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SYNTOPOGENOUS SPACES WITH PREORDER. V (EXTENSIONS)

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The present part of the series ([11]—[14]) deals with the extensions of preordered syntopogenous spaces. Our results give the generalizations of numerous notions and theorems concerning the classical theory of compactifications of ordered topological and proximity spaces and the completion of ordered uniform spaces (cf. [4], [5], [15]—[18]). Since in this paper no convexity or separation condition of the initial space is assumed, more general connections can be obtained even if only the particular topological, proximity or uniform case is considered. In the construction of these extensions the methods developed in [6]—[10] will be used, but the reader who is not familiar with this theory can find a short introduction to syntopogenous extension theory in Section 2, too.

In the first version of the manuscript of this article the proof of Theorems (4.3) and (4.7) was partly incorrect. I am very grateful to Prof. Á. Császár for pointing out these errors.

0. Introduction

A reflexive and transitive relation $\leqq$ on a set E is called a *preorder,* and the pair $(E, \leqq)$ is a *preordered space.* A preorder $\leqq$ is an *order* iff $x, y \in E$, $x \leqq y$ and $y \leqq x$ imply $x=y$. $X \subset E$ is said to be *increasing (decreasing)* iff $x \in X$, $x \leqq y$ $(y \leqq x)$ imply $y \in X$. $i(X)$ and $d(X)$ denote the smallest of all increasing and decreasing sets, respectively, containing an arbitrary $X \subset E$. The *graph* of $\leqq$ is defined by $G(\leqq) = \{(x, y) : x \leqq y\}$. If $\leqq'$ is another preorder on E then we say that $\leqq'$ is *finer* than $\leqq$ iff $G(\leqq') \subset G(\leqq)$. In this case every increasing or decreasing set of $(E, \leqq)$ is increasing or decreasing in $(E, \leqq')$. A mapping f of $(E, \leqq)$ into a preordered space $(E', \leqq')$ is *preorder preserving (inversing)* iff $x, y \in E$, $x \leqq y$ imply $f(x) \leqq' f(y)$ $(f(y) \leqq' f(x))$.

In respect both of terminology and notations concerning *syntopogenous spaces* we follow the monograph [6].

A *preordered syntopogenous space* is a triplet $(E, \mathscr{S}, \leqq)$ consisting of a set E, a syntopogenous structure $\mathscr{S}$ and a preorder $\leqq$ on E (cf. [1]—[3], [11]—[14]). If, for $< \in \mathscr{S}$, we define $G(<) = \{(x, y) \in E \times E : x < E - y \text{ is false}\}$, then $G(\mathscr{S}) = \bigcap \{G(<) : < \in \mathscr{S}\}$ is the graph of a preorder on E. $(E, \mathscr{S}, \leqq)$ (or shortly $\mathscr{S}$) is *increasing (decreasing)* iff $G(\leqq) \subset G(\mathscr{S})$ $\big(G(\leqq)^{-1} \subset G(\mathscr{S})\big)$ (cf. [11], [1]—[3]). In an arbitrary space $(E, \mathscr{S}, \leqq)$, $\mathscr{S}^u$ $(\mathscr{S}^l)$ denotes the finest of all increasing (decreasing) syntopogenous structures on $(E, \leqq)$ coarser than $\mathscr{S}$, and it is called the *upper (lower)* syntopogenous structure of $(E, \mathscr{S}, \leqq)$ (see [2]). If ^{a} is an elementary operation ([6], p. 80) then $(E, \mathscr{S}, \leqq)$ is said to be *^{a}-convex* iff $\mathscr{S} \sim (\mathscr{S}^u \vee \mathscr{S}^l)^a$ ([11], cf. [1]—[3]

for $^a={}^i$, ^{p} or ^{b}, see also [11], (4.2)). $(E, \mathcal{S}, \leqq)$ is called *symmetrizable* [11] iff $\mathcal{S}_0 < \mathcal{S} < \mathcal{S}_0^p$ for a symmetrical ^{i}-convex syntopogenous structure $\mathcal{S}_0$ on $(E, \leqq)$. In the present paper we shall consider only the special case in which $\mathcal{S}$ is perfect. Then $\mathcal{S} \sim \mathcal{S}_0^p$. A space $(E, \mathcal{S}, \leqq)$ is *continuous* [12] iff, for any $<\in\mathcal{S}$, there exists $<_1\in\mathcal{S}$ such that $A<B$ implies $i(A)<_1 i(B)$ and $d(A)<_1 d(B)$. $(E, \mathcal{S}, \leqq)$ is T_0*-preordered* iff $G(\leqq)=G(\mathcal{S}^u)\cap G(\mathcal{S}^{lc})$ (see [13]).

1. Relatively T_0- preordered spaces

A preordered syntopogenous space $(E', \mathcal{S}', \leqq')$ will be said to be *relatively T_0-preordered with respect to* $E\subset E'$, if

(1.1) $$(G(\mathcal{S}'^u)\cap G(\mathcal{S}'^{lc}))-(E\times E) \subset G(\leqq').$$

It is obvious that a T_0-preordered space is also relatively T_0-preordered with respect to each of its subspaces. A discretely ordered space $(E', \mathcal{S}', =)$ is relatively T_0-ordered with respect to $E\subset E'$ iff $[E', \mathcal{S}']$ is relatively separated with respect to E (cf. [6], p. 241).

(1.2) LEMMA. *Let $(E, \mathcal{S}, \leqq)$ and $(E', \mathcal{S}', \leqq')$ be preordered syntopogenous spaces such that $E\subset E'$, $\mathcal{S}'|E < \mathcal{S}$ and $G(\leqq)\subset G(\leqq')$. Then*

(1.2.1) *$\mathcal{S}'^u|E < \mathcal{S}^u$ and $\mathcal{S}'^l|E < \mathcal{S}^l$.*

(1.2.2) *If $(E', \mathcal{S}', \leqq')$ is T_0-preordered and $G(\leqq)=G(\leqq')\cap E\times E)$, then $(E, \mathcal{S}, \leqq)$ is also T_0-preordered.*

(1.2.3) *If $(E, \mathcal{S}, \leqq)$ is T_0-preordered, $\mathcal{S}'^u|E \sim \mathcal{S}^u$ and $\mathcal{S}'^l|E \sim \mathcal{S}^l$, moreover $(E', \mathcal{S}', \leqq')$ is relatively T_0-preordered with respect to E, then $(E', \mathcal{S}', \leqq')$ is also T_0-preordered and $G(\leqq)=G(\leqq')\cap(E\times E)$.*

PROOF. (1.2.1): The canonical injection of E into E' is continuous and preorder preserving, thus $\mathcal{S}'^u|E$ is increasing, $\mathcal{S}'^l|E$ is decreasing, and both of them are coarser than $\mathcal{S}$ (see [11], (1.1.4)).

(1.2.2): The canonical injection satisfies the conditions of [13], (1.9).

(1.2.3): It is easy to verify that $\mathcal{S}'^u|E\sim\mathcal{S}^u$ implies $G(\mathcal{S}'^u)\cap(E\times E)=G(\mathcal{S}^u)$, and $\mathcal{S}'^l|E\sim\mathcal{S}^l$ implies $G(\mathcal{S}'^{lc})\cap(E\times E)=G(\mathcal{S}^{lc})$. Hence $G(\mathcal{S}'^u)\cap G(\mathcal{S}'^{lc})\cap$ $\cap(E\times E)=G(\mathcal{S}^u)\cap G(\mathcal{S}^{lc})=G(\leqq)\subset G(\leqq')$. $(E', \mathcal{S}', \leqq')$ satisfies (1.1), therefore $G(\mathcal{S}'^u)\cap G(\mathcal{S}'^{lc})\subset G(\leqq')$. Finally $G(\leqq')\subset G(\mathcal{S}'^u)\cap G(\mathcal{S}'^{lc})$ (cf. [13], (1.0)), thus $G(\leqq')\cap(E\times E)=G(\mathcal{S}'^u)\cap G(\mathcal{S}'^{lc})\cap(E\times E)=G(\mathcal{S}^u)\cap G(\mathcal{S}^{lc})=G(\leqq)$. □

If $(E', \leqq')$ is a preordered space then, for each $\emptyset\neq E\subset E'$, a preorder $\leqq$ can be defined on E by

(1.3) $$G(\leqq) = G(\leqq')\cap(E\times E).$$

Conversely, if E' is a set, $E\subset E'$, and $\leqq$ is a preorder on E, then the definition

$$G(\leqq') = G(\leqq)\cup\{(x, x)\colon x\in E'-E\}$$

yields a preorder on E' satisfying (1.3). If, in addition, we consider a syntopogenous structure $\mathcal{S}'$ on E', $\leqq$ cannot be always extended onto E' so that the originating preordered syntopogenous space $(E', \mathcal{S}', \leqq')$ be relatively T_0-preordered with respect to E.

(1.4) EXAMPLE. Let $\mathcal{O}$ be the indiscrete syntopogenous structure of E' ([6], p. 95) and E be a preordered subset of E' such that $\emptyset \neq E'-E$, and $x \not\leqq y$ for at least one pair $(x, y) \in E \times E$. Then there is no preorder $\leqq'$ on E', for which $G(\leqq) = G(\leqq') \cap (E \times E)$ and $(E', \mathcal{O}, \leqq')$ is relatively T_0-preordered with respect to E. In fact, $G(\mathcal{O}) = E' \times E'$ would imply $(E' \times E') - (E \times E) \subset (G(\mathcal{O}^u) \cap G(\mathcal{O}^{lc})) - (E \times E) \subset G(\leqq')$, thus $x \leqq' u \leqq' y$ for some $u \in E'-E$, consequently $(x, y) \in G(\leqq') \cap (E \times E) = G(\leqq)$, which contradicts the choice of x and y. $\square$

The following theorem shows that a condition weaker than (1.3) can always be fulfilled by a suitable preorder on $[E', \mathcal{S}']$:

(1.5) THEOREM. *Let $[E', \mathcal{S}']$ be a syntopogenous space, $E \subset E'$, and $\leqq$ be a preorder on E. Then there exists a preorder $\leqq'$ on E' with the following conditions:*

(1.5.1) *$G(\leqq) \subset G(\leqq')$.*

(1.5.2) *$(E', \mathcal{S}', \leqq')$ is relatively T_0-preordered with respect to E.*

(1.5.3) *$\leqq'$ is the finest of all preorders on E' satisfying* (1.5.1) *and* (1.5.2).

PROOF. Let the ordering structure Φ (Ψ) consist of all $(\mathcal{S}', \mathcal{I})$-continuous ordering families φ (ψ) such that, for any $f \in \varphi$ $(g \in \psi)$, $f|E$ is preorder preserving ($g|E$ is preorder inversing). $\Phi \neq \emptyset$, $\Psi \neq \emptyset$, because the family of all constant functions is in $\Phi \cap \Psi$. If $x, y \in E'$, then define $x \leqq' y$ iff there exist points $x_0, x_1, \ldots, x_n \in E'$ such that

$$(1.5.4.)\quad \begin{cases} x = x_0,\ x_n = y, \quad \text{and for any} \quad 0 \leqq i < n, \\ (x_i, x_{i+1}) \in E \times E \Rightarrow x_i \leqq x_{i+1},\ (x_i, x_{i+1}) \notin E \times E \Rightarrow f(x_i) \leqq f(x_{i+1}) \quad \text{and} \\ g(x_{i+1}) \leqq g(x_i) \quad \text{for each} \quad f \in \varphi \in \Phi \quad \text{and} \quad g \in \psi \in \Psi. \end{cases}$$

It is obvious that $\leqq'$ is a preorder on E' such that $G(\leqq) \subset G(\leqq')$. Any $f \in \varphi \in \Phi$ is preorder preserving, and any $g \in \psi \in \Psi$ is preorder inversing, thus $\mathcal{S}_\Phi$ is decreasing and $\mathcal{S}_\Psi$ is increasing on $(E', \leqq')$ (see [6], (12.13), (12.24); [11], (1.5), (1.1.6)). We have $\mathcal{S}_\Phi < \mathcal{S}'$ and $\mathcal{S}_\Psi < \mathcal{S}'$ by [6], (12.33), hence $\mathcal{S}_\Phi < \mathcal{S}'^l$ and $\mathcal{S}_\Psi < \mathcal{S}'^u$. Suppose $(x, y) \notin E \times E$, $x \not\leqq' y$. Then either $f(y) < f(x)$ or $g(x) < g(y)$ for some $f \in \varphi \in \Phi$ or $g \in \psi \in \Psi$. Then $y <_{\varphi, \varepsilon} E' - x$ or $x <_{\psi, \varepsilon} E' - y$ for a suitable $\varepsilon > 0$, therefore $(x, y) \notin G(\mathcal{S}'^{lc})$ or $(x, y) \notin G(\mathcal{S}'^u)$. From this $(G(\mathcal{S}'^u) \cap G(\mathcal{S}'^{lc})) - (E \times E) \subset G(\leqq')$.

Suppose that $\leqq''$ is a preorder on $[E', \mathcal{S}']$ satisfying (1.5.1)—(1.5.2). We show $G(\leqq') \subset G(\leqq'')$. In fact, assume $x, y \in E'$, $x \leqq' y$. There exists $x_0, x_1, \ldots, x_n \in E'$ with (1.5.4). If $(x_i, x_{i+1}) \in E \times E$ then $x_i \leqq'' x_{i+1}$ by $G(\leqq) \subset G(\leqq'')$. If $(x_i, x_{i+1}) \notin E \times E$ and $x_i \not\leqq'' x_{i+1}$, then $x_i <_{\psi, \varepsilon} E' - x_{i+1}$ or $x_{i+1} <_{\varphi, \varepsilon} E' - x_i$, where ψ (φ) is an $(\mathcal{S}', \mathcal{I})$-continuous ordering family on E' consisting of preorder inversing (preserving) functions with respect to $\leqq''$, and $\varepsilon > 0$ is a suitable real number (see (1.5.2); [11], (1.12)). Since $G(\leqq) \subset G(\leqq'')$, the elements of $\psi|E$ $(\varphi|E)$ are preorder inversing (preserving), hence $\psi \in \Psi$ $(\varphi \in \Phi)$. We have $g(x_i) <_\varepsilon \mathbf{R} - g(x_{i+1})$ (or

$f(x_{i+1}) <_\varepsilon \mathbf{R} - f(x_i))$ for some $g \in \psi$ (or $f \in \varphi$), consequently $g(x_i) < g(x_{i+1})$ (or $f(x_{i+1}) < f(x_i)$), which contradicts the choice of x_i and x_{i+1}. Thus $x_i \leqq'' x_{i+1}$, and from here $x = x_0 \leqq'' x_1 \leqq'' \ldots \leqq'' x_n = y$, i.e. $x \leqq'' y$. □

(1.6) THEOREM. *Under the conditions of* (1.5) *let us consider the preorder* $\leqq'$ *determined by* (1.5.1)—(1.5.3), *and let* f *be a continuous mapping of* $(E', \mathscr{S}', \leqq')$ *into a preordered syntopogenous space* $(E'', \mathscr{S}'', \leqq'')$. *Suppose either*

(1.6.1) $(E'', \mathscr{S}'', \leqq'')$ *is* T_0*-preordered, or*

(1.6.2) f *is an injection and* $(E'', \mathscr{S}'', \leqq'')$ *is relatively* T_0*-preordered with respect to* $f(E)$.

In order that f *be preorder preserving it is necessary and sufficient that* $f|E$ *be also preorder preserving with respect to* $\leqq$.

PROOF. The necessity is evident by (1.5.1). For the verification of the sufficiency, let us define a preorder $\leqq_0$ on E' by the equivalence $x \leqq_0 y \Leftrightarrow f(x) \leqq'' f(y)$. Then $G(\leqq) \subset G(\leqq_0)$ is obvious. f is preorder preserving with respect to $\leqq_0$, hence $f^{-1}(\mathscr{S}''^u) < \mathscr{S}_1$ and $f^{-1}(\mathscr{S}''^l) < \mathscr{S}_2$, where $\mathscr{S}_1$ and $\mathscr{S}_2$ denote the upper and the lower syntopogenous structure of $(E', \mathscr{S}', \leqq_0)$, respectively (see [11], (1.1.4)). Suppose $(x, y) \in (E' \times E') - (E \times E)$ and $x \not\leqq_0 y$. Then $f(x) \not\leqq'' f(y)$, either (1.6.1) or (1.6.2) is satisfied, thus we have $f(x) <_1'' E'' - f(y)$ or $f(y) <_2'' E'' - f(x)$, where $<_1'' \in \mathscr{S}''^u$, $<_2'' \in \mathscr{S}''^l$. There exists $<_1' \in \mathscr{S}_1$ or $<_2' \in \mathscr{S}_2$ such that $f^{-1}(<_1'') \subset <_1'$ or $f^{-1}(<_2'') \subset <_2$, therefore $x <_1' E' - y$ or $y <_2' E' - x$. This shows that $(E', \mathscr{S}', \leqq_0)$ is relatively T_0-preordered with respect to E, hence $G(\leqq') \subset G(\leqq_0)$ by (1.5). From here $x, y \in E'$, $x \leqq' y$ imply $x \leqq_0 y$, i.e. $f(x) \leqq'' f(y)$, thus f is preorder preserving. □

2. Extension theoretical preliminaries

The syntopogenous space $[E', \mathscr{S}']$ (or the syntopogenous structure $\mathscr{S}'$) is said to be an *extension* of the syntopogenous space $[E, \mathscr{S}]$ (or shortly $\mathscr{S}$) if E is a dense subset of $[E', \mathscr{S}']$, and $\mathscr{S}'|E \sim \mathscr{S}$ (see e.g. [8]).

A filter base $\mathbf{r}$ is called *round* in $[E, \mathscr{S}]$ iff, for any $R \in \mathbf{r}$, there exists $R_1 \in \mathbf{r}$ such that $R_1 < R$ for some $< \in \mathscr{S}$ ([6], p. 240).

Suppose that $\mathbf{r}$ is an arbitrary filter base in $[E, \mathscr{S}]$, and define

$$\mathscr{S}(\mathbf{r}) = \{V \subset E\colon \exists R \in \mathbf{r},\ \exists < \in \mathscr{S},\ R < V\}.$$

The following statements are obvious:

(2.1) $\mathscr{S}(\mathbf{r})$ is a round filter in $[E, \mathscr{S}]$.

(2.2) If $\mathbf{s}$ is a round filter in $[E, \mathscr{S}]$ then $\mathscr{S}(\mathbf{s}) = \mathbf{s}$.

(2.3) If $\mathscr{S}_1$ is a syntopogenous structure on E such that $\mathscr{S}_1 < \mathscr{S}$ then $\mathscr{S}_1(\mathscr{S}(\mathbf{r})) = \mathscr{S}_1(\mathbf{r})$.

Introducing the notations $\mathscr{S}(\{A\}) = \mathscr{S}(A)$ and $\mathscr{S}(\{x\}) = \mathscr{S}(x)$ for $A \subset E$ and $x \in E$, $\mathscr{S}(A)$ and $\mathscr{S}(x)$ are the *neighbourhood filters* of A and x, respectively.

If $[E', \mathcal{S}']$ is an extension of the syntopogenous space $[E, \mathcal{S}]$ then the filters $\mathcal{S}'(x)(\cap)\{E\} = \{V \cap E: V \in \mathcal{S}'(x)\}$ $(x \in E')$ are round in $[E, \mathcal{S}]$, and they are called the *trace filters* of this extension (see e.g. [8], (1.5)).

One of the most important methods for the construction of an extension of a syntopogenous space was described in detail in [8], Theorem 3.1. In our further examinations we need a more general form of this theorem, which can be deduced immediately from [9], (2.3), (2.2), (1.3) and (2.4).

(2.4) THEOREM. *Let* $[E, \mathcal{S}]$ *be a syntopogenous space and* E' *be a set such that* $E \subset E'$. *Suppose that, for each* $x \in E'$, *a filter* $\mathbf{s}(x)$ *is given in* E *with the condition*

(2.4.1) $$\mathcal{S}(\mathbf{s}(x)) = \mathcal{S}(x) \quad \textit{for} \quad x \in E.$$

Put, for $A \subset E$,

(2.4.2) $$s(A) = \{x \in E': A \in \mathbf{s}(x)\}.$$

If $< \in \mathcal{S}$, *the definition*

(2.4.3) $$A <' B \Leftrightarrow \exists A_0, B_0 \subset E: A_0 < B_0, A \subset s(A_0), s(B_0) \subset B$$

yields a semi-topogenous order $<'$ *on* E'. *With the notation*

(2.4.4) $$s(<) = <'^q$$

the order family

(2.4.5) $$s(\mathcal{S}) = \{s(<): < \in \mathcal{S}\}$$

is a syntopogenous structure, which is an extension of $\mathcal{S}$, *and the trace filters are of the form* $\mathcal{S}(\mathbf{s}(x))$ $(x \in E')$. □

(2.5) COROLLARY. *Under the conditions of* (2.4) *let* $\mathcal{S}_1$ *be a syntopogenous structure on* E *such that* $\mathcal{S}_1 \prec \mathcal{S}$. *Then* $\mathcal{S}_1(\mathbf{s}(x)) = \mathcal{S}_1(x)$ *for any* $x \in E$, *and* $s(\mathcal{S}_1)$ *is an extension on* E' *of* $\mathcal{S}_1$ *coarser than* $s(\mathcal{S})$. □

(2.6) LEMMA. *Under the conditions of* (2.4) *suppose that the filters* $\mathbf{s}(x)$ $(x \in E')$ *are compressed in* $[E, \mathcal{S}]$, *and let* $\mathcal{S}'$ *be a syntopogenous structure on* E' *such that* $\mathcal{S}' \prec s(\mathcal{S})$. *Then*

(2.6.1) $<' \in \mathcal{S}'$, $X, Y \subset E'$, $X <' Y$ *imply* $X \subset s(Y \cap E)$ *and* $s(X \cap E) \subset Y$.

(2.6.2) $\mathcal{S}' \sim s(\mathcal{S}'|E)$.

(2.6.3) *If* $\mathcal{S} \sim \mathcal{S}^p$ *and* $\mathbf{s}(x)$ *is a Cauchy filter in* $[E, \mathcal{S}]$ *for any* $x \in E'$, *then* $s(\mathcal{S}) \sim$ $\sim s(\mathcal{S})^p$.

PROOF. (2.6.1): There exists $< \in \mathcal{S}$ such that $<' \subset s(<)$. From this

$$X \subset \bigcup_{j=1}^{n} s(X_j), \quad \bigcup_{j=1}^{n} s(Y_j) \subset Y, \quad X_j < Y_j$$

for some natural number n and sets X_j, Y_j $(1 \leqq j \leqq n)$. If $x \in X$ then $x \in s(X_j)$ for an index j, i.e. $X_j \in \mathbf{s}(x)$. From (2.4.1) it follows that $X_j \subset Y \cap E$, thus $Y \cap E \in \mathbf{s}(x)$ and $x \in s(Y \cap E)$. In fact, $y \in X_j$ implies $Y_j \in \mathcal{S}(y) = \mathcal{S}(\mathbf{s}(y)) \subset \mathbf{s}(y)$, that is $y \in s(Y_j) \cap$ $\cap E \subset Y \cap E$. Conversely, suppose $x \in s(X \cap E)$. Putting $<_1 \in \mathcal{S}$, $< \subset <_1^2$, one can choose sets Z_j $(1\leqq j\leqq n)$ such that $X_j<_1Z_j<_1Y_j$. Similarly to the above reasoning we get $X\cap E\subset(\bigcup_{j=1}^{n} s(X_j))\cap E=\bigcup_{j=1}^{n}(s(X_j)\cap E)\subset\bigcup_{j=1}^{n} Z_j$ by (2.4.1), thus $\bigcup_{j=1}^{n} Z_j\in\mathbf{s}(x)$. $\emptyset\notin\{Z_j\}(\cap)\mathbf{s}(x)$ for at least one index j, hence in view of the compressedness of $\mathbf{s}(x)$, we have $Y_j\in\mathbf{s}(x)$, i.e. $x\in s(Y_j)\subset Y$.

(2.6.2): Suppose $<'\in\mathcal{S}'$, $<_1'\in\mathcal{S}'$, $<'\subset<_1'^3$. Then $A<'B$ implies $A<_1'C<<_1'D<_1'B$ for some $C,D\subset E'$, therefore $C\cap E(<_1'|E)D\cap E$ and (2.6.1) give $A\subset s(C\cap E)$ and $s(D\cap E)\subset B$, thus $As(<_1'|E)B$. From here $<'\subset s(<_1'|E)$. Conversely, assume $<\in\mathcal{S}'|E$, $A_0, B_0\subset E$, $A_0<B_0$. If $<'\in\mathcal{S}'$, $<'|E=<$, then $A<'B$, $A_0\subset A\cap E$, $B\cap E\subset B_0$ for suitable sets $A,B\subset E'$. Suppose $<_1'\in\mathcal{S}'$, $<'\subset<_1'^3$, and put $A<_1'C<_1'D<_1'B$. Then $s(A_0)\subset s(A\cap E)\subset C$ and $D\subset s(B\cap E)\subset\subset s(B_0)$, consequently $s(A_0)<_1's(B_0)$. Because of (2.4.3)–(2.4.4) we obtain $s(<)\subset\subset<_1'^q=<_1'$.

(2.6.3): Suppose $<\in\mathcal{S}$, and choose $<_1\in\mathcal{S}$ such that $<\subset<_1^2$, finally assume $<_1^p\subset<_2\in\mathcal{S}$. We show $s(<)^p\subset s(<_2)$. If $As(<)^pB$ then $xs(<)B$ for any $x\in A$, i.e. $x\in s(A_x)$, $s(B_x)\subset B$, where $A_x<B_x$. One can find $C_x\subset E$ with $A_x<_1C_x<_1B_x$. Then $A_0=\bigcup_{x\in A} A_x<_2\bigcup_{x\in A} C_x=C_0$. It is clear that $A\subset\bigcup_{x\in A} s(A_x)\subset\subset s(A_0)$. At the same time $s(C_0)\subset B$ is also true, namely $y\in s(C_0)$ implies $P\cap C_x\neq\emptyset$ for at least one point $x\in A$, where P is a member of $\mathbf{s}(y)$ with the property that $X<_1Y$, $P\cap X\neq\emptyset$ imply $P\subset Y$. Thus $P\subset B_x$ and $y\in s(B_x)\subset B$. □

3. Preorders on $[E', s(\mathcal{S})]$

Throughout this section the following fixed notations will be used:

$(E,\mathcal{S},\leqq)$ is a given preordered syntopogenous space, $E\subset E'$, $\mathbf{s}(x)$ is a filter in E for any $x\in E'$ such that $\mathcal{S}(\mathbf{s}(x))=\mathcal{S}(x)$, provided $x\in E$. We write $\mathcal{S}'=s(\mathcal{S})$ (see (2.4)).

Let $\mathbf{P}$ denote the set of all pairs $(\mathcal{S}_1,\mathcal{S}_2)$ consisting of syntopogenous structures on E, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are coarser than $\mathcal{S}$, $\mathcal{S}_1$ is increasing and $\mathcal{S}_2$ is decreasing on $(E,\leqq)$ (or equivalently $\mathcal{S}_1<\mathcal{S}^u$ and $\mathcal{S}_2<\mathcal{S}^l$).

If $(\mathcal{S}_1,\mathcal{S}_2)\in\mathbf{P}$, we define the *preorder* $\leqq'$ *corresponding to* $(\mathcal{S}_1,\mathcal{S}_2)$ on E' as follows:

$$(3.1)\quad\begin{cases} x\leqq' y \text{ iff there exists a sequence } x_0, x_1, \dots, x_n \text{ of points of } E' \text{ such that} \\ x=x_0,\ x_n=y \text{ and, for any } 0\leqq i<n,\quad (x_i,x_{i+1})\in E\times E\Rightarrow x_i\leqq x_{i+1}, \\ \qquad (x_i,x_{i+1})\notin E\times E\Rightarrow \mathcal{S}_1(\mathbf{s}(x_i))\subset\mathbf{s}(x_{i+1}) \text{ and } \mathcal{S}_2(\mathbf{s}(x_{i+1}))\subset\mathbf{s}(x_i).\end{cases}$$

It is easy to see that $\leqq'$ is in fact a preorder.

(3.2) LEMMA. *If* $(\mathcal{S}_1,\mathcal{S}_2)\in\mathbf{P}$, $(\mathcal{S}_3,\mathcal{S}_4)\in\mathbf{P}$, $\mathcal{S}_1<\mathcal{S}_3$ *and* $\mathcal{S}_2<\mathcal{S}_4$, *then the preorder corresponding to* $(\mathcal{S}_3,\mathcal{S}_4)$ *is finer than the preorder corresponding to* $(\mathcal{S}_1,\mathcal{S}_2)$.

The next theorem characterizes the preorders of this type in the following way:

(3.3) THEOREM. *Suppose that* $\leqq'$ *is the preorder corresponding to* $(\mathcal{S}_1,\mathcal{S}_2)\in\mathbf{P}$, *and put* $\mathcal{S}_1'=s(\mathcal{S}_1)$, $\mathcal{S}_2'=s(\mathcal{S}_2)$ (cf. (2.5)). *Then*

(3.3.1) *Both* $\mathscr{S}_1'$ *and* $\mathscr{S}_2'$ *are coarser than* $\mathscr{S}'$, $\mathscr{S}_1'$ *is increasing and* $\mathscr{S}_2'$ *is decreasing on* $(E', \leqq')$.

(3.3.2) $G(\leqq)\subset G(\leqq')$.

(3.3.3) $\big(G(\mathscr{S}_1')\cap G(\mathscr{S}_2'^{c})\big)-(E\times E)\subset G(\leqq')$.

(3.3.4) $\leqq'$ *is the finest preorder satisfying* (3.3.1)—(3.3.3).

NOTE. From (3.3.1) and (3.3.3) it follows that $(E', \mathscr{S}', \leqq')$ is relatively T_0-preordered with respect to E.

PROOF. (3.3.1): The first statement issues from (2.5). Now we prove that

(3.3.5) $x\leqq' y$ *implies* $\mathscr{S}_1(\mathbf{s}(x))\subset\mathbf{s}(y)$ *and* $\mathscr{S}_2(\mathbf{s}(y))\subset\mathbf{s}(x)$ *for any* $x, y\in E'$.

Indeed, assume $x, y\in E'$, $x\leqq' y$, and let $x_0, x_1, \ldots, x_n$ be a sequence chosen in accordance with (3.1). If $(x_i, x_{i+1})\in E\times E$ then $(x_i, x_{i+1})\in G(\leqq)\subset G(\mathscr{S}_1)$. $V\in\mathscr{S}_1(x_i)$ implies $x_i<W<V$ for some $<\in\mathscr{S}_1$, thus $x_{i+1}\in W$, i.e. $\mathscr{S}_1(x_i)\subset\mathscr{S}_1(x_{i+1})$. From here $\mathscr{S}_1(\mathbf{s}(x_i))=\mathscr{S}_1(x_i)\subset\mathscr{S}_1(x_{i+1})=\mathscr{S}_1(\mathbf{s}(x_{i+1}))\subset\mathbf{s}(x_{i+1})$, so that $\mathscr{S}_1(\mathbf{s}(x_i))\subset \mathbf{s}(x_{i+1})$ for each $0\leqq i<n$. Suppose that $\mathscr{S}_1(\mathbf{s}(x))\subset\mathbf{s}(x_i)$ is already shown for some $0\leqq i<n$. Then we have $\mathscr{S}_1(\mathbf{s}(x))=\mathscr{S}_1(\mathscr{S}_1(\mathbf{s}(x)))\subset\mathscr{S}_1(\mathbf{s}(x_i))\subset\mathbf{s}(x_{i+1})$, and continuing the induction we arrive at $\mathscr{S}_1(\mathbf{s}(x))\subset\mathbf{s}(x_n)=\mathbf{s}(y)$. The other inclusion of (3.3.5) is dual.

If $x, y\in E'$; $xs(<)E'-y$, where $<\in\mathscr{S}_1$, then there exist sets $A, B\subset E$ such that $x\in s(A)$, $s(B)\subset E'-y$, $A<B$. Then $B\in\mathscr{S}_1(\mathbf{s}(x))$, but $B\notin\mathbf{s}(y)$, thus $x\not\leqq' y$ by (3.3.5). This shows $G(\leqq')\subset G(\mathscr{S}_1')$. Analogously $G(\leqq')^{-1}\subset G(\mathscr{S}_2')$.

(3.3.2) is trivial.

(3.3.3): If $(x, y)\in E\times E$ then $x\not\leqq' y$ implies either $\mathscr{S}_1(\mathbf{s}(x))\not\subset\mathbf{s}(y)$ or $\mathscr{S}_2(\mathbf{s}(y))\not\subset\mathbf{s}(x)$. E.g. in the first case there exist $A\in\mathbf{s}(x)$, $<\in\mathscr{S}_1$, $A<B$ such that $B\notin\mathbf{s}(y)$. Thus $x\in s(A)$, $s(B)\subset E'-y$, i.e. $xs(<)E'-y$, which means $(x, y)\notin G(\mathscr{S}_1')$. In the other case $(y, x)\notin G(\mathscr{S}_2')$, that is $(x, y)\notin G(\mathscr{S}_2'^{c})$.

(3.3.4): Suppose that $<''$ is a preorder on E' with the properties $G(\leqq)\subset G(\leqq'')$ and $\big(G(\mathscr{S}_1')\cap G(\mathscr{S}_2')\big)-(E\times E)\subset G(\leqq'')$, and put $x\leqq' y$. If $x_0, x_1, \ldots, x_n$ is a sequence determined by (3.1) then $(x_i, x_{i+1})\in E\times E$ gives $x_i\leqq'' x_{i+1}$. If $(x_i, x_{i+1})\notin E\times E$ then $x_i\leqq' x_{i+1}$ by the definition of $\leqq'$, thus $(x_i, x_{i+1})\in\big(G(\mathscr{S}_1')\cap G(\mathscr{S}_2')\big)-(E\times E)$ issues from (3.3.1), therefore $x_i\leqq'' x_{i+1}$ by hypothesis. Hence $x=x_0\leqq'' x_1\leqq''\ldots\leqq'' x_n=y$, i.e. $x\leqq'' y$. □

For compressed filters $\mathbf{s}(x)$ the converse is also true:

(3.4) THEOREM. *Suppose that* $\mathbf{s}(x)$ *is compressed in* $\mathscr{S}$ *for any* $x\in E'$. *Let* $\mathscr{S}_1', \mathscr{S}_2'$ *be syntopogenous structures and* $\leqq'$ *be a preorder on* E' *satisfying* (3.3.1)—(3.3.4). *Then* $\leqq'$ *is the preorder corresponding to the pair* $(\mathscr{S}_1'|E, \mathscr{S}_2'|E)\in\mathbf{P}$.

PROOF. The canonical injection of E into E' is preorder preserving by (3.3.2), thus from (3.3.1) and [11], (1.1) it follows that $(\mathscr{S}_1'|E, \mathscr{S}_2'|E)\in\mathbf{P}$ indeed. Hence the preorder corresponding to this pair is defined on E', and because of $\mathscr{S}_1'\sim s(\mathscr{S}_1'|E)$, $\mathscr{S}_2'\sim s(\mathscr{S}_2'|E)$ (cf. (2.6.2)) it is equal to $\leqq'$ (see (3.3)). □

It is the most important particular case of Theorems (3.3)—(3.4), in which $\mathscr{S}_1\sim\mathscr{S}^u$ and $\mathscr{S}_2\sim\mathscr{S}^l$.

(3.5) THEOREM. *Let* $\leqq'$ *be the preorder corresponding to the pair* $(\mathcal{S}^u, \mathcal{S}^l)$. *Then*

(3.5.1) $\mathcal{S}'^u|E \sim \mathcal{S}^u$ *and* $\mathcal{S}'^l|E \sim \mathcal{S}^l$.

(3.5.2) *If* $(E, \mathcal{S}, \leqq)$ *is* T_0*-preordered then* $(E', \mathcal{S}', \leqq')$ *is also* T_0*-preordered and* $G(\leqq) = G(\leqq') \cap (E \times E)$.

(3.5.3) *Every bounded preorder preserving (inversing)* $(\mathcal{S}, \mathcal{I})$*-continuous function has a bounded preorder preserving (inversing)* $(\mathcal{S}', \mathcal{I})$*-continuous extension.*

PROOF. (3.5.1): $\mathcal{S}^u \sim s(\mathcal{S}^u)|E \prec \mathcal{S}'^u|E \prec \mathcal{S}^u$ by (2.5), (3.3.1) and (1.2.1). The other case is similar.

(3.5.2): The statement issues from (3.5.1), (1.2.3) and the note after Theorem (3.3).

(3.5.3): Let f be a bounded $(\mathcal{S}, \mathcal{I})$-continuous function on E, and define a function f^* on E' as follows:

(3.5.4): $f^*(x) = \inf \{\sup f(A) : A \in \mathbf{s}(x)\}$ $(x \in E')$.

The boundedness of f^* is evident. After this we prove $f^*(x) = f(x)$ for $x \in E$. In fact, owing to the $(\mathcal{S}, \mathcal{I})$-continuity of f, for any $\varepsilon > 0$, there exists $V \in \mathcal{S}(x)$ with $\sup f(V) \leqq f(x) + \varepsilon$. From this $f^*(x) \leqq f(x)$ by $\mathcal{S}(x) = \mathcal{S}(\mathbf{s}(x)) \subset \mathbf{s}(x)$. But, for each $\varepsilon > 0$, there is $< \in \mathcal{S}$ such that $f^{-1}(<_{\varepsilon/2}) \sqsubset <$, thus if $A \in \mathbf{s}(x)$, $\sup f(A) \leqq f^*(x) + \varepsilon/2$, then $A \subset f^{-1}((-\infty, f^*(x) + \varepsilon/2]) < f^{-1}((-\infty, f^*(x) + \varepsilon)) = B$, therefore $B \in \mathcal{S}(\mathbf{s}(x))$, hence $f(x) \leqq \sup f(B) \leqq f^*(x) + \varepsilon$. This shows $f(x) \leqq f^*(x)$.

In order to verify that f^* is $(\mathcal{S}', \mathcal{I})$-continuous, put $\varepsilon > 0$, and choose $< \in \mathcal{S}$ such that $f^{-1}(<_{\varepsilon/3}) \sqsubset <$. Then $f^{*-1}(<_\varepsilon) \sqsubset s(<)$. Indeed, if $A, B \subset E'$, $A f^{*-1}(<_\varepsilon) B$ then $A \subset f^{*-1}((-\infty, p])$, $f^{*-1}((-\infty, p+\varepsilon)) \subset B$. Introducing the notations $A_0 = f^{-1}((-\infty, p+\varepsilon/3])$, $B_0 = f^{-1}((-\infty, p+2\varepsilon/3))$, we have $A_0 < B_0$. $x \in A$ implies $f^*(x) \leqq p$, thus there is $X \in \mathbf{s}(x)$ with $\sup f(X) \leqq p + \varepsilon/3$. Then $X \subset A_0$, i.e. $x \in s(A_0)$. If $y \in s(B_0)$ then $\sup f(B_0) \leqq p + 2\varepsilon/3$ implies $f^*(y) < p + \varepsilon$, that is $y \in B$. Because of $A \subset s(A_0)$, $s(B_0) \subset B$ we get $A s(<) B$. (Cf. [9], (3.3), (3.4).)

Let f be preorder preserving, and put $x, y \in E'$, $x \leqq' y$. Then $\mathcal{S}^l(\mathbf{s}(y)) \subset \mathbf{s}(x)$ (see (3.3.5)). For an arbitrary $\varepsilon > 0$, there exists $A \in \mathbf{s}(y)$ such that $\sup f(A) \leqq f^*(y) + \varepsilon/2$. It is clear that $A \subset f^{-1}((-\infty, f^*(y) + \varepsilon/2])$, thus with the notation $B = f^{-1}((-\infty, f^*(y) + \varepsilon))$ we have $A f^{-1}(<_{\varepsilon/2}) B$. Because of $f^{-1}(\mathcal{I}) \prec \mathcal{S}^l$ (see [11], (1.4), (1.1.4)), $A < B$ for some $< \in \mathcal{S}^l$. Therefore $B \in \mathcal{S}^l(\mathbf{s}(y)) \subset \mathbf{s}(x)$, hence $f^*(x) \leqq \sup f(B) \leqq f^*(y) + \varepsilon$. In view of the arbitrary choice of ε, we get $f^*(x) \leqq f^*(y)$, consequently f^* is also preorder preserving. The train of thought is similar when f is preorder inversing. □

(3.6) THEOREM. *Let* $\leqq'$ *be the preorder corresponding to the pair* $(\mathcal{S}^u, \mathcal{S}^l)$, *and suppose that the filters* $\mathbf{s}(x)$ *are compressed in* $\mathcal{S}$ *for any* $x \in E'$. *Then*

(3.6.1) $\mathcal{S}'^u \sim s(\mathcal{S}^u)$ *and* $\mathcal{S}'^l \sim s(\mathcal{S}^l)$.

(3.6.2) *If* $(E, \mathcal{S}, \leqq)$ *is symmetrical and* T_0*-preordered then, for any* $x, y \in E'$, $x \leqq' y$ *iff* $\mathcal{S}^u(\mathbf{s}(x)) \subset \mathbf{s}(y)$.

(3.6.3) *If* $(E, \mathcal{S}, \leqq)$ *is* ^{i}*-convex then* $(E', \mathcal{S}', \leqq')$ *is also* ^{i}*-convex.*

(3.6.4) *If* $\mathbf{s}(x)$ *is Cauchy in* $\mathcal{S}$ *for any* $x \in E'$, *and* $(E, \mathcal{S}, \leqq)$ *is* ^{p}*-convex, then* $(E', \mathcal{S}', \leqq')$ *is also* ^{p}*-convex.*

PROOF. (3.6.1) follows from (3.5.1) and (2.6.2). (3.6.2): (3.3.5) shows that $x\leqq' y$ implies $\mathscr{S}^u(\mathbf{s}(x))\subset\mathbf{s}(y)$. In order to see the inverse implication, first of all we prove

$$(3.6.5)\qquad \mathscr{S}^u(\mathbf{s}(x))\subset\mathbf{s}(y)\quad \textit{iff}\quad \mathscr{S}^l(\mathbf{s}(y))\subset\mathbf{s}(x).$$

Assume $x,y\in E'$, $\mathscr{S}^u(\mathbf{s}(y))\subset\mathbf{s}(y)$. If $\mathscr{S}^l(\mathbf{s}(y))\not\subset\mathbf{s}(x)$ then there exist $<\in\mathscr{S}^l$, $A\in\mathbf{s}(y)$ and $B\subset E$ such that $A<B$, but $B\notin\mathbf{s}(x)$. Suppose $<_1\in\mathscr{S}^l$, $<\subset<_1^2$ and $A<_1C<_1B$. Then $\emptyset\notin\{E-B\}(\cap)\mathbf{s}(x)$ implies $E-C\in\mathbf{s}(x)$, thus $E-A\in\in\mathscr{S}^{lc}(\mathbf{s}(x))=\mathscr{S}^u(\mathbf{s}(x))\subset\mathbf{s}(y)$, which contradicts the choice of A ((cf. [6], (15.50), [11], (1.8)). The converse is dual. Now, if $(x,y)\notin E\times E$ then $x\leqq' y$ issues from (3.1), therefore assume $x,y\in E$. Then $\mathscr{S}^u(x)=\mathscr{S}^u(\mathbf{s}(x))=\mathscr{S}^u(\mathscr{S}^u(\mathbf{s}(x)))\subset\mathscr{S}^u(\mathbf{s}(y))=\mathscr{S}^u(y)$ (cf. (2.5)), and similarly $\mathscr{S}^l(y)\subset\mathscr{S}^l(x)$. From this one can deduce $(x,y)\in G(\mathscr{S}^u)\cap\cap G(\mathscr{S}^{lc})$, i.e. $(x,y)\in G(\leqq)\subset G(\leqq')$, thus $x\leqq' y$.

(3.6.3): $(\mathscr{S}'^u\vee\mathscr{S}'^l)|E=(\mathscr{S}'^u|E)\vee(\mathscr{S}'^l|E)\sim\mathscr{S}^u\vee\mathscr{S}^l\sim\mathscr{S}$, $\mathscr{S}'^u\vee\mathscr{S}'^l<\mathscr{S}'$, thus we get $\mathscr{S}'^u\vee\mathscr{S}'^l\sim s(\mathscr{S})=\mathscr{S}'$ from (2.6.2).

(3.6.4): $\mathscr{S}\sim\mathscr{S}^p$, thus $\mathscr{S}'\sim\mathscr{S}'^p$ by (2.6.3). We have $(\mathscr{S}'^u\vee\mathscr{S}'^l)^p|E==((\mathscr{S}'^u\vee\mathscr{S}'^l)|E)^p=((\mathscr{S}'^u|E)\vee(\mathscr{S}'^l|E))^p\sim(\mathscr{S}^u\vee\mathscr{S}^l)^p\sim\mathscr{S}$, and $(\mathscr{S}'^u\vee\mathscr{S}'^l)^p<\mathscr{S}'^p\sim\sim\mathscr{S}'$, thus $(\mathscr{S}'^u\vee\mathscr{S}'^l)^p\sim\mathscr{S}'$ follows from (2.6.2). □

4. Preordered double compactification and completion

In nis paper [17] R. H. Redfield showed that any uniform ordered space has a uniform ordered completion. The analogous result for proximity ordered spaces is due to M. K. Singal and Sunder Lal [18] (see also Remark (4.10)): such a space can be always considered as a dense subspace of a compact proximity ordered space.

In order to the generalization let us recall that a doubly compact (doubly complete) syntopogenous space $[E^*,\mathscr{S}^*]$ is a *double compactification (completion)* of the syntopogenous space $[E,\mathscr{S}]$, if E is a dense subset of $[E^*,\mathscr{S}^{*s}]$ ($[E^*,\mathscr{S}^{*sb}]$), $\mathscr{S}^*|E\sim\mathscr{S}$, and $\mathscr{S}^*$ is relatively separated with respect to E (Császár [7], p. 5:, [6], p. 251).

Let $(E,\mathscr{S},\leqq)$ be a preordered syntopogenous space. By a *double compactification (completion)* of $(E,\mathscr{S},\leqq)$ we shall mean a preordered syntopogenous space $(E^*,\mathscr{S}^*,\leqq^*)$ such that

$[E^*,\mathscr{S}^*]$ is a *double compactification (completion)* of $[E,\mathscr{S}]$ and $\leqq^*$ satisfies the following conditions:

(4.1.1) $G(\leqq)\subset G(\leqq^*)$,

(4.1.2) $(E^*,\mathscr{S}^*,\leqq^*)$ *is relatively* T_0*-preordered with respect to* E,

and

(4.1.3) $\leqq^*$ *is the finest of all preorders on* E^* *fulfilling* (4.1.1)—(4.1.2).

(4.2) THEOREM. *Any preordered syntopogenous space* $(E,\mathscr{S},\leqq)$ *has double compactifications and completions. If* $(E^*,\mathscr{S}^*,\leqq^*)$ *and* $(E^{**},\mathscr{S}^{**},\leqq^{**})$ *are two double compactifications (completions) of* $(E,\mathscr{S},\leqq)$ *then there exists a unique isomorphism* f *of* $(E^*,\mathscr{S}^*,\leqq^*)$ *onto* $(E^{**},\mathscr{S}^{**},\leqq^{**})$ *such that* $f(x)=x$ *for each* $x\in E$.

PROOF. (1.5), [7], Theorem 1, [6], (16.74), and [7], Theorem 2, [6], (16.80), finally (1.6) of the present paper. □

(4.3) THEOREM. *The preordered syntopogenous space* $(E^*, \mathcal{S}^*, \leqq^*)$ *is a double compactification (completion) of* $(E, \mathcal{S}, \leqq)$ *iff* $E \subset E^*$, *and there exists a one-to-one correspondence between the points* $x \in E^* - E$ *and all non-convergent round compressed (round Cauchy) filters* $\mathbf{s}(x)$ *in* $[E, \mathcal{S}^s]$ *(in* $[E, \mathcal{S}^{sb}]$*) such that putting* $\mathbf{s}(x) = \{V \subset E : x \in V\}$ *for* $x \in E$, *we have*

(4.3.1) $\mathcal{S}^* \sim s(\mathcal{S})$
and
(4.3.2) $\leqq^*$ *is the preorder corresponding to* $(\mathcal{S}^u, \mathcal{S}^l)$.

PROOF. The construction of a double compactification of $[E, \mathcal{S}]$ can be found in [7], Theorem 1 (cf. [7], (10)). The statement concerning a completion of $[E, \mathcal{S}]$ is analogous ([6], (16.71)). In order to prove (4.3.2) let us denote by $\leqq^+$ the preorder corresponding to $(\mathcal{S}^u, \mathcal{S}^l)$. Then the upper and the lower syntopogenous structures of $(E^*, \mathcal{S}^*, \leqq^*)$ are $s(\mathcal{S}^u)$ and $s(\mathcal{S}^l)$, respectively, by (3.6.1). From (3.3) and (4.1.1)—(4.1.3) it follows that $\leqq^*$ is finer than $\leqq^+$. Conversely, denoting by $\mathcal{S}_1'$ and $\mathcal{S}_2'$ the upper and the lower syntopogenous structure of $(E^*, \mathcal{S}^*, \leqq^*)$, resp., $\mathcal{S}_1'$, $\mathcal{S}_2'$ and $\leqq' = \leqq^*$ satisfy (3.3.1)—(3.3.4). In fact, (3.3.1) and (3.3.2) are trivial. (3.3.3.) is also valid, because $(E^*, \mathcal{S}^*, \leqq^*)$ is relatively T_0-preordered with respect to E. Suppose that $\leqq''$ is another preorder on E^* satisfying (3.3.1)—(3.3.3), and let $\mathcal{S}_1''$ and $\mathcal{S}_2''$ denote the upper and the lower syntopogenous structures of $(E^*, \mathcal{S}^*, \leqq'')$. (3.3.1) implies $\mathcal{S}_1' < \mathcal{S}_1''$ and $\mathcal{S}_2' < \mathcal{S}_2''$, thus $(G(\mathcal{S}_1'') \cap G(\mathcal{S}_2''^c)) - (E \times E) \subset (G(\mathcal{S}_1') \cap \cap G(\mathcal{S}_2'^c) - (E \times E) \subset G(\leqq'')$, i.e. $(E^*, \mathcal{S}^*, \leqq'')$ is relatively T_0-preordered with respect to E. This and (3.3.2) show that (4.1.1)—(4.1.2) are fulfilled by $(E^*, \mathcal{S}^*, \leqq'')$, hence $\leqq^*$ is finer than $\leqq''$ (see (4.1.3)), and at the same time (3.3.4) is proved. Now Theorem (3.4) implies that $\leqq' = \leqq^*$ is the preorder corresponding to the pair $(\mathcal{S}_1'|E, \mathcal{S}_2'|E)$. But $\mathcal{S}_1'|E < \mathcal{S}^u$, $\mathcal{S}_2'|E < \mathcal{S}^l$ by (4.1.2) and (1.2), consequently $\leqq^+$ is finer than $\leqq^*$ (see (3.2)), that is $\leqq^* = \leqq^+$. □

(4.4) THEOREM. *If the preordered syntopogenous space* $(E, \mathcal{S}, \leqq)$ *is* ^{i}*-convex* (^{i}*-convex or* ^{p}*-convex) then its double compactification (completion)* $(E^*, \mathcal{S}^*, \leqq^*)$ *is also* ^{i}*-convex* (^{i}*- or* ^{p}*-convex). If, in addition,* $(E, \mathcal{S}, \leqq)$ *is* T_0*-[pre] ordered then the double compactification (completion) is also* T_0*-[pre]ordered, moreover* $G(\leqq) = G(\leqq') \cap (E \times E)$, *and* $\leqq^*$ *is the finest* [*pre*]*order on* E^* *satisfying these conditions.*

PROOF. The main part of the theorem can be verified on the basis of (4.3)' (3.6.3), (3.6.4) and (3.5.2). We need to see only that if $(E, \mathcal{S}, \leqq)$ is ^{i}-convex (^{i}- or ^{p}-convex) and T_0-ordered, then the double compactification (completion) is ordered. In this case $[E, \mathcal{S}]$ is T_0, thus so is $[E^*, \mathcal{S}^*]$, too (see [13], (1.15) and [6], (16.75)). From this $\mathcal{S}(\mathbf{s}(x)) \neq \mathcal{S}(\mathbf{s}(y))$ follows for any $x \neq y$ in E^*. Therefore it will be sufficient to prove that $x, y \in E^*$, $x \leqq^* y$ and $y \leqq^* x$ imply $\mathcal{S}(\mathbf{s}(x)) = \mathcal{S}(\mathbf{s}(y))$. First of all we show $\mathcal{S}(\mathbf{s}(x)) = \mathcal{S}^u(\mathbf{s}(x)) (\cap) \mathcal{S}^l(\mathbf{s}(x))$ for each $x \in E^*$. If $V \in \mathcal{S}(\mathbf{s}(x))$ then there exist $W \in \mathbf{s}(x)$ and $< \in \mathcal{S}$ such that $W < V$. Suppose $< \subset (<_1 \cup <_2)^{qa}$, where $^a = ^i$ or ^{p}, and $<_1 \in \mathcal{S}^u$, $<_2 \in \mathcal{S}^l$. Choose $<_1' \in \mathcal{S}^u$ and $<_2' \in \mathcal{S}^l$ with $<_1 \subset <_1'^2$, $<_2 \subset <_2'^2$, finally assume $<' \in \mathcal{S}$, $<_1' \subset <'$ and $<_2' \subset <'$. If $(E, \mathcal{S}, \leqq)$ is ^{i}-convex (i.e. $^a = ^i$), then

$$W = \bigcup_{j=1}^{n} (W_j \cap W_j'), \quad V = \bigcup_{j=1}^{n} (V_j \cap V_j'), \quad W_j <_1 V_j, \quad W_j' <_2 V_j' \quad (1 \leqq j \leqq n).$$

We have $W_j <_1' U_j <_1' V_j$ and $W_j' <_2' U_j' <_2' V_j'$ $(1 \leqq j \leqq n)$. For at least one index j, we get $\emptyset \notin \{W_j \cap W_j'\}(\cap)\mathbf{s}(x)$, hence $U_j \in \mathbf{s}(x)$, and $U_j' \in \mathbf{s}(x)$, because $\mathbf{s}(x)$ is compressed, consequently $V \supset V_j \cap V_j' \in \mathcal{S}^u(\mathbf{s}(x))(\cap)\mathcal{S}^l(\mathbf{s}(x))$. If $(E, \mathcal{S}, \leqq)$ is ^{p}-convex (i.e. ${}^a = {}^p$) and $[E^*, \mathcal{S}^*]$ is the completion of $[E, \mathcal{S}]$ (i.e. $\mathbf{s}(x)$ is Cauchy), then there exists $P \in \mathfrak{P}(<')\cap \mathbf{s}(x)$ (see [6], (15.34)). For $y \in P \cap W$ we have $y(<_1 \mathbf{U} \mathbf{U} <_2)^q V$, hence $y \in W_1 \cap W_2$, $V_1 \cap V_2 \subset V$, where $W_1 <_1 V_1$ and $W_1 <_2 V_2$. Assuming $W_1 <_1' U_1 <_1' V_1$, $W_2 <_2' U_2 <_2' V_2$, from $P \cap W_1 \cap W_2 \neq \emptyset$ the inclusions $P \subset U_1$, $P \subset U_2$ follow, thus $V \supset V_1 \cap V_2 \in \mathcal{S}^u(\mathbf{s}(x))(\cap)\mathcal{S}^l(\mathbf{s}(x))$. Therefore in both cases $\mathcal{S}(\mathbf{s}(x)) \subset \mathcal{S}^u(\mathbf{s}(x))(\cap)\mathcal{S}^l(\mathbf{s}(x))$. The converse is clear. Now, if $x \leqq^* y$ and $y \leqq^* x$, then (3.3.5) gives $\mathcal{S}^u(\mathbf{s}(x)) = \mathcal{S}^u(\mathbf{s}(y))$ and $\mathcal{S}^l(\mathbf{s}(x)) = \mathcal{S}^l(\mathbf{s}(y))$, consequently $\mathcal{S}(\mathbf{s}(x)) = \mathcal{S}(\mathbf{s}(y))$ and $x = y$. □

REMARK. As the double compactification (completion) of a symmetrical or topogenous (symmetrical, perfect or biperfect) syntopogenous space can be chosen also with the same property ([7], Theorem 4 and 5; [6], (16.74)), (4.4) gives a generalization of the results of Redfield [17] and Singal—Sunder Lal [18] (see also (4.10)).

(4.5) THEOREM. *The double compactification of a continuous symmetrical preordered syntopogenous space is also continuous.*

PROOF. Let $(E^*, \mathcal{S}^*, \leqq^*)$ be the double compactification of the continuous and symmetrical space $(E, \mathcal{S}, \leqq)$, and denote, for $A \subset E^*$,

$$i^*(A) = \{y \in E^* \colon x \leqq^* y \text{ for some } x \in A\}$$

and

$$d^*(A) = \{y \in E^* \colon y \leqq^* x \text{ for some } x \in A\}.$$

First of all we show that

$$(4.5.1) \qquad < \in \mathcal{S},\ A < B \ \textit{ imply } \ s(i(A)) \subset i^*(s(B)).$$

Assume $y \in s(i(A))$. If $y \in E$ then $y \in i(A)$, therefore $x \leqq y$ for some $x \in A \subset B$, thus $x \leqq^* y$ and $x \in B$, which means $y \in i^*(s(B))$. Suppose $y \notin E$. Then $\mathcal{S}(A)(\cap)(\cap)\mathcal{S}^l(\mathbf{s}(y)) = \mathbf{r}$ is a round filter in $[E, \mathcal{S}]$. In fact, $V \in \mathcal{S}^l(\mathbf{s}(y))$, $C \in \mathcal{S}(A)$ imply $d(W) \subset V$ for some $W \in \mathbf{s}(y)$ (see [11], (1.2)), thus in view of $i(A) \in \mathbf{s}(y)$ we get $\emptyset \neq i(A) \cap W$, i.e. $\emptyset \neq A \cap d(W) \subset C \cap V$. $\mathbf{r}$ can be included in a maximal round filter $\mathbf{z}$ in $[E, \mathcal{S}]$, which is compressed ([10], (5.2)). If $\mathbf{z} \to x \in E$ in $\mathcal{S}$, then $\mathbf{z} = \mathcal{S}(x) \subset \mathbf{s}(x)$ because $\mathcal{S}(x)$ is also maximal round in $[E, \mathcal{S}]$. If $\mathbf{z}$ is not convergent in $[E, \mathcal{S}]$ then $\mathbf{z} = \mathbf{s}(x)$ for some $x \in E^* - E$. In both cases we have $x \in s(B)$ and $x \leqq^* y$. Indeed, $B \in \mathcal{S}(A) \subset \mathbf{r} \subset \mathbf{z} \subset \mathbf{s}(x)$, and on the other hand $\mathcal{S}^l(\mathbf{s}(y)) \subset \mathbf{r} \subset \mathbf{z} \subset \mathbf{s}(x)$. By (3.6.5) $\mathcal{S}^u(\mathbf{s}(x)) \subset \mathbf{s}(y)$ is also true, hence $x \leqq^* y$ as we stated (see (3.1)). Thus $y \in i^*(s(B))$.

Now we show

$$(4.5.2) \qquad < \in \mathcal{S},\ A < B \ \textit{ imply } \ i^*(s(A)) \subset s(i(B)).$$

If $y \in i^*(s(A))$ then $A \in \mathbf{s}(x)$ and $x \leqq^* y$ for some $x \in E^*$. There exists an order $<_1 \in \mathcal{S}^u$ such that $i(A) <_1 i(B)$, because $(E, \mathcal{S}, \leqq)$ is continuous (see [12], (2.4.5)). From $i(A) \in \mathbf{s}(x)$ it follows that $i(B) \in \mathcal{S}^u(\mathbf{s}(x)) \subset \mathbf{s}(y)$ (see (3.3.5)), i.e. $y \in s(i(B))$.

Suppose $< \in \mathcal{S}$. Choose $<_1 \in \mathcal{S}$ such that $< \subset <_1^3$ and select $<_2 \in \mathcal{S}$, for which $A <_1 B$ implies $i(A) <_2 i(B)$. If $A, B \subset E^*$, $As(<)B$, then $A \subset \bigcup_{j=1}^{n} s(A_j)$,

$\bigcup_{j=1}^{n} s(B_j)\subset B$, where $A_j < B_j$ for any $1\leqq j\leqq n$. Put $A_j <_1 C_j <_1 D_j <_1 B_j$ $(1\leqq j\leqq n)$. Then $i^*(A)\subset \bigcup_{j=1}^{n} i^*(s(A_j))\subset \bigcup_{j=1}^{n} s(i(C_j))$ and $\bigcup_{j=1}^{n} s(i(D_j))\subset \bigcup_{j=1}^{n} i^*(s(B_j))\subset i^*(B)$ by (4.5.1)—(4.5.2). Owing to $i(C_j) <_2 i(D_j)$, we obtain $i^*(A)\, s(<_2)\, i^*(B)$.

The verification of the fact that, for $<\in\mathcal{S}$, there exists $<_2\in\mathcal{S}$ such that $As(<)B$ implies $d^*(A)\,s(<_2)\,d^*(B)$ is dual. □

REMARK. Observe that the symmetricity of $\mathcal{S}$ (as well as the compressedness of the filters $\mathbf{s}(x)$) was used only for the verification of (4.5.1). I do not know whether these conditions can be omitted. □

Further on let $(E,\mathcal{S},\leqq)$ be a symmetrizable preordered syntopological space. Then $[E,\mathcal{S}]$ is symmetrizable in the sense of [7] (see [11], ch. 4), thus it has a Császár's *compactification* ([7], Definition 3) (that is a compact symmetrizable syntopological extension $[E',\mathcal{S}']$ of $[E,\mathcal{S}]$, which is relatively separated with respect to E).

The preordered syntopological space $(E',\mathcal{S}',\leqq')$ will be said to be an *ordinary compactification* of $(E,\mathcal{S},\leqq)$, if $[E',\mathcal{S}']$ *is a compactification of* $[E,\mathcal{S}]$ (in the above sense), moreover the following conditions are satisfied:

(4.6.1) $(E',\mathcal{S}',\leqq')$ *is symmetrizable.*

(4.6.2) $G(\leqq)\subset G'(\leqq')$.

(4.6.3) *If* $x,y\in E'$, $(x,y)\notin E\times E$ *and* $x\not\leqq' y$, *then there exists a bounded* $(\mathcal{S}',\mathcal{H})$*-continuous preorder preserving function* f *with* $f(y)<f(x)$.

(4.6.4) $\leqq'$ *is the finest of all preorders on* E' *fulfilling* (4.6.2)—(4.6.3).

Suppose that $(E',\mathcal{S}',\leqq')$ and $(E'',\mathcal{S}'',\leqq'')$ are two ordinary compactifications of $(E,\mathcal{S},\leqq)$. $(E'',\mathcal{S}'',\leqq'')$ will be called *finer than* $(E',\mathcal{S}',\leqq')$ iff there is a continuous preorder preserving surjection h of $(E'',\mathcal{S}'',\leqq'')$ onto $(E',\mathcal{S}',\leqq')$ such that $h(x)=x$ for each $x\in E$. These two ordinary compactifications are *equivalent* iff h is an isomorphism.

Theorem 14 of [7] can be generalized as follows:

(4.7) THEOREM. *Let* $(E,\mathcal{S},\leqq)$ *be a symmetrizable preordered syntopological space. Disregarding equivalences, there exists a one-to-one correspondence between the ordinary compactifications of this space and those symmetrical* ^{i}*-convex syntopogenous structures* $\mathcal{S}_1$ *on* $(E,\leqq)$, *for which* $\mathcal{S}_1^p\sim\mathcal{S}$. *Among two ordinary compactifications of* $(E,\mathcal{S},\leqq)$, $(E'',\mathcal{S}'',\leqq'')$ *is finer than* $(E',\mathcal{S}',\leqq')$ *iff for the corresponding structures* $\mathcal{S}_1,\mathcal{S}_2$, *the relation* $\mathcal{S}_1<\mathcal{S}_2$ *holds. Two ordinary compactifications are equivalent iff each of them is finer than the other.*

PROOF. We show that an ordinary compactification $(E',\mathcal{S}',\leqq')$ corresponds to the symmetrical ^{i}-convex structure $\mathcal{S}_1$ with $\mathcal{S}_1^p\sim\mathcal{S}$ iff there exists a symmetrical ^{i}-convex syntopogenous structure $\mathcal{S}_1'$ on $(E',\leqq')$ such that $\mathcal{S}_1'^p\sim\mathcal{S}'$ and $(E',\mathcal{S}_1',\leqq')$ is a double compactification of $(E,\mathcal{S}_1,\leqq)$.

In fact, let $\mathcal{S}_1$ be symmetrical, ^{i}-convex and suppose $\mathcal{S}_1^p\sim\mathcal{S}$. Then $(E,\mathcal{S}_1,\leqq)$ has a double compactification $(E',\mathcal{S}_1',\leqq')$ which is symmetrical and ^{i}-convex by (4.4) and [7], Theorem 4. Putting $\mathcal{S}'=\mathcal{S}_1'^p$, $[E',\mathcal{S}']$ is a compactification of $[E,\mathcal{S}]$, and $(E',\mathcal{S}',\leqq')$ is symmetrizable. The validity of (4.6.2) is obvious. In order to prove (4.6.3) suppose $x,y\in E'$, $(x,y)\notin E\times E$ and $x\not\leqq' y$. Then $(x,y)\notin G(\mathcal{S}_1'^u)\cap\cap G(\mathcal{S}_1'^{lc})=G(\mathcal{S}_1'^{lc})$ (cf. (4.1.2), [11], (1.8)), thus there exists a preorder preserving

$(\mathscr{S}_1', \mathscr{I})$-continuous bounded function f on E' such that $yf^{-1}(<_\varepsilon)E'-x$ for some $\varepsilon>0$ (see [11], (1.12)). Then $f(y)<f(x)$, and f is also $(\mathscr{S}_1', \mathscr{I}^s)$-, and $(\mathscr{S}', \mathscr{H})$-continuous by $\mathscr{H}=\mathscr{I}^{sp}$. After this let $\leqq''$ be a preorder on E' satisfying (4.6.2)—(4.6.3). Put $x\leqq' y$. Denoting by Φ the set of all $(\mathscr{S}_1', \mathscr{I})$-continuous ordering families φ on E' such that $f|E$ is preorder preserving for any $f\in\varphi$, there exists a sequence $x_0, x_1, \dots$ $\dots, x_n$ of points of E' with the following properties for any $0\leqq i<n$:

$$(x_i, x_{i+1})\in E\times E \Rightarrow x_i \leqq x_{i+1}, \quad (x_i, x_{i+1})\notin E\times E \Rightarrow f(x_i) \leqq f(x_{i+1}) \quad \text{for} \quad f\in\varphi\in\Phi$$

(see (4.1.1)—(4.1.3), (1.5), (1.5.4)). We show $x_i\leqq'' x_{i+1}$ for any $0\leqq i<n$. If $(x_i, x_{i+1})\in E\times E$ then our statement follows from (4.6.2). If $(x_i, x_{i+1})\notin E\times E$, $x_i\not\leqq'' x_{i+1}$, then there exists an $(\mathscr{S}', \mathscr{H})$-continuous bounded preorder preserving function f on $(E', \leqq'')$ such that $f(x_{i+1})<f(x_i)$ by (4.6.3). In view of Theorem 13 of [7], f is $(\mathscr{S}_1', \mathscr{I}^s)$-, and *a fortiori* $(\mathscr{S}_1', \mathscr{I})$-continuous, moreover $f|E$ is preorder preserving with respect to $\leqq$ (see (4.6.2)). It is obvious that f can be included in an ordering family $\varphi\in\Phi$ and this contradicts the choice of x_i and x_{i+1}. Therefore $x=x_0\leqq''$ $\leqq'' x_1\leqq'' \dots \leqq'' x_n=y$, that is $x\leqq'' y$, hence $G(\leqq')\subset G(\leqq'')$. Thus (4.6.4) is also satisfied, and $(E', \mathscr{S}', \leqq')$ is in fact an ordinary compactification of $(E, \mathscr{S}, \leqq)$.

One can choose any double compactification of $(E, \mathscr{S}_1, \leqq)$ for this construction, the originating ordinary compactifications are equivalent by (4.2).

Conversely, let $(E', \mathscr{S}', \leqq')$ be an ordinary compactification of $(E, \mathscr{S}, \leqq)$. Disregarding equivalences, there exists a unique symmetrical ^{i}-convex syntopogenous structure $\mathscr{S}_1'$ on E' such that $\mathscr{S}_1'^p\sim\mathscr{S}'$ (see (4.6.1) and [7], Lemma 8). Suppose $\mathscr{S}_1=\mathscr{S}_1'|E$. Then $\mathscr{S}_1$ is symmetrical and ^{i}-convex (see (4.6.2), [11], (2.9)), moreover $[E', \mathscr{S}_1']$ is a double compactification of $[E, \mathscr{S}_1]$. We need to show that $(E', \mathscr{S}_1', \leqq')$ satisfies (4.1.1)—(4.1.3), and this is very similar to the above train of thought. (4.1.1) is trivial. If $x, y\in E'$, $(x, y)\notin E\times E$ and $x\not\leqq' y$, then $f(y)<f(x)$ for an $(\mathscr{S}', \mathscr{H})$-continuous preorder preserving bounded function f on $(E', \leqq')$. This is $(\mathscr{S}_1', \mathscr{I}^s)$-, and $(\mathscr{S}_1', \mathscr{I})$-continuous, too, therefore $f^{-1}(\mathscr{I})<\mathscr{S}_1'$, which implies $(x, y)\notin G(\mathscr{S}_1'^{lc})$, hence (4.1.2) is also fulfilled. Finally let $\leqq''$ be a preorder on E' satisfying (4.1.1)—(4.1.2). It is easy to verify that in this case $\leqq''$ has properties (4.6.2)—(4.6.3) (cf. [11], (1.12)), thus by (4.6.4) we obtain $G(\leqq')\subset G(\leqq'')$, that is (4.1.3) is also satisfied by $\leqq'$.

The remaining part of the theorem can be seen on the basis of (1.6), [7], Theo rems 13 and 14. □

(4.8) COROLLARY. *Among the ordinary compactifications of a symmetrizable preordered syntopological space there exists a finest one.*

PROOF. This finest ordinary compactification of $(E, \mathscr{S}, \leqq)$ corresponds to the structure $\mathscr{S}_\Phi^s$, where Φ is the ordering structure of all $(\mathscr{S}, \mathscr{H})$-continuous ordering families consisting of preorder preserving functions (see [11], (4.8)). □

As the following corollary shows, the notion of an ordinary compactification is a generalization of the *ordered Čech—Stone compactification* of a uniformizable ordered topological space (Nachbin [16], p. 104; McCallion [15]):

(4.9) COROLLARY. *Let* $(E, \mathscr{S}, \leqq)$ *be a symmetrizable ordered syntopological space such that*

(4.9.1) *if* $x, y \in E$, $x \not\equiv y$, *then* $f(y) < f(x)$ *for some* $(\mathcal{S}, \mathcal{H})$*-continuous bounded order preserving function* f *on* E (cf. [13], (1.13)).

Then there exists an ordinary compactification $(E', \mathcal{S}', \leqq')$ *of* $(E, \mathcal{S}, \leqq)$ *with the following properties:*

(4.9.2) $\leqq'$ *is an order and* $G(\leqq) = G(\leqq') \cap (E \times E)$.

(4.9.3) *If* $x, y \in E'$, $x \not\equiv' y$, *then* $f'(y) < f'(x)$ *for some* $(\mathcal{S}, \mathcal{H})$*-continuous bounded order preserving function* f' *on* E'.

(4.9.4) $\leqq'$ *is the finest of all preorders on* E' *satisfying* (4.9.2.)—(4.9.3).

In particular, the finest ordinary compactification of $(E, \mathcal{S}, \leqq)$ *is of this type.*

Proof. Under condition (4.9.1) there exists a symmetrical ^{i}-convex syntopogenous structure $\mathcal{S}_1$ on $(E, \leqq)$ such that $(E, \mathcal{S}_1, \leqq)$ is T_0-ordered ([13], (1.13)). Then the ordinary compactification corresponding to $\mathcal{S}_1$ has properties (4.9.2)—(4.9.4) (see (4.4), [13], (1.13)). The finest symmetrical ^{i}-convex structure $\mathcal{S}_0$ with $\mathcal{S}_0^p \sim \mathcal{S}$ is also T_0-ordered ([13], (1.6)). □

(4.10) Remark. In [18] M. K. Singal and Sunder Lal presented a generalization of Smirnov's compactification theory for completely regular ordered spaces. I would like to mention that *I cannot realize the proof of Theorems* 4, 5, 7, 9 *and* 10 *of this paper*. The common foundation of these theorems is Lemma 5 of [18], the proof of which is incorrect. In fact, in this proof the authors state that if δ is a quasi-proximity (in the sense of Pervin) on a set X, moreover an order $\leqq$ is defined on X by letting $x \leqq y$ iff $\{x\}\delta\{y\}$, finally δ^* is the coarsest (symmetrical) proximity finer than δ, then the implication

$(*)$ B is decreasing $\tau(\delta^*)$-closed, $A \cap B = \emptyset \Rightarrow a\bar{\delta}B$ for any $a \in A$

is valid. It is obviously equivalent to $(*)$ that any increasing $\tau(\delta^*)$-open set is open in the topology $\tau(\delta)$ induced by δ. However, in the preceding part of this series I gave an example ([14], Section 2), for which the above statement fails to be true.

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THE BILINEAR PRODUCT CATEGORY

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0. Introduction

The study of categorical duality theorems looks for inspiration to the simplest signficant duality theorem, that for finite dimensional vector spaces. In that case, the duality is intimately related to tensor products. One reasonable generalization of tensor products is to a bifunctor which is part of the structure of a closed category, but this approach has some disadvantages; closed categories are not as common as one could hope, and the two categories whose duals are to be studied must be subcategories of the same closed category. The bilinear product category is a more external construction which enables us to define bilinear maps involving objects from poorly related categories.

The dual of a category **T** is the category $\mathbf{T}^*$ obtained by reversing the direction of each morphism in **T**. By a duality, we mean a categorical equivalence between one category **A** and the dual $\mathbf{T}^*$ of another. Such a duality is given by a pair of functors $\varkappa: \mathbf{A} \to \mathbf{T}^*$ and $\lambda: \mathbf{T}^* \to \mathbf{A}$, whose composition in either order is naturally equivalent to an identify functor.

Examples of duality theorems are given by the Pontrjagin duality theorem, which states that the category of locally compact Abelian groups is dual to itself, and has various extensions and refinements given in van Kampen [17], Freyd [3], Kaplan [8] and Kaplansky [9]. A second example is the Gelfand duality between the categories of compact Hausdorff spaces and of commutative C^*-algebras with identity. Beyond these there is a duality of Stone between the categories of compact totally disconnected spaces and of Boolean algebras; a duality between the categories of compact convex sets and of Archimedian ordered vector spaces (cf. Semadeni [15]) and the dualities between the categories of real compact spaces and rings of the form $C(X)$ for some space X (cf. Gillman and Jerison [4]). With the development of category theory, a language appeared which clarified the connection between these theorems, and some unified results on duality have developed, such as those of Hofmann and Keimel [7]. In addition, the Pontrjagin and Gelfand duality theorems have been investigated categorically by Negrepontis [12] and Roeder [14], [15].

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1. Fibre objects and fibre morphisms

DEFINITION. Let $\sigma: \mathbf{A} \to \mathbf{C}$ and $\tau: \mathbf{T} \to \mathbf{C}$ be functors. A fibre object for (σ, τ) is an ordered pair (A, X) such that $A \in \mathrm{Ob}(\mathbf{A})$, $X \in \mathrm{Ob}(\mathbf{T})$ and $\sigma A = \tau X$. If (A, X) and (A', X') are fibre objects for (σ, τ) we say (m, m') is a fibre morphism from (A, X) to (A', X') and write $(m, m'): (A, X) \to (A', X')$ if $m \in \mathbf{A}(A, A')$, $m' \in \mathbf{T}(X, X')$ and $\sigma m = \tau m'$. The composition of fibre morphisms $(m, m'): (A, X) \to (A', X')$ and $(n, n'): (A', X') \to (A'', X'')$ is given by $(n, n')(m, m') = (nm, n'm')$.

The fibre objects and fibre morphisms for any (σ, τ) form a category which is the fibre product of the diagram

$$\begin{array}{ccc} & & \mathbf{A} \\ & & \downarrow{\scriptstyle \sigma} \\ \mathbf{T} & \xrightarrow{\tau} & \mathbf{C} \end{array}$$

in the quasicategory of all categories.

Bass [2] also defines a more general fibre product category. The category defined above embeds fully in Bass' fibre product category. In many important cases, this embedding is an equivalence. In fact, if σ and τ are concrete functors in the following sense, the canonical embedding of the fibre product category above into Bass' fibre product category is an equivalence.

DEFINITION. Functor $\emptyset: \mathbf{D} \to \mathbf{G}$ is concrete if $\emptyset$ is faithful and satisfies the ICLP (Isomorphism Codomain Lifting Property): Whenever $f: G \cong \emptyset D$ in G, there is a $D' \in \mathrm{Ob}(\mathbf{D})$ such that $\emptyset D' = G$ and a $g: D' \cong D$ in $\mathbf{D}$ such that $\emptyset g = f$.

The fibre objects defined in this section are necessary for the definition of bilinear product categories in the next section. Note that in some cases they may not exist.

2. The bilinear product category

The category we construct here will clarify the relation between duality and bilinear maps. It is an extension of a construction used by Grossman [6].

Let $\mathbf{C}$ be a category with finite products and an object P which satisfies the following conditions:

1. P is a generator of $\mathbf{C}$.

2. There is a natural epimorphism $m: 1_{\mathbf{C}} \to L_P$ where L_P is the left multiplication by P (i.e. the functor defined by $L_P C = P \times C$ and $L_P f = l_P \times f$.)

There is, of course, also a natural transformation $n: l_{\mathbf{C}} \to R_P$ where R_P is right multiplication by P since $R_P \cong L_P$.

Now let $\mathbf{A}$ and $\mathbf{T}$ be categories and $\sigma: \mathbf{A} \to \mathbf{C}$ and $\tau: \mathbf{T} \to \mathbf{C}$ be faithful functors, and choose a fibre object $E = ({}^*E, E^*)$ for (σ, τ). $\sigma^*E = \tau E^* \in \mathrm{Ob}(\mathbf{C})$ is also denoted by E.

DEFINITION. The bilinear product category of $\mathbf{A}$ and $\mathbf{T}$ with respect to σ, τ, and E is the category $\mathbf{B}_E(\mathbf{A}, \mathbf{T})$ with objects of the form (A, e, X) where $A \in \mathrm{Ob}(\mathbf{A})$, $X \in \mathrm{Ob}(\mathbf{T})$, and $e \in \mathbf{C}(\sigma A \times \tau X, E)$ satisfies the following properties:

1. For every $a \in \mathbf{C}(P, \sigma A)$, there is a (unique) $e_a \in \mathbf{T}(X, E^*)$ such that $\tau(e_a) = e(a \times 1)m: \tau X \to E$.

2. For every $t \in \mathbf{C}(P, \tau X)$, there is a (unique) $e_t \in \mathbf{A}(A, {}^*E)$ such that $\sigma(e_t) = e(1 \times t)n: \sigma A \to E$.

The morphisms in $\mathbf{B}_E(\mathbf{A}, \mathbf{T})$ are of the form $f \otimes g: (A, e, X) \to (A', e', X')$ which is taken to mean that $f \in \mathbf{A}(A, A')$, $g \in \mathbf{T}(X', X)$ and that $e'(\sigma f \times 1) = e(1 \times \tau g): \sigma A \times \tau X' \to E$. $f \otimes g = h \otimes k$ if and only if $f = h$ and $g = k$. The composition of morphisms in $\mathbf{B}_E(\mathbf{A}, \mathbf{T})$ is given by $(f \otimes g)(h \otimes k) = fh \otimes g$.

The uniqueness of e_a and e_t actually follows from the faithfulness of the functors σ and τ. The interchange of the roles of $\mathbf{A}$ and $\mathbf{T}$ dualizes the bilinear product category. (i.e. $\mathbf{B}_E(\mathbf{A}, \mathbf{T}) = \mathbf{B}_E(\mathbf{T}, \mathbf{A})^*$.) Therefore, every result concerning the relation between $\mathbf{A}$ and $\mathbf{B}_E(\mathbf{A}, \mathbf{T})$ dualizes to a result concerning the relation between $\mathbf{T}$ and $\mathbf{B}_E(\mathbf{A}, \mathbf{T})$.

Henceforth, $\mathbf{A}$ and $\mathbf{T}$ are assumed fixed and are deleted from the notation for our bilinear product categories (b. p. c.'s).

There is an obvious functor $\alpha_E: \mathbf{B}_E \to \mathbf{A} \times \mathbf{T}^*$ defined by $\alpha_E(A, e, X) = (A, X)$ and $\alpha_E(f \otimes g) = (f, g)$. In all that follows, let $\pi_1: \mathbf{A} \times \mathbf{T}^* \to \mathbf{A}$ and $\pi_2: \mathbf{A} \times \mathbf{T}^* \to \mathbf{T}^*$ be the canonical projection functors.

PROPOSITION 1. *If $m: E \to D$ is a fibre morphism, m induces a functor $\mu: \mathbf{B}_E \to \mathbf{B}_D$ defined by $\mu(A, e, X) = (A, |m|e, X)$ and $\mu(f \otimes g) = f \otimes g$. This functor satisfies $\alpha_D \mu = \alpha_E$, is faithful, and functorial (i.e. if $n: D \to G$ induces $\nu: \mathbf{B}_D \to \mathbf{B}_G$, then $nm: E \to G$ induces $\nu\mu: \mathbf{B}_E \to \mathbf{B}_G$ and l_E induces $l_{\mathbf{B}_E}$.) If m is mono then μ is an embedding of categories.*

This sets up a functor from the fibre product category to the quasicategory of all categories over $\mathbf{A} \times \mathbf{T}^*$.

The duality functors in the introduction all resolve through bilinear product categories. For example, the Pontrjagin duality can be described using the category of topological spaces as $\mathbf{C}$ and the category of locally compact Abelian groups as both $\mathbf{A}$ and $\mathbf{T}$. Letting ${}^*E = E^* = R/Z$, the circle group, and defining $\beta: \mathbf{A} \to \mathbf{B}_E$ by $\beta A = (A, \text{eval}, \hat{A})$ and $\beta f = f \otimes \hat{f}$ where $\hat{A}$ is the Pontrjagin character group of A and $\hat{f}$ is the map defined by $\hat{f}(g) = gf$, β is a left adjoint cross-section for $\pi_1 \alpha_E$. There is similarly a functor $\gamma: \mathbf{T}^* \to \mathbf{B}$ which is a right adjoint cross-section for $\pi_2 \alpha_E$. Notice in addition that $\pi_1 \alpha_E \gamma$ and $\pi_2 \alpha_E \beta$ are the character group functors yielding the duality between the categories. Thus the dual functors resolve through the b.p.c. As we see in the next section, this is not an exceptional case.

3. Naturally grounded categories

Hofman and Keimel [7] introduced naturally grounded categories. Their work suggested Theorem 1.

DEFINITIONS. A concrete category consists of a concrete functor $\emptyset: \mathbf{D} \to \mathbf{Ens}$ where $\mathbf{Ens}$ denotes the category of sets. $\emptyset: \mathbf{D} \to \mathbf{Ens}$ is a natural grounding if it is representable. A naturally grounded category is a category $\mathbf{D}$ with a natural grounding.

E.g. If $\emptyset: \mathbf{D} \to \mathbf{Ens}$ has a left adjoint, it is a natural grounding. Other natural groundings include the standard forgetful functors to $\mathbf{Ens}$ from the categories of finitely generated groups, Noetherian rings, connected spaces, etc.

THEOREM 1. *If* $\sigma\colon \mathbf{A}\to\mathbf{Ens}$ *and* $\tau\colon \mathbf{T}\to\mathbf{Ens}$ *are concrete natural groundings, and* $\emptyset\colon \mathbf{A}\to\mathbf{T}^*$ *and* $\Psi\colon \mathbf{T}^*\to\mathbf{A}$ *are functors with* $\emptyset$ *left adjoint to* Ψ, *then there is b.p.c.* $\mathbf{B}_E$ *with functors* $\beta_E\colon \mathbf{A}\to\mathbf{B}_E$ *and* $\gamma_E\colon \mathbf{T}^*\to\mathbf{B}_E$ *such that* $\emptyset=\pi_2\alpha_E\beta_E$ *and* $\Psi=\pi_1\alpha_E\gamma_E$. *(i.e. every pair of contravariant adjoint functors between concrete naturally grounded categories resolves through a b.p.c.).*

PROOF 1. Let $\sigma\cong\mathbf{A}(Z,-)$ and $\tau\cong\mathbf{T}(G,-)$. Then $\sigma\Psi G\cong\mathbf{A}(Z,\Psi G)\cong\mathbf{T}^*(\emptyset Z,G)=\mathbf{T}(G,\emptyset Z)\cong\tau\emptyset Z$. Let $^*E$ be chosen with $\sigma{}^*E=\mathbf{A}(Z,\Psi G)$. [$\sigma{}^*E\cong\sigma\Psi G$ lifts to $^*E\cong\Psi G$.] and E^* such that $\tau E^*=\mathbf{A}(Z,\Psi G)$. $E=({}^*E,E^*)$ determines b.p.c. $\mathbf{B}_E$ and functor $\alpha_E\to\mathbf{A}\times\mathbf{T}^*$. To define $\beta_E\colon\mathbf{A}\to\mathbf{B}_E$, let $\beta_E A=(A,e,\emptyset A)$ where $e\colon\sigma A\times\tau\emptyset A\to{}^*E$ is given by $e(a,t)=\Psi t.u_A.a\in\mathbf{A}(Z,\Psi G)=\sigma{}^*E$, where $a\in\big(\mathbf{A}(Z,A)\cong\sigma A,\ t\in\mathbf{T}(G,\emptyset A)\big)\cong\tau\emptyset A$, and $u_A\colon A\to\Psi\emptyset A$ is the front adjunction for $\emptyset\dashv\Psi$. With this definition, for all such a and t, $e_a=\emptyset a$ and $e_t=\Psi t.\,u_A$ as can be seen from the commutative diagrams:

$$\begin{array}{ccc} \sigma A & \cong & \mathbf{A}(Z,A) \\ \downarrow{\scriptstyle \sigma e_t} & & \downarrow{\scriptstyle t.u_A.(\,)} \\ \sigma\Psi G & \cong & \mathbf{A}(Z,\Psi G) \end{array} \qquad \begin{array}{ccccc} \tau\emptyset A & \cong & \mathbf{T}(G,\emptyset A) & \cong & \mathbf{A}(A,\Psi G) \\ \downarrow{\scriptstyle \tau e_a} & & \downarrow{\scriptstyle \emptyset a.(\,)} & & \downarrow{\scriptstyle (\,).a} \\ \tau\emptyset A & \cong & \mathbf{T}(G,\emptyset Z) & \cong & \mathbf{A}(Z,\Psi G) \end{array}$$

Then for $f\in\mathbf{A}(A,A')$, let $\beta_E f=f\otimes\emptyset f$ which is a morphism in $\mathbf{B}_E$ since

$$\begin{array}{ccccc} Z & \xrightarrow{a} & A & \xrightarrow{u_A} & \Psi\emptyset A \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Psi\emptyset f} \\ & & A & \xrightarrow{u_A} & \Psi\emptyset A \xrightarrow{\Psi t} \Psi G \end{array}$$

commutes. Thus $\pi_1\alpha_E\beta_E=1_{\mathbf{A}}$ and $\pi_2\alpha_E\beta_E=\emptyset$. Similar definitions give the functor $\gamma_E\colon\mathbf{T}^*\to\mathbf{B}_E$ such that $\pi_2\alpha_E\gamma_E=1_{\mathbf{T}^*}$ and $\pi_1\alpha_E\gamma_E=\Psi$.

LEMMA 2. *Continuing the hypotheses of Theorem* 1, $\pi_1\alpha_E\colon\mathbf{B}_E(\beta_E A,B)\to\mathbf{A}(A,\pi_1\alpha_E B)$ *is injective for every* $A\in\mathrm{Ob}(\mathbf{A})$ *and* $B\in\mathrm{Ob}(\mathbf{B}_E)$.

PROOF. Let $\pi_1\alpha_E(f\otimes g)=\pi_1\alpha_E(h\otimes k)$ i.e. $f=h$. Then commutativity of

$$\begin{array}{ccc} X & \overset{\tau g}{\underset{\tau k}{\rightrightarrows}} & A \\ \downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbf{A}(\tilde{A},{}^*E) & \xrightarrow{f^*} & \mathbf{A}(A,{}^*E) \end{array} \qquad (\tilde{A}=\pi_1\alpha_E B.)$$

guarantees that $\tau g=\tau k$. Thus $g=k$. (The assumption that P is a generator for the category $\mathbf{C}$ and that the transformations m and n are epimorphic in the definition of b.p.c.'s is used here to make the commutativity of the above diagram equivalent to the fact that $f\otimes g$ and $f\otimes k$ are morphisms in the b.p.c.).

The injectivity proved in Lemma 1 is the first step in relating the b.p.c.'s to fibred categories introduced by Gray [5]. This relationship is carried further in the next section.

4. Adjoints, cross-sections, and duality

In this section E is fixed, and we will delete it from the notation for our b.p.c.'s and functors.

When fibred categories are called into play, various lifting properties of morphisms can be very useful. In the case of b.p.c.'s, the lifting properties are intricately related to the existence of adjoint, and therefore dualizing functors. The first of the lifting properties is:

DEFINITION. Let $\Psi\colon \mathbf{C}\to\mathbf{D}$ be a functor. Ψ has the codomain lifting property (CLP) if for each $C\in\mathrm{Ob}(\mathbf{C})$ and $D\in\mathrm{Ob}(\mathbf{D})$, whenever $g\in\mathbf{D}(D,\Psi C)$ there is an object $C'\in\mathrm{Ob}(\mathbf{C})$ such that $\Psi C'=D$ and a morphism $f\in\mathbf{C}(C',C)$ such that $\Psi f=g$.

E.g. the forgetful functor from the category of topological spaces to the category of sets has the CLP. The next proposition shows that b.p.c.'s also have this property.

PROPOSITION 2. *If* $f\in\mathbf{A}(A,A')$, $(A',e',X')\in\mathrm{Ob}(\mathbf{B})$, *and* $e=e'(\sigma f\times 1)$, *then* $(A,e,X')\in\mathrm{Ob}(\mathbf{B})$ *and* $f\otimes 1\colon (A,e,X')\to(A',e',X')$.

PROOF. For $a\in\mathbf{C}(P,\sigma A)$, define $e_a=e'_{\sigma f.a.}\colon X'\to E^*$ and for $t\in\mathbf{C}(P,\tau X')$, define $e_t=e'_t.f.A\to {}^*E$.

DEFINITION. A full subcategory $\mathbf{J}$ of $\mathbf{D}$ will be called a lower ideal of $\mathbf{D}$ if for all objects D and D' of $\mathbf{D}$, whenever $D\in\mathrm{Ob}(\mathbf{J})$ and $\mathbf{D}(D',D)$ is nonempty, it follows that $D'\in\mathrm{Ob}(\mathbf{J})$.

Note that $\Psi\colon\mathbf{C}\to\mathbf{D}$ has the CLP if and only if the image of Ψ is a lower ideal of $\mathbf{D}$.

PROPOSITION 3. *If* $\Psi\colon\mathbf{C}\to\mathbf{D}$ *has the CLP and a left adjoint functor* $\emptyset\colon\mathbf{D}\to\mathbf{C}$, *then* Ψ *is surjective on objects and morphisms.*

PROOF. For any object $A\in\mathrm{Ob}(\mathbf{D})$, there is a front adjunction map $h_A\colon A\to\Psi\emptyset A$ and the image of Ψ is a lower ideal.

Therefore $\pi_1\alpha\colon\mathbf{B}\to\mathbf{A}$ will be surjective on objects and morphisms when it has a left adjoint, but more can be said than this:

PROPOSITION 4. *If* $\pi_1\alpha\colon B\to\mathbf{A}$ *has a left adjoint functor* $\beta\colon\mathbf{A}\to\mathbf{B}$ *then* $\pi_1\alpha$ *has a cross-section (i.e. there is a functor* $\lambda\colon\mathbf{A}\to\mathbf{B}$ *such that* $\pi_1\alpha\lambda=1_{\mathbf{A}}$.*)*.

PROOF. Let $A\in\mathrm{Ob}(\mathbf{A})$ and $\beta A=(A^+,e^+,A^\#)$, and let the front adjunction be $h_A\colon A\to A^+=\pi_1\alpha\beta A$. By Proposition 2, h_A lifts to a morphism $h_A\otimes 1\colon (A,e,A^\#)\to\to(A^+,e^+,A^\#)$, where $e=e^+(\sigma h\times 1)$. Define $\lambda\colon\mathbf{A}\to\mathbf{B}$ by $\lambda A=(A,e,A^\#)$ and for $k\in\mathbf{A}(A',A)$, $\lambda k=k\otimes k^\#$ where $\beta k=k^+\otimes k^\#$. λ is the desired cross-section.

THEOREM 2. *If* $\pi_1\alpha\colon\mathbf{B}\to\mathbf{A}$ *has a left adjoint functor* $\beta\colon\mathbf{A}\to\mathbf{B}$ *then the cross-section* $\lambda\colon\mathbf{A}\to\mathbf{B}$ *constructed in Proposition 4 is itself left adjoint to* $\pi_1\alpha$. *Furthermore, the front adjunction for* λ *is given by* $1\colon 1_{\mathbf{A}}\to 1_{\mathbf{A}}=\pi_1\alpha\lambda$.

PROOF. Let $\varrho_A=h_A\otimes 1\colon\lambda A\to\beta A$ as in Proposition 4. The naturality of p is simple to check coordinatewise. By the universal property of the front adjunction, there is a unique map $q_A\colon\beta A\to\lambda A$ such that $1_A=\pi_1\alpha(q_A)h_A$. The naturality of the equivalence $\mathbf{B}(\beta-,-)\cong\mathbf{A}(-,\pi_1\alpha-)$ can then be used to show that q_A is the back adjunction for λA and is therefore natural in A, and also to show that $p_A q_A=1_{\beta A}$. Thus p is a retraction, and so is $h_A=\pi_1\alpha(p_A)$. Also $1_A=\pi_1\alpha(q_A)h_A$, so that h_A is also a section and thus an isomorphism. Since h_A and 1 are isomorphisms, it is trivial to check that $p_A=h_A\otimes 1$ is an isomorphism in **B**. The last assertion can be shown from the naturality of the isomorphisms $\mathbf{A}(A,A)\cong\mathbf{B}(\beta A,\lambda A)\cong\mathbf{B}(\lambda A,\lambda A)$.

The isomorphism $\mathbf{B}(\beta A,B)\cong\mathbf{A}(A,\pi_1\alpha B)$ required by adjointness is the map $h\leftrightarrow\pi_1\alpha(h)$. Thus if β is an arbitrary cross-section to $\pi_1\alpha$, β is left adjoint to $\pi_1\alpha$ with front adjunction 1 if and only if $\pi_1\alpha$ maps $\mathbf{B}(\beta A,B)$ bijectively to $\mathbf{A}(A,\pi_1\alpha B)$ for each $A\in\mathrm{Ob}(\mathbf{A})$ and each $B\in\mathrm{Ob}(\mathbf{B})$. This can be phrased in terms of lifting properties. In the next three definitions, let $\Psi:\mathbf{D}\to\mathbf{C}$ and $\emptyset:\mathbf{C}\to\mathbf{D}$ be functors satisfying $\Psi\emptyset=1_{\mathbf{C}}$.

DEFINITIONS. Ψ has the restricted unique lifting property (RULP) with respect to $\emptyset$ if for every $C\in\mathrm{Ob}(\mathbf{C})$ and every $D\in\mathrm{Ob}(\mathbf{D})$, Ψ maps $\mathbf{D}(\emptyset C,D)$ injectively to $\mathbf{C}(C,\Psi D)$. Ψ has the restricted covering property (RCP) with respect to $\emptyset$ if for every $C\in\mathrm{Ob}(\mathbf{C})$ and every $D\in\mathrm{Ob}(\mathbf{D})$, Ψ maps $\mathbf{D}(\emptyset C,D)$ surjectively to $\mathbf{C}(C,\Psi D)$. Ψ has the restricted identity covering property (RICP) with respect to $\emptyset$ if for every $C\in\mathrm{Ob}(\mathbf{C})$ and every D in the fibre of C by Ψ, there is a morphism $f\in\mathbf{D}(\emptyset C,D)$ such that $\Psi f=1_C$.

THEOREM 3. *Consider the following statements:*

(1) $\pi_1\alpha\beta=1_A$ *and* β *is left adjoint to* $\pi_1\alpha$ *with front adjunction* 1.

(2) $\pi_1\alpha$ *has a cross-section* β.

(3) *There is a natural transformation* $r:\beta\pi_1\alpha\to 1_{\mathbf{A}}$ *such that* $\pi_1\alpha(r_B)=1_{\pi_1\alpha B}$ *for every* $B\in\mathrm{Ob}(\mathbf{B})$.

(4) $\pi_1\alpha$ *has the RCP with respect to* β.

(5) $\pi_1\alpha$ *has the RULP with respect to* β.

(6) $\pi_1\alpha$ *has the RICP with respect to* β.

Then (1) *is equivalent to the conjunction of* (2), (4), *and* (5); *which is equivalent to the conjunction of* (2), (3), *and* (5); *and to the conjunction of* (2), (5), *and* (6).

Theorem 2 summarizes the connection between cross-sections and left adjoints to $\pi_1\alpha:\mathbf{B}\to\mathbf{A}$. Each left adjoint is essentially a cross section with certain lifting properties. Of course, $\pi_2\alpha:\mathbf{B}\to\mathbf{T}^*$ has the appropriate dual properties. Henceforth we will consider only adjoints which are simultaneously cross-sections to these functors having 1 as their front and back adjunctions respectively.

DEFINITION. Let $\emptyset:\mathbf{C}\to\mathbf{D}$ and $\Psi:\mathbf{D}\to\mathbf{C}$ be functors. $\emptyset$ is lacs (left adjoint cross-section) to Ψ if $\emptyset$ is left adjoint to Ψ, $\Psi\emptyset=1_{\mathbf{C}}$ and the front adjunction map $1_{\mathbf{C}}\to\Psi\emptyset=1_{\mathbf{C}}$ is the identity. Racs (right adjoint cross-section) is defined similarly.

The relations between left adjoint cross-sections and dualities is clarified by a few observations from Krishnan [10].

DEFINITION. A pair $(\emptyset,\Psi)$ of contravariant functors $\emptyset:\mathbf{C}\to\mathbf{D}$ and $\Psi:\mathbf{D}\to\mathbf{C}$ is a Galois (resp. dual) correspondence between **C** and **D** if there are natural transformations (resp. isomorphisms) $u:1_{\mathbf{C}}\to\Psi\emptyset$ and $v:1_{\mathbf{D}}\to\emptyset\Psi$.

Given any Galois correspondence, there are full subcategories $\mathbf{C}'$ of **C** and $\mathbf{D}'$ of **D** such that the restrictions of $\emptyset$ and Ψ to these categories is a dual correspondence.

$\mathbf{C}'$ has as objects those $C\in\mathrm{Ob}(\mathbf{C})$ for which u_C is an isomorphism, $v_{\Psi C}$ is an isomorphism and $v_{\varnothing C}=(\varnothing u_C)^{-1}$. $\mathbf{D}'$ has a similar description.

THEOREM 4. *If* $\pi_1\alpha$ *has lacs* β *and* $\pi_2\alpha$ *has racs* γ, *then* $(\pi_2\alpha\beta, \pi_1\alpha\gamma)$ *is a Galois correspondence between* $\mathbf{A}$ *and* $\mathbf{T}$, *which induces a duality between the full subcategories of* $\mathbf{A}$ *and* $\mathbf{T}$ *described above.*

PROOF. We define the transformation $u=\pi_1\alpha*\bar{u}*\beta$ where this notation means: Let u_A be the image by $\pi_1\alpha$ of the front adjunction map of the object βA under the adjointness $\pi_2\alpha\dashv\gamma$. v has a similar description.

Note that since u and v are images by $\pi_1\alpha$ and $\pi_2\alpha$ respectively, we could say that they lift to transformations $\bar{u}\colon\beta\to\gamma\pi_2\alpha\beta$ and $\bar{v}\colon\beta\pi_1\alpha\beta\to\gamma$ which are defined by $\bar{u}=u\otimes 1$ and $\bar{v}=1\otimes v$.

Thus the existence of the lacs and racs functors sets up a Galois correspondence which restricts to the given duality theorem. We shall call the image of an object in $\mathbf{A}$ or $\mathbf{T}$ by the two functors in the Galois correspondence the Galois correspondent of that object.

5. Dualities over the category of sets

Many of the dualities mentioned in the introduction arise when we let $\mathbf{C}=\mathbf{Ens}$, the category of sets and $\sigma\colon\mathbf{A}\to\mathbf{C}$ and $\tau\colon\mathbf{T}\to\mathbf{C}$ be forgetful functors. In this case, P must be the singleton set and the transformations $m\colon 1_{\mathbf{C}}\cong L_p$ and $n\colon 1_{\mathbf{C}}\cong R_p$ are defined in the obvious manner.

Particular examples of categories of interest are given by such classes as (i) algebraic categories, meaning categories based on an algebraic theory Ω in the sense of Pareigis [13]; (ii) categories with optimal lifts (cf. Arbib and Maines [1]):

DEFINITION. A functor $\tau\colon\mathbf{T}\to\mathbf{Ens}$ has optimal lifts if whenever $S\in\mathrm{Ob}(\mathbf{Ens})$ and $\{(X_i, f_i)\}$ is any set (indexed by $i\in I$) of ordered pairs such that for each $X_i\in\mathrm{Ob}(\mathbf{T})$ and each $f_i\colon S\to\tau X_i$, there is a unique $X\in\mathrm{Ob}(\mathbf{T})$ such that $\tau X=S$ and there are unique maps $g_i\in\mathbf{T}(X, X_i)$ such that $\tau g_i=f_i$, and that furthermore for every $h\colon\tau Y\to S$, there is a morphism $k\in\mathbf{T}(Y, X)$ such that $\tau k=h$ if and only if there are morphisms $m_i\in\mathbf{T}(Y, X_i)$ such that $\tau m_i=f_i h$ for each $i\in I$.

Examples of categories with forgetful functors to **Ens** which have optimal lifts are the categories of topological spaces, uniform spaces, and preordered sets.

We complete our list of categories of interest with (iii) categories of Ω-algebras in categories with optimal lifts. If we take two categories of types (i), (ii) or (iii), or any combination thereof, and choose a fibre object E, it is possible to introduce "pointwise" structures to define Galois correspondents. Thus $\pi_1\alpha_E$ has a lacs and $\pi_2\alpha_E$ has a racs, and a duality theorem is assured (through the dual categories may be empty). These "pointwise" structures also produce a property related to separation of points:

DEFINITIONS. $A\in\mathrm{Ob}(\mathbf{A})$ is a separative if for every a and $a'\in\mathbf{C}(P, \sigma A)$ such that $a\neq a'$, there exists an $(A, e, X)\in\mathrm{Ob}(\mathbf{B})$ such that $e_a\neq e_{a'}$. A is extremely separative if $e_{\_}\colon\mathbf{C}(P, \tau A^{\#})\to\mathbf{A}(A, {}^*E)$ is injective, where $A^{\#}$ is the Galois correspondent of A.

When the adjoints of Section 5 exist, being separative is equivalent to the requirement that $e\_: \mathbf{C}(P, \sigma A) \to \mathbf{T}(A^{\#}, E^{*})$ be injective. If A is extremely separative, then $A^{\#}$ is separative. For dualizable A, A is extremely separative if and only if $A^{\#}$ is separative and vice versa. Thus the two conditions are "dual". With the pointwise construction mentioned above, all objects in each category are separative and extremely separative.

THEOREM 5. *If* $\mathbf{C}=\mathbf{Ens}$ *and the adjoints* β *and* γ *exist, and if* $A \in \mathrm{Ob}(\mathbf{A})$ *is extremely separative, then* A *is dualizable if and only if there is an* $X \in \mathrm{Ob}(\mathbf{T})$ *such that* $A = X^{\#}$ *and* u_A *is epimorphic.*

Occasionally, this theorem makes it simpler to determine whether an object is dualizable by comparison with known results in duality theory.

6. Examples in duality theory

EXAMPLE 1. Taking **A** and **T** to be the category of vector spaces over a field k, and choosing $^{*}E = E^{*} = k$, the full subcategories of dualizable objects are in both cases the category of finite dimensional vector spaces.

EXAMPLE II. Taking **A** to be the category of groups and **T** to be the category of topological groups, with $^{*}E = E^{*} = R/Z$, the circle group, we induce the duality between the category of Abelian groups and that of compact Abelian Groups.

EXAMPLE III. Taking both **A** and **T** as the category of topological groups with $^{*}E = E^{*} = R/Z$, one could hope to obtain the full Pontrjagin duality theorem from the adjoint functors, but because $\mathbf{C} = \mathbf{Ens}$, the adjoint functors constructed in the "pointwise" manner suggested in the last section will assign to a given topological group A, not its Pontrjagin character group A^{*}, but the group of continuous homomorphisms $f: A \to R/Z$ with the topology of pointwise convergence, rather than uniform convergence on compact sets. Theorem 4 still guarantees a duality theorem, however, and using Theorem 5 and a comparison with the known Pontrjagin theorem, we may deduce a little about its extent:

LEMMA 2. *If* $A \cong X^{\#}$ *for some topological group* X *and is naturally isomorphic to its second (Pontrjagin) character group* A^{**}, *then* A *is dualizable under the Galois correspondence of Section* 4.

In particular, then, every compact Abelian group is dualizable under the Galois correspondence formed here, since each such is the Galois correspondent of a discrete Abelian group and satisfies the Pontrjagin theorem. Of course, this means that the Galois correspondent of any compact Abelian group is also dualizable, but with the new topology, this group is in general not discrete. $(R/Z)^{\#}$ is a Hausdorff, non-discrete topological group with the algebraic structure of the group of integers. It is totally bounded but not compact, so the theorem extends past the compact Abelian groups.

PROPOSITION 5. *There is a self-dual subcategory of the category of topological groups which lies between the categories of compact Abelian groups and that of totally* *bounded Abelian groups, and the dual equivalence functors are given by providing character groups with the topology of pointwise convergence.*

The usual Pontrjagin duality theorem can be recaptured by using the category of topological spaces as **C** rather than the category of sets, and properly restricting the categories of topological groups one considers to set up the adjoint functors.

EXAMPLE IV. Let **T** be the category of topological spaces and **A** the category of complex-algebras with identity and involution satisfying the algebraic laws of a C^*-algebra. **A** is an algebraic category in the sense of Pareigis, and the category of C^*-algebras embeds fully into it as those algebras which admit a norm of the proper type. A Galois correspondence can be established as in Section 5 using $^*E=E^*=C$, the complex numbers. The standard Gelfand theorem shows that the duality holds at least for all compact Hausdorff spaces. In fact, using Theorem 5, and a comparison with the duality established in Gilmann and Jerison [4] it can be seen that every realcompact space is dualizable under this Galois correspondence. Thus the Gelfand theorem extends to a duality between the category of realcompact spaces and some category of complex-algebras which is a full subcategory of the category described above, namely those which are $C(X)$ for some topological space X.

Stone duality and the Gilmann—Jerison duality mentioned above can also be framed in the language of bilinear product categories.

EXAMPLE V. If a poor choice of categories is made for **A** and for **T**, the duality theorem of Section 4 may be a poor one; for if **T** is the category of topological spaces and **A** is the category of rings (with or without identity) and the Galois correspondence is set up as in Section 4 using $^*E=E^*=R$, the real numbers, the resulting categories of dualizable objects are empty.

Having framed the standard duality theorems in terms of b.p.c.s we now return to the formal theory.

7. Bilinear situations

The categorical situation of the existence of the proper adjoint functors is now formalized.

DEFINITION. A bilinear situation for categories **A** and **T** consists of a category **B**, together with a faithful functor $\alpha: \mathbf{B} \to \mathbf{A} \times \mathbf{T}^*$ and functors $\beta: \mathbf{A} \to \mathbf{B}$ which is lacs to $\pi_1 \alpha$ and $\gamma: \mathbf{T} \to \mathbf{B}$ which is racs to $\pi_2 \alpha$.

Bilinear situations recall the definition of the comma category for two functors (see [11]). Let ϱ_1 and ϱ_2 denote the two projections from the comma category.

PROPOSITION 6. *Let* $(\mathbf{B}, \alpha, \beta, \gamma)$ *be a bilinear situation for categories* **A** *and* **T**. *Then there is a faithful functor* $\varepsilon: \mathbf{B} \to (\beta \downarrow \gamma)$ *such that* $\varepsilon\beta$ *is left adjoint to* ϱ_1, ϱ_2 *is left adjoint to* $\varepsilon\gamma$ *and the diagram*

$$\begin{array}{ccc} \mathbf{B} & & \\ {\scriptstyle\varepsilon}\downarrow & \searrow^{\alpha} & \\ & & \mathbf{A} \times \mathbf{T}^* \\ (\beta\downarrow\gamma) & \nearrow_{\varrho_1 \times \varrho_2} & \end{array}$$

commutes.

PROOF. Define $\varepsilon: \mathbf{B} \to (\beta \downarrow \gamma)$ by $\varepsilon B = (\pi_1 \alpha B, u_B v_B, \pi_2 \alpha B)$ and $\varepsilon f = (\pi_1 \alpha f, \pi_2 \alpha f)$, where u_B is the front adjunction map for $\pi_2 \alpha \dashv \gamma$ and v_B is the back adjunction map for $\beta \dashv \pi_1 \alpha$. Clearly $(\varrho_1 \times \varrho_2)\varepsilon = \alpha$. To show $\varepsilon\beta \dashv \varrho_1$, for $A \in \mathrm{Ob}(\mathbf{A})$ and $D = (A', \beta A \xrightarrow{h} \gamma X, X) \in \mathrm{Ob}(\beta \downarrow \gamma)$, note that $\varepsilon \beta A = (A, \beta A \xrightarrow{u_A} \gamma \pi_2 \alpha \beta A, \pi_2 \alpha \beta A)$. Given $f \in \mathbf{A}(A, A') = \mathbf{A}(A, \varrho_1 D)$, let $f \leftrightarrow (f, g) \in (\beta \downarrow \gamma)(\varepsilon \beta A, D)$ where $g \in \mathbf{T}(\pi_2 \alpha \beta A, X)$ is the unique morphism such that

$$\begin{array}{ccc} \beta A & \xrightarrow{u_A} & \gamma\pi_2\alpha\beta A \\ & \underset{h.\beta f}{\searrow} & \downarrow{\scriptstyle \gamma g} \\ & & \gamma X \end{array}$$

commutes. Since such g exists uniquely, $(\beta \downarrow \gamma)(\varepsilon \beta A, D) \cong \mathbf{A}(A, \varrho_1 D)$. The naturality is from the commutative diagram

$$\begin{array}{ccccccc} \beta A'' & \xrightarrow{\beta r} & \beta \bar{A} & \xrightarrow{\beta f} & \beta A & \xrightarrow{\beta p} & \beta \tilde{A} \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \gamma\pi_2\alpha\beta A'' & \xrightarrow{\gamma\pi_2\alpha\beta r} & \gamma\pi_2\alpha\beta A & \xrightarrow{\gamma g} & \gamma X & \xrightarrow{q} & \gamma X \end{array}$$

Thus all the bilinear situations are "represented" in the comma categories of pairs of functors. This enables us to observe that if **A** and **T** are naturally grounded, bilinear situations give rise to bilinear product categories.

PROPOSITION 7. *Let* $(\mathbf{B}, \alpha, \beta, \gamma)$ *be a bilinear situation for naturally grounded categories* **A** *and* **T**. *Let* $\emptyset = \pi_2 \alpha \beta$ *and* $\Psi = \pi_1 \alpha \gamma$ *resolve as in Theorem* 1 *through b.p.c.* $\mathbf{B}_E$. *Then there is a functor* $\eta: (\beta \downarrow \gamma) \to \mathbf{B}_E$ *such that the diagrams*

$$\begin{array}{ccc} (\beta\downarrow\gamma) & & \\ \downarrow{\scriptstyle \eta} & \overset{\varrho_1\times\varrho_2}{\searrow} & \mathbf{A}\times\mathbf{T}^* \\ & \underset{\alpha_E}{\nearrow} & \\ \mathbf{B}_E & & \end{array} \quad \textit{and} \quad \begin{array}{ccccc} & & \mathbf{B} & & \\ & \overset{\beta}{\nearrow} & & \overset{\gamma}{\nwarrow} & \\ \mathbf{A} & & \downarrow{\scriptstyle \eta\varepsilon} & & \mathbf{T}^* \\ & \underset{\beta_E}{\searrow} & & \underset{\gamma_E}{\swarrow} & \\ & & \mathbf{B}_E & & \end{array}$$

commute.

PROOF. Define $\eta: (\beta \downarrow \gamma) \to \mathbf{B}_E$ by $\eta(A, m, X) = (A, e, X)$ where $e: \sigma A \times \tau X \cong \cong \mathbf{A}(Z, A) \times \mathbf{T}^*(X, G) \to {}^*E = \mathbf{A}(Z, \Psi G)$ is given by $e(a, t) = \pi_1 \alpha(\gamma t.m.\beta\alpha)$ and $\eta(f, g) = f \otimes g$. e is bilinear since $e_a = \tilde{m}.\emptyset a$ where $\tilde{m}: \emptyset A \to X$ is the unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{u_A} & \Psi\emptyset A \\ & \underset{\pi_1\alpha m}{\searrow} & \downarrow{\scriptstyle \Psi_{\tilde{m}}} \\ & & \Psi X \end{array}$$

commute, and $e_t = \Psi t.\pi_1 \alpha m$. That $f \otimes g: (A, e, X) \to (A', e', X')$ is immediate from the diagram

$$\begin{array}{ccccc} \beta Z & \xrightarrow{\beta a} & \beta A & \xrightarrow{m} & \gamma X \\ & & {\scriptstyle \beta f}\downarrow & & \downarrow{\scriptstyle \gamma g} \\ & & \beta A' & \xrightarrow{m'} & \gamma X' & \xrightarrow{\gamma t'} & \gamma G \end{array}$$

The first triangle in the statement of the proposition commutes trivially; for the second, $\beta_E A=(A, e, \emptyset A)$ where $e(a, t)=\Psi t.u_A.a$ and $\eta\varepsilon\beta A=(A, \tilde{e}, \emptyset A)$ where $\tilde{e}(a, t)=\pi_1\alpha(\gamma t.(uv)_{\beta A}.\beta A)$. Now $v_{\beta A}=1_{\beta A}$ and $\mathbf{T}^*(\emptyset A, \emptyset A)\cong\mathbf{B}(\beta A, \gamma\emptyset A)\cong$ $\cong\mathbf{A}(A, \Psi\emptyset A)$ where $1_{\emptyset A}\leftrightarrow u_{\beta A}\leftrightarrow u_A$, so that $e=\tilde{e}$, and $\beta_E f=f\times\emptyset f=\eta\varepsilon\beta f$. The proof for the third triangle is similar.

Since the definition of η and the first commutative diagram in Proposition 7 depend only on the adjunction $\emptyset\dashv\Psi$ and not on $\beta\dashv\pi_1\alpha$ or $\pi_2\alpha\dashv\gamma$, we can show the following corollary by the same proof.

COROLLARY 1. *If the conditions of Theorem* 1 *hold, there is a functor* $\eta_E:(\beta_E\downarrow\gamma_E)\to$ $\to\mathbf{B}_E$ *given by* $\eta_E(A, m, X)=(A, e, X)$ *with* $e(a, t)=\pi_1\alpha_E(\gamma_E t.m.\beta_E a)$, *and* $\eta_E(f, g)=f\otimes g$.

8. Adept representors

We now wish to make optimal use of the theory of representable functors in a way that characterizes b.p.c.s as familiar categories. Let $\sigma:\mathbf{A}\to\mathbf{Ens}$ and $\tau:\mathbf{T}\to\mathbf{Ens}$ be natural groundings. We denote by $\mathbf{A}*\mathbf{T}$ the fibre product category of Section 1.

DEFINITION 1. $E=(A_0, X_0)\in\mathrm{Ob}(\mathbf{A}*\mathbf{T})$ is an adept representor for $\mathbf{A}*\mathbf{T}$ if $\pi_1\alpha_E$ has lacs β_E, $\pi_2\alpha_E$ has racs γ_E and these functors satisfy the conditions:

1. $\tau\pi_2 \circ_E\beta_E\cong\mathbf{A}(-, A_0)$ by an isomorphism making

$$\begin{array}{ccc} \sigma A\times\tau\pi_2\alpha_E\beta_E A & \xrightarrow{\quad e \quad} & \sigma A_0 \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{A}(Z, A)\times\mathbf{A}(A, A_0) & \xrightarrow{\text{comp.}} & \mathbf{A}(Z, A_0) \end{array}$$

commuty for all $A\in\mathrm{Ob}(\mathbf{A})$.

2. $\sigma\pi_1\alpha_E\gamma_E\cong\mathbf{T}(-, X_0)$ by an isomorphism making

$$\begin{array}{ccc} \sigma\pi_1\alpha_E\gamma_E X\times\tau X & \xrightarrow{\quad e \quad} & \tau X_0 \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{T}(X, X_0)\times\mathbf{T}(G, X) & \xrightarrow{\text{comp.}} & \mathbf{T}(G, X_0) \end{array}$$

commute for every $X\in\mathrm{Ob}(\mathbf{T})$.

LEMMA 3. *The fibre object* E *established in Theorem* 1 *is an adept representor for* $\mathbf{A}*\mathbf{T}$ *if and only if* β_E *is left adjoint to* $\pi_1\alpha_E$ *and* γ_E *is right adjoint to* $\pi_2\alpha_E$.

PROOF. $\tau\emptyset\cong\mathbf{A}(-, {}^*E)$, $\sigma\Psi\cong\mathbf{T}(-, E^*)$ and β_E and γ_E are defined so that the squares

$$\begin{array}{ccc} \sigma A\times\tau\emptyset A & \xrightarrow{\quad e \quad} & \sigma\Psi G \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ \mathbf{A}(Z, A)\times\mathbf{A}(A, {}^*E) & \xrightarrow{\text{comp.}} & \mathbf{A}(Z, {}^*E) \end{array}$$

and

$$\begin{array}{ccc} \sigma\Psi X\times\tau X & \xrightarrow{\quad e \quad} & \sigma\Psi G \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ \mathbf{T}(X, E^*)\times\mathbf{T}(G, X) & \xrightarrow{\text{comp.}} & \mathbf{T}(G, E^*) \end{array}$$

commute.

When E is an adept representor, we have an alternate description of the category $\mathbf{B}_E$.

PROPOSITION 8. *If E is an adept representor for $\mathbf{A}*\mathbf{T}$, let $\varepsilon_E\colon \mathbf{B}_E\to(\beta_E{\downarrow}\gamma_E)$ be defined as in Proposition 6, and $\eta_E\colon(\beta_E{\downarrow}\gamma_E)\to\mathbf{B}_E$ as in Proposition 7, then $\varepsilon_E=\eta_E^{-1}$.*

PROOF. $\eta_E\varepsilon_E(A, e, X)=(A, \tilde{e}, X)$ where $\tilde{e}(a, t)=\Psi t.u_A.\beta_A$ for $a\in\mathbf{A}(Z, A)$ and $t\in\mathbf{T}^*(X, G)$. Consider the commutative diagram

$$\begin{array}{ccccc} \sigma A\times\tau X & \xrightarrow{\bar{u}\times 1} & \sigma\Psi X\times\tau X & \cong & \mathbf{T}^*(\emptyset Z, X)\times\mathbf{T}^*(X, G) \\ & {}_{e}\searrow & \swarrow_{ev} & & \Big\downarrow \\ & & \sigma\Psi G & \cong & \mathbf{A}(Z, \Psi G)\cong\mathbf{T}^*(\emptyset Z, G) \end{array}$$

Then $e(a, t)=ev(ua, t)=\Psi t.\bar{u}.a=\tilde{e}(a, t)$. Since $\eta_E\varepsilon_E(f\otimes g)=f\otimes g$, $\eta_E\varepsilon_E=1_{\mathbf{B}_E}$.

Secondly, $\varepsilon_E\eta_E(A, m, X)=\bar{u}\otimes\bar{v}\colon \beta A\to\gamma X$ where $m\colon\beta A\to\gamma X$ induces $e\colon\sigma A\times\times\tau X\to\tau\Psi G$ by $e(a, t)=\pi_1\alpha_E(\gamma_E t.m.\beta_E a)$. Let $m=m_1\otimes m_2$.

$$\begin{array}{ccccc} \mathbf{A}(Z, A)\times\mathbf{T}^*(X, G) & \cong & \sigma A\times\tau X & \xrightarrow{\ e\ } & \sigma\Psi G \\ \Big\downarrow & & \Big\downarrow u\times 1 & & \Big\downarrow\cong \\ \mathbf{A}(Z, X)\times\mathbf{T}^*(X, G) & \cong & \sigma\Psi X\times\tau X & & \mathbf{A}(Z, \Psi G) \\ \Big\downarrow\cong & & & & \Big\downarrow\cong \\ \mathbf{T}^*(\emptyset Z, X)\times\mathbf{T}^*(X, G) & & \xrightarrow{\quad\text{comp.}\quad} & & \mathbf{T}(\emptyset X, G) \end{array}$$

commutes, so that $\Psi t.u(a)=\Psi t, m_1.a$ for all $a\in\mathbf{A}(Z, A)$, $t\in\mathbf{T}^*(X, G)$. Now if $\Psi t.f=\Psi t.g$ for all such t, then $f=g$. Thus $u(a)=m_1.a$ and dually $v(t)=m_2.t$. Thus $u\otimes v=m$ and since $\varepsilon_E\eta_E(f, g)=(f, g)$, $\varepsilon_E\eta_E=1_{(\beta_E\downarrow\gamma_E)}$.

Let $\mathbf{Cat}/\mathbf{A}\times\mathbf{T}^*$ denote the quasicategory of all categories over $\mathbf{A}\times\mathbf{T}^*$ and $\mathbf{Bil}(\mathbf{A}, \mathbf{T})$ denote the quasicategory whose objects are bilinear situations $(\mathbf{B}, \alpha, \beta, \gamma)$ for $\mathbf{A}$ and $\mathbf{T}$ and whose morphisms $\lambda\colon(\mathbf{B}, \alpha, \beta, \gamma)\to(\mathbf{B}', \alpha', \beta', \gamma')$ are functors $\lambda\colon\mathbf{B}\to\mathbf{B}'$ such that the diagram

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\ \lambda\ } & B' \\ & {}_{\alpha}\searrow\ \swarrow_{\alpha'} & \\ & \mathbf{A}\times\mathbf{T}^* & \end{array}$$

commutes. Define $\zeta: \mathbf{Bil(A, T)} \to \mathbf{Cat/A \times T^*}$ by $\zeta(\mathbf{B}, \alpha, \beta, \gamma) = (\mathbf{B}, \alpha)$ and $\zeta(\lambda) = \lambda$. ζ is a full and faithful functor.

Also let $\xi: \mathbf{A * T} \to \mathbf{Cat/A \times T^*}$ be the functor given in Proposition 1, and let **AdRep** denote the full subcategory of $\mathbf{A * T}$ whose objects are the adept representors for $\mathbf{A * T}$. Let $\omega: \mathbf{AdRep} \to \mathbf{A * T}$ be inclusion and define $\theta: \mathbf{AdRep} \to \mathbf{Bil(A, T)}$ by $\theta(E) = (\mathbf{B}_E, \alpha_E, \beta_E, \gamma_E)$ and $\theta(m) = \mu$ as given in Proposition 1.

PROPOSITION 9.

$$\begin{array}{ccc} \mathbf{AdRep} & \xrightarrow{\theta} & \mathbf{Bil(A, T)} \\ \omega \downarrow & & \downarrow \zeta \\ \mathbf{A * T} & \xrightarrow{\xi} & \mathbf{Cat/A \times T} \end{array}$$

is a fibre product diagram.

PROOF. The equivalence between **AdRep** and the fibre product category is given by $E \leftrightarrow (E, (\mathbf{B}_E, \alpha_E, \beta_E, \gamma_E))$ and $m \leftrightarrow (m, \mu)$.

THEOREM 6. $\zeta\theta: \mathbf{AdRep} \to \mathbf{Cat/A \times T^*}$ *is a full embedding.*

PROOF. Faithfulness is immediate from the faithfulness of σ and τ and the adeptness of the representor. Let $\mu: \mathbf{B}_E \to \mathbf{B}_D$ with $\alpha_D \mu = \alpha_E$. Consider the back adjunction $\bar{v}: \beta_D \pi_1 \alpha_D \to 1$. Then $\tau\pi_2\alpha_D * \bar{v} * \mu\beta_E: \tau\pi_2\alpha_D\mu\beta_E \to \tau\pi_2\alpha_D\beta_D$. This induces a transformation $\mathbf{A}(-, E_1) \to \mathbf{A}(-, D_1)$. By the Yoneda lemma, this is m_{1*} for some $m_1 \in \mathbf{A}(E_1, D_1)$. Similarly one obtains $m_2 \in \mathbf{T}(E_2, D_2)$.

Let $(A, e, X) \in \mathrm{Ob}(\mathbf{B}_E)$ and $\mu(A, e, X) = (A, \hat{e}, X)$. Computing $\hat{e}$ by the use of the adeptness criterion, we find that $\sigma(m_1).e = \hat{e} = \tau(m_2).e$, for all $(A, e, X) \in \mathrm{Ob}(\mathbf{B}_E)$. Thus $\sigma(m_1) = \tau(m_2)$ and $m = (m_1, m_2): E \to D$ and m induces μ as in Proposition 1.

This theorem has interesting consequences when applied to standard categories. Again, let Ω be an algebraic theory in the sense of Pareigis, and denote by $\Omega\mathbf{C}$ the category of Ω-algebras in the category $\mathbf{C}$.

COROLLARY 1. *For any algebraic theories* Ω *and* Λ, $(\Omega \otimes \Lambda)\,\mathbf{Ens}$ *embeds fully in* $\mathbf{Cat}/\Omega\,\mathbf{Ens} \times \Lambda\,\mathbf{Ens}^*$.

COROLLARY 2. *Let* Ω *be an algebraic theory and* $\mathbf{C}$ *a concrete category with optimal lifts. Then* $\Omega\mathbf{C}$ *embeds fully in* $\mathbf{Cat}/\Omega\,\mathbf{Ens} \times \mathbf{C}^*$.

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ON AN INTERPOLATIONAL PROCESS

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1. Let $\tilde{S}_n[g]$ denote the trigonometric polynomial of degree at most n interpolating the function $g \in C_{2\pi}$ at $m=2n+1$ equidistant nodes,

$$t_i = \tau + \frac{2i\pi}{m}, \quad \tilde{S}_n[g](t_i) = g(t_i), \quad (i = 0, \pm 1, \pm 2, \ldots). \tag{1}$$

We consider the following expressions

$$\tilde{S}_{kn}[g](t) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \tilde{S}_n[g]\left(t + \frac{k-2j}{m}\pi\right), \tag{2}$$

which for $k=1$ and $k=2$ were first introduced by S. N. Bernstein [1]. If not noted otherwise we take $m>k$ such that the arguments of $\tilde{S}_n[g]$ in (2) ly within a period of length 2π.

In [5] the expressions

$$\tilde{M}_{kn}(t) = \sup_{\substack{g \in C_{2\pi} \\ g \not\equiv \text{const.}}} \frac{|\tilde{S}_{kn}[g](t) - g(t)|}{\omega\left(g, \frac{2\pi}{m}\right)}, \quad c_{kn} = \sup_{-\infty < t < \infty} \tilde{M}_{kn}(t) \tag{3}$$

were investigated $(\omega(g, \delta)$ denoting the modulus of continuity of $g)$.

It is stated in [5] (Theorem 2) that

THEOREM 1.

$$c_{1n} = 1 + \frac{1}{2m}\left(\operatorname{cosec}\frac{\pi}{2m} - 1\right).$$

Nevertheless the proof in [5] fails to be true. We hope to give a corrected version of the proof. Kis and Névai [5] deduced from this result that

$$\sup_{n \geq 0} c_{1n} = \lim_{n \to \infty} c_{1n} = 1 + \frac{1}{\pi}. \tag{4}$$

Further they proved

$$c_{2n} = \frac{5}{4}, \quad (n > 0). \tag{5}$$

Let us therefore focuse attention on the case $k>2$. We show for odd values of k

THEOREM 2.

$$c_{kn} \geqq \frac{1}{2} + \frac{2r+1}{2^{2r+1}} \binom{2r}{r}, \quad k = 2r+1.$$

This expression is strictly increasing as a function of r thus we immediately get $(r=1)$

$$c_{kn} \geqq \frac{5}{4}, \quad k \text{ odd}, \quad k \geqq 3. \tag{6}$$

Now let k be even. By a sligthly changed version of the previous proof we get

THEOREM 3.

$$c_{kn} \geqq \frac{1}{2} + \frac{2r+1}{2^{2r+1}} \binom{2r}{r}, \quad k = 2r.$$

This yields $(r=2)$

$$c_{kn} \geqq \frac{23}{16}, \quad k \text{ even}, \quad k \geqq 4. \tag{7}$$

REMARKS. The case $m=2n$ investigated in [3], [4] and [2] can be treated in a similar manner. Define as in [4]

$$M_{kn}(t) = \sup_{\substack{g \in C_{2\pi} \\ g \not\equiv \text{const.}}} \frac{|S_{kn}[g](t) - g(t)|}{\omega\left(g, \frac{\pi}{n}\right)}, \quad \bar{c}_{kn} = \sup_{-\infty < t < \infty} M_{kn}(t).$$

Starting with [3] (7) we get by similar arguments

$$\bar{c}_{kn} \geqq \frac{1}{2} + \frac{2r+1}{2^{2r+1}} \binom{2r}{r}, \quad k = 2r+1 \quad \text{or} \quad k = 2r \quad \text{resp.},$$

the only modification being that in both cases we have to evaluate $S_{kn}[f]\left(\frac{\pi}{2n}\right)$. The same estimation is valid for the numbers c_{kn}^* introduced in [5] too.

Stirling's formulae yields the asymptotic estimation

$$\frac{2r+1}{2^{2r+1}} \binom{2r}{r} \sim \sqrt{\frac{r}{\pi}}.$$

2. Proof of Theorem 1

We start with [5] (34), $n>0$, which can be written as

$$2\sigma_0(t) = 1 + d_n\left(\frac{\pi}{m} + t\right) + \sum_{i=2}^{n} \left[d_n\left(\frac{2i-1}{m}\pi + t\right) - d_n\left(\frac{2i-1}{m}\pi - t\right)\right].$$

By easy calculation we can show that for $0 \leqq t \leqq \pi/m$, $2 \leqq i \leqq n$,

$$d_n\left(\frac{2i-1}{m}\pi + t\right) - d_n\left(\frac{2i-1}{m}\pi - t\right) =$$

$$= (-1)^i \frac{1}{m} \cos\frac{mt}{2} \left\{\operatorname{cosec}\left(\frac{2i-1}{2m}\pi - \frac{t}{2}\right) - \operatorname{cosec}\left(\frac{2i-1}{2m}\pi + \frac{t}{2}\right)\right\},$$

(without loss of generality we can assume $\tau=0$) which yields

$$(-1)^i\left[d_n\left(\frac{2i-1}{m}\pi+t\right)-d_n\left(\frac{2i-1}{m}\pi-t\right)\right]\geqq$$

$$\geqq(-1)^{i+1}\left[d_n\left(\frac{2i+1}{m}\pi+t\right)-d_n\left(\frac{2i+1}{m}\pi-t\right)\right],$$

(contrary to [5] (35)) and therefore

$$(8)\qquad 2\sigma_0(t)\leqq 1+d_n\left(\frac{\pi}{m}+t\right)+d_n\left(\frac{3\pi}{m}+t\right)-d_n\left(\frac{3\pi}{m}-t\right)=$$

$$=1+\frac{1}{m}\cos\frac{mt}{2}\left\{\operatorname{cosec}\left(\frac{\pi}{2m}+\frac{t}{2}\right)+\operatorname{cosec}\left(\frac{3\pi}{2m}-\frac{t}{2}\right)-\operatorname{cosec}\left(\frac{3\pi}{2m}+\frac{t}{2}\right)\right\}=$$

$$=1+\frac{1}{m}\cos\frac{mt}{2}\left\{\operatorname{cosec}\frac{\pi}{2m}+\left[\operatorname{cosec}\left(\frac{\pi}{2m}+\frac{t}{2}\right)-\operatorname{cosec}\frac{\pi}{2m}\right]+\right.$$

$$\left.+\left[\operatorname{cosec}\left(\frac{3\pi}{2m}-\frac{t}{2}\right)-\operatorname{cosec}\left(\frac{3\pi}{2m}+\frac{t}{2}\right)\right]\right\}.$$

Now $(\operatorname{cosec} x-1/x)'$, $0<x\leqq\pi/m$, is an increasing function of x, which can be easily seen by its Taylor-series expansion. Thus evaluating this function at the end-points of the interval we get $\frac{1}{6}\leqq(\operatorname{cosec} x-1/x)'\leqq\frac{1}{3}$. Using the mean-value theorem we obtain, $0\leqq t\leqq\pi/m$,

$$\operatorname{cosec}\left(\frac{3\pi}{2m}-\frac{t}{2}\right)-\operatorname{cosec}\left(\frac{3\pi}{2m}+\frac{t}{2}\right)\leqq\left(\frac{3\pi}{2m}-\frac{t}{2}\right)^{-1}-\left(\frac{3\pi}{2m}+\frac{t}{2}\right)^{-1}-\frac{t}{6}=$$

$$=t\left[\left(\frac{3\pi}{2m}\right)^2-\left(\frac{t}{2}\right)^2\right]^{-1}-\frac{t}{6}\leqq t\left(\frac{m^2}{2\pi^2}-\frac{1}{6}\right),$$

$$\operatorname{cosec}\left(\frac{\pi}{2m}+\frac{t}{2}\right)-\operatorname{cosec}\frac{\pi}{2m}\leqq\left(\frac{\pi}{2m}+\frac{t}{2}\right)^{-1}-\left(\frac{\pi}{2m}\right)^{-1}+\frac{t}{6}=$$

$$=-\frac{t}{2}\left[\left(\frac{\pi}{2m}+\frac{t}{2}\right)\frac{\pi}{2m}\right]^{-1}+\frac{t}{6}\leqq t\left(-\frac{m^2}{\pi^2}+\frac{1}{6}\right).$$

Substituting this in (8) we have, $0\leqq t\leqq\pi/m$,

$$(9)\quad 2\sigma_0(t)\leqq 1+\frac{1}{m}\cos\frac{mt}{2}\left\{\operatorname{cosec}\frac{\pi}{2m}-t\frac{m^2}{2\pi^2}\right\}\leqq 1+\frac{1}{m}\cos\frac{mt}{2}\operatorname{cosec}\frac{\pi}{2m}.$$

This makes possible to correct the proof in [5] because using (9) and [5] (37), (38) we get

$$\sum_{i=-n}^{n}|\sigma_i(t)|\leqq 1+\frac{1}{2m}\cos\frac{mt}{2}\left\{\operatorname{cosec}\frac{\pi}{2m}-\sec\frac{t}{2}\right\}\leqq 1+\frac{1}{2m}\left\{\operatorname{cosec}\frac{\pi}{2m}-1\right\},$$

which coincides with (40) in [5].

3. Proof of Theorems 2 and 3

First we prove Theorem 2 for k odd, $k=2r-1$. It is easy to construct a function $f\in C_{2\pi}$ satisfying $f(\pi/m)=0$ and

$$f(t)=i,\quad \frac{2i-1}{m}\pi+\delta\leqq t\leqq\frac{2i+1}{m}\pi,\quad 1\leqq i\leqq\frac{k+1}{2},$$

$\delta>0$ being sufficiently small, $f\left(\frac{\pi}{m}+x\right)=f\left(\frac{\pi}{m}-x\right)$, $\omega\left(f,\frac{2\pi}{m}\right)=1$. Therefore we get

$$\text{(10)}\qquad 2^k\frac{\tilde S_{kn}[f]\left(\frac{\pi}{m}\right)-f\left(\frac{\pi}{m}\right)}{\omega\left(f,\frac{2\pi}{m}\right)}=\sum_{j=0}^{k}\binom{k}{j}\tilde S_n[f]\left(\frac{k+1-2j}{m}\pi\right)=$$

$$=\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{k+1-2j}{m}\pi\right)=2\sum_{j=0}^{r-1}\binom{2r-1}{j}(r-j).$$

But

$$\text{(11)}\qquad 2r\sum_{j=0}^{r-1}\binom{2r-1}{j}=r\sum_{j=0}^{2r-1}\binom{2r-1}{j}=r2^{2r-1}$$

and

$$\text{(12)}\qquad 2\sum_{j=0}^{r-1}\binom{2r-1}{j}j=(2r-1)2\sum_{j=1}^{r-1}\binom{2r-2}{j-1}=$$

$$=(2r-1)\left[\sum_{j=0}^{2r-2}\binom{2r-2}{j}-\binom{2r-2}{r-1}\right]=(2r-1)\left[2^{2r-2}-\binom{2r-2}{r-1}\right].$$

Substituting (12) and (11) in (10) we have Theorem 2 for $k=2r-1$.

Now in order to prove Theorem 3 let f denote a function satisfying

$$f(t_i)=i,\quad 1\leqq i\leqq k/2,\quad f(t_{1-i})=i,\quad 1\leqq i\leqq k/2+1,$$

k odd, $k=2r$, $r\geqq 1$. Of course we have for arbitrary $\varepsilon>0$ (m and k be given)

$$|\tilde S_{kn}[f](t)-\tilde S_{kn}[f](0)|<\varepsilon,$$

if $t>0$ is sufficiently small. We choose such t and now complete f as follows: $f(t)=0$, $f\in C_{2\pi}$, $\omega\left(f,\frac{2\pi}{m}\right)=1$. Thus we get

$$\frac{\tilde S_{kn}[f](t)-f(t)}{\omega\left(f,\frac{2\pi}{m}\right)}>\tilde S_{kn}[f](0)-\varepsilon,$$

$$2^k\tilde S_{kn}[f](0)=\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{k-2j}{m}\pi\right)=2\sum_{j=0}^{r}\binom{2r}{j}\left(r-j+\frac{1}{2}\right).$$

But now

$$\left(r+\frac{1}{2}\right)2\sum_{j=0}^{r}\binom{2r}{j} = \left(r+\frac{1}{2}\right)\left[\sum_{j=0}^{2r}\binom{2r}{j}+\binom{2r}{r}\right] = \left(r+\frac{1}{2}\right)\left[2^{2r}+\binom{2r}{r}\right],$$

$$2\sum_{j=0}^{r}\binom{2r}{j}j = 4r\sum_{j=0}^{r-1}\binom{2r-1}{j} = 2r\sum_{j=0}^{2r-1}\binom{2r-1}{j} = r2^{2r},$$

which completes the proof of Theorem 3.

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ON THE BANACH SPACE OF STRONGLY CONVERGENT TRIGONOMETRIC SERIES

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Introduction. Denote by C the Banach space of 2π-periodical real or complex valued continuous functions with the norm

$$\|f\|_C = \sup_{0\leq t\leq 2\pi} |f(t)|.$$

Using the familiar notations $S_N(f)$ means the N-th partial sum of the Fourier series of $f\in C$

(1.1)
$$\frac{a_0(f)}{2}+\sum_{n=1}^{\infty}(a_n(f)\cos nt+b_n(f)\sin nt) \equiv \sum_{n=0}^{\infty} A_n(f,t).$$

Very recently Tanovic—Miller [2] introduced a new definition of strong convergence, saying that the series $\sum_{n=0}^{\infty} c_n$ is strongly convergent to a limit T if

$$\lim_{M\to\infty}\frac{1}{M+1}\sum_{N=0}^{M}|(N+1)(S_N-T)-N(S_{N-1}-T)| = 0$$

where $S_N=\sum_{n=0}^{N} c_n$ and $S_{-1}=0$.

We shall denote by U, S and A the classes of functions f belonging to C whose Fourier series converge uniformly, strongly uniformly and absolutely on $[0, 2\pi]$, respectively. Tanovic—Miller showed *that the set* A *is a real subset of* S *which itself is a real subset of* U ([2], Theorem 4).

By the Fejér theorem we can see that if $f\in U$ then f is the sum of its Fourier series and in the cases $f\in S$ or $f\in A$ a similar conclusion is valid. Of course, if f is a sum of uniformly, strongly uniformly or absolutely convergent trigonometric series then $f\in U$, $f\in S$ or $f\in A$, respectively.

To show the difference between the uniform and strong uniform convergence we have the following trivial

LEMMA 1. *The function* $f\in C$ *belongs to the class* S, *that is*

(1.2)
$$\lim_{M\to\infty}\frac{1}{M+1}\left\|\sum_{N=0}^{M}|(N+1)(S_N(f)-f)-N(S_{N-1}(f)-f)|\right\|_C = 0$$

if and only if

(1.3)
$$\lim_{N\to\infty}\|f-S_N(f)\|_C = 0$$

and

$$\lim_{M\to\infty}\frac{1}{M+1}\Big\|\sum_{N=0}^{M} N|A_N(f)|\Big\|_C = 0 \tag{1.4}$$

are fulfilled.

It is known that U is a Banach space with the norm

$$\|f\|_U = \sup_N \|S_N(f)\|_C \tag{1.5}$$

and A is a Banach space with the norm

$$\|f\|_A = \frac{|a_0(f)|}{2}+\sum_{n=1}^{\infty}(|a_n(f)|+|b_n(f)|) \tag{1.6}$$

(see e.g. [1] p. 12).

Our aim is to find a suitable norm for S such that S would be a Banach space with this norm. Considering that $|(N+1)S_N(f)-NS_{N-1}(f)|\leqq N|A_N(f)|+|S_N(f)-f|+|f|$ $(N=0, 1, \ldots)$ we have the following inequality for any $M=0, 1, 2, \ldots$

$$\frac{1}{M+1}\Big\|\sum_{N=0}^{M}|(N+1)S_N(f)-NS_{N-1}(f)|\Big\|_C \leqq \frac{1}{M+1}\Big\|\sum_{N=0}^{M}N|A_N(f)|\Big\|_C+$$

$$+\frac{1}{M+1}\sum_{N=0}^{M}\|S_N(f)-f\|_C+\|f\|_C.$$

By Lemma 1 we can see that for any $f\in S$ the norm

$$\|f\|_S = \sup_M \frac{1}{M+1}\Big\|\sum_{N=0}^{M}|(N+1)S_N(f)-NS_{N-1}(f)|\Big\|_C \tag{1.7}$$

is finite.

THEOREM 1. *The set S is a Banach space with the norm* (1.7) *and for any $f\in S$*

$$\|f\|_U \leqq \|f\|_S. \tag{1.8}$$

Moreover for any $f\in A$

$$\|f\|_S \leqq 2\|f\|_A \tag{1.9}$$

holds.

Of course we have to show that S *is not a Banach space with the norm* (1.5) Turning to this direction we mention that for any non negative integer $N>M$ the inequality

$$\big\|S_N(f-S_M(f))\big\|_C \leqq \|S_N(f)-f\|_C+\|f-S_M(f)\|_C \tag{1.10}$$

holds.

Using the definition (1.5) the inequality (1.10) yields

LEMMA 2. *If $f\in U$ then $\lim_{M\to\infty}\|f-S_M(f)\|_U=0$.*

Let $f\in U\setminus S$. Lemma 2 shows that the sequence $\{S_M(f)\}_{M=0}^{\infty}$ has the Cauchy-property in the norm $\|\cdot\|_U$. As for any M, $S_M(f)\in S$ but $f\notin S$ we reached our aim. Moreover by Theorem 1 we have

(1.11) $$\sup_{\substack{f\in S\\ f\neq 0}} \frac{\|f\|_S}{\|f\|_U} = \infty.$$

If $f\in A$ then $\lim_{M\to\infty} \|f-S_M(f)\|_A=0$ is obvious. We can prove a similar result for the class S.

THEOREM 2. *If* $f\in S$ *then* $\lim_{M\to\infty} \|f-S_M(f)\|_S=0$.

Let $f\in S\setminus A$. Theorem 2 shows that the sequence $\{S_M(f)\}_{M=0}^{\infty}$ has the Cauchy property in the norm $\|\cdot\|_S$. As for any M, $S_M(f)\in A$ but $f\notin A$, we obtain that *A is not a Banach space with the norm* (1.7) and

(1.12) $$\sup_{\substack{f\in A\\ f\neq 0}} \frac{\|f\|_A}{\|f\|_S} = \infty.$$

Finally we mention that S is not a Banach algebra. To show this we have

THEOREM 3. *There exist functions* $f\in A$ *and* $g\in S$ *such that* $fg\notin U$.

PROOF OF THEOREM 1. First we prove the inequalities (1.8) and (1.9). By the identity $(M+1)S_M(f)=\sum_{N=0}^{M}\big((N+1)S_N(f)-NS_{N-1}(f)\big)$ we get that

$$\|S_M(f)\|_C \leqq \frac{1}{M+1}\Big\|\sum_{N=0}^{M} |(N+1)S_N(f)-NS_{N-1}(f)|\Big\|_C$$

and casting a glance at the definitions (1.5) and (1.7) we have (1.8). On the other hand for any $t\in[0, 2\pi]$ by (1.6) we have that for any $N=0, 1, \dots$

$$|(N+1)S_N(f, t)-NS_{N-1}(f, t)| = |S_N(f, t)+NA_N(f, t)| \leqq \|f\|_A+N\big(|a_N(f)|+|b_N(f)|\big)$$

and so for any $M=0, 1, \dots$

$$\frac{1}{M+1}\sum_{N=0}^{M} |(N+1)S_N(f, t)-NS_{N-1}(f, t)| \leqq \|f\|_A+\frac{1}{M+1}\sum_{N=0}^{M} N\big(|a_N(f)|+|b_N(f)|\big).$$

As the second term on the right hand side is less than $\|f\|_A$ by the definition (1.7) we get (1.9).

Let us assume that for any $n=1, 2, \dots$; $f_n\in S$ and the sequence $\{f_n\}_{n=1}^{\infty}$ has the Cauchy-property in the $\|\cdot\|_S$-norm, that is for any $\varepsilon(>0)$ there exists $\nu_1=\nu_1(\varepsilon)$ such that if $n, m\geqq \nu_1$ then

(2.1) $$\|f_n-f_m\|_S < \varepsilon.$$

By (2.1) and (1.8) we can see that the sequence $\{f_n\}_{n=1}^{\infty}$ has the Cauchy property in $\|\cdot\|_U$-norm, too. As U is a Banach space with the $\|\cdot\|_U$-norm we obtain that there exists a function $f\in U$ such that for any $\varepsilon(>0)$ there exists $\nu_2=\nu_2(\varepsilon)$ such that if $n\geqq \nu_2$ then

(2.2) $$\|f_n-f\|_U < \varepsilon.$$

Let M be a non-negative integer and δ be an arbitrary positive number. Assuming that $m \geqq \nu_2\left(\frac{\delta}{M+1}\right)$ we obtain by (2.2) that

$$\frac{1}{M+1}\left\|\sum_{N=0}^{M}|(N+1)S_N(f_m-f)-NS_{N-1}(f_m-f)|\right\|_C \leqq$$

$$\leqq \frac{1}{M+1}\sum_{N=0}^{M}\left((N+1)\|S_N(f_m-f)\|_C+N\|S_{N-1}(f_m-f)\|_C\right) \leqq$$

$$\leqq \frac{1}{M+1}\sum_{N=0}^{M}(2N+1)\|f_m-f\|_U < \delta.$$

Moreover if $n \geqq \nu_1(\varepsilon)$ and $m = \max\left(\nu_1(\varepsilon), \nu_2\left(\frac{\delta}{M+1}\right)\right)$ by (2.1) we get

$$\frac{1}{M+1}\left\|\sum_{N=0}^{M}|(N+1)S_N(f_n-f)-NS_{N-1}(f_n-f)|\right\|_C \leqq$$

$$\leqq \frac{1}{M+1}\left\|\sum_{N=0}^{M}|(N+1)S_N(f_n-f_m)-NS_{N-1}(f_n-f_m)|\right\|_C+$$

$$+\frac{1}{M+1}\left\|\sum_{N=0}^{M}|(N+1)S_N(f_m-f)-NS_{N-1}(f_m-f)|\right\|_C \leqq \|f_n-f_m\|_S+\delta < \varepsilon+\delta.$$

Hence

$$\|f_n-f\|_S \leqq \varepsilon \tag{2.3}$$

if $n \geqq \nu_1(\varepsilon)$.

Considering the identity

$$\sum_{N=0}^{M} NA_N(f) = \sum_{N=0}^{M}\left((N+1)S_N(f-f_n)-NS_{N-1}(f-f_n)\right)-$$

$$-\sum_{N=0}^{M} S_N(f_n-f)+\sum_{N=0}^{M} NA_N(f_n)$$

we have the inequality

$$\frac{1}{M+1}\left\|\sum_{N=0}^{M} N|A_N(f)|\right\|_C \leqq \|f-f_n\|_S+\frac{1}{M+1}\sum_{N=0}^{M}\|f_n-f\|_U+\frac{1}{M+1}\left\|\sum_{N=0}^{M} N|A_N(f_n)|\right\|_C.$$

Let now ε be an arbitrarily small positive number and $n = \max\left(\nu_1(\varepsilon/2), \nu_2(\varepsilon/2)\right)$. Then by (2.2) and (2.3) we obtain

$$\frac{1}{M+1}\left\|\sum_{N=0}^{M} N|A_N(f)|\right\|_C < \varepsilon+\frac{1}{M+1}\left\|\sum_{N=0}^{M} N|A_n(f_n)|\right\|_C.$$

As $f_n \in S$ by Lemma 1 (see (1.4)) we have that for any positive ε

$$\lim_{M\to\infty}\frac{1}{M+1}\left\|\sum_{N=0}^{M} N|A_N(f)|\right\|_C \leqq \varepsilon. \tag{2.4}$$

Since $f\in U$, condition (1.3) is fulfilled and by (2.4) we can see that (1.4) holds, too. Applying Lemma 1 we obtain that $f\in S$ and our proof is complete.

PROOF OF THEOREM 2. Our task is to estimate the difference

$$\|f-S_M(f)\|_S = \sup_\mu \left\|\frac{1}{\mu+1}\sum_{N=0}^{\mu}\left|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\right|\right\|_C.$$

Let M be a fixed non-negative integer. As

$$f-S_M(f) = \sum_{k=M+1}^{\infty} A_k(f) \tag{3.1}$$

the identity

$$S_N(f-S_M(f)) \equiv 0 \quad (N = 0, 1, 2, \dots, M) \tag{3.2}$$

is obvious, so we get

(3.3)

$$\frac{1}{\mu+1}\left\|\sum_{N=0}^{\mu}\left|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\right|\right\|_C = 0 \quad (\mu = 0, 1, \dots, M).$$

Using (3.1) again we can see

$$S_N(f-S_M(f)) = S_N(f)-S_M(f) \quad (N = M+1, M+2, \dots)$$

and

$$S_{N-1}(f-S_M(f)) = \begin{cases} 0 & \text{if } N = M+1 \\ S_{N-1}(f)-S_M(f) & \text{if } N = M+2, M+3, \dots. \end{cases}$$

So we have

$$(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f)) = NA_N(f)+S_N(f)-S_M(f) \tag{3.4}$$
$$(N = M+1, M+2, \dots).$$

For any $\mu=M+1, M+2, \dots$ by (3.2) and (3.4) we get

$$\sum_{N=0}^{\mu}\left|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\right| =$$

$$= \sum_{N=M+1}^{\mu}\left|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\right| =$$

$$= \sum_{N=M+1}^{\mu}\left|NA_N(f)+(S_N(f)-f)+(f-S_M(f))\right|.$$

So we obtain

$$\frac{1}{\mu+1}\left\|\sum_{N=0}^{\mu}\left|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\right|\right\|_C \leqq \tag{3.5}$$

$$\leqq \frac{1}{\mu+1}\left\|\sum_{N=M+1}^{\mu} N|A_N(f)|\right\|_C+\frac{1}{\mu+1}\sum_{N=M+1}^{\mu}\|S_N(f)-f\|_C+\|f-S_M(f)\|_C$$

$$(\mu = M+1, M+2, \dots).$$

As $f\in S$ by Lemma 1 we have (1.3) and (1.4), that is for any $\varepsilon(>0)$ there exist $\nu_1=\nu_1(\varepsilon)$ and $\nu_2=\nu_2(\varepsilon)$ such that if $k\geqq\nu_1$ then

$$\|f-S_k(f)\|_C<\varepsilon,$$

and if $\mu\geqq\nu_2$ then

$$\frac{1}{\mu+1}\Big\|\sum_{N=0}^{\mu} N|A_N(f)|\Big\|_C<\varepsilon.$$

Choosing $M\geqq\max\big(\nu_1(\varepsilon/3),\nu_2(\varepsilon/3)\big)$ by (3.5) we have

(3.6)

$$\frac{1}{\mu+1}\Big\|\sum_{N=0}^{\mu}\big|(N+1)S_N(f-S_M(f))-NS_{N-1}(f-S_M(f))\big|\Big\|_C<\varepsilon\quad(\mu=M+1, M+2, \dots).$$

Collecting (3.3) and (3.6) we can see that if $M\geqq\max\big(\nu_1(\varepsilon/3),\nu_2(\varepsilon/3)\big)$ then $\|f-S_M(f)\|_S\leqq\varepsilon$, so our proof is complete.

PROOF OF THEOREM 3. The series

$$\sum_{n=1}^{\infty}\frac{\sin nt}{n\log(n+1)}\quad(0\leqq t\leqq 2\pi)\tag{4.1}$$

converges uniformly in $[0, 2\pi]$ (See e.g. [1] p. 12.) Denoting by $g(t)$ the sum of the series (4.1) we have $\lim\limits_{N\to\infty}\|g-S_N(g)\|_C=0$. On the other hand for any $t\in[0, 2\pi]$

$$\frac{1}{M+1}\sum_{N=1}^{M}N|A_N(g,t)|\leqq\frac{1}{M+1}\sum_{N=1}^{M}\frac{1}{\log(N+1)}\to 0\quad\text{if}\quad M\to\infty,$$

so by Lemma 1 the function g belongs to the class S. Using the Euler's formula an easy computation shows that for any $N=1, 2, \dots$

$$S_N\big(ig(t)e^{iNt}\big)=-\frac{1}{2}e^{iNt}\sum_{n=1}^{2N}\frac{e^{-int}}{n\log(n+1)}\quad(t\in[0, 2\pi]),$$

and so we have

$$\big\|S_N\big(g(t)e^{iNt}\big)\big\|_C=\frac{1}{2}\sum_{n=1}^{2N}\frac{1}{n\log(n+1)}\quad(N=1, 2, \dots).\tag{4.2}$$

Let us assume that for any $f\in A$ we have $fg\in U$. By (1.5) we can write that for any $f\in A$ and $N=0, 1, \dots$

$$\|S_N(fg)\|_C\leqq\|fg\|_U.\tag{4.3}$$

Now we consider the linear continuous operators T_N from the Banach space A into the Banach space C:

$$T_Nf=S_N(fg)\quad(N=1, 2, \dots).$$

By (4.3) we may apply the Banach—Steinhaus theorem and get that there exists a positive constant K such that

$$\|S_N(fg)\|_C \leqq K\|f\|_A \quad (N=1,2,\ldots), \tag{4.4}$$

where K does not depend on f. Setting $f(t)=e^{int}$ $(\|f\|_A=2)$ from (4.4) we have

$$\left\|S_N\left(g(t)e^{iNt}\right)\right\|_C \leqq 2K \quad (N=1,2,\ldots),$$

and casting a glance at (4.2) we obtain a contradiction.

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ON THE CHARACTERIZATION OF THE DYADIC DERIVATIVE

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1. Introduction

Two essential open problems in dyadic analysis, as initiated by J. E. Gibbs (and his collaborators M. J. Millard and B. Ireland) [9, 10, 11] and further developed by P. L. Butzer and H. J. Wagner [2, 3, 4, 5], F. Schipp (and his collaborators J. Pál and P. Simon) [19, 20, 21, 25, 26], N. R. Ladhawala [14], R. Penney [23], C. W. Onneweer [15, 16, 17], Zheng-Wei-xing and Su-Wei-yi [32], and He Zelin [13] are the characterization and interpretation of the dyadic derivative. Gibbs and Ireland [12] gave a first, but rather abstract interpretation in the realm of locally compact abelian groups. However, just as the classical derivative may be associated with the slope of a tangent to a curve, or with the speed of an object — thus associated with basic geometric or physical notions — there is still no intuitive interpretation of the dyadic derivative (which may lie in the setting of these modern sciences such as information or signal theory which make use of dyadic Walsh analysis). Nevertheless, as a further step in this direction Skvorcov and Wade [28] derived a first characterization of the class of $f \in C[0, 1)$; they improved some earlier results due to Bočkarev [1], Butzer and Wagner [5] as well as to Schipp [27].

The aim of this paper is to give a rather complete characterization of the class of functions that are dyadic differentiable in dependence upon the discontinuities of the first kind (which means only jumps) of the function in question. The cases that the function has a finite number of jumps or an infinite number of jumps under the additional constraint that this set of discontinuities has only a finite number of cluster points are distinguished. Our main results state roughly that a function of either case is dyadic differentiable if and only if it is a piecewise constant.

Although this is a rather restrictive condition, the dyadic derivative is especially adapted to functions that have only a few or small intervals of constancy. It is even applicable to functions having a denumerable set of discontinuities like the well-known Dirichlet-function.

This paper will deal with the situation of functions defined on $[0, 1)$. The corresponding material for functions defined on the positive real axis $\mathbf{R}^+$ will be treated in a further paper.

The paper is divided into four sections. Section 2 is concerned with a summary of the fundamental properties of dyadic analysis, including dyadic representation, dyadic addition, the basic Walsh functions, some elements of Walsh—Fourier transforms, as well as the dyadic derivative. Section 3 deals with two characterization theorems. Finally, Section 4 is devoted to some representative examples which are worked out in detail.

2. Preliminaries

2.1. Dyadic addition. In the following, let $\mathbf{N}:=\{1, 2, 3, \ldots\}$, $\mathbf{P}:=\mathbf{N}\cup\{0\}$ and $\mathbf{Z}:=\{0, \pm 1, \pm 2, \ldots\}$. Each $k\in\mathbf{P}$ has a unique dyadic expansion

$$k = \sum_{j=0}^{\infty} k_j 2^j \quad (k_j\in\{0, 1\}). \tag{2.1}$$

Likewise each $x\in[0, 1)$ has a unique dyadic representation

$$x = \sum_{j=0}^{\infty} x_j 2^{-j} \quad (x_j\in\{0, 1\}) \tag{2.2}$$

if the finite expansion is chosen in case x belongs to the dyadic rationals $(=\text{D. R.})$, i.e., the set of all numbers of the form $x=p2^{-q}\in[0, 1)$, $p\in\mathbf{P}$, $q\in\mathbf{N}$. The dyadic sum of $x=\sum_{j=1}^{\infty} x_j 2^{-j}$ and $y=\sum_{j=1}^{\infty} y_j 2^{-j}$ is defined by

$$x\oplus y = \sum_{j=1}^{\infty} h_j 2^{-j}, \tag{2.3}$$

$$h_j := x_j \dot{+} y_j := (x_j+y_j)_{\mathrm{mod}\,2} = |x_j-y_j|. \tag{2.4}$$

In view of the uniqueness of the representation, dyadic addition is only defined for almost all $y\in[0, 1)$. For example, one easily sees that

$$x\oplus 2^{-j} = x-2^{-j}(x_j-(x_j\dot{+}1)), \tag{2.5}$$

and, setting

$$J_0 := \{j\in\mathbf{P},\, x_{j+1}=0\}, \quad J_1 := \{j\in\mathbf{P},\, x_{j+1}=1\}, \tag{2.6}$$

$$x\oplus 2^{-j-1} = \begin{cases} x+2^{-j-1}, & j\in J_0 \\ x-2^{-j-1}, & j\in J_1. \end{cases} \tag{2.7}$$

Formulas (2.5)—(2.7) will often be used later on in this paper.

2.2. Walsh functions. The functions which are taken as a basis for dyadic analysis are the Walsh functions $\psi_k(x)$ [31]. For $x\in[0, 1)$ — using Paley's enumeration [22] — they are given by

$$\psi_k(x) = \exp\left\{\pi i \sum_{j=0}^{\infty} k_j x_{j+1}\right\} = (-1)^{\sum_{j=0}^{\infty} k_j x_{j+1}}, \quad k\in\mathbf{P}, \tag{2.8}$$

and on $\mathbf{R}$ (set of all reals) by periodic extension. The ψ_k, $k\in\mathbf{P}$, form a complete, orthonormal system, and possess the important property

$$\psi_k(x\oplus y) = \psi_k(x)\psi_k(y) \quad (k\in\mathbf{P}) \tag{2.9}$$

for fixed $x\in[0, 1)$ and almost all $y\in[0, 1)$.

Denote by $L^p(0, 1)$, $1\leqq p\leqq\infty$, the set of all functions f of period 1 which are p-th power Lebesgue integrable and endowed with norm

$$\|f\|_p := \left\{\int_0^1 |f(x)|^p\, dx\right\}^{1/p}, \tag{2.10}$$

$L^\infty(0, 1)$ that of all essentially bounded functions f of period 1 with norm

(2.11) $$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in[0,1)} |f(x)|,$$

and finally $C^\oplus[0, 1]$ that of all dyadic continuous functions f of period 1, endowed with the usual sup norm, thus

(2.12) $$C^\oplus[0, 1] := \{f;\ \lim_{h\to 0} \|f(\cdot\oplus h)-f(\cdot)\|_{C^\oplus} := \lim_{h\to 0} \sup_{x\in[0,1)} |f(x\oplus h)-f(x)| = 0\}.$$

In the following $X=X(0, 1)$ always stands for one of the (Banach) spaces $L^p(0, 1)$, $1\leq p<\infty$, $L^\infty(0, 1)$, and $C^\oplus[0, 1]$, with norm $\|f\|_X$.

Denoting the Walsh—Fourier coefficients of $f\in X(0, 1)$ by

(2.13) $$f^\wedge(k) = \int_0^1 f(u)\psi_k(u)\,du \quad (k\in\mathbf{P}),$$

the formal series

(2.14) $$\sum_{k=0}^\infty f^\wedge(k)\psi_k(x)$$

is called the Walsh—Fourier series of f.

2.3. Dyadic differentiation. Further, the concept of dyadic differentiation, as defined by Butzer and Wagner [2], [4], is basic.

DEFINITION 3.1. a) Let $f\in X(0, 1)$. If there exists $g\in X(0, 1)$ such that

(2.15) $$\lim_{m\to\infty} \Big\| \sum_{j=0}^m 2^{j-1}[f(\cdot)-f(\cdot\oplus 2^{-j-1})]-g(\cdot)\Big\|_X = 0,$$

then g is called the first strong dyadic derivative of f, denoted by $g=D^{[1]}f$. Derivatives of higher order are defined successively by $(D^{[0]}f:=f)$

(2.16) $$D^{[r]}f = D^{[1]}(D^{[r-1]}f), \quad r\in\mathbf{N}.$$

b) Let f be defined on $[0, 1)$. If

(2.17) $$\sum_{j=0}^\infty 2^{j-1}[f(x)-f(x\oplus 2^{-j-1})] = c < \infty$$

for $x\in[0, 1)$, then c is called the first pointwise dyadic derivative of f at x, denoted by $f^{[1]}(x)$. Setting $f^{[0]}(x)=f(x)$, derivatives of higher order are given by

(2.18) $$f^{[r]}(x) = (f^{[r-1]})^{[1]}(x) \quad (r\in\mathbf{N}).$$

Some of the most important properties of the dyadic derivative are (for the proofs see [2, 3, 4]) listed below:

(i) $D^{[r]}$ is a closed, linear operator in $X(0, 1)$.

(ii) The Walsh functions ψ_k are infinitely differentiable, and

(2.19) $$D^{[r]}\psi_k = \psi_k^{[r]} = k^r\psi_k.$$

(iii) If $f, D^{(r)}f\in X(0, 1)$ for some $r\in\mathbf{N}$, then

(2.20) $$[D^{[r]}f]^\wedge(k) = k^r f^\wedge(k) \quad (k\in\mathbf{P}).$$

The pointwise counterpart of (2.20) can be found in [5]. Moreover, one can construct antidifferentiation operators to (2.15) and (2.17), respectively. This leads to the fundamental theorem of dyadic analysis; it is similar to the fundamental theorem of classical differential and integral calculus. For the strong version of the dyadic derivative the fundamental theorem is due to Butzer and Wagner [4], in the more difficult pointwise sense it is due to Schipp [25], [26]. The dyadic derivative gives information about the smoothness of a function. It allows one to study the rate of approximation of dyadic differentiable functions by partial sums of Walsh—Fourier series, the best degree of approximation in the Walsh setting, partial differential equations with the derivatives being understood in the dyadic sense, etc. One of the more important application lies, for instance, in the field of digital signal processing, cf. [6], [7], [8], [29].

3. Characterization theorems

Though the dyadic derivative has attained significance in various fields of dyadic (Walsh)-analysis, an interpretation of it or even a full characterization of the class of functions which are dyadic differentiable is still lacking. In this respect, however, a first result, due to Skvorcov and Wade [28], which generalizes an earlier one by Butzer and Wagner [5], states:

Let f be continuous on $[0, 1)$, *and let $f^{[1]}$ exist for all but countably many points $x \in (0, 1)$. Then f is constant.*

This result tells us that it is not reasonable to begin with functions that are continuous on the whole interval of definition $[0, 1)$. So we will deal with functions f that have infinitely many discontinuities. It seems convenient to divide this class of functions into two parts, namely those functions the set of discontinuity-points of which possesses either a finite or an inifinite number of cluster-points in $[0, 1)$. The first group of functions will be treated in our main theorem. For the second group it is clearly not possible to obtain an equivalent statement between dyadic differentiability and piecewise constancy, because the members of this class of functions cannot consist totally of piecewise (non-degenerate) constant functions. One could even say that in this event the intervals of constancy may degenerate to points. Nevertheless even those functions can be dyadic differentiable. In the next section we will show that Dirichlet's function, which is an important member of this 'exotic' class, is indeed dyadic differentiable. However, in case of f having a countable set of discontinuities which have at most a *finite* number of cluster-points we are able to prove the following

THEOREM. *Let f be defined and bounded on* $[0, 1)$, *possessing a countable set of discontinuities $x^{(k)}$, $k \in \mathbf{N}$, exclusively of first kind, which have at most a finite number of cluster-points in* $[0, 1)$. *Then f is pointwise dyadic differentiable except on a countable set in* $[0, 1)$ *if and only if the function is a piecewise constant on* $[0, 1)$.

PROOF. In the following it is assumed that the function is righthand continuous at the points of discontinuity. Although our results are also valid in case of lefthand continuity, they are slightly more complicated to establish.

At frist we will deal with the case that f has only one discontinuity, namely $x^{(1)} \in [0, 1)$. Suppose that f is dyadic differentiable on $[0, 1)$ except on a countable set $H\subset[0, 1)$. Then the series (2.17) converges for all $x\in[0, 1)\setminus H$, and therefore

$$\lim_{j\to\infty} 2^{j-1}[f(x)-f(x\oplus 2^{-j-1})] = 0. \tag{3.1}$$

If $x^{(1)}=0$, then f is continuous on $(0, 1)$, and with the help of the result of Skvorcov and Wade [28] cited above, one immediately concludes that f is constant on $(0, 1)$. Now suppose that $x^{(1)}>0$. Then f is continuous on $(0, x^{(1)})\cup(x^{(1)}, 1)$. If $x\in(0, x^{(1)})$, one can choose a sequence of integers $(j_n)_{n=1}^{\infty}$ with $j_n<j_{n+1}$, $n\in\mathbf{N}$, such that $x_{j_n}=0$ for all $n\in\mathbf{N}$. Noting (2.5) and (3.1), one deduces for all $x\notin H$

$$\lim_{n\to\infty}\frac{f(x)-f(x\oplus 2^{-j_n})}{2^{-j_n}} = \lim_{n\to\infty}\frac{f(x)-f(x+2^{-j_n})}{2^{-j_n}} = 0.$$

Likewise, if $x\notin$D.R., then there exists a sequence of integers $(\bar{j}_n)_{n=1}^{\infty}$ with $\bar{j}_n<\bar{j}_{n+1}$, $n\in\mathbf{N}$ such that $x_{\bar{j}_n}=1$ for all $n\in\mathbf{N}$. So similarly, one concludes for all $x\notin H\cup$D.R.

$$\lim_{n\to\infty}\frac{f(x)-f(x\oplus 2^{-\bar{j}_n})}{2^{-\bar{j}_n}} = \lim_{n\to\infty}\frac{f(x)-f(x-2^{-\bar{j}_n})}{2^{-\bar{j}_n}} = 0.$$

Therefore the upper and lower right Dini derivatives of f exist and are equal for $x\notin H\cup$D.R. (cf. [24], pp. 155). Since $H\cup$D.R. is countable, and f is continuous on $[0, x^{(1)})$, it follows as in [28] (using [24]) that f=const. on $(0, x^{(1)})$. Analogously one can verify that f is also constant on $(x^{(1)}, 1)$.

Conversely, suppose that f is a piecewise constant function on $[0, 1)$, possessing a jump at $x^{(1)}\in(0, 1)$. Obviously this function can be expressed as

$$f(x) = \begin{cases} A, & x\in[0, x^{(1)}) \\ B, & x\in[x^{(1)}, 1)\end{cases} \quad (A, B\in\mathbf{R}). \tag{3.2}$$

If $A=B$, the matter is trivial, because in this event f equals $A\psi_0$, which is clearly dyadic differentiable. So, let $A\neq B$; with J_0, J_1 defined as in (2.6) one has by (2.5), (2.7)

$$f(x\oplus 2^{-j-1}) = \begin{cases} f(x+2^{-j-1}), & j\in J_0 \\ f(x-2^{-j-1}), & j\in J_1.\end{cases} \tag{3.3}$$

It can be easily shown that for $x\in[0, x^{(1)})$ there exist $\bar{j}, \bar{\bar{j}}\in\mathbf{N}$ such that for the difference $f(x)-f(x\oplus 2^{-j-1})$ the following three relations are valid:

$$f(x)-f(x\oplus 2^{-j-1}) = \begin{cases} 0, & j\in J_1 \\ A-B, & j\in J_0;\ j\leqq\bar{j}, \quad x\in[0, x^{(1)}) \\ 0, & j\in J_0;\ j>\bar{j},\end{cases} \tag{3.4}$$

$$f(x)-f(x\oplus 2^{-j-1}) = \begin{cases} B-A, & j\in J_1; \\ 0, & j\in J_0,\end{cases} \quad x=x^{(1)} \tag{3.5}$$

$$f(x)-f(x\oplus 2^{-j-1}) = \begin{cases} B-A, & j\in J_1,\ j<\bar{\bar{j}} \\ 0, & j\in J_1;\ j\geqq\bar{\bar{j}}, \quad x\in[x^{(1)}, 1) \\ 0, & j\in J_0.\end{cases} \tag{3.6}$$

In view of these, the pointwise dyadic derivative (2.17) clearly exists for all $x\in[0,1)\setminus\{x^{(1)}\}$. If $x^{(1)}$ is a dyadic rational, f is dyadic differentiable there too. Hence f is dyadic differentiable almost everywhere on $[0,1)$. It is now obvious that both directions of the proof can easily be extended to the case of a finite number of discontinuities $x^{(k)}$, $1\leqq k\leqq n$.

Let us now assume that f possesses infinitely many discontinuities having in the first instance only one cluster point. Then, clearly, the limit $\lim_{k\to\infty} x^{(k)}=x_0\in[0,1]$ exists. Furthermore, the sequence may without loss of generality be assumed to be monotone, namely $x^{(k)}>x^{(k+1)}$. Now construct a sequence of intervals $I_k=(x^{(k+1)},x^{(k)})$ with $I_k\cap I_j=\emptyset$, $k\neq j$, so that the interval $[0,1)$, apart from the jumps $x^{(k)}$, is representable as a union of pointwise disjoint intervals:

$$[0,x_0)\cup\bigcup_{k=1}^{\infty}(x^{(k+1)},x^{(k)})\cup(x^{(1)},1)=[0,1)\setminus\bigcup_{k=1}^{\infty}\{x^{(k)}\}. \tag{3.7}$$

We can proceed as in the case of finitely many discontinuities. Suppose f is pointwise dyadic differentiable on $[0,1)$ except on a countable set $H\subset[0,1)$. Then one has (3.1) again for all $x\in[0,1)\setminus H$, where H is again the set of all points for which the derivative fails to exist. Now apply the above arguments to each of the intervals of the decomposition (3.7). It turns out that f is continuous on each of the intervals $[0,x_0)$, $(x^{(1)},1)$, I_k, $k\in\mathbf{N}$; similarly upper as well as lower right Dini-derivatives of f exist for $x\notin H\cup\text{D.R.}$, and they are as well equal to one another. So one can conclude that f is a constant on $[0,x_0)$, $(x^{(1)},1)$ as well as on each I_k, $k\in\mathbf{N}$.

Conversely, f is assumed to be a piecewise constant function possessing the discontinuities $x^{(k)}$, $k\in\mathbf{N}$, which are only jumping-points of f. Consequently f is constant on each of the intervals $[0,x_0)$, $(x^{(1)},1)$, and I_k, $k\in\mathbf{N}$, respectively. Now we are principally in the position to use the methods of the converse direction of the proof of the corresponding finite case. Suppose for example that $x\in I_k$, $k\in\mathbf{N}$. Obviously there exists $j_0\in\mathbf{N}$ such that for all $j>j_0$ the dyadic sum $x\oplus 2^{-j-1}$ belongs to I_k, I_{k+1} or I_{k-1}, respectively. (For $k=1$ define $I_0:=(x^{(1)},1)$.) Hence for $j>j_0$ we have the same situation as in the above converse direction. For example, if $x\oplus 2^{-j-1}$ belongs to I_k or I_{k+1}, respectively, one obtains eight relations for the difference $x\pm 2^{-j-1}$, quite similar to those in (3.4)—(3.6). Thus the difference $f(x)-f(x\oplus 2^{-j-1})$ vanishes for some $j>j^*$, since f is constant on $(0,x_0)$, $(x^{(1)},1)$ and on I_k, $k\in\mathbf{N}$. Consequently we have for all $x\in(0,x_0)\cup(x^{(1)},1)\cup I_k$, $k\in\mathbf{N}$

$$\sum_{j=0}^{\infty}2^{j-1}[f(x)-f(x\oplus 2^{-j-1})]<\infty.$$

Therefore the dyadic derivative can only fail to exist at the points $x_0, x^{(k)}$, $k\in\mathbf{N}$. Since $(x^{(k)})_{k=1}^{\infty}$ is countable, f is dyadic differentiable on $[0,1)$, except the countable set.

Let us finally discuss the case that the sequence of discontinuities $(x^{(k)})_{k=1}^{\infty}$ has a finite number of cluster points in $[0,1]$, say x_i, $1\leqq i\leqq n$. Then we can split up the sequence $(x^{(k)})_{k=1}^{\infty}$ into the subsequences $(x^{(i_k)})_{k=1}^{\infty}$, $1\leqq i\leqq n$, having the property that $\lim_{k\to\infty} x^{(i_k)}=x_i$, $1\leqq i\leqq n$. The proof of the result then follows along the same lines as the above proof concerning one cluster point.

For the particular case of a *finite* number of discontinuities of the function f it is quite clear that the statement of the Theorem remains valid if the pointwise derivative is replaced by the *strong* dyadic derivative. In this regard we have

COROLLARY. *Let f be a function on* $[0, 1)$ *possessing a finite number of discontinuities* $x^{(k)}$, $1 \leqq k \leqq n$, *in* $[0, 1)$ *and no points of discontinuity of the second kind (poles, etc.). Then f is dyadic differentiable on* $[0, 1)$ *in the strong sense if and only if* $f = \varphi$ *except on a countable set,* φ *being a piecewise constant function on* $[0, 1)$.

4. Examples

In this section we will present some examples which illustrate the applicability as well as the limits of the dyadic derivative. We begin our considerations with an example of a function that is piecewise constant and consequently is dyadic differentiable in both senses. Nevertheless this simple example will point out what the derivative actually effects. Note that the general theorems treated do not give any information about the real nature of the dyadic derivative.

Let g_1 be defined by

$$g_1(x) = \begin{cases} 3, & x \in [0, 1/4) \\ -1, & x \in [1/4, 1). \end{cases}$$

According to our theorem as well as the corollary, g_1 is differentiable in the strong as well as in the pointwise sense and, since the point of discontinuity is a dyadic rational, g_1 is dyadic differentiable everywhere on $[0, 1)$. Evaluating the derivative gives

$$D^{[1]} g_1(x) = g_1^{[1]}(x) = \begin{cases} 6, & x \in [0, 1/4) \\ -4, & x \in [1/4, 1/2) \\ -2, & x \in [1/2, 3/4) \\ 0, & x \in [3/4, 1). \end{cases}$$

This example shows that the derivative of a piecewise constant function can possess more points of discontinuity than the original function does have; this clearly results from the global character of the derivative.

Let us now consider a piecewise linear function, for example $g_2(x) = x\psi_1(x)$. Clearly the theorem tells us that g_2 cannot be dyadic differentiable since it is not piecewise constant. It is even *nowhere* dyadic differentiable on $[0, 1)$. Indeed, the terms of the series (2.17) are

$$2^{j-1}[x\psi_1(x) - (x \oplus 2^{-j-1})\psi_1(x \oplus 2^{-j-1})] = \begin{cases} (1/2)\psi_1(x)[x \oplus 1/2] + x, & j = 0 \\ -(1/4)\psi_1(x), & j \in J_0 \quad j \geqq 1 \\ (1/4)\psi_1(x), & j \in J_1 \end{cases}$$

noting that $\psi_1(2^{-j-1}) = -1$, $j = 0$, $\psi(2^{-j-1}) = 1$, $j \geqq 1$, as well as formula (2.9). So the series (2.17) does not converge for any $x \in [0, 1)$, since the necessity condition (3.1) is not satisfied. Thus g_2 is nowhere dyadic differentiable on $[0, 1)$.

Let us now present an example of a function the discontinuities of which have one cluster point in $[0, 1)$, namely

$$g_3(x) = \begin{cases} 1/2^n, & x\in[2^{-n-1}, 2^{-n}) \\ 0, & x = 0 \end{cases} \quad (n\in \mathbf{N}\cup\{0\}).$$

Since g_3 is bounded with cluster point $x=0$, g_3 is dyadic differentiable on $[0, 1)$ except on a countable set. Its dyadic derivative at $x=1/2$, for example, is given by $g_3^{[1]}(1/2)=1/2$; at $x=0$ it does not exist.

One may conjecture that a function which possesses a countable number of discontinuities, which lie *dense* in $[0, 1]$, is not dyadic differentiable. On the contrary, the dyadic derivative is especially suited for such exotic functions, although the general theorem is not applicable. One such example is Dirichlet's function

$$g_4(x) = \begin{cases} 0, & x\in[0, 1)\cap\mathbf{Q} \\ 1, & \text{elsewhere on } [0, 1), \end{cases}$$

where $\mathbf{Q}$ denotes the set of all rationals. g_4 is dyadic differentiable everywhere on $[0, 1)$ with $D^{[1]}g_4=g_4^{[1]}=0$. In fact, if $x\in[0, 1)\setminus\mathbf{Q}$, then $g_4(x\oplus 2^{-j-1})=0$ by (2.7). If $x\in[0, 1)\cap\mathbf{Q}$, then (2.7) and the definition of g_4 deliver $g_4(x\oplus 2^{-j-1})=0$. So $g_4(x)-g_4(x\oplus 2^{-j-1})=0$, for all $x\in[0, 1)$ and $j\in\mathbf{P}$. Hence $d^{[1]}(x)=0$, $x\in[0, 1)$.

For the strong dyadic derivative (2.15) with respect to the spaces $L^p(0, 1)$, $1\leqq p\leqq\infty$ one has on account of

$$\Big\|\sum_{j=0}^{n-1} 2^{j-1}[g_4(\cdot)-g_4(\cdot\oplus 2^{-j-1})]\Big\|_{L^p(0,1)} =$$

$$= \begin{cases} \Big\{\displaystyle\int_{[0,1)\setminus Q} \Big|\sum_{j=0}^{n-1} 2^{j-1}[g_4(x)-g_4(x\oplus 2^{-j-1})]\Big|^p dx\Big\}^{1/p}, & 1\leqq p<\infty, \\ \operatorname*{ess\,sup}_{x\in[0,1)} \Big|\displaystyle\sum_{j=0}^{n-1} 2^{j-1}[g_4(x)-g_4(x\oplus 2^{-j-1})]\Big|, & p=\infty \end{cases}$$

and in connection with the preceding remarks that $D^{[1]}g_4(x)=0$. Note that the space $C^{\oplus}[0, 1]$ is not permissible here since $g_4\notin C^{\oplus}[0, 1]$.

A more complicated function dyadic differentiable at every $x\in[0, 1)$ is

$$g_5(x) = \begin{cases} g_4(x)+1, & x\in[0, 1/2) \\ g_4(x)+3, & x\in[1/2, 3/4) \\ g_4(x)-1, & x\in[3/4, 1). \end{cases}$$

Another exotic function is $g_6(x)=1+\sum_{k=1}^{\infty}(1/k^2)\psi_k(x)$, $x\in[0, 1)$. Although it has discontinuities at each dyadic rational, it is still dyadic differentiable with $g_6^{[1]}(x)= =D^{[1]}g_6(x)=\sum_{k=1}^{\infty}(1/k)\psi_k(x)$ (see [6] for exact evaluation, [8] for its graph).

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INTEGRABILITY OF WALSH SERIES

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1. We shall use the following notations. Let f be a function integrable on $(0, 1)$ and periodic with period 1. We denote the dyadic addition by $\dotplus$ and the dyadic difference of a sequence (c_n) of real numbers with respect to an integer m by

$$\Delta_m(c_n) = c_{n\dotplus m} - c_n.$$

Let us put

$$_a A_{p,\alpha}(c_n) = \Big\{\sum_{m=1}^{\infty} m^{-p\alpha-1} \|\Delta_m(c_n)\|_a^p\Big\}^{1/p},$$

and

$$_a \tilde{A}_{p,\alpha}(f) = \Big\{\sum_{m=1}^{\infty} m^{-p\alpha-1} [Y_a(m)]^p\Big\}^{1/p},$$

where

$$\|\Delta_m(c_n)\|_a = \Big\{\sum_{n=0}^{\infty} |\Delta_m(c_n)|^a\Big\}^{1/a},$$

$$Y_a(m) = \|f(\psi_m - 1)\|_a = \Big[\int_0^1 |f(x)|^a |\psi_m(x) - 1|^a \, dx\Big]^{1/a}$$

for $0 < a < \infty$ and

$$\|\Delta_m(c_n)\|_\infty = \sup_n |\Delta_m(c_n)|,$$

$$Y_\infty(m) = \|f(\psi_m - 1)\|_\infty = \operatorname*{ess\,sup}_x \big|f(x)\big(\psi_m(x) - 1\big)\big|.$$

Here we have denoted the Walsh—Paley system by ψ_m: $m = 0, 1, 2, \ldots$.

Let us introduce the Beurling norm (cf. [3]). For the purpose, we set

$$W = \{w \in L^1(0, 1):\ w > 0,\ w\downarrow\}, \quad \text{and} \quad {}_a\|f\|_{p,w} = \|f w^{1/a - 1/p}\|_a.$$

Then the Beurling norm is defined by

$$_a\|f\|_p = \inf_{w \in W} \big[\|w\|_1^{1/p - 1/a}\, {}_a\|f\|_{p,w}\big].$$

A function space $_aL_p$ is the class of functions whose Beurling norms are finite. It is clear that $_aL_p \subseteqq L^p$ for $0 < p \leqq a$ and $_aL_a = L^a$.

We shall denote by K a positive constant depending at most on p, a and α which may be different from one occurrence to another.

The main results are as follows.

THEOREM 1. *Suppose that* $1<a\leqq 2$, $1/a+1/a'=1$, $1\leqq p\leqq a'$, *and* $\alpha=1/p-1/a'$. *For a given sequence* (c_n), *if* $\lim_{n\to\infty} c_n=0$ *and* ${}_aA_{p,\alpha}(c_n)<\infty$, *then there exists a function* f *such that*

$$_{a'}\|f\|_p \leqq K_a A_{p,\alpha}(c_n)$$

and its Walsh—Fourier expansion is

$$f\sim \sum_{n=0}^{\infty} c_n\psi_n.$$

The result holds also for the case $a=1$ *and* $1\leqq p<\infty=a'$.

THEOREM 2. *Suppose that* $1\leqq a\leqq 2$, $1/a+1/a'=1$, $0<p\leqq a$ *and* $0<\alpha$. *If a function* f *is integrable on* $(0, 1)$ *and* $f\sim \sum_{n=0}^{\infty} c_n\psi_n$, *then*

$$_{a'}A_{p,\alpha}(c_n) \leqq K_a\|f_{a,p,\alpha}\|_p,$$

where $f_{a,p,\alpha}(x)=f(x)\,x^{\alpha-1/p+1/a}$ *for* $x\in(0, 1)$.

REMARK. (I) Our theorems are the Walsh series analogue of Beurling's results in [3]. (II) The trigonometric series analogue was proved in [4] and the dual results to the above have been discussed in the previous paper [5]. (III) In the case of trigonometric series, we have used the higher order difference of (c_n) in our discussion. However, in the case of Walsh series, any higher order difference is essentially the same as the first one because of the dyadic addition. That is why we shall discuss only the first order difference.

2. The main part of the paper is to prove the following lemmas.

LEMMA 1. *If* $0<p\leqq a<\infty$ *or* $0<p<\infty=a$, *then* ${}_a\|f_{a,p,\alpha}\|_p\leqq K_a\tilde{A}_{p,\alpha}(f)$.

LEMMA 2. *If* $0<p\leqq a<\infty$ *and* $0<\alpha$, *then* ${}_a\tilde{A}_{p,\alpha}(f)\leqq K_a\|f_{a,p,\alpha}\|_p$.

PROOF OF LEMMA 1. (Cf. [5].) Let us suppose that ${}_a\tilde{A}_{p,\alpha}(f)<\infty$ and discuss firstly the case $0<p<a<\infty$. We define a decreasing function w by

$$w(x) = \sum_{m=1}^{2^k} m^{-p\alpha}Y^p(m), \quad \text{for} \quad 2^{-k} \leqq x < 2^{-k+1},$$

where $Y(m)=Y_a(m)$. Then we have

$$\int_0^1 w(x)\,dx = \sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}} w(x)\,dx = \sum_{k=1}^{\infty}\sum_{m=1}^{2^k} 2^{-k}m^{-p\alpha}Y^p(m) =$$

$$= Y^p(1)+\sum_{m=1}^{\infty}\sum_{k=m}^{\infty}\sum_{j=1}^{2^{m-1}} 2^{-k}(2^{m-1}+j)^{-p\alpha}Y^p(2^{m-1}+j) =$$

$$= Y^p(1)+\sum_{m=1}^{\infty}2^{-m+1}\sum_{j=1}^{2^{m-1}}(2^{m-1}+j)^{-p\alpha}Y^p(2^{m-1}+j) =$$

$$\leqq Y^p(1)+K\sum_{m=1}^{\infty}\sum_{j=1}^{2^{m-1}}(2^{m-1}+j)^{-p\alpha-1}Y^p(2^{m-1}+j) \leqq K\sum_{m=1}^{\infty} m^{-p\alpha-1}Y^p(m),$$

that is,

(1) $$\|w\|_1 \leqq K[{}_a\tilde{A}_{p,\alpha}(f)]^p,$$

and thus $w \in W$.

Let us estimate ${}_a\tilde{A}_{p,\alpha}(f)$:

(2) $$[{}_a\tilde{A}_{p,\alpha}(f)]^p = \sum_{m=1}^{\infty} m^{-p\alpha-1} Y^a(m) Y^{p-a}(m) =$$

$$= \int_0^1 |f(x)|^a \Big\{ \sum_{m=1}^{\infty} m^{-p\alpha-1} Y^{p-a}(m) |\psi_m(x)-1|^a \Big\} dx \geqq$$

$$\geqq \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} |f(x)|^a \Big\{ \sum_{m=1}^{2^k} m^{-p\alpha-1} Y^{p-a}(m) |\psi_m(x)-1|^a \Big\} dx =$$

$$= \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} |f(x)|^a M(x)\, dx, \quad \text{say.}$$

Now we have to evaluate the lower bound of $M(x)$ on $(2^{-k}, 2^{-k+1})$. For that purpose, we set $P=a/p$, and $1/P+1/Q=1$. Then, by Hölder inequality, we have

$$[w(x)]^{1/Q}[M(x)]^{1/P} \geqq \sum_{m=1}^{2^k} [m^{-p\alpha} Y^p(m)]^{1/Q} [m^{-p\alpha-1} Y^{p-a}(m) |\psi_m(x)-1|^a]^{1/P} =$$

$$= \sum_{m=1}^{2^k} m^{-p\alpha-p/a} |\psi_m(x)-1|^p \geqq K \sum_{m=2^{k-1}}^{2^k-1} m^{-p\alpha-p/a},$$

since $\psi_m(x)=-1$ if $2^{k-1} \leqq m < 2^k$ and $2^{-k} < x < 2^{-k+1}$. Consequently we have

$$[w(x)]^{1/Q}[M(x)]^{1/P} \geqq K(2^k)^{-p\alpha-p/a+1} \geqq K x^{p\alpha+p/a-1}, \quad \text{for} \quad 2^{-k} < x < 2^{-k+1}.$$

Therefore we get

$$M(x) \geqq K[w(x)]^{-P/Q} x^{P(p\alpha+p/a-1)} = K x^{a\alpha+1-a/p} [w(x)]^{1-a/p}.$$

Replacing $M(x)$ in (2) by the above estimation, we have

$$[{}_a\tilde{A}_{p,\alpha}(f)]^p \geqq K \int_0^1 |f(x)|^a x^{a\alpha+1-a/p} [w(x)]^{1-a/p}\, dx = K[{}_a\|f_{a,p,\alpha}\|_{p,w}]^a.$$

Using the above inequality and (1), we have

$$[\|w\|_1]^{1/p-1/a}{}_a\|f_{a,p,\alpha}\|_{p,w} \leqq K[{}_a\tilde{A}_{p,\alpha}(f)]^{p(1/p-1/a)+p/a} = K\,{}_a\tilde{A}_{p,\alpha}(f),$$

from which we have the conclusion.

Next we shall discuss the case $0 < p = a < \infty$. For this case, we need only to show the following inequality:

$$\int_0^1 |f(x)|^p x^{p\alpha}\, dx \leqq \sum_{m=1}^{\infty} m^{-p\alpha-1} \int_0^1 |f(x)|^p |\psi_m(x)-1|^p\, dx.$$

However, the right hand side is greater than

$$\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}|f(x)|^p\Big[\sum_{m=2^{k-1}}^{2^k-1}m^{-p\alpha-1}|\psi_m(x)-1|^p\Big]\,dx \geqq$$

$$\geqq K\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}}|f(x)|^p\Big[\sum_{m=2^{k-1}}^{2^k-1}m^{-p\alpha-1}\Big]\,dx \geqq K\int_0^1|f(x)|^p x^{p\alpha}\,dx,$$

which is the required inequality.

Finally, we shall discuss the case $0<p<\infty=a$. Set

$$w(x)=\sum_{m=1}^{2^k}m^{-p\alpha}Y^p(m)$$

for $2^{-k}\leqq x<2^{-k+1}$, where $Y(m)=Y_\infty(m)$. Then, by the same way as used in the first case, we have

$$\|w\|_1\leqq K[{}_\infty\tilde{A}_{p,\alpha}(f)]^p. \tag{3}$$

Now we have to prove that

$$[\|w\|_1]^{1/p}{}_\infty\|f_{\infty,p,\alpha}\|_{p,w}\leqq K_\infty\tilde{A}_{p,\alpha}(f).$$

By (3), we see that the above inequality follows from

$${}_\infty\|f_{\infty,p,\alpha}\|_{p,w}\leqq K,$$

which may be read as $\|f_{\infty,p,\alpha}w^{-1/p}\|_\infty\leqq K$. Consequently, it is enough to show that for $2^{-k}\leqq x<2^{-k+1}$,

$$|f(x)|x^{\alpha-1/p}\leqq K\Big\{\sum_{m=1}^{2^k}m^{-p\alpha}\big|f(x)\big(\psi_m(x)-1\big)\big|^p\Big\}^{1/p}$$

that is,

$$x^{\alpha-1/p}\leqq K\Big\{\sum_{m=1}^{2^k}m^{-p\alpha}|\psi_m(x)-1|^p\Big\}^{1/p}.$$

However, the sum in the right hand side is greater than

$$K\sum_{m=2^{k-1}}^{2^k-1}m^{-p\alpha}\geqq K(2^k)^{-p\alpha+1}\geqq Kx^{p\alpha-1}$$

for $2^{-k}\leqq x<2^{-k+1}$, which completes the proof.

Proof of Lemma 2. Let us first prove the required inequality for the case when $0<p<a<\infty$. For any $w\in W$ and any numbers ε and δ such that $0<\varepsilon<1<\delta$, we may find a function w^* which satisfies the following properties (cf. Beurling [2]):

(i) $w\leqq w^*$;

(ii) $x^\delta w^*(x)$ is increasing;

(iii) $x^\varepsilon w^*(x)$ is decreasing;

(iv) $\int_0^1 w^*(x)\,dx=K\int_0^1 w(x)\,dx$.

Let us put $P=a/p$, $1/P+1/Q=1$, $\alpha_1+\alpha_2=-p\alpha-1$, and $\alpha_2=-2/Q$. Then, by

Hölder inequality, we have

$$(4) \qquad [{}_a\tilde{A}_{p,\alpha}(f)]^p = \sum_{m=1}^{\infty}\{Y^p(m)[w^*(1/m)]^{-1/Q}m^{\alpha_1}\}\{[w^*(1/m)]^{1/Q}m^{\alpha_2}\} \leqq$$

$$\leqq \Big\{\sum_{m=1}^{\infty} Y^a(m)[w^*(1/m)]^{1-a/p}m^{P\alpha_1}\Big\}^{1/P}\Big\{\sum_{m=1}^{\infty} w^*(1/m)m^{-2}\Big\}^{1/Q} \leqq$$

$$\leqq K[\|w^*\|_1]^{1/Q}\Big\{\sum_{m=1}^{\infty} Y^a(m)[w^*(1/m)]^{1-a/p}m^{P\alpha_1}\Big\}^{1/P} =$$

$$= K[\|w^*\|_1]^{1/Q}\{S\}^{1/P}, \quad \text{say.}$$

According to the definition of $Y(m)=Y_a(m)$, we have

$$S = \int_0^1 |f(x)|^a\Big\{\sum_{m=1}^{\infty} m^{P\alpha_1}|\psi_m(x)-1|^a[w^*(1/m)]^{1-a/p}\Big\}dx =$$

$$= \sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}} |f(x)|^a\Big\{\sum_{m\leqq 1/x} + \sum_{m>1/x}\Big\}dx.$$

The first part of summation in the above will be denoted by J_1 and the second by J_2. If $2^{-k}<x<2^{-k+1}$, and $m<2^{k-1}$, then $\psi_m(x)=1$. Therefore we have

$$J_1 \leqq K \sum_{2^{k-1}\leqq m\leqq 1/x} m^{P\alpha_1}[w^*(1/m)]^{1-a/p}.$$

By the property (ii), we have, for $m\leqq 1/x$,

$$x^\delta w^*(x) \leqq m^{-\delta}w^*(1/m),$$

that is,

$$[w^*(1/m)]^{1-a/p} \leqq [m^\delta x^\delta w^*(x)]^{1-a/p}.$$

Consequently we have

$$J_1 \leqq Kx^{\delta(1-a/p)}[w^*(x)]^{1-a/p} \sum_{2^{k-1}\leqq m\leqq 1/x} m^{P\alpha_1+\delta(1-a/p)} \leqq Kx^{-P\alpha_1-1}[w^*(x)]^{1-a/p}.$$

By property (iii), we have for $1/x<m$,

$$[w^*(1/m)]^{1-a/p} \leqq [m^\varepsilon x^\varepsilon w^*(x)]^{1-a/p}.$$

Hence we have

$$J_2 \leqq Kx^{\varepsilon(1-a/p)}[w^*(x)]^{1-a/p} \sum_{1/x<m} m^{P\alpha_1+\varepsilon(1-a/p)}.$$

Since we may choose ε so that $P\alpha_1+\varepsilon(1-a/p)<-1$, we have

$$J_2 \leqq Kx^{-P\alpha_1-1}[w^*(x)]^{1-a/p}.$$

Now returning to S, we have

$$S \leqq K\int_0^1 |f(x)|^a x^{-P\alpha_1-1}[w^*(x)]^{1-a/p}\,dx \leqq K\int_0^1 |f(x)|^a x^{a\alpha-a/p+1}[w(x)]^{1-a/p}\,dx.$$

Replace S in (4) by the above estimation, then we have

$$[{}_a\tilde{A}_{p,\alpha}(f)]^p \leqq K[\|w\|_1]^{1/Q}[{}_a\|f_{a,p,\alpha}\|_{p,w}]^p,$$

which is the required inequality.

Finally we shall prove Lemma 2 for the case when $0<p=a<\infty$. For this case we need to show that

$$[{}_p\tilde{A}_{p,\alpha}(f)]^p \leqq K\|f_{p,p,\alpha}\|_p^p.$$

However, the left hand side is

$$\sum_{m=1}^{\infty} m^{-p\alpha-1}\int_0^1 |f(x)|^p|\psi_m(x)-1|^p\,dx \leqq K\sum_{k=1}^{\infty}\int_{2^{-k}}^{2^{-k+1}} |f(x)|^p\Big\{\sum_{m=2^{k-1}}^{\infty} m^{-p\alpha-1}\Big\}\,dx \leqq$$

$$\leqq K\int_0^1 |f(x)|^p x^{p\alpha}\,dx,$$

which is the required inequality.

3. Proof of Theorem 1. Since ${}_aA_{p,\alpha}(c_n)$ is finite, we have for each m

$$\Big(\sum_{n=0}^{\infty} |\Delta_m(c_n)|^a\Big)^{1/a} < \infty.$$

Consequently, there exists a function $F_m \in L^{a'}(0,1)$ whose Walsh—Fourier series is

$$F_m \sim \sum_{n=0}^{\infty} \Delta_m(c_n)\psi_n.$$

Write $E_m=\{x\in(0,1)\colon \psi_m(x)\neq 1\}$, and for $x\in E_m$

$$F_m(x) = f_m(x)\big(\psi_m(x)-1\big).$$

On the other hand, we have

$$F_m(\psi_l-1) \sim \sum_{n=0}^{\infty} \Delta_l(\Delta_m(c_n))\psi_n$$

and

$$F_l(\psi_m-1) \sim \sum_{n=0}^{\infty} \Delta_m(\Delta_l(c_n))\psi_n.$$

Since $\Delta_l(\Delta_m(c_n))=\Delta_m(\Delta_l(c_n))$, we have, for a.e. $x\in(0,1)$,

$$F_m(x)\big(\psi_l(x)-1\big) = F_l(x)\big(\psi_m(x)-1\big),$$

that is,

$$f_m(x)\big(\psi_m(x)-1\big)\big(\psi_l(x)-1\big) = f_l(x)\big(\psi_l(x)-1\big)\big(\psi_m(x)-1\big).$$

Therefore, we have $f_m(x)=f_l(x)$ for a.e. $x\in E_m\cap E_l$. Now we can define a function f by $f(x)=f_m(x)$ for $x\in E_m$. Since $\bigcup_m E_m=(0,1)$, f is well defined at almost every point of $(0,1)$. It is not hard to see that $f\in L^1(0,1)$. And we have, for each m and n

that $\Delta_m(c_{m \dotplus n}) = -\Delta_m(c_n)$. Therefore

$$F_m\psi_m \sim -\sum_{n=0}^{\infty} \Delta_m(c_n)\psi_n.$$

Thus $F_m(x) = -F_m(x)\psi_m(x)$ for a.e. $x \in (0, 1)$, that is, $F_m(x)=0$ for a.e. $x \in (0, 1) \setminus E_m$, from which we get

$$f(\psi_m - 1) \sim \sum_{n=0}^{\infty} \Delta_m(c_n)\psi_n.$$

By the Hausdorff—Young theorem, we have

$$\|f(\psi_m-1)\|_{a'} \leqq K\Big(\sum_{n=0}^{\infty} |\Delta_m(c_n)|^a\Big)^{1/a},$$

and so

$${}_{a'}\tilde{A}_{p,\alpha}(f) \leqq K_a A_{p,\alpha}(c_n).$$

By the application of Lemma 1, we get

$${}_{a'}\|f\|_p \leqq K_a A_{p,\alpha}(c_n) < \infty.$$

Let us denote the Walsh—Fourier coefficients of f by (c_n^*), then

$$f(\psi_m - 1) \sim \sum_{n=0}^{\infty} \Delta_m(c_n^*)\psi_n.$$

On the other hand, we have

$$f(\psi_m - 1) \sim \sum_{n=0}^{\infty} \Delta_m(c_n)\psi_n.$$

Therefore we get $\Delta_m(c_n^*) = \Delta_m(c_n)$ for each n, and hence $c_n^* = c_n - c_{n \dotplus m}^* - c_{n \dotplus m}$. Let $m \to \infty$, then we have $c_n^* = c_n$, which completes the proof.

PROOF OF THEOREM 2. We may suppose that ${}_a\|f_{a,p,\alpha}\|_p < \infty$. Then, by Lemma 2, we have ${}_a\tilde{A}_{p,\alpha}(f) < \infty$. Hence we have, for each m,

$$Y_a(m) = \|f(\psi_m - 1)\|_a < \infty.$$

By the Hausdorff—Young theorem, we have

$$\|\Delta_m(c_n)\|_{a'} \leqq KY_a(m).$$

Again by Lemma 2, we have finally

$${}_{a'}A_{p,\alpha}(c_n) \leqq K_a \tilde{A}_{p,\alpha}(f) \leqq K_a \|f_{a,p,\alpha}\|_p.$$

4. From Theorems 1 and 2, we have

THEOREM 3. *Suppose that* $1 \leqq p < 2$ *and* $\alpha = 1/p - 1/2$. *Then* (c_n) *is the sequence of Walsh—Fourier coefficients of a function* $f \in {}_2L_p$, *if and only if* ${}_2A_{p,\alpha}(c_n) < \infty$.

The following result is a generalization of Y. Okuyama's theorem in [6].

THEOREM 4. *Suppose that* $1 \leqq p < 2$, $f \in L^1(0, 1)$ *and*

$$f \sim \sum_{n=0}^{\infty} c_n \psi_n.$$

For a given sequence (a_n), *if* $\lim_{n \to \infty} a_n = 0$ *and*

$$\sum_{n=0}^{\infty} |\Delta_m(a_n)|^2 \leqq K \sum_{n=0}^{\infty} |\Delta_m(c_n)|^2, \tag{5}$$

and if

$$\int_0^1 x^{-p/2} \Big(\int_x^1 f^2(t)\, dt \Big)^{p/2} dx < \infty, \tag{6}$$

then there exists a function $g \in {}_2L_p$ *such that* $g \sim \sum_{n=0}^{\infty} a_n \psi_n$.

PROOF OF THEOREM 4. Let $1 \leqq p < 2$ and $\alpha = 1/p - 1/2$. Then, by Lemmas 1, 2, Theorems 1 and 2, we see that the finiteness of norms ${}_2\|f\|_p$, ${}_2A_{p,\alpha}(c_n)$ and ${}_2\tilde{A}_{p,\alpha}(f)$ are mutually equivalent. On the other hand, we have

$$[{}_2\tilde{A}_{p,\alpha}(f)]^p = \sum_{m=1}^{\infty} m^{-p\alpha-1} \Big(\int_0^1 |f(x)(\psi_m(x) - 1)|^2\, dx \Big)^{p/2} =$$

$$= \sum_{k=0}^{\infty} \sum_{m=2^k}^{2^{k+1}-1} m^{p/2-2} \Big(\int_{2^{-k-1}}^1 f^2(x)(\psi_m(x) - 1)^2\, dx \Big)^{p/2} \leqq$$

$$\leqq K \sum_{k=0}^{\infty} (2^k)^{p/2-1} \Big(\int_{2^{-k-1}}^1 f^2(x)\, dx \Big)^{p/2} \leqq K \int_0^1 x^{-p/2} \Big(\int_x^1 f^2(t)\, dt \Big)^{p/2} dx.$$

Now, from (5), we have

$${}_2A_{p,\alpha}(a_n) \leqq K\, {}_2A_{p,\alpha}(c_n) \leqq K\, {}_2\tilde{A}_{p,\alpha}(f) < \infty.$$

Consequently we have the conclusion from Theorem 3.

The following is a direct conclusion of Theorem 1.

THEOREM 5. *Suppose that* $1 < a \leqq 2$, $1/a + 1/a' = 1$, $1 \leqq p \leqq a'$ *and* $\alpha = 1/p - 1/a'$. *For a given sequence* (c_n), *if* $\lim_{n \to \infty} c_n = 0$ *and*

$$\sum_{n=0}^{\infty} |\Delta_m(c_n)|^a = O(m^\beta),$$

where $\beta < a\alpha$, *then there exists a function* $f \in {}_{a'}L_p$ *whose Walsh—Fourier coefficients are* c_n*'s.*

The corresponding Fourier series analogue, which is a generalization of the Boas theorem in [1] (cf. M. Sato [7]), is a corollary of our previous result in [4].

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A CHARACTERIZATION OF SEPARABLE POLYNOMIALS OVER A RING

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1. Introduction. Let R be a ring with 1 (not necessarily commutative) and $R[X]$ a polynomial ring in the indeterminate X such that $rX=Xr$ for each r in R. A monic polynomial $f(X)=X^n+a_{n-1}X^{n-1}+\dots+a_1X+a_0$ for some integer n in $R[X]$ is called a separable polynomial if $R[X]/(f(X))$ is a separable ring extension over R with a free basis $\{1, x, \dots, x^{n-1}\}$ such that $x^n=-a_{n-1}x^{n-1}-\dots-a_1x-a_0$, where $(f(X))$ is the ideal of $R[X]$ generated by $f(X)$ and $x=X+(f(X))$. When R is commutative, a lot of interesting properties of a separable polynomial have been found ([1], [2]). It has been known that $f(X)$ is separable over a commutative ring R if and only if the determinant det (T) of T is a unit in R where each entry of T is the trace $t(x^ix^j)$ of x^ix^j ([1], Theorem 4.4, p. 111). The purpose of the present paper is to generalize the above characterization of a separable polynomial over a commutative ring to a non-commutative ring finitely generated and projective over its center.

2. Basic definitions. Let s be an element in $R[x]$. Then $s=\Sigma_{i=0}^{n-1}r_ix^i$ for some r_i in R. We denote the i^{th}-projection map from $s\to r_i$ by π_i. Then $s=\Sigma_i\pi_i(s)x^i$. The *trace* t is defined by $t(s)=\Sigma_i\pi_i(sx^i)$. It is easy to see that t is an R-homomorphism from $R[x]$ to R. Let A be a subring with 1 of R. Then R is called a *separable extension* over A if there exist elements $\{a_i, b_i$ in R, $i=1, \dots, k$ for some integer $k\}$ such that $\Sigma a_ib_i=1$ and $s(\Sigma a_i\otimes b_i)=(\Sigma a_i\otimes b_i)s$ for each s in R, where $\otimes$ is over A. Such an $\Sigma(a_i\otimes b_i)$ is called a *separable idempotent* for R (see [1], [3], [4], or [5]). Throughout, we assume that $R[x]$ is a ring $(=R[X]/(f(X)))$ where $f(X)=$ $=X^n+a_{n-1}X^{n-1}+\dots+a_1X+a_0$ with a free basis $\{1, x, \dots, x^{n-1}\}$ over R, and that C is the center of R.

3. Separable polynomials. In this section, we shall characterize the separability of $f(X)$ in terms of the determinant of the matrix T of the trace t of $R[x]$. We begin with some properties of the trace t.

LEMMA 3.1. *Let T be an n by n matrix with the $(i+1, j+1)$-th entry $t(x^ix^j)$ for $i, j=0, 1, \dots, n-1$. If there exists an n by n matrix P over R such that $PT=I$, the identity matrix, then* (1) *there exist elements $\{z_i/i=0, 1, \dots, n-1\}$ in $R[x]$ such that $t(sz_i)=t(z_is)=\pi_i(s)$ for each s in $R[x]$ where π_i is the i-th projection of $R[x]$ to R, and* (2) *t is free over $R[x]$, that is, $(ts)(u)=t(su)=0$ for each u in $R[x]$, then $s=0$ in $R[x]$.*

[1] This paper was written during the first author's sabbatical leave at the University of Chicago, and he would like to thank Professor I. Herstein for his excellent lectures on Galois theory and Professor R. Swan on projective modules.

PROOF. (1) Let $P=[a_{ij}]$ for some a_{ij} in R such that $PT=I$. Then $\Sigma_k a_{ik} t(x^k x^j)=\delta_{ij}$ ($=0$ for $i\neq j$, and 1 for $i=j$). Hence $t((\Sigma_k a_{ik} x^k) x^j)=\delta_{ij}$. Now, let $z_i=\Sigma_k a_{ik} x^k$. We have that $t(z_i x^j)=\delta_{ij}$. Since x^j is in the center of $R[x]$ for each j, $t(z_i x^j)=t(x^j z_i)=\delta_{ij}=\pi_i(x^j)$. Hence, for s $(=\Sigma_i r_i x^i)$ in $R[x]$,

$$t(z_i s) = t(z_i(\Sigma_j x^j r_j)) = \Sigma_j(t(z_i x^j) r_j) = \Sigma r_j t(z_i x^j) = \Sigma r_j t(x^j z_i) = t(\Sigma_j r_j x^j z_i) = t(s z_i).$$

Also,

$$\Sigma_j r_j t(z_i x^j) = \Sigma_j r_j \pi_i(x^j) = \pi_i(\Sigma r_j x^j) = \pi_i(s).$$

Thus, $t(z_i s)=t(s z_i)=\pi_i(s)$ for each s in $R[x]$.

(2) Let $ts(u)=t(su)=0$ for an s in $R[x]$, and for each u in $R[x]$. Then $t(sz_i)=$ $=\pi_i(s)=0$ for each $i=0, \dots, n-1$ by part (1). But then $s=\Sigma_i \pi_i(s) x^i=0$.

Next we show that $PT=I$ as given in Lemma 3.1 is sufficient for the separability of $R[x]$ over R.

THEOREM 3.2. *If $PT=I$ as given in Lemma* 3.1, *then $R[x]$ is a separable extension over R.*

PROOF. Let $\{z_i\}$ be given as in Lemma 3.1. Then we claim that $\Sigma_i z_i \otimes x^i$ is a separable idempotent for $R[x]$. In fact,

$$t((1-\Sigma_i z_i x^i) u) = t(u) - t(\Sigma z_i x^i u) = \Sigma_i \pi_i(u x^i) - t(\Sigma z_i x^i u) = \Sigma t(z_i u x^i) - t(\Sigma z_i x^i u) = 0$$

for each u in $R[x]$, so $\Sigma_i z_i x^i = 1$ by part (2) of Lemma 3.1.

Moreover, for each u in $R[x]$,

$$\begin{aligned} u(\Sigma_i z_i \otimes x^i) &= \Sigma u z_i \otimes x^i = \Sigma_i(\Sigma_k \pi_k(u z_i) x^k) \otimes x^i = \Sigma_i(\Sigma_k t(z_k u z_i) x^k) \otimes x^i = \\ &= \Sigma_k(x^k \otimes \Sigma_i t(z_k u z_i) x^i) = \Sigma_k(x^k \otimes \Sigma_i t(z_i z_k u) x^i) = \Sigma_k(x^k \otimes \Sigma_i \pi_i(z_k u) x^i) = \\ &= \Sigma_k x^k \otimes z_k u = (\Sigma_k x^k \otimes z_k) u \end{aligned}$$

(by Lemma 3.1 (1)). Let $u=1$. We have $\Sigma_i z_i \otimes x^i = \Sigma_i x^i \otimes z_i$. Thus $u(\Sigma_i x^i \otimes z_i)=$ $=(\Sigma_i x^i \otimes z_i) u$ for each u in $R[x]$. Therefore, $R[x]$ is separable over R.

In order to show the converse of Theorem 3.2, we have the following lemma.

LEMMA 3.3. *If $R[x]$ is separable over R, then for each s in $R[x]$, $\pi_k(s)=$ $=t(z_k s)=t(s z_k')$ for some z_k and z_k' in $R[x]$ for $k=0, \dots, n-1$.*

PROOF. Let $\Sigma_i x_i \otimes y_i$, $i=1, \dots, k$ for some integer k be a separable idempotent for $R[x]$. Then $z_k=\Sigma_i \pi_k(x_i) y_i$ and $z_k'=\Sigma_i x_i \pi_k(y_i)$ are required to satisfy the equations in the lemma from the proof of Theorem 2.1 in [1], p. 92.

THEOREM 3.4. *If $R[x]$ is separable over R, then there exists a matrix P of order n such that $PT=I$.*

PROOF. Since $R[x]$ is separable over R, there exist elements z_i in $R[x]$ such that $\pi_i(s)=t(z_i s)$ for each s in $R[x]$ by Lemma 3.3. Let $z_i=\Sigma_k a_{ik} x^k$ for some a_{ik} in R, $i=0, \dots, n-1$. Then

$$\pi_i(x^j) = \delta_{ij} = t(z_i x^j) = t(\Sigma_k a_{ik} x^k x^j) = (\Sigma_k a_{ik}) t(x^k x^j)$$

which is the $(i+1, j+1)$-th entry of the matrix PT where $P=[a_{ij}]$ and $T=[t(x^i x^j)]$. Thus $PT=I$.

We remark that P is not unique in Theorem 3.4 since there exist z_k' $(=\Sigma_i x_i \pi_k(y_i))$ such that $\pi_i(s)=t(sz_i')$ by Lemma 3.3. Hence $\pi_i(x^j)=t(x^j z_i')=t(z_i' x^j)=\delta_{ij}$. Let $z_i'=\Sigma_k d_{ik} x^k$ for some d_{ik} in R. Then $P'T=I$ where $P'=[d_{ij}]$. However, for R finitely generated and projective over its center C, we shall show that P is unique.

THEOREM 3.5. *Let R be finitely generated and projective over C. Then $R[x]$ is separable over R if and only if T is invertible.*

PROOF. Since $R[x]$ is separable over R, there exists a matrix P of order n such that $PT=I$ by Theorem 3.5. Since R is finitely generated and projective over C, so is $R[x]$ over C (for $R[x]$ is free over R). Hence the C-algebra $\mathrm{Hom}_R(R[x], R[x])$ is also finitely generated and projective over C. Considering P and T as elements in $\mathrm{Hom}_R(R[x], R[x])$, we have $TP=I$ from $PT=I$ ([1], Exercise 10, p. 38). Thus T is invertible. The converse is clear by Theorem 3.2.

Now we generalize the characterization of a separable polynomial $f(X)$ in $R[X]$ in terms of the determinant of T from a commutative ring R to a non-commutative ring R finitely generated and projective over its center C.

THEOREM 3.6. *Let R be a non-commutative ring finitely generated and projective over its center C. Then, $R[x]$ is separable over R if and only if the determinant of T is a unit in C. In this case, the entries of P in Theorem 3.5 are in C.*

PROOF. Since $rt(x^i x^j)=t(rx^i x^j)=t(x^i x^j)r$ for each r in R, $t(x^i x^j)$ is in C for all i and j. Now, suppose the determinant of T, $\det(T)$, is a unit (and hence in C). Then T is invertible. Hence $R[x]$ is separable over R by Theorem 3.5. Conversely, if $R[x]$ is separable over R, then T is invertible by Theorem 3.5 again (note that T^{-1} is over R). Since T is over C, $\det(T)$ is defined. Suppose that $\det(T)$ is not a unit in C. There exists a maximal ideal M of C containing $\det(T)$. By hypothesis, R is finitely generated projective and faithful (for 1 is in R) over C, $MR\neq R$. Hence $\bar{P}\bar{T}=\bar{I}$ in $\mathrm{Hom}_{\bar{R}}(\bar{R}[x], \bar{R}[x])$, where $\bar{C}=C/MC$ and $\bar{R}=R/MR\neq 0$. But $\det(\bar{T})=0$ in $\bar{C}$, so that $\bar{P}\bar{T}=\bar{I}$ is a contradiction. Thus $\det(T)$ is a unit in C. In this case, $PT=I$, so

$$(*) \qquad \Sigma_k a_{ik} t(x^k x^j) = \delta_{ij}, \quad k, i, j = 0, \dots, n-1.$$

Since $\det(T)$ is a unit in C, the system of equations $(*)$ is solvable for a_{ik} in C by Cramer's rule. Thus P is over C.

We note that the determinant of a matrix over a non-commutative ring is not defined, so Theorems 3.4 and 3.6 are not equivalent.

We conclude the paper with two separable polynomials by using Theorem 3.6.

EXAMPLE 1. Let $f(X)=X^n-b$ for a b in C of R as given in Theorem 3.6. Then $T=[t_{ij}]$ where $t_{11}=n$, $t_{ij}=bn$ for $i+j=n+2$ and other t_{ij} are 0. Hence $\det(T)=$ $=n^n b^{n-1}$. Thus $f(X)$ is separable over R if and only if b and n are units.

EXAMPLE 2. Let $f(X)=X^n-X-b$ for a b in C of R as given in Theorem 3.6. Then $T=[t_{ij}]$ where $t_{11}=n$, $t_{ij}=n$ for $i+j=n+1$, and $t_{ij}=bn$ for $i+j=n+2$, and others are 0. Hence $\det(T)=n^n(b^{n-1}+(-1)^{n+1})$. Thus $f(X)$ is separable if and only if n and $(b^{n-1}+(-1)^{n+1})$ are units in C.

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NON-FISSILE RINGS WITH MIXED ADDITIVE GROUP

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Let R be a ring, R^+ the additive group of R, R_t the torsion subgroup of R^+. If R_t is a ring theoretic-direct summand of R, then R is said to be a fissile ring [1]. The following question was posed by S. Feigelstock [2] in a recent paper: Let G be a mixed group satisfying $p(G/G_t)\neq G/G_t$ for every prime p, $G_t=$ the torsion part of G. Must there exist a non-fissile (associative) ring R with $R^+=G$? In this short note we prove that the answer is affirmative. Our result generalizes [1] Theorem 7 and [2] Theorem 1.

PROPOSITION. *Let G be a mixed group, G_t its torsion subgroup, p a prime, G_p the p-torsion part of G. Suppose $G_p\neq 0$ and $p(G/G_t)\neq G/G_t$. Then there exists a non-fissile associative commutative ring R with $R^+=G$.*

PROOF. Let $\pi_1\colon G\to G/G_t$ and $\pi_2\colon G/G_t\to(G/G_t)/p(G/G_t)$ be the canonical projections. Fix a non-zero homomorphism $\varphi\colon (G/G_t)/p(G/G_t)\to Z/pZ$. Let $\mu\colon Z/pZ\otimes\otimes Z/pZ\to Z/pZ$ be the homomorphism induced by the multiplication in the field Z/pZ. Since $G_p\neq 0$, there exists an injective homomorphism $\iota\colon Z/pZ\to G$. Then the image of the composed mapping $\nu=\iota\mu\big((\varphi\pi_2\pi_1)\otimes(\varphi\pi_2\pi_1)\big)\colon G\otimes G\to G$ is isomorphic to Z/pZ, and therefore $\nu\neq 0$ and $\nu(G\otimes G)\subset G_t$. Define a multiplication on G via ν; this multiplication induces a commutative ring structure R on G. Since $R^3=0$, the ring R is associative. Finally, R is non-fissile, because if $R=R_t\oplus S$ is a ring direct sum, then $0\neq R^2=S^2\subset R_t$ and $S^2\cap R_t\neq 0$, a contradiction.

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ALMOST EVERYWHERE CONVERGENCE OF ORTHOGONAL SERIES

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1. Introduction. Yoneda [5] showed that if

(1.1) $$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e. on} \quad [0, 1],$$

then there exists a positive function $\delta(x)$ and, for every $\varepsilon>0$, there exists an integer $n(\varepsilon)$ such that,

(1.2) $$|f_n(x)-f(x)| < \varepsilon\delta(x) \quad \text{everywhere}$$

for $n>n(\varepsilon)$. This function $\delta(x)$ is termed a control function of (1.1). Wagner and Wilczyński [4] showed that (1.2) is equivalent to the well-known Egoroff's theorem and Taylor's theorem. We apply the control function to the a.e. convergence of orthogonal series.

2. Main theorems. Let $\{\varphi_n(x)\}_n$ be an orthogonal system on [0, 1] and

(2.1) $$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

be an orthogonal series which satisfies $\sum_{n=0}^{\infty} c_n^2<+\infty$. It is well known that series (2.1) converges in $L^2[0, 1]$ to an L^2-integrable function $f(x)$. Let $s_n(x)$ and $\sigma_n^\alpha(x)$ be the n-th partial sum and the (c, α)-mean of the series (2.1), respectively. Let $\{n_\nu\}_\nu$ be an increasing sequence of integers and set

$$\sigma_n^\alpha(\{n_\nu\}, x) = (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_{n_\nu}(x),$$

where $A_n^\alpha=\binom{n+\alpha}{n}$. Let $k>0$, $\gamma=\min(1, k/2)$. Put

$$\Delta_1 = \{(\alpha, \beta)|\beta > 1-\alpha/k,\ 0 < \alpha \leqq \gamma\},\quad \Delta_2 = \{(\alpha, \beta)|\gamma < \alpha < +\infty,\ \beta > 1-\gamma/k\},$$
$$\Delta = \Delta_1\cup\Delta_2.$$

The following are the main theorems.

THEOREM A. *Suppose that* $\sum_{n=0}^{\infty} c_n^2<+\infty$ *and* $(\alpha, \beta)\in\Delta$. *If*

(2.2) $$\lim_{n\to\infty} \sigma_n^\beta(x) = f(x) \quad \text{a.e. on} \quad [0, 1]$$

and (2.2) *has an L^p-integrable control function for some* $p>0$, *then*

$$\text{(2.3)} \qquad \lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(x)-f(x)|^k = 0 \quad a.e. \ on \quad [0,1]$$

and (2.3) *has an $L^{p^*/k}$-integrable control function, where* $p^*=\min(2,p)$.

THEOREM B. *Suppose that* $\sum_{n=0}^{\infty} c_n^2 \{\log\log(n+3)\}^2 < +\infty$ *and* $(\alpha,\beta)\in\Delta$. *Then*

$$\text{(2.4)} \qquad \lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(\{n_\nu\},x)-f(x)|^k = 0 \quad a.e. \ on \quad [0,1]$$

for each increasing sequence $\{n_\nu\}$ *of integers. Moreover,* (2.4) *has an $L^{2/k}$-integrable control function.*

3. Lemmas. To prove the main theorems we need the following lemmas. When $\lim_{n\to\infty} f_n(x)=f(x)$, a majorant function $f^*(x)=\sup_{n\geq 0}|f_n(x)|$ plays an important role in determining a class of control functions. Yoneda [6] proved the following lemma.

LEMMA 1. *If* (1.1) *holds and* $f^*(x)$ *is an L^p-integrable function for some positive* $p<+\infty$, *then* (1.1) *has an L^p-integrable control function.*

The following lemma is well known. $C, C_1, C_2, \ldots$ will denote positive constants not necessarily the same at each occurrence.

LEMMA 2 (Zygmund [7], p. 193). *If* $\sum_{n=0}^{\infty} c_n^2\{\log(n+2)\}^2 < +\infty$, *then* $s_n(x)$ *converges a.e. to an L^2-integrable function* $f(x)$ *and*

$$\int_0^1 \sup_{n\geq 0} |s_n(x)|^2\,dx \leq C \sum_{n=0}^{\infty} c_n^2\{\log(n+2)\}^2.$$

By Lemmas 1 and 2 we have a trivial consequence. If $\sum_{n=0}^{\infty} c_n^2\{\log(n+2)\}^2 < +\infty$, then $s_n(x)$ converges a.e. to an L^2-integrable function and its convergence has an L^2-integrable control function.

LEMMA 3. *If* $\sum_{n=0}^{\infty} c_n^2 < +\infty$, *then*

$$\text{(3.1)} \qquad \int_0^1 \Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)|^2\Big\}\,dx \leq C \sum_{n=0}^{\infty} c_n^2 \quad (\alpha > 1/2).$$

PROOF. It is a consequence of the proof of 2.6.2 in Alexits [1].

LEMMA 4 (Flett [2], p. 115). *Let* $r\geq k>1$, $\alpha>-1$ *and* $\beta\geq\alpha+k^{-1}-r^{-1}$. *Then*

$$\text{(3.2)} \qquad \Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\beta-1}(x)-\sigma_n^\beta(x)|^r\Big\}^{1/r} \leq C\Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^\alpha(x)|^k\Big\}^{1/k}.$$

LEMMA 5. *If* $\sum_{n=0}^{\infty} c_n^2 < +\infty$, *then*

$$\int_0^1 \Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\beta-1}(x)-\sigma_n^{\beta}(x)|^r\Big\}^{2/r} dx \leqq C \sum_{n=0}^{\infty} c_n^2 \tag{3.3}$$

for each $r \geqq 2$ *and* $\beta > 1 - r^{-1}$.

PROOF. Set $\varepsilon = \beta - (1-r^{-1})$ and $\alpha = \varepsilon + 2^{-1}$. Then we have $\beta = \alpha + 2^{-1} - r^{-1}$ and $\alpha > 1/2$. Apply Lemma 4 by setting $k=2$ in (3.2). Hence we have

$$\int_0^1 \Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\beta-1}(x)-\sigma_n^{\beta}(x)|^r\Big\}^{2/r} dx \leqq C \int_0^1 \Big\{\sum_{n=1}^{\infty} n^{-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^{\alpha}(x)|^2\Big\} dx.$$

From (3.1) we have (3.3).

LEMMA 6. *Suppose that* $\sum_{n=0}^{\infty} c_n^2 < +\infty$ *and* $(\alpha, \beta) \in \Delta$. *Then*

$$\int_0^1 \Big\{\sup_{n \geqq 0} (1/A_n^{\alpha}) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^k\Big\}^{2/k} dx \leqq C \sum_{n=0}^{\infty} c_n^2. \tag{3.4}$$

PROOF. Set $\gamma = \min(1, k/2)$. Two cases arise; one is $(\alpha, \beta) \in \Delta_1$, the other is $(\alpha, \beta) \in \Delta_2$. If $0 < \alpha \leqq \gamma$, then $k(1-\beta) < \alpha \leqq \gamma \leqq 1$, thus we can take $r(>1)$ which satisfies $k(1-\beta) < 1/r < \alpha$ and set $1/r + 1/s = 1$. Then, by Hölder's inequality,

$$(1/A_n^{\alpha}) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^k \leqq$$

$$\leqq (1/A_n^{\alpha})\Big\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^{rk}\Big\}^{1/r}\Big\{\sum_{\nu=1}^{n} \nu^{s/r}(A_{n-\nu}^{\alpha-1})^s\Big\}^{1/s} \leqq$$

$$\leqq (C_1/A_n^{\alpha})\Big\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^{rk}\Big\}^{1/r}\{n^{s/r}n^{(\alpha-1)s+1}\}^{1/s} =$$

$$= (C_1 n^{\alpha}/A_n^{\alpha})\Big\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^{rk}\Big\}^{1/r} \leqq$$

$$\leqq C_2\Big\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^{rk}\Big\}^{1/r}.$$

Hence it follows that

$$\int_0^1 \Big\{\sup_{n \geqq 0} (1/A_n^{\alpha}) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^k\Big\}^{2/k} dx \leqq \tag{3.5}$$

$$\leqq C_3 \int_0^1 \Big\{\sum_{\nu=1}^{\infty} \nu^{-1}|\sigma_\nu^{\beta-1}(x)-\alpha_\nu^{\beta}(x)|^{rk}\Big\}^{2/rk} dx.$$

Since $1/r<\alpha\leqq\gamma\leqq k/2$ and $k(1-\beta)<1/r$, we have $rk>2$ and $\beta>1-1/rk$. Apply Lemma 5 with a parameter rk instead of r. From (3.5), (3.4) holds.

Next we prove (3.4) for $(\alpha, \beta)\in\Delta_2$. Set

$$\sigma_n^{\alpha,\beta}(x) = (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^\beta(x)|^k.$$

By the well-known results of Alexits [1], p. 69, it follows that

$$\sup_{n\geqq 0} \sigma_n^{\alpha,\beta}(x) \leqq \sup_{n\geqq 0} \sigma_n^{\gamma,\beta}(x).$$

By the preceding argument, (3.4) is valid for (γ, β). Hence we obtain (3.4) with $(\alpha, \beta)\in\Delta_2$. This completes the proof.

LEMMA 7. *Suppose that* $\alpha>0$, $\beta>0$ *and* $k>0$; *and that*

$$\lim_{n\to\infty} \sigma_n^\beta(x) = f(x) \quad \text{a.e. on} \quad [0, 1] \tag{3.6}$$

where $f(x)$ *is an* L^2*-integrable function. If* (3.6) *has an* L^p*-integrable control function for some* $0<p<+\infty$, *then*

$$\lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^k = 0 \quad \text{a.e. on} \quad [0, 1] \tag{3.7}$$

and then (3.7) *has an* $L^{p^*/k}$*-integrable control function, where* $p^*=\min(2, p)$.

PROOF. Let $\delta(x)$ be a control function of (3.6). For any $\varepsilon>0$ there exists a positive integer $n(\varepsilon)$ such that $|\sigma_n^\beta(x)-f(x)| < \varepsilon\delta(x)$ everywhere for all $n>n(\varepsilon)$. When $\varepsilon=1$, there exists a positive integer n_1 such that $|\sigma_n^\beta(x)-f(x)|<\delta(x)$ everywhere for all $n>n_1$. Put $n_2=n(\varepsilon^{1/k})$. Without loss of generality we can take $n_2>n_1$. If $n>n_2$, then

$$(1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^k \leqq (1/A_n^\alpha) \sum_{\nu=0}^{n_1} A_{n-\nu}^{\alpha-1} \max\{|\sigma_\nu^\beta(x)-f(x)|^k\colon 0\leqq\nu\leqq n_1\}+$$

$$+(1/A_n^\alpha) \sum_{\nu=n_1+1}^{n_2} A_{n-\nu}^{\alpha-1}\{\delta(x)\}^k+\varepsilon\{\delta(x)\}^k.$$

Hence, there exists a positive integer $m(\varepsilon)$ such that

$$(1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^k < \varepsilon\delta_1(x)$$

everywhere, for all $n>m(\varepsilon)$, where

$$\delta_1(x) = \max\{|\sigma_\nu^\beta(x)-f(x)|^k\colon 0\leqq\nu\leqq n_1\}+2\{\delta(x)\}^k\in L^{p^*/k}.$$

Therefore $\delta_1(x)$ is a control function of (3.7). Lemma 7 is proved.

4. Proof of the main theorems. PROOF OF THEOREM A. For any $\varepsilon>0$ there exists a positive integer N such that

$$\sum_{n=N+1}^{\infty} c_n^2 < \varepsilon^3. \tag{4.1}$$

We consider two new series:

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \quad \text{where} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N, \\ 0 & \text{for} \quad n > N; \end{cases} \tag{4.2}$$

and

$$\sum_{n=0}^{\infty} b_n \varphi_n(x) \quad \text{where} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N, \\ c_n & \text{for} \quad n > N. \end{cases} \tag{4.3}$$

Let us denote the partial sum and the (c, β)-mean of the series (4.2) and (4.3) by $s_n(a; x)$, $\sigma_n^\beta(a; x)$ and $s_n(b; x)$, $\sigma_n^\beta(b; x)$, respectively. It is clear that

$$\sigma_n^\beta(x) = \sigma_n^\beta(a;\, x) + \sigma_n^\beta(b;\, x). \tag{4.4}$$

Using Lemma 6 with series (4.3) we obtain from (4.1) that

$$\int_0^1 \Big\{\sup_{n\geqq 0} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(b;\, x) - \sigma_\nu^\beta(b;\, x)|^k\Big\}^{2/k} dx \leqq C \sum_{n=0}^{\infty} b_n^2 = C \sum_{n>N}^{\infty} c_n^2 < C\varepsilon^3.$$

Hence
(4.5)

$$\operatorname{meas}\Big\{x\in[0,1]\colon \varlimsup_{n\to\infty} \Big\{(1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(b;\, x) - \sigma_\nu^\beta(b;\, x)|^k\Big\}^{1/k} > \varepsilon\Big\} \leqq C\varepsilon.$$

On the other hand, if $n>N$, then

$$\sigma_n^\beta(a;\, x) = (1/A_n^\beta) \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} s_\nu(a;\, x) = (1/A_n^\beta) \sum_{\nu=0}^{N} A_{n-\nu}^{\beta-1} \{s_\nu(a;\, x) - s_N(x)\} + s_N(x).$$

Therefore $\lim_{n\to\infty} \sigma_n^\beta(a; x) = s_N(x)$ a.e. on [0, 1] and so, we get

$$\lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(a;\, x) - \sigma_\nu^\beta(a;\, x)|^k = 0 \quad \text{a.e. on} \quad [0, 1]. \tag{4.6}$$

From (4.4), (4.5) and (4.6), it follows that

$$\lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_\nu^{\beta-1}(x) - \sigma_\nu^\beta(x)|^k = 0 \quad \text{a.e. on} \quad [0, 1]. \tag{4.7}$$

Since $\lim_{n\to\infty} \sigma_n^\beta(x) = f(x)$ a.e. from the hypothesis, we get (2.3).

At last we shall prove that (2.3) has an $L^{p^*/k}$-integrable control function. By Lemmas 1 and 6, (4.7) has an $L^{2/k}$-integrable control function. From the hypothesis, $\sigma_n^\beta(x)$ converges to $f(x)$ a.e. on [0, 1] and has an L^p-integrable control function, and so by Lemma 7 it follows that

$$\lim_{n\to\infty} (1/A_n^\alpha) \sum_{\nu=0}^{\infty} A_{n-\nu}^{\alpha-1} |\sigma_\nu^\beta(x) - f(x)|^k = 0$$

a.e. on [0, 1] and the above convergence has an $L^{p^*/k}$-integrable control function. Since

$$(1/A_n^\alpha)\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-f(x)|^k \leqq$$

$$\leqq C\Big\{(1/A_n^\alpha)\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(x)-\sigma_\nu^{\beta}(x)|^k+(1/A_n^\alpha)\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta}(x)-f(x)|^k\Big\},$$

we get the required result.

PROOF OF THEOREM B. Let $\{n_\nu\}_\nu$ be any increasing sequence of positive integers and set

$$\psi_0(x) = (1/d_0)\{c_0\varphi_0(x)+\ldots+c_{n_0}\varphi_{n_0}(x)\}, \tag{4.8}$$

where $d_0=(c_0^2+\ldots+c_{n_0}^2)^{1/2}$ and

$$\psi_\nu(x) = (1/d_\nu)\{c_{n_{\nu-1}+1}\varphi_{n_{\nu-1}+1}(x)+\ldots+c_{n_\nu}\varphi_{n_\nu}(x)\}, \tag{4.9}$$

where $d_\nu=(c_{n_{\nu-1}+1}^2+\ldots+c_{n_\nu}^2)^{1/2}$ $(\nu=1, 2, 3, \ldots)$. We consider the following orthogonal series:

$$\sum_{\nu=0}^{\infty} d_\nu\psi_\nu(x) \quad \text{where} \quad \sum_{\nu=0}^{\infty} d_\nu^2 < +\infty. \tag{4.10}$$

Since $\{n_\nu\}$ is an increasing sequence, we have

$$\sum_{\nu=0}^{\infty} d_\nu^2\{\log\log(\nu+3)\}^2 = \sum_{\nu=0}^{\infty}\{c_{n_{\nu-1}+1}^2+\ldots+c_{n_\nu}^2\}\{\log\log(\nu+3)\}^2 \leqq$$

$$\leqq \sum_{\nu=0}^{\infty}\Big\{c_{n_{\nu-1}+1}^2\big(\log\log(n_{\nu-1}+4)\big)^2+\ldots+c_{n_\nu}^2\big(\log\log(n_\nu+3)\big)^2\Big\} =$$

$$= \sum_{n=0}^{\infty} c_n^2\{\log\log(n+3)\}^2 < +\infty.$$

By Theorem 2.8.1 in [1], the sequence $\sigma_n^\beta(\psi, x)$ of the $(c, \beta>0)$-mean of the series (4.10) converges a.e. to an L^2-integrable function $f(x)$. Moreover

$$\int_0^1 \sup_{n\geqq 0}|\sigma_n^\beta(\psi, x)|^2\,dx \leqq C\sum_{\nu=0}^{\infty} d_\nu^2\{\log\log(\nu+3)\}^2 \leqq C\sum_{n=0}^{\infty} c_n^2\{\log\log(n+3)\}^2 < +\infty.$$

By Lemma 1, the convergence $\lim_{n\to\infty}\sigma_n^\beta(\psi, x)=f(x)$ has an L^2-integrable control function. Hence by Theorem A

$$\lim_{n\to\infty}(1/A_n^\alpha)\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(\psi, x)-f(x)|^k = 0 \quad \text{a.e. on} \quad [0, 1] \tag{4.11}$$

and (4.11) has an $L^{2/k}$-integrable control function. Since

$$\sum_{\nu=0}^{m} d_\nu\psi_\nu(x) = \sum_{n=0}^{n_m} c_n\varphi_n(x)$$

from (4.8) and (4.9), we get $s_m(\psi, x)=s_{n_m}(x)$ and $\sigma_\nu^{\beta-1}(\psi, x)=\sigma_\nu^{\beta-1}(\{n_\nu\}, x)$. Thus we have proved Theorem B.

Our method is applicable for the following result:

$$\int_0^1 \{\sup_{n\geqq 0}(1/A_n^\alpha)\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|\sigma_\nu^{\beta-1}(\{n_\nu\}, x)-f(x)|^k\}^{2/k}\,dx \leqq C\sum_{n=0}^\infty c_n^2\{\log\log(n+3)\}^2$$

for each $(\alpha, \beta)\in\Delta$ and for each increasing sequence of positive integers $\{n_\nu\}$.

The next corollaries follow immediately from Theorem B.

Corollary 1 (Sunouchi [3]). *If* $\sum_{n=0}^\infty c_n^2\{\log\log(n+3)\}^2<+\infty$, *then there exists an L^2-integrable function $f(x)$ such that*

$$\lim_{n\to\infty}(1/A_n^\alpha)\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|s_{n_\nu}(x)-f(x)|^k=0 \quad a.e.\ on\ [0,1]$$

for each $\alpha>0$, $k>0$ and for each increasing sequence of positive integers $\{n_\nu\}$ and that its convergence has an $L^{2/k}$-integrable control function.

Corollary 2. *If* $\sum_{n=0}^\infty c_n^2\{\log\log(n+3)\}^2<+\infty$, *then there exists an L^2-integrable function $f(x)$ such that*

$$\lim_{n\to\infty}(1/A_n^\alpha)\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^\alpha=0 \quad a.e.\ on\ [0,1]$$

for each (α, β) which satisfies the condition $0<\alpha\leqq 2$, $\beta>-1/2$ or $2<\alpha<+\infty$, $\beta>-1/\alpha$. Moreover, its convergence has an $L^{2/\alpha}$-integrable control function.

Corollary 3. *If* $\sum_{n=0}^\infty c_n^2\{\log\log(n+3)\}^2<+\infty$, *then there exists an L^2-integrable function $f(x)$ such that*

$$\lim_{n\to\infty}(n+1)^{-1}\sum_{\nu=0}^n|\sigma_\nu^\beta(x)-f(x)|=0$$

a.e. on $[0,1]$ for each $\beta>-1/2$. Moreover, its convergence has an L^2-integrable control function.

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ANOTHER GENERALIZATION OF A THEOREM OF A. KERTÉSZ

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We say a ring R is left s-unital if and only if $x \in Rx$ for all $x \in R$. It is known that if F is a finite subset of a left s-unital ring R then there exists an element e in R such that $x=ex$ for all $x \in F$ (see, e.g. [2, Theorem 1]).

In [3], a theorem of Kertész [1] has been generalized as follows: Let A be an ideal of a ring R, and B an additive subgroup of R such that $R=A+B$. If R/A is left s-unital and $AB=BA=0$, then R is the ringtheoretical direct sum of A and B^2.

The present objective is to generalize the above result as follows:

THEOREM 1. *Let A be an ideal of a ring R, and B an additive subgroup of R such that $R=A+B$. If R/A is left s-unital and there exist positive integers $k>h$ such that $B^kA \subseteqq B^k$, $AB^h \subseteqq B^h$ and $B^kA \cap B^{k+1}=0=AB^h \cap B^{h+1}$, then R is the ringtheoretical direct sum of A and B^{k+1}. If, furthermore, R is left s-unital (the right annihilator of R is 0) then $B^{k+1}=B^{h+1}$.*

In preparation for proving our theorem, we state the following

LEMMA 1. *Let A be an ideal of a ring R, and B an additive subgroup of R such that $R=A+B$.*

(1) *If R/A is idempotent then $R=A+B^n$ for any positive integer n. If, furthermore, $AB^h=0$ for some positive integer h, then $B^{h+1}=B^{h+2}$.*

(2) *If R/A is left s-unital and $B^kA=0=AB^h$ for some positive integers $k>h$, then $R=A \oplus B^k$ (ringtheoretical direct sum). If, furthermore, R is left s-unital then $B^k=B^h$.*

PROOF. (1) The first assertion is almost evident. Since $A+B=R=A+B^2$, $AB^h=0$ gives $B^{h+1}=B^{h+2}$.

(2) By (1), $R=A+B^k$ and $B^{h+1}=B^k=B^{2k}$. Now, let $a=x_1y_1+\ldots+x_ny_n$ $(x_i, y_i \in B^k)$ be an arbitrary element of $A \cap B^k$. Then, there exists $e \in B^k$ such that $x_i-ex_i \in A$ for all i. Then $a=(x_1-ex_1)y_1+\ldots+(x_n-ex_n)y_n+ea=0$. Thus, we obtain $R=A \oplus B^k$. Henceforth, we assume further that R is left s-unital. Again by (1), $A+B^k=R=A+B^h$, and then $B^k \subseteqq B^h+I$, where $I=A \cap (B^k+B^h)$. Obviously, $RI=(A+B^k)I=0$, and hence $I=0$. This proves $B^k \subseteqq B^h$, and similarly $B^h \subseteqq B^k$.

We are now ready to complete the proof of our theorem.

PROOF OF THEOREM 1. Obviously, $B^{k+1}A \subseteqq B^kA \cap B^{k+1}=0$ and $AB^{h+1}=0$. Hence, the assertion is immediate by Lemma 1 (2).

REMARK. Let A be an ideal of a ring R, and B an additive subgroup of R such that $R=A+B$. If R/A is left s-unital, then the following are equivalent: 1) $A=R$; 2) B is nil modulo A; 3) B is nilpotent modulo A. In fact, 1)$\Rightarrow$3)$\Rightarrow$2). We prove that

2) implies 1). Given $x \in R$, we can find $e \in B$ such that $x - ex \in A$. Since $e^n \in A$ for some positive integer n, we see that

$$x = (x-ex)+e(x-ex)+\ldots+e^{n-1}(x-ex)+e^n x \in A,$$

which proves $R=A$. Further, we can easily see that the following are equivalent: 1)′ $B^k R=0$ for some positive integer k; 2)′ B is nil modulo A and $B^k A=0$ for some positive integer k; 3)′ B is nilpotent.

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A NOTE ON MAPPINGS OF EXTREMALLY DISCONNECTED SPACES

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1. Introduction. In recent years various classes of non-continuous mappings between topological spaces were introduced by use of the concepts of semi-open sets, pre-open sets and α-sets in topological spaces. The purpose of this note is to investigate the interrelations among these classes of mappings and to apply obtained results in case of mappings of extremally disconnected topological spaces. In Section 2 we establish several relevant basic properties of pre-open sets and α-sets in topological spaces. In Section 3, pre-continuous and pre-open mappings are further investigated and their relations with semi-continuous and semi-open mappings are discussed. In particular, sufficient conditions for continuity and openness of mappings of α-topologies are given. Our main results are stated in the final section. Extremally disconnected topological spaces are characterized as the spaces whose semi-open sets are pre-open and it is shown that extremally disconnected spaces are pre-open hereditary and are invariant under semi-continuous pre-open surjections. We also improve several results due to Noiri [16]. It is proved that a semi-continuous function from an S-closed extremally disconnected space into a Hausdorff space is δ-closed.

The topological spaces are not assumed to satisfy any separation axioms except those explicitly stated.

2. Preliminary definitions and results. Throughout, (X, T) and (Y, S) will denote topological spaces (hereafter referred to as "spaces") and $T\text{-cl}(A)$ $(T\text{-int}(A))$ will denote the closure (interior) of a subset A of a space (X, T), although we may suppress the T when there is no possibility of confusion. A subset A of a space (X, T) is said to be regular-closed (regular-open) if $A=\text{cl}(\text{int}(A))$ $(A=\text{int}(\text{cl}(A)))$. The collection of all regular-closed (regular-open) subsets of (X, T) is denoted by $RC(X, T)$ $(RO(X, T))$. The topology T_s on X which has as its base $RO(X, T)$ is called the semiregularization topology of (X, T). A point $x\in X$ is in the θ-closure [24] of a subset A of (X, T) $(x\in T\text{-cl}_\theta(A))$ if $\text{cl}(U)\cap A\neq\emptyset$ for each open set U containing x. Clearly $T\text{-cl}(A)\subset T_s\text{-cl}(A)\subset T\text{-cl}_\theta(A)$ for each subset A of (X, T). A subset A of a space (X, T) is said to be semi-open [11] (resp. pre-open [12], α-set [14]) if $A\subset \text{cl}(\text{int}(A))$ (resp. $A\subset\text{int}(\text{cl}(A))$, $A\subset\text{int}(\text{cl}(\text{int}(A)))$). The collection of all subsets of a space (X, T) which are semi-open (resp. pre-open, α-sets) is denoted by $SO(X, T)$ (resp. $PO(X, T)$, T^α). It was observed in [14] that T^α is a topology on X and that $T\subset T^\alpha\subset SO(X, T)$. Moreover $T^\alpha=SO(X, T)\cap PO(X, T)$ [19]. Semi-closed sets and semi-closure are defined in a manner analogous to the corresponding concepts of closed sets and closure [4]. The collection of all semi-closed subsets of a space (X, T) is denoted by $SC(X, T)$ and the semi-closure of a subset A of (X, T) is denoted by $\text{scl}(A)$. Let A and B be nonempty families of subsets of spaces (X, T)

and (Y, S) respectively and let $f: (X, T) \to (Y, S)$ be a function. We say that f is AB-continuous (AB-open) if $f^{-1}(V) \in A\big(f(U) \in B\big)$ for each $V \in B$ $(U \in A)$. If $A = SO(X, T)$ (resp. $A = PO(X, T)$, $A = T^\alpha$) and $B = S$, then an AB-continuous function f is called semi-continuous [11] (resp. pre-continuous [12], α-continuous [13]). If $A = SO(X, T)$ (resp. $A = PO(X, T)$) and $B = SO(Y, S)$ (resp. $B = PO(Y, S)$), then an AB-continuous function f is called irresolute [5] (resp. pre-irresolute [19]). If $A = T$ and $B = SO(Y, S)$ (resp. $B = PO(Y, S)$, S^α), then an AB-open function f is called semi-open [1] (resp. pre-open [12]), α-open [13]). If $A = SO(X, T)$ (resp. $A = PO(X, T)$) aud $B = SO(Y, S)$ (resp. $B = PO(Y, S)$), then an AB-open function f is called pre-semi-open [5] (resp. p-open). We also need the following definitions. A function $f: (X, T) \to (Y, S)$ is said to be almost continuous (S and S) [21] (resp. θ-continuous [7], weakly continuous [10]) if for each $x \in X$ and for each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset \text{int}(\text{cl}(V))$ (resp. $f(\text{cl}(U)) \subset \text{cl}(V)$, $f(U) \subset \text{cl}(V)$). Almost continuity ($S$ and S) implies θ-continuity and θ-continuity implies weak continuity, but the converses are false. Also, it is known that α-continuity implies θ-continuity, but the converse is false. A function $f: (X, T) \to (Y, S)$ is said to be almost closed [21] if $f(U)$ is closed for each $U \in RC(X, T)$.

PROPOSITION 2.1. ([14]) *$SO(X, T^\alpha) = SO(X, T)$ for any space (X, T).*

The following proposition will be very useful in the sequel.

PROPOSITION 2.2. *If $A \in SO(X, T)$, then $T^\alpha - \text{cl}(A) = T - \text{cl}(A) = T_s - \text{cl}(A)$.*

PROOF. We have only to prove that $T_s - \text{cl}(A) \subset T^\alpha - \text{cl}(A)$ for $A \in SO(X, T)$. Let $x \notin T^\alpha - \text{cl}(A)$. Then there exists a $U \in T^\alpha$ such that $x \in U$ and $U \cap A = \emptyset$. This implies that $T - \text{int}(U) \cap T - \text{int}(A) = \emptyset$ and $T - \text{cl}(T - \text{int}(U)) \cap T - \text{int}(A) = \emptyset$. Consequently, $T - \text{int}(T - \text{cl}(T - \text{int}(U))) \cap T - \text{int}(A) = \emptyset$ and $T - \text{int}(T - \text{cl}(T - \text{int}(U))) \cap T - \text{cl}(T - \text{int}(A)) = \emptyset$. Since $A \in SO(X, T)$, $A \subset T - \text{cl}(T - \text{int}(A))$. So, $T - \text{int}(T - \text{cl}(T - \text{int}(U))) \cap A = \emptyset$. Since $U \in T^\alpha$, $x \in T - \text{int}(T - \text{cl}(T - \text{int}(U)))$. Hence $x \notin T_s - \text{cl}(A)$ and the proof is complete.

COROLLARY 2.3. *$T_s = (T^\alpha)_s$ for any space (X, T).*

PROOF. Since regular-closed sets are precisely semi-open closed sets, it follows from Propositions 2.1 and 2.2 that $RC(X, T) = RC(X, T^\alpha)$. This implies $T_s = (T^\alpha)_s$.

COROLLARY 2.4. *If A is a subset of a space (X, T), then*
(a) $T^\alpha - \text{int}(T^\alpha - \text{cl}(A)) = T - \text{int}(T - \text{cl}(A))$.
(b) $T^\alpha - \text{cl}(T^\alpha - \text{int}(T^\alpha - \text{cl}(A))) = T - \text{cl}(T - \text{int}(T - \text{cl}(A)))$.
(c) [19, Proposition 1] $T - \text{cl}(T - \text{int}(T - \text{cl}(A))) \subset T^\alpha - \text{cl}(A)$.

PROOF. (a) Let $A \subset X$. From Proposition 2.1 it follows that $SC(X, T^\alpha) = SC(X, T)$ so that $T^\alpha - \text{cl}(A) \in SC(X, T)$. By Proposition 2.2, $T^\alpha - \text{int}(F) = T - \text{int}(F)$ for each $F \in SC(X, T)$ so that $T^\alpha - \text{int}(T^\alpha - \text{cl}(A)) = T - \text{int}(T^\alpha - \text{cl}(A))$. Since $T - \text{int}(T^\alpha - \text{cl}(A)) = T - \text{int}(T - \text{cl}(A))$ [3, Lemma 1.2], we conclude that $T^\alpha - \text{int}(T^\alpha - \text{cl}(A)) = T - \text{int}(T - \text{cl}(A))$.

(b) This follows from (a) and Proposition 2.2.

(c) This is an immediate consequence of (b).

COROLLARY 2.5. *Let* (X, T) *be a space. Then*
(a) $PO(X, T^\alpha)=PO(X, T)$.
(b) [14, Proposition 10] $(T^\alpha)^\alpha=T^\alpha$.
(c) $T^\alpha-\mathrm{cl}(A)=T-\mathrm{cl}(A)=T_s-\mathrm{cl}(A)=T-\mathrm{cl}_\theta(A)$ *for each* $A\in PO(X, T)$.

PROOF. (a) This follows from Corollary 2.4 (a).

(b) Since $T^\alpha=SO(X, T)\cap PO(X, T)$ [19, Theorem 3], the result follows from Proposition 2.1 and (a).

(c) Let $A\in PO(X, T)$. Then $A\subset T-\mathrm{int}(T-\mathrm{cl}(A))$, and hence $T-\mathrm{cl}_\theta(A)\subset \subset T-\mathrm{cl}_\theta(T-\mathrm{int}(T-\mathrm{cl}(A)))=T-\mathrm{cl}(T-\mathrm{int}(T-\mathrm{cl}(A)))$ since the closure of an open set coincides with its closure [24]. By Corollary 2.4 (c) it follows that $T-\mathrm{cl}_\theta(A)\subset \subset T^\alpha-\mathrm{cl}(A)$. On the other hand, $T^\alpha-\mathrm{cl}(B)\subset T-\mathrm{cl}(B)\subset T_s-\mathrm{cl}(B)\subset T-\mathrm{cl}_\theta(B)$ for each $B\subset X$. Therefore, $T^\alpha-\mathrm{cl}(A)=T-\mathrm{cl}_\theta(A)$ and the result follows.

The following lemma is easily established.

LEMMA 2.6. *If* A *is a subset of a space* (X, T), *then*
(a) $\mathrm{int}(\mathrm{cl}(A))\subset \mathrm{scl}(A)$.
(b) $\mathrm{int}(\mathrm{scl}(A))=\mathrm{int}(\mathrm{cl}(A))$.

PROPOSITION 2.7. *Let* A *be a subset of a space* (X, T). *Then*
(a) $A\in PO(X, T)$ *if and only if* $\mathrm{scl}(A)=\mathrm{int}(\mathrm{cl}(A))$.
(b) $A\in PO(X, T)$ *if and only if* $\mathrm{scl}(A)\in RO(X, T)$.
(c) $RO(X, T)=PO(X, T)\cap SC(X, T)$.

PROOF. (a) Let $A\in PO(X, T)$. Then $\mathrm{scl}(A)\subset \mathrm{scl}(\mathrm{int}(\mathrm{cl}(A)))$ and since $\mathrm{int}(\mathrm{cl}(A))\in SC(X, T)$, $\mathrm{scl}(A)\subset \mathrm{int}(\mathrm{cl}(A))$. By Lemma 2.6 (a) it follows that $\mathrm{scl}(A)= =\mathrm{int}(\mathrm{cl}(A))$. The converse is obvious.

(b) Let $\mathrm{scl}(A)\in RO(X, T)$. Then $\mathrm{scl}(A)=\mathrm{int}(\mathrm{cl}(\mathrm{scl}(A)))$, and hence $\mathrm{scl}(A)\subset \subset \mathrm{int}(\mathrm{cl}(\mathrm{cl}(A)))=\mathrm{int}(\mathrm{cl}(A))$. By Lemma 2.6 (a) it follows that $\mathrm{scl}(A)=\mathrm{int}(\mathrm{cl}(A))$. By (a), $A\in PO(X, T)$. The converse follows from (a).

(c) This follows from (a) and (b).

LEMMA 2.8. *If* A *and* B *are subsets of a space* (X, T), $A\subset B\subset \mathrm{cl}(A)$, *and* $B\in PO(X, T)$, *then* $A\in PO(X, T)$.

PROOF. Since $A\subset B\subset \mathrm{cl}(A)$, $\mathrm{scl}(A)\subset \mathrm{scl}(B)\subset \mathrm{cl}(A)$. Proposition 2.7 (a) implies that $\mathrm{scl}(A)\subset \mathrm{int}(\mathrm{cl}(B))\subset \mathrm{cl}(A)$ since $B\in PO(X, T)$. Since $\mathrm{int}(\mathrm{cl}(A))=\mathrm{int}(\mathrm{cl}(B))$, we conclude that $A\in PO(X, T)$.

3. Pre-continuous and pre-open functions. Mashhour et al. [12] observed that pre-continuity coincides with almost continuiuty in the sense of Husain [9]. Rose [20] proved that a function $f\colon (X, T)\to(Y, S)$ is almost continuous in the sense of Husain if and only if $f(\mathrm{cl}(U))\subset \mathrm{cl}(f(U))$ for each $U\in T$. In the following proposition we offer some more characterizations of pre-continuous functions. The straightforward proof is omitted.

PROPOSITION 3.1. *The following statements are equivalent for a function* $f\colon (X, T)\to(Y, S)$.
(a) f *is pre-continuous.*
(b) $f(\mathrm{cl}(\mathrm{int}(A)))\subset \mathrm{cl}(f(A))$ *for each* $A\subset X$.
(c) $f(\mathrm{cl}(U))\subset \mathrm{cl}(f(U))$ *for each* $U\in SO(X, T)$.

In [25] Wilansky defined a function $f: (X, T) \to (Y, S)$ to be almost open if $f^{-1}(\mathrm{cl}(V)) \subset \mathrm{cl}(f^{-1}(V))$ for each $V \in S$. As it was observed in [12] pre-openness coincides with almost openness in the sense of Wilansky. In the following proposition we characterize pre-open functions.

PROPOSITION 3.2. *The following statements are equivalent for a function* $f: (X, T) \to (Y, S)$.

(a) *f is pre-open.*

(b) $f^{-1}(\mathrm{cl}(\mathrm{int}(B))) \subset \mathrm{cl}(f^{-1}(B))$ *for each* $B \subset Y$.

(c) $f^{-1}(\mathrm{cl}(V)) \subset \mathrm{cl}(f^{-1}(V))$ *for each* $V \in SO(Y, S)$.

Let $f: (Y, T) \to (Y, S)$ be a function. Then a function $f_\alpha: (X, T^\alpha) \to (Y, S)$ (resp. $f^\alpha: (X, T) \to (Y, S^\alpha)$, $f_*: (X, T^\alpha) \to (Y, S^\alpha)$) associated with $f: (X, T) \to (Y, S)$ is defined as follows: $f_\alpha(x) = f(x)$ (resp. $f^\alpha(x) = f(x)$, $f_*(x) = f(x)$) for each $x \in X$. Our next result immediately follows from Proposition 2.1 and Corollary 2.5 (a).

PROPOSITION 3.3. *Let $f: (X, T) \to (Y, S)$ be a function. Then*

(a) *f is pre-continuous (semi-continuous) if and only if f_α is pre-continuous (semi-continuous).*

(b) *f is pre-open (semi-open) if and only if f^α is pre-open (semi-open).*

(c) *f is pre-irresolute (irresolute) if and only if f_* is pre-irresolute (irresolute).*

(d) *f is p-open (pre-semi-open) if and only if f_* is p-open (pre-semi-open).*

Mashhour et al [13] have shown that every pre-continuous α-open function is pre-irresolute. The following proposition improves this result.

PROPOSITION 3.4. *If a function $f: (X, T) \to (Y, S)$ is pre-continuous and semi-open, then f is pre-irresolute.*

PROOF. Let $V \in PO(Y, S)$. Since f is semi-open, $f^{-1}(V) \subset f^{-1}(\mathrm{scl}(V)) \subset \mathrm{cl}(f^{-1}(V))$ by Theorem 2 of [17]. By Proposition 2.7 (b), $\mathrm{scl}(V) \in RO(Y, S)$ so that $f^{-1}(\mathrm{scl}(V)) \in PO(X, T)$ because f is pre-continuous. Lemma 2.8 implies that $f^{-1}(V) \in PO(X, T)$ and the result follows.

Since pre-continuous semi-open functions are pre-semi-open [18, Theorem 2.5], combining this result and Proposition 3.4 we obtain the following corollary.

COROLLARY 3.5. *A function $f: (X, T) \to (Y, S)$ is pre-irresolute and pre-semi-open if and only if f is pre-continuous and semi-open.*

PROPOSITION 3.6. *If a function $f: (X, T) \to (Y, S)$ is pre-open and semi-continuous, then f is p-open.*

PROOF. Let $U \in PO(X, T)$. Since f is semi-continuous, $f(U) \subset f(\mathrm{scl}(U)) \subset \mathrm{cl}(f(U))$ by Theorem 1.6 of [4]. By Proposition 2.7 (b), $\mathrm{scl}(U) \in RO(X, T)$ so that $f(\mathrm{scl}(U)) \in PO(Y, S)$ because f is pre-open. Lemma 2.8 implies that $f(U) \in PO(Y, S)$ and the result follows.

Since pre-open semi-continuous functions are irresolute [15, Theorem 1], combining this result and Proposition 3.6 we obtain the following corollary.

COROLLARY 3.7. *A function $f: (X, T) \to (Y, S)$ is irresolute and p-open if and only if f is semi-continuous and pre-open.*

PROPOSITION 3.8. *A function* $f: (X, T) \to (Y, S)$ *is pre-continuous and pre-open if and only if* f_* *is pre-continuous and pre-open.*

PROOF. Suppose that f is pre-continuous and pre-open and let $U \in T$. By Proposition 3.1, $f(\mathrm{cl}(U)) \subset \mathrm{cl}(f(U))$. Since f is pre-open, $f(U) \in PO(Y, S)$. Therefore $f(\mathrm{cl}(U)) \subset S^\alpha - \mathrm{cl}(f(U))$ by Corollary 2.5 (c) hence f^α is pre-continuous. Since $(f^\alpha)_\alpha = f_*$, it follows from Proposition 3.3 (a) that f_* is pre-continuous. Now, let $V \in S$. Since f is pre-open, $f^{-1}(\mathrm{cl}(V)) \subset \mathrm{cl}(f^{-1}(V))$. The pre-continuity of f implies that $f^{-1}(V) \in PO(X, T)$ so that $f^{-1}(\mathrm{cl}(V)) \subset T^\alpha - \mathrm{cl}(f^{-1}(V))$ by Corollary 2.5 (c). Therefore f_α is pre-open. Since $(f_\alpha)^\alpha = f_*$, it follows from Proposition 3.3 (b) that f_* is pre-open.

Conversely, if f_* is pre-continuous and pre-open, then f_α and f^α are pre-continuous and pre-open. It follows from Proposition 3.3 (a) and (b) that f is pre-continuous and pre-open.

PROPOSITION 3.9. *A function* $f: (X, T) \to (Y, S)$ *is semi-continuous and semi-open if and only if* f_* *is semi-continuous and semi-open:*

PROOF. Let $B \in S^\alpha$. From the definition of α-sets it follows that there exists a $V \in S$ such that $V \subset B \subset \mathrm{int}(\mathrm{cl}(V))$. By Proposition 2.7 (a), $V \subset B \subset \mathrm{scl}(V)$. Therefore, $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\mathrm{scl}(V))$ and since f is semi-open, $f^{-1}(V) \subset f^{-1}(B) \subset \subset \mathrm{cl}(f^{-1}(V))$ [17, Theorem 2]. Since f is semi-continuous, $f^{-1}(V) \in SO(X, T)$ so that $f^{-1}(B) \in SO(X, T)$ [11, Theorem 3]. Therefore f^α is semi-continuous. Since $(f^\alpha)_\alpha = f_*$, f_* is semi-continuous by Proposition 3.3 (a). Now, let $A \in T^\alpha$. Then there exists a $U \in T$ such that $U \subset A \subset \mathrm{scl}(U)$. Therefore $f(U) \subset f(A) \subset f((\mathrm{scl}(U))$ and since f is semi-continuous $f(U) \subset f(A) \subset \mathrm{cl}(f(U))$ [4, Theorem 1.6]. Since f is semi-open, $f(U) \in SO(Y, S)$ so that $f(A) \in SO(Y, S)$ [11, Theorem 3]. Therefore f_α is semi-open. Since $(f_\alpha)^\alpha = f_*$, it follows from Proposition 3.3 (b) that f_* is semi-open.

Conversely, if f_* is semi-continuous and semi-open, then f_α and f^α are semi-continuous and semi-open. It follows from Proposition 3.3 (a) and (b) that f is semi-continuous and semi-open.

As it was observed in [19] a function $f: (X, T) \to (Y, S)$ is α-continuous (α-open) if and only if $f_\alpha(f^\alpha)$ is continuous (open). Also, f is α-continuous (α-open) if and only if f is pre-continuous and semi-continuous (pre-open and semi-open).

Mashhour et al. [13] have given an alternative proof of the first part of the following proposition.

PROPOSITION 3.10. *Let* $f: (X, T) \to (Y, S)$ *be pre-open or semi-open. Then* f *is* α*-continuous if and only if* f_* *is continuous.*

PROOF. Suppose that f is pre-open and α-continuous. Then f is pre-open, pre--continuous and semi-continuous so that by Corollary 3.7, f is irresolute. It follows from Proposition 3.3 (c) that f_* is irresolute, and hence is semi-continuous. On the other hand, f_* is pre-open and pre-continuous by Proposition 3.8. Since f_* is pre--continuous and semi-continuous, f_* is α-continuous. By Corollary 2.5 (b), f_* is continuous.

Now, suppose that f is semi-open and α-continuous. Then f is semi-open, pre--continuous and semi-continuous. By Proposition 3.4 f is pre-irresolute. It follows from Proposition 3.3 (c) that f_* is pre-irresolute, and hence pre-continuous. On the other hand, f_* is semi-open and semi-continuous by Proposition 3.9. Since f_* is pre-continuous and semi-continuous, f_* is α-continuous, and hence is continuous by Corollary 2.5 (b).

Although the proof of the following proposition is not identical to that of Proposition 3.10, it is quite similiar, and hence is omitted.

PROPOSITION 3.11. *Let* $f: (X, T) \to (Y, S)$ *be pre-continuous or semi-continuous. Then* f *is* α*-open if and only if* f_* *is open.*

COROLLARY 3.12. *Let* $f: (X, T) \to (Y, S)$ *be a function. Then the following are equivalent:*

(a) f *is* α*-continuous and* α*-open.*

(b) f *is pre-continuous, pre-open, semi-continuous, and semi-open.*

(c) f *is pre-irresolute, p-open, irresolute, and pre-semi-open.*

(d) f_* *is continuous and open.*

4. Mappings of extremally disconnected spaces. Recall that a space (X, T) is said to be extremally disconnected (abbreviated as e.d.) if $A \in T$ for each $A \in RC(X, T)$. The following characterization of e.d. spaces will be very useful in the sequel.

PROPOSITION 4.1. *A space* (X, T) *is e.d. if and only if* $SO(X, T) \subset PO(X, T)$.

PROOF. Let $A \in SO(X, T)$. Then there exists a $U \in T$ such that $U \subset A \subset \mathrm{cl}(U)$. Since (X, T) is e.d., $\mathrm{cl}(U) \in T$ so that $U \subset A \subset \mathrm{int}(\mathrm{cl}(U))$. This shows that $A \in T^\alpha \subset \subset PO(X, T)$.

Conversely, let $A \in RC(X, T)$. Then $A \in SO(X; T)$ and by hypothesis, $A \in \in PO(X, T)$. Therefore, $A \subset \mathrm{int}(\mathrm{cl}(A))$ and since A is closed, $A \in T$. This shows that (X, T) is e.d.

PROPOSITION 4.2. *Extremally disconnectedness is pre-open hereditary.*

PROOF. Let A be a pre-open subset of an e.d. space (X, T). Then $A \subset A^*$, where $A^* = \mathrm{int}(\mathrm{cl}(A))$. Since extremally disconnectedness is open hereditary, A^* is e.d. Since $\mathrm{cl}_{A^*}(A) = A^* \cap \mathrm{cl}(A) = A^*$ and extremally disconnectedness is dense hereditary, A is an e.d. subspace of A^*. Therefore A is e.d. in (X, T).

Combining Propositions 4.1 and 4.2 we obtain the following corollary.

COROLLARY 4.3. *Extremally disconnectedness is semi-open hereditary.*

We now generalize the well known result that extremally disconnectedness is invariant under continuous open surjections.

PROPOSITION 4.4. *If* (X, T) *is an e.d. space and* $f: (X, T) \to (Y, S)$ *is a semi--continuous pre-open surjection, then* (Y, S) *is e.d.*

PROOF. Let $V \in SO(Y, S)$. Since f is semi-continuous and pre-open, f is irresolute by Corollary 3.7 so that $f^{-1}(V) \in SO(X, T)$. It follows from Proposition 4.1 that $f^{-1}(V) \in PO(X, T)$ because (X, T) is e.d. By Corollary 3.7, f is p-open and since f is surjective, $V \in PO(Y; S)$. Therefore, $SO(Y, S) \subset PO(Y, S)$ and by Proposition 4.1, (Y, S) is e.d.

The following result is also obtained by use of Proposition 4.1.

COROLLARY 4.5. *If a space* (X, T) *is e.d., then* $\mathrm{scl}(A)=\mathrm{cl}_\theta(A)$ *for each* $A\in PO(X, T)$.

PROOF. Since (X, T) is e.d., it follows from Proposition 4.1 that $SO(X, T)=T^\alpha$. Therefore, $\mathrm{scl}(A)=T^\alpha-\mathrm{cl}(A)$ for each $A\subset X$. Lemma 2.5 (c) implies that $\mathrm{scl}(A)=\mathrm{cl}_\theta(A)$ for each $A\in PO(X, T)$.

Our next result improves Lemma 4.1 of [16].

COROLLARY 4.6. *If a space* (X, T) *is e.d., then* $\mathrm{scl}(A)=\mathrm{cl}_\theta(A)$ *for each* $A\in SO(X, T)$.

Noiri established that a semi-continuous function from an e.d. space is weakly continuous [18, Theorem 3.2]. Since α-continuous functions are θ-continuous, and hence are weakly continuous, the first part of the following corollary improves this result.

COROLLARY 4.7. *Let a space* $(X, T)((Y, S))$ *be e.d. A function* $f: (X, T)\to(Y, S)$ *is semi-continuous (semi-open) if and only if* f *is* α*-continuous (*α*-open).*

PROOF. We have only to prove the necessity. Suppose that f is semi-continuous (semi-open). It follows from Proposition 4.1 that f is pre-continuous (pre-open). Therefore, f is α-continuous (α-open).

Combining Proposition 4.1, Corollary 3.5 (Corollary 3.7) and Proposition 3.10 (Proposition 3.11) we obtain the following improvement of Theorem 2.6 (Theorem 1.14) of [18].

COROLLARY 4.8. *If a space* $(X, T)((Y, S))$ *is e.d. and a function* $f: (X, T)\to(Y, S)$ *is semi-continuous and semi-open, then*

(a) f *is pre-irresolute and pre-semi-open (irresolute and p-open).*

(b) f_* *is continuous (open).*

A function $f: (X, T)\to(Y, S)$ is said to be *rc*-continuous if $f^{-1}(V)\in RC(X, T)$ for each $V\in RC(Y, S)$. Let $f: (X, T)\to(Y, S)$ be a function. Then a function $f^s: (X, T)\to(Y, S_s)$ associated with f is defined as follows: $f^s(x)=f(x)$ for each $x\in X$. In the following proposition we give a sufficient condition for a function to be *rc*-continuous.

PROPOSITION 4.9. *If a function* $f: (X, T)\to(Y, S)$ *is irresolute and* f^s *is pre-continuous, then* f *is rc-continuous.*

PROOF. Let $V\in RO(Y, S)$. The pre-continuity of f^s implies that $f^{-1}(V)\in PO(X, T)$. Since $V\in SC(Y, S)$ and f is irresolute, $f^{-1}(V)\in SC(X, T)$ [5, Theorem 1.4]. Therefore, $f^{-1}(V)\in PO(X, T)\cap SC(X, T)$. By Proposition 2.7 (c), $f^{-1}(V)\in RO(X, T)$. This shows that f is *rc*-continuous.

It is known that a function $f:(X, T)\to(Y, S)$ is almost continuous (S and S) if and only if f^s is continuous. Therefore the following result is an immediate consequence of Proposition 4.9.

COROLLARY 4.10. *If an irresolute function is almost continuous (S and S) or pre-continuous, then* f *is rc-continuous.*

PROPOSITION 4.11. *If a space* (X, T) *is e.d. and a function* $f: (X, T)\to(Y, S)$ *is irresolute, then* f *is rc-continuous.*

PROOF. Since f is irresolute, f is semi-continuous. It follows from the first part of Corollary 4.7 that f is pre-continuous. Therefore, by Corollary 4.10, f is *rc*-continuous.

The following characterization of S-closed subsets relative to a space (X, T) (briefly, S-sets) will be used as primitive [16, Lemma 2.1]. A subset A of a space (X, T) is said to be an S-set if every cover of A by regular-closed sets in (X, T) has a finite subcover.

The proof of the following proposition is straightforward and hence is omitted.

PROPOSITION 4.12. *rc-continuous functions preserve S-sets.*

Combining Propositions 4.11 and 4.12 we obtain Theorem 4.2 of [16].

COROLLARY 4.13. *If a space (X, T) is e.d. and a function f: $(X, T) \to (Y, S)$ is irresolute, then f preserves S-sets.*

Combining Corollary 4.10 and Proposition 4.12 we obtain the following improvement of Proposition 3.1 of [6].

COROLLARY 4.14. *An irresolute function preserves S-sets if it is almost continuous (S and S) or pre-continuous.*

Thompson [23] introduced the concept of S-closed spaces. We restate his definition as follows: A space (X, T) is S-closed if (X, T) is an S-set of itself. It is observed in [23] that S-closed Hausdorff spaces are e.d. This result was improved in [8] where it was proved that S-closed weakly Hausdorff spaces are e.d. (A space (X, T) is weakly Hausdorff if (X, T_s) is T_1). Cameron [2] introduced the concept of I-compact spaces and showed that I-compact spaces are precisely S-closed and e.d. spaces.

Our next result enables us to improve the main result of Noiri [16, Theorem 5.2] that a semi-continuous function from an S-closed Hausdorff space into a Hausdorff space is almost closed. A function $f: (X, T) \to (Y, S)$ is said to be δ-closed if $f: (X, T_s) \to (Y, S_s)$ is closed. Clearly, δ-closed functions are almost closed. We point out that the converse is false.

PROPOSITION 4.15. *If (X, T) is an I-compact space, (Y, S) is a Hausdorff space, and $f: (X, T) \to (Y, S)$ is a semi-continuous function, then f is δ-closed.*

PROOF. Since (X, T) is e.d. and f is semi-continuous, f is α-continuous by Corollary 4.7. It is known that a space is S-closed (resp. Hausdorff, e.d.) if and only if its semiregularization is S-closed (resp. Hausdorff, e.d.). Therefore (X, T_s) is I-compact and (Y, S_s) is Hausdorff. Since (X, T_s) is e.d., it is easily established that (X, T_s) is regular. But (X, T_s) is S-closed so that (X, T_s) is compact. Since f is α-continuous, f is θ-continuous. It is not difficult to show that this implies that $f: (X, T_s) \to (Y, S_s)$ is θ-continuous. Since θ-continuous functions into Hausdorff spaces have closed graphs and functions with closed graphs map compact sets onto closed sets, we conclude that $f: (X, T_s) \to (Y, S_s)$ is closed. Therefore f is δ-closed.

COROLLARY 4.16. *If (X, T) is an S-closed weakly Hausdorff space, (Y, S) is a Hausdorff space, and $f: (X, T) \to (Y, S)$ is a semi-continuous function, then f is δ-closed.*

PROPOSITION 4.17. *If* (X, T) *is an I-compact space,* (Y, S) *is a Hausdorff space, and* $f: (X, T)\to(Y, S)$ *is a semi-continuous bijection, then*

(a) f *is irresolute.*

(b) $f: (X, T_s)\to(Y, S_s)$ *is a homeomorphism.*

(c) (Y, S) *is I-compact.*

PROOF. (a) Since (X, T) is e.d. and f is semi-continuous, f is α-continuous, and hence is pre-continuous. It follows from Proposition 3.1 that $f(\mathrm{cl}(U))\subset\mathrm{cl}(f(U))$ for each $U\in SO(X, T)$. By Proposition 4.15, f is δ-closed so that $S_s-\mathrm{cl}(f(A))\subset\subset f(T_s-\mathrm{cl}(A))$ for each $A\subset X$. Proposition 2.2 implies that $S_s-\mathrm{cl}(f(U))\subset f(\mathrm{cl}(U))$ for each $U\in SO(X, T)$. Therefore, $f(\mathrm{cl}(U))\subset\mathrm{cl}(f(U))\subset S_s-\mathrm{cl}(f(U))\subset f(\mathrm{cl}(U))$. So, $f(\mathrm{cl}(U))=\mathrm{cl}(f(U))$ for each $U\in SO(X, T)$. Let $V\in S$. Since f is bijective, $\mathrm{cl}(f^{-1}(V))=f^{-1}(f(\mathrm{cl}(f^{-1}(V))))$. The semi-continuity of f implies that $f^{-1}(V)\in\in SO(X, T)$ so that $\mathrm{cl}(f^{-1}(V))=f^{-1}(f(\mathrm{cl}(f^{-1}(V))))=f^{-1}(\mathrm{cl}(f(f^{-1}(V))))=f^{-1}(\mathrm{cl}(V))$. By Proposition 3.2, f is pre-open and hence is irresolute by Corollary 3.7.

(b) By (a), f is irresolute and since (X, T) is e.d., f is rc-continuous by Proposition 4.11. This implies that $f:(X, T_s)\to(Y, S_s)$ is continuous. Since f is δ-closed, $f:(X, T_s)\to(Y, S_s)$ is a homeomorphism.

(c) This follows from (b).

The following consequence of Proposition 4.17 is an improvement of Corollary 5.5 of [16].

COROLLARY 4.18. *If* (X, T) *is an S-closed weakly Hausdorff space,* (Y, S) *is a Hausdorff space, and* $f: (X, T)\to(Y, S)$ *is a semi-continuous bijection, then*

(a) f *is irresolute.*

(b) $f: (X, T_s)\to(Y, S_s)$ *is a homeomorphism.*

(c) (Y, S) *is S-closed.*

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A STRONG INVARIANCE PRINCIPLE FOR REVERSE MARTINGALES

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1. Introduction

In a previous work (Scott and Huggins [5]) an embedding of reverse martingales in Brownian motion was obtained and used to give a law of the iterated logarithm for reverse martingales using properties of Brownian motion near the origin. This iterated logarithm law complements the iterated logarithm law for martingales in the same manner as results concerning the behaviour of Brownian motion as $t\to 0$ complement those concerning the behaviour of Brownian motion as $t\to\infty$. This becomes clearer from an examination of the embedding of doubly infinite martingales in Brownian motion given as Theorem 3 of Scott and Huggins [5]. Here we sharpen the underlying invariance principle of Scott and Huggins [5] and this result complements the corresponding martingale result of Jain, Jogdeo and Stout [3]. We then give, as an application of this invariance principle, integral tests for upper and lower functions of reverse martingales which again complement the result of Jain, Jogdeo and Stout. As we quite often follow the proofs of Jain, Jogdeo and Stout fairly closely with the appropriate changes necessary for the reverse martingale case in some places only a sketch of the proof is needed.

2. An almost sure invariance principle

We take $\{S_n, \mathscr{F}_n;\ n\geqq 1\}$ to be a reversed martingale on a probability space $(\Omega, \mathscr{F}, P)$. It is assumed here that $ES_1^2<\infty$. For $n\geqq 1$ let $X_n=S_n-S_{n+1}$, $V_n^2=\sum_{k=n}^{\infty} E\{X_k^2|\mathscr{F}_{k+1}\}$ and $s_n^2=EV_n^2$. Note that $S_\infty \underset{L_2}{\overset{\text{a.s.}}{=}} \lim_{n\to\infty} S_n$ always exists, $S_n-S_\infty=\sum_{k=n}^{\infty} X_k$ and $s_n^2\to 0$. Thus w.l.o.g. we may suppose $S_\infty=0$. (If not set $S_n=S_n-S_\infty$.)

THEOREM 2.1. *For fixed* $\alpha\geqq 0$ *let* $f_\alpha(t)=t(\log_2 t^{-1})^{-\alpha}$, $e^{-1}\geqq t\geqq 0$. *Suppose that*

$$V_n^2 \xrightarrow{\text{a.s.}} 0, \tag{2.2}$$

and for all $\delta>0$,

$$\lim_{n\to\infty} f_\alpha(V_n^2)^{-1} \sum_{k=n}^{\infty} E\{X_k^2 I(X_k^2 \geqq \delta f_\alpha(V_k^2))|\mathscr{F}_{k+1}\}=0 \quad a.s., \tag{2.3}$$

$$\sum_{k=1}^{\infty} f_\alpha(V_k^2)^{-1/2} E\{|X_k| I(X_k^2 \geqq \delta f_\alpha(V_k^2))|\mathscr{F}_{k+1}\}<\infty \quad a.s., \tag{2.4}$$

$$\sum_{k=1}^{\infty} f_\alpha(V_k^2)^{-2} E\{X_k^4 I(X_k^2 < \delta f_\alpha(V_k^2))|\mathscr{F}_{k+1}\}<\infty \quad a.s. \tag{2.5}$$

Let S be the random function defined on $[0, \infty)$ by

$$S(t) = \begin{cases} S_{n+1}, & V_{n+1}^2 \leqq t < V_n^2, \quad n \geqq 1 \\ S_1, & t \geqq V_1^2. \end{cases}$$

Then by extending our probability space if necessary (which we then rename $(\Omega, \mathcal{F}, P)$) there exists a Brownian motion $\{B(t), \mathcal{F}_t^*;\ t \geqq 0\}$ such that

$$|S(t) - B(t)| = o\big(t^{1/2}(\log_2 t^{-1})^{(1-\alpha)/2}\big) \quad a.s. \ as \quad t \to 0. \tag{2.6}$$

PROOF. Fix $\alpha \geqq 0$, let $f_\alpha = f$ and set

$$\tilde{X}_j = X_j I\big(X_j^2 < f(V_j^2)\big) \tag{2.7}$$

and

$$X_j^* = \tilde{X}_j - E\{\tilde{X}_j | \mathcal{F}_{j+1}\}. \tag{2.8}$$

We now set $S_n^* = \sum_{j=n}^{\infty} X_j^*$ and thus $\{S_n^*, \mathcal{F}_n;\ n \geqq 1\}$ is again a square integrable reversed martingale which we may embed in a Brownian motion. That is, using Theorem 2 of Scott and Huggins [5], there exists a Brownian motion $\{B(t), \mathcal{F}_t^*;\ t \geqq 0\}$ and a decreasing sequence of stopping times $\{\tau_n;\ n \geqq 1\}$ such that $B(\tau_n) = S_n^*$ a.s. and there exist σ-fields $\mathcal{G}_n$ such that

$$E\{\tau_n - \tau_{n+1} | \mathcal{G}_{n+1}\} = E\{(S_n^* - S_{n+1}^*)^2 | \mathcal{G}_{n+1}\} = E\{(S_n^* - S_{n+1}^*)^2 | \mathcal{F}_{n+1}\} \quad \text{a.s.}$$

Furthermore for $1 < p < \infty$ there exists a constant N_p depending only on p such that

$$E\{(\tau_n - \tau_{n+1})^{p/2} | \mathcal{G}_{n+1}\} \leqq N_p E\{|S_n^* - S_{n+1}^*|^p | \mathcal{G}_{n+1}\} = N_p E\{|S_n^* - S_{n+1}^*|^p | \mathcal{F}_{n+1}\} \quad \text{a.s.}$$

Letting $t_n = \tau_n - \tau_{n+1}$ we rewrite these as

$$E\{t_n | \mathcal{G}_{n+1}\} = E\{X_n^{*2} | \mathcal{G}_{n+1}\} = E\{X_n^{*2} | \mathcal{F}_{n+1}\} \quad \text{a.s.} \tag{2.9}$$

and

$$E\{t_n^{p/2} | \mathcal{G}_{n+1}\} \leqq N_p E\{|X_n^*|^p | \mathcal{G}_{n+1}\} = N_p E\{|X_n^*|^p | \mathcal{F}_{n+1}\} \quad \text{a.s.} \tag{2.10}$$

Now set

$$V_n^{*2} = \sum_{j=n}^{\infty} E\{X_j^{*2} | \mathcal{F}_{j+1}\} = \sum_{j=n}^{\infty} E\{t_j | \mathcal{G}_{j+1}\} \quad \text{a.s.}$$

LEMMA 2.11. *For a fixed $\alpha \geqq 0$ let $f = f_\alpha$ be as in the Theorem. If (2.2)—(2.5) hold then*

$$|V_n^{*2} - V_n^2| = o\big(f(V_n^2)\big) \quad a.s., \tag{2.12}$$

$$\Big|\sum_{j=n}^{\infty} X_j^* - \sum_{j=n}^{\infty} X_j\Big| = o\big(f(V_n^2)^{1/2}\big) \quad a.s., \tag{2.13}$$

$$|\tau_n - V_n^2| = o\big(f(V_n^2)\big) \quad a.s., \tag{2.14}$$

$$|V_n^2 - V_{n+1}^2| = o\big(f(V_n^2)\big) \quad a.s., \quad V_{n+1}^{-2} V_n^2 \xrightarrow{\text{a.s.}} 1. \tag{2.15}$$

PROOF. It is easy to see that

$$|E\{X_j^2-X_j^{*2}|\mathcal{F}_{j+1}\}| \leqq 2E\{X^2 I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}$$

and thus (2.3) implies (2.12). To prove (2.13) first note that

$$\Big|\sum_{j=n}^{\infty} X_j^* - \sum_{j=n}^{\infty} X_j\Big| \leqq \Big|\sum_{j=n}^{\infty} X_j^* - \sum_{j=n}^{\infty} \tilde{X}_j\Big| + \Big|\sum_{j=n}^{\infty} \tilde{X}_j - \sum_{j=n}^{\infty} X_j\Big|.$$

Now

$$\Big|\sum_{j=n}^{\infty} X_j^* - \sum_{j=n}^{\infty} \tilde{X}_j\Big| \leqq \sum_{j=n}^{\infty} E\{|X_j|\, I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}$$

and thus by Lemma 1 of Heyde [1] we have from (2.4) that

$$\Big|\sum_{j=n}^{\infty} X_j^* - \sum_{j=n}^{\infty} \tilde{X}_j\Big| = o\big(f(V_n^2)^{1/2}\big) \quad \text{a.s.} \tag{2.16}$$

Also

$$\Big|\sum_{j=n}^{\infty} \tilde{X}_j - \sum_{j=n}^{\infty} X_j\Big| = \Big|\sum_{j=n}^{\infty} X_j I(X_j^2 \geqq f(V_j^2))\Big|. \tag{2.17}$$

Let

$$Y_j = f(V_j^2)^{-1/2}\big[X_j I(X_j^2 \geqq f(V_j^2)) - E\{X_j I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}\big].$$

Clearly as V_j^2 is $\mathcal{F}_{j+1}$ measurable, $E\{Y_j|\mathcal{F}_{j+1}\}=0$ a.s. and thus $\{Y_j, \mathcal{F}_j;\, j\geqq 1\}$ is a reversed martingale difference sequence. Now

$$\sum_{j=1}^{\infty} E\{|Y_j||\mathcal{F}_{j+1}\} \leqq 2\sum_{j=1}^{\infty} f(V_j^2)^{-1/2} E\{|X_j| I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}$$

which is a.s. finite by (2.4). Therefore the reversed martingale analogue of Corollary (2.8.5) of Stout [6] stated as Lemma 2 of Scott and Huggins [5] is applicable with $p=1$ and thus $\sum_{j=1}^{\infty} Y_j$ converges a.s.

That is

$$\sum_{j=1}^{\infty} f(V_j^2)^{-1/2}\big[X_j I(X_j^2 \geqq f(V_j^2)) - E\{X_j I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}\big]$$

converges a.s. and again Lemma 1 of Heyde [1] implies

$$f(V_n^2)^{-1/2} \sum_{j=n}^{\infty} \big[X_j I(X_j^2 \geqq f(V_j^2)) - E\{X_j I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\}\big] \xrightarrow{\text{a.s.}} 0.$$

Now (2.4) implies, via Lemma 1 of Heyde [1],

$$f(V_n^2)^{-1/2} \sum_{j=n}^{\infty} E\{X_j I(X_j^2 \geqq f(V_j^2))|\mathcal{F}_{j+1}\} \xrightarrow{\text{a.s.}} 0$$

and hence

$$f(V_n^2)^{-1/2} \sum_{j=n}^{\infty} X_j I(X_j^2 \geqq f(V_j^2)) \xrightarrow{\text{a.s.}} 0.$$

From (2.17) we conclude that

(2.18) $$\left|\sum_{j=n}^{\infty}\tilde{X}_j-\sum_{j=n}^{\infty}X_j\right| = o\big(f(V_n^2)^{1/2}\big)$$

and combining (2.16) and (2.18) yields (2.13) as required.

We obtain (2.14) by observing that

$$E\{X_j^{*4}|\mathscr{F}_{j+1}\} \leqq 16E\big\{X_j^4 I(X_j^2<f(V_j^2))|\mathscr{F}_{j+1}\big\}$$

and hence

$$\sum_{j=1}^{\infty} f(V_j^2)^{-2}E\{X_j^{*4}|\mathscr{F}_{j+1}\}<\infty.$$

Thus

$$\sum_{j=1}^{\infty} E\{X_j^{*4}|\mathscr{F}_{j+1}\}<\infty.$$

Also the sequences $\sum_{j=1}^{\infty} f(V_j^2)^{-1}[t_j-E\{t_j|\mathscr{G}_{j+1}\}]$ and $\sum_{j=1}^{\infty}[t_j-E\{t_j|\mathscr{G}_{j+1}\}]$ converge a.s. using Lemma 2 of Scott and Huggins [5]. Once again using Lemma 1 of Heyde [1] we have

$$f(V_n^2)^{-1}\sum_{j=n}^{\infty}[t_j-E\{t_j|\mathscr{G}_{j+1}\}]\xrightarrow{\text{a.s.}} 0,$$

therefore

$$f(V_n^2)^{-1}(\tau_n-V_n^{*2})\xrightarrow{\text{a.s.}} 0$$

and with (2.12) this yields (2.14).

Now (2.15) follows easily by noting

$$|V_n^2-V_{n+1}^2| = E\{X_n^2|\mathscr{F}_{n+1}\} =$$

$$= E\big\{X_n^2 I(X_n^2\geqq \delta f(V_n^2))|\mathscr{F}_{n+1}\big\}+E\big\{X_n^2 I(X_n^2<\delta f(V_n^2))|\mathscr{F}_{n+1}\big\}\leqq o\big(f(V_n^2)\big)+\delta f(V_n^2)$$

by (2.3). Since δ is arbitrary we have the first part of (2.15). The second part of (2.15) is obtained by noting

$$|V_{n+1}^2/V_n^2-1| = |[(V_{n+1}^2-V_n^2)/f(V_n^2)][f(V_n^2)/V_n^2]|$$

and $f(V_n^2)/V_n^2\xrightarrow{\text{a.s.}} 0$. The Lemma is proven.

We now return to the proof of Theorem 1.

Since $\alpha\geqq 0$ is fixed throughout we continue to drop the subscripts from f_α. For $0<\delta<1$ define

(2.19) $$p_j = e^{-\delta j/(\log j)^\alpha},$$

and

(2.20) $$n_j = \inf\{n\colon V_n^2\leqq p_j\}.$$

Given $\varepsilon>0$, for all j sufficiently large (depending on $\omega\in\Omega$ and ε), we have

(2.21) $$\sup_{p_{j+1}\le t\le p_j} |S(t)-B(t)| \le \sup_{V^2_{n_{j+1}}\le t\le V^2_{n_{j-1}}} |S(t)-B(t)| \le$$

$$\sup_{n_{j+1}\le n\le n_{j-1}} \Big|\sum_{i=n+1}^{\infty} X_i - \sum_{i=n+1}^{\infty} X_i^*\Big| + R_B\big(V^2_{n_{j+1}}-\varepsilon f(V^2_{n_{j-1}}),\, V^2_{n_{j-1}}+\varepsilon f(V^2_{n_{j-1}})\big)$$

where R_B is defined for $0\le a<b<\infty$ by $R_B(a, b) = \max_{a\le u,\, v\le b} |B(u)-B(v)|$.

The inequality (2.21) follows from (2.14) and the facts that $S(t)=S_{n+1}$ on $[V^2_{n+1}, V^2_n)$ for $t\ge 0$, and $\sum_{i=n+1}^{\infty} X_i^*=B(\tau_{n+1})$.

Using (2.13) it follows that for all j sufficiently large (depending on ω)

(2.22)
$$\sup_{p_{j+1}\le t\le p_j} |S(t)-B(t)| \le o\big((f(V^2_{n_{j-1}}))^{1/2}\big)+R_B\big(V^2_{n_{j+1}}-\varepsilon f(V^2_{n_{j-1}}),\, V^2_{n_{j-1}}+\varepsilon f(V^2_{n_{j-1}})\big).$$

Elementary computations, using the definitions of f, p_j, n_j and (2.15) show that the interval involved in R_B in (2.22) is contained in the interval $[p_{j+2}, p_{j-1}]$. Thus for all j sufficiently large

(2.23) $$\sup_{p_{j+1}\le t\le p_j} |S(t)-B(t)| \le o\big((f(p_j))^{1/2}\big)+R_B(p_{j+2}, p_{j-1}) \quad \text{a.s.}$$

From Lemma 2.2 of Jain, Jogdeo and Stout [3] it now follows that for any $\nu>0$

$$P[R_B(p_{j+2}, p_{j-1}) > \nu p_j^{1/2}(\log_2 p_j^{-1})^{(1-\alpha)/2}] =$$
$$= P[R_B(0, 1) > \nu(p_{j-1}-p_{j+2})^{-1/2}p_j^{1/2}(\log_2 p_j^{-1})^{(1-\alpha)/2}].$$

Now for all j sufficiently large and δ sufficiently small we have

$$p_j^{-1}(p_{j-1}-p_{j+2}) \le 4\delta(\log_2 p_j^{-1})^{-\alpha}.$$

Hence for such large j and small δ

$$P[R_B(p_{j+2}, p_{j-1}) > \nu p_j^{1/2}(\log_2 p_j^{-1})^{(1-\alpha)/2}] \le P[R_B(0, 1) > \nu(4\delta)^{-1/2}(\log_2 p_j^{-1})^{1/2}] \le$$
$$\le c2^{-1}(\log_2 p_j^{-1})^{-1/2}\exp\{-2\log_2 p_j^{-1}\} \le cj^{-2}(\log j)^{2\alpha}$$

by choosing δ so that $\nu^2/8\delta\ge 2$. Thus for every $\nu>0$ $\sum_{j=1}^{\infty} P[R_B(p_{j+2}, p_{j-1})> >\nu p_j^{1/2}(\log_2 p_j^{-1})^{(1-\alpha)/2}]<\infty$. Therefore by the Borel—Cantelli lemma, almost surely,

(2.24) $$R_B(p_{j+2}, p_{j-1}) \le \nu p_j^{1/2}(\log_2 p_j^{-1})^{(1-\alpha)/2}.$$

The Theorem now follows from (2.23) and (2.24) as we may choose ν as small as we please. For convenience of use in the sequel we state the following.

THEOREM 2.25. *Suppose* $V_n^2 \xrightarrow{\text{a.s.}} 0$ *and, for fixed* $\alpha\ge 0$,

(2.26)
$$\sum_{k=1}^{\infty} V_k^{-2}(\log_2 V_k^{-2})^{\alpha} E\{X_k^2 I(X_k^2 > V_k^2)[\log V_k^{-2}(\log_2 V_k^{-2})^{2(\alpha+1)}]^{-1}|\mathcal{F}_{k+1}\} < \infty \quad a.s.$$

Let S be the random function defined as $[0, \infty)$ *obtained by setting*

$$S(t) = \begin{cases} S_{n+1}, & V_{n+1}^2 \leqq t < V_n^2, \\ S_1, & t \geqq V_1. \end{cases}$$

Then the conclusions of Theorem (2.1) *hold.*

PROOF. Firstly note that (2.26) implies

$$\sum_{k=1}^{\infty} [f_\alpha(V_k^2)]^{-1} E\{X_k^2 I(X_k^2 > \delta f_\alpha(V_k^2)) | \mathscr{F}_{k+1}\} < \infty \quad \text{a.s.} \tag{2.27}$$

for all $\delta > 0$. This implies (2.3) by Lemma 1 of Heyde [1] and (2.4) quite easily. To obtain (2.5) write $g_\alpha(V_j^2) = V_k^2[\log V_k^{-2}(\log_2 V_k^{-2})^{2(\alpha+1)}]^{-1}$. Then

$$E\{X_k^4 I(X_k^2 \leqq \delta f_\alpha(V_k^2)) | \mathscr{F}_{k+1}\} =$$

$$= E\{X_k^4 I(X_k^2 \leqq g_\alpha(V_k^2)) | \mathscr{F}_{k+1}\} + E\{X_k^4 I(g_\alpha(V_k^2) < X_k^2 < \delta f_\alpha(V_k^2)) | \mathscr{F}_{k+1}\} \leqq$$

$$\leqq g_\alpha(V_k^2) E\{X_k^2 I(X_k^2 \leqq g_\alpha(V_k^2)) | \mathscr{F}_{k+1}\} + \delta f_\alpha(V_k^2) E\{X_k^2 > (g_\alpha(V_k^2)) | \mathscr{F}_{k+1}\}.$$

Multiplying by $f_\alpha(V_k^2)^{-2}$ and summing over k we see that (2.5) is dominated by

$$\sum_{k=1}^{\infty} |V_k^2 - V_{k+1}^2| [V_k^{-2}(\log V_k^{-2})(\log_2 V_k^{-2})^2]^{-1} + \delta \sum_{k=1}^{\infty} f(V_k^2)^{-1} E\{X_k^2 I(X_k^2 > g_\alpha(V_k^2)) | \mathscr{F}_{k+1}\}.$$

Now

$$\sum_{k=1}^{\infty} |V_k^2 - V_{k+1}^2| [V_k^{-2}(\log V_k^{-2})(\log_2 V_k^{-2})^2]^{-1} \leqq$$

$$\leqq \sum_{k=1}^{\infty} |V_{k+1}^{-2} - V_k^{-2}| \left[V_{k+1}^{-2}(1+\varepsilon)^{-1}\left(\log (V_{k+1}^{-2}/(1+\varepsilon))\right)\left(\log_2(V_{k+1}^{-2}/(1+\varepsilon))\right)^2\right]^{-1} \leqq$$

$$\leqq (1+\varepsilon) \int_0^{\infty} \frac{dx}{x \log x (\log_2 x)^2} < \infty.$$

Thus (2.5) holds.

3. Integral tests for upper functions for reverse martingales

In this section we use our invariance principle to exploit properties of Brownian motion to obtain results for reverse martingales corresponding to the integral tests for upper functions of martingales and for lower sums of absolute maxima of martingales of Jain, Jogdeo and Stout.

THEOREM 3.1. *Suppose* $V_n^2 \xrightarrow{\text{a.s.}} 0$, *and*

$$\sum_{k=1}^{\infty} V_k^{-2}(\log_2 V_k^{-2}) E\{X_k^2 I(X_k^2 > V_k^2)[\log V_k^{-2}(\log_2 V_k^{-2})]^{-1} | \mathscr{F}_{k+1}\} < \infty \quad a.s. \tag{3.2}$$

Let $\varphi > 0$ be a non decreasing function, $\varphi(t) \to \infty$ as $t \to \infty$, then

$$P(|S_n| > V_n \varphi(V_n^{-2}) \text{ i.o}) = 0 \quad or \quad 1 \tag{3.3}$$

according as

$$(3.4)\qquad I(\varphi)=\int_0^{\infty}\frac{\varphi(t)}{t}\exp\{-\varphi^2(t)/2\}\,dt<\infty \quad \textit{or} \quad =\infty.$$

The proof of this theorem is an obvious adaptation of the proof of the analogous result of Jain, Jogdeo and Stout using the transformation $B(t)\to tB(t^{-1})$ on Kolmogorov's test for Brownian motion, see Ito and McKean [2] p. 163.

Now let Φ_ε denote the class of functions φ from $(0,\varepsilon)$ to $[0,\infty)$ such that $\varphi(t)\uparrow\infty$ as $t\downarrow 0$. We may restate Theorem (3.1) in a form corresponding to a proof based on (3.3) p. 533 of Jain and Taylor [4].

THEOREM (3.5). *Suppose* $V_n^2\xrightarrow{\text{a.s.}}0$ *and* (3.2) *holds. If* $\varphi\in\Phi_\varepsilon$ *then*

$$(3.6)\qquad P[|S_n|>V_n\varphi(V_n^2)\ i.o.]=0 \quad \textit{or} \quad 1$$

according as

$$(3.7)\qquad I'(\varphi)=\int_{0+}\frac{\varphi(t)}{t}\exp\{-\varphi^2(t)/2\}\,dt<\infty \quad \textit{or} \quad =\infty.$$

We are now concerned with the behaviour of $M_n=\max_{i\geq n}|S_i|$. Instead of the approach above we use the appropriate results for $M(t)=\max_{0\leq s\leq t}|B(s)|$ as $t\to 0$. Let Φ_ε be as before.

THEOREM (3.8). *Assume that* $V_n^2\xrightarrow{\text{a.s.}}0$, *and*

(3.9)

$$\sum_{k=1}^{\infty}V_k^{-2}(\log_2 V_k^{-2})^4E\{X_k^2I(X_k^2>V_k^2)(\log V_k^{-2})^{-1}(\log_2 V_k^{-2})^{-10}|\mathcal{F}_{k+1}\}<\infty \quad a.s.$$

Then for $\varphi\in\Phi_\varepsilon$

$$P[M_n<V_n\{\varphi(V_n^2)\}^{-1}\ i.o.]=0 \quad \textit{or} \quad 1$$

according as

$$(3.10)\qquad I_1(\varphi)=\int_{0+}\frac{\varphi^2(t)}{t}\exp\{-8(\varphi(t))^2/\pi^2\}\,dt<\infty \quad \textit{or} \quad =\infty.$$

The proof is again comparable to that of the analogous result of Jain, Jogdeo and Stout using the corresponding theorem for Brownian motion of Jain and Taylor [4] p. 547.

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REPRESENTATIONS OF INNER PRODUCTS IN THE SPACE OF POLYNOMIALS

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1. Introduction. This paper is based on [6]. We briefly summarize the main results of [6].

Suppose that $\{p_i\}_{i=1}^{\infty}$ is an orthonormal base in the space of polynomials in n real variables. Assume that the inner product $\langle\,.\,,.\,\rangle$ has the following property:

$$\langle p_i, p_j\rangle = \langle 1, p_i p_j\rangle, \quad i, j = 1, 2, \ldots.$$

Let μ be a positive Lebesgue—Stieltjes measure on R^n such that $\int\limits_{R^n} p^2\, d\mu > 0$ for any non-zero polynomial p. We provided a necessary and sufficient condition when the inner product $\langle\,.\,,.\,\rangle$ can be expressed in the form

$$\langle p_i, p_j\rangle = \int\limits_{R^n} p_i p_j \varrho\, d\mu$$

where ϱ is a square μ-integrable function. We also pointed out that this condition is satisfied if the measure μ is of the form

$$d\mu(x_1, x_2, \ldots, x_n) = d\psi(x_1)\, d\psi(x_2) \ldots d\psi(x_n)$$

where ψ is some bounded non-decreasing function with infinitely many points of increase.

The goal of this paper is to study the following problems:

(i) What is a condition on μ such that the function ϱ is unique?

(ii) What is a condition on μ such that ϱ can be chosen as a non-negative function?

We pay a special attention to the case $d\mu = d\psi(x_1) d\psi(x_2) \ldots d\psi(x_n)$.

We now summarize the results of this paper. In Section 3 we provide the complete answer to the first question. We partially answer the second question by showing some sufficient conditions on μ for which ϱ is unique and non-negative.

In Section 4 we consider orthogonal polynomials P_k with respect to a measure α and study a relation between the convergence of the series $\sum\limits_{k=1}^{\infty}\Big(\int\limits_{R^n} P_k\, d\beta\Big)^2$ and continuity points of the measures α and β.

In Section 5 we provide examples of a one dimensional measure μ such that $\langle f, g\rangle = \int\limits_{R} fg\varrho\, d\mu$ is an inner product in the space of polynomials for a unique function ϱ which changes sign. These examples are based on some properties of the Al-Salam—Carlitz polynomials. Finally, we apply the results from Section 4 to specify a class of measures β associated with the Al-Salam—Carlitz polynomials which define the spaces $L_2(R, \beta)$ such that

(i) polynomials are dense in $L_2(R, \beta)$,

(ii) there exists a non-negative function f from $L_2(R, \beta)$ which is not a limit of any convergent sequence of non-negative polynomials.

2. Notation. Let $\mathscr{B}(R^n)$ be the σ-field of Borel sets in R^n. Let $\mu\colon \mathscr{B}(R^n)\to R$ be a non-negative finite measure. We say that x_0 is a continuity point of μ iff $\mu(\{x_0\})=0$. By supp (μ) we denote the support of the measure μ, i.e., the smallest closed set A such that $\mu(A)=\mu(A\cap B)$ for any Borel set B. By $L_p(R^n, \mu)$ we mean the linear space of all μ-measurable complex functions f which have the finite norm

$$\|f\|_p = \Big(\int_{R^n} |f(x)|^p \, d\mu(x)\Big)^{1/p}, \quad p \geqq 1.$$

Let Π_n^∞ be the linear space of all polynomials in n real variables with real coefficients. If $\Pi_n^\infty\subset L_p(R^n, \mu)$ then by $\mathscr{M}(n, p, \mu)$ we denote the closure (in $L_p(R^n, \mu)$) of the space spanned by all monomials from Π_n^∞. We say that a polynomial sequence $\{p_k\}_{k=1}^\infty$ is μ-orthonormal iff

$$\int_{R^n} p_i p_j \, d\mu = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

3. Representations of inner products in the space of polynomials.* Let $\langle .\,, .\rangle\colon \Pi_n^\infty\times\Pi_n^\infty\to R$ be an inner product such that $\langle p, w\rangle=\langle 1, pw\rangle$ for any $p, w\in\Pi_n^\infty$. The comparison criterion of orthogonality from [6] states a necessary and sufficient condition for the existence of a function $\varrho\in L_2(R^n, \mu)$ such that

$$\langle p, w\rangle = \int_{R^n} pw\varrho \, d\mu \quad \text{for any} \quad p, w\in\Pi_n^\infty. \tag{1}$$

We assume throughout the rest of this paper that such a function ϱ exists.

We now answer the first question mentioned in the introduction.

THEOREM 1. *The function* ϱ *from* (1) *is unique iff* $\mathscr{M}(n, 2, \mu)=L_2(R^n, \mu)$.

PROOF. ($\Rightarrow$) Assume that $\mathscr{M}(n, 2, \mu)=L_2(R^n, \mu)$, i.e., monomials are linearly dense in $L_2(R^n, \mu)$. Suppose that two functions ϱ and ϱ_1 from $L_2(R^n, \mu)$ satisfy (1). Then

$$\int_{R^n} p(\varrho-\varrho_1)\, d\mu = 0 \quad \text{for any} \quad p\in\Pi_n^\infty. \tag{2}$$

We then conclude that the continuous linear functional $f\to\int_{R^n} f(\varrho-\varrho_1)\, d\mu$ vanishes on the linearly dense subset of $L_2(R^n, \mu)$. Thus it must vanish identically on $L_2(R^n, \mu)$, i.e., $\varrho=\varrho_1$ almost everywhere.

($\Leftarrow$) Suppose now that $\mathscr{M}(n, 2, \mu)\neq L_2(R^n, \mu)$. Hence, there exists a continuous linear functional $\mathbf{F}\colon L_2(R^n, \mu)\to C$ such that $\mathbf{F}(\Pi_n^\infty)=\{0\}$ and $\mathbf{F}$ is not

* This section and [6] form a part of the dissertation [8] written under the guidance of Professor S. Turski.

identically zero. Thus $\langle p, w\rangle = \int\limits_{R^n} pw\varrho\, d\mu + a\mathbf{F}(pw)$ for any $p, w\in\Pi_n^\infty$ and $a\in C$. Since the linear functional $f\to\int\limits_{R^n} f\varrho\, d\mu + aF(f)$ is continuous on $L_2(R^n, \mu)$, it can be represented, due to the Riesz theorem, as $\int\limits_{R^n} f\varrho_a\, d\mu$ for a unique function ϱ_a from $L_2(R^n, \mu)$. Thus

$$(3)\qquad \langle p, w\rangle = \int\limits_{R^n} pw\varrho_a\, d\mu \quad\text{for any}\quad p, w\in\Pi_n^\infty \quad\text{and}\quad a\in C.$$

This completes the proof. □

As an immediate consequence of (3) we get the following corollary.

COROLLARY 1. *If* $\mathcal{M}(n, 2, \mu)\neq L_2(R^n, \mu)$ *then* (1) *holds for infinitely many functions* ϱ *form* $L_2(R^n, \mu)$.

We now turn to the second problem. We first prove a sufficient condition that ϱ is non-negative.

THEOREM 2. *If* $\mathcal{M}(n, 4, \mu)=L_4(R^n, \mu)$ *then a function* $\varrho\in L_2(R^n, \mu)$ *which satisfies* (1) *is non-negative (almost everywhere).*

PROOF. Let $A\in\mathcal{B}(R^n)$ be an arbitrary set and let χ_A be its characteristic function. Consider a sequence of polynomials $\{w_k\}_{k=1}^\infty$ convergent to χ_A in $L_4(R^n, \mu)$. Then

$$\|w_k^2-\chi_A\|_2 = \|(w_k-\chi_A)^2+2\chi_A(w_k-\chi_A)\|_2 \leqq \|(w_k-\chi_A)^2\|_2+$$

$$+2\|w_k-\chi_A\|_2 \leqq \|w_k-\chi_A\|_4^2+2\mu(R^n)^{1/4}\|w_k-\chi_A\|_4.$$

Thus

$$\int\limits_{R^n} \chi_A\varrho\, d\mu = \lim_{k\to\infty}\int\limits_{R^n} w_k^2\varrho\, d\mu = \lim_{k\to\infty}\langle w_k, w_k\rangle \geqq 0$$

which yields that ϱ is non-negative (almost everywhere) as claimed. □

REMARK. The condition $\mathcal{M}(n, 4, \mu)=L_4(R^n, \mu)$ is stronger than $\mathcal{M}(n, 2, \mu)= =L_2(R^n, \mu)$, see [3]. It is well-known that if μ has a bounded support then polynomials are dense in $L_p(R^n, \mu)$ for $p>1$.

Let $\psi: R\to R$ be a bounded non-decreasing function with infinitely many points of increase. By $\psi^{(n)}$ we denote the Lebesque—Stieltjes measure corresponding to the integrator $d\psi(x_1)d\psi(x_2)\dots d\psi(x_n)$.

Due to Theorem 4 from [6], the inner product $\langle\, .\, , .\, \rangle$ can be represented in the form

$$(4)\qquad \langle p, w\rangle = \int\limits_{R^n} pw\gamma\, d\psi^{(n)}, \quad p, w\in\Pi_n^\infty,$$

for some functions $\psi: R\to R$ and $\gamma\in L_2(R^n, \psi^{(n)})$. We now specify Theorems 1 and 2 for $\mu=\psi^{(n)}$. We begin with the following lemma.

LEMMA 1. $\mathcal{M}(n, p, \psi^{(n)})=L_p(R^n, \psi^{(n)})$ *iff* $\mathcal{M}(1, p, \psi^{(1)})=L_p(R, \psi^{(1)})$.

PROOF. ($\Rightarrow$) Suppose that $f\in L_p(R, \psi^{(1)})$. Since the function $f: f(x_1, x_2, \dots$ $\dots, x_n)=f(x_1)$ belongs to $L_p(R^n, \psi^{(n)})$ there exists a sequence $\{w_k\}_{k=1}^\infty$ of polynomials such that $\int\limits_{R^n} |w_k(x_1, \ldots, x_n)-f(x_1)|^p \, d\psi^{(n)} \to 0$ as $k\to\infty$. Define now the following sequence of polynomials in one variable:

$$v_k(x) = [\psi^{(n-1)}(R^{n-1})]^{-1} \int\limits_{R^{n-1}} w_k(x, t_1, \ldots, t_{n-1})\, d\psi^{(n-1)}, \quad k = 1, 2, \ldots.$$

Due to Hölder inequality we get

$$\int\limits_R |v_k(x)-f(x)|^p \, d\psi =$$

$$= [\psi^{(n-1)}(R^{n-1})]^{-p} \int\limits_R \Big| \int\limits_{R^{n-1}} \big(w_k(x, t_1, \ldots, t_{n-1})-f(x)\big)\, d\psi^{(n-1)}\Big|^p d\psi(x) \leqq$$

$$\leqq [\psi^{(n-1)}(R^{n-1})]^{-1} \int\limits_{R^n} |w_k(x, t_1, \ldots, t_{n-1})-f(x)|^p \, d\psi^{(n)} \to 0$$

as $n\to\infty$. This yields that $\mathcal{M}(1, p, \psi^{(1)})=L_p(R, \psi^{(1)})$.

($\Leftarrow$) Recall that the set H_n of characteristic functions f of the form

$$f = \chi_{A_1\times A_2\times \ldots \times A_n}, \quad A_i\in\mathscr{B}(R), \tag{5}$$

is linearly dense in $L_p(R^n, \psi^{(n)})$. Therefore it is enough to verify that $H_n\subset \mathcal{M}(n, p, \psi^{(n)})$. We prove this by induction. The case $n=1$ is obvious. Thus assume that $H_n\subset\mathcal{M}(n, p, \psi^{(n)})$ for some $n\geqq 1$, i.e., for any function $u\in H_n$ there exists a sequence $\{w_k\}_{k=1}^\infty$ of polynomials such that $\|u-w_k\|_{n,p}\to 0$ as $k\to\infty$ where $\|y\|_{n,p}=\Big(\int\limits_{R^n} |y|^p \, d\psi^{(n)}\Big)^{1/p}$. Take any $f\in H_{n+1}$. The function f can be represented as

$$f(x_1, \ldots, x_n, x_{n+1}) = g(x_1, \ldots, x_n) h(x_{n+1})$$

where $g\in H_n$ and $h\in H_1$. Let $\{u_k\}_{k=1}^\infty$, $u_k=u_k(x_1, \ldots, x_n)$, and $\{v_k\}_{k=1}^\infty$, $v_k=v_k(x)$, be sequences of polynomials such that

$$\|g-u_k\|_{n,p} \to 0, \quad \|h-v_k\|_{1,p} \to 0 \quad (k\to\infty).$$

We get

$$\|f-u_kv_k\|_{n+1,p} = \|gh-u_kv_k\|_{n+1,p} = \|(g-u_k)(h-v_k)+u_k(h-v_k)+$$

$$+v_k(g-u_k)\|_{n+1,p} \leqq \|(g-u_k)(h-v_k)\|_{n+1,p}+\|u_k(h-v_k)\|_{n+1,p}+$$

$$+\|v_k(g-u_k)\|_{n+1,p} = \|g-u_k\|_{n,p}\|h-v_k\|_{1,p}+\|u_k\|_{n,p}\|h-v_k\|_{1,p}+$$

$$+\|v_k\|_{1,p}\|g-u_k\|_{n,p} \to 0, \quad k\to\infty.$$

Thus the sequence $\{u_kv_k\}_{k=1}^\infty$ is convergent to h in $L_p(R^{n+1}, \psi^{(n+1)})$. This completes the proof. □

COROLLARY 2. *The function* γ *from* (4) *is unique iff* $\mathcal{M}(1, 2, \psi^{(1)})=L_2(R, \psi^{(1)})$.

COROLLARY 3. *The function* γ *from* (4) *is unique and non-negative if* $\mathcal{M}(1, 4, \psi^{(1)})=L_4(R, \psi^{(1)})$.

We now give another sufficient condition on ψ such that γ is unique and non-negative.

THEOREM 3. *If for any non-negative function h from $L_2(R, \psi^{(1)})$ there exists a sequence $\{v_k\}_{k=1}^{\infty}$ of non-negative polynomials which is convergent to h in $L_2(R, \psi^{(1)})$ then γ is unique and non-negative.*

PROOF. Let $A_1, A_2, \dots, A_n$ be arbitrary Borel sets in R. The assumption of Theorem 3 implies that there exist sequences $\{w_k^{(i)}\}_{k=1}^{\infty}$ of non-negative polynomials which are convergent to the characteristic functions χ_{A_i} in $L_2(R, \psi^{(1)})$, $i=1, 2, \dots, n$. From the proof of Lemma 1 it follows that the sequence $\{u_k\}_{k=1}$ of polynomials in n separated variables,

$$u_k(x_1, \dots, x_n) = w_k^{(1)}(x_1) \dots w_k^{(n)}(x_n),$$

is convergent to $\chi_{A_1 \times A_2 \times \dots \times A_n}$ in $L_2(R^n, \psi^{(n)})$. Since the functional

$$f \to \int_{R^n} f\gamma \, d\psi^{(n)}$$

is continuous on $L_2(R^n, \psi^{(n)})$, we get

$$\int_{R^n} u_k \gamma \, d\psi^{(n)} \to \int_{R^n} \chi_{A_1 \times A_2 \times \dots \times A_n} \gamma \, d\psi^{(n)}, \quad k \to \infty.$$

Because $w_k^{(i)}$ is a non-negative polynomial it can be represented in the form

$$w_k^{(i)}(x) = [p_k^{(i)}(x)]^2 + [q_k^{(i)}(x)]^2$$

for some polynomials $p_k^{(i)}$ and $q_k^{(i)}$ $(i=1, 2, \dots, n;\ k=1, 2, \dots)$. Hence

$$\int_{R^n} u_k \gamma \, d\psi^{(n)} = \int_{R^n} \prod_{i=1}^{n} ([p_k^{(i)}(x_i)]^2 + [q_k^i(x_i)]^2) \gamma \, d\psi^{(n)}.$$

The integral in the right-hand side of this equality can be rewritten as the sum of positive terms $\int_{R^n} p^2 \gamma \, d\psi^{(n)}$ where $p \in \Pi_n^{\infty}$. Thus we conclude that

$$\int_{R^n} u_k \gamma \, d\psi^{(n)} > 0 \quad \text{and} \quad \int_{R^n} \chi_{A_1 \times A_2 \times \dots \times A_n} \gamma \, d\psi^{(n)} \geqq 0.$$

Since A_i are arbitrary Borel sets, the last inequality implies that γ is non-negative. To show that γ is unique, note that any f from $L_2(R, \psi^{(1)})$ can be represented as

$$f = (h_1 - h_2) + i(h_3 - h_4)$$

where h_1, h_2, h_3, h_4 are non-negative functions from $L_2(R, \psi^{(1)})$. Thus the assumption of Theorem 3 implies that $\mathcal{M}(1, 2, \psi^{(1)}) = L_2(R, \psi^{(1)})$. Applying Corollary 2 we complete the proof. □

REMARKS. (i) Due to the M. Riesz theorem (see [9]), Corollary 2 can be restated as follows. Consider the Hamburger moment problem

$$\mu_k = \int_R x^k \, d\psi(x), \quad k = 0, 1, \dots. \tag{6}$$

The function γ is unique if this problem is determined or ψ is its extremal solution.

(ii) The assumption of Corollary 3 implies the assumption of Theorem 3. Indeed, suppose that $\mathcal{M}(1,4,\psi^{(1)})=L_4(R,\psi^{(1)})$. Take any non-negative function h from $L_2(R,\psi^{(1)})$. Since $\sqrt{h}\in L_4(R,\psi^{(1)})$ we conclude that there exists a sequence $\{w_k\}_{k=1}^{\infty}$ of polynomials such that $\|\sqrt{h}-w_k\|_{1,4}\to 0$ as $k\to\infty$. Then

$$\|h-w_k^2\|_{1,2}=\|(\sqrt{h}-w_k)^2+2w_k(\sqrt{h}-w_k)\|_{1,2}\leqq\|(\sqrt{h}-w_k)^2\|_{1,2}+$$
$$+2\|w_k(\sqrt{h}-w_k)\|_{1,2}\leqq\|\sqrt{h}-w_k\|_{1,4}^2+2\|w_k\|_{1,4}\|\sqrt{h}-w_k\|_{1,4}\to 0,\quad k\to\infty.$$

Thus the assumption of Theorem 3 holds.

(iii) If supp $\psi^{(1)}$ is unbounded, we do not know how to verify the assumption of Corollary 2. Nevertheless we know that the determination of the moment problem (6) is its necessary condition. Moreover the existence of a positive number α such that

$$\int_R e^{-\alpha|x|}\,d\psi(x)<\infty$$

implies that $\mathcal{M}(1,p,\psi^{(1)})=L_p(R,\psi^{(1)})$ for $p\geqq 1$ (see [3]).

(iv) Due to [2] there exists an inner product $\langle .,.\rangle:\Pi_n^{\infty}\times\Pi_n^{\infty}\to R$ with the property $\langle f,g\rangle=\langle 1,fg\rangle$ for which it is not possible to choose a non-negative measure τ on $\mathcal{B}(R^n)$ such that $\langle f,g\rangle\equiv\int_{R^n} fg\,d\tau$. In particular it is not always possible to find a function ψ such that (4) holds for a non-negative functon γ from $L_2(R^n,\psi^{(n)})$.

From [7] we know that the function ψ in (4) can be chosen in such a way that $\psi^{(1)}$ is an indeterminate measure. Moreover, we may require that $\psi^{(1)}$ is extremal and then (4) holds for a unique γ from $L_2(R^n,\psi^{(n)})$. From Theorem 3 and Remark (iv) we now get the following theorem.

THEOREM 4. *There exists an extremal measure* $\psi^{(1)}$ *such that not every non-negative function from* $L_2(R,\psi^{(1)})$ *is the limit of a sequence of non-negative polynomials.*

Note that if $\psi^{(1)}$ is indeterminate and non-extremal then (4) holds for infinitely many functions γ. From Remark (iii) we get that $\mathcal{M}(1,4,\psi^{(1)})\neq L_4(R,\psi^{(1)})$ for any indeterminate measure $\psi^{(1)}$.

4. Orthogonal polynomials and continuity points of corresponding measures. Let $\{p_k\}_{k=1}^{\infty}$ be an α-orthonormal base of the space Π_n^{∞}. Suppose that $\mathcal{M}(n,4,\alpha)=L_4(R^n,\alpha)$. Let $\{r_k\}_{k=1}^{\infty}$ be a β-orthonormal base of Π_n^{∞}. Assume that the measure β is determinate.

THEOREM 5. *If* supp $\beta\setminus$supp $\alpha\neq\emptyset$ *or there exists a continuity point of* α *which is not continuity point of* β *then*

$$\sum_{k=1}^{\infty}\Big(\int_{R^n} p_k\,d\beta\Big)^2=\infty.$$

PROOF. Assume without loss of generality that $\deg r_1=0$. The coefficient c_{k1} in the series $p_k=\sum_{i=1}^{\infty}c_{ki}r_i$ is equal to

$$c_{k1}=\beta(R^n)^{-1/2}\int_{R^n}p_k\,d\beta.$$

Suppose by contradiction that the series $\sum_{k=1}^{\infty} c_{k1}^2$ is convergent. Then, by the comparison criterion of orthogonality, there exists a function ϱ from $L_2(R^n, \alpha)$ such that $\int_{R^n} w\varrho\, d\alpha = \int_{R^n} w\, d\beta$ for any $w \in \Pi_n^\infty$.

From Theorem 2 it follows that the function ϱ is non-negative. Since the measure β is determinate, we get $\varrho\, d\alpha = d\beta$. Thus we conclude that $\operatorname{supp}\beta \subset \operatorname{supp}\alpha$ and all continuity points of α are continuity points of β. This contradiction completes the proof. □

For a given $S \in \mathscr{B}(R^n)$ we define the measure α_S by $d\alpha_S = \chi_S\, d\alpha$. Suppose now that for some bounded Borel set B there exists an α_B-orthonormal base of Π_n^∞.

COROLLARY 4. *If x_0 is a continuity point of α then* $\sum_{k=1}^{\infty} p_k(x_0)^2 = \infty$.

PROOF. Let σ be a measure whose support consists of only one point x_0. Define the measure β by $d\beta = d\alpha_B + d\sigma$. From the comparison criterion of orthogonality it follows that $\sum_{k=1}^{\infty} \left(\int_{R^n} p_k \chi_B\, d\alpha\right)^2 < \infty$. Assume by contradiction that $\sum_{k=1}^{\infty} p_k(x_0)^2 < \infty$. Hence

$$\sum_{k=1}^{\infty}\left(\int_{R^n} p_k\, d\beta\right)^2 = \sum_{k=1}^{\infty}\left(\int_{R^n} p_k\chi_B\, d\alpha\right)^2 + 2\sum_{k=1}^{\infty} p_k(x_0)\int_{R^n} p_k\chi_B\, dx + \sum_{k=1}^{\infty} p_k(x_0)^2 =$$

$$= \sum_{k=1}^{\infty}\left(\int_{R^n} p_k\chi_B\, d\alpha\right)^2 + 2\sqrt{\sum_{k=1}^{\infty} p_k(x_0)^2 \sum_{k=1}^{\infty}\left(\int_{R^n} p_k\chi_B\, d\alpha\right)^2} + \sum_{k=1}^{\infty} p_k(x_0)^2 < \infty.$$

Due to Corollary 1.1 from [9] the measure β is determinate. Since x_0 is not a continuity point of β, Theorem 5 yields

$$\sum_{k=1}^{\infty}\left(\int_{R} p_k\, d\beta\right)^2 = \infty.$$

This contradiction completes the proof. □

The author is inclined to believe that Theorem 5 can be proved without th assumption that $\mathscr{M}(n, 4, \alpha) = L_4(R^n, \alpha)$.

5. Negative approximation properties of the Al-Salam—Carlitz polynomials. Let $a \in (0, \infty)$ and $q \in (0, 1)$. Al-Salam and Carlitz introduced in [1] polynomials V_n $(n = 0, 1, \dots)$ which satisfy the recursion formula

$$V_{n+1}(x) = \left(x - (1+a)\,q^{-n}\right)V_n(x) - a q^{1-2n}(1-q^n)V_{n-1}(x), \quad n = 0, 1, \dots$$

$$\left(V_{-1}(x) = 0, \quad V_0(x) = 1\right).$$

Due to Favard's theorem [5] there exists a non-decreasing bounded function $\psi = \psi_{a,q}$ such that

$$\int_{R} V_i(x)V_j(x)\, d\psi(x) = a^i q^{-i^2}[q]_i \delta_{ij}$$

where $[b]_0=1$, $[b]_k=[b]_{k-1}(1-bq^{k-1})$; $b\in R$, $k=1,2,\ldots$. From [1] we know that

$$\int_R V_i(x)V_j(x)\,d\beta(x) = Ka^iq^{-i^2}[q]_i\delta_{ij}$$

where $\beta=\beta_{a,q}$ is a step function defined by

$$d\beta(q^i) = a^iq^{i^2}/([q]_i[aq]_i) \quad i=0,1,\ldots$$

and $K=K_{a,q}=\sum_{i=0}^{\infty} a^iq^{i^2}/([q]_i[aq]_i)$, $aq^i\neq 1$.

It is easily seen that β is a non-decreasing function iff $aq<1$. The Hamburger moment problem corresponding to the polynomials V_i is determined iff $a\leqq q$ or $q^{-1}\leqq a$ (see [4]). In particular if $a\leqq q$ then the function ψ is unique (up to an arbitrary constant) and $\psi(x)=K^{-1}\beta(x)$. Furthermore, $\mathcal{M}(1,2,\psi^{(1)})=L_2(R,\psi^{(1)})$. Note that for every positive numbers a and a_0 such that $(1-aq^i)(1-a_0q^i)\neq 0$, $i=1,2,\ldots$, we have

$$d\beta_{a_0,q}(x) = \tau(x)\,d\beta_{a,q}(x)$$

where $\tau(q^{-k})=a_0^k[aq]_k/(a^k[a_0q]_k)$, $k=0,1,\ldots$. Thus $\tau\in L_2(R,\beta_{a,q}^{(1)})$. If $a_0>1/q$ and $a\leqq q$ then the function τ changes sign, although $\mathcal{M}(1,2,\beta_{a,q}^{(1)})=L_2(R,\beta_{a,q}^{(1)})$. By the contraposition of Theorem 3 we get

THEOREM 6. *If $aq<1$ then there exists a non-negative function h from $L_2(R,\beta_{a,q}^{(1)})$ which is not a limit of any convergent sequence of non-negative polynomials.*

In particular $a\leqq q$ implies that $\mathcal{M}(1,p,\beta_{a,q}^{(1)})\neq L_p(R,\beta_{a,q}^{(1)})$ for $p\geqq 4$, although $\mathcal{M}(1,p,\beta_{a,q}^{(1)})=L_p(R,\beta_{a,q}^{(1)})$ for $1\leqq p\leqq 2$. It would be interesting to know if $\mathcal{M}(1,p,\beta_{a,q}^{(1)})=L_p(R,\beta_{a,q}^{(1)})$ for $p\in(2,1)$.

As a final word observe that the contraposition of Theorem 3 yields two sufficient conditions for the existence of a non-negative function h from $L_2(R,\psi^{(1)})$ which is not a limit of any convergent sequence of non-negative polynomials. These two conditions are:

1° There exists a function $\tau\in L_2(R,\psi^{(1)})$ which changes sign such that $\langle f,g\rangle = \int_R fg\tau\,d\psi$ is an inner product in Π_1^∞.

2° There exists a determinate measure $\varphi^{(1)}$ such that

(i) $\varphi^{(1)}$ is singular with respect to $\psi^{(1)}$,

(ii) the series $\sum_{k=1}^{\infty}\left(\int_R p_k\,d\psi\right)^2$ is convergent where $\{p_k\}_{k=1}^{\infty}$ is $\varphi^{(1)}$-orthonormal base of Π_1^∞.

The condition 1° is obvious. Thus we prove the sufficiency of the condition 2°. Due to the comparison criterion of orthogonality, (ii) implies that there exists a function ϱ from $L_2(R,\psi^{(1)})$ such that $\int_R p\,d\varphi = \int_R p\varrho\,d\psi$ for any $p\in\Pi_1^\infty$. Suppose by contradiction that the assumption of Theorem 3 holds for ψ. This implies that ϱ is non-negative. Since the measure $\varphi^{(1)}$ is determinate we get $d\varphi=\varrho\,d\psi$ which contradicts (i).

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STRUCTURAL THEOREMS FOR MULTIPLICATIVE SYSTEMS OF FUNCTIONS

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§ 1. Introduction. The purpose of this paper is to prove some structural theorems for multiplicative systems and then a generalization of some basic theorems for independent random variables on uniformly bounded multiplicative systems. Such generalizations were obtained by many authors, but the method of this paper is simpler and gives, in general, stronger results. § 2 contains the definitions of multiplicative systems of different types. At the beginning of § 3 we prove a theorem, which shows that every multiplicative system, which satisfies some natural conditions can be represented as a conditional expectation of a strongly multiplicative system with respect to some σ-algebra. This explains why multiplicative systems have the structure similar to the structure of the Rademacher system, in particular it explains similar properties of lacunary trygonometric systems and the Rademacher system. At the end of that section there is a theorem which compares a uniformly bounded weakly multiplicative system with the Rademacher system. In § 4 various applications of the results from § 3 are considered. Convergence almost surely, moment inequalities, strong laws of large numbers and laws of the iterated logarithm are investigated.

§ 2. Definitions. DEFINITION 1. A sequence (X_i) of random variables is called a multiplicative system (MS system), if $EX_{i_1}X_{i_2}\dots X_{i_n}=0$ for every sequence of different indices. If moreover $EX_{i_0}^2X_{i_1}\dots X_{i_n}=0$ for every sequence of different indices, (X_i) is called a strongly multiplicative system (SMS system). If $EX_{i_1}X_{i_2}\dots X_{i_m}=0$ for every $m\leqq k$ and every sequence of indices $i_1<\dots<i_m$ then the sequence (X_i) is called an MS system of order k.

The definition of an MS system was introduced by Alexits [1] (MS system of order k by Serfling [22]). The definition of an SMS system [12] is more general than usual, for instance in [1]. The MS and SMS systems were studied by Alexits [1, 2], Alexits and Sharma [5], Móricz [15], Révész [19, 20], Takahashi [25], Jakubowski and Kwapień [12].

EXAMPLE 1. A sequence of bounded martingale differences is an MS system.

EXAMPLE 2. A sequence (X_i) of independent random variables in L^2 with expected value equal to zero is an SMS system.

EXAMPLE 3 [26]. A sequence of functions $X_k(t)=\sin 2\pi n_k t$ (resp. $Y_k(t)=\cos 2\pi n_k t$) for $t\in[0, 1]$ and $k=1, 2, \dots$ is an MS system for every sequence (n_k) of positive integers such that $\frac{n_{k+1}}{n_k}\geqq 2$, moreover it is an SMS system provided $\frac{n_{k+1}}{n_k}\geqq 3$.

DEFINITION 2. A sequence (X_i) of random variables is called a p-weakly multiplicative system (p—WMS system) for $p\in[1, 2]$, if $\sum |EX_{i_1}X_{i_2}\dots X_{i_k}|^p<\infty$, where the sum is taken over all combinations of indices.

Alexits, who introduced the notion of a WMS system [3] which coincides with our 1—WMS system, noticed [4] that every infinite sequence of orthonormal functions contains an infinite subsequence, which is a 1—WMS system (for an MS system this is not true [7, 8]).

The following idea is of great importance in this paper.

DEFINITION 3. Let X and Y be random variables with values in a Banach space F. We say that X is strongly dominated by Y if $Ef(X)\leqq Ef(Y)$ for every continuous, convex and nonnegative function $f\colon F\to R^+$, provided both integrals exist.

The idea of this domination is connected with a conditional expectation with respect to some σ-algebra in the following way:

THEOREM A (Hoffmann—Jørgensen [11]). *Let X, Y be Banach space valued random variables with finite first moments. Then X is strongly dominated by Y iff there exist random variables X' and Y' with distributions like X and Y respectively and a σ-algebra $\mathcal{N}$ such that $X'=E(Y'|\mathcal{N})$.*

We introduce also the definition of p-domination.

DEFINITION 4. Let X, Y be random variables with values in R^∞. Then X is p-dominated, $p\geqq 1$, by Y iff there exist constants K and C_p such that for every continuous, convex, non-negative function $f\colon R^\infty\to R^+$, which depends only on a finite number of coordinates,

$$Ef(X)\leqq C_p[Ef^p(KY)]^{1/p}$$

holds.

It is important for the applications that, if X is dominated by Y in the sense of Definition 3 or 4, then many properties which are true for Y are also true for X. This fact will be applied in § 4.

In this paper (r_n) will denote the Rademacher system on the interval [0, 1].

§ 3. Structural theorems. Let us start from a generalization of the theor m from [12].

THEOREM 1. *Let $X_1, \dots, X_n$ be an MS system of order $k\leqq n$ and $Y_1, \dots, Y_n$ be an SMS system. If $0<D^2 Y_i<\infty$ and $\sup|X_iY_i|\leqq\frac{k}{n}D^2Y_i$ for $i=1, 2, \dots, n$, then $(X_1, \dots, X_n)$ is strongly dominated by $(Y_1, \dots, Y_n)$.*

PROOF. Let Q be a distribution of $(Y_1, \dots, Y_n)$ and $\bar{P}$ be a distribution of $(X_1, \dots, X_n)$. Define a probability measure P as a distribution of vector $\left(\frac{n}{k}X_1, \dots\right.$ $\left.\dots, \frac{n}{k}X_n\right)$. Let $(\Omega, \mathcal{M}, M)$ be a measure space, where $\Omega=R^n\times R^n$, $\mathcal{M}=\mathcal{B}(R^n\times R^n)$ is a σ-algebra of Borel sets and the measure M is defined by $M(dx, dy)=f(x, y)\times$ $\times P(dx)\,Q(dy)$ where

$$f(x, y)=\frac{1}{\binom{n}{k}}\Big[\sum_{i_1<\dots<i_k}\prod_{m=1}^{k}(1+b_{i_m}x_{i_m}y_{i_m})\Big],\quad b_i=\frac{1}{D^2Y_i}.$$

M is a positive measure, since the support of the measure $\bar{P}(dx)Q(dy)$ is contained in the set $\left\{(x, y)\colon |x_i y_i|\, b_i \leqq \frac{k}{n},\ i=1, 2, \dots, n\right\}$ so the support of the measure $P(dx)Q(dy)$ is contained in the set $\{(x, y)\colon |x_i y_i| b_i \leqq 1,\ i=1, 2, \dots, n\}$. The condition that $X_1, \dots, X_n$ is an MS system of order k implies that

$$\forall y \in R^n, \quad \int_{R^n} f(x, y) P(dx) = 1. \tag{1}$$

This follows from

$$\int_{R^n} \binom{n}{k}^{-1} \Big[\sum_{i_1 < \dots < i_k} \prod_{m=1}^{k} (1 + b_{i_m} x_{i_m} y_{i_m})\Big] P(dx) =$$

$$= \binom{n}{k}^{-1} \sum_{i_1 < \dots < i_k} \int_{R^n} \Big[1 + \sum_{l=1}^{k} \sum_{\substack{i_1 < \dots < i_l \\ \{i_1, \dots, i_l\} \subset \{i_1, \dots, i_k\}}} b_{i_1} x_{i_1} y_{i_1} \dots b_{i_l} x_{i_l} y_{i_l}\Big] P(dx) =$$

$$= \binom{n}{k}^{-1} \binom{n}{k} = 1.$$

Hence M is a probability measure.

Since $Y_1, \dots, Y_n$ is an SMS system, we also have

$$\forall x \in R^n, \quad \int_{R^n} f(x, y) Q(dy) = 1. \tag{2}$$

Let us define the random vectors X and Y on the probability space $(\Omega, \mathcal{M}, P)$ as $X(x, y) = x$, $Y(x, y) = y$. Let A be an arbitrary Borel subset of R^n, then by (1), (2), we obtain

$$M(X \in A) = M(A \times R^n) = \int_A \int_{R^n} f(x, y) Q(dy) P(dx) = \int_A 1 P(dx) = P(A).$$

Analogously $M(Y \in A) = Q(A)$. This means that the distribution of the random vector X (resp. Y) is equal to P (resp. Q). Every point $x = (x_1, \dots, x_n) \in R^n$ satisfies the relations

$$\int_{R^n} y f(x, y) Q(dy) = \frac{k}{n} x. \tag{3}$$

This is a consequence of the relations (for $i_0 = 1, 2, \dots, n$)

$$\int_{R^n} y_{i_0} f(x, y) Q(dy) = \int_{R^n} \binom{n}{k}^{-1} \Big[\sum_{\substack{i_1 < \dots < i_k \\ i_0 \in \{i_1, \dots, i_k\}}} y_{i_0} \prod_{m=1}^{k} (1 + b_{i_m} x_{i_m} y_{i_m})\Big] Q(dy) +$$

$$+ \int_{R^n} \binom{n}{k}^{-1} \Big[\sum_{\substack{i_1 < \dots < i_k \\ i_0 \notin \{i_1, \dots, i_k\}}} y_{i_0} \prod_{m=1}^{k} (1 + b_{i_m} x_{i_m} y_{i_m})\Big] Q(dy) =$$

$$= \binom{n}{k}^{-1} \int_{R^n} \Big[\sum_{\substack{i_1 < \dots < i_k \\ i_0 \in \{i_1, \dots, i_k\}}} \Big(y_{i_0} + y_{i_0} \sum_{i_m \neq i_0} b_{i_m} x_{i_m} y_{i_m} + b_{i_0} x_{i_0} y_{i_0}^2 +$$

$$+y_{i_0}\sum_{l=2}^{k}\sum_{\substack{j_1<\dots<j_l\\ \{j_1,\dots,j_l\}\subset\{i_1,\dots,i_k\}}} b_{j_1}x_{j_1}y_{j_1}\dots b_{j_l}x_{j_l}y_{j_l})]\,Q(dy)=$$

$$=\binom{n}{k}^{-1}\binom{n-1}{k-1}b_{i_0}x_{i_0}\int_{R^n}y_{i_0}^2Q(dy)=\frac{k}{n}x_{i_0}.$$

Let $\mathcal{N}$ be a σ-algebra generated by the random vector X. Every set from $\mathcal{N}$ has a form $A\times R^n$, where $A\in\mathcal{B}(R^n)$. By (1), (3) we get

$$\int_{A\times R^n}Y\,dM=\int_{A\times R^n}yM(dx,dy)=\int_A\int_{R^n}yf(x,y)Q(dy)P(dx)=$$

$$=\int_A\frac{k}{n}xP(dx)=\int_A\Big(\int_{R^n}f(x,y)Q(dy)\Big)\frac{k}{n}xP(dx)=$$

$$=\int_{A\times R^n}\frac{k}{n}xM(dx,dy)=\int_{A\times R^n}\frac{k}{n}X\,dM,$$

i.e. $E(Y/\mathcal{N})=\frac{k}{n}X$. This ends the proof, because $\frac{k}{n}X$ has the distribution $\bar{P}$.

For $k=n$ we obtain the basic corollary.

COROLLARY 1 [12]. *Let* $X_1,\dots,X_n$ *be an MS system and* $Y_1,\dots,Y_n$ *be an SMS system. If, for every* i, $0<D^2Y_i<\infty$ *and* $\sup|Y_i|\leqq\frac{D^2Y_i}{\sup|Y_i|}$, *then* $(X_1,\dots,X_n)$ *is strongly dominated by* $(Y_1,\dots,Y_n)$.

For $k=1$ we get

COROLLARY 2. *Let* $X_1,\dots,X_n$ *be random variables such that* $EX_i=0$ *and* $\sup|X_i|\leqq\frac{1}{n}$ *for every* $i=1,2,\dots,n$. *Then* $(X_1,\dots,X_n)$ *is strongly dominated by* $(r_1,\dots,r_n)$.

The assumptions of Corollary 2 can be weakened if $n\geqq 8$.

THEOREM 2. *Let* $X=(X_1,\dots,X_n)$ *be a random vector with the expected value equal to zero, such that* $\sum_{i=1}^n X_i^2\leqq\frac{1}{8}$ *almost surely. Then* $(X_1,\dots,X_n)$ *is strongly dominated by* $(r_1,\dots,r_n)$.

PROOF. At the beginning let us assume that X has the two-point distribution $P(X=a)=P(X=-a)=\frac{1}{2}$, where $a=(a_1,\dots,a_n)$ and $\sum_{i=1}^n a_i^2\leqq\frac{1}{2}$. Let $f\colon R^n\to R^+$ be a convex, nonnegative function and define $g(x)=\frac{1}{2}[f(x)+f(-x)]$. Then g is a convex, nonnegative and symmetric function; moreover $Ef(X)=\frac{1}{2}[f(a)+f(-a)]=$

$=Eg(X)$ and $Ef(r_1, \dots, r_n)=Eg(r_1, \dots, r_n)$. So it is enough to prove that

(4) $$Eg(X_1, \dots, X_n) \leqq Eg(r_1, \dots, r_n)$$

for a convex, symmetric, nonnegative function g. For such a g there exists a supporting functional at the point a, i.e. there exist real numbers $b_1, b_2, \dots, b_n, c$ such that $g(a)=\sum_{i=1}^n b_i a_i+c$ and $g(x)\geqq \sum_{i=1}^n b_i x_i+c$. So it is enough to show (4) for a function h of the form $h(x)=g(x)-c$. Then $h(x)\geqq \left|\sum_{i=1}^n b_i x_i\right|$, $h(a)=\sum_{i=1}^n b_i a_i$ and $Eh(X)=h(a)$. Using the Khintchine inequality with the best constant [23] we obtain

$$Eh(X) = h(a) = \sum_{i=1}^n b_i a_i \leqq \Big(\sum_{i=1}^n a_i^2\Big)^{1/2}\Big(\sum_{i=1}^n b_i^2\Big)^{1/2} \leqq$$

$$\leqq \frac{1}{\sqrt{2}}\Big(\sum_{i=1}^n b_i^2\Big)^{1/2} \leqq E\left|\sum_{i=1}^n b_i r_i\right| \leqq Eh(r_1, \dots, r_n).$$

Now we prove the theorem for an arbitrary symmetric random vector such that $\sum_{i=1}^n X_i^2 \leqq \frac{1}{2}$ almost surely. Let A denote a set of symmetric, probability measures with support contained in the ball $\left\{x\colon \sum_{i=1}^n x_i^2 \leqq \frac{1}{2}\right\}$. A is a convex, closed subset of the set of all probability measures on the cube $[-1, 1]^n$. Extreme points of the set A are symmetric probability measures concentrated in two points. By the Choquet theorem [14] the set A has the Choquet property, i.e. for every measure μ, which belongs to A, there exists a probability measure P concentrated on extreme points of A such that $\mu=\int_{\mathrm{Ex}A} mP(dm)$. This ends the first part of the proof because it implies that for a nonnegative, convex function $f\colon R^n \to R^+$,

$$\int_{R^n} f(x)\mu(dx) = \int_{R^n} f(x) \int_{\mathrm{Ex}A} m(dx)P(dm) =$$

$$= \int_{\mathrm{Ex}A}\Big(\int_{R^n} f(x)m(dx)\Big)P(dm) \leqq \int_{\mathrm{Ex}A}\Big(\int_{R^n} f(x)Q(dx)\Big)P(dm) = \int_{R^n} f(x)Q(dx)$$

holds; where Q denotes a distribution of $(r_1, \dots, r_n)$. The distribution of the symmetric random vector X belongs to A, so X is strongly dominated by $(r_1, \dots, r_n)$.

Now we consider the case of an arbitrary random vector, which fulfils the assumptions of the theorem. For a convex, nonnegative function $f\colon R^n \to R^+$, there exist real numbers $b_1, \dots, b_n$ such that $f(x)\geqq \sum_{i=1}^n b_i x_i+c$ where $c=f(0)\geqq 0$. Hence $Ef(X)\geqq c$ and $Ef(-X)\geqq c$, because the expected value of X is equal to zero. The convexity of f implies $f(x)\leqq \frac{1}{2}[f(2x)+f(0)]$ and therefore

$$E[f(X)-c] \leqq \max\big(E[f(X)-c], E[f(-X)-c]\big) \leqq$$

$$\leqq \frac{1}{2}E[f(2X)+f(-2X)]-c \leqq Ef(r_1, \dots, r_n)-c$$

if $\sum_{i=1}^{n}(2X_i)^2 \leqq \frac{1}{2}$ almost surely. The last inequality is true, because the random vector $(2\bar{r}_1X_1, \ldots, 2\bar{r}_nX_n)$, where $(\bar{r}_1, \ldots, \bar{r}_n)$ is equidistributed with $(r_1, \ldots, r_n)$ and independent of X, is symmetric, so

$$\frac{1}{2}E[f(2X)+f(-2X)] = Ef(2\bar{r}_1X_1, \ldots, 2\bar{r}_nX_n) \leqq Ef(r_1, \ldots, r_n)$$

if $\sum_{i=1}^{n}(2\bar{r}_iX_i)^2 \leqq \frac{1}{2}$. This gives the desired result: $Ef(X_1, \ldots, X_n) \leqq Ef(r_1, \ldots, r_n)$.

EXAMPLE 4. For the symmetric case, the constant obtained in the proof of Theorem 2 is the best (for all n).

Let X be a random vector with distribution $P(X=(a, a))=P(X=(-a, -a))=\frac{1}{2}$, $a \in R$, and let $g: R^2 \to R^+$ be a function defined by $g(x_1, x_2)=\max(x_1+x_2, 0)$. Then $Eg(r_1, r_2)=\frac{1}{2}$ and $Eg(X_1, X_2)=|a|$ which implies for $a=\frac{1}{2}$: $Eg(r_1, r_2)= =Eg(X_1, X_2)$ and $X_1^2+X_2^2=\frac{1}{2}$ almost surely.

REMARK 1. Let X be a symmetric random vector with the distribution concentrated in two points $a=(a_1, a_2, \ldots, a_n)$ and $-a$. The relation $\sum_{i=1}^{n} a_i^2 \leqq 1$ is a necessary condition for the strong domination X by $(r_1, \ldots, r_n)$.

PROOF. If $C=\sum_{i=1}^{n} a_i^2 > 1$, then for a convex function f: $R^n \to R^+$ defined by $f(x)=C^{-1}\left|\sum_{i=1}^{n} a_ix_i\right|$ we obtain $Ef(X_1, \ldots, X_n)=1$ and

$$Ef(r_1, \ldots, r_n) = C^{-1}E\left|\sum_{i=1}^{n} a_ir_i\right| \leqq C^{-1}\left[E\left(\sum_{i=1}^{n} a_ir_i\right)^2\right]^{1/2} = C^{-1/2} < 1.$$

This means that X can not be strongly dominated by $(r_1, \ldots, r_n)$.

Even a small perturbation of an MS system can induce such a change of structure, that this system is not strongly dominated by $(r_1, \ldots, r_n)$.

EXAMPLE 5. Let $X=(X_1, \ldots, X_n)$ be a random vector with distribution

$$P_x(e_1, \ldots, e_n) = 2^{-n}\left[1+a\left(\prod_{i=1}^{n} e_i\right)\right]$$

where $e_i=\pm 1$, $i=1, 2, \ldots, n$ and $0<a<1$. Let us take an arbitrary nonnegative, measurable function f: $R^n \to R^+$. Then

$$Ef(X_1, \ldots, X_n) = \sum_{(e_1, \ldots, e_n)} f(e_1, \ldots, e_n)2^{-n}\left[1+a\left(\prod_{i=1}^{n} e_i\right)\right] =$$

$$= E\left\{f(r_1, \ldots, r_n)\left[1+a\left(\prod_{i=1}^{n} r_i\right)\right]\right\} = Ef(r_1, \ldots, r_n)+aE\left[f(r_1, \ldots, r_n)\prod_{i=1}^{n} r_i\right].$$

This implies that (X_i) is an MS system of order $n-1$ but since $EX_1 \dots X_n = a$ it is not an MS system. For the convex, nonnegative function $f\colon R^n \to R^+$ defined by

$$f(x_1, \dots, x_n) = \max\Big(0, \sum_{i=1}^{n} x_i - n + 1/2\Big)$$

we get

$$E\Big[f(r_1, \dots, r_n) \prod_{i=1}^{n} r_i\Big] = 2^{-(n+1)}$$

so $Ef(X_1, \dots, X_n) > Ef(r_1, \dots, r_n)$.

This means that X can not be strongly dominated by $(r_1, \dots, r_n)$. It is possible, however, to obtain weaker forms of domination for p-WMS systems.

THEOREM 3. *Let (X_i) be a uniformly bounded p-WMS system, $1 \leqq p \leqq 2$. Then (X_i) is p-dominated by the Rademacher system.*

Theorem 3 is an immediate consequence of the definition of a p-WMS system and the following lemma.

LEMMA 1. *Let $f\colon R^n \to R^+$ be a nonnegative convex function and $X_1, \dots, X_n$ be arbitrary random variables uniformly bounded by one. Then*

$$Ef(X_1, \dots, X_n) \leqq C_p[Ef^p(r_1, \dots, r_n)]^{1/p}$$

where

$$C_p = \Big[1 + \sum_{k=1}^{n} \sum_{i_1 < \dots < i_k} |EX_{i_1} X_{i_2} \dots X_{i_k}|^p\Big]^{1/p}, \quad 1 \leqq p \leqq 2.$$

PROOF. Every point $x = (x_1, \dots, x_n)$, $|x_i| \leqq 1$ of the n-dimensional cube may be written as the convex combination

$$(x_1, \dots, x_n) = \sum_{\substack{(e_1, \dots, e_n) \\ e_i = \pm 1}} \prod_{i=1}^{n} \Big(\frac{1 + x_i e_i}{2}\Big)(e_1, \dots, e_n) \tag{5}$$

of vertices $(e_1, \dots, e_n)$. Since $E\big[\prod_{i=1}^{n}(1 + x_i r_i)\big] = 1$ the sum of coefficients of this combination is equal to 1. For every i, $i = 1, 2, \dots, n$ we have

$$E\Big[r_i \prod_{j=1}^{n}(1 + x_j r_j)\Big] = E[r_i(1 + x_i r_i)] \prod_{\substack{j=1 \\ j \neq i}}^{n} E(1 + x_j r_j) = x_i.$$

So (5) is true.

Let P be a distribution of $(X_1, \dots, X_n)$. Since f is nonnegative and convex, using (5), we obtain for $p=1$.

$$\int_{R^n} f(x_1, \dots, x_n) P(dx) \leqq \sum_{\substack{(e_1, \dots, e_n) \\ e_i = \pm 1}} f(e_1, \dots, e_n) \int_{R^n} \prod_{i=1}^{n} \Big(\frac{1 + x_i e_i}{2}\Big) P(dx) =$$

$$= \sum_{\substack{(e_1, \dots, e_n) \\ e_i = \pm 1}} f(e_1, \dots, e_n) \frac{1}{2^n}\Big[1 + \sum_{k=1}^{n} \sum_{i_1 < \dots < i_k} \Big(\prod_{l=1}^{k} e_{i_l}\Big) E(X_{i_1} X_{i_2} \dots X_{i_k})\Big] \leqq$$

$$\leqq C_1 Ef(r_1, \dots, r_n).$$

Now we consider the case $p>1$. Let $w_{i_1\dots i_k}(t)=r_{i_1}(t)\dots r_{i_k}(t)$ be the Walsh system on [0, 1]. Using a part of the above estimations and Hölder inequality we have:

$$Ef(X_1, \dots, X_n) \leqq E\{f(r_1, \dots, r_n)[1+\sum_{k=1}^{n} \sum_{i_1<\dots<i_k} E(X_{i_1}\dots X_{i_k})w_{i_1\dots i_k}]\} \leqq$$

$$\leqq [Ef^p(r_1, \dots, r_n)]^{1/p}\{E[1+\sum_{k=1}^{n} \sum_{i_1<\dots<i_k} E(X_{i_1}\dots X_{i_k})w_{i_1\dots i_k}]^q\}^{1/q}.$$

For $p=2$ we calculate the second component of the product and obtain that it is equal to C_2. For $1<p<2$ we use the Riesz—Thorin interpolation theorem [26].

Since, for Walsh function w_α, $E[a_0+\sum_{\alpha\in I} a_\alpha w_\alpha]^2 \leqq a_0^2+\sum_{\alpha\in I} a^2$ and $\sup_t |a_0+ + \sum_{\alpha\in I} a_\alpha w_\alpha(t)| \leqq |a_0|+\sum_{\alpha\in I}|a_\alpha|$, then $[E(1+\sum_{\alpha\in I} a_\alpha w_\alpha)^q]^{1/q} \leqq (1+\sum_{\alpha\in I}|a_\alpha|^p)^{1/p}$, where $\frac{1}{p}+ +\frac{1}{q}+=1$. This finishes the proof of Lemma 1.

It is possible to obtain an analogous theorem for weakly multiplicative systems defined by Révész [21].

THEOREM 4. *Let (X_i) be a uniformly bounded sequence of random variables for which there exists a constant L such that*

$$E\prod_{i=1}^{n}(1+a_iX_i) \leqq L \quad (n=1, 2, \dots),$$

where (a_i) is an arbitrary sequence of real numbers such that $|a_i|\leqq 1$. Then (X_i) is 1-dominated by (r_i).

PROOF. An estimation like in Lemma 1 for $p=1$ is valid for such a system with the constant L instead of C_1, independent of n. (We make an evident change in the proof.)

This theorem shows that the same results which are obtained in § 4 for a 1-WMS system can be proved for a weakly multiplicative system defined by Révész.

§ 4. Properties of p-WMS systems ($1\leqq p\leqq 2$). In this part we use a notation

$$S_n=\sum_{i=1}^{n} a_iX_i, \quad A_n=(\sum_{i=1}^{n}|a_i|^2)^{1/2}$$

for a sequence (a_i) of real or complex numbers and (X_n) a p-WMS system, $1\leqq p\leqq 2$. The constants $C_p=(1+\sum |EX_{i_1}X_{i_2}\dots X_{i_k}|^p)^{1/p}$, where the sum is taken over all combinations of indices, are always constants for p-WMS systems connected with p-domination.

At first we consider the convergence problem and inequalities for moments.

THEOREM 5. *Let (u_n), $n=1, 2, \dots$, be a sequence of vectors in a Banach space and let (X_n) be a uniformly bounded 2-WMS system. If the series $\sum_{n=1}^{\infty} u_n r_n$ is con-*

vergent almost surely, then the series $\sum_{n=1}^{\infty} u_n X_n$ *is convergent almost surely and* $E\exp(\varepsilon\|\sum_{n=1}^{\infty} u_n X_n\|^2)<\infty$ *for every* $\varepsilon>0$.

PROOF. The method of proof is the same as in [12]. Without loss of generality we may assume that (X_n) is uniformly bounded by one. It is enough to show that

$$P(\max_{n\leq k\leq m}\|\sum_{i=n}^{k} u_i X_i\| > \delta)\xrightarrow[n,m\to\infty]{} 0,$$

for every $\delta>0$. Let us apply Theorem 3. If we take the convex function $f: R^n\to R^+$ defined by

$$f(x_1, \dots, x_n) = \max_{1\leq k\leq n}\|\sum_{i=1}^{k} v_i x_i\|$$

with properly chosen n and vectors $v_1, \dots, v_n$ we obtain

$$E(\max_{n\leq k\leq m}\|\sum_{i=n}^{k} u_i X_i\|) \leq C_2(E\max_{n\leq k\leq m}\|\sum_{i=n}^{k} u_i r_i\|^2)^{1/2}$$

for every $m>n$. If the series $\sum_{n=1}^{\infty} u_n r_n$ is convergent almost surely, then it is convergent in L^2 and [10],

$$E(\max_{n\leq k\leq m}\|\sum_{i=n}^{k} u_i r_i\|^2)\xrightarrow[n,m\to\infty]{} \infty.$$

Hence

$$E(\max_{n\leq k\leq m}\|\sum_{i=n}^{k} u_i X_i\|)\xrightarrow[n,m\to\infty]{} 0$$

which implies the desired result. Analogously, applying Theorem 3 to the convex function $f(x_1, \dots, x_n)=\exp(\varepsilon\|\sum_{i=1}^{n} u_i x_i\|^2)$, we have

$$E\exp(\varepsilon\|\sum_{i=1}^{n} u_i X_i\|^2) \leq C_2[E\exp(2\varepsilon\|\sum_{i=1}^{n} u_i r_i\|^2)]^{1/2}.$$

Now the statement of Theorem 5 is a consequence of a result of Kwapien [13]: if $\sum_{n=1}^{\infty} u_n r_n$ is convergent almost surely, then $E\exp(\varepsilon\|\sum_{n=1}^{\infty} u_n r_n\|^2)<\infty$ for every $\varepsilon>0$.

This theorem generalizes a result on the convergence of 1-WMS systems and real numbers proved by Alexits [4] and for 2-WMS systems by Móricz [16].

THEOREM 6. *Let* (X_n) *be a p-WMS system uniformly bounded by* $K\geq 1$ *and* (a_n) *be a sequence of complex numbers. Then, for every* $q\in[1, \infty)$, *there exists a constant* B_q *depending only on* C_p, K, q *such that for all* n, $E\max_{k\leq n}|S_k|^q\leq B_q A_n^q$.

PROOF. It is known [10] that for every $q\in[1, \infty)$,

$$E\max_{k\leqq n}\Big|\sum_{i=1}^{k} a_i r_i\Big|^q \leqq 2E\Big|\sum_{i=1}^{n} a_i r_i\Big|^q.$$

Hence by Theorem 3 and the Khnitchine inequality

$$E\max_{k\leqq n}|S_k|^q \leqq C_p K^q \big[E\big(\max_{k\leqq n}\big|\sum_{i=1}^{k} a_i r_i\big|^{pq}\big)\big]^{1/p} \leqq C_p K^q (2DA_n^{pq})^{1/p} = B_q A_n^q$$

where D is the constant from the Khintchine inequality.

REMARK 2. Under the above assumptions, obviously $E|S_n|^q \leqq B_q A_n^q$.

THEOREM 7. *Let (X_n) be a p-WMS system uniformly bounded by $K\geqq 1$ and let (a_n) be a sequence of real or complex numbers. Then, for every $\lambda>0$, the estimations*

(i) $$E\exp(\lambda|S_n|) \leqq 2C_p \exp\left(\frac{1}{2}pl\lambda^2K^2A_n^2\right)$$

and

(ii) $$E\exp(\lambda\max_{k\leqq n}|S_k|) \leqq 4^{1/p}C_p \exp\left(\frac{1}{2}pl\lambda^2K^2A_n^2\right)$$

hold, where $l=1$ or 2 for real or complex numbers (a_n), respectively.

PROOF. Let (X_n) be a p-WMS system and $a_n\in R$. Then using Theorem 3 we obtain, for $\lambda\in R$

$$[E\exp(\lambda S_n)]^p \leqq C_p^p E\exp\big(p\lambda K\sum_{k=1}^{n} a_k r_k\big) =$$

$$= C_p^p \prod_{k=1}^{n} E\exp(p\lambda K a_k r_k) = C_p^p \frac{1}{2^n}\prod_{k=1}^{n}[\exp(\lambda pKa_k)+$$

$$+\exp(-\lambda pKa_k)] \leqq C_p^p \prod_{k=1}^{n}\exp\left(\frac{1}{2}p^2\lambda^2K^2a_k^2\right) = C_p^p\exp\left(\frac{1}{2}p^2\lambda^2K^2A_n^2\right).$$

For obtaining the estimate with absolute value, we apply the inequality

$$E\exp(\lambda|S_n|) \leqq E\exp(\lambda S_n)+E\exp(-\lambda S_n).$$

Now let us take the complex case. Using the inequality

(6) $$\exp(|b+ic|) \leqq \frac{1}{2}\big[\exp(\sqrt{2}\,|b|)+\exp(\sqrt{2}\,|c|)\big], \quad b, c\in R$$

and representing a_n as a real combination $a_n=b_n+ic_n, b_n, c_n\in R$, from the proof for the real case, we get the desired result.

For the Rademacher system we have

$$E\exp(\lambda\max_{k\leq n}|\sum_{i=1}^{k}a_ir_i|)=\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}E(\max_{k\leq n}|\sum_{i=1}^{k}a_ir_i|^m)\leq$$

$$\leq 2\sum_{m=0}^{\infty}\frac{\lambda^m}{m!}E(|\sum_{i=1}^{n}a_ir_i|^m)=2E(\exp(\lambda|\sum_{i=1}^{n}a_ir_i|)).$$

If (a_n) is a sequence of real numbers, then by the above, Theorem 3 and inequality (i) for the Rademacher system, for a p-WMS system we get

$$E\exp(\lambda\max_{k\leq n}|S_k|)\leq C_p[E\exp(p\lambda K\max_{k\leq n}|\sum_{i=1}^{k}a_ir_i|)]^{1/p}\leq$$

$$\leq C_p[2E\exp(p\lambda K|\sum_{i=1}^{n}a_ir_i|)]^{1/p}\leq 4^{1/p}C_p\exp\left(\frac{1}{2}p\lambda^2K^2A_n^2\right).$$

The complex case is obtained by (6), like in (i). The proof is complete.

Part (i) of Theorem 7 was proved for MS systems by Azuma [6] and for 1-WMS systems Révész [21] adopted the proof of Azuma. Theorem 7 allows to find estimations, useful for the applications, for probabilities of type $P(|S_n|\geq y)$ and $P(\max_{k\leq n}|S_k|>y)$.

COROLLARY 4. *If* (X_n) *is a p-WMS system uniformly bounded by* $K\geq 1$ *and* (a_n) *is a sequence of real or complex numbers, then for every* $y>0$,

$$P(|S_n|>y)\leq 2C_p\exp\left(-\frac{y^2}{2plK^2A_n^2}\right),$$

and

$$P(\max_{k\leq n}|S_k|>y)\leq 4^{1/p}C_p\exp\left(-\frac{y^2}{2plK^2A_n^2}\right),$$

where $l=1$ *or* 2 *for real or complex numbers* (a_n), *respectively.*

PROOF. By Theorem 7 we have

$$P(|S_n|>y)\leq\frac{E\exp(\lambda|S_n|)}{\exp(\lambda y)}\leq 2C_p\exp\left(\frac{1}{2}pl\lambda^2K^2A_n^2-\lambda y\right)$$

for arbitrary $\lambda>0$. The right side has a minimum at the point $\lambda=\frac{y^2}{plK^2A_n^2}$, equal to $2C_p\exp\left(-\frac{y^2}{2plK^2A_n^2}\right)$. The proof of the second inequality is analogous.

For 1-WMS systems we have the following generalization of a result of Zygmund [26] obtained for the Rademacher system:

THEOREM 8. *Let* (X_n) *be a 1-WMS system uniformly bounded by* $K\geq 1$ *and* (a_n) *be a sequence of complex numbers such that* $A=\lim_{n\to\infty}A_n<\infty$. *Then there exist* *constants B, B' depending only on C_1 and K such that*

$$E\Big(\Big|\sum_{n=1}^{\infty} a_n X_n\Big| \ln^+ \Big|\sum_{n=1}^{\infty} a_n X_n\Big|\Big) \leqq BA \ln^+ A + B'.$$

PROOF. Our assertion follows from Theorem 3 applied to the convex function $f(x)=|x| \ln^+|x|$ and the result of Zygmund [26, XV. 5. 14]. If (a_n) are real numbers then

$$E\Big(\Big|\sum_{n=1}^{\infty} a_n X_n\Big| \ln^+ \Big|\sum_{n=1}^{\infty} a_n X_n\Big|\Big) \leqq C_1 E\Big(\Big|K \sum_{n=1}^{\infty} a_n r_n\Big| \ln^+ \Big|K \sum_{n=1}^{\infty} a_n r_n\Big|\Big) \leqq$$

$$\leqq C_1\big(DKA \ln^+ (KA)+D'\big) \leqq 2C_1 DKA \ln^+ A + C_1 D' + C_1 DK^2 \ln^+ K = BA \ln^+ A + B'$$

where D, D' are constants from the theorem for the Rademacher system [26]. The complex case follows from the real case and from the inequalities

$$(7) \qquad (x^2+y^2)^{1/2} \ln^+ (x^2+y^2)^{1/2} \leqq 3\big[\sqrt{2}\,|x| \ln^+ (\sqrt{2}\,|x|)+$$

$$+\sqrt{2}\,|y| \ln^+(\sqrt{2}\,|y|)\big] \leqq 6(x^2+y^2)^{1/2} \ln^+ (x^2+y^2)^{1/2}.$$

For a uniformly bounded SMS system the result of Zygmund was generalized by Móricz [15]. For an SMS system we can obtain the converse inequality.

REMARK 6. Let (Y_n) be an SMS system uniformly bounded by K such that $\inf_n D^2 Y_n = L > 0$. If (a_n) is a sequence of complex numbers such that $\lim_n A_n = A < \infty$ then there exist constants $\bar{B}, \bar{B}'$ depending only on K and L such that

$$\bar{B} A \ln^+ A - \bar{B}' \leqq E\Big(\Big|\sum_{n=1}^{\infty} a_n Y_n\Big| \ln^+ \Big|\sum_{n=1}^{\infty} a_n Y_n\Big|\Big).$$

PROOF. By assumption $\dfrac{D^2 Y_n}{\sup|Y_n|} \geqq \dfrac{L}{K}$ for $n=1, 2, \ldots$.
Since (Y_n) is an SMS system, by Corollary 1 we get

$$E\Big(\Big|\sum_{n=1}^{\infty} a_n Y_n\Big| \ln^+ \Big|\sum_{n=1}^{\infty} a_n Y_n\Big|\Big) \leqq E\Big(\Big|K^{-1}L \sum_{n=1}^{\infty} a_n r_n\Big| \ln^+ \Big|K^{-1} \sum_{n=1}^{\infty} a_n r_n\Big|\Big).$$

Now by the mentioned result of Zygmund and the inequality $\ln^+(ax) \geqq$ $\geqq a \ln^+ x - a \ln \dfrac{1}{a}$ for $0<a<1$ we get the desired result for the real case. The complex case is obtained by using inequalities (7).

Such a generalization in a weaker form was proved by Móricz [15]. Now we consider the strong laws of large numbers. Theorem 5 allows us to generalize the result of Alexits and Sharma [5] for 1-WMS systems (with the same proof).

THEOREM 9. *Let (X_n) be a 2-WMS system such that $|X_n| \leqq M_n$ where M_n is not decreasing. Choosing two sequences of real numbers (a_n) and (q_n) satisfying the conditions $q_n>0$, $q_n<q_{n+1}\to\infty$ and $\sum_{n=1}^{\infty} \dfrac{a_n^2}{q_n^2} < \infty$ we have $\dfrac{S_n}{q_n M_n} \xrightarrow[n\to\infty]{} 0$ almost surely.*

It is possible to prove more.

THEOREM 10. *Let* (X_n) *be a uniformly bounded 2-WMS system and let* (a_n) *be a sequence of complex numbers such that* $A_n \xrightarrow[n\to\infty]{} \infty$. *Then, for every* $q>0$,

$$\frac{S_n}{A_n(\ln A_n)^q} \xrightarrow[n\to\infty]{} 0$$

holds almost surely.

PROOF. We put $A_0=0$. Since the A_n increase to infinity, we can find, for every $m>1$, a sequence of indices (n_k) tending to infinity and such that $A^2_{n_k-1} \leqq m^k < A^2_{n_k}$ $(n_k = 1+\max\{1:\ A_1^2 \leqq m^k\})$. Fix arbitrary $\varepsilon>0$ and denote by K a uniform bound for (X_n). We take $K\geqq 1$. By Corollary 4

$$q_k = P(|S_{n_k}| > \varepsilon A_{n_k}(\ln A_{n_k})^q) \leqq 2C_2 \exp\left(-\frac{\varepsilon^2(\ln A_{n_k})^{2q}}{8K^2}\right).$$

On account of $\exp[-(\ln x)^{2q}] \leqq [\exp(-\ln x)]^{2q} = x^{-2q}$ where $q \geqq \frac{1}{2}$, $x\geqq e$; for $0<q<\frac{1}{2}$ and arbitrary $b>0$, there exists a B such that $\exp[-b(\ln x)^{2q}] \leqq B(\ln x)^{-1-2q}$ whenever $x\geqq e$, we have $q_k \leqq B_1\left[\left(\frac{1}{m}\right)^{\frac{\varepsilon^2}{8K^2}}\right]^k$ for $q\geqq\frac{1}{2}$ and $q_k \leqq B_2\left(\frac{1}{k\ln m}\right)^{1+2q}$ for $0<q<\frac{1}{2}$. So $\sum_{k=1}^{\infty} q_k < \infty$ and according to the Borel—Cantelli lemma

$$\frac{S_{n_k}}{A_{n_k}(\ln A_{n_k})^q} \xrightarrow[k\to\infty]{} 0$$

almost surely.

If we take an arbitrary n different from the elements of the sequence (n_k), then there exists k such that, $n_k<n<n_{k+1}$ and

$$\frac{S_n}{T_n} = \frac{S_{n_k}}{T_{n_k}}\frac{T_{n_k}}{T_n} + \frac{S_n - S_{n_k}}{T_n}$$

where $T_n = A_n(\ln A_n)^q$. Put

$$T_k' = (\ln A_{n_k})^q (A^2_{n_{k+1}-1} - A^2_{n_k})^{1/2}.$$

Then

$$\frac{|S_n|}{T_n} \leqq \frac{|S_{n_k}|}{T_{n_k}} + \max_{n_k<l<n_{k+1}} \frac{|S_l - S_{n_k}|}{T_k'}\frac{T_k'}{T_n}.$$

Applying again Corollary 4 we obtain

$$p_k = P\big(\max_{n_k<l<n_{k+1}} |S_l - S_{n_k}| \geqq T_k'\big) \leqq 4C_2 \exp\left[-\frac{(\ln A_{n_k})^{2q}}{8K^2}\right].$$

Using the same argument as before, we have $\sum_{k=1}^{\infty} p_k < \infty$, so from the Borel—Cantelli lemma the relation

$$\limsup_{k\to\infty}\left(\max_{n_k<l<n_{k+1}} \frac{|S_l - S_{n_k}|}{T_k'}\right) \leqq 1$$

holds almost surely. If $n_{k+1}>n_k$, then

$$\left(\frac{T'_k}{T_n}\right)^2 \leqq \left(\frac{T'_k}{T_{n_k}}\right)^2 = \frac{A^2_{n_{k+1}-1}-A^2_{n_k}}{A^2_{n_k}} \leqq \frac{m^{k+1}-m^k}{m^k} = m-1.$$

m is an arbitrary number greater than one, so taking $m\to 1$ we obtain the desired result.

Under the assumption of Theorem 10, $\frac{S_n}{A_n^2}\to 0$ holds almost surely. We can ask about convergence behaviour. Modifying the method of [18, 23], we obtain

THEOREM 11. *Let* (X_n) *be a p-WMS system uniformly bounded by* $K\geqq 1$ *and let* (a_n) *be a sequence of real or complex numbers such that the series* $\sum_{n=1}^{\infty} q^{A_n^2}$ *is convergent for every* $q\in(0,1)$. *Then, for every* $\varepsilon>0$ *and every* $0\leqq q<\exp\left(\frac{\varepsilon^2}{2plK^2}\right)$, *we have*

$$\sum_{n=1} q^{A_n^2} P\left(\sup_{k\geqq n}\frac{|S_k|}{A_k^2}\geqq\varepsilon\right)<\infty,$$

where $l=1$ *or* 2 *for the real or complex case, resp.*

PROOF. If $q<\exp\left(\frac{\varepsilon^2}{2plK^2}\right)$, then there exists $m>1$ such that $q<\exp\left(\frac{\varepsilon^2}{2plK^2m^2}\right)$. Let (n_k) be a sequence of indices satisfying $A^2_{n_k-1}\leqq m^k<A^2_{n_k}$. Then we have

$$\frac{A^4_{n_k}}{A^2_{n_{k+1}-1}} \geqq \frac{m^{2k}}{m^{k+1}} = m^{k-1}.$$

For every fixed $n\in N$ there exists k_0 such that $n_{k_0}\leqq n<n_{k_0+1}$. This gives, by Corollary 4

$$P\left(\sup_{l\geqq n}\frac{|S_l|}{A_l^2}\geqq\varepsilon\right) \leqq \sum_{k=k_0}^{\infty} P\left(\max_{n_k\leqq l<n_{k+1}}\frac{|S_l|}{A_l^2}\geqq\varepsilon\right) \leqq 4C_p\sum_{k=k_0}^{\infty}\exp\left(-\frac{\varepsilon^2A^4_{n_k}}{2plK^2A^2_{n_{k+1}-1}}\right) \leqq$$

$$\leqq 4C_p\sum_{k=k_0}^{\infty}\exp\left(-\frac{\varepsilon^2}{2plK^2}m^{k_0-1}\right)m^{k-k_0} \leqq 4C_p\exp\left(-\frac{\varepsilon^2}{2plK^2}m^{k_0-1}\right)+$$

$$+4C_p\sum_{k=k_0+1}^{\infty}\exp\left(-\frac{\varepsilon^2}{2plK^2}m^{k_0-1}\right)^{(m-1)(k-k_0)} =$$

$$=4C_p\frac{\exp\left(-\frac{\varepsilon^2}{2plK^2}m^{k_0-1}\right)}{1-\exp\left(-\frac{\varepsilon^2}{2plK^2}m^{k_0-1}(m-1)\right)} = 4C_pD_{k_0}\exp\left(-\frac{\varepsilon^2m^{k_0-1}}{2plK^2}\right),$$

where D_k tends monotonically to one. It follows from $A_n^2 \leqq A_{n_{k_0+1}-1}^2 \leqq m^{k_0+1}$ that

$$\exp\left(-\frac{\varepsilon^2 A_n^2}{2plK^2m^2}\right) \geqq \exp\left(-\frac{\varepsilon^2 m^{k_0-1}}{2plK^2}\right).$$

This, on account of the assumption of the theorem, implies our assertion

$$\sum_{n=n_1}^{\infty} q^{A_n^2} P\left(\sup_{l\geqq n} \frac{|S_l|}{A_l^2} \geqq \varepsilon\right) \leqq 4C_pD_1 \sum_{n=n_1}^{\infty} q^{A_n^2} \exp\left(-\frac{\varepsilon^2 A_n^2}{2plK^2m^2}\right) < \infty.$$

Another estimation for the rate of convergence can be obtained by Theorem 7 from the results of Móricz [18].

COROLLARY 5. *Let* (X_n) *be a uniformly bounded p-WMS system and* (a_n) *be a sequence of real or complex numbers such that* $A_n \to \infty$, $|a_n| = o(A_n)$. *Then we have*

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 (\ln A_n)^\beta}{A_n^2} P\left(\sup_{k\geqq n} \frac{|S_k|}{A_k(\ln A_k)^\alpha} \geqq 1\right) < \infty$$

for each choice of $\alpha > 0$ *and* $\beta > 0$.

Now we consider the law of the iterated logarithm. It was investigated by Gaposkin [9], Takahaski [25], Révész [19, 20, 21], Móricz [17]. The result of Révész [21] can be generalized.

THEOREM 12. *Let* (X_n) *be a uniformly bounded 2-WMS system and let* (a_n) *be a sequence of real numbers for which* $A_n \to +\infty$ *and* $a_n = o\left(\frac{A_n}{(\ln\ln A_n)^{1/2}}\right)$ *hold. Suppose that, for every* $m > 1$, $\liminf \frac{T^2(M_k)}{A_{M_k}^2} > 0$ *almost surely, where* $T^2(n) = \sum_{k=1}^{n} a_k^2 X_k^2$, $M_k = M_k(m)$ *is such that* $A_{M_k-1} < m^k \leqq A_{M_k}^2$. *Then*

$$\limsup_{n\to\infty} \frac{S_n}{(2T^2(n)\ln\ln A_n)^{1/2}} \leqq 1$$

almost surely.

The proof is the same as in [21]. We use Theorem 2 from [20] and Corollary 4. Another generalization of the law of the iterated logarithm is

THEOREM 13. *Let* (X_n) *be a p-WMS system uniformly bounded by* $K \geqq 1$. *I* (a_n) *is a sequence of real or complex numbers such that* $A_n \to \infty$, *then* (X_n) *satisfief*

$$\limsup_{n\to\infty} \frac{|S_n|}{(2A_n^2 \ln\ln A_n)^{1/2}} \leqq \sqrt{pl}\, K$$

almost surely, where $l=1$ *or* 2 *in the real or complex case, resp.*

In fact, this is Theorem 3 of [18]. (By Theorem 7 we have needed estimations.)

Estimations obtained by Theorem 13 and Móricz' result [17] are incomparable. More precisely, there exist p-WMS systems for which a better estimate is given by Theorem 13 and there exist such ones for which a better estimate is given by Móricz' result. An estimation for the rate of convergence in the law of the iterated logarithm can be got by Theorem 7 from the result of Móricz [18].

COROLLARY 6. *Under the condition of Corollary* 5, *for every* $\theta > 2K^2pl$, *we have*

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{A_n^2 \ln A_n} P\left(\sup_{k \geq n} \frac{|S_k|}{(\theta A_k^2 \ln \ln A_k)^{1/2}} \geq 1\right) < \infty,$$

where $l = 1$ *or* 2 *in the real or complex case, resp.*

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CHARACTERIZATIONS OF THE BROWN—McCOY RADICAL

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We characterize the Brown—McCoy radical in an arbitrary universal class of not necessarily associative rings or near-rings, as a lower radical. We also establish the corresponding upper radical representation with respect to an arbitrary universal class of alternative rings or near-rings, as well as the intersection property with respect to the upper class of this representation. Finally we exhibit a natural module counterpart for this ring radical.

1. Three more characterizations of the Brown—McCoy radical

A class $\mathscr{A}$ of not necessarily associative rings, or near-rings, is called a *universal class* if $\mathscr{A}$ is hereditary and homomorphically closed, that is, if

$$I \lhd A \in \mathscr{A} \Rightarrow I \in \mathscr{A} \quad \text{and} \quad A/I \in \mathscr{A}.$$

In this section we shall work in a universal class $\mathscr{A}$ of rings, or near-rings. We state for emphasis the following two well known characterizations of the Brown—McCoy radical class $\mathscr{G}$ in $\mathscr{A}$: $\mathscr{G}$ is the upper radical class of the class $\mathbf{S}(1)$ of simple rings, or near-rings, with unity (in which we include 0):

$$\mathscr{G} = \mathscr{U}\mathbf{S}(1) = \{A \in \mathscr{A} \mid A/I \neq 0 \Rightarrow A/I \notin \mathbf{S}(1)\};$$

and, for any ring (or near-ring) A in $\mathscr{A}$,

$$\mathscr{G}(A) = \bigcap \{M \lhd A \mid A/M \in \mathbf{S}(1)\}.$$

The following characterization sharpens the statement of Theorem 1 in [5].

THEOREM 1. *Let $\mathscr{A}$ be a universal class of not necessarily associative rings (or near-rings). The Brown—McCoy radical class $\mathscr{G}$ in $\mathscr{A}$ is the lower radical determined by (it does in fact coincide with) the uniquely determined largest homomorphically closed class of rings (near-rings) in $\mathscr{A}$ without unity.*

PROOF. We prove the theorem for rings; the proof carries over mutatis mutandis for near-rings. Denote by $\mathbf{E}$ the class of all rings of $\mathscr{A}$ with unity and set

$$\mathbf{H} = \{A \in \mathscr{A} \mid A/I \neq 0 \Rightarrow A/I \notin \mathbf{E}\}.$$

Clearly $\mathbf{H}$ is a class of rings without (non-trivial) unity and $\mathbf{H}$ is homomorphically closed. Moreover, if $\mathbf{Q}$ is any homomorphically closed subclass of $\mathscr{A}$ consisting of rings without unity, then for any $A \in \mathbf{Q}$, $A/I \neq 0 \Rightarrow A/I \in \mathbf{Q} \Rightarrow A/I \notin \mathbf{E} \Rightarrow A \in \mathbf{H}$. Hence

H is the uniquely determined largest homomorphically closed class of rings in $\mathscr{A}$ without unity. By definition of **H**, a ring A in **H** cannot have an ideal M such that $0 \neq A/M \in \mathbf{S}(1)$. Hence $\mathbf{H} \subseteq \mathscr{G}$.

Conversely, let $A \in \mathscr{G}$. Suppose that A has a homomorphic image $A/I \neq 0$ in **E**. By Zorn's Lemma A/I, having a unity, must have a maximal ideal K/I. But then we would have

$$0 \neq (A/I)/(K/I) \in \mathscr{G} \cap \mathbf{S}(1) = \{0\}!$$

Hence $A \in \mathbf{H}$, and consequently $\mathscr{G} \subseteq \mathbf{H}$ holds. □

After the lower radical characterization we characterize $\mathscr{G}$ as an upper radical.

THEOREM 2. *Let $\mathscr{A}$ be a universal class of alternative rings (or near-rings). The Brown—McCoy radical $\mathscr{G}$ in $\mathscr{A}$ is the upper radical determined by the uniquely determined largest hereditary class of rings (near-rings) in $\mathscr{A}$ with unity.*

PROOF. Again, denote by **E** the class of all rings (near-rings) of $\mathscr{A}$ with unity, and set

$$\mathbf{P} = \{A \in \mathscr{A} \mid 0 \neq I \lhd A \Rightarrow I \in \mathbf{E}\}.$$

We first show that **P** is hereditary, and for this purpose we distinguish between the two possible types of contents of $\mathscr{A}$.

Case 1: *alternative rings*. Let $A \in \mathbf{P}$ and $0 \neq J \lhd I \lhd A$. Denote by J^* the ideal generated by J in A. We shall show that $J = J^*$. Suppose then that $J \neq J^*$. Then J^* has a unity, as $A \in \mathbf{P}$, and therefore J^*/J has a unity. By Zorn's Lemma the ring J^*/J (with unity) has a maximal ideal L/J. By Andrunakievich's Lemma for alternative rings (see [3]) the ring J^*/J is a Baer radical ring, that is $J^*/J \in \beta$. But then

$$J^*/L \cong (J^*/J)/(L/J) \in \mathbf{S}(1) \cap \beta = \{0\},$$

contradicting the fact that L is a proper ideal of J^*. Thus we have that $J = J^* \in \mathbf{E}$; and we have shown that **P** is hereditary in this case.

Case 2: *near-rings*. Once again, let $A \in \mathbf{P}$ and $0 \neq J \lhd I \lhd A$. Since $I \in \mathbf{E} \subseteq \mathscr{A}$ and $\mathscr{A}$ is universal we have that $I/J \in \mathbf{E}$. Hence, applying proposition 5 of [1], we obtain that $J \lhd A$ and therefore $J \in \mathbf{E}$. This shows that **P** is hereditary in this case.

It easily follows that **P** is the uniquely determined largest hereditary class of rings (near-rings) in $\mathscr{A}$ with unity.

We now consider the upper radical

$$\mathscr{U}\mathbf{P} = \{A \in \mathscr{A} \mid A/I \neq 0 \Rightarrow A/I \notin \mathbf{P}\}.$$

For any $A \in \mathbf{P}$, $\mathscr{G}(A) = 0$; for otherwise $\mathscr{G}(A)$ would have a (non-trivial) unity. Therefore **P** is contained in the semisimple class

$$\mathscr{S}\mathscr{G} = \{A \in \mathscr{A} \mid \mathscr{G}(A) = 0\}.$$

Obviously also $\mathbf{S}(1) \subseteq \mathbf{P}$, and hence we have $\mathbf{S}(1) \subseteq \mathbf{P} \subseteq \mathscr{S}\mathscr{G}$, which implies that

$$\mathscr{G} = \mathscr{U}\mathscr{S}\mathscr{G} \subseteq \mathscr{U}\mathbf{P} \subseteq \mathscr{U}\mathbf{S}(1) = \mathscr{G}.$$

Thus we have that $\mathscr{G} = \mathscr{U}\mathbf{P}$. □

COROLLARY. *Let $\mathcal{A}$ be a universal class of alternative rings, or near-rings. For any $A\in\mathcal{A}$,*

$$\mathcal{G}(A)=\bigcap\{I\lhd A\mid A/I\in\mathbf{P}\}.$$

PROOF. The inclusions $\mathbf{S}(1)\subseteq\mathbf{P}\subseteq\mathcal{S}\mathcal{G}$ justify the inclusions in

$$\mathcal{G}(A)=\bigcap\{K\lhd A\mid A/K\in\mathcal{S}\mathcal{G}\}\subseteq\bigcap\{L\lhd A\mid A/L\in\mathbf{P}\}\subseteq\bigcap\{M\lhd A\mid A/M\in\mathbf{S}(1)\}=\mathcal{G}(A).$$

□

2. A natural module counterpart for the Brown—McCoy ring radical

We consider an arbitrary associative ring A and the category A-mod of left A-modules. We include as objects of A-mod all left A-module structures — even when A has a unity, the non-unital modules are included. In this general setting the first author has in [2] introduced the radical $\mathfrak{r}_3$: A-mod$\to A$-mod defined by the assignment

$$M\mapsto\mathfrak{r}_3(M)=\bigcap\{S<M\mid S\in\Delta(M)\}$$

where $\Delta(M)$ consists of all maximal submodules S of M such that there is an $a=a(S)\in A$ with the property that $ax-x\in S$ for all $x\in M$. The radical $\mathfrak{r}_3$ derives its importance from the fact that an A-module M is $\mathfrak{r}_3$-semisimple if and only if M is isomorphic to a subdirect product of irreducible A-modules M_α each having an $a_\alpha\in A$ which acts as a unital operator on M_α. In our final theorem we establish a relationship between the ring radical $\mathcal{G}$ and the module radical $\mathfrak{r}_3$ according to the same norm as that used by Szász [6] with respect to the Jacobson ring radical $\mathcal{J}$ and the Kertész module radical, $\mathfrak{r}_2$ in [2]. The additive group of the ring A may of course assume more than one left A-module structure. However, when referring to the radicals $\mathfrak{r}_2(A)$ and $\mathfrak{r}_3(A)$ of A, we shall have in mind the canonical A-module structure of A provided by the ring multiplication $A\times A^+\to A^+$, $(a,x)\mapsto ax$ in A.

THEOREM 3. *Let A be an associative ring. Then $\mathfrak{r}_3(A)=A$ in A-mod if and only if $\mathcal{G}(A)=A$ in the category of associative rings.*

PROOF. Let A be an arbitrary associative ring. Since clearly

$$\{M\lhd A\mid A/M\in\mathbf{S}(1)\}\subseteq\Delta(A),$$

we have that $\mathfrak{r}_3(A)\subseteq\mathcal{G}(A)$. Hence $\mathfrak{r}_3(A)=A$ implies that $\mathcal{G}(A)=A$. Conversely, suppose that $\mathcal{G}(A)=A$. Assume that $\mathfrak{r}_3(A)\neq A$. Then $\Delta(A)$ is nonempty. Let $L\in\Delta(A)$. Then L is a maximal left ideal of A and there is an $a\in A$ such that $ax-x\in L$ for all $x\in A$. Since $\mathcal{G}(A)=A$, we have the well known relation

$$a\in G(a)=\{ax-x+\Sigma x_i(ay_i-y_i)\mid x,x_i,y_i\in A\}.$$

The fact that $\mathcal{G}(a)$ is a twosided ideal of A implies that $ax\in\mathcal{G}(a)$ for all $x\in A$. Since $\mathcal{G}(a)\subseteq L$ in view of $ax-x\in L$ for all $x\in A$, we have that $ax\in L$ for all $x\in A$. But then $x\in L$ for all $x\in A$, so that $L=A$. This contradiction shows that $\mathfrak{r}_3(A)=A$. □

We are now also in a position to reaffirm the separation $\mathfrak{r}_2<\mathfrak{r}_3$ established in [4]. This inequality follows from our Theorem 3, the corresponding result for $\mathfrak{r}_2$ and $\mathcal{J}$ in [6] and the fact that $\mathcal{J}\neq\mathcal{G}$.

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АСИМПТОТИЧЕСКИЙ АНАЛИЗ РАСПРЕДЕЛЕНИЯ СТАТИСТИКИ ДИКСОНА

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Профессору К. Тандори к 60-летию со дня рождения

1. Пусть $X_1, \ldots, X_m$ и $Y_1, \ldots, Y_n$ — две независимые выборки из совокупностей с одной и той же непрерывной функцией распределения т.е. случайные величины $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ независимые в совокупности. Образуем вариационный ряд из X-ов: $X_{(1)} < X_{(2)} < \ldots < X_m$ и обозначим через S_k количеств Y-ов из второй выборки, попавших в полуинтервал $[X_{(k-1)}, X_{(k)})$ $(k=1, \ldots$ $\ldots, m+1)$, где положено $X_{(0)} = -\infty$, $X_{(m+1)} = +\infty$.

Образуем статистику

$$T_{m,n} = \sum_{k=1}^{m+1} a(k, S_k) \tag{1}$$

где $a(i, j)$ — некоторая целозначная функция.

Для получения другого представления статистики Tm, n рассмотрим вариационный ряд:

$$Z_{(1)} < Z_{(2)} < \ldots < Z_{(m+n)} \tag{2}$$

объединённой выборки $(X_1, \ldots, X_m, \ Y_1, \ldots, Y_n)$. Обозначим через R_k ранг величины $X_{(k)}$ в ряду (2), т. е. R_k такой номер, что $Z_{(R_k)} = X_{(k)}$.

Очевидно, что $1 \leqq R_1 < R_2 < \ldots < R_m \leqq m+n$ и $S_k = R_k - R_{k-1} - 1$ $(k=1, \ldots$ $\ldots, m+1)$, где для единообразия записи положено $R_0 = 0$, $R_{m+1} = n+m+1$. Подставляя это выражение S_k в равенство (1), получаем представление статистики Tm, n через ранги:

$$T_{m,n} = \sum_{k=1}^{m+1} a(k, R_k - R_{k-1} - 1). \tag{1°}$$

Очевидно (по классическому определению вероятности):

$$P(T_{m,n} = N) = \frac{r_{m,n}(N)}{\binom{m+n}{n}}, \tag{2°}$$

где $r_{m,n}(N)$ — число решений диофантова уравнения (ради удобства полагаем $\alpha_0 = 0$, $\alpha_{m+1} = n+m+1$):

$$\sum_{k=1}^{m+1} a(k, \alpha_k - \alpha_{k-1} - 1) = N \tag{3}$$

в целых $\alpha_1, \ldots, \alpha_m$, удовлетворяющих условию: $1 \leqq \alpha_1 < \alpha_2 < \ldots < \alpha_m \leqq m+n$.

В уравнении (3) произведём замену переменных по формулам

(4) $$X_1 = \alpha_1 - 1, \quad X_k = \alpha_k - \alpha_{k-1} - 1 \quad (k = 2, \ldots, m), \quad X_{m+1} = n + m - \alpha_m.$$

Тогда уравнение (3) примет вид:

(3°) $$\sum_{k=1}^{m+1} a(k, X_k) = N,$$

с условием на переменные $X_1, \ldots, X_{m+1}$ $(0 \leqq X_k \leqq n)$: $X_1 + \ldots + X_{m+1} = n$. Другими словами, $r_{m,n}(N)$ — число решений уравнения (3) — равно количеству решений в $0 \leqq X_1 < \ldots < X_{m+1} \leqq n$ следующей системы диофантовых уравнений:

(5) $$\sum_{k=1}^{m+1} a(k, X_k) = N, \quad \sum_{k=1}^{m+1} X_k = n.$$

Рассмотрии частный случай $a(k, X_k) = X_k^2$, который соответствует статистике Диксона [1, 2, 3]. В этом случае система (5) превращается в известную систему диофантовых уравнений Гильберта—Варинга:

(6) $$\begin{cases} X_1 + \ldots + X_{m+1} = n \\ X_1^2 + \ldots + X_{m+1}^2 = N. \end{cases}$$

В дальнейшем $r_{m,n}(N)$ обозначает число решений системы (6) в $0 \leqq X_k \leqq n$.

Нас будет интересовать поведение вероятности $P(T_{m,n} = N)$ в зависимости от величин m, n и N. Это, в силу равенства (2°), сводится к изучению величины $r_{m,n}(N)$.

Так как при $X_i \geqq 0$ имеем неравенства

$$X_1^2 + \ldots + X_{m+1}^2 \leqq (X_1 + \ldots + X_{m+1})^2 \leqq (m+1)(X_1^2 + \ldots + X_{m+1}^2),$$

то $r_{m,n}(N) = 0$ для $n^2 < N$ и $n^2 > (m+1)N$. Таким образом достаточно рассматривать m, n и N удовлетворяющие неравенствам

(7) $$N \leqq n^2 \leqq (m+1)N.$$

Далее, как видно из второго уравнения системы (6): $0 \leqq X_i^2 \leqq N$; т.е. число решений $r_{m,n}(N)$ системы (6) в $0 \leqq X_i \leqq n$ совпадает с числом её решений в $0 \leqq X_i \leqq [N^{1/2}]$.

Мы впередь считаем выполненным соотношение (7).

Если m фиксированное $(\geqq 8)$, а $N \to \infty$, то для величины $r_{m,n}(N)$ имеет место асимптотическая формула, установленная К. К. Марджанишвили [4].

Здесь исследуем асимптотическое поведение $r_{m,n}(N)$ в случае, когда $m \to \infty$ вместе с N (тогда, как следует из (7), n также стремится к ∞). При этом мы будем следовать пути, предложенному Г. А. Фрейманом [5] (и восходящему к А. Я. Хинчину [6]) при исследовании асимптотики числа решений уравнения

(8) $$X_1^s + \ldots + X_m^s = N,$$

когда m и $N \to \infty$.

Приведём здесь теорему Г. А. Фреймана [5].

Теорема. *Для числа $I_{m,N}$ решений уравнения* (8) *при* $m\to\infty$, $m<\gamma N$ *(где* $0<\gamma<1$ *а* N *— целое положительное число) имеет место следующая асимптотическая фоямула*

$$I_{m,N} = \frac{1}{\sqrt{2\pi m D\xi}} e^{\sigma N}\Phi^m(\sigma)\left[1+O\left(\frac{1}{m^{1/2-\varepsilon}}\right)\right], \tag{9}$$

где σ опеределяется соомношением

$$\frac{N}{m} = \frac{\sum_{x=1}^{\infty} x^s e^{-\sigma x^s}}{\sum_{x=1}^{\infty} e^{-\sigma x^s}} = M\xi, \quad \Phi(\sigma) = \sum_{x=1}^{\infty} e^{-\sigma x^s}, \tag{9°}$$

а ξ — случайная величина, принимающая значения X^s ($X=1, 2, \ldots$) с вероятностями $p_x=e^{-\sigma x^s}[\Phi(\sigma)]^{-1}$. В (9) *величина ε-сколь угодно малая положительная постоянная.*

В дальнейшем, путём некоторого видоизменения метода доказательства (а именно, получив оценки встречающихся тригонометрических сумм в более широкой зоне изменения аргумента и конкретизируя применение аппарата локальных предельных теорем теории вероятностей, состоящее в том, что были использованы асимптотические разложения), в работе [7] было получено уточнение теоремы А. Г. Фреймана; а именно доказано что в соотношении (9) остаточный член имеет порядок $O\left(\frac{1}{m}\right)$; причём этот результат не может быть улучшен.

2. Вернёмся к нашей основной задаче об асимптотике $r_{m,n}(N)$. Введём обозначения:

$$\begin{cases} \Phi(\sigma) = \sum_{x=0}^{\infty} e^{-\sigma x^2}, \quad p_x = e^{-\sigma x^2}[\Phi(\sigma)]^{-1} \quad (x = 0, 1, \ldots), \\ a_\nu = \frac{1}{\Phi(\sigma)} \sum_{x=0}^{\infty} x^\nu e^{-\sigma x^2} = M\xi^\nu, \quad (\nu = 1, 2, 3, 4). \end{cases} \tag{10}$$

Здесь σ — параметр, выбор которого будет сделан ниже.

Исходя из известного соотношения

$$\int_0^1 e^{2\pi i k\alpha}\, d\alpha = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \quad \text{целое,} \end{cases}$$

легко показать, что

$$r_{m,n}(N) = e^{\sigma N}\Phi^m(\sigma) \int_0^1 \int_0^1 \Big(\sum_{x=0}^{\infty} p_x e^{2\pi i(\alpha_1 x + \alpha_2 x^2)}\Big)^{m+1} e^{-2\pi i(\alpha_1 n + \alpha_2 N)}\, d\alpha_1\, d\alpha_2. \tag{11}$$

Асимптотика $r_{m,n}(N)$ будет получена путём анализа этого интегрального представления.

Нетрудно видеть (по определению p_x в (10)), что функция

$$(12)\qquad f(2\pi\alpha) = f(2\pi\alpha_1, 2\pi\alpha_2) = \sum_{x=0}^{\infty} p_x e^{2\pi i(\alpha_1 x + \alpha_2 x^2)}$$

участвующая под знаком интеграла в (11), является характеристической функцией случайного вектора $\xi = (\eta_1, \eta_2)$ с распределением вероятностей

$$(13)\qquad P\{\xi = (x, x^2)\} = p_x \quad (x = 0, 1, \ldots).$$

Очевидно, матрицей максимального шага (определение этого понятия см. например в [8]) этого случайного вектора будет

$$H = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Пусть $\xi_1, \ldots, \xi_{m+1}$ — независимые случайные векторы с тем же распределением вероятностей, что и у ξ; обозначая через $P_m(n, N) = P\{\xi_1 + \ldots + \xi_{m+1} = (n, N)\}$, по формуле обращения [8, 9] получим:

$$(14)\qquad P_m(n, N) = \frac{|H|}{(2\pi)^2} \int_I f^{m+1}(\alpha) e^{-i(\alpha, \vec{N})}\, d\alpha,$$

где $\vec{N} = (n, N)$, $I = \{(t_1, t_2)H^{-1}\colon -\pi \leqq t_\nu \leqq \pi;\ \nu = 1, 2\}$, $|H|$ детерминант матрицы H (он равен 2). Заметим, что в нашем случае в качестве области I можно взять множество

$$\left\{\alpha\colon -\pi \leqq \alpha_1 \leqq \pi, \quad -\frac{\pi}{2} \leqq \alpha_2 \leqq \frac{\pi}{2}\right\}.$$

Поскольку, также имеем

$$(15)\qquad P_m(n, N) = \frac{|H|}{4\pi^2} \int_I f^{m+1}(\alpha) e^{-i(\alpha, \vec{N})}\, d\alpha,$$

то, сопоставляя равенства (11), (14) и (15), получаем

$$(16)\qquad \frac{r_{m,n}(N)}{e^{\sigma N} \Phi^m(\sigma)} = P_m(n, N) = \frac{1}{2\pi^2} \int_I f^{m+1}(\alpha) e^{-i(\alpha, \vec{N})}\, d\alpha.$$

Будем анализировать величину $P_m(n, N)$, исходя из равенства (16) по схеме локальной пределной теоремы для суммы $\xi_1 + \ldots + \xi_m = S_m$.

Введём обозначения для числовых характеристик случайного вектора $\xi = (\eta_1, \eta_2)$; нетрудно убедиться, что имеют место следующие соотношения:

$$(17)\qquad \begin{cases} M\eta_1 = a_1, \quad M\eta_2 = a_2, \quad D\eta_1 = a_2 - a_1^2 = b_1^2, \quad D\eta_2 = a_4 - a_2^2 = b_2^2, \\ M[(\eta_1 - M\eta_1)(\eta_2 - M\eta_2)] = a_3 - a_1 a_2 = b_{12}, \\ \varrho = \dfrac{b_{12}}{b_1 b_2}. \end{cases}$$

Ещё примем обозначения:

$$(18)\qquad \begin{cases} \beta_{k1} = M|\eta_1 - M\eta_1|^k, \quad \beta_{k2} = M|\eta_2 - M\eta_2|^k, \\ u_1 = \dfrac{n-(m+1)a_1}{b_1\sqrt{m+1}}, \quad u_2 = \dfrac{N-(m+1)a_2}{b_2\sqrt{m+1}}, \\ Q(u_1, u_2) = \dfrac{1}{1-\varrho^2}(u_1^2 - 2\varrho u_1 u_2 + u_2^2), \\ u = (u_1, u_2) \\ q(\alpha) = q(\alpha_1, \alpha_2) = \alpha_1^2 + 2\varrho\alpha_1\alpha_2 + \alpha_2^2, \\ f_m(\alpha) = \left[f\left(\dfrac{\alpha_1}{b_1\sqrt{m+1}}, \dfrac{\alpha_2}{b_2\sqrt{m+1}}\right) e^{-i\left(\frac{\alpha_1 a_1}{b_1\sqrt{m+1}} + \frac{\alpha_2 a_2}{b_2\sqrt{m+1}}\right)}\right]^{+1} \end{cases}$$

В интеграле из равенства (16) произведём замену переменных $\alpha_\nu = t_\nu / b_\nu \sqrt{m+1}$ $(\nu=1, 2)$, получим:

$$(19)\qquad 2\pi^2 b_1 b_2 (m+1) P_m(n, N) = \int_I f_m(t) e^{-i(t,u)}\, dt,$$

где

$$I_m = \left\{t\colon |t_1| \leqq \pi b_1 \sqrt{m+1}, \quad |t_2| \leqq \frac{\pi}{2} b_2 \sqrt{m+1}\right\}.$$

Далее, используя (19) и известное равенство

$$(19^\circ)\qquad \varphi(u) \overset{\text{def}}{=} \frac{1}{2\pi\sqrt{1-\varrho^2}} e^{-\frac{1}{2}Q(u_1, u_2)} = \frac{1}{(2\pi)^2} \int_{R^2} e^{-i(u,t) - \frac{1}{2}q(t)}\, dt,$$

составим разность. (Начиная с этого момента, считаем параметр σ из соотношений (10) таким, что

$$(20)\qquad \frac{\sum\limits_{x=0}^{\infty} x^2 e^{-\sigma x^2}}{\sum\limits_{x=0}^{\infty} e^{-\sigma x^2}} = \frac{N}{m+1} \quad \left(\text{т.е. } \frac{N}{m+1} = a_2\right),$$

тогда u_2 из (18) будет равен 0. В том, что σ можно выбрать таким, нетрудно убедиться (см. например [5]).:

$$(20^\circ)\qquad \Delta_m \equiv \Delta_m(n, N) = 2\pi^2 \left[b_1 b_2 (m+1) P_m(n, N) - \frac{1}{\pi\sqrt{1-\varrho^2}} e^{-\frac{1}{2}Q(u_1, 0)} \right].$$

Представим Δ_m в виде суммы трёх интегралов

$$\Delta_m = I_1 + I_2 + I_3,$$

где

$$\mathscr{D}=\left\{t:\ \frac{M\left|\left(\frac{\xi-a}{b},\ t\right)\right|^3}{q(t)}\leqq\frac{\sqrt{m+1}}{8}\right\},\quad \frac{\xi-a}{b}=\left(\frac{\eta_1-a_1}{b_1},\ \frac{\eta_2-a_2}{b_2}\right),$$

$$(21)\qquad \begin{cases} I_1=\int\limits_{\mathscr{D}} e^{-iu_1t_1}\left[f_m(t)-e^{-\frac{1}{2}q(t)}\right]dt,\\ I_2=-\int\limits_{\mathbf{R}^2\setminus\mathscr{D}} e^{-iu_1t_1-\frac{1}{2}q(t)}\,dt,\\ I_3=\int\limits_{I_m\setminus\mathscr{D}} e^{-iu_1t_1}f_m(t)\,dt.\end{cases}$$

Прежде чем приступить к оцениванию интегралов I_ν, получим необходимые оценки для величин a_ν, b_ν, ϱ, $\beta_{k\nu}$ $(\nu=1, 2)$, которые были введены в (17) и (18). Для этого нам нужна следующая теорема Сонина.

Лемма 1 (Сонин, см. [11]). *Если функция $f(x)$ имеет непрерывную вторую производную в интервале $(Q, R]$, то*

$$\sum_{Q<x\leqq R} f(x)=\int\limits_Q^R f(x)\,dx+\varrho(R)f(R)-\varrho(Q)f(Q)-$$

$$-\sigma(R)f'(R)+\sigma(Q)f'(Q)+\int\limits_Q^R \sigma(x)f''(x)\,dx,$$

где

$$\varrho(x)=\frac{1}{2}-\{x\},\quad \sigma(x)=\int\limits_0^x \varrho(z)\,dz.$$

Далее, если $\int\limits_Q^\infty |f''(x)|\,dx<\infty$, *то при* R *больших имеем*:

$$\sum_{Q<x\leqq R} f(x)=C+\int\limits_Q^R f(x)\,dx+\varrho(R)f(R)-\sigma(R)f'(R)-\int\limits_R^\infty \sigma(x)\,f''(x)\,dx,$$

где величина C не зависит от R.

Всюду в дальнейшем C_0 означает достаточно малую постоянную, C_1 достаточно большую, а C — некоторую постоянную (если не оговорено особо).

Опираясь на эту лемму, после длинных, но несложных вычислений получим (причём, как будет видно из следующего, мы можем считать $\sigma\to+0$):

$$(22)\quad \begin{cases} \Phi(\sigma) = \dfrac{\pi^{1/2}}{2\sqrt{\sigma}}[1+O(\sqrt{\sigma})], \\ a_1 = \dfrac{1}{\sqrt{\pi\sigma}}[1+O(\sqrt{\sigma})], \quad a_2 = \dfrac{1}{2\sigma}[1+O(\sqrt{\sigma})], \\ a_3 = \dfrac{1}{\sqrt{\pi}\,\sigma^{3/2}}[1+O(\sqrt{\sigma})], \\ a_4 = \dfrac{3}{4\sigma^2}[1+O(\sqrt{\sigma})] = \dfrac{3N^2}{(m+1)^2}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \\ b_1^2 = \dfrac{\pi-2}{2\pi\sigma}[1+O(\sqrt{\sigma})], \quad b_2^2 = \dfrac{1}{2\sigma^2}[1+O(\sqrt{\sigma})], \\ \varrho = \dfrac{1}{\sqrt{\pi-2}}[1+O(\sqrt{\sigma})]. \end{cases}$$

Из найденного выражения для a_2 и выбора σ, таким что $N/m+1=a_2$ в частности, следует что

$$(23)\qquad \sigma = \frac{m+1}{2N}\left[1+O\left(\sqrt{\frac{m}{N}}\right)\right].$$

Подставляя это значение σ в (22), находим:

$$(23^\circ)\quad \begin{cases} a_1 = \sqrt{\dfrac{2N}{\pi(m+1)}}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \quad a_3 = \dfrac{1}{\sqrt{\pi}}\left(\dfrac{2N}{m+1}\right)^{3/2}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \\ b_1^2 = \dfrac{\pi-2}{\pi}\dfrac{N}{m+1}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \quad b_2^2 = \dfrac{2N^2}{(m+1)^2}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \\ \varrho = \dfrac{1}{\sqrt{\pi-2}}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right], \quad \Phi(\sigma) = \sqrt{\dfrac{\pi N}{2(m+1)}}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right]. \end{cases}$$

Оценим теперь β_{41} и β_{42}. $\beta_{41}=a_4-4a_1a_3+6a_1^2a_2-3a_1^4$. Используя (22) получим:

$$(24)\qquad \beta_{41} = \left(\frac{3}{4\sigma^2}-\frac{4}{\sqrt{\pi\sigma}}\frac{1}{\sqrt{\pi}\,\sigma^{3/2}}+\frac{6}{\pi\sigma 2\sigma}-\frac{3}{\pi^2\sigma^2}\right)[1+O(\sqrt{\sigma})] =$$

$$= \frac{3\pi^2-4\pi-12}{4\pi^2}\frac{1}{\sigma^2}[1+O(\sqrt{\sigma})] = \frac{3\pi^2-4\pi-12}{\pi^2}\frac{N^2}{(m+1)^2}\left[1+O\left(\sqrt{\frac{m}{N}}\right)\right],$$

также

(25)

$$\beta_{42} = a_8-4a_2a_6+6a_2^2a_4-3a_2^4 = \frac{c}{\sigma^4}[1+O(\sqrt{\sigma})] = c\,\frac{N^4}{(m+1)^4}\left[1+O\left(\sqrt{\frac{m}{N}}\right)\right].$$

Из (24) и (25) по неравенству между моментами, т.е. $\beta_{k\nu}^{1/k} \leqq \beta_{s\nu}^{1/s}$ $(s \geqq k)$ получим:

$$(26)\qquad \beta_{31} \leqq c_1\left(\frac{N}{m}\right)^{3/2}, \quad \beta_{32} \leqq c_1\left(\frac{N}{m}\right)^3.$$

Отсюда и из (23°), по неравенству моментов следует, что эти оценки точные в смысле порядка по N/m.

Тепер оценим

$$\beta_3(t) = M\left|\left(\frac{\xi-a}{b},\, t\right)\right|^3.$$

По неравенствам $|(x, y)| \leqq |x|\cdot|y|$, $(a^2+b^2)^2 \leqq 2(a^4+b^4)$ и неравенству между моментами, получим:

(27) $$\beta_3(t) \leqq |t|^3 M\left|\frac{\xi-a}{b}\right|^3 \leqq |t|^3\left(M\left|\frac{\xi-a}{b}\right|^4\right)^{3/4} =$$

$$= |t|^3\left\{M\left[\left(\frac{\eta_1-a_1}{b_1}\right)^2+\left(\frac{\eta_2-a_2}{b_2}\right)^2\right]^2\right\}^{3/4} \leqq$$

$$\leqq 2^{3/4}|t|^3\left\{M\left[\left(\frac{\eta_1-a_1}{b_1}\right)^4+\left(\frac{\eta_2-a_2}{b_2}\right)^4\right]\right\}^{3/4} = 2^{3/4}|t|^3\left\{\frac{\beta_{41}}{b_1^4}+\frac{\beta_{42}}{b_2^4}\right\}^{3/4}.$$

Отсюда, используя оценки (23°), (24) и (25), получим:

(28) $$\beta_3(t) \leqq c_1|t|^3.$$

Оценим $q(t)$; в силу оценки для ϱ из (23°) имеем:

(29) $$q(t) = t_1^2+2\varrho t_1t_2+t_2^2 = (1-\varrho)(t_1^2+t_2^2)+\varrho(t_1+t_2)^2 \geqq (1-\varrho)(t_1^2+t_2^2) \geqq c_0(t_1^2+t_2^2).$$

Теперь мы в состоянии оценить интеграл I_1 из (21). Согласно теореме А. Бикялиса [10], для $t\in\mathscr{D}$ (см. (21)) имеет место неравенство:

(29°) $$\left|f_m(t)-e^{-\frac{1}{2}q(t)}\right| \leqq \frac{c_1\beta_3(t)}{\sqrt{m+1}}\, e^{-\frac{q(t)}{4}}.$$

Отсюда, применяя оценки (28) и (29), находим:

(30) $$|I_1| \leqq \frac{c_1}{\sqrt{m+1}}\int_{\mathscr{D}}|t|^3e^{-c(t_1^2+t_2^2)}\,dt \leqq \frac{c_1}{\sqrt{m+1}}\int_{\mathbf{R}^2}|t|^3e^{-c|t|^2}\,dt = \frac{c_1}{\sqrt{m+1}}.$$

Далее, замечая (по (28) и (29)), что

(31) $$\mathscr{D}\supset\left\{t\colon \frac{c_1|t|^3}{c_0|t|^2}\leqq\frac{\sqrt{m+1}}{8}\right\} = \left\{t\colon |t|\leqq c_0\sqrt{m+1}\right\} \equiv \mathscr{D}_1,$$

оценим I_2:

(32) $$|I_2| \leqq \int_{\mathbf{R}^2\setminus\mathscr{D}} e^{-\frac{1}{2}q(t)}\,dt \leqq \int_{\mathbf{R}^2\setminus\mathscr{D}_1} e^{-c_0|t|^2}\,dt \leqq \frac{c_1}{\sqrt{m+1}}.$$

Переходим к оцениванию основного интеграла I_3. В силу (31) имеем:

$$\mathscr{D}\supset\mathscr{D}_1\supset\mathscr{D}_2\equiv\left\{t\colon |t_1|\leqq c_0\sqrt{m+1},\ |t_2|\leqq c_0\sqrt{m+1}\right\},$$

поэтому

$$|I_3| \leqq \int\limits_{I_m \setminus \mathscr{D}} |f_m(t)|\,dt \leqq \int\limits_{I_m \setminus \mathscr{D}_2} |f_m(t)|\,dt =$$

$$= \int\limits_{I_m \setminus \mathscr{D}_2} \left| f\left(\frac{t_1}{b_1 \sqrt{m+1}}, \frac{t_2}{b_2 \sqrt{m+1}} \right) \right|^{m+1} dt_1\,dt_2;$$

произведя замену переменных $t_\nu = \alpha_\nu b_\nu \sqrt{m+1}$ $(\nu=1, 2)$ и обозначив $B_1 = \left\{ \alpha\colon |\alpha_\nu| \leqq \frac{c_0}{b_\nu};\ \nu=1, 2 \right\}$, получим:

$$|I_3| \leqq b_1 b_2 (m+1) \int\limits_{I \setminus B_1} |f(\alpha)|^{m+1}\,d\alpha \tag{33}$$

(определение 1 см. перед формулой (15)).

Обращаясь к оценкам (23°) для b_1 и b_2, видим, что

$$B_1 \supset B = \left\{ \alpha\colon |\alpha_\nu| \leqq c_0 \left(\frac{m+1}{N} \right)^{\nu/2} \right\}. \tag{34}$$

Докажем следующую лемму.

Лемма 2. *При* $\alpha \in I \setminus B$ *имеет место неравенство*

$$|f(\alpha)| \leqq \chi, \tag{35}$$

где $\chi < 1$ *— абсолютная постоянная.*

Доказательство. Обозначим $Q = \left(c_1 \frac{N}{m+1} \right)^{1/2}$, где c_1 такое, что Q — целое, $u \geqq 2$. Тогда, в силу определения (34) области B, при некотором c_0 имеем

$$B \supset \mathscr{R} = \{\alpha\colon |\alpha_\nu| \leqq c_0 Q^{-\nu};\ \nu = 1, 2\}. \tag{36}$$

Оценку (35) мы выведем из следующего неравенства, доказанного в работе [12]: при $\alpha \in I \setminus \mathscr{R}$ имеет место неравенство

$$\frac{1}{Q+1} \left| \sum_{x=0}^{Q} e^{\alpha_1 x + \alpha_2 x^2} \right| \leqq \chi_1, \tag{37}$$

где $\chi_1 < 1$ постоянная (т.е. не зависит от Q).

Вспомним определение (12) характеристической функции $f(\alpha)$, и произведём последующие очевидные преобразования:

$$f(\alpha) = \sum_{x=0}^{\infty} P_x e^{i(\alpha_1 x + \alpha_2 x^2)} = \sum_{x=0}^{Q} (P_x - P_Q) e^{i(\alpha_1 x + \alpha_2 x^2)} +$$

$$+ P_Q \sum_{x=0}^{\infty} e^{i(\alpha_1 x + \alpha_2 x^2)} + \sum_{x=Q+1}^{\infty} P_x e^{i(\alpha_1 x + \alpha_2 x^2)}.$$

Отсюда, принимая во внимание: $P_Q \leqq P_x$ при $x \leqq Q$ получим

$$|f(\alpha)| \leqq \sum_{x=0}^{Q} |P_x - P_Q| + P_Q \left| \sum_{x=0}^{Q} e^{i(\alpha_1 x + \alpha_2 x^2)} \right| + \tag{38}$$

$$+ \sum_{x=Q+1}^{\infty} P_x = 1 - (Q+1) P_Q + P_Q \left| \sum_{x=0}^{Q} e^{i(\alpha_1 x + \alpha_2 x^2)} \right|.$$

Подставляя оценку (37) в неравенство (38), найдем:

$$|f(\alpha)| \leqq 1 - (Q+1) P_Q (1 - \chi_1) \quad (\alpha \in I \diagdown R). \tag{39}$$

Из (39) будет следовать (35), т.е. утверждение Леммы 2, если мы покажем, что $P_Q(Q+1) \not\to 0$, при $Q \to \infty$. Для этого запишем явный вид P_Q из (10), и воспользуемся оценкой (22) для $\Phi(\sigma)$, выражением для $Q = [cN/m+1]^{1/2}$ и формулой (23) для σ; в результате получим:

$$QP_Q = \frac{Qe^{-\sigma Q^2}}{\Phi(\sigma)} = \frac{\frac{c}{\sqrt{\sigma}} [1 + O(\sqrt{\sigma})] e^{-\sigma \frac{c}{\sigma} [1 + O(\sqrt{\sigma})]}}{\frac{c}{\sqrt{\sigma}} [1 + O(\sqrt{\sigma})]} = C[1 + O(\sqrt{\sigma})].$$

Лемма 2 доказана.

Завершим оценивание I_3. Обращаясь к соотнощениям (34) и (36), и после подставляя оценку (35) в неравенство (33), находим (учитывая, также, оценки (23°) для b_1 и b_2):

$$|I_3| \leqq b_1 b_2 (m+1) \chi^{m+1} \leqq \frac{c_1}{\sqrt{m+1}} N^{\frac{3}{2}} \chi^{m+1} \leqq \frac{c_1}{\sqrt{m+1}}, \tag{40}$$

при $m+1 \geqq c_1 \ln N \left(c_1 = 3/2 \ln \frac{1}{\chi} \right)$.

3. Собирая оценки (30), (32), (40) и вспоминая соотношение (20°) получаем теорему:

Теорема 1. *Пусть $m \to \infty$, n и N удовлетворяют условию $n^2/m+1 \leqq N \leqq n^2$. Сушествуют постоянные $\beta > 0$ и $c = c(\beta)$ такие, что при $\gamma m \leqq N \leqq e^{\beta m}$ ($\gamma > 1$ — произвольная постоянная) имеет место неравенство:*

$$\left| b_1 b_2 (m+1) P_m(n, N) - \frac{1}{\pi \sqrt{1-\varrho^2}} e^{-\frac{1}{2} \frac{u_1^2}{1-\varrho^2}} \right| \leqq \frac{c}{\sqrt{m}}. \tag{41}$$

Теперь вернёмся к исходной формулировке рассматриваемой задачи, т.е. к вопросу об асимптотике распределения $P(T_{m,n} = N)$ — статистики Дискона $T_{m,n}$. Обращаясь к равенствам (2°) и (16) находим:

$$P(T_{m,n} = N) = \frac{e^{\sigma N} \Phi^m(\sigma)}{\binom{m+n}{n}} P_{m,n}(N), \tag{42}$$

что, в соотношении с (41), даёт следующее утверждение

Теорема 2. *Пусть N, n и m натуральные числа, удовлетворяющие условиям Теоремы* 1. *Тогда существуют постоянные $\beta>0$ и $c=c(\beta)$ такие, что при $\gamma m \leqq N \leqq e^{\beta m}$ имеет место соотношение*

$$(43)\quad P(T_{m,n}=N)=\frac{e^{\sigma N}\Phi^m(\sigma)}{b_1 b_2(m+1)\binom{m+n}{n}}\left[\frac{1}{\pi\sqrt{1-\varrho^2}}e^{-\frac{u_1^2}{2(1-\varrho^2)}}+\Theta\frac{1}{\sqrt{m}}\right];\quad |\Theta|\leqq c.$$

Для полноты запишем эти утверждения и как асимптотическое выражение величины $r_{m,n}(N)$ — числа решений системы (6).

Теорема 3. *Пусть $N\leqq n^2\leqq(m+1)N$ и для достаточно большого постоянного w: $w\ln N\leqq m\leqq\gamma N$ ($\gamma<1$ — произвольная постоянная). Тогда существует $c=c(w)$ такая, что*

$$(44)\quad r_{m,n}(N)=\frac{e^{\sigma N}\Phi^m(\sigma)}{(m+1)b_1 b_2}\left[\frac{1}{\pi\sqrt{1-\varrho^2}}e^{-\frac{u_1^2}{2(1-\varrho^2)}}+\Theta\frac{1}{\sqrt{m}}\right];\quad |\Theta|\leqq c.$$

Теперь, накладывая более жёсткие условия на величины n, m и N, выведем более обозримые асимптотические формулы для рассматриваемых величин.

Так, например, если кроме условий Теоремы 3, выполняется еще условие

$$(45)\quad \frac{m^{3/2}}{N^{1/2}}\to 0,\quad \text{при}\quad m\to\infty;$$

то воспользовавшись соотношениями (23) и (23°), будем иметь (после несложных вычислений):

$$(46)\quad \begin{cases} \Phi^{m+1}(\sigma)=\left(\dfrac{\pi N}{2(m+1)}\right)^{\frac{m+1}{2}}\left[1+O\left(\dfrac{m^{3/2}}{N^{1/2}}\right)\right],\\ e^{\sigma N}=e^{\frac{m+1}{2}}\left[1+O\left(\dfrac{m^{3/2}}{N^{1/2}}\right)\right],\\ e^{\sigma N}\Phi^{m+1}(\sigma)=\left(\dfrac{\pi e N}{2(m+1)}\right)^{\frac{m+1}{2}}\left[1+O\left(\dfrac{m^{3/2}}{N^{1/2}}\right)\right],\\ b_1 c_2\sqrt{1-\varrho^2}=\dfrac{\sqrt{2(\pi-3)}}{\sqrt{\pi}}\left(\dfrac{N}{m+1}\right)^{3/2}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right],\\ \sqrt{1-\varrho^2}=\sqrt{\dfrac{\pi-3}{\pi-2}}\left[1+O\left(\sqrt{\dfrac{m}{N}}\right)\right]. \end{cases}$$

Из (44) и (46) легко получаем следующее утверждение:

Теорема 4. *Пусть $m\to\infty$, выполнены условия Теоремы 3 и $m^3=o(N)$; тогда*

(47)

$$r_{m,n}(N)=\left[\frac{\pi e}{2(m+1)}\right]^{\frac{m+1}{2}}\frac{\sqrt{m+1}}{\sqrt{2\pi(\pi-3)}}N^{\frac{m-2}{2}}\left[e^{-\frac{u_1^2}{2(1-\varrho^2)}}+O\left(\frac{1}{\sqrt{m}}+\frac{m^{3/2}}{\sqrt{N}}\right)\right].$$

Замечание. Очевидно, что в приведённых теоремах, как обычно бывает в локальных предельных теоремах, главные члены асимптотических формул по порядку не меньше остаточных, если $|u_1|/\sqrt{1-\varrho^2} \leqq \sqrt{(1-\varepsilon)\ln m}$, т.е. если

$$\left| \frac{n-(m+1)a_1}{\sqrt{1-\varrho^2}\, b_1 \sqrt{m+1}} \right| \leqq \sqrt{(1-\varepsilon)\ln m};$$

($\varepsilon > 0$ произвольное, фиксированное).

Если ограничиться рассмотрением таких значений n, что

$$|n-(m+1)a_1| \leqq \frac{c_1\sqrt{N}}{m^a}, \quad (a>0 \text{ — любая постоянная}) \tag{48}$$

т.е., как следует из оценки b_1 и p по (23°), значениями u_1: $|u_1| \leqq c_1/m^a$ (c_1 — любая постоянная), то справедлива, как легко убедиться, подставляя в (47) величину

$$e^{-\frac{u_1^2}{2(1-\varrho^2)}} = 1+O(u_1^2) = 1+O\left(\frac{1}{m^{2a}}\right),$$

следующая теорема:

Теорема 5. *Если в условиях Теоремы* 4 n *такое, что выполняется неравенство* (48); *то*

(49)

$$r_{m,n}(N) = \left[\frac{\pi e}{2(m+1)}\right]^{\frac{m+1}{2}} \sqrt{\frac{m+1}{2\pi(\pi-3)}}\, N^{\frac{m-2}{2}} \left[1+O\left(\frac{1}{m^{2a}}+\frac{1}{\sqrt{m}}+\frac{m^{3/2}}{N^{1/2}}\right)\right].$$

Замечание. В Теореме 5, которая выведена нами из Теорем 3 и 4, условие $w\ln N \leqq m$ (см. Теорему 3) может быть опущено. Это доказывается, следуя работе [4] (нужно оценить постоянные — достаточно даже грубых оценок — в асимптотических формулах приведённых там), в которой находится асимптотика для $r_{m,n}(N)$ при нерастущих m.

После сделанного замечания, из Теорем 2 и 5 следует следующая теорема, дающая удобную асимптотическую формулу для вероятностей $P(T_{m,n}=N)$ статистики Диксона.

Теорема 6. *Пусть* $m\to\infty$, $m=o(N^{1/3})$ *и* $|n-(m+1)a_1| \leqq \dfrac{c_1\sqrt{N}}{m^a}$, ($c_1$, $a>0$ *— произвольные фиксированные*), *тогда*

(50)

$$P(T_{m,n}=N) = \frac{m!\,n!}{(m+n)!}\left[\frac{\pi e}{2(m+1)}\right]^{\frac{m+1}{2}} \sqrt{\frac{m+1}{2\pi(\pi-3)}}\, N^{\frac{m-2}{2}} \left\{1+O\left(\frac{1}{m^{2a}}+\frac{1}{\sqrt{m}}+\frac{m^{3/2}}{\sqrt{N}}\right)\right\}.$$

Замечание. Применяя формулу Стирлинга, соотношение (50) можно переписать в виде (учитывая при этом, что при условии (48), $1/n=o(1/\sqrt{N})$):

(51)

$$P(T_{m,n}=N) = \frac{m^{m+\frac{1}{2}}\, n^{n+\frac{1}{2}}}{(m+n)^{m+n-\frac{1}{2}}}\left[\frac{\pi e}{2(m+1)}\right]^{\frac{m+1}{2}} N^{\frac{m-2}{2}} \left\{1+O\left(\frac{1}{m^{2a}}+\frac{1}{\sqrt{m}}+\frac{m^{3/2}}{\sqrt{N}}\right)\right\}.$$

4. В заключение сделаем несколько замечаний о результатах работ [3] и [13].

В работе [3] в виде Теоремы 8 приведено (без доказательства) следующее утверждение о статистике Диксона $T_{m,n}$.

Теорема. *Если* $n^2/m+1 \leqq k \leqq n^2$, *то существует абсолютная постоянная* $c>0$ *такая, что*

$$\sup_k |\Delta_{m,n}(k)| \leqq \frac{c}{\sqrt{m}}, \tag{52}$$

где

$$\begin{cases} \Delta_{m,n}(k) = \dfrac{(m+1)\sigma_1\sigma_2\binom{m+n}{n}}{(n+1)^{m+1}} P(T_{m,n}=k) - \dfrac{1}{\pi\sqrt{1-\varrho^2}} e^{-\frac{1}{2}Q(y_1,y_2)}, \\ Q(y_1,y_2) = \dfrac{1}{1-\varrho^2}(y_1^2-2\varrho y_1 y_2+y_2^2), \\ y_1 = \dfrac{n-(m+1)\mu_1}{\sigma_1\sqrt{m+1}}, \quad y_2 = \dfrac{k-(m+1)\mu_2}{\sigma_2\sqrt{m+1}}, \quad \mu_\nu = \dfrac{1}{n+1}\sum_{x=0}^{n} x^\nu, \\ \sigma_\nu^2 = \mu_{2\nu}-\mu_\nu^2 \quad (\nu=1,2,3,4), \quad \varrho = \dfrac{\mu_3-\mu_1\mu_2}{\sigma_1\sigma_2}. \end{cases} \tag{53}$$

Покажем, что эта теорема неверна; а именно, покажем что главный член асимптотической (при $n, m\to\infty$) формулы (52) меньше остаточного.

Действительно, как легко показать (см. [12] (Лемма 1.))

$$\mu_\nu = \frac{1}{\nu+1} n^\nu \left[1+O\left(\frac{1}{n}\right)\right], \quad \sigma_\nu = c(\nu) n^\nu \left[1+O\left(\frac{1}{n}\right)\right], \quad \frac{3\sqrt{5}}{8} \leqq \varrho \leqq \frac{\sqrt{15}}{4},$$

гдс $c(\nu)$ — постоянные зависящие от ν, их можно легко вычислить.

Подставим найденные выражения для μ_1 и σ_1 в равенство (53) для y_1:

(54)
$$y_1 = \frac{n-(m+1)\frac{1}{2}n\left[1+O\left(\frac{1}{n}\right)\right]}{cn\left[1+O\left(\frac{1}{n}\right)\right]\sqrt{m+1}} = \frac{-n\left(m-\frac{1}{2}\right)\left[1+O\left(\frac{1}{n}\right)\right]}{cn\sqrt{m+1}\left[1+O\left(\frac{1}{n}\right)\right]} = -c\sqrt{m}\,[1+o(1)].$$

Далее, учитывая оценку для ϱ, получим:

$$Q(y_1,y_2) \geqq \left(1+\frac{\sqrt{15}}{4}\right)^{-1}(y_1^2+y_2^2). \tag{55}$$

Из оценок (54) и (55) следует:

$$e^{-\frac{1}{2}Q(y_1,y_2)} \leqq e^{-c_0(y_1^2+y_2^2)} = e^{-c_0m}e^{-cy_2^2} \leqq e^{-c_0m} = O\left(\frac{1}{\sqrt{m}}\right).$$

Таким образом, то, что соотношение (52) не содержательно, доказано.

Замечание к результатам работы [13] состоит в следующем.

В этой работе для числа решений $r_{m,n}(N)$ системы диофантовых уравнений (6) приведены (в виде двух теорем; без доказательства) две асимптотические формулы. Из введённых там обозначений видно, что авторы при установлении этих формул использовали методы близкие кметодике получения Теорем 3 и 4 нашей работы. Однако элементарные вычисления показывают (даже если не сравнивать с нашими Теоремами 3 и 4), что главные члены асимптотических формул в обеих теоремах неверны.

Если в (1), (4) и (5) полагать $a(k, x_k)=X_k^s$; где $s\geqq 2$ фиксированное целое, то получим некоторую статистику $T_{m,n}^{(s)}$, которая при значении $s=2$ переходит в статистику Диксона; поэтому $T_{m,n}^{(s)}$ естественно назвать обобщённой статистикой Диксона. В этом случае вместо системы уравнений (6) получится следующая система:

(56)
$$\begin{cases} x_1+\ldots+x_{m+1} = n, \\ x_1^s+\ldots+x_{m+1}^s = N. \end{cases}$$

Соображения, аналогичные проведенным в случае $s=2$, позволяют получить теоремы, подобные доказанным и в этом случае (используя также идеи работ [5]).

5. Замечание о точности оценок. То что величина $1/\sqrt{m}$ в остаточном члене найденных формул (41), (43), (44), (47) не может быть заменена на величину более быстрого порядка следует из того, что при получении главного члена (оценка интеграла I_1) мы использовали теорему А. Бикялиса (неравенство (29°)), которая имеет уточнения типа асимптотических разложений по степеням $1/\sqrt{m}$.

Теперь дадим уточнение Теорем 5 и 6. Из определения функций $P_\nu(\omega)$ и $P_\nu(-\varphi(x))$, (см. [8—10]), найдём явное выражение для $P_1(-\varphi(x))$.

Пусть $\bar\eta_\nu=\dfrac{\eta_\nu-a_\nu}{b_\nu}$; используя (18) и (19°) имеем

$$6P_1(\omega) = M\left(\frac{\xi-a}{b},\, \omega\right)^3 = M(\bar\eta_1\omega_1+\bar\eta_2\omega_2)^3 =$$

$$= M(\bar\eta_1^3\omega_1^3+3\bar\eta_1^2\omega_1^2\bar\eta_2\omega_2+3\bar\eta_1\omega_1\bar\eta_2^2\omega_2^2+\bar\eta_2^3\omega_2^3) =$$

$$= \alpha_{13}\omega_1^3+3\alpha_{12,21}\omega_1^2\omega_2+3\alpha_{11,22}\omega_1\omega_2^2+\alpha_{23}\omega_2^3,$$

$$P_1(-\varphi(x)) = \alpha_{13}\frac{\partial^3\varphi}{\partial x_1^3}+3\alpha_{12,21}\frac{\partial^3\varphi}{\partial x_1^2\partial x_2}+3\alpha_{11,22}\frac{\partial^3\varphi}{\partial x_1\partial x_2^2}+\alpha_{23}\frac{\partial^3\varphi}{\partial x_2^3}.$$

Теперь хотим бовислить $P_1(-\varphi(x))$ по членам.

$$\frac{\partial\varphi}{\partial x_1}=\varphi\left(-\frac{1}{2}\frac{\partial Q}{\partial x_1}\right)=\varphi\left[-\frac{1}{1-\varrho^2}(x_1-\varrho x_2)\right],$$

$$\frac{\partial^2\varphi}{\partial x_1^2}=\varphi\left(-\frac{1}{2}\frac{\partial Q}{\partial x_1}\right)^2+\varphi\left(-\frac{1}{2}\frac{\partial^2 Q}{\partial x_1^2}\right)=$$

$$=\varphi\left\{\frac{1}{(1-\varrho^2)^2}(x_1-\varrho x_2)^2-\frac{1}{1-\varrho^2}\right\},$$

$$\frac{\partial^3\varphi}{\partial x_1^3}=\varphi\left(-\frac{1}{2}\frac{\partial Q}{\partial x_1}\right)\left\{\ \right\}+\varphi\left[\frac{2}{(1-\varrho^2)^2}(x_1-\varrho x_2)\right]=$$

$$=\varphi\left\{-\frac{1}{(1-\varrho^2)^3}(x_1-\varrho x_2)^3+\frac{3}{(1-\varrho^2)^2}(x_1-\varrho x_2)\right\},$$

$$\frac{\partial^3\varphi}{\partial x_1^2\partial x_2}=\varphi\left(-\frac{1}{2}\frac{\partial Q}{\partial x_2}\right)\left\{\ \right\}+\varphi\left[\frac{2}{(1-\varrho^2)}(-\varrho)(x_1-\varrho x_2)\right]=$$

$$=\varphi\left[-\frac{1}{1-\varrho^2}(x_2-\varrho x_1)\right]\left\{\frac{1}{(1-\varrho^2)^2}(x_1-\varrho x_2)^2-\frac{1}{1-\varrho^2}\right\}+$$

$$+\varphi\left[\frac{-2\varrho}{1-\varrho^2}(x_2-\varrho x_1)\right]=$$

$$=\varphi\left\{-\frac{(x_1-\varrho x_2)(x_2-\varrho x_1)}{(1-\varrho^2)^3}+\frac{(x_2-\varrho x_1)(1-2\varrho)}{(1-\varrho^2)^2}\right\},$$

$$\frac{\partial^2\varphi}{\partial x_1\partial x_2}=\varphi\left(-\frac{1}{2}\frac{\partial Q}{\partial x_2}\right)\left[-\frac{1}{1-\varrho^2}(x_1-\varrho x_2)\right]+\varphi\left(\frac{\varrho}{1-\varrho^2}\right)=$$

$$=\varphi\left(-\frac{1}{1-\varrho^2}(x_2-\varrho x_1)\right)\left[-\frac{1}{1-\varrho^2}(x_1-\varrho x_2)\right]+\varphi\left(\frac{\varrho}{1-\varrho^2}\right)=$$

$$=\varphi\left[\frac{(x_2-\varrho x_1)(x_1-\varrho x_2)}{(1-\varrho^2)^2}+\frac{\varrho}{1-\varrho^2}\right],$$

$$\frac{\partial^3\varphi}{\partial x_1\partial x_2^2}=\varphi\left(-\frac{1}{1-\varrho^2}(x_2-\varrho x_1)\right)\left[\frac{(x_2-\varrho x_1)(x_1-\varrho x_2)}{(1-\varrho^2)^2}+\frac{\varrho}{1-\varrho^2}\right]+$$

$$+\varphi\left[\frac{(x_1-\varrho x_2)-\varrho(x_2-\varrho x_1)}{(1-\varrho^2)^2}\right]=$$

$$=\varphi\left\{-\frac{(x_2-\varrho x_1)^2(x_1-\varrho x_2)}{(1-\varrho^2)^3}-\frac{2\varrho(x_2-\varrho x_1)-(x_1-\varrho x_2)}{(1-\varrho^2)^2}\right\}.$$

Именно

$$P_1(-\varphi(x)) = -\frac{\varphi(x)}{6}\left\{-\alpha_{13}\frac{(x_1-\varrho x_2)^3}{(1-\varrho^2)^3}+3\alpha_{13}\frac{x_1-\varrho x_2}{(1-\varrho^2)^2}+\right.$$

$$+3\alpha_{12,21}\frac{(1-2\varrho)(x_2-\varrho x_1)}{(1-\varrho^2)^2}-3\alpha_{12,21}\frac{(x_1-\varrho x_2)^2(x_2-\varrho x_1)}{(1-\varrho^2)^3}-$$

$$-3\alpha_{11,22}\frac{(x_2-\varrho x_1)^2(x_1-\varrho x_2)}{(1-\varrho^2)^3}-3\alpha_{11,22}\frac{2\varrho(x_2-\varrho x_1)-(x_1-\varrho x_2)}{(1-\varrho^2)^2}-$$

$$\left.-\alpha_{23}\frac{(x_2-\varrho x_1)^3}{(1-\varrho^2)^3}+3\alpha_{23}\frac{x_2-\varrho x_1}{(1-\varrho^2)^2}\right\}.$$

Отсюда для $P_1(-\varphi(x_1, 0))$ получим

$$P_1(-\varphi(x_1, 0)) = -\frac{\varphi(x_1, 0)}{6}\{Ax_1^3+Bx_1\},$$

где

$$A = -\frac{1}{(1-\varrho^2)^3}(-\alpha_{13}+3\alpha_{12,21}\varrho-3\alpha_{11,22}\varrho^2+\alpha_{23}\varrho^3),$$

$$B = -\frac{1}{(1-\varrho^2)^2}[3\alpha_{13}-3\alpha_{12,21}(1-\varrho)\varrho-3\alpha_{11,22}(2\varrho^2+1)-3\alpha_{23}\varrho].$$

Из (23°), (24)—(26) получим $|\alpha_{\cdot\,,\,\cdot}|\leqq c$, т.е. $|A|, |B|\leqq C$. Таким образом, если применить для выделения главного члена асимптотической формулы (41) разложение характеристической функции с двумя членами [10], то, аналогично Теореме 1, получим

$$(*)\qquad b_1b_2(m+1)P_m(n, N)-\varphi(u_1, 0)\left[1-\frac{Au_1^3+Bu_1}{6\sqrt{m+1}}\right] = O\left(\frac{1}{m}\right).$$

Применяя оценку для A и B, и считая $|u_1|\leqq C\,m^{-a}$, отсюда получим

$$\Delta_m(n, N) = O\left(\frac{1}{m^{1/2+a}}\right).$$

Таким образом в Теоремах 5 и 6, и в формуле (51) остаточный член можно заменить на

$$O\left(\frac{1}{m^{2q}}+\frac{1}{m^{1/2+a}}+\frac{m^{3/2}}{\sqrt{N}}\right).$$

Причем, как видно из $(*)$, слагаемоя $1/m^{1/2+a}$ не может быть заменено на меньшее.

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ACCUMULATION THEOREMS FOR PRIMES IN ARITHMETIC PROGRESSIONS

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1. Knapowski and Turán devoted two long series of papers [3, 4] to "comparative prime number theory", i.e. to problems connected with comparison of the number of primes in two residue classes l_1 and l_2 (mod q), where (as in the following always)

$$(l_1, q)=(l_2, q)=1, \quad l_1 \not\equiv l_2 \pmod q. \tag{1.1}$$

Let us introduce the notations

$$\varepsilon(n) = \varepsilon(n, q, l_1, l_2) = \begin{cases} 1 & \text{if } n \equiv l_1(q) \\ -1 & \text{if } n \equiv l_2(q) \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

$$\begin{cases} a_1(n) = \begin{cases} 1 & \text{if } n = \text{prime} \\ 0 & \text{otherwise,} \end{cases} & a_3(n) = \begin{cases} \log n & \text{if } n = p \\ 0 & \text{otherwise} \end{cases} \\ a_2(n) = \dfrac{\Lambda(n)}{\log n} = \begin{cases} 1/m & \text{if } n = p^m \\ 0 & \text{otherwise,} \end{cases} & a_4(n) = \Lambda(n). \end{cases} \tag{1.3}$$

In the first series [3], Knapowski and Turán investigated the sign changes of the functions $\Delta_i(x) = \Delta_i(x, q, l_1, l_2)$ where

$$\begin{aligned} \Delta_1(x) &\overset{\text{def}}{=} \pi(x, q, l_1) - \pi(x, q, l_2) = \sum_{n \le x} \varepsilon(n) a_1(n), \\ \Delta_2(x) &\overset{\text{def}}{=} \Pi(x, q, l_1) - \Pi(x, q, l_2) = \sum_{n \le x} \varepsilon(n) a_2(n), \\ \Delta_3(x) &\overset{\text{def}}{=} \Theta(x, q, l_1) - \Theta(x, q, l_2) = \sum_{n \le x} \varepsilon(n) a_3(n), \\ \Delta_4(x) &\overset{\text{def}}{=} \psi(x, q, l_1) - \psi(x, q, l_2) = \sum_{n \le x} \varepsilon(n) a_4(n). \end{aligned} \tag{1.4}$$

The second series [4] was devoted to "Chebyshev type problems" and "accumulation theorems". By "Chebyshev-type problems" they meant investigation of the asymptotic behaviour of weighted sums of all primes or prime-powers, i.e. asymptotic behaviour of expressions of the form

$$G_i(x) = \sum_{n=1}^{\infty} \varepsilon(n) a_i(n) W_x(n), \quad x \to \infty \tag{1.5}$$

with some weight functions $W_x(n)$ depending on a parameter x (or in same cases on two parameters).

The famous assertion of Chebyshev [1] from the year 1853 was

$$\lim_{x\to\infty} \sum_{p>2} (-1)^{(p-1)/2} e^{-p/x} = -\infty, \tag{1.6}$$

but he never published a proof for this. As it was shown by Hardy—Littlewood [2] and Landau [5], (1.6) is equivalent to the Riemann—Piltz conjecture

$$L(s, \chi_1) \neq 0, \quad s = \sigma + it, \quad \sigma > 1/2 \tag{1.7}$$

for the (unique) non-principal character χ_1 mod 4. They remarked that the same holds if $(-1)^{(p-1)/2} = \varepsilon(p, 4, 1, 3)$ is replaced by

$$(-1)^{(p-1)/2} \log p = \varepsilon(p, 4, 1, 3) \log p. \tag{1.8}$$

Knapowski and Turán investigated the above type problems with the weight functions

$$W_{k,\mu}(n) = \exp\left(-\frac{(\mu-\log n)^2}{4k}\right), \quad k \leqq \mu, \quad \mu \to \infty \tag{1.9}$$

and succeeded in this case in extending the results of Hardy—Littlewood and Landau for general moduli q, under some conditions (both for the parameters and for L-functions mod q). The advantage of the weight-function in (1.9) was that the knowledge of the behaviour of

$$G_i(k, \mu) = \sum_n \varepsilon(n) a_i(n) \exp\left(-\frac{(\mu-\log n)^2}{4k}\right) \tag{1.10}$$

made possible to get information about sums of type

$$\sum_{U<n\leqq V} \varepsilon(n) a_i(n) \tag{1.11}$$

with some

$$[U, V] \subset \left[e^{\mu-3\sqrt{\mu k}}, e^{\mu+3\sqrt{\mu k}}\right], \tag{1.12}$$

since the weight-function (1.9) gives completely negligible weights outside the mentioned interval. (For the precise connections between (1.10) and (1.11) see Lemmas 7 and 8 in the Appendix of the present paper.)

In all of their results for general moduli, Knapowski and Turán had to assume the so-called "Haselgrove-condition", i.e. that the L-functions mod q have no positive real zeros. They formulated this in the quantitative form (where we shall assume always $0 < A(q) \leqq 1$)

$$L(s, \chi, q) \neq 0 \quad \text{for} \quad 0 < \sigma < 1, \quad |t| < A(q). \tag{1.13}$$

(This condition was verified with explicit $A(q)$ by Haselgrove for all $q \leqq 10$, and by Spira for all $q \leqq 24$). Without an assumption of this type one cannot exclude e.g. the existence of a real zero β_0 (of a real L-function χ_1) for which all other zeros are in the halfplane $\sigma < \beta_0 - \varepsilon$. In this case one has

$$\Delta_4(x, q, l_1, l_2) = (\chi_1(l_2) - \chi_1(l_1)) \frac{x^{\beta_0}}{\beta_0} + O(x^{\beta_0-\varepsilon}) \tag{1.14}$$

thus $\Delta_4(x, q, l_1, l_2)$ is of constant sign if $\chi_1(l_1)\neq\chi_1(l_2)$, $x>x_0$. Knapowski and Turán could show infinitely many sign changes for $\Delta_4(x, q, l_1, l_2)$ under this condition, but in all other investigations for general l_1, l_2, q they had to assume besides (1.13) a kind of "finite Riemann—Piltz conjecture" i.e.

$$L(s, \chi, q) \neq 0 \quad \text{for} \quad \sigma > 1/2, \quad |t| \leqq D, \tag{1.15}$$

with some $D \geqq cq^{10}$, e.g. Under this additional assumption they succeeded in proving that $\Delta_2(x)$ changes sign infinitely often for all l_1, l_2 with (1.1), and that $\Delta_1(x)$ and $\Delta_3(x)$ change sign infinitely often if l_1 and l_2 are both quadratic non-residues or both quadratic residues mod q. (The latter condition, as well as (1.15) can be deleted if $l_1=1$.)

Although the above mentioned results (with the exception of that concerning the case $l_1=1$) follow very easily from a general theorem of Landau [6, § 197] (even without making use of (1.15)), the high importance of the results of Knapowski and Turán was that they could explicitly "localise" sign changes of $\Delta_i(x)$, extimate the number of sign changes from below and give upper estimates for the first sign change in explicit dependence on q and $A(q)$, whereas Landau's theorem is completely ineffective.

Now we shall formulate those theorems of Knapowski and Turán from the second series (also some of them implicitly contained in their works) which are in connection with the results of the present paper. In all of the following theorems the truth of the Haselgrove condition (1.13) is assumed, further in Theorems I, II, III and IV the finite Riemann—Piltz conjecture is needed with a value D satisfying

$$D \geqq cq^{15} \tag{1.16}$$

and

$$D \geqq \frac{12\pi}{A(q)}. \tag{1.17}$$

(The somwhat unnatural assumption (1.17) was needed for some technical purposes.) In (1.16) and in the following c will always denote some explicitly calculable positive absolute constant, which may have different values at various appearances.

THEOREM I. *For every* q, l_1, l_2 *with* (1.1), $q>c$ *and*

$$Y \geqq \exp\left(D^8 \exp(q^{10})\right) \tag{1.18}$$

there exist x *and* k *with*

$$x \in \left[Y^{1-\frac{2}{D}}, Y\right] \tag{1.19}$$

$$K \in \left[\frac{2\log Y}{D^2}, \frac{2\log Y}{D^2} + \sqrt{\log Y}\right] \tag{1.20}$$

such that

$$\sum_n \varepsilon(n, q, l_1, l_2)\Lambda(n)\exp\left(-\frac{\log^2(n/x)}{4k}\right) > Y^{\frac{1}{2}-\frac{8}{D}}. \tag{1.21}$$

If we additionally assume that l_1 *and* l_2 *are both quadratic non-residues then also*

$$\sum \varepsilon(p, q, l_1, l_2) \log p \exp\left(-\frac{\log^2(p/x)}{4k}\right) > Y^{\frac{1}{2}-\frac{8}{D}}. \tag{1.22}$$

THEOREM II. *If* $q>c$, $l_1 \not\equiv l_2(q)$ *are both quadratic residues, then for*

$$Y \geqq \exp\left(\exp\left(D^6\right)\right) \tag{1.23}$$

we have x *and* k *with* (1.19), (1.20), (1.22).

These theorems (Theorems I and II are contained in parts IV and VI, resp., of the second series [4]) implied (cf. Lemma 7)

THEOREM III. *Under the conditions of Theorem I there exist*

$$U_i, V_i \in \left[Y^{1-\frac{8}{D}}, Y^{1+\frac{8}{D}}\right] \tag{1.24}$$

such that for $i=2, 4$

$$\sum_{U_i<n\leqq V_i} \varepsilon(n, q, l_1, l_2) a_i(n) > Y^{\frac{1}{2}-\frac{10}{D}}. \tag{1.25}$$

Further, if l_1, l_2 *are quadratic non-residues, we have also for* $i=1, 3$

$$\sum_{U_i<p\leqq V_i} \varepsilon(p, q, l_1, l_2) a_i(p) > Y^{\frac{1}{2}-\frac{10}{D}}. \tag{1.26}$$

THEOREM IV. *Under conditions of Theorem* II *we have* U_1, V_1, U_3, V_3 *with* (1.24) *and* (1.26).

We may remark that in case of the infinite Riemann—Piltz conjecture we can choose in Theorems I, III and II, IV, resp.

$$D = \frac{\log^{1/8} Y}{\exp(q^{10})} \quad \text{and} \quad D = (\log\log Y)^{1/6}, \tag{1.27}$$

and this furnishes localisations of type $[Y \exp(-g(Y)), Y]$ and lower estimations of type $\sqrt{Y} \exp(-g(Y))$ with

$$g(Y) = c \exp(q^{10}) \log^{7/8} Y \quad \text{and} \quad g(Y) = \frac{c \log Y}{(\log\log Y)^{1/6}} \tag{1.28}$$

resp., which improve the results compared with the assumption of the finite Riemann—Piltz conjecture (which corresponds to $g(Y)=cD^{-1}\log Y$).

In the present work we shall show (cf. Theorems 1—3) that in Theorems I—IV it is possible

a) to drop the condition (1.17) (and to weaken (1.16) to $D=c_0 q^2 \log^6 q$),

b) to admit smaller values of Y in dependence on q, D and $A(q)$,

c) to improve localisation and lower estimation, roughly speaking one can take $g(Y) \approx c(q) \log^{1/2} Y$ supposing only the finite Riemann—Piltz conjecture (cf. (1.28) which is valid on the infinite Riemann—Piltz conjecture),

d) to assure that U_i, V_i would be really short intervals, i.e. $V_i - U_i = o(U_i)$.

We remark also that the role of l_1 and l_2 being symmetrical in all Theorems I—IV, we can trivially state the same inequalities as (1.21)—(1.22), (1.25)—(1.26) with $< -Y^{1-\frac{C}{D}}$ instead of $> Y^{1-\frac{C}{D}}$. Further on we remark that (1.22) and (1.26) trivially follow from (1.21) and (1.25), resp., if l_1 is a quadratic non-residue and l_2 a residue (but this harms the symmetrical role already, naturally).

Now in the remaining case of l_1 being a residue, l_2 a non-residue, (1.22) does not remain true if one assumes the (infinite) Riemann—Piltz conjecture, as shown by Theorem V. Supposing the Haselgrove condition (1.13) and the (infinite) Riemann—Piltz conjecture one has for any k, $x = e^{\mu} > cq^{50}$ with

$$(1.29) \qquad c\frac{\log q}{A^2(q)} \leqq k \leqq \mu$$

the inequality

$$(1.30) \qquad \sum_p \varepsilon(p, q, l_1, l_2) \log p \exp\left(-\frac{(\mu - \log p)^2}{4k}\right) < -c\sqrt{x}$$

if l_1 is a quadratic residue, l_2 a non-residue mod q.

For the proof see Theorem VI of part II of the second series [4]. On the other hand if the infinite Riemann—Piltz conjecture is false then at least in the case $l_1 = 1$ or $l_2 = 1$ they could prove an explicit theorem which we formulate now roughly as (for the precise formulation see Theorem IV of part II in [4])

THEOREM VI. *If there is a* $\varrho_0 = \beta_0 + i\gamma_0$ *zero of an* $L(s, \chi, q)$ *with* $\beta_0 > 1/2$, $\chi(1) \neq 1$, *then for* $Y > c(\varrho_0, q, A(q))$ *one has (with explicitly localised values x and k)*

$$(1.31) \qquad \sum_p \varepsilon(p, q, l, 1) \log p \exp\left(-\frac{(\mu - \log p)^2}{4k}\right) = \Omega_{\pm}\left(x^{\beta_0} \exp(-\log^{5/7} x)\right).$$

We shall show (cf. Theorem 4) that in Theorem V it is enough to assume the finite Riemann—Piltz conjecture, if we restrict k more strongly than in (1.29), further it is possible to calculate an asymptotic value for the left hand side of (1.30) (in dependence of k, μ and q). Concerning the problem dealt with in Theorem VI for general pairs l_1 and l_2 we are able to prove a similar theorem (cf. Theorem 5), but this holds only for $Y > Y_0$, an ineffective constant (although we can give good lower estimation and good localisation for the corresponding values x).

2. In Theorems 1—3 we shall assume always the truth of the Haselgrove condition (1.13) and the finite Riemann—Piltz conjecture (1.15) until a level D with

$$(2.1) \qquad D \geqq D_0 = c_0 q^2 \log^6 q$$

where c_0 is a sufficiently large absolute constant. First we shall formulate Theorems 1 and 2 which improve the results of Theorems I and II of Knapowski and Turán. In these theorems we shall have a parameter λ for which we suppose

$$(2.2) \qquad \lambda \geqq 20D_0.$$

We shall always assume the trivial condition (1.1) without mentioning it. We shall use the notation

$$l_1 \sim l_2 \tag{2.3}$$

if l_1 and l_2 are both quadratic non-residues or if they are both quadratic residues mod q. In the sequel c will denote a generic (explicitly calculable positive absolute) constant which might have other values at different appearences.

THEOREM 1. *Let us suppose for* q *the truth of the Haselgrove condition* (1.13), *the finite Riemann—Piltz conjecture* (1.15)—(2.1), *further* (2.2). *Then for every* $T>c$ *with*

$$L = \log T > \max\left(\frac{cq^2}{A(q)}\lambda \log^3 L,\ c\lambda^2\right) \tag{2.4}$$

there exist $x=e^\mu$, k *with*

$$x \in \left[T \exp\left(-\frac{cq^2}{A(q)}\lambda \log^3 \lambda\right),\ T\right], \tag{2.5}$$

$$k \in \left[\frac{\mu}{\lambda^2}-1,\ \frac{\mu}{\lambda^2}+1\right], \tag{2.6}$$

(2.7)

$$\sum_n \varepsilon(n, q, l_1, l_2)\Lambda(n) \exp\left(-\frac{\log^2(n/x)}{4k}\right) > \sqrt{x} \exp\left(-\frac{LD^2}{\lambda^2}-\frac{cq^2}{A(q)}\lambda \log^3 L\right) > x^{0.49}.$$

It is nearly trivial to show (cf. Lemma 6) that (essentially) the same estimate is valid if we replace $\Lambda(n)$ by $\log p$ and assume that l_1 is a quadratic non-residue. Beyond this we shall settle also the case of l_1 and l_2 being both quadratic residues.

THEOREM 2. *Assuming the conditions of Theorem* 1 *and* $l_1 \sim l_2$ *(cf.* (2.3)*), we have for every* $T>c$ *with* (2.4) *an* x *and* k *satisfying* (2.5)—(2.6) *such that*

(2.8)

$$\sum_p \varepsilon(p, q, l_1, l_2) \log p \exp\left(-\frac{\log^2(p/x)}{4k}\right) > \sqrt{x} \exp\left(-\frac{LD^2}{\lambda^2}-\frac{cq^2}{A(q)}\lambda \log^3 L\right) > x^{0.49}.$$

The application of Lemma 6 of the Appendix makes possible to obtain accumulation theorems for pime powers and in case of $l_1 \sim l_2$, also for primes.

Now we shall choose λ in such a way as to optimize the lower estimation as well as the localisation of the short intervals furnished by Lemma 7.

For this purpose we choose for $Y=e^{L_1}$

$$\lambda = \frac{A^{1/2}(q)}{q} \frac{L_1^{1/2}}{\log^{3/2} L_1}, \quad \log T = L = L_1\left(1-\frac{7}{2\lambda}\right). \tag{2.9}$$

Then an easy calculation shows that for

$$L_1 > \frac{D_0^5}{A(q)} \log^3 \frac{1}{A(q)} \tag{2.10}$$

we have (2.2), further

$$\frac{LD_0^2}{\lambda^2} \ll \frac{L_1}{\lambda}, \quad \frac{q^2}{A(q)} \lambda \log^3 L \ll \frac{L_1}{\lambda}, \quad \sqrt{\mu k} \leqq \frac{1.05 L_1}{\lambda}. \tag{2.11}$$

So we obtain from Theorems 1 and 2 with the aid of Lemma 7 of the Appendix the following

THEOREM 3. *Let us suppose for* q *the Haselgrove condition* (1.13) *and the finite Riemann—Piltz conjecture* (1.15)—(2.1) *and let* $Y > c$ *satisfy*

$$Y > \exp\left(\frac{D_0^5}{A(q)} \log^3 \frac{1}{A(q)}\right). \tag{2.12}$$

Further let us use the notation

$$f(Y) = \exp\left(-cqA^{-1/2}(q) \log^{1/2} Y (\log\log Y)^{3/2}\right). \tag{2.13}$$

Then in case of $l_1 \sim l_2$, *there exist*

$$U_1, V_1, U_3, V_3 \in [f(Y)Y, Y] \tag{2.14}$$

such that $U_i < V_i \leqq U_i + U_i \exp(-\log^{1/2} U_i)$ *and*

$$\sum_{U_1 < p \leqq V_1} \varepsilon(p, q, l_1, l_2) > f(Y)\sqrt{Y}, \tag{2.15}$$

$$\sum_{U_3 < p \leqq V_3} \varepsilon(p, q, l_1, l_2) \log p > f(Y)\sqrt{Y}. \tag{2.16}$$

Further we have for all pairs l_1, l_2 *with* (1.1)

$$U_2, V_2, U_4, V_4 \in [f(Y)Y, Y] \tag{2.17}$$

such that $U_i \leqq V_i \leqq U_i + U_i \exp(-\log^{1/2} U_i)$ *and*

$$\sum_{U_2 < p^m \leqq V_2} \varepsilon(p^m, q, l_1, l_2) \frac{1}{m} > f(Y)\sqrt{Y}, \tag{2.18}$$

$$\sum_{U_4 < n \leqq V_4} \varepsilon(n, q, l_1, l_2) \Lambda(n) > f(Y)\sqrt{Y}. \tag{2.19}$$

We remark that it would be possible to deduce (2.15)—(2.16) from (2.18)—(2.19) using the prime number theorem for arithmetic progression with best known error terms. However, we preferred to prove Theorem 2 (from which (2.15)—(2.16) is an easy consequence by making use of Lemma 7) in order to avoid these deep theorems. Beside this, Theorem 2 cannot be deduced from Theorem 1 using even the best known error terms.

Let us consider now the case, when l_1 is a quadratic residue and l_2 a non-residue. Our following theorem shows that even weaker conditions than the *finite* Riemann—Piltz conjecture can assure theorems similar to Theorem V, if we assume that $k, \mu \to \infty$ and $4 \leqq \mu/k \leqq c(q)$.

THEOREM 4. *Suppose the truth of the Haselgrove condition* (1.13) *for* q, *and let* k, λ, ω *satisfy* $\lambda \geqq 2$, $\omega \to \infty$ *and*

$$k \geqq \frac{5 \log q + \omega}{A^2(q)}. \tag{2.20}$$

Let us suppose that the zeros $\varrho = 1/2 + \delta + i\gamma$ *of all* $L(s, \chi, q)$ *functions with* $\chi(l_1) \neq \neq \chi(l_2)$ *and* $|\gamma| \leqq \lambda$ *satisfy*

$$\gamma^2 \geqq (2\lambda^2 + 3)\delta; \tag{2.21}$$

where l_1 *is a quadratic residue,* l_2 *a non-residue* mod q. *Then with* $\mu = k\lambda^2$ *we have for* $\omega \to \infty$

$$\frac{1}{2\sqrt{\pi k}} \sum_p \varepsilon(p, q, l_1, l_2) \log p \exp\left(-\frac{(\mu - \log p)^2}{4k}\right) \sim -\frac{N(q)}{2\varphi(q)} e^{\frac{k}{4} + \frac{\mu}{2}} \tag{2.22}$$

(uniformly in the variables k, λ *and* q*) where* $N(q)$ *denotes the number of solutions of* $x^2 \equiv 1 \pmod q$.

COROLLARY 1. *If the Haselgrove condition and the finite Riemann—Piltz conjecture are true until the level* λ *for all* $L(s, \chi, q)$ *functions with* $\chi(l_1) \neq \chi(l_2)$, *and* k *satisfies* (2.20) *then* (2.22) *is true, if* l_1 *is quadratic residue,* l_2 *a non-residue.*

Corollary 1 shows that if we assume the Haselgrove condition and the finite Riemann—Piltz conjecture up to a level D with

$$D \geqq 20D_0 = c_1 q^2 \log^6 q, \tag{2.23}$$

then the assertion of Theorem 2, (2.8), definitely does not hold in full generality if l_1 is a residue, l_2 a non-residue. We have namely the opposite inequality for all k, μ with

$$\frac{\mu}{D^2} \leqq k \leqq \frac{\mu}{4}, \quad k \geqq \frac{3 \log q + c_2}{A^2(q)} \tag{2.24}$$

i.e. if λ is chosen as

$$20D_0 \leqq \lambda \leqq D. \tag{2.25}$$

COROLLARY 2. *If the Haselgrove condition and the infinite Riemann—Piltz conjecture are true for all L-functions* mod q *with* $\chi(l_1) \neq \chi(l_2)$, *where* l_1 *is a quadratic residue,* l_2 *a non-residue* mod q, *further* μ *and* k *satisfy*

$$k \geqq \frac{5 \log q + \omega}{A^2(q)}, \quad \mu \geqq 4k \tag{2.26}$$

then (2.22) *is true for* $\omega \to \infty$.

We remark that if the infinite Riemann—Piltz conjecture is not true, then (2.21) implies $\mu/k \leqq c(q)$, thus Theorem 4 is not applicable if $\mu/k > c(q)$. This is due to the fact (mentioned in the preceding section) that in this case there is really no constant preponderance of primes $\equiv l_2$ over those $\equiv l_1$ in the above sense. It is relatively easy to show the following ineffective

THEOREM 5. *Suppose the truth of the Haselgrove condition* (1.13), *the existence of a zero* $\varrho_0=\beta_0+i\gamma_0$, $\beta_0-(1/2)=\delta_0>0$ *of an L-function* $\bmod q$ *with* $\chi(l_1)\neq\chi(l_2)$, *where* l_1 *and* l_2 *are arbitrary with* (1.1). *Let us suppose we have any* $\eta>0$ *and a (fixed)* $\lambda>0$ *with*

$$\lambda^2>\frac{\gamma_0^2}{\delta_0}-(1+\delta_0)\Leftrightarrow\gamma_0^2<(\lambda^2+1+\delta_0)\delta_0. \tag{2.27}$$

Then for every $Y>Y_0(q,\eta,\lambda)$ *(where* Y_0 *is an ineffective constant) we have an* $x=e^\mu$ *and* k *with*

$$x=e^\mu\in[Ye^{-2\pi/A(q)},Y],\quad \frac{\mu}{k}\in[\lambda^2,\lambda^2+\eta] \tag{2.28}$$

such that

$$\frac{1}{2\sqrt{\pi k}}\sum_p \varepsilon(p,q,l_1,l_2)\log p\exp\left(-\frac{(\mu-\log p)^2}{4k}\right)>(2-\eta)|a_{\varrho_0}|e^{k\beta_0^2+\mu\beta_0} \tag{2.29}$$

if

$$a_{\varrho_0}\overset{\text{def}}{=}\frac{1}{\varphi(q)}\sum_{\substack{\chi\\ L(\varrho_0,\chi)=0}}(\bar\chi(l_2)-\bar\chi(l_1))m_\chi(\varrho_0)\neq 0, \tag{2.30}$$

where $m_\chi(\varrho_0)$ *denotes the multiplicity of the zero* ϱ_0.

If we replace $\varepsilon(p)\log p$ *by* $\varepsilon(n)\Lambda(n)$, *then* (2.29) *holds for an arbitrary zero* ϱ_0 *with* $\beta_0\geqq 1/2$, *without condition* (2.27).

3. The main tool in proving Theorem 1 is a one-sided power-sum theorem of Knapowski and Turán (cf. Theorem 4.1 in part III of [3]) which we state now in a slightly improved form as follows. (For the modifications needed in the proof see the Appendix.)

LEMMA 1. *Let* b_j, z_j *be complex numbers* $(j=1,2,\dots,n)$ *with*

$$0<\varkappa\leqq|\arg z_j|\leqq\pi\quad(j=1,2,\dots,n), \tag{3.1}$$

$$|z_1|\geqq\dots\geqq|z_n|. \tag{3.2}$$

Then for any h *with* $1\leqq h\leqq n$ *and for any* $m\geqq 0$, *there exists an integer*

$$v\in\left[m,\ m+n\left(3+\frac{\pi}{\varkappa}\right)\right] \tag{3.3}$$

such that

$$\operatorname{Re}\sum_{j=1}^n b_jz_j^v>\frac{E}{2n+1}|z_h|^v\left(\frac{|z_h|}{|z_1|}\right)^{n\left(5+\frac{\pi}{\varkappa}\right)}\left(\frac{n}{24e\left(m+n\left(5+\frac{\pi}{\varkappa}\right)\right)}\right)^{2n} \tag{3.4}$$

where

$$E=\min_{l\geqq h}\left|\operatorname{Re}\sum_{j=1}^l b_j\right|. \tag{3.5}$$

Further we shall use the integral-formula ($k>0$ real, A arbitrary complex)

$$\frac{1}{2\pi i}\int_{(2)} e^{ks^2+As}\,ds = \frac{1}{2\sqrt{\pi k}}\exp\left(-\frac{A^2}{4k}\right). \tag{3.6}$$

Introducing the notations

$$F(s) = \frac{1}{\varphi(q)}\sum_{\chi(\mathrm{mod}\, q)}(\bar\chi(l_2)-\bar\chi(l_1))\frac{L'}{L}(s,\chi), \quad \mu = \log x, \tag{3.7}$$

we obtain by (3.6) (denoting the zeros of L-functions in $\sigma\geqq 0$ by $\varrho=\beta+i\gamma$)

(3.8)

$$I \overset{\mathrm{def}}{=} \frac{1}{2\sqrt{\pi k}}\sum_n \varepsilon(n)\Lambda(n)\exp\left(-\frac{(\mu-\log n)^2}{4k}\right) = \frac{1}{2\pi i}\int_{(2)}\sum_n \frac{\varepsilon(n)\Lambda(n)}{n^s}e^{ks^2+\mu s}\,ds =$$

$$= \frac{1}{2\pi i}\int_{(2)} F(s)e^{ks^2+\mu s}\,ds = \frac{1}{\varphi(q)}\sum_\chi(\bar\chi(l_2)-\bar\chi(l_1))\sum_{\substack{\varrho=\varrho(\chi)\\ \beta\geqq 0}} e^{k\varrho^2+\mu\varrho}+R$$

where (using $L'(s)/L(s)\ll\log(q(|t|+2))$ for $\sigma=-1/2$)

$$R \overset{\mathrm{def}}{=} \frac{1}{2\pi i}\int_{(-1/2)} F(s)e^{ks^2+\mu s}\,ds = O(1). \tag{3.9}$$

Although the main part of I is an infinite exponential sum, we can estimate easily the contribution of zeros with $|\gamma|\geqq 2\lambda$, since we have for any χ mod q

$$\sum_{\substack{\varrho=\varrho(\chi)\\ |\gamma|\geqq 2\lambda}} |e^{k\varrho^2+\mu\varrho}| \ll \sum_{n=[2\lambda]}^{\infty} e^{k(1-n^2)+\mu}\log(qn) = O(1) \tag{3.10}$$

owing to (2.6) and the well-known relation

$$\sum_{\substack{\varrho=\varrho(\chi)\\ n\leqq|\gamma|\leqq n+1}} 1 \ll \log(q(n+2)). \tag{3.11}$$

The main difficulty in applying Lemma 1 for the remaining finite exponential sum is that perhaps $E=0$ (this occurs probably relatively often in reality). This difficulty may be overcome by the following

LEMMA 2. *There exists a prime* $P\equiv l_1(q)$,

$$\frac{D_0}{2} < P\log^2 P < D_0 \tag{3.12}$$

such that

$$\frac{1}{\varphi(q)}\mathrm{Re}\sum_\chi(\bar\chi(l_2)-\bar\chi(l_1))\sum_{\substack{\varrho=\varrho(\chi)\\ \beta\geqq 0}} e^{\frac{\varrho^2}{P^2\log^2 P}}P^\varrho \gg P\log^2 P \tag{3.13}$$

where the constant implied by the $\ll$ *sign is independent from* c_0 *appearing in* (2.1).

This is a slightly modified form of Lemma VI of Knapowski and Turán [4, part IV]. (To obtain the modification, we have only to note that the finite Riemann—Piltz conjecture (1.15)—(2.1) and the shortened explicit prime number formula (cf. Prachar [7]) imply for $x \geqq q$

$$(3.14) \qquad \sum_{\substack{n \leqq x \\ n \equiv l(q)}} \Lambda(n) = \frac{x}{\varphi(q)} + O(\sqrt{x}\log^2 D) + O\left(\frac{x}{D}\log^2 x\right)$$

and this assures the existence of primes with (3.12).)

Actually (3.13) is true for every prime $\equiv l_1(q)$ with (3.12) but we do not need this fact.

In connection with Lemma 2 we have to note that in (3.13) the contribution of zeros with $|\gamma| > D_0$ is negligible, since in view of (3.11) we have for any χ mod q

$$(3.15) \qquad \sum_{\substack{\varrho = \varrho(\chi) \\ |\gamma| > D_0}} \left| e^{\frac{\varrho^2}{P^2 \log^2 P}} P^{\varrho} \right| \ll \sum_{n=[P\log^2 P]}^{\infty} e^{\frac{1-n^2}{P^2 \log^2 P}} P \log(nq) \ll 1$$

where the constants implied by the $\ll$ sign are independent from c_0 appearing in (2.1). This means that the estimation (3.13) is valid for any set of zeros which contains all zeros with $|\gamma| \leqq D_0$.

The next problem arises if we want to asure the argument condition (3.1) (possibly with a not too small value of $\varkappa$). For this purpose the following general lemma is useful.

LEMMA 3. *Let* $a_j \neq 0$ *be real numbers,* $j=1, \ldots, n$, $\frac{1}{n}\sum_{j=1}^{n} \frac{1}{|a_j|} \leqq \eta$. *Then for every* H *there exists an* y_0 *with*

$$(3.16) \qquad y_0 \in [H, H+\eta]$$

such that for any integer k *and all* $j=1, \ldots, n$

$$(3.17) \qquad |y_0 a_j - 2k\pi| \geqq \frac{1}{4n}.$$

This implies (with the aim of further applications in Theorem 4)

LEMMA 4. *Let* ϱ_j *be complex numbers,* $j=1, \ldots, n$, $\operatorname{Re} \varrho_j \geqq 0$, $\frac{1}{n}\sum_{j=1}^{n} \frac{1}{|\operatorname{Im} \varrho_j|} \leqq \eta$. *Then for arbitrary* $\omega \geqq 0$ *and* H *there exists a* B *with*

$$(3.18) \qquad B \in \left[H, H+\frac{3}{2}\eta\right]$$

such that for all $\varrho = \varrho_j$ $(j=1, \ldots, n)$

$$(3.19) \qquad \pi \geqq |\arg e^{\omega B \varrho^2 + B\varrho}| \geqq \frac{1}{8n},$$

$$(20) \qquad \pi \geqq |\arg e^{\omega B(\varrho/2)^2 + B\varrho/2}| \geqq \frac{1}{8n}.$$

LEMMA 5. *For $M \geqq q$ we have with*

$$N_q(M) = \sum_{\chi(\mathrm{mod}\, q)} \sum_{\substack{\varrho=\varrho(\chi) \\ |\gamma| \leqq M}} 1 \tag{3.21}$$

the inequality

$$\frac{1}{N_q(M)} \sum_{\chi(\mathrm{mod}\, q)} \sum_{\substack{\varrho=\varrho(\chi) \\ |\gamma| \leqq M}} \frac{1}{|\gamma|} \ll \frac{A^{-1}(q)+\log M}{M}. \tag{3.22}$$

Lemmas 3 and 4 are improved versions of the lemmas of Appendix VIII in Turán's book [8].

We shall prove them in our Appendix. To prove Lemma 5 we use (3.11) and the well known relation

$$N_q(M) \gg \varphi(q) M \log M. \tag{3.23}$$

From (3.11) we obtain

$$\sum_{\chi(\mathrm{mod}\, q)} \sum_{\substack{\varrho=\varrho(\chi) \\ 1 \leqq \gamma \leqq M}} \frac{1}{|\gamma|} \ll \varphi(q) \log^2 M \tag{3.24}$$

and the Haselgrove condition with (3.11) trivially implies

$$\sum_{\chi(\mathrm{mod}\, q)} \sum_{\substack{\varrho=\varrho(\chi) \\ |\gamma| \leqq 1}} \frac{1}{|\gamma|} \ll \varphi(q) \log q \cdot A^{-1}(q). \tag{3.25}$$

The required result follows from (3.23)—(3.25). □

Lemmas 4 and 5 show that in our case we can choose

$$H = \eta = \frac{cA^{-1}(q) \log \lambda}{\lambda}. \tag{3.26}$$

In view of (3.8)—(3.10), our task is to assure a good positive lower bound for (the real part of)

$$\mathscr{F} = \frac{1}{\varphi(q)} \sum_{\chi} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \sum_{\substack{\varrho=\varrho(\chi) \\ \beta \geqq 0, |\gamma| \leqq 2\lambda}} e^{k\varrho^2+\mu\varrho}. \tag{3.27}$$

$\mathscr{F}$ can be written in the form (by appropriate choice of k and μ)

$$\mathscr{F} = \mathscr{F}(\nu) = \sum_{\substack{\varrho \\ |\gamma| \leqq 2\lambda}} b_\varrho z_\varrho^\nu \tag{3.28}$$

where ϱ runs over zeros of all L-functions mod q,

$$b_\varrho = \frac{1}{\varphi(q)} \sum_{\substack{\chi \\ L(\varrho,\chi)=0}} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) e^{k_0\varrho^2+\mu_0\varrho} \tag{3.29}$$

with

$$k_0 = \frac{1}{P^2 \log^2 P}, \quad \mu_0 = \log P, \tag{3.30}$$

further

(3.31) $$z_\varrho = e^{B\varrho^2/\lambda^2 + B\varrho}$$

with the value of B furnished by Lemmas 4 and 5 (with the choice of H and η given by (3.26)). The variables k and μ will depend on ν in the following way

(3.32) $$k = k_0 + \frac{B\nu}{\lambda^2}, \quad \mu = \mu_0 + B\nu$$

and ν will be chosen appropriately with

(3.33) $$\nu \in \left[\frac{L-\mu_0}{B} - cq^2\lambda^2 \log^2 \lambda,\ \frac{L-\mu_0}{B}\right].$$

First we note that the number n of terms in (3.28) satisfies

(3.34) $$n < cq\lambda \log \lambda$$

(this follows from (3.11)). In view of Lemma 4 we can set

(3.35) $$\varkappa = \frac{1}{8n}.$$

After numbering the z_ϱ's according to (3.2), we choose h as the largest index (i.e. $z_h = z_{\varrho_0}$ as the smallest term) corresponding to a zero with

(3.36) $$|\gamma| \leqq D_0.$$

Taking into account the remark following (3.15), this choice assures

(3.37) $$\operatorname{Re} \sum_{j=1}^{l} b_j > 1 \quad (l = h, h+1, \ldots, n).$$

Owing to (2.1) ϱ_0 lies on the critical line and so

(3.38) $$|z_h| \geqq \exp\left(\frac{B}{2} - \frac{B}{\lambda^2} D_0^2\right)$$

and (owing to $\lambda \geqq 20 D_0$)

(3.39) $$\frac{|z_h|}{|z_1|} \geqq \exp\left(\frac{B}{2} - \frac{B}{\lambda^2} D_0^2 - B - \frac{B}{\lambda^2}\right) > e^{-B} \geqq e^{-3H}.$$

Now applying Lemma 1 for $\mathscr{F}(\nu)$ in (3.28) we obtain the existence of an ν with (3.33) such that, owing to (3.28)—(3.33),

(3.40) $$\operatorname{Re} \mathscr{F}(\nu) > \exp\left(\mu_0 + \frac{B\nu}{2} - (2n+1) 2 \log L - \frac{B\nu D_0^2}{\lambda^2} - \right.$$
$$\left. - 3H30n^2\right) < \sqrt{x} \exp\left(-\frac{LD_0^2}{\lambda^2} - \frac{cq^2}{A(q)} \lambda \log^3 L\right). \quad \square$$

4. If l_1 and l_2 are both quadratic non-residues then the corresponding assertion of Theorem 2 immediately follows from Theorem 1, taking into account the following.

LEMMA 6. *If* $1 \leqq k \leqq \dfrac{\log x}{10}$ *and* $g(n)$ *is a function with*

$$G(r) = \sum_{n \leqq r} g(n) \sim C r^{\alpha} \tag{4.1}$$

where $1/10 \leqq \alpha \leqq 1$, *then*

$$\frac{1}{2\sqrt{\pi k}} \Sigma g(n) \exp\left(- \frac{\log^2 \dfrac{n}{x}}{4k} \right) \sim C\alpha e^{k\alpha^2} x^{\alpha} \tag{4.2}$$

for $x \to \infty$ *(uniformly in* k*).*

The proof follows easily by partial summation (see formulae (53.3.7.)—(53.3.10) of [8]). □

The case of l_1 and l_2 being quadratic residues is much more complicated. First we note that the number of solutions of the congruences

$$x^2 \equiv l_1 (\mathrm{mod}\, q), \quad x^2 \equiv l_2 (\mathrm{mod}\, q) \tag{4.3}$$

is equal. Let us denote the solutions by $(N = N(q))$

$$\alpha_1', \ldots, \alpha_N' \quad \text{and} \quad \alpha_1'', \ldots, \alpha_N'', \tag{4.4}$$

resp. Let us introduce the notations $(\mu = \log x)$

$$I^* \overset{\mathrm{def}}{=} \frac{1}{2\sqrt{\pi k}} \left\{ \sum_n \varepsilon(n) \Lambda(n) \exp\left(- \frac{(\mu - \log n)^2}{4k} \right) - \right. \tag{4.5}$$

$$\left. - \sum_p \varepsilon(n^2) \Lambda(n) \exp\left(- \frac{(\mu - \log n^2)^2}{4k} \right) \right\},$$

(4.6)

$$F^*(s) \overset{\mathrm{def}}{=} \frac{1}{\varphi(q)} \left\{ \sum_{\chi} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \frac{L'}{L}(s, \chi) + \sum_{i=1}^{N} \sum_{\chi} (\bar{\chi}(\alpha_i') - \bar{\chi}(\alpha_i'')) \frac{L'}{L}(2s, \chi) \right\}.$$

Since the expression on the left hand side of (2.8) differs from $2\sqrt{\pi k} I^*$ only in respect of some prime powers of the form p^j, $j \geqq 3$, applying Lemma 6 for

$$G(r) = \sum_{n = p^j \leqq r,\, j > 2} \Lambda(n) + \sum_{n = p^j \leqq r,\, j > 1} \Lambda(n^2)$$

we have $C = 1$, $\alpha = 1/3$ and so Theorem 2 will follow if we can assure the estimate (3.40) for I^*. Analogously to (3.8) we obtain

$$I^* = \frac{1}{\varphi(q)} \left\{ \sum_{\chi} (\bar{\chi}(l_2) - \bar{\chi}(l_1)) \sum_{\substack{\varrho = \varrho(\chi),\, \beta \geqq 0 \\ |\gamma| \leqq 2\lambda}} e^{k\varrho^2 + \mu\varrho} + \right. \tag{4.7}$$

$$\left. + \frac{1}{2} \sum_{\chi} \sum_{i=1}^{N} (\bar{\chi}(\alpha_i') - \bar{\chi}(\alpha_i'')) \sum_{\substack{\varrho = \varrho(x),\, \beta \geqq 0 \\ |\gamma| \leqq 2\lambda}} e^{k(\varrho/2)^2 + \mu(\varrho/2)} \right\} + O(1) \overset{\mathrm{def}}{=} \mathscr{F}^* + O(1)$$

since similarly to (3.10) we have again

$$\sum_{\substack{\varrho=\varrho(\chi)\\|\gamma|\geqq 2\lambda}} e^{k(\varrho/2)^2+\mu(\varrho/2)} \ll \sum_{n=[\lambda]}^{\infty} e^{k\left(\frac{1}{4}-n^2\right)+\frac{\mu}{2}\log(qn)} = O(1). \tag{4.8}$$

Now using the notations (3.28)—(3.33) we can write the exponential sum $\mathscr{F}^*$ as

$$\mathscr{F}^* = \mathscr{F}^*(v) = \sum_{\substack{\varrho\\|\gamma|\leqq 2\lambda}} b_\varrho z_\varrho^v + \sum_{\substack{\varrho\\|\gamma|\leqq 2\lambda}} b_\varrho^*(z_\varrho^*)^v \tag{4.9}$$

where

$$b_\varrho^* = \frac{1}{2\varphi(q)} \sum_{\substack{\chi\\L(\varrho,\chi)=0}} \sum_{i=1}^{N} (\bar\chi(\alpha_i') - \bar\chi(\alpha_i''))e^{k_0(\varrho/2)^2+\mu_0(\varrho/2)}, \tag{4.10}$$

$$z_\varrho^* = e^{\lambda^{-2}B(\varrho/2)^2+B(\varrho/2)}. \tag{4.11}$$

Let us observe that the finite Riemann—Piltz conjecture (1.15)—(2.1) implies that the contribution of all z_ϱ^* terms with $|\gamma|\leqq D_0$ is

$$\ll N \sum_{n\leqq D_0} e^{k\left(\frac{1}{16}-\frac{n^2}{4}\right)+\frac{\mu}{4}\log(q(n+2))} \ll N \log q e^{\frac{k}{16}+\frac{\mu}{4}} \ll \sqrt[3]{x}. \tag{4.12}$$

Thus the remaining power-sum to be estimated is

$$\mathscr{F}^{**} = \mathscr{F}^{**}(v) = \sum_{\substack{\varrho\\|\gamma|\leqq 2\lambda}} b_\varrho z_\varrho^v + \sum_{\substack{\varrho\\D_0<|\gamma|\leqq 2\lambda}} b_\varrho^*(z_\varrho^*)^v, \tag{4.13}$$

where (3.34)—(3.35) are also valid. Further we have analogously to (3.15)

$$\sum_{D_0<|\gamma|\leqq 2\lambda} |b_\varrho^*| \ll \sum_{n=[P\log^2 P]}^{\infty} e^{\frac{1-n^2/4}{P^2\log^2 P}} P\log(qn) = O(1). \tag{4.14}$$

After denumerating the numbers z_ϱ and z_ϱ^* in $\mathscr{F}^{**}(v)$, and according to this b_ϱ and b_ϱ^*, let us choose $z_h = z_{\varrho_0}$ in exactly the same way as in Section 3 (i.e. z_h should belong to the first sum in $\mathscr{F}^{**}(v)$). Now using Lemma 2 (4.14) assures again for any $v\geqq h$ (where $\{b_j\} = \{b_\varrho\}_{|\gamma|\leqq 2\lambda} \cup \{b_\varrho^*\}_{D_0<|\gamma|\leqq 2\lambda}$)

$$\operatorname{Re}\sum_{j=1}^{v} b_j \geqq \operatorname{Re}\sum_{\varrho} b_\varrho - \sum_{\substack{\varrho\\D_0<|\gamma|}} |b_\varrho| - \sum_{\substack{\varrho\\D_0<|\gamma|\leqq 2\lambda}} |b_\varrho^*| > 1 \tag{4.15}$$

similarly to (3.15) and (3.37). Since we have trivially in this case also

$$|z_1| = \max(|z_\varrho|, |z_\varrho^*|) \leqq e^{\lambda^{-2}B+B}, \tag{4.16}$$

(3.39) is valid without any change and so the final estimate in (3.40) is also true for $\mathscr{F}^{**}(v)$, which proves Theorem 2 taking into account the remark following (4.6). □

What concerns the proof of Theorem 3, all the formulas (2.13)—(2.19) follow very easily from Lemma 7, except the condition

$$U < V \leqq U + U\exp(-\log^{1/2} U). \tag{4.17}$$

But we have only to observe that we can subdivide the original interval $[U, V]$ furnished by Lemma 7 into at most

$$\log Y \exp(\log^{1/2} U) = K \tag{4.18}$$

subintervals which already satisfy (4.17), and at least one of them, say $[U^{(j)}, V^{(j)}]$, satisfies already trivially

$$\sum_{U^{(j)} < n \leqq V^{(j)}} \varepsilon(n) a_i(n) \geqq \frac{1}{K} \sum_{U < n \leqq V} \varepsilon(n) a_i(n). \tag{4.19}$$

The loss of the factor K can be taken in consideration in the constant c appearing in $f(Y)$ in (2.13). □

5. To prove Theorem 4 we first note that similarly to (3.8)—(3.10) we have here

$$I = \frac{1}{\varphi(q)} \sum_{\chi} \left(\bar{\chi}(l_2) - \bar{\chi}(l_1)\right) \sum_{\substack{\varrho = \varrho(\chi) \\ \beta \geqq 0, |\gamma| \leqq \lambda}} e^{k\varrho^2 + \mu\varrho} + O\left(e^k \log(q\lambda)\right). \tag{5.1}$$

Now for every $\varrho = 1/2 + \delta + i\gamma$, $|\gamma| \leqq \lambda$ we have by (2.20)—(2.21)

$$k\left(\frac{1}{2}+\delta\right)^2 - K\gamma^2 + \frac{\mu}{2} + \mu\delta \leqq \frac{k}{4} + \frac{\mu}{2} + k\left\{\delta\left(\frac{3}{2} + \lambda^2\right) - \gamma^2\right\} \leqq \frac{k}{4} + \frac{\mu}{2} - \frac{K\gamma^2}{2}. \tag{5.2}$$

Thus we have for any χ with $\chi(l_1) \neq \chi(l_2)$ by (3.11) and (2.20)

$$\sum_{\substack{\varrho = \varrho(\chi) \\ \beta \geqq 0, |\gamma| \leqq \lambda}} |e^{k\varrho^2 + \mu\varrho}| \leqq \sum_{|\gamma| \leqq 1} + \sum_{1 < |\gamma| \leqq \lambda} \ll \tag{5.3}$$

$$\ll e^{k/4 + \mu/2} \left\{\log q e^{-kA^2(q)/2} + \sum_{1 \leqq n \leqq \lambda} e^{-kn^2/2} \log\left(q(n+2)\right)\right\} \ll$$

$$\ll e^{k/4 + \mu/2} \log q e^{-kA^2(q)/2} \ll q^{-1} e^{-\omega/2} e^{k/4 + \mu/2}$$

and so by (5.1) and (2.20)

$$I \ll \frac{e^{k/4 + \mu/2}}{q e^{\omega/2}} + e^k \mu. \tag{5.4}$$

Now using Lemma 6 we obtain for the contribution of prime powers p^j with $j \geqq 3$ to I the upper estimate

$$\frac{1}{2\sqrt{\pi k}} \left| \sum_{\substack{n = p^j \\ j \geqq 3}} \varepsilon(n) \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right) \right| \ll e^{k/9 + \mu/3}. \tag{5.5}$$

On the other hand since l_1 is a quadratic residue and l_2 a non-residue, we obtain again by Lemma 6

$$\frac{1}{2\sqrt{\pi k}} \sum_{n = p^2} \varepsilon(n) \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right) = \tag{5.6}$$

$$= \frac{1}{2\sqrt{\pi k}} \sum_{\substack{n = p^2 \\ n \equiv l_1(q)}} \Lambda(n) \exp\left(-\frac{(\mu - \log n)^2}{4k}\right) \sim \frac{N(q)}{2\varphi(q)} e^{k/4 + \mu/2}$$

since (using the notation in (4.3)—(4.4)) the prime number theorem for arithmetic progressions implies

$$\sum_{\substack{n=p^2\le x\\ n\equiv l_1(q)}} \Lambda(n) = \sum_{i=1}^{N(q)} \sum_{\substack{p\le\sqrt{x}\\ p\equiv\alpha_i'(q)}} \log p \sim \frac{N(q)}{\varphi(q)}\sqrt{x}. \tag{5.7}$$

Theorem 4 would follow now by easy calculation from formulas (5.1), (5.4), (5.5) and (5.6), but only for $\mu=k\lambda^2\gg q$, which is required for (5.7). But it is possible to assure (5.6) without this condition, only by (2.20); even (2.21) is superfluous for this purpose (further we do not need the prime number theorem for arithmetic progressions). We have namely similarly to (3.8)—(3.9) by Lemma 6

$$\frac{1}{2\sqrt{\pi k}} \sum_{p\equiv\alpha_i'(q)} \log p \exp\left(-\frac{(\mu-\log p^2)^2}{4k}\right) = \tag{5.8}$$

$$= \frac{1}{2\sqrt{\pi k}} \sum_{n\equiv\alpha_i'(q)} \Lambda(n) \exp\left(-\frac{(\mu-\log n^2)^2}{4k}\right) + O(e^{k/16+\mu/4}) =$$

$$= \frac{e^{k/4+\mu/2}}{2\varphi(q)} - \frac{1}{2\varphi(q)} \sum_{\chi} \bar\chi(\alpha_i') \sum_{\varrho=\varrho(\chi)} e^{k(\varrho/2)^2+\mu(\varrho/2)} + O(e^{k/16+\mu/4})$$

and similarly to (5.3) we have here by (2.20) (without using (2.21))

$$\sum_{\varrho=\varrho(\chi)} e^{k(\varrho/2)^2+\mu(\varrho/2)} \ll (\log q) e^{(k/4)(1-A^2(q))+\mu/2} \ll q^{-5/4} e^{-\omega} e^{k/4+\mu/2}. \quad \square \tag{5.9}$$

To prove Theorem 5 we can write again similarly to (3.8)—(3.10), using Lemma 6

(5.10)

$$\tilde I \overset{\text{def}}{=} \frac{1}{2\sqrt{\pi k}} \Sigma\varepsilon(p, q, l_1, l_2) \log p \exp\left(-\frac{(\mu-\log p)^2}{4k}\right) = \sum_{\substack{\varrho\\ |\gamma|\le\lambda}} a_\varrho e^{k\varrho^2+\mu\varrho} + O(e^{k/4+\mu/2}).$$

Now let us choose a number $\tilde\lambda\in[\lambda, \sqrt{\lambda^2+\eta_0}]$ such that the for $\varrho_i\ne\varrho_j$, $|\gamma_i|, |\gamma_j|\le\lambda$, the equality

$$\beta_i^2-\gamma_i^2+\tilde\lambda^2\beta_i = \beta_j^2-\gamma_j^2+\tilde\lambda^2\beta_j \tag{5.11}$$

should hold only for conjugate zeros, i.e. for $\beta_i=\beta_j$, $\gamma_i=-\gamma_j$ (this is clearly possible even for all zeros). Thus if

$$\beta^2-\gamma^2+\tilde\lambda^2\beta = S_\chi(\varrho) = S(\varrho) \tag{5.13}$$

assumes its maximum S' for ϱ' and $\bar\varrho'$, then with some positive $d=d(q, \tilde\lambda, \lambda, \varrho_0)$ we have by (5.10) and (2.27) for $\mu=k\tilde\lambda^2$

$$\tilde I = 2\,\mathrm{Re}\,\{a_{\varrho'} e^{k(\varrho')^2+\mu\varrho'}\} + O(e^{k(\beta'^2-\gamma'^2+\tilde\lambda^2\beta'-d)}) \tag{5.14}$$

since by (2.27) we have

$$S' \ge \left(\frac12+\delta_0\right)^2-\gamma_0^2+\tilde\lambda^2\left(\frac12+\delta_0\right) > \frac14+\frac{\tilde\lambda^2}{2}. \tag{5.15}$$

Now (5.14) assures (2.28)—(2.29) since

$$\arg\{a_{\varrho'}e^{k(\varrho')^2+\mu\varrho'}\} = \arg a_{\varrho'} + \mu\left(\gamma' + \frac{2\beta'\gamma'}{\tilde{\lambda}^2}\right), \tag{5.16}$$

and by (5.13) for $\mu > \mu_0$

$$|a_{\varrho'}e^{k(\varrho')^2+\mu\varrho'}| \geqq |a_{\varrho_0}e^{k\varrho_0^2+\mu\varrho_0}|. \quad \square \tag{5.17}$$

Appendix

What concerns the proof of Lemma 1 we have to notice that the following result is implicitly contained in Knapowski—Turán's work (cf. Theorem 4.1 in part III of [3]).

If b_j, z_j are complex numbers with

$$\varkappa \leqq |\arg z_j| \leqq \pi, \tag{1}$$

$$1 = |z_1| \geqq \ldots \geqq |z_n| \tag{2}$$

then for any $\delta > 0$ and $h \in [1, \ldots, n]$ with

$$|z_h| > \delta \tag{3}$$

and for any $m \geqq 0$ there exists an integer

$$v \in \left[m,\ m + n\left(3 + \frac{\pi}{\varkappa}\right)\right] \tag{4}$$

such that

$$\operatorname{Re} \sum_{j=1}^{n} b_j z_j^{v} > \frac{E}{2n+1}\left(\frac{|z_h| - \delta}{48}\right)^{2n} \delta^{m+n\left(3+\frac{\pi}{\varkappa}\right)} \tag{5}$$

where

$$E = \min_{v \geqq h} \left|\operatorname{Re} \sum_{j=1}^{v} b_j\right|. \tag{6}$$

Now setting

$$\delta = |z_h|\left(1 - \frac{2n}{m + n\left(5 + \frac{\pi}{\varkappa}\right)}\right) \tag{7}$$

unlike Turán's choice $\delta = |z_h| - 2n/(m + n(3 + \pi/\varkappa))$, we obtain the maximal value of the right hand side. Thus using the inequality

$$\left(1 - \frac{1}{x}\right)^{x-1} > e^{-1} \quad \text{for} \quad x > 2 \tag{8}$$

with $x=x_0=\dfrac{m+n\left(5+\frac{\pi}{\varkappa}\right)}{2n}$, we obtain

$$(9)\qquad \operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu} > \frac{E}{2n+1}|z_h|^{m+n\left(5+\frac{\pi}{\varkappa}\right)}\left(\frac{1}{48x_0}\right)^{2n}\left(1-\frac{1}{x_0}\right)^{(x_0-1)2n} >$$

$$> \frac{E}{2n+1}|z_h|^{m+n\left(5+\frac{\pi}{\varkappa}\right)}\left(\frac{1}{48ex_0}\right)^{2n}.$$

If we do not assume the normalisation (2) then we have the general form

$$(10)\qquad \operatorname{Re}\sum_{j=1}^{n} b_j z_j^{\nu} > |z_1|^{\nu}\frac{E}{2n+1}\left(\frac{1}{48ex_0}\right)^{2n}\left(\frac{|z_h|}{|z_1|}\right)^{m+n\left(5+\frac{\pi}{\varkappa}\right)} =$$

$$= \frac{E}{2n+1}\left(\frac{1}{48ex_0}\right)^{2n}|z_h|^{\nu}\left(\frac{|z_h|}{|z_1|}\right)^{m+n\left(5+\frac{\pi}{\varkappa}\right)-\nu}$$

which proves Lemma 1. □

To prove Lemma 3 we first note that if for a fixed $j=j_0$ the inequality (3.17) is false then k has to belong to an interval of length

$$(11)\qquad \frac{1}{2\pi}(\eta|a_{j_0}|+1/2n)$$

and so k can assume at most

$$(12)\qquad \left[\frac{\eta|a_{j_0}|+1/2n}{2\pi}\right]+1 < \frac{\eta|a_{j_0}|}{2}+\frac{3}{2}$$

integer values. For any fixed k the inequality

$$(13)\qquad |ya_{j_0}-2k\pi| < \frac{1}{4n}$$

can hold only for y values in an interval of length

$$(14)\qquad \frac{1}{2n|a_{j_0}|}.$$

Thus the total Lebesgue measure of y values for which (13) holds with any value of k is by (12)

$$(15)\qquad < \frac{1}{2n|a_{j_0}|}\left(\frac{\eta|a_{j_0}|}{2}+\frac{3}{2}\right) = \frac{\eta}{4n}+\frac{3}{4}\frac{1}{n}\frac{1}{|a_{j_0}|}.$$

Therefore the total Lebesgue measure of y values for which (3.17) can be false is

$$(16)\qquad < \frac{\eta}{4}+\frac{3}{4}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{|a_j|} \leqq \frac{\eta}{4}+\frac{3\eta}{4} = \eta$$

which proves Lemma 3. □

Lemma 4 is a special case of Lemma 3 since for $z=u+iv$, $u\geqq 0$, $\omega>0$

(17) $$\pi \geqq |\arg e^{\omega Bz^2+Bz}| \geqq \frac{1}{8n}$$

is equivalent to

(18) $$|Bv(1+2u\omega)-2k\pi| \geqq \frac{1}{4\cdot 2n}$$

and in this case we have for $\varrho_j=\beta_j+i\gamma_j$ $(\beta_j\geqq 0)$

$$\frac{1}{2n}\left\{\sum_{j=1}^{n}\left(\frac{1}{|\gamma_j|(1+2\beta_j\omega)}+\frac{1}{\frac{|\gamma_j|}{2}\left(1+2\frac{\beta_j}{2}\omega\right)}\right)\right\} < \frac{1}{2n}\sum_{j=1}^{n}\left(\frac{1}{|\gamma_j|}+\frac{2}{|\gamma_j|}\right) < \frac{3}{2}\eta. \quad \square$$

The following lemma gives the possibility of deducing Theorem 3 from Theorems 1 and 2.

LEMMA 7. *If* $1\leqq k\leqq \mu/16$, $\mu>c$, *then for arbitrary* l_1, l_2, q *there exist*

(20) $$[U_1, U_2] \subset \left[e^{\mu-3\sqrt{\mu k}}, e^{\mu+3\sqrt{\mu k}}\right]$$

such that for $\nu=0$ *or* 1, $i=3$ *or* 4, $\varepsilon(n)=\varepsilon(q, l_1, l_2, n)$,

(21) $$\sum_n \varepsilon(n)\frac{a_i(n)}{(\log n)^\nu} \geqq \frac{1}{(\mu+1)^\nu}\left\{\sum_n \varepsilon(n)a_i(n)\exp\left(-\frac{(\mu-\log n)^2}{4k}\right)-10\right\}$$

where $a_3(n)$, $a_4(n)$ *and* $\varepsilon(n)$ *are defined in* § 1.

Although the proof of the above lemma is contained in Turán's book [8, Lemma 54.2] we prefer to prove here the following more general

LEMMA 8. *Let* $a(n)$ *be a series of real numbers with*

(22) $$|A(x)| \overset{\text{def}}{=} \Big|\sum_{1\leqq n\leqq x} a(n)\Big| \leqq Cx$$

and $f(x)\in C^1[1, +\infty)$ *a non-negative function with*

(23) $$\lim_{x\to\infty} xf(x) = 0$$

(24) $$\begin{cases} f'(x) > 0 & \text{for } 1 < x < M,\\ f'(x) < 0 & \text{for } x > M.\end{cases}$$

Then for any $M_1\leqq M\leqq M_2$ *there exist*

(25) $$U, V\in[M_1, M_2]$$

such that

(26) $$f(M)\sum_{U<n\leqq V} a(n) \geqq \sum_{n=1}^{\infty} a(n)f(n) - C\Big(2M_1f(M_1)+2M_2f(M_2)+\int_{M_2}^{\infty} f\Big).$$

If we have additionaly

(27) $$\int_{M_2}^{\infty} f(x)\,dx \leqq M_2 f(M_2)$$

then (26) *simplifies to*

(28) $$f(M)\sum_{U<n\leqq V} a(n) \geqq \sum_{n=1}^{\infty} a(n)f(n) - C(2M_1 f(M_1) + 3M_2 f(M_2)).$$

Proof. Summation by parts and (1.3)—(1.4) give

(29) $$\sum_{n=1}^{\infty} a(n)f(n) = \int_1^{\infty} (-A(r)f'(r))\,dr = \int_1^{M_1} + \int_{M_1}^{M_2} + \int_{M_2}^{\infty}.$$

By (22)—(24) we have

(30) $$\Big|\int_1^{M_1}\Big| \leqq CM_1 \int_1^{M_1} f'(r)\,dr \leqq CM_1 f(M_1),$$

(31) $$\Big|\int_{M_2}^{\infty}\Big| \leqq C \int_{M_2}^{\infty} -rf'(r)\,dr = CM_2 f(M_2) + C\int_{M_2}^{\infty} f(r)\,dr,$$

(32) $$\int_{M_1}^{M_2} \leqq -\min_{M_1\leqq r\leqq M} A(r)\int_{M_1}^{M} f'(r)\,dr + \max_{M\leqq r\leqq M_2} A(r)\int_{M}^{M_2} (-f'(r))\,dr =$$
$$= f(M)\{\max_{M\leqq r\leqq M_2} A(r) - \min_{M_1\leqq r\leqq M} A(r)\} + R$$

where

(33) $$R = f(M_1)\min_{M_1\leqq r\leqq M} A(r) - f(M_2)\max_{M\leqq r\leqq M_2} A(r) \leqq CM_1 f(M_1) + CM_2 f(M_2). \quad \square$$

To prove Lemma 7 we consider the functions $(k, \mu>0)$

(34) $$f_{k,\mu}^{(\nu)}(x) = f^{(\nu)}(x) = \exp\left(-\frac{(\mu-\log x)^2}{4k}\right)\log^{\nu} x \quad (\nu = 0, 1)$$

(35) $$a(n) = \Lambda(n) \quad \text{or} \quad a(p) = \log p \quad (a(n) = 0 \text{ otherwise}).$$

Then (22) is satisfied with $C=2$, e.g., further it is easy to check (23) and (24) with

(36) $$M^{(0)} = e^{\mu}, \quad e^{\mu} < M^{(1)} < e^{\mu+\frac{2k}{\mu}}.$$

Further we shall show that (27) is satisfied too, if

(37) $$M_2^{(\nu)} = e^{\mu+m} \geqq e^{\mu+4k}.$$

Observing $1\leqq\dfrac{y}{4k}\leqq\dfrac{y}{2k}-1$ for $y\geqq m$, we obtain namely $(\nu=0, 1)$

(38) $$\int_{M_2^{(\nu)}}^{\infty} \log^{\nu} x \exp\left(-\frac{(\log x-\mu)^2}{4k}\right)dx = \int_m^{\infty} (\mu+y)^{\nu}\exp\left(\mu+y-\frac{y^2}{4k}\right)dy \leqq$$
$$\leqq \int_m^{\infty} (\mu+4k)^{\nu}\left(\frac{y}{2k}-1\right)\exp\left(\mu+y-\frac{y^2}{4k}\right)dy =$$
$$= (\mu+4k)^{\nu}\exp\left(\mu+m-\frac{m^2}{4k}\right) \leqq e^{\mu+m}(\mu+m)^{\nu}\exp\left(-\frac{m^2}{4k}\right).$$

Finally for $0<k\leqq\mu/16$, $\mu\geqq4$ we have $4k\leqq\sqrt{\mu k}\leqq\mu/4$ and so

$$M_1^{(\nu)}f(M_1^{(\nu)})\leqq1,\quad M_2^{(\nu)}f(M_2^{(\nu)})\leqq1 \tag{39}$$

if we choose

$$M_1^{(\nu)}=e^{\mu-3\sqrt{\mu k}},\quad M_2^{(\nu)}=e^{\mu+3\sqrt{\mu k}}.\quad\square \tag{40}$$

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ÜBER REELL-UNITÄRE FINSLER—OTSUKISCHE RÄUME

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§ 1. Einleitung

Die Übertragungstheorie der Finsler—Otsukischen $F-O_n$-Räume haben wir in unserem Aufsatz [2] begründet. Nach dieser Theorie ist ein $F-O_n$-Raum eine Mannigfaltigkeit der Linienelemente $(x^i, \dot{x}^i)$ $(i=1, 2, \ldots, n)$ in der die Parallelübertragung der Tensoren durch ein invariantes Differential von der Form:

$$(1.1)\qquad DT^a_b := P^a_i P^j_b \bar{D} T^i_j, \quad P^a_i := P^a_i(x, \dot{x}),$$

$$(1.1a)\qquad \bar{D}T^i_j := dT^i_j + (C_r{}^i{}_k T^r_j - C_j{}^s{}_k T^i_s)\, d\dot{x}^k + (\Gamma_r{}^i{}_k T^r_j - \bar{\bar{\Gamma}}_j{}^s{}_k T^i_s)\, dx^k$$

angegeben ist (wir haben das invariante Differential für einen gemischten Tensor zweiter Stufe angegeben; die Verallgemeinerung auf beliebige Tensoren ist analog zum Fall der Punkträume; vgl. [4], (2.14) und (2.15)). Von den in $\dot{x}^i$ von nullter Dimension homogenen Grundgrössen $\Gamma_j{}^i{}_k(x, \dot{x})$ und $\bar{\bar{\Gamma}}_j{}^i{}_k(x, \dot{x})$ bzw. $P^a_i(x, \dot{x})$ mit $\mathrm{Det}\,(P^a_i) \neq 0$, nehmen wir an, daß sie den Relationen

$$(1.2)\qquad \partial_k P^i_j - P^i_{j\|s}\, {}'\Gamma_o{}^s{}_k - {}'\Gamma_j{}^s{}_k P^i_s + {}''\Gamma_s{}^i{}_k P^s_j = 0, \quad {}_{\|s} := F\frac{\partial}{\partial \dot{x}^s},$$

$$(1.2a)\qquad {}'\Gamma_j{}^s{}_k := \Gamma_j{}^s{}_k - C_j{}^s{}_t \Gamma_p{}^t{}_k \dot{x}^p, \qquad (1.2b)\qquad {}''\Gamma_j{}^s{}_k := \bar{\bar{\Gamma}}_j{}^s{}_k - C_j{}^s{}_t \Gamma_p{}^t{}_k \dot{x}^p$$

genügen; dabei sind die ${}'\Gamma_s{}^i{}_k$ und ${}''\Gamma_s{}^i{}_k$ offenbar mit $\Gamma_s{}^i{}_k$ und $\bar{\bar{\Gamma}}_s{}^i{}_k$ äquivalente Übertragungsparameter; $F(x, \dot{x})$ ist die metrische Grundfunktion des Raumes, und der Index „o" bedeutet — wie gewöhnlich — die Kontraktion mit dem Einheitsvektor $l^i := \frac{\dot{x}^i}{F}$ (für die vollständige Theorie vgl. [2], (2.1)—(2.10), insbesondere die Formeln (2.8) a)—d)). Alle diese Größen sind von den Linienelementen $(x^i, \dot{x}^i)$ abhängig, $F(x, \dot{x})$ ist in den $\dot{x}^i$ homogen von erster bzw. in $C_j{}^s{}_k(x, \dot{x})$ homogen von (-1)-ter Dimension.

In den $F-O_n$-Räumen ist der metrische Grundtensor $g_{ij}(x, \dot{x})$ von der Grundfunktion $F(x, \dot{x})$ durch die Formeln

$$g_{ij}(x, \dot{x}) := \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \quad \dot{\partial}_i := \frac{\partial}{\partial \dot{x}^i}$$

bestimmt (vgl. z. B. [5], Kap. I. § 3) und aus g_{ij} und P^i_j können dann die Übertragungsparameter ${}'\Gamma_j{}^i{}_k$ und ${}''\Gamma_j{}^i{}_k$ aus der Forderung, daß die Übertragung metrisch sein soll, so bestimmt werden, wie das in [2] durchgeführt wurde.

Für $P_{ij} := g_{it} P^t_j$ *soll die Symmetrie in* (i, j) *gelten,* hingegen soll $P^i_o = l^i$ in dieser Arbeit — abgesehen von gewissen Spezialfällen — nicht vorausgesetzt sein.

Der Grundtensor $P^i_j(x, \dot{x})$ soll aber in den $F-O_n$-Räumen und auch in den Otsukischen Punkträumen immer einen eindeutig bestimmten inversen Tensor $Q^i_j(x, \dot{x})$ (vgl. [2], S. 122 und [4], S. 109) haben.

Im folgenden wollen wir solche $F-O_n$-Räume untersuchen, in denen $P^i_j = Q^i_j$ besteht. Wir geben die folgende

DEFINITION. Ein Finsler—Otsukischer Raum, in dem der Grundtensor $P^i_j(x, \dot{x})$ involutorisch ist, d. h.

$$P^i_j P^j_k = \delta^i_k \tag{1.3}$$

besteht, wird reell-unitärer Raum genannt. Den Raum selbst werden wir mit $F-O^*_n$-Raum bezeichnen.

In dem vorliegenden Artikel wollen wir die $F-O^*_n$-Räume eingehender untersuchen und einige charakteristische Eigenschaften von $F-O^*_n$-Räumen bestimmen. Dementsprechend geben wir im nächsten Paragraphen einige Grundrelationen der $F-O_n$-Räume an, die sich in den $F-O^*_n$-Räumen etwas vereinfachen, bzw. die die $F-O^*_n$-Räume charakterisieren werden. Im Paragraphen 3 werden wir dann weitere verschiedene charakteristische Eigenschaften der $F-O^*_n$-Räume zusammenstellen.

Im Paragraphen 4 wollen wir die höheren Differentiale der Eigenvektoren des Raumes untersuchen. Ein Eigenvektor des Raumes längs einer Folge $(x^i(s), \dot{x}^i(s))$ der Linienelemente ist der Vektor $V^i = V^i(s)$, falls die Gleichung

$$P^j_k(x(s), \dot{x}(s))V^k(s) = \tau(s)V^j(s), \quad \vec{V} \neq \vec{O} \tag{1.4}$$

besteht, wo $\tau(s)$ einen Skalar, nämlich die Eigenfunktion von $\vec{V}$ (vgl. [4], (5.2)) bedeutet. Aus (1.4) und (1.3) folgt leicht der schon in [3] angegebene, aber in den $F-O^*_n$-Räumen fundamentale

SATZ 1. *Die Eigenfunktion $\tau(s)$ kann in einem $F-O^*_n$-Raum nur $\tau(s)\equiv 1$, oder $\tau(s)\equiv -1$ sein.*

BEWEIS. Eine Kontraktion von (1.4) mit P^i_j gibt nach (1.3) im Hinblick auf (1.4) selbst: $V^i = \tau^2 V^i$, woraus die Behauptung des Satzes unmittelbar folgt.

In den $F-O^*_n$-Räumen sind also die Eigenvektoren immer durch eine Relation von der Form

$$P^i_j(x(s), \dot{x}(s))V^j(s) = \pm V^i(s), \quad \vec{V} \neq \vec{O} \tag{1.5}$$

gekennzeichnet. Es ist auch möglich, daß (1.5) nicht nur längs einer Folge der Linienelemente, sondern sogar für Vektor- und Tensorfelder besteht, d. h. es gilt statt (1.5)

$$P^i_j(x, \dot{x})V^j(x, \dot{x}) = \pm V^i(x, \dot{x}), \quad \vec{V} \neq \vec{O}. \tag{1.6}$$

Im Paragraphen 5 wollen wir die durch (1.6) charakterisierten 2-dimensionalen $F-O^*_2$-Räume bestimmen. Es wird sich zeigen, daß, wenn $P_{ij} \equiv P_{ji}$ besteht, in diesen Räumen der Tensor P^i_j von einem Parameter abhängig ist (vgl. unseren Satz 6).

§ 2. Grundrelationen über die kovarianten Ableitungen in den $F-O_n$-Räumen

Das invariante Differential (1.1a) läßt sich mit Hilfe der kovarianten Ableitungen in der Form:

$$\bar{D}T^i_j = \overset{*}{\nabla}_k T^i_j \bar{\omega}^k(d) + \overset{\circ}{\nabla}_k T^i_j\, dx^k, \quad \bar{\omega}^k(d) := \bar{D}l^k. \tag{2.1}$$

$$\overset{*}{\nabla}_k T^i_j := T^i_{j\|k} + A_s{}^i{}_k T^s_j - A_j{}^s{}_k T^i_s, \quad A_s{}^i{}_k := FC_s{}^i{}_k, \tag{2.2}$$

$$\overset{\circ}{\nabla}_k T^i_j := \partial_k T^i_j - T^i_{j\|s}{}'\Gamma_o{}^s{}_k + {}'\Gamma_s{}^i{}_k T^s_j - {}''\Gamma_j{}^s{}_k T^i_s \tag{2.3}$$

bestimmen (vgl. [2], Formeln (2.8) a)—d)); $A_s{}^i{}_k$ ist dabei im wesentlichen der Cartansche Torsionstensor des Raumes und ${}''\Gamma_j{}^s{}_k$ ist der in (j, k) symmetrische (vgl. [2], (2.15)) Übertragungsparameter des $F-O_n$-Raumes.

Die kovariante Ableitung (2.2) ist mit jener der Cartanschen Theorie identisch (vgl. [5], Kap IV. (1.20)), hingegen ist die kovariante Ableitung (2.3) — abgesehen von einigen Spezialfällen (vgl. unseren Satz 7.) — für die $F-O_n$-Räume kennzeichnend. In dieser kovarianten Ableitung benutzt man bei den kontravarianten Indizes die ${}'\Gamma_i{}^j{}_k$ bzw. bei den kovarianten Indizes die ${}''\Gamma_i{}^j{}_k$. Nur im zweiten Glied, bei $T^i_{j\|s}$, wird immer ${}'\Gamma_o{}^h{}_k$ verwendet. Die Formel (1.2) bedeutet also im allgemeinen nicht, daß $\overset{\circ}{\nabla}_k P^i_j = 0$, wenn aber $P^i_j Q^j_m = \delta^i_m$ kovariant abgeleitet wird, so bekommt man leicht:

$$(\overset{\circ}{\nabla}_k P^i_j + {}''\Gamma_j{}^t{}_k P^i_t) Q^j_m + P^i_j (\overset{\circ}{\nabla}_k Q^j_m - {}'\Gamma_t{}^j{}_k Q^t_m) \equiv \overset{\circ}{\nabla}_k \delta^i_m, \tag{2.4}$$

wo selbstverständlich $\overset{\circ}{\nabla}_k \delta^i_m = {}'\Gamma_m{}^i{}_k - {}''\Gamma_m{}^i{}_k$. Wenn wir jetzt von der kovarianten Ableitung $\overset{\circ}{\nabla}_k P^i_j$ die Größe: $\partial_k P^i_j - P^i_{j\|s}{}'\Gamma_o{}^s{}_k$ mittels (1.2) eliminieren, so erhalten wir

$$\overset{\circ}{\nabla}_k P^i_j = P^i_s \overset{\circ}{\nabla}_k \delta^s_j + P^s_j \overset{\circ}{\nabla}_k \delta^i_s, \quad \overset{\circ}{\nabla}_k \delta^i_s \equiv {}'\Gamma_s{}^i{}_k - {}''\Gamma_s{}^i{}_k. \tag{2.5}$$

Aus den Relationen (2.4) und (2.5) folgt dann wegen $Det(P^i_j) \neq 0$ die wichtige Gleichung:

$$\overset{\circ}{\nabla}_k Q^j_m = 0, \tag{2.6}$$

die auf Grund von (2.4) mit (1.2) äquivalent ist, wie das nach einer Kontraktion von (2.4) mit P^m_h im Hinblick auf (2.5) unmittelbar folgt. Es gilt also in den allgemeinen $F-O_n$-Räumen das

FUNDAMENTALLEMMA. *In den allgemeinen $F-O_n$-Räumen bestehen immer die Relationen* (2.5) *und* (2.6).

§ 3. Einige Relationen in den $F-O^*_n$-Räumen

In diesem Paragraphen wollen wir in den $F-O^*_n$-Räumen, in denen also $P^i_j = Q^i_j$ ist, einige mit $\overset{\circ}{\nabla}_k P^i_j$ im Zusammenhang stehende Formeln ableiten. Wegen (2.6) haben wir in den $F-O^*_n$-Räumen statt (2.5) nach unserem Hauptlemma den folgenden Satz:

SATZ 2. *In den $F-O_n^*$-Räumen gilt immer*

$$P_s^i \mathring{\nabla}_k \delta_j^s + P_j^s \mathring{\nabla}_k \delta_s^i = 0. \tag{3.1}$$

Da in den $F-O_n^*$-Räumen $P_j^s = Q_j^s$ ist, kann (3.1) in den $F-O_n^*$-Räumen offenbar auch in der äquivalenten Form

$$\mathring{\nabla}_k \delta_m^i = -P_m^t P_s^i \mathring{\nabla}_k \delta_t^s \tag{3.1a}$$

angegeben werden, was aus (3.1) nach Überschiebung mit P_m^j entsteht. Die Umkehrung von Satz 2 ist im allgemeinen nicht gültig, da aus (3.1) und (2.5) im Hinblick auf (2.6) nur der folgende Satz abgeleitet werden kann:

SATZ 3. *Besteht* (3.1), *so unterscheiden sich P_j^i und Q_j^i höchstens um einen Tensor mit verschwindender kovarianter Ableitung.*

BEWEIS. Nach (3.1) folgt aus (2.5), daß $\mathring{\nabla}_k P_j^i = 0$, was nach (2.6) tatsächlich

$$\mathring{\nabla}_k[(P_j^i - Q_j^i) + Q_j^i] \equiv \mathring{\nabla}_k(P_j^i - Q_j^i) = 0,$$

also die Behauptung des Satzes nach sich zieht.

Nehmen wir jetzt an, daß in einem $F-O_n$-Raum die Relation (3.1a) besteht, ferner

$$A^k(\mathring{\nabla}_k \delta_s^i) M_i^t = \delta_s^t \Phi(x, \dot{x}), \quad \Phi(x, \dot{x}) \neq 0 \tag{3.2}$$

bezüglich M_i^t eine eindeutige Lösung hat, wenn $\Phi(x, \dot{x})$ einen Skalar bedeutet. Es gilt der folgende

SATZ 4. *Bestehen in einem $F-O_n$-Raum* (3.1a), (3.2) *und* $\mathring{\nabla}_k P_j^i = 0$, *so ist dieser $F-O_n$-Raum ein $F-O_n^*$-Raum, d. h. $P_j^i = Q_j^i$.*

BEWEIS. Aus (3.1a) folgt nach einer Überschiebung mit Q_j^m, nach einer Vertauschung des Summationsindexes m:

$$P_s^i \mathring{\nabla}_k \delta_j^s = -Q_j^s \mathring{\nabla}_k \delta_s^i.$$

Substituiert man das in (2.5), so wird im Hinblick auf die Bedingungen des Satzes:

$$\mathring{\nabla}_k \delta_s^i (P_j^s - Q_j^s) = 0.$$

Nach einer Überschiebung dieser Gleichung mit $A^k M_i^t$ bekommt man nach (3.2) unmittelbar $P_j^t = Q_j^t$, w. z. b. w.

§ 4. Höhere Differentiale der Eigenvektoren

Es sei $V^i(x, \dot{x})$ ein Eigenvektorfeld in einem $F-O_n^*$-Raum. Das bedeutet das Bestehen von (1.6) längs jeder Folge $(x^i(s), \dot{x}^i(s))$ der Linienelemente. Offenbar kann in den $F-O_n^*$-Räumen angenommen werden, daß der Parameter „s" die Bogenlänge bedeutet. Wir wollen nun in diesem Paragraphen das Analogon der Untersuchungen von T. Otsuki bezüglich der höheren Differentiale der Eigenvektorfelder durchführen (vgl. [4], Theorem 5.7).

Nehmen wir also an, daß (1.6) bzw. (1.5) besteht. Bilden wir das invariante Differential $\bar{D}$ beider Seiten von (1.5), so wird nach (1.2a):

$$V^j dP^i_j + P^i_j dV^j + (A_s{}^i{}_k \varpi^k(d) + {}'\Gamma_s{}^i{}_k dx^k) P^s_j V^j = \pm \bar{D} V^i, \tag{4.1}$$

wo wir (1.1a) benützt haben, und statt $d\dot{x}^k$ die Formel

$$d\dot{x}^k = F(\varpi^k(d) - {}'\Gamma_o{}^k{}_t\, dx^t - \dot{x}^k dF^{-1}), \quad {}'\Gamma_o{}^k{}_t \equiv \Gamma_o{}^k{}_t \tag{4.2}$$

und $C_s{}^i{}_k \dot{x}^k = 0$ verwendet wurde (vgl. die Formeln (2.6)—(2.8) d) in [2]). Aus (4.1) eliminieren wir dP^i_j und dV^j. Nach (1.2) hat man wegen der Homogenität nullter Dimension in den $\dot{x}^i$ von P^i_j:

$$\begin{aligned} dP^i_j &\equiv \partial_k P^i_j dx^k + \dot{\partial}_k P^i_j d\dot{x}^k = \partial_k P^i_j dx^k + P^i_{j\|k}(\varpi^k(d) - {}'\Gamma_o{}^k{}_t dx^t) = \\ &= P^i_{j\|k}\varpi^k(d) + ({}'\Gamma_j{}^s{}_k P^i_s - {}''\Gamma_s{}^i{}_k P^s_j)\, dx^k \end{aligned} \tag{4.3}$$

und nach der Formel (2.7) von [2] wird

$$dV^j = \bar{D} V^j - (A_s{}^j{}_k \varpi^k(d) + {}'\Gamma_s{}^j{}_k\, dx^k) V^s. \tag{4.4}$$

Substituieren wir (4.3) und (4.4) in (4.1), so wird:

$$DV^i + \{\overset{*}{\nabla}_k P^i_j \varpi^k(d) + P^s_j (\overset{\circ}{\nabla}_k \delta^i_s)\, dx^k\} V^j = \pm \bar{D} V^i.$$

Eine Kontraktion mit P^h_i gibt nach einer gewissen Umformung der Indizes und im Hinblick auf (1.1), (2.1)—(2.3):

$$P^i_t D V^t = \pm D V^i + \psi^i_j V^j, \tag{4.5}$$

$$\psi^i_j := P^t_j (\overset{*}{\nabla}_k P^i_t) \varpi^k(d) - D\delta^i_j, \tag{4.6}$$

wo wir auch die aus (1.3) folgende Relation

$$P^t_j \overset{*}{\nabla}_k P^i_t \equiv -P^i_t \overset{*}{\nabla}_k P^t_j \tag{4.7}$$

beachtet haben. (Für die Operation $\overset{*}{\nabla}_k$ gilt die Leibnizsche Regel).

Bezüglich der höheren invarianten Differentiale beweisen wir den folgenden

SATZ 5. *Besteht* (1.5) *längs einer Folge* C_L: $(x^i(s), \dot{x}^i(s))$ *der Linienelemente, so ist längs* C_L:

$$P^i_t D^m V^t = \pm D^m V^i + \sum_{\varrho=0}^{m-1} \psi^{i(m)}_{j(\varrho)} D^\varrho V^j, \quad (m = 1, 2, \ldots) \tag{4.8}$$

wo die $\psi^{i(m)}_{j(\varrho)}$ *durch die folgenden rekursiven Formeln bestimmt sind:*

$$\psi^{i(1)}_{j(0)} := \psi^i_j, \quad (\text{vgl. } (4.6)), \tag{4.9}$$

$$\psi^{i(\sigma)}_{j(\varrho)} \equiv 0, \quad \textit{wenn} \quad \varrho \geqq \sigma, \quad \psi^{i(\sigma)}_{j(-1)} \equiv 0, \tag{4.9a}$$

$$\psi^{i(m+1)}_{j(\varrho)} = \delta^m_\varrho \psi^i_j + P^t_j D\psi^{i(m)}_{t(\varrho)} + (D\delta^s_j) P^i_t \psi^{t(m)}_{s(\varrho)} + P^i_t P^s_j \psi^{t(m)}_{s(\varrho-1)}, \quad (\varrho = 0, 1, \ldots, m). \tag{4.9b}$$

Die Größen $\psi^{i(m)}_{j(\varrho)}$ *sind in* (4.8)—(4.9b) *Tensoren vom Typ:* (1, 1), *(m, ϱ bezeichnen keine tensorielle Indizes).*

BEWEIS. Vor allem bemerken wir, daß dieser Satz aus dem Satz I von [3] abgeleitet werden könnte. Vollständigkeitshalber geben wir hier einen kurzen Beweis dieses Satzes, der auch dadurch bewiesen werden könnte, daß in [3]: $P^i_j=Q^i_j$ gesetzt würde. Offenbar entstehen dadurch gewisse Vereinfachungen. Wir wollen nun den Satz durch vollständige Induktion beweisen. Für $m=1$ stimmt (4.8) mit (4.5) überein, was eben aus (1.5) abgeleitet wurde. Der Satz 5 ist also für $m=1$ richtig. Angenommen, der Satz 5 gilt für irgendein $m\geqq 1$, so erhält man nach einer invarianten Ableitung $\bar{D}$ von beiden Seiten von (4.8)

$$(dP^i_j)D^mV^j+P^i_j dD^mV^j+\left(A_s{}^i{}_k\varpi^k(d)+{}'\Gamma_s{}^i{}_k\,dx^k\right)P^s_jD^mV^j=$$

$$=\pm\bar{D}D^mV^i+\sum_{\varrho=0}^{m-1}\left({}'\bar{D}\psi^{i(m)}_{j(\varrho)}D^\varrho V^j+\psi^{i(m)}_{j(\varrho)}\bar{D}D^\varrho V^j\right),$$

wo ${}'\bar{D}$ das allein mit ${}'\Gamma$ gebildete invariante Differential bezeichnet.[1] Jetzt beachten wir auf der linken Seite (4.3) und (4.4), auf der rechten Seite die Formeln

$${}'\bar{D}\psi^{i(m)}_{j(\varrho)}\equiv\bar{D}\psi^{i(m)}_{j(\varrho)}-(\bar{D}\delta^s_j)\psi^{i(m)}_{s(\varrho)},\qquad \bar{D}\delta^s_j\equiv\mathring{\nabla}_k\delta^s_j\,dx^k;$$

somit wird nach der Indexveränderung $i\to t$:

$$D^{m+1}V^t+\left(\overset{*}{\nabla}_kP^t_j\varpi^k(d)+P^s_j\bar{D}\delta^t_s\right)D^mV^j=\pm\bar{D}D^mV^t+ \tag{4.10}$$

$$+\sum_{\varrho=0}^{m-1}\left\{\left(\bar{D}\psi^{t(m)}_{j(\varrho)}-(\bar{D}\delta^s_j)\psi^{t(m)}_{s(\varrho)}\right)D^\varrho V^j+\psi^{t(m)}_{j(\varrho)}\bar{D}D^\varrho V^j\right\}.$$

Wir überschieben jetzt diese Gleichung mit P^i_t und wir wollen das invariante Differential $\bar{D}$ durch das invariante Differential D ausdrücken. Wir beachten auf der linken Seite die Relationen (4.6), (4.7) und die mit

$$\bar{D}\delta^i_j=-D\delta^i_j \tag{4.11}$$

äquivalente und in den $F-O^*_n$-Räumen gültige Identität (3.1a); auf der rechten Seite beachten wir wieder (4.11) und die wegen $P^i_j=Q^i_j$ bestehende Identität:

$$P^i_t\bar{D}\psi^{t(m)}_{j(\varrho)}=P^b_jD\psi^{i(m)}_{b(\varrho)},$$

wodurch im Hinblick auf (4.9b) die Formel

$$P^i_tD^{m+1}V^t=\pm D^{m+1}V^i+\sum_{\varrho=0}^{m}\psi^{i(m)}_{j(\varrho)}D^\varrho V^j$$

entsteht, die nach (4.9)—(4.9b) die Gültigkeit von (4.8) für $(m+1)$ statt m zeigt. Damit ist der Beweis von Satz 5 beendet.[2]

[1] Es ist selbstverständlich für die rein kontravarianten Tensoren $\bar{D}={}'\bar{D}$. $\psi^{i(m)}_{j(\varrho)}$ bedeutet einen (1, 1)-Tensor, da m und ϱ keine tensoriellen Indizes sind.

[2] Für eine etwas allgemeinere Form dieses Satzes vgl. die Arbeit [3]. Die Relation (1.3) ergibt mehrere Vereinfachungen für den Beweis. Vgl. [3], Satz I.

BEMERKUNG. Aus den Formeln (4.9)—(4.9b) bekommt man z. B. für $m=1$ und $\varrho=0, 1$

(4.12) $$\psi^{i(2)}_{j(0)} = P^t_j D\psi^i_t + (D\delta^s_j)P^i_t\psi^t_s, \quad \psi^{i(2)}_{j(1)} = \psi^i_j + P^i_t P^s_j \psi^t_s,$$

was aus (4.5) auch direkt — mit der angegebenen Methode — berechnet werden konnte.

Aus diesem Satz kann das folgende Korollar leicht bewiesen werden.

KOROLLAR I. *Ist längs einer Folge* C_L *der Linienelemente* $\overline{\omega}^k(d)=0$ *und* $D\delta^i_j=0$, *so vereinfacht sich* (4.8) *für die Eigenvektoren auf*

$$P^i_t D^m V^t = \pm D^m V^i, \quad (m = 0, 1, \dots; \; D^0 V^i := V^i).$$

BEWEIS. Es ist nach (4.9) und (4.6) in diesem Fall $\psi^{i(1)}_{j(\varrho)}=0$ und nach (4.12) wird auch $\psi^{i(2)}_{j(\varrho)}=0$ $(\varrho=0, 1)$. Angenommen, daß $\psi^{i(m)}_{j(\varrho)}=0$ für ein $m\geqq 2$, so folgt aus (4.9)—(4.9b), daß auch $\psi^{i(m+1)}_{j(\varrho)}=0$ ist, was nach (4.8) die Behauptung des Korollars beweist.

§ 5. Der zweidimensionale $F-O^*_2$-Raum

L. Berwald hat in seinem Aufsatz [2] in den Finslerräumen ein natürliches d. h. von $(x^i, \dot{x}^i)$ abhängiges orthonormiertes Zweibein $(\vec{l}, \vec{h})$ konstruiert und damit die zweidimensionalen Finslerräume untersucht. Da dieses Zweibein allein von der Grundfunktion $F(x, \dot{x})$ des Finslerraumes ausgehend konstruiert wurde, kann auch in den $F-O^*_2$-Räumen benutzt werden (vgl. [1] § 4, oder [5], Kapitel VI. § 6).[3]

Wir bestimmen in diesem Paragraphen zunächst die möglichen Typen der $F-O^*_2$-Räume. Es besteht der

SATZ 6. *In den* $F-O^*_2$-*Räumen mit symmetrischem* P_{ij}-*Tensor hat* P^i_j *die Form:*

(5.1) $$P^i_j = \alpha(l^i l_j - h^i h_j) \pm \sqrt{1-\alpha^2}(l^i h_j + h^i l_j), \quad |\alpha| \leqq 1,$$

wo α *einen frei wählbaren konstanten Parameter bedeutet, oder es ist* $P^i_j = -\delta^i_j$.

Vor dem Beweis des Satzes verweisen wir auf Satz VI. unserer Arbeit [3], wo statt (5.1) nur der Fall $\alpha=1$ als charakteristische Form für P^i_j bezeichnet ist. In [3] haben wir aber durchweg $P^i_0=l^i$ vorausgesetzt, und diese Bedingung ist auf Grund von (5.1) tatsächlich mit $\alpha=1$ äquivalent.

BEWEIS VOM SATZ 6. Vor allem verweisen wir auf die Tatsache, daß von der Form (5.1) folgt, daß P^i_j der Relation (1.3) wirklich genügt, da bekanntlich (vgl. [1], (4.1) (b))):

(5.2) $$l^i l_k + h^i h_k = \delta^i_k$$

besteht. Wir haben also nur noch zu beweisen, daß (5.1) die allgemeinste Form für P^i_j ist, falls P_{ij} in (i, j) symmetrisch ist und (1.3) besteht.

[3] In [5] ist $\vec{h}$ mit $\vec{m}$ bezeichnet.

Die allgemeinste Form für P^i_j, falls nur die Symmetrie von P^i_j vorausgesetzt wird, ist im $F-O_2$-Raum:

(5.3)
$$P^i_j = \alpha l^i l_j + \beta h^i h_j + \gamma(l^i h_j + h^i l_j),$$

wo $\vec{l}$ und $\vec{h}$ die Vektoren des Berwaldschen orthonormierten Zweibeins, α, β, γ aber Parameter bedeuten. Substituiert man (5.3) in (1.3), so wird

$$(\alpha^2+\gamma^2)\,l^i l_k + (\beta^2+\gamma^2)\,h^i h_k + (\alpha+\beta)\gamma(h^i l_k + l^i h_k) = \delta^i_k.$$

Vergleichen wir diese Gleichung mit (5.2), so folgt unmittelbar, daß entweder 1) $\alpha=-\beta$ und $\alpha^2+\gamma^2=\beta^2+\gamma^2=1$ besteht, oder 2) $\gamma=0$, $\alpha=\pm 1$, $\beta=\mp 1$ gültig ist. In beiden Fällen kann $\gamma=\pm\sqrt{1-\alpha^2}$ mit $|\alpha|\leqq 1$ genommen werden, da die Annahme über α auch den Typ 2), d. h. $\gamma=0$ umfaßt, falls dann $\alpha=-\beta=1$, oder $\alpha=-\beta=-1$ gesetzt wird. Der Typ 1), d. h. $\gamma=\pm\sqrt{1-\alpha^2}$ ergibt schon nach (5.3) den Typ (5.1), wenn noch $\alpha=-\beta$ beachtet wird. Die Formel (5.1) enthält aber auch den Typ 2), d. h. $\gamma=0$, $\alpha=\pm 1$, $\beta=\mp 1$, ferner $\gamma=0$, $\alpha=\beta=1$, was wir aber ausschließen können, da dann nach (5.3) $P^i_j=\delta^i_j$ wäre. Der $F-O^*_2$-Raum wäre also ein gewöhnlicher Finslerraum mit Cartanscher Übertragung, wie das aus (1.2) und (2.1)—(2.3) unmittelbar folgen würde. Der Typ 2) enthält noch $\gamma=0$, $\alpha=\beta=-1$, d. h. nach (5.3) und (5.2) $P^i_j=-\delta^i_j$, womit der Satz 6. vollständig bewiesen ist.

Wir beweisen noch den folgenden merkwürdigen

SATZ 7. *In dem* $F-O^*_2$*-Raum mit*

(5.4)
$$P^i_j = \delta^i_j - 2h^i h_j$$

definiert die Übertragung $\bar{D}$ *eine gewöhnliche Cartansche Übertragung* (vgl. [5], Kapitel III. §§ 1—2).

BEWEIS. Der Typ (5.4) entsteht aus (5.1), wenn in (5.1) $\alpha=1$ genommen wird, wie man aus (5.1) und (5.2) unmittelbar ersieht. Aus der Formel (2.22) unserer Arbeit [2], d. h. aus

(5.5)
$${}'\Gamma_j{}^i{}_k = {}''\Gamma_j{}^i{}_k + Q^i_t\,{}''\nabla_k P^t_j$$

folgt im Hinblick auf (5.4), daß ${}'\Gamma_j{}^i{}_k={}''\Gamma_j{}^i{}_k$ ist, weil ${}''\nabla_k$ wegen $P^i_o=l^i$ die Cartansche kovariante Ableitung bedeutet (vgl. [2], Satz 1 auf S. 124). Da die Cartansche kovariante Ableitung von h^i verschwindet, (vgl. [1], (4.5) oder [5], Kapitel VI. § 6), folgt aus (5.4) und (5.5), daß ${}'\Gamma_j{}^i{}_k={}''\Gamma_j{}^i{}_k$ besteht, woraus nach (2.1)—(2.3) die Behauptung von Satz 7. folgt.

BEMERKUNG. Die durch (1.1) festgelegte Übertragung D ist selbstverständlich von der Cartanschen Übertragung doch verschieden, da in (1.1) der durch (5.4) angegebene P^i_j-Tensor vorkommt, der in der gewöhnlichen Cartanschen Übertragung nicht vorhanden ist.

In unserem Aufsatz [3] haben wir die Eigenvektoren der $F-O_2$-bzw. $F-O^*_2$-Räume bestimmt. Wir wollen jetzt die Form der Eigenvektoren bezüglich des durch (5.1) gekennzeichneten P^i_j-Tensors bestimmen. In [3] ist nur der durch $\alpha=1$, d. h. $P^i_0=l^i$ gekennzeichnete Fall erledigt. Wir beweisen nun bezüglich des Typus (5.1) den

SATZ 8. *Hat in einem $F-O_2^*$-Raum der P_j^i-Tensor die Form* (5.1), *so sind die Eigenvektoren bezüglich des Zweibeins* $(\vec{l}, \vec{h})$ *von der Form:*

$$V^j = \pm\sqrt{1+\alpha}\, l^j + \sqrt{1-\alpha}\, h^j, \quad \tau = +1, \quad |\alpha| \leqq +1, \tag{5.6a}$$

$$V^j = \mp\sqrt{1-\alpha}\, l^j + \sqrt{1+\alpha}\, h^j, \quad \tau = -1, \quad |\alpha| \leqq +1, \tag{5.6b}$$

wo τ *die Eigenfunktion bedeutet.*

BEWEIS. Nehmen wir an, daß der Eigenvektor $\vec{V}$ die Beindarstellung

$$V^j = \varrho l^j + \sigma h^j \tag{5.7}$$

hat, wo ϱ, σ Skalare bedeuten. Schreibt man $P_j^i V^i = V^i$ auf, substituiert man dann (5.1) und (5.7) in diese Relation, so wird nach gewissen Umformungen:

$$[\varrho(\alpha-1) \pm \sigma\sqrt{1-\alpha^2}]\, l^i + [\sigma(-\alpha-1) \pm \varrho\sqrt{1-\alpha^2}]\, h^i = 0.$$

Die Koeffizienten von l^i und h^i gleich Null gesetzt, bekommt man ein homogenlineares Gleichungssystem für ϱ, σ. Die Lösung ist

$$\varrho : \sigma = \pm\sqrt{1+\alpha} : \sqrt{1-\alpha}$$

woraus nach (5.7) für V^j die Formel (5.6a) folgt.

Ist die Eigenfunktion: $\tau = -1$, und substituiert man (5.1) und (5.7) in $P_j^i V^j = -V^i$, so bekommt man nach einigen Umformungen:

$$[\varrho(1+\alpha) \pm \sigma\sqrt{1-\alpha^2}]\, l^i + [\sigma(1-\alpha) \pm \varrho\sqrt{1-\alpha^2}]\, h^i = 0.$$

Wie im vorigen Fall, geben die Koeffizienten von l^i und h^i gleich Null gesetzt ein homogen lineares Gleichungssystem für ϱ, σ. Die Lösung wird jetzt

$$\varrho : \sigma = \mp\sqrt{1-\alpha} : \sqrt{1+\alpha},$$

woraus nach (5.7) die Relation (5.6b) folgt.

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ON THE CURVATURE AND INTEGRABILITY OF HORIZONTAL MAPS

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1. Introduction. In the recent progress of connection theory (especially in Finsler geometry, see [9]) and its applications the vector bundle viewpoint plays a central role. Purely geometrically, from this point of view the notion of a (general) connection ([11], Definition 3) is based upon a Whitney decomposition of the tangent bundle of the total space of the considered vector bundle into the vertical subbundle and a *horizontal* one. Each of the horizontal subbundles can be obtained with the help of a (right, smooth) splitting of the canonical short exact sequence ([4], Vol. II, p. 335) constructed from the given vector bundle. These splittings are called *horizontal maps*. Hence the theory of connections may be developed starting from a horizontal map; a sketch of such an approach can be found in the author's paper [11], cf. also [2], [12]. In this note we are going to investigate mainly the integrability of a horizontal map. Here, *integrability* means that the image bundle of the horizontal map under discussion is an involutive distribution in the usual sense ([4], Vol. I, p. 134). Our main result gives a number of necessary and sufficient conditions for a horizontal map to be integrable. These criteria will be formulated in terms of *horizontal projection, vertical projection, almost product structure* and *curvature tensor field* induced by the horizontal map $\mathscr{H}$ in question. We shall also discuss some interesting relations between the *Dombrowski map K* belonging to $\mathscr{H}$ and the curvature tensor field, furthermore — in the linear case — between K and the usual curvature *form* induced by $\mathscr{H}$. These relations generalize some results of Dombrowski's important paper [1].

Grifone's and Vilms' works [5], [12] were very stimulating for the present investigations. A discussion of similar questions also occurs in Duc's excellent survey [2].

Notations, terminology and basic conventions are as in the monograph [4] and in the paper [11].

2. Preliminaries. For the convenience of the reader, we begin with some definitions and technical remarks needed in the subsequent considerations.

Let M be a manifold. $\mathscr{X}^p_q(M)$, $A^p(M, \tau_M)$, $\operatorname{End}\tau_M$ and $\operatorname{End}\mathscr{X}(M)$ denote the $C^\infty(M)$-module of (p, q) tensor fields on M, τ_M-valued p-forms on M, $\tau_M \to \tau_M$ (linear) bundle maps, and $\mathscr{X}(M)\to\mathscr{X}(M)$ $C^\infty(M)$-linear maps, respectively. It is known (e.g. from [7], Proposition 3.1) that the module of q-linear maps $\mathscr{X}(M)\times\ldots\times\mathscr{X}(M)\to\mathscr{X}(M)$ is isomorphic to the module $\mathscr{X}^1_q(M)$; in particular $\mathscr{X}^1_1(M)\cong\operatorname{End}\mathscr{X}(M)$. Now suppose that $f\in\operatorname{End}\tau_M$. Then $\forall p\in M: f|T_pM=f_p\in\operatorname{End}T_pM$, so the map $\bar{f}: p\in M\to\bar{f}(p):=f_p$ is an element of $A^1(M, \tau_M)$ and the correspondence $f\to\bar{f}$ defines a (natural) isomorphism $\operatorname{End}\tau_M\cong A^1(M, \tau_M)$. Because of $\operatorname{End}T_pM\cong(T_pM)^*\otimes T_pM$ one can also write $f\in\operatorname{Sec}(\tau^*_M\otimes\tau_M)=\mathscr{X}^1_1(M)$. Summarizing, we obtain the following.

LEMMA. $\mathscr{X}_1^1(M) \cong \operatorname{End} \mathscr{X}(M) \cong \operatorname{End} \tau_M \cong A^1(M, \tau_M)$.

We shall identify these isomorphic modules, without further reference.

DEFINITION 1. (See [3] or [7], Proposition 3.12). Let $f, g \in \operatorname{End} \tau_M$. Their *Nijenhuis-torsion* is the $(1, 2)$ tensor field $[f, g]\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ given by

$$[f, g](X, Y) := [fX, gY] + [gX, fY] - g[fX, Y] - f[gX, Y] -$$
$$- g[X, fY] - f[X, gY] + g \circ f[X, Y] + f \circ g[X, Y]$$

($[fX, gY]$ etc. are the usual Lie products of vector fields).

The introduction of the following useful concept was inspired by Haantjes's theorem ([3], Theorem III).

DEFINITION 2. Let $f \in \operatorname{End} \tau_M$. Suppose that $\forall p \in M\colon f_p =: f|T_pM$ has constant eigenvalues $\lambda_1, \ldots, \lambda_k$ with constant geometric multiplicity $(1 \leqq k \leqq n)$. Let us denote by $S(\lambda_i)$ those subbundles of τ_M whose fibers at a point $p \in M$ are the invariant subspaces belonging to λ_i. f is called *integrable* if the subbundles $S(\lambda_i)$ and the Whitney sums $S(\lambda_1) \oplus \ldots \oplus S(\lambda_j)$ $(j = 2, \ldots, k)$ are involutive distributions.

In the sequel $\xi = (E, \pi, B, F)$ will always denote a fixed vector bundle over the n-dimensional base manifold B. The sequence of vector bundles $0 \to V_\xi \xrightarrow{i} \tau_E \to$ $\xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \to 0$ (where V_ξ denotes the vertical subbundle of the tangent bundle τ_E) is a short exact sequence, which is also called the canonical exact sequence starting from ξ. So a horizontal map is a bundle map $\mathscr{H}\colon \pi^*(\tau_B) \to \tau_E$, while the horizontal projection, vertical projection and almost product structure mentioned above are

$$h := \mathscr{H} \circ \widetilde{d\pi}, \quad v := 1 - h, \quad P := 2h - 1$$

respectively. If $\alpha\colon V_\xi \to \xi$ is the canonical bundle map described e.g. in [4], Vol. I p. 291, then $K := \alpha \circ V$ is the Dombrowski-map induced by $\mathscr{H}$. (For details, see [11].) The endomorphisms $h, v, P \in \operatorname{End} \tau_E$ — according to the lemma — will also be interpreted as $(1, 1)$ tensor fields. In this case — for better clarity — we shall sometimes write $\tilde{h}, \tilde{v}, \tilde{P}$ instead of h, v and P. Observe (cf. [11], Section 5) that these endomorphisms satisfy the conditions of Definition 2, hence we can speak about their integrability.

3. Curvature of horizontal maps. Following Grifone's idea, which in turn was inspired by [3], we define the *curvature tensor field* of a horizontal map $\mathscr{H}$ as the Nijenhuis-torsion $R := -1/2\,[h, h]$. The geometric meaning of this construction will be clarified by the next results. There are, of course, other possibilities to define curvature for a horizontal map; see e.g. [8], [12] and — in the linear case — [10].

PROPOSITION 1. *Let $U \subset B$ be a trivializing neighbourhood for ξ and (x^i, y^α) the local coordinate system over $\pi^{-1}(U)$ described in* [11], *Section* 5. *If the functions $\Gamma_i^\alpha\colon \pi^{-1}(U) \to \mathbf{R}$ are the connection parameters of $\mathscr{H}$ with respect to (x^i, y^α) and $Z = Z^i \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial y^\alpha}$, $V = V^i \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial y^\alpha}$ are vector fields over $\pi^{-1}(U)$, then $R(Z, V) = Z^i V^j R_{ij}^\alpha \frac{\partial}{\partial y^\alpha}$, where*

$$R_{ij}^\alpha = \frac{\partial \Gamma_j^\alpha}{\partial x^i} - \frac{\partial \Gamma_i^\alpha}{\partial x^j} + \Gamma_j^\beta \frac{\partial \Gamma_i^\alpha}{\partial y^\beta} - \Gamma_i^\beta \frac{\partial \Gamma_i^\alpha}{\partial y^\beta}.$$

COROLLARY. $\forall Z, V \in \mathscr{X}(E)$: $R(Z, V) \in \mathscr{X}_V(E) := \operatorname{Sec} V_\xi$.

The proof is a standard, but lengthy calculation, so we omit it.

In view of $hX^h = X^h$, $hY^h = Y^h$ (where X^h and Y^h are the horizontal lifts of X and Y with respect to $\mathscr{H}$, see [11], Definition 2) we immediately obtain from the definition of R and of the Nijenhuis torsion

PROPOSITION 2 (cf. [13] and [2]). $\forall X, Y \in \mathscr{X}(B)$: $R(X^h, Y^h) = h[X^h, Y^h] - [X^h, Y^h]$.

Using this proposition and the obvious relation $K \circ h = o$ we have the following generalization of formula (23) of [1]:

PROPOSITION 3. *If K is the Dombrowski map belonging to $\mathscr{H}$ then*

$$\forall X, Y \in \mathscr{X}(B)\colon\ \alpha \circ R(X^h, Y^h) = -K \circ [X^h, Y^h],$$

that is the diagram

$$\begin{array}{ccc} E & \xrightarrow{R(X^h,\, Y^h)} & VE \\ {\scriptstyle [X^h,\, Y^h]}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TE & \xrightarrow{\ -K\ } & E \end{array}$$

commutes.

PROOF. On the one hand

$$K \circ R(X^h, Y^h) = K \circ (h[X^h, Y^h] - [X^h, Y^h]) = -K \circ [X^h, Y^h],$$

on the other hand

$$K \circ R(X^h, Y^h) = \alpha \circ v \circ R(X^h, Y^h) = \alpha \circ R(X^h, Y^h),$$

because of $R(X^h, Y^h) \in \mathscr{X}_V(E)$. □

Now we suppose that $\mathscr{H}$ satisfies the so-called homogeneity condition ([11], Definition 2). In this case $\mathscr{H}$ induces a $\nabla\colon \operatorname{Sec} \xi \to A^1(B, \xi)$ *linear connection*. It is well-known that the *curvature form* of ∇ is the mapping

$$\tilde{R}\colon \mathscr{X}(B) \times \mathscr{X}(B) \to \operatorname{End}\ \operatorname{Sec} \xi,$$

$$(X, Y) \mapsto \tilde{R}(X, Y) := \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X, Y]},$$

where $\operatorname{End} \operatorname{Sec} \xi$ is the module of the $C^\infty(B)$-linear maps $\operatorname{Sec} \xi \to \operatorname{Sec} \xi$ (see [4], Vol. II). What is the relation between the curvature tensor field R and the curvature form $\tilde{R}$? The answer to this very natural question is given in the

THEOREM 1. $\forall X, Y \in \mathscr{X}(B)$, $\sigma \in \operatorname{Sec} \xi$: $\alpha \circ R(X^h, Y^h) \circ \sigma = \tilde{R}(X, Y)(\sigma)$, *so we have the following commutative diagram:*

$$\begin{array}{ccc} B & \xrightarrow{\ \sigma\ } & E \\ {\scriptstyle \tilde{R}(X,\, Y)(\sigma)}\downarrow & & \downarrow{\scriptstyle R(X^h,\, Y^h)} \\ E & \xleftarrow{\ \alpha\ } & VE \end{array}$$

PROOF. Choosing a trivializing neighbourhood $U \subset B$, we again work in the coordinate system (x^i, y^α). It induces a framing $e_\alpha\colon U \to E$ (see [11], Section 5), so each section $\sigma\colon B \to E$ can be written locally in the form $\sigma^\alpha e_\alpha$. A straightforward (lengthy) calculation shows that over U

$$(\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})\sigma = X^i Y^j \tilde{R}^\alpha_{ij\beta} \sigma^\beta e_\alpha,$$

where

$$X = X^i \frac{\partial}{\partial u^i}, \quad Y = Y^j \frac{\partial}{\partial u^j}, \quad \tilde{R}^\alpha_{ij\beta} = \frac{\partial \Gamma^\alpha_{j\beta}}{\partial u^i} - \frac{\partial \Gamma^\alpha_{i\beta}}{\partial u^j} + \Gamma^\alpha_{i\delta}\Gamma^\delta_{j\beta} - \Gamma^\alpha_{j\delta}\Gamma^\delta_{i\beta},$$

and in the last expression the functions $\Gamma^\alpha_{i\beta}$ are defined by $\Gamma^\alpha_{i\beta} := \frac{\partial \Gamma^\alpha_i}{\partial y^\beta}$ (see [11], Th. 1). On the other hand,

$$R(X^h, Y^h) = (X^i Y^j \circ \pi) y^\beta \left[\frac{\partial \Gamma^\alpha_{j\beta}}{\partial u^i} \circ \pi - \frac{\partial \Gamma^\alpha_{i\beta}}{\partial u^j} \circ \pi + (\Gamma^\alpha_{i\delta} \circ \pi)(\Gamma^\delta_{j\beta} \circ \pi) - \right.$$

$$\left. - (\Gamma^\alpha_{j\delta} \circ \pi)(\Gamma^\delta_{i\beta} \circ \pi) \right] \frac{\partial}{\partial y^\alpha}.$$

The value of this vector field at the point $\sigma(x) \in \pi^{-1}(U)$ is

$$R(X^h, Y^h)[\sigma(x)] = X^i Y^j \sigma^\beta(x) \tilde{R}^\alpha_{ij\beta}(x) \left(\frac{\partial}{\partial y^\alpha} \right)_{\sigma(x)} \in T_{\sigma(x)} E,$$

consequently

$$\alpha_{\sigma(x)}[\tilde{R}(X^h, Y^h)[\sigma(x)]] = X^i Y^j \sigma^\beta(x) \tilde{R}^\alpha_{ij\beta}(x) \left[\alpha_{\sigma(x)} \left(\frac{\partial}{\partial y^\alpha} \right)_{\sigma(x)} \right] =$$

$$= X^i Y^j \tilde{R}^\alpha_{ij\beta}(x) \sigma^\beta(x) e_\alpha(x).$$

This is exactly a coordinate statement of the conclusion of the Theorem. □

COROLLARY. *In the linear case* $\forall X, Y \in \mathscr{X}(B)$, $\sigma \in \operatorname{Sec} \xi\colon K \circ [X^h, Y^h] \circ \sigma = -[\tilde{R}(X, Y)](\sigma)$.

This last relation is the immediate generalization of the above mentioned formula of Dombrowski's paper.

4. Integrability of horizontal maps. Now we turn to the study of the integrability of a horizontal map $\mathscr{H}$. The next result summarizes the situation.

THEOREM 2. *For a horizontal map* $\mathscr{H}\colon \pi^*(\tau_B) \to \tau_E$ *the following are equivalent:*

(1) $\mathscr{H}$ *is integrable.*

(2) *The curvature tensor field of* $\mathscr{H}$ *vanishes, that is* $\forall Z, V \in \mathscr{X}(E)\colon R(Z, V) = 0$.

(3) *The component functions* R^α_{ij} *of* R *vanish.*

(4) *The horizontal projection* h *is integrable.*

(5) $\tilde{v} \circ [\tilde{h}, \tilde{h}] = 0$.

(6) *The vertical projection* v *is integrable.*

(7) $[\tilde{v}, \tilde{v}]=0$.

(8) $\tilde{v}\circ[\tilde{v}, \tilde{v}]=0$.

(9) $[\tilde{P}, \tilde{P}]=0$.

(10) *The almost product structure* P *is integrable.*

PROOF. (a) We begin with the following technical remarks:

$$[\tilde{h}, \tilde{h}] = [\tilde{v}, \tilde{v}], \quad \operatorname{Im} \tilde{h} = \operatorname{Ker} \tilde{v} = \operatorname{Sec} \operatorname{Im} \mathscr{H}.$$

Indeed, $[\tilde{v}, \tilde{v}]=[1-\tilde{h}, 1-\tilde{h}]=[1, 1]-[\tilde{h}, 1]-[1, \tilde{h}]+[\tilde{h}, \tilde{h}]$. Here $[1, 1]=0$, as it immediately follows from the definition of the operation $[\ ,\ ]$, while $\forall Z_1, Z_2\in\mathscr{X}(E)$:

$$[\tilde{h}, 1](Z_1, Z_2) = [\tilde{h}Z_1, Z_2]+[Z_1, \tilde{h}Z_2]-[\tilde{h}Z_1, Z_2]-\tilde{h}[Z_1, Z_2]-[Z_1, \tilde{h}Z_2]-$$
$$-\tilde{h}[Z_1, Z_2]+\tilde{h}[Z_1, Z_2]+\tilde{h}[Z_1, Z_2] = 0 \Rightarrow [\tilde{h}, 1] = 0.$$

Similarly $[1, \tilde{h}]=0$, thus $[\tilde{v}, \tilde{v}]=[\tilde{h}, \tilde{h}]$. The second assertion is clear.

(b) We show: (2)⇔(1)⇔(5)⇔(8). Let $Z_1, Z_2\in\mathscr{X}(E)$. From the definition of the curvature we have:

$$[\tilde{h}Z_1, \tilde{h}Z_2] = -R(Z_1, Z_2)-\tilde{h}[Z_1, Z_2]+\tilde{h}[\tilde{h}Z_1, Z_2]+\tilde{h}[Z_1, \tilde{h}Z_2].$$

On the right hand side $-R(Z_1, Z_2)\in\mathscr{X}_V(E)$, while the other terms belong to $\mathscr{X}_H(E):=\operatorname{Sec}\operatorname{Im}\mathscr{H}$ because of (a). It implies that $[\tilde{h}Z_1, \tilde{h}Z_2]\in\mathscr{X}_H(E)\Leftrightarrow R(Z_1, Z_2)=0$ proving the equivalence (2)⇔(1). From this the implication (1)⇒(5) also follows. If — conversely — $\tilde{v}\circ[\tilde{h}, \tilde{h}]=0$, then (applying (a) again) $\operatorname{Im}[\tilde{h}, \tilde{h}]\subset\operatorname{Ker}\tilde{v}=\operatorname{Im}\tilde{h}=\mathscr{X}_H(E)$ hence

$$\frac{1}{2}[\tilde{h}, \tilde{h}](Z_1, Z_2) = [\tilde{h}Z_1, \tilde{h}Z_2]+\tilde{h}[Z_1, Z_2]-\tilde{h}[\tilde{h}Z_1, Z_2]-\tilde{h}[Z_1, \tilde{h}Z_2]\in\operatorname{Sec}\operatorname{Im}\mathscr{H}\Rightarrow$$
$$\Rightarrow [\tilde{h}Z_1, \tilde{h}Z_2]\in\operatorname{Sec}\operatorname{Im}\mathscr{H},$$

that is (1) holds. The equivalence (5)⇔(8) is obvious from $[\tilde{h}, \tilde{h}]=[\tilde{v}, \tilde{v}]$.

(c) We verify that the assertions (3), (4), (6), (7), (9) and (10) are equivalent to (2). Let us first observe that in case of the endomorphisms $h, v, P\in\operatorname{End}\tau_E$ the subbundles $S(\lambda_i)$ mentioned in Definition 2 are the following:

$$h\colon\ S(1) = \operatorname{Im} h = \operatorname{Im}\mathscr{H}, \quad S(0) = \operatorname{Im} v = V_\xi;$$

$$v\colon\ S(1) = V_\xi, \quad S(0) = \operatorname{Im}\mathscr{H}; \quad P\colon S(1) = \operatorname{Im}\mathscr{H}, \quad S(-1) = V_\xi.$$

Here, of course, the subbundle V_ξ is an involutive distribution, so we easily get the equivalences (2)⇔(4), (2)⇔(6), (2)⇔(10). The equivalence (2)⇔(3) is evident, (2)⇔(7) follows from (a). Finally — applying also (a) —

$$[\tilde{P}, \tilde{P}] = [2\tilde{h}-1, 2\tilde{h}-1] = 4[\tilde{h}, \tilde{h}]-2[1, \tilde{h}]-2[\tilde{h}, 1]+[1, 1] = -8R,$$

this establishes the equivalence of (2) and (9).

REMARK. From the proof we find that $R=-1/8\,[P, P]$. In a more special situation this was proved also by Stere Ianus [6].

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LOWER ESTIMATES FOR THE EIGENFUNCTIONS OF A LINEAR DIFFERENTIAL OPERATOR

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Let $G\subset\mathbf{R}$ be a bounded open interval, $n\in\mathbf{N}$, $q_1, \dots, q_n\in L^1(G)$ arbitrary complex functions, and consider the formal differential operator

$$Lu = u^{(n)}+q_1u^{(n-1)}+\dots+q_nu.$$

Let us recall the definition of the eigenfunctions of higher order:

Given a complex number λ, the function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. A function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \dots)$ of the operator L with the eigenvalue λ if the following two conditions are satisfied:

— u and its first $n-1$ derivatives are absolute continuous on every compact subinterval of G;

— there exists an eigenfunction u^* of order $m-1$ of the operator L with the eigenvalue λ such that for almost all $x\in G$

$$(Lu)(x) = \lambda u(x)+u^*(x). \tag{1}$$

Let us denote by $\mu_1, \dots, \mu_n$ the n-th roots of λ such that

$$\operatorname{Re}\mu_1 \geqq \dots \geqq \operatorname{Re}\mu_n$$

and put for brevity

$$n' = \left[\frac{n+1}{2}\right], \quad \mu = \mu_{n'}, \quad \varrho = \operatorname{Re}\mu_{n'}.$$

Obviously,

$$|\varrho| = \min\{|\operatorname{Re}\mu_p|: p = 1, \dots, n\}.$$

It is known (see [3], [5], [7], [8]) that for any eigenfunction u of order m of the operator L with some eigenvalue λ,

$$\|u\|_{L^\infty(G)} \leqq C_m(1+|\operatorname{Re}\mu_1|)^{1/p}\|u\|_{L^p(G)} \quad (p\in[1, \infty]);$$

the same estimate holds for $u^{(i)}$ $(i=1, \dots, n-1)$ instead of u if $|\lambda|$ is sufficiently large. Conversely, the estimates

$$\|u^{(i)}\|_{L^\infty(G)} \geqq C'_m(1+|\operatorname{Re}\mu_{n'}|)^{1/p}\|u^{(i)}\|_{L^p(G)} \quad (p\in[1, \infty])$$

are proved in the following two cases:

— $n\leqq 2$, m arbitrary (see [4], [8]),

— n arbitrary, $m=0$, $q_1=\dots=q_n\equiv 0$ (see [6]).

In the present paper we shall extend this last result to the general case $q_1, \dots$ $\dots, q_n\in L^1(G)$. The case $m\geqq 1$ remains open; however, a weaker form of it will be proved which is sometimes sufficient for the applications. We shall prove the following results:

THEOREM 1. *There exist positive constants* α, E *such that for any eigenfunction* u *of order* 0 *of the operator* L *with some eigenvalue* λ,

$$\left\|u^{(i)}(x)\exp\left(\alpha|\operatorname{Re}\mu_{n'}|\operatorname{dist}(x,\partial G)\right)\right\|_{L^\infty(G)} \leqq E\|u^{(i)}\|_{L^\infty(G)} \quad (i=0,\dots,n-1). \tag{2}$$

THEOREM 2. *There exists a positive constant* B *such that for any eigenfunction* u *of order* 0 *of the operator* L *with some eigenvalue* λ,

$$\|u^{(i)}\|_{L^\infty(G)} \geqq B(1+|\operatorname{Re}\mu_{n'}|)^{1/p}\|u^{(i)}\|_{L^p(G)} \quad (i=0,\dots,n-1,\quad p\in[1,\infty]). \tag{3}$$

THEOREM 3. *Suppose* $q_1\equiv 0$. *Then to any* $m\in\{0,1,\dots\}$ *and to any compact subinterval* K *of* G *there exist positive constants* C_m, R_m *such that for any eigenfunction* $u=u_m$ *of order* m *of the operator* L *with some eigenvalue* λ,

$$\|u^{(i)}\|_{L^\infty(K)} \leqq C_m e^{-|\operatorname{Re}\mu_{n'}|R_m}\|u^{(i)}\|_{L^\infty(G)} \quad (i=0,\dots,n-1). \tag{4}$$

If $v=v_m$ is an eigenfunction of order $\leqq m$ then we put for brevity $v_j=v_{j+1}^*$ if $j\in\{0,\dots,m-1\}$ and $v_j=0$ if $j\in\{-1,-2,\dots\}$. Furthermore we set $N=n(m+1)$ and $N'=n'(m+1)$. We recall that, by Proposition 2 in [9], there exist continuous functions f_k, F_r such that for any eigenfunction v_m of order $\leqq m$ of the operator L with some eigenvalue λ,

$$\sum_{k=N'-N}^{N'} f_k(\mu,t)v_m^{(i)}(x+kt) = \sum_{r=0}^{m}\int_{x+(N'-N)t}^{x+N't} D_3^i F_r(\mu,t,x-\tau)\sum_{s=1}^{n} q_s(\tau)v_{m-r}^{(n-s)}(\tau)\,d\tau \tag{5}$$

$$(i=0,\dots,n-1)$$

whenever $x+(N'-N)t\in G$ and $x+N't\in G$. (D_3 denotes differentiation with respect to the 3^{rd} variable.) Furthermore, introducing the notation

$$Q(\mu,t)=\exp\left((m+1)(\mu_1+\dots+\mu_{[(n+1)/2]})t\right), \tag{6}$$

there exists a constant C such that in the above formula for $t\geqq 0$

$$|f_0(\mu,t)-Q(\mu,t)|\leqq C|Q(\mu,t)|e^{-\varrho t},\quad |f_k(\mu,t)|\leqq C|Q(\mu,t)|e^{-k\varrho t}, \tag{7}$$
$$|D_3^i F_r(\mu,t,x-\tau)|\leqq C|\mu|^{i+(r+1)(1-n)}|Q(\mu,t)|e^{-\varrho|x-\tau|}.$$

(We have taken into account that G is bounded.)

PROOF OF THEOREM 1. We consider only the case $\varrho\geqq 0$. The case $\varrho<0$ is similar but one can also trace it back to the case $\varrho\geqq 0$, applying the transformation described in the introduction of [9]. In the sequel C will denote diverse constants independent of the choice of the eigenfunction u, the points $x, y\in G$ and the number R (see below).

Fixing $y\in G$ and $0<R\leqq(n')^{-1}\operatorname{dist}(y,\partial G)$ arbitrarily, let us set

$$G^*=G^*_{y,R}:=(y-R,\,y+R) \tag{8}$$

and

$$Q=Q_{y,R}:=\max\{\|q_s\|_{L^1(y-n'R,\,y+n'R)}:\ s=1,\dots,n\}. \tag{9}$$

Applying (5), (6), (7) for any $x \in G^*$ with $t = \operatorname{dist}(x, \partial G^*)$, we obtain for any $i \in \in \{0, \dots, n-1\}$

$$|u^{(i)}(x) e^{\varrho \operatorname{dist}(x, \partial G^*)}| \leqq C\|u^{(i)}\|_{L^\infty(G)} +$$

$$+ C_0 Q \sum_{s=1}^{n} |\mu|^{i+1-n} \|u^{(n-s)}(\tau) e^{\varrho(t-|x-\tau|)}\|_{L^\infty(G)} \leqq C\|u^{(i)}\|_{L^\infty(G)} +$$

$$+ C_0 Q \sum_{s=1}^{n} |\mu|^{i+1-n} \|u^{(n-s)}\|_{L^\infty(G \setminus G^*)} +$$

$$+ C_0 Q \sum_{s=1}^{n} |\mu|^{i+1-n} \|u^{(n-s)}(\tau) e^{\varrho \operatorname{dist}(\tau, \partial G^*)}\|_{L^\infty(G^*)}.$$

For $|\mu| \geqq 1$ hence we obtain

$$\|\mu^{-i} u^{(i)}(x) e^{\varrho \operatorname{dist}(x, \partial G^*)}\|_{L^\infty(G^*)} \leqq$$

$$\leqq C \sum_{s=1}^{n} \|\mu^{-n+s} u^{(n-s)}\|_{L^\infty(G)} + C_0 Q \sum_{s=1}^{n} \|\mu^{-n+s} u^{(n-s)}(\tau) e^{\varrho \operatorname{dist}(\tau, \partial G^*)}\|_{L^\infty(G^*)}$$

and

$$(1 - nC_0 Q) \sum_{i=0}^{n-1} \|\mu^{-i} u^{(i)}(x) e^{\varrho \operatorname{dist}(x, \partial G^*)}\|_{L^\infty(G^*)} \leqq C \sum_{i=0}^{n-1} \|\mu^{-i} u^{(i)}\|_{L^\infty(G)}.$$

Let us now fix $R = R_y$ such that

$$\text{(10)} \qquad \begin{cases} R_y \text{ is maximal with respect to the properties} \\ 0 < R \leqq (n')^{-1} \operatorname{dist}(y, \partial G) \quad \text{and} \quad 1 - nC_0 Q \geqq 1/2. \end{cases}$$

Then, if we take also into account that

$$\sum_{i=0}^{n-1} \|\mu^{-i} u^{(i)}\|_{L^\infty(G)} \leqq C\|u\|_{L^\infty(G)}$$

(see [7]), we obtain for any $i \in \{0, \dots; n-1\}$

$$\text{(11)} \qquad \|u^{(i)}(x) e^{\varrho \operatorname{dist}(x, \partial G^*)}\|_{L^\infty(G^*)} \leqq C|\mu|^i \|u\|_{L^\infty(G)}.$$

Let us now fix two points $a + R_0$, $b - R_0 \in (a, b) := G$ such that $a + R_0 \leqq b - R_0$ and

$$2nC_0 \max \{\|q_s\|_{L^1(a, a+2R_0)} : s = 1, \dots, n\} \leqq 1,$$

$$2nC_0 \max \{\|q_s\|_{L^1(b-2R_0, b)} : s = 1, \dots, n\} \leqq 1.$$

Then in the case when $y \in (a, a+R_0) \cup (b-R_0, b)$, $\operatorname{dist}(y, \partial G^*) = (n')^{-1} \operatorname{dist}(y, \partial G)$ by (8), (9), (10) and then (11) implies

$$\text{(12)} \qquad \|u^{(i)}(y) e^{(n')^{-1} \varrho \operatorname{dist}(y, \partial G)}\|_{L^\infty(G \setminus [a+R_0, b-R_0])} \leqq C|\mu|^i \|u\|_{L_\infty(G)}.$$

On the other hand, the compact set $[a+R_0, b-R_0]$ can be covered by finitely many intervals $\left(y_r - \frac{1}{2} R_{y_r}, y_r + \frac{1}{2} R_{y_r}\right)$. If $x \in \left(y_r - \frac{1}{2} R_{y_r}, y_r + \frac{1}{2} R_{y_r}\right)$ for some r then $\operatorname{dist}(x, \partial G^*) \geqq \frac{1}{2} R_{y_r} \geqq \frac{R_{y_r}}{b-a} \operatorname{dist}(x, \partial G)$, whence, putting $\alpha_0 = \min_r \left\{\frac{R_{y_r}}{b-a}\right\}$,

$$\text{(13)} \qquad \|u^{(i)}(x) e^{\alpha_0 \varrho \operatorname{dist}(x, \partial G)}\|_{L^\infty(a+R_0, b-R_0)} \leqq C|\mu|^i \|u\|_{L^\infty(G)}.$$

Setting $\alpha=\min\{(n')^{-1},\alpha_0\}$, (12) and (13) imply

$$\|u^{(i)}(x)e^{\alpha\varrho\operatorname{dist}(x,\partial G)}\|_{L^\infty(G)}\leqq C|\mu|^i\|u\|_{L^\infty(G)}. \tag{14}$$

Obviously, it suffices to prove the estimate (2) for $|\mu|$ sufficiently large. But then (2) follows from (14) beacuse by Theorem 3 of paper [7]

$$|\mu|^i\|u\|_{L^\infty(G)}\leqq C\|u^{(i)}\|_{L^\infty(G)}.$$

The theorem is proved.

PROOF OF THEOREM 2. It follows from the preceding theorem that

$$\|u^{(i)}\|_{L^p(G)}\leqq E\|u^{(i)}\|_{L^\infty(G)}\|e^{-\alpha|\varrho|\operatorname{dist}(x,\partial G)}\|_{L^p(G)}. \tag{15}$$

Furthermore, by easy computation

$$\|e^{-\alpha|\varrho|\operatorname{dist}(x,\partial G)}\|_{L^p(G)}\leqq(2/\alpha)^{1/p}|\varrho|^{-1/p}. \tag{16}$$

(15) and (16) imply (3) if $|\varrho|\geqq 1$. (3) being obvious for $|\varrho|<1$, the theorem is proved.

OPEN PROBLEM. It would be interesting to extend Theorems 1 and 2 for the case $m\geqq 1$.

PROOF OF THEOREM 3. As before, it suffices to consider the case $\varrho\geqq 0$. Obviously it suffices to prove the estimate (4) for $|\mu|$ sufficiently large.

Let us fix a compact subinterval K_1 of G such that $K\subset\operatorname{int}K_1$. In the sequel C will denote diverse constants independent of the choice of the eigenfunction u_m and the points $x,y\in G$ (see below). Set $R=(N')^{-1}\operatorname{dist}(K,\partial K_1)$. Fixing $y\in K$ arbitrarily, put $K_y=[y-R,y+R]$. For any $x\in K_y$, applying (5), (6) and (7) with $v_m=u_{m-j}$ ($j\in\{0,\dots,m\}$) and $t=\operatorname{dist}(x,\partial K_y)$,

$$|u_{m-j}^{(i)}(x)e^{\varrho\operatorname{dist}(x,\partial K_y)}|\leqq C\|u_{m-j}^{(i)}\|_{L^\infty(K_1)}+$$

$$+C\sum_{r=0}^{m-j}\sum_{s=2}^{n}|\mu|^{i+(r+1)(1-n)}\|u_{m-j-r}^{(n-s)}(\tau)e^{\varrho(\operatorname{dist}(x,\partial K_y)-|x-\tau|)}\|_{L^\infty(K_1)}\leqq$$

$$\leqq C\|u_{m-j}^{(i)}\|_{L^\infty(K_1)}+C\sum_{r=0}^{m-j}\sum_{s=2}^{n}|\mu|^{i+(r+1)(1-n)}\|u_{m-j-r}^{(n-s)}\|_{L^\infty(K_1\setminus K_y)}+$$

$$+C\sum_{r=0}^{m-j}\sum_{s=2}^{n}|\mu|^{i+(r+1)(1-n)}\|u_{m-j-r}^{(n-s)}(\tau)e^{\varrho\operatorname{dist}(\tau,\partial K_y)}\|_{L^\infty(K_y)}.$$

For $|\mu|\geqq 1$ hence we obtain

$$\|\mu^{j(1-n)-i}u_{m-j}^{(i)}(x)e^{\varrho\operatorname{dist}(x,\partial K_y)}\|_{L^\infty(K_y)}\leqq C\sum_{r=0}^{m}\sum_{s=2}^{n}\|\mu^{r(1-n)-n+s}u_{m-r}^{(n-s)}\|_{L^\infty(K_1)}+$$

$$+C|\mu|^{-1}\sum_{r=0}^{m}\sum_{s=2}^{n}\|\mu^{r(1-n)-n+s}u_{m-r}^{(n-s)}(\tau)e^{\varrho\operatorname{dist}(\tau,\partial K_y)}\|_{L^\infty(K_y)}$$

and

$$(1-C|\mu|^{-1})\sum_{j=0}^{m}\sum_{i=0}^{n-1}\|\mu^{j(1-n)-i}u_{m-j}^{(i)}(x)e^{\varrho\operatorname{dist}(x,\partial K_y)}\|_{L^\infty(K_y)}\leqq$$

$$\leqq C\sum_{j=0}^{m}\sum_{i=0}^{n-1}\|\mu^{j(1-n)-i}u_{m-j}^{(i)}\|_{L^\infty(K_1)}.$$

Applying now Theorem 1 from [9] (this in the only point where the condition $q_1 \equiv 0$ is essential) and Theorem 2 from [7],

$$\sum_{j=0}^{m} \sum_{i=0}^{n-1} \|\mu^{j(1-n)-i} u_{m-j}^{(i)}\|_{L^\infty(K_1)} \leqq C \|u_m\|_{L^\infty(G)};$$

therefore if $|\mu|$ is sufficiently large then

$$\|u_m^{(i)}(x) e^{\varrho \operatorname{dist}(x, \partial K_y)}\|_{L^\infty(K_y)} \leqq C |\mu|^i \|u_m\|_{L^\infty(G)}$$

for all $i \in \{0, \ldots, n-1\}$. Putting $x=y$ we obtain $\|u_m^{(i)}\|_{L^\infty(K)} e^{\varrho R} \leqq C |\mu|^i \|u_m\|_{L^\infty(G)}$. If $|\mu|$ is sufficiently large then $|\mu|^i \|u_m\|_{L^\infty(G)} \leqq C \|u_m^{(i)}\|_{L^\infty(G)}$ by Theorem 3 from [7]; hence

$$\|u_m^{(i)}\|_{L^\infty(K)} e^{\varrho R} \leqq C \|u_m^{(i)}\|_{L^\infty(G)}$$

i.e. (4) is shown (for $|\mu|$ sufficiently large). The theorem is proved.

OPEN PROBLEM. It would be useful to show that Theorem 3 remains valid under the condition $q_1 \in L^1(G)$ instead of $q_1 \equiv 0$.

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APPROXIMATION BY BERNSTEIN TYPE RATIONAL FUNCTIONS ON THE REAL AXIS

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The aim of this paper is to present a Bernstein type operator which generates a convergent approximation process on the whole real axis.

Such an extension of the Szász—Mirakyan operator

$$S_n(f;\, x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \quad (x \geqq 0,\ n > 0)$$

is known already. This extension generating convergent approximation process on the real axis is due to J. Gróf [4] and it is the following:

$$H_n(f;\, x) = \frac{1}{2\,\mathrm{ch}\,(nx)} \sum_{k=0}^{\infty} \left[f\left(\frac{k}{n}\right) + (-1)^k f\left(-\frac{k}{n}\right)\right] \frac{(nx)^k}{k!} \quad (-\infty < x < \infty,\ n > 0),$$

assuming that this series is convergent.

To have a Bernstein type discrete linear operator we start from the so-called Bernstein type rational functions

$$R_n(f;\, x) = \frac{1}{(1+a_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n x)^k = \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) r_{kn}(x)$$

$$(x \geqq 0,\ n = 1, 2 \ldots)$$

defined for functions f given on the positive half-axis, where a_n, b_n are positive numbers, satisfying the conditions $b_n \to \infty$, $a_n = b_n/n \to 0$, if $n \to \infty$.

If we choose $b_n = n^\beta$, $0 < \beta < 1$ (consequently $a_n = n^{\beta-1}$), then we may write

$$R_n(f;\, x;\, \beta) = \frac{1}{(1+n^{\beta-1}x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n^\beta}\right) \binom{n}{k} (n^{\beta-1}x)^k \quad (x \geqq 0,\ n = 1, 2, \ldots).$$

The positive linear operator R_n was introduced and investigated first by the author [1], [2] and later by Catherine Balázs and J. Szabados [3], V. Totik [5].

Let now f be a real function defined in $(-\infty, \infty)$. For this f we construct the n-th Bernstein type rational function as

$$(1)\quad R_n^*(f;\, x) = \frac{1}{(1+a_n x)^n + (1-a_n x)^n} \sum_{k=0}^{n} \left[f\left(\frac{k}{b_n}\right) + (-1)^k f\left(-\frac{k}{b_n}\right)\right] \binom{n}{k} (a_n x)^k =$$

$$= \sum_{k=0}^{n} \left[f\left(\frac{k}{b_n}\right) + (-1)^k f\left(-\frac{k}{b_n}\right)\right] r_{kn}^*(x) \quad (-\infty < x < \infty),$$

where $n>0$ is *even*, the positive numbers a_n, b_n satisfy the relations $b_n\to\infty$, $a_n=b_n/n\to 0$, if $n\to\infty$.

Clearly we have

$$r^*_{kn}(x)=\frac{(1+a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}r_{kn}(x)\quad (k=1,2,\dots,n). \tag{2}$$

This latter identity will be useful in our proofs.

Similarly, for $R_n(f;x)$ we can choose $b_n=n^\beta$, $0<\beta<1$ (then $a_n=n^{\beta-1}$) and use the notation

$$R^*_n(f;x;\beta)=\frac{1}{(1+n^{\beta-1}x)^n+(1-n^{\beta-1}x)^n}\sum_{k=0}^{n}\left[f\left(\frac{k}{n^\beta}\right)+\right. \tag{3}$$

$$\left.+(-1)^kf\left(-\frac{k}{n^\beta}\right)\right]\binom{n}{k}(n^{\beta-1}x)^k\quad(-\infty<x<\infty,\quad n>0\ \text{ is even}).$$

R^*_n is a discrete linear operator. The following relations are true: $R^*_n(c;x)=c$, if c is a constant function, $R^*_n(f;0)=0$ and

$$R^*_n(f;\pm\infty)=\lim_{x\to\pm\infty}R^*_n(f;x)=\frac{1}{2}\left[f\left(\frac{n}{b_n}\right)+f\left(-\frac{n}{b_n}\right)\right]\quad(n>0\text{ is even})$$

for any function f defined in $(-\infty,\infty)$.

Now we show that the sequence $R^*_n(f)$ defines indeed a convergent approximation process to f. First of all we state a theorem on the pointwise convergence.

THEOREM 1. *Let f be continuous in x and $-x$, furthermore $f(t)=O(e^{\alpha|t|})$ $(-\infty<t<\infty)$ for some $\alpha>0$. Then*

$$\lim_{n\to\infty}R^*_n(f;x)=f(x)\quad(n\text{ is even}). \tag{4}$$

The assumption of continuity at $-x$ cannot be generally omitted as our counterexample will show. This phenomenon is known for $H_n(f;x)$, too (cf. Gróf [4]).

For continuous functions the convergence is uniform in finite intervals and its speed can be estimated.

THEOREM 2. *Let f be continuous in $(-\infty,\infty)$ and $f(x)=O(e^{\alpha|x|})$ for some $\alpha>0$. Then for arbitrary fixed $A>0$, $\varepsilon>0$*

$$|f(x)-R^*_n(f;x)|\leqq c_1\omega_{[-A-\varepsilon,A+\varepsilon]}(f;\max\{a_n,b_n^{-1/2}\}) \tag{5}$$

$$(-A\leqq x\leqq A,\ n>0\text{ is even})$$

$\omega_{[-A-\varepsilon,A+\varepsilon]}(f;.)$ denotes the modulus of continuity of f in $[-A-\varepsilon,A+\varepsilon]$, $c_1=c_1(\alpha;A;\varepsilon)>0$ is a number independent of n.

For the class of continuous functions the choice $b_n=n^{2/3}$ ($a_n=n^{-1/3}$ in this case) seems to be optimal and then

$$\left|f(x)-R^*_n\left(f;x;\frac{2}{3}\right)\right|\leqq c_1\omega_{[-A-\varepsilon,A+\varepsilon]}(f;n^{-1/3}) \tag{6}$$

$$(-A\leqq x\leqq A,\ n>0\text{ is even}).$$

Next, a weighted estimation will be given on the real axis. Here the uniform continuity of f will be supposed. Then $\lim_{\delta\to 0}\omega(f;\delta)=0$ holds for the modulus of continuity in $(-\infty,\infty)$.

THEOREM 3. *Let f be uniformly continuous in $(-\infty,\infty)$ and $b_n=n^\beta$, $0<\beta\leqq 2/3$. Then*

$$(7)\qquad |f(x)-R_n^*(f;\,x;\,\beta)|\leqq 6\left(1+|x|^{\frac{1}{2(1-\beta)}}\right)\omega(f;\,|x|^{1/2}n^{-\beta/2})$$
$$(-\infty<x<\infty;\ n>0 \text{ is even}).$$

(7) is not valid if $2/3<\beta<1$, but probably an adequate inequality exists for this case, too.

Let $C[-\infty,\infty]$ denote the class of such continuous functions which have a finite limit $\lim_{|x|\to\infty} f(x)=f(\infty)$ at infinity. If $f\in C[-\infty,\infty]$ then f is uniformly continuous. A necessary and sufficient condition of the uniform convergence of $R_n^*(f)$ to f is $f\in C[-\infty,\infty]$. Namely we prove

THEOREM 4. *Suppose that $b_n=n^\beta$, $0<\beta\leqq 2/3$. We have*

$$(8)\qquad \lim_{n\to\infty}\ \sup_{-\infty<x<\infty}|f(x)-R_n^*(f;\,x;\,\beta)|=0\quad (n>0 \text{ is even})$$

if and only if $f\in C[-\infty,\infty]$.

We assume $0<\beta\leqq 2/3$ only because the proof of the sufficiency part of Theorem 4 is based on Theorem 3.

To give an estimation for the rate of convergence in the uniform case in $(-\infty,\infty)$ we use the so-called "modulus at infinity"

$$(9)\qquad \Omega(f;\,A)\overset{\text{def}}{=}\sup_{|x_1|\geqq|x_2|\geqq A}|f(x_1)-f(x_2)|\quad (A>0).$$

Obviously $\lim_{|x|\to\infty} f(x)<\infty$ exists if and only if $\lim_{A\to\infty}\Omega(f;A)=0$.

THEOREM 5. *Let $f\in C[-\infty,\infty]$ and $b_n=n^\beta$, $0<\beta\leqq 2/3$. Then*

$$(10)\qquad \sup_{-\infty<x<\infty}|f(x)-R_n^*(f;\,x;\,\beta)|\leqq c_2\inf_{A\geqq 10}\{\Omega(f;\,A)+A^{\frac{1}{1-\beta}}\omega(f;\,An^{-\beta/2})\}$$

where c_2 is a number independent of n.

We can obtain explicit estimates for the rate of convergence from (10). For example, considering the classes of functions for which

$$\omega(f;\,h)\leqq h^\alpha\quad (0<\alpha<1),\quad \Omega(f;\,A)\leqq A^{-a}\quad (a>0)$$

we have

$$\sup_{-\infty<x<\infty}|f(x)-R_n^*(f;\,x;\,\beta)|\leqq$$
$$\leqq\begin{cases} c_3 n^{-\frac{a\alpha}{2(\sqrt{a+\alpha+1}+1)^2}} & \text{if}\quad 0<a+\alpha\leqq 3,\quad \beta=\dfrac{\sqrt{a+\alpha+1}}{\sqrt{a+\alpha+1}+1}\\[2ex] c_4 n^{-\frac{a\alpha}{3(a+\alpha+3)}} & \text{if}\quad a+\alpha\geqq 3,\quad \beta=2/3.\end{cases}$$

The latter estimate tends to $O(n^{-\alpha/3})$ as $a\to\infty$. The above-mentioned orders of magnitude are the same we have got in [3] in approximating functions by $R_n(f; x; \beta)$ in $[0, \infty)$.

COUNTEREXAMPLE. A function f will be constructed which is continuous for all $x>0$, but f is not continuous at $-x$. It will be shown that $R_n^*(f; x)$ does not converge to $f(x)$, if $n\to\infty$.

Let H be the set of such negative numbers which can be written in the form $t=-p/2^{2m}$ where $p>0$ is odd, $m=1, 2, \ldots$. Define f by

$$f(t) = \begin{cases} 1, & \text{if } t\in H \\ 0 & \text{elsewhere.} \end{cases}$$

We consider that subsequence of $\{R_n^*(f; x)\}_{n=1}^{\infty}$ where $n=2^{3m}$. We choose $b_{2^{3m}}=2^{2m}$ (consequently $a_{2^{3m}}=2^{-m}$). If $x>0$, then $f(x)=0$ and f is continuous at x. Still we have by Lemma 2

$$R_{2^{3m}}^*(f; x) = \frac{1}{(1+2^{-m}x)^{2^{3m}}+(1-2^{-m}x)^{2^{3m}}} \sum_{k=0}^{2^{3m}} \left[f\left(\frac{k}{2^{2m}}\right)+\right.$$

$$\left.+(-1)^k f\left(-\frac{k}{2^{2m}}\right)\right]\binom{2^{3m}}{k}(2^{-m}x) =$$

$$= -\frac{1}{(1+2^{-m}x)^{2^{3m}}+(1-2^{-m}x)^{2^{3m}}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^{3m}-1} \binom{2^{3m}}{k}(2^{-m}x)^k \to -\frac{1}{2}$$

if $m\to\infty$. Now we turn to our lemmas and proofs.

LEMMA 1 (Balázs [1], Lemma 2.1). *If $x\geqq 0$, then the following identities hold:*

$$\sum_{k=0}^{n} r_{kn}(x) = 1 \quad (n = 1, 2, \ldots), \tag{11}$$

$$\sum_{k=0}^{n} (k-b_n x)\, r_{kn}(x) = \frac{-a_n b_n x}{1+a_n x}, \tag{12}$$

$$\sum_{k=0}^{n} (k-b_n x)^2 r_{kn}(x) = \frac{a_n^2 b_n^2 x^4+b_n x}{(1+a_n x)^2} \tag{13}$$

where $a_n=b_n/n$, $b_n\to 0$.

LEMMA 2. *Let $x>0$ be fixed, $b_n\to\infty$, $a_n=b_n/n\to 0$, if $n\to\infty$. Then*

$$\lim_{n\to\infty} \frac{(1+a_n x)^n}{(1+a_n x)^n+(1-a_n x)^n} = 1 \quad (n \text{ is even}) \tag{14}$$

where

$$0 < \frac{(1+a_n x)^n}{(1+a_n x)^n+(1-a_n x)^n} < 1. \tag{15}$$

Furthermore

$$\lim_{n\to\infty} \frac{(1-a_n x)^n}{(1+a_n x)^n+(1-a_n x)^n} = 0 \quad (n \text{ is even}) \tag{16}$$

where

$$0 < \frac{(1-a_n x)^n}{(1+a_n x)^n+(1-a_n x)^n} = \sum_{k=0}^{n} (-1)^k r_{kn}^*(x) \le e^{-b_n x}. \tag{17}$$

PROOF. Since $(1+a_n x)^n = \left(1+\frac{b_n x}{n}\right)^n$, therefore $(1+a_n x)^n < e^{b_n x}$, if $a_n < 1/x$ and $0 < (1-a_n x)^n < e^{-b_n x}$ (n is even, $a_n < 1/x$). Hence by $\lim_{n\to\infty} b_n = \infty$ the lemma follows.

LEMMA 3 (Balázs [1], Lemma 2.2). *If* $a_n = b_n/n \to 0$, $b_n \to \infty$ $(n\to\infty)$, $0 \le x \le A$, $A, \gamma, \delta > 0$ *are arbitrary fixed constants, then there exists a number* $c_5 = c_5(A, \gamma, \delta)$ *independent of* n *such that the inequality*

$$\sum_{\substack{k \\ |k/b_n - x| \ge \delta}} e^{\gamma \frac{k}{b_n}} r_{kn}(x) \le c_5 (a_n^2 x^4 + x/b_n) \tag{18}$$

holds.

LEMMA 4. *Let* f *be a continuous function in* $[0, \infty)$ *and* $f(x) = O(e^{\alpha x})$ $(x \ge 0)$ *for some* $\alpha > 0$, *and let* $\varepsilon > 0$ *be fixed. Then in any interval* $0 \le x \le A$ *the inequality*

$$|f(x) - R_n(f;\, x)| \le c_6 \omega_{[0, A+\varepsilon]}(f;\, \max\{a_n, b_n^{-1/2}\}) \quad (n = 1, 2, \dots) \tag{19}$$

is valid, where $\omega_{[0, A+\varepsilon]}(f;\, .)$ *denotes the modulus of continuity of* f *on* $[0, A+\varepsilon]$, $c_6 = c_6(A; \varepsilon; \alpha)$ *is a constant independent of* n. *Choosing* $b_n = n^{2/3}$ *(then* $a_n = n^{-1/3}$*) especially we have*

$$\left| f(x) - R_n\left(f;\, x;\, \frac{2}{3}\right)\right| \le c_6 \omega_{[0, A+\varepsilon]}(f;\, n^{-1/3}) \quad (0 \le x \le A). \tag{20}$$

PROOF. (20) is identical with Theorem 1 in [1]. (19) is contained in the proof of (20) based on (18).

LEMMA 5 (Balázs—Szabados [3], Theorem 1). *If* f *is uniformly continuous in* $[0, \infty)$, *and* $b_n = n^\beta$, $0 < \beta \le 2/3$, *then*

$$|f(x) - R_n(f;\, x;\, \beta)| \le 2\left(1 + x^{\frac{1}{2(1-\beta)}}\right)\omega^+\left(f;\, \sqrt{xn^{-\beta}}\right) \quad (0 \le x < \infty) \tag{21}$$

where $\omega^+(f;\, .)$ *denotes the modulus of continuity of* f *on* $[0, \infty)$.

LEMMA 6 (Balázs—Szabados [3], Theorem 2). *If* f *is continuous in* $[0, \infty)$ *and has a finite limit at infinity, then*

$$\sup_{0 \le x \le \infty} |f(x) - R_n(f;\, x;\, \beta)| \le c_7 \inf_{A \ge 10} \{\Omega^+(f;\, A) + A^{\frac{1}{1-\beta}} \omega^+(f;\, An^{-\beta/2})\} \tag{22}$$

provided $b_n=n^\beta$, $0<\beta\leqq 2/3$, *and* $\Omega^+(f;A)=\sup\limits_{A\leqq x_1\leqq x_2}|f(x_1)-f(x_2)|$ *is the modulus at positive infinity. Evidently* $\lim\limits_{x\to\infty} f(x)$ *exists and is finite if and only if* $\lim\limits_{A\to\infty}\Omega^+(f;A)=0$.

LEMMA 7 (Balázs—Szabados [3], Theorem 4). *Suppose that* $b_n=n^\beta$, $0<\beta\leqq 2/3$. *Then*

$$\lim_{n\to\infty}\sup_{0\leqq x<\infty}|f(x)-R_n(f;\,x;\,\beta)|=0 \tag{23}$$

holds if and only if f *is continuous in* $[0,\infty)$ *and* $\lim\limits_{x\to\infty} f(x)$ *exists and is finite.*

In our proofs we shall use the notation

$$f^-(x)\overset{\text{def}}{=}f(-x). \tag{24}$$

Trivially

$$R_n^*(f^-;\,-x)=R_n^*(f;\,x). \tag{25}$$

PROOF OF THEOREM 1. If $x=0$, then $R_n^*(f;0)=f(0)$. Suppose now $x>0$. By the continuity of f and f^- at x and $-x$ we have

$$R_n^*(f;\,x)=\sum_{k=0}^n\left[f\left(\frac{k}{b_n}\right)+(-1)^kf\left(-\frac{k}{b_n}\right)\right]r_{kn}^*(x)=$$

$$=\sum_{k=0}^n f(x)r_{kn}^*(x)+\sum_{k=0}^n\lambda\left(\frac{k}{b_n}\right)r_{kn}^*(x)+\sum_{k=0}^n f^-(x)(-1)^kr_{kn}^*(x)+$$

$$+\sum_{k=0}^n\mu\left(\frac{k}{b_n}\right)(-1)^kr_{kn}^*(x)=S_1+S_2+S_3+S_4,$$

where $\lambda\overset{\text{def}}{=}f-f(x)$ and $\mu\overset{\text{def}}{=}f^--f^-(x)$, and for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$ such that $|t-x|<\delta$ implies

$$|\lambda(t)|<\varepsilon\quad\text{and}\quad|\mu(t)|<\varepsilon. \tag{26}$$

Applying (2), (11) and Lemma 2 we get the following relations:

$$S_1=\frac{(1+a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}f(x)=$$

$$=f(x)-\frac{(1-a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}f(x)\to f(x),$$

if $n\to\infty$ and

$$S_3=\sum_{k=0}^n f^-(x)(-1)^kr_{kn}^*(x)=\frac{(1-a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}f^-(x)\to 0,$$

if $n\to\infty$. Using (2) and Lemma 2 again one can estimate S_2 in the following manner:

$$|S_2| \leqq \frac{(1+a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}\sum_{k=0}^{n}\left|\lambda\left(\frac{k}{b_n}\right)\right| r_{kn}(x) \leqq \sum_{\left|\frac{k}{b_n}-x\right|<\delta}\left|\lambda\left(\frac{k}{b_n}\right)\right| r_{kn}(x)+\sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta}.$$

On the one hand

$$\sum_{\left|\frac{k}{b_n}-x\right|<\delta}\left|\lambda\left(\frac{k}{b_n}\right)\right| r_{kn}(x) \leqq \varepsilon$$

holds by (26) and (11). On the other hand Lemma 3 can be used:

$$\sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta}\left|\lambda\left(\frac{k}{b_n}\right)\right| r_{kn}(x) \leqq \sum_{\left|\frac{k}{b_n}-x\right|\geqq\delta} e^{\alpha\frac{k}{b_n}} r_{kn}(x) \leqq c_5(a_n^2x^4+x/b_n)\to 0,$$

if $n\to\infty$. The proof of $\lim_{n\to\infty} S_4=0$ can be done in a completely analogous way. So we have proved $R_n^*(f;\,x)=\sum_{i=1}^{4} S_i\to f(x)$ if $n\to\infty$ (n is even, $x>0$). If $x<0$, then $-x>0$. Since f^- satisfies the conditions of Theorem 1, we can apply the above proved case of our Theorem, that is $\lim_{n\to\infty} R_n^*(f^-;\,-x)=f^-(-x)$ ($x<0$, n is even). By (24) and (25) this relation can be written in the equivalent form

$$\lim_{n\to\infty} R_n^*(f;\,x) = f(x) \quad (x<0,\ n \text{ is even}).$$

Proof of Theorem 2. First consider the case $0\leqq x\leqq A$. Since

$$\sum_{k=0}^{n}[r_{kn}^*(x)+(-1)^k r_{kn}^*(x)] = 1$$

and $f(x)=f^-(x)+f(x)-f(-x)$, we have

$$|f(x)-R_n^*(f;\,x)| \leqq \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{b_n}\right)\right| r_{kn}^*(x)+\left|\sum_{k=0}^{n}\left[f(x)-f\left(-\frac{k}{b_n}\right)\right](-1)^k r_{kn}^*(x)\right| \leqq$$

$$\leqq \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{b_n}\right)\right| r_{kn}^*(x)+\left|\sum_{k=0}^{n}\left[f^-(x)-f^-\left(\frac{k}{b_n}\right)\right](-1)^k r_{kn}^*(x)\right|+$$

$$+\left|\sum_{k=0}^{n}[f(x)-f(-x)](-1)^k r_{kn}^*(x)\right| = S_1+S_2+S_3.$$

We estimate S_1 and S_2 with the aid of (2) and Lemma 2 using the way of proving Lemma 4:

$$S_1 \leqq \frac{(1+a_nx)^n}{(1+a_nx)^n+(1-a_nx)^n}\sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{b_n}\right)\right| r_{kn}(x) \leqq$$

$$\leqq c_8\omega_{[0,\,A+\varepsilon]}(f;\,\max\{a_n,\,b_n^{-1/2}\}) \leqq c_8\omega_{[-A-\varepsilon,\,A+\varepsilon]}(f;\,\max\{a_n,\,b_n^{-1/2}\})$$

and

$$S_2 \leqq c_8\omega_{[0,\,A+\varepsilon]}(f^-;\,\max\{a_n,\,b_n^{-1/2}\}) \leqq$$

$$\leqq c_8\omega_{[-A-\varepsilon,\,A+\varepsilon]}(f;\,\max\{a_n,\,b_n^{-1/2}\}) \quad (n>0 \text{ is even},\ 0\leqq x\leqq A).$$

From (17) and the property $\omega(f;\lambda\delta)\leqq(\lambda+1)\,\omega(f;\delta)$ $(\lambda,\delta>0)$ of the modulus of continuity

$$S_3 \leqq e^{-b_n x}\omega_{[-A-\varepsilon,\,A+\varepsilon]}(f;\;2x) \leqq$$

$$\leqq e^{-b_n x}\left(\frac{2x}{\max\{a_n,\,b_n^{-1/2}\}}+1\right)\omega_{[-A-\varepsilon,\,A+\varepsilon]}(f;\;\max\{a_n,\,b_n^{-1/2}\}) \leqq$$

$$\leqq c_9\,\omega_{[-A-\varepsilon,\,A+\varepsilon]}(f;\;\max\{a_n,\,b_n^{-1/2}\}),$$

where $c_9=c_9(A;\varepsilon;\alpha)>0$ is a number idependent of n.

As regards the case $x<0$, f^- satisfies the conditions of Theorem 2 and $-x>0$, so as we have proved above

$$|f(x)-R_n^*(f;\;x)| = |f^-(-x)-R_n^*(f^-;\;-x)| \leqq$$

$$\leqq c_{10}\,\omega_{[-A-\varepsilon,\,A+\varepsilon]}\big(f^-;\;\max\{a_n,\,b_n^{-1/2}\}\big) = c_{10}\,\omega_{[-A-\varepsilon,\,A+\varepsilon]}\big(f;\;\max\{a_n,\,b_n^{-1/2}\}\big).$$

PROOF OF THEOREM 3. The proof runs similarly as in Theorem 2. The difference is in estimating S_1 and S_2; here we refer to Lemma 5 instead of Lemma 4. The estimation of S_3 happens as follows: if $x<n^{1-\beta}$, then by $\omega(f;\lambda\delta)\leqq(\lambda+1)\cdot\omega(f;\delta)$ $(\lambda,\delta>0)$

$$S_3 \leqq e^{-b_n x}\omega(f;\;2x) \leqq e^{-b_n x}(2x^{1/2}n^{\beta/2}+1)\,\omega(f;\;x^{1/2}n^{-\beta/2}).$$

Since the function $\varphi(x)=e^{-b_n x}2x^{1/2}n^{\beta/2}$ attains its maximum at $x_0=1/2n^{\beta}$ and this maximum is less than 1, therefore $S_3\leqq 2\omega(f;\;x^{1/2}n^{-\beta/2})$, if $x<n^{1-\beta}$. If $x\geqq n^{1-\beta}$, then we obtain by $0\leqq\sum_{k=0}^{n} r_{kn}^*(x)\leqq 1$

$$S_3 \leqq \omega(f;\;2x)\sum_{k=0}^{n} r_{kn}^*(x) \leqq 2\big(1+x^{\frac{1}{2(1-\beta)}}\big)\,\omega(f;\;x^{1/2}n^{-\beta/2}).$$

PROOF OF THEOREM 5. The convergence of the right hand side to zero and the inequality (10) in the cases $|x|\leqq A^2$ and $x\geqq A^2$ may be proved by the same way as in Lemma 6, if we take into consideration (2) and (15). Let now $x\leqq -A^2$. Then we get (10) similarly as in the case $x\geqq A^2$ by the facts $-x\geqq A^2$, $f^-\in C[-\infty,\infty]$, $f^-(\infty)=f(\infty)$, $\Omega(f^-;A)=\Omega(f;A)$ and $R_n^*(f^-;-x)=R_n^*(f;x)$.

PROOF OF THEOREM 4. $f\in C[-\infty,\infty]$ implies (8) as we have established in Theorem 5. Suppose now that (8) holds. Then f is continuous at every point $x\in(-\infty,\infty)$ being the uniform limit of continuous functions in a compact neighborhood of x. f is also bounded in $(-\infty,\infty)$, otherwise we could choose a sequence $\{x_m\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty}|x_m|=\infty$ and

(27) $$\lim_{m\to\infty}|f(x_m)| = \infty.$$

Let $\varepsilon>0$ be fixed and n_0 be such that for $n\geqq n_0$ (n is even)

(28) $$\sup_{-\infty<x<\infty}|f(x)-R_n^*(f;\;x)| < \varepsilon.$$

On the one hand from (28) follows the inequality

(29) $$\lim_{m\to\infty} |f(x_m) - R^*_{n_0}(f;\ x_m)| < \varepsilon.$$

On the other hand

$$\lim_{x\to\infty} R^*_n(f;\ x) = \frac{1}{2}\left[f\left(\frac{n}{b_n}\right) + f\left(-\frac{n}{b_n}\right)\right]$$

that is

$$\lim_{m\to\infty} R_{n_0}(f;\ x_m) = \frac{1}{2}[f(n_0^{1-\beta}) + f(-n_0^{1-\beta})]$$

thus from (27) the relation $\lim_{m\to\infty} |f(x_m) - R^*_{n_0}(f;\ x_m)| = \infty$ follows which contradicts (29).

The existence of the finite limit $\lim_{x\to\infty} f(x) = d$ can be shown by the way described in the proof of Lemma 7. We have to prove that $\lim_{x\to-\infty} f(x) = d$. Let $\delta > 0$ be an arbitrary, fixed number. There exists a $D = D(\delta)$ such that

(30) $$|f(x) - d| < \delta, \quad \text{if} \quad x \geqq D.$$

It is obvious from (8) that there exists n_1 such that

(31) $$|f(x) - R^*_n(f;\ x)| < \delta, \quad \text{if} \quad n \geqq n_1 \quad (n \text{ is even}).$$

Since

$$\lim_{x\to\infty} R^*_{n_1}(f;\ x) = \lim_{x\to-\infty} R^*_{n_1}(f;\ x) = \lim_{x\to\infty} R^*_n(f;\ -x) = \frac{1}{2}[f(n_1^{1-\beta}) + f(-n_1^{1-\beta})]$$

therefore one can find an $E \geqq D$ such that

(32) $$|R^*_{n_1}(f;\ x) - R^*_{n_1}(f;\ -x)| < \delta, \quad \text{if} \quad x \geqq E.$$

Collecting (30), (31) and (32) we get

$$|f(-x) - d| \leqq |f(-x) - R^*_{n_1}(f;\ -x)| + |R^*_{n_1}(f;\ -x) - R^*_{n_1}(f;\ x)| +$$
$$+ |R^*_{n_1}(f;\ x) - f(x)| + |f(x) - d| \leqq 4\delta \quad (x \geqq E),$$

hence $\lim_{x\to-\infty} f(x) = d$.

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ON THE DENSITY OF FINITE PACKINGS

J. M. WILLS (Siegen)

Introduction and results. Let $\mathscr{C}^d$ denote the set of compact convex sets in the euclidean d-space E^d, and for a $K\in\mathscr{C}^d$ let $V(K)$ denote its volume. In particular $V(B^d)=\omega_d$ for the unit ball $B^d=\{x\in E^d \mid |x|\leqq 1\}$. Further let $\mathscr{C}_k^d$ denote the set of $K\in\mathscr{C}^d$, which contains k translates $B_1^d, \ldots, B_k^d$ of B^d with $\operatorname{int}(B_i^d\cap B_j^d)=\emptyset$ for $i\neq j$. $\left(\text{So } \bigcup_{i=1}^{k} B_i^d\subset K.\right)$ For each $K\in\mathscr{C}_k^d$ let $\delta(K)=\dfrac{k\omega_d}{V(K)}$. We also put

$$\delta_k^d = \max_{K\in\mathscr{C}_k^d} \delta(K) = \max_{K\in\mathscr{C}_k^d} \frac{k\omega_d}{V(K)}$$

(the maximum exists by Blaschke's selection theorem). Clearly $\delta_1^d=1$ and $\delta_k^d<1$ for $k>1$. In the following we always assume $k>1$. δ_k^d can be interpreted as the maximal packing density of k unit balls. If the centres of $B_1^d, \ldots, B_k^d$ lie on a line-segment S_k of length $2(k-1)$ then S_k+B^d forms a "sausage" with $V(S_k+B^d)=2(k-1)\omega_{d-1}+\omega_d$. So

$$\sqrt{\frac{\pi}{2}}(d+1)^{-1/2} < \delta(S_k+B^d) = \frac{k\omega_d}{2(k-1)\omega_{d-1}+\omega_d} \leqq \delta_k^d < 1.$$

From this and the known upper bounds for the packing density δ^d of E^d by unit balls it follows that $\delta^d<\delta_k^d$ for $d\geqq 5$.

This led L. Fejes Tóth to the "sausage-conjecture" [3]:

$$\delta(S_k+B^d) = \delta_k^d \quad \text{for} \quad d\geqq 5.$$

For partial results see [1, 2, 6]. For $d=2$ δ_k^2 is attained for certain hexagonal arrangements of the densest circle packing [4].

For $d=3$ and 4 no general result can be expected and L. Fejes Tóth considered the problem hopeless [3]. Nevertheless some information is possible as explained in [7]. To elaborate let C_k be the convex hull of the centres of $B_1^d, \ldots, B_k^d$ in the best (resp. a best) arrangement, i.e. when δ_k^d is attained. Then, for $d=3$, there exist integers $k_0\leqq k_1$, such that

$$\dim C_k = 1 \quad \text{for} \quad k<k_0 \quad \text{and} \quad \dim C_{k_0} > 1,$$

$$\dim C_{k_1-1} = 1 \quad \text{and} \quad \dim C_k > 1 \quad \text{for} \quad k\geqq k_1$$

or, in other words, and only for $d=3$,

$$\delta_k^3 = \delta(S_k+B^3) \quad \text{for} \quad k<k_0 \quad \text{and} \quad \delta(S_{k_0}+B^3) < \delta_{k_0}^3,$$

$$\delta_{k_1-1}^3 = \delta(S_{k_1-1}+B^3) \quad \text{and} \quad \delta(S_k+B^3) < \delta_k^3 \quad \text{for} \quad k \geqq k_1.$$

In particular this means that for $k<k_0$ the "sausage" yields maximal density, but not for $k=k_0$ ("sausage-catastrophe"). The exact determination of k_0 resp. k_1 seems to be hard, but it is relatively easy to give good upper bounds.

In the following theorem we show that $k_0 \leqq 56$. The small differences in the densities suggest that either $k_0=56$ or at least k_0 is not much smaller than 56.

THEOREM. *There are* $P, Q \in \mathscr{C}^3$ *with* $P+B^3 \in \mathscr{C}_{55}^3$, $Q+B_3 \in \mathscr{C}_{56}^3$ *and*

$$\delta(P+B^3) = 0{,}6699 < 0{,}6707 = \delta(S_{55}+B^3) \leqq \delta_{55}^3,$$

$$\delta(S_{56}+B^3) = 0{,}6707 < 0{,}6710 = \delta(Q+B^3) \leqq \delta_{56}^3.$$

REMARKS. 1) The appropriate C_k are truncated tetrahedra, such that the balls in the facets form Groemer-packings. The whole packing can be considered as an analogue to a Groemer-packing [4] in the plane.

2) We checked several other truncated tetrahedra. E.g. for $k=59$ and 60 we have found "better" truncated tetrahedra than the corresponding sausages.

3) Truncated tetrahedra can also be used to find an upper bound for k_1 and good lower bounds for δ_k^3.

4) Obviously $\delta(S_{56}+B^3) < \delta(S_{55}+B^3)$.

Proof of the theorem. For a $K \in \mathscr{C}^3$ Steiner's formula for the parallel body $K+B^3$ yields (see e.g. [5], p. 31)

$$V(K+B^3) = V(K)+F(K)+M(K)+\frac{4\pi}{3},$$

where F denotes the surface area and M the integral of the mean curvature. In particular $V(S_k+B^3)=2\pi(k-1)+\frac{4\pi}{3}$ and

$$V(S_{55}+B^3) = 108\pi+\frac{4\pi}{3} = 343{,}48, \quad V(S_{56}+B^3) = 110\pi+\frac{4\pi}{3} = 349{,}76.$$

From Lemmas 3 and 4 we know that there are $P, Q \in \mathscr{C}^3$ with $k(P)=55$ and $k(Q)=56$, which means that $P \in \mathscr{C}_{55}^3$, $Q \in \mathscr{C}_{56}^3$. Further

$$V(P+B^3) = \frac{280}{3}\sqrt{2}+92\sqrt{3}+48{,}39+\frac{4\pi}{3} = 343{,}91,$$

$$V(Q+B^3) = \frac{292}{3}\sqrt{2}+92\sqrt{3}+48{,}39+\frac{4\pi}{3} = 349{,}58.$$

Since $55\omega_3=230{,}38$ and $56\omega_3=234{,}57$ we obtain

$$\delta(P+B^3) \leqq \frac{230{,}38}{343{,}91} \leqq 0{,}6699 < 0{,}6707 \leqq \frac{230{,}38}{343{,}48} \leqq \delta(S_{55}+B^3)$$

and

$$\delta(S_{56}+B^3) \leqq \frac{234{,}57}{349{,}76} \leqq 0{,}6707 < 0{,}6710 \leqq \frac{234{,}57}{349{,}58} \leqq \delta(Q+B^3).$$

So the theorem is proved if we prove Lemmas 3 and 4. For these we need Lemmas 1 and 2:

LEMMA 1. *For the regular tetrahedron T of edge-length 2 we have:*

$$V(T) = \frac{2}{3}\sqrt{2} = 0{,}942809, \quad F(T) = 4\sqrt{3} = 6{,}928203,$$

$$M(T) = 6 \arccos\left(-\frac{1}{3}\right) = 11{,}463796.$$

PROOF. The results can be found in Hadwiger [5] p. 37, if one observes that the circumradius $R(T)=\sqrt{\frac{3}{2}}$.

Clearly one obtains these results for V and F directly if one notes that one facet of T has area $\sqrt{3}$ and that the distance of a vertex of T to the opposite facet is $\sqrt{2}$.

One obtains $M(T)=12\alpha'=6\alpha$, where α denotes the exterior angle at an edge of T (such that the full angle is 2π) and α' such that the full angle is π. Easy calculation shows that $\cos\frac{\alpha}{2}=3^{-1/2}$, $\sin\frac{\alpha}{2}=\left(\frac{2}{3}\right)^{1/2}$, $\cos\alpha=-\frac{1}{3}$. To calculate the number of centers of unit balls in nT, $n=0, 1, 2, 3, \dots$ we consider the densest packing of unit balls generated by the 4 vertices of $T_1=\operatorname{conv}\{(0,0,0), (\sqrt{2}, \sqrt{2}, 0), (\sqrt{2}, 0, \sqrt{2}), (0, \sqrt{2}, \sqrt{2})\}$. Clearly T_1 is regular and has edge length 2, as required in Lemma 1. For this special T_1 we calculate the number k of centers in $nT_1=T_n$. For simplicity and to emphasize that k is a function on $\mathscr{C}^3$ we write for an integer $k=c$, $k(T_n)=c$ instead of $T_n\in\mathscr{C}^3_c$, as mentioned before.

LEMMA 2. $k(T_n)=\binom{n+3}{3}$ *for* $n=0, 1, 2, 3, \dots$.

PROOF. The lemma holds for $n=0, 1$. We assume that it holds for $\nu=0, 1, \dots, n-1$. Easy considerations show that $k(T_n)-k(T_{n-1})=\binom{n+2}{2}$. So $\binom{n-1+3}{3}+\binom{n+2}{2}=\frac{1}{6}(n+2)(n+1)n+\frac{1}{2}(n+2)(n+1)=\binom{n+3}{3}$. We now construct P and Q as follows:

P: Take T_7 as basic tetrahedron. From one of its vertices cut off T_5 and from the remaining three vertices cut off three translates T_3^1, T_3^2, T_3^3 of T_3. If we compactify this halfopen set, we obtain the truncated tetrahedron (resp. the irregular octahedron) shown in Fig. 1. We remark that $T_5\cap T_3^i=T_1^i$, $i=1, 2, 3$ are translates of T_1 whereas $T_3^i\cap T_3^j=\emptyset$ $i\neq j$.

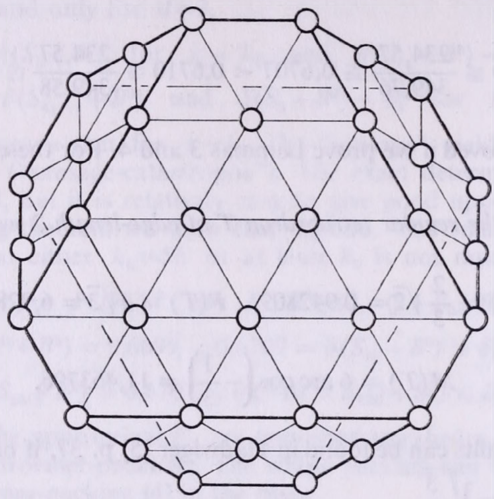

Fig. 1

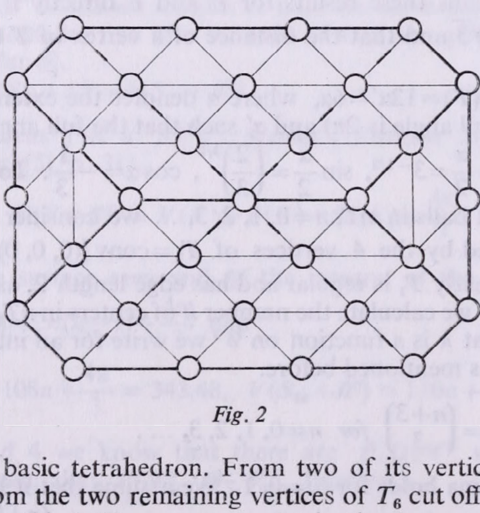

Fig. 2

Q: Take T_6 as basic tetrahedron. From two of its vertices cut off translates T_2', T_2'' of T_2 and from the two remaining vertices of T_6 cut off translates T_3', T_3'' of T_3. If we compactify this halfopen set we obtain the truncated tetrahedron (resp. the irregular octahedron) shown in Fig. 2.

In Fig. 1 and 2 the heavy lines show the visible edges, the dotted lines the other edges.

LEMMA 3. *We have*

$$k(Q) = 56, \quad V(Q) = \frac{292}{3}\sqrt{2}, \quad F(Q) = 92\sqrt{3}, \quad M(Q) = 48{,}39\ldots$$

PROOF. For k, V, F, M we write φ and have by the additivity $\varphi(Q) = \varphi(T_6) - 2\varphi(T_3) - 2\varphi(T_2) + 2\varphi(Q \cap T_3) + 2\varphi(Q \cap T_2)$.

So

$$k(P) = k(T_6) - 2k(T_3) - 2k(T_2) + 2k(Q \cap T_3) + 2k(Q \cap T_2) =$$
$$= \binom{9}{3} - 2\binom{6}{3} - 2\binom{5}{3} + 2\binom{5}{2} + 2\binom{4}{2} = 56.$$

$$V(Q) = V(T_6) - 2V(T_3) - 2V(T_2) = V(T)(6^3 - 2\cdot 3^3 - 2\cdot 2^3) = 146V(T) = \frac{292}{3}\sqrt{2},$$

$$F(Q) = F(T_6) - 2F(T_3) - 2F(T_2) + 2F(Q \cap T_3) + 2F(Q \cap T_2) =$$
$$= F(T)(6^2 - 2\cdot 3^2 - 2\cdot 2^2 + 3^2 + 2^2) = 23F(T) = 92\sqrt{3},$$

$$M(Q) = M(T_6) - 2M(T_3) - 2M(T_2) + 2M(Q \cap T_3) + 2M(Q \cap T_2) =$$
$$= M(T)(6 - 2\cdot 3 - 2\cdot 2) + 18\pi + 12\pi = 30\pi - 4M(T) =$$
$$= 30\pi - 24 \arccos\left(-\frac{1}{3}\right) = 48{,}39\ldots.$$

LEMMA 4. *We have*

$$k(P) = 55, \quad V(P) = \frac{280}{3}\sqrt{2}, \quad F(P) = F(Q) = 92\sqrt{3}, \quad M(P) = M(Q) = 48{,}39\ldots.$$

PROOF. The calculation of k, V, F, M can be done as in Lemma 3. But for P it is easier to use direct arguments, except for V which is simply additive.

$$V(P) = V(T_7) - V(T_5) - 3V(T_3) + 3V(T_1) = V(T)(7^3 - 5^3 - 3\cdot 3^3 + 3) = \frac{280}{3}\sqrt{2}.$$

To calculate $k(P)$ we remark that the lattice points in P lie in three parallel planes (see Fig. 1), those in the exterior planes are arranged in irregular hexagons each with 18 points, those in the interior plane being arranged in a regular hexagon with 19 points, so $k(P) = 18 + 19 + 18 = 55$.

The calculation of $F(P)$ is easy, if one realizes that bd P consists of 92 regular triangles of edge-length 2 (see Fig. 1), so $F(P) = 92\sqrt{3} = F(Q)$.

For M we need the exterior angles at the edges of P. If again α denotes the exterior angle at an edge of T, it is obvious that the only exterior angles at the edges of P are α and $\beta = \pi - \alpha$. α occurs at edges incident to two facets of the basic T_7 and at edges where T_5 and the T_3^i, $i = 1, 2, 3$ intersect. β occurs at all other edges. It is easy to see (Fig. 1) that the total edge-length of Q is 72, 12 corresponding to α and 60 to β. So $M(P) = \frac{1}{2}(12\alpha + 60(\pi - \alpha)) = 30\pi - 24\alpha = 30\pi - 4M(T) = M(Q)$.

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A PATHOLOGICAL APPROXIMATELY SMOOTH FUNCTION

M. J. EVANS (Raleigh) and P. D. HUMKE (Northfield)

A real valued function f defined on the real line is said to be *smooth at the point* $x \in R$ if

$$\lim_{h \to 0} (f(x+h)+f(x-h)-2f(x))/h = 0. \tag{1}$$

We call f *smooth* if it is smooth at each $x \in R$. The continuity properties of measurable smooth functions have been studied in detail. Neugebauer [3 and 4] showed that such a function f belongs to Baire class one and that $R \setminus C(f)$ is a nowhere dense countable set, where $C(f)$ denotes the set of points at which f is continuous. Evans and Larson [1] have recently shown that $R \setminus C(f)$ can be characterized as a *clairseme* (scattered) set.

If we replace the limit in (1) by an approximate limit, we arrive at the notion of *approximate smoothness* [4]. (To be precise, we will say that f is *approximately smooth at the point* $x \in R$ if for each $\varepsilon > 0$ the set $\{h: |f(x+h)+f(x-h)-2f(x)| \geqq \varepsilon |h|\}$ has 0 as a point of dispersion.) Larson [2] has recently shown that measurable approximately smooth functions are in Baire class one. However, $R \setminus C(f)$ need not be countable. Indeed, it may have large measure [4]. A natural question arises as to whether or not the first category, measure zero set $R \setminus AC(f)$ must be countable for a measurable approximately smooth function, where $AC(f)$ denotes the set of points at which f is approximately continuous. The purpose of the present paper is to show that $R \setminus AC(f)$ need *not* be countable.

THEOREM. *There is a measurable approximately smooth function which is approximately discontinuous at uncountably many points.*

PROOF. For each natural number n, let $\xi_n = 8^{-1}(n+1)^{-2}$ and let C denote the symmetric Cantor set determined by the sequence $\{\xi_n\}$ in $[0, 1]$. Then

$$C = \left\{ \sum_{m=1}^{\infty} k(m)L(m) \colon k(m) = 0 \text{ or } N(m)-1 \right\},$$

where $L(m) = 8^{-m}(m+1)!^{-2}$ and $N(m) = 8(m+1)^2$. This set C can also be defined geometrically as the intersection of a nested sequence of compact sets, I_n, where each I_n is the union of 2^n closed subintervals of I_{n-1}, each subinterval is of length $L(n)$, and each has exactly one endpoint in common with one of the intervals comprising I_{n-1}; more specifically,

$$I_n = \{ \sum_{m=1}^{\infty} k(m)L(m): k(m) = 0 \quad \text{or} \quad N(m)-1 \quad \text{for} \quad 1 \leqq m \leqq n,$$

$$\text{and} \quad 0 \leqq k(m) < N(m) \quad \text{for} \quad m > n\}.$$

For any specific natural number n, $R \setminus I_n$ consists of 2^n+1 open intervals whose union we denote by CI_n. We will define the function f in an inductive manner using the sets CI_n; the function f restricted to C will be identically zero. However, at the n^{th} stage of the construction, f will not be defined on all of CI_n but only on a certain relatively large (and increasingly larger with n) open subset of CI_n. This subset, denoted by J_n, is defined as follows.

Let I be the union of a finite collection of disjoint open intervals, say $I = \bigcup_{k \in P} (a_k, b_k)$, where P is a finite set of natural numbers. If $0 < r < (b_k - a_k)/2$ for all $k \in P$, then we set

$$I^r = \bigcup_{k \in P} (a_k + r, b_k - r).$$

Using this notation, we define

$$J_n = (CI_n)^{(n+1)L(n)}.$$

We are now in a position to define the function f and we begin by defining f on $\bigcup_{n=0}^{\infty} J_n = R \setminus C$.

Stage 0. The set J_0 consists of the two intervals $(-\infty, -1)$ and $(2, +\infty)$. We define f to be -1 on the former and 1 on the latter.

Stage 1. Partition $R \setminus J_0$ into three subintervals, each of length $L(0)=1$. Code the center open interval with the value 1 and the two outside open intervals with -1. Let f be defined on $J_1 \setminus J_0$ by giving $f(x)$ the value 1 or -1 according to the code assigned to the interval of the partition that contains x. Note that f has not been defined at -1 or 2 under this scheme. We define $f(-1)=-1$ and $f(2)=0$. Now, the function f has been defined on all of $J_1 = (-\infty, -1/16) \cup \cup (3/32, 39/32) \cup (17/16, +\infty)$.

Stage n. We now describe the inductive step, which is patterned after Stage 1. Suppose that f has been defined on J_n. Now, $R \setminus J_n$ consists of 2^n intervals, each centered on an interval of I_n, and each of length $(2n+3)L(n)$. Denote one such interval by I and partition I into $2n+3$ intervals of equal length. Code the middle open interval with the number 1 and then code each of the remaining $2n+2$ open intervals determined by this partition each with a -1 or a 1 so that the signs alternate. For $x \in J_{n+1} \setminus J_n$, let $f(x)$ be the value of the code assigned to the interval to which x belongs. Again, there are finitely many points x in $J_{n+1} \setminus J_n$ at which this procedure fails to assign a value to $f(x)$. At each such point, define $f(x)$ in such a way to make f continuous at x, if that is possible, and define $f(x)=0$ otherwise.

In this manner f is defined at every point of $\bigcup_{n=0}^{\infty} J_n = R \setminus C$, and we complete the definition by defining f to be 0 on C. We now verify that f has the desired properties.

For each x, let $S(x)=\{h: f(x+h)+f(x-h)=0\}$, and let

$$\varrho_x(t) = m(S(x)\cap[0, t])/t.$$

We will establish

CLAIM A. *For each natural number n and each $x\in I_n$*

$$\varrho_x(t) > n/(n+2)$$

for every $t\in[L(n-1), L(n-2)]$.

The approximate smoothness of f at each point of C will follow easily from this claim.

Case $n-1$: Let $x\in I_1$. By the symmetry of the construction we may assume that $x\in[0, 1/32]$. Note that $[L(0), L(-1)]=[1, 8]$ and set

$$V_1=[4L(1), L(0)-3L(1)] = [4/32, 29/32]$$

and

$$\mathscr{V} = [2L(0), \infty) = [2, \infty).$$

Let $T_1=V_1-x$ and $\mathscr{T}=\mathscr{V}-x$. Clearly, if $h\in\mathscr{T}$, then $f(x+h)=1$ and $f(x-h)=-1$, implying that $h\in S(x)$. Furthermore, if $h\in T_1$, then $f(x+h)=1$, $f(x-h)=-1$, and again $h\in S(x)$. Consequently, for $t\geqq 1$ we have

$$\varrho_x(t) \geqq m(T_1) = 25/64 > 1/3,$$

and, in particular, this holds for $t\in[1, 8]$.

(At this point the reader may wish to skip to the general case, but we include the cases for $n=2$ and 3 for motivational purposes.)

Case $n=2$: Let $x\in I_2$. We shall assume that $x\in[0, L(2)]=[0, 1/2304]$, but again the symmetry of the construction will allow our argument to be easily adapted to hold for x in any of the four component intervals forming I_2. Note that $[L(1), L(0)]=[1/32, 1]$ and set

$$V_1 = [5L(2), L(1)-4L(2)] = [5/2304, 68/2304],$$

$$V_2 = [L(1)+3L(2), 2L(1)] = [75/2304, 2/32],$$

$$\mathscr{V} = [3L(1), L(0)-3L(1)] = [3/32, 29/32],$$

and let $T_i=V_i-x$, $i=1, 2$, and $\mathscr{T}=\mathscr{V}-x$. For $h\in T_1$ we have $f(x+h)=-1$ and $f(x-h)=1$; for $h\in T_2$ we have $f(x+h)=1$ and $f(x-h)=-1$; and for $h\in\mathscr{T}$ we have $f(x+h)=1$ and $f(x-h)=-1$. Hence, $T_1\cup T_2\cup\mathscr{T}\subseteqq S(x)$. If we set

$$v_x(t) = \frac{m((T_1\cup T_2\cup\mathscr{T})\cap[0, t])}{t},$$

then for $t\in[L(1), L(0)]$ we clearly have $\varrho_x(t)\geqq v_x(t)$. However, an examination of v_x on $[L(1), L(0)]$ shows that it attains its absolute minimum at $t=3L(1)-x$.

In fact, for $t\in[L(1), L(0)]$, we have

$$\varrho_x(t) \geqq v_x(t) \geqq \frac{m((T_1\cup T_2)\cap[0, 3L(1)-x])}{3L(1)-x} \geqq \frac{m(T_1)+m(T_2)}{3L(1)} = 11/18 \geqq 1/2.$$

Case $n=3$: Let $x\in I_3$. We shall write our proof for $x\in[0, L(3)]=[0, 1/294, 912]$. Note that $[L(2), L(1)]=[1/2304, 1/32]$ and set

$$V_1 = [6L(3), L(2)-5L(3)] = [6/294, 912, 123/294,912],$$

$$V_2 = [L(2)+4L(3), 2L(2)] = [132/294, 912, 2/2304],$$

$$V_3 = [2L(2)+2L(3), 3L(2)] = [258/294, 912, 3/2304],$$

$$\mathscr{V} = [4L(2), L(1)-4L(2)] = [4/2304, 68/2304]$$

and let $T_i=V_i-x$, $i=1, 2, 3$, and $\mathscr{T}=\mathscr{V}-x$. For $h\in T_i$, $i=1, 2, 3$, we have $f(x+h)=(-1)^{i-1}$ and $f(x-h)=(-1)^i$; for $h\in\mathscr{T}$ we have $f(x+h)=1$ and $f(x-h)=-1$. Consequently, $(\bigcup_{i=1}^{3} T_i)\cup\mathscr{T}\subseteq S(x)$, and if we set

$$v_x(t) = \frac{m(((\bigcup_{i=1}^{3} T_i)\cup\mathscr{T})\cap[0, t])}{t},$$

then for $t\in[L(2), L(1)]$ we have $\varrho_x(t)\geqq v_x(t)$. An examination of v_x on $[L(2), L(1)]$ shows that it attains its absolute minimum at $t=4L(2)-x$. Hence for $t\in[L(2), L(1)]$ we have

$$\varrho_x(t) \geqq v_x(t) \geqq v_x(4L(2)-x) \geqq \Big(\sum_{i=1}^{3} m(T_i)\Big)/4L(2) = 367/512 > 3/5.$$

Case $n>3$: Let $x\in I_n$ and again for specificity assume $x\in[0, L(n)]$. Set

$$V_1 = [(n+3)L(n), L(n-1)-(n+2)L(n)],$$

$$V_2 = [L(n-1)+(n+1)L(n), 2L(n-1)],$$

and

$$V_k = [(k-1)L(n-1)+2L(n), kL(n-1)]$$

for $3\leqq k\leqq n$. Then set

$$\mathscr{V} = [(n+1)L(n-1), L(n-2)-(n+1)L(n-1)]$$

and let $T_i=V_i-x$, $i=1, 2, \ldots, n$, and $\mathscr{T}=\mathscr{V}-x$. For $h\in T_i$, $i=1, 2, \ldots, n$, we have $f(x+h)=(-1)^{i-1}$ and $f(x-h)=(-1)^i$ and hence $h\in S(x)$. If $h\in\mathscr{T}$, then $f(x+h)=1$, $f(x-h)=-1$ and again $h\in S(x)$. Consequently, if we set

$$v_x(t) = \frac{m(((\bigcup_{i=1}^{n} T_i)\cup\mathscr{T})\cap[0, t])}{t},$$

then for $t\in[L(n-1), L(n-2)]$ we have $\varrho_x(t)\geqq v_x(t)$. The function v_x on

$[L(n-1), L(n-2)]$ attains its absolute minimum at $t=(n+1)L(n-1)-x$ and so for $t\in[L(n-1), L(n-2)]$ we have

$$\varrho_x(t) \geqq v_x(t) \geqq v_x((n+1)L(n-1)-x) \geqq$$
$$\geqq \left(\sum_{i=1}^{n} m(T_i)\right)/(n+1)L(n-1) = 1-(8n^2+21n+10)/8(n+1)^3 > n/(n+2).$$

Claim A is, therefore, verified.

From Claim A it immediately follows that for each $x_0\in C$, $S(x_0)$ has density 1 at 0. Consequently, f is approximately smooth at x_0. Furthermore, f is clearly smooth at each point $x_0\in R\setminus C$. Hence f is approximately smooth.

Finally, f is not approximately continuous at any $x_0\in C$ because $f(x_0)=0$ and $m(\{x: |f(x)|\neq 1\})=0$, and the theorem is established.

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... we have

Note that $[L(2), L(1)]=[1/2304, 1/32]$ and set

and let ... For $h \in T_i$, $i=1, 2, 3$, we have

then for $t \in [L(2), L(1)]$ we have ... An examination of ... on $[L(2), L(1)]$ shows that it attains its absolute minimum at ... Hence for $t \in [L(2), L(1)]$ we have

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and

for $3 \leq k \leq n$. Then set

and let ... For $h \in T_i$, $i=1, 2, \ldots, n$, we have ... and hence $h \in S(x)$. If $h \in T$, then ... and again $h \in S(x)$. Consequently, if we set

then for $t \in [L(n-1), L(n-2)]$ we have ... The function ... on

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BALANCED PROJECTIVE AND COBALANCED INJECTIVE TORSION FREE GROUPS OF FINITE RANK

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0. Introduction

Let TF be the category of torsion free abelian groups of finite rank and homomorphisms. The objects in TF will be called simply "groups". An epimorphism $H \to G$ of groups is *balanced* if, for every type t, $H(t) \to G(t)$ is an epimorphism, where $A(t) = \{a \in A\colon \text{type } a \geqq t\}$ for any group A. An exact sequence (E): $0 \to K \to H \to G \to 0$ in TF is balanced if $H \to G$ is a balanced epimorphism. A group A is called *balanced projective* if A is projective with respect to all balanced exact sequences (E). Balanced projectives in the category of all abelian groups were characterized in 1976 by Hunter and Warfield in [5] and [8].

It has long been known (and is easy to show) that in the category of all torsion free abelian groups, the balanced projectives are exactly the completely decomposable groups ([4]). The same result holds in the category of *Butler groups*, the pure subgroups of finite rank completely decomposable groups ([3] or [1]). In Section 2 (Theorem A) we show that in TF, the balanced projectives are precisely the completely decomposable groups.

An exact sequence (E) in TF is balanced if and only if every rank-1 group is projective with respect to (E) ([4]). It follows from Theorem A that the projective class in TF generated by the class of all rank-1 groups is the class of completely decomposable groups.

The dual notions of cobalanced and cobalanced injective were first defined in [2].

A monomorphism $0 \to G \to H$ is called *cobalanced* if, for every type t, $0 \to G/G[t] \to H/H[t]$ is pure exact, where

$$A[t] = \cap \{\operatorname{Ker} f\colon f \in \operatorname{Hom}(A, X) \text{ for } X \text{ a rank-1 group of type } t\}.$$

An exact sequence $0 \to G \to H \to K \to 0$ is cobalanced if $0 \to G \to H$ is cobalanced. A group A is *cobalanced injective* if A is injective with respect to all cobalanced exact sequences.

It is easy to characterize the cobalanced injectives in $\overline{TF}$, the category of all torsion free abelian groups. If $A \in \overline{TF}$ is algebraically compact then A is clearly cobalanced injective, since A is injective with respect to all exact sequences in $\overline{TF}$. If $B \in \overline{TF}$ is a direct product of rank one groups, it follows from Lemma 3.2 that B is also cobalanced injective. Thus, direct summands of groups of the form $A \oplus B$ will be cobalanced injective. Conversely, for any $X \in \overline{TF}$ the sequence $0 \to X \to A \oplus B \to A \oplus B/X \to 0$ is cobalanced exact in TF where A is the pure injective hull of X and $B = \prod_{f \in \operatorname{Hom}(X, Q)} f(X)$. (Here the embedding of X is the direct sum of the natural embeddings.) Thus, cobalanced injectives in TF are direct summands of groups of the form $A\oplus B$.

As an easy consequence of Lemma 3.2, the class $\mathcal{C}'$ of exact sequences in TF for which the completely decomposable groups are injective contains the class $\mathcal{C}$ of cobalanced exact sequences. Example 3.3 shows that this containment is proper. Nonetheless, Theorem B implies that the injective class in TF generated by the rank-1 groups is precisely the class of completely decomposable groups.

Similar results on the projective and injective classes in TF generated by finite sets of rank one groups were obtained by the authors in [7], using different techniques.

Our notation generally follows [1]. Specifically $\sim$ and $\doteq$ denote quasi-isomorphism and quasi-equality. For groups G, H we let $QG=Q\otimes G$ and $Q\operatorname{Hom}(G,H)=Q\otimes\operatorname{Hom}(G,H)$. The localization of a group at a prime p is written G_p, and $\hat{Z}_p$, $\hat{Q}_p$ are the rings of p-adic integers and numbers. For notational convenience we write $\operatorname{type}(Z)=0$ and Z_{-p} for the subring of Q generated by Z and $\frac{1}{p}$.

1. Tensor products and homomorphisms

In this section we obtain several results on tensor products and homomorphisms to be used in Sections 2 and 3. The first proposition is of independent interest.

THEOREM 1.1. *Let $G, H_1, H_2, \dots$ be a countable collection of groups with $Z^2\otimes G\subseteq H_i\subseteq Q^2\otimes G$, and $H_i/(Z^2\otimes G)$ infinite for each i. Then there exists a rank-2 group A (regarding $Z^2\subseteq A\subseteq Q^2$) such that $(H_i+(A\otimes G))/(A\otimes G)$ is infinite for each i. Moreover, A can be chosen so that Q is a homomorphic image of A.*

PROOF. Denote $H_i'=H_i/(Z^2\otimes G)$, and fix the index i. If H_i' is not reduced, choose a prime p for which $Z(p^\infty)\subseteq H_i'$. Then there is a sequence $x_j=x_j(i,p)=(1,0)\otimes g_j+(0,1)\otimes h_j$ with g_j, h_j in G, satisfying the following properties:

1-a p-height$(x_j)=0$ in $Z^2\otimes G$.

1-b p-height$(x_j)=j$ in H_i.

1-c x_j-x_{j-1} is an element of $p^{j-1}(Z^2\otimes G)$.

In this case we will say H_i belongs to Case 1(p).

If H_i' is reduced, then since H_i' is infinite, there is a sequence $y_j=y_{ij}=(1,0)\otimes g_{ij}+(0,1)\otimes h_{ij}$ with g_{ij}, h_{ij} in G, satisfying,

2-a p_j-height$(y_j)=0$ in $Z^2\otimes G$ (p_j is the j-th prime).

2-b p_j-height$(y_j)=1$ in H_i for infinitely many j.

In this case we will say H_i belongs to Case 2.

We can now construct the rank-2 group A by specifying (see [4], Chapter 93)

$$(*)\qquad \hat{Z}_p\otimes A=\hat{Q}_p(1,a)\oplus\hat{Z}_p(0,1)\subseteq\hat{Z}_p\times Q^2,$$

where a is a non-zero p-adic integer which will be constructed as the limit of a sequence of integers $s_k=s_k(p)$, $k=0,1,\dots$, with $s_k(p)$ congruent to $s_{k+1}(p)$ modulo

p^k. The values of $s_0(p)$ $(1 \leqq s_0(p) < p)$ are determined by considering those H_i which belong to Case 2. By relabeling, call this set of groups $H_1, H_2, \ldots$, with associated sequences $(y_{1j}), (y_{2j}), \ldots$ satisfying 2-a and 2-b. Let N be the number of groups H_i. Then for $j=1, 2$ and for $j>N$ if N is finite, set $s_0(p_j)=1$. For $3 \leqq j \leqq N$, choose $s_0(p_j)$ so that p_j-height $(h_{ij}-s_0(p_j)g_{ij})=0$ in G for $1 \leqq i \leqq j$. To see that this is possible, note first that by 2-a, $\min(p_j\text{-height}(g_{ij}), p_j\text{-height}(h_{ij}))=0$ for each j. Also note that if $h_{ij}-bg_{ij}$ and $h_{ij}-cg_{ij}$ are both divisible by p_j for distinct integers $0 \leqq b, c < p_j$, then $(b-c)g_{ij}$ and therefore g_{ij} is divisible by p_j. Since p_j divides $h_{ij}-bg_{ij}$, it follows that p_j divides h_{ij}, a contradiction. Thus there is at most one integer b between 0 and p_j-1 such that p_j divides $h_{ij}-bg_{ij}$. Since $p_j \geqq j+2$ for $j \geqq 3$, there exists $1 \leqq s_0(p_j) \leqq p_j - 1$ such that p_j does not divide $h_{ij}-s_0(p_j)g_{ij}$ for $1 \leqq i \leqq j$.

For each prime p, the sequence $s_1(p), s_2(p), \ldots$ is constructed by considering the groups in Case 1(p). For ease of notation, assume that these groups have been relabelled $H_1(p), H_2(p), \ldots$, where we employ the convention, $H_j(p)=0$ for j greater than M=the number of groups in Case 1(p). We will assume p is fixed and delete it from the symbols where convenient. The integer $s_j(p)$, $j \geqq 1$, is chosen so that p-height $(h_j - s_j(p)g_j) \leqq j-1$ if $H_j(p)$ is not zero. This is possible by an argument similar to that in the preceding paragraph. If $H_j(p)=0$, $s_j(p)$ need only satisfy the requirement that p^{j-1} divides $s_j(p)-s_{j-1}(p)$.

The rank-2 group A is constructed according to (*). It remains to show that $H_i'=(H_i+A\otimes G)/A\otimes G$ is infinite for each i. First assume that $H_i=H_i(p)$ belongs to Case 1(p), and that H_i' is finite. Then $p^k H_i \in (A\otimes G)_p$ for some $k \geqq 0$. Again deleting reference to the prime p and the index i in the sequence $x_j=x_j(i,p)$, we have

(i) p^i divides $x_{i+k}=(1,0)\otimes g_{i+k}+(0,1)\otimes h_{i+k}$ in $A\otimes G$

(ii) p^i divides $(1, s_i(p))\otimes g_{i+k}$ in $A\otimes G$, hence

(iii) p^i divides $(0,1)\otimes(h_{i+k}-s_i(p)\,g_{i+k})$ in $A\otimes G$.

By the definition of A, this implies that

(iv) p^i divides $h_{i+k}-s_i(p)\,g_{i+k}$ in G.

However, by property 1-c, p^i divides $h_{i+k}-h_i$ and $g_{i+k}-g_i$. It follows that

(v) p^i divides $h_i-s_i(p)\,g_i$ in G, contradicting the choice of $s_i(p)$.

Finally, assume H_i belongs to Case 2 and that $H_i'=(H_i+A\otimes G)/A\otimes G$ is finite. Then for all but a finite number of primes p, $(H_i)_p \subseteq (A\otimes G)_p$. In particular, p_j divides x_{ij} in $A\otimes G$ for infinitely many j. However, for each j, p_j divides $(1, s_0(p_j))\otimes g_{ij}$. This implies that for infinitely many j, $(0,1)\otimes(h_{ij}-s_0(p_j)g_{ij})$ is divisible by p_j in $A\otimes G$. As above, it follows that p_j divides $h_{ij}-s_0(p_j)g_{ij}$ in G for infinitely many j. This contradicts the choice of $s_0(p_j)$. For the last statement of the proposition, note that $A/B \cong Q$, where B is the subgroup generated by $(0,1)$.

Corollary 1.2. *Given groups G, H, there is a rank-2 group A with Q a homomorphic image of A such that* $Q\,\mathrm{Hom}(H, Z^2\otimes G)=Q\,\mathrm{Hom}(H, A\otimes G)$.

This corollary is obtained by taking $H_1, H_2, \ldots$ in the theorem to be the countable number of homomorphic images of H in $Q^2 \otimes G$ which are not quasi-equal to $Z^2 \otimes G$.

Our final two results are probably well-known, but are included for the sake of convenience.

LEMMA 1.3. *Let H be a strongly indecomposable rank-2 p-local group. Then any p-local subgroup of H is either quasi-equal to H or is a free Z_p-module.*

PROOF. Since such an H must have p-rank 1 there is a free rank-2 Z_p-submodule $F \subseteq H$ with $H/F \cong Z(p^\infty)$. Let K be a p-local subgroup of H. If $(K+F)/F$ is finite then K is a free Z_p-module. Otherwise, $(K+F)/F \cong Z(p^\infty)$ and $K \doteq H$.

LEMMA 1.4. *If A is a p-local group with no free Z_p-summand then there exists a strongly indecomposable rank-2 p-local group H with* $\mathrm{Hom}(A, H) = 0$.

PROOF. There are uncountably many quasi-isomorphism classes of strongly indecomposable rank-2 p-local groups H. Therefore, we may choose an H which does not belong to any of the countably many quasi-isomorphism classes of the rank-2 factors of A. By Lemma 1.3, $\mathrm{Hom}(A, H) = 0$ for this H.

2. Balanced projectives

We begin by stating for reference the previously mentioned result of Butler.

LEMMA 2.1. *A balanced projective Butler group is completely decomposable.*

Let G be balanced projective and $G \sim \oplus \{G_i : 1 \leqq i \leqq n\}$ be a quasi-decomposition of G into strongly indecomposable components. Then each G_i is *almost balanced projective* in the sense that for any balanced epimorphism $B \to C \to 0$ the sequence $Q\,\mathrm{Hom}(G_i, B) \to Q\,\mathrm{Hom}(G_i, C) \to 0$ is exact. We show any strongly indecomposable almost balanced projective group must be rank-1. This result implies that the balanced projective group G is a Butler group, hence, by Lemma 2.1 is completely decomposable.

We first examine the local case.

LEMMA 2.2. *If G is (almost) balanced projective then G_p is (almost) balanced projective in the category of p-local torsion free groups of finite rank.*

PROOF. Assume G is balanced projective and $B \to C \to 0$ is a balanced epimorphism of p-local groups. Then $\mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to 0$ is exact. There are natural isomorphisms $\mathrm{Hom}(G, B) \cong \mathrm{Hom}(G_p, B)$, $\mathrm{Hom}(G, C) \cong \mathrm{Hom}(G_p, C)$. Therefore, $\mathrm{Hom}(G_p, B) \to \mathrm{Hom}(G_p, C) \to 0$ is exact. The proof for G almost balanced projective is similar.

LEMMA 2.3. *Let G be (almost) balanced projective. Then G is locally completely decomposable.*

PROOF. Suppose G is balanced projective and G_p is not completely decomposable for some prime p. Then $G_p = F \oplus D \oplus A$ where F is a free Z_p module, D is divisible and $A \neq 0$ is a reduced Z_p module with no free summand. It follows that all rank-1 factors of A are isomorphic to Q.

By Lemma 1.4 there exists a strongly indecomposable rank-2 p-local group H with $\mathrm{Hom}(A, H)=0$. Pick two epimorphisms $f\colon A\to Q$ and $g\colon H\to Q$ and construct the pullback $B=\{(a, h)\colon f(a)=g(h)\}$.

Projection of B onto the first component induces a balanced epimorphism $B\to A\to 0$ of p-local groups. This must split by Lemma 2.2, since A is summand of the p-local balanced projective group G_p.

Let $e\colon A\to B$ be a splitting map. By a rank argument, e followed by projection onto the second component of B is non-zero. This contradicts $\mathrm{Hom}(A, H)=0$. Thus, for each prime p, G_p must be completely decomposable. Again, the argument for almost balanced projective is similar.

LEMMA 2.4. *Let G be a group which is strongly indecomposable, homogeneous of type t, and almost balanced projective. Then* $\mathrm{rank}\, G=1$.

PROOF. Suppose $\mathrm{rank}\, G=n>1$. Let $X=X_1\oplus\ldots\oplus X_n$ be a full completely decomposable subgroup of G, homogeneous of type t. Since G is locally completely decomposable (Lemma 2.3) and homogeneous then G/X is reduced. Indeed, either G_p is reduced or G contains a p divisible element. In the latter case X is p divisible and $(G/X)_p\cong G_p/X_p=0$.

Since G is not quasi-equal to X, G/X is an infinite torsion group. Since G/X is reduced, we can write $G/X=T_1\oplus T_2$, where T_1 and T_2 are infinite.

Let $G_i=e^{-1}(T_i)$, $i=1, 2$, where $e\colon G\to G/X$ is the natural map. In view of the fact that $X\subseteqq G_i\subseteqq G$ each G_i is of rank-n and is homogeneous of type t. It follows that $G_1\oplus G_2\to G_1+G_2=G\to 0$ is balanced exact, and, therefore, quasi-splits. Since G is strongly indecomposable we must have $G\sim G_1$ or $G\sim G_2$. Neither of these possibilities can occur because each G/G_i is infinite. Thus $\mathrm{rank}\, G=n>1$ is impossible.

PROPOSITION 2.5. *Let G be a strongly indecomposable group which is almost balanced projective. Then* $\mathrm{rank}\, G=1$.

PROOF. Suppose $\mathrm{rank}\, G>1$. Let $t=$inner type G and let h be a height vector in t. By Lemma 2.4 G is not homogeneous so we can choose $G_0\subseteqq G$ maximal with respect to the property that G_0 is a pure subgroup of G with inner type greater than t. Let $t_0=$inner type G_0.

The maximality of G_0 implies that $\inf\{\mathrm{type}\, x, t_0\}=t$ for all $x\notin G_0$. Choose a height vector h_0 in t_0 such that $h_0(p)\geqq h(p)$ for all primes p. Next choose a full free subgroup F of G and write $G/F=\oplus T_p$ where T_p is a torsion p-group for each prime p. Define $T=\sum\{h(p)\text{-socle } T_p\colon p \text{ is a prime for which } h_0(p)>h(p)\}\oplus \oplus\sum\{T_p\colon h_0(p)=h(p)\}$. The $h(p)$-socle of T_p is $\{x\in T_p\colon p^{h(p)}x=o\}$.

Now let H be the inverse image of T in G (under the map $G\to G/F$). The construction of H guarantees that if $x\in H\setminus G_0$, then $\mathrm{type}_H\, x=\mathrm{type}_G\, x$. Indeed, since $\inf\{\mathrm{type}_G\, x, t_0\}=t$, the height vector for x in G agrees with h at all primes p such that $h_0(p)>h(p)$, at least up to type equivalence.

Further note, that since $h<h_0$, outer type $H\not\geqq t_0$. (Apply Theorem 1.10 of [1] to H/F.) In particular, there is no $0\neq x\in H$ with type $x\geqq t_0$.

By Corollary 1.2 there is a rank-2 group A with $q\colon A\to Q\to 0$ such that $Q\,\mathrm{Hom}(G_0, A\otimes H)=Q\,\mathrm{Hom}(G_0, H\oplus H)$. Thus, since inner type $G_0=t_0$, $\mathrm{Hom}(G_0, H\oplus H)=0=\mathrm{Hom}(G_0, A\otimes H)$.

Let $e\colon A\otimes H\to QG$ be the map induced by $a\otimes x\to q(a)x$ and let $i\colon G_0\to QG$ be inclusion. Consider $f=i\oplus e\colon G_0\oplus(A\otimes H)\to QG$. It is not hard to check that $G\subseteq \text{image } f$.

Thus, there is an epimorphism (which we will still call f) $f\colon f^{-1}(G)\to G\to 0$. This epimorphism is balanced since, if $x\in G_0$, then $f(x,o)=x$ and, if $x\in G\setminus G_0$, then $x=qy$ for some $q\in Q$ and $y\in H$. In this case $q=q(a)$ for some $a\in A$ and $f(0,a\otimes y)=x$. In either case, x is the image of an element of the same type.

Since G is almost balanced projective there exists $f'\colon G\to f^{-1}(G)$, a quasi-splitting map for f. Because f' is monic and $\operatorname{Hom}(G_0, A\otimes H)=0$, we must have $f'(G_0)$ quasi-equal to G_0. But then $G\sim G_0\oplus f'(G)\cap(A\otimes H)$ by the modular law. This is impossible for G is strongly indecomposable and $\operatorname{rank} G_0<\operatorname{rank} G$.

Lemma 2.1 and Proposition 2.5 together imply:

THEOREM A. *Let G be a balanced projective group in TF. Then G is completely decomposable.*

3. Cobalanced injectives

We begin by stating the previously mentioned result of Arnold and Vinsonhaler [2].

LEMMA 3.1. *A cobalanced injective Butler group is completely decomposable.*

Our plan in this section is the same as that of Section 2. A group G is called *almost cobalanced injective* if for any pure cobalanced embedding, $0\to B\to C$, the induced sequence $Q\operatorname{Hom}(C,G)\to Q\operatorname{Hom}(B,G)\to 0$ is exact. If G is cobalanced injective and $G\sim\oplus\{G_i\colon 1\le i\le n\}$ is a quasi-decomposition of G into strongly indecomposable summands, then each G_i is almost cobalanced injective. We show each G_i must be rank-1 (Proposition 3.5). Hence G is Butler and, by Lemma 3.1, is therefore completely decomposable. Our arguments in this section are somewhat more cumbersome than those of Section 2, in keeping with the tradition of injectives. Things would be considerably simplified by a short proof that cobalanced injectives in TF are locally completely decomposable.

The next lemma follows directly from Corollary 1.9 of [7].

LEMMA 3.2. *Let (E) be a cobalanced exact sequence in TF and X be a rank-1 group. Then X is injective with respect to (E).*

The following is an example of a non-cobalanced exact sequence (E) such that every rank one group is injective with respect to (E). A group G is called *cohomogeneous of cotype t* if cotypeset $G=\{t\}$. That is, the type of every rank one factor of G is t.

EXAMPLE 3.3. Let $p\ne q$ be primes and t be the type of Z_{-p}. Let G be a rank two group, homogeneous of type 0, cohomogeneous of type t (such groups are easy to construct). Choose $a,b\in G$ such that Za, Zb are disjoint pure cyclic subgroups of G. Regard $G\subseteq Qa\oplus Qb\subseteq Qa\oplus Qb\oplus Qc$ and let $H=\langle G\oplus Z_{-q}c, (b+c)/p\rangle$.

Consider the pure exact sequence (E): $0\to G\to H\to H/G\to 0$. The sequence (E) is not cobalanced since $G[t]=0$, $H[t]=Z_{-q}c$ and, thus, $0\to G/G[t]\to H/H[t]$ is not a pure embedding. However, since every rank-1 image of G is isomorphic to

Z_{-p}, it follows immediately that, for every rank-1 group X, $\operatorname{Hom}(H, X) \to \operatorname{Hom}(G, X)$ is epic.

We next prove the analogue to Lemma 2.4.

LEMMA 3.4. *Let G be a strongly indecomposable cobalanced injective group which is cohomogeneous of cotype t. Then* $\operatorname{rank} G = 1$.

PROOF. Let $\operatorname{rank} G = r$. Since G is cohomogeneous of cotype t there is an embedding $e: G \to X$, where $X = \oplus \{X_i : 1 \leqq i \leqq r, \operatorname{rank} X_i = 1, \operatorname{type} X_i = t\}$. We will show X/G is finite, so that $r = 1$ since G is strongly indecomposable.

Suppose that the torsion group X/G can be written $X/G = T_1 \oplus T_2$ with T_1, T_2 infinite and coprime $(\gcd\{\text{order } x_1, \text{order } x_2\} = 1$ for all $0 \neq x_1 \in T_1$, $0 \neq x_2 \in T_2)$. Let $f_i: X \to T_i$ be the composition $X \to X/G \to T_i$ and let $H_i = f_i^{-1}(T_i)$, $i = 1, 2$. Then $G \subseteq H_i \subseteq X$, so each H_i is cohomogeneous of cotype t. Put $H = H_1 \oplus H_2$ and form the diagonal embedding $d: G \to H$, $d(g) = (g, g) \in H_1 \oplus H_2$. Since T_1 and T_2 are coprime, d is a pure embedding. Moreover, because G and H are cohomogeneous of cotype t, d is cobalanced. But d cannot quasi-split. This follows since G is strongly indecomposable, $\operatorname{rank} G = \operatorname{rank} H_1 = \operatorname{rank} H_2$ and G is quasi-equal to neither H_1 nor H_2. Thus, X/G cannot equal $T_1 \oplus T_2$ where T_1 and T_2 are infinite and coprime.

To show X/G is finite it therefore suffices to eliminate the possibility that X/G is a divisible p-group. In this case $pX = X$ and $pG \neq G$. Let A be a rank-2 group, homogeneous of type 0, cohomogeneous of cotype = type Z_{-p}. Choose a pure embedding $Z \to A$ and an embedding $Z \to G/D_p$, where D_p is the maximal p-divisible subgroup of G. Construct the pushout (K, i, j):

$$\begin{array}{ccc} Z & \longrightarrow & G/D_p \\ \downarrow & & \downarrow j \\ A & \xrightarrow{\ i\ } & K \end{array}$$

and set $H = K \oplus X$.

Consider the embedding $(j\pi \oplus e): G \to H$, where $\pi: G \to G/D_p$ is the natural map.

First, $j\pi \oplus e$ is a pure embedding. This follows because $q\text{-height}_X(e(g)) = q\text{-height}_G(g)$ for all primes $q \neq p$ $((X/G)_q = 0)$ and, if $g \notin D_p$, $p\text{-height}_K(j\pi(g)) = p\text{-height}_G(g)$.

Next, $j\pi \oplus e$ is cobalanced. Indeed $G[s] = G$ unless $s \geqq t$ in which case $G[s] = 0$. If $s \geqq t$ then s is infinite at p and $K[s] = H[s] = 0$. It follows that, for any type s, $0 \to G/G[s] \to H/H[s]$ is pure exact.

Finally, $j\pi \oplus e$ cannot quasi-split. Note that $G(t) = 0$ since G is strongly indecomposable of rank greater than one and cohomogeneous of cotype t. Hence, $\operatorname{Hom}(X, G) = 0$. Assume $j\pi \oplus e$ quasi-splits. It follows that G is quasi-equal to $j\pi(G)$, a quasi-summand of K. Thus $j\pi$ is monic and $D_p = 0$. Identify G and A with $j(G)$ and $i(A)$ inside K. Write $K \doteq G \oplus U$, where U is a rank one subgroup of K, and let $f \in \operatorname{End}(K)$ be a quasi-projection of K onto G. Then there exists a non-zero integer m such that $f(g) = mg$ for all $g \in G$. It follows that $(f - m)$ induces a map from $A/G \cap A$ into G. But $A/G \cap A \cong Z_{-p}$ and $D_p = 0$. Consequently, $f(A) = mA$ and A is quasi-contained in G, a contradiction, since K is a pushout. This completes the proof.

PROPOSITION 3.5. *Let G be a strongly indecomposable almost cobalanced injective group. Then* $\operatorname{rank} G=1$.

PROOF. Suppose $\operatorname{rank} G>1$. By Lemma 3.4 we may assume that G is not cohomogeneous. Let G_0 be a pure subgroup of G minimal with respect to the property that $s_0=\text{outer type } G/G_0<s=\text{outer type } G$. Such a G_0 exists because G is not cohomogeneous.

Choose height vectors $h_0\in s_0$, $h\in s$ such that $h_0(p)\leqq h(p)$ for all primes p. Define a height vector h_1 by $h_1(p)=0$ if $h_0(p)=h(p)$, $h_1(p)=h(p)$ if $h_0(p)<h(p)$. Let s_1 be the type of h_1. Note that $s_1>0$ and if X is a pure rank-1 subgroup of G with $X\cap G_0=0$, then type $X\not\leqq s_1$.

Let A be a rank-2 group, homogeneous of type 0, cohomogeneous of type s_1, and pick $x\in G\setminus G_0$. There is an induced embedding $Z\to Zx\subseteq G$. Choose a pure embedding $0\to Z\to A$ and let (K, i, j) be the pushout

$$\begin{array}{ccc} Z & \longrightarrow & G \\ \downarrow & & \downarrow j \\ A & \xrightarrow[i]{} & K \end{array}.$$

Define $H=K\oplus(G/G_0)$ and form the embedding $e\colon G\to H$ where $e(x)=(j(x), x+G_0)$. Since $Z\to A$ is a pure embedding, so is j and therefore e.

To show e is cobalanced, first let t be a type with $t\not\geqq s_1$. Suppose $G_0\not\subseteq G[t]$. Then there is an embedding $G/(G_0\cap G[t])\to G/G_0\oplus G/G[t]$. This implies outer type $G/(G_0\cap G[t])\leqq\sup\{\text{outer type } G/G_0, \text{ outer type } G/G[t]\}\leqq\sup\{s_0, t\}$. But by the construction of s_1, if $t\not\geqq s_1$ then $\sup\{s_0, t\}\not\geqq s$. Thus outer type $G/(G_0\cap G[t])<s$, contradicting the choice of G_0. Therefore $G_0\subseteq G[t]$ and $G/G[t]\cong(G/G_0)/(G/G_0)[t]$. It follows that $0\to G/G[t]\to H/H[t]$ is a pure embedding, since G/G_0 is a summand of H.

Next let t be a type with $t\geqq s_1$. In this case we show $0\to G/G[t]\to K/K[t]$ is a pure embedding by showing $K[t]=G[t]$. Note that $K/G\cong A/Z$ is a rank-1 group of type s_1. This implies $K[t]\subseteq G$, so that $G[t]\subseteq K[t]\cap G=K[t]$. To show the reverse inclusion, observe that $K/G[t]$ is a homomorphic image of $A\oplus G/G[t]$, a group of outer type $\leqq t$. Thus outer type $K/G[t]\leqq t$ and $K[t]\subseteq G[t]$.

To complete the proof we will show that $e\colon G\to H$ cannot quasi-split. Write $e=e_1\oplus e_2$ where $e_1=j\colon G\to K$ and $e_2\colon G\to G/G_0$ is the factor map. Suppose $f_1\oplus f_2\colon K\oplus G/G_0\to G$ is a quasi-splitting of e. Then $(f_1\oplus f_2)(e_1\oplus e_2)=f_1e_1+f_2e_2=m$ for some non-zero integer m. Moreover, $f_2e_2\in\operatorname{End}(G)$ is not monic, hence is nilpotent since G is strongly indecomposable ([6]). Therefore f_1e_1 must be monic since the nilpotent elements of $\operatorname{End}(G)$ form an ideal ([6] again). Thus f_1e_1 is a quasi-isomorphism and $e_1=j\colon G\to K$ quasi-splits. Identify G with jG in K and write $K\doteq G\oplus U$ where U is a rank-1 subgroup of K. Let $g\colon K\to G$ be a quasi-projection. Note that $x\in gA\setminus G_0$, where $x\in G\setminus G_0$ is the element chosen above to obtain the embedding $Z\to G$ in the pushout diagram. Hence $(gA+G_0)/G_0$ is a non-zero subgroup of G/G_0. Since A is cohomogeneous of cotype s_1, this implies $s_0=$ $=\text{outer type } G/G_0\geqq s_1$, a contradiction. Therefore, $e\colon G\to H$ does not quasi-split, contradicting G almost cobalanced injective. We may conclude $\operatorname{rank} G=1$.

Proposition 3.5 and Lemma 3.1 together imply:

THEOREM B. *Let G be a cobalanced injective group in TF. Then G is completely decomposable.*

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EQUINORMALITY CHARACTERIZES THE COMPACTNESS

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1. A proximity space (X, δ) is called *equinormal* if its disjoint closed sets are far, i.e. $\bar{A} \cap \bar{B} = \emptyset$ implies $A\bar{\delta}B$ ([11]). As S. G. Mrówka showed, a metrizable space X is compact iff any compatible metric proximity is equinormal on X ([9]). Recently the author introduced the notion of the proximity hyperspaces and proved that a separated proximity space (X, δ) is compact iff its hyperspace $(E(X), E(\delta))$ is equinormal.

In this paper we will study certain products of syntopogenous spaces in which the "*SP*-normality" implies the compactness. Applying these results for proximity spaces, we will get that *the following conditions are equivalent for any separated proximity space* (X, δ):

(A) (X, δ) *is compact.*
(B) *The product of* (X, δ) *and its Smirnov compactification* (X^*, δ^*) *is equinormal.*
(B′) *The product* $(X \times Y, \delta \times \delta')$ *is equinormal for any compact separated proximity space* (Y, δ').
(C) *The power* $(X^{\mathfrak{m}}, \delta^{\mathfrak{m}})$ *is equinormal, where* $\mathfrak{m}$ *is the proximity weight of* (X, δ).
(D) *The topology of* $(X^{\mathfrak{m}}, \delta^{\mathfrak{m}})$ *is normal for any cardinal number* $\mathfrak{m}$.

REMARKS. 1. The equivalence of (A) and (D) follows from a stronger result of N. Noble [10]: a T_1-space X is compact iff the power $X^{\mathfrak{m}}$ is normal for any cardinal number $\mathfrak{m}$. However, in contrast to (C), in this case the cardinal numbers $\mathfrak{m} > \omega(X)$ are in fact necessary (e.g. consider a second countable non-compact totally bounded metric space X).

2. In (B) the normality of the topology $\tau(\delta \times \delta^*)$ is not sufficient (as it is well-known, this weaker condition characterizes the paracompactness of $\tau(\delta)$ (see [13])).

3. Statement (B) implies that $\delta \times \delta^*$ is the finest proximity compatible with the topology $\tau(\delta \times \delta^*)$, but this latter condition is satisfied whenever $\tau(\delta)$ is pseudo-compact and δ is the finest proximity compatible with it (see [4]), so that the weakening of condition (B) is not possible in this direction either.

2. We will define only those notions and notations concerning *syntopogenous spaces* which cannot be found in the monograph of Á. Császár [1].

The syntopogenous space $(X, \mathscr{S})$ will be called *SP-normal* if

$$\bar{A} \cap \bar{B} = \emptyset \quad \textit{implies} \quad A < C <^c X - B \quad \textit{for some} \quad < \in \mathscr{S}$$

(see [12]). (Let us remark that *SP*-normality is not the unique generalization of topological normality for syntopogenous spaces (see [4], [7]), and that it is equivalent to *strong normality* defined in [4].)

Observe that the SP-normality of $\mathcal{S}$ implies the normality of $\mathcal{S}^{tp}$ in the usual sense.

LEMMA 1. *Let* $(X_a, \mathcal{S}_a)$ *be a syntopogenous space for any* $a\in A$. *If* $(\bigtimes_{a\in A} X_a, \bigtimes_{a\in A} \mathcal{S}_a)$ *is SP-normal then the spaces* $(X_a, \mathcal{S}_a)$ *are SP-normal, too.*

The statement can be shown by using Lemma 6.1 of [5].

LEMMA 2. *A compact* T_2 *syntopogenous space is always SP-normal.*

PROOF. If $(X, \mathcal{S})$ is a compact T_2-space then so are $(X, \mathcal{S}^t)$ and $(X, \mathcal{S}^{tp})$, too. Therefore $(X, \mathcal{S}^{tp})$ is normal (in the usual sense), thus $\bar{A}\cap\bar{B}=\emptyset$ implies the existence of an $(\mathcal{S}^{tp}, \mathcal{H}^{tp})$-continuous real function f such that $f(X)\subset[0, 1]$, $f(A)=\{0\}$ and $f(B)=\{1\}$. By Theorem 13 of [3] f is $(\mathcal{S}^t, \mathcal{H}^t)$-, and *a fortiori* $(\mathcal{S}^t, \mathcal{H})$-continuous, consequently $(X, \mathcal{S}^t)$, and at the same time $(X, \mathcal{S})$ is SP-normal (see [12], Theorems 2 and 3).

Now (14.22) and (15.82) of [1] and Lemma 2 yield

LEMMA 3. *The product of an arbitrary family of compact* T_2 *syntopogenous spaces is SP-normal.*

Further on we need the notion of the simple compactification $(X^*, \mathcal{S}^*)$ of a syntopogenous space $(X, \mathcal{S})$ ([7]). This is a syntopogenous space determined uniquely (up to equivalences) by the following properties:

(a) X is a dense subset of $(X^*, \mathcal{S}^{*s})$;
(b) $\mathcal{S}^*|X\sim\mathcal{S}$;
(c) $(X^*, \mathcal{S}^*)$ is compact;
(d) $\{x\}$ is closed in $(X^*, \mathcal{S}^*)$ for any $x\in X^*-X$.

In the construction of $(X^*, \mathcal{S}^*)$ the round filters play an important role. A filter $\mathfrak{f}$ in X is called $\mathcal{S}$*-round* ([2], p. 240) if, for any $F\in\mathfrak{f}$, there exist $<\in\mathcal{S}$ and $F_1\in\mathfrak{f}$ such that $F_1<F$. For any filter base $\mathfrak{r}$, the filter

$$\mathcal{S}(\mathfrak{r}) = \{V\subset X\colon\ R<V \text{ for some } R\in\mathfrak{r}, <\in\mathcal{S}\}$$

is $\mathcal{S}$-round. If $x\in X$, we will write simply $\mathcal{S}(x)$ instead of $\mathcal{S}(\{\{x\}\})$.

Now let $\mathfrak{f}(x)=\mathcal{S}^s(x)$ for $x\in X$ and make a one-to-one correspondence $x\leftrightarrow\mathfrak{f}(x)$ between the points $x\in X^*-X$ and all compressed $\mathcal{S}^c$-round filters $\mathfrak{f}(x)$ without cluster points in $(X, \mathcal{S})$. Put $\mathfrak{s}(x)=\mathcal{S}(\mathfrak{f}(x))$ for every $x\in X^*$ and consider the sets $s(P)=\{x\in X^*\colon P\in\mathfrak{s}(x)\}$ for $P\subset X$. If $<\in\mathcal{S}$, we have a topogenous order $s(<)$ on X^* defined by $Es(<)F$ $(E, F\subset X^*)$ if and only if there exist a natural number m and sets $P_i, Q_i\subset X$ such that $P_i<Q_i$ $(1\le i\le m)$, moreover $E\subset\bigcup_{i=1}^m s(P_i)$ and $\bigcup_{i=1}^m s(Q_i)\subset F$. Then we have $\mathcal{S}^*\sim\{s(<)\colon <\in\mathcal{S}\}$.

In [7] we pointed out that this construction is a generalization of the Wallman-type compactification method. A further similarity is shown by

LEMMA 4. *If* $(X, \mathcal{S})$ *is an SP-normal* T_1*-space then* $(X^*, \mathcal{S}^*)$ *is a* T_2*-space.*

PROOF. For any point $x\in X$, the set $\{x\}$ is closed in $(X, \mathcal{S})$. Every filter $\mathfrak{f}(x)$, where $x\in X^*-X$, has a filter base consisting of closed sets in $(X, \mathcal{S})$ because it is $\mathscr{S}^c$-round, moreover it has no cluster points in $(X, \mathscr{S})$. Finally if $x, y \in X^* - X$, $x \neq y$ then $\mathfrak{f}(x) \neq \mathfrak{f}(y)$ implies $\emptyset \in \mathfrak{f}(x)(\cap)\mathfrak{f}(y)$ because $\mathfrak{f}(x)$ and $\mathfrak{f}(y)$ are maximal $\mathscr{S}^c$-round (see [7], (5.2.1)). Since $\mathscr{S}$ is SP-normal, we have disjoint sets $V \in \mathfrak{s}(x)$ and $W \in \mathfrak{s}(y)$ for any $x, y \in X^*$, $x \neq y$, hence $s(V)$ and $s(W)$ are disjoint $\mathscr{S}^*$-neighbourhoods of x and y respectively.

One of our main results is the following

LEMMA 5. *If $(X, \mathscr{S})$ is a non-compact T_1-space, then the product $(X \times X^*, \mathscr{S} \times \mathscr{S}^*)$ is not SP-normal.*

PROOF. Suppose $\mathscr{S} \times \mathscr{S}^*$ is SP-normal but $\mathscr{S}$ is not compact. Then there exists a point $p \in X^* - X$ (see Lemma (5.3) of [7]). Let us consider the sets $D = \{(x, x) : x \in X\}$ and $V = \bigcup \{s(X - F) : F \in \mathfrak{f}(p)\} \subset X^*$. Since $\mathscr{S} \times \mathscr{S}^*$ is a T_2-structure (see Lemmas 1, 4 and [1], (14.22)), the set D is closed and the set $X \times V$ is open with respect to $\mathscr{S} \times \mathscr{S}^*$. As $\mathfrak{f}(p)$ has no cluster point, we have $D \subset X \times V$. The SP-normality of $\mathscr{S} \times \mathscr{S}^*$ implies $D < X \times V$ for some $< \in \mathscr{S} \times \mathscr{S}^*$. Then there are orders $<_1, <_2 \in \mathscr{S}$ and a finite number of sets $P_i, Q_i \subset X$, $P_i^*, Q_i^* \subset X^*$ such that $D \subset \bigcup_{i=1}^{m} (P_i \times P_i^*)$, $\bigcup_{i=1}^{m} (Q_i \times Q_i^*) \subset X \times V$, where $P_i <_1 Q_i$ and $P_i^* s(<_2) Q_i^*$ $(1 \leq i \leq m)$ (see [1], (11.10)). There exist natural numbers n_i $(1 \leq i \leq m)$ and sets $R_{ij}, S_{ij} \subset X$ $(1 \leq j \leq n_i)$ for which $P_i^* \subset \bigcup_{j=1}^{n_i} s(R_{ij})$, $\bigcup_{j=1}^{n_i} s(S_{ij}) \subset Q_i^*$ and $R_{ij} <_2 S_{ij}$ $(1 \leq i \leq m,\ 1 \leq j \leq n_i)$. Finally suppose $<_3 \in \mathscr{S}$, $<_2 \subset <_3^2$ and put $R_{ij} <_3 T_{ij} <_3 S_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n_i$. It is easy to see that $X = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} R_{ij}$, consequently there are indices i, j such that $\emptyset \notin \mathfrak{f}(p)(\cap)\{R_{ij}\}$. Since $\mathfrak{f}(p)$ is compressed, this gives $T_{ij} \in \mathfrak{f}(p)$ and $S_{ij} \in \mathfrak{s}(p)$, i.e. $p \in s(S_{ij})$. Therefore $p \in V$, i.e. $p \in s(X - F)$ for some $F \in \mathfrak{f}(p)$, which is an obvious contradiction.

Let us agree that if $\mathfrak{m}$ is a cardinal number, then by $(X^{\mathfrak{m}}, \mathscr{S}^{\mathfrak{m}})$ we shall mean the product $\left(\bigtimes_{a \in A} X_a, \bigtimes_{a \in A} \mathscr{S}_a\right)$, where $\bar{\bar{A}} = \mathfrak{m}$ and $X_a = X$, $\mathscr{S}_a = \mathscr{S}$ for each $a \in A$. $|\mathscr{S}|^i$ will denote the ^{i}-weight of the syntopogenous space $(X, \mathscr{S})$.

LEMMA 6. *Let $(X, \mathscr{S})$ be a T_1 syntopogenous space. If $(X^{\mathfrak{m}}, \mathscr{S}^{\mathfrak{m}})$ is SP-normal for any cardinal number $\mathfrak{m} \leq |\mathscr{S}^t|^i$, then $(X, \mathscr{S})$ is compact.*

PROOF. Assume that $(X^{\mathfrak{m}}, \mathscr{S}^{\mathfrak{m}})$ is SP-normal for any $\mathfrak{m} \leq |\mathscr{S}^t|^i$ but $(X, \mathscr{S})$ is not compact. Then there exists an $\mathscr{S}^c$-round compressed filter $\mathfrak{f}$ without cluster points in $(X, \mathscr{S})$ (see Lemma (5.3) of [7]). Put $\mathscr{S}^t = \{<_0\}$ and choose an ^{i}-separator $\mathfrak{S}$ for $<_0$ and $\mathscr{S}^t$ with $\bar{\bar{\mathfrak{S}}} \leq |\mathscr{S}^t|^i$. It is easy to see that we can select a filter base $\mathfrak{r} \subset \{X - S : S \in \mathfrak{S}\}$ for $\mathfrak{f}$, and then we have $\bar{\bar{\mathfrak{r}}} \leq |\mathscr{S}^t|^i$. Put $\mathfrak{r} = \{R_a : a \in A\}$, $\bar{\bar{A}} = \mathfrak{m}$. In the product space $(X^{\mathfrak{m}}, \mathscr{S}^{\mathfrak{m}})$ let us consider the diagonal D and the set $Z = \bigtimes_{a \in A} \bar{R}_a$. Then D and Z are closed sets with respect to $\mathscr{S}$ (see Lemmas 1, 4 and [1] (14.22)), besides $D \cap Z = \emptyset$ because $\mathfrak{f}$ (and at the same time $\mathfrak{r}$) has no cluster point. Owing to the SP-normality of $\mathscr{S}^{\mathfrak{m}}$, we get $D <' X^{\mathfrak{m}} - Z$ for some $<' \in \mathscr{S}^{\mathfrak{m}}$. Then $D \subset \bigcup_{h=1}^{k} \left(\bigtimes_{a \in A} P_{ha}\right)$, $\bigcup_{h=1}^{k} \left(\bigtimes_{a \in A} Q_{ha}\right) \subset X^{\mathfrak{m}} - Z$, there exist a finite subset $A_0 \subset A$ and orders $<_a \in \mathcal{S}$ $(a \in A_0)$ such that $P_{ha} <_a Q_{ha}$ for $a \in A_0$ and $P_{ha} = Q_{ha} = X$ for $a \in A - A_0$ $(1 \leq h \leq k)$. Now let $<$ be an element of $\mathcal{S}$ with $\bigcup_{a \in A_0} <_a \subset <$. Then, with the notations $P_h = \bigcap_{a \in A_0} P_{ha}$ and $\bigcap_{a \in A_0} Q_{ha} = Q_h$, we have $P_h < Q_h$ for $1 \leq h \leq k$. Observe that the sets p_h $(1 \leq h \leq k)$ cover X, therefore $\emptyset \notin \mathfrak{f}(\bigcap)\{P_h\}$ for at least one index h. This implies $Q_h \in \mathfrak{f}$ by the compressedness of $\mathfrak{f}$. Suppose $y \in Q_h \cap \bigcap_{a \in A_0} \bar{R}_a$. Define a point $x \in X^{\mathfrak{m}}$ by letting $x = (x_a)_{a \in A}$, where $x_a \in \bar{R}_a$ for any $a \in A$, in particular $x_a = y$ for $a \in A_0$. Then $x \in (\bigtimes_{a \in A} Q_{ha}) \cap Z$ and this is impossible.

COROLLARY. *Let* $(X, \mathcal{S})$ *be a* T_1 *syntopogenous space and* $\mathfrak{m} = |\mathcal{S}^t|^i$. *If* $(X^{\mathfrak{m}}, \mathcal{S}^{\mathfrak{m}})$ *is SP-normal, then* $(X, \mathcal{S})$ *is compact.*

PROOF. Use Lemma 1, [1], (11.15) and Lemma 6.

Now we can prove our

THEOREM. *The following statements are equivalent for any* T_1 *syntopogenous space* $(X, \mathcal{S})$:

(1) $(X, \mathcal{S})$ *is a compact* T_2*-space.*
(2) $(X \times X^*, \mathcal{S} \times \mathcal{S}^*)$ *is SP-normal.*
(3) $(X^{\mathfrak{m}}, \mathcal{S}^{\mathfrak{m}})$ *is SP-normal, where* $\mathfrak{m} = |\mathcal{S}^t|^i$.
(4) *The topology of the space* $(X^{\mathfrak{m}}, \mathcal{S}^{\mathfrak{m}})$ *is normal for any cardinal number* $\mathfrak{m}$.

PROOF. (1)⇒(2): In this case we have $X = X^*$ and $\mathcal{S} \sim \mathcal{S}^*$ by Lemma 4 and property (a), thus Lemma 3 can be applied. (2)⇒(1): Use Lemmas 1, 4 and 5. (1)⇒(3) by Lemma 3, and conversely, (3)⇒(1) issues from Lemmas 1, 4 and the Corollary of Lemma 6. (1)⇔(4): Applying the equality $(\bigtimes_{a \in A} \mathcal{S}_a)^{tp} = (\bigtimes_{a \in A} \mathcal{S}_a^{tp})^{tp}$ ([1], (11.23)), we can state the following equivalences: $(X, \mathcal{S})$ is a compact T_2-space ⇔ $(X, \mathcal{S}^{tp})$ is a compact T_2-space ⇔ $(\mathcal{S}^{\mathfrak{m}})^{tp}$ is normal for any cardinal number $\mathfrak{m}$ (see [10], Corollary 2.2, cf. [1, pp. 134—135]).

3. In order to verify the equivalence of the statements (A)—(D) in Section 1, it is sufficient to observe that they correspond to conditions (1)—(4) in the particular case when $\mathcal{S} = \{<\}$ is the topogenous structure associated with the proximity δ on X. In fact, (1)↔(A) and (4)↔(D) are quite clear. A proximity is equinormal iff the corresponding symmetrical topogeneous structure is *SP*-normal. Properties (a)—(d) of $(X^*, \mathcal{S}^*)$ show that it agrees with the double compactification of $(X, \mathcal{S})$ whenever $\mathcal{S}$ is symmetrical (see [3]). Since now $\mathcal{S}$ is topogenous, too, $(X^*, \mathcal{S}^*)$ is associated with the Smirnov compactification of (X, δ), thus (2)↔(B). The equivalence of (A), (B) and (B′) is obvious. Finally the correspondence (3)↔(C) can be seen on the basis of [1, pp. 136—137 and 297].

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A NOTE ON THE NILRADICAL N OF A RING R AND A SUBDIRECT SUM REPRESENTATION OF THE RING R/N

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Throughout this note R will be an associative nonzero ring which need not be unitary, and N will denote the set of all nilpotent elements of R. We say that R is *locally unitary* if for any element x in R there is an idempotent e_x in R with $e_x x = xe_x = x$. If we say that two positive integers m, n are relatively prime, then we mean also that they can not be both equal to 1.

Our aim is to prove the following

THEOREM. *Let R be an associative nonzero ring satisfying conditions*

$$(1) \qquad (x+y)^m = x^m + y^m$$

(m being a given positive integer) and

$$(2) \qquad x^m = x^n$$

($n = n(x)$ being a positive integer relatively prime to m). Then N is an ideal of R, and $N=0$ if and only if R is locally unitary. Moreover, if $N \neq R$ then R/N is a subdirect sum of fields and satisfies the condition

$$(3) \qquad (x+N)^r = x+N \quad (x \in R),$$

where $r = r(x)$ is a positive integer given by

$$(4) \quad r(x) = n(x)-m+1 \quad \text{if} \quad n(x) > m, \quad \text{and} \quad r(x) = m-n(x)+1 \quad \text{otherwise.}$$

For $m=1$ the conditions (1) and (2) reduce to a single condition

$$(5) \qquad x^n = x \quad (n = n(x) > 1)$$

implying according to Jacobson's theorem, the commutativity of R ([1], Theorem 3.1.2). We note that in any case the two conditions (1) and (2) can be replaced by a single condition

$$(6) \qquad (x+y)^m = x^m + y^n$$

(m being a given, and $n = n(y)$ a positive integer relatively prime to m). In 1961 Herstein [2] proved that for an associative ring R satisfying condition (1) for a given integer $m > 1$, N is an ideal of R, and R/N is commutative. But we will not use this fact, and we shall prove in a very elementary manner, that N is an ideal of R if R is an associative ring satisfying (1) for an integer $m > 1$. (See Lemma 1.)

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E. Psomopoulos considered recently [4] an associative unitary ring R satisfying condition (6) for a fixed positive integer n relatively prime to m. He has shown that such a ring R is a subdirect sum of fields. Our Theorem improves this result of Psomopoulos.

In the preparation of the proof of our Theorem, we prove two lemmas.

LEMMA 1. *Let R be an associative nonzero ring satisfying condition* (1). *If* $m>1$, *then N is an ideal of R.*

PROOF. As we have noted, this lemma is contained in a well known result due to Herstein ([2], Theorem 2). But we can give a very elementary direct proof. Let a, $b\in N$. Since $m>1$, there is a positive integer k such that $a^{m^k}=b^{m^k}=0$, and in view of (1)

$$a, b\in N \Rightarrow (a-b)^{m^k} = a^{m^k} \pm b^{m^k} = 0 \Rightarrow a-b\in N.$$

Let now $x\in R$ and $a\in N$. Then obviously $ax\in N \Leftrightarrow xa\in N$, and we will prove that $ax\in N$. We can assume that $a\neq 0$. There exists an integer $t>1$ such that

$$a^t = 0, \quad a^{t-1} \neq 0. \tag{7}$$

Therefore, according to (1)

$$(ax+a^{t-1})^m = (ax)^m + (a^{t-1})^m,$$

i.e. in view of (7),

$$(ax)^j a^{t-1} = 0 \quad (j \geqq m-1).$$

Similarly,

$$a^{t-1}(xa)^j = 0 \quad (j \geqq m-1).$$

Moreover, in view of (7) we also have

$$a^{t-1}(ax)^j = (xa)^j a^{t-1} = 0 \quad (j \geqq 1).$$

Hence there is a minimal nonnegative integer $k=k_0$ satisfying for some positive integer j, at least one of following four equations

$$(ax)^j a^k = 0, \; a^k(ax)^j = 0, \; (xa)^j a^k = 0, \; a^k(xa)^j = 0. \tag{8}$$

If $k_0=0$ we understand the corresponding equation in (8) as $(ax)^j=0$ or $(xa)^j=0$, hence in this case $ax\in N$. We prove that in fact k_0 must be equal to 0.

The value $k_0>1$ can be taken neither in the first nor in the last equation in (8), since otherwise we would have either

$$(xa)^{j+1} a^{k_0-1} = x(ax)^j a^{k_0} = 0,$$

or

$$a^{k_0-1}(ax)^{j+1} = a^{k_0}(xa)^j x = 0.$$

But this value can be taken also neither in the second nor in the third equation in (8). Assume it can be taken in the second. Then surely

$$(ax)^j a^{k_0} \neq 0 \quad (j \geqq 1),$$

and for the minimal value k_1 of k in the first equation in (8) we have $k_1>k_0$. According to the second equation in (8), from

$$(a^{k_0}+(ax)^j)^m = (a^{k_0})^m + (ax)^{jm}$$

it follows

$$(ax)^j a^{(m-1)k_0} + \ldots + (ax)^{(m-2)j} a^{2k_0} + (ax)^{(m-1)j} a^{k_0} = 0. \tag{9}$$

After multiplication on the left by $(ax)^{j_1}$, where $(ax)^{j_1} a^{k_1} = 0$, we multiply (9) on the right by $a^{k_1-k_0-1}$ to get

$$(ax)^{j_1+(m-1)j} a^{k_1-1} = 0,$$

hence a contradiction to the assumption on k_1 to be minimal. In a similar way it can be shown that $k_0 > 1$ can not be taken in the third equation in (8).

LEMMA 2. *Let R be an associative nonzero ring. If R satisfies*

(2′) $\quad x^m = x^n \quad$ *($n = n(x)$ being a positive integer different from m)*

and if $N = 0$, then R is locally unitary and satisfies condition

$$x^{r'+1} = x \quad (x \in R), \tag{3′}$$

where $r' = r'(x)$ is given by

(4′) $r'(x) = m(n(x)-m)$ *if* $n(x) > m$, *and* $r'(x) = n(x)(m-n(x))$ *otherwise.*

If R is locally unitary and has properties (1) *and* (2), *then* $N = 0$.

PROOF. 1) Let R have property (2′) and let $N = 0$. For any element x in R we set $e_x = x^{r'}$, where $r' = r'(x)$ is given by (4′). In view of (2′) it is easy to see that e_x is an idempotent. Namely, for $n = n(x) > m$,

$$e_x^2 = x^{m(n-m)} x^{m(n-m)} = x^{m(n-m)+(n-m)} x^{(m-1)(n-m)} =$$

$$= x^{m(n-m)} x^{(m-1)(n-m)} = \ldots = x^{m(n-m)} x^{n-m} = x^{m(n-m)} = e_x.$$

For $n = n(x) < m$ we conclude similarly that $e_x^2 = e_x$. It is clear that $e_x x = x e_x = e_x x e_x$. Thereby,

$$(x^2 e_x - x^2)^{r'} = x^{r'} (x e_x - x)^{r'} = (e_x x e_x - e_x x)(x e_x - x)^{r'-1} = 0,$$

and hence

$$x^2 e_x - x^2 = 0.$$

But then from

$$(x e_x - x)^2 = x^2 - x^2 e_x = 0$$

follows

$$e_x x = x e_x = x,$$

hence R is locally unitary.

Since $e_x = x^{r'}$, where $r' = r'(x)$ is given by (4′), we have also (3′).

2) Let now R be an associative nonzero locally unitary ring satisfying (1) and (2), and let a be any element of N, i.e. $a^m = a^{n(a)} = 0$. Repeating in fact the reasoning of ([3], Claim 2), we will prove that $a = 0$. Assume $a \neq 0$. Then there is an integer $t > 1$ such that (7) is valid. Moreover, there is an idempotent e_a in R with $e_a a = a e_a = a$. Thereby, in view of (1)

$$(e_a + a^{t-1})^m = e_a^m + (a^{t-1})^m, \quad \text{i.e.} \quad e_a + m a^{t-1} = e_a,$$

hence

$$m a^{t-1} = 0. \tag{10}$$

But according to (2),

$$(e_a+a^{t-1})^n = (e_a+a^{t-1})^m = e_a, \quad n = n(e_a+a^{t-1}),$$

hence because of (7)

$$na^{t-1} = 0. \tag{11}$$

Since m and n are relatively prime, from (10) and (11) follows $a^{t-1}=0$, in contrary to (7). Hence $a=0$, i.e. $N=0$.

Now we get to the proof of the theorem. If an associative ring R satisfies (1) and (2), then for $m=1$, $N=0$, and for $m>1$, N is an ideal of R according to Lemma 1. If $N\neq R$, the ring R/N is also an associative nonzero ring satisfying (1) and (2) (with $x+N$ and $y+N$ instead of x and y, where $x, y\in R$). Moreover, R/N has no nonzero nilpotent elements, and thus in view of Lemma 2, R/N is also locally unitary. It is well known that R/N can be represented as a subdirect sum of subdirectly irreducible rings R_i $(i\in I)$ ([1], Lemma 2.2.3). By definition of the subdirect sum, for any $i\in I$ there is an epimorphism f_i: $R\to R_i$. Consequently, any R_i satisfies (1) and (2) (with $f_i(x+N)$, $f_i(y+N)$ instead of x, y, where $x, y\in R$) and is also locally unitary. Therefore, none of R_i's has nonzero nilpotent elements, and hence none of R_i's has proper zero divisors ([3], Lemma 3). But then each R_i satisfies the condition

$$(f_i(x+N))^r = f_i(x+N) \quad (x\in R),$$

where $r=r(x)$ is given by (4). Each R_i is hence a division ring, and thus a field, according to the "division ring case" of Jacobson's commutativity theorem ([1], Lemma 3.1.3). Consequently, R/N is a subdirect sum of fields and R/N satisfies (3), where $r=r(x)$ is given by (4). Especially, R/N is a commutative ring, a fact following from (1) if $m>1$ ([2], Theorem 2). Finally, aecording to Lemma 2, $N=0$ if and only if R is locally unitary.

From the Theorem follows immediately

COROLLARY 1. *Any locally unitary associative nonzero ring R satisfying* (1) *and* (2) *can be represented as a subdirect sum of fields. Moreover, for any element x in such a ring R we have $x^r=x$, where $r=r(x)$ is given by* (4).

As a special case of this corollary we have the main result of [4]:

COROLLARY 2. *Any unitary associative ring R satisfying* (6) *is a subdirect sum of fields.*

REMARK 1. The fields R_i in the subdirect sum representation of R/N in the theorem and of R in the corollaries are of very special kind. Because of (3) in the case of the theorem and because of $x^r=x$ in case of the corollaries, the fields R_i are algebraic over corresponding prime fields P_i of characteristic $p_i\neq 0$. Moreover, if $m>1$, then by (1) p_i divides 2^m-2 and thus the set $\{p_i\colon i\in I\}$ is finite. If m is not a power of p_i, then for any x_i in R_i from $(x_i+e_i)^m=x_i^m+e_i$ we have an algebraic equation of degree $\leqq m-1$ over P_i, hence R_i is a finite field of at most p_i^{m-1} elements. But the fields R_i with $m=p_i^k$ can be infinite if the set $\{n(x)\colon x\in R\}$ of integers $n(x)$ in (2) is not bounded.

REMARK 2. For an associative nonzero ring R with the property (6) we can have $N=R$ even in the case when n is a fixed integer. For instance, let F be any field and let R be the ring of all upper triangular matrices of order $m+1$ over F having zeros on the main diagonal, where $m>1$ is a given integer. Then $N=R$ and R is an associative nonzero ring having property (6) for any given integer $n>m$ relatively prime to m. Moreover, this ring R is not commutative.

REMARK 3. Let R_1 be the ring from Remark 2, and let R_2 be a subdirect sum of fields satisfying (6) for some integer $n>m$ relatively prime to m. Then the direct sum $R=R_1\oplus R_2$ of R_1 and R_2 satisfies the same condition and has the nilradical $N\cong N_1$, hence different from 0 and from R. Moreover, the ring R is not commutative.

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[illegible] and (2), then for $m=1$, $N=0$, and for $m>1$, N is an ideal of R according to Lemma 1. If $N\neq R$, the ring R/N is also an associative nonzero ring satisfying (1) and (2) (with $x+N$ and $y+N$ [illegible] x and y, where $x, y\in R$). Moreover, R/N has no nonzero nilpotent elements, and thus in view of Lemma 2, R/N is also locally unitary [illegible] sum of subdirectly [illegible] the subdirect sum, for any [illegible] there is an [illegible] and is also locally unitary [illegible] elements, and hence none of R_i's has proper zero divisors ([3], Lemma 3). But then each R_i satisfies the condition

$$(\bar{x}_i(x+N))^r = \bar{x}_i(x+N) \quad (x\in R),$$

where $r=r(x)$ is given by (4). Each R_i is hence a [illegible] according to the "division ring case" of Jacobson's commutativity theorem ([1], Lemma 3.1.3). Consequently, R/N is a subdirect sum of fields and R/N satisfies (3), where $r=r(x)$ is given by (4). Especially, R/N is a commutative ring, a fact following from (1) if $m=1$ ([2], Theorem 2). Finally, according to Lemma 2, $N=0$ if and only if R is locally unitary.

From the Theorem follows immediately

Corollary 1. *Any locally unitary associative nonzero ring R satisfying (1) and (2) can be represented as a subdirect sum of fields. Moreover, for any element x in such a ring R we have $x^r=x$, where $r=r(x)$ is given by (4).*

As a special case of this corollary we have the main result of [4]:

Corollary 2. *Any unitary associative ring R satisfying (6) is a subdirect sum of fields.*

Remark 1. The fields R_i in the subdirect sum representation of R/N in the theorem and of R in the corollaries are of very special kind. Because of (3) in the case of the theorem and because of $x^r=x$ in case of the corollaries, the fields R_i are algebraic over corresponding prime fields P_i of characteristic $p_i\neq 0$. Moreover, if $m>1$, then by (1) p_i divides 2^m-2 and thus the set $\{p_i: i\in I\}$ is finite. If m is not a power of p_i, then for any x_i in R_i from $(x_i+e_i)^m=x_i^m+e_i$ we have an algebraic equation of degree $\leq m-1$ over P_i, hence R_i is a finite field of at most p_i^{m-1} elements. But the fields R_i with $m=p_i^s$ can be infinite if the set $\{n(x): x\in R\}$ of integers $n(x)$ in (2) is not bounded.

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ON COUNTABLE CODIMENSIONAL SUBSPACES IN ULTRA-(DF) SPACES

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As it is known [3, p. 35] a linear subspace of finite codimension in an Ultra-(DF) space is again an Ultra-(DF) space. In this paper it is proved that an infinite countable codimensional subspace G of an Ultra-(DF) space E is an Ultra-(DF) space, provided G has the property (b), i.e. for every bounded subset B of E the codimension of G in the linear span of $G \cup B$ is finite. It is shown also that the property of Ultra-(DF) spaces is maintained in subspaces of infinite countable codimension, when the initial Ultra-(DF) space is sequentially complete and boundedly summing (in the sense of [1]), or an ultrabarrelled space. In particular the Valdivia's result of [9] is obtained.

Introduction

In [5] Grothendieck introduced the class of (DF) spaces using for their definition properties of the strong dual of a Fréchet space. A locally convex space E is said to be a (DF) *space* if it has a fundamental sequence of bounded sets and every strongly bounded subset M of E' which is the union of countably many equicontinuous sets is also equicontinuous. It is well-known that this definition can be replaced by the following one: E is a (DF) space if it is an $\aleph_0$-quasibarrelled space and has a fundamental sequence of bounded sets. As observed in [7, p. 401] a subspace of a (DF) space need not be a (DF) space. In [9] Valdivia proved that the (DF) property is hereditary to a subspace of finite codimension. The property is also maintained in subspaces of infinite countable codimension, when the initial (DF) space is sequentially complete, or a barrelled space. In [4] Ernst extended the notion of (DF) spaces to the class of all topological vector spaces (tvs) in the following way: A tvs E is an *Ultra-(DF) space* if it is an $\aleph_0$-quasiultrabarrelled space and has a fundamental sequence of bounded balanced sets.

Following [1] a sequence (U_j) of balanced absorbent subsets of a linear space E is called a *string* if $U_{j+1}+U_{j+1} \subset U_j$ for all $j \in \mathbf{N}$. A string (U_j) in a tvs E is *closed,* if every U_j is closed in E; *bornivorous,* if every U_j absorbs all bounded sets of E; *topological,* if every U_j is a neighbourhood of zero in E.

We recall [1] that a tvs E is *ultrabarrelled (quasiultrabarrelled)* if every closed (and bornivorous) string in E is topological; E is $\aleph_0$*-quasiultrabarrelled* if every bornivorous string, which is the intersection of $\aleph_0$ closed topological strings, is topological in E. In [1] the authors studied these spaces under the names quasibarrelled, barrelled. A fundamental sequence (B_n) of bounded closed and balanced sets we shall call an *f-sequence*. Without loss of generality we may assume that $B_n+B_n \subset B_{n+1}$ for all $n \in \mathbf{N}$.

Throughout, we consider only Hausdorff tvs over the field **K** of the real or complex scalars. A tvs E with the topology τ is denoted by (E, τ) or simply by E. By $(G, \tau|G)$ we shall denote a subspace of E endowed with the induced topology. For a subset A of (E, τ) by $\bar{A}$ we shall denote the closure of the set A with respect to the topology τ.

Results

We shall need the following

PROPOSITION 1 (see [6, Lemma 2.6]). *Let E be a tvs and G its subspace of finite codimension with a co-base $(x_1, x_2, ..., x_p)$. If (B_n) is an f-sequence in E, then $(\overline{B_n \cap G + 2^n\{\sum_{i=1}^{p} a_i x_i : |a_i| \leqq 1\}})$ is also an f-sequence in E.*

The locally convex space case was obtained also by Ruess [8], Theorem 2.5.

COROLLARY 1. *Let E be a tvs and G its subspace of infinite countable codimension in E with a co-base (x_n). If G has the property* (b), *then $(\overline{B_n \cap G + 2^n\{\sum_{i=1}^{n} a_i x_i : |a_i| \leqq 1\}})$ is an f-sequence in E, provided (B_n) is such a sequence in E.*

Recall that according to [1], p. 74, a tvs E is *boundedly summing* if for every bounded subset B of E there exists a sequence of scalars (a_n), $a_n \neq 0$, such that $\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} a_k B$ is bounded. Every metrizable tvs is boundedly summing. Almost convex, locally convex, locally pseudoconvex spaces are boundedly summing.

PROPOSITION 2. *A tvs (E, τ) is boundedly summing if and only if for every bounded and balanced subset B of E there exists a metrizable tvs Y and a one-to-one continuous linear map T of Y into E such that $T^{-1}(B)$ is bounded and $B \subset T(Y)$.*

PROOF. Suppose E is boundedly summing and let B be a bounded and balanced subset of E. There exists a sequence (a_n) of **K** with $a_n \neq 0$, $a_{n+1} \leqq a_n$, and such that $\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} a_k B$ is bounded. Let E_B be a linear span of B. Putting $V_i := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} a_{k_2 i} B$, $i \in \mathbf{N}$, we obtain a metrizable linear topology τ_B generated by the string (V_i). Clearly $\tau|E_B \leqq \tau_B$ and B is τ_B-bounded. Since every metrizable tvs is boundedly summing, the proof of the second part of our Proposition 2 is obvious.

COROLLARY 2. *Let E be a sequentially complete tvs and let (A_n) be a sequence of closed and balanced subsets of E whose union is E. Let B be a bounded balanced subset of E. If E is boundedly summing, or B is absolutely convex, then there exist $a \in \mathbf{K}$ and $m \in \mathbf{N}$ such that $B \subset a(A_m + A_m)$.*

PROOF. The case when B is absolutely convex is well-known, for instance see [8, Lemma 1.1]. We prove the boundedly summing space case. There exists a metrizable tvs Y and a one-to-one continuous linear map T of Y into E such that $T^{-1}(B)$ is bounded and $B \subset T(Y)$. By $\tilde{T}$ we denote the extension map of T from the completion $\tilde{Y}$ of Y into E. To conclude the proof it is enough to apply the fact that $\tilde{Y}$ is a Baire space and $\tilde{T}^{-1}(A_n)$ covers $\tilde{Y}$.

We shall need the following fact (cf. also [1, (8), (9), p. 267—268]).

PROPOSITION 3. *Let* (E, τ) *be an ultrabarrelled tvs and let* (B_n) *be a sequence of closed bounded and balanced subsets of E whose union is E and such that* $B_n+B_n\subset \subset B_{n+1}$ *for all* $n\in\mathbf{N}$. *Then* (B_n) *is an f-sequence in E.*

PROOF. Suppose that there exists a τ-bounded set B which is contained in no B_n. For every $n\in\mathbf{N}$ there exists $x_n\in n^{-1}B$, $x_n\notin B_n$ such that $x_n\to 0$. There exists a sequence (U_j^n) of τ-topological strings in E such that $x_n\notin B_n+U_2^n$. Putting $V_j:=\bigcap_{n=j}^{\infty}\overline{(B_{n-j+1}+U_{j+1}^n)}$ we obtain a τ-topological string, but $x_n\notin V_1$ for all $n\in\mathbf{N}$, a contradiction.

PROPOSITION 4. *Let* (E, τ) *be an* $\aleph_0$*-quasiultrabarrelled tvs with an f-sequence* (B_n) *and let* G *be its dense subspace of infinite countable codimension. If* G *has property* (b), *or if E is a sequentially complete boundedly summing space, or if E is ultrabarrelled, then* $(B_n\cap G)$ *is an f-sequence in E.*

PROOF. Our proof uses an idea from [8], Proposition 3.5. Let $F:=\bigcup_{n=1}^{\infty}\overline{B_n\cap G}$ and let γ be the finest linear topology on E agreeing with τ on the sets $\overline{B_n\cap G}$. Observe that $\gamma|F$ is the finest linear topology on F agreeing with $\tau|F$ on the sets $\overline{B_n\cap G}$. Indeed, let H be an algebraic complement of F in E and let ϑ be an arbitrary linear topology on H. If ϱ denotes the finest linear topology on F such that $\varrho|\overline{B_n\cap G}= =\tau|\overline{B_n\cap G}$, $n\in\mathbf{N}$, then the product topology $\alpha:=\varrho\times\vartheta$ satisfies $\alpha|F=\varrho$. Hence we obtain $\varrho=\gamma|F$. Now we prove that $\tau=\gamma$. Let (x_n) be a co-base of G in E. In view of Corollary 1, Corollary 2 and Proposition 3 the sequence of the sets $K_n:=\overline{B_n\cap G}+2^n\{\sum_{i=1}^{n}a_ix_i\colon |a_i|\leqq 1\}$ composes an f-sequence in E. Moreover, by Lemma 2.6 of [8], we have $\gamma|K_n=\tau|K_n$ for all $n\in\mathbf{N}$. Since E is an $\aleph_0$-quasiultrabarrelled space, in view of [1, (7), p. 87], we have $\tau=\gamma$. Applying [1, (11), p. 89] we obtain $E=\bar F=\bigcup_{n=1}^{\infty}\overline{B_n\cap G}$. We complete the proof using again Corollary 2 for the compact absolutely convex sets $2^n\{\sum_{i=1}^{n}a_ix_i\colon |a_i|\leqq 1\}$.

THEOREM 1. *Let E be an Ultra-(DF) space and G its subspace of infinite countable codimension. If* G *has property* (b), *or if* E *is a sequentially complete boundedly summing space, or if E is ultrabarrelled, then G is an Ultra-(DF) space.*

PROOF. Suppose G is closed in E. Let (x_n) be a co-base of G in E. For every $n\in\mathbf{N}$ let $G_n:=G+\operatorname{lin}\{x_1, x_2, \dots, x_n\}$. By Corollary 1 and Proposition 2 every bounded subset of E is contained in some G_n. Hence on account of [1, (7), p. 87] E is the strict inductive limit space of the sequence (G_n). This implies that G has a topological complement in E, and G is isomorphic to a quotient space of E by a closed subspace of countable dimension. Hence G is an Ultra-(DF) space. Let G be dense in E and let (U_j^n) be a sequence of topological closed strings in G such that $U_j=\bigcap_{n=1}^{\infty}U_j^n$ is bornivorous in E. Since $\bigcap_{n=1}^{\infty}\overline{U_j^n}\supset\overline{\bigcap_{n=1}^{\infty}U_j^n}=\overline{U}_j$ for all $j\in N$, then

$(\bigcap_{n=1}^{\infty} \bar{U}_j^n)$ is bornivorous in E. Hence $(\bigcap_{n=1}^{\infty} \bar{U}_j^n)$ is a topological string in E, and so (U_j) is topological in G. If G is neither closed nor dense in E, the proof of this case is an easy consequence of the above cases.

Since every locally convex tvs is a (DF) space if and only if it is an Ultra-(DF) space, [2, (5), p. 260], we obtain the following Valdivia's result of [9].

COROLLARY 3. *Let E be a sequentially complete locally convex (DF) space and let G be its subspace of countable codimension. Then G is a (DF) space.*

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PUNKTWEISE ABSCHÄTZUNGEN ZUR APPROXIMATION DURCH ALGEBRAISCHE POLYNOME

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1. Einleitung

Im Jahre 1951 verfeinerte A. F. Timan [11] den wohlbekannten Satz von Jackson in der Weise, daß er zu jedem $f \in C^r[-1, 1]$, $r \geqq 0$, eine Folge von Polynomen vom Grad $\leqq n$ angab, die für alle $|x| \leqq 1$ den Ungleichungen

$$(1.1)\qquad |f(x)-p_n(x)| \leqq c\Delta_n(x)^r \omega_1(f^{(r)}, \Delta_n(x))$$

genügen. Hierbei ist $\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$, und $\omega_1(f^{(r)}, \cdot)$ bezeichnet den ersten Stetigkeitsmodul der r-ten Ableitung von f. In der Folgezeit ist diese Aussage in einer Reihe von Arbeiten verbessert worden. Die Untersuchungen gingen dabei im wesentlichen in drei Richtungen:

A. *Konstruktion von Polynomen, die f simultan approximieren.* Ein typisches Resultat in dieser Richtung ist dabei (vgl. R. M. Trigub [12]) die Ungleichung

$$(1.2)\qquad |f^{(k)}(x)-p_n^{(k)}(x)| \leqq c\Delta_n(x)^{r-k}\omega_1(f^{(r)}, \Delta_n(x)), \quad 0 \leqq k \leqq r.$$

B. *Ersatz von $\Delta_n(x)$ durch $\Gamma_n(x) := \frac{\sqrt{1-x^2}}{n}$.* Hier wurde die Ungleichung

$$(1.3)\qquad |f(x)-p_n(x)| \leqq c\Gamma_n(x)^r \omega_1(f^{(r)}, \Gamma_n(x))$$

zuerst von S. A. Telyakovskiĭ [10] und I. E. Gopengauz [5] angegeben.

C. *Ersatz von $\omega_1(f^{(r)}, \cdot)$ durch einen Stetigkeitsmodul höherer Ordnung $\omega_s(f^{(r)}, \cdot)$, $s \geqq 2$.* Ein allgemeines Resultat in dieser Richtung stammt von Ju. A. Brudnyi [1]; die Ungleichung lautet

$$(1.4)\qquad |f(x)-p_n(x)| \leqq c\Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x)).$$

Im Jahre 1967 bewies I. E. Gopengauz [5, 6] Kombinationen von (1.2) und (1.3), sowie von (1.2) und (1.4). Er zeigte Ungleichungen vom Typ

$$(1.5)\qquad |f^{(k)}(x)-p_n^{(k)}(x)| \leqq c\Gamma_n(x)^{r-k}\omega_1(f^{(r)}, \Gamma_n(x)), \quad 0 \leqq k \leqq r,$$

und

$$(1.6)\qquad |f^{(k)}(x)-p_n^{(k)}(x)| \leqq c\Delta_n(x)^{r-k}\omega_s(f^{(r)}, \Delta_n(x)), \quad 0 \leqq k \leqq r.$$

Hieraus ergab sich die Frage nach weiteren Verallgemeinerungen und Kombinationsmöglichkeiten der oben skizzierten Aussagen. Beiträge in dieser Richtung stammen von R. A. DeVore [2, 3] und den Autoren [8], die in Verallgemeinerung von (1.3) Abschätzungen des Typs

$$(1.3') \qquad |f(x)-p_n(x)| \leqq c\Gamma_n(x)^r \omega_2(f^{(r)}, \Gamma_n(x)), \quad r \geqq 0,$$

bewiesen.

In der vorliegenden Arbeit soll zum einen gezeigt werden, daß eine Verallgemeinerung von (1.3) in Richtung auf (1.4) möglich ist, nämlich

$$(1.7) \qquad |f(x)-p_n(x)| \leqq c\Gamma_n(x)^r \omega_s(f^{(r)}, \Gamma_n(x)) \quad \text{für} \quad r \geqq s-2$$

erreicht werden kann. Zum anderen werden wir für $r \geqq s \geqq 1$ eine Kombination von (1.2), (1.3) und (1.4) beweisen und so zu einer Aussage vom Typ

$$(1.8) \qquad |f^{(k)}(x)-p_n^{(k)}(x)| \leqq c\Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)), \quad 0 \leqq k \leqq r-s,$$

gelangen. Die zugrundeliegenden Approximationsprozesse Q_n sind dabei sämtlich linear und nach einem einheitlichen Aufbauprinzip konstruiert.

In der gesamten Arbeit ist $\|f\| = \sup\{|f(x)|: |x| \leqq 1\}$. Alle Konstanten hängen nur von den angegebenen Indizes, nicht aber von f, x oder n ab.

2. Glättung von Funktionen

Ein wichtiges Hilfsmittel bei den Beweisen unserer Sätze wird das Prinzip der Glättung differenzierbarer Funktionen durch solche mit höherer Differenzierbarkeit sein. Das folgende Resultat geht auf M. W. Müller [9] zurück.

SATZ 2.1. *Für $f \in C^r[-1, 1]$, $r \in \mathbf{N}_0$, beliebiges $h \in (0, 2]$ und jede natürliche Zahl $r+s > r$ existiert eine Funktion $F_h = F_{h,r+s} \in C^{2r+s}[-1, 1]$ mit den Eigenschaften:*

$$\text{(i)} \qquad \|f^{(i)} - F_{h,r+s}^{(i)}\| \leqq c\omega_{r+s-i}(f^{(i)}, h), \quad 0 \leqq i \leqq r,$$

und

$$\text{(ii)} \qquad \|F_{h,r+s}^{(r+s)}\| \leqq ch^{-(r+s)} \omega_{r+s}(f, h).$$

Hierbei hängt die Konstante c nur von $r+s$ ab.

Für unsere weiteren Untersuchungen benötigen wir folgendes Korollar aus Satz 2.1, welches sich aus elementaren Eigenschaften höherer Glattheitsmoduln ergibt. In seiner Formulierung benutzen wir die zusätzliche Konvention $\omega_0(f, \cdot) = \|f\|$.

KOROLLAR 2.2. *Für natürliche Zahlen $r, s \geqq 0$, $f \in C^r[-1, 1]$ und beliebiges $h \in (0, 2]$ existiert eine Funktion $F_h = F_{h,r+s} \in C^{2r+s}[-1, 1]$, so daß gilt:*

$$\text{(i)} \qquad \|f^{(i)} - F_h^{(i)}\| \leqq c_{r,s} h^{r-i} \omega_s(f^{(r)}, h) \quad \textit{für} \quad 0 \leqq i \leqq r,$$

$$\text{(ii)} \qquad h^s \|F_h^{(r+s)}\| \leqq c_{r,s} \omega_s(f^{(r)}, h).$$

3. Folgerungen aus der Ungleichung von Dzjadyk

Wir werden unten verschiedentlich von folgender Ungleichung Gebrauch machen.

SATZ 3.1 (V. K. Dzjadyk [4, Theorem 7.1.3]). *Wenn für eine reelle Zahl p das algebraische Polynom p_n vom Grad n für alle Punkte $x \in [-1, 1]$ der Abschätzung*

$$|p_n(x)| \leqq M\Delta_n(x)^p, \quad M = \text{const.},$$

genügt, so erfüllt für eine beliebige natürliche Zahl k seine k-te Ableitung die Ungleichung

$$|p_n^{(k)}(x)| \leqq A_1 M\Delta_n(x)^{p-k},$$

wobei A_1 eine Konstante ist, die nur von p und k abhängt.

SATZ 3.2. *Sei $r \geqq 0$. Ist $f \in C^r[-1, 1]$ und p_n ein Polynom vom Grad $n \geqq r$, so daß für alle $|x| \leqq 1$ gilt:*

$$|f(x)-p_n(x)| \leqq c_r \Delta_n(x)^r \|f^{(r)}\|,$$

dann ist für $0 \leqq k \leqq r$

$$|f^{(k)}(x)-p_n^{(k)}(x)| \leqq M_r \Delta_n(x)^{r-k} \|f^{(r)}\|.$$

BEWEIS. Lemma 1 in R. M. Trigubs Arbeit [12] zeigt, daß für $n \geqq r$, $r \geqq 0$ und $f \in C^r[-1, 1]$ stets ein Polynom $q_n \in \Pi$ existiert, so daß für $0 \leqq k \leqq r$ und $|x| \leqq 1$ die Ungleichungen

$$|f^{(k)}(x)-q_n^{(k)}(x)| \leqq \frac{1}{2} A_r \Delta_n(x)^{r-k} \omega_1\big(f^{(r)}, \Delta_n(x)\big) \leqq A_r \Delta_n(x)^{r-k} \|f^{(r)}\|$$

gelten. Nun ist $p_n - q_n \in \pi_n$ und

$$|p_n(x)-q_n(x)| \leqq (c_r + A_r) \Delta_n(x)^r \|f^{(r)}\|.$$

Satz 3.1 impliziert für $0 \leqq k \leqq r$ die Abschätzungen

$$|p_n^{(k)}(x)-q_n^{(k)}(x)| \leqq B_r \Delta_n(x)^{r-k} \|f^{(r)}\|.$$

Also ergibt sich

$$|f^{(k)}(x)-p_n^{(k)}(x)| \leqq |f^{(k)}(x)-q_n^{(k)}(x)| + |q_n^{(k)}(x)-p_n^{(k)}(x)| \leqq$$

$$\leqq (A_r + B_r) \Delta_n(x)^{r-k} \|f^{(r)}\|. \quad \square$$

SATZ 3.3. *Sei $r \geqq 0$ eine natürliche Zahl und $P_n \colon C[-1, 1] \to \Pi_n$, $n \geqq r$, eine Folge linearer Operatoren, so daß gilt:*

(i) $\|P_n f\| \leqq c_2 \|f\|$ *für alle* $f \in C[-1, 1]$,

und

(ii) $|f(x)-P_n f(x)| \leqq c_1 \Delta_n(x)^r \|f^{(r)}\|$ *für alle* $|x| \leqq 1$ *und alle* $f \in C^r[-1, 1]$.

Dann gilt für $0 \leqq k \leqq r$ und alle $f \in C^r[-1, 1]$

$$\|(P_n f)^{(k)}\| \leqq c_3 \|f^{(k)}\|.$$

Wenn c_1 und c_2 nicht von n oder f abhängen, so auch c_3 nicht.

BEWEIS. Für $r=0$ ist die Aussage offensichtlich richtig; sei also $r\geqq 1$. Sei k fest in $\{1, \ldots, r\}$, x fest in $[-1, 1]$ gewählt und $h\in(0, 2]$ vorgegeben. Sei $r_k:=k$ und $s_k:=r-k$. Aufgrund von (ii) gilt für alle $f\in C^r[-1, 1]=C^{r_k+s_k}[-1, 1]$:

$$|f(x)-P_nf(x)| \leqq c_1\Delta_n(x)^r \|f^{(r_k+s_k)}\|.$$

Wir benutzen Korollar 2.2 mit $r=r_k$ und $s=s_k$. Also existiert eine Funktion $F_h=F_{h,r}\in C^{r_k+s_k}[-1, 1]$, so daß gilt:

$$\text{(i)} \qquad \|f^{(j)}-F_h^{(j)}\| \leqq c_{r_k,s_k} h^{r_k-j}\omega_{s_k}(f^{(r_k)}, h) \quad \text{für} \quad 0 \leqq j \leqq r_k$$

und

$$\text{(ii)} \qquad h^{s_k}\|F_h^{(r_k+s_k)}\| \leqq c_{r_k,s_k}\omega_{s_k}(f^{(r_k)}, h).$$

Damit ist

$$\begin{aligned}|f(x)-P_nf(x)| &\leqq |f(x)-F_h(x)|+|F_h(x)-P_nF_h(x)|+|P_n(F_h-f)(x)| \leqq \\ &\leqq \|f-F_h\|+c_1\Delta_n(x)^r\|F_h^{(r)}\|+c_2\|F_h-f\| \leqq \\ &\leqq (1+c_2)c_{r_k,s_k}h^{r_k}\omega_{s_k}(f^{(r_k)}, h)+c_1\Delta_n(x)^r c_{r_k,s_k}h^{-s_k}\omega_{s_k}(f^{(r_k)}, h).\end{aligned}$$

Wählen wir nun $h=\Delta_n(x)$, so ist also

$$\begin{aligned}|f(x)-P_nf(x)| &\leqq (1+c_2+c_1)c_{r_k,s_k}\Delta_n(x)^k\omega_{r-k}(f^{(k)}, \Delta_n(x)) \leqq \\ &\leqq 2^r c_4\Delta_n(x)^k\|f^{(k)}\|.\end{aligned}$$

Satz 3.2 impliziert nun

$$|f^{(j)}(x)-(P_nf)^{(j)}(x)| \leqq c_5\Delta_n(x)^{k-j}\|f^{(k)}\| \quad \text{für} \quad 0 \leqq j \leqq k.$$

Die spezielle Wahl $j=k$ liefert also

$$|f^{(k)}(x)-(P_nf)^{(k)}(x)| \leqq c_5\|f^{(k)}\|,$$

was wiederum

$$\|(P_nf)^{(k)}\| \leqq (1+c_5)\|f^{(k)}\| = c_3\|f^{(k)}\|$$

impliziert. Da diese Überlegung für $k=1, \ldots, r$ durchgeführt werden kann, ergibt sich die Behauptung. □

4. Ein Satz vom Brudnyĭ—Teljakovskiĭ-Typ

In diesem Abschnitt beweisen wir die Gültigkeit einer Kombination von (1.3) und (1.4). Wesentliches Hilfsmittel ist dabei Satz 4.2, dessen Beweis auch die Konstruktion der Operatoren Q_n enthält, für die wir dann im Satz 4.3 die gewünschte Ungleichung beweisen.

Zentral für die Konstruktion der Q_n wiederum ist das folgende Lemma, welches in ganz ähnlicher Form bei R. M. Trigub [12] zu finden ist.

LEMMA 4.1. *Seien* $n^*, m, p\in\mathbf{N}_0$ *mit* $n^*, m\geqq 1$. *Dann existiert ein Polynom* T_N *vom Grad* $N\leqq(2(m-1)+p)(n^*-1)/2$, *so daß für alle* $|x|\leqq 1$ *die Ungleichung*

$$|x^p-x^{p+2m}T_N(x^2)| \leqq \frac{c_{p,m}}{(n^*)^p}$$

erfüllt ist.

SATZ 4.2. *Sei* $r \geqq 0$, $s \geqq 1$, $n \geqq \max(4(r+1), r+s)$. *Dann gibt es lineare Operatoren* $Q_n = Q_n^{(r,s)}: C^r[-1,1] \to \Pi_n$, *so daß gilt:*

(i) $(Q_n f)^{(k)}(\pm 1) = f^{(k)}(\pm 1)$ *für alle* $f \in C^r[-1,1]$ *und* $0 \leqq k \leqq r$,

(ii) $|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqq A_{r,s} \Delta_n(x)^{r+s-k} \|f^{(r+s)}\|$

für alle $f \in C^{r+s}[-1,1]$, $|x| \leqq 1$ *und* $0 \leqq k \leqq r+s$.

BEWEIS. 1. Schritt: Konstruktion der Operatoren Q_n.

a) Sei $\bar{L}_n$: $C[-1,1] \to \Pi_n$ der lineare Operator von DeVore (oder ein anderer Operator), der für $n \geqq p-1$ der Ungleichung

$$|f(x) - \bar{L}_n f(x)| \leqq c_p \omega_p(f, \Delta_n(x)), \quad f \in C[-1,1], \quad |x| \leqq 1,$$

genügt. Setze $p = r+s$.

b) Sei R_1: $C^r[-1,1] \to \Pi_{2r+1}$ der Hermite-Interpolations-Operator, so daß für $0 \leqq k \leqq r$ gilt:

$$(R_1 f)^{(k)}(\pm 1) = f^{(k)}(\pm 1).$$

Dann hat $R_1 f$ die Darstellung (vgl. [10]):

$$R_1 f(x) = \sum_{i=0}^{r} (1-x^2)^i \{f^{(i)}(1) A_i(x) + f^{(i)}(-1) B_i(x)\},$$

wobei $A_i, B_i \in \Pi_{2(r-i)+1}$ und $\|A_i\| \leqq a_r$, $\|B_i\| \leqq b_r$ für $0 \leqq i \leqq r$.

c) Sei $R_{2,n}$: $C^r[-1,1] \to \Pi_n$ gegeben durch

$$R_{2,n}(f,x) = \sum_{i=0}^{r} (1-x^2)^{r+1+i-[i/2]} T_{N_i}(1-x^2) \{f^{(i)}(1) A_i(x) + f^{(i)}(-1) B_i(x)\}$$

mit A_i und B_i aus b) und den Trigub-Polynomen T_{N_i} aus Lemma 4.1, die zu den Wahlen $n^* = \left[\frac{n}{4(r+1)}\right]$, $m = r+1-\left[\frac{i}{2}\right]$ und $p = i$ gehören. Man verifiziert, daß $R_{2,n}$ wegen $n \geqq 4(r+1)$ tatsächlich in Π_n abbildet und daß ferner die Gleichungen $(R_{2,n} f)^{(k)}(\pm 1) = 0$, $0 \leqq k \leqq r$, erfüllt sind.

d) Wir definieren den linearen Operator R_n: $C^r[-1,1] \to \Pi_n$ durch $R_n := R_1 - R_{2,n}$ und die Boolesche Summe Q_n von R_n und $\bar{L}_n$ durch $Q_n := R_n \oplus \bar{L}_n := R_n + \bar{L}_n - R_n \circ \bar{L}_n$ (vgl. z. B. W. Gordon [7]). Dann gilt für jedes $f \in C^r[-1,1]$ und $0 \leqq k \leqq r$:

$$(Q_n f)^{(k)}(\pm 1) = (R_n f)^{(k)}(\pm 1) + (\bar{L}_n f)^{(k)}(\pm 1) - (R_n(\bar{L}_n f))^{(k)}(\pm 1) =$$

$$= f^{(k)}(\pm 1) + (\bar{L}_n f)^{(k)}(\pm 1) - (\bar{L}_n f)^{(k)}(\pm 1) = f^{(k)}(\pm 1).$$

Also erfüllen die Operatoren Q_n die Eigenschaft (i).

2. Schritt: Abschätzung von $|f^{(k)}(x) - (Q_n f)^{(k)}(x)|$. Wir haben für $0 \leqq k \leqq r+s$ die Differenz

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqq$$

$$\leqq |f^{(k)}(x) - (\bar{L}_n f)^{(k)}(x)| + |(R_n \circ (\mathrm{Id} - \bar{L}_n)(f))^{(k)}(x)| := \Pi_{1,k}(x) + \Pi_{2,k}(x)$$

zu betrachten.

Nach obiger Wahl von $\bar{L}_n$ (für $p=r+s$) ergibt sich, daß für $r\geqq 0$ und $n\geqq p-1=$ $=r+s-1$ die Ungleichung

$$|f(x)-\bar{L}_n f(x)|\leqq c_{r+s}\omega_{r+s}(f, \Delta_n(x))\leqq c_{r+s}\Delta_n(x)^{r+s}\|f^{(r+s)}\|$$

für $f\in C^{r+s}[-1, 1]$ gilt. Satz 3.2 impliziert nun für $n\geqq r+s$

$$\Pi_{1,k}(x)=|f^{(k)}(x)-(\bar{L}_n f)^{(k)}(x)|\leqq M_{r+s}\Delta_n(x)^{r+s-k}\|f^{(r+s)}\|$$

für $0\leqq k\leqq r+s$.

Unter Benutzung von Lemma 4.1 ergibt sich ferner für den zweiten Summanden

$$\Pi_{2,0}(x)=\Big|\sum_{i=0}^{r}(\sqrt{1-x^2})^i\big[(\sqrt{1-x^2})^i-(\sqrt{1-x^2})^{2(r+1)+i-2[i/2]}T_{N_i}(1-x^2)\big]\cdot$$

$$\cdot[(f-\bar{L}_n f)^{(i)}(1)A_i(x)+(f-\bar{L}_n f)^{(i)}(-1)B_i(x)]\Big|\leqq$$

$$\leqq\sum_{i=0}^{r}(\sqrt{1-x^2})^i\left|(\sqrt{1-x^2})^i-(\sqrt{1-x^2})^{2(r+1)+i-2[i/2]}T_{N_i}(1-x^2)\right|\cdot$$

$$\cdot M_{r+s}\left(\frac{1}{n^2}\right)^{r+s-i}\|f^{(r+s)}\|(a_r+b_r)\leqq$$

$$\leqq\sum_{i=0}^{r}c_{i,r}M_{r+s}(a_r+b_r)(\sqrt{1-x^2})^i\frac{1}{n^i}\Delta_n(x)^{r+s-i}\|f^{(r+s)}\|\leqq d_{r,s}\Delta_n(x)^{r+s}\|f^{(r+s)}\|.$$

Satz 3.1 liefert dann

$$\Pi_{2,k}(x)\leqq A_{r,s}\Delta_n(x)^{r+s-k}\|f^{(r+s)}\|\quad \text{für}\quad 0\leqq k\leqq r+s.\quad\square$$

Wir kommen nun zum Hauptergebnis dieses Abschnitts.

SATZ 4.3. *Es sei* $s\geqq 1$, $r\geqq s-2$ *und* $n\geqq 4(r+1)$. *Dann genügen die linearen Operatoren aus Satz* 4.2 *mit* $\Gamma_n(x)=\dfrac{\sqrt{1-x^2}}{n}$ *der Ungleichung*

$$|f(x)-Q_n f(x)|\leqq d_{r,s}\Gamma_n(x)^r\omega_s(f^{(r)}, \Gamma_n(x))$$

für alle $f\in C^r[-1, 1]$ *und alle* $|x|\leqq 1$.

BEWEIS. Sei x fest in $(-1, 1)$ und $h\in(0, 2]$ vorgegeben. Wir benutzen Korollar 2.2. Danach existiert eine Funktion $F_h\in C^{r+s}[-1, 1]$ mit

(i) $\|f^{(k)}-F_h^{(k)}\|\leqq c_{r,s}h^{r-k}\omega_s(f^{(r)}, h)$ für $0\leqq k\leqq r$,

sowie

(ii) $h^s\|F_h^{(r+s)}\|\leqq c_{r,s}\omega_s(f^{(r)}, h)$.

Unter Benutzung von F_h zerlegen wir $|f(x)-Q_n f(x)|$ in

$$|f(x)-F_h(x)|+|F_h(x)-Q_n F_h(x)|+|Q_n F_h(x)-Q_n f(x)|\leqq$$

$$\leqq c_{r,s}h^r\omega_s(f^{(r)}, h)+|F_h(x)-Q_n F_h(x)|+|Q_n F_h(x)-Q_n f(x)|.$$

Zur Abschätzung von $|F_h(x)-Q_nF_h(x)|$ betrachten wir zunächst ein beliebiges $g\in C^{r+s}[-1, 1]$. Satz 4.2 (dort $n\geqq\max\{4(r+1), r+s\}$) liefert

$$|g^{(k)}(x)-(Q_ng)^{(k)}(x)|\leqq A_{r,s}\Delta_n(x)^{r+s-k}\|g^{(r+s)}\| \quad \text{für} \quad 0\leqq k\leqq r+s.$$

Für $\sqrt{1-x^2}/n\geqq 1/n^2$ ergibt sich hieraus unmittelbar

$$|g(x)-Q_ng(x)|\leqq 2^{r+s}A_{r,s}\left(\frac{\sqrt{1-x^2}}{n}\right)^{r+s}\|g^{(r+s)}\|.$$

Für $\sqrt{1-x^2}/n<1/n^2$ und $x>0$ erhält man

$$|g(x)-Q_ng(x)|=\left|\int_x^1\int_{u_1}^1\dots\int_{u_r}^1(Q_ng-g)^{(r+1)}(u_{r+1})\,du_{r+1}\dots du_1\right|\leqq$$

$$\leqq\int_x^1\int_{u_1}^1\dots\int_{u_r}^1|(Q_ng-g)^{(r+1)}(u_{r+1})|\,du_{r+1}\dots du_1\leqq$$

$$\leqq\int_x^1\int_{u_1}^1\dots\int_{u_r}^1 A_{r,s}\Delta_n(u_{r+1})^{s-1}\|g^{(r+s)}\|\,du_{r+1}\dots du_1.$$

Wegen $\Delta_n(u_{r+1})\leqq\Delta_n(x)\leqq 2/n^2$ ergibt sich weiter

$$|g(x)-Q_ng(x)|\leqq A_{r,s}2^{s-1}\frac{1}{n^{2(s-1)}}\|g^{(r+s)}\|\int_x^1\int_{u_1}^1\dots\int_{u_r}^1 1\,du_{r+1}\dots du_1=$$

$$=A_{r,s}2^{s-1}\frac{1}{n^{2(s-1)}}\|g^{(r+s)}\|(1-x)^{r+1}\leqq A_{r,s}2^{s-1}\frac{1}{n^{2(s-1)}}\|g^{(r+s)}\|(\sqrt{1-x^2})^{2r+2}=$$

$$=A_{r,s}2^{s-1}\left(\frac{\sqrt{1-x^2}}{n}\right)^{r+s}\|g^{(r+s)}\|(n\sqrt{1-x^2})^{r+2-s}\leqq A_{r,s}2^{s-1}\left(\frac{\sqrt{1-x^2}}{n}\right)^{r+s}\|g^{(r+s)}\|.$$

Es ist festzustellen, daß wir soeben die einzige Stelle unseres Beweises passiert haben, an der von der Bedingung $r\geqq s-2$ Gebrauch gemacht wird.

Für $\sqrt{1-x^2}/n<1/n^2$ und $x<0$ gilt die gleiche Abschätzung. Damit ist also für alle $g\in C^{r+s}[-1, 1]$ die Ungleichung

$$|g(x)-Q_ng(x)|\leqq 2^{r+s}A_{r,s}\Gamma_n(x)^{r+s}\|g^{(r+s)}\|$$

bewiesen. Wählen wir nun für g die Funktion F_h vom Beginn dieses Beweises, so ergibt sich für $0<h\leqq 2$

$$|F_h(x)-Q_nF_h(x)|\leqq 2^{r+s}A_{r,s}\Gamma_n(x)^{r+s}\|F_h^{(r+s)}\|\leqq 2^{r+s}A_{r,s}\Gamma_n(x)^{r+s}c_{r,s}h^{-s}\omega_s(f^{(r)}, h).$$

Die verbleibende Größe $|Q_nF_h(x)-Q_nf(x)|$ ist gleich

$$|\bar{L}_n(F_h-f, x)+R_n\circ(\mathrm{Id}-\bar{L}_n)(F_h-f, x)|.$$

Zunächst ist wegen $\|\bar{L}_n\|\leqq c$ für alle $n\in\mathbf{N}$ und alle $h\in(0, 2]$

$$|\bar{L}_n(F_h-f, x)|\leqq\|\bar{L}_n\|\,\|F_h-f\|\leqq cc_{r,s}h^r\omega_s(f^{(r)}, h).$$

Der verbleibende Ausdruck $|R_n \circ (\mathrm{Id} - \bar{L}_n)(F_h - f, x)|$ kann ähnlich wie oben abgeschätzt werden (vgl. den letzten Teil des Beweises von Satz 4.2). Man erhält

$$|R_n \circ (\mathrm{Id} - \bar{L}_n)(F_h - f, x)| \leqq \sum_{i=0}^{r} (\sqrt{1-x^2})^i \frac{c_{i,r}}{n^i}(1+c_3)\,\|(F_h - f)^{(i)}\|\,(a_r + b_r)$$

mit der Konstanten c_3 aus Satz 3.3.

Unter Benutzung der Abschätzungen für $\|(F_h - f)^{(i)}\|$ für $0 \leqq i \leqq r$ ergibt sich, daß die zuletzt betrachtete Summe kleiner als oder gleich

$$\sum_{i=0}^{r} c_{i,r}(1+c_3)\,c_{r,s}(a_r+b_r)\,\Gamma_n(x)^i h^{r-i}\omega_s(f^{(r)}, h) \leqq e_{r,s} \sum_{i=0}^{r} \Gamma_n(x)^i h^{r-i}\omega_s(f^{(r)}, h)$$

ist; hierbei ist $e_{r,s}$ eine nur von r und s abhängige Konstante.

Zusammenfassend gilt also für $x \in (-1, 1)$ und jedes $h \in (0, 2]$ die Abschätzung

$$|f(x) - Q_n f(x)| \leqq c_{r,s} h^r \omega_s(f^{(r)}, h) + 2^{r+s} A_{r,s} \Gamma_n(x)^{r+s} c_{r,s} h^{-s} \omega_s(f^{(r)}, h) +$$

$$+ c \cdot c_{r,s} h^r \omega_s(f^{(r)}, h) + e_{r,s} \sum_{i=0}^{r} \Gamma_n(x)^i h^{r-i} \omega_s(f^{(r)}, h).$$

Wählt man nun $h = \Gamma_n(x) > 0$, so ergibt sich zunächst für $x \in (-1, 1)$ die Ungleichung

$$|f(x) - Q_n f(x)| \leqq d_{r,s} \Gamma_n(x)^r \omega_s(f^{(r)}, \Gamma_n(x)).$$

Wegen der Interpolationseigenschaft von Q_n an den Rändern des Intervalls gilt die behauptete Aussage auch für $x = \pm 1$. □

5. Simultanapproximation durch die Operatoren Q_n

Dieser Abschnitt enthält im Satz 5.4 eine quantitative Aussage zur Simultanapproximation durch die oben konstruierten Operatoren Q_n. In Satz 5.5 wird schließlich die oben angekündigte Kombination der Aussagen (1.2), (1.3) und (1.4) bewiesen.

Um nachzuweisen, daß für alle $f \in C^r[-1, 1]$ und $0 \leqq k \leqq r$ die Ableitungen $f^{(k)}$ durch $(Q_n f)^{(k)}$ approximiert werden und dies über punktweise Abschätzungen einzusehen, verwenden wir die folgende erweiterte Version der Ungleichung von Dzjadyk.

SATZ 5.1 (V. K. Dzjadyk [4, Theorem 7.1.3′]). *Wenn für festes $p > 0$ und eine natürliche Zahl $s \geqq 1$ das Polynom p_n vom Grad n für alle $x \in [-1, 1]$ der Ungleichung*

$$|p_n(x)| \leqq M \Delta_n(x)^p \omega_s(f, \Delta_n(x)), \quad f \in C[-1, 1],$$

genügt (M=const.), *so gilt für jede natürliche Zahl k, daß $p_n^{(k)}$ die Ungleichung*

$$|p_n^{(k)}(x)| \leqq AM\Delta_n(x)^{p-k}\omega_s(f, \Delta_n(x))$$

erfüllt; hierbei ist $A = A_{p,s,k}$ eine Konstante, die nicht von $x \in [-1, 1]$ und $n = 1, 2, \ldots$ abhängt.

Dieser Satz ermöglicht zunächst die folgende Erweiterung von Satz 3.2.

SATZ 5.2. *Es seien* $r \geqq 0$, $s \geqq 1$. *Ist* $f \in C^r[-1, 1]$ *und* p_n *ein Polynom vom Grad* $n \geqq r+s$, *so daß für alle* $x \in [-1, 1]$ *gilt:*

$$|f(x)-p_n(x)| \leqq c_{r,s} \Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x)),$$

so gilt für $0 \leqq k \leqq r$:

$$|f^{(k)}(x)-p_n^{(k)}(x)| \leqq M_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

BEWEIS. Für $r=0$ ist offenbar nichts zu zeigen. Sei also $r \geqq 1$. Nach Sätzen von R. M. Trigub [12, Lemma 1 und Bemerkung 1] (für den Fall $s=1, 2$) bzw. I. E. Gopengauz [6, S. 205] (für $s>2$) existieren zu $r, s \geqq 0$, $n \geqq r+s$ und $f \in C^r[-1, 1]$ Polynome $q_n \in \Pi_n$ mit

$$|f^{(k)}(x)-q_n^{(k)}(x)| \leqq A_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x))$$

für $0 \leqq k \leqq r$ und $|x| \leqq 1$.

Nun ist $p_n - q_n \in \pi_n$, und es gilt:

$$|p_n(x)-q_n(x)| \leqq (c_{r,s}+A_{r,s}) \Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x)).$$

Satz 5.1 zeigt nun, daß für $0 \leqq k \leqq r$ und $|x| \leqq 1$ gilt:

$$|p_n^{(k)}(x)-q_n^{(k)}(x)| \leqq B_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

Also ist

$$|f^{(k)}(x)-p_n^{(k)}(x)| \leqq |f^{(k)}(x)-q_n^{(k)}(x)|+|q_n^{(k)}(x)-p_n^{(k)}(x)| \leqq$$
$$\leqq (A_{r,s}+B_{r,s}) \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)),$$

und dies entspricht gerade obiger Behauptung. □

BEMERKUNG 5.3. Für $r \geqq s-2$ läßt sich unter Benutzung der Sätze 4.3 und 5.2 nun unmittelbar auf

$$|f^{(k)}(x)-(Q_n f)^{(k)}(x)| \leqq M_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)), \quad 0 \leqq k \leqq r,$$

schließen. Der Beweis des folgenden Satzes zeigt jedoch, daß die Argumentation mit Satz 4.3 ungünstig ist, da die Voraussetzung $r \geqq s-2$ sich im Falle der Simultanapproximation (mit Kontrollgröße $\Delta_n(x)$) als überflüssig erweist.

Ein Hauptergebnis dieses Abschnitts ist

SATZ 5.4. *Es sei* $r \geqq 0$, $s \geqq 1$. *Für die Operatoren* Q_n *aus Satz* 4.2 *bzw.* 4.3 *gilt für alle* $f \in C^r[-1, 1]$, *alle* $|x| \leqq 1$ *und alle* $n \geqq \max(4(r+1), r+s)$:

$$|f^{(k)}(x)-(Q_n f)^{(k)}(x)| \leqq \tilde{M}_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)), \quad 0 \leqq k \leqq r.$$

BEWEIS. Bis zur Abschätzung

$$|f(x)-\bar{L}_n f(x)| \leqq c_{r+s} \omega_{r+s}(f, \Delta_n(x))$$

verläuft der Beweis exakt wie der von Satz 4.2. Hieraus folgt

$$|f(x)-\bar{L}_n f(x)| \leqq c_{r+s} \Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x)).$$

Satz 5.2 impliziert für $0 \leqq k \leqq r$ und $n \geqq r+s$

$$\Pi_{1,k}(x) = |f^{(k)}(x) - (\bar{L}_n f)^{(k)}(x)| \leqq M_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

Zur Abschätzung von $\Pi_{2,k}(x)$ geht man ähnlich wie in Satz 4.2 vor. Zunächst ist wieder

$$\Pi_{2,0}(x) \leqq \sum_{i=0}^{r} (\sqrt{1-x^2})^i \left|(\sqrt{1-x^2})^i - (\sqrt{1-x^2})^{2(r+1)+i-2[i/2]} T_{N_i}(1-x^2)\right| \cdot$$
$$\cdot (a_r + b_r)\left(|(f-\bar{L}_n f)^{(i)}(1)| + |(f-\bar{L}_n f)^{(i)}(-1)|\right).$$

Hierbei sind a_r und b_r die Konstanten aus dem Beweis von Satz 4.2. Lemma 4.1 liefert für $0 \leqq i \leqq r$ mit $m_i = r+1-\left[\frac{i}{2}\right]$

$$\left|(\sqrt{1-x^2})^i - (\sqrt{1-x^2})^{2(r+1)+i-2[i/2]} T_{N_i}(1-x^2)\right| \leqq c_{i,m_i} \frac{1}{n^i};$$

aus der Abschätzung für die Größen $\Pi_{1,i}(x)$ ergibt sich

$$|(f-\bar{L}_n f)^{(i)}(\pm 1)| \leqq M_{r,s} \left(\frac{1}{n^2}\right)^{r-i} \omega_s\left(f^{(r)}, \frac{1}{n^2}\right) \quad \text{für} \quad 0 \leqq i \leqq r.$$

Wir erhalten also

$$\Pi_{2,0}(x) \leqq \sum_{i=0}^{r} (\sqrt{1-x^2})^i c_{i,m_i} \frac{1}{n^i} (a_r + b_r) M_{r,s} \left(\frac{1}{n^2}\right)^{r-i} \omega_s\left(f^{(r)}, \frac{1}{n^2}\right) \leqq$$

$$\leqq \sum_{i=0}^{r} c_{i,m_i} (a_r + b_r) M_{r,s} \Delta_n(x)^i \Delta_n(x)^{r-i} \omega_s(f^{(r)}, \Delta_n(x)) \leqq \tilde{c}_{r,s} \Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x)).$$

Für $r=0$ ist nichts mehr zu zeigen; für $r \geqq 1$ schließen wir aus

$$\Pi_{2,0}(x) = \left|(R_n \circ (\mathrm{Id} - \bar{L}_n)(f))(x)\right| \leqq \tilde{c}_{r,s} \Delta_n(x)^r \omega_s(f^{(r)}, \Delta_n(x))$$

mit Satz 5.1 auf

$$\Pi_{2,k}(x) = \left|(R_n \circ (\mathrm{Id} - \bar{L}_n)(f))^{(k)}(x)\right| \leqq A_{r,s,k} \tilde{c}_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

Hieraus ergibt sich dann

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqq$$
$$\leqq M_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)) + A_{r,s,k} \tilde{c}_{r,s} \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)),$$

was die in Satz 5.4 behaupteten Ungleichungen impliziert. □

Der zuletzt bewiesene Satz ist das zentrale Hilfsmittel, um auch unter Benutzung der Größe $\Gamma_n(x) = \sqrt{1-x^2}/n$ zu einer Aussage über Simultanapproximation zu gelangen. Dies ist das zweite Hauptergebnis dieses Abschnitts.

SATZ 5.5. *Es sei* $r \geqq s \geqq 1$ *und* $n \geqq 4(r+1)$. *Dann gilt für die Operatoren* Q_n *aus Satz* 4.2 *bzw.* 4.3 *die Ungleichung*

$$|f^{(k)}(x) - (Q_n f)^{(k)}(x)| \leqq c_{r,s} \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x))$$

für alle $f \in C^r[-1, 1]$, *alle* $x \in [-1, 1]$ *und* $0 \leqq k \leqq r-s$.

BEWEIS. Sei $r \geqq s \geqq 1$ und $n \geqq 4(r+1) > r+s$. Für die Operatoren Q_n gilt nach Satz 5.4 für $0 \leqq k \leqq r$ und $|x| \leqq 1$

$$|f^{(k)}(x)-(Q_n f)^{(k)}(x)| \leqq c_1 \Delta_n(x)^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

Für $\sqrt{1-x^2}/n \geqq 1/n^2$ folgt hieraus unmittelbar

$$|f^{(k)}(x)-(Q_n f)^{(k)}(x)| \leqq c_1 2^{r+s} \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)).$$

Sei also wieder $\sqrt{1-x^2}/n < 1/n^2$ und $0 < x < 1$. Für beliebiges $x \leqq u \leqq 1$ folgt wegen $\Delta_n(u) \leqq \Delta_n(x) < 2/n^2$ für die Differenz zwischen den r-ten Ableitungen

$$|f^{(r)}(u)-(Q_n f)^{(r)}(u)| \leqq 2^s c_1 \omega_s\left(f^{(r)}, \frac{1}{n^2}\right).$$

Für $0 \leqq k \leqq r-1$ ergibt sich aus den Interpolationseigenschaften von Q_n an den Rändern die Ungleichungskette

$$|f^{(k)}(x)-(Q_n f)^{(k)}(x)| = \left|\int_x^1 \int_{u_1}^1 \dots \int_{u_{r-k-1}}^1 (f-Q_n f)^{(r)}(u_{r-k})\, du_{r-k} \dots du_1\right| \leqq$$

$$\leqq \int_x^1 \int_{u_1}^1 \dots \int_{u_{r-k-1}}^1 2^s c_1 \omega_s\left(f^{(r)}, \frac{1}{n^2}\right) du_{r-k} \dots du_1 = 2^s c_1 (1-x)^{r-k} \omega_s\left(f^{(r)}, \frac{1}{n^2}\right) \leqq$$

$$\leqq 2^s c_1 \omega_s\left(f^{(r)}, \frac{1}{n^2}\right) (\sqrt{1-x^2})^{2r-2k} \leqq 2^s c_1 \left((n\sqrt{1-x^2})^{-1}+1\right)^s \omega_s(f^{(r)}, \Gamma_n(x)) \cdot$$

$$\cdot \Gamma_n(x)^{r-k} n^{r-k} (\sqrt{1-x^2})^{r-k} \leqq 4^s c_1 \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)) \left(n\sqrt{1-x^2}\right)^{r-k-s} \leqq$$

$$\leqq 4^s c_1 \Gamma_n(x)^{r-k} \omega_s(f^{(r)}, \Gamma_n(x)).$$

Die letzte Ungleichung erhält man dabei aus der Bedingung $0 \leqq k \leqq r-s$. Aus der Interpolationseigenschaft ergibt sich weiter, daß die bewiesene Ungleichung auch für $x=1$ richtig ist. Im Fall $-1 \leqq x < 0$ schließt man analog. □

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WEAK CLOSURE OF THE UNITARY ORBIT OF CONTRACTIONS

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Introduction. We consider an infinite dimensional complex separable Hilbert space H. A *shift* S on H is an operator defined by

$$Se_i = e_{i+1},$$

for some orthonormal basis (e_i) in H. If i belongs to the set of all integers, S is said to be a *bilateral shift*. If i belongs to the set of all positive integers, then S is said to be a *unilateral shift*.

Because any two shifts (of the same type) on H are unitarily equivalent, the set of all shifts on H can be generated by one given shift, namely, it is the unitary orbit of that given shift. Recall that if U is the group of unitary operators on H, then the unitary orbit of S, denoted by $U(S)$, is given by $U(S)=\{u^*Su\colon u\in U\}$. Halmos [3], 1973 studied the closure of $U(S)$ in different topologies, and he proved the following result.

HALMOS THEOREM. *The weak closure of the unitary orbit $U(S)$ of the (unilateral or bilateral) shift S is equal to the set of all contractions on H; in symbols, $WCU(S)=(HB)_1$, where WC=weak closure, $(HB)_1$=unit ball of H=set of all contractions on H.*

Halmos [3] asked, what can be said about $U(T_\alpha)$, where T_α is a *weighted shift* defined by

$$T_\alpha e_i = \alpha_i e_{i+1},$$

for some orthonormal basis (e_i) in H and $\alpha=(\alpha_i)$ is a bounded sequence of complex numbers called *weight sequence*. More generally, one may ask, what about $U(T)$, where T is an arbitrary contraction on H? The answer to this question is the destination of this work and is presented in Section 1. In Section 2 we answer the Halmos question as a special case. In Section 3 we prove some results about arbitrary weighted shifts, and Section 4 is devoted to an application to the disc algebra.

1. Weak closure of the unitary orbit of a contraction

In this section we give some conditions which are mutually equivalent and each is necessary and sufficient for the weak closure of the unitary orbit of a contraction to be equal to the set of all contractions on H.

Recall that the *spectrum* $\sigma(T)$ of an operator T is defined to be the set

$$\{\lambda\in\mathbf{C}\colon T-\lambda \text{ is not invertible}\},$$

where $\mathbf{C}$ is the complex plane. The spectral radius $|\sigma(T)|$ of T, is defined by the equality $|\sigma(T)|=\sup\{|\lambda|: \lambda\in\sigma(T)\}$.

It is known that if T is a contraction, i.e. $\|T\|\leqq 1$, then $\sigma(T)$ is included in the closed unit disc $\bar{D}$ (see [1]). If K is the ideal of all compact operators on H and if $L(H)$ denotes the algebra of all bounded linear operators on H, then $L(H)/K$ is called the Calkin algebra. Let π be the canonical map $L(H)\to L(H)/K$. Then the *essential spectrum* $\sigma_e(T)$ is defined to be $\sigma_e(T)=\sigma(\pi(T))$.

The numerical range $W(T)$ of T is defined to be $W(T)=\{\lambda=(Te, e): e\in H, \|e\|=1\}$. It is known that $W(T)$ is a convex subset of $\mathbf{C}$ and $\sigma(T)\subset\overline{W(T)}$, (see [1]). Clearly, if T is a contraction, then $\overline{W(T)}\subset\bar{D}$. The numerical radius $|W(T)|$ of T is defined by $|W(T)|=\sup\{|\lambda|: \lambda\in\overline{W(T)}\}$.

Now, we present the main result of this section.

THEOREM 1.1. *For any contraction T in $L(H)$, the following statements are equivalent.*

(a) $\overline{WCU(T)}=(HB)_1$;

(b) $\overline{W(T)}=\bar{D}$;

(c) $\sigma(T)\supset\partial\bar{D}$, *where* $\partial\bar{D}=$ *boundary of* $\bar{D}$;

(d) $\sigma_e(T)\supset\partial\bar{D}$.

We split the proof into lemmas, propositions and theorems.

LEMMA 1.2. *If T is a contraction such that $|1-(Te,f)|<\varepsilon$, with some unit vectors e, f and positive number ε, then $\|Te-f\|<\varepsilon+\sqrt{2\varepsilon}$.*

PROOF. Let $\beta=(Te,f)$ and $\lambda=|\beta|$. Since $\lambda\leqq 1$, we have $1-\varepsilon<|(Te,f)|=|\beta|=\lambda\leqq 1$, and $|\lambda-\beta|<|1-\beta|=|1-(Te,f)|<\varepsilon$. By the definition of β, $Te=\beta f-k$, where $(k,f)=0$. Thus $|\beta|^2+\|k\|^2=\|Te\|^2\leqq 1$, i.e. $\|k\|^2\leqq 1-|\beta|=1-\lambda^2=(1-\lambda)(1+\lambda)\leqq 2\varepsilon$. Therefore, $\|Te-f\|\leqq\|(\beta-1)f\|+\|k\|<\varepsilon+\sqrt{2\varepsilon}$.

LEMMA 1.3. *If T is a contraction and $\lambda\in\mathbf{C}$ with $|\lambda|=1$ is such that for any $\varepsilon>0$, we have $|(Te_\varepsilon, e_\varepsilon)-\lambda|<\varepsilon$ with some unit vector $e_\varepsilon\in H$, then $\|Te_\varepsilon-\lambda e_\varepsilon\|\to 0$ as $\varepsilon\to 0$, i.e., λ is an approximate eigenvalue of T.*

PROOF. Since $|(Te_\varepsilon, e_\varepsilon)-\lambda|<\varepsilon$, we have $|(\bar{\lambda}Te_\varepsilon, e_\varepsilon)-1|<\varepsilon$, and by Lemma 1.2 $\|\bar{\lambda}Te_\varepsilon-e_\varepsilon\|\to 0$ as $\varepsilon\to 0$, which is equivalent to the assertion.

LEMMA 1.4. *If $\gamma_1, \gamma_2, \dots, \gamma_n$ are distinct approximate eigenvalues of a contraction T such that $|\gamma_i|=1$, $i=1, 2, \dots, n$, then for any $\varepsilon>0$ there is an orthonormal system $(e_i(\varepsilon))$, $i=1, 2, \dots, n$, such that $\|Te_i(\varepsilon)-\gamma_i e_i(\varepsilon)\|<\varepsilon$, for every $i=1, \dots, n$.*

PROOF. We have three cases to consider: the case when γ_i, $i\leqq n$, are eigenvalues, the case when γ_i, $i\leqq n$, are approximate eigenvalues but not eigenvalues, and the case when some γ_i are eigenvalues and the other are approximate eigenvalues but not eigenvalues.

For the first case, it is known (see [1]) that the eigenspace M_{γ_i} of an eigenvalue γ_i with $|\gamma_i|=1$ of a contraction T reduces T, i.e. $T(M_{\gamma_i})\subset M_{\gamma_i}$ and $T(M_{\gamma_i}^{\perp})\subset M_{\gamma_i}^{\perp}$, so that if $i\neq j$ and $e_j\in M_{\gamma_j}$, $e_i\in M_{\gamma_i}$ where e_i, e_j are unit vectors then e_i, e_j are orthonormal.

In the second case we claim that there is an infinite-dimensional subspace $M_i(\varepsilon)$ of vectors f for which $\|(T-\gamma_i)f\|<\varepsilon\|f\|$, if $f\neq 0$. Indeed, let $T-\gamma_i=V_iP_i$ be the polar decomposition of $T-\gamma_i$. Since γ_i is not an eigenvalue but approximate eigenvalue, we have $\ker(T-\gamma_i)=\{0\}$ and thus $\ker P_i=\{0\}$. The spectral representation of P_i is qiven by the equality $P_i=\int_0^{\|P_i\|} t\,dE_i(\Delta)$ where $E_i(\Delta)H=E_i((0,\varepsilon))H$, $\varepsilon>0$ is an infinite dimensional subspace of H. Since for every $f\in E_i((0,\varepsilon))H$, $f\neq 0$, we have

$$\|(T-\gamma_i)f\| = \|V_iP_if\| \leqq \|P_if\| < \varepsilon\|f\|,$$

we can take $M_i(\varepsilon)=E_i((0,\varepsilon))H$.

Now, it is sufficient to construct, by induction, an orthonormal system $(e_i(\varepsilon))_{i=1}^n$ such that $e_i(\varepsilon)\in M_i(\varepsilon)$, $i\leqq n$.

For the third case assume that $\gamma_1,\dots,\gamma_k$ are eigenvalues, while $\gamma_{k+1},\dots,\gamma_n$ are approximate eigenvalues but not eigenvalues of T. Set $H'=(M_{\gamma_1}\oplus\dots\oplus M_{\gamma_k})^\perp$ and $T'=T|_{H'}$. Let $M_i'(\varepsilon)=\{f'\colon \|(T'-\gamma_i)f'\|<\varepsilon\|f'\|\}\cup\{0\}$, $k<i\leqq n$. Since γ_i is an approximate eigenvalue of T' in H' $(i>k)$, $\dim M_i'(\varepsilon)=\infty$ $(i>k)$. Building the required orthonormal systems for $i\leqq k$ and $i>k$ as above, their union will suit our purposes.

THEOREM 1.5. *For a contraction* $T\in L(H)$, $WCU(T)=(HB)_1$ *if and only if* $\overline{W(T)}=\bar{D}$.

PROOF. Since T is a contraction, we have $\overline{W(T)}\subset\bar{D}$. For the other inclusion, let $\lambda\in\bar{D}$ so that $\lambda I\in(HB)_1$. By the assumption there are unitaries u_n so that $u_n^*Tu_n$ converges weakly to λI, i.e., $(u_n^*Tu_nf,g)\to(\lambda f,g)$, for any $f,g\in H$. In particular $(u_n^*Tu_ne,e)\to(\lambda e,e)$, for any unit vector e in H. Thus $\lambda\in\overline{W(T)}$, i.e., $\bar{D}\subset\overline{W(T)}$.

Conversely, we assume that $\overline{W(T)}=\bar{D}$, and we want to show that $WCU(T)=(HB)_1$. Since T is a contraction, u^*Tu is a contraction for every $u\in U$, thus $U(T)\subset(HB)_1$, and since $(HB)_1$ is weakly closed, we have $WCU(T)\subset(HB)_1$.

For the other inclusion, it is known that the unitary operators are weakly dense in $(HB)_1$ and so the diagonal unitary operators are weakly dense in $(HB)_1$ (see [3]). Thus it is enough to approximate weakly $u=\sum\lambda_i(.,e_i)e_i$ with $|\lambda_i|=1$. Fix $i=1,\dots,n$, given $\varepsilon>0$, by Lemma 1.4 there exist orthonormal vectors $f_1,\dots,f_n$ so that

$$\|Tf_i-\lambda_if_i\| < \frac{\varepsilon}{n}. \tag{1}$$

Let V be unitary and $Vf_1=e_1,\dots,Vf_n=e_n$. Then, by (1), VTV^* is in the weak neighborhood of u determined by e_i, $i=1,2,\dots,n$ and ε. This implies that any contraction $A\in(HB)_1$ can be approximated weakly by VTV^* for some V in U, i.e. $(HB)_1\subset WCU(T)$. The theorem is proved.

PROPOSITION 1.6. *For any contraction T in $L(H)$,* $\overline{W(T)}=\bar{D}$ *if and only if* $\sigma(T)\supset\partial\bar{D}$.

PROOF. Since T is a contraction, $\overline{W(T)}\subset\bar{D}$. If $\sigma(T)\supset\partial\bar{D}$, then $\partial\bar{D}\subset\overline{W(T)}$, and since $\overline{W(T)}$ is convex, it contains the interior of $\bar{D}$ so that $\bar{D}\subset\overline{W(T)}$, i.e. $\overline{W(T)}=\bar{D}$.

Conversely, if $\overline{W(T)}=\bar{D}$, then by Lemma 1.3, every point of $\partial\bar{D}$ is an approximate eigenvalue of T, and therefore $\partial\bar{D}\subset\sigma(T)$.

PROPOSITION 1.7. *For any contraction T in $L(H)$, $\sigma(T)\supset\partial\bar{D}$ if and only if $\sigma_e(T)\supset\partial\bar{D}$.*

PROOF. One direction is clear, since $\sigma_e(T)\subset\sigma(T)$ for any T in $L(H)$. For the other direction, let $\lambda\in\partial\bar{D}\subset\sigma(T)$. Then $T-\lambda$ is not invertible. If we assume that λ is not in $\sigma_e(T)$ then $\pi(T)-\lambda$ is * invertible in the Calkin algebra, i.e., $T-\lambda$ is Fredholm. Since $T-\lambda$ is not invertible, $\dim\ker(T-\lambda)$ is not locally constant on a punctured disc of center λ and radius ε. But by applying Gohberg theorem ([1] page 146) on the Fredholm operator $T-\lambda$, one concludes that $\exists\varepsilon>0$: $\dim\ker\big(T-(\lambda+\gamma)\big)$ is constant for γ: $0<|\gamma|<\varepsilon$. This contradiction ends the proof of the proposition.

PROOF OF THEOREM 1.1. Theorem 1.5, Propositions 1.6 and 1.7 prove the theorem.

2. Weak closure of the unitary orbit of a contractive weighted shift

We consider a contractive unilateral weighted shift T_α, $\alpha=(\alpha_i)$. Since T_α is a contraction we have $\sup_i|\alpha_i|\leqq 1$. It is known that two unilateral weighted shifts T_α, T_β are unitarily equivalent if and only if $|\alpha_i|=|\beta_i|$, for every positive integer i (see [4] problem 75, page 46). So it means no loss of generality to consider sequences of positive numbers for sequences of weights. For $U(T_\alpha)$ we prove the following theorem.

THEOREM 2.1. *For any contractive weighted shift T_α with positive weights (α_i) on H we have $WCU(T_\alpha)=(HB)_1$ if and only if $\forall n\in\mathbf{N}$, $\varepsilon>0$; $\exists N\in\mathbf{N}$ such that $\alpha_{N+i}>1-\varepsilon$, $i=1, 2, \dots, n$, where $\mathbf{N}$=set of all positive integers.*

We divide the proof into several lemmas and propositions. The next proposition proves the sufficiency of the condition.

PROPOSITION 2.2. *If the sequence of weights (α_i), $i\in\mathbf{N}$ of the contractive weighted shift T_α is such that $\forall n\in\mathbf{N}$, $\varepsilon>0$; $\exists N\in\mathbf{N}$: $\alpha_{N+i}>1-\varepsilon$, $i\leqq n$, then $WCU(T_\alpha)=(HB)_1$.*

PROOF. The spectrum of a weighted shift is circular, i.e. if $\lambda\in\sigma(T_\alpha)$, the circle of radius $|\lambda|$ is also in $\sigma(T_\alpha)$. On the other hand, the hypothesis implies (see [4] problem 77 page 48) that the spectral radius $|\sigma(T_\alpha)|$ of T_α is 1. Thus $\partial\bar{D}\subset\sigma(T_\alpha)$. By Theorem 1.1, $WCU(T_\alpha)=(HB)_1$.

THEOREM 2.3. *If S is a shift and T is any contraction such that $|((S-T)e_i, e_{i+1})|<\varepsilon$ for $i<M$, where (e_i) is the orthonormal basis shifted by S, then for every $n\in\mathbf{N}$ we have*

$$|((S^n-T^n)e_i, e_{i+n})|<n\big(\varepsilon+\sqrt{2\varepsilon}\big),\quad i+n-1<M.$$

PROOF. From the assumption one concludes

$$\varepsilon>|(Se_i, e_{i+1})-(Te_i, e_{i+1})|=|1-(Te_i, e_{i+1})|. \tag{2}$$

Applying Lemma 1.2 to (2) one obtains

$$\|Te_i - e_{i+1}\| < \varepsilon + \sqrt{2\varepsilon}, \quad i < M.$$

Consequently,

$$\|(T^n - S^n)e_i\| = \|T^n e_i - e_{i+n}\| \leqq \|T^n e_i - T^{n-1} e_{i+1}\| + \|T^{n-1} e_{i+1} - T^{n-2} e_{i+2}\| + \ldots$$

$$\ldots + \|Te_{i+n-1} - e_{i+n}\| < n(\varepsilon + \sqrt{2\varepsilon}), \quad i+n-1 < M.$$

Hence the assertion follows.

THEOREM 2.4. *If S is a shift with respect to the orthonormal basis (f_i) and T_α is a contractive weighted shift with respect to the orthonormal basis (e_i) and having the positive weight sequence (α_i), $i \in \mathbf{N}$ such that*

$$|((S - T_\alpha) f_i, f_{i+1})| < \varepsilon, \quad i < M,$$

then there are chains of weights of length $n < M$, each weight of which is $> 1 - n(\varepsilon + \sqrt{2\varepsilon})$.

PROOF. Set $n < M$, $f_1 = \sum_0^\infty a_k e_k$, $f_{n+1} = \sum_0^\infty d_k e_k$. By Theorem 2.3,

$$1 - n(\varepsilon + \sqrt{2\varepsilon}) < |(T^n f_1, f_{n+1})| = \Big|\Big(\sum_0^\infty a_k T^n e_k, \sum_0^\infty d_k e_k\Big)\Big| =$$

$$< \Big|\Big(\sum_0^\infty \alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} a_k e_{k+n}, \sum_0^\infty d_k e_k\Big)\Big| \leqq \sum_0^\infty \alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} |a_k| \cdot |\bar{d}_{k+n}| \leqq$$

$$< \Big(\sum_0^\infty |a_k|^2\Big)^{1/2} \Big(\sum_0^\infty |d_{k+n}|^2\Big)^{1/2} \sup \alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} < \sup \alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1}.$$

Thus there is a positive integer N such that $\alpha_{N+1} \alpha_{N+2} \ldots \alpha_{N+n} > 1 - n(\varepsilon + \sqrt{2\varepsilon})$, and consequently $\alpha_{N+i} > 1 - n(\varepsilon + \sqrt{2\varepsilon})$, $i = 1, \ldots, n$.

PROPOSITION 2.5. *If T_α is a contractive weighted shift whose sequence of weights is (α_i), $i \in \mathbf{N}$ such that $WCU(T_\alpha) = (HB)_1$, then $\forall \varepsilon > 0$, $n \in \mathbf{N}$, $\exists N \in \mathbf{N}$ such that*

$$\alpha_{N+i} > 1 - n(\varepsilon + \sqrt{2\varepsilon}), \quad i \leqq n.$$

PROOF. Let S be a shift with respect to the orthonormal basis (f_i). Since $S \in (HB)_1$, there is a sequence (u_m) of unitaries such that, for every ε, M there is a number $K \in \mathbf{N}$ such that for $m > K$, we have

$$|((S - u_m^* T_\alpha u_m) f_i, f_{i+1})| < \varepsilon, \quad i < M.$$

But $u_m^* T_\alpha u_m$ is also a weighted shift (having the same sequence of weights and different orthonormal basis). Applying Theorem 2.4 we obtain, $\forall \varepsilon > 0$, $n \in \mathbf{N}$; $\exists N \in \mathbf{N}$ such that

$$\alpha_{N+i} > 1 - n(\varepsilon + \sqrt{2\varepsilon}), \quad i \leqq n.$$

PROOF OF THEOREM 2.1. Proposition 2.2 proves sufficiency and Proposition 2.5 proves necessity.

REMARK 2.6. Theorem 2.1 holds for contractive bilateral weighted shifts as well.

COROLLARY 2.7. *If T_α is a contractive weighted shift with positive weights such that $WCU(T_\alpha)=(HB)_1$ and if T is another weighted shift with positive weights such that $\|T_\alpha+T\|\leqq 1$, then $WCU(T_\alpha+T)=(HB)_1$.*

COROLLARY 2.8. *The set of all weighted shifts on H is weakly dense in the algebra of all bounded operators $L(H)$ on H.*

3. Arbitrary weighted shifts

In this section we consider two arbitrary weighted shifts T_α, T_β having the bounded positive sequences (α_i), (β_i) for weights. We tried to generalize results of Section 2 in the following direction. What are the conditions on (α_i), (β_i) in order that $WCU(T_\alpha)=WCU(T_\beta)$? This problem is still open, but we prove the following results.

PROPOSITION 3.1. *Let T_α, T_β be weighted shifts with bounded positive weight sequences (α_i), (β_i) respectively. If $\forall\varepsilon>0$, N, K; $\exists M$ such that $|\alpha_{k+i}-\beta_{M+i}|<\varepsilon$, $0\leqq i<N$, and if $\forall\varepsilon>0$, N, M; $\exists K$ such that $|\beta_{M+i}-\alpha_{K+i}|<\varepsilon$, $0\leqq i<N$, then $SCU(T_\alpha)=SCU(T_\beta)$ and in particular, $WCU(T_\alpha)=WCU(T_\beta)$, where SC denotes strong closure.*

PROOF. It is enough to show that the first condition implies $SCU(T_\alpha)\subset SCU(T_\beta)$ and $(WCU(T_\alpha)\subset WCU(T_\beta))$, since the other condition implies the opposite inclusion in a similar manner.

By the first condition, given $\varepsilon>0$, N and $K=1$ there is M such that

$$|\alpha_{1+i}-\beta_{M+i}|<\varepsilon,\quad 0\leqq i<N.$$

Let U_N be a unitary operator on H which takes $e_1, \dots, e_N$ into $f_M, \dots, f_{M+N-1}$ respectively, where (e_k) and (f_k) are the orthonormal bases with $T_\alpha e_k=\alpha_k e_{k+1}$ and $T_\beta f_k=\beta_k f_{k+1}$. Then $U_N^* T_\beta U_N e_{1+i}=\beta_{M+i}e_{2+i}$, and thus

$$\|(U_N^*T_\beta U_N-T_\alpha)e_{1+i}\| = |\beta_{M+i}-\alpha_{1+i}|<\varepsilon,\quad \text{for}\quad 0<i<N-1.$$

If we let $\varepsilon=\frac{1}{N}$, then we obtain

$$U_N^*T_\beta U_N e_j \to T_\alpha e_j,\quad j=1,2,\dots,\quad \text{as}\quad N\to\infty.$$

Thus $T_\alpha=s-\lim_N U_N^*T_\beta U_N$, which implies that $SCU(T_\beta)\supset SCU(T_\alpha)$ and in particular $WCU(T_\beta)\supset WCU(T_\alpha)$.

The following proposition gives necessary conditions for the equality $SCU(T_\alpha)=$ $=SCU(T_\beta)$.

PROPOSITION 3.2. *If T_α, T_β are weighted shifts having bounded positive weight sequences (α_i), (β_i) respectively and $SCU(T_\alpha)=SCU(T_\beta)$, then $\forall\varepsilon>0$, N, M; $\exists K$ such that $\prod_{i=0}^{N-1}\alpha_{k+i}>\prod_{i=0}^{N-1}\beta_{M+i}-\varepsilon$, and $\forall\varepsilon>0$, N, K; $\exists M$ such that*

$$\prod_{i=0}^{N-1}\beta_{M+i}>\prod_{i=0}^{N-1}\alpha_{k+i}-\varepsilon.$$

PROOF. It is enough to show that if (U_n) is a sequence of unitary operators such that $U_N^* T_\alpha U_n$ converges strongly to T_β then $\forall\varepsilon>0, N, M; \exists K$ such that $\prod_{i=0}^{N-1}\alpha_{K+i}>\prod_{i=0}^{N-1}\beta_{M+i}-\varepsilon$, since if $V_n^* T_\beta V_n$ converges strongly to T_α then the other condition follows in a similar manner.

Let (f_i) and (e_i) be the orthonormal bases shifted by T_β and T_α, respectively, and let ε, M, N be given. We may assume that $\varepsilon<\prod_{i=0}^{N-1}\beta_{M+i}$. Further we write $U_n f_M=\sum_{k=1}^{\infty} a_k^{(n)} e_k$, where $\sum_{k=1}^{\infty}|a_k^{(n)}|^2=1$.

Since the product is sequentially continuous in the strong operator topology, $U_n^* T_\alpha^N U_n$ converges strongly to T_β^N. Consequently $\exists L$ such that

$$\|T_\beta^N f_M-U_n^* T_\alpha^N U_n f_M\|<\varepsilon \quad \text{for} \quad n>L.$$

This implies that $\varepsilon>\|\beta_M\beta_{M+1}\dots\beta_{M+N-1}f_{M+N}-U_n^* T_\alpha^N U_n f_M\|$, and all the more, $\varepsilon>\prod_{i=0}^{N-1}\beta_{M+i}-\|U_n^* T_\alpha^N U_n f_M\|$ for $n>L$. Hence

$$\sum_{i=0}^{N-1}\beta_{M+i}-\varepsilon<\|T_\alpha^N U_n f_M\|=\Big\|\sum_{k=1}^{\infty} a_k^{(n)} T_\alpha^N e_k\Big\|\leqq\Big\|\sum_{k+1}^{\infty} a_k^{(n)}\alpha_k\alpha_{k+1}\dots\alpha_{k+N-1}e_{k+N}\Big\|\leqq$$

$$<\Big(\sum_{k=1}^{\infty}|a_k^{(n)}|^2\Big)^{1/2}\sup_k(\alpha_k\alpha_{k+1}\dots\alpha_{k+N-1})\leqq\sup_k(\alpha_k\alpha_{k+1}\dots\alpha_{k+N-1}).$$

This implies that there is a positive integer K such that

$$\prod_{i=0}^{N-1}\beta_{M+i}-\varepsilon<\alpha_K\alpha_{K+1}\dots\alpha_{K+N-1}=\prod_{i=0}^{N-1}\alpha_{K+i}.$$

REMARK 3.3. We cannot replace strong closure by weak closure in Proposition 3.2. simply because the product is not sequentially continuous in the weak operator topology.

4. Application to the disc algebra

In this section we present some corollaries of Theorem 1.1 and an application to the disc algebra.

COROLLARY 4.1. *If T is a contraction such that $WCU(T)=(HB)_1$ and if K is a compact operator with $\|T+K\|\leqq 1$, then $WCU(T+K)=(HB)_1$.*

COROLLARY 4.2. *If T is a contraction, S is a shift, then $WCU(T\oplus S)=(HB)_1$.*

PROOF. $T\oplus S$ is a contraction, so that $\sigma(T\oplus S)\supset\sigma(S)\supset\partial\bar{D}$.

COROLLARY 4.3. *If $WCU(T)=(HB)_1$, for some contraction T, and if $Q=T\oplus\oplus T\oplus\dots$, then $WCU(Q)=(HB)_1$.*

PROOF. Q is a contraction and $\sigma(Q)\supset\sigma(T)\supset\partial\bar{D}$.

Recall that the *disc algebra* A is the norm closure of the set of all polynomials defined on the boundary $\partial\bar{D}$ of the unit disc $\bar{D}$. For a given contraction T in $L(H)$ define the map $\Psi\colon A\to L(H)$ by the equality $\Psi(f)=f(T)$ for every $f\in A$. von Neumann [5] proved that Ψ is a contractive homomorphism. Now we prove the following result.

THEOREM 4.4. *For any contraction T in $L(H)$, $WCU(T)=(HB)_1$ if and only if Ψ is an isometry.*

PROOF. By the von Neumann theorem our condition reduces to

$$\|f(T)\| \geqq \|f\|_\infty = \sup_{z\in D}|f(z)|,$$

for any f in A. Let $f\in A$; since f is a norm limit of polynomials and using a result of Foias, and Mlak [2], $\sigma(f(T))=f(\sigma(T))$. By Theorem 1.1, $\sigma(T)\supset\partial\bar{D}$. Thus $\sigma(f(T))\supset f(\partial\bar{D})$, which implies that

$$\|f(T)\| \geqq |\sigma(f(T))| \geqq \sup\{|f(z)|\colon z\in\partial\bar{D}\} = \|f\|_\infty.$$

Conversely, if Ψ is an isometry then for any $f\in A$, $\|f\|_\infty=\|f(T)\|$. Let $|\lambda|=1$ and let $f_\varepsilon(z)=(z-\lambda(1+\varepsilon))^{-1}$. Then $f_\varepsilon\in A$, and $\|f_\varepsilon\|_\infty$ converges to ∞. Therefore,

$$\|f_\varepsilon(T)\| = \left\|(T-\lambda(1+\varepsilon))^{-1}\right\| \text{ converges to } \infty,$$

so that $\lambda\in\sigma(T)$, i.e. $\partial\bar{D}\subset\sigma(T)$ and by Theorem 1.1, $WCU(T)=(HB)_1$.

We conclude by the following.

PROPOSITION 4.5. *If T, A are as above, and if $\left|\overline{W(f(T))}\right|=\|f\|_\infty$, for every $f\in A$, then $WCU(T)=(HB)_1$.*

PROOF. Let $\lambda\in\partial\bar{D}$, $\lambda\notin\sigma(T)$. There is no loss of generality if we take $\lambda=1$. Let $f\in A$ be defined by $f(z)=\frac{1}{2}(z+1)$. Then $f(1)=1$ and $f(\bar{D})$ is the disc of center $\frac{1}{2}$ and radius $\frac{1}{2}$. Since $\sigma(f(T))=f(\sigma(T))\subset f(\bar{D})$, where as $1=f(1)$ is not in $\sigma(f(T))$, it follows that $|\sigma(f(T))|<1$. Since $\|f\|_\infty=1$,

$$|\sigma(f(T))| < \|f\|_\infty. \tag{3}$$

Now since

$$|\sigma(f(T))| \leqq \left|\overline{W(f(T))}\right| \leqq \|f(T)\| \leqq \|f\|_\infty,$$

by a theorem of Williams [6] (part of Theorem 3) the following are equivalent: (1) $|\sigma(T-\lambda)|=\|T-\lambda\|$, $\forall\lambda\in\mathbf{C}$, (2) $|w(T-\lambda)|=\|T-\lambda\|$, $\forall\lambda\in\mathbf{C}$). One concludes

$$|\sigma(f(T))| = \|f\|_\infty. \tag{4}$$

Since (4) contradicts (3), $\lambda\in\sigma(T)$, or $\partial\bar{D}\subset\sigma(T)$ and by Theorem 1.1, $WCU(T)=(HB)_1$.

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GENERALIZATION OF EISENSTEIN'S CONGRUENCE

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We shall use the letter p to denote an odd prime number. Fermat's little theorem asserts that the residue of 2^{p-1} modulo p is 1. Regarding the residue of 2^{p-1} modulo p^2, Eisenstein [1], proved the following.

THEOREM A. *For any integer* s, $1 \leqq s \leqq p-1$, *let* $\bar{s}$ *represent the inverse class of* s mod p. *Then*

$$2^{p-1} \equiv 1 + p(\bar{1} + \bar{3} + \ldots + \overline{p-2}) \pmod{p^2}.$$

In this note, Eisenstein's congruence is extended to include residue of 2^{p-1} modulo p^3. We state

THEOREM B. *Let* $p > 3$ *and define* $\lambda = \frac{p-1}{2}$. *For any integer* s, $1 \leqq s \leqq p-1$; *let* $\tilde{s}$ *be a representative of the inverse class of* s mod p^2. *Then*

$$2^{p-1} \equiv 1 + p(\tilde{1} + \tilde{3} + \ldots + \widetilde{p-2}) - p^2 \sum_{s=1}^{\lambda} \left\{ \theta_s + \left(\frac{2}{p}\right) \theta_{\lambda-s+1} \right\} 2^{s-1} \pmod{p^3}$$

where $\left(\frac{2}{p}\right)$ *stands for the Legendre's symbol and* $\theta_1, \theta_2, \ldots, \theta_\lambda$ *are the quadratic residues* mod p *in a certain order.*

REMARK. Since $p\tilde{s} \equiv p\bar{s} \pmod{p^2}$, it is clear that Theorem A follows from Theorem B.

PROOF OF THEOREM B. We start from the identity

$$(1) \qquad 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{(-1)^{n-1}}{n} = \sum_{s=1}^{n} (-1)^{s-1} \binom{n}{s} \frac{2^s - 1}{s}.$$

This identity is easily established through induction. Take $n = p$ in this identity and use the fact that $\binom{p}{s} = \frac{p}{s}\binom{p-1}{s-1}$; we get

$$(2) \qquad 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{p-1} - \frac{2^p - 2}{p} = p \sum_{s=1}^{p-1} (-1)^{s-1} \binom{p-1}{s-1} \frac{2^s - 1}{s^2}.$$

Since $\binom{p-1}{s-1} \equiv (-1)^{s-1} \pmod{p}$, the identity (2) implies that the numerator of the fraction (in reduced form)

(3) $$1-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{p-1}-\frac{2^p-2}{p}-p\sum_{s=1}^{p-1}\frac{2^s-1}{s^2}$$

is divisible by p^2. Multiplying the fraction (3) by $[(p-1)!]^2$ we find that p^2 divides the integer

(4) $$\tilde{1}-\tilde{2}+\tilde{3}-\dots-\widetilde{p-1}\,\frac{2^p-2}{p}-p\sum_{s=1}^{p-1}(2^s-1)\,\bar{s}^2$$

where $\bar{s}$ denotes inverse of s mod p.

But it is well known that [1]

$$\tilde{1}-\tilde{2}+\tilde{3}-\dots-\widetilde{p-1}\equiv 2(\tilde{1}+\tilde{3}+\dots+\widetilde{p-2}) \pmod{p^2}$$

and that

$$\bar{1}^2+\bar{2}^2+\dots+\overline{p-1}^2\equiv 0 \pmod{p}.$$

Using these informations in (4) we find that the integer

$$2(\tilde{1}+\tilde{3}+\dots+\widetilde{p-2})-\frac{2^p-2}{p}-2p\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2$$

is divisible by p^2. Cancelling the factor 2 and multiplying by p we find that

(5) $$2^{p-1}\equiv 1+p(\tilde{1}+\tilde{3}+\dots+\widetilde{p-2})-p^2\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2 \pmod{p^3}.$$

By Euler's criterion, $2^{(p-1)/2}\equiv\left(\frac{2}{p}\right) \pmod{p}$ i.e. $2^\lambda\equiv\left(\frac{2}{p}\right) \pmod{p}$. Hence $2^{\lambda+1}\equiv$ $\equiv 2\left(\frac{2}{p}\right)$, $2^{\lambda+2}\equiv 2^2\left(\frac{2}{p}\right)$, ..., $2^{p-2}\equiv 2^{(p-3)/2}\left(\frac{2}{p}\right) \pmod{p}$. This implies

$$\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2=\sum_{s=1}^{\lambda}2^{s-1}\bar{s}^2+\sum_{s=1}^{\lambda}2^{\lambda+s-1}\overline{(\lambda+s)}^2\equiv\sum_{1}^{\lambda}2^{s-1}\bar{s}^2+\sum_{1}^{\lambda}2^{s-1}\left(\frac{2}{p}\right)\overline{(\lambda+s)}^2\equiv$$

$$\equiv\sum_{1}^{\lambda}\left\{\bar{s}^2+\left(\frac{2}{p}\right)\overline{(\lambda+s)}^2\right\}2^{s-1}\equiv\sum_{1}^{\lambda}\left\{\bar{s}^2+\left(\frac{2}{p}\right)\overline{(\lambda-s+1)}^2\right\}2^{s-1} \pmod{p}$$

since $\lambda+s\equiv-(\lambda-s+1) \pmod{p}$. Let $\theta_1, \theta_2, \dots, \theta_\lambda$ be the residues of $\bar{1}^2, \bar{2}^2, \dots, \bar{\lambda}^2$ mod p, in this order. It is clear then that $\theta_1, \theta_2, \dots, \theta_\lambda$ are the quadratic residues modulo p. Moreover, from what precedes,

$$\sum_{s=1}^{p-1} 2^{s-1}\bar{s}^2 \equiv \sum_{s=1}^{\lambda}\left\{\theta_s+\left(\frac{2}{p}\right)\theta_{\lambda-s+1}\right\}2^{s-1} \pmod{p}.$$

Using this in (5) we get Theorem B.

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Since $\binom{p-1}{s-1} \equiv (-1)^{s-1} \pmod p$, the identity (2) implies that the numerator of the fraction (in reduced form)

$$(3) \qquad 1-\frac{1}{2}+\frac{1}{3}-\ldots-\frac{1}{p-1}-\frac{2^p-2}{p}-p\sum_{s=1}^{p-1}\frac{2^{s-1}}{s^2}$$

is divisible by p^2. Multiplying the fraction (3) by $((p-1)!)^2$ we find that p^2 divides

$$(4) \qquad \bar{1}-\bar{2}+\bar{3}-\ldots-\overline{p-1}-\frac{2^p-2}{p}-p\sum_{s=1}^{p-1}(2^{s-1})\bar{s}^2$$

where $\bar{s}$ denotes inverse of s mod p.

But it is well known that [1]

$$\bar{1}-\bar{2}+\bar{3}-\ldots-\overline{p-1} \equiv 2(\bar{1}+\bar{3}+\ldots+\overline{p-2}) \pmod{p^2}$$

and that

$$\bar{1}^2+\bar{2}^2+\ldots+\overline{p-1}^2 \equiv 0 \pmod p.$$

Using these informations in (4) we find that the integer

$$2(\bar{1}+\bar{3}+\ldots+\overline{p-2})-\frac{2^p-2}{p}-2p\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2$$

is divisible by p^2. Cancelling the factor 2 and multiplying by p we find that

$$(5) \qquad 2^{p-1} \equiv 1+p(\bar{1}+\bar{3}+\ldots+\overline{p-2})-p^2\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2 \pmod{p^3}.$$

By Euler's criterion, $2^{(p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod p$, i.e. $2^{\lambda} \equiv \left(\frac{2}{p}\right) \pmod p$. Hence $2^{\lambda+1} \equiv 2\left(\frac{2}{p}\right)$, $2^{\lambda+2} \equiv 2^2\left(\frac{2}{p}\right)$, ..., $2^{p-1} \equiv 2^{(p-1)/2}\left(\frac{2}{p}\right) \pmod p$. This implies

$$\sum_{s=1}^{p-1}2^{s-1}\bar{s}^2 = \sum_{s=1}^{\lambda}2^{s-1}\bar{s}^2+\sum_{s=1}^{\lambda}2^{\lambda+s-1}\overline{(\lambda+s)}^2 \equiv \sum_{s=1}^{\lambda}2^{s-1}\bar{s}^2+\sum_{s=1}^{\lambda}2^{s-1}\left(\frac{2}{p}\right)\overline{(\lambda+s)}^2 \equiv$$

$$\equiv \sum_{s=1}^{\lambda}\left\{\bar{s}^2+\left(\frac{2}{p}\right)\overline{(\lambda+s)}^2\right\}2^{s-1} \equiv \sum_{s=1}^{\lambda}\left\{\bar{s}^2+\left(\frac{2}{p}\right)\overline{(\lambda-s+1)}^2\right\}2^{s-1} \pmod p$$

since $\lambda+s \equiv -(\lambda-s+1) \pmod p$. Let $\theta_1, \theta_2, \ldots, \theta_\lambda$ be the residues of $\bar{1}^2, \bar{2}^2, \ldots, \bar{\lambda}^2$ mod p, in this order. It is clear then that $\theta_1, \theta_2, \ldots, \theta_\lambda$ are the quadratic

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ON THE EMBEDDING OF FINITE RINGS INTO MATRICES

A. SYCHOWICZ (Olsztyn)

In [1] Amitsur showed that every semiprime PI-ring is a subring of a matrix ring over a commutative ring. Cohn [7], Amitsur [2], Small [10] constructed examples of algebras over infinite fields satisfying polynomial identities and stronger and stronger finiteness conditions of different kind which could not be embedded into matrices with commutative entries. Other authors tried to characterize rings embeddable into matrices satisfying certain identities, e.g. [3], [8].

The Wedderburn—Artin theorems result that every finite ring satisfies a nontrivial polynomial identity and that every semiprime finite ring is a direct sum of finitely many matrix rings over fields and so it is embeddable into a matrix ring with commutative entries.

Examples of Cohn, Amitsur and Small mentioned above cannot be adopted to finite rings. Bergman [4] was the first who presented an example of a finite ring (having p^5 elements, p any prime) which could not be embedded into matrices over any commutative ring.

In this note we show that every ring which has p^2, p^3 or p^4 elements, is embeddable into a matrix ring with commutative entries.

Throughout this note R will be an associative finite ring with unity, R^+ its additive group, homomorphisms will be unitary. For convenience, rings embeddable into matrices over a commutative ring will be called embeddable rings.

At first we observe:

Proposition 1. *Finite embeddable rings can be embedded into matrices over a finite commutative ring.*

Proof. Let $R=\{r_1, \dots, r_k\}$ and let f be an embedding of R into matrices $M_n(C)$ over a commutative ring C. Let $f(r_\alpha)=c^\alpha_{ij}$, where $\alpha=1, \dots, k$ and $i, j=1, \dots, n$ and let C' be a subring in C generated by c^α_{ij}'s. Then $f(R)\subset M_n(C')$ so we can assume C is finitely generated by c^α_{ij}'s. By Theorem 1 [9] for every $0\neq c^\alpha_{ij}\in C$ there exists an ideal I^α_{ij} in C such that $c^\alpha_{ij}\notin I_{ij}$ and C/I^α_{ij} is finite. Considering the mapping

$$R \to M_n(C) \to M_n(\bigoplus_{i,j,\alpha} C/I^\alpha_{ij})$$

we see that this is a nontrivial embedding of R into matrices over a finite commutative ring $\bigoplus_{i,j,\alpha} C/I^\alpha_{ij}$.

Since R^+ is a finite group it is a direct sum of finitely many p-groups which are ideals in R.

Proposition 2. *Let* $R=R_{p_1}\oplus\ldots\oplus R_{p_k}$ *be a decomposition of* R *into a direct sum of prime components, where* p_i *are distinct primes. Then* R *is embeddable if and only if every* R_{p_i} *is embeddable.*

Proof. If f_i is a monomorphism of R_{p_i} into a ring M_i of matrices of rank s_i and $s=\prod_{i=1}^{k} s_i$ then there exists a natural embedding of M_i into $\overline{M}_i$-matrix ring of rank s with the same entries as M_i. Then f_i can be regarded as an embedding of R_{p_i} into $\overline{M}_i$. Since all M_i have the same rank s, the family of f_i gives the embedding of R into matrices of rank s over a commutative ring.

Now if R is embeddable, say into $M_n(C)$, by Proposition 1 we can assume C is finite and use a decomposition of C into prime components $C=\bigoplus_i C_{q_i}$. Then it is easy to see that every R_{p_i} occurs in $\bigoplus_i M_n(C_{q_i})$.

Thus we may confine our attention to p-rings. So let p be a fixed prime. Then R^+ is a finite p-group. We will use the following:

Theorem 1 (Fuchs [6], Theorem 120.1). *A multiplication* μ *on a p-group* A *is completely determined by the values* $\mu(a_i, a_j)$ *with* a_i, a_j *running over a p-basis of* A. *Moreover, any choice of* $\mu(a_i, a_j)\in A$ *with* a_i, a_j *from a p-basis of* A *— subject to the sole condition that* $o(\mu(a_i, a_j))\leq\min(o(a_i), o(a_j))$ *— extends to a multiplication on* A. *Further* μ *is associative (commutative) if it is associative (commutative) on a p-basis of* A.

It is known that a finite p-group is a direct sum of cyclic groups and that an element of maximal order generates a direct summand. Since 1 of R has maximal order we will assume 1 generates one of the maximal order summands.

Now we recall the example of a not embeddable finite ring.

Example (Bergman). Let p be any prime. Let R be a ring with basis $\{1, a, b, c\}$, where the order of 1 is p^2, the orders of a, b, c are p and the nonzero basis products are $aa=a$, $ab=b$, $ca=c$, $cb=b'1$. Then R is not embeddable.

A simple modification of this Example gives:

Proposition 3. *For every prime* p *and every integer* $n>5$ *there exists a not embeddable ring with* p^n *elements.*

Proof. Let S be a ring such that $S^+\simeq(1)\times(a)\times(b)\times(c)\times(s_1)\times\ldots\times(s_{n-5})$, where $o(1)=p^2$ and the orders of other generators are p. The elements a, b, c are multiplied as in the above Example and other products on generators are zero. Then S has p^n elements and contains, as a subring, the ring from the Example and so it is not embeddable.

Now let us examine embeddings of small rings. The following lemma will be useful in our investigations.

Lemma 1 (Eldridge [5]). *Let* R *be a ring with* p^n *elements,* n *any integer. If the decomposition of* R^+ *into a direct sum of cyclic groups has at most two direct summands, then* R *is commutative and so it is embeddable into matrices of order one over itself.*

Corollary 1. *A ring with* p *or* p^2 *elements is embeddable.*

Concerning a ring R^+ with p^3 elements, by Lemma 1 we have to consider only the case when R^+ is a direct sum of three cyclic groups of order p. The following is known:

PROPOSITION 4. *A free module of rank n over a subring generated by its unity can be embedded into matrices of order n over this ring.*

It follows that R can be embedded into matrices of order three over Z_p. Eldridge showed even more:

THEOREM 2 (Eldridge). *If R is a noncommutative ring of order p^3 then R is isomorphic to the ring of 2×2 upper triangular matrices over Z_p.*

Using Theorem 1 we can have an elementary proof of this fact.

COROLLARY 2. *A ring of order p^3 is embeddable into matrices of order at most two.*

Now let R be a ring with p^4 elements. By Lemma 1 and Proposition 4 we have immediately:

PROPOSITION 5. *If the decomposition of a ring R with p^4 elements into a direct sum of cyclic groups has one, two or four direct summands then R is embeddable.*

All we have to do is to consider the case when R^+ is a direct sum of a cyclic group of order p^2 and two cyclic groups of order p.

For our convenience, the symbol $\langle ..., ..., ..., ...\rangle$ will denote values of products aa, ab, ba, bb in R, respectively.

LEMMA 2. *Let R be a noncommutative ring with p^4 elements such that R^+ is a direct sum of a cyclic group of order p^2 and two cyclic groups of order p. Then $R^+\simeq \simeq(1)\times(a)\times(b)$, $o(1)=p^2$, $o(a)=o(b)=p$. Moreover, multiplication on R is determined by one of the following possibilities:*

$(*)$ $\langle 0, 0, a, b\rangle$,

$(*')$ $\langle 0, a, 0, b\rangle$,

$(**)$ $\langle pm_1\cdot 1, pm_2\cdot 1, pm_3\cdot 1, pm_4\cdot 1\rangle$, $m_i\in Z_p$, $m_2\neq m_3$.

PROOF. The first part is trivial. Hence every element of R can be uniquely written as $ka+lb+m\cdot 1$ for some $k, l\in Z_p$ and $m\in Z_{p^2}$. To determine multiplication in R it sufficies to determine it on generators a and b and by Theorem 1 we can write $aa=k_1a+l_1b+pm_1\cdot 1$, $ab=k_2a+l_2b+pm_2\cdot 1$, $ba=k_3a+l_3b+pm_3\cdot 1$, $bb=k_4a+l_4b+pm_4\cdot 1$ for some k_i, l_i, m_i from Z_p. Then $a(aa)=(aa)a$ if and only if $l_1\cdot ab=l_1\cdot ba$ and by the noncommutativity of R, $l_1=0$. Similarly $b(bb)= =(bb)b$ gives $k_4=0$. By $a(ab)=(aa)b$ we have $k_2l_2\cdot a=0$ so $k_2=0$ or $l_2=0$ and by $b(ba)=(bb)a$ we get $k_3=0$ or $l_3=0$. Thus we have the following possibilities:

(1) $\langle k_1a+pm_1\cdot 1, k_2a+pm_2\cdot 1, k_3a+pm_3\cdot 1, l_4b+pm_4\cdot 1\rangle$,

(2) $\langle k_1a+pm_1\cdot 1, k_2a+pm_2\cdot 1, l_3b+pm_3\cdot 1, l_4b+pm_4\cdot 1\rangle$,

(3) $\langle k_1a+pm_1\cdot 1, l_2b+pm_2\cdot 1, k_3a+pm_3\cdot 1, l_4b+pm_4\cdot 1\rangle$,

(4) $\langle k_1a+pm_1\cdot 1, l_2b+pm_2\cdot 1, l_3b+pm_3\cdot 1, l_4b+pm_4\cdot 1\rangle$

for some k_i, l_i, m_i from Z_p.

Case 1. By associativity of multiplication we have: $k_1m_2=k_2m_1$, $k_2=k_3$ or $aa=0$, $k_2=l_4$ or $ab=0$, $k_1m_3=k_3m_1$, $k_2m_3=k_3m_2$, $l_4=k_3$ or $ba=0$. Suppose $k_2\neq l_4$. Then $ab=0$. Since $ba\neq 0$ we get $(*)$ by $a\mapsto a-pm_3/k_3\cdot 1$, $b\mapsto k_3b-pm_4/k_3\cdot 1$, so we can assume $l_4=k_2$.

If $k_2\neq k_3$ then $k_2\neq 0$ and $a\mapsto a-pm_2/k_2\cdot 1$, $b\mapsto k_2b-pm_4/k_2\cdot 1$ gives us $(*')$.

If $k_2=k_3$ and $k_2\neq 0$ then $ab=ba$, a contradiction. So $k_2=0$ and consequently $k_1=0$ which gives $(**)$.

Case 2. From associativity of multiplication: $l_4=k_2$, $l_3=k_1$, $k_1m_2=k_2m_1$, $k_1m_4=k_2m_3$. It is easy to check that for $k_2=0$ we obtain $(**)$ (for $k_1=0$) or $(*')$ (for $k_1\neq 0$). If $k_2\neq 0$ we have $(*')$ (for $k_1=0$) and $(*')$ for $k_1\neq 0$ by $a\mapsto k_1a+k_1b-pm_1/k_1\cdot 1$ and $b\mapsto k_2b-pm_3/k_1\cdot 1$.

Case 3. This ring is anti-isomorphic to the ring of Case 2, so we get $(*)$ or $(**)$ again.

Case 4 is Case 1 with a and b instead of b and a, respectively.

In the proof of Theorem 3 we show that the rings $(*)$ and $(*')$ of Lemma 2 are embeddable into matrices of order two. Concerning the ring $(**)$ we have:

LEMMA 3. *Let R be the ring $(**)$ of Lemma 2. If R is embeddable into $M_n(C)$, C being commutative, then $p|n$.*

PROOF. We treat the generators a and b as matrices in $M_n(C)$. Let $(x)_{ij}$ be the (i, j)-th element of a matrix x. Of course $(ba)_{ii}-(ab)_{ii}=p(m_3-m_2)\cdot 1$ for $i=1, \ldots, n$. Adding these n equations we obtain $p(m_3-m_2)n\cdot 1=0$ in C. Hence $p|n$.

COROLLARY 3. *The ring $(**)$ of Lemma 2 for $p>2$ cannot be embedded into matrices of order two over any commutative ring.*

Finally we show:

THEOREM 3. *Every ring p^4 elements is embeddable.*

PROOF. Let R be a ring with p^4 elements. By Proposition 5 it suffices to consider the case when R^+ is a direct sum of a cyclic group of order p^2 and two cyclic groups of order p. By Lemma 2 $R^+\simeq(1)\times(a)\times(b)$, where $o(1)=p^2$, $o(a)=o(b)=p$ and we have three cases.

Case $(*)$. We show that the ring $(*)$ is embeddable into matrices of order two over some commutative ring C. Let the matrix $\begin{pmatrix} c & d \\ u & v \end{pmatrix}$ be the image of a and the matrix $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ the image of b in $M_2(C)$. Putting these matrices to the equations describing the products of a and b we get 16 equations, where — for simplification — we set $u=t$ and $c=d=v=x=y=z=0$. The coefficient t we treat as a generator of order p of C^+ and the resulting equations as relations describing multiplication in C. Thus we get an embedding of R into $M_2(C)$ via

$$a\to\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},\quad b\to\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix},$$

where $C^+\simeq(1)\times(t)$, $o(1)=p^2$, $o(t)=p$ and $tt=t$.

Similarly we treat Case $(*')$.

Case $(**)$. We embed R into matrices of order p over some commutative ring C. Using the same method as above we see that the mapping

$$a \mapsto \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ & a_{22} & & 0 \\ & & \ddots & \\ 0 & & & a_{pp} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} b_{11} & & & 0 \\ b_{21} & b_{22} & & \\ \vdots & 0 & \ddots & \\ b_{p1} & & & b_{pp} \end{pmatrix}$$

where a_{ij}, b_{kl} are generators of order p of C^+ with nonzero basis products $a_{ii}a_{ii}=pm_1\cdot 1$, $b_{ii}b_{ii}=pm_4\cdot 1$ for $i=1, \dots, p$ and $a_{11}b_{11}=b_{11}a_{11}=pm_3\cdot 1$, $a_{ii}b_{ii}=b_{ii}a_{ii}=pm_2\cdot 1$ for $i=2, \dots, p$ and $a_{1j}b_{1j}=b_{j1}a_{1j}=p(m_3-m_2)\cdot 1$ for $j=2, \dots, p$ gives an embedding of R into $M_p(C)$. By Lemma 3 this embedding is minimal with respect to the order of matrices.

COROLLARY 4. *The smallest not embeddable ring has order* 32.

PROOF. Let R be a ring of order m. If $m=p_1^{n_1}\dots p_k^{n_k}$ is the prime decomposition of m then $R\simeq \bigoplus_{i=1}^{k} R_i$, where R_i has order $p_i^{n_i}$ and our thesis results from Proposition 2, Theorem 3 and Example with $p=2$.

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Case 1. By associativity of multiplication we have [illegible] or [illegible] Suppose $k_2 \neq k_3$. Then [illegible] $=pm_2/k_3\cdot 1$, so we can assume $k_2=k_3$.

If $k_2 \neq k_3$ then [illegible] $=0$ and [illegible] $=pm_2/k_3\cdot 1$ gives us $(*)$.

If $k_2=k_3$ and [illegible] $\neq 0$ then [illegible], a contradiction. So $k_3=0$ and consequently $k_1=0$, which gives $(*)$.

Case 2. From associativity of multiplication: [illegible]. It is easy to check that for [illegible] we obtain $(*)$ (for $k_1 \neq 0$) or [illegible]

[illegible]

$(**)$ again.

Case 4. [illegible]

In the proof of Theorem 3 we show that the rings $(*)$ and $(*')$ of Lemma 2 are [illegible]

[illegible] *C being commutative, then p|n.*

PROOF. We treat the generators a and b as matrices in $M_n(C)$. Let $(x)_{ij}$ be the (i, j)-th element of a matrix x. Of course, [illegible] $=p(m_2-m_1)\cdot 1$ for $i=1, \dots, n$. Adding these n equations we obtain [illegible] $=0$ in C. Hence $p|n$.

COROLLARY [illegible]

[illegible]

PROOF. Let R be a ring with p^5 elements. By Pro[illegible] suffices to consider the case when R^+ is a [illegible]

[illegible] over some commutative ring C. [illegible]

[illegible] the image of b in $M_2(C)$. [illegible] equations describing the products of a and b we get 16 equations, where — for simplification — we set [illegible] The coefficient t we treat as [illegible] order p of C^+ and the resulting equations as relations describing multiplication in C. Thus we get an embedding of R into $M_2(C)$ via

[illegible]

where [illegible]

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ESTIMATION OF THE GREEN FUNCTION OF THE SINGULAR SCHRÖDINGER OPERATOR

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Dedicated to Professor L. Leindler on the occasion of his 50-th birthday

Let Ω be an arbitrary bounded domain in R^N $(N>3)$ having C^∞-smooth boundary, and q an arbitrary non negative function from the class $L_2(\Omega)$. Consider the Schrödinger operator

$$L = L(x, D) = -\Delta + q(x).$$

Denote $\hat{L}$ an arbitrary positive selfadjoint extension of the operator L from the domain $C_0^\infty(\Omega)$ with discrete spectrum. According to a well known theorem of K. O. Friedrichs there exists such an extension [3]. Denote $0<\lambda_1 \leqq \lambda_2 \leqq \dots$ the sequence of eigenvalues and $\{u_n\}_1^\infty$ the complete orthonormal system of eigenfunctions of the operator $\hat{L}$ in $L_2(\Omega)$. For any $s \geqq 0$ and $f \in L_2(\Omega)$ consider the s-th Riesz means of the spectral expansion of f:

$$(E_\lambda^s f)(x) \overset{\text{def}}{=} \sum_{\lambda_n<\lambda} n \left(1-\frac{\lambda_n}{\lambda}\right)^s (f, u_n)\, u_n(x).$$

The convergence properties of $E_\lambda^s f$ were investigated* in [5] for spherically symmetrical potential q in the case of functions from Liouville classes further in [9] for potential q of the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|} + q_1(x) (= q_0(x) + q_1(x)) \quad (x_0 \in \Omega),$$

where $0 \leqq a \in C^\infty(0, \infty)$ such that

$$t^k |a^{(k)}(t)| \leqq c_\tau t^{\tau-1} \quad (t>0;\ k = 1, 2, \dots, N),$$

$$q_1(x) \in C^N(\Omega), \quad |q_1(x)| \leqq c\,|x-x_0|^l \quad \left(l > \frac{N-4}{2}\right).$$

In the investigation of the convergence properties of $E_\lambda^s f$ an important role is played by the estimation of the Green function $G = G(x, y, \mu)$ of the operator $\hat{L}_\mu \overset{\text{def}}{=} \hat{L} - \mu^2 I$ (i.e. the kernel function of the operator $(\hat{L}-\mu^2 I)^{-1}$).

The aim of the present paper is to give estimation for the Green function G for more general potential than above, namely, we assume only that there exists a function $0 \leqq \omega(t)$ such that

(1) $\omega(t)$ increases,

* The Riesz summability of general orthogonal series was investigated by L. Leindler (e.g. [2]).

(2) $\omega(t)/t$ decreases,

(3) $\displaystyle\int_{+0} \frac{\omega(t)}{t}\,dt < \infty,$

(4) $|q(x)| \leqq \dfrac{\omega(|x-x_0|)}{|x-x_0|^2} \quad (x\in\Omega).$

This result is of some interest, because it seems to be not refinable from point of view of the order of singularity of q (cf. [13]). Define

$$Z \overset{\text{def}}{=} \left\{z\in\mathbf{C}\colon \left|\arg z-\frac{\pi}{2}\right| \leqq \frac{\pi}{2}-\varepsilon,\ \varepsilon>0 \text{ fixed}\right\},\quad B \overset{\text{def}}{=} \{x\in\mathbf{R}^N\colon |x-x_0|<R\},$$

$$B_1 \overset{\text{def}}{=} \left\{x\in\mathbf{R}^N\colon |x-x_0|<\frac{2}{3}R\right\},\quad B_0 \overset{\text{def}}{=} \left\{x\in\mathbf{R}^N\colon |x-x_0|<\frac{1}{3}R\right\},$$

where $0<R<\operatorname{dist}(x,\partial\Omega)\overset{\text{def}}{=}R^*$, and R will be chosen below to be small enough. We shall prove

THEOREM. *If (1)—(4) are fulfilled, then we have for* $N>3$

$$|G(x,y,\mu)| \leqq c_1\frac{1}{|x-y|^{N-2}}e^{-c_2|\mu|\,|x-y|} \quad (\mu\in Z;\ x,y\in B_0), \tag{5}$$

where the positive constants c_1 *and* c_2 *do not depend on* x, y *and* μ.

PROOF. For the proof we need some lemmas. Assume in this work that $x_0=0$ and use the notations

$$a(R) \overset{\text{def}}{=} \int_B \frac{\omega(|u|)}{|u|^N}\,du,\quad I \overset{\text{def}}{=} \int_B \frac{1}{|x-u|^{N-2}}\frac{1}{|y-u|^{N-2}}\frac{\omega(|u|)}{|u|^2}\,du,$$

$$I^* \overset{\text{def}}{=} \int_B \frac{1}{|x-u|^{N-2}}\frac{\omega(|u|)}{|u|^2}\,du,\quad r=|x-y|.$$

LEMMA 1. *For any* $x,y\in B$ *and* $0<R<R^*$

$$|I| \leqq c_3\,a(R)/r^{N-2}, \tag{6}$$

$$|I^*| \leqq c_4\,a(R). \tag{7}$$

The constants c_3 *and* c_4 *do not depend on* x, y *and* R.

PROOF. First we prove (7). Define

$$E_1 \overset{\text{def}}{=} B\cap\{u\in B\colon |x-u| \leqq |y-u|\},$$

$$E_2 \overset{\text{def}}{=} B\cap\{u\in B\colon |x-u| \geqq |y-u|\},\quad I_i^* = \int_{E_i}\cdots \quad (i=1,2).$$

It is enough to estimate I_1^*, because estimation of I_2^* is analogous. Consider

$$E_1' \overset{\text{def}}{=} \{u\in E_1\colon |u| \leqq |x-u|\},\quad E_1'' \overset{\text{def}}{=} \{u\in E_1\colon |u| \geqq |x-u|\}.$$

If $u\in E_1'$, then $\omega(|u|)\le\omega(|x-u|)$ and using the Hölder inequality at $p=N/(N-2)$, $q=N/2$, we obtain

$$I_1^{*\prime}\overset{\text{def}}{=}\int_{E_1'}\ldots\le\int_{E_1'}\frac{\omega^{1/p}(|x-u|)}{|x-u|^{N-2}}\frac{\omega^{1/q}(|u|)}{|u|^2}\,du\le[a(R)]^{1/q}[a(2R)]^{1/p}.$$

If $u\in E_1''$, then $\dfrac{\omega(|u|)}{|u|}\le\dfrac{\omega(|x-u|)}{|x-u|}$ and the Hölder inequality give at $p=N/(N-1)$, $q=N$:

$$I_1^{*\prime\prime}\overset{\text{def}}{=}\int_{E_1''}\ldots\le\int_{E_1''}\frac{\omega^{1/p}(|x-u|)}{|x-u|^{N-1}}\frac{\omega^{1/q}(|u|)}{|u|^1}\,du\le[a(2R)]^{1/p}[a(R)]^{1/q}.$$

Now we prove (4). If $u\in E_1$, then $r=|x-y|\le|x-u|+|u-y|\le 2|u-y|$, hence $|y-u|\ge r/2$. We get:

$$I_1\overset{\text{def}}{=}\int_{E_1}\le\left(\frac{2}{r}\right)^{N-2}\int_{E_1}\frac{1}{|x-u|^{N-2}}\frac{\omega(|u|)}{|u|^2}\,du\le\left(\frac{2}{r}\right)^{N-2}I^*.$$

Lemma 1 is proved.

Now we construct and estimate a special fundamental solution $E(x,y,\mu)$ of the operator $\hat{L}_\mu$ and then we shall estimate the difference between E and the Green function G of the operator $\hat{L}_\mu$. Hence we obtain the desired estimate (5) for G.

Using the method of E. E. Levi [11] first we construct a fundamental solution $E(x,y,\mu)$ of the operator $\hat{L}-\mu^2 I$, i.e. a function for which

$$(-\Delta+q(x)\cdot-\mu^2 I)E(x,y,\mu)=\delta(x-y)\quad(x,y\in B).$$

In the case $q\equiv 0$, the fundamental solution $E_0(x,y,\mu)$ which decreases exponentially for $\operatorname{Im}\mu>0$ is ([11], 13.7.2)

$$E_0(x,y,\mu)=c_N(\mu/r)^{N/2-1}H^{(1)}_{N/2-1}(r\mu).$$

Here $H_\nu^{(1)}(z)$ denotes the ν-th Hankel function of first order. Obviously, the exponentially decreasing fundamental solution E is the solution of the integral equation

$$E(x,y,\mu)=E_0(x,y,\mu)-\int_B E_0(x,u,\mu)E(u,y,\mu)q(u)\,du. \tag{8}$$

Indeed, it is well known that $L_0E_0(x,y,\mu)=\delta(x-y)$ for the operator $L_0\overset{\text{def}}{=}-\Delta-\mu^2 I$. Now we show that $L_\mu E(x,y,\mu)=\delta(x-y)$ for $L_\mu\overset{\text{def}}{=}L_0+qI$. Let $f,\varphi\in C_0^\infty(B)$ be arbitrary, then**

$$(\hat{E}f,L_\mu\varphi)=\int_B\Big[\int_B E_0(x,y,\mu)f(y)\,dy-$$

$$-\int_B\Big(\int_B E_0(x,u,\mu)E(u,y,\mu)q(u)\,du\Big)f(y)\,dy\Big]L_\mu\varphi(x)\,dx=$$

$$=\int_B f(y)\,dy\int_B E_0(x,y,\mu)L_\mu\varphi(x)\,dx-$$

$$-\int_B f(y)\,dy\int_B E(u,y,\mu)q(u)\,du\int_B E_0(x,u,\mu)L_\mu\varphi(x)\,dx.$$

** The operation ^ is defined by $(\hat{\varphi}f)(x)=\int_\Omega\varphi(x,y)f(y)\,dy$.

According to our assumption

$$\int_B E_0(x, y, \mu) L_\mu \varphi(x)\, dx = \int_B E_0(x, y, \mu)(L_0 + q\cdot)\, \varphi(x)\, dx =$$

$$= \int_B E_0(x, y, \mu) L_0 \varphi(x)\, dx + \int_B E_0(x, y, \mu)\, q(x) \varphi(x)\, dx =$$

$$= \varphi(y) + \int_B E_0(x, y, \mu)\, q(x) \varphi(x)\, dx.$$

Consequently

$$(\hat{E}f, L_\mu \varphi) = \int_B f(y) \Big[\varphi(y) + \int_B E_0(x, y, \mu) q(x) \varphi(x)\, dx\Big]\, dy -$$

$$- \int_B f(y) \int_B E(u, y, \mu)\, q(u)\, du \Big[\varphi(u) + \int_B E_0(x, u, \mu)\, q(x)\, \varphi(x)\, dx\Big]\, dy =$$

$$= (f, \varphi) + \int_B f(y)\, dy \Big[\int_B E_0(x, y, \mu)\, q(x)\, \varphi(x)\, dx -$$

$$- \int_B E(u, y, \mu) q(u) \varphi(u)\, du - \int_B E(u, y, \mu)\, q(u)\, du \int_B E_0(x, u, \mu)\, q(x)\, \varphi(x)\, dx\Big] =$$

$$= (f, \varphi) + \int_B f(y)\, dy \int_B q(x) \varphi(x)\, dx \Big[E_0(x, y, \mu) - E(x, y, \mu) -$$

$$- \int_B E_0(x, u, \mu)\, E(u, y, \mu)\, q(u)\, du\Big] = (f, \varphi),$$

i.e. $(\hat{E}f, L_\mu \varphi) = (f, \varphi)$ $(\varphi \in C_0^\infty(B))$, hence $L_\mu \hat{E} f = f$ for every $f \in C_0^\infty(B)$.
Now define

$$E_k(x, y, \mu) \overset{\text{def}}{=} E_0(x, y, \mu) - \int_B E_0(x, u, \mu)\, E_{k-1}(u, y, \mu)\, q(u)\, du,$$

$$F_0(x, y, \mu) \overset{\text{def}}{=} E_0(x, y, \mu), \quad F_k(x, y, \mu) \overset{\text{def}}{=} E_k(x, y, \mu) - E_{k-1}(x, y, \mu).$$

Obviously

$$E(x, y, \mu) = \sum_{k=0}^{\infty} F_k(x, y, \mu), \tag{9}$$

if the series is uniformly convergent. Furthermore

$$F_k(x, y, \mu) = -\int_B E_0(x, u, \mu)\, F_{k-1}(u, y, \mu)\, q(u)\, du. \tag{10}$$

Our first aim is to prove that the series (9) has good convergence properties. To this end we have to estimate the functions F_k. Define the functions f_k by the equation

$$F_k(x, y, \mu) = e^{-|\mathrm{Im}\,\mu|\,|x-y|} f_k(x, y, \mu).$$

Obviously, we have

$$|f_k(x, y, \mu)| \leqq c_5 \int_B \frac{1}{|x-u|^{N-2}} |f_{k-1}(u, y, \mu)| \frac{\omega(|u|)}{|u|^2} du, \tag{11}$$

because

$$|E_0(x, y, \mu)| \leqq c_6 \frac{1}{|x-y|^{N-2}} e^{-|\mathrm{Im}\,\mu|\,|x-y|} \quad (x, y \in B;\ \mu \in \mathbf{C}) \tag{12}$$

([6], 7.2.1(2), (5); 7.3.1(1)). First we prove the following estimate (by induction on k):

$$|f_k(x, y, \mu)| \leqq [c_0 a(R)]^k |x-y|^{2-N} \quad (x, y \in B;\ \mu \in \mathbf{C}), \tag{13}$$

where $c_0 = c_5 c_3$.

For $k=0$ (13) follows from (12), according to the definition of F_0 and that of f_0. Suppose (13) is fulfilled for $k-1$ and prove it for k. Using (11) and the induction hypothesis, we obtain

$$|f_k(x, y, \mu)| \leqq c_5 \int_B \frac{1}{|x-y|^{N-2}} |f_{k-1}(u, y, \mu)| \frac{\omega(|u|)}{|u|^2} du \leqq$$

$$\leqq c_5 [c_0 a(R)]^{k-1} \int_B \frac{1}{|u-y|^{N-2}} \frac{1}{|x-u|^{N-2}} \frac{\omega(|u|)}{|u|^2} du$$

hence the desired estimate (13) follows using (6).

From (9) and (13) we obtain

LEMMA 2. *For any* $x, y \in B$ *and* $\mu \in \mathbf{C}$

$$|E(x, y, \mu)| \leqq c_7 |x-y|^{2-N} e^{-|\mathrm{Im}\,\mu|\,|x-y|} \tag{14}$$

holds, if $R>0$ *is such that* $c_0 a(R) \leqq 0{,}5 c_7$ *does not depend on* R.

Next we estimate the gradient of E, namely we prove

LEMMA 3. *If* $R \geqq |x| \geqq \frac{3}{4} R$, $|y| \leqq \frac{1}{4} R$ *and* $\mu \in \mathbf{C}$, *then*

$$|\nabla_x E(x, y, \mu)| \leqq c_8(R). \tag{15}$$

The constant $c_8(R)$ *depends on* R *(it may increase as* $R \to +0$*).*

PROOF. From (8)

$$e^{|\mathrm{Im}\,\mu|\,|x-y|} |\nabla_x E(x, y, \mu)| \leqq \frac{c_9}{|x-y|^{N-1}} +$$

$$+ \int_B \frac{1}{|x-u|^{N-1}} \frac{1}{|y-u|^{N-2}} \frac{\omega(|u|)}{|u|^2} du \leqq \frac{c_9}{|x-y|^{N-1}} +$$

$$+ \int_{|u| \leqq R/2} + \int_{R/2 \leqq |u| \leqq (5/8)R} + \int_{(5/8)R \leqq |u| \leqq R} = \frac{c_9}{|x-y|^{N-1}} + \sum_{i=1}^{3} I_i.$$

Using (7) we obtain

$$I_1 \leqq \text{const} \int_{|u|\leqq R/2} \frac{1}{|u-y|^{N-2}} \frac{\omega(|u|)}{|u|^2} \leqq c(R),$$

because in this case also $|x-u| \geqq \text{const}\cdot R$;

$$I_2 \leqq \text{const}\, \frac{\omega(R)}{R^2} \int_{R/2\leqq|u|\leqq(5/8)R} \frac{du}{|u-y|^{N-2}} \leqq c(R),$$

because in this case also $|x-u|\geqq \text{const}\cdot R$; and at last

$$I_3 \leqq c(R) \int_{(5/8)R\leqq|u|\leqq R} \frac{du}{|x-u|^{N-1}} \leqq \tilde{c}(R),$$

because in this case $|y-u|\geqq\text{const}\cdot R$. Summarising our estimates that of (15) follows. Lemma 3 is proved.

We have proved after (8) that for the fundamental solution $E(x, y, \mu)$ we have

$$(16) \qquad (L(x, D)-\mu^2 I) \int_\Omega E(x, y, \mu) f(y)\, dy = f(x), \quad x\in B$$

for any $f\in C_0^\infty(\Omega)$, $\operatorname{supp} f\subset \bar{B}$.

Now define $E^*(x, y, \mu) \overset{\text{def}}{=} E(y, x, \mu)$ "the formal adjoint of E". Then we have for any $f\in C_0^\infty(\Omega)$, $\operatorname{supp} f\subset\bar{B}$, the equation

$$(17) \qquad \int_\Omega E^*(x, y, \mu)[L(y, D)-\mu^2 I] f(y)\, dy = f(x), \quad x\in B.$$

Indeed, for any $\varphi\in C_0^\infty(\Omega)$

$$\int_\Omega \varphi(x)\, dx \int_\Omega E^*(x, y, \mu) L_\mu f(y)\, dy = \int_\Omega L_\mu f(y)\, dy \int_\Omega E^*(x, y, \mu)\varphi(x)\, dx =$$

$$= \int_\Omega L_\mu f(x)\, dx \int_\Omega E(x, y, \mu)\,\varphi(y)\, dy =$$

$$= \int_\Omega f(x) \Big(L_\mu \int_\Omega E(x, y, \mu)\varphi(y)\, dy\Big)\, dx = \int_\Omega f(x)\,\varphi(x)\, dx = (f, \varphi),$$

and hence (17) follows.

Now let $\eta\in C_0^\infty(\Omega)$ be such that $\eta(x)=1$ for $x\in B_1$, and define

$$H(x, y, \mu) \overset{\text{def}}{=} \eta(x)\, E(x, y, \mu), \quad H^*(x, y, \mu) \overset{\text{def}}{=} \eta(y)\, E(y, x, \mu) = \eta(y)\, E^*(x, y, \mu),$$

$$K(x, y, \mu) \overset{\text{def}}{=} 2(\nabla_x \eta(x))\nabla_x E(x, y, \mu) + (\Delta_x \eta(x))\, E(x, y, \mu),$$

$$K^*(x, y, \mu) \overset{\text{def}}{=} 2(\nabla_y \eta(y))\nabla_y E^*(x, y, \mu) + (\Delta_y \eta(y))\, E^*(x, y, \mu).$$

Obviously

$$[L(x, D)-\mu^2 I]\, H(x, y, \mu) = \eta(x)[L(x, D)-\mu^2 I]\, E(x, y, \mu) - K(x, y, \mu),$$

$$[L(y, D)-\mu^2 I]\, H^*(x, y, \mu) = \eta(y)[L(y, D)-\mu^2 I]\, E^*(x, y, \mu) - K^*(x, y, \mu),$$

hence, for any $f \in C_0^\infty(\Omega)$ we get

$$\int_B H^*(x, y, \mu)[L(y, D)-\mu^2 I] f(y)\, dy = \int_B f(y)[L(y, D)-\mu^2 I] H^*(x, y, \mu)\, dy =$$

$$= \int_B f(y)\eta(y)[L(y, D)-\mu^2 I] E^*(x, y, \mu)\, dy - \int_B K^*(x, y, \mu) f(y)\, dy =$$

$$= f(x)\eta(x) - \hat{K}^* f(x),$$

i.e.

(18) $$\hat{H}^*(L-\mu^2 I) f(x) = f(x) - \hat{K}^* f(x) \quad (f \in C_0^\infty(\Omega), x \in B_0).$$

On the other hand, for any $f \in C_0^\infty(B)$ we have

$$[L(x, D)-\mu^2 I] \int_B H(x, y, \mu) f(y)\, dy =$$

$$= \eta(x)[L(x, D)-\mu^2 I] \int_B E(x, y, \mu) f(y)\, dy - \int_B K(x, y, \mu) f(y)\, dy =$$

$$= \eta(x) f(x) - \hat{K} f(x),$$

i.e.

(19) $$[L-\mu^2 I]\hat{H} f(x) = f(x) - \hat{K} f(x) \quad (f \in C_0^\infty(B), x \in B_0).$$

Hence we have for any $f \in C_0^\infty(B)$ and $\varphi \in C_0^\infty(B)$

(20) $$(\hat{H} f, L_\mu(x, D)\varphi) = (f - \hat{K} f, \varphi).$$

Let $f \in L_2(B)$ be arbitrary and $\{f_n\} \subset C_0^\infty(B)$ be such that $f_n \xrightarrow{L_2} f$. Then (20) follows for every $f \in L_2(B)$. Because $C_0^\infty(B) \subset C_0^\infty(\Omega) \subset D(\bar{L}_\mu)$, so we have $(\hat{H} f, \bar{L}_\mu \varphi) = (f - \hat{K} f, \varphi)$ hence $\hat{H} f \in D(\hat{L}_\mu^*)$ and $\hat{L}_\mu^* \hat{H} f = f - \hat{K} f$ for every $x \in \Omega$. Thus

(21) $$(\hat{L}_\mu^* \hat{H} f)(x) = (f - \hat{K} f)(x)$$

for every $f \in L_2(B)$ and $x \in \Omega$.

LEMMA 4. *Suppose $N > 3$ and $f \in L_2(\Omega)$. Then*

(22) $$\|qf\|_{L_p} \le c_{10} \|f\|_{L_2} \quad (p = 2N/(N+4)).$$

The constant c_{10} does not depend on f.

PROOF. Using the generalized Hölder inequality we get

$$\|qf\|_{L_p} \le \|q\|_{L_{p_1}} \|f\|_{L_{q_1}} \quad (p_1^{-1} + q_1^{-1} = p^{-1}).$$

On the other hand $q \in L_{N/2}(\Omega)$ because

$$\int_\Omega |q(x)|^{N/2}\, dx \le \int_\Omega \frac{\omega(|x-x_0|)}{|x-x_0|^N} \le \text{const.} \int_{+0} r^{N-1} \frac{\omega(r)}{r^N}\, dr = c \int_{+0} \frac{\omega(r)}{r}\, dr < \infty.$$

Choose $p_1 = N/2$, then $q_1^{-1} = p^{-1} - p_1^{-1} = 2^{-1} + 2N^{-1} - 2N^{-1} = 1/2$, i.e. $q_1 = 2$. Lemma 4 is proved.

LEMMA 5. *Let* $f\in W_p^2(\Omega)$, $p=2N/(N+4)$. *Then*

(23) $$\|qf\|_{L_p}\leqq c_{11}\|f\|_{W_p^2}.$$

PROOF. Using the imbedding $W_p^2\to L_2((N/p)-2=N/2)$, the estimate (23) follows from (22).

LEMMA 6. *Suppose* $\{f_n\}\subset W_p^2(\Omega)$, $f\in W_p^2(\Omega)$ *and* $f_n\xrightarrow{W_p^2}f\ (n\to\infty)$. *Then*

(24) $$\|Lf_n-Lf\|_{L_p}\to 0\quad (n\to\infty,\ p=2N/(N+4)).$$

PROOF. (24) follows from the following inequalities:

$$\|Lf_n-Lf\|_{L_p}\leqq\|\Delta(f_n-f)\|_{L_p}+\|q(f_n-f)\|_{L_p}\leqq c\|f_n-f\|_{W_p^2}+c_{11}\|f_n-f\|_{W_p^2}.$$

Lemma 6 is proved.

LEMMA 7. *For any* $f\in W_p^{2,\mathrm{loc}}(\Omega)$ *we have*

(25) $$\hat{H}^*(L-\mu^2I)f\overset{L_p}{=}f-\hat{K}^*f\quad (p=2N/(N+4)).$$

PROOF. Let $f\in W_p^{2,\mathrm{loc}}(\Omega)$ be arbitrary and $\{f_n\}\subset C_0^\infty(\Omega)$ be a sequence, for which $\|f_n-f\|_{W_2^p(B)}\to 0\ (n\to\infty)$. Using Lemma 6 we obtain

$$g_n\overset{\mathrm{def}}{=}(L-\mu^2I)f_n\xrightarrow{L_p}(L-\mu^2I)f\overset{\mathrm{def}}{=}g\quad (n\to\infty),$$

i.e. $\|g_n-g\|_{L_p}\to 0\ (n\to\infty)$, hence $\|\hat{H}^*g_n-\hat{H}^*g\|_{L_p}\to 0\ (n\to\infty)$. Using (18), (25) follows. Lemma 7 is proved.

LEMMA 8. *Suppose* $u,f\in L_2(\Omega)$ *and* $L_\mu(x,D)u=f$ *in the distribution sense (i.e. in* $D'(\Omega)$*). Then* $u\in W_p^{2,\mathrm{loc}}(\Omega)$ *if* $p=2N/(N+4)$.

PROOF. Obviously,

$$L_\mu(x,D)u(x)=-\Delta u(x)+q(x)u(x)-\mu^2u(x)=f(x),$$

hence by Lemma 5 we get $-\Delta u(x)=f(x)+\mu^2u(x)-q(x)u(x)\in L_p$, and taking into account Triebel [12], 6.4.1 and 2.3.4 we obtain $u\in W_p^{2,\mathrm{loc}}(\Omega)$. Lemma 8 is proved.
Denote by $\sigma(\hat{L})$ the spectrum of $\hat{L}$.

LEMMA 9. *Suppose* $f\in L_2(\Omega)$ *and* $\lambda\in\mathbf{C}\setminus\sigma(\hat{L})$. *Then* $(\hat{L}-\lambda I)^{-1}f\in W_p^{2,\mathrm{loc}}(\Omega)$.

PROOF. Let $g\overset{\mathrm{def}}{=}(\hat{L}-\lambda I)^{-1}f$. Then for every $\varphi\in C_0^\infty(\Omega)$ we have

$$(f,\varphi)=((\hat{L}-\lambda I)g,\varphi)=(g,(\hat{L}-\lambda I)\varphi).$$

Since $\varphi\in C_0^\infty(\Omega)\subset D(\hat{L})$, we have $(\hat{L}-\lambda I)\varphi=L(x,D)\varphi-\lambda\varphi$ hence $(f,\varphi)=(g,(L(x,D)-\lambda)\varphi)$ i.e. $[L(x,D)-\lambda I]g\overset{D'(\Omega)}{=}f$. Taking into account the assumption $f\in L_2(\Omega)$, Lemma 8 gives $g\in W_p^{2,\mathrm{loc}}(\Omega)$. Lemma 9 is proved.

LEMMA 10. *For any* $f\in L_2(\Omega)$ *and* $x\in B_0$

(26) $$\hat{G}_\mu f(x)-\hat{H}^*f(x)=\hat{K}^*\hat{G}_\mu f(x),\quad (\mu^2\in\mathbf{C}\setminus\sigma(\hat{L})).$$

PROOF. Let $f\in C_0^\infty(\Omega)$, then by Lemma 8 we get $\hat{G}_\mu f\in W_p^{2,\mathrm{loc}}(\Omega)$. According to Lemma 7

$$\hat{H}^*(L-\mu^2 I)\hat{G}_\mu f \overset{L_p}{=} \hat{G}_\mu f - \hat{K}^*\hat{G}_\mu f$$

i.e. $\hat{H}^* f = \hat{G}_\mu f - \hat{K}^*\hat{G}_\mu f$. If $f\in L_2(\Omega)$ then pick a sequence $\{f_n\}\subset C_0^\infty(\Omega)$ such that $f_n \xrightarrow{L_2} f$. Lemma 10 follows.

LEMMA 11. *For any* $f\in L_2(B)$ *and* $x\in\Omega$

$$\hat{G}_\mu f(x) - \hat{H}f(x) = \hat{G}_\mu \hat{K}f(x), \quad (\mu^2\in \mathbf{C}\setminus\sigma(\hat{L})). \tag{27}$$

PROOF. From (21) it follows for any $f\in L_2(B)$ $\hat{G}_\mu(L-\mu^2 I)\hat{H}f = \hat{G}_\mu f - \hat{G}_\mu\hat{K}f$. Lemma 11 is proved.

LEMMA 12. *For any* $f\in L_2(B)$ *and* $x\in B_0$

$$\hat{G}_\mu f(x) - \hat{H}^* f(x) = \hat{K}^*\hat{H}f(x) + \hat{K}^*\hat{G}_\mu\hat{K}f(x). \tag{28}$$

PROOF. This equality follows immediately from (26) and (27). Lemma 12 is proved.

LEMMA 13. *For every* $f\in L_2(B)$ *and* $x\in B_0$

$$|\hat{G}_\mu f(x) - \hat{H}^* f(x)| \leqq c_{12}\frac{1}{d(\mu)} e^{-c_{13}|\mathrm{Im}\,\mu|}\, \|f\|_{L_2(B)}, \tag{29}$$

where $d(\mu) \overset{\mathrm{def}}{=} \mathrm{dist}\,(\mu, \sigma(\hat{L}))$.

PROOF. We know that

$$|\hat{K}^*\hat{H}f(x)| \leqq ce^{-c_{13}|\mathrm{Im}\,\mu|}\|\hat{H}f\|_{L_1} \leqq ce^{-c_{13}|\mathrm{Im}\,\mu|}\|f\|_{L_2},$$

and

$$|\hat{K}^*\hat{G}_\mu\hat{K}f(x)| \leqq ce^{-c_{13}|\mathrm{Im}\,\mu|}\,\|\hat{G}_\mu\hat{K}f\|_{L_2} \leqq c\frac{1}{d(\mu)} e^{-c_{13}|\mathrm{Im}\,\mu|}\,\|\hat{K}f\|_{L_2},$$

hence

$$|\hat{K}^* G_\mu \hat{K} f(x)| \leqq c_{14}\frac{1}{d(\mu)} e^{-c_{13}|\mathrm{Im}\,\mu|}\,\|f\|_{L_2}.$$

Summarising our estimates we get (29). Lemma 13 is proved.

COROLLARY 1. *For any* $x\in B_0$, $y\in B$ *and* $\mu\in Z$

$$|G(x, y, \mu) - H(x, y, \mu)| \leqq c_{15}e^{-c_{16}|\mu|}. \tag{30}$$

COROLLARY 2. *For any* $x\in B_0$, $y\in B_0$, $\mu\in Z$

$$|G(x, y, \mu)| \leqq c_{16}\,|x-y|^{2-N}\, e^{-17\,|\mu|\,|x-y|},$$

and this is the statement of the Theorem.

The Theorem is proved.

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GENERAL LINEAR SUMMATION OF THE VILENKIN—FOURIER SERIES

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Some special summation of Vilenkin—Fourier series [7] of functions in $L^p(0, 1)$ $(1 \leqq p \leqq \infty)$ were investigated by S. Yano, A. Efimov, I. Yastrebova, S. Blumin, Su Weiyi and the author etc. (see [1]—[5]). In [1] Blumin also discussed the general summation. This paper is devoted to extending some results in [5] for the case $1 \leqq p \leqq \infty$ and some specific discussions for the case $1<p<\infty$.

The notions and the symbols in this paper are basically the same as in [5], but some of them have been revised.

§ 1. Notions and symbols

1. Let $\mathbf{N}:=\{1, 2, \ldots\}$, $\mathbf{P}:=\{0, 1, 2, \ldots\}$, $\{m_j\}_{j\in \mathbf{P}}$ be a sequence of integers each greater than 1, $\sup m_j<\infty$. $\mathbf{Z}_j:=\{0, 1, \ldots, m_j-1\}$, $M_0:=1$, $M_{j+1}:=m_jM_j$ $(j\in\mathbf{P})$.

2. If $x, y\in[0, 1)$ and their expansions respectively are $x=\sum_{j=0}^{\infty} x_j M_{j+1}^{-1}$, $y=\sum_{j=0}^{\infty} y_j M_{j+1}^{-1}$ $(x_j, y_j\in\mathbf{Z}_j)$, then let $x\ominus y:=\sum_{j=0}^{\infty}(x_j-y_j) \pmod{m_j} M_{j+1}^{-1}$.

3. Let $\varphi_k(x)=\exp\frac{2\pi i}{m_k}x_k$, $\psi_k(x):=\prod_{j=0}^{\infty}(\varphi_j(x))^{k_j}$, where $x=\sum_{j=0}^{\infty}x_jM_{j+1}^{-1}$ $(x_j\in\mathbf{Z}_j)$, $k=\sum_{j=0}^{\infty}k_jM_j\in\mathbf{P}$ $(k_j\in\mathbf{Z}_j)$. $\{\psi_k\}_{k\in\mathbf{P}}$ is called the Vilenkin function system.

4. $D_n(x):=\sum_{j=0}^{n-1}\psi_j(x)$, $F_n(x):=\frac{1}{n}\sum_{j=1}^{n}D_j(x)$ $(n\in\mathbf{N})$.

5. Let $f, g\in L^p[0, 1)$ $(1\leqq p\leqq\infty)$, $f^\wedge(k):=\int_0^1 f(t)\bar\psi_k(t)\,dt$ $(k\in\mathbf{N})$, $(f*g)(x):=$ $:=\int_0^1 f(x\ominus t)g(t)\,dt$, $S_n(f):=f*D_n$ $(n\in\mathbf{N})$.

6. $\omega(f, \delta):=\omega(L^p[0, 1), f, \delta):=\sup_{0\leqq k<\delta}\|f(\cdot\ominus h)-f(\cdot)\|_p$ $(f\in L^p[0, 1),\ 1\leqq p\leqq\infty$, $\delta>0)$.

7. Let G and H be functions or functionals defined on a set U. If there exist positive numbers α_1 and α_2 such that for all $x\in U$, α, $G(x)\leqq H(x)\leqq\alpha_2G(x)$, we will denote it by $G\sim H$.

8. Let $f\in L^p[0, 1)$ $(1\leqq p\leqq\infty)$, $T_r^{(\alpha)}(t):=\sum_{k=0}^{M_r-1}k^\alpha\psi_k(t)$ (α is real, $0^\alpha:=1$ if $\alpha<0$).

If there exists $g\in L^p[0,1)$ such that $\lim_{r\to\infty}\|T_r^{\langle\alpha\rangle}*f-g\|_p=0$, then if $\alpha>0$, g is called the (strong) derivative of order α of f in $L^p[0,1)$; if $\alpha<0$, g is called the (strong) integral of order $(-\alpha)$ of f in $L^p[0,1)$. In both cases g will be denoted by $T^{\langle\alpha\rangle}f$.

§ 2. The case $1\leqq p\leqq\infty$

Blumin [1] gave the following result:

If $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$, $K_n=1+\sum_{k=1}^{n-1}c_k\psi_k$ $(n\in\mathbf{N})$, then

$$\|f-f*K_n\|_p\leqq c\Big\{(1+\|K_n\|_1)E_n(f)+\sum_{l=0}^{r}\Big\|\sum_{k=1}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1E_{M_l}(f)\Big\} \tag{2.1}$$

where $M_r\leqq n<M_{r+1}$, $E_n(f):=\inf\{\|f-g\|_p\colon g^\wedge(k)=0\ (\mathbf{N}\ni k\geqq n)\}$ and $C>0$ is an absolute constant.

We will give another estimate by a simpler method. First we show

LEMMA 1. *If* $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$, $g\in L^1[0,1)$, $g^\wedge(k)=0$ $(k=0,1,2,\dots$ $\dots,M_r-1,\ r\in\mathbf{N})$, *then*

$$\|f*g\|_p\leqq\omega\Big(f,\frac{1}{M_r}\Big)\|g\|_1. \tag{2.2}$$

PROOF. Since for any $k\in\mathbf{P}$,

$$f^\wedge(k)g^\wedge(k)=(f^\wedge(k)-(S_{M_r}(f))^\wedge(k))g^\wedge(k),$$

by the uniqueness theorem of Fourier transform, we get $f*g=(f-S_{M_r}(f))*g$. Hence by [8]

$$\|f*g\|_p\leqq\|f-S_{M_r}(f)\|_p\|g\|_1\leqq\omega\Big(f,\frac{1}{M_r}\Big)\|g\|_1.$$

THEOREM 1. *If* $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$, $K_\varrho\in L^1[0,1)$ *(ϱ belongs to a set of indices),* $K_\varrho^\wedge(0)=1$, $K_\varrho^\wedge(k):=C_k(\varrho):=C_k$ $(k\in\mathbf{N})$, *then for any positive integer* r

$$\|f-f*K_\varrho\|_p\leqq\omega\Big(f,\frac{1}{M_r}\Big)+\sum_{l=0}^{r-1}\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1\omega\Big(f,\frac{1}{M_l}\Big)+ \tag{2.3}$$

$$+\sum_{l=r}^{\infty}\Big\|\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1\omega\Big(f,\frac{1}{M_l}\Big).$$

PROOF. Since the sequence of the M_r-th partial sums of a Vilenkin—Fourier series of an integrable function is convergent in $L^1[0,1)$ (see e.g. [7]), i.e. $K_\varrho=1+$ $+\sum_{l=0}^{\infty}\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k$, we have for all $s\geqq r$ $(s\in\mathbf{N})$

$$\|f-f*K_\varrho\|_p\leqq\|f-S_{M_r}(f)\|_p+\Big\|f*\sum_{l=0}^{r-1}\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_p+$$

$$+\Big\|f*\sum_{l=r}^{s}\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_p+\Big\|f*\sum_{l=s+1}^{\infty}\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_p.$$

By the above Lemma we have

$$(2.4)\qquad \|f-f*K_\varrho\|_p \leqq \omega\left(f,\frac{1}{M_r}\right)+\sum_{l=0}^{r-1}\omega\left(f,\frac{1}{M_l}\right)\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\varphi_k\Big\|_1+$$

$$+\sum_{l=r}^{s}\omega\left(f,\frac{1}{M_l}\right)\Big\|\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1+\omega\left(f,\frac{1}{M_s}\right)\Big\|\sum_{l=s+1}^{\infty}\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1.$$

Moreover, by [8]

$$\Big\|\sum_{l=s+1}^{\infty}\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1=\|K_\varrho-S_{M_s}(K_\varrho)\|_1\leqq\omega\left(L^1[0,1),K_\varrho,\frac{1}{M_s}\right).$$

In (2.4) letting $s\to\infty$, we get (2.3).

COROLLARY 1.1. *If* $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$, $K_n=1+\sum_{k=1}^{n-1}c_k\psi_k$ $(n\in\mathbf{N})$, *then*

$$(2.5)\qquad \|f-f*K_n\|_p\leqq\sum_{l=0}^{r-1}\omega\left(f,\frac{1}{M_l}\right)\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1+\omega\left(f,\frac{1}{M_r}\right)\left(1+\Big\|\sum_{k=M_r}^{n-1}c_k\psi_k\Big\|_1\right),$$

where $r\in\mathbf{P}$ *is defined by* $M_r\leqq n<M_{r+1}$.

NOTE. In some cases the estimation (2.5) is more accurate than (2.1). For example, let $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$ with $\omega\left(f,\frac{1}{M_j}\right)=\frac{1}{j+1}$ and the kernel $K_n=1+\left(1+\frac{1}{\ln n}\right)\psi_1+\sum_{j=2}^{M_r-1}\psi_j$ $(M_r\leqq n<M_{r+1})$, then by (2.1)

$$\|f-f*K_n\|_p\leqq c\left\{\sum_{l=0}^{r}\frac{1}{l+1}E_{M_l}(f)\right\}\sim c\left\{\sum_{l=0}^{r}\frac{1}{l+1}\right\}\frac{1}{\ln n}\sim c\frac{\ln\ln n}{\ln n},$$

and by (2.5)

$$\|f-f*K_n\|_p\leqq\omega\left(f,\frac{1}{M_0}\right)\Big\|\frac{\psi_1}{\ln n}\Big\|_1=\frac{1}{\ln n}.$$

If $T^{\langle\alpha\rangle}f\in L^p[0,1)$ $(\alpha\geqq0)$, then using the inequality (see [5])

$$\omega\left(f,\frac{1}{M_l}\right)\leqq M_l^{-\alpha}\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\right)\quad(l\in\mathbf{P})$$

we may give the following

COROLLARY 1.2. *If* $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$ *and* $T^{\langle\alpha\rangle}f\in L^p[0,1)$ $(\alpha\geqq0)$, *then for all* $r\in\mathbf{N}$

$$(2.6)\qquad \|f-f*K_\varrho\|_p\leqq\sum_{l=0}^{r-1}\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1M_l^{-\alpha}\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\right)+$$

$$+M_r^{-\alpha}\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_r}\right)+\sum_{l=r}^{\infty}\Big\|\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1M_l^{-\alpha}\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\right),$$

resp. for all $n\in\mathbf{N}$

$$\|f-f*K_n\|_p \leqq \sum_{l=0}^{r-1} \Big\| \sum_{k=M_l}^{M_{l+1}-1} (1-c_k)\psi_k \Big\|_1 M_l^{-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f, \frac{1}{M_l}\right)+ \tag{2.7}$$

$$+\left(1+\Big\| \sum_{k=M_l}^{n-1} c_k\psi_k \Big\|_1\right) M_r^{-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f, \frac{1}{M_r}\right)$$

where $r\in\mathbf{P}$ *and* $M_r\leqq n<M_{r+1}$.

In [5] we discussed the typical means of Vilenkin—Fourier series and gave the following result:

If $T^{\langle\alpha\rangle}f\in L^p[0,1)$, $K_n=\sum_{k=0}^{n-1}\left(1-\left(\frac{k}{n}\right)^\lambda\right)\psi_k$ ($\alpha\geqq 0$, $\lambda>0$, $n\in\mathbf{N}$, $p\geqq 1$), then

$$\|f-f*K_n\|_p = O(1)\frac{1}{M_r^\lambda}\sum_{l=0}^{r} M_l^{\lambda-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f, \frac{1}{M_l}\right), \tag{2.8}$$

where $r\in\mathbf{P}$ and $M_r\leqq n<M_{r+1}$.

Now we use (2.6) to deal with the Abel—Cartwright mean. We first give a lemma.

LEMMA 2. *We have*

$$\Big\|\sum_{k=M_l}^{M_{l+1}-1} c_k\psi_k\Big\|_1 = O(1)\Big[\sum_{k=M_l}^{M_{l+1}-3} k|\Delta^2 c_k| + M_l \max_{M_l\leqq k\leqq M_{l+1}-2}|\Delta c_k| + \max_{M_l\leqq k\leqq M_{l+1}-1}|c_k|\Big],$$

where $\Delta^2 c_k=c_k-2c_{k+1}+c_{k+2}$, $\Delta c_k=c_k-c_{k+1}$ $(k, l\in\mathbf{P})$.

PROOF. Applying Abel's transform twice, we get

$$\Big\|\sum_{k=M_l}^{M_{l+1}-1} c_k\psi_k\Big\|_1 = \Big\|\sum_{l=M_l}^{M_{l+1}-3} \Delta^2 c_k (k+1) F_{k+1} - \Delta c_{M_l} M_l F_{M_l} +$$

$$+\Delta c_{M_{l+1}-2}(M_{l+1}-1)F_{M_{l+1}-1} - c_{M_l}D_{M_l} + c_{M_{l+1}-1}D_{M_{l+1}}\Big\|_1.$$

Taking into account $\|D_{M_l}\|_1$ and $\|F_k\|_1=O(1)$ $(l, k\in\mathbf{P}$, see e.g. [8]), we get Lemma 2.

THEOREM 2. *If* $f\in L^p[0,1)$ $(1\leqq p\leqq\infty)$ *and* $T^{\langle\alpha\rangle}f\in L^p[0,1)$ $(\alpha\geqq 0)$ *and*

$$K_\varrho=\sum_{k=0}^{\infty}\varrho^{k\lambda}\psi_k \quad (\lambda>0)$$

then

$$\|f-f*K_\varrho\|_p = O(1)\frac{1}{M_r^\lambda}\sum_{l=0}^{r} M_l^{\lambda-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f, \frac{1}{M_l}\right) \quad (\varrho\to 1-0), \tag{2.9}$$

where $r\in\mathbf{P}$ *and* $\frac{1}{M_r^\lambda}\leqq 1-\varrho<\frac{1}{M_{r-1}^\lambda}$.

It is interesting to see the similarity between (2.8) and (2.9).

PROOF. By (2.6) (with $c_k:=\varrho^{k^\lambda}$ $(k\in\mathbf{N})$)

$$\|f-f*K_\varrho\|_p \leqq \sum_{l=0}^{r-1}\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1 M_l^{-\alpha}\,\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\Big)+M_r^{-\alpha}\,\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_r}\Big)+$$

$$+\sum_{l=r}^{\infty}\Big\|\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1 M_l^{-\alpha}\,\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\Big).$$

By Lemma 2,

$$\Big\|\sum_{k=M_l}^{M_{l+1}-1}c_k\psi_k\Big\|_1=O(1)\Big(\sum_{k=M_l}^{M_{l+1}-3}k|\Delta^2c_k|+M_l\max_{M_l\leqq k\leqq M_{l+1}-2}|\Delta c_k|+\max_{M_l\leqq k\leqq M_{l+1}-1}|c_k|\Big),$$

$$\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1=O(1)\Big(\sum_{k=M_l}^{M_{l+1}-3}k|\Delta^2c_k|+$$

$$+M_l\max_{M_l\leqq k\leqq M_{l+1}-2}|\Delta c_k|+\max_{M_l\leqq k\leqq M_{l+1}-1}|1-c_k|\Big).$$

Furthermore,

$$\max_{M_l\leqq k\leqq M_{l+1}-1}|c_k|=\max_{M_l\leqq k\leqq M_{l+1}-1}\varrho^{k^\lambda}=\varrho^{M_l^\lambda},$$

regarding k as a continuous variable and using the mean value theorem we have

$$\max_{M_l\leqq k\leqq M_{l+1}-2}|\Delta c_k|=\max_{M_l\leqq k\leqq M_{l+1}-2}|\varrho^{k^\lambda}-\varrho^{(k+1)^\lambda}|\leqq\max_{M_l\leqq k\leqq M_{l+1}-1}|\varrho^{k^\lambda}\ln\varrho\lambda k^{\lambda-1}|=$$

$$=O(1)\,\varrho^{M_l^\lambda}(1-\varrho)M_l^{\lambda-1}=O(1)\,\varrho^{M_l^\lambda}\frac{M_l^{\lambda-1}}{M_r^\lambda},$$

$$\max_{M_l\leqq k\leqq M_{l+1}-2}|\Delta^2c_k|\leqq 2\max_{M_l\leqq k\leqq M_{l+1}-1}\big(|\varrho^{k^\lambda}\ln^2\varrho\lambda^2k^{2\lambda-2}|+|\varrho^{k^\lambda}\ln\varrho\lambda(\lambda-1)k^{\lambda-2}|\big)=$$

$$=O(1)\,\varrho^{M_l^\lambda}\frac{M_l^{2\lambda-2}}{M_r^{2\lambda}},$$

$$\max_{M_l\leqq k\leqq M_{l+1}-1}|1-c_k|=\max_{M_l\leqq k\leqq M_{l+1}-1}(1-\varrho^{k^\lambda})\leqq\max_{0\leqq k\leqq M_{l+1}}|\varrho^{k^\lambda}\ln\varrho\lambda k^{\lambda-1}M_{l+1}|=$$

$$=O(1)\frac{M_l^\lambda}{M_r^\lambda}\varrho^{M_l^\lambda}.$$

Thus we have

$$\sum_{l=0}^{r-1}\Big\|\sum_{k=M_l}^{M_{l+1}-1}(1-c_k)\psi_k\Big\|_1 M_l^{-\alpha}\,\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\Big)= \tag{2.10}$$

$$=O(1)\sum_{l=0}^{r-1}\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\Big)M_l^{-\alpha}\varrho^{M_l^\lambda}\Big(\frac{M_l^\lambda}{M_r^\lambda}+\frac{M_l^\lambda}{M_r}+\frac{M_l^{2\lambda}}{M_r^{2\lambda}}\Big)=$$

$$=O(1)\frac{1}{M_r^\lambda}\sum_{l=0}^{r-1}M_l^{\lambda-\alpha}\,\omega\Big(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\Big),$$

and $\left(\text{taking account that } 1-\frac{1}{M_{r-1}^\lambda}<\varrho\leqq 1-\frac{1}{M_r^\lambda} \text{ and } \left(1-\frac{1}{M_r^\lambda}\right)^{M_r^\lambda}\to\frac{1}{e} \text{ as } r\to\infty\right)$

$$\text{(2.11)}\qquad \sum_{l=r}^{\infty}\Big\|\sum_{k=M_l}^{M_{l+1}-1} c_k\psi_k\Big\|_1 M_l^{-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_l}\right)=$$

$$=O(1)\,\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_r}\right)M_r^{-\alpha}\sum_{l=r}^{\infty}\varrho^{M_l^\lambda}\left(1+\frac{M_l^\lambda}{M_r^\lambda}+\frac{M_l^{2\lambda}}{M_r^{2\lambda}}\right)=$$

$$=O(1)\,\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_r}\right)M_r^{-\alpha}\sum_{l=r}^{\infty}\left(1-\frac{1}{M_r^\lambda}\right)^{M_r^\lambda(M_l^\lambda/M_r^\lambda)}\left(\frac{M_l^\lambda}{M_r^\lambda}\right)^2=O(1)\,M_r^{-\alpha}\,\omega\left(T^{\langle\alpha\rangle}f,\frac{1}{M_r}\right).$$

Substituting (2.10) and (2.11) into (2.6) we get (2.9).

The following corollaries are immediate consequences of Theorem 2 (cf. [5]).

COROLLARY 2.1. *If* $f\in X[0,1)$ $(X[0,1):=L^p[0,1)$ $(1\leqq p<\infty)$ *or* $WC[0,1)$,

$$WC[0,1):=\{f\in L^\infty[0,1):\ \sup_{0\leqq x<1}|f(x\oplus h)-f(x)|\to 0 \text{ as } h\to 0\})$$

then $\|f-f*K_\varrho\|_X\to 0$ $(\varrho\to 1-0)$.

COROLLARY 2.1. *If* $T^{\langle\alpha\rangle}f\in \mathrm{Lip}\,\beta$ *(i.e.* $\omega(T^{\langle\alpha\rangle}f,\delta)=O(\delta^\beta)$ *as* $\beta\to 0)$ $(\alpha\geqq 0,\ \beta>0)$ *then*

$$\|f-f*K_\varrho\|_p=\begin{cases}O\left((1-\varrho)^{(\alpha+\beta)/\lambda}\right) & \text{if } \alpha+\beta<\lambda\\ O\left((1-\varrho)\ln\frac{1}{1-\varrho}\right) & \text{if } \alpha+\beta=\lambda\\ O(1-\varrho) & \text{if } \alpha+\beta>\lambda.\end{cases}$$

§ 3. The case $1<p<\infty$

In this case using the Littlewood—Paley theorem and the strong multiplier theorem (see [2], Theorem 1), we may obtain further results. Blumin [2] gave the following result:

Let $f\in L^p[0,1)$ $(1<p<\infty)$ and $\{c_k\}$ be a sequence such that

$$\text{(3.1)}\qquad |c_k|\leqq B,\qquad \sum_{j=M_{l-1}}^{M_l-2}|c_j-c_{j+1}|<B\quad (k\in\mathbf{P},\ l\in\mathbf{N}).$$

Then $\sum_{k=0}^{\infty}c_k f^\wedge(k)\psi_k=g\in L^p[0,1)$ and $\|g\|_p\leqq A_p\|f\|_p$, where A_p is a constant depending only on p.

LEMMA 3. *If* $f_k\in L^p[0,1)$ $(1\leqq p<\infty,\ k=0,1,\dots,n,\ n\in\mathbf{P})$, *then*

$$\text{(3.2)}\qquad \Big(\sum_{k=0}^{n}\|f_k\|_p^{s_2}\Big)^{1/s_2}\leqq\Big\|\Big(\sum_{k=0}^{n}|f_k|^2\Big)^{1/2}\Big\|_p\leqq\Big(\sum_{k=0}^{n}\|f_k\|_p^{s_1}\Big)^{1/s_1},$$

where $s_1=\min(2,p)$, $s_2=\max(2,p)$.

PROOF. Suppose $1\leqq p\leqq 2$. Since $\left(\sum_{k=0}^{n}|a_k|\right)^{\alpha}\leqq\sum_{k=0}^{n}|a_k|^{\alpha}$ as $\alpha\leqq 1$, we have

$$\text{(3.3)}\qquad \|(\sum_{k=0}^{n}|f_k|^2)^{1/2}\|_p=(\int_0^1(\sum_{k=0}^{n}|f_k|^2)^{p/2})^{1/p}\leqq(\int_0^1\sum_{k=0}^{n}|f_k|^p)^{1/p}=(\sum_{k=0}^{n}\|f_k\|_p^p)^{1/p}.$$

On the other hand, since $\|f_1+f_2\|_q\geqq\|f_1\|_q+\|f_2\|_q$ as $q\leqq 1$,

$$\text{(3.4)}\qquad \|(\sum_{k=0}^{n}|f_k|^2)^{1/2}\|_p=(\int_0^1(\sum_{k=0}^{n}|f_k|^2)^{p/2})^{(2/p)(1/2)}\geqq(\sum_{k=0}^{n}(\int_0^1|f_k|^{2(p/2)})^{2/p})^{1/2}=$$

$$=(\sum_{k=0}^{n}\|f_k\|_p^2)^{1/2}.$$

Similarly we may prove that if $2\leqq p<\infty$, then

$$\text{(3.5)}\qquad (\sum_{k=0}^{n}\|f_k\|_p^p)^{1/p}\leqq\|(\sum_{k=0}^{n}|f_k|^2)^{1/2}\|_p\leqq(\sum_{k=0}^{n}\|f_k\|_p^p)^{1/p}.$$

LEMMA 4. *Let $f\in L^p[0,1)$ $(1<p<\infty)$, $c_0=c_0^*$, $c_k=c_l^*\geqq 0$ $(M_{l-1}\leqq k<M_l,\ l\in\mathbf{N})$, $\{c_l^*\}$ a bounded nondecreasing sequence. Then $g=\sum_{k=0}^{\infty}c_k f^{\wedge}(k)\psi_k\in L^p[0,1)$ and*

$$\text{(3.6)}\qquad \tilde{A}_p\left(\sum_{l=0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_l}\right)\right]^{s_2}\right)^{1/s_2}\leqq\|g\|_p\leqq A_p\left(\sum_{l=0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}$$

where $a_0=c_0^*$, $a_l=\sqrt{c_l^{*2}-c_{l-1}^{*2}}$ $(l\in\mathbf{N})$, $s_1=\min(2,p)$, $s_2=\max(2,p)$ and A_p, $\tilde{A}_p>0$ are constants depending on p.

PROOF. Let $\Delta_0=f^{\wedge}(0)$, $\Delta_l=\sum_{k=M_{l-1}}^{M_l-1}f^{\wedge}(k)\psi_k$ $(l\geqq 1)$. By the generalized Littlewood—Paley theorem (see [6] p. 58—59)

$$\|S_{M_s}(g)\|_p\sim\|(\sum_{l=0}^{s}c_l^{*2}|\Delta_l|^2)^{1/2}\|_p=\|(\sum_{l=0}^{s}c_l^{*2}(\sum_{j=l}^{\infty}|\Delta_j|^2-\sum_{j=l+1}^{\infty}|\Delta_j|^2)^{1/2})\|_p=$$

$$=\|(\sum_{l=0}^{s}c_l^{*2}\sum_{j=l}^{\infty}|\Delta_j|^2-\sum_{l=0}^{s}c_l^{*2}\sum_{j=l+1}^{\infty}|\Delta_j|^2)^{1/2}\|_p=$$

$$=\|(\sum_{l=0}^{s}c_l^{*2}\sum_{j=l}^{\infty}|\Delta_j|^2-\sum_{l=1}^{s+1}c_{l-1}^{*2}\sum_{j=l}^{\infty}|\Delta_j|^2)^{1/2}\|_p=$$

$$=\|(\sum_{l=1}^{s}(c_l^{*2}-c_{l-1}^{*2})\sum_{j=l}^{\infty}|\Delta_j|^2+c_0^{*2}\sum_{j=0}^{\infty}|\Delta_j|^2-c_s^{*2}\sum_{j=s+1}^{\infty}|\Delta_j|^2)^{1/2}\|_p=$$

$$=\|(\sum_{l=0}^{s}a_l^2\sum_{j=l}^{\infty}|\Delta_j|^2-c_s^{*2}\sum_{j=s+1}^{\infty}|\Delta_j|^2)^{1/2}\|_p,$$

thus

$$\|S_{M_s}(g)\|_p \leqq B_p \Big\|\Big(\sum_{l=0}^{s} a_l^2 \sum_{j=l}^{\infty} |\Delta_j|^2\Big)^{1/2}\Big\|_p$$

and

$$\|S_{M_s}(g)\|_p \geqq \tilde{B}_p \Big\|\Big(\sum_{l=0}^{s} a_l^2 \sum_{j=l}^{\infty} |\Delta_j|^2\Big)^{1/2}\Big\|_p - \tilde{B}_p \Big\|\Big(c_s^{*2} \sum_{j=s+1}^{\infty} |\Delta_j|^2\Big)^{1/2}\Big\|_p$$

(B_p, $\tilde{B}_p > 0$ are constants depending on p.)

By Lemma 3,

$$\Big\|\sum_{l=0}^{s} a_l^2 \sum_{j=l}^{\infty} |\Delta_j|^2)^{1/2}\Big\|_p \leqq \Big(\sum_{l=0}^{s} a_l^{s_1} \Big\|\Big(\sum_{j=l}^{\infty} |\Delta_j|^2\Big)^{1/2}\Big\|_p^{s_1}\Big)^{1/s_1}.$$

Using Littlewood—Paley theorem again and by [8]

$$\|S_{M_s}(g)\|_p \leqq A_p \Big(\sum_{l=0}^{\infty} a_l^{s_1} \Big\|\sum_{k=M_l}^{\infty} f^{\wedge}(k)\varphi_k\Big\|_p^{s_1}\Big)^{1/s_1} \leqq A_p \left(\sum_{l=0}^{s}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}.$$

Similarly we get

$$\|S_{M_s}(g)\|_p \geqq \tilde{A}_p \left(\sum_{l=0}^{s}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_2}\right)^{1/s_2} - B_p' c_s^* \,\omega\left(f, \frac{1}{M_s}\right),$$

where $B_p' > 0$ is a constant depending on p.

Letting $s \to \infty$, we get (by $\|S_{M_s}(g)\|_p \to \|g\|_p$ $(s \to \infty)$, see e.g. [7])

$$\tilde{A}_p \left(\sum_{l=0}^{\infty}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_2}\right)^{1/s_2} \leqq \|g\|_p \leqq A_p \left(\sum_{l=0}^{\infty}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}$$

THEOREM 3. *Let* $f \in L^p[0, 1)$ $(1 < p < \infty)$, $K_\varrho \in L^1[0, 1)$, $K_\varrho^{\wedge}(k) = 1$ $(0 \leqq k < M_{l_0},\ l_0 \in \mathbf{P})$.
(1) *If there exists a nonincreasing sequence* $\{c_l^*\}$, $c_0 = c_0^*$, $c_k = c_l^* < 1$ $(m_{l-1} \leqq k < M_l,$ $l \in \mathbf{N})$ *such that* $\left\{\dfrac{1 - K_\varrho^{\wedge}(k)}{1 - c_k}\right\}_{k=M_{l_0}}^{\infty}$ *satisfies* (3.1), *then*

$$\|f - f * K_\varrho\|_p \leqq A_p \left(\sum_{l=l_0}^{\infty}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}. \tag{3.7}$$

(2) *If* $K_\varrho^{\wedge}(j) \neq 1$ $(j \in \mathbf{N},\ j \geqq M_{l_0})$ *and there exists a nonincreasing sequence* $\{c_l^*\}$, $c_0 = c_0^*$, $c_k = c_l^* < 1$ $(M_{l-1} \leqq k < M_l,\ l \in \mathbf{N})$ *such that* $\left\{\dfrac{1 - c_k}{1 - K_\varrho^{\wedge}(k)}\right\}_{k=M_{l_0}}^{\infty}$ *satisfies* (3.1) *then*

$$\|f - f * K_\varrho\|_p \geqq \tilde{A}_p \left(\sum_{l=l_0}^{\infty}\left[a_l\omega\left(f, \frac{1}{M_l}\right)\right]^{s_2}\right)^{1/s_2}. \tag{3.8}$$

In (1) and (2), A_p and $\tilde{A}_p$ are positive constants depending on p,

$$a_l = \sqrt{(1 - c_l^*) - (1 - c_{l-1}^*)^2} \quad (l > l_0), \quad a_{l_0} = 1 - c_{l_0}^*, \quad s_1 = \min(2, p), \quad s_2 = \max(2, p).$$

PROOF OF (1). Since

$$\|f-f*K_\varrho\|_p = \Big\|\sum_{k=M_{l_0}}^{\infty}(1-K_\varrho^\wedge(k))f^\wedge(k)\psi_k\Big\|_p = \Big\|\sum_{k=M_{l_0}}^{\infty}\frac{1-K_\varrho^\wedge(k)}{1-c_k}(1-c_k)f^\wedge(k)\psi_k\Big\|_p,$$

we have by Lemmas 3 and 4

$$\|f-f*K_\varrho\|_p \leqq B_p\Big\|\sum_{k=M_{l_0}}^{\infty}(1-c_k)f^\wedge(k)\psi_k\Big\|_p \leqq A_p\left(\sum_{l=l_0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}.$$

Using the equality

$$\Big\|\sum_{k=M_{l_0}}^{\infty}(1-c_k)f^\wedge(k)\psi_k\Big\|_p = \Big\|\sum_{k=M_{l_0}}^{\infty}\frac{1-c_k}{1-K_\varrho^\wedge(k)}((1-K_\varrho^\wedge(k))\psi_k\Big\|_p,$$

we may prove (2) in the same way.

THEOREM 4. *Let* $f\in L^p[0,1)$ $(1<\varrho<\infty)$, $K_\varrho\in L^1[0,1)$, $\{K_\varrho^\wedge(k)\}$ *a nonincreasing sequence and* $K_\varrho^\wedge(k)=1$ $(k<M_{l_0})$, $K_\varrho^\wedge(j)<1$ $(j\in\mathbf{N},\ j\geqq M_{l_0},\ l_0\in\mathbf{P})$. *Then*

$$(3.9)\quad \tilde A_p\left(\sum_{l=l_0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_{l+1}}\right)\right]^{s_2}\right)^{1/s_2} \leqq \|f-f*K_\varrho\|_p \leqq A_p\left(\sum_{l=l_0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}$$

where A_p *and* $\tilde A_p$ *are positive constants depending on* p, $s_1=\min(2,p)$, $s_2=\max(2,p)$,

$$a_{l_0}=1-K_\varrho^\wedge(M_{l_0}),\quad a_l=\sqrt{(1-K_\varrho^\wedge(M_l))^2-(1-K_\varrho^\wedge(M_{l-1}))^2}\quad (l>l_0).$$

PROOF. Applying Theorem 3, we take $\{c_l^*\}=\{K_\varrho^\wedge(M_l)\}$ $(l\geqq l_0)$, then $\left\{\frac{1-K_\varrho^\wedge(k)}{1-c_k}\right\}_{k=M_{l_0}}^{\infty}$ satisfies (3.1). In fact

$$\left|\frac{1-K_\varrho^\wedge(k)}{1-c_k}\right| = \left|\frac{1-K_\varrho^\wedge(k)}{1-K_\varrho^\wedge(M_l)}\right| \leqq 1\quad (M_{l-1}\leqq k<M_l,\ l>l_0),$$

$$\sum_{k=M_{l-1}}^{M_l-2}\left|\frac{1-K_\varrho^\wedge(k)}{1-c_k}-\frac{1-K_\varrho^\wedge(k+1)}{1-c_{k+1}}\right| = \frac{1}{1-K_\varrho^\wedge(M_l)}\sum_{k=M_{l-1}}^{M_l-2}(K_\varrho^\wedge(k)-K_\varrho^\wedge(k+1)) =$$

$$= \frac{1}{1-K_\varrho^\wedge(M_l)}(K_\varrho^\wedge(M_{l-1})-K_\varrho^\wedge(M_l-1)) \leqq 1\quad (l>l_0).$$

Thus by Theorem 3 $\|f-f*K_\varrho\|_p \leqq A_p\left(\sum_{l=l_0}^{\infty}\left[a_l\,\omega\left(f,\frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}$, where $a_{l_0}=1-$ $-c_{l_0}^*=1-K_\varrho^\wedge(M_{l_0})$, $a_l=\sqrt{(1-c_l^*)^2-(1-c_{l-1}^*)^2}=\sqrt{(1-K_\varrho^\wedge(M_l))^2-(1-K_\varrho^\wedge(M_{l-1}))^2}$ $(l>l_0)$.

On the other hand we take $c_l^*=K_\varrho^\wedge(M_{l-1})$ $(l\geqq l_0)$, then $\left\{\frac{1-c_k}{1-K_\varrho^\wedge(k)}\right\}_{k=M_{l_0}}^{\infty}$

satisfies (3.1). In fact

$$\left|\frac{1-K_\varrho\hat{}(M_{l-1})}{1-K_\varrho\hat{}(k)}\right| \leqq 1 \quad (M_{l-1} \leqq k < M_l,\ l > l_0),$$

$$\sum_{k=M_{l-1}}^{M_l-2}\left|\frac{1-K_\varrho\hat{}(M_{l-1})}{1-K_\varrho\hat{}(k)} - \frac{1-K_\varrho\hat{}(M_{l-1})}{1-K_\varrho\hat{}(k+1)}\right| = \left(1-K_\varrho\hat{}(M_{l-1})\right)\left(\frac{1}{1-K_\varrho\hat{}(M_{l-1})} - \right.$$

$$\left. - \frac{1}{1-K_\varrho\hat{}(M_l-1)}\right) = 1 - \frac{1-K_\varrho\hat{}(M_{l-1})}{1-K_\varrho\hat{}(M_l-1)} \leqq 1 \quad (l > l_0).$$

Applying Theorem 3, we may get

$$\|f-f*K_\varrho\|_p \geqq \tilde{A}_p\left(\sum_{l=l_0+1}^{\infty}\left[a_{l-1}\,\omega\left(f, \frac{1}{M_l}\right)\right]^{s_2}\right)^{1/s_2} = \tilde{A}_p\left(\sum_{l=l_0}^{\infty}\left[a_l\,\omega\left(f, \frac{1}{M_{l+1}}\right)\right]^{s_2}\right)^{1/s_2}.$$

This completes the proof.

The following Theorems 5 and 6 are direct applications of Theorem 4.

THEOREM 5. *If* $f\in L^p[0, 1)$ $(1<p<\infty)$, $K_n = \sum_{k=0}^{n-1}\left(1-\left(\frac{k}{n}\right)^\lambda\right)\psi_k$ $(\lambda>0,\ n\in\mathbf{N})$, *then*

$$\tilde{A}\frac{1}{M_r^\lambda}\left(\sum_{l=0}^{r}\left[M_l^\lambda\,\omega\left(f, \frac{1}{M_{l+1}}\right)\right]^{s_2}\right)^{1/s_2} \leqq \|f-f*K_n\|_p \leqq A\frac{1}{M_r^\lambda}\left(\sum_{l=0}^{r}\left[M_l^\lambda\,\omega\left(f, \frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1}$$

where $r\in\mathbf{P}$, $M_{r-1}\leqq n<M_r$ *and* A, $\tilde{A}>0$ *are constants depending on* p, λ.

THEOREM 6. *If* $f\in L^p[0, 1)$ $(1<p<\infty)$, $K_\varrho = \sum_{k=0}^{\infty}\varrho^{k^\lambda}\psi_k$ $(\lambda>0,\ 0<\varrho<1)$, *then*

$$\tilde{A}\frac{1}{M_r^\lambda}\left(\sum_{l=0}^{r}\left[M_l^\lambda\,\omega\left(f, \frac{1}{M_{l+1}}\right)\right]^{s_2}\right)^{1/s_2} \leqq \|f-f*K_\varrho\|_p \leqq A\frac{1}{M_r^\lambda}\left(\sum_{l=0}^{r}\left[M_l^\lambda\,\omega\left(f, \frac{1}{M_l}\right)\right]^{s_1}\right)^{1/s_1},$$

where $r\in\mathbf{P}$, $\frac{1}{M_r^\lambda}\leqq 1-\varrho<\frac{1}{M_{r-1}^\lambda}$ *and* A, $\tilde{A}>0$ *are constants depending on* p, λ, ϱ.

Here we only give the proof of Theorem 6. Apply Theorem 4. Since

$$a_0 = 1-\varrho^{M_0^\lambda} = 1-\varrho \sim \frac{M_0^\lambda}{M_r^\lambda}, \quad a_l = \sqrt{(1-\varrho^{M_l^\lambda})^2-(1-\varrho^{M_{l-1}^\lambda})^2} \sim \left(\frac{M_l}{M_r}\right)^\lambda \varrho^{M_l^\lambda} \quad (l\in\mathbf{N})$$

and

$$\sum_{l=r+1}^{\infty}\left(\frac{M_l}{M_r}\right)^\lambda \varrho^{M_l^\lambda}\,\omega\left(f, \frac{1}{M_l}\right) \leqq \omega\left(f, \frac{1}{M_{r+1}}\right)\sum_{l=r+1}^{\infty}\left(\frac{M_l}{M_r}\right)^\lambda\left(1-\frac{1}{M_r^\lambda}\right)^{M_r^\lambda(M_l/M_r)^\lambda} \sim$$

$$\sim \omega\left(f, \frac{1}{M_{r+1}}\right) \leqq \omega\left(f, \frac{1}{M_r}\right),$$

substituting these into (3.9), we obtain the desired result.

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CONVEX CURVES IN GEAR

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Introduction

When looking at two wheels in gear we notice the essential property that no rotation of one of them can be performed without rotating the other. But it is equally obvious that they are not too convex. Can we construct convex wheels that are in gear? This paper answers (locally) this question, but has no ambition to have any technical relevance!

Let us consider two Jordan (closed) curves C_1, C_2 in the half-plane $\mathbf{R}\times\mathbf{R}_+$, both containing the origin $\mathbf{O}=(0,0)$. ($\mathbf{R}_+=[0,\infty)$, $\mathbf{R}_-=(-\infty,0]$.) Suppose $C_1-\{\mathbf{O}\}$ lies in the bounded domain with frontier C_2. Such ordered pairs of curves (C_1, C_2) will shortly be called *supporting curves (C_2 supports C_1)*. According to intuitive evidence we say that (locally at $\mathbf{O}$) C_i *can rotate around* $(0, a)$ if there is a neighbourhood of $\mathbf{O}$ in $\mathbf{R}_-\times\mathbf{R}_+$ or in $\mathbf{R}_+\times\mathbf{R}_+$ within which C_j meets the circle with centre $(0, a)$ and radius a only in $\mathbf{O}$ ($\{i,j\}=\{1,2\}$). (In fact the intuition would impose here a stronger condition, which however turns out to be superfluous for all later purposes.) Consequently, we say that C_1 and C_2 are *in gear with respect to* $(0, a_1)$ *and* $(0, a_2)$ $(a_1\leqq a_2)$ if C_i cannot rotate around $(0, a_i)$ $\left(i\in\{1,2\}\right)$ and that C_1 and C_2 are *in perfect gear* if they are in gear with respect to any pair of (correctly ordered) points on the positive y-axis.

If the curvatures of C_1 and C_2 at $\mathbf{O}$ exist and are different, then C_1 and C_2 are not in gear with respect to any pair of points. If it happens that C_1 and C_2 have a common centre c of curvature at $\mathbf{O}$, then they possibly are in gear with respect to c and c only.

Our main result is that, in a certain sense, in general two supporting convex curves are in perfect gear.

Circles tangent to convex curves

In a Baire space, "most" means "all, except those in a set of first category", i.e. "those in a residual set".

Let $\mathcal{K}$ be the space of all (closed) convex curves in $\mathbf{R}^2$ and $\mathcal{C}$ the subspace of all those lying in $\mathbf{R}\times\mathbf{R}_+$ and containing $\mathbf{O}$. It is easily seen that both of them, endowed with Hausdorff's metric, are Baire spaces. It is not difficult to modify the proof of Klee's result [3] (see also Gruber [2]), which says that most curves in $\mathcal{K}$ are differentiable and strictly convex, to demonstrate that most curves in $\mathcal{C}$ have the same properties.

We shall consider the lower and upper radii of curvature $\varrho_i^{\pm}(p)$ and $\varrho_s^{\pm}(p)$ at an arbitrary point p on C, in both directions (for a definition see [1], p. 14). If all four of them are equal we write $\varrho(p)$ for the common value. About $\mathscr{K}$ we know the following: On one hand [5]

for most curves in $\mathscr{K}$, $\varrho(p)=\infty$ *a.e.*

On the other hand [6]

for most curves in $\mathscr{K}$, $\varrho_i^{\pm}(p)=0$ *and* $\varrho_s^{\pm}(p)=\infty$ *at most points p.*

How behaves C at $\mathbf{O}$ for most curves $C\in\mathscr{C}$? From a probabilistic point of view it can be expected that the bahaviour at $\mathbf{O}$ is of the first kind, while from a topological point of view the second seems more plausible.

THEOREM 1. *For most curves in* $\mathscr{C}$, $\varrho_i^{\pm}(\mathbf{O})=0$ *and* $\varrho_s^{\pm}(\mathbf{O})=\infty$.

PROOF. Let

$$\mathscr{A}=\{C\in\mathscr{C}\colon \varrho_i^{+}(\mathbf{O})=\varrho_i^{-}(\mathbf{O})=0\}.$$

Denote by $\mathscr{C}^*$ the set of all curves in $\mathscr{C}$ supporting some convex curve Γ composed of a semicircle and a segment of the y-axis. Let $\mathscr{C}_n$ be the subset of all curves in $\mathscr{C}$ for which Γ has diameter $2n^{-1}$.

It is rather obvious that $\mathscr{C}^*$ is the complement of $\mathscr{A}$ and that $\mathscr{C}^*=\bigcup_n \mathscr{C}_n$. We show that $\mathscr{C}_n$ is nowhere dense in $\mathscr{C}$, for every n. Since $\mathscr{C}_n$ is easily seen to be closed, it suffices to find in its complement a set dense in $\mathscr{C}$. This is provided by the family of all convex polygons in $\mathbf{R}\times\mathbf{R}_+$ which have $\mathbf{O}$ as a vertex and have no edge on the x-axis. Thus, $\mathscr{A}$ is residual.

In an analogous manner one shows that the set

$$\mathscr{B}=\{C\in\mathscr{C}\colon \varrho_s^{+}(\mathbf{O})=\varrho_s^{-}(\mathbf{O})=\infty\}$$

is residual, the set of convex curves dense in $\mathscr{C}$ being this time the family of all convex polygons in $\mathbf{R}\times\mathbf{R}_+$ having $\mathbf{O}$ as interior point of an edge.

Hence $\mathscr{A}\cap\mathscr{B}$ is residual, which proves the theorem.

As a consequence we get the following result from [7]. Denote by $\mathscr{C}^0$ the subspace of $\mathscr{C}$ all the elements of which are circles. $\mathscr{C}\times\mathscr{C}^0$ is then obviously a Baire space.

COROLLARY. *Most pairs of curves in* $\mathscr{C}\times\mathscr{C}^0$ *intersect each other in every neighbourhood of* $\mathbf{O}$ *at some point different from* $\mathbf{O}$.

PROOF. Consider two topological spaces $\mathscr{X}$ and $\mathscr{Y}$, where $\mathscr{Y}$ has a countable basis, and two sets $A\subset\mathscr{X}$, $B\subset\mathscr{Y}$. We use the following known result ([4], Theorem 15.3):

$A\times B$ *is of first category in* $\mathscr{X}\times\mathscr{Y}$ *if and only if* A *or* B *is of first category.*

In our case, we know by Theorem 1 that most curves in $\mathscr{C}$ intersect every circle from $\mathscr{C}^0$ in a point different from $\mathbf{O}$ and as close to $\mathbf{O}$ as we want. Then, the preceding result (with $B=\mathscr{Y}=\mathscr{C}^0$) yields the Corollary.

Convex curves in perfect gear

We easily see that the set $\mathcal{D}\subset\mathcal{C}^2$ of all pairs of supporting convex curves is a Baire space. Let $(C_1, C_2)\in\mathcal{D}$.

Let now $\varrho_i^\pm(C_k)$ and $\varrho_s^\pm(C_k)$ denote $\varrho_i^\pm(\mathbf{O})$ and $\varrho_s^\pm(\mathbf{O})$ calculated for the curve C_k. We observe the following: If

$$\varrho_i^-(C_2) < a_1, \quad \varrho_i^+(C_2) < a_1$$

and

$$\varrho_s^-(C_1) > a_2, \quad \varrho_s^+(C_1) > a_2,$$

then C_1 and C_2 are in gear with respect to $(0, a_1)$ and $(0, a_2)$.

Thus, if

$$\varrho_i^\pm(C_2) = 0 \quad\text{and}\quad \varrho_s^\pm(C_1) = \infty,$$

then C_1 and C_2 are in perfect gear. We also remark that the converse holds too.

THEOREM 2. *Most pairs of supporting convex curves are in perfect gear.*

PROOF. We prove that for most pairs of curves $(C_1, C_2)\in\mathcal{D}$,

$$\varrho_i^\pm(C_2) = 0 \quad\text{and}\quad \varrho_s^\pm(C_1) = \infty.$$

The argument parallels that of Theorem 1:

Let

$$\mathcal{A} = \{(C_1, C_2)\in\mathcal{D}\colon \varrho_i^+(C_2) = \varrho_i^-(C_2) = 0\}.$$

Let $\mathcal{D}^*$ be the set of all pairs $(C_1, C_2)\in\mathcal{D}$, such that C_2 supports some convex curve Γ composed by a semicircle and a segment of the y-axis. Let $\mathcal{D}_n$ be the subset of all pairs in $\mathcal{D}^*$ for which Γ has diameter $2n^{-1}$. $\mathcal{D}_n$ is obviously closed. The family of all pairs of supporting convex polygons, both admitting $\mathbf{O}$ as a vertex and having no edge on the x-axis, is dense in $\mathcal{D}$. Hence

$$\mathcal{A} = \mathcal{D} - \bigcup_{n=1}^{\infty} \mathcal{D}_n$$

is residual. Analogously,

$$\mathcal{B} = \{(C_1, C_2)\in\mathcal{D}\colon \varrho_s^+(C_1) = \varrho_s^-(C_1) = \infty\}$$

is residual and most pairs of $\mathcal{D}$ belong to $\mathcal{A}\cap\mathcal{B}$.

Theorems 1 and 2 extend in an obvious way to higher dimensions and the proofs present no difficulty. Also, one can consider pairs of supporting convex surfaces of different dimensions; it is still true that most such pairs of surfaces are in perfect gear.

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ON THE CONTROL OF INSTABLE SYSTEMS

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Dedicated to Professor László Leindler on his 50th birthday

1. In the last years the control theory of singular systems has been developed intensively (see [3]). Recently A. Bensoussan showed that the optimal cost and the optimal control of a singular system may happen stable with respect to a singular perturbation ([4]). Further results on this field were obtained by A. Haraux, F. Murat and the author in [5]—[8].

Developing the method of Bensoussan we shall prove some general results concerning the system $\varepsilon Az+|z|^n=v$, $n\geqq 1$ real. Surprisingly, this system behaves better than the system for which the method was originally created.

The author is deeply indebted to Professor J.-L. Lions for useful consultations and advices and also grateful to Professor F. Murat for fruitful discussions.

2. Throughout this paper we shall use the following notations and conditions: Ω denotes an arbitrary measure space and $\|\ \|_p$ the usual norm of the real Banach space $L^p=L^p(\Omega)$ $(1\leqq p\leqq\infty)$. For some $1\leqq r<\infty$, $1\leqq n<\infty$ fixed, A denotes an arbitrary linear operator $A\colon L^{rn}\to L^r$, $D(A)$ dense in L^{rn}. For some $0\leqq N<\infty$, $z_d\in L^{rn}$, $\mathcal{U}_{ad}\subset L^r$ fixed we put

$$J\colon L^r\times L^{rn}\to\mathbf{R},\quad J(v,z)=\|z-z_d\|_{rn}^{rn}+N\|v\|_r^r,$$

$$J_\varepsilon=\inf\{J(v,z)\,|\,v\in\mathcal{U}_{ad},\ z\in D(A),\ v=\varepsilon Az+|z|^n\}\quad(\varepsilon\in\mathbf{R},\ \varepsilon\neq 0)$$

and for each $\varepsilon\neq 0$ we fix u_ε, y_ε such that

$$u_\varepsilon\in\mathcal{U}_{ad},\ y_\varepsilon\in D(A),\ u_\varepsilon=\varepsilon Ay_\varepsilon+|y_\varepsilon|^n\quad\text{and}\quad J(u_\varepsilon,y_\varepsilon)\leqq J_\varepsilon+|\varepsilon|.\tag{1}$$

The present paper is devoted to the study of the behavior of the sequences J_ε, u_ε, y_ε when ε tends to 0.

REMARK. The system $\varepsilon Az+|z|^n$ is instable in general; this can be shown by the method presented in [3]. Some deeper results of this type are proved in [2].

Put

$$J_0=\inf\{J(v,z)\,|\,v\in L^r,\ z\in L^{rn},\ v=|z|^n\};$$

one can easily see that

$$\begin{cases}\text{for}\quad y_0\in L^{rn}\quad\text{and}\quad u_0=|y_0|^n_r\quad J(u_0,y_0)=J_0\quad\text{iff}\\ |y_0-z_d|^{rn-1}\operatorname{sgn}(y_0-z_d)+N|y_0|^{rn-1}\operatorname{sgn}(y_0)=0\quad\text{a.e.}\end{cases}\tag{2}$$

Hence u_0 and y_0 are determined uniquely if $rn>1$ or if $N\neq 1$. If $rn=N=1$ then there are several solutions.

In what follows we shall assume that

(3) $$u_0 \in \operatorname{int} \mathcal{U}_{ad}$$

(in case $rn=N=1$ at least for one u_0). We note that (3) and the density of $D(A)$ imply

$$J_0 = \inf \{J(v, z) \big| v \in \mathcal{U}_{ad}, z \in D(A), v = |z|^n\}.$$

3. In this section we prove the following result:

THEOREM 1. *Assume* $y_0 \in D(A)$ *and* $u_0^{r-1} \in D(A^*)$. *Then*

$$J_\varepsilon = J_0 + O(\varepsilon).$$

Furthermore

$$\|y_\varepsilon - y_0\|_{rn}^{rn} = O(\varepsilon) \quad \textit{if} \quad rn \geqq 2$$

and

$$\|u_\varepsilon - u_0\|_r^r = O(\varepsilon) \quad \textit{if} \quad r \geqq 2 \quad \textit{and} \quad N > 0.$$

We remark that the adjoint of A is defined uniquely as an operator

$$A^*: L^{r/(r-1)} \to L^{rn/(rn-1)} \left(\frac{1}{1-1} := \infty\right).$$

We need three lemmas for the proof of this theorem.

LEMMA 1. *For any* $1 \leqq p < \infty$ *there exists a constant* $C_p \geqq 0$ *such that*

$$|a+b|^p - |a|^p - p\,|a|^{p-1} (\operatorname{sgn} a)\, b \geqq C_p |b|^p \quad (\forall a, b \in \mathbf{R}).$$

Moreover, $C_p > 0$ *if* $p \geqq 2$.

PROOF. Dividing by $|b|^p$ and putting

$$C_p = \inf_{x \in \mathbf{R}} \{|1+x|^p - |x|^p - p\,|x|^{p-1} \operatorname{sgn} x\},$$

it suffices to show that $C_p \geqq 0$ if $p \geqq 1$ and $C_p > 0$ if $p \geqq 2$. The case $p=1$ is obvious. If $p>1$, then the function $f(x) \equiv |x|^p$ is convex and $f'(x) \equiv p\,|x|^{p-1} \operatorname{sgn} x$. Therefore $C_p \geqq 0$.

Consider now the case $p \geqq 2$, then $f''(x) \equiv p(p-1)|x|^{p-2}$. Applying the Taylor formula,

$$|1+x|^p - |x|^p - p\,|x|^{p-1} \operatorname{sgn} x = \frac{p(p-1)}{2} \xi^2$$

for some $x < \xi < x+1$. Hence

$$\inf_{|x|>2} \{|1+x|^p - |x|^p - p\,|x|^{p-1} \operatorname{sgn} x\} \geqq \frac{p(p-1)}{2} \geqq 1.$$

On the other hand, the function $|x|^p$ is strictly convex whence the continuous function $|1+x|^p - |x|^p - p\,|x|^{p-1} \operatorname{sgn} x$ is strictly positive. Therefore

$$\inf_{|x| \leqq 2} \{|1+x|^p - |x|^p - p\,|x|^{p-1} \operatorname{sgn} x\} > 0$$

and the lemma is proved. □

LEMMA 2. $J_\varepsilon \leqq J_0 + O(\varepsilon)$.

PROOF. Using the conditions $u_0 \in \operatorname{int} \mathcal{U}_{ad}$, $y_0 \in D(A)$ we have for $|\varepsilon|$ sufficiently small

$$J_\varepsilon \leqq J(\varepsilon A y_0 + |y_0|^n, y_0) = \int_\Omega (|y_0 - z_d|^{rn} + N|u_0 + \varepsilon A y_0|^r)\, dx =$$

$$= J(u_0, y_0) + N \int_\Omega (|u_0 + \varepsilon A y_0|^r - |u_0|^r)\, dx =$$

$$= J_0 + rN \int_\Omega |u_0 + \theta \varepsilon A y_0|^{r-1} |\varepsilon A y_0|\, dx =$$

$$= J_0 + \varepsilon r N \int_\Omega (u_0 + |A y_0|)^{r-1} |A y_0|\, dx = J_0 + O(\varepsilon).$$

We have applied the triangle inequality for $r=1$ and the Lagrange inequality for $r>1$ (hence $|\theta| \leqq 1$).

LEMMA 3.

$$J_\varepsilon \geqq J_0 + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r - O(\varepsilon).$$

PROOF. Using (1), (2) and applying Lemma 1 three times,

$$J_\varepsilon + |\varepsilon| \geqq J(u_\varepsilon, y_\varepsilon) = \int_\Omega |(y_0 - z_d) + (y_\varepsilon - y_0)|^{rn}\, dx + N \int_\Omega |u_0 + (u_\varepsilon - u_0)|^r\, dx \geqq$$

$$\geqq \int_\Omega (|y_0 - z_d|^{rn} + rn|y_0 - z_d|^{rn-1} \operatorname{sgn}(y_0 - z_d)(y_\varepsilon - y_0) + C_{rn}|y_\varepsilon - y_0|^{rn})\, dx +$$

$$+ N \int_\Omega (u_0^r + r u_0^{r-1}(u_\varepsilon - u_0) + C_r |u_\varepsilon - u_0|^r)\, dx =$$

$$= J(u_0, y_0) + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r +$$

$$+ rN \int_\Omega (u_0^{r-1}(u_\varepsilon - u_0) - n|y_0|^{rn-1} \operatorname{sgn}(y_0)(y_\varepsilon - y_0))\, dx =$$

$$= J(u_0, y_0) + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r + \varepsilon r N \int_\Omega u_0^{r-1} A y_\varepsilon\, dx +$$

$$+ rN \int_\Omega |y_0|^{rn-n} \{|y_\varepsilon|^n - |y_0|^n - n|y_0|^{n-1} \operatorname{sgn}(y_0)(y_\varepsilon - y_0)\}\, dx \geqq$$

$$\geqq J(u_0, y_0) + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r + \varepsilon r N \int_\Omega u_0^{r-1} A y_\varepsilon\, dx =$$

$$= J_0 + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r + \varepsilon r N \int_\Omega y_\varepsilon A^*(u_0^{r-1})\, dx \geqq$$

$$\geqq J_0 + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + NC_r \|u_\varepsilon - u_0\|_r^r - |\varepsilon| r N \|y_\varepsilon\|_{rn} \|A^*(u_0^{r-1})\|_{rn/(rn-1)}$$

and the lemma follows because by Lemma 2

$$\|y_\varepsilon - z_d\|_{rn}^{rn} \leqq J(u_\varepsilon, y_\varepsilon) \leqq J_\varepsilon + |\varepsilon| \leqq J_0 + O(\varepsilon)$$

whence

$$\|y_\varepsilon\|_{rn} = O(1).$$

Finally, Theorem 1 is an immediate consequence of Lemmas 1—3.

REMARK. One can easily see that the above proof equally works if we replace the condition (3) by the weaker one

$$u_0+\varepsilon A y_0 \in \mathscr{U}_{ad} \quad \text{for } |\varepsilon| \text{ sufficiently small.} \tag{4}$$

4. In this section we turn to the more general case when $y_0 \notin D(A)$ or $u_0^{r-1} \notin D(A^*)$. The following result will be proved:

THEOREM 2. *Assume* $u_0^{r-1} \in \overline{D(A^*)}$ *(the closure of* $D(A^*)$ *in* $L^{r/(r-1)}$*). Then* $J_\varepsilon \to J_0$. *Furthermore*

$$\|y_\varepsilon - y_0\|_{rn} \to 0 \quad \textit{if} \quad rn \geqq 2$$

and

$$\|u_\varepsilon - u_0\|_r \to 0 \quad \textit{if} \quad r \geqq 2 \quad \textit{and} \quad N > 0.$$

REMARK. We note that the condition $u_0^{r-1} \in \overline{D(A^*)}$ is obviously satisfied if A is a closed (or closable) operator because then $D(A^*)$ is dense in $L^{r/(r-1)}$. □

We need two new lemmas.

LEMMA 4. $\overline{\lim}\, J_\varepsilon \leqq J_0$.

PROOF. Fix a sequence $(z_m) \subset D(A)$ such that $\|z_m - y_0\|_{rn} \to 0$. We can assume that $|z_m|^n \in \operatorname{int} \mathscr{U}_{ad}$ for all m because for any $f, g \in L^{rn}$

$$\int_\Omega \big||f|^n - |g|^n\big|^r dx \leqq \int_\Omega |n|f+\theta(g-f)|^{n-1}|f-g||^r dx \leqq n^r (\|f\|_{rn} + \|g\|_{rn})^{rn-r} \|f-g\|_{rn}^r$$

by the Lagrange or the triangle inequality $(0<\theta<1)$. Then

$$\overline{\lim_\varepsilon}\, J_\varepsilon \leqq \inf_m \overline{\lim_\varepsilon}\, J(\varepsilon A z_m + |z_m|^n, z_m) = \inf_m J(|z_m|^n, z_m) \leqq$$

$$\leqq \overline{\lim_m}\, J(|z_m|^n, z_m) = J(u_0, y_0) = J_0.$$

We applied twice the inequality obtained from the above one for f and g replacing n, r by rn, 1. The lemma is proved. □

LEMMA 5.

$$J_\varepsilon \geqq J_0 + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + N C_r \|u_\varepsilon - u_0\|_r^r - o(1).$$

PROOF. Repeating the first part of the proof of Lemma 3 we obtain

$$J_\varepsilon + |\varepsilon| \geqq J_0 + C_{rn} \|y_\varepsilon - y_0\|_{rn}^{rn} + N C_r \|u_\varepsilon - u_0\|_r^r + \varepsilon r N \int_\Omega (A y_\varepsilon) u_0^{r-1}\, dx.$$

To finish the proof we show that

$$\varepsilon N \int_\Omega (A y_\varepsilon) u_0^{r-1}\, dx \geqq o(1). \tag{5}$$

Fix for each $\varepsilon \neq 0$ a function z_ε such that

$$z_\varepsilon \in D(A^*), \quad \left\|\sqrt{|\varepsilon|}\, A^* z_\varepsilon\right\|_{rn/(rn-1)} \leqq 1$$

and

$$\|u_0^{r-1}-z_\varepsilon\|_{r/(r-1)} \leqq |\varepsilon| + \inf\left\{\|u_0^{r-1}-z\|_{r/(r-1)} \middle| z\in D(A^*),\ \left\|\sqrt{|\varepsilon|}\,A^*z\right\|_{rn/(rn-1)} \leqq 1\right\}.$$

It follows from this construction and from the condition $u_0^{r-1}\in\overline{D(A^*)}$ that

(6) $$\|\varepsilon A^* z_\varepsilon\|_{rn/(rn-1)} \to 0 \quad\text{and}\quad \|u_0^{r-1}-z_\varepsilon\|_{r/(r-1)} \to 0.$$

Furthermore the quantities

(7) $$\|y_\varepsilon\|_{rn} \quad\text{and}\quad \|\varepsilon NAy_\varepsilon\|_r$$

are bounded because, by Lemma 4,

$$\|y_\varepsilon - z_d\|_{rn}^{rn} + N\,\|\varepsilon Ay + |y_\varepsilon|^n\|_r^r = J(u_\varepsilon, y_\varepsilon) \leqq J_\varepsilon + |\varepsilon| \leqq J_0 + o(1).$$

Now (5) follows from (6) and (7) because

$$\varepsilon N\int_\Omega (Ay_\varepsilon)u_0^{r-1}\,dx = \varepsilon N\int_\Omega (Ay_\varepsilon)z_\varepsilon\,dx + \varepsilon N\int_N (Ay_\varepsilon)(u_0^{r-1}-z_\varepsilon)\,dx =$$

$$= N\int_\Omega y_\varepsilon(\varepsilon A^* z_\varepsilon)\,dx + \int_\Omega (\varepsilon NAy_\varepsilon)(u_0^{r-1}-z_\varepsilon)\,dx \geqq$$

$$\geqq -N\|y_\varepsilon\|_{rn}\,\|\varepsilon A^* z_\varepsilon\|_{rn/(rn-1)} - \|\varepsilon NAy_\varepsilon\|_r\,\|u_0^{r-1}-z_\varepsilon\|_{r/(r-1)}$$

and the lemma is proved.

Finally, Theorem 2 follows from Lemmas 1, 4 and 5.

5. We formulate some open problems connected with the results of this **paper:**

The functions y_0 and u_0 are determined uniquely if $r>1$ but the convergences $y_\varepsilon\to y_0$ and $u_\varepsilon\to u_0$ were proved only if $r\geqq 2$. It would be interesting to study the case $1<r<2$.

The condition $u_0\in\operatorname{int}\mathscr{U}_{ad}$ played important role in the proofs. What can be said in general, for example if $u_0\in\operatorname{ext}\mathscr{U}_{ad}$?

It would be useful to find an asymptotic development for J_ε, y_ε and u_ε (cf. [1]).

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and

$$\|u_0^{-1}-z_\varepsilon\|_{L^{r/(r-1)}} \le |\varepsilon| + \inf\left\{\|u_0^{-1}-z\|_{L^{r/(r-1)}} : z\in D(A^*),\ \left\|\sqrt{|\varepsilon|}\,A^*z\right\|_{L^{r/(r-1)}} \le 1\right\}.$$

It follows from this construction and from the condition $u_0^{-1}\in \overline{D(A^*)}$ that

(6) $$\|\varepsilon A^* z_\varepsilon\|_{L^{r/(r-1)}} \to 0 \quad\text{and}\quad \|u_0^{-1}-z_\varepsilon\|_{L^{r/(r-1)}} \to 0.$$

Furthermore the quantities

(7) $$\|y_\varepsilon\|_{L^r} \quad\text{and}\quad \|\varepsilon N A y_\varepsilon\|_{L^r}$$

are bounded because, by Lemma 4,

$$\|y_\varepsilon\|^r_{L^r} + \|\varepsilon N A y_\varepsilon\|^r_{L^r} \le J(u_\varepsilon, y_\varepsilon) \le J_\varepsilon + |\varepsilon| \le J_0 + o(1).$$

Now (5) follows from (6) and (7) because

$$\varepsilon N\int_\Omega (Ay_\varepsilon)u_0^{-1}\,dx = \varepsilon N\int_\Omega (Ay_\varepsilon)z_\varepsilon\,dx + \varepsilon N\int_\Omega (Ay_\varepsilon)(u_0^{-1}-z_\varepsilon)\,dx =$$

$$= N\int_\Omega y_\varepsilon(\varepsilon A^* z_\varepsilon)\,dx + \int_\Omega (\varepsilon N A y_\varepsilon)(u_0^{-1}-z_\varepsilon)\,dx \ge$$

$$\ge -N\|y_\varepsilon\|_{L^r}\|\varepsilon A^* z_\varepsilon\|_{L^{r/(r-1)}} - \|\varepsilon N A y_\varepsilon\|_{L^r}\|u_0^{-1}-z_\varepsilon\|_{L^{r/(r-1)}}$$

and the lemma is proved.

Finally, Theorem 2 follows from Lemmas 3, 4 and 5.

5. We formulate some open problems connected with the results of this paper: The functions y_0 and u_0 are determined uniquely if $r=1$, but the convergences $y_\varepsilon\to y_0$ and $u_\varepsilon\to u_0$ were proved only if $r\ge 2$. It would be interesting to study the case $1<r<2$.

The condition $u_0\in \operatorname{int}\mathscr{W}_{ad}$ played important role in the proofs. What can be said in general, for example if $u_0\in \operatorname{ext}\mathscr{W}_{ad}$?

It would be useful to find an asymptotic development for J_ε, y_ε and u_ε (cf. [1]).

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(L^1, H)-TYPE ESTIMATIONS FOR SOME OPERATORS WITH RESPECT TO THE WALSH—PALEY SYSTEM

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1. Introduction. Let $(w_n,\ n\in\mathbf{N}:=\{0, 1, \dots\})$ be the Walsh—Paley system and let $r_0, r_1, \dots$ represent the Rademacher functions. (For more details see e.g. [3].) If f is a Lebesgue integrable function defined on $[0, 1]$, then denote by $S_n f$ and $\sigma_n f$ the n-th $(n\in\mathbf{N})$ partial sum, resp. $(C, 1)$ mean of the Walsh—Fourier series of f. Furthermore, let $D_n:=\sum_{k=0}^{n-1} w_k\ (k=1, 2, \dots)$ be the n-th Dirichlet kernel. The Fejér kernels are denoted by $K_n:=\frac{1}{n}\sum_{k=1}^{n} D_k\ (n=1, 2, \dots)$.

The so-called dyadic Hardy space H is defined as follows [2]. A function $a\in L^\infty[0, 1]$ is called an atom, if either $a=1$ or a has the following properties:

$$\text{(1)}\qquad \text{i) } \operatorname{supp} a \subset I_a,\quad \text{ii) } \|a\|_\infty \leqq |I_a|^{-1},\quad \text{iii) } \int_0^1 a=0,$$

where $I_a\subset[0, 1]$ is a dyadic interval and $|I_a|$ denotes its length. We say that the function f belongs to H, if f can be represented as $f=\sum_{i=0}^{\infty}\lambda_i a_i$, where a_i's are atoms and for the coefficients λ_i $(i\in\mathbf{N})$ $\sum_{i=0}^{\infty}|\lambda_i|<\infty$ is true. It is well-known that H is a Banach space with respect to the norm

$$\|f\| := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f=\sum_{0}^{\infty}\lambda_i a_i\in H$.

Throughout this work, $C>0$ will denote an absolute constant which will not necessarily be the same at different occurrences.

2. In this note we shall examine the following operators introduced earlier by G. I. Sunouchi [5], [6]:

$$\text{i)}\qquad U_p f:=\left(\sum_{n=1}^{\infty}\frac{|S_n f-\sigma_n f|^p}{n}\right)^{1/p}\qquad (p\geqq 1,\ f\in L^1[0, 1]),$$

$$\text{ii)}\qquad Tf:=\Big(\sum_{n=0}^{\infty}|S_{2^n}f-\sigma_{2^n}f|^2\Big)^{1/2}\qquad (f\in L^1[0, 1]).$$

He has proved that these operators (U_p for all $p\geqq 2$) as mappings from $L^r[0, 1]$

into $L^r[0, 1]$ are bounded, if $1<r<\infty$. The analogous statements for $r=1$ are not true and thus it is of interest to study what happens in this case. First we prove

THEOREM 1. *The operator* $T: H \to L^1[0, 1]$ *is bounded.*

The secondary part of above mentioned Sunouchi's result is that $\|Tf\|_r$ and $\|U_2 f\|_r$ $\left(f \in L^r[0, 1],\ 1<r<\infty,\ \int_0^1 f=0\right)$ are equivalent to $\|f\|_r$. The following question remains open: is $\|f\|$ equivalent to $\|Tf\|_1$ $(f \in H)$? For U_p we shall prove a statement of negative character, i.e.

THEOREM 2. *The operator* U_p $(p \geqq 1)$ *as mapping from* H *into* $L^1[0, 1]$ *is not bounded.*

We remark that W. R. Wade [7] obtained certain bounds for the L^1 norm of Tf and $U_2 f$, if f belongs to $L^r[0, 1]$ for some $r>1$.

3. Let $q=(q_n,\ n \in \mathbf{N})$ be an arbitrary sequence of ± 1 and define the operator T_q as follows:

$$T_q f := \sum_{n=0}^{\infty} q_n r_n (S_{2^n} f - \sigma_{2^n} f) \quad (f \in L^1[0, 1]).$$

By Khinchin's inequality we have

$$\|Tf\|_1 \sim \int_0^1 \|T_{r(t)} f\|_1 \, dt$$

($\sim$ stands for the equivalence and $r(t) := (r_n(t),\ n \in \mathbf{N})$ $(t \in [0, 1])$), therefore to show Theorem 1 it is enough to prove that

$$\|T_q f\|_1 \leqq C \|f\| \quad (f \in H). \tag{2}$$

PROOF OF (2). Since $\|T_q f\|_2 = \|Tf\|_2$ $(f \in L^2[0, 1])$, so by Sunouchi's result $T_q: L^2[0, 1] \to L^2[0, 1]$ is (in q uniformly) bounded. From this it follows by standard argument (more precisely see e.g. [4]) that we need only to prove the following estimation:

$$\sup \int_{[0,1] \setminus I_a} |T_q a| < \infty, \tag{3}$$

where the supremum is taken over all atoms $a \in H$.

To this end let $a \in H$ be an atom and $|I_a| = 2^{-N}$ for some $N \in \mathbf{N}$, i.e. by (1) we have $\operatorname{supp} a \subset I_a$, $\|a\|_\infty \leqq 2^N$ and $\int_0^1 a = 0$. If $n \in \mathbf{N}$ is an arbitrary but fixed natural number, then it is well-known [1] that

$$D_{2^n}(x) - K_{2^n}(x) = \frac{2^n - 1}{2^{n+1}} D_{2^n}(x) - \sum_{k=0}^{n-1} 2^{k-n-1} D_{2^n}(x \dotplus 2^{-k-1})$$

($x \in [0, 1]$, $\dotplus$ denotes the dyadic addition in $[0, 1]$). On the other hand [3]

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \leqq x < 2^{-n}) \\ 0 & (2^{-n} \leqq x < 1), \end{cases}$$

from which we get for $x\in[0,1]\setminus I_a$ that

$$S_{2^n}a(x)-\sigma_{2^n}a(x)=\frac{2^n-1}{2^{n+1}}\int_0^1 a(t)D_{2^n}(x\dotplus t)\,dt-\sum_{k=0}^{n-1}2^{k-n-1}\cdot$$

$$\cdot\int_0^1 a(t)D_{2^n}(x\dotplus 2^{-k-1}\dotplus t)\,dt=\begin{cases}0 & (\text{if } n\le N \text{ or } n>N \text{ and } x\dotplus 2^{-k-1}\notin I_a \\ & (k=0,\dots,n-1))\\ -2^{k-n-1}\int_0^1 a(t)D_{2^n}(x\dotplus 2^{-k-1}\dotplus t)\,dt & \\ & (\text{if } n>N \text{ and } x\dotplus 2^{-k-1}\in I_a \text{ for some} \\ & k=0,\dots,N-1).\end{cases}$$

Therefore for $x\in I_a\dotplus 2^{-k-1}\,(k=0,\dots,N-1)$ it can be stated that

$$T_q a(x)=-\sum_{n=N+1}^{\infty}q_n r_n(x)2^{k-n-1}\int_0^1 a(t)D_{2^n}(x\dotplus 2^{-k-1}\dotplus t)\,dt,$$

i.e.

$$\int_{[0,1]\setminus I_a}|T_q a|\le\sum_{n=N+1}^{\infty}2^{N-n-1}\int_0^1|a(t)|\Big(\int_0^1 D_{2^n}(x\dotplus 2^{-k-1}\dotplus t)\,dx\Big)dt\le\sum_{n=0}^{\infty}2^{-n-2}=1/2.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let us denote by df the dyadic derivative of a Walsh polynomial f, which is defined by $dw_k=kw_k$ [1]. If G_n $(n\in\mathbf{N})$ stands for $D_{2^{n+1}}-D_{2^n}$, then $\|G_n\|=1$ and for all $N=2^M, 2^M+1,\dots,2^{M+1}-1$ $(M,N\in\mathbf{N})$ we have

$$S_N G_n-\sigma_N G_n=\begin{cases}N^{-1}dG_n & (M>n)\\ 0 & (M<n)\\ N^{-1}d(D_N-D_{2^n}) & (M=n).\end{cases}$$

From this it follows that

$$U_pG_n=\Big(\sum_{M=0}^{\infty}\sum_{k=0}^{2^M-1}\frac{|S_{2^M+k}G_n-\sigma_{2^M+k}G_n|^p}{2^{M+k}}\Big)^{1/p}=$$

$$=\Big(\sum_{k=0}^{2^n-1}\frac{|d(D_{2^n+k}-D_{2^n})|^p}{(2^n+k)^{p+1}}+\sum_{M=n+1}^{\infty}\sum_{k=0}^{2^M-1}\frac{|dG_n|^p}{(2^M+k)^{p+1}}\Big)^{1/p}\ge$$

$$\ge C\Big(2^{-(p+1)n}\sum_{k=1}^{2^n-1}|d(D_{2^n+k}-D_{2^n})|^p+|dG_n|^p\sum_{M=n+1}^{\infty}2^{-pM}\Big)^{1/p}\ge$$

$$\ge C\Big(2^{-(p+1)n}\sum_{k=1}^{2^n}|d(D_{2^n+k}-D_{2^n})|^p\Big)^{1/p}=C\Big(2^{-(p+1)n}\sum_{k=1}^{2^n}\Big|\sum_{j=2^n}^{2^n+k-1}jw_j\Big|^p\Big)^{1/p}=$$

$$=C\Big(2^{-(p+1)n}\sum_{k=1}^{2^n}\Big|\sum_{j=0}^{k-1}(2^n+j)w_j\Big|^p\Big)^{1/p}=C\Big(2^{-n}\sum_{k=1}^{2^n}\Big|\sum_{j=0}^{k-1}(1+j2^{-n})w_j\Big|^p\Big)^{1/p}\ge$$

$$\ge C2^{-n}\sum_{k=1}^{2^n}|D_k+2^{-n}dD_k|=C2^{-n}\sum_{k=1}^{2^n}|D_k+2^{-n}k(D_k-K_k)|=$$

$$=C2^{-n}\sum_{k=1}^{2^n}|(1+k2^{-n})D_k-k2^{-n}K_k|\ge C\Big(2^{-n}\sum_{k=1}^{2^n}|D_k|-2^{-n}\sum_{k=1}^{2^n}k2^{-n}|K_k|\Big).$$

Now we can estimate $\|U_pG_n\|_1$ from below as follows:

$$\|U_pG_n\|_1 \geqq C\Big(2^{-n}\sum_{k=1}^{2^n}\|D_k\|_1 - 2^{-2n}\sum_{k=1}^{2^n} k\,\|K_k\|_1\Big) \geqq C\Big(2^{-n}\sum_{k=1}^{2^n}\|D_k\|_1 - 2\Big),$$

since $\|K_k\|_1 \leqq 2$ $(k=1, 2, \ldots)$ [3]. If we take into consideration that [3] $2^{-n}\sum_{k=1}^{2^n}\|D_k\|_1 \geqq$ $\geqq Cn$ $(n\in\mathbf{N})$, the statement of Theorem 2 follows.

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ANY FOUR INDEPENDENT EDGES OF A 4-CONNECTED GRAPH ARE CONTAINED IN A CIRCUIT

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L. Lovász [2] raised the following problem.

Conjecture. Suppose G is a k-connected graph ($k \geqq 2$), $e_1, e_2, \dots, e_k \in E(G)$ are independent edges, and if k is odd then $G - \{e_1, e_2, \dots, e_k\}$ is connected. Then G contains a circuit using all the edges $e_1, e_2, \dots, e_k$.

This conjecture is proved for $k=3$ by Lovász [3; 6. § 67]. In general, R. Häggkvist and C. Thomassen [1] proved a slightly weaker statement that the same conclusion follows if G is $(k+1)$-connected.

Now we prove that the conjecture of Lovász holds for $k=4$.

THEOREM. *In a 4-connected graph, any four independent edges are contained in a circuit.*

This result effects on a conjecture of Erdős and Gallai. Using this theorem, L. Pyber [4] proved that every graph of n vertices can be covered by $1,5n$ circuits or edges. (Without this result, a greater constant could be proved by the method of Pyber.)

PROOF OF THE THEOREM. Let us fix the 4-connected graph G and the independent edges $x_1y_1, x_2y_2, x_3y_3, x_4y_4 \in E(G)$. By 4-connectivity (using Menger's theorem), there exist four vertex-disjoint paths from the vertices x_1, y_1, x_3, y_3 to the vertices x_2, y_2, x_4, y_4. These paths P_1, P_2, P_3, P_4 with the edges x_1y_1, x_2y_2, x_3y_3, x_4y_4 constitute one or two circuits. In the first case it is a desired circuit, so without loss of generality, we may suppose that the paths P_1, P_2, P_3 and P_4 lead from x_1, y_1, x_3 and y_3 to x_2, y_2, x_4 and y_4, respectively; P_1, P_2 and the edges x_1, y_1, x_2, y_2 constitute the circuit C_1, P_3, P_4 and the edges x_3y_3, x_4y_4 constitute the circuit C_2.

Now again by 4-connectivity and Menger's theorem, there exist four vertex-disjoint paths Q_1, Q_2, Q_3, Q_4 from C_1 to C_2. The circuits C_1, C_2 and the paths Q_1, Q_2, Q_3, Q_4 constitute a subgraph H. In what follows, we deal with this subgraph H.

We introduce some notation. The paths are denoted by the sequence of labelled vertices in them. For a path P from x to y, $[xy]$, $[xy)$, $(xy]$, (xy) denote the vertex-sets $V(P)$, $V(P)-\{y\}$, $V(P)-\{x\}$, $V(P)-\{x, y\}$, respectively. The subpaths of P_1, P_2, P_3 and P_4 are called arcs.

First make a very simple observation which however is used several times.

Fact 1. If two vertex-disjoint paths Q_i connect the same pair of paths P_j then deleting the inner points and the edges of the arcs between the endpoints of these paths we get a desired circuit.

So we may suppose that the paths Q_1, Q_2, Q_3, Q_4 lead from the arcs P_1, P_1, P_2, P_2 to the arcs P_4, P_3, P_3, P_4, respectively. Let $w_1, w_2, z_1, z_2 \in V(C_1)$, $z_4, w_3, w_4, z_3 \in V(C_2)$ be the endpoints of the paths Q_1, Q_2, Q_3, Q_4, respectively. If the vertices w_3 and z_4 do not separate the vertices w_4 and z_3 in C_2 then the disjoint subpaths $w_3 z_4$ and $w_4 z_3$ of C_2 with Q_1, Q_2, Q_3 and Q_4 constitute two paths such that both paths can replace one arc of C_1 and this new circuit is a desired one.

So we may suppose that the vertices w_3 and z_4 separate the vertices w_4 and z_3 in C_2. Now without loss of generality, we may suppose that we have the subgraph H in Figure 1. (We drew the subgraph H so that the figure should show the large symmetry of the situation.) Of course, it may occur that $w_1 = x_1$, $w_2 = x_2$, $w_3 = x_3$, $w_4 = x_4$, $z_1 = y_1$, $z_2 = y_2$, $z_2 = y_3$ or $z_4 = y_4$.

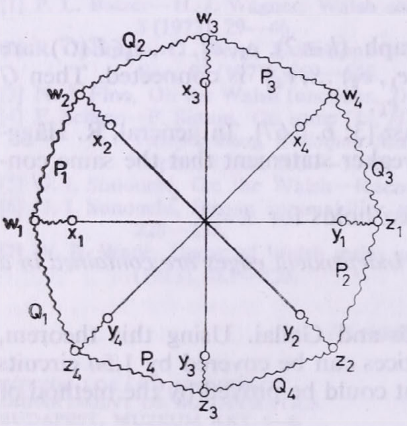

Fig. 1

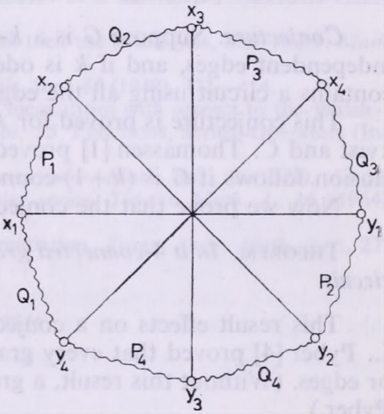

Fig. 2

Suppose that H is a subgraph as in Figure 1 such that the sum of the lengths of the arcs w_1x_1, w_2x_2, w_3x_3, w_4x_4, z_1y_1, z_2y_2, z_3y_3, z_4y_4 is minimum. Suppose that e.g. $w_1 \neq x_1$. Then the path $w_1x_1y_1z_1$ contains inner vertices and there is a path in $G - \{w_1, z_1\}$ from $(w_1x_1y_1z_1)$ to the remaining part of $H - \{w_1, z_1\}$ by 4-connectivity. By symmetry, we may assume that this path leads from a vertex $u \in (w_1 x_1]$. If this path leads to a vertex $v \in [w_1 w_2] \cup [w_1 z_4]$ then adding this path to H and deleting the inner vertices and the edges of the arc vw_1 we obtain a subgraph like in Figure 1 such that the path ux_1 is shorter than w_1x_1, a contradiction. If this path leads to a vertex in $(w_3x_3]$, $(w_3w_4]$ or $[w_4x_4]$ ($[y_4z_4)$, $(z_4z_3]$ or $[y_3z_3]$, resp.) then this path and Q_2 (Q_1 resp.) are two vertex-disjoint paths from the arc P_1 to the arc P_3 (P_4 resp.) and we are done by Fact 1. If this path leads to a vertex v in (w_4z_1) ((z_3z_2), resp.) then the path uvw_4 (uvz_3, resp.) and Q_2 (Q_1, resp.) are two vertex-disjoint paths from P_1 to P_3 (P_4, resp.) and we are ready by Fact 1 again. The other possibilities can be settled by the axial symmetry of Figure 1 with the axis $w_1x_1y_1z_1$.

So we may assume that $x_1 = w_1$, $x_2 = w_2$, $x_3 = w_3$, $x_4 = w_4$, $y_1 = z_1$, $y_2 = z_2$, $y_3 = z_3$, $y_4 = z_4$, like in Figure 2.

Now by 4-connectivity, there is a path P in $G-\{x_2, y_1, y_4\}$ from $(x_2 x_1 y_4)$ to the remaining part of $H-\{x_2, y_1, y_4\}$. By symmetry, we may assume that P leads from $[x_1 y_4)$. We distinguish two cases.

Case 1. P starts at x_1.

If P leads to a vertex $v\in(x_3x_4]\cup(x_4y_1)$ then P or P together with the path vx_4 and Q_2 are two vertex-disjoint paths from P_1 to P_3 and we are done by Fact 1. If P leads to $(y_3y_2]\cup(y_2y_1)$ then we are done by axial symmetry. So in this case P leads to a neighbouring subpath of the circuit $x_1x_2x_3x_4y_1y_2y_3y_4x_1$, i.e. to $(x_2x_3]$ or $(y_4y_3]$.

Case 2. P leads from a vertex $u\in(x_1y_4)$.

If P leads to a vertex v in (x_4y_1) ($[x_4x_3)$, resp.) the paths x_1u, P_1vx_4 (x_1u, P, resp.) constitute a further path from P_1 to P_3 and we are done by Fact 1. If P leads to a vertex $v\in(x_2x_3]$ then the paths and edges P, vx_3, x_3y_3, y_3y_2, y_2x_2, x_2x_1, x_1y_1, y_1x_4, x_4y_4, y_4u constitute a desired circuit. If P leads to $[y_1y_2)$ or $[y_2y_3)$ then we are done by symmetry.

So in both cases, we obtained

Fact 2. Only neighbouring segments of the circuit $x_1x_2x_3x_4y_1y_2y_3y_4x_1$ are connected by any path openly disjoint to H.

Without loss of generality, we may assume that there is such a path P from $[x_1y_4)$ to $(y_4y_3]$. Let $u\in[x_1y_4)$ and $v\in(y_4y_3]$ be the points nearest to x_1 and y_3 in the path x_1y_4 and y_4y_3, respectively, which occur as the endpoint of such a path. (It may happen that u and v belong to different paths.) Choose H (with the constraints $x_1=w_1, \dots, y_4=z_4$) so that the sum of the lengths of paths ux_1 and vy_3 should be the minimum.

By 4-connectivity, there is a path R in $G-\{u, v, x_4\}$ from (uy_4v) to the remaining part of H. R does not lead from (y_4v) to $[x_1u)$ by the definition of u. If R leads from $x\in(uy_4]$ to $y\in[x_1u)$ then replacing the path xy of H by R we obtain a subgraph H_0 such that there is a path from y to $(y_4y_3]$ (via u), a contradiction to the choice of H. Similarly, R does not lead from (uy_4v) to $(vy_3]$. But according to Fact 2, only neighbouring segments can be connected by R. Without loss of generality, we may assume that R leads from $x\in(uy_n]$ to $y\in(x_1x_2]$. Now let R_1 be a path from $(y_4y_3]$ to u. (There exists such an R_1 by the definition of u.) If R and R_1 are vertex-disjoint then y_4x with R and R_1 with ux_1 are vertex-disjoint paths from P_4 to P_1 and we are finished by Fact 1. And if R and R_1 have a vertex in common then $R_1\cup R_2$ contains a path from $(x_1x_2]$ to $(y_4y_3]$, a contradiction to Fact 2.

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