}), \Gamma\big) = H_{13},$$

$$v_{\sigma^q, \Gamma} = \nu_1, \quad v^*_{\sigma^q, \Gamma} = \nu_2, \quad v\big(L(\sigma^q), M(\sigma^q), \Gamma, q-1\big) = \nu_3, \quad v_{\sigma^{q-1}, \Gamma} = \nu_4,$$

$$\pi_{\sigma^q, \Gamma} = \pi_1, \quad \pi^*_{\sigma^q, \Gamma} = \pi_2, \quad \pi\big(L(\sigma^q), M(\sigma^q), \Gamma, q-1\big) = \pi_3 \quad \text{and} \quad \pi_{\sigma^{q-1}, \Gamma} = \pi_4.$$

Observe that $\nu_1, \nu_2, \nu_3, \nu_4$ and $\pi_1, \pi_2, \pi_3, \pi_4$ are isomorphisms.

Now let $m_1\colon H_2 \to H_3$, $k_1\colon H_4 \to H_3$, $m_2\colon H_7 \to H_8$, $m_3\colon H_{11} \to H_{12}$ and $k_3\colon H_{13} \to H_{12}$ be homomorphisms induced by the inclusions $\dot\sigma^q \subset \big(\dot\sigma^q, |M(\sigma^q)|\big)$, $\big(\sigma^{q-1}, \dot\sigma^{q-1}\big) \subset \big(\dot\sigma^q, |M(\sigma^q)|\big)$, $N_3 \subset (N_3, N_4)$, $L(\sigma^q) \subset \big(L(\sigma^q), M(\sigma^q)\big)$ and $\big(K(\sigma^{q-1}), L(\sigma^{q-1})\big) \subset \big(L(\sigma^q), M(\sigma^q)\big)$ respectively. Observe that k_1, k_3, $m_1\partial^1_\Gamma$ and $m_3\partial^3_\Gamma$ are isophisms.

Now for $A \in \big(K(\sigma^{q-1})\big)^0$ let

$$v\big(\mathrm{st}_{K(\sigma^{q-1})}(A)\big) = \mathrm{st}_{L(\sigma^q)}(A).$$

Thus we obtain a simplicial mapping $v\colon N_5 \to N_3$ and obviously $v(N_6) \subset N_4$. Let $k_2\colon H_9 \to H_8$ be the homomorphism induced by the simplicial mapping $v\colon (N_5, N_6) \to (N_3, N_4)$. k_2 is clearly an isomorphism.

Next, for $U \in \omega\big(K(\sigma^q)\big)$ let

$$w(U) = U \cap \dot\sigma^q \in \omega\big(L(\sigma^q)\big).$$

Thus we obtain a simplicial isomorphism $w\colon N_2 \to N_3$ and let $w_*\colon H_6 \to H_7$ be the isomorphism induced by w.

Now consider the diagram

$$\begin{array}{ccccccccc}
H_1 & & \xrightarrow{\partial^1_\Gamma} & & H_2 & \xrightarrow{m_1} & H_3 & \xleftarrow{k_1} & H_4 \\
\downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_4} \\
H_5 & \xrightarrow{\partial^2_\Gamma} & H_6 & \xrightarrow{w_*} & H_7 & \xrightarrow{m_2} & H_8 & \xleftarrow{k_2} & H_9 \\
\uparrow{\scriptstyle \nu_1} & & & & \uparrow{\scriptstyle \nu_2} & & \uparrow{\scriptstyle \nu_3} & & \uparrow{\scriptstyle \nu_4} \\
H_{10} & & \xrightarrow{\partial^3_\Gamma} & & H_{11} & \xrightarrow{m_3} & H_{12} & \xleftarrow{k_3} & H_{13}
\end{array}$$

which is clearly commutative.

Now denoting the boundary isomorphism of the triad $(\sigma^q;\ \sigma^{q-1}, |M(\sigma^q)|)$ in the Čech homology theory over Γ by $[\sigma^q:\ \sigma^{q-1}]^1$ (see [6] p. 79) we get

$$[\sigma^q:\ \sigma^{q-1}]^1 = k_1^{-1} m_1 \partial_\Gamma^1. \tag{13}$$

Further we have

$$[\sigma^q:\ \sigma^{q-1}]^1(\gamma^* s^q) = \gamma^* s^{q-1} \tag{14}$$

(see [6] p. 80).

On the other hand by the induction hypothesis one has

$$\nu_4^{-1}\pi_4(\gamma^* s^{q-1}) = \{\gamma t^{q-1}\}. \tag{15}$$

Consider now the homology class $\{\gamma t^q\} \in H_{10}$. Then for the chain γt^q of $K(\sigma^q)$ we have

$$\Delta \gamma t^q = \gamma t^{q-1} + \sum_{i=1}^{q} \gamma_i t_i^{q-1} \tag{16}$$

where $t_1^{q-1}, \ldots, t_q^{q-1}$ are orientations of the $(q-1)$-simplexes of $M(\sigma^q)$ and for $i=1, \ldots, q$ either $\gamma_i = \gamma$ or $\gamma_i = -\gamma$. $\Delta \gamma t^q$ is considered here as a cycle in $L(\sigma^q)$. (16) evidently implies

$$m_3 \partial_\Gamma^3(\{\gamma t^q\}) = \overline{\Delta \gamma t^q} = \overline{\gamma t^{q-1}} \in H_{12}. \tag{17}$$

However

$$k_3(\{\gamma t^{q-1}\}) = \overline{\gamma t^{q-1}} \tag{18}$$

where on the left side of this equality γt^{q-1} is considered as a chain of $K(\sigma^{q-1})$ and on the right side as a chain of $L(\sigma^q)$. Now (17) and (18) imply

$$k_3^{-1} m_3 \partial_\Gamma^3(\{\gamma t^q\}) = \{\gamma t^{q-1}\} \in H_{13}$$

and thus taking also (13), (14), (15) and the commutativity of the preceding diagram into account we get

$$\nu_1^{-1}\pi_1(\gamma^* s^q) = (m_3 \partial_\Gamma^3)^{-1} k_3 \nu_4^{-1} \pi_4 [\sigma^q:\ \sigma^{q-1}]^1 (\gamma^* s^q) = \{\gamma t^q\}.$$

Consequently $\pi_1(\gamma' s^q) = \nu_1(\{\gamma t^q\})$ as required. The proof of the lemma is complete. □

35. PROPOSITION. *Let q be a positive integer less than $n+1$. Let σ^q, $D^0 < \ldots < D^q$, s^q, t^q, Γ and γ be the same as in Lemma* 34. *Moreover let*

$$\partial_\Gamma^1 \colon H_q^1(\sigma^q, \dot{\sigma}^q, \Gamma) \to \tilde{H}_{q-1}^1(\dot{\sigma}^q, \Gamma) \quad \text{and} \quad \partial_\Gamma^3 \colon H_q(K(\sigma^q), L(\sigma^q), \Gamma) \to \tilde{H}_{q-1}(L(\sigma^q), \Gamma)$$

be the same as in the second part of the proof of Lemma 34. *Then*

$$\pi^*_{\sigma^q, \Gamma}(\partial_\Gamma^1(\gamma^* s^q)) = \nu^*_{\sigma^q, \Gamma}(\{\Delta \gamma t^q\}).$$

PROOF. Keeping the notations of the second part of the proof of Lemma 34 recall that

$$\{\Delta \gamma t^q\} = \partial_\Gamma^3(\{\gamma t^q\}). \tag{19}$$

Also, by Lemma 34 we have

$$\pi_1(\gamma^* s^q) = \nu_1(\{\gamma t^q\}). \tag{20}$$

Thus according to (19) and (20) taking also the commutativity of the second diagram in the proof of Lemma 34 into account we get the required equality $\pi_2 \partial_I^1(\gamma^* s^q) = \nu_2(\{\Delta \gamma t^q\})$. □

36. PROPOSITION. *Let q, σ^q, $D^0 < \ldots < D^q$ and s^q be the same as in Lemma 34. Then the relations*

$$h(q, \sigma^q, \dot{\sigma}^q)(gs^q) = g^* s^q \quad \text{and} \quad h'(q, \sigma^q, \dot{\sigma}^q) g' s^q = g'^* s^q$$

hold for each $g \in G$ and $g' \in G'$.

PROOF. By symmetry we only need to verify the first relation. We again proceed by induction.

Let $g \in G$. Let $q=0$ and let $f\colon \sigma^0 \to P_0$ be the only map of σ^0 into P_0. Let $H_0(f)\colon H_0(\sigma^0) \to H_0(P_0)$ be the homomorphism induced by f. Since the homomorphism $h\colon H \to H^1(., G)$ is an extension of h_0 it follows that the diagram

$$\begin{array}{ccc} H_0(P_0) & \xleftarrow{H_0(f)} & H_0(\sigma^0) \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h(0, \sigma^0, \dot{\sigma}^0)} \\ H_0^1(P_0, G) & \xleftarrow{f_*} & H_0^1(\sigma^0, G) \end{array}$$

is commutative.

However f_* is an isomorphism. Moreover we have $H_0(f)(gs^0) = g$ (see [6] p. 79) and $f_*(g^* s^0) = g^*$ and also $h_0(g) = g^*$ (see Section 29). Consequently $h(0, \sigma^0, \dot{\sigma}^0)(gs^0) = g^* s^0$ as required.

Now suppose that $q > 0$ and that the assertion is true if we replace q by $q-1$. Let σ^{q-1} be the $(q-1)$-face of σ^q spanned by the vertices $D^1 < \ldots < D^q$ and let s^{q-1} be the ordered simplex thus obtained. The diagram

$$\begin{array}{ccc} H_{q-1}(\sigma^{q-1}, \dot{\sigma}^{q-1}) & \xleftarrow{[s^q : s^{q-1}]} & H_q(\sigma^q, \dot{\sigma}^q) \\ \downarrow{\scriptstyle h(q-1, \sigma^{q-1}, \dot{\sigma}^{q-1})} & & \downarrow{\scriptstyle h(q, \sigma^q, \dot{\sigma}^q)} \\ H_{q-1}^1(\sigma^{q-1}, \dot{\sigma}^{q-1}, G) & \xleftarrow{[s^q : s^{q-1}]^1} & H_q^1(\sigma^q, \dot{\sigma}^q, G) \end{array}$$

is clearly commutative and $[s^q : s^{q-1}]^1$ is an isomorphism.

However $[s^q : s^{q-1}](gs^q) = gs^{q-1}$ and $[s^q : s^{q-1}]^1(g^* s^q) = g^* s^{q-1}$ (see [6] p. 80). Moreover by the induction hypothesis we have $h(q-1, \sigma^{q-1}, \dot{\sigma}^{q-1})(gs^{q-1}) = g^* s^{q-1}$ and thus by the commutativity of the latter diagram we get $h(q, \sigma^q, \dot{\sigma}^q)(gs^q) = g^* s^q$ indeed. □

37. PROPOSITION. *Let q be a positive integer. Let σ^q, $D^0 < \ldots < D^q$ and s^q be the same as in Lemma 34. Let*

$$\partial^0\colon H_q(\sigma^q, \dot{\sigma}^q) \to \tilde{H}_{q-1}(\dot{\sigma}^q)$$

and

$$\partial^{0\prime}\colon H'_q(\sigma^q, \dot{\sigma}^q) \to \tilde{H}'_{q-1}(\dot{\sigma}^q)$$

be boundary homomorphisms. Moreover for $\Gamma=G$ *or* $\Gamma=G'$ *let* ∂^1_Γ *be the same as in the second part of the proof of Lemma* 34. *Then for every* $g\in G$ *and* $g'\in G'$ *we have*

$$h(q-1, \dot\sigma^q, \emptyset)\partial^0(gs^q) = \partial^1_G(g^* s^q)$$

and

$$h'(q-1, \dot\sigma^q, \emptyset)\partial^{0\prime}(g' s^q) = \partial^1_{G'}(g'^* s^q).$$

PROOF. By symmetry we clearly need to verify only the first equality.
However, since the diagram

$$\begin{array}{ccc} H_q(\sigma^q, \dot\sigma^q) & \xrightarrow{\partial^0} & \tilde H_{q-1}(\dot\sigma^q) \\ \downarrow{\scriptstyle h(q,\sigma^q,\dot\sigma^q)} & & \downarrow{\scriptstyle \tilde h(q-1,\dot\sigma^q,\emptyset)} \\ H^1_q(\sigma^q, \dot\sigma^q, G) & \xrightarrow{\partial^1_G} & \tilde H^1_{q-1}(\dot\sigma^q, G) \end{array}$$

— where $\tilde h(q-1, \dot\sigma^q, \emptyset)$ is the restriction of $h(q-1, \dot\sigma^q, \emptyset)$ to $\tilde H_{q-1}(\dot\sigma^q)$ — is commutative, the assertion is a direct consequence of Proposition 36. □

38. *Let* q, σ^q, s^q *and* t^q *be the same as in Lemma* 34. *Then Lemma* 34 *and Proposition* 36 *show that for every* $g\in G$ *and* $g'\in G'$ *we have*

$$\pi_{\sigma^q, G} h(q, \sigma^q, \dot\sigma^q)(gs^q) = \nu_{\sigma^q, G}(\{gt^q\})$$

and

$$\pi_{\sigma^q, G'} h'(q, \sigma^q, \dot\sigma^q)(g' s^q) = \nu_{\sigma^q, G'}(\{g' t^q\}).$$

39. *Let* q *be a positive integer and let* σ^q, s^q *and* t^q *be the same as in Lemma* 34. *Let* ∂^0 *and* $\partial^{0\prime}$ *be the same as in Proposition* 37. *Then Propositions* 35 *and* 37 *show that for every* $g\in G$ *and* $g'\in G'$ *we have*

$$\pi^*_{\sigma^q, G} h(q-1, \dot\sigma^q, \emptyset)\partial^0(gs^q) = \nu^*_{\sigma^q, G}(\{\Delta gt^q\})$$

and

$$\pi^*_{\sigma^q, G'} h'(q-1, \dot\sigma^q, \emptyset)\partial^{0\prime}(g' s^q) = \nu^*_{\sigma^q, G'}(\{\Delta g' t^q\}).$$

40. Let K be a euclidean complex contained in R^n. Let Γ be an abelian group and q a nonnegative integer. We introduce a homomorphism

$$\mu_{q,K,\Gamma}\colon \tilde H_q(K, \Gamma) \to \tilde\Delta^q(|K|, \Gamma)$$

where $|K|$ is the space of K as follows.

Let $\vartheta\in\tilde H_q(K, \Gamma)$ and $z^q\in\vartheta$. For any nonnegative integer y let z^q_y be the barycentric subdivision of the cycle z^q of order y.

$$\mathfrak{z}^q = (z^q = z^q_0, z^q_1, \ldots)$$

is obviously a proper cycle and the homology class of this cycle clearly belongs to $\tilde\Delta^q(|K|, \Gamma)$. The element of $\tilde\Delta^q(|K|, \Gamma)$ is clearly independent of the special choice of $z^q\in\vartheta$; it depends only on ϑ. We shall denote it by $\mu_{q,K,\Gamma}(\vartheta)$. Thus we obtain a mapping

$$\mu_{q,K,\Gamma}\colon \tilde H_q(K, \Gamma) \to \tilde\Delta^q(|K|, \Gamma)$$

which is a homomorphism, or even more it is an isomorphism (see [2] p. 168).

It is important that the diagram

$$\begin{array}{ccc} \tilde{\Delta}^q(|K|, \Gamma) & \xrightarrow{\eta_{q,|K|,\Gamma}} & \tilde{H}_q^1(|K|, \Gamma) \\ \uparrow{\scriptstyle \mu_{q,K,\Gamma}} & & \downarrow{\scriptstyle \tilde{\pi}(K,\emptyset,\Gamma,q)} \\ \tilde{H}_q(K, \Gamma) & \xrightarrow{\tilde{\nu}(K,\emptyset,\Gamma,q)} & \tilde{H}_q(N_{\omega(K)}, \Gamma) \end{array}$$

— where $\tilde{\pi}(K, \emptyset, \Gamma, q)$ and $\tilde{\nu}(K, \emptyset, \Gamma, q)$ are restrictions of the isomorphisms $\pi(K, \emptyset, \Gamma, q)$ and $\nu(K, \emptyset, \Gamma, q)$ (see Section 32) to $\tilde{H}_q^1(|K|, \Gamma)$ and $\tilde{H}_q(K, \Gamma)$ respectively — is commutative (see also Section 24). This fact follows immediately from the fundamental identity 3.11 of [2] p. 140.

41. Let K, Γ and q be the same as in Section 40. Let z^q be a q-cycle of K over Γ. For any nonnegative integer w let z_w^q be the barycentric subdivision of the cycle z^q of order w.

Now let y be a nonnegative integer. Then the body of z_y^q (see [2] p. 21) lies in $|K|$ and we have

$$z_y^q - z_{y+1}^q = \Delta R z_y^q$$

(see [6] p. 197) where the body of the $q+1$-chain Rz_y^q of R^n lies in $|K|$ as well.

42. Let K and K' be euclidean complexes contained in R^n such that $|K| \cap |K'| = \emptyset$. Let z^r be an r-cycle of K over G and $z'^{r'}$ an r'-cycle of K' over G'. Suppose that z^r is a normal cycle if $r=0$ (see [2] p. 51) and $z'^{r'}$ is a normal cycle if $r'=0$. Let

$$\mathfrak{z}^r = (z^r = z_0^r, \ldots, z_y^r, \ldots)$$

and

$$\mathfrak{z}'^{r'} = (z'^{r'} = z_0'^{r'}, \ldots, z_y'^{r'}, \ldots)$$

where z_y^r and $z_y'^{r'}$ are barycentric subdivisions of order y of the cycles z^r and $z'^{r'}$ respectively. Then the observations in Section 41 show that for the proper cycles $\mathfrak{z}^r$ and $\mathfrak{z}'^{r'}$ of $|K|$ and $|K'|$ respectively we have

$$\mathfrak{v}(\mathfrak{z}^r, \mathfrak{z}'^{r'}) = \mathfrak{v}(z^r, z'^{r'})$$

(see also [3] p. 86).

43. Now we are going to prove the fundamental assertion of Section 31, namely that the relation $\mathfrak{v}' = \mathfrak{v}_{\sigma,\dot{\sigma}'}$ holds.

First let us take the ordering $D^0 < D^1 < \ldots < D^{z'}$ of the vertices of σ' for which $D^i = B^{i-1}$ for $i = 1, \ldots, z'$ (see Section 22). Thus we get an ordered simplex s' for which $|s'| = \sigma'$. Let $t'^{z'}$ be the orientation of σ' taking the value $+1$ on the sequence $|D^0, \ldots, D^{z'}|$. We then have obviously

$$(t'^{z'} : t_1'^{r'}) = 1. \tag{21}$$

Now introduce the notations η, η', μ and μ' for the mappings $\eta_{r,\dot{\sigma},G}$, $\eta_{r',\dot{\sigma}',G'}$, $\mu_{r,L(\sigma),G}$ and $\mu_{r',L(\sigma'),G'}$ respectively and observe that the mappings $\pi^*_{\sigma,G}$ and $\pi^*_{\sigma',G'}$ are isomorphisms (see Section 32). Thus Section 39 and the commutativity of the diagram in Section 40 show that the relations

$$h(r, \dot{\sigma}, \emptyset)\partial(gs) = \eta\mu(\{\Delta g t^z\}) \tag{22}$$

and

$$h'(r', \dot{\sigma}', \emptyset)\partial'(g's') = \eta'\mu'(\{\Delta g' t'^{z'}\}) \tag{23}$$

hold for every $g \in G$ and $g' \in G'$.

Next, let $u \in \tilde{H}_r(\dot{\sigma})$ and $u' \in H'_r(\dot{\sigma}')$. Then u and u' can be represented in the form

$$u = \partial(gs) \tag{24}$$

and

$$u' = \partial'(g's') \tag{25}$$

where $g \in G$ and $g' \in G'$.

We have seen in Section 30 that

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = \mathfrak{v}(h(r, \dot{\sigma}, \emptyset)(u), h'(r', \dot{\sigma}', \emptyset)(u'))$$

and thus by (22), (23) and Section 27 we have

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = \mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(\partial(gs), \partial'(g's')) =$$

$$= \mathfrak{v}(h(r, \dot{\sigma}, \emptyset)\partial(gs), h'(r', \dot{\sigma}', \emptyset)\partial'(g's')) =$$

$$= \mathfrak{v}(\eta\mu(\{\Delta g t^z\}), \eta'\mu'(\{\Delta g' t'^{z'}\})) = \mathfrak{v}(\mu(\{\Delta g t^z\}), \mu'(\{\Delta g' t'^{z'}\})).$$

Consequently taking also the definition of

$$\mathfrak{v}(\mu(\{\Delta g t^z\}), \mu'(\{\Delta g' t'^{z'}\}))$$

(see Sections 23 and 42) into account we get

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = \mathfrak{v}(\Delta g t^z, \Delta g' t'^{z'}) = g t^z \times \Delta g' t'^{z'}$$

where $g t^z$ and $\Delta g' t'^{z'}$ are chains in R^n in relative general position. However among the r'-faces of the z'-simplex σ' only σ'_1 intersects the simplex σ and thus by (21) and by Definition 23 one has

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = g t^z \times \Delta g' t'^{z'} = g t^z \times g' t_1'^{r'} = t^z \times t_1'^{r'} \mathfrak{v}_1(g, g').$$

Now according to Sections 22 and 21 we get

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = t^z \times t_1'^{r'} \mathfrak{v}_1(g, g') = \mathfrak{v}^*(gs, g's'_1) = \mathfrak{v}'(\partial(gs), \tilde{m}_*^{-1} k_*(g's'_1)). \tag{26}$$

On the other hand $g's'_1 = [\sigma' : \sigma'_1](g's')$ where $[\sigma' : \sigma'_1] = k_*^{-1}\tilde{m}_*\partial'$ (see [6] p. 79) and thus

$$\tilde{m}_*^{-1} k_*(g's'_1) = \tilde{m}_*^{-1} k_* k_*^{-1} \tilde{m}_* \partial'(g's') = \partial'(g's').$$

Hence by (26), (24) and (25) we obtain

$$\mathfrak{v}_{\dot{\sigma}, \dot{\sigma}'}(u, u') = \mathfrak{v}'(\partial(gs), \partial'(g's')) = \mathfrak{v}'(u, u')$$

as required.

The proof of the existence theorem of the theory of linking is complete. □

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ON CERTAIN RATIONAL INTERPOLATORY OPERATORS

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Introduction

We define certain trigonometric as well as algebraic rational interpolating functions depending on several parameters. It turns out that they have fairly good approximating properties not only in order but taking into account the acting constants, too.

The structure of this paper is as follows.

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Analogous problems for trigonometric and algebraic polynomial operators were treated by O. Kis and J. Szabados [4]. We use some of their results.

Trigonometric case

I. Definitions. Remarks

1. Definitions. Let

$$Q_{klm}(g, t) := \frac{S_{klm}(g, t)}{\sigma_{klm}(t)}, \tag{1}$$

where k and m are positive, l is a nonnegative integer, t is a real variable, g is a real valued 2π-periodic continuous function of t (shortly $g \in \tilde{C}$), further

$$S_{klm}(g, t) := \sum_{i=1}^{m} g(t_{im})\, s_{klm}(t - t_{im}), \tag{2}$$

$$t_{im} := \frac{2\pi i}{m}, \quad i = 0, \quad \pm 1, \quad \pm 2, \ldots, \tag{3}$$

$$s_{klm}(t) := \left(\frac{\sin \frac{mt}{2}}{m \sin \frac{t}{2}} \right)^k \cos^l \frac{t}{2}, \tag{4}$$

$$\sigma_{klm}(t) := \sum_{i=1}^{m} s_{klm}(t - t_{im}). \tag{5}$$

Generally, k and l are ("small") fixed numbers and we shall consider $|S_{klm}(g, t) - g(t)|$ when m tends to infinity.

In what follows, sometimes we omit the unnecessary parameters.

2. It is easy to see that

$$Q_{klm}(g, t) = \frac{\sum_{i=1}^{m} (-1)^{ik} g(t_i) \frac{\cos^l \frac{t - t_i}{2}}{\sin^k \frac{t - t_i}{2}}}{\sum_{i=1}^{m} (-1)^{ik} \frac{\cos^l \frac{t - t_i}{2}}{\sin^k \frac{t - t_i}{2}}}.$$

3. THEOREM. *If k, l and m are such that*

$$n_{klm} := \frac{km - k + l}{2} \tag{6}$$

is an integer (which will be supposed from now on) then $S_{klm}(g, t)$ and $\sigma_{klm}(t)$ are trigonometric polynomials of order $\leqq n$ and

$$Q_{klm}(g, t_i) = g(t_i), \quad i = 0, \pm 1, \pm 2, \ldots .$$

4. It is well known (see also (53), (49) and (50)) that

$$\sigma_{10m}(t) = 1, \quad m = 1, 3, 5, \ldots, \quad \sigma_{11m}(t) = 1, \quad m = 2, 4, 6, \ldots; \tag{8}$$

$$\sigma_{20m}(t) = 1, \quad m = 1, 2, 3, \ldots . \tag{9}$$

Hence

$$Q_{10m}(g, t) = S_{10m}(g, t), \quad m = 1, 3, 5, \ldots,$$

and

$$Q_{11m}(g, t) = S_{11m}(g, t), \quad m = 2, 4, 6, \ldots,$$

are trigonometric interpolatory polynomials,

$$Q_{20m}(g, t) = S_{20m}(g, t), \quad m = 1, 2, 3, \ldots$$

are the so called Jackson polynomials ([3]).

II. Estimations

From now on we suppose that

$$\sigma_{klm}(t) > 0.$$

5. THEOREM. *If* $k \geqq 2$ *and* $l \geqq 1$ *then*

$$|Q_{klm}(g, t) - g(t)| \leqq \frac{\pi L_{klm}(t) + k L_{k-1,l-1,m}(t)}{\pi \sigma_{klm}(t)} \omega\left(g, \frac{2\pi}{m}\right) \tag{10}$$

where $\omega(g, \delta)$ *is the modulus of continuity of* g *further*

$$L_{klm}(t) := \sum_{i=1}^{m} |s_{klm}(t - t_i)|. \tag{11}$$

6. REMARK. One can see that $\sigma_{klm}(t) > 0$ if
a) both k and l are even;
b) m is odd, $l=0$, further $k=1$ or $k=3$;
c) m is even, $l=1$, further $k=1$ or $k=3$
(see (4) and (5) further (8), (54) and (55), respectively).

7. THEOREM. *Let*

$$c_{klm} := \min_{t} \sigma_{klm}(t). \tag{12}$$

If $k \geqq 3$ *and* $l \geqq 1$, *then*

$$|Q_{klm}(g, t) - g(t)| \leqq \frac{\pi + k}{\pi c_{klm}} \omega\left(g, \frac{2\pi}{km}\right). \tag{13}$$

If $k \geqq 4$ *and even,* $l \geqq 2$ *and even then*

$$|Q_{klm}(g, t) - g(t)| \leqq \left(1 + \frac{k}{\pi c_{klm}}\right) \omega\left(g, \frac{2\pi}{km}\right). \tag{14}$$

If $k \geqq 3$ *and odd then*

$$|Q_{klm}(g, t) - g(t)| \leqq \left(\frac{1}{2} + \frac{1}{2c_{klm}}\right) \omega\left(g, \frac{2\pi}{m}\right). \tag{15}$$

8. REMARK. By (12), (15), (11) and (64) $c_{klm} \leqq 1$. Using this fact, the estimation (14) is better than (13) whenever both k and l are even; on the other hand, (14) is better than (13) for "good" g, finally (15) is better than (13) whenever g is "bad".
These properties can be deduced from (17) and (18) for $k=3$ and $l=1$.

9. THEOREM. *The following estimations hold true:*

$$|Q_{30m}(g,t)-g(t)| \leqq \frac{3}{2}\,\omega\left(g,\frac{2\pi}{m}\right), \quad m=1,3,5,\ldots, \tag{16}$$

$$|Q_{31m}(g,t)-g(t)| \leqq \frac{3}{2}\,\omega\left(g,\frac{2\pi}{m}\right), \quad m=2,4,6,\ldots, \tag{17}$$

$$|Q_{31m}(g,t)-g(t)| \leqq \left(2+\frac{6}{\pi}\right)\omega\left(g,\frac{2\pi}{3m}\right), \quad m=2,4,6,\ldots, \tag{18}$$

$$|Q_{42m}(g,t)-g(t)| \leqq \left(1+\frac{12}{\pi-\frac{\pi}{m^2}}\right)\omega\left(g,\frac{\pi}{2m}\right), \quad m=2,3,4,\ldots. \tag{19}$$

III. Saturation. On the order of approximation

In this chapter we prove a saturation theorem, at least for $k\geqq 3$. Further we show that the approximation order of our operators is the best possible for a certain function-class.

10. THEOREM. *Let* $k\geqq 3$ *further* $c_{klm}\geqq\gamma_{kl}>0$ *with* $m\geqq m_0(k,l)$. *Then for fixed* k *and* l

$$\|Q_{klm}(g)-g\| = o\left(\frac{1}{m}\right) \tag{20}$$

holds true iff $g=\text{const}$. *Further,*

$$\|Q_{klm}(g)-g\| = O\left(\frac{1}{m}\right) \tag{21}$$

iff $g\in\text{Lip}\,1$.

(As usual, $\|g\|:=\max|g(t)|$.)

It is worthwile to mention that $c_{10m}=c_{11m}=c_{20m}=1$, $c_{30m}=\frac{1}{2}+\frac{1}{2m^2}$, $c_{31m}=\frac{1}{2}$, $c_{40m}=\frac{1}{3}+\frac{2}{3m^2}$ and $c_{42m}=\frac{1}{3}-\frac{1}{3m^2}$ (see (8), (9), (12), (76), (77), (116) and (78), respectively), i.e. the condition $c_{klm}\geqq\gamma_{kl}$ can be satisfied in many cases.

11. To obtain that the approximation order achieved by our operators Q_{klm} for a certain function-class is the best possible one, we quote the following statement.

For a given $g\in\tilde{C}$ *and the fixed integer* $n\geqq 1$, *the trigonometric polynomials* $U_n(g,t)$ *and* $V_n(g,t)$ *can be chosen such that* $\|g(t)-W_n(g,t)\|$ *is the smallest possible. Here*

$$W_n(g,t):=\frac{U_n(g,t)}{V_n(g,t)} \tag{22}$$

aud it is unique. If, after all the possible simplifications, the orders of the numerator and the denominator of W_n equal $n-\nu$ and $n-\mu$, respectively, then W_n can be uniquely characterized by the following alternating property. There exist at least $N:=4n+2-\min(\nu,\mu)$ points $\tau_1, \tau_2, \ldots, \tau_N$ with $0\leqq\tau_1<\tau_2<\ldots<\tau_N<2\pi$ such that

$$W_n(g,\tau_i)-g(\tau_i)=(-1)^{i+\alpha}\|W_n(g,t)-g(t)\|, \quad i=1,2,\ldots,N,$$

where $\alpha=0$ or 1 (for the proof see e.g. N. I. Achieser [1; 31, 40 and 34]).

Using the above statement we shall prove as follows.

12. THEOREM. *Let $\Omega_q(\delta)$ be a positive q^{th} modulus of smoothness ($q=1,2,\ldots$, fixed) with*

$$\lim_{\delta=+0}\frac{\delta^q}{\Omega_q(\delta)}=0. \tag{23}$$

Then one can find a function $g\in\tilde{C}(\Omega_q)$ such that

$$\overline{\lim_{n\to\infty}}\frac{\|W_n(g)-g\|}{\Omega_q\left(\frac{1}{n}\right)}\geqq 1. \tag{24}$$

Here, as usual, $g\in\tilde{C}(\Omega_q)$ iff $\omega_q(g,\delta)/\Omega_q(\delta)$ is bounded; $\omega_q(g,\delta)$ is the qth modulus of smoothness of g.

Algebraic case

I. Definitions. Remarks

13. DEFINITION. Let

$$R_{klm}(f,x):=Q_{klm}(f\circ\cos,\arccos x) \tag{25}$$

where $-1\leqq x\leqq 1$ and f is a real valued continuous function of x (shortly $f\in C$).

14. THEOREM. *$R_{klm}(f,x)$ is a rational function. The degree of its numerator and denominator is $\leqq n$; further*

$$R_{klm}(f,x_{im})=f(x_{im}), \quad i=0,1,\ldots,\left[\frac{m}{2}\right], \tag{26}$$

where

$$x_{im}:=\cos\frac{2\pi i}{m}, \quad i=0,1,\ldots,\left[\frac{m}{2}\right]. \tag{27}$$

15. REMARKS. It is easy to see, that $R_{10m}(f,x)$, $m=1,3,5,\ldots$, and $R_{11m}(f,x)$, $m=2,4,6,\ldots$, are the corresponding Lagrange interpolatory polynomials; further $R_{20m}(f,x)$, $m=1,2,3,\ldots$, are the Hermite—Fejér step-parabolas.

If, instead of (3), we define $t_{im}:=\frac{2i-1}{m}\pi$, then conditions (26) hold true for $x_{im}:=\cos\frac{2i-1}{m}\pi$, $i=1, 2, \ldots, \left[\frac{m+1}{2}\right]$, which, for even m, are the Chebyshev nodes.

II. Estimations

In Sections 16—24 and 49—58 let

$$x = \cos t, \quad t = \arccos x. \tag{28}$$

We state the following estimations.

16. THEOREM. *If* $k\geqq 2$, *then for* $|x|\leqq 1$

$$|R_{klm}(f, x)-f(x)| \leqq \frac{\pi L_{klm}(t)+kL_{k-1,lm}(t)}{\pi\sigma_{klm}(t)}\,\omega\left(f, \frac{2\pi}{km}\right). \tag{29}$$

17. THEOREM. *If* $k\geqq 3$, *then*

$$|R_{klm}(f, x)-f(x)| \leqq \frac{\pi+k}{\pi c_{klm}}\,\omega\left(f, \frac{2\pi}{km}\right), \quad |x| \leqq 1. \tag{30}$$

If $k\geqq 4$ *and both* k *and* l *are even then*

$$|R_{klm}(f, x)-f(x)| \leqq \left(1+\frac{k}{\pi c_{klm}}\right)\omega\left(f, \frac{2\pi}{km}\right), \quad |x| \leqq 1. \tag{31}$$

If $k\geqq 3$ *and odd then*

$$|R_{klm}(f, x)-f(x)| \leqq \left(\frac{1}{2}+\frac{1}{2c_{klm}}\right)\omega\left(f, \frac{2\pi}{m}\right), \quad |x| \leqq 1. \tag{32}$$

18. THEOREM. *If* $l=0$ *and* m *is odd or* $l=1$ *and* m *is even, then we have for* $|x|\leqq 1$

$$|R_{3lm}(f, x)-f(x)| \leqq \left(2+\frac{6}{\pi}\right)\omega\left(f, \frac{2\pi}{3m}\right) \tag{33}$$

and

$$|R_{3lm}(f, x)-f(x)| \leqq \frac{3}{2}\,\omega\left(f, \frac{2\pi}{m}\right). \tag{34}$$

Further, for arbitrary m

$$|R_{40m}(f, x)-f(x)| \leqq \left(1+\frac{12}{\pi}\right)\omega\left(f, \frac{\pi}{2m}\right), \quad |x| \leqq 1. \tag{35}$$

19. THEOREM. *If* $k \geqq 3$ *then*

$$|R_{klm}(f,x)-f(x)| \leqq \frac{\pi L_{klm}(t)+kL_{k-1,l+1,m}(t)}{\pi\sigma_{klm}(t)}\,\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right)+ \tag{36}$$

$$+\frac{2\pi^2 L_{klm}(t)+k^2 L_{k-2,l,m}(t)}{2\pi^2\sigma_{klm}(t)}\,\omega\left(f,\frac{4\pi^2}{k^2m^2}|x|\right), \quad |x| \leqq 1.$$

20. THEOREM. *For* $k \geqq 4$

$$|R_{klm}(f,x)-f(x)| \leqq \frac{\pi+k}{\pi c_{klm}}\,\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right)+\frac{2\pi^2+k^2}{2\pi^2 c_{klm}}\,\omega\left(f,\frac{4\pi^2|x|}{k^2m^2}\right), \quad |x| \leqq 1. \tag{37}$$

When $k \geqq 4$ *further both* k *and* l *are even, then for* $|x| \leqq 1$

$$|R_{klm}(f,x)-f(x)| \leqq \left(1+\frac{k}{\pi c_{klm}}\right)\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right)+\left(1+\frac{k^2}{2\pi^2 c_{klm}}\right)\omega\left(f,\frac{4\pi^2|x|}{k^2m^2}\right). \tag{38}$$

If $k \geqq 3$ *is odd then for* $|x| \leqq 1$

$$|R_{klm}(f,x)-f(x)| \leqq \left(\frac{1}{2}+\frac{1}{2c_{klm}}\right)\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\left(1+\frac{\pi+2}{\pi c_{klm}}\right)\omega\left(f,\frac{2\pi^2}{m^2}|x|\right). \tag{39}$$

21. THEOREM. *Let* $l=0$ *and* m *be odd or* $l=1$ *and* m *be even. Then for* $|x| \leqq 1$

$$|R_{3lm}(f,x)-f(x)| \leqq \frac{3}{2}\,\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\left(3+\frac{4}{\pi}\right)\omega\left(f,\frac{2\pi^2|x|}{m^2}\right). \tag{40}$$

Further, for arbitrary m *and* $|x| \leqq 1$

$$|R_{40m}(f,x)-f(x)| \leqq \left(1+\frac{12}{\pi}\right)\omega\left(f,\frac{\pi\sqrt{1-x^2}}{2m}\right)+\left(1+\frac{24}{\pi^2}\right)\omega\left(f,\frac{\pi^2|x|}{4m^2}\right). \tag{41}$$

22. THEOREM. *Let* m *be even. If* $k \geqq 3$ *then*

$$|R_{klm}(f,x)-f(x)| \leqq \tag{42}$$

$$\leqq \frac{2\pi L_{klm}(t)+2kL_{k-1,l+1,m}(t)+k|x|L_{k-2,l,m}(t)}{2\pi\sigma_{klm}(t)}\,\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right), \quad |x| \leqq 1.$$

If $k \geqq 4$ *then*

$$|R_{klm}(f,x)-f(x)| \leqq \frac{2\pi+2k+k|x|}{2\pi c_{klm}}\,\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right), \quad |x| \leqq 1. \tag{43}$$

If $k \geqq 4$ *further both* k *and* l *are even, then*

$$|R_{klm}(f,x)-f(x)| \leqq \left(1+\frac{2k+|x|}{2\pi c_{klm}}\right)\omega\left(f,\frac{2\pi}{km}\sqrt{1-x^2}\right), \quad |x| \leqq 1. \tag{44}$$

If $k \geqq 3$ is odd, then

$$|R_{klm}(f, x) - f(x)| \leqq \left(1 + \frac{2+3|x|}{2c_{klm}}\right) \omega\left(f, \frac{2\pi}{m}\right) \sqrt{1-x^2}, \quad |x| \leqq 1. \tag{45}$$

Finally, for $|x| \leqq 1$

$$|R_{31m}(f, x) - f(x)| \leqq (3+3|x|)\,\omega\left(f, \frac{2\pi}{m}\sqrt{1-x^2}\right); \tag{46}$$

$$|R_{40m}(f, x) - f(x)| \leqq \left(3 + \frac{6}{\pi}(2+|x|)\right) \omega\left(f, \frac{\pi}{2m}\sqrt{1-x^2}\right). \tag{47}$$

III. Saturation. On the order of approximation

Again, the statements are analogous to the trigonometric case.

23. THEOREM. *Let $k \geqq 3$ further $c_{klm} \geqq \gamma_{kl} > 0$ with $m \geqq m_0(k, l)$. Then for fixed k and l*

$$\|R_{klm}(f) - f\| = o\left(\frac{1}{m}\right) \quad \textit{iff} \quad f = c.$$

Further, with $f(x) = f(\cos t) := g(t)$,

$$\|R_{klm}(f) - f\| = O\left(\frac{1}{m}\right) \quad \textit{iff} \quad g \in \operatorname{Lip} 1.$$

(Again, $\|f\| := \max_{-1 \leqq x \leqq 1} |f(x)|$.)

24. The counterpart of Sections 11—12 can be expressed by the following statement proved in Szabados [6].

If

$$\lim_{\delta \to 0} \frac{\delta}{\Omega_1(\delta)} = 0$$

then there exists a function $f \in C(\Omega_1)$ for which

$$\overline{\lim_{n \to \infty}} \frac{\|r_n(f) - f\|}{\Omega_1\left(\frac{1}{n}\right)} \geqq 1.$$

Here $r_n(f, x)$ is the best approximating rational function to f of degree $\leqq n$, $C(\Omega_1)$ is analogous to $\tilde{C}(\Omega_1)$.

Proofs for the trigonometric case

25. PROOF OF THEOREM 3. By [4, Sections 18—23]

$$s_{klm}(t) = b_{0klm} + 2\sum_{j=1}^{n} b_{jklm}\cos jt \tag{48}$$

where

$$b_{jklm} = \frac{1}{2^l m^k}\sum_{i=0}^{l}\binom{l}{i} a_{n-i-j,k,m}, \quad j = 0, 1, \ldots, n, \tag{49}$$

and

$$a_{jkm} = \begin{cases} \sum_{i=0}^{[j/m]} (-1)^i \binom{k}{i}\binom{j+k-1-im}{k-1}, & \text{if } 0 \leqq j \leqq km-k, \\ 0, & \text{if } j < 0 \text{ or } j > km-k. \end{cases} \tag{50}$$

By (2) and (48)

$$S_{klm}(g,t) = b_{0klm}\sum_{i=1}^{m} g(t_{im}) + 2\sum_{j=1}^{n} b_{jklm}\cos jt \sum_{i=1}^{m} g(t_{im})\cos jt_{im} + \tag{51}$$

$$+ 2\sum_{j=1}^{n} b_{jklm}\sin jt \sum_{i=1}^{m} g(t_{im})\sin jt_{im},$$

i.e. $S_{klm}(g, t)$ is a trigonometric polynomial of order at most n. The same holds for $\sigma_{klm}(t)$ using

$$\sigma_{klm}(t) = S_{klm}(1, t) \tag{52}$$

(see (5) and (2)). (7) is a simple consequence of (1)—(5).

26. REMARK. Using (3) we get

$$\sum_{i=1}^{m}\cos jt_{im} = \begin{cases} m & \text{if } m|j, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sum_{i=1}^{m}\sin jt_{im} = 0.$$

By these identities, (52) and (51) we gain

$$\sigma_{klm}(t) = mb_{0klm} + 2m\sum_{i=1}^{[n/m]} b_{im,k,l,m}\cos imt. \tag{53}$$

Using (53), (49) and (50) we get (8) and (9) and the following relations:

$$\sigma_{30m}(t) = \frac{3}{4} + \frac{1}{4m^2} + \left(\frac{1}{4} - \frac{1}{4m^2}\right)\cos mt, \quad m = 1, 3, 5, \ldots, \tag{54}$$

$$\sigma_{31m}(t) = \frac{3}{4} + \frac{1}{4}\cos mt, \quad m = 2, 4, 6, \ldots, \tag{55}$$

$$\sigma_{40m}(t) = \frac{2}{3} + \frac{1}{3m^2} + \left(\frac{1}{3} - \frac{1}{3m^2}\right)\cos mt, \quad m = 1, 2, 3, \ldots, \tag{56}$$

$$\sigma_{42m}(t) = \frac{2}{3} - \frac{1}{6m^2} + \left(\frac{1}{3} + \frac{1}{6m^2}\right)\cos mt, \quad m = 1, 2, 3, \ldots. \tag{57}$$

We use them later.

27. PROOF OF THEOREM. 5. Let us suppose that m is even, say. (The odd values can be settled analogously.) Further, let us denote by $h=h_m(t)$ that index for which

$$t_{hm} \leqq t < t_{h+1,\,m}$$

(i.e., $h=[mt(2\pi)^{-1}]$) and let

$$u_m(t) := h_m(t) - \frac{m}{2} + 1, \quad v_m(t) := h_m(t) + \frac{m}{2}. \tag{59}$$

Using $g, s_{klm} \in \tilde{C}$, by (2)—(5)

$$S_{klm}(g, t) = \sum_{i=u}^{v} g(t_{im})\, s_{klm}(t - t_{im}), \quad \sigma_{klm}(g, t) = \sum_{i=u}^{v} s_{klm}(t - t_{im}).$$

Using these and (1), we get

$$Q_{klm}(g, t) - g(t) = \frac{Y_{klm}(g, t)}{\sigma_{klm}(t)}, \tag{60}$$

where

$$Y_{klm}(g, t) := \sum_{i=u}^{v} \big(g(t_{im}) - g(t)\big) s_{klm}(t - t_{im}). \tag{61}$$

Here

$$|g(t_{im}) - g(t)| \leqq \left(1 + \frac{km}{2\pi}|t - t_{im}|\right) \omega\left(g, \frac{2\pi}{km}\right). \tag{62}$$

Considering that $\left|\sin \frac{mt}{2}\right| \leqq 1$ further $|\cot t| \leqq 1/t$ whenever $0 < |t| \leqq \pi/2$, we have, using (4), the inequality

$$|s_{klm}(t)| \leqq \frac{2}{m|t|}\, |s_{k-1,\,l-1,\,m}(t)| \quad \text{if} \quad 0 < |t| \leqq \pi. \tag{63}$$

In (61) $|t-t_{im}|\leqq\pi$ (see (58) and (59)), i.e. using (60)–(63), (11) and the conditions of Theorem 5, we get (10).

28. LEMMA. *If* $k\geqq 2$ *then*

$$\|L_{klm}\| = 1. \tag{64}$$

29. PROOF OF THE LEMMA. Using (4) and (11), $|s_{klm}(t)|$ and $L_{klm}(t)$ are decreasing functions of k and l. If k and l are even, $s_{klm}(t)\geqq 0$ (see (4)), i.e. by (5) and (11)

$$L_{klm}(t) = \sigma_{klm}(t), \tag{65}$$

from where, considering (9), we get

$$L_{klm}(t) \leqq 1 \quad \text{whenever} \quad k \geqq 2. \tag{66}$$

Now by (3) and (4)

$$s_{klm}(t_{im}) = \begin{cases} 0 & \text{if} \quad 1 \leqq i < m, \\ 1 & \text{if} \quad i = m. \end{cases}$$

(Actually, if $t/2\pi$ is an integer, we have to use the definition

$$s_{klm}(t) := \lim_{\tau \to t} s_{klm}(\tau),$$

from where we get $s_{klm}(0)=1$ by (4). Using that $s_{klm}\in\bar{C}$, obviuously $s_{klm}(2\pi)=1$, too.) By these relations and (11) we obtain

$$L_{klm}(0) = 1. \tag{67}$$

Now (64) comes from (66) and (67).

30. PROOF OF THEOREM 7. If $k\geqq 3$ and $l\geqq 1$, (13) is a consequence of (10), (64) and (12). If $k\geqq 4$, even and $l\geqq 2$, even, (14) comes from (65), (64) and (12).

To prove (15), let

$$y_{iklm}(t) := \begin{cases} \sum_{j=u}^{i} s_{klm}(t-t_{jm}), & \text{if} \quad u \leqq i \leqq h, \\ \sum_{j=i}^{v} s_{klm}(t-t_{jm}), & \text{if} \quad h < i \leqq v. \end{cases} \tag{68}$$

From (61), using Abel transformation,

$$\begin{aligned} Y_{klm}(g, t) = &\sum_{i=u}^{h-1} \left(g(t_{im}) - g(t_{i+1,m})\right) y_{iklm}(t) + \\ &+ \left(g(t_{hm}) - g(t)\right) y_{hklm}(t) + \left(g(t_{h+1,m}) - g(t)\right) y_{h+1,k,l,m}(t) + \\ &+ \sum_{i=h+2}^{v} \left(g(t_{im}) - g(t_{i-1,m})\right) y_{iklm}(t). \end{aligned} \tag{69}$$

Here the absolute value of the differences $(g(.)-g(.))$ is less than or equal to $\omega(g, 2\pi/m)$ (see (3) and (58)), i.e.

$$|Y_{klm}(t)| \leqq \omega\left(g, \frac{2\pi}{m}\right) \sum_{i=u}^{v} |y_{iklm}(t)|. \tag{70}$$

By (4), (3) and (58) we get the relations

$$\operatorname{sgn} s_{klm}(t-t_{im}) = (-1)^{h-i}, \quad i = u, u+1, \ldots, h, \tag{71}$$

$$\operatorname{sgn} s_{klm}(t-t_{im}) = (-1)^{h+i+1}, \quad i = h+1, h+2, \ldots, v, \tag{72}$$

$$|s(t-t_u)| \leqq |s(t-t_{u+1})| \leqq \ldots \leqq |s(t-t_h)|, \tag{73}$$

$$|s(t-t_{h+1})| \geqq |s(t-t_{h+2})| \geqq \ldots \geqq |s(t-t_v)|. \tag{74}$$

So we conclude that $\operatorname{sgn} y_{iklm}(t) = \operatorname{sgn} s_{klm}(t-t_i)$ (see (68)), from where by (68), (71), (72), (5) and (11),

$$\sum_{i=u}^{v} |y_{iklm}(t)| = \sum_{i=u}^{h} |y_{iklm}(t)| + \sum_{i=h+1}^{v} |y_{iklm}(t)| = \tag{75}$$

$$= \sum_{i=u}^{h} (-1)^{h-i} \sum_{j=u}^{i} s_{klm}(t-t_{jm}) + \sum_{i=h+1}^{v} (-1)^{h+i+1} \sum_{j=i}^{v} s_{klm}(t-t_{jm}) =$$

$$= \sum_{i=0}^{[m/4-1/2]} s_{klm}(t-t_{h-2i,m}) + \sum_{i=0}^{[m/4-1/2]} s_{klm}(t-t_{h+1+2i,m}) =$$

$$= \frac{1}{2}\sigma_{klm}(t) + \frac{1}{2} L_{klm}(t).$$

Now, using (60), (70), (75), (64) and (12), we obtain (15).

31. PROOF OF THEOREM 9. By (12) and (54)

$$c_{30m} = \frac{1}{2} + \frac{1}{2m^2}. \tag{76}$$

Using this and (15) we get the relation

$$|Q_{30m}(g, t) - g(t)| \leqq \left(\frac{1}{2} + \frac{1}{1+\frac{1}{m^2}}\right) \omega\left(g, \frac{2\pi}{m}\right),$$

which is slightly better than (16). According to (55)

$$c_{31m} = \frac{1}{2}. \tag{77}$$

By (77), (15) and (13) one can obtain (17) and (18). Using (57) we get

$$c_{42m} = \frac{1}{3} - \frac{1}{3m^2}, \tag{78}$$

from where by (14) follows (19).

32. In Sections 32—43 we verify Theorem 10. The proof is based on Somorjai's paper [5].

In what follows $d_0, d_1, d_2, \ldots$ are positive numbers depending only on k and l,

$$h(T) := g(2\pi T), \tag{79}$$

$$v_m(T) = v_{klm}(T) := s_{klm}(2\pi T), \tag{80}$$

$$\tilde{M}(g, t) := M(h, T) := \overline{\lim_{u \to T}} \left| \frac{h(u)-h(T)}{u-T} \right| \quad \text{for} \quad t = 2\pi T, \tag{81}$$

$$\tilde{M}(g) := M(h) := \sup_{T \in [0,1]} M(h, t). \tag{82}$$

We use the following lemmas.

33. LEMMA. *$M(h, T)=0$ for arbitrary $T\in[0, 1]$ iff h is a constant.*

34. LEMMA. *$M(h)<\infty$ iff $h\in$ Lip 1.*

For the proof of these lemmas see [5, Lemma 2].

35. LEMMA. *If k and l are fixed then with a suitable small $\varrho>0$*

$$\frac{1}{2} < s_{klm}(t) \leqq 1 \quad \text{whenever} \quad |t| < \frac{\varrho}{m}. \tag{83}$$

36. PROOF OF LEMMA 35. The right hand side is obvious by (4), $|\sin mt| \leqq m|\sin t|$ and $|\cos t \leqq 1$. To get the other part, first we remark that one can suppose $t \geqq 0$ because s_{klm} is an even function. Now if ϱ is small enough, by

$$t - \frac{t^3}{6} \leqq \sin t \leqq t, \quad 1 - \frac{t^2}{2} \leqq \cos t \tag{84}$$

we get

$$s_{klm}(t) \geqq \left(1 - \frac{m^2 t^2}{24}\right)^k \left(1 - \frac{t^2}{2}\right)^l > \left(1 - \frac{\varrho^2}{24}\right)^k \left(1 - \frac{\varrho^2}{2}\right)^l > \frac{1}{2}. \tag{85}$$

37. LEMMA. *Let $\varrho>0$ be small enough. Then if $|t-t_j| \leqq \frac{\varrho}{m}$, we have*

$$\frac{1}{2} < s(t-t_i) \leqq \begin{cases} d_1 \left| \frac{\varrho}{j-i} \right|^k, & \text{if} \quad 0 < |j-i| \leqq \frac{m}{2}, \\ d_1 \left(\frac{\varrho}{m-|j-i|} \right)^k, & \text{if} \quad \frac{m}{2} < |j-i| < m. \end{cases} \tag{86}$$

38. PROOF OF LEMMA 37. Using (3) we get

$$\left|\sin m\frac{t+t_i}{2}\right| = \left|\sin m\frac{t-t_j}{2}\right|.$$

Now by (4), $|\cos t|\leqq 1$ and $\frac{2}{\pi}\leqq\frac{\sin t}{t}\leqq 1$ $\left(-\frac{\pi}{2}\leqq t\leqq\frac{\pi}{2}\right)$ we obtain

$$|s(t-t_i)| \leqq \left|\frac{\sin\frac{m}{2}(t-t_j)}{m\sin\frac{t-t_i}{2}}\right|^k \leqq \left|\frac{\pi}{2}\frac{t-t_j}{t-t_i}\right|^k = \left|\frac{\pi}{2}\frac{m(t-t_j)}{m(t-t_j)+2\pi(j-i)}\right|^k \tag{87}$$

from where we get the first estimation. The second one can be proved analogously. The left hand side comes from Lemma 35.

39. LEMMA. *We have the estimation*

$$|\sigma_{klm}(t)| \leqq 1 \quad \textit{whenever} \quad k\geqq 2. \tag{88}$$

40. PROOF OF LEMMA 39. The statement immediately follows from (5), (11) and (64).

41. LEMMA. *There exists a* $d_2>0$ *such that*

$$\overline{\lim_{m\to\infty}}\, m\|Q_{klm}(g)-g\| \geqq d_2\tilde{M}(g) = d_2M(h) \tag{89}$$

whenever $k\geqq 2$.

42. PROOF OF LEMMA 41. By (79)—(82)

$$Q_{klm}(g, 2\pi T)-g(2\pi T) = \left\{\frac{\sum_{i=1}^{n} h\left(\frac{i}{m}\right)-h(t)v_{klm}\left(T-\frac{i}{m}\right)}{\sigma_{klm}(2\pi T)}\right\}, \tag{90}$$

$T\in[0, 1]$. Now, by Lemmas 33, 34, 35, 37 and 39 we get, using Somorjai's argument in [5; Lemma 3] that

$$\overline{\lim_{m=\infty}}\, m \max_{0\leqq T\leqq 1}|\{\ldots\}| \geqq d_2M(h) \tag{91}$$

which by (79), (81), (82) and (90) gives (89).

43. PROOF OF THEOREM 10. When $g=c$ (or $g\in$Lip 1) we get (20) (or (21)) using Theorem 7 and the condition $c_{klm}\geqq\gamma_{kl}>0$. On the other hand, if we suppose (20) (or (21)) then we get the corresponding property of g, using 33, 34 and Lemma 42.

44. In Sections 44—48 we prove Theorem 12. The ideas are similar to those developed by D. Newman and J. Szabados (see e.g. [6]).

First we prove the following

LEMMA. *Let* $k_1=1$, k_2, k_3, ... *be a sequence of positive integer numbers with*

$$\Omega_q(3^{-k_{i+1}}) \leqq \frac{1}{2}\,\Omega_q(3^{-k_i}), \quad i = 1, 2, 3, \ldots, \tag{92}$$

$$\sum_{i=1}^{j-1} 3^{qk_i}\,\Omega_q(3^{-k_i}) \leqq 3^{qk_j}\,\Omega_q(3^{-k_j}), \quad j = 2, 3, 4, \ldots \tag{93}$$

(*these can be attained by* $\Omega_q(\delta)>0$ $(\delta>0)$ $\lim_{\delta\to 0}\Omega_q(\delta)=0$ *and* (23)),

$$g_i(t) := \cos 3^{k_i} t, \quad i = 1, 2, 3, \ldots, \tag{94}$$

$$g(t) := \sum_{i=1}^{\infty} \Omega_q(3^{-k_i})\, g_i(t). \tag{95}$$

Then

$$g \in C(\Omega_q). \tag{96}$$

45. PROOF OF LEMMA 44. By (92) and (94), $g\in\tilde{C}$. Let $0<\delta\leqq 2\pi$ and j be chosen such that

$$3^{k_j} \leqq \frac{1}{\delta} < 3^{k_{j+1}} \tag{97}$$

(when $\delta>\frac{1}{3}$, $j:=0$ and $k_0:=-2$). Obviously

$$\omega_q(g, \delta) \leqq \sum_{k=1}^{\infty} \Omega_q(3^{-k_i})\,\omega_q(g_i, \delta). \tag{98}$$

If $j\geqq 1$, by $\omega_q(g_i, \delta)\leqq 3^{qk_i}\delta^q$, (93), (97) and the inequality

$$\frac{\Omega_q(\Delta)}{\Delta^q} \leqq 2^q \frac{\Omega_q(\delta)}{\delta^q}, \quad 0 < \delta \leqq \Delta,$$

we can write

$$\sum_{i=1}^{j} \Omega_q(3^{-k_i})\,\omega_q(g_i, \delta) \leqq \sum_{i=1}^{j} 3^{qk_i}\delta^q\,\Omega_q(3^{-k_i}) \leqq 2\delta^q 3^{qk_j}\,\Omega_q(3^{-k_j}) \leqq 2^{q+1}\Omega(\delta). \tag{99}$$

Using $\omega_q(g_i, \delta)\leqq 2^q$, (92), (97) and the monotonicity of $\Omega_q(\delta)$, we get

$$\sum_{i=j+1}^{\infty} \Omega_q(3^{k_i})\,\omega_q(g_i, \delta) \leqq 2^q \sum_{i=j+1}^{\infty} \Omega_q(3^{-k_i}) \leqq 2^{q+1}\Omega_q(3^{-k_{j+1}}) \leqq 2^{q+1}\Omega_q(\delta). \tag{100}$$

By (98)—(100)

$$\omega_q(g, \delta) \leqq 2^{q+2}\Omega_q(\delta),$$

which proves (96), as it was stated.

46. LEMMA. *If* $\{k_i\}$ *and* $g(t)$ *are as in Section* 44 *and* $n:=3^{k_{j-1}}$, $j=2, 3, 4, \ldots$; *then*

$$W_n(g, t):= \sum_{i=1}^{j-1} \Omega_q(3^{-k_i})\, g_i(t) \tag{101}$$

is the best approximating rational trigonometric function to g of order $3^{k_{j-1}}$, *as defined by* (22).

47. PROOF OF LEMMA 46. We shall count the corresponding Chebyshev alternations. By (95) and (101)

$$g(t)-W_n(g, t) = \sum_{i=j}^{\infty} \Omega_q(3^{-k_i})\, g_i(t),$$

from where, using (94),

$$|g(t)-W_n(g, t)| \leqq \sum_{i=j}^{\infty} \Omega_q(3^{-k_i}).$$

Now let

$$\tau_p:= \frac{p\pi}{3^{k_j}}, \quad p = 1, 2, \ldots, 2\cdot 3^{k_j}.$$

Then

$$g(\tau_p)-W_n(g, \tau_p) = (-1)^p \sum_{i=j}^{\infty} \Omega_q(3^{-k_i}),$$

which means

$$\|g-W_n(g)\| = \sum_{i=j}^{\infty} \Omega_q(3^{-k_i}). \tag{102}$$

Obviously $0<\tau_1<\tau_2<\tau_3<\ldots\leqq 2\pi$, and the number of τ_j's equals $2\cdot 3^{k_j}$. On the other hand, the values of n, ν, μ and N (see Section 11) are

$$n = 3^{k_{j-1}}, \quad \nu = 0, \quad \mu = n \quad \text{and} \quad N = 4\cdot 3^{k_{j-1}}+2,$$

respectively. By $k_j>k_{j-1}$ we get

$$N < \frac{4}{3}\, 3^{k_j}+2 \leqq 2\cdot 3^{k_j},$$

which, by Section 11, gives Lemma 46.

48. PROOF OF THEOREM 12. By (102)

$$\|g-W_n(g)\| \geqq \Omega_q\left(\frac{1}{n}\right), \quad n = 3^{k_i}, \quad i = 1, 2, 3, \ldots.$$

This estimation and (96) prove (24).

Proofs for the algebraic case

49. PROOF OF THEOREM 14. Let

$$T_i(x) := \cos it, \quad i = 0, 1, 2, \ldots; \quad t = \arccos x, \tag{103}$$

be the i-th Chebyshev polynomial,

$$p_{0klm}(x) := s_{klm}(t), \tag{104}$$

$$p_{iklm}(x) := s_{klm}(t-t_{im})+s_{klm}(t+t_{im}), \quad i = 1, 2, \ldots, \left[\frac{m-1}{2}\right], \tag{105}$$

further, for even m

$$p_{m/2\,klm}(x) := s_{klm}(t-\pi). \tag{106}$$

By (103)—(106), (48), (3) and (27) we get

$$p_{0klm}(x) = b_{0klm}+2\sum_{j=1}^{n} b_{jklm}T_j(x), \tag{107}$$

$$p_{iklm}(x) = 2b_{0klm}+4\sum_{j=1}^{n} b_{jklm}T_j(x_i)\,T_j(x), \quad i = 1, 2, \ldots, \left[\frac{m-1}{2}\right], \tag{108}$$

and if m is even,

$$p_{m/2\,klm}(x) = b_{0klm}+2\sum_{j=1}^{n}(-1)^j b_{jklm}T_j(x). \tag{109}$$

The expressions (107)—(109) are polynomials of degree $\leqq n$. Using the definitions of $Q_{klm}(g, t)$, $R_{klm}(f, x)$ and $p_{iklm}(x)$ further (27) and (53), we get

$$R_{klm}(f, x) = \frac{\sum_{i=0}^{[m/2]} f(x_{im})p_{iklm}(x)}{mb_{0klm}+2m\sum_{i=1}^{[n/m]} b_{im,k,l,m}T_{im}(x)}, \tag{110}$$

from where we conclude that $R_{klm}(f, x)$ is a rational function, its numeator and denominator are polynomials of degree $\leqq n$. Finally, by (25), (7), (3) and (27) we get (26).

50. PROOF OF THEOREM 16. Let m be even, say (the case $m=$odd is similar). By (60) and (61)

$$R_{klm}(f, x)-f(x)=\frac{Z_{klm}(f, x)}{\sigma_{klm}(t)}, \tag{111}$$

where

$$Z_{klm}(f, x):=\sum_{i=u}^{v}\left(f(x_{im})-f(x)\right)s_{klm}(t-t_{im}). \tag{112}$$

By [4, (64)]

$$|f(x_{im})-f(x)| \leqq \left(1+\frac{km}{\pi}\left|\sin\frac{t-t_{im}}{2}\right|\right)\omega\left(f, \frac{2\pi}{km}\right), \tag{113}$$

further from (4) we get

$$m\left|\sin\frac{t-t_{im}}{2}\right|\cdot|s_{klm}(t-t_{im})| \leqq |s_{k-1,l,m}(t-t_{im})|. \tag{114}$$

Now, using (111)—(114) and (11), we obtain (29).

Let us remark that in (11) the index i can vary between u and $u+m-1$ where u is an arbitrary integer (see (59), say)

51. PROOF OF THEOREM 17. (30) is a consequence of (29), (64) and (12). (31) can be obtained by (65). Finally, using (15), (25) and

$$\omega(f\circ\cos, \delta) \leqq \omega(f, \delta) \tag{115}$$

we conclude inequality (32).

52. PROOF OF THEOREM 18. (33) follows from (30), (76) and (77), (34) from (32). Finally, using (31) and

$$c_{40m}=\frac{1}{3}+\frac{2}{3m^2} \tag{116}$$

(see (56)), we get

$$|R_{40m}(f, x)-f(x)| \leqq \left(1+\frac{12}{\pi}\frac{1}{1+\frac{2}{m^2}}\right)\omega\left(f, \frac{\pi}{2m}\right) \tag{117}$$

which is slightly better than (35).

53. PROOF OF THEOREM 19. By [4, (87)] we have

$$|f(x_{im})-f(x)| \leqq \left(1+\frac{km}{\pi}\left|\sin\frac{t-t_{im}}{2}\right|\cdot\left|\cos\frac{t-t_{im}}{2}\right|\right)\omega\left(f, \frac{2\pi\sqrt{1-x^2}}{km}\right)+ \tag{118}$$

$$+\left(1+\frac{k^2m^2}{2\pi^2}\sin^2\frac{t-t_{im}}{2}\right)\omega\left(f, \frac{4\pi^2|x|}{k^2m^2}\right).$$

Using (4), we can write

$$m\left|\sin\frac{t-t_{im}}{2}\right|\cdot\left|\cos\frac{t-t_{im}}{2}\right|\cdot|s_{klm}(t-t_{im})|\leqq|s_{k-1,l+1,m}(t-t_{im})| \tag{119}$$

and

$$m^2\sin^2\frac{t-t_{im}}{2}|s_{klm}(t-t_{im})|\leqq|s_{k-2,l,m}(t-t_{im})|. \tag{120}$$

Now, by (111), (112), (118)—(120) and (11) we obtain (36).

54. PROOF OF THEOREM 20. By (12), (36) and (64) we get (37). (38) can be obtained using (65). To get (39), we write

$$\begin{aligned}Z_{klm}(f,x)=&\sum_{i=u}^{h-1}\left(f(x_{im})-f(x_{i+1,m})\right)y_{iklm}(t)+\\&+\left(f(x_{hm})-f(x)\right)y_{hklm}(t)+\left(f(x_{h+1,m})-f(x)\right)y_{h+1,klm}(t)+\\&+\sum_{i=h+2}^{v}\left(f(x_{im})-f(x_{i-1,m})\right)y_{iklm}(t).\end{aligned} \tag{121}$$

By [4, (93), (95)] we have

$$\begin{aligned}|f(x_{im})-f(x_{i-1,m})|\leqq&\,\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\\&+\left(2+\frac{2m}{\pi}\left|\sin\frac{t_{im}-t}{2}\right|\cdot\left|\cos\frac{t_{im}-t}{2}\right|\right)\omega\left(f,\frac{2\pi^2}{m^2}|x|\right),\end{aligned} \tag{122}$$

$$|f(x_{hm})-f(x)|\leqq\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\omega\left(f,\frac{2\pi^2}{m^2}|x|\right), \tag{123}$$

$$|f(x_{h+1,m})-f(x)|\leqq\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\omega\left(f,\frac{2\pi^2}{m^2}|x|\right). \tag{124}$$

Using (75), (119) and (121)—(124), we can write as follows

$$\begin{aligned}|Z_{klm}(f,x)|\leqq&\left(\frac{1}{2}\sigma_{klm}^{(t)}+\frac{1}{2}L_{klm}(t)\right)\omega\left(f,\frac{2\pi}{m}\sqrt{1-x^2}\right)+\\&+\left(\sigma_{klm}(t)+L_{klm}(t)+\frac{2}{\pi}\sum_{i=0}^{[m/4-1/2]}s_{k-1,l+1,m}(t-t_{h-2i,m})+\right.\\&\left.+\frac{2}{\pi}\sum_{i=0}^{[m/4-1/2]}s_{k-1,l+1,m}(t-t_{h+1+2i,m})\right)\omega\left(f,\frac{2\pi^2}{m^2}|x|\right).\end{aligned}$$

Using this (111), (64), (12) and (11) we get (39).

55. PROOF OF THEOREM 21. (40) comes from (39), (76) and (77); (41) can be obtained using (38) and (116).

56. Proof of Theorem 22. First we prove the estimation (42). If $x=\pm 1$, the statement is obvious. On the other hand, if $x\in(-1, 1)$, by [4, 96)] we get

$$|f(x_{im})-f(x)| \leqq \omega\left(f, \frac{2\pi}{km}\sqrt{1-x^2}\right)\left(1+\frac{km}{\pi}\left|\sin\frac{t-t_{im}}{2}\right|\cdot\left|\cos\frac{t-t_{im}}{2}\right|+\right.$$

$$\left.+\frac{km}{\pi}|\cot t|\sin^2\frac{t-t_{im}}{2}\right).$$

Using this, (119) and

$$\frac{m}{|\sin t|}\sin^2\frac{t-t_{im}}{2}|s_{klm}(t-t_{im})| \leqq \frac{1}{2}|s_{k-2,l,m}(t-t_{im})|,$$

(which comes from (2)) we obtain (42).

To get (43), we have to use (42), (64) and (12); (44) comes from (65).

To obtain (45) let us use [4, (98), (99)] which are as follows:

$$|f(x_{im})-f(x_{i-1,m})| \leqq \left(2+3|\cot t|\cdot\left|\sin\frac{t_{im}-t}{2}\right|\right)\omega\left(f, \frac{2\pi}{m}\sqrt{1-x^2}\right),$$

$$|f(x_{hm})-f(x)| \leqq \left(2+|\cot t|\cdot\left|\sin\frac{t_{hm}-t}{2}\right|\right)\omega\left(f, \frac{2\pi}{m}\sqrt{1-x^2}\right),$$

$$|f(x_{h+1,m})-f(x)| \leqq \left(2+|\cot t|\cdot\left|\sin\frac{t_{h+1,m}-t}{2}\right|\right)\omega\left(f, \frac{2\pi}{m}\sqrt{1-x^2}\right).$$

By these estimations, (121), (75) and

$$\frac{\left|\sin\frac{t_{im}-t}{2}\right|}{|\sin t|}|s_{klm}(t-t_{im})| \leqq \frac{1}{2}|s_{k-1,l,m}(t-t_{im})|$$

(see (4)) we have

$$|Z_{klm}(f, x)| \leqq \omega\left(f, \frac{2\pi}{m}\sqrt{1-x^2}\right)\left(\sigma_{klm}(t)+L_{klm}(t)+\right.$$

$$\left.+\frac{3}{2}|x|\sum_{i=0}^{[m/4-1/2]}s_{k-1,l,m}(t-t_{h-2i,m})+\frac{3}{2}|x|\sum_{i=0}^{[m/4-1/2]}s_{k-1,l,m}(t-t_{n+1+2i,m})\right).$$

This inequality, (111), (64), (11) and (12) gives (45).

Finally, (46) can be obtained using (45), (77) and (78); (47) comes from (44) and (116).

57. Proof of Theorem 23. It is similar to the proof of Theorem 10, if we apply the relation $f(x)=f(\cos t)=g(t)$. We omit the details (see e.g. [2]).

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SPECTRAL MEASURES WITH SINGULARITIES

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Introduction

It is a well-known fact that there is a one-to-one correspondence between closed normal operators and strongly countably additive selfadjoint spectral measures in a Hilbert space or, more generally, between closed spectral operators of scalar type and strongly countably additive spectral measures in a Banach space [1]. However, the large majority of (bounded or closed) linear operators have so called spectral singularities. As it was proved in [5], for every closed operator T in a complex Banach space X there is a uniquely determined smallest closed subset of the extended spectrum $\sigma(T)$ such that a reasonable kind of (generalized) spectral measure for T cannot be defined for the Borel subsets of this set. More precisely, in the terminology of Section 1 below, there is a smallest one (the set of the spectral singularities) among the closed sets $K\subset\sigma(T)$, for which T has a $D(K, T)$-resolution of the identity.

In this paper we are concerned in a sense with the converse problem: given a (generalized) spectral measure of the above type, does there always exist a closed operator having it as its (generalized) resolution of the identity? To this end, the definition of a $D(K)$-spectral measure in Section 1 is made independent of the presence of an operator T (as contrasted with a $D(K, T)$-spectral measure). Nevertheless, we show in Proposition 1 that if T has a $D(K)$-resolution of the identity E, and $K_1=$ $=K\cap\sigma(T)$, then E extends to the $D(K_1, T)$-resolution of the identity for T. In Section 2 we give an example of a one-parameter continuous family $\{E_p\}$ of (in a sense nontrivial) $D(\{0\})$-spectral measures in a Hilbert space, for which the existence of an operator T_p having E_p as its $D(\{0\})$-resolution of the identity depends on the fact how rapidly the norms $|E_p(e)|$ tend to ∞ as the distance of e to 0 tends to 0. Here we use an idea of Harvey [3] in a modified form, and note that he did not say anything about the non-existence of T_p in the case of rapid growth.

In Section 3 we present necessary and sufficient conditions that, given a $D(K)$-spectral measure satisfying certain regularity conditions (in a Banach space X), there exists a closed operator T having it as its $D(K)$-resolution of the identity. These conditions are close in spirit to those given by Nel [6] for the characterization of closed spectral operators in the sense of Bade [1]. However, serious new technical problems stem from the fact that a $D(K)$-spectral measure is not necessarily uniformly bounded, and will be overcome by extending a method (originally for a Hilbert space and under more restrictive conditions) of Lyantse [4] for dealing with certain types of similar spectral measures.

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1.

Let X denote a complex Banach space, and let $B(X)$ and $C(X)$ denote the set of bounded and closed linear operators acting in X, respectively. Let A be a Boolean algebra (with respect to the usual set operations) of Borel subsets of the one-point compactification $\overline{C}$ of the complex plane C. A homomorphism E of A into a Boolean algebra of projections in $B(X)$ is called an A-spectral measure if $E(u)=I$ (where u and I are the identities in A and $B(X)$, respectively), and E is countably additive on A in the strong operator topology. Such an E is called an A-resolution of the identity for the operator $T\in C(X)$ if, in addition,

$$E(a)T\subset TE(a), \quad \sigma(T, E(a)X)\subset\bar{a} \quad (a\in A).$$

Here σ denotes the extended spectrum, $\bar{a}$ is the closure of a, and $\sigma(T, E(a)X)$ is the extended spectrum of the restriction $T|E(a)X\in C(E(a)X)$. We say that T is A-spectral, if it has an A-resolution of the identity E.

Let K be a closed set in $\overline{C}$, and let

$$D(K)=\{b \text{ Borel set: } \bar{b}\cap K=\emptyset \text{ or else } b^0\supset K\},$$

$$B_K=\{b \text{ Borel set: } b\cap K=\emptyset \text{ or else } b\supset K\},$$

where b^0 is the interior of b. Then $D(K)$ is an algebra, and B_K is a σ-algebra. Instead of B_K-spectral, etc., we also say K-spectral, etc. If $T\in C(X)$ is given, then let $\sigma=\sigma(T)$, let K be a closed subset of σ, and let

$$D(K, T)=\{b \text{ Borel set: } \overline{b\sigma}\cap K=\emptyset \text{ or else } (b\sigma)^r\supset K\},$$

where $b\sigma$ is $b\cap\sigma$ and h^r denotes the interior of the set $h\subset\sigma$ in the relative topology of σ (as a subset of $\overline{C}$). $D(K, T)$ is clearly an algebra. It was shown in [5] that T has at most one $D(K, T)$-resolution of the identity, and there is a smallest one among the sets $K\subset\sigma$, for which the operator T is $D(K, T)$-spectral. This smallest K is called the set of the spectral singularities or the smallest critical set for T. If $K=\emptyset$, and the $D(\emptyset)$-resolution of the identity E for T has the property that $E(\{\infty\})=0$, then the operator T is spectral in the sense of Dunford and Bade [1].

PROPOSITION 1. *If the operator $T\in C(X)$ has a $D(K)$-resolution of the identity E, and $K_1=K\sigma(T)=K\sigma$, then E can be extended to the $D(K_1, T)$-resolution of the identity F for T.*

PROOF. For the Borel set b in $\overline{C}$ let

$$F(b)=\begin{cases}E(b\sigma) & \text{if } \overline{b\sigma}\cap K_1=\emptyset, \quad (*)\\ E(b\cup\varrho) & \text{if } (b\sigma)^r\supset K_1, \quad (**)\end{cases}$$

where $\varrho=\overline{C}\setminus\sigma=\sigma^c$. In the first case

$$\overline{b\sigma}\cap K=b\sigma\cap(K_1\cup K\varrho)=\emptyset,$$

so $b\sigma\in D(K)$. In the second case let o denote an open set in $\overline{C}$. Then

$$(b\cup\varrho)^0=\cup\{o: o\subset b\cup\varrho\}\supset\cup\{o: o\sigma\subset b\sigma\}\cap\sigma=\cup\{o\sigma: o\sigma\subset b\sigma\}=(b\sigma)^r\supset K\sigma.$$

Further, $(b\cup\varrho)^0\supset K\varrho$, hence $(b\cup\varrho)^0\supset K$. So F is well-defined, $F(\sigma)=I$, and $F(b)+F(b^c)=I$ for every $b\in D(K_1,T)$. It is not hard to see that F is a $D(K_1,T)$-spectral measure commuting with T. We show first that F is multiplicative. If b_1, b_2 satisfy $(*)$, then b_1b_2 satisfies $(*)$, and

$$F(b_1)F(b_2)=E(b_1\sigma)E(b_2\sigma)=E(b_1b_2\sigma)=F(b_1b_2).$$

With understandable shorter notations, if $b_1\in(*), b_2\in(**)$, then $b_2^c\in(*), b_1b_2\in(*)$, and

$$F(b_1)F(b_2)=E(b_1\sigma)(I-E(b_2^c\sigma))=E(b_1\sigma)-E(b_1b_2^c\sigma)=E(b_1b_2\sigma)=F(b_1b_2).$$

If $b_1, b_2\in(**)$, then $b_1b_2\in(**)$, so

$$F(b_1)F(b_2)=E(b_1\cup\varrho)E(b_2\cup\varrho)=E(b_1b_2\cup\varrho)=F(b_1b_2).$$

If $b_1\in(*)$ and $b_2\in(**)$ are disjoint, then $b_1\cup b_2\in(**)$, and

$$F(b_1\cup b_2)=E(b_1\cup b_2\cup\varrho)=E(b_1\sigma\cup b_2\cup\varrho)=E(b_1\sigma)+E(b_2\cup\varrho)=F(b_1)+F(b_2).$$

We similarly prove the finite additvity in the remaining case, and the countable additivity of F is a consequence of that of E. If $b\in(*)$, then

$$\sigma(T,F(b)X)=\sigma(T,E(b\sigma)X)\subset\overline{b\sigma}\subset\bar{b}.$$

So it remains to show that

$$\sigma(T,F(b)X)=\sigma(T,E(b\cup\varrho)X)\subset\bar{b}\quad\text{for}\quad b\in(**).$$

Since $(b\sigma)^r\supset K\sigma$, the closed sets $\overline{\sigma\setminus b}$ and K are disjoint. So there is an open set $o(b)\subset\bar{C}$ such that

$$o(b)\supset K,\quad\overline{o(b)}\cap\overline{\sigma\setminus b}=\emptyset.\qquad(***)$$

Let $S=\overline{o(b)}$. Then the restriction of E to the σ-algebra B_S is the (unique) S-resolution of the identity for T. By the theory of the S-spectral operators (cf. [5]), then $E(S\cup\sigma)=I$, hence $E(S^c\varrho)=0$. We have

$$b\cup\varrho=b\sigma\cup S\varrho\cup S^c\varrho,$$

where the sets on the right-hand side are pairwise disjoint. So

$$E(b\cup\varrho)=E(b\sigma\cup S\varrho),$$

and this projection commutes with T. Therefore

$$\sigma(T,F(b)X)=\sigma(T,E(b\sigma\cup S\varrho)X)\subset\bar{b}\cup S\sigma.$$

By $(***)$, $S\sigma=\overline{o(b)}\sigma\subset\sigma b\subset b$. Hence $\sigma(T,F(b)X)\subset\bar{b}$, so F is the $D(K_1,T)$-resolution of the identity for T.

Finally, we show that F is an extension of E. If $\bar{b}\cap K=\emptyset$, then there is a closed neighbourhood $S(b)$ of K such that $\bar{b}\cap S(b)=\emptyset$. The restriction of E to the σ-algebra $B_{S(b)}$ is the $S(b)$-resolution of the identity for T. Hence we have again $E(S(b)^c\varrho)=0$, so

$$E(b\varrho)=E(b\varrho)E(S(b)^c\varrho)=0.$$

Therefore

$$F(b) = E(b\sigma) = E(b\sigma)+E(b\varrho) = E(b) \quad (\bar{b}\cap K = \emptyset)$$

and, taking complements, we obtain $F(b)=E(b)$ for each $b\in D(K)$. The proof is complete.

COROLLARY. *Each* $T\in C(X)$ *has at most one* $D(K)$*-resolution of the identity.*

Since $D(K)$ is, as a rule, not a σ-algebra, a $D(K)$-spectral measure E is not necessarily uniformly bounded. The question naturally arises whether, given a $D(K)$-spectral measure E, there always exists a closed operator T having E as its $D(K)$-resolution of the identity. If $\{\infty\}\in D(K)$ and $E(\{\infty\})X$ is a nonzero subspace of finite linear dimension, then it is clear that there is no $T\in C(X)$ such that

$$\sigma(T, E(\{\infty\})X) = \{\infty\}.$$

Therefore, we shall assume in what follows that $\{\infty\}\in D(K)$ implies $E(\{\infty\})=0$. Further, to avoid a trivial case, we shall also assume that $K\neq\bar{C}$.

2.

In this section we shall exhibit an example of a one-parameter continuous family $\{E_p\}$ of $D(\{0\})$-spectral measures for which the existence of an operator T_p with E_p as its $D(\{0\})$-resolution of the identity depends on how rapidly the norms $|E_p(e)|$ tend to ∞ as the distance of e to 0 converges to 0. In our example the behaviour changes at the values $p=1/2$ and $3/2$. However, it is clear that these values play no distinguished role: similar examples could be constructed showing that the rapidity of the growth is decisive, as it can be seen from the discussion below.

EXAMPLE 1. A Hilbert space H and a one-parameter family E_p $(0<p<\infty)$ of $D(\{0\})$-spectral measures with values in $B(H)$ and with the following properties:

1° for every $b\in D(\{0\})$, $p_0>0$ we have

$$\lim_{p\to p_0} |E_p(b)-E_{p_0}(b)| = 0;$$

2° for $0<p<3/2$ there is exactly one, and for $p\geqq 3/2$ there is no closed operator $T=T_p$ having $E=E_p$ as its $D(\{0\})$-resolution of the identity;

3° for $0<p<1/2$ the bounded operator T_p is, in fact, spectral in the sense of Dunford [1; XV. 2. 5].

Let $J=[0, 1]$, $X=L_2(J)$, and let H be the Hilbert space direct sum $X\oplus C$. Let p be a positive number, and let $k(b, t)$ denote the value of the characteristic function $k(b)=k(b, \cdot)$ of the set b in C at the point t. Let $e\subset J$ be a Borel set satisfying $0\notin\bar{e}$, let $h=(x, c)\in H$, and define

$$E(e)h = \begin{bmatrix} k(e,t) & t^{-p}k(e,t) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ c \end{bmatrix} = \begin{bmatrix} (x(t)+ct^{-p})k(e,t) \\ 0 \end{bmatrix}.$$

If $a\in D(\{0\})$, $0\notin\bar{a}$, then let $E(a)=E(aJ)$, and for $b=\bar{C}\setminus a$ let $E(b)=I-E(a)$. Then E is a $D(\{0\})$-spectral measure.

Assume that E is the $D(\{0\})$-resolution of the identity for $T\in C(H)$. If $b\in D(\{0\})$ contains the union of J with a neighbourhood of 0, then $\sigma(T)=\sigma(T, E(b)H)\subset\bar{b}$, hence $\sigma(T)\subset J$. So $T\in B(H)$. Let $K=\{0\}\cap\sigma(T)$, and let the $D(K, T)$-resolution of the identity for T be denoted again by E. Then $E(\sigma(T))=I$, so $E(\bar{C}\setminus J)=0$. Thus we may restrict ourselves to considering the projections $E(e)$ with e as above, and $E(d)=I-E(e)$ where $d=e^c=J\setminus e$. If e is a Borel set as above, then the subspace $E(e)H$ is $L_2(eJ)$, that is the space of those functions $f\in X$, for which $f=fk(e)$. The closed linear subspace generated by all these $E(e)H$ is clearly X. By assumption, T leaves each $E(e)H$ invariant. Since T is bounded, the subspace X is T-invariant.

For each e as above the operator $T_e=T|E(e)H$ is a bounded spectral operator in Dunford's sense. Hence, $T_e=S_e+N_e$, where $S_e\in B(L_2(eJ))$ is the scalar part determined by the resolution of the identity $E|E(e)H$, and N_e is the radical part of T_e. For every $x\in L_2(eJ)=E(e)H$ and every Borel set $b\subset e$ we have $E(b)x=k(b)x$, hence $(S_e x)(t)=tx(t)$.

The Boolean algebra of projections

$$B = \{E(b)|E(e)H\colon\ b \text{ Borel set},\quad b\subset e\}$$

in the subspace $E(e)H$ is σ-complete (cf. [1; XVII. 3. 1 and XVII. 3. 10]). For the function $x_0\colon e\to\{1\}$ the set

$$\{E(b)x_0\colon\ E(b)|E(e)H\in B\}$$

is clearly fundamental in $E(e)H$, i.e. its closed linear span is all of $L_2(eJ)$. By [1; XVII. 3. 14], the uniformly closed operator algebra A, generated by B, consists of those operators in $B(E(e)H)$, which commute with every element of B. Hence $N_e\in A$. By [1; XVII. 2.13], every element of A is a spectral operator of scalar type, for $L_2(eJ)$ is weakly complete. Since N_e is quasinilpotent, we obtain that $N_e=0$. By the continuity of T, we see that

$$(Tx)(t) = tx(t)\quad (x\in X).$$

Now let

$$T(0, 1) = (r(t), q)\in H.$$

Let e be as above, and let $d=e^c=J\setminus e$. By assumption, $\sigma(T, E(d)H)\subset\bar{d}$. Hence for $z\notin\bar{d}$ the map

$$T-z\colon\ E(d)H\to E(d)H$$

is bijective. Thus for every $(y, j)\in H$ there is an $(x, c)\in H$ such that

$$(T-z)E(d)(x, c) = E(d)(y, j),$$

i.e.

$$\begin{bmatrix}(t-z)x(t)k(d, t)-(t-z)ct^{-p}k(e, t)+cr(t)\\ c(q-z)\end{bmatrix} = \begin{bmatrix}y(t)k(d, t)-jt^{-p}k(e, t)\\ j\end{bmatrix}.$$

If $q\neq 0$, then there is a set d as above such that $q\notin\bar{d}$, hence for $z=q$ and $j\neq 0$ the equality above cannot be satisfied. Therefore $q=0$.

By assumption, T commutes with each $E(e)$. So

$$\begin{bmatrix} r(t)k(e,t) \\ 0 \end{bmatrix} = E(e)T(0,1) = TE(e)(0,1) = T\begin{bmatrix} t^{-p}k(e,t) \\ 0 \end{bmatrix} = \begin{bmatrix} t^{-p+1}k(e,t) \\ 0 \end{bmatrix}.$$

Hence for every e such that $0 \notin \bar{e}$

$$r(t)k(e,t) = t^{-p+1}k(e,t).$$

If $p \geqq 3/2$, then this equality contradicts $r \in L_2(J)$. So for these values of p there is no operator T having E as its $D(\{0\})$-resolution of the identity.

For $0<p<3/2$ we have $r(t)=t^{-p+1}\in L_2(J)$. So T has to act in H as follows:

$$T(x,c) = \begin{bmatrix} tx(t)+ct^{-p+1} \\ 0 \end{bmatrix}.$$

On the other hand, it is easily seen that T so defined has E as its $D(\{0\})$-resolution of the identity. Moreover, if $0<p<1/2$, then E extends to the σ-algebra of all Borel subsets of J in such a way that the extended mapping $\bar{E}$ is a (uniformly bounded) spectral measure in the sense of Dunford. Indeed, if e is a Borel subset of J such that $0 \notin e$, and $(x,c)\in H$, then let

$$\bar{E}(e)(x,c) = \begin{bmatrix} (x(t)+ct^{-p})k(e,t) \\ 0 \end{bmatrix} \in H.$$

For the set $d=e^c=J\setminus e$ let $\bar{E}(d)=I-\bar{E}(e)$. It is immediately seen that $\bar{E}$ is a strongly countably additive spectral measure on the Borel subsets of J, and T is the spectral operator of scalar type (uniquely) determined by the spectral measure $\bar{E}$.

Finally, if $0<p, p_0$, and E_p denotes the $D(\{0\})$-spectral measure E for the value p, then for any Borel subset e of J such that $0 \notin \bar{e}$ and $(x,c)\in H$ we have

$$(E_p(e)-E_{p_0}(e))(x,c) = \begin{bmatrix} c(t^{-p}-t^{-p_0})k(e,t) \\ 0 \end{bmatrix}.$$

Hence $\lim_{p\to p_0} |E_p(e)-E_{p_0}(e)|=0$, so $E_p(e)$ continuously depends on the value of p. From this, the same holds with e replaced by $d=J\setminus e$.

REMARK. The least elementary part of the proof, i.e. the proof of the fact that $N_e=0$, can be obtained in a more elementary way. Indeed, let $x_e(t)\equiv 1$ $(t\in e)$, and let

$$g = N_e x_e \in L_2(eJ), \quad e_n = \{t\in e\colon |g(t)| \leqq n\}.$$

If b is a bounded Borel function on e and $f\in L_2(eJ)$, then $bf\in L_2(eJ)$, and

$$N_e(bf) = N_e b(S_e)f = b(S_e)N_e f = bN_e f, \tag{1}$$

where $b(S_e)=\int_e b(t)E(dt)$. With the notations

$$x_n = x_e k(e_n), \quad g_n = gk(e_n),$$

we have $|g_n(t)| \leqq n$ $(t \in e)$. By (1),

$$N_e x_n = N_e(x_e k(e_n)) = k(e_n) g = g_n.$$

Assuming that $N_e^k x_n = g_n^k$ for a positive integer k, we obtain by (1)

$$N_e^{k+1} x_n = N_e(g_n^k x_n) = g_n^k N_e x_n = g_n^{k+1}.$$

Since N_e is quasinilpotent,

$$\lim_{k \to \infty} |N_e^k x_n|^{1/k} \leqq \lim_{k \to \infty} |N_e^k|^{1/k} = 0. \tag{2}$$

Denoting the $L_k(eJ)$-norm of g_n by $|g_n|_k$, we obtain from (2)

$$|g_n|_\infty = \lim_{k \to \infty} |g_n|_{2k} = \lim_{k \to \infty} \left(\int_e |g_n(t)|^{2k} dt \right)^{1/2k} = \lim_{k \to \infty} (|g_n^k|_2)^{1/k} = 0.$$

So in $L_2(eJ)$ we have

$$N_e x_e = g = \lim_{n \to \infty} g_n = 0.$$

By (1), for any bounded Borel function b on e

$$N_e b = N_e(b x_e) = b N_e x_e = 0.$$

By the continuity of N_e, we obtain $N_e = 0$.

3.

Let X be a complex Banach space, K a closed proper subset of $\bar{C}$, and E a $D(K)$-spectral measure in X, for which $E(\{\infty\}) = 0$ if $\{\infty\} \in D(K)$. Assume that $X = X_0 \oplus Y$, where (b will denote always an element of $D(K)$)

$$X_0 = \cap \{E(b^c) X : \bar{b} \subset K^c\}, \quad Y = \overline{\cup \{E(b) X : \bar{b} \subset K^c\}}. \tag{3}$$

Since for $x = x_0 + y$ $(x_0 \in X_0,\ y \in Y)$ and $\bar{b} \subset K^c$ we have $E(b)x = E(b)E(b^c)x_0 + E(b)y = E(b)y$, the second equality in (3) implies

$$Y = \overline{\cup \{E(b) Y : \bar{b} \subset K^c\}}.$$

Let $P(b) = E(b)|Y$ $(b \in D(K))$, then P is a $D(K)$-spectral measure in Y, for which $P(\{\infty\}) = 0$ if $\{\infty\} \in D(K)$. Let

$$D(P) = \{b \in D(K) : \bar{b} \subset K^c,\ b \text{ is bounded}\}.$$

We have then

$$Y = \overline{\cup \{P(b) Y : b \in D(P)\}}.$$

Indeed, for any y in Y there is a sequence $y_n = E(b_n) y_n$ such that $\bar{b}_n \subset K^c$ and $\lim_n y_n = y$. Further, for any n we have, by the countable additivity of E on $D(K)$, $y_n = E(b_n) y_n = \lim_k P(b_n^k) y_n$, where $b_n^k = b_n \cap \{z \in C : |z| \leqq k\} \in D(P)$.

LEMMA 1. *Let* $\{A(b): b\in D(P)\}\subset B(Y)$ *be such that* $b_1\subset b_2$, $b_1, b_2\in D(P)$ *imply*

$$P(b_1)A(b_2) = A(b_2)P(b_1) = A(b_1). \tag{4}$$

For every $b\in D(P)$ *and* $y=P(b)y$ *let* $A_0y=A(b)y$. *Then the operator* A_0 *is well-defined, linear and closable with domain* $D(A_0)=\cup\{P(b)Y: b\in D(P)\}$, *and its closure* $A\in C(Y)$ *satisfies*

$$AP(b)\supset P(b)A \quad (b\in D(K)).$$

PROOF. Let $b_1, b_2\in D(P)$ and $y=P(b_i)y$ $(i=1, 2)$. Then $b_1b_2=b_1\cap b_2\in D(P)$, and $y=P(b_1b_2)y$. Hence $A(b_1)y=A(b_1)P(b_1b_2)y=A(b_1b_2)y=A(b_2)P(b_1b_2)y=$ $=A(b_2)y$. So A_0 is well-defined on $D(A_0)$. Since $b_1, b_2\in D(P)$ imply $b_1\cup b_2\in D(P)$, and $A(b_1\cup b_2)$ is, by assumption, linear, A_0 is linear. Since $D(A_0)$ is dense in Y, A_0 is closable if and only if the domain of its dual $D(A_0')$ is a total set of linear functionals on Y (cf. Goldberg [2; II. 2. 11]).

Let $y\in D(A_0)$. Then $y=P(b)y$ for some $b\in D(P)$. Since $P(c)\in B(Y)$, its dual $P'(c)$ belongs to $B(Y')$ for every $c\in D(P)$. Let $y'=P'(c)y'$ for some $c\in D(P)$. Then

$$y'A_0y = P'(c)y'A(b)y = y'P(c)A(b)y = y'P(c)P(b)A(b)y = y'A(bc)y =$$

$$= y'A(c)P(b)y = A(c)'y'y.$$

Since $A(c)'y'\in Y'$, we obtain that

$$P'(c)Y'\subset D(A_0') \quad (c\in D(P)).$$

Assume now that for some $y\in Y$

$$P'(c)y'y = 0 \quad (c\in D(P),\ y'\in Y').$$

Then $y'P(c)y=0$, hence $E(c)y=P(c)y=0$ for every c in $D(P)$. By the countable additivity of E on $D(K)$, we obtain that $E(c)y=0$ for every $c\in D(K)$ such that $\bar{c}\subset K^c$. Hence $y\in X_0\cap Y$, so $y=0$. We have proved that $D(A_0')$ is total in Y', i.e. A_0 is closable in Y.

Let A denote the closure of A_0 in Y, let $y_0\in D(A_0)$ and $c\in D(P)$. Then $y_0=P(b)y_0$ for some $b\in D(P)$, and

$$P(c)Ay_0 = P(c)A(b)y_0 = A(b)P(bc)y_0 = A(b)P(c)P(b)y_0 = AP(c)y_0.$$

If $y\in D(A)$, then there is a sequence $\{y_n\}\subset D(A_0)$ such that $\lim_n y_n=y$ and $\lim_n Ay_n=Ay$. Hence

$$P(c)Ay = \lim_n P(c)Ay_n = \lim_n AP(c)y_n.$$

Since $\lim_n P(c)y_n=P(c)y$, and A is closed,

$$P(c)Ay = AP(c)y \quad (c\in D(P)). \tag{5}$$

If $d\in D(K)$, $\bar{d}\subset K^c$, and $d_k=d\cap\{z\in C\colon |z|\leqq k\}$, then the countable additivity of E implies

$$\lim_k P(d_k)y_1 = P(d)y_1 \quad (y_1\in Y).$$

Then, by (5), $\lim_k AP(d_k)y=\lim_k P(d_k)Ay=P(d)Ay$ for every y in $D(A)$. Since A is closed,

$$P(d)Ay = AP(d)y \quad (y\in D(A)).$$

Taking the complements of the sets d as above, we obtain

$$AP(b)\supset P(b)A \quad (b\in D(K)).$$

The proof is complete.

Let F denote the class of such Borel functions $f\colon C\setminus K\to C$ that are bounded on each set $b\in D(P)$. If $f\in F$, then let

$$f(b)=\int_b f(z)P(dz) \quad (b\in D(P)).$$

Since $P|P(b)Y$ is a spectral measure in $P(b)Y$, defined on the σ-algebra of all Borel subsets of b, it is uniformly bounded by a constant $k(b)$ for each $b\in D(P)$. Hence $f(b)\in B(Y)$, and

$$|f(b)|\leqq 4k(b)\sup\{|f(z)|\colon z\in b\}|P(b)|.$$

Further, if $b_1\subset b_2$, $b_1, b_2\in D(P)$, then

$$P(b_1)f(b_2)=f(b_2)P(b_1)=f(b_1).$$

For every $b\in D(P)$ and $y=P(b)y$ let $A_0(f)y=f(b)y$. By Lemma 1, the operator $A_0(f)$ has the closure $A(f)\in C(Y)$. If $f_1(z)=z$ $(z\in C\setminus K)$, then let $S=A(f_1)$.

Following Lyantse [4; pp. 106—107], for two families $\{A_i(b)\colon b\in D(P)\}$ $(i=1,2)$, satisfying condition (4) of Lemma 1, we define the following operations:

$$\{A_1(b)\}+\{A_2(b)\}=\{A_1(b)+A_2(b)\},\ \{A_1(b)\}\{A_2(b)\}= \tag{6}$$

$$=\{A_1(b)A_2(b)\},\ z\{A_1(b)\}=\{zA_1(b)\} \quad (z\in C).$$

It is immediately seen that the right-hand side families also satisfy condition (4) of Lemma 1. According to Lemma 1, they define closed operators in Y, which will be denoted by A_1+A_2, A_1A_2, zA_1, and which satisfy

$$A_1+A_2=\overline{A_1^0+A_2^0},\quad A_1A_2=\overline{A_1^0A_2^0},\quad zA_1=\overline{zA_1^0}, \tag{7}$$

where A_i^0 denote the restriction of A_i to

$$\cup\{P(b)Y\colon b\in D(P)\}.$$

An application of these definitions then defines

$$A(f_1)+A(f_2),\quad A(f_1)A(f_2),\quad zA(f_1)$$

for every $f_1, f_2\in F$. With these definitions one can prove the following lemmas similarly as Theorem 6 in [4].

LEMMA 2. *The mapping $f\to A(f)$ is an algebra homomorphism of F (with the usual operations) onto an algebra of operators in $C(Y)$ (with the operations defined above) satisfying $A(f_0)=I$ for the function $f_0(z)\equiv 1$ $(z\in C\setminus K)$.*

The proof will be omitted.

Let $c\in D(K)$. The set function

$$P_c(b) = P(b)|P(c)Y \quad (b\in D(K))$$

is then clearly a $D(K)$-spectral measure in $P(c)Y$, satisfying

$$\cap\{P_c(b)P(c)Y:\ b^0\supset K\}\subset\cap\{E(b)Y:\ b^0\supset K\} = \{0\}.$$

Making use of this fact, it is easily seen that Lemma 1 remains valid, if we replace Y by $P(c)Y$ and the measure P by P_c (with $D(P_c)=D(P)$).

Let $\{A_c(b):\ b\in D(P)\}\subset B(P(c)Y)$ satisfy condition (4) of this modification of Lemma 1. Let A_c be the minimal closed operator in $P(c)Y$, determined by $A_c(b)$ as in Lemma 1. Let

$$\sigma = \cup\{\sigma(A_c|P_c(b)P(c)Y):\ b\in D(P)\}(\subset C),$$

and let $H(A_c)$ denote the family of such functions $h:\ C\to C$ that are holomorphic in a neighbourhood (in C) of σ. Let $h\in H(A_c)$, $b\in D(P)$, and let

$$h(b) = (2\pi i)^{-1}\int_{g(b)} h(z)(z-A_c|P_c(b)P(c)Y)^{-1}\,dzP_c(b),$$

where $g(b)$ is the positively oriented boundary of a Cauchy domain (cf. [7; p. 288]) containing the set $\sigma(A_c|P_c(b)P(c)Y)$, and $g(b)$ lies in the domain of holomorphy of h. As in [4; pp. 108—109], it is easily seen that $\{h(b):\ b\in D(P)\}\subset B(P(c)Y)$ satisfies condition (4) of the modified Lemma 1, so it determines uniquely the operator $h(A_c)\in C(P(c)Y)$. With the suitably modified operations (6) and (7) we obtain the definitions of $h_1(A_c)+h_2(A_c)$, $h_1(A_c)h_2(A_c)$ and $zh_1(A_c)$ $(h_1, h_2\in H(A_c),\ z\in C)$. We shall omit again the easy proof of the following (cf. [4; Theorem 5])

LEMMA 3. *The mapping $h\to h(A_c)$ is an algebra homomorphism of $H(A_c)$ onto an algebra of operators in $C(P(c)Y)$ satisfying $h_0(A_c)=I$, $h_1(A_c)=A_c$ for the functions $h_0(z)\equiv 1$, $h_1(z)\equiv z$ $(z\in C)$.*

We are now in a position to prove

THEOREM 1. *Let X be a complex Banach space and let E be a $D(K)$-spectral measure in X satisfying the conditions stated at the beginning of Section 3. The operator $T\in C(X)$ has E as its $D(K)$-resolution of the identity if and only if*

$$T = T|X_0\oplus T|Y$$

(in the sense that $x=x_0+y\in D(T)$ if and only if $x_0\in D(T|X_0)$ and $y\in D(T|Y)$), where $\sigma(T, X_0)\subset K$, further $T|Y$ is the closure of an operator S_0+N_0, where S_0 is the restriction to

$$Y_0 = \cup\{E(b)Y:\ b\in D(P)\}$$

of the operator $S=A(f_1)$, $N_0:\ Y_0\to Y$ is a linear operator leaving each $E(b)Y$ $(b\in D(P))$ invariant and such that $N_0|E(b)Y$ $(b\in D(P))$ is quasinilpotent, finally

(i) *for* $c\in D(K)\setminus D(P)$, $c_1=c\cup\{\infty\}$, $z\notin\bar{c}_1$, $d_n=\{z\in C\colon |z|\leqq n,\ \text{dist}\ (z,K)\geqq n^{-1}\}$, *the sequence* $\{R_n\in B(E(d_nc)Y)\}$ *defined by*

$$R_n=\sum_{k=0}^{\infty}N_0^k\int_{d_nc}(z-v)^{-k-1}E(dv)|E(d_nc)Y$$

is bounded,

(ii) *if* $c\in D(K)$ *is bounded and* $c^0\supset K$, *then the sequence*

$$W_n=\int_{d_nc}vE(dv)|E(d_nc)Y+N_0|E(d_nc)Y$$

is bounded.

PROOF. *Sufficiency.* Let P be as before, and let

$$T(b)=(S_0+N_0)P(b)\quad(b\in D(P)).$$

By assumption, $T(b)\in B(Y)$. For $b_1,b_2\in D(P)$, $b_1\subset b_2$ we have

$$P(b_1)T(b_2)=P(b_1)(S_0+N_0)P(b_2)=(S_0+N_0)P(b_2)P(b_1)=$$
$$=T(b_2)P(b_1)=T(b_1).$$

Here we have made use of the facts that

$$P(b_1)S_0P(b_2)=P(b_1)f_1(b_2)=f_1(b_2)P(b_1)=S_0P(b_2)P(b_1),$$

$$P(b_1)N_0P(b_2)=P(b_1)N_0\big(P(b_1)+P(b_2\setminus b_1)\big)=N_0P(b_1)+$$

$$+P(b_1)P(b_2\setminus b_1)N_0P(b_2\setminus b_1)=N_0P(b_1)=N_0P(b_2)P(b_1).$$

So the family $\{T(b)\colon b\in D(P)\}$ satisfies condition (4) of Lemma 1, therefore the operator $T_1=\overline{S_0+N_0}$ exists, and is closed in Y. Let $T_0\in C(X_0)$ satisfy $\sigma(T_0)\subset K$, and let $T=T_0\oplus T_1$. First we show that

$$E(b)T\subset TE(b)\quad(b\in D(K)).\tag{8}$$

It is sufficient to prove this for the case $\bar{b}\subset K^c$. Assuming this, let $x_0\in D(T_0)$. Then $T_0x_0\in X_0$, so

$$E(b)Tx_0=E(b)E(b^c)T_0x_0=0=T_0E(b)E(b^c)x_0=TE(b)x_0.$$

By the definition of T_1, Lemma 1 yields that

$$E(b)T_1=P(b)T_1\subset T_1P(b)=T_1E(b).$$

So we have (8), and it remains to show that

$$\sigma\big(T,E(b)X\big)\subset\bar{b}\quad(b\in D(K)).$$

First let $b\in D(P)$. Then $E(b)X_0=E(b)E(b^c)X_0=\{0\}$, and $T_1|E(b)Y=(S_0+N_0)|E(b)Y\in B\big(E(b)Y\big)$. Here $S_0|E(b)Y$ is a bounded spectral operator of scalar type with resolution of the identity $E|E(b)Y$ and, by assumption, $N_0|E(b)Y$

is a quasinilpotent commuting with $E|E(b)Y$. So $T_1|E(b)Y$ is a bounded spectral operator with spectrum contained in $\bar{b}$. Hence

$$\sigma(T, E(b)X) = \sigma(T_1, E(b)Y) \subset \bar{b} \quad (b \in D(P)).$$

Let $c \in D(K) \setminus D(P)$, $c_1 = c \cup \{\infty\}$, $z \notin \overline{c_1}$, and d_n as in assumption (i). Then $d_n \in D(P)$, so, by the preceding paragraph, and with the notation of (i) (cf. [1; XV. 5.2])

$$R_n = (z - T_1|E(d_n c)Y)^{-1},$$

where $(z - T_1|E(d_n c)Y)^{-1} \in B(E(d_n c)Y)$.

Let T_c denote the operator $T_1|E(c)Y = T_1|P(c)Y$, which belongs to $C(P(c)Y)$. Let

$$Y_c = \cup \{P(b)P(c)Y: b \in D(P)\},$$

and let $S_c = S_0|Y_c$, $N_c = N_0|Y_c$. Then T_c is determined by S_c and N_c exactly as is T_1 by S_0 and N_0. In particular,

$$T_c = \overline{S_c + N_c}.$$

For $d \in D(P)$ we have $\sigma(T_c, P(d)P(c)Y) = \sigma(T_1, E(dc)Y) \subset \overline{dc} \subset \bar{c} \cap C$. Since $z \notin \overline{c_1}$, the function $g(v) = (z - v)^{-1}$ is holomorphic on a neighbourhood of $\bar{c} \cap C$. By the holomorphic functional calculus, defined above, $g \in H(T_c)$, and the operator $g(T_c) \in C(P(c)Y)$ satisfies in the sense of (7)

$$g(T_c)(z - T_c) = (z - T_c)\, g(T_c) = I \in B(P(c)Y). \tag{9}$$

Further, $g(T_c)|E(dc)Y = (z - T_1|E(dc)Y)^{-1}$, hence

$$R_n = g(T_c)|E(d_n c)Y.$$

By assumption (i), there is an $M > 0$ such that for every n

$$|g(T_c)|P(d_n c)Y| \leqq M.$$

Since $\cup \{P(d_n)P(c)Y: n = 1, 2, \ldots\}$ is dense in $P(c)Y$, we obtain that $g(T_c) \in B(P(c)Y)$ with norm $\leqq M$. By (9), it is easily seen that

$$\sigma(T_1, E(c)Y) = \sigma(T_c) \subset \overline{c_1}.$$

Since $E(c)X = E(c)X_0 \oplus E(c)Y$, we have

$$T|E(c)X = T_0|E(c)X_0 \oplus T_1|E(c)Y.$$

Now if $\bar{c} \subset K^c$, then $E(c)X_0 = \{0\}$, and $\overline{c_1} = \bar{c}$. Hence

$$\sigma(T, E(c)X) = \sigma(T_1, E(c)Y) \subset \bar{c} \quad (\bar{c} \subset K^c).$$

On the other hand, if $c^0 \supset K$, then $E(c)X_0 = X_0$, so

$$\sigma(T_0, E(c)X_0) = \sigma(T_0) \subset K.$$

Hence

$$\sigma(T, E(c)X) = \sigma(T_0, E(c)X_0) \cup \sigma(T_1, E(c)Y) \subset K \cup \overline{c_1} \subset \overline{c_1} \quad (c^0 \supset K).$$

Further, if c is unbounded, then $\overline{c_1}=\bar{c}$, so we have proved that $\sigma(T, E(c)X)\subset\bar{c}$. If c is bounded, then $\overline{c_1}$ is the disjoint union of $\bar{c}$ and $\{\infty\}$. We show that in this case $T|E(c)X\in B(E(c)X)$. By assumption (ii), there is an $M>0$ such that for every n

$$|T_c|P(d_n c)Y| = |(S_c+N_c)|P(d_n c)Y| \leqq M.$$

So we obtain as above that $T_c\in B(P(c)Y)=B(E(c)Y)$. Since $K\subset c^0$, K is also bounded, thus $T_0\in B(X_0)$. Hence

$$T|E(c)X = T_0|E(c)X_0\oplus T_c\in B(E(c)X).$$

So we have proved that E is the $D(K)$-resolution of the identity of the operator T.

Necessity. Let E be the $D(K)$-resolution of the identity of $T\in C(X)$. For every $x_0\in X_0\cap D(T)$ and $b\in D(K)$ such that $b^0\supset K$ we have

$$Tx_0 = TE(b)x_0 = E(b)Tx_0,$$

since T and $E(b)$ commute. Hence X_0 is T-invariant, and $T|X_0\in C(X_0)$. Since

$$X_0 = \cap\{E(b)X:\ b\in D(K),\quad b^0\supset K,\quad b \text{ is closed}\},$$

and for every such b the subspace $E(b)X$ consists of exactly those $x\in X$, the local spectrum $\sigma_T(x)$ of which is contained in b (cf. [5]), we obtain that

$$X_0 = \{x\in X:\ \sigma_T(x)\subset K\}.$$

Hence (cf. [8; IV. 3. 8]) $\sigma(T, X_0)\subset K$.

Let $y_0\in Y_0\cap D(T)$. Then $y_0=E(b)y_0$ for some $b\in D(P)$, and $E(b)Ty_0= =TE(b)y_0=Ty_0$. So T leaves Y_0 invariant, and $T|Y_0\subset T\in C(X)$ implies that the closure $\overline{T|Y_0}$ exists and belongs to $C(Y)$. For every b as above, the restriction $T|E(b)Y$ is a bounded spectral operator in $E(b)Y=P(b)Y$, so

$$T|E(b)Y = S_0|E(b)Y+N_b,$$

where $N_b\in B(P(b)Y)$ is a quasinilpotent operator commuting with $P|P(b)Y$. Let $N(b)=N_bP(b)$, and for every $b\in D(P)$ and $y=P(b)y$ let

$$N_0y = N(b)y = N_by.$$

By Lemma 1, N_0 is well-defined, linear and closable, maps Y_0 into Y, leaves each $E(b)Y$ invariant, and $N_0|E(b)Y=N_b$ is quasinilpotent ($b\in D(P)$). Further,

$$\overline{T|Y_0} = \overline{S_0+N_0}.$$

With the notations of (i) we have

$$|R_n| = |(z-T|E(d_n c)Y)^{-1}| \leqq |(z-T|E(c)X)^{-1}| \quad (n = 1, 2, \ldots),$$

so the sequence $\{R_n\}$ is bounded. With the notations of (ii) we have that

$$T|E(c)X\in B(E(c)X),$$

and

$$|W_n| = |T|E(d_n c)Y| \leqq |T|E(c)X| \quad (n = 1, 2, \ldots).$$

By the proof of the sufficiency, and with the notations there, the operator $\tilde{T}$ defined by

$$\tilde{T} = T|X_0 \oplus T_1 = T|X_0 \oplus \overline{T|Y_0}.$$

is $D(K)$-spectral with the $D(K)$-resolution of the identity E. On the other hand, it is clear that $\tilde{T} \subset T$.

Assume now that $x \in D(T) \setminus D(\tilde{T})$. For each $b \in D(K)$ then $E(b)x \in D(T)$. Further, for some $d \in D(K)$ such that $\overline{d} \neq \overline{C}$ we have then $E(d)x \notin D(\tilde{T})$. Indeed, since we have excluded the trivial case $K = \overline{C}$, there is d as above such that $\overline{d^c} \neq \overline{C}$. If we had $E(d)x$, $E(d^c)x \in D(\tilde{T})$, then we should have $x \in D(\tilde{T})$, a contradiction.

Let $z \in C$, $z \notin \overline{d}$. Since E is the $D(K)$-resolution of the identity of both $\tilde{T}$ and T, the mappings

$$z - \tilde{T}|E(d)X\colon\ E(d)X \cap D(\tilde{T}) \to E(d)X,$$

$$z - T|E(d)X\colon\ E(d)X \cap D(T) \to E(d)X$$

are bijections. However, $z - T|E(d)X$ is a proper extension of $z - \tilde{T}|E(d)X$, a contradiction. Hence $T = \tilde{T}$, therefore Y is T-invariant, $T|Y = \overline{T|Y_0} \in C(Y)$, and

$$T = T|X_0 \oplus T|Y.$$

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OPERATOR MOMENT THEOREMS FOR C^*-ALGEBRAS

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Dedicated to Professor I. Kátai on his 50th birthday

Introduction

Given a (complex) C^*-algebra A, a (multiplicative) *-subsemigroup G in A such that the linear span of it is norm dense in A, and a complex valued function f on G, we have obtained in a previous work [4, Theorem 1] a *necessary and sufficient condition* for f to have a positive linear extension to A. This condition is a new variant of positive definiteness with an additional norm boundedness with respect to the algebra A and reads as follows: *There exists a positive constant M such that*

$$\frac{1}{M}\left|\sum_g c_g f(g)\right|^2 \leqq \sum_{g,h} \bar{c}_h c_g f(h^*g) \leqq M\left\|\sum_g c_g g\right\|^2$$

for any finite sequence $\{c_g\}$ of complex numbers, indexed by elements of G.

We shall refer to this result as Theorem 0.

Theorem 1 of the present note generalizes this theorem to operator valued, namely, $B(H)$-valued functions f where $B(H)$ denotes the C^*-algebra of bounded linear operators on a (complex) Hilbert space H.

In Corollary 1 we characterize the positive linear maps on C^*-algebras, by their multiplicative structure.

As a consequence, in the case when A is a commutative C^*-algebra, we prove, in Theorem 4, a common generalization of the operator moment theorems of Sz.-Nagy and Stinespring. We need for this the Naimark dilation theorem in a more general case (Theorem 2) which has a consequence (Theorem 3) answering a question of Berberian about positive definite operator valued functions on locally compact Abelian groups in [1].

Positive linear operator valued functions

THEOREM 1. *Let f: $G \to B(H)$ be an operator valued function on a (multiplicative) *-semigroup G in a C^*-algebra A such that its linear span is norm dense in A. f has a (necessarily unique) positive linear extension φ to A if and only if there exists a positive constant M such that the inequalities*

$$\frac{1}{M}\left|\sum_g c_g (f(g)x,x)\right|^2 \leqq \|x\|^2 \sum_{g,h} \bar{c}_h c_g (f(h^*g)x,x) \leqq M\|x\|^4\left\|\sum_g c_g g\right\|^2 \tag{1}$$

hold for any finite sequence $\{c_g\}$ of complex numbers indexed by elements of G, and for every x in H.

PROOF. Let φ be a positive linear (hence continuous) extension of f to A. For any $x\in H$ let $\varphi_{x,x}$ be the positive linear functional on A defined for $a\in A$ by $\varphi_{x,x}(a)=(\varphi(a)x, x)$. Thus we have by the Schwarz type inequality for positive linear functionals on C^*-algebras (see [7] p. 22, Proposition 2.15):

$$|\varphi_{x,x}(a)|^2 \leqq \|\varphi_{x,x}\|\,\varphi_{x,x}(a^*a) = \|\varphi_{x,x}\|(\varphi(a^*a)x, x) \leqq$$
$$\leqq \|\varphi\|\|x\|^2(\varphi(a^*a)x, x),$$

and then for $a=\sum_g c_g g\in A$:

$$\Big|\sum_g c_g(f(g)x, x)\Big|^2 = |(\varphi(a)x, x)|^2 \leqq \|\varphi\|\|x\|^2(\varphi(a^*a)x, x) =$$
$$= \|\varphi\|\|x\|^2 \sum_{g,h} \bar{c}_h c_g (f(h^*g)x, x) \leqq \|\varphi\|\|x\|^4\|\varphi(a^*a)\| \leqq$$
$$\leqq \|\varphi\|^2\|x\|^4\|a^*a\| = \|\varphi\|^2\|x\|^4\|a\|^2 = \|\varphi\|^2\|x\|^4\Big\|\sum_g c_g g\Big\|^2.$$

To prove the sufficiency of (1) we consider the scalar valued function $f_x=(f(.)x, x)$ on G for a fixed x in H. Condition (1) then says, by Theorem 0, that there is a (necessarily unique) positive linear extension φ_x of f_x to A such that $\|\varphi\|_x\leqq M^{1/2}\|x\|^2$. Define for any pair (x, y) of elements in H a function

$$\varphi_{x,y} := \frac{1}{4}\{\varphi_{x+y}+i\varphi_{x+iy}-\varphi_{x-y}-i\varphi_{x-iy}\}$$

on A, which is linear, bounded in norm by $2M^{1/2}(\|x\|^2+\|y\|^2)$ and satisfies $\varphi_{x,x}=\varphi_x$ for any x in H. For a fixed element a in A we state: $\varphi_{x,y}(a)$ is a bounded bilinear functional of (x, y). This is clearly the case when $a=g$ is in G, by definition:

$$\varphi_{x,y}(g) = \frac{1}{4}\{(f(g)(x+y), x+y)+i(f(g)(x+iy), x+iy) -$$
$$-(f(g)(x-y), x-y)-i(f(g)(x-iy), x-iy)\} = (f(g)x, y)$$

hence when $a=\sum_g c_g g$ is in the linear span of G. But the last *-subalgebra is norm dense in A, by assumption, where $\varphi_{x+x',y}(a)-\varphi_{x,y}(a)-\varphi_{x',y}(a)$ vanishes, such that it vanishes also for any a in A. The homogeneity of $\varphi_{x,y}(a)$ in x has a similar proof. The properties of $\varphi_{x,y}(a)$ in y are seen also by a similar argument. The boundedness follows by

$$\|\varphi_{x,x}(a)\| = \|\varphi_x(a)\| \leqq 4M^{1/2}\|x\|^2$$

for any x in H.

Summing up — by a Theorem of Riesz — we have a bounded linear operator on H say $\varphi(a)$, such that

$$\varphi_{x,y}(a) = (\varphi(a)x, y) \quad (x, y\in H)$$

holds for any a in A. The map $a\mapsto\varphi(a)$ of A into $B(H)$ is linear, bounded and positive also by its extension property

$$(\varphi(g)x, y) = \varphi_{x,y}(g) = (f(g)x, y) \quad (x, y\in H)$$

since $(\varphi(.)x, x)=\varphi_{x,x}(.)$ is a positive linear functional on A for any x in H extending $(f(.)x, x)$. We have thus the desired extension of f to A and the proof is complete.

COROLLARY 1. *Let f: $A\to B(H)$ be an operator valued function on a C^*-algebra A. f is positive linear (hence bounded) if and only if there is a positive constant M such that*

$$\frac{1}{M}\left|\sum_a c_a(f(a)x, x)\right|^2 \leqq \|x\|^2 \sum_{a,b} \bar{c}_b c_a(f(b^*a)x, x) \leqq M\|x\|^4 \left\|\sum_a c_a a\right\|^2$$

holds for any finite sequence $\{c_a\}$ of complex numbers indexed by elements of A, and for any $x\in H$.

Naimark dilation of operator measures

THEOREM 2. *Let R be a ring of subsets of a (nonempty) set Ω and let F: $R\to B(H)$ be a positive operator valued function on R such that*

(i) $F(A\cup B)=F(A)+F(B)$ *if* $A, B\in R$ *and* $A\cap B=\emptyset$;

(ii) $\sup\{\|F(A)\|: A\in R\}<\infty$.

Then there exist a Hilbert space K, a bounded linear operator V of K into H, and an (orthogonal) projector valued finitely additive measure E on K with

(iii) $E(A\cap B)=E(A)E(B)$ *for* A, B *in* R

and

(iv) $F(A)=VE(A)V^*$ *holds for any* A *in* R.

Moreover, if F is a measure in the weak sense, i.e. if

(i') $\left(F\left(\bigcup_{n=1}^{\infty} A_n\right)x, x\right)=\sum_{n=1}^{\infty}(F(A_n)x, x)$ $(x\in H)$

holds for any sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint A_n in R with $\bigcup_{n=1}^{\infty} A_n\in R$, then E is a spectral measure.

PROOF. We need the Schwarz inequality for finitely many positive operators $\{P_n\}$ on H (due to Kadison [2]):

$$\left\|\sum_n P_n x_n\right\|^2 \leqq \left\|\sum_n P_n\right\| \sum_n (P_n x_n, x_n), \tag{2}$$

where $\{x_n\}$ is also a finite sequence of vectors in H, to prove

$$\left\|\sum_n F(A_n)x_n\right\|^2 \leqq \left\|F\left(\bigcup_n A_n\right)\right\| \sum_{m,n}(F(A_m\cap A_n)x_m, x_n) \tag{3}$$

for any finite sequence $\{A_n\}$ in R and $\{x_n\}$ in H.

Let $A=\bigcup_n A_n$, $A_n^0=A\cap A_n$, $A_n^1=A\setminus A_n$ and $A^{(k)}=\bigcap_n A_n^{k_n}$ where $k_n=0$ or 1 for any n. We have then a disjoint (finite) sequence $\{A^{(k)}\}$ in R such that

$$A_n = \cup\{A^{(k)}\colon\ k_n = 0\} \text{ for any } n,$$

hence

$$A_m\cap A_n = \cup\{A^{(k)}\colon\ k_m = 0 = k_n\} \text{ for any } m, n.$$

As a consequence, we also have (by (2))

$$\Big\|\sum_n F(A_n)x_n\Big\|^2 = \Big\|\sum\{F(A^{(k)})x_n\colon\ k_n = 0\}\Big\|^2 = \Big\|\sum_{(k)} F(A^{(k)})\big(\sum x_n\colon\ k_n = 0\big)\Big\|^2 \leqq$$

$$\leqq \Big\|\sum_{(k)} F(A^{(k)})\Big\|\Big(\sum_{(k)} F(A^{(k)})\Big)\big(\sum x_m\colon\ k_m = 0\big), \big(\sum x_m\colon\ k_n = 0\big)\Big) =$$

$$= \Big\|F\Big(\bigcup_{(k)} A^{(k)}\Big)\Big\|\sum_{(k)}\sum\{(F(A^{(k)})x_m, x_n)\colon\ k_m = 0 = k_n\} =$$

$$= \Big\|F\Big(\bigcup_n A_n\Big)\Big\|\sum_{m,n}(F(A_m\cap A_n)x_m, x_n).$$

Now let X be the linear space of functions on R into H with finite support, a general element of which is of the form $\sum_n A_n\otimes x_n$ (finite sum), where $A_n\otimes X_n$ denotes the function with value $X_n\in H$ in $A_n\in R$ and 0 otherwise. We have by (3) a semi-inner product on X defined by

$$\Big\langle\sum_n A_n\otimes x_n, \sum_m B_m\otimes y_m\Big\rangle = \sum_{m,n}(F(A_n\cap B_m)x_n, y_m). \tag{4}$$

A Hilbert space K with inner product $\langle\cdot,\cdot\rangle$ is given thus from X first by factoring with respect to the null-space of $\langle\cdot,\cdot\rangle$ and then completing this factor space with respect to the norm inherited from the inner product. For simplicity, the image X in K will be denoted by the same symbol.

The map V of K into H defined on X by

$$V\Big(\sum_n A_n\otimes x_n\Big) := \sum_n F(A_n)x_n \tag{5}$$

is linear and bounded by (3) and (ii), since this is so on a dense set of K. Define an operator valued function $E\colon\ R\to B(K)$ on R given on X by

$$E(A)\Big(\sum_n A_n\otimes x_n\Big) := \sum_n (A\cap A_n)\otimes x_n. \tag{6}$$

E is a contraction, since on a dense set of K

$$\Big\|E(A)\Big(\sum_n A_n\otimes x_n\Big)\Big\|^2 = \sum_{m,n}(F(A\cap A_m\cap A_n)x_n, x_m) =$$

$$= \Big\langle\sum_n (A\cap A_n)\otimes x_n, \sum_n A_n\otimes x_n\Big\rangle \leqq \Big\|\sum_n (A\cap A_n)\otimes x_n\Big\|\Big\|\sum_n A_n\otimes x_n\Big\| =$$

$$= \Big\|E(A)\Big(\sum_n A_n\otimes x_n\Big)\Big\|\Big\|\sum_n A_n\otimes x_n\Big\|.$$

A simple calculation shows that $E(A)$ is a self-adjoint projection. To prove (iv) we have to show

$$E(A)V^*x = A\otimes x \quad (A\in R, x\in H)$$

which is plain by observing (on X) that (by (4))

$$\langle \sum_n A_n\otimes x_n, E(A)V^*x\rangle = \langle E(A)\big(\sum_n A_n\otimes x_n, V^*x\rangle =$$

$$= \big(V\big(\sum_n (A\cap A_n)\otimes x_n, x\big) = \sum_n \big(F(A\cap A_n)x_n, x\big) = \langle \sum_n A_n\otimes x_n, A\otimes x\rangle.$$

Indeed, we have for any A in R and x in H

$$VE(A)V^*x = V(A\otimes x) = F(A)x,$$

proving (iv). Our construction gives a spectral measure in the case when F is not only a bounded finitely additive function but a measure in a weak sense. The proof is complete.

As a corollary we answer a question of Berberian [1] about weakly continuous positive definite, operator valued functions on a locally compact Abelian group replacing his k-dilation of an operator valued measure by ours.

THEOREM 3. *Let T: $G\to B(H)$ be an operator valued function on a locally compact Abelian group G which is weakly continuous and positive definite:*

(v) $G\ni g\mapsto (T_gx, x)$ *is continuous for every x in H,*

(vi) $0\leqq \sum_{m,n} (T_{g_m^{-1}g_n}x, x)$ *for any finite sequence $\{g_n\}$ in G.*

Then T is strongly continuous: the map $G\ni g\to T_gx$ is continuous on G for any x in H.

PROOF. Let Ω be the locally compact Abelian group which is the dual group of G in the Pontrjagin duality theory. As is shown in [1], there is an operator measure F on Ω such that

$$(T_gx, x) = \int_\Omega \hat{g}(t)\big(F(dt)x, x\big) \quad (g\in G, x\in H)$$

(with $\hat{g}(t)=t(g)$ for any $t\in\Omega$) and for which $\|F(dt)\|\leqq\|T_e\|$ holds, where e is the identity in G.

Let E be the corresponding spectral measure on some Hilbert space K (with (iv) for weak Borel sets A in Ω) by Theorem 2. Define the map S of G into $B(K)$ by

$$(S_gx, x) = \int_\Omega \hat{g}(t)\big(E(dt)x, x\big) \quad (g\in G, x\in H).$$

Then we have (as usual integration process gives) a weakly continuous unitary representation of G on K satisfying

$$T_g = VS_gV^* \quad (g\in G)$$

with V: $K\to H$ in Theorem 2. Hence for g, h in G, and x in H

$$\|T_gx-T_hx\|^2 \leqq \|V\|^2\|(S_g-S_h)V^*x\|^2 = \|V\|^2\big((S_g-S_h)V^*x, (S_g-S_h)V^*x\big) =$$

$$= 2\|V\|^2\{\|V^*x\|^2 - \mathrm{Re}(S_gV^*x, S_hV^*x)\} = 2\|V\|^2\{\|V^*x\|^2 - \mathrm{Re}(VS_{h^{-1}}S_gV^*x, x)\} =$$

$$= 2\|V\|^2\{\|V^*x\|^2 - \mathrm{Re}(VS_{h^{-1}g}V^*x, x)\} = 2\|V\|^2\{\|V^*x\|^2 - \mathrm{Re}(T_{h^{-1}g}x, x)\}.$$

Considering $h \to g$ in G we have by assumption

$$\mathrm{Re}(T_{h^{-1}g}x, x) \to \mathrm{Re}(T_e x, x) = \mathrm{Re}(VS_eV^*x, x) = \|V^*x\|^2$$

since $S_e = I$, the identity operator on K, we have $T_h x \to T_g x$ indeed. The proof is complete.

An operator moment theorem

We have a common generalization of theorems of Sz.-Nagy [5], [6] and Stinespring [3], in the case of commutative C^*-algebras.

THEOREM 4. *Let Ω be a locally compact Hausdorff space, G a (multiplicative) $*$-subsemigroup in $C_0(\Omega)$, the complex valued continuous functions vanishing at the infinity such that its linear span is norm dense in $C_0(\Omega)$. Let further $f\colon G \to B(H)$ be operator valued function on G with values operators on a Hilbert space H.*

Then there is a spectral measure $E(\cdot)$ on Ω, with values on some Hilbert space K and a continuous operator V from K into H such that

$$(7) \qquad (f(g)x, x) = \int_\Omega g(t)(E(dt)V^*x, V^*x) \quad (g \in G, x \in H)$$

if and only if there is a positive constant M such that (1) *holds for any finite sequence $\{c_g\}$ of complex numbers indexed by elements of G.*

PROOF. Since the right hand side of (7) defines an extension of f to a positive linear function φ of $C_0(\Omega)$ into $B(H)$, the necessity of (1) is obvious by the proof of Theorem 1. Conversely, assuming (1) we have by Theorem 1 a positive linear extension $\varphi\colon C_0(\Omega) \to B(H)$ of f and then a measure $F(\cdot)$ as (Berberian has proved) in the proof of Theorem 3. This has a spectral measure (as a Naimark dilation) by Theorem 2 with the desired property (7).

The proof is complete.

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STRONGLY ALMOST SUMMABLE AND STRONGLY ALMOST CONVERGENT SEQUENCES

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1. Let S be the set of all real or complex sequences and let l_∞, c, c_0 denote the Banach spaces of bounded, convergent and null sequences $x=\{x_k\}_{k=0}^{\infty}$, respectively, normed as usual by $\|x\|=\sup_{k\geq 0}|x_k|$. Let D be the shift operator on S, that is, $Dx=\{x_k\}_{k=1}^{\infty}$, $D^2x=\{x_k\}_{k=2}^{\infty}$ and so on. It may be recalled that (see Banach [1]) the Banach limit L is a non-negative linear functional on l_∞ such that L is invariant under the shift operator (that is, $L(Dx)=L(x)$ for all $x\in l_\infty$) and that $L(e)=1$ where $e=\{1, 1, \ldots\}$. A sequence $x\in l_\infty$ is called almost convergent (see Lorentz [4]) if all Banach limits of x coincide. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [4] proved that

$$\hat{c}=\left\{x:\ \lim_{m\to\infty}\frac{1}{m+1}\sum_{i=0}^{m}x_{n+i}\ \text{exists uniformly in } n\right\}.$$

Several authors including Lorentz [4], King [3] and Nanda [8, 10] have studied almost convergent sequences. Just as convergence gives rise to absolute and strong convergence, it was quite natural to expect that almost convergence must give rise to the concepts of absolute almost and strong almost convergence. Absolutely almost and strongly almost convergent sequences have been introduced and discussed in a natural way by Das, Kuttner and Nanda [2] and Maddox [7] respectively.

The summability methods of real or complex sequences by infinite matrices are of three types (see Maddox [6], p. 185) — ordinary, absolute and strong. In the same vein, it is expected that the concept of almost convergence must give rise to three types of summability methods — almost, absolutely almost and strongly almost. The almost summable sequences have been discussed by King [3], Schaefer [12] and some others. More recently Das, Kuttner and Nanda [2] and Nanda [9] have introduced the concept of absolute almost summability along with the concept of absolute almost convergence. The spaces of strongly almost summable sequences were defined and studied by Nanda [8], but the definition introduced there is not very satisfactory and seems to be unnatural. By specialising the infinite matrix in the definition introduced in [8] we do not get strongly almost convergent sequences.

In Section 2 we introduce an alternative definition of strong almost summability. This definition seems to be more natural and contains the definition of strong almost convergence as introduced by Maddox [7] as a special case. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [5].

Let $\hat{c}_0$ denote the set of all sequences which are almost convergent to zero. In [10] the spaces $\hat{c}$ and $\hat{c}_0$ were extended to $\hat{c}(p)$ and $\hat{c}_0(p)$ in the same manner as l_∞, c

and c_0 were extended to $l_\infty(p)$, $c(p)$ and $c_0(p)$ respectively (see Maddox [5, 6]). In Section 3 we generalise the spaces $\hat{c}(p)$ and $\hat{c}_0(p)$ with the help of infinite matrices.

2. Let $A=(a_{nk})$ be an infinite matrix of non-negative real numbers and let $p=\{p_k\}$ be a sequence of positive real numbers. (These assumptions are made throughout.) We write $A_n(x)=\sum_k a_{nk}|x_k|^{p_k}$ if the series converges for each n and $A_{m,n}(x)=\sum_k a_{mk}|x_{k+n}|^{p_k}$ if the series converges for each m and n. (Here and afterwards summation without limits runs from 1 to ∞.) We define

$$[\hat{A}, p]_0 = \{x: A_{mn}(x) \to 0 \quad \text{as} \quad m \to \infty \quad \text{uniformly in } n\},$$

$$[\hat{A}, p] = \{x: A_{mn}(x-l_e) \to 0 \quad \text{as} \quad m \to \infty \text{ for some } l \text{ uniformly in } n\},$$

$$[\hat{A}, p]_\infty = \{x: \sup_{m,h} A_{mn}(x) < \infty\}.$$

The sets $[\hat{A}, p]_0$, $[\hat{A}, p]$ and $[\hat{A}, p]_\infty$ will be called the spaces of strongly almost summable to zero, strongly almost summable and strongly almost bounded, respectively. If $[\hat{c}]$ denotes the set of all strongly almost convergent sequences, then (see Maddox [7])

$$[\hat{c}] = \{x: \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{k+n}-l| = 0 \text{ uniformly in } n\}.$$

Note that if $A=(c, 1)$ and $p_k=1$ for all k, then $[\hat{A}, p]=[\hat{c}]$. Also in this case $[\hat{A}, p]_\infty=l_\infty$. For if $A=(c, 1)$ and $p_k=1$ for all k, then

$$A_{mn}(x) = \frac{1}{m} \sum_{k=1}^{m} |x_{k+n}| = \frac{1}{m} \sum_{k=n+1}^{n+m} |x_k|.$$

Now

$$\sup_{m,n} A_{mn}(x) \leqq \sup_m \frac{\sup_k |x_k|}{m} \sum_{k=n+1}^{n+m} 1 = \sup_k |x_k|.$$

and

$$\sup_{m,n} A_{mn}(x) \geqq \sup_{1,n} A_{1,n}(x) = \sup_n |x_{n+1}|.$$

We have

THEOREM 1. *If $p\in l_\infty$, then $[\hat{A}, p]_0$, $[\hat{A}, p]$ and $[\hat{A}, p]_\infty$ are linear spaces over the complex field* **C**.

PROOF. We consider only $[\hat{A}, p]$. If $H=\sup p_k$ and $K=\max(1, 2^{H-1})$, we have (see Maddox [5], p. 346)

$$|a_k+b_k|^{p_k} \leqq K(|a_k|^{p_k}+|b_k|^{p_k}) \tag{1}$$

and for $\lambda\in\mathbf{C}$,

$$|\lambda|^{p_k} \leqq \max(1, |\lambda|^H). \tag{2}$$

Suppose $x_k\to l[\hat{A}, p]$ and $y_k\to l'[\hat{A}, p]$ and $\lambda, \mu\in\mathbf{C}$. Then from (1) and (2) it follows that

$$A_{mn}(\lambda x+\mu y-(\lambda l+\mu l')e) \leqq K \sup |\lambda|^{p_k} A_{mn}(x-le)+K \sup |\mu|^{p_k} A_{mn}(y-l'e).$$

This implies that $\lambda x+\mu y\to\lambda l+\mu l'[\hat{A}, p]$ and completes the proof.

The following theorem gives inclusion of $[\hat{A}, p]$ spaces. We have been able to prove inclusion only for a special class.

THEOREM 2. *Suppose that* $\|A\|<\infty, 0<p_k\leqq q_k$ *and* $\frac{q_k}{p_k}$ *is bounded. Then* $[\hat{A}, q]\subset [\hat{A}, p]$.

PROOF. Write $w_{k,n}=|x_{k+n}-l|^{q_k}$, $\lambda_k=\frac{p_k}{q_k}$, so that $0<\lambda\leqq\lambda_k\leqq 1$. Define

$$u_{k,n}=w_{k,n}(w_{k,n}\geqq 1)=0\ \ (w_{k,n}<1),\quad v_{k,n}=0\ \ (w_{k,n}\geqq 1)=w_{k,n}(w_{k,n}<1).$$

So

$$w_{k,n}=u_{k,n}+v_{k,n},\quad w_{k,n}^{\lambda_k}=u_{k,n}^{\lambda_k}+v_{k,n}^{\lambda_k}.$$

Now it follows that $u_{k,n}^{\lambda_k}\leqq u_{k,n}\leqq w_{k,n}$, $v_{k,n}^{\lambda_k}\leqq v_{k,n}^{\lambda}$. We have the inequality (see Maddox [5], p. 251)

$$\sum_k a_{mk}w_{k,n}^{\lambda_k}\leqq\sum_k a_{mk}w_{k,n}+\left(\sum_k a_{mk}v_{k,n}\right)^{\lambda}\|A\|^{1-\lambda}.$$

Therefore, if $x\in[\hat{A}, q]$, then $x\in[\hat{A}, p]$ and this completes the proof of the theorem.

If x is strongly almost summable to l we write $x_k\to l[\hat{A}, p]$. A pair (A, p) will be called strongly almost regular if $x_k\to l$ implies that $x_k\to l[\hat{A}, p]$. We now characterise the class of strongly almost regular matrices and consider the uniqueness of generalised limits of strongly almost summable sequences.

Let X and Y be any two nonempty subsets of the space S of all sequences. If $x\in X$ implies that $\left\{\sum_k a_{nk}x_k\right\}\in Y$, we say that A defines a matrix transformation from X into Y, and we write $A: X\to Y$. (X, Y) denotes the class of matrices A such that $A: X\to Y$. It is known that $A\in(c_0, c_0)$ if and only if $\|A\|=\sup_m\sum_k a_{mk}<\infty$ and $a_{mk}\to 0$ $(m\to\infty,\ k$ fixed).

THEOREM 3. *Suppose that* $A\in(c_0, c_0)$ *and* $p=\{p_k\}$ *converges to a positive limit. Then* $x_k\to l$, $x_k\to l[\hat{A}, p]$, $x_k\to l'[\hat{A}, p]$ *imply* $l=l'$ *if and only if*

$$\sum_k a_{mk}\to 0\quad (m\to\infty). \tag{3}$$

PROOF. *Necessity*. Suppose that $A\in(c_0, c_0)$ *and* $\{p_k\}$ is bounded. Let $x_k\to l$ imply that $x_k\to l\,[\hat{A}, p]$ uniquely. We have $e\to 1\,[\hat{A}, p]$. Therefore condition (3) must hold, for otherwise $e\to 0\,[\hat{A}, p]$ which contradicts the uniqueness of l.

Sufficiency. Suppose that condition (3) holds, $A\in(c_0, c_0)$ and $p_k\to r>0$. Further assume that $x_k\to l$ implies $x_k\to l[\hat{A}, p]$ and $x_k\to l'[\hat{A}, p]$ where $|l-l'|=a>0$. Then we get

$$\lim_{m\to\infty}\sum a_{mk}u_{nk}=0\quad\text{(uniformly in } n) \tag{4}$$

where

$$u_{n,k}=|x_{k+n}-l|^{p_k}+|x_{k+n}-l'|^{p_k}.$$

By the assumption $u_{nk}\to a^r$. Since $A\in(c_0, c_0)$, $u_{nk}\to a^r$ implies that

$$\sum_k a_{mk}|u_{nk}-a^r|\to 0\quad\text{as}\quad m\to\infty\quad\text{(uniformly in } n). \tag{5}$$

But we have

$$a^r \sum_k a_{mk} \leqq \sum_k a_{mk} u_{nk} + \sum_k a_{mk} |u_{nk} - a^r|. \tag{6}$$

By (4), (5) and (6) it follows that

$$\lim_{m \to \infty} \sum_k a_{mk} = 0.$$

Since this contradicts (3), we must have $l = l'$ and this completes the proof.

THEOREM 4. *Let* $0 < \theta \leqq p_k \leqq H < \infty$. *Then* (A, p) *is strongly almost regular if and only if* $A \in (c_0, c_0)$.

To prove Theorem 4 we require the following result.

LEMMA 1 (see Maddox [5], p. 347). *If* $p_k, q_k > 0$, *then* $c_0(q) \subset c_0(p)$ *if and only if* $\liminf_{k \to \infty} \frac{p_k}{q_k} > 0$.

PROOF OF THEOREM 4. *Necessity*. Suppose that (A, p) is strongly almost regular. Therefore

$$|x_k - l|^{1/p_k} \to 0 \Rightarrow \sum_k a_{mk} |x_{n+k} - l| \to 0 \quad \text{as} \quad m \to \infty$$

uniformly in n. Again since $1/p_k \geqq 1/H > 0$, by Lemma 1,

$$x_k \to l \Rightarrow |x_k - l|^{1/p_k} \to 0.$$

Thus

$$x_k \to l \Rightarrow \sum_k a_{mk} (x_{n+k} - l) \to 0 \quad \text{as} \quad m \to \infty$$

uniformly in n and therefore $A \in (c_0, c_0)$.

Sufficiency. Since $p_k \geqq \theta > 0$, by Lemma 1

$$x_k \to l \Rightarrow |x_k - l|^{p_k} \to 0.$$

Again we have $A \in (c_0, c_0)$. Therefore $x_k \to l[\hat{A}, p]$ and this completes the proof.

3. Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers and let $\{p_k\}$ be a bounded sequence of positive real numbers. We write $B_{mn}(x) = \sum_k a_{mk} x_{k+n}$ if the series converges for each m and n. We define

$$(\hat{A}, p)_0 = \{x: |B_{mn}(x)|^{p_m} \to 0 \quad \text{as} \quad m \to \infty \text{ uniformly in } n\},$$

$$(\hat{A}, p) = \{x: |B_{mn}(x - le)|^{p_m} \to 0 \quad \text{as} \quad m \to \infty \text{ for some } l \text{ uniformly in } n\},$$

$$(\hat{A}, p)_\infty = \{x: \sup_{m,n} |B_{mn}(x)|^{p_m} < \infty\}.$$

Note that if $A = (c, 1)$, then $(\hat{A}, p)_0 = \hat{c}_0(p)$, $(\hat{A}, p) = \hat{c}(p)$ and $(\hat{A}, p)_\infty = \hat{m}(p)$. The space $\hat{m}(p)$ was introduced and discussed in [11], but it may be remarked here that if $p_m = p$ for all m, then $\hat{m}(p) = l_\infty$ and this follows from the following inequalities:

$$\sup_{m,n} \left| \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \right| \leqq \sup_n |x_n|$$

and

$$\sup_{m,n}\left|\frac{1}{m+1}\sum_{i=0}^{m}x_{n+i}\right| \leqq \sup_{n}|x_n|\frac{1}{m+1}\sum_{i=0}^{m}1 = \sup_{n}|x_n|.$$

We may, without loss of generality, assume that $0<p_m\leqq 1$, for if $0<p_m<\infty$ and $\sup p_m<\infty$, then $0<\frac{p_m}{\sup p_m}\leqq 1$.

It is clear that $(\hat{A},p)_0\subset(\hat{A},p)$ and $(\hat{A},p)_0\subset(\hat{A},p)_\infty$. But we have been able to prove $(\hat{A},p)\subset(\hat{A},p)_\infty$ only for a special case. We have

THEOREM 5. *$(\hat{A},p)\subset(\hat{A},p)_\infty$ if*

$$\sup_{m}\left|\sum_{k}a_{mk}\right|^{p_m}<\infty. \tag{7}$$

PROOF. Suppose that $x\in(\hat{A},p)$ and (7) holds. We have

$$|B_{mn}(x)|^{p_m} = B_{mn}(x-le+le)|^{p_m} \leqq \tag{8}$$

$$\leqq |B_{mn}(x-le)|^{p_m}+|l\sum_{k}a_{mk}|^{p_m} \leqq |B_{mn}(x-le)|^{p_m}+\sup|l|^{p_m}\left|\sum_{k}a_{mk}\right|^{p_m}.$$

Therefore $x\in(\hat{A},p)_\infty$ and this completes the proof.
We have

THEOREM 6. *$(\hat{A},p)_0$ is a linear topological space paranormed by h defined by*

$$h(x)=\sup_{m,n}|B_{mn}(x)|^{p_m}.$$

$(\hat{A},p)_\infty$ is paranormed by h if $\inf p_m>0$. If (7) holds, then $(\hat{A},p)$ is paranormed with the same paranorm h. $(\hat{A},p)_0$ and $(\hat{A},p)_\infty$ are complete in their paranorm topologies. (A,p) is complete if

$$\left|\sum_{k}a_{mk}\right|^{p_m}\to 0 \quad as \quad m\to\infty \tag{9}$$

holds.

PROOF. This is routine verification and can be obtained by using standard techniques. If (7) holds, then by Theorem 5, $(\hat{A},p)\subset(\hat{A},p)_\infty$ and hence $(\hat{A},p)$ has the same paranorm h. If (9) holds, then (7) also holds and it follows from the inequality (8), that $(\hat{A},p)=(\hat{A},p)_0$ and therefore the completeness of $(\hat{A},p)$ follows from the completeness of $(\hat{A},p)_0$.

THEOREM 7. *If $0<p_m<q_m\leqq 1$, then $(\hat{A},q)_\infty$ is a closed subspace of $(\hat{A},p)_\infty$.*

PROOF. It is clear that $(\hat{A},q)_\infty$ is a subset of $(\hat{A},p)_\infty$. To show that $(\hat{A},q)_\infty$ is closed, suppose that $x^i\in(\hat{A},q)_\infty$, $x^i\to x$ and $x\in(\hat{A},p)_\infty$. Then for every ε, $0<\varepsilon<1$, there is N such that for all m,n

$$|B_{mn}(x^i-x)|^{p_m}<\varepsilon \quad \text{for} \quad i>N.$$

Now

$$|B_{mn}(x^i-x)|^{q_m}<|B_{mn}(x^i-x)^{p_m}<\varepsilon$$

for $i>N$. Therefore $x\in(A,q)_\infty$ and this completes the proof.

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A NOTE ON CONVERGENCE OF WALSH—FOURIER SERIES OF A FUNCTION ON A CLASS V_α

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1. Introduction. Wiener [3] proved that *if f is a function of V_2, then*

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)\exp(ikx) = 1/2\,\{f(x+0)+f(x-0)\} \tag{1}$$

everywhere where $\hat{f}(k)$ is the k-th Fourier coefficient of f. Siddigi [2] improved Wiener's result for V_α, $1 \leqq \alpha < \infty$. The above result just cited points out that *the Fourier series of any function of V_α, $1 \leqq \alpha < \infty$, converges everywhere.*

The purpose of this paper is to prove the following

THEOREM. *If f is a function of V_α, $1 \leqq \alpha < \infty$, then*

$$\sum_{k=0}^{\infty} \hat{f}(k)\, w_k(x) \begin{cases} = f(x), & \text{if } f \text{ is continuous at } x, \\ \text{diverges otherwise,} \end{cases}$$

where $\hat{f}(k)$ is the k-th Walsh—Fourier coefficient of f and $w_0, w_1, \ldots$ denote Walsh functions.

2. Definitions and notations. The dyadic group which is the set of all 0—1 sequences $(t_1, t_2, \ldots)$, is a compact abelian group [1]. Throughout this paper, I_n^p denotes the subset of the dyadic group such that

$$\sum_{k=1}^{n} t_k/2^k = p/2^n.$$

I_n^p is said to be a *dyadic interval of rank n.* $I_n(x)$ denotes the dyadic interval of rank n which contains x. For convenience I_0^0 denotes the dyadic group.

An element $(t_1, t_2, \ldots)$ of the dyadic group is said to be a dyadic rational if $\lim_{k\to\infty} t_k = 0$ and a conjugate dyadic rational if $\lim_{k\to\infty} t_k = 1$. We shall find it convenient to use the following notation:

$$x = (t_1, t_2 \ldots) = \begin{cases} (\sum_{k=1}^{\infty} t_k/2^k)^-, & \text{if } x \text{ is a conjugate dyadic rational,} \\ (\sum_{k=1}^{\infty} t_k/2^k), & \text{otherwise.} \end{cases}$$

When $x \in I_n^p$ and $y \in I_n^q$ for sufficiently large n and $p<q$, we shall write $x<y$. A function f is said to be a function of V_α, $1 \leqq \alpha < \infty$ if it satisfies

$$\sup_{\Delta} \left\{ \left(\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|^\alpha \right\}^{1/\alpha} < \infty \right.$$

where Δ is a division of the dyadic group, that is,

$$\Delta: \quad x_0 = 0 < x_1 < x_2 < \ldots < x_n = 1^-.$$

In general when I is an arbitrary dyadic interval, set

$$V_\alpha(f; I) = \sup_{\Delta} \left\{ \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|^\alpha \right\}^{1/\alpha},$$

where Δ is an arbitrary division of I.

We say that $f(x)$ *is continuous at* x_0 *if*

$$\lim_{h \to 0} f(x_0 + h) = f(x_0).$$

3. Proof of the theorem. We need the following lemma whose proof is in [3].

LEMMA. *If f is a function of V_α, $1 \leqq \alpha < \infty$, and x is a continuity point of f, then*

$$\lim_{n \to \infty} V_\alpha(f; I_n(x)) = 0.$$

PROOF OF THE THEOREM. From the hypothesis, f can have no discontinuities of the first kind. We can see this fact quite similar to [2]. Therefore we can assume that

$$f(x) = \begin{cases} f(x+0) = f(x+0) = \lim_{h \to 0} f(x+h), \text{ if } x \text{ is a dyadic rational,} \\ f(x-0) = f(x+0) = \lim_{h \to 0} f(x+h), \text{ if } x \text{ is a conjugate dyadic rational,} \\ \frac{1}{2}\{f(x+0) + f(x-0)\}, \text{ otherwise.} \end{cases}$$

When $x_0 = (t_1, t_2, \ldots)$ is neither a dyadic rational nor a conjugate dyadic rational and f is not continuous at x_0, set

$$f^*(x) = \begin{cases} f(x) + \theta, & \text{for} \quad x < x_0, \\ f(x) - \theta, & \text{for} \quad x_0 < x, \\ \frac{1}{2}\{f(x_0+0) + f(x_0-0)\}, & \text{for} \quad x = x_0, \end{cases}$$

where

$$\theta = \frac{1}{2}\{f(x_0+0) - f(x_0-0)\}.$$

Therefore we have

$$S_{2^n}(x_0) = \sum_{k=0}^{2^n-1} \hat{f}(k)\, w_k(x_0) = 2^n \int_{I_n(x_0)} f(x)\, dx =$$

$$= 2^n \int_{I_n(x_0)} f^*(x)\, dx + 2^n \int_{I_n(x_0)\cap[0,\, x_0)} -\theta\, dx + 2^n \int_{I_n(x_0)\cap(x_0,\, 1^-]} \theta\, dx =$$

$$= 2^n \int_{I_n(x_0)} f^*(x)\, dx - 2^n\theta \Big\{ \sum_{k=n+1}^{\infty} t_k/2^k - 1/2^n + \sum_{k=n+1}^{\infty} t_k/2^k \Big\} =$$

$$= f(x_0) + 2^n \int_{I_n(x_0)} \{f^*(x) - f(x_0)\}\, dx + \theta 2^n \Big\{1/2^n - 2 \sum_{k=n+1}^{\infty} t_k/2^k\Big\} =$$

$$= f(x_0) + o(1) + \{f(x_0+0) - f(x_0-0)\}\Big\{1/2 - \sum_{k=1}^{\infty} t_k/2^k\Big\}.$$

Since x_0 is neither a dyadic rational nor a conjugate dyadic rational, 0 and 1 appear infinitely many times in the sequence $\{t_k\}$ respectively. Hence the sequence

$$\Big\{1/2 - \sum_{k=1}^{\infty} t_{n+k}/2^k\Big\}_n$$

diverges for each k. Therefore the sequence $\{S_{2^n}(x_0)\}$ diverges.

Next, when x_0 is a continuity point of f, set

$$\varphi(x) \equiv \varphi_{x_0}(x) = f(x_0 \dot{+} x) - f(x_0).$$

For $2^n \leqq m < 2^{n+1}$, we have

$$S_m(x_0) - f(x_0) = \int_{I_0^0} \varphi(x) D_m(x)\, dx =$$

$$= \int_{I_0^0} \varphi(x) D_{2^n}(x)\, dx + \int_{I_0^0} \varphi(x) \Big(\sum_{k=2^n}^{m=1} w_k(x)\Big)\, dx =$$

$$= 2^n \int_{I_n^0} \varphi(x)\, dx + \sum_{p=0}^{2^n-1} \Big\{ \sum_{k=0}^{m-2^n-1} w_k(p/2^n)\Big\}\Big\{ \int_{I_{n+1}^{2p}} \varphi(x)\, dx - \int_{I_{n+1}^{2n+1}} \varphi(x)\, dx\Big\} =$$

$$= 2^n \int_{I_0^0} \varphi(x)\, dx + \sum_{p=0}^{2^n-1} D_{m-2^n}(p/2^n) \times$$

$$\times \Big\{ \int_{I_{n+1}^0} \varphi(p/2^n \dot{+} x) - \varphi(p/2^n \dot{+} 1/2^{n+1} \dot{+} x)\, dx\Big\},$$

where $D_m(x)=\sum_{k=0}^{m-1} w_k(x)$. Therefore by Hölder inequality we have

$$|S_m(x_0)-f(x_0)| \leqq \sum_{j=0}^{n-1}\sum_{k=2^j}^{2^{j+1}-1} 2^{n-j}\times \tag{1}$$

$$\times\Big\{\Big|\int_{I_{n+1}^0}\varphi(k/2^n\dotplus x)-\varphi(k/2^n\dotplus 1/2^{n+1}\dotplus x)\,dx\Big|\Big\}+o(1)\leqq$$

$$\leqq o(1)+\sum_{j=0}^{n-1}\big[2^{n-j}2^{j/\alpha'}/2^{n+1}\times$$

$$\times \sup_{x\in I_{n+1}^0}\Big\{\sum_{k=2^j}^{2^{j+1}-1}|\varphi(k/2^n\dotplus x)-\varphi(k/2^n\dotplus 1/2^{n+1}\dotplus x)|^\alpha\Big\}^{1/\alpha}\big]\leqq$$

$$\leqq o(1)+\sum_{j=0}^{n-1}2^{-j/\alpha}\sup_{x\in I_{n+1}^0}\Big\{\sum_{k=2^j}^{2^{j+1}-1}|\varphi(k/2^n\dotplus x)-\varphi(k/2^n\dotplus 1/2^{n+1}\dotplus x)|^\alpha\Big\}^{1/\alpha},$$

where $1/\alpha+1/\alpha'=1$. It is easy to see that

$$\sup_{x\in I_{n+1}^0}\Big\{\sum_{k=2^j}^{2^{j+1}-1}|\varphi(k/2^n\dotplus x)-\varphi(k/2^n\dotplus 1/2^{n+1}\dotplus x)|^\alpha\Big\}^{1/\alpha}=$$

$$=\sup_{x\in I_{n+1}^0}\sum_{k=2^j}^{2^{j+1}-1}|f(x_0\dotplus k/2^n+x)-f(x_0\dotplus k/2^n\dotplus 1/2^{n+1}\dotplus x)|^\alpha\Big\}^{1/\alpha}\leqq$$

$$\leqq V_\alpha(f;\,I_{n-j-1}(x_0)\setminus I_{n-j}(x_0)).$$

Set

$$|S_m(x_0)-f(x_0)|\leqq o(1)+\sum_{j=0}^{n-1}2^{-j/\alpha}V_\alpha(f;I_{n-j-1}(x_0)\setminus I_{n-j}(x_0))=$$

$$=o(1)+\sum_{j=0}^{[n/2]}+\sum_{[n/2]+1}^{n}\equiv o(1)+A_n+B_n.$$

From (1) for each $\varepsilon>0$, there exists an integer $k(\varepsilon)$ such that for $k>k(\varepsilon)$,

$$V_\alpha(f;\,I_{k-1}(x_0)\setminus I_k(x_0))<\varepsilon.$$

Take n so large that $k(\varepsilon)\leqq[n/2]\leqq n-j\leqq n$, then we get

$$A_n\leqq\varepsilon\sum_{j=0}^{\infty}2^{-j/\alpha}=\varepsilon/(1-2^{-1/\alpha}).$$

On the other hand, we have

$$B_n=\sum_{[n/2]+1}^{n}2^{-j/\alpha}V_\alpha(f;\,I_{n-j-1}(x_0)\setminus I_{n-j}(x_0))=$$

$$=2^{([n/2]+1)(-1/\alpha)}V_\alpha(f;I_0^0)\Big\{\sum_{k=0}^{\infty}2^{-k/\alpha}\Big\}\leqq 2^{-[n/2]/\alpha}V_\alpha(f;I_0^0)\{1/(1-2^{-1/\alpha})\}.$$

Take n so large that $B_n<\varepsilon$. There exists an integer $k'(\varepsilon)>k(\varepsilon)$ such that

$$A_n+B_n < \varepsilon(1+1/(1-2^{-1/\alpha})),$$

for $n>k'(\varepsilon)$. Thus we proved that

$$\lim_{m\to\infty} S_m(x_0) = f(x_0).$$

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ALMOST SURE INVARIANCE PRINCIPLES FOR THE EMPIRICAL PROCESS OF LACUNARY SEQUENCES

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1. Introduction

Let $\{n_k, k \geqq 1\}$ be a lacunary sequence of real numbers, i.e.

$$n_{k+1}/n_k \geqq q > 1.$$

Write $\langle \varepsilon \rangle = \varepsilon - [\varepsilon]$ for the fractional part of ε, where $[\varepsilon]$ denotes the largest integer not exceeding ε.

Let $\{[0, 1], \mathscr{J}, P\}$ be the unit interval with Lebesgue measurability and Lebesgue measure P. Then $\{\langle n_k w \rangle, k \geqq 1\}$ can be considered as a sequence of random variables (on the probability space $\{[0, 1], \mathscr{J}, P\}$) with asymptotically uniform distribution. For fixed s and t with $0 \leqq s \leqq t \leqq 1$ write $L = [s, t)$, $\varrho = t - s$,

$$x_k = x_k(s, t) = x_k(s, t, w) = 1\{s \leqq \langle n_k w \rangle < t\} - \varrho = 1_L(n_k w) - \varrho \tag{1.1}$$

where $1_L(\cdot)$ is extended with period 1, and

$$\Gamma(s, t) = \lim_{N \to \infty} N^{-1} E\Big(\sum_{j,k \leqq N} x_j(0, s) x_k(0, t)\Big). \tag{1.2}$$

Corollary 1 below shows that under the hypothesis of Theorem 1 the limit in (1.2) always exists and

$$\Gamma(s, t) = \lim_{N \to \infty} N^{-1} E\Big(\sum_{j,k \leqq N} x_{j+M}(0, s) x_{k+M}(0, t)\Big) = \tag{1.3}$$

$$= \lim_{M, N \to \infty} N^{-1} E\Big(\sum_{j,k \leqq N} x_{j+M}(0, s) x_{k+M}(0, t)\Big) = s(1 - t)$$

for $0 \leqq s \leqq t \leqq 1$. Thus it is a well defined covariance function.

The empirical process $\{R(s, t), 0 \leqq s \leqq 1, t \geqq 0\}$ of the sequence $\{n_k, k \geqq 1\}$ is defined by

$$R(s, t) = \sum_{k \leqq [t]} x_k(0, s), \quad 0 \leqq s \leqq 1, \quad t \geqq 0$$

by (1.1).

A separable Gaussian process $K(s, t)$ on $[0, 1] \times [0, \infty)$ is called a Kiefer process if for all $s < s' \in [0, 1]$ and $t, t' \geqq 0$

$$K(s, 0) = K(0, t) = K(1, t) = 0,$$

$$EK(s, t) = 0, \quad E\big(K(s, t) K(s', t')\big) = \min(t, t') \cdot s(1 - s').$$

* Supported by the Thai Government while studying at the University of Illinois.

For fixed s and t, respectively, $K(s, t)$ is proportional to a Brownian motion, respectively, to a Brownian Bridge.

We write

$$\psi_k = n_{k+1}/n_k, \quad k \geqq 1. \tag{1.4}$$

The purpose of the present paper is to show that the empirical process $R(s, t)$ can be approximated by a Kiefer process $K(s, t)$ in the form

$$\sup_{0\leqq s\leqq 1}\sup_{0\leqq t\leqq T}|R(s,t)-K(s,t)| \ll g(T)(T\log\log T)^{1/2} \quad \text{a.s.} \tag{1.5}$$

where $g(x)\to 0$ as $x\to\infty$ with the rate depending on ψ_k in (1.4). This problem is actually set up by Berkes and Philipp in their joint paper [6]. Our main result is the following:

THEOREM 1. *Let* $\{n_k, k\geqq 1\}$ *be a lacunary sequence of real numbers, and let* $R=\{R(s,t)\}$ *be its corresponding empirical process.*

(i) *Assume that*

$$\psi_k \gg (\log k)^{\Phi_k} \tag{1.6}$$

where $\Phi_k\to\infty$ *monotonically as* $k\to\infty$ *and* Φ_k/k *is decreasing for large* k. *Then without changing the distribution of* R *we can redefine* R *on a richer probability space on which there exists a Kiefer process* K *such that* (1.5) *holds with*

$$g(x) = \big(\Phi(\log x)\big)^{-1/18}.$$

Here $\Phi(x)$ *is any (nondecreasing) extension of* Φ_k *over* R^+.

(ii) *Assume that* Φ_k *in* (1.6) *satisfies*

$$\Phi_k \gg (\log k)^{\beta} \quad \text{for some } \beta > 0. \tag{1.7}$$

Then the conclusion in (i) *holds with* $g(x)=(\log x)^{-\lambda}$ *for some* $\lambda>0$.

(iii) *Assume that* $\psi_k\to\infty$ *and*

$$\psi_{k+1}-\psi_k \gg e^{-k^{\beta}} \tag{1.8}$$

for some $0<\beta<\frac{1}{63}$. *Then the conclusion in* (ii) *holds.*

REMARKS AND EXAMPLES. 1) Suppose that ψ_k satisfies

$$\psi_k/1-e^{-k^{\alpha}} \nearrow \infty \quad (\text{as } k\to\infty) \tag{1.9}$$

for some $0<\alpha<\frac{1}{63}$. Then the same conclusion as above holds since (1.9) implies (1.8). To prove this we write

$$\frac{\psi_{k+1}-\psi_k}{1-e^{-(k+1)^{\alpha}}} = \frac{\psi_{k+1}}{1-e^{-(k+1)^{\alpha}}} - \frac{\psi_k}{1-e^{-k^{\alpha}}} + \psi_k\left(\frac{1}{1-e^{-k^{\alpha}}} - \frac{1}{1-e^{-(k+1)^{\alpha}}}\right).$$

Using (1.9) we see that

$$\psi_{k+1}-\psi_k \gg \frac{1-e^{-(k+1)^{\alpha}}}{1-e^{-k^{\alpha}}} - 1 \gg e^{-k^{\alpha}} - e^{-(k+1)^{\alpha}} \gg \frac{e^{-(k+1)^{\alpha}}}{(k+1)^{1-\alpha}} \gg e^{-k^{\beta}}$$

where $\alpha<\beta<\frac{1}{63}$;

2) It is clear that all functions of the form

$$\psi_k = \ln_N k \quad (= \underbrace{\log\log \dots \log}_{N \text{ times}} k)$$

satisfy (1.8).

3) As for another result, it follows from the proof of Theorem 1 immediately that the conclusion in (ii) holds if ψ_k satisfies

$$\sum_{k=1}^{\infty} \psi_k^{-1} < \infty.$$

Apart from minor differences in the conditions, Theorem 1 is proved in our Ph. D. thesis [9]. It is natural to ask if condition (1.8) in (iii) can be omitted, i.e. Theorem 1 holds (with some $g(x) \to 0$ in (1.5)) under the mere condition $\psi_k \to \infty$. Unfortunately, we were not able to decide this question.

2. Proof of Theorem 1

2.1. *Preliminaries.* One of the main tools in the proof of Theorem 1 is the following theorem which is proved in Berkes and Philipp [5].

THEOREM 2. *Let $\{X_k, k \geqq 1\}$ be a sequence of random variables with values in $\mathbf{R}^{d_k}$, $k \geqq 1$ and let $\{F_k, k \geqq 1\}$ be a nondecreasing sequence of σ-fields such that X_k is F_k-measurable. Finally, let $\{G_k, k \geqq 1\}$ be a sequence of probability distributions on $\mathbf{R}^{d_k}$ with characteristic functions $g_k(u)$, $u \in \mathbf{R}^{d_k}$. Suppose that for some nonnegative numbers λ_k, δ_k and $T_k \geqq 10 d_k$*

$$E|E\{\exp(i\langle u, X_k\rangle)|F_{k-1}\} - g_k(u)| \leqq \lambda_k$$

for all u with $|u| \leqq T_k$, and that

$$G_k\{u\colon |u| > \frac{1}{4} T_k\} \leqq \delta_k.$$

Then without changing its distribution we can redefine the sequence $\{X_k, k \geqq 1\}$ on a richer probability space together with a sequence $\{Y_k, k \geqq 1\}$ of independent random variables such that Y_k has distribution G_k and

$$P\{|X_k - Y_k| \geqq \alpha_k\} \leqq \alpha_k \quad (k = 1, 2, \dots)$$

where $\alpha_1 = 1$ and

$$\alpha_k = 10\, d_k T_k^{-1} \log T_k + 4\lambda_k^{1/2} T_k^{d_k} + \delta_k \quad (k \geqq 2).$$

We need some more results.

Let $f(x)$, $-\infty < x < \infty$, be a measurable function such that

$$f(x+1) = f(x), \quad \int_0^1 f(x)\, dx = 0, \quad \|f\|_2 = \Big(\int_0^1 f^2(x)\, dx\Big)^{1/2} < \infty. \tag{2.1}$$

Let

$$\sum_{k=1}^{\infty}(a_k\cos 2\pi kx+b_k\sin 2\pi kx)\sim f \tag{2.2}$$

be the Fourier expansion of f. Thus

$$\begin{cases} a_k=2\int_0^1 f(x)\cos 2\pi kx\,dx, \quad b_k=2\int_0^1 f(x)\sin 2\pi kx\,dx, \\ \|f\|_2^2=\frac{1}{2}\sum_{k=1}^{\infty}(a_k^2+b_k^2). \end{cases} \tag{2.3}$$

The following lemma is Lemma (3.3) of Berkes [2].

LEMMA 1. *Let* $f(x)$ *satisfy* (2.1) *and*

$$\frac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2+b_k^2)\leqq A^2n^{-2\alpha} \tag{2.4}$$

for some constants $A>1$ *and* $0<\alpha\leqq 1$ *where* a_k *and* b_k *are defined as in* (2.3). *Let* $1\leqq m_1<m_2<\ldots<m_n$ *be arbitrary real numbers such that* $m_{k+1}/m_k\geqq q>1$ *for* $1\leqq k\leqq n-1$. *Then we have for any real number* a

$$\int_a^{a+1}(f(m_1x)+\ldots+f(m_nx))^2\,dx=n\|f\|_2^2+T$$

where

$$|T|\leqq\frac{4q}{q-1}\|f\|_2^2+C\cdot A\left(\frac{1}{q^{1/2}-1}+\frac{1}{q^{\alpha}-1}+\frac{\|f\|_2}{\log q}\right)(\|f\|_2^2+\|f\|_2)n$$

with an absolute constant C.

From this lemma, Berkes [2] then proved

COROLLARY 1. *Let* $f(x)$ *satisfy* (2.1) *and* (2.4). Let $1\leqq m_1<m_2<\ldots$ *be a sequence of real numbers such that* $m_{k+1}/m_k\to\infty$. *Then we have*

$$\int_a^{a+1}\Big(\sum_{v=j+1}^{j+n}f(m_vx)\Big)^2\,dx\sim n\cdot\|f\|_2^2 \quad as \quad n\to\infty$$

uniformly in j *and in* a.

Now if we apply Corollary 1 to the functions

$$f(x)=1_{[0,\,s)}(x)-s \quad \text{and} \quad f(x)=1_{[s,\,t)}(x)-(t-s)$$

together with the equation

$$\Big(\sum_{n\leqq N}x_n(0,s)\Big)^2+\Big(\sum_{n\leqq N}x_n(0,t)\Big)^2-2\Big(\sum_{m,\,n\leqq N}x_m(0,s)x_n(0,t)\Big)=\Big(\sum_{n\leqq N}x_n(s,t)\Big)^2,$$

we can get the results in (1.3) immediately.

LEMMA 2. *Let $f(x)$ be as in Lemma* 1. *Then we have for integer $m \geqq \lambda \geqq 1$*

$$\|f(\lambda x) - E\{f(\lambda x)|M_m\}\|_2 \leqq C_1 \left(\frac{m}{\lambda}\right)^{-\alpha/3} (\|f\|_2 + A)$$

for some absolute constant C_1 where $M_m = \sigma\left(\left[\frac{k}{m}, \frac{k+1}{m}\right); \quad k = 0, 1, \ldots, m-1\right)$.

This is implicit in Lemma (3.1) of Berkes [2].

The following simple result is Lemma (3.2) of Berkes [2].

LEMMA 3. *Let $g(x)$, $-\infty < x < \infty$, be a function such that*

$$g(x+1) = g(x), \quad \int_0^1 g(x)\,dx = 0.$$

Then we have

$$\left|\int_a^b g(\lambda x)\,dx\right| \leqq \frac{2}{\lambda} \int_0^1 |g(x)|\,dx$$

for any real numbers $a < b$ and any $\lambda > 0$.

We need one more well-known result whose proof can be found in Chow and Teicher [7], page 272.

LEMMA 4. *For any δ in $[0, 1]$*

$$\left|e^{it} - \sum_{j=0}^{n} (it)^j/j!\right| \leqq \frac{2^{1-\delta}|t|^{n+\delta}}{(1+\delta)(2+\delta)\ldots(n+\delta)}.$$

2.2. *An exponential bound.* PROPOSITION 1. *Let $\{n_k, k \geqq 1\}$ be a lacunary sequence with*

$$n_{k+1}/n_k \geqq q > 1.$$

Let $H \geqq 0, N \geqq 1$ be integers, $0 < \alpha \leqq \frac{1}{3}$, and let $R \geqq 1$. Suppose that $\varrho \geqq N^{-1/6\alpha}$. Then as $N \to \infty$

$$P\left\{\left|\sum_{k=H+1}^{H+N} x_k\right| \geqq BR\varrho^{\alpha}(N \log\log N)^{1/2}\right\} \ll \exp(-10R\varrho^{-\alpha}\log\log N) + R^{-6}N^{-3/2}$$

where both B (>1) and the constant implied by $\ll$ depend only on q.

This is a generalization of Proposition 4.2.1 of Philipp [14].

2.3. *Proof of Theorem* 1 (case (ii)). With no loss of generality we shall assume, for all three cases, that $n_k \geqq 0$ for all k. We can also assume, for the present case, that $\beta < 1$.

2.3.1. *Basic notation.* Put

$$\varepsilon = \frac{\beta}{100}. \tag{2.5}$$

We then define integers, for each $k \in \mathbf{Z}^+$,

(2.6) $t_k = [\exp(k^{1-\varepsilon})]$, $N_k = t_{k+1} - t_k$, $r_k = \left[\frac{1-\varepsilon}{4} \beta \log k / \log 2\right]$, and $d_k = 2^{r_k}$.

Let

$$s_j = s_{jk} = j2^{-r_k}, \quad 0 \leqq j \leqq d_k.$$

We divide the set of positive integers into consecutive blocks

$$H_1 < H_2 < \ldots < H_k < \ldots \quad \text{(say)}$$

(without gaps) in such a way that H_k contains N_k consecutive integers. Define the random vectors X_k in $\mathbf{R}^{d_k}$ via

$$X_k(j) = N_k^{-1/2} \sum_{v \in H_k} x_v(0, s_j), \quad 1 \leqq j \leqq d_k \tag{2.7}$$

where x_k is defined in (1.1). We denote for each $k \in \mathbf{Z}^+$ the matrix

$$\Gamma_k = (\Gamma(s_i, s_j) : 1 \leqq i, j \leqq d_k),$$

where $\Gamma(s, t)$ is defined in (1.3), i.e. $\Gamma(s, t) = s(1-t)$ for $0 \leqq s \leqq t \leqq 1$.

2.3.2. *The central limit theorem.* Put

$$\alpha = \frac{1}{21}, \quad \hat{\alpha} = \frac{1}{63}. \tag{2.8}$$

Define integers, for $k \in \mathbf{Z}^+$,

$$p_k = [N_k^{\alpha}], \quad q_k = [N_k^{\hat{\alpha}}], \quad l_k = [N_k / p_k + q_k]. \tag{2.9}$$

We then divide the block H_k into small blocks

$$H_{k1} < I_{k1} < H_{k2} < I_{k2} < \ldots < H_{kl_k} < I_{kl_k} < I_{k(l_k+1)}$$

(say) so that

$$\operatorname{card}(H_{kj}) = p_k, \quad \operatorname{card}(I_{kj}) = q_k, \quad 1 \leqq j \leqq l_k,$$

and

$$\operatorname{card}(I_{k(l_k+1)}) = N_k - l_k(p_k + q_k).$$

Write

$$W_j = \Big(\sum_{v \in H_{kj}} x_v(0, s_1), \ldots, \sum_{v \in H_{kj}} x_v(0, s_{d_k}) \Big). \tag{2.10}$$

We need to define some more integers. For $k > 1$ we define the integer m_j^k for $1 \leqq j \leqq l_k$ so that

$$2^{m_j^k} \leqq n_{t_k + jp_k + (j-1)q_k + [q_{k-1}/2]} < 2^{m_j^k + 1} \tag{2.11}$$

and put

$$2^{m_0^k} = 2^{m_{l_{k-1}}^{k-1}}.$$

We are going to prove a central limit theorem for a martingale difference sequence. Basically the proof follows the idea of the proof of the central limit theorem

for martingale cases appearing, for example, in Chow and Teicher [7], page 313. Thus we need to define some more σ-fields. Let

$$\mathscr{J}_k = \sigma\big([m\cdot 2^{-m_{l_k}^k}, (m+1)2^{-m_{l_k}^k}); \ 0 \leqq m < 2^{m_{l_k}^k}\big) \tag{2.12}$$

$$F_j^k = \sigma\big([m\cdot 2^{-m_j^k}, (m+1)2^{-m_j^k}; \ 0 \leqq m \leqq 2^{-m_j^k} \quad (1 \leqq j \leqq l_k)$$

and put $F_0^k = \mathscr{J}_{k-1}$. Note that $\mathscr{J}_k = F_{l_k}^k = F_0^{k+1}$. Write

$$W_j' = \big(E\{W_j(1)|F_j^k\}, \ldots, E\{W_j(d_k)|F_j^k\}\big) = \big(W_j'(1), \ldots, W_j'(d_k)\big) \quad (1 \leqq j \leqq l_k) \tag{2.13}$$

by (2.10), and write

$$X_k' = \big(N_k^{-1/2} \sum_{j \leqq l_k} W_j'(1), \ldots, N_k^{-1/2} \sum_{j \leqq l_k} W_j'(d_k)\big). \tag{2.14}$$

PROPOSITION 2. *Let* $\mathscr{J}_0 = \{\emptyset, [0, 1)\}$. *We have as* $k \to \infty$

$$E\,|E\{\exp(i\langle u, X_k'\rangle|\mathscr{J}_{k-1}\} - \exp\big(-\frac{1}{2}\langle u, \Gamma_k u\rangle\big)\,| \ll \exp\big(-k^{(2/3)(1-\varepsilon)\beta}\big)$$

uniformly in $u \in \mathbf{R}^{d_k}$ *with* $|u| \leqq T_k$ *where*

$$T_k = [\exp(k^{(1-\varepsilon)\beta/4})]. \tag{2.15}$$

Fix $u = (u_1, \ldots, u_{d_k}) \in \mathbf{R}^{d_k}$ with $|u| \leqq T_k$. Define the step function f of period 1 by

$$f(x) = f_{k,u}(x) = \sum_{j \leqq d_k} u_j\big(1_{[0, s_j)}(x) - s_j\big), \ x \in [0, 1). \tag{2.16}$$

LEMMA 5. $\langle u, \Gamma_k u\rangle = \|f\|_2^2$.

PROOF. By Lemma 1 with $A = T_k d_k$ and $\alpha = \frac{1}{2}$ we have

$$\|f\|_2^2 = N^{-1} E\big(\sum_{n \leqq N} f(n_{n+M}x)\big)^2 - \frac{T_M}{N} =$$

$$= \sum_{i, j \leqq d_k} u_i u_j N^{-1} E\big(\sum_{m, n \leqq N} x_{m+M}(0, s_i)\, x_{n+M}(0, s_j)\big) - \frac{T_M}{N}$$

where $\frac{T_M}{N} \to 0$ as $M, N \to \infty$. Letting $M, N \to \infty$ we obtain

$$\|f\|_2^2 = \sum_{i, j \leqq d_k} u_i u_j \Gamma_k(s_i, s_j) = \langle u, \Gamma_k u\rangle$$

by (1.3).

Write, by (2.16) and (2.12),

$$Y_j = \sum_{v \in H_{kj}} f(n_v x), \qquad 1 \leqq j \leqq l_k, \tag{2.17}$$

$$\varphi_j = E\,\{Y_j|F_j^k\}, \qquad 1 \leqq j \leqq l_k, \tag{2.18}$$

$$V_j = \varphi_j - E\{\varphi_j|F_{j-1}^k\}, \quad 1 \leqq j \leqq l_k, \tag{2.19}$$

and

$$S_j = V_1 + \ldots + V_j, \qquad 1 \leqq j \leqq l_k, \quad S_0 = 0. \tag{2.20}$$

We observe that, by (2.14) and (2.18),

$$\langle u, X_k' \rangle = N_k^{-1/2} \sum_{j \leqq l_k} \varphi_j \tag{2.21}$$

since, by (2.13),

$$\langle u, W_j' \rangle = \sum_{n \leqq d_k} u_n W_j'(n) = \sum_{n \leqq d_k} u_n \sum_{v \in H_{kj}} E\{1_{[0, s_n)}(n_v x) - s_n | F_j^k\} =$$

$$= \sum_{v \in H_{kj}} \sum_{n \leqq d_k} u_n E\{1_{[0, s_n)}(n_v x) - s_n | F_j^k\} = \sum_{v \in H_{kj}} E\{f(n_v x) | F_j^k\} = \varphi_j, \quad 1 \leqq j \leqq l_k.$$

LEMMA 6. $\|E\{\varphi_j | F_{j-1}^k\}\|_1 \leqq 2^{m_{j-1}^k + 1} T_k d_k \sum_{v \in H_{kj}} \frac{1}{n_v}, \quad 1 \leqq j \leqq l_k.$

PROOF. Let $A = [n 2^{-m_{j-1}^k}, (n+1) 2^{-m_{j-1}^k})$ for some $0 \leqq n < 2^{m_{j-1}^k}$.

$$\left| P^{-1}(A) \int_A f(n_v x) \right| \leqq \frac{2 P^{-1}(A)}{n_v} \|f\|_1 \leqq \frac{2 \cdot 2^{m_{j-1}^k}}{n_v} T_k d_k, \quad v \in H_{kj},$$

by Lemma 3. Thus

$$\|E\{\varphi_j | F_{j-1}^k\}\|_1 = \|E\{E\{Y_j | F_j^k\} | F_{j-1}^k\}\|_1 = \|E\{Y_j | F_{j-1}^k\}\|_2 \leqq$$

$$\leqq \sum_{v \in H_{kj}} \frac{2 \cdot 2^{m_{j-1}^k}}{n_v} T_k d_k = 2^{m_{j-1}^k + 1} T_k d_k \sum_{v \in H_{kj}} \frac{1}{n_v}.$$

LEMMA 7. $\|\varphi_j - Y_j\|_2 \leqq 2 C_1 T_k d_k \sum_{v \in H_{kj}} \left(\frac{2^{m_j^k}}{n_v} \right)^{-1/6}, \quad 1 \leqq j \leqq l_k,$ *where* C_1 *is the constant occurring in Lemma* 2.

PROOF. By Lemma 2,

$$\|\varphi_j - Y_j\|_2 = \left\| E\left\{ \sum_{v \in H_{kj}} f(n_v x) | F_j^k \right\} - \sum_{v \in H_{kj}} f(n_v x) \right\|_2 \leqq$$

$$\leqq \sum_{v \in H_{kj}} \|E\{f(n_v x) | F_j^k\} - f(n_v x)\|_2 \leqq 2 C_1 T_k d_k \sum_{v \in H_{kj}} \left(\frac{2^{m_j^k}}{n_v} \right)^{-1/6}, \quad 1 \leqq j \leqq l_k,$$

since $A = T_k d_k$ and $\|f\|_2 \leqq T_k d_k$.

LEMMA 8. $\|E\{Y_j^2 | F_{j-1}^k\} - p_k \|f\|_2^2\|_1 \leqq 8 T_k^2 d_k^3 \left(\sum_{v \in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + \sum_{u < v \in H_{kj}} \frac{n_u}{n_v} \right)$ *where* p_k *is defined in* (2.9).

PROOF. Let A be as in the proof of Lemma 6. Thus

$$P^{-1}(A) \int_A Y_j^2 = \sum_{v \in H_{kj}} P^{-1}(A) \int_A f^2(n_v x) + 2 \sum_{u < v \in H_{kj}} P^{-1}(A) \int_A f(n_u x) f(n_v x),$$

and

$$\left| \sum_{v \in H_{kj}} P^{-1}(A) \int_A f^2(n_v x) - p_k \|f\|_2^2 \right| \leqq \sum_{v \in H_{kj}} P^{-1}(A) \left| \int_A f^2(n_v x) - \|f\|_2^2 \right| \leqq$$

$$\leqq \sum_{v \in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} 4 \|f\|_2^2 \leqq 4 T_k^2 d_k^2 \sum_{v \in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v}$$

by Lemma 3. (So far the proof is based on the proof of Lemma (3.3) of Berkes [2].) Now

$$\left|\int_A f(n_u x)f(n_v x)\right| = \frac{1}{n_u}\left|\int_{n_u A} f(x)f\left(\frac{n_v}{n_u}x\right)\right| \leqq 4P(A)T_k d_k^2 \frac{n_u}{n_v}\|f\|_1 \leqq 4T_k^2 d_k^3 P(A)\frac{n_u}{n_v}$$

for all $u \neq v$ by Lemma 3. Thus

$$\left\|E\{Y_j^2|F_{j-1}^k\}-p_k\|f\|_2^2\right\|_1 \leqq 4T_k^2 d_k^2 \sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + 8T_k^2 d_k^3 \sum_{u<v\in H_{kj}} \frac{n_u}{n_v} \leqq$$

$$\leqq 8T_k^2 d_k^3\left(\sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + \sum_{u<v\in H_{kj}} \frac{n_u}{n_v}\right).$$

We have some more simple facts before we proceed the proof of Proposition 2. We observe that, by (2.8), (2.9), and (2.11),

$$\sum_{j\leqq l_k}\sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} \leqq C_2, \tag{2.22}$$

$$\sum_{j\leqq l_k}\sum_{v\in H_{kj}} \left(\frac{2^{m_j^k}}{n_v}\right)^{-1/6} \leqq C_3, \tag{2.23}$$

$$\sum_{j\leqq l_k}\sum_{u<v\in H_{kj}} \frac{n_u}{n_v} \leqq C_4 N_k \psi_{t_k}^{-1} \tag{2.24}$$

where C_2, C_3, C_4 are absolute contants.

Finally if we put

$$p_k' = N_k/l_k, \tag{2.25}$$

then

$$0 < p_k' - p_k \leqq 2q_k \tag{2.26}$$

since

$$l_k > \frac{N_k}{2p_k} - 1 \geqq \frac{N_k}{2N_k^\alpha} - 1 \geqq \frac{N_k^{1-\alpha}}{2} - 1 > N_k^\alpha \geqq p_k$$

and since

$$\frac{N_k}{p_k+q_k} \leqq l_k+1 \Rightarrow N_k < l_k(p_k+q_k)+2p_k \Rightarrow \frac{N_k}{l_k} < p_k+q_k+\frac{2p_k}{l_k}$$

$$\Rightarrow p_k' - p_k < q_k + \frac{2p_k}{l_k}.$$

PROOF OF PROPOSITION 2. Let

$$\mathrm{I} = E\left|E\{\exp(i\langle u, x_k'\rangle)|\,\mathcal{J}_{k-1}\} - \exp\left(-\frac{1}{2}\langle u, \Gamma_k u\rangle\right)\right|.$$

Thus by (2.21), Lemma 5, (2.18), (2.19), Lemma 6 and (2.22),

$$\text{(2.27)}\qquad \mathrm{I} \leqq E\left|E\left\{\exp\left(iN_k^{-1/2}\sum_{j\leqq l_k}\varphi_j\right)\middle|\mathscr{J}_{k-1}\right\}-E\left\{\exp\left(iN_k^{-1/2}\sum_{j\leqq l_k}V_j\right)\middle|\mathscr{J}_{k-1}\right\}\right|+$$

$$+E\left|E\left\{\exp\left(iN_k^{-1/2}\sum_{j\leqq l_k}V_j\middle|\mathscr{J}_{k-1}\right\}-\exp\left(-\frac{1}{2}\|f\|_2^2\right)\right|\leqq$$

$$\leqq N_k^{-1/2}\sum_{j\leqq l_k}\|E\{\varphi_j|F_{j-1}^k\}\|_1+\mathrm{II}\leqq N_k^{-1/2}\sum_{j\leqq l_k}\sum_{v\in H_{kj}}2T_k\,d_k\cdot\frac{2^{m_{j-1}^k}}{n_v}+\mathrm{II}\leqq$$

$$\leqq 2C_2T_k\,d_kN_k^{-1/2}+\mathrm{II}.$$

By (2.20) and (2.25),

$$\mathrm{II}=E\left|E\left\{\exp\left(iN_k^{-1/2}\sum_{j\leqq l_k}V_j\right)\middle|\mathscr{J}_{k-1}\right\}-\exp\left(-\frac{1}{2}\|f\|_2^2\right)\right|=$$

$$=E\left|\exp\left(-\frac{1}{2}\|f\|_2^2\right)\sum_{j\leqq l_k}\left(E\left\{\exp\left(iN_k^{-1/2}S_j+\frac{jp_k'\|f\|_2^2}{2N_k}\right)\middle|\mathscr{J}_{k-1}\right\}-\right.\right.$$

$$\left.\left.-E\left\{\exp\left(iN_k^{-1/2}S_{j-1}+\frac{(j-1)p_k'\|f\|_2^2}{2N_k}\right)\middle|\mathscr{J}_{k-1}\right\}\right)\right|\leqq$$

$$\leqq\exp\left(-\frac{1}{2}\|f\|_2^2\right)\sum_{j\leqq l_k}E\left|E\left\{\exp\left(iN_k^{-1/2}S_j+\frac{jp_k'\|f\|_2^2}{2N_k}\right)-\right.\right.$$

$$\left.\left.-\exp\left(iN_k^{-1/2}S_{j-1}+\frac{(j-1)p_k'\|f\|_2^2}{2N_k}\right)\middle|\mathscr{J}_{k-1}\right\}\right|\leqq\exp\left(-\frac{1}{2}\|f\|_2^2\right)\sum_{j\leqq l_k}\mathrm{III}_j$$

(say). By (2.12), (2.18)—(2.20), (2.25),

$$\mathrm{III}_j=E\left|E\left\{\exp\left(iN_k^{-1/2}S_j+\frac{jp_k'\|f\|_2^2}{2N_k}\right)-\exp\left(iN_k^{-1/2}S_{j-1}+\right.\right.\right.$$

$$\left.\left.\left.+\frac{(j-1)p_k'\|f\|_2^2}{2N_k}\right)\middle|\mathscr{J}_{k-1}\right\}\right|\leqq$$

$$\leqq E\left|E\left\{\exp\left(iN_k^{-1/2}S_j+\frac{jp_k'\|f\|_2^2}{2N_k}\right)-\exp\left(iN_k^{-1/2}S_{j-1}+\right.\right.\right.$$

$$\left.\left.\left.+\frac{(j-1)p_k'\|f\|_2^2}{2N_k}\right)\middle|F_{j-1}^k\right\}\right|\leqq$$

$$\leqq E\left|\exp\left(iN_k^{-1/2}S_{j-1}+\frac{jp_k'\|f\|_2^2}{2N_k}\right)E\left\{\exp\left(iN_k^{-1/2}V_j\right)-\right.\right.$$

$$\left.\left.-\exp\left(-\frac{p_k'\|f\|_2^2}{2N_k}\right)\middle|F_{j-1}^k\right\}\right|\leqq$$

$$\leqq\exp(jp_k'\|f\|_2^2/2N_k)E\left|E\left\{\exp\left(iN_k^{-1/2}V_j\right)-\exp\left(-\frac{p_k'\|f\|_2^2}{2N_k}\right)\middle|F_{j-1}^k\right\}\right|\leqq$$

$$\leqq\exp\left(jp_k'\|f\|_2^2/2N_k\right)\mathrm{IV}_j\leqq\exp\left(\|f\|_2^2/2\right)\mathrm{IV}_j.$$

Put

$$\mu = \frac{5}{12}. \tag{2.28}$$

By Lemma 4 and (2.28), we have

$$\mathrm{IV}_j = E\left|E\left\{\exp\left(iN_k^{-1/2}V_j\right) - \exp\left(-P_k'\|f\|_2^2/2N_k\right)\middle|F_{j-1}^k\right\}\right| =$$

$$= E\left|E\left\{\left(\exp\left(iN_k^{-1/2}V_j\right) - 1 - iN_k^{-1/2}V_j + \frac{V_j^2}{2N_k}\right) - \right.\right.$$

$$\left.\left. - \left(\exp\left(-p_k'\|f\|_2^2/2N_k\right) - 1 + \frac{p_k'\|f\|_2^2}{2N_k}\right)\middle|F_{j-1}^k\right\} - \frac{1}{2N_k}\left(E\left\{V_j^2|F_{j-1}^k\right\} - p_k'\|f\|_2^2\right)\right| \leqq$$

$$\leqq E\left|\exp\left(iN_k^{-1/2}V_j\right) - 1 - iN_k^{-1/2}V_j + \frac{V_j^2}{2N_k}\right| + \left|\exp\left(-p_k'\|f\|_2^2/2N_k\right) - 1 + \right.$$

$$\left. + \frac{p_k'\|f\|_2^2}{2N_k}\right| + \frac{1}{2N_k}E\left|E\left\{V_j^2|F_{j-1}^k\right\} - p_k'\|f\|_2^2\right| \leqq$$

$$\leqq N_k^{-1}\int\limits_{\{|V_j|>N_k^{1/2-\mu}\}} V_j^2 + N_k^{-3/2}\int\limits_{\{|V_j|\leqq N_k^{1/2-\mu}\}} |V_j|^3 + \left(\frac{p_k'\|f\|_2^2}{N_k}\right)^2 + \frac{1}{2N_k}\overline{V}_j$$

(say). We have

$$N_k^{-1}\int\limits_{\{|V_j|>N_k^{1/2-\mu}\}} V_j^2 \leqq 2p_k^2T_k^2d_k^2N_k^{-1}\cdot\frac{\|V_j\|_2^2}{N_k^{1-2\mu}} \leqq$$

$$\leqq 4T_k^4d_k^4p_k^4N_k^{-1-1+2\mu} \leqq 4T_k^4d_k^4N_k^{4\alpha-2+2\mu}$$

by (2.19), (2.17), (2.18) and (2.9). For the other integral we have

$$N_k^{-3/2}\int\limits_{\{|V_j|\leqq N_k^{1/2-\mu}\}} |V_j|^3 \leqq N_k^{-3/2+3/2-3\mu} = N_k^{-3\mu}.$$

By (2.19), (2.20), (2.17), (2.9), Lemma 8, (2.29), Lemma 6, and Lemma 7,

$$\overline{V}_j = E\left|E\left\{V_j^2|F_{j-1}^k\right\} - P_k'\|f\|_2^2\right| \leqq$$

$$\leqq E\left|E\left\{V_j^2|F_{j-1}^k\right\} - E\left\{\varphi_j^2|F_{j-1}^k\right\}\right| + E\left|E\left\{\varphi_j^2|F_{j-1}^k\right\} - \right.$$

$$\left. - E\left\{Y_j^2|F_{j-1}^k\right\}\right| + E\left|E\left\{Y_j^2|F_{j-1}^k\right\} - P_k'\|f\|_2^2\right| \leqq$$

$$\leqq 3p_kT_kd_k\|V_j - \varphi_j\|_1 + 2p_kT_kd_k\|\varphi_j - Y_j\|_1 +$$

$$+ E\left|E\left\{Y_j^2 - p_k\|f\|_2^2|F_{k-1}^k\right\}\right| + \left|p_k - p_k'\right|\|f\|_2^2 \leqq$$

$$\leqq 3p_kT_kd_k\left(\|E\{\varphi_j|F_{j-1}^k\}\|_1 + \|\varphi_j - Y_j\|_2\right) +$$

$$+ 8T_k^2d_k^3\left(\sum_{v\in H_{kj}}\frac{2^{m_{j-1}^k}}{n_v} + \sum_{u<v\in H_{kj}}\frac{n_u}{n_v}\right) + 2T_k^2d_kq_k \leqq$$

$$\leqq 3p_k T_k d_k \left(2T_k d_k \sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + 2C_1 T_k d_k \sum_{v\in H_{kj}} \left(\frac{2^{m_j^k}}{n_v}\right)^{-1/6}\right) +$$

$$+ 8T_k^2 d_k^3 \left(\sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + \sum_{u<v\in H_{kj}} \frac{n_u}{n_v}\right) + 2T_k^2 d_k^2 q_k.$$

Now by (2.27), (2.9), (2.8), (2.26), (2.22), (2.23), (2.24) and (2.28),

$$\mathrm{I} \leqq 2C_2 T_k d_k N_k^{-1/2} + \exp\left(-\frac{1}{2}\|f\|_2^2\right) \sum_{j\leqq l_k} \exp\left(\frac{1}{2}\|f\|_2^2\right) \left\{4T_k^4 d_k^4 N_k^{4\alpha-2+2\mu} +\right.$$

$$+ N_k^{-3\mu} + (p_k')^2 T_k^4 d_k^4 N_k^{-2} + N_k^{-1} \left\{9p_k T_k^2 d_k^3 \sum_{v\in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} +\right.$$

$$\left.\left. + 6C_1 p_k T_k^2 d_k^2 \sum_{v\in H_{kj}} \left(\frac{2^{m_j^k}}{n_v}\right)^{-1/6} + 8T_k^2 d_k^2 \sum_{u<v\in H_{kj}} \frac{n_u}{n_v} + 2T_k^2 d_k^2 N_k^{-1/63}\right\}\right\} \ll$$

$$\ll T_k^4 d_k^4 (N_k^{-1/2} + N_k^{4\alpha-2+2\mu+1-\alpha} + N_k^{1-\alpha-3\mu} + N_k^{1-\alpha+2\alpha-2} + N_k^{-1+\alpha} + N_k^{-1+\alpha} +$$

$$+ N_k^{-1+1} \psi_{t_k}^{-1} + N_k^{-1+1-\alpha+1/63}) \ll T_k^4 d_k^4 \exp(-k^{3/4(1-\varepsilon)\beta}) \ll \exp(-k^{2/3(1-\varepsilon)\beta}).$$

2.3.3. *Introduction of an independent sequence of random vectors.* By Theorem 2 there exists a sequence $\{Y_k', k>1\}$ of independent random vectors whose distributions are the Gaussian distributions with mean 0 and covariance matrix Γ_k such that

$$P\{|X_k' - Y_k'| \geqq e^{-k\theta}\} \leqq e^{-k\theta} \tag{2.29}$$

for some $0<\theta<\frac{1-\varepsilon}{5}\beta$ (see (2.5) for the definition of ε). Also see (7.5) in Berkes and Philipp [6] for the estimate of δ_k in Theorem 2. Hence by (2.14), (2.9), Lemma 2 and (2.23) we have

$$P\left\{\left|X_k' - N_k^{-1/2}\left(\sum_{j\leqq l_k} W_j(1), \ldots, \sum_{j\leqq l_k} W_j(d_k)\right)\right| \geqq e^{-k\theta}\right\} \leqq \tag{2.30}$$

$$\leqq e^{k\theta} \cdot N_k^{-1/2} \sum_{j\leqq l_k} \|W_j - W_j'\|_2 \leqq$$

$$\leqq e^{k\theta} \cdot N_k^{-1/2} \sum_{j\leqq l_k} d_k \sum_{V\in H_{kj}} 3C_1 \left(\frac{2^{m_j^k}}{n_v}\right)^{-1/6} \leqq 3C_1 C_3 d_k e^{k\theta} N_k^{-1/2} \ll e^{-k\theta}$$

where C_1, C_3 are constants in Lemma 2 and in (2.23) respectively.

Put

$$Z_j = \left(\sum_{v\in I_{kj}} x_v(0, s_1), \ldots, \sum_{v\in I_{kj}} x_v(0, d_k)\right) = (Z_j(1), \ldots, Z_j(d_k)), \quad 1 \leqq j \leqq l_k+1.$$

LEMMA 9. $\left\|\left(\sum_{j\leqq l_k+1} Z_j(1), \ldots, \sum_{j\leqq l_k+1} Z_j(d_k)\right)\right\|_2 \leqq C_5 d_k N_k^{31/63}$ *for some absolute constant* C_5.

PROOF. Fix m in $\{1, \ldots, d_k\}$. Thus it is enough to prove

$$\Big\| \sum_{j \leqq l_k+1} Z_j(m) \Big\|_2 \leqq C_6 d_k N_k^{31/63}.$$

Now

$$\Big\| \sum_{j \leqq l_k} Z_j(m) \Big\|_2^2 = \sum_{j \leqq l_k} E(Z_j^2(m)) + 2 \sum_{i<j \leqq l_k} E(Z_i(m) Z_j(m)).$$

By Lemma 1, (2.9) and (2.6),

$$E(Z_j^2(m)) \leqq q_k + 8 + 2C \cdot 2 \cdot 2q_k \leqq 81(C+1)^2 q_k$$

where C is the constant occurring in Lemma 1.

As in the proof of Lemma 8 we have by Lemma 3

$$\big|E(Z_i(m) Z_j(m))\big| \leqq 8 \sum_{\substack{u \in I_{ki} \\ v \in I_{kj}}} \frac{n_u}{n_v}, \quad 1 \leqq i < j \leqq l_k.$$

Thus

$$\Big\| \sum_{j \leqq l_k} Z_j(m) \Big\|_2^2 \leqq 9(C+1) l_k q_k + 16 \sum_{i<j \leqq l_k} \sum_{\substack{u \in I_{ki} \\ v \in I_{kj}}} \frac{n_u}{n_v} \leqq 10^2 (C+1)^2 l_k q_k$$

since

$$\sum_{i<i \leqq l_k} \sum_{\substack{u \in I_{ki} \\ v \in I_{kj}}} \frac{n_u}{n_v} \leqq \frac{1}{16}.$$

Now

$$\Big\| \sum_{j \leqq l_k+1} Z_j(m) \Big\|_2 \leqq \Big\| \sum_{j \leqq l_k} Z_j(m) \Big\|_2 + \|Z_{l_k+1}(m)\|_2 \leqq 10(C+1) l_k^{1/2} q_k^{1/2} + 2p_k \leqq C_5 l_k^{1/2} q_k^{1/2}$$

for some absolute constant C_5 since by (2.9) and (2.8),

$$p_k \leqq N_k^{\alpha} \ll N_k^{(1-\alpha)/2} \ll l_k^{1/2} q_k^{1/2}.$$

Hence

$$\Big\| \sum_{j \leqq l_k+1} Z_j(m) \Big\|_2 \leqq C_5 N_k^{(1-\alpha)/2+\delta/2} \leqq C_5 N_k^{31/63}$$

by (2.8).

Lemma 9 implies

$$P\Big\{\Big|X_k - N_k^{-1/2}\Big(\sum_{j \leqq l_k} W_j(1), \ldots, \sum_{j \leqq l_k} W_j(d_k)\Big)\Big| \geqq e^{-k\theta}\Big\} \leqq \tag{2.31}$$

$$\leqq e^{k\theta} \Big\| N_k^{-1/2} \Big(\sum_{j \leqq l_k+1} Z_j(1), \ldots, \sum_{j \leqq l_k+1} Z_j(d_k) \Big) \Big\|_2 \ll e^{k\theta} N_k^{-1/2} N_k^{31/63} d_k \ll e^{-k\theta}$$

by (2.8) and (2.9).

By (2.29)—(2.31) we have

$$P\{|X_k - Y_k'| > 3e^{-k\theta}\} \ll e^{-k\theta}.$$

Hence by Borel—Cantelli Lemma we have

$$|X_k - Y_k'| \ll e^{-k\theta} \quad \text{a.s.}$$

The rest of the proof of Theorem 1 (ii) can be done as in Berkes and Philipp [6] starting from Section 5 there. There are, however, some changes that need to be specified. We apply the exponential bound, namely Proposition 4.2.1, in Philipp [14] instead of the one we have here. Proposition 1 will be applied to the case (i). Note that Proposition 4.2.1 in [14] satisfies the condition (5.2) in Berkes and Philipp [6]. We replace α and 4ε in Lemma 5.2 in [6] by 1/8 and 2ε, respectively. Thus we have

$$\max_{1 \leqq j \leqq 2^{r_k}} \sup_{s_j \leqq s < s_{j+1}} |R(s, t_k) - R(s_j, t_k)| \ll t_k^{1/2} (\log t_k)^{-2\varepsilon} \quad \text{a.s.}$$

Lemma 5.2 in [6] remains unchanged. For the properties of Kiefer processes in Section 6 of [6], we use the fact that $\sigma^2(s, t) \ll |s-t|$ for $0 \leqq s,\ t \leqq 1$. This follows from (1.3). For the definition of "σ" see [6]. Now replace 1/8 in Lemma 6.2 of [6] by 2ε and then conclude that

$$\max_{1 \leqq j \leqq 2^{r_k}} \sup_{s_j \leqq s < s_{j+1}} |K(s, t_k) - K(s_j, t_k)| \ll t_k^{1/2} (\log t_k)^{-2\varepsilon} \quad \text{a.s.}$$

Lemmas 6.1 and 6.3 in [6] remain unchanged.

2.4. *Proof of Theorem* 1 (case (iii)). All definitions in Section 2.3 remain the same as here except that we use β in (1.8) instead of the one in (1.7). We also assume without loss of generality that $\beta < 1$. We put

$$M_k = \exp(k^{(1-\varepsilon)/4}), \quad \varepsilon_k = \exp(-q_k/2) \tag{2.32}$$

where q_k is defined in (2.9).

All we need here is to improve Lemma 8 since the upper bound there is not small enough. Here we have to prove

LEMMA 10. $\|E\{Y_j^2 | F_{j-1}^k\} - p_k \|f\|_2^2\|_1 \leqq 32 T_k^2 d_k^3 \Big(\sum_{v \in H_{kj}} \frac{2^{m_{j-1}^k}}{n_v} + k p_k M_k^{-1/2} \log^2 M_k \Big)$.

PROOF. From the proof of Lemma 8 we see that to get the desired approximation we need to calculate the integral

$$\int_A f(n_u x) f(n_v x) \quad \text{for} \quad u < v \in H_{kj}.$$

Let the Fourier expansion of f be as in (2.2), and put

$$\tilde{f}(x) = \sum_{n \leqq M_k} (a_n \cos 2\pi n x + b_n \sin 2\pi n x).$$

Therefore, for each $a\in\mathbf{R}$ and each $1<\lambda_1<\lambda_2$, we have

$$\left|\int_a^{a+1} f(\lambda_1 x)f(\lambda_2 x)\right| \leqq \int_a^{a+1} |f(\lambda_1 x)-\tilde{f}(\lambda_1 x)|\cdot|f(\lambda_2 x)| + \tag{2.33}$$

$$+\int_a^{a+1} |f(\lambda_2 x)-\tilde{f}(\lambda_2 x)|\cdot|\tilde{f}(\lambda_1 x)| + \left|\int_a^{a+1} \tilde{f}(\lambda_1 x)\tilde{f}(\lambda_2 x)\right| \leqq$$

$$\leqq 2\|f-\tilde{f}\|_2\cdot\|f\|_2 + 2\|f-\tilde{f}\|_2\cdot\|\tilde{f}\|_2 + \left|\int_a^{a+1} \tilde{f}(\lambda_1 x)\tilde{f}(\lambda_2 x)\right| \leqq$$

$$\leqq 4M_k^{-1/2}\cdot T_k^2 d_k^2 + \left|\int_a^{a+1} \tilde{f}(\lambda_1 x)\tilde{f}(\lambda_2 x)\right|,$$

and

$$\left|\int_a^{a+1} \tilde{f}(\lambda_1 x)\tilde{f}(\lambda_2 x)\right| \leqq \sum_{m,n\leqq M_k} \left|a_m a_n \int_a^{a+1} \cos 2\pi(m\lambda_1 \pm n\lambda_2)x\right| + \tag{2.34}$$

$$+\sum_{m,n\leqq M_k} \left|b_m a_n \int_a^{a+1} \sin 2\pi(m\lambda_1 \pm n\lambda_2)x\right| + \sum_{m,n\leqq M_k} \left|a_m b_n \int_a^{a+1} \sin 2\pi(m\lambda_1 \pm n\lambda_2)x\right| +$$

$$+\sum_{m,n\leqq M_k} \left|b_m b_n \int_a^{a+1} \cos 2\pi(m\lambda_1 \pm n\lambda_2)x\right|.$$

Put $m_u=n_u/2^{m_j^k-1}$, $u\in H_{kj}$. Thus for some $a\in\mathbf{R}$ we have

$$P^{-1}(A)\int_A f(n_u x)f(n_v x) = \int_a^{a+1} f(m_u x)f(m_v x).$$

This implies

$$\sum_{u<v\in H_{kj}} \left|P^{-1}(A)\int_A f(n_u x)f(n_v x)\right| = \sum_{u<v\in H_{kj}} \left|\int_a^{a+1} f(m_u x)f(m_v x)\right| = \tag{2.35}$$

$$= \left(\sum_{\mathrm{I}}+\sum_{\mathrm{II}}+\sum_{\mathrm{III}}\right)\left|\int_a^{a+1} f(m_u x)f(m_v x)\right|$$

(say) where the summations in $\sum_{\mathrm{I}}$, $\sum_{\mathrm{II}}$ and $\sum_{\mathrm{III}}$ extend over those $u<v$ in H_{kj} for which $\left(\frac{n_v}{n_u}\geqq M_k\right)$, $\left(\frac{n_v}{n_u}<M_k \text{ and } \left|\frac{n_v}{n_u}-\frac{p}{q}\right|\geqq\varepsilon_k \text{ for all } 1\leqq p,q\leqq M_k\right)$, and $\left(\frac{n_v}{n_u}<M_k \text{ and } \left|\frac{n_v}{n_u}-\frac{p}{q}\right|<\varepsilon_k \text{ for some } 1\leqq p,q\leqq M_k\right)$ respectively. By the fact that $\psi_k\to\infty$ as $k\to\infty$ and by the proof of Lemma 8, we obtain

$$\sum_{\mathrm{I}}\left|\int_a^{a+1} f(m_u x)f(m_v x)\right| \leqq 4T_k^2 d_k^3\frac{n_u}{n_v} \leqq 4T_k^2 d_k^3 p_k M_k^{-1}. \tag{2.36}$$

By (2.33), (2.34), (2.32) and the fact that $\frac{n_v}{n_u} > M_k$ if $v \geqq u+k$ for all large k, we must have

$$\sum_{\text{II}} \left| \int_a^{a+1} f(m_u x) f(m_v x) \right| \leqq 4k p_k T_k^2 d_k^2 M_k^{-1/2} + 4\varepsilon_k^{-1} m_J^{-1} k p_k T_k d_k \log^2 M_k \leqq \tag{2.37}$$

$$\leqq 12 k p_k T_k^2 d_k^2 M_k^{-1/2}$$

where J is the first element of H_{kj}.

Finally, by the condition (1.10) and by (2.33), (2.34) we have

$$\sum_{\text{III}} \left| \int_a^{a+1} f(m_u x) f(m_v x) \right| \leqq \sum_{\text{III}} \left(4M_k^{-1/2} T_k^2 d_k^2 + \left| \int_a^{a+1} \tilde{f}(m_u x) \tilde{f}(m_v x) \right| \right) \leqq \tag{2.38}$$

$$\leqq 4 p_k^2 T_k^2 d_k^2 M_k^{-1/2} + \sum_{\text{III}} \left| \int_a^{a+1} \tilde{f}(m_u x) \tilde{f}(m_v x) \right|,$$

and

$$\sum_{\text{III}} \left| \int_a^{a+1} \tilde{f}(m_u x) \tilde{f}(m_v x) \right| \leqq 4 \sum_{\substack{1 \leqq p, q \leqq M_k \\ (p,q)=1}} \sum_{\substack{u < v \in H_{kj} \\ \left| \frac{p}{q} - \frac{n_v}{n_u} \right| < \varepsilon_k}} \sum_{n=1}^{\infty} \frac{T_k d_k}{n^2 p q} +$$

$$+ 4 T_k d_k k p_k M_k^2 m_J^{-1} \log^2 M_k \leqq 8 T_k d_k k \sum_{1 \leqq p, q \leqq M_k} \frac{1}{pq} + 4 T_k d_k k p_k M_k^2 m_J^{-1} \log^2 M_k \leqq$$

$$\leqq 16 k T_k d_k \log^2 M_k .$$

Now the Lemma follows by the proof of Lemma 8 together with (2.35)—(2.38).

Replacing Lemma 8 by Lemma 10 we can finish the proof of Theorem 1, Case (iii) as in Section 2.3.

2.5. *Proof of Theorem* 1 (case (i)). There is no loss of generality in assuming that $\Phi_k \ll k^\beta$, otherwise we can always replace Φ_k by $\min(\Phi_k, k^\beta)$. Put

$$\lambda = \frac{1}{18} \quad \text{and} \quad \alpha = \frac{2}{9}, \tag{2.39}$$

and

$$\mu = \frac{1}{2}\left(\frac{1}{3\alpha} - 1\right), \quad \tilde{\alpha} = \frac{\alpha}{2}. \tag{2.40}$$

We then define integers

$$t_k = [\exp(k/\Phi_k^{\tilde{\alpha}})], \quad N_k = t_{k+1} - t_k, \quad T_k = k^3, \tag{2.41}$$

$$r_k = \left[\log\left(\frac{1}{12}\Phi_k - 3\right) / \log 2 \right], \quad d_k = 2^{r_k},$$

for each $k \in \mathbf{Z}^+$. Since Φ_k is increasing and Φ_k/k is decreasing, $t_k < t_{k+1}$ for all large k. We also have

$$t_{k+1} \ll t_k, \quad N_k \ll t_k/\Phi_k^{\tilde{\alpha}}. \tag{2.42}$$

As in Section 2.3 we can construct a sequence $\{Y_k', k \geqq 1\}$ of independent random vectors whose distributions are the Gaussian distributions with mean 0 and covariance matrix Γ_k such that

$$|X_k - Y_k'| \ll k^{-2} \ll \Phi_k^{-\alpha} \quad \text{a.s.}$$

Using the following five lemmas we can finish the proof of Theorem 1, Case (i) as in Section 2.3.

LEMMA 11. $\sum_{k \leqq m} \frac{t_k^{1/2}}{\Phi_m^{\alpha}} \ll \frac{t_m^{1/2}}{\Phi_m^{\tilde{\alpha}}}$ *as* $m \to \infty$.

PROOF. (The proof is based on the idea of the proof of Lemma 3.1 of Marcus and Philipp [13].) Let

$$s_k = \Phi_k^{\tilde{\alpha}}/k, \quad u_k = \Phi_k^{-\tilde{\alpha}}.$$

Thus

$$\frac{u_{k+1}}{2} \leqq \frac{1}{s_{k+1}} - \frac{1}{s_k} \leqq u_{k+1}. \tag{2.43}$$

To prove (2.43) we first write

$$1 = \frac{1}{s_{k+1}u_{k+1}} - \frac{1}{s_k u_k} = \frac{(s_k - s_{k+1})u_k - s_{k+1}(u_{k+1} - u_k)}{s_{k+1}s_k u_k u_{k+1}}. \tag{2.44}$$

But

$$(u_k - u_{k+1})/u_{k+1} \leqq (s_k - s_{k+1})/s_{k+1} \tag{2.45}$$

since, by the assumptions on Φ_k, we have

$$(\Phi_{k+1}/k+1)^{2\tilde{\alpha}} \leqq (\Phi_k/k)^{2\tilde{\alpha}} \left(\frac{k+1}{k}\right)^{1-2\tilde{\alpha}}.$$

This implies

$$\left(\frac{1}{\Phi_k^{\tilde{\alpha}}} - \frac{1}{\Phi_{k+1}^{\tilde{\alpha}}}\right)\Phi_{k+1}^{\tilde{\alpha}} \leqq \left(\frac{\Phi_k^{\tilde{\alpha}}}{k} - \frac{\Phi_{k+1}^{\tilde{\alpha}}}{k+1}\right) \cdot \frac{k+1}{\Phi_{k+1}^{\tilde{\alpha}}},$$

and this is the inequality in (2.45). Now (2.44) and (2.45) imply

$$1 \leqq (s_{k+1}^{-1} - s_k^{-1})(u_{k+1}^{-1} + u_k^{-1}) \leqq 2(s_{k+1}^{-1} - s_k^{-1})/u_{k+1}.$$

Therefore

$$\frac{u_{k+1}}{2} \leqq s_{k+1}^{-1} - s_k^{-1}.$$

Also, from (2.44),

$$1 \geqq \frac{s_k - s_{k+1}}{s_{k+1}s_k u_{k+1}} \geqq (s_{k+1}^{-1} - s_k^{-1})/u_{k+1}.$$

Hence

$$s_{k+1}^{-1} - s_k^{-1} \leqq u_{k+1}.$$

Next we prove

(2.46) $$t_{k+1}^{1/2}u_{k+1} \ll t_{k+1}^{1/2} - t_k^{1/2} \ll t_{k+1}^{1/2}u_{k+1}.$$

Equation (2.46) is true since, by (2.43), (2.41)

$$t_{k+1}^{1/2}u_{k+1} \ll t_{k+1}^{1/2}\left(1-\exp\left(\frac{1}{2}(s_k^{-1}-s_{k+1}^{-1})\right)\right) \ll t_{k+1}^{1/2} - t_k^{1/2} \ll$$

$$\ll t_{k+1}^{1/2}\left(1-\exp\left(\frac{1}{2}(s_k^{-1}-s_{k+1}^{-1})\right)\right) \ll t_{k+1}^{1/2}u_{k+1}.$$

Now by summation by parts and (2.46), we have

$$\sum_{k\leqq m} t_k^{1/2}u_k^2 = \Big(\sum_{k\leqq m} t_k^{1/2}u_k\Big)u_m + \sum_{k<m}\Big(\sum_{j\leqq k} t_j^{1/2}u_j\Big)(u_k-u_{k+1}) \ll$$

$$\ll t_m^{1/2}u_m + \sum_{k\leqq m} t_k^{1/2}u_k\cdot u_k^2 \ll t_m^{1/2}u_m$$

since $u_k \leqq 1/2$ for all large k and since, by (2.47),

$$u_k - u_{k+1} \leqq u_{k+1}\frac{s_k - s_{k+1}}{s_{k+1}} \leqq u_k\left(\frac{s_k}{s_{k+1}}-1\right) = u_k\left[\frac{k+1}{k}\left(\frac{\Phi_k}{\Phi_{k+1}}\right)^{\tilde{\alpha}}-1\right] \leqq$$

$$\leqq u_k\left(\frac{k+1}{k}-1\right) = u_k/k \leqq u_k\cdot u_k^2$$

for all large k.

LEMMA 12. $$P\Big\{\max_{1\leqq j\leqq 2^{r_k}}\sup_{s_j\leqq s<s_{j+1}}|R(s,t_k)-R(s_j,t_k)| \gg (t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha}\Big\} \ll k^{-2}.$$

PROOF. From the proof of Lemma 5.1 of Berkes and Philipp [6], we have here

$$P(E_k) \ll \sum_{r<v\leqq m+1}\exp\left(-2^{\alpha v}\log\frac{k}{\Phi_k^{\tilde{\alpha}}}\right)+k^{-2} \ll \left(\frac{k}{\Phi_k^{\tilde{\alpha}}}\right)^{-d_k^{\alpha}+1}+k^{-2} \ll k^{-2}.$$

On E_k^c we have

$$F(0,t_k,s_j,s) \ll (t_k\log\log t_k)^{1/2}\cdot d_k^{-\alpha} \ll (t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha}.$$

Note that in the proof of Lemma 12 we have used Proposition 1 since by (2.39) and (2.40) we have (5.2) in Berkes and Philipp [6] holds with $\beta=1/2$. This fact will be used in the proof of Lemma 13 below also.

LEMMA 13. $$P\Big\{\max_{t_k\leqq t\leqq t_{k+1}}\sup_{0\leqq s\leqq 1}|R(s,t)-R(s,t_k)| \gg (t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha/4}\Big\} \ll k^{-2}.$$

PROOF. By the proof of Lemma 5.2 of Berkes and Philipp [6] and by (2.42) Lemma 13 is proved.

LEMMA 14. $$P\Big\{\max_{1\leqq j\leqq 2^{r_k}}\sup_{s_j\leqq s<s_{j+1}}|K(s,t_k)-K(s_j-t_k)| \gg (t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha}\Big\} \ll k^{-2}.$$

PROOF. From the proof of Lemma 6.2 of Berkes and Philipp [6] we have here

$$\psi(u) = \text{constant}\cdot 2^{-r_k/2}\cdot u^{1/2}.$$

If we let $x=\Phi_k^{1/2-\alpha}(\log\log t_k)^{1/2}$, then the probability in question is

$$\ll d_k\cdot k\cdot\int_x^\infty e^{-u^2/2}\,du\ll d_k\cdot k\cdot e^{-x^2/2}\ll$$

$$\ll\Phi_k\cdot k\cdot\exp(-\Phi_k^{1-2\alpha}\log\log t_k)\ll\Phi_k\cdot k\cdot\left(\frac{k}{\Phi_k^{\bar a}}\right)^{-\Phi_k^{1-2\alpha}}\ll k^{-2},$$

and

$$x\cdot 2^{-r_k/2}\ll(\log\log t_k)^{1/2}/\Phi_k^{\alpha}.$$

LEMMA 15. $P\{\sup_{t_k\leqq t\leqq t_{k+1}}\sup_{0\leqq s\leqq 1}|K(s,t)-K(s,t_k)|\gg(t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha/4}\}\ll k^{-2}$.

PROOF. In the proof of Lemma 6.3 of Berkes and Philipp [6], we take

$$\psi(u)=\text{constant}\cdot(t_{k+1}-t_k)^{1/2}u^{1/2}$$

and let $x=3(\log\log t_k)^{1/2}$. Therefore the probability in question is

$$\ll\int_x^\infty e^{-u^2/2}\,du\ll e^{-x^2/2}\ll e^{-3\log\log t_k}\ll k^{-2}.$$

Also

$$x(t_{k+1}-t_k)^{1/2}\ll N_k^{1/2}(\log\log t_k)^{1/2}\ll(t_k\log\log t_k)^{1/2}/\Phi_k^{\alpha/4}.$$

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ON THE CONVERGENCE AND SATURATION PROBLEM OF A SEQUENCE OF DISCRETE LINEAR OPERATORS OF EXPONENTIAL TYPE IN $L_p(-\infty, \infty)$ SPACES

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1. Introduction

The problem of determining the optimal order of approximation of a function f by a sequence of operators allows the following interpretations: one keeps the approximation process fixed and varies the properties of the function f to obtain the optimal order. The result is that the approximation by the given process will be optimal for all functions belonging to a certain class, the so-called saturation class. This interpretation was posed by J. Favard in 1947. The first results of this type are due to G. Alexits [14], E. Hille [16], A. Zygmund [15].

The first saturation result of discrete type operators is obtained by J. Szabados [2] ($p=\infty$, 2π-periodic case), and by V. Popov—J. Szabados [3] ($1 \leqq p < \infty$, 2π-periodic case).

2. The purpose of the paper

We consider the following discrete linear operators of entire exponential type:

$$I_\sigma(f, x) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin^2 \frac{\sigma}{2}(x - t_k)}{\left[\frac{\sigma}{2}(x - t_k)\right]^2}, \quad k = 0, \pm 1, \pm 2, \ldots; \quad t_k = \frac{2k\pi}{\sigma}$$

for a fixed, bounded function $f(x)$ in $L_p(-\infty, \infty)$.

The operators of the analogous type were considered by P. Butzer [12] in connection with generalizations of the Shannon sampling theorem.

The purpose of this paper is to give different types of estimates for

$$\|I_\sigma(f) - f\|_{L_p(-\infty,\infty)}, \quad 1 \leqq p \leqq \infty$$

and to solve the saturation problem for this approximation process. The operators $I_\sigma(f, x)$ possess the following interpolation properties:

$$I_\sigma(f, t_k) = f(t_k), \quad I'_\sigma(f, t_k) = 0$$

and

$$I_\sigma(1, x) \equiv 1, \quad \forall x \in (-\infty, \infty).$$

3. Notations

(3.1) $$F_\sigma(x) = \frac{2}{\sigma\pi}\frac{\sin^2\frac{\sigma x}{2}}{x^2}$$

is the Fejér kernel,

(3.2) $B_{\sigma,p}$ is the linear space of entire functions of exponential type $\sigma>0$, which belong to L_p.

(3.3) The operators $Q_\sigma: L_p \to B_{2\sigma,p}$, where

$$Q_\sigma(f,x) = c_\sigma \int_{-\infty}^{\infty} f(x+t)F_\sigma^2(t)\,dt, \quad c_\sigma = 1/\int_{-\infty}^{\infty} F_\sigma^2(t)\,dt$$

are the Jackson means.

(3.4) The operators $\Omega_\sigma: L_p \to B_{\sigma,p}$, where

$$\Omega_\sigma(f,x) = \int_{-\infty}^{\infty} f(x+t)F_\sigma(t)\,dt$$

are the Fejér means.

Obviously

(3.5) $$I_\sigma(f,x) = \frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty} f(t_k)F_\sigma(x-t_k), \quad t_k = \frac{2k\pi}{\sigma}, \quad k = 0, \pm1, \pm2, \ldots,$$

(3.6) $$\begin{cases} \|g\|_{L_p'} = \begin{cases} \operatorname{ess\,sup}\,(|g(x)|,\ x\in(-\infty,\infty)), & p=\infty \\ \|g\|_{L_p}, & 1 \leqq p < \infty \end{cases} \\ \|g\|_{L_\infty} = \sup\,(|g(x)|,\ x\in(-\infty,\infty)), \end{cases}$$

(3.7) $$Ff(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itx}f(t)\,dt, \quad F^{-1}f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}f(t)\,dt$$

are the Fourier and the converse of Fourier transforms, resp.

4. Two lemmas

LEMMA A. *If* $1\leqq p\leqq\infty$, $h>0$, $x_k=a+kh$, $k=0, \pm1, \pm2, \ldots$ *and* $g_\sigma(z)\in B_{\sigma,p}$, *then*

(4.1) $$(1-h\sigma)\|g_\sigma\|_{L_p} \leqq \left(h\sum_{k=-\infty}^{\infty}|g_\sigma(x_k)|^p\right)^{1/p} \leqq (1+h\sigma)\|g_\sigma\|_{L_p}.$$

PROOF. The right inequality is in fact Theorem 3.3.1 from [1], pp. 122—123. On the other hand

(4.2) $$\int_{-\infty}^{\infty}|g_\sigma(x)|^p\,dx = h\sum_{k=-\infty}^{\infty}|g_\sigma(\xi_k)|^p, \quad x_k<\xi_k<x_{k+1}$$

and from the proof of the just quoted theorem

$$-\left(h\sum_{-\infty}^{\infty}|g_\sigma(\xi_k)|^p\right)^{1/p}+\left(h\sum_{-\infty}^{\infty}|g_\sigma(x_k)|^p\right)^{1/p}\leqq -h\sigma\|g_\sigma\|_{L_p} \tag{4.3}$$

from where the left side of (4.1) follows.

LEMMA B. *If $f(x)$ is a fixed bounded function, then*

$$\begin{cases}\text{a)} & I_\sigma(f,x)\in B_{\sigma,\infty}\\ \text{b)} & \|I_\sigma(f,x)\|_{L_\infty}\leqq\|f\|_{L_\infty}.\end{cases} \tag{4.4}$$

PROOF. From the inequality

$$|I_\sigma(f,x)|\leqq\|f\|_{L_\infty}\sum_{k=-\infty}^{\infty}\frac{\sin^2\frac{\sigma}{2}(x-t_k)}{\left[\frac{\sigma}{2}(x-t_k)\right]^2}=\|f\|_{L_\infty} \tag{4.5}$$

follows b). Let $z=x+iy$ be a complex number. The series $I_\sigma(f,z)$ is uniformly convergent in every compact domain in the complex plane which follows for example from the representation

$$\left|\sin^2\frac{\sigma}{2}(x+iy-t_k)\right|=\left|\sum_{l=0}^{\infty}\frac{(iy)^l}{l!}\left[\sin^2\frac{\sigma}{2}x\right]^{(l)}(x-t_k)\right|\leqq\sum_{l=0}^{\infty}\frac{(\sigma|y|)^l}{l!}=e^{\sigma|y|}. \tag{4.6}$$

From the theorem of Weierstrass it follows that $I_\sigma(f,z)$ is an entire function. We have from (3.5) applying Bernstein's inequality and Lemma A

(4.7)

$$|I_\sigma(f,z)|\leqq\frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)||F_\sigma(z-t_k)|=\frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)|\left|\sum_{l=0}^{\infty}\frac{z}{l!}F_\sigma^{(l)}(-t_k)\right|\leqq$$

$$\leqq\|f\|_{L_\infty}\frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty}|F_\sigma^{(l)}(-t_k)|\frac{|z|^l}{l!}=\frac{2\pi}{\sigma}\|f\|_{L_\infty}\sum_{l=0}^{\infty}\frac{|z|^l}{l!}\sum_{k=-\infty}^{\infty}|F_\sigma^{(l)}(-t_k)|\leqq$$

$$\leqq\|f\|_{L_\infty}(1+2\pi)\|F_\sigma\|_{L_1}\sum_{l=0}^{\infty}\frac{|\sigma z|^l}{l!}=(1+2\pi)\|f\|_{L_\infty}\|F_\sigma\|_{L_1}e^{\sigma|z|}.$$

From (4.7) it follows that $I_\sigma(f,z)\in B_{\sigma,\infty}$.

5. Convergence of $I_\sigma(f,x)$ to $f(x)$

Applying the inequality

$$|F_\sigma(x)|\leqq\min\left(\frac{2}{\pi\sigma x^2},\frac{\sigma}{2\pi}\right) \tag{5.1}$$

the first theorem is obtained with a technique similar to that in [3].

THEOREM 1. *If* $f(x)\in L_\infty$, *then*

$$\|f(x)-I_\sigma(f,x)\|_{L_\infty}=O(1)\sum_{i=1}^{\infty} i^{-2}\omega\left(f,\frac{i}{\sigma}\right)_{L_\infty}; \tag{5.2}$$

and

a) *if* $\omega(f,\delta)_{L_\infty}\to 0$, $\delta\to 0_+$, *then* $\|f(x)-I_\sigma(f,x)\|_{L_\infty}\to 0$, $\sigma\to\infty$,

b) *if* $f(x)\in \mathrm{Lip}\,1$, *then* $|f(x+h)-f(x)|=O(h)$ *and*

$$\|f(x)-I_\sigma(f,x)\|_{L_\infty}=O\left(\frac{\ln\sigma}{\sigma}\right),\quad \sigma\to\infty$$

i.e. the expected optimal order $O\left(\frac{1}{\sigma}\right)$ *is not obtained.*

The following theorem is obtained by the definition of $\tau(f,\delta)_{L_1}$ (see [6]) and with a similar technique.

THEOREM 2. *If* $f(x)\in L_1\cap L_\infty$, *then*

$$\|f(x)-I_\sigma(f,x)\|_{L_1}=O(1)\left(\frac{1}{\sigma}\int_{1/2}^{\sigma}\tau\left(f,\frac{1}{u}\right)_{L_1}du+\tau\left(f,\frac{1}{\sigma}\right)_{L_1}+\frac{\|f\|_{L_1}}{\sigma}\right) \tag{5.3}$$

and

a) *if* $\tau(f,\delta)_{L_1}\to 0$, $\delta\to 0_+$, *then* $\|f(x)-I_\sigma(f,x)\|_{L_1}\to 0$, $\sigma\to\infty$,

b) *if* $\tau(f,\delta)_{L_1}=O(\delta)$, $\delta\to 0_+$, *then* $\|f(x)-I_\sigma(f,x)\|_{L_1}=O\left(\frac{\ln\sigma}{\sigma}\right)$, $\sigma\to\infty$.

These estimates do not give the expected optimal order $O\left(\frac{1}{\sigma}\right)$. In order to get a better estimate we prove Theorem 3 which also settles the case $1\leqq p\leqq\infty$.

6. Definition of the Hilbert transform

DEFINITION 1. The Hilbert transform of a function $f\in L_p$, $1\leqq p<\infty$ is defined as the Cauchy principal value of the integral

$$Hf(x)=\mathrm{PV}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x+u)}{u}du\right]=-\mathrm{PV}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-v)}{v}dv\right] \tag{6.1}$$

whenever it exists (see Definition 8.0.1, [7], page 306).

According to Theorem 8.1.6, [7], the Hilbert transform Hf of a function $f\in L_p$, $1\leqq p<\infty$ exists almost everywhere.

By Proposition 8.3.1, [7], if $f\in L_p$, $1<p\leqq 2$, then

$$Hf(x)=F^{-1}[i\,\mathrm{sign}\,(*)Ff(*)](x)\quad \text{a.e.}\quad (\text{see } 3.7). \tag{6.2}$$

DEFINITION 2 ([9], p. 172). If $f(t)$ is such that $\frac{f(t)}{1+|t|}\in L_2(-\infty,\infty)$ then the Hilbert transform is (following N. Achiezer)

$$HAf(x)=(x-i)F^{-1}\left[i\,\mathrm{sign}\,(*)F\left(\frac{f(\cdot)}{(\cdot)-i}\right)(*)\right](x). \tag{6.3}$$

By N. Achiezer [9], p. 173, if $f\in L_2(-\infty,\infty)$ then

$$Hf(x)=HAf(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\,dt. \tag{6.4}$$

PROPOSITION 6.1. *If* $f\in L_p\cap\left\{f:\ \frac{f(t)}{1+|t|}\in L_2\right\}$ *then* $(1\leqq p<\infty)$

$$Hf(x)=HAf(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\,dt. \tag{6.5}$$

PROOF. By the F. Riesz inequality we have (for ψ_ε see [1], p. 7)

$$\|H(f\psi_\varepsilon)-Hf\|_{L_p}\leqq c\|f\psi_\varepsilon-f\|_{L_p}\to 0 \quad \text{as} \quad \varepsilon\to 0,\quad 1<p<\infty$$

where $\psi_\varepsilon(x)=\sqrt{2\pi}F\varphi_\varepsilon(x)$. On the other hand for $p=1$

$$\operatorname{mes}\{x,\,|H(f\psi_\varepsilon)(x)-Hf(x)|>M\}\leqq\frac{c}{M}\|f\psi_\varepsilon-f\|_{L_1}\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

It follows that there exists a sequence such that (cf. Proposition 0.1.10, [7])

$$H(f\psi_{\varepsilon_k})(x)\xrightarrow{\text{a.e.}}Hf(x);\quad \varepsilon_k\to 0,\quad k\to\infty.$$

From (6.3) we have because $\frac{f(t)}{1+|t|}\in L_2$ (applying Plancherel's theorem)

$$\left\|F^{-1}\left[i\,\operatorname{sign}(*)\,F\left(\frac{(f\psi_{\varepsilon k}-f)(\cdot)}{(\cdot)-i}\right)(*)\right]\right\|_{L_2}=\left\|\frac{(f\psi_{\varepsilon_k}-f)(\cdot)}{(\cdot)-i}\right\|_{L_2}\xrightarrow[k\to\infty]{}0\Rightarrow$$

$$\Rightarrow HA(f\psi_{\varepsilon_{k_1}})(x)\xrightarrow{\text{a.e.}}HAf(x) \quad \text{as} \quad k_1\to\infty.$$

Now one can write that (from (6.4) for $f\psi_\varepsilon(x)$)

$$Hf(x)=HAf(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\,dt \quad \text{a.e.}$$

DEFINITION 3. $W^1_\infty=\{g$ is a locally absolutely continuous function $g(x)\in L_\infty$, $g'(x)\in L'_\infty\}$. $(W^1_\infty)'=\{h^*$: there exists $h\in W^1_\infty$, so that $h'=h^*$ a.e.$\}$. Obviously $(W^1_\infty)'\not\subset\bigcup_{1\leqq p<\infty}L_p$ and $(W^1_\infty)'\cap\left(\bigcup_{1\leqq p<\infty}L_p\right)\neq\emptyset$. Now we define the Hilbert transform for the class $(W^1_\infty)'$. Let $f(x)\in W^1_\infty$. Then $f'(x)\psi_\varepsilon(x)\in L_2(-\infty,\infty)$ where $\psi_\varepsilon(x)=\sqrt{2\pi}F\varphi_\varepsilon(x)$ (see [1], p. 7). We have

$$\psi_\varepsilon(x)\in B_{\varepsilon,p},\quad 1\leqq p\leqq\infty;\quad \psi_\varepsilon(x)\in S\subset L_p,\quad |\psi_\varepsilon(x)|\leqq\int_{-\infty}^{\infty}\varphi_\varepsilon(t)\,dt=1,$$

$$\psi_\varepsilon(x)\to 1,\quad \varepsilon\to 0,\quad |\psi'_\varepsilon(x)|\leqq\varepsilon\left(\psi_\varepsilon\in B_{\varepsilon,\infty}\right).$$

On the other hand

$$\|H(f\psi_\varepsilon')\|_{L_2} = \|f\psi_\varepsilon'\|_{L_2} \leqq \sup_x |f(x)| \|\psi_\varepsilon'\|_{L_2} \leqq \varepsilon \|\psi_\varepsilon\|_{L_2} \|f\|_{L_\infty} =$$

$$= \varepsilon \|f\|_{L_\infty} \left(\int_{-\varepsilon}^{\varepsilon} \left[\frac{1}{\varepsilon} \varphi\left(\frac{v}{\varepsilon}\right) \right]^2 dv \right)^{1/2} = \sqrt{\varepsilon}\, \|f\|_{L_\infty} \|\varphi_1\|_{L_2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By Proposition 0.1.10, [7], p. 4 it follows that there exists a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ such that

$$H(f\psi_{\varepsilon_k}')(x) \to 0 \quad \text{as} \quad k \to \infty \quad \text{a.e.}; \quad \varepsilon_k \to 0, \quad k \to \infty. \tag{6.6}$$

According to (6.4) we have

$$H(f'\psi_\varepsilon)(x) = HA(f'\psi_\varepsilon)(x) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f'\psi_\varepsilon(t)}{t-i}\, dt. \tag{6.7}$$

Using (6.3) and applying Plancherel's theorem we obtain

$$\left\| F^{-1}\left[i \operatorname{sign}(*) F\left(\frac{f'\psi_\varepsilon(\cdot)}{(\cdot)-i} \right)(*) \right] - F^{-1}\left[i \operatorname{sign}(*) F\left(\frac{f'(\cdot)}{(\cdot)-i} \right)(*) \right] \right\|_{L_2} =$$

$$= \left(\int_{-\infty}^{\infty} \frac{|f'\psi_\varepsilon(t) - f'(t)|^2}{1+t^2}\, dt \right)^{1/2} \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{6.8}$$

by Lebesgue's theorem. The previous formula can be given in the form

$$H(f'\psi_\varepsilon)(x) = (x-i) R_\varepsilon(x) + \tau_\varepsilon. \tag{6.9}$$

So it has been established that

$$R_\varepsilon \xrightarrow[\varepsilon \to 0]{L_2} R, \quad R(x) = F^{-1}\left[i \operatorname{sign}(*) F\left(\frac{f'(\cdot)}{(\cdot)-i} \right)(*) \right](x). \tag{6.10}$$

On the other hand we get

$$\pi\tau_\varepsilon = \int_{-\infty}^{\infty} \frac{f'\psi_\varepsilon(t)}{t-i}\, dt = -\int_{-\infty}^{\infty} \frac{f(t)\psi_\varepsilon'(t)}{t-i}\, dt + \int_{-\infty}^{\infty} \frac{f(t)\psi_\varepsilon(t)}{(t-i)^2}\, dt = I_1(\varepsilon) + I_2(\varepsilon). \tag{6.11}$$

Applying Cauchy—Buniakowski's inequality yields

$$|I_1(\varepsilon)| \leqq \left(\int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2}\, dt \right)^{1/2} \sqrt{\varepsilon}\, \|\varphi_1\|_{L_2} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6.12}$$

By Lebesgue's theorem for the second integral it follows

$$\int_{-\infty}^{\infty} \frac{f(t)\psi_\varepsilon(t)}{(t-i)^2}\, dt \xrightarrow[\varepsilon \to 0]{} \int_{\infty-}^{\infty} \frac{f(t)}{(t-i)^2}\, dt = \pi\tau. \tag{6.13}$$

DEFINITION 4. If $f(x)\in W_\infty^1$ then

$$HN(f')(x) = (x-i)R(x)+\tau$$

where $\|R_\varepsilon - R\|_{L_2}\to 0$, $\varepsilon\to 0$ and $\tau = \lim_{\varepsilon\to 0}\tau_\varepsilon$.

PROPOSITION 6.2. *If* $f(x)\in W_\infty^1$ *and* $f'(x)\in L_p$ $(1\leqq p<\infty)$ *then* $HN(f')(x) = H(f')(x)$ *(see Proposition* 6.1*) so* $HN(f')$ *is an extension of the classical definition in* $(W_\infty^1)'\cup\left(\bigcup_{1\leqq p<\infty} L_p\right)$.

DEFINITION 5. If $g(x)\in L_\infty$, then

$$HRg(x+h)-HRg(x) = HN\left[\left(\int_0^{0+h} g(t)\,dt\right)'\right](x).$$

PROPOSITION 6.3. *If* $g(x)\in L_p\cap L_\infty$ $(1\leqq p<\infty)$ *then*

$$HRg(x+h)-HRg(x) = HN\left[\left(\int_0^{0+h} g(t)\,dt\right)'\right](x) = H\left[\left(\int_0^{0+h} g(t)\,dt\right)'\right](x) =$$

$$= Hg(x+h)-Hg(x)$$

so $HRg(x+h)-HRg(x)$ *is an extension of the classical definition* $Hg(x+h)-Hg(x)$ *in* $\bigcup_{1\leqq p\leqq\infty} L_p$.

Now we can write Hf' in $(W_\infty^1)'\cup\left(\bigcup_{1\leqq p<\infty} L_p\right)$ instead of $HN(f')$ and $Hg(x+h)-Hg(x)$ instead of $HRg(x+h)-HRg(x)$ in $\bigcup_{1\leqq p\leqq\infty} L_p$.

7. Some lemmas

We proceed with the proof of several lemmas. The first one is concerned with the Fejér means (see 3.4).

LEMMA 1. *If* $f(x)\in W_p^1$ *and* $H(f')\in L_p'$ *then*

$$\sigma\|\Omega_\sigma(f)-f\|_{L_p} = O(1)\|H(f')\|_{L_p'}.$$

PROOF. The following propositions are used (we have a change of signs because of Definition 1).

Let $1<p<\infty$ and $f\in L_p$. Then

$$H(Hf)(x) = -f(x) \quad \text{a.e.} \tag{7.1}$$

(see Proposition 8.2.10 in [7] and Definition 1).

We have for the Hilbert transform on the real line

$$HF_\sigma(x) = \frac{1}{\pi\sigma}\,\frac{\sin\sigma x-\sigma x]}{x^2} \tag{7.2}$$

(see Definition 6.1 and [7], p. 519, Table 3).

Let $f\in L_p$, $1<p<\infty$ and $g\in L_q$. Then

$$\int_{-\infty}^{\infty} Hf(x)\,g(x)\,dx = -\int_{-\infty}^{\infty} f(x)\,Hg(x)\,dx, \quad \frac{1}{p}+\frac{1}{q}=1, \tag{7.3}$$

(see [7], 8.2.1, p. 316 and Definition 6.1).

Let $f\in L_p$, $1<p<\infty$. Then $f\in W_p^1$ if and only if $Hf\in W_p^1$; and if $f(x)=\Phi(x)$ a.e. with $\Phi\in AC_{\text{loc}}$, $\Phi'\in L_p$ and $Hf(x)=\psi(x)$ a.e. with $\psi\in AC_{\text{loc}}$, $\psi'\in L_p$ then $H(\Phi')(x)=\psi'(x)$ a.e. (cf. Proposition 8.3.7, [7]).

There are 3 cases in the proof:

a) $1<p<\infty$. We have (Propositions 8.2.10 and 8.2.1 with change of signs because of Definition 6.1)

$$-\Omega_\sigma(f,x) = -\int_{-\infty}^{\infty} f(x+t)\,F_\sigma(t)\,dt = \int_{-\infty}^{\infty} f(x+t)\,H(HF_\sigma)(t)\,dt = \tag{7.4}$$

$$= -\int_{-\infty}^{\infty} Hf(x+t)\,HF_\sigma(t)\,dt = \frac{1}{\pi\sigma}\int_{-\infty}^{\infty} Hf(x+t)\,\frac{\sigma t-\sin\sigma t}{t^2}\,dt =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Hf(x+t)}{t}\,dt - \frac{1}{\pi\sigma}\int_{-\infty}^{\infty} Hf(x+t)\,\frac{\sin\sigma t}{t^2}\,dt =$$

$$= -f(x) - \frac{1}{\pi\sigma}\int_{-\infty}^{\infty} Hf(x+t)\,\frac{\sin\sigma t}{t^2}\,dt.$$

This implies

$$\Omega_\sigma(f,x)-f(x) = \frac{1}{\pi\sigma}\int_0^{\infty}[Hf(x+t)-Hf(x-t)]\,\frac{\sin\sigma t}{t^2}\,dt. \tag{7.5}$$

For $t>0$ integrating by parts gives

$$\left|\int_t^{\infty}\frac{\sin\sigma u}{u^2}\,du\right| \leqq \frac{|\cos\sigma t|}{\sigma t^2}+\frac{2}{\sigma}\frac{1}{t^2} = O\left(\frac{1}{\sigma t^2}\right) \tag{7.6}$$

from where and (7.5)

$$\pi\sigma[\Omega_\sigma(f,x)-f(x)] = \int_0^{1/\sigma}\int_{-t}^{t}(Hf)'(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt - \tag{7.7}$$

$$-\int_{1/\sigma}^{\infty}[Hf(x+t)-Hf(x-t)]\,d\int_t^{\infty}\frac{\sin\sigma u}{u^2}\,du = \int_0^{1/\sigma}\int_{-t}^{t}(Hf)'(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt -$$

$$-\left[[Hf(x+t)-Hf(x-t)]\int_t^{\infty}\frac{\sin\sigma u}{u^2}\,du\right|_{1/\sigma}^{\infty} + \int_{1/\sigma}^{\infty}\int_t^{\infty}\frac{\sin\sigma u}{u^2}\,du[(Hf)'(x+t)+$$

$$+(Hf)'(x+t)]\,dt = \int_0^{1/\sigma}\int_{-t}^{t}(Hf)'(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt + \left[Hf\left(x+\frac{1}{\sigma}\right)-Hf\left(x-\frac{1}{\sigma}\right)\right]\times$$

$$\times\int_{1/\sigma}^{\infty}\frac{\sin\sigma u}{u^2}\,du + \int_{1/\sigma}^{\infty}\int_t^{\infty}\frac{\sin\sigma u}{u^2}\,du[(Hf)'(x+t)+(Hf)'(x-t)]\,dt.$$

After taking the L_p-norm we obtain

$$\pi\sigma\|\Omega_\sigma(f,x)-f(x)\|_{L_p} \leqq 2\|(Hf)'\|_{L_p}\int_0^{1/\sigma}\frac{\sin\sigma t}{t}\,dt+ \tag{7.8}$$

$$+\frac{2}{\sigma}\|(Hf)'\|_{L_p}\left|\int_{1/\sigma}^{\infty}\frac{\sin\sigma u}{u^2}\,du\right|+6\|(Hf)'\|_{L_p}\int_{1/\sigma}^{\infty}\frac{1}{\sigma u^2}\,du \leqq$$

$$\leqq 10\|(Hf)'\|_{L_p} = 10\|H(f')\|_{L_p}$$

(cf. Proposition 8.3.7).

When $1<p<\infty$, $\|H(f')\|_{L_p}$ can be estimated by $\|f'\|_{L_p}$ given above (Proposition 8.1.9, [7], p. 312).

b) $f(x)\in W_1^1$ and $H(f')(x)\in L_1$. We have $f*\varphi_\varepsilon\in W_p^1$. By a) it follows that

$$\pi\sigma\|\Omega_\sigma(f*\varphi_\varepsilon,x)-f*\varphi_\varepsilon(x)\|_{L_1} \leqq 10\|H(f*\varphi_\varepsilon)'\|_{L_1}. \tag{7.9}$$

If $H(f')(x)\in L_1$, then according to Proposition 8.2.2, [7], p. 316 we have the following. Let $f\in L_p$ and $g\in L_1$. Then for $1<p<\infty$

$$H(f*g) = Hf*g \quad \text{a.e.} \tag{7.10}$$

For $p=1$ this remains true if one furthermore supposes $Hf\in L_1$. Now we get

$$\|H(f*\varphi_\varepsilon)'\|_{L_1} = \|H(f'*\varphi_\varepsilon)\|_{L_1} = \|H(f')*\varphi_\varepsilon\|_{L_1} \leqq \|H(f')\|_{L_1}. \tag{7.11}$$

By Fatou's theorem we obtain (Proposition 0.1.1 in [7])

$$\pi\sigma\|\Omega_\sigma(f,x)-f(x)\|_{L_1} \leqq 10\|H(f')\|_{L_1}, \tag{7.12}$$

since

$$|\Omega_\sigma(f*\varphi_\varepsilon,x)-f*\varphi_\varepsilon(x)| \to |\Omega_\sigma(f,x)-f(x)| \quad \text{as} \quad \varepsilon\to 0. \tag{7.13}$$

c) $f(x)\in W_\infty^1$ and $H(f')\in L'_\infty$. We have $f\psi_\varepsilon\in W_p^1$, $p\geqq 1$ and by a), (7.5) we obtain

$$\Omega_\sigma(f\psi_\varepsilon,x)-f\psi_\varepsilon(x) = \frac{1}{\pi\sigma}\int_0^{\infty}\int_{-t}^{t}H(f\psi_\varepsilon)'(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt. \tag{7.14}$$

From (6.5), (6.6), (6.7), (6.9), Definition 4 and Proposition 6.2 we have

$$H(f\psi_\varepsilon)'(x) = (x-i)R_\varepsilon(x)+P_\varepsilon(x)+r_\varepsilon, \tag{7.15}$$

where

$$\|R_\varepsilon-R\|_{L_2}\to 0, \quad \|P_\varepsilon\|_{L_2}\to 0, \quad r_\varepsilon\to r, \quad \text{as} \quad \varepsilon\to 0, \tag{7.16}$$

and

$$H(f')(x) = (x-i)R(x)+r. \tag{7.17}$$

Substituting $\psi_0(x)\equiv 1$ we get the following equality (like in (7.7))

$$(7.18)\qquad \Omega_\sigma(f\psi_\varepsilon, x)-f\psi_\varepsilon(x)=\frac{1}{\pi\sigma}\Big[\int_0^{1/\sigma}\int_{-t}^{t} H(f\psi_\varepsilon)'(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt+$$

$$+\int_{-1/\sigma}^{1/\sigma} H(f\psi_\varepsilon)'(x+v)\,dv\int_{1/\sigma}^{\infty}\frac{\sin\sigma u}{u^2}\,du+\int_{1/\sigma}^{\infty}\int_{t}^{\infty}\frac{\sin\sigma u}{u^2}\,du\times$$

$$\times\big[H(f\psi_\varepsilon)'(x+t)-H(f\psi_\varepsilon)'(x-t)\big]\,dt\Big]=\frac{1}{\pi\sigma}[I_1^\varepsilon(x)+I_2^\varepsilon(x)+I_3^\varepsilon(x)].$$

For the first integral we get

(7.19)

$$|I_1^\varepsilon(x)-I_1^0(x)|\leqq\int_0^{1/\sigma}\int_{-t}^{t}[|x+v-i|\,|R_\varepsilon(x+v)-R(x+v)|+P_\varepsilon(x+v)+|r_\varepsilon-r|]\,dv\times$$

$$\times\frac{\sin\sigma t}{t^2}\,dt\leqq\|R_\varepsilon-R\|_{L_2}\int_0^{1/\sigma}\Big(\int_{-t}^{t}[1+(x+v)^2]\,dv\Big)^{1/2}\frac{\sin\sigma t}{t^2}\,dt+$$

$$+\|P_\varepsilon\|_{L_2}\int_0^{1/\sigma}\sqrt{2t}\,\frac{\sin\sigma t}{t^2}\,dt+|r_\varepsilon-r|\int_0^{1/\sigma}2t\,\frac{\sin\sigma t}{t^2}\,dt\to 0\quad\text{as}\quad\varepsilon\to 0.$$

Analogously

$$(7.20)\qquad |I_2^\varepsilon(x)-I_2^0(x)|\to 0,\quad |I_3^\varepsilon(x)-I_3^0(x)|\to 0\quad\text{as}\quad\varepsilon\to 0.$$

Since

$$(7.21)\qquad \Omega_\sigma(f\psi_\varepsilon, x)\to\Omega_\sigma(f,x)\quad\text{and}\quad f\psi_\varepsilon(x)\to f(x)\quad\text{as}\quad\varepsilon\to 0$$

we establish from (7.14), (7.15) and (7.16)

$$(7.22)\qquad \Omega_\sigma(f,x)-f(x)=\frac{1}{\pi\sigma}\Big[\int_0^{1/\sigma}\int_{-t}^{t}H(f')(x+v)\,dv\,\frac{\sin\sigma t}{t^2}\,dt+$$

$$+\int_{-1/\sigma}^{1/\sigma}H(f')(x+v)\,dv\int_{1/\sigma}^{\infty}\frac{\sin\sigma u}{u^2}\,du+\int_{1/\sigma}^{\infty}\int_{t}^{\infty}\frac{\sin\sigma u}{u^2}\,du[H(f')(x+t)+$$

$$+H(f')(x-t)]\,dt,$$

and after having the L'_∞-norm in (7.22) we complete the proof.

LEMMA 2. *If* $f(x)\in W_p^1$, *then*

a) $$\|f(x)-Q_\sigma(f,x)\|_{L_p}=O\left(\frac{1}{\sigma}\right)\|f'\|_{L'_p},$$

b) $$\|Q'_\sigma(f,x)\|_{L_p}=O(1)\|f'\|_{L'_p}$$

where $Q_\sigma(f,x)$ *is the Jackson mean* (see (3.3)).

PROOF. From the condition $c_\sigma \|F_\sigma^2\|_{L_1}=1$ it follows that $c_\sigma = O\left(\frac{1}{\sigma}\right)$. Applying Hölder—Minkowski's inequality in the following chain

$$f(x)-Q_\sigma(f,x) = c_\sigma \int_{-\infty}^{\infty} (f(x)-f(x+t))F_\sigma^2(t)\,dt = c_\sigma \int_{-\infty}^{\infty}\int_{t}^{0} f'(x+u)\,du\,F_\sigma^2(t)\,dt$$

we get

$$\|f(x)-Q_\sigma(f,x)\|_{L_p} = O\left(\frac{1}{\sigma}\right)\|f'\|_{L_p'} \int_{-\infty}^{\infty} |t|\,F_\sigma^2(t)\,dt = O\left(\frac{1}{\sigma}\right)\|f'\|_{L_p'} \times$$

$$\times\left(\int_0^{1/\sigma} tF_\sigma^2(t)\,dt + \int_{1/\sigma}^{\infty} tF_\sigma^2(t)\,dt\right) = O\left(\frac{1}{\sigma}\right)\|f'\|_{L_p'}.$$

Applying the same inequality to

$$(Q_\sigma(f,x+h)-Q_\sigma(f,x))/h = c_\sigma \int_{-\infty}^{\infty} \frac{1}{h}\int_0^h f'(x+t+v)\,dv\,F_\sigma^2(t)\,dt$$

we obtain from Fatou's lemma

$$\left\|\lim_{h\to 0}\frac{Q_\sigma(f,x+h)-Q_\sigma(f,x)}{h}\right\|_{L_p} \leqq \liminf_{h\to 0}\left\|\frac{Q_\sigma(f,x+h)-Q_\sigma(f,x)}{h}\right\|_{L_p} \leqq \|f'\|_{L_p'}$$

which gives the second part of the lemma.

Lemma 3 is an application of the Riesz—Thorin interpolation theorem. Let

$$L_{\sigma,p,t_k} = \left\{f:\ \|f\|_{\sigma,p,t_k} = \left(\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)|^p\right)^{1/p} < \infty\right\};$$

then we prove the following

LEMMA 3. *If* $f(x)\in W_p^1$, *then*

$$\|f-Q_\sigma(f)\|_{\sigma,p,t_k} = O\left(\frac{1}{\sigma}\right)\|f'\|_{L_p'}.$$

PROOF. By Fubini's theorem we get after changing the order of integration

$$\frac{1}{\sigma}|f(t_k)-Q_\sigma(f,t_k)| \leqq \frac{c_\sigma}{\sigma}\left[\left|\int_0^{\infty}\int_0^{t} f'(u+t_k)\,du\,F_\sigma^2(t)\,dt\right| + \right. \tag{7.23}$$

$$\left. + \left|\int_{-\infty}^{0}\int_{t}^{0} f'(u+t_k)\,du\,F_\sigma^2(t)\,dt\right|\right] = \frac{c_\sigma}{\sigma}\left|\int_0^{\infty} f'(u+t_k)\int_u^{\infty} F_\sigma^2(t)\,dt\,du\right| +$$

$$+\frac{c_\sigma}{\sigma}\left|\int_{-\infty}^{0} f'(u+t_k)\int_{-\infty}^{u} F_\sigma^2(t)\,dt\,du\right|.$$

On the other hand

$$\int_u^\infty F_\sigma^2(t)\,dt = \frac{4}{\pi^2\sigma^2}\int_u^\infty \frac{\sin^4 \sigma t/2}{t^4}\,dt \leqq \frac{4}{\pi^2\sigma^2}\int_u^\infty \frac{1}{t^4}\,dt = \frac{4}{3\pi^2}\frac{1}{\sigma^2 u^3},$$

and by

$$\int_0^\infty = \sum_{l=0}^\infty \int_{2l\pi/\sigma}^{2(l+1)\pi/\sigma}$$

estimating the integrals we obtain

a) $$\left|\int_0^{2\pi/\sigma} f'(u+t_k)\int_u^\infty F_\sigma^2(t)\,dt\,du\right| = O(\sigma)\int_0^{2\pi/\sigma}|f'(u+t_k)|\,du, \quad l=0,$$

b)

$$\left|\int_{2l\pi/\sigma}^{2(l+1)\pi/\sigma} f'(u+t_k)\int_u^\infty F_\sigma^2(t)\,dt\,du\right| = \frac{\sigma}{6\pi^5 l^3}\int_{2l\pi/\sigma}^{2(l+1)\pi/\sigma}|f'(u+t_k)|\,du, \quad l \geqq 1.$$

Thus the following inequality is established:

$$\frac{1}{\sigma}|f(t_k)-Q_\sigma(f,t_k)| = O\left(\frac{1}{\sigma}\right)\sum_{l=0}^\infty \frac{1}{(l+1)^3}\int_{2l\pi/\sigma<|u|\leqq 2(l+1)\pi/\sigma}|f'(u+t_k)|\,du.$$

After summing over k we get

(7.24)

$$\sum_{k=-\infty}^\infty \frac{1}{\sigma}|f(t_k)-Q_\sigma(f,t_k)| = O\left(\frac{1}{\sigma}\right)\sum_{l=0}^\infty \frac{1}{(l+1)^3}\sum_{k=-\infty}^\infty\left(\int_{2l\pi/\sigma}^{2(l+1)\pi/\sigma}|f'(u+t_k)|\,du+\right.$$

$$\left.+\int_{-2(l+1)\pi/\sigma}^{-2l\pi/\sigma}|f'(u+t_k)|\,du\right) = O\left(\frac{1}{\sigma}\right)\sum_{l=0}^\infty\frac{1}{(l+1)^3}\|f'\|_{L_1} = O\left(\frac{1}{\sigma}\right)\|f'\|_{L_1}.$$

The case $p=\infty$ is a trivial one.

Let us consider the linear operator

$$T_\sigma(g,x) = c_\sigma\int_{-\infty}^\infty\int_x^{x+t} g(u)\,du\,F_\sigma^2(t)\,dt$$

and the norms

$$\|g\|_{\sigma,p,t_k} = \begin{cases}\left(\frac{1}{\sigma}\sum_{k=-\infty}^\infty |g(t_k)|^p\right)^{1/p}, & 1\leqq p<\infty\\ \sup|g(t_k)|, \quad -\infty<k<\infty, & p=\infty.\end{cases}$$

Then the above estimates can be stated as follows:

(7.25)

$$\begin{cases}\|T_\sigma(g,x)\|_{\sigma,1,t_k} = O\left(\frac{1}{\sigma}\right)\|g\|_{L_1'}; \quad T_\sigma: L_1' \to L_{\sigma,1,t_k}\\ \|T_\sigma(g,x)\|_{\sigma,\infty,t_k} = O\left(\frac{1}{\sigma}\right)\|g\|_{L_\infty'}; \quad T_\sigma: L_\infty' \to L_{\sigma,\infty,t_k}.\end{cases}$$

Hence by the Riesz—Thorin interpolation theorem we get

$$\|T_\sigma(g,x)\|_{\sigma,p,t_k} = O\left(\frac{1}{\sigma}\right)\|g\|_{L'_p}, \quad 1 \leqq p \leqq \infty \tag{7.26}$$

which is equivalent to the statement of the lemma.

LEMMA 4. *If the function $f(x)$ is bounded, then*

$$\|I_\sigma(f,x)\|_{L_p} = O(1)\left(\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)|^p\right)^{1/p}. \tag{7.27}$$

PROOF. The case $p=\infty$ has been proved in Lemma B of this paper. By Hölder's inequality we get

(7.28)

$$|I_\sigma(f,x)| \leqq \frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)||F_\sigma(x-t_k)| \leqq O\left(\frac{1}{\sigma}\right)\left(\sum_{k=-\infty}^{\infty}|f(t_k)|^p F_\sigma(x-t_k)\right)^{1/p} \times$$

$$\times\left(\sum_{k=-\infty}^{\infty}F_\sigma(x-t_k)\right)^{(p-1)/p} = O(\sigma^{-1/p})\left(\sum_{k=-\infty}^{\infty}|f(t_k)|^p F_\sigma(x-t_k)\right)^{1/p}.$$

By B. Levi's theorem we obtain

$$\|I_\sigma(f,x)\|_{L_p} = O(\sigma^{-1/p})\left(\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|f(t_k)|^p F_\sigma(x-t_k)\,dx\right)^{1/p} = \tag{7.29}$$

$$= O(\sigma^{-1/p})\left(\sum_{k=-\infty}^{\infty}|f(t_k)|^p\int_{-\infty}^{\infty}F_\sigma(x-t_k)\,dx\right)^{1/p} = O(1)\left(\frac{1}{\sigma}\sum_{k=-\infty}^{\infty}|f(t_k)|^p\right)^{1/p}$$

which completes the proof of the lemma.

LEMMA 5. *If $t_\sigma(x)\in B_{\sigma,p}$, then*

$$\|I_\sigma(t_\sigma,x)-\Omega_\sigma(t_\sigma,x)\|_{L_p} = O\left(\frac{1}{\sigma}\right)(\|t'_\sigma\|_{L_p}+\|H(t'_\sigma)\|_{L_p}), \tag{7.30}$$

where $H(t'_\sigma)$ is the Hilbert transform of the function t'_σ.

PROOF. Let first $t_\sigma(x)\in B_{\sigma,1}$. Then

$$\begin{cases} Ft'_\sigma(u) = iuFt_\sigma(u), \\ FH(t'_\sigma)(u) = iu\cdot i\operatorname{sgn} uFt_\sigma(u) = -|u|\,Ft_\sigma(u). \end{cases} \tag{7.31}$$

Let us consider the expression (analogously to [9], page 187)

(7.32)
$$\sin\alpha t'_\sigma(x)+\cos\alpha H(t'_\sigma)(x)=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(iu\sin\alpha-|u|\cos\alpha)Ft_\sigma(u)e^{iux}\,du=$$

$$=\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(u\sin\alpha+i|u|\cos\alpha)Ft_\sigma(u)e^{iux}\,du=$$

$$=\frac{i}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}e^{i(x-\alpha/\sigma)u}e^{i\alpha u/\sigma}[u\sin\alpha+i|u|\cos\alpha]\,Ft_\sigma(u)\,du=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}e^{i(x-\alpha/\sigma)u}V(u)\,Ft_\sigma(u)\,du$$

where

(7.33)
$$V(u)=ie^{i\alpha u/\sigma}(u\sin\alpha+i|u|\cos\alpha)$$

has the Fourier series

(7.34)
$$V(u)=2\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{\sin^2\dfrac{\alpha-k\pi}{2}}{(\alpha-k\pi)^2}e^{ik\pi u/\sigma}.$$

From (7.32), (7.33) and (7.34) we obtain

(7.35)
$$\sin\alpha t'_\sigma(x)+\cos\alpha H(t'_\sigma)(x)=2\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{\sin^2\dfrac{\alpha-k\pi}{2}}{(\alpha-k\pi)^2}t_\sigma\left(x+\frac{k\pi-\alpha}{\sigma}\right).$$

If $t_\sigma\in B_{\sigma,p}$, $1\leqq p\leqq\infty$ using (7.35) for $t_\sigma\psi_\varepsilon\in B_{\sigma+\varepsilon,1}$ and the fact that $H[(t_\sigma\psi_{\varepsilon_k})']\xrightarrow[\text{a.e.}]{}H(t'_\sigma)$ we obtain the interpolation formula

(7.36)
$$\sin\alpha t'_\sigma(x)+\cos\alpha\,H(t'_\sigma)(x)=2\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{\sin^2\dfrac{\alpha-k\pi}{2}}{(\alpha-k\pi)^2}t_\sigma\left(x+\frac{k\pi-\alpha}{\sigma}\right),\quad\text{a.e.}$$

We see from (7.36) that there exists a function $A(x)$ such that

$$H(t'_\sigma)(x)\overset{\text{a.e.}}{=}A(x),\quad A(x)\in B_{\sigma,p}.$$

Let now $t_\sigma(x)\in B_{\sigma,1}$ again. If $\alpha=\sigma x$ in (7.36) then

(7.37)
$$\sin\sigma x t'_\sigma(x)+\cos\sigma x H(t'_\sigma)(x)=\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\dfrac{\sigma x-k\pi}{2}}{(\sigma x-k\pi)^2}t_\sigma\left(\frac{k\pi}{\sigma}\right)=$$

$$=-\sigma I_\sigma(t_\sigma,x)+\sigma\sum_{k=-\infty}^{\infty}\frac{2\sin^2\dfrac{\sigma x-k\pi}{2}}{(\sigma x-k\pi)^2}t_\sigma\left(\frac{k\pi}{\sigma}\right).$$

THEOREM (Civin P., see [1], p. 111). *If $\omega_\sigma(t)$ is a continuous function with period 2σ and a is such that the Fourier series*

$$(7.38)\qquad e^{iax}\omega_\sigma(x)=\sum_{k=-\infty}^{\infty} c_k^\sigma e^{ik\pi x/\sigma},\quad |x|<\sigma,\quad c_k^\sigma=\frac{1}{2\sigma}\int_{-\sigma}^{\sigma}\omega_\sigma(u)e^{i(a-k\pi/\sigma)u}\,du$$

is uniformly convergent, i.e. $\sum_{k=-\infty}^{\infty}|c_k^\sigma|<\infty$, *moreover* $f\in L_1$ *and* $Ff\in L_1$ *then*

$$(7.39)\qquad F^{-1}(\omega_\sigma Ff)=\sum_{k=-\infty}^{\infty} c_k^\sigma f\left(x+\frac{k\pi}{\sigma}-a\right)$$

and the series on the right is uniformly convergent.

Here $Ft_\sigma=\omega_\sigma$ is a continuous function which can be extended continuously out of $(-\sigma,\sigma)$. Also $f=F_\sigma$, $F(F_\sigma)\in L_1$ which is trivial.

On the other hand

$$\sum_{k=-\infty}^{\infty}\left|t_\sigma\left(\frac{k\pi}{\sigma}\right)\right|=\frac{\sigma}{\pi}\sum_{k=-\infty}^{\infty}\left|t_\sigma\left(\frac{k\pi}{\sigma}\right)\right|\frac{\pi}{\sigma}\leqq\frac{\sigma}{\pi}(1+\pi)\|t_\sigma\|_{L_1}<\infty$$

from Lemma A. Now we can write

$$(7.40)\qquad \Omega_\sigma(t_\sigma,x)=\sqrt{2\pi}\;t_\sigma * F_\sigma(x)=\sqrt{2\pi}\;F^{-1}(Ft_\sigma FF_\sigma)=\sqrt{2\pi}\;F^{-1}(\omega_\sigma FF_\sigma)=$$

$$=\sqrt{2\pi}\sum_{k=-\infty}^{\infty}c_k^\sigma F_\sigma\left(x+\frac{k\pi}{\sigma}\right)=\sum_{k=-\infty}^{\infty}\frac{\pi}{\sigma}t_\sigma\left(\frac{k\pi}{\sigma}\right)F_\sigma\left(x-\frac{k\pi}{\sigma}\right)=$$

$$=\sum_{k=-\infty}^{\infty}\frac{2\sin^2\dfrac{\sigma x-k\pi}{2}}{(\sigma x-k\pi)^2}\,t_\sigma\left(\frac{k\pi}{\sigma}\right).$$

On the other hand

$$(7.41)\qquad \Omega_\sigma(t_\sigma,x)=\sqrt{2\pi}\;t_\sigma * F_\sigma(x)=\int_{-\infty}^{\infty}t_\sigma(x\pm t)F_\sigma(t)\,dt.$$

Now from (7.37) and (7.40) we get

$$(7.42)\qquad \sigma\Omega_\sigma(t_\sigma,x)-\sigma I_\sigma(t_\sigma,x)=\sin\sigma x\,t_\sigma'(x)+\cos\sigma x H(t_\sigma')(x)$$

if we assume that $t_\sigma(x)\in B_{\sigma,1}$.

If $1<p\leqq\infty$ for $t_\sigma\psi_\varepsilon(x)\in B_{\sigma+\varepsilon,1}$ one may write

$$(7.43)\qquad (\sigma+\varepsilon)\Omega_{\sigma+\varepsilon}(t_\sigma\psi_\varepsilon,x)-(\sigma+\varepsilon)I_{\sigma+\varepsilon}(t_\sigma\psi_\varepsilon,x)=$$

$$=\sin[(\sigma+\varepsilon)x](t_\sigma\Psi_\varepsilon)'(x)+\cos[(\sigma+\varepsilon)x]H[(t_\sigma\psi_\varepsilon)'](x).$$

The limit in (7.43) as $\varepsilon_k\to 0$ $(k\to\infty)$ gives

$$(7.44)\qquad \sigma\Omega_\sigma(t_\sigma,x)-\sigma I_\sigma(t_\sigma,x)=\sin\sigma x\,t_\sigma'(x)+\cos\sigma x H(t_\sigma')(x)$$

for $t_\sigma(x)\in B_{\sigma,p}$, where $1\leqq p\leqq\infty$.

Taking the L_p-norm yields

$$\sigma\|\Omega_\sigma(t_\sigma, x)-I_\sigma(t_\sigma, x)\|_{L_p} \leqq \|t'_\sigma(x)\|_{L_p}+\|H(t'_\sigma)(x)\|_{L_p} \tag{7.45}$$

which ends the proof.

8. The main theorem of convergence

THEOREM 3. *If* $f(x)\in L_p\cap L_\infty$ *then*

$$\|f-I_\sigma(f)\|_{L_p} = \begin{cases} O(1)\left(\tau\left(f, \frac{1}{\sigma}\right)_{L_1}+\omega\left(Hf, \frac{1}{\sigma}\right)_{L_1}\right), & p=1 \\ O(1)\tau\left(f, \frac{1}{\sigma}\right)_{L_p}, & 1<p<\infty \\ O(1)\left(\tau\left(f, \frac{1}{\sigma}\right)_{L_\infty}+\omega\left(Hf, \frac{1}{\sigma}\right)_{L'_\infty}\right), & p=\infty \end{cases} \tag{8.1}$$

where $Hf(x)$ *is the Hilbert transform of* $f(x)$.

PROOF OF THEOREM 3. As an intermediate approximation tool we use the Steklov transform

$$f_\sigma(x) = \frac{\sigma}{2}\int_{-1/\sigma}^{1/\sigma} f(x+t)\,dt, \tag{8.2}$$

which has the following properties:

$$\|f-f_\sigma\|_{L_p} = O\left(\omega\left(f, \frac{1}{\sigma}\right)_{L_p}\right),$$

$$\|f'_\sigma\|_{L'_p} = O(\sigma)\omega\left(f, \frac{1}{\sigma}\right)_{L'_p}, \quad \|H(f'_\sigma)\|_{L'_p} = O(\sigma)\omega\left(Hf, \frac{1}{\sigma}\right)_{L'_p}.$$

Let $1\leqq p<\infty$. By Lemmas 1, 2, 4, 5 and the remark above we get

$$\begin{aligned} \|f(x)-I_\sigma(f, x)\|_{L_p} &\leqq \|f(x)-f_\sigma(x)\|_{L_p}+\|f_\sigma(x)-\Omega_\sigma(f_\sigma, x)\|_{L_p}+ \\ &+\|\Omega_\sigma(f_\sigma-Q_\sigma(f_\sigma), x)\|_{L_p}+\|\Omega_\sigma(Q_\sigma(f_\sigma), x)-I_\sigma(Q_\sigma(f_\sigma), x)\|_{L_p}+ \\ &+\|I_\sigma(Q_\sigma(f_\sigma)-f_\sigma, x)\|_{L_p}+\|I_\sigma(f_\sigma-f, x)\|_{L_p} = \end{aligned} \tag{8.3}$$

$$= O(1)\left\{\omega\left(f, \frac{1}{\sigma}\right)_{L_p}+\frac{1}{\sigma}\|H(f'_\sigma)\|_{L'_p}+\frac{1}{\sigma}\|f'_\sigma\|_{L'_p}+\frac{1}{\sigma}\|H[Q'_\sigma(f_\sigma)]\|_{L_p}+\right.$$

$$\left.+\frac{1}{\sigma}\|Q'_\sigma(f_\sigma)\|_{L_p}+\|Q_\sigma(f_\sigma)-f_\sigma\|_{\sigma, p, t_k}+\|f_\sigma-f\|_{\sigma, p, t_k}\right\}.$$

By Lemma 3 it follows that

$$\|Q_\sigma(f_\sigma, x)-f_\sigma(x)\|_{\sigma, p, t_k} = O\left(\frac{1}{\sigma}\right)\|f'_\sigma\|_{L'_p}. \tag{8.4}$$

On the other hand

(8.5)

$$\|f_\sigma - f\|_{\sigma,p,t_k} = \left(\sum_{k=-\infty}^{\infty} \frac{1}{\sigma}|f_\sigma(t_k)-f(t_k)|^p\right)^{1/p} = O(1)\left[\sum_{k=-\infty}^{\infty} \frac{2\pi}{\sigma}\omega^p\left(f, t_k, \frac{1}{\sigma}\right)\right]^{1/p} =$$

$$= O(1)\left(\int_{-\infty}^{\infty} \omega^p\left(f, x, \frac{2\pi+1}{\sigma}\right)dx\right)^{1/p} = O(1)\tau\left(f, \frac{1}{\sigma}\right)_{L_p}$$

$\left(\text{see [6] for } \tau\left(f, \frac{1}{\sigma}\right)_{L_p}\right)$. By Lemma 2 it follows that

(8.6) $$\|Q'_\sigma(f_\sigma, x)\|_{L_p} = O(1)\|f'_\sigma\|_{L'_p}.$$

From Proposition 8.2.2 it follows $(1<p<\infty)$

(8.7) $$\|H[Q'_\sigma(f_\sigma)](x)\|_{L_p} = \|H[Q_\sigma(f'_\sigma)](x)\|_{L_p} = \|Q_\sigma[H(f'_\sigma)](x)\|_{L_p} \leqq \|H(f'_\sigma)\|_{L_p}.$$

If $H(f'_\sigma)\in L_1$ then the above estimate holds for $p=1$ too. In (8.7) we use that

(8.8) $$Q'_\sigma(f_\sigma, x) = Q_\sigma(f'_\sigma, x).$$

Namely, for $1\leqq p<\infty$ we have from Proposition 0.1.10 in [7]

(8.9) $$\left\|\frac{Q_\sigma(f_\sigma, x+h)-Q_\sigma(f_\sigma, x)}{h} - Q_\sigma(f'_\sigma, x)\right\|_{L_p} =$$

$$= \left\|c_\sigma \int_{-\infty}^{\infty} \frac{\int_0^h [f'_\sigma(x+t+v)-f'_\sigma(x+t)]\,dv}{h} F_\sigma^2(t)\,dt\right\|_{L_p} \leqq$$

$$\leqq \frac{1}{h}\int_0^h \|f'_\sigma(x+v)-f'_\sigma(x)\|_{L_p}\,dv \xrightarrow[h\to 0]{} 0,$$

from where

(8.10) $$Q'_\sigma(f_\sigma, x) \overset{\text{a.e.}}{=} Q_\sigma(f'_\sigma, x).$$

Let now $p=\infty$. We have for $f_\sigma\in W^1_\infty$

a) $$\frac{f_\sigma(x+h+t)-f_\sigma(x+t)}{h} F_\sigma^2(t) \xrightarrow[\text{a.e.}]{h\to 0} f'_\sigma(x+t)F_\sigma^2(t),$$

b) $$\left|\frac{f_\sigma(x+h+t)-f_\sigma(x+t)}{h} F_\sigma^2(t)\right| \leqq \|f'_\sigma\|_{L'_\infty}|F_\sigma^2(t)|\in L_1$$

and from Proposition 0.1.3 $(h_k \to 0,\ k \to \infty)$

$$\lim_{k\to\infty} \frac{Q_\sigma(f, x+h_k)-Q_\sigma(f, x)}{h_k} = \lim_{k=\infty} c_\sigma \int_{-\infty}^{\infty} \frac{f(x+h_k+t)-f(x+t)}{h_k} F_\sigma^2(t)\, dt =$$

$$= c_\sigma \int_{-\infty}^{\infty} f'(x+t) F_\sigma^2(t)\, dt.$$

When $p=\infty$ by (7.36) we have

$$H[Q'_\sigma(f_\sigma)](x) = \lim_{\varepsilon\to 0} H[Q'_\sigma(f_\sigma\psi_\varepsilon)] \quad \text{a.e.} \tag{8.11}$$

Since by Definition 4 and by (6.6) we have

$$H[(f_\sigma\psi_\varepsilon)'](x) = (x-i)R_\varepsilon(x)+P_\varepsilon(x)+r_\varepsilon, \quad H(f'_\sigma)(x) = \tag{8.12}$$

$$= (x-i)R(x)+r \quad \text{a.e.},$$

$$\|R_\varepsilon-R\|_{L_2} \to 0, \quad \|P_\varepsilon\|_{L_2} \to 0, \quad r_\varepsilon \to r \quad \text{as} \quad \varepsilon \to 0, \tag{8.13}$$

it follows that

$$\lim_{\varepsilon\to 0} Q_\sigma(H[(f_\sigma\psi_\varepsilon)']) = Q_\sigma[H(f'_\sigma)]. \tag{8.14}$$

From (8.7), (8.11) and (8.14) we get

$$H[Q'_\sigma(f_\sigma)](x) \overset{\text{a.e.}}{=} Q_\sigma[H(f'_\sigma), x]. \tag{8.15}$$

From (8.15) follows (8.7) in the case $p=\infty$.

Applying F. Riesz's inequality yields $(1<p<\infty)$

(8.16)

$$\|Hf(t+h)-Hf(t)\|_{L_p} = O(1)\|f(t+h)-f(t)\|_{L_p} \Rightarrow \omega\left(Hf, \frac{1}{\sigma}\right)_{L_p} = O(1)\,\omega\left(f, \frac{1}{\sigma}\right)_{L_p}$$

which completes the proof of Theorem 3.

9. The saturation problem of $I_\sigma(f, x)$

The paper [3] is the first case where the saturation problem of a discreate linear operator is solved in $L_p[0, 2\pi]$ spaces, $1 \leqq p < \infty$. Analogously to (7.36) one may prove the following interpolation formula for $t_\sigma(x) \in B_{\sigma,p}$, $1 \leqq p \leqq \infty$:

$$\sin\alpha t'_\sigma(x) - \sigma\cos\alpha t_\sigma(x) = \sigma \sum_{k=-\infty}^{\infty} (-1)^{k-1} \frac{\sin^2\alpha}{(\alpha-k\pi)^2}\, t_\sigma\left(\frac{k\pi-\alpha}{\sigma}+x\right). \tag{9.1}$$

On the other hand

$$(9.2)\qquad \sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{\sin^2\sigma x}{(\sigma x-k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}\right)=\sum_{k=-\infty}^{\infty}\frac{\sin^2\sigma x}{(\sigma x-k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}\right)-$$

$$-8\sum_{l=-\infty}^{\infty}\frac{\sin^2\frac{\sigma x}{2}\cos^2\frac{\sigma x}{2}}{(\sigma x-2l\pi)^2}I_\sigma\left(f,\frac{2l\pi}{\sigma}\right)=I_{2\sigma}(I_\sigma(f),x)-2\cos^2\frac{\sigma x}{2}I_\sigma(f,x)$$

from where we get by (9.1) for $\alpha=\sigma x$

$$(9.3)\qquad \frac{\sin\sigma x I'_\sigma(f,x)}{\sigma}=I_{2\sigma}(I_\sigma(f),x)-I_\sigma(f,x)$$

using that

$$I_\sigma\left(f,\frac{2l\pi}{\sigma}\right)=f\left(\frac{2l\pi}{\sigma}\right).$$

By (7.36) for $\alpha=\sigma x$ and $\alpha=0$ we get

$$(9.4)\qquad \sin\sigma x I'_\sigma(f,x)+\cos\sigma x H[I'_\sigma(f)](x)=\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\frac{\sigma x-k\pi}{2}}{(\sigma x-k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}\right),$$

$$(9.5)\qquad H(I'_\sigma(f))(x)=\sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\frac{k\pi}{2}}{(k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}+x\right).$$

We can easily see that

$$(9.6)\qquad \sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\frac{\sigma x-k\pi}{2}}{(\sigma x-k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}\right)=$$

$$=\sigma\sum_{l=-\infty}^{\infty}\frac{2\sin^2\frac{\sigma x-(2l+1)\pi}{2}}{[\sigma x-(2l+1)\pi]^2}I_\sigma\left(f,\frac{(2l+1)\pi}{\sigma}\right)-\frac{\sigma I_\sigma(f,x)}{2}.$$

Analogously

$$(9.7)\qquad \sigma\sum_{k=-\infty}^{\infty}(-1)^{k-1}\frac{2\sin^2\frac{k\pi}{2}}{(k\pi)^2}I_\sigma\left(f,\frac{k\pi}{\sigma}+x\right)=-\sigma\frac{I_\sigma(f,x)}{2}+$$

$$+\sigma\sum_{l=-\infty}^{\infty}\frac{2\sin^2(2l+1)\frac{\pi}{2}}{[(2l+1)\pi]^2}I_\sigma\left(f,x+\frac{(2l+1)\pi}{\sigma}\right).$$

Assuming that $I_\sigma(f,x)\in B_{\sigma,1}$ by the interpolation formula (7.39) for the convolution

$$(9.8)\qquad \left[\frac{\sin^2\frac{\sigma}{2}\left(\circ-\frac{\pi}{\sigma}\right)}{(\sigma(\circ)-\pi)^2}*I_\sigma\left(f,\circ+\frac{\pi}{\sigma}\right)\right](x)$$

we obtain

$$(9.10)\qquad \sum_{k=-\infty}^{\infty} \frac{\sin^2 \frac{\sigma}{2}\left(x-\frac{k\pi}{\sigma}-\frac{\pi}{\sigma}\right)}{\left[\sigma\left(x-\frac{k\pi}{\sigma}\right)-\pi\right]^2} I_\sigma\left(f, \frac{k\pi}{\sigma}+\frac{\pi}{\sigma}\right) =$$

$$= \sum_{k=-\infty}^{\infty} I_\sigma\left(f, x+\frac{k\pi}{\sigma}+\frac{\pi}{\sigma}\right) \frac{\sin^2 \frac{\sigma}{2}\left(-\frac{k\pi}{\sigma}-\frac{\pi}{\sigma}\right)}{(-k\pi-\pi)^2}.$$

Let us denote the two sides of (9.10) by L and R. Then

$$\text{a)}\qquad L = \frac{I_\sigma(f,x)}{4} + \sum_{l=-\infty}^{\infty} I_\sigma\left(f, (2l+1)\frac{\pi}{\sigma}\right) \frac{\sin^2 \frac{\sigma}{2}\left(x-(2l+1)\frac{\pi}{\sigma}\right)}{(\sigma x-(2l+1)\pi)^2},$$

$$\text{b)}\qquad R = \frac{I_\sigma(f,x)}{4} + \sum_{l=-\infty}^{\infty} I_\sigma\left(f, x+(2l+1)\frac{\pi}{\sigma}\right) \frac{\sin^2 (2l+1)\frac{\pi}{2}}{[(2l+1)\pi]^2}$$

and from this and (9.6), (9.7) it follows that the right side of (9.4) is equal to the right side of (9.5). Thus by (9.4) and (9.5) we obtain

$$(9.11)\qquad \sin \sigma x I'_\sigma(f,x) = (1-\cos \sigma x) H[I'_\sigma(f)](x)$$

if $I_\sigma(f,x) \in B_{\sigma,1}$.

If $f(x)$ is bounded, then $I_\sigma(f,x) \in B_{\sigma,\infty}$ and $\psi_\varepsilon I_\sigma(f)(x) \in B_{\sigma+\varepsilon,1}$. Applying the interpolation formula (7.39) for the convolution

$$(9.12)\qquad \left(\frac{\sin^2 \frac{\sigma}{2}\left(\circ-\frac{\pi}{\sigma}\right)}{(\sigma(\circ)-\pi)^2} * \left[I_\sigma\left(f, \circ+\frac{\pi}{\sigma}\right)\psi_\varepsilon\left(\circ+\frac{\pi}{\sigma}\right)\right]\right)(x)$$

we get

$$(9.13)\qquad \sum_{k=-\infty}^{\infty} \frac{\sin^2 \frac{\sigma}{2}\left(x-\frac{k\pi}{\sigma+\varepsilon}-\frac{\pi}{\sigma}\right)}{\left[\sigma\left(x-\frac{k\pi}{\sigma+\varepsilon}\right)-\pi\right]^2} I_\sigma\left(f, \frac{k\pi}{\sigma+\varepsilon}+\frac{\pi}{\sigma}\right)\psi_\varepsilon\left(\frac{k\pi}{\sigma+\varepsilon}+\frac{\pi}{\sigma}\right) =$$

$$= \sum_{k=-\infty}^{\infty} \psi_\varepsilon\left(x+\frac{k\pi}{\sigma}+\frac{\pi}{\sigma}\right) I_\sigma\left(f, x+\frac{k\pi}{\sigma}+\frac{\pi}{\sigma}\right) \frac{\sin^2 \frac{\sigma}{2}\left(-\frac{k\pi}{\sigma}-\frac{\pi}{\sigma}\right)}{(k\pi+\pi)^2}.$$

Taking the limit, when $\varepsilon \to 0$ we obtain (9.10) and from here (9.11).

We solve the saturation problem for $I_\sigma(f)$ in the space

$$(9.14)\qquad X_p = \left\{f(x) \in L_p \cap L_\infty, \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x+t)\,dt \xrightarrow[\varepsilon\to 0]{} f(x)\right\}.$$

THEOREM 4. *If* $f(x)\in X_p$, *then*

$$\|f(x)-I_\sigma(f,x)\|_{L_p}=O\left(\frac{1}{\sigma}\right) \tag{1}$$

if and only if

a) $p=1$, *both* $f(x)$ *and* $Hf(x)$ *are of bounded variation,*
b) $1<p<\infty$, $f(x)\in W_p^1$,
c) $p=\infty$, *both* $f(x)$ *and* $Hf(x)$ *belong to the class* Lip 1.

$$\|f(x)-I_\sigma(f,x)\|_{L_p}=o\left(\frac{1}{\sigma}\right) \tag{2}$$

if and only if $f(x)$ *is a constant.*

PROOF. We need the following facts:

$$I_{2\sigma}(I_{2\sigma}(f),x)=I_{2\sigma}(f,x), \tag{9.15}$$

$$\text{if}\quad f(x)\in W_p^1\quad\text{then}\quad \tau(f,h)_{L_p}=O(h)\|f\|_{L_p'},\quad \omega(f,h)_{L_p}=O(h)\|f'\|_{L_p'}, \tag{9.16}$$

$$\text{if}\quad \gamma_\sigma(x)\in B_{\sigma,p}\quad (1\leqq p\leqq\infty) \tag{9.17}$$

then by the interpolation formula (7.36) we obtain for $\alpha=\frac{\pi}{2}$ and $\alpha=0$ the inequalities

$$\|\gamma_\sigma'\|_{L_p}\leqq c\sigma\|\gamma_\sigma\|_{L_p},\quad \|H[\gamma_\sigma']\|_{L_p}\leqq c\sigma\|\gamma_\sigma\|_{L_p}.$$

A) Let $\|I_\sigma(f)-f\|_{L_p}=O\left(\frac{1}{\sigma}\right)$.

a) $p=1$. We have $I_\sigma(f)\in L_1$ and by Lemma A and Theorem 3.3.5, $I_\sigma(f)\in L_p$, $1\leqq p\leqq\infty$. By Proposition 8.3.7 we may interchange the order of taking the Hilbert transform and differentiation. Using Theorem 3 we get

$$\|I_{2\sigma}(I_\sigma(f),x)-I_\sigma(f,x)\|_{L_1}=\|I_{2\sigma}(I_\sigma(f)-I_{2\sigma}(f),x)-(I_\sigma(f)- \tag{9.18}$$

$$-I_{2\sigma}(f))(x)\|_{L_1}=O(1)\left[\tau\left(I_\sigma(f)-I_{2\sigma}(f),\frac{1}{\sigma}\right)_{L_1}+\omega\left(HI_\sigma-HI_{2\sigma},\frac{1}{\sigma}\right)_{L_1}\right]=$$

$$=O\left(\frac{1}{\sigma}\right)(\|I_\sigma'-I_{2\sigma}'\|_{L_1}+\|H(I_\sigma')-H(I_{2\sigma}')\|_{L_1})=O(1)\|I_\sigma-I_{2\sigma}\|_{L_1}=O\left(\frac{1}{\sigma}\right),\quad \sigma\to\infty.$$

b) $1<p<\infty$. Analogoosly to a) using Theorem 3 we get

(9.19)

$$\|I_{2\sigma}(I_\sigma(f),x)-I_\sigma(f,x)\|_{L_p}=\|I_{2\sigma}(I_\sigma(f)-I_{2\sigma}(f),x)-(I_\sigma(f)-I_{2\sigma}(f))(x)\|_{L_p}=$$

$$=O(1)\tau\left(I_\sigma(f)-I_{2\sigma}(f),\frac{1}{\sigma}\right)_{L_p}=O\left(\frac{1}{\sigma}\right),\quad \sigma\to\infty.$$

c) $p=\infty$. From (9.18) and Definition 5 and Proposition 6.3 we obtain

(9.20)

$$\|I_{2\sigma}(I_\sigma(f),x)-I_\sigma(f,x)\|_{L_\infty}=\|I_{2\sigma}[I_\sigma(f)-I_{2\sigma}(f),x]-(I_\sigma(f)-I_{2\sigma}(f))(x)\|_{L_\infty}=$$

$$=O(1)\left[\tau\left(I_\sigma(f)-I_{2\sigma}(f),\frac{1}{\sigma}\right)_{L_\infty}+\omega\left(HI_\sigma(f)-HJ_{2\sigma}(f),\frac{1}{\sigma}\right)_{L'_\infty}\right]=$$

$$=O(1)\|I_\sigma(f)-I_{2\sigma}(f)\|_{L_\infty}=O\left(\frac{1}{\sigma}\right),$$

since

(9.21)

$$\begin{cases}H(I_{2\sigma}-I_\sigma)(t+h)-H(I_{2\sigma}-I_\sigma)(t)=H\left(\left[\int_0^{0+h}(I_{2\sigma}-I_\sigma)(f)(v)\,dv\right]'\right)(t),\\ \left\|H\left(\left[\int_0^{0+h}(I_{2\sigma}-I_\sigma)(f)(v)\,dv\right]'\right)(t)\right\|_{L_\infty}\leqq\sigma\|(I_{2\sigma}-I_\sigma)(f)(t+h)-(I_{2\sigma}-I_\sigma)(f)(t)\|_{L_\infty}.\end{cases}$$

Now from (9.19)—(9.21) and using (9.3) we get

$$\|\sin\sigma x I'_\sigma(f,x)\|_{L_p}=O(1). \tag{9.22}$$

From (9.22) and by Lemma A we obtain

$$\left(1+\frac{\pi}{5}\right)\|\sin\sigma x I'_\sigma(f,x)\|_{L_p}\geqq\left(\frac{\pi}{5\sigma}\sum_{k=-\infty}^{\infty}\left|\sin\left(\frac{\pi}{2}+\frac{k\pi}{5}\right)I'_\sigma\left(f,\frac{\pi}{2\sigma}+\frac{k\pi}{5\sigma}\right)\right|^p\right)^{1/p}\geqq \tag{9.23}$$

$$\geqq\sin\frac{\pi}{14}\left(\frac{\pi}{5\sigma}\sum_{k=-\infty}^{\infty}\left|I'_\sigma\left(f,\frac{\pi}{2\sigma}+\frac{k\pi}{5\sigma}\right)\right|^p\right)^{1/p}\geqq\sin\frac{\pi}{14}\left(1-\frac{\pi}{5}\right)\|I'_\sigma(f)\|_{L_p}\Rightarrow$$

$$\Rightarrow\|I'_\sigma(f)\|_{L_p}\leqq\text{const}\,\|\sin\sigma x I'_\sigma(f,x)\|_{L_p},$$

from where it follows that

$$\|I'_\sigma(f,x)\|_{L_p}=O(1). \tag{9.24}$$

Analogously from (9.11) and by Lemma A we obtain

$$\|H[I'_\sigma(f)]\|_{L_p}=O(1). \tag{9.25}$$

In this way by (9.24) and (9.25) we obtain

$$\omega(f,h)_{L_p}=\sup_{|t|<h}\|f(x+t)-f(x)\|_{L_p}\leqq\sup_{|t|<h}\|f(x+t)-I_\sigma(f,x+t)\|_{L_p}+ \tag{9.26}$$

$$+\|f(x)-I_\sigma(f,x)\|_{L_p}+\sup_{|t|<h}\|I_\sigma(f,x+t)-I_\sigma(f,x)\|_{L_p}=O\left(\frac{1}{\sigma}+\omega(I_\sigma(f),h)_{L_p}\right)=$$

$$=O\left(\frac{1}{\sigma}+h\|I'_\sigma\|_{L_p}\right)=O\left(\frac{1}{\sigma}+h\right)=O(h)\quad\left(\sigma=\frac{1}{h}\right).$$

Substituting $\sigma=\frac{1}{h}$ and by [1], p. 177 Theorems 4.8, 4.8.1 and 4.8.2 we get

a) $1<p<\infty$, $f(x)\in W_p^1$,
b) $p=1$, $f(x)$ is of bounded variation,
c) $p=\infty$, $f(x)\in \mathrm{Lip}\,1$.

Let $p=1$. Since $f(x)\in L_1\cap L_\infty \Rightarrow f(x)\in L_2$, $\tau(f,\delta)_{L_1}$ is bounded because $f(x)\in BV$. On the other hand $I_\sigma(f)\in B_{\sigma,1}\subset B_{\sigma,2}$ and we obtain

$$(9.27)\quad \int_{-\infty}^{\infty}\chi_{[-\sqrt{\sigma}+t,\sqrt{\sigma}+t]}(u)\,|Hf(u)-H[I_\sigma(f)](u)|\,du \leqq \Big(\int_{-\sqrt{\sigma}+t}^{\sqrt{\sigma}+t}1\,du\Big)^{1/2}\|Hf(u)-$$

$$-H[I_\sigma(f)](u)\|_{L_2}=\sqrt{2}\,\sigma^{1/4}\|f-I_\sigma(f)\|_{L_2}\leqq\sqrt{2}\,\sigma^{1/4}\tau\Big(f,\frac{1}{\sigma}\Big)_{L_2}\leqq$$

$$=\sqrt{2}\sup|f|\sigma^{1/4}\sqrt{\tau\Big(f,\frac{1}{\sigma}\Big)_{L_1}}=c(f)\sigma^{-1/4}$$

for every real t, from where

$$\|H(f)-H[I_\sigma(f)]\|_{L_1[-\sqrt{\sigma}+t,\sqrt{\sigma}+t]}=O(1)\frac{1}{\sigma^{1/4}}.$$

Thus we obtain

$$(9.28)\quad \omega(Hf,h)_{L_1\left[\frac{-\sqrt{\sigma}}{2},\frac{\sqrt{\sigma}}{2}\right]}=\sup_{|t|<h}\|Hf(x+t)-Hf(x)\|_{L_1\left[\frac{-\sqrt{\sigma}}{2},\frac{\sqrt{\sigma}}{2}\right]}\leqq$$

$$\leqq\sup_{|t|<h}\|Hf(x+t)-H[I_\sigma(f)](x+t)\|_{L_1\left[\frac{-\sqrt{\sigma}}{2},\frac{\sqrt{\sigma}}{2}\right]}+\|Hf(x)-H[I_\sigma(f)](x)\|_{L_1\left[\frac{-\sqrt{\sigma}}{2},\frac{\sqrt{\sigma}}{2}\right]}+$$

$$+\sup_{|t|<h}\|H[I_\sigma(f)](x+t)-H[I_\sigma(f)](x)\|_{L_1}=O(1)\Big(\frac{1}{\sigma^{1/4}}+\big\|(H[I_\sigma(f)])'\big\|_{L_1}h\Big)=$$

$$=O(1)(\sigma^{-1/4}+\|H([I_\sigma(f)]')\|_{L_1}h)=O(1)(\sigma^{-1/4}+h),$$

and if $\sigma\to\infty$ we get

$$(9.29)\quad \omega(Hf,h)_{L_1}=O(h)$$

and from Theorem 4.8.2 of [7], Hf is of bounded variation. Let $p=\infty$. We have $\|f(x)-I_\sigma(f)(x)\|_{L_\infty}=O(\sigma^{-1})$, $\sigma\to\infty$. On the other hand by $H[I_\sigma(f)](x+h)-$ $-H[I_\sigma(f)](x)=H\Big[\Big(\int_0^{0+h}I_\sigma(v)\,dv\Big)'\Big](x)$ and the interpolation formula (7.36) it follows that $H[I_\sigma(f)](x+h)-H[I_\sigma(f)](x)$ is a continuous function. By the Definitions 4 and 5 we have

$$(9.30)\quad HI_\sigma(f)(x+h)-HI_\sigma(f)(x)\overset{\text{a.e.}}{=}\lim_{\varepsilon_k\to0}\left(H[I_\sigma(f)\psi_{\varepsilon_k}](x+h)-H[I_\sigma(f)\psi_{\varepsilon_k}](x)\right)=$$

$$=\lim_{\varepsilon_k\to0}\int_x^{x+h}(H[I_\sigma(f)\psi_{\varepsilon_k}])'(v)\,dv=\lim_{\varepsilon_k\to0}\int_x^{x+h}H[(I_\sigma(f)\psi_{\varepsilon_k})'](v)\,dv=$$

$$=\int_x^{x+h}H[I_\sigma'(f)](v)\,dv=O(h)\quad\text{(see (9.25))},$$

because of $\|H([I_\sigma(f)\psi_{\varepsilon_k}]')\|_{L_\infty} \leqq c(\sigma+1)\|f\|_{L_\infty}$ and Lebegue's theorem. We have

$$|Hf(x+h)-Hf(x)| \leqq \tag{9.31}$$

$$\leqq |Hf(x+h)-Hf(x)-HI_\sigma(f)(x+h)+HI_\sigma(f)(x)|+O(h),$$

and taking the limit when $\sigma\to\infty$ in (9.31) we obtain

$$|Hf(x+h)-Hf(x)| = O(h) \tag{9.32}$$

for a.e. x (there exists $\varphi(x)$: $\|f(x) \overset{\text{a.e.}}{=} \varphi(x)$, $\varphi(x)\in \text{Lip } 1$).

B) Conversely, if a), b) and c) of Theorem 4 hold then by Theorem 3 it follows

$$\|f(x)-I_\sigma(f)(x)\|_{L_p} = O\left(\frac{1}{\sigma}\right), \quad \sigma\to\infty. \tag{9.33}$$

The proof of (2) from Theorem 4 is analogous to (1) only instead of O we write o and using Theorem 2.2.19 from [8].

NOTE. Let f be a 2π-periodic, bounded function. By [9], p. 144 we get

$$I_n(f,x) = \frac{1}{n^2}\sum_{k=0}^{n-1} f(q_k)\frac{\sin^2\frac{n}{2}(x-q_k)}{\sin^2\frac{x-q_k}{2}}, \quad q_k = \frac{2k\pi}{n}, \tag{9.34}$$

$$k = 0, 1, \ldots, n-1; \quad n = 1, 2, \ldots.$$

By [7], p. 334, [9], p. 173 and [13] we have

$$HAf(x) = (x-i)F^{-1}\left[i\,\text{sign}\,(*)F\left(\frac{f(\circ)}{(\circ)-i}\right)(*)\right](x) = \frac{x-i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\frac{dt}{x-t}, \tag{9.35}$$

$$H_{cl.}f(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\cot\frac{u}{2}\,du. \tag{9.36}$$

By Definitions 4 and 5 we get

$$Hf(x+h)-Hf(x) = HA\left(\left[\int_0^{0+h} f(v)\,dv\right]'\right)(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t+h)-f(t)}{t-i}\,dt. \tag{9.37}$$

On the other hand if $\int_0^{2\pi} f(v)dv=0$ we obtain by [7], p. 335,

$$H_{cl.}f(x) = -\frac{1}{\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x-u)\left(\frac{1}{u-2k\pi}+\frac{1}{2k\pi}\right)du-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x-u)}{u}\,du= \tag{9.38}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-u)}{u}\,du$$

and

$$HAf(x)-H_{cl.}f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-i}{t-i}-1\right)\frac{1}{x-t}f(t)\,dt = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-i}\,dt. \tag{9.39}$$

From (9.34), (9.38) and (9.39) we obtain

(9.40)

$$H_{c1.}f(x+h)-H_{c1.}f(x)=H_{c1.}\left[\left(\int_0^{0+h} f(v)\,dv\right]'\right)(x)=HA\left[\left(\int_0^{0+h} f(v)\,dv\right)'\right](x)+$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t+h)-f(t)}{t-i}\,dt=Hf(x+h)-Hf(x).$$

By (9.40) one may obtain the saturation result of [2].

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A FIFTH-ORDER ITERATIVE METHOD FOR SOLVING EQUATIONS

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In this paper a fifth-order method for finding simple zeros of nonlinear algebraic and transcendental equations is described, which requires no explicit evaluation of higher derivatives and costs only two function and two derivative evaluations per iteration. Comparison of computational efficiency is made with Newton's method. This method can be generalized for solving nonlinear operator equations. Its convergence is proved under several assumptions.

I. A fifth-order iterative method for solving nonlinear algebraic and transcendental equations

1. A fifth-order method. In order to obtain solution of nonlinear algebraic and transcendental equations

$$f(x) = 0 \tag{1}$$

we consider

$$\varphi(x) = x - u(x) - \frac{f(x-u(x))}{f'\left(x-u(x)-\dfrac{f(x-u(x))}{2f'(x)}\right)} \tag{2}$$

where $u(x)=f(x)/f'(x)$.

An effective way of deducing the properties of (2) is to expand $\varphi(x)$ as a power series in u and use J. F. Traub's comparison series method [1].

Developing $\varphi(x)$ in a power series in u, we have

$$\varphi(x) = x-u-A_2u^2-(2A_2^2-A_3)u^3-(A_4-5A_2A_3+5A_2^3)u^4-$$

$$-(-A_5+6A_2A_4+3A_3^2-21A_2^2A_3+12A_2^4)u^5+O(u^6),$$

where $A_r(x)=f^{(r)}(x)/r!f'(x)$.

Using

$$E_6(x) = x-u-A_2u^2-(2A_2^2-A_3)u^3-(A_4-5A_2A_3+5A_2^3)u^4-$$

$$-(-A_5+6A_2A_4+3A_3^2-21A_2^2A_3+14A_2^4)u^5,$$

we derive the relation

$$\varphi(x)-E_6(x) = 2A_2^4u^5+O(u^6). \tag{3}$$

Thus $\varphi(x)$ is fifth-order and we have a fifth-order iterative method:

$$x_{n+1} = x_n - u(x_n) - \frac{f(x_n - u(x_n))}{f'\left(x_n - u(x_n) - \dfrac{f(x_n - u(x_n))}{2f'(x_n)}\right)}; \quad n = 0, 1, 2, \ldots \tag{4}$$

which requires two function and two derivative evaluations per iteration.

The asymptotic error constant C is

$$C = 2A_2^4(\alpha) \tag{5}$$

where α is the simple zero of (1).

2. Computational efficiency. In order to obtain an assessment of the computational efficiency of method (4), we shall make use of Traub's efficiency index [1]

$$E = \ln p/a$$

where p is the order of the process and a is the cost per iterative step of computing the function and derivative values needed to form the iterative formula.

If we let the cost of evaluating f be 1 and that of computing a derivative w, then for our method (4) we find

$$E = \frac{\ln 5}{2+2w}.$$

For Newton's method we have $E = \ln 2/(1+w) = \ln 4/(2+2w)$, and hence our method (4) will always be superior.

II. A fifth-order iterative method for solving nonlinear operator equations

1. The iterative method (4) can be extended for solving nonlinear operator equations of the form

$$P(x) = 0 \tag{6}$$

where P is a twice continuously Fréchet differentiable operator which maps the Banach space X into a Banach space Y. Then the iteration becomes

$$x_{n+1} = y_{n+1} - U_n^{-1} P(y_{n+1}) \tag{7}$$

where

$$y_{n+1} = x_n - \Gamma_n P(x_n), \quad U_n = P'\left(y_{n+1} - \frac{1}{2}\Gamma_n P(y_{n+1})\right), \quad \Gamma_n = [P'(x_n)]^{-1}.$$

We now prove a convergence theorem for (7).

THEOREM 1. *Let us assume that the following conditions are satisfied:*
1) *For the initial point* x_0 *the operator* $\Gamma_0 = [P'(x_0)]^{-1}$ *exists;*
2) $\|\Gamma_0 P(x_0)\| \leqq \eta_0$;

3) $\|\Gamma_0 P''(x)\| \leqq M_0$, $\|\Gamma_0 P''(x') - \Gamma_0 P''(x'')\| \leqq N_0 \|x' - x''\|^\alpha$, $0 < \alpha \leqq 1$ *for every* x', x'' *of the sphere*

$$S_0 = \left\{x: \|x - x_0\| \leqq \frac{4 - 2h_0 - h_0^2}{4 - 4h_0 - h_0^2} \cdot \frac{\eta_0}{1 - c_0 h_0^{3+\alpha}}\right\}, \quad h_0 = M_0 \eta_0;$$

4) $$h_0\left(1 + \frac{h_0}{4}\right) < 1, \quad g_0 = \frac{4 - 2h_0 - h_0^2}{4 - 4h_0 - h_0^2} h_0 = \left(1 + \frac{h_0}{2\left(1 - h_0\left(1 + \frac{h_0}{4}\right)\right)}\right) h_0 < 1;$$

5) $k_0^{3+\alpha} = \dfrac{c_0}{1 - g_0} h_0^{3+\alpha} < 1$, *where*

$$c_0 = \frac{1}{1 - g_0}\left\{\frac{\left(1 + \frac{h_0}{4}\right) h_0^{1-\alpha}}{8\left[1 - h_0\left(1 + \frac{h_0}{4}\right)\right]^2} + \frac{N_0 h_0^\alpha}{2^{3+2\alpha}(1+\alpha) M_0^{1+\alpha}\left[1 - h_0\left(1 + \frac{h_0}{4}\right)\right]} + \right.$$

$$\left. + \frac{N_0 h_0^\alpha}{2^{2+\alpha}(2+\alpha)(1+\alpha) M_0^{1+\alpha}\left[1 - h_0\left(1 + \frac{h_0}{4}\right)\right]^{2+\alpha}}\right\}.$$

Then equation (6) *has a solution* x^* *in* S_0, *the sequence* $\{x_n\}$ *defined by* (7) *converges to* x^*, *and the rate of convergence is given by the following inequality:*

$$\|x_n - x^*\| \leqq \frac{4 - 2h_0 - h_0^2}{4 - 4h_0 - h_0^2} (1 - g_0)^n \frac{k_0^{(4+\alpha)^n - 1} \eta_0}{1 - c_0 h_0^{3+\alpha}}. \tag{8}$$

PROOF. This theorem is proved by mathematical induction. We first show that conditions like 1) to 5) are fulfilled at x_1, and by induction at all x_n.

From the conditions of the theorem and using Taylor—Graves' formula [2], we obtain

$$\begin{cases} \|y_1 - x_0\| \leqq \eta_0, \\ \|x_1 - y_1\| = \|U_0^{-1} P(y_1)\| \leqq \dfrac{h_0 \eta_0}{2\left[1 - h_0\left(1 + \dfrac{h_0}{4}\right)\right]}, \\ \|x_1 - x_0\| \leqq \|x_1 - y_1\| + \|y_1 - x_0\| \leqq \dfrac{4 - 2h_0 - h_0^2}{4 - 4h_0 - h_0^2} \eta_0. \end{cases} \tag{9}$$

Condition 1. We have

$$\left\|\Gamma_0\left(P'(x_0) - P'(x_1)\right)\right\| \leqq \frac{4 - 2h_0 - h_0^2}{4 - 4h_0 - h_0^2} h_0 = g_0 < 1.$$

By Banach's lemma, the operator

$$K_0 = [I - \Gamma_0(P'(x_0) - P'(x_1))]$$

has an inverse, and

$$\|K_0^{-1}\| \leqq \frac{1}{1-g_0}. \tag{10}$$

Thus the operator $\Gamma_1 = K_0^{-1}\Gamma_0$ exists and 1) holds at x_1.

Condition 2. Using Taylor—Graves' formula,

$$P(x_1) = P(y_1) + P'(y_1)(x_1 - y_1) + \frac{1}{2}P''(y_1)(x_1 - y_1)^2 + \tag{11}$$

$$+ \int_0^1 [P''(y_1 + \tau(x_1 - y_1)) - P''(y_1)](1-\tau)\, d\tau (x_1 - y_1)^2.$$

Taking norms and using the conditions of the theorem and (7) we get

$$\|\Gamma_0 P(x_1)\| \leqq \frac{\left(1+\frac{h_0}{4}\right)}{8\left[1 - h_0\left(1+\frac{h_0}{4}\right)\right]^2} h_0^4 \eta_0 + \frac{N_0 h_0^{3+2\alpha}\eta_0}{2^{3+2\alpha}(1+\alpha)M_0^{1+\alpha}\left[1 - h_0\left(1+\frac{h_0}{4}\right)\right]} + \tag{12}$$

$$+ \frac{N_0 h_0^{3+2\alpha}\eta_0}{2^{2+\alpha}(2+\alpha)(1+\alpha)\, M_0^{1+\alpha}\left[1 - h_0\left(1+\frac{h_0}{4}\right)\right]^{2+\alpha}}.$$

Now from (10) and (12) we get

$$\|\Gamma_1 P(x_1)\| = \|K_0^{-1}\Gamma_0 P(x_1)\| \leqq c_0 h_0^{3+\alpha}\eta_0 = \eta_1 \tag{13}$$

i.e. condition 2) at point x_1 is satisfied.

Conditions 3), 4) and 5). Since, by 4) and 5), we have $c_0 h_0^{3+\alpha} < 1$, it follows that $\eta_1 < \eta_0$. For any $x \in S_0$, we get:

$$\|\Gamma_1 P''(x)\| = \|K_0^{-1}\Gamma_0 P''(x)\| \leqq \frac{1}{1-g_0} M_0 = M_1,$$

$$\|\Gamma_1 P''(x') - \Gamma_1 P''(x'')\| = \|K_0^{-1}\Gamma_0(P''(x') - P''(x''))\| \leqq$$

$$\leqq \frac{1}{1-g_0} N_0 \|x' - x''\|^\alpha = N_1 \|x' - x''\|^\alpha.$$

Next, from the inequality

$$h_1 = M_1\eta_1 = \frac{M_0}{1-g_0} c_0 h_0^{3+\alpha}\eta_0 = \frac{c_0 h_0^{3+\alpha}}{1-g_0} h_0 < h_0,$$

we deduce

$$h_1\left(1+\frac{h_1}{4}\right) < h_0\left(1+\frac{h_0}{4}\right) < 1,$$

$$g_1 = \left(1+\frac{h_1}{2\left[1-h_1\left(1+\frac{h_1}{4}\right)\right]}\right) h_1 < \left(1+\frac{h_0}{2\left[1-h_0\left(1+\frac{h_0}{4}\right)\right]}\right) h_0 = g_0 < 1,$$

$$c_1 = \frac{1}{1-g_1}\left\{\frac{\left(1+\frac{h_1}{4}\right)h_1^{1-\alpha}}{8\left[1-h_1\left(1+\frac{h_1}{4}\right)\right]^2} + \frac{N_1 h_1^{\alpha}}{2^{3+2\alpha}(1+\alpha)M_1^{1+\alpha}\left[1-h_1\left(1+\frac{h_1}{4}\right)\right]} + \right.$$

$$\left. + \frac{N_1 h_1^{\alpha}}{2^{2+\alpha}(2+\alpha)(1+\alpha)M_1^{1+\alpha}\left[1-h_1\left(1+\frac{h_1}{4}\right)\right]^{2+\alpha}}\right\} < c_0, \quad k_1^{3+\alpha} < k_0^{3+\alpha} < 1.$$

We consider the ball

$$S_1 = \left\{x: \|x-x_1\| \leqq \frac{4-2h_1-h_1^2}{4-4h_1-h_1^2}\cdot\frac{\eta_1}{1-c_1 h_1^{3+\alpha}}\right\}$$

and note that the following inequalities hold for any $x \in S_1$:

$$\|x-x_0\| \leqq \|x-x_1\| + \|x_1-x_0\| \leqq$$

$$\leqq \frac{4-2h_1-h_1^2}{4-4h_1-h_1^2}\cdot\frac{\eta_1}{1-c_1 h_1^{3+\alpha}} + \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}\eta_0 \leqq$$

$$\leqq \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}\left(1+\frac{c_0 h_0^{3+\alpha}}{1-c_0 h_0^{3+\alpha}}\right)\eta_0 \leqq \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}\cdot\frac{\eta_0}{1-c_0 h_0^{3+\alpha}}.$$

Thus all conditions 1)—5) at x_1 are satisfied. Now it can be shown by induction on n that the sequence $\{x_n\}$ is welldefined and that conditions 1)—5) at x_n are verified for all n. We obtain also the inequalities

$$h_n \leqq \frac{c_{n-1}h_{n-1}^{3+\alpha}}{1-g_{n-1}}h_{n-1} \leqq \frac{c_{n-1}h_{n+1}^{3+\alpha}}{1-g_0}h_{n-1}, \quad \eta_n \leqq (1-g_0)^n k_0^{(4+\alpha)^n-1}\eta_0,$$

$$\|x_{n+1}-x_n\| \leqq \frac{4-2h_n-h_n^2}{4-4h_n-h_n^2}\eta_n \leqq \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}(1-g_0)^n k_0^{(4+\alpha)^n-1}\eta_0,$$

from which we deduce

$$\|x_{n+p}-x_n\| \leqq \sum_{i=n}^{n+p-1}\|x_{i+1}-x_i\| \leqq$$

$$\leqq \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}(1-g_0)^n k_0^{(4+\alpha)^n-1}\eta_0\frac{1-(1-g_0)^p k_0^{(3+\alpha)p}}{1-c_0 h_0^{3+\alpha}}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in S_0; hence, $\lim_{n\to\infty} x_n = x^*$ exists, $x^* \in S_0$ and the error bound (8) holds.

The proof now follows precisely as in [3].

Thus the theorem is completely proved.

REMARK. If the conditions of Theorem 1 are satisfied for $\alpha = 1$, then the rate of convergence is given by the inequality

$$\|x_n - x^*\| \leqq \frac{4-2h_0-h_0^2}{4-4h_0-h_0^2}(1-g_0)^n \frac{k_0^{5^n-1}\eta_0}{1-c_0h_0^4}.$$

2. In order to avoid the explicit computation of derivatives we consider now the iterative method defined by the formula

$$x_{n+1} = y_{n+1} - [P(y_{n+1}, y_{n+1} - T_n P(y_{n+1}))]^{-1} P(y_{n+1}) \tag{14}$$

where

$$y_{n+1} = x_n - T_n P(x_n), \quad T_n = [P(x_n, x_n - cP(x_n))]^{-1},$$

$P(x, y)$ is the first divided difference of P [4], c a real parameter. Conditions of the convergence of the process (14) which are also sufficient conditions for the existence of a solution of equation (6) are given by the following.

THEOREM 2. *Let us assume that the following conditions are fulfilled:*

1) *For the initial point* x_0 *the operator*

$$T_0 = [P(x_0, x_0 - cP(x_0))]^{-1}$$

exists and $\|T_0\| \leqq B_0$;

2) $\|P(x_0\| \leqq \eta_0$;

3) $\|P(x, y)\| \leqq N, \quad \|P(x, y, z)_1\| \leqq M, \quad \|P(x, y, z)_2\| \leqq M,$

$$\|P(x, y, z)_1 - P(x, y, \tilde{z})_1\| \leqq K\|z - \tilde{z}\|$$

in $\bar{G}_0$ *defined by:* $\|x - x_0\| \leqq r_0 + |c|\,\eta_0$, *where*

$$r_0 = \frac{k_0 B_0 \eta_0}{1 - c_0 l_0^2 h_0^4},$$

and $P(x, y, z)_1$, $P(x, y, z)_2$ *are the second divided differences of* P [4];

$$4)\ h_0 = B_0^2 M \eta_0, \quad h_0(1 + g_0 - g_0 h_0) < 1, \quad k_0 h_0 (2 + |c| N) < 1, \quad g_0 = 1 + \frac{|c|}{B_0},$$

$$k_0 = 1 + \frac{g_0 h_0}{1 - h_0 - g_0 h_0 (q + h_0)};$$

$$5)\ s_0^2 = \frac{l_0^2 h_0^4}{[1 - k_0 h_0 (2 + |c| N)]^2} = c_0^2 l_0^2 h_0^4 < 1, \quad c_0 = \frac{1}{1 - k_0 h_0 (2 + |c| N)},$$

$$l_0^2 = \frac{g_0^2}{1 - h_0(1 + g_0 + g_0 h_0)} \left\{ \frac{1 + g_0 + g_0 h_0}{1 - h_0(1 + g_0 + g_0 h_0)} + \frac{K}{B_0 M^2} \left(1 + \frac{1}{1 - h_0(1 + g_0 + g_0 h_0)} \right) \cdot \right.$$

$$\left. \cdot \left(g_0 + \frac{g_0 h_0}{1 - h_0(1 + g_0 + g_0 h_0)} \right) \right\}.$$

Then equation (6) *possesses a solution* x^* *which belongs to the sphere* $G_0=\{x\colon \|x-x_0\|\leqq r_0\}$, *and the sequence of approximate solutions* x_n *defined by* (14) *converges to* x^*. *The error estimate is given by the formula*

$$\|x_n-x^*\| \leqq k_0\left(\frac{1}{c_0}\right)^n (s_0)^{(5^n-1)/4}\frac{B_0\eta_0}{1-c_0l_0^2h_0^4}. \tag{15}$$

PROOF. From the conditions of Theorem 2 and using the Newton interpolating formula [4] we obtain

(16)

$$\|y_1-x_0\| \leqq B_0\eta_0,\ \|x_1-y_1\| = \left\|[P(y_1, y_1-T_0P(y_1))]^{-1}P(y_1)\right\| \leqq \frac{g_0h_0B_0\eta_0}{1-h_0(1+g_0+g_0h_0)},$$

$$\|x_1-x_0\| \leqq \|x_1-y_1\|+\|y_1-x_0\| \leqq k_0B_0\eta_0.$$

We now show that conditions like 1)—5) are fulfilled at x_1.

Condition 1. We have

$$\left\|T_0(P(x_0, x_0-cP(x_0))-P(x_1, x_1-cP(x_1)))\right\| =$$

$$= \left\|-T_0\left[P(x_1, x_0, x_1-cP(x_1))_2(x_1-x_0)+P(x_0, x_1-cP(x_1), x_0-cP(x_0))_1\cdot\right.\right.$$

$$\left.\left.\cdot(x_1-x_0-c(P(x_1)-P(x_0)))\right]\right\| \leqq k_0h_0(2+|c|N) < 1.$$

By Banach's lemma, the operator

$$K_0 = I-T_0\left[P(x_0, x_0-cP(x_0))-P(x_1, x_1-cP(x_1))\right]$$

has an inverse, and

$$\|K_0^{-1}\| \leqq \frac{1}{1-k_0h_0(2+|c|N)}.$$

Also, $T_1=K_0^{-1}T_0$. Therefore

$$\|T_1\| \leqq \frac{B_0}{1-k_0h_0(2+|c|N)} = B_1 \tag{17}$$

and 1) holds at x_1.

Condition 2. Using the Newton interpolating formula, we have

$$P(x_1) = P(y_1)+P(y_1, x_0-cP(x_0))(x_1-y_1)+P(y_1, x_0-cP(x_0), x_1)_1\cdot \tag{18}$$

$$\cdot(x_1-x_0+cP(x_0))(x_1-y_1).$$

Taking norms and using the conditions of the theorem and (14) we get

$$\|P(x_1)\| \leqq \frac{g_0^2}{1-h_0(1+g_0+g_0h_0)}\left\{\frac{1+g_0+g_0h_0}{1-h_0(1+g_0+g_0h_0)}+\right. \tag{19}$$

$$\left.+\frac{K}{B_0M^2}\left(1+\frac{1}{1-h_0(1+g_0+g_0h_0)}\right)\left(g_0+\frac{g_0h_0}{1-h_0(1+g_0+g_0h_0)}\right)\right\}h_0^4\eta_0 = l_0^2h_0^4\eta_0 = \eta_1$$

i.e. condition 2) at x_1 is satisfied.

Conditions 3)—5) are proved precisely as in Theorem 1.

Thus all conditions 1)—5) at x_1 are satisfied. Now it can be shown by induction on n that the sequence $\{x_n\}$ is welldefined and that conditions 1)—5) at x_n are verified for all n. We obtain also the inequalities

$$l_n^2 \leqq l_{n-1}^5, \quad s_n^2 \leqq s_{n-1}^2, \quad B_n = \frac{B_{n-1}}{1-k_{n-1}h_{n-1}(2+|c|N)},$$

$$h_n \leqq s_{n-1}^2 h_{n-1}, \quad \eta_n \leqq \left(\frac{1}{c_0}\right)^{2n} (s_0^2)^{(5^n-1)/4}\eta_0,$$

from which we deduce

$$\|x_{n+p}-x_n\| \leqq k_0\left(\frac{1}{c_0}\right)^n (s_0^2)^{(5^n-1)/4} \frac{B_0\eta_0}{1-c_0 l_0^2 h_0^4}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in G_0; hence, $\lim_{n\to\infty} x_n = x^*$ exists, $x^* \in G_0$ and the error bound (15) holds.

Since the condition

$$\|P(x_n)\| \leqq \eta_n \leqq \left(\frac{1}{c_0}\right)^{2n} (s_0^2)^{(5^n-1)/4}\eta_0$$

holds, it follows that

$$\lim_{n\to\infty} \|P(x_n)\| = 0.$$

Since the continuity of P and $x_n \to x^*$ we have

$$\lim_{n\to\infty} P(x_n) = P(x^*) = 0$$

and x^* is a solution of (6).

Thus, the theorem is completely proved.

REMARK 1. One special case of (14) is that for $c=0$. We have

$$x_{n+1} = y_{n+1} - [P(y_{n+1}, y_{n+1}-T_nP(y_{n+1}))]^{-1}P(y_{n+1}) \tag{20}$$

where

$$y_{n+1} = x_n - T_nP(x_n), \quad T_n = [P'(x_n)]^{-1}.$$

REMARK 2. If P is a real-valued function of one variable, then (14) becomes

$$x_{n+1} = y_{n+1} - \frac{y_{n+1}-v_{n+1}}{f(y_{n+1})-f(v_{n+1})} f(y_{n+1}) \tag{21}$$

where

$$y_{n+1} = x_n - \frac{cf^2(x_n)}{f(x_n)-f(x_n-cf(x_n))},$$

$$v_{n+1} = y_{n+1} - \frac{cf(x_n)f(y_{n+1})}{f(x_n)-f(x_n-cf(x_n))}.$$

The iterative method (21) is of interest, since it exhibits the same rate of convergence as Jarratt's method [5], Richard F. King's method [6] while not requiring any derivatives of f.

If $c=0$, from (20), we obtain the following method in one dimension:

$$x_{n+1} = y_{n+1} - \frac{y_{n+1}-v_{n+1}}{f(y_{n+1})-f(v_{n+1})} f(y_{n+1}) \tag{22}$$

where

$$y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad v_{n+1} = y_{n+1} - \frac{f(y_{n+1})}{f'(x_n)}.$$

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BIHARMONIC FUNCTIONS WITH PRESCRIBED FINE NORMAL DERIVATIVE ON THE MARTIN BOUNDARY

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1. Introduction. The Dirichlet problem for the bi-Laplacian is that of finding a real-valued function u on an open subset R of the n-dimensional Euclidean space satisfying the equation $\Delta^2 u=0$, where Δ is the Laplacian, with prescribed values on the boundary of R along with prescribed normal derivative on the boundary. We will consider the following modification of this problem. We will assume that R is bounded and connected with Green function G on $R\times R$ (see [4] for terminology and definitions). We will denote the Martin boundary of R by R_M. Limits of functions on $R\cup R_M$ in the fine topology will be denoted by f-lim (see [5] for the definition of the fine topology on $R\cup R_M$). If $\xi\in R$, $\mu(\xi, \cdot)$ will denote the harmonic measure on R_M associated with the point ξ. Such measures are supported by the set of minimal boundary points in R_M. We will take a fixed $\xi_0\in R$ as a reference point. The class of subsets of R_M of $\mu(\xi, \cdot)$ measure zero is independent of ξ so that the phrase "a.e. on R_M" does not depend upon ξ and means "except for a subset of R_M of $\mu(\xi, \cdot)$ measure zero." Likewise the class of real-valued functions on R_M which are measurable and p-th power integrable relative to $\mu(\xi, \cdot)$ is independent of ξ and will be denoted by $L_p(R_M, \mu(\xi_0, \cdot))$. Potentials of positive measures on R are positive superharmonic functions on R which Doob [2] has shown to have fine limits zero a.e. on R_M. If u is a function on $R\cup R_M$, $\eta\in R_M$, and

$$\text{f-lim}_{\xi\to\eta}\frac{u(\xi)-u(\eta)}{G(\xi_0, \xi)}$$

is defined, then u is said to have this fine limit, denoted by $\frac{du}{dG}(\eta)$, as its "fine normal derivative at η." If u is a strictly positive superharmonic function R and η is a minimal point of R_M, then Naïm [5] has shown that $u/G(\xi_0, \cdot)$ has a strictly positive, possibly infinite, fine limit at η; if u is defined to be zero at any such point, then u has a fine normal derivative there. In this context, the Dirichlet problem for the bi-Laplacian to be considered here is that of finding a function u on R with continuous fourth partial derivatives such that (i) $\Delta^2 u=0$ on R, (ii) $\text{f-lim}_{\xi\to\eta} u(\xi)=0$ a.e. on R_M, and (iii) $du/dG=f$ a.e. on R_M where f is in a rather general class of functions on R_M. In a more general treatment of the Dirichlet problem, the zero boundary condition in (ii) would be replaced by a function but our methods do not apply to this more general problem.

2. The Dirichlet problem. Since $\mu(\xi, \cdot)\ll\mu(\xi_0, \cdot)$, $\xi\in R$, there is a nonnegative measurable function $\chi_\xi(\cdot)$ on R such that $\mu(\xi, A)=\int_A \chi_\xi(\eta)\mu(\xi_0, d\eta)$ for any

Borel set $A \subset R_M$. The function $\chi_\xi(\eta)$ is jointly measurable on $R \times R_M$. Letting $\lambda(d\xi)$ denote Lebesgue measure on R, we define a measure ν_0 on R_M by putting

$$\nu_0(A) = \int_A \int_R \chi_\xi(\eta)\lambda(d\xi)\mu(\xi_0, d\eta) = \int_R \mu(\xi, A)\lambda(d\xi).$$

Since $\mu(\xi, R_M)=1$ and R is bounded, $\nu_0(R_M)=\lambda(R)<+\infty$. We will let $L_2(R_M, \nu_0)$ and $L_2(R, \lambda)$ denote the classes of square-integrable functions on R_M and R with inner products $(\cdot, \cdot)_M$ and $(\cdot, \cdot)$ and norms $\|\cdot\|_M$ and $\|\cdot\|$, respectively. Note that $L_2(R_M, \nu_0) \subset L_2(R_M, \mu(\xi_0, \cdot))$ for if $f \in L_2(R_M, \nu_0)$, then $|f|^2 \in L_1(R_M, \nu_0)$ and it follows from Fubini's Theorem that

$$\int_R \int_{R_M} |f(\eta)|^2 \mu(\xi, d\eta)\lambda(d\xi) = \int_{R_M} |f(\eta)|^2 \nu_0(d\eta) < +\infty;$$

thus, $\int_{R_M} |f(\eta)|^2 \mu(\xi, d\eta) < +\infty$ a.e. (λ) and since this integral as a function of ξ is harmonic on R, it is finite for all $\xi \in R$. It can also be shown that $L_p(R_M, \nu_0) \subset L_p(R_M, \mu(\xi_0, \cdot))$, $p \geqq 1$. For $f \in L_2(R_M, \nu_0) \subset L_2(R_M, \mu(\xi_0, \cdot)) \subset L_1(R_M, \mu(\xi_0, \cdot))$, there is associated a harmonic function $H_f \in L_2(R, \lambda)$ with $\|H_f\| \leqq \|f\|_M$ and the map $f \to H_f$ from $L_2(R_M, \nu_0)$ into $L_2(R, \lambda)$ is linear.

We define a nonnegative kernel on $R_M \times R_M$ by putting

$$K(\eta, \eta') = \frac{\int_R \chi_\xi(\eta)\chi_\xi(\eta')\lambda(d\xi)}{\left(\int_R \chi_\varrho(\eta)\lambda(d\varrho)\right)\left(\int_R \chi_\varrho(\eta')\lambda(d\varrho)\right)},$$

zero if either factor in the denominator is zero, and formally putting $Kg(\eta) = \int_{R_M} g(\eta')K(\eta, \eta')\nu_0(d\eta')$, $\eta \in R_M$, whenever g is a function on R_M. If $f, g \in L_2^+(R_M, \nu_0)$, then

$$(H_f, H_g) = \int_{R_M} f(\eta)Kg(\eta)\nu_0(d\eta)$$

by Tonelli's Theorem. If f is an arbitrary element of $L_2(R_M, \nu_0)$ and $g \in L_2^+(R_M, \nu_0)$, then $H_f = H_{f+} - H_{f-}$ and

$$\left|\int_{R_M} f(\eta)Kg(\eta)\nu_0(d\eta)\right| = |(H_f, H_g)| < +\infty.$$

It follows from standard L_2 theory that $Kg \in L_2(R_M, \nu_0)$ whenever $g \in L_2^+(R_M, \nu_0)$ and therefore $Kg \in L_2(R_M, \nu_0)$ for arbitrary $g \in L_2(R_M, \nu_0)$. Since $|(f, Kg)_M| = |(H_f, H_g)| \leqq \|H_f\|\|H_g\| \leqq \|f\|_M\|g\|_M$ for all $f, g \in L_2(R_M, \nu_0)$, $\|K\|_M \leqq 1$.

If we put $\psi(\eta) = \int_R \chi_\xi(\eta)\lambda(d\xi)$, $\eta \in R_M$, then $\psi \in L_1(R_M, \mu(\xi_0, \cdot))$ and

$$\int_R H_f(\xi)H_g(\xi)\lambda(d\xi) = \int_{R_M} f(\eta)[\psi(\eta)Kg(\eta)]\mu(\xi_0, d\eta) \tag{1}$$

for $f, g \in L_2(R_M, \nu_0)$. If $\nu_0(A) > 0$, then $\mu(\xi, A)$ is a positive harmonic function and

$$\int_A \psi(\eta)\mu(\xi_0, d\eta) = \int_R \mu(\xi, A)\lambda(d\xi) = \nu_0(A) > 0$$

and it is easy to see that $\psi>0$ a.e. on R_M relative to ν_0. If we put $\mu_g(A)=\int_A H_g(\xi)\lambda(d\xi)$, $g\in L_2(R_M, \nu_0)$, then μ_g is a finite signed measure since $H_g\in L_2(R, \lambda)\subset L_1(R, \lambda)$. It follows from (1) that

$$\int_R H_f(\xi)\mu_g(d\xi)=\int_{R_M} f\psi Kg\mu(\xi_0, d\eta), \quad f, g\in L_2(R_M, \nu_0). \tag{2}$$

We now let u_g be the Green potential of μ_g, that is $u_g(\eta)=\int_R G(\eta, \eta')\mu_g(d\eta')$, $\eta\in R$. If $g\in L_2^+(R_M, \nu_0)$ with $\|g\|_M\neq 0$, then $H_g>0$ on R, μ_g is a measure, and u_g is a strictly positive superharmonic function on R for which $\text{f-lim}_{\xi\to\eta} u_g(\xi)=0$ a.e. on R_M. If we put $u_g=0$ on R_M and η is a minimal point of R_M, then

$$\text{f-lim}_{\xi\to\eta}\frac{u_g(\xi)-u_g(\eta)}{G(\xi_0, \xi)}$$

is defined, possibly infinite, and positive; that is, the fine normal derivative du_g/dG is defined a.e. on R_M and $0<du_g/dG\leqq+\infty$. According to Theorem 8.1 of [3]

$$\int_R H_f(\eta)\mu_g(d\eta)=\int_{R_M}\frac{du_g}{dG} f\mu(\xi_0, d\eta) \tag{3}$$

whenever $f\in L_2(R_M, \nu_0)$. Of course, (3) is trivially true if $\|g\|_M=0$. By (8.3′) of [3],

$$\mu_g(R)=\int_{R_M}\frac{du_g}{dG}\mu(\xi_0, d\eta).$$

Since $\mu_g(R)=\int_R H_g(\xi)\lambda(d\xi)<+\infty$, du_g/dG is finite a.e. on R_M. It follows from (2) and (3) that

$$\int_{R_M}\frac{du_g}{dG} f\mu(\xi_0, d\eta)=\int_{R_M} f\psi Kg\mu(\xi_0, d\eta)$$

for all $f\in L_2(R_M, \nu_0)$. It also follows from (1) that the integral on the right is finite for all $f\in L_2(R_M, \nu_0)$ and therefore $du_g/dG\in L_2(R_M, \nu_0)$ by standard L_2 theory; moreover, $du_g/dG=\psi Kg$ a.e. on R_M whenever $g\in L_2^+(R_M, \nu_0)$. This result can be extended to arbitrary $g\in L_2(R_M, \nu_0)$ by considering the decomposition $g=g^+-g^-$. Since the signed measure μ_g has a density H_g, $\Delta u_g=-\varkappa_n H_g$ on R, where $\varkappa_n$ is a constant depending upon the dimensionality of the space, and $\Delta^2 u_g=0$ on R. We have therefore proved the following.

THEOREM 1. *If f is a real-valued function on R_M such that $f\psi^{-1}$ is in the range of K, then there is a function u on R such that*

(i) $\Delta^2 u=0$ *on* R,

(ii) $\text{f-lim}_{\xi\to\eta} u(\xi)=0$ *a.e. on* R_M, *and*

(iii) $du/dG=f$ *a.e. on* R_M.

No claim is made that the solution is unique at this time although this is undoubtedly the case. Since $f\psi^{-1}$ must be in the range of K, the extent of the result is not clear. We will see in the next section that the range of K is dense in $L_2(R_M, \nu_0)$.

3. Boundary functions. Since the range of the operator K determines the applicability of Theorem 1, we will examine the operator K further. We first note that K is a symmetric operator on $L_2(R_M, \nu_0)$ for if $f, g \in L_2(R_M, \nu_0)$, then $(f, Kg)_M = (H_f, H_g) = (Kf, g)_M$. Suppose $f \in L_2(R_M, \nu_0)$ and $Kf = 0$. Then $(H_f, H_f) = (f, Kf)_M = 0$ implies $H_f = 0$ a.e. (λ) on R but since H_f is harmonic, $H_f = 0$ on R and $f = 0$ a.e. on R_M. Therefore, $f = 0$ a.e. (ν_0). This shows that zero is not an eigenvalue of K or, in other words, the null space of K is trivial. It follows that K is one-to-one. Using the fact that K is symmetric with trivial null space, it is easy to show that the range of K is dense in $L_2(R_M, \nu_0)$.

Moreover, the operator K can be seen to be completely continuous as follows. Consider $L_2(R_M \times R_M, \nu_0 \times \nu_0)$. For $f, g \in L_2(R_M, \nu_0)$, we know $|(f, Kg)_M| \leqq \|f\|_M \|g\|_M$. Thus, if F is the indicator function of a measurable rectangle in $R_M \times R_M$, then

$$\int_{R_M} \int_{R_M} K(\eta, \eta') |F(\eta, \eta')| \nu_0(d\eta) \nu_0(d\eta') \leqq \|F\|. \tag{4}$$

Using a monotone class argument, this inequality can be extended to indicator functions of measurable subsets of $R_M \times R_M$ and then to simple functions. Since $\nu_0 \times \nu_0$ is a finite measure, each $F \in L_2(R_M \times R_M, \nu_0 \times \nu_0)$ can be approximated in mean by a sequence of simple functions and (4) can be extended to any such F. By standard L_2 theory,

$$\int_{R_M} \int_{R_M} K^2(\eta, \eta') \nu_0(d\eta) \nu_0(d\eta') < +\infty$$

and this implies that K is completely continuous.

Since K is completely continuous, there is a complete orthonormal sequence of eigenfunctions $\{\varphi_n\}$ for K corresponding to the sequence of real eigenvalues $\{\mu_n\}$. It is also known [1] that g is in the range of K if and only if

$$\sum_{n=1}^{\infty} \mu_n^{-2} \left| \int_{R_M} g(\eta) \varphi_n(\eta) \nu_0(d\eta) \right|^2 < +\infty.$$

THEOREM 2. *Suppose f is a real-valued function on R_M. Then $f\psi^{-1}$ is in the range of K if and only if*

$$\sum_{n=1}^{\infty} \mu_n^{-2} \left| \int_{R_M} f(\eta) \varphi_n(\eta) \mu(\xi_0, d\eta) \right|^2 < +\infty.$$

Given f satisfying the condition of this theorem, the function g for which $f = \psi K g$ has the expansion $g \sim \sum_{n=1}^{\infty} c_n \varphi_n$ where $c_n = \mu_n^{-1} \int_{R_M} f(\eta) \varphi_n(\eta) \mu(\xi_0, d\eta)$.

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AN ASYMPTOTIC FUGLEDE THEOREM FOR GENERATORS OF C_0 GROUPS

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In his paper [7] Moore proved the following theorem:

Let N be a normal operator on a Hilbert space H. Then for every positive number $\varepsilon>0$ there exists $\delta>0$, such that for every operator $T\in B(H)$ with $\|T\|\leqq 1$ and $\|NT-TN\|<\delta$, we have $\|N^*T-TN^*\|<\varepsilon$. If $N=R+iS$, where R, S are commuting self-adjoint operators in $B(H)$, we consider the linear operators L, $M\in B(B(H))$ defined by $L(T)=RT-TR$, $M(T)=ST-TS$ for $T\in B(H)$. It is known that these operators are Hermitian in the sense of Vidav (i.e. e^{itL}, e^{itM}, $t\in\mathbf{R}$ are groups of isometries on $B(H)$). Moreover, $LM=ML$ and $L(T)+iM(T)=NT-TN$, $L(T)-iM(T)=N^*T-TN^*$ $(T\in B(H))$. So that we can obtain a strengthening of Moore's theorem, by considering two commuting Hermitian operators A, B on a Banach space X and showing that when $x\in X$, $\|x\|\leqq 1$ and $\|Ax+iBx\|$ is small, then $\|Ax-iBx\|$ is small, too. A generalization of this kind was given in [2] (see also [3]) where Moore's technique was developed to obtain a stronger result.

This technique is based on the Cauchy integral formula for a holomorphic function and its derivative. We want to generalize the Fuglede theorem further, taking A and B to be the (unbounded) generators of two commuting C_0 groups of operators, e^{tA}, e^{sB} $(t, s\in\mathbf{R})$ on a Banach space X. It is not possible to use the Cauchy integral formula in this case, because we can not construct an appropriate holomorphic function from e^{tA}, e^{sB} of the kind considered in [2] or [7]. However, we can achieve our aim by making use of the generalized Cauchy formula:

Let K be a bounded closed and convex domain in $\mathbf{C}$ with partially smooth boundary dK. Then for every complex function $F\in C^1(K)$ one has the formula

$$F(z)=\frac{1}{2\pi i}\int_{dK}\frac{F(w)\,dw}{w-z}-\frac{1}{\pi}\iint_K\frac{\partial F(w)}{\partial\bar{w}}\frac{du\,dv}{w-z} \tag{1}$$

where $z\in\operatorname{Int}K$, $w=u+iv$ $(u, v\in\mathbf{R})$ *and* $\dfrac{\partial}{\partial\bar{w}}=\dfrac{1}{2}\left(\dfrac{\partial}{\partial u}+i\dfrac{\partial}{\partial v}\right)$.

This representation easily follows from the Green integral formula for complex functions (see [4], Theorem 1.2.1, or [8], Ch. II, § 5, Theorem 3).

So let now e^{tA}, e^{sB} $(t, s\in\mathbf{R})$ be two commuting C_0 groups of operators on a Banach space X with generators A, B. Let $x\in X$ be such that

$$x\in D(A^2)\cap D(B^2),\quad Ax, Bx\in D(A)\cap D(B),\quad ABx=BAx. \tag{2}$$

For $f\in X'$ (the dual of X) with $\|f\|=1$ we consider the complex function $F(z)=F(t+is)=f(e^{tA}e^{sB}x)$ $(z=t+is,\ t, s\in\mathbf{R})$ defined on the whole complex plane $\mathbf{C}$.

We denote the complex variable also by $w=u+iv$, $u, v\in R$ and use the standard notations

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial t}-i\frac{\partial}{\partial s}\right),\quad \frac{\partial}{\partial \bar{w}}=\frac{1}{2}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right).$$

It is easy to see that under the assumption (2), the function $\boldsymbol{F}$ on $\mathbf{C}$ has all partial derivatives of second order and they are continuous (this is proved as in [1]). We have

$$2F'_z(z)=2\frac{\partial F(z)}{\partial z}=f\left(e^{tA}e^{sB}(Ax-iBx)\right),\quad 2F'_z(0)=f(Ax-iBx);$$

$$2F'_{\bar{w}}(w)=2\frac{\partial F(w)}{\partial \bar{w}}=f\left(e^{uA}e^{vB}(Ax+iBx)\right),\quad 2F'_{\bar{w}}(0)=f(Ax+iBx).$$

So in order to compare $|f(Ax-iBx)|$ and $|f(Ax+iBx)|$, we use formula (1) for $\boldsymbol{F}$ and find $\dfrac{\partial \boldsymbol{F}}{\partial z}$ from there. We have to prepare beforehand an estimate for the difference $F'_{\bar{w}}(w)-\boldsymbol{F}'_{\bar{w}}(z)$. Let the domain K be as in (1). We apply the mean value theorem to the real and imaginary parts of $F'_{\bar{w}}$ (considered as a function of two real variables) and the points $w, z\in K$. Let $w-z=re^{i\theta}$ $(r>0, 0\leqq\theta\leqq 2\pi)$, so that $u-t=r\cos\theta$, $v-s=r\sin\theta$. For $c=p+iq$, $p, q\in R$ we have

$$2\frac{\partial F'_{\bar{w}}(c)}{\partial u}=f\left(e^{pA}e^{qB}(A(Ax+iBx))\right),\quad 2\frac{\partial F'_{\bar{w}}(c)}{\partial v}=f\left(e^{pA}e^{qB}(B(Ax+iBx))\right);$$

and

$$F'_{\bar{w}}(w)-F'_{\bar{w}}(z)=(u-t)\left(\operatorname{Re}\frac{\partial F'_{\bar{w}}(a)}{\partial u}+i\operatorname{Im}\frac{\partial F'_{\bar{w}}(b)}{\partial u}\right)+$$

$$+(v-s)\left(\operatorname{Re}\frac{\partial F'_{\bar{w}}(a)}{\partial v}+i\operatorname{Im}\frac{\partial F'_{\bar{w}}(b)}{\partial v}\right)$$

where $a, b\in K$. Thus we obtain

$$|F'_{\bar{w}}(w)-F'_{\bar{w}}(z)|\leqq 2^{-1/2}rm(K)h(x), \tag{3}$$

where

$$h(x)=\|A(Ax+iBx)\|+\|B(Ax+iBx)\|$$

and

$$m(K)=\sup\{\|e^{pA}e^{qB}\|:\ p, q\in R,\ p+iq\in K\}.$$

Now we are ready to find F'_z from (1). The first integral on the right hand side in (1) is a holomorphic function of z, so

$$\frac{\partial}{\partial z}\int_{dK}\frac{F(w)\,dw}{w-z}=\int_{dK}\frac{F(w)\,dw}{(w-z)^2}. \tag{4}$$

We now consider the second integral in (1):

$$G(z)=\iint_K\frac{F'_{\bar{w}}(w)\,du\,dv}{w-z}=\iint_K\frac{F'_{\bar{w}}(w)e^{-i\theta}\,du\,dv}{r} \tag{5}$$

(here $w-z=re^{i\theta}$). To find its partial derivatives we follow a rule considered by Mihlin ([5], p. 81; [6], Ch. II, §8; see also [9]):

$$\frac{\partial G(z)}{\partial t} = \iint_K F'_{\bar{w}}(w)\frac{\partial}{\partial t}\left(\frac{1}{w-t-is}\right)du\,dv - F'_{\bar{w}}(z)\int_0^{2\pi} e^{-i\theta}\cos\theta\,d\theta, \tag{6a}$$

$$\frac{\partial G(z)}{\partial s} = \iint_K F'_{\bar{w}}(w)\frac{\partial}{\partial s}\left(\frac{1}{w-t-is}\right)du\,dv - F'_{\bar{w}}(z)\int_0^{2\pi} e^{-i\theta}\sin\theta\,d\theta. \tag{6b}$$

The singular integral

$$\iint_K F'_{\bar{w}}(w)\frac{\partial}{\partial t}\left(\frac{1}{w-t-is}\right)du\,dv = \iint_K F'_{\bar{w}}(w)\frac{du\,dv}{(w-z)^2} = \iint_K F'_{\bar{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2}$$

is convergent in the sense of Cauchy, i.e. the limit

$$\lim_{\varepsilon\to 0}\iint_{K\setminus K_\varepsilon} F'_{\bar{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2} = \iint_K F'_{\bar{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2} \tag{7}$$

exists, where K_ε is a circle with centre at z and radius $\varepsilon>0$ taken so small that $K_\varepsilon\subset \operatorname{Int} K$ (see [6], ch. II). For convenience of the reader we shall prove this convergence here. We have

$$\iint_{K_\varepsilon}\frac{F'_{\bar{w}}(w)e^{-2i\theta}\,du\,dv}{r^2} = \iint_{K_\varepsilon}\frac{\left(F'_{\bar{w}}(w)-F'_{\bar{w}}(z)\right)e^{-2i\theta}\,du\,dv}{r^2} + F'_{\bar{w}}(z)\iint_{K_\varepsilon}\frac{e^{-2i\theta}\,du\,dv}{r^2}.$$

According to (3)

$$\left|\iint_{K_\varepsilon}\frac{\left(F'_{\bar{w}}(w)-F'_{\bar{w}}(z)\right)e^{-2i\theta}\,du\,dv}{r^2}\right| \leqq \frac{1}{\sqrt{2}}m(K)h(x)\iint_{K_\varepsilon}\frac{du\,dv}{r} = \tag{8}$$

$$= \frac{1}{\sqrt{2}}m(K)h(x)\int_0^{2\pi}\int_0^{\varepsilon} dr\,d\theta = \sqrt{2}\,\pi m(K)h(x)\varepsilon$$

which tends to zero when ε tends to zero. Also

$$\iint_{K_\varepsilon}\frac{e^{-2i\theta}\,du\,dv}{r^2} = \lim_{\delta\to 0}\int_0^{2\pi}\int_\delta^{\varepsilon}\frac{e^{-2i\theta}\,dr\,d\theta}{r} = \lim_{\delta\to 0}\log\frac{\varepsilon}{\delta}\int_0^{2\pi} e^{-2i\theta}\,d\theta = 0. \tag{9}$$

As

$$\iint_K = \iint_{K\setminus K_\varepsilon} + \iint_{K_\varepsilon}$$

we see that (7) holds.

The same is true for the integral

$$\iint_K F'_{\bar{w}}(w)\frac{\partial}{\partial s}\left(\frac{1}{w-t-is}\right)du\,dv = i\iint_K F'_{\bar{w}}(w)\frac{du\,dv}{(w-z)^2} = i\iint_K F'_{\bar{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2}.$$

We note that later we shall make use of the way the estimate in (8) and the computation in (9) were made.

From (6a) and (6b) we obtain

$$\frac{\partial G(z)}{\partial z}=\frac{1}{2}\left(\frac{\partial G(z)}{\partial t}-i\frac{\partial G(z)}{\partial s}\right)=\iint\limits_K F'_{\overline{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2}- \tag{10}$$

$$-\frac{1}{2}F'_{\overline{w}}(z)\int\limits_0^{2\pi}e^{-i\theta}(\cos\theta-i\sin\theta)\,d\theta.$$

Finally by (1), (4), (5) and (10) we have

$$\frac{\partial F(z)}{\partial z}=\frac{1}{2\pi i}\int\limits_{dK}\frac{F(w)\,dw}{(w-z)^2}-\frac{1}{\pi}\iint\limits_K F'_{\overline{w}}(w)\frac{e^{-2i\theta}\,du\,dv}{r^2}, \tag{11}$$

as

$$\int\limits_0^{2\pi}e^{-i\theta}(\cos\theta-i\sin\theta)\,d\theta=\int\limits_0^{2\pi}e^{-2i\theta}\,d\theta=0.$$

Now we are ready to make the desired estimates. We put $z=0$ in (11) and take the domain K to be a circle K_ϱ with radius $\varrho>0$ and centre at the origin. Denoting $m(\varrho)=m(K_\varrho)$ (see (3)) we have

$$\left|\frac{1}{2\pi i}\int\limits_{dK_\varrho}\frac{F(w)\,dw}{w^2}\right|=\frac{1}{2\pi}\left|\int\limits_0^{2\pi}F(\varrho e^{i\theta})\frac{e^{-i\theta}\,d\theta}{\varrho}\right|\leqq\|x\|\frac{m(\varrho)}{\varrho}\quad(w=\varrho e^{i\theta}),$$

and

$$\frac{1}{\pi}\iint\limits_{K_\varrho}F'_{\overline{w}}(w)r^{-2}e^{-2i\theta}\,du\,dv=\frac{1}{\pi}\iint\limits_{K_{1/\varrho}}F'_{\overline{w}}(w)r^{-2}e^{-2i\theta}\,du\,dv+$$

$$+\frac{1}{\pi}\iint\limits_{K_\varrho\setminus K_{1/\varrho}}F'_{\overline{w}}(w)r^{-2}e^{-2i\theta}\,du\,dv\quad\text{when}\quad\varrho\geqq 1.$$

Because $\iint\limits_{K_{1/\varrho}}r^{-2}e^{-2i\theta}\,du\,dv=0$ (as in (9)), we obtain

$$\left|\frac{1}{\pi}\iint\limits_{K_{1/\varrho}}F'_{\overline{w}}(w)r^{-2}e^{-2i\theta}\,du\,dv\right|=\left|\frac{1}{\pi}\iint\limits_{K_{1/\varrho}}(F'_{\overline{w}}(w)-F'_{\overline{w}}(0))r^{-2}e^{-2i\theta}\,du\,dv\right|\leqq$$

$$\leqq 2^{1/2}h(x)m(1/\varrho)/\varrho$$

(in the same way as in (8), according to (3)). Also

$$\left|\frac{1}{\pi}\iint\limits_{K_\varrho\setminus K_{1/\varrho}}F'_{\overline{w}}(w)r^{-2}e^{-2i\theta}\,du\,dv\right|\leqq\frac{1}{\pi}\int\limits_0^{2\pi}\int\limits_{1/\varrho}^{\varrho}|F'_{\overline{w}}(w)|r^{-1}\,dr\,d\theta\leqq$$

$$\leqq 2\|Ax+iBx\|m(\varrho)\ln\varrho,$$

as

$$F'_{\overline{w}}(w)=f\left(e^{uA}e^{vB}(Ax+iBx)\right)/2.$$

Finally

$$|F_z'(0)| \leqq \|x\| m(\varrho)/\varrho + 2^{1/2} h(x) m(1/\varrho)/\varrho + 2\|Ax+iBx\| m(\varrho) \ln \varrho.$$

As $F_z'(0)=f(Ax-iBx)/2$ and $f\in X'$ is arbitrary with norm one, we obtain the following theorem:

THEOREM. *Let e^{tA}, $e^{sB}(t, s\in R)$ be two commuting C_0 groups of operators on the Banach space X with generators A, B. Let $x\in X$ satisfy the condition* (2). *Then*

$$\|Ax-iBx\| \leqq 2\|x\| m(\varrho)/\varrho + 2^{3/2} h(x) m(1/\varrho)/\varrho + 4\|Ax+iBx\| m(\varrho) \ln \varrho, \tag{12}$$

for every $\varrho\geqq 1$, where $h(x)=\|A(Ax+iBx)\|+\|B(Ax+iBx)\|$ and

$$m(r) = \sup \{\|e^{tA}e^{sB}\| : t, s\in R, |t+is| \leqq r\}, \quad r > 0.$$

Note that since

$$Ax = (Ax+iBx)/2+(Ax-iBx)/2, \quad Bx = i(Ax-iBx)/2 - i(Ax+iBx)/2,$$

we have under the assumptions of the theorem also the inequality

$$\max(\|Ax\|, \|Bx\|) \leqq \tag{13}$$

$$\leqq \|x\| m(\varrho)/\varrho + 2^{1/2} h(x) m(1/\varrho)/\varrho + \|Ax+iBx\| (1/2+2m(\varrho) \ln \varrho) \quad (\varrho \geqq 1).$$

The above theorem enables us to obtain a generalization of the Moore asymptotic version of Fuglede's theorem.

COROLLARY 1. *Let A, B, $m(\varrho)$ be as in the theorem and let $m(\varrho)/\varrho\to 0$ as $\varrho\to\infty$. Then for every two positive numbers L, ε, there exists $\delta>0$, such that each time when $x\in X$ is as in* (2) *and $\|x\|$, $h(x)\leqq L$, the inequality $\|Ax+iBx\|<\delta$ implies $\|Ax-iBx\|<\varepsilon$. (In view of* (13), *$\|Ax-iBx\|$ can be replaced by $\max(\|Ax\|, \|Bx\|)$.)*

PROOF. For given $L, \varepsilon>0$, we take $\varrho>1$ so large that $m(\varrho)/\varrho<\varepsilon/(12L)$. Then we put $\delta=\varepsilon/(8m(\varrho)\ln\varrho)$. It is clear from (12) that this δ serves our purpose. (Note that $m(1/\varrho)\leqq m(\varrho)$ for $\varrho\geqq 1$).

Obviously, when A, B are bounded, the condition "$h(x)\leqq L$" can be removed from the above corollary.

COROLLARY 2. *Let $m(\varrho)\leqq m$ (a constant) for $\varrho>0$. Then (under the conditions of the theorem) minimizing the right hand side in* (12) *for $\varrho\geqq 1$ we come to the inequality*

$$\|Ax-iBx\| \leqq \begin{cases} 4m\|Ax+iBx\| & \text{if} \quad \|x\|+2^{1/2}h(x) \leqq 2\|Ax+iBx\|, \\ 4m\|Ax+iBx\|\,(1+\ln((\|x\|+2^{1/2}h(x))/(2\|Ax+iBx\|))) & \text{otherwise.} \end{cases} \tag{14}$$

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VECTOR-VALUED IDEMPOTENT MEASURES

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1. Introduction

Let S be a compact semitopological semigroup and let $\mathscr{A}$ be a Banach algebra with a closed cone K which contains the identity i and which is closed under multiplication such that its norm is additive on the positive cone K, i.e. $\|x+y\|=\|x\|+\|y\|$ if $x, y \in K$. The algebra of continuous functions from S to $\mathscr{A}$ endowed with the uniform norm is denoted by $C(S, \mathscr{A})$. If $\mathscr{A}=\mathscr{C}$, the set of all complex numbers, we simply write $C(S)$. The set of Borel sets in S is denoted by $B(S)$ and the algebra of weakly compact measures m: $B(S)\to\mathscr{A}$ which represent weakly compact operators T: $C(S)\to\mathscr{A}$ is denoted by $MW(S, \mathscr{A})$, (see, for example, [2] or [12]). A measure m in $MW(S, \mathscr{A})$ is called a *probability measure* if m is positive and the total variation of m is 1 i.e. $V(m)(S)=1$, where $V(m)(E)=\sup\{\Sigma\|m(e_i)\|: \{e_i\}$ are Borel partitions of $E\}$ for $E\in B(S)$. The set of all these probability measures is denoted by $PW(S, \mathscr{A})$. It is shown in [7] that the support of an idempotent measure in $PW(S, \mathscr{A})$ is a closed topologically simple subsemigroup and that $PW(S, \mathscr{A})$ is a compact semitopological semigroup with respect to the weak operator topology and convolution operation.

Scalar-valued idempotent measures on semigroups have been studied, for example, by Collins [8], Pym [11] and the author [4]. In this paper we shall show how some results in the scalar-valued case continue to hold in the vector-valued case. In particular we show in Corollary 2.5 that if S is a compact group then the support, supp μ, of an idempotent μ in $PW(S, \mathscr{A})$ is a compact subgroup and $\mu=\mu(S)\cdot m_{\text{supp}\,\mu}$, where $m_{\text{supp}\,\mu}$ is the normalized Haar measure on supp μ.

2. Idempotent measures

2.1. NOTATION. For each $f\in C(S)$ and each pair $x, y\in S$, we write ${}_xf(y)=f(xy)$ and $f_x(y)=f(yx)$. If $\mu\in MW(S, \mathscr{A})$ we put $\mu(f)=\int f(x)\,d\mu(x)$ and define ${}_\mu f(x)=\mu(f_x)$, $f_\mu(x)=\mu({}_xf)$. As we can see from [12, Theorem 2.1] that the functions ${}_\mu f$ and f_μ: $S\to\mathscr{A}$ are continuous. We denote ${}_\mu f_\mu={}_\mu(f_\mu)=({}_\mu f)_\mu$.

2.2. LEMMA. *Let μ be an idempotent in $PW(S, \mathscr{A})$ and let $f\in C^+(S)$. Then ${}_\mu f_\mu$ is constant on* supp μ.

PROOF. Since f is continuous and supp μ is compact, f attains its supremum on supp μ at a (say).

Let $F(x)=f(a)-f(x)$. Then $F(x)\geqq 0$ on supp μ. Recall that K is closed under multiplication. Hence

$$ {}_\mu F_\mu(x)=\int\int F(yxz)\,d\mu(y)\,d\mu(z)\in K $$

for $x \in \operatorname{supp} \mu$, since μ is positive and $\operatorname{supp} \mu$ is a semigroup. Therefore ${}_\mu f_\mu(a) \geqq {}_\mu f_\mu(x)$ $(x \in \operatorname{supp} \mu)$. Hence, since μ is an idempotent and ${}_\mu f_\mu(a)\mu(S) = {}_\mu f_\mu(a)$,

$$\|{}_\mu f_\mu(a)\| = \|{}_{\mu\mu} f_{\mu\mu}(a)\| =$$

$$= \left\|\int\int {}_\mu f_\mu(xay)\, d\mu(x)\, d\mu(y)\right\| \leqq \left\|\int\int {}_\mu f_\mu(a)\right\| dV(\mu)(x)\, dV(\mu)(y) = \|{}_\mu f_\mu(a)\|.$$

Since ${}_\mu f_\mu : S \to \mathscr{A}$ is continuous, we see $\|{}_\mu f_\mu(a)\| = \|{}_\mu f_\mu(xay)\|$ and so ${}_\mu f_\mu(a) = {}_\mu f_\mu(xay)$ $(x, y \in \operatorname{supp} \mu)$. So ${}_\mu f_\mu$ attains its supremum everywhere on $(\operatorname{supp} \mu)a(\operatorname{supp} \mu) \supset K_\mu$ where K_μ is the minimal ideal of the compact semitopological semigroup $\operatorname{supp} \mu$. By symmetry ${}_\mu f_\mu$ attains its infimum on K_μ, and as $K_\mu \neq \emptyset$, ${}_\mu f_\mu$ is constant on $\operatorname{supp} \mu$.

The next lemma can now be proved by the same arguments as in [11, Lemma 3].

2.3. LEMMA. *Let μ be an idempotent in $PW(S, \mathscr{A})$, then $\mu\nu\mu = \mu$ for any ν in $PW(S, \mathscr{A})$ with $\operatorname{supp} \nu \subset \operatorname{supp} \mu$, and $\nu(S) = i$.*

Before we come to the decomposition of idempotent measures let us recall some facts concerning the structure of the minimal ideal K_μ of $\operatorname{supp} \mu$. Proofs of our assertions may be found in Berglund and Hofmann [1].

There must exist an idempotent $e \in K_\mu$. Furthermore $(\operatorname{supp} \mu)e = K_\mu e$ and $e(\operatorname{supp} \mu) = eK_\mu$ are compact semigroups, while $e(\operatorname{supp} \mu)e = eK_\mu e$ is a compact group and is denoted by G_μ. Denote by $I(S)$ the set of idempotents in S. Then $E_\mu = I(K_\mu e)$, $F_\mu = I(eK_\mu)$ are compact subsemigroups and $e_1 e_2 = e_1$, $f_1 f_2 = f_2$ for $e_1, e_2 \in E_\mu$ and $f_1, f_2 \in F_\mu$. Furthermore $E_\mu G_\mu F_\mu = K_\mu$, and each $x \in K_\mu$ has a unique expression in the form $x = e'x'f'$ with $e' \in E_\mu$, $x' \in G_\mu$ and $f' \in F_\mu$. Since $F_\mu e \subset$ $\subset eK_\mu e = G_\mu$, $K_\mu e = E_\mu G_\mu F_\mu e = E_\mu G_\mu$. Similarly we see that $eK_\mu = G_\mu F_\mu$.

2.4. THEOREM. *For idempotent μ in $PW(S, \mathscr{A})$,*

$$\mu = \mu_E \mu_G \mu_F,$$

where $\mu_E, \mu_G, \mu_F \in PW(S, \mathscr{A})$, $\operatorname{supp} \mu_E \subset E_\mu$, $\operatorname{supp} \mu_F \subset F_\mu$, $\mu_G = i \cdot m_G$ and m_G is the (normalized) Haar measure of the compact group G_μ.

PROOF. Let $e^2 = e \in K_\mu$ and let $\delta_e(E) = i$ if $e \in E$ and $\delta_e(E) = 0$ otherwise. Then $\operatorname{supp}(\mu\delta_e) = (\operatorname{supp} \mu)e = K_\mu e = E_\mu G_\mu$. Note that $T = E_\mu G_\mu$ is a compact left simple semigroup and so every $x \in T$ has a unique expression in the form $x = e'g'$ with $e' \in E_\mu$, $g' \in G_\mu$. Given $\psi \in C(E_\mu)$, define ψ_T on T by $\psi_T(e'g') = \psi(e')$ $(e' \in E_\mu,$ $g' \in G_\mu)$. Then ψ_T is well defined and is continuous on T. The projection μ_E of $\mu\delta_e$ on E_μ is given by $\mu_E(\psi) = \mu\delta_e(\psi_T)$ $(\psi \in C(E_\mu))$. Then clearly $\operatorname{supp} \mu_E = E_\mu$ and we can consider $\mu_E \in PW(S, \mathscr{A})$ by [12, Theorem 1.4]. For $f \in C(S)$,

$$\int_S f(xyz)\, d\mu\delta_e\mu_G = \int_S \int_{G_\mu} f(xez) \cdot i\, d\mu(x)\, d\mu_G(z) = \int_S \int_{G_\mu} f(xz)\, d\mu(x)\, d\mu_G(z),$$

since m_G is the Haar measure of G_μ. Therefore $\mu\mu_G = \mu\delta_e\mu_G$. Now

$$\int\int\int f(xyz)\, d\mu(x)\, d\delta_e(y)\, d\mu_G(z) = \int\int f(xez)\, i\, d\mu(x)\, d\mu_G(z) =$$

$$= \int\int f(xez)\, d\mu(x)\, d\mu_G(z) = \int\int f(xz)\, d\mu_E(x)\, d\mu_G(z),$$

whence $\mu\mu_G = \mu\delta_e\mu_G = \mu_E\mu_G$.

Similarly $\mu_G\mu = \mu_G\mu_F$, where μ_F is the projection of $\delta_e\mu$ on F_μ. Thus, since $G_\mu \subset \operatorname{supp} \mu$, by Lemma 2.3,

$$\mu = \mu\mu_G\mu_G\mu = \mu_E\mu_G\mu_G\mu_F = \mu_E\mu_G\mu_F,$$

completing the proof.

2.5. COROLLARY. *Let S be a compact group and let μ be an idempotent. Then* supp μ *is a compact group and* $\mu=\mu(S)\cdot m_{\operatorname{supp}\mu}$.

PROOF. All we need is to observe that $(\mu(S))^2=\mu(S)$ and that the identity e is the only idempotent in S; so $E_\mu=F_\mu=\{e\}$, $\mu_E=\mu_F=\mu(S)\delta_e$. The other part of the proof is clear.

2.6. REMARKS. For idempotent μ in $PW(S, \mathscr{A})$, if $\operatorname{supp}\mu\cap K(S)\neq\emptyset$, then there is an idempotent $e\in K_\mu\cap K(S)$. Let

$$H=\{\mu\in PW(S, A)\colon \mu^2=\mu,\ \operatorname{supp}\mu\cap K(S)\neq\emptyset \text{ and } \mu_G=i\cdot m_G\},$$

where we decompose μ with respect to an idempotent $e\in K_\mu\cap K(S)$. Then μ is in the minimal ideal $K(PW(S, \mathscr{A}))$ iff $\mu\in H$.

A natural partial ordering of the set $I(S)$ of idempotents in S is defined by $e\leqq f$ if $ef=fe=e$. An element e in $I(S)$ is called a *primitive idempotent* if e is a minimal non-zero element of the partially ordered set $(I(S), \leqq)$. The set of primitive idempotents in S is denoted by $\Pi(S)$. If $K(S)$ is not a group, then $\Pi(PW(S, \mathscr{A}))=K(PW(S, \mathscr{A}))$. If $K(S)$ is a group then $\mu\in\Pi(PW(S, \mathscr{A}))$ iff for each closed subsemigroup S' containing $\operatorname{supp}\mu$, $K(S')$ satisfies either (a) $K(S')=K(S)$ or (b) $\operatorname{supp}\mu\cap K(S')\neq\emptyset$ and $\mu_G=i\cdot m_G$, where we regard μ as an idempotent measure on S'.

Proofs of our assertions may be formulated in the same way as in the paper by the author [4].

3. Normality of $PW(S, \mathscr{A})$

It is shown in [6] that if μ is a positive measure in $MW(S, \mathscr{A})$, then the semivariation $S_C(\mu, E)$ and the total variation $V(m)(E)$ are the same, where

$$S_C(\mu, E)=\sup\{\|\Sigma\alpha_i\mu(e_i)\|\colon \alpha_i \textit{ are complex numbers, } |\alpha_i|\leqq 1 \textit{ and } \{e_i\} \textit{ are Borel partitions of } E\}$$

for $E\in B(S)$. Our next theorem is a generalization of Theorem 5.4 of Brooks and Lewis [2] from $\mathscr{C}$ to $\mathscr{A}$.

3.1. THEOREM. *If μ is an extreme point of $PW(S, \mathscr{A})$, then* supp μ *is a singleton.*

PROOF. We claim that if $A, B\in B(S)$ and $A\cap B=\emptyset$, then either $\mu(A)=0$ or $\mu(B)=0$. Suppose $\mu(A)\neq 0$ and $\mu(B)\neq 0$. Define, for $E\in B(S)$, $\mu_1(E)=\mu(A\cap E)$ and $\mu_2(E)=\mu(E\setminus A)$. Then μ_1, μ_2 are additive, with finite semivariation and

$$S_C(\mu_1, E)=S_C(\mu, A\cap E),\quad S_C(\mu_2, E)=S_C(\mu, E\setminus A).$$

Therefore μ_1, μ_2 are positive measures in $MW(S, \mathscr{A})$ [12, Theorem 1.4]. Now, since the norm is additive on K and the semivariation and variation are the same for μ, we note that $V(\mu)$ is additive [12, Theorem 1.10] and so

$$1=S_C(\mu, S)=V(\mu)(S)=V(\mu)(A)+V(\mu)(S\setminus A)=V(\mu_1)(S)+V(\mu_2)(S).$$

Note that $V(\mu_1)(S)=\|\mu_1(S)\|=\|\mu(A)\|\neq 0$ and that $V(\mu_2)(S)=\|\mu_2(S)\|=$ $=\|\mu(S\setminus A)\|\geqq\|\mu(B)\|\neq 0$, since μ is a positive measure, and the norm is additive on the positive cone. Let $\nu_1=\mu_1/V(\mu_1)(S)$, $\nu_2=\mu_2/V(\mu_2)(S)$. Then ν_1, ν_2 are in $PW(S, \mathscr{A})$ and

$$\mu = V(\mu_1)(S)\nu_1+V(\mu_2)(S)\nu_2.$$

Since μ is an extreme point, we see $\mu=\nu_1=\nu_2$. Consequently $0\neq\mu(A)=\nu_2(A)=0$, a contradiction.

Now for arbitrary $A\in B(S)$, A and $S\setminus A$ are disjoint and so either $\mu(A)=0$ or $\mu(A)=\mu(S)$. If $x, y\in\operatorname{supp}\mu$ and $x\neq y$, let N_x, N_y be disjoint open sets containing x and y respectively. Then either $\mu(N_x)=0$ or $\mu(N_y)=0$, and so either $S_C(\mu, N_x)=V(\mu)(N_x)=0$ or $S_C(\mu, N_y)=V(\mu)(N_y)=0$. Therefore either x or y does not belong to supp μ. We conclude that supp μ is a singleton.

Let

$$\tilde{S}_+=\{\mu\in PW(S, \mathscr{A})\colon \operatorname{supp}\mu \text{ } \textit{is a singleton}\}.$$

Then, since $PW(S, \mathscr{A})$ is convex and compact in the weak operator topology [7, Theorem 3.2], we have the next Corollary by Krein—Milman Theorem.

3.2. COROLLARY. *$PW(S, \mathscr{A})=\overline{CO}\tilde{S}_+$, where $\overline{CO}$ denotes the closed convex hull.*

The following theorem can now be proved by similar arguments as in [3].

3.3. THEOREM. *The semigroup $PW(S, \mathscr{A})$ is normal iff for each $x\in S$, $\mu\in PW(S, \mathscr{A})$ there are $\varrho, \lambda\in PW(S, \mathscr{A})$ such that $\mu\delta_x=\varrho\mu$ and $\delta_x\mu=\mu\lambda$, where $\delta_x\in\tilde{S}_+$ with support at x.*

We conclude this paper with a remark, an analog of [3, Theorem 2.1], that if S is a union of subsemigroups each of which has an identity element then $PW(S, \mathscr{A})$ is normal implies that S is commutative. In particular, if S is a group then $PW(S, \mathscr{A})$ is normal implies that S is commutative.

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DIFFERENTIATION OF SUPERADDITIVE PROCESSES IN L_p

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1. Introduction

In this paper differentiation theorems are proved for superadditive processes in L_p $(1 \leqq p \leqq \infty)$ with respect to semigroups of positive linear operators on L_p, under certain additional hypotheses. The results obtained below generalize Sato's local ergodic theorems and Akcoglu—Krengel's differentiation theorems for additive or superadditive processes.

2. Definitions and theorems

Let $(X, \mathfrak{F}, \mu)$ be a σ-finite measure space and $T=\{T_t\}=\{T_t\}_{t>0}$ a semigroup of positive linear operators on $L_p(X)=L_p(X, \mathfrak{F}, \mu)$ for some fixed p, with $1 \leqq p \leqq \infty$. We call T *locally integrable* if for each $f \in L_p(X)$ the vector valued function $t \mapsto T_t f$ is Bochner integrable with respect to the Lebesgue measure on every finite interval. By a *process in* $L_p(X)$ we mean a family $F=\{F_t\}=\{F_t\}_{t>0}$ of functions in $L_p(X)$. A process F is called *positive* if $F_t \in L_p^+(X)$ for all $t>0$, *increasing* if $F_t \leqq F_s$ for all $0<t<s$, *linearly bounded* if $\sup_{0<s<t} s^{-1}\|F_s\|_p < \infty$ for some $t>0$, and *superadditive* [resp. *additive*] if $F_{t+s} \geqq F_t + T_t F_s$ [resp. $F_{t+s}=F_t+T_tF_s$] for all $t, s>0$.

It is easily seen (see e.g. [10]) that $\{T_t\}$ decomposes X into two measurable sets P and N with the properties that

(i) if $f \in L_p(N)$ then $\|T_t f\|_p=0$ for all $t>0$,
(ii) if $f \in L_p^+(P)$ and $\|f\|_p>0$ then $\|(T_t f)1_P\|_p>0$ for some $t>0$, where 1_P denotes the indicator function of P.

Since $\{T_t\}$ is zero on $L_p(N)$, it is direct to see that there are many positive superadditive processes in $L_p(N)$ for which $\text{q-}\lim_{t \to 0} \frac{1}{t} F_t$ does not exist a.e. on N, where $q\text{-}\lim_{t \to 0}$ means that the limit is taken as t approaches to zero through a dense countable set in the interval $(0, 1)$. However, the situation is different on P; we shall prove the following theorems.

THEOREM 1. *Assume that* $1 \leqq p < \infty$ *and that* $\{T_t\}_{t>0}$ *is locally integrable. Then for any positive superadditive process* F *in* $L_p(X)$, *the limit* $\text{q-}\lim_{t \to 0} \frac{1}{t} F_t$ *exists and is finite a.e. on* P.

THEOREM 2. *Assume that* $p=1$ *and that* $\{T_t\}_{t>0}$ *is locally integrable. Then for any positive process* F *in* $L_\infty(X)$ *that is superadditive with respect to the adjoint semigroup* $\{T_t^*\}$ *acting on* $L_\infty(X)$, *the limit* $\text{q-}\lim_{t\to 0}\frac{1}{t}F_t$ *exists and is finite a.e. on* P^*, *where* $X=P^*+N^*$ *is the above decomposition determined by the adjoint semigroup* $\{T_t^*\}$ *on* $L_\infty(X)$.

We note that the decomposition $X=P^*+N^*$ is identical with the one $X=C+D$ given in Theorem 3 of [10] (cf. also [11]). Theorems 1 and 2 contain local ergodic theorems of Sato [10] and differentiation theorems of Akcoglu—Krengel [3], [4] for additive or superadditive processes. See also [8], [7] and [5]. The proofs of Theorems 1 and 2 are dependent upon the use of the results of Akcoglu—Krengel [3], together with arguments of McGrath [9], Sato [10] and Feyel [8].

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3. Proofs of Theorems 1 and 2

The arguments in Theorems 1 and 4 of Sato [10] (see also McGrath [9]) show that for the proofs of Theorems 1 and 2 it is enough to establish the next proposition; and thus in this section we shall only prove the proposition.

PROPOSITION. *Assume that* $\{T_t\}$ *is a strongly continuous semigroup of positive linear contractions on* $L_1(X)$. *Then for any positive superadditive process* F *in* $L_1(X)$, *the limit* $\text{q-}\lim_{t\to 0}\frac{1}{t}F_t$ *exists and is finite a.e. on* P.

Before proceeding toward the proof of the proposition we note that in the following proof we first show that $F_t=0$ a.e. on $P\setminus C$ for all $t>0$. Using this, one may obtain the conclusion of the proposition if the second theorem in Feyel [8] is applied. But we prefer to give a direct and complete proof, because Feyel's argument applies a tauberian theorem. On the other hand, Akcoglu [1] recently gave an alternative direct proof of Feyel's theorem. Our argument is different from Akcoglu's and somewhat simpler. Thus we think that our approach remains to be of some interest

PROOF. (I) We may assume without loss of generality that $P=X$. Let $X=C+D$ denote the decomposition given in Theorem 3 of [10] (cf. also [2]). We shall prove that $F_t\in L_1(C)$ for all $t>0$. To do this we first note that $T_tf\in L_1(C)$ for all $f\in L_1(X)$ and all $t>0$. Write

$$F_t = G_t+H_t \quad \text{with} \quad G_t\in L_1(C) \quad \text{and} \quad H_t\in L_1(D).$$

Then we have

$$G_{t+s} = F_{t+s} \geqq F_t+T_tF_s = G_t+T_tG_s+T_tH_s \quad \text{on} \quad C \tag{1}$$

for all $t, s>0$. Since $G=\{G_t\}$ is increasing, it follows that the function $t\mapsto G_t$ $(\in L_1(C))$ is continuous with respect to the norm topology of $L_1(C)$ at every point, with the possible exception of a countable subset of the interval $(0, \infty)$. Thus, given

a $\delta>0$ we may choose an $s>\delta$ such that $\lim_{t\to s}\|G_t-G_s\|_1=0$. Then we get that $\lim_{t\to 0}\|G_{t+s}\|_1=\|G_s\|_1$. On the other hand, since q-$\lim_{t\to 0} T_t^*1=1$ on C (this is an easy consequence of the fact that there exists an $h\in L_1(C)$ satisfying $h>0$ a.e. on C and $\lim_{t\to 0}\|T_th-h\|_1=0$ (see e.g. [2])),

$$\lim_{t\to 0}\|T_tG_s\|_1=\lim_{t\to 0}\int_C G_s(T_t^*1)\,d\mu=\int G_s\,d\mu=\|G_s\|_1.$$

Hence, by (1), we have

$$\lim_{t\to 0}\|G_t\|_1=0=\lim_{t\to 0}\|T_tH_s\|_1. \tag{2}$$

In particular, the second equality of (2) together with the assumption $P=X$ proves that $\|H_s\|_1=0$, and hence $H_t=0$ for all t with $0<t\leqq\delta$ $(<s)$. Since $\delta>0$ is arbitrary, it follows that $H_t=0$ for all $t>0$ and that $F_t=G_t\in L_1(C)$ for all $t>0$.

(II) By step (I) we may and will assume without loss of generality that $C=P=X$. Let us fix an $h\in L_1(C)$ such that $h>0$ a.e. on C and $\lim_{t\to 0}\|T_th-h\|_1=0$. Define

$$F_t(h)=\min\left\{F_t,\ \int_0^t T_sh\,ds\right\}\quad\text{for}\quad t>0.$$

Then $F(h)=\{F_t(h)\}$ is a positive superadditive process in $L_1(C)$. Let B denote the set of all positive binary rational numbers, and put

$$F_t^n(h)=\sum_{j=0}^{k-1}T_{2^{-n}}^jF_{2^{-n}}(h)\quad\text{for}\quad t=k2^{-n}\in B.$$

It is direct to check that $F_t(h)\geqq F_t^n(h)\geqq F_t^{n+1}(h)$ and

$$\|F_t^n(h)\|_1\leqq k\|F_{2^{-n}}(h)\|_1=t2^n\|F_{2^{-n}}(h)\|_1\leqq t\|h\|_1<\infty.$$

Thus, $F_t^n(h)$ converges in $L_1(C)$ to a function E_t as $n\to\infty$; and further $E_{t+s}=E_t+T_tE_s$ and $\|E_t-E_s\|_1\leqq|t-s|\|h\|_1$ for all t and s in B. By continuity E_t can be defined for all $t>0$, and the process $E=\{E_t\}$ in $L_1^+(C)$ is additive and satisfies

$$\frac{1}{t}\|E_t\|_1\leqq\liminf_n 2^n\|F_{2^{-n}}(h)\|_1\quad\text{for all}\quad t>0. \tag{3}$$

Let $\tilde{F}=\{\tilde{F}_t\}$, where we set $\tilde{F}_t=F_t(h)-E_t$ for all $t>0$. Then we shall prove that

$$\lim_n 2^n\|\tilde{F}_{2^{-n}}\|_1=0. \tag{4}$$

For this purpose, let us write $\alpha=\limsup_n 2^n\|F_{2^{-n}}(h)\|_1$ and take a subsequence $\{t_m\}$ of the sequence $\{2^{-n}\}_{n\geqq 1}$ such that

$$\alpha=\lim_m\frac{1}{t_m}\|F_{t_m}(h)\|_1.$$

Here it may be assumed without loss of generality that weak-$\lim_m \frac{1}{t_m} F_{t_m}(h)=g$ exists in $L_1(C)$, because $0\leqq F_{t_m}(h)\leqq \int_0^{t_m} T_s h\, ds$ and $\lim_{s\to 0}\|T_s h-h\|_1=0$. Since $\int g\, d\mu=\alpha$ and since q-$\lim_{t\to 0} T_t^* 1=1$ a.e. on $C=X$ (see step (I)), given an $\varepsilon>0$ we can choose an $r>0$ so that $\int g(T_r^* 1)d\mu>\alpha-\varepsilon$. Then choose an $N\geqq 1$ so that

$$\int \frac{1}{t_m} F_{t_m}(h)(T_r^* 1)\, d\mu > \alpha-2\varepsilon \quad \text{for all} \quad m\geqq N.$$

It follows that if $r>t=k2^{-n}\in B$ and $2^{-n}=t_m$ for some $m\geqq N$ then $T_{2^{-n}}^{*j} 1\geqq T_r^* 1$ for all $0\leqq j\leqq k$; and thus we have

$$\frac{1}{t}\|E_t^n\|_1=\frac{1}{t}\sum_{j=0}^{k-1}\int F_{2^{-n}}(h)(T_{2^{-n}}^{*j}1)\, d\mu=\frac{1}{k}\sum_{j=0}^{k-1}\int 2^n F_{2^{-n}}(h)(T_{2^{-n}}^{*j}1)\, d\mu>\alpha-2\varepsilon.$$

Letting $n\to\infty$ and then $\varepsilon\to 0$, we observe that $\frac{1}{t}\|E_t\|_1\geqq\alpha$ for all $t\in B$ with $t<r$. Thus

$$\liminf_{t\to 0}\frac{1}{t}\|E_t\|_1=\liminf_{\substack{t\to 0\\ t\in B}}\frac{1}{t}\|E_t\|_1\geqq\alpha,$$

and (3) yields

$$\lim_{t\to 0}\frac{1}{t}\|E_t\|_1=\lim_n 2^n\|F_{2^{-n}}(h)\|_1=\alpha.$$

This proves (4), as

$$\lim_n 2^n\|\tilde{F}_{2^{-n}}\|_1=\lim_n\left(2^n\|F_{2^{-n}}(h)\|_1-2^n\|E_{2^{-n}}\|_1\right)=0.$$

(III) We shall prove that

(5) $$\text{q-}\lim_{t\to 0}\frac{1}{t}\tilde{F}_t=0 \quad \text{a.e. on} \quad C=P=X.$$

For this purpose, fix any $N\geqq 1$ and set $A_N=\left\{T_{2^{-N}}^* 1>\frac{1}{2}\right\}$. By step (II), given an $\varepsilon>0$ there exists an $n\geqq N$ such that $2^n\|\tilde{F}_{2^{-n}}\|_1<\varepsilon$. Let us write $t_0=2^{-n}$ and define, for each $t>0$,

$$\tilde{H}_t=(I-T_t)\frac{1}{t_0}\int_0^{t_0}\tilde{F}_s\, ds+\frac{1}{t_0}\int_0^t T_s\tilde{F}_t\, ds.$$

By [3], $\tilde{H}=\{\tilde{H}_t\}$ is a positive additive process in $L_1(C)$ which satisfies $\tilde{H}_t\geqq\left(1-\frac{t}{t_0}\right)\tilde{F}_t$ for each $0<t<t_0$. An easy calculation shows that

$$\frac{1}{t_0}\|\tilde{H}_{t_0}\|_1<3\varepsilon.$$

On the other hand, if $t_0=k2^{-m}$ with $m>n$, then

$$\frac{1}{t_0}\|\tilde{H}_{t_0}\|_1=\frac{1}{k}\sum_{j=0}^{k-1}\|T_{2^{-m}}^{j}(2^m\tilde{H}_{2^{-m}})\|_1=\frac{1}{k}\sum_{j=0}^{k-1}\int 2^m\tilde{H}_{2^{-m}}(T_{2^{-m}}^{*j}1)\,d\mu\geqq$$

$$\geqq\frac{1}{2}\int_{A_N}2^m\tilde{H}_{2^{-m}}\,d\mu,$$

because $T_{2^{-m}}^{*j}1\geqq T_{2^{-n}}^{*}1\geqq T_{2^{-N}}^{*}1>\frac{1}{2}$ on A_N for each $0\leqq j\leqq k-1$. Therefore

$$\int_{A_N}2^m\tilde{H}_{2^{-m}}\,d\mu<6\varepsilon\quad(m>n). \tag{6}$$

By a similar argument we have

$$\lim_m\int_{A_M}2^m\tilde{H}_{2^{-m}}\,d\mu=0$$

for each set $A_M=\left\{T_{2^{-M}}^{*}1>\frac{1}{2}\right\}$. Since $\tilde{H}=\{\tilde{H}_t\}$ is increasing and $\lim_M A_M=X$, this demonstrates that $\lim_{t\to 0}\|\tilde{H}_t\|_1=\lim_m\|\tilde{H}_{2^{-m}}\|_1=0$. Hence we may apply Akcoglu—Krengel's differentiation theorem [3] to infer that $\text{q-}\lim_{t\to 0}\frac{1}{t}\tilde{H}_t=\tilde{h}$ exists a.e. on X, and by (6), $\int_{A_N}\tilde{h}\,d\mu<6\varepsilon$. Since $\varepsilon>0$ is arbitrary and $\text{q-}\limsup_{t\to 0}\frac{1}{t}\tilde{F}_t\leqq\tilde{h}$ a.e. on X, it follows that $\text{q-}\lim_{t\to 0}\frac{1}{t}\tilde{F}_t=0$ a.e. on A_N and hence on X.

(IV) By step (III) and again by the Akcoglu—Krengel theorem [3], $\text{q-}\lim_{t\to 0}\frac{1}{t}F_t(h)=\text{q-}\lim_{t\to 0}\frac{1}{t}E_t$ exists a.e. on $C=P=X$. Suppose $\text{q-}\liminf_{t\to 0}\frac{1}{t}F_t<h$ a.e. on an $A\in\mathfrak{F}$ with $\mu(A)>0$. Since $\text{q-}\lim_{t\to 0}\frac{1}{t}\int_0^t T_s h\,ds=h$ a.e. on X by the local ergodic theorem, it then follows that

$$\text{q-}\lim_{t\to 0}\frac{1}{t}F_t(h)=\text{q-}\liminf_{t\to 0}\frac{1}{t}F_t\quad\text{a.e. on } A.$$

But this implies that $\text{q-}\lim_{t\to 0}\frac{1}{t}F_t$ exists a.e. on A. To see this, suppose

$$\text{q-}\liminf_{t\to 0}\frac{1}{t}F_t<\text{q-}\limsup_{t\to 0}\frac{1}{t}F_t\quad\text{a.e.}$$

on some $B\in\mathfrak{F}$ with $B\subset A$ and $\mu(B)>0$. Then it follows that

$$\text{q-}\lim_{t\to 0}\frac{1}{t}F_t(h)=\min\left\{h,\ \text{q-}\limsup_{t\to 0}\frac{1}{t}F_t\right\}>\text{q-}\liminf_{t\to 0}\frac{1}{t}F_t\quad\text{a.e. on } B,$$

which is a contradiction. Here we may replace h by Nh with $N \geqq 1$. Then, letting $N \to \infty$, we see that $\text{q-}\lim_{t \to 0} \frac{1}{t} F_t = f$ exists a.e. on X. Lastly, it remains to prove that $f < \infty$ a.e. on X. To do this, let $2^{-n} = k2^{-m}$ with $m > n$. As in step (III), we then have

$$2^n \| F_{2^{-n}} \|_1 \geqq \int 2^m F_{2^{-m}}(T^*_{2^{-n}} 1)\, d\mu,$$

and hence by Fatou's lemma,

$$\int f(T^*_{2^{-n}} 1)\, d\mu \leqq 2^n \| F_{2^{-n}} \|_1 < \infty \quad (n \geqq 1),$$

which immediately implies that $f < \infty$ a.e. on $C = P = X$.

The proof is complete.

4. A remark

In Theorems 1 and 2 the positivity hypothesis imposed on the superadditive process F may be replaced by the following boundedness hypothesis: $\sup_{0<s<t} \frac{1}{s} \| F_s^- \|_p < \infty$ for some $t > 0$, where F_s^- denotes the negative part of F_s. (See e.g. [3] or [8].)

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MONOCHROMATIC DIRECTED WALKS IN ARC-COLORED DIRECTED GRAPHS

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All graphs considered here are directed and have no loops. For a directed graph D, let $V(D)$ denote the set of vertices of D, $E(D)$ its set of arcs, and $\chi(D)$ the chromatic number of D. D is *symmetric* iff $(x, y)\in E(D)\Leftrightarrow(y, x)\in E(D)$. A directed walk of length k in D is a sequence of k arcs (not necessarily distinct), $e_1, e_2, \ldots, e_k$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for $i=1, 2, \ldots, k-1$. The directed walk above is called a directed path if all the $k+1$ vertices incident with its arcs are distinct. An arc-coloring of D is a mapping of $E(D)$ into a set C of colors. A subgraph of D is monochromatic if all its arcs have the same color.

Gallai [5] and Roy [7] proved independently the first result connecting the chromatic number of a directed graph with the maximal length of a directed path in it; Every directed graph D contains a directed path of length $\chi(D)-1$. Chvátal [2] noticed that the result of Gallai and Roy implies the following extension of a result of Busolini [1]:

THEOREM A (Chvátal). *Let D be a directed graph and let k, r be positive integers such that $\chi(D)>k^r$; then in any arc-coloring of D with r colors, D contains a monochromatic directed path (and hence a monochromatic directed walk) of length k.*

In view of this theorem, the following two definitions seem natural:

DEFINITION 1. An arc-coloring of a directed graph D is k-free ($k\geqq 2$) if D does not contain a monochromatic directed walk of length k (i.e., if no directed path of length k and no directed cycle whatsoever is monochromatic).

Define also:

$$C_k(D) = \min\{r\text{: there exists a } k\text{-free arc-coloring of } D \text{ with } r \text{ colors}\}.$$

DEFINITION 2. For $k, h\geqq 2$

$$\underline{C}_k(h)=\min\{C_k(D)\text{: } D \text{ is a directed graph and } \chi(D)=h\},$$

$$\overline{C}_k(h)=\max\{C_k(D)\text{: } D \text{ is a directed graph and } \chi(D)=h\}.$$

To avoid trivialities we shall consider from now on only directed graphs D for which $\chi(D)\geqq 2$, i.e., $E(D)\neq\emptyset$. By Theorem A, for every directed graph D and every $k\geqq 2$:

$$C_k(D) \geqq \lceil \log_k \chi(D)\rceil, \tag{1}$$

where $\lceil y \rceil$ denotes the smallest integer $\geq y$, and thus

$$\bar{C}_k(h) \geq \underline{C}_k(h) \geq \lceil \log_k h \rceil, \tag{2}$$

for all $k, h \geq 2$.

In this paper we determine $\bar{C}_k(h)$ exactly for all $k, h \geq 2$, and show that

$$\lceil \log_k h \rceil = \underline{C}_k(h) \leq \bar{C}_k(h) \leq \lceil \log_k h + \log_k \log_k h + 4 \rceil.$$

We also show that if D is symmetric then $C_k(D) = \bar{C}_k(\chi(D))$.

We begin with the following definition:

DEFINITION 3. Let k, r be two positive integers, $k \geq 2$. Let D be a directed graph, and M a k-free arc-coloring of D with r colors $1, 2, \ldots, r$. For $v \in V(D)$ and $1 \leq i \leq r$, denote by $l_M(v, i)$ the maximum length of a monochromatic directed walk of color i beginning at v. ($l_M(v, i) = 0$ if no such walk exists.) With each vertex v of D associate the vector $l_M(v) = (l_M(v, 1), l_M(v, 2), \ldots, l_M(v, r))$. (We shall usually omit the index M, whenever there is no danger of confusion.)

Note that each component of $l_M(v)$ is a nonnegative integer smaller than k, since M is k-free.

The following lemma is a trivial consequence of Definition 3:

LEMMA 1. *Let D be a directed graph, and M a k-free arc-coloring of D with r colors $1, 2, \ldots, r$. Suppose $(v, v') \in E(D)$. If the color of (v, v') under M is i, then:*

$$l_M(v', i) < l_M(v, i).$$

REMARK 1. Suppose D is a directed graph, and M is a k-free arc-coloring of D with colors $1, 2, \ldots, r$. If v, v' are adjacent vertices of D, then $l(v) \neq l(v')$, by Lemma 1. Thus, the mapping $v \to l(v) (v \in V(D))$ is a proper vertex-coloring of D, with at most k^r different colors. Thus $\chi(D) \leq k^r$. This yields a proof of inequalities (1) and (2) which does not depend on Theorem A.

DEFINITION 4. For positive integers k, r let $P(k, r)$ denote the set of all functions $b: \{1, \ldots, r\} \to \{0, \ldots, k-1\}$. If $b, c \in P(k, r)$ write $b \leq c$ if $b(t) \leq c(t)$ for all $t, 1 \leq t \leq r$. Clearly $\leq$ is a partial order on $P(k, r)$. An $AC(k, r)$ is an antichain in $P(k, r)$, i.e,, a set $F \subset P(k, r)$ such that for every two vectors (=functions) $b, c \in F$ there are indices $1 \leq s, r \leq r$ such that $b(s) < c(s)$ and $c(t) < b(t)$.

We denote by $N(k, r)$ the maximal cardinality of an $AC(k, r)$.

E. Sperner [8] proved:

$$N(2, r) = \binom{r}{[r/2]},$$

and De Bruijn, Tenbergen and Kruywijk [3] proved the following generalization of Sperner's result:

THEOREM B (De Bruijn, Tenbergen and Kruywijk). *$N(k, r)$ is the number of all vectors $b \in P(k, r)$ satisfying*

$$\sum_{i=1}^{r} b(i) = \left[\frac{1}{2}(k-1)r\right],$$

i.e., $N(k, r)$ is the coefficient of $x^{[(k-1)r/2]}$ in

$$(1+x+x^2+\ldots+x^{k-1})^r.$$

As $(1+x+\ldots+x^{k-1})=(1-x^k)/(1-x)$, it follows easily that

$$N(k, r) = \sum_{i=0}^{[p/k]} (-1)^i \binom{r}{i} \binom{r+p-ki-1}{r-1}, \quad \text{where} \quad p = \left[\frac{1}{2}(k-1)r\right].$$

We shall now establish an upper bound for $C_k(D)$.

LEMMA 2. *For every directed graph D and for every integer $k \geqq 2$:*

$$C_k(D) \leqq \min \{r: N(k, r) \geqq \chi(D)\}. \tag{3}$$

PROOF. Given an integer r such that $N(k, r) \geqq \chi(D)$, we shall exhibit a k-free arc-coloring of D with r colors. Choose an $AC(k, r)$ $\{b_1, b_2, \ldots, b_{\chi(D)}\}$ of size $\chi(D)$.

Let $\gamma: V(D) \to \{1, \ldots, \chi(D)\}$ be a fixed proper vertex coloring of D. Define an arc-coloring $M: E(D) \to \{1, \ldots, r\}$ as follows: If $(v, w) \in E(D)$, let

$$M(v, w) = \min \{t: 1 \leqq t \leqq r,\ b_{\gamma(v)}(t) > b_{\gamma(w)}(t)\}.$$

($M(v, w)$ is well defined, since $\gamma(v) \neq \gamma(w)$, and $\{b_1, \ldots, b_{\chi(D)}\}$ is an $AC(k, r)$.)

It remains to show that M is k-free. Suppose (v_0, v_1), (v_1, v_2), $\ldots$, (v_{m-1}, v_m) is a monochromatic directed walk of length m in D, say, of color i. By the definition of M, $k > b_{\gamma(v_0)}(i) > b_{\gamma(v_1)}(i) > \ldots > b_{\gamma(v_m)}(i) \geqq 0$, and thus $m \leqq k-1$. Therefore M is k-free and (3) follows. □

Combining (1) and Lemma 2 we obtain:

THEOREM 1. *For every directed graph D and every $k \geqq 2$:*

$$\lceil \log_k \chi(D) \rceil \leqq C_k(D) \leqq \min \{r: N(k, r) \geqq \chi(D)\}.$$

The next lemma shows that both bounds in Theorem 1 are best possible, and that the upper bound is attained whenever D is symmetric.

LEMMA 3. *Let $h \geqq 2$ be an integer.*

(i) *There exists a directed graph T with $\chi(T) = h$ such that*

$$C_k(T) = \lceil \log_k h \rceil \tag{4}$$

for every $k \geqq 2$.

(ii) *If G is symmetric and $\chi(G) = h$, then*

$$C_k(G) = \min \{r: N(k, r) \geqq h\}. \tag{5}$$

for every $k \geqq 2$.

PROOF. (i) Given $h \geqq 2$, let T be a transitive tournament on h vertices, that is: $V(T) = \{v_1, v_2, \ldots, v_n\}$, and $E(T) = \{(v_i, v_j): h \geqq i > j \geqq 1\}$. Obviously $\chi(T) = h$ and thus, by Theorem 1, for every $k \geqq 2$, $C_k(T) \geqq \lceil \log_k h \rceil$. In order to establish (4) we shall exhibit, for every $k \geqq 2$, a k-free arc-coloring of T with r colors, where

$$r = \lceil \log_k h \rceil. \tag{6}$$

Given $k \geqq 2$, define r by (6). Obviously $k^r \geqq h$. For $1 \leqq i \leqq h$, let $b_i \in P(k, r)$ be the k-ary representation of the integer $i-1$, i.e., $b_i=(\alpha_1, \ldots, \alpha_r)$, where $0 \leqq \alpha_v < k$ for $1 \leqq v \leqq r$ and $i-1 = \sum_{v=1}^{r} \alpha_v k^{v-1}$. (Note that $0 \leqq i-1 < k^r$.) Define a k-free arc-coloring of T with r colors 1, 2, ..., r exactly as in the proof of Lemma 2:

For $h \geqq i > j \geqq 1$, color the arc (v_i, v_j) of T with color s, where

$$s = \min \{t: 1 \leqq t \leqq r, \ b_i(t) > b_j(t)\}.$$

(Such an s exists since b_i represents a larger number than b_j.) The argument used in the proof of Lemma 2 shows that this coloring is indeed k-free. This establishes (4).

(ii) Let G be a symmetric directed graph satisfying $\chi(G)=h$. By Theorem 1, for every $k \geqq 2$,

$$C_k(G) \leqq \min \{r: \ N(k, r) \geqq h\}.$$

In order to establish (5) we will show that for $k \geqq 2$, if there exists a k-free arc-coloring of G with r colors, then $N(k, r) \geqq h$.

Given $k \geqq 2$, suppose M is a k-free arc-coloring of G with r colors 1, ..., r. By Dilworth's Theorem (see [4]), the partially ordered set $P(k, r)$ is the union of $N(k, r)$ chains $H_1, H_2, \ldots, H_{N(k,r)}$. Define a vertex-coloring f: $V(G) \to \{1, 2, \ldots, N(k, r)\}$ as follows: If $v \in V(G)$ let

$$f(v) = \min \{t: 1 \leqq t \leqq N(k, r), \ l_M(v) \in H_t\}.$$

Since G is symmetric, Lemma 1 implies that if $(v, w) \in E(G)$ then neither $l_M(v) \leqq l_M(w)$ nor $l_M(w) \leqq l_M(v)$ holds. This means that $f(v) \neq f(w)$ and that f is a proper vertex-coloring of G with $N(k, r)$ colors. Thus $N(k, r) \geqq h$ and (5) follows. □

Combining Lemma 3 and Theorem 1 we obtain:

THEOREM 2. *For every two integers* $k, h \geqq 2$:

$$C_k(h) = \lceil \log_k h \rceil, \tag{7}$$

$$\bar{C}_k(h) = \min \{r: \ N(k, r) \geqq h\}. \ \square \tag{8}$$

REMARK 2. We can prove that there are positive constants c_1, c_2, say $c_1=1$ and $c_2=4$, such that:

$$\min \{r: \ N(k, r) \geqq h\} \leqq \lceil \log_k h + c_1 \log_k \log_k h + c_2 \rceil \tag{9}$$

for every $k, h \geqq 2$.

This shows that $\bar{C}_k(h)$ is not very far from $\underline{C}_k(h)$ and thus the lower and upper bounds for $C_k(D)$, given in Theorem 1, are quite close.

The proof of (9) depends on the trivial estimate

$$N(k, r) \geqq |P(k, r)| / ((k-1) \cdot r + 1) \geqq k^{r-1}/r.$$

We omit the detailed proof of (9), since it is rather lengthy and not very complicated.

REMARK 3. It is well known that the problem of deciding whether the chromatic number of a given undirected graph G is greater than 3 is NP-Complete, even under

rather severe restrictions on G (see [6, p. 191]). Since $\bar{C}_2(3)=3<4=\bar{C}_2(4)$, part (ii) of Lemma 3 implies that the problem of deciding whether $C_2(D)\leqq 3$ is NP-complete, even if D is a directed symmetric graph.

REMARK 4. Let T be a transitive tournament on h vertices and let G be a complete symmetric directed graph on h vertices. By Lemma 3, for every $k\geqq 2$

$$C_k(T) = \underline{C}_k(h) \leqq \bar{C}_k(h) = C_k(G).$$

Clearly G can be obtained from T by adding $h(h-1)/2$ arcs, one at a time. If H' is obtained from H by adding one arc, then clearly

$$C_k(H) \leqq C_k(H') \leqq C_k(H)+1.$$

It follows that for every $k, h\geqq 2$ and for every m satisfying $\underline{C}_k(h)\leqq m\leqq \bar{C}_k(h)$, there is a directed graph D that satisfies $\chi(D)=h$ and $C_k(D)=m$.

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A NEW CHARACTERIZATION OF HERMITE POLYNOMIALS

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In 1913 Edmund Landau [4] proved that if f is continuous together with its first and second order derivatives in the interval $[0, 1]$, and if $\|f\|=1$, $\|f''\|=4$, then

$$\|f'\| \leqq 4. \tag{1.1}$$

He showed that 4 is the best constant and the result is not necessarily true for an interval of length <1.

Landau's problem was generalized in may directions. We formulate some of these results as follows. Let $n\geqq 2$. We consider the class of functions $f(x)$ from R to R $(R=(-\infty, \infty))$ that are bounded and have a bounded n^{th} derivative $f^{(n)}(x)$. Some of these theorems may now be spelled out as follows.

THEOREM A (Landau). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \quad \|f''\| \leqq 8, \tag{1.2}$$

then

$$\|f'\| \leqq 4. \tag{1.3}$$

THEOREM B (G. E. Silov). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \quad \|f'''\| \leqq 24 \tag{1.4}$$

then

$$\|f'\| \leqq 3, \quad \|f''\| \leqq 12. \tag{1.5}$$

THEOREM C (G. E. Silov). *If $f(x)$ is such that*

$$\|f\| \leqq 1, \quad \|f^{(\text{iv})}\| \leqq \frac{384}{5}, \tag{1.6}$$

then

$$\|f'\| \leqq \frac{16}{5}, \quad \|f''\| \leqq \frac{48}{5}, \quad \|f'''\| \leqq \frac{192}{5}. \tag{1.7}$$

As remarked by Prof. I. J. Schoenberg [5] a lively activity on this subject culminated in 1939 with Kolmogorov's remarkable paper [3] where Landau's R-problem was solved for all values of n. Moreover, he gave another proof of this theorem using Euler splines. The analogue of Kolmogorov's theorem for the half line R_+ has been established by A. Cavaretta and I. J. Schoenberg [6]. For further generalization of Landau's inequality to linear operators we refer to the work of E. Hille [2] and references in it.

In 1957 J. Balázs and P. Turán [1] obtained certain interesting inequalities which arise from their consideration of (0, 2) interpolation on π abscissas $(\pi_n(x)=(1-x^2)\cdot P'_{n-1}(x))$, $P_n(x)$ being the Legendre polynomial of degree n. By (0, 2) interpolation they mean the problem of finding interpolatory polynomial $R_n(x)$ of degree $\leqq 2n-1$ for which

$$R_n(x_{kn})=\alpha_{kn}, \quad R''_n(x_{kn})=\beta_{kn}, \quad k=1, 2, \ldots, n, \tag{1.8}$$

are prescribed. From this consideration they proved the following theorem.

THEOREM D (J. Balázs and P. Turán). *Let n be even and assume that for a polynomial $Q_{2n-1}(x)$ of degree $\leqq 2n-1$ we have*

$$|Q_{2n-1}(x_{kn})| \leqq A, \quad |Q''_{2n-1}(x_{kn})| \leqq B, \quad k=1, 2, \ldots, n. \tag{1.9}$$

Then for $-1\leqq x\leqq 1$ we have

$$|Q_{2n-1}(x)| \leqq \pi n A+\frac{\pi^5 B}{n}, \tag{1.10}$$

$$|Q'_{2n-1}(x)| \leqq \pi^8 n^{5/2} A+\pi^5 B n^{1/2}. \tag{1.11}$$

For analogous theorems based on the other choice of nodes we refer to the work of the author [7].

The object of this paper is to consider the following problem: Let $f(x)$ be a polynomial of degree $\leqq n$ such that

$$\|f(x)\|^2=\|P_n(x)\|^2=\int_{-\infty}^{\infty} P_n^2(t)e^{-t^2}\,dt. \tag{1.12}$$

Next, suppose r is a fixed positive integer $r\geqq 2$. Given $\|P_n(x)\|$ and $\|P_n^{(r)}(x)\|$ the problem is to estimate $\|P_n^{(q)}(x)\|$ for $q=1, 2, \ldots, r-1$. We limit ourselves to the case $r=2$, $r=3$ and $r=4$. Now we state the main results of this paper as follows:

THEOREM 1. *Let $P_n(x)$ be a given polynomial of degree $\leqq n$. Let the norm of $P_n(x)$, $P''_n(x)$ be given by* (1.12). *Then we have the following inequality:*

$$\|P'_n\|^2 \leqq \frac{1}{2(2n-1)}\|P''_n\|^2+\frac{2n^2}{2n-1}\|P_n\|^2. \tag{1.13}$$

Equality holds if and only if $P_n(x)=cH_n(x)$ where $H_n(x)$ is the Hermite polynomial of degree n orthogonal to all polynomials of degree $\leqq n-1$ with weight function e^{-x^2} on $(-\infty, \infty)$.

THEOREM 2. *Let $P_n(x)$ be a given algebraic polynomial of degree $\leqq n$. Let $\|P_n\|$, $\|P'''\|$ be given by* (1.12). *Then we have*

$$\|P'_n\|^2 \leqq \frac{1}{4}\frac{\|P'''_n\|^2}{3n^2-6n+2}+\frac{2n^2(2n-3)}{3n^2-6n+2}\|P_n\|^2, \tag{1.14}$$

and

$$\|P''_n\|^2 \leqq \frac{(2n-1)\|P'''_n\|^2}{2(3n^2-6n+2)}+\frac{4n^2(n-1)^2\|P_n\|^2}{3n^2-6n+2}. \tag{1.15}$$

Equality holds again for only $P_n(x)=cH_n(x)$.

THEOREM 3. *Let $P_n(x)$ be a given algebraic polynomial of degree $\leqq n$. Let $\|P_n\|$, $\|P_n^{(iv)}\|$ be given by* (1.12). *Then we have*

$$\|P_n'\|^2 \leqq \frac{n^2(4n^3-9n^2+10n-1)\|P_n\|^2}{(2n-1)(2n^3-9n^2+11n-3)} + \frac{\|P_n^{(v)}\|^2}{16(2n^3-9n^2+11n-3)}, \tag{1.16}$$

and

$$\|P_n''\|^2 \leqq \frac{2n^2(n-1)^2(2n-5)\|P_n\|^2}{(2n^3-9n^2+11n-3)} + \frac{\|P_n^{(iv)}\|^2}{16(2n^3-9n^2+11n-3)}, \tag{1.17}$$

and

$$\|P_n'''\|^2 \leqq A_n\|P_n\|^2 + B_n\|P_n^{(iv)}\|^2 \tag{1.18}$$

where

$$A_n = \frac{4n^2(n-1)^2(n-2)^2(4n^3-18n^2+24n-11)}{(2n-1)(3n^2-12n+11)(2n^3-9n^2+11n-3)},$$

$$B_n = \frac{(n-1)^2(n-2)^2+4(2n-1)(2n-3)(2n^3-9n^2+11n-3)}{8(2n-1)(3n^2-12n+11)(2n^3-9n^2+11n-3)}.$$

PROOF OF THEOREM 1. Proof of Theorem 1 is based on the following identity:

$$(4n-2)\int_{-\infty}^{\infty}(P_n'(x))^2e^{-x^2}\,dx = 4n^2\int_{-\infty}^{\infty}P_n^2(x)e^{-x^2}\,dx + \int_{-\infty}^{\infty}(P_n''(x))^2e^{-x^2}\,dx - \tag{2.1}$$

$$-\int_{-\infty}^{\infty}(P_n''(x)-2xP_n'(x)+2nP_n(x))^2e^{-x^2}\,dx.$$

Since the differential equation satisfied by the well known Hermite polynomial of degree n is given by

$$H_n''(x)-2xH_n'(x)+2nH_n(x) = 0 \tag{2.2}$$

it is clear that (1.13) becomes equality only if $P_n(x)=cH_n(x)$. Now we turn to prove the identity (2.1). On integration by parts we have

$$\int_{-\infty}^{\infty}(P_n'(x))^2e^{-x^2}\,dx = -\int_{-\infty}^{\infty}P_n(x)P_n''(x)e^{-x^2}\,dx + 2\int_{-\infty}^{\infty}xP_n(x)P_n'(x)e^{-x^2}\,dx, \tag{2.3}$$

and

$$\int_{-\infty}^{\infty}(P_n'(x))^2e^{-x^2}\,dx = 2\int_{-\infty}^{\infty}x^2(P_n'(x))^2e^{-x^2}\,dx - 2\int_{-\infty}^{\infty}xP_n'(x)P_n''(x)e^{-x^2}\,dx. \tag{2.4}$$

On multiplying (2.3) by $4n$ and (2.4) by (-2) and then adding we obtain

$$(4n-2)\int_{-\infty}^{\infty}(P_n'(x))^2e^{-x^2}\,dx = 4\int_{-\infty}^{\infty}xP_n'(x)P_n''(x)e^{-x^2}\,dx -$$

$$-4n\int_{-\infty}^{\infty}P_n(x)P_n''(x)e^{-x^2}\,dx - 4\int_{-\infty}^{\infty}x^2(P_n'(x))^2e^{-x^2}\,dx +$$

$$+8n\int_{-\infty}^{\infty}xP_n(x)P_n'(x)e^{-x^2}\,dx.$$

But it is easy to verify that this is equivalent to (2.1). This proves Theorem 1.

PROOF OF THEOREM 2. From (1.13) we have

$$\|P_n''\|^2 \leqq \frac{1}{2(2n-3)}\|P_n'''\|^2+\frac{2(n-1)^2}{2(n-3)}\|P_n'\|^2 \leqq$$

$$\leqq \frac{1}{2(2n-3)}\|P_n'''\|^2+\frac{2(n-1)^2}{2n-3}\left\{\frac{\|P_n''\|^2}{2(2n-1)}+\frac{2n^2}{2n-1}\|P_n\|^2\right\}.$$

In other words

$$\left(1-\frac{(n-1)^2}{(2n-3)(2n-1)}\right)\|P_n''\|^2 \leqq \frac{\|P_n'''\|^2}{2(2n-3)}+\frac{4n^2(n-1)^2\|P_n\|^2}{(2n-3)(2n-1)}.$$

But this is equivalent to (1.15). Next, we turn to prove (1.14). Here we make use of (1.13) and (1.15). We have

$$\|P_n'\|^2 \leqq \frac{1}{2(2n-1)}\|P_n''\|^2+\frac{2n^2}{2n-1}\|P_n\|^2 \leqq$$

$$\leqq \frac{1}{2(n-1)}\left\{\frac{(2n-1)\|P_n'''\|^2}{2(3n^2-6n+2)}+\frac{4n^2(n-1)^2\|P_n\|^2}{3n^2-6n+2}\right\}+\frac{2n^2}{2n-1}\|P_n\|^2.$$

This proves (1.14) as well. A simple computation shows that equality holds in each case for $P_n(x)=cH_n(x)$. The proof of Theorem 3 is similar to the proof of Theorem 2, so we omit the details.

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AN ALMOST SUBIDEMPOTENT RADICAL PROPERTY

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1. Almost subidempotent radical properties are radical properties with idempotent radical rings. Following V. A. Andrunakievič [1] a hereditary almost subidempotent radical property is simply called subidempotent. Instances of subidempotent radical properties are the von Neumann [14], [4], regular class, the complementary radical properties of supernilpotent radical properties (V. A. Andrunakievič [1], [2]) and many others. Instances of nonhereditary almost subidempotent radicals are the radical property of de la Rosa [15], the two radical properties of F. A. Szász [17], [18], and the classes $I_{\mathscr{Y}}$ of all idempotent $\mathscr{Y}$-radical rings, where $\mathscr{Y}$ is a weakly supernilpotent radical property (G. Tzintzis [20]).

In the present paper we will study an interesting new radical property $\mathscr{X}$, which is the upper radical property determined by the class E of all rings without admissible ideals which are not ideals.

At first, it is proved that $\mathscr{X}$ is a non trivial almost subidempotent radical property with $\mathscr{X} \prec \mathscr{P}$, where $\mathscr{P}$ is the T. L. Jenkins [10] radical property, and that $\mathscr{X}$ is far removed from hereditariness in the sense that for every $\mathscr{X}$-radical ring $R \neq 0$ there exists an ideal $I \neq 0$ which is not an $\mathscr{X}$-radical ring.

Even, it is proved that $\mathscr{X}$ is also far removed from hereditariness in the sense of N. Divinsky and A. Suliński [8].

Next, examining E-rings, that is, rings R contained in E, it is proved that for an $\mathscr{F}$-radical idempotent E-ring R, where $\mathscr{F}$ is the upper radical property determined by all fields, for each ideal A, A^2 is an idempotent ring and $A^2 = \sum_{a \in A} (a)_R^2$ holds.

However, it is shown by an example that there exist $\mathscr{F}$-radical idempotent rings R with the above properties which are not E-rings. As a corollary it follows that each Ψ-radical idempotent ring R is $\mathscr{X}$-radical, where Ψ is the upper radical property determined by all rings without nonzero transfinitely nilpotent ideals (G. Tzintzis [19]). Restricting our interest to $\mathscr{G}$-radical idempotent E-rings, where $\mathscr{G}$ is the radical of B. Brown and N. H. McCoy [3], it is proved that $\mathscr{B}(R) = \operatorname{Ann} R = Z(R)$ holds, where $\mathscr{B}$ and $Z(R)$ are the Baer's lower radical property and the center of R respectively. From this last result, it is obtained that all commutative $\mathscr{G}$-radical idempotent rings are $\mathscr{X}$-radical and, more generally, that if R is an idempotent $\mathscr{G}$-radical ring with the property for each prime homomorphic image $R' \neq 0$ to hold $Z(R') \neq 0$, then R is an $\mathscr{X}$-radical ring. Also, many other known and unknown corollaries are obtained. Finally, the question is raised whether $\mathscr{X}$ coincides with some almost subidempotent radical property of the form $I_{\mathscr{Y}}$, where $\mathscr{Y}$ is some known weakly supernilpotent radical property.

2. Throughout this paper all rings considered will be associative. The termi-

nology and basic results of radical theory can be found in [6], [1], [2]. With the symbols $(a)_{(a)_R}$, $\langle a\rangle$ and $A \trianglelefteqq R$ we mean the ideal of $(a)_R$ generated by a, the additive subgroup generated by a, and that A is an ideal of R, respectively.

Suppose that E is the class of all rings R without admissible ideals which are not ideals. Evidently, $E \neq \emptyset$ holds, since the class E contains both the class of all prime simple rings and the class of all zerorings. Moreover, it is known that E also contains all hereditarily idempotent rings. The class E is hereditary as it can be shown by a straightforward reasoning.

Now, let $\mathscr{X}$ be the upper radical property determined by the class E. Since E contains the class of all zero rings, $\mathscr{X}$ consists of idempotent rings. Evidently, $\mathscr{X} \trianglelefteqq \mathscr{P}$ holds, where $\mathscr{P}$ is the T. L. Jenkins [10] radical property, that is, the upper radical property determined by the class of all prime simple rings. Also, $\mathscr{X}$ is not a trivial radical property, that is, $\mathscr{X} \neq \{\{0\}\}$, since, as it is shown, the following commutative idempotent and $\mathscr{B}$-radical ring is $\mathscr{X}$-radical.

EXAMPLE 2.1 ([6], Example 3). It is well known that the commutative F-algebra A, with basis the set $\{x_a | 0<a<1\}$ and with multiplication $x_a x_b = x_{a+b}$ if $a+b<1$ and $x_a x_b = 0$ if $a+b \trianglerighteqq 1$, where a is a real number and F an arbitrary field, thought of as a ring is idempotent and $\mathscr{B}$-radical. In Proposition 2.6 we shall see that A is $\mathscr{X}$-radical.

Now, with the following proposition we can show that $\mathscr{X}$ is far removed from hereditariness.

PROPOSITION 2.2. *For every nonzero $\mathscr{X}$-radical ring there exists a nonzero ideal which is not $\mathscr{X}$-radical.*

PROOF. Obviously, if R is a nonzero $\mathscr{X}$-radical ring then it contains U, W such that

$$0 \subset W \subset U \subset R$$

and $W \trianglelefteqq U$, $U \trianglelefteqq R$, $W \ntrianglelefteqq R$ hold. Consequently, if W^* is the ideal of R generated by W, then by Andrunakievič's lemma, $W^{*3} \subseteq W$ and since W is not an ideal of R, $W^{*3} \subset W \subset W^*$. This latter means that the nonzero ideal W^* of R is not an idempotent ring and consequently is not an $\mathscr{X}$-radical ring.

Next, we study E-rings.

PROPOSITION 2.3. *The following statements are equivalent.*

(1) $R \in E$,

(2) $(a)_R = (a)_{(a)_R} \quad \forall a \in R$,

(3) $(a)_R = (a)_R^2 + \langle a\rangle \quad \forall a \in R$,

(4) $Ra \cup aR \subseteq (a)_R^2 + \langle a\rangle \quad \forall a \in R$.

PROOF. (1)⇒(2)⇒(3)⇒(4). If $R \in E$ then $(a)_{(a)_R} \trianglelefteqq R \Rightarrow (a)_R \subseteq (a)_{(a)_R} \Rightarrow (a)_R = (a)_{(a)_R}$, $\forall a \in R$. Now, the subset $X = (a)_R^2 + \langle a\rangle$ is a two-sided ideal of $(a)_R$. Indeed, X is an additive group and $X \subseteq (a)_R$ holds. Moreover, we have

$$(a)_R X = (a)_R[(a)_R^2 + \langle a\rangle] = (a)_R^3 + (a)_R\langle a\rangle \subseteq (a)_R^2 \subseteq (a)_R^2 + \langle a\rangle = X$$

and similarly $X(a)_R \subseteq X$. Consequently, $(a)_{(a)_R} \subseteq (a)_R^2 + \langle a\rangle \subseteq (a)_R = (a)_{(a)_R}$, $\forall a \in R$. Finally, from this latter it is evident that $Ra \cup aR \subseteq (a)_R^2 + \langle a\rangle$, $\forall a \in R$ holds.

Conversely, (4)$\Rightarrow$(1). Indeed, if $R \notin E$, then there are subrings U and W such that $W \trianglelefteq U \trianglelefteq R$ but $W \not\trianglelefteq R$. Further, also $W \trianglelefteq W^* \trianglelefteq R$ holds, where W^* denotes the ideal of R generated by W. Now, there exists an element $a \in W$ such that $Ra \cup aR \not\subset W$. Thus, by (4) we have $Ra \cup aR \subset (a)_R^2 + \langle a\rangle \subset (a)_R^2 + W$. We shall get a contradiction if we exhibit $(a)_R^2 \subset W$. In fact, we have

$$(a)_R^2 = \left(\langle a\rangle + R\langle a\rangle + \langle a\rangle R + R\langle a\rangle R\right)^2 \subset$$

$$\subset WW^* + W^*W + R\langle a\rangle W^* \subset W + RaW^* \subset W + \left((a)_R^2 + \langle a\rangle\right)W^* \subset$$

$$\subset W + W^{*2}W^* + WW^* \subset W + W^{*3} \subset W.$$

This contradiction proves the validity of (1).

LEMMA 2.4. *If the ring R contains $a \neq 0$ such that $aR \subseteq (a)_R^2 + \langle a\rangle$ $(Ra \subseteq (a)_R^2 + \langle a\rangle)$ and $aR \not\subseteq (a)_R^2$ $(Ra \not\subseteq (a)_R^2)$ hold, then there exists a non trivial surjective ring-homomorphism*

$$\varphi\colon R \to \frac{(k_0)_Z}{(\lambda_0)_Z} \trianglelefteq \frac{Z}{(\lambda_0)_Z}, \quad \lambda_0 \neq 1, \quad k_0 \geq 1, \quad k_0 | \lambda_0.$$

PROOF. Evidently, for each $r \in R$ there exist $x \in (a)_R^2$ and $k \in Z$ such that $ar = x + ka$. We thus define the following mapping:

$$\varphi\colon R \ni r \to k + (\lambda_0)_Z \in \frac{Z}{(\lambda_0)_Z},$$

where λ_0 is the smallest positive integer, if such exist, with the property $\lambda_0 a \in (a)_R^2$, or $\lambda_0 = 0$ in the other case. In every case it is $\lambda_0 \neq 1$ since $aR \not\subseteq (a)_R^2 \Rightarrow (a)_R^2 \subset (a)_R \Rightarrow$ $\Rightarrow a \notin (a)_R^2$. Now, it is easy to show that φ is well defined and also a ring-homomorphism. Finally, φ is not trivial since $aR \not\subseteq (a)_R^2$, whence there exists $k_0 \in Z$, $k_0 \geq 1$, $k_0 \neq \lambda_0$, $k_0 | \lambda_0$, such that $0 \neq \varphi(R) = \dfrac{(k_0)_Z}{(\lambda_0)_Z} \trianglelefteq \dfrac{Z}{(\lambda_0)_Z}$, $\lambda_0 \neq 1$.

COROLLARY 2.5. *If R is an idempotent ring and contains $a \neq 0$ such that $aR \subseteq (a)_R^2 + \langle a\rangle$ $(Ra \subseteq (a)_R^2 + \langle a\rangle)$ and $aR \not\subseteq (a)_R^2$ $(Ra \not\subseteq (a)_R^2)$ hold, then there exists a non trivial surjective ring-homomorphism*

$$\varphi\colon R \to \frac{Z}{(\lambda_0)_Z}, \quad \lambda_0 \neq 1.$$

PROOF. Indeed, in this case $\dfrac{(k_0)_Z}{(\lambda_0)_Z}$, where $\lambda_0 \neq 1$, $k_0 \geq 1$, $k_0 \neq \lambda_0$, $k_0 | \lambda_0$, must be an idempotent ring. Consequently, $(k_0^2)_Z + (\lambda_0)_Z = (k_0)_Z$ must hold, which means that there exist integers e, π such that $k_0 = ek_0^2 + \pi\lambda_0$ holds. Now, if $\lambda_0 = k_0 s_0$, where $s_0 \neq 1$ since $\lambda_0 \neq k_0, \Rightarrow 1 = ek_0 + \pi s_0 \Rightarrow (e, s_0) = 1$ which implies that for each $t \in Z$ there exists a unique k mod s_0 such that $ke \equiv t \bmod s_0 \Rightarrow kek_0 \equiv tk_0 \bmod s_0 k_0$. Thus, each element $tk_0 \bmod s_0 k_0$ has a representative of the form kek_0, $k \in Z$, and

so we can construct the mapping

$$\frac{(k_0)_Z}{(s_0k_0)_Z} \ni ke k_0+(s_0k_0)_Z \to k+(s_0)_Z \in \frac{Z}{(s_0)_Z}, \quad s_0 \neq 1,$$

which, as it is easy to show, is a surjective ring homomorphism.

PROPOSITION 2.6. *Every idempotent $\mathscr{B}$-radical ring R is $\mathscr{X}$-radical.*

PROOF. Suppose that there exists an idempotent $\mathscr{B}$-radical ring $R\neq 0$ which is not $\mathscr{X}$-radical. Then, evidently, R could be homomorphically mapped on a nonzero E-ring which will also be idempotent and $\mathscr{B}$-radical. Thus, without any restriction of generality we can suppose R to be an E-ring. Also, it is evident that $\operatorname{Ann}_l R\neq R$ holds, since $R^2=R$. Consequently, there exists an ideal A of R such that $A\supset \operatorname{Ann}_l R$, $A^{n_0}\subseteq \operatorname{Ann}_l R$ and $A^{n_0-1}\nsubseteq \operatorname{Ann}_l R$, for some integer $n_0\geqq 2$ ([6], p. 56). Obviously, $A^{n_0+1}=0$ holds, and there must exist elements $a_1, a_2, \ldots, a_{n_0-1}\in A$ and $r\in R$ with $a_1a_2\ldots a_{n_0-1}r\neq 0$, since $A^{n_0-1}\nsubseteq \operatorname{Ann}_l R$. If n_0 is even and $n_0\geqq 4$ we take $b=a_1a_2\ldots a_{(n_0+2)/2}$, where $(n_0+2)/2\leqq n_0-1$ since $n_0\geqq 4\Leftrightarrow 2n_0-2\geqq n_0+2\Leftrightarrow n_0-1\geqq \geqq (n_0+2)/2$. Consequently, by $(b)_R\subseteq A^{(n_0+2)/2}\Rightarrow (b)_R^2\subseteq A^{n_0+2}\subseteq A^{n_0+1}=0$ and by Proposition 2.3 we have $(b)_R=(b)_R^2+\langle b\rangle=\langle b\rangle$. Therefore, since obviously, $br\neq 0$ holds, by Corollary 2.5 there must exist a non trivial surjective ring-homomorphism $\varphi\colon R\to Z/(\lambda_0)_Z$, where $\lambda_0\neq 1$, which is contradictory to being a $\mathscr{B}$-radical ring. If now, n_0 is odd taking $b=a_1a_2\ldots a_{(n_0+1)/2}$, where $(n_0+1)/2\leqq n_0-1$ since $n_0\geqq 3\Leftrightarrow$ $\Leftrightarrow 2n_0-2\geqq n_0+1\Leftrightarrow n_0-1\geqq (n_0+1)/2$, we are heading for the same contradiction. Thus, $A^2\subseteq \operatorname{Ann}_l R$. Now, it is evident that $R/A\neq 0$, since if $A=R$ then $R^3=A^3=0$ would hold. Consequently there exists an ideal A_1 of R with $A_1\supset A$ and $A_1^2\subseteq A$ since R/A is a $\mathscr{B}$-radical ring. A_1 is a nilpotent ideal, since $A_1^6=(A_1^2)^3\subseteq A^3=0$. Moreover, if $A_1^2\nsubseteq \operatorname{Ann}_l R$ held, then the choice of A_1 for A as above would lead to a contradiction. Thus, $A_1^2\subseteq \operatorname{Ann}_l R$. Continuing in this way we can construct an ascending chain of ideals $\operatorname{Ann}_l R\subset A\subset A_1\subset A_2\subset\ldots\subseteq A_a\subset\ldots$, where for the limit ordinal numbers a we define $A_a=\bigcup_{c<a} A_c$, thus preserving the property $A_a^2\subseteq \operatorname{Ann}_l R$, for every ordinal number a. Now, if the set R has the ordinal number v, then after at most v steps this chain must terminate. We may then consider the smallest ordinal τ such that $A_\tau=A_{\tau+1}=\ldots$. This ideal A_τ and R are different, since otherwise $R^3=A_\tau^3=0$, a contradiction.

Consequently, there exists an ideal B of R with $B\supset A_\tau$ and $B^2\subseteq A_\tau$. B is also a nilpotent ideal, since $B^5=B^4\cdot B=(B^2)^2B\subseteq A_\tau^2\cdot B\subseteq \operatorname{Ann}_l R\cdot B=0$, but $B^2\nsubseteq \operatorname{Ann}_l R$. Thus, by Corollary 2.5, there exists a non trivial surjective ring-homomorphism $\varphi\colon R\to Z/(\lambda_0)_Z$ where $\lambda_0\neq 1$, a contradiction.

PROPOSITION 2.7. *For the idempotent $\mathscr{F}$-radical ring R, the following conditions are equivalent:*

(1) $R\in E$,

(2) $aR\cup Ra\subseteq (a)_R^2$, $\forall a\in R$.

PROOF. (2)$\Rightarrow$(1). It is evident by Proposition 2.3.

(1)$\Rightarrow$(2). Indeed, by Proposition 2.3, $aR\cup Ra\subseteq (a)_R^2+\langle a\rangle$, $\forall a\in R$ holds. Now, if there exists an $a\in R$ such that $aR\nsubseteq (a)_R^2$ or $Ra\nsubseteq (a)_R^2$ then, by Corollary 2.5,

there exists a non trivial surjective ring-homomorphism $\varphi\colon R\to Z/(\lambda_0)_Z$ $\lambda_0\neq 1$. Consequently, if the prime $p|\lambda_0$ then $R\to Z/(\lambda_0)_Z\to \dfrac{Z/(\lambda_0)_Z}{(p)_Z/(\lambda_0)_Z}\cong Z(p)$, which is a contradiction, since R is an $\mathscr{F}$-radical ring.

COROLLARY 2.8. *For the $\mathscr{F}$-radical ring R with unity the following conditions are equivalent:*

(1) $R\in E$,

(2) *R is a hereditarily idempotent ring.*

PROOF. (2)$\Rightarrow$(1) is trivial.

(1)$\Rightarrow$(2). Evidently, R is an idempotent ring, since it contains a unity element. Consequently, by Proposition 2.7, $aR\cup Ra\subseteq(a)_R^2$, $\forall a\in R$ holds, and it is implied that $a=1\cdot a\in(a)_R^2$, $\forall a\in R$.

In general, we have the following

COROLLARY 2.9. *For the $\mathscr{F}$-radical ring R which is also radical with respect to the F. Szász* [18] *radical property $\mathscr{S}$, the following conditions are equivalent:*

(1) $R\in E$,

(2) *R is a hereditarily idempotent ring.*

PROOF. Indeed, $\mathscr{S}$ is an almost subidempotent radical property and so R is an idempotent ring. Consequently, by Proposition 2.7, $aR\cup Ra\subseteq(a)_R^2$, $\forall a\in R$ holds. Now, if there exists an $a\in R$ such that $a\notin(a)_R^2$, then the nonzero homomorphic image $R/(a)_R^2$ will contain nonzero two-sided annihilators, a contradiction.

COROLLARY 2.10. *The antisimple and $\mathscr{S}$-radical ring R is $\mathscr{X}$-radical.*

PROOF. In fact, if there exists an antisimple and $\mathscr{S}$-radical ring $R\neq 0$ which is not $\mathscr{X}$-radical, then there exists a nonzero antisimple and $\mathscr{S}$-radical ring $K\in E$. Evidently, K is idempotent as an $\mathscr{S}$-radical ring and even $\mathscr{F}$-radical, since $\mathscr{B}_\varphi\subseteq\mathscr{F}$, where $\mathscr{B}_\varphi$ is the antisimple radical class. Consequently, by Corollary 2.9, K is a hereditarily idempotent ring, a contradiction.

OBSERVATION 2.11. The existence of nonzero antisimple and $\mathscr{S}$-radical rings is an open question which was set by F. Szász in his paper [18], Problem 3.

PROPOSITION 2.12. *If the ring R is* (1) *idempotent,* (2) *$\mathscr{F}$-radical,* (3) *E-ring, then A^2 is an idempotent ring and $A^2=\sum\limits_{a\in A}(a)_R^2$ holds, for every ideal A of R.*

PROOF. By Proposition 2.3, $(a)_R=(a)_R^2+\langle a\rangle$, $\forall a\in R$, holds, since R is an E-ring. Consequently, we have $(a)_R^2=((a)_R^2+\langle a\rangle)^2\subseteq(a)_R^3+\langle a^2\rangle\subseteq(a)_R^2$, $\forall a\in R$, which implies $(a)_R^2=(a)_R^3+\langle a^2\rangle$, $\forall a\in R$. Therefore, if we take an $a\in R$, then for each $r\in R$, there exists $y\in(a)_R^3$ and $\lambda\in Z$ such that $ar=y+\lambda a^2$, since by Proposition 2.7, $aR\subseteq(a)_R^2$ holds. Now, taking the ideal $I[x]=\{f(x)|f(x)\in Z[x]$ with $f(a)a\in(a)_R^3\}$ of $Z[x]$ we construct the mapping

$$\varphi\colon R\ni r\to \lambda x+I[x]\in Z[x]/I[x]$$

where $ar=y+\lambda a^2$. It is easy to show that φ is well-defined and is also a zero ring-homomorphism. Consequently, $aR\subseteq(a)_R^3$, $\forall a\in R$ holds. Thus, it is evident that $a^2\in(a)_R^3$, from which it is implied that $(a)_R^2=(a)_R^3$ and $((a)_R^2)^2=(a)_R^4=(a)_R^3(a)_R=$ $=(a)_R(a)_R=(a)_R^2$.

Now, suppose A to be an ideal of R. Evidently, $(a)_R^2\subseteq A^2$, $\forall a\in A$, whence $\sum_{a\in A}(a)_R^2\subseteq A^2$. Conversely, if $a, b\in A$ then $ab\in(a)_R^2$, since $aR\subseteq(a)_R^2$, $\forall a\in R$, and consequently, $A^2\subseteq\sum_{a\in A}(a)_R^2$. Finally, A^2 is an idempotent ideal of R, since on the one hand we have $(A^2)^2\supseteq((a)_R^2)^2=(a)_R^2$, $\forall a\in A$, $\Rightarrow(A^2)^2\supseteq\sum_{a\in A}(a)_R^2=A^2$ and on the other hand $(A^2)^2\subseteq A^2$.

COROLLARY 2.13. *If R is an idempotent $\mathscr{F}$-radical E-ring, then* $R=\sum_{a\in A}(a)_R^2$.

COROLLARY 2.14. *Every idempotent Ψ-radical ring R is $\mathscr{X}$-radical.*

PROOF. If there exists an idempotent Ψ-radical E-ring $R\neq 0$, then it must contain a nonzero transfinitely nilpotent ideal A. However, by Proposition 2.12, A^2 is an idempotent ideal, since $\Psi<\mathscr{F}$, consequently, $A^2=0$. Therefore, R is a $\mathscr{B}$-radical ring, since every nonzero homomorphic image is also an idempotent Ψ-radical E-ring, which by Proposition 2.6, is impossible.

OBSERVATION 2.15. Corollary 2.14 is a generalization of Proposition 2.6. Indeed, $\mathscr{B}<L_2<\Psi\leqq\mathscr{B}_\varphi$ holds, where L_2 is the upper radical property determined by all almost nilpotent rings (L. C. A. van Leeuwen and G. A. P. Heyman [13]).

By the next example we will show that there exist idempotent $\mathscr{F}$-radical rings R with the properties $(A^2)^2=A^2$ and $A^2=\sum_{a\in A}(a)_R^2$, for every ideal A of R, which are not E-rings. This means that the properties:

(1) $aR\cup Ra\subseteq(a)_R^2$, $\forall a\in R$, and
(2) $(A^2)^2=A^2$, $A^2=\sum_{a\in A}(a)_R^2$, for every ideal A of R,

are not equivalent.

EXAMPLE 2.16. Let us consider the matrix ring

$$R=\left\{\begin{pmatrix}r & a\\ 0 & r\end{pmatrix}\middle|\, r,a\in A\right\},$$

where A is an $\mathscr{F}$-radical simple prime ring. Evidently, the subset

$$H=\left\{\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\middle|\, a\in A\right\}$$

is a zero-ring ideal of R with $R/H\cong A$. Consequently, R is an $\mathscr{F}$-radical ring. Also, R is an idempotent ring and its only ideals are $\{0\}$, H and R. Indeed, if $I\neq 0$ is an ideal of R and $0\neq\begin{pmatrix}r & a\\ 0 & r\end{pmatrix}\in I$, then if $r\neq 0$ there exists an element $b\in A$ such that $rb\neq 0$, since A is a simple prime ring, whence we have

$$\sum\begin{pmatrix}\varrho & a\\ 0 & \varrho\end{pmatrix}\begin{pmatrix}r & a\\ 0 & r\end{pmatrix}\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\begin{pmatrix}\varrho' & 0\\ 0 & \varrho'\end{pmatrix}\in I\Rightarrow\sum\begin{pmatrix}0, & \varrho rb\varrho'\\ 0, & 0\end{pmatrix}\in I\Rightarrow$$

$$\Rightarrow\begin{pmatrix}0, & \Sigma\varrho rb\varrho'\\ 0, & 0\end{pmatrix}\in I,\ \forall\varrho,\varrho'\in A,\ \Rightarrow H\subset I\Rightarrow H=I \text{ or } I=R,$$

while if $r=0\Rightarrow a\neq 0$ and we have

$$\Sigma\begin{pmatrix}\varrho & 0\\ 0 & \varrho\end{pmatrix}\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}\varrho' & 0\\ 0 & \varrho'\end{pmatrix}\in I\Rightarrow \Sigma\begin{pmatrix}0 & \varrho a\varrho'\\ 0 & 0\end{pmatrix}\in I\Rightarrow$$

$$\begin{pmatrix}0 & \Sigma\varrho a\varrho'\\ 0 & 0\end{pmatrix}\in I, \forall\varrho,\varrho'\in A\Rightarrow H\subset I\Rightarrow H=I \text{ or } I=R.$$

Moreover,

$$0=H^2=\sum_H\left(\left(\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\right)_R^2\right), \quad \text{and} \quad R=R^2=\sum_R\left(\left(\begin{pmatrix}r & a\\ 0 & r\end{pmatrix}\right)_R^2\right).$$

However, $R\notin E$ since if we suppose $R\in E$, by Proposition 2.7,

$$\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}R\subseteqq\left(\left(\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\right)_R^2=0, \quad \forall a\in A,\right.$$

whence

$$\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}\varrho & b\\ 0 & \varrho\end{pmatrix}=0, \quad \forall b,\varrho\in A\Rightarrow$$

$$\Rightarrow\begin{pmatrix}0 & a\varrho\\ 0 & 0\end{pmatrix}=0, \quad \forall\varrho\in A\Rightarrow a\varrho=0, \quad \forall\varrho\in A\Rightarrow a\in\mathrm{Ann}_l A$$

$$\forall a\in A\Rightarrow A^2=0,$$

a contradiction.

LEMMA 2.17. *If in the $\mathcal{G}$-radical ring R an element a is contained such that $a\in Z(R)$ and $(a)_R=R$, then R is not idempotent.*

PROOF. We suppose that R is an idempotent ring. Then, $R=R^2=R\cdot R=$ $=(aR+Za)(aR+Za)\subseteqq aR$ which implies $R=aR$. Consequently, $a\notin I$ for any ideal I of R with $I\neq R$, that is, $a+I\neq 0$ and, evidently, $a+I\in Z(R/I)\neq 0$.

Now, a must be a zero divisor, since otherwise, by Lemma 6 [5], R will contain a unit element which is contradictory to R being a $\mathcal{G}$-radical ring. Consequently, if $b\neq 0$ such that $ab=0$, we have for every $x\in R$,

$$xb=(ar)b=(ra)b=r(ab)=r\cdot 0=0$$

and

$$bx=b(ar)=(ba)r=(ab)r=0\cdot r=0$$

which mean that $b\in\mathrm{Ann}\,R$, that is, $\mathrm{Ann}\,R\neq 0$. Now, $R/\mathrm{Ann}\,R\neq 0$ since we have supposed R to be an idempotent ring. Consequently, for the nonzero idempotent and $\mathcal{G}$-radical ring $R/\mathrm{Ann}\,R$, $\mathrm{Ann}\,(R/\mathrm{Ann}\,R)\neq 0$ holds, since $0\neq a+\mathrm{Ann}\,R\in$ $\in Z(R/\mathrm{Ann}\,R)$ and $(a+\mathrm{Ann}\,R)_{R/\mathrm{Ann}\,R}=R/\mathrm{Ann}\,R$. However, if $c+\mathrm{Ann}\,R\in$ $\in\mathrm{Ann}\,(R/\mathrm{Ann}\,R)\Rightarrow cR\subseteqq\mathrm{Ann}R\Rightarrow (cR)R=cR^2=cR=0$, and similarly $Rc=0$, that is, $c\in\mathrm{Ann}\,R\Rightarrow c+\mathrm{Ann}\,R=0\Rightarrow\mathrm{Ann}\,(R/\mathrm{Ann}\,R)=0$, a contradiction.

LEMMA 2.18. *If R is a $\mathcal{G}$-radical E-ring, then for every $0\neq a\in Z(R)$ the ideal $(a)_R$ is not an idempotent ring.*

PROOF. By Proposition 2.3, $(a)_{(a)_R}=(a)_R$, since R is an E-ring. Thus, the ring $\bar{R}=(a)_R$, $a\neq 0$, is, evidently, a $\mathcal{G}$-radical ring and contains an element $a\in Z((a)_R)=$ $=Z(\bar{R})$ such that $(a)_{\bar{R}}=(a)_{(a)_R}=(a)_R=\bar{R}$. Consequently, by Lemma 2.17, the ring $\bar{R}=(a)_R$ is not idempotent.

PROPOSITION 2.19. *If R is an idempotent $\mathscr{G}$-radical E-ring, then* $\mathscr{B}(R)=\operatorname{Ann} R=$ $=Z(R)$.

PROOF. Let $a\in Z(R)$. If $a^2\neq 0$, then by Lemma 2.18, the ideal $(a^2)_R$ can not be an idempotent ring. But, by Proposition 2.12, $(a)_R^2=(a^2)_R$ is an idempotent ideal of R, a contradiction. Thus, $a^2=0$, that is, $(a)_R^2=0$ and, by Proposition 2.7, $aR\cup Ra\subseteq(a)_R^2=0$. Consequently, $a\in\operatorname{Ann} R$, $\forall a\in Z(R)\Rightarrow Z(R)\subseteq\operatorname{Ann} R\Rightarrow Z(R)=$ $=\operatorname{Ann} R$. Now, suppose $b\in\mathscr{B}(R)$. Evidently then, $(b)_R^2$ is a $\mathscr{B}$-radical ring, since $\mathscr{B}$ is a hereditary radical property, and, by Proposition 2.12, idempotent. Consequently, if $(b)_R^2\neq 0$, by Proposition 2.6, $(b)_R^2$ is an $\mathscr{X}$-radical ring, which is contradictory to R being an E-ring. Thus, $(b)_R^2=0$ and, by Proposition 2.7, $bR\cup Rb\subseteq(b)_R^2=0\Rightarrow$ $\Rightarrow b\in\operatorname{Ann} R$, $\forall b\in\mathscr{B}(R)\Rightarrow\mathscr{B}(R)\subseteq\operatorname{Ann} R\Rightarrow\mathscr{B}(R)=\operatorname{Ann} R$.

COROLLARY 2.20. *If R is an idempotent $\mathscr{G}$-radical ring and for its every nonzero homomorphic image R', $Z(R')\supset\operatorname{Ann} R'$ holds, then R is an $\mathscr{X}$-radical ring.*

PROOF. Indeed, if R is not an $\mathscr{X}$-radical ring, then at least one of its nonzero homomorphic images, R', will be an E-ring. Consequently, by Proposition 2.19, $Z(R')=\operatorname{Ann} R'$, a contradiction.

COROLLARY 2.21. *Every commutative $\mathscr{G}$-radical idempotent ring R is $\mathscr{X}$-radical.*

PROOF. Indeed, since R is an idempotent ring, for every nonzero homomorphic image R' of R

$$Z(R')=R'\supset\operatorname{Ann} R'$$

holds, whence, by Corollary 2.20, R is an $\mathscr{X}$-radical ring.

We can slightly generalize Corollary 2.20 by the following.

COROLLARY 2.22. *If R is an idempotent $\mathscr{G}$-radical ring and for its every nonzero prime homomorphic image R', $Z(R')\neq 0$ holds, then R is an $\mathscr{X}$-radical ring.*

PROOF. If R is not an $\mathscr{X}$-radical ring, then there exists its nonzero homomorphic image $R'\in E$. Now, if R'' is a prime homomorphic image of R', then R'' is also a prime homomorphic image of R; consequently $Z(R'')\neq 0$, since $R''\in E$. But then, by Proposition 2.19, $\mathscr{B}(R'')=\operatorname{Ann} R''=Z(R'')\neq 0$ which is contradictory to R'' being a prime ring. Thus, R' is a $\mathscr{B}$-radical ring, whence by Proposition 2.6, R' is an $\mathscr{X}$-radical ring, a contradiction.

COROLLARY 2.23. *Every idempotent $\mathscr{G}$-radical P. I.-ring R is $\mathscr{X}$-radical.*

PROOF. If R' is a nonzero prime homomorphic image of R, then, since R' is also a P. I.-ring, $Z(R')\cap A\neq 0$, for every nonzero ideal A of R' ([16], theorem 2). Consequently, we have $Z(R')\neq 0$ whence, by Corollary 2.22, R is an $\mathscr{X}$-radical ring.

COROLLARY 2.24. *For every $\mathscr{G}$-radical prime simple ring A, $Z(A)=0$.*

PROOF. Evidently, A is an idempotent E-ring and $\mathscr{B}$-semisimple.

OBSERVATION 2.25. Corollary 2.24 generalizes Lemma 1 [11], which uses the Jacobson radical property. At this point we must note that Corollary 2.24 can be proved by combining N. Divinsky's Lemma 6 [5] with the well known result that the

center of a prime ring is without nonzero zero-divisors. However, this argument cannot be used for classes larger than the class of prime simple rings, contrary to Proposition 2.19, which gives, as below, the same result for other, larger classes as well.

COROLLARY 2.26. *For every von Neumann regular $\mathscr{G}$-radical ring R, $Z(R)=0$ holds.*

PROOF. Indeed, R is an idempotent E-ring and $\mathscr{B}$-semisimple.

COROLLARY 2.27. *For every hereditarily idempotent $\mathscr{G}$-radical ring R, $Z(R)=0$ holds.*

PROOF. Indeed, R is an idempotent E-ring and $\mathscr{B}$-semisimple.

We can also formulate, in relation with the above, the following.

COROLLARY 2.28. *Every von Neumann regular $\mathscr{G}$-radical ring R is $\mathscr{N}_g$-radical, where $\mathscr{N}_g$ denotes the generalized nil radical.*

PROOF. If R is a von Neumann regular $\mathscr{G}$-radical ring and does not have nonzero nilpotent elements, then by Lemma 1 [9], $ax\in Z(R)$ for each $a\neq 0$ with $a=axa$, that is, $Z(R)\neq 0$. But, this last result contradicts Corollary 2.28.

Consequently, R is an $\mathscr{N}_g$-radical ring, since its every homomorphic image is also a von Neumann regular $\mathscr{G}$-radical ring.

More generally

COROLLARY 2.29. *If every homomorphic image $R'\neq 0$ of an idempotent $\mathscr{G}$-radical E-ring R contains nonzero idempotent elements, then R is an $\mathscr{N}_g$-radical ring.*

PROOF. If $R'\neq 0$ does not have nonzero nilpotent elements, then, by Lemma 1 [9], $e\in Z(R')$ for every idempotent element $e\in R'$. Consequently, by Proposition 2.19, $e\in Z(R')=\mathscr{B}(R')=\operatorname{Ann} R'$, whence $e=e^2=0$, a contradiction.

COROLLARY 2.30. *Every $\mathscr{G}$-radical simple ring, with nonzero idempotent elements, is $\mathscr{N}_g$-radical.*

OBSERVATION 2.31. Corollary 2.30 is also given by Theorem 8 or Corollary 9 ([12], pp. 441—442) using essentially N. Divinsky's Lemma 6 [5].

If we use this lemma we can prove Corollaries 2.28, 2.29, 2.30, but here we wanted to prove them using the result of Proposition 2.19, for the center $Z(R)$ of such a ring, and Lemma 1 of A. Forsythe and N. H. McCoy [9].

3. From what we have said, it is evident that $I_\Psi\leqq\mathscr{X}\leqq I_\mathscr{P}$ holds. It is now natural to ask: Does $\mathscr{X}$ coincide with some of the known almost subidempotent radical properties I_Ψ, $I_\mathscr{L}$, $I_{\mathscr{B}_\varphi}$, $I_\mathscr{P}$, and T_U where $\mathscr{L}$ and T_U are the Levitzki's radical property and the upper radical property determined by all unequivocal rings (N. Divinsky [7]), respectively?

The ring S_{ω_0} (N. Divinsky [7]) is a hereditarily idempotent ring and simultaneously $\mathscr{P}$-radical. Consequently, $\mathscr{X}<I_\mathscr{P}$ and $I_{\mathscr{B}_\varphi}<I_\mathscr{P}$. Also, it is implied that $T_U\not\equiv\mathscr{X}$ and, by Example 3 [6], that $\mathscr{X}\not\equiv T_U$. Thus, the question is restricted to the radical properties I_Ψ, $I_\mathscr{L}$, $I_{\mathscr{B}_\varphi}$.

About this open problem we can say the following:

LEMMA 3.1. *If the ring R has ideals I, J such that R/I is an $\mathscr{X}$-radical ring and R/J is an E-ring, then $I+J=R$.*

PROOF. Indeed, if $I+J\neq R$, then

$$0 \neq \frac{R/I}{I+J/I} \cong \frac{R}{I+J} \cong \frac{R/J}{I+J/J} \neq 0,$$

a contradiction.

PROPOSITION 3.2. *If for the idempotent $\mathscr{P}$-radical ring R, R/I happens to be a nonzero $\mathscr{X}$-radical ring for some ideal I which has at least one of the following properties:* (1) *nilpotent* (2) *$\mathscr{B}$-radical* (3) *almost nilpotent* (4) *commutative* (5) *P. I.-ring* (6) *simple prime* (7) *Noetherian* (8) *Artinian* (9) *Ψ-radical, then R is an $\mathscr{X}$-radical ring.*

PROOF. If R is not an $\mathscr{X}$-radical ring, then there exists an ideal J such that $0\neq R/J\in E$. Consequently, by Lemma 3.1, $I+J=R$ whence we have $0\neq I/I\cap J\cong \cong I+J/J\cong R/J\in E$, a contradiction since the nonzero $I/I\cap J$ in cases (1) and (3) is not an idempotent ring, in cases (2), (4), (5) and (9) is an $\mathscr{X}$-radical ring (Proposition 2.6, Corollaries 2.21, 2.23, and 2.14), in cases (6) and (7) is not a $\mathscr{P}$-radical ring, and finally, in case (8) is an $\mathscr{X}$-radical ring (as Artinian $\mathscr{G}$-radical is a $\mathscr{B}$-radical ring).

COROLLARY 3.3. *If for the idempotent $\mathscr{P}$-radical ring R we have $C(R)\neq R$, where $C(R)$ is its commutator ideal and $C(R)$ has at least one of properties* (1), (2), (3), (4), (5), (6), (7), (8), (9), *then R is an $\mathscr{X}$-radical ring.*

PROOF. Indeed, the ring $R/C(R)\neq 0$ is commutative, idempotent and $\mathscr{G}$-radical, whence, by Corollary 2.21, it is an $\mathscr{X}$-radical ring.

PROPOSITION 3.4. *If the idempotent $\mathscr{G}$-radical ring R is subdirectly irreducible with idempotent heart $H\neq R$ and R/H is an $\mathscr{X}$-radical ring, then R is also $\mathscr{X}$-radical.*

PROOF. If R is not an $\mathscr{X}$-radical ring and J is its ideal with $0\neq R/J\in E$, then, by Lemma 3.1, $H+J=R$. But since $H\neq R\Rightarrow J\neq 0\Rightarrow J\supseteq H\Rightarrow J=R$, a contradiction.

COROLLARY 3.5. *If the idempotent $\mathscr{G}$-radical ring R is subdirectly irreducible with idempotent heart $H\neq R$ and $C(R)=H$ then R is an $\mathscr{X}$-radical ring.*

COROLLARY 3.6. *If $\mathscr{X}\leqq I_{\mathscr{B}_\varphi}$ then for every idempotent $\mathscr{G}$-radical ring R containing as an ideal a prime simple ring C with R/C an $\mathscr{X}$-radical ring, C is a ring-direct summand of R.*

PROOF. Evidently, if $C=R$ then Corollary 3.6 trivially holds. Now suppose $C\neq R$. If C^* is the two-sided annihilator of C in R, then R/C^* is an idempotent $\mathscr{G}$-radical and subdirectly irreducible ring with idempotent heart $\frac{C\oplus C^*}{C^*}\cong C$. Moreover,

$$\frac{R/C^*}{C\oplus C^*/C^*} \cong \frac{R}{C\oplus C^*} \cong \frac{R/C}{C\oplus C^*/C}$$

is an $\mathscr{X}$-radical ring. Consequently, if we have $C \oplus C^* \neq R$, then, by Proposition 3.4, R/C^* is an $\mathscr{X}$-radical ring, a contradiction.

COROLLARY 3.7. *If the idempotent $\mathscr{P}$-radical ring R contains an ideal I with the droperties:*

(1) *I is not a $\mathscr{P}$-radical ring*
(2) *R/I is an $\mathscr{X}$-radical ring*

then $\mathscr{X} \not\equiv I_{\mathscr{B}_\varphi}$.

PROOF. Suppose that $\mathscr{X} \subseteq I_{\mathscr{B}_\varphi}$ Since I is not a $\mathscr{P}$-radical ring, it contains an ideal J with I/J a prime simple ring. By V. A. Andrunakievič's lemma, J is also an ideal of R. Consequently, the ring R/J has the prime simple ideal I/J, whence, by Corollary 3.6, I/J is a ring-direct summand of R/J. But this means that R is not a $\mathscr{P}$-radical ring, a contradiction.

COROLLARY 3.8. *If $\mathscr{X} \subseteq I_{\mathscr{B}_\varphi}$ and the commutator ideal $C(R)$ of an idempotent $\mathscr{G}$-radical ring R, is a prime simple ring, then $C(R)$ is a ring-direct summand of R.*

COROLLARY 3.9. *If the commutator ideal $C(R)$ is not a $\mathscr{P}$-radical ring for some idempotent $\mathscr{P}$-radical ring R, then $\mathscr{X} \not\equiv I_{\mathscr{B}_\varphi}$.*

PROPOSITION 3.10. *If for each pair of ideals A, B of a ring R, $A \subseteq B$ or $B \subseteq A$ holds, and R contains an ideal $I \neq R$ with R/I an $\mathscr{X}$-radical ring, then R is also an $\mathscr{X}$-radical ring.*

PROOF. If R is not an $\mathscr{X}$-radical ring and J is its ideal with $0 \neq R/J \in E$, then $I+J=R \Rightarrow I \subseteq J \Rightarrow J=R$, a contradiction.

COROLLARY 3.11. *If we have $A \subseteq B$ or $B \subseteq A$ for each pair of ideals A, B of an idempotent $\mathscr{G}$-radical ring R and $C(R) \neq R$ for its commutator ideal, then R is an $\mathscr{X}$-radical ring.*

PROPOSITION 3.12. *If the $\mathscr{S}$-radical ring R has, as an ideal, an $\mathscr{L}$-radical ring I with $0 \neq R/I$ an $\mathscr{X}$-radical ring, then R is also an $\mathscr{X}$-radical ring.*

PROOF. If R is not an $\mathscr{X}$-radical ring and J is its ideal with $0 \neq R/J \in E$, then $I+J=R$, whence $0 \neq R/J = I+J/J \cong I/I \cap J \in \mathscr{L}$, a contradiction since R/J is a nonzero $\mathscr{S}$-radical ring (F. Szász [18]).

COROLLARY 3.13. *If for the $\mathscr{S}$-radical and $\mathscr{G}$-radical ring R its commutator ideal $C(R)$ is an $\mathscr{L}$-radical ring, then R is an $\mathscr{X}$-radical ring.*

Finally, we formulate another open problem.

Is $\mathscr{X}$ the class of all idempotent $\mathscr{G}$-radical rings R with $Z(R') \neq 0$ for every nonzero prime homomorphic image R' of R?

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SIGNIERTE ZELLENZERLEGUNGEN. II

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Einleitung

In unserer Arbeit *Signierte Zellenzerlegungen. I* [4] befassten wir uns mit den q-regulären p-Eckzerlegungen der Sphäre. Der Gegenstand der vorliegenden Arbeit ist die Untersuchung ähnlicher Zerlegungen der topologischen Kreisscheibe.[1] Um unser Ziel schildern zu können, müssen wir einige Begriffe erklären.

Es bezeichnen p und q ganze Zahlen ($p \geqq 3$, $q \geqq 2$),[2] und unter einer *p-Eckzerlegung* verstehen wir eine solche Zerlegung K der Kreisscheibe, wo jede *Zelle* (Flächenstück) das topologische Bild eines p-Ecks ist. Dabei soll der Durchschnitt zweier Zellen, falls er nicht leer ist, nur aus Eckpunkten und Kanten bestehen. Diejenigen Ecken und Kanten, die am Rande der Kreisscheibe liegen, bilden das *Randpolygon* $V=[v_1v_2\ldots v_n]$ ($n \geqq 2$) von K. Die Kreisscheibe sei immer mit Orientierung versehen, und wir nehmen an, daß im Falle $n>2$ die Reihenfolge $v_1, v_2, \ldots, v_n$ der Orientierung entspricht. K heißt *signiert*, falls zu jeder Zelle X von K eine ganze Zahl $\sigma(X)$ als *Signatur* zugeordnet ist. Ähnlich zu [4] wird nur die *Heawoodsche Signierung* angewendet. Bei dieser kann $\sigma(X)$ nur die Werte 1 und -1 annehmen. Das durch die Funktion σ signierte K wird mit K_σ bezeichnet. $s_\sigma(x)$ ist die Summe der Signaturen derjenigen Zellen, die mit der Ecke x inzidieren. Ist x eine innere Ecke und ist $s_\sigma(x) \equiv 0 \pmod q$, so heißen σ und K_σ in x *q-regulär*. Sind sämtliche inneren Ecken q-regulär, dann sagen wir, daß K_σ eine *q-reguläre p-Eckzerlegung* ist. Im folgenden soll K_σ immer eine solche Zerlegung bezeichnen, und wenn anders nicht gesagt wird, verstehen wir unter einer Zerlegung immer ein K_σ. Es soll $s_\sigma(v_i)$ der zu v_i gehörige Winkelwert von K_σ heißen. Ist kein Mißverständnis zu befürchten, dann setzen wir statt $s_\sigma(v_i)$ kurz $\langle v_i \rangle$. Die Folge $\langle v_1 \rangle, \langle v_2 \rangle, \ldots, \langle v_n \rangle$ ist die *Winkelwertfolge* von K_σ.

Wir wollten zuerst die Winkelwertfolgen von q-regulären p-Eckzerlegungen charakterisieren. Jedoch stellte sich heraus, daß die modulo q Reste der Winkelwerte sich einfacher behandeln lassen. Einen beliebigen Repräsentanten der zu $\langle v_i \rangle$ gehörigen Restklasse modulo q werden wir einen zur Ecke v_i gehörigen *Randwert* nennen und mit r_i bezeichnen; $r_1, r_2, \ldots, r_n$ ist eine zu K_σ gehörige *Randwertfolge*. Nun ist unser Hauptziel die Charakterisierung der Randwertfolgen von q-regulären p-Eckzerlegungen.

Unsere Arbeit ist folgendermaßen aufgebaut:

§1. Es werden die *gerichteten Kantenfolgen* des regulären Mosaiks $\{p, q\}$ und die *Winkelwerte* solcher Folgen definiert. Die Sätze (2.7) und (3.3) von [4] zeigen

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[1] Die Ergebnisse von [4] wenden wir eigentlich nicht an, sie dienen uns nur als Wegweiser.

[2] Die meisten Ergebnisse lassen sich auch auf die Fälle $p=2$, $q \geqq 2$ erweitern. Wir haben den Wert $p=2$ deshalb ausgeschlossen, weil die bei den regulären $\{2, q\}$ Mosaiken auftretenden Mehrfachkanten eine Erschwerung unserer Ausdrucksweise verursachen würden.

nähmlich, daß zwischen den Randwertfolgen der Zerlegungen und den Winkelwertfolgen von *geschlossenen* Kantenfolgen eine enge Beziehung besteht. Wir geben mit Hilfe der Bewegungsgruppe[3] des Mosaiks $\{p, q\}$ eine Charakterisierung dieser Winkelwertfolgen an (Satz 1.2).

§2 und §3. Wir beweisen die Hauptsätze: Es wird eine Charakterisierung der Randwertfolgen von q-regulären p-Eckzerlegungen angegeben. (Sätze 2.1, 3.2, 3.3).

§4. Es werden die Ergebnisse von §2 und §3 auf *multi-p-gonale* Zerlegungen erweitert. (Bei diesen ist die Eckenanzahl einer jeden Zelle ein Vielfaches von p (Sätze 4.1, 4.2, 4.3).

§5. Wir befassen uns mit solchen Signierungen, wo jede Zelle die Signatur 1 erhält (vollpositive Signierungen). Es ergibt sich, daß im Falle $(p-2)(q-2)<4$ jede durch beliebige Heawoodsche Signierung erzeugte Randwertfolge auch durch vollpositive Signierungen darstellbar ist (s. (5.1)). Ferner wird gezeigt, daß dies im Falle $(p-2)(q-2)\geqq 4$ nicht zutrffit. (Satz 5.1).

§1. Die Kantenfolgen der regulären Mosaike

(1.1) Es bezeichnet $\{p, q\}$ $(p\geqq 3, q\geqq 2)$ das reguläre Mosaik, indem die Zellen reguläre p-Ecke sind, und wo in jeder Ecke q Zellen zusammenlaufen. Wir betrachten die Trägerfläche des Mosaiks immer mit Orientierung versehen. Diese Fläche ist die Sphäre, die euklidische, bzw. die hyperbolische Ebene je nachdem in $(p-2)(q-2)\lessgtr 4$ das Zeichen $<$, $=$, bzw. $>$ gilt (s. [2]). Wegen $p\geqq 3$ existieren in $\{p, q\}$ keine Mehrfachkanten; deshalb wird eine Kante eindeutig durch ihre Endpunkte bestimmt. Mit xy bezeichnen wir jene *gerichtete* Kante, deren Anfangspunkt x und Endpunkt y ist.

Die (gerichtete) *Kantenfolge*

$$S = (a_0 a_1 \dots a_n a_{n+1}) \quad (n \geqq 1) \tag{1}$$

von $\{p, q\}$ besteht aus den Ecken $a_0, a_1, \dots, a_n, a_{n+1}$, und aus den gerichteten Kanten $a_0a_1, a_1a_2, \dots, a_na_{n+1}$ von $\{p, q\}$. Eine Ecke darf in der Folge auch mehrmals vorkommen. Da jedoch in $\{p, q\}$ keine Schlingen existieren, bestehen $a_i\neq a_{i+1}$ $(i=0, 1, \dots, n)$. Ist $a_na_{n+1}=a_0a_1$, (d.h. sind $a_n=a_0$ und $a_{n+1}=a_1$), so ist S *geschlossen*. Wir werden in diesem Falle statt (1) auch kurz

$$S = [a_1 a_2 \dots a_n] \quad (n \geqq 2, \quad a_0 = a_n, \quad a_{n+1} = a_1) \tag{2}$$

schreiben. Wir machen folgendes Übereinkommen: Es sei $[a_2a_3 \dots a_na_1]=[a_1a_2 \dots a_n]$, d.h. eine zyklische Umordnung der Ecken soll die geschlossene Folge nicht verändern. (Wir bezeichnen mit einem Ausdruck der Form (2) sowohl das Randpolygon einer Zerlegung bzw. einer Zelle, als auch die Zelle selbst. Dies führt jedoch zu keinem Mißverständnis.)

Die *Winkelwerte* der Kantenfolge (1) sind jene ganze Zahlen

$$w_i = \langle a_{i+1}a_ia_{i-1}\rangle \quad (0 \leqq w_i \leqq q; \quad i = 1, \dots, n)$$

[3] Die Gruppe der *eigentlichen* Bewegungen.

die die folgende Eigenschaft haben: dreht man die Kante $a_i a_{i+1}$ um a_i mit dem Drehwinkel $w_i(2\pi/q)$ in der positiven Richtung so fällt sie mit $a_i a_{i-1}$ zusammen (s. Fig. 1.) Außer dem Fall $a_{i+1}=a_{i-1}$, ist w_i eindeutig bestimmt. Ist kein Mißverständnis zu befürchten, dann wird statt $\langle a_{i+1} a_i a_{i-1}\rangle$ kurz $\langle a_i\rangle$ gesetzt. $w_1, w_2, \ldots, w_n$ ist die *Winkelwertfolge* von (1).

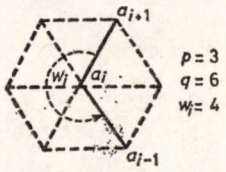

Fig. 1

Die Richtigkeit folgender Behauptung ist trivial: Gelten für die Folge ganzer Zahlen $u_1, u_2 \ldots, u_n$ $(n \geqq 1)$ die Beschränkungen

$$0 \leqq u_i \leqq q \quad (i = 1, 2, \ldots, n),$$

dann kann man zu einer beliebigen Anfangskante $a_0 a_1$ eine (und nur eine) Kantenfolge $S=(a_0 a_1 \ldots a_n a_{n+1})$ so konstruieren, daß $u_1, u_2, \ldots, u_n$ die Winkelwertfolge von S sei.

(1.2) Wir wollen mit den Winkelwerten von $S=(a_0 a_1 \ldots a_n a_{n+1})$ und mit geeignet gewählten Generatoren der *Bewegungsgruppe* $\mathscr{B}(p, q)$ des Mosaiks $\{p, q\}$ diejenige Bewegung ausdrücken, durch welche die Kante $a_0 a_1$ auf den Platz von $a_n a_{n+1}$ versetzt wird (d.h. a_0 den Platz von a_n und a_1 den Platz von a_{n+1} annimt). (Diese Bewegung ist durch $a_0 a_1$ und $a_n a_{n+1}$ eindeutig bestimmt.)

Die Tatsache, daß die Bewegung ϱ die Kante xy an den Platz der Kante $x'y'$ versetzt, bezeichnen wir mit

$$xy|\varrho = x'y'. \tag{3}$$

Aus (3) folgt $x'y'|\varrho^{-1}=xy$. Gilt neben (3) auch $x'y'|\varrho'=x''y''$, so soll $\varrho\varrho'$ diejenige Bewegung bezeichnen, für die $xy|\varrho\varrho'=x''y''$ besteht. Mit dieser Bezeichnungsweise ist unsere Aufgabe die Bestimmung jener Bewegung ω, für die

$$a_0 a_1|\omega = a_n a_{n+1} \tag{4}$$

gilt.

Zu der Bestimmung der Generatoren soll eine Kante $b_0 b_1$ von $\{p, q\}$ fest ausgewählt werden. Ferner bezeichne $X=[x_1 x_2 \ldots x_p]$ jene positiv-orientierte Zelle von $\{p, q\}$ für die

$$b_0 b_1 = x_1 x_2 \tag{5}$$

besteht. Es sei α, bzw. β jene Drehung von $\mathscr{B}(p, q)$, für die

$$x_1 x_2|\alpha = x_2 x_1, \quad \text{bzw.} \quad x_1 x_2|\beta = x_2 x_3 \tag{6}$$

gilt. Bezeichnet e das Einselement von $\mathscr{B}(p, q)$, dann gelten $\alpha^2=e$ und $\beta^p=e$.

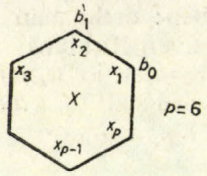

Fig. 2

(1.3) Es wird sich ergeben, daß α und β Generatoren von $\mathscr{B}(p, q)$ sind. Zuerst wollen wir jene Drehung δ mit α und β ausdrücken, für die

$$x_2x_3|\delta = x_2x_1 \tag{7}$$

besteht. Nach (6) und $\alpha^2=e$ ist $x_2x_3|\beta^{-1}\alpha=x_2x_1$ es ist also

$$\delta = \beta^{-1}\alpha = (\alpha\beta)^{-1}. \tag{8}$$

Ferner besteht für δ die Relation $\delta^q=e$, und dies ergibt nach (8) $(\alpha\beta)^q=e$. Daher erfüllen α und β die Relationen

$$\alpha^2 = \beta^p = (\alpha\beta)^q = e. \tag{9}$$

Jedoch sind diese eben die definierenden Relationen von $\mathscr{B}(p,, q)$. Setzt man nämlich $T=\alpha$, $S=\alpha\beta$, $E=e$, so ergibt (9)

$$S^q = T^2 = (ST)^p = E$$

(s. (4.43) S. 38 von [1]). (Es sei bemerkt, daß dieselbe Behauptung auch aus unseren Betrachtungen des Abschnitts (1.5) folgen wird.) Wir betonen, daß die Generatoren α und β allein durch die festgewählte Kante $b_0\,b_1$ bestimmt sind.

(1.4) Es sei

$$\tilde{S} = (b_0b_1 \ldots b_mb_{m+1}) \quad (m \geqq 1)$$

eine Kantenfolge, deren Anfangskante $b_0\,b_1$ ist, die aber sonst beliebig angenommen werden darf. Die Winkelwerte von $\tilde{S}$ seien die Zahlen

$$u_i = \langle b_{i+1}b_ib_{i-1}\rangle \quad (i = 1, 2, \ldots, m).$$

Wir werden die mit

$$b_0b_1|\omega_i = b_ib_{i+1} \quad (i = 1, 2, \ldots, m) \tag{10}$$

definierte Bewegung ω_i durch α, β und $u_1, u_2, \ldots, u_i$ ausdrücken $(i=1, 2, \ldots, m)$.

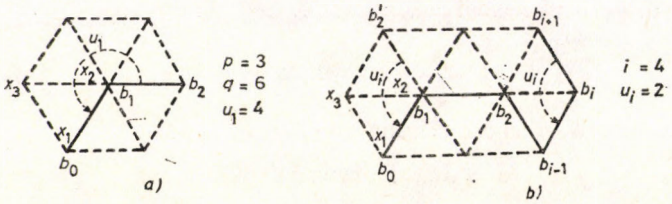

Fig. 3

1) Es sei $i=1$. ω_1 ist durch $b_0 b_1|\omega_1=b_1 b_2$ definiert. Nach (7) ist $b_1 b_2|\delta^{u_1}=$ $=b_1 b_0$, woraus nach (6) und (9) $b_1 b_2|\delta^{u_1}\alpha=b_0 b_1$ folgt (s. Fig. 3a)). Es gilt daher $b_0 b_1|(\delta^{u_1}\alpha)^{-1}=b_1 b_2$, und dies ergibt nach (8) $\omega_1=\alpha\delta^{-u_1}=\alpha(\alpha\beta)^{u_1}$. Mit der Bezeichnung

$$\beta_u = \alpha(\alpha\beta)^u \tag{11}$$

sind also

$$\omega_1 = \beta_{u_1} \quad \text{und} \quad b_0 b_1|\beta_{u_1} = b_1 b_2. \tag{12}$$

2) Jetzt drücken wir ω_i durch ω_{i-1} und u_i aus ($i\geqq 2$). Nach (10) gilt $b_{i-1}b_i|\omega_{i-1}^{-1}=b_0 b_1$. Es sei b_2' diejenige Ecke, für die

$$b_i b_{i+1}|\omega_{i-1}^{-1} = b_1 b_2' \tag{13}$$

besteht (s. Fig. 3b)). Dann ist $\langle b_2' b_1 b_0\rangle=\langle b_{i+1}b_i b_{i-1}\rangle=u_i$. Aus (12) bekommt man $b_0 b_1|\beta_{u_i}=b_1 b_2'$, und (13) ergibt so $b_0 b_1|\beta_{u_i}\omega_{i-1}=b_i b_{i+1}$. Nach (10) besteht also $\omega_i=\beta_{u_i}\omega_{i-1}$ für jedes $i\geqq 2$. Dies ergibt mit (12)

$$\omega_i = \beta_{u_i}\beta_{u_{i-1}}\dots\beta_{u_1} \quad (i = 1, 2, \dots, m). \tag{14}$$

(1.5) Nun sind wir in der Lage, die in (1.2) gestellte Aufgabe zu lösen. Dazu wird die Kantenfolge $\tilde{S}$ so gewählt, daß $S=(a_0 a_1 \dots a_n a_{n+1})$ eine Teilfolge von $\tilde{S}$ bilden soll. Es wird mit einem $m>n$, $m-n=j$ gesetzt und $b_{j+i}=a_i$ ($i=0, 1, \dots$ $\dots, n, n+1$) gewählt. (Dies kann mit genügend großem m bei beliebiger Lage von $b_0 b_1$ und S erreicht werden.) (S. Fig. 4.) Es gelten dann

$$b_j b_{j+1} = a_0 a_1,\ b_m b_{m+1} = a_n a_{n+1} \quad \text{und} \quad u_{j+i} = w_i = \langle a_{i+1} a_i a_{i-1}\rangle \tag{15}$$
$$(i = 1, 2, \dots, n).$$

Nach (10) bestehen dann

$$a_0 a_1|\omega_j^{-1} = b_0 b_1 \quad \text{und} \quad b_0 b_1|\omega_m = a_n a_{n+1},$$

und so gilt

$$a_0 a_1|\omega_j^{-1}\omega_m = a_n a_{n+1}.$$

Es besteht also für die in (4) definierte Bewegung ω die Relation

$$\omega = \omega_j^{-1}\omega_m.$$

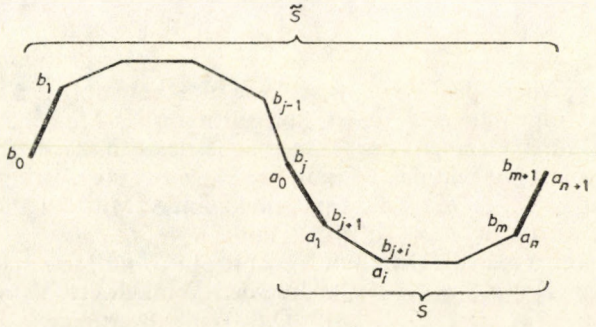

Fig. 4

Nach (14) und (15) bekommt man so die Relation

$$\omega = \omega_j^{-1}\beta_{w_n}\beta_{w_{n-1}}\dots\beta_{w_1}\omega_j. \tag{16}$$

Durch geeignete Wahl von a_0a_1 kann erreicht werden, daß ω mit einer beliebig vorgegebenen Bewegung von $\mathscr{B}(p, q)$ zusammenfällt, so beweisen (11), (14) und (16) die schon erwähnte Tatsache, daß α und β Generatoren der Gruppe $\mathscr{B}(p, q)$ sind (vgl. (1.3)).

Wir werden aus (16) auch zwei weitere — für uns wichtige — Folgerungen ziehen.

1) Besteht für die Winkelwerte $w_1, w_2, \dots, w_n$ von S die Relation

$$\beta_{w_n}\beta_{w_{n-1}}\dots\beta_{w_1} = e, \tag{17}$$

dann gilt nach (16) auch $\omega=e$, und dies ist laut (4) mit $a_n\,a_{n+1}=a_0\,a_1$ gleichwertig. Es folgt also aus (17), daß S geschlossen ist.

2) Umgekehrt, ist S geschlossen, dann ist $\omega=e$, und so folgt aus (16) die Relation (17).

In Betracht des letzten Absatzes von (1.1) können wir so folgenden Satz aussagen:

SATZ 1.1. *Die Zahlenfolge* $u_1, u_2, \dots, u_n$ $(n\geqq 1)$ *ist dann und nur dann Winkelwertfolge einer geschlossenen Kantenfolge, wenn sie den Bedingungen* $0\leqq u_i\leqq q$ $(i=1, 2, \dots, n)$ *und der Relation*

$$\beta_{u_n}\beta_{u_{n-1}}\dots\beta_{u_1} = e \quad (\beta_u = \alpha(\alpha\beta)^u) \tag{18}$$

genügt.

(1.6) Offensichtlich wird das Bestehen der Relation (18) durch eine zyklische Umordnung der Faktoren nicht beeinträchtigt. Wir werden zeigen, daß aus (18) auch

$$\beta_{u_1}\beta_{u_2}\dots\beta_{u_n} = e \tag{19}$$

folgt. Dazu brauchen wir folgende einfache Behauptung:

$$\text{Ist } u'\equiv u \pmod q, \text{ dann gilt } \beta_{u'}=\beta_u. \tag{20}$$

(Dies folgt unmittelbar aus $\beta_u=\alpha(\alpha\beta)^u$ und $(\alpha\beta)^q=e$.) Nehmen wir nun an, daß die Zahlenfolge $u_1, u_2, \dots, u_n$ der Relation (18) genügt, und seien die Zahlen $w_1, w_2, \dots$ $\dots, w_n$ durch $w_i\equiv u_i \pmod q$; $0\leqq w_i<q$ $(i=1, 2, \dots, n)$ definiert. Laut (20) besteht dann auch

$$\beta_{w_n}\beta_{w_{n-1}}\dots\beta_{w_1} = e. \tag{21}$$

Nach Satz 1.1 existiert dann eine geschlossene Kantenfolge $S=(a_0a_1 \dots a_na_{n+1})$, deren Winkelwertfolge $w_1, w_2, \dots, w_n$ ist. Spiegelt man das Mosaik an einem solchen Großkreis (bzw. einer Geraden) der Trägerfläche des Mosaiks, der mit einer Kante des Mosaiks inzidiert, so geht das Mosaik in sich selbst über. Bei einer solchen Spiegelung sei $S'=(a_0', a_1', \dots, a_n'a_{n+1}')$ das Spiegelbild von S. Mit S ist auch S' geschlossen. Für die Winkelwerte $w_i'=\langle a_{i+1}'a_i'a_{i-1}'\rangle$ und $w_i=\langle a_{i+1}a_ia_{i-1}\rangle$ gilt $w_i'=q-w_i$ $(i=1, 2, \dots, n)$. Kehrt man die Durchlaufsrichtung in S' um, so entsteht die geschlossene Kantenfolge $\bar{S}=(a_{n+1}'a_n' \dots a_1'a_0')$. Für den Winkelwert $\bar{w}_i=\langle a_{i-1}'a_i'a_{i+1}'\rangle$ von $\bar{S}$ gilt $\bar{w}_i=q-w_i'=w_i$ $(i=1, 2, \dots, n)$. Die Winkelwertfolge von $\bar{S}$ ist jedoch $\bar{w}_n, \bar{w}_{n-1}, \dots, \bar{w}_1$, und dies stimmt daher mit der Folge $w_n, w_{n-1}, \dots, w_1$ überein.

Wendet man den Satz 1.1 auf $\bar{S}$ an, so ergibt sich $\beta_{w_1}\beta_{w_2}\dots\beta_{w_n}=e$, und laut (20) folgt daraus (19).

Zusammenfassend können wir folgenden Satz formulieren:

SATZ 1.2. *Mit den definierenden Relationen*

$$\alpha^2=\beta^p=(\alpha\beta)^q=e$$

erzeugen α und β die Bewegungsgruppe $\mathscr{B}(p,q)$ des regulären Mosaiks $\{p,q\}$. Die Winkelwertfolge $w_1, w_2, \dots, w_n$ $(n\geqq 1)$ einer jeden geschlossenen Kantenfolge von $\{p,q\}$ genügt, mit der Bezeichnung

$$\beta_u=\alpha(\alpha\beta)^u,$$

der Relation

$$\beta_{w_1}\beta_{w_2}\dots\beta_{w_n}=e.$$

Umgekehrt, genügt die Zahlenfolge $w_1, w_2, \dots, w_n$ dieser Relation und der Beschränkungen $0\leqq w_i\leqq q$ $(i=1,2,\dots,n)$, dann existiert eine geschlossene Kantenfolge in $\{p,q\}$ deren Winkelwertfolge $w_1, w_2, \dots, w_n$ ist.

§2. Charakterisierung der Randwertfolgen. I

(2.1) Sätze (2.7) und (3.3) von [4] geben uns Anlaß zur folgenden Fragestellung. Kann man die Randwertfolgen von q-regulären p-Eckzerlegungen in derselben Weise charakterisieren, wie dies im Satz 1.2 für die Winkelwertfolgen geschlossener Kantenfolgen von $\{p,q\}$ geschieht? Die Antwort ist bejahend.

In diesem §2 wird nur der "notwendige" Teil der Charakterisierung behandelt. Wir beweisen nämlich folgenden Satz.

SATZ 2.1. *Es soll die Bewegunggsruppe $\mathscr{B}(p,q)$ von $\{p,q\}$ durch die definierenden Relationen*

(1) $$\alpha^2=\beta^p=(\alpha\beta)^q=e$$

angegeben werden. Dann genügt jede Randwertfolge $r_1, r_2, \dots, r_n$ einer q-regulären p-Eckzerlegung K_σ, mit den Bezeichnung

(2) $$\beta_u=\alpha(\alpha\beta)^u,$$

der Relation

(3) $$\beta_{r_1}\beta_{r_2}\dots\beta_{r_n}=e.$$

BEMERKUNG. Da eine Winkelwertfolge zugleich eine Randwertfolge ist, besteht (2) auch für Winkelwertfolgen von Zerlegungen.

BEWEIS. Die Behauptung (20) von (1.6) hat zur Folge, daß das Produkt von (3) für jede Randwertfolge von K_σ dasselbe Element von $\mathscr{B}(p,q)$ ergibt. Daher dürfen wir das Produkt von (3) *Randprodukt* von K_σ nennen und mit $\Pi(K_\sigma)$ bezeichnen.

Der Beweis geht durch Induktion bezüglich der Anzahl der Zellen von K_σ.

1) Besteht K_σ aus einer einzigen mit ε $(\varepsilon\in\{1,-1\})$ signierten Zelle, so ist $\Pi(K_\sigma)=\beta_\varepsilon^p$. Da nach (2) und (1) $\beta_1=\beta$, $\beta_{-1}=\alpha(\alpha\beta)^{-1}=\alpha\beta^{-1}\alpha$ sind, besteht nach (1)

(4) $$\beta_\varepsilon^p=e \quad (\varepsilon\in\{1,-1\}).$$

2) Nehmen wir an, daß (3) für Zerlegungen mit weniger als m Zellen ($m \geqq 2$) richtig ist, und jetzt soll K_σ aus m Zellen bestehen. Es sei $V=[v_1 v_2 \dots v_n]$ ($n \geqq 2$) das Randpolygon, und $r_1, r_2, \dots, r_n$ sei eine Randwertfolge von K_σ. Nach [10] S. 137 existiert in jeder — aus mehreren Zellen bestehenden — Zerlegung eine solche Zelle, deren Entfernung eine mit der Kreisscheibe homeomorphe Figur zurückläßt. Es soll $X=[x_1 x_2 \dots x_p]$ eine solche Zelle (mit positivem Umlauf) von K_σ bezeichnen, und K'_σ sei die zurückbleibende Zerlegung.

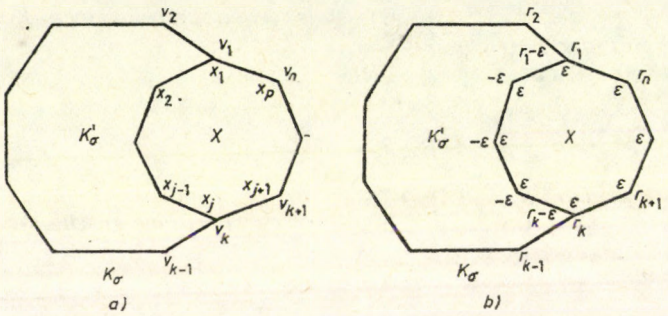

Fig. 5

Bezeichnet V' das Randpolygon von K'_σ, so haben X und V', X und V, sowie V und V' je eine Kantenfolge gemeinsam. Man kann annehmen, daß diese bei X und V' $(x_1 x_2 \dots x_j)$ $(2 \leqq j \leqq p)$, bei X und V $(x_j x_{j+1} \dots x_p x_1)=(v_k v_{k+1} \dots v_n v_1)$ $(2 \leqq k \leqq n)$, schließlich bei V und V' $(v_1 v_2 \dots v_{k-1} v_k)$ sind. Es besteht dann

$$n-k=p-j \tag{5}$$

(s. Fig 5a)). Bezeichnet ε die Signatur von X in K_σ, und betrachtet man X als einzellige Zerlegung, so ist $\overset{1}{\varepsilon}, \overset{2}{\varepsilon}, \dots, \overset{p}{\varepsilon}$ eine Randwertfolge von X. Laut (4) ist dann $\Pi(X)=e$.

Aus den Randwerten von K_σ und X erzeugen wir eine Randwertfolge für K'_σ: Ist $k>2$, so seien in den Ecken $v_2, \dots, v_{k-1}$ die Randwerte $r_2, \dots, r_{k-1}$. Ist $j>2$, dann existieren die Ecken $x_2, \dots, x_{j-1}$, und K_σ ist in diesen Ecken q-regulär. Deswegen wählen wir in K'_σ für alle diese Ecken den Randwert $-\varepsilon$. Endlich sei für die Ecke v_1, bzw. v_k der Randwert $r_1-\varepsilon$, bzw. $r_k-\varepsilon$ gewählt (s. Fig. 5b)).

Die zu K_σ und K'_σ gehörigen Randprodukte sind

$$\Pi(K_\sigma)=\beta_{r_1}\beta_{r_2}\dots\beta_{r_k}\beta_\varepsilon^{n-k} \quad (2 \leqq k \leqq n), \tag{6}$$

$$\Pi(K'_\sigma)=\begin{cases}\beta_{r_1-\varepsilon}\beta_{r_k-\varepsilon}\beta_{-\varepsilon}^{j-2} & \text{falls} \quad k=2,\\ \beta_{r_1-\varepsilon}\beta_{r_2}\dots\beta_{r_{k-1}}\beta_{r_k-\varepsilon}\beta_{-\varepsilon}^{j-2} & \text{falls} \quad k>2\end{cases} \qquad (2 \leqq j \leqq p). \tag{7}$$

Zur Umformung von $\beta_{r_1-\varepsilon}$ und $\beta_{r_k-\varepsilon}$ brauchen wir die sich aus (1) und (2) einfach ergebende Relation

$$\beta_{i+j}=\beta_i\alpha\beta_j. \tag{8}$$

Daraus folgen

$$\beta_{r_1-\varepsilon}=\beta_{-\varepsilon}\alpha\beta_{r_1} \quad \text{und} \quad \beta_{r_k-\varepsilon}=\beta_{r_k}\alpha\beta_{-\varepsilon}.$$

Nach der Induktionsannahme ist aber $\Pi(K'_\sigma)=e$, und so folgt daraus laut (7)

$$\beta_{-\varepsilon}\alpha\beta_{r_1}\beta_{r_2}\dots\beta_{r_k}\alpha\beta_{-\varepsilon}^{j-1}=e.$$

Da nach 1) $\beta_{-\varepsilon}=\alpha\beta_\varepsilon^{-1}\alpha$ ist, gelangt man zur Relation

$$\beta_{r_1}\beta_{r_2}\dots\beta_{r_k}=\alpha\beta_{-\varepsilon}^{-j}\alpha=\beta_\varepsilon^j.$$

Mit dieser ergeben (6), (5) und (4)

$$\Pi(K_\sigma)=\beta_\varepsilon^{n-k+j}=\beta_\varepsilon^p=e.$$

Damit ist der Beweis beendet.

Es sei bemerkt, daß mit Beachtung der Sätze (2.7) und (3.3) von [4] aus dem Satz 1.2 der Satz 2.1 unmittelbar folgt.

§3. Charakterisierung der Randwertfolgen. II

(3.1) Hier wollen wir beweisen: falls eine Zahlenfolge $u_1, u_2, \dots, u_n$ $(n\geqq 2)$ der Relation $\beta_{u_1}\beta_{u_2}\dots\beta_{u_n}=e$ genügt, dann existiert ein K_σ mit der Randwertfolge $u_1, u_2, \dots, u_n$. Wir brauchen dazu den folgenden, von Gruppen unabhängigen Satz.

SATZ 3.1. *Zu einer jeden geschlossenen Kantenfolge* $S=[a_1a_2\dots a_n]$ $(n\geqq 2)$ *des Mosaiks* $\{p, q\}$ *kann man eine q-reguläre p-Eckzerlegung* K_σ *mit folgender Eigenschaft konstruieren: Ist* $V=[v_1v_2\dots v_n]$ *das Randpolygon von* K_σ, *so besteht*

$$(1)\qquad \langle v_i\rangle\equiv\langle a_i\rangle \pmod q \quad (i=1,2,\dots,n;\quad \langle a_i\rangle=\langle a_{i+1}a_ia_{i-1}\rangle).$$

BEWEIS. Bezeichne $|S|$ die Länge der Folge S $(|S|=n)$. Wir werden den Beweis durch Induktion bezüglich $|S|$ führen. Vorher wollen wir jedoch einen Spezialfall erledigen den man ohne Induktion beweisen kann.

1) Es sei angenommen, daß die Ecken $a_1, a_2, \dots, a_n$ alle verschieden sind. Diesmal bildet S einen gerichteten Kreis des Mosaiks. Ist $\{p, q\}$ *endlich*, dann zerlegt S das Mosaik in zwei endliche Teile. Einer davon, der mit K bezeichnet sei, hat die Eigenschaft, daß der positive Umlauf der Zellenränder der zu K gehörigen Zellen dem Umlauf von S entspricht. Gibt man allen Zellen von K die Signatur 1, so entsteht eine q-reguläre p-Eckzerlegung K_σ, deren Winkelwerte mit den Winkelwerten von S übereinstimmen. Ist $\{p, q\}$ *unendlich*, dann bezeichne K den endlichen der Stücke, in die $\{p, q\}$ durch S zerlegt wurde. Entspricht der Umlauf von S dem positiven Umlauf der Zellenränder von K, dann können wir ebenso verfahren wie bei dem endlichen Fall. Besitzt jedoch K nicht diese Eigenschaft, dann wird ein gewünschtes K_σ dadurch erreicht, daß man einer jeden Zelle von K die Signatur -1 gibt.

2) Zur Begründung des Induktionsverfahrens betrachten wir die Fälle $2\leqq n\leqq 4$ einzeln. Die Konstruktionen der Zerlegungen werden durch Figuren angegeben. In diesen werden die inneren Eckpunkte zweiten Grades mit je einem Strich, die Signaturen mit $+$ und $-$ Zeichen veranschaulicht. Ferner werden zu den Randecken die Winkelwerte dazugeschrieben. Über den Figuren der Zerlegungen findet man die Skizzen der entsprechenden Kantenfolgen (s. Fig. 6).

Beschreibung der Figuren: In a) besteht K_σ aus zwei p-Ecken, die entlang $p-1$ Kanten inzidieren, in b) ebenfalls aus zwei p-Ecken, die aber entlang $p-2$ Kanten

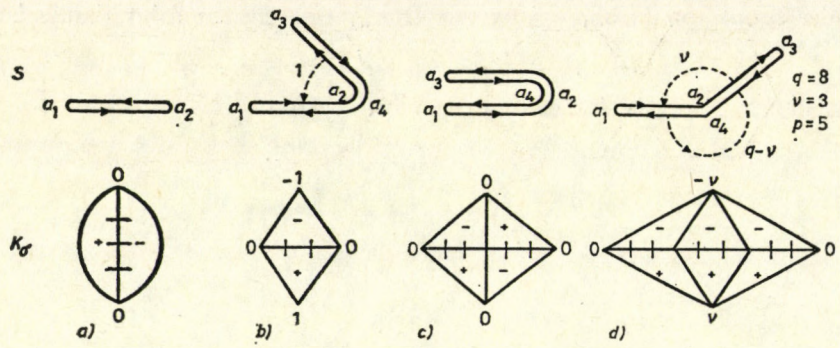

Fig. 6

inzidieren, in c) aus zwei, in d) aus ν Vierecken vom Typ b), die sich auf — aus der Figur ersichtlichen Weise — einander anschließen.

Ist $n=2$, also $S=[a_1a_2]$, dann gibt die Figur a) ein gewünschtes K_σ an. Im Falle $n=3$ müssen die Ecken von S alle verschieden sein. (Dies ist nur bei $p=3$ möglich.) Ein K_σ wird dann durch das Verfahren 1) angegeben. Der Fall $n=4$ kann bei jedem $p\geqq 3$ vorkommen. In Betracht von 1) darf angenommen werden, daß mindestens zwei von den Ecken a_1, a_2, a_3, a_4 zusammenfallen. Es sei $a_2=a_4$. Dann ist für $i=1$ und $i=3$ $\langle a_i\rangle=0$. Ferner besteht mit einem ganzen ν $(0\leqq\nu<q)$

$$\langle a_i\rangle=\begin{cases}\nu, & \text{falls } i=2 \text{ ist}\\ q-\nu, & \text{falls } i=4 \text{ ist.}\end{cases}$$

Jetzt geben für $\nu=1$, $\nu=0$, bzw. $\nu>1$ die Figuren b), c) bzw. d) je ein gewünschtes K_σ an. Wir können also behaupten, daß der Satz für $n\leqq 4$ richtig ist.

3) Es sei $n\geqq 5$. Wir nehmen an, daß der Satz für alle solche Kantenfolgen besteht, deren Längen kleiner als n sind, und es sei $|S|=n$. In Betracht von 1) darf angenommen werden, daß in $S=[a_1a_2\ldots a_n]$ zwei Ecken zusammenfallen. Es sei

$$a_k=a_l \quad (1<k<l\leqq n).$$

Da keine Schlingen vorhanden sind, ist

$$k+1<l.$$

S wird durch die Ecke a_k (bzw. a_l) in die folgenden zwei geschlossenen Kantenfolgen zerlegt (s. Fig. 7a))

$$S_1=[\overset{a_l}{\overbrace{a_k a_{l+1}}}, a_{l+2}\ldots a_n a_1 a_2\ldots a_{k-1}], \quad \text{(es sind auch } l=n, \text{ und } k-1=1 \text{ möglich)},$$

$$S_2=[\overset{a_k}{\overbrace{a_l a_{k+1}}}\ldots a_{l-1}], \quad \text{(es ist auch } k+1=l-1 \text{ möglich)}.$$

Es gelten: $|S_1|\geqq 2$, $|S_2|\geqq 2$, $|S_1|+|S_2|=|S|$. Ferner wird angenommen, daß $|S_1|\leqq|S_2|$ ist. (Dies kann durch eine geeignete zyklische Umordnung der Bezifferung der Ecken immer erreicht werden.) Wegen $n\geqq 5$ besteht dann

$$|S_2|\geqq 3. \tag{1}$$

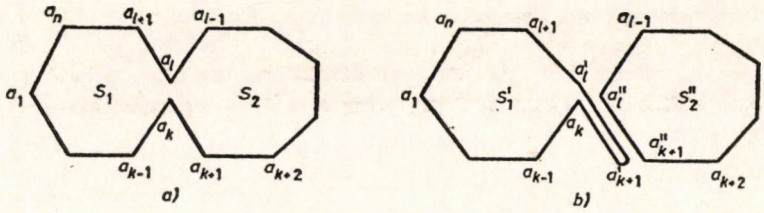

Fig. 7

Wir führen für die Ecken a_{k+1}, a_l auch einige neue Bezeichnungen ein, und zwar seien die Zeichen a'_{k+1}, a''_{k+1}, a'_l, a''_l, a folgendermaßen definiert:

$$a'_{k+1} = a''_{k+1} = a_{k+1}, \quad a'_l = a''_l = a = a_l.$$

Mit diesen konstruieren wir die geschlossenen Kantenfolgen

$$S'_1 = [a_k a'_{k+1} a'_l a_{l+1} \dots a_n a_1 \dots a_{k-1}],$$

und

$$S''_2 = [a''_l a''_{k+1} a_{k+2} \dots a_{l-1}]$$

(s. Fig. 7b)). Es gelten $|S'_1| = |S_1| + 2$, $|S''_2| = |S_2|$. So folgt nach (1)

(2) $$|S'_1| < |S_1| + |S_2| = |S|, \quad \text{und es gilt auch} \quad |S''_2| < |S|.$$

Ist $i \notin \{k+1, l\}$, dann stimmt der zu a_i gehörige Winkelwert von S'_1, bzw. S''_2 mit demjenigen von S überein. Ist $i \in \{k+1, l\}$, dann gelten

(3) $$\begin{cases} \langle a'_{k+1} \rangle = \langle a'_l a'_{k+1} a_k \rangle = \langle a_k a_{k+1} a_k \rangle = 0, \\ \langle a''_{k+1} \rangle = \langle a_{k+2} a''_{k+1} a''_l \rangle = \langle a_{k+2} a_{k+1} a_k \rangle = \langle a_{k+1} \rangle. \end{cases}$$

(4) $$\begin{cases} \langle a'_l \rangle = \langle a_{l+1} a'_l a'_{k+1} \rangle = \langle a_{l+1} a a_{k+1} \rangle, \\ \langle a''_l \rangle = \langle a''_{k+1} a''_l a_{l-1} \rangle = \langle a_{k+1} a a_{l-1} \rangle, \\ \langle a_l \rangle = \langle a_{l+1} a_l a_{l-1} \rangle = \langle a_{l+1} a a_{l-1} \rangle. \end{cases}$$

Nach (3) gilt

(5) $$\langle a'_{k+1} \rangle + \langle a''_{k+1} \rangle = \langle a_{k+1} \rangle.$$

Ferner, da die Relation

$$\langle a_{l+1} a a_{k+1} \rangle + \langle a_{k+1} a a_{l-1} \rangle \equiv \langle a_{l+1} a a_{l-1} \rangle \pmod{q}$$

bei beliebiger Lage der Ecken richtig ist, besteht nach (4)

(6) $$\langle a'_l \rangle + \langle a''_l \rangle \equiv \langle a_l \rangle \pmod{q}.$$

4) In Betracht von (2) besteht nach der Induktionsannahme unser Satz für S'_1 und S''_2. Es soll K'_{σ_1}, bzw. K''_{σ_2} eine dem Satz entsprechende Zerlegung für S'_1, bzw. S''_2 bezeichnen. Man kann die Randpolygone

$$V'_1 = [v_k v'_{k+1} v'_l v_{l+1} \dots v_n v_1 \dots v_{k-1}] \quad \text{und} \quad V''_2 = [v''_l v''_{k+1} v_{k+2} \dots v_{l-1}]$$

dieser Zerlegungen so annehmen, daß diese konvexe Polygone derselben Ebene sind und noch den folgenden Bedingungen genügen: Die Reihenfolgen der Ecken entsprechen der positiven Umlaufsrichtung der Ebene, und es besteht $v'_{k+1}=v''_{k+1}$, $v'_l=v''_l$. Dann inzidieren V'_1 und V''_2 entlang der Kante $v'_{k+1}v'_l=v''_{k+1}v''_l$ (s. Fig. 8). Führen wir ferner die Zeichen v_{k+1}, v_l durch

$$v_{k+1}=v'_{k+1}, \quad v_l=v'_l$$

ein, dann berandet das Polygon

$$V=[v_1\dots v_{k-1}v_kv_{k+1}v_{k+2}\dots v_{l-1}v_lv_{l+1}\dots v_n]$$

(mit positivem Umlauf) eine Zerlegung $K_\sigma=K'_{\sigma_1}\cup K''_{\sigma_2}$, wobei $K=K'\cup K''$, $\sigma\equiv\sigma_1$ in K' und $\sigma\equiv\sigma_2$ in K'' ist. Offensichtlich ist auch K_σ eine q-reguläre p-Eckzerlegung. Für die Winkelwerte $\langle v_i\rangle$ $(i=1, 2, \dots, n)$ von K_σ bestehen nach der Induktionsannahme und laut (5) und (6)

$$\langle v_i\rangle\equiv\langle a_i\rangle \pmod q \text{ falls } i\notin\{k+1, l\},$$

$$\langle v_{k+1}\rangle=\langle v''_{k+1}\rangle+\langle v'_{k+1}\rangle\equiv\langle a''_{k+1}\rangle+\langle a'_{k+1}\rangle\equiv\langle a_{k+1}\rangle \pmod q,$$

$$\langle v_l\rangle=\langle v'_l\rangle+\langle v''_l\rangle\equiv\langle a'_l\rangle+\langle a''_l\rangle\equiv\langle a_l\rangle \pmod q.$$

Wir haben somit gezeigt, daß K_σ bezüglich S die im Satz vorgeschriebenen Eigenschaften besitzt. Damit ist der Beweis beendet.

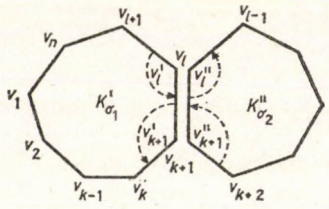

Fig. 8

(3.2) Nun können wir den noch fehlenden Teil der Charakterisierung der Randwertfolgen angeben. Es sei angenommen, daß die Zahlenfolge $u_1, u_2, \dots, u_n$ $(n\geqq 2)$ der Relation $\beta_{u_1}\beta_{u_2}\dots\beta_{u_n}=e$ des Satzes 2.1 genügt. Die Zahlen $w_1, w_2, \dots, w_n$ definieren wir durch

$$w_i\equiv u_i \pmod q, \quad 0\leqq w_i<q \quad (i=1, 2, \dots, n).$$

Nach (1.6) (20) besteht dann auch $\beta_{w_1}\beta_{w_2}\dots\beta_{w_n}=e$. Laut Satz 1.2 existiert dann eine geschlossene Kantenfolge $S=[a_1a_2\dots a_n]$ von $\{p, q\}$, für die die Relationen $\langle a_i\rangle=w_i$ $(i=1, 2, \dots, n)$ bestehen. Es gelten also auch $\langle a_i\rangle\equiv\langle u_i\rangle \pmod q$ $(i=1, 2, \dots, n)$. Nach Satz 3.1 gibt es zu S eine Zerlegung K_σ mit der Eigenschaft $\langle v_i\rangle\equiv\langle a_i\rangle \pmod q$ $(i=1, 2, \dots, n)$. Es besteht dann auch $\langle v_i\rangle\equiv u_i \pmod q$ $(i=1, 2, \dots, n)$. Wir können so folgenden Satz aussagen:

SATZ 3.2. *Genügt die Zahlenfolge $u_1, u_2, \dots, u_n$ $(n\geqq 2)$ der in Satz 2.1 definierten Relation*

(6) $$\beta_{u_1}\beta_{u_2}\dots\beta_{u_n}=e \quad (\beta_u=\alpha(\alpha\beta)^u),$$

dann kann eine solche q-reguläre p-Eckzerlegung konstruiert werden, für die $u_1, u_2, \ldots, u_n$ *eine Randwertfolge ist.*

Den Inhalt der Sätze 2.1 und 3.2 kann man kurz zusammenfassen:

SATZ 3.3. *Die Folge* $u_1, u_2, \ldots, u_n$ $(n \geqq 2)$ *von ganzen Zahlen ist dann und nur dann Randwertfolge einer q-regulären p-Eckzerlegung, wenn die Relation* (6) *besteht.*

§4. Multi-p-gonale Zerlegungen

(4.1) In der Literatur kommen neben den p-Eckzerlegungen auch *multi-p-gonale* Zerlegungen vor (s. [3], [7], [8], [9]). Bei diesen ist die Eckenanzahl einer jeden Zelle ein ganzzahliges Vielfaches von p. Man definiert die Signierung, Regularität, Randwertfolge dieser Zerlegungen in derselben Weise, wie dies für die p-Eckzerlegungen angegeben wurde. Die Charakterisierung ihrer Randwertfolgen kann einfach auf die Sätze 2.1 und 3.2 zurückgeführt werden. Nach (2.1) (4) ist nämlich mit beliebigem ganzen v

$$\beta_\varepsilon^{vp} = e \quad (\varepsilon \in \{-1, 1\}).$$

Der Satz 3.2 sagt dann die Existenz einer solchen q-regulären p-Eckzerlegung K_σ^v des vp-Ecks K^v aus, für die die Folge

$$\overset{1}{\varepsilon}, \overset{2}{\varepsilon}, \ldots, \overset{vp}{\varepsilon} \quad (v \geqq 2)$$

eine Randwertfolge ist. Ersetzt man nun in einer q-regulären multi-p-gonalen Zerlegung eine jede Zelle mit der Eckenanzahl vp $(v \geqq 2)$ durch ein K_σ^v, so entsteht eine q-reguläre p-Eckzerlegung, deren Randwertfolgen mit denjenigen der multi-p-gonalen Zerlegung übereinstimmen. Nach Satz 2.1 besteht dann der folgende

SATZ 4.1. *Ist* $r_1, r_2, \ldots, r_n$ *eine Randwertfolge einer q-regulären multi-p-gonalen Zerlegung* $\tilde{K}_\sigma$ *dann besteht, mit den Bezeichnungen des Satzes* 2.1 *die Relation*

$$\Pi(\tilde{K}_\sigma) = \beta_{r_1}\beta_{r_2}\ldots\beta_{r_n} = e. \tag{1}$$

Da aber jede p-Eckzerlegung zugleich eine multi-p-gonale Zerlegung ist, gilt der Satz 3.2 trivialerweise auch für multi-p-gonale Zerlegungen. In Betracht von Satz 4.1 besteht also auch der Satz 3.3 für multi-p-gonale Zerlegungen.

BEMERKUNG. Nach unserem Wissen war Motzkin der erste, der in Form einer Gruppenrelation eine Bedingung für Winkelwertfolgen angegeben hatte. In [9] werden mit einer Gruppe von Ordnung 24 solche vollpositiv signierten 3-regulären multitrigonalen Zerlegungen behandelt, wo der Grad jeder Ecke $\leqq 3$ ist.

A. I. Medianik befaßt sich in [8] ebenfalls mit den vorher erwähnten Zerlegungen: Es wird eine Charakterisierung der Winkelwertfolgen mit Hilfe der Gruppe $A_4 \cong \mathscr{B}(3, 3)$ angegeben.

(4.2) Im Falle $q=2$ kann die Relation (1) durch eine einfachere Bedingung ersetzt werden. Wegen $-1 \equiv 1 \pmod 2$ können wir uns auf solche Signierungen beschränken, wo jede Zelle die Signatur 1 bekommt. Die 2-Regularität its dann mit der Bedingung äquivalent, daß sämtliche inneren Ecken von geradem Grade sind.

Ferner besteht entweder $r_i \equiv 0$ oder $r_i \equiv 1 \pmod 2$, und so gilt entweder $\beta_{r_i} = \alpha$ oder $\beta_{r_i} = \beta$. Man kann daher den Randprodukt von (1) in der Form

$$\Pi(\tilde{K}) = \alpha^{s_1}\beta^{t_1}\alpha^{s_2}\beta^{t_2}\dots\alpha^{s_k}\beta^{t_k}$$

schreiben, wobei im Falle $k=1$: $s_1 \geqq 0$, $t_1 \geqq 0$, $s_1+t_1=n$ $(n \geqq 2)$ gelten, während im Falle $k \geqq 2$: $s_1 \geqq 0$, $s_i \geqq 1$ $(i=2, 3, \dots, k)$, $t_i \geqq 1$ $(i=1, 2, \dots, k-1)$, $t_k \geqq 0$, und $s_1+t_1+s_2+t_2+\dots+s_k+t_k=n$ bestehen.

Nun bekommt man von $\alpha^2=(\alpha\beta)^2=e$ ausgehend nacheinander $\beta=\alpha\beta^{-1}\alpha$, $\beta^t=\alpha\beta^{-t}\alpha=\alpha^s\beta^{t(-1)^s}\alpha^s$ und endlich

$$\alpha^s\beta^t = \beta^{t(-1)^s}\alpha^s \tag{2}$$

für beliebigen Werte von s und t. Mit den Bezeichnungen

$$\mu_i = s_1+s_2+\dots+s_i \text{ und } \tau_i = (-1)^{\mu_1}t_1+(-1)^{\mu_2}t_2+\dots+(-1)^{\mu_i}t_i \quad (1 \leqq i \leqq k) \tag{3}$$

ergibt (2) — durch einen Induktionsschluß — die Relationen

$$\alpha^{s_1}\beta^{t_1}\alpha^{s_2}\beta^{t_2}\dots\alpha^{s_i}\beta^{t_i} = \beta^{\tau_i}\alpha^{\mu_i} \quad (i = 1, 2, \dots, k, \quad k \geqq 1).$$

Die Relation (1) nimmt so die Form

$$\Pi(\tilde{K}) = \beta^{\tau_k}\alpha^{\mu_k} = e$$

an. Dies besteht jedoch dann und nur dann, wenn $\tau_k \equiv 0 \pmod p$ und $\mu_k \equiv 0 \pmod 2$ gelten.

Wir können jetzt die angekündigte Vereinfachung des Satzes 4.1 formulieren. Statt der Randwerten wollen wir lieber die Grade der Randecken auftreten lassen. Dem Randwert $r \equiv 0$, bzw. $r \equiv 1 \pmod 2$ entspricht eine Randecke von ungeradem, bzw. geradem Grade. Wir werden kurz von *ungeraden*, bzw. *geraden* Ecken sprechen.

SATZ 4.2. *Es sei* $V=[v_1v_2\dots v_n]$ *das Randpolygon einer 2-regulären multi-p-gonalen Zerlegung* $(p \geqq 3)$.

1) *Enthält V nur ungerade, bzw. gerade Ecken, dann gilt*

$$n \equiv 0 \pmod 2, \quad \text{bzw.} \quad n \equiv 0 \pmod p. \tag{4}$$

2) *Enthält V sowohl ungerade wie auch gerade Ecken, dann darf angenommen werden, daß diese die folgende Anordnung einnehmen: Die ersten* s_1 *Ecken sind ungerade, die nachfolgenden* t_1 *gerade, u.s.w. auf* s_i *ungerade Ecken folgen* t_i *gerade Ecken* $(i=1, 2, \dots, k, s_i \geqq 1, t_i \geqq 1, k \geqq 1)$. *In diesem Falle bestehen mit* $\mu_i=s_1+s_2+\dots+s_i$ $(i=1, 2, \dots, k, k \geqq 1)$

$$\sum_{i=1}^{k} s_i \equiv 0 \pmod 2 \quad \textit{und} \quad \sum_{i=1}^{k} (-1)^{\mu_i}t_i \equiv 0 \pmod p. \tag{5}$$

Umgekehrt, bestehen für die Randecken die in 1), *bzw.* 2) *formulierten Annahmen, und sind die Bedingungen* (4), *bzw.* (5) *erfüllt, so existiert eine 2-reguläre p-Eckzerlegung, deren Randecken die vorgeschriebene Anordnung einnehmen.*

BEMERKUNG. Fleischner und Roy geben in [3] eine Charakterisierung der Randwertfolgen von 2-regulären trigonalen Zerlegungen an. Diese ist mit einem Spezialfall unseres Satzes 4.2 äquivalent.

Jaeger gibt in [7] eine mit dem Satz 4.2 äquivalente Charakterisierung der Randwertfolgen von 2-regulären multi-p-gonalen Zerlegungen ($p \geqq 3$) an.

(4.3) Die 2-Eckzerlegungen besitzen eine sehr einfache Struktur, dagegen führt die Untersuchung der q-regulären multi-2-gonalen Zerlegungen zu nichttrivialen Ergebnissen. Wir haben bisher den Wert $p=2$ aus unseren Überlegungen ausgeschlossen. Es kann jedoch gezeigt werden, daß der Satz 4.1 auch für $p=2$ gültig ist. Aus diesem Satz läßt sich (im Falle $p=2$) eine sehr einfache Relation für die Randwertfolgen ableiten. Diese kann dann, auch unabhängig von Satz 4.1, mit dem gleichen Induktionsschluß bewiesen werden, mit welchem wir den Satz 2.1 erhalten haben. Ist ferner eine Zahlenfolge, die dieser Relation genügt, angegeben, dann kann in einfacher Weise eine solche q-reguläre multi-2-gonale Zerlegung konstruiert werden, für welche die gegebene Folge eine Randwertfolge ist. Hier wollen wir nur das Endergebnis mitteilen:

SATZ 4.3. *Die Zahlenfolge $u_1, u_2, \ldots, u_n$ ($n \geqq 2$) ist dann und nur dann Randwertfolge einer q-regulären multi-2-gonalen Zerlegung ($q \geqq 2$), wenn die Bedingungen*

$$n \equiv 0 \pmod 2 \quad \textit{und} \quad u_1 - u_2 + u_3 - u_4 + \ldots + u_{n-1} - u_n \equiv 0 \pmod q$$

erfüllt sind.[4]

Im Fall $q=2$ wird die zweite Kongruenz von jeder 2-regulären multi-2-gonalen Zerlegung trivialerweise erfüllt. Denn es ist in diesem Fall

$$u_1 - u_2 + u_3 - u_4 + \ldots + u_{n-1} - u_n \equiv \sum_{i=1}^{n} u_i \pmod 2,$$

und $u_1 + u_2 + \ldots + u_n \equiv 0 \pmod 2$ ist gleichbedeutend mit der Behauptung, daß die Anzahl der Randecken von ungeradem Grade gerade ist. Da jedoch jetzt sämtliche inneren Ecken von geradem Grade sind, ist die Bedingung äquivalent mit der Forderung, daß die Anzahl sämtlicher Ecken von ungeradem Grade gerade sei.

§5. Vollpositive Signierungen

(5.1) Eine Signierung heißt *vollpositiv*, falls jede Zelle die Signatur 1 besitzt. B. Grünbaum hat uns auf die folgende Tatsache aufmerksam gemacht:[5] Gibt man dem Schlegelschen Diagramm des regulären Polyeders $\{p, q\}$ die vollpositive Signierung, dann entsteht eine q-reguläre p-Eckzerlegung P_+ eines p-Ecks. (P_+ besitzt um 1 weniger Zellen als $\{p, q\}$.) (S. Fig. 9.) Die Winkelwerte von P_+ sind alle gleich $q-1$, die Folge $r_i = -1$ ($i=1, 2, \ldots, p$) ist also eine Randwertfolge von P_+. Sie ist zugleich auch die Randwertfolge jener Zerlegung, die aus einer einzigen — mit -1 signierten — Zelle besteht. Ersetzt man in einer beliebig-signierten q-regulären p-Eckzerlegung K_σ jede mit -1 signierte Zelle durch ein P_+, so entsteht eine vollpositiv-signierte q-reguläre p-Eckzerlegung K'_+. Die Zerlegungen K'_+ und K_σ haben ein gemeinsames Randpolygon und übereinstimmende Randwertfolgen. Es kann also

[4] Offensichtlich genügt jede multi-2-gonale Zerlegung der ersten Kongruenz.
[5] Briefliche Mitteilung (1968).

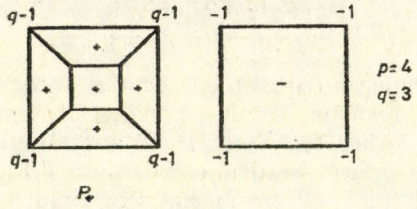

Fig. 9

falls $\{p, q\}$ endlich und $q \geqq 3$ ist, jede Randwertfolge, die durch Heawoodsche Signierung erzeugt wurde, auch durch vollpositive Signierung verwirklicht werden. Im Falle $q=2$ folgt dies unmittelbar aus $-1 \equiv 1 \pmod 2$.

(5.2) Es stellt sich nun die Frage: läßt sich ein ähnliches Verhalten auch bei unendlichen $\{p, q\}$ finden? Wir zeigen, daß dies nicht der Fall ist. Es wird nämlich folgender Satz bewiesen:

SATZ 5.1. *Im Falle* $(p-2)(q-2) \geqq 4$ *kann die Randwertfolge* $r_i = -1$ $(i=1, 2, \ldots, p)$ *durch vollpositiv-signierte q-reguläre p-Eckzerlegungen nicht erzeugt werden.*

BEWEIS. Nehmen wir an, daß die vollpositiv-signierte q-reguläre p-Eckzerlegung K_+ das Randpolygon $V=[v_1 v_2 \ldots v_p]$ und die Randwertfolge $r_i = -1$ $(i=1, 2, \ldots, p)$ besitzt. Die Winkelwertfolge von K_+ hat dann folgende Form:

$$\langle v_i \rangle = \lambda_i q - 1 \quad (\lambda_i \geqq 1, \quad i = 1, 2, \ldots, p).$$

Bezeichne k, e, bzw. f die Anzahl der Kanten, (sämtlicher) Ecken, bzw. Zellen (Flächenstücke) von K_+. Da mit jeder Ecke mindestens q Kanten inzidieren, gilt $eq \geqq 2k$. Ferner besteht $fp+p=2k$. Es ist also

$$e \leqq \frac{2k}{q} \quad \text{und} \quad f = \frac{2k}{p} - 1.$$

Aus der Eulerschen Gleichung $e+f=k+1$, bekommt man so

$$\frac{2k}{q} + \frac{2k}{p} - 1 \geqq k+1,$$

woraus

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$$

folgt. Dies steht jedoch mit $(p-2)(q-2) \geqq 4$ in Widerspruch.

So sehen wir, daß im Falle $(p-2)(q-2) \geqq 4$ nicht alle Randwertfolgen durch vollpositive Signierungen erzeugt werden können. Es ergibt sich daher folgendes

PROBLEM. Man soll im Falle $(p-2)(q-2) \geqq 4$ die Charakterisierung der Randwertfolgen der vollpositiv-signierten q-regulären p-Eckzerlegungen angeben.

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ON THE REACHABILITY SET OF A STRING

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Consider a string along the segment [0, 1] fixed at the two ends 0 and 1 and controlled at some point $0<a<1$ with a function $u(t)$. If we denote by $y(x, t)$ the distance of the point x of the string from the equilibrium state at time t, then we have the equation

$$\varrho(x)\, y_{tt} = y_{xx} + \delta(x-a)\, u(t), \quad x\in(0,1), \quad t>0, \tag{1/a}$$

with boundary condition

$$y(0, t) = y(1, t), \quad t>0, \tag{1/b}$$

where $a\in[0, 1]$ (cf. [1—3]).

Here $\varrho(x)$ is the linear density of the substance. (1/a) is the equation of motion, when the density of external force is nonzero only in a small neighborhood of the point a and it is equal to $u(t)$ at the moment t.

We can investigate two problems here: the controllability of the string and the reachable movement states. The first question (investigated by A. G. Butkovskii in [3]) is to find conditions for the position of the point $0<a<1$ such that for any initial conditions

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x) \tag{2}$$

given in some function space, the string can be relaxed in a finite time $T<\infty$, i.e. the control $u(t)$ can be given such that

$$y(x, T) = y_t(x, T) = 0.$$

The other question (investigated in [12]) refers to the structure of the reachability set $D_a(T) := \{(y(\cdot, T), y_t(\cdot, T))$: $u(t)$ runs over some space of controls$\}$.

In this case we suppose in general that

$$y(x, 0) = y_t(x, 0) = 0. \tag{3}$$

In both cases we can apply the Fourier method following the treatment of [12]. Arguing formally, we are looking for the solution $y(x, t)$ in the form

$$y(x, t) = \sum_n c_n(t)\, v_n(x), \tag{4}$$

where $\{v_n\}$ are the solutions of the boundary value problem

$$v_n''(x) + \lambda_n \varrho(x) v_n(x) = 0, \quad v_n(0) = v_n(1) = 0. \tag{5}$$

Suppose $\varrho \in C^2[0, 1]$, $\varrho > 0$. Then for the function

$$w_n(t) := v_n(\varphi(t)) \sqrt[4]{\varrho(\varphi(t))}$$

$$\left(\varphi := f^{-1}\colon \left[0, \int_0^1 \sqrt{\varrho}\right] \to [0, 1], \quad f(x) := \int_0^x \sqrt{\varrho}\right)$$

the equations

$$w_n'' + q w_n + \lambda_n w_n = 0 \quad \text{on } \left[0, \int_0^1 \sqrt{\varrho}\right], \quad q = \frac{-\left(\sqrt[4]{\varrho \circ \varphi}\right)''}{\sqrt[4]{\varrho \circ \varphi}},$$

$$w_n(0) = w_n\left(\int_0^1 \sqrt{\varrho}\right) = 0$$

hold. By [4, pp. 74 and 91] we know that

$$\lambda_n = \left(n \frac{\pi}{h}\right)^2 + O(1) \quad \text{where} \quad h := \int_0^1 \sqrt{\varrho} \tag{6}$$

and (w_n) forms an orthonormal basis in $L^2(0, h)$; so (v_n) forms also an orthonormal basis in $L^2_\varrho(0, 1)$.

$$\left(\int_{\varphi^{-1}(0)}^{\varphi^{-1}(1)} [(v_n \circ \varphi) \sqrt[4]{\varrho \circ \varphi}]\, [(f \circ \varphi) \sqrt[4]{\varrho \circ \varphi}] = \int_{\varphi^{-1}(0)}^{\varphi^{-1}(1)} (v_n \circ \varphi)(f \circ \varphi)(\varrho \circ \varphi)\, \varphi' = \int_0^1 f v_n \varrho.\right)$$

From (1/a) and (1/b) we get for any $z \in C_0^\infty((0, 1) \times (0, T))$

$$\int_0^1 \int_0^T \varrho(x)\, y_{tt}(x, t)\, z(x, t)\, dx\, dt = \int_0^T \int_0^1 y_{xx}(x, t)\, z(x, t)\, dx\, dt + \int_0^T z(a, t)\, u(t)\, dt.$$

Twofold (formal) integration by parts gives

$$\int_0^T y_{tt}(x, t)\, z(x, t)\, dt = [y_t(x, t)\, z(x, t) - y(x, t)\, z_t(x, t)]_{t=0}^{t=T} + \int_0^T y(x, t)\, z_{tt}(x, t)\, dt.$$

We have no information about $y_t(x, T)$ and $y(x, T)$, hence it is convenient to suppose $z(x, T) = z_t(x, T) = 0$ and then the term in the brackets is $y_1(x) z(x, 0) - y_0(x) z_t(x, 0)$. Analogously,

$$\int_0^1 y_{xx}(x, t)\, z(x, t)\, dx = [y_x(x, t)\, z(x, t) - y(x, t)\, z_x(x, t)]_{x=0}^{x=1} + \int_0^1 y(x, t)\, z_{xx}(x, t)\, dx;$$

since $y_x(0, t)$ and $y_x(1, t)$ are unknown, we suppose that $z(0, t) = z(1, t) = 0$. The above arguments justify the following interpretation of the solution of (1/a) and (1/b).

DEFINITION 1. The solution of (1/a)—(1/b) is a function $y(x, t)\in L^2((0, 1)\times(0, T))$ for which the relation

$$(7)\qquad \int_0^1 \varrho(x)\,[y_1(x)\,z(x,0)-y_0(x)\,z_t(x,0)]\,dx+$$

$$+\int_0^1\int_0^T y(x,t)\,[\varrho(x)\,z_{tt}(x,t)-z_{xx}(x,t)]\,dt\,dx=\int_0^T z(a,t)\,u(t)\,dt$$

holds for all $z\in C^2([0, 1]\times[0, T])$ with the properties $z(x, T)=z_t(x, T)=0$, $z(0, t)=$ $=z(1, t)=0$.

In [12] the following result is proved.

THEOREM A [12]. *Let* $y_0\in H^1(0, 1)$, $y_1\in L^2(0, 1)$, $\varrho\in C^2(0, 1)$, $\varrho>0$. *Then for any control* $u\in L^2(0, T)$ *the problem* (1/a)—(1/b) *has a unique solution* $y(x, t)=$ $=\sum c_n(t)v_n(x)$; *this sum converges in* $L^2((0, 1)\times(0, T))$ *and* y *has such a representation for which the sum converges in* $L^2(0, 1)$ *for all* t. *Further we have*

$$(8)\qquad c_n(t)=c_{n,0}\cos\sqrt{\lambda_n}\,t+c'_{n,0}\frac{\sin\sqrt{\lambda_n}\,t}{\sqrt{\lambda_n}}+v_n(a)\int_0^t u(\tau)\frac{\sin\sqrt{\lambda_n}\,(t-\tau)}{\sqrt{\lambda_n}}\,d\tau,$$

where

$$y_0(x)=\sum c_{n,0}v_n(x),\quad y_1(x)=\sum c'_{n,0}v_n(x)$$

are the corresponding Fourier series. With the additional assumption

$$\sum\lambda_n|c_{n,0}|^2<\infty$$

we have also that $y\in H^1((0, 1)\times(0, T))$, $x\mapsto y(x, t)\in H^1(0, 1)$ *for all* t *and the trajectory* $t\mapsto(y(x, t), y_t(x, t))$ *is a continuous maping from* $[0, T]$ *to* $Y:=H^1(0, 1)\oplus$ $\oplus L^2(0, 1)$. *Also,* $y(\cdot, t)\in H^1_{(x)}(0, 1)$, $y(x, \cdot)\in H^1_{(t)}(0, T)$.

REMARK 1. The system $\{v'_n/\sqrt{\lambda_n}\}$ is orthonormal in $L^2(0, 1)$

$$\left(\int v'_nv'_m=-\int v_nv''_m=\lambda_m\int v_nv_m=\lambda_m\cdot\delta_{nm}\right),$$

hence the condition $\sum\lambda_n|c_{n,0}|^2<\infty$ is fulfilled iff the series $y_0(x)=\sum c_{n,0}v_n(x)$ can be differentiated term by term: $y'_0(x)=\sum c_{n,0}\sqrt{\lambda_n}\,\frac{v'_n(x)}{\sqrt{\lambda_n}}$. In this case the partial derivatives of $y(x, t)$ can be taken also term by term (cf. [12]).

Let the reachability sets be

$$D_a(T):=\{(y(\cdot, T), y_t(\cdot, T)):\quad u\in L^2(0, T)\}$$

and

$$B_a(T):=\{(\sqrt{\lambda_n}\,c_n(T)+ic'_n(T)):\quad u\in L^2(0, T)\}.$$

THEOREM B [12]. *Suppose* $\sum\lambda_n|c_{n,0}|^2<\infty$. *Then there exists a closed subspace* $Z\subset Y$ *and an isomorphism* J: $Z\to l_2$ *for which* $J(D_a(T))=B_a(T)$ *for all* $T<\infty$.

On the basis of Theorem B it is proved in [12] the following

THEOREM C [12]. *Let* $\varrho \equiv 1$, $a=p/q$ *rational*, $(p, q)=1$, *and suppose* (3) *is fulfilled. Then*

(i) $D_a(T_1) = D_a(T_2)$ *if* $2\dfrac{q-1}{q} \leqq T_1 < T_2$,

(ii) $D_a(T_1) \supseteqq D_a(T_2)$ *if* $T_1 < T_2 \leqq 2\dfrac{q-1}{q}$,

(iii) $D_a(T) \subset Z$ *is closed and of infinite codimension in* Z *for arbitrary* $T < \infty$.

The aim of the present paper is to extend Theorem C in different directions, namely for irrational a and for non-homogeneous strings ($\varrho \not\equiv 1$) in Theorems 1 and 2 below.

THEOREM 1. *Let* $0<a<1$ *be irrational*, $\varrho=1$ *and suppose* (3) *is fulfilled. Then*

(i) $D_a(T)=\overline{D_a(T)}$ *if* $T<2$ *and* $D_a(T)$ *is of infinite codimension in* Z *for every* $0<T<\infty$,

(ii) $D_a(T_1) \neq D_a(T_2)$ *if* $T_1 < T_2 \leqq 2$,

(iii) $D_a(T_1) = D_a(T_2)$ *if* $2 \leqq T_1 \leqq T_2$,

(iv) $D_a(T) \neq \overline{D_a(T)}$ *if* $2 \leqq T$.

PROOF. We use essentially that for $\varrho \equiv 1$ we have

$$\lambda_n = (\pi n)^2, \quad v_n(x) = \frac{1}{\sqrt{2}} \sin n\pi x.$$

Proof of (i) is based on a result of S. A. Avdonin [8]. For its formulation we need the following notions.

DEFINITION 2. Let $\Lambda := \{\lambda_n : n \in \mathbf{Z}\}$ be a separable set (i.e. $\inf\limits_{n \neq m} |\lambda_n - \lambda_m| > 0$) lying in some strip $|\operatorname{Im} z| \leqq c < \infty$. A partition $\Lambda = \bigcup\limits_{j \in \mathbf{Z}} \Lambda_j$ of Λ is said to be an A-partition if there exists an increasing sequence $\{\alpha_j : j \in \mathbf{Z}\}$ for which

$$\Lambda_j = \{\lambda_n : \alpha_j \leqq \operatorname{Re} \lambda_n < \alpha_{j+1}\}$$

and

$$\sup_j (\alpha_{j+1} - \alpha_j) < \infty, \quad \inf_j \alpha_j = -\infty, \quad \sup_j \alpha_j = +\infty.$$

DEFINITION 3 [10]. An entire function of exponential type is said to be of sine type if its zeros lie in a strip $|\operatorname{Im} z| \leqq c < \infty$ and there exists $y_0 \in \mathbf{R}$ such that

$$0 < \inf_{x \in \mathbf{R}} |f(x+iy_0)| \leqq \sup_{x \in \mathbf{R}} |f(x+iy_0)| < +\infty.$$

THEOREM D (S. A. Avdonin [8]). *Let* Λ *be the set of zeros of a function of sine type, the indicator diagram of which has horizontal width equal to* a (*i.e.* $\overline{G}_F=[0, ia]$; cf. [9]). *Let* $\{\delta_\lambda : \lambda \in \Lambda\}$ *be an arbitrary bounded set of complex numbers for which there exists an A-partition* $\Lambda = \bigcup\limits_j \Lambda_j$ *such that*

$$\Big| \sum_{\lambda \in \Lambda_j} \operatorname{Re} \delta_\lambda \Big| \leqq d\alpha_j, \tag{9}$$

where $d<1/4$. *If the set* $\{\lambda+\delta_\lambda\colon \lambda\in\Lambda\}$ *is separable, then the sequence* $\{e^{i(\lambda+\delta_\lambda)x}\colon \lambda\in\Lambda\}$ *is a Riesz basis in* $L^2(0, a)$.

For the proof of (i) first remark the following trivial fact. Given any $\varepsilon>0$, denote by $\{n_k\}$ a monotone increasing sequence running over the set $\{n\in\mathbf{N}\colon |\sin(n\pi a)|<\varepsilon\}$. Then for every $N\in\mathbf{N}$ there exists an $\varepsilon>0$ for which

$$n_{k+1}-n_k > N \quad (k=1,2,\ldots). \tag{10}$$

For $T<2$ there exists an $N\in\mathbf{N}$ such that $(2/T)N>N+2$. Fix an $\varepsilon>0$ such that (10) be fulfilled for this N. We shall prove the existence of a series $(\mu_j)\subset\{n\in\mathbf{Z}\colon |\sin(n\pi a)|\geqq\varepsilon\}$, for which $\{\exp(i\mu_j\pi x)\}$ forms a Riesz basis in $L^2(0, T)$. Let $n>0$ and define

$$\left(\widehat{\frac{2n}{T}}\right) := \begin{cases} \left[\frac{2n}{T}\right] \text{ if } \left[\frac{2n}{T}\right] \neq \left(\widehat{\frac{2(n-1)}{T}}\right) \text{ and } \left[\frac{2n}{T}\right] \notin \{n_k\}, \\ \left[\frac{2n}{T}\right]+1 \quad \text{otherwise,} \end{cases}$$

$$\{d_r\colon r\in\mathbf{N}\} := \mathbf{N}\setminus\{n_k\}\setminus\left\{\left(\widehat{\frac{2n}{T}}\right)\colon n\in\mathbf{N}\right\}.$$

Obviously

$$d_{r+1}-d_r < 2N. \tag{11}$$

Let

$$\Lambda_j := \left\{\frac{2\pi}{T}n\colon 100jN \leqq \frac{2n}{T} < 100(j+1)N\right\} \quad (j\geqq 0)$$

Denote the elements of $\{d_r\colon r\in\mathbf{N}\}$ belonging to $(T/2\pi)\Lambda_j$ by $d_{r_j}, d_{r_j+1}, \ldots, \ldots, d_{r_{j+1}-1}$ and define

$$\left(\widetilde{\frac{2n}{T}}\right) := \begin{cases} \left(\widehat{\frac{2n}{T}}\right) \text{ if } n\in\frac{T}{2\pi}\Lambda_j \text{ and } \left(\widehat{\frac{2n}{T}}\right) < d_{r_j}, \\ \left(\widehat{\frac{2n}{T}}\right)-1 \text{ if } d_{r_j} < \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+1} \text{ and } \left(\widehat{\frac{2n}{T}}\right)-1 \notin \{n_k\}, \\ \left(\widehat{\frac{2n}{T}}\right)-2 \text{ if } d_{r_j} < \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+1} \text{ and } \left(\widehat{\frac{2n}{T}}\right)-1 \in \{n_k\}. \end{cases}$$

If

$$\sum_j := \sum_{\substack{n\in\frac{T}{2\pi}\Lambda_j \\ \left(\widehat{\frac{2n}{T}}\right)<d_{r_j+1}}} \left(\frac{2n}{T}-\left(\widetilde{\frac{2n}{T}}\right)\right) < 0,$$

then set

$$\left(\widecheck{\frac{2n}{T}}\right) := \left(\widehat{\frac{2n}{T}}\right) \quad \text{for} \quad d_{r_j+1} < \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+2};$$

and if $\sum_j \geqq 0$, then define

$$\left(\widetilde{\frac{2n}{T}}\right) := \begin{cases} \left(\widehat{\frac{2n}{T}}\right) - 1 & \text{if } d_{r_j+1} < \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+2} \text{ and } \left(\widehat{\frac{2n}{T}}\right) \notin \{n_k\}, \\ \left(\widehat{\frac{2n}{T}}\right) - 2 & \text{if } d_{r_j+1} < \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+2} \text{ and } \left(\widehat{\frac{2n}{T}}\right) \in \{n_k\}. \end{cases}$$

Continue this process investigating the sign of the sum

$$\sum_{\substack{n \in \frac{T}{2\pi} \Lambda_j \\ \left(\widehat{\frac{2n}{T}}\right) < d_{r_j+2}}} \left(\frac{2n}{T} - \left(\widetilde{\frac{2n}{T}}\right) \right)$$

etc. until reaching $d_{r_{j+1}}$. At last set for $n<0$, $\Lambda_{-j} := -\Lambda_j$ and $\left(\widetilde{\frac{2n}{T}}\right) := -\left(\widetilde{\frac{-2n}{T}}\right)$. Then, according to the construction above one can apply the Avdonin's Theorem, consequently $\left\{\exp\left[i\left(\widetilde{\frac{2n}{T}}\right)\pi x\right]\right\}_{n\in\mathbf{Z}}$ is a Riesz basis in $L^2(0, T)$, hence $\{\mu_j : j \in \mathbf{Z}\} := \left\{\left(\widetilde{\frac{2n}{T}}\right) : n \in \mathbf{Z}\right\}$ is satisfactory.

Now we show that $B_a(T)$ is closed. According to Theorem B this is equivalent with the statement that $D_a(T)$ is closed. Suppose that for some $u_k \in L^2(0, T)$ the sequence $\{\alpha_n(u_k) : n \in \mathbf{N}\}$ converges in l_2. Then the sequences

$$\left\{\int_0^T u_k(\tau) \sin n\pi(T-\tau)\, d\tau\right\}_{n\in N_e}, \quad \left\{\int_0^T u_k(\tau) \cos n\pi(T-\tau)\, d\tau\right\}_{n\in N_e},$$

converge in $l_2(N_e)$, where $N_e := \mathbf{N} \setminus \{n_k\}$. Hence the sequence

$$\left\{\int_0^T u_k(\tau)\, e^{i\left(\widetilde{\frac{2n}{T}}\right)(T-t)\pi}\, d\tau\right\}_{n\in\mathbf{Z}}$$

converges in l_2 too, i.e. there exists $u \in L^2(0, T)$ such that $u_k \to u$ in $L^2(0, T)$ and hence $D_a(T)$ is closed.

For the proof of $\operatorname{codim}_{\mathbf{Z}} D_a(T) = \infty$ it is enough to prove that $\operatorname{codim}_{l_2} B_a(T) = \infty$. To this consider a sequence $\{n_k\} \subset \mathbf{N}$ for which $\{\sin \pi n_k a\}_{k\in\mathbf{N}} \in l_2$. Then a series of the form $\{\alpha_{n_k}(u) : u \in L^2(0, T)\}$ belongs to $l_1(\{n_k\})$ only, which is of infinite codimension in $l_2(\{n_k\})$, hence $\operatorname{codim}_{l_2} B_a(T) = \infty$.

Proof of (ii). Consider the characteristic function $\chi_{[T_1, T_2]}$ of the segment $[T_1, T_2]$ in $L^2(0, 2)$. Then if there were a $u \in L^2(0, T_1) \subset L^2(0, 2)$ such that $\alpha_n(u) - \alpha_n(\chi_{[T_1, T_2]}) = 0$ $(n = 0, 1, \ldots)$, then all the Fourier coefficients of $u - x_{[T_1, T_2]} \in L^2(0, 2)$ would be zeros, hence $u \equiv \chi_{[T_1, T_2]}$, a contradiction.

Proof of (iii). It can be seen from (7) and (8) that for $T \geqq 2$

$$\{(\alpha_n(u)) : u \in L^2(0, T)\} \equiv \left\{\left(c_n \sin\left[\frac{n}{2}\right]\pi\right) : (c_n) \in l_2\right\}.$$

Proof of (iv). It is trivial that the finite sequences belong to $B_a(T)$ if $T \geqq 2$. On the other hand $B_a(T) \neq l_2$ as we have proved in (i), consequently $B_a(T) \neq \overline{B_a(T)}$.

THEOREM 2. *Suppose* $\varrho \in C^2[0,1]$, $\varrho(x) > 0$ $(0 \leqq x \leqq 1)$ *and* $c'_{n,0} = 0$ *for all* n. *Then for almost all* $a \in (0,1)$ *we have*

(i) $D_a(T) = \overline{D_a(T)}$ *if* $T < \hat{T}$ *and* $\operatorname{codim}_Z D_a(T) = \infty$ *for every* $0 < T < \infty$,

(ii) $D_a(T_1) \neq D_a(T_2)$ *if* $T_1 < T_2 \leqq \hat{T}$,

(iii) $D_a(T) \neq \overline{D_a(T)}$ *if* $T \geqq \hat{T}$,

(iv) $D_a(T_1) = D_a(T_2)$ *if* $\hat{T} \leqq T_1 \leqq T_2$, *where* $\hat{T} := 2\int_0^1 \sqrt{\varrho(x)}\, dx$.

PROOF. We showed in (6) that

$$\sqrt{\lambda_n} = \frac{2n\pi}{\hat{T}} + O\left(\frac{1}{n}\right). \tag{12}$$

On the other hand we have

$$w_n(x) = \sin\frac{2\pi n}{\hat{T}}x + O\left(\frac{1}{n}\right) \tag{13}$$

uniformly in $x \in [0, \hat{T}/2]$. Indeed, [4, p. 58, Theorem 1] shows that the equation $y'' + qy + \lambda y = 0$ has two linearly independent solutions y_1, y_2 such that

$$\begin{cases} y_k(x) = e^{i(-1)^k\sqrt{\lambda}x}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right] \\ y'_k(x) = i(-1)^k e^{i(-1)^k\sqrt{\lambda}x}\left[1 + O\left(\frac{1}{\sqrt{\lambda}}\right)\right] \end{cases} \quad (k = 1, 2). \tag{14}$$

Now w_n must be a linear combination of y_1 and y_2:

$$w_n(x) = y_1(x)\,y_2(0) - y_2(x)\,y_1(x) = \sin\sqrt{\lambda_n}\,x\left(1 + O\left(\frac{1}{\sqrt{\lambda_n}}\right)\right) = \sin\frac{2\pi n}{\hat{T}}x + O\left(\frac{1}{n}\right).$$

Finally we get

$$v_n(x) = w_n(f(x))\,\varrho^{-1/4}(x) = \varrho^{-1/4}(x)\sin\left(\frac{2\pi n}{\hat{T}}f(x)\right) + O\left(\frac{1}{n}\right) \tag{15}$$

uniformly in $x \in [0,1]$.

The next step in the proof is to show that $(e^{i\sqrt{\lambda_n}x})$ forms a Riesz basis in $L^2(0, \hat{T})$ in its closed linear hull. We shall use the following known results.

THEOREM E (Levinson [5]). *Suppose that* $(e^{i\mu_n x})$ *is complete in* $L^2(0,a)$. *If* $\mu_1, \mu_2, \ldots, \mu_N$ *is changed by any different real numbers* $\mu'_1, \mu'_2, \ldots, \mu'_N$ *which are also different from* $\mu_{N+1}, \mu_{N+2}, \ldots$ *then* $(e^{i\mu'_n x})_1^N \cup (e^{i\mu_n x})_{N+1}^{\infty}$ *is also complete in* $L^2(0,a)$.

THEOREM F (Bari [6]). *Let* (φ_n) *be an orthonormal basis in some Hilbert space* H *and* $(\psi_n) \subset H$ *be such that* $\|\psi_n\| = 1$, $\sum \|\varphi_n - \psi_n\|^2 < 1$. *Then* (ψ_n) *is a Riesz basis in* H.

THEOREM G (Bari [6]). *Let (φ_n) be a Riesz basis in some Hilbert space H and $(\psi_n)\subset H$ be such that $\|\psi_n\|=1$, $\sum\|\varphi_n-\psi_n\|^2<\infty$. Then (ψ_n) is minimal iff it is complete and in this case (ψ_n) is also a Riesz basis in H.*

Returning to the proof of Theorem 2, the estimate

$$\int_0^{\hat{T}}\left|e^{\sqrt{\lambda_n}x}-e^{i\frac{2\pi n}{\hat{T}}x}\right|^2dx=\int_0^{\hat{T}}\left|e^{io\left(\frac{1}{n}\right)x}-1\right|^2dx=2\hat{T}\left(1-\frac{\sin O\left(\frac{1}{n}\right)\hat{T}}{O\left(\frac{1}{n}\right)\hat{T}}\right)=O\left(\frac{1}{n^2}\right)$$

shows by Theorem F that for large N the system

$$\left(e^{i\frac{2\pi n}{\hat{T}}x}\right)_{n=-N}^{N}\cup\left(e^{i\sqrt{\lambda_n}x},\,e^{-i\sqrt{\lambda_n}x}\right)_{n=N+1}^{\infty}$$

is a Riesz basis in $L^2(0,a)$. From Theorem E we see that

$$\left(e^{i\sqrt{\lambda_n}x}\right)_{n=0}^{\infty}\cup\left(e^{-i\sqrt{\lambda_n}x}\right)_{n=1}^{\infty}$$

is complete hence by Theorem G it is a Riesz basis in $L^2(0,\hat{T})$.

(iv) Now suppose $\hat{T}\leqq T_1\leqq T_2$. We have

$$\sqrt{\lambda_n}\,c_n(T)+ic_n'(T)=iv_n(a)\int_0^T u(T-\tau)\,e^{-i\sqrt{\lambda_n}\tau}\,d\tau.$$

It is well known that if (φ_n) is a complete system in some Hilbert space H then it is a Riesz basis iff for any finite sum $\|\sum c_n\varphi_n\|^2\asymp\sum|c_n|^2$ holds. Taking into account this fact and the above considerations it follows that

$$\left(e^{i\sqrt{\lambda_0}x},\sin\sqrt{\lambda_n}x,\cos\sqrt{\lambda_n}x\right)_{n=1}^{\infty} \tag{16}$$

is a Riesz basis in $L^2(0,\hat{T})$ or in any segment of length $\hat{T}$. Now

$$V_{L^2(N\hat{T},(N+1)\hat{T})}\left(\sin\sqrt{\lambda_n}x,\cos\sqrt{\lambda_n}x:\ n\geqq 1\right)=V_{L^2(N\hat{T},(N+1)\hat{T})}\left(e^{i\sqrt{\lambda_n}x},e^{-i\sqrt{\lambda_n}x}:n\geqq 1\right)=$$

$$=\tau_{N\hat{T}}\left\{V_{L^2(0,\hat{T})}\left(e^{i\sqrt{\lambda_n}x},e^{-i\sqrt{\lambda_n}x}:\ n\geqq 1\right)\right\}=\tau_{N\hat{T}}\left\{V_{L^2(0,\hat{T})}\left(\sin\sqrt{\lambda_n}x,\cos\sqrt{\lambda_n}x:n\geqq 1\right)\right\}$$

where

$$\tau_{N\hat{T}}:\ L^2(0,\hat{T})\to L^2\left(N\hat{T},(N+1)\hat{T}\right)$$

is the shift in the arguments of the functions. We see that the moment space

$$\left\{(c_n)=\left(\int_{N\hat{T}}^{(N+1)\hat{T}}u(\tau)\,e^{i\sqrt{\lambda_n}\tau}\,d\tau\right)_{n=0}^{\infty}:\ u\in L^2\left(N\hat{T},(N+1)\hat{T}\right)\right\}$$

is the set of all sequences in l_2 such that $\alpha\,\mathrm{Re}\,c_0+\beta\,\mathrm{Im}\,C_0$ with α, β independent of N and such that $\alpha^2+\beta^2>0$. This moment space will be $B_a(T)$ for any $T\geqq\hat{T}$.

(ii) It is widely known [7, Ch. XI, Corollary 5.2] that the eigenfunctions of a Sturm—Liouville operator have only finitely many zeros. Hence for all $0<a<1$ except for a denumerable set we may suppose $v_n(a)\neq 0$ for all n. Let $T_1<T_2\leqq\hat{T}$.

If the function $\chi_{[T_1, T_2]}$ has the same coefficients $\int_0^T \chi_{[T_1, T_2]}(\tau) e^{-i\sqrt{\lambda_n}\tau}\, d\tau$ as some function in $L^2(0, T_1)$, then their difference, being real, has only zero coefficients with respect to (16) which is a Riesz basis in $L^2(0, \hat{T})$, a contradiction.

The cases (i) and (iii) can be proved similarly to (i) and (iv) of Theorem 1, resp. Theorem 2 is proved.

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THE GENERAL SOLUTION OF A FUNCTIONAL EQUATION ARISING IN INFORMATION THEORY

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1. Introduction

An axiomatic characterization of the so-called entropy of degree α leads to the investigation of the following functional equation:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi(p_i q_j)=\sum_{i=1}^{n}\varphi(p_i)\sum_{j=1}^{m}\varphi(q_j) \tag{0}$$

where $\varphi\colon [0,1]\to\mathbf{R}$ is the unknown function, n and m are fixed natural numbers greater than one and equation (0) holds for all $p_i, q_j\in[0,1]$, $i=1,\dots,n$, $j=1,\dots,m$, $\sum_{i=1}^{n}p_i=\sum_{j=1}^{m}q_j=1$. (See equation (8) in [2].) In [2] the general solution of (0) has been given when $n\geqq 3$ and $m\geqq 3$. If $n\geqq 3$, $m=2$, $\varphi\colon [0,1]\to\mathbf{R}$ is a solution of (0) and $\varphi(q_0)+\varphi(1-q_0)\neq 1$ for some $q_0\in[0,1]$ then, applying the method given in [2], (0) can be reduced to

$$f(pq)+f(p(1-q))=f(p)[f(q)+f(1-q)] \tag{1}$$

which holds for all $p, q\in[0,1]$, and for the unknown function $f\colon [0,1]\to\mathbf{R}$ we have that the function $f-\varphi$ is a restriction to $[0,1]$ of a function $a\colon \mathbf{R}\to\mathbf{R}$ which is additive, that is, $a(x+y)=a(x)+a(y)$ for all $x, y\in\mathbf{R}$.

The subject of this paper is to give all solutions of (1) by showing that they can be expressed by the solutions of the fundamental Chauchy-equations

$$f(xy)=f(x)f(y)\quad (x, y\in[0,1]) \tag{2}$$

and

$$a(x+y)=a(x)+a(y)\quad (x, y\in\mathbf{R}). \tag{3}$$

In our opinion the investigation of equation (1) may be interesting only from the point of view of the theory of functional equations. ((1) can be obtained by adding the fundamental Cauchy-equations $f(pq)=f(p)f(q)$ and $f(p(1-q))=f(p)f(1-q)$.) This is the reason why we do not deal with the consequences of our main result concerning the characterization of certain information measures.

2. The general solution of (1)

Obviously, if $f\colon [0,1]\to\mathbf{R}$ is a constant solution of (1); then $f(p)=0$ for all $p\in[0,1]$ or $f(p)=1$ for all $p\in[0,1]$. In both cases f satisfies (2). With the substitution $q=1$, (1) implies $f(p)[f(1)+f(0)-1]=f(0)$ for all $p\in[0,1]$. Therefore if $f\colon [0,1]\to\mathbf{R}$ is a non-constant solution of (1) then $f(0)=0$ and $f(1)=1$. In this

case define the set L by

$$L = \{\alpha\in[0, 1]: f(t\alpha) = f(t)f(\alpha) \text{ for all } t\in[0, 1]\}.$$

This set will play an important role in our discussions. First we prove the following

LEMMA 1. *If* f: $[0, 1]\to\mathbf{R}$ *is a non-constant solution of* (1) *and* $f(\alpha_0)=0$ *for some* $0<\alpha_0\in L$ *then*

$$f(t) = \begin{cases} 0 & \text{if } 0 \leqq t < 1, \\ 1 & \text{if } t = 1. \end{cases} \tag{4}$$

We remark that the function f defined by (4) satisfies (2).

PROOF. Since $f(\alpha_0)=0$, by the definition of L, $f(t)=0$ for all $t\in[0, \alpha_0]$. (1) implies $1-\alpha_0\in L$, that is, $f(p(1-\alpha_0))=f(p)f(1-\alpha_0)$ for all $p\in[0, 1]$. Thus by induction, we get $f(p(1-\alpha_0)^k)=f(p)(f(1-\alpha_0))^k$ for all $p\in[0, 1]$ and natural numbers k. Choose p and k so that $p(1-\alpha_0)^k\in[0, \alpha_0]$ and $f(p)\neq 0$. Thus we have $f(1-\alpha_0)=0$, consequently $f(t)=0$ for all $t\in[0, 1-\alpha_0]$. Now we have proved that $f(t)=0$ for all $t\in\left[0, \frac{1}{2}\right]$. If $t\in\left]\frac{1}{2}, 1\right]$ then, from (1), $f(pt)=f(p)f(t)$ for all $p\in[0, 1]$. Again, by induction, $f(pt^k)=f(p)(f(t))^k$ for all $p\in[0, 1]$ and natural numbers k. Choosing p and k so that $pt^k\in\left[0, \frac{1}{2}\right]$ and $f(p)\neq 0$ we obtain $f(t)=0$ for all $t\in[0, 1[$. □

We remark that if f: $[0, 1]\to\mathbf{R}$ is a non-constant solution of (1) and it is not of the form (4) then, by Lemma 1, $f(\alpha)\neq 0$ for all $0<\alpha\in L$. In the following lemma we describe the structure of L, in this case.

LEMMA 2. *Suppose that* f: $[0, 1]\to\mathbf{R}$ *is a non-constant solution of* (1), *and it is not of the form* (4). *Then* L *has the following properties:*

(i) $0, \frac{1}{2}, 1\in L$.

(ii) *If* $\alpha\in L$ *then* $1-\alpha\in L$.

(iii) *If* $\alpha\in L\cap\left[0, \frac{1}{2}\right]$ *then* $2\alpha\in L$.

(iv) *If* $\alpha, \beta\in L$ *then* $\alpha\beta\in L$.

(v) *If* $\alpha, \beta\in L$, $\beta\neq 0$, $\frac{\alpha}{\beta}\in[0, 1]$ *then* $\frac{\alpha}{\beta}\in L$.

(vi) *If* $\alpha, \beta\in L$, $\beta\geqq\alpha$ *then* $\beta-\alpha\in L$.

(vii) *If* $\alpha, \beta\in L$, $\alpha+\beta\in[0, 1]$ *then* $\alpha+\beta\in L$.

PROOF. (i) $f(0)=0$ and $f(1)=1$ imply that $0, 1\in L$. $\frac{1}{2}\in L$ follows from (1) with $q=\frac{1}{2}$. (ii) is a trivial consequence of (1). Since $\frac{1}{2}\in L$,

$$f\left(\frac{1}{2}\right)f(t\cdot 2\alpha) = f(t\alpha) = f(t)f(\alpha) = f(t)f(2\alpha)f\left(\frac{1}{2}\right)$$

for all $t\in[0, 1]$. By Lemma 1, $f\left(\frac{1}{2}\right)\neq 0$ thus we have (iii). For all $t\in[0, 1]$,

$$f(t\alpha\beta) = f(t\alpha)f(\beta) = f(t)f(\alpha)f(\beta) = f(t)f(\alpha\beta)$$

which implies (iv). For the proof of (v), let $t\in[0, 1]$. Then

$$f\left(t\cdot\frac{\alpha}{\beta}\right)f(\beta) = f(t\alpha) = f(t)f(\alpha) = f(t)f\left(\frac{\alpha}{\beta}\right)f(\beta).$$

By Lemma 1, $f(\beta)\neq 0$, thus we have (v). (vi) is trivial if $\beta=0$. If $\beta\neq 0$ then, by (v), $\frac{\alpha}{\beta}\in L$ and, by (ii) and (iv), we get $\beta-\alpha=\beta\left(1-\frac{\alpha}{\beta}\right)\in L$. Finally, (vii) follows from the statements we have proved and from the relation

$$\alpha+\beta = 2\cdot\begin{cases}\beta-\frac{1}{2}(\beta-\alpha) & \text{if } \beta\geqq\alpha\\ \alpha-\frac{1}{2}(\alpha-\beta) & \text{if } \beta<\alpha.\end{cases}\quad\square$$

In what follows we will show that the form of the solutions of (1) depends only on the fact that L contains an interval of positive length or not.

LEMMA 3. *Suppose that* f: $[0, 1]\to\mathbf{R}$ *is a non-constant solution of* (1), *and it is not of the form* (4). *If there exist* $u, v\in L$, $u<v$ *such that* $[u, v]\subset L$ *then* $L=[0, 1]$, *that is,* f *is a solution of* (2).

PROOF. Applying Lemma 2, by (v), $\left[\frac{u}{v}, 1\right]\subset L$ and, by (iv), $\left[\left(\frac{u}{v}\right)^k, 1\right]\subset L$ for all natural numbers k. Since $0\in L$, we obtain that $[0, 1]\subset L$. $\square$

The most complicated part of the determination of the solutions of (1) is contained in the following

LEMMA 4. *If* f: $[0, 1]\to\mathbf{R}$ *is a non-constant solution of* (1), *and* L *does not contain an interval of positive length then* f *is a restriction to* $[0, 1]$ *of a function* a: $\mathbf{R}\to\mathbf{R}$ *satisfying* (3).

PROOF. Because of the assumptions, f is not of the form (4). Define

$$F(t, x) = f(tx) - f(t)f(x)$$

on $[0, 1]\times[0, 1]$. Then (1) implies

$$F(t, x) + F(t, 1-x) = 0 \tag{5}$$

for all $t, x\in[0, 1]$ and, particularly

$$F\left(t, \frac{1}{2}\right) = 0 \tag{6}$$

for all $t\in[0, 1]$. First we prove that

$$f(x+y)[F(t, x)+F(t, y)] = F(t, x+y)[f(x)+f(y)] \tag{7}$$

holds for all $t\in[0,1]$ and $(x,y)\in\Delta$ where $\Delta=\{(x,y): x,y,x+y\in[0,1]\}$. Let $t\in[0,1]$ and $(x,y)\in\Delta$. Then (7) holds if $txy=0$. Suppose that $txy>0$ and write $p=t(x+y)$, $q=\dfrac{tx}{t(x+y)}$ in (1). Then

$$f(tx)+f(ty)=f(t(x+y))\left[f\left(\frac{x}{x+y}\right)+f\left(\frac{y}{x+y}\right)\right] \tag{8}$$

particularly

$$f(x)+f(y)=f(x+y)\left[f\left(\frac{x}{x+y}\right)+f\left(\frac{y}{x+y}\right)\right]. \tag{9}$$

It follows from the definition of F and from (8) and (9) that

$$f(x+y)[F(t,x)+F(t,y)]=f(x+y)[f(tx)+f(ty)]-$$

$$-f(x+y)f(t)[f(x)+f(y)]=f(t(x+y))f(x+y)\left[f\left(\frac{x}{x+y}\right)+f\left(\frac{y}{x+y}\right)\right]-$$

$$-f(t)f(x+y)[f(x)+f(y)]=f(t(x+y))[f(x)+f(y)]-$$

$$-f(t)f(x+y)[f(x)+f(y)]=F(t,x+y)[f(x)+f(y)],$$

that is, (7) is verified.

Let $t\in[0,1]$, $(x,y)\in\Delta$ and write $1-x-y$ instead of y in (7). Then

$$f(1-y)[F(t,x)+F(t,1-x-y)]=F(t,1-y)[f(x)+f(1-x-y)].$$

This, using (5), can be written in the form

$$f(1-y)[F(t,x)-F(t,x+y)]=-F(t,y)[f(x)+f(1-x-y)]. \tag{10}$$

With the help of (10) and (7), $F(t,x+y)$ can be eliminated, thus

$$f(1-y)[f(x+y)-f(x)-f(y)]F(t,x)= \tag{11}$$

$$=[(f(x)+f(y))(f(x)+f(1-x-y))-f(1-y)f(x+y)]F(t,y)$$

holds for all $t\in[0,1]$ and $(x,y)\in\Delta$.

Let $1>y\in L$, $x\in[0,1]\setminus L$ and $(x,y)\in\Delta$. Then, by the definition of L, there exists $t_0\in[0,1]$ such that $F(t_0,x)\neq 0$ and $F(t_0,y)=0$. By Lemma 1 and (ii) of Lemma 2, $f(1-y)\neq 0$ thus (11) implies that

$$f(x+y)=f(x)+f(y) \tag{12}$$

holds for all $x\in[0,1]\setminus L$, $y\in L$, $(x,y)\in\Delta$. ((12) obviously holds if $y=1$, $x=0$.)

Let now $x\in L, y\in L$, $x+y<1$. Then there exists $x_0\in[0,1-x-y]$ such that $x+x_0\in[0,1]\setminus L$, since in the opposite case L would contain the interval $[x,1-y]$. Applying Lemma 2, (vii) implies $x_0\in[0,1]\setminus L$, $x+y\in L$ and it follows from (vi) that $x+x_0+y\in[0,1]\setminus L$. Thus, by (12),

$$f(x+y)=f(x+y+x_0)-f(x_0)=f(x+x_0)+f(y)-f(x_0)=f(x)+f(y).$$

If $x,y\in L$ and $x+y=1$ then $\dfrac{x}{2}+\dfrac{y}{2}<1$, $\dfrac{x}{2}$, $\dfrac{y}{2}\in L$ therefore

$$f\left(\frac{1}{2}\right)f(x+y)=f\left(\frac{x}{2}+\frac{y}{2}\right)=f\left(\frac{x}{2}\right)+f\left(\frac{y}{2}\right)=f\left(\frac{1}{2}\right)[f(x)+f(y)].$$

Since $f\left(\frac{1}{2}\right)\neq 0$, in this case we have (12), too. Summarizing, we have proved that (12) holds if $(x, y)\in\Delta$ and $y\in L$. Particularly,

$$f\left(x+\frac{1}{2}\right)=f(x)+f\left(\frac{1}{2}\right) \tag{13}$$

for all $x\in\left[0, \frac{1}{2}\right]$.

Let now $y=\frac{1}{2}$ in (10). Then $F(t, x)=F\left(t, x+\frac{1}{2}\right)$, that is, $f\left(tx+\frac{t}{2}\right)=$ $=f(t)f\left(x+\frac{1}{2}\right)+f(tx)-f(t)f(x)$, if $t\in[0, 1]$, $x\in\left[0, \frac{1}{2}\right]$. By (13), this implies

$$f\left(tx+\frac{t}{2}\right)=f(tx)+f(t)f\left(\frac{1}{2}\right)=f(tx)+f\left(\frac{t}{2}\right)$$

therefore $f(x+y)=f(x)+f(y)$ for all $x, y\in\left[0, \frac{1}{2}\right]$. If $(x, y)\in\Delta$ then $\frac{x}{2}, \frac{y}{2}\in\left[0, \frac{1}{2}\right]$ thus

$$f\left(\frac{1}{2}\right)f(x+y)=f\left(\frac{x}{2}+\frac{y}{2}\right)=f\left(\frac{x}{2}\right)+f\left(\frac{y}{2}\right)=f\left(\frac{1}{2}\right)[f(x)+f(y)].$$

Since $f\left(\frac{1}{2}\right)\neq 0$ (because of Lemmas 1 and 2) we have $f(x+y)=f(x)+f(y)$ for all $(x, y)\in\Delta$. Finally, we can apply the extension theorem of Daróczy and Losonczi [1] and the proof is complete. □

Summarizing the results contained by the lemmas we can formulate our main theorem:

THEOREM. *If* f: $[0, 1]\to\mathbf{R}$ *is a solution of* (1) *then either* f *is a solution of* (2) *or* f *is a restriction to* $[0, 1]$ *of a function* a: $\mathbf{R}\to\mathbf{R}$ *satisfying* (3).

We remark that the regular solutions of (2) and (3) (say the solutions bounded on a measurable set of positive measure) are well-known thus those of (1) can easily be obtained from the general solutions.

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POINTWISE ESTIMATES FOR THE DERIVATIVES OF A POLYNOMIAL WITH REAL ZEROS

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Introduction

Markov's inequality asserts that

$$\|p_n'\|_{[-1,1]} \leqq n^2 \|p_n\|_{[-1,1]} \tag{1}$$

for any polynomial $p_n \in \Pi_n$. (Π_n denotes the set of real algebraic polynomials of degree at most n, and $\|\cdot\|_A$ denotes the supremum norm on A.)

In 1940 P. Erdős [1] offered the following refinement of Markov's inequality: If $p_n \in \Pi_n$ and p_n has all its roots in $\mathbf{R} \setminus (-1, 1)$ then

$$\|p_n'\|_{[-1,1]} \leqq \frac{en}{2} \|p_n\|_{[-1,1]} \tag{2}$$

and this result cannot be improved.

P. Borwein in [2] proved the following theorem: If p_n is a polynomial of degree n and p_n has at least $n-k$ roots in $\mathbf{R} \setminus (-1, 1)$ then there is a constant $c(\leqq 9)$ so that

$$\|p_n'\|_{[-1,1]} \leqq cn(k+1) \|p_n\|_{[-1,1]}.$$

Up to the constant this result is the best possible, J. Szabados in [3] constructed polynomials p_n of degree n with $n-k$ zeros in $\mathbf{R} \setminus (-1, 1)$ and with k zeros in $(-1, 1)$ so that

$$\|p_n'\|_{[-1,1]} \geqq \frac{nk}{2} \|p_n\|_{[-1,1]}. \tag{3}$$

But Borwein's result can be improved in the interval $[-a, a]$, where $0 < a < 1$, by establishing a Bernstein type inequality. Bernstein's inequality asserts that

$$|p_n'(x)| \leqq \frac{n}{\sqrt{1-x^2}} \|p_n\|_{[-1,1]}, \quad |x| < 1$$

for any polynomial $p_n \in \Pi_n$.

P. Erdős in [1] proved that if $p_n \in \Pi_n$ has no root in the unit circle then

$$|p_n'(x)| \leqq \frac{c\sqrt{n}}{(1-x^2)^2} \|p_n\|_{[-1,1]}, \quad |x| < 1$$

where c is an absolute positive constant.

It can also be proved that if $p_n \in \Pi_n$ and p_n has all roots in $\mathbf{R} \setminus (-1, 1)$ then

$$|p_n'(x)| \leqq \frac{c\sqrt{n}}{\sqrt{1-x^2}} \|p_n\|_{[-1,1]}, \quad |x| < 1.$$

On the basis of the last two results, my conjecture is that if $p_n \in \Pi_n$ has all but at most k roots in $\mathbf{R} \setminus (-1, 1)$ then there is an absolute positive constant c so that

$$|p_n'(x)| \leqq \frac{c\sqrt{n(k+1)}}{\sqrt{1-x^2}} \|p_n\|_{[-1,1]}, \quad |x| < 1.$$

But I can prove only the following theorem:

THEOREM 1. *Denote by P_n^k the class of those polynomials in Π_n which have all but at most k roots in $\mathbf{R} \setminus (-1, 1)$. Then $p_n \in P_n^k$ and $x \in (-1, 1)$ imply*

$$|p_n'(x)| \leqq \frac{c(k+1)^2\sqrt{n+1}}{1-x^2} \|p_n\|_{[-1,1]}$$

where c is an absolute constant independent of k and n.

For the proof we need some lemmas.

LEMMA 1. *Denote by T_k the kth Chebyshev polynomial (which has leading coefficient 2^{k-1}) and let $c_1 > 0$ be a fixed constant. Then*

a) $$\left|T_k\left(\pm 1 \pm \frac{2c_1}{k^2}\right)\right| \leqq c_2 \|T_k\|_{[-1,1]} \quad (= c_2),$$

b) $$\left|T_k'\left(\pm 1 \pm \frac{2c_1}{k^2}\right)\right| \leqq c_2' k^2 \|T_k\|_{[-1,1]} \quad (= c_2' k^2)$$

where c_2 and c_2' depend only on c_1.

The proof of a) comes simply from the formula

$$T_k(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^k + (x-\sqrt{x^2-1})^k\right], \quad |x| \geqq 1.$$

The proof of b) comes from a) and Markov's inequality.

LEMMA 2. *If $q \in \Pi_k$ and $x \in \mathbf{R} \setminus (-1, 1)$ then*

a) $$\frac{|q(x)|}{\|q\|_{[-1,1]}} \leqq \frac{|T_k(x)|}{\|T_k\|_{[-1,1]}} \quad (= |T_k(x)|),$$

b) $$\frac{|q'(x)|}{\|q\|_{[-1,1]}} \leqq \frac{|T_k'(x)|}{\|T_k\|_{[-1,1]}} \quad (= |T_k'(x)|).$$

The proof can be found in [4] pp. 41—42.

From Lemmas 1 and 2 the following consequence is easily obtained.

COROLLARY 1. *If* $p\in\Pi_k$ *and* $a<b(\in\mathbf{R})$ *then for an arbitrary* $c_1>0$ *there exist suitable* c_2 *and* c_2' *so that*

a) $$\|p\|_{\left[a-c_1\frac{b-a}{k^2},\, b+c_1\frac{b-a}{k^2}\right]} \leqq c_2\|p\|_{[a,b]},$$

b) $$\|p'\|_{\left[a-c_1\frac{b-a}{k^2},\, b+c_1\frac{b-a}{k^2}\right]} \leqq c_2'\frac{2}{b-a}k^2\|p\|_{[a,b]}.$$

LEMMA 3. *Let* $y\in\mathbf{R}$ *be a fixed real number. Then there exists a polynomial* $p_n^*\in P_n^k$ *maximizing* $\dfrac{|p_n'(y)|}{\|p_n\|_{[-1,1]}}$ *in* P_n^k.

PROOF. Let

$$K(y)=\sup_{p\in P_n^k}\frac{|p'(y)|}{\|p\|_{[-1,1]}}.$$

$K(y)$ is a finite number ($K(y)\leqq n^2$ according to Markov's inequality). Let $p_n^j\in P_n^k$ be a sequence of polynomials so that

$$\frac{|p_n^{j\prime}(y)|}{\|p_n^j\|_{[-1,1]}}>K(y)-\frac{1}{j},\quad j=1,2,\ldots.$$

We may suppose that $\|p_n^j\|_{[-1,1]}=1$. (We can normalize every p_n^j to have norm 1, while the values $\dfrac{|p_n^{j\prime}(y)|}{\|p_n^j\|_{[-1,1]}}$ do not change.) As Π_n is a finite dimensional normed space the sequence $\{p_n^j\}_{j=1}^{\infty}$ has a subsequence (let this be simply $\{p_n^j\}_{j=1}^{\infty}$ itself) so that

$$\lim_{j\to\infty}\|p_n^*-p_n^j\|_{[-1,1]}=0$$

where $p_n^*\not\equiv 0$ is the limit of the polynomials p_n^j. As in a finite dimensional normed space two arbitrary norms are equivalent, the sup norm and the "maximal absolute value of the coefficients" norm are also equivalent in the space Π_n, so $\lim_{j\to\infty}(p_n^{*\prime}(y)-p_n^{j\prime}(y))=0$. Consequently

$$K(y)=\lim_{j\to\infty}\frac{|p_n^{j\prime}(y)|}{\|p_n^j\|_{[-1,1]}}=\frac{|p_n^{*\prime}(y)|}{\|p_n^*\|_{[-1,1]}}.$$

Lastly we have to prove that $p_n^*\in P_n^k$. Suppose indirectly that p_n^* has at least $k+1$ roots (with multiplicity) none of which is in $\mathbf{R}\setminus(-1,1)$. But then, according to the wellknown Rouche's Theorem p_n^j has also at least $k+1$ roots none of which is in $\mathbf{R}\setminus(-1,1)$ if j is large enough (as $\|p_n^j-p_n^*\|_{[-1,1]}\to 0$ if $j\to\infty$). As $p_n^j\in P_n^k$, this is a contradiction. Q.E.D.

In the sequel let $y=0$ and

$$\frac{|p_n^{*\prime}(0)|}{\|p_n^*\|_{[-1,1]}}=\sup_{p\in P_n^k}\frac{|p'(0)|}{\|p\|_{[-1,1]}}.$$

LEMMA 4. *The number of real* a*'s for which* $|a|>1$ *and* $p_n^*(a)=0$ *is at most one.*

PROOF. Suppose indirectly that $p_n^*(a)=p_n^*(b)=0$, where $a\neq b$ are real, $|a|>1$ and $|b|>1$. Consider the polynomial

$$v(x) = p_n^*(x)\frac{x^2}{(x-a)(x-b)}.$$

Then $\deg v=\deg p_n^*$ and

1. p^* and v have the same multiplicity of roots at ± 1,
2. if δ is real, $|\delta|>1$ and p_n^* has h repeated roots at δ, then v has at least $h-1$ repeated roots at δ,
3. if $|p_n^*(\delta)|=\|p_n^*\|_{[-1,1]}$ for a number $\delta\in[-1, 1]$, then $\operatorname{sgn} v(\delta)=\operatorname{sgn}(ab)\cdot \cdot\operatorname{sgn} p_n^*(\delta)$, as by definition $p_n^{*\prime}(0)\neq 0$, so $|p_n^*(0)|\neq\|p_n^*\|_{[-1,1]}$,
4. $v'(0)=0$.

Then, however, the polynomial $p_n^*-\varepsilon \operatorname{sgn}(ab)v$ contradicts the maximality of p_n^* if $\varepsilon>0$ is small enough. Q.E.D.

LEMMA 5. *If p_n^* has a root at a real a for which $|a|>1$, then the multiplicity of a is at most one.*

PROOF. Suppose indirectly that there are at least two repeated roots at a real a, for which $|a|>1$. Consider the polynomial

$$v(x) = p_n^*(x)\frac{x^2}{(x-a)^2}.$$

Evidently $\deg v=\deg p_n^*$. Examine the polynomial $p_n^*-\varepsilon v$.

1. $$\|p_n^*-\varepsilon v\|_{[-1,1]} = \left\|p_n^*(x)\left(1-\varepsilon\frac{x^2}{(x-a)^2}\right)\right\|_{[-1,1]} < \|p_n^*\|_{[-1,1]}$$

if $\varepsilon>0$ is small enough. (Observe that $\dfrac{x^2}{(x-a)^2}>0$ if $x\in[-1, 1]$ and $x\neq 0$, and $|p_n^*(0)|\neq\|p_n^*\|_{[-1,1]}$, otherwise $p_n^{*\prime}(0)=0$, but $p_n^{*\prime}(0)\neq 0$ because of the maximality of p_n^*.)

Now let $\varepsilon_0>0$ be a fixed real number for which $\varepsilon_0<|a|-1$ and p_n^* has no roots in $\{(a-\varepsilon_0, a)\cup(a, a+\varepsilon_0)\}$. Then $p_n^*-\varepsilon v$ has at least $s-2$ roots (with multiplicity) in

$$\{\mathbf{R}\setminus(-1, 1)\}\setminus\{(a-\varepsilon_0, a)\cup(a, a+\varepsilon_0)\}$$

if p_n^* has exactly s roots (with multiplicity) in $\mathbf{R}\setminus(-1, 1)$. We show that $p_n^*-\varepsilon v$ has two roots in $(a-\varepsilon_0, a)\cup(a, a+\varepsilon_0)$ if $\varepsilon>0$ is small enough. Indeed

$$(p_n^*-\varepsilon v)(x) = p_n^*(x)\left(1-\varepsilon\frac{x^2}{(x-a)^2}\right)$$

and $1-\varepsilon\dfrac{x^2}{(x-a)^2}$ has two roots, namely $x_1=a\dfrac{1}{1+\sqrt{\varepsilon}}$ and $x_2=a\dfrac{1}{1-\sqrt{\varepsilon}}$. Thus $x_1\neq x_2\in(a-\varepsilon_0, a)\cup(a, a+\varepsilon_0)$ if $\varepsilon>0$ is small enough. So

2. $p_n^*-\varepsilon v$ has at least $(s-2)+2=s$ roots in $\mathbf{R}\setminus(-1, 1)$ if p_n^* has exactly s roots in $\mathbf{R}\setminus(-1, 1)$.

3. $(p_n^*-\varepsilon v)'(0)=p_n^{*\prime}(0)-\varepsilon v'(0)=p_n^{*\prime}(0)$ because $v'(0)=0$.

According to 1, 2, and 3 the polynomial $p_n^*-\varepsilon v$ contradicts the maximality of p_n^* if $\varepsilon>0$ is small enough. Q.E.D.

On the basis of Lemmas 4 and 5 p_n^* is of the form

$$p_n^*(x) = (x+1)^j(x-1)^{h-j}q(x)$$

where $q(x)\in\Pi_{k+1}$ and $h:=\deg p_n^*-\deg q$. It can also be supposed that $2j\leqq h$, otherwise the polynomial $p_n^*(-x)$ can be considered. As $p_n^*\in P_n^k$, we have $h\leqq n$. Further

$$p_n^{*\prime}(x) = j(x+1)^{j-1}(x-1)^{h-j}q(x)+(h-j)(x+1)^j(x-1)^{h-j-1}q(x)+$$
$$+(x+1)^j(x-1)^{h-j}q'(x).$$

So

$$p_n^{*\prime}(0) = j(-1)^{h-j}q(0)+(h-j)(-1)^{h-j-1}q(0)+(-1)^{h-j}q'(0) = \tag{4}$$
$$= (-1)^{h-j-1}(h-2j)\,q(0)+(-1)^{h-j}q'(0).$$

Using the earlier notations, we prove the following.

LEMMA 6. *We have*

a) $$(h-2j)|q(0)| \leqq c'\sqrt{n+1}\,(k+1)^2\|p_n^*\|_{[-1,1]},$$

b) $$|q'(0)| \leqq c''\sqrt{n+1}\,(k+1)^2\|p_n^*\|_{[-1,1]}$$

where c', c'' are absolute positive constants.

PROOF. a) Consider the interval

$$I=\left[\frac{-1}{3\sqrt{h+1}}\left(2+\frac{1}{(k+1)^2}\right),\ \frac{-1}{3\sqrt{h+1}\,(k+1)^2}\right]\subset[-1,0].$$

Applying a) of Corollary 1 to the polynomial $q(x)$, we get $|q(0)|\leqq c_2\|q\|_I$, $c_2>0$, so there exists a $\xi\in I$ for which $|q(\xi)|\geqq\frac{|q(0)|}{c_2}$. Further

$$|p_n^*(\xi)| = |(\xi^2-1)^j(\xi-1)^{h-2j}q(\xi)| \geqq$$
$$\geqq\left(1-\frac{1}{h+1}\right)^j\left(1+(h-2j)\cdot\frac{1}{3\sqrt{h+1}\,(k+1)^2}\right)|q(\xi)|.$$

As $j\leqq h$, $\left(1-\frac{1}{h+1}\right)^j>\frac{1}{e}$. Hence

$$|p_n^*(\xi)|\geqq\frac{1}{e}(h-2j)\frac{1}{3\sqrt{h+1}\,(k+1)^2}|q(\xi)|\geqq\frac{1}{c'}(h-2j)\frac{1}{\sqrt{n+1}\,(k+1)^2}|q(0)|$$

and therefore

$$c'\sqrt{n+1}\,(k+1)^2\|p_n^*\|_{[-1,1]}\geqq(h-2j)|q(0)|.$$

By this the proof of a) of Lemma 6 is complete.

b) Applying b) of Corollary 1 to the polynomial $q(x)$, we have

$$|q'(0)| \leqq c_2' \cdot 2 \cdot \frac{3}{2} \sqrt{h+1}\,(k+1)^2 \|q\|_I \leqq c_2'' \sqrt{n+1}\,(k+1)^2 \|q\|_I, \quad c_2'' > 0,$$

so there exists a $\xi \in I$ for which

$$|q(\xi)| \geqq \frac{|q'(0)|}{c_2'' \sqrt{n+1}\,(k+1)^2}.$$

Thus

$$|p_n^*(\xi)| = |(\xi^2-1)^j(\xi-1)^{h-2j} q(\xi)| \geqq \left(1-\frac{1}{h+1}\right)^j |q(\xi)| \geqq \frac{1}{e} \frac{|q'(0)|}{c_2'' \sqrt{n+1}\,(k+1)^2}$$

and therefore

$$c'' \sqrt{n+1}\,(k+1)^2 \|p_n^*\|_{[-1,1]} \geqq |q'(0)|$$

where c'' is an absolute constant.

By this the proof of b) of Lemma 6 is also complete.

LEMMA 7. *Let* $p \in P_n^k$. *Then*

$$|p'(0)| \leqq c \sqrt{n+1}\,(k+1)^2 \|p\|_{[-1,1]}.$$

PROOF. On the basis of Lemmas 4 and 5 it may be supposed that

$$p(x) = p_n^*(x) = (x+1)^j (x-1)^{h-j} q(x)$$

where $q \in \Pi_{k+1}$. Using both parts of Lemma 6 and (4), we have

$$|p'(0)| \leqq (c'+c'') \sqrt{n+1}\,(k+1)^2 \|p\|_{[-1,1]} \leqq c \sqrt{n+1}\,(k+1)^2 \|p\|_{[-1,1]}.$$

Q.E.D.

The proof of Theorem 1 is now straightforward. Let $x \in (-1, 1)$ and $p \in P_n^k$. We may suppose that $0 \leqq x < 1$, otherwise the polynomial $p(-x)$ can be considered. Consider the interval $[2x-1, 1]$ with midpoint x. By a simple linear transformation we have

$$|p'(x)| \leqq \frac{c \sqrt{n+1}\,(k+1)^2}{1-x} \|p\|_{[2x-1,1]} \leqq \frac{2c \sqrt{n+1}\,(k+1)^2}{1-x^2} \|p\|_{[2x-1,1]} \leqq$$

$$\leqq \frac{2c \sqrt{n+1}\,(k+1)^2}{1-x^2} \|p\|_{[-1,1]}.$$

(Let $ay+b$ map $[-1, 1]$ one to one onto $[2x-1, 1]$ in such a way that $-1 \to 2x-1$. Then $x \to 0$, $a = 1-x$, $b = x$ and we can apply Lemma 7 to the polynomial $r(y) := p(ay+b) \in P_n^k$.) By this the proof of Theorem 1 is complete.

Now we give an estimate for higher derivatives.

THEOREM 2. *We have*

$$|p_n^{(m)}(x)| \leqq c_m \left(\frac{\sqrt{n+1}\,(k+1)^2}{1-x^2}\right)^m \|p_n\|_{[-1,1]}, \quad |x| < 1$$

for any polynomial $p_n \in P_n^k$, *where* $c_m > 0$ *depends only on* m.

PROOF. We prove by induction on m. According to Theorem 1 Theorem 2 is true if $m=1$. Suppose that Theorem 2 is true for $m-1$, so

$$|p_n^{(m-1)}(x)| \leqq c_{m-1}\left(\frac{\sqrt{n+1}\,(k+1)^2}{1-x^2}\right)^{m-1}\|p_n\|_{[-1,1]}, \quad |x|<1. \tag{5}$$

Observe that if $p_n\in P_n^k$ then $q_n(x):=p_n\left(\frac{x}{2}\right)$ is also in P_n^k. So

$$|q_n'(0)| \leqq c\sqrt{n+1}\,(k+1)^2\|q_n\|_{[-1,1]},$$

therefore

$$|p_n'(0)| \leqq 2c\sqrt{n+1}\,(k+1)^2\|p_n\|_{\left[-\frac{1}{2},\frac{1}{2}\right]}. \tag{6}$$

If $p_n\in P_n^k$, then $p_n^{(m-1)}\in P_{n-m+1}^{k+m-1}$, so according to (5) and (6)

$$|p_n^{(m)}(0)| = |p_n^{(m-1)\prime}(0)| \leqq 2c\sqrt{n+1}\,(k+m)^2\|p_n^{(m-1)}\|_{\left[-\frac{1}{2},\frac{1}{2}\right]} \leqq$$

$$\leqq 2c\sqrt{n+1}\,(k+m)^2 c_{m-1}\left(\sqrt{n+1}\,(k+1)^2\right)^{m-1}\left(1-\left(\frac{1}{2}\right)^2\right)^{-(m-1)}\|p_n\|_{[-1,1]} \leqq$$

$$\leqq c_m'\left(\sqrt{n+1}\,(k+1)^2\right)^m\|p_n\|_{[-1,1]}.$$

Now let $0\leqq x<1$ (the case $-1<x\leqq 0$ is similar to this). Consider the interval $[2x-1, 1]$ with midpoint x. Let $ay+b$ map $[-1, 1]$ one to one onto $[2x-1, 1]$ in such a way that $-1\to 2x-1$. Then

$$a=1-x, \quad b=x, \quad q_n(y):=p_n(ay+b)\in P_n^k \quad \text{if} \quad p_n\in P_n^k,$$

and

$$|p_n^{(m)}(x)| = (1-x)^{-m}|q_n^{(m)}(0)| \leqq (1-x)^{-m}c_m'\left(\sqrt{n+1}\,(k+1)^2\right)^m\|q_n\|_{[-1,1]} =$$

$$= (1-x)^{-m}c_m'\left(\sqrt{n+1}\,(k+1)^2\right)^m\|p_n\|_{[2x-1,1]} \leqq$$

$$\leqq c_m\left(\frac{1}{1-x^2}\right)^m\left(\sqrt{n+1}\,(k+1)^2\right)^m\|p_n\|_{[-1,1]}.$$

Q.E.D.

COROLLARY 2. *Let $a\in(0,1)$ be a fixed real number. Then*

$$\|p_n^{(m)}\|_{[-a,a]} \leqq c_{m,a}\left(\sqrt{n+1}\,(k+1)^2\right)^m\|p_n\|_{[-1,1]} \quad \text{if} \quad p_n\in P_n^k$$

where $c_{m,a}$ depends only on m and a.

In particular if $k=0$, $m>0$, then

$$|p_n^{(m)}(x)| \leqq c_m n^{m/2}\frac{1}{(1-x^2)^m}\|p_n\|_{[-1,1]} \quad \text{if} \quad p_n\in P_n^0, \quad |x|<1, \tag{7}$$

so

$$\|p_n^{(m)}\|_{[-a,a]} \leqq c_{m,a} n^{m/2} \|p_n\|_{[-1,1]} \quad \text{if} \quad p_n \in P_n^0, \quad 0 < a < 1, \tag{8}$$

where $c_{m,a}$ depends only on m and a.

P. Erdős proved in [1] that

$$\|p_n'\|_{[-a,a]} \leqq c_{1,a} \sqrt{n} \, \|p_n\|_{[-1,1]} \tag{9}$$

if $p_n \in P_n^0$ and $0<a<1$, and this result is the best possible. He did not examine the case of higher derivatives.

THEOREM 3. *Let* $m \leqq n$ *and* $a \in (0, 1)$. *Then*

$$c_m' n^{m/2} \leqq \sup_{p \in P_n^0} \frac{\|p^{(m)}\|_{[-a,a]}}{\|p\|_{[-1,1]}} \leqq c_{m,a} n^{m/2}$$

where $c_{m,a}$ *and* c_m' *are positive constants.*

PROOF. The estimate for the upper bound comes simply from (8). Now let

$$p(x) = (x-1)^{k^2}(x+1)^{k^2+k} = (x^2-1)^{k^2}(x+1)^k \in P_{2k^2+k}^0 ,$$

where k is a positive integer and let $2k^2+k \geqq m \geqq 2$ be a fixed positive integer. (The case $m=1$ was proved by P. Erdős [1].) We shall verify that there exists a $c_m>0$ depending only on m so that

$$|p^{(m)}(0)| \geqq c_m k^m \|p\|_{[-1,1]}$$

if k is large enough, $k \geqq k_0(m)$. Writing $x=dk^{-1}$ we get

$$|p(x)| = \left(1-\frac{d^2}{k^2}\right)^{k^2} (1+dk^{-1})^k < e^{-d^2+d} < e \quad \text{if} \quad x \in [-1, 1]$$

so $\|p\|_{[-1,1]} < e$.

Now we give a lower bound for the value $|p^{(m)}(0)|$ if k is large enough $(k \geqq k_0(m))$. We have

$$p(x) = (x^2-1)^{k^2}(x+1)^k = \left(\sum_{i=0}^{k^2} \binom{k^2}{i} x^{2i}(-1)^{k^2-i}\right)\left(\sum_{j=0}^{k} \binom{k}{j} x^j\right),$$

and

$$|p^{(m)}(0)| = m! \, |b_m|, \quad \text{where} \quad p(x) = \sum_{i=0}^{2k^2+k} b_i x^i.$$

So

$$|p^{(m)}(0)| = m! \left| \sum_{\substack{2i+j=m \\ 0 \leqq i \leqq k^2 \\ 0 \leqq j \leqq k}} (-1)^{k^2-i} \binom{k^2}{i} \binom{k}{j} \right| = m! \left| \sum_{\substack{2i+j=m \\ 0 \leqq i \leqq k^2 \\ 0 \leqq j \leqq k}} (-1)^{i} \binom{k^2}{i} \binom{k}{j} \right|.$$

We want to prove that $|p^{(m)}(0)| \geqq c_m k^m$ for a suitable $c_m>0$ depending only on m if k is large enough $(k \geqq k_0(m))$, so it is sufficient to prove that

$$\frac{1}{k^m} \left| \sum_{\substack{2i+j=m \\ 0 \leqq i \leqq k^2 \\ 0 \leqq j \leqq k}} (-1)^{i} \binom{k^2}{i} \binom{k}{j} \right| > c_m$$

for a suitable $c_m>0$ depending only on m if k is large enough $(k\geqq k_0(m))$. We have

$$\frac{1}{k^m}\left|\sum_{\substack{2i+j=m\\0\leqq i\leqq k^2\\0\leqq j\leqq k}}(-1)^i\binom{k^2}{i}\binom{k}{j}\right|=\frac{1}{k^m}\left|\sum_{s=0}^{m}\alpha_s k^s\right|\geqq|\alpha_m|-\left|\sum_{s=0}^{m-1}\alpha_s k^{s-m}\right|\geqq|\alpha_m|-\frac{\gamma_m}{k}$$

where $\gamma_m>0$ is a constant depending only on m, and k is large enough $(k\geqq k_0(m))$. By this it is sufficient to prove that $|\alpha_m|>0$ if k is large enough $(k\geqq k_0(m))$. But if $m\geqq 2$ and k is large enough $(k\geqq k_0(m))$, then

$$\sum_{\substack{2i+j=m\\0\leqq i\leqq k^2\\0\leqq j\leqq k}}$$

has at least two terms and if $k\geqq m$ and $i=0$, $j=m$, then $2i+j=m$, $0\leqq i\leqq k^2$ and $0\leqq j\leqq k$, so

$$|\alpha_m|=\left|\sum_{\substack{2i+j=m\\0\leqq i\leqq k^2\\0\leqq j\leqq k}}(-1)^i\frac{1}{i!}\frac{1}{j!}\right|=\frac{1}{m!}\left|\sum_{\substack{2i+j=m\\0\leqq i\leqq k^2\\0\leqq j\leqq k}}(-1)^i\frac{m!}{i!\,j!}\right|=$$

$$=\frac{1}{m!}\left|(-1)^0\cdot 1+\sum_{\substack{2i+j=m\\0<i\leqq k^2\\0\leqq j\leqq k}}(-1)^i\frac{m!}{i!\,j!}\right|=\frac{1}{m!}\left|1+\sum_{\substack{2i+j=m\\0<i\leqq k^2\\0\leqq j\leqq k}}(-1)^i\frac{(i+j)!}{i!\,j!}\frac{m!}{(i+j)!}\right|=$$

$$=\frac{1}{m!}\cdot|1+lm|\left(\geqq\frac{1}{m!}\right)$$

where l is an integer because $\dfrac{(i+j)!}{i!\,j!}$ is an integer and if $0<i$ then $i+j<2i+j=m$ and so $\dfrac{m!}{(i+j)!}$ is divisible by m. We used the condition $m\geqq 2$ in the last inequality. So indeed $|p^{(m)}(0)|\geqq c_m k^m\|p\|_{[-1,1]}$ if m is fixed and k is large enough $(k\geqq k_0(m))$, where c_m is a positive constant depending only on m. Now let $k=\left[\frac{1}{2}\sqrt{n}\right]$. Then $2k^2+k<n$, so $p\in P_n^0$. If n is large enough, $n\geqq n_0(m)$, then according to the previous argument we have

$$|p^{(m)}(0)|\geqq c_m\left(\left[\frac{1}{2}\sqrt{n}\right]\right)^m\|p\|_{[-1,1]}\geqq c'_m n^{m/2}\|p\|_{[-1,1]},\quad c'_m>0.$$

If $m\leqq n\leqq n_0(m)$ then $n^{-m/2}\geqq c_m>0$ for a certain constant c_m depending only on m. Then for the polynomial $p(x)=(x+1)^n$ we have $\dfrac{|p^{(m)}(0)|}{\|p\|_{[-1,1]}}=c'_m>0$, so

$$|p^{(m)}(0)|=c'_m\|p\|_{[-1,1]}\geqq c'_m c_m n^{m/2}\|p\|_{[-1,1]}.\quad \text{Q.E.D.}$$

In the sequel c_j will always denote a suitable positive constant depending only on j. P. Borwein [2] proved the following theorem: If $p_n\in\Pi_n$ has at least $n-k$ zeros in

$\mathbf{R}\setminus(-1, 1)$ then

$$\|p_n^{(j)}\|_{[-1,1]} \leqq c_j \frac{n!\,(k+j)!}{(n-j)!\,k!} \|p_n\|_{[-1,1]}.$$

This can be written also in the form

$$\|p_n^{(j)}\|_{[-1,1]} \leqq c_j (n(k+1))^j \|p_n\|_{[-1,1]}.$$

Now we prove that, up to the constant c_j, this result cannot be improved.

THEOREM 4. *Let $k \geqq 0$ be a fixed integer. If $n \geqq k, j$, then polynomials $p_n \in P_n^k$ having $n-k$ real roots at -1 and k real roots in $[-1, 1)$ can be constructed so that*

$$\|p_n^{(j)}\|_{[-1,1]} \geqq c_j (n(k+1))^j \|p_n\|_{[-1,1]}.$$

First we need the following.

LEMMA 8. *Denote by Q_n^k the class of those polynomials in Π_n which have at least $n-k$ roots at -1. Consider that polynomial $q_n^* \in Q_n^k$ for which*

$$\frac{|q_n^{*\prime}(1)|}{\|q_n^*\|_{[-1,1]}} = \sup_{q_n \in Q_n^k} \frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}.$$

(Such a polynomial does exist, this can be proved similarly to Lemma 3.)

Then q_n^* has k simple roots in $(-1, 1)$: $(-1 <) \delta_1 < \delta_2 < \ldots \delta_k (< 1)$ and q_n^* attains its maximum modulus (the value $\pm\|q_n^*\|_{[-1,1]}$) on each of the intervals $[\delta_0, \delta_1], [\delta_1, \delta_2], \ldots$ $\ldots, [\delta_k, \delta_{k+1}]$ (where $\delta_0 := -1, \delta_{k+1} := 1$).

PROOF. We need the following statement.

STATEMENT 1. *Let $1 \leqq k \leqq n$, $x_1 = x_2 = \ldots = x_{n-k} < x_{n-k+1} < \ldots < x_n \leqq x_{n+1}$ and $b_1, b_2, \ldots, b_{n+1}$ be arbitrary real numbers. Then there exists a polynomial $p \in \Pi_n$ for which*

a) $$p^{(j)}(x_1) = b_{j+1}, \quad 0 \leqq j \leqq n-k-1,$$

b) $$p(x_i) = b_i, \quad n-k+1 \leqq i \leqq n,$$

c) $$p'(x_{n+1}) = b_{n+1}.$$

PROOF OF STATEMENT 1. Choose $\eta(x) \in \Pi_{n-1}$ which satisfies conditions a) and b). It can be constructed by Hermite interpolation. Let $\eta'(x_{n+1}) = b^*$ and let $\omega(x) = \prod_{i=1}^{n} (x - x_i)$. As $\omega'(x)$ has all its zeros in $[x_1, x_n)$, $\omega'(x_{n+1}) = \hat{b} \neq 0$. Then the polynomial

$$p(x) = \eta(x) + \frac{b_{n+1} - b^*}{\hat{b}} \omega(x)$$

satisfies conditions a), b) and c) and $\deg p \leqq n$. Q.E.D.

PROOF OF LEMMA 8. Denote the number of values $a \in [-1, 1]$ for which $|q_n^*(a)| = \|q_n^*\|_{[-1,1]}$ by $r (\leqq k+1)$. Let these values be $a_1 < a_2 < \ldots < a_r$. Construct a polynomial $v \in \Pi_{n-k+r}$ for which

1. v has $n-k$ repeated roots at -1,
2. if $\delta\in[-1,1]$ and $q_n^*(\delta)=\|q_n^*\|_{[-1,1]}$ then $v(\delta)=-1$,
3. if $\delta\in[-1,1]$ and $q_n^*(\delta)=-\|q_n^*\|_{[-1,1]}$ then $v(\delta)=1$,
4. $v'(1)=\operatorname{sgn} q_n^{*\prime}(1)$.

We can construct such a polynomial according to Statement 1. If $n-k+r\leqq \leqq\deg q_n^*$, then the polynomial $q_n^*+\varepsilon v$ with a small $\varepsilon>0$ contradicts the maximality of q_n^*. Hence

$$r>\deg q_n^*-(n-k), \tag{10}$$

so $q_n^{*\prime}$ has at least $r-1\big(\geqq\deg q_n^*-(n-k)\big)$ roots in $(-1,1)$ (namely $a_1, a_2, \ldots, a_{r-1}$) and $n-k-1$ roots at -1 if $k<n$, and $r-2$ roots in $(-1,1)$ (namely $a_2, a_3, \ldots, \ldots, a_{r-1}$) and $n-k(=0)$ roots at -1 if $k=n$. If $q_n^*(a_i)=q_n^*(a_{i+1})$ for an i $(1\leqq i\leqq \leqq r-1)$ then q_n^* has a root in (a_i, a_{i+1}). Then $q_n^{*\prime}$ has at least

$$\deg q_n^*-(n-k)+(n-k-1)+1=\deg q_n^*$$

roots in $\mathbf{R}$, a contradiction. Thus $q_n^*(a_i)=-q_n^*(a_{i+1})$ $(1\leqq i\leqq r-1)$, therefore q_n^* has a root on each of the intervals $(a_1, a_2), (a_2, a_3), \ldots, (a_{r-1}, a_r)$. Hence q_n^* has at least $(n-k)+r-1$ roots, thus $r\leqq\deg q_n^*-(n-k)+1$. By (10) $r\geqq\deg q_n^*-(n-k)+1$, therefore $r=\deg q_n^*-(n-k)+1$. Thus q_n^* has $n-k$ repeated zeros at -1 and $r-1$ zeros in $(-1,1)$. As $r=\deg q_n^*-(n-k)+1$, the roots lying in $(-1,1)$ are simple and there is no root in $\mathbf{C}\setminus[-1,1)$.

Now we prove that $\deg q_n^*=n$ (so $r=k+1$) and by this the proof of Lemma 8 will be complete. If $\deg q_n^*<n$, we can consider the polynomial $(x+1)q_n^*(x)$ the degree of which is at most n. Evidently

$$\|(x+1)q_n^*(x)\|_{[-1,1]}\leqq 2\|q_n^*\|_{[-1,1]},$$

and equality is true if and only if $|q_n^*(1)|=\|q_n^*\|_{[-1,1]}$. Further

$$[(x+1)q_n^*(x)]'(1)=q_n^*(1)+2q_n^{*\prime}(1).$$

$q_n^*(1)q_n^{*\prime}(1)>0$, otherwise q_n^* or $q_n^{*\prime}$ would have a root in $[1,\infty)$, but it is already known that q_n^* and $q_n^{*\prime}$ have no root in $[1,\infty)$. Then, however, the polynomial $(x+1)q_n^*(x)$ contradicts the maximality of q_n^*. Q.E.D.

From Lemma 8 we get information concerning q_n^*. This information is sufficient to prove the unicity of q_n^*, which has leading coefficient 1.

LEMMA 9. *If $q_n, s_n\in Q_n^k$ have leading coefficient 1 and are such that*

$$\frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}}=\frac{|s_n'(1)|}{\|s_n\|_{[-1,1]}}=\sup_{p_n\in Q_n^k}\frac{|p_n'(1)|}{\|p_n\|_{[-1,1]}}$$

then $q_n=s_n$.

PROOF. According to Lemma 8 we have

$$q_n(x)=(x+1)^{n-k}\prod_{i=1}^{k}(x-\varrho_i),\quad s_n(x)=(x+1)^{n-k}\prod_{i=1}^{k}(x-\gamma_i)$$

where $-1<\varrho_1<\ldots<\varrho_k<1$ and $-1<\gamma_1<\ldots<\gamma_k<1$.

Suppose indirectly that $\varrho_i > \gamma_i$ for some i. Choose the largest i for which $\varrho_i > \gamma_i$, and η so that $\|\eta q_n\|_{[-1,1]} = \|s_n\|_{[-1,1]}$. The sign of η will be chosen later. From the equioscillation of q_n we can deduce that $\eta q_n - s_n$ has at least $k-i$ roots in $[\beta, 1]$ where β is the first point greater than ϱ_i where ηq_n achieves its maximum modulus (the value $\pm\|\eta q_n\|_{[-1,1]}$). From the equioscillation of s_n we can deduce that $\eta q_n - s_n$ has at least $(n-k)+(i-1)$ roots in $[-1, \alpha]$ where α is the largest point less than γ_i where s_n achieves its maximum modulus (the value $\pm\|s_n\|_{[-1,1]}$). Observe that if we choose the sign of η so that $\operatorname{sgn} \eta q_n(\beta) = -\operatorname{sgn} s_n(\alpha)$, then $\eta q_n - s_n$ must have 2 roots in $[\alpha, \beta]$ and if $\eta q_n - s_n$ has a root at $\alpha \neq -1$ or $\beta \neq 1$, then there are at least two repeated roots. So $\eta q_n - s_n$ has at least $(n-k)+(k-i)+(i-1)+2 = n+1$ roots in $\mathbf{R}$, a contradiction. So indeed $\varrho_i = \gamma_i$ and hence $s_n = \eta q_n$. Evidently $\eta = 1$, as the leading coefficient of q_n and s_n is 1. Q.E.D.

Now we can speak about the roots $(-1 <)\varrho_1 < \varrho_2 < \ldots < \varrho_k (< 1)$ of the polynomial $q_n \in Q_n^k$ for which

$$\frac{|q_n'(1)|}{\|q_n\|_{[-1,1]}} = \sup_{p \in Q_n^k} \frac{|p'(1)|}{\|p\|_{[-1,1]}}.$$

We shall examine the roots ϱ_i.

The upper bound for ϱ_i can be obtained from [2] in the case $n > 2k$, $k \geqq 1$. Denote the polynomial $T_{2k}(x-1)$ by $B_{2k}(x)$. B_{2k} is the $(2k)^{\text{th}}$ Chebyshev polynomial shifted to the interval $[0, 2]$. Denote the k roots lying in $(1, 2)$ of B_{2k} by $(1 <)\alpha_1 < \alpha_2 < \ldots < \alpha_k (< 2)$. Let

$$t_k(x) = \prod_{i=1}^{k} (x - \alpha_i), \quad g_n(x) = \left(x + \frac{2m}{k}\right)^{m+k} t_k(x)$$

and

$$h_n(x) = \frac{1}{\left(1 + \frac{m}{k}\right)^n} g_n\left(\left(1 + \frac{m}{k}\right)x + \left(1 - \frac{m}{k}\right)\right)$$

where $n = m + 2k$. Then according to [2]

$$h_n(x) = (x+1)^{m+k} \prod_{i=1}^{k} (x - \beta_i)$$

where $(-1 <)\beta_1 < \beta_2 < \ldots < \beta_k (< 1)$, further we have $\varrho_i \leqq \beta_i$. By definition if $1 \leqq i \leqq k$ then

$$\alpha_i = 1 + \cos \frac{2(k-i)+1}{2k} \frac{\pi}{2}$$

and

$$\left(1 + \frac{m}{k}\right)\beta_i + \left(1 - \frac{m}{k}\right) = \alpha_i,$$

hence

$$\beta_i = \frac{k}{m+k}\left(1 + \cos \frac{(2(k-i)+1)\pi}{4k}\right) + \frac{m-k}{m+k},$$

so

$$\varrho_i \leqq \beta_i = \frac{n-2k}{n-k} + \frac{k}{n-k} \cos \frac{(2(k-i)+1)\pi}{4k}.$$

For $i=k$ according to the inequality $\cos x \leqq 1-\frac{x^2}{2}+\frac{x^4}{24}$,

$$\varrho_k \leqq \beta_k \leqq 1-\frac{\pi^2}{32}\frac{1}{(n-k)k}+\frac{\pi^4}{6144}\frac{1}{(n-k)k^3}=$$

$$=1-\frac{1}{(n-k)k}\left[\frac{\pi^2}{32}-\frac{\pi^4}{6144}\frac{1}{k^2}\right]<1-\frac{0.29}{(n-k)k}<1-\frac{0.29}{nk}.$$

So we have $\varrho_k \leqq 1-\frac{c_1}{nk}$ if $k \geqq 1$, $2k<n$, with e.g. $c_1=0.29$. Now we give a lower bound for ϱ_i when

$$\cos\frac{(2(k-i)+3)\pi}{2k}>\frac{1}{2}, \quad \text{i.e.} \quad k \geqq i > \left[\frac{2}{3}k+\frac{3}{2}\right] =: l \tag{11}$$

and $n \geqq k \geqq 1$.

Let $D_k(x)=\frac{1}{2^{k-1}}T_k(x)$ where $T_k(x)$ is the kth Chebyshev polynomial in $[-1, 1]$ with leading coefficient 2^{k-1}. Then $D_k(x)=\prod_{i=1}^{k}(x-v_i)$ where

$$v_i=\cos\frac{2(k-i)+1}{k}\frac{\pi}{2}, \quad -1<v_1<v_2<\ldots<v_k<1.$$

Let $n=m+k$ and $p(x)=\left(x+\frac{m-k}{2k}\right)^m D_k(x)$, where

$$|D_k(x)| \leqq \|D_k\|_{[-1,1]}=\frac{1}{2^{k-1}} \leqq 2|x-1|^k \quad \text{if} \quad -1 \leqq x \leqq \frac{1}{2}.$$

$|D_k(x)| \leqq 2|x-1|^k$ is trivial if $x<-1$. We have

$$\left(\left(x+\frac{m-k}{2k}\right)^m(x-1)^k\right)'=\left(x+\frac{m-k}{2k}\right)^{m-1}(x-1)^{k-1}\left[k\left(x+\frac{m-k}{2k}\right)+m(x-1)\right].$$

This expression has only one root in $\left(-\frac{m-k}{2k}, 1\right)$, namely $x=\frac{m+k}{2(m+k)}=\frac{1}{2}$. So for $-\frac{m-k}{2k} \leqq x \leqq \frac{1}{2}$ we get

$$|p(x)| \leqq 2\left|\left(x+\frac{m-k}{2k}\right)^m(x-1)^k\right| \leqq 2\left(\frac{1}{2}+\frac{m-k}{2k}\right)^m\left(\frac{1}{2}\right)^k.$$

Hence

$$\|p\|_{\left[-\frac{m-k}{2k}, \frac{1}{2}\right]} \leqq 2\left(\frac{m}{2k}\right)^m\left(\frac{1}{2}\right)^k.$$

Let $v_0:=-\frac{m-k}{2k}\left(\leqq\frac{1}{2}\right)$ and $v_{k+1}:=1$. If $l+1\leqq i\leqq k$, then

$$\|p\|_{[v_{i-1},v_i]}<\|p\|_{[v_i,v_{i+1}]}. \tag{12}$$

If $l\leqq i\leqq k$, then

$$\|p\|_{\left[v_0,\frac{1}{2}\right]}\leqq 2\left(\frac{1}{2}+\frac{m-k}{2k}\right)^m\left(\frac{1}{2}\right)^k=\left(\frac{1}{2}+\frac{m-k}{2k}\right)^m\left(\frac{1}{2}\right)^{k-1}<\|p\|_{[v_i,v_{i+1}]}. \tag{13}$$

Let $ay+b$ map $[-1,1]$ one to one onto $\left[-\frac{m-k}{2k},1\right]$ in such a way that $-1\to-\frac{m-k}{2k}$. Then $a=\frac{m+k}{4k}$, $b=\frac{3k-m}{4k}$. Let

$$r(y)=p\left(\frac{m+k}{4k}y+\frac{3k-m}{4k}\right).$$

r has m repeated roots at -1 and for the remaining roots $\eta_1<\eta_2<\ldots<\eta_k$ we have $v_i=a\eta_i+b$, so

$$\eta_i=\frac{1}{a}v_i-\frac{b}{a}=\frac{4k}{m+k}\cos\frac{2(k-i)+1}{2k}\pi+\frac{m-3k}{m+k}.$$

Let $\eta_0:=-1$ and $\eta_{k+1}:=1$, then according to (12) and (13) we get

$$\|r\|_{[\eta_{i-1},\eta_i]}<\|r\|_{[\eta_i,\eta_{i+1}]}\quad\text{if}\quad l<i\leqq k \tag{14}$$

and

$$\|r\|_{[\eta_0,\eta_l]}<\|r\|_{[\eta_i,\eta_{i+1}]}\quad\text{if}\quad l\leqq i\leqq k. \tag{15}$$

LEMMA 10. *We have* $\varrho_i\geqq\eta_{i-1}$ *if* $l<i\leqq k$.

PROOF. Suppose indirectly that there is an i, $l<i\leqq k$ so that $\varrho_i<\eta_{i-1}$. Choose α in such a way that

$$\|\alpha q_n\|_{[-1,\eta_{i-1}]}=\|r\|_{[-1,\eta_{i-1}]}.$$

Then according to (14), (15) and the equioscillation of q_n, $r-\alpha q_n$ has at least $(n-k)+(i-1)$ roots in $[-1,\varrho_i)$ and at least $k-i+2$ roots in $(\varrho_i,1)$. So $r-\alpha q_n\not\equiv 0$ (this is trivial), $r-\alpha q_n$ has at least $(n-k)+(i-1)+(k-i+2)=n+1$ roots in $\mathbf{R}$ and the degree of $r-\alpha q_n$ is at most n, a contradiction. So $\varrho_i\geqq\eta_{i-1}$ if $l<i\leqq k$. Q.E.D.

As $\cos x\geqq 1-\frac{x^2}{2}$, we get

$$\varrho_i\geqq\eta_{i-1}=\frac{4k}{m+k}\cos\frac{2(k-i)+3}{2k}\pi+\frac{m-3k}{m+k}\geqq$$

$$\geqq 1-\frac{(2(k-i)+3)^2\pi^2}{2\cdot 4k^2}\frac{4k}{m+k}=1-\frac{(2(k-i)+3)^2\pi^2}{2nk}$$

if $l<i\leqq k$ $(m+k=n)$. So if i is fixed and $k\geqq k-i>l$, then there is a $c_i>0$ so that

$$\varrho_{k-i} > 1-c_i\frac{1}{nk}. \tag{16}$$

If $k\geqq 6$ then

$$\varrho_k \geqq \eta_{k-1} = \frac{4k}{m+k}\cos\frac{3}{2k}\pi+\frac{m-3k}{m+k} \geqq$$

$$\geqq \frac{4k}{m+k}\left(1-\frac{9\pi^2}{8k^2}\right)+\frac{m-3k}{m+k} = 1-\frac{9\pi^2}{2k(m+k)} = 1-\frac{9}{2}\pi^2\frac{1}{kn}$$

so

$$1-\varrho_k \leqq \frac{9}{2}\pi^2\frac{1}{kn}.$$

According to the Mean Value Rolle's Theorem there is a $\xi\in(\varrho_k, 1)$ for which

$$|q_n'(\xi)| = \left|\frac{q_n(1)-q_n(\varrho_k)}{1-\varrho_k}\right| = \frac{|q_n(1)|}{1-\varrho_k} = \frac{\|q_n\|_{[-1,1]}}{1-\varrho_k} \geqq$$

$$\geqq \|q_n\|_{[-1,1]}\frac{1}{\frac{9}{2}\pi^2\frac{1}{kn}} = \|q_n\|_{[-1,1]}\frac{2}{9\pi^2}kn.$$

The polynomial r gives a better result. We know that

$$|r(1)| = \|r\|_{[-1,1]}, \quad r(y) = p\left(\frac{m+k}{4k}y+\frac{3k-m}{4k}\right),$$

$$\frac{|r'(1)|}{\|r\|_{[-1,1]}} = \left|\frac{r'(1)}{r(1)}\right| = \frac{m+k}{4k}\left|\frac{p'(1)}{p(1)}\right| = \frac{m+k}{4k}\left[m\frac{1}{1+\frac{m-k}{2k}}+k^2\right] =$$

$$= \frac{m}{2}+\frac{m+k}{4k}k^2 > \frac{1}{4}nk$$

so

$$|r'(1)| > \frac{1}{4}nk\|r\|_{[-1,1]}$$

(compare with the counterexample in [3]).

Now we can construct polynomials $p_n\in P_n^k$ having $n-k(\geqq 0)$ real roots at -1 and k real roots in $[-1, 1)$ so that

$$\|p_n^{(j)}\|_{[-1,1]} \geqq c_j(n(k+1))^j\|p_n\|_{[-1,1]}$$

if $k\geqq 0$ is a fixed integer and $n\geqq k, j$.

Consider the case $k\geqq 3(j+2)$. Then $l<k-j$ $\left(\text{as } l=\left[\frac{2}{3}k+\frac{3}{2}\right]\right)$. Hence if $k-i\geqq k-j$ then $\varrho_{k-i}>1-\frac{c_i}{nk}$. From the equioscillation of q_n and the Mean Value

Theorem we can deduce that there are $\beta_1^1 < \beta_2^1 < \ldots < \beta_j^1$ in $(\varrho_{k-j+1}, 1)$ so that

$$|q_n'(\beta_i^1)| \geqq c_j^{-1} nk \|q_n\|_{[-1,1]} \quad (1 \leqq i \leqq j)$$

and

$$\operatorname{sgn} q_n'(\beta_i^1) = -\operatorname{sgn} q_n'(\beta_{i+1}^1) \quad (1 \leqq i < j).$$

Using the Mean Value Theorem again, we can find $\beta_1^2 < \beta_2^2 < \ldots < \beta_{j-1}^2$ in (β_1^1, β_j^1) so that

$$|q_n''(\beta_i^2)| \geqq c_j^{-2}(nk)^2 \|q_n\|_{[-1,1]} \quad (1 \leqq i \leqq j-1)$$

and

$$\operatorname{sgn} q_n''(\beta_i^2) = -\operatorname{sgn} q_n''(\beta_{i+1}^2) \quad (1 \leqq i < j-1).$$

Iterating the process, using always the Mean Value Theorem on suitable intervals, we get lastly

$$|q_n^{(J)}(\beta_1^j)| \geqq c_j^{-j}(nk)^j \|q_n\|_{[-1,1]} \tag{17}$$

where $\beta_1^j \in (\varrho_{k-j+1}, 1)$. As $k \geqq 1$ in this case, we have

$$\|q_n^{(J)}\|_{[-1,1]} \geqq c_j'(n(k+1))^j \|q_n\|_{[-1,1]}.$$

In the case $k < 3(j+2)$ let $p_n(x) := (x+1)^n$ $(p_n \in P_n^0 \subset P_n^k)$. Then

$$p_n^{(J)}(x) = n(n-1)\ldots(n-j+1)(x+1)^{n-j}.$$

So indeed

$$\|p_n^{(J)}\|_{[-1,1]} \geqq c_j n^j \|p_n\|_{[-1,1]} \geqq c_j'(n(k+1))^j \|p_n\|_{[-1,1]}, \quad \text{as} \quad j \leqq n.$$

By this the proof of Theorem 4 is complete.

THEOREM 5. *We have*

$$\sup_{p_n \in P_n^k} \frac{\left\| p_n^{(J)}(x)\sqrt{1-x^2} \right\|_{[-1,1]}}{\|p_n\|_{[-1,1]}} \geqq c_j(n(k+1))^{J-1/2}$$

and

$$\sup_{p_n \in P_n^k} \frac{\|p_n^{(J)}(x)(1-x^2)^{J/2}\|_{[-1,1]}}{\|p_n\|_{[-1,1]}} \geqq c_j'(n(k+1))^{J/2}$$

if $1 \leqq j \leqq n$ *and* $0 \leqq k \leqq n$.

We prove only the first inequality, the proof of the second one is similar.

Case 1. $n > 2k$, $k \geqq 1$. *Subcase* a) $k \geqq 3(j+2)$. Then $l < k-j$ $\left(\text{as } l = \left[\frac{2}{3}k + \frac{3}{2}\right]\right)$.

Hence if $k-i \geqq k-j$, we have

$$1 - \frac{c_j}{nk} \geqq \varrho_i \geqq 1 - \frac{c_j'}{nk}, \quad 0 < c_j < c_j'.$$

In the same way as in Theorem 4 (but instead of the interval $(\varrho_{k-j+1}, 1)$ we choose first the interval $(\varrho_{k-j}, \varrho_k)$) we have

$$\left\| q_n^{(J)}(x)\sqrt{1-x^2} \right\|_{[-1,1]} \geqq c_j'^{-J}(kn)^J \sqrt{1 - \left(1 - c_j \frac{1}{nk}\right)^2} \|q_n\|_{[-1,1]} \geqq$$

$$\geqq c_j'^{-J}(kn)^J c_j''(nk)^{-1/2} \|q_n\|_{[-1,1]} \geqq c_j'''(n(k+1))^{J-1/2} \|q_n\|_{[-1,1]}.$$

Subcase b) $k<3(j+2)$. Then $k^{j-1/2}<c_j$. Let $p_n(x):=(x+1)^n$ $(p_n\in P_n^0\subset P_n^k)$. Then

$$\left\|p_n^{(j)}(x)\sqrt{1-x^2}\right\|_{[-1,1]}\geqq\frac{n!}{(n-j)!}\left(\frac{2n+2}{n+2}\right)^{n-j}\sqrt{1-\left(\frac{n}{n+2}\right)^2}\geqq$$

$$\geqq c_j n^j\left(\frac{n+1}{n+2}\right)^{n+1}\cdot 2^n\cdot 2^{-j}\sqrt{\frac{4n+4}{(n+2)^2}}\geqq c_j' n^j e^{-1}\|p_n\|_{[-1,1]}c'\frac{1}{\sqrt{n}}\geqq$$

$$\geqq c_j'' n^{j-1/2}\|p_n\|_{[-1,1]}\geqq c_j'''\big(n(k+1)\big)^{j-1/2}\|p_n\|_{[-1,1]},\quad \text{as}\quad j\leqq n.$$

Case 2. $n\leqq 2k$ or $k=0$. If $k=0$, let $p_n(x):=(x+1)^n$ like in Subcase b) of Case 1. If $n\leqq 2k$, $c_j'=c_j\cdot 4^j$ is suitable if c_j is so in the case $k=\left[\frac{n-1}{2}\right]$. Q.E.D.

PROBLEM. Is it true that

$$\sup_{p_n\in P_n^k}\frac{\left\|p'(x)\sqrt{1-x^2}\right\|_{[-1,1]}}{\|p\|_{[-1,1]}}\leqq c_1\sqrt{n(k+1)}\ ?$$

If the answer is yes, then

$$\sup_{p_n\in P_n^k}\frac{\left\|p^{(j)}(x)\sqrt{1-x^2}\right\|_{[-1,1]}}{\|p\|_{[-1,1]}}\leqq c_j\big(n(k+1)\big)^{j-1/2}.$$

Indeed, if $p_n\in P_n^k$ then $p_n^{(j-1)}\in P_{n-j+1}^{k+j-1}$ so

$$\left\|p_n^{(j)}(x)\sqrt{1-x^2}\right\|_{[-1,1]}=\left\|p_n^{(j-1)'}(x)\sqrt{1-x^2}\right\|_{[-1,1]}\leqq c_1\sqrt{n(k+j)}\ \|p_n^{(j-1)}\|_{[-1,1]}\leqq$$

$$\leqq c_j\sqrt{n(k+1)}\ \big(n(k+1)\big)^{j-1}\|p_n\|_{[-1,1]}$$

on the basis of the Theorem in [2].

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ON UNIVERSAL FUNCTIONS

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Let $\mathcal{F} \subset L^0(0, 1)$ be a function space endowed with some topology. We call $f \in \mathcal{F}$ a universal function in $\mathcal{F}$ if there exists $\{\lambda_n\} \subset \mathbf{R}$, $\lambda_n \to 0$, such that

$$\left\{\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right\} \subset \mathcal{F}$$

is dense in $\mathcal{F}$.

I. Joó [3], [4] generalized Marcinkiewicz's well-known theorem (see [2] or [5, p. 118, Theorem 3.3]). He proved [4]:

Let $\{\lambda_n\}_1^\infty \subset \mathbf{R}-\{0\}$, $\lambda_n \to 0$. Then for each $f \in L^1(0, 1)$, except for a set of first category, the following statement is valid: for any $0 \leqq p < 1$ and $F \in L^p(0, 1)$ there exists a subsequence $\{\lambda_{n_k}\}_1^\infty$ of $\{\lambda_n\}_1^\infty$ for which

$$\lim_{k\to\infty} \frac{f(x+\lambda_{n_k})-f(x)}{\lambda_{n_k}} = F(x)$$

in the sense of the L^p-metric.

I. Joó stated in [4] the following problem: does the theorem remain valid for $p \geqq 1$ or not?

In this paper we give a negative answer to this problem.

THEOREM.* *If* $p \geqq 1$, *then there is no universal function in* $L^p(0, 1)$.

PROOF. The idea of the proof is that if f is a universal function in $L^p(0, 1)$ $(p \geqq 1)$, then $f \not\equiv$ const. On the other hand we will show that if $f \not\equiv$ const, then the differential quotients of f can not approximate the zero function in $L^p(0, 1)$-norm.

Since f is not constant, by Lebesgue's density theorem there exist $I_1=[a, b]$, $I_2=[c, d]$, $I_1 \cap I_2 = \emptyset$, $|I_1|=|I_2|$, $c_1 < c_2$ such that

$$|I_1 \cap (f \leqq c_1)| > 0.9|I_1|, \tag{1}$$

$$|I_2 \cap (f \geqq c_2)| > 0.9|I_2|. \tag{2}$$

We can also suppose $b < c$ since otherwise we consider $-f$ instead of f.

On the other hand for any function universal in $L^p(0, 1)$ we have

$$\int_0^1 \left|\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right|^p dx \to 0 \quad (n \to \infty) \tag{3}$$

* A similar solution to Joó's problem was given independently by Z. Buczolich [6].

for some sequence $\lambda_n \to 0$. We will prove that for $p \geqq 1$ (3) contradicts (1) and (2). Without loss of generality we may suppose $\lambda_n > 0$.

Indeed, for $p \geqq 1$ the inequality

$$\left(\frac{\sum x_n^p}{n}\right)^{1/p} \geqq \frac{\sum x_n}{n} \quad (x_n \geqq 0)$$

holds; applying it we get

$$(4) \quad \int_0^1 \left|\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right|^p dx \geqq \sum_{k=1}^{[1/\lambda_n]} \int_0^{\lambda_n} \left|\frac{f(x+k\lambda_n)-f(x+(k-1)\lambda_n)}{\lambda_n}\right|^p dx \geqq$$

$$\geqq \int_0^{\lambda_n} \sum_{k=n_1+1}^{n_2} \left|\frac{f(x+k\lambda_n)-f(x+(k-1)\lambda_n)}{\lambda_n}\right|^p dx =$$

$$= \int_0^{\lambda_n} \sum_{k=n_1+1}^{n_2} \frac{|f(x+k\lambda_n)-f(x+(k-1)\lambda_n)|^p}{n_2-n_1} \frac{n_2-n_1}{|\lambda_n^p|} dx \geqq$$

$$\geqq \int_0^{\lambda_n} \left(\sum_{k=n_1+1}^{n_2} \frac{|f(x+k\lambda_n)-f(x+(k-1)\lambda_n)|}{n_2-n_1}\right)^p \frac{n_2-n_1}{|\lambda_n^p|} dx \geqq$$

$$\geqq \int_0^{\lambda_n} |f(x+n_2\lambda_n)-f(x+n_1\lambda_n)|^p \frac{(n_2-n_1)^{1-p}}{|\lambda_n^p|} dx$$

for arbitrary $0 < n_1 < n_2 < \frac{1}{|\lambda_n|}$. Cover I_1 with disjoint intervals of length λ_n, beginning at a and define

$$a' := \left(\left[\frac{a}{\lambda_n}\right]+1\right)\lambda_n, \quad c'-a' := \left(\left[\frac{c-a}{\lambda_n}\right]+1\right)\lambda_n,$$

$$m := \min\{j:\ a+j\lambda_n > a+|I_1|\}.$$

Apply (4) for

$$n_1 = n_{1,i} = \frac{a'}{\lambda_n}+i-1; \quad n_2 = n_{2,i} = \frac{c'}{\lambda_n}+i-1 \quad (i = 1, \ldots, m)$$

and summarize the resulting inequalities. We get

$$(5) \quad m \int_0^1 \left|\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right|^p dx \geqq$$

$$\geqq \int_0^{\lambda_n} \sum_{k=1}^{m} |f(x+n_{2,k}\lambda_n)-f(x+n_{1,k}\lambda_n)|^p \frac{(n_{2,1}-n_{1,1})^{1-p}}{|\lambda_n^p|} dx =$$

$$= \int_{a'}^{a'+m\lambda_n} |f(x+c'-a')-f(x)|^p \frac{(n_{2,1}-n_{1,1})^{1-p}}{|\lambda_n^p|} dx.$$

It is obvious that $f(x)\geqq c_2$, $f(x+c'-a')\leqq c_1$ implies

$$f(x+c'-a')-f(x)\geqq c_2-c_1. \tag{6}$$

On the other hand, it is easy to see that

$$\left|[a', a'+m|\lambda_n|]\cap(f\leqq c_1)\right| > 0.9|I_1|-2|\lambda_n| > 0.8|I_1|$$

and

$$\left|[c', c'+m|\lambda_n|]\cap(f\geqq c_2)\right| > 0.9|I_2|-5|\lambda_n| > 0.8|I_2|$$

if n is large enough. Consequently (6) holds on a set of measure $>0.6|I_1|$ and hence, by (5) we have

$$m\int_0^1\left|\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right|^p dx \geqq 0.6|I_1|(c_2-c_1)^p\frac{\left(\frac{c'-a'}{|\lambda_n|}\right)^{1-p}}{|\lambda_n^p|} =$$

$$= 0.6|I_1|(c_2-c_1)^p\frac{(c'-a')^{1-p}}{|\lambda_n|}.$$

If n is large enough then $c'-a'>\frac{c-a}{2}$ and $m|\lambda_n|<2|I_1|$, hence we obtain

$$\int_0^1\left|\frac{f(x+\lambda_n)-f(x)}{\lambda_n}\right|^p dx > 0.3(c_2-c_1)^p\left(\frac{c-a}{2}\right)^{1-p} > 0$$

for $n>n_0$. Since the right hand side does not depend on n, we get the desired contradiction.

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A JACKSON-TYPE ESTIMATION ON THE REAL LINE BY INTERPOLATION

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1. Introduction. Preliminary results

1.1. In their paper [1] O. Kis and P. Vértesi defined a simple interpolatory process to prove the Jackson theorem for $f\in\tilde{C}$ (i.e. f is a 2π-periodic continuous function).

Namely, let

$$t_k = t_{kn} = \frac{2k\pi}{2n+1} \quad (k = 0, \pm 1, \pm 2, \ldots;\ n = 0, 1, 2, \ldots). \tag{1.1}$$

The corresponding fundamental polynomials of trigonometric interpolation are

(1.2)

$$l_k(t) = l_{kn}(t) = \frac{\sin\frac{2n+1}{2}(t-t_k)}{(2n+1)\sin\frac{1}{2}(t-t_k)} \quad (k = 0, \pm 1, \pm 2, \ldots, \pm n;\ n = 0, 1, 2, \ldots).$$

Now if

$$u_k(t) = u_{kn}(t) = 4l_k(t)^3 - 3l_k(t)^4 \quad (k = 0, \pm 1, \pm 2, \ldots, \pm n;\ n = 0, 1, 2, \ldots), \tag{1.3}$$

we obviously have the relation $u_k(t_j)=\delta_{kj}$.

Further for

$$p_n(t) = p_n(f, t) = \sum_{k=-n}^{n} f(t_k)\, u_k(t), \tag{1.4}$$

we have $p_n(t_k)=f(t_k)$, and, as it was proved in [1], for $t\in\mathbf{R}$

$$|f(t) - p_n(t)| = O\left(\omega\left(f, \frac{1}{n}\right)\right), \quad \text{if} \quad n\in\mathbf{N}, \tag{1.5}$$

where

$$\omega(f, \delta) = \sup_{|t_1 - t_2|\leq\delta} |f(t_1) - f(t_2)| \tag{1.6}$$

is the usual modulus of continuity. I.e., using the interpolatory trigonometric polynomials $p_n(f, t)$ of degree $\leq 4n$, we can prove the well-known Jackson estimation.

1.2. To extend this result (see Theorem 2.1) we need some definitions.

DEFINITION 1. The entire function g is of exponential type σ $(0\leq\sigma<\infty)$ (shortly $g\in E_\sigma$) iff for any fixed $\varepsilon>0$

$$|g(z)| = O(e^{(\sigma+\varepsilon)|z|}) \quad \text{if} \quad z\in\mathbf{C}. \tag{1.7}$$

DEFINITION 2. We say that $g \in B_\sigma$ iff $g \in E_\sigma$ moreover g is bounded on the real line.

DEFINITION 3. Let f: $\mathbf{R} \to \mathbf{R}$. We say that $f \in \mathrm{UCB}(\mathbf{R})$ iff f is uniformly continuous and bounded.

It is easy to see that definition (1.6) and the well-known properties of the modulus of continuity can be extended for the function class UCB (**R**), too (see [3; 3.1]).

For further purposes we quote the following statement of Phragmen—Lindelöf type (see [3; p. 194]).

If $g \in B_\sigma$ *and* $|g(x)| \leqq K$ *for* $x \in \mathbf{R}$, *then*

$$|g(x+iy)| \leqq Ke^{\sigma|y|} \tag{1.8}$$

for any $x+iy=z \in \mathbf{C}$.

1.3. A possible extension of the trigonometric interpolation based on the nodes (1.1) and the function class $\tilde{C}$ was given by R. Gervais, Q. I. Rahman and G. Schmeisser [2], and A. F. Timan [3; p. 196].

Let with a real α_σ (for every fixed $\sigma > 0$)

$$x_{k\sigma} = x_{k\sigma}(\alpha_\sigma) = \frac{k\pi}{\sigma} + \alpha_\sigma, \quad k \in \mathbf{Z}. \tag{1.9}$$

Then for the entire function

$$g_{k\sigma}(z) = \begin{cases} \dfrac{\sin(\sigma(z-x_{k\sigma}))}{\sigma(z-x_{k\sigma})} + \dfrac{1}{k\pi}\sin(\sigma(z-x_{k\sigma})), & \text{if} \quad k \neq 0, \\ \dfrac{\sin(\sigma(z-\alpha_\sigma))}{\sigma(z-\alpha_\sigma)}, & \text{if} \quad k = 0 \end{cases} \tag{1.10}$$

we have

(i) $g_{k\sigma} \in B_\sigma$, $k \in \mathbf{Z}$,

(ii) $g_{k\sigma}(x_{j\sigma}) = \delta_{kj}$, $j \in \mathbf{Z}$,

(iii) $g'_{k\sigma}(\alpha_\sigma) = 0$,

(iv) it is the only function satisfying conditions (i)—(iii) (see [3; 4.3.11]).

Further, by [2; Lemma 5], if $|\alpha_\sigma| \leqq H$, where H is independent of σ, then for $x \in \mathbf{R}$

$$\text{(v)} \qquad \sum_{k=-\infty}^{\infty} |g_{k\sigma}(x)| \leqq c_H[\log(\sigma+2) + \log(|x|+2)]$$

with a fixed $c_H > 0$, and by [2; Lemma 6], if $|\alpha_\sigma| \leqq \dfrac{\pi}{2\sigma}$, $\varepsilon > 0$ for $x \in \mathbf{R}$,

$$\text{(vi)} \qquad \sum_{k=-\infty}^{\infty} \frac{|g_{k\sigma}(x)|}{[\log(|x_{k\sigma}|+3)]^{1+\varepsilon}} < c_\varepsilon \log(\sigma+2), \quad \text{if} \quad \sigma \geqq 1,$$

where c_ε is independent of x and σ.

Using these fundamental functions, for arbitrary $G\in B_\sigma$ we have the *unique* representation (see [3; p. 197])

$$G(z)=\frac{\sin\sigma(z-\alpha_\sigma)}{\sigma}G'(\alpha_\sigma)+\sum_{k=-\infty}^{\infty}G(x_{k\sigma})\,g_{k\sigma}(z). \tag{1.11}$$

To obtain the "Lagrange interpolation" for other functions, let

$$L_\sigma(f,z,c)=c\frac{\sin\sigma(z-\alpha_\sigma)}{\sigma}+\sum_{k=-\infty}^{\infty}f(x_{k\sigma})\,g_{k\sigma}(z), \tag{1.12}$$

where $f\in\mathrm{UCB}(\mathbf{R})$ and c is fixed (see [2; §3]).

Also in [2; §3], the following properties were essentially proved

(1) $L_\sigma(f,\cdot,c)\in E_\sigma$,

(2) $L_\sigma(f,x_{k\sigma},c)=f(x_{k\sigma})$, $k\in\mathbf{Z}$,

(3) $L'_\sigma(f,\alpha_\sigma,c)=c$,

(4) $|L_\sigma(f,x,c)|=O(\log|x|)$, as $x\to\pm\infty$,

(5) it is the only function satisfying (1)—(4).

Moreover, the next statements hold true.

THEOREM A (see [2; Theorem 1]). *Let* $f\in\mathrm{UCB}(\mathbf{R})$ *with* $\lim_{\sigma\to\infty}\omega\left(f,\frac{1}{\sigma}\right)\log\sigma=0$. *Then* $L_\sigma(f,\cdot,c_\sigma)$ *with* $\alpha_\sigma=0$ $(\sigma>0)$ *uniformly tends to* f *on all compact subintervals of the real line as* $\sigma\to\infty$. *Here* $c_\sigma=\varphi'_\sigma(f,\alpha_\sigma)$, *where*

$$\varphi_\sigma(f,x)=\frac{1}{\mu}\int_{-\infty}^{\infty}\left[\frac{\sin(\sigma(x-t)/4)}{x-t}\right]^4 f(t)\,dt,$$

and

$$\mu=\int_{-\infty}^{\infty}\left[\frac{\sin(\sigma t/4)}{t}\right]^4 dt.$$

THEOREM B (see [2; Theorem 2]). *If* $f\in\mathrm{UCB}(\mathbf{R})$ *and*

$$\lim_{\sigma\to\infty}\omega\left(F,\frac{1}{\sigma}\right)\log\sigma=0,$$

where

$$F(x)=F_\varepsilon(x)=[\log(|x|+3)]^{1+\varepsilon}f(x)$$

($\varepsilon>0$, *fixed), then* $L_\sigma(f,\cdot,c_\sigma)$ *with* $\alpha_\sigma=0$ $(\sigma>0)$ *uniformly tends to* f *on the whole real line as* $\sigma\to\infty$. *Here* c_σ *is the same as above.*

2. New results

2.1. First we define a new interpolatory process for $f\in \mathrm{UCB}(\mathbf{R})$ which corresponds to the Kis—Vértesi operator. For $\sigma>0$ and $k\in\mathbf{Z}$, let

$$q_{k\sigma}(z)=\frac{\sin\left(\sigma(z-x_{k\sigma})\right)}{\sigma(z-x_{k\sigma})}, \tag{2.1}$$

$$u_{k\sigma}(z)=4q_{k\sigma}(z)^3-3q_{k\sigma}(z)^4, \tag{2.2}$$

$$U_\sigma(f,z)=\sum_{k=-\infty}^{\infty} f(x_{k\sigma})u_{k\sigma}(z), \quad f\in\mathrm{UCB}(\mathbf{R}). \tag{2.3}$$

The following properties are obvious:

$q_{k\sigma}\in B_\sigma$, $u_{k\sigma}\in B_{4\sigma}$, $q_{k\sigma}(x_{j\sigma})=\delta_{kj}$, $u_{k\sigma}(x_{j\sigma})=\delta_{kj}$, $U_\sigma(f,x_{k\sigma})=f(x_{k\sigma})$ $(k,j\in\mathbf{Z})$ and $U_\sigma(f)\in B_{4\sigma}$ (see Lemma 3.2), i.e. $U_\sigma(f)$ is analogous to (1.4), indeed. Moreover, we prove

THEOREM 2.1. *For arbitrary* $f\in\mathrm{UCB}(\mathbf{R})$ *we have the estimation*

$$\sup_{x\in\mathbf{R}}|f(x)-U_\sigma(f,x)|=O(\omega(f,1/\sigma)), \quad \textit{if} \quad \sigma>0. \tag{2.4}$$

(The "O" can be estimated by 28.)

Theorem 2.1 gives an interpolatory proof for [3; §5.1.3 (11)], when $k=1$.

2.2. Let us see a simple application of (2.4). As P. Vértesi remarked, instead of (1.12) we can use the simpler expression

$$L_\sigma(f,z)=\sum_{k=-\infty}^{\infty} f(x_{k\sigma})g_{k\sigma}(z), \quad f\in\mathrm{UCB}(\mathbf{R}) \tag{2.5}$$

without loosing its nice approximating property if we apply another function $\varphi_\sigma(f)$ having not only the good approximating property but for which $\varphi'_\sigma(f,\alpha_\sigma)=0$ (compare Theorems A and B). Our operator U_σ, using estimation (2.4), seems to be adequate. Namely, we prove

THEOREM 2.2. *Let* $f\in\mathrm{UCB}(\mathbf{R})$ *and* $|\alpha_\sigma|\leqq H$, *where* H *is independent of* σ. *Then* $L_\sigma(f)$ *has the properties* (1)—(5) *(with* $c=0$*). Moreover,* $L_\sigma(f)$ *uniformly tends to* f *on all compact subintervals of the real line as* $\sigma\to\infty$ *whenever* $\lim_{\sigma\to\infty}\omega\left(f,\frac{1}{\sigma}\right)\log\sigma=0$.

As another application of the operator U_σ, we have

THEOREM 2.3. *If* $f\in\mathrm{UCB}(\mathbf{R})$, $|\alpha_\sigma|\leqq\frac{\pi}{2\sigma}$ *and* $\lim_{\sigma\to\infty}\omega\left(F,\frac{1}{\sigma}\right)\log\sigma=0$ *where* $F(x)={}_\varepsilon F(x)=[\log(|x|+3)]^{1+\varepsilon}f(x)$ *(*$\varepsilon>0$, *fixed), then* $L_\sigma(f)$ *uniformly tends to* f *on the whole real line as* $\sigma\to\infty$.

3. Proofs

3.1. First we prove a useful lemma which slightly generalizes [3; p. 194] and [2; Lemma 3].

LEMMA 3.1. *Let* $g\in E_{n\sigma}$ $(n\in\mathbf{N})$ *with*

$$\lim_{x\to\pm\infty}\frac{g(x)}{x}=0 \tag{3.1}$$

and

$$g(x_{k\sigma})=g'(x_{k\sigma})=\ldots=g^{(n-1)}(x_{k\sigma})=0,\quad k\in\mathbf{Z}. \tag{3.2}$$

Then

$$g(z)=c\sin^n(\sigma(z-\alpha_\sigma)), \tag{3.3}$$

where c *is a constant.*

PROOF. $g(z)(z-\alpha_\sigma)^{-1}\in E_{n\sigma}$, i.e. by (1.8), $|g(z)(z-\alpha_\sigma)^{-1}|=O(e^{n\sigma|y|})$ from where

$$|g(z)|=O\big((1+|z|)e^{n\sigma|y|}\big). \tag{3.4}$$

By (3.2),

$$h(z)=\frac{g(z)}{\sin^n(\sigma(z-\alpha_\sigma))} \tag{3.5}$$

is an entire function. Let us estimate its absolute value. For $|\operatorname{Im} z|\leqq\frac{\pi}{2\sigma}$ take that node $x_{j\sigma}$ for which $|\operatorname{Re} z-x_{j\sigma}|\leqq\frac{\pi}{2\sigma}$. Then, by (3.4) and the maximum principle applied on the square

$$S_j=\left\{(x,y)\,\middle|\,x_{j\sigma}-\frac{\pi}{2\sigma}\leqq x\leqq x_{j\sigma}+\frac{\pi}{2\sigma},\ -\frac{\pi}{2\sigma}\leqq y\leqq\frac{\pi}{2\sigma}\right\}$$

we get

$$|h(z)|=O\big((1+|z|)e^{n\sigma|y|}\big)=O(1+|z|),\quad \text{if}\quad z\in S_j. \tag{3.6}$$

On the other hand, for $|\operatorname{Im} z|>\frac{\pi}{2\sigma}$ we have (denoting $\operatorname{Im} z$ by y)

$$|\sin(\sigma z)|=\left|\frac{e^{i\sigma z}-e^{-i\sigma z}}{2i}\right|\geqq\frac{\big||e^{i\sigma z}|-|e^{-i\sigma z}|\big|}{2}=\frac{e^{\sigma|y|}-e^{-\sigma|y|}}{2} \tag{3.7}$$

whence

$$|h(z)|=\frac{O\big((1+|z|)e^{n\sigma|y|}\big)}{\text{const}\,e^{n\sigma|y|}}=O(1+|z|). \tag{3.8}$$

Then, using (3.6), (3.8) and the generalized Liouville theorem, $h(z)=a_0+a_1z$. Now, by (3.1), we get $a_1=0$, which was to be proven.

3.2. Here we prove

LEMMA 3.2. *For any arbitrary fixed number $r>1$ we have*

$$\sum_{k=-\infty}^{\infty} q_{k\sigma}^r \in B_{r\sigma}. \tag{3.9}$$

PROOF. To estimate $\sum_{k=-\infty}^{\infty} q_{k\sigma}^r(z)$, first let $|\operatorname{Im} z| \leqq \frac{\pi}{2\sigma}$. Then on the square S_j we get with $z=x+iy$

(3.10)

$$\left|\sum_{k=-\infty}^{\infty} q_{k\sigma}(z)^r\right| \leqq \sum_{k=-\infty}^{\infty} \frac{|\sin(\sigma(z-x_{k\sigma}))^r|}{\sigma^r|z-x_{k\sigma}|^r} \leqq \frac{|\sin(\sigma(z-x_{j\sigma}))^r|}{\sigma^r|z-x_{j\sigma}|^r} + \sum_{k\neq j} \frac{e^{r\pi/2}}{\sigma^r|z-x_{k\sigma}|^r} \leqq$$

$$\leqq K_1(r) + \sum_{k\neq j} \frac{K_2(r)}{\sigma^r|\operatorname{Re} z - x_{k\sigma}|^r} \leqq K_1(r) + K_2(r) \sum_{k\neq j} \frac{1}{\sigma^r\left(\frac{2|j-k|-1}{2\sigma}\pi\right)^r} \leqq$$

$$\leqq K_1(r) + K_2(r) K_3(r) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^r} = K_4(r),$$

using the inequality

$$|x - x_{k\sigma}| \geqq \frac{2|j-k|-1}{2\sigma}\pi, \quad \text{if} \quad k \neq j. \tag{3.11}$$

For $|\operatorname{Im} z| > \frac{\pi}{2\sigma}$ take that node $x_{j\sigma}$ for which $|\operatorname{Re} z - x_{j\sigma}| \leqq \frac{\pi}{2\sigma}$. Then using (3.11) we obtain

$$\left|\sum_{k=-\infty}^{\infty} q_{k\sigma}(z)^r\right| \leqq \sum_{k=-\infty}^{\infty} \frac{|\sin(\sigma(z-x_{k\sigma}))^r|}{\sigma^r|z-x_{k\sigma}|^r} \leqq \tag{3.12}$$

$$\leqq \frac{e^{r\sigma|y|}}{\sigma^r} \sum_{k=-\infty}^{\infty} \frac{1}{|z-x_{k\sigma}|^r} \leqq \frac{e^{r\sigma|y|}}{\sigma^r}\left[\left(\frac{2\sigma}{\pi}\right)^r + \sum_{k\neq j} \frac{1}{|\operatorname{Re} z - x_{k\sigma}|^r}\right] \leqq$$

$$\leqq \frac{e^{r\sigma|y|}}{\sigma^r}\left[\left(\frac{2\sigma}{\pi}\right)^r + \sum_{k\neq j} \frac{1}{\left(\frac{2|j-k|-1}{2\sigma}\pi\right)^r}\right] \leqq K_5(r)\, e^{r\sigma|y|}.$$

Now by (3.10) and (3.12) we get (3.9), using the Weierstrass theorem on locally uniformly convergent series of regular functions.

Using the above argument it is easy to get the following

COROLLARY 3.3. *If $|f(x)| \leqq K$ for $x \in \mathbf{R}$, then $\sum_{k=-\infty}^{\infty} f(x_{k\sigma}) q_{k\sigma}^r \in B_{r\sigma}$. In particular $U_\sigma(f) \in B_{4\sigma}$. If $|f(x)| \leqq K(1+|x|)^{r-1-\delta}$ $(r>1,\ \delta>0)$ for $x \in \mathbf{R}$, then $\sum_{k=-\infty}^{\infty} f(x_{k\sigma}) q_{k\sigma}^r$ is an entire function, in particular $U_\sigma(f)$ is an entire function, if $|f(x)| \leqq K(1+|x|)^{2-\delta}$ $(\delta>0)$ for $x \in \mathbf{R}$.*

We will need the following strengthening of Lemma 3.2 for $r\geqq 2$. Let $z=x+iy$. Then we have

LEMMA 3.4. *If* $r\geqq 2$, *then*

$$\left|\sum_{k=-\infty}^{\infty} q_{k\sigma}(z)^r\right| \leqq \frac{3e^{r\sigma|y|}}{(\sigma|y|)^{r-1}}, \quad \text{if} \quad y\to\pm\infty.$$

In particular if $R(z)=U_\sigma(1,z)=\sum_{k=-\infty}^{\infty} u_{k\sigma}(z)$, *then*

$$|R(z)| \leqq \frac{21e^{4\sigma|y|}}{(\sigma|y|)^3}, \quad \text{if} \quad y\to\pm\infty. \tag{3.13}$$

PROOF. First we verify the inequality

$$\sum_{k=-\infty}^{\infty} \frac{1}{1+\left(\frac{k\pi}{\sigma y}\right)^2} \leqq 1+\sigma|y|, \quad \text{if} \quad \sigma>0, \quad y\in\mathbf{R}, \quad y\neq 0. \tag{3.14}$$

Indeed,

$$\sum_{k=-\infty}^{\infty} \frac{1}{1+\left(\frac{k\pi}{\sigma y}\right)^2} = 1+2\sum_{k=1}^{\infty} \frac{1}{1+\left(\frac{k\pi}{\sigma y}\right)^2} \leqq 1+2\int_0^{\infty} \frac{1}{1+\left(\frac{x\pi}{\sigma y}\right)^2}dx =$$

$$= 1+2\left[\arctan\left(\frac{x\pi}{\sigma y}\right)\right]_{x=0}^{x=\infty} y\frac{\sigma}{\pi} = \begin{cases} 1+\sigma y, & \text{if} \quad y>0, \\ 1-\sigma y, & \text{if} \quad y<0. \end{cases}$$

Now we may assume that $|y|>\frac{1}{\sigma}$. Then, using (3.11) and (3.14), we get

$$\left|\sum_{k=-\infty}^{\infty} q_{k\sigma}(z)^r\right| \leqq \sum_{k=-\infty}^{\infty} \frac{e^{r\sigma|y|}}{\sigma^r|z-x_{k\sigma}|^r} = \tag{3.15}$$

$$= \frac{e^{r\sigma|y|}}{\sigma^r}\sum_{k=-\infty}^{\infty}\left(\frac{1}{|z-x_{k\sigma}|^2}\right)^{r/2} = \frac{e^{r\sigma|y|}}{\sigma^r}\sum_{k=-\infty}^{\infty}\left[\frac{1}{(x-x_{k\sigma})^2+y^2}\right]^{r/2} \leqq$$

$$\leqq \frac{e^{r\sigma|y|}}{\sigma^r|y|^r}\sum_{k=-\infty}^{\infty}\frac{1}{1+\left(\frac{x-x_{k\sigma}}{y}\right)^2} \leqq \frac{e^{r\sigma|y|}}{\sigma^r|y|^r}\left[1+2\sum_{k=1}^{\infty}\frac{1}{1+\left(\frac{2k-1}{2\sigma y}\pi\right)^2}\right] \leqq$$

$$\leqq \frac{e^{r\sigma|y|}}{\sigma^r|y|^r}(1+2\sigma|y|) \leqq \frac{3e^{r\sigma|y|}}{(\sigma|y|)^{r-1}}.$$

3.3. Now we prove

LEMMA 3.5. *We have the relation* $R(z)=1$ *for arbitrary* $z\in\mathbf{C}$.

PROOF. First we verify conditions (3.1) and (3.2) for the function $S(z):=R(z)-1$. Indeed (compare [1; §5]),

$$u'_{k\sigma} = 12(q_{k\sigma}^2 q'_{k\sigma} - q_{k\sigma}^3 q'_{k\sigma}), \quad \text{if} \quad k\in\mathbf{Z}, \tag{3.16}$$

from where by $q'_{k\sigma}(x_{k\sigma})=0$ $(k\in\mathbf{Z})$

$$u'_{k\sigma}(x_{j\sigma}) = 0, \quad \text{if} \quad k,j\in\mathbf{Z}, \tag{3.17}$$

$$u''_{k\sigma} = 12(2q_{k\sigma}q_{k\sigma}'^2 + q_{k\sigma}^2 q''_{k\sigma} - 3q_{k\sigma}^2 q_{k\sigma}'^2 - q_{k\sigma}^3 q''_{k\sigma}), \tag{3.18}$$

if $k\in\mathbf{Z}$, thus again

$$u''_{k\sigma}(x_{j\sigma}) = 0, \quad \text{if} \quad k,j\in\mathbf{Z}. \tag{3.19}$$

Further, we get

$$u'''_{k\sigma} = 12(2q_{k\sigma}'^3 + 6q_{k\sigma}q'_{k\sigma}q''_{k\sigma} + q_{k\sigma}^2 q'''_{k\sigma} - 6q_{k\sigma}q_{k\sigma}'^3 - 9q_{k\sigma}^2 q'_{k\sigma}q''_{k\sigma} - q_{k\sigma}^3 q'''_{k\sigma}), \quad \text{if} \quad k\in\mathbf{Z}, \tag{3.20}$$

i.e.

$$u'''_{k\sigma}(x_{j\sigma}) = \begin{cases} \dfrac{24\sigma^3}{\pi^3}\dfrac{(-1)^{j-k}}{(j-k)^3}, & \text{if} \quad k\neq j, \\ 0, & \text{if} \quad k=j. \end{cases} \tag{3.21}$$

By Corollary 3.3, (3.17) and (3.19), $S(z)\in B_{4\sigma}$, and

$$R(x_{j\sigma})-1 = R'(x_{j\sigma}) = R''(x_{j\sigma}) = 0 \quad \text{for} \quad j\in\mathbf{Z}.$$

Now we state that $R'''(x_{j\sigma})=0$ $(j\in\mathbf{Z})$, too. Indeed, by (3.21)

$$R'''(x_{j\sigma}) = \sum_{k=-\infty}^{\infty} u'''_{k\sigma}(x_{j\sigma}) = \lim_{M\to\infty} \sum_{\substack{k=j-M \\ k\neq j}}^{j+M} u'''_{k\sigma}(x_{j\sigma}) = \tag{3.22}$$

$$= \frac{24\sigma^3}{\pi^3} \lim_{M\to\infty} \left\{ \left[\frac{(-1)^M}{M^3} + \frac{(-1)^M}{(-M)^3} \right] + \ldots + \left[\frac{-1}{1} + \frac{-1}{-1} \right] \right\} = 0.$$

By Lemma 3.1 we get that $R(z)=1+c\sin^4(\sigma(z-\alpha_\sigma))$. Applying (3.7) and (3.13), c must be equal to zero.

3.4. PROOF OF THEOREM 2.1. The proof is similar to [1; §2]. First let us see some simple estimations:

$$|q_{k\sigma}(x)| \leqq 1, \quad \text{if} \quad x\in\mathbf{R}, \tag{3.23}$$

$$|u_{k\sigma}(x)| \leqq 4|q_{k\sigma}(x)|^3 + 3|q_{k\sigma}(x)|^4, \quad \text{if} \quad x\in\mathbf{R}, \tag{3.24}$$

$$|f(x)-f(x_{k\sigma})| \leqq \omega(f,|x-x_{k\sigma}|) \leqq (\sigma|x-x_{k\sigma}|+1)\omega\left(f,\frac{1}{\sigma}\right), \tag{3.25}$$

$$|q_{k\sigma}(x)| = \left|\frac{\sin(\sigma(x-x_{k\sigma}))}{\sigma(x-x_{k\sigma})}\right| \leqq \frac{1}{\sigma|x-x_{k\sigma}|}, \quad \text{if} \quad x\neq x_{k\sigma}. \tag{3.26}$$

Then, by Lemma 3.5 and (3.11) we get

$$(3.27)\quad |f(x)-U_\sigma(f,x)| = |U_\sigma(f(x)-f,x)| = \Big|\sum_{k=-\infty}^{\infty}(f(x)-f(x_{k\sigma}))u_{k\sigma}(x)\Big| \leqq$$

$$\leqq \omega\left(f,\frac{1}{\sigma}\right)\left[7(\sigma|x-x_{j\sigma}|+1)+\sum_{k\neq j}(\sigma|x-x_{k\sigma}|+1)\left(\frac{4}{\sigma^3|x-x_{k\sigma}|^3}+\frac{3}{\sigma^4|x-x_{k\sigma}|^4}\right)\right] \leqq$$

$$\leqq \omega\left(f,\frac{1}{\sigma}\right)\left[7\frac{\pi}{2}+7+\frac{32}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}+\frac{112}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}+\right.$$

$$\left.+\frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\right] \leqq 28\omega\left(f,\frac{1}{\sigma}\right).$$

3.5. PROOF OF THEOREM 2.2. The first part is obvious (see (1.12) and (1)—(5)). To prove the estimation we use that by (3.17) $U'_{\sigma/4}(f, y_{k,\sigma/4})=0$, if $k\in\mathbf{Z}$, where $y_{j,\sigma/4}$ are the interpolatory nodes for the operator $U_{\sigma/4}$. Choosing these nodes such that $y_{0,\sigma/4}=x_{0\sigma}$ (where $x_{j\sigma}$ are the interpolatory nodes for the operator L_σ), we get (see (2.4), (1.11) and (v))

$$(3.28)\quad |f(x)-L_\sigma(f,x)| \leqq |f(x)-U_{\sigma/4}(f,x)|+|U_{\sigma/4}(f,x)-L_\sigma(f,x)| \leqq$$

$$\leqq c_1\omega\left(f,\frac{1}{\sigma}\right)+|L_\sigma(U_{\sigma/4}(f)-f,x)| \leqq c_1\omega\left(f,\frac{1}{\sigma}\right)(1+c_H(\log(\sigma+2)+\log(|x|+2))),$$

as it was stated.

3.6. PROOF OF THEOREM 2.3. The proof is similar to that of [2; Theorem 2], if we use that $(\log(|x|+3))^{1+\varepsilon}\in\mathrm{Lip}\,1$, moreover, $U_{\sigma/4}(F_\varepsilon)$ is an entire function (see Corollary 3.3), from where (3.27) holds true for F_ε. We omit the details.

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ON LOCALLY REPEATED VALUES OF CERTAIN ARITHMETIC FUNCTIONS. II

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§1. Introduction

Let $v(n)$ denote the number of distinct prime factors of n. In Part I of this paper [5], we studied the equation

$$n+v(n) = m+v(m), \quad n \neq m \tag{1.1}$$

and generalizations where v is replaced with a more general arithmetic function. In this part, we study the distribution of the n for which $v(n) \approx v(n+1)$ and the n for which $\Phi(n)=\Phi(n+1)$, where Φ is Euler's function.

It seems reasonable to conjecture that there is a positive constant c, such that the number of $n \leqq x$ with

$$v(n) = v(n+1) \tag{1.2}$$

is $(c_1+o(1))x/\sqrt{\log\log x}$. Indeed, from the Erdős—Kac theorem, most integers n satisfy

$$|v(n)-\log\log n| \leqq K\sqrt{\log\log n} \tag{1.3}$$

where K is some large constant. In fact, the asymptotic density of the n which satisfy (1.3) is exactly

$$\frac{1}{\sqrt{2\pi}} \int_{-K}^{K} e^{-t^2/2}\, dt,$$

which is nearly 1 if K is large. Thus if n and $n+1$ both satisfy (1.3) and if we view $v(n)$ and $v(n+1)$ as "independent events", then the "probability" that (1.2) holds should be at least $(2K\sqrt{\log\log n})^{-1}$. Summing these probabilities would then give order of magnitude $x/\sqrt{\log\log x}$ solutions n of (1.2) with $n \leqq x$, thus supporting the conjecture. A refinement of this heuristic argument even suggests that $c_1=(2\sqrt{\pi})^{-1}$.

It is not even known, however, if (1.2) has infinitely many solutions. Our principal result in this paper is that a slight weakening of (1.2) has at least the "correct" order of magnitude for the number of solutions $n \leqq x$.

THEOREM 1. *There are absolute constants $c_2, c_3>0$ such that for $x \geqq 3$, the number of $n \leqq x$ with*

$$|v(n)-v(n+1)| \leqq c_2 \tag{1.4}$$

is at least $c_3 x/\sqrt{\log\log x}$.

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The proof of Theorem 1 is based on the fundamental lemma of the combinatorial sieve. By using Selberg's lower bound sieve (see Halberstam and Richert [6], Theorem 7.4 which gives Lemma 1 below with $u=4.43$) and a more careful argument, Theorem 1 can be proved with $c_2=3$. However, the proof would be longer and not involve any essentially new ideas, so we present here only the simpler version in Theorem 1.

It is to be remarked that our proof of Theorem 1 easily gives the same result with Ω in place of ν, where $\Omega(n)$ is the number of prime factors of n counting multiplicity. If $d(n)$ denotes the number of divisors of n, our proof also shows that the number of $n\leqq x$ with $d(n)/d(n+1)=2^i$ where i is an integer with $|i|\leqq c_2$ is at least $c_3x/\sqrt{\log\log x}$. Recently, Heath-Brown [7] proved that $d(n)=d(n+1)$ has at least $c_4x/(\log x)^7$ solutions $n\leqq x$ where $c_4>0$. He announced that his method also works for $\Omega(n)=\Omega(n+1)$. It is not unlikely that some refinement of his idea would also work for (1.2).

A result somewhat weaker than Theorem 1 can be immediately obtained using a special case of a theorem of Barban and Vinogradov (see Elliott [2], Theorem 20.1). From this theorem, for $x>10^7$ and $z>0$,

$$\frac{1}{x}\cdot\#\{n\leqq x:\ |\nu(n+1)-\nu(n)|\leqq z\sqrt{2\log\log x}\}=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-z}^{z}e^{-t^2/2}\,dt+O\left(\frac{1}{\sqrt{\log\log x}}\left(1+\frac{\log\log\log x}{e^{t^2/2}\log\log\log\log x}\right)\right).$$

Applying this result to values of z near 0 gives

$$\#\left\{n\leqq x:\ |\nu(n+1)-\nu(n)|\leqq c_5\frac{\log\log\log x}{\log\log\log\log x}\right\}\asymp\frac{x\log\log\log x}{\sqrt{\log\log x}\,\log\log\log\log x},$$

for some constant $c_5>0$. We are endebted to R. R. Hall and G. Tenenbaum for this observation.

Some changes in the proof of Theorem 1 give the following result which we state without proof.

THEOREM 1′. *For each positive integer k, there are absolute constants $c_2(k)$, $c_3(k)>0$ such that for $x\geqq 3$, the number of $n\leqq x$ with*

$$\max\{\nu(n),\nu(n+1),\ldots,\nu(n+k)\}-\min\{\nu(n),\nu(n+1),\ldots,\nu(n+k)\}\leqq c_2(k)$$

is at least $c_3(k)x/(\log\log x)^{k/2}$.

In the third paper in this series, we shall complement Theorem 1 with an upper bound result of the same order of magnitude and also give an upper bound for the frequency of solutions of (1.1).

We shall complete this paper with a result concerning Euler's function $\Phi(n)$. In [3], it is shown that the asymptotic density of the n with $\Phi(n)<\Phi(n+1)$ is 1/2 and the same for the n with $\Phi(n)>\Phi(n+1)$. Thus as a corollary, the number of $n\leqq x$ with

$$\Phi(n)=\Phi(n+1) \tag{1.5}$$

is $o(x)$. The following result implies, for example, the stronger assertion that the solutions of (1.5) have a bounded sum of reciprocals.

THEOREM 2. *For large x, the number of solutions of* (1.5) *not exceeding x is at most $x/\exp\{(\log x)^{1/3}\}$.*

The proof of Theorem 2, which is based on the argument in [8], can also be used to show the same result for the equation $\sigma(n)=\sigma(n+1)$, where σ is the sum of the divisors function. We conjecture that for every $\varepsilon>0$ and $x\geqq x_0(\varepsilon)$ the equations $\Phi(n)=\Phi(n+1)$, $\sigma(n)=\sigma(n+1)$ each have at least $x^{1-\varepsilon}$ solutions $n\leqq x$. We cannot prove, however, that there are even infinitely many solutions for either equation.

§2. Preliminaries for Theorem 1

In this section, we record three results which will be used in the proof of Theorem 1.

LEMMA 1. *Let $p_0(n)$ denote the least odd prime factor of n if n is not a power of 2 and let $p_0(2^k)=1$. Then there are real numbers $u>1$, x_0, and $c_6>0$ such that if a, b, a_0, b_0 are nonnegative integers satisfying*

$$ab \neq 0, \quad ab_0-a_0b = 1$$

and if $x\geqq\max(x_0, a^u, b^u)$, then

$$\#\{n\leqq x: p_0((an+a_0)(bn+b_0))>x^{1/u},\ (an+a_0)(bn+b_0)\not\equiv ab \bmod 2\}>$$

$$> c_6\frac{ab}{\Phi(a)\Phi(b)}\frac{x}{\log^2 x}.$$

PROOF. This follows from the "fundamental lemma" of the combinatorial sieve (see Halberstam and Richert [6], Theorem 2.5). Note that if $2|ab$, then the sieve result implies that we may insist that $(an+a_0)(bn+b_0)$ be odd, while if $2\nmid ab$, then $(an+a_0)(bn+b_0)$ is even for all n.

LEMMA 2. *Let $\pi(x,t)$ denote the number of $n\leqq x$ with $\nu(n)=t$. Then uniformly for $x\geqq 3$ and integers t satisfying*

$$|t-\log\log x| < (\log\log x)^{2/3}, \tag{2.1}$$

we have

$$\pi(x,t)=\frac{e^{-c^2/2}}{\sqrt{2\pi}}\frac{x}{\sqrt{\log\log x}}\left(1+O\left(\frac{|c+c^3|}{\sqrt{\log\log x}}\right)+O\left((\log\log x)^{-1}\right)\right)$$

where c is defined by the equation: $t=\log\log x+c\sqrt{\log\log x}$.

PROOF. From a result of Sathe [9] and Selberg [10], for any $B>0$ we have uniformly for $x\geqq 3$ and integers t with $1\leqq t\leqq B\log\log x$,

$$\pi(x,t)=\frac{x}{\log x}\frac{(\log\log x)^{t-1}}{(t-1)!}F\left(\frac{t-1}{\log\log x}\right)\left(1+O\left(\frac{t}{(\log\log x)^2}\right)\right)$$

where

$$F(z) = \frac{1}{\Gamma(1+z)} \prod_p \left(1+\frac{z}{p-1}\right)\left(1-\frac{1}{p}\right)^z.$$

If t satisfies (2.1), then

$$F\left(\frac{t-1}{\log\log x}\right) = 1+O\left(\frac{|c|}{\sqrt{\log\log x}}\right)+O((\log\log x)^{-1}),$$

$$\frac{(\log\log x)^{t-1}}{(t-1)!} = \frac{e^{-c^2/2}}{\sqrt{2\pi}} \frac{\log x}{\sqrt{\log\log x}} \left(1+O\left(\frac{|c+c^3|}{\sqrt{\log\log x}}\right)+O((\log\log x)^{-1})\right),$$

which proves the lemma. Note that a somewhat weaker version of this lemma follows from [4].

LEMMA 3. *Let us put*

$$S(n) = \sum_{p|n} \frac{1}{p}.$$

There are numbers x_1 and Δ, $\eta>0$ such that if $x>x_1$, then there are at least ηx nitegers $n\leqq x$ with

$$S(n) \leqq \frac{1}{10} \quad \text{and} \quad |\nu(n)-\log\log x| \leqq \Delta\sqrt{\log\log x}. \tag{2.2}$$

PROOF. For any t, let $D(t)$ denote the asymptotic density of the integers n with $S(n)\leqq t$. By the Erdős—Wintner theorem (see Elliott [2], Theorem 5.1) $D(t)$ exists for every t. It is easy to see that $D(t)$ is strictly increasing on $[0, \infty)$. Thus $D(1/10)>0$.

By the Erdős—Kac theorem (see Elliott [2], Theorem 12.3) the number of $n\leqq x$ with $|\nu(n)-\log\log x|>\Delta\sqrt{\log\log x}$ is

$$\left(\frac{2}{\sqrt{2\pi}} \int_\Delta^x e^{-t^2/2}\,dt+o(1)\right)x.$$

Let $0<\eta<D(1/10)$ be arbitrary and choose Δ so that

$$\frac{2}{\sqrt{2\pi}} \int_\Delta^x e^{-t^2/2}\,dt < D\left(\frac{1}{10}\right)-\eta.$$

Then for large x, at least ηx integers $n\leqq x$ satisfy (2.2).

§3. Proof of Theorem 1

Basically, the idea of the proof of Theorem 1 is to construct many pairs n, $n+1$ from a fixed choice of integers a, b where $b|n$, $a|n+1$, $\nu(a)=\nu(b)$, and $\nu(n/b)$, $\nu((n+1)/a)$ are small. Then $\nu(n)\approx\nu(n+1)$. To show there are many such n for a given choice of a, b, we use Lemma 1. To show there are many pairs a, b, we use Lemmas 2 and 3.

Let x be large. Let $u>1$ denote the number defined in Lemma 1 and let $v=u+2$. Let $\mathscr{A}$ denote the set of integers a satisfying

$$a \leqq x^{1/v}, \quad S(a) \leqq \frac{1}{10}, \quad |\nu(a)-\log\log x| \leqq 2\Delta\sqrt{\log\log x}, \tag{3.1}$$

where S and Δ are defined in Lemma 3. Then (with η given in Lemma 3)

$$\sum_{a\in A}\frac{1}{a} > \frac{\eta}{2v}\log x. \tag{3.2}$$

Indeed, if $x^{1/4v}<y\leqq x^{1/v}$ and x is large, then by Lemma 3, the number of members of $\mathscr{A}$ that do not exceed y is at least ηy. (We use the fact that $\log\log y=\log\log x+$ $+O(1)$). Thus (3.2) follows from partial summation.

For $a\in\mathscr{A}$, let $\mathscr{B}(a)$ denote the set of integers b with

$$b \leqq x^{1/v}, \quad (a,b)=1, \quad \nu(b)=\nu(a). \tag{3.3}$$

Finally, for $a\in\mathscr{A}$ and $b\in\mathscr{B}(a)$, let $g(a,b)$ denote the number of integer solutions y_1, y_2 of

$$ay_1-by_2=1, \quad 0<y_1\leqq\frac{x}{a}, \quad p_0(y_1y_2)>x^{1/v}, \quad ab\not\equiv y_1y_2 \mod 2. \tag{3.4}$$

For such a quadruple a, b, y_1, y_2 let

$$n=by_2, \quad n+1=ay_1. \tag{3.5}$$

Then $n+1\leqq x$ and by (3.1), (3.3), and (3.4), we have $(a,y_1)=(b,y_2)=1$. Thus from (3.3), (3.4), and (3.5)

$$|\nu(n+1)-\nu(n)| = |\nu(a)+\nu(y_1)-\nu(b)-\nu(y_2)| = |\nu(y_1)-\nu(y_2)| < \nu+1.$$

We shall then take $c_2=\nu+1$ in Theorem 1.

We next note that for a given integer n, there is at most one quadruple a, b, y_1, y_2 satisfying (3.1), (3.3), (3.4), and (3.5). Thus if $f(x)$ denotes the number of $n\leqq x$ satisfying (1.4) with $c_2=\nu+1$, then

$$f(x) \geqq \sum_{a\in\mathscr{A}}\sum_{b\in\mathscr{B}(a)} g(a,b). \tag{3.6}$$

Note that Lemma 1 immediately gives a lower bound for each $g(a,b)$. Indeed, if positive integers y_1, y_2 satisfy $ay_1-by_2=1$, then y_1, y_2 are of the form

$$y_1 = bm+b_0, \quad y_2 = am+a_0,$$

where $m=0, 1, 2, \ldots,$

$$0 \leqq b_0 < b, \quad 0 \leqq a_0 < a, \quad \text{and} \quad ab_0 - ba_0 = 1.$$

Thus from Lemma 1,

$$g(a,b) > c_6 \frac{ab}{\Phi(a)\Phi(b)} \frac{x/ab}{\log^2(x/ab)} - 1 \geqq c_6 \frac{x}{ab \log^2 x}.$$

Therefore, from (3.6)

$$f(x) \geqq c_6 \frac{x}{\log^2 x} \sum_{a \in \mathscr{A}} \frac{1}{a} \sum_{b \in \mathscr{B}(a)} \frac{1}{b}. \tag{3.7}$$

We now estimate the inner sum. If $\mathscr{B}(a)(y)$ denotes the number of members of $\mathscr{B}(a)$ below y, then, for $y \leqq x^{1/\nu}$,

$$\mathscr{B}(a)(y) = \sum_{\substack{b \leqq y \\ (a,b)=1 \\ \nu(b)=\nu(a)}} 1 \geqq \sum_{\substack{b \leqq y \\ \nu(b)=\nu(a)}} 1 - \sum_{p|a} \sum_{\substack{b \leqq y \\ p|b \\ \nu(b)=\nu(a)}} 1 \geqq$$

$$\geqq \pi(y, \nu(a)) - \sum_{p|a} \left(\pi\left(\frac{y}{p}, \nu(a)\right) + \pi\left(\frac{y}{p}, \nu(a)-1\right) \right).$$

For $x^{1/4\nu} < y \leqq x^{1/\nu}$ and $p \leqq \sqrt{y}$,

$$\pi\left(\frac{y}{p}, \nu(a)\right) \sim \pi\left(\frac{y}{p}, \nu(a)-1\right) \sim \frac{1}{p} \pi(y, \nu(a))$$

uniformly, by Lemma 2 and (3.1). Therefore for large x, and $x^{1/4\nu} < y \leqq x^{1/\nu}$,

$$\mathscr{B}(a)(y) \geqq \left(1 - \sum_{p|a} \frac{3}{p}\right) \pi(y, \nu(a)) - 2 \sum_{\substack{p|a \\ p > \sqrt{y}}} \frac{y}{p} = (1 - 3S(a)) \pi(y, \nu(a)) + O(\sqrt{y}).$$

Thus from (3.1), we have

$$\mathscr{B}(a)(y) \geqq \frac{1}{2} \pi(y, \nu(a))$$

for large x and $x^{1/4\nu} < y \leqq x^{1/\nu}$. From this inequality, (3.1), and Lemma 2, we have

$$\sum_{b \in \mathscr{B}(a)} \frac{1}{b} > (4\sqrt{2\pi}\, \nu e^{2A^2})^{-1} \frac{\log x}{\sqrt{\log \log x}}$$

for large x. Combined with (3.2), and (3.7), we thus have

$$f(x) \gg \frac{x}{\sqrt{\log \log x}},$$

which was to be proved.

§4. Proof of Theorem 2

In this section, we outline the proof of Theorem 2. It is nearly the same as the proof in [8] dealing with amicable numbers.

For simplicity of notation, we put

$$l = \exp\{(\log x)^{1/3}\},\ L = \exp\left\{\frac{1}{8}(\log x)^{2/3}\log\log x\right\}.$$

Let $P(n)$ denote the largest prime factor of n. From deBruijn's estimate [1], the number of $n \leqq x$ with $P(n) < L^2$ is $o(x/l)$. Thus we may assume that

(i) $P(n) \geqq L^2$ and $P(n+1) \geqq L^2$.

It is easy to see that the number of $n \leqq x$ divisible by a non-trivial power exceeding l^3 is $o(x/l)$, so we may assume that

(ii) if k^a divides n or $n+1$ where $a \geqq 2$, then $k^a \leqq l^3$.

Now we show that we may assume that

(iii) $n/P(n) \geqq L$, $(n+1)/P(n+1) \geqq L$.

Let $n=mp$, $n+1=m'p'$ where $p=P(n)$, $p'=P(n+1)$. From (i), (ii), and (1.5) we have

$$\Phi(m)(p-1) = \Phi(m')(p'-1).$$

Using this equation and $mp+1=m'p'$, we have

$$p'(\Phi(m)m' - m\Phi(m')) = \Phi(m) - m\Phi(m') + m\Phi(m). \tag{4.1}$$

Assuming $n \geqq 4$, (1.5) implies that $m > 1$ and $m' > 1$. Thus the parenthetical expression in (4.1) is not 0, for otherwise

$$\frac{\Phi(m)}{m} = \frac{\Phi(m')}{m'},$$

contradicting $(m, m') = 1$, $m > 1$, $m' > 1$. Thus m, m' determine p' and p. So the number of $n \leqq x$ for which (1.5), (i), and (ii) hold, but (iii) fails is at most the number of pairs m, m' where either

$$m < L,\ m' \leqq (x+1)/L^2 \quad \text{or} \quad m \leqq x/L^2,\ m' < L,$$

which is $O(x/L) = o(x/l)$.

Continuing with the notation $m = n/P(n)$, $m' = (n+1)/P(n+1)$, we now show we may assume that

(iv) $P(\Phi(m)) \geqq l^4$, $P(\Phi(m')) \geqq l^4$.

Suppose $P(\Phi(m)) < l^4$. We have

$$\Phi(m) = \prod_{q^a \| m} (q-1)q^{a-1} = a_1 a_2 \dots a_t, \tag{4.2}$$

where q denotes prime and each a_i is some $q-1$ or q. Thus the function Φ not only maps m to the integer $\Phi(m)$ but also gives a factorization of $\Phi(m)$ as $a_1 a_2 \dots a_t$

(order of factors is unimportant) where at most one $a_i=1$. It is easy to see (cf. [8]) that there are never 3 distinct integers m_1, m_2, m_3 such that not only are $\Phi(m_1)=\Phi(m_2)=\Phi(m_3)$, but the factorizations $a_1a_2\dots a_t$ given by (4.2) are the same.

If $f(k)$ denotes the number of unordered factorizations of k into factors exceeding 1, then the number of unordered factorizations of k into factors where at most one factor is 1 is $2f(k)$ (for $k>1$). Let $N(z)$ denote the number of m with $1<\Phi(m)\leqq z$ and $P(\Phi(m))<l^4$. Thus

$$N(z) \leqq 4 \sum_{\substack{k\leqq z \\ P(k)<l^4}} f(k).$$

This sum is identical with the sum in the first display on p. 186 in [8]. From the argument there (see (6)), for all large x and $z\geqq L$,

$$N(z) \leqq z/l^2. \tag{4.3}$$

If $n\leqq x$ satisfies (1.5) and (i)—(iii), then

$$L \leqq m = \frac{n}{P(n)} \leqq \frac{x}{P(n)}.$$

Thus the number of such n with the first inequality in (iv) failing is by (4.3) at most

$$\sum_{p\leqq x/L} N(x/p) \leqq \sum_{p\leqq x/L} \frac{x}{pl^2} = o(x/l).$$

Similarly, the number of n which the second inequality in (iv) fails is $o(x/l)$. Thus we may assume (iv).

Finally, we may assume that

(v) $P(n) > P(n+1)$.

Indeed, the case $P(n)<P(n+1)$ can be treated in the same way so that there is no loss of generality in assuming (v).

Let $n=mp$, $n+1=m'p'$ where $p=P(n)$, $p'=P(n+1)$ and assume $n\leqq x$, n satisfies (1.5) and (i)—(v). From (iv) there is a prime $r\geqq l^4$ with $r|\Phi(m)$, so that $r|\Phi(n)=\Phi(n+1)$. Thus from (ii) there are primes q, q' with $q|m$, $q'|n+1$ with $q\equiv q'\equiv 1 \bmod r$. Thus

$$n+1 = mp+1 \equiv 0 \bmod q'. \tag{4.4}$$

Since $m|n$, $q'|n+1$, we have $(m, q')=1$, so that (4.4) puts p in a certain residue class $a(m, q') \bmod q'$. Also, $p>q'$ by (v). Thus the number of $n\leqq x$ which satisfy (1.5) and (i)—(v) is at most

$$\sum_{r\geqq l^4} \sum_{\substack{q\equiv 1(r) \\ q\leqq x}} \sum_{\substack{m\equiv 0(q) \\ m\leqq x}} \sum_{\substack{q'\equiv 1(r) \\ q'\leqq x+1}} \sum_{\substack{p\equiv a(m,q')(q') \\ q'<p\leqq x/m}} 1 \leqq$$

$$\leqq \sum_r \sum_q \sum_m \sum_{q'} \frac{x}{mq'} \ll \sum_r \sum_q \sum_m \frac{x\log x}{rm} \ll$$

$$\ll \sum_r \sum_q \frac{x\log^2 x}{rq} \ll \sum_r \frac{x\log^3 x}{r^2} \ll \frac{x\log^3 x}{l^4} = o(x/l).$$

This completes the proof of Theorem 2.

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ON THE CONTROL OF STRONGLY NONLINEAR HYPERBOLIC SYSTEMS

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The control theory of singular distributed systems was systematically studied by J. L. Lions in his book [1]. In general the nonlinear terms had polynomial growth and the author raised the problem of the control of strongly nonlinear systems i.e. when the nonlinear term is more general, e.g. of exponential type.

In [2] and [3], by developing the method used in [1] this problem was solved for some systems of elliptic and parabolic type. Using the L^p $(1<p<\infty)$ regularity results known for these equations, it was not necessary to make any assumption on the nonlinear term. For the hyperbolic equations, however, we have no L^p-regularity results except $p=2$, therefore the above method cannot be applied directly.

The purpose of this paper is to overcome this difficulty. We will show that, making some new and very general assumptions on the growth of the nonlinear term (which are satisfied, for example for the functions having polynomial or exponential growth), a singular system of optimality can be derived in this case too.

Throughout this paper let Ω be a bounded open subset of $\mathbf{R}^n$ $(n\in\mathbf{N})$ with boundary Γ of class C^∞, T a positive number and set $Q=\Omega\times]0,T[$, $\Sigma=\Gamma\times]0,T[$.

We are given a function $f\in C^1(\mathbf{R})$ such that

$$\sup_{x\in\mathbf{R}}\frac{|f'(x)|}{1+|f(x)|}<\infty, \tag{1}$$

$$\sup_{\substack{x,y\in\mathbf{R}\\ |x-y|\leqq 1}}\frac{|f(x)|}{1+|f(y)|}<\infty. \tag{2}$$

We remark that these conditions are satisfied for example if f is a polynomial or an exponential function. The assumption (1) is an improvement of another growth condition used by N. Barbu [6] and D. Tiba [7].

Let $\mathcal{U}_{ad}$ be a convex, closed subset of $L^2(Q)$, $y^0\in H_0^1(\Omega)$, $y^1\in L^2(\Omega)$ arbitrary given functions. We shall say that a pair (v,z) is admissible if

$$\begin{cases} z''-\Delta z-f(z)=v \quad \text{in} \quad \mathcal{D}'(Q),\\ z\in L^\infty(0,T;\ H_0^1(\Omega)),\quad z'\in L^\infty(0,T;\ L^2(\Omega)),\\ z(0)=y^0,\quad z'(0)=y^1,\quad v\in\mathcal{U}_{ad},\quad f(z)\in L^2(Q). \end{cases} \tag{3}$$

Let us fix $z_d\in L^2(Q)$ and a number $N>0$ arbitrarily and set

$$\mathcal{J}(v,z)=\frac{1}{2}\|f(z)-z_d\|^2_{L^2(Q)}+\frac{N}{2}\|v\|^2_{L^2(Q)}. \tag{4}$$

An admissible pair (u, y) will be called optimal if

$$\mathcal{J}(u, y) = \inf\{\mathcal{J}(v, z) | (v, z) \text{ is admissible}\}. \tag{5}$$

We shall always assume that

(6) there exists at least one admissible pair.

(This is satisfied for example if $\mathcal{U}_{ad} = L^2(Q)$ and $y^0 = y^1 = 0$). Then the following theorems will be proved:

THEOREM 1. *There exists at least one optimal pair.*

THEOREM 2. *Suppose that one of the following three assumptions is satisfied:*

(i) $n = 1$,
(ii) $\mathcal{U}_{ad} = L^2(Q)$,
(iii) f *and* f' *are bounded.*

Then to any optimal pair (u, y) *there exists a function* p *such that the following singular system of optimality holds:*

$$y'' - \Delta y - f(y) = u \quad in \quad \mathcal{D}'(Q), \tag{7}$$

$$p'' - \Delta p - f'(y)\, p = (f(y) - z_d)\, f'(y) \quad in \quad \mathcal{D}'(Q), \tag{8}$$

$$\int_Q (p + Nu)(v - u)\, dx\, dt \geqq 0 \quad \forall v \in \mathcal{U}_{ad}, \tag{9}$$

$$y \in L^\infty(0, T;\ H_0^1(\Omega)),\ y' \in L^\infty(0, T;\ L^2(\Omega)), \tag{10}$$

$$y(0) = y^0, \quad y'(0) = y^1, \tag{11}$$

$$p \in L^\infty(0, T;\ L^2(\Omega)),\ p' \in L^\infty(0, T;\ H^{-1}(\Omega)), \tag{12}$$

$$p(T) = p'(T) = 0, \quad p = 0 \quad on \quad \Sigma, \tag{13}$$

$$u \in \mathcal{U}_{ad}. \tag{14}$$

PROOF OF THEOREM 1. Let (v_k, z_k) be a sequence of admissible pairs such that $\mathcal{J}(v_k, z_k) \to \inf\{\mathcal{J}(v, z) | (v, z) \text{ is admissible}\}$; then the sequences

$$\|v_k\|_{L^2(Q)}, \quad \|f(z_k)\|_{L^2(Q)}, \quad \|z_k'' - \Delta z_k\|_{L^2(Q)}$$

are bounded by (3) and (4). Applying the usual a priori estimates for the wave equation we conclude that

$$\|z_k\|_{L^\infty(0, T;\ H_0^1(\Omega))} \quad \text{and} \quad \|z_k'\|_{L^\infty(0, T;\ L^2(\Omega))}$$

are also bounded whence by [4], Ch. 1, Theorem 5.1, z_k is precompact in $L^2(Q)$. Using also the Riesz lemma, there exists a subsequence of (v_k, z_k) such that

$$\begin{aligned}
v_k &\to u \quad \text{in} \quad L^2(Q) \quad \text{weakly},\\
z_k &\to y \quad \text{in} \quad L^\infty(0, T;\ H_0^1(\Omega)) \quad \text{weakly-star},\\
z_k' &\to y' \quad \text{in} \quad L^\infty(0, T;\ L^2(\Omega)) \quad \text{weakly-star},\\
z_k &\to y \quad \text{in} \quad Q \quad \text{almost everywhere},\\
f(z_k) &\to f(y) \quad \text{in} \quad L^2(Q) \quad \text{weakly}.
\end{aligned}$$

(In the last step we applied [4], Chapter 1, Lemma 1.3). Hence it follows that (u, y) is admissible and $\mathscr{J}(u, y) \leqq \lim \mathscr{J}(v_k, z_k)$ i.e. (u, y) is optimal. □

PROOF OF THEOREM 2. Fixing an optimal pair (u, y) arbitrarily, for each $\varepsilon > 0$ set

$$\mathscr{J}_\varepsilon(v, z) = \mathscr{J}(v, z) + \frac{1}{2\varepsilon} \|z'' - \Delta z - f(z) - v\|^2_{L^2(Q)} + \frac{1}{2} \|z - y\|^2_{L^2(Q)} + \frac{1}{2} \|v - u\|^2_{L^2(Q)}.$$

A pair (v, z) will be called ε-admissible if

$$\begin{cases} z'' - \Delta z \in L^2(Q), \\ z \in L^\infty(0, T;\ H_0^1(\Omega)), \quad z' \in L^\infty(0, T;\ L^2(\Omega)), \\ z(0) = y^0, \quad z'(0) = y^1, \quad v \in \mathscr{U}_{\text{ad}}, \quad f(z) \in L^2(Q), \end{cases} \tag{15}$$

and ε-optimal if (15) and

$$\mathscr{J}_\varepsilon(v, z) = \inf \{\mathscr{J}_\varepsilon(w, x) | (w, x) \quad \text{is } \varepsilon\text{-admissible}\} \tag{16}$$

are satisfied.

Obviously, every admissible pair is also ε-admissible. Repeating the reasoning of Theorem 1 one can easily see that for each $\varepsilon > 0$ there exists at least one ε-optimal pair.

Let us fix an ε-optimal pair $(u_\varepsilon, y_\varepsilon)$ for each $\varepsilon > 0$.

LEMMA 1. *If* $\varepsilon \to 0$ *then*

$$u_\varepsilon \to u \quad \text{in} \quad L^2(Q) \quad \textit{strongly}, \tag{17}$$

$$y_\varepsilon \to y \quad \text{in} \quad L^\infty(0, T;\ H_0^1(\Omega)) \quad \textit{strongly}, \tag{18}$$

$$y'_\varepsilon \to y' \quad \text{in} \quad L^\infty(0, T;\ L^2(\Omega)) \quad \textit{strongly}, \tag{19}$$

$$f(y_\varepsilon) \to f(y) \quad \text{in} \quad L^2(Q) \quad \textit{strongly}. \tag{20}$$

PROOF. It follows from the estimate

$$\mathscr{J}_\varepsilon(u_\varepsilon, y_\varepsilon) \leqq \mathscr{J}_\varepsilon(u, y) = \mathscr{J}(u, y) \tag{21}$$

that the sequences

$$\|u_\varepsilon\|_{L^2(Q)}, \quad \|f(y_\varepsilon)\|_{L^2(Q)}, \quad \|y_\varepsilon\|_{L^\infty(0, T; H_0^1(\Omega))}, \quad \|y'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))}$$

are bounded and that

$$\|y''_\varepsilon - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon\|_{L^2(Q)} \to 0. \tag{22}$$

Therefore any subsequence of $(u_\varepsilon, y_\varepsilon)$ has another subsequence such that

$$u_\varepsilon \to \hat{u} \quad \text{in} \quad L^2(Q) \quad \text{weakly}, \tag{23}$$

$$y_\varepsilon \to \hat{y} \quad \text{in} \quad L^\infty(0, T;\ H_0^1(\Omega)) \quad \text{weakly-star}, \tag{24}$$

$$y'_\varepsilon \to \hat{y}' \quad \text{in} \quad L^\infty(0, T;\ L^2(\Omega)) \quad \text{weakly-star}, \tag{25}$$

$$y_\varepsilon \to \hat{y} \quad \text{in} \quad L^2(Q) \quad \text{strongly}, \tag{26}$$

$$y_\varepsilon \to \hat{y} \quad \text{almost everywhere in } Q, \tag{27}$$

$$f(y_\varepsilon) \to f(\hat{y}) \quad \text{in} \quad L^2(Q) \quad \text{weakly}. \tag{28}$$

It follows from (22), (23), (24), (25) and (28) that $(\hat{u}, \hat{y})$ is admissible. Using also (21) and the optimality of (u, y) we obtain

$$\mathcal{J}(u, y) \leqq \mathcal{J}(\hat{u}, \hat{y}) \leqq \underline{\lim}\, \mathcal{J}(u_\varepsilon, y_\varepsilon) \leqq \overline{\lim}\, \mathcal{J}(u_\varepsilon, y_\varepsilon) \leqq \mathcal{J}(u, y)$$

where

$$(\hat{u}, \hat{y}) = (u, y) \tag{29}$$

and

$$\mathcal{J}(u_\varepsilon, y_\varepsilon) \to \mathcal{J}(u, y). \tag{30}$$

Now (17) and (20) follow from (23), (28), (29) and (30). Furthermore (26) and (29) yield

$$y_\varepsilon \to y \quad \text{in} \quad L^2(Q) \quad \text{strongly.} \tag{31}$$

Using again the a priori estimates for the wave equation, (18) and (19) follow from (17), (20), (22) and (31). □

Let us now set

$$p_\varepsilon = -\frac{1}{\varepsilon}(y_\varepsilon'' - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon) \quad (\in L^2(Q)). \tag{32}$$

LEMMA 2. *For all* $v \in \mathcal{U}_{ad}$ *we have*

$$\int_Q (p_\varepsilon + Nu_\varepsilon + u_\varepsilon - u)(v - u_\varepsilon)\, dx\, dt \geqq 0. \tag{33}$$

PROOF. By the ε-optimality of $(u_\varepsilon, y_\varepsilon)$ we have

$$\frac{\mathcal{J}_\varepsilon(u_\varepsilon + t(v - u_\varepsilon), y_\varepsilon) - \mathcal{J}_\varepsilon(u_\varepsilon, y_\varepsilon)}{t} \geqq 0$$

for all $0 < t < 1$. Letting $t \to 0$ (33) follows without any difficulty. □

The following observation will play an important role in the sequel.

LEMMA 3. *If* $\varepsilon \to 0$ *then*

$$f'(y_\varepsilon) \to f'(y) \quad \textit{in} \quad L^2(Q) \quad \textit{strongly.} \tag{34}$$

PROOF. It follows from (20) that the sequence $|f(y_\varepsilon)|^2$ is uniformly absolutely integrable. Using property (1) then $|f'(y_\varepsilon)|^2$ and also $|f'(y_\varepsilon) - f'(y)|^2$ is uniformly absolutely integrable. On the other hand, it follows from (18) (or from (31)) that any subsequence of y_ε has another subsequence such that $f'(y_\varepsilon) \to f'(y)$ almost everywhere in Q. We can therefore apply a theorem of Vitali [5] whence the lemma follows. □

REMARK. The application of Vitali's theorem is not a usual tool in such investigations. Nevertheless the above lemma is unecessary in the case when $f(x) = e^x$. Indeed, in this case $f' = f$ and (34) coincides with (20).

In the following lemma both assumptions (1) and (2) on the growth of f will be needed.

LEMMA 4. *We have*

$$(35)\qquad \int_Q p_\varepsilon(\xi''-\Delta\xi-f'(y_\varepsilon)\,\xi)\,dx\,dt=\int_Q (f(y_\varepsilon)-z_d)\,f'(y_\varepsilon)\,\xi+(y_\varepsilon-y)\,\xi\,dx\,dt$$

whenever $\xi\in C^2(\bar{Q})$, $\xi(0)=\xi'(0)=0$ *in* Ω *and* $\xi=0$ *on* Σ.

PROOF. It follows from (2) that $(u_\varepsilon, y_\varepsilon+s\xi)$ is ε-admissible for all $0<s<1$. Being $(u_\varepsilon, y_\varepsilon)$ ε-optimal we have therefore

$$(36)\qquad \frac{d}{ds}\mathscr{J}_\varepsilon(u_\varepsilon, y_\varepsilon+s\xi)|_{s=0}=0$$

provided that this derivative exists. To show this, write

$$\frac{\mathscr{J}_\varepsilon(u_\varepsilon, y_\varepsilon+s\xi)-\mathscr{J}_\varepsilon(u_\varepsilon, y_\varepsilon)}{s}=\int_Q\frac{|f(y_\varepsilon+s\xi)-z_d|^2-|f(y_\varepsilon)-z_d|^2}{2s}\,dx\,dt+$$

$$+\int_Q\frac{|y_\varepsilon''-\Delta y_\varepsilon-f(y_\varepsilon+s\xi)-u_\varepsilon+s\xi''-s\Delta\xi|^2-|y_\varepsilon''-\Delta y_\varepsilon-f(y_\varepsilon)-u_\varepsilon|^2}{2\varepsilon s}\,dx\,dt+$$

$$+\int_Q\frac{|y_\varepsilon-y+s\xi|^2-|y_\varepsilon-y|^2}{2s}\,dx\,dt$$

and try to pass to the limit by the use of Lebesgue's dominated convergence theorem. One can readily see that the integrands tend to

$$(f(y_\varepsilon)-z_d)\,f_\varepsilon'(y_\varepsilon)\,\xi,\quad -p_\varepsilon(\xi''-\Delta\xi-f'(y_\varepsilon)\,\xi),\quad (y_\varepsilon-y)\,\xi,$$

respectively, almost everywhere in Q. According to (36) this provides (35) if we show that the integrands are L^1-bounded in all the three cases.

In the first case, applying the Lagrange mean value theorem, for some $0<\theta<1$ we have

$$\left|\frac{|f(y_\varepsilon+s\xi)-z_d|^2-|f(y_\varepsilon)-z_d|^2}{2s}\right|=$$

$$=\left|(f(y_\varepsilon+\theta s\xi)-z_d)\,f'(y_\varepsilon+\theta s\xi)\,\xi\right|\leqq(|f(y_\varepsilon+\theta s\xi)|+|z_d|)|f'(y_\varepsilon+\theta s\xi)|\cdot|\xi|\leqq$$

$$\leqq C(1+|f(y_\varepsilon)|+|z_d|)(1+|f(y_\varepsilon)|)\in L^1(Q)$$

with some constant C depending only on $\|\xi\|_{L^\infty(Q)}$; we used (1), (2) and the boundedness of ξ.

In the second case the integrand is equal to

$$\frac{1}{\varepsilon}(y_\varepsilon''-\Delta y_\varepsilon-u_\varepsilon+\theta s\xi''-\theta s\Delta\xi-f(y_\varepsilon+\theta s\xi))(\xi''-\Delta\xi-f_\varepsilon'(y+\theta s\xi)\,\xi)$$

and its absolute value may be estimated by

$$\frac{1}{\varepsilon}(|y_\varepsilon''-\Delta y|+|u_\varepsilon|+C+C|f(y_\varepsilon)|)(C+C|f(y_\varepsilon)|)\in L^1(Q).$$

Finally, the third integrand is obviously L^1-bounded. □

REMARK. Up to this point none of the assumptions (i), (ii), (iii) was used.

LEMMA 5. $\|p_\varepsilon\|_{L^2(Q)}$ *is bounded.*

PROOF. If condition (ii) is satisfied then (33) yields $p_\varepsilon = u - u_\varepsilon - Nu_\varepsilon$, and the emma follows from (17).

In the remaining cases we can argue as in the proof of [1], Chapter 2, Theorem 2.1. Indeed, putting

$$m_\varepsilon = f'(y_\varepsilon), \quad F_\varepsilon = (f(y_\varepsilon) - z_d) f'(y_\varepsilon) + (y_\varepsilon - y),$$

in the case (i) m_ε is bounded in $L^2(Q)$, F_ε is bounded in $L^1(Q)$, in the case (ii), m_ε is bounded in $L^\infty(Q)$, F_ε is bounded in $L^2(Q)$. Applying Theorem 2.2 of [1], Ch. 2, in both cases we obtain that p_ε is bounded in $L^\infty(0, T; L^2(\Omega))$, p'_ε is bounded in $L^\infty(0, T; H^{-1}(\Omega))$ whence the lemma follows. □

Now we can finish the proof of Theorem 2. Indeed, passing to a subsequence we may assume by Lemma 5 that

$$p_\varepsilon \to p \quad \text{in} \quad L^2(Q) \quad \text{weakly.} \tag{37}$$

Using (17), (20), (34) and (37), by letting $\varepsilon \to 0$ from (33), (35) we obtain (9) and

$$\int_Q p(\xi'' - \Delta\xi - f'(y)\,\xi)\,dx\,dt = \int_Q (f(y) - z_d) f'(y)\,\xi\,dx\,dt \tag{38}$$

whenever $\xi \in C^2(\bar{Q})$, $\xi(0) = \xi'(0) = 0$, $\xi = 0$ on Σ, which is equivalent to (8), (12) and (13). Finally, (7), (10), (11) and (14) follow from the admissibility of the pair (u, y). □

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ON THE EQUATION $F(x, y)=z^m$ OVER FUNCTION FIELDS

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1. Introduction

Let $\mathbf{C}$ be the complex number field and let $R=\mathbf{C}[t]$ be the polynomial ring over $\mathbf{C}$. Let $P(X, Y)$ be a form of degree d with coefficients in R and without multiple factors, and let $c\neq 0$ be in R. In [12] W. M. Schmidt proved that all the solutions of the equation

$$F(x, y) = cz^m \tag{1}$$

in comprime polynomials $x, y, z\in R$ have bounded degrees, provided $d\geqq 3$, $m\geqq 4$ or $d\geqq 5$, $m\geqq 3$ or $d\geqq 17$, $m=2$. More precisely, he proved

$$\max(\deg x, \deg y, \deg z) \leqq \gamma(d, m)\,\mathrm{Deg}\,(c, F),$$

where $\mathrm{Deg}\,(c, F)$ is the maximum of the degrees of c and of the coefficients of F. The constant γ is computable. In particular, if $d=m$, then we have $F(\eta, \xi)=c$, where $\eta=x/z$ and $\xi=y/z$ are elements of the function field $\mathbf{C}(t)$. So (1) can be considered as a generalization of the Thue equation.

In this paper we improve Schmidt's theorem and extend it to the case of any algebraic function field of one variable over an algebraically closed ground field of characteristic 0. Moreover, we suppose that in (1) m is also unknown.

In order to be able to state our result, we shall introduce some concepts and notations. Let k be an algebraically closed field of characteristic zero, and let $k(t)$ denote the rational function field over k. Let $\mathbf{K}$ be an arbitrary finite extension of $k(t)$, and let S be a finite set of additive valuations of $\mathbf{K}/k$ containing all the infinite valuations.[1] Define $O_{\mathbf{K},S}$ as the set of $\alpha\in\mathbf{K}$ having $v(\alpha)\geqq 0$ for every additive valuation $v\notin S$ of $\mathbf{K}/k$. They form a ring.[2] The (additive) height of $x\in\mathbf{K}$ is defined by

$$H_{\mathbf{K}}(x) = -\sum_{v} \min(0, v(x)),$$

where v runs through the valuations of $\mathbf{K}/k$ with value group $\mathbf{Z}$, the rational integers (cf. [8] and [11]). In the special case when $\mathbf{K}=k(t)$ and $x\in k[t]$, then $H_{\mathbf{K}}(x)=\deg x$. Given a vector $\mathbf{x}=(x_1, \ldots, x_n)\in\mathbf{K}^n$, write

$$\mathbf{v}(\mathbf{x}) = \min(0, v(x_1), \ldots, v(x_n)).$$

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[1] The valuation $v_\infty=-\deg$ of $k(t)/k$ with $v_\infty(t)=-1$ will be called the infinite valuation. Extensions of v_∞ to $\mathbf{K}$ will also be called "infinite valuations". The other valuations of $\mathbf{K}/k$ which are extensions of the valuations of $k(t)/k$ corresponding to the elements of k will be called "finite".

[2] The elements of $O_{\mathbf{K},S}$ are called the S-integers of $\mathbf{K}$.

We define the height of $\mathbf{x}$ by

$$H_{\mathbf{K}}(\mathbf{x}) = -\sum_{v} \mathbf{v}(\mathbf{x}) \quad \text{(cf. [11])}.$$

If $f(x)$ is a polynomial with coefficients in $\mathbf{K}$, let $\mathbf{v}(f)$ be defined in terms of the vector whose components are the coefficients of $f(x)$. We define the height $H_{\mathbf{K}}(f)$ of a polynomial in the obvious way. We say that a divisor $\mathfrak{X}$ of $\mathbf{K}/k$ is integral if $v_{\mathfrak{P}}(\mathfrak{X}) \geqq 0$ for every prime divisor $\mathfrak{P}$ of $\mathbf{K}/k$.[3] The greatest common divisor $[\mathfrak{X}_1, \mathfrak{X}_2]$ of the integral divisors $\mathfrak{X}_1, \mathfrak{X}_2$ is the integral divisor

$$[\mathfrak{x}_1, \mathfrak{x}_2] = \prod_{\mathfrak{P}} \mathfrak{P}^{\min(v_{\mathfrak{P}}(\mathfrak{x}_1),\, v_{\mathfrak{P}}(\mathfrak{x}_2))}$$

where the product is taken over all prime divisors $\mathfrak{P}$ of $\mathbf{K}$. Two integral divisors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are said to be relatively prime if $[\mathfrak{M}_1, \mathfrak{M}_2] = \mathfrak{n}$, where $\mathfrak{n}$ is the unit divisor. We define the numerator $\mathfrak{z}(\omega)$ of $0 \neq \omega \in \mathbf{K}$ to be the divisor

$$\prod_{v_{\mathfrak{P}}(\omega) > 0} \mathfrak{P}^{v_{\mathfrak{P}}(\omega)}$$

(the product being taken over all $\mathfrak{P}$ for which $v_{\mathfrak{p}}(\omega) > 0$). We say that $0 \neq \omega_1 \in O_{\mathbf{K},S}$ and $0 \neq \omega_2 \in O_{\mathbf{K},S}$ are relatively prime if $[\mathfrak{z}(\omega_1), \mathfrak{z}(\omega_2)] = \mathfrak{n}$. In particular, if $\mathbf{K} = k(t)$ and $O_{\mathbf{K},S} = k[t]$, then this means that $\mathfrak{z}(\omega_1) = \omega_1$ and $\mathfrak{z}(\omega_2) = \omega_2$ are relatively prime polynomials.

Next, consider the equation

$$F(x, y) = cz^m \tag{2}$$

where $F(X, Y)$ is a binary form with coefficients in $\mathbf{K}$ and $0 \neq c \in \mathbf{K}$. Let $\mathbf{L}$ be the splitting field of $f(X) = c^{-1}F(X, 1)$ over $\mathbf{K}$. $F(X, Y)$ factorizes into linear factors over $\mathbf{L}$. Further let δ be the degree of $\mathbf{L}$ over $\mathbf{K}$, let $g(\mathbf{L})$ denote the genus of $\mathbf{L}/k$, and let $|S|$ be the cardinality of the set S.

THEOREM. *Suppose that the form $F(X, Y)$ has at least three distinct simple linear factors*[4] *over $\mathbf{L}$. Then all solutions x, y, z, m of the equation* (2) *with $z \in O_{\mathbf{K},S}$, $m > 3$ and relatively prime $x, y \in O_{\mathbf{K},S}$ satisfy*

$$\max\{H_{\mathbf{K}}(x), H_{\mathbf{K}}(y), H_{\mathbf{K}}(t)\} < 1000\, M^5,$$

and if $z \notin k$ then $m < M + N$, where $M = 2N^3 H_{\mathbf{K}}(f) + 2N(\delta|s| + g(\mathbf{L}))$, $N = \deg F$.

If in (2) $0 \neq z \in k$, then we get the Thue equation $F^*(x, y) = c$, where $F^* = z^{-m}F$ is a binary form with coefficients in $\mathbf{K}$ and $H_{\mathbf{K}}(F^*) = H_{\mathbf{K}}(F)$. In this case W. M. Schmidt [11] and R. C. Mason [6] gave explicit bounds for $H_{\mathbf{K}}(x)$ and $H_{\mathbf{K}}(y)$ which depend only on $H_{\mathbf{K}}(F)$, $g(\mathbf{K})$ and $|S|$. These results of Schmidt and Mason were extended by K. Győry [4] to decomposable form equations in an arbitrary number of variables. Further, R. C. Mason [9] proved an effective version of an analogue over function fields of Schmidt's [10] well-known ineffective finiteness criterion concerning

[3] Write $v_{\mathfrak{P}}(\mathfrak{X}) = a$ if $\mathfrak{p}$ occurs with exponent a in the representation of $\mathfrak{X}$ as a product of prime divisors.

[4] A linear factor $\alpha X + \beta Y$ of $F(X, Y)$ is called simple if its square does not divide $F(X, Y)$.

norm form equations over $\mathbf{Z}$. We note that Schmidt obtained effective bounds even for the solutions in $\mathbf{K}$ of the Thue equation.

If in particular $\mathbf{K}=k(t)$ and $O_{\mathbf{K},S}=k[t]$ we have the following corollary.

COROLLARY. *Suppose that the form $F(X, Y)$ has at least three distinct simple linear factors over $\mathbf{L}$. Then all solutions x, y, z, m of the equation (2) with $z\in k[t]$, $m>3$ and relatively prime polynomials $x, y\in k[t]$ satisfy*

$$\max\{\deg x, \deg y, \deg z\} \leqq 1000\, M_1^5,$$

and if $z\notin k$ then $z<M_1+N$, where $M_1=2N^3H_{\mathbf{K}}(f)+2N(\delta+g(\mathbf{L}))$.

2. Auxiliary results

Let k, $\mathbf{K}$, $\mathbf{L}$, δ be as above. For the following height estimates we refer to W. M. Schmidt [11]. We have

$$\max\{H_{\mathbf{K}}(xy), H_{\mathbf{K}}(x+y)\} \leqq H_{\mathbf{K}}(x)+H_{\mathbf{K}}(y), \quad x, y\in\mathbf{K},$$

$$H_{\mathbf{K}}(x^n) = |n|\,H_{\mathbf{K}}(x), \quad n\in\mathbf{Z}, \quad x\in\mathbf{K},$$

$$H_L(\mathbf{x}) = \delta H_{\mathbf{K}}(\mathbf{x});\ H_{\mathbf{K}}(x_i) \leqq H_{\mathbf{K}}(\mathbf{x}) \leqq \sum_{i=1}^{n} H_{\mathbf{K}}(x_i), \quad \mathbf{x}=(x_1, \ldots, x_n)\in\mathbf{K}^n.$$

Write $h(X)=h_0\prod_{i=1}^{d}(X-\beta_i)$, where $0\neq h_0\in\mathbf{K}$, $\beta_1, \ldots, \beta_d\in\mathbf{K}$ and $\beta_i\neq\beta_j$ for $i\neq j$. Then $H_{\mathbf{K}}(\beta_i)\leqq H_{\mathbf{K}}(h)$, $i=1, \ldots, d$.

LEMMA 1 (R. C. Mason [8]). *Suppose that γ_1, γ_2 and γ_3 are non-zero elements of $\mathbf{K}$ with $\gamma_1+\gamma_2+\gamma_3=0$, and such that $v(\gamma_1)=v(\gamma_2)=v(\gamma_3)$ for every valuation v not in a given finite set S. Then either γ_2/γ_3 lies in k or*

$$H_{\mathbf{K}}(\gamma_2/\gamma_3) \leqq 2g(\mathbf{K})-2+|s|.$$

LEMMA 2. *Let $n\geqq 1$ be a rational integer, and let $0\neq\tau$ be in $\mathbf{K}$. Then $\mathbf{K}_1=\mathbf{K}(\sqrt[n]{\tau})$ has genus*

$$g(\mathbf{K}_1) \leqq [\mathbf{K}_1\colon \mathbf{K}](g(\mathbf{K})+H_{\mathbf{K}}(\tau)).$$

PROOF. In the case $n=2$, W. M. Schmidt [11] showed that

$$g(\mathbf{K}_1) \leqq 2g(\mathbf{K})+H_{\mathbf{K}}(\tau)-1.$$

We shall follow his proof. We have (see R. C. Mason [8])

$$g(\mathbf{K}_1) = \frac{1}{2}\sum_V (e_V-1)+[\mathbf{K}_1\colon \mathbf{K}](g(\mathbf{K})-1)+1, \tag{3}$$

where V runs through the valuations of $\mathbf{K}_1/k$ and e_V is the ramification index of V over $\mathbf{K}$. Suppose that the principal divisor (τ) in $\mathbf{K}$ is of the form

$$(\tau) = \mathfrak{P}_1^{a_1}\ldots\mathfrak{P}_r^{a_r}/q_1^{b_1}\ldots q_s^{b_s}$$

with distinct prime divisors $\mathfrak{P}_1, \ldots, \mathfrak{P}_r, q_1, \ldots, q_s$ and positive integers $a_1, \ldots, a_r$, $b_1, \ldots, b_s$. If a prime divisor $\mathfrak{P}$ will ramify in the extension field $\mathbf{K}_1$, then $\mathfrak{P}$ is equal to some $\mathfrak{P}_i$ or q_j. Write $V|v$ if the valuation V of $\mathbf{K}_1/k$ is an extension of the valuation v of $\mathbf{K}/k$. It is well-known that

$$\sum_{V|v} e_V = [\mathbf{K}_1 : \mathbf{K}] \tag{4}$$

so that by (4)

$$\frac{1}{2}\sum_{V}(e_V - 1) \leqq \frac{1}{2}(r+s)([\mathbf{K}_1 : \mathbf{K}]-1) \leqq \frac{1}{2}\Big(\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j\Big)([\mathbf{K}_1 : \mathbf{K}]-1) =$$

$$= \frac{1}{2}\{H_{\mathbf{K}}(\tau) + H_{\mathbf{K}}(\tau^{-1})\}([\mathbf{K}_1 : \mathbf{K}]-1) \leqq [\mathbf{K}_1 : \mathbf{K}] H_{\mathbf{K}}(\tau).$$

By (3) we have

$$g(\mathbf{K}_1) \leqq [\mathbf{K}_1 : \mathbf{K}] H_{\mathbf{K}}(\tau) + [\mathbf{K}_1 : \mathbf{K}](g(\mathbf{K})-1)+1.$$

LEMMA 3 (R. C. Mason [8]). *Let* $P(X) = a\prod_{i=1}^{n}(X-\alpha_i)$ *be a polynomial in* X *of degree* n *and suppose that* $0 \neq a \in \mathbf{K}$, $\alpha_1, \ldots, \alpha_n \in \mathbf{K}$, $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Then all solutions in* $\mathbf{K}$ *of the equation* $P(x) = y^m$ *satisfy* $H_{\mathbf{K}}(x) \leqq 35 H_{\mathbf{K}}(P) + 8g(\mathbf{K})$ *provided* $n=2$, $m \geqq 5$ *or* $n \geqq 3$, $m \geqq 4$.

3. Proof of the Theorem

We may assume without loss of generality that $F(X, Y)$ has the representation

$$F(x, y) = ac^{-1}(x-\alpha_1 y)^{r_1} \ldots (x-\alpha_n y)^{r_n} \tag{5}$$

with $a \neq 0$, $n \geqq 3$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Indeed if $F(1, 0) = 0$ then we choose $\beta \in k$ with $F(1, \beta) \neq 0$ and put $X' = X$, $Y' = Y - \beta X$ and

$$G(X', Y') = F(X', Y' + \beta X') = F(X, Y).$$

It is easy to see that $G(X', Y')$ can be written in the form (5) and we consider the equation $C^{-1}G(x', y') = z^m$ instead of (2). Moreover

$$H_{\mathbf{K}}(x) = H_{\mathbf{K}}(x'), \; H_{\mathbf{K}}(y) \leqq H_{\mathbf{K}}(y') + H_{\mathbf{K}}(x') \quad \text{and} \quad H_{\mathbf{K}}(F) = H_{\mathbf{K}}(G).$$

Put $\mathbf{L} = \mathbf{K}(\alpha_1, \ldots, \alpha_n)$. Let S_1 be the set of all extensions of elements of S to $\mathbf{L}$ (we renormalize all $v \in S_1$ to have value group $\mathbf{Z}$). For $0 \neq \xi \in \mathbf{L}$, $h_1(\xi)$ denote the set of all valuations v of $\mathbf{L}/k$ for which $v(\xi) < 0$; and $h_2(\xi)$ denote the set of all valuations v of $\mathbf{L}/k$ for which $v(\xi) > 0$. It is clear that

$$\max(|h_1(\xi)|, |h_2(\xi)|) \leqq H_{\mathbf{L}}(\xi).$$

Put $\mathcal{H}(\xi) = h_1(\xi) \cup h_2(\xi)$ for $0 \neq \xi \in \mathbf{L}$, and $S_2 = S_1 \cup h_1(\alpha_1) \cup \ldots \cup h_1(\alpha_n)$. Define $O_{\mathbf{L}, S_1}$ (resp. $O_{\mathbf{L}, S_2}$) as the set of $\alpha \in \mathbf{L}$ having $v(\alpha) \geqq 0$ for every valuation $v \notin S_1$ (resp. $v \notin S_2$) of $\mathbf{L}$. Then $O_{\mathbf{K}, S} \subseteq O_{\mathbf{L}, S_1} \subseteq O_{\mathbf{L}, S_2}$ and

$$|S_2| \leqq \delta |S| + nH_{\mathbf{L}}(f) \quad (\delta = [\mathbf{L} : \mathbf{K}]).$$

Let $x, y, z \in O_{\mathbf{K},S}$ and $m \in \mathbf{N}$ be an arbitrary but fixed solution of (2) with $m>3$ and relatively prime x, y. Put

$$\Omega = h_1(a) \cup h_2(z) \cup S_2 \cup \Big(\bigcup_{1 \leqq i < j \leqq n} \mathscr{H}(\alpha_i - \alpha_j).$$

Then we have

$$|\Omega| \leqq H_{\mathbf{K}}(z) + \delta |s| + n^2 H_{\mathbf{K}}(f).$$

For brevity, let us set $\beta_i = x - \alpha_i y$, $i=1, \ldots, n$. For any three suffixes i, j, k, we have

$$\beta_i(\alpha_j - \alpha_k) + \beta_j(\alpha_k - \alpha_i) + \beta_k(\alpha_i - \alpha_j) = 0. \tag{6}$$

If v is a valuation of $\mathbf{L}/k$ with $v \notin \Omega$, then $v \notin S_2$ and

$$v(\beta_i) \geqq \min\big(v(x), v(y) + v(\alpha_i)\big) \geqq 0, \quad i = 1, \ldots, n.$$

Further $v(a) \geqq 0$. On the other hand, $v(z) \leqq 0$. Moreover, from (2) and (5) we obtain

$$v(a) + \sum_{i=1}^{n} r_i v(\beta_i) = m v(z).$$

In this way we get $v(\beta_1) = \ldots = v(\beta_n) = 0$ and

$$v\big(\beta_i(\alpha_j - \alpha_k)\big) = v\big(\beta_j(\alpha_k - \alpha_i)\big) = v\big(\beta_k(\alpha_i - \alpha_j)\big) = 0.$$

Applying Lemma 1 to (6), we have

$$H_{\mathbf{L}}(\beta_i/\beta_j) \leqq H_{\mathbf{L}}\left(\frac{\beta_i(\alpha_j - \alpha_k)}{\beta_j(\alpha_k - \alpha_i)}\right) + H_{\mathbf{L}}\left(\frac{\alpha_k - \alpha_i}{\alpha_j - \alpha_k}\right) \leqq 2g(\mathbf{L}) + |\Omega| + 4H_{\mathbf{L}}(f). \tag{7}$$

We define integral divisors $\mathfrak{X}_i$, $\mathfrak{M}_i$ $(i=1, \ldots, n)$ such that

$$(\beta_i) = \mathfrak{X}_i / \mathfrak{M}_i$$

with relatively prime $\mathfrak{X}_i$, $\mathfrak{M}_i$ (using brackets to indicate principal divisors in $\mathbf{L}$). We suppose that $v_{\mathfrak{P}}(\mathfrak{X}_i) > 0$, $v_{\mathfrak{P}}(\mathfrak{X}_j) > 0$ for some prime divisor $\mathfrak{P}$ and $i \neq j$. If $\alpha_i \alpha_j \neq 0$, then we get

$$1 \leqq \min\big(v_{\mathfrak{P}}(\mathfrak{X}_i), v_{\mathfrak{P}}(\mathfrak{X}_j)\big) = \min\big(v_{\mathfrak{P}}(x - \alpha_i y), v_{\mathfrak{P}}(x - \alpha_j y)\big) \leqq \tag{8}$$

$$\leqq v_{\mathfrak{P}}\big(y(\alpha_i - \alpha_j)\big) = v_{\mathfrak{P}}(y) + v_{\mathfrak{P}}(\alpha_i - \alpha_j)$$

and

$$1 \leqq \min\big(v_{\mathfrak{P}}(\mathfrak{X}_i), v_{\mathfrak{P}}(\mathfrak{X}_j)\big) = \tag{9}$$

$$= \min\big(v_{\mathfrak{P}}(\alpha_j(x - \alpha_i y)) + v_{\mathfrak{P}}(\alpha_j^{-1}); \; v_{\mathfrak{P}}(\alpha_i(x - \alpha_j y)) + v_{\mathfrak{P}}(\alpha_i^{-1})\big) \leqq$$

$$\leqq v_{\mathfrak{P}}\big(x(\alpha_i - \alpha_j)\big) + \max\big(v_{\mathfrak{P}}(\alpha_j^{-1}), v_{\mathfrak{P}}(\alpha_i^{-1})\big) =$$

$$= v_{\mathfrak{P}}(x) + v_{\mathfrak{P}}(\alpha_i - \alpha_j) + \max\big(v_{\mathfrak{P}}(\alpha_j^{-1}), v_{\mathfrak{P}}(\alpha_i^{-1})\big).$$

Let w denote the valuation of $\mathbf{K}/k$ for which $v_{\mathfrak{P}} | w$ (i.e. $v_{\mathfrak{P}}$ is an extension of the valuation w of $\mathbf{K}/k$). Then $v_{\mathfrak{P}}(\omega) = e_{v_{\mathfrak{P}}} w(\omega)$ for any $\omega \in \mathbf{K}$, where $e_{v_{\mathfrak{P}}}$ is the ramification index of the $v_{\mathfrak{P}}$ over w. Since x and y are relatively prime S-integers in $\mathbf{K}$, therefore if $w(x) > 0$ then $w(y) \leqq 0$. This means that $v_{\mathfrak{P}}(x) > 0$ and $v_{\mathfrak{P}}(y) > 0$ cannot hold

simultaneously. By (8) and (9) we get

$$1 \leqq \min\left(v_{\mathfrak{p}}(\mathfrak{X}_i), v_{\mathfrak{p}}(\mathfrak{X}_j)\right) \leqq \tag{10}$$

$$\leqq \max\left\{v_{\mathfrak{p}}(\alpha_i-\alpha_j);\ v_{\mathfrak{p}}(\alpha_i-\alpha_j)+\max\left(v_{\mathfrak{p}}(\alpha_i^{-1}), v_{\mathfrak{p}}(\alpha_j^{-1})\right)\right\}.$$

If $\alpha_i=0$, then also

$$1 \leqq \min\left(v_{\mathfrak{p}}(\mathfrak{X}_i), v_{\mathfrak{p}}(\mathfrak{X}_j)\right) = \min\left(v_{\mathfrak{p}}(x), v_{\mathfrak{p}}(x-\alpha_j y)\right) \leqq \tag{11}$$

$$\leqq v_{\mathfrak{p}}(\alpha_i y) = v_{\mathfrak{p}}(\alpha_j)+v_{\mathfrak{p}}(y) \leqq v_{\mathfrak{p}}(\alpha_j),$$

since $v_{\mathfrak{p}}(x)>0$ and so $v_{\mathfrak{p}}(y)\leqq 0$.

Put $\mathfrak{X}_{ij}=[\mathfrak{X}_i, \mathfrak{X}_j]$. Then (10) and (11) imply that

$$\sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_{ij}) = \sum_{\mathfrak{p}} \min\left(v_{\mathfrak{p}}(\mathfrak{X}_i), v_{\mathfrak{p}}(\mathfrak{X}_j)\right) \leqq \tag{12}$$

$$\leqq \sum_{\mathfrak{p}} \max\left(0, v_{\mathfrak{p}}(\alpha_i-\alpha_j)\right)+\sum_{\mathfrak{p}}\left(\max\left(0, v_{\mathfrak{p}}(\alpha_i^{-1})\right)+\max\left(0, v_{\mathfrak{p}}(\alpha_j^{-1})\right)\right) \leqq 4H_{\mathbf{L}}(f).$$

Let us return to the inequality (7). Write $\mathfrak{M}_{ij}=[\mathfrak{M}_i, \mathfrak{M}_j]$, $\mathfrak{X}_i=\mathfrak{X}_i'\mathfrak{X}_{ij}$, $\mathfrak{X}_j=\mathfrak{X}_j'\mathfrak{X}_{ij}$, $\mathfrak{M}_i=\mathfrak{M}_i'\mathfrak{M}_{ij}$ and $\mathfrak{M}_j=\mathfrak{M}_j'\mathfrak{M}_{ij}$, where $\mathfrak{X}_i'$, $\mathfrak{X}_j'$, $\mathfrak{M}_i'$ and $\mathfrak{M}_j'$ are integral divisors. Then

$$(\beta_i\beta_j^{-1}) = \mathfrak{X}_i'\mathfrak{M}_j'/\mathfrak{X}_j'\mathfrak{M}_i',$$

and $\mathfrak{X}_i'\mathfrak{M}_j'$, $\mathfrak{X}_j'\mathfrak{M}_i'$ are relatively prime. By definition of the $H_{\mathbf{L}}(\alpha)$ we have

$$H_{\mathbf{L}}(\beta_i/\beta_j) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_i'\mathfrak{M}_j') \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_j'\mathfrak{M}_i'). \tag{13}$$

By applying (7), (12) and (13) we get

$$H_{\mathbf{L}}(\beta_i) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_i) \sum_{\mathfrak{p}}\left(v_{\mathfrak{p}}(\mathfrak{X}_{ij})+v_{\mathfrak{p}}(\mathfrak{X}_i')\right) \leqq \tag{14}$$

$$\leqq \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_{ij})+\sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_i')+\sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{M}_j') =$$

$$= \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_{ij})+\sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_i'\mathfrak{M}_j') = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\mathfrak{X}_{ij})+H_{\mathbf{L}}(\beta_i/\beta_j) \leqq$$

$$\leqq (n^2+10)H_{\mathbf{L}}(f)+2g(\mathbf{L})+\delta|s|+H_{\mathbf{L}}(z),$$

and similarly for $H_{\mathbf{L}}(\beta_j)$. From (2) we now have

$$mH_{\mathbf{L}}(z) \leqq H_{\mathbf{L}}(a)+\sum_{k=1}^{n} r_k H_{\mathbf{L}}(\beta_k) \leqq 2N^3H_{\mathbf{L}}(f)+2Ng(\mathbf{L})+N\delta|s|+NH_{\mathbf{L}}(z), \tag{15}$$

where $N=r_1+\ldots+r_n=\deg F$. If $z\in O_{\mathbf{K},S}\setminus k$, then for some v we have $v(z)\neq 0$, and $H_{\mathbf{L}}(z)\geqq 1$. Therefore from (15) we get $m\leqq N$ or

$$m-n \leqq (m-N)H_{\mathbf{L}}(z) \leqq M,$$

where $M=2N^3H_{\mathbf{L}}(f)+2N(\delta|S|+g(\mathbf{L}))$. Consequently, in the case $z\in O_{\mathbf{K},S}\setminus k$ we have

$$m \leqq M+N. \tag{16}$$

We recall that $(\beta_i)=\frac{\mathfrak{X}_i}{\mathfrak{M}_i}$ with relatively prime $\mathfrak{X}_i, \mathfrak{M}_i$ for which $v(\mathfrak{M}_i)=0$ holds for all $v\notin S_2$. Similarly, let

$$(a) = \mathfrak{U}_1/\mathfrak{U}_2, \quad (z) = \mathfrak{M}_1/\mathfrak{M}_2, \quad (\alpha_i) = \mathfrak{B}_i^{(1)}/\mathfrak{B}_i^{(2)} \quad (i = 1, \ldots, n)$$

with integral divisors $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{B}_i^{(1)}$ and $\mathfrak{B}_i^{(2)}$ such that

$$[\mathfrak{U}_1, \mathfrak{U}_2] = [\mathfrak{W}_1, \mathfrak{W}_2] = [\mathfrak{B}_i^{(1)}, \mathfrak{B}_i^{(2)}] = \mathfrak{n},$$

where $\mathfrak{n}$ is the unit divisor; moreover $v(\mathfrak{U}_2)=v(\mathfrak{B}_i^{(2)})=v(\mathfrak{W}_2)=0$ for all $v\notin S_2\cup\mathscr{H}(a)$. First suppose $z\in O_{\mathbf{K},S}\setminus k$. Then (2) and (7) give

$$\mathfrak{U}_1\mathfrak{X}_1^{r_1}\ldots\mathfrak{X}_n^{r_n} = \mathfrak{X}\mathfrak{W}^m, \tag{17}$$

where $\mathfrak{X}, \mathfrak{W}$ are integral divisors such that $v(\mathfrak{X})=0$ for all $v\notin S_2\cup\mathscr{H}(a)=S_3$, and $v(\mathfrak{X})\leqq m-1$ for all $v\notin S_3$.

Let $\mathfrak{P}$ be a prime divisor and suppose that $\mathfrak{P}|\mathfrak{X}_i$ and $\mathfrak{P}|\mathfrak{X}_j$ for some i, j with $i\neq j$. By (10) and (11) we have

$$\min\{v_{\mathfrak{p}}(\mathfrak{X}_i), v_{\mathfrak{p}}(\mathfrak{X}_j)\} \leqq 3H_{\mathbf{L}}(f). \tag{18}$$

We can assume without loss of generality that $r_1=r_2=r_3=1$. Write

$$(\alpha_i-\alpha_j) = \mathfrak{D}_{ij}^{(1)}/\mathfrak{D}_{ij}^{(2)} \quad (i\neq j),$$

where $\mathfrak{D}_{ij}^{(1)}, \mathfrak{D}_{ij}^{(2)}$ are relatively prime integral divisors and $v(\mathfrak{D}_{ij}^{(2)})=0$ for every $v\notin S_2$. From (18) we get

$$[\mathfrak{X}_i, \mathfrak{X}_j] \,|\, \mathfrak{B}_i^{(1)}\mathfrak{B}_j^{(1)}\mathfrak{D}_{ij}^{(1)}$$

so that for the g.c.d. of any two factors $\mathfrak{X}_i^{r_i}, \mathfrak{X}_j^{r_j}$ on the right hand side of (17)

$$[\mathfrak{X}_i^{r_i}, \mathfrak{X}_j^{r_j}] \,\Big|\, \Big(\prod_{i=1}^{n}\mathfrak{B}_i^{(1)}\Big)^{2N}\prod_{1\leqq i<j\leqq n}(\mathfrak{D}_{ij}^{(1)})^{2N} = \mathfrak{D}.$$

By the construction of $\mathfrak{B}_i^{(1)}$ and $\mathfrak{D}_{ij}^{(1)}$

$$\sum_v v(\mathfrak{D}) = 2N\sum_v\sum_{i=1}^{n} v(\mathfrak{B}_i^{(1)}) + 2N\sum_v\sum_{1\leqq i<j\leqq n} v(\mathfrak{D}_{ij}^{(1)}) \leqq \tag{19}$$

$$\leqq 2N\sum_{i=1}^{n} H_{\mathbf{L}}(\alpha_i) + 2N\sum_{1\leqq i<j\leqq n} H_{\mathbf{L}}(\alpha_i-\alpha_j) \leqq 2Nn^2H_{\mathbf{L}}(f).$$

Hence we can write $\mathfrak{X}_i=\mathfrak{X}_{i1}^m\mathfrak{X}_{i2}$ $(i=1, 2, 3)$, where $\mathfrak{X}_{i1}$ and $\mathfrak{X}_{i2}$ are integral divisors and $\mathfrak{X}_{i2}|(\mathfrak{U}_1\mathfrak{X}\mathfrak{D})^m$. Similarly, write $\mathfrak{M}_i=\mathfrak{M}_{i1}^m\mathfrak{M}_{i2}$ $(i=1, 2, 3)$, where $0\leqq v_{\mathfrak{p}}(\mathfrak{M}_{i2})\leqq$ $\leqq m-1$ for every prime divisor $\mathfrak{P}$. Then

$$(\beta_i) = (x-y\alpha_i) = \mathfrak{X}_{i3}^m\mathfrak{X}_{i4}, \quad i = 1, 2, 3,$$

where $\mathfrak{X}_{i3}=\mathfrak{X}_{i1}/\mathfrak{M}_{i1}$ and $\mathfrak{X}_{i4}=\mathfrak{X}_{i2}/\mathfrak{M}_{i2}$. Write

$$a_i = \sum_v v(\mathfrak{X}_{i3}), \quad i = 1, 2, 3.$$

By Riemann's theorem (see [11]) there is a non-zero $y_i \in \mathbf{L}$ with

$$v(y_i) \geqq v(\mathfrak{X}_{i3}) \quad \text{for} \quad v \neq v_0,$$

$$v(y_i) \geqq v_0(\mathfrak{X}_{i3}) - a_i - g(\mathbf{L}),$$

where v_0 is an arbitrary valuation picked in advance. We shall follow an argument of W. M. Schmidt [11]. Define z_i by the equation

$$x - \alpha_i y = y_i^m z_i, \quad i = 1, 2, 3.$$

We have

(20)

$$H_{\mathbf{L}}(z_i) = H_{\mathbf{L}}(1/z_i) = -\sum_v \min\left(0, v(y_i^m/\beta_i)\right) = -\sum_v \min\left(0, mv((y_i)/\mathfrak{X}_{i3}) - v(\mathfrak{X}_{i4})\right) \leqq$$

$$\leqq \sum_v v(\mathfrak{X}_{i4}) - m \cdot \min\left(0, -a_i - g(\mathbf{L})\right) \leqq \sum_v v(\mathfrak{X}_{i4}) + mg(\mathbf{L}) + m \cdot \max\left(0, a_i\right) =$$

$$= mg(\mathbf{L}) + \max\left(\sum_v v(\mathfrak{X}_{i4}), \sum_v v(\mathfrak{X}_{i3}^m \mathfrak{X}_{i4})\right) = mg(\mathbf{L}) + \max\left(\sum_v v(\mathfrak{X}_{i4}), \sum_v v(\beta_i)\right) =$$

$$= mg(\mathbf{L}) + \max\left(\sum_v v(\mathfrak{X}_{i4}), 0\right) \leqq$$

$$\leqq mg(\mathbf{L}) + \max\left(\sum_v v(\mathfrak{X}_{i2}) - \sum_v v(\mathfrak{M}_{i2}), 0\right) \leqq mg(\mathbf{L}) + \sum_v v(\mathfrak{X}_{i2}).$$

By the construction of $\mathfrak{X}_{i2}$ and by (19)

$$\sum_v v(\mathfrak{X}_{i2}) \leqq m \sum_v v(\mathfrak{U}_1) + m \sum_v v(\mathfrak{X}) + m \sum_v v(\mathfrak{D}) \leqq$$

$$\leqq mH_{\mathbf{L}}(a) + m \sum_{v \in S_3} (m-1) + m \sum_v v(\mathfrak{D}) < M^3.$$

By definition of z_i $(i=1, 2, 3)$ we get

$$z_1(\alpha_3 - \alpha_2) y_1^m + z_2(\alpha_1 - \alpha_3) y_2^m + z_3(\alpha_2 - \alpha_1) y_3^m = 0,$$

or $\alpha(y_1^m + \beta y_2^m) = y_3^m$, where

$$\alpha = z_1 z_3^{-1}(\alpha_3 - \alpha_2)(\alpha_1 - \alpha_2)^{-1} \quad \text{and} \quad \beta = z_2 z_1^{-1}(\alpha_1 - \alpha_3)(\alpha_3 - \alpha_2)^{-1}.$$

Write

$$\xi = y_1 y_2^{-1}, \quad \beta_0 = \sqrt[m]{\beta}, \quad \mathbf{L}_0 = \mathbf{L}(\beta_0) \quad \text{and} \quad \eta = y_3 y_2^{-1}.$$

Then $\alpha(\xi^m + \beta) = \eta^m$, where $m > 3$ and $\xi, \eta \in \mathbf{L} \subseteq \mathbf{L}_0$. Now we can apply Lemmas 3 and 2 and we obtain

$$H_{\mathbf{L}_0}(\xi) \leqq 35\left(H_{\mathbf{L}_0}(\alpha) + H_{\mathbf{L}_0}(\alpha\beta)\right) + 8g(\mathbf{L}_0),$$

or

$$H_{\mathbf{L}_0}(\xi) \leqq 35H_{\mathbf{L}}(\alpha) + 35H_{\mathbf{L}}(\alpha\beta) + 8\left(g(\mathbf{L}) + H_{\mathbf{L}}(\beta)\right) < 320M^3. \tag{21}$$

By construction of ξ we have

$$H_{\mathbf{L}}(\beta_1/\beta_2) = H_{\mathbf{L}}(z_1 z_2^{-1} \xi^m) \leqq mH_{\mathbf{L}}(\xi) + H_{\mathbf{L}}(z_1) + H_{\mathbf{L}}(z_2) < 325M^4. \tag{22}$$

We have proved already (see (12) and (14)) that

$$\max\{H_L(\beta_1), H_L(\beta_2)\} \leqq 4H_L(f)+H_L(\beta_1/\beta_2) < 326\,M^4. \tag{23}$$

We note that by (14) this also holds in the case $z\in k$. Since $\beta_1=x-\alpha_1 y$ and $\beta_2=$ $=x-\alpha_2 y$, we have

$$H_L(y) \leqq H_L(\beta_1)+H_L(\beta_2)+2H_L(f) < 653M^4, \tag{24}$$

and

$$H_L(x) \leqq H_L(y)+H_L(f)+H_L(\beta_1) < 1000M^4. \tag{25}$$

Further, if $z\in O_{K,S}\setminus k$,

$$H_L(z) = \frac{1}{m}H_L(F(x, y)) \leqq \tag{26}$$

$$\leqq \frac{1}{m}(N+1)(H_L(f)+\max\{H_L(x), H_L(y)\}) < 1000M^5.$$

Finally, (16) and (24)—(26) imply our Theorem.

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RADON—NIKODYM DERIVATIVE IN THE CASE OF A RANDOM SCHRÖDINGER-TYPE EQUATION

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A new approach to boundary value problems of random partial differential equations was given by Yu. A. Rozanov [6], [7]. In our previous paper [1] we have given necessary and sufficient conditions for the existence and uniqueness of the solution of Schrödinger-type equation with deterministic and random potential. Definitions 1—6 and Theorem 1 contain the necessary information on the terminology and results concerning this topic. We have to notice that the object of our investigations has nothing in common with the prosperuous theory of Schrödinger operators with stochastic potential, developed by Moser, Pastur, Sinai etc. (cf. e.g. [5]).

In this paper we investigate these random equations from statistical point of view. The first step towards statistical inferences is the computation of R—N derivative of measures generated by operators with different potentials. Conditions which guarantee the equivalence of measures (mutual absolute continuity) are known for some local (e.g. differential) operators (cf. [2], [4]). Since in the Schrödinger equation the operator is not local the results above cannot be used in this case. However, using elementary considerations concerning finite dimensional Gaussian distributions we can compute the R—N derivative for the Schrödinger type equation, too. This will be done in this paper. More precisely we shall study a little more general system

$$L\xi + (F, \xi)\, g = \eta, \tag{1}$$

$$\xi_\Gamma^+ = \zeta, \tag{2}$$

(the notations will be given later).

Before formulating our results, let us look over some notations, definitions and theorems which can be found in [1], [6]—[8].

Let $S, T \subset \mathbf{R}^d$ be open domains and Γ the boundary of S. Let $D(T)$ denote the space of infinitely differentiable functions with compact support on T. We assume that $D(T)$ is endowed with the usual topology. $D'(T)$ denotes the space of the linear continuous functionals defined on $D(T)$. Let $F(T) \subset D'(T)$ be a Hilbert space with the scalar product $\langle , \rangle_F$, in which (φ, f) is continuous in f for every fixed $\varphi \in D(T)$. ((φ, f) means the value of f at φ.)

DEFINITION 1. The space $F(T)$ is called local iff $f, g \in F(T)$, $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ implies $\langle f, g \rangle_F = 0$.

DEFINITION 2. The operator $L: D(T) \to F(T)$ is called local iff $\operatorname{supp} L\varphi \subset \operatorname{supp} \varphi$ holds for every $\varphi \in D(T)$.

Let the above $F(T)$ be a local Hilbert space, $L: D(T) \to F(T)$ a linear continuous local (e.g. differential) operator and $L^*: F(T) \to D'(T)$ the Banach adjoint of L. Set

$$X(T)=\{x\in D'(T)\colon x \text{ continuous with respect to the norm } \sqrt{\langle L\varphi, L\varphi\rangle_F}=\|L\varphi\|_F\},$$

$$X(\Gamma)=\{x\in X(T)\colon \operatorname{supp} x\subset\Gamma\},\quad X(S)=\{x\in X(T)\colon \operatorname{supp} x\subset S\},$$

$$X^+(\Gamma)=\{x\in X(\Gamma)\colon\ x=L^*g,\ g\in F(T),\ \operatorname{supp}\ g\cap S=\emptyset\}.$$

Observe that $X(T)=L^*F(T)$, where $X(T)$ is a Hilbert space with the scalar product $\langle x, y\rangle=\langle L^*f, L^*d\rangle=\langle f, g\rangle_F$. We always assume that $X(S)$ is separable. Denoting by $\mathbf{H}$ the space $L^2(\Omega, \mathscr{A}, P)$ let $\xi, \xi_1\colon X(T)\to\mathbf{H}$, $\eta\colon F(T)\to\mathbf{H}$, $\zeta\colon X^+(\Gamma)\to\mathbf{H}$ be random linear continuous functionals [8] (r.l.c.f.). The action of them is denoted by (x, ξ), (f, η), (x, ζ), respectively. Let ξ_Γ^+ denote the restriction of ξ onto $X^+(\Gamma)$ and $g\in F(T)$, $F\in X(T)$.

DEFINITION 3. The random functionals ξ_1 and ξ are called the solution on $\bar{S}$ of the systems (1), (2) and

$$L\xi=\eta, \tag{3}$$

$$\xi_\Gamma^+=\zeta, \tag{4}$$

respectively, iff $\forall f\in F(S)$, $\forall x\in X^+(\Gamma)$ the following equations hold:

$$(L^*f, \xi_1)=(F, \xi_1)\langle f, g\rangle_F=(f, \eta)\quad (\text{resp. } (L^*f, \xi)=(f, \eta))$$

$$(x, \xi_1)=(x, \zeta)\quad (\text{resp. } (x, \xi)=(x, \zeta)).$$

The problem (3)—(4) is uniquely solvable (see [6], [7]). From now on ξ always denotes this unique solution, and $u\in X(S)$ the solution of $Lu=g$, $u_\Gamma^+=0$, which — by the same reason — exists uniquely, too. Let us recall our previous result.

THEOREM 1. *Equations* (1)—(2) *are uniquely solvable on* $\bar{S}$, *iff* $1+\langle u, F\rangle\neq 0$ *and then* $\xi_1=\xi-(\tilde{F}, \xi)u$, *where* $\tilde{F}=F/(1+\langle u, F\rangle)$ (cf. [1]).

In the following ξ_1 always denotes the solution of (1)—(2). The next definition plays a fundamental role.

DEFINITION 4. We say $B\colon X(T)\to X(T)$ is the correlation operator of ξ, if the relation

$$\langle Bx, y\rangle=E[(x, \xi)(y, \xi)]$$

holds.

We shall denote the correlation operator of ξ, ξ_1, η by B, B_1, B_η, respectively. They are positive, completely continuous operators. The generated scalar products and norms will be denoted by $\{\}$, $\{\}_1$, $\{\}_\eta$, $\|\|$, $\|\|_1$, $\|\|_\eta$ respectively. (E.g. $\sqrt{\langle Bx, x\rangle}=\sqrt{\{x, x\}}=|x|$.) Let us recall the definition of the Hilbert—Schmidt operator.

DEFINITION 5. A completely continuous self-adjoint operator is called H—S operator iff the relation $\sum_i \lambda_i^2<\infty$ holds for its complete eigenvalue system λ_i.

DEFINITION 6. A r.l.c.f. ξ is Gaussian iff (x, ξ) is normally distributed $\forall x\in X$ (see [8]).

Now we are in the position to formulate our results.

THEOREM 2. *Suppose that the solution of* (3)—(4) *is a Gaussian r.l.c.f. with zero expectation and F, g satisfy the conditions of Theorem* 1 *and the correlation operators* B, B_1 *have* 0 *kernel. Then*

a) *the uniquely existing solution of* (1)—(2) *is also a Gaussian r.l.c.f. with zero expectation, and*

b) *the measures* Q_ξ, Q_{ξ_1} *generated by* ξ, ξ_1 *on* $X(S)$ *are equivalent (mutually absolutely continuous).*

PROOF. a) is the immediate consequence of Theorem 1. The proof of b) is based on the following

LEMMA. *Let X be a separable Hilbert space,* ξ, ξ_1 *Gaussian r.l.c.f. with zero mean. Suppose the correlation operators* B, B_1 *have zero kernel. Then the well known Hajek—Feldman theorem states that the measures generated by* ξ, ξ_1 *on X are equivalent iff* $I-B^{-1}B_1$ *is a* H—S *operator having no eigenvalues equal to* 1 *in the* $\{\overline{X}, ||\}$ *space Denote if* $\{u_i\}_{i=i}^{\infty}$ *the orthonormal (in the* $||$ *norm) eigenvector system of* $I-B^{-1}B_1$, *and* λ_i *the corresponding eigenvalues. Then the R−N derivative is given by*

$$\frac{dQ_{\xi_1}}{dQ_\xi}(x) = p(x) = \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{n} \sqrt{1-\lambda_i}} \exp\left\{-\frac{1}{2}\sum_{k=1}^{n} \frac{\lambda_k}{1-\lambda_k} \langle x, u_k\rangle^2\right\}.$$

(See [3]. We use the formulation of Rozanov [8].)

The statement of the theorem is obvious, when $I-B^{-1}B_1$ has a finite rank. The essence of Theorem 1 is that $\xi_1=(I-C)\xi$, where C is (in general skew) projection onto the space spanned by F and u. So we can use the obvious special case of the Lemma mentioned above.

More formally let A: $X(T)\to X(T)$ be the following operator: $Ax=x-\langle \tilde{F}, x\rangle u$; then $\xi_1=A\xi$. Thus $\langle B_1 x, y\rangle = E\big((x, A\xi)(y, A\xi)\big)=E\big((A^*x, \xi)(A^*y, \xi)\big)=$ $=\langle BA^*x, A^*y\rangle=\langle ABA^*x, y\rangle$. Hence $B_1=ABA^*$. Thus $D=I-B^{-1}B_1=I-B^{-1}\cdot$ $\cdot ABA^*$ and

$$Dx = \{B^{-1}u, x\}\,\tilde{F} + \{\tilde{F}, A^*x\}\,B^{-1}u.$$

It is clear that D is a bounded operator in the $||$ norm, and its range is a one or two dimensional subspace spanned by F, $B^{-1}u$. So D is a H—S operator. We only have to check that it has no eigenvalue equal to 1. Since $\frac{1}{\text{const}}\,F$, const g define the same equation (1) as F and g, without loss of generality we may assume that $|B^{-1}u|=1$. Choosing $\tilde{F}$, $B^{-1}u$ as basis in the range of D, taking A from above and using $\langle u, x\rangle=$ $=\{B^{-1}u, x\}$ we get the determinant $\det(D-I)$ in the following form:

$$\det(D-I) = \begin{vmatrix} \{B^{-1}u, \tilde{F}\}-1, & (1-B^{-1}u, \tilde{F}\})\,|\tilde{F}|^2 \\ 1 \quad , & \{B^{-1}u, \tilde{F}\}-1-|\tilde{F}|^2 \end{vmatrix} = \big(1-\{B^{-1}u, \tilde{F}\}\big)^2 = (1-\langle u, \tilde{F}\rangle)^2.$$

The right hand side of this formula is equal to zero iff $\frac{1}{1+\langle u, F\rangle}=0$ which always fails. On the other hand $\det(D-I)\neq\infty$ because of the condition in Theorem 1. So the equivalence of measures Q_ξ and Q_{ξ_1} is proved.

Now let us compute the R—N derivative. Applying Lemma 1 we get

$$\frac{dQ_{\xi_1}}{dQ_\xi}(x) = p(x) = [(1-\lambda_1)(1-\lambda_2)]^{-1/2} \exp\{[(-2)(1-\lambda_1)(1-\lambda_2)]^{-1} \cdot$$

$$\cdot [(\lambda_1 - \lambda_1\lambda_2)\langle x, u_1\rangle^2 + (\lambda_2 - \lambda_1\lambda_2)\langle x, u_2\rangle^2]\}$$

where λ_i $(i=1, 2)$ are the eigenvalues, u_i are the normed eigenvectors of D. So we have to determine λ_i and u_i. Denoting by $\sigma = 1 - \{B^{-1}u, \tilde{F}\}$ and $\beta = |\tilde{F}|^2$ one gets:

$$|D-\lambda I| = \begin{vmatrix} \{B^{-1}u, \tilde{F}\} - \lambda, & 1 - \{B^{-1}u, \tilde{F}\}|\tilde{F}|^2 \\ 1 \quad , & \{B^{-1}u, \tilde{F}\} - |\tilde{F}|^2 - \lambda \end{vmatrix} = \lambda^2 + (\beta + 2\sigma - 2)\lambda + (1-\sigma)^2 - \beta$$

and

$$\lambda_{1,2} = 1 - \sigma - \frac{\beta \pm \sqrt{\beta^2 + 4\sigma\beta}}{2}.$$

Except the case when $\tilde{F} = \text{const}\, B^{-1}u$, D has two different real eigenvalues (by the Cauchy—Schwarz inequality). Since $(1-\lambda_1)(1-\lambda_2) = \sigma^2$ the R—N derivative has the form

$$p(x) = |\sigma|^{-1} \exp\left\{-(2-\sigma)^{-2} \sum_{i=1}^{2} [(2\sigma - 2\sigma^2 + \beta + (-1)^{i+1}\sqrt{\beta^2 + 4\sigma\beta})] \langle x, u_i\rangle^2\right\}.$$

From equation $(D-\lambda I)x = 0$ we have the eigenvectors

$$u_i = \tilde{F} + (\lambda_i - 1 + \sigma) B^{-1}u.$$

So the final form of the R—N derivative is

$$p(x) = |\sigma|^{-1} \exp\{\langle x, F\rangle \langle x, B^{-1}u\rangle - \frac{1}{2}\langle x, F\rangle^2\}. \tag{5}$$

Let us drop the assumption that B and B_1 have zero kernel. So for the equivalence of Q_ξ, Q_{ξ_1} it is necessary and sufficient that $\operatorname{Ker} B = \operatorname{Ker} B_1$ holds. It is easy to see that $\operatorname{Ker} B = L^* \operatorname{Ker} B_\eta$.

PROPOSITION. $\operatorname{Ker} B = \operatorname{Ker} B_1 \langle = \rangle F \in \operatorname{Ker} B$ *or* $g \perp \operatorname{Ker} B_\eta$.

It is obvious that $x \in \operatorname{Ker} B$ is equivalent to $(x, \xi) = 0$. The same is true for B_1 and ξ_1. Theorem 1 states that $(x, \xi) - (x, \xi_1) = (\tilde{F}, \xi)\langle x, u\rangle$. Therefore $(x, \xi) = 0$ and $(x, \xi_1) = 0$ are simultaneously valid if and only if $(\tilde{F}, \xi)\langle x, u\rangle = 0$ for every $x \in \operatorname{Ker} B$. If only $(F, \xi) \neq 0$ this can be satisfied iff $u \perp \operatorname{Ker} B$, which is equivalent to $g \perp \operatorname{Ker} B_\eta$. This proves our statement.

Let us drop even the assumption of zero expectation. Then $E\xi = E\xi_1 = E(\tilde{F}, \xi)u = L^* E(f, \xi)g$, and so by point 1 in [8, Ch. II, 3.§] and Theorems 2 and 3 in [8, Ch. II, 3.§.2] the measures Q_ξ and Q_{ξ_1} are equivalent iff $E\xi - E\xi_1$ is continuous in norm. If $E(f, \xi) \neq 0$ then this condition requires exactly the continuity of g in $\| \|_\eta$ norm.

Summarizing our results we can state

THEOREM 3. *Suppose the solution of the system* (3)—(4) *is a Gaussian r.l.c.f. on the separable Hilbert space* $X(S)$, *and the system* (1)—(2) *has a unique solution on* S.

The measures generated by these solutions are equivalent if and only if at least one of the following conditions holds:

1. $(F, \xi)=0$,
2. $g \perp \operatorname{Ker} B_\eta$ *and* $E(F, \xi)=0$,
3. $g \perp \operatorname{Ker} B_\eta$ *and continuous in* $\| \|_\eta$ *norm.*

In this case the R—N *derivative exists and it can be written in the form*

$$p(x+E\xi_1) = |\sigma|^{-1} \exp \{\langle x, F\rangle \langle x, B^{-1}u\rangle - \frac{1}{2}\langle x, F\rangle^2 -$$

$$- \sigma E(F, \xi) \langle x, u\rangle + \frac{1}{2}\sigma^2 E(F, \xi)^2 |u|^2\}.$$

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COUNTEREXAMPLES IN THE THEORY OF ORTHOGONAL SERIES

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1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a positive measure space and $\{\varphi_k(x): k=1,2,\ldots\}$ an orthonormal system (ONS) defined on X. Consider the orthogonal series

$$\sum_{k=1}^{\infty} a_k \varphi_k(x) \tag{1.1}$$

where $\{a_k: k=1,2,\ldots\}$ is a sequence of numbers. Denote the partial sums by

$$s_n(x) = \sum_{k=1}^{n} a_k \varphi_k(x)$$

and the first arithmetic means of the series (1.1) by

$$\sigma_n(x) = \frac{1}{n}\sum_{k=1}^{n} s_k(x) = \sum_{k=1}^{n}\left(1-\frac{k-1}{n}\right) a_k \varphi_k(x) \quad (n=1,2,\ldots).$$

The following results are well-known (see [3]).

THEOREM A. *If*

$$\sum_{k=1}^{\infty} a_k^2 < \infty, \tag{1.2}$$

then

$$\lim_{n\to\infty} \frac{s_n(x)}{\log n} = 0 \quad a.e.$$

THEOREM B. *Under the condition* (1.2),

$$\lim_{p\to\infty} \frac{s_{2^p}(x)}{\log p} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\sigma_n(x)}{\log\log n} = 0 \quad a.e.$$

For definiteness, the logarithms are to the base 2. The next theorem can be proved in a routine way.

THEOREM C. *If*

$$\sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty, \tag{1.3}$$

then

$$\lim_{n\to\infty} \frac{s_n(x)}{n\log n} = 0 \quad a.e. \tag{1.4}$$

For example, a short proof is the following. By (1.3), the Rademacher—Menshov theorem implies that the series

$$\sum_{k=2}^{\infty} \frac{a_k \varphi_k(x)}{k \log k}$$

converges a.e., whence the Kronecker lemma yields (1.4).

In terms of probability theory, (1.4) expresses a strong law: the first arithmetic mean of the orthogonal random variables $a_1\varphi_1(x), \ldots, a_n\varphi_n(x)$ with mean 0 and variances $a_1^2, \ldots, a_n^2$ is of order of magnitude $o(\log n)$ under condition (1.3).

2. Counterexamples

We are going to show that the above estimates are exact in the sense that if conditions (1.2) and (1.3) are not satisfied and a certain kind of monotonicity condition involving the $|a_k|$ is satisfied, then the corresponding conclusions do not hold any longer.

In the counterexamples the underlying measure space is the unit interval (0, 1) with the Borel measurable subsets as $\mathscr{F}$ and the Lebesgue measure as μ denoted by "mes".

A function $f(x)$ defined in (0, 1) is called a step function if (0, 1) can be divided into a finite number of disjoint subintervals such that $f(x)$ takes on a constant value on each of these subintervals. Further, a subset of (0, 1) is called simple if it can be represented as the union of finitely many disjoint subintervals.

The first theorem is due to Csernyák [1], who proved it in a more general form.

THEOREM D. *If the sequence* $\{|a_k|: k=1, 2, \ldots\}$ *is nonincreasing and*

$$\sum_{k=1}^{\infty} a_k^2 = \infty, \tag{2.1}$$

then there exists an ONS $\{\varphi_k(x): k=1, 2, \ldots\}$ *of step functions in* (0, 1) *such that*

$$\limsup_{n\to\infty} \frac{|s_n(x)|}{\log n} = \infty \quad a.e. \tag{2.2}$$

THEOREM 1. *If the sequence* $\{A_n^2 = \sum_{k=2^n+1}^{2^{n+1}} a_k^2: n=0, 1, \ldots\}$ *is nonincreasing and* (2.1) *is satisfied, then there exists an* ONS $\{\varphi_k(x)\}$ *of step functions in* (0, 1) *such that*

$$\limsup_{p\to\infty} \frac{|s_{2^p}(x)|}{\log p} = \infty \quad a.e. \tag{2.3}$$

and

$$\limsup_{p\to\infty} \frac{|\sigma_{2^p}(x)|}{\log p} = \infty \quad a.e. \tag{2.4}$$

In particular, by (2.4),

$$\limsup_{n\to\infty} \frac{|\sigma_n(x)|}{\log\log n} = \infty \quad \text{a. e.}$$

THEOREM 2. *If the sequence* $\{|a_k|/k\}$ *is nonincreasing and*

$$\sum_{k=1}^{\infty} \frac{a_k^2}{k^2} = \infty, \tag{2.5}$$

then there exists an ONS $\{\varphi_k(x)\}$ *of step functions in* (0, 1) *such that*

$$\limsup_{n\to\infty} \frac{|s_n(x)|}{n\log n} = \infty \quad a.\,e. \tag{2.6}$$

3. Auxiliary results

The proofs of our theorems are based on the following basic result of Menshov [2].

LEMMA 1. *Let* $N>2$ *be an integer. Then there exist an* ONS $\{\varphi_k(x)\colon k=1, 2, \ldots, N\}$ *of step functions in* (0, 1) *and a simple subset* E *of* (0, 1) *such that* $\operatorname{mes} E \geqq C_1$ *and for every* $x\in E$ *there exists an integer* $n=n(x)$, $2<n\leqq N$, *such that*

$$\varphi_k(x) \geqq 0 \quad (k = 1, 2, \ldots, n)$$

and

$$\sum_{k=2}^{n-1} \varphi_k(x) \geqq C_2\sqrt{N}\log N.$$

Actually, Menshov proved this inequality with $\sum_{k=1}^{n}$ instead of $\sum_{k=2}^{n-1}$, but this sharper inequality immediately follows from his proof.

Here C_1 and C_2 denote absolute positive constants.

Given a function $f(x)$ defined in (0, 1) and two subsets E and $I=(a, b)$ of (0, 1), we set

$$f(I;\ x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & \text{if} \quad x\in I, \\ 0 & \text{otherwise.} \end{cases}$$

Further, by $E(I)$ we denote the set into which E is carried over by the linear transformation $\bar{x}=(b-a)x+a$, $x\in(0, 1)$. Clearly,

$$\operatorname{mes} E(I) = \operatorname{mes} E \cdot \operatorname{mes} I.$$

LEMMA 2. *Let* $N>2$ *be an integer and* $\{a_n\colon n=1, 2, \ldots, N\}$ *a sequence of numbers such that* $\{|a_n|\}$ *is nonincreasing. Then there exist an* ONS $\{\psi_n(x)\colon k=1, 2, \ldots, N\}$ *of step functions in* (0, 1) *and a simple subset* F *of* (0, 1) *such that*

$$\operatorname{mes} F \geqq C_1 \min\{1, Na_N^2\} \tag{3.1}$$

and for every $x \in F$ *there exists an integer* $n=n(x)$, $2<n\leqq N$, *such that*

$$a_k \psi_k(x) \geqq 0 \quad (k = 1, 2, \ldots, n) \tag{3.2}$$

and

$$\sum_{k=2}^{n-1} a_k \psi_k(x) \geqq C_2 \log N. \tag{3.3}$$

PROOF. Let $I \subset (0, 1)$ with $\operatorname{mes}(I) = \min\{1, Na_N^2\}$. Apply Lemma 1 and set

$$\psi_k(x) = \frac{\operatorname{sign} a_k}{\sqrt{\operatorname{mes} I}} \varphi_k(I; x) \quad (k = 1, 2, \ldots, N)$$

and $F=E(I)$. The statement of Lemma 2 is clearly satisfied.

LEMMA 3. *Let* N_1, N_2 *be nonnegative integers,* $N_2>N_1+2$, *and let* $\{a_k: k=2^{N_1}+1, \ldots, 2^{N_2}\}$ *be a sequence of rational numbers different from* 0 *such that the sequence*

$$\left\{A_n^2 = \sum_{k=2^n+1}^{2^{n+1}} a_k^2 : \; n = N_1, \ldots, N_2-1\right\}$$

is nonincreasing. Then there exist an ONS $\{\chi_k(x): k=2^{N_1}+1, \ldots, 2^{N_2}\}$ *of step functions in* (0, 1) *and a simple subset* H *of* (0, 1) *such that*

$$\operatorname{mes} H \geqq C_1 \min\{1, (N_2-N_1) A_{N_2-1}^2\} \tag{3.4}$$

and for every $x \in H$ *there exists an integer* $p=p(x)$, $N_1+1<p\leqq N_2$, *such that*

$$a_k \chi_k(x) \geqq 0 \quad (k = 2^{N_1}+1, \ldots, 2^p) \tag{3.5}$$

and

$$\sum_{k=2^{N_1+1}+1}^{2^{p-1}} a_k \chi_k(x) \geqq C_2 \log (N_2-N_1). \tag{3.6}$$

PROOF. Since the a_k are rational numbers, there exist positive integers r_k and Q such that

$$\frac{a_k^2}{A_n^2} = \frac{r_k}{Q} \quad (k = 2^n+1, \ldots, 2^{n+1}; \; n = N_1, \ldots, N_2-1).$$

Clearly,

$$\sum_{k=2^n+1}^{2^{n+1}} r_k = Q \quad (n = N_1, \ldots, N_2-1).$$

Apply Lemma 2 with $\{A_n: n=N_1, \ldots, N_2-1\}$ and $N=N_2-N_1$. Let $\{I_r: r=1, 2, \ldots, Q\}$ be a partition of (0, 1) into disjoint subintervals of equal length. Define

$$H = \bigcup_{r=1}^{Q} F(I_r)$$

and

$$\chi_k(x) = \frac{A_n}{a_k} \sum_{r=s_{k-1}+1}^{s_k} \psi_n(I_r; x),$$

where

$$s_k = \begin{cases} 0 & \text{for } k = 2^n, \\ r_{2^n+1}+\ldots+r_k & \text{for } k = 2^n+1, \ldots, 2^{n+1};\ n = N_1, \ldots, N_2-1. \end{cases}$$

It is not hard to check that (3.1)—(3.3) imply (3.4)—(3.6).

The next orthogonalization lemma was actually proved several times in papers dealing with divergence theorems. (See, e.g. [3].) However, we cannot cite any paper where it would have been stated explicitly.

LEMMA 4. *Let* $\{f_k(x)\colon k=1,2,\ldots\}$ *be a system of step functions in* $(0,1)$, $\{k_N\colon N=1,2,\ldots\}$ *an increasing sequence of integers,* $k_1=0$, *and* $\{F_N\colon N=1,2,\ldots\}$ *a sequence of simple subsets of* $(0,1)$ *such that* $\{f_k(x)\colon k=k_N+1,\ldots,k_{N+1}\}$ *is an* ONS *for each* $N=1,2,\ldots$. *Then there exist an* ONS $\{g_k(x)\colon k=1,2,\ldots\}$ *of step functions in* $(0,1)$ *and a sequence* $\{G_N\colon N=1,2,\ldots\}$ *of stochastically independent simple subsets of* $(0,1)$ *such that*

$$\operatorname{mes} G_N = \operatorname{mes} F_N \quad (N = 1, 2, \ldots)$$

and for each $N=1,2,\ldots$ *there is a mapping* $x \to x'$ *from* G_N *to* F_N *such that*

$$g_k(x) = f_k(x') \quad (k = k_N+1, \ldots, k_{N+1}).$$

4. Proof of Theorem 1

Without loss of generality, we may assume that the coefficients a_k are rational numbers different from 0. Otherwise, we can choose a sequence $\{\tilde{a}_k\colon k=1,2,\ldots\}$ of rational numbers different from 0 such that

$$\left\{\tilde{A}_n = \sum_{k=2^n+1}^{2^{n+1}} \tilde{a}_k^2\colon n = 0, 1, \ldots\right\}$$

is nonincreasing and

$$|\tilde{a}_k - a_k| \leqq \frac{1}{k^2} \quad (k = 1, 2, \ldots).$$

Then for every ONS $\{\varphi_k(x)\}$,

$$\sum_{k=1}^{\infty} |\tilde{a}_k - a_k|\,|\varphi_k(x)| < \infty \quad \text{a.e.}$$

Consequently, if we prove (2.3) and (2.4) with $\tilde{s}_{2^p}(x)$ and $\tilde{\sigma}_{2^p}(x)$ corresponding to the sequence $\{\tilde{a}_k\}$, then we are done.

By (2.1),

$$\sum_{n=0}^{\infty} A_n^2 = \infty.$$

Taking into account that the sequence $\{A_n^2\}$ is nonincreasing, we can conclude that

$$\sum_{N=1}^{\infty} 2^N A_{2^{N+1}-1}^2 = \infty.$$

There exists a nonincreasing sequence $\{\varepsilon_N: N=1, 2, \ldots\}$ of positive numbers tending to 0 as $N\to\infty$ such that

$$\sum_{N=1}^{\infty} \varepsilon_N^2 2^N A_{2^{N+1}-1}^2 = \infty. \tag{4.1}$$

We apply Lemma 3 for the coefficients $\{\varepsilon_N a_k: k=2^{2^N}+1, \ldots, 2^{2^{N+1}}\}$ for each $N=2, 3, \ldots$. Thus, we get an ONS $\{\chi_k^{(N)}(x): k=2^{2^N}+1, \ldots, 2^{2^{N+1}}\}$ of step functions in $(0, 1)$ and a simple subset H_N of $(0, 1)$ such that

$$\operatorname{mes} H_N \geqq C_1 \min\{1, \varepsilon_N^2 2^N A_{2^{N+1}-1}^2\} \tag{4.2}$$

and for every $x\in H_N$ there exists an integer $p=p(x)$, $2^N+1<p\leqq 2^{N+1}$ such that

$$a_k \chi_k^{(N)}(x) \geqq 0 \quad (k = 2^{2^N}+1, \ldots, 2^p) \tag{4.3}$$

and

$$\sum_{k=2^{2^N+1}+1}^{2^{p-1}} \varepsilon_N a_k \chi_k^{(N)}(x) \geqq C_2 \log 2^N. \tag{4.4}$$

We take an arbitrary ONS $\{\chi_k^{(1)}(x): k=1, 2, \ldots, 2^{2^2}\}$ of step functions in $(0, 1)$. Applying Lemma 4, we obtain an ONS $\{\varphi_k(x): k=1, 2, \ldots\}$ of step functions in $(0, 1)$ and a sequence $\{G_N: N=2, 3, \ldots\}$ of stochastically independent simple subsets of $(0, 1)$ such that

$$\operatorname{mes} G_N = \operatorname{mes} H_N \quad (N = 2, 3, \ldots) \tag{4.5}$$

and for each $N=2, 3, \ldots$ there is a mapping $x\to x'$ from G_N to H_N such that

$$\varphi_k(x) = \chi_k^{(N)}(x') \quad (k = 2^{2^N}+1, \ldots, 2^{2^{N+1}}). \tag{4.6}$$

We consider the following identities:

$$2\sigma_{2^p}(x) - \sigma_{2^{p-1}}(x) = \sum_{k=1}^{2^{p-1}} a_k \varphi_k(x) + \sum_{k=2^{p-1}+1}^{2^p} \left(2 - \frac{k-1}{2^{p-1}}\right) a_k \varphi_k(x)$$

and

$$2\sigma_{2^{2^N+1}}(x) - \sigma_{2^{2^N}}(x) = \sum_{k=1}^{2^{2^N+1}} a_k \varphi_k(x) - \sum_{k=2^{2^N}+1}^{2^{2^N+1}} \left(\frac{k-1}{2^{2^N}} - 1\right) a_k \varphi_k(x).$$

Hence

$$[2\sigma_{2^p}(x) - \sigma_{2^{p-1}}(x)] - [2\sigma_{2^{2^N+1}}(x) - \sigma_{2^{2^N}}(x)] =$$

$$= \sum_{k=2^{2^N+1}+1}^{2^{p-1}} a_k \varphi_k(x) + \sum_{k=2^{p-1}+1}^{2^p} \left(2 - \frac{k-1}{2^{p-1}}\right) a_k \varphi_k(x) + \sum_{k=2^{2^N}+1}^{2^{2^N+1}} \left(\frac{k-1}{2^{2^N}} - 1\right) a_k \varphi_k(x).$$

By (4.3) and (4.6), for every $x\in G_N$ the last two sums on the right-hand side are nonnegative, and therefore by (4.4), with a suitable $x'\in H_N$,

$$2\sigma_{2^p}(x) - \sigma_{2^{p-1}}(x) - 2\sigma_{2^{2^N+1}}(x) + \sigma_{2^{2^N}}(x) \geqq \sum_{k=2^{2^N+1}+1}^{2^{p-1}} a_k \chi_k^{(N)}(x') \geqq \frac{C_2}{\varepsilon_N} \log 2^N.$$

Hence

$$\max\left\{\frac{|\sigma_{2^p}(x)|}{\log p}, \frac{|\sigma_{2^p-1}(x)|}{\log(p-1)}, \frac{|\sigma_{2^{2^N}+1}(x)|}{N}, \frac{|\sigma_{2^{2^N}}(x)|}{N}\right\} \geqq \frac{C_2}{16\varepsilon_N}. \tag{4.7}$$

By (4.1), (4.2), and (4.5),

$$\sum_{N=2}^{\infty} \operatorname{mes} G_N = \infty.$$

Thus, the Borel—Cantelli lemma implies that

$$\operatorname{mes}(\limsup_{N\to\infty} G_N) = 1.$$

Consequently, for almost every $x\in(0, 1)$ relation (4.7) is satisfied for infinitely many values of N and this proves (2.4).

The fulfillment of (2.3) follows from (4.4) in a simpler way. We omit the details.

5. Proof of Theorem 2

By (2.5) and the monotonicity of $\{|a_k|/k\}$ we have

$$\sum_{N=1}^{\infty} 2^{-N} a_{2^{N+1}}^2 = \infty.$$

There exists a nonincreasing sequence $\{\varepsilon_N: N=1, 2, \ldots\}$ of positive numbers tending to 0 as $N\to\infty$ such that

$$\sum_{N=1}^{\infty} \varepsilon_N^2 2^{-N} a_{2^{N+1}}^2 = \infty. \tag{5.1}$$

We apply Lemma 2 for the coefficients $\{\varepsilon_N a_k/k: k=2^N+1, \ldots, 2^{N+1}\}$ for each $N=1, 2, \ldots$. Thus we obtain an ONS $\{\psi_k^{(N)}(x): k=2^N+1, \ldots, 2^{N+1}\}$ of simple functions in (0, 1) and a simple subset F_N of (0, 1) such that

$$\operatorname{mes} F_N \geqq C_1 \min\left\{1, \varepsilon_N^2 2^N \frac{a_{2^{N+1}}^2}{2^{2N+2}}\right\} \tag{5.2}$$

and for every $x\in F_N$ there exists an integer $n=n(x)$, $2^N<n\leqq 2^{N+1}$ such that

$$a_k \psi_k^{(N)}(x) \geqq 0 \quad (k = 2^N+1, \ldots, n) \tag{5.3}$$

and

$$\sum_{k=2^N+1}^{n} \varepsilon_N \frac{a_k}{k} \psi_k^{(N)}(x) \geqq C_2 \log 2^N. \tag{5.4}$$

From (5.3) and (5.4) it follows that, for $x\in F_N$,

$$\sum_{k=2^N+1}^{n} a_k \psi_k^{(N)}(x) \geqq \frac{C_2}{\varepsilon_N} 2^N \log 2^N. \tag{5.5}$$

Applying Lemma 4 we get an ONS $\{\varphi_k(x): k=1, 2, \ldots\}$ of step functions in $(0, 1)$ and a sequence $\{G_N: N=1, 2, \ldots\}$ of stochastically independent simple subsets of $(0, 1)$ such that

$$\text{mes } G_N = \text{mes } F_N \quad (N = 1, 2, \ldots) \tag{5.6}$$

and for each $N=1, 2, \ldots$ there is a mapping $x \to x'$ from G_N to F_N such that

$$\varphi_k(x) = \psi_k^{(N)}(x') \quad (k = 2^N+1, \ldots, 2^{N+1}). \tag{5.7}$$

By (5.5), for every $x \in G_N$, with an appropriate $x' \in F_N$, we have

$$s_n(x) - s_{2^N}(x) = \sum_{k=2^N+1}^{n} a_k \psi_k^{(N)}(x') \geqq \frac{C_2}{\varepsilon_N} 2^N \log 2^N,$$

whence

$$\max\left\{\frac{|s_n(x)|}{n \log n}, \frac{|s_{2^N}(x)|}{2^N \log 2^N}\right\} \geqq \frac{C_2}{8\varepsilon_N}. \tag{5.8}$$

By (5.1), (5.2), and (5.6), $\sum_{N=1}^{\infty} \text{mes } G_N = \infty$, and by the Borel—Cantelli lemma, $\text{mes}\,(\limsup_{N\to\infty} G_N) = 1$. Consequently, for almost every $x \in (0, 1)$ inequality (5.8) is satisfied for infinitely many values of N, and this proves (2.6).

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TRANSVERSALS FOR ORDINAL INTERVALS

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A *transversal* of a family of sets is a one to one choice function.

If I is a closed interval $[\alpha, \beta]$ of ordinals, we write $\iota(I)=\alpha$ and $\tau(I)=\beta$. We also write $\mu(I)=\beta-\alpha$, i.e., $\mu(I)$ is the ordinal satisfying $\tau(I)=\iota(I)+\mu(I)$. The *length* $\lambda(I)$ of I is $\mu(I)+1$. If $\gamma<\iota(I)$ then $I-\gamma$ is defined as $[\iota(I)-\gamma, \tau(I)-\gamma]$.

The aim of this paper is to prove a conjecture of A. P. Huhn (Szeged, Hungary).

THEOREM. *If $\mathcal{I}$ is a set of closed intervals of ordinals such that $\lambda(I)\neq\lambda(J)$ whenever $I, J\in\mathcal{I}$ and $I\neq J$, then $\mathcal{I}$ has a transversal.*

(This result was proved in [1, p. 12] under the assumption that $\lambda(I)<\omega^k$ for each $I\in\mathcal{I}$ for some natural number k.)

PROOF. Note first that the result is easy for a set of finite intervals: choose elements from intervals by order of the lengths of the intervals; since the k-th interval contains at least k elements, it has an element different from those chosen so far.

Let $I\in\mathcal{I}$. Write

$$\mu(I) = \omega^{\alpha_1}\cdot k_1+\omega^{\alpha_1}\cdot k_2+\ldots+\omega^{\alpha_{t-1}}\cdot k_{t-1}+k_t,$$

where $\alpha_1>\alpha_2>\ldots>\alpha_{t-1}>0$ and $k_i<\omega$. We define the set

$$\Xi(I) = \{\omega^{\alpha_1}\cdot l_1+\ldots+\omega^{\alpha_{t-1}}\cdot l_{t-1}+l_t\colon\ l_i\leqq k_i \quad \text{for} \quad 0<i\leqq t\}$$

and $\theta(I)=\{\iota(I)+\xi\colon\ \xi\in\Xi(I)\}$. Since $\theta(I)\subseteq I$ for each $I\in\mathcal{I}$ it suffices to prove:

(A) The set $\theta(\mathcal{I})=\{\theta(I)\colon\ I\in\mathcal{I}\}$ has a transversal.

Since the sets $\theta(I)$ are finite, it suffices, by compactness, to show that $\theta(\mathcal{F})$ has a transversal for each finite subfamily $\mathcal{F}$ of $\mathcal{I}$. For any such $\mathcal{F}$ let $\zeta(\mathcal{F})=\min\{\alpha\colon\ \tau(I)<\omega^\alpha$ for each $I\in\mathcal{F}\}$. Since $\mathcal{F}$ is finite, $\zeta(\mathcal{F})$ is a successor ordinal. We prove our assertion by induction on $\zeta(\mathcal{F})$. Let $\zeta(\mathcal{F})=\alpha+1$ and assume that for any finite family $\mathcal{H}$ of closed intervals of different lengths such that $\zeta(\mathcal{H})\leqq\alpha$, the family $\theta(\mathcal{H})$ has a transversal.

For each $I\in\mathcal{F}$ define:

$$m^\alpha(I) = \max\{k\colon\ \iota(I)\leqq\omega^\alpha\cdot k\} \quad \text{and} \quad n^\alpha(I) = \min\{k\colon\ \tau(I)<\omega^\alpha\cdot(k+1)\}.$$

Also write $\hat{I}=[m^\alpha(I), n^\alpha(I)]$. For every $\beta<\omega^\alpha$ let

$$\mathcal{F}_\beta = \{I\in\mathcal{F}\colon\ \mu(I)=\omega^\alpha\cdot k+\beta \quad \text{for some} \quad k<\omega\}$$

and $\mathscr{K}_\beta=\{\hat{I}\colon I\in\mathscr{F}_\beta\}$. Notice that if $I\in\mathscr{F}_\beta$ then

$$\mu(I)=\omega^\alpha\big(n^\alpha(I)-m^\alpha(I)\big)+\beta.$$

If $I,J\in\mathscr{F}_\beta$ and $I\neq J$ then $n^\alpha(I)-m^\alpha(I)\neq n^\alpha(J)-m^\alpha(J)$, since otherwise clearly $\lambda(I)=\lambda(J)$. Thus $\mathscr{K}_\beta$ is a set of finite intervals of different lengths, and hence has a transversal f_β. For each $I\in\mathscr{F}_\beta$ let $Z_\beta(I)=[\iota(I),\iota(I)+\beta]$ if $f_\beta(\hat{I})=m^\alpha(I)$ and $Z_\beta(I)=[\omega^\alpha\cdot f_\beta(\hat{I}),\ \omega^\alpha\cdot f_\beta(\hat{I})+\beta]$ otherwise. Also define $W_\beta(I)=Z_\beta(I)-\omega^\alpha\cdot f_\beta(\hat{I})$. For each $k<\omega$ let

$$\mathscr{A}_k=\{W_\beta(I)\colon\ \beta<\omega^\alpha\ \text{ and }\ I\in\mathscr{F}_\beta\ \text{ and }\ f_\beta(\hat{I})=k\}.$$

Since each f_β is a transversal, and since $\lambda\big(W_\beta(I)\big)=\beta+1$ whenever $I\in\mathscr{F}_\beta$, it follows that all lengths of intervals in $\mathscr{A}_k$ are different. Also, $\zeta(\mathscr{A}_k)<\alpha$. Hence, by the induction hypothesis, $\theta(\mathscr{A}_k)$ has a transversal g_k. We define a transversal h of $\theta(\mathscr{F})$ as follows: let $I\in\mathscr{F}$, and write $\mu(I)=\omega^\alpha\cdot l+\beta$, where $\beta<\omega^\alpha$. If $k=f_\beta(\hat{I})$, define:

$$h\big(\theta(I)\big)=\omega^\alpha\cdot k+g_k\big(\theta(W_\beta(I))\big).$$

It is easily checked that $h\big(\theta(I)\big)\in\theta(I)$ and that h is a transversal, proving (A) and thus proving the theorem.

Reference

[1] P. Hajnal, One-to-one mappings between well ordered sets, preprint (Szeged, 1983) (in Hungarian).

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[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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ON LAGRANGE AND HERMITE INTERPOLATION. I

Y. E. MUNEER (Jeddah)

1. Introduction

Consider a triangular matrix

$$A = A(T)\colon [x_{v,n}]_{v=1}^{n}, \quad n = 1, 2, 3, \ldots \tag{1.1}$$

where

$$x_{v,n} = \cos\frac{2v-1}{2n}\pi \quad (1 \leqq v \leqq n) \tag{1.2}$$

are the roots of the first kind Tchebysev polynomial

$$T_n(x) = \cos(n \arccos x) \quad (n = 1, 2, 3, \ldots). \tag{1.3}$$

If there is no danger of misunderstanding we shall drop the second index of $x_{v,n}$ for all $1 \leqq v \leqq n$.

Corresponding to the matrix (1.1), we consider the following matrices:

$$M\colon [m_v]_{v=1}^{n}, \tag{1.4}$$

where $m_v = 1$, and

$$Y\colon [f_v]_{v=1}^{n}. \tag{1.5}$$

Here $f_v = f(x_v)$, where $f(x)$ is a well defined function in the interval $[-1, 1]$.

It is well known from the theory of interpolation that there exists a polynomial

$$L_{n-1}(x, Y, A) = \sum_{v=1}^{n} f_v l_v(x) \tag{1.6}$$

of degree at most $n-1$ which satisfies [5]

$$L_{n-1}(x_v, Y, A_v) = f_v \quad (1 \leqq v \leqq n). \tag{1.7}$$

Here

$$l_v(x) = \frac{T_n(x)}{T_n'(x_v)(x - x_v)}. \tag{1.8}$$

Many mathematicians have investigated the convergence of (1.6), e.g, L. Fejér [2], Grünwald [4], Turán [10], O. Kis [7]. Most of them proved the convergence to the function and its first derivative only. In the following theorem we prove the convergence to higher derivatives, and an estimate order of approximation is given, too.

THEOREM 1.1. *Let* $f(x)$ *be an arbitrary real function defined in the interval* $[-1, 1]$. *Let* $f \in C^{(2q)}$ $[-1, 1]$ *where* q *is a finite positive integer. Then for all* $x \in [-1, 1]$ *the inequality*

$$|L_{n-1}^{(s)}(x, Y, A) - f^{(s)}(x)| = O(1)\,\omega\left(\frac{1}{n}, f^{(2q)}\right)\frac{\log n}{n^{2q-2s}} \quad (s = 0, 1, 2, \ldots q) \tag{1.9}$$

holds true. i.e.,

$$L_{n-1}^{(q)}(x, Y, A) \to f^{(q)}(x) \quad (n \to \infty)$$

uniformly if and only if

$$\omega\left(\frac{1}{n}, f^{(2q)}\right) = o(1)[\log n]^{-1}. \tag{1.10}$$

If $f \in C^{(q)}[-1, 1]$ *then*

$$|L_{n-1}^{(s)}(x, Y, A) - f_{(x)}^{(s)}| = O(1)\frac{\omega\left(\frac{1}{n-1}, f^{(q)}\right)}{n^{q-s}(1-x^2)^{s/2}}\log n \quad (|x| < 1, s = 0, 1, \ldots, q) \tag{1.11}$$

for any finite integer $q \geqq 0$.

Now suppose that the endpoints $x_0 = 1$ and $x_{n+1} = -1$ of the closed interval $[-1, 1]$ are added to the elements of the matrix (1, 1), then we shall have the matrix

$$A^* = A^*(T)\colon [x_\nu]_{\nu=0}^{n+1} \quad (n = 1, 2, 3, \ldots). \tag{1.12}$$

Corresponding to this matrix suppose we have the matrices

$$M^*\colon [m_\nu]_{\nu=0}^{n+1} \tag{1.13}$$

where

$$m_\nu = 1 \quad (1 \leqq \nu \leqq n),$$

$$m_0 = m_{n+1} = r = \begin{cases} q/2 & \text{if } q \text{ if even integer} \\ \dfrac{q+1}{2} & \text{if } q \text{ if odd integer} \end{cases} \tag{1.14}$$

and

$$Y^*\colon [f_\nu^{(s)}]_{\nu=0}^{n+1} \quad 0 \leqq s \leqq m_\nu - 1 \tag{1.15}$$

where $f_\nu^{(s)}$ are the real values of the function $f^{(s)}(x)$ defined in the closed interval $[-1, 1]$.

Hence there exists an interpolation polynomial $H_m(x, Y^*, A^*)$ of degree at most m,

$$m = \begin{cases} n+q-1 & \text{if } q \text{ is even} \\ n+q & \text{if } q \text{ is odd} \end{cases} \tag{1.16}$$

which satisfies [5]

$$\begin{cases} H_m(x_\nu) = f_\nu & (1 \leqq \nu \leqq n), \\ H_m^{(s)}(1) = f_0^{(s)} & (0 \leqq s \leqq r-1), \\ H_m^{(s)}(-1) = f_{n+1}^{(s)} & (0 \leqq s \leqq r-1). \end{cases} \tag{1.17}$$

It is not difficult to show that the explicit form of $H_m(x, Y^*, A^*)$ is

$$H_m(x, Y^*, A^*) = \sum_{k=0}^{r-1} f_0^{(k)} r_{0,k}(x) + \sum_{k=0}^{r-1} f_{n+1}^{(k)} r_{n+1,k}(x) + \sum_{\nu=1}^{n} f_\nu \left(\frac{1-x^2}{1-x_\nu^2}\right)^r l_\nu(x), \tag{1.18}$$

where $l_\nu(x)$ is given by (1.8) and

$$r_{0,k}(x) = \left(\frac{1+x}{2}\right)^r T_n(x) \sum_{i=0}^{r-k-1} C_i^{(0,k)} (x-1)^{k+1}, \tag{1.19}$$

$$r_{n+1,k}(x) = \left(\frac{1-x}{2}\right)^r T_n(x) \sum_{i=0}^{r-k-1} C_i^{(n+1,k)} (x+1)^{k+1}. \tag{1.20}$$

The real coefficients $C_i^{(0,k)}$ and $C_i^{(n+1,k)}$ can simply be evaluated since we have

$$r_{0,k}^{(i)}(1) = \delta_{ki} = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k, \end{cases} \qquad r_{n+1,k}^{(i)}(-1) = \delta_{ki} \tag{1.21}$$

and

$$r_{0,k}^{(i)}(-1) = r_{n+1,k}(1) = 0 \quad (0 \leqq i \leqq r-k-1, \, 0 \leqq k \leqq r-1). \tag{1.22}$$

The convergence of the higher derivatives of this interpolation process was not investigated before. In the following we present a convergence theorem of this type.

THEOREM 1.2. *Let* $f \in C^{(q)}$ $[-1, 1]$, *then for all* $x \in [-1, 1]$,

$$|H_m^{(s)}(x, Y^*) - f^{(s)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right) \frac{\log n}{n^{q-s}} \quad (s = 0, 1, \ldots q) \tag{1.23}$$

holds true for all fixed integer $q \geqq 0$ (m *is given by* (1.16)).

If $f \in C^{(q+1)}$ $[-1, 1]$ *and* q *is even, then for any* $x \in (-1, 1)$

$$|H_{n+q-1}^{(q+1)}(x, Y^*, A^*) - f^{(q+1)}(x)| = O(1)\omega\left(\frac{1}{n}, f^{(q+1)}\right) \frac{\log n}{\sqrt{1-x^2}}. \tag{1.24}$$

Now, if we include one endpoint of the interval $[-1, 1]$, say $x_0 = 1$, to the nodes (1.1), we are going to have

$$A^{**}(T)\colon [x_\nu]_{\nu=0}^n \tag{1.25}$$

where

$$m_\nu = 1 \quad (1 \leqq \nu \leqq n), \tag{1.26}$$

$$m_0 = r = \begin{cases} q/2 & \text{if } q \text{ is an even integer} \\ \dfrac{q+1}{2} & \text{if } q \text{ is an odd integer} \end{cases} \tag{1.27}$$

and

$$Y^{**}\colon [f_\nu^{(s)}]_{\nu=0}^n \quad 0 \leqq s \leqq m_\nu - 1, \quad 0 \leqq \nu \leqq n. \tag{1.28}$$

Thus an interpolation process $H_m(x, Y^{**}, A^{**})$ of minimal degree $m=n+r-1$ exists uniquely [4], which satisfies

$$\begin{cases} H_m(x_\nu, Y^{**}, A^{**}) = f_\nu & (1 \leqq \nu \leqq n) \\ H_m^{(s)}(1, Y^{**}, A^{**}) = f_0^{(s)} & (0 \leqq s \leqq r-1). \end{cases} \tag{1.29}$$

It is easy to show that

$$H_m(x, Y^{**}, A^{**}) = \sum_{k=0}^{r-1} f_0^{(k)} r_{0,k}(x) + \sum_{\nu=1}^{n} \left(\frac{1-x}{1-x_\nu}\right)^r l_\nu(x), \tag{1.30}$$

where $l_\nu(x)$ is given by (1.8) and

$$r_{0,k}(x) = T_n(x) \sum_{i=0}^{r-k-2} C_i^{(0,k)} (x-1)^{k+i}. \tag{1.31}$$

The real coefficients $C_i^{(0,k)}$ can easily be evaluated since

$$r_{0,k}^{(i)}(1) = \delta_{ki} = \begin{cases} 1 & \text{if} \quad k = i \\ 0 & \text{if} \quad k \neq i \end{cases} \quad (0 \leqq i, k \leqq r-1). \tag{1.32}$$

The result on the above operator is stated in the following

THEOREM 1.3. *Let* $f \in C^{(q)}[-1, 1]$, *then for all* $x \in (-1, 1]$

(1.33)

$$|H_m^{(s)}(x, Y^{**}, A^{**}) - f^{(s)}(x)| = O(1)\,\omega\left(\frac{1}{m}, f^{(q)}\right) \frac{\log n}{(1+x)^{r+q} n^{q-s}} \quad (s = 0, 1, 2, \ldots, q)$$

where

$$m = \begin{cases} n + \dfrac{q-2}{2} & \text{if } q \text{ is even} \\ n + \dfrac{q-1}{2} & \text{if } q \text{ is odd.} \end{cases}$$

2. Preliminaries

In this part we introduce the most important formulae and definitions needed for our proofs in the rest of the paper.

It is obvious from (1.3) and (1.2) that

$$T_n'(x_\nu) = \frac{(-1)^\nu n}{\sqrt{1-x_\nu^2}} \quad (1 \leqq \nu \leqq n, n = 1, 2, 3, \ldots), \tag{2.1}$$

$$\sum_{\nu=0}^{n} l_\nu(x)^2 \leqq 2 \quad (|x| \leqq 1), \tag{2.2}$$

i.e.

$$|l_\nu(x)| \leqq \sqrt{2} \quad (\nu = 1, \ldots, n, |x| \leqq 1), \tag{2.3}$$

and for the Lebesgue function [6]

$$\frac{\log n}{8\sqrt{\pi}} < \max_{-1\leq x\leq 1} \sum_{v=1}^{n} |l_v(x)| \leq \frac{2}{\pi}\log n + C. \tag{2.4}$$

We shall use the well known S. Bernstein's and Markov's inequalities which are given as follows respectively:

For any polynomial $g_k(x)$ of real coefficients and of degree k, we have [1]

$$g_k^{(q)}(x) = O(1)\frac{n^q}{(1-x^2)^{q/2}} \max_{-1\leq x\leq 1} |q_k(x)| \tag{2.5}$$

where q is a finite positive integer, and [8]

$$g_k^{(q)}(x) = O(1)n^{2q} \max_{-1\leq x\leq 1} |g_k(x)|. \tag{2.6}$$

All over the fourth section we are going to apply the following:

THEOREM (I. E. Gopengaus [3]). *Let* $f\in C^{(q)}[-1, 1]$. *Then there exists a polynomial* $G_m(x, f)$ *of degree at most* m $(m\geq 4q+5)$ *such that*

$$|G_m^{(i)}(x)-f^{(i)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\left(\frac{\sqrt{1-x^2}}{m}\right)^{q-i} \quad (i = 0, 1, ..., q) \tag{2.7}$$

holds true for all fixed integer $q\geq 0$ *and for all* $x\in[-1, 1]$.

3. Some lemmas

LEMMA 3.1. *Let* $g_k(x)$ *be a polynomial of real coefficients and of degree* k. *Suppose*

$$Q_{k+q/2}(x) \overset{\text{def}}{=} (1-x^2)^{q/2}g_k(x), \tag{3.1.1}$$

then for all $x\in[-1, 1]$ *and* q *even, positive fixed integer, the following inequeality is true*:

$$|Q_{k+q/2}^{(q)}(x)| \leq C_q k^q \max_{-1\leq x\leq 1} |g_k(x)|. \tag{3.1.2}$$

The C_q *are real constants depending on* q *only.*

PROOF. The proof is by induction. Suppose $q=2$. Then we have from (3.1.1)

$$|Q_{k+1}''(x)| = |[(1-x^2)g_k(x)]''| \leq 2|g_k(x)|+4|g_k'(x)|+|(1-x^2)g_k''(x)|. \tag{3.1.3}$$

Using the inequalities (2.5-6) into (3.1.3) we get

$$|Q_{k+1}''(x)| \leq (2+C_1k^2) \max_{-1\leq x\leq 1} |g_k(x)| \leq C_2k^2 \max_{-1\leq x\leq 1} |g_k(x)| \tag{3.1.4}$$

where C_1 and C_2 are constans depending on q only. Thus (3.1.2) is true for $q=2$.

Suppose (3.1.2) is true for $q=2r$. Now, we have from (3.1.1) the following:

$$|Q_{k+r+1}^{(2r+2)}(x)| \leqq \left|\{(1-x^2)^{r+1}g_k(x)\}^{(2r+2)}\right| = \tag{3.1.5}$$

$$= \left|\{(1-x^2)Q_{k+r}(x)\}^{(2r+2)}\right| \leqq 2\binom{2r+2}{2r}|Q_{k+r}^{(2r)}(x)| +$$

$$+4\binom{2r+2}{2r+1}|Q_{k+r}^{(2r+1)}(x)| + (1-x^2)|Q_{k+r}^{(2r+2)}(x)|.$$

Since (3.1.2) is true for $q=2r$, we eliminate (3.1.2) and use (2.5-6) to obtain

$$|Q_{k+r+1}^{(2r+2)}(x)| \leqq C_{r+1}k^{2r+2}\max_{-1\leqq x\leqq 1}|g_k(x)|, \tag{3.1.6}$$

where C_{r+1} is a real valued constant depending on r only.

Thus (3.1.6) shows that (3.1.2) is also true for $q=2r+2$, and hence the lemma is proved.

LEMMA 3.2. *Let $g_k(x)$ be a polynomial of real coefficients and of degree k. Let q be a fixed positive odd integer. Suppose we have*

$$Q_{k+(q+1)/2}(x) \overset{\text{def}}{=} (1-x^2)^{(q+1)/2}g_k(x). \tag{3.2.1}$$

Then for all $x\in[-1, 1]$ the inequality

$$|Q_{k+(q+1)/2}^{(q)}(x)| \leqq c^{*q-1}\max_{-1\leqq x\leqq 1}|g_{k+1}^*(x)| \tag{3.2.2}$$

holds true. c_q is a real valued constant independent of k and

$$g_{k+1}^*(x) = (1-x^2)g_k'(x) - x(q+1)g_k(x). \tag{3.2.3}$$

PROOF. Differentiating (3.2.1) we obtain

$$Q_{k+(q+1)/2}^{(q)}(x) = [(1-x^2)^{(q+1)/2}g_k(x)]^{\prime(q-1)} =$$

$$= [(1-x^2)^{(q-1)/2}g_{k+1}^*(x)]^{(q-1)} \overset{\text{def}}{=} Q_{k+(q-1)/2}^{*(q-1)}(x)$$

where

$$g_{k+1}^*(x) = (1-x^2)g_k'(x) - x(q+1)g_k(x). \tag{3.2.4}$$

Applying (3.1.2) for $q-1$ which is an even positive integer we get

$$|Q_{k+(q-1)/2}^{*(q-1)}(x)| \leqq c_q^*k^{q-1}\max_{-1\leqq x\leqq 1}|g_{k+1}^*(x)|, \tag{3.2.5}$$

where c_q^* is a real constant independent of k. Hence it is obvious that (3.2.3) and (3.2.5) give the lemma.

LEMMA 3.3. *For all $x\in[-1, 1]$ and all $l_\nu(x)$ defined in (1.8) the following inequality holds:*

$$\sum_{\nu=1}^{n}\frac{1-x^2}{\sqrt{1-x_\nu^2}}|l_\nu'(x)| = O(1)n\log n. \tag{3.3.1}$$

PROOF. Let $x\in[x_{j+1}, x_j]$ $(0\leqq j<n)$. It is obvious from (1.2-3) that

$$1-x^2 = 1-x_j^2 = O(1)\left(\frac{2j-1}{n}\right)^2, \tag{3.3.2}$$

$$\frac{1-x^2}{1-x_v^2} = O(1) \quad \text{for} \quad v = j, j+1, \tag{3.3.3}$$

and

(3.3.4)

$$|x-x_v| = \begin{cases} O(1)\dfrac{|j-v|\cdot|j+v-1|}{2} & \text{for all} \quad 1 \leqq v \leqq j-1 \\ O(1)\dfrac{|j-v+1|\cdot|j+v|}{2} & \text{for all} \quad j+1 \leqq v \leqq \left[\dfrac{n}{2}\right] \\ O(1)\dfrac{|j-(n-v)+1|\cdot|j+(n-v)|}{n^2} & \text{for all} \quad \left[\dfrac{n}{2}\right]+1 \leqq v \leqq n. \end{cases}$$

Now,

$$\sum_{v=1}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| = \sum_{v=1}^{j-1} + \sum_{v=j}^{j+1} + \sum_{v=j+2}^{[n/2]} + \sum_{v=[n/2]+1}^{n}. \tag{3.3.5}$$

It is obvious from (1.8) that

$$\sum_{v=0}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| \leqq \sum_{v=1}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}} \frac{|T_n'(x)|}{|T_n'(x_v)(x-x_v)|} + \tag{3.3.6}$$

$$+ \sum_{v=1}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}\,|T_n'(x_v)|\,(x-x_v)^2}.$$

Using (1.3) together with the Bernstein inequality (2.5) for $q=1$, we reduce (3.3.6) to give

$$\sum_{v=1}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| \leqq \sum_{v=1}^{n} \frac{\sqrt{1-x^2}}{|x-x_v|^2} + \frac{1}{n} \sum_{v=1}^{n} \frac{1-x^2}{(x-x_v)^2}. \tag{3.3.8}$$

Now,

$$\sum_{v=1}^{n} = \sum_{v=1}^{j-1} + \sum_{v=j}^{j+1} + \sum_{j+2}^{[n/2]} + \sum_{[n/2]+1}^{n}. \tag{3.3.9}$$

Using (3.3.2) and (3.3.4) into (3.3.8) for $1\leqq v\leqq j-1$ we obtain

$$\sum_{v=1}^{j-1} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| = O(1)n \sum_{v=1}^{j-1} \frac{j}{|j-v||j+v-1|} + \tag{3.3.10}$$

$$+ O(1)n \sum_{v=1}^{j-1} \frac{j^2}{|j-v|^2|j+v-1|^2} = O(1)n\left\{\sum_{v=1}^{j-1} \frac{1}{|j-v|} + \sum_{v=1}^{j-1} \frac{1}{|j-v|^2}\right\} = O(1)n\log n.$$

Similarly, we can use (3.3.2) and (3.3.4) into (3.3.8) for $j+2\leqq v\leqq n$ to obtain

$$\sum_{v=j+2}^{[n/2]} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| = O(1)n\log n. \tag{3.3.11}$$

Likewise, on using (3.3.2) and (3.3.4) into (3.3.8) for $\left[\frac{n}{2}\right]+1 \leqq v \leqq n$, we can simply obtain

$$\sum_{v=[n/2]+1}^{n} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| = O(1) n \log n. \tag{3.3.12}$$

Using Bernstein's inequality (2.5), the inequalities (2.3) and (3.3.3) we obtain

$$\sum_{v=j}^{j+1} \frac{1-x^2}{\sqrt{1-x_v^2}} |l_v'(x)| = O(1) n. \tag{3.3.13}$$

Adding (3.3.6-8) we obtain the lemma.

4. Proof of the theorems

4.1. PROOF OF THEOREM 1.1. Let $G_{n-1}(x, f)$ be the Gopengaus polynomial of degree at most $n-1$. It is obvious from (1.6) that

$$G_{n-1}(x, f) = \sum_{v=1}^{n} G_{n-1}(x_v) l_v(x), \tag{4.1.1}$$

where $l_v(x)$ is given by (1.8). Therefore

$$G_{n-1}(x, f) - L_{n-1}(x, Y, A) = \sum_{v=1}^{n} [G_{n-1}(x_v, f) - f_v] l_v(x). \tag{4.1.2}$$

Applying Markov's inequality (2.6) we obtain

$$|G_{n-1}^{(s)}(x, f) - L_{n-1}^{(s)}(x, Y, A)| = O(1) n^{2s} \max_{-1 \leqq x \leqq 1} |G_{n-1}(x) - L_{n-1}(x)|. \tag{4.1.3}$$

Using (2.4) and (2.7) we obtain for $f \in C^{(2q)}[-1, 1]$

$$|G_{n-1}(x, f) - L_{n-1}(x, Y, A)| = O(1) \omega\left(\frac{1}{n}, f^{(2q)}\right) \frac{\log n}{n^{2q}} \tag{4.1.4}$$

for all $n \geqq 8q+6$ and $q \geqq 0$ fixed integer. Therefore (4.1.3-4) give

$$|G_{n-1}^{(s)}(x, f) - L_{n-1}^{(s)}(x, Y, A)| = O(1) \omega\left(\frac{1}{n}, f^{(2q)}\right) \frac{\log n}{n^{2q-2s}} \tag{4.1.5}$$

for all $n \geqq 8q+6$ and $s = 0, 1, 2, \ldots, 2q$. Using the triangle inequality we get

$$\begin{aligned} |L_{n-1}^{(s)}(x, Y, A) - f^{(s)}(x)| \leqq & |L_{n-1}^{(s)}(x, Y, A) - G_{n-1}^{(s)}(x, f)| + \\ & + |G_{n-1}^{(s)}(x, f) - f^{(s)}(x)|. \end{aligned} \tag{4.1.6}$$

Thus using (2.7) and (4.1.5) into (4.1.6) we obtain

$$|L_{n-1}^{(s)}(x, Y, A) - f^{(s)}(x)| = O(1) \omega\left(\frac{1}{n-1}, f^{(2q)}\right) \frac{\log n}{n^{2q-2s}} \tag{4.1.7}$$

for all $n \geqq 8q+6$, the $q \geqq 0$ fixed integer and $s=0, 1, 2, \ldots, 2q$.

It is obvious that if $\omega\left(\frac{1}{n-1}, f^{(2q)}\right)=o(1)\left[\log(n-1)\right]^{-1}$ then $L_{n-1}^{(q)}(x, Y, A) \to$ $\to f^{(q)}(x)$ uniformly in the closed interval $[-1, 1]$.

If we apply Bernstein's inequality (2.5) on (4.1.2) we can simply obtain

$$|L_{n-1}^{(s)}(x, Y, A)-G_{n-1}^{(s)}(x, f)| = \tag{4.1.8}$$

$$= O(1)\frac{n^s}{(1-x^2)^{s/2}} \max_{-1 \leqq x \leqq 1} |L_{n-1}(x, Y, A)-G_{n-1}(x, f)|.$$

Using (2.4) and (2.7) we show for $f \in C^{(q)}[-1, 1]$ that (4.1.2) reduces to

$$|L_{n-1}(x, Y, A)-G_{n-1}(x, f)| = O(1)\frac{\omega\left(\frac{1}{n-1}, f^{(q)}\right)}{n^q}\log n \tag{4.1.9}$$

for all $n \geqq 4q+6$ and $q \geqq 0$ fixed integer. Using (2.7), (4.1.8-9) and the triangle inequality we obtain

$$|L_{n-1}^{(s)}(x, Y, A)-f^{(s)}(x)| = O(1)\frac{\omega\left(\frac{1}{n-1}, f^{(q)}\right)}{n^{q-s}(1-x^2)^{s/2}}\log n, \tag{4.1.10}$$

for all $n \geqq 4q+6$, $q \geqq 0$ fixed integer and $s=0, 1, \ldots, q$.

4.2. PROOF OF THEOREM 1.2. Let $G_m(x, f)$ be the Gopengaus polynomial of degree at most m, where

$$m = n+2r-1 = \begin{cases} n+q-1 & \text{if } q \text{ is even} \\ n+q & \text{if } q \text{ is odd.} \end{cases} \tag{4.2.1}$$

It is obvious from (2.7) and (1.18) that

$$|G_m^{(q)}(x, f)-H_m^{(q)}(x, Y^*, A^*)| = \left|\left\{\sum_{\nu=1}^{n}[G_m(x_\nu, f)-f(x_\nu)]\frac{1-x^2}{1-x_\nu^2}l_\nu(x)\right\}^{(q)}\right|. \tag{4.2.2}$$

Now suppose q is even, then

$$|G_m^{(q)}(x, f)-H_m^{(q)}(x, Y^*, A^*)| = \left|\{(1-x^2)^{q/2}g_{n-1}(x)\}^{(q)}\right| = |Q_{n+q/2-1}^{(q)}(x)|, \tag{4.2.3}$$

where

$$g_{n-1}(x) = \sum_{\nu=1}^{n}[G_m(x_\nu)-f_\nu]\frac{l_\nu(x)}{(1-x_\nu^2)^{q/2}}. \tag{4.2.4}$$

Now using lemma (3.1) into (4.2.3) we obtain

$$|G_m^{(q)}(x, f)=H_m^{(q)}(x, Y^*, A^*)| = O(1)n^q \max_{-1 \leqq x \leqq 1} |g_{n-1}(x)|. \tag{4.2.5}$$

Let $f \in C^{(q)}$ ($[-1, 1]$) and q a fixed even integer. Therefore using (2.7) and (2.4) into (4.2.4) we obtain

$$|g_{n-1}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{n^q}, \tag{4.2.6}$$

where $m=n+q+1$. Eliminating (4.2.6) into (4.2.5) we get

$$|G_m^{(q)}(x,f)-H_m^{(q)}(x,Y^*,A^*)| = O(1)\omega\left(\frac{1}{m},f^{(q)}\right)\log n. \tag{4.2.7}$$

Now, suppose q is odd. Then we get from (1.15), (1.18) and (2.7) the following:

$$|G_m^{(q)}(x,f)-H_m^{(q)}(x,Y^*\,A^*) = \left|\{(1-x^2)^{(q+1)/2}g_{n-1}(x)\}^{(q)}\right| = |Q_{n-1+(q+1)/2}^{(q)}(x)|. \tag{4.2.8}$$

Thus using Lemma 3.2 we obtain

$$|G_m^{(q)}(x,f)-H_m^{(q)}(x,Y^*,A^*)| = O(1)n^{q-1}\max_{-1\leqq x\leqq 1}|g_n^*(x)|, \tag{4.2.9}$$

where

$$g_n^*(x) = (1-x^2)g'_{n-1}(x)-x(q+1)g_{n-1}(x) \tag{4.2.10}$$

and

$$g_{n-1}(x) = \sum_{\nu=1}^{n}[G_m(x_\nu)-f_\nu]\frac{l_\nu(x)}{(1-x_\nu^2)^{(q+1)/2}}. \tag{4.2.11}$$

If $f\in C^{(q)}[-1,1]$ and q is odd, then using (2.7) into (4.2.11) we obtain

$$|g_{n-1}(x)| = O(1)\frac{\omega\left(\frac{1}{n+q},f^{(q)}\right)}{n^q}\sum_{\nu=1}^{n}\frac{|l_\nu(x)|}{\sqrt{1-x_\nu^2}}. \tag{4.2.12}$$

Similarly if $f\in C^{(q)}[-1,1,]$ and q is odd, then using (2.7) into (4.2.11) we get

$$|(1-x^2)g'_{n-1}(x)| = O(1)\frac{\omega\left(\frac{1}{n+q},f^{(q)}\right)}{n^q}\sum_{\nu=1}^{n}\frac{1-x^2}{\sqrt{1-x_\nu^2}}l'_\nu(x)|. \tag{4.2.13}$$

Using (1.3), (2.3) and Cauchy's inequality into (4.2.12) we get

$$|g_{n-1}(x)| = O(1)\frac{\left(\frac{1}{n+q},f^{(q)}\right)}{n^{q-1}} \tag{4.2.14}$$

for $n\geqq 3q+5$ and q odd integer. While using Lemma 3.3 in (4.2.13) we obtain

$$|(1-x^2)g'_{n-1}(x)| = O(1)\omega\left(\frac{1}{n+q},f^{(q)}\right)\frac{\log n}{n^{q-1}}, \tag{4.2.15}$$

for q odd integer. Eliminating (4.2.16) into (4.2.9) we get

$$|G_m^{(q)}(x,f)-H_m^{(q)}(x,Y^*,A^*)| = O(1)\omega\left(\frac{1}{n+q},f^{(q)}\right)\log n \tag{4.2.17}$$

for all $n\geqq 3q+5$ and q an odd integer.

Consider the triangle inequality

$$|H_m^{(q)}(x,Y^*,A^*)-f^{(q)}(x)| \leqq |H_m^{(q)}(x,Y^*,A^*)-G_m^{(q)}(x,f)-f^{(q)}(x)|. \tag{4.2.18}$$

Using (2.7) together with (4.2.7) and (4.2.17) we obtain

$$|H_m^{(q)}(x, Y^*, A^*) - f^{(q)}| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\log n, \tag{4.2.19}$$

for all

$$m = \begin{cases} n+q-1, & \text{if } q \text{ is even} \\ n+q, & \text{if } q \text{ is odd} \end{cases} \geqq 4q+5.$$

Now let

$$f(x) - H_n(x) = Z(x) \Rightarrow Z^{(q)}(x) = f^{(q)} - H_n^{(q)}(x). \tag{4.2.20}$$

It is obvious from (1.18) that

$$Z(x_\nu) = Z(x_{\nu+1}) = 0 \quad (0 \leqq \nu \leqq n). \tag{4.2.21}$$

Hence applying Rolle's theorem, we can find a $\xi_{1,\nu}$ such that for $x_{\nu+1} < \xi_{1,\nu} < x_\nu$,

$$Z'(\xi_{1,\nu}) = 0 \quad (0 \leqq \nu \leqq n). \tag{4.2.22}$$

Once more we can apply Rolle's theorem to obtain $\xi_{2,\nu-1}$ such that

$$Z''(\xi_{2,\nu-1}) = 0, \quad \xi_{1,\nu-2} < \xi_{2,\nu-1} < \xi_{1,\nu-1} \quad (1 \leqq \nu \leqq n). \tag{4.2.23}$$

Successively we can apply Rolle's theorem, and find $\xi_{q-s_1,\nu-(q-s-1)}$ such that for $\xi_{q-s_1-1,\nu-q+s_1+2} < \xi_{q-s_1,\nu-q+s_1+1} < \xi_{q-s_1-1,\nu-q+s_1+1} = 0$,

$$Z^{(q-s_1)}(\xi_{q-s_1,\nu-(q-s_1-1)}) = 0 \tag{4.2.24}$$

where $q-s_1-1 \leqq \nu \leqq n$, $s_1 = 1, 2, 3, \ldots, q-1$.

It is obvious from (5.2.22-24) that

$$x_{\nu+1} < \xi_{1,\nu} < \ldots < \xi_{q-1,\nu-q+2} < \ldots < \xi_{1,\nu-q+2} < x_{\nu-q+2}. \tag{4.2.25}$$

Now, let $x \in [x_{\nu+1}, x_{\nu-q+2}]$ $(0 \leqq \nu \leqq n)$. Integrating between $\xi_{q-1\,\nu-q+2}$ and x, we obtain from (4.2.19-20) the following:

$$\left| \int_{q-1,\nu-q+2}^{x} Z^{(q)}(t)\,dt \right| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\log n\,|x - \xi_{q-1,\nu-q+2}|. \tag{4.2.26}$$

It is obvious from (4.2.25) that (4.2.26) reduces to

$$|Z^{(q-1)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\log n\,|x_\nu - x_{\nu+1}| = \tag{4.2.27}$$

$$= O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{n}.$$

Integrating (4.2.27) between $\xi_{q-2,\nu-q+3}$ and x we get

$$|Z^{(q-2)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{n^2}. \tag{4.2.28}$$

Successively integrating we produce

$$|Z^{(q-s)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{n^s}, \tag{4.2.29}$$

where $s_1=1, 2, \ldots, q-1$. Write $s=q-s_1$ into (4.2.9) to obtain

$$|Z^{(s)}(x)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{n^{q-s}} \tag{4.2.30}$$

for $s=1, 2, 3, \ldots, q$. Integrating $|Z'(x)|$ in (4.2.30) between $x_{\nu+1}$ and x we get (1.23).

For q even fixed integer we will have the operator (1.18) with $m=n+q-1$. Now consider the Gopengaus polynomial $G_m(x, f)$ of degree at most $m=n+q-1$. It is obvious from (1.18) that

$$|G_m^{(q+1)}(x)-H_m^{(q+1)}(x, Y^*, A^*)| = \left|\{(1-x^2)^{q/2}g_{n-1}(x)\}\right|^{(q+1)} = \left|\{Q_{k+q/2}^{(q)}\}'\right|, \tag{4.2.31}$$

where

$$g_{n-1}(x) = \sum_{\nu=1}^{n}[G_m(x_\nu)-f_\nu]\frac{l_\nu(x)}{(1-x_\nu^2)^{q/2}}. \tag{4.2.32}$$

Using S. Bernstein inequality (2.5) into (4.2.31) we get

$$|G_m^{(q+1)}(x)-H_m^{(q+1)}(x, Y^*, A^*)| \leqq \frac{n}{\sqrt{1-x^2}}\max_{-1\leqq x\leqq 1}|Q_{k+q/2}^{(q)}(x)|. \tag{4.2.33}$$

Using Lemma 3.1 we reduce (4.2.33) to

$$|G_m^{(q+1)}(x)-H_m^{(q+1)}(x, Y^*, A^*)| \leqq \frac{n}{\sqrt{1-x^2}}\max_{-1\leqq x\leqq 1}|Q_{k+q/2}^{(q)}(x)|. \tag{4.2.34}$$

It is obvious from (2.7) and (4.2.32) that for $f\in C^{(q-1)}(-1, 1])$

$$|g_{n-1}(x) = O(1)\omega\left(\frac{1}{n}, f^{(q)}\right)\frac{\log n}{n^{q+1}} \tag{4.2.35}$$

for all $m=n+q+1\geqq 4q+9$ i.e., $n\geqq 3q+8$.

Thus (4.2.35) when substituted in (4.2.34) yields

$$|G_m^{(q+1)}(x)-H_m^{(q+1)}(x, Y^*, A^*)| = O(1)\omega\left(\frac{1}{m}, f^{(q)}\right)\frac{\log n}{\sqrt{1-x^2}} \tag{4.2.36}$$

for all $n\geqq 3q+8$. and $m=n+q-1$.

Thus (2.7) and (4.2.36) together with the triangle inequality

$$|H_m^{(q+1)}(x, Y^* A^*)-f^{(q+1)}(x)| \leqq$$

$$\leqq |H_m^{(q+1)}(x, Y^*, A^*)-G_m^{(q+1)}(x)|+|G_m^{(q+1)}(x, f)-f^{(q+1)}(x)|, \tag{4.2.36}$$

give the proof of (1.24).

PROOF OF THEOREM 1.3. The proof of this theorem is carried out similarly to that done for Theorem 1.2.

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A ONE-SIDED ADMISSIBLE IDEAL RADICAL WHICH IS ALMOST SUBIDEMPOTENT

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1. The purpose of the present paper, which in some sense is a continuation of the author's paper [10], is to study the upper radical property $\mathscr{X}_r$ determined by the class E_r of all rings without admissible right ideals which are not right ideals.

At first, it is proved that $\mathscr{X}_r$ is a nonhereditary almost subidempotent radical property and that $\mathscr{X} < \mathscr{X}_r$ holds, where $\mathscr{X}$ is the upper radical property determined by the class E of all rings without admissible ideals which are not ideals (G. Tzintzis [10]). Next, it is proved that $\mathscr{X}_r$ coincides with the upper radical determined by the class of all division rings and of all zerorings, which means that $\mathscr{X}_r$ coincides with the class $I_{\mathscr{D}}$ of all idempotent $\mathscr{D}$-radical rings where $\mathscr{D}$ is the upper radical determined by all division rings (G. Tzintzis [9]).

From this last result it is evident that $\mathscr{X}_r$ is independent of the choice of right or left ideals. However, by examples it is shown that the classes E_r, E_l (the left analogue of E_r) and E are mutually incomparable. Finally, E_r-rings are examined.

2. Throughout this paper all rings considered will be associative. The terminology and basic results of radical theory can be found in [1], [2], [6]. With the symbol $\langle a \rangle$ we denote the additive subgroup generated by a and, whenever B is a ring, with $A \underset{r}{\leqq} B$ $(A \underset{l}{\leqq} B,\ A \leqq B)$ we mean that A is a right (left, two-sided) ideal of B. Finally, by the symbol $|a)_R$ $((a)_R)$ we mean the right (two-sided) ideal of R generated by the element a of R. Suppose E_r to be the class of all rings R where if $A \underset{r}{\leqq} B$ and $B \underset{r}{\leqq} R$ then $A \underset{r}{\leqq} R$. Evidently, $E_r \neq \emptyset$. Also, E_r is hereditary. Now, suppose $\mathscr{X}_r$ to be the upper radical property determined by the class E_r. Evidently, $\mathscr{X}_r \leqq \mathscr{D}$, where $\mathscr{D}$ is the upper radical property determined by all division rings, since every division ring is an element of the class E_r.

LEMMA 2.1. *If the ring R contains $a \neq 0$ and $I \underset{r}{\leqq} R$ such that:*

(1) $aI \subset I$, (2) $\exists n_0 \geqq 2$: $a^{n_0} \in I$, (3) $\exists r \in R$: $a^{n_0-1} r \notin I + \langle a^{n_0-1} \rangle$

then $R \notin E_r$.

PROOF. It is easy to show that $A = \{r \mid r \in R,\ ar \in I\}$ is a right ideal of R and that $I + \langle a^{n_0-1} \rangle$ is a right ideal of A but not of R.

COROLLARY 2.2. *If the ring R contains $a \neq 0$ and there exists $r \in R$ such that $a^{n_0-1} r \notin |a^{n_0})_R + \langle a^{n_0-1} \rangle$, for some integer $n_0 \geqq 2$, then $R \notin E_r$.*

PROOF. By Lemma 2.1, provided that $I = |a^{n_0})_R$.

LEMMA 2.3. *If the ring* R *contains* $a\neq 0$ *such that for some integer* $n_0\geqq 2$, (1) $a^{n_0-1}R\nsubseteq |a^{n_0})_R$ *and*

$$(2)\qquad R/\Pi \ncong \frac{(k_0)_Z}{(\lambda_0)_Z}\neq 0,\quad k_0|\lambda_0,\ \lambda_0\geqq 0,\quad \lambda_0\neq 1,\quad \forall \Pi\leqq R,\quad \Pi\neq R,$$

hold, then there exists an $r\in R$ *such that* $a^{n_0-1}r\notin |a^{n_0})_R+\langle a^{n_0-1}\rangle$.

PROOF. Suppose that $a^{n_0-1}r\in |a^{n_0})_R+\langle a^{n_0-1}\rangle\ \forall r\in R$. Then for each $r\in R$ there exist elements $\varrho\in R$ and $m,k\in \mathbf{Z}$ depending on r such that $a^{n_0-1}r=a^{n_0}\varrho+ma^{n_0}+ka^{n_0-1}$, that is, $a^{n_0-1}(r-i)=ka^{n_0-1}$, where $i\in |a)_R$. By (1) it is evident that $a^{n_0-1}\notin |a^{n_0})_R$, whence, if λ_0 is the smallest positive integer, whenever it exists, or otherwise $\lambda_0=0$, $(\lambda_0)_Z\subset \mathbf{Z}$ must hold. Now, if we construct the mapping $\varphi: R\ni r\to k+(\lambda_0)_Z\in \mathbf{Z}/(\lambda_0)_Z\neq 0$, where $a^{n_0-1}(r-i)=ka^{n_0-1}$, for some $i\in |a)_R$, it is easy to show that φ is well defined and also a ring homomorphism. The kernel of this homomorphism is the set $\Pi=\{r\,|\,a^{n_0-1}r\in |a^{n_0})_R\}$ which by (1) can not coincide with R. Consequently,

$$R/\Pi \cong \frac{(k_0)_Z}{(\lambda_0)_Z}\neq 0,$$

a contradiction.

LEMMA 2.4. *For each* $a\in R$ *the following statements are equivalent:*

$$(1)\qquad a^{n_0-1}R\nsubseteq |a^{n_0})_R,\qquad (2)\quad |a)_R+\mathrm{Ann}_r\{a^{n_0-1}\}\neq R.$$

PROOF. (1)$\Rightarrow$(2). Indeed, if we suppose that (2) does not hold, then for each $r\in R$ there exist $i\in |a)_R$ and $b\in \mathrm{Ann}_r\{a^{n_0-1}\}$ such that $r=i+b$. Consequently,

$$a^{n_0-1}r = a^{n_0-1}(i+b) = a^{n_0-1}i+a^{n_0-1}b = a^{n_0-1}i\in |a^{n_0})_R,$$

a result which contradicts (1).

(2)$\Rightarrow$(1). Similarly, if we suppose that $a^{n_0-1}r\in |a^{n_0})_R$, $\forall r\in R$ then $a^{n_0-1}r=a^{n_0}\varrho+ka^{n_0}\Rightarrow a^{n_0-1}(r-i)=0$, where $i\in |a)_R,\Rightarrow r-i\in \mathrm{Ann}_r\{a^{n_0-1}\}\Rightarrow r\in |a)_R+\mathrm{Ann}_r\{a^{n_0-1}\}\Rightarrow R=|a)_R+\mathrm{Ann}_r\{a^{n_0-1}\}$, a contradiction.

PROPOSITION 2.5. *If the ring* R *contains* $a\neq 0$ *such that, for some integer* $n_0\geqq 2$,

$$(1)\qquad |a)_R+\mathrm{Ann}_r\{a^{n_0-1}\}\neq R$$

and

$$(2)\qquad R/\Pi \ncong \frac{(k_0)_Z}{(\lambda_0)_Z}\neq 0,\quad k_0|\lambda_0,\ \lambda_0\geqq 0,\quad \lambda_0\neq 1,\quad \forall \Pi\leqq R,\quad \Pi\neq R$$

hold, then $R\notin E_r$.

PROOF. By Lemmas 2.1, 2.3, 2.4.

COROLLARY 2.6. *If* R *is a simple prime and* $\mathcal{N}_g$*-radical ring then it is* $\mathcal{X}_r$*-radical.*

PROOF. Since R is an $\mathcal{N}_g$-radical ring, there exists a nonzero $a\in R$ such that $a^{n_0}=0$, $a^{n_0-1}\neq 0$, for some integer $n_0\geqq 2$. For this element, $|a)_R+\mathrm{Ann}_r\{a^{n_0-1}\}\neq R$ holds. Indeed, if we suppose the converse, then $a^{n_0}=0\Rightarrow a\in\mathrm{Ann}_r\{a^{n_0-1}\}\Rightarrow|a)_R\subseteq$ $\subseteq\mathrm{Ann}_r\{a^{n_0-1}\}\Rightarrow\mathrm{Ann}_r\{a^{n_0-1}\}=R\Rightarrow R^2=0$, a contradiction. Now, it does not hold that $R\cong\frac{(k_0)_{\mathbf{Z}}}{(\lambda_0)_{\mathbf{Z}}}$ for some nonnegative integers k_0, λ_0, where $k_0|\lambda_0$, $\lambda_0\neq 1$, since otherwise $\frac{(k_0)_{\mathbf{Z}}}{(\lambda_0)_{\mathbf{Z}}}\cong\frac{\mathbf{Z}}{(p)_{\mathbf{Z}}}$ for some prime p where $\lambda_0=pk_0$ and $p\nmid k_0$, which is contradictory to R being $\mathcal{N}_g$-radical. Consequently, by Proposition 2.5, $R\notin E_r$ and as a simple prime it must be an $\mathcal{X}_r$-radical ring.

COROLLARY 2.7. *We have* $\mathcal{X}_r\not\equiv\mathcal{T}$, *where* $\mathcal{T}$ *is the upper radical property determined by the class of all matrix rings over division rings.*

PROOF. Indeed, the 2×2 matrix ring over division ring is a simple prime $\mathcal{N}_g$-radical ring and simultaneously $\mathcal{T}$-semisimple.

LEMMA 2.8. *If* R *is a nonzero ring without divisors of zero and with the property* $|a)_R=|a^2)_R\ \forall a\in R$, *then* R *is a division ring.*

PROOF. Suppose I to be a nonzero right ideal of R. Since, for every $a\in I$, $a\neq 0$, we have $|a)_R=|a^2)_R$, it follows that $a=ae$, where $e\in|a)_R$. Consequently, by Divinsky's Lemma 6 [5], e is a left unity element of R. Thus we have $R=eR\subseteq|a)_R\subseteq$ $\subseteq I\subseteq R$, that is, $R=I$. Hence R is a division ring.

PROPOSITION 2.9. *Every* $\mathcal{D}$*-radical ring* $R\neq 0$ *contains an element* $a\neq 0$ *with* $a\notin|a^2)_R$.

PROOF. If we suppose that the $\mathcal{D}$-radical ring $R\neq 0$ has the property $|a)_R=$ $=|a^2)_R$, $\forall a\in R$, then, evidently, every homomorphic image R' of R must have the same property. Consequently, by Lemma 2.8, every homomorphic image R' cannot be an integral domain. This cannot be true since, by supposition, R is a fully idempotent ring (V. A. Andrunakievič and J. M. Rjabuhin [4]) and thus the homomorphic images R/I_a, where $a\neq 0$ and I_a is the maximal ideal of R with respect to $a\notin I_a$, are integral domains ([4], Theorem 2).

LEMMA 2.10. *If the* $\mathcal{D}$*-radical ring* R *has a nonzero homomorphic image* R' *such that for any* $a\in R'$ *and* $a\notin|a^2)_R$, $a\in\mathrm{Ann}_l R'$ *holds, then* R' *is a zeroring.*

PROOF. The $\mathcal{D}$-radical homomorphic image $\bar{R}=R'/\mathrm{Ann}_l R'$ has the property $\bar{r}\in|\bar{r}^2)_{\bar{R}}\ \forall\bar{r}=r+\mathrm{Ann}_l R'$. Indeed, if $r\in|r^2)_{R'}\Rightarrow\bar{r}\in|\bar{r}^2)_{\bar{R}}$ and if $r\notin|r^2)_{R'}\Rightarrow r\in\mathrm{Ann}_l R'\Rightarrow$ $\Rightarrow\bar{r}=0\Rightarrow\bar{r}\in|\bar{r}^2)_{\bar{R}}$. Consequently, by Proposition 2.9, $\bar{R}=0$, that is $\mathrm{Ann}_l R'=R'$, which means that $(R')^2=0$.

COROLLARY 2.11. *Every nonzero homomorphic image* R' *of an idempotent* $\mathcal{D}$*-radical ring* R *contains an element* $a\neq 0$ *with the following properties:*

(1) $a\notin|a^2)_{R'}$

(2) $\mathrm{Ann}_r\{a\}\neq R'$.

PROOF. By Proposition 2.9 and Lemma 2.10.

LEMMA 2.12. *If R is an idempotent $\mathscr{D}$-radical ring, then the conditions $|a)_R + \mathrm{Ann}_r\{a\} \neq R$ and $aR + \mathrm{Ann}_r\{a\} \neq R$ are equivalent.*

PROOF. Indeed, if the first condition holds the second is obvious. Conversely, if the second condition holds but not the first, then $a \notin aR + \mathrm{Ann}_r\{a\}$ and if $I_a \supseteq aR + \mathrm{Ann}_r\{a\}$ is a right ideal of R maximal with respect to the property $a \notin I_a$, then since $\forall i \in I_a$ and $\forall r = aq + ka + b$, where $b \in \mathrm{Ann}_r\{a\}$, we have $ri = (aq + ka + b)i = a(qi + ki) + bi \in aR + \mathrm{Ann}_r\{a\} \supseteq I_a$, I_a is a two-sided ideal of R, and if $I \supset I_a$ is a right ideal of R, then $a \in I \Rightarrow |a)_R + \mathrm{Ann}_r\{a\} \supseteq I \Rightarrow R = I$. Consequently, since R is an idempotent ring, R/I_a is a prime simple ring and also, as we will show in the following, a division ring, which is contradictory to R being a $\mathscr{D}$-radical ring. In fact, R/I_a does not have divisors of zero, since if $\bar{a}, \bar{b} \in R/I_a$, $\bar{a} \neq 0$, $\bar{b} \neq 0$ and $\bar{a}\bar{b} = 0$, then the nonzero right ideal $U = \{\bar{b} | \bar{a}\bar{b} = 0\}$ of R/I_a is equal to R/I_a, whence we have $\mathrm{Ann}_l(R/I_a) \neq 0 \Rightarrow \mathrm{Ann}_l(R/I_a) = R/I_a \Rightarrow (R/I_a)^2 = 0$, a contradiction. Now, it is evident that $|\bar{a})_{R/I_a} = |(\bar{a})^2)_{R/I_a}$ for every $\bar{a} \in R/I_a$, whence by Lemma 2.8 R/I_a is a division ring.

LEMMA 2.13. *If the $\mathscr{D}$-radical ring R has the property $|a^2)_R = |a^3)_R$ $\forall a \in R$, then R is an $\mathscr{N}_g$-radical ring.*

PROOF. Indeed, if we suppose that $\mathscr{N}_g(R) \neq R$, then there exists a nonzero homomorphic image A of R without divisors of zero (V. A. Andrunakievič and J. M. Rjabuhin [4], [6], pp. 153—154). Moreover, it is evident that $|a^2)_A = |a^3)_A$ $\forall a \in A$. Thus with the same proof as in Lemma 2.8, A is a division ring which is contradictory to R being a $\mathscr{D}$-radical ring.

LEMMA 2.14. *If R is an idempotent $\mathscr{D}$-radical ring, then there exists an element $a \in R$, $a \neq 0$ with the properties $a \notin |a^2)_R$ and $\mathrm{Ann}_r\{a\} \neq R$ such that $|a)_R + \mathrm{Ann}_r\{a\} \neq R$ holds.*

PROOF. By Corollary 2.11, there exists a nonzero element $a \in R$ with the properties $a \notin |a^2)_R$ and $\mathrm{Ann}_r\{a\} \neq R$. Now we suppose that $|a)_R + \mathrm{Ann}_r\{a\} = R$, or equivalently by Lemma 2.12 $aR + \mathrm{Ann}_r\{a\} = R$ for all such elements. Then $|b^2)_R = |b^3)_R$ $\forall b \in R$. Indeed, if $|b)_R = |b^2)_R$, then it is obvious that $|b^2)_R = |b^3)_R$. If $b \notin |b^2)_R$ and $\mathrm{Ann}_r\{b\} = R$, then $b^2 = 0 \Rightarrow |b^2)_R = 0 = |b^3)_R$. Finally, if $b \notin |b^2)_R$ and $\mathrm{Ann}_r\{b\} \neq R$, then by assumption we have $bR + \mathrm{Ann}_r\{b\} = R$, whence $b = br + c$, where $c \in \mathrm{Ann}_r\{b\}, \Rightarrow b^2 = b(br + c) = b^2r = b^2(br' + c') = b^3r' \Rightarrow |b^2)_R = |b^3)_R$. Consequently, by Lemma 2.13, R is an $\mathscr{N}_g$-radical ring, that is, it contains at least one nonzero nilpotent element b (V. A. Andrunakievič and J. M. Rjabuhin [3]). Since $|b^2)_R = |b^3)_R$, it follows that $b^2 = 0 \Rightarrow b \in \mathrm{Ann}_r\{b\} \Rightarrow bR \subseteq \mathrm{Ann}_r\{b\} \Rightarrow \mathrm{Ann}_r\{b\} = R$ (indeed, $b \notin |b^2)_R = 0$). Thus, $b \in \mathrm{Ann}_l R$ for all nonzero nilpotent elements b of R, a result which means that $\mathrm{Ann}_l R = \mathscr{N}_r(R) = R$ (V. A. Andrunakievič and J. M. Rjabuhin [3]), that is $R^2 = 0$, a contradiction.

PROPOSITION 2.15. *If R is an idempotent $\mathscr{D}$-radical ring, then R is an $\mathscr{X}_r$-radical ring.*

PROOF. By Corollary 2.11 and Lemma 2.14, $|a)_R + \mathrm{Ann}_r\{a\} \neq R$ holds for at least one element $a \neq 0$, $a \in R$, with the properties $a \notin |a^2)_R$ and $\mathrm{Ann}_r\{a\} \neq R$. Consequently, by Proposition 2.5 it follows that $R \notin E_r$. Indeed, since R is an idem-

potent $\mathscr{D}$-radical ring it is easy to show that it cannot be homomorphically mapped onto a nonzero ring $\frac{(k_0)_{\mathbf{Z}}}{(\lambda_0)_{\mathbf{Z}}}$, $k_0|\lambda_0$. Now, by Corollary 2.11 and since every nonzero homomorphic image R' of R is also an idempotent and $\mathscr{D}$-radical ring, it follows that $R' \notin E_r$ which means that R is an $\mathscr{X}_r$-radical ring.

COROLLARY 2.16. *For the radical property $\mathscr{X}_r$ the following holds:*
(1) *$\mathscr{X}_r$ coincides with the upper radical property $\mathscr{U}$ determined by the class of all division rings and zerorings.*
(2) *$\mathscr{X}_r = I_{\mathscr{D}}$, where $I_{\mathscr{D}}$ is the class of all idempotent and $\mathscr{D}$-radical rings.*
(3) *$\mathscr{X}_r$ is a nonhereditary almost subidempotent radical property.*

PROOF. (1) $\mathscr{X}_r \leqq \mathscr{U}$, since all division rings and all zerorings are contained in E_r. Conversely, if $R \in \mathscr{U}$ then R is an idempotent $\mathscr{D}$-radical ring, whence, by Proposition 2.15, $R \in \mathscr{X}_r$.

(2) By (1), $\mathscr{X}_r \leqq I_{\mathscr{D}}$ and by Proposition 2.15, $I_{\mathscr{D}} \leqq \mathscr{X}_r$.

(3) By (2), $\mathscr{X}_r$ is an almost subidempotent radical property and the following example shows that it is non-hereditary.

EXAMPLE 2.17 ([6], example 3). Let F be some field and let A be the commutative F-algebra with basis set $\{x_a | 0 < a < 1\}$, where a is a real number.

Multiplication of basic elements is defined as follows:

$$x_a x_b = \begin{cases} x_{a+b} & \text{if} \quad a+b < 1 \\ 0 & \text{if} \quad a+b \geqq 1. \end{cases}$$

Thought of as a ring, it is well known that $A \in \mathscr{N} \leqq \mathscr{D}$, where $\mathscr{N}$ is the nil-radical property of G. Koethe [7], and also that it is an idempotent ring. Consequently, by Corollary 2.16, A is an $\mathscr{X}_r$-radical ring. However, $(x_a)_A^n = 0$ for some $n \in \mathbf{N}$, $n > \frac{1}{a}$, which means that the nonzero ideal $(x_a)_A$ of A is not an $\mathscr{X}_r$-radical ring, and consequently $\mathscr{X}_r$ is a nonhereditary radical property.

Evidently, by Corollary 2.16 it follows that $\mathscr{X}_r = \mathscr{X}_l$, where $\mathscr{X}_l$ is the upper radical property determined by the class E_l.

Finally, by the following examples it is shown that regarding the relationship between the classes E_r, E_l we have

$$E_r \nsubseteqq E_l, \quad E_l \nsubseteqq E_r, \quad E_r \cap E_l \neq \emptyset.$$

EXAMPLE 2.18. Let A be a division ring where $A^+ \ncong \mathbf{Z}/(p)_{\mathbf{Z}}$ $\forall p$. Defining for the group direct sum $R = A \oplus A$ the multiplication

$$(a, r)(b, s) = \big(a(b+s), r(b+s)\big), \qquad a, b, r, s \in A,$$

R becomes an associative ring and the following hold: the subset $B = \{(a, 0) | a \in A\}$ is a right ideal of R and as a ring it is isomorphic with A; also, the subset $C = \{(-r, r) | r \in A\}$ is a two-sided ideal of R with $R/C \cong A$, $C + B = R$, $C \cap B = \{(0, 0)\}$, $C^+ \cong A^+$, and with all its elements right annihilators of R.

Now we can show that $R \in E_l$ and $R \notin E_r$. Indeed, if we have $I_1 \underset{l}{\leqq} I_2$ and $I_2 \underset{l}{\leqq} R$, then, if $I_2 \nsubseteqq C$ holds, there exist $(a, r) \in I_2$ and $(a, r) \notin C$, that is, with $a + r \neq 0$.

Consequently, since I_2 is a left ideal of R, $(b, s)(a, r)\in I_2$ for every $b, s\in A, \Rightarrow$ $\Rightarrow (b(a+r), s(a+r))\in I_2 \Rightarrow R \subsetneqq I_2 \Rightarrow R = I_2 \Rightarrow I_1 \underset{l}{\leqq} R$.

Also, if $I_2 \subseteq C \Rightarrow I_1 \subseteq C \Rightarrow RI_1 = \{(0, 0)\} \Rightarrow I_1 \underset{l}{\leqq} R$. However, if I is any nontrivial additive subgroup of C, it is easy to show that I is a right ideal of C but not a right ideal of R.

EXAMPLE 2.19. Changing the multiplication in Example 2.18 to $(a, r)(b, s) = ((a+r)b, (a+r)s)\ \forall a, b, r, s\in A$, we have $R\in E_r$ and $R\notin E_l$ for the corresponding ring R.

EXAMPLE 2.20. If A is a division ring, then it is contained in both E_r and E_l.

3. Next, we will examine E_r-rings.

PROPOSITION 3.1. *The following statements are equivalent:*

(1) $R\in E_r$,

(2) $|a)_R = |a)_{|a)_R}, \quad \forall a\in R$,

(3) $|a)_R = |a^2)_R + \langle a\rangle, \quad \forall a\in R$,

(4) $aR \subseteq |a^2)_R + \langle a\rangle, \quad \forall a\in R$.

PROOF. The same as in Proposition 2.3 [10].

The following powers of rings with ordinal numbers as indices are known (G. Krause and T. H. Lenegan [8]). If r is a nonlimit ordinal number, then $R_r = \bigcap_n R_{r-1}^n$, while if r is a limit ordinal number, then $R_r = \bigcap_{c<r} R_c$. It is evident that for every ring R there exists an ordinal number π depending on R such that R_π is the maximal idempotent ideal of R. Consequently, by Proposition 2.15, if $R\in E_r$, then $\mathscr{D}(R)_\pi = 0$, where $\mathscr{D}(R)_\pi$ is the maximal idempotent ideal of $\mathscr{D}(R)$, since otherwise $\mathscr{D}(R)_\pi \neq 0$ is an $\mathscr{X}_r$-radical ring.

PROPOSITION 3.2. *For every E_r-ring R, $\mathscr{B}(R) = \mathscr{N}_g(R)$ holds.*

PROOF. If a is a nilpotent element of R, i.e. $a^{n_0} = 0$, then by Proposition 3.1 (3) it is implied that

$$|a)_R = \langle a^{2^{m-1}}\rangle + \langle a^{2^{m-2}}\rangle + \ldots + \langle a^2\rangle + \langle a\rangle,$$

where $2^m \geqq n_0$. Consequently, since $|a)_R$ is a commutative ring, $|a)_R^{n_0} = |a^{n_0})_R = 0$, which means that $a\in\mathscr{B}(R)$. Therefore we have $\mathscr{B}(R) = \mathscr{N}_g(R)$.

LEMMA 3.3. *If R is an $\mathscr{F}$-radical E_r-ring and it contains a nonzero element a with $a^2 = 0$, then if at least one of the following properties:*

(1) $\operatorname{ord} a = 0$

(2) $\operatorname{ord} a = p$, p *prime*

(3) R^+ *divisible*

(4) R^+ *elementary*

(5) $R^2 = R$

holds, a must be a left annihilator of R.

PROOF. By Proposition 3.1 (4) and Lemma 2.3, it is known that there exists a surjective ring-homomorphism

$$\varphi\colon R \ni r \to k+(\lambda_0) \in \frac{(k_0)_Z}{(\lambda_0)_Z}, \quad \lambda_0 \neq 1, \quad k_0 | \lambda_0$$

where $ar=ka$ and λ_0 is the order of a. Consequently, if ord $a=0$ holds, then we will have $\varphi(R)=(k_0)_Z$ which, in case $(k_0)_Z \neq 0$, is contradictory to R being an $\mathscr{F}$-radical ring. Thus $(k_0)_Z=0$, which means that a is a left annihilator of R. Now if ord $a=p$, where p is a prime integer, then it is evident that k_0 must be 1 or p. Consequently, if $k_0=1$, we will have $\varphi(R)=Z/(p)_Z$ which is contradictory to R being an $\mathscr{F}$-radical ring. Therefore $k_0=p$, i.e. a is a left annihilator of R.

Continuing, in case when R^+ is a divisible group, since the group $\frac{(k_0)_Z}{(\lambda_0)_Z}$ cannot be divisible, $\frac{(k_0)_Z}{(\lambda_0)_Z}=0$, whence a is a left annihilator of R.

For the next case, when R^+ is an elementary group, since R is an $\mathscr{F}$-radical ring, if $\frac{(k_0)_Z}{(\lambda_0)_Z} \neq 0$, it must be nilpotent. Consequently, there exists an integer $n \geqq 2$ such that $\lambda_0 | k_0^n$, that is $k_0^n = \lambda_0 m$, and since $\lambda_0 = k_0 s$, $s \neq 1$, we have

$$k_0^n = k_0 sm \Rightarrow k_0^{n-1} = sm \Rightarrow (k_0, s) \neq 1 \Rightarrow \lambda_0 = p^2 z,$$

for some prime p, a contradiction. Consequently, $\frac{(k_0)_Z}{(\lambda_0)_Z}=0$, which means that a is a left annihilator of R.

Finally, if R is an idempotent ring, then $\frac{(k_0)_Z}{(\lambda_0)_Z}$ is zero, since otherwise $\frac{(k_0)_Z}{(\lambda_0)_Z}$ will not be an $\mathscr{F}$-radical ring, a contradiction. Consequently, a is a left annihilator of R.

COROLLARY 3.4. *If R is an $\mathscr{F}$-radical E_r-ring and $\mathscr{N}_g(R) \neq 0$, then* $\mathrm{Ann}_l R \neq 0$.

PROOF. Indeed, since $\mathscr{N}_g(R) \neq 0$, there exists a nonzero element a with $a^2=0$. Consequently, if ord $a=0$, then, by Lemma 3.3, a is a left annihilator of R, that is $\mathrm{Ann}_l R \neq 0$. Now if ord $a=m>1$ and $m=pk$, where p is a prime, then it is evident that $ka \neq 0$ and $(ka)^2=0$, whence by Lemma 3.3, ka is a left annihilator of R.

LEMMA 3.5. *If in the $\mathscr{F}$-radical E_r-ring R the nonzero element a is nilpotent, i.e. $a^{n_0}=0$, with $n_0<4$, and its additive group R^+ is divisible (elementary, torsion-free), then $aR \subseteqq (a^2)_R$ for every $a \in R$.*

PROOF. Indeed, if $n_0=2$, this is evident by Lemma 3.3. Now suppose $n_0=3$. If R^+ is divisible or elementary, then, by Lemma 3.3, it has been shown that $R/I \ncong \frac{(k_0)_Z}{(\lambda_0)_Z} \neq 0$ for every ideal $I \neq R$. Consequently, by Lemma 2.3, $aR \subseteqq (a^2)_R$. Also, if R is a torsion-free ring and there exists an $r \in R$ such that $ar=a^2\varrho+\lambda a^2+\mu a$, where $\mu a \notin (a^2)_R$, then since, by Lemma 3.3, a^2 is a left annihilator of R, we will have $\mu a^2=0$, a contradiction. Consequently, $aR \subseteqq (a^2)_R$.

PROPOSITION 3.6. *If for the $\mathscr{F}$-radical E_r-ring R its additive group R^+ is divisible (elementary, torsion-free), then* $\mathscr{N}_g(R)^2 \subseteqq \mathrm{Ann}_l R$.

PROOF. Let R^+ be a divisible (elementary, torsion-free) group. If a is a nilpotent element, $a^{n_0}=0$, $a^{n_0-1}\neq 0$, $n_0>1$ and n_0 is an even natural integer, then for the nonzero element $a^{n_0/2}$, $(a^{n_0/2})^2=0$ holds, whence, by Lemma 3.3, we have $a^{n_0/2}R=$ $=0\Rightarrow a^{n_0/2+1}=0\Rightarrow \frac{n_0}{2}+1\geqq n_0\Rightarrow n_0\leqq 2\Rightarrow n_0=2\Rightarrow aR=0$. Similarly, if n_0 is an odd integer, then for the nonzero element $a^{(n_0+1)/2}$, $(a^{(n_0+1)/2})^2=0$ holds, whence, by Lemma 3.3, we have $a^{(n_0+1)/2}R=0\Rightarrow a^{(n_0+1)/2+1}=0\Rightarrow \frac{n_0+1}{2}+1\geqq n_0\Rightarrow n_0\leqq 3\Rightarrow n_0=3\Rightarrow a^2R=0$ and by Lemma 3.5, $aR^2=(aR)R\subseteqq |a^2)_R R=\langle a^2\rangle R=0$.

Consequently, for every nilpotent element $a\in R$, $aR^2=0$ holds, whence we have

$$\mathscr{N}_g(R)^2 R = \mathscr{N}_g(R)\big(\mathscr{N}_g(R)R\big) \subseteqq \mathscr{N}_g(R)R^2 = 0,$$

that is $\mathscr{N}_g(R)^2 \subseteqq \mathrm{Ann}_l R$.

LEMMA 3.7. *If the $\mathscr{F}$-radical E_r-ring R is idempotent, then $|a^2)_R=|a^2)_R^2$ holds for every $a\in R$.*

PROOF. The same as in Proposition 2.12 [10].

PROPOSITION 3.8. *If the $\mathscr{F}$-radical E_r-ring R is idempotent, then $\mathscr{D}(R)=\mathrm{Ann}_l R$.*

PROOF. Indeed, for every $a\in\mathscr{D}(R)$, by Lemma 3.7, we have $|a^2)_R=|a^2)_R^2$, which, in case $a^2\neq 0$, contradicts $\mathscr{D}(R)_\pi=0$ for some ordinal number π. Consequently, $a^2=0$ and, by Lemma 3.3, $aR=0$ for every $a\in\mathscr{D}(R)$, which implies $\mathscr{D}(R)=\mathrm{Ann}_l R$.

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ON GENERALIZED TAKAGI FUNCTIONS

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§ 1. Introduction. After the famous Weierstrass' nowhere differentiable function, Takagi [8] in 1903 gave another simple example which has nowhere finite derivative. This example is defined as follows: Set

$$\varphi(x) = \begin{cases} 2x & \text{for} \quad 0 \leqq x \leqq 1/2 \\ 2-2x & \text{for} \quad 1/2 \leqq x \leqq 1, \end{cases}$$

and

$$\varphi^{(1)}(x) = \varphi(x), \quad \varphi^{(n)}(x) = \varphi(\varphi^{(n-1)}(x)), \quad n = 1, 2, \ldots.$$

Then, Takagi proved that the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}\varphi^{(n)}(x), \quad 0 \leqq x \leqq 1,$$

has nowhere finite one-side derivative. Since then many people have investigated nowhere differentiable functions from various point of views.

In this paper, we investigate a generalization of the Takagi function such as

$$f(x) = \Sigma c_n \varphi^{(n)}(x),$$

and give necessary and sufficient conditions in order that f is nowhere differentiable, differentiable on a set of continuum or absolutely continuous, resp. A part of these results was investigated in [3] in more general situations. We also discuss a modification of the function $\varphi^{(n)}$ such as

$$\varphi_*^{(n)}(x) = \varphi^{(n)}(x) - 1/2$$

and show that $\{\varphi_*^{(n)}\}_{n=1}^{\infty}$ is a concrete example of the so called multiplicative system [1] but not a strongly multiplicative system. Using some results by the Hungarian school [2], [3], [10], [11], [12], and [13], we get some properties of the function defined by

$$f_*(x) = \Sigma c_n \varphi_*^{(n)}(x).$$

One of them gives another proof of Theorem 2.2 in [5].

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§ 2. Main results. First we establish some properties of $\varphi_*^{(n)}$ or $f_*(x)$.

THEOREM 1. *$\{\varphi_*^{(n)}\}_{n=1}^{\infty}$ forms a multiplicative system but not a strongly multiplicative system (for the terminologies we refer to the book by Alexits* [1]*), more precisely, we have*

$$\text{(i)} \quad \int_0^1 \varphi_*^{(n_1)} \dots \varphi_*^{(n_r)} \, dx = 0 \quad \textit{for} \quad n_1 < \dots < n_r,$$

$$\text{(ii)} \quad \int_0^1 |\varphi_*^{(n)}|^2 \, dx = 1/12,$$

$$\text{(iii)} \quad \int_0^1 |\varphi_*^{(n)}|^2 \varphi_*^{(n_1)} \dots \varphi_*^{(n_r)} \, dx = \begin{cases} (-1)^r 2^{-2(n_r - n + 1)}/3 & \textit{if} \quad n < n_1 < \dots < n_r, \ \textit{and} \\ 0 & \textit{if} \quad n \geqq \min(n_1, \dots, n_r), \end{cases}$$

$$\text{(iv)} \quad 1/24 \leqq \int_0^1 |\varphi_*^{(n)}|^2 \prod_{k=1}^{p} (1 + \eta_k \varphi_*^{(n+k)}) dx \leqq 1/8,$$

where $\eta_k = 1$ or -1, and

(v) *if a Borel set E has positive Lebesgue measure, then there exists a positive constant c such that*

$$\int_E |\varphi_*^{(n)}|^2 \, dx \geqq c$$

holds for all n,

$$\text{(vi)} \quad \Big\{M_n = \sum_{k=n}^{\infty} c_k \varphi_*^{(k)}\Big\}, \quad \textit{where} \quad \{c_k\} \in l_2, \quad \textit{form a reversed martingale.}$$

By virtue of Theorem 1, some convergence properties follow immediately from [3], [10] and [12]. We state them as Theorem A for later convenience.

THEOREM A. (i) *$\{\varphi_*^{(n)}\}$ or its product system forms a convergence system i.e. L^2 expansion implies almost sure convergence.*

(ii) *If there exists a Borel set E with positive Lebesgue measure such that $\Big\{\sum_{n=1}^{m} c_n \varphi_*^{(n)}(x)\Big\}_{m=1}^{\infty}$ is uniformly bounded on E, then $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$.*

(iii) *If there exists an interval I which satisfies one of the following conditions:*

(a) *$\Big\{\sum_{n=1}^{m} c_n \varphi_*^{(n)}(x)\Big\}_{m=1}^{\infty}$ is uniformly bounded from above (or from below) on I,*

(b) *$f_*(x) = \sum c_n \varphi_*^{(n)}(x)$ converges almost surely and essentially bounded from above (or from below) on I, or*

(c) *$f_*(x) = \sum c_n \varphi_*^{(n)}(x)$ converges for all $x \in I$, then $\sum_{n=1}^{\infty} |c_n| < +\infty$.*

REMARK. The proof of (iii) (c) follows from (b) as follows. Set

$$N_k = \Big\{x; \ \forall p > \forall q \geqq k, \quad \Big|\sum_{n=q}^{p} c_n \varphi^{(n)}(x)\Big| \leqq 1\Big\}.$$

Then, it follows by the assumption that

$$I = \bigcup_{k=1}^{\infty} N_k,$$

and N_k is a closed set. So, by Baire's category theorem, at least one of N_k contains an open interval and $f(x)$ is bounded on it. By (b), we get the conclusion.

As a corollary of Theorem A, we have some convergence properties for $\{\varphi^{(n)}\}$, that is,

COROLLARY. *If there exists an interval I which satisfies one of the following conditions:*

(a) $\{\sum_{n=1}^{n} c_n\varphi^{(n)}(x)\}_{n=1}^{\infty}$ *is uniformly bounded on* I,

(b) (Hata—Yamaguti Theorem 2.2 [5]) $f(x)=\sum_{n=1}^{\infty} c_n\varphi^{(n)}(x)$ *converges for all* $x\in I$,

then $\sum_{n=1}^{\infty} |c_n|<+\infty$.

The above Theorem implies that

$$f(x) = \sum_{n=1}^{\infty} c_n\varphi^{(n)}(x)$$

defines a continuous function on [0, 1] if and only if $\{c_n\}\in l^1$.

Next, we will establish some properties of the continuous function $f(x)=$ $= \sum_{n=1}^{\infty} c_n\varphi^{(n)}(x)$.

The derivative of $\varphi^{(n)}(x)$ is $2^n X_n(x)$, where $X_n(x)$ is the Rademacher function which forms independent random variables on ([0, 1], dx). Therefore, we have a slightly different theorem compared to that of [11].

THEOREM 2. *Assume that* $\{c_n\}\in l^1$. *Then,*

(i) $f(x)$ *is absolutely continuous with the derivative*

$$f'(x) = \sum_{n=1}^{\infty} 2^n c_n X_n(x) \quad a.s.$$

if and only if $\{2^n c_n\}\in l^2$.

(ii) $f(x)$ *is differentiable on a set of continuum and the range of the derivative is the whole line but there exists no derivative almost surely if and only if* $\{2^n c_n\}\notin l^2$ *and* $\lim_{n\to\infty} 2^n c_n=0$,

(iii) $f(x)$ *has nowhere finite derivative if and only if* $\overline{\lim}_{n\to\infty} 2^n|c_n|>0$.

In [16], Zygmund introduced smooth functions. A continuous function $f(x)$ defined on (0, 1) is called smooth if

$$f(x+h)+f(x-h)-2f(x) = o(h)$$

holds for all $x\in(0, 1)$.

THEOREM 3. $f(x)=\sum_{n=1}^{\infty} c_n\varphi^{(n)}(x)$ *is smooth in the sense of Zygmund if and only if* $f(x)=ax(1-x)$.

This means that $f(x)$ above has a second derivative if and only if $f(x)=ax(1-x)$ as pointed out in Theorem 3.1 in [5].

The uniform modulus of continuity of the function $f(x)$ is easily obtained:

$$|f(x+h)-f(x)| \leqq |h| \sum_{n=1}^{n(h)} 2^n|c_n| + \sum_{n>n(h)} |c_n|,$$

where $n(h)=[\log_2 1/|h|]$, the integer part of $\log_2 1/|h|$. The more precise uniform or local modulus of continuity like Brownian motion paths are obtained for $f(x)$ with restricted conditions.

THEOREM 4. (The uniform modulus of continuity.) *If there exists* $K>0$ *such that* $2^n|c_n| \leqq K$ *holds for all* n *and* $\{2^n c_n\} \notin l^2$, *then*

(i) $$\overline{\lim_{|x-y|\downarrow 0}} \frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)} = 1, \quad \textit{and}$$

(ii) $$\underline{\lim_{|x-y|\downarrow 0}} \frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)} = -1,$$

where σ_u *is a non-increasing continuous function satisfying the relation*

$$\sigma_u(2^{-p}) = \sum_{n=1}^{p} 2^n|c_n|.$$

In case of the Takagi function i.e. $c_n=2^{-n}$, we have $\sigma_u(h)=\log_2 1/h$.

THEOREM 5. (The local modulus of continuity.) *Under the same conditions as in Theorem* 4, *we have*

(i) $$\overline{\lim_{h\to 0}} \frac{f(x+h)-f(x)}{h\sigma_l(|h|)\sqrt{2\log\log\sigma_l(|h|)}} = 1 \quad a.s.\ (x),$$

and

(ii) $$\underline{\lim_{h\to 0}} \frac{f(x+h)-f(x)}{h\sigma_l(|h|)\sqrt{2\log\log\sigma_l(|h|)}} = -1 \quad a.s.\ (x),$$

where σ_l *is a non-increasing continuous function satisfying the relation*

$$\sigma_l(2^{-p}) = \sqrt{\sum_{n=1}^{p} 2^{2n}|c_n|^2},$$

and a.s. means that it holds except a set of Lebesgue measure 0.

In case of the Takagi function, we have $\sigma_l(h)=\sqrt{\log_2 1/h}$.

§ 3. Proofs of the theorems. We denote a point $x\in[0, 1)$ by the binary series $x=\sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $\varepsilon_k=0$ or 1 and for the uniqueness of the expression we adapt the right hand side for $\sum_{k=q+1}^{\infty} 2^{-k}$ as 2^{-q}. This assumption is sometimes essential. Let $\{X_n(x)\}$ be the Rademacher system of functions, that is,

$$X_n(x) = 1-2\varepsilon_n \quad \text{for} \quad x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}.$$

Using these terminologies we can easily get the expression

$$\varphi_*(x) = -X_1(x) \sum_{k=2}^{\infty} 2^{-k} X_k(x),$$

and taking into account the relation

$$\varphi_*^{(n)}(x) = \varphi_*(2^{n-1}x - [2^{n-1}X]),$$

we have

LEMMA 1. $\varphi_*^{(n)}(x) = -2^{n-1} X_n(x) \sum_{k=n+1}^{\infty} 2^{-k} X_k(x)$.

Now we can give the proof of Theorem 1.

PROOF OF THEOREM 1. We easily get (i) and (ii) from Lemma 1. So, we give the proof of (iii), (iv) and (v). By Lemma 1, for $n<n_1<\ldots<n_r$ we have

$$\int_0^1 |\varphi_*^{(n)}|^2 \varphi_*^{(n_1)} \ldots \varphi_*^{(n_r)}\, dx =$$

$$= 2^{-2(n_1-n)} \int_0^1 X_{n_1} \sum_{k=1}^{\infty} 2^{-k-1-(n_1-n)} X_{n_1+k} \varphi_*^{(n_1)} \ldots \varphi_*^{(n_r)}\, dx =$$

$$= -2^{-2(n_1-n)} \int_0^1 |\varphi_*^{(n_1)}|^2 \varphi_*^{(n_2)} \ldots \varphi_*^{(n_r)}\, dx = (-1)^2 2^{-2(n_2-n)} \int_0^1 |\varphi_*^{(n_2)}|^2 \varphi_*^{(n_3)} \ldots \varphi_*^{(n_r)}\, dx =$$

$$\vdots$$

$$= (-1)^r 2^{-2(n_r-n)} \int_0^1 |\varphi_*^{(n_r)}|^2 dx = (-1)^r 2^{-2(n_r-n)-2}/3.$$

Using this formula, we have

$$\int_0^1 |\varphi_*^{(n)}|^2 \prod_{k=1}^{p} (1+\eta_k \varphi_*^{(n+k)})\, dx = \int_0^1 |\varphi_*^{(n)}|^2 dx +$$

$$+ \sum_{r=1}^{p} \sum_{1\le k_1<\ldots<k_r\le p} \eta_{k_1} \ldots \eta_{k_r} \int_0^1 |\varphi_*^{(n)}|^2 \varphi_*^{(n+k_1)} \ldots \varphi_*^{(n+k_r)} dx \equiv \mathrm{I} + \mathrm{II}.$$

$$\mathrm{II} = \sum_{r=1}^{p} \sum_{1\le k_1<\ldots<k_r\le p} \eta_{k_1} \ldots \eta_{k_r} (-1)^r 2^{-2k_r-2}/3 =$$

$$= -\sum_{k=1}^{p} \eta_k 2^{-2k-2}/3 + \sum_{r=2}^{p} \sum_{k_r=r}^{p} \sum_{1\le k_1<\ldots<k_r} (-1)^r \eta_{k_1} \ldots \eta_{k_r} 2^{-2k_r-2}/3 =$$

$$= -\sum_{k=1}^{p} \eta_k 2^{-2k-2}/3 - \sum_{k=2}^{p} \eta_k \Big\{ \prod_{j=1}^{k-1} (1-\eta_j) - 1 \Big\} 2^{-2k-2}/3 =$$

$$= -\eta_1/48 - \sum_{k=2}^{p} \eta_k \prod_{j=1}^{k-1} (1-\eta_j) 2^{-2k-2}/3 = -\sum_{k=1}^{\min(p,q)} \eta_k 2^{-k-3}/3,$$

where

$$q = \begin{cases} \min\{j;\ \eta_1 = -1, \ldots, \eta_{j-1} = -1, \eta_j = 1\}, \\ p+1 \quad \text{if} \quad \{\ldots\} = \emptyset. \end{cases}$$

Since we have

$$\sum_{k=1}^{p} 2^{-k-3}/3 = 1/24 - 2^{-p-3}/3,$$

it follows that $1/24 \leqq \mathrm{I} + \mathrm{II} \leqq 1/8$.

To show (v), first we point out that the map φ from [0, 1] to [0, 1] preserves Lebesgue measure. Setting

$$I_n = \{x;\ 1/2 - |E|/4 \leqq \varphi^{(n)}(x) \leqq 1/2 + |E|/4\},$$

where $|E|$ denotes the Lebesgue measure of E, we have

$$|I_n| = |E|/2 \quad \text{and} \quad |E \cap I_n^c| \geqq |E|/2.$$

Therefore

$$\int_E |\varphi_*^{(n)}|^2 dx = \int_E |\varphi^{(n)}(x) - 1/2|^2 dx \geqq \int_{E \cap I_n^c} |\varphi^{(n)}(x) - 1/2|^2 dx \geqq 2^{-5}|E|^3 = c > 0.$$

Before starting the proofs of Theorems 2—5, we prepare some lemmas. Set

$$x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k},\ x+h = \sum_{k=1}^{\infty} \varepsilon_k' 2^{-k} \quad \text{and}$$

$$k_0 = \begin{cases} \max\{k;\ \varepsilon_1 = \varepsilon_1', \ldots, \varepsilon_k = \varepsilon_k'\} \\ 0 \quad \text{if} \quad \{\ldots\} = \emptyset. \end{cases}$$

Assume that h satisfies

$$2^{-p-1} < h \leqq 2^{-p}, \quad p = 0,1,2, \ldots .$$

Then, by the definitions we have

LEMMA 2. (i) $0 \leqq k_0 \leqq p$,
(ii) $\varepsilon_{k_0+1} = 0$ *and* $\varepsilon_{k_0+1}' = 1$,
(iii) *if* $k_0 + 2 \leqq p$, *then* $\varepsilon_k' = 0$ *and* $\varepsilon_k = 1$ *for* $k_0 + 2 \leqq k \leqq p$.

By Lemma 1, we have

$$\varphi^{(n)}(x+h) - \varphi^{(n)}(x) = 2^{n-1} \sum_{k=n+1}^{\infty} \big(X_n(x)X_k(x) - X_n(x+h)X_k(x+h)\big)2^{-k}.$$

Using this formula together with Lemma 2, we get

LEMMA 3.

$$f(x+h) - f(x) = h \sum_{n=1}^{k_0} 2^n c_n X_n(x) +$$

$$+ \Big(\sum_{k=p+1}^{\infty} (1 - \varepsilon_k' - \varepsilon_k) 2^{-k}\Big) \sum_{n=k_0+1}^{p} 2^n c_n X_n(x) +$$

$$+ \sum_{n=p+1}^{\infty} c_n 2^{n-1} \sum_{k=n+1}^{\infty} \big(X_n(x)X_k(x) - X_n(x+h)X_k(x+h)\big)2^{-k} = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

LEMMA 4. *If* $2^n|c_n|=O(1)$ *(resp.* $o(1)$*), then* $\mathrm{III}=O(h)$ *(resp.* $o(h)$*) uniformly in* x.

Now we start proving Theorem 2.

PROOF OF THEOREM 2. First, we notice that by the assumption for the expression of x, k_0 goes to infinity when $h\downarrow 0$. This is not always true when $h\uparrow 0$. In this case, we have to adapt the left hand side in the expression $\sum_{k=q+1}^{\infty} 2^{-k}$ as 2^{-q}. Since $f(x)$ is a continuous function under the condition $\{c_n\}\in l^1$, all lemmas are still valid under this convention.

Now assume $\{2^n c_n\}\in l^2$. Since $X_n(x)$ is a sequence of independent identically distributed random variables, $\sum 2^n c_n X_n$ converges almost surely. So, we get

$$\lim_{h\downarrow 0} \sum_{n=k_0+1}^{p} 2^n c_n X_n(x) = 0 \quad \text{a.s.}$$

Combining this with Lemmas 3 and 4, we have

$$\lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} = \sum_{n=1}^{\infty} 2^n c_n X_n(x) \quad \text{a.s.}$$

Since we have the left derivative analogously,

$$f'(x) = \sum_{n=1}^{\infty} 2^n c_n X_n(x) \quad \text{a.s.}$$

Conversely, $\sum 2^n c_n X_n(x)$ converges in L^2, so in L^1, and it follows that

$$\int_0^x f'(t)\,dt = \lim_{p\to\infty} \sum_{n=1}^{p} 2^n c_n \int_0^x X_n(t)\,dt = \sum_{n=1}^{\infty} c_n \varphi^{(n)}(x) = f(x),$$

i.e. $f(x)$ is absolutely continuous. We next assume that $\{2^n c_n\}\notin l^2$ and $\lim_{n\to\infty} 2^n c_n = 0$.

For an irrational $x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, let

$$x+h_m = \sum_{k=1}^{m} \varepsilon_k 2^{-k} + \sum_{k=m+1}^{\infty} (1-\varepsilon_k) 2^{-k},$$

so

$$h_m = \sum_{k=m+1}^{\infty} X_k(x) 2^{-k} \quad (\neq 0).$$

Then we have

$$f(x+h_m)-f(x) = h_m \sum_{n=1}^{m} 2^n c_n X_n(x).$$

Since $\{2^n c_n\}\notin l^2$ and $2^n|c_n|$ is bounded, $S_m = \sum_{n=1}^{\infty} 2^n c_n X_n(x)$ satisfies Kolmogorov's law of iterated logarithm [6]. Therefore $f(x)$ has no finite derivative almost surely.

But S_m can converge conditionally to any finite number, say α, and at this point we have

$$\sum_{k_0=1}^{p} 2^n c_n X_n(x) \sum_{k=p+1}^{\infty} (1-\varepsilon'_k-\varepsilon_k)2^{-k} = o(h),$$

and by Lemma 4,

$$\lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{k_0\to\infty} \sum_{n=1}^{k_0} 2^n c_n X_n(x) = \alpha.$$

At this point, k_0 goes to infinity even if $h\downarrow 0$. So $f'(x)$ exists at this point and equals α. When we assume $\overline{\lim}_{n\to\infty} 2^n|c_n|>0$, using the same terminologies, we have

$$f(x+h_m)-f(x) = h_m \sum_{n=1}^{m} 2^n c_n X_n(x)$$

for all x. Since $\sum_{n=1}^{m} 2^n c_n X_n(x)$ does not converge for all x, it follows that $f(x)$ does not have finite derivative at any point.

PROOF OF THEOREM 3. By Zygmund's theorem [16], a smooth function is differentiable on a set of continuum. It follows by Theorem 3 that $\lim_{n\to\infty} 2^n c_n = 0$. In Lemma 4, setting $x=2^{-m}$ and $h=2^{-p}$, we have

$$f(x+h)+f(x-h)-2h(x) = -2h\big(2^m c_m - \sum_{n=m+1}^{p} 2^n c_n\big)+o(h).$$

By smoothness, we have

$$2^m c_m = \sum_{n=m+1}^{\infty} 2^n c_n, \quad m = 1, 2, \ldots$$

which implies $c_m = a4^{-m}$, and $f(x)=ax(1-x)$.

PROOF OF THEOREM 4. Since we have

$$|f(x+h)-f(x)| \leqq |h| \sum_{n=1}^{n(h)} 2^n|c_n| + \sum_{n>n(h)} |c_n| \leqq |h| \sum_{n=1}^{p} 2^n|c_n| + O(h),$$

where $n(h)=[\log_2 1/|h|]$ and $2^{-p-1}<|h|\leqq 2^{-p}$, we can easily get the upper bound. To derive the lower bound, we remark first that $f(x)$ is symmetric with respect to the line $x=1/2$, so if necessary considering the function $-f(x)$, we can assume that $c_n\geqq 0$ for infinitely many n.

Now look at a point

$$x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}, \quad \varepsilon_k = \begin{cases} 0 & \text{if} \quad c_n \geqq 0 \\ 1 & \text{if} \quad c_n < 0. \end{cases}$$

Setting $x_m = \sum_{k=1}^{m} \varepsilon_k 2^{-k}$ and $h_m = 2^{-m-1}$, we have

$$f(x_m+h_m)-f(x_m) = h_m \sum_{n-1}^{m+1} 2^n|c_n| + O(h_m) = h_m \sigma_u(h_m) + O(h_m).$$

This leads to

$$\overline{\lim_{|x-y|\downarrow 0}} \frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)} \geqq 1.$$

PROOF OF THEOREM 5. Since we have

$$0 \leqq \sum_{k=p+1}^{\infty} (1-\varepsilon_k'-\varepsilon_k)2^{-k} \leqq h,$$

it follows by Lemmas 3 and 4 that

$$|f(x+h)-f(x)| \leqq h \max\left(|S_p(x)|,\ |S_{k_0}(x)|\right) + O(h),$$

where $S_m(x)=\sum_{k=1}^{m} 2^n c_n X_n(x)$. Since Kolmogorov's law of iterated logarithm [9] tells us

$$\overline{\lim_{m\to\infty}} \frac{S_m(x)}{\sqrt{2s_m \log\log s_m}} = 1 \quad \text{a.s.},$$

where $s_m=\sum_{n=1}^{m} 2^{2n}c_n^2$, it follows the upper bound. To get the lower bound, again taking into account the law of iterated logarithm, for almost every $x=\sum_{k=1}^{\infty}\varepsilon_k 2^{-k}$ there exists a subsequence $i_1<i_2<\ldots\to+\infty$ such that

$$\lim_{m\to\infty} \frac{S_{i_m}(x)}{\sqrt{2s_{i_m}\log\log s_{i_m}}} = 1.$$

If $\varepsilon_{i_m}=0$, setting

$$x+h_m=\sum_{k=1}^{i_m-1}\varepsilon_k 2^{-k}+2^{-i_m}+2^{-i_m-1},$$

we have $2^{-i_m-1}<h_m<2^{-i_m+1}$, and by Lemma 4

$$f(x+h_m)-f(x)=h_m S_{i_m-1}+O(h_m).$$

If $\varepsilon_{i_m}=1$, setting $h_m=2^{-i_m}$, we have

$$f(x-h_m)-f(x)=-h_m S_{i_m-1}+O(h_m).$$

Since at least one of the above cases happens infinitely often, we conclude that

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h\sigma_l(|h|)\sqrt{2\log\log\sigma_l(|h|)}} \geqq 1.$$

The second part of Theorem 5 is a consequence of the symmetry of f with respect to the line $x=1/2$.

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ИССЛЕДОВАНИЕ СХОДИМОСТИ РАСШИРЕННОГО ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА КРЫЛОВА—ШТАЕРМАНА

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1. Пусть

$$x_k^{(n)} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, 2, \ldots, n, \ n = 1, 2, \ldots \tag{1}$$

и C — множество всех функций $f(x)$, непрерывных в $[-1, 1]$. Обозначим через $H_n(f, x)$ многочлен степени $2n-1$, однозначно определяемый из условий

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad H_n'(f, x_k^{(n)}) = 0, \quad k = 1, 2, \ldots, n,$$

где $f \in C$. Как известно, процесс $\{H_n(f, x)\}$ называется интерполяционным процессом Эрмита—Фейера. Л. Фейер [1] доказал, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение $H_n(f, x) \to f(x)$, $n \to \infty$. Н. М. Крылов и И. Я. Штаерман [2] удлинили процесс Эрмита—Фейера в том смысле, что они заменили полином $H_n(f, x)$ степени $2n-1$ на полином $P_n(f, x)$ степени $4n-1$, однозначно определяемый из условий:

$$P_n(f, x_k^{(n)}) = f(x_k^{(n)}), \quad P_n^{(i)}(f, x_k^{(n)}) = 0, \quad i = 1, 2, 3, \quad k = 1, 2, \ldots, n,$$

где $P_n^{(i)}(f, x)$ — производная порядка i от $P_n(f, x)$. Н. М. Крылов и И. Я. Штаерман [2] доказали, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение $P_n(f, x) \to f(x)$, $n \to \infty$. В 1965 г. появилась статья автора [3], в которой изучался процесс $\{H_n(f, x)\}$ для узлов

$$x_0^{(n+2)} = 1, \quad x_k^{(n+2)} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n, \quad x_{n+1}^{(n+2)} = -1, \quad n = 1, 2, \ldots. \tag{2}$$

Оказалось, что этот процесс, построенный для $f(x) = |x|$, расходится при $x = 0$. Позже [4—6] было доказано, что процесс $\{H_n(f, x)\}$, построенный при узлах (2) для $f(x) = x^2$ и $f(x) = |x|$, расходится всюду в $(-1, 1)$. При $f(x) = x$ этот процесс расходится во всех точках $x \neq 0$ из $(-1, 1)$. Л. Кук и Т. М. Миллс [7] доказали, что процесс $\{P_n(f, x)\}$, построенный при узлах (2) для $f(x) = (1-x^2)^3$ расходится при $x = 0$. Естественно встал вопрос о построении всюду расходящегося в $[-1, 1]$ процесса Крылова—Штаермана. Этот вопрос решен в [10]. Полином $P_n(f, x)$ имеет довольно сложный вид. Для упрощения вычисленый мы строим процесс Крылова—Штаермана при узлах[1]

$$x_k^{(2t)} = \cos\frac{(2k-1)\pi}{4t}, \quad k = 1, 2, \ldots, 2t; \quad x = 0, t = 1, 2, \ldots. \tag{3}$$

[1] Матрица узлов (3) рассматривалась в [8] и [11] для процессов Эрмита—Фейера и Егервари—Турана.

При этом расширенный полином $\Pi_n(f, x)$ Крылова—Штаермана степени $8t+3$ однозначно определяется из условий

$$\Pi_n(f, x_k^{(2t)}) = f(x_k^{(2t)}), \quad \Pi_n^{(i)}(f, x_k^{(2t)}) = 0, \quad i = 1, 2, 3, \; k = 1, 2, \ldots, n,$$

$$\Pi_n(f, 0) = f(0), \quad \Pi_n^{(i)}(f, 0) = 0, \quad i = 1, 2, 3, \; n = 2t.$$

2. В [10] доказана следующая теорема

Теорема 1. *Процесс* $\{\Pi_n(f, x)\}$, *построенный при узлах* (3) *для* $f(x)=x^2$, *расходится во всех точках* $x \neq 0$ *из* $[-1, 1]$.

В этой работе продолжается исследование интерполяционного процесса $\{\Pi_n(f, x)\}$.

Теорема 2. *Расширенный интерполяционний процесс* $\{\Pi_n(f, x)\}$ *Крылова—Штаермана, построенный при узлах* (3) *для* $f(x)=x^r$, $r \geqq 4$, *сходится равномерно в* $[-1, 1]$.

Доказательство. Мы рассмотрим лишь случай нечетного r, $r=2m+1$.[2] При четном r доказательство проще. Рассмотрим разность

$$R_n(f, x) = \Pi_n(f, x) - P_n(f, x). \tag{4}$$

Из определения полиномов $\Pi_n(f, x)$ и $P_n(f, x)$ следует, что точки $\{x_k^{(n)}\}_{k=1}^n$ являются корнями четвертой кратности $R_n(f, x)$. Учтем еще, что узлы $\{x_k^{(n)}\}_{k=1}^n$ расположены симметрично относительно точки $x=0$ и что $f(x)=x^{2m+1}$ — нечетная функция. Поэтому

$$R_n(f, x) = T_n^4(x)\, x(a+bx^2), \quad T_n(x) = \cos n \arccos x, \tag{5}$$

где a и b определяются из равенств

$$[T_n^4(x)\, x(a+bx^2)]'_{x=0} = -P_n'(f, 0), \quad [T_n^4(x)\, x(a+bx^2)]^{(3)}_{x=0} = -P_n^{(3)}(f, 0). \tag{6}$$

С помощью дифференциального уравнения для $T_n(x)$ из (6) получим

$$a = -P_n'(f, 0), \quad b = -2n^2 P_n'(f, 0) - \frac{P_n^{(3)}(f, 0)}{6}. \tag{7}$$

Найдем теперь a и b для $f(x)=x^{2m+1}$. Г. Н. Ладен [9] доказал,[1] что при любой системе узлов $\{x_k\}_{k=1}^n$ интерполяционный полином Крылова—Штаермана $P_n(f, x)$ может быть вычислен по формуле

$$P_n(f, x) = \sum_{k=1}^{n} f(x_k)\, U_k(x)\, [l_k(x)]^4, \tag{8}$$

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[2] Случай четного r рассмотрен в [14].

[3] *Примечание при корректуре (26. 10. 1987.)*. Формулы (8)—(9) не принадлежат Ладену. Они находятся в [2].

где

$$(9)\qquad U_k(x) = 1-2(x-x_k)\frac{\omega''(x_k)}{\omega'(x_k)}+\frac{(x-x_k)^2}{2}\left[5\left(\frac{\omega''(x_k)}{\omega'(x_k)}\right)^2-\frac{4\omega^{(3)}(x_k)}{3\omega'(x_k)}\right]+$$

$$+\frac{(x-x_k)^3}{6}\left[-15\left(\frac{\omega''(x_k)}{\omega'(x_k)}\right)^3+10\,\frac{\omega''(x_k)\omega^{(3)}(x_k)}{(\omega'(x_k))^2}-\frac{\omega^{(4)}(x_k)}{\omega'(x_k)}\right],$$

$$(10)\qquad l_k(x) = \frac{\omega(x)}{(x-x_k)\omega'(x_k)},$$

$$\omega_n(x) = \omega(x) = \prod_{k=1}^{n}(x-x_k).$$

Из (9) с помощью дифференциального уравнения для $T_n(x)$, после элементарных вычислений, получим, что

$$(11)\qquad U_k(0) = 1+\frac{2(n^2+2)x_k^2}{3(1-x_k^2)}+\frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2};$$

$$(12)\qquad U_k'(0) = -\frac{2(2n^2+1)x_k}{3(1-x_k^2)}-\frac{(4n^2+1)x_k^3}{2(1-x_k^2)^2};$$

$$(13)\qquad U_k''(0) = \frac{4(n^2-1)}{3(1-x_k^2)}+\frac{4n^2x_k^2}{(1-x_k^2)^2};$$

$$(14)\qquad U_k'''(0) = -\frac{(4n^2-1)x_k}{(1-x_k^2)^2}.$$

При узлах (1)

$$(15)\qquad l_k(0) = (-1)^{k+t}\sqrt{1-x_k^2}/nx_k;$$

$$(16)\qquad l_k'(0) = (-1)^{k+t}\sqrt{1-x_k^2}/nx_k^2;$$

$$(17)\qquad l_k''(0) = (-1)^{k+t+1}(n^2x_k^2-2)\sqrt{1-x_k^2}/nx_k^3.$$

Из (8) следует, что

$$P_n'(z^{2m+1}, 0) = \sum_{k=1}^{n} x_k^{2m+1}\left(U_k'(0)l_k^4(0)+4U_k(0)l_k^3(0)l_k'(0)\right)$$

Отсюда и из (15), (16) получим, что

$$(18)\qquad P_n'(z^{2m+1}, 0) = \sum_{k=1}^{n} x_k^{2m+1}U_k'(0)\frac{(1-x_k^2)^2}{n^4x_k^4}+4\sum_{k=1}^{n}x_k^{2m+1}U_k(0)\frac{(1-x_k^2)^2}{n^4x_k^5} \equiv$$

$$\equiv \Sigma_1+\Sigma_2.$$

Из (11) и (12) вытекает, что

$$\sum_1 = -\frac{2(2n^2+1)}{3n^4}\sum_{k=1}^{n} x_k^{2m-2}(1-x_k^2)-\frac{4n^2+1}{2n^4}\sum_{k=1}^{n} x_k^{2m}; \tag{19}$$

$$\sum_2 = \frac{4}{n^4}\sum_{k=1}^{n} x_k^{2m-4}\left(1+\frac{2(n^2+2)x_k^2}{3(1-x_k^2)}+\frac{(2n^2+1)x_k^4}{3(1-x_k^2)^2}\right)(1-x_k^2)^2. \tag{20}$$

Введем обозначения $\sum_{k=1}^{n} x_k^{2m-2}=\theta_1 n$, $\sum_{k=1}^{n} x_k^{2m}=\theta_2 n$. Тогда из (19), (20) выводим, что

$$\sum_1 = -\frac{4}{3}(\theta_1-\theta_2)-\frac{2\theta_2}{n}+O\left(\frac{1}{n^3}\right); \tag{21}$$

$$\sum_2 = \frac{8(\theta_1-\theta_2)}{3n}+\frac{8\theta_2}{3n}+O\left(\frac{1}{n^3}\right). \tag{22}$$

В силу (18), (21—22), получим

$$a = -P_n'(z^{2m+1}, 0) = \frac{2\theta_2}{3n}-\frac{4\theta_1}{3n}. \tag{23}$$

Найдем теперь b. С помощью формулы Лейбница из (8) вытакает

$$P_n^{(3)}(z^{2m+1}, x) = \sum_{k=1}^{n} x_k^{2m+1}U_k^{(3)}(x)\big(l_k(x)\big)^4+3\sum_{k=1}^{n} x_k^{2m+1}U_k^{(2)}(x)\big(l_k^4(x)\big)'+ \tag{24}$$

$$+3\sum_{k=1}^{n} x_k^{2m+1}U_k'(x)\big(l_k^4(x)\big)''+\sum_{k=1}^{n} x_k^{2m+1}U_k(x)\big(l_k(x)\big)^{(3)} \equiv \sum_{i=1}^{4} S_i.$$

Из (14) и (15) выводим, что

$$S_1 = -\frac{4n^2-1}{n^4}\sum_{k=1}^{n} x_k^{2m-2}. \tag{25}$$

Отсюда следует

$$S_1 = O\left(\frac{1}{n}\right). \tag{26}$$

Найдем S_2. Для этого заметим, что

$$\big(l_k^4(x)\big)' = 4l_k^3(x)l_k'(x). \tag{27}$$

Поэтому из (13) и (16) вытекает

$$S_2 = \frac{12}{n^4}\left(4n^2\sum_{k=1}^{n} x_k^{2m-2}+\frac{4(n^2-1)}{3}\sum_{k=1}^{n} x_k^{2m-4}(1-x_k^2)\right). \tag{28}$$

Поскольку $m\geqq 2$, то из (28) следует, что

$$S_2 = O\left(\frac{1}{n}\right). \tag{29}$$

Переходим к S_3. Из (27) получаем $(l_k(x))''=12l_k^2(x)(l_k'(x))^2+4l_k^3(x)l_k''(x)$. Отсюда и из (24), (12) выводим, что

$$S_3 = l_1+l_2+l_3+l_4, \tag{30}$$

где

$$l_1 = -\frac{24(2n^2+1)}{n^4}\sum_{k=1}^{n} x_k^{2m-2}(1-x_k^2);\ l_2 = -\frac{18(4n^2+1)}{n^4}\sum_{k=1}^{n} x_k^{2m-2}, \tag{31}$$

$$l_3 = \frac{8(2n^2+1)}{n^4}\Big(n^2\sum_{k=1}^{n} x_k^{2m-2}(1-x_k^2)+2\sum_{k=1}^{n} x_k^{2m-4}(x_k^2-1)\Big); \tag{32}$$

$$l_4 = \frac{6(4n^2+1)}{n^4}\sum_{k=1}^{n} x_k^{2m-2}(n^2x_k^2-2).$$

Очевидно, что $l_1=O\left(\frac{1}{n}\right)$, $l_2=O\left(\frac{1}{n}\right)$. Кроме того, легко получить, что

$$l_3 = 16n\theta_1-16n\theta_2+O\left(\frac{1}{n}\right),\quad l_4 = 24n\theta_2+O\left(\frac{1}{n}\right).$$

Стало быть, из (30) получаем

$$S_3 = 16n\theta_1+8n\theta_2+O\left(\frac{1}{n}\right). \tag{33}$$

Из (10), после простых вычислений, найдем

$$l_k'''(0) = 3(-1)^{k+t+1}(n^2x_k^2-2)\sqrt{1-x_k^2}/nx_k^4.$$

Отсюда и из (11), (24) вытекает

$$S_4 = \frac{24}{n^4}\Bigg(\sum_{k=1}^{n} x_k^{2m-6}(1-x_k^2)+\frac{2(n^2+2)}{3}\sum_{k=1}^{n} x_k^{2m-4}(1-x_k^2)+\frac{2n^2+1}{3}\sum_{k=1}^{n} x_k^{2m-2}- \tag{34}$$

$$-2\sum_{k=1}^{n} x_k^{2m-6}(1-x_k^2)^2(n^2x_k^2-2)-\frac{4(n^2+2)}{3}\sum_{k=1}^{n} x_k^{2m-4}(1-x_k^2)(n^2x_k^2-2)-\frac{2(2n^2+1)}{3}\times$$

$$\times\sum_{k=1}^{n} x_k^{2m-2}(n^2x_k^2-2)\Bigg).$$

Поэтому

$$S_4 = -32n\theta_1+O\left(\frac{1}{n}\right). \tag{35}$$

Из (24), (26), (29), (33), (35) выводим

$$P_n^{(3)}(z^{2m+1},0) = -16n\theta_1+8n\theta_2+O\left(\frac{1}{n}\right). \tag{36}$$

Из (7) и (36) следует, что

$$b = O\left(\frac{1}{n}\right). \tag{37}$$

В силу (5), (37) имеем

$$\|\Pi_n(f,x)-P_n(f,x)\| \leqq \frac{c}{n}, \quad f(x)=x^{2m+1}, \quad \|f\| = \max_{-1\leqq x\leqq 1}|f(x)|, \quad m \geqq 2.$$

Поэтому

$$\|\Pi_n(f,x)-f(x)\| \leqq \|P_n(f,x)-f(x)\| + \frac{c}{n}.$$

Согласно теореме Крылова—Штаермана $\lim_{n\to\infty}\|P_n(f,x)-f(x)\|=0$. Поэтому

$$\lim_{n\to\infty}\|\Pi_n(f,x)-f(x)\| = 0, \quad f(x)=x^{2m+1}, \quad m\geqq 2.$$

В связи с теоремой 2 интересна

Теорема 3. *Расширенный интерполяционный процесс* $\{\Pi_n(f,x)\}$ *Крылова—Штаермана, построенный при узлах* (3) *для* $f(x)=x^3$, *расходится во всех точках* $x\neq 0$ *из* $[-1,1]$.

Доказательство. Функция $f(x)=x^3$ — частный случай функции $f(x)=x^{2m+1}$, когда $m=1$. Поэтому можно пользоваться формулами из доказательства теоремы 2 с учетом, что $m=1$. Из формул (18—20) получим

$$a = -P_n(z^3,0) = -\frac{1}{n}-\frac{4}{n^2}+O\left(\frac{1}{n^3}\right). \tag{38}$$

Далее, из (25) видно, что

$$S_1 = O\left(\frac{1}{n}\right). \tag{39}$$

Из (28) следует, что

$$S_2 = 16+O\left(\frac{1}{n}\right). \tag{40}$$

Из (31) вытекает, что при $m=1$, $l_1=O\left(\frac{1}{n}\right)$, $l_2=O\left(\frac{1}{n}\right)$. Из (32) имеем $l_3=8n-32+O\left(\frac{1}{n}\right)$, $l_4=12n+O\left(\frac{1}{n}\right)$. Поэтому

$$S_3 = 20n-32+O\left(\frac{1}{n}\right). \tag{41}$$

Найдем S_4. Из (34) получим, что при $m=1$

$$S_4 = \frac{24}{n^4}\left(\sum_{n=1}^{n}\frac{1}{x_k^4}-2\sum_{k=1}^{n}\frac{1}{x_k^2}+\frac{2(n^2+2)}{3}\sum_{k=1}^{n}\left(\frac{1}{x_k^2}-1\right)-\frac{2n^4}{3}-\right. \tag{42}$$

$$\left.-\frac{4(n^2+2)}{3}\sum_{k=1}^{n}\left(n^2-\frac{2}{x_k^2}-n^2x_k^2+2\right)-\frac{2n^5}{3}\right)+O\left(\frac{1}{n}\right).$$

Воспользуемся теперь тождествами [12], [10]

$$\sum_{k=1}^{n} \frac{1}{x_k^2} = n^2, \quad \sum_{k=1}^{n} \frac{1}{x_k^4} = \frac{n^4+2n^2}{3}. \tag{43}$$

Тогда из (42) получим

$$S_4 = 72 - 32n + O\left(\frac{1}{n}\right). \tag{44}$$

Из (39—41), (44), (24) выводим

$$P_k^{(3)}(z^3, 0) = -12n + 56 + O\left(\frac{1}{n}\right). \tag{45}$$

Из (7), (38), (45) найдем

$$b = -17\frac{1}{3} + O\left(\frac{1}{n}\right). \tag{46}$$

В [5] доказано, что для любого $x\in[-1, 1]$ существует такая последовательность четных натуральных чисел $\{n_k\}$, что $\lim_{k\to\infty} T_{n_k}^2(x)=1$. Отсюда и из (5), (38), (46) вытекает, что при любом $x\in[-1, 1]$

$$\lim_{k\to\infty} R_{n_k}(f, x) = -\frac{52}{3}x^3, \quad f(x) = x^3. \tag{47}$$

Согласно теореме Крылова—Штаермана при $f(x)=x^3$

$$\lim_{n\to\infty} P_n(f, x) = x^3. \tag{48}$$

Из (4) и (47—48) следует, что всюду в $[-1, 1]$

$$\lim_{k\to\infty} \Pi_{n_k}(z^3, x) = -\frac{49}{3}x^3.$$

Стало быть, процесс $\{\Pi_n(z^3, x)\}$ расходится во всех точках $x\neq 0$ из $[-1, 1]$. Сходимость процесса $\{\Pi_n(z^3, x)\}$ при $x=0$ следует непосредственно из определения полиномов $\Pi_n(f, x)$.

Замечание. Любопытно, что процесс Эрмита—Фейера, построенный при узлах (3) для $f(x)=x^3$, сходится равномерно в $[-1, 1]$. Это утверждение непосредственно вытекает из следствия 2 статьи [10].

Теорема 4. *Расширенный интерполяционный процесс* $\{\Pi_n(f, x)\}$ *Крылова—Штаермана, построенный при узлах* (3) *для* $f(x)=x$, *расходится во всех точках* $x\neq 0$ *из* $[-1, 1]$.

Доказательство. Функция $f(x)=x$ получается из функции $f(x)=x^{2m+1}$ когда $m=0$. Поэтому будем пользоваться формулами из доказательства теоремы 2 с учетом, что $m=0$. Из (19) получим, что при $m=0$

$$\Sigma_1 = -\frac{2(2n^2+1)}{3n^4}\sum_{k=1}^{n}\left(\frac{1}{x_k^2}-1\right) - \frac{4n^2+1}{2n^3} = -\frac{4}{3} - \frac{2}{3n} - \frac{2}{3n^2} + O\left(\frac{1}{n^3}\right). \tag{49}$$

Из (20) получим, что при $m=0$

$$\sum_2 = -\frac{4}{n^4}\left(\sum_{k=1}^{n}\frac{1}{x_k^4}+\frac{2(n^2+2)}{3}\sum_{k=1}^{n}\frac{1-x_k^2}{x_k^2}+\frac{n(2n^2+1)}{3}\right). \tag{50}$$

Отсюда с помощью тождеств (43) выведем

$$\sum_2 = 4-\frac{8}{3n}+O\left(\frac{1}{n^3}\right). \tag{51}$$

Из (18), (50), (51) получим

$$a = -P_n'(z,0) = -\frac{8}{3}+\frac{10}{3n}+\frac{2}{3n^2}+O\left(\frac{1}{n^3}\right). \tag{52}$$

Из (25) и (43) следует, что

$$S_1 = -4+O\left(\frac{1}{n^2}\right). \tag{53}$$

С помощью тождеств (43) из (28) выводим, что при $m=0$

$$S_2 = \frac{112}{3}+\frac{16n^2}{3}+\frac{16}{3n^2}. \tag{54}$$

Из (31) при $m=0$ получим

$$l_1 = -\frac{24}{n^4}(2n^2+1)\sum_{k=1}^{n}(x_k^{-2}-1) = -48+O\left(\frac{1}{n}\right). \tag{55}$$

Таким же образом

$$l_2 = -72+O\left(\frac{1}{n^2}\right). \tag{56}$$

Из (32) следует

$$l_3 = \frac{16n^2}{3}-16n+\frac{40}{3}+O\left(\frac{1}{n}\right). \tag{57}$$

Из (32) получим

$$l_4 = 24n-48+O\left(\frac{1}{n}\right). \tag{58}$$

Из (55—58) следует

$$S_3 = \sum_{i=1}^{4} l_i = \frac{16n^2}{3}+8n-\frac{464}{3}+O\left(\frac{1}{n}\right). \tag{59}$$

Из (34) при $m=0$ найдем

$$S_4 = \frac{24}{n^4}\left(\sum_{k=1}^{n}\left(\frac{1}{x_k^6}-\frac{2}{x_k^4}+\frac{1}{x_k^2}\right)+\right.$$

$$+\frac{2(n^2+2)}{3}\sum_{k=1}^{n}\left(\frac{1}{x_k^4}-\frac{1}{x_k^2}\right)+\frac{n^2(2n^2+1)}{3}-2\sum_{k=1}^{n}\frac{(1-x_k^2)^2}{x_k^6}\times$$

$$\left.\times(n^2x_k^2-2)-\frac{4(n^2+2)}{3}\sum_{k=1}^{n}\frac{(1-x_k^2)(n^2x_k^2-2)}{x_k^4}-\frac{2(2n^2+1)}{3}\sum_{k=1}^{n}\frac{n^2x_k^2-2}{x_k^2}\right).$$

Отсюда с помощью тождеств (43) и тождества [13]

$$\sum_{k=1}^{n} \frac{1}{x_k^6} = \frac{n^2(2n^4+5n^2+8)}{15},$$

после несложных преобразований, получим

$$S_4 = -\frac{16n^2}{3} + \frac{368}{3} + O\left(\frac{1}{n}\right). \tag{60}$$

Из (24), (53—54), (59—60) получим

$$P_n^{(3)}(z, 0) = \frac{16n^2}{3} + 8n + \frac{4}{3} + O\left(\frac{1}{n}\right). \tag{61}$$

Из (6), (52), (61) следует

$$b = -\frac{56n^2}{9} + \frac{16n}{3} + \frac{10}{9} + O\left(\frac{1}{n}\right). \tag{62}$$

Из (52) и (62) ясно, что при $n \to \infty$

$$a \to -\frac{8}{3}, \quad b \to -\infty. \tag{63}$$

Повторяя рассуждение, которое было использовано при окончании доказательства теоремы 3, с помощью (5), (63) получим, что для любого $x \neq 0$ из $[-1, 1]$ существует такая последовательность натуральных чисел $\{n_k\}$, что $\lim_{k\to\infty} \Pi_{n_k}(z, x) = -\infty$ для $0 < x \leqq 1$ и $\lim_{k\to\infty} \Pi_{n_k}(z, x) = \infty$ для $-1 \leqq x < 0$.

В этой работе рассматривается процесс Крылова—Штаермана при узлах Чебышева. Хорошо известно, что полиномы Чебышева — частный случай полиномов Якоби $P^{(\alpha,\beta)}(x)$ когда $\alpha = \beta = -\frac{1}{2}$. Вероятно, было бы интересно рассматривать расширенный интерполяционный процесс Крылова—Штаермана при корнях полиномов Якоби.

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UNE CONTRIBUTION À LA STABILITÉ DES ÉQUATIONS DIFFÉRENTIELLES ORDINAIRES DANS LES ESPACES LOCALEMENT CONVEXES

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1. Introduction et définitions

Soit E un espace vectoriel topologique complèt localement convexe du type de Hausdorff. On dénote chaque voisinage convexe, balancé et absorbant U de l'origine de E, par $\|\cdot\|_U$ la fonctionelle de Minkowski qui correspond à U, c'est-à-dire que $\|x\|_U=\inf\{\lambda: \lambda>0,\ x\in\lambda U\}$.

Il est évident que $\{x: \|x\|_U<1\}\subset U\subset\{x: \|x\|_U\leqq 1\}$.

Soit $\mathcal{U}$ un système fondamental de voisinages de l'origine 0, alors $\{\|\cdot\|_U: U\in\mathcal{U}\}$ est une famille des semi-normes de E qui engendre la topologie localement convexe de E.

B étant un ensemble borné de E, soit $M_E(B)$ la famille de tous les voisinages $U\in\mathcal{U}$ telle que $B\subset K+U$ pour un certain sous-ensemble totalement borné K de E.

DÉFINITION [1]. Considérons la fonction $f: J\times\Omega\to E$, $J=[0,\infty)$, $\Omega\subset E$. On dit que la fonction f est une b-condensation si pour chaque ensemble compact $I\subset J$ et un ensemble borné $B\subset\Omega$ (B n'étant-pas totalement borné), $M_E(B)$ est un sous-ensemble propre de $M_E(f(I\times B))$.

On doit à Reich [4] le lemme suivant.

LEMME 1. *Soit K un sous-ensemble non-vide semi-complet convexe d'un espace vectoriel topologique convexe E. Si une b-condensation continue $f: K\to K$ a une image bornée, alors elle a un point fixe.*

On considère l'équation différentielle

$$\dot{x}=f(t,x),\quad x(t_0)=x_0 \tag{1}$$

où $f\in C(J\times E, E)$ et dans chaque ensemble borné $\Delta\in J\times E$, $f(\Delta)$ est borné. En utilisant le lemme 1 nous obtenons le lemme suivant dû à S. B. Agase [1] et à V. M. Millionscikov [3] concernant l'existence de la solution de (1).

LEMME 2. *Soit la fonction continue $f: J\times E\to E$ telle que dans chaque ensemble borné $\Delta\subset J\times E$, $f(\Delta)$ est borné. On définit l'opérateur $T: C(J,E)\to C(J,E)$ par*

$$T(x)(t)=x_0+\int_t^t f(s,x(s))\,ds,\quad t_0\in J,\quad x_0\in E.$$

Alors,

(i) *T est continu et a une image bornée, et*

(ii) *si la fonction f est une b-condensation, alors T est une b-condensation de $C(I,E)$, où $I=[t_0, t_0+a]$, $0<a\leqq 1$.*

L' intégral qui figure dans le Lemme 2 est défine comme dans [3].

2. Théorèmes de la stabilité

THÉORÈME 1. *Soit* $f\in C(J\times E, E)$ *où* $f(\Delta)$ *est borné et une b-condensation dans chaque* $\Delta\in J\times E$ *borné et soit de plus pour* $(t, x)\in J\times E$,

$$\|f(t, x)\|_U \leqq g(t, \|x\|_U), \quad \forall U\in\mathscr{U}, \tag{2}$$

où $g\in C(J\times R^+, R^+)$. *Soit* $r(t)=r(t, t_0, u_0)$ *la solution maximale (voir* [2]*) de l'équation différentielle scalaire*

$$\dot{u} = g(t, u), \quad u(t_0) = u_0 \geqq 0, \tag{3}$$

qui est définie pour $t\geqq t_0$. *Si* $x(t)=x(t, t_0, x_0)$ *est une solution de* (1) *définie pour* $t\geqq t_0$ *telle que*

$$\|x(t_0)\|_U \leqq u_0, \quad \forall U\in\mathscr{U},$$

alors

$$\|x(t)\|_U \leqq r(t), \quad t\geqq t_0, \quad \forall U\in\mathscr{U}.$$

DÉMONSTRATION. Soit $x(t)$ une solution de (1) (Lemme 2) définie pour $t\geqq t_0$ et telle que $\|x(t_0)\|_U\leqq u_0$, $\forall U\in\mathscr{U}$. — On voit que si $m(t)=\|x(t)\|_U$, alors $m(t_0)\leqq u_0$. Pour $h>0$ suffisamment petit on a

$$m(t+h)-m(t) = \|x(t+h)\|_U - \|x(t)\|_U = \big\|x(t)+hf\big(t, x(t)\big)+\varepsilon(h)\big\|_U - \|x(t)\|_U$$

où $\varepsilon(h)/h\to 0$ quand $h\to 0$.

Comme $\|x\|_U$ est une fonction localement Lipschitzienne par rapport à x (voir [2])

$$D^+\|x(t)\|_U \leqq \|\dot{x}(t)\|_U = \big\|f\big(t, x(t)\big)\big\|_U$$

on obtient, par l'hypothèse (2), l'inégalité

$$D^+m(t) \leqq g\big(t, m(t)\big), \quad t\in J.$$

En utilisant le théorème de comparaison ([2], Théorème 1.4.1), on reçoit le résultat desiré. □

THÉORÈME 2. *Si on suppose que* $g(t, 0)=0$, $f(t, 0)=0$ *dans le Théorème* 1 *et que la solution nulle* $u=0$ *de* $\dot{u}=g(t, u)$ *est stable (asymptotiquement stable) et l'inégalité* (2) *est valable pour* $(t, x)\in J\times U_0$, *où* $U_0\in\mathscr{U}$ *et pour tous* $U\in\mathscr{U}$, *alors la solution nulle de* $\dot{x}=f(t, x)$ *est aussi stable (asymptotiquement stable).*

DÉMONSTRATION. La stabilité de la solution nulle, $u=0$ entraîne que nous pouvons choisir un $\varepsilon>0$ tel que $r(t_0)=\varepsilon$ et $r(t)<1$, $\forall t\geqq t_0$.

On choisit $r(t)$ comme la solution maximale de $\dot{u}=g(t, u)$, $u(t_0)=\varepsilon$. On peut conclure du Théorème 1 que

$$\|x(t)\|_{U_0} \leqq r(t) < 1, \quad \forall t \geqq t_0,$$

donc $x(t)\in U_0$, $\forall t\geqq t_0$.

Pour chaque voisinage $U\in\mathscr{U}$, on considère le voisinage $U'=U_0\cap U$, alors on a

$$\|x(t)\|_{U'} \leqq r(t) < 1,$$

donc $x(t)\in U'\subset U$. Par conséquent, la solution $x=0$ est stable. — Si, en outre, $r(t)\to 0$ quand $t\to\infty$, il s'ensuit que $\|x(t)\|_U\to 0$, $t\to\infty$ pour tous les voisinages $U\in\mathcal{U}$. Ainsi la solution $x=0$ est asymptotiquement stable. □

REMARQUES. a) Le Théorème 2 reste valable quand si l'on suppose au lieu de la stabilité (stabilité asymptotique) de $u=0$, qu'il y a un voisinage $\bar{B}(\varepsilon)$ tel que pour chaque solution de (3) qui parte de ce voisinage, reste dans la boule $B(1)=\{u\colon |u|<1\}$ et en outre tend vers zero quand $t\to\infty$.

b) Les stabilités peuvent-être considerées uniformes.

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ON A TRIGONOMETRIC SUM AND ITS APPLICATION

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1. Introduction

In many cases the distribution of values of multiplicative functions in short intervals has a close connection with the module of

$$(*) \qquad \sum_{n \leqq x^{1/3}} \psi\left(\frac{x}{n^2}\right)$$

where $\psi(u)=u-[u]-1/2$. The main purpose of our paper is to give a good estimation for this sum.

Using the Fourier expansion of $\psi(u)$, we get

$$\sum_{N \leqq n \leqq N'} \psi\left(\frac{x}{n^2}\right) \ll \frac{N}{\Delta} + \sum_{q \leqq \log_2 \Delta} 2^{-q} \Big| \sum_{2^q \leqq m \leqq \beta_q} \sum_{N \leqq n \leqq N'} e^{2\pi i x m/n^2} \Big| +$$
$$+ \sum_{p \leqq \Delta} p^{-2} \Delta^{-1} \Big| \sum_{p\Delta \leqq m \leqq \gamma_p} \sum_{N \leqq n \leqq N'} e^{2\pi i x m/n^2} \Big|,$$

where $1 \leqq N \leqq N' \leqq 2N$, $\Delta > 1$ is arbitrary,

$$\beta_q = \begin{cases} 2^{q+1}, & \text{if} \quad 2^{q+1} \leqq \Delta, \\ \Delta & \text{if} \quad 2^q \leqq \Delta < 2^{q+1}, \end{cases}$$

and

$$\gamma_p = \begin{cases} (p+1)\Delta, & \text{if} \quad p+1 \leqq \Delta, \\ \Delta^2 \quad , & \text{if} \quad p \leqq \Delta < p+1. \end{cases}$$

(see P. G. Schmidt [1]). Let

$$S_1 = S_1(M, N) = \sum_{\substack{M \leqq m \leqq M' \\ M' \leqq 2M}} \sum_{\substack{N \leqq n \leqq N' \\ N' \leqq 2N}} e^{2\pi i x m/n^2}.$$

Hence it is obvious that an estimate for the sum $(*)$ essentially depends on the estimates for sums of type S_1.

NOTATION. $f(x) \ll g(x)$ means that $f(x)=O(|g(x)|)$, $e(f(x))$ means $e^{2\pi i f(x)}$.

2. Preliminaries

LEMMA 1. *Let* $f(x_1, x_2)$ *be real in a region* D *contained in the rectangle* $a_i \leqq x_i \leqq b_i$, $i=1, 2$. *Then*

$$\sum_{(x_1, x_2)\in D} e\big(f(x_1, x_2)\big) \ll \frac{(b_1-a_1)(b_2-a_2)}{q^{1/2}} +$$

$$+\left(\frac{(b_1-a_1)(b_2-a_2)}{q}\right)^{1/2} \Big\{\sum_{h=1}^{q-1} \Big|\sum_{(x_1, x_2)\in D(h)} e\big(f_1(x_1, x_2)\big)\Big|\Big\}^{1/2},$$

where

$$D(h) = \{(x_1, x_2)\in D \,|\, (x_1, x_2)\in D,\ (x_1+h, x_2)\in D\},$$

$$f_1(x_1, x_2) = f(x_1+h, x_2) - f(x_1, x_2).$$

LEMMA 2. *Let* $f(x)$ *be an infinitely many times differentiable function on* $[a, b]$. *Let* H *be a positive number such that*

$$c_1 H \leqq f''(x) \leqq c_2 H, \quad c_1, c_2 > 0.$$

Then

$$\sum_{a\leqq n\leqq b} e\big(f(n)\big) = \sum_{\alpha\leqq v\leqq \beta} |f''(n_v)|^{1/2} e\big(f(n_v) - vn_v\big) +$$

$$+ O(H^{-1/2}) + O\big(\log(\beta-\alpha+2)\big),$$

where $\alpha = \min\big(|f'(a)|, |f'(b)|\big)$, $\beta = \max\big(|f'(a)|, |f'(b)|\big)$, *the variable* v *runs over the integers and* n_v *is defined by* $f'(n_v) = v$.

LEMMA 3. *Let* $f(x)$ *be twice differentiable on* $[a, b]$ *and let* $0 < |f'(x)| \leqq 1$, $c_1 H \leqq f''(x) \leqq c_2 H$. *Then*

$$\sum_{a\leqq n\leqq b} e\big(f(n)\big) \ll 1 + H^{-1/2}.$$

These lemmas are wellknown, see e.g. [1], [2].
We need two definitions.

DEFINITION 1. The real function $g(x_1, \ldots, x_k)$ is said to be an approximation of degree r to the real function $f(x_1, \ldots, x_k)$ in D if f and g possess partial derivatives up to the order r in D and

$$|f_{x_{i_1}\ldots x_{i_m}} - g_{x_{i_1}\ldots x_{i_m}}| < \varepsilon |g_{x_{i_1}\ldots x_{i_m}}|$$

for all $(x_1, \ldots, x_k)\in D$, $m\leqq r$ and for some sufficiently small number $\varepsilon\in[0, 1/2]$. We then write

$$f \underset{D}{\sim}^{r} g.$$

DEFINITION 2. We say that (l_0, l_1) is a pair of exponents of dimension n if $0\leqq l_0$, $l_1 - l_0 \leqq \frac{n+1}{2}$ and if to every set of real numbers $S_1, \ldots, S_k$ such that $S_1, \ldots, S_k \neq 0$,

$$\sum_{i=1}^{k} (\mu+\mu_i) S_i + \mu + \mu' + 1 \neq 0,$$

where μ and μ' are any nonnegative integers and $\mu_i=0$ or 1, there exists an integer $r\geqq k+4$, depending only on S_i such that the inequality

$$\sum_{(x_1,\ldots,x_k)\in D} e\big(f(x_1,\ldots,x_k)\big) \ll \prod_{i=1}^{k} z_i^{l_0} a_i^{1-l_1}$$

holds with respect to $S_1, \ldots, S_k$ whenever the following conditions are satisfied:

$$D \subset R = [a_i < x_i \leqq a_i' \leqq 2a_i;\ i=1,\ldots,k],$$

$$z_i = |ys_i| \cdot a_1^{-s_1}\ldots a_k^{-s_k} \cdot a_i^{-1} > 1, \quad i=1,\ldots,k,$$

$$f \overset{r}{\underset{D}{\sim}} y x_1^{-s_1}\ldots x_k^{-s_k}.$$

Srinivasan proved in [2] that the functions $f(x_1, x_2)$ and $\varphi(v) =: f(n_v) - vn_v$ (see Lemmas 1, 2) can be approximated in degree ∞ by functions of the form $x_1^{\alpha_1} x_2^{\alpha_2}$ and v^{β}, respectively, if f can be approximated in a similar manner.

3. The estimate for $S(M, N)$

Let

$$S_1 = S_1(M,N) = \sum_{M\leqq m\leqq M'} \sum_{N\leqq n\leqq N'} e(xmn^{-2}), \tag{1}$$

where $M<M'\leqq 2M$, $N<N'\leqq 2N$, $M_0=M'-M$. Using Abel's summation formula, we get

$$S_1 \ll (xM)^{-1/2}N^2|S_2| + (xM)^{-1/2}N^2M_0 + M_0\log(MN), \tag{2}$$

where

$$S_2 = \sum_{(m,n)\in D_0} e\big(c_1\cdot(xmn^{-2})\big)$$

(c_1 is a constant). A region D_0 is contained in the rectangle

$$R_0 = \begin{bmatrix} M \leqq m \leqq M' \\ N \leqq n \leqq N' \end{bmatrix}, \quad N_1 = \frac{1}{8} xMN^{-3}, \quad N_1 < N_1' \leqq 2N_1.$$

Using Lemma 1, we obtain

$$S_2 \ll \frac{xMN^{-3}M_0}{q^{1/2}} + \left(\frac{xMN^{-3}M_0}{q^{4/3}}\right)^{3/4} \Big\{\sum_{h_1=1}^{q-1}\Big(\sum_{h_2=1}^{q^2-1}|S_3|\Big)^{1/2}\Big\}^{1/2}, \tag{3}$$

where

$$S_3 = \sum_{(m,n)\in D_0(h_1,h_2)} e\big(\varphi(m,n)\big),$$

$$\varphi(m,n) \underset{D_0(h_1,h_2)}{\overset{\infty}{\sim}} c_2 h_1 h_2 x^{1/3} m^{1/3} n^{-4/3} \quad (c_2 = \text{const.}).$$

We have

$$\Big\{\sum_{h_1=1}^{q-1}\Big(\sum_{h_2=1}^{q^2-1}|S_3|\Big)^{1/2}\Big\}^{1/2} \ll \Big\{\sum_{h_1\leqq q^{1/2}}\Big(\sum_{h_2\leqq q}|S_3|\Big)^{1/2}\Big\}^{1/2} +$$

$$+\Big\{\sum_{q^{1/2}\leqq h_1\leqq q}\Big(\sum_{h_2\leqq q^2}|S_3|\Big)^{1/2}\Big\}^{1/2}+\Big\{\sum_{h_1\leqq q}\Big(\sum_{q\leqq h_2\leqq q^2}|S_3|\Big)^{1/2}\Big\}^{1/2}\ll$$

$$\ll (xMN^{-3}M_0)^{1/4}q^{1/2}+\Big\{\sum_{\substack{h_1\leqq q\\ h_1h_2>q^{1/2}}}\Big(\sum_{h_2\leqq q^2}|S_3|\Big)^{1/2}\Big\}^{1/2}.$$

Hence

$$S_2 \ll \frac{xMN^{-3}M_0}{q^{1/2}}+\frac{(xMN^{-3}M_0)^{3/4}}{q}\Big\{\sum_{\substack{h_1<q\\ h_1h_2>q^{1/2}}}\Big(\sum_{h_2<q^2}|S_3|\Big)^{1/2}\Big\}^{1/2}. \tag{4}$$

Now we apply Lemma 2 for $x_1=n$, $x_2=m$. We have

$$S_3 \ll (h_1h_2)^{-1/2}x^{3/2}M^{3/2}N^5|S_4|+(h_1h_2)^{-1/2}x^{3/2}M^{3/2}N^5M_0+M_0\log(MN), \tag{5}$$

where

$$S_4=\sum_{(m,n)\in D_1}e(\varphi(m,n)), \quad D_1\subset R_1=\begin{bmatrix}M\leqq m\leqq M'\\ N_2\leqq n\leqq N_2'\end{bmatrix},$$

$$h_1h_2x^{-2}N^7M^{-2}\ll N_2\ll h_1h_2x^{-2}N^7M^{-2}, \quad N_2<N_2'\leqq 2N_2,$$

$$\varphi(m,n)\overset{\infty}{\underset{D_1}{\sim}}c_3(h_1h_2)^{3/7}x^{1/7}m^{1/7}n^{4/7}, \quad c_3=\text{const.}$$

We apply Lemma 1 to the sum S_4 over m and obtain:

$$S_4\ll\frac{h_1h_2x^{-2}N^7M^{-2}M_0}{q^{1/2}}+ \tag{6}$$

$$+\left(\frac{h_1h_2x^{-2}N^7M^{-2}M_0}{\varrho}\right)^{1/2}\Big(\sum_{p=1}^{\varrho-1}|S_5|\Big)^{1/2}$$

where

$$S_5=\sum_{(m,n)\in D_1(p)}e(\varphi_1(m,n)), \quad D_1(p)\subset D_1,$$

$$\varphi_1(m,n)\overset{\infty}{\underset{D_1(p)}{\sim}}c_4(h_1h_2)^{3/7}pm^{-6/7}n^{4/7}x^{1/7}, \quad c_4=\text{const.}$$

Let

$$z_1=c_5(h_1h_2)^{3/7}x^{1/7}pM^{-13/7}N_2^{4/7}=c_5'h_1h_2x^{-1}pM^{-3}N^4,$$

$$z_2=c_6(h_1h_2)^{3/7}x^{1/7}pM^{-6/7}N_2^{-3/7}=c_6'xpN^{-3}.$$

Since $N\leqq x^{1/3}$ we have $z_2>1$. We shall distinguish the cases $z_1>1$, $z_2\leqq 1$.

a) $z_1>1$. Then for $\varphi_1(m,n)$ the conditions in Definition 2 (for $n=2$) are satisfied, and we have for every pair of exponents (l_0,l_1) of dimension 2

$$S_5\ll(z_1z_2)^{l_0}(M_0N_2)^{1-l_1}\ll(h_1h_2)^{1+l_0-l_1}p^{2l_0}x^{-2+2l_1}N^{7-7l_1+l_0}M_0^{1-l_1}M^{-2-3l_0+2l_1}. \tag{7}$$

b) $z_1\leqq 1$. Then we apply Lemma 3 to the inner sum S_5 over m:

$$S_5\ll N_2\big(1+(h_1h_2)^{-3/14}x^{-1/14}p^{-1/2}M^{10/7}N_2^{-2/7}\big)\ll(h_1h_2)^{1/2}x^{-3/2}p^{-1/2}N^5. \tag{8}$$

Hence, from (7) and (8) we obtain

$$S_5 \ll (h_1h_2)^{1+l_0-l_1}p^{2l_0}x^{-2+2l_1}N^{7+l_0-7l_1}M_0^{1-l_1}M^{-2-3l_0+2l_1}+(h_1h_2)^{1/2}x^{-1/2}p^{-1/2}N^5.$$

But then

$$S_4 \ll h_1h_2x^{-2}N^7M^{-2}M_0\varrho^{-1/2}+(h_1h_2)^{3/4}x^{-7/4}M^{-1}M_0^{1/2}N^6\varrho^{-1/4}+ \tag{9}$$

$$+\left(\frac{h_1h_2x^{-2}N^7M^{-2}M_3}{\varrho}\right)^{1/2}\cdot(h_1h_2)^{\frac{1+l_0-l_1}{2}}\varrho^{1/2+l_0}\left(\frac{N^{7+l_0-7l_1}M_0^{1-l_1}}{x^{2-l_1}M^{2+3l_0-2l_1}}\right)^{1/2}.$$

Put

$$\varrho =: \left((h_1h_2)^{l_1-l_0}x^{-2l_1}N^{7l_1-l_0}M^{3l_0}M_0^{-l_1}\right)^{\frac{1}{1+2l_0}}.$$

Choosing a suitable pair of exponents (l_0, l_1), we have to guarantee the fulfilment of the inequality

$$1 < \varrho < M. \tag{10}$$

Since $h_1h_2>q^{1/2}$ and $h_1<q$, $h_2<q^2$, condition (10) is satisfied whenever

$$1 < \left(q^{\frac{l_1-l_0}{2}}x^{-2l_1}N^{7l_1-l_0}M^{3l_0}M_0^{-l_1}\right)^{\frac{1}{1+2l_0}} \leqq \tag{11}$$

$$\leqq \left(q^{3(l_1-l_0)}x^{-2l_1}N^{7l_1-l_0}M^{3l_0}M_0^{-l_1}\right)^{\frac{1}{1+2l_0}} < M.$$

So, if (11) is satisfied then

$$S_4 \ll (h_1h_2)^{3/4}x^{-7/4}N^6M^{-1}M_0^{1/2}\varrho^{-1/4}+ \tag{12}$$

$$+\left((h_1h_2)^{2+5l_0-l_1}x^{-4-8l_0+l_1}N^{14+29l_0-7l_1}\frac{M_0^{2+4l_0+l_1}}{M^{4+11l_0}}\right)^{\frac{1}{2(1+2l_0)}}.$$

Now, from (4)—(5) we obtain

$$S_2 \ll xMN^{-3}M_0q^{-1/2}+ \tag{13}$$

$$+\frac{(xMN^{-3}M_0)^{3/4}}{q}\{q^{5/8}x^{3/8}M^{3/8}M_0^{1/4}N^{-5/4}+M_0^{1/4}q\log(MN)+$$

$$+(q^{19+41l_0-3l_1}x^{-1-l_0+2l_1}N^{4+8l_0-7l_1}M^{2+l_0}M_0^{2+4l_0+l_1})^{\frac{1}{16(1+2l_0)}}+$$

$$+(q^{11+25l_0-3l_1}x^{-1-2l_0+l_1}N^{4+9l_0-7l_1}M^{-1-5l_0}M_0^{2+4l_0+l_1})^{\frac{1}{8(1+2l_0)}}.$$

Put

$$q =: \left(x^{3+6l_0-l_1}N^{-10-21l_0+7l_1}M^{3+9l_0}M_0^{-l_1}\right)^{\frac{1}{7+17l_0-3l_1}}. \tag{14}$$

Then for $1<q<(xMN^{-3})^{1/2}$ we have

$$S_2 \ll x^{\frac{11+28l_0-5l_1}{2(7+17l_0-3l_1)}}N^{-\frac{32+81l_0-11l_1}{2(7+17l_0+3l_1)}}M^{\frac{11+25l_0-6l_1}{2(7+17l_0-3l_1)}}M_0^{\frac{14+34l_0-5l_1}{2(7+17l_0-3l_1)}}+ \tag{15}$$

$$+x^{\frac{54+135l_0-24l_1}{8(7+17l_0-3l_1)}}N^{-\frac{166+413l_0-63l_1}{8(7+17l_0-3l_1)}}M^{\frac{54+146l_0-27l_1}{8(7+17l_0-3l_1)}}M_0^{\frac{56+136l_0-5l_1}{8(7+17l_0-3l_1)}}+$$

$$+R(x, M_0, M, N)+x^{3/4}M^{3/4}M_0N^{-9/4}\log(MN),$$

where

$$R(x, M_0, M, N) =$$

$$= q^{\frac{3+9l_0-3l_1}{16(1+2l_0)}} x^{\frac{11+23l_0+2l_1}{16(1+2l_0)}} N^{-\frac{32+64l_0+7l_1}{16(1+2l_0)}} M^{\frac{14+25l_0}{16(1+2l_0)}} M_0^{\frac{14+26l_0+l_1}{16(1+2l_0)}}.$$

From (2), (15) we obtain

The sum $\sum^{(1)}$ is empty when $m_0 \leqq 1$. But we have $m_0 \leqq 1$ for $N \leqq x^{3/10}$. Similarly, the sum $\sum^{(2)}$ is empty, if $\delta = 1$. We note that the inner sums in $\sum^{(j)}$ are sums of type S_1. Obviously (see (2)):

$$\sum^{(1)} \ll \max_{M \leqq m_0} |S_1| \cdot \log m_0 \ll \tag{23}$$

$$\ll \max_{M \leqq m_0} \frac{1}{M} [(xM)^{-1/2} N^2 |S_2| + (xM)^{-1/2} N^2 M + M \log (MN)] \ll$$

$$\ll \max_{M \leqq m_0} \frac{1}{M} [(xM)^{-1/2} N^2 \cdot M \cdot xMN^{-3} + (xM)^{-1/2} N^2 M \log (MN)] \ll$$

$$\ll \left(x^{1/2} N^{-1} m_0^{1/2} + x^{1/2} N^2 + \log (m_0 N)\right) \cdot \log m_0.$$

For the sum $\sum^{(2)}$ we apply the method of exponent pairs (of dimension 1). Applying Theorem 1 ([1]) we obtain

$$\sum^{(2)} \ll \left(\frac{x}{N}\right)^{\frac{1+2\lambda}{4+6\lambda}} \delta^{\frac{1+\lambda+\varkappa}{4+6\lambda}}, \tag{24}$$

where $(\varkappa, \lambda)$ is a pair of exponents (of dimension 1) with the condition $\lambda < 2\varkappa$.

We shall estimate the sums $\sum^{(3)}$ by using our Theorem, since the conditions (A), (11) are satisfied.

We have

$$\sum^{(3)} \ll \sum_{\log_2 \delta \leqq q \leqq \log_2 \Delta} 2^{-q} \Big| \sum_{2^q \leqq m \leqq \beta_q} \sum_{N \leqq n \leqq N'} e(xmn^{-2}) \Big| \ll \tag{25}$$

$$\ll x^{\frac{4+11l_0-2l_1}{2(7+17l_0-3l_1)}} N^{-\frac{4+13l_0+l_1}{2(7+17l_0-3l_1)}} \Delta^{\frac{4+8l_0-2l_1}{2(7+17l_0-3l_1)}} +$$

$$+ x^{\frac{26+67l_0-12l_0}{8(7+17l_0-3l_1)}} N^{-\frac{54+141l_0-15l_1}{8(7+17l_0-3l_1)}} \Delta^{\frac{26+78l_0-12l_1}{8(7+17l_0-3l_1)}} +$$

$$+ x^{1/4} N^{-1/4} \Delta^{1/4} \log (\Delta N) + x^{-1/2} N^2 + R(x, \Delta, \Delta, N) x^{-1/2} \Delta^{-3/2} N^2.$$

We can get similar estimation for $\sum^{(4)}$.

We take $(l_0, l_1) = \left(\frac{23}{250}, \frac{56}{250}\right)$ as it was chosen by Srinivasan in [3]. From (20) we have

$$\Delta_1 = x^{-\frac{1141}{5018}} N^{\frac{5301}{5018}}, \quad \Delta_2 = x^{-\frac{73690}{232060}} N^{\frac{314870}{232060}}. \tag{26}$$

Hence

$$\Delta = \begin{cases} \Delta_2, & \text{if } N < x^{0.29944}, \\ \Delta_1, & \text{if } N \geqq x^{0.29944}. \end{cases} \tag{27}$$

Next, it is obvious that condition (B) is satisfied. Now we shall define δ. We have

$$\delta_1 = x^{\frac{414463}{81031}} N^{-\frac{1470571}{81031}}, \quad \delta_2 = x^{-\frac{138509}{468560}} N^{\frac{471528}{468560}}. \tag{28}$$

So

$$\delta = \begin{cases} \delta_1, & \text{if} \quad N \leqq x^{\frac{414463}{1470571}}, \\ 1, & \text{if} \quad x^{\frac{414463}{1470571}} < N < x^{\frac{138509}{471528}}, \\ \delta_2, & \text{if} \quad N \geqq x^{\frac{138509}{471528}}. \end{cases} \tag{29}$$

It is clear that $\delta \leqq \Delta$ for $N \geqq x^{0.278514}$. We have seen that $\sum^{(1)}$ is not empty only if $m_0 > 1$. We have $m_0 = x^{-0.9234} N^{2.87569}$, such that $m_0 < 1$ if $N \leqq x^{0.3}$. In the interval $x^{0.3} \leqq N \leqq x^{1/3}$,

$$\sum^{(1)} \ll x^{0.0383} N^{0.437845} + x^{-1/2} N^2 \ll x^{0.19}.$$

Furthermore, for $(\varkappa, \lambda) = \left(\frac{1029}{3278}, \frac{1820}{3278}\right)$

$$\sum^{(2)} \ll \left(\frac{x}{N}\right)^{\frac{1+2\lambda}{4+6\lambda}} \left(x^{\frac{414465}{81031}} N^{-\frac{1470571}{81031}} + 1\right)^{\frac{1+\lambda+\varkappa}{4+6\lambda}} \ll x^{1.59194} N^{-4.914794},$$

if $N \geqq x^{0.25}$.

To give an estimation for $\sum_1$ we shall use (18) by choosing $(k, l) = \left(\frac{194}{758}, \frac{445}{758}\right)$. (This pair of exponents was used by P. G. Schmidt [1]). We have

$$\sum_1 \ll x^{0.2037772} X^{0.0519217}.$$

The optimal value of X can be computed from the equation

$$x^{1.59194} X^{-4.914794} = x^{0.2037772} X^{0.0514217},$$

i.e. $X = x^{0.2794873}$. Then for $N \geqq x^{0.2937}$ we have

$$\sum_1 + \sum^{(1)} + \sum^{(2)} \ll x^{0.21829}. \tag{30}$$

Such an estimation can be proved in the interval

$$x^{0.2937} \leqq N \leqq x^{1/3}.$$

THEOREM 1. *Let* x, M_0, M, N *be positive numbers,* $N \leqq x^{1/3}$, $M_0 \leqq M$, *and* (l_0, l_1) *a pair of exponents of dimension* 2. *Let*

$$q =: (x^{3+6l_0-l_1} N^{-10-21l_0+7l_1} M^{3+9l_0} M_0^{-l_1})^{\frac{1}{7+17l_0-3l_1}}.$$

If (11) *and the condition*

$$1 < q < (xMN^{-3})^{1/2} \tag{A}$$

are satisfied, then

$$S_1 = \sum_{M \leqq m \leqq M+M_0} \sum_{N \leqq n \leqq N'} e(xmn^{-2}) \ll \tag{16}$$

$$\ll x^{\frac{4+11l_0-2l_1}{2(7+17l_0-3l_1)}} N^{-\frac{4+13l_0+l_1}{2(7+17l_0-3l_1)}} M^{\frac{4+8l_0-3l_1}{2(7+17l_0-3l_1)}} M_0^{\frac{14+34l_0-5l_1}{2(7+17l_0-3l_1)}} +$$

$$+ x^{\frac{26+67l_0-12l_1}{8(7+17l_0-3l_1)}} N^{-\frac{54+141l_0-15l_1}{8(7+17l_0-3l_1)}} M^{\frac{26+78l_0-15l_1}{8(7+17l_0-3l_1)}} M_0^{\frac{56+136l_0-21l_1}{8(7+17l_0-3l_1)}} +$$

$$+ x^{1/4} M M_0^{1/4} N^{-1/4} \log(MN) + x^{-1/2} M^{-1/2} M_0 N^2 + R(x, M_0, M, N) x^{-1/2} M^{-1/2} N^2.$$

4. The estimate for $\sum \psi\left(\frac{x}{n^2}\right)$

We take an X, $x^{1/4} \leqq X \leqq x^{1/3}$. Then we have

$$\sum_{n \leqq x^{1/3}} \psi\left(\frac{x}{n^2}\right) = \sum_{n \leqq X} + \sum_{X < n \leqq x^{1/3}} = \Sigma_1 + \Sigma_2. \tag{17}$$

If (k, l) is a pair of exponents of dimension 1 such that $1/5 < k < l/2$, then (see [1])

$$\Sigma_1 \ll x^{\frac{k}{1+k}} X^{\frac{l-2k}{1+k}}. \tag{18}$$

Next, we have for any $\Delta > 1$

$$\sum_{N \leqq n \leqq N'} \psi\left(\frac{x}{n^2}\right) \ll \frac{N}{\Delta} + \sum_{q \leqq \log_2 \Delta} 2^{-q} \Big| \sum_{2^q \leqq m \leqq \beta_q} \sum_{N \leqq n \leqq N'} e(xmn^{-2}) \Big| + \tag{19}$$

$$+ \sum_{p \leqq \Delta} p^{-2} \Delta^{-1} \Big| \sum_{p\Delta \leqq m \leqq \gamma_p} \sum_{N \leqq n \leqq N'} e(xmn^{-2}) \Big|.$$

The inner sums in (19) can be estimated by using Theorem 1. Let Δ_1, Δ_2 be the numbers defined by

$$x^{\frac{4+11l_0-2l_1}{2(7+17l_0-3l_1)}} N^{-\frac{4+13l_0+l_1}{2(7+17l_0-3l_1)}} \Delta_1^{\frac{18+42l_0-8l_1}{2(7+17l_0-3l_1)}-1} = \frac{N}{\Delta_1},$$

$$x^{\frac{26+67l_0-12l_1}{8(7+17l_0-3l_1)}} N^{-\frac{54+141l_0-15l_1}{8(7+17l_0-3l_1)}} \Delta_2^{\frac{26+78l_0-12l_1}{8(7+17l_0-3l_1)}} = \frac{N}{\Delta_2}.$$

We get

$$\Delta_1 = N^{\frac{18+47l_0-5l_1}{18+42l_0-8l_1}} x^{-\frac{4+11l_0-2l_1}{18+42l_0-8l_1}}, \quad \Delta_2 = N^{\frac{110+277l_0-39l_1}{82+214l_0-36l_1}} x^{-\frac{26+67l_0-12l_1}{82+214l_0-36l_1}}. \tag{20}$$

Put $\Delta = \min(\Delta_1, \Delta_2)$. Now we shall consider the validity of the conditions (A) and (11) in Theorem 1.

From (14) we have $q \leqq (xMN^{-3})^{1/2}$ if

$$x^{3+6l_0-l_1} N^{-(10+21l_0-7l_1)} M^{3+9l_0-l_1} \leqq (xMN^{-3})^{\frac{7+17l_0-3l_1}{2}} \Leftrightarrow$$

$$\Leftrightarrow x^{1+5l_0-l_1} N^{-1-9l_0-5l_1} \geqq M^{-1+l_0+l_1}.$$

But the last inequality is satisfied if $l_1<1/4$. Next,

$$q > 1 \Leftrightarrow M > x^{-\frac{3+6l_0-l_1}{3+9l_0-l_1}} N^{\frac{10+21l_0-7l_1}{3+9l_0-l_1}} =: m_0.$$

Hence, condition (A) is equivalent to the condition

(A′) $$M > m_0, \quad l_1 < 1/4.$$

Now we define δ_1, δ_2 by

$$x^{\frac{(3+6l_0-l_1)(l_1-l_0)}{7+17l_0-3l_1}-4l_1} N^{-\frac{(10+21l_0-7l_1)(l_1-l_0)}{7+17l_0-3l_1}+14l_1-2l_0} = \delta_1^{-\frac{(3+9l_0-l_1)(l_1-l_0)}{7+17l_0-3l_1}+2l_1-6l_0},$$

(21)
$$x^{\frac{3(l_1-l_0)(3+6l_0-l_1)}{7+17l_0-3l_1}-2l_1} N^{-\frac{3(l_1-l_0)10+21l_0-7l_1)}{7+17l_0-3l_1}+7l_1-l_0} = \delta_2^{-\frac{3(l_1-l_0)(3+9l_0-l_1)}{7+17l_0-3l_1}+1-l_0+l_1}.$$

We suppose (l_0, l_1) is such that the condition

(B)
$$\begin{cases} (3+9l_0-l_1)(l_1-l_0)+(6l_0-2l_1)(7+17l_0-3l_1) \geqq 0, \\ (1-l_0+l_1)(7+17l_0-3l_1)-(3l_1-3l_0)(3+9l_0-l_1) \geqq 0 \end{cases}$$

is satisfied.

Then let $\delta=\max(1, \delta_1, \delta_2)$. In this case condition (11) of Theorem 1 is satisfied. Now, from (19) we obtain for $\delta \leqq \Delta$:

(22)
$$\sum_{N \leqq n \leqq N'} \psi\left(\frac{x}{n^2}\right) \ll \frac{N}{\Delta} + \sum_{q \leqq \log_2 m_0} + \sum_{q \leqq \log_2 \delta} + \sum_{\log_2 \delta \leqq q \leqq \log_2 \Delta} +$$

$$+ \sum_{p \leqq \Delta}' = \frac{N}{\Delta} + \sum{}^{(1)} + \sum{}^{(2)} + \sum{}^{(3)} + \sum{}^{(4)}.$$

It has remained to estimate the contribution of the sums $\sum^{(3)}$ and $\sum^{(4)}$. Let us assume that $N \leqq x^{0.29944}$. The first term on the right hand side of (25) is $\frac{N}{\Delta_2}$ according to the definition of Δ_2, the second term is less than

$$\ll \frac{N}{\Delta_1} \cdot \left(\frac{\Delta_2}{\Delta_1}\right)^{\frac{26+78l_0-12l_1}{8(7+17l_0-3l_1)}} \ll \frac{N}{\Delta_1} \ll \frac{N}{\Delta_2},$$

since $\Delta_2 \leqq \Delta_1$. So the contribution of the first two terms in (25) is estimated by

(31) $$\ll \frac{N}{\Delta_2} \ll x^{0.317547} N^{-0.356847} \ll x^{0.21822},$$

if $x^{0.27948} \leqq N \leqq x^{0.29944}$.

In the case $N > x^{0.29944}$ for these two terms we get

(32) $$\ll \frac{N}{\Delta_1} \ll x^{0.22758} N^{-0.056397} \ll x^{0.21}.$$

The contribution of the last three terms is $\ll x^{0.20}$.

Collecting these estimations (30)—(32), we have

$$\sum_{n \leqq x^{1/3}} \psi\left(\frac{x}{n^2}\right) \ll x^{0.21829}. \tag{33}$$

This inequality improves an earlier result due to P. G. Schmidt [1], who derived a similar inequality with $x^{0.2215\ldots}$ instead of $x^{0.21829}$.

From (33) we can deduce some improvements of earlier results for the distribution of square-free numbers in short intervals, and for the summation function of the number of non-isomorphic Abelian groups of given order.

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ON SOME PROPERTIES OF A SET OF PROBABILITY MEASURES

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Let $(\Omega, \mathcal{A}, P)$ be a probability space. By (S, ϱ) we denote a metric space, and ζ stands for the σ-field generated by the open sets of S. Throughout this note S is assumed to be a nondegenerate, separable and complete space. A mapping $X: \Omega \to S$ such that $X^{-1}(\zeta) \subset \mathcal{A}$, is called a random element (r.e.). The set of all r.e. is denoted by $\mathcal{X}$. By P_X we denote the distribution of the r.e. X $(P_X(B) = P[X \in B]$ for every $B \in \zeta)$.

On the set $\mathcal{P}(S)$ of probability measures we can define a Levy—Prochorow metric

$$L(\nu, \tau) = \inf\{\varepsilon > 0: \nu(A) \leqq \tau(A^\varepsilon) + \varepsilon \quad \text{and} \quad \tau(A) \leqq \nu(A^\varepsilon) + \varepsilon, A \in \zeta\} \tag{1}$$

where $A^\varepsilon = \{x: d^s(x, A) = \inf_{y \in A} \varrho(x, y) < \varepsilon\}$.

Convergence in this metric and weak convergence of measures coincide. Let

$$r(X, Y) = \inf\{\varepsilon > 0: P[\varrho(X, Y) > \varepsilon] < \varepsilon\} \tag{2}$$

and

$$r_1(X, Y) = E\frac{\varrho(X, Y)}{1 + \varrho(X, Y)},$$

be two metrics introduced in the space $\mathcal{X}$ (here $E(\cdot)$ denotes the mean value). The convergence with respect to r and r_1 is equivalent to each other and to the convergence in probability [3]. It can be proved [3] that if S is a normed linear space then there does not exist a norm in $\mathcal{X}$ for which the convergence in norm is equivalent to the convergence in probability.

It is known [1] that

$$L(P_X, P_Y) \leqq r(X, Y). \tag{3}$$

This fact yields that the convergence in probability implies the weak convergence.

The following result is well known [1], [4].

THEOREM 1. *If* $L(\nu_n, \nu) \to 0$ *as* $n \to \infty$ *where* $\nu_n, \nu \in \mathcal{P}(S)$, *then there exists a probability space* $(\langle 0, 1\rangle, \mathcal{B}, \mu)$, *a sequence of r.e.* $\{X_n, n \geqq 1\}$ *and r.e.* X, *such that* $\mu_{X_n} = \nu_n$, $\mu_X = \nu$ *and*

$$X_n \xrightarrow{\text{a.s.}} X \quad \text{as} \quad n \to \infty, \quad \text{i.e.} \quad \mu\{\omega: \lim_{n \to \infty} X_n(\omega) = X(\omega)\} = 1.$$

In [2] and [6] another relation between the metrics L and r was proved.

THEOREM 2. *If* $\tau, \nu \in \mathscr{P}(S)$ *and* $L(\tau, \nu)=\varepsilon$, *then there exist two sequences of r.e.* $\{X_n, n\geqq 1\}$ *and* $\{Y_n, n\geqq 1\}$ *defined on the probability space* $(\langle 0, 1\rangle, \mathscr{B}, \mu)$ *with values in arbitrary separable metric space* S, *such that*

$$\mu_{X_n} = \tau, \quad \mu_{Y_n} = \nu \quad \text{and} \quad \lim_{n\to\infty} r(X_n, Y_n) = \varepsilon.$$

Moreover, if (S, ϱ) *is an inner regular space (i.e. for any measure on* (S, ϱ) *and an arbitrary set* $A\in\zeta$, $\nu(A)=\sup\{\nu(K): K\subset A\ K\text{-compact}\}$*), then there exist r.e.* X *and* Y *such that* $\mu_X=\tau$, $\mu_Y=\nu$ *and* $r(X, Y)=\varepsilon$.

DEFINITION 1. The set

$$(a, b) = \{x: \varrho(a, x)+\varrho(x, b) = \varrho(a, b)\} \tag{4}$$

in a metric space (S, ϱ) will be called a segment (a, b).

If $L(\tau, \nu)=c$ and $0<t<1$ then we can ask under what conditions there exists a probability measure η defined on (S, ζ) such that

$$L(\tau, \eta) = ct \quad \text{and} \quad A(\eta, \nu) = (1-t)c.$$

It is easy to see that such a probability measure need not exist in general.

EXAMPLE. Let (S, ϱ) be a metric space, where $S=\{0, 1/2\}$ and

$$\varrho(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/2 & \text{if } x \neq y. \end{cases}$$

Let us put

$$\mu_x(A) = \begin{cases} 0 & \text{if } x\notin A \\ 1 & \text{if } x\in A \end{cases} \quad \text{for any } x\in S.$$

It can be easily seen that $L(\mu_0, \mu_{1/2})=1/2$ and there does not exist a measure η on (S, ϱ) such that

$$L(\mu_0, \eta) = 1/4 \quad \text{and} \quad L(\eta, \mu_{1/2}) = 1/4.$$

In what follows we will assume that (S, ϱ) satisfies the following condition:

(A) For every $0<t<1$ and every $x, y\in S$ such that $\varrho(x, y)=c<1$ there exists $z\in S$ such that $\varrho(x, z)=tc$ and $\varrho(z, y)=(1-t)c$.

LEMMA. *Let* X *and* Y *be r.e. defined on* $(\langle 0, 1\rangle, \mathscr{B})$ *with values in the separable, complete metric space* (S, ϱ). *If* (S, ϱ) *satisfies condition* (A) *then for every* $0<t<1$ *there exists a r.e.* Z *such that*

$$\varrho(X(\omega), Z(\omega)) = tc \quad \text{and} \quad \varrho(Z(\omega), Y(\omega)) = (1-t)c$$

for $\omega\in D=\{\omega: \varrho(X(\omega), Y(\omega))=c\}$.

PROOF. We choose a sequence of Borel subsets $S_{i_1, i_2, \ldots, i_k}$ satisfying the following conditions [4]:

(i) $S_{i_1, i_2, \ldots, i_k} \cap S_{i'_1, i'_2, \ldots, i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \leqq s \leqq k$,

(ii) $\bigcup_{i_k=1}^{\infty} S_{i_1, i_2, \ldots, i_{k-1}, i_k} = S_{i_1, i_2, \ldots, i_{k-1}}$, $\bigcup_{i_1=1}^{\infty} S_{i_1} = S$,

(iii) $d(S_{i_1, i_2, \ldots, i_k}) \leqq 1/2^k$ where $d(B)$ denotes the diameter of $B\subset S$.

Let

$$A_{i_1,i_2,\ldots,i_k} = D \cap \{\omega\colon d^s(X(\omega), S_{i_1,i_2,\ldots,i_k}) \leqq tc, \quad d^s(Y(\omega), S_{i_1,i_2,\ldots,i_k}) \leqq (1-t)c\}$$

and

$$A'_{i_1,i_2,\ldots,i_k} = A'_{i_1,i_2,\ldots,i_{k-1}} \cap A_{i_1,i_2,\ldots,i_k} \setminus \bigcup_{s=1}^{i_k-1} A_{i_1,i_2,\ldots,i_{k-1},s}$$

where $A'_{i_1} = A_{i_1} \setminus \bigcup_{s=1}^{i_1-1} A_s$. Then

$$A'_{i_1,i_2,\ldots,i_k} \in \mathscr{B}, \quad A'_{i_1,i_2,\ldots,i_k} \cap A_{i'_1,i'_2,\ldots,i'_k} = \emptyset$$

if $i_s \neq i'_s$ for some $1 \leqq s \leqq k$ and, by condition (A)

$$\bigcup_{i_k=1}^{\infty} A'_{i_1,i_2,\ldots,i_k} = A'_{i_1,i_2,\ldots,i_{k-1}}, \quad \bigcup_{i_1=1}^{\infty} A'_{i_1} = D.$$

If $z_{i_1,i_2,\ldots,i_k} \in S_{i_1,i_2,\ldots,i_k}$ and

$$Z_n(\omega) = z_{i_1,i_2,\ldots,i_n} \quad \text{for} \quad \omega \in A'_{i_1,i_2,\ldots,i_n}$$

then for every ω the sequence $\{Z_n(\omega), n \geqq 1\}$ satisfies the Cauchy condition and therefore converges to some $Z(\omega) \in S$. By definition of $Z_n(\omega)$ we have

$$\varrho\{X(\omega), Z_n(\omega)\} \leqq tc + 1/2^n$$

and

$$\varrho\{Y(\omega), Z_n(\omega)\} \leqq (1-t)c + 1/2^n$$

for every $\omega \in D$. Therefore

$$\varrho(X(\omega), Z(\omega)) \leqq tc \quad \text{and} \quad \varrho(Y(\omega), Z(\omega)) \leqq (1-t)c \quad (\omega \in D),$$

whence

$$\varrho(X(\omega), Z(\omega)) = tc \quad \text{and} \quad \varrho(Y(\omega), Z(\omega)) = (1-t)c \quad (\omega \in D).$$

THEOREM 3. *If X and Y are r.e. defined on $(\langle 0, 1\rangle, \mathscr{B}, \mu)$ with values in the separable, complete metric space S, satisfying condition* (A) *and such that $r(X, Y) = c$, then for every $0 < t < 1$ there exists a r.e. Z such that $r(X, Z) = tc$ and $r(Z, Y) = (1-t)c$.*

PROOF. Let $c(\omega) = \varrho(X(\omega), Y(\omega))$ and $C = \{\omega\colon c(\omega) > c\}$. Since $r(X, Y) = c$ we have $\mu(C) \leqq c$. Further there exist sets A and B such that, $A \cup B = C$, $A \cap B = \emptyset$, $\mu(A) \leqq tc$ and $\mu(B) \leqq (1-t)c$.

Let $z_i \in S$, $i = 1, 2, \ldots$ be a dense subset in S and $A_i = \{\omega\colon \varrho(X(\omega), z_i) < tc$ and $\varrho(z_i, Y(\omega)) < (1-t)c\}$. Then $A_i \in \mathscr{B}$ and $\bigcup_{i=1}^{\infty} A_i = D_1 = \{\omega\colon \varrho(X(\omega), Y(\omega)) < c\}$.

If $B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i$ then

$$\bigcup_{k=1}^{\infty} B_k = D_1 \quad \text{and} \quad B_k \cap B_l = \emptyset \quad \text{for} \quad k \neq 1.$$

By the Lemma, there exists a r.e. $Z^*(\omega)$ on the set $D=\{\omega: \varrho(X(\omega), Y(\omega))=c\}$ such that

$$\varrho(X(\omega), Z^*(\omega)) = tc \quad \text{and} \quad \varrho(Z^*(\omega), Y(\omega)) = (1-t)c.$$

Put

$$Z(\omega) = \begin{cases} X(\omega) & \text{for} \quad \omega\in B \\ Y(\omega) & \text{for} \quad \omega\in A \\ z_i & \text{for} \quad \omega\in B_i \\ Z^*(\omega) & \text{for} \quad \omega\in D. \end{cases}$$

Hence for every $\varepsilon>0$

$$\mu[\varrho(X, Z) > tc+\varepsilon] \leqq \mu(A) \leqq tc+\varepsilon$$

and

$$\mu[\varrho(Z, Y) > (1-t)c+\varepsilon] \leqq \mu(B) \leqq (1-t)c+\varepsilon.$$

Then we have

$$r(X, Z) = tc \quad \text{and} \quad r(Z, Y) = (1-t)c.$$

THEOREM 4. *Let the separable, complete metric space (S, ϱ) satisfy condition* (A). *If $\tau, \nu\in\mathscr{P}(S)$ and $L(\tau, \nu)=c$ then for every $0<t<1$ there exist probability measures $\eta\in\mathscr{P}(S)$ such that $L(\tau, \eta)=tc$ and $L(\eta, \nu)=(1-t)c$.*

PROOF. By Theorem 2 there exist r.e. X and Y defined on $(\langle 0, 1\rangle, \mathscr{B}, \mu)$ such that $\mu_X=\tau$, $\mu_Y=\nu$ and $r(X, Y)=c$. Therefore by Theorem 3 there exists a r.e. Z such that

$$r(X, Z) = tc \quad \text{and} \quad r(Z, Y) = (1-t)c.$$

Since $L(\mu_X, \mu_Z)\leqq r(X, Z)$ and $L(\mu_Z, \mu_Y)\leqq r(Z, Y)$, we have $L(\tau, \mu_Z)=tc$ and $L(\mu_Z, \nu)=(1-t)c$. The assumption that either $L(\tau, \mu_Z)<tc$ or $L(\mu_Z, \nu)<(1-t)c$ implies $L(\tau, \nu)<c$. This contradicts the assumption $L(\tau, \nu)=c$ and therefore completes the proof.

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AFFINELY EMBEDDABLE CONVEX SETS

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A subset A of the real projective plane P^2 s defined to be convex in P^2, if there ts a line L in P^2 disjoint from A and A is a convex subset of the affine plane $P^2 \setminus L$; cf. [2] and [4].

Let $\mathscr{A}_r$ be a finite collection of mutually disjoint convex sets in P^2. In [3], N. H. Kuiper determines conditions under which there exists a line in P^2 meeting every element of $\mathscr{A}_r$. Presently, we determine when there exists a line in P^2 not meeting any element of $\mathscr{A}_r$, or equivalently, when each element of $\mathscr{A}_r$ is convex (and thus contained) in the same affine restriction of P^2.

This problem, interestingly enough, is related to Helly's theorem, and for pointing out this connection and how it is obtained, we wish to express our thanks to Dr. I. Bárány.

The connection is based upon the extension of polarity from R^3 to P^2, Since P^2 is obtained from S^2 by identifying antipodal points, a convex set $A \subset P^2$ is obtained by identifying disjoint connected sets $S^+(A)$ and $S^-(A)$ in S^2 with $S^+(A) = -S^-(A)$. As there is a convex cone $C(A) \subset R^3$ with $C(A) \cap S^2 = S^+(A)$ and the polar of a convex cone is again a convex cone, the polar of $C(A)$ determines a set $B \subset P^2$ which we call the polar of A. The set B is semi-convex, that is, any two points of B can be joined by a line segment contained in B; cf. [2]. If B is disjoint from a line of P^2 then of course, B is convex.

Let $\mathscr{B}_r = \{B_1, B_2, \ldots, B_r\}$ be the family of polars of $\mathscr{A}_r = \{A_1, A_2, \ldots, A_r\}$. Then by polarity, $\bigcap_{l=1}^{r} B_l \neq \emptyset$ if and only if there is a line in P^2 not meeting any $A_i \in \mathscr{A}_r$.

Now our main result for $\mathscr{A}_r$ restated for $\mathscr{B}_r$ is as follows.

If for any three B_i, B_j and B_k in $\mathscr{B}_r$, $r \geqq 4$, no connected (convex) component of $B_i \cap B_j$ is contained in B_k then $\bigcap_{l=1}^{r} B_l \neq \emptyset$.

Preliminaries

We denote the points and lines of P^2 by $p, q, \ldots$ and $L, M, \ldots$ respectively, and the line spanned by distinct points p and q by $\langle p, q \rangle$.

If A is a subset of P^2 and L is a line in P^2 disjoint from A then A is contained in the affine restriction $P^2 \setminus L$, and we denote by $H_L(A)$ the *convex hull* of A in $P^2 \setminus L$. As stated, A is then *convex* (in P^2) if $H_L(A) = A$. If A is convex and disjoint from the lines M and N then $H_M(A) = A = H_N(A)$.

If $\mathscr{A}_r=\{A_1, A_2, \ldots, A_r\}$ is a collection of $r\geqq 2$ convex sets in P^2, we say that $\mathscr{A}_r$ is *affinely embeddable* if there is a line L in P^2 such that $L\cap\bigcup_{i=1}^{r} A_i=\emptyset$; that is, if each $A_i\to\mathscr{A}_r$ is convex in the same affine restriction $P^2\setminus L$.

We wish to determine necessary and sufficient conditions for $\mathscr{A}_r$ to be affinely embeddable. There is a necessary condition independent of r but the sufficient conditions which we determine depend on whether $r=2$, $r=3$ or $r\geqq 4$.

1. LEMMA. *Let $\mathscr{A}_r$ be a collection of r convex sets of P^2; $r\geqq 2$. If $\mathscr{A}_r$ is affinely embeddable then the intersection of any two elements of $\mathscr{A}_r$ is connected.*

PROOF. Let $\{A_j, A_k\}\subset\mathscr{A}_r$ such that $A_j\cap A_k$ is disconnected. Since both sets are convex, $A_j\cap A_k$ consists of two components. Choose points p and q, one from each component. Then

$$\langle p, q\rangle = (\langle p, q\rangle\cap A_j)\cup(\langle p, q\rangle\cap A_k)$$

and every line in P^2 meets $A_j\cup A_k$; thus $\mathscr{A}_r$ is not affinely embeddable. □

We note that the necessary condition in Lemma 1 is not sufficient. In fact, even if the elements of $\mathscr{A}_r$ are mutually disjoint, $\mathscr{A}_r$ need not be affinely embeddable as the following trivial example shows: Let p and q be distinct points; then they are the end points of two disjoint open segments of $\langle p, q\rangle$, say B_1 and B_2. Then $A_1=B_1$ and $A_2=\bar{B}_2$ are convex in P^2, $A_1\cap A_2=\emptyset$, $A_1\cup A_2=\langle p, q\rangle$ and $\{A_1, A_2\}$ is not affinely embeddable. Next we note the following trivial sufficient condition for $\mathscr{A}_r$ to be affinely embeddable.

2. LEMMA. *Let $\mathscr{A}_r$ be a collection of $r\geqq 2$ closed convex sets of P^2 such that there is an A_i and an A_j in $\mathscr{A}_r$ with the property that one of their common supporting lines meets neither $A_i\cap A_j$ nor any other element $\mathscr{A}_r$. Then $\mathscr{A}_r$ is affinely embeddable.*

3. LEMMA. *Let A_1 and A_2 be closed convex sets in P^2 such that $A=A_1\cap A_2$ is connected. Then $\{A_1, A_2\}$ is affinely embeddable.*

PROOF. If $A=\emptyset$ then Lemma 3 follows from a proof by Kuiper in [3]. Hence let $A\neq\emptyset$. Since A_i is convex, there is a line M_i such that $M_i\cap A_i=\emptyset$; $i=1, 2$. We may assume that $M_1\neq M_2$, the sets A_1, A_2 and A are distinct, $M_1\cap\operatorname{int} A_2\neq\emptyset$ and $M_2\cap\operatorname{int} A_1\neq\emptyset$. Let $M_1\cap M_2=\{q\}$ and denote by Q and Q' the closed half-planes of P^2 bounded by M_1 and M_2; since $A\cap(M_2\cup M_2)=\emptyset$, $A\subset\operatorname{int} Q'$ say. Let $L\subset Q'$, $M_1\neq L\neq M_2$ and set $A_i^*=Q\cap A_i$; $i=1, 2$.

As $A\subset\operatorname{int} Q'$ and $M_1\cap\operatorname{int} A_2\neq\emptyset\neq M_2\cap\operatorname{int} A_1$, A_1^* and A_2^* are non-empty disjoint closed convex sets in $P^2\setminus L$. Hence (cf. [1]) there exist distinct lines $\tilde{N}_1$ and $\tilde{N}_2$ in $P^2\setminus L$ such that

i) each is a common supporting line of A_1^* and A_2^* and

ii) each separates A_1^* and A_2^* in $P^2\setminus L$.

Setting $\tilde{M}_i=M_i\setminus L$ $(i=1, 2)$, our construction can be easily followed in the affine plane $P^2\setminus L$: $\tilde{M}_1$ and $\tilde{M}_2$ are parallel lines, $Q\setminus L$ is the parallel strip bounded by $\tilde{M}_1$ and $\tilde{M}_2$ and $A_1^*\cup A_2^*\subset Q\setminus L$.

Let $N_i\subset P^2$ such that $N_i\setminus L=\tilde{N}_i$ $(i=1, 2)$ and let $N_1\cap N_2=\{p\}$. Let P and P' denote the closed half-planes of P^2 determined by N_1 and N_2. By (ii), $\tilde{N}_1\cap\tilde{N}_2=$

$=\{p\}\subset Q\setminus L$ and thus $p\neq q$ and $p\in\operatorname{int} Q$. Finally by i) and ii), A_1^* and A_2^* are both contained in P' say.

If $q\in P$ then $\langle p, q\rangle\subset P\cap Q$ and it is clear that $\langle p, q\rangle\cap(A_1\cup A_2)=\emptyset$, hence $\{A_1, A_2\}$ is affinely embeddable. So let $q\in P'$. Then ii) readily yields that N_1 or N_2, say N_1, does not meet $(M_1\cap A_2^*)\cup(M_2\cap A_1^*)$. As

$$\operatorname{bd} A_1^*\subset\operatorname{bd} A_1\cup M_2 \quad\text{and}\quad \operatorname{bd}A_2^*\subset\operatorname{bd} A_2\cup M_1,$$

we now obtain that N_1 is a common supporting line of A_1 and A_2.

Since $M_2\cap A_1^*=M_2\cap A_1$ and $M_1\cap A_1\neq\emptyset$, we can rewrite $N_1\cap(M_2\cap A_1^*)\neq\emptyset$ as

iii) $N_1\cap[(M_1\cup M_2)\cap A_1]=\emptyset$.

As $N_1\cap A_1$ is a line segment (possibly a point) and A_2 is convex, iii) together with $N_1\cap A_1^*\subset\operatorname{int} Q$ and $A\subset\operatorname{int} Q'$ imply that $N_1\cap A=\emptyset$. Thus by Lemma 2, $\{A_1, A_2\}$ is affinely embeddable. □

To simplify our considerations, we assume henceforth that $\mathscr{A}_r=\{A_1, A_2, \ldots, A_r\}$ is a collection of r mutually disjoint closed convex sets in P^2; $r\geqq 3$.

Let $A_i\neq A_j$ in $\mathscr{A}_r$. By Lemma 3, $\{A_i, A_j\}$ is affinely embeddable and thus

$$\mathscr{L}_{ij}=\{L\subset P^2 | L\cap(A_i\cup A_j)=\emptyset\}\neq\emptyset.$$

Let $M\in\mathscr{L}_{ij}$. Then A_i and A_j are disjoint closed convex sets in $P^2\setminus M$ and there is a line $\tilde{N}\subset P^2\setminus M$ which strictly separates A_i and A_j in $P^2\setminus M$. Let $N\subset P^2$ such that $N\setminus M=\tilde{N}$. Clearly $N\neq M$, $N\in\mathscr{L}_{ij}$ and any connected set in the space of lines of P^2 which contains M and N necessarily contains a line which meets $A_i\cup A_j$. Thus $\mathscr{L}_{ij}$ is not connected (in the space of lines of P^2); it has two components, say $\mathscr{L}_{ij}^m$ and $\mathscr{L}_{ij}^n$, and $\mathscr{L}_{ij}=\mathscr{L}_{ij}^m\cup\mathscr{L}_{ij}^n$. We note that if $\{M, M'\}\subset\mathscr{L}_{ij}^m$ and $\{N, N'\}\subset\mathscr{L}_{ij}^n$, then $H_M(A_i\cup A_j)=H_{M'}(A_i\cup A_j)$ and $H_N(A_i\cup A_j)=H_{N'}(A_i\cup A_j)$. Thus we set

$$H_m(i,j)=H_M(A_i\cup A_j) \quad\text{and}\quad H_n(i,j)=H_N(A_i\cup A_j)$$

and call these closed sets the *m-hull* and the *n-hull* of $A_i\cup A_j$ respectively.

Suppose that the points of $A_i\cup A_j$ are not collinear and neither A_i nor A_j is a singleton. Then A_i and A_j have four distinct common supporting lines, two supporting $H_m(i,j)$ and two supporting $H_n(i,j)$, which we denote by M_{ij}, M_{ji} and N_{ij}, N_{ji} respectively. We set

$$M_{ij}^*=M_{ij}\cap H_m(i,j), \quad M_{ij}^0=M_{ij}\setminus M_{ij}^*,$$

$$N_{ij}^*=N_{ij}\cap H_n(i,j), \quad N_{ij}^0=N_{ij}\setminus N_{ij}^*$$

and define similarly $M_{ji}=M_{ji}^*\cup M_{ji}^0$ and $N_{ji}=N_{ji}^*\cup N_{ji}^0$. Thus with M and N fixed we have symmetry between i and j, and with i and j fixed we have symmetry between M and N. Therefore we shall state some results only for one of M or N and a fixed permutation of i and j.

Next we examine the case $r=3$.

4. LEMMA. *Let A_1, A_2 and A_3 be mutually disjoint closed convex sets in P^2. Then $\mathscr{A}_3=\{A_1, A_2, A_3\}$ is affinely embeddable if and only if $A_k\cap H_m(i,j)$ or $A_k\cap\cap H_n(i,j)$ is connected for $\{i,j,k\}=\{1,2,3\}$.*

PROOF. Let $\mathscr{A}_3$ be affinely embeddable. Then there is a line L such that $L \cap (A_1 \cup A_2 \cup A_3) = \emptyset$ and we may assume that $L \in \mathscr{L}_{12}^m \cap \mathscr{L}_{13}^m \cap \mathscr{L}_{23}^m$. Thus

$$H_m(1, 2) \cup H_m(1, 3) \cup H_m(2, 3) \subset H_L(A_1 \cup A_2 \cup A_3)$$

and $A_k \cup H_m(i, j) \subset H_L(A_1 \cup A_2 \cup A_3)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $\{A_k, H_m(i, j)\}$ is affinely embeddable and $A_k \cap H_m(i, j)$ is connected by 1.

Let $A_k \cap H_m(i, j)$ be connected. Then $\{A_k, H_m(i, j)\}$ is affinely embeddable by Lemma 3. Since $A_1 \cup A_2 \cup A_3 \subset A_k \cup H_m(i, j)$, $\mathscr{A}_3$ is affinely embeddable. □

For $r \geqq 4$, even if $A_k \cap H_m(i, j)$ is connected for every triple A_i, A_j and A_k of $\mathscr{A}_r$, $\mathscr{A}_r$ need not be affinely embeddable; cf. Fig. 1.

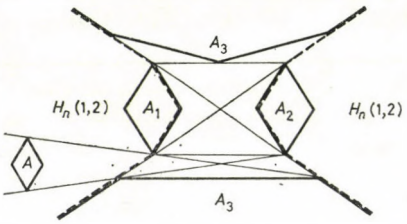

Fig. 1

In Fig. 1, we also note $A_4 \subset H_n(1, 2)$. Surprisingly, this is the reason why $\mathscr{A}_4$ is not affinely embeddable.

The main theorem

THEOREM. *Let $\mathscr{A}_r = \{A_1, A_2, \ldots, A_r\}$ be a collection of $r \geqq 4$ mutually disjoint closed convex sets in P^2 with the property that no element of $\mathscr{A}_r$ is contained in a convex hull of any other two elements of $\mathscr{A}_r$. Then $\mathscr{A}_r$ is affinely embeddable.*

In preparation for the proof, we henceforth assume the hypotheses of the main theorem.

We first note that our claim is immediate if $\mathscr{A}_r$ contains distinct A_i and A_j which do not have four distinct common supporting lines. For then there exists a common supporting line T such that $A_i \subset T$ say. If T meets a third set $A_k \in \mathscr{A}_r$ then it is clear that either $A_i \subset H_m(j, k)$ or $A_i \subset H_n(j, k)$; a contradiction. Thus T meets no other element of $\mathscr{A}_r$, and $\mathscr{A}_r$ is then affinely embeddable by Lemma 2.

So we assume that any two distinct A_i and A_j in $\mathscr{A}_r$ have four distinct common supporting lines M_{ij}, M_{ji}, N_{ij} and N_{ji}. We note the following evident facts about $H_m(i, j)$ and $H_n(i, j)$:

5. $M_{ij} \cup N_{ij} \subset H_m(i, j) \cup H_n(i, j)$.

6. Any line in P^2 meets $H_m(i, j) \cup H_n(i, j)$.

Let $A \subset P^2$ be convex with disjoint non-empty connected subsets B and C. Let L be a line such that $L \subset \text{rel int } A \neq \emptyset$ and L contains neither B nor C. Then $A \setminus L$ consists of two components, and we say that L *separates B and C in A* if B is contained in the closure of one component and C in the closure of the other.

In the following, we always assume that A_i, A_j, A_k and A_l are distinct elements of $\mathscr{A}_r$.

7. LEMMA. *$M_{ij}^0 \cup M_{ji}^0 \subset \operatorname{int} H_n(i,j)$ and $N_{ij}^0 \cup N_{ji}^0 \subset \operatorname{int} H_m(i,j)$.*

PROOF. Because of the various symmetries, we need only to show $M_{ij}^0 \subset \subset \operatorname{int} H_n(i,j)$, say. We recall that

$$M_{ij}^* = M_{ij} \cap H_m(i,j) \quad \text{and} \quad M_{ij}^0 = M_{ij} \setminus M_{ij}^*.$$

Thus M_{ij}^0 is a (relatively) open line segment and $M_{ij}^0 \cap H_m(i,j) = \emptyset$. So by 5, $M_{ij}^0 \subset \subset H_n(i,j)$ and since M_{ij}^0 is relatively open, $M_{ij}^0 \subset \operatorname{int} H_n(i,j)$. □

8. LEMMA. *If $L \in \mathscr{L}_{ij}^m$ then L meets each of N_{ij}^*, N_{ji}^*, M_{ij}^0 and M_{ji}^0.*

PROOF. If $L \in \mathscr{L}_{ij}^m$ then $L \cap H_m(i,j) = \emptyset$. Since $H_m(i,j)$ contains $M_{ij}^* \cup M_{ji}^*$ by definition and $N_{ij}^0 \cup N_{ji}^0$ by Lemma 7, the claim is immediate. □

9. LEMMA. *M_{ij} separates A_i and A_j in $H_n(i,j)$.*

PROOF. Let $L \in \mathscr{L}_{ij}^m$. Then $L \cap H_m(i,j) = \emptyset$ and 6 yield that

$$L \cap [H_n(i,j) \setminus (A_i \cup A_j)] \neq \emptyset.$$

Clearly L separates A_i and A_j in $H_n(i,j)$. As M_{ij} is a limit of lines in L_{ij}^m, the claim thus follows. □

We note that Lemma 9 states that the line segment $M_{ij} \cap H_n(i,j)$ separates A_i and A_j in $H_n(i,j)$. Since $M_{ij}^0 \subset M_{ij} \cap H_n(i,j)$, we simply say that M_{ij}^0 is a *separating line segment* of A_i and A_j in $H_n(i,j)$, and similarly so is M_{ji}^0. Also N_{ij}^0 and N_{ji}^0 are separating line segments of A_i and A_j in $H_m(i,j)$.

10. LEMMA. *Let $A_k \cap H_n(i,j)$ be non-empty and connected such that A_k does not meet both N_{ij}^* and N_{ji}^*. Then $H_n(i,j)$ contains one of N_{ik}^0, N_{ki}^0, M_{ik}^0 or M_{ki}^0 and one of N_{jk}^0, N_{kj}^0, M_{jk}^0 or M_{kj}^0.*

PROOF. Since $A_k \cap H_n(i,j)$ is connected, $\{A_k, H_n(i,j)\}$ is affinely embeddable by Lemma 3. Let L be a line such that

$$L \cap (A_k \cup H_n(i,j)) = \emptyset \quad \text{and} \quad H = H_L(A_k \cup H_n(i,j)).$$

Then $A_i \cup A_k \subset H$ and exactly one of the hulls of A_i and A_k, say $H_m(i,k)$, is contained in H. Then $L \cap (H_m(i,k) \cup H_n(i,j)) = \emptyset$ yields that the two hulls are affinely embeddable and thus by Lemma 1, $H_m(i,k) \cap H_n(i,j)$ is connected. Because H is convex and so does not contain any line, it follows that $H_m(i,k)$ contains every line segment in H with end points in A_i and A_k.

By hypothesis, we may assume that $A_k \cap N_{ij}^* = \emptyset$, say.

If $A_k \cap \operatorname{int} H_n(i,j) = \emptyset$ then N_{ij} is a common supporting line of A_i and A_k which separates $H_n(i,j)$ and A_k in H. But then N_{ij} also separates A_i and A_k in H and, since $A_i \cup A_k \subset H_m(i,k) \subset H$, in $H_m(i,k)$ as well. Thus by Lemma 9, N_{ij} is either N_{ik} or N_{ki} and the line segment of N_{ij}^* bounded by A_i and A_k is either N_{ik}^0 or N_{ki}^0. Hence $N_{ik}^0 \subset N_{ij}^* \subset H_n(i,j)$ or $N_{ki}^0 \subset N_{ij}^* \subset H_n(i,j)$.

If $A_k \cap \operatorname{int} H_n(i,j) \neq \emptyset$, we set $A_k' = A_K \cap H_n(i,j)$. Then A_i and A_k' have four common supporting lines, one of which necessarily separates A_i and A_k' in

$H_L(A_i \cup A_k') \subset H_n(i,k)$ and supports A_k' outside of N_{ij}^*; we denote this line by T. Since A_k is convex, $T \cap A_k' \neq \emptyset$, and $T \cap (A_k' \cap N_{ij}^*) = \emptyset$, it follows that

$$T \cap A_k = T \cap (A_k \setminus N_{ij}^*)$$

and thus T is also a supporting line of A_k. Then T separates A_i and A_k in $H_m(i,k)$, so T is N_{ik} or N_{ki}, and as in the preceding N_{ik}^0 or N_{ki}^0 is in $H_n(i,j)$.

Replacing A_i by A_j in the preceding yields that N_{jk}^0 or N_{kj}^0 is in $H_n(i,j)$. □

11. LEMMA. *Let $A_k \cap H_n(i,j)$ be disconnected. Then $H_n(i,j)$ contains N_{ik}^0 or N_{ki}^0 and M_{ik}^0 or M_{ki}^0 (and N_{jk}^0 or N_{kj}^0, and M_{jk}^0 or M_{kj}^0).*

PROOF. Since A_k and $H_n(i,j)$ are convex, $A_k \cap H_n(i,j)$ consists of two components. Since $A_k \cap (A_i \cup A_j) = \emptyset$, one of these components (say A_k^{ij}) meets only N_{ij}^* and the other (say A_k^{ji}) meets only N_{ji}^*. Let

$$b \in A_k^{ij}, \quad c \in A_k^{ji} \quad \text{and} \quad L \in \mathscr{L}_{ik}.$$

Obviously, $\langle b,c\rangle \cap A_k \not\subset H_n(i,j)$ and $\langle b,c\rangle \setminus A_k \subset H_n(i,j)$. Hence L meets the line segment in $H_n(i,j)$ with end points b and c. Then $L \cap A_i = \emptyset$ yields either

(1) L separates A_i from A_k^{ij} but not from A_k^{ji} in $H_n(i,j)$, or

(2) L separates A_i from A_k^{ji} but not from A_k^{ij} in $H_n(i,j)$.

Clearly the sets $\mathscr{L}_1 = \{L \in \mathscr{L}_{ik} | (1)\}$ and $\mathscr{L}_2 = \{L \in \mathscr{L}_{ik} | (2)\}$ are non-emtpy, connected and disjoint. Since $\mathscr{L}_{ik} = \mathscr{L}_1 \cup \mathscr{L}_2$, we may assume that

(3) $\mathscr{L}_1 = \mathscr{L}_{ik}^m$ and $\mathscr{L}_2 = \mathscr{L}_{ik}^n$.

From (1) and (2), $H_m(i,k) \cap H_n(i,j)$ and $H_n(i,k) \cap H_n(i,j)$ are both disconnected. But now (1) and (3) imply that A_i and A_k^{ji} are in the same component of $H_m(i,k) \cap H_n(i,j)$, and (2) and (3) imply that A_i and A_k^{ij} are in the same component of $H_n(i,k) \cap H_n(i,j)$.

Restricting our attention to A_i, A_k^{ji} and $H_m(i,k)$, we now argue as in the proof of Lemma 10. □

12. Let $\lambda = i$ or j. As in Lemma 11, if $A_k \cap H_n(i,j)$ is disconnected we denote by A_k^{ij} [A_k^{ji}] the component of $A_k \cap H_n(i,j)$ meeting N_{ij}^* [N_{ji}^*]; then A_λ and A_k^{ij} [A_λ and A_k^{ji}] are in the same component of $H_n(\lambda,k) \cap H_n(i,j)$ [$H_m(\lambda,k) \cap H_n(i,j)$]. By Lemma 11, we may also assume that

12.1 if $M_{\lambda k} \neq N_{ji}$ then $M_{\lambda k}^0$ is a separating line segment of A_λ and A_k^{ij} in $H_n(\lambda,k) \cap \cap H_n(i,j)$; otherwise, $M_{\lambda k}^0$ is the line segment of N_{ij}^* bounded by A_λ and A_k^{ij}, and

12.2 if $N_{\lambda k} \neq N_{ji}$ then $N_{\lambda k}^0$ is a separating line segment of A_λ and A_k^{ji} in $H_m(\lambda,k) \cap H_n(i,j)$; otherwise, $N_{\lambda k}^0$ is the line segment of N_{ji}^* bounded by A_λ and A_k^{ji}. From this labelling, it is clear that

$$N_{ik}^0 \cap N_{jk}^0 = \emptyset$$

and if a line meets both N_{ik}^0 and N_{jk}^0 then it meets A_k^{ji}.

Let $N \in \mathscr{L}_{ik}^n$. Since $N \cap H_n(i,k) = \emptyset$, it follows from Lemmas 7, 8 and the preceding that

12.3 $N \cap M_{ik}^0 = \emptyset$ and $N \cap N_{ik}^0 \neq \emptyset$,

and hence $N\cap N^0_{jk}=\emptyset$. Since $H_n(i,k)$ is convex, any point of $P^2\setminus H_n(i,k)$ lies on a line disjoint from $H_n(i,k)$; that is, on a line of $\mathscr{L}^n_{ik}$. Therefore

12.4 $N^0_{jk}\subset H_n(i,k)$.

From 12.2, we also note that either $N^0_{ik}\subset N^*_{ji}$ or N^0_{ik}, N^*_{ji}, ∂A_i and ∂A^{ji}_k enclose a closed region in $H_m(i,k)$. Since $N\cap(A_i\cup A^{ji}_k)=\emptyset$ for $N\in\mathscr{L}^n_{ik}$, 12.3 implies that

12.5 N meets N^*_{ji} between A_i and A^{ji}_k.

Thus $N^0_{ji}=N_{ji}\setminus N^*_{ji}\subset H_N([A_i\cap N^*_{ji}]\cup[A^{ji}_k\cap N^*_{ji}])\subset H_n(i,k)$.

Choosing $M\in\mathscr{L}^m_{ik}$, a similar argument yields that $N^0_{ij}\subset H_m(i,k)$ and thus

12.6 $N^0_{ij}\cap N^0_{ji}\subset H_m(i,k)\cap H_n(i,k)$.

Finally we note the following important consequence of 12.4 and 12.5:

12.7 either $N^0_{jk}\subset N^*_{ji}$ or N^0_{jk}, N^*_{ji}, ∂A_j and ∂A^{ji}_k enclose a closed region in $H_n(i,k)\cap H_n(i,j)$.

13. LEMMA. *Always under the hypotheses of the main theorem,* $A_k\cap\cap(M^0_{ij}\cup M^0_{ji}\cup N^0_{ij}\cup N^0_{ji})=\emptyset$ *for* $\{A_i,A_j,A_k\}\subset\mathscr{A}_r$.

PROOF. By symmetry, it suffices to show that $A_k\cap N^0_{ij}=\emptyset$.

Suppose that $A_k\cap N^0_{ij}$ contains a point p. Then $A_k\cap N^*_{ij}=\emptyset$ follows because A_k is convex, $A_k\cap(A_i\cup A_j)=\emptyset$, and $N_{ij}=N^*_{ij}\cup N^0_{ij}$ meets both A_i and A_j. Thus $A_k\cap H_n(i,j)$ is connected, and by Lemma 3

$$\mathscr{L}=\{L\subset P^2\,|\,L\cap(A_k\cup H_n(i,j))=\emptyset\}\neq\emptyset.$$

Let $L\in\mathscr{L}$. By 6, $L\cap(H_m(i,j)\cup H_n(i,j))\neq\emptyset$ and thus L meets $H_m(i,j)$ Since $L\cap H_n(i,j)=\emptyset$, $L\in\mathscr{L}^n_{ij}$ and hence L meets both M^*_{ij} and M^*_{ji} by Lemma 8 Since $p\notin L$, L separates p from either A_i or A_j in $H_m(i,j)$. Let $\mathscr{L}_i[\mathscr{L}_j]$ denote the set of lines in $\mathscr{L}$ which separate p from $A_i[A_j]$ in $H_m(i,j)$. Clearly

$$\mathscr{L}=\mathscr{L}_i\cup\mathscr{L}_j,\quad\mathscr{L}_i\cap\mathscr{L}_j=\emptyset$$

and $\mathscr{L}_i$ and $\mathscr{L}_j$ are connected (in the space of lines of P^2). Since $\mathscr{L}$ itself is connected if and only if $A_k\cap H_n(i,j)\neq\emptyset$;

(1) either $\mathscr{L}=\mathscr{L}_i$ or $\mathscr{L}=\mathscr{L}_j$ if $A_k\cap H_n(i,j)\neq\emptyset$, and

(2) $\mathscr{L}_i\neq\emptyset\neq\mathscr{L}_j$ if $A_k\cap H_n(i,j)=\emptyset$.

As N_{ij} and N_{ji} are supporting lines of $H_n(i,j)$ and neither A_i nor A_j is a singleton,

$$N_{ij}\cap N_{ji}=N^0_{ij}\cap N^0_{ji}=\{q\}$$

and $q\in\operatorname{int}H_m(i,j)$. Let $a_i\in A_i$ and $b_j\in A_j$ denote the end points of N^0_{ij}. Then either

(3) p is between a_i and q (or $p=q$) and the line $\langle b,p\rangle$ meets A_i for any $b\in A_j$; or

(4) p is between b_j and q (or $p=q$) and the line $\langle a,p\rangle$ meets A_j for any $a\in A_i$.

CASE 1: $\mathscr{L}$ is disconnected.

If (3) then we choose $L\in\mathscr{L}_i$. Since $L\cap(A_i\cup A_j)=\emptyset$, we may assume that $L\in\mathscr{L}_{ik}^m$ and $H_L(i,k)=H_m(i,k)$. Let $b\in A_j$. By (3),

$$\langle b,p\rangle=\langle b,p,a\rangle \quad \text{for some} \quad a\in A_i.$$

As L separates p from A_i in $H_m(i,j)$, L separates p from a in $\langle b,p,a\rangle\cap H_m(i,j)$. Thus L meets the segment of $\langle b,p,a\rangle$ with end points p and a which does not contain b, and

$$b\in\langle b,p,a\rangle\cap H_L(i,k)\subset H_m(i,k).$$

As b is any point of A_j, $A_j\subset H_m(i,k)$; a contradiction.

If (4), then we choose $L\in\mathscr{L}_i$ and obtain that $A_i\subset H_m(j,k)$; again a contradiction.

CASE 2: $\mathscr{L}$ is connected.

From Case 1, it suffices to show that (3) [(4)] implies that $\mathscr{L}=\mathscr{L}_i[\mathscr{L}_j]$. By (1), $A_k\cap H_n(i,j)\neq\emptyset$. Thus $A_k\cap N_{ij}^*=\emptyset$ yields that there is a point $p'\in A_k\cap N_{ji}^*$. As $p'\notin A_i\cup A_j$, there are two supporting lines of $A_i[A_j]$ passing through p'.

If (3), we consider p' and A_i. One of the supporting lines of A_i through p' is N_{ji}, and let T denote the other. Let $\mathscr{P}$ denote the closed half-plane of P^2 determined by N_{ji} and T which contains A_i. The segment of N_{ij}^0 with end-points a_i and q is contained in $\mathscr{P}$, and thus $p\in\mathscr{P}$ and

(5) the line $\langle p,p'\rangle\subset\mathscr{P}$ meets A_i.

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ denote the distinct closed segments of $\langle p,p'\rangle$ with end points p and p'. Since $\{p,p'\}\subset A_k$ and A_k is convex, we may assume that $\mathscr{S}_1\subset A_k$. Then $A_i\cap A_k=\emptyset$ and (5) imply that

$$\langle p,p'\rangle\cap A_i\subset\mathscr{S}_2.$$

Since $\langle p,p'\rangle\cap A_i$ is a segment, $\mathscr{S}_2\setminus A_i$ consists of two segments $\mathscr{S}_{21}$ and $\mathscr{S}_{22}$ say. Let p' be an end point of $\mathscr{S}_{21}$ and p be an end point of $\mathscr{S}_{22}$. Then $\{p',A_i\}\subset H_n(i,j)$ and $p\notin H_n(i,j)$ yield that

$$\mathscr{S}_{21}\subset H_n(i,j) \quad \text{and} \quad \mathscr{S}_{22}\subset H_m(i,j).$$

Thus

(6) $\langle p,p'\rangle\setminus(A_k\cup H_n(i,j))\subset\mathscr{S}_{22}$.

If $L\in\mathscr{L}$ then L necessarily meets $\mathscr{S}_{22}$ by (6) and thus L separates p and $\langle p,p'\rangle\cap A_i$ in $H_m(i,j)$. Since A_i is convex and $L\cap A_i=\emptyset$, it follows that $L\in\mathscr{L}_i$.

If (4), then we consider p' and A_j and argue similarly to obtain $\mathscr{L}=\mathscr{L}_j$. Thus $A_k\cap N_{ij}^0=\emptyset$. □

We note that Lemma 13 is the result upon which the proof of the main theorem is founded. Our proof is by induction, and Lemmas 14 and 15 are preliminary steps which are direct consequences of Lemma 13.

14. LEMMA. *Let* $\{A_i,A_j,A_k\}\subset\mathscr{A}_r$, *and let* H *be a convex set in* P^2 *such that* $H_n(i,j)\subset H$ *and* N_{ij} *is a supporting line of* H. *If* $A_k\cap N_{ij}^*=\emptyset$ *then* $\{A_k,H\}$ *is affinely embeddable.*

PROOF. If $A_k \cap N_{ij}^* = \emptyset$ then by Lemma 13, $N_{ij} = N_{ij}^* \cap N_{ij}^0$ is a supporting line of H which is disjoint from A_k. Since A_k is closed, there is a line arbitrarily close to N_{ij} which does not meet $A_k \cup H$. □

15. LEMMA. *Since* $r \geqq 4$, *each* $\{A_i, A_j, A_k\} \subset \mathscr{A}_r$ *is affinely embeddable.*

PROOF. By Lemma 3, it is sufficient to show that $A_k \cap H_n(i,j)$ or $A_k \cap H_m(i,j)$ is connected. Suppose that $A_k \cap H_n(i,j)$ is disconnected (with the labelling of Lemma 12).

Let $\mathscr{P}$ and $\mathscr{P}'$ denote the two closed half-planes of P^2 determined by N_{ij} and N_{ji}. As N_{ij} and N_{ji} are supporting lines of $H_n(i,j)$, $H_n(i,j) \subset \mathscr{P}$ say. Since $N_{ij} \cap \cap N_{ji} = N_{ij}^0 \cap N_{ji}^0 = \{q\}$, any line through q in $\mathscr{P}$ meets both A_i and A_j and it is clear that

(1) $\mathscr{P} \setminus H_n(i,j) \subset H_m(i,j)$.

We note that A_k meets both N_{ij}^* and N_{ji}^*, $\mathscr{P} \setminus A_k$ is disconnected and

$$A_k' = \mathscr{P}' \cap A_k$$

is connected. By Lemma 13, $q \notin A_k$ and thus $\mathscr{P}' \setminus (A_k' \cup \{q\})$ consists of two distinct components, say $\mathscr{P}_1'$ and $\mathscr{P}_2'$. Let $p \in A_k'$. Then $\langle p, q \rangle \subset \mathscr{P}'$ and $\langle p, q \rangle$ meets both $\mathscr{P}_1'$ and $\mathscr{P}_2'$. We next note that for any $p_1 \in \mathscr{P}_1'$ and $p_2 \in \mathscr{P}_2'$,

(2) $\langle p_1, p_2 \rangle \cap A_k' \neq \emptyset$.

Let $T \subset P^2$ such that $T \cap (A_k \cup \{q\}) = \emptyset$. Then by (2), T meets either $\mathscr{P}_1'$ or $\mathscr{P}_2'$ and it follows that

(3) $\mathscr{P}_1' \subset H_T(A_k' \cup \{q\}) \subseteqq H_T(A_k \cup \{q\})$ or $\mathscr{P}_2' \subseteqq H_T(A_k \cup \{q\})$.

Now let $L \in \mathscr{L}_{ik} = \mathscr{L}_{ik}^m \cup \mathscr{L}_{ik}^n$. Thus $H_L(i,k)$ is $H_m(i,k)$ or $H_n(i,k)$ and $L \cap H_L(i,k) = \emptyset$. Since $A_k \cup \{q\} \subset H_L(i,k)$ by 12.6, we have that $H_L(A_k \cup \{q\}) \subset \subset H_L(i,k)$. Thus by (3), either $\mathscr{P}_1' \subset H_L(i,k)$ or $\mathscr{P}_2' \subset H_L(i,k)$. Clearly $\{L \in \mathscr{L}_{ik} | \mathscr{P}_1' \subset H_L(i,k)\}$ and $\{L \in \mathscr{L}_{ik} | \mathscr{P}_2' \subset H_L(i,k)\}$ are non-empty, connected and disjoint, and thus we may assume that say

(4) $\mathscr{P}_1' \subset H_n(i,k)$ and $\mathscr{P}_2' \subset H_m(i,k)$.

Since $r \geqq 4$, there is an $A_l \in \mathscr{A}_r \setminus \{A_i, A_j, A_k\}$. By Lemma 13, $A_l \cap (N_{ij}^0 \cup N_{ji}^0) = \emptyset$. As $\mathscr{P} \setminus H_n(i,j)$ is a region bounded by A_i, A_j, N_{ij}^0 and N_{ji}^0, (1) and $A_l \not\subset H_m(i,j)$ yield that $A_l \cap (\mathscr{P} \setminus H_n(i,j)) = \emptyset$ and thus $A_l \subset \mathscr{P}' \cup H_n(i,j)$. Next $A_l \cap A_k = \emptyset$ and (2) imply that either $A_l \cap \mathscr{P}_1' = \emptyset$ or $A_l \cap \mathscr{P}_2' = \emptyset$. Thus we may assume that

(5) $A_l \subset \mathscr{P}_1' \cup H_n(i,j) \subset H_n(i,k) \cup H_n(i,j)$.

We now consider $H_n(i,k) \cap H_n(i,j)$. From 12.1, we observe that either M_{ik}^0, N_{ij}^*, ∂A_i and ∂A_k^{ij} enclose a closed region $\mathscr{R}$ in $H_n(i,k) \cap H_n(i,j)$, or $M_{ik}^0 \subset N_{ij}^*$. In the latter case we set $\mathscr{R} = M_{ik}^0$ and note that $\mathscr{R}$ is still contained in $H_n(i,k)$ by Lemma 7. By 12.7, either N_{jk}^0, N_{ji}^*, ∂A_j and ∂A_k^{ji} enclose a closed region $\mathscr{S}$ in $H_n(i,k) \cap H_n(i,j)$, or $N_{jk}^0 \subset N_{ji}^*$. Then we set $\mathscr{S} = N_{jk}^0$ in the latter case and note that now $\mathscr{S} \subset H_n(i,k)$ by 12.4. Thus altogether

(6) $\mathscr{P}_1' \cup \mathscr{R} \cup \mathscr{S} \subset H_n(i,k)$

and

$$(7)\quad (\mathscr{P}_1' \cup H_n(i,j)) \setminus H_n(i,k) = (\mathscr{P}_1' \cup H_n(i,j)) \setminus (\mathscr{P}_1' \cup \mathscr{R} \cup \mathscr{S}) \subset H_n(i,j).$$

Suppose that $A_l \cap (\mathscr{P}_1' \cup \mathscr{R} \cup \mathscr{S}) \neq \emptyset$. Since $A_l \subset H_n(i,k)$ by hypothesis, (6) and (4) imply that

$$A_l \cap [(\partial\mathscr{R}' \cup \partial\mathscr{S}) \setminus \mathscr{P}_1'] \neq \emptyset.$$

But

$$(\partial\mathscr{R} \cup \partial\mathscr{S}) \setminus \mathscr{P}_1' \subset \partial A_i \cup \partial A_i \cup \partial A_k \cup M_{ik}^0 \cup N_{jk}^0$$

and $A_l \cap (A_i \cup A_j \cup A_k) = \emptyset$ then yield that $A_l \cap (M_{ik}^0 \cup N_{jk}^0) \neq \emptyset$; a contradiction by Lemma 13. Thus by (5) and (7),

$$A_l \subset (\mathscr{P}_1' \cup H_n(i,j)) \setminus (\mathscr{P}_1' \cup \mathscr{R} \cup \mathscr{S}) \subset H_n(i,j);$$

again a contradiction. Hence $A_k \cap H_n(i,j)$ is connected. □

Proof of the main theorem

First we consider the case $r=4$. By Lemma 15, $\{A_1, A_2, A_3\}$ is affinely embeddable. Let $L \subset P^2$ such that $L \cap (A_1 \cup A_2 \cup A_3) = \emptyset$ and set $H = H_L(A_1 \cup A_2 \cup A_3)$. We may assume that $L \in \mathscr{L}_{ij}^n$ and hence $H_n(i,j) \subset H$ for $\{i,j\} \subset \{1,2,3\}$. By hypothesis, $H \neq H_n(i,j)$ and thus each of A_1, A_2 and A_3 meets the boundary of H. As H is a "triangular" region with "vertices" A_1, A_2 and A_3, each $H_n(i,j)$ has a supporting line which supports A_i, A_j and H, say N_{ij}. Since $A_4 \cap (A_1 \cup A_2 \cup A_3) = \emptyset$, we thus obtain that if $A_4 \cap H \neq \emptyset$ then A_4 necessarily meets some $N_{ij} \cap H$. As A_4 is convex and H is a convex "triangular" region, A_4 meets at most two of the $N_{ij} \cap H$. Let $A_4 \cap (N_{12} \cap H) = \emptyset$ say. Then

$$N_{12}^* = N_{12} \cap H_n(1,2) \subset N_{12} \cap H$$

implies that $A_4 \cap N_{12}^* = \emptyset$. Thus by Lemma 14, $\{A_4, H\}$ is affinely embeddable. Since $\bigcup_{s=1}^{4} A_s \subset A_4 \cup H$, $\mathscr{A}_4$ is affinely embeddable.

We now assume that $r>4$ and that any proper subset of $\mathscr{A}_r = \{A_1, A_2, \ldots, A_r\}$ is affinely embeddable. Let $L \subset P^2$ such that $L \cap \bigcup_{s=1}^{r-1} A_s = \emptyset$ and set $H = H_L(\bigcup_{s=1}^{r-1} A_s)$. Again we assume that $L \in \mathscr{L}_{ij}^n$ and $H_n(i,j) \subset H$ for $\{i,j\} \subset \{1,2,\ldots,r-1\}$. By hypothesis, $H \neq H_n(i,j)$ and thus at least three elements of $\mathscr{A}_r' = \mathscr{A}_r \setminus \{\mathscr{A}_r\}$ meet the boundary of H.

If exactly three elements of $\mathscr{A}_r'$, say A_1, A_2 and A_3, meet ∂H then H is a convex "triangular" region with "vertices" A_1, A_2 and A_3. Now the preceding argument yields that $\{A_r, H\}$, and hence $\mathscr{A}_r$, is affinely embeddable.

Let at least four elements of $\mathscr{A}_r'$ meet ∂H. Then there exist pairs A_i, A_j in $\mathscr{A}_r'$ such that A_i, A_j and H have a common supporting line. We call such sets A_i and A_j *adjacent*. Since at least four elements of $\mathscr{A}_r'$ meet ∂H, there exist four sets, say A_1, A_2, A_3 and A_4, in $\mathscr{A}_r'$ such that each of the pairs A_1, A_2; A_2, A_3; A_3, A_4 is adjacent. Let N_{12}, N_{23} and N_{34} be the corresponding supporting lines of H. Let $H' = H_L(\bigcup_{s=1}^{4} A_s) \subseteq H$. Clearly N_{12}, N_{23} and N_{34} are also supporting lines of the "quad-

rangle" H' with "vertices" A_1, A_2, A_3 and A_4. As A_r and H' are both convex, it again follows that A_r meets at most two of the $N_{ij}\cap H'$ and thus $A_r\cap(N_{12}\cap H')=\emptyset$, say. Since N_{12} is a supporting line of $H_n(1, 2)$, H' and H, and A_1 and A_2 are adjacent; we obtain that

$$N_{12}^* = N_{12}\cap H_n(1, 2) \subseteq N_{12}\cap H' \subseteq N_{12}\cap H$$

and $A_r\cap N_{12}^*=\emptyset$. Thus by Lemma 14 $\{A_r, H\}$, and hence $\mathscr{A}_r$, is affinely embeddable. □

As a point of interest, we suspect that the following also implies that $\mathscr{A}_r$ ($r\geqq 3$) is affinely embeddable. For $A_i\neq A_j$ in $\mathscr{A}_r$, either $A_l\cap H_m(i, j)=\emptyset$ for all $A_l\in \in\mathscr{A}_r\setminus\{A_i, A_j\}$ or $A_l\cap H_n(i, j)=\emptyset$ for all $A_l\in\mathscr{A}_r\setminus\{A_i, A_j\}$.

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DECOMPOSITION OF LINEAR SINGULARLY PERTURBED SYSTEMS

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In this paper, the following system of differential equations will be considered:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + f_1; \quad \varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + f_2, \tag{1}$$

where $x_i \in R^{n_i}$, $A_{ij} = A_{ij}(t, \varepsilon)$ are matrix functions, $f_i = f_i(t, \varepsilon)$ are vector functions $(i, j = 1, 2)$, $t \in R$, $\varepsilon > 0$ is a small parameter. Such systems appear in some problems of mechanics [1, 2] and control [3—6].

The object of our investigation is to obtain a transformation, allowing to reduce (1) to a system of the form

$$\dot{y}_1 = A_1(t, \varepsilon)y_1 + f(t, \varepsilon); \quad \varepsilon\dot{y}_2 = A_2(t, \varepsilon)y_2 \tag{2}$$

and to discuss some applications in stability, boundary value and control problems.

Recently, the decomposition in boundary value problems and in optimal control problems has received great attention; the reader can find comprehensive surveys in [3—8]. Note that for systems with constant coefficients our transformation is equivalent to the one in [9], but for systems with variable coefficients is different from the transformation in [10].

1. The transformation

Throughout this paper, the matrices A_{ij}, $(A_{22}^{(0)})^{-1}$ and the functions f_i are assumed to have high order continuous and bounded derivatives with respect to t and ε for $t \in R$, $\varepsilon \in [0, \varepsilon_0]$ and, therefore, they can be represented as asymptotic expansions

$$A_{ij} = A_{ij}^{(0)}(t) + \varepsilon A_{ij}^{(1)}(t) + \ldots + \varepsilon^k A_{ij}^{(k)}(t) + \varepsilon^{k+1} A_{ij}^{(k+1)}(t, \varepsilon), \tag{1.1}$$

$$f_i = f_i^{(0)}(t) + \varepsilon f_i^{(1)}(t) + \ldots + \varepsilon^k f_i^{(k)}(t) + \varepsilon^{k+1} f_i^{(k+1)}(t, \varepsilon) \tag{1.2}$$

with smooth coefficients.

Let us suppose that (1) conforms with the following hypotheses.

(i) There exist $\alpha \geqq 0$ and $K > 0$ such that the fundamental matrix $U(t, s)$ of the equation $\dot{x}_1 = A_1^{(0)}x_1$, $A_1^{(0)} = A_{11}^{(0)} - A_{12}^{(0)}(A_{22}^{(0)})^{-1}A_{21}^{(0)}$ satisfies

$$|U(t, s)| \leqq K \exp[-\alpha(t-s)], \quad -\infty < t \leqq s < \infty. \tag{1.3}$$

(ii) There exist $\beta>0$, $N>0$ such that the fundamental matrix $V(t, s\,\varepsilon)$ of the equation $\varepsilon\dot{x}_2=(A_2^{(0)}+\varepsilon A_2^{(1)})x_2$, $A_2^{(0)}=A_{22}^{(0)}$, $A_2^{(1)}=A_{22}^{(1)}+(A_{22}^{(0)})^{-1}A_{21}^{(0)}A_{12}^{(0)}$ satisfies

$$|V(t, s, \varepsilon)| \leqq N\exp[-\varepsilon^{\nu}\beta(t-s)], \quad -\infty < s \leqq t < \infty, \quad 0 < \varepsilon \leqq \varepsilon_0 \tag{1.4}$$

where $\nu=-1, 0$; $\beta>\alpha$ if $\nu=0$.

Let us introduce new variables y_1, y_2 by the formulae $y_2=x_2-Hx_1-h$, $y_1=x_1-\varepsilon Py_2$ where $H=H(t, \varepsilon)$, $P=P(t, \varepsilon)$ are bounded matrix functions and and $h=h(t, \varepsilon)$ is a vector function such that y_1, y_2 satisfy (2), where

$$A_1 = A_{11}+A_{12}H, \quad A_2 = A_{22}-\varepsilon HA_{12}, \quad f=f_1+A_{12}h.$$

This is possible if H, P and h are bounded solutions of the equations

$$\varepsilon\dot{H}+\varepsilon H(A_{11}+A_{12}H) = A_{21}+A_{22}H, \tag{1.5}$$

$$\varepsilon\dot{P}+PA_2 = \varepsilon A_1P+A_{12}, \tag{1.6}$$

$$\varepsilon\dot{h}+\varepsilon Hf_1 = A_2h+f_2, \tag{1.7}$$

on R.

It is clear that the hyperplane $x_2=Hx_1+h$ is an invariant manifold of (1). Some existence, stability and asymptotic expansion problems for invariant manifolds of linear systems were discussed in [1, 11].

We shall show that there exist bounded solutions of (1.5)—(1.7) and they may be represented as asymptotic expansions.

Let us introduce the following notations:

$$\begin{cases} H^{(0)} = -(A_{22}^{(0)})^{-1}A_{21}^{(0)}; \quad H^{(1)} = -(A_{22}^{(0)})^{-1}[A_{21}^{(1)}+A_{22}^{(1)}H^{(0)}-\dot{H}^{(0)}-H^{(0)}A_1^{(0)}], \\ H^{(i)} = -(A_{22}^{(0)})^{-1}\Big[A_{21}^{(i)}+\sum_{j=1}^{i}A_{22}^{(j)}H^{(i-j)}-\dot{H}^{(i-1)}-\sum_{j=0}^{i-1}H^{(i-j-1)}A_1^{(j)}\Big] \end{cases} \tag{1.8}$$

where $A_1^{(i)}=A_{11}^{(i)}+\sum_{j=0}^{i}A_{12}^{(j)}H^{(i-j)}$, $i=1, \ldots, k$ and let

$$H = \sum_{i=0}^{k}\varepsilon^iH^{(i)}+F. \tag{1.9}$$

Then F is a bounded solution of the equation

$$\varepsilon\dot{F}+\varepsilon F(A_{11}+A_{12}H_k+A_{12}F) = \varepsilon^{k+1}A_3^{(k+1)}+(A_{22}-\varepsilon H_kA_{12})F \tag{1.10}$$

where

$$H_k = \sum_{i=0}^{k}\varepsilon^iH^{(i)}; \quad \varepsilon^{k+1}A_3^{(k+1)} = A_{21}+A_{22}H_k-\varepsilon\dot{H}_k-\varepsilon H_k(A_{11}+A_{12}H_k) = O(\varepsilon^{k+1})$$

and therefore F is the solution of the equation

$$F(t, \varepsilon) = \frac{1}{\varepsilon}\int_{-\infty}^{t}V_k(t, s, \varepsilon)[\varepsilon^{k+1}A_3^{(k+1)}(s, \varepsilon)-\varepsilon F(s, \varepsilon)A_{12}(s, \varepsilon)F(s, \varepsilon)]U_k(s, t, \varepsilon)\,ds \tag{1.11}$$

where $V_k(t, s, \varepsilon)$ and $U_k(t, s, \varepsilon)$ are the fundamental matrices of the equations $\varepsilon\dot{x}_2=(A_{22}-\varepsilon H_kA_{12})x_2$ and $\dot{x}_1=(A_{11}+A_{12}H_k)x_1$, resp.

Note that for sufficiently small ε, (1.3) and (1.4) imply the existence of $\alpha_1 \geqq \alpha$ and $\beta_1 \leqq \beta$ ($\beta_1 > 0$, $\beta_1 > \alpha_1$ if $\nu = 0$) such that

$$\begin{cases} |U_k(t, s, \varepsilon)| \leqq K \exp[-\alpha_1(t-s)], & -\infty < t \leqq s < \infty \\ |V_k(t, s, \varepsilon)| \leqq N \exp[-\beta_1 \varepsilon^{\nu}(t-s)], & -\infty < s \leqq t < \infty. \end{cases} \tag{1.12}$$

LEMMA 1. *Let the hypotheses* (i) *and* (ii) *hold. Then there exist* $D > 0$ *and* ε_1 $(0 < \varepsilon_1 \leqq \varepsilon_0)$ *such that* (1.11) *has a unique continuous solution* F *for which*

$$|F(t, \varepsilon)| \leqq \varepsilon^{k-\nu} D, \quad t \in R, \quad \varepsilon \in (0, \varepsilon_1]. \tag{1.13}$$

PROOF. Let B be the set of matrix functions F such that they are continuous and satisfy (1.13), and let d be the metric on B defined by $d(F, \bar{F}) = \sup_t |F(t, \varepsilon) - \bar{F}(t, \varepsilon)|$. Note that with respect to d, B is a complete metric space.

Now define an operator T on B by setting

$$(TF)(t, \varepsilon) = \frac{1}{\varepsilon} \int_{-\infty}^{t} V_k(t, s, \varepsilon)[\varepsilon^{k+1} A_3^{(k+1)}(s, \varepsilon) - \varepsilon F(s, \varepsilon) A_{12}(s, \varepsilon) F(s, \varepsilon)] U_k(s, t, \varepsilon)\, ds. \tag{1.14}$$

Our intention is to prove that T is a contraction mapping of B into itself.

It is a straightforward exercise now that the improper integral in (1.14) converges and TF is continuous on R. Also, using (1.12) and (1.14) it is easy to see that

$$|(TF)(t, \varepsilon)| \leqq \frac{KN}{\beta_1 - \alpha_1 \varepsilon^{-\nu}} [\varepsilon^{k-\nu} a + \varepsilon b (\varepsilon^{k-\nu} D)^2] \leqq \varepsilon^{k-\nu} D$$

where $a = \sup_t |A_3|$, $b = \sup_t |A_{12}|$, if ε_1 is sufficiently small and D is any positive solution of the inequality

$$\frac{KN}{\beta_1 - \alpha_1 \varepsilon_1^{-\nu}} (a + \varepsilon_1 b D^2) \leqq D.$$

Thus TF satisfies the boundedness condition required by (1.13). To prove that T is a contraction mapping, we reason as follows. Let $F, \bar{F} \in B$. Then, by (1.14)

$$|(TF)(t, \varepsilon) - (T\bar{F})(t, \varepsilon) \leqq \Big| \int_{-\infty}^{t} V_k(t, s, \varepsilon)\big[\bar{F}(s, \varepsilon) A_{12}(s, \varepsilon)\big(\bar{F}(s, \varepsilon) - F(s, \varepsilon)\big) +$$

$$+ \big(\bar{F}(s, \varepsilon) - F(s, \varepsilon)\big) A_{12}(s, \varepsilon)\big] U_k(s, t, \varepsilon)\, ds \Big| \leqq$$

$$\leqq \int_{-\infty}^{t} |V_k(t, s, \varepsilon)| 2\varepsilon^{k-\nu} D b d(F, \bar{F}) |U_k(s, t, \varepsilon)|\, ds \leqq \frac{2KN}{\beta_1 - \alpha_1 \varepsilon^{-\nu}} \varepsilon^{k-2\nu} D b d(F, \bar{F}).$$

From this last inequality it easily follows that T is a contraction mapping if $\dfrac{2KNDb}{\beta_1 - \alpha_1 \varepsilon_1^{-\nu}} \varepsilon_1^{k-2\nu} < 1$.

Thus T is a contraction mapping of B into itself and so by the Banach Contraction Mapping Principle, T must have a unique fixed point $F_0 \in B$, F_0 is the required solution of (1.11).

As a corollary, we obtain that $H=H_k+\varepsilon^{k-\nu}H^{(k+1)}(t,\varepsilon)$, where $\varepsilon^{k-\nu}H^{(k+1)}=F_0$.

Analogous facts are valid for P and h. Let $U_H(t,s,\varepsilon)$ and $V_H(t,s,\varepsilon)$ be the fundamental matrices of the equations $\dot{x}_1=(A_{11}+A_{12}H)x_1$ and $\varepsilon\dot{x}_2=(A_{22}-\varepsilon HA_{12})x_2$, resp., where $H=H_k+F_0$. By (1.12) and (1.13), there exist $\alpha_0\geqq\alpha_1$, $\beta_0\leqq\beta_1$ ($\beta_0>0$, $\beta_0>\alpha_0$ if $\nu=0$) and ε_2 such that

$$(1.15)\quad \begin{cases} |U_H(t,s,\varepsilon)| \leqq K\exp[-\alpha_0(t-s)], & -\infty<t\leqq s<\infty \\ |V_H(t,s,\varepsilon)| \leqq N\exp[-\beta_0\varepsilon^{\nu}(t-s)], & -\infty<s\leqq t<\infty, \quad 0<\varepsilon\leqq\varepsilon_2\leqq\varepsilon_1. \end{cases}$$

Now for P and h we have the exact formulas

$$P(t,\varepsilon)=-\frac{1}{\varepsilon}\int_t^{\infty}U_H(t,s,\varepsilon)A_{12}(s,\varepsilon)V_H(s,t,\varepsilon)\,ds,$$

$$h(t,\varepsilon)=\frac{1}{\varepsilon}\int_{-\infty}^{t}V_H(t,s,\varepsilon)[f_2(s,\varepsilon)-\varepsilon H(s,\varepsilon)f_1(s,\varepsilon)]\,ds.$$

Let

$$P_k=\sum_{i=0}^{k}\varepsilon^i P^{(i)}(t),\quad h_k=\sum_{i=0}^{k}\varepsilon^i h^{(i)}(t)$$

where

$$(1.16)\quad P^{(0)}=A_{12}^{(0)}(A_{22}^{(0)})^{-1};\quad P^{(i)}=\Big[A_{12}^{(i)}+\sum_{j=0}^{i-1}A_1^{(j)}P^{(i-j-1)}-\dot{P}^{(i-1)}-$$

$$-\sum_{j=1}^{i}P^{(i-j)}A_2^{(j)}\Big](A_{22}^{(0)})^{-1},\quad A_2^{(i)}=A_{22}^{(i)}-\sum_{j=0}^{i-1}H^{(i-j-1)}A_{12}^{(j)},$$

$$(1.17)\quad h^{(0)}=-(A_{22}^{(0)})^{-1}f_2^{(0)};\quad h^{(i)}=-(A_{22}^{(0)})^{-1}\Big[f_2^{(i)}+\sum_{j=1}^{i-1}A_2^{(i)}h^{(j-i)}-$$

$$-\dot{h}^{(i-1)}-\sum_{j=0}^{i-1}H^{(j)}f_1^{(i-j-1)}\Big],\quad i=1,\ldots,k.$$

Then for the errors $P-P_k$, $h-h_k$ we obtain

$$P-P_k=-\frac{1}{\varepsilon}\int_t^{\infty}U_H(t,s,\varepsilon)\varepsilon^{k+1}A_4^{(k+1)}(s,\varepsilon)V_H(s,t,\varepsilon)\,ds,$$

$$h-h_k=\frac{1}{\varepsilon}\int_{-\infty}^{t}V_H(t,s,\varepsilon)\varepsilon^{k+1}f_3^{(k+1)}(s,\varepsilon)\,ds$$

where $\varepsilon^{k+1}A_4^{(k+1)}=A_{12}+\varepsilon A_1P_k-\varepsilon\dot{P}_k-P_kA_2=O(\varepsilon^{k+1})$, $\varepsilon^{k+1}f_3^{(k+1)}=A_2h_k+f_2-\varepsilon\dot{h}_k-\varepsilon Hf_1=O(\varepsilon^{k+1})$.

Finally, from (1.15) we obtain

$$|P-P_k|\leqq\frac{KNa_0}{\beta_0-\alpha_0\varepsilon^{-\nu}}\varepsilon^{k-\nu},\quad |h-h_k|\leqq\frac{KNb_0}{\beta_0-\alpha_0\varepsilon^{-\nu}}\varepsilon^{k-\nu}$$

where $a_0=\sup_t|A_4^{(k+1)}(t,\varepsilon)|$, $b_0=\sup_t|f_3^{(k+1)}(t,\varepsilon)|$.

Now we can summarize our results in the following

THEOREM 1. *Under the assumptions of Lemma* 1 *there exist matrices*

$$H = \sum_{i=0}^{k} \varepsilon^i H^{(i)}(t) + \varepsilon^{k-\nu} H^{(k+1)}(t, \varepsilon), \quad P = \sum_{i=0}^{k} \varepsilon^i P^{(i)}(t) + \varepsilon^{k-\nu} P^{(k-1)}(t, \varepsilon)$$

and a function

$$h = \sum_{i=0}^{k} \varepsilon^i h^{(i)}(t) + \varepsilon^{k-\nu} h^{(k+1)}(t, \varepsilon)$$

where $H^{(i)}$, $P^{(i)}$, $h^{(i)}$ $(i=0, 1, \ldots, k)$ *are defined by* (1.8), (1.16), (1.17), $H^{(k+1)}$, $P^{(k+1)}$, $h^{(k+1)}$ *are continuous on* R *and bounded on* $R\times(0, \varepsilon_2]$, *such that* (1) *has an invariant manifold* $x_2=Hx_1+h$. *The transformation* $y_2=x_2-Hx_1-h$, $y_1=x_1-\varepsilon P y_2$ *reduces* (1) *to the decoupled form* (2).

REMARK 1. All the assumptions were formulated for $t\in R$, but for many problems they can be realized only at finite or semi-infinite intervals. But the argument below and Theorem 1 may hold true for the latter case too, if the functions on the right side of (1) with the necessary properties can be extended to the whole real axis.

2. The stability problem

Let $f_i=0$. Using H and P it is possible to introduce new variables y_1, y_2 by the formulae $y_1=x_1-\varepsilon P y_2=(I+\varepsilon PH)x_1-\varepsilon P x_2$, $y_2=-Hx_1+x_2$, satisfying

$$\dot{y}_1 = (A_{11}+A_{12}H)y_1, \tag{2.1}$$

$$\varepsilon\dot{y}_2 = (A_{22}-\varepsilon HA_{12})y_2. \tag{2.2}$$

Here the first equation describes the behavior of (1) on the invariant manifold of slow motions and the second one provides a similar information for the invariant manifold of fast motions.

From (1.16) we obtain that the stability of the system (1) is equivalent to the stability of the equation (2.1).

Let us consider the design of stabilizing the feedback controllers problem [12]. We have the equations

$$\dot{x}_1 = C_{11}x_1+C_{12}x_2+B_1u, \quad \varepsilon\dot{x}_2 = C_{21}x_1+C_{22}x_2+B_2u \tag{2.3}$$

where $C_{ij}=C_{ij}(t, \varepsilon)$, $B_i=B_i(t, \varepsilon)$, $u\in R^n$. Supposing $u=K_1(t, \varepsilon)x_1+K_2(t, \varepsilon)x_2$ we can see that the system (2.3) is analogous to (1) where $A_{ij}=C_{ij}+B_iK_j$, $f_i=0$ $(i, j=1, 2)$.

If C_{ij} and B_i do not depend on t and are analytic with respect to ε then H is defined by $\varepsilon H(A_{11}+A_{12}H)=A_{21}+A_{22}H$ as a convergent series. For the stabilization it is necessary and sufficient that the equations

$$\dot{y}_1 = (A_{11}+A_{12}H)y_1, \quad \varepsilon\dot{y}_2 = (A_{22}-\varepsilon HA_{12})y_2$$

were asymptotically stable.

Let us take for example the problem of stabilization for the linear autonomous gyroscopic system

$$\dot{x} = y, \quad \varepsilon\dot{y} = -[G+\varepsilon B]y - \varepsilon Cx + \varepsilon B_2 u \tag{2.4}$$

where x and y are m-vectors, G is a skew-symmetric matrix, $\det G \neq 0$, B is a positive definite symmetric matrix. Let $u=Kx$; then the system (2.4) has an invariant manifold $y=H(\varepsilon)x=G^{-1}(B_2K-C)x+\varepsilon^2$. On the latter the motion is described by the equation $\dot{x}=H(\varepsilon)x$. If K is chosen such that the matrix $G^{-1}(B_2K-C)$ is stable, then the problem of the stabilization of (2.4) is solved. It is necessary to note that in this case $v=0$ and results of [12] cannot be used.

3. Initial and boundary value problems

If we have initial conditions $x_i(t_0)=x_i^0$ for (1), then for (2) we obtain the initial conditions

$$y_1(t_0) = \big(I+\varepsilon P(t_0,\varepsilon)H(t_0,\varepsilon)\big)x_1^0 - \varepsilon P(t_0,\varepsilon)x_2^0, \quad y_2(t_0) = x_2^0 - H(t_0,\varepsilon)x_1^0 - h(t_0,\varepsilon).$$

If we have a boundary value problem for (1):

$$D_1\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix} + D_2\begin{pmatrix}x_1(1)\\x_2(1)\end{pmatrix} = d$$

where D_i are constant matrices and d is a constant vector then for (2) we obtain coupled conditions, which can be decoupled in some cases.

EXAMPLE. Consider the following boundary value problem for the scalar x:

$$\varepsilon\ddot{x} + a(t)\dot{x} + b(t)x = f(t), \quad a(t) \geqq a_0 > 0,$$

$$\alpha_{i1}x(0) + \alpha_{i2}x(1) + \beta_{i1}\dot{x}(0) + \beta_{i2}\dot{x}(1) = \gamma_i, \quad i = 1, 2$$

where $a(t)$, $b(t)$, $f(t)$ are smooth functions on $[0, 1)$, α_{ij}, β_{ij}, γ_i are numbers, $\beta_{22}\neq 0$.

Let us consider the equivalent system

$$\dot{x}_1 = x_2, \quad \varepsilon\dot{x}_2 = -bx_1 - ax_2 + f.$$

For this case we have $v=-1$, $\beta=a_0$, $\alpha=0$ and therefore there exists the transformation $y_1=(I+\varepsilon PH)x_1-\varepsilon Px_2$, $y_2=-Hx_1+x_2-h$ and for the new variables y_1, y_2 we obtain the boundary value problem

$$\dot{y}_1 = H(t,\varepsilon)y_1 + h(t,\varepsilon); \quad \varepsilon\dot{y}_2 = -[a(t)+\varepsilon H(t,\varepsilon)]y_2,$$

$$\varkappa_{i1}y_1(0) + \varkappa_{i2}y_1(1) + \mu_i y_2(0) = \delta_i,$$

$$\varkappa_{i1} = \alpha_{i1} + \beta_{i1}H(0,\varepsilon); \quad \varkappa_{i2} = \alpha_{i2} + \beta_{i2}H(1,\varepsilon),$$

$$\mu_i = \varepsilon\alpha_{i1}P(0,\varepsilon) + \beta_{i1}[1+\varepsilon H(0,\varepsilon)P(0,\varepsilon)],$$

$$\delta_i = \gamma_i - \beta_{i1}h(0,\varepsilon) - \beta_{i2}h(1,\varepsilon) + O\big(e^{-\frac{a_0}{\varepsilon}}\big); \quad i = 1, 2$$

which can be decoupled to the boundary value problem

$$\dot{y}_1 = H(t,\varepsilon)y_1 + h(t,\varepsilon),$$

$$[\varkappa_{21} - \mu_2\mu_1^{-1}\varkappa_{11}]y_1(0) + [\varkappa_{22} - \mu_2\mu_1^{-1}\varkappa_{12}]y_1(1) = \delta_2 - \mu_2\mu_1^{-1}\delta_1$$

and the initial value problem

$$\varepsilon\dot{y}_2 = -[a(t) + \varepsilon H(t,\varepsilon)]y_2,$$

$$y_2(0) = \mu_1^{-1}[\delta_1 - \varkappa_{11}y_1(0) - \varkappa_{12}y_1(1)].$$

For $H^{(i)}$, $h^{(i)}$ and $P^{(i)}$, according to (1.8), (1.16), (1.17) we have

$$H^{(0)} = -a^{-1}b; \quad H^{(1)} = -a^{-1}[\dot{H}^{(0)} + (H^{(0)})^2]; \quad H^{(2)} = -a^{-1}[\dot{H}^{(1)} + 2H^{(0)}H^{(1)}]; \ \ldots,$$

$$h^{(0)} = a^{-1}f; \quad h^{(1)} = -a^{-1}[\dot{h}^{(0)} + H^{(0)}h^{(0)}]; \quad h^{(2)} = -a^{-1}[\dot{h}^{(1)} + H^{(0)}h^{(1)} + H^{(1)}h^{(0)}]; \ \ldots,$$

$$P^{(0)} = -a^{-1}; \quad P^{(1)} = -a^{-1}[2H^{(0)}P^{(0)} - \dot{P}^{(0)}]; \ \ldots.$$

4. A linear optimal control problem

Let us consider the problem of minimization of the functional

$$I_\varepsilon = \frac{1}{2}x'(1)Fx(1) + \frac{1}{2}\int_0^1 [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]\,dt \tag{4.1}$$

under the restrictions

$$\begin{cases} \dot{y} = A_1(t)y + A_2(t)z + B_1(t)u, & y(0) = y^0, \quad y \in R^n \\ \varepsilon\dot{z} = A_3(t)y + A_4(t)z + B_2(t)u, & z(0) = z^0, \quad z \in R^m \end{cases} \tag{4.2}$$

where $u \in R^k$,

$$x = \begin{pmatrix} y \\ z \end{pmatrix}; \quad Q = Q' = \begin{pmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{pmatrix} \geqq 0; \quad F(\varepsilon) = F' = \begin{pmatrix} F_1 & F_2 \\ F_2' & F_3 \end{pmatrix} \geqq 0, \quad R = R' > 0.$$

The boundary value problem by the maximum principle is represented as

$$\begin{cases} \dot{y} = A_1y + S_1\eta_1 + A_2z + S_2\eta_2; \quad \dot{\eta}_1 = Q_1y - A_1'\eta_1 + Q_2z - A_3'\eta_2 \\ \varepsilon\dot{z} = A_3y + S_2'\eta_1 + A_4z + S_3\eta_2; \quad \varepsilon\dot{\eta}_2 = Q_2'y - A_2'\eta_1 + Q_3z - A_4'\eta_2 \\ y(0) = y^0; \quad z(0) = z^0 \\ \eta_1(1) + F_1y(1) + F_2z(1) = 0; \quad \varepsilon\eta_2(1) + F_2'y(1) + F_3z(1) = 0 \end{cases} \tag{4.3}$$

where $S_1 = B_1R^{-1}B_1'$, $S_2 = B_1R^{-1}B_2'$, $S_3 = B_2R^{-1}B_2'$ [6].

We can represent (4.3) in the form (1), where

$$x_1 = \begin{pmatrix} y \\ \eta_1 \end{pmatrix}; \quad x_2 = \begin{pmatrix} z \\ \eta_2 \end{pmatrix}; \quad f_1 = 0, \quad f_2 = 0; \quad A_{11} = \begin{pmatrix} A_1 & S_1 \\ Q_1 & -A_1' \end{pmatrix};$$

$$A_{12} = \begin{pmatrix} A_2 & S_2 \\ Q_2 & -A_3' \end{pmatrix}; \quad A_{21} = \begin{pmatrix} A_3 & S_2' \\ Q_2' & -A_2' \end{pmatrix}; \quad A_{22} = \begin{pmatrix} A_4 & S_3 \\ Q_3 & -A_4' \end{pmatrix}.$$

Assume that all eigenvalues of A_{22} have nonzero real parts for $0 \leqq t \leqq 1$, then A_{22} has m stable eigenvalues λ, $\operatorname{Re}\lambda \leqq -2\beta < 0$, and m unstable ones, $\operatorname{Re}\lambda \geqq 2\beta > 0$. According to [13] sufficient conditions for that are as follows. For every $t \in [0, 1]$ the pair $\{A_4, B_2\}$ has the property of complete controllability, the pair $\{C, A_4\}$ has that of complete observability ($CC' = Q_3$) and $\det A_4 \neq 0$.

Under these assumptions the fundamental matrix $V(t, s, \varepsilon)$ of the equation $\varepsilon \dot{x}_2 = A_{22} x_2$ satisfies the inequality $|V(t, s, \varepsilon)| \leqq N \exp[-\varepsilon^{-1}\beta|t-s|]$ and we can prove, as earlier, that there exist matrices H and P such that for the new variables $y_1 = (I + \varepsilon PH)x_1 - \varepsilon P x_2$, $y_2 = -Hx_1 + x_2$ we obtain

$$\dot{y}_2 = (A_{11} + A_{12}H)\, y_2, \tag{4.4}$$

$$\varepsilon \dot{y}_2 = (A_{22} - \varepsilon H A_{12})\, y_2. \tag{4.5}$$

Note that H and P can be found as asymptotic expansions by (1.8), (1.16). Now we consider the system (4.5) in the form

$$\begin{cases} \varepsilon \dot{y}_{21} = (A_4 + \varepsilon C_{11})\, y_{21} + (S_3 + \varepsilon C_{12})\, y_{22}; \\ \varepsilon \dot{y}_{22} = (Q_3 + \varepsilon C_{21})\, y_{21} + (-A_4' + \varepsilon C_{22})\, y_{22}; \end{cases} \quad \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix} = y_2; \quad \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = -HA_{12}. \tag{4.6}$$

The last system can be reduced to the block diagonal form by the transformation

$$\begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where M and L, like H and P for (1), satisfy the equations

$$\varepsilon \dot{M} + M[A_4 + \varepsilon C_{11} + (S_3 + \varepsilon C_{12})\, M] = Q_3 + \varepsilon C_{21} + [-A_4' + \varepsilon C_{22}]\, M,$$

$$\varepsilon \dot{L} + L[-A_4' + \varepsilon C_{22} - M(S_3 + \varepsilon C_{12})] = [A_4 + \varepsilon C_{11} + (S_3 + \varepsilon C_{12})\, M]\, L + S_3 + \varepsilon C_{12}.$$

Moreover, M and L can be found from these equations as asymptotic expansions:

$$M = \sum_{i=0}^{k} \varepsilon^i M^{(i)}(t) + \varepsilon^{k+1} M^{(k+1)}(t, \varepsilon); \quad L = \sum_{i=0}^{k} \varepsilon^i L^{(i)}(t) + \varepsilon^{k+1} L^{(k+1)}(t, \varepsilon).$$

For $M^{(0)}$ we have the Riccati equation

$$M^{(0)} A_4 + A_4' M^{(0)} + M^{(0)} S_3 M^{(0)} = Q_3.$$

Under our assumptions this equation has a unique negative definite solution, and the eigenvalues λ of $A_4 + S_3 M^{(0)}$ have negative real parts: $\operatorname{Re}\lambda \leqq -2\beta < 0$ [13].

For $M^{(i)}$ $(i = 1, \ldots, k)$ and $L^{(i)}$ $(i = 0, 1, \ldots, k)$ we have the linear equations

$$M^{(i)} A_4 + A_4' M^{(i)} = R_i,$$

$$-L^{(0)}(A_4' + M^{(0)} S_3) - (A_4 + S_3 M^{(0)})\, L^{(0)} = S_3,$$

$$-L^{(i)}(A_4' + M^{(0)} S_3) - (A_4 + S_3 M^{(0)})\, L^{(i)} = G_i,$$

where R_i, G_i are known.

Thus (4.6) can be reduced to

$$\varepsilon \dot{v}_1 = K v_1, \tag{4.7}$$

$$\varepsilon \dot{v}_2 = N v_2, \tag{4.8}$$

where

$$K = A_4 + \varepsilon C_{11} + (S_3 + \varepsilon C_{12}) M; \quad N = -A_4' + \varepsilon C_{22} - M(S_3 + \varepsilon C_{12}).$$

Eigenvalues of the matrix K have negative real parts, those of N have positive real parts. Note that (4.7) describes the behavior of the solutions of (4.6) on the invariant manifold of stability, and (4.8) provides a similar information for the unstable manifold.

M and L also permit the decomposition of the boundary conditions. As a result one gets the regular boundary value problem and two singularly perturbed initial value problems. Using the final transformation

$$\begin{pmatrix} y \\ \eta_1 \\ z \\ \eta_2 \end{pmatrix} = \begin{pmatrix} I & O & \varepsilon D_{11} & \varepsilon D_{12} \\ O & I & \varepsilon D_{21} & \varepsilon D_{22} \\ H_1 & H_2 & E_{11} & E_{12} \\ H_3 & H_4 & E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ v_1 \\ v_2 \end{pmatrix}; \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = y_1; \quad \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} = H,$$

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} = P \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix}; \quad \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = (I + \varepsilon HP) \begin{pmatrix} I & L \\ M & I + ML \end{pmatrix}$$

we obtain the boundary conditions

$$w_1(0) + \varepsilon D_{11}(0, \varepsilon)\, v_1(0) = y^0,$$

$$H_1(0, \varepsilon)\, w_1(0) + H_2(0, \varepsilon)\, w_2(0) + E_{11}(0, \varepsilon)\, v_1(0) = z^0,$$

$$p_1 w_1(1) + p_2 w_2(1) + p_3 v_2(1) = 0; \quad p_i = p_i(1, \varepsilon),$$

$$q_1 w_1(1) + q_2 w_2(1) + q_3 v_2(1) = 0; \quad q_i = q_i(1, \varepsilon),$$

where

$$p_1 = F_2 H_1, \quad p_2 = F_1 + F_2 H_2, \quad p_3 = \varepsilon F_1 D_{22} + F_2 E_{12},$$

$$q_1 = F_2' + F_3 H_1 + \varepsilon H_3, \quad q_2 = F_3 H_2 + \varepsilon H_2, \quad q_3 = \varepsilon F_2' D_{12} + F_3 E_{12} + \varepsilon E_{22}.$$

Finally, we obtain the boundary value problem

$$\dot{w}_1 = (A_1 + A_2 H_1 + S_2 H_3)\, w_1 + (S_1 + A_2 H_2 + S_2 H_4)\, w_2,$$

$$\dot{w}_2 = (Q_1 + Q_2 H_1 - A_3' H_3)\, w_1 + (-A_1' + Q_2 H_2 - A_3' H_4)\, w_2,$$

$$(I - \varepsilon D_{11} E_{11}^{-1} H_1)\, w_1 - \varepsilon D_{11} E_{11}^{-1} H_2 w_2 = y^0 - \varepsilon D_{11} E_{11}^{-1} z^0, \quad \text{when} \quad t = 0$$

$$(p_1 - p_3 q_3^{-1} q_1)\, w_1 + (p_2 - p_3 q_3^{-1} q_2)\, w_2 = 0, \quad \text{when} \quad t = 1,$$

and two initial value problems

$$\varepsilon \dot{v}_1 = K v_1; \quad v_1(0) = (E_{11}(0, \varepsilon))^{-1} [z^0 - H_1(0, \varepsilon)\, w_1(0) - H_2(0, \varepsilon)\, w_2(0)],$$

$$\varepsilon \dot{v}_2 = N v_2; \quad v_2(1) = -q_3^{-1}(1, \varepsilon) [q_1(1, \varepsilon)\, w_1(1) + q_2(1, \varepsilon)\, w_2(1)].$$

EXAMPLE. Consider the following linear optimal control problem:

$$\dot{y} = z; \qquad y(0) = y^3,$$
$$\varepsilon\dot{z} = -a(t)\,z - b(t)\,y + u; \quad z(0) = z^0,$$
$$I_\varepsilon = \frac{1}{2}\left(y^2(1)+z^2(1)\right)+\frac{1}{2}\int_0^1 \left(y^2(t)+z^2(t)+u^2(t)\right)dt \to \min$$

where y, z, u, a, b are scalars.

The appropriate boundary value problem is

$$\dot{y} = z; \quad \dot{\eta}_1 = y + b\eta_2; \quad \varepsilon\dot{z} = -by - az + \eta_2; \quad \varepsilon\dot{\eta}_2 = -\eta_1 + z + a\eta_2,$$
$$y(0) = y^0; \quad z(0) = z^0; \quad \eta_1(1) + y(1) = 0; \quad \varepsilon\eta_2(1) + q(1) = 0.$$

We can reduce this singularly perturbed problem to the regular boundary value problem

$$\dot{w}_1 = H_1 w_1 + H_2 w_2; \quad \dot{w}_2 = (1 + bH_3)\,w_1 + bH_4 w_2,$$
$$[1 - \varepsilon abd^{-3}]\,w_1(0) + \varepsilon d^{-3} w_2(0) = y^0 + \varepsilon d^{-1} z^0 + O(\varepsilon^2); \quad d = \sqrt{1+a^2},$$
$$[1 + \varepsilon abd^{-1}(1 + ab + bd)]\,w_1(1) + [1 - \varepsilon d^{-1}(1 + ab + bd)]\,w_2(1) + O(\varepsilon^2) = 0,$$
$$H_1 = -abd^{-2} + \varepsilon d^{-4}[-ab^2 - ab\dot{a} + d^2\dot{b}] + O(\varepsilon^2); \quad H_2 = d^{-2} + \varepsilon d^{-4}(b + \dot{a}) + O(\varepsilon^2),$$
$$H_3 = bd^{-2} + \varepsilon d^{-4}[1 + a^2 + b^2 - b\dot{a}] + O(\varepsilon^2); \quad H_4 = ad^{-2} + \varepsilon d^{-4}(b - \dot{a})\,a + O(\varepsilon^2)$$

and to two initial value problems

$$\varepsilon\dot{v}_1 = -[(1 + c\varepsilon)\,d + O(\varepsilon^2)]\,v_1,$$
$$v_1(0) = (1 + \varepsilon c)^{-1}[z^0 - H_1(0, \varepsilon)\,w_1(0) - H_2(0, \varepsilon)\,w_2(0) + O(\varepsilon^2)],$$
$$\varepsilon\dot{v}_2 = [(1 + \varepsilon c)\,d + O(\varepsilon^2)]\,v_2,$$
$$v_2(1) = -2d\left[1 - \varepsilon\left(\frac{1}{2}\dot{a}d^{-2} + c + a + d\right)\right]\times$$
$$\times[(H_1(1, \varepsilon) + \varepsilon H_3(1, \varepsilon))\,w_1(1) + (H_2(1, \varepsilon) + \varepsilon H_4(1, \varepsilon)) + w_2(1) + O(\varepsilon^2)].$$

5. Systems with small time lag

Now let us consider the system [14]

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, \\ \varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + C_1x_1(t-\mu) + C_2x_2(t-\mu) \end{cases} \tag{5.1}$$

where $x_i \in R^{n_i}$, $A_{ij} = A_{ij}(\varepsilon)$, $C_i = C_i(\varepsilon)$ are analytic matrices in ε, ε and μ are small positive parameters. As earlier, we shall try to find an invariant manifold for (5.1) in the form $x_2 = Hx_1$, where $H = H(\varepsilon, \mu)$. For H we have

$$\varepsilon H(A_{11} + A_{12}H) = A_{21} + A_{22}H + C_1 e^{-\mu(A_{11}+A_{12}H)} + C_2 e^{-\mu(A_{11}+A_{12}H)}H. \tag{5.2}$$

If $\det(A_{22}^{(0)}+C_2^{(0)})\neq 0$ then H can be determined from (5.2) as a series

$$H=\sum_{m,n=0}^{\infty}\varepsilon^m\mu^n H^{(m,n)}$$

where

$$H^{(0,0)}=-(A_{22}^{(0)}+C_2^{(0)})^{-1}(A_{21}^{(0)}+C_1^{(0)}),$$

$$H^{(1,0)}=-(A_{22}^{(0)}+C_2^{(0)})^{-1}[C_1^{(1)}+A_{21}^{(1)}-H^{(0,0)}(A_{11}^{(0)}+A_{12}^{(0)}H^{(0,0)})],$$

$$H^{(0,1)}=(A_{22}^{(0)}+C_2^{(0)})^{-1}[C_1^{(0)}(A_{11}^{(0)}+A_{12}^{(0)}H^{(0,0)})+C_2^{(0)}(A_{11}^{(0)}+A_{12}^{(0)}H^{(0,0)})H^{(00)}].$$

For the variables x_1, $y_2=x_2-Hx_1$ we obtain the system

$$\dot{x}_1=(A_{11}+A_{12}H)x_1+A_{12}y_2, \tag{5.3}$$

$$\varepsilon\dot{y}_2=(A_{22}-\varepsilon HA_{12})y_2+C_2y_2(t-\mu). \tag{5.4}$$

Let us suppose that the integral

$$z=\int_t^{\infty}e^{(A_{11}+A_{12}H)(t-s)}A_{12}y_2(s)\,ds \tag{5.5}$$

exists for any solution $y_2=y_2(t)$ of (5.4). Then by setting $x_1=y_1-z$ we obtain two independent equations

$$\dot{y}_1=(A_{11}+A_{12}H)y_1, \tag{5.6}$$

$$\varepsilon\dot{y}_2=(A_{22}-\varepsilon HA_{12})y_2+C_2y_2(t-\mu). \tag{5.7}$$

If, for example, $\operatorname{Re}\lambda_i\leqq -2\beta<0$, $i=1,\ldots,n_2$ where λ_i are eigenvalues of $A_{22}^{(0)}+C_2^{(0)}$, then (5.4) is asymptotically stable, the integral (5.5) is convergent [15] and the stability of (5.1) is equivalent to the stability of (5.6).

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AN ESTIMATE OF THE RATE OF CONVERGENCE OF THE CONJUGATE FOURIER SERIES OF FUNCTIONS OF BOUNDED VARIATION

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1. Let f be a 2π-periodic function of bounded variation on $[-\pi, \pi]$. Suppose

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1}$$

then the series conjugate to (1.1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \tag{1.2}$$

We denote by $s_n(x)$ and $\tilde{s}_n(x)$ the nth partial sums of the series (1.1) and (1.2) respectively. We write

$$g_x(t) = f(x+t)+f(x-t)-f(x+0)-f(x-0),$$

$$\psi_x(t) = \frac{1}{2}(f(x+t)-f(x-t)),$$

$$\tilde{f}(x) = \lim_{\varepsilon \to 0+} \tilde{f}(x, \varepsilon) = \lim_{\varepsilon \to 0+} \left\{ \frac{-2}{\pi} \int_{\varepsilon}^{\pi} \frac{\psi_x(t)}{2 \tan t/2} \, dt \right\},$$

and

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos t/2 - \cos(n+1/2)\, t}{2 \sin t/2}.$$

2. Concerning convergence of Fourier series of functions of bounded variation the well-known Dirichlet—Jordan theorem ([3], Vol 1, p. 57) reads as follows:

THEOREM A. *Suppose that $f(x)$ is a function of bounded variation on $[-\pi, \pi]$. Then at every point x_0,*

$$\lim_{n \to \infty} \left\{ s_n(x_0) - \frac{1}{2}[f(x_0+0)+f(x_0-0)] \right\} = 0.$$

in particular, $\lim_{n \to \infty} \{s_n(x_0) - f(x_0)\} = 0$, *at every point of continuity of* $f(x)$.

Recently R. Bojanic [1] gave an estimate of the rate of convergence of Fourier series of functions of bounded variation in the following form:

THEOREM B. *Let* $f(x)$ *be of bounded variation, then*

$$\left|s_n(x)-\frac{1}{2}\{f(x+0)+f(x-0)\}\right| \leqq \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(g_x), \quad |x| \leqq \pi$$

where $V_0^y(g_x)$ *is the total variation of* g_x *on* $[0, y]$, $y\in[0, \pi]$.

The analogous criterion for the convergence of the conjugate series was given by W. H. Young ([2], see also ([3], p. 59)).

THEOREM C. *If* $f(x)$ *is of bounded variation on* $[-\pi, \pi]$, *a necessary and sufficient condition for the convergence of the conjugate series* (1.2) *at* x *is the existence of the integral*

$$\tilde{f}(x) = \frac{-2}{\pi}\int_0^{\pi} \frac{\psi_x(t)}{2\tan t/2}\,dt = \lim_{h\to 0+}\left\{\frac{-2}{\pi}\int_h^{\pi} \frac{\psi_x(t)}{2\tan t/2}\,dt\right\},$$

which represents then the sum of (1.2).

Our object is to obtain an estimate of the rate of convergence of the conjugate series of a function $f(x)$ of bounded variation. We prove the following theorem:

THEOREM. *If* $f(x)$ *is a function of bounded variation on* $[-\pi, \pi]$, *then*

$$\left|\tilde{s}_n(x)-\tilde{f}\left(x, \frac{\pi}{n}\right)\right| \leqq \frac{3.3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x),$$

where $V_0^t(\psi_x)$ *denotes the total variation of* ψ_x *on the interval* $[0, t]$, $t\in[0, \pi]$.

Since $V_0^t(\psi_x)$ is continuous when $f(x)$ is continuous, it follows that

$$\frac{1}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x) \to 0 \quad \text{as} \quad n\to\infty.$$

Thus we deduce the following result ([3], p. 59) from our theorem.

COROLLARY. *If* $f(x)$ *is of bounded variation on* $[-\pi, \pi]$, *then* $\tilde{s}_n^*(x)-\tilde{f}\left(x, \frac{\pi}{n}\right)\to 0$ *as* $n\to\infty$ *at every point* x *of continuity of* $f(x)$ *and is bounded at every point of discontinuity, where*

$$\tilde{s}_n^*(x) = \frac{-2}{\pi}\int_0^{\pi} \psi_x(t)\left(\tilde{D}_n(t)-\frac{1}{2}\sin nt\right)dt.$$

3. We need the following lemma to prove our theorem:

LEMMA. *Let* $\Lambda(y)=\int_y^{\pi} \frac{\cos(n+1/2)t}{2\sin t/2}\,dt$, *then*

$$|\Lambda(y)| \leqq \frac{\pi}{ny}, \quad 0 < y \leqq \pi.$$

PROOF. Applying the second mean-value theorem

$$\Lambda(y) = \frac{1}{2\sin y/2}\int_y^\xi \cos(n+1/2)\,t\,dt = \frac{1}{2\sin y/2}\left[\frac{\sin(n+1/2)\,t}{n+1/2}\right]_y^\xi \quad (y < \xi < \pi).$$

Thus

$$|\Lambda(y)| \leqq \frac{2}{n}\cdot\frac{1}{2\sin y/2} \leqq \frac{\pi}{ny}.$$

4. PROOF OF THE THEOREM. Since

$$\tilde{s}_n(x) = \frac{-2}{\pi}\int_0^\pi \psi_x(t)\,\tilde{D}_n(t)\,dt,$$

we have

$$\tilde{s}_n(x) - \tilde{f}\left(x, \frac{\pi}{n}\right) = \frac{-2}{\pi}\int_0^\pi \frac{\psi_x(t)}{2\sin t/2}\left(\cos t/2 - \cos(n+1/2)\,t\right)dt +$$

$$+\frac{2}{\pi}\int_{\pi/n}^\pi \frac{\psi_x(t)}{2\tan t/2}\,dt = \frac{-2}{\pi}\int_0^{\pi/n} \frac{\psi_x(t)}{2\sin t/2}\left(\cos t/2 - \cos(n+1/2)\,t\right)dt +$$

$$+\frac{2}{\pi}\int_{\pi/n}^\pi \frac{\psi_x(t)}{2\sin t/2}\cos\frac{2n+1}{2}\,t\,dt = I_1 + I_2, \quad \text{say.}$$

Using the inequality $|\tilde{D}_n(t)| \leqq n$, we get

(4.1)

$$|I_1| \leqq \frac{2n}{\pi}\int_0^{\pi/n} |\psi_x(t)|\,dt = \frac{2n}{\pi}\int_0^{\pi/n} |\psi_x(t) - \psi_x(0)|\,dt \leqq \frac{2n}{\pi}\int_0^{\pi/n} V_0^t(\psi_x)\,dt \leqq 2V_0^{\pi/n}(\psi_x).$$

Now

$$\int_{\pi/n}^\pi \frac{\psi_x(t)\cos(n+1/2)\,t}{2\sin t/2}\,dt = -\int_{\pi/n}^\pi \psi_x(t)\,d\Lambda(t) =$$

$$= -[\psi_x(t)\,\Lambda(t)]_{\pi/n}^\pi + \int_{\pi/n}^\pi \Lambda(t)\,d\psi_x(t),$$

so by virtue of the above lemma

$$|I_2| \leqq \frac{2}{\pi}\left|\psi_x\left(\frac{\pi}{n}\right)\Lambda\left(\frac{\pi}{n}\right)\right| + \frac{2}{\pi}\int_{\pi/n}^\pi |\Lambda(t)|\,|d\psi_x(t)| \leqq \frac{2}{\pi}V_0^{\pi/n}(\psi_x) + \frac{2}{n}\int_{\pi/n}^\pi \frac{|d\psi_x(t)|}{t} =$$

$$= \frac{2}{\pi}V_0^{\pi/n}(\psi_x) + \frac{2}{n}\int_{\pi/n}^\pi \frac{dV_0^t(\psi_x)}{t} = \frac{2}{\pi}V_0^{\pi/n}(\psi_x) + \frac{2}{n}\left[\frac{V_0^t(\psi_x)}{t}\right]_{\pi/n}^\pi + \frac{2}{n}\int_{\pi/n}^\pi \frac{V_0^t(\psi_x)}{t^2}\,dt =$$

$$= \frac{2}{n}\frac{V_0^\pi(\psi_x)}{\pi} + \frac{2}{n}\int_{\pi/n}^\pi \frac{V_0^t(\psi_x)}{t^2}\,dt.$$

Now

$$\int_{\pi/n}^{\pi} \frac{V_0^t(\psi_x)}{t^2}\,dt = \frac{1}{\pi}\int_1^n V_0^{\pi/t}(\psi_x)\,dt = \frac{1}{\pi}\sum_{k=1}^{n-1}\int_k^{k+1} V_0^{\pi/t}(\psi_x)\,dt \leqq \frac{1}{\pi}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x).$$

Hence

$$|I_2| = \frac{2}{n\pi}V_0^{\pi}(\psi_x) + \frac{2}{n\pi}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x) \leqq \frac{4}{n\pi}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x). \tag{4.2}$$

Thus from (4.1) and (4.2) we have

$$\left|\tilde{s}_n(x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| \leqq \frac{2}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x) + \frac{4}{n\pi}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x) \leqq \frac{3.3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(\psi_x).$$

This completes the proof of our theorem.

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A NOTE ON SMALL RINGS

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Throughout R will denote a ring with identity and all modules are unitary left R-modules unless specified otherwise. A module N is said to be small in M if for a submodule K of M, $K+N=M$ implies $K=M$ [1]. We say a module M is small if it can be embedded as a small submodule of some module [4]. A ring R is said to be a small ring or small if it is small as a module over itself.

It is easily seen that the class of small modules is closed under the formation of submodules, homomorphic images and finite direct sums. It is known that a module M is small iff it is small in its injective hull $E(M)$ [4]. Also, the sum of all submodules of M that are small in M is the Jacobson radical $J(M)$ of M [3].

We now give some examples of small rings.

PROPOSITION 1. *A direct product of proper commutative domains is a small ring.*

PROOF. Let $\{R_i\}_{i\in I}$ be a set of proper commutative domains. We put $R=\prod_i R_i$. For each $i\in I$, let Q_i be the field of quotients of R_i. Suppose $R^A\subseteqq\prod Q_i$ such that

$$R+A=\prod Q_i. \tag{1}$$

Since each R_i is a proper commutative domain, $R_i\neq Q_i$. For each $i\in I$, let $x_i\in Q_i\setminus R_i$. Now $(x_i)\in\prod Q_i$. From (1) it follows that there exist $(r_i)\in R$ and $(a_i)\in A$ such that $(r_i)+(a_i)=(x_i)$. That is, $r_i+a_i=x_i$ for each $i\in I$. By the choice of x_i, a_i is non zero for each i. Since $a_i\in Q_i$, there exists $r_i'\in R_i$ with $r_i'a_i\neq 0$ and $y_i=r_i'a_i\in R_i$. We thus have $(r_i')\in R$ and $0\neq(r_i')(a_i)=(r_i'a_i)=(y_i)\in A\cap R$. From (1) $(y_i)\prod R_i+(y_i)A=(y_i)\prod Q_i$. Since each R_i is a commutative domain, this reduces to $\prod R_i(y_i)+(y_i)A=\prod y_iQ_i=\prod Q_i$. Thus $A\supseteqq\prod Q_i$ or $A=\prod Q_i$. Hence R is small.

COROLLARY ([6]). *Every proper commutative domain is small.*

REMARK. Proposition 1 shows that there exist small rings which are not domains. Further not every domain is small. Cozzins [2] contructs a pli-pri domain (principal left ideal and principal right ideal domain) with the property that it has a unique simple module which is injective. Hence this domain is not small.

DEFINITION. A ring is called a local ring if its nonunits form an ideal.

PROPOSITION 2. *A proper local domain is small.*

PROOF. Let R be a local domain with Jacobson radical $J\neq 0$. Then R is not injective. Let Q be the injective hull of R. Suppose $R^A\subseteqq R^Q$ such that $R+A=Q$.

Clearly $A \neq 0$ and since R is essential in Q, $A \cap R \neq 0$. Let $0 \neq a \in A \cap R$. Since Q is divisible, $Q = aQ = aR + aA$. Hence $1 = ar + ar'$ with $r \in R$ and $r' \in A$. It follows that $ar' = 1 - ar \in R \cap A$. R being a local ring either ar or ar' is a unit. If ar is a unit in R, i.e. $ar \notin J$, then $a \notin J$ and so a is a unit and $R = Ra \subseteq A$. If, instead, ar' is a unit then $R = Rar' \subseteq A$. Thus, in both cases $R \subseteq A$ and therefore $A = Q$.

We next extend Propositions 6.1 and 6.4 of Pareigis in [6] to a finite direct sum of commutative domains. The proof is also on similar lines.

PROPOSITION 3. *Let $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ where each R is a proper commutative domain. Then the following are equivalent:*

(i) *$J(M) = M$ iff M is injective.*

(ii) *R_i is a Dedekind domain for $i = 1, 2, \ldots, n$.*

PROOF. (i)⇒(ii). Let M be an injective module. Then $J(M) = M$. Therefore M has no maximal submodules. Thus, every factor module of M has no maximal submodules. That is, every factor module M' of M satisfies $J(M') = M'$. By hypothesis M' is injective. This shows R is hereditary. Hence each R_i is a hereditary ring and being already a domain each R_i is a Dedekind domain.

(ii)⇒(i). By Proposition 1, R is small. Therefore if M is injective, $J(M) = M$ by Theorem 4.8 of [6]. Now suppose M is not injective. R is hereditary since each R_i is and R has a semi-simple classical ring of quotients. By Theorem 3.4 [5], M is not divisible. Therefore there exists a regular element r in R such that $rM \neq M$, or $RrM \neq M$. Let $r = r_1 + r_2 + \ldots + r_n$ with $r_i \in R_i$. Each r_i is non zero and $R_i/R_i r_i$ is artinian. Hence

$$R/Rr = \bigoplus_{i=1}^{n} R_i/R_i r_i \quad \text{is artinian.}$$

This implies there exist maximal ideals $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_m$ of R such that $\mathscr{P}_1 \mathscr{P}_2 \ldots \mathscr{P}_m \subseteq \subseteq Rr$. Thus there exists a maximal ideal $\mathscr{P}$ of R such that $\mathscr{P} M \neq M$. Therefore $J(M) \neq M$.

PROPOSITION 4. *Let $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ where each R_i is a proper noetherian commutative domain. Then the following are equivalent:*

(i) *A finitely generated module is projective if its socle is zero.*

(ii) *R_i is a Dedekind domain for $i = 1, 2, \ldots, n$.*

PROOF. (i)⇒(ii) since each R_i is a domain, socle of $R_i = \mathrm{Soc}_{R_i} R_i = 0$. Hence $\mathrm{Soc}_R R = 0$. This implies every free R-module has zero socle and so also every projective module. Let M be a finitely generated submodule of a projective module P. Since Soc P is zero, Soc M is zero also. By hypothesis M is projective. Hence R is semihereditary. Being also noetherian R is hereditary. Each R_i is therefore hereditary and thus a Dedekind domain.

(ii)⇒(i). R is hereditary since each R_i is. Further R has a semisimple classical ring of quotients. By Theorem 6.1 of [5], if M is any finitely generated module, $M = T(M) \oplus N_1 \oplus N_2 \oplus \ldots \oplus N_k$ where $T(M)$ is the torsion submodule of M and N_i's are isomorphic to ideals of R. But R being hereditary each N_i is projective. If M is not projective, then $T(M)$ is nonzero. So there exists a regular element r in R and a non zero element m in M such that $rm = 0$. This implies that $Rrm = 0$. As in the proof of Proposition 4, there exist maximal ideals $\mathscr{P}_1, \ldots, \mathscr{P}_s$ of R such that

$\mathscr{P}_1 \mathscr{P}_2 \ldots \mathscr{P}_s \subseteq Rr$. Thus there exists a maximal ideal $\mathscr{P}$ of R and a non zero element m' in M such that $\mathscr{P}m' = 0$. Then Rm' is a simple submodule of M.

By using the above propositions, Propositions 6.2 and 6.5 of Paraigis in [6] can also be extended to a finite direct sum of Dedekind domains.

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ON LATTICES WHOSE CONGRUENCES FORM STONE LATTICES*

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1. Introduction

This paper solves the following problem posed by G. Grätzer [2].

PROBLEM 1.1 (Problem III.5 of [2]). Investigate lattices whose congruences form a Stone lattice.

As a particular case, the solution of the problem when the lattice is weakly modular, is also deduced.

It is known that the lattice of congruences on any complete relatively complemented lattice is a Stone lattice (M. F. Janowitz [5]). The same is the case for complete section complemented weakly modular lattices (Iqbalunnisa [4]). In [6] and [7] further generalizations of these results were obtained. We shall give a solution that is applicable to lattices in general and derive a particular solution for weakly modular lattices.

DEFINITION 1.1 [2]. An element a^* of a lattice L with the least element 0 is called a pseudocomplement of $a \in L$ iff $a \wedge a^* = 0$ and for $x \in L$, $a \wedge x = 0$ implies $x \leqq a^*$.

In a pseudocomplemented lattice every element has a pseudocomplement.

DEFINITION 1.2 [2]. A distributive lattice L with pseudocomplementation is called a Stone algebra iff it satisfies the Stone identity $a^* \vee a^{**} = 1$ for all $a \in L$.

DEFINITION 1.3 [2]. A congruence relation $\mathscr{H}$ of a lattice L is called separable iff for any $a, b \in L$, $a < b$, there exists a sequence $a = x_0 < x_1 < \ldots < x_n = b$ such that for each i, $0 \leqq i < n$, $x_i \equiv x_{i+1}(\mathscr{H})$ or $u \equiv v(\mathscr{H})$ for no proper subquotient u/v of x_{i+1}/x_i.

For further information we refer to [2].

2. A solution to Problem 1.1

The solution that we give is actually a modification of a result of G. Grätzer and E. T. Schmidt (see [2], Theorem III. 4.9) which characterizes lattices whose congruences form Boolean lattices.

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THEOREM 2.1. *For a lattice L, its congruence lattice $C(L)$ is a Stone lattice iff for every congruence relation $\mathscr{H}$ its pseudocomplement $\mathscr{H}^*$ is separable and L satisfies the following condition:*

(C) $a/b \underset{\omega}{\approx} c/d$, $a, b, c, d \in L$, $a > b$, $c > d$ *imply the existence of a proper subquotient* c'/d' *of* c/d *satisfying* $c' \equiv d' (\mathscr{H}^{**}_{a/b})$.

PROOF. Let $C(L)$ be a Stone lattice. Let $a/b \underset{\omega}{\approx} c/d$, $a, b, c, d \in L$, $a \neq b$. Now $\mathscr{H}^* a/b \vee \mathscr{H}^{**} a/b = \iota$ and so $c \equiv d$ $(\mathscr{H}^* a/b \vee \mathscr{H}^{**} a/b)$. Hence there is a sequence $d = x_0 < x_1 < \ldots < x_n = c$ such that for each i, $0 \leqq i < n$, $x_i \equiv x_{i+1} (\mathscr{H}^* a/b)$ or $x_i \equiv x_{i+1} (\mathscr{H}^{**} a/b)$. If for all i, $0 \leqq i < n$, $x_i \equiv x_{i+1} (\mathscr{H}^* a/b)$. then $c \equiv d (\mathscr{H}^* a/b)$. But $a/b \underset{\omega}{\approx} c/d$. Therefore $a \equiv b (\mathscr{H}^* a/b)$, which implies $a \equiv b (\mathscr{H} a/b \wedge \mathscr{H}^* a/b)$. That is, $a \equiv b (\omega)$, which contradicts $a \neq b$. Therefore $x_i \equiv x_{i+1} (\mathscr{H}^{**} a/b)$ for some i, $0 \leqq i < n$, proving L satisfies (C).

Let $\mathscr{H}$ be an congruence relation of L. To show $\mathscr{H}^*$ is separable, let $a, b \in L$, $a < b$. Then $a \equiv b (\mathscr{H}^* \vee \mathscr{H}^{**})$, which implies the existence of a sequence $a = x_0 < x_1 < \ldots < x_n = b$ such that for each i, $0 \leqq i < n$, $x_i \equiv x_{i+1} (\mathscr{H}^*)$ or $x_i \equiv x_{i+1} (\mathscr{H}^{**})$. It will be shown that the same sequence establishes the separability of $\mathscr{H}^*$. Indeed, if for some i, $x_i \neq x_{i+1} (\mathscr{H}^*)$ and $u, v \in [x_i, x_{i+1}]$ $u \equiv v (\mathscr{H}^*)$, then $x_i \equiv x_{i+1} (\mathscr{H}^{**})$ and so $u \equiv v (\mathscr{H}^{**})$, implying $u \equiv v (\mathscr{H}^* \wedge \mathscr{H}^{**})$, that is, $u = v$.

Conversely, let L be a lattice satisfying (C) and let $\mathscr{H}$ be a congruence relation on L. First it will be shown that the binary relation Φ defined by, $a \equiv b (\Phi)$ iff $u \equiv v (\mathscr{H}^*)$ for no proper subquotient u/v of $a \vee b / a \wedge b$, is a congruence relation and in fact $\Phi = \mathscr{H}^{**}$.

To verify the first part, it is clear that Φ is a reflexive binary relation and satisfies the condition that $a \equiv b (\Phi)$ iff $a \wedge b \equiv a \vee b (\Phi)$ for any $a, b \in L$.

Let $a \leqq b \leqq c$, $a \equiv b (\Phi)$ and $b \equiv c (\Phi)$. If on the contrary $a \not\equiv c (\Phi)$, then $u \equiv v (\mathscr{H}^*)$ for some proper subquotient u/v of c/a.

Since $u \equiv v (\mathscr{H} c/a)$ and $\mathscr{H} c/a = \mathscr{H} b/a \vee \mathscr{H} c/b$, there exists a proper subquotient x/y of u/v satisfying $x/y \underset{\omega}{\approx} c/b$ or $x/y \underset{\omega}{\approx} b/a$.

Let $x/y \underset{\omega}{\approx} c/b$. Then $b < c$. By (C) there exists a proper subquotient c_1/b_1 of c/b satisfying

$$c_1 \equiv b_1 \quad (\mathscr{H}^{**} x/y). \tag{2.1}$$

But $x, y \in [v, u]$ and $u \equiv v (\mathscr{H}^*)$, therefore $x \equiv y (\mathscr{H}^*)$. Hence $\mathscr{H} x/y \leqq \mathscr{H}^*$, which implies $\mathscr{H}^* x/y \geqq \mathscr{H}^{**}$ and $\mathscr{H}^{**} x/y \leqq \mathscr{H}^{***} = \mathscr{H}^*$ by Theorem VIII. 2.1 of [1]. That is,

$$\mathscr{H}^{**} x/y \leqq \mathscr{H}^*. \tag{2.2}$$

Now (2.1) and (2.2) imply $c_1 \equiv b_1 (\mathscr{H}^*)$, contradicting $b \equiv c (\Phi)$, since $c_1, b_1 \in [b, c]$ and $b_1 < c_1$.

Further, if $a, b, c \in L$, $a \leqq b$, $a \equiv b (\Phi)$ but $a \wedge c \not\equiv b \wedge c (\Phi)$, then there exists a proper subquotient u/v of $b \wedge c / a \wedge c$ satisfying $v \equiv u (\mathscr{H}^*)$. Now $u/v \underset{\omega}{\approx} b \wedge c / a \wedge c \underset{\omega}{\approx} b/a$, hence $u/v \underset{\omega}{\approx} b/a$ which implies $a < b$ and by (C), $b_1 \equiv a_1 (\mathscr{H}^{**} u/v)$ for some proper subquotient b_1/a_1 of b/a. But $u \equiv v (\mathscr{H}^*)$ and so $\mathscr{H} u/v \leqq \mathscr{H}^*$ which implies $\mathscr{H}^{**} u/v \leqq \mathscr{H}^*$. Thus $b_1 \equiv a_1 (\mathscr{H}^*)$, a contradiction to $a \equiv b (\Phi)$.

By duality, $a \vee c \equiv b \vee c(\Phi)$. Hence Φ is a congruence relation by Lemma I. 3.8 of [2]. $\mathscr{H}^* \wedge \Phi = \omega$ is trivial. Now if $\mathscr{H}^* \wedge \Psi = \omega$, $a \equiv b(\Psi)$, $u, v \in [a \wedge b, a \vee b]$ and $u \equiv v(\mathscr{H}^*)$, then $u \equiv v(\Psi)$, hence $u = v$, showing that $\Psi \leqq \mathscr{H}^*$. Thus $\Phi = \mathscr{H}^{**}$ as asserted.

To complete the proof of the theorem, let L satisfy (C) and let for each congruence $\mathscr{H}$ of L, $\mathscr{H}^*$ be separable. Let Φ be the congruence constructed as above for the given congruence $\mathscr{H}$ of L. For $a, b \in L$, $a < b$, let $a = x_0 < x_1 < \ldots < x_n = b$ be the chain establishing the separability of $\mathscr{H}^*$. Then for each i, $0 \leqq i < n$, either $x_i \equiv x_{i+1}(\mathscr{H}^*)$ or by the definitions of separability and of Φ, $x_i \equiv x_{i+1}(\Phi)$.

Therefore $a \equiv b(\mathscr{H}^* \vee \Phi)$, and so $\mathscr{H}^* \vee \Phi = \iota$. That is $\mathscr{H}^* \vee \mathscr{H}^{**} = \iota$. This holds for any congruence $\mathscr{H}$ on L, proving $C(L)$ is a Stone lattice as $C(L)$ is always a distributive lattice.

It is obvious that every weakly modular lattice satisfies (C) of the above theorem. But the converse is not true. In fact, the lattice of Fig. 1 satisfies (C) of the theorem but it is not weakly modular.

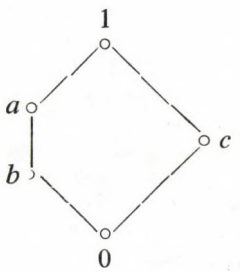

Fig. 1

The following theorem gives a characterization of lattices satisfying (C) of the above theorem and it is analogous to Theorem 4 of [3].

THEOREM 2.2. *Let L be any lattice and $\mathscr{H}$ be a congruence relation of L. Let Φ be the binary relation on L defined by $x \equiv y(\Phi)$ iff $u \equiv v(\mathscr{H}^*)$ for no proper subquotient u/v of $x \vee y / x \wedge y$. Then L satisfies* (C) *of Theorem* 2.1 *iff for any congruence relation $\mathscr{H}$ of L the binary relation Φ as defined above is a congruence relation of L.*

PROOF. If L satisfies (C) of Theorem 2.1, then by the converse part of the proof of that theorem, Φ is a congruence relation of L for any congruence $\mathscr{H}$ of L.

Conversely, if possible let L does not satisfy (C) of Theorem 2.1 Then there. exist $a, b, c, d \in L$, $a > b$, $c > d$ such that $a/b \underset{\omega}{\approx} c/d$, but no proper subquotient c_1/d_1 of c/d satisfies

$$c_1 \equiv d_1 \quad (\mathscr{H}^{**} a/b). \tag{2.3}$$

Consider $\mathscr{H} = \mathscr{H}^* a/b$. By hypothesis, the binary relation Φ defined by $x \equiv y(\Phi)$ iff $u \equiv v(\mathscr{H}^*)$ for no proper subquotient u/v of $x \vee y / x \wedge y$, is a congruence relation. It is clear from (2.3) and the definition of Φ that $c \equiv d(\Phi)$. But $a/b \underset{\omega}{\approx} c/d$, which implies $a \equiv b(\Phi)$. Now $\mathscr{H} a/b \leqq \mathscr{H}^{**} a/b$ by Theorem VIII. 2.1 of [1]. Con-

sequently $a \equiv b(\mathscr{H}^{**} a/b)$. That is, $a \equiv b(\mathscr{H}^*)$ and therefore $u \equiv v(\mathscr{H}^*)$ for every subquotient u/v of a/b, which contradicts $a \equiv b(\Phi)$. This completes the proof.

Using Theorem 1 of [3], the above theorem can be restated as follows:

THEOREM 2.3. *Let L be any lattice and $\mathscr{H}$ be a congruence relation of L. Let Φ be the binary relation of L defined by $x \equiv y(\Phi)$ iff for every proper subquotient u/v of $x \vee y/x \wedge y$ there exists a proper quotient u_1/v_1 in L such that $u_1/v_1 \underset{\omega}{\approx} u/v$ and $u_1 \equiv v_1(\mathscr{H})$. Then L satisfies* (C) *of Theorem* 2.1 *iff for any congruence $\mathscr{H}$ of L the binary relation Φ as defined above is a congruence relation of L.*

3. Application to weakly modular lattices

The following theorem is a particular case of Theorem 2.1 and it generalizes the corollary of Theorem 1 of [4].

THEOREM 3.1. *Let L be a weakly modular lattice. Then $C(L)$ is a Stone lattice iff for any congruence $\mathscr{H}$ of L and $a, b \in L$, $a > b$, there exists a finite chain $a = a_1 > a_2 > \ldots > a_n = b$ such that for each i, $1 \leqq i < n$, either no proper subquotient u/v of a_i/a_{i+1} satisfies $u \equiv v(\mathscr{H})$ or every proper subquotient u/v of a_i/a_{i+1} contains a proper subquotient u_1/v_1 satisfying $u_1 \equiv v_1(\mathscr{H})$.*

PROOF. Let $C(L)$ be a Stone lattice. Let $a, b \in L$ $a > b$. By Theorem 2.1, $\mathscr{H}^*$ is separable for any congruence $\mathscr{H}$ of L. Hence there exists a finite chain $a = a_1 > a_2 > \ldots > a_n = b$ such that for each i, either $a_i \equiv a_{i+1}(\mathscr{H}^*)$ or $u \equiv v(\mathscr{H}^*)$ for no proper subquotient u/v of a_i/a_{i+1}, $i = 1, \ldots, n-1$.

In the first case no proper subquotient u/v of a_i/a_{i+1} satisfies $u \equiv v(\mathscr{H})$, for, otherwise we get $u \equiv v(\mathscr{H} \wedge \mathscr{H}^*)$ for a proper subquotient u/v of a_i/a_{i+1}, a contradiction.

Consider the second case. Since L is weakly modular, by Corollary 1 of Theorem 1 of [3], $u \equiv v(\mathscr{H}^*)$ iff no proper subquotient u_1/v_1 of u/v satisfies $u_1 \equiv v_1(\mathscr{H})$. But no such proper subquotient u/v of a_i/a_{i+1} exists. This means that for every proper subquotient u/v of a_i/a_{i+1} there exists a proper subquotient u_1/v_1 of u/v satisfying $u_1 \equiv v_1(\mathscr{H})$.

Conversely, let a weakly modular lattice L satisfy the conditions of the theorem. To show $C(L)$ is a Stone lattice, by Theorem 2.1, it suffices to show that $\mathscr{H}^*$ is separable for any congruence $\mathscr{H}$ of L. But for any congruence $\mathscr{H}$ and $a, b \in L$, $a > b$, by hypothesis, there exists a finite chain $a = a_1 > a_2 > \ldots > a_n = b$ such that for each i, $1 \leqq i < n$, one of the following conditions holds:

(3.1) no proper subqoutient u/v of a_i/a_{i+1} satisfies $u \equiv v(\mathscr{H})$,

(3.2) for every proper subquotient u/v of a_i/a_{i+1} there exists a proper subquotient u_1/v_1 of u/v such that $u_1 \equiv v_1(\mathscr{H})$.

But (3.1) means $a_i \equiv a_{i+1}(\mathscr{H}^*)$ and (3.2) means $u \equiv v(\mathscr{H}^*)$ for no proper subquotient u/v of a_i/a_{i+1}. Hence $\mathscr{H}^*$ is separable. This proves the theorem.

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MULTIPLICATIVE LATTICES AND FRAMES

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A *multiplicative lattice* is a complete lattice L endowed with an associative multiplication which distributes over arbitrary joins (i.e. $a\cdot\bigvee_i b_i=\bigvee_i a\cdot b_i$) and has a two-sided identity e (i.e. $e\cdot a=a\cdot e=e$). A *frame* is a complete lattice L in which finite meets distribute over arbitrary joins. Any frame is a multiplicative lattice with $\wedge$ as a multiplication. Multiplicative lattices intend to generalize lattices of ring ideals (see e.g. M. Ward and R. P. Dilworth [9], K. Keimel [8], or D. D. Anderson [1]) while frames are generalized open set lattices of topological spaces (see P. T. Johnstone [6]). Recently, A. Joyal and M. Tierney [7] and F. Borceaux [3] have considered connections between these two concepts and C. J. Mulvey stressed that multiplicative lattices can replace frames in non-commutative situations. Our attention is mainly focused on the commutative case and the main result generalizes the classical Krull theorem to continuous multiplicative lattices.

Multiplicative lattices will be briefly called m-lattices. An *integral* m-lattice (im-lattice) is an m-lattice L in which e is the greatest element 1 of L. If the multiplication is commutative, we will speak about cm- and cim-lattices. The following result is well known.

PROPOSITION 1. *Frames coincide with idempotent im-lattices.*

PROOF. Let L be an im-lattice. Since $a\cdot b\leqq a\cdot 1=a$ and similarly $a\cdot b\leqq b$ it holds

$$a\cdot b\leqq a\wedge b \tag{1}$$

for any $a, b\in L$. Further, $a\wedge b=(a\wedge b)\cdot(a\wedge b)\leqq a\cdot b$. Hence L is a frame.

This implies that frames form a variety of im-lattices. Here, im-lattices are algebras of type $\{\vee, e, \cdot\}$ and thus homomorphisms of im-lattices are maps preserving $\vee$, e and $\cdot$. Any im-lattice L has a maximal frame quotient $\varrho_L: L\to L$. Hence the category $\mathscr{F}$ of frames is reflective in the category $\mathscr{IM}$ of im-lattices, where ϱ is the corresponding reflection. $\mathscr{IM}$ is reflective in the category $\mathscr{M}$ of m-lattices because integrality is given by the implication $(\forall x)(x\leqq e)$ (hence im-lattices form a quasivariety of m-lattices).

Recall that if L is an im-lattice then $\vee$-semilattice quotients of L correspond to closure operators $f: L\to L$ (i.e. $a\leqq f(a)$, $a\leqq b\Rightarrow f(a)\leqq f(b)$ and $f(f(a))=f(a)$, see [7]). Hence quotients of L correspond to closure operators $f: L\to L$ such that

$$f(a)=f(b),\ f(c)=f(d)\Rightarrow f(a\cdot c)=f(b\cdot d).$$

If L is a frame then the next proposition is contained in [6].

PROPOSITION 2. *Let L be an im-lattice. Then frame quotiensts of L correspond to closure operators $f: L \to L$ such that*

$$f(a \cdot b) = f(a) \wedge f(b) \tag{2}$$

for any $a, b \in L$.

PROOF. Let $h: L \to X$ be a frame quotient of L. Since h is $\vee$ preserving, the mapping $h_*: X \to L$, $h_*(x) = \vee \{a \in L \setminus h(a) \leqq x\}$ is $\wedge$ preserving. Hence the corresponding closure operator $h_* \cdot h: L \to L$ has property (2). Since $h \cdot h_*$ is the identity on X, (2) implies that $h: L \to X$ is a frame quotient.

A frame is called *spatial* if it is isomorphic to an open set lattice of a topological space. An element $x \in L$ is *prime* if $x \neq 1$ and $y \wedge z \leqq x \Rightarrow y \leqq x$ or $z \leqq x$ for any $y, z \in L$. It is well known that a frame L is spatial if and only if it has enough primes, i.e. if

$$a = \wedge \{x \in L \setminus a \leqq x,\ x \text{ prime}\}$$

for any $a \in L$. The category $\mathscr{SF}$ of spatial frames is a reflective subcategory of $\mathscr{F}$ (see [6]). Hence $\mathscr{SF}$ is reflective in $\mathscr{IM}$. The following proposition explicitly describes the corresponding reflection.

An element $x \neq 1$ of an m-lattice L is called *m-prime* if

$$y \cdot z \leqq x \Rightarrow y \leqq x \quad \text{or} \quad z \leqq x$$

for any $y, z \in L$. Put

$$p(a) = \wedge \{x \in L \setminus a \leqq x,\quad x \text{ m-prime}\}$$

for any $a \in L$. Then $p: L \to L$ is a closure operator. Let L_p be the complete lattice of all p-closed elements a (i.e. $p(a) = a$).

PROPOSITION 3. *Let L be an im-lattice. Then $p: L \to L_p$ is the maximal spatial frame quotient of L.*

PROOF. It is easy to see that p satisfies condition (2). The map $p: L \to L_p$ is a corresponding frame quotient. An element x is m-prime in L iff $x = p(x)$ and x is prime in L_p. To prove it let $x \in L$ be m-prime. Then $x = p(x)$ and $p(y) \wedge p(z) \leqq x$ implies $y \cdot z \leqq p(y) \cdot p(z) \leqq p(y) \wedge p(z) \leqq x$. Hence $y \leqq x$ or $z \leqq x$ and thus $p(y) \leqq x$ or $p(z) \leqq x$. Conversely, let $x \in L_p$ be prime in L_p. Then $y \cdot z \leqq x$ implies $p(y) \wedge p(z) = p(y \cdot z) \leqq p(x)$ and therefore $y \leqq p(y) \leqq x$ or $z \leqq p(z) \leqq x$. The consequence is that $p: L \to L_p$ is a spatial frame quotient of L.

Let $h: L \to X$ be a spatial frame quotient of L. If $h(a) \neq h(b)$ then there is a prime $x \in X$ such that $h(a) \leqq x$, $h(b) \not\leqq x$ or $h(a) \not\leqq x$, $h(b) \leqq x$. Then $h_*(x)$ is m-prime in L and therefore $p(a) \neq p(b)$. This proves that $p: L \to L_p$ is the maximal spatial frame quotient of L.

If L is a frame then we get the usual spatial reflection of L.

COROLLARY 1. *Any im-lattice having enough m-primes is a spatial frame.*

An element $x \neq 1$ of an m-lattice L is called *m-semiprime* if $y^2 \leqq x \Rightarrow y \leqq x$ for any $y \in L$ (see [8]). Any m-prime element is m-semiprime and any intersection of m-semiprime elements is m-semiprime. If L is a frame then any of its elements is

m-semiprime. Recall that if a, b are elements of a complete lattice L then b is *way below* a if

$$\{x_i\}_{i\in I} \text{ directed}, \quad a \leqq \bigvee_{i\in I} x_i \Rightarrow b \leqq x_i \text{ for some } i\in I.$$

We write $b \ll a$. A complete lattice L is *continuous* if

$$a = \vee\{b\in L \setminus b \ll a\}$$

for any $a\in L$ (see [5]).

PROPOSITION 4. *Let L be a continuous im-lattice and $a\in L$ m-semiprime. Then $a\in L_p$.*

PROOF. Let a be an m-semiprime element of a continuous im-lattice L. Assume that $a<p(a)$. Then there is $a_1\ll p(a)$ such that $a_1 \not\leqq a$. Hence $a_1^2 \not\leqq a$. There is $a_2\ll a_1^2$ such that $a_2\not\leqq a$. In this way we get a chain

$$p(a)\gg a_1\gg a_2\gg \ldots \gg a_n\gg \ldots$$

such that $a_{n+1}^2\leqq a_n$ and $a_n\not\leqq a$ for any natural number n. Let

$$A = \{x\in L \setminus x\geqq a_n \text{ for some } n\}.$$

Let $x_1\leqq x_2\leqq \ldots \leqq x_m\leqq \ldots$ be a chain in $L-A$ and put $x=\bigvee_i x_i$. If $x\in A$ then $a_n\leqq x$ for some n, i.e. $a_{n+1}\leqq x_m$ for some m because $a_{n+1}\ll a_n$. This is a contradiction and therefore $x\in L-A$. Following the Zorn lemma, there is a maximal element $a\leqq z\in L-A$. Since $p(a)\in A$, it holds $p(a)\not\leqq z$. Hence z is not m-prime and thus there are $u, v\in L$ such that $u\cdot v\leqq z$, $u\not\leqq z$ and $v\not\leqq z$. Therefore $u\vee z, v\vee z\in A$, i.e. $a_n\leqq u\vee z$, $v\vee z$ for some n. We have

$$a_{n+1}\leqq a_n^2\leqq (u\vee z)\cdot(v\vee z) = (u\cdot v)\vee(u\cdot z)\vee(z\cdot v)\vee z^2\leqq z,$$

which is a contradiction to $z\notin A$.

This proposition generalizes the well known fact that any continuous frame is spatial (see [6] p. 311 or [5] p. 259). Recall that an element x of a complete lattice L is *compact* if $x\ll x$. A complete lattice is *algebraic* if

$$a = \vee\{x\in L \setminus x\leqq a, x \text{ compact}\}$$

for any $a\in L$. Any algebraic lattice is continuous. Keimel [8] proved Proposition 4 for algebraic lattices.

Let L be an m-lattice and $a\in L$. Then the *radical* $r(a)$ of a is defined as follows:

$$r(a) = \vee\{x\in L \setminus x^n\leqq a \text{ for some } n\}$$

(see L. Fuchs [4]). This generalizes the corresponding notion for commutative ring ideals. We say that a continuous m-lattice L is *m-continuous* if

(3) $\quad b \ll a$, $\{x_i\}_{i\in I}$ directed, $\quad a^n \leqq \bigvee_{i\in I} x_i \Rightarrow b^m \leqq x_i$ for some m and $i\in I$.

This weakens the condition $b\ll a\Rightarrow b^n\ll a^n$ which holds for commutative ring ideals.

LEMMA 1. *A continuous m-lattice L is m-continuous iff r preserves directed joins.*

PROOF. Condition (3) says that $r(\vee x_i) \leqq \vee r(x_i)$ for any directed set $\{x_i\}$.

PROPSOSITION 5. *Let L be an m-continuous cim-lattice. Then $r: L \to L$ is a closure operator.*

PROOF. If is evident that $a \leqq r(a)$ and $a \leqq b$ imply $r(a) \leqq r(b)$. Further, $r(a) = r(a^m)$ for any $a \in L$. The set $\{x \in L \setminus x^n \leqq a \text{ for some } n\}$ is directed because $x^n \leqq a$, $y^m \leqq a$ imply $(x \vee y)^{n+m} = \bigvee_{i+j=n+m} x^i \cdot y^j \leqq a$. Hence Lemma 1 gives that

$$rr(a) = r(\vee \{x \in L \setminus x^n \leqq r(a) \text{ for some } n\}) = \vee \{r(x) \setminus x^n \leqq r(a) \text{ for some } n\} =$$

$$= \vee \{r(x^n) \setminus x^n \leqq r(a)\} \leqq r(a).$$

THEOREM 1. *Let L be an m-continuous cim-lattice. Then $p = r$.*

PROOF. It is easy to see that $r(a) \leqq p(a)$ for any $a \in L$. Let $x^2 \leqq r(a)$ and $y \ll x$. It was shown in the proof of Proposition 5 that the set $\{z \setminus z^n \leqq a\}$ is directed. Following the m-continuity, $y^m \leqq z^n$ for some m and n. Hence $y \leqq r(a)$ and therefore $x \leqq r(a)$. We have proved that a is m-semiprime. Proposition 4 yields that $r(a) = pr(a) \geqq p(a)$.

Theorem 1 says that the radical of a is the intersection of all primes containing a. It thus generalizes the classical Krull theorem for commutative ring ideals. It also covers the generalizations of the Krull theorem given by L. Fuchs [4] and D. D. Anderson [1]. Fuchs considers a complete lattice L satisfying the ascending chain condition (i.e. any element of L is compact) while Anderson deals with a certain class of algebraic lattices. Theorem 1 implies the corresponding theorem of K. E. Aubert [2] for x-ideals, as well.

COROLLARY 2. *Let L be an m-continuous cim-lattice. Then the complete lattice L_p is continuous.*

PROOF. It follows from Lemma 1 and Theorem 1 that p preserves directed joins. Hence L_p is a $\wedge$-sublattice of L which is closed with respect to directed joins. Following [5], L_p is continuous.

D. D. Anderson [1] pointed out that algebraic cim-lattices form a convenient setting for an abstract commutative ideal theory. We have shown that some results can be extended to continuous cim-lattices. From the point of view of ideal theory, this is an excessive generality. The following example touches the remaining cases.

EXAMPLE 1. The unit interval [0, 1] is an m-continuous cim-lattice with respect to $\vee$ and $\cdot$. It is not algebraic because 0 is the only compact element. [0, 1] has no m-prime elements. Hence Theorem 1 says that $\lim_{n \to \infty} x^n = 0$ for any $0 \leqq x < 1$.

Let $\mathscr{CM}$ ($\mathscr{CIM}$) be the category of cm-lattices (cim-lattices). We know from the beginning that $\mathscr{F} \subset \mathscr{CIM} \subset \mathscr{CM}$ is the sequence of reflective subcategories.

PROPOSITION 6. *$\mathscr{F} \subset \mathscr{CIM} \subset \mathscr{CM}$ is the sequence of coreflective subcategories.*

PROOF. The inclusion $\{x \in L \setminus x \leqq e\} \to L$ is a coreflection of a cm-lattice L in $\mathscr{CIM}$. Let L be a cim-lattice and $i(L)$ be the set of all of its idempotent elements. Then the inclusion $i(L) \to L$ is a coreflection of L in $\mathscr{F}$. It is evident that if X is a frame

and $h: X \to L$ a homomorphism then $h(X) \subseteq i(L)$. Hence the only thing to verify is that $i(L)$ is closed in L with respect to $\vee$, e and $\cdot$. For e and $\cdot$ this is evident and

$$\left(\bigvee_{j \in J} a_j\right)^2 = \bigvee_{j,k \in J} (a_j \cdot a_k) \geqq \bigvee_{j \in J} a_j^2 = \bigvee_{j \in J} a_j \quad \text{for any} \quad a_j \in i(M), \quad j \in J.$$

This yields that the inclusion $\mathscr{F} \subset \mathscr{CM}$ preserves colimits (see [7]).

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ON THE DISTRIBUTION OF RESIDUES OF PRODUCTS OF INTEGERS

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1. In this paper, our aim is to generalize the problem of the least quadratic non-residue modulo p and to prove a related result which may have applications independently of the original problem.

Throughout this paper, we use the following notations: Let $\{x\}=x-[x]$ denote the fractional part of x and let $\|x\|=\min(\{x\}, 1-\{x\})$ denote the distance from x to the nearest integer. Further denote $e^{2\pi ix}$ by $e(x)$. $\left(\frac{n}{p}\right)$ denotes the Legendre symbol modulo p with the convention $\left(\frac{0}{p}\right)=0$.

First we will show that if p is a prime number, the positive integers u, v are large in terms of p, and $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v$ are integers (satisfying (1) and (2) below) then the residues of the products $a_x b_y$ are uniformly distributed in the interval $(0, p)$:

THEOREM 1. *If p is a prime number, u, v, S, T are integers with $1 \leqq u, v \leqq p-1$, $1 \leqq T \leqq p$, furthermore, $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v$ are integers with*

$$a_i \not\equiv a_j \pmod{p} \quad (\text{for } 1 \leqq i < j \leqq u), \tag{1}$$

$$b_i \not\equiv b_j \pmod{p} \quad (\text{for } 1 \leqq i < j \leqq v), \tag{2}$$

and we denote the number of solutions of

$$a_x b_y \equiv n \pmod{p}, \quad 1 \leqq x \leqq u, \ 1 \leqq y \leqq v$$

by $f(n)$, then we have

$$\left| \sum_{n=S+1}^{S+T} f(n) - \frac{uvT}{p} \right| < 2(puv)^{1/2} \log p.$$

Section 2 will be devoted to the proof of this theorem.

Putting $S=0$, $T=[2p^{3/2}u^{-1/2}v^{-1/2} \log p]+1$ in Theorem 1, we obtain that

$$\sum_{n=1}^{T} f(n) > \frac{uvT}{p} - 2(puv)^{1/2} \log p >$$

$$> \frac{uv \cdot 2p^{3/2}u^{-1/2}v^{-1/2} \log p}{p} - 2(puv)^{1/2} \log p = 0$$

hence

COROLLARY. *If* $u, v, a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v$ *are defined in the same way as in Theorem* 1, *then there exist integers* n, x, y *with*

$$1 \leqq x \leqq u, \quad 1 \leqq y \leqq v, \tag{3}$$

$$a_x b_y \equiv n \pmod{p}, \quad 1 \leqq n \leqq [2p^{3/2}u^{-1/2}v^{-1/2}\log p] + 1. \tag{4}$$

(So that for $u \gg p, v \gg p$, there exist x, y, n satisfying (3), (4) and $1 \leqq n \ll p^{1/2}\log p$)
In Section 3, it will be shown that Theorem 1 is nearly best possible:

THEOREM 2. *If* p *is a prime number with* $p > 900$, *then putting* $u = v = T = \frac{p-1}{2}$, *there exist integers* $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v$ *such that* (1) *and* (2) *hold and*

$$\left| \sum_{n=S+1}^{S+T} f(n) - \frac{uvT}{p} \right| > \frac{1}{28}(puv)^{1/2} \left(\sim \frac{1}{28} p^{3/2} \right).$$

On the other hand, the corollary above seems to be far from the best possible. In fact, perhaps, also the following much sharper assertion is true:

CONJECTURE. If $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, p is a prime number large enough in terms of ε, ε_1 and ε_2, $u > \varepsilon_1 p$, $v > \varepsilon_2 p$ and $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v$ are defined in the same way as in Theorem 1, then there exist n, x, y with

$$a_x b_y \equiv n \pmod{p}, \quad 1 \leqq n < p^{\varepsilon}.$$

The corollary and conjecture above generalize the problem of the least quadratic non-residue modulo p. In fact, if $a_1, a_2, \ldots, a_u$ are the quadratic residues while $b_1, b_2, \ldots, b_v$ are the quadratic non-residues modulo p, then the numbers of the form $a_x b_y$ are the quadratic non-residues modulo p, so that the conjecture above would imply that the least quadratic non-residue is less than p^{ε}.

2. In this section, we prove Theorem 1. We shall need the following well-known lemma:

LEMMA. *Let* M *and* N *be integers with* $M < N$, *and let* α *be a non-integral real number. Then*

$$\left| \sum_{k=M}^{N} e(k\alpha) \right| \leqq \frac{1}{2\|\alpha\|}.$$

(See, e.g., [3], p. 23.)
Clearly, we have

$$\frac{1}{p} \sum_{k=0}^{p-1} e\big(k(a_x b_y - n)/p\big) = \begin{cases} 1, & \text{if } a_x b_y \equiv n \pmod{p} \\ 0, & \text{if } a_x b_y \not\equiv n \pmod{p} \end{cases}$$

and thus for all n,

$$f(n) = \frac{1}{p} \sum_{x=1}^{u} \sum_{y=1}^{v} \sum_{k=0}^{p-1} e\big(k(a_x b_y - n)/p\big)$$

so that

$$\sum_{n=S+1}^{S+T} f(n) = \frac{1}{p} \sum_{n=S+1}^{S+T} \sum_{x=1}^{u} \sum_{y=1}^{v} \sum_{k=0}^{p-1} e\big(k(a_x b_y - n)/p\big) =$$

$$= \frac{1}{p} \sum_{n=S+1}^{S+T} \sum_{x=1}^{u} \sum_{y=1}^{v} 1 + \frac{1}{p} \sum_{k=1}^{p-1} \sum_{n=S+1}^{S+T} \sum_{x=1}^{u} \sum_{y=1}^{v} e\big(k(a_x b_y - n)/p\big) =$$

$$= \frac{uvT}{p} + \frac{1}{p} \sum_{k=1}^{p-1} \sum_{n=S+1}^{S+T} \sum_{x=1}^{u} \sum_{y=1}^{v} e\big(k(a_x b_y - n)p\big)$$

hence

$$\text{(5)} \qquad \Big| \sum_{n=S+1}^{S+T} f(n) - \frac{uvT}{p} \Big| = \frac{1}{p} \Big| \sum_{k=1}^{p-1} \Big(\Big(\sum_{n=S+1}^{S+T} e(-kn/p) \Big) \sum_{x=1}^{u} \sum_{y=1}^{v} e\big(ka_x b_y/p\big) \Big) \Big| \leqq$$

$$\leqq \frac{1}{p} \sum_{k=1}^{p-1} \Big(\Big| \sum_{n=S+1}^{S+T} e(-kn/p) \Big| \sum_{x=1}^{u} \Big| \sum_{y=1}^{v} e(ka_x b_y/p) \Big| \Big).$$

By the lemma above, here we have

$$\text{(6)} \qquad \Big| \sum_{n=S+1}^{S+T} e(-kn/p) \Big| \leqq \frac{1}{2\|-k/p\|} = \frac{1}{2\|k/p\|}.$$

Furthermore, by the Cauchy—Schwarz inequality for $1 \leqq k \leqq p-1$ we have

$$\text{(7)} \qquad \sum_{x=1}^{u} \Big| \sum_{y=1}^{v} e(ka_x b_y/p) \Big| \leqq \Big(u \sum_{x=1}^{u} \Big| \sum_{y=1}^{v} e(ka_x b_y/p) \Big|^2 \Big)^{1/2} \leqq$$

$$\leqq \Big(u \sum_{j=1}^{p-1} \Big| \sum_{y=1}^{v} e(jb_y/p) \Big|^2 \Big)^{1/2} = \Big(u \sum_{j=0}^{p-1} \sum_{y=1}^{v} \sum_{z=1}^{v} e\big(j(b_y - b_z)/p\big) \Big)^{1/2} =$$

$$= \Big(u \sum_{y=1}^{v} \sum_{z=1}^{v} \Big(\sum_{j=0}^{p-1} e\big(j(b_y - b_z)/p\big) \Big) \Big)^{1/2} = \Big(u \sum_{y=1}^{v} p \Big)^{1/2} = (uvp)^{1/2}$$

since by (1) and (2), $ka_x \not\equiv ka_y \pmod p$ for $x \neq y$ and

$$\sum_{j=0}^{p-1} e\big(j(b_y - b_z)/p\big) = \begin{cases} p & \text{for} \quad y = z \\ 0 & \text{for} \quad y \neq z. \end{cases}$$

It follows from (5), (6) and (7) that for $p > 2$,

$$\Big| \sum_{n=S+1}^{S+T} f(n) - \frac{uvT}{p} \Big| \leqq \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{2\|k/p\|} (uvp)^{1/2} =$$

$$= p^{-1/2} (uv)^{1/2} \cdot 2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k/p} = (puv)^{1/2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \leqq$$

$$\leqq (puv)^{1/2} \left(1 + \log \frac{p-1}{2} \right) < 2\,(puv)^{1/2} \log p$$

since

$$\sum_{k=1}^{n} \frac{1}{k} \leqq 1 + \log n \quad \text{for} \quad n = 1, 2, \ldots,$$

while for $p=2$,

$$\left|\sum_{n=S+1}^{S+T} f(n)-\frac{uvT}{p}\right| \leqq \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{2\|k/p\|}(uvp)^{1/2} =$$

$$=\frac{1}{2}\cdot\frac{1}{2\|1/2\|}(2uv)^{1/2}=2^{-1/2}(uv)^{1/2}<2(2uv)^{1/2}\log 2=2(puv)^{1/2}\log p$$

and this completes the proof of Theorem 1.

3. In this section, we prove Theorem 2.

Let $a_1, a_2, \ldots, a_{(p-1)/2}$ be the set of the quadratic residues modulo p (in the interval $(0, p)$) while let $b_1, b_2, \ldots, b_{(p-1)/2}$ be the set of the quadratic non-residues. Then clearly, for $(n, p)=1$, the congruence

$$a_x b_y \equiv n \pmod p \tag{8}$$

can be solved if and only if $\left(\frac{n}{p}\right)=-1$ and if this condition holds, then for all $1\leqq y\leqq\frac{p-1}{2}$ there is a uniquely determined $1\leqq x\leqq\frac{p-1}{2}$ for which (8) holds. Thus

$$f(n)=\begin{cases}\dfrac{p-1}{2} & \text{for} \quad (n,p)=1, \quad \left(\dfrac{n}{p}\right)=-1\\ 0 \quad \text{for} \quad (n,p)>1 \quad \text{or} \quad (n,p)=1, \left(\dfrac{n}{p}\right)=+1\end{cases}$$

so that for all S,

$$\sum_{n=S+1}^{S+T} f(n)=\sum_{\substack{S<n\leqq S+T\\ \left(\frac{n}{p}\right)=-1}}\frac{p-1}{2}=\frac{p-1}{2}\sum_{\substack{S<n\leqq S+T\\ (u,p)=1}}\frac{1}{2}\left(1-\left(\frac{n}{p}\right)\right)=$$

$$=\frac{p-1}{4}\sum_{\substack{S<n\leqq S+T\\ (n,p)=1}}1-\frac{p-1}{4}\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)$$

hence

$$\left|\sum_{n=S+1}^{S+T} f(n)-\frac{uvT}{p}\right| = \tag{9}$$

$$=\left|\left(\frac{p-1}{4}\sum_{\substack{S<n\leqq S+T\\ (n,p)=1}}1-\frac{p-1}{4}\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)\right)-\frac{(p-1)^3}{8p}\right| =$$

$$=\left|-\frac{p-1}{4}\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)+\frac{p-1}{4}\left(\sum_{\substack{S<n\leqq S+T\\ (n,p)=1}}1-\frac{p-1}{2}\right)+\left(\frac{p-1}{4}\cdot\frac{p-1}{2}-\frac{(p-1)^3}{8p}\right)\right| \geqq$$

$$\geqq\frac{p-1}{4}\left|\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)\right|-\frac{p-1}{4}-\frac{(p-1)^2}{8}\left(1-\frac{p-1}{p}\right)=$$

$$=\frac{p-1}{4}\left|\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)\right|-\frac{p-1}{4}\left(1+\frac{p-1}{2p}\right)\geqq\frac{p-1}{4}\left|\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)\right|-\frac{p-1}{2}.$$

By Theorem 6 in [1] (see also [2]), there exists an integer S such that

$$\left|\sum_{n=S+1}^{S+T}\left(\frac{n}{p}\right)\right| \geqq \frac{1}{\pi}(\sqrt{p}-1/\sqrt{p}) > \frac{1}{\pi}\cdot\frac{1}{2}\sqrt{p} > \frac{1}{7}\sqrt{p}.$$

For such an integer S, we obtain from (9) that for $p>900$ we have

$$\left|\sum_{n=S+1}^{S+T} f(n)-\frac{uvT}{p}\right| > \frac{p-1}{4}\cdot\frac{1}{7}\sqrt{p}-\frac{p-1}{2} =$$

$$= \frac{p-1}{28}(\sqrt{p}-14) > \frac{p-1}{28}\cdot\frac{1}{2}\sqrt{p} = \frac{1}{28}\left(\left(\frac{p-1}{2}\right)^2 p\right)^{1/2} = \frac{1}{28}(uvp)^{1/2}$$

which completes the proof of Theorem 2.

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ON UNIVERSAL FUNCTIONS AND SERIES

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Introduction

It is well known that there exists an $f: [0, 1] \to \mathbf{R}$ and a sequence $\{h_n\}$; $h_n \to 0$ such that the functions $g_n(x) := \dfrac{f(x+h_n)-f(x)}{h_n}$ are dense in $L^0[0, 1] = S[0, 1]$ (the space of measurable functions of [0, 1]). (See [7] and [8].)

It was proved by I. Joó in [6] that if $0 < p < 1$ then there is an $f \in L^p(\Omega)$; $\Omega \subset \mathbf{R}^m$ and $\mathbf{h}_n \to 0$ such that the functions g_n are dense in L^p. He stated the problem for $p \geqq 1$. (Cf. also A. Bogmér and A. Sövegjártó [2] for the case $\Omega = [0, 1]$.)

In the first part of this paper we study the multidimensional case of this problem.

THEOREM 1. *Let $\Omega \subset \mathbf{R}^m$ be an open set, and let $p \geqq 1$. Then for an arbitrary $f \in L^p(\Omega)$ and $\{\mathbf{h}_n \in \mathbf{R}^m\}$ with $|\mathbf{h}_n| \to 0$, the functions $g_n(\mathbf{x}) = \dfrac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{|\mathbf{h}_n|}$ are not dense in $L^p(\Omega)$.*

We note that the case $p > 1$ was also considered by M. Horváth [4]. After learning my results he gave a shorter proof using a stronger apparatus.

In the second part of the paper we consider a similar problem concerning universal series. Let $\{\varphi_l\}$ be a complete orthonormal system in L^2. The series $\sum_1^\infty c_l \varphi_l$ is called universal in L^p if its partial sums are dense in $L^p[0, 1]$. Talaljan proved that for an arbitrary complete orthonormal system $\{\varphi_l\}$ there exists a series $\sum_1^\infty c_l \varphi_l$ universal in $L^0[0, 1]$.

I. Joó [5] generalized this theorem of Talaljan, proving that the series $\sum_1^\infty c_l \varphi_l$ is universal in $L^0[0, 1]$ for a residual set of sequences $\{c_l\} \in C_0$, where C_0 is the space of sequences tending to 0 endowed with the metric $d(\{c_l\}, \{c_l'\}) = \sup_l |c_l - c_l'|$. Later in [6] I. Joó showed that the theorem is not true for $p \geqq 1$. Instead of the L^p limit of partial sums we study the effect of a T, $\{\alpha_{nk}\}$ Toeplitz summation process.

Let T, $\{\alpha_{nk}\}$ be a permanent Toeplitz summation process. We want to define in this case the "partial sum" as the function

$$g_n(\{c_l\}, x) := \lim_{M \to \infty} \sum_{k=1}^{M} \alpha_{nk} \sum_{l=1}^{k} c_l \varphi_l(x).$$

In order to ensure the existence of this limit in L^2 we also suppose that T has the

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following property:

(T, 4) $$\sum_{k=1}^{\infty} |\alpha_{nk}| \cdot k < +\infty \quad (n = 1, 2, \ldots).$$

(This auxiliary assumption is slightly stronger than the well-known condition

(T, 3) $$\sum_{k=1}^{\infty} |\alpha_{nk}| < +\infty.$$

However, (T, 4) is satisfied by all the well-known summation processes; e.g. Cesàro or Abel—Poisson.)

THEOREM 2. *Let* $p \geqq 1$ *and* $1 = \frac{1}{p} + \frac{1}{q}$. *Suppose that* $\{\varphi_l\}$ *is a complete orthonormal system of* $L^p[0, 1]$ *such that* $\|\varphi_1\|_q < +\infty$. *Let* T *be a Toeplitz summation process with properties* (T, i) $(i = 1, \ldots, 4)$. *Then for any* $\{c_l\} \in C_0$ *the sequence of functions*

$$g_n(\{c_l\}, x) = \sum_{k=1}^{\infty} \alpha_{nk} \sum_{l=1}^{k} c_l \varphi_l(x)$$

is not dense in $L^p[0, 1]$.

REMARKS. If $p \geqq 2$ then we obviously have $\|\varphi_l\|_q < +\infty$. We do not know if this extra condition can be deleted if $1 \leqq p < 2$. This unsolved problem for partial sums was posed by I. Joó [6].

Before formulating Theorem 3 we mention that for any universal sequence $\sum_{l=1}^{\infty} c_l \varphi_l$ we can find a Toeplitz summation process which makes it convergent. Obviously the universality of $\sum_{l=1}^{\infty} c_l \varphi_l$ means that for any $f \in L^p$ we can choose n_k such that

$$\sum_{l=1}^{n_k} c_l \varphi_l(x) \xrightarrow{L^p} f \quad \text{when} \quad k \to +\infty.$$

Then we can define a summation process T, $\{\alpha_{km}\}$ by

$$\alpha_{km} := \begin{cases} 1 & \text{for} \quad m = n_k \\ 0 & \text{otherwise} \end{cases}$$

and for this process the functions $g_n(\{c_l\}, x)$ tend to f.

THEOREM 3. *Let* $0 \leqq p < 1$ *and* $\{\varphi_l\}$ *a complete orthonormal system in* $L^2[0, 1]$. *Let* T *be a Toeplitz summation process with properties* (T, i) $(i = 1, \ldots, 4)$ *and let* $g_n(\{c_l\}, x)$ *be defined as in Theorem* 2. *Then the set*

$$H := \{\{c_l\} | c_l \in C_0;\ \{g_n(\{c_l\}, x)\} \text{ is dense in } L^p[0, 1]\}$$

is residual in C_0.

In the above results we looked for estimations of different norms. Now we state a result on the L^1 norms of a complete orthonormal system. Obviously for

any $\varepsilon>0$ and $\alpha>0$ we can find an orthonormal system $\{\varphi_n\}$ in $L^2[0, 1]$ such that $\sum_{n=1}^{\infty}\|\varphi_n\|_1^{\alpha}<\varepsilon$. However, we have

THEOREM 4. *If* $\{\varphi_n\}$ *is a complete orthonormal system in* $L^2[0, 1]$ *then* $\sum_{n=1}^{\infty}\|\varphi_n\|_1^2=+\infty$.

REMARK. For the Haar system we have $\sum_{n=1}^{\infty}\|\varphi_n\|_1^2=+\infty$, but for every $\varepsilon>0$, $\sum_{n=1}^{\infty}\|\varphi_n\|_1^{2+\varepsilon}<+\infty$. Indeed, the functions of the Haar system are: $\chi_0^{(0)}\equiv 1$,

$$\chi_1^{(1)}:=\begin{cases}1 & \text{if } x\in[0;\ 1/2)\\ -1 & \text{if } x\in(1/2;\ 1]\\ 0 & \text{otherwise,}\end{cases}$$

$$\chi_n^{(k)}:=\begin{cases}\sqrt{2^n} & \text{if } x\in\left[\dfrac{2k-2}{2^{n+1}};\ \dfrac{2k-1}{2^{n+1}}\right)\\ -\sqrt{2^n} & \text{if } x\in\left(\dfrac{2k-1}{2^{n+1}};\ \dfrac{2k}{2^{n+1}}\right] \quad (k=1, 2, \ldots, 2^n).\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{n=0}^{\infty}\sum_{k=1}^{2^n}\|\chi_n^{(k)}\|_1^2=\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\frac{1}{2^n}=+\infty$$

but

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\|\chi_n^{(k)}\|_1^{2+\varepsilon}=\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\left(\frac{\sqrt{2^n}}{2^n}\right)^{2+\varepsilon}=\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\frac{1}{2^n\cdot 2^{n/2\cdot\varepsilon}}=\sum_{k=1}^{\infty}\frac{1}{2^{n/2\cdot\varepsilon}}<+\infty.$$

PROOF OF THEOREM 1. Suppose that $\{\tilde{g}_n(\mathbf{x}),\ n\in\mathbf{N}\}$ is dense in $L^p[\Omega]$. Then there exists a subsequence n_k such that $\tilde{g}_{n_k}(\mathbf{x})\xrightarrow{L_p}0$. Let $g_k':=\tilde{g}_{n_k}$ and let $\mathbf{e}$ be a cluster point of the sequence $\left\{\dfrac{\mathbf{h}_k'}{|\mathbf{h}_k'|};\ k\in\mathbf{N}\right\}$ and let k_n be a subsequence such that $d\left(\mathbf{e};\dfrac{\mathbf{h}_{k_n}'}{|\mathbf{h}_{k_n}'|}\right)\to 0$. We denote $\mathbf{h}_n=\mathbf{h}_{k_n}'$ and $g_n=g_{k_n}'$ for brevity.

We choose a cube K in Ω with an edge parallel to $\mathbf{e}$. We suppose that $\mathbf{e}$ is the m-th unit vector and K is the unit cube $\square^m=[0, 1]^m$.

Let

$$H_x:=\{(x_1, \ldots, x_{m-1}, x)\,|\,x_i\in[0, 1],\quad i=1, \ldots, m-1\},$$

$$A_{t,q}:=\{\mathbf{x}\in\square^m\,|\,q\leqq f(\mathbf{x})\leqq t\}$$

and

$$A_{t,q}':=\{\mathbf{x}\in A_{t,q}\,|\,\mathbf{x}\quad\text{is a density point of } A_{t,q}\}$$

for $q<t$; $q, t\in\mathbf{Q}$. Let λ_m denote the Lebesgue measure in $\mathbf{R}^m$.

According to the Lebesgue density theorem and the measurability of f, $\lambda_m(A_{t,q}\setminus A_{t,q}')=0$ and

$$\lambda_m\Big(\bigcup_{\substack{t,q\in\mathbf{Q}\\ q<t}}A_{t,q}\setminus A_{t,q}'\Big)=0.$$

Let

$$B = \square^m \setminus \bigcup_{\substack{t,q\in\mathbf{Q}\\ q<t}} (A_{t,q} \setminus A'_{t,q}).$$

From the previous remarks we get $\lambda_m(B)=1$. Let χ_B be the characteristic function of B. We have

$$1 = \lambda_m(B) = \int_0^1 \int_{H_{x_m}} \chi_B(x_1, x_2, \ldots, x_m)\, d\lambda_{m-1}\, d\lambda.$$

By Fubini's theorem for almost every $x_m\in(0, 1)$

$$\chi^*(x_m) := \int_{H_{x_m}} \chi_B(x_1, \ldots, x_{m-1}, x_m)\, d(x_1, x_2, \ldots, x_{m-1}) = \lambda_{m-1}(B\cap H_{x_m}) = 1$$

because $\int_0^1 \chi^*(x_m)\, d\lambda(x_m)=1=\lambda_m(B)$. If $\chi^*(x_m)=1$ then $\chi_B(x_1, \ldots, x_{m-1}, x_m)=1$, λ_{m-1} almost everywhere on H_{x_m}. Let $0<x_0<1$ be such that $\lambda_{m-1}(B\cap H_{x_0})=1$. For $\mathbf{y}\in\mathbf{R}^m$ we denote

$$E_{\mathbf{y}} := \{(x_1, \ldots, x_m) | x_i = y_i,\ i = 1, \ldots, m-1,\ x_m\in[0, 1]\}.$$

Similarly, by Fubini's theorem we get $\lambda_{m-1}(\tilde{B}_{x_0})=1$ where

$$\tilde{B}_{x_0} := \{x_1, \ldots, x_{m-1}, x_0) | \lambda(B\cap E_{(x_1, \ldots, x_{m-1}, x_0)}) = 1\}.$$

Let $B_{x_0} := B\cap\tilde{B}_{x_0}$. Now $\lambda_{m-1}(H_{x_0}\cap B)=1$ and hence

$$\lambda_{m-1}(B_{x_0}) = \lambda_{m-1}(B\cap H_{x_0}\cap\tilde{B}_{x_0}) = 1.$$

PROPOSITION. *For every* $\mathbf{y}\in B_{x_0}\cap A_{t,q}$ *and* $\mathbf{z}\in E_{\mathbf{y}}\cap B$ *we have* $\mathbf{z}\in A_{t,q}$.

PROOF. If $\mathbf{z}\in A_{t,q}$ then $f(\mathbf{z})>t$ or $f(\mathbf{z})<q$. Suppose, for example, that $s>f(\mathbf{z})>r>t>q$ $(s, r, t, q\in\mathbf{Q})$. So $\mathbf{z}\in A_{s,r}$ and $\mathbf{z}\in B$ and thus $\mathbf{z}\in A'_{r,s}$.

We denote by $K_\delta^{\mathbf{y}}$ the cube with centre $\mathbf{y}$ and with length of edges δ. By Lebesgue's density theorem we can choose δ so small that

$$\lambda_m(K_\delta^{\mathbf{y}}\cap A_{t,q}) > 0.99\lambda_m(K_\delta^{\mathbf{y}}) = 0.99\delta^m$$

and

$$\lambda_m(K_\delta^{\mathbf{z}}\cap A_{r,s}) > 0.99\delta^m$$

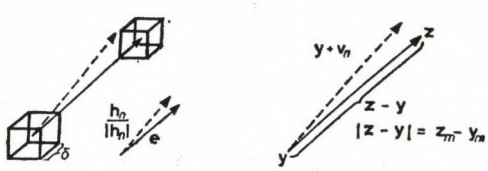

Fig. 1

Since $\frac{\mathbf{h}_n}{|\mathbf{h}_n|}\to\mathbf{e}$, $\left|\mathbf{e}-\frac{\mathbf{h}_n}{|\mathbf{h}_n|}\right|<\varepsilon$ if n is large enough. We have

$$|\mathbf{v}_n+\mathbf{y}-\mathbf{z}| = |\mathbf{v}_n-\mathbf{e}(y_m-z_m)| = (z_m-y_m)\left|\frac{\mathbf{h}_n}{|\mathbf{h}_n|}-\mathbf{e}\right| < \varepsilon.$$

Let $\tilde{K}^{\mathbf{x}}_{\delta,n}$ denote the cube with centre $\mathbf{x}$ and with an edge parallel to $\mathbf{h}_n$ and of length δ.

Since $\tilde{K}^{\mathbf{x}}_{\delta,n}$ is the image of $K^{\mathbf{x}}_{\delta}$ by a rotation which brings $\mathbf{e}$ into $\frac{\mathbf{h}_n}{|\mathbf{h}_n|}$ we have $\left(\text{for } \delta<\frac{1}{\sqrt{m}}\right)$ $\tilde{K}^{\mathbf{x}}_{\delta,n}\subset K^{\mathbf{x}}_{\delta+\varepsilon}$ and hence $\lambda_m(\tilde{K}^{\mathbf{x}}_{\delta,n}-K^{\mathbf{x}})\leqq(1+\varepsilon)^m\delta^m-\delta^m$. Similarly $\tilde{K}^{\mathbf{y}+\mathbf{v}_n}_{\delta,n}\subset K^{\mathbf{z}}_{\delta+2\varepsilon}$ and consequently $\lambda_m(\tilde{K}^{\mathbf{y}+\mathbf{v}_n}_{\delta,n}-K^{\mathbf{z}}_{\delta})\leqq(1+2\varepsilon)^m\delta^m-\delta^m$ because we get $\tilde{K}^{\mathbf{y}+\mathbf{v}_n}_{\delta,n}$ from $K^{\mathbf{z}}_{\delta}$ by composition of a translation and rotation described above.

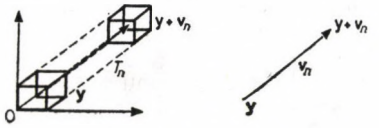

Fig. 2

Let T_n be the convex hull of $\tilde{K}^{\mathbf{y}}_{\delta,n}\cup\tilde{K}^{\mathbf{y}+\mathbf{v}_n}_{\delta,n}$. Evidently

$$\int_{\Omega}\left|\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{\mathbf{h}_n}\right|^p \geqq \int_{T_n}\left|\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{|\mathbf{h}_n|}\right|^p := A.$$

We choose a basis in $\mathbf{R}^m$ such that the m-th unit vector is parallel to $\frac{\mathbf{h}_n}{|\mathbf{h}_n|}$, the origin is one of the common vertices of T_n and $\tilde{K}^{\mathbf{x}}_{\delta,n}$, and T_n is in the first quadrant. (See fig. 2.)

Let

$$T_{x_m} = \{(x_1, \ldots, x_{m-1}, x_m)|\, x_i\in(0,\delta),\ i=1,\ldots,m-1\},$$

$$h_n := |\mathbf{h}_n| \quad\text{and}\quad 0\leqq u<v<\left[\frac{|\mathbf{v}_n|+\delta}{h_n}\right] := L.$$

We have

$$A = \int_0^{|\mathbf{v}_n|+\delta}\int_{T_{x_m}}\left|\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{h_n}\right|^p d(x_1,\ldots,x_{m-1})\,dx_m \geqq$$

$$\geqq \int_0^{h_n}\sum_{i=1}^{\left[\frac{|\mathbf{v}_n|+\delta}{h_n}\right]-1}\int_{T_{x_m}}\left|\frac{f(\mathbf{x}+i\mathbf{h}_n)-f(\mathbf{x}+(i-1)\mathbf{h}_n)}{h_n}\right|^p d(x_1\ldots x_{m-1})\,dx_m \geqq$$

$$\geqq \int_0^{h_n}\sum_{i=u+1}^{v}\int_{T_{x_m}}\left|\frac{f(\mathbf{x}+i\mathbf{h}_n)-f(\mathbf{x}+(i-1)\mathbf{h}_n)}{h_n}\right|^p d(x_1,\ldots,x_{m-1})\,dx_m =$$

$$= \int_0^{h_n} \int_{T_{x_m}} \sum_{i=u+1}^{v} \frac{|f(\mathbf{x}+i\mathbf{h}_n)-f(\mathbf{x}+(i-1)\mathbf{h}_n)|^p}{v-u} \cdot \frac{v-u}{h_n^p} d(x_1, \ldots, x_{m-1})\, dx_m \geqq$$

$$\geqq \int_0^{h_n} \int_{T_{x_m}} \frac{\left(\sum_{i=u+1}^{v} |f(\mathbf{x}+i\mathbf{h}_n)-f(\mathbf{x}+(i-1)\mathbf{h}_n)|\right)^p}{(v-u)^p} \cdot \frac{v-u}{h_n^p} d(x_1, \ldots, x_{m-1})\, dx_m \geqq$$

$$\geqq \int_0^{h_n} \int_{T_{x_m}} \frac{|f(\mathbf{x}+\mathbf{h}_n v)-f(\mathbf{x}+\mathbf{h}_n u)|^p}{h_n^p} \cdot (v-u)^{1-p} d(x_1, \ldots, x_{m-1})\, dx_m.$$

In these estimations we used the inequality

$$\sqrt[p]{\frac{\Sigma a_i^p}{n}} \geqq \frac{\Sigma a_i}{n} \quad (p \geqq 1 \quad \text{and} \quad a_i \geqq 0).$$

Let

$$1 \leqq u \leqq \left[\frac{\delta}{h_n}\right] - 1 = J \quad \text{and} \quad v_u = u + \left[\frac{|\mathbf{v}_n|}{h_n}\right],$$

then $\mathbf{x}+\mathbf{h}_n u \in \tilde{K}_{\delta,n}$ and $\mathbf{x}+\mathbf{h}_n v_u \in \tilde{K}_{\delta,n}^{\mathbf{y}+\mathbf{v}_n}$ whenever $\mathbf{x} \in \cup\{K_{x_m} | x_m \in [0, h_n]\}$. Summing up the previous estimations for $k=1, \ldots, J$ we obtain

$$J \int_{\Omega} \left|\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{h_n}\right|^p d\lambda_m \geqq$$

$$\geqq \int_0^{h_n} \int_{T_{x_m}} \sum_{u=1}^{J} \left|\frac{f(\mathbf{x}+\mathbf{h}_n v_u)-f(\mathbf{x}+\mathbf{h}_n u)}{h_n}\right|^p (v_u-u)^{1-p} d\lambda_m \geqq$$

$$\geqq \int_{h_n}^{\delta-h_n} \int_{T_{x_m}} \frac{\left|f\left(\mathbf{x}+\left[\frac{|\mathbf{v}_n|}{h_n}\right]h_n\right)-f(\mathbf{x})\right|^p}{h_n^p} \left(\frac{|\mathbf{v}_n|}{h_n}\right)^{1-p} d\lambda_m := B.$$

We need a lower estimation for $\left|f\left(\mathbf{x}+\left[\frac{|\mathbf{v}_n|}{h_n}\right]\mathbf{h}_n\right)-f(\mathbf{x})\right|^p$. Let

$$\hat{T} = \cup\{T_{x_m} | x_m \in [h_n;\ \delta - h_n]\}$$

then

$$\lambda_m(\hat{T} \cap A_{t,q}) \geqq \lambda_m(\hat{T}) - \left(\lambda_m(\hat{T} \setminus K_\delta^{\mathbf{y}}) + \lambda_m(K_\delta^{\mathbf{y}} \setminus A_{t,q})\right) \geqq$$

$$\geqq \lambda_m(\hat{T}) - \left(\lambda_m(\tilde{K}_{\delta,n}^{\mathbf{y}} \setminus K_\delta^{\mathbf{y}}) + \lambda_m(K_\delta^{\mathbf{y}} \setminus A_{t,q})\right) \geqq \delta^m - 2\delta^{m-1}h_n -$$

$$-\left((1+\varepsilon)^m \delta^m - \delta^m + 0.01\delta^m\right) \geqq \frac{3}{4}\delta^m$$

if h_n and ε is small enough.

Similarly, if

$$\tilde{T} = \cup\left\{T'_{x_m} | x_m \in \left[|\mathbf{v}_n| - \frac{\delta}{2} + h_n;\ |\mathbf{v}_n| + \frac{\delta}{2} - h_n\right]\right\}$$

then

$$\lambda_m(\tilde{T}\cap A_{r,s}) \geqq \lambda_m(\tilde{T}) - \left(\lambda_m(\tilde{K}_{\delta,n}^{\mathbf{v}_n+\mathbf{y}} \setminus K_\delta^{\mathbf{z}}) + \lambda_m(K_\delta^{\mathbf{z}} \setminus A_{r,s})\right) \geqq$$

$$\geqq \delta^m - 2\delta^{m-1}h_n - \left((1+2\varepsilon)^m\delta^m - \delta^m + 0.01\,\delta^m\right) \geqq \frac{3}{4}\delta^m$$

for small h_n and ε.

This immediately implies

$$\left|f\left(\mathbf{x}+\left[\frac{|\mathbf{v}_n|}{h_n}\right]\mathbf{h}_n\right)-f(\mathbf{x})\right| > r-t$$

on a set of Lebesgue measure greater than

$$\frac{3}{4}\,\delta^m + \frac{3}{4}\,\delta^m - (\delta^m - 2h_n\delta^{m-1}) \geqq \frac{1}{2}\,\delta^m.$$

So we get

$$\int_\Omega \left|\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{|\mathbf{h}_n|}\right|^p \geqq \frac{1}{J}\cdot\frac{1}{2}\cdot\delta^m|r-t|^p\,\frac{1}{|\mathbf{h}_n|^p}\left[\frac{|\mathbf{v}_n|}{|\mathbf{h}_n|}\right]^{1-p} \geqq$$

$$\geqq \frac{1}{\left[\dfrac{\delta}{|\mathbf{h}_n|}\right]}\cdot\frac{1}{2}\cdot\delta^m|r-t|^p\,\frac{1}{|\mathbf{h}_n|^p}\left(\frac{|\mathbf{v}_n|}{|\mathbf{h}_n|}-1\right)^{1-p} \geqq \frac{|\mathbf{h}_n|}{\delta}\cdot\frac{1}{2}\,\delta^m|r-t|^p\,\frac{1}{|\mathbf{h}_n|^p}\,\frac{(|\mathbf{v}_n|-|\mathbf{h}_n|)^{1-p}}{|\mathbf{h}_n|^{1-p}} \geqq$$

$$\geqq \frac{1}{2}\,\delta^{m-1}|r-t|^p(|\mathbf{v}_n|-|\mathbf{h}_n|)^{1-p} > \varrho > 0$$

for all n. Since $g_n \xrightarrow[L^p(\Omega)]{} 0$, this yields a contradiction proving our proposition.

Let

$$\tilde{\tilde{B}} = \{(x_1, \ldots, x_m)|(x_1, \ldots, x_m)\in B \quad \text{and} \quad (x_1, \ldots, x_{m-1}, x_0)\in B_{x_0}\}.$$

From $\lambda_{m-1}(B_{x_0})=1$ and $\lambda_m(B)=1$ we conclude that $\lambda_m(\tilde{\tilde{B}})=1$. For every j we choose sequences t_j and q_j of rational numbers tending to $f(\mathbf{z})$ from above and below, respectively. Hence $f(\mathbf{z}')\in A_{t_j,q_j}$ if $f(\mathbf{z})\in A_{t_j,q_j}$ and we have $f(\mathbf{z})=$ $=f(z_1, \ldots, z_{m-1}, x_0)=f(\mathbf{z}')$ for all $\mathbf{z}\in\tilde{\tilde{B}}$. From this it follows

$$\frac{f(\mathbf{x}+\mathbf{h}_n)-f(\mathbf{x})}{|\mathbf{h}_n|} = \frac{f(\mathbf{x}+\mathbf{h}_n+\alpha\mathbf{e})-f(\mathbf{x}+\alpha\mathbf{e})}{|\mathbf{h}_n|}$$

for $\alpha\in\mathbf{R}$ and for almost every $\mathbf{x}$ such that each argument is in the interior of the cube K. Now this property of f implies that the function

$$g(x_1, \ldots, x_m) = \begin{cases} 0 & \text{if} \quad x_m\in[0;\ 1/2] \\ 1 & \text{if} \quad x_m\in(1/2;\ 1] \end{cases}$$

cannot be approximated by the functions $\dfrac{f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})}{|\mathbf{h}|}$. This completes the proof of the theorem. For the case when $0\leqq p<1$ see [6].

Before proving Theorem 2 we recall the properties of a permanent T, $\{\alpha_{nk}\}$ Toeplitz summation process:

(T, 1) $$\sup_n \Omega_n^* := \sup_n \left(\sum_{k=1}^{\infty} |\alpha_{nk}|\right) = \Omega < +\infty,$$

(T, 2) $$\lim_{n \to +\infty} \Omega_n := \lim_{n \to \infty} \left(\sum_{k=1}^{\infty} \alpha_{nk}\right) = 1,$$

(T, 3) $$\lim_{n \to +\infty} \alpha_{nk} = 0 \quad (k = 1, 2, \ldots).$$

We also need the following property:

(T, 4) $$\sum_{k=1}^{\infty} |\alpha_{nk}| \cdot k < +\infty \quad (n = 1, 2, \ldots)$$

(T, 4) implies that for all $\{c_l\} \in C_0$ and for every n, the functions $\sum_{k=1}^{M} \alpha_{nk} \sum_{l=1}^{k} c_l \varphi_l(x)$ tend to a function $g_n(\{c_l\}, x)$ in L^2 norm as M tends to infinity. Indeed, $\|\varphi_l\|_2 = 1$ and if $\{c_l\} \in C_0$ then there exists N such that for $l > N$, $|c_l| < 1$. Then for $M_1, M_2 > N$

$$\left\| \sum_{k=M_1}^{M_2} \alpha_{nk} \sum_{l=1}^{k} c_l \varphi_l(x) \right\|_2 \leqq \sum_{k=M_1}^{M_2} |\alpha_{nk}| \left(\left\| \sum_{l=1}^{N} c_l \varphi_l(x) \right\|_2 + \left\| \sum_{l=N+1}^{k} c_l \varphi_l(x) \right\|_2 \right) \leqq$$

$$\leqq \left(\sum_{k=M_1}^{M_2} |\alpha_{nk}| \right) \left\| \sum_{l=1}^{N} c_l \varphi_l(x) \right\|_2 + \sum_{k=M_1}^{M_2} |\alpha_{nk}| (k - N) < \varepsilon$$

if N is big enough.

PROOF OF THEOREM 2. Suppose indirectly that $\{\tilde{c}_l\}$ is a sequence for which the functions $g_n(\{\tilde{c}_l\}, x)$ are dense in $L^p[0, 1]$. We define $\{c_l\}$ by $c_1 = 1$ and $c_l = \tilde{c}_l$ $(l = 2, \ldots)$. Then the functions $g_n(\{c_l\}, x)$ are also dense because

$$\sum_{k=1}^{\infty} \alpha_{nk} \sum_{l=1}^{k} \tilde{c}_l \varphi_l(x) - \sum_{k=1}^{\infty} \alpha_{nk} \left(\varphi_1(x) + \sum_{l=2}^{k} c_l \varphi_l(x) \right) =$$

$$= \sum_{k=1}^{\infty} \alpha_{nk} \left((c_1 - 1) \varphi_1(x) \right) \xrightarrow{L^p} (c_1 - 1) \varphi_1(x) \quad \text{as} \quad n \to +\infty.$$

Let g_{k_j} be a subsequence such that $\|g_{k_j}(x)\|_p \to 0$. Then

$$1 \leftarrow \sum_{l=1}^{\infty} \alpha_{k_j, l} = \langle g_{k_j};\ \varphi_1 \rangle = \int g_{k_j} \varphi_1 \leqq \|g_{k_j}\|_p \|\varphi_1\|_q < \varepsilon \quad \text{if} \quad \|g_{k_j}\|_p \leqq \frac{\varepsilon}{\|\varphi_1\|_q},$$

a contradiction.

PROOF OF THEOREM 3. (i) First we show that $H \neq \emptyset$.

It is known that there exists a sequence $\{c_l\} \in C_0$ such that the functions $\sum_{l=1}^{n} c_l \varphi_l(x)$ are dense in $L^p[0, 1]$. (See [6] and the introduction of this paper.) We can assume that $|c_l| < 1$ $(l = 1, 2, \ldots)$. We are going to define a sequence $\{d_l\}$ for which $\{g_n(\{d_l\}, x)\}$ will be dense.

Let A be a countable dense set in $L^p[0, 1]$, and let $\{f_i\}$ be an enumeration of A which contains every function of A infinitely many times.

We define the numbers $\tilde{N}_i$ by induction. Put $d_1=c_1$ and $\tilde{N}_1=1$. Suppose $j>1$ and $\tilde{N}_i$ have been defined for $i<j$. By the density of the functions $\sum_{l=1}^{n} c_l\varphi_l(x)$ there exists $N_j>\tilde{N}_j$ such that

$$\Delta_p\Big(f_j-\Big(\sum_{l=\tilde{N}_{j-1}+1}^{N_j} c_l\varphi_l+\sum_{l=1}^{\tilde{N}_{j-1}} d_l\varphi_l\Big)\Big)<\frac{1}{2^j}.$$

Let $d_l := c_l$, $l=\tilde{N}_{j-1}+1, \ldots, N_j$ and

$$\eta_j := \sup_{i\leqq N_j}\Delta_p\Big(\sum_{k=1}^{i} d_k\varphi_k\Big).$$

We choose K_j such that for $n>K_j$ we have

$$|\alpha_{nk}|^p<\frac{1}{2^j\cdot N_j\cdot\eta_j}\quad (k=1, \ldots, N_j)$$

(here we use (T, 3)) and

$$\Big|\sum_{k=N_j}^{\infty}\alpha_{nk}-1\Big|^p<\frac{1}{2^j\cdot\eta_j}$$

(we use (T, 3) and (T, 2)). Let $n=K_j$. By (T, 4) we can choose $\tilde{N}_j>N_j$ such that

$$\sum_{k=\tilde{N}_j}^{\infty}|\alpha_{K_jk}|\,k<\left(\frac{1}{3\cdot 2^j}\right)^{2/p}$$

and

$$\sum_{k=N_j}^{\infty}|\alpha_{K_jk}|<\frac{1}{2^j\eta_j}.$$

We define $d_l=0$ for $l=N_j+1, \ldots, \tilde{N}_j$. In this way we define d_l for every l and $\tilde{N}_i$ for every i. Then

$$\Delta_p\Big(g_{Kj}(\{d_l\}, x)-\sum_{l=1}^{\tilde{N}_j} d_l\varphi_l\Big)=\Delta_p\Big(\sum_{k=1}^{\infty}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l-\sum_{l=1}^{\tilde{N}_j} d_l\varphi_l\Big)=$$

$$=\Delta_p\Big(\sum_{k=1}^{N_j}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l+\sum_{k=N_j+1}^{\tilde{N}_j}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l-\sum_{l=1}^{N_j} d_l\varphi_l+\sum_{k=N_j}^{\infty}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l\Big)\leqq$$

$$\leqq\Delta_p\Big(\sum_{k=1}^{N_j}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l\Big)+\Big(\sum_{k=N_j}^{\tilde{N}_j}\alpha_{K_jk}-1\Big)^p\Delta_p\Big(\sum_{l=1}^{N_j} d_l\varphi_l\Big)+\Delta_p\Big(\sum_{k=\tilde{N}_j}^{\infty}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l\Big)\overset{\text{def}}{=}C.$$

We must be careful with the estimation of the third term of C. We have to avoid the estimation

$$\Delta_p\Big(\sum_{k=\tilde{N}_j}^{\infty}\alpha_{K_jk}\sum_{l=1}^{k} d_l\varphi_l\Big)\leqq\sum_{k=\tilde{N}_j}^{\infty}|\alpha_{K_jk}|^p\,k$$

because the convergence of this series is worse than that of $\sum_{k=\bar{N}_j}^{\infty} |\alpha_{K_j k}| k$. Instead we use the following estimation:

$$\Big\| \sum_{k=\bar{N}_j}^{\infty} \alpha_{K_j k} \sum_{l=1}^{k} d_l \varphi_l \Big\|_1 \leqq \Big\| \sum_{k=\bar{N}_j}^{\infty} \alpha_{K_j k} \sum_{l=1}^{k} d_l \varphi_l \Big\|_2 \leqq$$

$$\leqq \sum_{k=\bar{N}_j}^{\infty} |\alpha_{K_j k}| \cdot \sum_{l=1}^{k} |d_l| \, \|\varphi_l\|_2 \leqq \sum_{k=\bar{N}_j}^{\infty} |\alpha_{K_j k}| \cdot k \leqq \left(\frac{1}{3 \cdot 2^j}\right)^{2/p}.$$

We need the following simple lemma:

LEMMA 1. *If* $\int_0^1 |f| < \varepsilon$, $0 < \varepsilon < 1$, $0 < p < 1$ *then* $\int_0^1 |f|^p \leqq 3\varepsilon^{\frac{1}{2}}$.

PROOF. We have

$$\varepsilon > \int_{[0,1]} |f| \geqq \int_{\{x \,|\, |f(x)| \geqq \sqrt{\varepsilon}\}} |f| \geqq \sqrt{\varepsilon}\, \lambda\big(\{x \,|\, |f(x)| \geqq \sqrt{\varepsilon}\big),$$

$$\int_{[0,1]} |f|^p = \int_{\{x \,|\, |f(x)| \geqq 1\}} |f|^p + \int_{\{x \,|\, \sqrt{\varepsilon} \leqq f(x) \leqq 1\}} |f|^p +$$

$$+ \int_{\{x \,|\, |f(x)| < \sqrt{\varepsilon}\}} |f|^p \leqq \int_{[0,1]} |f| + \sqrt{\varepsilon} + (\sqrt{\varepsilon})^p \leqq 3\varepsilon^{p/2}.$$

Now we turn to the estimation of C:

$$C \leqq \sum_{k=1}^{N_j} |\alpha_{K_j k}|^p \eta_j + \Big(\Big| \sum_{k=N_j}^{\infty} \alpha_{K_j k} - 1\Big| + \sum_{k=\bar{N}_j}^{\infty} |\alpha_{K_j k}|\Big)^p \eta_j +$$

$$+ 3\left(\frac{1}{3} \cdot \frac{1}{2^j}\right)^{(2/p)\cdot(p/2)} \leqq \frac{1}{2^j} + \frac{2}{2^j} + \frac{1}{2^j} \leqq \frac{1}{2^{j-2}}.$$

Hence

$$\Delta_p(f_j - g_{K_j}(\{d_l\}, x)) \leqq \Delta_p\Big(g_{K_j}(\{d_l\}, x) - \sum_{l=1}^{\bar{N}_j} d_l \varphi_l\Big) +$$

$$+ \Delta_p\Big(f_j - \sum_{l=1}^{\bar{N}_j} d_l \varphi_l\Big) \leqq \frac{1}{2^{j-2}} + \frac{1}{2^j}.$$

So by the choice of f_i we get the desired density.

(ii) In order to prove that H is residual we use the lemma of I. Joó [5]:

LEMMA 2. *Let E be a complete metrizable topological vector space and F a separable metrizable topological vector space. Further let $L_n: E \to F$ be a sequence of continuous linear operators having the following properties:*

a) *There exists a countable $H_1 \subset E$ such that for every $h \in M_1$ the sequence $\{L_n(h)\}$ converges in F and the limit points are dense in F.*

b) *For a dense subset H_2 of E, O_F (the origin in F) is a cluster point of the sequence $\{L_n(h)\}$ for every $h \in H_2$.*

Then the set H of those elements h of E for which the sequence $\{L_n(h)\}$ is dense in F, is residual.

We want to apply this lemma with $E=C_0$, $F=L^p[0, 1]$ and $L_n(\{c_n\})=$ $=\sum_{k=1}^{\infty}\alpha_{nk}\sum_{l=1}^{k}c_l\varphi_l(x)$. It is easy to show if we have the above conditions and $\sum c_l\varphi_l \xrightarrow{L^2} f$ then $T\left(\sum c_l\varphi_l(x)\right)\xrightarrow{L^2} f$. (The summation process remains also permanent for functions.)

Condition a) of Lemma 2 follows from the fact that $L^2\cap L^p$ is dense in L^p and for H_1 we can take the sequences of Fourier coefficients of a countable dense subset of $L^2\cap L^p$. The L^2 convergence of these series implies the L^p convergence.

We have to check condition b). Let $\sum_1^k c_l\varphi_l$ be a sequence which is dense in $L^p[0, 1]$. Let $\varepsilon>0$ be given. So for an arbitrary $\{d_l\}\in C_0$ we can choose an N such that $|d_l|<\varepsilon$ $(l\geqq N)$ and for the above $\{c_l\}$; $|c_l|<\varepsilon$ $(l\geqq N)$. Let

$$d_l' = \begin{cases} d_l & l \leqq N \\ c_l & l > N. \end{cases}$$

Then $d_{c_0}(d_l'; d_l)<2\varepsilon$ and by the previous remark $\left\{\sum_1^k d_l'\varphi_l\right\}$ is dense in $L^p[0, 1]$. We can apply the first part of this proof by choosing $N_1>N$ at the beginning. (Our process will also work with this restriction because the functions $\sum_N^k d_l'\varphi_l$ are also dense in $L^p[0, 1]$.) Hence we get a sequence $\{d_l''\}$ for which $T\left(\sum_{l=1}^k d_l''\varphi_l\right)$ is dense in $L^p[0, 1]$ and naturally the origin will be also a cluster point. $d_{c_0}(\{d_l''\}; \{d_l\})<$ $<2\varepsilon$. So we can define in this way the dense set H_2 in C_0.

PROOF OF THEOREM 4. Suppose that

$$\sum_{n=1}^{\infty}\|\varphi_n\|_1^2 < +\infty.$$

Hence we have an N such that

$$\sum_{n=N+1}^{\infty}\|\varphi_n\|_1^2 < \frac{1}{4}.$$

By the Riemann—Lebesgue lemma

$$\lim_{M\to\infty}\int_0^1 \varphi_n(x)\sin(M\pi x)\,dx = 0$$

for any n. So we can choose an M such that

$$\sum_{n=1}^{N}\left(\int_0^1 \varphi_n(x)\sin(M\pi x)\,dx\right)^2 < \frac{1}{4}.$$

Now we shall use that $\{\varphi_n\}$ is a complete system. By the Parseval formula

$$\frac{1}{2} = \int_0^1 (\sin(M\pi x))^2\,dx = \sum_{n=1}^{\infty}\Big(\int_0^1 \sin(M\pi x)\,\varphi_n(x)\,dx\Big) \leqq$$

$$\leqq \sum_{n=1}^{N}\Big(\int_0^1 \sin(M\pi x)\,\varphi_n(x)\,dx\Big)^2 + \sum_{n=N+1}^{\infty}\Big(\int_0^1 |\sin(M\pi x)|\cdot|\varphi_n(x)|\,dx\Big)^2 <$$

$$< \frac{1}{4} + \sum_{n=N+1}^{\infty}\|\varphi_n\|_1^2 < \frac{1}{4}+\frac{1}{4},$$

a contradiction.

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ON (0, 2) INTERPOLATION FOR THE LAGUERRE ABSCISSAS

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In [1], the exact condition on the parameters $\alpha, \beta (> -1)$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ were found in order to have a unique solution for the (0, 2) interpolation based on the roots of these polynomials. In this paper we solve the similar problem for the Laguerre polynomials $L_n^{(\alpha)}(x)$ $(\alpha > -1)$. We mention that the case $\alpha = -1$ was settled by J. Prasad and R. B. Saxena [2], who also solved the problem of determining the fundamental polynomials. They mention (without proof) that the solution is unique in the case $-1 < \alpha < 1$ as well.

In this paper we settle the problem of uniqueness for all $\alpha > -1$. Let

$$(0 <) x_1 < x_2 < \ldots < x_n \tag{1}$$

be the roots of the Laguerre polynomial $L_n^{(\alpha)}(x)$ $(\alpha > -1)$, and let $f(x)$ be a continuous function on the half-line $[0, \infty)$. The (0, 2) interpolating polynomials $R_n(f, x)$ of degree at most $2n-1$ associated with $f(x)$ are defined by the relations

$$R_n(f, x_k) = f(x_k), \quad R_n''(f, x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{2}$$

THEOREM 1. *Let $\alpha > -1$ and n be a positive integer. The problem of* (0, 2) *interpolation is uniquely solvable for the nodes* (1) *if and only if*

$$P_n^{\left(\alpha, \frac{\alpha+1}{2} - n\right)}\left(-\frac{1}{3}\right) \neq 0, \tag{3}$$

where $P_n^{(\alpha,\beta)}(x)$ is the generalized Jacobi polynomial of degree n with parameters α, β.

PROOF. If polynomials satisfying (2) exist, then they can be written in the form

$$R_n(f, x) = \sum_{k=1}^{n} f(x_k) r_k(x),$$

where $r_k(x)$ are polynomials of degree at most $2n-1$ satisfying

$$r_k(x_j) = \delta_{kj}, \quad r_k''(x_j) = 0, \quad k, j = 1, 2, \ldots, n. \tag{4}$$

Hence the unique solvability of the (0, 2) interpolation is equivalent to the statement that if for a polynomial $r(x)$ of degree at most $2n-1$ we have

$$r(x_k) = r''(x_k) = 0, \quad k = 1, 2, \ldots, n \tag{5}$$

then $r(x) \equiv 0$. From now on let $L_n^{(\alpha)}(x)$ denote the Laguerre polynomials normalized

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by $L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$, and write $r(x)=L_n^{(\alpha)}(x)q(x)$ (by (5)), where $q(x)$ is a polynomial of degree at most $n-1$. The conditions in (5) imply that

$$r''(x_k) = 2L_n^{(\alpha)'}(x_k)q'(x_k) + L_n^{(\alpha)''}(x_k)q(x_k) = 0, \quad k = 1, 2, \ldots, n. \tag{6}$$

From the differential equation of the Laguerre polynomials (cf. G. Szegö [3], (5.1.2))

$$xL_n^{(\alpha)''}(x)+(\alpha+1-x)L_n^{(\alpha)'}(x)+nL_n^{(\alpha)}(x) = 0$$

we get

$$x_k L_n^{(\alpha)''}(x_k)+(\alpha+1-x_k)L_n^{(\alpha)'}(x_k) = 0, \quad k = 1, 2, \ldots, n \tag{7}$$

and thus (6) takes the form

$$2x_k q'(x_k)+(x_k-\alpha-1)q(x_k) = 0, \quad k = 1, 2, \ldots, n.$$

This means that

$$2xq'(x)+(x-\alpha-1)q(x) = a_n L_n^{(\alpha)}(x) \tag{8}$$

with some a_n. Let $q(x)=\sum_{k=0}^{n-1} a_k x^k$, then because of

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}$$

(cf. [3], (5.1.6)) we get the following linear system for $\{a_k\}_{k=0}^n$:

$$\begin{cases} -(\alpha+1)a_0 - \binom{n+\alpha}{n}a_n = 0 \\ a_{k-1}+(2k-\alpha-1)a_k - \binom{n+\alpha}{n-k}\frac{(-1)^k}{k!}a_n = 0, \quad k = 1, 2, \ldots, n-1 \\ a_{n-1} - \binom{n+\alpha}{0}\frac{(-1)^n}{n!}a_n = 0. \end{cases} \tag{9}$$

The determinant of this system is

$$D_n(\alpha) = \begin{vmatrix} -(\alpha+1) & 0 & 0 & 0 & \ldots & 0 & -\binom{n+\alpha}{n} \\ 1 & 1-\alpha & 0 & 0 & \ldots & 0 & \binom{n+\alpha}{n-1}\frac{-1}{1!} \\ 0 & 1 & 3-\alpha & 0 & \ldots & 0 & -\binom{n+\alpha}{n-2}\frac{(-1)^2}{2!} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0\ldots 1 & 2n-3-\alpha & & -\binom{n+\alpha}{1}\frac{(-1)^{n-1}}{(n-1)!} \\ 0 & 0 & 0 & 0\ldots 0 & 1 & & -\binom{n+\alpha}{0}\frac{(-1)^n}{n!} \end{vmatrix}.$$

Expanding this determinant with respect to the last column, we get

$$(10)\qquad (-1)^{n+1}\binom{n+\alpha}{n}+(-1)^{n+1}\binom{n+\alpha}{n-1}(-1-\alpha)+$$

$$+(-1)^{n+1}\binom{n+\alpha}{n-2}\frac{1}{2!}(-1-\alpha)(1-\alpha)+\ldots+(-1)^{n+1}\binom{n+2}{0}\frac{1}{n!}\cdot$$

$$\cdot(-1-\alpha)(1-\alpha)(3-\alpha)\ldots(2n-3-\alpha)=$$

$$=(-1)^{n+1}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-2)^k\frac{\alpha+1}{2}\frac{\alpha-1}{2}\ldots\left(\frac{\alpha+3}{2}-k\right)}{k!}=$$

$$=(-1)^{n+1}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{\frac{\alpha+1}{2}}{k}(-2)^k=(-1)^{n+1}3^n P_n^{\left(\alpha,\frac{\alpha+1}{2}-n\right)}\left(-\frac{1}{3}\right).$$

(see [3], (4.3.2)). Thus (3) is equivalent to the unique solvability of the system (9), which proves our Theorem 1.

REMARK. In [2], the authors prove that the uniqueness is true for the roots of $L_n^{(-1)}(x)=\frac{1}{n}xL_n^{(0)'}(x)$ if $n\geqq 2$. We mention that the proof given in [2] can be significantly simplified. Just like in (8), one arrives at the differential equation

$$(11)\qquad 2q'(x)+q(x)=a_nL_n^{(0)'}(x)$$

with the initial condition

$$(12)\qquad 2q'(0)-(n-1)q(0)=0$$

(see also [2], p. 564), where $q(x)$ is a polynomial of degree at most $n-1$. The simplest way to proceed from this point is to *solve* (11)—(12). Evidently

$$(13)\qquad q(x)=e^{-x/2}\left[b_n+\frac{a_n}{2}\int_{-\infty}^{x}L_n^{(0)'}(t)e^{t/2}dt\right].$$

Here, as it is easily shown, $e^{-x/2}\int_{-\infty}^{x}L_n^{(0)'}(t)e^{t/2}dt$ is a polynomial, thus in order for $g(x)$ to be a polynomial we must have $b_n=0$. Also,

$$(14)\quad q(0)=\frac{a_n}{2}\int_{-\infty}^{0}L_n^{(0)'}(t)e^{t/2}dt,\quad q'(0)=-\frac{a_n}{4}\int_{-\infty}^{0}L_n^{(0)'}(t)e^{t/2}dt+\frac{a_n}{2}L_n^{(0)'}(0),$$

i.e. by (12) (see also [2], Lemma 3.1)

$$(15)\qquad a_n\left\{L_n^{(0)'}(0)-\frac{n}{2}\int_{-\infty}^{0}L_n^{(0)'}(t)e^{t/2}dt\right\}=\frac{n(3^n-3)}{2}a_n=0.$$

Hence $a_n=0$, which proves that $q(x)\equiv 0$.

Although Theorem 1 gives a necessary and sufficient condition for the uniqueness in a compact form, it does not give a practical information on those n's for which there is uniqueness. A sufficient condition of this type is given in the next theorem. Let us introduce the notation

$$\gamma = \frac{\alpha+1}{2}. \tag{16}$$

THEOREM 2. *If*

$$n > \begin{cases} \dfrac{(3\alpha-1)(\alpha+3)}{8} & \text{when } \gamma \text{ is an integer} \\ \dfrac{\gamma+\left(2\gamma+\dfrac{1}{2}\right)[\gamma]-\dfrac{1}{2}[\gamma]^2}{\gamma-[\gamma]} & \text{otherwise,} \end{cases} \tag{17}$$

then the $(0, 2)$ *interpolation is uniquely solvable.*

REMARK. In particular, if $|\alpha|<1$, then we have uniqueness for all $n\geqq 2$. This is mentioned in [2] but no proof is given. In fact, we have uniqueness for $\alpha=1$, too, provided $n\geqq 2$.

PROOF OF THEOREM 2. As we have seen in the proof of Theorem 1, uniqueness holds if and only if (see (10))

$$D_n(\alpha) = (-1)^{n+1} \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{\gamma}{k}(-2)^k \neq 0. \tag{18}$$

Let

$$a_k = \binom{n+\alpha}{n-k}\binom{\gamma}{k}2^k, \quad k = 0, \ldots, n. \tag{19}$$

First we prove that under condition (17), we have

$$\begin{cases} 0 \leqq a_{k-1} < a_k, & \text{for} \quad k = 1, \ldots, \gamma; \text{ if } \gamma \text{ is an integer,} \\ \text{and for} \quad k = 1, \ldots, [\gamma]+1; & \text{otherwise.} \end{cases} \tag{20}$$

Conditions (17) imply $n\geqq[\gamma]+1$, i.e. the left hand side inequalities here hold. As for the right hand side inequalities, an easy calculation shows that the latter is equivalent to

$$n > \frac{(2\alpha+5)k-k^2-\alpha-3}{\alpha+3-2k} \tag{21}$$

for the k's specified in (20). Now, since the function

$$\frac{(2\alpha+5)x-x^2-\alpha-3}{\alpha+3-2x}$$

is monotone increasing in the interval $[1, \gamma+1)$, the maximum of the right hand side of (21) is attained when $k=\gamma$ (if γ is an integer) or $k=[\gamma]+1$ (otherwise). Substituting these values for k, we get exactly the right hand side of (17). This proves (20).

Now if γ is an integer then evidently

$$|D_n(\alpha)| = \left|\sum_{k=0}^{\gamma}(-1)^k a_k\right| \geqq a_\gamma - a_{\gamma-1} > 0.$$

If γ is not an integer then

$$\operatorname{sgn} a_k = (-1)^{k+[\gamma]+1} \quad (k = [\gamma]+1,\ [\gamma]+2,\ \ldots,\ n),$$

and thus by (20)

$$|D_n(\alpha)| = \left|\sum_{k=0}^{n}(-1)^k a_k\right| \geqq \sum_{k=[\gamma]+1}^{n}|a_k| - \left|\sum_{k=0}^{[\gamma]}(-1)^k a_k\right| > a_{[\gamma]+1} - a_{[\gamma]} > 0.$$

This completely proves Theorem 2.

We cannot state that the conditions given in (17) are best possible. However, some conditions must be imposed on n and α, as the next theorem shows.

THEOREM 3. *For an arbitrary* $n \geqq 4$, *there exist* α_i *such that*

$$D_n(\alpha_i) = 0, \quad i = 1, 2, \ldots, \left[\frac{\sqrt{n}}{2}\right], \tag{22}$$

i.e. there are no unique solutions in these cases.

PROOF. By Theorem 2, (18), (19) and (20) we have

$$\operatorname{sgn} D_n(2i-1) = \operatorname{sgn}\sum_{k=0}^{i}(-1)^k a_k = (-1)^i,$$

whenever the positive integer i satisfies $\frac{(3i-1)(i+3)}{2} < n$. There are at least $\left[\frac{\sqrt{n}}{n}\right]+1$ such i's. Since $D_n(\alpha)$ is a continuous function of α, there are $a_i \in (2i-1, 2i+1)$, $i = i, 2, \ldots, \left[\frac{\sqrt{n}}{2}\right]$ such that (22) holds. Q.E.D.

Finding the polynomials $r_k(x)$ satisfying (4) is usually a difficult problem. In [2], the authors gave the formulas in the case $\alpha = -1$. We have developed a method of solving this problem when α is an odd integer (and, of course, the corresponding condition on n in (17) is satisfied). However, the formulas involved are so complicated that we present only the case $\alpha = 1$.

THEOREM 4. *In case* $\alpha = 1$, *we have for* $n \geqq 2$

$$r_k(x) = xe^{-x/2}L_n^{(1)}(x)\left\{\frac{-(n-1)x_k^2 + 2(2n-1)x_k - 2(n+1)}{2(n-1)x_k L_n^{(1)\prime}(x_k)^2}\int_{-\infty}^{x} L_n^{(1)}(t)t^{-2}e^{t/2}dt - \right.$$

$$\left. - \int_{-\infty}^{x} \frac{x_k x l_k''(t) + 2x_k(t-2)l_k'(t) + [x_k^2 - (n+4)x_k + 6]l_k(t)}{2x_k t^2 L_n^{(1)\prime}(x_k)} e^{t/2}dt\right\} + \frac{l_k(x)L_n^{(1)\prime}(x)}{L_n^{(1)\prime}(x_k)}$$

$$(k = 1, 2, \ldots, n;\ -\infty < x < 0)$$

where

$$l_k(x) = \frac{L_n^{(1)}(x)}{L_n^{(1)'}(x_k)(x-x_k)}, \quad k = 1, \ldots, n \tag{23}$$

are the fundamental functions of Lagrange interpolation.

PROOF. Let

$$r_k(x) = L_n^{(1)}(x)q_k(x) + \frac{l_k(x)L_n^{(1)'}(x)}{L_n^{(1)'}(x_k)}, \tag{24}$$

then the first group of conditions in (4) is satisfied for any polynomial $q_k(x)$ of degree at most $n-1$. The second group of conditions means that

$$r_k''(x_j) = L_n^{(1)''}(x_j)q_k(x_j) + 2L_n^{(1)'}(x_j)q_k'(x_j) +$$

$$+ \frac{l_k''(x_j)L_n^{(1)'}(x_j) + 2l_k'(x_j)L_n^{(1)''}(x_j) + l_k(x_j)L_n^{(1)'''}(x_j)}{L_n^{(1)'}(x_k)} = 0, \quad k, j = 1, \ldots, n.$$

On using (7) we get

$$2x_jq_k'(x_j) + (x_j-2)q_k(x_j) + \frac{x_jl_k''(x_j) + 2(x_j-2)l_k'(x_j) + \left(x_k - n - 4 + \frac{6}{x_k}\right)l_k(x_j)}{L_n^{(1)'}(x_k)} = 0,$$

$$k, j = 1, \ldots, n.$$

Hence

$$2xq_k'(x) + (x-2)q_k(x) + \frac{x_kxl_k''(x) + 2x_k(x-2)l_k'(x) + [x_k^2 - (n+4)x_k + 6]l_k(x)}{x_kL_n^{(1)'}(x_k)} =$$

$$= C_kL_n^{(1)}(x), \quad k = 1, \ldots, n.$$

Introducing the notation

$$h_k(x) = C_kL_n^{(1)}(x) - \frac{x_kxl_k''(x) + 2x_k(x-2)l_k'(x) + [x_k^2 - (n+4)x_k + 6]l_k(x)}{x_kL_n^{(1)'}(x_k)}, \tag{25}$$

$$k = 1, \ldots, n,$$

the solutions of this differential equation can be written as

$$q_k(x) = xe^{-x/2}\left\{d_k + \frac{1}{2}\int_{-\infty}^{x} h_k(t)t^{-2}e^{t/2}dt\right\}, \quad k = 1, \ldots, n; \quad -\infty < x < 0.$$

Here $d_k = 0$, since otherwise $q_k(x) \to \infty$ exponentially as $x \to \infty$, which is impossible. Thus by integration by parts

$$q_k(x) = \frac{1}{2}xe^{-x/2}\int_{-\infty}^{x} h_k(t)t^{-2}e^{t/2}dt = \tag{26}$$

$$= -\frac{1}{2}h_k(x) + \frac{1}{2}xe^{-x/2}\int_{-\infty}^{x}\left[h_k'(t) + \frac{1}{2}h_k(t)\right]t^{-1}e^{t/2}dt,$$

$$k = 1, \ldots, n; \quad -\infty < x < 0.$$

Integration by parts shows that the functions

$$xe^{-x/2}\int_{-\infty}^{x} t^i e^{t/2}\,dt, \quad i = 0, 1, \ldots$$

are all polynomials. On the other hand, the function

$$\varphi(x) = xe^{-x/2}\int_{-\infty}^{x} t^{-1}e^{t/2}\,dt, \quad -\infty < x < 0,$$

is not a polynomial, since

$$\varphi'(x) = 1+e^{-x/2}\left(1-\frac{x}{2}\right)\int_{-\infty}^{x} t^{-1}e^{t/2}\,dt, \quad -\infty < x < 0,$$

and thus $\lim_{x\to 0-} \varphi'(x) = -\infty$, which is impossible. Since we know that there is a polynomial solution, we must have

$$h_k'(0)+\frac{1}{2}h_k(0) = 0, \quad k = 1, \ldots, n \tag{27}$$

in (26). We obtain from (23) successively that

$$l_k(0) = -\frac{n+1}{x_k L_n^{(1)\prime}(x_k)}, \quad k = 1, \ldots, n,$$

$$l_k'(0) = \frac{(n+1)(nx_k-2)}{2x_k^2 L_n^{(1)\prime}(x_k)}, \quad k = 1, \ldots, n,$$

$$l_k''(0) = \frac{(n+1)[-n(n-1)x_k^2+6nx_k-12]}{6x_k^3 L_n^{(1)\prime}(x_k)}, \quad k = 1, \ldots, n.$$

Thus from (25)

$$h_k(0) = (n+1)c_k\frac{4x_k l_k'(0)-[x_k^2-(x+4)x_k+6]l_k(0)}{x_k L_n^{(1)\prime}(x_k)},$$

$$h_k'(0) = -\frac{n(n+1)}{2}c_k+\frac{3x_k l_k''(0)-[x_k^2-(n+2)x_k+6]l_k'(0)}{x_k L_n^{(1)\prime}(x_k)},$$

$$h_k'(0)+\frac{1}{2}h_k(0) = \frac{1-n^2}{2}c_k+\frac{6x_k l_k''(0)-[x_k^2-(n+4)x_k+6][l_n(0)+2l_k'(0)]}{2x_k L_n^{(1)\prime}(x_k)} = 0,$$

$$k = 1, 2, \ldots, n.$$

Hence

$$c_k = \frac{-n(n-1)x_k^2+6nx_k-12-[x_k^2-(x+4)x_k+6][(n-1)x_k-2]}{(n-1)x_k^3 L_n^{(1)\prime}(x_k)^2} =$$

$$= \frac{-(n-1)x_k^2+2(2n-1)x_k-2(n+1)}{(n-1)x_k^2 L_n^{(1)\prime}(x_k)^2}, \quad k = 1, \ldots, n.$$

Substituting this into (25), the theorem follows from (24) and (26).

The drawback of Theorem 4 is that it represents $r_k(x)$ on $(-\infty, 0)$, while a representation on $(0, \infty)$ would be more important. We could transform the formulas into such a form, but this would lead to complicated relations.

If the (0, 2) interpolation is uniquely solvable, then any polynomial $p_n(x)$ of degree at most $2n-1$ can be written in the form

$$p_n(x) = \sum_{k=1}^{n} \left(p_n(x_k) r_k(x) + p_n''(x_k) \varrho_k(x)\right)$$

where ϱ_k are polynomials of degree at most $2n-1$ such that

$$\varrho_k(x_j) = 0, \quad \varrho_k''(x_j) = \delta_{kj}, \quad k, j = 1, 2, \ldots, n.$$

THEOREM 5. *In case* $\alpha = 1$, *we have for* $n \geqq 2$

$$\varrho_k(x) = \frac{1}{2} L_n^{(1)}(x) x e^{-x/2} \int_{-\infty}^{x} \left[\frac{x_k}{L_n^{(1)\prime}(x_k)} l_k(t) + \frac{(n-1)x_k - 2}{(x-1)x_k L_n^{(1)\prime}(x)^2} L_n^{(1)}(t)\right] t^{-2} e^{t/2} dt,$$

$$k = 1, \ldots, n; \quad -\infty < x < 0.$$

PROOF. Let

$$\varrho_k(x) = L_n^{(1)}(x) t_k(x), \quad k = 1, \ldots, n, \tag{28}$$

then

$$\varrho_k''(x_j) = L_n^{(1)\prime\prime}(x_j) t_k(x_j) + 2 L_n^{(1)\prime}(x_j) t_k'(x_j) = \delta_{kj}, \quad k, j = 1, \ldots, n,$$

i. e.

$$2x_j t_k'(x_j) + (x_j - 2) t_k(x_j) = \frac{x_j}{L_n^{(1)\prime}(x_j)} \delta_{kj}, \quad k, j = 1, \ldots, n.$$

Hence

$$2x t_k'(x) + (x-2) t_k(x) = \frac{x_k}{L_n^{(1)\prime}(x_k)} l_k(x) + \bar{c}_k L_n^{(1)}(x), \quad k = 1, \ldots, n.$$

Solving this differential equation we get

$$t_k(x) = \frac{1}{2} x e^{-x/2} \left\{\bar{d}_k + \int_{-\infty}^{x} \left[\frac{x_k}{L_n^{(1)\prime}(x_k)} l_k(t) + \bar{c}_k L_n^{(1)}(t)\right] t^{-2} x^{t/2} dt, \right. \tag{29}$$

$$k = 1, \ldots, n.$$

By the same argument as in the proof of Theorem 4, we must have $\bar{d}_k = 0$, and as in (27),

$$\frac{x_k}{L_n^{(1)\prime}(x_k)} l_k'(0) + \bar{c}_k L_n^{(1)\prime}(0) + \frac{x_k}{2L_n^{(1)\prime}(x_k)} l_k(0) + \frac{1}{2} \bar{c}_k L_n^{(1)}(0) =$$

$$= \frac{(n+1)(nx_k - 2)}{2x_k L_n^{(1)\prime}(x_k)^2} - \frac{n+1}{2L_n^{(1)\prime}(x_k)^2} + \bar{c}_k \left(\frac{n+1}{2} - \frac{n(n+1)}{2}\right) =$$

$$= \frac{(n+1)[(n-1)x_k - 2]}{2x_k L_n^{(1)\prime}(x_k)^2} - \frac{n^2-1}{2} \bar{c}_k = 0,$$

whence

$$\bar{c}_k = \frac{(n-1)x_k - 2}{(n-1)x_k L_n^{(1)\prime}(x_k)^2}, \quad k = 1, \ldots, n.$$

Substituting this into (29), we get the statement from (28).

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STRONG CONVERGENCE OF CERTAIN MEANS WITH RESPECT TO THE WALSH—FOURIER SERIES

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1. Introduction. It is known [1] that the Walsh—Paley system is not a Schauder basis in $L^1[0, 1]$. Moreover, there exists a function in the (dyadic) Hardy space $H^1[0, 1]$, the partial sums of which are not bounded in $L^1[0, 1]$. In this article we shall prove that some means of the L^1-norms of these partial sums can be convergent for all elements of $H^1[0, 1]$. For the trigonometric analogue of this statement see the work of B. Smith [5]. (In the proof we follow his method.) The sharpness of our theorem is also investigated.

2. We recall briefly some notations and definitions. First of all denote w_n $(n=0, 1, \ldots)$ the n-th Walsh—Paley function, i.e. let

$$w_1(t) := \begin{cases} 1 & (0 \leqq t < 1/2) \\ -1 & (1/2 \leqq t < 1), \end{cases} \qquad w_1(t) = w_1(t+1) \quad \text{(for all real } t\text{)}$$

and

$$w_{2^n}(t) := w_1(2^n t) \quad (0 \leqq t \leqq 1, \ n = 0, 1, \ldots).$$

If $n = \sum_{i=0}^{\infty} n_i 2^i$ $(n_i = 0, 1)$ is the dyadic representation of $n = 0, 1, \ldots$ then let

$$w_n := \prod_{k=0}^{\infty} w_{2^k}^{n_k}.$$

It is well-known that $(w_n, n=0, 1, \ldots)$ is a complete orthonormal system. (For more details see e.g. [1].) For $f \in L^1 := L^1[0, 1]$ let $\hat{f}(n)$ be the n-th Walsh—Fourier coefficient of f, i.e.

$$\hat{f}(n) := \int_0^1 f w_n \quad (n = 0, 1, \ldots).$$

Furthermore, we denote by D_n the n-th Dirichlet kernel with respect to $(w_n, n = =0, 1, \ldots)$:

$$D_n := \sum_{k=0}^{n-1} w_k \quad (n = 0, 1, \ldots).$$

Later we shall often use the following assertions (see [4]):

$$D_n = w_n \sum_{k=0}^{\infty} n_k w_{2^k} D_{2^k} \quad (n = \sum_{k=0}^{\infty} n_k 2^k = 0, 1, \ldots), \tag{1}$$

$$D_{2^k}(t) = \begin{cases} 2^k & (0 \leqq t < 2^{-k}) \\ 0 & (2^{-k} \leqq t < 1) \end{cases} \quad (k = 0, 1, \ldots). \tag{2}$$

The so-called Hardy space plays an important part in the further investigations. Let $Q(f)$ be the quadratic variation of $f \in L^1$, i.e.

$$Q(f) := \Big(\sum_{n=0}^{\infty} (S_{2^{n+1}}(f) - S_{2^n}(f))^2\Big)^{1/2},$$

where

$$S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k \quad (n = 1, 2, \ldots).$$

Then the space $H^1 := H^1[0, 1]$ is defined by $H^1 := \{f \in L^1 : Q(f) \in L^1\}$. It is well-known (see [2]) that the elements of H^1 can be represented as linear combinations of so-called atoms. A function $a \in L^\infty[0, 1]$ is called an atom, if either $a = 1$ or $\int_0^1 a = 0$ and there is a dyadic interval $I_a \subset [0, 1]$ such that supp $a \subset I_a$ and $|a| \leqq |I_a|^{-1}$ ($|I_a|$ is the length of I_a.) Then $f \in L^1$ belongs to H^1 if and only if there exist real coefficients α_i and atoms a_i $(i = 0, 1, \ldots)$ so that

$$\sum_{i=0}^{\infty} |\alpha_i| < +\infty \quad \text{and} \quad f = \sum_{i=0}^{\infty} \alpha_i a_i.$$

3. It is known in the Walsh—Fourier analysis (see [1]) that the system $(w_n, n = 0, 1, \ldots)$ is not a Schauder basis in L^1. Moreover, there exists a function $f \in H^1$ such that the L^1-norms of the partial sums $S_n(f)$ $(n = 1, 2, \ldots)$ are not bounded. However, the following theorem shows that certain means of the $\|S_n(f)\|_1$'s can be convergent for all $f \in H^1$.

THEOREM. *If* $f \in H^1$, *then*

$$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{-1} \|S_k(f)\|_1 = \|f\|_1. \tag{3}$$

Let n be a natural number for which $2^{N-1} \leqq n < 2^N$ $(N = 2, 3, \ldots)$ holds. Then

$$\left| \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|S_k(f)\|_1 - \|f\|_1 \right| = \left| \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} (\|S_k(f)\|_1 - \|f\|_1) + \right.$$

$$\left. + \|f\|_1 \left(\frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} - 1 \right) \right| \leqq \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|f - S_k(f)\|_1 + o(1) \quad (n \to +\infty),$$

i.e.

$$\lim_{n \to +\infty} \frac{1}{\log n} \sum_{k=1}^{n-1} k^{-1} \|f - S_k(f)\|_1 = 0 \tag{4}$$

implies our statement. On the other hand

$$\frac{1}{\log n}\sum_{k=1}^{n-1} k^{-1}\|f-S_k(f)\|_1 \leqq 2N^{-1}\sum_{k=0}^{N-1}\sum_{j=0}^{2^k-1}(2^k+j)^{-1}\|f-S_{2^k+j}(f)\|_1 \leqq$$

$$\leqq 2N^{-1}\sum_{k=0}^{N-1} 2^{-k}\sum_{j=0}^{2^k-1}\|f-S_{2^k+j}(f)\|_1 =: 2N^{-1}\sum_{k=0}^{N-1} d_k(f).$$

If we denote by $E_n(f)$ the arithmetic mean of $d_{n+1}(f), \ldots, d_{2n}(f)$, i.e.

$$E_n(f) := n^{-1}\sum_{k=n+1}^{2n} d_k(f) \quad (n = 1, 2, \ldots),$$

then it is clear that

$$\lim_{n\to+\infty} E_n(f) = 0 \tag{5}$$

is sufficient for (4) to hold.

The statement of the theorem cannot hold for all $f\in L^1$. To this end let $(\alpha_k, k=1, 2, \ldots)$ be a sequence of real numbers of bounded variation, i.e.

$$\sum_{k=1}^{\infty}|\alpha_k-\alpha_{k+1}| < +\infty,$$

and take the function f defined by

$$f := \alpha_k(D_{2^{k+1}}-D_{2^k}). \tag{6}$$

Since $\|D_{2^k}\|_1=1$ $(k=0, 1, \ldots)$ (see (2)), by means of Abel transformation it follows that $f\in L^1$. On the other hand if $n=1, 2, \ldots$ and $j=0, \ldots, 2^n-1$, then

$$S_{2^n+j}(f) = \sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})+\alpha_n(D_{2^n+j}-D_{2^n}) = \sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})+\alpha_n w_{2^n} D_j,$$

from which

$$\|S_{2^n+j}(f)\|_1 \geqq |\alpha_n|\|D_j\|_1 - \|\sum_{k=1}^{n-1}\alpha_k(D_{2^{k+1}}-D_{2^k})\|_1 \geqq$$

$$\geqq |\alpha_n|\|D_j\|_1 - \Big\|\sum_{k=2}^{n-1}(\alpha_{k-1}-\alpha_k)D_{2^k}-\alpha_1 D_2+\alpha_{n-1}D_{2^n}\Big\|_1 \geqq$$

$$\geqq |\alpha_n|\|D_j\|_1 - \Big(\sum_{k=2}^{n-1}|\alpha_k-\alpha_{k-1}|+|\alpha_1|+|\alpha_{n-1}|\Big) = |\alpha_n|\|D_j\|_1+O(1) \quad (n\to+\infty)$$

follows. This leads to

$$2^{-n}\sum_{j=0}^{2^n-1}\|S_{2^n+j}(f)\|_1 \geqq 2^{-n}|\alpha_n|\sum_{j=0}^{2^n-1}\|D_j\|_1+O(1) \quad (n\to+\infty).$$

Since there exists an absolute constant $C>0$ such that (see [3])

$$n^{-1}\sum_{j=0}^{n-1}\|D_j\|_1 \geqq C\log n \quad (n\to+\infty), \tag{7}$$

therefore

$$2^{-n}\sum_{j=0}^{2^n-1}\|S_{2^n+j}(f)\|_1 \geqq C\,n|\alpha_n|+O(1) \quad (n\to+\infty).$$

Hence, if $\lim_{+\infty} n\cdot|\alpha_n|=+\infty$, then it is easy to prove that

$$\lim_{n\to+\infty}(\log n)^{-1}\sum_{k=1}^{n}k^{-1}\|S_k(f)\|_1=+\infty.$$

For example the sequence $\alpha_n:=n^{-1/2}$ $(n=1, 2, \ldots)$ satisfies the conditions required above.

On the other hand there exists a function $f\in L^1\setminus H^1$ such that (3) holds. Indeed, if we take in (6)

$$\alpha_k := (k\cdot\log k)^{-1} \quad (k=2, 3, \ldots) \quad \text{and} \quad \alpha_1:=0,$$

then $f\in L^1$ and

$$Q(f)\|_1=\sum_{n=0}^{\infty}\int_{2^{-n-1}}^{2^{-n}}Q(f)\geqq\sum_{n=2}^{\infty}\int_{2^{-n-1}}^{2^{-n}}|S_{2^{n+1}}(f)|-S_{2^n}(f)|=$$

$$=\sum_{n=2}^{\infty}\alpha_n\int_{2^{-n-1}}^{2^{-n}}|D_{2^{n+1}}-D_{2^n}|=1/2\sum_{n=2}^{\infty}(n\log n)^{-1}=+\infty,$$

i.e. $f\notin H^1$. Fruthermore, if $n=1, 2, \ldots$ and $j=0, \ldots, 2^n-1$, then

$$\|f-S_{2^n+j}(f)\|_1=\Big\|\sum_{k=n}^{\infty}\alpha_k(D_{2^{k+1}}-D_{2^k})-\alpha_n w_{2^n}D_j\Big\|_1\leqq$$

$$\leqq\Big\|\sum_{k=n+1}^{\infty}(\alpha_{k-1}-\alpha_k)D_{2^k}-\alpha_n D_{2^n}\Big\|_1+\alpha_n\|D_j\|_1\leqq\sum_{k=n+1}^{\infty}|\alpha_{k-1}-\alpha_k|+\alpha_n+O(n\cdot\alpha_n)=$$

$$=o(1) \quad (n\to+\infty).$$

(Here we used the fact (see [3]) that $\|D_j\|_1=O(\log j)$ $(j\to+\infty)$.) From this (4) follows evidently.

Finally, we remark that $\lim_{k\to+\infty}d_k(f)=0$ cannot be true for all $f\in H^1$. Indeed,

$$f:=\sum_{k=1}^{\infty}k^{-2}(D_{2^{k^3}+1}-D_{2^{k^3}})\in H^1$$

and

$$d_{k^3}(f)=2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|S_{2^{k^3}+j}(f)-f\|_1\geqq$$

$$\geqq 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|S_{2^{k^3}+j}(f)-S_{2^{k^3}}(f)\|_1-\|S_{2^{k^3}}(f)-f\|_1.$$

Since $\|f-S_{2^k}(f)\|_1=o(1)$ $(k\to+\infty)$ (see [3]) thus applying (7) we get

$$d_{k^3}(f)\geqq k^{-2}\cdot 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|D_{2^{k^3}+j}-D_{2^{k^3}}\|_1+o(1)=$$

$$=k^{-2}\cdot 2^{-k^3}\sum_{j=0}^{2^{k^3}-1}\|D_j\|_1+o(1)\geqq C\cdot k+o(1)\quad (k\to+\infty).$$

4. For the proof of the theorem we need the following

LEMMA. *Let* $a\in H^1$ *be an atom. Then for all* $n=1, 2, \ldots$ *we have*

$$\|S_n(a)-a\|_1\leqq 12|\hat{a}(n)|\log|I_a|^{-1}+e_n(a),$$

where $\lim_{n\to+\infty} e_n(a)=0$ *and* $|e_n(a)|\leqq 2$.

PROOF. Since for $a=1$ the lemma is trivial, we may suppose that $a\neq 1$. Let $I_a=[k2^{-m}, (k+1)2^{-m})$ $(m=0, 1, \ldots$ and $k=0, \ldots, 2^m-1)$ and $x\in[0,1]\setminus I_a$. Then by (1) and (2)

$$S_n(a)(x)-a(x)=S_n(a)(x)=\int_0^1 a(t)w_n(x\dotplus t)\sum_{j=0}^{\infty}n_j w_{2^j}(x\dotplus t)D_{2^j}(x\dotplus t)\,dt=$$

$$=w_n(x)\sum_{j=0}^{m-1}n_j w_{2^j}(x\dotplus k2^{-m})\int_{I_a}a(t)w_n(t)D_{2^j}(x\dotplus t)dt=$$

$$=w_n(x)\sum_{j=0}^{j(x)}n_j w_{2^j}(x\dotplus k2^{-m})2^j\int_{I_a}aw_n,$$

where $j(x)$ denotes the maximum of indices $j=0, \ldots, m-1$ such that $D_{2^j}(x\dotplus t)=2^j$ $(t\in I_a)$. ($\dotplus$ stands for the dyadic addition.) If $x\in[s2^{-m}, (s+1)2^{-m})$ $(s=0, \ldots, 2^m-1$, $s\neq k)$, then $2^{j(x)}\leqq 2^m|s-k|^{-1}$, therefore

$$\int_{[0,1]\setminus I_a}|S_n(a)-a|=\sum_{\substack{s=0\\ s\neq k}}^{2^m-1}\int_{s2^{-m}}^{(s+1)2^{-m}}|S_n(a)-a|\leqq$$

$$=2\sum_{\substack{s=0\\ s\neq k}}^{2^m-1}|\hat{a}(n)||s-k|^{-1}\leqq 4|\hat{a}(n)|\sum_{j=1}^{2^m}j^{-1}<12\,|\hat{a}(n)|\,m.$$

On the other hand

$$\int_{I_a}|S_n(a)-a|\leqq|I_a|^{1/2}\|S_n(a)-a\|_2=:e_n(a)=o(1)\quad (n\to+\infty)$$

and

$$e_n(a)\leqq 2^{-m/2}\big(\|S_n(a)\|_2+\|a\|_2\big)\leqq 2^{1-m/2}\|a\|_2\leqq 2.$$

This completes the proof of Lemma.

PROOF OF THE THEOREM. Let $f\in H^1$ be an arbitrary function and consider an atomic decomposition of f:

$$f=\sum_{i=0}^{\infty}\alpha_i a_i,\quad \sum_{i=0}^{\infty}|\alpha_i|<+\infty.$$

Furthermore, introduce the notation $I_i:=I_{a_i}$ $(i=0, 1, \ldots)$ and rearrange the above decomposition of f as follows:

$$f=\sum_{s=0}^{\infty}\sum_{|I_i|=2^{-s}}\alpha_i a_i.$$

If n is a natural number, then

$$E_n(f)=n^{-1}\sum_{k=n+1}^{2n}d_k(f)=n^{-1}\sum_{k=n+1}^{2n}2^{-k}\sum_{j=0}^{2^k-1}\Big\|\sum_{s=0}^{\infty}\sum_{|I_i|=2^{-s}}\alpha_i\big(S_{2^k+j}(a_i)-a_i\big)\Big\|_1\leqq$$

$$\leqq\sum_{s=0}^{\infty}n^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\big\|S_{2^k+j}(a_i)-a_i\big\|_1=$$

$$=\sum_{s=0}^{2n}+\sum_{s=2n+1}^{\infty}=:A_n+B_n.$$

To the estimation of B_n we remark that if $a\in H^1$ is an atom, then $\hat{a}(m)=0$ for all $m=0, 1, \ldots, |I_a|^{-1}-1$. Hence, all of the partial sums of the a_i's in B_n are equal to zero, therefore

$$B_n\leqq\sum_{s=2n+1}^{\infty}\sum_{|I_i|=2^{-s}}|\alpha_i|=o(1)\quad(n\to+\infty).$$

Let us decompose A_n into two further parts as

$$A_n=\sum_{s=0}^{n}+\sum_{s=n+1}^{2n}=:A_{n1}+A_{n2}.$$

Then applying the lemma and Cauchy—Schwarz inequality we get

$$A_{n1}\leqq 12\sum_{s=0}^{n}sn^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}|\hat{a}_i(2^k+j)|+o(1)\leqq$$

$$\leqq 12\sum_{s=0}^{2n}sn^{-1}\sum_{k=n+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k/2}\|a_i\|_2+o(1)\leqq$$

$$\leqq 12\sum_{s=0}^{n}sn^{-1}\sum_{k=n+1}^{2n}2^{(s-k)/2}\sum_{|I_i|=2^{-s}}|\alpha_i|+o(1)=o(1)\quad(n\to+\infty).$$

Furthermore,

$$A_{n2}=\sum_{s=n+1}^{2n}n^{-1}\sum_{k=n+1}^{s}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\|S_{2^k+j}(a_i)-a_i\|_1+$$

$$+\sum_{s=n+1}^{2n}n^{-1}\sum_{k=s+1}^{2n}\sum_{|I_i|=2^{-s}}|\alpha_i|2^{-k}\sum_{j=0}^{2^k-1}\|S_{2^k+j}(a_i)-a_i\|_1=:A_{n2}^1+A_{n2}^2$$

and A_{n2}^1, A_{n2}^2 can be estimated by the same method as above. Thus

$$A_{n2}^1 \leqq \sum_{s=n+1}^{2n} n^{-1} \sum_{k=n+1}^{s} \sum_{|I_i|=2^{-s}} |\alpha_i| = n^{-1} \sum_{s=n+1}^{2n} (s-n) \sum_{|I_i|=2^{-s}} |\alpha_i| \leqq$$

$$\leqq \sum_{s=n+1}^{\infty} \sum_{|I_i|=2^{-s}} |\alpha_i| = o(1) \quad (n \to +\infty)$$

and

$$A_{n2}^2 \leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{k=s+1}^{2n} \sum_{|I_i|=2^{-s}} |\alpha_i| \, 2^{-k} \sum_{j=0}^{2^k-1} |\hat{a}_i(2^k+j)| + o(1) \leqq$$

$$\leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{k=s+1}^{2n} \sum_{|I_i|=2^{-s}} |\alpha_i| \, 2^{(s-k)/2} + o(1) \leqq$$

$$\leqq 12 \sum_{s=n+1}^{2n} sn^{-1} \sum_{|I|=2^{-s}}^{2n} |\alpha_i| + o(1) = o(1) \quad (n \to +\infty).$$

This completes the proof of the theorem.

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SELECTIVE AND BI-SELECTIVE DERIVATIVES

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1. Introduction. Selective and bi-selective derivatives were introduced by R. J. O'Malley in [2] and [3]. O'Malley proved that bi-selective derivatives are of honorary Baire class two ([3], Proposition 3), and asked if every selective derivative is Baire 1 on a dense open set. In Section 3 we prove that even a Lipschitz function can have a selective derivative everywhere which is not Baire 1 on any subinterval (Theorem 1).

In Section 4 we show that if the lower bi-selective derivative of f is greater than negative infinity on a set H then f is approximately differentiable almost everywhere on H (Theorem 2). This is a generalization of a result of [2], where the lower selective derivative is considered on $H=\mathbf{R}$. As a consequence, we obtain that the set of points at which the bi-selective derivative is infinite is always of measure zero (Corollary 2).

In Section 5 we introduce the notion of approximate selective derivative. We show that this derivative depends in an essential way on the selection (Corollary 3). Moreover, supposing CH, we can find an f such that every function is the approximative selective derivative of f with a suitable selection.

2. Notation and definitions. The Lebesgue outer measure of the set $H\subset\mathbf{R}$ will be denoted by $\lambda(H)$. If I is an interval, we put $|I|=\lambda(I)$. $\overline{H}$ denotes the closure of H. $[x, y]$ will denote the closed interval with end-points x and y. We allow $x>y$, so that $[x, y]=[y, x]$. The point x is called a point of dispersion of the set $H\subset\mathbf{R}$ if

$$\lim_{y\to x}\frac{\lambda(H\cap[x,y])}{y-x}=0.$$

Let f be a function defined on H. Suppose that there is a subset $H'\subset H$ such that x is a point of dispersion of $H\setminus H'$ and the finite limit

$$\lim_{\substack{y\to x\\ y\in H'}}(f(y)-f(x))/(y-x)$$

exists. Then we say that f is approximately differentiable at x and the value of the limit is denoted by $f'_{\mathrm{app}}(x)$. It is uniquely defined provided that x is not a point of dispersion of H. Note that in this definition H need not be measurable.

By a selection s we mean an interval function $s[x, y]$ such that

(1) $$x<s[x,y]<y \quad \text{for every} \quad 0\leqq x<y\leqq 1.$$

The selective derivative of $f: [0, 1] \to \mathbf{R}$ at $x \in [0, 1]$ with respect to s is defined by

$$sf'(x) = \lim_{y \to x} \frac{f(s[x, y]) - f(x)}{s[x, y] - x}.$$

By a bi-selection b we mean a pair of interval functions $s[x, y]$ and $v[x, y]$ such that s satisfies (1). The lower bi-selective derivative of f at x with respect to b is defined by

$${}_bf'(x) = \liminf_{y \to x} \frac{v[x, y] - f(x)}{s[x, y] - x}.$$

The upper bi-selective derivative, ${}^bf'(x)$ is defined similarly. If ${}_bf'(x) = {}^bf'(x)$ then this common value is called the bi-selective derivative of f and is denoted by $bf'(x)$. Obviously, the selective derivative is the special case of the bi-selective derivative, namely when $v[x, y] = f(s[x, y])$.

3. A selective derivative which is Baire 1 on no subinterval. THEOREM 1. *There exist a Lipschitz function f on $[0, 1]$ and a selection s such that sf' exists everywhere and it is not Baire 1 on any subinterval of $[0, 1]$.*

Our construction is based on the following lemma.

LEMMA 1. *Let C denote the Cantor ternary set. There exist a function g defined on $[0, 1]$ and a selection s such that*

(i) *g is Lipschitz,*
(ii) *g is continuously differentiable in $[0, 1] \setminus C$,*
(iii) *$sg'(x)$ exists at every $x \in C$,*
(iv) *sg' is not continuous at any $x \in C$.*

PROOF. Let h be a C^1 function on $[0, 1]$ such that $h(x) = 0$ if $x \in [0, 1/3]$, $h(1/2) = 2$ and $h(x) = x - 1$ if $x \in [2/3, 1]$. We define $g(x) = 0$ if $x \in C$, and $g(x) = (b-a)h\left(\frac{x-a}{b-a}\right)$, if (a, b) is an interval contiguous to C and $x \in (a, b)$. Since $h \in C^1$, h is Lipschitz and this easily implies that g is Lipschitz as well.

We denote by B the set of right hand side end-points of the intervals contiguous to C.

Now let $0 \leqq x < y \leqq 1$ be arbitrary. We choose a point $s = s[x, y]$ such that $s \in (x, y)$, and

$$s = (x+y)/2 \quad \text{if} \quad x, y \in [0,1] \setminus C;$$

$$g(s) = 0 \quad \text{if} \quad x \in [0,1] \setminus C, \ y \in C \setminus B;$$

$$\frac{g(s) - g(y)}{s - y} = 1 \quad \text{if} \quad x \in [0,1] \setminus C, \ y \in B;$$

$$g(s) = 0 \quad \text{if} \quad x \in C \setminus B; \quad \text{and}$$

$$\frac{g(s) - g(x)}{s - x} = 1 \quad \text{if} \quad x \in B. \tag{2}$$

The existence of such an s in the case of (2) follows from the following consideration. Let $x\in B$ and $y>x$ be fixed. One can easily check that for every $\delta>0$ there is an interval (a, b) contiguous to C such that

$$x<a<b<x+\delta \quad \text{and} \quad b-a = a-x.$$

Let $p(z)=g(z)-(z-x)$, then $p(a+(b-a)/3)<0$ and

$$p((a+b)/2) = 2(b-a) - ((a+b)/2-x) = (a-x)/2>0.$$

Therefore, there is a point $s\in(a, b)$ such that $p(s)=0$, i.e. (2) holds. Moreover, if $y\in C$ and we choose δ small enough then, by the continuity of g, $g(s)$ will be so small that

$$0 > \frac{g(s)-g(y)}{s-y} > x-y$$

will hold. It is easy to check that $sf'(x)=0$ if $x\in C\setminus B$ and $sf'(x)=1$ if $x\in B$. This completes the proof of Lemma 1.

LEMMA 2. *Suppose that the functions* $f_1, f_2, \ldots$ *are uniformly bounded and uniformly Lipschitz on* [0, 1]. *Let*

$$\sum_{i=1}^{\infty} |a_i| <\infty \quad \textit{and} \quad f = \sum_{i=1}^{\infty} a_i f_i.$$

If each f_i *is differentiable at* $x\in[0, 1]$ *then so is* f, *and*

$$f'(x) = \sum_{i=1}^{\infty} a_i f_i'(x).$$

The easy proof is left to the reader.

PROOF OF THEOREM 1. We shall denote

$$aC+b = \{ax+b;\ \ x\in C\} \quad \text{for every} \quad a, b\in\mathbf{R}.$$

We define, by induction, a sequence of intervals $I_n=[a_n, b_n]$ $(n=1, 2, \ldots)$ such that, denoting $C_n=(b_n-a_n)C+a_n$, we have $I_n\cap \bigcup_{j=1}^{n-1} C_j=\emptyset$ for every $n=1, 2, \ldots$, and

(3) for each subinterval $J\subset[0, 1]$ there is an n with $I_n\subset J$.

Let g denote the function constructed in Lemma 1 and define for every n

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x\notin I_n, \\ (b_n-a_n)\, g\left(\dfrac{x-a_n}{b_n-a_n}\right) & \text{if} \quad x\in I_n. \end{cases}$$

Then the functions f_n are uniformly Lipschitz on [0, 1]. We define $f=\sum_{n=1}^{\infty} f_n/2^n$. Then f is Lipschitz and, by Lemma 2 and Lemma 1 (ii), f is differentiable at every $x\in[0, 1]\setminus \bigcup_{n=1}^{\infty} C_n$.

Let s denote the selection defined in Lemma 1. We define the selections s_n by

$$s_n[x, y] = \begin{cases} (x+y)/2 & \text{if} \quad x \notin I_n \quad \text{or} \quad y \notin I_n \\ a_n + (b_n - a_n)\, s\left[\dfrac{x-a_n}{b_n-a_n}, \dfrac{y-a_n}{b_n-a_n}\right] & \text{if} \quad x, y \in I_n. \end{cases}$$

Finally, let the selection t be defined by

$$t[x, y] = \begin{cases} t(x+y)/2 & \text{if} \quad x, y \notin \bigcup_{n=1}^{\infty} C_n, \\ s_n[x, y] & \text{if} \quad x \in C_n,\ y \notin \bigcup_{j=1}^{\infty} C_j \quad \text{or} \quad y \in C_n,\ x \notin \bigcup_{j=1}^{\infty} C_j, \\ s_k[x, y] & \text{if} \quad x \in C_n,\ y \in C_m \quad \text{and} \quad k = \min(n, m). \end{cases}$$

Let n be fixed, and put $h = \sum_{k \neq n} f_k/2^k$. By Lemma 2, h is differentiable at each point of C_n and

$$h'(x) = \sum_{k \neq n} f_k'(x)/2^k \quad \text{on} \quad C_n.$$

Since f_k' is continuous at every point of C_n for $k \neq n$ and the functions f_k' are uniformly bounded, the restriction $h'|_{C_n}$ is continuous. If $x \in C_n$ and

$$|y - x| < \operatorname{dist}\left(x, \bigcup_{j<n} C_j\right)$$

then $t[x, y] = s_n[x, y]$. Therefore

$$tf_n'(x) = s_n f_n'(x) \quad \text{and} \quad tf'(x) = \frac{1}{2^n} s_n f_n'(x) + h'(x)$$

for every $x \in C_n$. Thus tf' exists everywhere and the restriction $tf'|_{C_n}$ is not continuous at any point of C_n. By (3), this implies that tf' is not Baire 1 in any subinterval of [0, 1] which completes the proof.

4. Functions with ${}_bf' > -\infty$. THEOREM 2. *Let f be defined on* [0, 1], *let b be a bi-selection, and suppose that ${}_bf'(x) > -\infty$ holds at each point x of a subset $H \subset [0, 1]$. Then*

(i) *$f|_H$ is approximately differentiable at almost every $x \in H$, and*

(ii) *${}_bf'(x) \leqq (f|_H)'_{\mathrm{app}}(x) \leqq {}^bf'(x)$ holds almost everywhere on H.*

LEMMA 3. *Let $g\colon [0, 1] \to \mathbf{R}$ be arbitrary and let*

$$P_n = \left\{x;\ \frac{v[x, y] - g(x)}{s[x, y] - x} > 0 \quad \text{if} \quad 0 < |y - x| < \frac{1}{n}\right\}.$$

Suppose that x is a right hand side accumulation point of P_n, y is a left hand side accumulation point of P_n, $0 < y - x < \dfrac{1}{n}$, and $\min({}_bg'(x), {}_bg'(y)) > -\infty$. Then $g(x) < g(y)$.

PROOF. First we remark that, if $x, y \in P_n$ and $0<y-x<\frac{1}{n}$ then $g(x)<g(y)$. Indeed, we have

$$\frac{v[x, y]-g(x)}{s[x, y]-x}>0 \quad \text{and} \quad \frac{v[x, y]-g(y)}{s[x, y]-y}>0$$

from which $g(x)<v[x, y]<g(y)$.

As for the rest of the proof, we refer to [2], Lemma 2, which is our lemma with $v[x, y]=g(s[x, y])$. One can easily check that the argument used in [2], Lemma 2 applies to our more general case as well.

PROOF OF THEOREM 2. Let A denote the set of points $x \in H$ at which $f|_H$ is not approximately differentiable, and let

$$A_n=\left\{x \in A;\ \frac{v[x, y]-f(x)}{s[x, y]-x}>-n \quad \text{if} \quad 0<|y-x|<\frac{1}{n}\right\} \quad (n=1, 2, \ldots).$$

Since ${}_bf'(x)>-\infty$ for every $x \in H$, $A=\bigcup_{n=1}^{\infty} A_n$. In order to show (i) we have to prove $\lambda(A_n)=0$ for every n. Suppose this is not true and let $\lambda(A_n)>0$.

Let D denote the set of bilateral accumulation points of A_n. Then $D=\bar{A}_n \setminus M$, where M is countable and hence $\lambda(D)>0$. Let $g(x)=f(x)+nx$ for $x \in [0, 1]$. By Lemma 3, $g(x)<g(y)$ whenever $0<y-x<\frac{1}{n}$ and $x, y \in D$. Let I be an interval with $|I|<\frac{1}{n}$ and $\lambda(A_n \cap I)>0$. Then g is increasing on $D \cap I$ and hence g is approximately differentiable at a.e. $x \in D \cap I$. Since $A_n \setminus D$ is countable, there is a point $x \in A_n$ such that g is approximately differentiable at x. Then f is also approximately differentiable at x which contradicts $x \in A$. This proves (i).

Next we put

$$K=\{x \in H;\ {}_bf'(x)>(f|_H)'_{\mathrm{app}}(x)\},$$

and

$$K_r=\{x \in K;\ {}_bf'(x)>r>(f|_H)'_{\mathrm{app}}(x)\} \quad (r \in \mathbf{R}),$$

We prove that $\lambda(K_r)=0$ for every r. Taking $f(x)-rx$ instead of f, we can suppose that $r=0$.

Let

$$K_{0,n}=\left\{x \in K_0;\ \frac{v[x, y]-f(x)}{s[x, y]-x}>0 \quad \text{if} \quad 0<|y-x|<\frac{1}{n}\right\},$$

then $K_0=\bigcup_{n=1}^{\infty} K_{0,n}$. Suppose $\lambda(K_{0,n})>0$ for some n. Let I be an interval with $|I|<\frac{1}{n}$, then f is increasing on $K'=K_{0,n} \cap I$. We can choose I such that $\lambda(K')>0$ holds. Let x be a density point of K'. Then $x \in K_0$ implies $(f|_H)'_{\mathrm{app}}(x)<0$, and hence $\frac{f(y)-f(x)}{y-x}<0$ for some $y \in K'$, a contradiction. This proves that $K_{0,n}$ is of measure zero for every n and then so is K_0.

Finally, $K=\bigcup\{K_r;\ r \text{ is rational}\}$ and thus $\lambda(K)=0$ and hence ${}_bf'(x)\leqq (f|_H)'_{\text{app}}(x)$ holds a.e. in H. The other inequality of (ii) can be shown similarly.

COROLLARY 1. *Let b be a bi-selection and suppose that ${}_bf'(x)>-\infty$ almost everywhere in* [0, 1]. *Then*

(i) *f is measurable,*

(ii) *f is approximatively differentiable almost everywhere,*

(iii) ${}_bf'(x)\leqq f'_{\text{app}}(x)\leqq {}^bf'(x)$ *for a.e. $x\in[0,1]$.*

REMARK. If ${}_bf'(x)>-\infty$ holds *everywhere* in [0, 1] then f is BVG. This can be shown by using Lemma 3; see [2], Theorem 4.

COROLLARY 2. *For an arbitrary function f: $[0,1]\to\mathbf{R}$ and for every bi-selection b, the set $H=\{x;\ bf'(x)=\infty\}$ is of measure zero.*

PROOF. By Theorem 2, $f|_H$ is approximately differentiable a.e. in H, and $(f|_H)'_{\text{app}}(x)\geqq {}_bf'(x)=\infty$ holds a.e. in H.

REMARK. Corollary 2 generalizes a well-known result on ordinary derivatives ([4], p. 236). For ordinary derivatives one can say more: the set of points x at which

$$\lim_{y\to x}\left|\frac{f(y)-f(x)}{y-x}\right|=\infty$$

is of measure zero for every f ([4], p. 270). However, the analogous statement is false for bi-selective or even for selective derivatives.

Let f be a function not bounded from above on any interval. We can define then a selection s such that

$$\frac{f(s[x,y])-f(x)}{s[x,y]-x}>\frac{1}{y-x}\quad\text{and}\quad\frac{f(s[x,y])-f(y)}{s[x.\,y]-y}<-\frac{1}{y-x}$$

for every $x<y$, and hence

$$\lim_{y\to x}\left|\frac{f(s[x,y])-f(x)}{s[x,y]-x}\right|=\infty$$

holds everywhere.

5. A note on approximate selective derivatives. It seems natural to define approximate selective derivatives by

$$sf'_{\text{app}}(x)\overset{\text{def}}{=}\lim_{y\to x}\text{app}\,\frac{f(s[x,y])-f(x)}{s[x,y]-x},$$

where s is a selection and f is defined on [0, 1].

An important property of selective (or bi-selective) derivatives is that they are "almost" independent of the selection (or bi-selection). More exactly, if b_1f' and b_2f' exist in [0, 1] then $b_1f'=b_2f'$ nearly everywhere (see [1], Theorem 3 and [3], Theorem 1).

Aus our next result shows, the approximate selective derivative does not possess this property.

$\varkappa$ will denote the smallest cardinal for which there is a subset $H\subset[0,1]$ with $\lambda(H)>0$ and card $H=\varkappa$.

THEOREM 3. *Suppose* $H\subset[0,1]$, *and* card $H=\varkappa$. *Let* f *be defined on* $[0,1]$ *and let* $g(x)$ *be a bilateral derived number of* f *at* x *for every* $x\in H$. *Then there is a selection* s *such that*

$$sf'_{\mathrm{app}}(x) = g(x)$$

holds for every $x\in H$.

PROOF. Let $\{x_\alpha;\ \alpha<\varkappa\}$ be a well-ordering of H. For $x, y\in[0,1]\setminus H$, $x\neq y$ we define $s[x,y]=(x+y)/2$. If $x\in H$, $y\in[0,1]\setminus H$ or $x=x_\alpha$, $y=x_\beta$ and $\alpha<\beta$ then we choose $s=s[x,y]$ such that

$$\left|\frac{f(s)-f(x)}{s-x}-g(x)\right| < |y-x|.$$

Since $g(x)$ is a bilateral derived number of f at x, we can find such an s. In this way we have defined a selection s. If $x\in H$, $x=x_\alpha$ then, obviously,

$$\lim_{\substack{y\to x\\ y\in H_\alpha}} \frac{f(s[x,y])-f(x)}{s[x,y]-x} = g(x),$$

where

$$H_\alpha = [0,1]\setminus\{x_\beta;\ \beta\leqq\alpha\}.$$

By the definition of $\varkappa$, $\lambda(\{x_\beta;\ \beta\leqq\alpha\})=0$ and hence H_α is of full measure. Therefore $sf'_{\mathrm{app}}(x)=g(x)$ and the proof is finished.

COROLLARY 3. *There is a function* f *and there are selections* s, t *such that*

$$\lambda(\{x;\ sf'_{\mathrm{app}}(x)\neq tf'_{\mathrm{app}}(x)\})>0.$$

PROOF. Choose a subset $H\subset[0,1]$ with $\lambda(H)>0$ and card $H=\varkappa$, and let f be such that 0 and 1 are both bilateral derived numbers of f at each $x\in H$. Then Theorem 3 applies.

PROBLEM. Can $sf'_{\mathrm{app}}(x)\neq tf'_{\mathrm{app}}(x)$ hold everywhere?

We remark that if $\varkappa$ equals continuum (which is true under CH) then the answer is affirmative. In fact, in this case we can take $H=[0,1]$ in the proof above. If f is such that every real number is a bilateral derived number of f at each x then, by Theorem 3, for every g there is a selection s with $sf'_{\mathrm{app}}(x)=g(x)$ $(x\in[0,1])$. We do not know if such a "universal" function can be given in ZFC.

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OSCILLATORY PROPERTIES OF ARITHMETICAL FUNCTIONS. II

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1. Introduction and statement of results

In the first paper of this series, [5], we have proved that the real-valued function $f(x)$ has at least $c(f)\log Y$ sign-changes in the interval $(0, Y]$ provided the analytic continuation of its Mellin transform

$$F(s)=\int_0^\infty f(x)x^{-s-1}dx \tag{1.1}$$

has all singularities of the form

$$P_\nu(s-\varrho_\nu)\log(s-\varrho_\nu)+F_\nu(s) \tag{1.2}$$

where $F_\nu(s)$ is meromorphic at $s=\varrho_\nu$, and P_ν is a polynomial (or $P_\nu\equiv 0$).

Using the method of [3] it is possible to generalize it for a much wider class of functions $f(x)$. Before we state our theorem let us recall that for any real function $f(x)$, $x>0$ we define the number $V(f, Y)$ of sign-changes in the interval $(0, Y]$ as follows:

$$V(f,Y)=\sup\{N:\ \exists\{x_i\}_{i=1}^N,\ 0<x_1<\ldots<x_N\leqq Y, \tag{1.3}$$
$$f(x_i)\neq 0,\ \operatorname{sgn}f(x_i)\neq\operatorname{sgn}f(x_{i+1}),\ 1\leqq i<N\}.$$

Moreover, we shall say that $V(f, Y)>h(Y)$ with combined oscillation of size $g(x)$ if there exists a series $\{x_i\}_{i=1}^{h(Y)}$ with $\operatorname{sgn}f(x_i)\neq\operatorname{sgn}f(x_{i+1})$ and

$$|f(x_i)|\geqq g(x_i). \tag{1.4}$$

Under these notations our theorem may be formulated as follows:

THEOREM. *Let* $f(x)$ *be real for* $x>0$ *and suppose that* $\int_0^\infty f(x)x^{-s-1}dx$ *converges absolutely for* $\sigma\geqq\sigma_1$ *and represents in that half-plane a function* $F(s)$ *having the following properties:*

(1) $F(s)$ *is regular for* $\sigma>\Theta$ *but not in any half-plane* $\sigma>\Theta-\varepsilon$ *with* $\varepsilon>0$,

(2) *there exists a denumerable (finite or infinite) set of singularities of* $F(s)$, $S=\{\varrho_\nu=\beta_\nu\pm i\gamma_\nu\}$, $\gamma_\nu>0$, *without finite limit point satisfying* $\Theta-c_0\leqq\beta_\nu\leqq\Theta$ *for some* $c_0>0$ *and such that* $F(s)$ *can be continued as a meromorphic function in the open set* D *obtained by making cuts* $s=\sigma\pm i\gamma_n$, $\sigma\leqq\beta_n$, *in the half-plane* $\sigma>\Theta-c_0$.

(3) *For* $|s-\varrho_\nu|\leqq\eta_\nu$, $\eta_\nu>0$, $s\in D$:

$$F(s)=(s-\varrho_\nu)^{a_\nu}\sum_{t=0}^{k_\nu}g_{\nu t}(s-\varrho_\nu)\log^t(s-\varrho_\nu)+F_\nu(s-\varrho_\nu), \tag{1.5}$$

where g_{vt} *and* F_v *are regular for* $|z|<\eta_v$, k_v *is a nonnegative integer,* a_v *an arbitrary complex number and* $g_{vk_v}(0)\neq 0$.

Let $\gamma=\min_{\beta_v=\Theta}\gamma_v$ *and* $\gamma=\infty$ *if* $\beta_v<\Theta$ *for all* $v=1, 2, \ldots$. *Under these conditions we have*

$$\varliminf_{Y\to\infty}\frac{V(f,Y)}{\log Y}\geqq\frac{\gamma}{\pi}, \tag{1.6}$$

and every interval of the form

$$[Y^{1-\varepsilon}, Y], \quad Y>Y_0(\varepsilon) \tag{1.7}$$

contains at least one sign-change of $f(x)$. *The sign-changes in* (1.6) *and* (1.7) *are combined with an oscillation of size*

$$x^{\Theta-\varepsilon} \tag{1.8}$$

for arbitrary $\varepsilon>0$.

Our theorem has immediate applications to the theory of distribution of prime numbers. Oscillatory properties of the difference $\pi(x)-\operatorname{li} x$ has been discussed in details in [3]. Further applications are the following.

COROLLARY 1. *If* $(l, q)=1$ *and all L-functions* (mod q) *have no real zeros in* [1/2, 1) *then for the difference*

$$\Delta_1(x)=\Pi(x, q, l)-\frac{1}{\varphi(q)}\operatorname{li} x=\sum_{n\equiv l(\mathrm{mod}\ q)}\frac{\Lambda(n)}{\log n}-\frac{\operatorname{li} x}{\varphi(q)} \tag{1.9}$$

we have

$$\varliminf_{Y\to\infty}\frac{V(\Delta_1, Y)}{\log Y}>0 \tag{1.10}$$

with combined oscillation of the size $x^{1/2-\varepsilon}$, $\varepsilon>0$. *Moreover, every interval of the form*

$$[Y^{1-\varepsilon}, Y], \quad Y\geqq Y(\varepsilon) \tag{1.11}$$

contains at least one sign-change of $\Delta_1(x)$.

The same assertions are true for the functions

$$\Delta_2(x)=\Pi(x; q, l_1)-\Pi(x; q, l_2) \tag{1.12}$$

where $l_1\not\equiv l_2$ (mod q),

$$\Delta_3(x)=\pi(x; q, l)-\frac{\operatorname{li} x}{\varphi(q)} \tag{1.13}$$

where l is a quadratic nonresidue,

$$\Delta_4(x)=\pi(x; q, l_1)-\pi(x; q, l_2) \tag{1.14}$$

where l_1 and l_2 are both quadratic residues or they are both nonresidues (mod q),

$$\Delta_5(x)=\pi(x; q, 1)-\frac{\operatorname{li} x}{\varphi(q)} \tag{1.15}$$

and

$$\Delta_6(x) = \pi(x; q, 1) - \pi(x; q, l) \tag{1.16}$$

for any $l \not\equiv 1 \pmod{q}$.

Still another field of applications of the Theorem is offered by the theory of factorization in algebraic number fields. Let K denote an arbitrary algebraic number field, R_K its ring of integers and $H(K)=\{X_1=E, X_2, \ldots, X_h\}$ its classgroup (E denotes the unit class). Let further M denote the set of all irreducible algebraic integers in K and let $M(x)$ and $\zeta(s, M)$ denote the counting function of M and the associated zeta-function respectively, i.e.

$$M(x) = \sum_{\substack{N(aR_K) \leq x \\ a \in M}} 1, \tag{1.17}$$

where from each set of associated integers only one is counted,

$$\zeta(s, M) = \sum_{\substack{aR_K \\ a \in M}} |N(a)|^{-s}, \tag{1.18}$$

and $N=N_{K/Q}$ denotes the norm-function.

If V denotes the set of all sequences $[d_1, \ldots, d_h]$, $d_i \in N \cup \{0\}$ such that $X_1^{d_1} \ldots X_h^{d_h}$ equals E; moreover, the product $X_1^{e_1} \ldots X_h^{e_h}$, $0 \leq e_i \leq d_i$ is equal to E if and only if either all e_i's are zero or $e_i = d_i$ holds for $i=1, 2, \ldots, h$, then

$$\zeta(s, M) = \sum_{[d_1, \ldots, d_h] \in V} \prod_{i=1}^{h} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{m_1 \geq 1} \cdots \sum_{m_k \geq 1} \frac{P_i(m_1 s) \ldots P_i(m_k s)}{m_1 \ldots m_k} \tag{1.19}$$

where

$$P_i(s) = \sum_{\mathfrak{p} \in X_i} N\mathfrak{p}^{-s} \tag{1.20}$$

(compare [4]). The summation in (1.20) is taken over all prime ideals from the class X_i.

We define

$$T(x) = \frac{1}{2\pi i} \int_{\mathscr{C}} \zeta(s, M) \frac{x^s}{s} ds; \tag{1.21}$$

$\mathscr{C}$ denotes the curve of integration consisting of the segment $[1/4, 1-\varepsilon_0]$ of the lower side of the real axis, the circumference $C(1, \varepsilon_0)$ and the segment $[1-\varepsilon_0, 1/4]$ of the upper side of real axis, where

$$\varepsilon_0 = 1/2 \min_{\chi \in \widehat{H(K)}} \min_{\substack{\varrho \\ \zeta_K(\varrho, \chi)=0}} |1-\varrho|. \tag{1.22}$$

$T(x)$ attains real values only and it was proved in [4] that it is the main term in the asymptotic formula for $M(x)$. $T(x)$ is a rather complicated function of x but for large x its behaviour is described by

$$T(x) \sim x \sum_{r=1}^{\infty} \frac{W_r(\log \log x)}{(\log x)^r} \tag{1.23}$$

(compare [4, Theorem 1]) where W_r are polynomials, $\deg W_1 = D-1$ and $\deg W_r \leqq D$ for $r \geqq 2$; here D is the Davenport's constant of K (see [6]).

Now we can say more about the difference

$$\Delta_7(x) = M(x) - T(x). \tag{1.24}$$

COROLLARY 2. *If the Dedekind zeta function of the Hilbert class field K_H of K does not vanish in the segment* $[1/2, 1)$ *and has at least one simple zero in the half-plane* $\sigma > 1/2$, *then*

$$\lim_{Y\to\infty} \frac{V(\Delta_7, Y)}{\log Y} > 0 \tag{1.25}$$

with combined oscillations of size $x^{1/2}$. Moreover, every interval of the form $[Y^{1-\varepsilon}, Y]$, $Y \geqq Y(\varepsilon)$ contains a sign-change of $\Delta_7(x)$.

In the same way as in the proof of Corollary 2 we can treat analogous remainder terms in the asymptotic formulae for the counting functions of certain subsets of R_K (compare [4]). There are some examples of such subsets: the sets F_k, $k=1, 2, \ldots$ of all algebraic integers from K which have at most k factorization into irreducibles, the sets G_k, $k=1, 2, \ldots$ of all algebraic integers from K which have at most k such factorizations of distinct lengths, $F_k' = F_k \cap Z$, $G_k' = G_k \cap Z$, and many others.

Oscillatory properties of the associated remainders depend of course on the analytic properties of the involved zeta-functions. These zeta-functions belong to the ring Ω (see [1]) which is the smallest ring containing all Dirichlet series with abscissas of absolute convergence <1 and also functions of the form $\zeta_K^w(s, \chi)$, $\log^k \zeta_K(a.\chi)$, where K denotes a certain number field, χ is a Hecke character, $w \in C$, $\operatorname{Re} w \geqq 0$ if $\chi = \chi_0$ (principal character) and k is a natural number. Hence such zeta functions have analytic continuation into the half-plane $\sigma > \sigma_0$, $\sigma_0 < 1$ and a slightly extended version of our theorem is applicable in all these cases.

It would be interesting to prove a stronger form of Corollary 2 assuming only that $\zeta_{K_H}(s)$ does not vanish in the segment $[1/2, 1]$.

The authors hope to return to this problem on another occasion.

2. Some auxiliary results

Let us introduce the following notations:

$$\mathbf{N} = \{1, 2, 3, \ldots\},$$

$$l = \log Y,$$

$$x = e^{\alpha l} = e^v, \quad \alpha_1 \leqq \alpha \leqq 1,$$

$$n = bl \ (n \text{ an integer}), \quad 0 < b < \frac{\alpha_1 |\varrho|}{10}, \quad \varrho = \beta + i\gamma, \quad \gamma > 0,$$

$0 < \eta < |\varrho|/10$,

$g(z)$ a regular function for $|z| \leqq \eta$ with $g(0) \neq 0$, $k \in \mathbf{N}^* = \{0, 1, 2, \ldots\}$,

B an arbitrary complex number, but $B \notin \mathbf{N}$ if $k = 0$.

Let $L_a(r)$ consist of the segment $[-a, -r]$ on the lower side of the real axis, the circumference $C(0, r)$ and the segment $[-r, -a]$ on the upper side of the real axis, $0<r\leqq a$, r, a real; $L(r)=L_\infty(r)$.

Let $L_a(r; \varrho)=\{s: s-\varrho\in L_a(r)\}$. Let

$$\tilde{\Gamma}(\omega, B)=\frac{1}{2\pi i}\int\limits_{L(r)} e^{z\omega} z^{B-1}\,dz$$

which is convergent for every B and every $\omega\in C$ with $\operatorname{Re}\omega>0$, the value of $\tilde{\Gamma}(\omega, B)$ being independent of r by Cauchy's theorem.

Let

$$\tilde{\Gamma}_j(\omega, B)=\frac{d^j\tilde{\Gamma}(\omega, B)}{dB^j}=\frac{1}{2\pi i}\int\limits_{L(r)} e^{z\omega} z^{B-1}\log^j z dz, \quad j\in N^*$$

and let $\tilde{\Gamma}(B)=\tilde{\Gamma}(1, B)=\pi^{-1}(\sin\pi B)\Gamma(B)$, according to Hankel's formula (see [7]).

LEMMA 1. *For $l\to\infty$ we have*

$$I(x)=\frac{1}{2\pi i}\int\limits_{L_\eta(r;\,\varrho)}\frac{x^s g(s-\varrho)\log^k(s-\varrho)(s-\varrho)^{B-1}}{s^n}\,ds= \tag{2.1}$$

$$=\frac{x^\varrho}{\varrho^n l^B}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\log l)^{k-j}A_j(\alpha)$$

with

$$A_j(\alpha)=g(0)\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B\right)+ \tag{2.2}$$

$$+\frac{1}{l}\left\{g'(0)\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B+1\right)+g(0)\frac{b}{2\varrho^2}\tilde{\Gamma}_j\left(\alpha-\frac{b}{\varrho}, B+2\right)\right\}+O(l^{-2})$$

where the constant in the O-symbol may depend on all parameters α_1, b, η, B, k, j, ϱ and on the function g.

PROOF. Writing $\omega=s-\varrho$ and $z=l\omega$ we have

$$I(x)=\frac{x^\varrho}{\varrho^n}\frac{1}{2\pi i}\int\limits_{L_\eta(r)}\frac{e^{v\omega}g(\omega)\log^k\omega\,\omega^{B-1}}{\left(1+\frac{\omega}{\varrho}\right)^n}\,d\omega= \tag{2.3}$$

$$=\frac{x^\varrho}{\varrho^n l^B}\frac{1}{2\pi i}\int\limits_{L_{\eta l}(r)}\frac{e^{zz}g(z/l)(\log z-\log l)^k z^{B-1}}{\left(1+\frac{z}{l\varrho}\right)^n}\,dz=$$

$$\frac{x}{\varrho^n l^B}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\log l)^{k-j}\frac{1}{2\pi i}\int\limits_{L_{\eta l}(r)}\frac{e^{zz}g(z/l)\log^j z\,z^{B-1}}{\left(1+\frac{z}{l\varrho}\right)^n}\,dz,$$

where we denote the integral by $A(\alpha)$.

To evaluate $A(\alpha)$ we note that for $h\in C$, $|h|\leqq 1/10$:

$$(1+h)^{-1} = e^{-h+h^2/2+O(h^3)},\ (1+h)^{-1} \leqq e^{2|h|}, \tag{2.4}$$

further $|z/lQ|\leqq|\eta/Q|\leqq 1/10$ for $z\in L_{\eta l}(r)$. Since we can choose r arbitrary with $0<r\leqq\eta l$ we choose $r=1$ and consider the shortened integral $A^*(\alpha)$ on $L^*=L_{\log^2 l}(1)$.

Then by (2.4), $b<\alpha_1|Q|/10$, and the regularity of g we have

$$\left|A(\alpha)-\frac{1}{2\pi i}\int_{L^*}\frac{e^{\alpha z}g(z/l)\log^j z z^{B-1}}{\left(1+\frac{z}{lQ}\right)^n}dz\right| = \tag{2.5}$$

$$= O\Big(\int_{\log^2 l}^{\eta l} e^{-\alpha x+2b|x/l|Q|}\log^j x x^{B-1}dx\Big) = O(e^{-(\alpha_1\log^2 l)/2}).$$

Further again by (2.4) we have

$$A^*(\alpha) = \frac{1}{2\pi i}\int_{L^*}\exp\left(\alpha z-\frac{bz}{Q}+\frac{bz^2}{2lQ^2}+O\left(\frac{z^3}{l^2}\right)\right)g(z/l)\log^j z z^{B-1}dz = \tag{2.6}$$

$$= \frac{1}{2\pi i}\int_{L^*} e^{\alpha z-bz/Q}\left(1+\frac{bz^2}{2lQ^2}+O\left(\frac{z^3}{l^2}\right)\right)\left(g(0)+g'(0)\frac{z}{l}+O\left(\frac{z^2}{l^2}\right)\right)\log^j z z^{B-1}dz =$$

$$= \frac{1}{2\pi i}\int_{L^*} e^{z(\alpha-b/Q)}\left\{g(0)+g(0)\frac{bz^2}{2lQ^2}+g'(0)\frac{z}{l}\right\}\log^j z z^{B-1}dz+O(1/l^2) =$$

$$= \frac{1}{2\pi i}\int_{L(1)} e^{z\left(\alpha-\frac{b}{Q}\right)}\left\{g(0)+\frac{1}{l}\left(g'(0)z+\frac{g(0)bz^2}{2Q^2}\right)\right\}\log^j z z^{B-1}dz+O(1/l^2),$$

similarly to (2.5). Now (2.6) yields the required result (2.2).

LEMMA 2. *If* $q\in\mathbf{N}$ *then* $\tilde{\Gamma}_0(\omega, q)=0$ *for* $\operatorname{Re}\omega>0$. *If* $q\notin\mathbf{N}$ *then for* $\Delta<\Delta_0(b, Q, q)$ *we hawe* $|\tilde{\Gamma}_0(\alpha-b/Q, q)|>\Delta$ *for* $\alpha\in\bigcup_{r=1}^{r(\Delta)}(h_r, h_r')\subset[\alpha_1, 1]$ *where* $r(\Delta)<\infty$, $h_r'\leqq h_{r+1}(r=1,\dots, r(\Delta)-1)$, $H(\Delta)=1-\alpha_1-\sum_{r=1}^{r(\Delta)}(h_r'-h_r)\to 0$ *as* $\Delta\to 0$. *If* $q\in\mathbf{N}$ *then the above assertion holds for* $\tilde{\Gamma}_1(\alpha-b/Q,)$ *in place of* $\tilde{\Gamma}_0(\alpha-b/Q, q)$.

PROOF. If $q\in\mathbf{N}$ it is sufficient to show that for real $\omega>0$ we have $\tilde{\Gamma}_0(\omega, q)=0$, since for fixed a $\tilde{\Gamma}_0(\omega, a)$ is a regular function of ω in the half-plane $\operatorname{Re}\omega>0$. But for $\omega\in R^+$ we have with $w=z\omega$

$$\tilde{\Gamma}_0(\omega, q) = \frac{\omega^{-q}}{2\pi i}\int_{L(r\omega)} e^w w^{q-1}dw = \frac{\Gamma(q)\sin(q\pi)}{\pi\omega^q} = 0. \tag{2.7}$$

If $q \notin \mathbf{N}$ then in view of (2.7), $\tilde{\Gamma}_0(\omega, q) \neq 0$ for $\omega \in R^+$. Since $\tilde{\Gamma}_0(\omega, q)$ is a regular function for $\operatorname{Re} \omega > 0$ the relation

$$\tilde{\Gamma}_0\left(\alpha - \frac{b}{\varrho}, q\right) = 0 \tag{2.8}$$

holds at most for finitely many $\alpha_1', \ldots, \alpha_k' \in [\alpha_1, 1]$.

Let

$$\xi_0(\alpha) = \left|\tilde{\Gamma}_0\left(\alpha - \frac{b}{\varrho}, q\right)\right|.$$

Now we have clearly for every non-negative continuous real function $\xi(\alpha)$ the relation $\xi(\alpha) \geqq \Delta$ for $\alpha \in \bigcup_{r=1}^{r(\Delta)} (h_1, h_r') \subset [\alpha_1, 1]$ where $h_r' \leqq h_{r+1}$ for $r = 1, \ldots, r(\Delta) - 1$, $r(\Delta) < \infty$, $\lim_{\Delta \to 0} \left\{1 - \alpha_1 - \sum_{r=1}^{r(\Delta)} (h_r' - h_r)\right\} = 0$, if $\xi(\alpha)$ has only finitely many zeros in $[\alpha_1, 1]$. If $q \in \mathbf{N}$ then the same is true for $\tilde{\Gamma}_1\left(\alpha - \frac{b}{\varrho}, q\right)$ since then for $\omega \in R^+$

$$\tilde{\Gamma}_1(\omega, q) = \frac{d\tilde{\Gamma}(\omega, q)}{dq} = \frac{d\left\{\dfrac{\Gamma(q) \sin(\pi q)}{\pi \omega^q}\right\}}{dq} = \frac{\Gamma(q) \cos(\pi q)}{\omega^q} \neq 0. \tag{2.9}$$

Now Lemmas 1 and 2 clearly imply

LEMMA 3. *Let* $m \in \mathbf{N}^*$, $x = e^{\alpha l}$, $0 < \alpha_1 \leqq \alpha \leqq 1$, $l \to \infty$. *Then there exists a* $\Delta = \Delta(\alpha_1, b, \varrho, B, g, m)$ *independent of* l *such that for* $\alpha \in \bigcup_{r=1}^{r(\Delta)} (h_r, h_r') \subset [\alpha_1, 1]$ *where* $r(\Delta) < \infty$, $1 - \alpha_1 - \sum_{r=1}^{r(\Delta)} (h_r' - h_r) \leqq \frac{\alpha_1}{m}$, $h_r' \leqq h_{r+1}$ *for* $r = 1, \ldots, r(\Delta) - 1$, *the relations*

$$I(x) = \frac{(-1)^k (-k)^j x^q (\log l)^{k-j}}{\varrho^n l^B} \left(g(0) \tilde{\Gamma}_j\left(\alpha - \frac{b}{\varrho}, B\right) + O\left(\frac{1}{l}\right)\right), \tag{2.10}$$

$$\left|g(0) \tilde{\Gamma}_j\left(\alpha - \frac{b}{\varrho}, B\right)\right| \geqq \Delta$$

hold, where $j = j(B) = 0$ *if* $B \notin \mathbf{N}$ *and* $j = j(B) = 1$ *if* $B \in \mathbf{N}$.

LEMMA 4. *If*

$$J = \left(x_0, x_0 \exp\left(\frac{2\pi}{\gamma}(1 + c)\right)\right) \subset \bigcup_{r=1}^{r(\Delta)} (Y^{h_r}, Y^{h_r'})$$

where $c > 0$ *is an arbitrary constant then for sufficiently large* Y *there exist* $x_1, x_2, x_3 \in J$ *such that*

$$\operatorname{sgn} \operatorname{Re} x_\mu \neq \operatorname{sgn} \operatorname{Re} x_{\mu+1} \ (\mu = 1, 2), \quad |\operatorname{Re} I(x_\mu)| \gg \frac{x_\mu^\beta (\log l)^{k - j(B)}}{|\varrho|^n j^{\operatorname{Re} B}} \ (1 \leqq \mu \leqq 3). \tag{2.11}$$

PROOF. In order to prove (2.11) we have only to note that for $x \in J$ we have

$$\log x = \log x_0 + O(1), \quad \text{i.e.} \quad \alpha = \frac{\log x_0}{l} + O\left(\frac{1}{l}\right). \tag{2.12}$$

Therefore we have for $x \in J$

$$I(x) = \frac{(-1)^k(-k)^j(\log l)^{k-j}}{\varrho^n l^B} g(0)\tilde{I}_j\left(\frac{\log x_0}{l} - \frac{b}{\varrho}, B\right) x^{\beta+i\gamma}\left(1+O\left(\frac{1}{l}\right)\right), \tag{2.13}$$

which implies (2.11).

3. Proof of the Theorem

We are entitled to assume $\Theta > 0$, since otherwise we work with $f'(x) = f(x) \cdot x^C$ with suitably chosen C. Also we can assume that $\gamma > 0$ since otherwise we have nothing to prove.

Since the proof is in many aspect similar to that of Theorem 1 in part I [5], we shall be brief at these places.

Similarly to [5, Section 3] we define for an arbitrary function $h(x)$ the operation δ by

$$\delta h(x) = \int_0^x \frac{h(\xi)}{\xi} d\xi, \tag{3.1}$$

and denote by δ_n the n times iterated operation δ. Then we have

$$\delta_n f(x) = \frac{1}{2\pi i} \int_{(\sigma_1)} F(s) \frac{x^s}{s^n} \mathrm{ds}. \tag{3.2}$$

Let us choose an $\eta' > 0$ in such a way that $\eta' < c_0$, $\eta' < \Theta/2$ and that the following region and line, resp.,

$$\sigma > \Theta - \eta', \quad |t| \leqq \gamma \quad \text{and} \quad \sigma = \Theta - \eta' \tag{3.3}$$

should contain no singularity except $\Theta \pm i\gamma$, if $\gamma < \infty$.

If $\gamma = \infty$ let η' be defined so that $\eta' < c_0$, $\eta' < \Theta/2$ and that the segment $[\Theta - \eta', \Theta]$ should be free of singularities of $F(s)$.

We shall choose later on a sufficiently large constant T, so that there should be no singularity ϱ of $F(s)$ on the broken line L' defined by

$$L' = \begin{cases} \sigma = \sigma_1 & \text{if} \quad |t| \geqq T, \\ \theta - \eta' \leqq \sigma \leqq \sigma_1 & \text{if} \quad |t| = T, \\ \sigma = \Theta - \eta' & \text{if} \quad |t| \leqq T, \end{cases} \tag{3.4}$$

but there should be at least one singularity to the right of L'. Let $\varrho_1 = \beta_1 + i\gamma_1$ denote the lowest singularity above the real axis with $\beta > \Theta - \eta'$. If there are more of them then let us choose that with maximal β. Let for $\nu \geqq 1$ $\varrho_{\nu+1} = \beta_{\nu+1} + i\gamma_{\nu+1}$ denote the singularity with minimal $\gamma > \gamma_\nu$, among those with $\beta \geqq \beta_\nu$, $\gamma < T$. If there are more of them then let us choose that with maximal β.

Let us suppose that we obtain exactly m singularities $\varrho_1, \ldots, \varrho_m$ in such a way.

Let us choose η with $\Theta-\eta' \leqq \beta_1-\eta$ and so that the domains

$$\{\beta_\nu-\eta \leqq \sigma < \beta_\nu,\ \gamma_\nu \leqq |t| \leqq \gamma_{\nu+1}\} \tag{3.5}$$

should be free of singularities of $F(s)$, for $\nu=1, 2, \ldots, m$, further let

$$\eta < \min_{1\leqq\nu\leqq m} \min\left(\eta_\nu, |\varrho_\nu|/20, \frac{\gamma_\nu-\gamma_{\nu-1}}{2}\right) \tag{3.6}$$

where we define $\gamma_0=0$, $\gamma_{m+1}=T$.

Let us choose now with an $r<\eta$, defining $\gamma_{m+1}=T+\eta$, $\beta_0=\Theta+\eta-\eta'$, the following broken line:

$$\begin{aligned} L = {} & \{\sigma=\sigma_1,\ T \leqq |t|\} \cup \{\sigma_1 \geqq \sigma \geqq \beta_m-\eta,\ |t| = T\} \\ & \bigcup_{\nu=m}^{1} \big[\{\sigma=\beta_\nu-\eta,\ \gamma_{\nu+1}-\eta \geqq |t| \geqq \gamma_\nu\} \cup L_\eta(r, \varrho_\nu) \cup \\ & \cup \{\sigma=\beta_\nu-\eta,\ \gamma_\nu \geqq |t| \geqq \gamma_\nu-\eta\} \cup \{\beta_\nu-\eta \geqq \sigma \geqq \beta_{\nu-1}-\eta,\ |t| = \gamma_\nu-\eta\}\big] \cup \\ & \cup \{\sigma=\Theta-\eta',\ \gamma_1-\eta \geqq |t|\}. \end{aligned} \tag{3.7}$$

Then $F(s)$ is regular on L and to the right of L, so we obtain with the notation $L^+=\{s\in L:\ \mathrm{Re}\, s\geqq 0\}$

$$\delta_n f(x) = \int_L F(s)\frac{x^s}{s^n}\,ds = 2\,\mathrm{Re}\int_{L^+} F(s)\frac{x^s}{s^n}\,ds \tag{3.8}$$

since L is symmetric to the real axis and $F(\bar{s})=\overline{F(s)}$.

Using the notation

$$J_\nu(x) = \int_{L_\eta(r,\varrho_\nu)} F(s)\frac{x^s}{s^n}\,ds \tag{3.9}$$

we have by easy calculation

$$\left|\delta_n f(x) - 2\sum_{\nu=1}^{m} \mathrm{Re}\, J_\nu(x)\right| \ll \frac{x^{\sigma_1}}{nT^{n-1}} + \frac{x^{\Theta-\eta'}}{(\Theta/2)^n} + \sum_{\nu=1}^{m}\frac{x^{\beta_\nu-\eta}}{(|\varrho_\nu|-2\eta)^n}. \tag{3.10}$$

Let

$$n = [b\log Y],\quad x = Y^\alpha,\quad \sqrt{b} \leqq \alpha \leqq 1. \tag{3.11}$$

If we fix b satisfying

$$b < b_0 = b_0(\eta') < \min\left(1/100, (|\varrho_1|/10)^2\right) \tag{3.12}$$

where b_0 is chosen sufficiently small (but independently of T) then with a positive constant $d_1=d_1(\eta', b)$

$$\frac{x^{\Theta-\eta'}}{(\Theta/2)^n} \ll \frac{x^{\beta_1}}{|\varrho_1|^n} Y^{-d_1}. \tag{3.13}$$

Further we have by $|\varrho_1|\leqq\ldots\leqq|\varrho_n|$ and (3.6)

$$\frac{x^{\beta_\nu-\eta}}{(|\varrho_\nu|-2\eta)^n} \leqq \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} x^{-\eta} e^{4n\eta/|\varrho_\nu|} \leqq \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-\eta\sqrt{b}(1-4\sqrt{b}/|\varrho_1|)} < \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-\eta\sqrt{b}/2}. \tag{3.14}$$

Now we fix T satisfying

$$T = T_0(b, \eta, \eta') \tag{3.15}$$

where T_0 is chosen sufficiently large. Then with a positive constant $d_2 = d_2(\eta', \eta, b, T)$

$$\frac{x^{\sigma_1}}{nT^{n-1}} \ll \frac{x^{\beta_1}}{|\varrho_1|^n} \cdot Y^{-d_2}. \tag{3.16}$$

So we have by (3.10)—(3.16) with $d_3 = \min(d_1, d_2, \eta\sqrt{b}/2) > 0$

$$\left|\delta_n f(x) - 2 \sum_{\nu=1}^{m} \operatorname{Re} J_\nu(x)\right| \ll \max_{1 \leqq \nu \leqq m} \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_3}. \tag{3.17}$$

From now on we consider b, η, η', T (hence $\varrho_1, \ldots, \varrho_m$) as fixed constants. Further we restrict ourselves for the sake of convenience for those values of Y for which $b \log Y$ is an integer. But it is clearly sufficient to show $V(Y) \geqq c \log Y$ $(Y \geqq Y_0)$ for these values of Y, since this implies $V(Y) \geqq c \log Y + O(1)$ for all Y.

Similarly to [5, (3.12)—(3.17)] there exists a small positive constant $d_4 = d_4(b, \varrho_1, \ldots, \varrho_m)$ such that we have disjoint intervals $(e_\nu, e'_\nu) \subset [\sqrt{b}, 1]$ (where $e_\nu = e'_\nu$ is possible), $\nu = 1, \ldots, m$ with total length at least

$$1 - 2\sqrt{b} \tag{3.18}$$

so that for $1 \leqq \nu \leqq m$

$$\max_{\substack{1 \leqq \mu \leqq m \\ \mu \neq \nu}} \frac{x^{\beta_\mu}}{|\varrho_\mu|^n} < \frac{x^k}{|\varrho_\nu|^n} Y^{-d_4} \quad \text{if} \quad x \in (Y^{e_\nu}, Y^{e'_\nu}). \tag{3.19}$$

Let ν be fixed and let us consider from now on always $x \in (Y^{e_\nu}, Y^{e'_\nu})$. Then we have with $d_5 = \min(d_3, d_4) > 0$

$$\left|\delta_n f(x) - 2 \sum_{\mu=1}^{m} \operatorname{Re} J_\mu(x)\right| \ll \frac{x^{\beta_\nu}}{|\varrho_\nu|^n} Y^{-d_5}. \tag{3.20}$$

Denoting the integral $I(x)$ in (2.1) of Lemma 1 by $I_{\mu t}(x)$ in case of $\varrho = \varrho_\mu$, $g = g_{\mu t}$, $B = B_\mu = a_\mu + 1$, $k = t$, we have by the regularity of $F_\mu(s)$

$$J_\mu(x) = \sum_{t=0}^{k_\mu} I_{\mu t}(x). \tag{3.21}$$

Taking into account Lemma 3 we obtain

$$\left|\delta_n f(x) - 2 \operatorname{Re} I_{\nu k_\nu}(x)\right| \ll \frac{x^{\beta_\nu}}{|\varrho_\nu|^n l^{\operatorname{Re} a_\nu}} (\log l)^{k_\nu - 1 - j(a_\nu)} \tag{3.22}$$

where as before

$$j(a_\nu) = \begin{cases} 1 & \text{if} \quad a_\nu \in \mathbf{N}^*, \\ 0 & \text{if} \quad a_\nu \notin \mathbf{N}^*. \end{cases}$$

Thus in view of (3.22) and Lemma 3 the oscillatory behaviour of $\delta_n f(x)$ is completely described by that of $I_{vk_v}(x)$, which is given by Lemma 4 and this clearly yields that $\delta_n f(x)$ has at least

$$2\left[\frac{\left(e'_v - e_v - \frac{2\sqrt{b}}{m}\right)\log Y}{(2\pi/\gamma_v)(1+b)}\right] \geqq 2\left[\frac{\left(e'_v - e_v - \frac{2\sqrt{b}}{m}\right)\log Y}{(2\pi/\gamma_1)(1+b)}\right] \tag{3.23}$$

sign changes in the interval $[Y^{e_v}, Y^{e'_v}]$, if Y is large enough.

This yields in view of (3.18)

$$V(\delta_n f, Y) \geqq (1-5\sqrt{b})\,\frac{\gamma_1}{\pi}\log Y. \tag{3.24}$$

Since for any real function $h(x)$ we have

$$V(h, Y) \geqq V(\delta h, Y), \tag{3.25}$$

inequality (3.24) clearly proves (1.6) if $\gamma < \infty$ since then $\gamma = \gamma_1$ and b can be chosen arbitrarily small. If $\gamma = \infty$ we have only to note that for every constant C we have $\gamma_1 > C$ if we choose η' so small at the beginning that the domain $\sigma \geqq \Theta - \eta'$, $|t| \leqq C$ should be free of singularities of $F(s)$.

The assertions (1.7) and (1.8) follow completely in the same way as in [5, Section 3]. To show (1.7) we need, apart from (3.22) and Lemma 4, the trivial observation that if for any function $h(x)$, δh has at least $k+1$ sign changes in $[A, B]$ then h has at least k sign changes.

To show (1.8) we have to work with the functions

$$\bar{f}(x) = \begin{cases} f(x) \pm x^{\Theta-\varepsilon}, & x \geqq 1, \\ 0, & 0 \leqq x \leqq 1, \end{cases}$$

$$\bar{F}(s) = F(s) \pm (s - \Theta - \varepsilon)^{-1}$$

and choose at the beginning $\eta' < \varepsilon/2$.

4. Proof of Corollary 1

We shall prove Corollary 1 in case $\Delta_1(x) = \Pi(x;\, q, l) - \dfrac{\operatorname{li} x}{\varphi(q)}$ only. The other proofs are very similar using the fact that the corresponding generating functions have non-real singularities in $\sigma \geqq 1/2$ (see Grosswald [1]) further that Corollary 1 is true for $\Delta_5(x)$ and $\Delta_6(x)$ if the Generalized Riemann Hypothesis is true which follows from the work [2] of Ingham (although he treated only the special case $q=1$). Let

$$f(x) = \begin{cases} 0 \quad \text{for} \quad 0 \leqq x \leqq 2, \\ \Pi(x;\, q, l) - \dfrac{1}{\varphi(q)} \displaystyle\int_2^x \frac{du}{\log u} \quad \text{for} \quad x > 2. \end{cases} \tag{4.1}$$

Then the corresponding $F(s)$ function is

$$F(s) = \frac{1}{s\varphi(q)} \sum_{\chi(\mathrm{mod}\, q)} \chi(l) \log \left\{(s-1)^{\varepsilon_\chi} L(s, \chi)\right\} + h(s) \tag{4.2}$$

where

$$\varepsilon_\chi = \begin{cases} 1 & \text{for} \quad \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

and $h(s)$ is holomorphic in the whole complex plane.

In the half-plane $\sigma > 0$ $F(s)$ has singularities of the form

$$\frac{k_\varrho}{s\varphi(q)} \log(s-\varrho) + F_\varrho(s) \tag{4.4}$$

where ϱ denotes a zero of $L(s, \chi)$ for some χ, and k_ϱ is a complex number. Moreover, $k_\varrho > 0$ for at least one zero with $\mathrm{Re}\, \varrho \geqq 1/2$ (compare [1]). The results (1.10) and (1.11) therefore follow from our Theorem.

5. Proof of Corollary 2

Let

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \leqq x \leqq 1, \\ M(x) - T(x) & \text{for} \quad x > 1. \end{cases} \tag{5.1}$$

Then

$$F(s) = \frac{1}{s} \zeta(s, M) - \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{1}{s-z} \frac{\zeta(z, M)}{z}\, dz. \tag{5.2}$$

Using (1.5) and (5.2) it is easy to prove that $F(s)$ is regular in the half-plane $\sigma > 1/2$ with the cuts $s = \sigma \pm i\gamma$, $1/2 \leqq \sigma \leqq \beta$ where $\varrho = \beta \pm i\gamma$ is a zero of $\zeta_{K_H}(s)$. At these points $F(s)$ has singularities of the form (1.5) with $a_\nu = 0$. At least one of these singularities is non-trivial. Indeed, (1.19) implies that in the neighbourhood of ϱ_0, the simple zero of $\zeta_{K_H}(s)$ with $\mathrm{Re}\, \varrho_0 > 1/2$, we have

$$F(s) = \frac{R}{h^D S} \log^D(s-\varrho_0) + \sum_{j=0}^{D-1} g_j(s) \log^j(s-\varrho_0), \tag{5.3}$$

where $g_j(s)$, $j = 0, 1, \ldots, D-1$ are regular near ϱ_0 and

$$R = \sum_{\substack{[d_1, \ldots, d_h] \in V \\ d_1 + \ldots + d_h = D}} \frac{1}{d_1! \ldots d_h!} > 0. \tag{5.4}$$

This proves Corollary 2.

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A NOTE ON INVARIANT MEASURES

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Introduction. Let X be a compact Hausdorff space. There exist a uniform structure $\mathfrak{A}$ on X (we may suppose that it contains only symmetric subsets of $X\times X$) such that $\mathfrak{A}$ induces the topology of X. If $U\in\mathfrak{A}$ and $x\in X$ then we denote by $U(x)$ the section of U at x i.e. $U(x)=\{y|(x, y)\in U\}$.

Denote by $C(X, X)$ the set of continuous mappings from X to X. Suppose that $G\subset C(X, X)$ is a group with the composition of mappings as the product. Moreover suppose that at every point of X the elements of G are equicontinuous i.e. for every fixed $x\in X$ and for every $U\in\mathfrak{A}$ there exists a $V\in\mathfrak{A}$ such that for every $g\in G$, $y\in V(x)$ we have $(gy, gx)\in U$.

The topology of G. First we try to define a topology on G such that G becomes a topological group.

Equip $C(X, X)$ and G with the topology of the uniform convergence on X. (This means for example that for any $g\in G$ the sets having the form $U(g)=$ $=\{h\in G|(gx, hx)\in U$ for every $x\in X\}$ (where $U\in\mathfrak{A}$) generate a neighbourhood-base in G of g.)

Denote by $\bar{G}$ the closure of G in $C(X, X)$. We will show that the composition of mappings defines a product operation on $\bar{G}$ and in such a way $\bar{G}$ becomes a compact topological group.

First we show that the elements of $\bar{G}$ remain equicontinuous at every point of X. Namely, let $x\in X$ be an arbitrary point and $U\in\mathfrak{A}$ be a set. There exists a $W\in\mathfrak{A}$ such that $W\cdot W\cdot W\subset U$. Using the equicontinuity of the elements of G we can find a set $V\in\mathfrak{A}$ such that for every $y\in V(x)$ and $g\in G$ we have $(gy, gx)\in W$. Now let $y\in V(x)$ be an arbitrary point and, for a given $\gamma\in\bar{G}$, pick up an element $g\in G$ of $W(\gamma)$. Then $(gx, \gamma x)\in W$, $(gy, \gamma y)\in W$, $(gx, gy)\in W$. This means that $(\gamma x, \gamma y)\in$ $\in W\cdot W\cdot W\subset U$ thus the elements of $\bar{G}$ are equicontinuous at x.

PROPOSITION 1. *The elements of* $\bar{G}$ *are uniformly equicontinuous on* X (*i.e. for every* $U\in\mathfrak{A}$ *there exists a* $V\in\mathfrak{A}$ *such that if* $x, y\in X$ *and* $(x, y)\in V$ *then for every* $\gamma\in\bar{G}$ *we have* $(\gamma x, \gamma y)\in U$).

PROOF. First consider a uniform neighbourhood $W\in\mathfrak{A}$ such that $W\cdot W\subset U$. Since the elements of $\bar{G}$ are equicontinuous, for every $x\in X$ there exists a set $V_x\in\mathfrak{A}$ such that if $y\in V_x(x)$ and $\gamma\in\bar{G}$ then $(\gamma y, \gamma x)\in W$. For every set V_x there exists another set $V_x'\in\mathfrak{A}$ such that $V_x'\cdot V_x'\subset V_x$. Since $\bigcup_{x\in X} V_x'(x)=X$ there exists a finite subcovering, $V_{x_i}'(x_i)$, $i=1, \ldots, n$. Let $V=\bigcup_{i=1}^{n} V_{x_i}'$. If $(x, y)\in V$ then there

exists an index i such that $x \in V'_{x_i}(x_i)$ so $(y, x_i) \in V_{x_i}$. If $\gamma \in \bar{G}$ then $(\gamma y, \gamma x_i) \in W$ and $(\gamma x, \gamma x_i) \in W$ so $(\gamma y, \gamma x) \in W \cdot W \subset U$.

PROPOSITION 2. *$\bar{G}$ is a topological group.*

PROOF. First consider the product operation. Let $\gamma, \delta \in \bar{G}$. We show that $\gamma \cdot \delta \in \bar{G}$. Let $U \in \mathfrak{U}$. There exists a $V \in \mathfrak{U}$ for which $V \cdot V \subset U$. Pick up an arbitrary element $g \in G$, $g \in V(\gamma)$. Using the uniform equicontinuity of the elements of G there exists a W such that for every $(x, y) \in W$ and $h \in G$ we have $(hx, hy) \in V$. Let h be an element of $W(\delta)$. In this case if $x \in X$ then $(hx, \delta x) \in W$ consequently $(ghx, g\delta x) \in V$ and $(\gamma\delta x, g\delta x) \in V$ so $(\gamma\delta x, ghx) \in U$ thus $\gamma\delta \in \bar{G}$.

Observe that we have shown essentially that for every $\gamma, \delta \in \bar{G}$ and $U \in \mathfrak{U}$ there exist $V \in \mathfrak{U}$, $W \in \mathfrak{U}$ such that for every $h \in W''(\delta)$, $g \in V^*(\gamma)$ we have $gh \in U^*(\gamma\delta)$. (For any $\alpha \in \bar{G}$, $U \in \mathfrak{U}$, $U^*(\alpha)$ denotes the set $\{\beta \in \bar{G} | (\alpha x, \beta x) \in U$ for every $x \in X\}$.) Thus the product operation is continuous on G.

Next consider the inverse operation. Let $\gamma \in \bar{G}$ be an arbitrary element. We show that its range is just the whole X. Let $y \in X$ be a point. For any $U \in \mathfrak{U}$ consider the following closed set: $\overline{\{g^{-1}y | g \in G, (gs, \gamma s) \in U \text{ for every } s \in X\}}$. Since X is compact the intersection of these closed sets is not empty. Let z be an element of this intersection. Choose an arbitrary set $U \in \mathfrak{U}$ and let $U_1 \in \mathfrak{U}$ be another set for which $U_1 \circ U_1 \subset U$. Using the uniform equicontinuity of the elements of G on X there exists a set $V \in \mathfrak{U}$ such that $(gt, gs) \in U_1$ if $(t, s) \in V$ and $g \in G$. Since z belongs to $\overline{\{g^{-1}y | g \in G, (gs, \gamma s) \in U_1 \text{ for every } s \in X\}}$ there exists a $g \in G$ such that $(g^{-1}y, z) \in V$ and $(gz, \gamma z) \in U_1$. From this it follows that $(y, gz) \in U_1$ so $(y, \gamma z) \in U$. This holds for any $U \in \mathfrak{U}$, thus $\gamma z = y$. Next we show that γ^{-1} exists. Let $x \in X$, $U, U_1 \in \mathfrak{U}$ be such that $U_1 \circ U_1 \circ U_1 \subset U$. Using Proposition 1 define a set W for U_1 i.e. $(gs, gt) \in U_1$ if $(s, t) \in W$, $g \in \bar{G}$. Let $y \in X$ be a point for which $\gamma y \in W(\gamma x)$. Since $\gamma \in \bar{G}$ there exists a $g \in G$ such that $(gx, \gamma x) \in W$, $(gy, \gamma y) \in W$. Applying g^{-1} to $gx, \gamma x, gy, \gamma y$ we obtain that $(x, g^{-1}\gamma x) \in U_1$, $(y, g^{-1}\gamma y) \in U_1$, $(g^{-1}\gamma y, g^{-1}\gamma x) \in U_1$. So $(x, y) \in U$. From this it immediately follows that if $\gamma x = \gamma y$ then $x = y$. Moreover, γ^{-1} is continuous.

We have to show that $\gamma^{-1} \in \bar{G}$. Let $U \in \mathfrak{U}$ be a set. Using Proposition 1 we can find a set $V \in \mathfrak{U}$ for U i.e. if $(t, s) \in V$, $g \in \bar{G}$ then $(gt, gs) \in U$. Consider an element $g \in G$ of $V(\gamma)$. In this case $(\gamma s, gs) \in V$ for any $s \in X$. Applying g^{-1} we obtain $(g^{-1}\gamma s, s) \in U$. Putting $s = \gamma^{-1}t$ we have $(g^{-1}t, \gamma^{-1}t) \in U$ for every $t \in X$. Thus $\gamma^{-1} \in \bar{G}$.

Finally concerning the continuity of the inverse operation it is enough to prove that this holds at the unit element of $\bar{G}$ owing to the continuity of the product operation. Let $U \in \mathfrak{U}$ and apply again Proposition 1. We obtain a set $V \in \mathfrak{U}$. Now if $\gamma \in V^*(e)$ then $(\gamma x, x) \in V$ for every $x \in X$. Applying γ^{-1} to γx and x we obtain that $(x, \gamma^{-1}x) \in U$ thus $\gamma^{-1} \in U^*(e)$.

Now we analyse the topology on $\bar{G}$.

PROPOSITION 3. *$\bar{G}$ is a compact topological group.*

PROOF. It remains to show that $\bar{G}$ is a compact space. But this is a direct consequence of Theorem 7.17 of Kelley [1].

PROPOSITION 4. *$\bar{G}$ is a Hausdorff space.*

PROOF. If $g \neq h$ then there exists an $x \in X$ such that $gx \neq hx$. X is T_2 so there exists a $V \in \mathfrak{A}$ such that $V(gx) \cap V(hx) = \emptyset$. The neighbourhoods $V^*(g)$ and $V^*(h)$ are obviously disjoint.

The invariant linear mapping. By the preceding arguments, $\bar{G}$ can be considered as a compact Hausdorff space which is a topological group. Thus there exists an invariant measure on it. Denote this by μ. We can suppose that $\mu(\bar{G}) = 1$. If $f \in C(X)$ then we can define an invariant linear mapping by means of the integral $x \mapsto \int_{\bar{G}} f(\gamma x) d\mu(\gamma)$. Denote $\tilde{f}(x) = \int_{\bar{G}} f(\gamma x) d\mu(\gamma)$.

PROPOSITION 5. *If $f \in C(X)$ then $\tilde{f}(x)$ is continuous.*

PROOF. For every $\varepsilon > 0$ we can define a $U \in \mathfrak{A}$ such that if $(x, y) \in U$ then $|f(x) - f(y)| < \varepsilon$. Now fix a point $x \in X$. There exists a $V \in \mathfrak{A}$ for which if $y \in V(x)$ and $\gamma \in \bar{G}$ then we have $(\gamma y, \gamma x) \in U$. Thus in this case $|f(\gamma y) - f(\gamma x)| < \varepsilon$. Consequently

$$\left| \int_{\bar{G}} f(\gamma x) d\mu(\gamma) - \int_{\bar{G}} f(\gamma y) d\mu(\gamma) \right| \leqq \varepsilon \cdot \mu(\bar{G}) = \varepsilon.$$

So $\tilde{f}$ is continuous.

The mapping $f \to \tilde{f}$ is obviously linear. But its range is not the whole $C(X)$. So we have to define a factor space of X. Define a relation on X as follows: $x \sim y$ iff $\bar{G}x \ni y$. Observe that since $\bar{G}$ is compact $\bar{G}x$ is also compact.

PROPOSITION 6. *The relation $x \sim y$ is an equivalence relation.*

PROOF. It is obviously reflexive. We show that if $y \in \bar{G}x$ then $\bar{G}y = \bar{G}x$. From this follows the symmetry and transitivity. Obviously $\bar{G}y \subset \bar{G}x$. Let $V \in \mathfrak{A}$. There exists a $U \in \mathfrak{A}$ and a $g \in G$ such that for every $z \in U(y)$ and for every $h \in G$ we have $(hz, hy) \in V$ and for which $(gx, y) \in U$. If $h^{-1} = g$ and $z = gx$ then we obtain $(x, g^{-1}y) \in V$. Thus $x \in \bar{G}y$ so $\bar{G}x \subset \bar{G}y$, consequently $\bar{G}y = \bar{G}x$.

By means of the equivalence relation $x \sim y$ we can define a factor space Y. Denote by ϱ the projection from X onto Y; $\varrho: X \to Y$. $\varrho(x) = \bar{G}x$ as a point of Y. Equip Y with the finest topology with respect to which the projection ϱ is continuous. Since for every $f \in C(X)$ the function $\tilde{f}$ is constant on every equivalence class we can consider it as a function defined on Y. Denote this by f_G i.e. if $y = \bar{G}x$ then $f_G(y) = \tilde{f}(x)$. f_G is obviously continuous.

PROPOSITION 7. *ϱ is an open mapping, Y is a compact Hausdorff space.*

PROOF. First we show that ϱ is open. To this aim we have to prove that if $A \subset X$ is open then $B = \bigcup_{z \in A} \bar{G}z$ is also open. Suppose that $s \in \bar{G}z$, $z \in A$ and $U \in \mathfrak{A}$, $U(z) \subset A$. There exists a $\gamma \in \bar{G}$ and a $V \in \mathfrak{A}$ such that $s = \gamma z$ and $(\gamma^{-1}x, \gamma^{-1}y) \in U$ if $(x, y) \in V$. Now if t is an arbitrary point which belongs to $V(s)$ then $(\gamma^{-1}s, \gamma^{-1}t) \in U$ thus $\gamma^{-1}t \in U(z) \subset A$ consequently $t \in B$. This means that $V(s) \subset B$ i.e. ϱ is open.

Next we show that Y is T_2. Let $x, y \in X$ be such points for which $\bar{G}x \cap \bar{G}y = \emptyset$. There exist $W, V, U \in \mathfrak{A}$ such that $W(x) \cap \cap \bar{G}y = \emptyset$, $U \cdot U \subset W$ and $(\gamma s, \gamma t) \in U$ if $(s, t) \in V$ and $\gamma \in \bar{G}$. Let P, Q be open sets such that $x \in P \subset U(x)$, $y \in Q \subset V(y)$. The sets $A = \bigcup_{z \in P} \bar{G}z$, $B = \bigcup_{z \in Q} \bar{G}z$ are open and $\varrho(A)$, $\varrho(B)$ are neighbourhoods

of $\varrho(x)$ and $\varrho(y)$ so it is enough to prove that $A\cap B=\emptyset$. But if s were a point in $A\cap B$ then we could find such points $z_1\in U(x)$, $z_2\in V(y)$ for which $\bar{G}z_1=\bar{G}z_2$. This means that there would be a $\gamma\in\bar{G}$ for which $z_1=\gamma z_2$. The relation $(y, z_2)\in V$ implies $(\gamma y, \gamma z_2)\in U$. Comparing this with $(x, z_1)\in U$ it would follow $(x, \gamma y)\in W$, a contradiction. Thus $\varrho(A)$ and $\varrho(B)$ are disjoint open sets.

Now if $\varphi\in C(Y)$ then the function $f(x)=\varphi(\varrho(x))$, $x\in X$, is obviously continuous and if $y=\bar{G}x$ then

$$f_G(y) = \int_{\bar{G}} f(\gamma x)d\mu(\gamma) = \int_{\bar{G}} \varphi(\varrho(\gamma x))d\mu(\gamma) = \varphi(\gamma).$$

Thus we have proved the following theorem.

THEOREM. *Let X be a compact Hausdorff space and let $G\subset C(X, X)$ be a group under the composition of mappings as the product and suppose that the elements of G are equicontinuous at every point of X. Then there exists a compact Hausdorff space Y and a linear G-invariant mapping from $C(X)$ onto $C(Y)$.*

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PROXIMITIES, SCREENS, MEROTOPIES, UNIFORMITIES. I

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0. Introduction. The purpose of this paper is to investigate properties and interrelations of a number of well-known and some new topological categories (the latter concept is always understood in the sense of [11]). Besides those mentioned in the title, we shall deal with closures, topologies, with certain generalizations of them, and with a series of subcategories of the above categories. For a part of them, a unified treatment is possible with the help of the concept of a (semi-)topogenous order; in this respect, we refer to [2].

If $\mathbf{C}$ is a concrete category, the objects of $\mathbf{C}$ will be called *structures in* $\mathbf{C}$; if F denotes the forgetful functor from $\mathbf{C}$ into **Set**, $\mathscr{S}$ is a structure in $\mathbf{C}$, and $F(\mathscr{S})=X$, then we shall say that $\mathscr{S}$ is a *structure on* X. If m is a morphism of $\mathbf{C}$ from $\mathscr{S}$ to $\mathscr{S}'$, we say that the map $F(m)$ from $F(\mathscr{S})$ into $F(\mathscr{S}')$ is $(\mathscr{S}, \mathscr{S}')$-*continuous*. If $\mathscr{S}$ and $\mathscr{S}'$ are structures in $\mathbf{C}$ on X, then $\mathscr{S}$ is said to be *finer* than $\mathscr{S}'$, or $\mathscr{S}'$ *coarser* than $\mathscr{S}$, iff id_X is $(\mathscr{S}, \mathscr{S}')$-continuous; $\mathscr{S}$ and $\mathscr{S}'$ are said to be *equivalent* iff both of them are finer than the other one.

In most cases, we shall check the fact that a category is topological with the help of the following lemma whose proof is left to the reader:

(0.1) LEMMA. *Let* $\mathbf{C}$ *be a concrete category with the following properties:*

(a) *If* $f: X\to Y$ *is a map and* $\mathscr{S}_0$ *is a structure in* $\mathbf{C}$ *on* Y, *then there is a structure* $f^{-1}(\mathscr{S}_0)=f_{\mathbf{C}}^{-1}(\mathscr{S}_0)$ *in* $\mathbf{C}$ *on* X *such that* f *is* $(f^{-1}(\mathscr{S}_0), \mathscr{S}_0)$-*continuous and* $f^{-1}(\mathscr{S}_0)$ *is coarser than* $\mathscr{S}$ *whenever* $\mathscr{S}$ *is a structure in* $\mathbf{C}$ *on* X *such that* f *is* $(\mathscr{S}, \mathscr{S}_0)$-*continuous.*

(b) *If* Σ *is a class of structures in* $\mathbf{C}$ *on* X, *then there is a structure* $\sup\Sigma=\sup_{\mathbf{C}}\Sigma=\sup_{\mathbf{C},X}\Sigma$ *in* $\mathbf{C}$ *on* X *such that it is finer than each* $\mathscr{S}\in\Sigma$ *and coarser than any other structure in* $\mathbf{C}$ *on* X *finer than each* $\mathscr{S}\in\Sigma$.

(c) *If* $f: X\to Y$, $g: Z\to X$, *and* $\mathscr{S}_0$ *is a structure in* $\mathbf{C}$ *on* Y, *then*

$$(f\circ g)^{-1}(\mathscr{S}_0) = g^{-1}\big(f^{-1}(\mathscr{S}_0)\big).$$

(d) *If* $\mathscr{S}_i$ $(i\in I)$ *are structures in* $\mathbf{C}$ *on* X *and* $g: Z\to X$, *then*

$$g^{-1}\big(\sup\{\mathscr{S}_i: i\in I\}\big) = \sup\{g^{-1}(\mathscr{S}_i): i\in I\}.$$

Then $\mathbf{C}$ *is a topological category, and if* $f_i: X\to Y_i$ *and structures* $\mathscr{S}_i$ *in* $\mathbf{C}$ *on* Y_i *are given for* $i\in I$, *then*

$$\sup\{f_i^{-1}(\mathscr{S}_i): i\in I\}$$

is an initial structure in $\mathbf{C}$ *corresponding to the structures* $\mathscr{S}_i$ *and the maps* f_i. □

In most cases we shall deal with concrete categories **C** satisfying (0.1) (a)—(d), moreover,

(e) $\mathrm{id}_X^{-1}(\mathscr{S})=\mathscr{S}$ for any structure $\mathscr{S}$ in **C** on X,

(f) $\sup \Sigma=\mathscr{S}$ whenever $\mathscr{S}$ is a structure in **C** on X and $\Sigma=\{\mathscr{S}\}$.

We shall say that **C** is a *strongly topological category* iff it fulfils (0.1) (a)—(d) and (e), (f) for suitable definitions of $f^{-1}(\mathscr{S})$ and $\sup \Sigma$. If, in a topological category, equivalent structures necessarily coincide (in particular, if the category is properly fibred), then the category is strongly topological.

If **C** satisfies (0.1) (a)—(d), then, for any set X, $\sup_{\mathbf{C},X}\emptyset$ is a structure in **C** on X that is coarser than any structure in **C** on X; it is called the *standard indiscrete structure* in **C** on X.

Now we formulate two more lemmas in order to obtain bireflective and bicoreflective subcategories, respectively.

(0.2) LEMMA. *Let* **C** *be a strongly topological category and* **C′** *a full subcategory of* **C** *such that*

(a) *if* $\mathscr{S}_0$ *is a structure in* **C′**, *then* $f^{-1}(\mathscr{S}_0)$ *is a structure in* **C′** *as well,*

(b) *if* $\mathscr{S}_i$ *is a structure in* **C′** *for* $i\in I$, *then* $\sup\{\mathscr{S}_i\colon i\in I\}$ *is a structure in* **C′**.

If further **C″** *is a full subcategory of* **C** *such that* $\mathbf{C}'\subset\mathbf{C}''\subset\mathbf{C}$, *then* **C′** *is bireflective in* **C″**; *the reflection of a structure* $\mathscr{S}$ *in* **C″** *on* X *is the supremum of all structures in* **C′** *coarser than* $\mathscr{S}$, *with the reflector* id_X. *By* (0.1), **C′** *is strongly topological with the operations*

$$f_{\mathbf{C}'}^{-1}(\mathscr{S}_0)=f_{\mathbf{C}}^{-1}(\mathscr{S}_0),\quad \sup_{\mathbf{C}'}\{\mathscr{S}_i\colon i\in I\}=\sup_{\mathbf{C}}\{\mathscr{S}_i\colon i\in I\}.\quad\square$$

(0.3) LEMMA. *Let* **C** *be a strongly topological category and* **C′** *a full subcategory of* **C** *such that*

(a) *for any structure* $\mathscr{S}$ *in* **C**, *there is a structure* $\mathscr{S}^q$ *in* **C′** *finer than* $\mathscr{S}$ *and coarser than any structure in* **C′** *finer than* $\mathscr{S}$,

(b) $f_{\mathbf{C}}^{-1}(\mathscr{S})^q=f_{\mathbf{C}}^{-1}(\mathscr{S}^q)$ *for every structure* $\mathscr{S}$ *in* **C**,

(c) $\mathscr{S}^q=\mathscr{S}$ *for every structure* $\mathscr{S}$ *in* **C′**.

Then **C′** *is bicoreflective in any full subcategory* **C″** *of* **C** *such that* $\mathbf{C}'\subset\mathbf{C}''\subset\mathbf{C}$; *the coreflection of* $\mathscr{S}$ *is* $\mathscr{S}^q$ *with the coreflector* id_X. **C′** *is strongly topological with the operations*

$$f_{\mathbf{C}'}^{-1}(\mathscr{S}_0)=f_{\mathbf{C}}^{-1}(\mathscr{S}_0),\quad \sup_{\mathbf{C}'}\{\mathscr{S}_i\colon i\in I\}=(\sup_{\mathbf{C}}\{\mathscr{S}_i\colon i\in I\})^q.\quad\square$$

The following lemmas (where (0.5) and (0.6) follow from (0.2) and (0.3), respectively) enable us to define covariant functors for strongly topological categories:

(0.4) LEMMA. *Let* **C** *and* **D** *be concrete categories and, for every structure* $\mathscr{S}$ *in* **C** *on* X, *let a structure* $F(\mathscr{S})$ *in* **D** *on* X *be given such that any* $(\mathscr{S},\mathscr{S}')$*-continuous map is* $(F(\mathscr{S}),F(\mathscr{S}'))$*-continuous whenever* $f\colon X\to Y$, $\mathscr{S}$ *and* $\mathscr{S}'$ *are structures in* **C** *on* X *and* Y, *respectively. If we define* $F(f)=f$ *for every* $(\mathscr{S},\mathscr{S}')$*-continuous map* $f\colon X\to Y$, *then* F *is a covariant functor from* **C** *into* **D**. $\square$

(0.5) LEMMA. *If, under the hypotheses of* (0.4), **C** *and* **D** *are strongly topological,*

and F *satisfies, for arbitrary structures* $\mathscr{S}_i$ *in* $\mathbf{C}$,

(a) $f_{\mathbf{D}}^{-1}(F(\mathscr{S}_0)) = F(f_{\mathbf{C}}^{-1}(\mathscr{S}_0))$,

(b) $\sup_{\mathbf{D}}\{F(\mathscr{S}_i)\colon i\in I\} = F(\sup_{\mathbf{C}}\{\mathscr{S}_i\colon i\in I\})$,

then the full subcategory $\mathbf{D}_1'$ *whose objects have the form* $F(\mathscr{S})$ *is strongly topological with the operations*

$$f_{\mathbf{D}_1'}^{-1}(\mathscr{T}_0) = f_{\mathbf{D}}^{-1}(\mathscr{T}_0), \quad \sup_{\mathbf{D}_1'}\{\mathscr{T}_i\colon i\in I\} = \sup_{\mathbf{D}}\{\mathscr{T}_i\colon i\in I\},$$

and $\mathbf{D}_1'$ *is bireflective in every full subcategory* $\mathbf{D}''$ *of* $\mathbf{D}$ *such that* $\mathbf{D}_1'\subset\mathbf{D}''\subset\mathbf{D}$. □

(0.6) LEMMA. *If, under the hypotheses of* (0.5), $\mathbf{C}'$ *and* $\mathbf{D}'$ *are full subcategories of* $\mathbf{C}$ *and* $\mathbf{D}$, *respectively, and* $\mathscr{S}^q$ *is defined for every structure in* $\mathbf{C}$, $\mathscr{T}^r$ *for every structure in* $\mathbf{D}$ *so that* (0.3) (a)—(c) *are fulfilled for* $\mathbf{C}$, $\mathscr{S}^q$, *and* $\mathbf{C}'$, *and similar conditions hold for* $\mathbf{D}$, $\mathscr{T}^r$, *and* $\mathbf{D}'$, *if further*

(a) $F(\mathscr{S}^q) = F(\mathscr{S})^r$,

(b) $f_{\mathbf{D}}^{-1}(F(\mathscr{S}^q)) = f_{\mathbf{D}}^{-1}(F(\mathscr{S}))^r$

for every structure $\mathscr{S}$ *in* $\mathbf{C}$, *then the full subcategories* $\mathbf{D}_1'$ *and* $\mathbf{D}_2'$ *of* $\mathbf{D}$ *whose objects are the structures* $F(\mathscr{S})$ *where* $\mathscr{S}$ *is a structure in* $\mathbf{C}$ *and* $\mathbf{C}'$, *respectively, are strongly topological with the operations*

$$f_{\mathbf{D}_1'}^{-1}(\mathscr{T}_0) = f_{\mathbf{D}}^{-1}(\mathscr{T}_0), \quad \sup_{\mathbf{D}_1'}\{\mathscr{T}_i\colon i\in I\} = \sup_{\mathbf{D}}\{\mathscr{T}_i\colon i\in I\},$$

$$f_{\mathbf{D}_2'}^{-1}(\mathscr{T}_0) = f_{\mathbf{D}}^{-1}(\mathscr{T}_0), \quad \sup_{\mathbf{D}_2'}\{\mathscr{T}_i\colon i\in I\} = (\sup_{\mathbf{D}}\{\mathscr{T}_i\colon i\in I\})^r,$$

and $\mathbf{D}_2'$ *is bicoreflective in every full subcategory* $\mathbf{D}''$ *of* $\mathbf{D}$ *such that* $\mathbf{D}_2'\subset\mathbf{D}''\subset\mathbf{D}_1'$. □

For a set X, we denote by $\exp X$ the set of all subsets of X. A *non-empty* subset of $\exp X$ is called a *system of sets in* X. A system of sets $\mathfrak{a}$ in X is said to be *ascending in* X iff $A\in\mathfrak{a}$, $A\subset B\subset X$ implies $B\in\mathfrak{a}$. For $\mathfrak{a}\subset\exp X$, we set

$$\sec\mathfrak{a} = \{B\in\exp X\colon B\cap A\neq\emptyset \quad \text{for} \quad A\in\mathfrak{a}\}$$

(more precisely, one should write $\sec_X\mathfrak{a}$). Clearly $\sec\sec\mathfrak{a}$ is the smallest ascending system in X that contains $\mathfrak{a}$, hence $\sec\sec\mathfrak{a}=\mathfrak{a}$ iff $\mathfrak{a}$ is ascending in X. If $\mathfrak{a}$ and $\mathfrak{b}$ are systems of sets in X, we set

$$\mathfrak{a}|\mathfrak{b} = \{A\cap B\colon A\in\mathfrak{a}, B\in\mathfrak{b}\}.$$

If $B\subset X$, we write $\mathfrak{a}|B$ instead of $\mathfrak{a}|\{B\}$; $\mathfrak{a}|B$ is said to be the *trace* of $\mathfrak{a}$ in B.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two systems of sets in X, we write $\mathfrak{a}<\mathfrak{b}$, iff for every $A\in\mathfrak{a}$, there is a $B\in\mathfrak{b}$ such that $B\subset A$. In this case we say that $\mathfrak{a}$ is *coarser* than $\mathfrak{b}$, or $\mathfrak{b}$ is *finer* than $\mathfrak{a}$. If $\mathfrak{b}$ is ascending in X, then $\mathfrak{a}<\mathfrak{b}$ iff $\mathfrak{a}\subset\mathfrak{b}$. We write $\mathfrak{a}\sim\mathfrak{b}$ iff $\mathfrak{a}<\mathfrak{b}$ and $\mathfrak{b}<\mathfrak{a}$. If $\mathfrak{a}$ is a system of sets in X, and $f\colon X\to Y$, $g\colon Z\to X$, we set

$$f(\mathfrak{a}) = \{fA)\colon A\in\mathfrak{a}\}, \quad g^{-1}(\mathfrak{a}) = \{g^{-1}(A)\colon A\in\mathfrak{a}\}.$$

A system of sets $\mathfrak{a}$ in X is said to be *fixed* or *free* according as $\bigcap\mathfrak{a}\neq\emptyset$ or $\bigcap\mathfrak{a}=\emptyset$; $\mathfrak{a}$ is said to be *fixed at* $x\in X$ iff $x\in\bigcap\mathfrak{a}$.

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1. Semi-topogenous orders. If $f: X \to Y$ and $<'$ is a semi-topogenous order on Y, then a semi-topogenous order $f^{-1}(<')$ on X is defined according to [2], (6.2). If $<$ is a semi-topogenous order on X, let us call f $(<, <')$-*continuous* iff $f^{-1}(<') \subset <$.

(1.1) THEOREM. *The semi-topogenous orders and the* $(<, <')$*-continuous maps constitute a (properly fibred topological, hence a) strongly topological category* **Stpg** *with the operations* $f^{-1}(<')$ *defined in* [2], (6.2) *and*

$$\sup\{<_i: i \in I\} = \bigcup_{i \in I} <_i$$

for $I \neq \emptyset$,

$$\sup\{<_i: i \in I\} = <_{\emptyset, X}$$

for $I = \emptyset$.

PROOF. [2], (6.3) and (6.5) show that **Stpg** is a concrete category. It is strongly topological by [2], (6.2), (2.4), (6.3), (6.7) and (cf. [2], (6.19))

$$g^{-1}(<_{\emptyset, X}) = <_{\emptyset, Z}$$

for $g: Z \to X$. □

(1.2) THEOREM. *Let* **Tpg**, **Pstpg**, **Ptpg**, **Symstpg**, **Symtpg** *denote the full subcategories of* **Stpg** *whose objects are the topogenous orders, perfect semi-topogenous orders, perfect topogenous orders, symmetric semi-topogenous orders, symmetric topogenous orders, respectively. Then each of them is a bicoreflective subcategory of any full subcategory of* **Stpg** *in which it is contained, with the respective coreflections* $<^q$, $<^p$, $<^{qp}$, $<\cup<^c$, $<^s$ *of* $<$. *Each of these categories is strongly topological with* $f^{-1}(<')$ *defined in* [2], (6.2) *and*

$$\sup\{<_i: i \in I\} = <^q, <^p, <^{qp}, <, <^q,$$

respectively, where $< = \bigcup_{i \in I} <_i$.

PROOF. (0.3) can be applied according to [2], (3.6), (6.10); (4.4), (6.13); (4.17)' (6.10), (6.13); (2.15), (6.7), (6.8), (2.12); (3.35), (6.12), (2.12), (3.41); taking into account that $<_{\emptyset, X}$ is topogenous, perfect, symmetric on X. □

Let us recall that a semi-topogenous order $<$ is said to be *idempotent* iff $< \subset <^2$.

(1.3) THEOREM. *In each of the categories* **Stpg**, **Tpg**, **Pstpg**, **Ptpg**, **Symstpg**, **Symtpg**, *the idempotent objects constitute a bireflective subcategory that is strongly topological.*

PROOF. The hypotheses of (0.2) are fulfilled by [2], (6.5), (6.18),

$$\bigcup_{i \in I} <_i^2 \subset \left(\bigcup_{i \in I} <_i\right)^2,$$

and [2], (3.53), (4.23), taking into account that $<_{\emptyset, X}$ is idempotent on X. □

2. Semi-closures. Let X be a set and $c: \exp X \to \exp X$. Consider the following conditions:

C1. $c(\emptyset) = \emptyset$,
C2. $c(A) \supset A$ for $A \subset X$,
C3. $A \subset B \subset X$ implies $c(A) \subset c(B)$,
C4. $c(A \cup B) = c(A) \cup c(B)$ for $A, B \subset X$,
C5. $c(c(A)) = c(A)$ for $A \subset X$.

If c fulfils C1, C2, C3, it is said to be a *semi-closure* on X; if it fulfils C4, too, it is a *closure* on X ([1], 14.A.1); a c fulfilling C1 to C5 is a *topological closure* or a topology ([1], 15.A.1).

If c and c' are semi-closures on X and Y, respectively, a map $f: X \to Y$ is said to be (c, c')-*continuous* iff

$$f(c(A)) \subset c'(f(A)) \quad \text{for} \quad A \subset X$$

([1], 16.A.1 for closures). It would be easy to check directly that the semi-closures and (c, c')-continuous maps constitute a (properly fibred topological, hence) strongly topological category **Scl**. However, we can reduce this problem to our results on semi-topogenous orders.

(2.1) LEMMA. *Let c be a semi-closure on X and set, for $A, B \subset X$,*

$$A <' B \quad \textit{iff} \quad c(A) \subset B.$$

Then $<'$ is a semi-topogenous order on X such that $<_c = <'^c$ is perfect. Conversely, if $<$ is a perfect semi-topogenous order on X, define, for $A \subset X$, $c_<(A)$ to be the intersection of all sets B such that $A <^c B$. Then $c_<$ is a semi-closure on X, and

$$c_{<_c} = c, \quad <_{c_<} = <,$$

whenever c is a semi-closure, $<$ is a perfect semi-topogenous order on X.

PROOF. $<'$ is clearly a semi-topogenous order, and

$$A = \bigcap_{i \in I} A_i, \quad B = \bigcap_{i \in I} B_i, \quad A_i <' B_i \quad \text{for} \quad i \in I$$

imply

$$c(A) \subset \bigcap_{i \in I} c(A_i) \subset \bigcap_{i \in I} B_i = B,$$

whence $<'^c$ is perfect.

If $<$ is a perfect semi-topogenous order on X, then $c_<$ is easily seen to be a semi-closure on X, and $A <^c c_<(A)$ for $A \subset X$ so that $A <^c B$ iff $c_<(A) \subset B$.

Now if c is a semi-closure, then

$$c_{<_c}(A) = \bigcap \{B \subset X: A <_c^c B\} = \bigcap \{B \subset X: c(A) \subset B\} = c(A),$$

and if $<$ is a perfect semi-topogenous order, then

$$A <_{c_<} B \Leftrightarrow c_<(X - B) \subset X - A \Leftrightarrow X - B <^c X - A \Leftrightarrow A < B. \quad \square$$

(2.2) LEMMA. *Let c, c' be semi-closures on X and Y, respectively, and $f: X \to Y$. Then f is (c, c')-continuous iff it is $(<_c, <_{c'})$-continuous.*

PROOF. $Af^{-1}(<_{c'})B$ means $f(A) <_{c'} Y - f(X - B)$ by [2], (6.2), whence, if f is (c, c')-continuous, then $f(X - B) <_{c'}^c Y - f(A)$, $f(c(X - B)) \subset c'(f(X - B)) \subset$

$\subset Y-f(A)$, $c(X-B)\subset X-A$, $A<_c B$. Thus $f^{-1}(<_{c'})\subset <_c$ and f is $(<_c, <_{c'})$-continuous.

Conversely, if f is $(<_c, <_{c'})$-continuous, then $f(A)<^c_{c'} B'$ implies

$$f(A)<^c_{c'} Y-f\big(X-f^{-1}(B')\big),$$

whence

$$f\big(X-f^{-1}(B')\big)<_{c'} Y-f(A)=Y-f\big(X-(X-A)\big),$$

$$X-f^{-1}(B')<_c X-A,\ A<^c_c f^{-1}(B'),\ c(A)\subset f^{-1}(B'),\ f\big(c(A)\big)\subset B'.$$

Thus $f\big(c(A)\big)\subset c'\big(f(A)\big)$, and f is (c, c')-continuous. □

(2.3) COROLLARY. *The functor* F *defined by* $F(c)=<_c$ *according to* (0.4) *induces an isomorphism from* **Scl** *onto* **Pstpg**. □

Another description of $c_<$ is given by the following:

(2.4) LEMMA. *If* $<$ *is a perfect semi-topogenous order on* X, $x\in X$, $A\subset X$, *then*

(a) $x\in c_<(A)$ *iff* $V\cap A\neq\emptyset$ *whenever* $x<V$,

i.e.

(b) $x\notin c_<(A)$ *iff* $x<X-A$.

PROOF. $x<V\subset X-A$ implies $A\subset X-V<^c X-\{x\}$, hence $c_<(A)\subset X-\{x\}$, $x\notin c_<(A)$. On the other hand, $x\notin c_<(A)$ implies

$$c_<(A)\subset X-\{x\},\ A<^c X-\{x\},\ x<X-A$$

so that $x<V$, $V\cap A=\emptyset$ for $V=X-A$. □

(2.5) THEOREM. *The category* **Scl** *is (properly fibred topological, hence) strongly topological. If* $f: X\to Y$, c' *is a semi-closure on* Y, $c=f^{-1}(c')$, *then*

$$x\in c(A)\quad \textit{iff}\quad f(x)\in c'\big(f(A)\big)\quad (x\in X,\ A\subset X).$$

If c_i *is a semi-closure on* X *for* $i\in I\neq\emptyset$, *and* $c=\sup\{c_i: i\in I\}$, *then*

$$c(A)=\bigcap_{i\in I} c_i(A)\quad \textit{for}\quad A\subset X.$$

For $c=\sup\emptyset$, $c(\emptyset)=\emptyset$, $c(A)=X$ *if* $A\neq\emptyset$.

PROOF. For $c=f^{-1}(c')=c_<$, $<=f^{-1}(<_{c'})$, we have by (2.4) $x\notin c(A)$ iff $x<X-A$, i.e. iff

$$f(x)<_{c'} Y-f\big(X-(X-A)\big)=Y-f(A),$$

thus iff $f(x)\notin c'\big(f(A)\big)$.

Now let $c=c_<$ for $<=\big(\bigcup_{i\in I}<_{c_i}\big)^p$, $I\neq\emptyset$. Then by (2.4) $x\notin c(A)$ iff $x<X-A$, i.e. iff ([2], (4.7)) $x<_{c_i}X-A$ for at least one $i\in I$, hence iff $x\notin c_i(A)$ for some i.

For $c=c_<$, $<=<_{\emptyset,X}$, $A\neq\emptyset$, $x<_{\emptyset,X}X-A$ does not hold for any $x\in X$, hence $c(A)=X$. □

(2.6) THEOREM. ([3], (5.2)). *A semi-closure* c *is a closure iff* $<_c$ *is a (perfect) topogenous order.*

PROOF. Assume c is a closure and $A_i <_c^c B_i$, $i=1,2$. Then $c(A_i)\subset B_i$, $c(A_1\cup A_2)=c(A_1)\cup c(A_2)\subset B_1\cup B_2$, thus $A_1\cup A_2 <_c^c B_1\cup B_2$. Since $<_c$ is perfect, $A_1\cap A_2 <_c^c B_1\cap B_2$ is obvious for the same sets A_i, B_i. Hence $<_c^c$ is topogenous and so is $<_c$ by [2], (3.17), (3.20).

On the other hand, if $<_c$ is topogenous, then $x\notin c(A)\cup c(B)$ implies by (2.4) $x<X-A$, $x<X-B$, hence

$$x < (X-A)\cap(X-B) = X-(A\cup B), \quad x\notin c(A\cup B).$$

Thus $c(A\cup B)\subset c(A)\cup c(B)$. The opposite inclusion follows from C3. □

(2.7) COROLLARY. *The full subcategory of* **Scl** *the objects of which are the closures is a category* **Cl** *isomorphic to* **Ptpg**, *the isomorphism is defined by* $F(c)=<_c$. *Hence* **Cl** *is bicoreflective in* **Scl** *and strongly topological.*

PROOF. (2.6) and (1.2). □

(2.8) THEOREM. *The coreflection* c^q *in* **Cl** *of a semi-closure* c *is defined by*

$$x\in c^q(A) \quad \textit{iff} \quad A=\bigcup_1^n A_i,\ n\in N \quad \textit{implies} \quad x\in c(A_i) \quad \textit{for some} \quad i.$$

PROOF. By (2.7) $c^q=c_{<_c^q}$. Now by (2.4) $x\notin c^q(A)$ iff $x<_c^q X-A$, i.e. iff $X-A=\bigcap_1^n B_i$ and $x<_c B_i$ for every i (cf. [2], (3.6)), hence iff there are $n\in\mathbf{N}$ and sets A_i such that $A=\bigcup_1^n A_i$ and $x<_c X-A_i$, or in other words $x\notin c(A_i)$ $(i=1,\ldots,n)$. □

(2.9) THEOREM. *A semi-closure* c *is a topology iff* $<_c$ *is (perfect,) topogenous and idempotent.*

PROOF. For a topology c and $A<_c B$, we have $X-B<_c^c X-A$, hence $c(X-B)\subset X-A$, or

$$c(X-B)\subset c(X-B)=c\big(c(X-B)\big)\subset X-A, \quad X-B<_c^c C<_c^c X-A$$

for $C=c(X-B)$. Thus $A<_c X-C<_c B$, and $<_c\subset<_c^2$.

Conversely $<_c\subset<_c^2$ implies that, for $x\notin c(A)$, we have $x<_c X-A$, hence $x<_c V<_c X-A$ for some V, and $c(A)\subset X-V$, $x\notin c(X-V)$, $x\notin c\big(c(A)\big)\subset c(X-V)$. Therefore $c\big(c(A)\big)\subset c(A)$ and the opposite inclusion is contained in C3. □

(2.10) COROLLARY. *The category* **Top** *of topologies and continuous maps is a bireflective subcategory of* **Cl** *and it is strongly topological.*

PROOF. (2.9) and (1.3). □

The content of (2.10) is well-known, of course (cf. [1], 14.A.9, 16.A.6); if c' is the reflection in **Top** of a closure c, then a set A is c'-closed iff $c(A)=A$. c'-closed and c'-open sets are said to be c-closed and c-open, respectively.

3. Semi-proximities. Let X be a set and δ a relation on $\exp X$. Consider the following conditions:

P1. $A\delta B$ implies $B\delta A$,

P2. $\emptyset\bar{\delta}X$ ($\bar{\delta}$ means non-δ),

P3. $A\cap B\neq\emptyset$ implies $A\delta B$ for $A, B\subset X$,

P4. $A\delta B$, $A\subset A'\subset X$, $B\subset B'\subset X$ implies $A'\delta B'$,

P5. $A\cup B\delta C$ implies $A\delta C$ or $B\delta C$.

A relation δ satisfying P1—P4 is said to be a *semi-proximity* on X; it is a *proximity* on X iff it fulfils P5, too ([1], 25.A.1).

If δ and δ' are semi-proximities on X and Y, respectively, a map $f: X\to Y$ is said to be (δ, δ')-*continuous* iff $A\delta B$ implies $f(A)\delta' f(B)$ ([1], 25.A.7 for proximities).

In order to see that the semi-proximities and (δ, δ')-continuous maps constitute a strongly topological category, we can use again the results on semi-topogenous orders.

(3.1) LEMMA. *If δ is a semi-proximity on X, then*

$$A<_\delta B \quad \text{iff} \quad A\bar{\delta}X-B$$

defines a symmetric semi-topogenous order on X. Conversely, if $<$ is a symmetric semi-topogenous order on X, then

$$P\bar{\delta}_< Q \quad \text{iff} \quad P< X-Q$$

defines a semi-proximity $\delta_<$ on X. We have

$$\delta_{<\delta}=\delta, \quad <_{\delta_<}=<$$

for any semi-proximity δ and any symmetric semi-topogenous order $<$ on X, respectively.

PROOF. The straightforward argument is contained in [2], pp. 63—64. □

(3.2) LEMMA. *If δ and δ' are semi-proximities on X and Y, respectively, then $f: X\to Y$ is (δ, δ')-continuous iff it is $(<_\delta, <_{\delta'})$-continuous.*

PROOF. If f is (δ, δ')-continuous, then

$$A<_{f^{-1}(<_{\delta'})}B$$

implies

$$f(A)<_{\delta'} Y-f(X-B), \quad f(A)\,\bar{\delta}'f(X-B),$$

hence $A\bar{\delta}X-B$, $A<_\delta B$ so that f is $(<_\delta, <_{\delta'})$-continuous.

Conversely if f is $(<_\delta, <_{\delta'})$-continuous, then $f(P)\bar{\delta}'f(Q)$ implies

$$f(P)<_{\delta'} Y-f(Q) = Y-f(X-(X-Q)),$$

hence $P<_\delta X-Q$, $P\bar{\delta}Q$, so that f is (δ, δ')-continuous. □

(3.3) COROLLARY. *The category* **Sprox** *of semi-proximities and (δ, δ')-continuous maps is isomorphic to the category* **Symstpg**, *the isomorphy is established with the*

help of the functor $F(\delta)=<_\delta$. *Hence* **Sprox** *is (properly fibred topological, consequently) strongly topological.* □

(3.4) LEMMA. *In the category* **Sprox**, *we have, for* $f: X\to Y$ *and a semi-proximity* δ' *on* Y,

$$Pf^{-1}(\delta')Q \quad \textit{iff} \quad f(P)\delta' f(Q)$$

for $P, Q\subset X$. *If* δ_i *is a semi-proximity on* X *for* $i\in I$, *and* $\delta=\sup\{\delta_i: i\in I\}$, *then*

$$P\delta Q \quad \textit{iff} \quad P\delta_i Q \ \textit{ for every } \ i\in I$$

provided $I\neq\emptyset$, *and*

$$P\delta Q \quad \textit{iff} \quad P\neq\emptyset\neq Q$$

for $I=\emptyset$.

PROOF. If $<'=<_{\delta'}$, $<=f^{-1}(<')$, $\delta=\delta_<$, then $P\bar{\delta}Q$ iff $P<X-Q$, hence iff $f(P)<'Y-f(Q)$, i.e. iff $f(P)\bar{\delta}'f(Q)$.

If $I\neq\emptyset$, $<_i=<_{\delta_i}$, $<=\bigcup_{i\in I}<_i$, $\delta=\delta_<$, then $P\bar{\delta}Q$ iff $P<X-Q$, hence iff $P<_i X-Q$ for some $i\in I$, i.e. iff $P\bar{\delta}_i Q$ for at least one i. On the other hand, $P\bar{\delta}_{<_{\mathfrak{o},X}}Q$ iff $P<_{\mathfrak{o},X}X-Q$, hence iff $P=\emptyset$ or $Q=\emptyset$. □

(3.5) LEMMA ([3], p. 49). *A semi-proximity* δ *is a proximity iff* $<_\delta$ *is a (symmetric) topogenous order.* □

(3.6) COROLLARY. *The full subcategory* **Prox** *of* **Sprox** *whose objects are the proximities is isomorphic to* **Symtpg**, *hence strongly topological and bicoreflective in* **Sprox**.

PROOF. (3.3), (3.5), (1.2). □

(3.7) LEMMA. *For a semi-proximity* δ *on* X, *the coreflection* δ^q *in* **Prox** *is defined by* $A\delta^q B$ *iff, for* $A=\bigcup_1^m A_i$, $B=\bigcup_1^n B_j$, $m, n\in\mathbf{N}$, *there are* i *and* j *such that* $A_i\delta B_j$.

PROOF. For $\delta^q=\delta_<$, $<=<_\delta^q$, we have $A\bar{\delta}^q B$ iff $A<X-B$, hence, by [2], (3.6), iff

$$A=\bigcup_1^m A_i,\ X-B=\bigcap_1^n (X-B_j),\ A_i<X-B_j$$

for every i, j. □

(3.8) COROLLARY. *In the category* **Prox**, *if* δ_k *is a proximity on* X *for* $k\in I\neq\emptyset$, *we have, for* $\delta=\sup\{\delta_k: k\in I\}$, $A\delta B$ *iff, for* $A=\bigcup_1^m A_i$, $B=\bigcup_1^n B_j$, $m, n\in\mathbf{N}$, *there are* i, j *such that* $A_i\delta_k B_j$ *for every* $k\in I$.

PROOF. (0.3), (3.4), (3.7). □

It is well-known that proximities induce closures ([1], 25.A.1). More generally, semi-proximities induce semi-closures; this is the consequence of the fact that a (symmetric) semi-topogenous order $<$ induces a perfect semi-topogenous order $<^p$ ([2], (4.4)).

(3.9) THEOREM. *If δ is a semi-proximity on X, then $c_\delta = c_{\prec_\delta^p}$ is a semi-closure on X given by*

$$x \in c_\delta(A) \quad \text{iff} \quad \{x\}\,\delta A.$$

If $g: Z \to X$, then $g^{-1}(c_\delta) = c_{g^{-1}(\delta)}$. If δ_i is a semi-proximity on X for $i \in I$, then

$$\sup\{c_{\delta_i}: i \in I\} = c_\delta \quad \text{for} \quad \delta = \sup\{\delta_i: i \in I\}.$$

PROOF. c_δ is a semi-closure by (3.1) and (2.1). By

(2.4) $$x \notin c_\delta(A) \quad \text{iff} \quad x \prec_\delta^p X - A \quad \text{iff} \quad x \prec_\delta X - A \quad \text{iff} \quad \{x\}\,\bar{\delta} A.$$

By (2.5) and (3.4) $x \in g^{-1}(c_\delta)(A)$ iff $g(x) \in c_\delta(g(A))$ iff $\{g(x)\}\delta g(A)$ iff $\{x\} g^{-1}(\delta) A$ iff $x \in c_{g^{-1}(\delta)}(A)$.

Let us denote $\delta = \sup\{\delta_i: i \in I\}$, $c = \sup\{c_{\delta_i}: i \in I\}$. Then $c = c_\delta$ if $I = \emptyset$, because then $x \in c(A)$ iff $A \neq \emptyset$ iff $\{x\}\delta A$ iff $x \in c_\delta(A)$ by (2.5) and (3.4). In the case $I \neq \emptyset$, $x \in c(A)$ iff $x \in c_{\delta_i}(A)$ for $i \in I$ iff $\{x\}\delta_i A$ for $i \in I$ iff $\{x\}\delta A$ iff $x \in c_\delta(A)$, by (2.5) and (3.4) again.

(3.10) THEOREM. *If δ is a semi-proximity on X, then $c_{\delta^q} = (c_\delta)^q$. If δ is a proximity then c_δ is a closure. For a proximity δ on X and $g: Z \to X$, we have*

$$g^{-1}_{\mathrm{Cl}}(c_\delta) = c_{g^{-1}_{\mathbf{Prox}}(\delta)} = c_{g^{-1}_{\mathbf{Sprox}}(\delta)} = g^{-1}_{\mathrm{Scl}}(c_\delta).$$

If δ_i is a proximity on X for $i \in I$, $\delta = \sup_{\mathbf{Prox}}\{\delta_i: i \in I\}$, then

$$\sup_{\mathrm{Cl}}\{c_{\delta_i}: i \in I\} = c_\delta.$$

PROOF. By (3.9), (3.7) and (2.8), $x \in c_{\delta^q}(A)$ iff $\{x\}\delta^q A$ iff $A = \bigcup_1^n A_i$, $n \in \mathbf{N}$ implies $\{x\}\delta A_i$ for some i iff the same condition implies $x \in c_\delta(A_i)$ for some i iff $x \in (c_\delta)^q(A)$.

If δ is a proximity, then by (3.7) $\delta = \delta^q$, hence $c_\delta = c_{\delta^q} = (c_\delta)^q$, and c_δ is a closure by (3.7) and (2.8).

The equalities concerning g^{-1} follow from (1.1), (1.2), (2.3), (2.6), (3.3), (3.6). The same statements furnish, together with (3.9), and [2], (8.4), (8.9), (3.41), (2.12), the equality for sup. □

The fact that c_δ is a closure provided δ is a proximity is well-known, of course ([1], 25.A.1).

We say that a semi-proximity *induces* the semi-closure c, or that δ is *compatible* with c, iff $c = c_\delta$. The question arises which semi-closures can be induced by semi-proximities. In order to answer this question, we first solve the corresponding problem for semi-topogenous orders.

(3.11) THEOREM. *If $\prec'$ is a symmetric semi-topogenous order on X, then $\prec = \prec'^p$ satisfies the following condition:*

SO. $$x, y \in X, \quad x \prec X - \{y\} \quad \text{implies} \quad y \prec X - \{x\}.$$

Conversely, if $\prec$ is a perfect semi-topogenous order, define $\prec' = \prec \cap \prec^c$. Then $\prec'$ is a symmetric semi-topogenous order such that $\prec'^p \subset \prec$; $\prec'^p = \prec$ iff $\prec$ satisfies SO, *and then $\prec'$ is the finest of all symmetric semi-topogenous orders $\prec''$ satisfying $\prec = \prec''^p$. If $\prec$ is topogenous then so is $\prec'$.*

PROOF. If $<'$ is a symmetric semi-topogenous order and $<=<'^p$, then $x, y\in X$, $x<X-\{y\}$ implies $x<'X-\{y\}$, hence $y<'X-\{x\}$ and $y<X-\{x\}$.

Let now $<$ be a perfect semi-topogenous order on X, and $<'=<\cap<^c$. By [2], (2.6), (2.4), (2.14), $<'$ is a symmetric semi-topogenous order obviously coarser than $<$, so that $<'^p\subset<$. Now $A<B$ implies $x<B$ for $x\in A$, hence $X-B<^cX-\{x\}$ and $y<^cX-\{x\}$, $x<X-\{y\}$ for $y\in X-B$. If $<$ satisfies SO, we have $y<X-\{x\}$ for $x\in A$, $y\in X-B$, hence $X-B<X-\{x\}$ because $<$ is perfect, therefore $x<^cB$. Thus $A<B$ implies $x<'B$ for $x\in A$, and $A<'^pB$, so that $<=<'^p$.

If $<''$ is a symmetric semi-topogenous order such that $<=<''^p$, then $A<''B$ implies $A<B$ and $X-B<''X-A$, hence $X-B<X-A$ and $A<^cB$, so that $<''\subset<'$.

If $<$ is a topogenous order, then $<\cap<^c$ is topogenous as well by [2], (3.4) and (3.5). □

(3.12) COROLLARY. *If δ is a semi-proximity on X, then the semi-closure c induced by δ satisfies the condition*

SC. $\qquad x\in c(\{y\})$ *implies* $y\in c(\{x\})$ *for* $x, y\in X$.

Conversely, if c is a semi-closure, then δ_c^* defined by

$$A\delta_c^*B \quad \text{iff} \quad A\cap c(B)\neq\emptyset \quad \text{or} \quad c(A)\cap B\neq\emptyset$$

is a semi-proximity on X such that $c_{\delta_c^*}$ is coarser than c; δ_c^* is compatible with c iff c satisfies SC, *and then δ_c^* is the finest semi-proximity that induces c. If c is a closure then δ_c^* is a proximity.*

PROOF. For $c=c_<$, $<=<_\delta^p$, SC clearly says the same as SO, and for $\delta_c^*=\delta_{<'}$, $<'=<_c\cap<_c^c$, δ_c^* can be described in the above manner:

$$A\overline{\delta_c^*}B\Leftrightarrow A<'X-B\Leftrightarrow A<_cX-B \quad \text{and} \quad A<_c^cX-B\Leftrightarrow$$

$$\Leftrightarrow B<_c^cX-A \quad \text{and} \quad A<_c^cX-B\Leftrightarrow c(B)\subset X-A \quad \text{and} \quad c(A)\subset X-B.$$

If c is a closure, then $<_c$ and $<'$ are topogenous by (2.6) and (3.11), and δ_c^* is a proximity by (3.5). □

(3.13) LEMMA. *If δ and δ' are semi-proximities on X and Y, respectively, and $f: X\to Y$ is (δ, δ')-continuous then it is $(c_\delta, c_{\delta'})$-continuous as well.*

PROOF. (3.2), [2], (6.13) and (4.11), and (2.2) of the present paper. □

(3.14) COROLLARY. *$F(\delta)=c_\delta$ defines a covariant functor from* **Sprox** *onto the full subcategory of* **Scl** *whose objects are the semi-closures satisfying* SC. *F carries* **Prox** *onto the intersection of* **Cl** *and this subcategory.* □

The condition SC is not always fulfilled; for a topology c, SC holds iff this topology satisfies the separation axiom S_1 ([4], p. 93, or axiom R_0 in [7]). The topology is T_1 iff it is T_0 and S_1, hence a T_0-space that is not T_1 certainly does not satisfy S_1, i.e. its topology does not fulfil SC.

If a semi-closure c satisfies a condition somewhat stronger than SC, we can construct another compatible semi-proximity, in general distinct from δ_c^*:

(3.15) THEOREM. *Let c be a semi-closure on X, and define δ_c by*

$$A\delta_c B \quad \text{iff} \quad c(A)\cap c(B)\neq\emptyset.$$

Then δ_c is a semi-proximity such that c_{δ_c} is coarser than c. δ_c is compatible with c iff c satisfies

SC′ $\qquad x\notin c(A) \quad \text{implies} \quad c(\{x\})\cap c(A)=\emptyset \quad \text{for} \quad x\in X, \quad A\subset X.$

If c is a closure then δ_c is a proximity.

PROOF. δ_c is obviously a semi-proximity and $x\in c(A)$ implies $\{x\}\delta_c A$, i.e. $x\in c_{\delta_c}(A)$ for $x\in X$, $A\subset X$. δ_c is compatible with c iff, conversely, $x\notin c(A)$ implies $\{x\}\bar{\delta}_c A$ for $x\in X$, $A\subset X$, which is precisely SC′. If c is a closure then δ_c clearly fulfils P5. □

SC′ obviously implies SC (put $A=\{y\}$). The converse is not true in general.

(3.16) EXAMPLE. Let $X=\mathbf{R}^2$, and, for $z_0=(x_0, y_0)\in X$, $\varepsilon>0$,

$$V_\varepsilon(z_0)=\{(x,y)\in X:\ x=x_0 \quad \text{or} \quad y=y_0 \quad \text{and} \quad |x-x_0|<\varepsilon\}.$$

Define $z\in c(A)$ iff $V_\varepsilon(z)\cap A\neq\emptyset$ for every $\varepsilon>0$. Then c is a closure on X. c satisfies SC because, for $z_0=(x_0, y_0)$, $z_1=(x_1, y_1)$, clearly

$$z_0\in c(\{z_1\}) \Leftrightarrow x_0=x_1 \Leftrightarrow z_1\in c(\{z_0\}).$$

However, for $z_0=(0,0)$, $A=\{(x,1):\ x>0\}$, $z_1=(0,1)$, we have $z_0\notin c(A)$, $z_1\in c(\{z_0\})\cap c(A)$, so that c fails to satisfy SC′. □

In some important cases, SC implies SC′. A semi-closure c on X is said to be *separated* iff $c(\{x\})=\{x\}$ for $x\in X$. Now, a separated semi-closure obviously fulfils SC′. Somewhat less obvious is the following:

(3.17) THEOREM. *If a topology is S_1, i.e. if it fulfils* SC, *then it satisfies* SC′ *as well.*

PROOF. For an S_1-topology c on X and $x, y\in X$, $y\in c(\{x\})\cap c(A)$ implies $x\in c(\{y\})\subset c(c(A))=c(A)$. □

Now we investigate some conditions under which a semi-proximity induces a semi-closure fulfilling SC′.

(3.18) LEMMA. *For a semi-proximity δ on X, c_δ satisfies* SC′ *iff δ fulfils:*

SP′ $\qquad \{x\}\delta A \quad \text{implies} \quad c_\delta(\{x\})\cap c_\delta(A)=\emptyset \quad \text{for} \quad x\in X, \quad A\subset X.$ □

A semi-proximity δ on X is said to be *separated* iff $\{x\}\delta\{y\}$ implies $x=y$ for $x, y\in X$. δ is separated iff c_δ is separated, so that a separated semi-proximity fulfils SP′.

A stronger condition than SP′ is, for a semi-proximity δ on X:

SP″ $\qquad A\bar{\delta}B$ implies $c_\delta(A)\cap c_\delta(B)=\emptyset$ for $A, B\subset X$.

(3.19) LEMMA. *A semi-proximity δ on X satisfies* SP″ *iff it satisfies* SP′ *and it is coarser than δ_{c_δ}.*

PROOF. A δ satisfying SP″ obviously fulfils SP′, and SP″ precisely says that $A\bar{\delta}B$ implies $A\bar{\delta}_{c_\delta}B$. □

SP″ is strictly stronger than SP′:

(3.20) EXAMPLE. Let c denote the usual topology of the real line; then δ_c^* satisfies SP′ by (3.19), (3.17), and (3.18), but it is strictly finer than δ_c. □

In this example, δ_c^* is obviously separated. On the other hand, if $X=\mathbf{R}^2$ and c is the topology induced by the pseudo-metric $\sigma(z_1, z_2)=|x_1-x_2|$ for $z_i=(x_i, y_i)$, then δ_c is a proximity satisfying SP″ without being separated.

4. Separation axioms for proximities. Separatedness, SP′, SP″ are some kinds of separation axioms for semi-proximities. For proximities, it is worthwhile to formulate some more conditions.

(4.1) LEMMA. *For a proximity δ on X, c_δ is a topology iff δ satisfies*

TP $\qquad \{x\}\delta A \quad \textit{implies} \quad \{x\}\bar{\delta}c_\delta(A).$

PROOF. Since $\{x\}\bar{\delta}c_\delta(A)$ implies $\{x\}\bar{\delta}A$, TP says that $c_\delta(c_\delta(A))=c_\delta(A)$. □

Let us observe that TP obviously implies SP′: $y\in c_\delta(A)$ and $\{x\}\bar{\delta}c_\delta(A)$ yield $\{x\}\bar{\delta}\{y\}$, $y\notin c_\delta(\{x\})$. On the other hand, TP and SP″ are independent of each other.

(4.2) EXAMPLE. Let $X=\mathbf{R}^2$, $z_0=(x_0, y_0)\in X$, and, for $\varepsilon>0$,

$$V_\varepsilon(z_0) = \{(x, y)\in X: |x-x_0| < \varepsilon \text{ and } y = y_0 \text{ or } |y-y_0| < \varepsilon \text{ and } x = x_0\}.$$

Set $z\in c(A)$ iff $V_\varepsilon(z)\cap A\neq\emptyset$ for every $\varepsilon>0$ ($z\in X$, $A\subset X$). Then c is a separated closure so that, by (3.19) and (3.15), δ_c satisfies SP″. On the other hand, c is not a topology: for $z=(0, 0)$, $A=\{(x, y): x>0, y>0\}$, we have $z\notin c(A)$, $z\in c(c(A))$. □

Conversely, Example (3.20) presents a TP-proximity that is not SP″.

A condition obviously stronger than both TP and SP″ is contained in [17]:

LO $\qquad A\bar{\delta}B \quad \text{implies} \quad c_\delta(A)\bar{\delta}c_\delta(B) \quad \text{for} \quad A, B\subset X.$

If a topology has a compatible (semi-)proximity, then it has a compatible LO-proximity; in fact, the topology is S_1 by (3.12), and then δ_c is compatible by (3.17) and (3.15), further δ_c clearly is an LO-proximity (and the finest compatible LO-proximity by (3.19)). On the other hand, if δ_c^* is strictly finer than δ_c, then δ_c^* is TP without being an LO-proximity.

There are also separated SP″- and TP-proximities that are not LO-proximities:

(4.3) EXAMPLE. Let c be the usual topology on $X=\mathbf{R}$, and set

$$A\bar{\delta}B \quad \text{iff} \quad c(A)\cap c(B)=\emptyset \quad \text{and either } A \text{ or } B \text{ is countable.}$$

δ is obviously a semi-proximity; it is a proximity because, if $A\bar{\delta}C$, $B\bar{\delta}C$, then $c(A\cup B)\cap c(C)=\emptyset$ and either C is countable or both A and B are so. Clearly $x\notin c(A)$ iff $\{x\}\bar{\delta}A$ so that δ is compatible with c. Since δ is coarser than δ_c, δ is SP″ by (3.15)—(3.19). Also δ is TP by (4.1). However, if $A=[0, 1]$, $B=\mathbf{Q}\cap(2, 3)$, we have $A\bar{\delta}B$ but $c_\delta(A)\bar{\delta}c_\delta(B)$. □

A further condition has been introduced in [10]:

R $\{x\}\bar{\delta}A$ implies the existence of a C such that $\{x\}\bar{\delta}X-C$, $C\bar{\delta}A$.

It is shown in [10] that the topology c_δ is regular whenever δ is an R-proximity. Conversely, if c is a regular topology, then δ_c^* and δ_c are compatible R-proximities ([5], (3.2) and (3.5)). On the other hand, a proximity inducing a regular topology need not be an R-proximity: consider Example (4.3); for $x=0$, $A=X-(-1, 1)$, we have $\{x\}\bar{\delta}A$, but $\{x\}\bar{\delta}X-C$ implies that C contains an interval containing 0, hence it is uncountable, and so $C\delta A$. If c is a T_3-topology such that $\delta_c^*\neq\delta_c$ (e.g. if c is the usual topology of $\mathbf{R}$), then δ_c^* is a separated R-proximity without being SP″.

An axiom stronger than R characterizes the classical Efremovich proximities ([8]):

EF $A\bar{\delta}B$ implies the existence of a C such that $A\bar{\delta}X-C$, $C\bar{\delta}B$.

An EF-proximity induces a completely regular topology, and every completely regular topology admits a compatible EF-proximity ([19], (3.14) and (2.10)).

Two more kinds of proximities can be interpolated between R- and EF-proximities. For this purpose, we recall that, if δ is an EF-proximity on X, there is a compact S_2-extension (Y, c') of the topological space (X, c_δ) such that

$$A\delta B \quad \text{iff} \quad c'(A)\cap c'(B)\neq\emptyset \tag{4.6}$$

([21]; [4], (6.4.8) and (6.4.4)). Now a proximity δ on X is said to be an RE-*proximity* iff there is a regular extension (Y, c') of (X, c_δ) such that (4.6) is fulfilled ([5]), and it is said to be an RC-*proximity* iff the same is true for a regular-closed extension (Y, c') (a regular topology is said to be *regular-closed* iff every regular filter is fixed; a *regular filter* is a filter that is open and closed; a filter is *open (closed)* iff it is generated by a filter base composed of open (closed) sets). Since an RE-proximity is obviously LO, we have the following diagram for proximities:

$$\begin{array}{ccccccc} \text{EF} & \Rightarrow & \text{RC} & \Rightarrow & \text{RE} & \Rightarrow & \text{LO} \Rightarrow \text{SP}'' \\ & & & & \Downarrow & & \Downarrow \quad \Downarrow \\ & & & & \text{R} & \Rightarrow & \text{TP} \Rightarrow \text{SP}' \end{array}$$

None of the above implications can be reversed. Moreover, there is a separated R- and SP″-proximity that is not LO:

(4.7) EXAMPLE. Let $X=\mathbf{R}$, δ_0 the EF-proximity on X induced by the Euclidean metric, and

$A\bar{\delta}B$ iff $A\bar{\delta}_0 B$ and either A or B contains a finite number of integers only.

Then δ is a proximity on X. It is clearly separated, R and SP″ (because $c_\delta=c_{\delta_0}$); however, if

$$A=\bigcup\{(4k, 4k+1):\ k\in\mathbf{N}\},\quad B=\bigcup\{(4k+2, 4k+3):\ k\in\mathbf{N}\},$$

then $A\bar{\delta}B$ but $c_\delta(A)\delta c_\delta(B)$, so that δ is not LO. □

In [5], (8.3), there is an example of an R- and LO-proximity that is not RE.

Further separation axioms for proximities are introduced and discussed in [6].

5. Subcategories of Prox. By (3.4) and (3.8), the fundamental operations in the strongly topological category **Prox** are defined as follows:

$$Af^{-1}(\delta')B \quad \text{iff} \quad f(A)\delta' f(B)$$

whenever $f\colon X \to Y$ and δ' is a proximity on Y, and if δ_k is a proximity on X for $k \in K$ and

$$\delta = \sup \{\delta_k\colon k \in K\},$$

then $A\bar{\delta}B$ iff $A = \bigcup_1^m A_i$, $B = \bigcup_1^n B_j$, $m, n \in \mathbf{N}$, and there is, for every pair (i, j), a $k \in K$ such that $A_i \bar{\delta}_k B_j$.

In particular,

$$c_{f^{-1}(\delta')}(A) = f^{-1}(c_{\delta'}(f(A))),$$

$x \in c_\delta(A)$ iff $A = \bigcup_1^m A_i$ implies $x \in \bigcap_{k \in K} c_{\delta_k}(A_i)$ for at least one i whenever $\delta = \sup \{\delta_k\colon k \in K\}$, consequently

$$c_\delta(A) \subset \bigcap_{k \in K} c_{\delta_\alpha}(A).$$

With the help of these formulas, a straightforward calculation shows according to (0.2):

(5.1) THEOREM. *Let* **Q—Prox** *denote the full subcategory of* **Prox** *the objects of which are the proximities satisfying the condition* Q *where* Q *is one of the following conditions:*

$$\text{SP}' \quad \text{TP}, \quad \text{R}, \quad \text{SP}'', \quad \text{LO}, \quad \text{EF}.$$

Then $f^{-1}(\delta')$ *is a structure in* **Q—Prox** *provided* δ' *is a structure in* **Q—Prox**, *and* $\sup \{\delta_k\colon k \in K\}$ *is a structure in* **Q—Prox** *provided each* δ_k *is a structure in* **Q—Prox**. *Consequently* **Q—Prox** *is a bireflective subcategory of every full subcategory* **C** *of* **Prox** *such that* **Q—Prox** $\subset$ **C** $\subset$ **Prox**. □

Most complicated is the calculation in the case of the formula for sup and Q=EF; this can be found e.g. in [4], (3.1.21).

The case of RE-proximities is more delicate and will be discussed later (see Section 8).

In general, the reflections corresponding to the subcategories **Q—Prox** can be obtained according to (0.2). Moreover, we can say:

(5.2) THEOREM. *If* δ *is a separated* TP*-proximity on* X, *then the* **LO—Prox** *reflection* δ' *if* δ *is given by*

$$A\delta' B \quad \textit{iff} \quad c_\delta(A)\,\delta\, c_\delta(B),$$

and $c_{\delta'} = c_\delta$.

PROOF. δ' is obviously a proximity coarser than δ. Hence $\{x\}\delta A$ implies $\{x\}\delta' A$. Conversely, if $\{x\}\bar{\delta}A$ then $x \notin c_\delta(A) = c_\delta(c_\delta(A))$, $\{x\}\bar{\delta}c_\delta(A)$, $c_\delta(\{x\}) = \{x\}$, hence

$\{x\}\overline{\delta'}A$. Therefore $c_{\delta'}=c_\delta$, and δ' is LO:

$$A\overline{\delta'}B \Rightarrow c_\delta(A)\,\overline{\delta}c_\delta(B) \Rightarrow c_\delta\big(c_\delta(A)\big)\,\overline{\delta}c_\delta\big(c_\delta(B)\big) \Rightarrow$$
$$\Rightarrow c_\delta\big(c_{\delta'}(A)\big)\,\overline{\delta}c_\delta\big(c_{\delta'}(B)\big) \Rightarrow c_{\delta'}(A)\,\overline{\delta'}\,c_{\delta'}(B).$$

If δ'' is an LO-proximity coarser than δ, then $A\overline{\delta''}B$ implies $c_{\delta''}(A)\overline{\delta''}c_{\delta''}(B)$, hence $c_\delta(A)\overline{\delta}c_\delta(B)$ by $c_\delta(C)\subset c_{\delta''}(C)$ for $C\subset X$, so that δ'' is coarser than δ'. □

Rather surprisingly, separatedness is quite essential in (5.2):

(5.3) EXAMPLE. Let $X\subset\mathbf{R}^2$ be defined by

$$X=\{(0,0)\}\cup\{(x,y)\in\mathbf{R}^2\colon x\neq 0\},$$

and let c be the topology induced by the pseudo-metric

$$\sigma\big((x,y),(x',y')\big)=|x-x'|.$$

Define δ by

$$A\delta B \quad\text{iff}\quad c(A)\cap c(B)=\emptyset$$

and either A or B is finite. Then δ is a proximity compatible with c.

If we define δ' like in (5.2), i.e.

$$A\delta' B \quad\text{iff}\quad c(A)\delta c(B),$$

then $\{z_0\}\overline{\delta'}\{z_1\}$ for $z_0=(0,0)$, $z_1=(1,0)$, because $c\big(\{z_0\}\big)=\{z_0\}$, $c\big(\{z_1\}\big)=E_1$ where

$$E_a=\{(x,y)\in\mathbf{R}^2\colon x=a\}\quad(a\in\mathbf{R}),$$

and $\{z_0\}\overline{\delta}E_1$. On the other hand, $x\neq 0$ implies $c\big(\{(x,y)\}\big)=E_x$ and $E_x\delta E_1$ so that $(x,y)\in c_{\delta'}\big(\{z_1\}\big)$ for $x\neq 0$, and $c_{\delta'}\big(\{z_0\}\big)\delta' c_{\delta'}\big(\{z_1\}\big)$ because

$$z_0\in c\big(c_{\delta'}\big(\{z_0\}\big)\big)\cap c\big(c_{\delta'}\big(\{z_1\}\big)\big).$$

Thus $\{z_0\}\overline{\delta'}\{z_1\}$ although $c_{\delta'}\big(\{z_0\}\big)\delta' c_{\delta'}\big(\{z_1\}\big)$, and δ' is not an LO-proximity.

δ' is not even compatible with c; in fact, for $z_2=(2,0)$, $z_2\notin c\big(\{z_1\}\big)$ but $z_2\in c_{\delta'}\big(\{z_1\}\big)$.

The **LO—Prox**-reflection of δ is not compatible with c either. Moreover, any LO-proximity δ'' coarser than δ is incompatible with c: $z_2\notin c_{\delta''}\big(\{z_1\}\big)$ would imply $c_{\delta''}\big(\{z_2\}\big)\overline{\delta}''c_{\delta''}\big(\{z_1\}\big)$, hence $c_\delta\big(\{z_2\}\big)\overline{\delta}c_\delta\big(\{z_1\}\big)$ although $E_2\delta E_1$. □

6. Screens. We introduce a new topological category, strongly related to a known one, that turns out to be useful in the study of some kinds of proximities.

Let X be a set. A collection $\mathfrak{S}$ of filters in X such that, for every $x\in X$, there is a filter $\mathfrak{s}\in\mathfrak{S}$ fixed at x, will be called a *screen* on X. If $\mathfrak{S}$ and $\mathfrak{S}'$ are screens on X and Y, respectively, a map $f\colon X\to Y$ will be said to be $(\mathfrak{S},\mathfrak{S}')$-*continuous* iff, for any $\mathfrak{s}\in\mathfrak{S}$, there is $\mathfrak{s}'\in\mathfrak{S}'$ such that $\mathfrak{s}'<f(\mathfrak{s})$.

Observe that, for screens as systems of sets in $\exp X$, the relation "finer" derived from this continuity is distinct from the one defined for systems of sets. In the case $X=\emptyset$, $\{\emptyset\}$ is to be considered as a filter in X so that $\big\{\{\emptyset\}\big\}$ is a screen on $\emptyset$.

(6.1) THEOREM. *The screens and* $(\mathfrak{S},\mathfrak{S}')$-*continuous maps constitute a concrete category* **Scr**. □

(6.2) LEMMA. *Let $\mathfrak{S}'$ be a screen on Y and $f: X\to Y$. The filters in X generated by filter bases of the form $f^{-1}(\mathfrak{s}')$, $\mathfrak{s}'\in\mathfrak{S}'$, constitute a screen $f^{-1}(\mathfrak{S}')$ on X such that f is $(f^{-1}(\mathfrak{S}'), \mathfrak{S}')$-continuous, and $f^{-1}(\mathfrak{S}')$ is coarser than every screen $\mathfrak{S}$ scuh that f is $(\mathfrak{S}, \mathfrak{S}')$-continuous.*

PROOF. A system $f^{-1}(\mathfrak{s}')$, where $\mathfrak{s}'\in\mathfrak{S}'$, is a filter base iff $f(X)\in\operatorname{sec}\mathfrak{s}'$. For $x\in X$, there is a filter $\mathfrak{s}'\in\mathfrak{S}'$ fixed at $f(x)$, hence $f^{-1}(\mathfrak{s}')$ is a filter base in this case and it generates a filter fixed at x. Hence $f^{-1}(\mathfrak{S}')$ is a screen on X.

If the filter $\mathfrak{s}\in f^{-1}(\mathfrak{S}')$ is generated by $f^{-1}(\mathfrak{s}')$, $\mathfrak{s}'\in\mathfrak{S}'$, then $\mathfrak{s}'<f(\mathfrak{s})$ so that f is $(f^{-1}(\mathfrak{S}'), \mathfrak{S}')$-continuous.

If $\mathfrak{S}$ is a screen on X such that f is $(\mathfrak{S}, \mathfrak{S}')$-continuous, then $\mathfrak{s}\in\mathfrak{S}$ implies the existence of $\mathfrak{s}'\in\mathfrak{S}'$ such that $\mathfrak{s}'<f(\mathfrak{s})$; now $f^{-1}(\mathfrak{s}')<f^{-1}(f(\mathfrak{s}))<\mathfrak{s}$, so that $f^{-1}(\mathfrak{S}')$ is coarser than $\mathfrak{S}$. □

(6.3) LEMMA. *If $\mathfrak{S}'$ is a screen on Y, $f: X\to Y$, $g: Z\to X$, then $g^{-1}(f^{-1}(\mathfrak{S}'))=h^{-1}(\mathfrak{S}')$ for $h=f\circ g$.* □

(6.4) LEMMA. *Let $\mathfrak{S}_i$ be a screen on X for $i\in I$, $\mathfrak{S}_i\neq\mathfrak{S}_j$ for $i\neq j$, and $\mathfrak{S}$ denote $\{\{X\}\}$ if $I=\emptyset$, and, for $I\neq\emptyset$, the collection of all filters on X generated by centred systems* (i.e. systems having the finite intersection property) *of the form $\bigcup_{i\in I}\mathfrak{s}_i$ where $\mathfrak{s}_i\in\mathfrak{S}_i$. Then $\mathfrak{S}=\sup\{\mathfrak{S}_i: i\in I\}$ is a screen on X, finer than every $\mathfrak{S}_i$, and coarser than every screen on X finer than every $\mathfrak{S}_i$.*

PROOF. The case $I=\emptyset$ is obvious. If $I\neq\emptyset$, $x\in X$, let $\mathfrak{s}_i\in\mathfrak{S}_i$ be a filter fixed at x. Then $\bigcup_{i\in I}\mathfrak{s}_i$ is a centred system fixed at x so that it generates a filter fixed at x. Hence $\mathfrak{S}$ is a screen on X.

If $\mathfrak{s}$ is the filter generated by $\bigcup_{i\in I}\mathfrak{s}_i$, $\mathfrak{s}_i\in\mathfrak{S}_i$, then $\mathfrak{s}_i\subset\mathfrak{s}$, hence $\mathfrak{S}$ is finer than every $\mathfrak{S}_i$. If $\mathfrak{S}'$ is finer than every $\mathfrak{S}_i$ and $\mathfrak{s}'\in\mathfrak{S}'$, let $\mathfrak{s}_i\in\mathfrak{S}_i$ be coarser than $\mathfrak{s}'$. Then $\bigcup_{i\in I}\mathfrak{s}_i$ is centred because a finite intersection of its elements contains a subset that is a finite intersection of elements of $\mathfrak{s}'$; hence $\bigcup_{i\in I}\mathfrak{s}_i$ generates a filter $\mathfrak{s}\in\mathfrak{S}$ coarser than $\mathfrak{s}'$, and $\mathfrak{S}$ is coarser than $\mathfrak{S}'$. □

(6.5) LEMMA. *Let $\mathfrak{S}_i$ be a screen on X for $i\in I$, $g: Z\to X$. Then*

$$\sup\{g^{-1}(\mathfrak{S}_i): i\in I\} = g^{-1}(\sup\{\mathfrak{S}_i: i\in I\}).$$

PROOF. The case $I=\emptyset$ is obvious. If $I\neq\emptyset$, choose $\mathfrak{s}_i\in\mathfrak{S}_i$, let $\mathfrak{s}$ be the filter generated by $\bigcup_{i\in I}\mathfrak{s}_i$ (or $\exp X$ if the latter is not centred), $\mathfrak{s}_i'$ the filter generated by $g^{-1}(\mathfrak{s}_i)$ (or $\exp Z$ if $g^{-1}(\mathfrak{s}_i)$ is not a filter base), $\mathfrak{s}'$ the filter generated by $\bigcup_{i\in I}\mathfrak{s}_i'$ (or $\exp Z$ if this system is not centred), $\mathfrak{s}''$ the filter generated by $\bigcup_{i\in I}g^{-1}(\mathfrak{s}_i)$ (or $\exp Z$ in the case of a non-centred system). Clearly $\mathfrak{s}_i'\sim g^{-1}(\mathfrak{s}_i)$ for $i\in I$, hence

$$\bigcup_{i\in I}\mathfrak{s}_i' \sim \bigcup_{i\in I}g^{-1}(\mathfrak{s}_i), \quad \mathfrak{s}'\sim\mathfrak{s}''\sim g^{-1}(\mathfrak{s}),$$

so that $\mathfrak{s}'$ is the filter generated by $g^{-1}(\mathfrak{s})$ iff the latter is a filter base. □

(6.6) COROLLARY. *The category* **Scr** *is strongly topological.* □

Let us now establish a connection between the categories **Scr** and **Prox**.

(6.7) THEOREM. *Let* $\mathfrak{S}$ *be a screen on* X, *and define, for* $P, Q \subset X$,

$P \delta Q$ *iff there is a filter* $\mathfrak{s} \in \mathfrak{S}$ *such that* $P, Q \in \operatorname{sec} \mathfrak{s}$,

and, for $A, B \subset X$,

$A < B$ *iff* $A \in \operatorname{sec} \mathfrak{s}$, $\mathfrak{s} \in \mathfrak{S}$ *implies* $B \in \mathfrak{s}$.

Then δ *is a proximity on* X, *and* $<$ *is the symmetric topogenous order associated with* δ (i.e. $< = <_\delta$).

PROOF. P3 is valid because $x \in P \cap Q$ implies $P, Q \in \operatorname{sec} \mathfrak{s}$ for $\mathfrak{s} \in \mathfrak{S}$ fixed at x. P5 follows from the fact that $P \cup Q \in \operatorname{sec} \mathfrak{s}$ iff $P \in \operatorname{sec} \mathfrak{s}$ or $Q \in \operatorname{sec} \mathfrak{s}$ (whenever $\mathfrak{s}$ is a filter). P1, P2, P4 are obvious.

Clearly, $<$ defined above coincides with $<_\delta$ (see (3.1)). □

We shall say that $\mathfrak{S}$ *induces* the proximity δ and the symmetric topogenous order $<$ defined in (6.7), and we write $\delta = \delta(\mathfrak{S})$, $< = <(\mathfrak{S})$. We also say that $\mathfrak{S}$ is *compatible* with δ or $<$.

(6.8) THEOREM. *If* $\mathfrak{S}$ *and* $\mathfrak{S}'$ *are screens on* X *and* Y, *respectively, and* $f: X \to Y$ *is* $(\mathfrak{S}, \mathfrak{S}')$*-continuous, then it is* $(\delta(\mathfrak{S}), \delta(\mathfrak{S}'))$*-continuous as well.*

PROOF. If $P \delta(\mathfrak{S}) Q$, and $P, Q \in \operatorname{sec} \mathfrak{s}$ for a filter $\mathfrak{s} \in \mathfrak{S}$, then choose $\mathfrak{s}' \in \mathfrak{S}'$ such that $\mathfrak{s}' < f(\mathfrak{s})$. Clearly $f(P), f(Q) \in \operatorname{sec} \mathfrak{s}'$. □

The question arises whether, conversely, every proximity can be induced by a screen. In order to answer this question, let us say, if δ is a semi-proximity on X, $<$ is the symmetric semi-topogenous order associated with δ, and $\mathfrak{a}$ is a system of sets in X, that $\mathfrak{a}$ is *compressed* with respect to δ or $<$ (briefly δ-compressed or $<$-compressed) iff $P, Q \in \operatorname{sec} \mathfrak{a}$ implies $P \delta Q$, or equivalently, iff $A < B$, $A \in \operatorname{sec} \mathfrak{a}$ implies $X - B \notin \operatorname{sec} \mathfrak{a}$ (for EF-proximities and filter bases see [4], p. 186; also [2], p. 224). If $\mathfrak{a}$ is δ-compressed and $\mathfrak{a} < \mathfrak{b}$, then $\mathfrak{b}$ is δ-compressed as well. If $f: X \to Y$ is (δ, δ')-continuous and $\mathfrak{a}$ is δ-compressed, then $f(\mathfrak{a})$ is δ'-compressed: if $P, Q \subset Y$, $P, Q \in \operatorname{sec} f(\mathfrak{a})$, then $f^{-1}(P), f^{-1}(Q) \in \operatorname{sec} \mathfrak{a}$, hence $f^{-1}(P) \delta f^{-1}(Q)$, $f(f^{-1}(P)) \delta' f(f^{-1}(Q))$, and $f(f^{-1}(P)) \subset P$, $f(f^{-1}(Q)) \subset Q$.

(6.9) LEMMA. *If* $\mathfrak{S}$ *is a screen on* X, *then every* $\mathfrak{s} \in \mathfrak{S}$ *is* $\delta(\mathfrak{S})$*-compressed.* □

Now we can prove the following important lemma:

(6.10) LEMMA. *If* δ *is a proximity on* X, $P \delta Q$, *then there exists a* δ*-compressed filter* $\mathfrak{s}$ *such that* $P, Q \in \operatorname{sec} \mathfrak{s}$.

PROOF. Consider the collection $\mathfrak{F}$ of all filters $\mathfrak{s}$ in X such that $S \cap P \delta S \cap Q$ for every $S \in \mathfrak{s}$. Since the union of a chain in $\mathfrak{F}$ belongs to $\mathfrak{F}$, and $\{\{X\}\} \in \mathfrak{F}$, by the Kuratowski—Zorn lemma, there is a filter $\mathfrak{s}$ that is maximal in $\mathfrak{F}$. Clearly $P, Q \in \operatorname{sec} \mathfrak{s}$.

We show that $\mathfrak{s}$ is δ-compressed. Assume indirectly $M, N \in \operatorname{sec} \mathfrak{s}$, $M \delta N$.

If $S \in \mathfrak{s}$ implies

$$S \cap P \cap (X - M) \, \delta S \cap Q \cap (X - M),$$

then $S\cap(X-M)\neq\emptyset$ for $S\in\mathfrak{s}$, hence $\mathfrak{s}|X-M$ is a filter base and generates a filter $\mathfrak{s}'\in\mathfrak{F}$, $\mathfrak{s}'\supset\mathfrak{s}$. By the maximality of $\mathfrak{s}$, we have $\mathfrak{s}'=\mathfrak{s}$, in particular $X\cap(X-M)\in\mathfrak{s}$, contrary to the assumption $M\in\operatorname{sec}\mathfrak{s}$. Thus there must exist a set $S_1\in\mathfrak{s}$ such that

$$S_1\cap P\cap(X-M)\,\bar{\delta}\,S_1\cap Q\cap(X-M).$$

Similarly $N\in\operatorname{sec}\mathfrak{s}$ implies the existence of $S_2\in\mathfrak{s}$ such that

$$S_2\cap P\cap(X-N)\,\bar{\delta}\,S_2\cap Q\cap(X-N).$$

For $S=S_1\cap S_2\in\mathfrak{s}$, we have

$$S\cap P\cap(X-M)\,\bar{\delta}\,S\cap Q\cap(X-M),\quad S\cap P\cap(X-N)\,\bar{\delta}\,S\cap Q\cap(X-N).$$

Define $P'=S\cap P$, $Q'=S\cap Q$ so that

$$P'\cap(X-M)\,\bar{\delta}\,Q'\cap(X-M),\quad P'\cap(X-N)\,\bar{\delta}\,Q'\cap(X-N).$$

The assumption $M\bar{\delta}N$ implies $M\cap N=\emptyset$, hence

$$P'\cap N\bar{\delta}Q'\cap(X-M),\quad P'\cap M\bar{\delta}Q'\cap(X-N),$$

further

$$P'\cap M\bar{\delta}Q'\cap N,\quad P'\cap N\bar{\delta}Q'\cap M.$$

Since δ satisfies P5, we have

$$P'\cap M\bar{\delta}Q',\quad P'\cap N\bar{\delta}Q',$$

$$P'\cap(X-M)\cap(X-N)\,\bar{\delta}\,(Q'\cap(X-M))\cup(Q'\cap(X-N))=Q',$$

finally, by P5 again, $P'\bar{\delta}Q'$, contrary to the assumption. □

(6.11) THEOREM. *If δ is a proximity on X, then the collection of all δ-compressed fielters is a screen $\mathfrak{S}$ on X such that $\delta=\delta(\mathfrak{S})$.*

PROOF. If $x\in X$, the trivial ultrafilter

$$\dot{x}=\{S\in\exp X\colon x\in S\}$$

is obviously δ-compressed so that $\mathfrak{S}$ is a screen. $\delta(\mathfrak{S})$ is finer than δ because every filter $\mathfrak{s}\in\mathfrak{S}$ is δ-compressed, and $\delta(\mathfrak{S})$ is coarser than δ by (6.10). □

Let us denote by $\mathfrak{S}(\delta)$ the screen described in (6.11).

In general, there exist several screens inducing the same proximity δ. In this direction, let us first observe:

(6.12) THEOREM. *If δ is a semi-proximity on X, then every δ-compressed filter $\mathfrak{s}$ contains a minimal δ-compressed filter $\mathfrak{s}_0$.*

PROOF. Let $\{\mathfrak{s}_i\colon i\in I\}$ be a chain of δ-compressed filters, and $\mathfrak{s}_0=\bigcap_{i\in I}\mathfrak{s}_i$. Then $\mathfrak{s}_0$ is a filter in X; we show that $\mathfrak{s}_0$ is δ-compressed. In fact, assume $P\bar{\delta}Q$, and, indirectly, $P, Q\in\operatorname{sec}\mathfrak{s}_0$. Since $X-P$, $X-Q\notin\mathfrak{s}_0$, there are $i,j\in I$ such that $X-P\notin\mathfrak{s}_i$, $X-Q\notin\mathfrak{s}_j$, so that $P\in\operatorname{sec}\mathfrak{s}_0$, $Q\in\operatorname{sec}\mathfrak{s}_j$. If, say, $\mathfrak{s}_i\subset\mathfrak{s}_j$, then $P, Q\in\operatorname{sec}\mathfrak{s}_i$ contrary to the fact that $\mathfrak{s}_i$ is δ-compressed. □

(6.13) THEOREM. *For an arbitrary proximity δ on X, the minimal δ-compressed filters constitute a screen $\mathfrak{S}_0$ such that $\delta=\delta(\mathfrak{S}_0)$.*

PROOF. Let $\mathfrak{S}$ be the screen composed of all δ-compressed filters; by (6.11), $\delta=\delta(\mathfrak{S})$. By (6.12), for $x\in X$, there is a filter $\mathfrak{s}_0\in\mathfrak{S}_0$ fixed at x so that $\mathfrak{S}_0$ is a screen. $\mathfrak{S}_0\subset\mathfrak{S}$ implies that $\mathfrak{S}_0$ is finer than $\mathfrak{S}$, and (6.12) says that $\mathfrak{S}_0$ is coarser than $\mathfrak{S}$. By (6.8) $\delta(\mathfrak{S}_0)=\delta(\mathfrak{S})$. □

Some types of proximities can be characterized by the fact that they admit special types of compatible screens. In order to establish a result of this kind, if δ is a proximity on X, let us agree in saying that $A\subset X$ is *δ-open* or *δ-closed* iff A is c_δ-open or c_δ-closed, respectively. A *filter* $\mathfrak{s}$ in X will be said to be *δ-open* or *δ-closed* iff it is generated by a filter base composed of δ-open or δ-closed sets, respectively. A *screen* $\mathfrak{S}$ on X will be said to be *open* iff its elements are $\delta(\mathfrak{S})$-open filters.

(6.14) LEMMA. *If $\mathfrak{S}$ is an open screen on X, then $\delta=\delta(\mathfrak{S})$ is an* LO*-proximity, hence c_δ is a topology, and any filter $\mathfrak{s}\in\mathfrak{S}$ fixed at $x\in X$ coincides with the c_δ-neighbourhood filter of x.*

PROOF. Suppose $\mathfrak{s}\in\mathfrak{S}$, $c_\delta(P)\in\operatorname{sec}\mathfrak{s}$. Then, for $S\in\mathfrak{s}$, there is a c_δ-open $G\in\mathfrak{s}$ such that $G\subset S$. $P\cap G=\emptyset$ would imply $P\subset X-G$, $c_\delta(P)\subset c_\delta(X-G)=X-G$, so that $P\cap G\neq\emptyset$, and, *a fortiori*, $P\cap S\neq\emptyset$. Hence $P\in\operatorname{sec}\mathfrak{s}$.

Similarly, $c_\delta(Q)\in\operatorname{sec}\mathfrak{s}$ implies $Q\in\operatorname{sec}\mathfrak{s}$. Consequently $c_\delta(P)\delta c_\delta(Q)$ implies $P\delta Q$, so that δ is an LO-proximity. By (4.1), c_δ is a topology.

Let $\mathfrak{s}\in\mathfrak{S}$ be a filter fixed at $x\in X$. Since $\mathfrak{s}$ is a δ-open, i.e. a c_δ-open filter, each $S\in\mathfrak{s}$ is a c_δ-neighbourhood of x. Conversely, if V is a c_δ-neighbourhood of x, then $\{x\}\delta X-V$, and $\{x\}\in\operatorname{sec}\mathfrak{s}$ implies $V\in\mathfrak{s}$. □

If $\mathfrak{s}$ is a filter in a topological space, then all neighbourhoods of all elements of $\mathfrak{s}$ constitute an open filter coarser than $\mathfrak{s}$, the *neighbourhood filter* of $\mathfrak{s}$.

(6.15) LEMMA. *If δ is an* LO*-proximity on X, and $\mathfrak{s}$ is a δ-compressed filter, then the c_δ-neighbourhood filter of $\mathfrak{s}$ is δ-compressed as well.*

PROOF. If $P\in\operatorname{sec}\mathfrak{s}'$ for the c_δ-neighbourhood filter $\mathfrak{s}'$ of $\mathfrak{s}$, then $c_\delta(P)\in\operatorname{sec}\mathfrak{s}$, otherwise $X-c_\delta(P)$ would belong to $\mathfrak{s}$ and to $\mathfrak{s}'$. Similarly $Q\in\operatorname{sec}\mathfrak{s}'$ implies $c_\delta(Q)\in\operatorname{sec}\mathfrak{s}$. Thus $P, Q\in\operatorname{sec}\mathfrak{s}'$ implies $c_\delta(P), c_\delta(Q)\in\operatorname{sec}\mathfrak{s}$, hence $c_\delta(P)\delta c_\delta(Q)$ and $P\delta Q$ because δ satisfies LO. □

(6.16) LEMMA. *If δ is an* LO*-proximity, then every minimal δ-compressed filter is δ-open.*

PROOF. By (6.15), a minimal δ-compressed filter coincides with its c_δ-neighbourhood filter. □

(6.17) THEOREM. *A proximity δ is* LO *iff there is an open screen compatible with δ.*

PROOF. If: (6.14). Only if: (6.13) and (6.16). □

Another type of proximities that can be characterized with the help of screens is that of RE-proximities. For this purpose, if δ is a semi-proximity on X, let us say that a filter $\mathfrak{s}$ in X is *δ-round* iff $S\in\mathfrak{s}$ implies the existence of $S_1\in\mathfrak{s}$ such that

$S_1 \bar{\delta} X - S$; if $\prec$ is the symmetric semi-topogenous order associated with δ, we can say equivalently that $S \in \mathfrak{s}$ implies the existence of $S_1 \in \mathfrak{s}$ such that $S_1 \prec S$ (see, for EF-proximities, [4], p. 250). If δ is TP, then a δ-round filter is clearly δ-open. Let us say that a *screen* $\mathfrak{S}$ is *round* iff every $\mathfrak{s} \in \mathfrak{S}$ is a $\delta(\mathfrak{S})$-round filter, i.e. iff $S_0 \in \mathfrak{s}_0 \in \mathfrak{S}$ implies the existence of $S_1 \in \mathfrak{s}_0$ such that $\mathfrak{s} \in \mathfrak{S}$, $S_1 \in \sec \mathfrak{s}$ implies $S_0 \in \mathfrak{s}$. It is easy to observe that a round screen is nothing else than a "preregular system of filters" in the terminology of [5]. Therefore (5.12) of [5] states that δ is an RE-proximity iff it admits a compatible round screen.

Further results on screens will follow in the second part of the paper.

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ON LOCALLY ORDER-POLYNOMIALLY COMPLETE MODULAR LATTICES

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1. Introduction

Let L be a lattice. $F_k(L)=L^{(L^k)}$ is the set of all k-place functions on L. If we define pointwise meet and join operation on $F_k(L)$, then $F_k(L)$ becomes a lattice. The elements of the sublattice $P_k(L)$ of $F_k(L)$ generated by the projections and the constant functions will be called k-place polynomial functions on L. If $f\in F_k(L)$ has the property that for every finite subset $M\subseteq L^k$ there exists a $p\in P_k(L)$ such that f and p coincide on M, then p is called a local polynomial function. $f\in F_k(L)$ is called order-preserving if $a_i\leqq b_i$, $i=1, ..., k$ implies $f(a_1, ..., a_k)\leqq f(b_1, ..., b_k)$. L is called (locally) order-polynomially complete iff every order-preserving function on L is a (local) polynomial function.

The first characterization of finite order-polynomially complete lattices was given in Wille [6]. For finite modular lattices he proved the following

THEOREM A. *A finite modular lattice L is (locally) order-polynomially complete if and only if L is simple and relatively complemented (i.e. an irreducible projective geometry).*

This theorem suggests the following question: is every locally order-polynomially complete modular lattice relatively complemented? In [2], Fried proved that the answer is yes if L has locally finite length. Our main result is a construction of a locally order-polynomially complete modular lattice wich is not relatively complemented. To prove that our example is locally order-polynomially complete we need the following useful result of Dorninger [1]:

THEOREM B. *A lattice L is locally order-polynomially complete if and only if* (1) *L is simple and* (2) *for all $a, b\in L$, $a\leqq b$ and all 1-place polynomial functions p, q with $p(b)=q(a)$ there exists a 1-place polynomial function r such that $r(a)\leqq p(a)$ and $r(b)\geqq q(b)$.*

It is an easy consequence of this theorem that every simple, relatively complemented lattice is locally order-polynomially complete (for finite lattices see Wille [7]).

2. Preliminaries

By a 1-translation of a lattice L we mean a unary polynomial-function on L that is either the identity function id (x) or a constant function or is obtained from one of the two lattice operations by fixing one of the arguments. By a translation of L we mean a unary polynomial that is the composition of 1-translations. Therefore if $t(x)$ is a translation but not a constant function then $t(x)$ may be written in the

form $t(x)=(\ldots((x\vee a_1)\wedge a_2)\vee a_3\ldots)\wedge a_n$ where each $a_i\in L$ or is the empty symbol. We say that a translation is a unary polynomial function of degree 1. If $f(x)$ and $g(x)$ are unary polynomial functions of degree n resp. m, then the degree of the functions $f(x)\vee g(x)$ and $f(x)\wedge g(x)$ is $n+m$.

If two intervals $[a, b]$ and $[c, d]$ in a lattice are such that $a=b\wedge c$ and $d=b\vee c$, then each is said to be transpose (perspective) of the other. $[a, b]$ and $[c, d]$ are said to be projective if there exists intervals $[a, b]=[x_0, y_0], [x_1, y_1],\ldots, [x_n, y_n]=[c, d]$ such that any two successive intervals are transposes of each other. In a modular lattice, any two projective intervals are isomorphic. A well-known property of modular lattices (see [3], p. 133) is expressed in the following

LEMMA 1. *Let $t(x)$ be a translation of a modular lattice L and let $a, b\in L$, $a<b$ such that $t(a)\neq t(b)$. Then there exists a proper subinterval $[a', b']$ of $[a, b]$ such that $[a', b']$ and $[t(a), t(b)]$ are projective.*

Let D be an arbitrary distributive lattice with 0 and 1. Take the subposet of D^3 consisting of all ordered triples (a, b, c) such that $a\wedge b=a\wedge c=b\wedge c$. This poset is a modular lattice $M_3[D]$ (see Schmidt [4]). The elements $i=(1, 1, 1)$, $u=(1, 0, 0,)$ $v=(0, 1, 0)$, $w=(0, 0, 1)$ and $o=(0, 0, 0)$ form a diamond, M_3. The interval $[0, u]$, i.e. the ideal $(u]$ is isomorphic to D. Similarly, $(v]\cong(w]\cong D$.

Let us take two bounded lattices L_1 and L_2. Suppose that L_1 has a principal dual ideal J_1, L_2 has a principal ideal J_2 and $J_1\cong J_2$. Let $\varphi: x\to x'$ denote this isomorphism. We can construct a lattice L as follows: L is the set of all $x\in L_1$ and $x\in L_2$; we identify x with x' for all $x\in J_1$; $x\leqq y$ has unchanged meaning if $x, y\in L_1$ or $x, y\in L_2$ and $x<y$, $x, y\notin J_1=J_2$ iff $x\in L_1$, $y\in L_2$ and there exists a $z\in J$ such that $x<z$ in L_1, and $z<y$ in L_2. It is easy to see that L is a modular lattice if so are L_1 and L_2. This is the so-called Hall—Dilworth construction.

3. Modular lattices of finite length

By Theorem B, a locally order-polynomially complete lattice is simple. A direct proof is the following (see Wille [7]): let Θ be a non trivial congruence relation of L and $a, b, c, d\in L$ such that $a\not\leqq b$, $c>d$, $(a, b)\in\Theta$ and $(c, d)\notin\Theta$. We define a mapping $f: L\to L$ by

$$f(x) := \begin{cases} c, & \text{if } a\leqq x \\ d, & \text{if } a\not\leqq x \end{cases}$$

then f is an order-preserving function and it cannot be a local polynomial function, namely $(a, b)\in\Theta$ but $(c, d)=(f(a), f(b))\notin\Theta$.

PROPOSITION. *Let $p(x)$ be a polynomial function on a modular lattice L of locally finite length. If the interval $[u, v]$ is a complemented sublattice then $[p(u), p(v)]$ is complemented, too.*

PROOF. We prove this statement by induction on the degree of $p(x)$. (The degree of $p(x)$ is the number of occurrences of the variable x; the constant function has degree 0.) If $p(x)$ has degree 1 (i.e. it is a translation) then by Lemma 1 $[p(u), p(v)]$ is isomorphic to a subinterval of a complemented modular lattice $[u, v]$, hence $[p(u), p(v)]$ is complemented. Assume that the assertion is proved for polynomials

of degree $<n$ ($n>1$), and let $p(x)$ be a polynomial function of degree n. Then $p(x)$ has one of the following two decompositions, $p(x)=q(x)\vee r(x)$ or $p(x)=q(x)\wedge r(x)$ where the degree of q and r is less than n. Denote $p(u)$ by s then $q'(x)=q(x)\vee s$ is a polynomial function and has the same degree as q. Then by our assumption $[q'(u), q'(v)]$ is complemented. On the other hand $q'(u)=q(u)\vee s=q(u)\vee p(u)=p(u)$ and $q'(v)=q(v)\vee s=q(v)\vee p(u)$, hence $[p(u), p(u)\vee q(v)]$ is a complemented interval of finite length. Then the unit element (i.e. $p(u)\vee q(v)$) is the join of atoms of this interval. Similarly, $p(u)\vee r(v)$ is the join of atoms in $[p(u), p(u)\vee r(v)]$. We claim that $p(v)=(p(u)\vee q(v))\vee(p(u)\vee r(v))$ is the join of atoms in $[p(u), p(v)]$, hence this interval is complemented.

THEOREM C (Fried [2]). *Let L be a modular lattice of locally finite length. L is locally order-polynomially complete iff each interval of L is an irreducible projective geometry.*

PROOF. If each interval is an irreducible projective geometry then L is a relatively complemented, simple lattice; hence L is locally order-polynomially complete.

Conversely, let us assume that L is locally order-polynomially complete. If L is not relatively complemented then L contains a triple a, b, c such that $a\prec b\prec c$ and b has no relative complement in $[a, c]$. Let $u\prec v$ be any two elements of L and define

$$f(x) := \begin{cases} a, & \text{if } x \leqq u \\ c, & \text{if } x \not\leqq u. \end{cases}$$

Then $f(x)$ is an order-preserving function and by the Proposition, f cannot be a local polynomial function. Let $[a, b]$ be an interval of L and let c_1, c_2 ($c_1\neq c_2$) be two atoms of $[a, b]$. To prove that $[a, b]$ is an irreducible projective geometry, it is enough to show that there exists an atom d of $[a, b]$ such that $d\neq c_1, c_2$ and $d\leqq c_1\vee c_2$. Since L is a simple modular lattice, the intervals $[a, c_1]$ and $[a, c_2]$ are projective in L, hence they are projective in some interval $[\bar{a}, \bar{b}]$, where $\bar{a}\leqq a\prec b\leqq\bar{b}$. The interval $[\bar{a}, \bar{b}]$ is again a complemented modular lattice of finite length, therefore $[\bar{a}, \bar{b}]$ is the direct product of irreducible projective geometries ([3], p. 212). The projectivity of $[a, c_1]$ and $[a, c_2]$ in $[\bar{a}, \bar{b}]$ yields that these intervals belong to the same irreducible component, i.e. $[a, c_1\vee c_2]$ is a subinterval of an irreducible projective geometry, therefore there exists a d with $a\prec d\prec c_1\vee c_2$.

4. The construction

We prove the following:

THEOREM D. *There exists a locally order-polynomially complete modular lattice which is not relatively complemented.*

Let Q be the interval $[0, 1]$ of rational numbers. First, we define two unary operations on Q:

$$f(x) := \begin{cases} 2x, & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x, \end{cases} \qquad g(x) := \begin{cases} 2x-1, & \text{if } \frac{1}{2} \leqq x \leqq 1 \\ 0, & \text{if } x < \frac{1}{2}, \end{cases}$$

and we consider the unary algebra $\mathbf{Q}=\langle Q; f, g\rangle$. Let id be the identity map on Q and define $f^0=g^0=\mathrm{id}$. Then apart from the constant maps the polynomial functions on $\mathbf{Q}$ are of the form $p(x)=g^{k_1}f^{l_1}g^{k_2}\ldots g^{k_r}f^{k_r}$, where $k_i\geqq 0$, $l_i\geqq 0$.

LEMMA 2. *To each* $a, b\in Q$, $a<b$ *there exists a* 1*-place polynomial function* $p(x)$ *on* $\mathbf{Q}$ *such that* $p(a)=0$ *and* $p(b)=1$.

PROOF. If $0\leqq a<b\leqq 1$ then for suitable k and n $(n\geqq 1,\ k=0, \ldots, 2^{n-1})$ $a\leqq\frac{k}{2^n}<\frac{k+1}{2^n}\leqq b$. Therefore if we have a $p(x)$ such that $p\left(\frac{k}{2^n}\right)=0$ and $p\left(\frac{k+1}{2^n}\right)=1$ then by the order-preserving property of polynomials we get $p(a)=0$, $p(b)=1$, i.e. we can assume that $a=\frac{k}{2^n}$ and $b=\frac{k+1}{2^n}$. We prove the lemma by induction on n. If $n=1$ then we have:

$$f\colon \left[0, \frac{1}{2}\right]\to[0, 1] \quad \text{and} \quad g\colon \left[\frac{1}{2}, 1\right]\to[0, 1].$$

Assume that the statement is proved for $n-1$. The following two cases arise:

(1) $$b=\frac{k+1}{2^n}\leqq\frac{1}{2}, \quad \text{then} \quad f(a)=\frac{k}{2^{n-1}}, \quad f(b)=\frac{k+1}{2^{n-1}}\leqq 1.$$

By our assumption there exists a polynomial function $p(x)$ such that $p\left(\frac{k}{2^{n-1}}\right)=0$, $p\left(\frac{k+1}{2^{n-1}}\right)=1$, thus $\bar{p}=pf$ satisfies $\bar{p}(a)=0$, $\bar{p}(b)=1$.

(2) $$a=\frac{k}{2^n}\geqq\frac{1}{2}, \quad \text{then} \quad g(a)=\frac{k}{2^{n-1}}-1, \quad g(b)=\frac{k+1}{2^{n-1}}-1$$

and we have a polynomial function $p(x)$ such that $p\left(\frac{k}{2^{n-1}}-1\right)=0$, $p\left(\frac{k+1}{2^{n-1}}-1\right)=1$. Then $\bar{p}=pg$ satisfies $\bar{p}(a)=0$ and $\bar{p}(b)=1$.

Now we consider the modular lattice $M_3[Q]$. The zero resp. unit of this lattice is denoted by o resp. i. $M_3[Q]$ has three elements u, v, w such that o, u, v, w, i form a diamond, M_3.

We take $M_3[Q]$ in three pairwise disjoint copies L_1, L_2 and L_3. Then o_k, u_k, v_k, w_k, i_k denote the elements corresponding to o, u, v, w, i by the isomorphism $M_3[Q]\cong L_k$.

$J_1=[u_1, i_1]$ is a principal dual ideal of L_1 and $J_1\cong Q$. Similarly, $J_2=[o_2, w_2]$ is a principal ideal of L_2 isomorphic to Q. Therefore $J_1\cong J_2$, we can apply the Hall—Dilworth construction and we get the following modular lattice $L_{1,2}$:

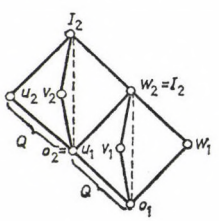

Fig. 1

Let S be the sublattice of L_{12} consisting of all elements $x \vee y$ where $x \leqq u_2$ and $y \leqq w_1$. The interval $[o_1, u_2]$ is isomorphic to Q hence $S \cong Q \times Q$. Similarly, L_3 contains a sublattice $T := \{x \vee y;\ x \leqq u_3,\ y \leqq w_3\}$ isomorphic to $Q \times Q$. Consequently we have an isomorphism $\varphi: S \to T$ with $\varphi(u_2) = u_3$, $\varphi(w_1) = w_3$. Now, we apply a gluing construction (similar to the Hall—Dilworth construction) by identifying the corresponding elements by φ. This construction was first defined in [4], see Fig. 2.

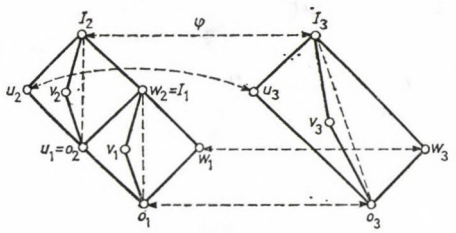

Fig. 2

In this way we get a modular lattice L.

L:

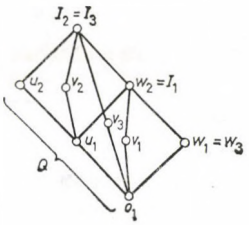

Fig. 3

To prove that L is modular, we have to show that L does not contain a pentagon generated by a, b, c. But L contains four sublattices generated by $\{u_2, v_1, x;\ 0 \leqq x \leqq v_3\}$, $\{u_2, w_1, x;\ 0 \leqq x \leqq v_3\}$, $\{v_2, v_1, x;\ 0 \leqq x \leqq v_3\}$ and $\{v_2, w_1, x;\ 0 \leqq x \leqq v_3\}$ which are all isomorphic to $M_3[Q]$, hence they are all modular sublattices. If L contains a pentagon generated by a, b, c then it is easy to see that a, b, c are contained in one of these sublattices, a contradiction.

The lattice L contains the diamonds $(o_k, u_k, v_k, w_k, i_k)$, $k = 1, 2, 3$. We identify $[o_1, u_2]$ with Q, and define two polynomial functions on L:

$$\bar{f}(x) = (((x \vee v_1) \wedge w_1) \vee v_3) \wedge u_2, \quad \bar{g}(x) = (((x \vee v_2) \wedge w_1) \vee v_3) \wedge u_2.$$

It is easy to show that the restrictions of these functions to Q are exactly the functions f and g defined above.

We prove that L is locally order-polynomially complete. Let $a, b \in L$, $a < b$. By the gluing construction there exists a $c \in L$ such that $a \leqq c \leqq b$ and $a, c \in L_{1,2}$, $c, b \in L_3$ or conversely $a, c \in L_3$, $c, b \in L_{1,2}$. On the other hand $a < b$ implies that either $a < c$ or $c < b$. If $a, c \in L_{1,2}$, $a < c$ then either $a \wedge u_2 < c \wedge u_2$ or $a \wedge w_1 < c \wedge w_1$. Similarly if $c, b \in L_3$, $c < b$ then either $c \wedge u_2 < b \wedge u_2$ or $c \wedge w_1 < b \wedge w_1$. The intervals

$[o_1, u_2]$ and $[o_1, w_1]$ are projective and therefore we have a 1-place polynomial function t on L such that $t(a)<t(b)\leqq u_2$. (The second case, $a, c\in L_3$ is similar.) By Lemma 2 there exists a polynomial function $s(x)$ satisfying $u_2=st(b)$, $o_1=st(a)$. In L we have the polynomial function $d(x)=(x\vee v_3)\wedge w_3$ which satisfies $d(u_2)=w_3$, $d(o_1)=o_1$. Finally let $r(x)=st(x)\vee dst(x)$, then we have

$$r(a) = st(a)\vee dst(a) = o_1\vee o_1 = o_1,$$

$$r(b) = st(b)\vee dst(b) = u_2\vee d(u_2) = u_2\vee w_3 = i_2.$$

We have to show that the conditions of Theorem B are satisfied. Indeed, let Θ be a congruence relation of L such that $a\equiv b(\Theta)$. Then we obtain $o_1=r(a)\equiv \equiv r(b)=i_2(\Theta)$, i.e. L is a simple lattice. If p, q are arbitary 1-place polynomial functions then with the given polynomial function $r(x)$ we get condition (2). The theorem is proved.

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ON THE HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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0. Introduction. In [3], I. Bihari defined the half-linear second order differential equations (D. Eqs)

$$(p(t)x')' + q(t)f(x, p(t)x') = 0 \tag{0.1}$$

for the unknown function $x=x(t)$ where the functions $p(t)$, $q(t)$ are continuous on some interval $I=[a, b)$ $(-\infty<a<b\leqq\infty)$, $p(t)>0$ and the function $f(x, y)$ satisfies the following relations:

(Bi) $f(x, y)$ is defined on R^2 and is Lipschitzian on every bounded domain in R^2,

(Bii) $xf(x, y)>0$ if $x\neq 0$ (consequently $f(0, y)=0$ for all $y\in R$),

(Biii) $f(\lambda x, \lambda y)=\lambda f(x, y)$ for all $\lambda\in R$, $(x, y)\in R^2$.

In a series of papers he studied the properties of the solutions of (0.1). For example, in [3], [4], [7] the basic qualitative properties have been proved, in [5], [10] a linear second order D. Eq. was perturbed by a half-linear term and the asymptotic behaviour was described, in [4], [9], [11] the Bôcher function and the Bôcher pairs — known for the linear second order differential equations — have been generalized to (0.1), and in [8] he found estimations on the eigenvalues and on the distances between the consecutive zeros of the solutions of (0.1).

The most characteristic requirement (Biii) finds its explanation in the Prüfer transformation technique where the polar functions $\varphi=\varphi(t)$, $\varrho=\varrho(t)$ are introduced by $x=\varrho\sin\varphi$, $px'=\varrho\cos\varphi$ and the relation (Biii) is necessary to derive a D. Eq. for φ independent of ϱ.

The restriction (Bi) ensures that the solution of any initial value problem posed at $t=t_0\in I$ exists, is unique and can be continued through the whole interval I.

The restriction (Bii) means that (0.1) has only one singular point, namely at $x=0$, $x'=0$. The sign condition $xf(x, y)>0$ is taken from the linear case $f(x, y)=x$.

The adjective "half-linear" wants to express the closeness of the qualitative properties of the solutions of (0.1) to the ones of the homogeneous linear second order D. Eqs. Among them the most characteristic is the following: if $x(t)$ is a solution of (0.1) and λ is a constant different from 0 then the function $\lambda x(t)$ is also a solution of (0.1).

In the lieterature we can find the nonlinear D. Eq.

$$(p(t)x'^{*}_{n})' + q(t)x^{*}_{n} = 0, \quad n > 0, \tag{0.2}$$

where $u^{*n}=|u|^n \operatorname{sgn} u$, as the Euler—Lagrangean differential equation for the minimum problem

$$\int_a^b \{p(t)|u'|^{n+1}-q(t)|u|^{n+1}\}\,dt \geqq 0$$

(see [1] and the references therein).

Let us observe that the differential equation (0.2) can be written as

$$(p^{1/n}x')'+\frac{1}{n}qp^{1/n-1}x^{*n}|x'|^{1-n}=0 \quad \text{if} \quad x'\neq 0. \tag{0.3}$$

Hence this equation has the form (0.1) with $f(x, y)=x^{*n}|y|^{1-n}$. If $n=1$ (0.3) (and also (0.2)) is the well-known linear Sturm—Liouville D. Eq. On the other hand $f(x, y)$ statisfies (Bii), (Biii) with the exception $y=0$. The restriction (Bi) is violated in any neighbourhood of $(0; 0)$. In spite of this in [14] we have proved that (0.3) enjoys almost every properties what (0.1) has.

Another example for half-linear D. Eqs. can be found in S. Fučik's paper [19] where the author dealt with the boundary value problem $u(0)=u(\pi)$ for

$$u''(\tau)+\mu u^+(\tau)-\nu u^-(\tau)=0 \tag{0.4}$$

with $u^+=\max\{u, 0\}$, $u^-=\max\{-u, 0\}$, $\mu>0$, $\nu>0$. Let $f(x, y)$ be defined by

$$f(x, y)=\begin{cases}\mu x & \text{if} \quad x\geqq 0\\ \nu x & \text{if} \quad x\leqq 0.\end{cases} \tag{0.5}$$

Clearly, (0.4) is a particular case of (0.1) with $p(t)=q(t)\equiv 1$. However $f(x, y)$ in (0.5) does not satisfy (Biii) with $\lambda=-1$ if $\mu\neq\nu$.

The aim of our paper is to relax the restrictions (Bi)—(Biii) so that the set of D. Eqs. (0.1) should cover also the D. Eqs. like (0.3), (0.4). The first step in this direction was made in [16]. In Section 1 we introduce the generalized sine function $S(\varphi)$, then in Section 2 we define the generalized polar coordinates with the aid of $S(\varphi)$. In Section 3 the Prüfer transformation known for the linear second order D. Eqs. is generalized to (0.1) and the problem of linear dependence/independence of the solutions is studied. In Section 4 we deal with Sturmian type theorems and we show also some connections between the zeros of the solutions and those of the derivatives as "strong interlacing properties" of the solutions which are not common even in the case of linear second order D. Eqs. Finally in the Appendix two lemmas on the properties of $f(x, y)$ are proved.

In connection with (0.2) we call the reader's attention to the excellent survey paper of P. R. Beesack [2]. Concerning the latest works on (0.2) we refer to J. Mirzov [23], [24], I. Bihari [12] and to the author's papers [15], [17]. Further progresses related to (0.1) were made by M. F. Estabrooks and J. W. Macki [18] and G. B. Gustafson [20].

1. Preliminary results. We shall be concerned with the D. Eq. (0.1) but we impose another set of restrictions on $f(x, y)$. Let the function $f(x, y)$ have the following properties:

(i) $f(x, y)$ is defined and continuous on $\Omega = R \times R_0$, where $R_0 = R \setminus \{0\}$;
(ii) $xf(x, y) > 0$ if $xy \neq 0$;
(iii) $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in R^+$, $(x, y) \in \Omega$;
(iv) let $f(x, y)$ be sufficiently smooth such that the solutions of (0.1) depend continuously and uniquely on the initial conditions posed at $t = t_0 \in I$ as $x(t_0) = x_0$, $x'(t_0) = x_0'$, $(x_0, x_0') \in \Omega$;
(v) the functions $F_+(T)$, $F_-(T)$ defined by

$$F_+(T) = Tf(T, 1), \quad F_-(T) = -Tf(-T, -1) \tag{1.1}$$

satisfy the relations

$$\int_{-\infty}^{\infty} \frac{dT}{1+F_+(T)} < \infty, \quad \int_{-\infty}^{\infty} \frac{dT}{1+F_-(T)} < \infty, \quad \lim_{|T| \to \infty} F_{\pm}(T) = \infty.$$

It is clear that by (i)—(ii), $F_{\pm}(T) > 0$ for $T \neq 0$. The basic properties of $f(x, y)$ are (i)—(v). Moreover we shall assume sometimes that

(vi) $f(-x, -y) = -f(x, y)$ for $(x, y) \in \Omega$.

This is a special case of (Biii) with $\lambda = -1$, where the zero and the negative values of λ are allowed. If (vi) holds then clearly $F_+(T) = F_-(T)$ and we shall use the notation $F(T)$ for them. The most important consequence of this assumption is that if $x(t)$ is a solution of (0.1) and $\lambda \neq 0$ is a constant then $\lambda x(t)$ is again a solution of (0.1) while in general we can state this only for positive λ's.

Comparing the assumptions (i)—(iv) with (Bi)—(Biii) we observe that the newly introduced assumptions are less restrictive. The assumption (v) seems to be an extra requirement. However, from (Bi) it follows that $f(1, y)$ is continuous at $y = 0$, from (Bii) $f(1, 0) > 0$, hence by (1.1) and (Biii)

$$\lim_{|T| \to \infty} \frac{F(T)}{T^2} = \lim_{|T| \to \infty} f\left(1, \frac{1}{T}\right) = f(1, 0) > 0 \tag{1.2}$$

and the integrals in (v) are certainly convergent. The assumption (iv) is obviously such what we need but it gives no clear restriction on $f(x, y)$ itself. In what follows we replace this by the following

(iv′) the functions $\log\left(1+F_+(T)\right)$ and $\log\left(1+F_-(T)\right)$ are uniformly Lipschitzian on $-\infty < T < \infty$.

In the Appendix we shall show that (iv′) is always valid if (Bi)—(Biii) hold. Moreover, in the Appendix we prove that (iv′) and the convergence of the two integrals in (v) imply also the limit relation in (v).

Following the main idea of [16] we define the generalized sine function $S = S(\varphi)$ as the solution of the D. Eq.

$$S'' + f(S, S') = 0 \tag{1.3}$$

satisfying the initial conditions $S(0) = 0$, $S'(0) = 1$. In what follows we give a constructive method to have a uniquely defined function $S(\varphi)$ also in the case when

the uniqueness of this initial value problem can not be guaranteed. Let us consider the generalized tangent function

$$T(\varphi) = \frac{S(\varphi)}{S'(\varphi)} \quad \text{provided} \quad S'(\varphi) \neq 0. \tag{1.4}$$

Then $T(0)=0$ and by (1.3), (iii), (1.1)

$$T'(\varphi) = \begin{cases} 1+F_+(T) & \text{if} \quad S'(\varphi) > 0 \\ 1+F_-(T) & \text{if} \quad S'(\varphi) < 0. \end{cases} \tag{1.5}$$

Let the numbers $\pi_{1/2}, \pi_{-1/2}, \pi_1, \pi_{3/2}, \pi_2$ be defined by

$$\begin{cases} \pi_{1/2} = \int\limits_0^\infty \dfrac{d\tau}{1+F_+(\tau)}, \quad \pi_{-1/2} = \int\limits_{-\infty}^0 \dfrac{d\tau}{1+F_+(\tau)}, \\ \pi_1 = \pi_{1/2} + \int\limits_{-\infty}^0 \dfrac{d\tau}{1+F_-(\tau)}, \quad \pi_{3/2} = \pi_1 + \int\limits_0^\infty \dfrac{d\tau}{1+F_-(\tau)}, \\ \pi_2 = \pi_{3/2} + \pi_{-1/2}. \end{cases} \tag{1.6}$$

The integrals here are finite due to (v). By the way we remark that if (vi) holds then $\pi_1 = \pi_{-1/2} + \pi_{1/2}$, $\pi_{3/2} = \pi_1 + \pi_{1/2}$, $\pi_2 = 2\pi_1$.

By (1.5) the function $T(\varphi)$ fulfils the relation

$$\int\limits_0^{T(\varphi)} \frac{d\tau}{1+F_+(\tau)} = \varphi, \quad -\pi_{-1/2} < \varphi < \pi_{1/2} \tag{1.7}$$

and this establishes a one-to-one connection between the values of $\varphi \in (-\pi_{-1/2}, \pi_{1/2})$ and $T \in (-\infty, \infty)$. Similarly we define $T(\varphi)$ on $(\pi_{1/2}, \pi_{3/2})$ by the relation

$$\int\limits_0^{T(\varphi)} \frac{d\tau}{1+F_-(\tau)} = \varphi - \pi_1. \tag{1.8}$$

Then owing to (1.6) the function $T(\varphi)$ has been defined on $(-\pi_{-1/2}, \pi_{3/2}) \setminus \{\pi_{1/2}\}$. Finally let $T(\varphi)$ be periodic with the period π_2:

$$T(\varphi + k\pi_2) = T(\varphi) \quad \text{for} \quad k = \pm 1, \pm 2, \ldots. \tag{1.9}$$

With the aid of $T(\varphi)$ and (1.4) we are able to determine $S(\varphi)$. Let $0 < \varepsilon \leqq \varphi < \pi_{1/2}$ where ε is sufficiently small, then by (1.4)

$$S(\varphi) = S(\varepsilon) \exp\left(\int\limits_\varepsilon^\varphi \frac{d\varphi}{T(\varphi)} \right) = S(\varepsilon) \exp \int\limits_{T(\varepsilon)}^{T(\varphi)} \left(\frac{d\tau}{\tau[1+F_+(\tau)]} \right).$$

Since $\log T(\varepsilon) = -\int\limits_{T(\varepsilon)}^1 d\tau/\tau$ we obtain

$$S(\varphi) = S'(\varepsilon) \exp\left(-\int\limits_{T(\varepsilon)}^1 \frac{d\tau}{\tau} + \int\limits_{T(\varepsilon)}^{T(\varphi)} \frac{d\tau}{\tau(1+F_+(\tau))} \right) =$$

$$= S'(\varepsilon) \exp\left(-\int\limits_{T(\varepsilon)}^1 \frac{f(\tau,1)\,d\tau}{1+F_+(\tau)} + \int\limits_1^{T(\varphi)} \frac{d\tau}{\tau(1+F_+(\tau))} \right)$$

and by $\varepsilon \to +0$

$$S(\varphi) = \exp\left(- \int_0^1 \frac{f(\tau, 1)\, d\tau}{1+F_+(\tau)} + \int_1^{T(\varphi)} \frac{d\tau}{\tau(1+F_+(\tau))} \right) \quad \text{for} \quad 0 < \varphi < \pi_{1/2}. \tag{1.10}$$

By (v), $S(\pi_{1/2})$ is finite and since $S'(\varphi)=S(\varphi)/T(\varphi)$ therefore $S'(\pi_{1/2})=0$ and $S'(\varphi)$ decreases from 1 to 0 as φ increases from 0 to $\pi_{1/2}$. We know already by (1.8) that $T(\pi_1)=0$ and $S(\pi_1)=0$ hence $S(\varphi)>0$ and $S'(\varphi)<0$ on $(\pi_{1/2}, \pi_1)$.

Let $S'(\pi_1)=-K_1$, $K_1>0$. By similar way as above we find the values of $S(\varphi)$ as

$$S(\varphi) = K_1 \exp\left(- \int_{-1}^0 \frac{f(-\tau, -1)}{1+F_-(\tau)}\, d\tau - \int_{T(\varphi)}^{-1} \frac{d\tau}{\tau(1+F_-(\tau))} \right) \quad \text{for} \quad \pi_{1/2} < \varphi < \pi_1. \tag{1.11}$$

Comparing this with (1.10) at the common point $\varphi=\pi_{1/2}$ we get the value of K_1 as

$$K_1 = \exp\left(\int_{-\infty}^{-1} \frac{d\tau}{\tau(1+F_-(\tau))} + \int_{-1}^0 \frac{f(-\tau, -1)\, d\tau}{1+F_-(\tau)} - \right. \tag{1.12}$$

$$\left. - \int_0^1 \frac{f(\tau, 1)\, d\tau}{1+F_+(\tau)} + \int_1^{\infty} \frac{d\tau}{\tau(1+F_+(\tau))} \right).$$

By the same way we obtain for $S(\varphi)$

$$S(\varphi) = -K_1 \exp\left(\int_0^1 \frac{f(-\tau, -1)\, d\tau}{1+F_-(\tau)} + \int_1^{T(\varphi)} \frac{d\tau}{\tau(1+F_-(\tau))} \right) \quad \text{if} \quad \pi_1 < \varphi < \pi_{3/2} \tag{1.13}$$

and

$$S(\varphi) = -K_2 \exp\left(\int_{-1}^0 \frac{f(\tau, 1)\, d\tau}{1+F_+(\tau)} - \int_{T(\varphi)}^{-1} \frac{d\tau}{\tau(1+F_+(\tau))} \right) \quad \text{if} \quad \pi_{3/2} < \varphi < \pi_2 \tag{1.14}$$

where $K_2=S'(\pi_2)$. The value of K_2 can be determined by equating the limits of $S(\varphi)$ at $\varphi=\pi_{3/2}$ from the left and from the right hand side by (1.13) and (1.14), resp., i.e.

$$K_1 \exp\left(\int_0^1 \frac{f(-\tau, -1)\, d\tau}{1+F_-(\tau)} + \int_1^{\infty} \frac{d\tau}{\tau(1+F_-(\tau))} \right) = \tag{1.15}$$

$$= K_2 \exp\left(\int_{-1}^0 \frac{f(\tau, 1)\, d\tau}{1+F_+(\tau)} - \int_{\infty}^{-1} \frac{d\tau}{\tau(1+F_+(\tau))} \right).$$

The constants K_1 and K_2 are characteristic quantities for the half-linear D. Eqs. Assuming (vi) we have by (1.12) and (1.15)

$$K_1 = \exp\left(\int_{-\infty}^{-1} \frac{d\tau}{\tau(1+F(\tau))} - \int_{-1}^1 \frac{f(\tau, 1)\, d\tau}{1+F(\tau)} + \int_1^{\infty} \frac{d\tau}{\tau(1+F(\tau))} \right), \quad K_2 = K_1^2 \tag{1.16}$$

if (vi) holds.

Let us consider the functions $S(\varphi+\pi_2)$ and $K_2S(\varphi)$. Since $K_2>0$ and $S(\pi_2)=0$, $S'(\pi_2)=K_2$, both functions satisfy the same initial conditions at $\varphi=0$. Hence by (iv) the relation

$$S(\varphi) = K_2S(\varphi-\pi_2) \tag{1.17}$$

holds for every φ. If we do not want to apply the uniqueness then we could use (1.17) as the extension of the function $S(\varphi)$ from the interval $[0, \pi_2]$ to $(-\infty, \infty)$. Now $S(\varphi)$ is not necessarily periodic because in general we do not have $K_2=1$. On the other hand we can observe that $S(\varphi)$ and $S'(\varphi)$ never vanish simultaneously, $S(\varphi)$ is concave on $[0, \pi_1]$ and convex on $[\pi_1, \pi_2]$.

Under the assumption (vi) we can state more than in (1.17), namely the following relations are true:

$$S(\varphi) = -K_1S(\varphi-\pi_1), \quad T(\varphi) = T(\varphi-\pi_1) \tag{1.18}$$

if (vi) holds.

2. Generalized polar coordinates. Now with the aid of the functions $S(\varphi)$, $S'(\varphi)$ we shall define the polar coordinates on the plane (x_1, x_2). Let the curve i be of the parametric form $x_1=S'(\varphi)$, $x_2=S(\varphi)$ for $-\infty<\varphi<\infty$. Then the curve i is spiralling around the origin $(0;0)$ infinitely many times, approaching it if $K_2<1$, going away if $K_2>1$ as $\varphi\to\infty$. In the particular case $K_2=1$ we have a closed curve.

Let $\varphi=\varphi_0$ be fixed. Then the points $P_\varrho=(\varrho S'(\varphi_0); \varrho S(\varphi_0))$ $(\varrho>0)$ are on the half-ray starting from the origin $O(0;0)$ and going through the point $P=(S'(\varphi_0); S(\varphi_0))$. Thus to a point $P_\varrho=(x_1; x_2)$ of this half-ray we can order the pair (φ_0, ϱ_0) by the relations $\varrho_0=\varrho$ and

$$x_1 = \varrho_0S'(\varphi_0), \quad x_2 = \varrho_0S(\varphi_0). \tag{2.1}$$

In what follows we shall say that φ_0 is the angle between the positive half-axis x_1 and the half-ray $\overrightarrow{OP}$. The name "angle" is taken freely from the geometry of the plane. Especially for the points of i we have $\varrho_0=1$ hence the curve i plays the role of the usual unit circle.

We mention here another two special cases. By making use of the definition of the function $S(\varphi)$ in Section 1 we obtain

$$\begin{cases} \text{if } x_2=0 \text{ then } \varphi_0 \equiv 0 \pmod{\pi_2} \text{ for } x_1>0 \text{ and} \\ \qquad\qquad\quad \varphi_0 \equiv \pi_1 \pmod{\pi_2} \text{ for } x_1<0 \\ \text{if } x_1=0 \text{ then } \varphi_0 \equiv \pi_{1/2} \pmod{\pi_2} \text{ for } x_2>0 \text{ and} \\ \qquad\qquad\quad \varphi_0 \equiv \pi_{3/2} \pmod{\pi_2} \text{ for } x_2<0. \end{cases} \tag{2.2}$$

Let us consider the half-plane $x_2>0$. By (2.1) $S(\varphi_0)>0$ and $T(\varphi_0)=x_2/x_1$ provided $x_1\neq0$. By the strict monotonicity of $T(\varphi)$ there is exactly one value φ_0 in the interval $(0, \pi_1)$ satisfying (2.1). A similar statement holds if $x_2<0$. Thus we obtain that all points but the origin O of the plane (x_1, x_2) have a unique pair (φ_0, ϱ_0) satisfying $0\leqq\varphi_0<\pi_2$ and $\varrho_0>0$. In our further investigations we shall remove the origin $O=(0;0)$ from the plane (x_1, x_2) and the rest will be denoted by P which is topologically a ring.

Let us consider the interval $[\pi_2, 2\pi_2)$. Then we find again that for each point $(x_1; x_2)\in P$ there is a unique pair (φ_1, ϱ_1) satisfying (2.1) and the inequalities $\pi_2\leqq \leqq\varphi_1<2\pi_2$, $\varrho_2>0$. By (1.17) the connection between the pairs (φ_0, ϱ_0) and (φ_1, ϱ_1) is

$$\varphi_1 = \varphi_0+\pi_2; \quad \varrho_1 = K_2^{-1}\varrho_0.$$

Hence, in general, to each point $(x_1; x_2)$ of P there are infinitely many pairs $\ldots, (\varphi_{-1}, \varrho_{-1}), (\varphi_0, \varrho_0), (\varphi_1, \varrho_1), (\varphi_2, \varrho_2), \ldots$ satisfying

$$x_1 = \varrho_k S'(\varphi_k), \quad x_2 = \varrho_k S(\varphi_k), \tag{2.3}$$

$$\varphi_k = \varphi_0+k\pi_2, \quad \varrho_k = K_2^{-k}\varrho_0, \quad k = \pm 1, \pm 2, \ldots. \tag{2.4}$$

Now a pair $(\varphi, \varrho)=(\varphi_k, \varrho_k)$ defined by (2.4) will be called polar coordinates of the point $(x_1; x_2)\in P$.

3. Generalized Prüfer transformation. The pioneering work to use the Prüfer transformation for studying the qualitative properties of linear second order differential equations is due to E. Kamke [22]. Later I. Bihari [3] extended the Kamke's method to the half-linear second order differential equations under the restrictions (Bi)—(Biii) by making use of the usual polar transformation

$$x_1 = \varrho\cos\varphi, \quad x_2 = \varrho\sin\varphi.$$

His requirement on the smoothness of the function $f(x, y)$, namely $f(x, y)$ be Lipschitzian, goes back to using of this transformation.

The main tool for our investigations will be the generalized Prüfer transformation. Let $x=x(t)$ be a solution of (0.1) satisfying the initial conditions $x(t_0)=x_0$, $x'(t_0)=x_0'$, $(x_0'; x_0)\in P$. Then let us consider the curve $\big(p(t)x'(t); x(t)\big)$ on the plane P with the parameter $t\in I$. Define the continuous functions $\varphi(t)$, $\varrho(t)$ as the generalized polar coordinates of the point of the curve:

$$x(t) = \varrho(t)S\big(\varphi(t)\big), \quad p(t)x'(t) = \varrho(t)S'\big(\varphi(t)\big). \tag{3.1}$$

Since by (2.3) every point has infinitely many polar coordinates, for the sake of uniqueness we choose among them that one for which $\varphi_0=\varphi(t_0)\in[0, \pi_2)$. Clearly $\varrho_0=\varrho(t_0)>0$. Then by this agreement and the continuity, the functions $\varphi(t)$, $\varrho(t)$ are uniquely defined. Making use of the relations $\frac{d}{dt}x(t)=x'(t)=\frac{1}{p}\varrho S'(\varphi)$ and (0.1), (1.3) we obtain a linear system for the unknowns $\varphi'=\frac{d}{dt}\varphi$, $\varrho'=\frac{d}{dt}\varrho$:

$$\varrho' S+\varrho S'\varphi' = \frac{1}{p}\varrho S', \quad \varrho' S'-\varrho f(S, S')\varphi' \equiv -qf(\varrho S, \varrho S').$$

Since $\varrho(t_0)=\varrho_0>0$ we can suppose $\varrho(t)>0$ (at least in some neighbourhood of $t=t_0$). Then by (iii) we have $f(\varrho S, \varrho S')=\varrho f(S, S')$ and we can solve this system:

$$\begin{cases} \varphi' = \dfrac{1}{p}[1-G(\varphi)]+qG(\varphi) & (3.2\varphi) \\ \varrho' = \varrho\left(\dfrac{1}{p}-q\right)\dfrac{G(\varphi)}{T(\varphi)}, & (3.2\varrho) \end{cases}$$

where

$$G(\varphi)=\begin{cases}\dfrac{F_+(T(\varphi))}{1+F_+(T(\varphi))} & \text{if} \quad S'(\varphi)>0,\\ \dfrac{F_-(T(\varphi))}{1+F_-(T(\varphi))} & \text{if} \quad S'(\varphi)<0,\\ 1 & \text{if} \quad S'(\varphi)=0.\end{cases} \tag{3.3}$$

By (1.9), $G(\varphi)$ is periodic with period π_2. Assumptions (i), (v) imply that $G(\varphi)$ is continuous for all $\varphi\in R$ and it takes on values between 0 and 1, moreover $G(\varphi)=0$ if $\varphi\equiv 0, \pi_1 \pmod{\pi_2}$ and $G(\varphi)=1$ if $\varphi\equiv\pi_{1/2}, \pi_{3/2} \pmod{\pi_2}$.

In the particular case when (vi) holds, by (1.18) $G(\varphi)$ is periodic with period π_1.

We claim that the function $G(\varphi)/T(\varphi)$ in (3.2ϱ) is continuous. Since $G(\varphi)$ is continuous we have to consider only the case when $T(\varphi)\to 0$. Taking into consideration the relations (3.3), (1.1) we obtain

$$\lim_{T(\varphi)\to 0}\frac{G(\varphi)}{T(\varphi)}=\lim_{T\to 0}\frac{F_\pm(T)}{T}=\lim_{T\to 0}\pm f(\pm T,\ \pm 1)=\pm f(0,\ \pm 1)=0.$$

Let $G(\varphi)/T(\varphi)$ be 0 if $T(\varphi)=0$, then we get a continuous function for $-\infty<\varphi<\infty$ which is periodic with period π_2. Hence (3.2ϱ) can be solved uniquely.

Now the D. Eq. (3.2φ) has also a unique solution if (0.1) has. Suppose the contrary. If there existed two solutions $\varphi_1(t)$, $\varphi_2(t)$ with the same initial value, then solving the D. Eq. (3.2ϱ) replacing $\varphi=\varphi_1(t)$ first we got $\varrho=\varrho_1(t)$ and then similarly $\varrho_2(t)$. The functions $\varrho_1(t)\,S(\varphi_1(t))$ and $\varrho_2(t)\,S(\varphi_2(t))$ would be two different solutions of (0.1), a contradiction.

If we have the assumption (iv′) then we claim that (3.2φ) is Lipschitzian with respect to φ. Let us restrict ourselves to an interval $[\varphi_1, \varphi_2]\subset(-\pi_{-1/2}, \pi_{1/2})$. By (iv′) there is a constant C_1 such that

$$|\log(1+F_\pm(T_2))-\log(1+F_\pm(T_1))|\leqq C_1(T_2-T_1) \quad \text{for} \quad -\infty<T_1<T_2<\infty \tag{3.4}$$

holds. Let k be a natural number sufficiently large and let ψ_i be defined by

$$\psi_i=\varphi_1+(\varphi_2-\varphi_1)\frac{i}{k} \quad \text{for} \quad i=0, 1, \ldots, k.$$

Then by (3.3)

$$G(\psi_i)-G(\psi_{i-1})=\frac{F_+(T(\psi_i))-F_+(T(\psi_{i-1}))}{[1+F_+(T(\hat{\psi}_i))]^2} \quad \text{with some} \quad \hat{\psi}_i\in(\psi_{i-1}, \psi_i),$$

because $F(T)$ is continuous on $[T(\varphi_1), T(\varphi_2)]$. On the other hand by (3.4)

$$|\log(1+F_+(T(\psi_i)))-\log(1+F_+(T(\psi_{i-1})))|=$$

$$=\frac{|F_+(T(\psi_i))-F_+(T(\psi_{i-1}))|}{1+F_+(T(\hat{\hat{\psi}}_i))}\leqq C_1(T(\psi_i)-T(\psi_{i-1}))$$

where $\hat{\hat{\psi}}_i \in (\psi_{i-1}, \psi_i)$, hence

$$|G(\varphi_2)-G(\varphi_1)| \leqq \sum_{i=1}^{k} |G(\psi_i)-G(\psi_{i-1})| \leqq$$

$$\leqq C_1(\varphi_2-\varphi_1)\left\{\frac{1}{k}\sum_{i=1}^{k}\frac{1+F_+(T(\hat{\hat{\psi}}_i))}{[1+F_+(T(\hat{\psi}_i))]^2}\,\frac{T(\psi_i)-T(\psi_{i-1})}{\psi_i-\psi_{i-1}}\right\}.$$

By the continuity of $F_+(T)$ and by (1.5) the expression in the braces on the right hand side tends to 1 as $k\to\infty$. Thus $G(\varphi)$ is Lipschitzian with the same constant C_1 as the functions $\log(1+F_\pm(T))$ with respect to T. Hence under assumption (iv′) we have the uniqueness of the solutions of the system (3.2) with arbitrary initial conditions φ_0, ϱ_0 provided $\varrho_0>0$.

Not only the uniqueness but also the continuous dependence of the solutions on the initial values can be investigated by using the system (3.2) instead of (0.1). Hence if there is any doubt how to define the unique solutions of (0.1) we should consider the corresponding system (3.2) for $\varphi(t)$ and $\varrho(t)$ where the problems can be settled easier and then the solution to (0.1) will be given as $x(t)=\varrho(t)S(\varphi(t))$. This is the way, e.g., if we should define the solution $x(t)=\text{const}\neq 0$ on some interval, because $x'(t)\equiv 0$ and $f(x, y)$ is usually not defined for these values. Now by (2.1), (2.2) we have $\varphi(t)\equiv\pi_{1/2}\,(\text{mod}\,\pi_2)$ or $\pi_{3/2}\,(\text{mod}\,\pi_2)$. Hence $\varphi'(t)\equiv 0$. By (3.2), $\varphi'(t)\equiv 0$ is possible if and only if $q(t)\equiv 0$ on this interval. Then also the relation $\varrho'(t)\equiv 0$ holds, i.e. $\varrho(t)=\text{const.}$, consequently $x(t)\equiv\text{const.}$ happens only if $q(t)\equiv 0$ on that interval.

Let us observe that (3.2φ) is independent of ϱ, actually it depends only on φ and, of course, on t. In our paper the proofs of the qualitative properties of the solutions of (0.1) will make use of (3.2φ) only. The D. Eq. (3.2ϱ) is linear in ϱ, hence it can be solved easily by quadrature if we have already solved (3.2φ) for $\varphi=\varphi(t)$. It is clear that the function $\varrho(t)$ will never vanish, i.e. $\varrho(t)>0$ for every $t\in I$. A consequence of this statement is that $x(t)$ and $x'(t)$ are *never vanishing simultaneously*. In this context we can not speak about the function $x(t)\equiv 0$ as a solution of (0.1). This function is excluded from the set of the solutions of (0.1).

Finally we remark that the requirement of the continuity of $q(t)$ can be relaxed by admitting piecewise continuous functions. Then the uniqueness and the continuous dependence of the solutions on the initial values remain valid which can be seen by using the system (3.2).

Now we continue our investigations with some definitions.

DEFINITION 3.1. Let $x_1(t)$, $x_2(t)$ be two solutions of (0.1). They are linearly dependent if there exists a positive constant λ such that $x_2(t)=\lambda x_1(t)$ for every $t\in I$.

DEFINITION 3.2. Under assumption (vi), two solutions $x_1(t)$ and $x_2(t)$ are linearly dependent if there is a constant $\lambda\neq 0$ (not necessarily positive) such that $x_2(t)=\lambda x_1(t)$ for every $t\in I$. Otherwise they are linearly independent.

COROLLARY 3.1. *If two solutions $x_1(t)$, $x_2(t)$ of* (0.1) *are linearly dependent then the Wronskian $W=W(x_1, x_2)=x_1'x_2-x_1x_2'$ is identically zero on the interval I.*

This statement is trivial but in some cases the converse is also true.

THEOREM 3.1. *Under assumption* (vi) *the Wronskian* $W(x_1, x_2)$ *of two solutions* $x_1(t), x_2(t)$ *of* (0.1) *is either identically zero or it is never zero on* I. *In the first case the solutions are linearly dependent while in the second case they are independent.*

PROOF. Let us suppose that there is a value $t_0 \in I$ such that

$$W|_{t=t_0} = x_1'(t_0)x_2(t_0) - x_1(t_0)x_2'(t_0) = 0.$$

Then the system of linear equations

$$\lambda_1 x_1(t_0) - \lambda_2 x_2(t_0) = 0, \quad \lambda_1 x_1'(t_0) - \lambda_2 x_2'(t_0) = 0 \tag{3.5}$$

for the unknowns λ_1, λ_2 has a nontrivial solution (λ_1, λ_2). Now we have $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ because otherwise it would be either $x_1(t_0) = x_1'(t_0) = 0$ or $x_2(t_0) = x_2'(t_0) = 0$ and none of these relations is possible. Let $\lambda = \lambda_1/\lambda_2$. Then we have $x_2(t_0) = \lambda x_1(t_0)$, $x_2'(t_0) = \lambda x_1'(t_0)$, $\lambda \neq 0$. First we suppose $\lambda > 0$. Let φ_{10}, ϱ_{10} be the polar coordinates of the point $(p(t_0)x_1'(t_0); x_1(t_0))$ on the plane P. Then φ_{10}, $\lambda\varrho_{10}$ are the corresponding polar coordinates of the point $(p(t_0)x_2'(t_0); x_2(t_0))$. Now let $x_i(t) = \varrho_i(t)\, S(\varphi_i(t))$ $(i = 1, 2)$ be defined by (3.1). Then the functions $\varphi_i(t)$, $\varrho_i(t)$ are solutions of (3.2) satisfying the initial conditions $\varphi_i(t_0) = \varphi_{10}$, $\varrho_1(t_0) = \varrho_{10}$, $\varrho_2(t_0) = \lambda\varrho_1(t_0)$. Owing to the uniqueness we have $\varphi_2(t) \equiv \varphi_1(t)$, $\varrho_2(t) \equiv \lambda\varrho_1(t)$ on I, consequently $x_2(t) = \lambda x_1(t)$ on I, i.e. the solutions $x_1(t)$, $x_2(t)$ are linearly dependent and the Wronskian is identically zero.

In the case $\lambda < 0$ we consider the function $x_3 = -x_2$. Then $x_3(t)$ is also a solution of (0.1) and $x_3(t_0) = \bar{\lambda}x_1(t_0)$, $x_3'(t_0) = \bar{\lambda}x_1'(t_0)$ where $\bar{\lambda} = -\lambda$. Since $\bar{\lambda} > 0$ we find again $x_3(t) \equiv \bar{\lambda}x_1(t)$ on I, hence $x_2(t) = \lambda x_1(t)$, i.e. the solutions $x_1(t)$, $x_2(t)$ are linearly dependent.

If the Wronskian is never vanishing on I then there is no λ such that $x_2(t) \equiv \lambda x_1(t)$ on I, consequently the solutions $x_1(t)$, $x_2(t)$ are linearly independent, which completes the proof.

REMARK 3.1. Without assuming (vi), in the case when the Wronskian W of two solutions $x_1(t)$, $x_2(t)$ of (0.1) vanishes at some $t_0 \in I$, we consider the quantity $\lambda = \lambda_1/\lambda_2$ where λ_1, λ_2 are non-trivial solutions of the system (3.5). If $\lambda > 0$ then by the same way as in the proof of Theorem 3.1 we obtain $x_2 \equiv \lambda x_1$ and $W = 0$ on I and the solutions x_1, x_2 are linearly dependent. But in the case $\lambda < 0$ we can not conclude that the solutions are linearly dependent. By an example we show that the Wronskian W of two solutions may be identically zero on some interval and the solutions are not linearly dependent.

E 3.1. Let

$$f(x, y) = \begin{cases} x & \text{if} \quad xy > 0 \\ \dfrac{1}{4}x & \text{if} \quad x \geqq 0, \quad y < 0 \\ x & \text{if} \quad x \leqq 0, \quad y > 0, \end{cases}$$

and

$$q(t) = \begin{cases} 4 & \text{if} \quad \dfrac{\pi}{2} \leqq t < \pi \\ 1 & \text{elsewhere.} \end{cases}$$

Then the functions

$$x_1(t) = \sin t, \quad 0 \leqq t \leqq \pi, \quad x_2(t) = \begin{cases} -\sin t, & 0 \leqq t \leqq \dfrac{\pi}{2} \\ \cos 2t, & \dfrac{\pi}{2} \leqq t \leqq \pi \end{cases}$$

are two linearly independent solutions of the half-linear D. Eq. $x''+q(t)f(x, x')=0$ on $[0, \pi]$. Now the Wronskian $W(x_1, x_2)\equiv 0$ on $\left[0, \frac{\pi}{2}\right]$ and

$$W(x_1, x_2) = \cos t(1+2\sin^2 t) < 0 \quad \text{on} \quad \left(\frac{\pi}{2}, \pi\right).$$

4. Sturmian theorems. The Sturmian theorems are the most powerful tools in studying the qualitative properties of the solutions of linear second order D. Eqs. Now we are going to generalize these theorems to (0.1). First we have to establish the connection between the zeros of a solution $x(t)$ and the polar function $\varphi(t)$. Let $t_0, t_1, t_2, \ldots$ be consecutive zeros of the solution $x(t)$. Then by (3.1) $S(\varphi(t_i))=0$ for $i=0, 1, 2, \ldots$. If we make the restriction $\varphi(t_0)\in[0, \pi_2)$ then by (2.1), (2.2) we have $\varphi(t_0)=0$ if $x'(t_0)>0$ and $\varphi(t_0)=\pi_1$ if $x'(t_0)<0$. Thus $\varphi(t)$ is uniquely defined. Now let us assume that $\varphi(t_0)=0$. By (3.2) $\varphi'(t_0)=1/p(t_0)>0$ because $G(0)=0$. Hence $\varphi(t)$ is strictly increasing in some neighbourhood of $t=t_0$. We claim that $\varphi(t)>0$ if $t\in(t_0, \infty)\cap I$. Suppose the contrary. Then there would be a value $\bar{t}\in(t_0, \infty)\cap I$ such that $\varphi(\bar{t})\leqq 0$. Let $\hat{t}$ be defined by $\hat{t}=\sup\{t: \varphi(s)>0, t_0<s<t, t_0<t\leqq\bar{t}\}$. The continuity of $\varphi(t)$ implies $\varphi(\hat{t})=0$, and also $\varphi'(\hat{t})\leqq 0$. From (3.2φ) it follows $\varphi'(\hat{t})=1/p(\hat{t})>0$ which is a contradiction. Therefore $\varphi(t)>0$ for $t\in(t_0, \infty)\cap I$.

Let us consider the value $\varphi(t_1)$. We show that $\varphi(t_1)=\pi_1$. We know already that $S(\varphi(t_1))=0$ and $\varphi(t_1)>0$. Since t_1 is the next zero to t_0 and $\varphi(t)$ is continuous hence by (2.2) $\varphi(t_1)=\pi_1$ and also the inequality $x'(t_1)<0$ holds. We have again $\varphi'(t_1)>0$ and $\varphi(t)>\pi_1$ for $t\in(t_1, \infty)\cap I$ and $0<\varphi(t)<\pi_1$ if $t_0<t<t_1$. Step by step we find in general the following relations:

$$\varphi(t_i) = \begin{cases} \dfrac{i}{2}\pi_2 & \text{if} \quad i = 0, 2, 4, \ldots \\ \pi_1+\dfrac{i-1}{2}\pi_2 & \text{if} \quad i = 1, 3, 5, \ldots, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \varphi(t_i) < \varphi(t) < \varphi(t_{i+1}) & \text{for} \quad t_i < t_i < t_{i+1}, \quad i = 0, 1, 2, \ldots \\ \varphi(t) > \varphi(t_i) & \text{for} \quad t > t_i, \quad i = 0, 1, 2, \ldots \end{cases} \tag{4.2}$$

First we generalize the interlacing property of the zeros of the solutions known in the theory of linear second order D. Eqs. (see e.g. in [21], p. 335).

THEOREM 4.1. *Let x_1 and x_2 be linearly independent solutions of* (0.1). *Suppose that t_0, t_1, t_2 are three consecutive zeros of $x_1(t)$. Then the solution $x_2(t)$ has at least one zero in (t_0, t_2) and at most three zeros in $[t_0, t_2]$.*

PROOF. Let $\varphi_i(t)$ be the polar angle function belonging to the solution $x_i(t)$ $(i=1, 2)$. We may assume that $x_1'(t_0)>0$ and then $\varphi_1(t_0)=0$. By (4.1) we have $\varphi_1(t_1)=\pi_1$, $\varphi_1(t_2)=\pi_2$. $\varphi_1(t)$ is a solution of (3.2φ). Let $\hat{\varphi}_1(t)=\varphi_1(t)+\pi_2$; this is also a solution of (3.2φ). Concerning $\varphi_2(t)$ we can suppose $\varphi_2(t_0)\in(0, \pi_2)$. Now the possibility $\varphi_2(t_0)=0$ is excluded because in this case we would have $x_1(t_0)=$ $=x_2(t_0)=0$ and $\lambda=x_2'(t_0)/x_1'(t_0)>0$, and by Remark 3.1 the solutions $x_1(t)$, $x_2(t)$ would be linearly dependent, contrary to our assumption.

Since among the values $\varphi_1(t_0)$, $\varphi_2(t_0)$, $\hat{\varphi}_1(t_0)$ the relations

$$0 = \varphi_1(t_0) < \varphi_2(t_0) < \hat{\psi}_1(t_0) = \pi_2$$

hold, and $\varphi_1(t)$, $\varphi_2(t)$, $\hat{\varphi}_1(t)$ are solutions of (3.2φ), hence the uniqueness of these solutions implies $\varphi_1(t)<\varphi_2(t)<\hat{\varphi}_1(t)$ for $t\geqq t_0$. Applying these inequalities at $t=t_2$ we find $\pi_2<\varphi_2(t_2)<2\pi_2$. Since $\varphi_2(t_0)<\pi_2$ and $\varphi_2(t)$ is continuous, $\varphi_2(t)$ takes on the value π_2 somewhere in (t_0, t_2). On the other hand the values of $\varphi_2(t)$ on $[t_0, t_2]$ may vary in $(0, 2\pi_2)$ thus $\varphi_2(t)$ may take on the values π_1, π_2, $\pi_1+\pi_2$, hence the solution $x_2(t)$ may have at most three zeros in $[t_0, t_2]$ which completes the proof.

If (0.1) is linear, i.e. if $f(x, y)=x$ then the interlacing property of the zeros says that under the conditions of Theorem 4.1 the solution $x_2(t)$ has exactly two zeros in (t_0, t_2), more precisely one in (t_0, t_1) and another in (t_1, t_2). We provide here an example to show that in general the statement of Theorem 4.1 is sharp, i.e. if $x_1(t)$ has three consecutive zeros t_0, t_1, t_2 then there may also be a linearly independent solution which has either exactly one zero in (t_0, t_2) or it has exactly three zeros in $[t_0, t_2]$.

EXAMPLE 4.1. Let $f(x, y)$ be the same as in Example 3.1 and let

$$q(t) = \begin{cases} 4, & \dfrac{\pi}{2} \leqq t \leqq \dfrac{5\pi}{4} \\ 1 & \text{elsewhere.} \end{cases}$$

Then

$$x_1(t) = \begin{cases} \sin t, & 0 \leqq t \leqq \pi \\ -\dfrac{1}{2}\sin 2t, & \pi \leqq t \leqq \dfrac{5\pi}{4} \\ \dfrac{1}{2}\cos\left(t-\dfrac{\pi}{4}\right), & \dfrac{5}{4}\pi \leqq t \leqq 2\pi \end{cases}$$

and

$$x_2(t) = \begin{cases} -\sin t, & 0 \leqq t \leqq \pi/2 \\ \cos 2t, & \dfrac{\pi}{2} \leqq t \leqq \pi \\ -\cos t, & \pi \leqq t \leqq 5\pi/4 \\ \sqrt{\dfrac{5}{2}}\sin\left(\dfrac{t}{2}+\psi_0-\dfrac{\pi}{8}\right), & \dfrac{5\pi}{4} \leqq t \leqq \hat{t}_2 \\ -\sqrt{\dfrac{5}{8}}\sin(t-\hat{t}_2), & \hat{t}_2 \leqq t \leqq \hat{t}_2+\pi \end{cases}$$

where $\tan\psi_0=2$, $\hat{t}_2=\frac{9\pi}{4}-2\psi_0$, are two linearly independent solutions of $x''+q(t)f(x,x')=0$ on the interval $[0,2\pi]$. The solution $x_1(t)$ has three zeros, namely $t_0=0$, $t_1=\pi$ and $t_2=7\pi/4$. The solution $x_2(t)$ has zeros at $\hat{t}_0=0$, $\hat{t}_1=3\pi/4$ and $\hat{t}_2=4.8543$. Since $\hat{t}_2<t_2$ thus $x_2(t)$ has exactly three zeros in $[t_0,t_2]$.

Let $x_\varepsilon(t)$ be the solution of our D. Eq. with the initial conditions $x_\varepsilon(0)=\varepsilon$, $x'_\varepsilon(0)=1$ for sufficiently small ε's. It is clear that $\lim_{\varepsilon\to 0} x_\varepsilon(t)=x_1(t)$; hence the zeros of $x_1(t)$ will be changed to $t_0(\varepsilon)$, $t_1(\varepsilon)$, $t_2(\varepsilon)$. If we define $\varphi_\varepsilon(t)$ as the polar angle function corresponding to the solution $x_\varepsilon(t)$ then from the initial conditions we obtain $T(\varphi_\varepsilon(0))=\varepsilon$. Then by the uniqueness of the solutions of (3.2φ), $\varphi_\varepsilon(t)$ is strictly increasing with ε increasing. Hence by (4.1) $t_0(\varepsilon)<0$, $t_1(\varepsilon)<t_1=\pi$, and $t_2(\varepsilon)<t_2$ if $\varepsilon>0$. Let ε be small enough such that the inequality $t_2(\varepsilon)>\hat{t}_2$ still holds. Then let us choose the solutions $x_2(t)$ and $x_\varepsilon(t)$ as x_1 and x_2 in Theorem 4.1. We find that $x_\varepsilon(t)$ has only one zero, namely $t_1(\varepsilon)$, in the interval $[\hat{t}_0,\hat{t}_2]$.

Let us observe that the method used in the proof of Theorem 4.1 can be applied also to the cases when the solution $x_1(t)$ has several consecutive zeros. Without any proof we give the following generalization.

THEOREM 4.2. *Let x_1, x_2 be two linearly independent solutions of* (0.1). *If the solution $x_1(t)$ has the consecutive zeros $t_0, t_1, \ldots, t_i$ where $i\geqq 1$ then x_2 has at least $i-1$ zeros on (t_0,t_i) and at most $i+1$ zeros on $[t_0,t_i]$.*

The strange phenomenon, reflected by the three possibilities in Theorem 4.1, disappears if we have assumption (vi). In this case it remains only one possibility which is formulated in the next theorem.

THEOREM 4.3. *Let x_1, x_2 be two linearly independent solutions of* (0.1). *Let t_0, t_1 be consecutive zeros of x_1. Then under assumption* (vi), *x_2 vanishes exactly once on (t_0,t_1).*

PROOF. Since $x_1(t)$ and $-x_1(t)$ are solutions of (0.1) we can suppose $x'_1(t_0)>0$ hence $\varphi_1(t_0)=0$ and $\varphi_1(t_1)=\pi_1$. By Theorem 3.1 we have $x_2(t_0)\neq 0$, $x_2(t_1)\neq 0$. Again we can suppose that $x_2(t_0)>0$. Hence $\varphi_2(t_0)\in(0,\pi_1)$. Let $\varphi_3(t)=\varphi_1(t)+\pi_1$ then φ_1, φ_2, φ_3 are solutions of (3.2φ). The uniqueness implies that $\varphi_1(t)<\varphi_2(t)<\varphi_3(t)$ for every $t\in I$. Applying this inequality at $t=t_1$ we obtain $\pi_1<\varphi_2(t_1)<2\pi_1$. Then there is exactly one value $\bar{t}$ with $\varphi_2(\bar{t})=\pi_1$ in (t_0,t_1) hence $x_2(\bar{t})=0$ which was to be proved.

LEMMA 4.1. *Let t_i, t_{i+1}, t_{i+2} be three consecutive zeros of a solution $x(t)$ of* (0.1). *Then*

$$\pi_2 \leqq \int_{t_i}^{t_{i+2}} \max\left\{\frac{1}{p(t)}, q(t)\right\} dt.$$

PROOF. This is a simple consequence of (4.1) and (3.2).

COROLLARY 4.1. *The zeros of the solutions of* (0.1) *are not accumulating on any finite interval $J\subset I$.*

PROOF. Let us consider a solution $x(t)$ of (0.1) which has consecutive zeros $t_i, t_{i+1}, t_{i+2}, \ldots, t_{i+2k-1}, t_{i+2k}$ on J. Then by repeated applications of Lemma 4.1

we get

$$k\pi_2 \leqq \int_{t_i}^{t_{i+2k}} \max\left\{\frac{1}{p(t)}, q(t)\right\} dt \leqq \int_J \max\left\{\frac{1}{p}, q\right\} dt.$$

The integrand on the right hand side is finite thus k is bounded from above, i.e. on J there are only finitely many zeros, as it was stated.

The last theorems and Corollary 4.1 enable us to classify Eqs. (0.1) as oscillatory or nonoscillatory ones. We say that a solution $x(t)$ of (0.1) is *oscillatory* at $t=\infty$ if $x(t)$ has infinitely many zeros on $I=[a, \infty)$. Let us denote them by $t_0, t_1, t_2, \ldots$. Then we have $x(t_i)=0$, $i=0, 1, 2, \ldots$ and $\lim_{i\to\infty} t_i=\infty$. By Theorem 4.1. every other solution of (0.1) is also oscillatory. In this case we say that the differential equation (0.1) is oscillatory. If a solution $x(t)$ has a finite number of zeros or it has no one at all then $x(t)$ is *nonoscillatory*. Again by Theorem 4.1 we find that every solution of (0.1) is also nonoscillatory, hence in this case (0.1) is nonoscillatory. An open interval $J\subset I$ is said to be a positive (negative) *disconjugacy interval* of (0.1) if there is a solution $x(t)$ such that $x(t)>0$ $(x(t)<0)$ on J. Under assumption (vi) the two kinds of disconjugacy intervals coincide, therefore in this case we can omit the adjectives.

By our definition a D. Eq. of the form (0.1) with given coefficients $p(t)$, $q(t)$ is either oscillatory or nonoscillatory. Even in the linear case it is usually difficult to decide whether the actual D. Eq. is oscillatory or not. There are many known criteria for this purpose and we can expect even newer ones. We hope however that many of them can be generalized to our half-linear D. Eqs. A sufficient criterion for oscillation is given in the next theorem (see A. Wintner [26] for the linear case, I. Bihari [6] for the half-linear one).

THEOREM 4.4. *The D. Eq.* (0.1) *is oscillatory if* $q(t)\geqq 0$, *and* $\int^{\infty} dt/p(t)=\infty$, $\int^{\infty} q(t)dt=\infty$.

PROOF. Let us consider the solution $x(t)$ of (0.1) with the initial conditions $x(t_0)=0$, $x'(t_0)=1$ for some $t_0\in I$. Then the polar angle function $\varphi(t)$ satisfies the D. Eq. (3.2φ) and we may assume $\varphi(t_0)=0$. By (3.2φ) $\varphi'(t)\geqq 0$, hence $\varphi(t)$ is nondecreasing. By (4.1) $\varphi(t)$ tends to infinity if and only if $x(t)$ is oscillatory.

Suppose the contrary, i.e. let $x(t)$ be non-oscillatory. Then $\varphi(t)$ is bounded on I. Hence there is a finite number φ^* such that $\varphi^*=\lim_{t\to\infty}\varphi(t)$ and by (3.2φ) we obtain

$$\varphi^* = \int_{t_0}^{\infty} \left\{\frac{1}{p(t)}\big(1-G(\varphi(t))\big)+q(t)G\big(\varphi(t)\big)\right\} dt.$$

Now we may have two possibilities: a) $G(\varphi^*)>0$, b) $G(\varphi^*)=0$. In case a) let t^0 be chosen so large that $G\big(\varphi(t)\big)>\frac{1}{2}G(\varphi^*)$ if $t\geqq t^0$. Then

$$\varphi^* > \int_{t^0}^{\infty} q(t)\frac{G(\varphi^*)}{2} dt$$

which contradicts our assumption on the integral of $q(t)$. In case b) we choose t^0 again so large that $G(\varphi(t))<\frac{1}{2}$ for $t\geqq t^0$. Then we have

$$\varphi^* > \frac{1}{2}\int_{t_0}^{\infty}\frac{dt}{p(t)}$$

which is again a contradiction. Hence φ^* can not be finite, i.e. the D. Eq. is oscillatory, as it was stated.

We remark here that A. Wintner did not assume the relation $q(t)\geqq 0$. He needed only the integral condition $\int^{\infty} q(t)dt=\infty$. As it was pointed out by G. B. Gustafson [20] this criterion can not be generalized automatically to every half-linear D. Eq. (0.1). Namely he constructed a half-linear differential equation $(p(t)\equiv 1, q(t), f(x, y))$ which was not oscillatory in spite of the relation $\int^{\infty} q(t)dt=$ $=\infty$. Hence if we want to generalize the Wintner's result to the half-linear D. Eqs. we must impose further restrictions either on $q(t)$ or on $f(x, y)$ or on both functions simultaneously (see [20]).

Now we are going to investigate the zeros of the derivative of the solutions. Let $x(t)$ be a solution of (0.1). Suppose that $x(t)$ has consecutive zeros at $t_0, t_1, t_2, \ldots$. By Rolle's theorem the function $x'(t)$ vanishes on the intervals $I_1=(t_0, t_1)$, $I_2=$ $=(t_1, t_2)$, ... Let t_1' denote the zero on I_1. It may happen that $x'(t)$ has several zeros on I_1. In this case we denote the smallest one by t_{1-}' the largest one by t_{1+}', if necessary. The same convention holds for other intervals, too. By (3.1) we have $S'(\varphi(t_1'))=0$. If $\varphi(t_0)=0$ and $\varphi(t_1)=\pi_1$ as in (4.1), then from the construction of $S(\varphi)$ it follows that $\varphi(t_1')=\pi_{1/2}$ and similarly $\varphi(t_2')=\pi_{3/2}$ or in general

$$\begin{cases} \varphi(t_i') = \pi_{1/2}+\dfrac{i}{2}\pi_2 & \text{if} \quad i = 0, 2, 4, \ldots \\ \varphi(t_i') = \pi_{3/2}+\dfrac{i-1}{2}\pi_2 & \text{if} \quad i = 1, 3, 5, \ldots . \end{cases} \tag{4.3}$$

By (3.2) $\varphi'(t_i')=q(t_i')$, $i=1, 2, \ldots$. Hence if we have $q(t)>0$ on I_i then $x'(t)$ has only one zero t_i' on I_i.

The next theorems provide more insight into the connections between the solutions. Let us observe first that if $t=t_1'$ is a common zero of the derivatives of two solutions $x_1(t), x_2(t)$ then by Remark 3.1 x_1, x_2 are linearly dependent if $x_1(t_1')x_2(t_1')>$ >0 and under the assumption (vi) also in the case $x_1(t_1')x_2(t_1')<0$.

THEOREM 4.5. *Let* x_1, x_2 *be solutions of* (0. 1). *Suppose that* $x_1(t_0)=x_1'(t_1')=0$, $x_1'(t_0)>0$ *and* $x_2(t_0)>0$, $x_2'(t_0)>0$. *Then there is a* $t^*\in(t_0, t_1')$ *such that* $x_2'(t^*)=0$. *Some converse statement is also true: if there is a* $t^*\in(t_0, t_{1-}')$ *where* $x_2(t^*)>0$ *and* $x_2'(t^*)=0$ *then* $x_2(t_0)>0$.

PROOF. Let $\varphi_1(t), \varphi_2(t)$ be the polar angle functions defined by (3.1). Under our conditions we have $\varphi_1(t_0)=0$ and by (4.3) $\varphi_1(t_1')=\pi_{1/2}$ and $\varphi_2(t_0)\in(0, \pi_{1/2})$. Hence $\varphi_1(t_0)<\varphi_2(t_0)$. From the uniqueness of the solutions of (3.2φ) it follows that $\varphi_1(t)<\varphi_2(t)$ everywhere, especially at $t=t_1'$, $\varphi_1(t_1')=\pi_{1/2}<\varphi_2(t_1')$. $\varphi_2(t)$ is

continuous thus it takes on the value $\pi_{1/2}$ somewhere in the interval (t_0, t_1'), say at $t=t^*$, i.e. $\varphi_2(t^*)=\pi_{1/2}$. Then by (3.1) we have $x_2'(t^*)=0$. On the other hand if we have a solution $x_2(t)$ with $x_2(t^*)>0$, $x_2'(t^*)=0$ then we have a polar function $\varphi_2(t)$ such that $\varphi_2(t^*)=\pi_{1/2}=\varphi_1(t_{1-}')>\varphi_1(t^*)$. Hence $\varphi_1(t_0)=0<\varphi_2(t_0)$ and by (4.2) $\varphi_2(t_0)<\pi_1$ hence $x_2(t_0)>0$, which was to be proved.

THEOREM 4.6. *Let* x_1, x_2 *be solutions of* (0.1). *If there exist* t_1', t_1 *such that* $t_1'<t_1$, $x_1'(t_1')=x_1(t_1)=0$, $x_1'(t_1)<0$, *and* $x_2(t_1')>0$, $x_2'(t_1')<0$ *then there is a* $t^*\in(t_1', t_1)$ *with the property* $x_2(t^*)=0$, $x_2'(t^*)<0$. *The converse of this statement is also true: if there is a* $t^*\in(t_1', t_1)$ *such that* $x_2(t^*)=0$, $x_2'(t^*)<0$ *then* $x_2(t_1')>0$, $x_2'(t_1')<0$.

PROOF. It goes along the same lines as the proof of Theorem 4.5, hence it is omitted.

The following theorems can be considered as a counterpart of the last two theorems.

THEOREM 4.7. *Let* x_1, x_2 *be solutions of* (0.1). *Suppose that* x_1 *satisfies the relations* $x_1(t_0)=0$, $x_1'(t_0)>0$, $x_1'(t_{1+}')=0$ *where* t_{1+}' *is the largest zero of* $x_1'(t)$ *on* I_1. *If for some* $\hat{t}>t_{1+}'$ *the relations* $x_2(\hat{t})>0$, $x_2'(\hat{t})=0$ *hold then there is a* $t^*\in(t_0, \hat{t})$ *satisfying* $x_2(t^*)=0$, $x_2'(t^*)>0$.

PROOF. Now the polar angle function $\varphi_1(t)$ satisfies the relations $\varphi_1(t_0)=0$, $\varphi_1(t_{1+}')=\pi_{1/2}$ and $\varphi_1(\hat{t})>\pi_{1/2}$. The last inequality holds because t_{1+}' is the largest value where $\varphi_1(t_1')=\pi_{1/2}$ can happen. The polar function $\varphi_2(t)$ takes on the value $\pi_{1/2}$ at $\hat{t}$, hence $\pi_{1/2}=\varphi_2(\hat{t})<\varphi_1(\hat{t})$. Then the uniqueness implies $\varphi_2(t)<\varphi_1(t)$ everywhere in I, consequently $\varphi_2(t_0)<\varphi_1(t_0)=0$. The continuous function $\varphi_2(t)$ assumes the value 0 on $(t_0, \hat{t})$, say at t^*, hence $\varphi_2(t^*)=0$ and $x_2(t^*)=0$ as we stated.

THEOREM 4.8. *Let* x_1, x_2 *be solutions of* (0.1). *Let* $t_1', t_1(>t_1')$ *be associated with* x_1 *satisfying the relations* $x_1'(t_1')=0$, $x_1(t_1)=0$, $x_1'(t_1)<0$. *If there is a value* $\hat{t}>t_1$ *with* $x_2(\hat{t})=0$, $x_2'(\hat{t})<0$ *then there is a* $t^*\in(t_{1+}', \hat{t})$ *such that* $x_2'(t^*)=0$, $x_2(t^*)>0$.

PROOF. Comparing with the proof of Theorem 4.7 we do not need any new idea hence we omit the proof.

Theorems 4.5 and 4.6 are "forward" theorems in the sense that we have assumptions on $x_2(t)$ at the starting point of the interval in question and t^* is larger than this value. Consequently, Theorems 4.7, 4.8 are "backward" theorems. It is not difficult to formulate the "forward" version of Theorems 4.7, and 4.8. Moreover, similar statements hold on the interval $I_2=(t_1, t_2)$ where $x_2(t_2')<0$. Here we shall not formulate all these statements but the content of them will be called as the *strong interlacing properties* of the zeros and the extremants of the solutions of (0.1).

We say "strong" because the usual interlacing properties of the zeros formulated in Theorem 4.1 are consequences of the "strong" interlacing properties. Namely, if $x_1(t)$ is a solution which has three consecutive zeros t_0, t_1, t_2, then we have to show that any linearly independent solution $x_2(t)$ has at least one zero in (t_0, t_2) and at most three zeros in $[t_0, t_2]$. We show this by indirect argumentation. If $x_2(t)$ had no zero on (t_0, t_2) then $x_2(t)$ would be either positive or negative there. We can suppose $x_2(t)>0$ on (t_0, t_2). Then let us consider the value $x_2'(t_1')$. If $x_2'(t_1')<0$

then by Theorem 4.6 $x_2(t)$ has a zero in (t_1', t_1). If $x_2'(t_1')>0$ then by the backward version of Theorem 4.6, $x_2(t)$ has a zero on (t_0, t_1') contradicting the indirect assumption, i.e. $x_2(t)$ has at least one zero on (t_0, t_2).

Now we prove that $x_2(t)$ can not have four or more zeros on $[t_0, t_2]$. First we observe that $x_1(t)$ has no zeros on (t_0, t_1), hence by the previous result $x_2(t)$ can not have three zeros on $[t_0, t_1]$. Thus $x_2(t)$ may have at most two zeros on $[t_0, t_1]$ and another two on $[t_1, t_2]$. We denote them as $\hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3$ if $x_2'(\hat{t}_0)>0$ or $\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4$ $x_2'(\hat{t}_1)<0$. Applying (the forward or backward version of) Theorem 4.6 at the zero of $x_2'(t)$ in $(\hat{t}_0, \hat{t}_1)$ or in $(\hat{t}_3, \hat{t}_4)$ we get a zero for $x_1(t)$ in $(\hat{t}_0, \hat{t}_1)\subset[t_0, t_1]$ or in $(\hat{t}_3, \hat{t}_4)\subset[t_1, t_2]$ which contradicts our assumption on $x_1(t)$, namely the values t_0, t_1, t_2 are consecutive zeros of $x_1(t)$.

In the theorems above we have investigated two solutions of the same D. Eq. However, for the applications it is more important to get information about two solutions of two D. Eqs. This is the situation concerning the Sturmian comparison theorems known for linear second order D. Eqs. Now we are to generalize them to the half-linear D. Eqs. of the form (0.1).

Let us consider the following two half-linear D. Eqs:

$$(p(t)x')'+q(t)f(x, p(t)x') = 0, \tag{4.4$_1$}$$

$$(P(t)X')'+Q(t)f(X, P(t)X') = 0, \tag{4.4$_2$}$$

where the functions p, P are positive and continuous, q, Q are piecewise continuous on I. Then (4.4$_2$) is a *Sturmian majorant* of (4.4$_1$) on I if the inequalities

$$p(t)\geqq P(t), \quad q(t)\leqq Q(t) \quad \text{on} \quad I \tag{4.5}$$

hold. Let $x(t)$ and $X(t)$ be solutions of (4.4$_1$) and (4.4$_2$), respectively, satisfying the initial conditions at $t^0\in I$

$$\begin{cases} x(t^0) = \varrho_0 S(\varphi_0), & p(t^0)x'(t^0) = \varrho_0 S'(\varphi_0), \quad \varrho_0>0, \\ X(t^0) = \varrho_1 S(\Phi_0), & P(t^0)X'(t^0) = \varrho_1 S'(\Phi_0), \quad \varrho_1>0. \end{cases} \tag{4.6}$$

Then the first generalized Sturmian comparison theorem reads as follows.

THEOREM 4.9. *Let* (4.4$_2$) *be a Sturmian majorant of* (4.4$_1$) *and let* $x(t)$, $X(t)$ *be solutions of* (4.4$_1$) *and* (4.4$_2$), *respectively, satisfying the initial conditions* (4.6) *with* $0\leqq\varphi_0\leqq\Phi_0<\pi_1$. *Let* $x(t)$ *have exactly* n *zeros, say* $t_1, t_2, \ldots, t_n$ *on some interval* $[t^0, t^1]\subset I$. *Then* $X(t)$ *has at least* n *zeros on* $[t^0, t^1]$, *say, at* $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n$ *satisfying the relations* $\hat{t}_i\leqq t_i$ *for* $i=1, 2, \ldots, n$. *Moreover, for the zeros* $t_i', \hat{t}_i'$ *of* $x'(t)=0$ *and* $X'(t)=0$, *resp., the inequality* $\hat{t}_i'\leqq t_{i+}'$ *holds for* $i=2, \ldots, n$ *and also for* $i=1$ *if* $0\leqq\varphi_0\leqq\Phi_0\leqq\pi_{1/2}$. *A similar statement is true also in the case* $\pi_1\leqq\varphi_0\leqq$ $\leqq\Phi_0<\pi_2$.

PROOF. Let $\varphi(t)$, $\Phi(t)$ be the polar angle functions of the solutions $x(t)$, $X(t)$, resp. Then by (4.6) we have $\varphi(t^0)=\varphi_0\leqq\Phi_0=\Phi(t^0)$. By (3.2$\varphi$) and (4.4$_2$), $\Phi(t)$ is a solution of the differential equation

$$\Phi' = \frac{1}{P(t)}(1-G(\Phi))+Q(t)G(\Phi). \tag{4.7}$$

Let us denote the function on the right hand side of (4.7) by $G_1(t, \Phi)$ and similarly the one of (3.2φ) by $g_1(t, \varphi)$. Then (4.5) implies $g_1(t, \psi) \leqq G_1(t, \psi)$ on I for every ψ. Hence from the standard theory of differential inequalities (see e.g. Corollary 4.2 on p. 27 in [21]) it follows the inequality $\varphi(t) \leqq \Phi(t)$ on $[t^0, \infty) \cap I$. Let us consider the case $0 \leqq \varphi_0 \leqq \Phi_0 < \pi_1$. Then by (4.1) we have $\varphi(t_1) = \pi_1$ and $\Phi(t_1) \geqq \varphi(t_1) = \pi_1$. By the continuity of Φ there is a $\hat{t}_1 \leqq t_1$ with $\Phi(\hat{t}_1) = \pi_1$ or equivalently $X(\hat{t}_1) = 0$. Similarly, from the relations $\pi_2 = \varphi(t_2) \leqq \Phi(t_2)$ we have again a $\hat{t}_2$ in $(\hat{t}_1, t_2]$ such that $\Phi(\hat{t}_2) = \pi_2$, $X(\hat{t}_2) = 0$, and so on. The corresponding inequalities for t_i' and $\hat{t}_i'$ can be proved in the same manner.

REMARK 4.1. In Theorem 4.9 the strict inequalities $\hat{t}_i < t_i$ $(i = 1, 2, \ldots)$ and also $\hat{t}_i' < t_i'$ $(i = 1, 2, \ldots, n)$ hold if either $\varphi_0 < \Phi_0$ or $\varphi_0 = \Phi_0$ and $q(t) < Q(t)$ on some open interval in (t^0, t_1) or $\varphi_0 = \Phi_0$, $q(t) \equiv Q(t)$ everywhere on $[t^0, t_1]$ and $p(t) > P(t)$ and $q(t) \not\equiv 0$ on some open interval in (t^0, t_1). To get the sharp inequality $\hat{t}_1' < t_1'$, we should replace the interval (t^0, t_1) in the conditions above with (t^0, t_1').

The proof of these statements runs essentially on the same lines as in the last part of the proof of Theorem 3.1 in [21] on p. 335.

If we want to apply Theorem 4.9, checking the conditions $0 \leqq \varphi_0 \leqq \Phi_0 < \pi_1$ (and similarly $\pi_1 \leqq \varphi_0 \leqq \Phi_0 < \pi_2$) is not convenient because in (4.6) the functions $S(\varphi)$, $S'(\varphi)$ appear. Let $\tilde{T}(\varphi)$ be defined by

$$\tilde{T}(\varphi) = \frac{S'(\varphi)}{S(\varphi)} = \frac{1}{T(\varphi)}.$$

Since $T(0) = T(\pi_1) = 0$, $\tilde{T}(\varphi)$ can not be defined for $\varphi = 0$ and $\varphi = \pi_1$. But it is clear that $\tilde{T}(\varphi)$ is a positive decreasing function on $(0, \pi_{1/2})$ and negative decreasing on $(\pi_{1/2}, \pi_1)$. Letting $\tilde{T}(\pi_{1/2}) = 0$ we obtain a continuous, strictly decreasing function on $(0, \pi_1)$. Hence the condition $0 \leqq \varphi_0 \leqq \Phi_0 < \pi_1$ in Theorem 4.9 is equivalent to the following:

$$(4.8)\quad \begin{cases} \text{if} \quad x(t^0) > 0 \quad \text{then} \quad X(t^0) > 0 \quad \text{and} \quad \dfrac{p(t)x'(t^0)}{x(t^0)} \geqq \dfrac{P(t^0)X'(t^0)}{X(t^0)} \\ \text{if} \quad x(t^0) = 0 \quad \text{then} \quad x'(t^0) > 0 \quad \text{and either} \quad X(t^0) > 0 \\ \qquad\qquad\quad \text{or} \quad X(t^0) = 0, \quad X'(t^0) > 0. \end{cases}$$

A similar set of inequalities can be given in the case $\pi_1 \leqq \varphi_0 \leqq \Phi_0 < \pi_2$. If assumption (vi) holds then the functions $x(t)$ and $-x(t)$ are at the same time solutions of (0.1) or not, hence the zeros of the equation $x(t) = 0$, $-x(t) = 0$ or $x'(t) = 0$, $-x'(t) = 0$ are the same. Thus in this case it is sufficient to check the condition $0 \leqq \varphi_0 \leqq \Phi_0 < \pi_1$ and we have either $x(t^0) = 0$ (then in Theorem 4.9 the assumption $0 = \varphi_0 \leqq \Phi_0$ always holds) or $x(t^0) \neq 0$ (then we can drop the sign requirements on $x(t^0)$, $X(t^0)$ in the first line of (4.8) while in the second line we replace the inequalities $x'(t^0) > 0$, $X'(t^0) > 0$ by $x'(t^0) \neq 0$, $X'(t^0) \neq 0$, resp.)

The second important theorem of comparison type for linear D. Eqs. is due to G. N. Watson [25] who called it Sturmian type theorem. It gives information also on the values of the solutions.

THEOREM 4.10. *Let* (4.4_2) *be a Sturmian majorant of* (4.4_1) *and* $p(t) \equiv P(t)$. *Let* $x(t)$, $X(t)$ *be solutions of* (4.4_1), (4.4_2), *resp. such that* $x(t^0) = \lambda X(t^0)$, $x'(t^0) =$

$=\lambda X'(t^0)$ *with some* $\lambda>0$. *Suppose* $X(t)\neq 0$ *on* $(\hat{t}_0, \hat{t}_1)$ *if* $X(t^0)\neq 0$ *where* $\hat{t}_0<t^0<\hat{t}_1$ *or* $X(t)\neq 0$ *on* $(\hat{t}_0, t^0)\cup(t^0, \hat{t}_1)$ *if* $X(t^0)=0$. *Then* $x'(t)/x(t)\gtreqless X'(t)/X(t)$ *according to* $t\gtreqless t^0$, $t\in(\hat{t}_0, \hat{t}_1)$ *and*

$$\frac{x(t)}{x(t^0)}\geqq\frac{X(t)}{X(t^0)} \quad \textit{for} \quad t\in(\hat{t}_0, \hat{t}_1) \quad \textit{if} \quad X(t_0)\neq 0$$

and

$$\operatorname{sgn}(t-t^0)\left[\frac{x(t)}{x'(t)}-\frac{X(t)}{X'(t^0)}\right]\geqq 0 \quad \textit{for} \quad t\in(\hat{t}_0, \hat{t}_1) \quad \textit{if} \quad X(t^0)=0.$$

REMARK 4.1. We call the reader's attention to the fact that in Theorem 4.10 the values $\hat{t}_0, \hat{t}_1$ are not necessarily zeros of $X(t)$ as it was usual in the former theorems.

PROOF. We define again the functions $\varphi(t)$, $\Phi(t)$ as in the proof of Theorem 4.9. We have again $\varphi(t^0)=\Phi(t^0)$, $\varphi(t)\geqq\Phi(t)$ on $[t^0, \hat{t}_1]$ and similarly $\varphi(t)\geqq\Phi(t)$ on $[\hat{t}_0, t^0]$. Let us suppose first $X(t^0)\neq 0$, hence $\varphi(t^0)=\Phi(t^0)\in(0, \pi_1)\cup(\pi_1, \pi_2)$. For $X(t)\neq 0$ on $(\hat{t}_0, \hat{t}_1)$ we have $0<\Phi(t)<\pi_1$ or $\pi_1<\Phi(t)<\pi_2$. Taking into consideration that $\tilde{T}(\varphi)$ decreases we obtain $\tilde{T}(\varphi(t))\gtreqless\tilde{T}(\Phi(t))$ or equivalently

$$\frac{x'(t)}{x(t)}\gtreqless\frac{X'(t)}{X(t)} \quad \text{for} \quad t\gtreqless t^0, \quad t\in(\hat{t}_0, \hat{t}_1) \tag{4.9}$$

provided $X(t^0)\neq 0$. Then an integration of this inequality over $[t^0, t]\subset[t^0, \hat{t}_1]$ or over $[t, t^0]\subset(\hat{t}_0, t^0]$ yields the desired inequality $x(t)/x(t^0)\geqq X(t)/X(t^0)$.

The inequality (4.9) remains valid if $X(t^0)=0$, too, which can be shown by the same way as above when it was $X(t^0)\neq 0$, considering independently the intervals $(\hat{t}_0, t^0)$ and $(t^0, \hat{t}_1)$. Let us integrate (4.9) over $[t', t]\subset(t^0, \hat{t}_1)$ then we get

$$\frac{x(t)}{x(t')}\geqq\frac{X(t)}{X(t')}. \tag{4.10}$$

Since

$$\lim_{t'\to t^0+0} x(t')/(t'-t^0)=x'(t^0), \quad \lim_{t'\to t^0+0} X(t')/(t'-t^0)=X'(t^0)$$

from (4.10) by letting $t'\to t^0$ we obtain that $x(t)/x'(t^0)\geqq X(t)/X'(t^0)$ for $t^0<t<\bar{t}_1$. The same method works for $\hat{t}_0<t<t^0$, only here we should have a care with the sign of $t'-t^0$ in the inequalities. Thereby the proof is complete.

REMARK 4.2. From the inequality (4.9) we can derive more than we had in Theorem 4.10. A stronger result is that the function $x(t)/X(t)$ is nondecreasing on $[t^0, \hat{t}_1)$ and nonincreasing on $(\hat{t}_0, t^0]$.

REMARK 4.3. If we have the sharp inequality $x'(t^0)/x(t^0)>X'(t^0)/X(t^0)$ among the initial values at $t=t^0$ instead of the equality as in Theorem 4.10, then under the assumptions of Theorem 4.10 on $p, P, q, Q, \hat{t}_1$ the sharp inequalities

$$\frac{x'(t)}{x(t)}>\frac{X'(t)}{X(t)}, \quad \frac{x(t)}{x(t^0)}>\frac{X(t)}{X(t^0)} \quad \text{for} \quad t^0\leqq t<\hat{t}_1$$

hold provided $x(t^0)X(t^0)>0$.

5. Appendix. Our first aim is to show that the restrictions (Bi)—(Biii) imply (iv'). Since now (vi) also holds we have to prove that the function $L(T)=\log(1+F(T))$ is uniformly Lipschitzian. By (Bi) there is a constant $C>0$ such that

$$|f(x,y)-f(x_1,y_1)| \leqq C(|x-x_1|+|y-y_1|) \quad \text{if} \quad |x|, |y|, |x_1|, |y_1| \leqq 1. \tag{5.1}$$

Let us consider the function $\theta(T)=|T|/(1+F(T))$. Then $\theta(T)$ is continuous on $(-\infty, \infty)$ and by (1.2) $\lim_{|T|\to\infty} \theta(T)=0$, hence $\theta(T)$ is bounded. Let M be defined by

$$M = \max\{\theta(T);\ -\infty < T < \infty\}. \tag{5.2}$$

Now we can formulate our statement as a lemma.

LEMMA 5.1. *Let the constants C and M be defined by* (5.1) *and* (5.2). *Then under the restrictions* (Bi)—(Biii) *the inequality*

$$|L(T_2)-L(T_1)| < C(2M+3)(T_2-T_1), \quad -\infty < T_1 < T_2 < \infty \tag{5.3}$$

holds.

PROOF. Since $F(0)=0$ it suffices to consider the cases when $0 \leqq T_1 < T_2$ or $T_1 < T_2 \leqq 0$. We may suppose that the first one holds. If $0 \leqq T_1 < T_2 \leqq 1$ then making use of the well-known inequality $|\log(1+x)-\log(1+y)| < |x-y|$ for $x>0$, $y>0$ we obtain by (5.1) and by $F(T)=f(T^2, T)$

$$|L(T_2)-L(T_1)| < |F(T_2)-F(T_1)| < C(T_2^2-T_1^2+T_2-T_1) < 3C(T_2-T_1)$$

hence (5.3) holds for $0 \leqq T_1 < T_2 \leqq 1$.

Now we suppose $1 \leqq T_1 < T_2$. Let k be a sufficiently large integer and choose $\tau_i = T_1 + i(T_2-T_1)/k$ for $i=0, 1, \ldots, k$. Then by the continuity of $F(t)$ we get

$$L(T_2)-L(T_1) = \sum_{i=1}^{k} [L(\tau_i)-L(\tau_{i-1})] = \sum_{i=1}^{k} \frac{F(\tau_i)-F(\tau_{i-1})}{1+F(\tau_i')}$$

with some $\tau_i' \in (\tau_{i-1}, \tau_i)$. Making use of (Biii) and (5.1) we obtain

$$\frac{1}{\tau_i^2}|F(\tau_i)-F(\tau_{i-1})| = \left|f\left(1, \frac{1}{\tau_i}\right) - f\left(\frac{\tau_{i-1}^2}{\tau_i^2}, \frac{\tau_{i-1}}{\tau_i^2}\right)\right| <$$

$$< C\left(1-\frac{\tau_{i-1}^2}{\tau_i^2}+\frac{1}{\tau_i}-\frac{\tau_{i-1}}{\tau_i^2}\right) < C\left(\frac{2}{\tau_i}+\frac{1}{\tau_i^2}\right)(\tau_i-\tau_{i-1})$$

hence

$$|L(T_2)-L(T_1)| < 2C\sum_{i=1}^{k} \frac{\tau_i}{1+F(\tau_i')}(\tau_i-\tau_{i-1}) + C\sum_{i=1}^{k} \frac{\tau_i-\tau_{i-1}}{1+F(\tau_i')}.$$

Letting $k \to \infty$ we have by (5.2)

$$|L(T_2)-L(T_1)| \leqq 2C\int_{T_1}^{T_2} \frac{\tau d\tau}{1+F(\tau)} + C\int_{T_1}^{T_2} \frac{d\tau}{1+F(\tau)} < (2CM+C)(T_2-T_1),$$

hence the estimate (5.3) holds again.

If $0 \leqq T_1 < 1 < T_2$ then we have

$$|L(T_2)-L(T_1)| \leqq |L(1)-L(T_1)|+|L(T_2)-L(1)| <$$
$$< 3C(1-T_1)+(2CM+C)(T_2-1) < C(2M+3)(T_2-T_1).$$

Thus the inequality (5.3) is true for $T_1 \geqq 0$, $T_2 \geqq 0$. The proof in the case $T_1 < T_2 \leqq 0$ goes similarly, hence (5.3) is valid for arbitrary T_1, T_2, which proves Lemma 5.1.

In assumption (v) we suppose that $\lim_{|T|\to\infty} F_{\pm}(T) = \infty$. In the next lemma we show that if we have assumption (iv′) then this limit-relation will be a consequence of the integrals in (v).

LEMMA 5.2. *Assumption* (vi′) *and the relation* $\int_0^{\infty} dT/(1+F_+(T)) < \infty$ *imply* $\lim_{T\to\infty} F_+(T) = \infty$.

PROOF. Suppose the contrary. Then there would be a sequence $\{T_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} T_k = \infty$ and $\lim_{k\to\infty} F_+(T_k) < \infty$. Then we select T_k's so that $T_{k+1} > T_k + 1$ holds for every $k \geqq 1$. Applying (iv′) we have a constant C_1 such that

$$\left|\log(1+F_+(T)) - \log(1+F_+(T_k))\right| < C_1(T-T_k),$$

hence

$$\frac{1}{1+F_+(T)} > \frac{1}{1+F_+(T_k)} e^{-C_1(T-T_k)} \quad \text{for} \quad T > T_k.$$

Then

$$\int_0^{\infty} \frac{dT}{1+F_+(T)} > \sum_{k=1}^{\infty} \int_{T_k}^{T_k+1} \frac{e^{-C_1(T-T_k)}}{1+F_+(T_k)}\, dT = \frac{1-e^{-C_1}}{C_1} \sum_{k=1}^{\infty} \frac{1}{1+F_+(T_k)} = \infty$$

which contradicts our integral assumption, hence $\lim_{T\to\infty} F_+(T) = \infty$.

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