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ARITHMETIC FUNCTIONS SATISFYING A CONGRUENCE PROPERTY

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1. An arithmetic function $f(n)$ is multiplicative (resp. additive), if

$$f(nm) = f(n)f(m) \qquad (\text{resp. } f(nm) = f(n) + f(m))$$

for any pair n, m of relatively prime positive integers, and completely multiplicative (resp. completely additive), if the above equality holds for any pair n, m.

The problem concerning the characterization of an integer-valued power function as an integer-valued multiplicative function satisfying a congruence property was studied by several authors. In 1966, M. V. Subbarao [17] proved that if an integer-valued multiplicative function $f(n)$ satisfies the congruence

$$f(n+m) \equiv f(m) \pmod{n} \tag{1.1}$$

for every positive integer n and m, then there is a non-negative integer α such that

$$f(n) = n^{\alpha} \qquad (n = 1, 2, \dots). \tag{1.2}$$

In [3], A. Iványi extended this result proving that if an integer-valued completely multiplicative function $f(n)$ satisfies (1.1) for a fixed positive integer m and for every positive integer n, then $f(n)$ is also of the same form (1.2). Furthermore A. Iványi also showed that the same assertion can be deduced from the congruence

$$f(n+m) \equiv f(n) + f(m) \pmod{n} \tag{1.3}$$

instead of (1.1) for an integer-valued multiplicative function $f(n)$ and for every positive integer n, m.

In the space of sequences $\{x_n\}$ we define the operators E, I and Δ as follows:

$$Ix_n := x_n, \quad Ex_n := x_{n+1} \quad \text{and} \quad \Delta x_n := x_{n+1} - x_n.$$

If $P(x) = a_0 + a_1 x + \dots + a_k x^k$ is an arbitrary polynomial with integer coefficients, then we extend the above definition as follows:

$$P(E)x_n := a_0 x_n + a_1 x_{n+1} + \dots + a_k x_{n+k}.$$

Let **P**, **N** denote the set of all primes resp. positive integers. For any subsets X, Y of **N** we shall denote by $K(P, X, Y)$ the set of all integer-valued multiplicative functions $f(n)$ for which

$$P(E)f(n+m) \equiv P(E)f(m) \pmod{n} \tag{1.4}$$

holds for every $n \in X$ and $m \in Y$. It is obvious that

$$f_\alpha(n) := n^\alpha \quad (n = 1, 2, \ldots) \tag{1.5}$$

is a solution of (1.4) for every non-negative integer α and for every triplet (P, X, Y). In this case $P(x) \equiv 1$ for example, from the result of Subbarao, we have

$$K(P, \mathbf{N}, \mathbf{N}) = \{f_0, f_1, f_2, \ldots\},$$

where f_α is defined in (1.5).

Recently, some authors were interested in characterizing all those triplets (P, X, Y) for which

$$K(P, X, Y) = \{f_0, f_1, f_2, \ldots\}. \tag{1.6}$$

In [11]–[14] B. M. Phong obtained some results concerning this problem. He proved that (1.6) holds for the following cases:

(i) $P(x) = 1, \quad X = \mathbf{N}, \quad Y = \mathbf{P};$

(ii) $P(x) = (x-1)^k, \quad x + \mathbf{N}, \quad Y = \mathbf{P};$

(iii) $P(x) = x^M - 1, \quad X = \mathbf{N}, \quad Y = \mathbf{P},$

where k, M are fixed positive integers. B. M. Phong and J. Fehér in [16] improved the results of Subbarao and Iványi mentioned above showing that (1.6) also holds for $P(x) = 1$, $X = \mathbf{N}$, $Y = \{B\}$ with some positive integer B.

In [4] we asked for a characterization of those integer-valued multiplicative functions $f(n)$ which satisfy

$$f(An + B) \equiv C \pmod{n} \qquad \text{for every} \qquad n \in \mathbf{N}, \tag{1.7}$$

where $A \geqq 1$, $b \geqq 1$ and $C \neq 0$ are fixed integers. We considered this problem with $A \in \mathbf{P}$, proving that there are a non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that $f(n) = \chi_A(n) n^\alpha$ for all $n \in \mathbf{N}$ which are prime to A. The general case has been proved by B. M. Phong [15].

For a fixed integer $k \geqq 1$ we define $N_k(n) := m$ if $n = m^k h$ and h is a k-free power number. In [5] we proved that if an integer-valued multiplicative function $f(n)$ satisfies the congruence

$$f(n + B) \equiv f(B) \pmod{N_k(n)}$$

for every $n \in \mathbf{N}$, where k, B are fixed positive integers, then $f(n)$ is of the same form (1.2). Condition (1.3) was weakened by the author in [5], namely we relaxed the congruence (1.3) to

$$f\left(n^k + m\right) \equiv f\left(n^k\right) + f(m) \pmod{n}$$

for every $n, m \in \mathbf{N}$, where k is a given positive integer.

The problem concerning characterization of integer-valued additive functions as real-valued additive functions satisfying some congruence properties (mod 1) have been studied by I. Kátai [6]–[8], I. Kátai and M. van Rossum-Wijsmuller [9] and Robert Styer [18]. For example, I. Kátai [7] proved that if the real-valued completely additive functions f_1, f_2, f_3 and f_4 satisfy the congruence

$$f_1(n) + f_2(n+1) + f_3(n+2) + f_4(n+3) \equiv 0 \pmod{1}$$

for every $n \in \mathbf{N}$, then

$$f_1(n) \equiv f_2(n) \equiv f_3(n) \equiv f_4(n) \equiv 0 \pmod{1}$$

for every $n \in \mathbf{N}$.

A similar problem concerning characterization of a zero-function as an integer-valued additive function satisfying a congruence property has been studied by K. Kovács [10]. She proved that if an integer-valued completely additive function $f(n)$ satisfies the congruence

$$f(An + B) \equiv C \pmod{n} \quad (n = 1, 2, \dots) \tag{1.8}$$

for some integers $A \geqq 1$, $B \geqq 1$, C then

$$f(n) = 0 \tag{1.9}$$

for all $n \in \mathbf{N}$ which are prime to A. P. V. Chung [1] extended this result for integer-valued additive functions, proving that if an integer-valued additive function satisfies (1.8), then (1.9) holds.

Our purpose in this paper is to investigate the same problem concerning characterization of a zero-function as a real-valued additive function satisfying a congruence property. The exact assertion will be formulated in Sections 2,3,4, and in Section 5 we investigate multiplicative functions.

2. In this part we investigate the following problem: Determine those functions $H : \mathbf{N} \to \mathbf{N}$ for which for fixed integers $A \geqq 1$, $B \geqq 1$, C and for real-valued additive function $f(n)$ the congruence

$$f(An + B) \equiv C \pmod{H(n)} \quad (n = 1, 2, \dots)$$

implies $f(n)=0$ for all $n \in \mathbf{N}$ which are prime to A.

Presently we are unable to determine all solutions.

For a fixed $A \in \mathbf{N}$ let $\mathcal{H}=\mathcal{H}(A)$ denote the set of all functions $H: \mathbf{N} \to \to \mathbf{N}$ which satisfy the following conditions:

a) $H(m) | H(nm)$ for every $m, n \in \mathbf{N}$;

b) For each positive integer M coprime to A we have

$$\limsup \frac{1}{s} h\left(\frac{M^{\varphi(A)s}-1}{A}\right)=\infty .$$

We shall prove the following

THEOREM 1. *Let $A \geqq 1$, $B \geqq 1$, C be integers and let $f(n)$ be a real-valued additive function. Assume that for some $H \in \mathcal{H}(A)$ the congruence*

$$f(An+B) \equiv C \pmod{H(n)} \tag{2.1}$$

holds for each $n \in \mathbf{N}$. Then

$$f(n)=0 \tag{2.2}$$

for all $n \in \mathbf{N}$ which are prime to A.

PROOF. We first deduce from (2.1) that

$$f(ab)=f(a)+f(b) \tag{2.3}$$

for all $a, b \in \mathbf{N}$ which are prime to A, furthermore

$$f(B)=C. \tag{2.4}$$

Let $a, b \in \mathbf{N}$ and $(ab, A)=1$. From condition b) of $\mathcal{H}$ there is a positive integer s such that

$$H\left[\frac{(ab)^{\varphi(A)s}-1}{A}\right]>|f(ab)-f(aB)-f(b)+C| . \tag{2.5}$$

Let $m:=\left((ab)^{\varphi(A)s}-1\right)/A$. Since $(m, ab)=1$ we can choose positive integers x, y, u and v such that

$$ax=1+Amy, \qquad (x, abB)=1 \tag{2.6}$$

and

$$bu=B+Amv, \qquad (u, abx)=1. \tag{2.7}$$

By using (2.1), (2.6), (2.7) and condition a) of $\mathcal{H}$ we have

$$f(aB)+f(x)=f(axB)=f(B+ABmy)\equiv C \pmod{H(m)},$$
$$f(b)+f(u)=f(bu)=f(B+Amv)\equiv C \pmod{H(m)}$$

and

$$f(ab)+f(x)+f(u)=f(axbu)=f(B+AmT)\equiv C \pmod{H(m)},$$

where $T:=Amyv+By+v$. These imply

$$f(ab)-f(aB)-f(b)+C\equiv 0 \pmod{H(m)},$$

which together with (2.5) proves that

$$f(ab)+C=f(aB)+f(b). \tag{2.8}$$

Thus we have proved that (2.8) holds if $(ab,A)=1$.

Applying (2.8) with $a=b=1$, we have $f(B)=C$. So (2.4) is true.

In order to prove (2.3) it is enough to show that

$$f\left(q^k\right)=kf(q) \qquad (k=1,2,\ldots) \tag{2.9}$$

holds for each prime q coprime to A. It is obvious that (2.9) holds for $k=1$.

Let q be a prime for which $(q,A)=1$. Assume that $q^\alpha \| B$ for some integer $\alpha \geqq 0$. Applying (2.8) with $a=q$ and $b=1$, we get

$$f(q)+C=f\left(q^{\alpha+1}\right)-f\left(q^\alpha\right)+f(B),$$

which together with (2.4) shows that

$$f\left(q^{\alpha+1}\right)-f\left(q^\alpha\right)=f(q). \tag{2.10}$$

Assume now that (2.9) holds for k. Then we apply (2.8) with $a=q$ and $b=q^k$ to obtain

$$f\left(q^{q+1}\right)+C=f\left(q^{\alpha+1}\right)-f\left(q^\alpha\right)+f(B)+f\left(q^k\right)$$

This together with (2.4), (2.10) and the induction hypothesis implies that

$$f\left(q^{k+1}\right)=f(q)+f\left(q^k\right)=(k+1)f(q).$$

Thus, we have proved (2.9), and so (2.3) holds.

Since $f(n)$ is an additive function and (2.3) holds for all positive integers a, b which are prime to A, it follows that

(2.11) $$f(ab) = f(a) + f(b) \quad \text{if} \quad (a, b, A) = 1.$$

Applying (2.1) with $n = Bm$ and using the fact $(B, Am + 1, A) = 1$ for every $m \in \mathbf{N}$, we get from (2.1), (2.4) and (2.11) that

(2.12) $$f(Am + 1) \equiv 0 \pmod{H(m)}$$

for every $m \in \mathbf{N}$.

Let n be a positive integer for which $(n, A) = 1$. From Euler's theorem we get $n^{\varphi(A)s} \equiv 1 \pmod{A}$ for every $s \in \mathbf{N}$. Using condition b) of $\mathcal{H}$, there is an $s_0 \in \mathbf{N}$ such that

(2.13) $$\frac{1}{s_0} H\left(\frac{n^{\varphi(A)s_0} - 1}{A}\right) > |f(n)| \cdot \varphi(A).$$

On the other hand, from (2.11) and (2.1.2) we have

$$s\varphi(A)f(n) = f\left(n^{\varphi(A)s}\right) \equiv 0 \quad \left(\bmod\ H\left(\frac{n^{\varphi(A)s} - 1}{A}\right)\right)$$

for every $s \in \mathbf{N}$. This and (2.13) imply that $f(n) = 0$. Thus, we have proved that for each $n \in \mathbf{N}$ coprime to A (2.2) holds. This completes the proof of Theorem 1.

We shall illustrate Theorem 1 with some corollaries.

COROLLARY 1. *For each $n \in \mathbf{N}$ we denote by $H_0(n)$ the product of all distinct prime divisors of n. Then $H_0 \in \mathcal{H}(A)$ for every $A \in \mathbf{N}$.*

REMARK. It is obvious that the results of K. Kovács [10] and P. V. Chung [1] mentioned in Section 1 are consequences of Corollary 1.

COROLLARY 2. *Let $\varphi(n)$ denote the Euler totient function. Then $\varphi \in \in \mathcal{H}(A)$ for every $A \in \mathbf{N}$.*

For the proofs of the corollaries we need the following result.

LEMMA 1. *Let a be a fixed positive integer. For each prime p coprime to a we denote by $a(p)$ the least positive integer x such that $a^x \equiv 1 \pmod{p}$. Then*

$$u(p) := \frac{p-1}{a(p)}$$

is unbounded on the set $\mathbf{P}$. Here $\mathbf{P}$ denotes the set of all primes.

PROOF. Let us assume that $u(p)$ is a bounded function on this set $\mathbf{P}$ and so it has only a finite number of distinct values $k_1, \ldots, k_t$. From the result of K. Zsigmondy [19] there is a prime p such that

$$a(p) = 6(6k_1 + 1) \ldots (6k_t + 1) + 6$$

and so

$$p = u(p)a(p) + 1 = 6u(p)(6k_1 + 1)\dots(6k_t + 1) + (6u(p) + 1). \tag{2.14}$$

Since $u(p) = k_i$ for some integer i ($1 \leqq i \leqq t$), and so by (2.14)

$$p = 6k_i(6k_1 + 1)\dots(6k_t + 1) + (6k_i + 1)$$

is divisible by $6k_i + 1$. This is a contradiction since p is a prime, thus the function $u(p)$ cannot be bounded.

PROOF OF THE COROLLARIES. It is obvious that the functions H_0, φ satisfy condition a) of $\mathcal{H}(A)$. Let M be a positive integer for which $(M, A) = 1$. Putting $a = M^{\varphi(A)}$, for all primes $p > A$ we have

$$\frac{1}{a(p)} H_0\left(\frac{a^{a(p)} - 1}{A}\right) \geqq \frac{p}{a(p)} > \frac{p-1}{a(p)} = u(p)$$

and

$$\frac{1}{a(p)} \varphi\left(\frac{a^{a(p)} - 1}{A}\right) \geqq \frac{p-1}{a(p)} = u(p)$$

which, using Lemma 1, imply that H_0, φ satisfy condition b) of $\mathcal{H}(A)$. Thus we have proved Corollaries 1 and 2.

The following theorem is a consequence of Theorem 1.

THEOREM 2. *Let $A \geqq 1$, $B \geqq C$ be integers and let $f(n)$, $g(n)$ be real-valued completely additive functions. Assume that for some $H \in \mathcal{H}(A)$ the congruence*

$$f(An + B) \equiv g(n) + C \pmod{H(n)} \tag{2.15}$$

holds for each $n \in \mathbf{N}$. Then $g(n) = 0$ for every $n \in \mathbf{N}$ and $f(n) = 0$ for all $n \in \mathbf{N}$ which are prime to A.

PROOF. We first prove Theorem 2 in the case $f = g$. Assume that a real-valued completely additive function $g(n)$ satisfies the congruence

$$g(An + B) \equiv g(n) + C \pmod{H(n)} \tag{2.16}$$

for every $n \in \mathbf{N}$. Applying (2.16) with $n = Bm$ and using the fact that $H(m)|H(Bm)$, we have

$$g(Am + 1) \equiv g(m) + C \pmod{H(m)} \tag{2.17}$$

for every $m \in \mathbf{N}$.

We shall deduce from (2.17) that

$$g(Am+k) \equiv g(m)+C \pmod{H(m)} \tag{2.18}$$

for every $k, m \in \mathbf{N}$. It is obvious from (2.17) that (2.18) holds for $k=1$. Assume that (2.18) is true for k. Then, using the complete additivity of g, we have

$$g(Am+1)+g(Am+k) = g\left[A\left(Am^2+km+m\right)+k\right] \equiv$$
$$\equiv g\left(Am^2+km+m\right)+C \equiv g(m)+C+g(Am+k+1) \pmod{H(m)}$$

which together with (2.17) and (2.18) implies

$$g(Am+k+1) \equiv g(m)+C \pmod{H(m)}.$$

This shows that (2.18) holds for $k+1$, and so (2.18) also holds for every $k, m \in \mathbf{N}$.

Let $n \geqq 2$ be an integer. We apply (2.18) with $k = A(n-1)m$ to get

$$g(Anm) \equiv g(m)+C \pmod{H(m)},$$

consequently

$$g(n) \equiv C - g(A) \pmod{H(m)}. \tag{2.19}$$

Since the condition b) of $\mathcal{H}(A)$

$$\limsup_{m\to\infty} H(m) = \infty,$$

from (2.19) it follows that $g(n) = C - g(A)$, and so

$$g(n) = 0 \quad \text{for every} \quad n \in \mathbf{N}. \tag{2.20}$$

We now prove Theorem 2. Assume that the real-valued completely additive functions f and g satisfy (2.15) for some $H \in \mathcal{H}(A)$, i.e.

$$f(An+B) \equiv g(n)+C \pmod{H(n)}$$

for every $n \in \mathbf{N}$. Replacing n in this congruence with Bm and using the complete additivity of f and g, we have

$$f(Am+1) \equiv g(m)+C+g(B)-f(B) \pmod{H(m)} \tag{2.21}$$

for every $m \in \mathbf{N}$. Using the complete additivity of f and g we deduce from (2.21) that

$$2f(Am+1) = f\left[A\left(Am^2+2m\right)+1\right] \equiv$$
$$\equiv g(Am+2)+g(m)+C+g(B)-f(B) \pmod{H(m)},$$

which together with (2.21) gives that

$$g(Am+2) \equiv g(m) + C + g(B) - f(B) \pmod{H(m)} \tag{2.22}$$

holds for every $m \in \mathbf{N}$. As we have seen above, from (2.22) we get

$$g(n) = 0 \tag{2.23}$$

for every $n \in \mathbf{N}$. Thus, by (2.15) and (2.23) it follows that

$$f(An+B) \equiv C \pmod{H(n)} \tag{2.24}$$

holds for every $n \in \mathbf{N}$. Using Theorem 1, by (2.2.4) we have

$$f(n) = 0 \tag{2.25}$$

for all $n \in \mathbf{N}$ which are prime to A. By (2.23) and (2.25) the proof of Theorem 2 is finished.

REMARK. It follows from Theorem 2 that if the real-valued completely additive functions f and g satisfy the congruence

$$f(An+B) \equiv g(n) + C \pmod{n}$$

then $g(n) = 0$ $(n = 1, 2, \ldots)$ and $f(n) = 0$ for all positive integers n which are prime to A. This result in the case $A = 1$, $f = g$ and $C = 0$ was proved by K. Kovács [10, Theorem 4] using the additional assumption that f is integer-valued.

3. In this part we shall consider the following problem: Determine all polynomials $P(x)$ with real coefficients such that for any integer-valued completely additive function $f(n)$ the congruence

$$P(E)f(m+n) \equiv P(E)f(m) \pmod{n} \qquad (m, n = 1, 2, \ldots)$$

implies that $f(n) = 0$ for all $n \in \mathbf{N}$.

We shall prove the following result.

THEOREM 3. *Let $f(n)$ be an integer-valued completely additive function. Let $P(x)$ be a non-zero polynomial with rational coefficients for which there exist suitable integers $A_P \neq 0$, $M_P \geqq 1$ such that*

$$A_P P(E)f(m+n) \equiv A_P P(E)f(m) \pmod{n} \tag{3.1}$$

for every integer $m \geqq M_P$ and $n \geqq 1$. Then $f(n) = 0$ for every $n \in \mathbf{N}$.

In the special case $P(x) = (x-1)^k$ for some integer $k \geqq 1$, Theorem 3 also holds under the assumption that $f(n)$ is a real-valued additive function.

THEOREM 4. *Let $f(n)$ be an integer-valued additive function and let $k \geqq 0$, $M \geqq 1$ be integers. If*

$$\Delta^k f(m+n) \equiv \Delta^k f(m) \pmod{n} \tag{3.2}$$

holds for every integer $m \geqq M$ and $n \geqq 1$, then $f(n) = 0$ for every $n \in \mathbf{N}$.

For the proof of Theorem 4 we shall use the following results.

LEMMA 2. *Let $f(n)$ be an integer-valued arithmetic function and let $k, M, Q \in \mathbf{N}$. If $\Delta^k f(n)$ satisfies*

$$\Delta^k f(m+Q) \equiv \Delta^k f(m) \pmod{Q} \tag{3.3}$$

for all integers $m \geqq M$, then

$$\Delta^{k-s} f(m+tQ) - \Delta^{k-s} f(m) \equiv \sum_{j=0}^{s-1} \binom{m-M}{j} \Delta_f^{k-s+j}(Q,t) \pmod{Q} \tag{3.4}$$

holds for every $s = 1, \ldots, k$ and for all integers $m \geqq M$, $t \geqq 0$, where

$$\Delta_f^i(Q,t) := \Delta^i f(M+tQ) - \Delta^i f(M) \qquad (i = 0, 1, 2, \ldots), \tag{3.5}$$

with $\Delta^0 f(n) := f(n)$.

REMARK. This lemma in the case $M = 1$ was proved by B. M. Phong [12] (see Lemma 1 of [12]).

PROOF. Let $t \geqq 0$ be a fixed integer. It is obvious that (3.4) holds for $m = M$. Let $m > M \geqq 1$. We shall prove (3.4) by induction on s.

Using (3.3), we have

$$\sum_{i=M}^{m-1} \Delta^k f(i+tQ) \equiv \sum_{i=M}^{m-1} \Delta^k f(i) \pmod{Q},$$

and so

$$\Delta^{k-1} f(m+tQ) - \Delta^{k-1} f(M+tQ) \equiv \Delta^{k-1} f(m) - \Delta^{k-1} f(M) \pmod{Q}$$

which proves (3.4) in the case $s = 1$.

Assume that $s < k$ and (3.4) holds for s. By using (3.4), we have

$$\Delta^{k-(s-1)} f(m+tQ) - \Delta^{k-(s+1)} f(M+tQ) -$$
$$-\Delta^{k-(s+1)} f(m) + \Delta^{k-(s+1)} f(M) =$$
$$= \sum_{i=M}^{m-1} \left\{ \Delta^{k-s} f(i+tQ) - \Delta^{k-s} f(i) \right\} \equiv \sum_{i=M}^{m-1} \sum_{j=0}^{s-1} \binom{i-M}{j} \Delta_f^{k-s+j}(Q,t) =$$

$$= \sum_{j=0}^{s-1} \Delta_f^{k-s+j}(Q,t) \sum_{i=M}^{m-1} \binom{i-M}{j} = \sum_{j=0}^{s-1} \Delta_f^{k-s+j}(Q,t) \binom{m-M}{j+1} =$$

$$= \sum_{j=1}^{s} \Delta_f^{k-(s+1)+j}(Q,t) \binom{m-M}{j} \pmod{Q},$$

and so

$$\Delta^{k-(s+1)} f(m+tQ) - \Delta^{k-(s+1)} f(m) \equiv$$

$$\equiv \sum_{j=0}^{s} \binom{m-M}{j} \Delta_f^{k-(s+1)+j}(Q,t) \pmod{Q}.$$

This shows that (3.4) holds for $s+1$, and so we have proved (3.4). This completes the proof of Lemma 2.

PROOF OF THEOREM 4. Assume that an integer-valued additive function $f(n)$ satisfies (3.2) for all integers $m \geqq M$ and $n \geqq 1$, where $k \geqq 0$ is an integer. If $k=0$, then (3.2) implies that

$$f(m+n) \equiv f(m) \pmod{n} \tag{3.6}$$

holds for all integers $m \geqq M$ and $n \geqq 1$. Putting $m=M$ in (3.6) we have $f(n+M) \equiv f(M) \pmod{n}$ for every $n \in \mathbf{N}$, which, using Theorem 1 implies that $f(n)=0$ for every $n \in \mathbf{N}$. Thus, Theorem 4 holds for $k=0$.

Assume now that $k \geqq 1$. Let $p>k$ be a prime. By using Lemma 2 in case $Q=p$ we get from (3.2) that

$$\Delta^{k-s} f(m+tp) - \Delta^{k-s} f(m) \equiv \sum_{j=0}^{s-1} \binom{m-M}{j} \Delta_f^{k-s+j}(p,t) \pmod{p} \tag{3.7}$$

for $s=1,\dots,k$ and for all $m \geqq M$ and $t \in \mathbf{N}$. Applying (3.7) with $m = M+ip$ and $t=1$, where $i \geqq 0$ is an integer, we have

$$\Delta^{k-s} f(M+(i+1)p) - \Delta^{k-s} f(M+ip) \equiv \tag{3.8}$$

$$\equiv \sum_{j=0}^{s-1} \binom{ip}{j} \Delta_f^{k-s+j}(p,1) \equiv \Delta_f^{k-s}(p,1) \pmod{p}$$

since $p>k>s-1 \geqq j$ and so $\binom{ip}{j} \equiv 0 \pmod{p}$ for $j=1,\dots,s-1$ and for all $i \in \mathbf{N}$. From (3.8) we get

$$\Delta_f^{k-s}(p,t) \equiv t\Delta_f^{k-s}(p,1) \pmod{p} \tag{3.9}$$

for $s = 1, \dots, k$ and for all $t \in \mathbf{N}$. Thus (3.9) implies $\Delta_f^{k-s}(p, \ell p) \equiv \equiv 0 \pmod p$ and so by applying (3.7) with $t = \ell p$ we have

$$\Delta^{k-s} f(m + \ell p^2) - \Delta^{k-s} f(m) \equiv 0 \pmod p$$

for all integers $m \geqq M$ and $\ell \geqq 0$. In the case $k = s$ the last congruence gives

$$f(m + \ell p^2) \equiv f(m) \pmod p \tag{3.10}$$

for all integers $m \geqq M$ and $\ell \geqq 0$. Since (3.10) holds for each prime $p > k$, so (3.10) holds for every prime $p > k$.

We deduce from (3.10) that $f(n)$ is a completely additive function.

Let a, b be positive integers. Let $q > \max(M, ab)$ be a fixed prime. Then for each prime $p > \max(q, k)$ there are positive integers x, y such that

$$ax = q + p^2 y \quad \text{and} \quad (x, ab) = 1. \tag{3.11}$$

Using (3.10) and (3.11), we have

$$f(a) + f(x) = f(ax) = f(q + p^2 y) \equiv f(q) \pmod p$$

and

$$f(ab) + f(x) = f(abx) = f(qb + p^2 by) \equiv f(qb) - f(q) + f(b) \pmod p.$$

These imply $f(ab) \equiv f(a) + f(b) \pmod p$ which shows that $f(ab) = f(a) + + f(b)$, since there exist infinitely many primes $p > \max(q, k)$. Thus, we proved that $f(n)$ is a completely additive function.

Finally, let $p > \max(k, M)$ be an arbitrary prime. Applying (3.10) with $m = p^2 n (> M)$ and $\ell = 1$, using the complete additivity of f, we get $f(n + + 1) \equiv f(n) \pmod p$ for every $n \in \mathbf{N}$. This shows that $f(n) \equiv 0 \pmod p$ for every $n \in \mathbf{N}$ and for every prime $p > \max(k, M)$, consequently $f(n) = 0$ for every $n \in \mathbf{N}$. This completes the proof of Theorem 4.

PROOF OF THEOREM 3. Assume that an integer-valued completely additive function f satisfies (3.1) for every integer $m \geqq M$ and $n \geqq 1$. We shall denote by I_f the set of all non-zero polynomials P with rational coefficients for which there are suitable integers $A_P \neq 0$ and $M_P \geqq 1$ such that

$$A_P P(E) f(m + n) \equiv A_P P(E) f(m) \pmod n$$

holds for every integer $m \geqq M_P$ and $n \geqq 1$. By the assumption of Theorem 3 it follows that $I_f \neq \emptyset$. It is also obvious that

(i) $cP(x) \in I_f$ for every $P \in I_f$ and $c \in \mathbf{Q}$,

(ii) $P(x) + P'(x) \in I_f$ for every $P, P' \in I_f$,

(iii) $xP(x) \in I_f$ for every $P \in I_f$.

Thus (i)–(iii) show that I_f is an ideal in $\mathbf{Q}[x]$. Here $\mathbf{Q}$ denotes the set of rational numbers.

Let

$$S(x) = c_0 + c_1 x + \ldots + c_k x^k \qquad (c_k = 1)$$

be a polynomial of minimum degree in I_f. If $k = 0$, then our Theorem 3 follows from Theorem 4. In the following we assume that $k \geqq 1$. Let

$$S(x) = (x - \Theta_1) \ldots (x - \Theta_k).$$

From the fundamental theorem of symmetric polynomials it follows that for each integer $s \geqq 1$ the polynomial

$$\prod_{j=1}^{k} \frac{x^s - \Theta_j^s}{x - \Theta_j}$$

has rational coefficients, consequently

$$Q_s(x^s) := (x^s - \Theta_1^s) \ldots (x^s - \Theta_k^s) \in I_f.$$

Furthermore we have

$$(3.13) \qquad Q_s(E^s) f(sn) = Q_s(1) f(s) + Q_s(E) f(n).$$

Since $Q_s(x^s) \in I_f$, there are integers $M_s \geqq 1$, $A_s \neq 0$ such that

$$(3.14) \qquad A_s Q_s(E^s) f(n+m) \equiv A_s Q_s(E^s) f(m) \pmod{n}$$

for all $n \in \mathbf{N}$, $m \geqq M_s$. Applying (3.14) with n, m given by ns and ms respectively from (3.13) and (3.14) we have

$$A_s Q_s(E^s) f(s(n+m)) \equiv A_s Q_s(E^s) f(sm) \pmod{n}$$

and so

$$A_s Q(1) f(s) + A_s Q_s(E) f(n+m) \equiv A_s Q_s(1) f(s) + A_s Q_s(E) f(m) \pmod{n}$$

for every $n \in \mathbf{N}$, $m \geqq M_s$. The last congruence shows that

$$A_s Q_s(E) f(n+m) \equiv A_s Q_s(E) f(m) \pmod{n}$$

for all $n \in \mathbf{N}$, $m \geqq M_s$, i.e.

$$(3.15) \qquad Q_s(x) \in I_f.$$

Thus $\delta(x) := (S(x), Q_s(x)) \in I_f$, and so $\deg \delta(x) = k$, $S(x) = Q_s(x)$ for every $s \in \mathbf{N}$. This implies that $\{\Theta_1, \dots, \Theta_k\} = \{\Theta_1^s, \dots, \Theta_k^s\}$ for every $s \in \in \mathbf{N}$, whence $\Theta_i = 0$ or 1 for each $i = 1, \dots, k$. Then there is an integer $\ell \geqq 0$ such that $S(x) = x^\ell (x-1)^{k-\ell}$ and so from the definition of I_f we have

$$A_S \Delta^{k-\ell} f(n+m) \equiv A_S \Delta^{k-\ell} f(m) \pmod{n} \tag{3.16}$$

for non-zero integer A_S and for all integers $n \geqq 1$, $m \geqq M_s + \ell$. From (3.16), using Theorem 4, we get that $A_S f(n) = 0$ for all $n \in \mathbf{N}$, which together with $A_S \neq 0$ implies $f(n) = 0$ for all $n \in \mathbf{N}$. Thus the proof of Theorem 3 is finished.

4. L. Lovász, A. Sárközy and M. Simonovits [2] have considered the class of complex-valued additive functions $f(n)$ for which

$$P(E)f(n) = 0 \qquad \text{for every} \qquad n \in \mathbf{N}, \tag{4.1}$$

where $P(x)$ is an arbitrary polynomial with complex coefficients. They showed that an additive function $f(n)$ satisfies (4.1) if and only if there exists an integer B such that $f(n+B) = f(n)$ for all $n \in \mathbf{N}$. From this result it follows that a completely additive function $f(n)$ satisfying (4.1) has to be a zero-function, i.e. $f(n) = 0$ for all $n \in \mathbf{N}$. We hope that the same conclusion holds if we weaken the condition (4.1) to

$$P(E)f(n) \equiv 0 \pmod{n} \quad (n = 1, 2, \dots)$$

for some polynomial P with real coefficients.

Here we shall prove this for the special case when $P(x) = (x-1)^Q$ where Q is a prime power.

THEOREM 5. *Let $Q = q^\alpha$ ($\alpha \geqq 1$ is an integer) be a prime power. If a real-valued completely additive function $f(n)$ satisfies the congruence*

$$\Delta^Q f(n) \equiv 0 \pmod{n} \tag{4.2}$$

for every $n \in \mathbf{N}$, then

$$f(n) = 0 \tag{4.3}$$

for every $n \in \mathbf{N}$.

PROOF. We note that from a result of I. Kátai [6, Lemma 2.1] the congruence (4.2) implies that $f(n)$ is an integer for all $n \in \mathbf{N}$. Thus, in the following we may assume that $f(n)$ is an integer-valued function.

In order to prove (4.3) it is enough to show that for each $n \in \mathbf{N}$ we have

$$f(n) \equiv 0 \pmod{q^k} \qquad (k = 0, 1, 2, \dots). \tag{4.4}$$

As we noted above, the congruence (4.4) holds for $k = 0$. Assume that (4.4) holds for $k(\geqq 0)$. We shall prove that (4.4) also holds for $k+1$. Let

$$F(n) := \frac{f(n)}{q^k} \qquad (n = 1,2,\dots).$$

From (4.2) we have

$$q^k\left\{F(n) - \binom{Q}{1}F(n+1) + \dots + (-1)^{Q-1}\binom{Q}{Q-1}F(n+Q-1) + \right.$$
$$\left. +(-1)^Q F(n+Q)\right\} \equiv 0 \pmod{n}$$

and so, replacing n by $q^{\alpha+k}m$,

$$F\left(q^{k+\alpha}m\right) + (-1)^Q F\left(q^{k+\alpha}m + Q\right) \equiv 0 \pmod{q}, \tag{4.5}$$

since $Q = q^\alpha$ and

$$\binom{q^\alpha}{1} \equiv \dots \equiv \binom{q^\alpha}{q^\alpha - 1} \equiv 0 \pmod{q}.$$

It is obvious that there is a $\beta \in \mathbf{N}$ such that $k + 2\alpha + \beta \equiv 0 \pmod{q}$ and so by applying (4.5) with $m = q^\beta N$, we have

$$F(N) + (-1)^Q F\left(q^{k+\beta}N + 1\right) \equiv 0 \pmod{q} \tag{4.6}$$

for every positive integer N.

We can deduce from (4.6), as in the proof of Theorem 2, that

$$F\left(q^{k+\beta}m + 2\right) \equiv F(m) \pmod{q}$$

holds for all $m \in \mathbf{N}$, from which it is easy to see that

$$F(n) \equiv -F\left(q^{k+\beta}\right) \pmod{q}$$

for all $n \in \mathbf{N}$. This together with $n = 2^q$ show that $F\left(q^{k+\beta}\right) \equiv 0 \pmod{q}$, and so $F(n) \equiv 0 \pmod{q}$ for all $n \in \mathbf{N}$. Thus, we have proved that

$$f(n) = q^k F(n) \equiv 0 \pmod{q^{k+1}}$$

holds for all $n \in \mathbf{N}$, i.e. (4.4) holds for $k+1$. So, by induction on k, (4.4) is true for every $k \in \mathbf{N}$.

From (4.4) the proof of Theorem 5 follows easily.

5. For a positive integer n we shall denote by n^* the product of all distinct prime divisors of n (In Corollary 1 we denoted n^* by $H_0(n)$.)

In this part we consider the problem of determining those integer-valued multiplicative functions $f(n)$ for which the congruence $f(An+1) \equiv C \pmod{n^*}$ holds for every $n \in \mathbf{N}$. Here $A \geqq 1$ and $C \neq 0$ are integers.

We shall prove the following

THEOREM 6. *Let $A \geqq 1$ and $C \neq 0$ be integers. Assume that an integer-valued multiplicative function $f(n)$ satisfies the congruence*

$$f(An+1) \equiv C \pmod{n^*} \tag{5.1}$$

for every $n \in \mathbf{N}$. Then there exist a non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that

$$f(n) = \chi_A(n) n^{\alpha}$$

for all $n \in \mathbf{N}$ which are prime to A.

COROLLARY 3. *Let $A \in \mathbf{N}$. If the integer-valued multiplicative functions $f(n)$ and $g(n)$ satisfy the congruence*

$$f(An+m) \equiv g(m) \pmod{n^*} \tag{5.2}$$

for every $n, m \in \mathbf{N}$, then there are non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that

$$f(n) = g(n) = \chi_A(n) n^{\alpha}$$

for all $n \in \mathbf{N}$ which are prime to A.

We shall use the following results in the proof of Theorem 6.

LEMMA 3. *Assume that the conditions of Theorem 6 are satisfied. Then*

$$C = 1 \tag{5.3}$$

and

$$f(ab) = f(a) f(b) \tag{5.4}$$

for all $a, b \in \mathbf{N}$ which are prime to A.

PROOF. Let $a, b \in \mathbf{N}$ be such that $(ab, A) = 1$. We choose a prime p such that

$$p > \max(a, b, |C|, |f(a)f(b) - Cf(ab)|). \tag{5.5}$$

Since $(ab, pA) = 1$, hence there are positive integers x, y, u and v such that

$$ax = 1 + Apy, \qquad (x, ab) = 1 \tag{5.6}$$

and

$$bu = 1 + Apv, \qquad (u, abx) = 1. \tag{5.7}$$

Then we have

$$abxu = 1 + ApT, \tag{5.8}$$

where $T := y + v + Apyv$. Applying (5.1) with n given by py, pv and pT, respectively, from (5.6), (5.7) and (5.8) we have

$$f(a)f(x) = f(ax) = f(1 + Apy) \equiv C \pmod p,$$
$$f(b)f(u) = f(bu) = f(1 + Apv) \equiv C \pmod p$$

and

$$f(ab)f(x)f(u) = f(abxu) = f(1 + Apt) \equiv C \pmod p,$$

because $p|(pm)^*$ for every $m \in \mathbf{N}$. These together with (5.5) imply that

$$f(x) \not\equiv 0 \pmod p, \quad f(a) \equiv f(ab)f(u) \pmod p$$

and so

$$f(a)f(b) \equiv f(ab)f(b)f(u) \equiv C \cdot f(ab) \pmod p.$$

The last congruence and (5.5) show that

$$C \cdot f(ab) = f(a)f(b). \tag{5.9}$$

Thus, we have proved that (5.9) holds for all $a, b \in \mathbf{N}$ which are prime to A.

By applying (5.9) with $a = b = 1$, we get (5.3) and (5.4). The proof of Lemma 3 is finished.

LEMMA 4. *Assume that the conditions of Theorem 6 are satisfied. Then there is a non-negative integer α such that $|f(n)| = n^\alpha$ for all $n \in \mathbf{N}$ which are prime to A.*

PROOF. In order to prove Lemma 4 it is enough to show that

$$f(q) = \pm q^{\alpha(q)} \tag{5.10}$$

and

$$\alpha(p) = \alpha(q) \tag{5.11}$$

for all primes p, q which are prime to A, where $\alpha(p) \geqq 0$, $\alpha(q) \geqq 0$ are integers.

Let q be a prime for which $(q, A) = 1$. Assume on the contrary that there is a prime $Q \neq q$ such that

$$Q | f(q). \tag{5.12}$$

Since $(q, QA) = 1$, by using the Euler's theorem we have

$$q^{\varphi(AQ)} = 1 + AQm \tag{5.13}$$

with some $m \in \mathbf{N}$, where φ denotes the Euler's totient function. By using Lemma 3 and the fact $(q, A) = 1$, we have

$$f(q)^{\varphi(AQ)} = f\left(Q^{\varphi(AQ)}\right) = f(1 + AQm).$$

This together with (5.1) and (5.2) show that

$$0 \equiv f(q)^{\varphi(AQ)} = f(1 + AQm) \equiv C = 1 \pmod{Q},$$

which is a contradiction. So, we proved (5.10).

We now prove (5.11). Let p and q be distinct primes with $(pq, A) = 1$ and let $\alpha(p) \geqq \alpha(q)$. Then it is well known that there is a positive integer s such that

$$H + H(s) := \left(\frac{(pq^s)^{2\varphi(A)} - 1}{A}\right)^* p^{2(\alpha(p) - \alpha(q))\varphi(A)} \tag{5.14}$$

(see e.g. [20]). Thus, we have $(pq^s)^{2\varphi(A)} = 1 + AHK$ with some $k \in \mathbf{N}$. Applying (5.1) and using Lemma 3, we get

$$f(p)^{2\varphi(A)} f(q)^{2\varphi(A)s} = f\left[(pq^s)^{2\varphi(A)}\right] = f(1 + AHK) \equiv C \pmod{H}.$$

This and (5.10) imply that

$$1 \equiv p^{2\alpha(p)\varphi(A)} q^{2\alpha(q)\varphi(A)s} \equiv p^{2(\alpha(p) - \alpha(q))\varphi(A)} \pmod{H},$$

which together with (5.14) shows that $\alpha(p) = \alpha(q)$. Thus, we have proved (5.11). This completes the proof of Lemma 4.

We now prove Theorem 6. Assume that (5.1) holds for every $n \in \mathbf{N}$, where $A \geqq 1$, $C \neq 0$ are integers and $f(n)$ is an integer-valued multiplicative function. Then by Lemma 4, there is a non-negative integer α such that

$$|f(n)| = n^\alpha \tag{5.15}$$

for all $n \in \mathbf{N}$ which are prime to A.

Now we define the function $g(n)$ as follows:

$$g(n) := \frac{f(n)}{n^\alpha} \qquad (n = 1, 2, \dots)$$

where $\alpha \geqq 0$ is defined in (5.15). By using Lemma 3, we have

$$g(ab) = g(a)g(b) \tag{5.16}$$

for all $a, b \in \mathbf{N}$ which are prime to A, furthermore $g(n) \in \{-1, 1\}$ for every $n \in \mathbf{N}$. Since $f(n) = g(n)n^{\alpha}$, we get from (5.1) that

$$g(An+1) \equiv g(An+1)(An+1)^{\alpha} = f(An+1) \equiv 1 \pmod{n^*} \tag{5.17}$$

holds for every $n \in \mathbf{N}$. We shall prove that the arithmetical function $g(n)$ is periodic (mod A).

Let t_1 and t_2 be positive integers for which

$$(t_1 t_2, A) = 1 \quad \text{and} \quad t_1 \equiv t_2 \pmod{A}.$$

Then there are positive integers s_0, n_1 and n_2 satisfying the conditions

$$s_0 t_1 = 1 + A n_1 \quad \text{and} \quad s_0 t_2 = 1 + A n_2.$$

Let β be a positive integer such that $2^{\beta} \nmid n_1$ and $2^{\beta} \nmid n_2$. Then for every $m \in \mathbf{N}$ we also have

$$\left(s_0 + 2^{\beta} A m\right) t_1 = 1 + A\left(n_1 + 2^{\beta} t_1 m\right) \tag{5.18}$$

and

$$\left(s_0 + 2^{\beta} A m\right) t_2 = 1 + A\left(n_2 + 2^{\beta} t_2 m\right). \tag{5.19}$$

By using (5.16), (5.17), (5.18) and (5.19), we have

$$g\left(s_0 + 2^{\beta} A m\right) g(t_1) \equiv 1 \quad \left(\bmod \left(n_1 + 2^{\beta} t_1 m\right)^*\right)$$

and

$$g\left(s_0 + 2^{\beta} A m\right) g(t_2) \equiv 1 \quad \left(\bmod \left(n_2 + 2^{\beta} t_2 m\right)^*\right).$$

Since $2^{\beta} \nmid n_1$, $2^{\beta} \nmid n_2$ we have

$$\left(n_i + 2^{\beta} t_i m\right)^* > 2 \qquad (i = 1, 2),$$

and so we get from the above congruences that

$$g\left(s_0 + 2^{\beta} A m\right) \cdot g(t_1) = 1 = g\left(s_0 + 2^{\beta} A m\right) \cdot g(t_2).$$

This implies $g(t_1) = g(t_2)$. It is well-known that if $g(n)$ satisfies (5.16) and

$$g(t_1) = g(t_2) \quad \text{for} \quad t_1 \equiv t_2 \pmod A, \quad (t_1 t_2, A) = 1,$$

then

$$g(n) = \chi_A(n) \tag{5.20}$$

holds for all $n \in \mathbf{N}$ which are prime to A. From this $f(n) = \chi_A(n)n^\alpha$ holds for all $n \in \mathbf{N}$ which are prime to A. Theorem 6 is proved.

Proof of the Corollary. Assume that the integer-valued multiplicative functions f and g satisfy the congruence (5.2), i.e. $f(An + m) \equiv$ $\equiv g(m) \pmod{n^*}$ holds for every $n, m \in \mathbf{N}$. Applying the above congruence with $m = 1$, by using Theorem 6, we see that there are a non-negative integer α and a real-valued Dirichlet character $\chi_A \pmod A$ such that $f(n) =$ $= \chi_A(n)n^\alpha$ for all $n \in \mathbf{N}$ which are prime to A, Thus, we have

$$\begin{aligned} g(m) \equiv f(An+m) = \chi_A(An+m)(An+m)^\alpha = \\ = \chi_A(m)m^\alpha \pmod{n^*} \end{aligned} \tag{5.21}$$

for every $n, m \in \mathbf{N}$ which are prime to A. Since

$$\limsup_{\substack{n\to\infty \\ (n,A)=1}} n^* = \infty,$$

it follows from (5.21) that $g(m) = \chi_A(m)m^\alpha$ for all $m \in \mathbf{N}$ which are prime to A. So, our Corollary is proved.

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FROZEN TIME METHODS FOR CONDITIONALLY STABLE PROBLEMS IN SINGULAR PERTURBATION THEORY

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I. Introduction

Let us consider the singularly perturbed linear system

$$\mu x' = A(t)x, \qquad \left(x' := \frac{dx}{dt}\right) \tag{1}$$

where μ is a small parameter, and $A(t)$ is a matrix function defined on $\mathbf{R}$. In [1], the uniform stability of the system (1) was considered under the following conditions: The eigenvalues of $A(t)$ satisfy $\operatorname{Re}\lambda(t) < -\gamma$, $A(t)$ and $A'(t)$ are bounded functions $\left(|A'(t)| \leqq \beta \text{ for any } t \in [0,\infty)\right)$.

Under these conditions the following is valid:

LEMMA 1. *If X represents the fundamental matrix of* (1) $(X(0) = I)$ *then the following inequality is satisfied:*

$$\left|X(t)X^{-1}(s) - \exp\{A(s)(t-s)/\mu\}\right| \leqq L(\mu,\alpha)\exp\{-\alpha(t-s)/\mu\} \tag{2}$$

where $t \geqq s$, $0 < \alpha < \gamma$, $\mu \in (0,\mu_0]$, $\mu_0 = \alpha^2/\beta K$, $\sigma = \gamma - \alpha$, $L(\mu,\alpha) = 2\mu K^2\beta/e^2(\alpha^2 - \mu K\beta)$, K is constant depending only on β and γ.

This approximation (2) is used to estimate an upper bound for the parameter μ in problems of uniform asymptotic stability; see for example [1] and [2]. In [2], this result is used to obtain conditions of controllability for singularly perturbed control systems. The central aim of this work is to generalize Lemma 1 to conditionally stable systems.

II. Previous results

We will assume that (1) satisfies the following properties:

(C1) The function $A(t)$ is defined on $\mathbf{R}$. $A(t)$ is a function of class C^1 and $|A(t)| \leqq M$, $|A'(t)| \leqq \beta$, for $t \in \mathbf{R}$.

(C2) For each t, k eigenvalues of $A(t)$ satisfy $\operatorname{Re}\lambda(t) \leqq -\gamma$, and $n-k$ eigenvalues satisfy $\operatorname{Re}\lambda(t) \geqq \gamma$.

In [3] it is proven that conditions (C1) and (C2) are enough to prove the following

LEMMA 2. *Conditions* (C1) *and* (C2) *with* $k = n$ *imply for* $0 < \alpha < \min(2M, \gamma)$ *and* $\xi \geqq 0$, *that*

$$\left|\exp\{A(s)\xi\}\right| \leqq (2M/\alpha)^{n-1} \exp\{(-\gamma + \alpha)\xi\}.$$

LEMMA 3. *Under conditions* (C1) *and* (C2), *there exists a* C^1 *function* $T(t)$ *such that*

$$D(t) := T^{-1}(t)A(t)T(t) = \operatorname{diag}(A_1(t), A_2(t)) \tag{3}$$

and the following properties hold:

(P1) *The eigenvalues of* $A_1(t)$ *and* $A_2(t)$ *satisfy* $\operatorname{Re}\lambda(t) \leqq -\gamma$, *and* $\operatorname{Re}\lambda(t) \geqq \gamma$, *resp.;*

(P2) *For any* $t \in \mathbf{R}$,

$$|T^{-1}(t)T'(t)| \leqq a|A(t)|\,|A'(t)| \leqq a\beta M \tag{4}$$

where a is a constant depending only on n and γ.

(P3) $|T(t)|\,|T^{-1}(t)| \leqq b$, *for* $t \in \mathbf{R}$, *where b is a constant.*

If we apply the change of variable $x = T(t)z$ *then* (1) *reduces to the system*

$$\mu z' = \left[D(t) - \mu T^{-1}(t)T'(t)\right] z. \tag{5}$$

THEOREM 4. *If conditions* (C1) *and* (C2) *are satisfied then there exist a fundamental matrix* $X(t)$ *of* (1) *and constants* $H \geqq 1$, μ_0, *such that for* $\mu \in (0, \mu_0)$ *and* $t \geqq s$ *one has:*

$$\left|X(t)PX^{-1}(s)\right|, \left|X(s)QX^{-1}(t)\right| \leqq H \exp\{-(\gamma - \alpha)(t - s)/\mu\} \tag{6}$$

where P and Q are the following projections:

$$P := \left|\begin{array}{c:c} I_k & 0 \\ \hline 0 & 0 \end{array}\right|, \qquad Q := I - P. \tag{7}$$

Finally we need a previously proven result [6]:

THEOREM 5. *If the system* (1) *has the exponential dichotomy* (6), *and* $B(t)$ *is a bounded continuous matrix, such that*

$$\delta = \sup_{\mathbf{R}} |B(t)| \leqq (\gamma - \alpha)/36K^5,$$

then the system $\mu y' = \left[A(t) + B(t)\right] y$ *has the exponential dichotomy:*

$$\left|Y(t)PY^{-1}(s)\right|, \left|Y(s)QY^{-1}(t)\right| \leqq 12K^3 \exp\{-(\gamma - \alpha - 6K^3)(t - s)/\mu\}$$

for $t \geqq s$, *and*

$$\left|X(t)PX^{-1}(s) - Y(t)PY^{-1}(s)\right| \leqq$$
$$\leqq (5K/2)^8 \delta(\gamma - \alpha)^{-1} \exp\{-(\gamma - \alpha)(t - s)/2\mu\},$$
$$\left|X(t)QX^{-1}(s) - Y(t)QY^{-1}(s)\right| \leqq$$
$$\leqq (5K/2)^8 \delta(\gamma - \alpha)^{-1} \exp\{-(\gamma - \alpha)(t - s)/2\mu\}.$$

III. The main result

The proof of our main theorem is based upon the theory of exponential dichotomies as presented in [3].

Let us define $\theta(t,s) := \exp\big(D(s)t/\mu\big)$, $W(t,s) := T(t)\theta(t,s)$; we can now formulate the central result of our investigation.

THEOREM 6. *If the system* (1) *satisfies* (C1) *and* (C2) *the for* $t \geqq s$ *we have*

$$(8)\qquad \big|X(t)PX^{-1}(s) - W(t,s)PW(s,s)\big| \leqq R(\mu,\alpha)\exp\{-\gamma(t-s)/4\mu\}$$

where $R(\mu,\alpha) = a\beta M\Big(L(\mu,\gamma) + (5K/2)^8 a\beta\mu^2 M(\gamma-\alpha)^{-1}\Big)$, *and*

$$(8)\qquad \big|X(t)QX^{-1}(s) - W(t,s)QW(s,s)\big| \leqq R(\mu,\alpha)\exp\{-\gamma(t-s)/4\mu\}$$

for $s \geqq t$.

Note that estimate (8) is a generalization of (2), since if in condition (C2) $k = 0$, then in Theorem 6 we have $T(t) = I$, $P = I$, $W(t,s) = \exp\{A(s)t/\mu\}$.

IV. Proof of Theorem 6

We first prove a lemma concerning the fundamental solution of the system

$$(9)\qquad v' = D(t)v.$$

This system can be written as two systems of lower dimension:

$$(10)\qquad \mu v_1' = A_1(t)v_1,$$

$$(11)\qquad \mu v_2' = A_2(t)v_2.$$

Let V_1 and V_2 represent the fundamental matrices of (10) and (11). According to Lemma 3, for $0 < \alpha < \min(2M,\gamma)$ we have the estimates

$$(12)\qquad \big|V_1(t)V_1^{-1}(s)\big| \leqq K\exp\big\{-(\gamma-\alpha)(t-s)/\mu\big\},\qquad t \geqq s$$

$$(13)\qquad \big|V_2(t)V_2^{-1}(s)\big| \leqq K\exp\big\{(\gamma-\alpha)(t-s)/\mu\big\},\qquad t \geqq s$$

for all $\mu > 0$ and $K = 2M/\alpha$. Clearly $V(t) = \operatorname{diag}\big(V_1(t), V_2(t)\big)$ is the fundamental matrix of (9) and

$$(14)\qquad \big|V(t)PV^{-1}(s)\big|, \big|V(s)QV^{-1}(t)\big| \leqq K\exp\big\{-(\gamma\alpha)(t-s)/\mu\big\}.$$

Using Lemma 1 in [1] we can immediately prove the following

LEMMA 7.

$$\left|V(t)PV^{-1}(s)-\theta(t,s)P\theta^{-1}(s,s)\right| \leqq L(\mu,\alpha)\exp\{-\sigma(t-s)/\mu\}$$

for $t \geqq s$, *and*

$$\left|V(t)QV^{-1}(s)-\theta(t,s)Q\theta^{-1}(s,s)\right| \leqq L(\mu,\alpha)\exp\{-\sigma(t-s)/\mu\}$$

for $s \geqq t$, $L(\mu,\alpha)=\mu K^2\beta/e^2(\alpha^2-\mu K\beta)$ *and* $0<\alpha<\gamma$, $\sigma=\gamma-\alpha$, $\mu\in(0,\mu_0]$, $\mu_0=\alpha^2/\beta K$.

PROOF OF THEOREM 6. We abbreviate $I := \left|X(t)PX^{-1}(s) - -W(t,s)PW(s,s)\right|$. Then we have

$$I=\left|T^{-1}(t)\left[T(t)X(t)PX^{-1}(s)T(s)-\theta(t,s)P\theta^{-1}(s,s))\right]T(s)\right|.$$

We denote by $Z(t)$ the fundamental matrix of the system (4), and we recall that $Z(t)=T(t)X(t)$. By the property (P2) we obtain

$$I \leqq a\beta M\left|Z(t)PZ^{-1}(s)-\theta(t,s)P\theta^{-1}(s,s)\right| = a\beta M(I_1+I_2)(t,s)$$

where

$$I_1(t,s) := \left|Z(t)PZ^{-1}(s)-V(t)PV^{-1}(s)\right|,$$
$$I_2(t,s) := \left|V(t)PV^{-1}(s)-\theta(t,s)P\theta^{-1}(s,s)\right|.$$

According to Theorem 1, we have

$$I_1(t,s) \leqq (5K/2)^8 a\beta\mu^2 M(\gamma-\alpha)^{-1}\exp\{-\gamma(t-s)/2\mu\}. \tag{15}$$

Theorem 3 gives us the estimate

$$I_2(t,s) \leqq L(\mu,\gamma)\exp\{-(\gamma-\alpha)(t-s)/2\mu\}. \tag{16}$$

Defining $R(\mu,\alpha)=a\beta M\left[L(\mu,\gamma)+(5K/2)^8 a\beta M\mu^2 M(\gamma-\alpha)^{-1}\right]$ we complete the proof of Theorem 6.

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A CHARACTERIZATION OF SOME ARITHMETICAL MULTIPLICATIVE FUNCTIONS

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1. Introduction and results

An arithmetical function $f(n) \not\equiv 0$ is said to be multiplicative if $(m,n) = 1$ implies that

$$f(mn) = f(m)f(n)$$

and it is completely multiplicative if the above relation holds for all positive integers m and n. Let $\mathcal{M}$ and $\mathcal{M}^*$ denote the set of complex-valued multiplicative and completely multiplicative functions, respectively.

In 1980, J. L. Mauclaire and Leo Murata [6] showed that if $f \in \mathcal{M}$ satisfies the conditions

$$(1.1) \qquad |f(n)| = 1 \qquad (n = 1, 2, \dots)$$

and

$$(1.2) \qquad \sum_{n \leqq x} |f(n+1) - f(n)| = o(x),$$

then $f \in \mathcal{M}^*$. Trivially, (1.1) and (1.2) hold for functions of the type $f(n) = n^{i\tau}$, where τ is a real number. I. Kátai [3] conjectured that $f(n) = n^{i\tau}$ are the only multiplicative functions that satisfy (1.1) and (1.2). This conjecture remains open, a few partial results are known. For such results we refer the reader to A. Hildebrand [1], [2] and I. Kátai [5].

In [7], improving the above result of Mauclaire and Murata, we proved that if $f \in \mathcal{M}$ satisfies (1.1) and the condition

$$(1.3) \qquad \sum_{n \leqq x} |f(An+B) - Cf(n)| = o(x)$$

for some positive integers A, B and a non-zero complex number C, then

$$f(p^k) = f(p)^k \qquad (k = 1, 2, \dots)$$

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for each prime p coprime to $2AB$, furthermore in the case $(2, AB) = 1$ we also have

$$f(2^k) = \left(\frac{f(A)}{C}\right)^{k-1} f(2)^k \qquad (k = 1, 2, \dots)$$

and $f(A)^2 = C^2$.

In this paper we shall consider the problem of characterizing the functions $f \in \mathcal{M}$ that satisfy the conditions (1.1) and

$$\sum_{n \leqq x} \frac{|f(An + B) - Cf(n)|}{n} = o(\log x), \tag{1.4}$$

where A, B are fixed positive integers and C is a non-zero complex number. A complete solution of (1.4) is not known, since the condition (1.4) is stronger than (1.2).

We shall prove the following results.

THEOREM 1. *Let A, B be positive integers and let C be a non-zero complex number. If $f \in \mathcal{M}$ satisfies* (1.1) *and* (1.4), *then there are functions f_1 and F such that*

$$f(n) = f_1(n)F(n) \qquad (n = 1, 2, \dots), \tag{1.5}$$

$$f_1 \in \mathcal{M}^*, \qquad F \in \mathcal{M}, \tag{1.6}$$

$$F(An) = F(A)F(n) \qquad (n = 1, 2, \dots), \tag{1.7}$$

$$F(n + B) = \frac{f(A)}{C} F(n) \qquad (n = 1, 2, \dots), \tag{1.8}$$

$$f^2(A) = C^2 \tag{1.9}$$

and

$$f(A) = C \qquad \textit{if} \qquad 2 | AB. \tag{1.10}$$

COROLLARY 1. *If $f \in \mathcal{M}$ satisfies the conditions* (1.1) *and* (1.3), *then $f = f_1 F$, where f, f_1, F satisfy the conditions* (1.6)–(1.10).

REMARKS. (I) All solutions of (1.8) for $F \in \mathcal{M}$ have been determined in [8]. From our proof it follows that

$$F(p^\gamma) = F\big[(p^\gamma, B)\big] \qquad (\gamma = 0, 1, 2, \dots) \tag{1.11}$$

for all odd primes p and

$$F(2^\beta) = \frac{f(A)}{C} F\big[(2^\beta, B)\big] \qquad (\beta = 1, 2, \dots). \tag{1.12}$$

(II) It is obvious that $f(n) = (-1)^{n-1}$ is a multiplicative function and it satisfies (1.1) and (1.4) with $A = B = -C = 1$. In this case we have $\frac{f(A)}{C} = -1$. In other words, from (1.1) and (1.4) it does not always follow that $f(A) = C$.

THEOREM 2. *Let A, B be positive integers and let C be a non-zero complex number. Assume that $f \in \mathcal{M}$ and $g \in \mathcal{M}^*$ satisfy the conditions*

$$|f(n)| = 1, \qquad g(n) \neq 0, \qquad (n = 1, 2, \ldots) \tag{1.13}$$

and

$$\sum_{n \leqq x} \frac{|g(An + B) - Cf(n)|}{n} = o(\log x). \tag{1.14}$$

Let $A' = A/(A, B)$ and $B' = B/(A, B)$. Then the following assertions hold:

(a) *If $f(A' + 1) = g(A' + 1)$, then*

$$f \in \mathcal{M}^* \tag{1.15}$$

and there is a Dirichlet character $\chi_{A'} \pmod{A'}$ such that

$$g(n) = \chi_{A'}(n) f(n) \tag{1.16}$$

for every positive integer n coprime to A'.

(b) *If $f(A' + 1) \neq g(A' + 1)$, then $2 \nmid (A' + 1)B'$ and there are $f_1 \in \mathcal{M}^*$ and a Dirichlet character $\chi_{2A'} \pmod{2A'}$ such that*

$$f(n) = (-1)^{n-1} f_1(n) \qquad (n = 1, 2, \ldots) \tag{1.17}$$

and

$$g(n) = \chi_{2A'}(n) f(n) \tag{1.18}$$

for every positive integer n coprime to $2A'$.

Several authors have considered the problem of finding similar conditions concerning $f(n + 1) - f(n)$ that imply $f(n) = n^{i\tau}$ for some real number τ. Improving some results of I. Kátai [4], E. Wirsing showed[1] that if $f \in \mathcal{M}$ satisfies $|f(n)| \equiv 1$ and

$$f(n + 1) - f(n) \to 0 \quad \text{as} \quad n \to \infty,$$

then $f(n) = n^{i\tau}$ for some real constant τ. Applying our theorems and using this result of E. Wirsing, we get the following

[1] Personal communication to I. Kátai on September 3, 1984.

THEOREM 3. *Assume that $f \in \mathcal{M}$ satisfies the conditions*

$$|f(n)| = 1 \qquad (n = 1, 2, \dots)$$

and

$$f(An + B) - Cf(n) \to \infty \quad as \quad n \to \infty \tag{1.19}$$

for some positive integers A, B and a non-zero complex number C. Then there are a real constant τ and a function $F \in \mathcal{M}$ such that

$$f(n) = n^{i\tau} F(n) \qquad (n = 1, 2, \dots), \tag{1.20}$$

where f, F satisfy the conditions (1.6)–(1.10).

THEOREM 4. *Assume that $f \in \mathcal{M}$ and $g \in \mathcal{M}^*$ satisfy the conditions*

$$|f(n)| = 1, \qquad g(n) \neq 0 \qquad (n = 1, 2, \dots) \tag{1.21}$$

and

$$g(An + B) - Cf(n) \to 0 \quad as \quad n \to \infty \tag{1.22}$$

for some positive integers A, B and a non-zero complex number C. Let $A' = A/(A, B)$, $B' = B/(A, B)$. Then the following assertions hold:

(i) *If $f(A' + 1) = g(A' + 1)$, then there are a real constant τ and a Dirichlet character $\chi_{A'}$ $(\bmod A')$ such that*

$$f(n) = n^{i\tau} \qquad (n = 1, 2, \dots) \tag{1.23}$$

and

$$g(n) = \chi_{A'}(n) n^{i\tau} \tag{1.24}$$

for every positive integer n coprime to A'.

(ii) *If $f(A' + 1) \neq g(A' + 1)$, then $2 \nmid (A' + 1)B'$ and there are a real constant τ and a Dirichlet character $\chi_{2A'}$ $(\bmod 2A')$ such that*

$$f(n) = (-1)^{n-1} n^{i\tau} \qquad (n = 1, 2, \dots) \tag{1.25}$$

and

$$g(n) = \chi_{2A'}(n) n^{i\tau} \tag{1.26}$$

for every positive integer n coprime to $2A'$.

We note that by writing a multiplicative function f of modulus 1 in the form $f = e^{2\pi i h}$, where h is a real-valued additive function, one can reformulate results involving multiplicative functions of modulus 1 in term of real-valued additive functions reduced modulo 1. For example, one easily sees in this way that from Theorem 3 we have

COROLLARY 2. *Let A, B be positive integers and let D be a real constant. If a real-valued completely additive function h satisfies*

$$\lim_{n\to\infty}\big\| h(An+B)-h(n)-D\big\| = 0, \tag{1.27}$$

where $\|u\|$ denotes the distance of u to the nearest integer, then there is a real constant τ such that

$$\big\| h(n)-\tau\log n\big\| = 0 \qquad (n=1,2,\dots). \tag{1.28}$$

2. Proof of Theorem 1

Assume that $f \in \mathcal{M}$ satisfies (1.1) and (1.4), where A, B are positive integers and C is a non-zero complex number. For an arbitrary positive integer n, let $D(n) = D_B(n)$ denote the product of prime power factors of B composed from the prime divisors of n, i.e., $D(n)|B$, $\big(D(n), B/D(n)\big) = 1$ and every prime divisor of $D(n)$ is a divisor of n.

For each positive integer Q we define the sequence

$$R = R(AQ) = \{R_k\}_{k=1}^{\infty}$$

by the initial term $R_1 = 1$ and by the formula

$$R_k = 1 + AQ + \ldots + (AQ)^{k-1} \tag{2.1}$$

for every integer $k \geqq 2$. Moreover, let

$$T_k = T_k(n,Q) = (AQ)^k D(Q)n + BR_k(AQ). \tag{2.2}$$

Then for every integer $k \geqq 1$ we have

$$T_{k+1}(n,Q) = AQT_k(n,Q) + B \tag{2.3}$$

and

$$\big(QD(Q), T_k(n,Q)/D(Q)\big) = 1. \tag{2.4}$$

Thus, by using (1.4), (2.2), (2.3), (2.4) and the multiplicativity of f, we have

$$\sum_{n\leqq x}\frac{1}{n}\left|f\big(T_1(n,Q)\big) - Cf\big(QD(Q)n\big)\right| = o(\log x)$$

and

$$\sum_{n\leqq x}\frac{1}{n}\left|f\left(T_k(n,Q)\right)-\Delta(Q)f\left(T_{k-1}(n,Q)\right)\right|=o(\log x)$$

for each fixed integer $k\geqq 2$, where

$$\Delta(Q):=Cf(QD(Q))/f(D(Q)).$$

These imply that

$$\sum_{n\leqq x}\frac{1}{n}\left|f\left(T_k(n,Q)\right)-C\Delta(Q)^{k-1}f\left(QD(Q)n\right)\right|=o(\log x) \tag{2.5}$$

for each fixed integer $k\geqq 1$.

We first deduce from (2.5) that if the positive integers k, P and Q satisfy the conditions

$$\left(P,R_k(AQ)\right)=1 \tag{2.6}$$

and

$$\left(PD(Q)+B,R_k(AQ)\right)=1, \tag{2.7}$$

then

$$f\left(A^{k-1}Q^kPD(Q)\right)=\Delta(Q)^{k-1}f(QD(Q)P). \tag{2.8}$$

Assume that (2.6) and (2.7) hold. Let $R_k=R_k(AQ)$. Then

$$\left(R_k,(AQ)^kPD(Q)+B\right)=\left(R_k,PD(Q)+B\right)=1. \tag{2.9}$$

Considering

$$n:=PR_kM(m)=PR_k(APQR_km+1)$$

and taking into account (2.5), using (1.1), (1.4), (2.6), (2.7), (2.9) and the multiplicativity of f, we get

$$\frac{1}{PR_k\cdot(APQR_k+1)}\Big|f\left(A^{k-1}Q^kPD(Q)\right)-$$
$$-\Delta(Q)^{k-1}f\left(QD(Q)P\right)\Big|\sum_{m\leqq x}\frac{1}{m}\leqq$$
$$\leqq\sum_{m\leqq x}\frac{1}{PR_kM(m)}\Big|f\left(A^{k-1}Q^kPD(Q)M(m)\right)-$$

$$-\Delta(Q)^{k-1}f\big(QD(Q)PM(m)\big)\Big| \leqq$$

$$\leqq \sum_{m\leqq x} \frac{1}{PR_k|C|M(m)} \Big| f\big((AQ)^k PD(Q)M(m)+B\big)-$$

$$-Cf\big(A^{k-1}Q^k PD(Q)M(m)\big)\Big| +$$

$$+\sum_{m\leqq x} \frac{1}{PR_k|C|M(m)} \Big| f\big((AQ)^k PD(Q)M(m)R_k + BR_k\big)-$$

$$-C\Delta(Q)^{k-1}f\big(QD(Q)PM(m)R_k\big)\Big| \leqq o\big(\log(PR_kM(x))\big) = o(\log x),$$

which implies that

$$f\big(A^{k-1}Q^kPD(Q)\big) - \Delta(Q)^{k-1}f\big(QD(Q)P\big) = o(1).$$

This proves (2.8).

It is easily seen that (2.6) and (2.7) hold for every positive integer k if $P=1$ and $2B|Q$. Thus, for each prime p coprime to $2AB$, by applying (2.8) with $Q=2B$ and $Q=2pB$, we have

(2.10)
$$f(p^k) = \frac{f\big[A^{k-1}(2pB)^kD(2pB)\big]}{f\big[A^{k-1}(2B)^kD(2B)\big]} = \left(\frac{\Delta(2pB)}{\Delta(2B)}\right)^{k-1} \frac{f(2pB)}{f(2B)} = f(p)^k$$

for every positive integer k. Thus, by using (2.10) and $f\in\mathcal{M}$ we have

$$f(mn)=f(m)f(n) \quad \text{if} \quad (m,n,2AB)=1. \tag{2.11}$$

Since $\big(R_k(AQ),A\big)=1$ for every positive integer k, from (2.11) we can relax the conditions (2.6) and (2.7) to

$$\big(P, R_k(AQ), 2B\big)=1 \tag{2.12}$$

and

$$\big(PD(Q)+B, R_k(AQ), 2\big)=1. \tag{2.13}$$

In other words, if the positive integers k, P and Q satisfy conditions (2.12) and (2.13), then

$$f\big(A^{k-1}Q^kPD(Q)\big) = \Delta(Q)^{k-1}f\big(QD(Q)P\big). \tag{2.14}$$

We now prove that

$$f(AP)=f(A)f(P) \qquad (P=1,2,\ldots). \tag{2.15}$$

First consider the case when $2|AB$. In this case, by applying (2.14) with $k=2$ and $Q=1$, we have

$$f(AP) = Cf(P) \quad \text{if} \quad (P, 1+A, B) = 1. \tag{2.16}$$

and

$$f(A) = C. \tag{2.17}$$

Thus, by using (2.16), (2.17) and the multiplicativity of f, it is obvious that (2.15) holds.

Suppose now that $2 \nmid AB$. In this case (2.12) and (2.13) hold for every even positive integer Q with $\big(P, R_k(AQ), B\big) = 1$. Thus, by applying (2.14) with $k = Q = 2$, we have

$$f(A2^2P) = Cf(2)f(2P) \quad \text{if} \quad (P, 1+2A, B) = 1.$$

This with the multiplicativity of f implies that

$$f(A2^2P) = Cf(2)f(2P) \qquad (P = 1, 2, \ldots),$$

consequently $f(AP) = f(A)f(P)$ holds for every odd positive integer P. So, (2.15) holds for every positive integer P, because in this case $2 \nmid A$. Thus, we have proved (2.15).

On the other hand, it follows from (2.12)–(2.14) that if the positive integers k, Q satisfy

$$\big(D(Q) + B, R_k(AQ), 2\big) = 1 \tag{2.18}$$

then

$$f\big(A^{k-1}Q^kD(Q)\big) = C^{k-1}\frac{f\big(QD(Q)\big)^k}{f\big(D(Q)\big)^{k-1}}. \tag{2.19}$$

Thus, if (2.18) holds, then from (2.15) and (2.19) we have

$$f\big(Q^kD(Q)\big) = \left(\frac{C}{f(A)}\right)^{k-1}\frac{f\big(QD(Q)\big)^k}{f\big(D(Q)\big)^{k-1}}. \tag{2.20}$$

It is easily seen that (2.18) holds in the following cases:

(i) $Q = 1$, $k = 3$;

(ii) $Q = 1$, $k = 2$ if $2|AB$;

(iii) $Q = 2$ and every positive integer k;

(iv) $Q = 2p$, $(p,2) = 1$ and every positive integer k.
Thus, from (2.20) we get

$$f(A)^2 = C^2, \tag{2.21}$$

$$f(A) = C \quad \text{if} \quad 2|AB \tag{2.22}$$

$$f\left(2^k D(2)\right) = \left(\frac{C}{f(A)}\right)^{k-1} \frac{f\left(2D(2)\right)^k}{f\left(D(2)\right)^{k-1}} \tag{2.23}$$

and

$$f\left(p^k D(p)\right) = \frac{f\left(pD(p)\right)^k}{f\left(D(p)\right)^{k-1}} \quad \text{if} \quad (p,2) = 1. \tag{2.24}$$

Now we write $f(n)$ as

$$f(n) = f_1(n)F(n), \tag{2.25}$$

where $f_1(n)$ is a completely multiplicative function defined as follows:

$$f_1(p) = \begin{cases} \frac{f(pD(p))}{f(D(p))} & \text{if } (p,2) = 1 \\ \frac{C}{f(A)} \frac{f(2D(2))}{f(D(2))} & \text{if } p = 2. \end{cases} \tag{2.26}$$

From (2.24), (2.25) and (2.26) it is easily seen that

$$F(p^\alpha D(p)) = F(D(p)) \qquad (\alpha = 0, 1, 2, \dots)$$

and so

$$F(p^\gamma) = F[(p^\gamma, B)] \qquad (\gamma = 0, 1, 2, \dots) \tag{2.27}$$

for every prime $p \neq 2$. Moreover, from (2.23), (2.25) and (2.26), we have

$$F(2^\beta) = \frac{C}{f(A)} F[(2^\beta, B)] \qquad (\beta = 1, 2, \dots). \tag{2.28}$$

By using (2.22), (2.27) and (2.28) one can check that

$$F(n + B) = \frac{C}{f(A)} F(n) \qquad (n = 1, 2, \dots). \tag{2.29}$$

Thus, from (2.15), (2.21), (2.22) and (2.29) we get the assertions (1.5)–(1.10) of Theorem 1. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Assume that $f \in \mathcal{M}$ and $g \in \mathcal{M}^*$ satisfy conditions (1.13) and (1.14), that is,

$$|f(n)| = 1, \quad g(n) \neq 0 \quad (n = 1, 2, \dots) \tag{3.1}$$

and

$$\sum_{n \leqq x} \frac{1}{n} \big| g(An + B) - Cf(n) \big| = o(\log x) \tag{3.2}$$

for some positive integers A, B and a non-zero complex number C. Let $A' = A/(A,B)$, $B' = B/(A,B)$ and $C' = C/g[(A,B)]$. Since $g \in \mathcal{M}^*$, we have

$$g(An + B) - Cf(n) = g[(A,B)]g(A'n + B') - Cf(n)$$

which with (3.2) implies

$$\sum_{n \leqq x} \frac{1}{n} \big| g(A'n + B') + B') - C'f(n) \big| = o(\log x).$$

Thus, it is enough to prove Theorem 2 under the condition $(A,B) = 1$.

Assume that (3.1), (3.2) hold and $(A,B) = 1$. For each fixed positive integer N, we have

$$(AN + 1)(An + B) = A\big[(AN + 1)n + BN\big] + B,$$

and so, by using the complete multiplicativity of g, we obtain

$$f\big[(AN + 1)n + BN\big] - g(AN + 1)f(n) =$$
$$= -\frac{1}{C}\left\{g\left[A\big((AN + 1)n + BN\big) + B\right] - Cf\big[(AN + 1)n + BN\big]\right\} +$$
$$+ \frac{1}{C}g(AN + 1)\big\{g(An + B) - Cf(n)\big\},$$

which with (3.1) and (3.2) implies that

$$\sum_{n \leqq x} \frac{1}{n} \Big| f\big[(AN + 1)n + BN\big] - g(AN + 1)f(n) \Big| = o(\log x). \tag{3.3}$$

Applying Theorem 1 when A, B, C are replaced by $AN+1$, BN and $g(AN+1)$, respectively, we see that there are $f_1 \in \mathcal{M}^*$ and $F \in \mathcal{M}$ such that

$$f(n) = f_1(n)F(n), \tag{3.4}$$

$$F(n+BN) = \frac{f(AN+1)}{g(AN+1)}F(n) \quad (n = 1,2,\ldots), \tag{3.5}$$

$$f^2(AN+1) = g^2(AN+1) \tag{3.6}$$

and

$$f(AN+1) = g(AN+1) \quad \text{if} \quad 2|(AN+1)BN. \tag{3.7}$$

Since (3.5)–(3.7) hold for each positive integer N, it follows that (3.5)–(3.7) also hold for every positive integer N.

a) We first consider the case when $f(A+1) = g(A+1)$. In this case from (3.5) we have $F(n+B) = F(n)$ $(n = 1,2,\ldots)$ and so

$$F(n+BN) = F(n) \qquad (n = 1,2,\ldots). \tag{3.8}$$

This with (3.5) shows that

$$f(AN+1) = g(AN+1) \qquad (N = 1,2,\ldots) \tag{3.9}$$

and

$$F(n) = \chi_B(n) \tag{3.10}$$

for all positive integers n which are prime to B. (Here χ_B denotes the Dirichlet character mod B.) Let

$$H(n) := \frac{g(n)}{f(n)} \qquad (n = 1,2,\ldots).$$

Then, by (3.9) we have $H(AN+1) = 1$ $(N = 1,2,\ldots)$, which gives $H = \chi_A \pmod{A}$. Thus we have proved that

$$g(n) = \chi_A(n)f(n) \tag{3.11}$$

for every positive integer n coprime to A.

We shall prove that in our case

$$f \in \mathcal{M}^*. \tag{3.12}$$

Indeed, from (3.4), (3.10), (3.11) and using the fact $g \in \mathcal{M}^*$, we have

$$f(n) = \begin{cases} f_1(n)\chi_B(n) & \text{if } (n,B)=1 \\ \frac{g(n)}{\chi_A(n)} & \text{if } (n,A)=1, \end{cases}$$

which implies $f \in \mathcal{M}^*$, since $(A,B)=1$. Thus we have proved assertion (a) of Theorem 2.

b) Assume now $f(A+1) \neq g(A+1)$. Then from (3.6) and (3.7) we have $2 \nmid (A+1)B$ and

$$f(A+1) = -g(A+1), \quad F(n+B) = -F(n) \quad (n=1,2,\ldots).$$

These imply

$$F(n+BN) = (-1)^N F(n) \qquad (n=1,2,\ldots) \tag{3.13}$$

and

$$f(AN+1) = (-1)^N g(AN+1) \quad (N=1,2,\ldots). \tag{3.14}$$

Thus, from (3.13) we get $F(n+2B) = F(n)$ $(n=1,2,\ldots)$, and so

$$F(n) = \chi_{2B}(n) \tag{3.15}$$

for every positive integer n coprime to $2B$. We also get from (3.14)

$$g(n) = \chi_{2A}(n)f(n) \tag{3.16}$$

for every positive integer n coprime to $2A$. Since $(A,B)=1$ and $g \in \mathcal{M}^*$, from (3.4), (3.15) and (3.16) it follows that

$$F(n) = \frac{g(n)}{\chi_{2A}(n)f_1(n)} = \chi_{2B}(n),$$

which implies

$$F(nm) = F(n)F(m) \quad \text{if} \quad (n,m,2)=1.$$

In our case we have $2 \nmid B$ and so from (3.13) we get

$$F(B)F(m+1) = -F(B)F(m) \quad (m=1,2,\ldots)$$

since $(B, m+1, 2) = 1$. This shows that

$$F(n) = (-1)^{n-1} \qquad (n=1,2,\ldots). \tag{3.17}$$

By (3.4), (3.16) and (3.17) the proof of assertion (b) of Theorem 2 is finished.

Theorem 2 is proved.

4. Proofs of Theorems 3 and 4

Assume that $f \in \mathcal{M}$ satisfies

$$|f(n)| = 1 \qquad (n = 1, 2, \dots) \tag{4.1}$$

and

$$f(An + B) - Cf(n) \to 0 \quad \text{as} \quad n \to \infty \tag{4.2}$$

for some positive integers A, B and a non-zero complex number C. Then conditions (1.1) and (1.4) of Theorem 1 are satisfied, and so

$$f(n) = f_1(n)F(n) \tag{4.3}$$

where $f_1 \in \mathcal{M}^*$, $F \in \mathcal{M}$ satisfy (1.6)–(1.10). It is obvious that there are positive constants M_1, M_2 such that

$$M_1 < |F(n)| = \big|F[(n, B)]\big| < M_2 \tag{4.4}$$

for every n. From (1.6)–(1.8) we have

$$\begin{aligned} f(An + B) - Cf(n) &= f_1(An + B)F(An + B) - Cf_1(n)F(n) = \\ &= f_1(An + B)\frac{C}{f(A)}F(An) - Cf_1(n)F(n) = \\ &= \frac{CF(An)}{f(A)}\{f_1(An + B) - f_1(An)\}, \end{aligned}$$

which with (4.2) and (4.4) implies

$$f_1(An + B) - f_1(An) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.5}$$

By replacing n in (4.5) by Bm, we get

$$f_1(Am + 1) - f_1(Am) \to 0 \quad \text{as} \quad m \to \infty. \tag{4.6}$$

We shall prove that for each positive integer K

$$f_1(Am + K) - f_1(Am) \to 0 \quad \text{as} \quad m \to \infty. \tag{4.6}$$

It is obvious that (4.7) holds for $K = 1$. Assume that (4.7) holds for K. We have

$$\begin{aligned} f_1(Am + K + 1) - f_1(Am) = -\frac{1}{f_1(Am)}\{f_1(A^2m^2 + KAm + Am + K) - \\ - f_1(A^2m^2 + KAm + Am)\} + \frac{f_1(Am + 1)}{f_1(Am)}\{f_1(Am + K) - f_1(Am)\} + \\ + \{f_1(Am + 1) - f_1(Am)\}, \end{aligned}$$

which shows that (4.7) holds for $K+1$, because

$$\frac{1}{M_2} < |f_1(n)| = \frac{1}{|F(n)|} < \frac{1}{M_1} \quad (n = 1, 2, \ldots).$$

So we have proved that (4.7) holds for each positive integer K.

Finally, by applying (4.7) with $K = A$, we get

$$f_1(A)\{f_1(m+1) - f_1(m)\} \to 0 \quad \text{as} \quad m \to \infty,$$

which with the result of E. Wirsing implies that

$$f_1(n) = n^{i\tau} \qquad (n = 1, 2, \ldots) \tag{4.8}$$

holds for some real constant τ. By (4.3) and (4.8) the proof of Theorem 3 is finished.

We now prove Theorem 4. Assume that $f \in \mathcal{M}$ and $g \in \mathcal{M}^*$ satisfy

$$|f(n)| = 1, \quad g(n) \neq 0 \quad (n = 1, 2, \ldots) \tag{4.9}$$

and

$$g(An+B) - Cf(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.10}$$

Let $D = (A, B)$, $A' = A/D$, $B' = B/D$ and $C' = C/g(D)$. Since $g \in \mathcal{M}^*$, from (4.10) we have

$$g(A'n+B') - C'f(n) \quad \text{as} \quad n \to \infty. \tag{4.11}$$

Then the conditions (1.13) and (1.4) of Theorem 2 are satisfied, and so we can apply Theorem 2.

(i) If $f(A'+1) = g(A'+1)$, then by Theorem 2 we have $f \in \mathcal{M}^*$ and $g(n) = \chi_{A'}(n)f(n)$. Thus, using (4.11) we get

$$f(A'n+B') - \frac{C'}{\chi_{A'}(B')}f(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.12}$$

By using $f \in \mathcal{M}^*$ and Theorem 3 it follows that

$$f(n) = n^{i\tau} \qquad (n = 1, 2, \ldots) \tag{4.13}$$

for some real number τ. Thus, part (i) of Theorem 4 is proved.

(ii) If $f(A'+1) \neq g(A'+1)$, then from Theorem 2 we have

$$f(n) = (-1)^{n-1}f_1(n), \quad g(n) = \chi_{2A'}(n)f(n), \tag{4.14}$$

where $f_1 \in \mathcal{M}^*$. It also follows from Theorem 2 that $2 \nmid (A'+1)B'$. From (4.11) and (4.14) we have

$$\chi_{2A'}(2A'n+B')f(2A'n+B') - C'f(2n) \to 0 \quad \text{as} \quad n \to \infty$$

which implies

$$f_1(2A'n+B') - \left(-\frac{Cf_1(2)}{\chi_{2A'}(B')}\right) f_1(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.15}$$

Since $f_1 \in \mathcal{M}^*$, we can apply Theorem 3 to (4.15) and get

$$f_1(n) = n^{i\tau} \qquad (n = 1, 2, \ldots)$$

for some real constant τ. Thus we have proved that

$$f(n) = (-1)^{n-1} n^{i\tau} \qquad (n = 1, 2, \ldots)$$

and

$$g(n) = \chi_{2A'}(n) f(n) = \chi_{2A'}(n) n^{i\tau}$$

for every positive integer n coprime to $2A'$.

Theorem 4 is proved.

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A NOTE ON HERMITE–FEJÉR INTERPOLATION ON EQUIDISTANT NODES

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1. Introduction

Let $M = \{x_{k,n} : k = 1, 2, \dots, n;\ n = 1, 2, 3, \dots\}$ be a triangular matrix such that, for each n,

$$-1 \leqq x_{1,n} < x_{2,n} < x_{3,n} < \dots < x_{n,n} \leqq 1.$$

(We will write x_k for $x_{k,n}$ when there is no ambiguity.) For $f : [-1, 1] \to (-\infty, \infty)$ and any $n \in \{1, 2, 3, \dots\}$ there is a unique polynomial $H_{2n-1}(f, x)$ such that the degree of $H_{2n-1}(f, x)$ is $2n - 1$ (or less) and

$$H_{2n-1}(f, x_k) = f(x_k), \quad H'_{2n-1}(f, x_k) = 0 \quad (k = 1, 2, \dots, n).$$

The interpolation polynomial $H_{2n-1}(f, x)$ is known as the Hermite–Fejér interpolation (or HFI) polynomial of degree $2n - 1$. L. Fejér [3] showed that if M is chosen so that $\{x_{k,n} : k = 1, 2, \dots, n\}$ is the set of zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $-1 \leqq x \leqq 1$, then, for all $f \in C([-1, 1])$, we have

$$\lim_{n\to\infty} \|H_{2n-1}(f) - f\| = 0,$$

where $\|\cdot\|$ is the uniform norm on the space $C([-1, 1])$. Thus the Weierstrass approximation theorem can be proved using Hermite–Fejér interpolation polynomials.

In 1958, D. L. Berman [1] studied the HFI polynomials when the nodes are equidistant: that is

$$x_{k,n} = -1 + 2(k-1)/(n-1) \qquad (k = 1, 2, \dots, n).$$

Berman proved that if the nodes are equidistant and $g(x) \equiv x$ $(-1 \leqq x \leqq 1)$, then the sequence $\{H_{2n-1}(g, x) : n = 1, 2, 3, \dots\}$ *diverges* if $0 < |x| < 1$. If $|x| = 1$ then $H_{2n-1}(g, x) = g(x)$ since $x_{1,n} = -1$ and $x_{n,n} = 1$ for all n. Berman proved also that $\{H_{2n-1}(g, 0) : n = 1, 2, 3, \dots\}$ converges to $g(0)$. Thus the question of convergence of $\{H_{2n-1}(g, x) : n = 1, 2, 3, \dots\}$

is resolved for all $x \in [-1,1]$. Berman's result shows that HFI polynomials based on equidistant nodes provide a very poor approximating method even for very simple functions such as $g(x) = x$.

The purpose of this note is to point out that the convergence of $\{H_{2n-1}(g,0) : n = 1,2,3,\ldots\}$ is not an isolated phenomenon. We will prove the following result.

THEOREM. *If* $f : [-1,1] \to (-\infty,\infty)$ *is bounded on* $[-1,1]$ *and continuous at* 0, *then*

$$\lim_{n\to\infty} H_{2n-1}(f,0) = f(0).$$

We note that some related results were stated (but not proved) by Runck [4, p.1212]. For example, Runck's results imply that the sequence $\{H_{2n-1}(f,0) : n = 1,2,3,\ldots\}$ of linear functionals is uniformly bounded. However the theorem above does not appear to be an immediate consequence of Runck's results.

The argument presented in Section 2 below shows that if $f \in C\big([-1,1]\big)$ has modulus of continuity

$$\omega(f;\delta) = \sup\big\{\,|f(x) - f(y)| : \ x,y \in [-1,1],\ |x-y| \leqq \delta\big\},$$

then

$$\big|H_{2n-1}(f,0) - f(0)\big| = O(1)\sum_{t=1}^{n} t^{-2}\,\omega(f;t/n) = O(1)\,\omega(f;(\log n)/n).$$

Similar estimates for $\| H_{2n-1}(f) - f \|$ were found by Bojanic [2] and Vértesi [5] when the nodes of interpolation are zeros of the Chebyshev polynomials.

2. Proof of the theorem

In this section we prove the theorem using a lemma; the proof of the lemma is rather technical and so is presented in the final section of the paper. We will also employ the following definition. If $f : [-1,1] \to (-\infty,\infty)$ is bounded, and $0 < \delta \leqq 1$, define

$$\omega_0(f;\delta) = \sup\big\{\,|f(t) - f(0)| : \ |t| \leqq \delta\big\}.$$

Note that f is continuous at 0 if and only if $\lim\limits_{\delta\to 0}\omega_0(f;\delta) = 0$.

Now, it is well known that

$$H_{2n-1}(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x),$$

where

$$h_k(x) = \left[1 - \frac{w''(x_k)}{w'(x_k)}(x - x_k)\right] \left(l_k(x)\right)^2, \tag{2.1}$$

$$l_k(x) = \frac{w(x)}{(x - x_k)w'(x_k)}, \tag{2.2}$$

$$w(x) = \prod_{i=1}^{n}(x - x_i) \tag{2.3}$$

and

$$x_k = x_{k,n} = -1 + 2(k-1)/(n-1) \qquad (k = 1, 2, \ldots, n). \tag{2.4}$$

Also,

$$\sum_{k=1}^{n} h_k(x) \equiv 1 \qquad (-1 \leqq x \leqq 1). \tag{2.5}$$

We may assume that $n = 2m$. For if $n = 2m + 1$, then $x_{m+1,n} = 0$, and hence $H_{2n-1}(f, 0) = f(0)$ for all n. If $n = 2m$, by using (2.5) it is found that

$$\left| H_{2n-1}(f, 0) - f(0) \right| \leqq \sum_{k=1}^{2m} \left| f(x_k) - f(0) \right| \left| h_k(0) \right|. \tag{2.6}$$

For $m + 1 \leqq k \leqq 2m$, let $k = m + t$. The lemma of the next section implies that there is a constant K so that

$$|h_{m+t}(0)| \leqq K t^{-2}, \tag{2.7}$$

and thus, if $1 < m_0 < m$, we have

$$\sum_{k=m+1}^{2m} \left| f(x_k) - f(0) \right| \left| h_k(0) \right| \leqq K \sum_{t=1}^{m} \left| f(x_{m+t}) - f(0) \right| t^{-2} =$$

$$= K \left(\sum_{t=1}^{m_0} + \sum_{t=m_0+1}^{m} \left| f(x_{m+t}) - f(0) \right| t^{-2} \right) <$$

$$< K\left(\frac{\pi^2}{6}\omega_0\left(f;\frac{2m_0-1}{2m-1}\right)+\omega_0(f;1)\sum_{t=m_0+1}^{\infty}t^{-2}\right).$$

Since $\sum_{t=m_0+1}^{\infty} t^{-2}$ can be made arbitrarily small by choosing m_0 large enough, following which $\omega_0\left(f;\frac{2m_0-1}{2m-1}\right)$ can be made arbitrarily small by choosing m sufficiently large, it follows that

$$\lim_{m\to\infty}\left(\sum_{k=m+1}^{2m}\left|f(x_k)-f(0)\right|\left|h_k(0)\right|\right)=0.$$

Similarly,

$$\lim_{m\to\infty}\left(\sum_{k=1}^{m}\left|f(x_k)-f(0)\right|\left|h_k(0)\right|\right)=0,$$

and so the theorem follows from (2.6).

3. An important estimate

This section is devoted to proving the estimate in (2.7).

LEMMA. *There exists a constant K such that for $n=2m$,*

$$\left|h_{m+t}(0)\right| \leqq Kt^{-2} \qquad (t=1,2,\ldots,m;\ m=1,2,3,\ldots). \tag{3.1}$$

PROOF. After considerable calculation based on (2.1)–(2.4), we find that

$$h_{m+t}(0)=\left[1+(2t-1)\sum_{i=m-t+1}^{m+t-1}i^{-1}\right]\left(l_{m+t}(0)\right)^2, \tag{3.2}$$

where

$$|l_{m+t}(0)|=\frac{\left((2m)!\right)^2}{(m!)^2(2t-1)\,2^{4m-1}(m+t-1)!\,(m-t)!}. \tag{3.3}$$

Equation (3.3) can be written as

$$\log|l_{m+t}(0)| = 2\log\Gamma(2m+1)-2\log\Gamma(m+1)-\log\Gamma(m+t)-\log\Gamma(m-t+1)-$$
$$-(4m-1)\log 2-\log(2t-1).$$

On employing the estimate (see Whittaker and Watson [6, pp. 251–253])

$$0 < \log\Gamma(x) - \left(\left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi\right) < \frac{1}{12x} \qquad (x > 0),$$

for the gamma function, we obtain

$$\begin{aligned}\log|l_{m+t}(0)| &< (4m+1)\log(2m+1) - (2m+1)\log(m+1)-\\ &-(m+t-1/2)\log(m+t) - (m-t+1/2)\log(m-t+1)-\\ &-(4m-1)\log 2 - \log(2t-1) + 1 - \log 2\pi + \frac{1}{6(2m+1)} =\\ &= 2m\log m - (m+t-1/2)\log(m+t)-\\ &-(m-t+1/2)\log(m-t+1) - \log t + O(1)\end{aligned}$$

as $m \to \infty$, where $O(1)$ is independent of t. Thus, since

$$\sum_{i=m-t+1}^{m+t-1} i^{-1} \leqq \log\frac{m+t-1/2}{m-t+1/2},$$

we obtain from (3.2)

$$\log\left|h_{m+t}(0)\right| + 2\log t \leqq \log\left(1 + (2t-1)\log\frac{m+t-1/2}{m-t+1/2}\right) + 4m\log m-$$

$$-(2m+2t-1)\log(m+t)-(2m-2t+1)\log(m-t+1)+O(1) = \Psi(m,c)+O(1)$$

as $m \to \infty$, where $c = t - 1/2$, and

$$\text{(3.4)} \qquad \Psi(m,c) = \log\left(1 + 2c\log\frac{m+c}{m-c}\right) - 2\big((m+c)\log(m+c) - $$

$$- 2m\log m + (m-c)\log(m-c)\big).$$

Since $\lim_{m\to\infty} \Psi(m,c) = 0$ for fixed c, it follows that $\Psi(m,c)$ is bounded above for $c = 1/2, 3/2$ and $m \geqq c + 1/2$. Hence (3.1) will be established if we can show that $\Psi(m,c)$ is bounded above for $c = 5/2, 7/2, 9/2, \ldots$, and $m \geqq$ $\geqq c + 1/2$. Note that if $c + 1/2 \leqq m \leqq c^2$, then $2c\log\dfrac{m+c}{m-c} > 1$, and so

$$\log\left(1 + 2c\log\frac{m+c}{m-c}\right) \leqq 1 + \log\left(2c\log\frac{m+c}{m-c}\right).$$

Hence, for $c+1/2 \leqq m \leqq c^2$, we can say

$$\text{(3.5)}\qquad \Psi(m,c) \leqq 1+\log 2c+\log\log\frac{m+c}{m-c}-2\big((m+c)\log(m+c)-$$
$$-2m\log m+(m-c)\log(m-c)\big) := \Phi(m,c).$$

We consider three cases.

Case 1: $c+1/2 \leqq m \leqq \sqrt{3}c$. For $m \geqq c+1/2$, the functions $\log\log\frac{m+c}{m-c}$ and $-2\big((m+c)\log(m+c)-2m\log m+(m-c)\log(m-c)\big)$ are decreasing and increasing, respectively, and so from (3.5) we have

$$\text{(3.6)}\qquad \Psi(m,c) \leqq 1+\log 2c+\log\log(4c+1)-$$
$$-2c\left(\sqrt{3}\log\frac{2}{3}+\log\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) \to -\infty, \quad \text{as } c\to\infty.$$

Case 2: $\sqrt{3}c \leqq m \leqq c^2$. From (3.5) we obtain

$$\frac{\partial\Phi}{\partial m}=2\log\frac{m^2}{m^2-c^2}-\frac{2c}{(m^2-c^2)\log\frac{m+c}{m-c}},$$

which is positive if and only if

$$c^{-1}(m^2-c^2)\log\left(\left(1-c^2/m^2\right)^{-1}\right)\log\frac{1+c/m}{1-c/m} \geqq 1.$$

Now, $(m^2-c^2)/m^3$ is a decreasing function of m for $m \geqq \sqrt{3}c$, and so

$$c^{-1}(m^2-c^2)\log\left(\left(1-c^2/m^2\right)^{-1}\right)\log\frac{1+c/m}{1-c/m} >$$
$$> c^{-1}(m^2-c^2)\frac{c^2}{m^2}\frac{2c}{m}=2c^2\,\frac{m^2-c^2}{m^3} \geqq 2c^2\,\frac{c^4-c^2}{c^6} > 1.$$

Thus, if $\sqrt{3}c \leqq m \leqq c^2$,

$$\text{(3.7)}\qquad \Psi(m,c) \leqq \Phi(c^2,c)=$$
$$=1+\log 2c+\log\log\frac{c+1}{c-1}+2c^2\log\frac{c^2}{c^2-1}-2c\log\frac{c+1}{c-1}=$$
$$=O(1), \quad \text{as} \quad c\to\infty.$$

Case 3: $m>c^2$. For fixed c, $-2\big((m+c)\log(m+c)-2m\log m+(m-c)\log(m-c)\big)$ is an increasing function for $m \geqq c+1/2$ that approaches 0 as $m\to\infty$, and so it is always negative. Therefore, if $m \geqq c^2$, we have by (3.4),

$$\text{(3.8)}\qquad \Psi(m,c) < \log\left(1+2c\log\frac{c+1}{c-1}\right)=O(1), \quad \text{as } c\to\infty.$$

From (3.6)–(3.8) it follows that $\Psi(m,c)$ is bounded above for $c = 5/2, 7/2, 9/2, \ldots$ and $m \geqq c+1/2$, and so (3.1) is established.

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A DINI–DAX THEOREM

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The following theorem is fundamental in analysis:

DINI-THEOREM *(classical version). Let Y be a compact topological space and $f_n : Y \to \mathbf{R}$, $n \in \mathbf{N}$ a sequence of continuous functions such that $f_1(y) \leqq f_2(y) \leqq \dots$ for all $y \in Y$. If there exists a continuous function $f : Y \to \mathbf{R}$ such that $\lim_{n\to\infty} f_n(y) = f(y)$ for all $y \in Y$, then f_n is uniformly convergent to f.*

In the literature several generalizations of this theorem can be found. The version below is taken from [2]:

In the sequel, let X be a nonvoid set, Y a compact topological space, and $f : X \times Y \to \mathbf{R}$ a function such that all sets

$$Y_\alpha(x) := \{y \in Y : f(x,y) \leqq \alpha\}, \quad \alpha \in \mathbf{R}, \quad x \in X$$

are closed (i.e., the function f is lower-semicontinuous in the second variable).

DINI-THEOREM *(modern version). The "Dini-condition"*

$$\text{(Di)} \quad \forall x_1, x_2 \in X \;\; \exists x_0 \in X \;\; \forall y \in Y : \; f(x_0,y) \geqq \max\big(f(x_1,y), f(x_2,y)\big)$$

implies $\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y)$.

REMARK. Set $f(n,y) = f_n(y) - f(y)$ to obtain the classical version.

In 1977, when I attended a lecture of Heinz König on mathematical economics, I heard about a counterpart of the Dini-Theorem which König called

"DAX-THEOREM". *The "Dax-condition"*

$$\text{(Da)} \quad \forall y_1, y_2 \in Y \;\; \exists y_0 \in Y \;\; \forall x \in X : \; f(x,y_0) \leqq \min\big(f(x,y_1), f(x,y_2)\big)$$

implies $\inf_{y\in Y} \sup_{x\in X} f(x,y) = \sup_{x\in X} \inf_{y\in Y} f(x,y)$.

The following theorem generalizes the Dini–Dax Theorem (take $\tau = \pm\infty$). We write $\operatorname{med}(\alpha,\beta,\gamma)$ for the middle of the three numbers α, β, γ.

THEOREM. *Suppose that for some number* $\tau \in \mathbf{R} \cup \{\pm\infty\}$ *the conditions*

$$(\mathrm{D}^\tau) \forall x_1, x_2 \in X \quad \exists x_0 \in X \quad \forall y \in Y : f(x_0, y) \geqq \operatorname{med}\big(f(x_1, y), f(x_2, y), \tau\big),$$

$$(\mathrm{D}_\tau) \ \forall y_1, y_2 \in Y \quad \exists y_0 \in Y \quad \forall x \in X : f(x, y_0) \leqq \operatorname{med}\big(f(x, y_1), f(x, y_2), \tau\big)$$

are satisfied. Then $\inf_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y)$.

PROOF. Obviously, $f_* := \sup_{x \in X} \inf_{y \in Y} f(x,y) \leqq f^* := \inf_{y \in Y} \sup_{x \in X} f(x,y)$. Hence, we may assume $f_* < \infty$. Let $\alpha \in \mathbf{R}$ with $\alpha > f_*$. We show by induction that the assertions

$$(n) \qquad Y_\alpha(A) := \bigcap_{x \in A} Y_\alpha(x) \neq \emptyset \quad \text{for all} \quad A \subset X \quad \text{with} \quad |A| = n$$

hold true for every $n \in \mathbf{N}$:

From $\alpha > f_*$ we infer (1). Now let (n) be satisfied, and let $A \subset X$ with $|A| = n + 1$. Choose $x_1, x_2 \in A$ with $x_1 \neq x_2$ and set $E = A - \{x_1, x_2\}$. Then (n) implies $Y_\alpha(E \cup \{x\}) \neq \emptyset$ for every $x \in X$. In case $\tau \leqq f_*$ we choose $y_i \in Y_\alpha(E \cup \{x_i\})$, $i \in \{1, 2\}$, and $y_0 \in Y$ according to (D_τ). In case $\tau > f_*$ we choose x_0 according to (D^τ) and $y_0 \in Y_\alpha(E \cup \{x_0\})$. In both cases we have $y_0 \in Y_\alpha(A)$, so $(n + 1)$ is proved.

Now, as the closed sets $Y_\alpha(x)$, $\alpha > f_*$, $x \in X$ have the finite intersection property, there exists a $y^* \in \bigcap\{Y_\alpha(x) : x \in X, \alpha > f_*\}$ as Y is compact. This implies $f^* \leqq \sup_{x \in X} f(x, y^*) \leqq f_*$.

EXAMPLE. Let $X = Y = [0, 1]$ and $f(x, y) = 1 - y$, or x for $x \geqq y$, or $x < y$, respectively.

Here the conditions (D^τ) and (D_τ) hold for $\tau = \frac{1}{2}$ with $x_0 = \operatorname{med}\big(x_1, x_2, \frac{1}{2}\big)$ and $y_0 = \operatorname{med}\big(y_1, y_2, \frac{1}{2}\big)$, and Y is compact with respect to the coarsest topology such that all sets $Y_\alpha(x)$ are closed. ($Y_\alpha(x) \neq 0$ implies $Y_\alpha(x) = [1 - \alpha, x]$ for $1 - x \leqq \alpha < x$ and $Y_\alpha(x) \supset \big[\frac{1}{2}, 1\big]$ otherwise.) Hence, our Theorem applies.

Observe that for $x_1 = \frac{1}{2}$ and $x_2 = \frac{3}{4}$, say, there is no $x_0 \in X$ such that

$$f(x_0, y) \geqq \frac{1}{2} f(x_1, y) + \frac{1}{2} f(x_2, y) \quad \text{for all} \quad y \in Y.$$

So, despite some similarity with the minimax theorem of Ky Fan – König [1], [2], our Theorem is independent of it, and we also see that the Dini-condition (Di), and similarly (Da), is violated. Finally, Sion's minimax theorem [3] does not apply, since not all sets $Y_\alpha(x)$ are closed in the euclidean topology.

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A COUNTEREXAMPLE ON MONOTONE MÜNTZ APPROXIMATION

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§1. Introduction

Let $C^N_{[0,1]}$ be the class of all real continuous functions in $[0,1]$ which have N continuous derivatives, $C_{[0,1]} = C^0_{[0,1]}$, Δ^k be the class of all kth monotone functions on $[0,1]$, that is,

$$\Delta^k = \left\{ f \in C_{[0,1]} : \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh) \geqq 0,\, h > 0,\, x \in [0, 1-kh] \right\}.$$

For $f \in C_{[0,1]}$, let $\|f\| = \max_{x\in[0,1]} |f(x)|$.

From Müntz theorem (cf.[2]), it is well-known that the combinations of $\{x^{\lambda_n}\}$ for $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, are dense if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty. \tag{1}$$

On the other hand, monotone approximation has been studied by many scholars for a long time. Results showed that for any sequence of real numbers $\Lambda = \{\lambda_n\}$ with (1) and

$$0, 1, \ldots, k+1 \in \Lambda, \tag{2}$$

$$\lambda_n \to \infty,\ n \to \infty, \tag{3}$$

the kth monotone combinations of $\{x^{\lambda_n}\}$ are dense in $C_{[0,1]} \cap \Delta^k$. Readers can refer to the references [1], [3]–[9], [11], [13]–[16] for monotone approximation by ordinary polynomials and related quantitative estimates, and to [10] for monotone approximation by Müntz polynomials.

Given a sequence of real numbers $\Lambda = \{\lambda_n\}_{n=0}^{\infty}$ satisfying (1), (2)[1] , (3), write

$$\Pi_n(\Lambda) = \left\{ P(x) = \sum_{k=0}^{n} a_k x^{\lambda_k} \right\},$$

[1] For convenience, we assume that $\lambda_j = j$ for $j = 0, 1, \ldots, k+1$.

for $f \in C_{[0,1]}$,

$$E_n(f,\Lambda) = \inf_{P \in \Pi_n(\Lambda)} \|f - P\|,$$

for $f \in C_{[0,1]} \cap \Delta^k$,

$$E_n^{(k)}(f,\Lambda) = \inf_{P \in \Pi_n(\Lambda) \cap \Delta^k} \|f - P\|,$$

in particular for $\Lambda = N = \{n\}_{n=0}^{\infty}$,

$$E_n(f,N) = E_n(f), \qquad E_n^{(k)}(f,N) = E_n^{(k)}(f).$$

On the comparison between $E_n(f)$ and $E_n^{(k)}(f)$, G. G. Lorentz and K. L. Zeller [6] showed that for $k \geqq 1$, there exists a function $f \in C_{[0,1]} \cap \Delta^k$ such that

$$\limsup_{n\to\infty} \frac{E_n^{(k)}(f)}{E_n(f)} = +\infty.$$

Now we ask if the corresponding result that the monotone Müntz approximation is worse than the ordinary Müntz approximation in the order still holds true. The present paper will construct a counterexample to confirm this fact.

§2. Result and proof

THEOREM. *Let $k \geqq 1$, and let Λ satisfy* (1)–(3). *Then there exists a function $f \in C_{[0,1]} \cap \Delta^k$ such that*

$$\limsup_{n\to\infty} \frac{E_n^{(k)}(f,\Lambda)}{E_n(f,\Lambda)} = +\infty.$$

LEMMA 1. *Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to zero. Set*

$$P_n(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{k-2}} P_n^*(x_{k-1})\, dx_{k-1}$$

for $k \geqq 2$ or $P_n(x) = P_n^*(x)$ for $k = 1$, where

$$P_n^*(x) = x(x - \varepsilon_n), \qquad \hat{P}_n(x) = (-1)^k P_n(1 - x).$$

Then there exists a function $f_n(x)$ satisfying the following properties:

$$f_n(x) \in \Pi_{N_n^*}(\Lambda) \cap \Delta^k, \tag{4}$$

where N_n^* is a natural number depending on ε_n,

$$\|f_n\| = O(1), \tag{5}$$

$$f_n^{(k)}(1) = 0, \tag{6}$$

$$\|f_n - \hat{P}_n\| \sim \varepsilon_n^\lambda, \tag{7}$$

where

$$\lambda = \lambda(k) = \begin{cases} 2, & k = 1, \\ 3, & k \geqq 2, \end{cases}$$

and by $A_n \sim B_n$, we mean that there exists a positive constant M independent of n such that $M^{-1} \leqq A_n/B_n \leqq M$.

PROOF. Set

$$g_n(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{k-2}} g_n^*(x_{k-1})\, dx_{k-1}$$

for $k \geqq 2$ or $g_n(x) = g_n^*(x)$ for $k = 1$, where

$$g_n^*(x) = \begin{cases} e_n(x) - h_n(x), & x \in [0, 2\varepsilon_n), \\ P_n^*(x), & x \in [2\varepsilon_n, 1], \end{cases}$$

$$e_n(x) = \varepsilon_n x \exp\left(\frac{x - 2\varepsilon_n}{\varepsilon_n}\right), \quad h_n(x) = 4^{-1}e^{-2}\varepsilon_n^{-1}x(x - 2\varepsilon_n)^2.$$

It is clear that for $k = 1$, $\|g_n - P_n\| \sim \varepsilon_n^2$. For $k \geqq 2$, we also can easily deduce that[2]

$$\|g_n - P_n\| \leqq M_1\varepsilon_n^3.$$

On the other hand let $\delta_i = \frac{2k-i}{2k}$, $i = 1, 2, \ldots, k$,

$$\|g_n - P_n\| \geqq \int_{\delta_1}^1 dx_1 \int_{\delta_2}^{\delta_1} dx_2 \cdots \int_{\delta_{k-2}}^{\delta_{k-1}} dx_{k-2} \int_{\varepsilon_n/2}^{\varepsilon_n} M_2\varepsilon_n^2 dx_{k-1}.$$

In fact, we need to show $g_n^*(x) - P_n^*(x) \geqq 0$, $0 \leqq x \leqq 2\varepsilon_n$, whence the desired inequality above follows. Put $y = \frac{2\varepsilon_n - x}{\varepsilon_n}$, then

$$g_n^*(x) - P_n^*(x) = \varepsilon_n x\left(e^{-y} - \frac{1}{4}e^{-2}y^2 + y + 1\right) := \varepsilon_n x\varphi(y).$$

[2] In the whole paper, we denote by $M_j(x)$, $j = 1, 2, \ldots$, positive constants depending only upon x.

We need to prove that $\varphi(y) \geqq 0$ for $0 \leqq y \leqq 2$. Since $\varphi(0) = 0$, it suffices to prove that $\varphi'(y) \geqq 0$, $0 \leqq y \leqq 2$. Now $\varphi''(y) = e^{-y} - \frac{1}{2}e^{-2} > 0$ for $0 \leqq \leqq y \leqq 2$. Therefore $\varphi'(y)$ is increasing and $\varphi'(0) = 0$, thus our proof of the above inequality is complete. Now we get

$$\|g_n - P_n\| \sim \varepsilon_n^\lambda, \tag{8*}$$

Also

$$g_n^{(k)}(0) = 0, \tag{9*}$$

$$g_n^{(k)}(x) \in \Delta^1, \tag{10*}$$

and

$$g_n(x) \in C^k_{[0,1]}. \tag{11*}$$

Let $\hat{g}_n(x) = (-1)^k g_n(1-x)$. Then from (8*)–(11*),

$$\|\hat{g}_n - \hat{P}_n\| \sim \varepsilon_n^\lambda, \tag{8}$$

$$\hat{g}_n^{(k)}(1) = 0, \tag{9}$$

$$-\hat{g}_n^{(k)}(x) \in \Delta^1, \tag{10}$$

and

$$\hat{g}_n(x) \in C^k_{[0,1]}. \tag{11}$$

Since $\lambda_n \to \infty$, $n \to \infty$, and $\sum_{n=1}^\infty 1/\lambda_n = \infty$, we may choose an N such that for all $n \geqq N$, $\lambda_n > k$, and

$$\sum_{n=N+1}^\infty \frac{1}{\lambda_n - k} = \infty.$$

Let $\Lambda' = \{0, \lambda_n - k\}_{n=N+1}^\infty$. Because of (10) and (11), applying the result on monotone approximation from [10], we can find a generalized polynomial

$$-\tilde{q}(x) = -a_0 - \sum_{j=N+1}^{N_n^*} a_j x^{\lambda_j - k} \in \Delta^1 \tag{12}$$

such that

$$\|\hat{g}_n^{(k)}(x) - \tilde{q}(x)\| < \frac{\varepsilon_n^4}{2}, \tag{13}$$

where N_n^* is a natural number depending on ε_n. Write $q(x) = \tilde{q}(x) - \tilde{q}(1)$. Now (13), (9) together with the fact $-\tilde{q} \in \Delta^1$ imply

$$-q(x) \in \Delta^1 \cap \Pi_{N_n^*-k}(\Lambda'), \tag{14}$$

$$q(1) = 0, \tag{15}$$

and

$$\|\hat{g}_n^{(k)}(x) - q(x)\| < \varepsilon_n^4. \tag{16}$$

Let

$$f_n(x) = (-1)^k \int_0^{1-x} dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{k-1}} q(1-x_k)\, dx_k,$$

then

$$\|f_n - \hat{g}_n\| \leqq \|q - \hat{g}_n^{(k)}\| < \varepsilon_n^4$$

due to $g_n^*(0) = 0$. From (14)–(16), we now have (4)–(6). Meanwhile by (8),

$$\|f_n - \hat{P}_n\| \leqq \|f_n - \hat{g}_n\| + \|\hat{g}_n - \hat{P}_n\| \leqq \varepsilon_n^4 + O(\varepsilon_n^\lambda) = O(\varepsilon_n^\lambda),$$

for sufficiently large n,

$$\|f_n - \hat{P}_n\| \geqq M_3\varepsilon_n^\lambda - \varepsilon_n^4 \geqq M_4\varepsilon_n^\lambda,$$

that is (7). Lemma 1 is proved. □

LEMMA 2 (D. J. Newman [12]). *Let Λ be a finite set of positive numbers,*

$$p(t) = a_0 + \sum_{\lambda \in \Lambda} a_\lambda e^{-\lambda t}.$$

Then

$$\frac{2}{3} \sum_{\lambda \in \Lambda} \lambda \leqq \sup_p \frac{\|p'(t)\|}{\|p(t)\|} \leqq 11 \sum_{\lambda \in \Lambda} \lambda.$$

Let

$$\rho_n = \sum_{j=0}^{n} \lambda_j, \quad \lambda_j \in \Lambda, \qquad \tilde{\rho}_n = \max\{\rho_n^k, n^k\}.$$

LEMMA 3. *Set $\varepsilon_n = \tilde{\rho}_n^{-1} n^{-2}$. Take $\{n_m\}$ to be a subsequence of natural numbers satisfying $n_1 = 1$, $n_{m+1} > \max\{n_m, N^*_{n_m}\}$, where N^*_n is the number appearing in* (12). *Then for any $r(x) \in \Pi_{n_m}(\Lambda) \cap \Delta^k$ the following inequality holds:*

$$\left\| \sum_{j=1}^{m} n_j^{-1} f_{n_j}(x) - r(x) \right\| \geqq M_5(k) n_m \varepsilon_{n_m}^2.$$

PROOF. Write

$$F_m(x) = \sum_{j=1}^{m} n_j^{-1} f_{n_j}(x), \quad S_m(x) = \sum_{j=1}^{m-1} n_j^{-1} f_{n_j}(x) + n_m^{-1} \hat{P}_{n_m}(x),$$

then by (6), (7) and the facts $S_m(x) \in \Pi_{n_m}(\Lambda)$, $\hat{P}_n^{(k)}(1) = P_n^{(k)}(0) = -\varepsilon_n$, for any $r(x) \in \Pi_{n_m}(\Lambda) \cap \Delta^k$ we have

$$M_6 \varepsilon_{n_m}^2 n_m^{-1} \leqq M_7 \varepsilon_{n_m}^{2-\lambda} \|F_m(x) - S_m(x)\| \leqq n_m^{-1} \varepsilon_{n_m} \left| P_{n_m}^{(k)}(0) \right| =$$

$$= \varepsilon_{n_m} \left| S_m^{(k)}(1) \right| \leqq \varepsilon_{n_m} \left| S_m^{(k)}(1) - r^{(k)}(1) \right|.$$

Applying Lemma 2, we get

$$\left| S_m^{(k)}(1) - r^{(k)}(1) \right| \leqq 11^k \tilde{\rho}_{n_m} \|S_m - r\| \leqq 11^k \tilde{\rho}_{n_m} \left(\|S_m - F_m\| + \|F_m - r\| \right),$$

thus for sufficiently large m,

$$\|F_m(x) - r(x)\| \geqq M_8(k) \varepsilon_{n_m}^{1-\lambda} \tilde{\rho}_{n_m}^{-1} \|S_m - F_m\| \geqq M_9(k) n_m \varepsilon_{n_m}^2. \quad \square$$

PROOF OF THE THEOREM. Set $n_1 = 1$,

$$n_{m+1} = 2[n_m + N^*_{n_m} + \varepsilon_{n_m}^{-2} + 1]. \tag{17}$$

Define

$$f(x) = \sum_{m=1}^{\infty} n_m^{-1} f_{n_m}(x).$$

Let $n_m \geqq k+1$. In view of Lemma 1, $f(x) \in C_{[0,1]} \cap \Delta^k$. From (5), (17) and Lemma 3,

$$E_{n_m}^{(k)}(f, \Lambda) = \min_{r \in \Pi_{n_m}(\Lambda) \cap \Delta^k} \left\| \sum_{j=m+1}^{\infty} n_j^{-1} f_{n_j}(x) + \sum_{j=1}^{m} n_j^{-1} f_{n_j}(x) - r(x) \right\| \geqq$$

$$\geqq \min_{r \in \Pi_{n_m}(\Lambda) \cap \Delta^k} \|F_m - r\| - O\left(\sum_{j=m+1}^{\infty} n_j^{-1} \right) \geqq M_{10}(k) n_m \varepsilon_{n_m}^2 - O(\varepsilon_{n_m}^2).$$

On the other hand, by (4), (5), (7) and (17),

$$E_{n_m}(f,\Lambda) \leqq E_{n_m}(f_{n_m},\Lambda) + \sum_{j=m+1}^{\infty} n_j^{-1} + E_{n_m}\left(\sum_{j=1}^{m-1} f_{n_j},\Lambda\right) \leqq$$

$$\leqq \left\| f_{n_m}(x) - \hat{P}_{n_m}(x)\right\| + O(\varepsilon_{n_m}^2) + 0 = O(\varepsilon_{n_m}^2)$$

since

$$\sum_{j=1}^{m-1} f_{n_j}(x) \in \Pi_{n_m}(\Lambda), \quad \hat{P}_{n_m}(x) \in \Pi_{n_m}(\Lambda).$$

Combining these estimates we get

$$\limsup_{m\to\infty} \frac{E_{n_m}^{(k)}(f,\Lambda)}{n_m E_{n_m}(f,\Lambda)} > 0,$$

thus the Theorem is proved. □

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CERTAINES MÉTHODES DE SOMMATION DE SÉRIES DE FOURIER DONNANT LE MEILLEUR ORDRE D'APPROXIMATION

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0. Introduction

Soit

$$L_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)U_n(t-x)\,dt, \quad f \in C_{2\pi}, \quad n \in \mathbf{N}, \tag{0.1}$$

une suite d'opérateurs de convolution dans l'espace $C_{2\pi}$; $U_n(t)$ est une fonction continue 2π-périodique et paire ayant un nombre pair de changements de signe sur l'intervalle $]-\pi,\pi[$.

DÉFINITIONS [1]. On dit que l'opérateur (0.1), où n est fixé, appartient à la classe S_{2m} ($m \in \mathbf{N}$) si son noyau $U_n(t)$ change de signe sur $]-\pi,\pi[$ au plus $2m$ fois.

On dit que les opérateurs (0.1) sont polynomiaux si pour tout n les fonctions $U_n(t)$ et $L_n(f;x)$ sont des polynômes trigonométriques d'ordre $\leqq n$; on suppose aussi que $L_n(1;x) \equiv 1$, c'est-à-dire,

$$U_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_{k,n}\cos kt. \tag{0.2}$$

P. P. Korovkin a montré [2] que l'ordre d'approximation des fonctions continues 2π-périodiques par une suite d'opérateurs linéaires polynomiaux de classe S_{2m} ne peut pas être meilleur que $1/n^{2m+2}$ déjà sur le système de fonctions $1, \cos x, \sin x, \ldots, \cos(m+1)x, \sin(m+1)x$. Il est intéressant de construire une suite d'opérateurs du type indiqué donnant un ordre exact d'approximation égal à $1/n^{2m+2}$ pour toute fonction ayant sa dérivée d'ordre $2m+2$ bornée.

A. I. Kovalenko a montré [3] que les opérateurs (0.1)–(0.2) avec les noyaux

$$U_{n+m}(t) = C_{n+m}\prod_{s=1}^{m}(\cos t - \cos\alpha_s^{(n)})L_n(t), \tag{0.3}$$

où

$$L_n(t) = \left| \sum_{k=0}^{n} \varphi\left(\frac{k}{n}\right) e^{ikt} \right|^2, \qquad C_{n+m} = \text{const},$$

peuvent donner le meilleur ordre d'approximation pour les fonctions de classe $C^{(2m+2)}$ si la fonction $\varphi(t)$ vérifie les conditions suivantes: a) $\varphi(t) \equiv$ $\equiv 0$ pour $t \in \mathbf{R} \setminus]0,1[$, b) $\varphi^{(2m)}(t) \in C]-\infty,+\infty[$, c) $\varphi^{(2m+2)}(t) \in C]0,1[$, d) $\left|\varphi^{(k)}(t)\right| \leqq M$, $k = 0,1,\dots,2m+2$, pour $t \in]0.1[$. Les symboles $\alpha_s^{(n)}$, $s = 1,\dots,m$, $n \in \mathbf{N}$, sont des nombres réels qui doivent être trouvés. En particulier, on peut poser $\varphi(t) = \sin^{2m+1} \pi t$ pour $t \in [0,1]$; le choix d'une autre fonction satisfaisante à a)–d) aboutit aux difficultés presque inabordables.

J. Szabados [4] a étendu ces résultats sur une famille d'opérateurs qui est plus vaste que celle donnée par les conditions a)–d). Formulons ses assertions dans une forme un peu simplifiée.

Soit $\{K_n(t)\}_{n\in\mathbf{N}}$ une suite de fonctions continues paires non négatives et 2π-périodiques pour lesquelles il existe une fonction non négative $\varphi_m(x)$ définie sur $[0,+\infty[$ telle que:

$$\max_{0\leqq x\leqq n\pi} \left[(1+x^{4m+2})|K_n(x/n) - \varphi_m(x)|\right] = o(1/n) \qquad (n\to\infty) \tag{0.4}$$

et

$$0 < \int_0^{+\infty} x^{4m+2} \varphi_m(x)\,dx < \infty. \tag{0.5}$$

Posons

$$\mu_{k,n} = \frac{n^{k+1}}{\pi} \int_{-\pi}^{\pi} \left|\sin^k \frac{t}{2}\right| \cdot K_n(t)\,dt \qquad (k,n\in\mathbf{N}). \tag{0.6}$$

On peut montrer que pour n suffisamment grand le système d'équations linéaires

$$\sum_{k=0}^{m} \mu_{2j+2k,n} \lambda_k^{(n)} = \delta_{j0} \qquad (j=0,1,\dots,m) \tag{0.7}$$

(δ_{j0} est le symbole de Kronecker) possède une solution unique: $\{\lambda_k^{(n)}\}_{k=0}^{m}$ pour laquelle $\lim_{n\to\infty} \lambda_k^{(n)} = \lambda_k$ ($k = 0,1,\dots,m$). De plus, pour tout n naturel la fonction

$$\psi_{m,n}(t) = \sum_{k=0}^{m} \lambda_k^{(n)} n^{2k+1} \sin^{2k} \frac{t}{2} \tag{0.8}$$

possède $2m$ racines simples sur $]-\pi,\pi]$: $\pm q_{jn}$, $j = 1,\dots,m$, pour lesquelles $\lim_{n\to\infty} nq_{jn} = c_j \neq 0$.

THÉORÈME DE SATURATION (J. Szabados [4]). *Des opérateurs linéaires du type* (0.1) *et de classe* S_{2m} *avec les noyaux:*

$$\overline{U}_n(t) = K_n(t)\psi_{m,n}(t) \qquad (n \in \mathbf{N}) \tag{0.9}$$

sont identiques sur l'ensemble des polynômes trigonométriques d'ordre $\leqq m$ *et vérifient les conditions suivantes:*

i) $\lim_{n\to\infty} n^{2m+2}\|f(x) - L_n(f;x)\|_{C_{2\pi}} = 0$ *si et seulement si* f *est un polynôme trigonométrique d'ordre* $\leqq m$;

ii) $\|f(x) - L_n(f;x)\|_{C_{2\pi}} = O(1/n^{2m+2})$ *si et seulement si* $f^{(2m+1)} \in$ $\in \operatorname{Lip} 1$.*

De plus,

iii) *si* $f^{(2m+2)}(x) \in C_{2\pi}$, *alors*

$$\lim_{n\to\infty} n^{2m+2}\big[L_n(f;x) - f(x)\big] = \tag{0.10}$$
$$= \frac{2^{2m+1}}{m+1}\left(\sum_{k=0}^{m} \mu_{2m+2k+2}\lambda_k\right)\left(\sum_{k=1}^{m+1} a_{k,m} f^{(2k)}(x)\right),$$

où

$$a_{k,m} = \frac{(-1)^{k+m+1}}{(2k)!}\left[x\binom{x+m}{2m+1}\right]^{(2k)}_{x=0} \qquad (k = 1,\dots,m+1), \tag{0.11}$$

$$\mu_{2k} = \frac{2^{1-2k}}{\pi}\int_0^\infty x^{2k}\varphi_m(x)\,dx \qquad (k = 1,\dots,m+1), \tag{0.12}$$

et les coefficients λ_k *sont la solution du système d'équations linéaires*

$$\sum_{k=0}^{m} \mu_{2j+2k}\lambda_k = \delta_{j0} \quad (j = 0,1,\dots,m); \qquad \lambda_k = \lim_{n\to\infty}\lambda_k^{(n)}. \tag{0.13}$$

Si les opérateurs considérés sont polynomiaux (voir (0.2)), alors $\lambda_{k,n} =$ $= L_n(\cos kt; 0)$ $(k \leqq n)$, d'où, d'après ii),

$$\lambda_{k,n} = 1 + O(1/n^{2m+2}), \quad n \to \infty \qquad (k = 1,2,\dots). \tag{0.14}$$

* Remarquons que $f^{(2m+1)} \in \operatorname{Lip}_M 1 \Leftrightarrow f^{(2m+1)}$ est absolument continue et

$$|f^{(2m+2)}(x)| \leqq M$$

presque partout ([5], p.155).

Une construction des opérateurs en question d'après les formules (0.8) et (0.9) aboutit à un résultat évalué avec peine. Il est plus raisonnable d'utiliser l'expression (0.3), où la fonction $L_n(t)$ est connue, et trouver les valeurs $\alpha_s^{(n)}$ en se basant sur l'égalité (0.14). D'ailleurs, ce procédé sera efficace si $\alpha_s^{(n)} = \gamma_s/n$, où $\gamma_s = \text{const}$. Nous verrons plus tard que ce cas peut avoir lieu. Le seul exemple concret des opérateurs extrémaux du type indiqué a été donné dans l'article de P. L. Butzer et E. L. Stark [6], où on a considéré les opérateurs (0.3) de classe S_2 avec la fonction $\varphi(t) = \sin^3 \pi t$. Quand-même, leurs calculs sont longs et complexes.

V. A. Baskakov [7] a proposé de construire les opérateurs de classe S_{2m} à l'aide du théorème des résidus. Comme généralisation de cette idée, l'auteur propose d'utiliser le principe de la décomposition de la fonction méromorphe en fractions rationnelles simples ce qui permet d'obtenir asser facilement les opérateurs en question.

Dans cet article on considère, en particulier, des suites nouvelles d'opérateurs extrémaux de classe S_2 et une suite d'opérateurs extrémaux de classe S_4 du type (0.3) avec la fonction $\varphi(t) = \sin^5 \pi t$.

1. Un schéma général de construction d'opérateurs de classe S_{2m} donnant le meilleur ordre d'approximation

Pour un calcul efficace des valeurs d'opérateurs polynomiaux du type (0.1), nous aurons besoin de l'assertion suivante. Elle peut être considérée comme un cas particulier du théorème de G. Mittag-Leffler sur la décomposition d'une fonction méromorphe en fractions simples ([8], p.249).

PROPOSITION 1.1. *Soit*

$$g(z) = f(z)\Big/\prod_{i=1}^{q}(\cos z - \cos t_i)^{l_i}, \quad z \in \mathbf{C},\ 0 \leqq t_1 < \ldots < t_q \leqq \pi,\ l_i \in \mathbf{N},$$

où $f(z)$ est une fonction entière paire 2π-périodique telle que $g(z) = o(1)$ quand $|\operatorname{Im} z| \to \infty$. Posons $\Delta = \{x = \pm t_i + 2k\pi : k \in \mathbf{Z}, i = 1, \ldots, q\}$. Alors,

$$\text{(1.1)}\quad g(z) = \sum_{k\in\mathbf{Z}}\sum_{i=1}^{q}\sum_{s=1}^{l_i}\left[\frac{a_s^{(i)}}{(z - t_i - 2k\pi)^s} + \frac{(-1)^s a_s^{(i)}}{(z + t_i - 2k\pi)^s}\right], \quad z \in \mathbf{C}\setminus\Delta,$$

où

(1.2)

$$G_{ki}^{+}(z) = \sum_{s=1}^{l_i}\frac{a_s^{(i)}}{(z - t_i - 2k\pi)^s} \quad et \quad G_{ki}^{-}(z) = \sum_{s=1}^{l_i}\frac{(-1)^s a_s^{(i)}}{(z + t_i - 2k\pi)^s}, \quad k \in \mathbf{Z},$$

sont les parties singulières du développement de $g(z)$ en série de Laurent respectivement aux voisinages des points $\pm t_i + 2k\pi$, $k \in \mathbf{Z}$, $i = 1, \ldots, q$.

DÉMONSTRATION. Pour i fixé posons: $\gamma_{\rho,k} : z = t_i + 2k\pi + \rho e^{it}$, $0 \leqq t \leqq 2\pi$; $\gamma'_{\rho,k} : z = -t_i + 2k\pi + \rho e^{it}$, $0 \leqq t \leqq 2\pi$; $\gamma_\rho = \gamma_{\rho,0}$, $\gamma'_\rho = \gamma'_{\rho,0}$ $(\rho > 0)$. La fonction $f(z)$ étant paire et périodique, on a:

$$a_{sk}^{(i)} = (1/2\pi i) \int_{\gamma_{\rho,k}} (z - t_i - 2k\pi)^{s-1} g(z)\, dz =$$
$$= (1/2\pi i) \int_{\gamma_\rho} (z - t_i)^{s-1} g(z)\, dz =$$
$$= a_s^{(i)} = (-1)^s (1/2\pi i) \int_{\gamma'_{\rho,k}} (z + t_i - 2k\pi)^{s-1} g(z)\, dz.$$

Donc, (1.2) sont les parties singulières du développement de $g(z)$ aux voisinages des points $\pm t_i + 2k\pi$.

Il est facile de voir que la série (1.1) converge sur $\mathbf{C} \setminus \Delta$ quelles que soient les valeurs $a_s^{(i)}$. De plus, pour tout compact $K \subset \mathbf{C}$ il existe un reste de la série (1.1) qui converge uniformément dans K (puisqu'il existe une série numérique majorée). Donc, la fonction $\widetilde{g}(z) = \sum_{k \in \mathbf{Z}} \sum_{i=1}^{q} \left[G_{ki}^{+}(z) + G_{ki}^{-}(z) \right]$ est méromorphe.

Les fonctions $g(z)$ et $\widetilde{g}(z)$ ayant les mêmes singularités, la différence $h(z) = g(z) - \widetilde{g}(z)$ est une fonction entière. De plus, pour $0 \leqq \operatorname{Re} z < 2\pi$ et $s = 1, \ldots, l_i$ on a

$$\sum_{k \in \mathbf{Z}} \left| (z - t_i - 2k\pi)^{-s} + (-1)^s (z + t_i - 2k\pi)^{-s} \right| \leqq$$
$$\leqq \sum_{|k| \leqq m} + 4 \sum_{k=m+1}^{+\infty} \left| 4(k-3)^2 \pi^2 \right|^{-1}.$$

Alors, les fonctions $g(z)$ et $\widetilde{g}(z)$ sont périodiques et elles tendent vers zéro quand $z \to \infty$ en restant dans la bande $0 \leqq \operatorname{Re} z < 2\pi$. D'après le théorème de Liouville, $h(z) \equiv 0$. Donc, $g(z) \equiv \widetilde{g}(z)$. La proposition 1.1 est démontrée.

REMARQUE. Les coefficients $a_s^{(i)}$ peuvent être trouvés à l'aide de la formule:

$$g(z) = \mu_i(z) \Big/ \left[2 \sin \frac{z - t_i}{2} \right]^{l_i} = \tag{1.3}$$
$$= \left[1 + \frac{l_i}{24}(z - t_i)^2 + \frac{l_i(5l_i + 2)}{5760}(z - t_i)^4 + \ldots \right] \sum_{k=0}^{\infty} \mu_i^{(k)}(t_i)(z - t_i)^{k - l_i} / k!.$$

Soit

$$\cos^{2l}\frac{nt}{2}=\sum_{\nu=0}^{l}c_\nu^{(l)}\cos\nu nt,\qquad l\in\mathbf{N},\tag{1.4}$$

où $c_0^{(l)}=2^{-2l}\binom{2l}{l}$ et $c_\nu^{(l)}=2^{-2l+1}\binom{2l}{l-\nu}$, $\nu=1,\dots,l$; donc,

$$\cos kt\cdot\cos^{2l}\frac{nt}{2}=c_0^{(l)}\cos kt+2^{-1}\sum_{\nu=1}^{l}c_\nu^{(l)}\big[\cos(\nu n-k)t+\cos(\nu n+k)t\big].$$

Posons

$$F_k(z)=c_0^{(l)}e^{ikz}+2^{-1}\sum_{\nu=1}^{l}c_\nu^{(l)}\left(e^{i(\nu n-k)z}+e^{i(\nu n+k)z}\right)\qquad(k,l\in\mathbf{N}).\tag{1.5}$$

PROPOSITION 1.2. *Soit*

$$g_n(z)=W_r(z)\Big/\prod_{i=1}^{q}\big(\cos z-\cos[(2k_i-1)\pi/n]\big)^{l_i}\quad(n,k_i,l_i\in\mathbf{N}\setminus\{0\}),\tag{1.6}$$

où $W_r(z)$ est un polynôme trigonométrique pair d'ordre r à coefficients réels, $r<l_1+\ldots+l_q$, $1\leqq k_1<k_2<\ldots<k_q\leqq(n+1)/2$. Posons

$$B_N=\int_{-\pi}^{\pi}\cos^{2l}\frac{nt}{2}\cdot g_n(t)\,dt,\tag{1.7}$$

$$U_N(t)=\pi B_N^{-1}\cos^{2l}\frac{nt}{2}\cdot g_n(t)=\frac{1}{2}+\sum_{k=1}^{N}\lambda_{k,N}\cos kt$$

$$\big(l\in\mathbf{N},\quad 2l\geqq\max_i l_i,\quad N=nl+r-(l_1+\ldots+l_q)\big).$$

Ensuite, soit

$$\Theta(z)=\sum_{i=1}^{q}\sum_{s=1}^{l_i}\left\{\frac{a_s^{(i)}}{[z-(2k_i-1)\pi/n]^s}+\frac{(-1)^s a_s^{(i)}}{[z+(2k_i-1)\pi/n]^s}\right\},\tag{1.8}$$

où $a_s^{(i)}$ sont les coefficients du développement de $g_n(z)$ d'après la formule (1.1), et soient $F_k(z)$, $k\in\mathbf{N}$, les fonctions entières définies par les formules (1.4) et (1.5).

Alors

$$B_N=2\pi i\sum_{i=1}^{q}\operatorname{Res}\big((2k_i-1)\pi/n;F_0\cdot\Theta\big)\tag{1.9}$$

et pour $k \leqq n$

(1.10)
$$\lambda_{k,N} = \left[\sum_{i=1}^{q} \operatorname{Res}\big((2k_i - 1)\pi/n; F_k \cdot \Theta\big)\right] \Big/ \sum_{i=1}^{q} \operatorname{Res}\big((2k_i - 1)\pi/n; F_0 \cdot \Theta\big),$$

où $\operatorname{Res}\big((2k_i - 1)\pi/n; F_k \cdot \Theta\big)$ *est le résidu de la fonction* $F_k \cdot \Theta$ *au point* $(2k_i - 1)\pi/n$.

DÉMONSTRATION. Tous les coefficients $a_s^{(i)}$ dans l'expression (1.8) sont réels (voir (1.3)). Ensuite, puisque $\operatorname{Re} F_k^{(j)}(t) = \left[\cos kt \cdot \cos^{2l} \frac{nt}{2}\right]_t^{(j)}$ pour $t \in \mathbf{R}$, $j \in \mathbf{N}$, alors

$$\operatorname{Re} F_k^{(j)}\big(\pm(2k_i - 1)\pi/n\big) = 0 \quad \text{pour} \quad j = 0, 1, \ldots, 2l - 1.^*$$

Donc, tous les coefficients des parties singulières des développements des fonctions $F_k(z) \cdot \Theta(z)$ ($k \in \mathbf{N}$) aux voisinages des points $\pm(2k_i - 1)\pi/n$ sont imaginaires purs. De plus, les fonctions $\operatorname{Im} F_k(t)$ étant impaires, on aura:

$$F_k^{(j)}\big((2k_i - 1)\pi/n\big) = (-1)^{j+1} F_k^{(j)}\big(-(2k_i - 1)\pi/n\big)$$

pour $j = 0, 1, \ldots, 2l - 1$. Donc, d'après (1.8),

(1.11)
$$\operatorname{Res}\big((2k_i - 1)\pi/n; F_k \cdot \Theta\big) = \operatorname{Res}\big(-(2k_i - 1)\pi/n; F_k \cdot \Theta\big) \quad (i = 1, \ldots, q).$$

Ensuite, considérons un contour C sur le plan $\mathbf{C}$ qui est la réunion des bords des demi-cercles $\gamma_R = \{z = Re^{it}, 0 \leqq t \leqq \pi\}$ de rayon suffisamment grand $R > 0$ et $\gamma_{\rho,i}^{+} = \{z = (2k_i - 1)\pi/n + \rho e^{it}, 0 \leqq t \leqq \pi\}$, $\gamma_{\rho,i}^{-} = \{z = -(2k_i - 1)\pi/n + \rho e^{it}, 0 \leqq t \leqq \pi\}$, $i = 1, \ldots, q$, de rayon suffisamment petit $\rho > 0$ et des segments sur l'axe réel liant les extrémités de ces demi-cercles. Puisque pour tout naturel $k > 1$

$$\int_{\gamma_{\rho,i}^{\pm}} \big(z \mp (2k_i - 1)\pi/n\big)^{-k} dz = \rho^{1-k}(1 - k)^{-1}\big[(-1)^{k-1} - 1\big] \in \mathbf{R},$$

et pour $k = 1$ cette intégrale est égale à πi, alors, d'après (1.11),

$$\lim_{\rho \to +0} \operatorname{Re} \int_{\gamma_{\rho,i}^{\pm}} F_k(z)\Theta(z)\, dz = \pi i \operatorname{Res}\big((2k_i - 1)\pi/n; F_k \cdot \Theta\big).$$

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* De plus, $F_0^{(j)}(\pm(2k_i - 1)\pi/n) = 0$ pour $j = 0, 2, 4, \ldots, 2l - 2$.

D'autre part, puisque $\max\limits_{z\in\gamma_R}|\Theta(z)| \to 0$ $(R \to +\infty)$ et pour $a > 0$

$$\left|\int_{\gamma_R} e^{iaz}\,dz\right| \leqq 2R\int_0^{\pi/2} e^{-aR\sin t}\,dt \leqq a^{-1}\pi(1-e^{-aR}) < a^{-1}\pi,$$

on a:

$$\lim_{R\to+\infty}\int_{\gamma_R} F_k(z)\Theta(z)\,dz = 0 \quad \text{pour} \quad k \leqq n.$$

Alors, en intégrant la fonction $F_k \cdot \Theta$ le long du contour C et en faisant tendre $\rho \to +0$ et $R \to +\infty$, on obtient:

$$\operatorname{Re}\int_{-\infty}^{+\infty} F_k(t)\Theta(t)\,dt = 2\pi i\sum_{i=1}^{q}\operatorname{Res}\big((2k_i-1)\pi/n; F_k\cdot\Theta\big) \quad (k \leqq n).$$

L'assertion de la proposition 1.2 découle des formules suivantes:

$$B_N = \int_{-\pi}^{\pi}\cos^{2l}\frac{nt}{2}\cdot g_n(t)\,dt = \operatorname{Re}\int_{-\infty}^{+\infty} F_0(t)\Theta(t)\,dt$$

et

$$\lambda_{k,N} = B_N^{-1}\int_{-\pi}^{\pi}\cos kt\cdot\cos^{2l}\frac{nt}{2}\cdot g_n(t)\,dt = B_N^{-1}\cdot\operatorname{Re}\int_{-\infty}^{+\infty} F_k(t)\Theta(t)\,dt.$$

REMARQUE. En remplaçant $\cos^{2l}\frac{nt}{2}$ par $\sin^{2l}\frac{nt}{2}$, on peut obtenir une assertion analogue à la proposition 1.2.

2. Opérateurs de classe S_2 de P. L. Butzer et E. L. Stark

Appliquons la construction de A. I. Kovalenko pour la fonction $\varphi(t) = \sin^3 \pi t$ si $t \in [0,1]$.

LEMME 2.1 ([6], p.455). *Pour tout naturel $n \geqq 1$ on a:*

(2.1) $$L_n(t) = \left|\sum_{k=0}^{n}\sin^3\frac{k\pi}{n}\cdot e^{ikt}\right|^2 =$$

$$= \sin^6\frac{\pi}{n}\cdot\cos^2\frac{nt}{2}\cdot\frac{\left(\cos t + 2\cos\frac{\pi}{n}\right)^2}{\left(\cos t-\cos\frac{\pi}{n}\right)^2\left(\cos t-\cos\frac{3\pi}{n}\right)^2}.$$

Remarquons que $L_n(t)$ est un polynôme trigonométrique d'ordre $n-2$.

LEMME 2.2. *Les fonctions*

$$(2.2)\qquad \begin{cases} K_n(t) = \left(n^8 \sin^6 \dfrac{\pi}{n}\right)^{-1} L_n(t) \quad et \\ \varphi_1(x) = 144 \cos^2 \dfrac{x}{2} \Big/ (x^2 - \pi^2)^2 (x^2 - 9\pi^2)^2 \end{cases}$$

vérifient les conditions de J. Szabados (0.4) *et* (0.5) *pour* $m = 1$.

DÉMONSTRATION. La fonction $\varphi_1(x)$ est continue et bornée sur $[0, +\infty[$; de plus, $\varphi_1(x) = O(1/x^8)$ quand $x \to +\infty$. On en déduit (0.5) (pour $m = 1$). Ensuite, on a uniformément sur $[0, 4\pi]$:

$$(1 + x^6)\big| K_n(x/n) - \varphi_1(x)\big| = (1 + x^6)\varphi_1(x) o(1/n) = o(1/n) \quad (n \to +\infty).$$

D'autre part, uniformément par rapport à $x \in [4\pi, n\pi]$,

$$(1 + x^6)|K_n(x/n) - \varphi_1(x)| = O(1)(x^2/n^2)(1 + x^6)/(x^2 - \pi^2)^2 (x^2 - 9\pi^2)^2 = \\ = O(1/n^2) = o(1/n) \qquad (n \to +\infty).$$

Le lemme 2.2 est démontré.

LEMME 2.3. *Soit*

(2.3)

$$g_n(z) = \frac{\left(\cos z + 2\cos \frac{\pi}{n}\right)^2 (\cos z - \cos\alpha)}{\left(\cos z - \cos \frac{\pi}{n}\right)^2 \left(\cos z - \cos \frac{3\pi}{n}\right)^2}, \quad où \quad \alpha \in \mathbf{R}, \quad n \in \mathbf{N} \setminus \{0\}.$$

Alors, en posant $\Delta_n = \{x = \pm 2\pi/n \pm \pi/n + 2k\pi : k \in \mathbf{Z}\}$, *on a pour* $n \geqq 4$ (voir (1.1)):

$$(2.4)\ g_n(z) = \sum_{k\in\mathbf{Z}} \sum_{i=1}^{2} \sum_{s=1}^{2} \left\{ \frac{a_s^{(i)}}{\left[z - \frac{(2i-1)\pi}{n} - 2k\pi\right]^s} + \frac{(-1)^s a_s^{(i)}}{\left[z + \frac{(2i-1)\pi}{n} - 2k\pi\right]^s} \right\}, \\ z \in \mathbf{C} \setminus \Delta_n,$$

où

$$(2.5)\ a_2^{(1)} = 9\left(\cos \frac{\pi}{n} - \cos\alpha\right) \Big/ 16 \sin^6 \frac{\pi}{n},$$

$$(2.6)\ a_1^{(1)} = a_2^{(1)} \left[-\frac{2}{3} \operatorname{tg} \frac{\pi}{n} + \operatorname{cosec} \frac{2\pi}{n} - \operatorname{ctg} \frac{\pi}{n} - \sin \frac{\pi}{n} \Big/ \left(\cos \frac{\pi}{n} - \cos\alpha\right)\right],$$

$$(2.7)\ a_2^{(2)} = \left(\cos \frac{3\pi}{n} - \cos\alpha\right) \Big/ 16 \sin^6 \frac{\pi}{n},$$

$$(2.8)\ a_1^{(2)} = a_2^{(2)} \left[-2 \operatorname{tg} \frac{\pi}{n} - \left(\sin \frac{3\pi}{n} \Big/ 2\cos \frac{\pi}{n} \cdot \sin^2 \frac{\pi}{n}\right) - \operatorname{ctg} \frac{3\pi}{n} - \\ - \sin \frac{3\pi}{n} \Big/ \left(\cos \frac{3\pi}{n} - \cos\alpha\right)\right].$$

DÉMONSTRATION. Soit

$$g_n(z) = \mu_n(z) \Big/ \left[2 \sin\left(\frac{z}{2} - \frac{\pi}{2n} \right) \right]^2,$$

d'où

$$\mu_n'(z) = \mu_n(z) \left[\frac{-2\sin z}{\cos z + 2\cos\frac{\pi}{n}} + \frac{2\sin z}{\cos z - \cos\frac{3\pi}{n}} - \right.$$
$$\left. - \operatorname{ctg}\left(\frac{z}{2} + \frac{\pi}{2n} \right) - \frac{\sin z}{\cos z - \cos\alpha} \right].$$

Alors, d'après la proposition 1.1 et la formule (1.3), $a_2^{(1)} = \mu_n(\pi/n)$ et $a_1^{(1)} = \mu_n'(\pi/n)$, d'où il découle (2.5) et (2.6). D'autre part, soit

$$g_n(z) = \mu_n^*(z) \Big/ \left[2 \sin\left(\frac{z}{2} - \frac{3\pi}{2n} \right) \right]^2.$$

Alors, d'après (1.3), $a_2^{(2)} = \mu_n^*(3\pi/n)$ et $a_1^{(2)} = \mu_n^{*\prime}(3\pi/n)$, d'où on obtient (2.7) et (2.8). Le lemme 2.3 est démontré.

LEMME 2.4. *Soit* (voir (2.3))

$$(2.9) \quad L_{n-1}(f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) U_{n-1}(t-x)\,dt, \quad f \in C_{2\pi},\ x \in \mathbf{R}\ (n \geqq 4),$$

où

$$(2.10) \qquad U_{n-1}(t) = \pi B_{n-1}^{-1} \cos^2 \frac{nt}{2} \cdot g_n(t) = \frac{1}{2} + \sum_{k=1}^{n-1} \lambda_{k,n-1} \cos kt,$$

$$(2.11) \qquad B_{n-1} = \int_{-\pi}^{\pi} \cos^2 \frac{nt}{2} \cdot g_n(t)\,dt.$$

Alors (voir (2.5)–(2.8)),

(2.12)

$$B_{n-1} = n\pi\left(a_2^{(1)} + a_2^{(2)}\right) = n\pi \left[9\cos\frac{\pi}{n} + \cos\frac{3\pi}{n} - 10\cos\alpha \right] \Big/ 16 \sin^6 \frac{\pi}{n}$$

et

$$(2.13) \quad \lambda_{k,n-1} = \left(a_2^{(1)} + a_2^{(2)}\right)^{-1} \left[(1 - k/n) \left(a_2^{(1)} \cos\frac{k\pi}{n} + a_2^{(2)} \cos\frac{3k\pi}{n} \right) - \right.$$

$$-n^{-1}\left(a_1^{(1)}\sin\frac{k\pi}{n}+a_1^{(2)}\sin\frac{3k\pi}{n}\right)\Big] =$$

$$=\left[9\cos\frac{\pi}{n}+\cos\frac{3\pi}{n}-10\cos\alpha\right]^{-1}\times$$

$$\times\Big\{(1-k/n)\left[9\cos\frac{k\pi}{n}\left(\cos\frac{\pi}{n}-\cos\alpha\right)+\cos\frac{3k\pi}{n}\left(\cos\frac{3\pi}{n}-\cos\alpha\right)\right]+$$

$$+n^{-1}\Big[\left(9\sin\frac{\pi}{n}\sin\frac{k\pi}{n}+\sin\frac{3\pi}{n}\sin\frac{3k\pi}{n}\right)+$$

$$+9\left(\cos\frac{\pi}{n}-\cos\alpha\right)\left(\frac{2}{3}\operatorname{tg}\frac{\pi}{n}-\operatorname{cosec}\frac{2\pi}{n}+\operatorname{ctg}\frac{\pi}{n}\right)\sin\frac{k\pi}{n}+$$

$$+\left(\cos\frac{3\pi}{n}-\cos\alpha\right)\left(2\operatorname{tg}\frac{\pi}{n}+\operatorname{ctg}\frac{3\pi}{n}+\sin\frac{3\pi}{n}\Big/2\cos\frac{\pi}{n}\cdot\sin^2\frac{\pi}{n}\right)\sin\frac{3k\pi}{n}\Big]\Big\}.$$

DÉMONSTRATION. Posons

$$\Theta_1(z)=\sum_{i=1}^{2}\sum_{s=1}^{2}\left\{\frac{a_s^{(i)}}{[z-(2i-1)\pi/n]^s}+\frac{(-1)^s a_s^{(i)}}{[z+(2i-1)\pi/n]^s}\right\}. \tag{2.14}$$

Ensuite, puisque $\cos kt\cdot\cos^2\frac{nt}{2} = 2^{-1}\{\cos kt + 2^{-1}[\cos(n-k)t + +\cos(n+k)t]\}$, alors (voir (1.5)),

$$F_k(t)=2^{-1}\left[e^{ikt}+2^{-1}\left(e^{i(n-k)t}+e^{i(n+k)t}\right)\right]\qquad(k\in\mathbf{N}), \tag{2.15}$$

d'où $F_k(m\pi/n)=\frac{i}{2}\sin\frac{mk\pi}{n}$ et $F_k'(m\pi/n)=-\frac{i}{2}(n-k)\cos\frac{mk\pi}{n}$. Donc,

$$\operatorname{Res}\big((2j-1)\pi/n;F_k\cdot\Theta_1\big) = \tag{2.16}$$

$$=-\frac{i}{2}a_2^{(j)}(n-k)\cos\frac{(2j-1)k\pi}{n}+\frac{i}{2}a_1^{(j)}\sin\frac{(2j-1)k\pi}{n},\quad j=1,2,$$

en particulier, $\operatorname{Res}\big((2j-1)\pi/n;F_0\cdot\Theta_1\big) = -\frac{i}{2}na_2^{(j)}$. En tenant compte des formules (1.9), (1.10) et (2.5)–(2.8), on obtient les égalités (2.12) et (2.13). Le lemme 2.4 est démontré.

LEMME 2.5.* *Pour que dans* (2.13) *on ait*

$$\lambda_{k,n-1}=1+O(n^{-4})\qquad(n\to\infty), \tag{2.17}$$

il faut et il suffit que $\alpha=\pm\sqrt{5}\frac{\pi}{n}\quad(-\pi<\alpha\leqq\pi)$.

---

* Voir [6], lemme 6.

DÉMONSTRATION. Posons $\alpha = \gamma\frac{\pi}{n}$. On aura:

1) $$a_2^{(2)}/a_2^{(1)} = \frac{9-\gamma^2}{9(1-\gamma^2)}\left(1-\frac{9+\gamma^2}{12}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right)\Big/$$
$$\Big/\left(1-\frac{1+\gamma^2}{12}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right)=$$

(2.18) $$=\frac{9-\gamma^2}{9(1-\gamma^2)}\left(1-\frac{2}{3}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right),$$

d'où

(2.19) $$1+a_2^{(2)}/a_2^{(1)} = \frac{18-10\gamma^2}{9(1-\gamma^2)}-\frac{2}{27}\cdot\frac{9-\gamma^2}{1-\gamma^2}\cdot\frac{\pi^2}{n^2}+O(n^{-4}),$$

et

(2.20) $$1+9a_2^{(2)}/a_2^{(1)} = 1+\frac{\gamma^2-9}{\gamma^2-1}-\frac{2}{3}\cdot\frac{\gamma^2-9}{\gamma^2-1}\cdot\frac{\pi^2}{n^2}+O(n^{-4}).$$

2) $$\sin\frac{\pi}{n}\Big/\left(\cos\frac{\pi}{n}-\cos\gamma\frac{\pi}{n}\right)=$$
$$=\frac{n}{\pi}\frac{2}{\gamma^2-1}\left(1-\frac{\pi^2}{6n^2}+O(n^{-4})\right)\left(1+\frac{\gamma^2+1}{12}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right)=$$
$$=\frac{n}{\pi}\left(\frac{2}{\gamma^2-1}+\frac{\pi^2}{6n^2}+O(n^{-4})\right).$$

Donc,

$$a_1^{(1)}/a_2^{(1)} = -\frac{2}{3}\operatorname{tg}\frac{\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}-\operatorname{ctg}\frac{\pi}{n}-\sin\frac{\pi}{n}\Big/\left(\cos\frac{\pi}{n}-\cos\gamma\frac{\pi}{n}\right)=$$

(2.21) $$=\frac{n}{\pi}\left(-\frac{1}{2}-\frac{2}{\gamma^2-1}-\frac{\pi^2}{6n^2}+O(n^{-4})\right).$$

3) $$-\sin\frac{3\pi}{n}\Big/\left(\cos\frac{3\pi}{n}-\cos\gamma\frac{\pi}{n}\right)=\frac{n}{\pi}\left(-\frac{6}{\gamma^2-9}-\frac{\pi^2}{2n^2}+O(n^{-4})\right).$$

D'autre part,

(2.22) $$-\sin\frac{3\pi}{n}\Big/2\cos\frac{\pi}{n}\cdot\sin^2\frac{\pi}{n}=$$
$$=-\frac{n}{\pi}3\left(1-\frac{3}{2}\frac{\pi^2}{n^2}+O(n^{-4})\right)\Big/2\left(1-\frac{2}{3}\frac{\pi^2}{n^2}+O(n^{-4})\right)\times$$
$$\times\left(1-\frac{\pi^2}{6n^2}+O(n^{-4})\right)=\frac{n}{\pi}\left(-\frac{3}{2}+\frac{\pi^2}{n^2}+O(n^{-4})\right).$$

Par conséquent,

$$(2.23)\qquad a_1^{(2)}/a_2^{(2)} = -2\operatorname{tg}\frac{\pi}{n} - \left(\sin\frac{3\pi}{n}\Big/2\cos\frac{\pi}{n}\cdot\sin^2\frac{\pi}{n}\right) - $$
$$-\operatorname{ctg}\frac{3\pi}{n} - \sin\frac{3\pi}{n}\Big/\left(\cos\frac{3\pi}{n} - \cos\gamma\frac{\pi}{n}\right) =$$
$$= \frac{n}{\pi}\left(-\frac{11}{6} - \frac{6}{\gamma^2-9} - \frac{\pi^2}{2n^2} + O(n^{-4})\right).$$

Ensuite, d'après (2.13), (2.18), (2.21) et (2.23), on a:

$$(2.24)\qquad \lambda_{k,n-1} = 1 - \frac{k}{n} - \left(1-\frac{k}{n}\right)\frac{\pi^2k^2}{2n^2}\cdot\frac{1+9a_2^{(2)}/a_2^{(1)}}{1+a_2^{(2)}/a_2^{(1)}} + O(n^{-4}) -$$
$$-\frac{k}{n}\left(1+a_2^{(2)}/a_2^{(1)}\right)^{-1}\left[\left(-\frac{1}{2} - \frac{2}{\gamma^2-1} - \frac{\pi^2}{6n^2} + O(n^{-4})\right)\times\right.$$
$$\times\left(1 - \frac{\pi^2k^2}{6n^2} + O(n^{-4})\right) +$$
$$+\left(\frac{9-\gamma^2}{9(1-\gamma^2)} - \frac{2}{27}\cdot\frac{9-\gamma^2}{1-\gamma^2}\cdot\frac{\pi^2}{n^2} + O(n^{-4})\right)\times$$
$$\left.\times\left(-\frac{11}{6} - \frac{6}{\gamma^2-9} - \frac{\pi^2}{2n^2} + O(n^{-4})\right)\left(3 - \frac{9}{2}\cdot\frac{\pi^2k^2}{n^2} + O(n^{-4})\right)\right].$$

En vertu de (2.19) et (2.20), le coefficient devant $1/n^2$ dans le développement (2.24) est égal à zéro si et seulement si

$$\lim_{n\to\infty}\left(1+9a_2^{(2)}/a_2^{(1)}\right) = 0 \iff \gamma^2 = 5,\quad \gamma = \pm\sqrt{5}.$$

(Il est facile de voir que le coefficient devant $1/n$ est égal à zéro pour tout γ.) Maintenant, soit $\gamma^2 = 5$. Alors,

$$\left(1+a_2^{(2)}/a_2^{(1)}\right)^{-1} = \frac{9}{8}\left(1 - \frac{\pi^2}{12n^2} + O(n^{-4})\right),$$
$$\left(1+9a_2^{(2)}/a_2^{(1)}\right)\Big/\left(1+a_2^{(2)}/a_2^{(1)}\right) = \frac{3}{4}\frac{\pi^2}{n^2} + O(n^{-4}),$$

et

$$(2.25)\qquad \lambda_{k,n-1} = 1 - \frac{k}{n} - \left(1-\frac{k}{n}\right)\frac{\pi^2k^2}{2n^2}\left(\frac{3}{4}\cdot\frac{\pi^2}{n^2} + O(n^{-4})\right) -$$

$$-\frac{k}{n}\cdot\frac{9}{8}\left(1-\frac{\pi^2}{12n^2}+O(n^{-4})\right)\left[\left(-1-\frac{\pi^2}{6n^2}+O(n^{-4})\right)\times\right.$$
$$\times\left(1-\frac{\pi^2k^2}{6n^2}+O(n^{-4})\right)+\left(-\frac{1}{9}+\frac{2}{27}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right)\times$$
$$\left.\times\left(-\frac{1}{3}-\frac{\pi^2}{2n^2}+O(n^{-4})\right)\left(3-\frac{9}{2}\cdot\frac{\pi^2k^2}{n^2}+O(n^{-4})\right)\right]=$$
$$=1-\frac{k}{n}+O(n^{-4})-\frac{k}{n}\cdot\frac{9}{8}\left(1-\frac{\pi^2}{12n^2}+O(n^{-4})\right)\left(-\frac{8}{9}-\frac{2}{27}\cdot\frac{\pi^2}{n^2}+O(n^{-4})\right)=$$
$$=1+O(n^{-4}).$$

Le lemme 2.5 est démontré.

REMARQUE. On a montré dans [6] (lemme 7) que les constantes de Lebesgue des opérateurs (2.9)–(2.11), où $\alpha = \pm\sqrt{5}\frac{\pi}{n}$, sont uniformément bornées.

THÉORÈME 2.1. *Pour les opérateurs* (2.9)–(2.11) *avec* $\alpha = \pm\sqrt{5}\frac{\pi}{n}$ *et pour toute fonction* $f(x) \in C_{2\pi}$ *telle que* $f^{(4)}(x) \in C_{2\pi}$*, l'égalité suivante aura lieu:*

$$(2.26)\quad \lim_{n\to\infty} n^4\left[L_{n-1}(f;x)-f(x)\right] = -\frac{3}{8}\pi^4\left[f^{(4)}(x)+f^{(2)}(x)\right],\quad x\in\mathbf{R}.^*$$

DÉMONSTRATION. Soit

$$\Phi_k(z) = 72z^{2k}(1+e^{iz})/2^{2k}\pi(z^2-\pi^2)^2(z^2-9\pi^2)^2,\quad z\in\mathbf{C},\quad k\in\mathbf{N}.$$

Alors

$$\mathrm{Res}(\pm\pi;\Phi_k) = -\frac{9i}{32\pi^7}\left(\frac{\pi}{2}\right)^{2k},\quad \mathrm{Res}(\pm3\pi;\Phi_k) = -\frac{i}{32\pi^7}\left(\frac{3\pi}{2}\right)^{2k}.$$

Ensuite, en appliquant le théorème des résidus pour le même contour C que nous avons utilisé dans la proposition 1.2, nous aurons (voir (0.12) et (2.2)):

$$(2.27)\quad \mu_{2k} = \frac{2^{1-2k}}{\pi}\int_0^{+\infty} x^{2k}\varphi_1(x)\,dx = \mathrm{Re\,v.p.}\int_{-\infty}^{+\infty}\Phi_k(x)\,dx =$$
$$= \mathrm{Re}\left[\pi i\sum_{j=0}^{3}\mathrm{Res}(-3\pi+2j\pi;\Phi_k)\right] = 16^{-1}\pi^{-6}(\pi/2)^{2k}[3^{2k}+9],$$
$$k=0,1,2,3.$$

* La formule (2.26) montre que l'égalité (80) dans [3] n'est pas correcte.

Donc, $\mu_0 = 5/8\pi^6$, $\mu_2 = 9/32\pi^4$, $\mu_4 = 45/128\pi^2$ et $\mu_6 = 369/512$. Alors, d'après (0.13) pour $m = 1$, on obtient $\lambda_0 = 5/2\pi^2$ et $\lambda_1 = -2\pi^4$, d'où

$$\mu_4\lambda_0 + \mu_6\lambda_1 = -\frac{9}{16}\pi^4.$$

D'autre part (voir (0.11)), $a_{1,1} = a_{2,1} = 1/6$. Alors, l'égalité (2.26) découle de la formule (0.10). Le théorème 2.1 est démontré.

Puisque $L_{n-1}(\cos kt; x) - \cos kx = (\lambda_{k,n-1} - 1)\cos kx$ ($k \leqq n-1$) (voir (2.10)), nous avons l'assertion suivante:

COROLLAIRE 2.1. *Sous les conditions du Théorème* 2.1 *on aura:*

$$(2.28)\qquad \lim_{n\to\infty} n^4(1-\lambda_{k,n-1}) = \frac{3}{8}\pi^4(k^4-k^2) \qquad (k \in \mathbf{N},\ k \leqq n-1).^*$$

Du théorème de saturation de J. Szabados nous obtenons encore un résultat.

COROLLAIRE 2.2. *Sous les conditions du Théorème* 2.1 *on aura:*

i) $\varliminf_{n\to\infty} n^4\|L_{n-1}(f;x) - f(x)\|_{C_{2\pi}} = 0$ *si et seulement si* $f(x)$ *est un polynôme trigonométrique d'ordre* $\leqq 1$;

ii) $\|L_{n-1}(f;x) - f(x)\|_{C_{2\pi}} = O(n^{-4})$ *si et seulement si* $f^{(3)} \in \operatorname{Lip} 1$.

3. Deuxième exemple d'opérateurs extrémaux de classe S_2

LEMME 3.1 ([4], p.190). *Les fonctions*

$$(3.1)\qquad \begin{cases} K_n(x) = n^{-8}\cos^4\dfrac{nx}{2}\left(\cos x - \cos\dfrac{\pi}{n}\right)^{-4} & \text{et} \\ \varphi_1^*(x) = 16\cos^4\dfrac{x}{2}\cdot\left(x^2-\pi^2\right)^{-4} & (n \in \mathbf{N}\setminus\{0\}) \end{cases}$$

vérifient les conditions de J. Szabados (0.4) *et* (0.5) *pour* $m = 1$.

LEMME 3.2. *Soit*

$$(3.2)\qquad g_n(z) = \frac{\cos z - \cos\alpha}{\left(\cos z - \cos\frac{\pi}{n}\right)^4} = l_n(z)\Big/\left[2\sin\left(\frac{z}{2}-\frac{\pi}{2n}\right)\right]^4, \quad n \in \mathbf{N}\setminus\{0\}.$$

Alors, en posant $\Delta_n = \{x = 2\pi t \pm \pi/n : t \in \mathbf{Z}\}$, *on a* (voir (1.1)):

(3.3)

$$g_n(z) = \sum_{k\in\mathbf{Z}}\sum_{s=1}^{4}\left[\frac{a_s}{\left(z-\frac{\pi}{n}-2k\pi\right)^s} + \frac{(-1)^s a_s}{\left(z+\frac{\pi}{n}-2k\pi\right)^s}\right], \quad z \in \mathbf{C}\setminus\Delta_n\ (n \geqq 2),$$

* Cette formule est énoncée dans [6], p.459, avec une erreur: il faut mettre $-k^2$ au lieu de $+k^2$.

où

$$(3.4)\qquad a_4 = A_n, \quad a_3 = B_n, \quad a_2 = (3C_n + A_n)/6, \quad a_1 = (D_n + B_n)/6,$$

et

$$(3.5)\qquad A_n = l_n(\pi/n) = \left(\sin\frac{\pi}{n}\right)^{-4}\left(\cos\frac{\pi}{n} - \cos\alpha\right),$$

$$(3.6)\qquad B_n = l_n'(\pi/n) = A_n\left[\frac{-\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\alpha} - 2\operatorname{ctg}\frac{\pi}{n}\right],$$

$$(3.7)\qquad C_n = l_n''(\pi/n) = A_n\left[\frac{5}{\sin^2\frac{\pi}{n}} - 4 + \frac{3\cos\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\alpha}\right],$$

$$(3.8)\qquad D_n = l'''(\pi/n) = A_n\left[\frac{-9 + 7\sin^2\frac{\pi}{n}}{\sin\frac{\pi}{n}\left(\cos\frac{\pi}{n} - \cos\alpha\right)} + 8\operatorname{ctg}\frac{\pi}{n} - 15\frac{\cos\frac{\pi}{n}}{\sin^3\frac{\pi}{n}}\right].$$

DÉMONSTRATION. D'après la proposition 1.1, il suffit de vérifier (3.4). Mais ces égalités découlent des formules (1.3), (1.2) et du développement

$$l_n(z) = A_n + B_n\left(z - \frac{\pi}{n} - 2k\pi\right) + \frac{C_n}{2}\left(z - \frac{\pi}{n} - 2k\pi\right)^2 + $$
$$+\frac{D_n}{6}\left(z - \frac{\pi}{n} - 2k\pi\right)^3 + \dots .$$

LEMME 3.3. *Soit*

$$(3.9)\quad L_{2n-3}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)U_{2n-3}(t-x)\,dt, \quad f \in C_{2\pi},\ x \in \mathbf{R}\ (n \geqq 2),$$

où (voir (3.2))

$$(3.10)\qquad U_{2n-3}(t) = \pi B_{2n-3}^{*-1}\cos^4\frac{nt}{2}\cdot g_n(t) = \frac{1}{2} + \sum_{k=1}^{2n-3}\lambda_{k,2n-3}\cos kt,$$

$$(3.11)\qquad B_{2n-3}^{*} = \int_{-\pi}^{\pi}\cos^4\frac{nt}{2}\cdot g_n(t)\,dt.$$

Alors, en posant $b_n = B_n/A_n$, $c_n = C_n/A_n$ *et* $d_n = D_n/A_n$ (voir (3.5)–(3.8)), *on a:*

$$(3.12)\qquad B_{2n-3}^{*} = \frac{\pi n^3}{6}A_n\left(1 + \frac{1+3c_n}{2n^2}\right),$$

et

$$\lambda_{k,2n-3} = 24\left[2 + (1 + 3c_n)/n^2\right]^{-1} \times \tag{3.13}$$
$$\times \left[-\frac{1}{16n^3}(b_n + d_n)\sin\frac{k\pi}{n} + \left(-\frac{3}{8}\frac{k}{n} + \frac{1}{4}\right)\cdot\frac{1+3c_n}{6n^2}\cdot\cos\frac{k\pi}{n} - \right.$$
$$\left. -\frac{k}{n^2}\left(\frac{1}{4} - \frac{3}{16}\frac{k}{n}\right) b_n \sin\frac{k\pi}{n} + \frac{1}{48}\left(4 - 6\frac{k^2}{n^2} + 3\frac{k^3}{n^3}\right)\cos\frac{k\pi}{n}\right].$$

DÉMONSTRATION. Posons

$$\Theta_2(z) = \sum_{s=1}^{4}\left[\frac{a_s}{\left(z - \frac{\pi}{n}\right)^s} + \frac{(-1)^s a_s}{\left(z + \frac{\pi}{n}\right)^s}\right], \qquad z \in \mathbf{C}. \tag{3.14}$$

Puisque

$$\cos kt \cdot \cos^4\frac{nt}{2} = \frac{3}{8}\cos kt + \frac{1}{4}\left[\cos(n-k)t + \cos(n+k)t\right] +$$
$$+\frac{1}{16}\left[\cos(2n-k)t + \cos(2n+k)t\right],$$

alors (voir (1.5))

$$F_k(t) = \frac{3}{8}e^{ikt} + \frac{1}{4}\left[e^{i(n-k)t} + e^{i(n+k)t}\right] + \frac{1}{16}\left[e^{i(2n-k)t} + e^{i(2n+k)t}\right] = \tag{3.15}$$
$$= F_k(\pi/n) + F_k'(\pi/n)(z - \pi/n) + F_k''(\pi/n)(z-\pi/n)^2/2 +$$
$$+ F_k'''(\pi/n)(z-\pi/n)^3/6 + \dots,$$

où

$$F_k(\pi/n) = \frac{3i}{8}\sin\frac{k\pi}{n}, \qquad F_k'(\pi/n) = \frac{(3k-2n)i}{8}\cos\frac{k\pi}{n},$$
$$F_k''(\pi/n) = \frac{k(4n-3k)i}{8}\sin\frac{k\pi}{n},$$
$$F_k'''(\pi/n) = \frac{(-4n^3 - 3k^3 - 6nk^2)i}{8}\cos\frac{k\pi}{n}.$$

En particulier, $F_0(\pi/n) = F_0''(\pi/n) = 0$, $F_0'(\pi/n) = -in/4$, $F_0'''(\pi/n) = -in^3/2$. Alors, d'après (3.4) et (3.14), on a:

$$\operatorname{Res}(\pi/n; F_k\cdot\Theta_2) = F_k(\pi/n)(B_n + D_n)/6 + F_k'(\pi/n)(3C_n + A_n)/6 +$$
$$+ F_k''(\pi/n)B_n/2 + F_k'''(\pi/n)A_n/6,$$

d'où $\operatorname{Res}(\pi/n; F_0\cdot\Theta_2) = -\frac{in^3 A_n}{12}\left(1 + \frac{1+3c_n}{2n^2}\right)$. Maintenant, l'assertion du lemme **3.3** découle des formules (1.9) et (1.10).

LEMME 3.4. *Pour que dans* (3.13) *avec* (3.5)–(3.8) *on ait*

$$\lambda_{k,2n-3} = 1 + O(n^{-4}) \qquad (n \to \infty), \tag{3.16}$$

il faut et il suffit que $\alpha = \pm\sqrt{(2\pi^2+3)/(2\pi^2-3)} \cdot \frac{\pi}{n} \quad (-\pi < \alpha \leqq \pi)$.

DÉMONSTRATION. Posons $\alpha = \gamma\frac{\pi}{n}$. On a:

$$b_n = B_n/A_n = -\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}} - 2\operatorname{ctg}\frac{\pi}{n} = \tag{3.17}$$

$$= -\frac{n}{\pi}\frac{2}{\gamma^2-1}\left(1 - \frac{\pi^2}{6n^2} + \frac{\gamma^2+1}{12}\frac{\pi^2}{n^2} + O(n^{-4})\right) -$$

$$-2\frac{n}{\pi}\left(1 - \frac{\pi^2}{3n^2} + O(n^{-4})\right) = -\frac{n}{\pi}\left(\frac{2}{\gamma^2-1} + 2\right) + \frac{\pi}{2n} + O(n^{-3}).$$

$$c_n = C_n/A_n = \frac{5}{\sin^2\frac{\pi}{n}} - 4 + \frac{3\cos\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}} = \tag{3.18}$$

$$= \frac{n^2}{\pi^2}5\left(1 + \frac{\pi^2}{3n^2} + O(n^{-4})\right) - 4 -$$

$$-\frac{n^2}{\pi^2}\frac{6}{\gamma^2-1}\left(1 - \frac{\pi^2}{2n^2} + O(n^{-4})\right)\left(1 + \frac{\gamma^2+1}{12}\cdot\frac{\pi^2}{n^2} + O(n^{-4})\right) =$$

$$= \frac{n^2}{\pi^2}\left(5 + \frac{6}{\gamma^2-1}\right) - \frac{11\gamma^2+1}{6(\gamma^2-1)} + O(n^{-2}).$$

Ensuite,

$$b_n + d_n = (B_n + D_n)/A_n = \tag{3.19}$$

$$= \frac{6\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}} + 6\operatorname{ctg}\frac{\pi}{n} - 15\frac{\cos\frac{\pi}{n}}{\sin^3\frac{\pi}{n}} - \frac{9}{\sin\frac{\pi}{n}\left(\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}\right)} =$$

$$= \frac{n}{\pi}\cdot\frac{12}{\gamma^2-1} + \frac{\pi}{n} + O(n^{-3}) + \frac{6n}{\pi} - 2\frac{\pi}{n} + O(n^{-3}) - 15\frac{n^3}{\pi^3} + O(n^{-1}) +$$

$$+\frac{n^3}{\pi^3}\left(-\frac{18}{\gamma^2-1}\right) + \frac{n}{\pi}\left(-\frac{3}{2} - \frac{6}{\gamma^2-1}\right) + O(n^{-1}) =$$

$$= \frac{n^3}{\pi^3}\left(-\frac{18}{\gamma^2-1} - 15\right) + \frac{n}{\pi}\left(\frac{6}{\gamma^2-1} + \frac{9}{2}\right) + O(n^{-1}).$$

Donc, d'après (3.13),

$$\lambda_{k,2n-3} =$$

$$= \left[\frac{1}{12} + \frac{1}{8\pi^2}\left(5 + \frac{6}{\gamma^2 - 1}\right) + \left(\frac{1}{24} - \frac{11\gamma^2 + 1}{48(\gamma^2 - 1)}\right)\frac{1}{n^2} + O(n^{-4})\right]^{-1} \times$$

$$\times \left\{\left[\frac{1}{12} + \frac{1}{8\pi^2}\left(5 + \frac{6}{\gamma^2 - 1}\right) + \left(\frac{1}{24} - \frac{11\gamma^2 + 1}{48(\gamma^2 - 1)}\right)\frac{1}{n^2}\right] + \right.$$

$$+\frac{k}{n}\left[\frac{1}{\pi^2}\left(\frac{9}{8} \cdot \frac{1}{\gamma^2 - 1} + \frac{15}{16}\right) - \frac{3}{8} \cdot \frac{1}{2\pi^2}\left(5 + \frac{6}{\gamma^2 - 1}\right)\right] +$$

$$+\frac{k}{n^3}\left[-\frac{1}{16}\left(\frac{6}{\gamma^2 - 1} + \frac{9}{2}\right) + \frac{3}{8}\left(\frac{11}{12} + \frac{1}{\gamma^2 - 1} - \frac{1}{6}\right)\right] +$$

$$+\frac{k^3}{n^3}\left[-\left(\frac{3}{16} \cdot \frac{1}{\gamma^2 - 1} + \frac{5}{32}\right) + \frac{3}{32}\left(5 + \frac{6}{\gamma^2 - 1}\right) - \right.$$

$$\left. -\frac{3}{8}\left(\frac{1}{\gamma^2 - 1} + 1\right) + \frac{1}{16}\right] +$$

$$\left. +\frac{k^2}{n^2}\left[-\frac{1}{16}\left(5 + \frac{6}{\gamma^2 - 1}\right) + \frac{1}{2}\left(\frac{1}{\gamma^2 - 1} + 1\right) - \frac{1}{8} - \frac{\pi^2}{24}\right] + O(n^{-4})\right\}.$$

Les coefficients devant $\frac{k}{n}$, $\frac{k}{n^3}$ et $\frac{k^3}{n^3}$ étant nuls, nous avons:

$$\lambda_{k,2n-3} =$$

$$= 1 + \left[\frac{1}{12} + \frac{1}{8\pi^2}\left(5 + \frac{6}{\gamma^2 - 1}\right)\right]^{-1} \cdot \frac{k^2}{n^2}\left[\frac{1}{16} - \frac{\pi^2}{24} + \frac{1}{8(\gamma^2 - 1)}\right] + O(n^{-4}).$$

Donc,

(3.20) $3 - 2\pi^2 + 6/(\gamma^2 - 1) = 0 \iff \gamma = \pm\sqrt{(2\pi^2 + 3)/(2\pi^2 - 3)} \approx \pm 1{,}16.$

Le lemme 3.4 est démontré.

Maintenant, appliquons le théorème de saturation de J. Szabados.

THÉORÈME 3.1. *Pour les opérateurs* (3.9)–(3.11) *avec*

$$\alpha = \sqrt{(2\pi^2 + 3)/(2\pi^2 - 3)} \cdot \frac{\pi}{n}$$

et pour toute fonction $f(x) \in C_{2\pi}$ *telle que* $f^{(4)} \in C_{2\pi}$, *l'égalité suivante aura lieu:*

(3.21)

$$\lim_{n\to\infty} n^4 \left[L_{2n-3}(f;x) - f(x)\right] = -\frac{\pi^4}{6} \cdot \frac{4\pi^2 - 9}{16\pi^2 + 7}\left[f^{(4)}(x) + f^{(2)}(x)\right], \quad x \in \mathbf{R}.$$

DÉMONSTRATION. Soient

$$\varphi(z) = \frac{1}{8}e^{2iz} + \frac{1}{2}e^{iz} + \frac{3}{8} = -\frac{i}{4}(z \mp \pi) - \frac{i}{12}(z \mp \pi)^3 + \dots ;$$

$$\operatorname{Re}\varphi(x) = \cos^4\frac{x}{2} \qquad \text{si} \qquad x \in \mathbf{R},$$

et

$$\Phi_k(z) = 16z^{2k}\varphi(z)/2^{2k}\pi(z^2-\pi^2)^4, \qquad z \in \mathbf{C}, \quad k \in \mathbf{N}.$$

Alors, en posant $\nu_{k,\pm}(z) = 16z^{2k}/2^{2k}\pi(z\pm\pi)^4$, on obtient:

$$\operatorname{Res}(\pm\pi;\Phi_k) = -\frac{i}{12}\nu_{k,\pm}(\pm\pi) - \frac{i}{8}\nu''_{k,\pm}(\pm\pi) =$$
$$= -i(1/24)\pi^{-7}(12k^2-30k+15+2\pi^2)(\pi/2)^{2k}.$$

En appliquant le théorème des résidus pour le contour C (voir la proposition 1.2) et en tenant compte de (0.12) et (3.1), nous aurons:

$$(3.22) \qquad \mu_{2k} = \frac{2^{1-2k}}{\pi}\int_0^{+\infty} x^{2k}\varphi_1^*(x)\,dx = \operatorname{Re}\text{v.p.}\int_{-\infty}^{+\infty}\Phi_k(x)\,dx =$$
$$= \operatorname{Re}\big[2\pi i\operatorname{Res}(\pi;\Phi_k)\big] = (1/12)\pi^{-6}(12k^2-30k+15+2\pi^2)(\pi/2)^{2k},$$
$$k = 0,1,2,3.$$

Donc, on a:

(3.23)
$$\begin{cases} \mu_0 = \dfrac{15+2\pi^2}{12\pi^6}, \quad \mu_2 = \dfrac{-3+2\pi^2}{48\pi^4}, \quad \mu_4 = \dfrac{3+2\pi^2}{192\pi^2}, \quad \mu_6 = \dfrac{33+2\pi^2}{768} \\ \text{et} \qquad \Delta = \mu_0\mu_4 - \mu_2^2 = \dfrac{16\pi^2+7}{768\pi^8}. \end{cases}$$

Alors, d'après (0.13) pour $m = 1$, on a:

$$(3.24) \qquad \mu_4\lambda_0 + \mu_6\lambda_1 = (\mu_4^2 - \mu_2\mu_6)/\Delta = -\frac{\pi^4}{4}\cdot\frac{4\pi^2-9}{16\pi^2+7}.$$

D'autre part, d'après (0.11), $a_{1,1} = a_{2,1} = 1/6$. Alors, l'égalité (3.21) découle de la formule (0.10) et de (3.24). Le théorème 3.1 est démontré.

COROLLAIRE 3.1. *Sous les conditions du Théorème* 3.1 *on aura:*

$$(3.25) \quad \lim_{n\to\infty} n^4(1-\lambda_{k,2n-3}) = \frac{\pi^4}{6}\cdot\frac{4\pi^2-9}{16\pi^2+7}(k^4-k^2) \quad (k\in\mathbf{N},\ k \leqq 2n-3).$$

COROLLAIRE 3.2. *Sous les conditions du Théorème* 3.1 *on aura:*

i) $\lim\limits_{n\to\infty} n^4\|L_{2n-3}(f;x)-f(x)\|_{C_{2\pi}} = 0$ *si et seulement si* $f(x)$ *est un polynôme trigonométrique d'ordre* $\leqq 1$;

ii) $\|L_{2n-3}(f;x)-f(x)\|_{C_{2\pi}} = O(n^{-4})$ *si et seulement si* $f^{(3)} \in \operatorname{Lip} 1$.

4. Troisième exemple d'opérateurs extrémaux de classe S_2

Nous allons considérer encore un exemple d'opérateurs extrémaux de classe S_2, où nous utiliserons directement la méthode de J. Szabados.

LEMME 4.1. *Les fonctions*

$$K_n(t) = \cos^2 \frac{nt}{2} \Big/ n^8 \left(\cos t - \cos \frac{\pi}{n}\right)^2 \left(\cos t - \cos \frac{3\pi}{n}\right)^2, \tag{4.1}$$

et

$$\widetilde{\varphi}_1(x) = 16 \cos^2 \frac{x}{2} / (x^2 - \pi^2)^2 (x^2 - 9\pi^2)^2 \tag{4.2}$$

vérifient les conditions de J. Szabados (0.4) *et* (0.5) *pour* $m = 1$.

La démonstration de ce lemme est identique à celle du lemme 2.2.

LEMME 4.2. *Soit*

(4.3)

$$g_n(z) = 1 \Big/ \left(\cos z - \cos \frac{\pi}{n}\right)^2 \left(\cos z - \cos \frac{3\pi}{n}\right)^2, \quad z \in \mathbf{C},\ n \in \mathbf{N} \setminus \{0\}.$$

Alors, en posant $\Delta_n = \{x = 2\pi t \pm 2\pi/n \pm \pi/n : t \in \mathbf{Z}\}$ *on aura pour* $n \geqq 4$*:*

$$g_n(z) = \sum_{k \in \mathbf{Z}} \sum_{i=1}^{2} \sum_{s=1}^{2} \left\{ \frac{\overline{a}_s^{(i)}}{\left[z - \frac{(2i-1)\pi}{n} - 2k\pi\right]^s} + \frac{(-1)^s \overline{a}_s^{(i)}}{\left[z + \frac{(2i-1)\pi}{n} - 2k\pi\right]^s} \right\}, \tag{4.4}$$
$$z \in \mathbf{C} \setminus \Delta_n,$$

où

$$\overline{a}_2^{(1)} = 1/16 \sin^6 \frac{\pi}{n} \cdot \cos^2 \frac{\pi}{n}, \tag{4.5}$$

$$\overline{a}_1^{(1)} = \overline{a}_2^{(1)} \left[\operatorname{cosec} \frac{2\pi}{n} - \operatorname{ctg} \frac{\pi}{n}\right], \tag{4.6}$$

$$\overline{a}_2^{(2)} = 1/16 \sin^6 \frac{\pi}{n} \cdot \cos^2 \frac{\pi}{n} \cdot \left(4 \cos^2 \frac{\pi}{n} - 1\right)^2, \tag{4.7}$$

$$\overline{a}_1^{(2)} = \overline{a}_2^{(2)} \left[- \operatorname{ctg} \frac{3\pi}{n} - \sin \frac{3\pi}{n} \Big/ 2 \cos \frac{\pi}{n} \cdot \sin^2 \frac{\pi}{n}\right]. \tag{4.8}$$

DÉMONSTRATION. Soit

$$g_n(z) = \mu_n(z) \Big/ \left[2 \sin \left(\frac{z}{2} - \frac{\pi}{2n}\right)\right]^2 = \nu_n(z) \Big/ \left[2 \sin \left(\frac{z}{2} - \frac{3\pi}{2n}\right)\right]^2.$$

Alors, d'après (1.3), $\overline{a}_2^{(1)} = \mu_n(\pi/n)$, $\overline{a}_1^{(1)} = \mu_n'(\pi/n)$, $\overline{a}_2^{(2)} = \nu_n(3\pi/n)$, $\overline{a}_1^{(2)} = \nu_n'(3\pi/n)$, d'où on obtient (4.5)–(4.8).

LEMME 4.3. *Soit* (voir (4.3))

$$U_{n-4}(t)=\pi B_{n-4}^{-1}\cos^2\frac{nt}{2}\cdot g_n(t)=\frac{1}{2}+\sum_{k=1}^{n-4}\lambda^*_{k,n-4}\cos kt,\quad n\in\mathbf{N}\setminus\{0\}\tag{4.9}$$

où

$$B_{n-4}=\int_{-\pi}^{\pi}\cos^2\frac{nt}{2}\cdot g_n(t)\,dt.\tag{4.10}$$

Alors,

$$B_{n-4}=n\pi\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)=\tag{4.11}$$

$$=n\pi\left(\sin^2\frac{3\pi}{n}+\sin^2\frac{\pi}{n}\right)\Big/16\sin^6\frac{\pi}{n}\cdot\cos^2\frac{\pi}{n}\cdot\sin^2\frac{3\pi}{n},$$

$$\lambda^*_{k,n-4}=\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)^{-1}\left[(1-k/n)\left(\overline{a}_2^{(1)}\cos\frac{k\pi}{n}+\overline{a}_2^{(2)}\cos\frac{3k\pi}{n}\right)-\right.\tag{4.12}$$

$$\left.-n^{-1}\left(\overline{a}_1^{(1)}\sin\frac{k\pi}{n}+\overline{a}_1^{(2)}\sin\frac{3k\pi}{n}\right)\right]=$$

$$=\left(\sin^2\frac{3\pi}{n}+\sin^2\frac{\pi}{n}\right)^{-1}\left\{(1-k/n)\left(\sin^2\frac{3\pi}{n}\cdot\cos\frac{k\pi}{n}+\sin^2\frac{\pi}{n}\cdot\cos\frac{3k\pi}{n}\right)+\right.$$

$$+n^{-1}\left[\sin^2\frac{3\pi}{n}\left(-\operatorname{cosec}\frac{2\pi}{n}+\operatorname{ctg}\frac{\pi}{n}\right)\sin\frac{k\pi}{n}+\right.$$

$$\left.\left.+\sin^2\frac{\pi}{n}\left(\operatorname{ctg}\frac{3\pi}{n}+\sin\frac{3\pi}{n}\Big/2\cos\frac{\pi}{n}\cdot\sin^2\frac{\pi}{n}\right)\sin\frac{3k\pi}{n}\right]\right\}.$$

La démonstration de ce lemme est identique à celle du lemme 2.4. D'après (4.1), (4.3), (4.9) et (4.11),

$$K_n(t)=n^{-8}\pi^{-1}B_{n-4}U_{n-4}(t)=n^{-7}\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)U_{n-4}(t).\tag{4.13}$$

Donc, puisque

$$\lambda^*_{k,n-4}=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos kt\cdot U_{n-4}(t)\,dt,$$

on obtient (voir (0.6) et (0.7)):

(4.14)

$$\begin{cases}\mu_{0,n}=n^{-6}\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right),\quad \mu_{2,n}=n^{-4}\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)(1-\lambda^*_{1,n-4})/2,\\[2mm] \mu_{4,n}=n^{-2}\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)(3-4\lambda^*_{1,n-4}+\lambda^*_{2,n-4})/8,\\[2mm] \mu_{6,n}=\left(\overline{a}_2^{(1)}+\overline{a}_2^{(2)}\right)(10-15\lambda^*_{1,n-4}+6\lambda^*_{2,n-4}-\lambda^*_{3,n-4})/32,\end{cases}$$

et

$$\lambda_0^{(n)} = \mu_{4,n}/(\mu_{0,n}\mu_{4,n} - \mu_{2,n}^2), \quad \lambda_1^{(n)} = -\mu_{2,n}/(\mu_{0,n}\mu_{4,n} - \mu_{2,n}^2). \tag{4.15}$$

Posons (voir (0.8), (0.9) et (4.1))

$$\overline{U}_n(t) = K_n(t)\left[n\lambda_0^{(n)} + n^3\lambda_1^{(n)}(1-\cos t)/2\right] = \tag{4.16}$$
$$= K_n(t)\cdot(1/2)n^3(-\lambda_1^{(n)})\left[\cos t - (1 - 2n^{-2}\mu_{4,n}/\mu_{2,n})\right].$$

THÉORÈME 4.1. *Soit*

$$L_{n-3}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\overline{U}_n(t)\,dt, \quad f \in C_{2\pi},\ n \in \mathbf{N},\ n \geqq 3, \tag{4.17}$$

la suite d'opérateurs de classe S_2 avec le noyau (4.16).

Alors, pour toute fonction $f(x) \in C_{2\pi}$ telle que $f^{(4)}(x) \in C_{2\pi}$, l'égalité suivante aura lieu:

$$\lim_{n\to\infty} n^4\left[L_{n-3}(f;x) - f(x)\right] = -\frac{3}{8}\pi^4\left[f^{(4)}(x) + f^{(2)}(x)\right], \quad x \in \mathbf{R}. \tag{4.18}$$

La démonstration est identique à celle du Théorème 2.1.

COROLLAIRE 4.1. *Sous les conditions du Théorème* 4.1 *on aura:*

$$\lim_{n\to\infty} n^4(1 - \lambda_{k,n-3}) = \frac{3}{8}\pi^4(k^4 - k^2), \tag{4.19}$$

où $\lambda_{k,n-3} = L_{n-3}(\cos kt; 0)$ $(k \in \mathbf{N},\ k \leqq n-3)$.

COROLLAIRE 4.2. *Sous les conditions du Théorème* 4.1 *on aura:*

i) $\lim\limits_{n\to\infty} n^4\|L_{n-3}(f;x) - f(x)\|_{C_{2\pi}} = 0$ *si et seulement si $f(x)$ est un polynôme trigonométrique d'ordre $\leqq 1$;*

ii) $\|L_{n-3}(f;x) - f(x)\|_{C_{2\pi}} = O(n^{-4})$ *si et seulement si* $f^{(3)} \in \operatorname{Lip} 1$.

Maintenant, trouverons une formule asymptotique approchée pour la valeur (voir (4.16)):

$$A_n = 1 - 2n^{-2}\mu_{4,n}/\mu_{2,n}. \tag{4.20}$$

LEMME 4.4. *On a* (voir (4.12)):

$$\lambda_{k,n-4}^{*} = 1 - \frac{9}{10}k^2\frac{\pi^2}{n^2} - \frac{24}{25}k^2\frac{\pi^4}{n^4} + \frac{3}{8}k^4\frac{\pi^4}{n^4} - \frac{168}{125}k^2\frac{\pi^6}{n^6} + \tag{4.21}$$
$$+\frac{4}{5}k^4\frac{\pi^6}{n^6} - \frac{41}{400}k^6\frac{\pi^6}{n^6} + o(n^{-6}) \quad (n\to\infty),\ 0 \leqq k \leqq n-4.$$

En particulier,

$$(4.22)\qquad \begin{cases} \lambda^*_{0,n-4} = 1, \\ \lambda^*_{1,n-4} = 1 - \dfrac{9}{10}\dfrac{\pi^2}{n^2} - \dfrac{117}{200}\dfrac{\pi^4}{n^4} - \dfrac{1293}{2000}\dfrac{\pi^6}{n^6} + o(n^{-6}), \\ \lambda^*_{2,n-4} = 1 - \dfrac{18}{5}\dfrac{\pi^2}{n^2} + \dfrac{54}{25}\dfrac{\pi^4}{n^4} + \dfrac{108}{125}\dfrac{\pi^6}{n^6} + o(n^{-6}). \end{cases}$$

DÉMONSTRATION. On a:

$$\overline{a}_2^{(2)}/\overline{a}_2^{(1)} = \left(1 + 2\cos\frac{2\pi}{n}\right)^{-2} = \frac{1}{9}\left(1 + \frac{8}{3}\frac{\pi^2}{n^2} + \frac{40}{9}\frac{\pi^4}{n^4} + \frac{272}{45}\frac{\pi^6}{n^6} + O(n^{-8})\right),$$

$$1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)} = \frac{10}{9}\left(1 + \frac{4}{15}\frac{\pi^2}{n^2} + \frac{4}{9}\frac{\pi^4}{n^4} + \frac{136}{225}\frac{\pi^6}{n^6} + O(n^{-8})\right),$$

$$\left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1} = \frac{9}{10}\left(1 - \frac{4}{15}\frac{\pi^2}{n^2} - \frac{28}{75}\frac{\pi^4}{n^4} - \frac{1304}{3375}\frac{\pi^6}{n^6} + O(n^{-8})\right).$$

Donc, il est facile de voir que

$$(4.23)\qquad \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1}\left[\cos\frac{k\pi}{n} + \left(\overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)\cos\frac{3k\pi}{n}\right] =$$

$$= \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1}\left[\left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right) - \frac{k^2\pi^2}{n^2} + \frac{5}{12}\cdot\frac{k^4\pi^4}{n^4} - \frac{4}{3}\frac{k^2\pi^4}{n^4} - \right.$$

$$\left. - \frac{41}{360}\frac{k^6\pi^6}{n^6} + \frac{k^4\pi^6}{n^6} - \frac{20}{9}\frac{k^2\pi^6}{n^6} + O(n^{-8})\right] =$$

$$= 1 - \frac{9}{10}\frac{k^2\pi^2}{n^2} - \frac{24}{25}\frac{k^2\pi^4}{n^4} + \frac{3}{8}\frac{k^4\pi^4}{n^4} - \frac{168}{125}\frac{k^2\pi^6}{n^6} + \frac{4}{5}\frac{k^4\pi^6}{n^6} - \frac{41}{400}\frac{k^6\pi^6}{n^6} + O(n^{-8})$$

$$(n \to \infty).$$

Puisque ($0 \leqq k \leqq n - 4$)

$$(4.24)\qquad \lambda^*_{k,n-4} = \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1}\left[\cos\frac{k\pi}{n} + \left(\overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)\cos\frac{3k\pi}{n}\right] + B_{kn}$$

où

$$(4.25)\, B_{kn} = \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1}\left[-\frac{k}{n}\left(\cos\frac{k\pi}{n} + \left(\overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)\cos\frac{3k\pi}{n}\right) - \right.$$

$$\left. - \frac{1}{n}\left[\left(\overline{a}_1^{(1)}/\overline{a}_2^{(1)}\right)\sin\frac{k\pi}{n} + \left(\overline{a}_1^{(2)}/\overline{a}_2^{(2)}\right)\left(\overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)\sin\frac{3k\pi}{n}\right],\right.$$

il suffit de montrer, d'après (4.23), que $B_{kn} = o(n^{-6})$.

On a:

$$\overline{a}_1^{(1)}/\overline{a}_2^{(1)} = \operatorname{cosec}\frac{2\pi}{n} - \operatorname{ctg}\frac{\pi}{n} = -\operatorname{ctg}\frac{2\pi}{n} = $$
$$= -\frac{n}{\pi}\left(\frac{1}{2} - \frac{2}{3}\frac{\pi^2}{n^2} - \frac{8}{45}\frac{\pi^4}{n^4} + O(n^{-6})\right),$$

$$\overline{a}_1^{(2)}/\overline{a}_2^{(2)} = -\frac{\sin\frac{3\pi}{n}}{2\cos\frac{\pi}{n}\cdot\sin^2\frac{\pi}{n}} - \operatorname{ctg}\frac{3\pi}{n} = -\frac{n}{\pi}\left(\frac{3}{2} - \frac{\pi^2}{n^2} - \frac{\pi^4}{5n^4} + O(n^{-6})\right) -$$
$$-\frac{n}{\pi}\left(\frac{1}{3} - \frac{\pi^2}{n^2} - \frac{3}{5}\frac{\pi^4}{n^4} + O(n^{-6})\right) = -\frac{n}{\pi}\left(\frac{11}{6} - 2\frac{\pi^2}{n^2} - \frac{4}{5}\frac{\pi^4}{n^4} + O(n^{-6})\right),$$
$$\left(\overline{a}_1^{(2)}/\overline{a}_2^{(2)}\right)\left(\overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right) = -\frac{n}{\pi}\cdot\frac{11}{54}\left(1 + \frac{52}{33}\frac{\pi^2}{n^2} + \frac{544}{495}\frac{\pi^4}{n^4} + O(n^{-6})\right).$$

Donc,

$$B_{kn} = \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1}\cdot\frac{k}{n}\left[-\frac{10}{9} - \frac{8}{27}\frac{\pi^2}{n^2} - \frac{40}{81}\frac{\pi^4}{n^4} + k^2\frac{\pi^2}{n^2} - \right.$$
$$-\frac{5}{12}k^4\frac{\pi^4}{n^4} + \frac{4}{3}k^2\frac{\pi^4}{n^4} + O(n^{-6}) +$$
$$+\frac{1}{2} - \frac{2}{3}\frac{\pi^2}{n^2} - \frac{8}{45}\frac{\pi^4}{n^4} - \frac{1}{12}k^2\frac{\pi^2}{n^2} + \frac{1}{240}k^4\frac{\pi^4}{n^4} + \frac{1}{9}k^2\frac{\pi^4}{n^4} +$$
$$\left. + \frac{11}{18} + \frac{26}{27}\frac{\pi^2}{n^2} + \frac{544}{810}\frac{\pi^4}{n^4} - \frac{11}{12}k^2\frac{\pi^2}{n^2} + \frac{33}{80}k^4\frac{\pi^4}{n^4} - \frac{13}{9}k^2\frac{\pi^4}{n^4} + O(n^{-6})\right] =$$
$$= \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1} O(n^{-7}) = \left(1 + \overline{a}_2^{(2)}/\overline{a}_2^{(1)}\right)^{-1} o(n^{-6}) = o(n^{-6}).$$

Le lemme 4.4 est démontré.

PROPOSITION 4.1. *On a* (voir (4.20)):

$$A_n = 1 - 2n^{-2}\mu_{4,n}/\mu_{2,n} = 1 - \frac{5}{2}\frac{\pi^2}{n^2} - \frac{7}{24}\frac{\pi^4}{n^4} + o(n^{-4}) = \tag{4.26}$$
$$= \cos\left[\frac{\sqrt{5}\pi}{n}\left(1 + \frac{4}{15}\frac{\pi^2}{n^2} + o(n^{-2})\right)\right].$$

DÉMONSTRATION. D'après (4.14),

$$A_n = 1 - 2n^{-2}\mu_{4,n}/\mu_{2,n} = 1 - (3 - 4\lambda^*_{1,n-4} + \lambda^*_{2,n-4})/2(1 - \lambda^*_{1,n-4}) =$$
$$= 1 - \frac{9}{4}\frac{\pi^4}{n^4}\left(1 + \frac{23}{30}\frac{\pi^2}{n^2} + o(n^{-2})\right)\Big/\frac{9}{10}\frac{\pi^2}{n^2}\left(1 + \frac{13}{20}\frac{\pi^2}{n^2} + \frac{431}{600}\frac{\pi^4}{n^4} + o(n^{-4})\right) =$$
$$= 1 - \frac{5}{2}\frac{\pi^2}{n^2} - \frac{7}{24}\frac{\pi^4}{n^4} + o(n^{-4}).$$

D'autre part, soit

$$1-\frac{5}{2}\frac{\pi^2}{n^2}-\frac{7}{24}\frac{\pi^4}{n^4}+o(n^{-4})=\cos\left[\sqrt{5}\frac{\pi}{n}\left(1+\beta\frac{\pi^2}{n^2}+o(n^{-2})\right)\right]=$$
$$=1-\frac{5}{2}\frac{\pi^2}{n^2}\left(1+2\beta\frac{\pi^2}{n^2}+o(n^{-2})\right)+\frac{25}{24}\frac{\pi^4}{n^4}+o(n^{-4})=$$
$$=1-\frac{5}{2}\frac{\pi^2}{n^2}+\left(\frac{25}{24}-5\beta\right)\frac{\pi^4}{n^4}+o(n^{-4}).$$

Alors, $\beta=4/15$. La proposition 4.1 est démontrée.

REMARQUE. L'exemple considéré montre que le terme oscillant du noyau n'est pas toujours présenté sous la forme $\cos t-\cos\gamma\frac{\pi}{n}$, où $\gamma=$ $=$ const. C'est pourquoi les opérateurs considérés dans les deux paragraphes précédents présentent un intérêt particulier.

5. Opérateurs extrémaux de classe S_4

Appliquons la construction de A. I. Kovalenko pour la fonction $\varphi(t)=$ $=\sin^5\pi t$ si $t\in[0,1]$.

LEMME 5.1. *Pour tout naturel $n\geqq 1$ on a:*

$$L_n(t)=\left|\sum_{k=0}^{n}\sin^5\frac{k\pi}{n}\cdot e^{ikt}\right|^2= \tag{5.1}$$

$$=\sin^{10}\frac{\pi}{n}\cdot\cos^2\frac{nt}{2}\cdot\left(\cos t-\cos\frac{\pi}{n}\right)^{-2}\left(\cos t-\cos\frac{3\pi}{n}\right)^{-2}\left(\cos t-\cos\frac{5\pi}{n}\right)^{-2}\times$$
$$\times\left[\cos^2 t-\cos\frac{\pi}{n}\left(16\sin^2\frac{\pi}{n}-13\right)\cdot\cos t+8\cos^2\frac{\pi}{n}\left(2-3\sin^2\frac{\pi}{n}\right)\right]^2.$$

DÉMONSTRATION. En tenant compte des formules

$$16\sin^5 x=\sin 5x-5\sin 3x+10\sin x,\quad \cos 3x=4\cos^3 x-3\cos x,$$
$$\sin 3x=3\sin x-4\sin^3 x,$$

on obtient

$$\sum_{k=0}^{n}\sin^5\frac{k\pi}{n}\cdot e^{ikt}=-\frac{i}{32}\sum_{k=0}^{n-1}\left(e^{ik\pi/n}-e^{-ik\pi/n}\right)^5\cdot e^{ikt}=$$
$$=\frac{1}{32}(1+e^{int})\left[\frac{\sin\frac{5\pi}{n}}{\cos t-\cos\frac{5\pi}{n}}-\frac{5\sin\frac{3\pi}{n}}{\cos t-\cos\frac{3\pi}{n}}+\frac{10\sin\frac{\pi}{n}}{\cos t-\cos\frac{\pi}{n}}\right]=$$

$$= \frac{1}{32}(1+e^{int})\left(\cos t-\cos\frac{\pi}{n}\right)^{-1}\left(\cos t-\cos\frac{3\pi}{n}\right)^{-1}\left(\cos t-\cos\frac{5\pi}{n}\right)^{-1}\times$$
$$\times\left\{16\sin^5\frac{\pi}{n}\cdot\cos^2 t-2\left[\cos\frac{2\pi}{n}\cdot\cos\frac{\pi}{n}\cdot 16\sin^5\frac{\pi}{n}+\right.\right.$$
$$+5\sin\frac{3\pi}{n}\cdot\cos\frac{2\pi}{n}\left(\cos\frac{\pi}{n}-\cos\frac{3\pi}{n}\right)-$$
$$\left.-10\sin\frac{\pi}{n}\cos\frac{\pi}{n}\left(\cos\frac{2\pi}{n}-\cos\frac{4\pi}{n}\right)\right]\cos t+\left[16\sin^5\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cos\frac{3\pi}{n}+\right.$$
$$\left.\left.+20\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{3\pi}{n}\left(\cos\frac{\pi}{n}\cdot\cos\frac{2\pi}{n}-\cos\frac{3\pi}{n}\right)\right]\right\}=$$
$$= e^{int/2}\cdot\cos\frac{nt}{2}\cdot\sin^5\frac{\pi}{n}\cdot\left[\cos^2 t-\cos\frac{\pi}{n}\cdot\left(16\sin^2\frac{\pi}{n}-13\right)\cos t+\right.$$
$$\left.+8\cos^2\frac{\pi}{n}\left(2-3\sin^2\frac{\pi}{n}\right)\right]\times$$
$$\times\left(\cos t-\cos\frac{\pi}{n}\right)^{-1}\left(\cos t-\cos\frac{3\pi}{n}\right)^{-1}\left(\cos t-\cos\frac{5\pi}{n}\right)^{-1},$$

d'où il découle (5.1).

LEMME 5.2. *Les fonctions*

$$(5.2)\quad \begin{cases} K_n(t)=\left(n^{12}\sin^{10}\frac{\pi}{n}\right)^{-1}L_n(t) \qquad et \\ \varphi_2(x)=57600\cos^2\frac{x}{2}\Big/(x^2-\pi^2)^2(x^2-9\pi^2)^2(x^2-25\pi^2)^2 \end{cases}$$

vérifient les conditions de J. Szabados (0.4) *et* (0.5) *pour* $m=2$.

La démonstration de ce lemme est identique à celle du lemme 2.2.

LEMME 5.3. *Soit*

$$(5.3)\quad g_n(z)=\frac{(\cos z-\cos\alpha)(\cos z-\cos\beta)}{\left(\cos z-\cos\frac{\pi}{n}\right)^2\left(\cos z-\cos\frac{3\pi}{n}\right)^2\left(\cos z-\cos\frac{5\pi}{n}\right)^2}\times$$
$$\times\left[\cos^2 z-\cos\frac{\pi}{n}\left(16\sin^2\frac{\pi}{n}-13\right)\cos z+8\cos^2\frac{\pi}{n}\left(2-3\sin^2\frac{\pi}{n}\right)\right]^2,$$

où $\alpha,\beta\in\mathbf{R}$, $n\in\mathbf{N}\setminus\{0\}$.

Alors, en posant $\Delta_n=\{x=2k\pi\pm(2i-1)\pi/n;\ i=1,2,3;\ k\in\mathbf{Z}\}$, *on a pour* $n\geqq 5$ (voir (1.1)):

$$(5.4)\ g_n(z)=\sum_{k\in\mathbf{Z}}\sum_{i=1}^{3}\sum_{s=1}^{2}\left\{\frac{a_{si}}{\left[z-\frac{(2i-1)\pi}{n}-2k\pi\right]^s}+\frac{(-1)^s a_{si}}{\left[z+\frac{(2i-1)\pi}{n}-2k\pi\right]^s}\right\},$$
$$z\in\mathbf{C}\setminus\Delta_n,$$

où

$$a_{21} = 25\left(\cos\frac{\pi}{n} - \cos\alpha\right)\left(\cos\frac{\pi}{n} - \cos\beta\right)\Big/ 64\sin^{10}\frac{\pi}{n}, \tag{5.5}$$

$$a_{11} = a_{21}\left[-\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\alpha} - \frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\beta} + \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{2\cos\frac{\pi}{n}\cdot\sin\frac{3\pi}{n}} - \right.$$
$$\left. - \operatorname{ctg}\frac{\pi}{n} - \operatorname{tg}\frac{\pi}{n}\cdot\frac{15 - 16\sin^2\frac{\pi}{n}}{5\left(3 - 4\sin^2\frac{\pi}{n}\right)}\right], \tag{5.6}$$

$$a_{22} = 25\left(\cos\frac{3\pi}{n} - \cos\alpha\right)\left(\cos\frac{3\pi}{n} - \cos\beta\right)\Big/ 256\sin^{10}\frac{\pi}{n}, \tag{5.7}$$

$$a_{12} = a_{22}\left[-\frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n} - \cos\alpha} - \frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n} - \cos\beta} - \frac{3 - 4\sin^2\frac{\pi}{n}}{2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}} + \right.$$
$$\left. + \frac{3 - 4\sin^2\frac{\pi}{n}}{4\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cdot\cos\frac{2\pi}{n}} - \operatorname{ctg}\frac{3\pi}{n} + \operatorname{tg}\frac{\pi}{n}\cdot\frac{3}{10}\cdot\frac{5 - 8\sin^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n} - 1/2}\right], \tag{5.8}$$

$$a_{23} = \left(\cos\frac{5\pi}{n} - \cos\alpha\right)\left(\cos\frac{5\pi}{n} - \cos\beta\right)\Big/ 256\sin^{10}\frac{\pi}{n}, \tag{5.9}$$

$$a_{13} = a_{23}\left[-\frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n} - \cos\alpha} - \frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n} - \cos\beta} - \right.$$
$$- \frac{16\cos^4\frac{\pi}{n} - 12\cos^2\frac{\pi}{n} + 1}{2\sin\frac{3\pi}{n}\cdot\cos\frac{\pi}{n}} - \frac{16\cos^4\frac{\pi}{n} - 12\cos^2\frac{\pi}{n} + 1}{\sin\frac{4\pi}{n}} -$$
$$\left. - \operatorname{ctg}\frac{5\pi}{n} - \operatorname{tg}\frac{\pi}{n}\cdot\frac{32\cos^4\frac{\pi}{n} - 24\cos^2\frac{\pi}{n} + 7}{\left(2\cos^2\frac{\pi}{n} - 1\right)\left(4\cos^2\frac{\pi}{n} - 1\right)}\right]. \tag{5.10}$$

DÉMONSTRATION. Soit

$$g_n(z) = \mu_n(z)\Big/\left[2\sin\left(\frac{z}{2} - \frac{\pi}{2n}\right)\right]^2 = \nu_n(z)\Big/\left[2\sin\left(\frac{z}{2} - \frac{3\pi}{2n}\right)\right]^2 =$$
$$= \pi_n(z)\Big/\left[2\sin\left(\frac{z}{2} - \frac{5\pi}{2n}\right)\right]^2.$$

Alors, d'après la proposition 1.1 et la formule (1.3), on a:

$$a_{21} = \mu_n(\pi/n), \quad a_{11} = \mu_n'(\pi/n), \quad a_{22} = \nu_n(3\pi/n), \quad a_{12} = \nu_n'(3\pi/n),$$
$$a_{23} = \pi_n(5\pi/n), \quad a_{13} = \pi_n'(5\pi/n).$$

On en déduit les expressions (5.5)–(5.10) en tenant compte des égalités suivantes:

$$\cos 3x = \cos x(1 - 4\sin^2 x),$$

$$\sin 3x = \sin x(3 - 4\sin^2 x) = \sin x(4\cos^2 x - 1),$$
$$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x =$$
$$= \cos x(16\cos^4 x - 20\cos^2 x + 5),$$
$$\sin 5x = \sin x(16\cos^4 x - 12\cos^2 x + 1),$$
$$3 - 10\sin^2 x + 8\sin^4 x = (4\sin^2 x - 3)(2\sin^2 x - 1) =$$
$$= -\sin 3x \cdot \cos 2x / \sin x,$$
$$128\cos^8 x - 192\cos^6 x + 96\cos^4 x - 18\cos^2 x + 1 =$$
$$= (2\cos^2 x - 1)(4\cos^2 x - 1)(16\cos^4 x - 12\cos^2 x + 1) =$$
$$= \cos 2x \cdot \sin 3x \cdot \sin 5x / \sin^2 x.$$

LEMME 5.4. *Soit* (voir (5.3))

$$(5.11)\qquad L_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)U_n(t-x)\,dt, \qquad f \in C_{2\pi},\ x \in \mathbf{R}\ (n \geqq 5),$$

où

$$(5.12)\qquad U_n(t) = \pi B_n^{-1}\cos^2\frac{nt}{2}\cdot g_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\lambda_{k,n}\cos kt,$$

$$(5.13)\qquad B_n = \int_{-\pi}^{\pi}\cos^2\frac{nt}{2}\cdot g_n(t)\,dt,$$

Alors (voir (5.5)–(5.10))

$$(5.14) B_n = n\pi(a_{21} + a_{22} + a_{23}) = n\pi\Big[100\Big(\cos\frac{\pi}{n} - \cos\alpha\Big)\Big(\cos\frac{\pi}{n} - \cos\beta\Big) +$$
$$+25\Big(\cos\frac{3\pi}{n} - \cos\alpha\Big)\Big(\cos\frac{3\pi}{n} - \cos\beta\Big) +$$
$$+\Big(\cos\frac{5\pi}{n} - \cos\alpha\Big)\Big(\cos\frac{5\pi}{n} - \cos\beta\Big)\Big]\Big/256\sin^{10}\frac{\pi}{n}$$

et

$$(5.15)\qquad \lambda_{k,n} = (a_{21} + a_{22} + a_{23})^{-1}\times$$
$$\times\Big[(1 - k/n)\Big(a_{21}\cos\frac{k\pi}{n} + a_{22}\cos\frac{3k\pi}{n} + a_{23}\cos\frac{5k\pi}{n}\Big) -$$
$$-n^{-1}\Big(a_{11}\sin\frac{k\pi}{n} + a_{12}\sin\frac{3k\pi}{n} + a_{13}\sin\frac{5k\pi}{n}\Big)\Big].$$

DÉMONSTRATION. Posons

$$\Theta_*(z) = \sum_{i=1}^{3} \sum_{s=1}^{2} \left\{ \frac{a_{si}}{[z-(2i-1)\pi/n]^s} + \frac{(-1)^s a_{si}}{[z+(2i-1)\pi/n]^s} \right\}, \tag{5.16}$$

et

$$F_k(z) = 2^{-1}\left[e^{ikz} + 2^{-1}\left(e^{i(n-k)z} + e^{i(n+k)z}\right)\right] \qquad (k \in \mathbf{N}),$$

d'où $F_k(m\pi/n) = \frac{i}{2}\sin\frac{mk\pi}{n}$, $F_k'(m\pi/n) = -\frac{i}{2}(n-k)\cos\frac{mk\pi}{n}$. Donc,

$$\operatorname{Res}\big((2j-1)\pi/n; F_k\cdot\Theta_*\big) =$$
$$= -\frac{i}{2}a_{2j}(n-k)\cos\frac{(2j-1)k\pi}{n} + \frac{i}{2}a_{1j}\sin\frac{(2j-1)k\pi}{n}, \quad j = 1,2,3.$$

D'après les formules (1.9), (1.10) et (5.5)–(5.10), on obtient les égalités (5.14) et (5.15). Le lemme 5.4 est démontré.

LEMME 5.5. *Pour que dans* (5.15) *on ait*

$$\lambda_{k,n} = 1 + O(n^{-6}) \qquad (n \to \infty), \tag{5.17}$$

il faut et il suffit que $\alpha = \pm\sqrt{\left(35-8\sqrt{7}\right)/3}\frac{\pi}{n} \approx \pm 2{,}15\frac{\pi}{n}$ *et* $\beta = \pm\sqrt{\left(35+8\sqrt{7}\right)/3}\frac{\pi}{n} \approx \pm 4{,}33\frac{\pi}{n}$ $(-\pi < \alpha,\ \beta \leqq \pi)$.

DÉMONSTRATION. Posons $\alpha = \gamma\frac{\pi}{n}$ et $\beta = \delta\frac{\pi}{n}$. On aura:

$$\frac{\cos\frac{3\pi}{n} - \cos\gamma\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}} =$$
$$= \frac{\gamma^2-9}{\gamma^2-1}\left[1 - (\gamma^2+9)\frac{\pi^2}{12n^2} + \frac{\gamma^4+9\gamma^2+81}{360n^4}\pi^4 + O(n^{-6})\right] \times$$
$$\times \left[1 + \frac{\gamma^2+1}{12}\cdot\frac{\pi^2}{n^2} + \frac{3\gamma^4+8\gamma^2+3}{720}\cdot\frac{\pi^4}{n^4} + O(n^{-6})\right] =$$
$$= \frac{\gamma^2-9}{\gamma^2-1}\left[1 - \frac{2}{3}\frac{\pi^2}{n^2} - \frac{\gamma^2-5}{30}\cdot\frac{\pi^4}{n^4} + O(n^{-6})\right].$$

Donc,

$$a_{22}/a_{21} = \frac{1}{4}\cdot\frac{\cos\frac{3\pi}{n} - \cos\gamma\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\gamma\frac{\pi}{n}}\cdot\frac{\cos\frac{3\pi}{n} - \cos\delta\frac{\pi}{n}}{\cos\frac{\pi}{n} - \cos\delta\frac{\pi}{n}} = \tag{5.18}$$
$$= \frac{1}{4}\cdot\frac{\gamma^2-9}{\gamma^2-1}\cdot\frac{\delta^2-9}{\delta^2-1}\left[1 - \frac{4}{3}\frac{\pi^2}{n^2} + \left(\frac{7}{9} - \frac{\gamma^2+\delta^2}{30}\right)\frac{\pi^4}{n^4} + O(n^{-6})\right].$$

D'une manière analogue on obtient:

$$(5.19)\qquad a_{23}/a_{21} = \frac{1}{100}\cdot\frac{\cos\frac{5\pi}{n}-\cos\gamma\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\gamma\frac{\pi}{n}}\cdot\frac{\cos\frac{5\pi}{n}-\cos\delta\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\delta\frac{\pi}{n}} =$$
$$= \frac{1}{100}\cdot\frac{\gamma^2-25}{\gamma^2-1}\cdot\frac{\delta^2-25}{\delta^2-1}\left[1-\frac{4\pi^2}{n^2}+\left(\frac{107}{15}-\frac{\gamma^2+\delta^2}{10}\right)\frac{\pi^4}{n^4}+O(n^{-6})\right].$$

Considerons l'expression:

$$(5.20)\qquad D_{kn} = \left[1+(a_{22}/a_{21})+(a_{23}/a_{21})\right]^{-1}\times$$
$$\times\left[\cos\frac{k\pi}{n}+(a_{22}/a_{21})\cos\frac{3k\pi}{n}+(a_{23}/a_{21})\cos\frac{5k\pi}{n}\right].$$

Il est facile de voir que la différence $\lambda_{k,n}-D_{kn}$ est une expression impaire par rapport à π/n. En posant

$$(5.21)\qquad A=\frac{1}{4}\cdot\frac{\gamma^2-9}{\gamma^2-1}\cdot\frac{\delta^2-9}{\delta^2-1}\quad\text{et}\quad B=\frac{1}{100}\cdot\frac{\gamma^2-25}{\gamma^2-1}\cdot\frac{\delta^2-25}{\delta^2-1},$$

nous aurons:

$$(5.22)\qquad D_{kn} = \left[1-\frac{k^2\pi^2}{2n^2}+\frac{k^4\pi^4}{24n^4}+O(n^{-6})+\right.$$
$$+A\left(1-\frac{4}{3}\frac{\pi^2}{n^2}+\left(\frac{7}{9}-\frac{\gamma^2+\delta^2}{30}\right)\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$
$$\times\left(1-\frac{9k^2\pi^2}{2n^2}+\frac{81k^4\pi^4}{24n^4}+O(n^{-6})\right)+$$
$$+B\left(1-\frac{4\pi^2}{n^2}+\left(\frac{107}{15}-\frac{\gamma^2+\delta^2}{10}\right)\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$
$$\left.\times\left(1-\frac{25k^2\pi^2}{2n^2}+\frac{625k^4\pi^4}{24n^4}+O(n^{-6})\right)\right]\Big/$$
$$\Big/\left[1+A\left(1-\frac{4}{3}\frac{\pi^2}{n^2}+\left(\frac{7}{9}-\frac{\gamma^2+\delta^2}{30}\right)\frac{\pi^4}{n^4}+O(n^{-6})\right)+\right.$$
$$\left.+B\left(1-\frac{4\pi^2}{n^2}+\left(\frac{107}{15}-\frac{\gamma^2+\delta^2}{10}\right)\frac{\pi^4}{n^4}+O(n^{-6})\right)\right]=$$
$$=\Bigg\{1+(1+A+B)^{-1}\left[\left(-\frac{k^2}{2}-\frac{4}{3}A-\frac{9k^2}{2}A-4B-\frac{25k^2}{2}B\right)\frac{\pi^2}{n^2}+\right.$$
$$+\left(\frac{k^4}{24}+A\left[\left(\frac{7}{9}-\frac{\gamma^2+\delta^2}{30}\right)+\frac{81k^4}{24}+6k^2\right]+\right.$$

$$+ \; B\left[\left(\frac{107}{15}-\frac{\gamma^2+\delta^2}{10}\right)+\frac{625k^4}{24}+50k^2\right]\Big)\frac{\pi^4}{n^4}+O(n^{-6})\Big]\Big\}\times$$

$$\times\Big\{1-(1+A+B)^{-1}\Big[\left(-\frac{4}{3}A-4B\right)\frac{\pi^2}{n^2}+$$

$$+\left[(1+A+B)^{-1}\left(-\frac{4}{3}A-4B\right)^2-\frac{7}{9}A-\frac{107}{15}B+\right.$$

$$\left.+\frac{\gamma^2+\delta^2}{30}(A+3B)\right]\frac{\pi^4}{n^4}+O(n^{-6})\Big]\Big\}=$$

$$=1+(1+A+B)^{-1}\left(-\frac{k^2}{2}-\frac{9k^2}{2}A-\frac{25k^2}{2}B\right)\frac{\pi^2}{n^2}+$$

$$+(1+A+B)^{-1}\left\{\frac{k^4}{24}+A\left(\frac{7}{9}-\frac{\gamma^2+\delta^2}{30}+\frac{81k^4}{24}+6k^2\right)+\right.$$

$$+B\left(\frac{107}{15}-\frac{\gamma^2+\delta^2}{10}+\frac{625k^4}{24}+50k^2\right)-$$

$$-\frac{7}{9}A-\frac{107}{15}B+\frac{\gamma^2+\delta^2}{30}(A+3B)+$$

$$\left.+(1+A+B)^{-1}\left(-\frac{4}{3}A-4B\right)\frac{k^2}{2}(1+9A+25B)\right\}\frac{\pi^4}{n^4}+O(n^{-6})=$$

$$=1-(1+A+B)^{-1}\frac{k^2}{2}(1+9A+25B)\frac{\pi^2}{n^2}+$$

$$+(1+A+B)^{-1}\left\{\frac{k^4}{24}(1+81A+625B)+2k^2(3A+25B)+\right.$$

$$\left.+(1+A+B)^{-1}\left(-\frac{4}{3}A-4B\right)\frac{k^2}{2}(1+9A+25B)\right\}\frac{\pi^4}{n^4}+O(n^{-6}).$$

Supposons que $1+A+B\neq 0$; alors

$$(5.23)\; D_{kn}=1+O(n^{-6})\iff 1+9A+25B=0,\quad 1+81A+625B=0,$$
$$3A+25B=0\iff A=-1/6,\quad B=1/50.$$

D'après (5.22), cela est aussi vrai lorsque $1+A+B=0$. Alors, en vertu de (5.21), pour ces valeurs des A et B on a:

$$5\gamma^2\delta^2-29(\gamma^2+\delta^2)=-245,\qquad \gamma^2\delta^2+23(\gamma^2+\delta^2)=623.$$

Donc,

(5.24)

$$D_{kn}=1+O(n^{-6})\iff\gamma^2=(35-8\sqrt{7})/3,\ \delta^2=(35+8\sqrt{7})/3\quad(\gamma^2<\delta^2).$$

Maintenant, vérifions que pour les γ et δ indiqués on a:

$$(5.25)\qquad \lambda_{k,n}-(1-k/n)D_{kn}=k/n+O(n^{-6})\qquad (n\to\infty).$$

On aura:

$$(5.26)\qquad a_{22}/a_{21}=-\frac{1}{6}+\frac{2}{9}\frac{\pi^2}{n^2}+O(n^{-6}),$$

$$a_{23}/a_{21}=\frac{1}{50}-\frac{2}{25}\frac{\pi^2}{n^2}+\frac{24}{250}\frac{\pi^4}{n^4}+O(n^{-6}),$$

$$1+(a_{22}/a_{21})+(a_{23}/a_{21})=\frac{64}{75}\left(1+\frac{1}{6}\frac{\pi^2}{n^2}+\frac{9}{80}\frac{\pi^4}{n^4}+O(n^{-6})\right),$$

$$\left[1+(a_{22}/a_{21})+(a_{23}/a_{21})\right]^{-1}=\frac{75}{64}\left(1-\frac{1}{6}\frac{\pi^2}{n^2}-\frac{61}{720}\frac{\pi^4}{n^4}+O(n^{-6})\right).$$

De plus,

$$(5.27)\qquad -\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\delta\frac{\pi}{n}}=$$

$$=-\frac{n}{\pi}\left(\frac{2}{\gamma^2-1}+\frac{2}{\delta^2-1}\right)-\frac{\pi}{3n}-\left(\frac{1}{180}+\frac{\gamma^2+\delta^2}{120}\right)\frac{\pi^3}{n^3}+O(n^{-5})=$$

$$=-\frac{2}{3}\frac{n}{\pi}-\frac{\pi}{3n}-\frac{\pi^3}{5n^3}+O(n^{-5}),$$

$$-\frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n}-\cos\delta\frac{\pi}{n}}=$$

$$=-\frac{n}{\pi}\left(\frac{6}{\gamma^2-9}+\frac{6}{\delta^2-9}\right)-\frac{\pi}{n}-\frac{3\pi^3}{n^3}\left(\frac{1}{20}+\frac{\gamma^2+\delta^2}{120}\right)+O(n^{-5})=$$

$$=\frac{3}{4}\frac{n}{\pi}-\frac{\pi}{n}-\frac{11}{15}\frac{\pi^3}{n^3}+O(n^{-5}),$$

$$-\frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n}-\cos\delta\frac{\pi}{n}}=$$

$$=-\frac{n}{\pi}\left(\frac{10}{\gamma^2-25}+\frac{10}{\delta^2-25}\right)-\frac{5}{3}\frac{\pi}{n}-\frac{\pi^3}{n^3}\left(\frac{25}{36}+\frac{\gamma^2+\delta^2}{24}\right)+O(n^{-5})=$$

$$=\frac{25}{12}\frac{n}{\pi}-\frac{5}{3}\frac{\pi}{n}-\frac{5}{3}\frac{\pi^3}{n^3}+O(n^{-5}).$$

Ensuite,

$$(5.28)\, T_1=a_{11}/a_{21}=-\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}-\cos\delta\frac{\pi}{n}}+\frac{1}{\sin\frac{2\pi}{n}}-\operatorname{ctg}\frac{\pi}{n}+$$

$$+\frac{1}{2\cos\frac{\pi}{n}\cdot\sin\frac{3\pi}{n}}-\operatorname{tg}\frac{\pi}{n}\cdot\frac{15-16\sin^2\frac{\pi}{n}}{15-20\sin^2\frac{\pi}{n}}=$$

$$=-\frac{2}{3}\frac{n}{\pi}-\frac{\pi}{3n}-\frac{\pi^3}{5n^3}+O(n^{-5})+\frac{1}{2}\frac{n}{\pi}+\frac{\pi}{3n}+\frac{7}{45}\frac{\pi^3}{n^3}+O(n^{-5})-$$
$$-\frac{n}{\pi}+\frac{\pi}{3n}+\frac{1}{45}\frac{\pi^3}{n^3}+O(n^{-5})+\frac{1}{6}\frac{n}{\pi}+\frac{\pi}{3n}+\frac{19}{45}\frac{\pi^3}{n^3}+O(n^{-5})-$$
$$-\frac{\pi}{n}-\frac{3}{5}\frac{\pi^3}{n^3}-\frac{22}{45}\frac{\pi^5}{n^5}+O(n^{-7})=-\frac{n}{\pi}-\frac{\pi}{3n}-\frac{\pi^3}{5n^3}+O(n^{-5}).$$

(5.29)
$$T_2=a_{12}/a_{22}=-\frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{3\pi}{n}}{\cos\frac{3\pi}{n}-\cos\delta\frac{\pi}{n}}-\frac{3-4\sin^2\frac{\pi}{n}}{2\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}}+$$
$$+\frac{3-4\sin^2\frac{\pi}{n}}{4\sin\frac{\pi}{n}\cdot\cos\frac{\pi}{n}\cdot\cos\frac{2\pi}{n}}-\operatorname{ctg}\frac{3\pi}{n}+\operatorname{tg}\frac{\pi}{n}\cdot\frac{3}{10}\cdot\frac{5-8\sin^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}-1/2}=$$
$$=\frac{3}{4}\frac{n}{\pi}-\frac{\pi}{n}-\frac{11}{15}\frac{\pi^3}{n^3}+O(n^{-5})-\frac{3}{2}\frac{n}{\pi}+\frac{\pi}{n}+\frac{\pi^3}{5n^3}+O(n^{-5})+$$
$$+\frac{3}{4}\frac{n}{\pi}+\frac{\pi}{n}+\frac{7}{5}\frac{\pi^3}{n^3}+O(n^{-5})-\frac{1}{3}\cdot\frac{n}{\pi}+\frac{\pi}{n}+\frac{3}{5}\frac{\pi^3}{n^3}+O(n^{-5})-$$
$$-3\frac{\pi}{n}-\frac{11}{5}\frac{\pi^3}{n^3}+O(n^{-5})=-\frac{1}{3}\cdot\frac{n}{\pi}-\frac{\pi}{n}-\frac{11}{15}\frac{\pi^3}{n^3}+O(n^{-5}).$$

(5.30)
$$-\operatorname{tg}\frac{\pi}{n}\cdot\frac{32\cos^4\frac{\pi}{n}-24\cos^2\frac{\pi}{n}+7}{\left(2\cos^2\frac{\pi}{n}-1\right)\left(4\cos^2\frac{\pi}{n}-1\right)}=$$
$$=-\frac{\pi}{n}\left(1+\frac{\pi^2}{3n^2}+\frac{2}{15}\frac{\pi^4}{n^4}+O(n^{-6})\right)\cdot 15\left(1-\frac{8}{3}\frac{\pi^2}{n^2}+\frac{136}{45}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$
$$\times(1/3)\left(1+\frac{10}{3}\frac{\pi^2}{n^2}+\frac{22}{3}\frac{\pi^4}{n^4}+O(n^{-6})\right)=$$
$$=-5\frac{\pi}{n}-5\frac{\pi^3}{n^3}-\frac{82}{9}\frac{\pi^5}{n^5}+O(n^{-7});$$

(5.31)
$$T_3=a_{13}/a_{23}=-\frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n}-\cos\gamma\frac{\pi}{n}}-\frac{\sin\frac{5\pi}{n}}{\cos\frac{5\pi}{n}-\cos\delta\frac{\pi}{n}}-$$
$$-\frac{16\cos^4\frac{\pi}{n}-12\cos^2\frac{\pi}{n}+1}{2\sin\frac{3\pi}{n}\cdot\cos\frac{\pi}{n}}-\frac{16\cos^4\frac{\pi}{n}-12\cos^2\frac{\pi}{n}+1}{\sin\frac{4\pi}{n}}-\operatorname{ctg}\frac{5\pi}{n}-$$

$$-\operatorname{tg}\frac{\pi}{n}\cdot\frac{32\cos^4\frac{\pi}{n}-24\cos^2\frac{\pi}{n}+7}{\left(2\cos^2\frac{\pi}{n}-1\right)\left(4\cos^2\frac{\pi}{n}-1\right)}=$$

$$=\frac{25}{12}\frac{n}{\pi}-\frac{5}{3}\frac{\pi}{n}-\frac{5}{3}\frac{\pi^3}{n^3}+O(n^{-5})-\frac{5}{6}\cdot\frac{n}{\pi}+\frac{5}{3}\frac{\pi}{n}+\frac{7}{9}\frac{\pi^3}{n^3}+O(n^{-5})-$$

$$-\frac{5}{4}\frac{n}{\pi}+\frac{5}{3}\frac{\pi}{n}+\frac{13}{9}\frac{\pi^3}{n^3}+O(n^{-5})-\frac{1}{5}\cdot\frac{n}{\pi}+\frac{5}{3}\frac{\pi}{n}+\frac{25}{9}\frac{\pi^3}{n^3}+O(n^{-5})-$$

$$-5\frac{\pi}{n}-5\frac{\pi^3}{n^3}+O(n^{-5})=-\frac{1}{5}\cdot\frac{n}{\pi}-\frac{5}{3}\frac{\pi}{n}-\frac{5}{3}\frac{\pi^3}{n^3}+O(n^{-5}).$$

D'après les formules (5.15), (5.24), (5.26), (5.28), (5.29) et (5.31), on obtient:

$$(5.32)\quad \lambda_{k,n}=1-\frac{k}{n}+O(n^{-6})-n^{-1}\left[1+(a_{22}/a_{21})+(a_{23}/a_{21})\right]^{-1}\times$$

$$\times\left[T_1\sin\frac{k\pi}{n}+(a_{22}/a_{21})T_2\sin\frac{3k\pi}{n}+(a_{23}/a_{21})T_3\sin\frac{5k\pi}{n}\right]=$$

$$=1-\frac{k}{n}-n^{-1}\cdot\frac{75}{64}\left(1-\frac{1}{6}\frac{\pi^2}{n^2}-\frac{61}{720}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times k\left\{-\left(1+\frac{1}{3}\frac{\pi^2}{n^2}+\frac{1}{5}\frac{\pi^4}{n^4}+O(n^{-6})\right)\left(1-\frac{k^2\pi^2}{6n^2}+\frac{k^4\pi^4}{120n^4}+O(n^{-6})\right)-\right.$$

$$-\frac{1}{6}\left(1-\frac{4}{3}\frac{\pi^2}{n^2}+O(n^{-6})\right)(-1/3)\left(1+3\frac{\pi^2}{n^2}+\frac{11}{5}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times 3\left(1-\frac{3k^2\pi^2}{2n^2}+\frac{27k^4\pi^4}{40n^4}+O(n^{-6})\right)+$$

$$+\frac{1}{50}\left(1-\frac{4\pi^2}{n^2}+\frac{24}{5}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times(-1/5)\left(1+\frac{25}{3}\frac{\pi^2}{n^2}+\frac{25}{3}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\left.\times 5\left(1-\frac{25k^2\pi^2}{6n^2}+\frac{125k^4\pi^4}{24n^4}+O(n^{-6})\right)\right\}+O(n^{-6})=$$

$$=1-\frac{k}{n}-\frac{k}{n}\cdot\frac{75}{64}\left(1-\frac{\pi^2}{6n^2}-\frac{61}{720}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times\left\{-1-\frac{\pi^2}{3n^2}-\frac{\pi^4}{5n^4}+\frac{k^2\pi^2}{6n^2}-\frac{k^4\pi^4}{120n^4}+\frac{k^2\pi^4}{18n^4}+\right.$$

$$+\frac{1}{6}\left(1+\frac{5}{3}\frac{\pi^2}{n^2}-\frac{9}{5}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times\left(1-\frac{3}{2}\frac{k^2\pi^2}{n^2}+\frac{27}{40}\frac{k^4\pi^4}{n^4}+O(n^{-6})\right)-$$

$$-\frac{1}{50}\left(1+\frac{13}{3}\frac{\pi^2}{n^2}-\frac{101}{5}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times\left(1-\frac{25}{6}\frac{k^2\pi^2}{n^2}+\frac{125}{24}\frac{k^4\pi^4}{n^4}+O(n^{-6})\right)\Bigg\}+O(n^{-6})=$$

$$=1-\frac{k}{n}-\frac{k}{n}\cdot\frac{75}{64}\left(1-\frac{\pi^2}{6n^2}-\frac{61}{720}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times\Bigg\{-1-\frac{\pi^2}{3n^2}-\frac{\pi^4}{5n^4}+\frac{k^2\pi^2}{6n^2}-\frac{k^4\pi^4}{120n^4}+\frac{k^2\pi^4}{18n^4}+\frac{1}{6}+$$

$$+\frac{5}{18}\frac{\pi^2}{n^2}-\frac{3}{10}\frac{\pi^4}{n^4}-\frac{k^2\pi^2}{4n^2}+\frac{9}{80}\cdot\frac{k^4\pi^4}{n^4}-\frac{5}{12}\frac{k^2\pi^4}{n^4}-\frac{1}{50}-$$

$$-\frac{13}{150}\frac{\pi^2}{n^2}+\frac{101}{250}\frac{\pi^4}{n^4}+\frac{k^2\pi^2}{12n^2}-\frac{5}{48}\frac{k^4\pi^4}{n^4}+\frac{13}{36}\frac{k^2\pi^4}{n^4}\Bigg\}+O(n^{-6})=$$

$$=1-\frac{k}{n}+\frac{k}{n}\left(1-\frac{\pi^2}{6n^2}-\frac{61}{720}\frac{\pi^4}{n^4}+O(n^{-6})\right)\times$$

$$\times\left(1+\frac{\pi^2}{6n^2}+\frac{9}{80}\frac{\pi^4}{n^4}+O(n^{-6})\right)+O(n^{-6})=$$

$$=1-\frac{k}{n}+\frac{k}{n}\left(1+O(n^{-6})\right)+O(n^{-6})=1+O(n^{-6}).$$

Le lemme 5.5 est démontré.

THÉORÈME 5.1. *Pour les opérateurs* (5.11)–(5.13) *avec* $\alpha = \pm\sqrt{(35-8\sqrt{7})/3}\frac{\pi}{n}$ *et* $\beta = \pm\sqrt{(35+8\sqrt{7})/3}\frac{\pi}{n}$ *et pour toute fonction* $f(x)\in C_{2\pi}$ *telle que* $f^{(6)}(x)\in C_{2\pi}$ *l'égalité suivante aura lieu:*

(5.33)

$$\lim_{n\to\infty} n^6\left[L_n(f;x)-f(x)\right] = \frac{5}{16}\pi^6\left[f^{(6)}(x)+5f^{(4)}(x)+4f^{(2)}(x)\right],\quad x\in\mathbf{R}.$$

DÉMONSTRATION. Soit

$$\Phi_k(z)=28800z^{2k}(1+e^{iz})/2^{2k}\pi(z^2-\pi^2)^2(z^2-9\pi^2)^2(z^2-25\pi^2)^2,$$
$$z\in\mathbf{C},\ k\in\mathbf{N}.$$

Alors

$$\operatorname{Res}(\pm\pi;\Phi_k)=-i\frac{100}{512\pi^{11}}\left(\frac{\pi}{2}\right)^{2k},\quad \operatorname{Res}(\pm3\pi;\Phi_k)=-i\frac{25}{512\pi^{11}}\left(\frac{3\pi}{2}\right)^{2k},$$

$$\operatorname{Res}(\pm5\pi;\Phi_k)=-i\frac{1}{512\pi^{11}}\left(\frac{5\pi}{2}\right)^{2k}.$$

Ensuite, en appliquant le théorème des résidus, comme nous avons fait dans la démonstration de la proposition 1.2, nous aurons (voir (0.12) et (5.2)):

$$(5.34)\qquad \mu_{2k} = \frac{2^{1-2k}}{\pi}\int_0^{+\infty} x^{2k}\varphi_2(x)\,dx = \operatorname{Re}\text{v.p.}\int_{-\infty}^{+\infty}\Phi_k(x)\,dx =$$

$$= \operatorname{Re}\left[\pi i\sum_{j=0}^{5}\operatorname{Res}(-5\pi+2j\pi;\Phi_k)\right] = 2^{-8}\pi^{-10}(\pi/2)^{2k}(100+25\cdot 3^{2k}+5^{2k}),$$

$$k = 0,1,2,3,4,5.$$

Donc, en posant $K = 50\cdot 2^{-8}\pi^{-10}$, on a:

$$\mu_0 = 2{,}52K,\quad \mu_2 = 7K(\pi/2)^2,\quad \mu_4 = 55K(\pi/2)^4,$$

$$\mu_6 = 679K(\pi/2)^6,\quad \mu_8 = 11095K(\pi/2)^8,\quad \mu_{10} = 224839K(\pi/2)^{10}.$$

Soit $(\lambda_0,\lambda_1,\lambda_2)$ la solution du système d'équations (0.13) (pour $m=2$). Alors, il est facile de voir que

$$S = \mu_6\lambda_0+\mu_8\lambda_1+\mu_{10}\lambda_2 = \frac{\pi^6}{64}(679\Lambda_0+11095\Lambda_1+224839\Lambda_2),$$

où $(\Lambda_0,\Lambda_1,\Lambda_2)$ est la solution du système d'équations suivant:

$$(5.35)\qquad \begin{cases} 2{,}52\Lambda_0+\ \ 7\Lambda_1+\ \ \ \ 55\Lambda_2 = 1,\\ 7\Lambda_0+\ 55\Lambda_1+\ \ 679\Lambda_2 = 0,\\ 55\Lambda_0+679\Lambda_1+11095\Lambda_2 = 0.\end{cases}$$

En éliminant Λ_0 des deux dernières équations, nous obtenons: $\Lambda_1 = -\frac{70}{3}\Lambda_2$, d'où $\Lambda_0 = 239\Lambda_1+5663\Lambda_2 = \frac{259}{3}\Lambda_2$. Alors, en tenant compte de la première équation, on obtient:

$$(5.36)\qquad \Lambda_2 = 300/32768 = 3\cdot 5^2/2^{13},\quad \Lambda_1 = -7\cdot 5^3/2^{12},$$

$$\Lambda_0 = 37\cdot 7\cdot 5^2/2^{13}.$$

Ensuite, puisque $679 = 7\cdot 97$, $679\Lambda_0+5335\Lambda_1+65863\Lambda_2 = 0$, on a:

$$(5.37)\qquad S = \frac{\pi^6}{64}(5760\Lambda_1+158976\Lambda_2) = \frac{\pi^6}{64}24576\Lambda_2 = 384\pi^6\Lambda_2 = \frac{225}{64}\pi^6.$$

D'autre part, $x\binom{x+2}{5} = (x^6-5x^4+4x^2)/120$, d'où, d'après (0.11), $a_{1,2} = 1/30$, $a_{2,2} = 1/24$, $a_{3,2} = 1/120$. Donc, en vertu de (0.10) et (5.37), on obtient l'égalité (5.33). Le théorème 5.1 est démontré.

Corollaire 5.1. *Sous les conditions du Théorème* 5.1 *on aura:*

$$(5.38)\qquad \lim_{n\to\infty} n^6(1-\lambda_{k,n}) = \frac{5}{16}\pi^6(k^6-5k^4+4k^2)\qquad (k\in\mathbf{N},\ k\leqq n).$$

Corollaire 5.2. *Sous les conditions du Théorème* 5.1 *on aura:*

i) $\varliminf_{n\to\infty} n^6\|L_n(f;x)-f(x)\|_{C_{2\pi}} = 0$ *si et seulement si* $f(x)$ *est un polynôme trigonométrique d'ordre* $\leqq 2$;

ii) $\|L_n(f;x)-f(x)\|_{C_{2\pi}} = O(n^{-6})$ *si et seulement si* $f^{(5)}\in \operatorname{Lip}1$.

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APPROXIMATION OF PERIODIC CONTINUOUS FUNCTIONS BY LOGARITHMIC MEANS OF FOURIER SERIES

XIANLIANG SHI and QIYU SUN (Hangzhou)

1. Introduction

Let $f(x)$ be a periodic continuous function with period 2π. Denote by $S_k(x)$ the k-th partial sum of its Fourier series. For any positive integer k and $0 < r < 1$, let

$$\left(\log\frac{1}{1-r}\right)^k = \sum_{n=0}^{\infty} L_n^{(k)} r^n.$$

Define the logarithmic means of order k of $f(x)$ by

$$L_r^{(k)}(f,x) = \left(\log\frac{1}{1-r}\right)^{-k} \sum_{n=k}^{\infty} L_n^{(k)} S_n(x) r^n.$$

For any modulus of continuity $\omega(t)$ define

$$H[\omega] = \{f \in C_{2\pi} : \omega(f,\delta) \leqq C\omega(\delta)\},$$

where $C_{2\pi}$ denotes the class of periodic continuous functions with period 2π and $\omega(f,\delta) = \sup\limits_{|x'-x|\leqq\delta} |f(x') - f(x)|$.

Recently, Mazhar considered the approximation problem by logarithmic means of order 1 and proved the following (see [1]).

THEOREM M. *If $f(x) \in H[\omega]$, then*

$$L_r^{(1)}(f,x) - f(x) = O\left(\left(\log\frac{1}{1-r}\right)^{-1} \int_{1-r}^{\pi} \frac{\omega(t)}{t}\,dt\right) \quad (r \to 1-0).$$

Furthermore, there exists a function $f_0(x) \in H[\omega]$ such that

$$f_0(0) - L_r^{(1)}(f_0,0) \geqq C\left(\log\frac{1}{1-r}\right)^{-1} \int_{1-r}^{\pi} \frac{\omega(t)}{t}\,dt.$$

In fact we proved this result much earlier. In [2] we obtained the following

THEOREM S1. *Let $f(x) \in C_{2\pi}$ and denote $\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$. Then we have*

$$L_r^{(1)}(f,x) - f(x) = \frac{1}{2}\left(\log\frac{1}{1-r}\right)^{-1}\int_{1-r}^{\pi}\frac{\varphi_x(t)}{t}\,dt + O\left(\frac{\omega(f,\pi)}{\log\frac{1}{1-r}}\right) + O(\omega(f,1-r)),$$

where O is independent of f and r.

In [2], we also discussed the saturation problem. The saturation degree of the logarithmic means is $O\left(\left(\log\frac{1}{1-r}\right)^{-1}\right)$ and the saturation class is determined by the following two conditions (see [2]):

(i) $\omega(f,t) = O\left(\left(\log\frac{1}{t}\right)^{-1}\right) \quad (t \to 0+)$,

(ii) $\overline{\lim_{\varepsilon\to 0+}}\left|\int_{\varepsilon}^{\pi}\frac{\varphi_x(t)}{t}\,dt\right| < c < +\infty.$

In this paper we will establish an asymptotic expansion for logarithmic means and show that condition (i) above can be dropped. Furthermore, we generalize this result to logarithmic means of order k.

2. Asymptotic expansion

THEOREM 1. *For any $f(x) \in C_{2\pi}$ and positive integer k, the asymptotic expansion*

$$L_r^{(k)}(f,x) - f(x) = \frac{1}{\left(\log\frac{1}{1-r}\right)^k}\int_{1-r}^{\frac{1}{2}}\frac{\varphi_x(t)}{t}\left(\sum_{j=0}^{k-1}\left(\log\frac{1}{t}\right)^j \cdot A_j^{(k)}\right)dt + O\left(\omega(f,\pi)\left(\log\frac{1}{1-r}\right)^{-k}\right) + O(\omega(f,1-r)),$$

holds where $A_j^{(k)}$ are some constants and $A_{k-2j}^{(k)} = 0$ $\left(0 \leqq j \leqq \left[\frac{k}{2}\right]\right)$.

PROOF. We have

(1) $$L_r^{(k)}(f,x) - f(x) =$$

$$= \frac{1}{\pi\left(\log\frac{1}{1-r}\right)^k}\int_0^\pi \frac{\varphi_x(t)}{2\sin\frac{t}{2}}\operatorname{Im}\left\{e^{i\frac{t}{2}}\log^k\frac{1}{1-re^{it}}\right\}dt =$$

$$= O(\omega(f,1-r)) + \frac{1}{2\pi\left(\log\frac{1}{1-r}\right)^k}\int_{1-r}^\pi \frac{\varphi_x(t)}{\sin\frac{t}{2}}\operatorname{Im}\left\{e^{i\frac{t}{2}}\log^k\frac{1}{1-re^{it}}\right\}dt =$$

$$= O(\omega(f,1-r)) + O\left(\omega(f,\pi)\left(\log\frac{1}{1-r}\right)^{-k}\right) +$$

$$+\frac{1}{2\pi\left(\log\frac{1}{1-r}\right)^k}\int_{1-r}^{\frac{1}{2}} \frac{\varphi_x(t)}{\sin\frac{t}{2}}\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^k(\log\Delta)^{k-2j-1}\Theta^{2j+1}(-1)^{j+1}\cos\frac{t}{2}\,dt,$$

where

$$\Delta = |1-re^{it}|, \quad \Theta = \arcsin\frac{r\sin t}{|1-re^{it}|}.$$

Notice that

$$\Delta^2 = \left|1-re^{it}\right|^2 = (1-r)^2 + 4r\sin^2\frac{1}{2}t = 4r\sin^2\frac{1}{2}t\left(1+\frac{(1-r)^2}{4r\sin^2\frac{1}{2}t}\right).$$

We get

(2) $$\log\Delta = \frac{1}{2}\log\left(4r\sin^2\frac{1}{2}t\right) + \frac{1}{2}\log\left(1+\frac{(1-r)^2}{4r\sin^2\frac{1}{2}t}\right) =$$

$$= \frac{1}{2}\log\left(4r\sin^2\frac{1}{2}t\right) + O\left(\left(\frac{1-r}{t}\right)^2\right).$$

On the other hand we have

(3) $$\Theta = \arcsin\frac{r\sin t}{2\sin\frac{t}{2}r^{\frac{1}{2}}\sqrt{1+\frac{(1-r)^2}{4r\sin^2\frac{t}{2}}}} =$$

$$= \arcsin\left(r^{\frac{1}{2}}\cos\frac{1}{2}t\right) + O\left(\left(\frac{1-r}{t}\right)^2\cdot\frac{t}{(1-r)+t^2}\right) =$$

$$= \left(\frac{1}{2}\pi - \frac{1}{2}t\right) + O(1-r) + O\left(\frac{1-r}{t}\right).$$

Combining the estimates (1), (2) and (3), we get

$$L_r^{(k)}(f,x) - f(x) =$$

$$= \frac{1}{\pi\left(\log\frac{1}{1-r}\right)^k} \int_{1-r}^{\frac{1}{2}} \frac{\varphi_x(t)}{t} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \left(\frac{1}{2}\log\frac{1}{t}\right)^{k-2j-1} (-1)^{k+j+1}\left(\frac{\pi}{2}\right)^{2j+1} dt+$$

$$+O(\omega(f,1-r)) + O\left(\left(\log\frac{1}{1-r}\right)^{-k}\omega(f,\pi)\right)$$

Thus the proof of Theorem 1 is complete.

3. Saturation class

Now we prove the following saturation theorem.

THEOREM 2. *Let* $f \in C_{2\pi}$. *Then*

1) $L_r^{(k)}(f,x) - f(x) = o\left(\left(\log\frac{1}{1-r}\right)^{-k}\right)$ $(r \to 1-0)$ *uniformly if and only if* $f(x)$ *is a trigonometric polynomial of order* $\leqq k-1$.

2) *The following two assertions are equivalent:*

(i) $L_r^{(k)}(f,x) - f(x) = O\left(\left(\log\frac{1}{1-r}\right)^{-k}\right)$ $(r \to 1-0)$ *uniformly;*

(ii) $\overline{\lim\limits_{\varepsilon\to+0}}\left|\int_{\varepsilon}^{\pi}\frac{\varphi_x(t)}{t}(\log t)^{k-1}\,dt\right| < c < +\infty.$

PROOF. The proof of 1) is easy, so we omit the details here. Now we prove 2). By Theorem 1 it is sufficient to show that conditions (i) and (ii) imply $\omega(f,1-r) = O\left(\left(\log\frac{1}{1-r}\right)^{-k}\right)$ independently.

Now we prove that condition (ii) implies $\omega(f,1-r) = O\left(\left(\log\frac{1}{1-r}\right)^{-k}\right)$ first. Let a_μ, b_μ be the Fourier coefficients of f. Without loss of generality we may assume $a_\mu = b_\mu = 0$ when $0 \leqq \mu \leqq k-1$. Therefore for any small positive $h > 0$,

$$\varphi_x(2h) = -4\sum_{\mu=k}^{\infty}\sin^2\mu h(a_\mu\cos\mu x + b_\mu\sin\mu x) = \tag{4}$$

$$= \frac{4}{\pi}\left(\log\frac{1}{h}\right)^k \int_{-\pi}^{\pi}\left[f(x+t) - L_{1-h}^{(k)}(f,x+t)\right] \sum_{\mu=k+1}^{\infty}\frac{\sin^2\mu h\cos\mu t}{\sum\limits_{\nu=k}^{\mu-1}L_\nu^{(k)}(1-h)^\nu}\,dt.$$

We observe from the proof of Theorem 1 that

(5)

$$\left| L_r^{(k)}(f,x) - f(x) - \frac{1}{\left(\log \frac{1}{1-r}\right)^k} \int_{\varepsilon(1-r)}^{\pi} \frac{\varphi_x(t)}{t} \sum_{j=0}^{\left[\frac{k-1}{2}\right]} A_j^k \left(\log \frac{1}{t}\right)^{k-2j-1} dt \right| \leqq$$

$$\leqq \varepsilon\omega(f,1-r) + C_\varepsilon \left(\log \frac{1}{1-r}\right)^{-k} + C_\varepsilon (1-r)^{-k} \int_{\varepsilon(1-r)}^{\frac{1}{2}} \frac{\omega(f,t)}{t^2} \left(\log \frac{1}{t}\right)^{k-1} dt$$

where ε is a given small positive number. Therefore by (5) and condition (ii), (4) reduces to

$$|\varphi_x(2h)| \leqq \int_0^\pi \left| \sum_{\mu=k+1}^{\infty} \frac{\sin^2 \mu h \cos \mu t}{\sum_{\nu=k}^{\mu-1} L_\nu^{(k)} (1-h)^\nu} \right| dt \times$$

$$\times \left\{ 2\varepsilon\omega(f,h) \left(\log \frac{1}{h}\right)^k + C_\varepsilon h \int_h^{\frac{1}{2}} \frac{\omega(f,t)}{t^2} \left(\log \frac{1}{t}\right)^{k-1} dt + \right.$$

$$\left. + C_\varepsilon \int_h^{\frac{1}{2}} \frac{\omega(f,t)}{t} \left(\log \frac{1}{t}\right)^{k-3} dt \right\}.$$

We claim that

$$\int_0^\pi \left| \sum_{\mu=k+1}^{\infty} \frac{\sin^2 \mu h \cos \mu t}{\sum_{\nu=k}^{\mu-1} L_\nu^{(k)} (1-h)^\nu} \right| dt \leqq C_0 \left(\log \frac{1}{h}\right)^{-k}. \tag{6}$$

Assume for a moment that (6) holds true, then we get

$$|\varphi_x(2h)| \leqq 2C_0\varepsilon\omega(f,h) + c\left(\log \frac{1}{h}\right)^{-k} + ch\left(\log \frac{1}{h}\right)^{-k} \cdot$$

$$\cdot \int_h^{\frac{1}{2}} \frac{\omega(f,t)}{t^2} \left(\log \frac{1}{t}\right)^{k-1} dt + c\left(\log \frac{1}{h}\right)^{-k} \int_h^{\frac{1}{2}} \frac{\omega(f,t)}{t} \left(\log \frac{1}{t}\right)^{k-3} dt$$

and by $\varepsilon = \frac{1}{4C_0}$, we get

$$(7)\quad \omega(f,h) \leqq c\left(\log\frac{1}{h}\right)^{-k} + ch\left(\log\frac{1}{h}\right)^{-k}\int_h^{\frac{1}{2}}\frac{\omega(f,t)}{t^2}\left(\log\frac{1}{t}\right)^{k-1}dt+$$

$$+c\left(\log\frac{1}{h}\right)^{-k}\int_h^{\frac{1}{2}}\frac{\omega(f,t)}{t}\left(\log\frac{1}{t}\right)^{k-3}dt.$$

Denote

$$g_m(t) = \omega(f,t)\left(\log\frac{1}{t}\right)^m \quad \text{and} \quad M_m = \sup_{0<t\leqq\frac{1}{2}} g_m(t)$$

for $0 \leqq m \leqq k$. Therefore we have that the right side of (7) is

$$\leqq c\left(\log\frac{1}{h}\right)^{-k} + ch\left(\log\frac{1}{h}\right)^{-k}\int_h^{\frac{1}{2}}\frac{M_{m-1}}{t^2}\left(\log\frac{1}{t}\right)^{k-1-m+1}dt+$$

$$+c\left(\log\frac{1}{h}\right)^{-k}\int_h^{\frac{1}{2}}\frac{M_{m-1}}{t}\left(\log\frac{1}{t}\right)^{k-3-m+1}dt \leqq$$

$$\leqq c\left(\log\frac{1}{h}\right)^{-k} + c\left(\log\frac{1}{h}\right)^{-m}M_{m-1}$$

and $M_m \leqq CM_{m-1} + C$ for $1 \leqq m \leqq k$. Hence

$$\omega(f,t) \leqq (C + CM_0)\left(\log\frac{1}{t}\right)^{-k}$$

holds. Therefore the problem is reduced to prove (6). Before we start to prove (6), we will use the following estimate about the coefficient of $\left(\log\frac{1}{1-r}\right)^{-k}$:

$$\sum_{\nu=k}^{\mu} L_\nu^{(k)}(1-h)^\nu \sim \begin{cases} (\log\mu)^k, & \text{when } \mu < \frac{1}{h}, \\ (\log\frac{1}{h})^k, & \text{when } \mu \geqq \frac{1}{h}, \end{cases}$$

$$L_\mu^{(k)} \sim \mu^{-1}(\log\mu)^{k-1}$$

and

$$\left| L_\mu^{(k)} - L_{\mu+1}^{(k)} \right| \leqq c\mu^{-2}(\log \mu)^{k-1}$$

where $A \sim B$ means $C^{-1}A \leqq B \leqq CA$ for some absolute constant C. Therefore for $-2h < t < 5h$ we have

$$\left| \sum_{\mu=k+1}^{\infty} \frac{\cos \mu t}{\sum\limits_{k}^{\mu-1} L_\nu^{(k)}(1-h)^\nu} \right| \leqq \sum_{\mu=k+1}^{\infty} \frac{\left|\sin\left(\mu+\frac{1}{2}\right)t\right|}{\left|\sin \frac{1}{2}t\right|} \frac{L_\mu^{(k)}(1-h)^\mu}{\left(\sum\limits_{\nu=k}^{\mu-1} L_\nu^{(k)}(1-h)^\nu\right)^2} \leqq$$

$$\leqq c\left(\sum_{\mu<\frac{1}{t}} + \sum_{\mu\geqq\frac{1}{t}}\right) \frac{\left|\sin\left(\mu+\frac{1}{2}\right)t\right|}{t} \frac{L_\mu^{(k)}(1-h)^\mu}{\left(\sum\limits_{\nu=k}^{\mu-1} L_\nu^{(k)}(1-h)^\nu\right)^2} = \mathrm{I} + \mathrm{II}.$$

We see

$$\mathrm{I} \leqq c \sum_{\mu<\frac{1}{t}} (\log \mu)^{k-1}(1-h)^\mu \left(\sum_{\nu=k}^{\mu-1} L_\nu^{(k)}(1-h)^\nu\right)^{-2} \leqq$$

$$\leqq c \sum_{\mu\leqq\frac{1}{h}} (\log \mu)^{k-1}(\log \mu)^{-2k} + \sum_{\frac{1}{h}<\mu<\frac{1}{t}} (\log \mu)^{k-1}(1-h)^\mu \left(\log \frac{1}{h}\right)^{-2k} \leqq$$

$$\leqq ch^{-1}\left(\log \frac{1}{h}\right)^{-k-1}.$$

On the other hand, we have

$$\mathrm{II} \leqq \frac{c}{t} \sum_{\mu\geqq\frac{1}{t}} \mu^{-1}(\log \mu)^{k-1}(1-h)^\mu \left(\log \frac{1}{h}\right)^{-2k} \leqq$$

$$\leqq \frac{c}{t}\left(\log \frac{1}{h}\right)^{-2k} \int\limits_{\nu\geqq\frac{1}{2t}} (\log \nu)^{k-1} \exp(\nu \log(1-h))\, d\nu \leqq ch^{-1}\left(\log \frac{1}{h}\right)^{-k-1}.$$

Hence for $0 < t < 3h$, we get

$$\left| \sum_{\mu=k+1}^{\infty} \frac{\cos \mu t}{\sum\limits_{\nu=k}^{\mu-1} L_\nu^{(k)}(1-h)^\nu} \right| \leqq ch^{-1}\left(\log \frac{1}{h}\right)^{-k-1}. \tag{8}$$

Now, let us start to prove (6). By (8) we have for $0 < t < 3h$,

$$\left|\sum_{\mu=k+1}^{\infty} \frac{\sin^2 \mu h \cos \mu t}{\sum_k^{\mu-1} L_\nu^{(k)}(1-h)^\nu}\right| \leqq$$

$$\leqq \left|\sum_{\mu>k} \frac{\cos \mu t}{\sum_k^{\mu-1} L_\nu^{(k)}(1-h)^\nu}\right| + \left|\sum_{\mu>k} \frac{\cos \mu(t+2h)}{\sum_k^{\mu-1} L_\nu^{(k)}(1-h)^\nu}\right| + \left|\sum_{\mu>k} \frac{\cos \mu(t-2h)}{\sum_\nu^{k-1} L_\nu^{(k)}(1-h)^\nu}\right| \leqq$$

$$\leqq ch^{-1}\left(\log \frac{1}{h}\right)^{-k-1}.$$

For $t > 3h$, by using Abel transform we get

$$\sum_{\mu>k} \frac{\sin^2 \mu h \cos \mu t}{\sum_k^{\mu-1} L_\nu^{(k)}(1-h)^\nu} =$$

$$= \left(\sum_{\mu<\frac{1}{t}} + \sum_{\frac{1}{t}\leqq\mu<\frac{1}{h}} + \sum_{\mu\geqq\frac{1}{h}}\right) \Delta\left(\frac{1}{\sum_k^{\mu-1} L_\nu^{(h)}(1-h)^\nu}\right)\left\{\frac{\sin\left(\mu+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} -\right.$$

$$\left. -\frac{1}{2}\left(\frac{\sin\left(\mu+\frac{1}{2}\right)(t+2h)}{\sin\frac{1}{2}(t+2h)} + \frac{\sin\left(\mu+\frac{1}{2}\right)(t-2h)}{\sin\frac{1}{2}(t-2h)}\right)\right\} =$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

where $\Delta(a_\mu) = a_\mu - a_{\mu+1}$. We see that

$$\left|\sum\nolimits_1\right| \leqq c \sum_{\mu<\frac{1}{t}} \mu^{-1}(\log\mu)^{k-1}(\log\mu)^{-2k} h\mu t^{-1} \leqq ct^{-2}\left(\log\frac{1}{t}\right)^{-k-1} h,$$

$$\left|\sum\nolimits_2\right| \leqq c \sum_{\frac{1}{t}\leqq\mu<\frac{1}{h}} \mu^{-1}(\log\mu)^{k-1}(\log\mu)^{-2k} h\mu t^{-1} \leqq ct^{-1}\left(\log\frac{1}{h}\right)^{-k-1}.$$

By using Abel transform again to $\sum_3$, we see that

$$\left|\sum\nolimits_3\right| \leqq ct^{-2} \sum_{\mu\geqq\frac{1}{h}} \left|\Delta^2\left(\frac{1}{\sum_k^{\mu} L_\nu^{(k)}(1-h)^\nu}\right)\right| \leqq$$

$$\leqq ct^{-2}\sum_{\mu\geqq\frac{1}{h}}\left|L_\mu^{(k)}-L_{\mu+1}^{(k)}\right|(1-h)^\mu\cdot\left(\log\frac{1}{h}\right)^{-2k}+$$

$$+ct^{-2}\sum_{\mu\geqq\frac{1}{h}}L_\mu^{(k)}(1-h)^\mu\cdot h\left(\log\frac{1}{h}\right)^{-2k}+$$

$$+ct^{-2}\sum_{\mu\geqq\frac{1}{h}}\frac{L_\mu^{(k)}(1-h)^\mu\left(L_{\mu+1}^{(k)}(1-h)^{\mu+1}+L_{\mu+2}^{(k)}(1-h)^{\mu+2}\right)}{\left(\log\frac{1}{h}\right)^{3k}}\leqq$$

$$\leqq ct^{-2}h\left(\log\frac{1}{h}\right)^{-k-1}.$$

Therefore for $t>3h$, we have

$$(9)\quad\left|\sum_{\mu>k}\frac{\sin^2\mu h\cos\mu t}{\sum\limits_{k}^{\mu-1}L_\nu^{(k)}(1-h)^\nu}\right|\leqq cht^{-2}\left(\log\frac{1}{t}\right)^{-k-1}+ct^{-1}\left(\log\frac{1}{h}\right)^{-k-1}+$$

$$+ct^{-2}h\left(\log\frac{1}{h}\right)^{-k-1}.$$

Taking (8) and (9) into (6), we get

$$\int_0^\pi\left|\sum_{\mu>k}\frac{\sin^2\mu h\cos\mu t}{\sum\limits_{\nu=k}^{\mu-1}L_\nu^{(k)}(1-h)^\nu}\right|dt\leqq c\int_o^{5h}h^{-1}\left(\log\frac{1}{h}\right)^{-k-1}dt+$$

$$+c\int_h^\pi\left(ht^{-2}\left(\log\frac{1}{t}\right)^{-k-1}+t^{-1}\left(\log\frac{1}{h}\right)^{-k-1}\right)dt\leqq c\left(\log\frac{1}{h}\right)^{-k}.$$

Hence (6) holds and condition (ii) implies $\omega(f,t)=O\left(\left(\log\frac{1}{t}\right)^{-k}\right)$. Similarly we can prove that condition (i) implies $\omega(f,t)=O\left(\left(\log\frac{1}{t}\right)^{-k}\right)$. Therefore the proof of Theorem 2 is completed by the asymptotic expansion of $L_r^{(k)}(f,x)-f(x)$.

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SOME RESULTS ON COMMUTATIVITY AND ANTI-COMMUTATIVITY IN RINGS

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Let R denote an arbitrary ring. For each $x, y \in R$, denote the anti-commutator $xy + yx$ by $x \circ y$; and define the anti-center $\widehat{Z}$ by

$$\widehat{Z} = \{x \in R |\ x \circ y = 0 \quad \text{for all} \quad y \in R\}.$$

Call R anti-commutative if $\widehat{Z} = R$.

Motivated by recent commutativity and anti-commutativity results for rings in which anti-commutators satisfy certain power constraints [3,6,8], we study rings satisfying constraints defined in terms of $\widehat{Z}$. Our results are analogues of well-known commutativity theorems.

As usual, we shall denote by $[x, y]$ the commutator $xy - yx$, and by Z the center of R. The symbols N, $J(R)$ and $C(R)$ will denote respectively the set of nilpotent elements, the Jacobson radical and the commutator ideal; and for $S \subseteqq R$, $A(S)$ will denote the two-sided annihilator of S. The ring of integers will be indicated by $\mathbf{Z}$.

1. An anti-commutativity theorem

Our first result is a direct analogue of a theorem of Putcha, Wilson, and Yaqub [7].

THEOREM 1. *Suppose that for each $x, y \in R$, there exists $w = w(x, y) \in \in \widehat{Z}$ such that $(x \circ y)^2 w = x \circ y$. Then R is anti-commutative.*

PROOF. Call a ring R satisfying our hypotheses a ρ-ring. Note that, by a standard argument, idempotents must lie in Z.

We employ the Jacobson structure theory, assuming first that R is a division ring. Let $x, y \in R$ and choose $w \in \widehat{Z}$ such that

$$(1) \qquad (x \circ y)^2 w = x \circ y.$$

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It follows that $((x \circ y)w)^2 = -(x \circ y)w$, so that $-(x \circ y)w$ is idempotent and $(x \circ y)w \in Z$. Thus for all $z \in R$, $(x \circ y)wz = z(x \circ y)w$, hence $wz(x \circ y) = -w(x \circ y)z$. If $w = 0$, then $x \circ y = 0$ by (1); otherwise $z(x \circ y) = -(x \circ y)z$, in which case $x \circ y \in \widehat{Z}$. Thus, $x \circ y \in \widehat{Z}$ for all $x, y \in R$. Now if $u \in \widehat{Z}$, we get $u \circ 1 = 2u = 0$; thus, if char $R \neq 2$, $\widehat{Z} = \{0\}$, and $x \circ y = 0$ for all $x, y \in R$. On the other hand, if char $R = 2$, $x \circ y = [x, y]$ and $\widehat{Z} = Z$; hence $[x, y] \in Z$ for all $x, y \in R$ — a condition known to imply commutativity in division rings. Since commutativity and anti-commutativity are the same in characteristic 2, ρ-division rings are always anti–commutative.

The primitive case reduces as usual to studying 2×2 matrices; and since $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ do not satisfy our hypotheses, all primitive ρ-rings are division rings, hence all semi-simple ρ-rings are anti-commutative.

Now suppose R is a ρ-ring with $J(R) \neq \{0\}$. Then $R/J(R)$ is anti-commutative, so for any $x, y \in R$ we have $x \circ y \in J(R)$. Choosing $w \in \widehat{Z}$ such that $(x \circ y)^2 w = x \circ y$, we see that $-(x \circ y)w$ is an idempotent in $J(R)$. Thus $(x \circ y)w = 0$ and $x \circ y = 0$.

2. An analogue of a theorem of Herstein

A deep theorem of Herstein [4] reads as follows:

THEOREM H. *Let R be a ring such that for each $x \in R$, there exists $p(t) \in \mathbf{Z}[t]$ for which $x - x^2 p(x) \in Z$. Then R is commutative.*

Our second theorem, which is our main theorem, has a similar hypothesis involving the anti-center.

THEOREM 2. *If the ring R has the property that*

$$(\dagger) \qquad \begin{cases} \textit{for each} \quad x \in R \quad \textit{there exists} \quad p(t) \in \mathbf{Z}[t] \\ \textit{such that} \quad x - x^2 p(x) \in \widehat{Z}, \end{cases}$$

then R is a subdirect product of commutative and anti-commutative rings.

PROOF. Let R be an arbitrary ring satisfying (†). Since $\widehat{Z}$ is an additive subgroup of R, for each $x \in R$ we get polynomials $q(t)$ of arbitrarily high co-degree such that $x - q(x) \in \widehat{Z}$; and it is immediate that $N \subseteq \widehat{Z}$. It is now easy to show that N is an ideal; and a standard structure theorem shows that $\frac{R}{N}$ is a subdirect product of rings without zero divisors, each satisfying (†).

Assume, then, that R satisfies (†) and has no zero divisors. If $Z \neq \{0\}$, let $0 \neq z \in Z$ and $u \in \widehat{Z}$. Then $zu = uz = -uz$, so $2uz = 0 = 2u$; hence

$\widehat{Z} \subseteqq Z$ and R is commutative by Theorem H. On the other hand, if $Z = \{0\}$, noting that $u^2 \in Z$ for each $u \in \widehat{Z}$ shows that for each $x \in R$ we have $p(t) \in \in \mathbf{Z}[t]$ for which $(x - x^2p(x))^2 \in Z = \{0\}$; thus $x - x^2p(x) = 0$ and R is again commutative by Theorem H.

Returning to the case of arbitrary R with (†), we now have $\frac{R}{N}$ commutative, so that $C(R) \subseteqq N \subseteqq \widehat{Z}$. Hence for each $x, y \in R$ we have $(xy - yx)x + x(xy - yx) = 0$, so that

$$x^2 \in Z \quad \text{for all} \quad x \in R \tag{2}$$

and

$$x \circ y \in Z \quad \text{for all} \quad x, y \in R. \tag{3}$$

Taking $p(t) \in \mathbf{Z}[t]$ for which $x - x^2p(x) \in \widehat{Z}$ and using (2), we get $xy + yx = = x^2p(x)y + yx^2p(x) = x^2p(x)y + x^2yp(x)$ for all $y \in R$; and it follows at once that R is 0-commutative (i.e. $xy = 0$ implies $yx = 0$).

We can write R as a subdirect product of subdirectly irreducible rings with (†); thus, we assume henceforth that R is subdirectly irreducible, and proceed to show that R is either commutative or anti-commutative. We make use of the following lemmas.

LEMMA 1 [1, Lemma 2]. *Let R be a subdirectly irreducible 0-commutative ring with heart H, and let D be the set of zero divisors of R. Then $D = A(H)$.*

LEMMA 2. *Let R be a ring satisfying (†) and having at least one element which is not a zero divisor. Then $\widehat{Z} \subseteqq Z$.*

PROOF. Let $u \in \widehat{Z}$, and let x be an element of R which is not a zero divisor. Then by (2), $x^2u - ux^2 = 0$. On the other hand, since $u \in \widehat{Z}$, $x^2u + ux^2 = 0$; hence $2x^2u = 0 = 2u$, and $u \in Z$.

LEMMA 3. *Let R be any ring satisfying (†). If $x, w \in R$ and $x(xw + + wx) = 0$, then $xw + wx = 0$.*

PROOF. Choose $p(t) \in \mathbf{Z}[t]$ for which $x - x^2p(x) \in \widehat{Z}$, and write $p(t) = g(t^2) + th(t^2)$. Then

$$xy + yx = x^2g(x^2)y + yx^2g(x^2) + x^3h(x^2)y + yx^3h(x^2) \quad \text{for all} \quad y \in R; \tag{4}$$

and since $x^2 \in Z$, we have

$$xy + yx = 2x^2g(x^2)y + x^2h(x^2)(xy + yx) \quad \text{for all} \quad y \in R. \tag{5}$$

Since $x(xw + wx) = 0$, left-multiplying (5) by x and taking $y = w$ gives

$$2x^3g(x^2)w = 0. \tag{6}$$

Taking $y = x$ in (5) gives $2x^2 = 2x^3g(x^2) + 2x^4h(x^2)$; thus (5) and (6) yield $xw + wx = 2x^2g(x^2)w = (2x^3g(x^2) + 2x^4h(x^2))g(x^2)w = 0$.

COMPLETION OF PROOF OF THEOREM 2. If R has an element which is not a zero divisor, then Lemma 2 and Theorem H imply that R is commutative. Thus assume that $R = D$, in which case $R = A(H)$ by Lemma 1. Suppose that there exist $a, b \in R$ with $a \circ b \neq 0$. Then, by (3) and Lemma 3, $(a \circ b)R$ is a nonzero two-sided ideal; hence $H \subseteqq (a \circ b)R$. Let $0 \neq h \in H$, so that $h = (a \circ b)r$ for some $r \in R$. Since $ah = 0$, we get $a(a \circ b)r = 0$; and applying Lemma 3 to the ring $\frac{R}{A(r)}$ gives $(a \circ b)r = 0$ — a contradiction. Therefore, R is anti-commutative.

3. Two theorems on periodic rings

Our final two theorems deal with periodic rings. The first generalizes Herstein's theorem ([5], [2, Theorem 2]) that a periodic ring with $N \subseteqq Z$ must be commutative; the second extends a result of the author [2, Theorem 3].

THEOREM 3. *If R is a periodic ring with $N \subseteqq Z \cup \widehat{Z}$, then R is a subdirect product of commutative rings and anti-commutative rings. Hence, $x^2 \in Z$ for all $x \in R$.*

PROOF. For $u, v \in N$, $uv = \pm vu$; therefore, N is an additive subgroup of R. Since N is the union of the subgroups $N \cap Z$ and $N \cap \widehat{Z}$, we see that $N \subseteqq Z$ or $N \subseteqq \widehat{Z}$. In the first case R is commutative by Herstein's result. In the second, since for each $x \in R$ there exists $n > 1$ for which $x - x^n \in N$ [2,(P_2)], the conclusion follows by Theorem 2.

THEOREM 4. *Let R be a periodic ring, and suppose that for each $x \in$ $\in R$ and $u \in N$, there either exists $n > 1$ for which $[x, u]^n = [x, u]$ or there exists $m > 1$ for which $(x \circ u)^m = x \circ u$. Then R is a subdirect product of commutative rings and anti-commutative rings.*

PROOF. For each $u \in N$ and $x \in R$, $ux = 0$ if and only if $xu = 0$; and it follows as in the proof of [2, Theorem 3] that N is an ideal. Thus, for each $u \in N$ and $x \in R$, $x \circ u$ and $[x, u]$ are in N, so that $x \circ u = 0$ or $[x, u] = 0$. By appealing to the fact that a group cannot be the union of two proper subgroups, we conclude that $N \subseteqq Z \cup \widehat{Z}$; and the theorem follows from Theorem 3.

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A FLEXIBLE MINIMAX THEOREM

S. SIMONS (Santa Barbara)

Dedicated to Professor Heinz König

Introduction

The purpose of this paper is to unify a number of minimax theorems that use hypotheses that are superficially very different.

The important role of *connectedness* in minimax theorems was first noted by Wu [29], followed by Tuy [27,28], who was able to generalize Sion's minimax theorem [24]. Based on Joó's result [8], Stachó [25] and Komornik [16] proved minimax theorems for "interval spaces". These results were unified by Kindler–Trost [12].

Minimax conditions that use *algebraic* conditions were considered by Fan [1], König [17], Neumann [19], Irle [7], Lin–Quan [18], Kindler [11] and Simons [20].

Minimax theorems that *mix* both connectedness and algebraic conditions were considered by Terkelsen [26], Geraghty–Lin [2,4,5], Kindler [11] and Simons [21].

Kindler [11] was the first to observe that the algebraic conditions *force* conditions akin to connectedness.

In this paper, we give results that unify all the ideas mentioned above, as well as other ideas due to Ha [6] and Simons [22,23].

The basic minimax theorem is Theorem 1 which has a simple proof using a *compactness* condition (1.1), a *condition on* Y, (1.2) and a *condition on* X, (1.3).

There are obvious *topological* situations in which (1.2) holds — see (8.2). Lemma 2 gives a *set-theoretic* situation in which (1.2) holds — in Remarks 3, we show that, to within ε, Lemma 2 encompasses all the *algebraic* situations mentioned above.

Lemmas 4 and 5 give *topological* situations (which will require that X be an interval space) in which (1.3) holds. Lemma 6 gives a *set-theoretic* situation in which (1.3) holds — in Remarks 7, we show that, to within ε again, Lemma 6 encompasses all the *algebraic* situations mentioned above.

The reader will undoubtedly notice the similarity between the hypotheses (2.2) and (6.1). In Remarks 7, we give a common result from which both

Lemma 2 and Lemma 6 can be derived. (We have not used this in the text for clarity of exposition.)

Let X and Y be nonempty sets and $f : X \times Y \to \mathbf{R}$. If $\gamma \in \mathbf{R}$ we define multifunctions $\underline{\gamma|}$ from X into 2^Y and $\overline{|\gamma}$ from Y into 2^X by

$$\forall x \in X, \quad \underline{\gamma|}x := \{y : y \in Y,\ f(x,y) \leqq \gamma\}$$

and

$$\forall y \in Y, \quad \overline{|\gamma}\, y := \{x : x \in X,\ f(x,y) > \gamma\}.$$

For convenience, we write $LE(W,\gamma)$ for $\bigcap_{w \in W} \underline{\gamma|}w$.

The author would like to thank Professor Jürgen Kindler for an interesting discussion on minimax theorems and for suggesting that he incorporate [12] into an earlier version of this work.

The joining of sets and pseudoconnectedness

We say that sets H_0 and H_1 are *joined* by a set H if

$$H \subset H_0 \cup H_1,\ H \cap H_0 \neq \emptyset \quad \text{and} \quad H \cap H_1 \neq \emptyset.$$

We say that a family $\mathcal{H}$ of sets is *pseudoconnected* if,

(0.1) $\quad H_0,\ H_1, H \in \mathcal{H}$ and H_0 and H_1 joined by $H \Rightarrow H_0 \cap H_1 \neq \emptyset$.

Any family of closed connected subsets of a topological space is pseudoconnected. So also is any family of open connected subsets. In Lemma 2 we give a situation related to minimax theorems in which a certain family of sets is *automatically* pseudoconnected.

THEOREM 1. *Let Y be a topological space, and $\mathcal{B}$ be a nonempty subset of* $\mathbf{R}$ *such that* $\inf \mathcal{B} = \sup_X \inf_Y f$. *Suppose that,* $\forall \beta \in \mathcal{B}$ *and finite subsets* W *of* X *(with the convention* $LE(\emptyset,\beta) = Y$*),*

(1.1) $\quad \forall x \in X, \quad \underline{\beta|}x$ *is closed and compact,*

(1.2) $\quad \left\{\underline{\beta|}x \cap LE(W,\beta)\right\}_{x \in X}$ *is pseudoconnected*

and,

(1.3) $\quad \begin{cases} \forall x_0,\ x_1 \in X,\ \exists x \in X \quad \text{such that} \\ \underline{\beta|}x_0 \text{ and } \underline{\beta|}x_1 \text{ are joined by } \underline{\beta|}x \cap LE(W,\beta). \end{cases}$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. Let $\beta \in \mathcal{B}$. Let V be a nonempty finite subset of X. We can write $V = \{x_0, x_1\} \cup W$. Let x be as in (1.3). It follows that $\underline{\beta\rfloor} x_0 \cap$ $\cap\, LE(W,\beta)$ and $\underline{\beta\rfloor} x_1 \cap LE(W,\beta)$ are joined by $\underline{\beta\rfloor} x \cap LE(W,\beta)$. From (1.2) and (0.1), $LE(V,\beta) \neq \emptyset$. The result follows from (1.1) and the finite intersection property.

Sufficient conditions for (1.2)

In our next result, W does not necessarily have to be finite.

LEMMA 2. *Let* $W \subset X$ *and* $\beta \in \mathbf{R}$. *Suppose that,*

(2.1) $\quad \forall\, \gamma > \beta$ *and* $x \in X$, $\underline{\gamma\rfloor} x \cap LE(W,\beta)$ *is closed and compact,*

and, whenever $\delta > \gamma$, $\exists\, N \geqq 1$ *and* $\gamma_0, \dots, \gamma_N \in \mathbf{R}$ *such that*

$$(2.2)\quad \begin{cases} & \gamma_0 = \delta,\ \gamma_N = \gamma \ \text{ and,} \\ & \forall y_0, y_1 \in Y,\ \exists\, y \in Y \ \text{ such that, } \forall\, n \in \{1,\dots,N\}, \\ (2.2.1) & \lceil\overline{\gamma_n}\, y \subset \lceil\overline{\gamma_{n-1}}\, y_0 \cup \lceil\overline{\beta}\, y_1, \\ (2.2.2) & \lceil\overline{\gamma_n}\, y \subset \lceil\overline{\beta}\, y_0 \cup \lceil\overline{\gamma_{n-1}}\, y_1, \\ (2.2.3) & \lceil\overline{\beta}\, y \subset \lceil\overline{\beta}\, y_0 \cup \lceil\overline{\beta}\, y_1, \\ (2.2.4) & \lceil\overline{\delta}\, y \subset \lceil\overline{\delta}\, y_0 \cup \lceil\overline{\delta}\, y_1. \end{cases}$$

Then

$$(1.2)\quad \left\{\underline{\beta\rfloor} x \cap LE(W,\beta)\right\}_{x\in X} \quad \textit{is pseudoconnected.}$$

PROOF. Suppose that the result fails. Then $\exists\, x_0, x_1, x \in X$ such that, writing $T := \underline{\beta\rfloor} x \cap LE(W,\beta)$,

$$(2.3)\quad T \subset \underline{\beta\rfloor} x_0 \cup \underline{\beta\rfloor} x_1,$$

$$(2.4)\quad \underline{\beta\rfloor} x_0 \cap \underline{\beta\rfloor} x_1 \cap T = \emptyset,$$

and, for $i = 0, 1$,

$$(2.5)\quad u_i \in \underline{\beta\rfloor} x_i \cap T.$$

From (2.1) and (2.4), $\exists\,\gamma > \beta$ such that

$$(2.6)\qquad \underline{\gamma}\rfloor x_0 \cap \underline{\gamma}\rfloor x_1 \cap T = \emptyset.$$

From (2.5) and (2.6), $u_0 \notin \underline{\gamma}\rfloor x_1$. Let $\delta := f(x_1,u_0) \vee f(x_0,u_1) > \gamma$,

$$U_0 := \underline{\beta}\rfloor x_0 \cap \underline{\delta}\rfloor x_1 \cap T \ni u_0 \quad\text{and}\quad U_1 := \underline{\delta}\rfloor x_0 \cap \underline{\beta}\rfloor x_1 \cap T \ni u_1.$$

Choose N and $\gamma_0,\dots,\gamma_N$ as in (2.2). Then, from (2.6),

$$U_0 \subset \underline{\delta}\rfloor x_1 = \underline{\gamma_0}\rfloor x_1 \quad\text{and}\quad U_0 \cap \underline{\gamma_N}\rfloor x_1 = U_0 \cap \underline{\gamma}\rfloor x_1 = \emptyset.$$

Thus, $\forall\, t \in U_0$, $\exists\,!\,g_0(t) \in \{1,\dots,N\}$ such that

$$(2.7)\qquad g_0(t) \leqq n \leqq N \Rightarrow t \notin \underline{\gamma_n}\rfloor x_1 \quad\text{and}\quad n = g_0(t) \Rightarrow t \in \underline{\gamma_{n-1}}\rfloor x_1.$$

Similarly, $\forall\, t \in U_1$, $\exists\,!\,g_1(t) \in \{1,\dots,N\}$ such that

$$g_1(t) \leqq n \leqq N \Rightarrow t \notin \underline{\gamma_n}\rfloor x_0 \quad\text{and}\quad n = g_1(t) \Rightarrow t \in \underline{\gamma_{n-1}}\rfloor x_0.$$

We fix $y_i \in U_i$ to maximize $g_i(y_i)$ and choose $y \in Y$ as in (2.2). From (2.2.3), $y \in T$. From (2.3), we can suppose without loss of generality that $y \in \underline{\beta}\rfloor x_0$. From (2.2.4) since $y_i \in \underline{\delta}\rfloor x_1$, $y \in \underline{\delta}\rfloor x_1$. Thus $y \in U_0$. Let $n := g_0(y_0)$. From (2.7), $y_0 \in \underline{\gamma_{n-1}}\rfloor x_1$. Since $y_1 \in U_1$, $y_1 \in \underline{\beta}\rfloor x_1$. From (2.2.1), $y \in$ $\in \underline{\gamma_n}\rfloor x_1$. From (2.7), $n < g_0(y)$. This contradiction of the maximality of $g_0(y_0)$ completes the proof of the Lemma.

REMARKS 3. In the context of minimax theorems, various authors have introduced conditions that imply (2.2).

Inspired by a result of Fan [1], König [17] introduced the condition:

$$(3.1)\qquad \begin{cases} \forall\, y_0, y_1 \in Y,\ \exists\, y \in Y \quad \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq \left[f(x,y_0) + f(x,y_1)\right]/2. \end{cases}$$

(3.1) was weakened by Neumann [19], who also showed that it sufficed that his condition hold "to within ε". (See the discussion on Irle's theorem below.)

Neumann's condition was further weakened by Geraghty–Lin [2,4,5] and Lin–Quan [18], who introduced the condition:

(3.2)

$$\begin{cases} \exists\, s \in (0,1) \ \textit{such that,}\ \forall\, y_0, y_1 \in Y,\ \exists\, y \in Y \ \textit{such that,} \\ x \in X \Rightarrow f(x,y) \leqq (1-s)\left[f(x,y_0) \vee f(x,y_1)\right] + s\left[f(x,y_0) \wedge f(x,y))\right]. \end{cases}$$

(To see this take $s := 1/2$).

Simons [20] weakened (3.2) to the "penalty condition":
(3.3)

$$\begin{cases} \exists \textit{ a nondecreasing function } \pi : \mathbf{R}^+ \to \mathbf{R}^+ \textit{ such that} \\ \lambda > 0 \Rightarrow \pi(\lambda) > 0 \textit{ and } \forall\, y_0, y_1 \in Y,\ \exists\, y \in Y \textit{ such that,} \\ x \in X \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1) - \pi\left(|f(x,y_0) - f(x,y_1)|\right). \end{cases}$$

(To see this take $\pi(\lambda) := s\lambda$. Much smaller choices of π are possible, for instance, $\pi(\lambda) := e^{-1/\lambda^2}$).

Simons [20] weakened (3.3) to the "upward condition":
(3.4)

$$\begin{cases} \forall \varepsilon > 0,\ \exists \eta > 0 \textit{ such that, } \forall\, y_0, y_1 \in Y,\ \exists\, y \in Y \textit{ such that,} \\ x \in X \textit{ and } |f(x,y_0) - f(x,y_1)| \geqq \varepsilon \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1) - \eta \\ \textit{and } x \in X \Rightarrow f(x,y) \leqq f(x,y_0) \vee f(x,y_1). \end{cases}$$

(To see this take $\eta := \pi(\varepsilon)$.)

We now show that *if* $\beta < \gamma < \delta$ *then* (3.4) *implies* (2.2): We set $\varepsilon := \gamma - \beta$, choose η as in (3.4) and $\gamma_0, \dots, \gamma_N \in [\gamma, \delta]$ with $\gamma_0 = \delta$, $\gamma_N = \gamma$ and, $\forall\, n \in \{1, \dots, N\}$, $\gamma_{n-1} - \gamma_n \leqq \eta$. Let $y_0, y_1 \in Y$ and choose $y \in Y$ as in (3.4). Suppose that $f(x,y_0) \leqq \gamma_{n-1}$ and $f(x,y_1) \leqq \beta$. We distinguish two cases:

Case 1: $f(x,y_0) \leqq \gamma$. Then $f(x,y) \leqq \gamma \vee \beta = \gamma \leqq \gamma_n$.

Case 2: $f(x,y_0) > \gamma$. Then $f(x,y_0) - f(x,y_1) \geqq \varepsilon$ hence, from (3.4),

$$f(x,y) \leqq \gamma_{n-1} \vee \beta - \eta = \gamma_{n-1} - \eta \leqq \gamma_n.$$

Thus $f(x,y_0) \leqq \gamma_{n-1}$ and $f(x,y_1) \leqq \beta \Rightarrow f(x,y) \leqq \gamma_n$, from which (2.2.1) follows. We can prove similarly that (2.2.2) holds. Finally, $f(\cdot,y) \leqq \leqq f(\cdot,y_0) \vee f(\cdot,y_1))$ gives (2.2.3) and (2.2.4).

Irle [7] introduced the concept of an *averaging function* φ (a suitable real function defined on a suitable subset of $\mathbf{R} \times \mathbf{R}$) and considered a condition of the form:

$$\begin{cases} \forall \varepsilon > 0 \textit{ and } y_0, y_1 \in Y,\ \exists\, y \in Y \textit{ such that,} \\ x \in X \Rightarrow f(x,y) \leqq \varphi(f(x,y_0),\ f(x,y_1)) + \varepsilon. \end{cases}$$

We see that, in common with the situation already described for Neumann's result, it suffices that Irle's condition hold "to within ε". However, if φ is a suitable averaging function or, more generally, *mean function* in the sense of Kindler [11] then

$$\text{(3.5)} \qquad \begin{cases} \forall\, y_0, y_1 \in Y,\ \exists\, y \in Y \textit{ such that,} \\ x \in X \Rightarrow f(x,y) \leqq \varphi(f(x,y_0),\ f(x,y_1)) \end{cases}$$

implies that (2.2) holds if $\beta < \gamma < \delta$.

Irle's minimax theorem was generalized by Simons [22], however it complicates the proof immensely to have to deal with "to within ε" conditions. In this paper, we shall follow the philosophy of Kindler [11] and not consider "to within ε" conditions. We hope that this simplification will show the underlying structures more clearly.

Using the same method of proof as that used in Lemma 2, one can establish the following more general result:

LEMMA 2′. *Let $T \subset Y$ and $\beta, \gamma \in \mathbf{R}$ with $\beta \leqq \gamma$. Suppose that, $\forall\, \delta > \gamma$, $\exists\, N \geqq 1$ and $\gamma_0, \dots, \gamma_N \in \mathbf{R}$ such that $\gamma_0 = \delta$, $\gamma_N = \gamma$ and $\forall\, y_0, y_1 \in T$, $\exists\, y \in T$ such that, $\forall\, n \in \{1, \dots, N\}$, (2.2.1), (2.2.2) and (2.2.4) hold. Let $x_0, x_1 \in X$ and $\underline{\beta\rfloor}x_0$ and $\underline{\beta\rfloor}x_1$ be joined by T. Then*

$$\underline{\gamma\rfloor}x_0 \cap \underline{\gamma\rfloor}x_1 \cap T \neq 0.$$

Kindler [11] was the first to observe that there are conditions resembling connectedness that are automatic in certain minimax theorems. He defines two concepts, *φ-connectedness* and *Γ-connectedness* and uses φ- connectedness to establish a general minimax theorem. We will not discuss φ-connectedness further since it involves a *mean function* φ, and the philosophy of this paper is to work as much as possible with the intrinsic properties of X, Y and f and avoid additional functions. The precise definition of Γ-connectedness is: *if* $\sup_X \inf_Y f < \beta < \gamma < \infty$, *$W$ is a finite subset of X, $x_0, x_1 \in X$, and $\underline{\beta\rfloor}x_0$* and *$\underline{\beta\rfloor}x_1$ are joined by $LE(W, \beta)$, then $\underline{\gamma\rfloor}x_0 \cap$ $\cap\, \underline{\gamma\rfloor}x_1 \cap LE(W, \gamma) \neq 0$.* Thus Lemma 2′ can be used to give a sufficient condition for Γ-connectedness and, in fact, for a more general concept in which W is not restricted to be finite.

Sufficient conditions for (1.3)

We suppose throughout this section that $Z \subset Y$.

LEMMA 4. *Let X be a topological space, $\beta \in \mathbf{R}$, $x_0, x_1 \in X$, and C be a connected subset of X such that*

$$C \ni x_0,\ x_1 \text{ and, } \forall\, x \in C,\ \ \underline{\beta\rfloor}x \subset \underline{\beta\rfloor}x_0 \cup \underline{\beta\rfloor}x_1. \tag{4.1}$$

Suppose that

$$\forall\, y \in Z,\ \{x : x \in C,\ f(x, y) < \beta\} \text{ is open in } C \tag{4.2}$$

and

$$\forall\, x \in C,\ \exists\, y \in Z \text{ such that } f(x, y) < \beta. \tag{4.3}$$

Then $\exists x \in X$ *such that*

$$\underline{\beta\rfloor}x_0 \text{ and } \underline{\beta\rfloor}x_1 \text{ are joined by } \underline{\beta\rfloor}x \cap Z. \tag{4.4}$$

PROOF. We can suppose that

$$\underline{\beta\rfloor}x_0 \cap \underline{\beta\rfloor}x_1 \cap Z = \emptyset, \tag{4.5}$$

for otherwise (4.4) follows with $x := x_0$. For $i = 0, 1$, let

$$C_i := \left\{x : x \in C,\ \underline{\beta\rfloor}x \cap Z \subset \underline{\beta\rfloor}x_i\right\} \ni x_i. \tag{4.6}$$

From (4.1) and (4.5),

$$C_i = \left\{x : x \in C,\ \underline{\beta\rfloor}x \cap \underline{\beta\rfloor}x_{1-i} \cap Z = \emptyset\right\}. \tag{4.7}$$

From (4.3), (4.5) and (4.6),

$$C_0 \cap C_1 = \emptyset. \tag{4.8}$$

We can suppose that

$$C_0 \cup C_1 = C, \tag{4.9}$$

for if $x \in C \setminus (C_0 \cup C_1)$ then (4.4) follows from (4.1) and (4.7). Let $x \in C$. We now prove that

$$x \in C_0 \Leftrightarrow \exists\, y \in \underline{\beta\rfloor}x_0 \cap Z \text{ such that } f(x,y) < \beta. \tag{4.10}$$

($\Rightarrow$) If $x \in C_0$ and y is as in (4.3) then $y \in \underline{\beta\rfloor}x \cap Z$. From (4.6), $y \in$ $\in \underline{\beta\rfloor}x_0 \cap Z$, as required. ($\Leftarrow$) If y is as in the right-hand side of (4.10) then $y \in \underline{\beta\rfloor}x \cap \underline{\beta\rfloor}x_0 \cap Z$. From (4.7), $x \notin C_1$. From (4.9) $x \in C_0$. This completes the proof of (4.10). From (4.2) and (4.10), C_0 is open in C. Similarly, C_1 is open in C. Then (4.8) and (4.9) contradict the connectedness of C. This contradiction completes the proof of the Lemma.

LEMMA 5. *Let* X *be a topological space,* $\beta \in \mathbf{R}$, $x_0, x_1 \in X$, *and* C *be a connected subset of* X *such that*

$$C \ni x_0, x_1 \quad \text{and,} \quad \forall\, x \in C,\ \underline{\beta\rfloor}x \subset \underline{\beta\rfloor}x_0 \cup \underline{\beta\rfloor}x_1. \tag{4.1}$$

Let Y *be a compact topological space,*

$$\{(x,y) : x \in C,\ y \in Z,\ f(x,y) \leqq \beta\} \text{ be closed in } C \times Y, \tag{5.1}$$

and

$$\forall\, x \in C,\ \underline{\beta}\rfloor x \cap Z \neq \emptyset. \tag{5.2}$$

Then $\exists x \in X$ *such that*

$$\underline{\beta}\rfloor x_0 \text{ and } \underline{\beta}\rfloor x_1 \text{ are joined by } \underline{\beta}\rfloor x \cap Z. \tag{4.4}$$

PROOF. Even though (5.2) is weaker than (4.3), we can proceed as in the proof of Lemma 4 up to (4.9). Instead of (4.10), we have: $\forall\, x \in C$,

$$x \in C_0 \Leftrightarrow \exists\, y \in \underline{\beta}\rfloor x_0 \cap Z \quad \text{such that} \quad f(x,y) \leqq \beta. \tag{5.3}$$

Let x_λ be a net of elements of $C_0, x \in C$ and $x_\lambda \to x$. From (5.3),

$$\exists\, y_\lambda \in \underline{\beta}\rfloor x_0 \cap Z \quad \text{such that} \quad f(x_\lambda, y_\lambda) \leqq \beta.$$

Since Y is compact, by passing to an appropriate subnet, we can suppose that $\exists\, y \in Y$ such that $y_\lambda \to y$. Then $(x_\lambda, y_\lambda) \to (x,y)$ and $(x_0, y_\lambda) \to \to (x_0, y)$. From (5.1), $y \in Z$, $f(x,y) \leqq \beta$ and $f(x_0,y) \leqq \beta$. From (5.3), $x \in C_0$. Thus C_0 is closed in C. Similarly, C_1 is closed in C. Then (4.8) and (4.9) contradict the connectedness of C. This contradiction completes the proof of the Lemma.

LEMMA 6. *Let* $\alpha, \beta \in \mathbf{R}$ *and* $\alpha < \beta$. *Suppose that,* $\forall\, \zeta < \alpha$, $\exists\, N \geqq 1$ *and* $\alpha_0, \dots, \alpha_n \leqq \beta$ *such that*

$$(6.1)\quad \begin{cases} & \alpha_0 = \zeta,\ \alpha_N = \alpha \text{ and,} \\ & \forall\, t_0, t_1 \in X,\ \exists\, x \in X \text{ such that, } \forall\, n \in \{1, \dots, N\}, \\ (6.1.1) & \underline{\alpha_n}\rfloor x \subset \underline{\alpha_{n-1}}\rfloor t_0 \cup \underline{\beta}\rfloor t_1, \\ (6.1.2) & \underline{\alpha_n}\rfloor x \subset \underline{\beta}\rfloor t_0 \cup \underline{\alpha_{n-1}}\rfloor t_1, \\ (6.1.3) & \underline{\beta}\rfloor x \subset \underline{\beta}\rfloor t_0 \cup \underline{\beta}\rfloor t_1, \\ (6.1.4) & \underline{\zeta}\rfloor x \subset \underline{\zeta}\rfloor t_0 \cup \underline{\zeta}\rfloor t_1. \end{cases}$$

Suppose that

$$\forall\, x \in X,\ \underline{\alpha}\rfloor x \cap Z \neq \emptyset, \tag{6.2}$$

Let

$$x_0, x_1 \in X,\ \inf f(x_0, Z) > -\infty \text{ and } \inf f(x_1, Z) > -\infty. \tag{6.3}$$

Then $\exists x \in X$ *such that*

$$\underline{\beta}\rfloor x_0 \text{ and } \underline{\beta}\rfloor x_1 \text{ are joined by } \underline{\beta}\rfloor x \cap Z. \tag{4.4}$$

PROOF. From (6.3), we can choose $\zeta \in \mathbf{R}$ such that $\underline{\zeta\rfloor} x_0 \cap Z = \underline{\zeta\rfloor} x_1 \cap$ $\cap Z = \emptyset$. From (6.2), $\zeta < \alpha$. Let $N \geqq 1$ and $\alpha_0 \ldots, \alpha_N$ satisfy (6.1). If $t \in X$ and $\underline{\zeta\rfloor} t \cap Z = \emptyset$ then, from (6.2),

$$\underline{\alpha_0\rfloor} t \cap Z = \underline{\zeta\rfloor} t \cap Z = \emptyset \text{ and } \underline{\alpha_N\rfloor} t \cap Z = \underline{\alpha\rfloor} t \cap Z \neq \emptyset.$$

Thus $\exists\, ! g(t) \in \{1, \ldots, N\}$ such that

$$g(t) \leqq n \leqq N \Rightarrow \underline{\alpha_n\rfloor} t \cap Z \neq \emptyset \text{ and } n = g(t) \Rightarrow \underline{\alpha_{n-1}\rfloor} t \cap Z = \emptyset. \tag{6.4}$$

For $i = 0, 1$ let $U_i := \left\{ t : t \in X,\ \underline{\zeta\rfloor} t \cap Z = \emptyset,\ \underline{\beta\rfloor} t \cap Z \subset \underline{\beta\rfloor} x_i \right\} \ni x_i$.
We fix $t_i \in U_i$ to maximize $g(t_i)$ and choose $x \in X$ to satisfy (6.1.1)–(6.1.4). From (6.1.4),

$$\underline{\zeta\rfloor} x \cap Z = \emptyset. \tag{6.5}$$

From (6.1.3), $\underline{\beta\rfloor} x \cap Z \subset \left(\underline{\beta\rfloor} t_0 \cap Z \right) \cup \left(\underline{\beta\rfloor} t_1 \cap Z \right)$. Since $t_i \in U_i$,

$$\underline{\beta\rfloor} x \cap Z \subset \underline{\beta\rfloor} x_0 \cup \underline{\beta\rfloor} x_1. \tag{6.6}$$

We next prove that

$$\underline{\beta\rfloor} x \cap \underline{\beta\rfloor} x_1 \cap Z \neq \emptyset. \tag{6.7}$$

If $x \notin U_0$ then, from (6.5), $\underline{\beta\rfloor} x \cap Z \not\subset \underline{\beta\rfloor} x_0$ and (6.7) follows from (6.6). If, on the other hand, $x \in U_0$ we set $n := g(t_0)$. From the assumed maximality of $g(t_0)$, $g(x) \leqq n$. From (6.4),

$$\underline{\alpha_n\rfloor} x \cap Z \neq \emptyset \text{ and } \underline{\alpha_{n-1}\rfloor} t_0 \cap Z = \emptyset.$$

From (6.1.1), $\underline{\alpha_n\rfloor} x \cap \underline{\beta\rfloor} t_1 \cap Z \neq \emptyset$. (6.7) follows since $\alpha_n \leqq \beta$ and $t_1 \in U_1$. This completes the proof of (6.7). We can prove similarly that $\underline{\beta\rfloor} x \cap \underline{\beta\rfloor} x_0 \cap$ $\cap Z \neq \emptyset$. The result follows from (6.6).

REMARKS 7. The numbering of the statements in these remarks is chosen to correspond with the numbering of the statements in Remarks 3. The credits are identical.

$$\begin{cases} \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq [f(t_0,y) + f(t_1,y)]/2 \end{cases} \tag{7.1}$$

implies

$$\begin{cases} \exists\, s \in (0,1) \quad \textit{such that,} \quad \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq (1-s)[(t_0,y) \vee f(t_1,y)] + s[f(t_0,y) \wedge f(t_1,y)] \end{cases} \tag{7.2}$$

which implies

$$(7.3)\qquad \begin{cases} \exists \textit{ a nondecreasing function } \pi : \mathbf{R}^+ \to \mathbf{R}^+ \quad \textit{such that} \\ \lambda > 0 \Rightarrow \pi(\lambda) > 0 \\ \textit{and} \qquad \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) + \pi(|f(t_0,y) - f(t_1,y)|) \end{cases}$$

which implies
(7.4)

$$\begin{cases} \forall\, \varepsilon > 0,\ \exists\, \eta > 0 \textit{ such that, } \forall\, t_0, t_1 \in X,\ \exists\, x \in X \textit{ such that,} \\ y \in Y \textit{ and } |f(t_0,y) - f(t_1,y)| \geqq \varepsilon \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) + \eta \\ \textit{and} \qquad y \in Y \Rightarrow f(x,y) \geqq f(t_0,y) \wedge f(t_1,y) \end{cases}$$

which implies that (6.1) holds if $\zeta < \alpha < \beta$. If φ is a suitable averaging or mean function

$$(7.5)\qquad \begin{cases} \forall\, t_0, t_1 \in X,\ \exists\, x \in X \quad \textit{such that,} \\ y \in Y \Rightarrow f(x,y) \geqq \varphi(f(t_0,y), f(t_1,y)) \end{cases}$$

also implies that (6.1) holds if $\zeta < \alpha < \beta$.

The following more abstract result can be used to prove both Lemma 2 and Lemma 6. *Let U and V be nonempty sets, $B : U \to 2^V$, and $\forall\, n \in \{1, \ldots, N\}$, $D_n : U \to 2^V$. Let $D_0 = \emptyset$. Suppose that,*

$$\begin{aligned} &\forall\, t_0, t_1 \in U,\ \exists\, u \in U \quad \textit{such that,} \quad \forall\, n \in \{1, \ldots, N\}, \\ &\qquad D_{n-1}t_0 = \emptyset \quad \textit{and} \quad Bu \cap Bt_1 = \emptyset \Rightarrow D_n u = \emptyset, \\ &\qquad D_{n-1}t_1 = \emptyset \quad \textit{and} \quad Bu \cap Bt_0 = \emptyset \Rightarrow D_n u = \emptyset, \\ \text{and} &\qquad\qquad Bu \subset Bt_0 \cup Bt_1. \end{aligned}$$

Suppose also that $\{Bu\}_{u \in U}$ is pseudoconnected and, $\forall\, u \in U$, $D_N u \neq \emptyset$. Then $\forall\, u_0, u_1 \in U$, $Bu_0 \cap Bu_1 \neq \emptyset$.

We note, finally, that (4.1) automatically holds if, $\forall\, y \in Y$, $f(\cdot, y)$ is quasiconcave in the sense of interval spaces.

Applications of Theorem 1

For Theorems 8 and 9, we suppose that Y is a topological space, $\mathcal{B}$ is a nonempty subset of $\mathbf{R}$, $\inf \mathcal{B} = \sup_X \inf_Y f$ and, $\forall\, \beta \in \mathcal{B}$,

$$(8.1)\qquad \forall\, x \in X,\ \underline{\beta\rfloor} x \quad \text{is nonempty, closed and compact,}$$

and *either*

$$(8.2) \qquad \forall \ \text{nonempty finite subsets} \quad V \text{ of } X,\ LE(V,\beta) \quad \text{is connected}$$

or

$$(8.3) \qquad \begin{cases} \forall\, \delta > \gamma > \beta \quad \text{and} \quad x \in X,\ \underline{\gamma}\rfloor x \quad \text{is closed and} \\ \exists\, N \geqq 1 \quad \text{and} \quad \gamma_0, \ldots, \gamma_N \in \mathbf{R} \quad \text{such that (2.2) holds.} \end{cases}$$

(The choice can depend on β.) We point out that the "nonempty" assumption in (8.1) automatically holds if *either*, $\forall\, \beta \in \mathcal{B}$, $\beta > \sup_X \inf_Y f$ *or*, $\forall\, x \in X$, $\min f(x, Y)$ exists.

THEOREM 8. *Let Y be compact, X be a topological space and, $\forall\, \beta \in \mathcal{B}$ and $x_0, x_1 \in X, \exists$ a connected subset C of X such that*

$$(4.1) \qquad C \ni x_0,\ x_1 \ \text{and},\ \forall\, x \in C,\ \ \underline{\beta}\rfloor x \subset \underline{\beta}\rfloor x_0 \cup \underline{\beta}\rfloor x_1$$

and

$$\{(x,y) : x \in C,\ y \in Y,\ f(x,y) \leqq \beta\} \ \textit{is closed in}\ C \times Y.$$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. Let $\beta \in \mathcal{B}$. By assumption, (1.1) holds and, from Lemma 2 if necessary, if W is finite then (1.2) holds. From Lemma 5 with $Z := Y$,

$$\text{if } W = \emptyset \text{ then (1.3) holds.}$$

Now suppose that $n \geqq 1$ and

$$\text{if card } W \leqq n - 1 \text{ then (1.3) holds.}$$

From the proof of Theorem 1, if card $V \leqq n + 1$ then $LE(V, \beta) \neq \emptyset$. Thus

$$\text{if card } W \leqq n \text{ and } Z = LE(W, \beta) \text{ then (5.2) holds.}$$

From Lemma 5,

$$\text{if card } W \leqq n \text{ then (1.3) holds.}$$

Thus we have proved by induction that

$$\text{if } W \text{ is finite then (1.3) holds.}$$

The result follows from Theorem 1.

THEOREM 9. *Suppose that either*

$$(9.1)\quad \begin{cases} \forall\,\beta \in \mathcal{B},\ \beta > \sup_X \inf_Y f,\ X \textit{ is a topological space and,} \\ \forall\, x_0, x_1 \in X,\ \exists \textit{ a connected subset } C \textit{ of } X \\ \textit{such that } (4.1) \textit{ holds and} \\ \forall\, y \in Y,\ \{x : x \in C,\ f(x,y) < \beta\} \textit{ is open in } C. \end{cases}$$

or,

$$(9.2)\quad \begin{cases} \forall\,\beta \in \mathcal{B},\ \beta > \sup_X \inf_Y f, \\ \forall\,\zeta < \alpha < \beta,\ \exists\, N \geqq 1 \textit{ and } \alpha_0, \dots, \alpha_N \leqq \beta \textit{ such that } (6.1) \textit{ holds} \\ \textit{and } \forall\, x \in X,\ \inf f(x,Y) > -\infty. \end{cases}$$

Then

$$\min_Y \sup_X f = \sup_X \inf_Y f.$$

PROOF. By assumption, $\forall\,\beta \in \mathcal{B}$, (1.1) holds and, from Lemma 2 if necessary, if W is finite then (1.2) holds. From Lemma 4 or Lemma 6 with $Z := Y$,

$$\text{if } \beta \in \mathcal{B} \text{ and } W = \emptyset \text{ then } (1.3) \text{ holds.}$$

Now suppose that $n \geqq 1$ and

$$\text{if } \beta \in \mathcal{B} \text{ and card } W < n - 1 \text{ then } (1.3) \text{ holds.}$$

If $\beta \in \mathcal{B}$, we choose $\alpha \in \mathcal{B}$ such that $\alpha < \beta$. From the proof of Theorem 1 with β replaced by α, if card $V \leqq n+1$ then $LE(V,\alpha) \neq \emptyset$. Thus

$$\text{if } \beta \in \mathcal{B},\ \text{card } W \leqq n \text{ and } Z = LE(W,\beta) \text{ then } (4.3) \text{ and } (6.2) \text{ hold.}$$

From Lemma 4 or Lemma 6,

$$\text{if } \beta \in \mathcal{B} \text{ and card } W \leqq n \text{ then } (1.3) \text{ holds.}$$

Thus we have proved by induction that

$$\text{if } \beta \in \mathcal{B} \text{ and } W \text{ is finite then } (1.3) \text{ holds.}$$

The result follows from Theorem 1.

REMARKS 10. The minimax theorems referred to in the introduction that depend only on *connectedness* follow from either Theorem 8–(8.2) or Theorem 9–(8.2, 9.1). Those that depend on *algebraic* conditions, and their *set-theoretic* generalizations follow from Theorem 9–(8.3, 9.2). Those that *mix* algebraic conditions and connectedness follow from Theorem 9–(8.2, 9.2). Theorem 8–(8.3) and Theorem 9–(8.3, 9.1) give new results. We remark, finally, that in Theorem 8 and Theorem 9–(9.1), C can depend on β.

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ON OPTIMAL AVERAGES

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0. Introduction

We begin this paper with some notations:

i) The symbol $\vee$ (resp. $\wedge$) stands for the maximum (resp. the minimum).

ii) $\chi(B)$ will denote the characteristic function of the set B.

iii) $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0, \infty)$, $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, $\overline{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{\infty\}$.

Convention: $0 \cdot \infty = 0$.

Let $(\Omega, \mathcal{F})$ be any measurable space, i.e. Ω is a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω.

From now on measurable sets will be called events as in probability theory. The complement of an event B will be denoted by B'.

DEFINITION 0.1. A set function $p : \mathcal{F} \to [0,1]$ will be called the optimal measure if it satisfies the following properties:

P1. $p(\emptyset) = 0$ and $p(\Omega) = 1$.

P2. p is F-additive, i.e. $p(B \cup E) = p(B) \vee p(E)$ for all B and $E \in \mathcal{F}$.

P3. p is continuous from above, i.e. $p\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \lim\limits_{n\to\infty} p(E_n)$ whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence.

The triple $(\Omega, \mathcal{F}, p)$ will be referred to as the optimal measure space.

For all events B and C, with $B \subset C$, the identity

$$p(C \setminus B) = p(C) - p(B) + p(C \setminus B) \wedge p(B) \tag{0.1}$$

holds, and especially

$$p(B') = 1 - p(B) + p(B) \vee p(B'), \quad B \in \mathcal{F}. \tag{0.2}$$

In fact it is obvious, via the F-additivity of p, that

$$\begin{aligned} p(B) + p(C \setminus B) &= p(B) \vee p(C \setminus B) + p(B) \wedge p(C \setminus B) = \\ &= p(C) + p(B) \wedge p(C \setminus B). \end{aligned}$$

LEMMA 0.1. *Let $(B_n) \subset \mathcal{F}$ be any sequence tending increasingly to an event B, and p an optimal measure. Then*

P4. $p(B) = \lim_{n\to\infty} p(B_n)$.

PROOF. The lemma will be proved if we show that for some $n_0 \geqq 1$, $p(B) = p(B_n)$ whenever $n \geqq n_0$. Assume that for all $n \geqq 1$, $p(B) \neq p(B_n)$, which is equivalent to $p(B_n) < p(B)$, for all $n \geqq 1$. This inequality however implies that $p(B) = p(B \setminus B_n)$ for each $n \geqq 1$. But since $(B \setminus B_n)$ tends decreasingly to $\emptyset$ as $n \to \infty$, we must have that $p(B) = 0$, which is impossible. This contradiction allows us to conclude on the validity of the lemma.

It is clear that p is monotonic and σ-subadditive. Let us point out that in fuzzy sets theory, set functions satisfying properties P1, P2 and P4 are called possibility measures. (For more about these functions, see [3,5,6].)

The following examples of optimal measure have been provided by Prof. M. Laczkovich to whom I would like to express my gratitude for his advices and helpful assistance.

EXAMPLE 0.1. i) Let Ω be a nonempty set, $(x_n) \subset \Omega$ be a fixed sequence, $\mathcal{F}$ a σ-algebra of Ω and $(\alpha_n) \subset [0,1]$, $\alpha_n \downarrow 0$, a given sequence. The function $p : \mathcal{F} \to [0,1]$, defined by $p(B) = \max\{\alpha_n : x_n \in B\}$ is an optimal measure.

ii) We remark that if $\Omega = [0,1]$ and $\mathcal{F}$ is a σ-algebra of $[0,1]$ containing the Borel sets, then every optimal measure defined on $\mathcal{F}$ can be obtained as in i). In fact we first prove that if $B \in \mathcal{F}$ and $p(B) = c > 0$, then there is an $x \in B$, $p(\{x\}) = c$. Let us show that there exists a nested sequence of intervals

$$I_0 \supset I_1 \supset I_2 \supset \dots \quad \text{such that} \quad |I_n| = 2^{-n}$$

and $p(B \cap I_n) = c$, for every $n = 0, 1, 2, \dots$. Let $I_0 = [0,1]$. If I_n has been defined then let $I_n = E \cup H$ where E and H are non-overlapping intervals with $|E| = |H| = 2^{-(n+1)}$. Obviously, we may choose $I_{n+1} = E$ or H. By the continuity from above we have $p\left(\bigcap_{n=1}^{\infty}(B \cap I_n)\right) = c > 0$. In particular, $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) \neq \emptyset$. This implies that $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) = \{x\}$ and $p(\{x\}) = c$. Fix $c > 0$. Then the set $\{x : p(\{x\}) \geqq c\}$ is finite. Assume that there is an infinite sequence $(x_k) \subset [0,1]$ such that $p(\{x_k\}) \geqq c$. Thus denoting $B_k = \{x_k, x_{k+1}, \dots\}$, it is clear that $\bigcap_{k=1}^{\infty} B_k = \emptyset$; but this contradicts the fact that $p(B_k) \geqq c$. Consequently the set $E_n = \left\{x : p(\{x\}) \geqq \frac{1}{n}\right\}$ is finite for all $n \geqq 1$. Hence there is a sequence $(x_n) \subset [0,1]$ such that $p(\{x_n\}) \downarrow 0$ and every point x with $p(\{x\}) > 0$ is contained in (x_n). Therefore for all $B \in \mathcal{F}$, $p(B) = \max\{p(\{x_n\}) : x_n \in B\}$ which is just the above optimal measure.

EXAMPLE 0.2. Let $(\Omega, \mathcal{F})$ be a measurable space. Clearly if $p_0 : \mathcal{F} \to \{0,1\}$ is a σ-additive measure, then $p_0(B \cup C) = p_0(B) + p_0(C) = p_0(B) \vee$

$\vee\, p_0(C)$ for all B and $C \in \mathcal{F}$. Hence p_0 is an optimal measure. One can easily show that p_0 is the only set function which is at the same time a σ-additive and optimal measure.

REMARK 0.1. The collection $M = \{B \in \mathcal{F} : p(B) < p(\Omega)\}$ is a σ-ideal, whenever p is an optimal measure.

In comparison with the mathematical expectation, we shall define a non-linear functional which may provide us with many well-known results in probability theory. Their proofs are carried out the same way as in probability theory.

1. Optimal average of nonnegative measurable simple functions

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be an arbitrary nonnegative measurable simple function, where $(B_i)_{i=1}^{n} \subset \mathcal{F}$ is a partition of Ω. Denote by $\mathbf{1} = \chi(\Omega)$.

DEFINITION 1.1. The quantity $I(s) = \bigvee_{i=1}^{n} b_i p(B_i)$ will be referred to as the "optimal average" of s, and for $B \in \mathcal{F}$

$$I_B(s) := I(s\chi(B)) = \bigvee_{i=1}^{n} b_i p(B_i \cap B)$$

as the optimal average of s on B.

It is well-known that in general a measurable simple function has many decompositions. The question thus arises whether or not the optimal average depends on the decomposition of the simple function. The following result gives a satisfactory answer.

THEOREM 1.0. *Let* $\sum_{i=1}^{n} b_i \chi(B_i)$ *and* $\sum_{k=1}^{m} c_k \chi(C_k)$ *be two decompositions of a measurable simple function* $s \geqq 0$, *where* $(B_i)_{i=1}^{n}$ *and* $(C_k)_{k=1}^{m} \subset \mathcal{F}$ *are partitions of* Ω. *Then*

$$\bigvee_{i=1}^{n} b_i p(B_i) = \bigvee_{k=1}^{m} c_k p(C_k).$$

PROOF. Since $B_i = \bigcup_{k=1}^{m} (B_i \cap C_k)$ and $C_k = \bigcup_{i=1}^{n} (B_i \cap C_k)$, the F-additivity of p implies that

$$p(B_i) = \bigvee_{k=1}^{m} p(B_i \cap C_k) \quad \text{and} \quad p(C_k) = \bigvee_{i=1}^{n} p(B_i \cap C_k).$$

Thus

$$\bigvee_{k=1}^{m} c_k p(C_k) = \bigvee_{k=1}^{m} \bigvee_{i=1}^{n} \{c_k p(B_i \cap C_k)\},$$

and

$$\bigvee_{i=1}^{n} b_i p(B_i) = \bigvee_{i=1}^{n} \bigvee_{k=1}^{m} \{b_i p(B_i \cap C_k)\}.$$

Clearly, if $B_i \cap C_k \neq \emptyset$ then $b_i = c_k$. If $B_i \cap C_k = \emptyset$ then $p(B_i \cap C_k) = 0$. Thus by the associativity and commutativity, we have that

$$\bigvee_{i=1}^{n} b_i p(B_i) = \bigvee_{k=1}^{m} c_k p(C_k).$$

This completes the proof.

PROPOSITION 1.1. *Let s and $\overline{s}$ be two nonnegative measurable simple functions, $b \in \mathbf{R}_+$ and $B \in \mathcal{F}$ be arbitrary. Then we have:*

1. $I(b\mathbf{1}) = b$.
2. $I(\chi(B)) = p(B)$.
3. $I(b\, s) = bI(s)$.
4. $I_B(s) = 0$ *if* $p(B) = 0$.
5. $I(s) = I_B(s)$ *if* $p(B') = 0$.
6. $I(s) \leqq I(\overline{s})$ *if* $s \leqq \overline{s}$ *on* Ω.
7. $I(s + \overline{s}) \leqq I(s) + I(\overline{s})$.
8. $I_B(s) = \lim_{n\to\infty} I_{B_n}(s)$, *where* $(B_n) \subset \mathcal{F}$ *tends monotonically to* B *as* $n \to \infty$.
9. $I(s \vee \overline{s}) = I(s) \vee I(\overline{s})$.

PROOF. Parts 1–5 and 8 are obvious.

6. By Theorem 1.0 we may assume that $s = \sum_{i=1}^{n} b_i\chi(B_i)$ and $\overline{s} = \sum_{i=1}^{n} c_i\chi(B_i)$, where $s = \overline{s}$. Then

$$I(s) = \bigvee_{i=1}^{n} b_i p(B_i) \leqq \bigvee_{i=1}^{n} c_i p(B_i) = I(\overline{s}).$$

7. By Theorem 1.0 we may assume that $s = \sum_{i=1}^{n} b_i\chi(B_i)$ and $\overline{s} = \sum_{i=1}^{n} c_i\chi(B_i)$. Hence

$$I(s + \overline{s}) = \bigvee_{i=1}^{n} (b_i + c_i) p(B_i) \leqq I(s) + I(\overline{s}).$$

9. By Theorem 1.0 we may assume that $s = \sum_{i=1}^{n} b_i\chi(B_i)$ and $\overline{s} = \sum_{i=1}^{n} c_i\chi(B_i)$. Hence

$$I(s \vee \overline{s}) = \bigvee_{i=1}^{n}(b_i \vee c_i)p(B_i) = \left[\bigvee_{i=1}^{n} b_i p(B_i)\right] \vee \left[\bigvee_{i=1}^{n} c_i p(B_i)\right] = I(s) \vee I(\overline{s}).$$

This ends the proof.

2. The optimal average of nonnegative measurable functions

PROPOSITION 2.0. *Let $f \geqq 0$ be any bounded measurable function. Then $\sup_{s \leqq f} I(s) = \inf_{\overline{s} \geqq f} I(\overline{s})$, (where s and $\overline{s}$ denote nonnegative measurable simple functions).*

PROOF. Let f be a measurable function such that $0 \leqq f \leqq b$ where $b \in \mathbf{R}_+$ is some constant. Let $E_k = \left(\frac{kb}{n} \leqq f < \frac{(k+1)b}{n}\right)$, $k = 0, 1, 2, \ldots, n$. Clearly $(E_k)_{k=0}^{n} \subset \mathcal{F}$ is a partition of Ω. Define the following measurable simple functions:

$$s_n = \frac{b}{n}\sum_{k=0}^{n} k\chi(E_k), \quad \overline{s}_n = \frac{b}{n}\sum_{k=0}^{n}(k+1)\chi(E_k).\cdot$$

Obviously $s_n \leqq f \leqq \overline{s}_n$. Then we can easily observe that

$$\sup_{s \leqq f} I(s) \geqq I(s_n) = \frac{b}{n}\bigvee_{k=0}^{n} kp(E_k)$$

and

$$\inf_{\overline{s} > f} I(\overline{s}) \leqq I(\overline{s}_n) = \frac{b}{n}\bigvee_{k=0}^{n}(k+1)p(E_k) \leqq I(s_n) + \frac{b}{n}.$$

Hence

$$0 \leqq \inf_{\overline{s} \geqq f} I(\overline{s}) - \sup_{s \leqq f} I(s) \leqq I(\overline{s}_n) - I(s_n) \leqq \frac{b}{n}.$$

Letting $n \to \infty$, the result follows.

From now on measurable functions will be referred to as random variables (abbreviated "r.v.'s"), just as in probability theory.

DEFINITION 2.1. Let $f \geqq 0$ be any r.v. The quantity $Af = \sup_{s \leqq f} I(s)$ will be called the "optimal average" of f (with s denoting nonnegative measurable simple functions). We shall say that f has finite optimal average if $Af < \infty$.

For $B \in \mathcal{F}$, the optimal average of f on B will be defined as $A_B f :=$ $:= A(f\chi(B)) = \sup_{s \leqq f} I_B(s)$. (The notation "$A$" stands for optimal average in comparison with "E" standing for the mathematical expectation).

PROPOSITION 2.1. *Let $f \geqq 0$ and $g \geqq 0$ be r.v.'s, $b \in \mathbf{R}_+$ and $B \in \mathcal{F}$ be arbitrary. Then*

1. $A(b\mathbf{1}) = b$.
2. $A(\chi(B)) = p(B)$.
3. $A(bf) = bAf$.
4. $A_B f = 0$ *if* $p(B) = 0$.
5. $Af \leqq Ag$ *if* $f \leqq g$.
6. $A(f + g) \leqq Af + Ag$.
7. $A_B f = Af$ *if* $p(B') = 0$.
8. $A(f \vee g) = Af \vee Ag$.

(The proof is immediate from Proposition 1.1 via Definition 2.1.)

PROPOSITION 2.2 (optimal Markov inequality). *Let $cf \geqq 0$ be a r.v. such that $Af < \infty$. Then for every real number $x > 0$, we have*

$$xp(f \geqq x) \leqq Af.$$

PROOF. By Proposition 2.1/5, 3 and 2 it follows that

$$Af \geqq A(f\chi(f \geqq x)) \geqq xp(f \geqq x).$$

DEFINITION 2.2. A property is said to hold "almost surely" (abbreviated a.s.) if the set of points where it fails to hold is a set of optimal measure zero.

PROPOSITION 2.3. *Let $f \geqq 0$ be a r.v. with finite optimal average. Then $f < \infty$ a.s.*

PROOF. Let $E = (f = \infty)$. It is easily seen that $E_k = (f \geqq k)$, $k \geqq 1$, is a decreasing sequence of events and $E = \bigcap_{k=1}^{\infty} E_k$. Then for each $k \geqq 1$, the optimal Markov inequality yields $p(E) \leqq p(E_k) \leqq \frac{1}{k} Af$ implying that $p(E) = 0$ as $k \to \infty$.

PROPOSITION 2.4. *Let $f \geqq 0$ be a r.v. Then $Af = 0$ if and only if $f = 0$ a.s.*

PROOF. Assume that $Af = 0$. Denote $E_n = \left(f \geqq \frac{1}{n}\right)$, $n \geqq 1$. Then the optimal Markov inequality yields $\frac{1}{n} p(E_n) \leqq 0$, $n \geqq 1$. Thus $p(E_n) = 0$

for all $n \geqq 1$. Consequently, $p(f > 0) = p\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$, since E_n tends increasingly to $(f > 0)$. This means that $f = 0$ a.s.

Conversely, assume that $f = 0$ a.s. Then all measurable simple functions s, satisfying $0 \leqq s \leqq f$, are equal to zero. Thus Definition 2.1 via Proposition 1.1/1 implies that $Af = 0$, ending the proof.

PROPOSITION 2.5. *Let $f \geqq 0$ be a r.v. with finite optimal average.*

i) *If $\frac{1}{p(E)} A_E f \geqq c$ for all $E \in \mathcal{F}$, $p(E) > 0$, where $c \in \mathbf{R}_+$ is a given constant, then $f \geqq c$ a.s.*

ii) *If $\frac{1}{p(E)} A_E f \leqq d$ for all $E \in \mathcal{F}$, $p(E) > 0$, where $d \in \mathbf{R}_+$ is a given constant, then $f \leqq d$ a.s.*

PROOF. i) Let $b \geqq 0$ be an arbitrary constant with $b < c$ and denote $B = (f < b)$. If the inequality $p(B) > 0$ were to hold, then we would have had, by assumption via Proposition 2.1, that

$$c \leqq \frac{1}{p(B)} A_B f \leqq \frac{1}{p(B)} A_B(b\mathbf{1}) = b < c$$

which is impossible. Thus $p(B) = 0$. Since $(f < c - \frac{1}{n})$ tends increasingly to $(f < c)$ as $n \to \infty$, it follows that $p(f < c) = 0$, i.e. $f \geqq c$ a.s.

ii) This part can be similarly proved.

COROLLARY 2.6. *Let $f \geqq 0$ be a r.v. with finite optimal average. If $c \leqq \frac{1}{p(E)} A_E f \leqq d$ for all $E \in \mathcal{F}$, $p(E) > 0$, where c and $d \in \mathbf{R}_+$ are two given constants, then $c \leqq f \leqq d$ a.s.*

3. Some convergence theorems

Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space.

THEOREM 3.1 (optimal monotone convergence). i) *Let (f_n) be an increasing sequence of nonnegative r.v.'s and $\lim_{n\to\infty} f_n = f$. Then $Af = \lim_{n\to\infty} Af_n$.*

ii) *Let (g_n) be a decreasing sequence of r.v.'s and $\lim_{n\to\infty} g_n = g$, such that $g_1 \leqq b$ for some $b \in \mathbf{R}_+$. Then $Ag = \lim_{n\to\infty} Ag_n$.*

PROOF. i) By the monotonicity of the optimal average it is clear that $\lim_{n\to\infty} Af_n \leqq Af$. So we just need to show that

$$Af \leqq \lim_{n\to\infty} Af_n. \tag{3.1}$$

If $\lim_{n\to\infty} Af_n = \infty$ then (3.1) holds. We may thus assume that $\lim_{n\to\infty} Af_n < \infty$. Inequality (3.1) will be proved if we show that for each measurable simple function s such that $0 \leqq s \leqq f$, the inequality

$$I(s) \leqq \lim_{n\to\infty} Af_n \tag{3.1'}$$

holds. In fact let $c \in [0,1)$ be an arbitrary but fixed constant. Denote by $E_n = (f_n \geqq cs)$, $n \geqq 1$, where $0 \leqq s \leqq f$. Obviously (E_n) is an increasing sequence of events and $\Omega = \lim_{n\to\infty} E_n$. Let $\tau(E) = I_E(c,s)$, $E \in \mathcal{F}$. Again by the monotonicity of the optimal average, we have that

$$Af_n \geqq A(f_n\chi(E_n)) \geqq \tau(E_n).$$

Consequently, from Proposition 1.1/8 and 3 it follows that

$$\lim_{n\to\infty} Af_n \geqq \tau(\Omega) = cI(s)$$

which leads to (3.1)′.

ii) Let $\varepsilon > 0$ be an arbitrary fixed real number. Denote $B_n = (g_n < g + \varepsilon\mathbf{1})$, $n \geqq 1$. Obviously

$$B_1 \subset B_2 \subset \dots, \quad \Omega = \bigcup_{n=1}^{\infty} B_n, \quad \bigcap_{n=1}^{\infty} B'_n = \emptyset$$

and $\lim_{n\to\infty} p(B') = 0$.

It is clear that for all $n \geqq 1$,

$$g_n \leqq (g + \varepsilon\mathbf{1}) \vee (b\chi(B'_n)),$$

and hence $Ag_n \leqq [A(g+\varepsilon\mathbf{1})] \vee [bp(B'_n)]$, $n \geqq 1$. But since $p(B'_n) \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} Ag_n \leqq \varepsilon + Ag$. Thus $\lim_{n\to\infty} Ag_n \leqq Ag$. But as on the other hand $Ag \leqq \lim_{n\to\infty} Ag_n$, the result of part ii) follows.

This completes the proof.

We shall here below give an example showing the reason why the optimal monotone convergence theorem fails to hold for all decreasing sequences of r.v.'s.

EXAMPLE 3.1. Let $\Omega = \mathbf{N}$ and $\mathcal{F}$ be the σ-algebra of all subsets of $\mathbf{N}$. Define the set function $p : \mathcal{F} \to [0,1]$ by $p(B) = \frac{1}{\min B}$. It is not difficult to see that p is an optimal measure. Now define the following r.v.'s:

$$g_n(\omega) = \begin{cases} 0, & \omega < n \\ \omega, & \omega \geqq n. \end{cases}$$

Obviously (g_n) tends decreasingly to zero as $n \to \infty$. Let us show that $Ag_n = 1$ for all $n \geqq 1$. Obviously, $Ag_n \geqq np(\{n\}) = 1$. On the other hand, let $0 \leqq s \leqq g_n$ where $s = \sum_{i=1}^{m} b_i\chi(B_i)$. For $i = 1,\dots,m$, denote $\omega_i = \min B_i$. Then $p(B_i) = \frac{1}{\omega_i}$ and $b_i \leqq \omega_i$ for all $i = 1,\dots,m$. Hence for every $i = 1,\dots,m$ the inequality $b_ip(B_i) \leqq 1$ holds. Consequently $I(s) \leqq 1$, $0 \leqq s \leqq g_n$. Thus $Ag_n \leqq 1$ for all $n \geqq 1$.

LEMMA 3.2 (optimal Fatou). i) *Let* $f_n \geqq 0$ $(n \geqq 1)$ *be r.v.'s. Then*

$$A\left(\liminf_{n\to\infty} f_n\right) \leqq \liminf_{n\to\infty} Af_n.$$

ii) *Let* (g_n), $n \geqq 1$, *be a uniformly bounded sequence of nonnegative r.v.'s. Then*

$$\limsup_{n\to\infty} Ag_n \leqq A\left(\limsup_{n\to\infty} g_n\right).$$

PROOF. i) By definition, $\liminf_{n\to\infty} f_n = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} f_k$. Let $f_n^* = \bigwedge_{k=n}^{\infty} f_k$, $f = \liminf_{n\to\infty} f_n$. Clearly (f_n^*) is an increasing sequence. The optimal monotone convergence theorem implies that

$$A\left(\liminf_{n\to\infty} f_n\right) \leqq \liminf_{n\to\infty} Af_n.$$

ii) By definition, $\limsup_{n\to\infty} g_n = \bigwedge_{n=1}^{\infty} \bigvee_{k=1}^{\infty} g_k$. Let $g_n^* = \bigvee_{k=n}^{\infty} g_k$. Clearly (g_n^*), $n \geqq 1$ is a decreasing sequence such that $g_1^* \leqq b$ for some $b \in \mathbf{R}_+$. Then the optimal monotone convergence theorem yields the result to be proved. This completes the proof.

THEOREM 3.3 (optimal dominated convergence). *Let* $f_n \geqq 0$ $(n \geqq 1)$ *be a uniformly bounded sequence of r.v.'s. Then* $Af = \lim_{n\to\infty} Af_n$, *where* $\lim_{n\to\infty} f_n = f$ *a.s.*

PROOF. The optimal Fatou lemma via the assumption implies that

$$A\left(\liminf_{n\to\infty} f_n\right) \leqq \liminf_{n\to\infty} Af_n \leqq \limsup_{n\to\infty} Af_n \leqq A\left(\limsup_{n\to\infty} f_n\right).$$

By assumption $f = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$ a.s. Consequently,

$$Af \leqq \liminf_{n\to\infty} Af_n \leqq \limsup_{n\to\infty} Af_n \leqq Af$$

meaning that $Af = \lim_{n\to\infty} Af_n$. This ends the proof.

PROPOSITION 3.4. *Let* $f \geqq 0$ *be a bounded r.v. Then for any positive real number* ε, *there exists a number* $\delta > 0$ *such that whenever* $p(E) < \delta$, $E \in \mathcal{F}$, *the inequality* $A_E f < \varepsilon$ *holds.*

PROOF. By assumption $0 \leqq f \leqq b$ a.s. for some $b > 0$. Then by Proposition 2.1 we have for $\delta < \frac{\varepsilon}{b}$ that

$$A_E f \leqq bp(E) < b\delta < \varepsilon.$$

In the following example we shall show that Proposition 3.4 does not hold for unbounded random variables.

EXAMPLE 3.2. Let $(\mathbf{N}, \mathcal{F}, p)$ be the optimal measure space defined in Example 3.1. Define the r.v. $f(\omega) = \omega$, $\omega \in \mathbf{N}$. Clearly, $Af \geqq 1$. Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$, $0 \leqq s \leqq f$. Denote $\omega_i = \min B_i$. Then $p(B_i) = \frac{1}{\omega_i}$ and $b_i \leqq \leqq \omega_i$ $(1 \leqq i \leqq n)$. Thus $I(s) \leqq 1$ and hence $Af \leqq 1$. Consequently $Af = 1$. On the other hand, there is no $\delta > 0$ such that $p(E) < \delta$ implies $A_E f < 1$. Indeed $A_{\{\omega\}} f = 1$ for every ω, and $p(\{\omega\}) \to 0$ as $\omega \to \infty$.

DEFINITION 3.1. We say that a sequence of r.v.'s (f_n), $n \geqq 1$, converges in optimal measure to a r.v. f, if for any constant $\varepsilon > 0$, $\lim_{n \to \infty} p(|f - f_n| \geqq \geqq \varepsilon) = 0$ (abbreviated $f_n \underset{p}{\longrightarrow} f$).

THEOREM 3.5 (optimal Riesz). *Let (f_n), $n \geqq 1$, be a sequence of r.v.'s which converges in optimal measure to a r.v. f. Then there exists a subsequence (f_{n_k}), $k \geqq 1$, which converges to f a.s.*

(The proof is the same as that of the original Riesz theorem.)

4. $\mathcal{A}^\alpha$-spaces, $1 \leqq \alpha \leqq \infty$

Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space.

DEFINITION 4.1. Let $f : \Omega \to \overline{\mathbf{R}}$ be any r.v. We say that:
i) $f \in \mathcal{A}^\infty$ if $p(|f| \leqq b) = 1$ for some $b \in \mathbf{R}_+$.
ii) $f \in \mathcal{A}^1$ if $A|f| < \infty$.
iii) For $\alpha \in (1, \infty)$, $f \in \mathcal{A}^\alpha$ if $|f|^\alpha \in \mathcal{A}^1$.
Define the following quantities:
i) $\|f\|_{\mathcal{A}^\alpha} = \{A(|f|^\alpha)\}^{1/\alpha}$ where $f \in \mathcal{A}^\alpha$, $1 \leqq \alpha < \infty$.
ii) $\|f\|_{\mathcal{A}^\infty} = \inf(b > 0 : p(|f| \leqq b) = 1)$, $f \in \mathcal{A}^\infty$.
It is not difficult to see for $\alpha \in [1, \infty]$ that
i) $\|f\|_{\mathcal{A}^\alpha} \geqq 0$ and $\|f\|_{\mathcal{A}^\alpha} = 0$ if and only if $f = 0$ a.s.
ii) $\|cf\|_{\mathcal{A}^\alpha} = |c| \cdot \|f\|_{\mathcal{A}^\alpha}$, for all $c \in \mathbf{R}$.

LEMMA 4.1. i) $A|fg| \leqq \|f\|_{\mathcal{A}_1} \cdot \|g\|_{\mathcal{A}^\infty}$ *whenever* $f \in \mathcal{A}^1$, $g \in \mathcal{A}^\infty$.

ii) *Let α and $\beta \in (1, \infty)$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then $A|fg| \leqq \|f\|_{\mathcal{A}^\alpha} \cdot \cdot \|g\|_{\mathcal{A}^\beta}$ (called the optimal Hölder inequality), whenever $f \in \mathcal{A}^\alpha$ and $g \in \in \mathcal{A}^\beta$.*

iii) $\|f + g\|_{\mathcal{A}^\alpha} \leqq \|f\|_{\mathcal{A}^\alpha} + \|g\|_{\mathcal{A}^\alpha}$ *(called the optimal Minkowski inequality) whenever f and $g \in \mathcal{A}^\alpha$, with $\alpha \in [1, \infty]$.*

(Again the proof is as in the classical case.)

We have thus shown that $||f||_{\mathcal{A}^\alpha}$ $(1 \leqq \alpha \leqq \infty)$ is a norm. It is easily checked that $\mathcal{A}^\alpha$ is a vector space, $1 \leqq \alpha \leqq \infty$.

DEFINITION 4.2. Let $(f_m) \subset \mathcal{A}^\alpha$, $1 \leqq \alpha \leqq \infty$, be a sequence of r.v.'s.

i) We say that (f_m) is a Cauchy sequence in $\mathcal{A}^\alpha$ if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that for all m and $n \geqq n_0$, $||f_m - f_n||_{\mathcal{A}^\alpha} < \varepsilon$.

ii) We say that (f_m) converges to a r.v. f in $\mathcal{A}^\alpha$-norm if $||f_m - f||_{\mathcal{A}^\alpha} \to 0$ as $m \to \infty$.

REMARK 4.1. If (f_m) converges to f in $\mathcal{A}^\alpha$-norm, $1 \leqq \alpha \leqq \infty$, it is not difficult to see, via the optimal Minkowski inequality, that $f \in \mathcal{A}^\alpha$ and $\lim\limits_{m\to\infty} ||f_m||_{\mathcal{A}^\alpha} = ||f||_{\mathcal{A}^\alpha}$.

REMARK 4.2. For any $\alpha \in (1, \infty)$, we have $\mathcal{A}^\infty \subset \mathcal{A}^\alpha \subset \mathcal{A}^1$.

THEOREM 4.2. *For each $\alpha \in [1, \infty]$, $\mathcal{A}^\alpha$ is a Banach space (i.e. any Cauchy sequence in $\mathcal{A}^\alpha$ converges to a r.v. in $\mathcal{A}^\alpha$-norm).*

PROOF. $\mathcal{A}^\infty$ being trivially a Banach space, we shall prove the theorem for $\alpha \in [1, \infty)$. Let (f_n) be a Cauchy sequence in $\mathcal{A}^\alpha$. Then we can choose a subsequence $n_1 < n_2 < \ldots < n_k < \ldots$ such that for all $n \geqq n_k$,

$$||f_n - f_{n_k}||_{\mathcal{A}^\alpha} < 4^{-k}. \tag{4.1}$$

Then it is clear, by the optimal Markov inequality, that

$$p\left(|f_n - f_{n_k}| > 2^{-k}\right) 2^{-k\alpha} \leqq 2^{-k}, \tag{4.2}$$

for all $n \geqq n_k$. Denote

$$g = |f_{n_1}| + \sum_{k=1}^{\infty} \left| f_{n_{k+1}} - f_{n_k} \right|, \quad S_m = |f_{n_1}| + \sum_{k=1}^{m} \left| f_{n_{k+1}} - f_{n_k} \right|, \quad m \geqq 1.$$

Then by means of the optimal Minkowski inequality, one can observe that

$$||S_m||_{\mathcal{A}^\alpha} \leqq ||f_{n_1}||_{\mathcal{A}^\alpha} + \frac{1}{3} =: c < \infty, \quad m \geqq 1.$$

Since S_m tends increasingly to g, the optimal monotone convergence theorem yields

$$\lim_{m\to\infty} AS_m^\alpha = Ag^\alpha.$$

Thus $Ag^\alpha \leqq c^\alpha < \infty$, i.e. $g \in \mathcal{A}^\alpha$ and hence $0 \leqq g < \infty$ a.s. Let $E = (g < < \infty)$. Then for all $\omega \in E$, $f_{n_1}(\omega) + \sum\limits_{k=1}^{\infty} \left(f_{n_{k+1}}(\omega) - f_{n_k}(\omega)\right)$ is finite.

Denote

$$f = \begin{cases} f_{n_1} + \sum\limits_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right) & \text{on } E \\ 0 & \text{on } E' \end{cases}$$

and

$$s_m = f_{n_1} + \sum_{k=1}^{m} (f_{n_{k+1}} - f_{n_k}) = f_{n_m}, \quad m \geqq 1.$$

Thus $\lim_{m\to\infty} s_m = f$ a.s. and $f \in \mathcal{A}^\alpha$. Let us show that

$$p\left(|f_{n_k} - f| > 2^{-k}\right) \leqq 2^{-k}. \tag{4.3}$$

Indeed, $p\left(\left|f_{n_k} - f_{n_j}\right| > 2^{-k}\right) < 2^{-k}$ for all $j > k$ (cf. 4.2)) and $f_{n_j} \to f$ a.s., when $j \to \infty$. Let us fix k and denote $B_j = \left(\left|f_{n_k} - f_{n_j}\right| > 2^{-k}\right)$. Clearly $p(B_j) < 2^{-k}$ for all $j \geqq k$, and then inequality (4.3) follows from

$$\left(|f_{n_k} - f| > 2^{-k}\right) \subset \bigcup_{m=1}^{\infty} \bigcap_{j=m}^{\infty} B_j$$

and

$$p\left(\bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} B_j\right) \leqq 2^{-k}.$$

The theorem will be proved if we show that $f_{n_k} - f \to 0$ in $\mathcal{A}^\alpha$-norm as $k \to \infty$. (For, an integer $N_0 = N_0(\varepsilon)$ can be found such that whenever $n_k \geqq N_0$ and $n \geqq N_0$,

$$\|f - f_n\|_{\mathcal{A}^\alpha} \leqq \|f - f_{n_k}\|_{\mathcal{A}^\alpha} + \|f_{n_k} - f_n\|_{\mathcal{A}^\alpha} < \varepsilon$$

where $\varepsilon > 0$ is an arbitrary real number.) Assume that there exists a positive number ε such that for infinitely many k, $A\left(|f_{n_k} - f|^\alpha\right) > \varepsilon$. Let k_0 be an integer satisfying both (4.3) and $\varepsilon > 2^{-k_0}$. Then there exists a measurable simple function s with the properties that $0 \leqq s \leqq \left|f_{n_{k_0}} - f\right|^\alpha$ and $As > \varepsilon$. Hence a constant $b \in \mathbf{R}_+$ and an event B can be found, with $b\chi(B) \leqq s$, such that $bp(B) > \varepsilon$ and $b\chi(B) \leqq \left|f_{n_{k_0}} - f\right|^\alpha$. Clearly $B \subset \left(\left\|f_{n_{k_0}} - f\right|^\alpha \geqq b\right)$ and thus

$$p\left(\left|f_{n_{k_0}} - f\right|^\alpha \geqq b\right) \geqq p(B). \tag{4.4}$$

For all $k \geqq k_0$ with $b^{1/\alpha} > 2^{-k}$, one can easily check that

$$\left(\left|f_{n_{k_0}} - f\right|^\alpha \geqq b\right) \subset \left(\left|f_{n_{k_0}} - f_{n_k}\right| \geqq b^{1/\alpha} - 2^{-k}\right) \cup \left(|f_{n_k} - f| \geqq 2^{-k}\right)$$

yielding

$$p\left(\left|f_{n_{k_0}} - f\right|^\alpha > b\right) \leqq p\left(\left|f_{n_{k_0}} - f_{n_k}\right| \geqq b^{1/\alpha} - 2^{-k}\right) + p\left(|f_{n_k} - f| \geqq 2^{-k}\right).$$

Consequently by (4.4) and (4.3) we have that

$$(4.5)\quad p(B) < p\left(\left|f_{n_{k_0}} - f\right|^{\alpha} > b\right) \leqq 2^{-k} + p\left(\left|f_{n_{k_0}} - f_{n_k}\right| \geqq b^{1/\alpha} - 2^{-k}\right).$$

By the optimal Markov inequality, via (4.5) and (4.1), it follows that

$$(4.6)\qquad 2^{-k_0} \geqq A\left(\left|f_{n_{k_0}} - f_{n_k}\right|^{\alpha}\right) \geqq \left(b^{1/\alpha} - 2^{-k}\right)^{\alpha}\left(p(B) - 2^{-k}\right)$$

for all $k \geqq k_0$ with $b^{1/\alpha} > 2^{-k}$. Now letting $k \to \infty$ in (4.6), it ensures that $2^{-k_0} \geqq bp(B) > \varepsilon$ which contradicts the choice of k_0. This allows us to conclude on the validity of the theorem.

5. The optimal Fubini theorem

Let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1,2$, be two optimal measure spaces and let us denote the smallest σ-algebra containing $\mathcal{F}_1 \times \mathcal{F}_2$ by $\mathcal{S} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. For each $\omega_1 \in \Omega_1$ (resp. $\omega_2 \in \Omega_2$) we define ω_1 (resp. $\omega_2 \in \Omega_2$) cross-section by $E_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in E\}$ (resp. $E^{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in E\}$), where $E \in \mathcal{S}$.

DEFINITION 5.1. Let f be any r.v. defined on $(\Omega_1 \times \Omega_2, \mathcal{S})$. For each $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the functions

i) $f_{\omega_1} : \Omega_2 \to \overline{\mathbf{R}}$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$ and

ii) $f_{\omega_2} : \Omega_1 \to \overline{\mathbf{R}}$ defined by $f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$

will be called ω_1-section and ω_2-section of f, respectively. Notice that for any optimal measure space $(\Omega, \mathcal{F}, p)$, we shall adopt the symbol "$\wr$" (in comparison with the symbol "$\int$" of the Lebesgue integral) to designate the optimal average, i.e.

$$Ag = \underset{\Omega}{\wr}\, g\, dp, \quad A_B g = \underset{B}{\wr}\, g\, dp$$

whenever $g \geqq 0$ is a r.v. and $B \in \mathcal{F}$.

THEOREM 5.1. *For every $E \in \mathcal{S}$, define the functions*

$$m_E : \Omega_1 \to \overline{\mathbf{R}}_+ \quad by \quad m_E(\omega_1) = p_2(E_{\omega_1})$$

and

$$m^E : \Omega_2 \to \overline{\mathbf{R}}_+ \quad by \quad m^E(\omega_2) = p_1(E^{\omega_2}).$$

Then

i) *m_E is $\mathcal{F}_1$-measurable.*

ii) *m^E is F_2-measurable.*

iii) $\int_{\Omega_1} m^E \, dp_1 = \int_{\Omega_2} m^E \, dp_2$.

Furthermore, define the function $p_1 \times p_2 : \mathcal{S} \to [0,1]$ *by*

$$(p_1 \times p_2)(E) = \int_{\Omega_1} m_E \, dp_1 = \int_{\Omega_2} m^E \, dp_2.$$

Then $p_1 \times p_2$ *is an optimal measure such that*

$$(p_1 \times p_2)(B \times D) = p_1(B) \cdot p_2(D),$$

for all $B \in \mathcal{F}_1$ *and* $D \in \mathcal{F}_2$.

PROOF. Let $\overline{\mathcal{S}}$ denote the collection of all $E \in \mathcal{S}$ for which properties i)–iii) of the theorem hold. It is enough to prove that $\overline{\mathcal{S}}$ is a σ-algebra containing $\mathcal{S}$. The proof is as in the classical case (cf. [4]) except the following claim:

For all E_1 and $E_2 \in \overline{\mathcal{S}}$, $E = E_1 \cup E_2 \in \overline{\mathcal{S}}$. By definition and property $P2$ we observe that

$$m_E(\omega_1) = m_{E_1}(\omega_1) \vee m_{E_2}(\omega_1)$$

and

$$m^E(\omega_2) = m^{E_1}(\omega_2) \vee m^{E_2}(\omega_2).$$

Thus,

$$\int_{\Omega_1} m_E \, dp_1 = \int_{\Omega_1} m_{E_1} \, dp_1 \vee \int_{\Omega_1} m_{E_2} \, dp_1 =$$

$$= \int_{\Omega_2} m^{E_1} \, dp_2 \vee \int_{\Omega_2} m^{E_2} \, dp_2 = \int_{\Omega_2} m^E \, dp_2.$$

Hence $E \in \overline{\mathcal{S}}$, since obviously m_E (resp. m^E) is $\mathcal{F}_1$- (resp. $\mathcal{F}_2$-) measurable.

THEOREM 5.2 (optimal Fubini). *Let* $(\Omega_1, \mathcal{F}_1, p_1)$ *and* $(\Omega_2, \mathcal{F}_2, p_2)$ *be two optimal measure spaces and let* $f \in \mathcal{A}^1(\Omega_1 \times \Omega_2, \mathcal{S}, p_1 \times p_2)$ *be any r.v. Then,*

1. *The* ω_1*-section* $|f_{\omega_1}| : \Omega_2 \to \overline{\mathbf{R}}_+$ *belongs to* $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$ *almost surely on* Ω_1. *The function* $\varphi : \Omega_1 \to \overline{\mathbf{R}}_+$, *defined by* $\varphi(\omega_1) = \int_{\Omega_2} |f_{\omega_1}| \, dp_2$, *belongs to* $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$.

2. *The* ω_2*-section* $|f_{\omega_2}| : \Omega_1 \to \overline{\mathbf{R}}_+$ *belongs to* $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$ *almost surely on* Ω_2. *The function* $\psi(\Omega_2) \to \overline{\mathbf{R}}_+$, *defined by* $\psi(\omega_2) = \int_{\Omega_1} |f_{\omega_2}| \, dp_1$, *belongs to* $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$.

3. *Furthermore,*

$$\int_{\Omega_1 \times \Omega_2} |f| \, d(p_1 \times p_2) = \int_{\Omega_1} \left(\int_{\Omega_2} |f| \, dp_2 \right) dp = \int_{\Omega_2} \left(\int_{\Omega_1} |f| \, dp_1 \right) dp_2.$$

(The proof follows from Theorem 5.1, using the same techniques as in the proof of the original Fubini theorem; cf. [4].)

6. Illustrations

i) Let Ω be the information set of creativity of Esther, $\mathcal{F}$ a σ-algebra of subsets of Ω. Let B_1 be the information set of her ability in mathematics with b_1 some corresponding evaluation, B_2 the information set of her ability in physics with b_2 some corresponding evaluation, etc. Let $\{B_i\}_{i=1}^n \subset \mathcal{F}$ be such that $\Omega = \bigcup_{i=1}^n B_i$. Then $0 \leqq s = \sum_{i=1}^n b_i \chi(B_i)$ may be viewed as her creativity function. To see in which subject she will be more creative, we may proceed as follows: We define (more precisely, we seek) an optimal measure and then take the optimal average of s accordingly. If the optimal average equals the infinity we may say that we cannot decide. If however the optimal average is finite then there must exist an i_0, $1 \leqq i_0 \leqq n$, such that $As = b_{i_0} p(B_{i_0})$ and thus say that Esther will be more creative in the subject corresponding to B_{i_0}.

ii) The occurrence of an event of informations stored in a given gene may be predicted the same way as above.

iii) Assume that given an input data we have got different outputs by using different statistical means. We thus have to decide which output does fit the input data. We may randomize these outputs and then proceed as above.

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ON NUMBERS WITH A LARGE PRIME POWER FACTOR

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1. Introduction. The problem of estimating the largest prime factor $P(u,k)$ dividing one of the numbers $u, u+1, \dots, u+k$, where u and k are positive integers, has a long history (see for instance [5], pp. 134–135). In 1974, Jutila [1] proved that for

$$k^{3/2} \leqq u \leqq k^{c_6 (\log k)^{1/2}/\log\log k}, \tag{1}$$

we have

$$P(u,k) \gg k^{1+c_5 \Lambda(k,u)}, \tag{2}$$

where c_5, c_6 are positive constants and $\Lambda(z,x) = (\log z/\log x)^2$. Jutila obtains this result by giving an upper bound for the exponential sum

$$\sum_{p \leqq N} e\left(\frac{x}{p}\right), \tag{3}$$

where $e(x) = \exp(2\pi i x)$ for real x, and p runs over the primes (as always in the sequel). By application of the theory of linear forms in logarithms, Tijdeman [6] and Shorey [4] were able to prove that for

$$u \geqq k^{c_6 (\log k)^{1/2}/\log\log k}, \tag{4}$$

we have

$$P(u,k) \gg k \log k \frac{\log\log k}{\log\log\log k}. \tag{5}$$

Our main result allows an improvement of that situation.

In this paper we will be concerned with large prime power factors of numbers in short intervals. For this reason let $P_j(u,k)$ be the largest prime p such that p^j divides one of the numbers $u, u+1, \dots, u+k$. We agree that all constants c_i occurring in the sequel are positive.

THEOREM 1. *Let j be a positive integer, and $\lambda > 0$. Then for some positive constants $\delta > 0$ and k_0, both depending only on j and λ, and all $k > k_0$ satisfying $k^{1+\lambda} \leqq u \leqq k^{\delta(\log k/\log\log k)^{1/2}}$, we have*

$$P_j(u,k) \gg k^{\frac{1}{j}+c_7\Lambda(k,u)},$$

where c_7 and the constant implied by $\gg$ only depend on j and λ.

REMARK. Notice that the case $j = 1$ in Theorem 1 improves Jutila's result with respect to the upper end of the permissible range of u in (1). In fact, for

$$u = k^{c_6(\log k)^{1/2}/\log\log k},$$

i.e. where the ranges in (1) and (4) meet, (2) is better than (5). By our theorem, (2) also holds for

$$k^{c_6(\log k)^{1/2}/\log\log k} \leqq u \leqq k^{\delta(\log k/\log\log k)^{1/2}},$$

thus improving on (5) in this range. Incidentally, for

$$u = k^{\delta(\log k/\log\log k)^{1/2}},$$

(2) and (5) roughly match (neglecting the constant δ).

Our main tool will be the following exponential sum estimate, which generalizes the one for (3).

THEOREM 2. *Let $j \geqq 1$, $N \leqq x^{1/j}$. Then there is a positive constant c_8 satisfying*

$$\sum_{p\leqq N} e\left(\frac{x}{p^j}\right) \ll \left(N^{1-c_8\Lambda(N,x)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}}\right)(\log x)^3.$$

The proof of Theorem 2 uses the methods of Vinogradov and van der Corput. Jutila's application of Vinogradov's complicated combinatorial argument, however, will be replaced by Vaughan's identity.

In order to show Theorem 1, we need a result on

$$D_j(\sigma) = \operatorname{card}\left\{p < P : 0 \leqq \left\{\frac{x}{p^j}\right\} < \sigma\right\},$$

where $0 < \sigma \leqq 1$, and $\{z\}$ denotes the fractional part of the real number z.

THEOREM 3. *Let $j \geqq 1$, $P \leqq x^{1/j}$. Then there is a positive constant c_9 such that*

$$D_j(\sigma) = \sigma\pi(P) + O\big(\big(P^{1-c_9\Lambda(P,x)} + P^{\frac{j+2}{2}}x^{-\frac{1}{2}}\big)(\log x)^4\big).$$

This follows from Theorem 2 by Vinogradov's Fourier series method.

Theorem 3 may be used to deal with a problem of Erdös concerning prime divisors of binomial coefficients. This is done in [3].

2. The exponential sum: Jutila's method. In this and the following section, all explicit and implicit constants may only depend on j.

LEMMA 1. *Let $j \geqq 1$, P and P' be integers satisfying $0 \leqq P' < P < T^{\frac{1}{j+1}+\frac{1}{100j^3}}$. Then*

$$\sum_{x=P}^{P+P'} e\left(\frac{T}{x^j}\right) \ll P^{1-c_{10}\Lambda(P,T)}.$$

PROOF. Clearly we may assume

$$P > (2j)^{100j^2}. \tag{6}$$

We want to apply Theorem 1 of [2] by setting

$$N = P,\ f(x) = \frac{T}{x^j},\ n = \left[100j^2\frac{\log T}{\log P}\right],$$

$$c_0 = \frac{1}{500j^5},\ c_1 = 1 - \frac{1}{50j^2},\ c_2 = 1 - \frac{1}{25j^2},\ c_3 = \frac{1}{20j^2},\ c_4 = \frac{1}{60j^2},$$

$$\{s_j : 1 \leqq j \leqq r\} = \left\{s \in \mathbf{Z} : \left(1 + \frac{1}{5j^2}\right)\frac{\log T}{\log P} \leqq s \leqq \left(1 + \frac{1}{j}\right)\frac{\log T}{\log P}\right\}.$$

By the condition of the lemma, we have

$$\frac{\log T}{\log P} > \left(\frac{1}{j+1} + \frac{1}{100j^3}\right)^{-1}. \tag{7}$$

This implies

$$\left(\left(1 + \frac{1}{j}\right) - \left(1 + \frac{1}{5j^2}\right)\right)\frac{\log T}{\log P} > 1 + \frac{1}{3j} > 1,$$

hence $\{s_j\} \neq \emptyset$. Moreover, (7) yields

$$r \geqq \left(\left(1 + \frac{1}{j}\right) - \left(1 + \frac{1}{5j^2}\right)\right)\frac{\log T}{\log P} - 1 > c_0 100j^2\frac{\log T}{\log P} \geqq c_0 n.$$

Clearly, $2 \leqq s_j \leqq n$. It remains to check conditions a) and b) of Theorem 1 in [2]. Obviously,

$$\left|\frac{1}{m!}f^{(m)}(x)\right| = \frac{Tj(j+1)\dots(j+m-1)}{m!\,|x|^{j+m}},$$

thus

$$\frac{T}{|x|^{j+m}} \leqq \left|\frac{1}{m!}f^{(m)}(x)\right| \leqq j^m \frac{T}{|x|^{j+m}}. \tag{8}$$

Therefore, we have for $P = N \leqq x \leqq N + P = 2P$

$$\left|\frac{1}{(n+1)!}f^{(n+1)}(x)\right| \leqq j^{n+1}\frac{T}{P^{j+n+1}} = P^{(n+1)\frac{\log j}{\log P}+\frac{\log T}{\log P}-(j+n+1)} = P^{-c_1'(n+1)}$$

with

$$c_1' = 1 + \frac{j}{n+1} - \frac{\log j}{\log P} - \frac{\log T}{(n+1)\log P} > c_1$$

by (6) and the definition of n and c_1. This proves a).

Similarly, for $P \leqq x \leqq 2P$ and $s \in \{s_j\}$

$$\left|\frac{1}{s!}f^{(s)}(x)\right| \leqq P^{s\frac{\log j}{\log P}+\frac{\log T}{\log P}-(j+s)} = P^{-c_3's},$$

where

$$c_3' = 1 + \frac{j}{s} - \frac{\log j}{\log P} - \frac{\log T}{s\log P} > c_3.$$

This yields the upper bound in b). By (8), we have for $P \leqq x \leqq 2P$ and $s \in \{s_j\}$

$$\left|\frac{1}{s!}f^{(s)}(x)\right| \geqq \frac{T}{(2P)^{j+s}} = P^{-c_2's},$$

where by (6), (7) and the initial definitions

$$c_2' = \left(1+\frac{j}{s}\right)\left(\frac{\log 2}{\log P}+1\right) - \frac{\log T}{s\log P} \leqq c_2.$$

This proves the lower bound of b). Now Theorem 1 of [2] implies our lemma.

LEMMA 2. *Let $j \geqq 1$, P and P' be integers satisfying $0 \leqq P' < P$, $P \geqq T^{\frac{1}{j+1}+\frac{1}{100j^3}} > 0$. Then*

$$\sum_{x=P}^{P+P'} e\left(\frac{T}{x^j}\right) \ll P^{j+1}T^{-1}.$$

PROOF. Without loss of generality, let

$$P \geqq (2j)^{100j}. \tag{9}$$

For $f(x) = \frac{T}{x^j}$ and $P \leqq x \leqq P + P'$, $f'(x)$ is obviously monotonic,

$$-f'(x) \geqq \frac{T}{(2P)^{j+1}}, \quad \text{and} \quad |f'(x)| \leqq j\frac{T}{P^{j+1}} < jP^{-\frac{1}{100j}} \leqq \frac{1}{2}$$

by (9) and the condition of the lemma. Thus by Lemma 4.8 and Lemma 4.2 of [7]

$$\sum_{x=P}^{P+P'} e\left(\frac{T}{x^j}\right) = \int_P^{P+P'} e\left(\frac{T}{x^j}\right)dx + O(1) \ll \frac{P^{j+1}}{T}.$$

Lemmas 1 and 2 imply

LEMMA 3. *Let $j \geqq 1$, P and P' be integers satisfying $0 \leqq P' < P$. Then, for $T > 0$,*

$$\sum_{x=P}^{P+P'} e\left(\frac{T}{x^j}\right) \ll P^{1-c_{10}\Lambda(P,T)} + P^{j+1}T^{-1}.$$

LEMMA 4. *Let $j \geqq 1$, $1 \leqq M \leqq x^{1/j}$. Then*

$$\sum_{m \leqq M} e\left(\frac{x}{m^j}\right) \ll M^{1-c_{11}\Lambda(M,x)} + M^{j+1}x^{-1}.$$

PROOF. Let $0 \leqq \kappa < 1$. Then

$$\left|\sum_{m<M} e\left(\frac{x}{m^j}\right)\right| \leqq \left|\sum_{m<M^\kappa} e\left(\frac{x}{m^j}\right)\right| +$$

$$+ \sum_{\substack{\nu \geqq 0 \\ M^\kappa 2^\nu \leqq M}} \left|\sum_{M^\kappa 2^\nu \leqq m < \min(M^\kappa 2^{\nu+1}, M)} e\left(\frac{x}{m^j}\right)\right| \leqq M^\kappa + \sum_\nu R_\nu,$$

say. By Lemma 3

$$R_\nu \ll (M^\kappa 2^\nu)^{1-c_{10}\Lambda(M^\kappa 2^\nu, x)} + (M^\kappa 2^\nu)^{j+1}x^{-1} \ll$$

$$\ll 2^\nu M^{\kappa - c_{10}\kappa^3\Lambda(M,x)} + 2^{(j+1)\nu}M^{(j+1)\kappa}x^{-1}.$$

Since $0 \leqq \nu \leqq \frac{(1-\kappa)\log M}{\log 2}$ in $\sum R_\nu$, we have $\sum 2^\nu \leqq 2M^{1-\kappa}$ and $\sum 2^{(j+1)\nu} \leqq$ $\leqq 2^{j+1}M^{(j+1)(1-\kappa)}$. By choosing κ close to 1, this gives the lemma.

Lemma 5. *Let* $2 \leqq M \leqq M' \leqq \min(2M, N) \leqq x$, $B \geqq 0$. *Then*

$$T := \sum_{M<m\leqq M'} \left| \sum_{B<n\leqq \frac{N}{m}} \Lambda(n) e\left(\frac{x}{(mn)^j}\right) \right|^2 \ll$$

$$\ll (N^2 M^{-1-c_{10}\Lambda(M,x)} + N^{j+2}(Mx)^{-1} + N)(\log N)^3,$$

where $\Lambda(n)$ *denotes von Mangoldt's function.*

Proof. We have

$$T = \sum_{M<m\leqq M'} \sum_{B<n_1\leqq \frac{N}{m}} \sum_{B<n_2\leqq \frac{N}{m}} \Lambda(n_1)\Lambda(n_2) e\left(x\left(\frac{1}{(mn_1)^j} - \frac{1}{(mn_2)^j}\right)\right) =$$

$$= \sum_{B<n_1\leqq \frac{N}{M}} \sum_{B<n_2\leqq \frac{N}{M}} \Lambda(n_1)\Lambda(n_2) \sum_{\substack{M<m\leqq M' \\ m\leqq \frac{N}{n_1}, m\leqq \frac{N}{n_2}}} e\left(\frac{x_1}{m^j}\right)$$

with $x_1 = x\left(\frac{1}{n_1^j} - \frac{1}{n_2^j}\right)$. Thus

$$T \ll (\log N)^2 \sum_{n_1\leqq \frac{N}{M}} \sum_{n_2\leqq \frac{N}{M}} \left| \sum_{\substack{M<m\leqq M' \\ m\leqq \frac{N}{n_1}, m\leqq \frac{N}{n_2}}} e\left(\frac{x_1}{m^j}\right) \right| =$$

$$= (\log N)^2 \left(\sum_{\substack{n_1 \\ |x_1|<M'}} \sum_{n_2} |\ \dots\ | + \sum_{\substack{n_1 \\ |x_1|\geqq M'}} \sum_{n_2} |\ \dots\ | \right) = (\log N)^2 (T_1 + T_2),$$

say. Set $A = \frac{N}{M}$; then

$$T_1 \ll M \sum_{\substack{n_1\leqq A \\ |x_1|<M'}} \sum_{n_2\leqq A} 1 \ll M \sum_{\substack{n_1 \\ 0<n_1\leqq n_2\leqq A \\ 0\leqq x_1<M'}} \sum_{n_2} 1.$$

We have

$$\sum_{\substack{n_1 \\ 0<n_1\leqq n_2\leqq A \\ 0\leqq x_1<M'}} \sum_{n_2} 1 = \sum_{n_2\leqq A} \sum_{n_2\left(1+\frac{M'n_2^j}{x}\right)^{-1/j} \leqq n_1\leqq n_2} 1 \leqq$$

$$\leqq \sum_{n_2 \leqq A} \left(n_2 \left(1 - \frac{1}{\left(1 + \frac{M' n_2^j}{x}\right)^{1/j}} \right) + 1 \right) \leqq$$

$$\leqq \sum_{n_2 \leqq A} \left(n_2 \left(1 - \frac{1}{1 + \frac{M' n_2^j}{xj}} \right) \right) + A \leqq$$

$$\leqq \sum_{n_2 \leqq A} \left(n_2^{j+1} \frac{M'}{xj} \right) + A \ll \frac{M'}{x} A^{j+2} + A.$$

This implies

$$T_1 \ll M \left(\frac{M'}{x} \left(\frac{N}{M} \right)^{j+2} + \frac{N}{M} \right) \ll \frac{N^{j+2}}{M^j x} + N. \tag{10}$$

It remains to consider T_2. By Lemma 3, we have for $|x_1| \geqq M'$

$$\Big| \sum_{\substack{M \leqq m < M' \\ m \leqq \frac{N}{n_1}, m \leqq \frac{N}{n_2}}} e\left(\frac{x_1}{m^j}\right) \Big| = \left| \sum_m e\left(\frac{|x_1|}{m^j}\right) \right| \ll$$

$$\ll M^{1 - c_{10} \Lambda(M, |x_1|)} + M^{j+1} |x_1|^{-1} \ll M^{1 - c_{10} \Lambda(M, x)} + M^{j+1} |x_1|^{-1}.$$

Hence

$$T_2 \ll \sum_{\substack{n_1 \leqq \frac{N}{M} \\ |x_1| \geqq M'}} \sum_{n_2 \leqq \frac{N}{M}} \left(M^{1 - c_{10} \Lambda(M,x)} + M^{j+1} |x_1|^{-1} \right) \ll$$

$$\ll N^2 M^{-1 - c_{10} \Lambda(M,x)} + M^{j+1} \sum_{\substack{n_1 \leqq \frac{N}{M} \\ |x_1| \geqq M'}} \sum_{n_2 \leqq \frac{N}{M}} \frac{1}{|x_1|}.$$

For arbitrary $A \geqq 1$,

$$S := \sum_{\substack{n_1 \leqq A \\ |x_1| \geqq M'}} \sum_{n_2 \leqq A} \frac{1}{|x_1|} = 2 \sum_{\substack{n_1 \\ 0 < n_1 < n_2 \leqq A \\ x_1 \geqq M'}} \sum_{n_2} \frac{1}{x_1} = 2 \sum_{0 < \Delta < A} \sum_{\substack{n_1 \\ 0 < n_1 < n_2 \leqq A \\ n_2 - n_1 = \Delta, x_1 \geqq M'}} \sum_{n_2} \frac{1}{x_1}.$$

Obviously, for $0 < n_1 < n_2$

$$x_1 = x \frac{n_2^j - n_1^j}{(n_1 n_2)^j} = x \frac{\Delta (n_2^{j-1} + n_2^{j-2} n_1 + \ldots + n_2 n_1^{j-2} + n_1^{j-1})}{(n_1 n_2)^j} > \frac{x \Delta j}{n_2^{j+1}}.$$

Thus

$$S \ll \frac{1}{x} \sum_{0<\Delta<A} \frac{1}{\Delta} \sum_{2 \leqq n_2 \leqq A} n_2^{j+1} \ll \frac{1}{x} A^{j+2} \log A.$$

Since $A = \frac{N}{M}$, we get

$$T_2 \ll N^2 M^{-1-c_{10}\Lambda(M,x)} + N^{j+2} \log N (Mx)^{-1}.$$

Together with (10), this proves the lemma.

3. The exponential sum: Vaughan's method. Proof of Theorem 2. As a corollary to Vaughan's well-known identity (see [8], [9], [10]), we have

LEMMA 6. *Let $U \geqq 2$, $V \geqq 2$, $UV \leqq N$, and $f(x)$ a complex-valued function with $|f(x)| = 1$ for real x. Then*

$$\sum_{n \leqq N} \Lambda(n) f(n) \ll V + (\log N) S_1 + |S_2|,$$

where $\Lambda(n)$ denotes von Mangoldt's function, and

$$S_1 = \sum_{t \leqq UV} \max_{w>0} \Bigg| \sum_{w \leqq r \leqq \frac{N}{t}} f(rt) \Bigg|, \quad S_2 = \sum_{U<m<\frac{N}{V}} \sum_{V<n \leqq \frac{N}{m}} \sum_{\substack{d \leqq U \\ d \mid m}} \mu(d) \Lambda(n) f(mn).$$

LEMMA 7. *Let $j \geqq 1$, $2 \leqq N \leqq x^{1/j}$. Then*

$$\sum_{n \leqq N} \Lambda(n) e\left(\frac{x}{n^j}\right) \ll \left(N^{1-c_{12}\Lambda(N,x)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}}\right) (\log N)^2 (\log x)^2.$$

PROOF. We apply Lemma 6 with $f(n) = e\left(\frac{x}{n^j}\right)$. First we consider S_2. By splitting up $\sum_m$ into intervals $M \leqq m < 2M$, we get

$$S_2 \ll (\log N) \max_{U<M<M' \leqq \min(2M, \frac{N}{V})} |S_3|,$$

where

$$S_3 = \sum_{M \leqq m < M'} \left(\sum_{V<n \leqq \frac{N}{m}} \Lambda(n) f(mn) \right) \left(\sum_{\substack{d \leqq U \\ d \mid m}} \mu(d) \right).$$

By Cauchy's inequality

$$|S_3| \leqq \left(\sum_m \left| \sum_{V<n\leqq \frac{N}{m}} \Lambda(n) f(mn) \right|^2 \right)^{1/2} \left(\sum_m \left(\sum_{\substack{d\leqq U \\ d|m}} \mu(d) \right)^2 \right)^{1/2} =$$

$$= T_1^{1/2} T_2^{1/2},$$

say. For $N \leqq x$, Lemma 5 implies

$$T_1 \ll (N^2 M^{-1-c_{10}\Lambda(M,x)} + N^{j+2}(Mx)^{-1} + N)(\log N)^3.$$

Moreover,

$$T_2 \leqq \sum_{M\leqq m<M'} \left(\sum_{\substack{d\leqq U \\ d|m}} 1 \right)^2 = \sum_{d_1\leqq U} \sum_{d_2\leqq U} \sum_{\substack{M\leqq m<M' \\ m\equiv 0 \bmod d_1, m\equiv 0 \bmod d_2}} 1 \leqq$$

$$\leqq 2M \sum_{d_1\leqq U} \sum_{d_2\leqq U} \frac{(d_1,d_2)}{d_1 d_2} \leqq 2M \sum_{b\leqq U} b \sum_{\substack{d_1\leqq U \\ d_1\equiv 0 \bmod b}} \sum_{\substack{d_2\leqq U \\ d_2\equiv 0 \bmod b}} \frac{1}{d_1 d_2} \ll M(\log N)^3.$$

Together we get

$$S_2 \ll (\log N) \max_{U<M\leqq \frac{N}{V}} (T_1 T_2)^{1/2} \ll \tag{11}$$

$$\ll (\log N)^4 \max_{U<M\leqq \frac{N}{V}} \left(NM^{-c'_{10}\Lambda(M,x)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}} + (NM)^{\frac{1}{2}} \right) \ll$$

$$\ll (\log N)^4 (NU^{-c'_{10}\Lambda(U,x)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}} + NV^{-\frac{1}{2}}).$$

It remains to estimate S_1. For $1 \leqq w \leqq \frac{N}{t}$, we have by Lemma 4

$$\left| \sum_{w\leqq r\leqq \frac{N}{t}} e\left(\frac{x}{(rt)^j} \right) \right| \leqq \left| \sum_{r\leqq \frac{N}{t}} e\left(\frac{x/t^j}{r^j} \right) \right| + \left| \sum_{r<w} e\left(\frac{x/t^j}{r^j} \right) \right| \ll$$

$$\ll \left(\frac{N}{t} \right)^{1-c_{11}\Lambda(\frac{N}{t}, \frac{x}{t^j})} + \left(\frac{N}{t} \right)^{j+1} \left(\frac{x}{t^j} \right)^{-1} + w^{1-c_{11}\Lambda(w, \frac{x}{t^j})}.$$

The function $g(y) = y^{1-c'(\log y)^2}$ is increasing for $1 \leqq y < \exp\left((3c')^{-1/2}\right)$. Without loss of generality, we may assume that $c_{11} \leqq \frac{j^2}{3}$. Then we have for $N \leqq x^{1/j}$

$$\frac{N}{t} \leqq \left(\frac{x}{t^j}\right)^{1/j} = \exp\left(\frac{\log \frac{x}{t^j}}{j}\right) \leqq \exp\left(\left(\frac{3c_{11}}{(\log \frac{x}{t^j})^2}\right)^{-1/2}\right).$$

Therefore,

$$\max_{w>0}\left|\sum_{w\leqq r\leqq \frac{N}{t}} e\left(\frac{x}{(rt)^j}\right)\right| \ll \left(\frac{N}{t}\right)^{1-c_{11}\Lambda(\frac{N}{t},\frac{x}{t^j})} + N^{j+1}(xt)^{-1}.$$

Thus, for $UV < N$,

$$S_1 \ll \sum_{t\leqq UV}\left(\left(\frac{N}{t}\right)^{1-c_{11}\Lambda(\frac{N}{t},\frac{x}{t^j})} + N^{j+1}(xt)^{-1}\right) \ll$$

$$\ll N^{1-c_{11}\Lambda(\frac{N}{UV},x)} \sum_{t\leqq UV} t^{-1+c_{11}\Lambda(\frac{N}{UV},x)} + N^{j+1}x^{-1}\sum_{t\leqq UV}\frac{1}{t} \ll$$

$$\ll N^{1-c_{11}\Lambda(\frac{N}{UV},x)}\Lambda\left(\frac{N}{UV},x\right)^{-1}(UV)^{c_{11}\Lambda(\frac{N}{UV},x)} + N^{j+1}x^{-1}\log UV.$$

Combining this with (11), Lemma 6 finally yields for $U = V = N^{1/3}$, $N \leqq \leqq x^{1/j}$, and sufficiently small c's

$$\sum_{n\leqq N}\Lambda(n)e\left(\frac{x}{n^j}\right) \ll N^{1/3} + (\log N)^2\left(N^{1-c''_{10}\Lambda(N,x)}\Lambda(N,x)^{-1} + N^{j+1}x^{-1}\right) +$$

$$+(\log N)^4\left(N^{1-c_{13}\Lambda(N,x)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}} + N^{5/6}\right) \ll$$

$$\ll \left(N^{1-c_{12}\Lambda(N,x)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}}\right)(\log N)^2(\log x)^2.$$

PROOF OF THEOREM 2. By Chebyshev's theorem

$$\sum_{n\leqq N}\Lambda(n)e\left(\frac{x}{n^j}\right) = \sum_{p}\sum_{\substack{a \\ p^a\leqq N}}\log p\, e\left(\frac{x}{p^{aj}}\right) =$$

$$= \sum_{p \leqq N} \log p \, e\left(\frac{x}{p^j}\right) + O\left(\log N \pi\left(\sqrt{N}\right)\right) = \sum_{p \leqq N} \log p \, e\left(\frac{x}{p^j}\right) + O\left(\sqrt{N}\right).$$

Define

$$h(N) = \left(N^{1-c_{12}\Lambda(N,x)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}}\right)(\log N)^2(\log x)^2.$$

By partial summation and Lemma 7, this yields

$$\left|\sum_{p \leqq N} e\left(\frac{x}{p^j}\right)\right| \leqq \left|\sum_{p \leqq N} \log p \, e\left(\frac{x}{p^j}\right)\right| \frac{1}{\log N} +$$

$$+\left|\int_2^N \sum_{p \leqq t}\left(\log p \, e\left(\frac{x}{p^j}\right)\right) \frac{dt}{t(\log t)^2}\right| \ll$$

$$\ll \frac{h(N)}{\log N} + \int_2^N \frac{h(t)}{t(\log t)^2} dt + \frac{\sqrt{N}}{\log N} \ll$$

$$\ll \frac{h(N)}{\log N} + (\log x)^2 \int_2^N t^{-c_{12}\Lambda(t,x)} dt + (\log x)^2 x^{-1/2} \int_2^N t^{j/2} dt \ll$$

$$\ll \frac{h(N)}{\log N} + (\log x)^2\left(\sqrt{N} + \int_{\sqrt{N}}^N t^{-c_{12}\Lambda(t,x)} dt\right) + N^{\frac{j+2}{2}} x^{-\frac{1}{2}} (\log x)^2 \ll$$

$$\ll \frac{h(N)}{\log N} + (\log x)^2\left(\sqrt{N} + N^{-\frac{1}{2}c_{12}\Lambda(\sqrt{N},x)} N\right) \ll \frac{h(N)}{\log N},$$

with a new constant $c_8 > 0$ in the exponent. This proves Theorem 2.

4. Vinogradov's Fourier series method. Proof of Theorem 3. The following method may be found in [11], p. 32.

Let $0 < \Delta < \frac{1}{4}$, A and B such that $0 \leqq B - A \leqq 1 - 2\Delta$. Then there is a function $\psi(z)$ with period 1,

$$\psi(z) = \begin{cases} 1 & \text{for } A \leqq z \leqq B, \\ 0 & \text{for } B + \Delta \leqq z \leqq 1 + A - \Delta, \end{cases}$$

and $0 \leqq \psi(z) \leqq 1$ otherwise, satisfying

$$\psi(z) = B - A + \Delta + \sum_{m=1}^{\infty}(a_m \cos 2\pi m z + b_m \sin 2\pi m z),$$

where

$$|a_m| \leqq \frac{2}{\pi m}, \ |b_m| \leqq \frac{2}{\pi m}, \ |a_m| \leqq \frac{2}{\pi^2 m^2 \Delta}, \ |b_m| \leqq \frac{2}{\pi^2 m^2 \Delta}.$$

Let

$$T(A,B)=\operatorname{card}\left\{p\leqq P: A\leqq\left\{\frac{x}{p^j}\right\}<B\right\}.$$

Then we have

$$T(A,B)\leqq\sum_{p\leqq P}\psi\left(\frac{x}{p^j}\right)\leqq T(A-\Delta,B+\Delta). \tag{12}$$

By construction of ψ,

$$\sum_{p\leqq P}\psi\left(\frac{x}{p^j}\right)=(B-A+\Delta)\pi(P)+\sum_{m=1}^{\infty}\left(a_m S_m^{(1)}+b_m S_m^{(2)}\right), \tag{13}$$

where

$$S_m^{(1)}+iS_m^{(2)}=\sum_{p\leqq P}e\left(\frac{mx}{p^j}\right).$$

Theorem 2 implies for $P\leqq x^{1/j}$ and $i=1,2$

$$S_m^{(i)}\ll\left(P^{1-c_8\Lambda(P,xm)}+P^{\frac{j+2}{2}}(xm)^{-\frac12}\right)\left(\log(xm)\right)^3.$$

Therefore,

$$R:=\sum_{m=1}^{\infty}(a_m S_m^{(1)}+b_m S_m^{(1)})\ll\left|\sum_{m\leqq\Delta^{-2}}\right|+\left|\sum_{m>\Delta^{-2}}\right|\ll$$

$$\ll\sum_{m\leqq\Delta^{-2}}\left(\frac{1}{m}P^{1-c_8\Lambda(P,xm)}(\log xm)^3\right)+$$

$$+P^{\frac{j+2}{2}}x^{-\frac12}\sum_{m\leqq\Delta^{-2}}\left(m^{-3/2}(\log xm)^3\right)+$$

$$+\frac{P}{\Delta}\sum_{m>\Delta^{-2}}\frac{(\log xm)^3}{m^2}+P^{\frac{j+2}{2}}x^{-\frac12}\Delta^{-1}\sum_{m>\Delta^{-2}}\frac{(\log xm)^3}{m^{5/2}}=$$

$$=S_1+P^{\frac{j+2}{2}}x^{-\frac12}S_2+\frac{P}{\Delta}S_3+P^{\frac{j+2}{2}}x^{-\frac12}\Delta^{-1}S_4$$

with the obvious definition of the S_i. Now set $\Delta=P^{-\Lambda(P,x)}$. Then for $\gamma\leqq\frac19$

$$\gamma\Lambda(P,x)\leqq\Lambda(P,x\Delta^{-2}).$$

Thus

$$S_1 \ll P^{1-c_{14}\Lambda(P,x)}\left(\log x\Delta^{-2}\right)^3 \sum_{m \leqq \Delta^{-2}} \frac{1}{m} \ll P^{1-c_{14}\Lambda(P,x)}(\log x)^4.$$

Clearly,

$$S_2 \ll \left(\log x\Delta^{-2}\right)^3 \sum_{m \leqq \Delta^{-2}} m^{-3/2} \ll (\log x)^3.$$

Moreover,

$$S_3 \ll \sum_{\Delta^{-2} < m \leqq x} \frac{(\log xm)^3}{m^2} + \sum_{\substack{m > \Delta^{-2} \\ m > x}} \frac{(\log xm)^3}{m^2} \ll$$

$$\ll (\log x)^3 \sum_{m > \Delta^{-2}} \frac{1}{m^2} + \sum_{m > \Delta^{-2}} \frac{(\log m)^3}{m^2} \ll$$

$$\ll (\log x)^3 \Delta^2 + \left(\log \Delta^{-2}\right)^3 \Delta^2 \ll (\log x)^3 \Delta^2.$$

Since $S_4 \ll S_3$, we get

$$R \ll \left(P^{1-c_{14}\Lambda(P,x)} + P^{\frac{j+2}{2}} x^{-\frac{1}{2}}\right) (\log x)^4.$$

By (13), we have

$$\sum_{p \leqq P} \psi\left(\frac{x}{p^j}\right) = (B - A + \Delta)\pi(P) + R.$$

With (12), this implies

$$T(A,B) \leqq (B - A + \Delta)\pi(P) + |R| \tag{14}$$

and

$$T(A - \Delta, B + \Delta) \geqq (B - A + \Delta)\pi(P) - |R|. \tag{15}$$

Choosing $A - \Delta$, A respectively $B, B + \Delta$ instead of A, B in (14), we get

$$T(A - \Delta, A) \leqq 2\Delta\pi(P) + |R'|,$$

respectively

$$T(B, B + \Delta) \leqq 2\Delta\pi(P) + |R''|,$$

where R' and R'' are defined in the obvious way, and both have the upper bounds given above for R. Applying (15) yields

$$T(A,B) = T(A-\Delta, B+\Delta) - T(A-\Delta, A) - T(B, B+\Delta) \geqq$$
$$\geqq (B-A)\pi(P) - 3\Delta\pi(P) - |R| - |R'| - |R''| \geqq$$
$$\geqq (B-A)\pi(P) + O\left(\left(P^{1-c_{14}\Lambda(P,x)} + P^{\frac{j+2}{2}}x^{-\frac{1}{2}}\right)(\log x)^4\right).$$

Setting $A = 0$ and $B = \sigma$, this and (14) prove Theorem 3.

5. Proof of Theorem 1. For given u and k, let

$$k < (2P)^j \leqq u. \tag{16}$$

Moreover, let

$$G = \left(P^{1-c_9\Lambda(P,u)} + P^{\frac{j+2}{2}}u^{-\frac{1}{2}}\right)(\log u)^4.$$

If we define $D = c_{15}GP^{-1}\log P$ with a suitable constant c_{15}, then, by Theorem 3, there is a prime p, $P \leqq p \leqq 2P$ such that

$$1 - D \leqq \left\{\frac{u}{p^j}\right\} < 1. \tag{17}$$

Let

$$u = np^j + u_{j-1}p^{j-1} + \ldots + u_0$$

with $n \in \mathbf{N}_0$ and $0 \leqq u_i < p$ for $0 \leqq i < j$. Then

$$\left\{\frac{u}{p^j}\right\} = \frac{u_{j-1}}{p} + \ldots + \frac{u_0}{p^j},$$

since

$$\frac{u_{j-1}}{p} + \ldots + \frac{u_0}{p^j} \leqq \frac{p-1}{p}\left(1 + \frac{1}{p} + \ldots + \frac{1}{p^{j-1}}\right) < 1.$$

Hence

$$\left\{\frac{u}{p^j}\right\} = \frac{u}{p^j} - n.$$

By (17), this implies

$$(1-D)p^j \leqq u - np^j < p^j,$$

thus

$$u < (n+1)p^j \leqq u + Dp^j \leqq u + D(2P)^j.$$

This proves Theorem 1, if (16) and $D(2P)^j \leqq k$ are satisfied. Let

$$P = k^{\frac{1}{j}(1+c_{16}\Lambda(k,u))},$$

where

$$c_{16} = \min\left\{\frac{c_9}{2j^3}, \frac{1}{6}\lambda(1-\lambda)(1+\lambda)^2\right\}$$

(and we assume $\lambda < 1$ without loss of generality). Clearly, the lower bound in (16) holds for sufficiently large k. By the condition of the theorem,

$$(2P)^j \leqq 2^j k^{1+c_{16}\Lambda(k,u)} \leqq 2^j u^{\frac{1+c_{16}\Lambda(k,u)}{1+\lambda}} \leqq u$$

for sufficiently large k, since

$$\frac{1+c_{16}\Lambda(k,u)}{1+\lambda} \leqq \frac{1+\frac{1}{6}\lambda(1+\lambda)^2\frac{1}{(1+\lambda)^2}}{1+\lambda} < 1.$$

In the remainder of the proof we will show

(18) $$D(2P)^j \leqq k.$$

By definition of D and P, we have

$$D(2P)^j = c_{15}2^j GP^{j-1}\log P \leqq$$

$$\leqq c_{15}2^j\Big(k^{\frac{1}{j}(1+c_{16}\Lambda(k,u))(j-c_9\Lambda(k^{\frac{1}{j}(1+c_{16}\Lambda(k,u))},u))}+$$

$$+k^{\frac{3}{2}(1+c_{16}\Lambda(k,u))}u^{-\frac{1}{2}}\Big)(\log u)^5.$$

For sufficiently large k (only depending on j), we have $2^j c_{15} \leqq \frac{1}{2}\log u$. Hence in order to prove (18) it suffices that

(19) $$k^{\frac{1}{j}(1+c_{16}L)(j-c_9 j^{-2}(1+c_{16}L)^2L)}(\log u)^6 \leqq k$$

and

(20) $$k^{\frac{3}{2}(1+c_{16}L)}u^{-\frac{1}{2}}(\log u)^6 \leqq k$$

with $L := \Lambda(k,u)$. We will prove Theorem 1 for

$$\delta = \sqrt{\frac{c_9}{18j^3}}.$$

By the condition of the theorem, we thus have

$$\frac{18j^3 \log\log k}{c_9 \log k} = \frac{\log\log k}{\delta^2 \log k} \leqq L \leqq \left(\frac{1}{1+\lambda}\right)^2, \tag{21}$$

and

$$\log\log u \leqq \frac{3}{2} \log\log k \tag{22}$$

for sufficiently large k.

First we show that (19) is satisfied. By taking logarithms, it suffices to have

$$6 \log\log u \leqq (c_9 j^{-3} - c_{16}) L \log k,$$

which by (21), (22) and $c_{16} \leqq c_9/2j^3$ obviously holds.

Similarly, (20) can be dealt with. Again taking logarithms, it is sufficient by (22) that

$$(1 + 3c_{16}L)\frac{\log k}{\log u} + \frac{18 \log\log k}{\log u} \leqq 1.$$

For large k (depending only on λ), the second summand on the left is smaller than $\lambda/2$. Now

$$c_{16} \leqq \frac{\lambda}{6}(1-\lambda)(1+\lambda)^2$$

and (21) imply

$$(1 + 3c_{16}L)\frac{\log k}{\log u} \leqq 1 - \frac{\lambda}{2},$$

which proves (20) and thus Theorem 1.

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WEIGHTED (0,2) INTERPOLATION ON THE EXTENDED TCHEBYCHEFF NODES OF SECOND KIND

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1. Introduction

In 1955 P. Turán initiated the study of what he called (0,2) interpolation. By (0,2) interpolation we mean an interpolation process where function values and second derivatives are prescribed at a given set of points. In a series of papers [2, 3, 7] P. Turán and his associates studied the (0,2) process on the zeros x_k of $(1-x^2)P'_{n-1}(x)$, where $P_n(x)$ is the n^{th} Legendre polynomial. Subsequently J. Balázs [1] considered the problem of weighted (0,2) interpolation, which required of finding the polynomial $p(x)$ when the values of $p(x)$ and $[(\rho(x)p(x)]''$ are prescribed at given abscissas, $\rho(x)$ being the given weight function. The problem of Balázs was modified by A. K. Varma and S. K. Gupta [6].

THEOREM 1.1 (Varma and Gupta). *If n is even, then to prescribed values y_{k0}, y_{k2} $(k=1,2,\dots,n)$ there is a uniquely determined polynomial $R_n(x)$ of degree $\leqq 2n-1$, such that*

$$(1.1)\qquad R_n(x_k)=y_{k0},\qquad \left[\left(1-x^2\right)^{\alpha}R_n(x)\right]''_{x=x_k}=y_{k2}$$

for $k=1,2,\dots,n$ and $\alpha\neq 3/4$, $\alpha\neq 9/4$, $\alpha>0$.

If n is odd there is in general no unique polynomial $R_n(x)$ of degree $\leqq$ $\leqq 2n-1$, which satisfies (1.1). For $\alpha=9/4$ or $\alpha=3/4$, the interpolatory polynomial $R_n(x)$ does not exist uniquely either for n even or n is odd.

As regards the convergence we have the following. Let $f(x)$ be a continuous function in the closed interval $[-1,+1]$ and let it satisfy the Zygmund condition

$$|f(x+h)-2f(x)+f(x-h)=o(h)\quad in\quad (-1,+1).$$

For $\alpha=7/4$ the sequence of polynomials $\{R_n(x,f)\}$ with $y_{ko}=f(x_k)$ converges uniformly to $f(x)$ in every closed interval $-1+\varepsilon\leqq x\leqq 1-\varepsilon$ ε being fixed $(0<\varepsilon<1)$, provided

$$|y_{k2}|=\frac{o(n)}{\left(1-x_k^2\right)^{3/4}},\quad k=1,2,\dots,n.$$

Our aim in this paper is to modify the above problem. On including the case $\alpha = 3/4$ our study revealed that for n even the interpolatory polynomials exist uniquely and if $f(x)$ belongs to the Zygmund class, the sequence of polynomials converges uniformly to $f(x)$ in every closed interval $-1+\varepsilon \leqq x \leqq 1-\varepsilon$ $(0<\varepsilon<1)$, ε being fixed.

A similar problem on the zeros of $T_n(x)$ [Tchebycheff polynomial of first kind] has been studied by Endeuanya [4] where he proved the existence and gave the explicit forms only.

2. Preliminaries

Let

(2.1) $$x_k = \cos\frac{k\pi}{n+1}, \quad k=1,2,\dots,n, \quad x_0=+1 \quad \text{and} \quad x_{n+1}=-1$$

be the zeros of $(1-x^2)U_n(x)$,

(2.2) $$U_n(x) = \frac{\sin(n+1)\Theta}{\sin\Theta}, \quad x=\cos\Theta$$

being the Tchebycheff polynomial of second kind. $U_n(x)$ satisfies the differential equation

(2.3) $$(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

It is easy to verify that

(2.4) $$U_n(1) = n+1 = (-1)^n U_n(-1).$$

For n even, we have

(2.5) $$(1-x_k^2)U_n'(x_k) = n+1.$$

Let

(2.6) $$l_k(x) = \frac{U_n(x)}{(x-x_k)U_n'(x_k)}, \quad k=1,2,\dots,n$$

be the fundamental polynomials of Lagrange interpolation which satisfy the conditions

(2.7) $$l_k(x_j) = \begin{cases} 0, & k\neq j \\ 1, & k=j \end{cases}.$$

We can further verify that

(2.8) $$l_k'(x_k) = \frac{3x_k}{2(1-x_k^2)}, \quad k=j,$$

(2.9) $$l_k'(x_j) = \frac{U_n'(x_j)}{(x_j-x_k)U_n'(x_k)}, \quad k\neq j.$$

3. Existence problem

We shall prove the following theorem.

THEOREM 3.1. *If n is even and x_k are given by (2.1) then to prescribed numbers f_k $(k=0,1,\dots,n+1)$ and f_k'' $(k=1,2,\dots,n)$ there is a uniquely determined polynomial $Q_n(x)$ of degree $\leqq 2n+1$ such that*

$$(3.1)\qquad \begin{cases} Q_n(x_k)=f_k, \quad k=0,1,\dots,n+1 \\ \left[\left(1-x^2\right)^{3/4}Q_n(x)\right]''_{x-x_k}=f_k'', \quad k=1,2,\dots,n. \end{cases}$$

If n is odd there is in general no unique polynomial satisfying the above conditions.

The polynomial $Q_n(x)$ of degree $\leqq 2n+1$ must have the form

$$(3.2)\qquad Q_n(x)=\sum_{k=0}^{n+1} f_k r_k(x)+\sum_{k=1}^{n} f_k'' s_k(x),$$

where $r_k(x)$ $(k=0,1,\dots,n+1)$ and $s_k(x)$ $(k=1,2,\dots,n)$ are the fundamental polynomials each of degree $\leqq 2n+1$ of our weighted (0,2) interpolation satisfying the following conditions:

$$(3.3)\qquad \begin{cases} r_k(x_j)=\begin{cases}0, & k\neq j\\ 1, & k=j\end{cases}, \quad (k,j=0,1,\dots,n+1), \\ \left[\left(1-x^2\right)^{3/4}r_k(x)\right]''_{x=x_j}=0,\ k=0,1,\dots,n+1,\ j=1,2,\dots,n, \end{cases}$$

$$(3.4)\qquad \begin{cases} s_k(x_j)=0, \quad k=1,2,\dots,n, \quad j=0,1,\dots,n+1, \\ \left[\left(1-x^2\right)^{3/4}s_k(x)\right]''_{x=x_j}=\left[\begin{matrix}0, & k\neq j\\ 1, & k=j\end{matrix}\right., \quad (k,j=1,2,\dots,n). \end{cases}$$

In the sequel we shall need the following lemmas which can be easily proved.

LEMMA 3.1. *We have for $k=1,2,\dots,n$*

(i) $\left[\left(1-x^2\right)^{3/4}U_n(x)\right]'_{x=x_k}=\left(1-x_k^2\right)^{3/4}U_n'(x_k),$

(ii) $\left[\left(1-x^2\right)^{3/4}U_n(x)\right]''_{x=x_k}=0.$

LEMMA 3.2.

$$\int_{-1}^{1} U_n(x)\,dx = 0 \quad or \quad \frac{2}{n+1}$$

according as n is odd or even.

LEMMA 3.3.

$$\frac{\left[\left(1-x^2\right)^{7/4} l_k^2(x)\right]'_{x=x_j}}{1-x_k^2} = \begin{cases} \dfrac{2\left(1-x_j^2\right)^{7/4} l_k'^2(x_j)}{1-x_k^2}, & k \neq j \\ 2\left(1-x_k^2\right)^{3/4}\left[l_k''(x_k)+l_k'^2(x_k)\right] - \\ -14x_k\left(1-x_k^2\right)^{-1/4} l_k'(x_k) - \\ -\dfrac{7}{2}\left(1-x_k^2\right)^{-1/4} + \dfrac{21}{4}x_k^2\left(1-x_k^2\right)^{-5/4}, & k = j. \end{cases}$$

PROOF OF THEOREM 3.1. For the proof of the Theorem 3.1 it is sufficient to show that if

$$f_k = 0 \quad (k = 0,1,\ldots,n+1) \quad \text{and} \quad f_k'' = 0 \quad (k = 1,2,\ldots,n)$$

then the only polynomial $Q_n(x)$ of degree $\leqq 2n+1$ satisfying the conditions (3.1) is identically zero. We write

$$Q_n(x) = U_n(x)q_{n+1}(x) \tag{3.5}$$

where $q_{n+1}(x)$ is a polynomial of degree $\leqq n+1$. The condition

$$\left[\left(1-x^2\right)^{3/4} Q_n(x)\right]''_{x=x_k} = 0, \quad k = 1,2,\ldots,n$$

requires that

$$\left[\left(1-x^2\right)^{3/4} U_n(x)q_{n+1}(x)\right]''_{x=x_k} = 0 \quad \text{for} \quad k = 1,2,\ldots,n,$$

which on using Lemma 3.1 implies $q_{n+1}'(x_k) = 0$, $k = 1,2,\ldots,n$. From this we have

$$q_{n+1}'(x) = c_1 U_n(x) \quad \text{or} \quad q_{n+1}(x) = c_1\int_{-1}^{x} U_n(t)\,dt + c_2,$$

where c_1 and c_2 are arbitrary constants. Now $q_{n+1}(-1) = 0$ gives $c_2 = 0$ and $q_{n+1}(1) = 0$ requires

$$c_1 \int_{-1}^{x} U_n(t)\,dt = 0.$$

We first consider n to be even, then owing to Lemma 3.2, we get $c_1 = 0$ and as c_2 is already zero we get $q_{n+1}(x) = 0$. Therefore from (3.5), $Q_n(x)$ is identically zero.

But if n is odd then

$$Q_n(x) = c_1 U_n(x) \int_{-1}^{x} U_n(t)\,dt$$

and therefore for n odd the problem has no unique solution.

From now onwards we shall take n to be even.

4. Explicit representation of the fundamental polynomials

For the explicit forms of the fundamental polynomials we have

THEOREM 4.1. *For n even the fundamental polynomials $r_k(x)$ ($k = 0, 1, \ldots, n+1$) satisfying the conditions* (3.3) *are given by*

$$r_0(x) = \frac{U_n(x)}{2} \int_{-1}^{x} U_n(t)\,dt, \tag{4.1}$$

$$r_{n+1}(x) = \frac{U_n(x)}{2} \int_{x}^{1} U_n(t)\,dt \tag{4.2}$$

and for $k = 1, 2, \ldots, n$ we have

$$r_k(x) = \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + \tag{4.3}$$

$$+\frac{U_n(x)}{(1-x_k^2)U_n'(x_k)}\left[\int_{-1}^{x}(1-t^2)\frac{l_k'(x_k)l_k(t)-l_k'(t)}{t-x_k}\,dt + \right.$$

$$\left. +\frac{13x_k^2+14}{8(1-x_k^2)}\int_{-1}^{x} l_k(t)\,dt + c_3\int_{-1}^{x} U_n(t)\,dt\right],$$

where c_3 is a constant given by

$$c_3 = -\frac{n+1}{2}\left[\int_{-1}^{1}(1-t^2)\frac{l_k'(x_k)l_k(t)-l_k'(t)}{(t-x_k)}\,dt + \frac{13x_k^2+14}{8(1-x_k^2)}\int_{-1}^{1} l_k(t)\,dt\right]. \tag{4.4}$$

THEOREM 4.2. *For n even the fundamental polynomials $s_k(x)$ $(k = 1,2,\ldots,n)$ satisfying the condition (3.4) are given by*

$$s_k(x) = \frac{U_n(x)}{2\left(1-x_k^2\right)^{3/4}U_n'(x_k)}\left[\int_{-1}^{x} l_k(t)\,dt + c_4\int_{-1}^{x} U_n(t)\,dt\right], \tag{4.5}$$

where c_4 is a constant given by

$$c_4 = -\frac{n+1}{2}\int_{-1}^{1} l_k(t)\,dt. \tag{4.6}$$

The conditions (3.3) and (3.4) can be easily verified on using (2.8), (2.9), Lemma 3.1, Lemma 3.2 and Lemma 3.3.

5. Convergence

Before we begin estimating the fundamental polynomials, we will simplify the integral occurring in $r_k(x)$.

LEMMA 5.1. *We have*

$$\int_{-1}^{x}\left(1-t^2\right)\frac{l_k'(x_k)l_k(t)-l_k'(t)}{t-x_k}\,dt = \frac{(1-x^2)l_k'(x)}{2} - \frac{(x+x_k)(1-x^2)U_n'(x)}{2(n+1)} -$$

$$-\frac{x_k(1-x^2)l_k(x)}{2\left(1-x_k^2\right)} - \frac{1+2x_k^2}{1-x_k^2}\int_{-1}^{x} l_k(t)\,dt + \frac{n(n+2)}{2}\int_{-1}^{x} l_k(t)\,dt -$$

$$-\frac{n(n+2)}{2(n+1)}\int_{-1}^{x}(t+x_k)U_n(t)\,dt - \frac{2x_k}{n+1}\int_{-1}^{x} U_n(t)\,dt.$$

PROOF. On using (2.8) we get

$$l_k'(x_k)l_k(t) - l_k'(t) = \frac{3tl_k(t) - 2(1-t^2)l_k'(t)}{2\left(1-x_k^2\right)} - \frac{(t-x_k)\left[3l_k(t) + 2(t+x_k)l_k'(t)\right]}{2\left(1-x_k^2\right)}. \tag{5.1}$$

But from (2.3) and (2.6) we have

$$3tl_k(t) - 2(1-t^2)l_k'(t) = (t-x_k)\left[(1-t^2)l_k''(t) - 3tl_k'(t) + n(n+2)l_k(t)\right]. \tag{5.2}$$

On applying (5.2) in (5.1), we get

$$l_k'(x_k)l_k(t) - l_k'(t) = (t-x_k)\frac{(1-t^2))l_k''(t) - 3tl_k'(t) + n(n+2)l_k(t)}{2\left(1-x_k^2\right)} - \frac{(t-x_k)\left[3l_k(t) + 2(t+x_k)l_k'(t)\right]}{2\left(1-x_k^2\right)}. \tag{5.3}$$

Thus on using (5.3) we easily get the required form.

COROLLARY 5.1. *We have*

$$\int_{-1}^{1}(1-t^2)\frac{l_k'(x_k)l_k(t) - l_k'(t)}{t-x_k}\,dt = -\frac{1+2x_k^2}{1-x_k^2}\int_{-}^{1} l_k(t)\,dt + \frac{n(n+2)}{2}\int_{-1}^{1} l_k(t)\,dt - \frac{n(n+2)}{2(n+1)}\int_{-1}^{1}(t+x_k)U_n(t)\,dt - \frac{4x_k}{(n+1)^2}.$$

Let $f(x)$ belong to $C[-1,+1]$ and $f_k = f(x_k)$. Then the sequence of polynomials $Q_n(f,x)$ has the following representation:

$$Q_n(f,x) = \sum_{k=0}^{n+1} f(x_k)r_k(x) + \sum_{k=1}^{n} f_k'' s_k(x), \tag{5.4}$$

where f_k'' are arbitrary real numbers and $r_k(x)$, $s_k(x)$ are the fundamental polynomials of our weighted (0,2) interpolation each of degree $\leqq 2n+1$.

Now we state the following convergence theorem.

THEOREM 5.1. *Let $f(x)$ satisfy the Zygmund condition*

$$|f(x+h)-2f(x)+f(x-h)|=o(h) \tag{5.5}$$

in $(-1,+1)$. *Then the sequence of polynomials* $Q_n(f,x)$ *converges uniformly to* $f(x)$ *in every closed interval* $-1+\varepsilon \leqq x \leqq 1-\varepsilon$, ε *being fixed* $(0<\varepsilon<1)$, *provided*

$$|f_k''|=\frac{o(n)}{\left(1-x_k^2\right)^{3/4}}, \quad k=1,2,\dots,n. \tag{5.6}$$

For the proof of this theorem the estimates of the fundamental polynomials are obtained in Sections 8 and 9 and we complete the proof of the theorem in Section 10.

6. Preliminaries for the estimation of the fundamental polynomials

First we shall mention some known results concerning the Tchebycheff polynomial of second kind $U_n(x)$ (proofs can be obtained from Varma and Gupta [6]) which will be needed for estimating the fundamental polynomials.

LEMMA 6.1. *For all x in $-1\leqq x\leqq 1$ we have*

(i) $|l_k(x)|\leqq 4$,

(ii) $\left|(1-x^2)^{1/2}l_k'(x)\right|\leqq 4n$,

(iii) $\left|\int_{-1}^{x}U_n(t)\,dt\right|\leqq \dfrac{2}{n+1}$,

(iv) $\left|\int_{-1}^{x}tU_n(t)\,dt\right|\leqq \dfrac{1}{2(n-2)}$,

(v) $\left|\int_{-1}^{x}l_k(t)\,dt\right|\leqq \dfrac{24\left(1-x_k^2\right)^{1/2}}{n+1}, \quad k=1,2,\dots,n,$

(vi) $\left|\int_{-1}^{x}l_k(t)\,dt\right|\leqq 4\left(1-x_k^2\right), \quad k=1,2,\dots,n$

and

(vii) $|U_n(x)|\leqq n+1$.

LEMMA 6.2. *We have*

$$\text{(i)} \qquad \frac{(1-x^2)|l_k(x)|}{1-x_k^2} \leqq 6,$$

$$\text{(ii)} \qquad \frac{(1-x^2)\,l_k^2(x)}{1-x_k^2} \leqq 24.$$

For the estimation of the fundamental polynomial $r_k(x)$, it is necessary to express it in a more convenient form. So we give the following alternative representation.

7. Alternative representation of $r_k(x)$

LEMMA 7.1. *We have*

$$\text{(7.1)} \qquad r_k(x) = \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + t(x) + \frac{U_n(x)}{n+1}\Bigg\{q(x) - \\ -\frac{(n+1)}{2}q(1)\int_{-1}^{x} U_n(t)\,dt\Bigg\},\ k=1,2,\dots,n,$$

where

(7.2)

$$t(x) = \frac{U_n(x)}{n+1}\left[\frac{(1-x^2)l_k'(x)}{2} - \frac{(x+x_k)(1-x^2)U_n'(x)}{2(n+1)} - \frac{x_k(1-x^2)l_k(x)}{2\left(1-x_k^2\right)}\right],$$

$$\text{(7.3)} \qquad q(x) = \frac{3\left(2-x_k^2\right)}{8\left(1-x_k^2\right)}\int_{-1}^{x} l_k(t)\,dt + \frac{n(n+2)}{2}\int_{-1}^{x} l_k(t)\,dt - \\ -\frac{n(n+2)}{2(n+1)}\int_{-1}^{x}(t+x_k)U_n(t)\,dt - \frac{2x_k}{n+1}\int_{-1}^{x} U_n(t)\,dt.$$

Using (2.5) and the expression for $r_k(x)$ as given in (4.3), we have the desired form on putting the value of integral from Lemma 5.1 and Corollary 5.1.

8. Estimation of $r_k(x)$

LEMMA 8.1. *For all x such that $-1+\varepsilon \leqq x \leqq 1-\varepsilon$, we have the following estimation:*

$$\text{(i)}\quad |r_k(x)| \leqq \frac{65}{\varepsilon^{3/2}}, \quad k=1,2,\dots,n,$$

$$\text{(ii)}\quad \sum_{k=1}^{n} |r_k(x)| \leqq \frac{65n}{\varepsilon^{3/2}}, \quad k=1,2,\dots,n.$$

PROOF. From (7.1), we get

$$|r_k(x)| \leqq \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + |t(x)| + 2\left|\frac{U_n(x)}{n+1}q(x)\right|.$$

We first evaluate $t(x)$. From (7.2), we get

$$|t(x)| \leqq \left|\frac{(1-x^2)U_n(x)l_k'(x)}{2(n+1)}\right| + \left|\frac{U_n(x)(x+x_k)(1-x^2)U_n'(x)}{2(n+1)^2}\right| + \left|\frac{U_n(x)x_k(1-x^2)l_k(x)}{2(n+1)\left(1-x_k^2\right)}\right|.$$

Now using Lemma 6.1 (ii) and (vii), Lemma 6.2 (i) and the fact that

$$\left|(1-x^2)^{1/2}U_n(x)\right| \leqq 1, \quad \left|(1-x^2)U_n'(x)\right| \leqq n,$$

we have $|t(x)| \leqq 6$. Further from (7.3), we have

$$2\left|\frac{U_n(x)q(x)}{n+1}\right| \leqq \left|\frac{3\left(2-x_k^2\right)U_n(x)}{4(n+1)\left(1-x_k^2\right)}\int_{-1}^{x} l_k(t)\,dt\right| + \left|\frac{n(n+2)U_n(x)}{(n+1)}\int_{-1}^{x} l_k(t)\,dt\right| +$$
$$+\left|\frac{n(n+2)U_n(x)}{(n+1)^2}\int_{-1}^{x}(t+x_k)U_n(t)\,dt\right| + \left|\frac{4x_kU_n(x)}{(n+1)^2}\int_{-1}^{x}U_n(t)\,dt\right| =$$
$$= J_1+J_2+J_3+J_4.$$

On using (v), (vi) and (vii) of Lemma 6.1, we have

$$J_1 = \left| \frac{3\left(2 - x_k^2\right) U_n(x)}{4(n+1)\left(1 - x_k^2\right)} \int\limits_{-1}^{x} l_k(t)\, dt \right| \leqq 6,$$

$$J_2 = \left| \frac{n(n+2) U_n(x)}{n+1} \int\limits_{-1}^{x} l_k(t)\, dt \right| \leqq \frac{24}{\varepsilon^{3/2}}.$$

Similarly on using (iii), (iv) and (vii) of Lemma 6.1, we have

$$J_3 = \left| \frac{n(n+2) U_n(x)}{(n+1)^2} \int\limits_{-1}^{x} (t + x_k) U_n(t)\, dt \right| \leqq \left| \frac{n(n+2) U_n(x)}{(n+1)^2} \int\limits_{-1}^{x} t U_n(t)\, dt \right| +$$

$$+ \left| \frac{n(n+2) x_k U_n(x)}{(n+1)^2} \int\limits_{-1}^{x} U_n(t)\, dt \right| \leqq 2 + 2 = 4,$$

$$J_4 = \left| \frac{4 x_k U_n(x)}{(n+1)^2} \int\limits_{-1}^{x} U_n(t)\, dt \right| \leqq 1.$$

Hence

$$2 \left| \frac{U_n(x) q(x)}{(n+1)} \right| \leqq 6 + \frac{24}{\varepsilon^{3/2}} + 4 + 1 \leqq \frac{35}{\varepsilon^{3/2}}.$$

Thus on using Lemma 6.2 (ii) we get

$$|r_k(x)| \leqq 24 + 6 + \frac{35}{\varepsilon^{3/2}} \leqq \frac{65}{\varepsilon^{3/2}}$$

and

$$\sum_{k=1}^{n} |r_k(x)| \leqq \frac{65n}{\varepsilon^{3/2}}.$$

LEMMA 8.2. *We have*

(i) $|r_0(x)| \leqq 1$,

(ii) $|r_{n+1}(x)| \leqq 1$.

9. Estimation of $s_k(x)$

LEMMA 9.1. *We have for* $-1+\varepsilon \leqq x \leqq 1-\varepsilon$,

$$\sum_{k=1}^{n} \frac{|s_k(x)|}{(1-x_k^2)^{3/4}} \leqq \frac{24}{\varepsilon^{3/2}(n+1)}, \qquad k=1,2,\dots,n.$$

PROOF. Using the expression (4.5) and (iii), (v) of Lemma 6.1, we have

$$|s_k(x)| \leqq \left| \frac{(1-x_k^2)^{1/4} U_n(x)}{2(n+1)} \int_{-1}^{x} l_k(t)\,dt \right| +$$

$$+ \left| \frac{(1-x_k^2)^{1/4} U_n(x)}{4} \int_{-1}^{1} l_k(t)\,dt \int_{-1}^{x} U_n(t)\,dt \right| \leqq$$

$$\leqq \left| \frac{(1-x_k^2)^{1/4} U_n(x)}{n+1} \int_{-1}^{x} l_k(t)\,dt \right| \leqq \frac{24(1-x_k^2)^{3/4}}{\varepsilon^{3/2}(n+1)^2}.$$

Hence,

$$\sum_{k=1}^{n} \frac{|s_k(x)|}{(1-x_k^2)^{3/4}} \leqq \frac{24}{\varepsilon^{3/2}(n+1)}.$$

10. Lemmas on approximating polynomials

LEMMA 10.1 (FREUD). *Let* $f(x)$ *be a continuous function in the interval* $[-1,+1]$ *satisfying the Zygmund condition* (5.4) *in* $(-1,+1)$. *Then there exists a sequence of polynomials* $\varphi_n(x)$ *satisfying the following conditions:*

$$|f(x)-\varphi_n(x)| = o\left(\frac{1}{n}\right)\left[(1-x^2)^{1/2} + \frac{1}{n}\right], \tag{10.1}$$

$$|\varphi_n'(x)| = o(\log n), \tag{10.2}$$

$$|\varphi_n''(x)| = o(n)\, \min\left[(1-x^2)^{-1/2}, n\right], \tag{10.3}$$

which holds uniformly in $[-1,+1]$.

We shall prove

LEMMA 10.2. *If the sequence of polynomials* $\varphi_n(x)$ *is defined as in Lemma* 10.1, *then the following hold true:*

(i) $$\sum_{k=1}^{n} \left| \left(1-x_k^2\right)^{3/4} \varphi_n''(x_k) s_k(x) \right| \leqq \frac{c_5}{\varepsilon^{3/2} n^2},$$

(ii) $$\sum_{k=1}^{n} \left| \frac{\varphi_n'(x_k) s_k(x)}{\left(1-x_k^2\right)^{1/4}} \right| \leqq \frac{c_6 \log n}{\varepsilon^{3/2}(n+1)},$$

(iii) $$\sum_{k=1}^{n} \left| \frac{\varphi_n(x_k) s_k(x)}{\left(1-x_k^2\right)^{5/4}} \right| \leqq \frac{c_8 \log n}{\varepsilon^{3/2}(n+1)},$$

The above results can be easily verified.

PROOF OF THEOREM 5.1. Let $\varphi_n(x)$ be a sequence of polynomials of degree at most n satisfying the condition (10.1)–(10.3). Then in view of uniqueness theorem we have the identity

$$\varphi_n(x) = \sum_{k=0}^{n+1} \varphi_n(x_k) r_k(x) + \sum_{k=1}^{n} \left[\left(1-x^2\right)^{3/4} \varphi_n(x) \right]''_{x=x_k} s_k(x). \tag{10.4}$$

We have

$$|Q_n(f,x) - f(x)| \leqq |Q_n(f,x) - \varphi_n(x)| + |\varphi_n(x) - f(x)|. \tag{10.5}$$

Now

$$|Q_n(f,x) - \varphi_n(x)| = \left| \sum_{k=0}^{n+1} (f(x_k) - \varphi_n(x_k)) r_k(x) + \right.$$

$$\left. + \sum_{k=1}^{n} \left(f_k'' - \left[\left(1-x^2\right)^{3/4} \varphi_n(x) \right]''_{x=x_k} s_k(x) \right| \leqq$$

$$\leqq \sum_{k=0}^{n+1} |(f(x_k) - \varphi_n(x_k)) r_k(x)| +$$

$$+ \sum_{k=1}^{n} |f_k'' s_k(x)| + \sum_{k=1}^{n} \left| \left[\left(1-x^2\right)^{3/4} \varphi_n(x) \right]''_{x=x_k} s_k(x) \right| = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

We estimate the sums $\sum_1$, $\sum_2$ and $\sum_3$ as follows:

On using (10.1), Lemma 8.1 and Lemma 8.2 we get

$$\sum\nolimits_1 = \sum_{k=0}^{n+1} |r_k(x)(f(x_k) - \varphi_n(x_k))| \leqq$$

$$\leqq |r_0(x)|\,|f(x_0)-\varphi_n(x_0)| + \sum_{k=1}^{n} |r_k(x)|\,|f(x_k)-\varphi(x_k)| +$$

$$+|r_{n+1}(x)|\,|f(x_{n+1})-\varphi(x_{n+1})| =$$

$$= o\left(\frac{1}{n^2}\right) + \sum_{k=1}^{n} o\left(\frac{1}{n}\right)\left[\left(1-x^2\right)^{1/2}+\frac{1}{n}\right]|r_k(x)| =$$

$$= o\left(\frac{1}{2}\right) + o\left(\frac{1}{n}\right)\left[\left(1-x^2\right)^{1/2}+\frac{1}{n}\right]\sum_{k=1}^{n}|r_k(x)| =$$

$$= o\left(\frac{1}{n^2}\right) + o\left(\frac{1}{n}\right)\left[\left(1-x^2\right)^{1/2}+\frac{1}{n}\right]\frac{65n}{\varepsilon^{3/2}} = o\left(\frac{1}{n^2}\right) + o(1) = o(1).$$

For $\sum_2$ we apply (5.6) and Lemma 9.1 and get

$$\sum\nolimits_2 = \sum_{k=1}^{n} |f_k'' s_k(x)| \leqq \sum_{k=1}^{n} \frac{o(n)}{\left(1-x_k^2\right)^{3/4}} |s_k(x)| = o(1).$$

Lastly on using (i), (ii) and (iii) of Lemma 10.2 we get

$$\sum\nolimits_3 = \sum_{k=1}^{n} \left| \left[\left(1-x^2\right)^{3/4}\varphi_n(x)\right]''_{x=x_k} s_k(x)\right| \leqq$$

$$\leqq \sum_{k=1}^{n} \left|\left(1-x_k^2\right)^{3/4}\varphi_n''(x_k)s_k(x)\right| +$$

$$+\sum_{k=1}^{n} \left|3x_k\left(1-x_k^2\right)^{-1/4}\varphi_n'(x_k)s_k(x)\right| +$$

$$+\sum_{k=1}^{n} \left|\frac{3}{4}\left(x_k^2-2\right)\varphi_n(x)\left(1-x_k^2\right)^{-5/4}s_k(x)\right|$$

$$\leqq \frac{c_5}{\varepsilon^{3/2}n^2} + \frac{c_9 \log n}{\varepsilon^{3/2}(n+1)} + \frac{c_{10}\log n}{\varepsilon^{3/2}(n+1)} = o(1).$$

Hence

$$|Q_n(f,x)-\varphi_n(x)| = o(1).$$

Thus from (10.5), on using (10.1), we have

$$|Q_n(f,x)-f(x)| = o(1).$$

This completes the proof of the theorem.

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QUANTUM CENTRAL LIMIT THEOREMS FOR WEAKLY DEPENDENT MAPS I

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§0. Introduction

Recall [3] that a *stochastic process* over a *-algebra $\mathcal{B}$, indexed by a set T, is a triple

$$\{\mathcal{A}, \varphi, (j_t)_{t\in T}\} \tag{0.1}$$

where $\mathcal{A}$ is a *-algebra (unless otherwise specified, all algebras in the present paper are complex, associative, with identity); φ a state on $\mathcal{A}$; and $j_t : \mathcal{B} \to \mathcal{A}$ a *-homomorphism. Every classical stochastic process (X_t) $(t \in T)$, from a probability space $(\Omega, \mathcal{F}, P)$ to a state space $(S, \mathcal{O})$ (a measurable space) naturally defines a structure as described above by choosing

$$\mathcal{A} = L^\infty(\Omega, \mathcal{F}, P)\ ;\ \mathcal{B} = L^\infty(S, \mathcal{B}),$$
$$j_t : f \in L^\infty(S, \mathcal{B}) \to j_t(f) := f \circ X_t \in L^\infty(\Omega, \mathcal{F}, P)$$

and φ to be the integral with respect to the P-measure. Conversely, every triple of the form (0.1) with $\mathcal{A}$ and $\mathcal{B}$ abelian, determines a (unique up to isomorphism) classical stochastic process.

Now let T be a subset of the natural integers. The classical law of large numbers (resp. central limit theorem) studies the asymptotic behaviour (for $N \to \infty$) of the normalized sums

$$\frac{1}{N}\sum_{j=1}^{N} f(X_j) \quad \left(\text{resp. } \frac{1}{\sqrt{N}}\sum_{j=1}^{N}[f(X_j) - \overline{f}(X_j)]\right)$$

where $f \in L^\infty(S, \mathcal{O})$ and

$$\overline{f}(X_j) = \int_\Omega f(X_j)\, dP.$$

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In the present algebraic context the analogue of these sums are

$$\frac{1}{N}\sum_{h=1}^{N} j_h(b) \quad \left(\text{resp. } \frac{1}{\sqrt{N}}\sum_{h=1}^{N}\left[j_h(b)-\varphi(j_h(b))\right]\right)$$

where $b \in \mathcal{B}$ and the study of the asymptotics of these sums, for $N \to \infty$, is the object of the *algebraic (or quantum) laws of large numbers (resp. central limit theorem)*.

In the paper by Giri and von Waldenfels [9] the first quantum central limit theorems for independent random variables was proved under the assumption that, for $h \neq k$ the algebras $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ commute. Previous results, by [11], [12] even if phrased in a quantum mechanical language are essentially classic in nature. In von Waldenfels [17] this result was extended to the case in which $\mathcal{B}$ is a $\mathbf{Z}_2$-graded algebra, the j_k are graded homomorphisms, and for $h \neq k$ the odd elements of $j_h(\mathcal{B})$ and $j_k(\mathcal{B})$ anticommute.

In these papers it was shown that, like in the classical quantum central limit theorem the limit distributions are Gaussian measures, in the quantum case the limit states are the quantum analogues of the Gaussian measures, i.e. the quasi-free states arising naturally in quantum field theory (cf. [13]). It was also shown that the usual Heisenberg commutation relation in unbounded form (or anticommutation, in the Fermi case) arise naturally from the quantum central limit theorems (cf. [18] for a simple proof). A proof of the Giri–von Waldenfels result, using cumulants techniques and an elegant noncommutative calculus of formal power series is due to Hegerfeldt [19].

Fannes and Quaegebeur [8] and, in a different context, Accardi and Bach [1] extended the central limit theorem to maps. If one starts from product maps on the CCR or the CAR algebra, the limit maps are the quasi-free maps introduced by Demoen, Vanheuverzwijn, Verbeure [20] and Evans, Lewis [5,6].

Motivated by the goal of extending the central limit theorem to quantum Markov chains, Accardi and Bach [1] extended the central limit theorem to non-independent random variables (i.e. to states φ which do not factorize on products of the form $a_{k_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_n}$ with $k_1 < k_2 < \ldots < k_n$ and $a_{k_m} \in$ $\in j_{k_m}(\mathcal{B})$) (for more details on this, cf. the remarks preceding Definition (1.2) below, where the basic strategy of [1] is outlined). In the present paper we take up the method of [1] and extend it to include the case in which the algebras $j_k(\mathcal{B})$ are not assumed to simply commute or anticommute, but to satisfy a more general commutation relation of the form

$$j_h(b)j_k(b') = \sigma_{h,k}(b,b')j_k(b')j_h(b) + \varepsilon_{h,k}(b,b')\,; \qquad h > k. \tag{0.2}$$

The first deduction of the CCR *in bounded form* from a quantum central limit theorem was given in [2], where the quantum harmonic oscillator was shown to be central limit of quantum Bernoulli processes.

In a series of papers starting from 1988, Goderis, Verbeure and Vets have deduced the CCR in bounded form from quantum central limit theorems in much more general conditions and with a new technique which allows only L^1-decay of correlations in the dependent case. Moreover in their techniques, the order structure of the index set is not relevant, hence their results include the case of a multidimensional index set (e.g. $\mathbf{Z}^d$). On the other hand, for these techniques, the commutativity of random variables localized on different sites of the lattice seems to be essential, while the consideration of the very general commutation relation (0.2) is a main gool of the present paper. Under such general commutation relation our results are new even in the product (i.e. totally independent) case.

In order to appreciate the generality of the commutation relations (0.2), let us examine some particular cases.

EXAMPLE 1. $\varepsilon_{h,k} \equiv 0$ and $\sigma_{h,k} \equiv +1$ for all $h, k \in \mathbf{N}$. This is the commuting case considered by Giri and von Waldenfels [9], and also in the papers by Goderis, Verbeure, Vets [10].

EXAMPLE 2. $\mathcal{B}$ is $\mathbf{Z}_2$-graded, $\varepsilon_{h,k} \equiv 0$ and $\sigma_{h,k} \equiv -1$ on odd elements. This is the anticommuting case of von Waldenfels [17].

EXAMPLE 3. Let H be a pre-Hilbert space with scalar product $< \cdot, \cdot >$; $\mathcal{A} = W(H)$ is the Weyl C^*-algebra of the canonical commutation relations over H with symplectic form $\operatorname{Im}\langle f, f' \rangle$ $(f, f' \in H)$. It is then given a family of pre-Hilbert subspaces $H_k \subseteqq H$ (not necessarily mutually orthogonal) such that each H_k is isomorphic to a single pre-Hilbert space H_0. Fix such an isomorphism $J_k : H_0 \to H_k$ and let $\mathcal{B} = W(H_0)$ be the Weyl C^*-algebra over H_0; for each $k \in \mathbf{N}$ define

$$j_k(W(f_0)) = W(J_k f_0) \ ; \ f_0 \in H_0 \ .$$

Then (0.2) holds with $\varepsilon_{h,k} \equiv 0$, $B = \{W(f_0) : f_0 \in H_0\}$, and

$$\sigma_{h,k}(W(f_0),\ W(g_0)) = \exp 2i \operatorname{Im}\langle J_h f_0,\ J_k g_0 \rangle \ .$$

EXAMPLE 4. Let $H, (H_k), H_0, W(H_0), W(H)$ be as in Example 3 above. Suppose that both $W(H)$ and $W(H_0)$ act on Hilbert spaces $\mathcal{H}, \mathcal{H}_0$ respectively so that the field operators exist and admit a common invariant dense domain $\mathcal{D}$ (resp. $\mathcal{D}_0$). Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the *-algebra of the polynomials in the fields, defined on the invariant domain $\mathcal{D}$ (resp. $\mathcal{D}_0$). Then if $A(f)$, $A^+(g)$ (resp. $A_0(f_0)$, $A^+(g_0)$ $(f, g \in H,\ f_0, g_0 \in H_0)$) denote the annihilation and creation operators in $\mathcal{A}$ (resp. $\mathcal{B}$), then the maps

$$j_k(A_0(f_0)) := A_0(J_k f_0); \quad j_k(A_0^+(f_0)) := A_0^+(J_k f_0)$$

define embeddings $j_k : \mathcal{B} \to \mathcal{A}$. If

$$B = \{A_0(f_0)), \ A_0^+(g_0) \ : \ f_0, g_0 \in H_0\}$$

then (0.2) holds with $\sigma_{h,k} \equiv 1$ and

$$\begin{aligned} \varepsilon_{h,k}(A_0(f_0), \ A_0^+(g_0)) &= \langle J_h f_0, J_k g_0 \rangle, \\ \varepsilon_{h,k}(A_0^+(g_0), \ A_0(f_0)) &= -\langle J_k g_0, J_h f_0 \rangle, \\ \varepsilon_{h,k}(A_0(f_0), A_0(g_0)) = \varepsilon_{h,k}(A_0^+(f_0), \ A_0^+(g_0)) &= 0. \end{aligned}$$

EXAMPLE 5. Example 4 can be modified in a obvious way to obtain the Fermion case.

EXAMPLE 6. Let $H, (H_k), (J_k), H_0$ be as in Example 4 above and $\mathcal{F}(\mathcal{F}_0)$ be the full Fock space (i.e. the tensor algebra) over $H(H_0)$ and let, for $f, g \in$ $\in H$ $l(g)$, $l^*(f)$ denote the free annihilation and creation operators (defined as in [16], cf. also [15] or [7]). Similarly one defines $l_0(f_0), l^*(g_0)$ $(f_0, g_0 \in$ $\in H_0)$. Let $\mathcal{A} \subseteqq \mathcal{B}(\mathcal{F})$ $(\mathcal{B} \subseteqq \mathcal{B}(\mathcal{F}_0))$ denote the algebra generated by the family

$$\{l^*(f), l(g) : f, g \in H\}$$

(resp. $\{l_0^*(f_0), l_0(g_0) : f_0, g_0 \in H_0\}$). Then for each $k \in \mathbf{N}$, the maps

$$j_k(l_0(f_0)) = l(J_k f_0) \ ; \ j_k(l_0^*(g_0)) = l^*(J_k g_0)$$

define embeddings $j_k : \mathcal{B} \to \mathcal{A}$. If

$$B = \{l_0(f_0), l^*(g_0) : f_0, g_0 \in H_0\}$$

then (0.2) holds with $\sigma_{h,k} \equiv 0$ and

$$\varepsilon_{h,k}(l_0(f_0), l_0^*(g_0) = \langle J_h f_0, J_k g_0 \rangle \ .$$

The above examples show that the variety of situations that can be covered by our results is very wide.

REMARK. The paper has been split into two parts: in Part I all the preliminary estimates are established; in Part II these estimates are put together to obtain the main results, i.e. the three theorems stated at the end of Section §1.

§1. Notations, definitions and statement of the main results

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be associative algebras, assume that $\mathcal{A}$ and $\mathcal{C}$ have an identity denoted, when no confusion can arise, by the same symbol 1. Let B be a subset of $\mathcal{B}$, usually it will be a subset of generators of the algebra. Let for each $t \in \mathbf{R}_+$ be given a homorphism $j_t : \mathcal{B} \to \mathcal{A}$, such that for each $t, s \in \mathbf{R}_+$, $t \neq s$ and $b, b' \in B$, there exist two scalars $\sigma(t,s,b,b')$ and $\varepsilon(t,s,b,b')$ satisfying

$$j_t(b)j_s(b') = \sigma(t,s,b,b')j_s(b')j_t(b) + \varepsilon(t,s,b,b'). \tag{1.1}$$

Notice that if $\mathcal{A}$, $\mathcal{B}$ are *-algebras and E is a state on $\mathcal{A}$, then the triple $\{\mathcal{A}, (j_t), E\}$ is a stochastic process over $\mathcal{B}$ in the sense of [3].

Following the notations of [1], we denote $\mathcal{S}_p$ the family of all p-permutations and $\mathcal{P}_{k,p}$ the family of all ordered partitions $(S_1, \dots, S_p)$ of the set $\{1, \dots, k\}$ into exactly p non-empty subsets ($k \in \mathbf{N}$ and $p \leqq k$). The partition $(S_1, \dots, S_p)$ is ordered with order "<" in the following sense: $S_i < S_j$ if and only if $\min\{r : r \in S_i\} < \min\{r : r \in S_j\}$ and each set S_j has the natural order. If some S_h has only one elements, we shall call it a *singleton.*

For each $(S_1, \dots, S_p) \in \mathcal{P}_{k,p}$ and $T \in \mathbf{R}_+$, denote $[S_1, \dots, S_p]_T$ the set of all k-tuples $(t_1, \dots, t_k) \in [0,T)^k$ such that

(i) for each $j = 1, 2, \dots, p$ and $i, i' \in S_j$, we have $t_i = t_{i'}$;

(ii) for each $j, j' = 1, 2, \dots, p$, $j \neq j'$, $i \in S_j$ and $i' \in S_{j'}$, we have $t_i \neq t_{i'}$.

The elements of $[S_1, \dots, S_p]_T$ can be identified to the functions t from $\{1, \dots k\}$ to $[0,T)$ which are constant on the elements of the partition $(S_1, \dots, S_p)$ and which take exactly p different values.

Similarly for each $N \in \mathbf{N}$, we denote by $[S_1, \dots, S_p]_N$ the set of all maps $\alpha : \{1, \dots, k\} \to \{1, \dots, N\}$ such that

(i) for each $j = 1, \dots, p$ and $i, i' \in S_j$, we have $\alpha(i) = \alpha(i')$;

(ii) for each $j, j' = 1, 2, \dots, p$, $j \neq j'$, $i \in S_j$ and $i' \in S_j$, we have $\alpha(i) \neq \alpha(i')$.

Throughout the paper, we shall denote by ν either the Lebesgue measure on $\mathbf{R}$ or the counting measure on $\mathbf{Z}$, both characterized by translation invariance and

$$\nu([0,T)) = T \ ; \qquad T \in \mathbf{R} \text{ or } \mathbf{N}.$$

We shall use the notations

$$S_T(b) = \int_{[0,T)} j_s(b)\nu(ds) \quad ; \quad T \in \mathbf{R}_+ \tag{1.2}$$

so that, if ν is the counting measure

(1.2a) $$S_N(b) = \sum_{k=1}^{N} j_k(b) \quad ; \quad N \in \mathbf{N}$$

for each $b \in \mathcal{B}$. Moreover, we assume that

(i) on $\mathcal{C}$, there is a semi-norm $|\cdot|$ and it is given a map $E : \mathcal{A} \to \mathcal{C}$ with property

(1.3a) $$E(1) = 1;$$

(ii) for each $k \in \mathbf{N}$, $b_1, \dots, b_k \in B := \{b : E(j_t(b)) = 0 \text{ for each } t \in \mathbf{R}_+\}$, there exists a positive constant $C(b_1, \dots, b_k) \in \mathbf{R}_+$, such that, for each $p \leqq k, (S_1, \dots, S_p) \in \mathcal{P}_{k,p}$, $(s_1, \dots, s_p) \in \mathbf{R}_+^p$

(1.3b) $$|E(j_{s_1}(b_{S_1}) \dots j_{s_p}(b_{S_p}))| \leqq C(b_1, \dots, b_k)$$

where and in the following, for $S_j = \{i_1, \dots, i_r\}$, we use the notations

(1.4) $$b_{S_j} = b_{i_1} \cdots b_{i_r}$$

DEFINITION 1.1. We call $E : \mathcal{A} \to \mathcal{C}$ an *FP-mixing map* (FP meaning "faster than polynomial") if there exist two functions $d, \delta : \mathbf{R}_+ \to \mathbf{R}_+$, (resp. d, $\delta : \mathbf{N} \to \mathbf{N}$) satisfying

(i) for each $q > 0$

(1.5a) $$d_T \to \infty, \quad \frac{d_T}{T^q} \to 0, \quad \text{as } T \to \infty$$

i.e. d_T tends to infinity more slowly than any power of T;

(ii) for each $q > 0$,

(1.5b) $$\delta_T \cdot T^q \to 0 \quad \text{as } T \to \infty$$

i.e. δ_T tends to zero more rapidly than any polynomial fuction.

(iii) for each $k \in \mathbf{N}$, $x \in \mathbf{R}$ (resp. $x \in \mathbf{N}$), $b_1, \dots, b_k \in B$, one has

(1.6) $$|E(M_x N_{x+d_T}) - E(M_x)E(N_{x+d_T})| \leqq C(b_1, \dots, b_k)\delta_T$$

where the constants $C(b_1, \dots, b_k)$ can be taken equal to those in (1.3b) and

(1.7) $$M_x := j_{s_1}(b_{S_1}) \dots j_{s_q}(b_{S_q}),$$

(1.8) $$N_{x+d_T} := j_{s_{q+1}}(b_{S_{q+1}}) \dots j_{s_p}(b_{S_p})$$

with $q \leqq p = 1, \dots, k$, $(S_1, \dots, S_p) \in \mathcal{P}_{k,p}$ and $(s_1, \dots, s_p) \in \mathbf{R}_+^p$ (resp. $(k_1, \dots, k_p) \in \mathbf{N}^p$), such that $s_1 < \dots < s_p$ (resp. $k_1 < \dots < k_p$) and

$$s_j \leqq x, \qquad j = 1, \dots, q, \tag{1.9}$$

$$s_j \geqq x + d_T, \qquad j = q + 1, \dots, p, \tag{1.10}$$

i.e. the correlations between observables which are localized in intervals $I, J \subset \mathbf{R}_+$ whose distance is greater than d_T decay at a rate which is faster than δ_T.

In the following, E will always denote an FP-mixing map satisfying (1.3a) and (1.3b). Since any state with exponential decay of the correlations is FP-mixing and since it is known that ergodic quantum Markov chains are exponentially mixing (cf. [2]), a Corollary of our results is that the central limit theorem holds for ergodic quantum Markov chains on countable tensor products of matrix algebras.

The basic idea of the proofs is the same as in [1], i.e. a quantum generalization of Bernstein's method to prove the central limit theorem for weakly dependent random variables. The idea is that, if the correlations decay sufficiently fast (conditions (1.6) and (1.5b)), then the blocks of random variables which are separated by a gap of length d_T become asymptotically independent. Moreover condition (1.5a) implies that, neglecting blocks of length d_T, we make an error which becomes negligible in the limit.

The present paper extends the results of [1] and corrects two errors in that paper: one, noted by von Waldenfels, is that in the formula (1.3) of [1] a combinatorial factor $(1/p!)$ was omitted. The other, noted by Verbeure, is that in the expression of the correlation function in Theorem (1.1) of [1], the term arising from the fact that the correlations at different times do not vanish (the term F of formula (1.16) of the present paper), was omitted due to an error in the proof of Lemma (2.2) of [1].

We are grateful to the above mentioned authors for pointing out these errors. The results of the present paper show however that the technique of the proof, developed in [1], was correct and applicable to a much more general situation, like the present one.

In the proofs we have tried to understand the analogies between the techniques used in the present paper and those developed by the authors to deal with the weak coupling and low density problems (cf. [3], [4] and the Remark (6.6a) in the following).

Since the proofs are long and technical, we formulate here the main results. In order to do that we need the following:

DEFINITION 1.2. We say that $f : \mathbf{R}_+^2 \to \mathcal{C}$ is $\mathbf{s} - \mathbf{L}^1(\mathcal{C}, \mathbf{d}\nu)$ if it is bounded and for each $s \in \mathbf{R}_+$, $f(\cdot, s) \in L^1([s, \infty), d\nu, \mathcal{C})$, the functions

$$s \longmapsto \int_{[s,\infty)} f(t,s)\, \nu(dt)\ , \qquad t \longmapsto \int_{[0,t)} f(t,s)\, \nu(ds) \tag{1.11a}$$

are bounded; the limit

$$\lim_{T\to\infty} \int_{[s,T)} f(t,s)\ \nu(dt) \tag{1.11b}$$

is uniform in s.

Moreover if the first integral of (1.11a) is not only bounded but also independent of s, we say that $f(\cdot,\cdot)$ is $\mathbf{S}-\mathbf{L}^1(\mathcal{C},\mathbf{d}\nu)$.

In particular, if ν is the counting (or the Lebesgue) measure, we denote $s-L^1(\mathcal{C},d\nu)$ by $s-L^1(\mathcal{C},dn)$ $(s-L^1(\mathcal{C},dt))$ and the same for $S-L^1(\mathcal{C},d\nu)$.

REMARK. If there exists an $\{f(k)\}_{k=1}^{\infty}\subset \mathbf{R}_+$ satisfying

(i) $|\varepsilon(h,r)|\leqq f(h-r)$ for each $r<h\leqq N$;

(ii) the series $\sum_{k=1}^{\infty} f(k)$ converges;

then, ε is $S-L^1(\mathcal{C},dn)$.

The meaning of this assumption is best understood by looking at (1.1) in the particular case in which $\sigma(t,s,b,b')=1$. In this case we immediately recognize that the condition $\varepsilon\in L^1(\mathcal{C},dn)$ is a condition of asymptotic abelianness, i.e. if s and t are very far apart, then $j_t(b)$ and $j_s(b')$ *almost commute.*

THEOREM 1.3. *Let E be an FP-mixing map and let $B\subset\mathcal{B}$ be a set of elements satisfying the commutation relation* (1.1) *and the mean zero condition*

$$E(j_t(b))=0\ ,\qquad \forall\ t\in T,\ b\in B.$$

If for each $b,b'\in B$, $\varepsilon(\cdot,\cdot,b,b')$ is $s-L^1(\mathcal{C},d\nu)$ and $\sigma(\cdot,\cdot,b,b')$ is bounded then, for each $b_1,\dots,b_k\in B$ and $a>\frac{1}{2}$ or $a=\frac{1}{2}$ and k odd,

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\int_{[0,T)^k} E(j_{t_1}(b_1)\dots j_{t_k}(b_k))\nu(dt_1)\dots\nu(dt_k)=0. \tag{1.12}$$

REMARK. If E is a stationary state, i.e. $E(j_t(b))=E_0(b)$ on B, independent of t, for some state E_0 and $a=1$, (1.12) is simply the *law of large numbers.* If $a=\frac{1}{2}$ and k is odd, (1.12) is the *first half* of the central limit theorem, i.e. the vanishing of odd moments for mean zero Gaussian state.

Moreover

THEOREM 1.4. *In the assumptions of Theorem* 1.3, *suppose that $T\subset\subset\mathbf{N}$ and that for each $b,b'\in B$,*

(i) $\sigma(\cdot,\cdot,b,b')=\sigma(b,b')$ *(i.e. $\sigma(h,k,b,b')$ does not depent on h,k).*

(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in* $S - L^1(C,dn)$ *in the sense of Definition* 1.1.

(iii) *The limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{h=1}^{N} E(j_h(b\cdot b')) =: C(bb') \tag{1.13}$$

exists.

Then for each $n\in\mathbf{N}$ *and* $b_1,\dots,b_{2n}\in B$, *the central limit*

$$\lim_{N\to\infty} \frac{1}{N^n} E(S_N(b_1)\dots S_N(b_{2n})) \tag{1.14}$$

exists and if we denote

$$f(b,b') := \sum_{k=h+1}^{\infty} \varepsilon(k,h,b,b'), \tag{1.15}$$

$$F(b,b') := \sum_{k=h+1}^{\infty} E\big(j_h(b)j_k(b')\big), \tag{1.16}$$

$$C_0(b,b') := C(bb') + F(b,b') + F(b',b) + f(b,b'), \tag{1.17}$$

then the limit (1.14) *is equal to*

$$\begin{aligned}\frac{1}{n!}\sum_{p.p.}\sum_{\pi\in\mathcal{S}_n} &\sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n})\times \\ &\times C_0(b_{i_{\pi(1)}},b_{j_{\pi(1)}})\cdots C_0(b_{i_{\pi(n)}},b_{j_{\pi(n)}})\end{aligned} \tag{1.18}$$

where, as usual $\sum_{p.p.}$ *means the sum over all ordered pairs partition of* $\{1,\dots,2n\}$, *i.e. all pairs* $\{i_1,j_1,\dots,i_n,j_n\}$ *such that*

$$\{i_1,j_1,\dots,i_n,j_n\} = \{1,\dots,2n\}, \tag{1.19a}$$

$$i_h < j_h, \quad \textit{for any } h=1,\dots,n, \tag{1.19b}$$

$$j_1 < j_2 < \dots < j_n \tag{1.19c}$$

and the $\sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n})$ *is a product of* σ*-factors.*

In the continuous analogue of Theorem 1.4 a qualitatively new phenomenon arises.

THEOREM 1.5. *In the continuous case, with the assumptions of Theorem* 1.3, *assume that for each* $b,b'\in B$,

(i) $\sigma(\cdot,\cdot,b,b') = \sigma(b,b')$,

(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in* $S - L^1(C,dt)$ *in the sense of Definition* 1.1.

Then for each $n \in \mathbf{N}$ *and* $b_1,\dots,b_{2n} \in B$, *the central limit*

$$\lim_{T\to\infty} \frac{1}{\nu([0,T))^n} \int\limits_{[0,T)^{2n}} E(j_{t_1}(b_1)\dots j_{t_{2n}}(b_{2n}))dt_1\dots dt_{2n} \tag{1.20}$$

exists. Moreover if we denote

$$f(b,b') := \int\limits_{[h,\infty)} ds\varepsilon(s,h,b,b'), \tag{1.21}$$

$$F(b,b') := \int\limits_{[h,\infty)} dsE\big(j_h(b)j_s(b')\big) \tag{1.22}$$

and

$$C_0(b,b') := F(b,b') + F(b',b) + f(b,b') \tag{1.23}$$

then the limit (1.20) *is equal to*

(1.24)
$$\frac{1}{n!}\sum_{p.p.} \sigma(i_1,j_1,\dots,i_n,j_n;b_1,\dots,b_{2n}) \times C_0(b_{i_{\pi(1)}},b_{j_{\pi(1)}})\cdots C_0(b_{i_{\pi(n)}},b_{j_{\pi(n)}})$$

REMARK. Notice that in the continuous case there is no analogue of condition (iii) in Theorem 1.4. This is because this condition is on products of pairs and we shall show that in the continuous case, only the partitions made up entirely of singletons survive in the limit.

§2. Some technical lemmata

In this section we introduce some notations and prove some lemmata needed in the following sections.

LEMMA 2.1. *Let* E *be as specified in Section* 1 *and let* $a \geqq \frac{1}{2}$, $p \leqq k \in \mathbf{N}$, $b_1,\dots,b_k \in B$, $(S_1,\dots,S_p) \in \mathcal{P}_{k,p}$. *Assume that either of the following conditions is satisfied:*

(i) $ak > p$;

(ii) $(S_1,\dots,S_p)$ *contains exactly* q *singletons with* $q \geqq 1$, $a > \frac{1}{2}$ *or* $a = \frac{1}{2}$ *and* k *odd.*

Then

(2.1)
$$\lim_{T\to\infty} \frac{1}{\nu([0,T))^{ak}} \int\limits_{0\leqq s_1<\ldots<s_p\leqq T} |E(j_{s_1}(b_{S_1})\ldots j_{s_p}(b_{S_p}))|\cdot\nu(ds_1)\ldots\nu(ds_p)=0.$$

PROOF. (i) If $ak>p$, then by (1.6),

$$\frac{1}{\nu([0,T))^{ak}} \int\limits_{0\leqq s_1<\ldots<s_p\leqq T} |E(j_{s_1}(b_{S_1})\ldots j_{s_p}(b_{S_p}))|\nu(ds_1)\ldots\nu(ds_p)\leqq \tag{2.2}$$
$$\leqq \frac{1}{\nu([0,T))^{ak}}C(b_1,\ldots,b_k)\nu([0,T))^p\to 0$$

(ii) Assuming that $(S_1,\ldots,S_p)$ contains exactly q singletons which correspond to the indices $j_1<\ldots<j_q$, we define the set

$$A(T,d_T,p,\{j_n\}_{n=1}^q):=\Big\{(s_1,\ldots,s_p)\in[0,T)^p: s_1<\ldots<s_p \tag{2.3}$$
$$\text{and for each } r=1,\ldots,q,\text{ either } s_{j_r}-s_{j_r-1}\leqq d_T \text{ or } s_{j_r+1}-s_{j_r}\leqq d_T\Big\}$$

denoting ν^p the product measure $(\otimes\nu)^p$, then, for each $\{j_n\}_{n=1}^q$, the quantity $\nu^p(A(T,d_T,p,\{j_n\}_{n=1}^q))$ can be written as

$$\int_0^T \nu(ds_1)\int_{s_1}^T \nu(ds_2)\ldots\int_{s_n-1}^T \nu(ds_n)\chi^{\varepsilon(1)}_{[s_{j_1-1},s_{j_1-1}+d_T)}(s_{j_1})\times \tag{2.4}$$
$$\times\chi^{\varepsilon(2)}_{[s_{j_1},s_{j_1}+d_T)}(s_{j_1+1})\times$$
$$\times\ldots\times\chi^{\varepsilon(2q-1)}_{[s_{j_q-1},s_{j_q-1}+d_T)}(s_{j_q})\times\chi^{\varepsilon(2q)}_{[s_{j_q},s_{j_q}+d_T)}(s_{j_q+1})$$

where $\varepsilon\in\{0,1\}^{2q}$ is determined uniquely by the rule

$$\varepsilon(2r-1)=\begin{cases}0\,, & \text{if } s_{j_r}-s_{j_r-1}>d_T;\\ 1\,, & \text{if } s_{j_r}-s_{j_r-1}\leqq d_T;\end{cases} \tag{2.5a}$$

$$\varepsilon(2r)=\begin{cases}0\,, & \text{if } s_{j_r+1}-s_{j_r}>d_T;\\ 1\,, & \text{if } s_{j_r+1}-s_{j_r}\leqq d_T\end{cases} \tag{2.5b}$$

and where, by definition, for any set I, $\chi_I^0=1$; $\chi_I^1=\chi_I$. From the definition of $A(T,d_T,p,\{j_h\}_{h=1}^q)$, it follows that $\varepsilon(2n-1)+\varepsilon(2n)\geqq 1$ for each $n=1,\ldots,q$. Notice that for each $n=1,\ldots,q$, the product

$$\chi^{\varepsilon(2n-1)}_{[s_{jn-1},s_{jn-1}+d_T)}(s_{j_n})\times\chi^{\varepsilon(2n)}_{[s_{j_n},s_{j_n}+d_T)}(s_{j_n+1}) \tag{2.6}$$

surely depends on s_{j_n} but not necessarily on s_{j_n+1} or s_{j_n-1}. So if we denote

(2.7)
$$F(s_1,\dots,s_p\ ;\ \{j_n\}_{n=1}^q,\varepsilon):=\prod_{n=1}^q \chi^{\varepsilon(2n-1)}_{[s_{j_n-1},s_{j_n-1}+d_T)}(s_{j_n})\times\chi^{\varepsilon(2n)}_{[s_{j_n},s_{j_n}+d_T)}(s_{j_n+1})$$

then F depends on $s_{j_1},\dots,s_{j_q}$, but not necessarily on the other variables, i.e. F is a function which depends on at least the q variables $s_{j_1},\dots,s_{j_q}$. In the multiple integral (2.4), if the variable s_j does not appear in any characteristic function, then we majorize the corresponding integral with $\nu([0,T))$. If it appears we would like to majorize the correspoding integral with the factor $\nu([0,d_T))$. However in doing so we should keep in mind that some of the characteristic functions can coincide. This can happen only if, for some $r=1,\dots,q$,

$$j_{r-1}=j_r-1. \tag{2.8}$$

In this case we have the factor

(2.9)
$$\chi_{[s_{j_{r-1}-1},s_{j_{r-1}-1}+d_T]}(s_{j_{r-1}})^{\varepsilon(2(r-1)-1)}\cdot\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_{r-1}+1})^{\varepsilon(2(r-1))}$$
$$\cdot\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})^{\varepsilon(2r-1)}\cdot\chi_{[s_{j_r},s_{j_r}+d_T]}(s_{j_r+1})^{\varepsilon(2r)}.$$

So if

$$\varepsilon(2r)=\varepsilon(2(r-1)-1)=0 \tag{2.10}$$

then the product (2.9) becomes

$$\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_{r-1}+1})\cdot\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})$$

and in the view of (2.5a) and (2.5b), this is equal to $\chi_{[s_{j_{r-1}},s_{j_{r-1}}+d_T]}(s_{j_r})$. Thus, if both conditions (2.8) and (2.10) are satisfied, then from two singletons we get only one characteristic function.

Since there are q singletons, the worst case is when we get only $\left[\frac{q+1}{2}\right]$ characteristic functions. This is clear if q is even and, if $q=2m+1$ is odd, then after having formed m pairs, the remaining term will surely produce a characteristic function, because if condition (2.8) is satisfied by three indices, say j_{r-2},j_{r-1},j_r, then condition (2.10) cannot be simultaneously verified for the two pairs (j_{r-2},j_{r-1}), (j_{r-1},j_r). In conclusion, if q is odd we have at least $m+1=\frac{q+1}{2}$ different characteristic functions.

As a consequence of this we obtain the estimate

$$\nu^p(A(T,d_T,p,\{j_h\}_{h=1}^q))\leqq\nu([0,T))^{p-[\frac{q+1}{2}]}\cdot\nu([0,d_T))^{[\frac{q+1}{2}]}. \tag{2.11}$$

Therefore if we denote
(2.12)

$$A(t, d_T, p, q) := \sum_{1 \leqq j_1 < \ldots < j_q \leqq p;\ \{S_{j_h}\}_{h=1}^{q} \text{ are singletons}} A\left(T, d_T, p, \{j_h\}_{h=1}^{q}\right)$$

then

$$\text{(2.12a)} \qquad \nu^p(A(t, d_T, p, q)) \leqq \binom{p}{q} \cdot \nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]}.$$

Moreover by (2.12a) we obtain

$$\text{(2.13)} \quad \frac{1}{\nu([0,T))^{ak}} \int_{A(T,d_T,p,q)} |E(j_{s_1}(b_{S_1}) \ldots j_{s_p}(b_{S_p}))| \nu(ds_1) \ldots \nu(ds_p) \leqq$$

$$\leqq \frac{1}{\nu([0,T))^{ak}} \nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]} \cdot C(b_1, \ldots, b_k) \cdot \binom{p}{q}.$$

But if $(S_1, \ldots, S_p)$ contains exactly q singletons, then there are $p - q$ non-sigletons, therefore

$$\text{(2.14)} \qquad k = \sum_{j=1}^{p} |S_j| \geqq q + 2(p-q) = 2p - q$$

and this implies that

$$\text{(2.14a)} \qquad p \leqq \frac{1}{2}(k+q) \leqq \frac{k}{2} + \left[\frac{q+1}{2}\right].$$

Therefore, if $a > \frac{1}{2}$, we have

$$\text{(2.15)} \qquad \frac{1}{\nu([0,T))^{ak}} \nu([0,T))^{p-[\frac{q+1}{2}]} \cdot \nu([0,d_T))^{[\frac{q+1}{2}]} \leqq$$

$$\leqq \frac{1}{\nu([0,T))^{k(a-\frac{1}{2})}} \nu([0,d_T))^{[\frac{q+1}{2}]} \to 0$$

as $T \to \infty$. Moreover, $p, k \in \mathbf{N}$ implies that in (2.14a) it is possible to have equality only if k is even, therefore if $a = \frac{1}{2}$ and k is odd, then

$$p < \frac{k}{2} + \left[\frac{q+1}{2}\right] \quad \text{i.e.} \quad p - [\frac{q+1}{2}] < \frac{k}{2}.$$

Thus
(2.16)

$$\frac{1}{\nu([0,T))^{ak}}\nu([0,T))^{p-[\frac{q+1}{2}]}\cdot\nu([0,d_T))^{[\frac{q+1}{2}]}=\frac{\nu([0,d_T))^{[\frac{q+1}{2}]}}{\nu([0,T))^{\frac{k}{2}-(p-[\frac{q+1}{2}])}}\to 0$$

as $T\to\infty$. Define now the set

$$A_p=\{(s_1,\dots,s_p)\in\mathbf{R}_+^p : s_1<\dots<s_p\leqq T\}\backslash A(T,d_T,p)$$

by (2.3) and (2.12). One knows that for each $(s_1,\dots,s_p)\in A_p$ there exists a $q\leqq p$, $1\leqq j_1<\dots<j_q\leqq p$ such that

$$|s_{j_r}-s_{j_r-1}|>d_T \quad\text{and}\quad |s_{j_r+1}-s_{j_r}|>d_T,\quad \forall\, r=1,\dots,q. \tag{2.17a}$$

Therefore the mean zero condition ($b_j\in B$, $j=1,\dots,k$) and (1.6) imply that

$$\frac{1}{\nu([0,T))^{ak}}\int\limits_{A_p}|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))|\nu(ds_1)\dots\nu(ds_p)\leqq \tag{2.17}$$

$$\leqq\frac{1}{\nu([0,T))^{ak}}\nu([0,T))^p\cdot O(\delta_T)\cdot C(b_1,\dots,b_k)\longrightarrow 0,\quad\text{as}\quad T\to\infty.$$

Putting together ((2.16) and (2.17) we obtain that (2.1) is equal to

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\left(\int\limits_{A_p}+\int\limits_{A(T,d_T,p,q)}\right)|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))|\cdot \tag{2.18}$$

$$\cdot\nu(ds_1)\dots\nu(ds_p)=0.$$

COROLLARY 2.2. *Let E be as specified in Section 1 and let $a>\frac{1}{2}$, $p\leqq$ $\leqq k\in\mathbf{N}$, $b_1,\dots,b_k\in B$, $(S_1,\dots,S_p)\in\mathcal{P}_{k,p}$. Then*

$$\lim_{T\to\infty}\frac{1}{\nu([0,T))^{ak}}\int\limits_{0\leqq s_1<\dots<s_p\leqq T}|E(j_{s_1}(b_{S_1})\dots j_{s_p}(b_{S_p}))|\cdot \tag{2.19}$$

$$\cdot\nu(ds_1)\dots\nu(ds_p)=0.$$

PROOF. We distinguish two cases:
i) if $p\leqq k/2$ then $ak>p$;
ii) if $p>k/2$ then there exist singletons among $S_1,\dots,S_p$.
The proof follows that of Lemma 2.1.

§3. Normal order in abstract algebras

In this section we generalize some techniques widely used in quantum field theory and known under the name of *normal order* or *Wick order*. In an abstract setting, the problem giving rise to these techniques can be formulated as follows: one starts from elements a_x ($x \in \mathbf{N}$) of an algebra $\mathcal{Q}$, satisfying some commutation relations of the form (3.3); one considers products of the a_x, of the form (3.1) and, by repeated application of the commutation relations (3.3), one wants to write the product (3.1) in such a way that the indices $x_1, \dots, x_n$ appear in a preassigned order. In the present paper, the preassigned order will be the increasing one (3.2). The *normal order*, usually considered by the physicists is different: the indices x_j take only the values 0 (corresponding to creation operators) and 1 (corresponding to annihilation operators), and one wants to write the product (3.1) as a sum of products in which all the zeros are to the left of all the ones and the original order among the zeros and among the ones is preserved. In that case the factor ε corresponds to a scalar product and the factor σ corresponds to 1 (Boson case) or -1 (Fermi case). The situation considered by us corresponds to a *time ordering*. The basic techniques are the same in both cases. The techniques developed below are a natural generalization of those, introduced by the authors, to deal with the weak coupling and the low density problem (cf. [3], [4]).

DEFINITION 3.1. For any algebra $\mathcal{Q}$, $n, N \in \mathbf{N}$, $\{a_h\}_{h=1}^N \subset \mathcal{Q}$, $1 \leqq x_1, \dots, x_n \leqq N$, we say that the product

$$a_{x_1} \cdots a_{x_n} \tag{3.1}$$

is *ordered* if the indices $\{x_h\}_{h=1}^n$ are ordered, i.e.

$$x_1 \leqq x_2 \leqq \ldots \leqq x_n. \tag{3.2}$$

In the following we shall investigate the ordered form of products of the form (3.1), where the a_j satisfy the commutation relations

$$a_x \cdot a_y = \sigma(x,y) \cdot a_y \cdot a_x + \varepsilon(x,y), \quad \forall \; x, y \in \mathbf{N}, \; x \neq y \tag{3.3}$$

with σ, ε in the center of $\mathcal{Q}$.

For each $n, N \in \mathbf{N}$, $n \leqq N$ and $1 \leqq x_1, \dots, x_n \leqq N$, there exists a unique n-permutation $\pi \in \mathcal{S}_n$ (the permutation group on $\{1, \dots, n\}$) such that π is a composition of k consecutive exchanges

$$x_{\pi(1)} \leqq x_{\pi(2)} \leqq \cdots \leqq x_{\pi(n)} \tag{3.4}$$

and for any other n-permutation π', if π' is a composition of k' consecutive exchanges with $k' < k$, then π' does *not* satisfy (3.4). An exchange is called *consecutive* if it exchanges two consecutive indices and leaves the remaining ones fixed.

In the following for any given $x = \{x_1, \dots, x_n\}$, we shall denote this permutation by π^x.

LEMMA 3.2. *In the notations* (1.2a),

$$S_T(b_1)\cdots S_T(b_n) = \tag{3.5}$$
$$= \sum_{p=1}^{n} \sum_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}} \int_{[S_1,\dots,S_p]_T} j_{t_1}(b_1)\cdots j_{t_n}(b_n)\nu(t_1)\cdots\nu(t_n).$$

PROOF. (3.5) is an immediate consequence of

$$[0,T)^n = \bigcup_{p=1}^{n} \bigcup_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}} \bigcup_{t\in[S_1,\dots,S_p]_T} \{t\}. \tag{3.6}$$

LEMMA 3.3. *For each* $n, N \in \mathbf{N}$, $n \leqq N$ *and* $1 \leqq x_1, \dots, x_n \leqq N$, *the ordered form of the product* (3.1) *is equal to*

(3.7)
$$\sum_{m=0}^{[n/2]} \sum_{1\leqq p_1<\dots<p_m\leqq n} \sum_{(q_1,\dots,q_m)}^{*} \prod_{h=1}^{m} \varepsilon(x_{p_h}, x_{q_h})\sigma(x_1,\dots,x_n)\cdot a_{x_{\pi^x(r_1)}}\cdots a_{x_{\pi^x(r_{n-2m})}}$$

where and in the following

i) *for each fixed* m *and* $1 \leqq p_1 < \dots < p_m \leqq n$, $\sum^*_{(q_1,\dots,q_m)}$ *means the sum over all* $1 \leqq q_1, \dots, q_m \leqq n$ *satisfying*

$$\operatorname{card}(\{q_h\}_{h=1}^m) := |\{q_h\}_{h=1}^m| = m, \tag{3.8a}$$
$$\{q_h\}_{h=1}^m \subset \{1,\dots,n\} \setminus \{p_h\}_{h=1}^m, \tag{3.8b}$$
$$p_h < q_h, \quad \forall\, h = 1,\dots,m \tag{3.8c}$$

and

$$x_{p_h} > x_{q_h}, \quad \forall\, h = 1,\dots,m; \tag{3.8d}$$

ii) *for each fixed* m *and* $\{p_h, q_h\}_{h=1}^m$,

$$\{r_h\}_{h=1}^{n-2m} := \{1,\dots,n\} \setminus \{p_h, q_h\}_{h=1}^m, \quad r_1 < \dots < r_{n-2m} \tag{3.9a}$$

and $x_{\pi^x(r_1)} \leqq \dots \leqq x_{\pi^x(r_{n-2m})}$;

iii) $\sigma(x_1,\dots,x_n)$ *is a product factor of the form* $\sigma(x_i, x_j)$.

PROOF. From the commutation relation (3.3) we know that in the ordered form of the product (3.1), some elements of $\{a_{x_h}\}_{h=1}^n$ will be used to produce an ε-factor and in order to get an ε-factor we use two elements of $\{a_{x_h}\}_{h=1}^n$, therefore the number m of ε-factors can be equal to $0, 1, \dots, [n/2]$.

For each fixed m, let $\{p_h\}_{h=1}^m$ denote indices $\{x_h\}_{h=1}^n$ such that a_{p_h} is used to produce an ε-factor with some element a_{q_h} with $q_h > p_h$. By relabeling the order, one may suppose that $p_1 < p_2 < \ldots < p_m$. If $p_1 < p_2 < \ldots < p_m$ and $q_1, q_2, \ldots, q_m \subset \{1, \ldots, n\}$ are chosen as above then obviously:

— q_h cannot be in $\{p_h\}_{h=1}^m$ for any $h = 1, \ldots, m$, i.e. $\{q_h\}_{h=1}^m \subset \subset \{1, \ldots, n\} \setminus \{p_h\}_{h=1}^m$;

— q_h cannot be equal to another $q_{h'}$, i.e. $|\{q_h\}_{h=1}^m| = m$.

— $q_h > p_h$ for each $h = 1, \ldots, m$;

— if for some $i < j$, $x_i < x_j$ then we do not exchange the order of the two elements a_{x_i} and a_{x_j}, so there is no factor $\varepsilon(x_i, x_j)$, i.e. $x_{p_h} > x_{q_h}$ for each $h = 1, \ldots, m$.

For each fixed m and $\{p_h, q_h\}_{h=1}^m$, denoting $\{r_h\}_{h=1}^{n-2m} := \{1, \ldots, n\} \setminus \setminus \{p_h, q_h\}_{h=1}^m$, the $\{a_{r_h}\}_{h=1}^{n-2m}$ are not used to produce ε-factors, therefore in order to bring their product to the ordered form one can apply the restriction of the permutation π^x to the set $\{r_h\}_{h=1}^{n-2m}$. Thus one obtains the product $a_{x_{\pi^x(r_1)}} \cdots a_{x_{\pi^x(r_{n-2m})}}$, where, by the definition of π^x (cf. (3.4)),

$$x_{\pi^x(r_1)} < \ldots < x_{\pi^x(r_{n-2m})}.$$

Since each exchange gives rise to one σ-factor, eventually we obtain a factor $\sigma(x_1, \ldots, x_n)$ which is a product of some factors $\sigma(x_i, x_j)$.

As a special case of Lemma 3.3, for each $T > 0$, $n \in \mathbf{N}$, $\{t_h\}_{h=1}^n \subset [0, T]$ and $b_1, \ldots, b_n$, we can obtain the ordered form of the product

$$j_{t_1}(b_1) \cdots j_{t_n}(b_n). \tag{3.9}$$

COROLLARY 3.4. *In the notations of Lemma* 3.3, *for each* $T > 0$, $n \in \in \mathbf{N}$, $t := \{t_h\}_{h=1}^n \subset [0, T]$ *and* $b_1, \ldots, b_n$, *the ordered form of the product* (3.9) *is equal to*

$$\sum_{m=0}^{[n/2]} \sum_{1 \leqq p_1 < \ldots < p_m \leqq n} \sum_{(q_1, \ldots, q_m)}^{*} \prod_{h=1}^{m} \varepsilon(t_{p_h}, t_{q_h}, b_{p_h}, b_{q_h}) \tag{3.10}$$
$$\sigma(t_1, \ldots, t_n, b_1, \ldots, b_n) \cdot j_{t_{\pi^t(r_1)}}(b_{\pi^t(r_1)}) \cdots j_{t_{\pi^t(r_{n-2m})}}(b_{\pi^t(r_{n-2m})}).$$

Moreover we have the following

COROLLARY 3.5. *In the notations of Lemma* 3.3, *for each* $T > 0$, $n \in \in \mathbf{N}$, *and* $b_1, \ldots, b_n$, *the product*

$$S_T(b_1) \cdots S_T(b_n) \tag{3.11a}$$

is equal to

$$\sum_{p=1}^{n} \sum_{(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}} \int_{t \in [S_1, \ldots, S_p]_T} \sum_{m=0}^{[n/2]} \sum_{1 \leqq p_1 < \ldots < p_m \leqq n} \sum_{(q_1, \ldots, q_m)}^{*} \tag{3.11b}$$

$$\prod_{h=1}^{m}\varepsilon(t_{p_h},t_{q_h},b_{p_h},b_{q_h})\sigma(t_1,\dots,t_n,b_1,\dots,b_n)\cdot j_{t_{\pi^t(r_1)}}(b_{\pi^t(r_1)})\cdots$$
$$\cdots j_{t_{\pi^t(r_{n-2m})}}(b_{\pi^t(r_{n-2m})})\nu(t_1)\cdots\nu(t_n).$$

PROOF. Corollary 3.5 follows immediately from Lemma 3.1 and Corollary 3.4.

REMARK. Notice that since $(S_1,\dots,S_p)$ is an ordered partition, one has that *inside* each S_h, $h=1,\dots,p$, π^t will keep the order, i.e. if $l_h < k_h \in S_h$, $h \in \{1,\dots,p\}$ and $l_h < k_h$,then $\pi^t(l_h) < \pi^t(k_h)$ and $t_{\pi^t(l_h)} = t_{\pi^t(k_h)}$. In other terms, π^t acts on the blocks S_j, keeping the order inside each block.

Now let us consider Corollary 3.5 from another point of view: for each $p=1,\dots,n$ and any n-permutation π, let $[S_1,\dots,S_p]_T^\pi$ be the subset of $t \in [S_1,\dots,S_p]_T$ such that $\pi^t=\pi$ with π^t defined by (3.4), i.e. $t_{\pi(1)} \leqq \leqq \dots \leqq t_{\pi(n)}$. Clearly we have

(3.12)
$$S_T(b_1)\dots S_T(b_n)=\sum_{p=1}^{n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}}\sum_{\pi\in\mathcal{S}_n}\sum_{t\in[S_1,\dots,S_p]_T^\pi} j_{t_1}(b_1)\dots j_{t_n}(b_n).$$

For each $t\in[S_1,\dots,S_p]_T^\pi$, the product $j_{t_1}(b_1)\dots j_{t_n}(b_n)$ is not ordered (unless π is the identity) and the permutation which makes the product $j_{t_1}(b_1)\cdots j_{t_n}(b_n)$ ordered is π. Therefore we can write the product as

$$j_{t_{\pi^{-1}(1)}}(b_1)\cdots j_{t_{\pi^{-1}(n)}}(b_n) \tag{3.13}$$

with $t\in I_T(S_1,\dots,S_p)$.

In the following for each fixed partition $(S_1,\dots,S_p)\in\mathcal{P}_{n,p}$, we shall use the notation

$$\mathcal{S}_n^{(S,p)} := \{\pi\in\mathcal{S}_n:\ [S_1,\dots,S_p]_T^\pi \ \text{ non-empty}\}. \tag{3.14}$$

That is, $\mathcal{S}_n^{(S,p)}$ consists of all permutations on $\{1,\dots,n\}$ which permute among themselves the blocks S_j, considered as individual objects. Thus $\mathcal{S}_n^{(S,p)}$ is isomorphic to $\mathcal{S}_p$.

Applying Corollary 3.4 to the product (3.13) we find the following result.

LEMMA 3.6. *In the notations of Lemma* 3.3, *for each* $T>0$, $n\in\mathbf{N}$ *and* $b_1,\dots,b_n$, *the product* (3.11a) *is equal to*

$$\sum_{p=1}^{n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}}\sum_{\pi\in\mathcal{S}_n^{(S,p)}}\int_{t\in I_T(S_1,\dots,S_p)}\sum_{m=0}^{[n/2]}\sum_{1\leqq p_1<\dots<p_m\leqq n}\sideset{}{^*}\sum_{(q_1,\dots,q_m)} \tag{3.15}$$
$$\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$
$$\sigma(t_1,\dots,t_n,b_1,\dots,b_n)\cdot j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\nu(t_1)\cdots\nu(t_n).$$

REMARK. Notice that (3.11b) and (3.15) are two different ways to write the product (3.11a).

PROOF. From the definition of $[S_1,\dots,S_p]_T^\pi$ and the identity

$$\int\limits_{t\in[S_1,\dots,S_p]_T^\pi} j_{t_1}(b_1)\cdots j_{t_n}(b_n)\nu(t_1)\cdots\nu(t_n) = \tag{3.16}$$
$$= \sum_{t\in I_T(S_1,\dots,S_p)} j_{t_{\pi^{-1}(1)}}(b_1)\cdots j_{t_{\pi^{-1}(n)}}(b_n)\nu(t_1)\cdots\nu(t_n)$$

immediately follows.

For each fixed partition $(S_1,\dots,S_p)\in\mathcal{P}_{n,p}$, the sum $\sum^*_{(q_1,\dots,q_m)}$ is the same as in Lemma (3.2), the only difference being that (3.8d) is replaced by

$$t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\quad h=1,\dots,m. \tag{3.17}$$

It will be useful, in the following, to perform the summation first in the m, p_h, q_h indices and then in the π, t indices. This goal is achieved in the following lemma

LEMMA 3.7. *For each $T>0$, $n\in\mathbf{N}$ and $b_1,\dots,b_n$, the product* (3.11a) *is equal to*

(3.18)

$$\sum_{p=1}^{n}\sum_{m=0}^{[n/2]}\sum_{1\leqq p_1<\dots<p_m\leqq n}\sideset{}{'}\sum_{(q_1,\dots,q_m)}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{n,p}}\sum_{\pi\in\mathcal{S}_n^{(S,p)}}\int\limits_{t\in I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}$$
$$\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\cdot\sigma(t_1,\dots,t_n,b_1,\dots,b_n)$$
$$j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\nu(t_1)\cdots\nu(t_n)$$

where and in the following, $\sum'_{(q_1,\dots,q_m)}$ means summation over all $1\leqq$ $\leqq q_1,\dots,q_m\leqq n$ satisfying the conditions (3.8a), (3.8b) *and* (3.8c) *(but without the condition* (3.8d)*), and*

$$I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi):= \tag{3.19}$$
$$:=\left\{t\in I_T(S_1,\dots,S_p):\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m\right\}.$$

PROOF. The Lemma is proved with the following procedure: First we choose m and $\{p_h,q_h\}_{h=1}^m$ as in Lemma (3.6) but without the condition (3.17). Second, for each fixed m and $\{p_h,q_h\}_{h=1}^m$ and $\pi\in\mathcal{S}_n^{(S,p)}$, we define $I_T(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)$ as (3.19).

§4. The negligible terms

In analogy with weak coupling and low density limit, the following lemma corresponds to proving that, in these limits, the so called "type II terms" tend to zero. Recall that, in this analogy, the index k (resp. t) is interpreted as time, the ε-factor as scalar product, and the type II terms are those products of ε-factors which contain at least one factor of the form $\varepsilon(h,k)$ with $k-h \geqq 2$.

LEMMA 4.2. *Suppose that* $\varepsilon_1, \varepsilon_2 : \mathbf{N}^2 \to \mathcal{C}$ *are in* $s - L^1(\mathcal{C}, dn)$, *then*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1\leqq k_1<k_2<k_3<k_4\leqq N} |\varepsilon_1(k_4,k_2)|\cdot|\varepsilon_2(k_3,k_1)| = 0 \tag{4.1a}$$

and

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{1\leqq k_1<k_2<k_3<k_4\leqq N} |\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2)| = 0. \tag{4.1b}$$

PROOF. Since $|\cdot|$ is a semi-norm on $\mathcal{C}$, we may suppose that the $\varepsilon_j(h,k)$ are positive numbers. Thus ε_i are $s - L^1(\mathbf{R}_+, dn)$, $i = 1,2$ and, because of (1.11a), there exists a finite constant M_i such that

$$\max\left\{\sum_{k=h+1}^{\infty} \varepsilon_i(k,h),\ \sum_{k=0}^{h} \varepsilon_i(h,k),\ \ h\in\mathbf{N}\right\} \leqq M_i < +\infty. \tag{4.2}$$

Then, because of (4.2), for each $\eta > 0$, there exists a $K \in \mathbf{N}$ such that for any $h_1, h_2 \in \mathbf{N}$

$$\sum_{k=K+h_1}^{\infty} \varepsilon_1(k,h_1) + \sum_{k=K+h_2}^{\infty} \varepsilon_2(k,h_2) < \eta. \tag{4.3}$$

We rewrite

$$\frac{1}{N^2} \sum_{1\leqq k_1<k_2<k_3<k_4\leqq N} \varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1)$$

as

(4.4)

$$\frac{1}{N^2}\sum_{k_1=1}^{N-3}\ \sum_{k_2=k_1+1}^{N-2}\left(\sum_{k_3=k_2+1}^{k_1+K} + \sum_{k_3=k_1+K+1}^{N-1}\right)\sum_{k_4=k_3+1}^{N} \varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1).$$

Then the first term of (4.4) becomes

$$\begin{aligned}(4.4a)\qquad &\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\sum_{k_4=k_3+1}^{N}\varepsilon_1(k_4,k_2)\leqq\\ &\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\sum_{k_4=k_2+1}^{\infty}\varepsilon_1(k_4,k_2)\leqq\\ &\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\cdot M_1.\end{aligned}$$

Notice that on the right hand side of (4.4a), $k_1 < k_3 \leqq k_1 + K$, hence one has $k_1 < k_2 < k_1 + K$. This implies that the right hand side of (4.4a) is less than or equal to

$$\begin{aligned}(4.4b)\qquad &\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{k_1+K-1}\sum_{k_3=k_2+1}^{k_1+K}\varepsilon_2(k_3,k_1)\cdot M_1\leqq\\ &\leqq\frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{k_1+K-1}M_2\cdot M_1=\frac{1}{N^2}\sum_{k_1=1}^{N-3}K\cdot M_2\cdot M_1\end{aligned}$$

and this tends to zero as $N \to \infty$. By (4.3), the second term of (4.4) is majorized by

$$(4.4c)\qquad \eta\cdot M_1\cdot\frac{(N-3)^2}{N^2}$$

therefore

$$(4.5)\qquad \limsup_{N\to\infty}\frac{1}{N^2}\sum_{1\leqq k_1<k_2<k_3<k_4\leqq N}\varepsilon_1(k_4,k_2)\varepsilon_2(k_3,k_1)\leqq\eta\cdot M_1$$

and since $\eta > 0$ is arbitrary, this proves (4.1a).

In order to prove (4.1b) we rewrite

$$\frac{1}{N^2}\sum_{1\leqq k_1<k_2<k_3<k_4\leqq N}\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2)$$

as

$$(4.6)\quad \frac{1}{N^2}\sum_{k_1=1}^{N-3}\sum_{k_2=k_1+1}^{N-2}\sum_{k_3=k_2+1}^{N-1}\left(\sum_{k_4=k_3+1}^{k_1+K}+\sum_{k_4=k_1+K+1}^{N}\right)\varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2).$$

By the same arguments as above, (4.6) is dominated by

$$\eta \cdot M_2 + M_1 \cdot M_2 \frac{K}{N^2}(N-3). \tag{4.6a}$$

Hence,

$$\limsup_{N\to\infty} \frac{1}{N^2} \sum_{1\leqq k_1<k_2<k_3<k_4\leqq N} \varepsilon_1(k_4,k_1)\varepsilon_2(k_3,k_2) \leqq \eta \cdot M_2 \tag{4.7}$$

and this implies (4.2a) by the arbitrariness of $\eta > 0$.

LEMMA 4.3. *Suppose that* $\varepsilon_1, \varepsilon_2 : \mathbf{N}^2 \to C$ *are in* $s - L^1(C, dn)$, *then*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{1\leqq h_1<k_1,h_2<k_2\leqq N \\ |\{h_j,k_j\}_{j=1}^2|<4}} \left|\varepsilon_1(k_1,h_1)\right| \cdot \left|\varepsilon_2(k_2,h_2)\right| = 0. \tag{4.8}$$

REMARK. The condition $\left|\{h_j,k_j\}_{j=1}^2\right| < 4$ means that in the sum $\sum_{1\leqq h_1<k_1,h_2<k_2\leqq N}$ some of the indices h_j, k_j are equal.

PROOF. Let us denote

$$\sum_2 := \frac{1}{N^2} \sum_{\substack{1\leqq h_1<k_1,h_2<k_2\leqq N \\ |\{h_j,k_j\}_{j=1}^2|<4}} \left|\varepsilon_1(k_1,h_1)\right| \cdot \left|\varepsilon_2(k_2,h_2)\right|$$

and discuss separately all the possibilities according to which indices are equal.

i) If $h_1 = h_2$ then

$$\sum_2 = \frac{1}{N^2} \sum_{h_1=1}^{N} \sum_{k_1=h_1+1}^{N} \left|\varepsilon_1(k_1,h_1)\right| \sum_{k_2=h_1+1}^{N} \left|\varepsilon_2(k_2,h_1)\right|. \tag{4.9}$$

Since $\varepsilon_1, \varepsilon_2$ are in $s - L^1(C, dn)$ one has, in the notation (4.3b),

$$\sum_2 \leqq M_1 \cdot M_2 \cdot \frac{1}{N^2} \sum_{h_1=1}^{N} 1 = \frac{M_1 \cdot M_2}{N} \longrightarrow 0. \tag{4.10}$$

ii) If $h_1 = k_2$ then

$$\sum_2 = \frac{1}{N^2} \sum_{h_1=1}^{N} \sum_{k_1=h_1+1}^{N} \left|\varepsilon_1(k_1,h_1)\right| \sum_{h_2=1}^{h_1-1} \left|\varepsilon_2(h_1,h_2)\right|. \tag{4.11}$$

Changing the order of summation on the right hand side of (4.11), we find that

$$\sum_2 = \frac{1}{N^2}\sum_{h_2=1}^{N}\sum_{h_1=h_2+1}^{N}\sum_{k_1=h_1+1}^{N}\left|\varepsilon_1(k_1,h_1)\right|\cdot\left|\varepsilon_2(h_1,h_2)\right| \leqq \tag{4.12}$$

$$\leqq M_1\cdot\frac{1}{N^2}\sum_{h_2=1}^{N}\sum_{h_1=h_2+1}^{N}\left|\varepsilon_2(h_1,h_2)\right| \leqq M_1M_2\cdot\frac{1}{N}\to 0.$$

The cases $k_1 = h_2$ and $k_1 = k_2$ follow from the same arguments.

Now we prove the generalization of Lemma 4.2 to the case of a product of n ε-factors. These products are analogues of the type II terms of [3].

LEMMA 4.3. *For each* $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n : \mathbf{N}^2\to\mathcal{C}$ *which are in* $s-L^1(\mathcal{C},dn)$, *we have*

$$\lim_{N\to\infty}\frac{1}{N^n}\sum_{\substack{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=1}^n|<2n}}\prod_{j=1}^{n}\left|\varepsilon_j(k_j,h_j)\right| = 0. \tag{4.13}$$

REMARK. *The condition* $\left|\{h_j,k_j\}_{j=1}^n\right| < 2n$ *means that in the sum* $\sum_{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N}$ *some of the indices* h_j, k_j *are equal.*

PROOF. Let us first consider the case in which h_1, k_1 are free indices, i.e.

$$\{h_1,k_1\}\cap\{h_j,k_j\}_{j=2}^n = \emptyset. \tag{4.14}$$

Then, since $\varepsilon_1(k_1,h_1)$ is in $s-L^1(\mathcal{C},dn)$, it follows that

$$\sum_n := \frac{1}{N^n}\sum_{\substack{1\leqq h_1<k_1,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=1}^n|<2n}}\prod_{j=1}^{n}\left|\varepsilon_j(h_j,k_j)\right| = \tag{4.15}$$

$$= \frac{1}{N}\sum_{h_1=1}^{N}\sum_{k_1=h_1+1}^{N}\varepsilon(k_1,h_1)\sum_{\substack{1\leqq h_2<k_2,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=2}^n|<2n-2}}\prod_{j=2}^{n}\left|\varepsilon_j(h_j,k_j)\right| \leqq$$

$$\leqq M_1\cdot\frac{1}{N^{n-1}}\sum_{\substack{1\leqq h_2<k_2,\dots,h_n<k_n\leqq N\\ |\{h_j,k_j\}_{j=2}^n|<2n-2}}\prod_{j=2}^{n}\left|\varepsilon_j(h_j,k_j)\right|.$$

Thus if (4.14) is true then (4.15) and the induction gives our proof. Therefore we may assume that (4.14) is *not true*, i.e. that there exists a $j = 2, \ldots, n$ such that

$$\{h_1, k_1\} \cap \{h_j, k_j\} \neq \emptyset. \tag{4.16}$$

In any case, because of (4.1),

$$\frac{1}{N} \sum_{1 \leqq h_m < k_m \leqq N} |\varepsilon_m(k_m, h_m)| \leqq M_m \tag{4.17}$$

therefore

$$\sum_n \leqq C_1 \frac{1}{N^2} \sum_{\substack{1 \leqq h_1 < k_1, h_j < k_j \leqq N \\ |\{h_1, k_1, h_j, k_j\}| < 4}} \left|\varepsilon_1(h_1, k_1)\right| \left|\varepsilon_j(h_j, k_j)\right| \tag{4.18}$$

for some constant C_1 and the statement follows from Lemma 4.2.

§5. The non-negligible terms

LEMMA 5.1. *Suppose that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n : \mathbf{N}^2 \to C$ are $S - L^1(C, dn)$ and that $F_1, \ldots, F_m$: $\mathbf{N} \to \mathbf{C}$ are such that the limits*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F_j(k) := F_j \quad , \quad j = 1, 2, \ldots, m, \tag{5.1}$$

exist. Then, for each $\{i_1, \ldots, i_m\} \subset \mathbf{N}$, we have

(5.2)

$$\lim_{N \to \infty} \frac{1}{N^{n+m}} \sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < k_{i_1+1} < k_{i_1+1}' < \ldots < \\ < k_{i_2} < k_{i_2}' < r_2 < \ldots < r_m < \ldots < k_n < k_n' \leqq N}} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{(n+m)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

where

$$f_i := \sum_{k=h+1}^{\infty} \varepsilon_i(k, h), \qquad i = 1, 2, \ldots, n. \tag{5.3}$$

PROOF. For $n+m=1$, (5.2) is clearly true. Suppose that (5.2) is true for $m+n \leqq q$. We have, for $m+n=q+1$

(5.4)

$$\frac{1}{N^{q+1}} \sum_{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < r_m < \ldots < k_n < k_n' \leqq N} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{m+n}} \sum_{k_n'=2n+m}^{N} \sum_{k_n=2n+m-1}^{k_n'-1}$$

$$\sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{n+m}} \sum_{k_n'=2n+m}^{N} \sum_{k_n=2n+m-1}^{k_n'-1} \varepsilon_n(k_n', k_n)(k_n-1)^{n+m-1} \cdot$$

$$\cdot \frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) =$$

$$= \frac{1}{N^{n+m}} \sum_{k_n'=2n+m}^{N} \left(\sum_{k_n=2n+m-1}^{K+1} + \sum_{k_n=K+2}^{k_n'-1} \right) \varepsilon_n(k_n', k_n)(k_n-1)^{n+m-1} \cdot$$

$$\cdot \frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k_1' < \ldots < k_{i_1} < k_{i_1}' < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k_{n-1}' \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j).$$

For each $\eta > 0$, we take K such that

(5.5)

$$\left| \frac{1}{K^{n+m-1}} \sum_{1 \leqq k_1 < k_1' < \ldots < k_{n-1} < k_{n-1}' \leqq K} \prod_{i=1}^{n-1} \varepsilon_i(k_i', k_i) \cdot \prod_{j=1}^{m} F_j(r_j) \right.$$
$$\left. - \frac{1}{(n+m-1)!} \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \right| \leqq \eta$$

and

(5.5a)

$$\left| \sum_{k=K+h+1}^{\infty} \varepsilon_i(k, h) \right| \leqq \eta.$$

First of all let us see the absolute value of the first term on the right hand side of (5.4), i.e.

$$\text{(5.6)}\qquad \Bigg| \frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)} \varepsilon_n(k'_n,k_n)\cdot(k_n-1)^{n+m-1}$$

$$\frac{1}{(k_n-1)^{n+m-1}} \cdot \sum_{\substack{1\leqq k_1<k'_1<\ldots<k_{i_1}<k'_{i_1}<r_1<\ldots< \\ <r_m<\ldots<k_{n-1}<k'_{n-1}\leqq k_n-1}} \prod_{i=1}^{n-1} \varepsilon_i(k'_i,k_i)\cdot \prod_{j=1}^{m} F_j(r_j)\Bigg| .$$

By the assumption of the induction, (5.6) is dominated by, with a constant M,

$$\text{(5.7)}\qquad M^{m+n-1}\frac{1}{N} \sum_{k'_n=2n+m}^{N} \left| \sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)} \varepsilon_n(k'_n,k_n)\right| =$$

$$= M^{m+n-1}\frac{1}{N}\left(\sum_{k'_n=2n+m}^{K+2} + \sum_{k'_n=K+2}^{N}\right) \left| \sum_{k_n=2n+m-1}^{(K+1)\wedge(k'_n-1)} \varepsilon_n(k'_n,k_n)\right| .$$

The first term on the right hand side of (5.7) is equal to

$$\text{(5.8)}\qquad O(1)\frac{K+2}{N} \longrightarrow 0, \quad \text{as } N\to\infty.$$

The second term on the right hand side of (5.7) is equal to

$$\text{(5.9)}\qquad M^{m+n-1}\frac{1}{N} \sum_{k'_n=K+2}^{N} \left| \sum_{k_n=2n+m-1}^{(K+1)} \varepsilon_n(k'_n,k_n)\right| \leqq$$

$$\leqq M^{m+n-1}\frac{1}{N} \sum_{k_n=2n+m-1}^{(K+1)} \sum_{k'_n=K+2}^{N} |\varepsilon_n(k'_n,k_n)| =$$

$$= O(1)\frac{K+1}{N} \longrightarrow 0, \quad \text{as } N\to\infty.$$

Let us now consider the second term of (5.4) and rewrite it as

$$\text{(5.10)}\qquad \frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=K+2}^{k'_n-1} \varepsilon_n(k'_n,k_n)(k_n-1)^{n+m-1} .$$

$$\cdot \frac{1}{(n+m-1)!}\prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i +$$

$$+\frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=K+2}^{k'_n-1} \varepsilon_n(k'_n, k_n)(k_n-1)^{n+m-1}$$

$$\left(\frac{1}{(n+m-1)!}\prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i - \right.$$

$$\left. -\frac{1}{(k_n-1)^{n+m-1}} \sum_{\substack{1 \leqq k_1 < k'_1 < \ldots < k_{i_1} < k'_{i_1} < r_1 < \ldots < \\ < r_m < \ldots < k_{n-1} < k'_{n-1} \leqq k_n - 1}} \prod_{i=1}^{n-1} \varepsilon_i(k'_i, k_i) \cdot \prod_{j=1}^{m} F_j(r_j) \right).$$

By (5.4) one knows that the absolute value of the second term of (5.10) is less than or equal to

$$(5.11) \qquad \eta \cdot \frac{1}{N^{n+m}} \sum_{k'_n=2n+m}^{N} \sum_{k_n=K+2}^{k'_n-1} |\varepsilon_n(k'_n, k_n)|(k_n-1)^{n+m-1} = \eta \cdot O(1).$$

Moreover, since n, m, K are fixed so that the limit, as $N \to \infty$, of the first term of (5.10) is equal to the limit of the following quantity:
(5.12)

$$\frac{1}{(n+m-1)!}\prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \cdot \frac{1}{N^{n+m}} \sum_{k'_n=2}^{N} \sum_{k_n=1}^{k'_n-1} \varepsilon_n(k'_n, k_n)(k_n-1)^{n+m-1}.$$

Exchanging the order of summations in (5.12), it becomes
(5.12a)

$$\frac{1}{(n+m-1)!}\prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n-1} f_i \cdot \frac{1}{N^{n+m}} \sum_{k_n=1}^{N-1} (k_n-1)^{n+m-1} \sum_{k'_n=k_n+1}^{N} \varepsilon_n(k'_n, k_n).$$

Letting N tend to infinity, we obtain the limit of (5.12a):

$$(5.13) \qquad \frac{1}{(n+m)!}\prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

and this ends the proof.

More generally we have the following

LEMMA 5.2. *With the same notations and assumptions as in Lemma* 5.1,

$$\lim_{N\to\infty}\frac{1}{N^{n+m}} \tag{5.14}$$

$$\sum_{\substack{1\leqq k_1<k'_1<\ldots<k_{i_1}<k'_{i_1}<r_1<k_{i_1+1}<k'_{i_1+1}<\ldots< \\ <k_{i_2}<k'_{i_2}<r_2<\ldots<r_m<\ldots<k_n<k'_n\leqq N \\ k'_{j_1}\leqq k_{j_1}+d_N,\ldots,k'_{j_p}\leqq k_{j_p}+d_N}} \varepsilon_i(k'_i,k_i)\cdot\prod_{j=1}^{m}F_j(r_j)$$

$$=\frac{1}{(n+m)!}\prod_{j=1}^{m}F_j\cdot\prod_{i=1}^{n}f_i$$

where $d_N\to\infty$ *and* $N-d_N\to\infty$ *as* $N\to\infty$.

PROOF. Notice that the only difference is that on the left hand side of (5.14), $k'_{j_h}-k_{j_h}\leqq d_N$ for each $h=1,\ldots,p$ but on the left hand side of (5.2), $k'_{j_h}-k_{j_h}$ can be greater than d_N ($\leqq N-1$) for some $h\in\{1,\ldots,p\}$. Since the series $\sum_{k=h}^{\infty}\varepsilon(h,k)$ converges and $d_N\to\infty$, we know that

$$\lim_{N\to\infty}\sum_{k=h}^{N-1}\varepsilon(h,k)=\lim_{N\to\infty}\sum_{k=h}^{d_N}\varepsilon(h,k). \tag{5.15}$$

This ends the proof.

In the continuous case, the analogue of Lemmata 5.1 and 5.2 are the following

LEMMA 5.3. *Let* $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n:\mathbf{N}^2\to C$ *be in* $S-L^1(C,dt)$ *and* $F_1,\ldots,F_m$: $\mathbf{R}_+\to\mathbf{C}$ *such that the limits*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T F_j(t)dt=F_j\quad,\quad j=1,2,\ldots,m \tag{5.16}$$

exist, then for each $\{i_1,\ldots,i_m\}\subset\mathbf{N}$ *we have*

$$\lim_{T\to\infty}\frac{1}{T^{n+m}}\int\limits_{\substack{0\leqq t_1<t'_1<\ldots<t_{i_1}<t'_{i_1}<s_1<t_{i_1+1}<t'_{i_1+1}<\ldots< \\ t_{i_2}<t'_{i_2}<s_2<\ldots<s_m<\ldots<t_n<t'_n\leqq T}} \tag{5.17}$$

$$\prod_{i=1}^{n}\varepsilon_i(t'_i,t_i)\cdot\prod_{j=1}^{m}F_j(s_j)dt_1\ldots dt_n\;\;dt'_1\ldots dt'_n\;\;ds_1\ldots ds_p=$$

$$=\frac{1}{(n+m)!}\cdot\prod_{j=1}^{m}F_j\cdot\prod_{i=1}^{n}f_i$$

where

(5.18) $$f_i := \int_{[s,\infty)} \varepsilon_i(t,s)\nu(dt), \quad i = 1,\dots,n.$$

LEMMA 5.4. *Let* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n : \mathbf{N}^2 \to C$ *be in* $S - L^1(C, dt)$ *and let* $F_1, \dots, F_m$: $\mathbf{R}_+ \to \mathbf{C}$ *be such that the limits*

(5.19) $$\lim_{T\to\infty} \frac{1}{T} \int_0^T F_j(t)dt = F_j \quad , \quad j = 1, 2, \dots, m$$

exist. Then for each $\{i_1, \dots, i_m\} \subset \mathbf{N}$ *and* $\{j_1, \dots, j_p\} \subset \{0, 1, \dots, n\}$, *we have*

(5.20) $$\lim_{T\to\infty} \frac{1}{T^{n+m}} \int\limits_{\substack{0 \leqq t_1 < t'_1 < \dots < t_{i_1} < t'_{i_1} < s_1 < t_{i_1+1} < t'_{i_1+1} < \dots < \\ t_{i_2} < t'_{i_2} < s_2 < \dots < s_m < \dots < t_n < t'_n \leqq T \\ t'_{j_h} \leqq t_{j_h} + d_T, \ h = 1, \dots, p}}$$

$$\prod_{i=1}^{n} \varepsilon_i(t'_i, t_i) \cdot \prod_{j=1}^{m} F_j(s_j) dt_1 \dots dt_n \ \ dt'_1 \dots dt'_n \ \ ds_1 \dots ds_p =$$

$$= \frac{1}{(n+m)!} \cdot \prod_{j=1}^{m} F_j \cdot \prod_{i=1}^{n} f_i$$

where $d_T \to \infty$, *and* $T - d_T \to \infty$ *as* $T \to \infty$.

The proofs of these two lemmata are the same as those of Lemmata 5.1 and 5.2.

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BIFURCATIONS IN A PREDATOR-PREY MODEL WITH MEMORY AND DIFFUSION. I: ANDRONOV–HOPF BIFURCATION

M. CAVANI (Cumaná) and M. FARKAS* (Budapest)

1. Introduction

We start off with the model

$$(1.1)\qquad \begin{cases} \dot{N} = \varepsilon N(1 - N/K) - \beta NP/(\beta + N), \\ \dot{P} = -P(\gamma + \delta P)/(1 + P) + \beta NP/(\beta + N) \end{cases}$$

where dot means differentiation with respect to time t; $N(t)$ and $P(t)$ are the quantities of prey and predator, respectively; $\varepsilon > 0$ is the specific growth rate of prey in the absence of predation and without environmental limitation; in the absence of predators the prey population grows logistically to carrying capacity $K > 0$; the functional response of the predator is of Holling's type (see [11, 12]) with satiation coefficient or conversion rate $\beta > 0$; the specific mortality of predators in absence of prey

$$(1.2)\qquad E(P) = (\gamma + \delta P)/(1 + P)$$

depends on the quantity of predators, $\gamma > 0$ is the mortality at low density and $\delta > 0$ is the limiting, maximal mortality (the natural assumption is $\gamma < \delta$).

This system seems to us a fairly realistic one if neither hereditary effects nor spatial distribution are taken into account. The Holling type functional response is widely used and has a vast literature, and if $\gamma = \delta$ then the mortality of predator reduces to a constant (see e.g. [9]). The advantage of the present model over the more often used models is that here the predator mortality is neither a constant nor an unbounded function, still, it is increasing with quantity.

First we study the stability of equilibria of this system and possible bifurcations.

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It is reasonable to assume that the present level of predator quantity effects instantaneously the growth of prey, on the other hand, the growth of predator is influenced by past values of prey quantity. Therefore, secondly, we replace N in the second equation of (1.1) by its time average over the past. We shall be concerned, primarily, in the destabilising effect of the influence of the past and in the character of the possible bifurcations.

Finally, we shall assume that predator and prey undergo Fickian diffusion in space.

Accordingly, in Section 2 conditions for stable equilibria of system (1.1) will be established. In Section 3 an Andronov–Hopf bifurcation will be calculated at a special constellation of the parameters. In Section 4 the delay will be introduced, and conditions for stability will be established. In Section 5 the Andronov–Hopf bifurcation will be calculated when the delay is increased. The study of the reaction-diffusion equation built upon (1.1) will be accomplished in a subsequent paper.

2. Stability of equilibrium points

Clearly, the positive quadrant of the N, P plane is invariant for system (1.1), and one may prove, similarly as it was done in [9], that all solutions with non-negative initial conditions stay bounded in $t \in [0, \infty)$.

On the boundary of the positive quadrant the system has two equilibrium points: (0,0) and $(K, 0)$. A simple linear stability analysis shows that (0,0) is always unstable, and that $(K, 0)$ is asymptotically stable if

$$\gamma > \beta K/(\beta + K), \tag{2.1}$$

and unstable if

$$\gamma < \beta K/(\beta + K). \tag{2.2}$$

Note that (2.2) is equivalent to $0 < \beta\gamma/(\beta - \gamma) < K$ and implies $\gamma < \beta$ and $\gamma < K$.

However, for reasonable parameter configurations we may establish the global stability of $(K, 0)$.

THEOREM 2.1. *If*

$$\gamma \geqq \beta \quad \text{and} \quad \delta \geqq \beta \tag{2.3}$$

then $(K, 0)$ is globally asymptotically stable with respect to the positive quadrant of the N, P plane.

PROOF. Decreasing the first term on the right hand side of the second equation of (1.1) by writing β for γ and δ we get that

$$\dot{P} \leqq -\beta P(1 - N/(\beta + N)) < -cP$$

for some $c > 0$, since $N(t)$ is bounded in $t \in [0, \infty)$. As a consequence, any solution $P(t)$ corresponding to non-negative initial conditions tends to zero as t tends to infinity. Thus, the omega limit set Ω of every solution with positive initial conditions is contained in $\{(N, 0) : N \geqq 0\}$. But for $N > K$ we have $\dot{N} < 0$, so, $\Omega \subset \{(N, 0) : 0 \leqq N \leqq K\}$. Taking into account that $(0, 0) \notin \Omega$ and that Ω is a nonempty, closed, invariant set we get that $\Omega = \{(K, 0)\}$. □

Note that since the right hand side of (2.1) is less than β, the first inequality of (2.3) implies (2.1), i.e. it implies the asymptotic stability of $(K, 0)$. The intuitive meaning of $\gamma \geqq \beta$ is clear: the minimal mortality of the predator is high compared to the conversion rate; this leads to the extinction of the predator. If we assume that the mortality of the predator grows with its quantity, i.e. $\delta > \gamma$ then the first inequality of (2.3) implies the second.

THEOREM 2.2. *If*

$$\gamma < \beta \leqq \delta \tag{2.4}$$

and

$$K \leqq \beta\gamma/(\beta - \gamma) \tag{2.5}$$

then $(K, 0)$ *is globally asymptotically stable with respect to the positive quadrant of the* N, P *plane.*

Note that if $\beta > \gamma$ then (2.5) with a strict inequality is equivalent to (2.1), so if (2.5) is strict we know that the equilibrium is *locally* asymptotically stable.

PROOF. First, consider the case when (2.5) is strict, i.e. (2.1) holds. This implies that an $\eta > 0$ exists such that $\gamma > \beta(K + \eta)/(\beta + K + \eta)$, and so if $N(t) \leqq K + \eta$ then applying (2.4)

$$\dot{P}(t) < -\left(\gamma - \frac{\beta N(t)}{\beta + N(t)}\right) P(t) \leqq -\left(\gamma - \frac{\beta(K + \eta)}{\beta + K + \eta}\right) P(t).$$

But the set $\{(N, P) : 0 < N \leqq K + \eta,\ P > 0\}$ is positively invariant since $\dot{N} < 0$ if $N = K + \eta$, $P \geqq 0$. So if the initial values satisfy $N(0) \leqq K + \eta$, $P(0) > 0$ then $P(t) \to 0$ exponentially as $t \to \infty$. If $N(0) > K + \eta$ then

$$\dot{N}(t) < -\varepsilon\eta N(t) \quad \text{while} \quad N(t) > K + \eta.$$

So N will be equal to $K + \eta$ in finite time, and then $P(t) \to 0$ as before. From here on one may repeat the proof of the previous theorem to complete the proof for this case.

Secondly, assume that (2.5) is an equality, i.e. $\gamma = \beta K/(\beta + K)$. We substitute this value into system (1.1) and move the origin into $(K, 0)$ by the coordinate transformation $n = N - K$, $p = P$. We get the system in the form

$$(2.6)\quad \begin{cases} \dot{n} = -\varepsilon(n+K)n/K - \beta(n+K)p/(\beta+K+n), \\ \dot{p} = -p(\beta K/(\beta+K) + \delta p)/(1+p) + \beta(n+K)p/(\beta+K+n). \end{cases}$$

Now, we use the positive definite Liapunov function

$$V(n,p) = (\beta/K)n^2 + (\beta+K)p^2.$$

If we denote the derivative of V with respect to the system (2.6) by $\dot{V}$ we have

$$\begin{gathered} -(1/2)\dot{V}(n,p)(\beta+K+n)(1+p) = n^2(n+K)(\beta+K+n)(p+1)\beta\varepsilon/K^2+ \\ +np(n+K)(p+1)\beta^2/K + p^2(\delta(\beta+K)p+\beta K)(\beta+K+n)- \\ -\beta(\beta+K)p^2(n+K)(p+1), \end{gathered}$$

and applying (2.4) a simple calculation shows that $\dot{V}(n,p) < 0$ for $n \geqq 0$, $p > 0$. This means that all solutions with positive initial conditions either tend (in principle) to $(n,p) = (0,0)$ or leave the $n \geqq 0$, $p > 0$ quadrant through the line $n = 0$ in finite time. Now, the strip $\{(n,p) : -K < < n < 0,\ p > 0\}$ is positively invariant and if $-K < n(t) < 0$ then applying (2.4)

$$\begin{aligned} \dot{p}(t) &\leqq -p(t)(\beta K/(\beta+K) + \beta p(t))/(1+p(t))+ \\ &\quad +\beta(n(t)+K)p(t)/(\beta+K+n(t)) = \\ &= -p(t)\left(\frac{\beta K}{\beta+K} - \frac{\beta(K+n(t))}{\beta+K+n(t)} + \frac{\beta^2 p(t)}{(1+p(t))(\beta+K)}\right) < 0. \end{aligned}$$

Thus, once, in the strip, $p(t)$ is monotone decreasing and $p(t) \to \alpha \geqq 0$, $t \to \infty$. If $\alpha > 0$ were the case then

$$\dot{p}(t) < -p(t)\frac{\alpha\beta^2}{\left(1+p(t_0)\right)(\beta+K)}, \quad t > t_0$$

would hold for some $t_0 > 0$, and this would imply that p tends to zero exponentially contradicting the assumption $\alpha > 0$. So $p(t)$ tends to zero, and the proof of the previous theorem can be repeated again. □

Note that, as a corollary, conditions (2.4), (2.5) imply that system (1.1) has no equilibrium point in the positive quadrant $N > 0$, $P > 0$.

We now turn to the case when an equilibrium point exists with positive coordinates. Making the right hand sides of system (1.1) equal to zero we get that the prey null-cline is the parabola

$$P = H_1(N) := (K - N)(\beta + N)\varepsilon/(\beta K)$$

and the predator null-cline is the hyperbola

$$P = H_2(N) := \frac{(\beta - \gamma)N - \beta\gamma}{(\delta - \beta)N + \beta\delta}.$$

To have a reasonable concave down predator curve we have to assume $\delta \geqq \beta$, so since the case when also $\gamma \geqq \beta$ has been treated in Theorem 2.1 we shall assume in the sequel that (2.4) holds. In the special case when $\delta = \beta$ the predator curve is the straight line

$$P = H_3(N) := (\beta - \gamma)N/\beta^2 - \gamma/\beta.$$

Since $H_1(N) > 0$ if and only if $-\beta < N < K$ and $H_2(N) > 0$ ($H_3(N) > 0$) if and only if $N > \beta\gamma/(\beta - \gamma)$ the system has an equilibrium point (at least) with positive coordinates $(\overline{N}, \overline{P})$ if and only if

$$\beta\gamma/(\beta - \gamma) < \overline{N} < K \tag{2.7}$$

(cf. condition (2.5); this shows again that if (2.4) and (2.5) hold then there is no equilibrium in the interior of the positive quadrant). The stability of the positive equilibrium can partly be settled by linear stability analysis.

THEOREM 2.3. *Assume that*

$$\gamma < \beta \leqq \delta, \tag{2.4}$$

$$\beta\gamma/(\beta - \gamma) < K, \tag{2.8}$$

and denote a positive equilibrium of system (1.1) *by* $(\overline{N}, \overline{P})$, $\overline{N} > 0$, $\overline{P} > 0$. *If* $K \leqq \beta$ *then system* (1.1) *has a single positive equilibrium and it is asymptotically stable; if* $0 < (K - \beta)/2 \leqq \overline{N}$ *then* $(\overline{N}, \overline{P})$ *(which may not be the only positive equilibrium) is asymptotically stable.*

PROOF. If $K \leqq \beta$ then H_1 is monotone decreasing in the interval $(0, K)$. Since H_2 is monotone increasing in $(\beta\gamma/(\beta - \gamma), \infty)$ this yields the uniqueness of the positive equilibrium.

The Jacobian of the right hand sides of system (1.1) evaluated at $(\overline{N},\overline{P})$ is

$$J\left(\overline{N},\overline{P}\right)=\begin{bmatrix} \frac{\varepsilon\overline{N}\left(K-\beta-2\overline{N}\right)}{K\left(\beta+\overline{N}\right)} & -\frac{\beta\overline{N}}{\beta+\overline{N}} \\ \frac{\beta^2\overline{P}}{\left(\beta+\overline{N}\right)^2} & -\frac{(\delta-\gamma)\overline{P}}{\left(1+\overline{P}\right)^2} \end{bmatrix}.$$

$$\operatorname{Tr} J\left(\overline{N},\overline{P}\right)=\frac{\varepsilon\overline{N}\left(K-\beta-2\overline{N}\right)}{K\left(\beta+\overline{N}\right)}-\frac{(\delta-\gamma)\overline{P}}{\left(1+\overline{P}\right)^2},$$

$$\det J\left(\overline{N},\overline{P}\right)=\frac{\beta\overline{N}\,\overline{P}}{\beta+\overline{N}}\left[-\frac{\varepsilon(\delta-\gamma)\left(K-\beta-2\overline{N}\right)}{K\beta\left(1+\overline{P}\right)^2}+\frac{\beta^2}{\left(\beta+\overline{N}\right)^2}\right].$$

If $K<\beta$ then, in view of (2.4), clearly, $\operatorname{Tr}<0$ and $\det>0$, i.e. $(\overline{N},\overline{P})$ is asymptotically stable indeed. If $(K-\beta)/2\leqq\overline{N}$ the same applies. □

Note that in case $\beta<K$ we have an interval $N\in(0,(K-\beta)/2)$ where the Allée effect holds, i.e. the increase of the prey quantity is beneficial to its growth rate. In this case the sufficient condition of stability $\overline{N}\geqq(K-\beta)/2$ is, in fact, the "Rosenzweig–MacArthur graphical criterion", cf. [6,7]. In our case when $0<\overline{N}<(K-\beta)/2$ then the equilibrium may still be stable.

3. The case $\delta=\beta$

In this section we assume that (2.4) and (2.8) hold with the equality valid in the former. This, as we have seen, ensures the existence of a positive equilibrium. In this special case the coordinates of the positive equilibrium can be determined explicitly and an Andronov–Hopf bifurcation can be calculated by hand. Now the equilibrium point $(\overline{N},\overline{P})$ is the intersection of the parabola $P=H_1(N)$ with the straight line $P=H_3(N)$. We get that

$$\text{(3.1)}\qquad \overline{N}=(1/2)\Big(K-\beta-(1-\gamma/\beta)K/\varepsilon+ +\left(\left(K-\beta-(1-\gamma/\beta)K/\varepsilon\right)^2+4K(\beta+\gamma/\varepsilon)\right)^{1/2}\Big).$$

Assuming that $K>\beta$, the sufficient condition of asymptotic stability proved in Theorem 2.3 is $(K-\beta)/2\leqq\overline{N}$. Substituting (3.1) into this condition we get that $(\overline{N},\overline{P})$ is asymptotically stable if

$$\text{(3.2)}\quad g(K,\beta,\gamma,\varepsilon):=(1-(1-\gamma/\beta)2/\varepsilon)K^2+2(\beta+\beta/\varepsilon+\gamma/\varepsilon)K+\beta^2\geqq 0.$$

This, obviously, means that *if* $\varepsilon \geqq 2(1-\gamma/\beta)$ *then* $(\overline{N},\overline{P})$ *is asymptotically stable (for arbitrary* $K>0$*).* On the other hand if

$$\varepsilon < 2(1-\gamma/\beta) \tag{3.3}$$

then $g(K,\beta,\gamma,\varepsilon) \geqq 0$ for $0<K\leqq K^*$, and $g(K,\beta,\gamma,\varepsilon)<0$ for $K>K^*$ where

$$K^* = \left(\beta(\varepsilon+1)+\gamma+\left(4\varepsilon\beta^2+(\beta+\gamma)^2\right)^{1/2}\right)/(2(1-\gamma/\beta)-\varepsilon). \tag{3.4}$$

Thus, we have arrived at a corollary of Theorem 2.3.

COROLLARY 3.1. *If* $\gamma<\beta=\delta$, (2.8) *and* (3.3) *hold, and* $0<K\leqq K^*$ *then* $(\overline{N},\overline{P})$ *is asymptotically stable.*

Now, let us turn to the most interesting case when (3.3) holds and $K>K^*$. Then g is negative and $(\overline{N},\overline{P})$ lies on the up-going branch of the prey isocline, i.e. to the left from the maximum, in the Allée effect zone. An easy calculation shows that in this case $\det J(\overline{N},\overline{P})>0$ always. On the other hand

$$\begin{aligned} &\operatorname{Tr} J(\overline{N},\overline{P}) = \\ &= \varepsilon\left(1-2\overline{N}/K\right)-\varepsilon\beta\left(\beta^2+\beta-\gamma\right)\left(1-\overline{N}/K\right)/\left(\left(\beta-\gamma\right)\left(\beta+\overline{N}\right)\right), \end{aligned} \tag{3.5}$$

hence

$$\begin{aligned} &\operatorname{sgn}\operatorname{Tr} J(\overline{N},\overline{P}) = \\ &= \operatorname{sgn}\left(1-\overline{N}/\left(K-\overline{N}\right)-\beta\left(\beta^2+\beta-\gamma\right)/\left(\left(\beta-\gamma\right)\left(\beta+\overline{N}\right)\right)\right). \end{aligned}$$

Thus, $(\overline{N},\overline{P})$ is asymptotically stable, resp. unstable if

$$\beta\left(\beta^2+\beta-\gamma\right)/\left(\left(\beta-\gamma\right)\left(\beta+\overline{N}\right)\right)+\overline{N}/\left(K-\overline{N}\right)>1, \quad \text{resp.} \quad <1.$$

Substituting $\overline{N}$ from (3.1) and introducing the notations

$$\begin{aligned} &A = 2(\beta-\gamma)/\left(\varepsilon\beta^2\right)-1/\beta,\ B = 1+\left(\beta^2+\beta-\gamma\right)/(\beta-\gamma)+2\gamma/(\varepsilon\beta), \\ &C = \left(\beta^2+\beta-\gamma\right)/(\beta-\gamma),\ D = (\beta\varepsilon+\gamma)/\varepsilon,\ E = (\varepsilon\beta-(\beta-\gamma))/(2\varepsilon\beta) \end{aligned}$$

we get that the condition of asymptotic stability can be written in the form

$$\left((EK-\beta/2)^2+DK\right)^{1/2} < BK/(AK+C)-EK+\beta/2 \tag{3.6}$$

(and we have instability if the inequality sign is reversed). Because of (3.3), clearly, $A, B, C, D > 0$. The last condition of stability makes sense only if the right hand side is positive. In this case it can be brought to the equivalent form

$$a_2 K^2 + a_1 K + a_o < 0$$

where

$$a_2 = A(AD + 2BE),\ a_1 = 2ACD + 2BCE - \beta AB - B^2,\ a_0 = C(CD - \beta B).$$

Now, an easy calculation shows that the right hand side of (3.6) is positive if

$$EAK^2 + (CE - B - A\beta/2)K - C\beta/2 < 0$$

and this holds if $K \in [0, \widetilde{K})$ where $\widetilde{K} = \infty$ if $E \leqq 0$, and

$$0 < \widetilde{K} = \frac{1}{2EA}\Big[B + A\beta/2 - CE +$$
$$+ \left(B^2 + (A\beta/2 + CE)^2 + 2B(A\beta/2 - CE)\right)^{1/2}\Big]$$

if $E > 0$. If we assume that $\beta \leqq \varepsilon/2$ (which is a fairly reasonable assumption taking into account that ε is the maximum growth rate of prey, and β is the predation rate) then it is easy to see that $a_2 > 0$, $a_0 < 0$, i.e. the equation

$$a_2 K^2 + a_1 K + a_0 = 0 \tag{3.7}$$

has two real roots of different signs. Let us denote the positive root by K_b. Clearly, if $0 < K < K_b$ then $(\overline{N}, \overline{P})$ is asymptotically stable; if $K_b < K$ then it is unstable. Since the determinant of the Jacobian stays positive, and the trace is changing its sign, the loss of stability happens by some kind of an Andronov–Hopf bifurcation. The parameters $\varepsilon, \beta, \gamma$ will be considered fixed, $K > 0$ will play the role of the bifurcation parameter. According to what has been established above we nay expect stability for some $K > K^*$ only if $K^* < \widetilde{K}$. If $K^* \geqq \widetilde{K}$ then the equilibrium point is unstable for all $K > K^*$.

THEOREM 3.2. *If $\gamma < \beta = \delta$, (2.8) holds,*

$$\beta \leqq \varepsilon/2 < 1 - \gamma/\beta, \tag{3.8}$$

$K^* < \widetilde{K}$, and $K_b \in \left(K^*, \widetilde{K}\right)$ then the equilibrium point $\left(\overline{N}(K), \overline{P}(K)\right)$ of system (1.1) undergoes an Andronov–Hopf bifurcation at $K = K_b$; the bifurcation is supercritical, resp. subcritical according as the number

$$\rho = (1/\omega)\big[\,\big(-\left(1 + r^2\right)\left(s\omega C/\left(\beta + \overline{N}\right) + \varepsilon/K_b\right) + \tag{3.9}$$

$$+rs\beta^2\left(\beta^4/\left((\beta-\gamma)^2(\beta+\overline{N})^2\right)-C\right)/(\beta+\overline{N})^2)\cdot$$
$$\cdot(\left(1-r^2+2rsC\right)\omega/\left(\beta+\overline{N}\right)-$$
$$-\beta^2\left(r\beta\left(1-\beta^2/\left((\beta-\gamma)\left(\beta+\overline{N}\right)\right)\right)/(\beta-\gamma)-sC\right)/(\beta+\overline{N})^2+$$
$$+2r\varepsilon/K_b)+\left(\left(1+r^2\right)\omega/\left(\beta+\overline{N}\right)+\right.$$
$$+r\beta^3\left(1-\beta^2/\left((\beta-\gamma)\left(\beta+\overline{N}\right)\right)\right)/\left((\beta-\gamma)\left(\beta+\overline{N}^2\right)^2\right)\right)\cdot$$
$$\cdot(\left(\left(r^2-1\right)sC-2r\right)\omega/\left(\beta+\overline{N}\right)+$$
$$+\beta^2\left(1-rs\left(\beta^4/\left((\beta-\gamma)^2(\beta+\overline{N})^2\right)-C\right)\right)/(\beta+\overline{N})^2+$$
$$+\left(r^2-1\right)\varepsilon/K_b)]+s\left(1+r^2\right)3\omega/(\beta+\overline{N})^2+$$
$$+\beta^2\left(2rsC-1-3r^2\left(1-\beta^4/\left((\beta-\gamma)\left(\beta+\overline{N}\right)\right)\right)\right)/(\beta+\overline{N})^3$$

where $\overline{N}=\overline{N}(K_b)$, $\omega=\left(\det J\left(\overline{N}(K_b),\overline{P}(K_b)\right)\right)^{1/2}$,

$$r=-\varepsilon\beta^3\left(1-\overline{N}/K_b\right)/(\omega(\beta-\gamma)\left(\beta+\overline{N}\right),$$

$$s=\varepsilon\beta\left(1-\overline{N}/K_b\right)/\left(\omega\left(\beta+\overline{N}\right)\right),$$

is negative, resp. positive.

Proof. Denote the characteristic polynomial of system (1.1) at $(\overline{N}(K),\overline{P}(K))$ by

$$p(\lambda)=\lambda^2-\mathrm{Tr}\,(K)\lambda+\det(K)$$

where $\mathrm{Tr}\,(K)=\mathrm{Tr}\,J\left(\overline{N}(K),\overline{P}(K)\right)$, $\det(K)=\det J\left(\overline{N}(K),\overline{P}(K)\right)$ are given at the end of Section 2 (to be read with $\delta=\beta$). We have seen that $\det(K)>0$ for $K>0$, and that $\mathrm{Tr}\,(K)<0$ for $K\in(0,K_b)$, $\mathrm{Tr}\,(K)>$ >0 for $K\in\left(K_b,\widetilde{K}\right)$. $\mathrm{Tr}\,(K_b)=0$ (this is actually equation (3.7) with the positive root substituted into it). Denote the roots of p by $\lambda_1(K)$, $\lambda_2(K)$. Clearly $\mathrm{Re}\,\lambda_i(K)\gtrless 0$ according as $K\gtrless K_b$. At $K=K_b$, $\mathrm{Re}\,\lambda_i(K_b)=$ $=0$ and $\mathrm{Im}\,\lambda_i(K_b)=\pm i\omega$ where $\omega=(\det(K_b))^{1/2}$. To establish the statement about the occurrence of the bifurcation we have to show that the transversality condition

$$d\mathrm{Re}\,\lambda_1(K_b)/dK=(1/2)\,d\mathrm{Tr}\,(K_b)/dK>0$$

holds. Introducing the notation $f(K)=\overline{N}(K)/K$ we get from (3.5) that

$$\mathrm{Tr}\,(K)=-\varepsilon+\varepsilon(1-f(K))(2-\beta C/\left(\beta+\overline{N}(K)\right),$$

hence

$$d\mathrm{Tr}\,(K)/dK = -\varepsilon f'(K)\left(2-\beta C/\left(\beta+\overline{N}(K)\right)\right) + \\ +\varepsilon(1-f(K))\beta C\overline{N}'(K)/\left(\beta+\overline{N}(K)\right)^2.$$

From (3.1) we have

$$f^2(K) - (2E - \beta/K)f(K) - D/K = 0.$$

Differentiating

$$f'(K) = \beta(f(K) - D/\beta)/(K^2(2f(K) - 2E + \beta/K)) < 0$$

since $2f(K) - 2E + \beta/K > 0$ because of (3.1), and $f(K) - D/\beta < 0$ because, clearly, $f(K) < 1$, and $D/\beta = (\varepsilon\beta+\gamma)/(\varepsilon\beta) > 1$. From (3.8) we get $C < 2$, thus $2 - \beta C/\left(\beta+\overline{N}(K)\right) > 0$. We are going to show that $\overline{N}'(K) > 0$. We know that $\overline{N}(K)$ is the solution of $H_1(N) = H_3(N)$ (see Fig. 1), i.e.

$$\varepsilon\left(K-\overline{N}(K)\right)\left(\beta+\overline{N}(K)\right) \equiv \beta K\left((\beta-\gamma)\overline{N}(K)/\beta^2 - \gamma/\beta\right). \tag{3.10}$$

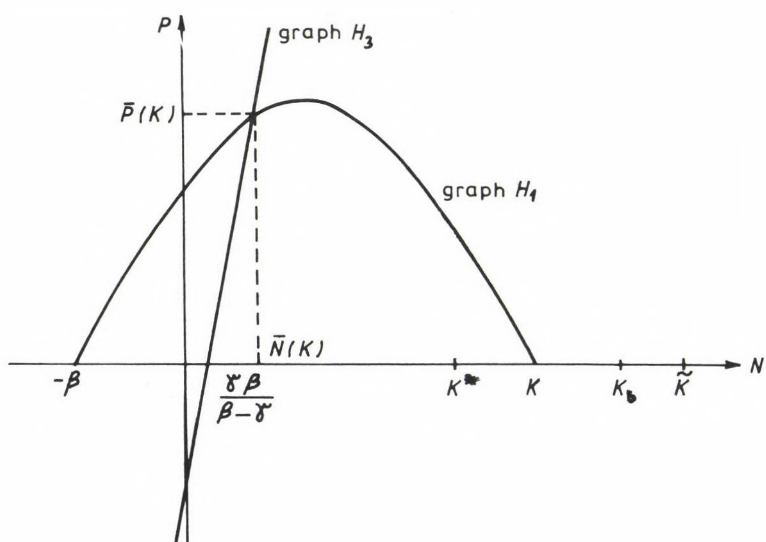

Fig. 1

Differentiating the last identity we get

$$\left(\beta+\overline{N}(K)\right)\overline{N}(K)\varepsilon/(\beta K^2) = \\ = \overline{N}'(K)\left(K(\beta-\gamma) + \varepsilon\beta^2 - \varepsilon\beta K + \varepsilon\beta 2\overline{N}(K)\right)/(\beta^2 K).$$

Thus, $\overline{N}'(K) > 0$ iff

$$K(\beta-\gamma)+\varepsilon\beta^2-\varepsilon\beta K+2\varepsilon\beta\overline{N}(K)>0. \tag{3.11}$$

Expressing K from (3.10) in terms of $\overline{N}$ we get

$$K=\varepsilon\beta\left(\beta+\overline{N}\right)\overline{N}/\left(\varepsilon\beta\left(\beta+\overline{N}\right)+\beta\gamma-(\beta-\gamma)\overline{N}\right).$$

Note that since $K>0$, we have

$$\varepsilon\beta\left(\beta+\overline{N}\right)+\beta\gamma-(\beta-\gamma)\overline{N}>0. \tag{3.12}$$

Substituting the expression for K into (3.11) yields the condition

$$\beta\left(1+\overline{N}\varepsilon\right)+\overline{N}\varepsilon\beta^3/\left(\varepsilon\beta\left(\beta+\overline{N}\right)+\beta\gamma-(\beta-\gamma)\overline{N}\right)>0$$

which holds true, indeed, in view of (3.12). Thus, $d\mathrm{Tr}\,(K)/dK>0$, i.e. all the conditions of the Hopf bifurcation theorem hold (see e.g. [8]). Transforming system (1.1) into normal form and applying Bautin's formula (see [1]) we get (3.9) and this completes the proof of the theorem. □

EXAMPLE. Set $\beta=\delta=0.106$, $\gamma=0.008$, $\varepsilon=1.8000$. These values satisfy the conditions of Theorem 3.2. $\left(K^*,\widetilde{K}\right)=(12.46,\ 31.07)$, and $K_b=$ $=12.92$. Note that for $K\in(K^*,K_b)=(12.46,\ 12.92)$ the asymptotically stable equilibrium $\left(\overline{N}(K),\overline{P}(K)\right)$ is in the Allée effect zone (like the case shown on Fig. 1).

$$\left(\overline{N}(K),\overline{P}(K)\right)=(6.40,\ 55.77),\quad \text{and}\quad \rho=-1.62\cdot10^{-4}.$$

Thus, at $K=K_b$ the equilibrium undergoes a supercritical Andronov–Hopf bifurcation, i.e. for $K>12.92$ (not too large) the system has a small amplitude orbitally asymptotically stable periodic solution.

4. The model with memory

We get a more realistic model if in the second equation of (1.1) we replace the present value of prey by the time average of prey quantity over the past. We follow Cushing [3], MacDonald [10] and Farkas [5] (see also Szabó [13]) in assuming that the influence of the past is fading away exponentially. Accordingly, instead of $N(t)$ the function

$$Q(t):=\int_{-\infty}^{t}N(\tau)a\exp\left(-a(t-\tau)\right)d\tau,\quad a>0 \tag{4.1}$$

will be introduced. Here the exponential weight function satisfies

$$\int_{-\infty}^{t} a\exp(-a(t-\tau))\,d\tau = \int_{0}^{\infty} a\exp(-as)\,ds = 1.$$

The smaller $a > 0$ is the longer is the time interval in the past in which the values of N are taken into account, i.e. $1/a$ is the "measure of the influence of the past".

Thus, (1.1) will be replaced by the integro-differential equation

$$\text{(4.2)}\qquad \begin{cases} \dot{N} = \varepsilon N(1-N/K) - \beta NP/(\beta+N), \\ \dot{P} = -P(\gamma+\delta P)/(1+P) + \beta PQ/(\beta+Q) \end{cases}$$

where Q is in given by (4.1). It can be easily shown that on the interval $t \in [0,\infty)$ (4.2) is equivalent to the ordinary differential system

$$\text{(4.3)}\qquad \begin{cases} \dot{N} = \varepsilon N(1-N/K) - \beta NP/(\beta+N) \\ \dot{P} = -P(\gamma+\delta P)/(1+P) + \beta PQ/(\beta+Q) \\ \dot{Q} = a(N-Q) \end{cases}$$

(see [4]). This system has the following equilibrium points. The origin $(0,0,0)$ which is unstable and of no interest, the point $(K,0,K)$ which is asymptotically stable if $\beta K/(\beta+K) < \gamma$ and unstable if $\beta K/(\beta+K) > \gamma$, and one or more equilibria with positive coordinates if and only if $\beta K/(\beta+$ $+K) > \gamma$, or equivalent iff

$$\text{(4.4)}\qquad 0 < \beta\gamma/(\beta-\gamma) < K$$

(cf. (2.7)). The coordinates $(\overline{N},\overline{P},\overline{Q})$ of an equilibrium are determined by the conditions $\overline{Q} = \overline{N}$, $\overline{P} = H_1(\overline{N}) = H_2(\overline{N})$ where H_1 and H_2 are the functions introduced in Section 2 describing the prey and the predator null--clines, respectively. From the equality of H_1 and H_2 we get that $\overline{N}$ must be a positive root of the cubic polynomial

$$\text{(4.5)}\qquad q(N) = N^3 + q_2N^2 + q_1N + q_0$$

where

$$q_0 = -\beta^2K(\varepsilon\delta+\gamma)/(\varepsilon(\delta-\beta)),$$

$$q_1 = \beta K(\beta-\gamma)/(\varepsilon(\delta-\beta)) - \beta\delta(K-\beta)/(\delta-\beta) - \beta K,$$

$$q_2 = \beta\delta/(\delta-\beta) - (K-\beta).$$

We assume, as before, that (2.4) holds (this time) with a strict inequality sign, i.e.

$$\gamma < \beta < \delta. \tag{4.6}$$

So the constant term in the cubic polynomial is negative, thus, there is either one or three positive roots. $(\overline{N}, \overline{P}, \overline{N})$ denotes one of these, $\overline{N} > 0$, $\overline{P} > 0$. As in (2.7) we must have

$$\beta\gamma/(\beta-\gamma) < \overline{N} < K. \tag{4.7}$$

In order to check the stability of this equilibrium we linearize the system, introduce the notations

$$\begin{cases} \eta = \varepsilon/(K\beta), \quad \Theta_1 = \beta\overline{N}/\left(\beta+\overline{N}\right), \quad \Theta_2 = K-\beta-2\overline{N}, \\ \Theta_3 = \left(K-\overline{N}\right)/\left(\beta+\overline{N}\right), \quad \Theta_4 = \left((\delta-\beta)\overline{N}+\beta\delta\right)^2/(\delta-\gamma), \end{cases} \tag{4.8}$$

and obtain for the coefficient matrix and for the characteristic equation, respectively,

$$A = \begin{bmatrix} \eta\Theta_1\Theta_2 & -\Theta_1 & 0 \\ 0 & -\eta\Theta_3\Theta_4 & \eta\beta^2\Theta_3 \\ a & 0 & -a \end{bmatrix},$$

$$\begin{aligned} &\lambda^3 + (a+\eta(\Theta_3\Theta_4-\Theta_1\Theta_2))\lambda^2 + \left(a\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)-\right.\\ &\left.-\eta^2\Theta_1\Theta_2\Theta_3\Theta_4\right)\lambda + a\eta\Theta_1\Theta_3\left(\beta^2-\eta\Theta_2\Theta_4\right) = 0. \end{aligned} \tag{4.9}$$

Applying the Routh–Hurwitz criterion the eigenvalues have negative real parts if and only if the following inequalities hold:

$$a+\eta(\Theta_3\Theta_4-\Theta_1\Theta_2) > 0, \tag{4.10}$$

$$a(\Theta_3\Theta_4-\Theta_1\Theta_2) - \eta\Theta_1\Theta_2\Theta_3\Theta_4 > 0, \tag{4.11}$$

$$a\eta\Theta_1\Theta_3\left(\beta^2-\eta\Theta_2\Theta_4\right) > 0 \tag{4.12}$$

and

$$\begin{aligned} M(a) := (\Theta_3\Theta_4-\Theta_1\Theta_2)a^2 + \Big(\eta(\Theta_3\Theta_4-\Theta_1\Theta_2)^2 - \beta^2\Theta_1\Theta_3\Big)a-\\ -\eta^2(\Theta_3\Theta_4-\Theta_1\Theta_2)\Theta_1\Theta_2\Theta_3\Theta_4 > 0. \end{aligned} \tag{4.13}$$

Clearly, $\eta, \Theta_1, \Theta_3, \Theta_4 > 0$. Three cases can be distinguished.

Case 1: $\Theta_2 < 0$. This means that $(\overline{N}, \overline{P})$ lies on the descending branch of the prey null-cline of system (1.1). In this case the inequalities (4.10)–(4.12) hold true. If

$$\eta(\Theta_3\Theta_4 - \Theta_1\Theta_2)^2 - \beta^2\Theta_1\Theta_3 \geqq 0 \tag{4.14}$$

then (4.13) holds for all $a > 0$ and $(\overline{N}, \overline{P}, \overline{N})$ is asymptotically stable. If (4.14) does not hold then since the constant term of the quadratic polynomial M is positive, this polynomial either has no real root or has two roots of the same sign. If M has no real roots or has two negative roots then (4.13) holds again for all $a > 0$ and the equilibrium is asymptotically stable. If M has two positive roots, $0 < a_1 < a_0$, say, then the equilibrium is asymptotically stable for large values of a, i.e. for small delays. Using a as a bifurcation parameter, the equilibrium is losing its stability by an Andronov–Hopf bifurcation when a is decreased below a_0, i.e. the delay is increased. However, if a is decreased further below a_1 the equilibrium regains its stability.

Case 2: $\Theta_2 = 0$. The point $(\overline{N}, \overline{P})$ is at the maximum point of the prey null-cline of system (1.1). Again (4.10)–(4.12) hold true. Now, (4.13) is equivalent to

$$a > \beta^2\Theta_1/\Theta_4 - \eta\Theta_3\Theta_4.$$

If the right hand side of this inequality is negative or zero (which taking into account that $\eta = \varepsilon/(K\beta)$ roughly means that the specific growth rate of prey is large enough) then $(\overline{N}, \overline{P}, \overline{N})$ is asymptotically stable for all $a > 0$. A more interesting situation arises if

$$a_0 := \beta^2\Theta_1/\Theta_4 - \eta\Theta_3\Theta_4 > 0.$$

In this case the equilibrium is losing its stability if a is decreased below a_0. This loss of stability occurs again by an Andronov–Hopf bifurcation.

Case 3: $\Theta_2 > 0$. This means that $(\overline{N}, \overline{P})$ is in the Allée effect zone, i.e. on the ascending branch of the prey null-cline of system (1.1). In this case (4.10)–(4.12) are not satisfied automatically.

Let us assume that

$$\Theta_3\Theta_4 - \Theta_1\Theta_2 > 0 \tag{4.15}$$

and

$$\beta^2 - \eta\Theta_2\Theta_4 > 0. \tag{4.16}$$

These inequalities imply (4.10) and (4.12). On the other hand (4.13), (4.15) and (4.16) imply (4.11), thus, (4.13), (4.15) and (4.16) together form a sufficient condition of asymptotic stability of the equilibrium.

If (4.15) and (4.16) hold then the polynomial M has a single positive root $a_0 > 0$, $M(a) > 0$ for $a > a_0$, and $M(a) < 0$ for $0 < a < a_0$. If a is decreased below a_0 then the equilibrium $(\overline{N}, \overline{P}, \overline{N})$ undergoes an Andronov–-Hopf bifurcation.

EXAMPLE. Set $\beta = 0.1$, $\gamma = 0.01$, $\delta = 0.1055$, $\varepsilon = K = 1$. The polynomial (4.5) is now $q(N) = N^3 + 1.018N^2 - 0.190N - 0.210$. The only positive equilibrium of system (1.1) $(\overline{N}, \overline{P}) = (0.448,\ 3.025)$ is in the Allée effect zone. For these values of the parameters (4.15) and (4.16) hold, $\Theta_2 > 0$, and the positive root of M is $a_0 = 0.51$. At a_0 (4.11) still holds true, i.e. a_0 is the critical point of the bifurcation.

The Andronov–Hopf bifurcation of the equilibrium

We are going to treat the three cases of the last section together under the additional assumptions (4.15) and (4.16). (In the first and second cases these inequalities hold automatically.)

DEFINITION 5.1. The positive parameters $\beta, \gamma, \delta, \varepsilon, K$ are called *admissible* if (4.6), (4.7), (4.15) and (4.16) hold, the polynomial (4.5) has a single positive root, and the polynomial M in (4.13) has a simple positive root a_0 such that $M(a) > 0$ for $a > a_0$.

Note that if the parameters are admissible then for $a > a_0$ system (4.3) has a single asymptotically stable equilibrium $(\overline{N}, \overline{P}, \overline{N})$ in the closed positive octant. Note also that the conditions imposed imply that at $a = a_0$ (4.11) is still valid.

THEOREM 5.1. *Suppose that the parameters of system* (4.3) *are admissible then as the bifurcation parameter a is decreased at a_0 the equilibrium $(\overline{N}, \overline{P}, \overline{N})$ undergoes an Andronov–Hopf bifurcation.*

PROOF. At a_0 the characteristic equation (4.9) assumes the form

$$\left(\lambda^2 + a_0\eta(\Theta_3\Theta_4 - \Theta_1\Theta_2) - \eta^2\Theta_1\Theta_2\Theta_3\Theta_4\right) \times \\ \times (\lambda + a_0 + \eta(\Theta_3\Theta_4 - \Theta_1\Theta_2)) = 0.$$

The eigenvalues are

$$\lambda_0(a_0) = -a_0 - \eta(\Theta_3\Theta_4 - \Theta_1\Theta_2) < 0, \quad \lambda_{1,2}(a_0) = \pm i\omega$$

where

$$\omega = \left(a_0\eta(\Theta_3\Theta_4 - \Theta_1\Theta_2) - \eta^2\Theta_1\Theta_2\Theta_3\Theta_4\right)^{1/2} \tag{4.17}$$

(the expression under the root is positive because of (4.11)). A routine calculation shows that

$$\frac{d\mathrm{Re}\,\lambda_1(a_0)}{da} = -\frac{(\Theta_3\Theta_4 - \Theta_1\Theta_2)\left(\omega^2/a_0 + a_0 + \Theta_3\Theta_4 - \Theta_1\Theta_2\right)}{2\left(\omega^2 + (a_0 + \Theta_3\Theta_4 - \Theta_1\Theta_2)^2\right)} < 0.$$

In [2] the Poincaré–Liapunov constant of this bifurcation has been determined whose sign decides about the supercriticality, resp. subcriticality of the bifurcation.

EXAMPLE. Assuming the parameter values of the Example at the end of Section 4: $\beta = 0.100$, $\gamma = 0.010$, $\delta = 0.1055$, $\varepsilon = K = 1$, we have $(\overline{N}, \overline{P}, \overline{N}) = (0.448,\ 3.025,\ 0.448)$, $a_0 = 0.510$. The parameters are admissible and the Poincaré–Liapunov constant is $\rho = -0.207$. This means that a $0 < \alpha < a_0$ exists such that for $a \in (a_0 - \alpha,\ a_0)$ system (4.3) has small amplitude orbitally asymptotically stable periodic solutions with approximate period $2\pi/\omega = 27.8$.

6. Discussion

We have introduced an autonomous (time independent) predator-prey model (1.1) which we consider a fairly realistic one in this category. The growth of prey is restricted by the carrying capacity K of the environment, the functional response of the predator is of Holling's type, i.e. it is growing with increasing prey quantity but is bounded. The mortality of the predator in the absence of prey is a growing but bounded function of the predator quantity. We have shown (Theorem 2.1) that if the mortality of the predator is high compared to the predation rate then the predator dies out. We have also shown (Theorem 2.2) that even in the case when the predation rate is higher than the minimal specific mortality of the predator (but lower than the maximal mortality) the predator dies out provided that the carrying capacity is low. If in this case the carrying capacity is higher but not too high then we have a positive locally stable equilibrium (Theorem 2.3). In a special case it has been shown that the increase of the carrying capacity destabilizes the equilibrium and generates small amplitude periodic oscillations. This happens somewhere in the interior of the Allée effect zone, i.e. when in the neighborhood of the equilibrium the increase of prey quantity is beneficial to its growth rate. A criterion has been given for the stability of these periodic solutions.

We have introduced infinite delay into the model (4.2) assuming that the predator's growth rate depends on past quantities of prey in an exponentially decreasing way. If now the equilibrium lies on the descending branch of the prey null-cline (i.e. in the neighbourhood of the equilibrium point the increase of prey quantity has an adverse effect on its growth rate) then either the equilibrium stays stable for arbitrary large delay or it loses its stability at some value of delay but regains its stability if the delay is increased further. If the equilibrium lies on the ascending branch of the prey null-cline (i.e. in the neighbourhood of the equilibrium the increase of the prey quantity is beneficial to its growth rate), in the Allée effect zone then the increase of delay destabilizes the system and causes the occurrence of periodic oscillations. In the Ph. D. thesis of the first author conditions are given for the stability of the bifurcating periodic solutions. In a second paper we shall introduce spatial distribution into the same model assuming that prey and predator are diffusing according to Fick's law with different diffusion coefficients.

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ALTERNATIVE THEOREMS AND SADDLEPOINT RESULTS FOR CONVEX PROGRAMMING PROBLEMS OF SET FUNCTIONS WITH VALUES IN ORDERED VECTOR SPACES*

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1. Introduction

In the last decade there were two directions, among others, to generalize the classical nonlinear programming problem. In the first of them the problem functions take their values in ordered vector spaces and are defined, as usual, in some topological vector spaces (see for example Zowe [19]). In the second one scalar problems for set mappings were investigated, that is the case when the "variables" are measurable subsets of a measurable space (cf. [18], [10], [11], [13], [14], and [2–5]). A new direction is to bring these two types of problem classes together, namely to investigate set mappings with values in ordered vector spaces (see Lai and Lin [15]). In [15] the usual optimality conditions and duality theorems were proved for this last problem class in the convex case. In our paper (also for the convex case) some alternative theorems and saddlepoint results are given for the above problem class. Our results generalize Theorem 3.1 and Theorem 3.2 of [2] as well as Theorem 3.4 of [18] on the one hand, and are analogous to the classical convex alternative and saddlepoint theorems on the other hand (see for example Berge and Ghouila-Houri [1] and Mangasarian [17]). In order to get our results, it is convenient to introduce first in Section 2 some notations and basic properties of ordered vector spaces and set mappings (see [15–16], [18] and [19]).

2. Preliminaries

Throughout the paper, let Y and Z be locally convex Hausdorff vector spaces over a real field $\mathbf{R}$ and ordered by the closed convex pointed cones C and D respectively. We say that an ordered vector space Y is an *ordered*

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complete vector lattice (see Zowe [19]) if $\inf(x,y)$ exists for any x and y in Y and $\inf B$ exists for any nonempty subset $B \subset Y$ which is bounded from below. For convenience, we assume further that the cones C and D are normal with nonempty interior. Here a *cone* C is said to be *normal* if there exists a neighbourhood system $\{V\}$ of origin Θ such that $V = (V + C) \cap \cap (V - C)$. Hence the order structure and the topological structure in a topological vector space are related by the normal cone. From now on assume that Y and Z are ordered complete vector lattices. We adopt the following notations (see [15–16]).

For any $x, y \in Y$,

$$x \leqq_C y \quad \text{if} \quad y - x \in C,$$
$$x <_C y \quad \text{if} \quad y - x \in C \setminus \{\Theta\},\ \Theta \quad \text{denotes the zero vector,}$$
$$x \ll_C y \quad \text{if} \quad y - x \in \overset{\circ}{C}, \quad \text{the set of interior points of } C.$$

Let C^* be the dual cone of C, that is,

$$C^* = \{y^* \in Y^* \mid \langle y^*, y\rangle \geqq 0 \quad \text{for all} \quad y \in C\}$$

where Y^* is the topological dual of Y. We define

$$B^+(Z,Y) = \{W \in B(Z,Y) \mid W(D) \subset C\}$$

where $B(Z,Y)$ is the space of all continuous linear operators from Z to Y.

Now, let (X, Γ, μ) be an atomless finite measure space with $L^1(X,\Gamma,\mu)$ separable, and let $\mathcal{S} \subset \Gamma$ be a subfamily of measurable subsets of X. Then consider the mappings $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$. Many authors had investigated programming problems for set mappings, see, for example, [2–6], [9–15] and [18]. We restate some crucial properties of set mappings as follows.

For any measurable set $\Omega \in \Gamma$, there corresponds a characteristic function $\chi_\Omega \in L^\infty(X,\Gamma,\mu) \subset L^1(X,\Gamma,\mu)$, since $\mu(X) < \infty$. By the separability of $L^1(X,\Gamma,\mu)$, for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0,1]$, there exist sequences, $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ, such that

$$\text{(1)} \qquad \chi_{\Omega_n} \xrightarrow{w^*} \lambda\chi_{\Omega\setminus\Lambda} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda\setminus\Omega}.$$

In [18, Proposition 3.2], Morris showed that (1) implies

$$\text{(2)} \qquad \chi_{\Omega_n \cup \Lambda_n \cup (\Omega\cap\Lambda)} \xrightarrow{w^*} \lambda\chi_\Omega + (1-\lambda)\chi_\Lambda.$$

Now we can define the convexity of the subfamily $\mathcal{S} \subset \Gamma$ (see [13–14], also [10] and [2–5]). Denote the above sequence $\{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ by

$\{M_n\}$ and call it a *Morris sequence*. A *subfamily* $\mathcal{S} \subset \Gamma$ of measurable sets in X is said to be *convex* if for any $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$, there exists in S a Morris sequence $\{M_n\}$ *in* $\mathcal{S}$ having property (2).

A *set function*, $F : \mathcal{S} \to Y$, is said to be C-*convex* on the convex subfamily $\mathcal{S}$ if, for any $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$,

$$\overline{\lim_{n\to\infty}} F(M_n) \leqq_C \lambda F(\Omega) + (1-\lambda)F(\Lambda)$$

holds for any Morris sequence $\{M_n\} \subset \mathcal{S}$ associated with $(\Omega, \Lambda, \lambda)$.

By these preparations, we can now proceed to establish Farkas type theorems for convex set functions in the next section.

3. Farkas type theorems

The following theorem is an analogue of the convex versions of the well-known Gordan and Ky Fan Theorems in finite dimensions. It is also a generalization of Lemma 3.1 of [2].

THEOREM 3.1. *Let* $\mathcal{S}$ *be a convex subfamily of* Γ, *and let* $F : \mathcal{S} \to Y$ *be a* C-*convex set function. Then the system*

$$F(\Omega) \ll_C \Theta \tag{3}$$

has no solution in $\mathcal{S}$ *if and only if there exists* $y^* \neq \Theta$, $y^* \in C^*$ *such that*

$$\langle y^*, F(\Omega)\rangle \geqq 0 \quad \textit{for all} \quad \Omega \in \mathcal{S}. \tag{4}$$

PROOF. It can be carried out by the same proof as Lemma 3.1 in [1]. For completeness we include here our proof as follows.

The sufficiency part is trivial. Indeed, if (4) holds for some $y^* \neq \Theta$ in C^*, then $\langle y^*, F(\Omega)\rangle \geqq 0$ implies that $F(\Omega) \notin -\overset{\circ}{C}$. Hence the system (3) has no solution.

For the necessity part, we denote the set

$$A = \{y \in Y \mid \text{there exists } \Omega \in \mathcal{S} \text{ such that } F(\Omega) \ll_C y\}.$$

Obviously $\Theta \notin A$, since (3) has no solution.

First, we claim that the set A is convex in Y.

Indeed, let y and $\overline{y}$ be in A. Then there exist $\Omega, \Lambda \in \mathcal{S}$ such that $F(\Omega) \ll_C y$, and $F(\Lambda) \ll_C \overline{y}$. Let $\lambda \in [0,1]$ be an arbitrary number. Since $\mathcal{S}$ is convex and F is a C-convex set function on $\mathcal{S}$, there exists a Morris sequence $\{M_n\}$ in $\mathcal{S}$ corresponding to $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$, such that

$$\overline{\lim_{n\to\infty}} F(M_n) \leqq_C \lambda F(\Omega) + (1-\lambda)F(\Lambda) \ll_C \lambda y + (1-\lambda)\overline{y}.$$

Thus $F(M_n) \ll_C \lambda y + (1-\lambda)\overline{y}$ holds for all, except possibly a finite number of n. It follows that $\lambda y + (1-y)\overline{y} \in A$. Hence A is convex.

Further, since $\overset{\circ}{C} \neq \emptyset$, $\overset{\circ}{A} \neq \emptyset$. Since $\Theta \notin A$, applying the separation theorem, we see that there exists a nonzero element $y^* \in Y^*$ such that

$$(5) \qquad \langle y^*, y\rangle \geqq 0 \quad \text{for all} \quad y \in A.$$

Since $F(\Omega) \ll_C F(\Omega) + c\ (= y)$ for any $c \in \overset{\circ}{C}$, it follows from (5) that $\langle y^*, F(\Omega) + c\rangle \geqq 0$, or equivalently

$$(6) \qquad \langle y^*, F(\Omega)\rangle + \langle y^*, c\rangle \geqq 0.$$

Finally, we want to show that $y^* \in C^*$. Otherwise there would be a $\overline{c} \in C$ such that $\langle y^*, \overline{c}\rangle < 0$. By the continuity of y^*, there is a $c_0 \in \overset{\circ}{C}$ such that $\langle y^*, c_0\rangle < 0$. Since C is a cone, $nc_0 \in \overset{\circ}{C}$ for any positive integer n. Thus (6) becomes

$$0 \leqq \langle y^*, F(\Omega)\rangle + n\langle y^*, c_0\rangle < 0$$

for sufficiently large n which is a contradiction. Hence $y^* \in C^*$.

Note that y^* is a continuous linear functional. Letting $c \to \Theta$, we obtain (4). □

If the system $\left\{\begin{matrix} F(\Omega) \ll_C \Theta \\ G(\Omega) <_D \Theta \end{matrix}\right\}$ has no solution in $\mathcal{S}$, it follows that the system $\left\{\begin{matrix} F(\Omega) \ll_C \Theta \\ G(\Omega) \ll_D \Theta. \end{matrix}\right\}$ has no solution in $\mathcal{S}$, too. Thus the following corollary follows immediately from the necessity part of Theorem 3.1.

COROLLARY 3.1. *Let $\mathcal{S}$ be a convex subfamily of Γ. Suppose that $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ are C-convex and D-convex set functions, respectively. If the system*

$$\left\{\begin{matrix} F(\Omega) \ll_C \Theta \\ G(\Omega) <_D \Theta \end{matrix}\right\}$$

has no solution in $\mathcal{S}$, then there exists $(y^*, z^*) \neq (\Theta, \Theta)$ in $C^* \times D^*$ such that

$$\langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geqq 0 \quad \textit{for all} \quad \Omega \in \mathcal{S}.$$

The following theorem is an analogue of the convex version of the Farkas theorem as well as that of the Motzkin theorem of the alternative.

THEOREM 3.2. *Let $\mathcal{S}$ be a convex subfamily of Γ. Suppose that $F : \mathcal{S} \to$ $\to Y$ is C-convex and $G : \mathcal{S} \to Z$ is D-convex set function, $G(\widetilde{\Omega}) \ll_D \Theta$ for some $\widetilde{\Omega}$. Then the system of inequalities*

$$\text{(7)} \qquad \begin{Bmatrix} F(\Omega) \ll_C \Theta \\ G(\Omega) \leqq_D \Theta \end{Bmatrix}$$

has no solution in $\mathcal{S}$ if and only if there exists $W_0 \in B^+(Z,Y)$ for which there is no $\Omega \in \mathcal{S}$ satisfying

$$F(\Omega) + W_0(G(\Omega)) \ll_C \Theta.$$

PROOF. The sufficiency part is trivial. We only prove the necessity part. Suppose that the system (7) has no solution in $\mathcal{S}$, then so does the system $F(\Omega) \ll_C \Theta$, $G(\Omega) <_D \Theta$. It follows from Corollary 3.1 that there exists $(y^*, z^*) \neq (\Theta, \Theta)$ in $C^* \times D^*$ such that

$$\langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geqq 0 \quad \text{for all} \quad \Omega \in \mathcal{S}.$$

The linear functional y^* in the above expression must be nonzero. For if $y^* = \Theta$, then $z^* \neq \Theta$ and $\langle z^*, z\rangle > 0$ for any nonzero z in $\overset{\circ}{D}$. Since there is an $\widetilde{\Omega} \in \mathcal{S}$ such that $G(\widetilde{\Omega}) \ll_D \Theta$, we have

$$0 > \langle z^*, G(\widetilde{\Omega})\rangle = \langle \Theta, F(\widetilde{\Omega})\rangle + \langle z^*, G(\widetilde{\Omega})\rangle \geqq 0,$$

which is a contradiction. Hence $y^* \neq \Theta$. It follows that $\langle y^*, y\rangle > 0$ for any $y \in \overset{\circ}{C}$. Since $\overset{\circ}{C} \neq \emptyset$, we can choose a nonzero $y_0 \in \overset{\circ}{C}$ such that $\langle y^*, y_0\rangle = 1$. Define an operator $W_0 : Z \to Y$ by $W_0(z) = \langle z^*, z\rangle y_0$. Then $W_0 \in B^+(Z,Y)$ and

$$\begin{aligned} \langle y^*, F(\Omega) + W_0(G(\Omega))\rangle &= \langle y^*, F(\Omega)\rangle + \langle y^*, W_0(G(\Omega))\rangle = \\ &= \langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle \geqq 0, \end{aligned}$$

whence we have

$$\text{(8)} \qquad \langle y^*, F(\Omega) + W_0(G(\Omega))\rangle \geqq 0 \quad \text{for all} \quad \Omega \in \mathcal{S}.$$

Since $\langle y^*, y\rangle > 0$ for all $y \in \overset{\circ}{C}$, it follows from (8) that $F(\Omega) + W_0(G(\Omega)) \notin$ $\notin -\overset{\circ}{C}$ for all $\Omega \in \mathcal{S}$. This shows that there does not exist $\Omega \in \mathcal{S}$ satisfying

$$F(\Omega) + W_0(G(\Omega)) \ll_C \Theta. \quad \square$$

COROLLARY 3.2. *If $Y = \mathbf{R}$ in Theorem 3.2, then — assuming the convexity as well as the Slater type conditions — the system*

$$\left\{ \begin{array}{l} F(\Omega) < 0 \\ G(\Omega) \leqq_D \Theta \end{array} \right\}$$

has no solution in $\mathcal{S}$ if and only if there exists $z^* \in D^*$ such that

$$F(\Omega) + \langle z^*, G(\Omega) \rangle \geqq 0 \quad \text{for any} \quad \Omega \in \mathcal{S}.$$

If $Y = Z = \mathbf{R}$ in Theorem 3.2, then Corollary 3.2 reduces to Theorem 3.1 of [2].

4. Saddlepoint theorems for convex programming problems

Consider the programming problem:

(P) minimize $F(\Omega)$ subject to $\Omega \in \mathcal{S} \subset \Gamma$, and $G(\Omega) \leqq_D \Theta$ where $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ are C-convex and D-convex set mappings, respectively.

Recall that $\Omega_0 \in \mathcal{S}$ is a *weak minimal point* of problem (P) if there does not exist $\Omega \in \mathcal{S}$ with $G(\Omega) \leqq_D \Theta$ such that $F(\Omega) \ll_C F(\Omega_0)$. Denote the Lagrangian by $L(\Omega, W) = F(\Omega) + W(G(\Omega))$, $W \in B(Z, Y)$. We say that (Ω_0, W_0) is a *weak saddle point* of $L(\Omega, W)$ if $W_0 \in B^+(Z, Y)$, $\Omega_0 \in \mathcal{S}$ and there are no $W \in B^+(Z, Y)$ and $\Omega \in \mathcal{S}$ satisfying

$$L(\Omega_0, W_0) \ll_C L(\Omega_0, WW) \quad \text{and} \quad L(\Omega, W_0) \ll_C L(\Omega_0, W_0),$$

respectively.

Applying theorems proved in Section 3, we get the following theorems related to saddlepoint optimality conditions for problem (P).

THEOREM 4.1. *Let $\mathcal{S}$ be a convex subfamily in Γ, and let $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ be C-convex and D-convex set functions, respectively. Assume further that there exists an $\overline{\Omega} \in \mathcal{S}$ such that $G(\overline{\Omega}) \ll_D \Theta$. If Ω_0 is a weak minimal point of problem (P), then there exists $W_0 \in B^+(Z, Y)$ such that (Ω_0, W_0) is a weak saddle point of the Lagrangian $L(\Omega, W)$.*

PROOF. Let $\Omega_0 \in \mathcal{S}$ be a weak minimal point of (P). Then there is no $\Omega \in \mathcal{S}$ satisfying $F(\Omega) \ll_C F(\Omega_0)$ and $G(\Omega) \leqq_D \Theta$. That is, the system of inequalities

$$\left\{ \begin{array}{l} F(\Omega) - F(\Omega_0) \ll_C \Theta \\ G(\Omega) \leqq_D \Theta \end{array} \right\}$$

has no solution. By Theorem 3.2, there exists $W_0 \in B^+(Z,Y)$ for which there is no $\Omega \in \mathcal{S}$ satisfying

$$(9) \qquad F(\Omega) - F(\Omega_0) + W_0(G(\Omega)) \ll_C \Theta.$$

Since $W_0 \in B^+(Z,Y)$ and $G(\Omega_0) \leqq_D \Theta$, we have

$$(10) \qquad W_0(G(\Omega_0)) \leqq_C \Theta.$$

Setting $\Omega = \Omega_0$ in (9), we have that $W_0(G(\Omega_0)) \ll_C \Theta$ is not true. That is,

$$(11) \qquad W_0(G(\Omega_0)) \notin (-C)^\circ.$$

From(10) and (11), we obtain

$$(12) \qquad W_0(G(\Omega_0)) \in \partial(-C)$$

where $\partial(-C)$ is the boundary of $-C$. It follows from (9) and (10) that

$$F(\Omega) - F(\Omega_0) + W_0(G(\Omega)) - W_0(G(\Omega_0)) \notin (-C)^\circ$$

for all $\Omega \in \mathcal{S}$. Hence there is no $\Omega \in \mathcal{S}$ satisfying $L(\Omega, W_0) \ll_C L(\Omega_0, W_0)$.

On the other hand we want to show also that there is no $W \in B^+(Z,Y)$ satisfying

$$L(\Omega_0, W_0) \ll_C L(\Omega_0, W).$$

Suppose on the contrary that there is a $W \in B^+(Z,Y)$ satisfying $L(\Omega_0, W_0) \ll_C L(\Omega_0, W)$, that is, $W_0(G(\Omega_0)) - W(G(\Omega_0)) \in -\overset{\circ}{C}$. Then, from (12),

$$W(G(\Omega_0)) \in \overset{\circ}{C} + W_0(G(\Omega_0)) \subset \overset{\circ}{C} + \partial(-C).$$

Evidently, $W(G(\Omega_0)) \in (-C)$, so that

$$W(G(\Omega_0)) \in \left\{\overset{\circ}{C} + \partial(-C)\right\} \cap (-C).$$

This is a contradiction since

$$(13) \qquad \left\{\overset{\circ}{C} + \partial(-C)\right\} \cap (-C) = \emptyset.$$

We claim that (13) holds. Indeed, if $y_1 \in \overset{\circ}{C}$, $y_2 \in \partial(-C)$ and $y_1 + y_2 \in -C$, then

$$y_2 \in -C - y_1 \subset -C - \overset{\circ}{C} \subset -\overset{\circ}{C}$$

since C is a convex pointed cone, and $C + \overset{\circ}{C} \subset \overset{\circ}{C}$. It follows that $y_2 \in \in (-C)^\circ \cap \partial(-C)$. This is impossible since $(-C)^\circ \cap \partial(-C) = \emptyset$. Therefore, there does not exist $W \in B^+(Z, Y)$ satisfying $L(\Omega_0, W_0) \ll_C L(\Omega_0, W)$. Hence (Ω_0, W_0) is a weak saddle point of $L(\Omega, W)$. $\square$

If $Y = \mathbf{R}$, then the weak minimal point of (P) coincides with the usual minimal point. Thus, from Theorem 4.1 we will have the following corollary.

COROLLARY 4.1. *Let the assumptions be as in Theorem* 4.1 *with* $Y = = \mathbf{R}$. *Then* Ω_0 *is a minimal point of* (P) *if and only if there exists* $z_0^* \in D^*$ *such that* (Ω_0, z_0^*) *is a saddle point of the Lagrangian* $L(\Omega, z^*)$.

PROOF. *Necessity.* As $Y = \mathbf{R}$ in Theorem 4.1, and Ω_0 is a minimal point of (P), there does not exist Ω in $\mathcal{S}$ satisfying $F(\Omega) < F(\Omega_0)$ with $G(\Omega) \leqq_D \Theta$, or equivalently $F(\Omega) \geqq F(\Omega_0)$ for all $\Omega \in \mathcal{S}$ with $G(\Omega) \leqq_D \Theta$, and by Theorem 3.2, there exists $z_0^* \in D^*$ for which there is no $\Omega \in \mathcal{S}$ satisfying

$$F(\Omega) - F(\Omega_0) + \langle z_0^*, G(\Omega)\rangle < 0.$$

That is, there exists $z_0^* \in D^*$ such that

$$F(\Omega) - F(\Omega_0) + \langle z_0^*, G(\Omega)\rangle \geqq 0 \tag{14}$$

holds for all $\Omega \in \mathcal{S}$. It follows that for $\Omega = \Omega_0$ in (14), we have

$$\langle z_0^*, G(\Omega_0)\rangle \geqq 0. \tag{15}$$

Since Ω_0 is a feasible solution for (P), (15) implies that $\langle z_0^*, G(\Omega_0)\rangle = 0$. We claim that (Ω_0, z_0^*) is a saddle point of the Lagrangian

$$L(\Omega, z^*) = F(\Omega) + \langle z^*, G(\Omega)\rangle.$$

Indeed, since for all $z^* \in D^*$, $\langle z^*, G(\Omega_0)\rangle \leqq 0 = \langle z_0^*, G(\Omega_0)\rangle$, we obtain

$$F(\Omega_0) + \langle z^*, G(\Omega_0)\rangle \leqq F(\Omega_0) + \langle z_0^*, G(\Omega_0)\rangle = F(\Omega_0) \leqq F(\Omega) + \langle z_0^*, G(\Omega)\rangle$$

for all $(\Omega, z^*) \in \mathcal{S} \times D^*$ (by (14)). Thus the necessity part is proved.

Sufficiency. Let $(\Omega_0, z_0^*) \in \mathcal{S} \times D^*$ be a saddle point of the Lagrangian $L(\Omega, z^*) = F(\Omega) + \langle z^*, G(\Omega)\rangle$, that is, for all $(\Omega, z^*) \in \mathcal{S} \times D^*$,

$$F(\Omega_0) + \langle z^*, G(\Omega_0)\rangle \leqq F(\Omega_0) + \langle z_0^*, G(\Omega_0)\rangle \leqq F(\Omega) + \langle z_0^*, G(\Omega)\rangle.$$

Since $z_0^* \in D^*$, for all feasible solutions Ω of (P) we have $\langle z_0^*, G(\Omega)\rangle \leqq 0$. In particular $\langle z_0^*, G(\Omega_0)\rangle \leqq 0$. The first saddlepoint inequality implies that

$$\langle z^*, G(\Omega_0)\rangle \leqq \langle z_0^*, G(\Omega_0)\rangle \quad \text{for all} \quad z^* \in D^*.$$

But D^* is a cone, hence $\Theta \in D^*$, and so $\langle z_0^*, G(\Omega_0)\rangle \geqq 0$. Therefore $\langle z_0^*, G(\Omega_0)\rangle = 0$. Thus, from the second saddlepoint inequality and by $\langle z_0^*, G(\Omega)\rangle \leqq 0$, we obtain

$$F(\Omega_0) \leqq F(\Omega) + \langle z_0^*, G(\Omega)\rangle \leqq F(\Omega)$$

for all $\Omega \in \mathcal{S}$ feasible for (P). That is Ω_0 is a minimal point of (P). The sufficiency is proved. □

REMARK 4.1. The Slater type condition $G\left(\overline{\Omega}\right) \ll_D \Theta$ was implicitly used as a preassumption of the above corollary. Nevertheless, it was used only for proving the necessity part.

REMARK 4.2. If $Y = Z = \mathbf{R}$ in Theorem 4.1, Corollary 4.1 reduces to Theorem 3.2 of [2].

If in Theorem 4.1 we omit condition $G\left(\overline{\Omega}\right) \ll_D \Theta$, we can prove only a Fritz John type saddle point (a generalized saddle point, see definition later) which is the necessary optimality condition.

To do this, first we introduce the concepts of the generalized Lagrangian and its weak saddle point for problem (P) as follows.

DEFINITION. The function

$$\widetilde{L}(\Omega, y^*, z^*) = \langle y^*, F(\Omega)\rangle + \langle z^*, G(\Omega)\rangle, \quad \Omega \in \mathcal{S},\ y^* \in Y^*,\ z^* \in Z^*$$

is called the *generalized Lagrangian* (the *Fritz John type Lagrangian)*) for problem (P).

DEFINITION. A triple $(\Omega_0, y_0^*, z_0^*) \in \mathcal{S} \times C^* \times D^*$ is called a *weak saddle point* of $\widetilde{L}(\Omega, y^*, z^*)$ if for all $\Omega \in \mathcal{S}$ and $z^* \in D^*$

$$\widetilde{L}(\Omega_0, y_0^*, z^*) \leqq \widetilde{L}(\Omega_0, y_0^*, z_0^*) \leqq \widetilde{L}(\Omega, y_0^*, z_0^*)$$

holds.

THEOREM 4.2. *Let $\mathcal{S}$ be a convex subfamily in Γ, and let $F : \mathcal{S} \to Y$ and $G : \mathcal{S} \to Z$ be C-convex and D-convex set function, respectively. If Ω_0 is a weak minimal point of problem* (P), *then there exists $(y_0^*, z_0^*) \neq (\Theta, \Theta)$ in $C^* \times D^*$ such that $\langle z_0^*, G(\Omega_0)\rangle = 0$ and (Ω_0, y_0^*, z_0^*) is a weak saddle point of the generalized Lagrangian $\widetilde{L}(\Omega, y^*, z^*)$.*

PROOF. If Ω_0 is a weak minimal point of (P), then the system

$$\left\{\begin{array}{l} F(\Omega) - F(\Omega_0) \ll_C \Theta \\ G(\Omega) \leqq_D \Theta \end{array}\right\}$$

has no solution. It follows that the system

$$\left\{\begin{array}{l} F(\Omega) - F(\Omega_0) \ll_C \Theta \\ G(\Omega) <_D \Theta \end{array}\right\}$$

is inconsistent, too. This satisfies the conditions of Corollary 3.1, so there exists an element $(y_0^*, z_0^*) \neq (\Theta, \Theta)$ in $C^* \times D^*$ such that

$$(16) \qquad \langle y_0^*, F(\Omega) - F(\Omega_0)\rangle + \langle z_0^*, G(\Omega)\rangle \geqq 0$$

for all $\Omega \in \mathcal{S}$. Setting $\Omega = \Omega_0$ in the above inequality we have $\langle z_0^*, G(\Omega_0)\rangle \geqq$ $\geqq 0$. Since $G(\Omega_0) \leqq_D \Theta$, we have $\langle z_0^*, G(\Omega_0)\rangle \leqq 0$. It follows that

$$(17) \qquad \langle z_0^*, G(\Omega_0)\rangle = 0.$$

The point (Ω_0, y_0^*, z_0^*) is a weak saddle point of $\widetilde{L}(\Omega, y^*, z^*)$.

Indeed, by the feasibility of Ω_0 in (P), for arbitrary $z^* \in D^*$ we have $\langle z^*, G(\Omega_0)\rangle \leqq 0 = \langle z_0^*, G(\Omega_0)\rangle$, thus

$$\widetilde{L}(\Omega_0, y_0^*, z^*) = \langle y_0^*, F(\Omega_0)\rangle + \langle z^*, G(\Omega_0)\rangle \leqq \langle y_0^*, F(\Omega_0)\rangle + \langle z_0^*, G(\Omega_0)\rangle = \\ = \widetilde{L}(\Omega_0, y_0^*, z_0^*)$$

and so $\widetilde{L}(\Omega_0, y_0^*, z^*) \leqq \widetilde{L}(\Omega_0, y_0^*, z_0^*)$. From (16) with any $\Omega \in \mathcal{S}$ and (17) we have

$$\widetilde{L}(\Omega_0, y_0^*, z_0^*) = \langle y_0^*, F(\Omega_0)\rangle + \langle z_0^*, G(\Omega_0)\rangle = \\ = \langle y_0^*, F(\Omega_0)\rangle \leqq \langle y_0^*, F(\Omega)\rangle + \langle z_0^*, G(\Omega)\rangle = \widetilde{L}(\Omega, y_0^*, z_0^*),$$

and so $\widetilde{L}(\Omega_0, y_0^*, z_0^*) \leqq \widetilde{L}(\Omega, y_0^*, z_0^*)$ for all $\Omega \in \mathcal{S}$. Consequently,

$$\widetilde{L}(\Omega_0, y_0^*, z^*) \leqq \widetilde{L}(\Omega_0, y_0^*, z_0^*) \leqq \widetilde{L}(\Omega, y_0^*, z_0^*) \quad \text{for all} \quad (\Omega, z^*) \in \mathcal{S} \times D^*.$$

So (Ω_0, y_0^*, z_0^*) is a weak saddle point of the generalized Lagrangian $\widetilde{L}(\Omega, y^*, z^*)$. □

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AXES OF SYMMETRY FOR PLANE CURVES

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1. Introduction and the functions Φ_α

In this paper we consider the class $\mathfrak{M}$ of plane curves described in [4] p. 70 and in [3]. This class contains all ovals and rosettes [1].

We give necessary and sufficient conditions for a curve in $\mathfrak{M}$ to have axes of symmetry. Next, we present a simple way of determining the equation of the curve which has axes of symmetry. For this purpose we introduce the special family of functions $\{\Phi\}_{\alpha\in\mathbf{R}}$ where $\mathbf{R} = (-\infty, +\infty)$. Let C_L be the set of all positive continuous periodic functions on $\mathbf{R}$ with period $L > 0$. Let us fix a function $k \in C_L$ satisfying conditions (A)–(C) in [3] and let $K(t) = \int_0^t k(s)\,ds$. Then, we put

$$\phi_\alpha(s) = K^{-1}\big(\alpha j - K(s)\big), \tag{1.1}$$

where $\alpha \in \mathbf{R}$ and K^{-1} is the inverse function of K.

For the functions Φ_α the following relations hold:

$$\phi_\alpha\big(\phi_\alpha(s)\big) = s, \tag{2.1}$$

$$\phi(s+L) = \phi(s) - L, \tag{3.1}$$

$$\phi_\alpha'(s) = -\frac{k(s)}{k\big(\phi_\alpha(s)\big)}. \tag{4.1}$$

The functions Φ_α are associated with the functions φ_α defined by formula (4) in [4].

The functions φ_α have geometric applications [1], [2], [3], [4]. The set of all functions $\{\varphi_\alpha\}_{\alpha\in\mathbf{R}} \cup \{\phi_\alpha\}_{\alpha\in\mathbf{R}}$ with the composition of functions is a non-commutative group.

2. The axis of symmetry

Consider a curve $z = r_f \in \mathfrak{M}$ ((17), [4]), i.e.

$$\mathbf{R} \in s \longrightarrow r_f(s) = \int_0^s k(t)f(t)e^{iK(t)}\,dt, \quad f, k \in C_L.$$

Let $T_s = e^{iK(s)}$ be the unit tangent vector at the point $r_f(s)$. One can see that the vector $e^{i\left(\alpha j - K(s)\right)}$ is the image of the vector T_s by the symmetry-map with axis spanned by the half straight line

$$\lambda \longrightarrow \lambda e^{i\frac{1}{2}\alpha j}, \quad \lambda > 0.$$

In the set of all points on the curve r_f with tangent vector equal to $e^{i(\alpha j - K(s))}$ we choose a point A which is the first before the point $r_f(s)$. Let d denote the length of the arc between A and $r_f(s)$. Then we define the function

$$F\colon s \longrightarrow s - d. \tag{3.1}$$

This is equal to the function ϕ_α defined by (1.1). We have:

THEOREM 1. *The curve $r_f \in \mathfrak{M}$ has an axis of symmetry parallel to the vector $e^{i\left(\alpha j + \frac{\pi}{2}\right)}$ if and only if $f(s) = f\left(\phi_\alpha(s)\right)$ for all $s \in \mathbf{R}$.*

PROOF. First, we prove that the vector $r_f(s) - r_f\left(\phi_\alpha(s)\right)$ for all $s \in \mathbf{R}$ is parallel to the vector $e^{\frac{1}{2}i\alpha j}$. In fact, setting

$$h(s) = \left[r_f(s) - r_f\left(\phi_\alpha(s)\right), e^{\frac{1}{2}i\alpha j}\right],$$

where $[z, w]$ is the determinant of the complex numbers z and w, we have $h'(s) = 0$. This means that $h(s) \equiv C$, but if $K(s_0) = \frac{1}{2}\alpha j$, then $T_{s_0} = e^{iK(s_0)} = T_{\phi_\alpha(s_0)}$ and $h(s_0) = 0$. Therefore, $h(s) \equiv 0$ and the vector $r_f(s) - r_f\left(\phi_\alpha(s)\right)$ is parallel to the vector $e^{\frac{1}{2}\alpha j}$ for all $s \in \mathbf{R}$.

Next, we verify that the points $\frac{1}{2}\Big(r_f(s) + r_f\left(\phi_\alpha(s)\right)\Big)$ belong to the straight line parallel to the vector $ie^{\frac{1}{2}i\alpha j}$. For this purpose we consider the function

$$g(t, s) = \left[r_f(s) + r_f\left(\phi_\alpha(s)\right) - r_f\left(\phi_\alpha(t)\right),\, e^{\frac{i}{2}\left(\alpha j + \frac{\pi}{2}\right)}\right].$$

Obviously, $g(t,t) = 0$ for all $t \in \mathbf{R}$. Computing the partial derivatives we obtain $\frac{\partial}{\partial s} g(t,s) = 0 = \frac{\partial}{\partial t} g(t,s)$. Thus, $g(t,s) \equiv 0$ and the points $\frac{1}{2}\Big(r_f(s) + r_f\big(\phi_\alpha(s)\big)\Big)$ belong to the axis of symmetry of the curve. This completes the proof.

3. Fourier series and axes of symmetry

Let $f(s) \in C_L$ and let the Fourier series of $f(s)$ ([3], Theorem 1) be given by the formula

$$f(s) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n}{j} K(s) + B_n \sin \frac{n}{j} K(s) \right). \tag{1.3}$$

It is easy to compute that

$$f\big(\phi_\alpha(s)\big) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(P_n \cos \frac{n}{j} K(s) + Q_n \sin \frac{n}{j} K(s) \right), \tag{2.3}$$

where $P_n = A_n \cos \alpha n + B_n \sin \alpha n$ and $Q_n = A_n \sin \alpha n - B_n \cos \alpha n$.

Now, let us examine the equation

$$f(s) = f\big(\Phi_\alpha(s)\big). \tag{3.3}$$

Comparing formulas (1.3) and (2.3) we obtain

THEOREM 2. *The necessary and sufficient condition for a function $f \in C_L$ to be a solution of the equation* (3.3) *is the following:*

$$A_n \sin \frac{1}{2} n\alpha = B_n \cos \frac{1}{2} n\alpha \quad \textit{for all integers} \quad n.$$

Let a function $f \in C_L$ be a solution of the equation

$$f\big(\phi_\alpha(s)\big) = f\big(\phi_\beta(s)\big). \tag{4.3}$$

In the sequel we assume that $\alpha, \beta \in [0, 2\pi)$ because

$$f\big(\phi_{\alpha+2\pi}(s)\big) = f\big(\phi_\alpha(s)\big). \tag{5.3}$$

Then the following two relations are satisfied.

LEMMA 1. *If $\alpha-\beta$ is not commensurable with 2π and the function $f\in C_L$ satisfies equation* (4.5), *then f is a constant function.*

Let (l,m) denote the greatest common divisor of the integers l and m.

LEMMA 2. *If $\alpha-\beta$ is commensurable with 2π, i.e. $\alpha-\beta=2\pi\frac{1}{m}$, $(l,m)=1$ and $f\in C_L$ is a solution of equation* (4.5), *then $A_n=B_n=0$ whenever m does not divide n ($m\nmid n$).*

PROOFS. Comparing the coefficients of the Fourier series of $f(\phi_\alpha(s))$ and $f(\phi_\beta(s))$ we obtain

$$\left(A_n\sin\frac{\alpha+\beta}{2}n-B_n\cos\frac{\alpha+\beta}{2}n\right)\sin\frac{\alpha-\beta}{2}n=0$$

and

$$\left(A_n\cos\frac{\alpha+\beta}{2}n+B_n\sin\frac{\alpha+\beta}{2}n\right)\sin\frac{\alpha-\beta}{2}n=0.$$

Hence, if $\alpha-\beta$ is not commensurable with 2π we immediately obtain $A_n=B_n=0$, for $n=1,2,\dots$. This means that $f(s)=\frac{A_0}{2}$, so it is a constant function.

If $\alpha-\beta=2\pi\frac{1}{m}$, then

$$\sin\frac{\alpha-\beta}{2}n=0\quad\text{for}\quad m\mid n.$$

Thus, $A_n=B_n=0$ for $m\nmid n$. This completes the proof.

Lemma 2 has the following extension. Let $[m_1,m_2,\dots,m_p]$ denote the least common multiple of the integers $m_1,m_2,\dots,m_p$. We assume that $f\in C_L$ is a solution of the following system of equations:

$$f(s)=f\left(\phi_{\alpha_0}(s)\right)=f\left(\phi_{\alpha_1}(s)\right)=\dots=f\left(\phi_{\alpha_p}(s)\right),\tag{6.3}$$

where $\alpha_0,\alpha_1,\dots,\alpha_p$ belong to $[0,2\pi)$; moreover we have (5.3).

LEMMA 3. *$f\in C_L$ ($f\not\equiv$ const.) is a solution of the system* (6.3) *with $\alpha_v-\alpha_0=2\pi\frac{l_v}{m_v}$, $(l_v,m_v)=1$, $v=1,2,\dots,p$, if and only if*

1° *$A_n=B_n=0$ whenever $m_1\nmid n$ or $m_2\nmid n$ or ... or $m_p\nmid n$,*

2° *$A_n\sin\frac{1}{2}n\alpha_0=B_n\cos\frac{1}{2}n\alpha_0$ whenever $[m_1,m_2,\dots,m_p]\mid n$.*

Lemma 3 implies

THEOREM 3. *Let $\alpha_0,\alpha_1,\alpha_2,\dots,\alpha_p$ be real numbers such that*

$$\alpha_v-\alpha_0=2\pi\frac{l_v}{m_v},\quad(l_v,m_v)=1,\quad v=1,2,\dots,p.$$

Then the curve $r_f \in \mathfrak{M}$ has the axes of symmetry parallel to the vectors

$$e^{i\left(\alpha_0 j+\frac{\pi}{2}\right)}, e^{i\left(\alpha_1 j+\frac{\pi}{2}\right)}, \dots, e^{i\left(\alpha_p j+\frac{\pi}{2}\right)}$$

if and only if the function f satisfies the equations (6.3).

Let the Fourier series of $f \in C_L$ be given by (1.3). Lemma 3 allows us to reformulate Theorem 3 in the following way.

THEOREM 4. *For any numbers $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_p$ satisfying the assumptions of Theorem* 3 *the curve r_f has the axes of symmetry if and only if the Fourier coefficients of $f(s)$ satisfy the conditions*

1° $A_n = B_n = 0$, *whenever* $m_1 \nmid n$ *or* $m_2 \nmid n$ *or* ... *or* $m_p \nmid n$,

2° $A_n \sin \frac{1}{2} n\alpha_0 = B_n \cos \frac{1}{2} n\alpha_0$, *whenever* $[m_1, m_2, \dots, m_p] \mid n$.

The last theorem gives a simple method of defining curves with given axes of symmetry.

EXAMPLE. We assume that $k(s) \equiv 1$, $L = 2\pi$, $j = 1$. Putting $\alpha_0 = \frac{1}{7}\pi$, $\alpha_1 = \frac{17}{21}\pi$ and $\alpha_2 = \frac{19}{35}\pi$ we have $\alpha_1 - \alpha_0 = 2\pi\frac{1}{3}$ and $\alpha_2 - \alpha_0 = 2\pi\frac{1}{5}$ and [3,5]= 15. Thus, the function of the form

$$f(s) = \frac{1}{2}A_0 + \sum_{\mu=1}^{i} A_{15\mu}\left(\cos 15\mu s + \tan\frac{15}{14}\pi\mu s \sin 15\mu s\right)$$

determines the curve r_f having axes of symmetry parallel to the vectors $e^{i\frac{9}{14}\pi}$, $e^{i\frac{55}{42}\pi}$, $e^{i\frac{73}{70}\pi}$.

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QUANTUM CENTRAL LIMIT THEOREMS FOR WEAKLY DEPENDENT MAPS. II

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Introduction

In Part I ([21]), three central limit theorems have been stated: the first one (Theorem (1.3)) includes a law of large numbers and is a vanishing result; the second one (Theorem (1.4)) is a central limit theorem for processes with discrete parameter and the third result (Theorem (1.5)) is the extension of the second one to the case of continuous parameter. Our central limit theorems are *deformations* of the usual quantum central limit theorems, considered up to now, in three different ways:

i) The factor $\sigma(b,b')$ in the commutation relations

$$j_t(b)j_s(b') = \sigma(t,s,b,b')j_s(b')j_t(b) + \varepsilon(t,s,b,b') \tag{$*$}$$

($b, b' \in B \subset \mathcal{B}$; B is the set of algebraic generators of $\mathcal{B}$) are not restricted to the values ± 1.

ii) The factor $\varepsilon(t,s,b,b')$ in $(*)$ is not required to vanish identically.

iii) Independence is replaced by weak dependence.

First of all the states that we obtain in the limit are of *Gaussian type*, in the sense that their odd momenta vanish and the even ones are given by weighted sums of products of pair correlations. However, while the ε-factor simply produces a shift in the correlation function of the limiting state (cf. (1.17) of Part I); the σ-factor can give rise to more interesting phenomena. In fact, if all the $\sigma(b,b')$ are present, then the final expressions (1.18) or (1.24) in Part I differ from the usual expressions for the even momenta of a Gaussian state only for the presence of a combinatorial factor. However, if some of the $\sigma(b,b')$ are allowed to vanish, then the sums (1.18) and (1.24) will no longer be over *all* pair partitions, but only over a subset of them. Now it is well known that the notion of *free independence,* recently introduced by Voiculescu [16], leads naturally, by means of *free central limit theorems* to a notion of *free Gaussianity* characterized, in terms of momenta, precisely by the property that the summation in the expression of even momenta is

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taken *not over all the pair partitions, but on a special sub-class of them* (cf. [15]).

The following conjectures are therefore natural:

I. By imposing that some of the factors $\sigma(b,b')$ vanish on some ordered pairs of generators, the expressions (1.18), (1.24) define free Gaussian states.

II. All the states with vanishing odd moments and with even moments given by weighted sums of products of pair correlations (*the sum not necessarily ranging over all the pair partitions, but only over a subset of them*) can be obtained by the present central limit theorems, by appropriately choosing the algebra $\mathcal{B}$ and the factors σ, ε.

The replacement of independence by weak dependence causes a new qualitative phenomenon: now the singletons give non-trivial contributions to the final state.

§6. Proof of the main theorems

After the preliminaries in §2, §3, §4 and §5 (cf. [21]), we can prove our main theorems.

PROOF OF THEOREM (1.3). First we prove Theorem (1.3) for the counting measure, in this case by Lemma (3.7),

$$(6.1)\qquad \frac{1}{\nu([0,T))^{ak}}\int_{[0,T)^k} E\big(j_{t_1}(b_1)\dots j_{t_k}(b_k)\big)\,\nu(dt_1)\dots\nu(dt_k)=$$

$$=\frac{1}{N^{ak}}E\big(S_N(b_1)\cdots S_N(b_k)\big)=$$

$$=\frac{1}{N^{ak}}\sum_{p=1}^{n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}}\sum_{t\in[S_1,\dots,S_p]_N}E\big(j_{t_1}(b_1)\cdots j_{t_k}(b_k)\big)=$$

$$=\frac{1}{N^{ak}}\sum_{p=1}^{k}\sum_{m=0}^{[k/2]}\sum_{1\le p_1<\dots<p_m\le k}\sideset{}{'}\sum_{(q_1,\dots,q_m)}\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{k,p}}$$

$$\sum_{\pi\in\mathcal{S}_k^{(S,p)}}\sum_{t\in I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$

$$\sigma(t_1,\dots,t_k,b_1,\dots,b_k)\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)$$

where t takes discrete values and $I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)$ is defined by (3.19). Clearly the asymptotic behaviour of (6.1) is determined by the sum in t. By the boundedness of ε, σ and formula (1.3b), we know that the right hand side of (6.1) is majorized by

$$\frac{1}{N^{ak}} C_1(b_1, \ldots, b_k) \cdot N^p. \tag{6.2}$$

Since $a > \frac{1}{2}$ or $a = \frac{1}{2}$ and k odd, $p \leqq k/2$ implies that

$$\frac{1}{N^{ak}} \sum_{t \in [S_1, \ldots, S_p]_N} E\big(j_{t_1}(b_1) \cdots j_{t_k}(b_k)\big) \leqq C \cdot N^{p-ak} \longrightarrow 0 \tag{6.3}$$

we need only to consider the case $p > k/2$. For clarity we write the sum

$$\sum_{t \in I_N(S_1, \ldots, S_p, \{p_h, q_h\}_{h=1}^m, \pi)} \tag{6.4}$$

in the notation of Section (.3), i.e.

$$\sum_{\substack{t \in I_N(S_1, \ldots, S_p) \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}} = \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_k \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{(S,p)} \tag{6.5}$$

where we use the superscript (S, p) to mean that the sum in t runs over $[S_1, \ldots, S_p]_N$.

In the following by the boundedness of the σ-factors we can neglect the factor $\sigma(t_1, \ldots, t_k, b_1, \ldots, b_k)$. Now we introduce a procedure to eliminate, step by step the ε- factors, by repeated use of inequality (4.2).

i) Suppose that there exists a $h' \in \{1, \ldots, m\}$ such that

$$\{t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}\} \subset \{t_{r_1}, \ldots, t_{r_{k-2m}}\} \tag{6.6}$$

i.e., to the same pair of times there corresponds an ε-factor and also a product of two operators. For example, in the product

$$j_{t_1}(b_1) j_{t_2}(b_2) j_{t_1}(b_3) j_{t_2}(b_4) = \tag{6.7}$$

$$= \sigma(t_1, t_2, b_2, b_3) j_{t_1}(b_1 \cdot b_3) j_{t_2}(b_2 \cdot b_4) + \varepsilon(t_2, t_1, b_2, b_3) j_{t_1}(b_1) j_{t_2}(b_4)$$

the factor with ε is of type i).

In this case by the uniform boundedness of ε-factors, we can neglect $\varepsilon(t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}, b_{p_{h'}}, b_{q_{h'}})$. Therefore without loss of generality we can suppose that there is no $h' \in \{1, \ldots, m\}$ such that (6.6) is valid.

ii) Suppose that there exists a $h' \in \{1, \ldots, m\}$ such that

$$\{t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}\} \cap \{t_{r_1}, \ldots, t_{r_{k-2m}}\} = \emptyset \tag{6.8}$$

i.e., there is a pair of times which has produced an ε-factor and which is not in correspondence with any operator factor; e.g.

$$\varepsilon(t_1, t_2, b_1, b_2) j_{t_3}(b_3)$$

with $t_3 > t_1 > t_2$.

In this case, since $\varepsilon(\cdot, \cdot, b, b')$ is $s - L^1(\mathcal{C}, dn)$ we know that the quantity

$$\frac{1}{N^{ak}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_k \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{(S,p)} \left| \prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \tag{6.9}$$

$$\left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\right) \right|$$

is dominated by

(6.10)

$$\frac{1}{N^{ak-1}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq \widehat{t_{p_{h'}}} \leqq \cdots \leqq \widehat{t_{q_{h'}}} \leqq \cdots \leqq t_k \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{(S',p')} \cdot \frac{1}{N} \sum_{l_1=1}^{N} \sum_{l_2=l_1+1}^{N} \left| \varepsilon(l_2, l_1, b_{p_{h'}}, b_{q_{h'}}) \right|$$

$$\left| \prod_{\substack{1 \leqq h \leqq m \\ h \neq h'}} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \cdot$$

$$\cdot \left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\right) \right| \leqq$$

$$\leqq M \cdot \frac{1}{N^{ak-1}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq \widehat{t_{p_{h'}}} \leqq \cdots \leqq \widehat{t_{q_{h'}}} \leqq \cdots \leqq t_k \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\ldots,m}}^{(S',p')}$$

$$\left| \prod_{\substack{1 \leqq h \leqq m \\ h \neq h'}} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \cdot \left| E\left(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\right) \right|$$

where M is a constant and (S',p') is defined as in (6.5) for the partition $(S'_1,\dots,S_{p'})$ obtained by taking away from the partition $(S_1,\dots,S_p)$ all the $t_{\pi^{-1}(p_{h'})}, t_{\pi^{-1}(q_{h'})}$ which satisfy (6.8). Suppose that there exist r elements of $\{1,\dots,m\}$ such that (6.8) is valid, then by relabeling the indices we find that (6.9) is dominated by

(6.11)

$$\frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\dots\leqq t_{k-2r}\leqq N \\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m-r}}^{(S',p')} M^r \cdot \left|\prod_{h=1}^{m-r} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right|.$$

iii) By step ii) the estimate of (6.9) is reduced to an estimate of expressions of the form (6.11), where for each $h=1,\dots,m-r$, one and only one of $\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\}$ is in $\{t_{r_1},\dots,t_{r_{k-2m}}\}$. Let us deal separately with the possible cases. If

(6.12a) $$t_{\pi^{-1}(p_1)} \notin \{t_{r_1},\dots,t_{r_{k-2m}}\}$$

and

(6.12b) $$t_{\pi^{-1}(q_1)} \in \{t_{r_1},\dots,t_{k_{n-2m}}\}$$

then (6.11) is majorized by

(6.13)

$$\frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\dots\leqq \widehat{t_{\pi^{-1}(p_1)}}\leqq\dots\leqq t_{k-2r}\leqq N \\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^r \cdot \left|\prod_{h=2}^{m-r} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\right| \cdot \sum_{l=t_{\pi^{-1}(q_1)}+1}^{\infty} \left|\varepsilon(l, t_{\pi^{-1}(q_1)}, b_{p_1}, b_{q_1})\right| \leqq$$

$$\leqq \frac{1}{N^{ak-r}} \sum_{\substack{1\leqq t_1\leqq\dots\leqq \widehat{t_{\pi^{-1}(p_1)}}\leqq\dots\leqq t_{k-2r}\leqq N \\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}.$$

$$\cdot\Big|\prod_{h=2}^{m-r}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\Big|\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)\Big|.$$

Moreover since $t_{\pi^{-1}(q_1)}$ is equal to some t_{r_j}, if we omit the index $t_{\pi^{-1}(q_1)}$ form the right hand side of (6.13), nothing will change (since t_{r_j} remains), i.e. (6.11) is majorized by

$$\text{(6.14)}\qquad \frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\cdots\leqq\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}\,.$$

$$\cdot\Big|\prod_{h=2}^{m-r}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\Big|\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\Big|.$$

Similarly, if

$$\text{(6.15a)}\qquad t_{\pi^{-1}(p_1)}\in\{t_{r_1},\dots,t_{r_{k-2m}}\}$$

and

$$\text{(6.15b)}\qquad t_{\pi^{-1}(q_1)}\notin\{t_{r_1},\dots,t_{r_{k-2m}}\}$$

then by the same argument, (6.11) is majorized by

(6.16)

$$\frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r}\cdot\Big|\prod_{h=2}^{m-r}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\Big|$$

$$\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{n-2m}}}(b_{\pi(r_{n-2m})})\big)\Big|\cdot\sum_{l=1}^{t_{\pi^{-1}(q_1)}+1}\big|\varepsilon(t_{\pi^{-1}(p_1)},l,b_{p_1},b_{q_1})\big|\leqq$$

$$\leqq\frac{1}{N^{ak-r}}\sum_{\substack{1\leqq t_1\leqq\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\leqq\cdots\leqq\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{k-2r}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=2,\dots,m-r}}^{(S',p')} M^{r+1}\,.$$

$$\cdot\Big|\prod_{h=2}^{m-r}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\Big|\Big|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{k-2m}}}(b_{\pi(r_{k-2m})})\big)\Big|.$$

Iterating the procedure we can eliminate the remaining pairs $(\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\})$ for $h = 2, \ldots, m - r$. Then, relabeling the remaining indices, we find the following majorization of (6.11):

(6.17)

$$\frac{1}{N^{ak-r}} \sum_{1 \leqq t_1 \leqq \cdots \leqq t_{k-2r-(2m-2r)} \leqq N}^{(S',p')} M^m \cdot \Big| E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{k-2m}}(b_{\pi(r_{k-2m})})\big) \Big| =$$

$$= \frac{1}{N^{ak-r}} \sum_{1 \leqq t_1 \leqq \cdots \leqq t_{k-2m} \leqq N}^{(S',p')} M^m \cdot \Big| E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{r_{k-2m}}}(b_{\pi(k-2m)})\big) \Big|$$

where $r_1, \ldots, r_{k-2m}$ are defined by (3.9a). By the definition of r, after (6.10), one has $r \leqq m$ and in any case $ak - r \geqq ak - m = \delta(n - 2m)$, where $\delta := \frac{ak-m}{k-2m}$, so that:

(1) if $a > \frac{1}{2}$ then $\delta > \frac{1}{2}$;

(2) if $a = \frac{1}{2}$ and k is odd, then $k - 2m$ is odd and $\delta = \frac{1}{2}$.

In case (1), the right hand side of (6.17) tends to zero because of Corollary (2.2), with δ replacing a and $k - 2m$ replacing k. In case (2), since the time indices in the sum are in increasing order, we can apply Lemma (2.1). Moreover, since $k - 2m$ is odd and the different times t_j are only p', one can only have:

$$p' < \frac{k-2m}{2}, \quad \text{or} \quad p' > \frac{k-2m}{2}.$$

The former case corresponds to $ak > p$ in Lemma (2.1); in the latter case, there must be at least a singleton (i.e. some t_j not equal to any t_h for $j \neq h$). Therefore we can apply condition (ii) of Lemma (2.1). Thus, in case (2) effectively (6.17) tends to zero. Therefore in both cases we conclude that (6.17) tends to zero as $N \to \infty$ and this ends the proof for the discrete case.

If ν is the Lebesgue measure, the situation is simpler because one needs only to consider the case $p = k$ (i.e. all singletons) since the Lebesgue measure on $\mathbf{R}^k$ is equal to zero on the subsets of those $(t_1, \ldots, t_k)$ such that for some indices $1 \leqq j \neq h \leqq k$, $t_j = t_h$. This implies that both cases i) and iii) in the proof of Theorem (1.3) correspond to sets of measure zero, and therefore we need only to consider the situation ii) in the proof of Theorem

(1.3) and replace the sum

(6.18a)
$$\sum_{\substack{1\leqq t_1\leqq\cdots\leqq t_k\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)}$$

by the integral

(6.18b)
$$\int_{\substack{1\leqq t_1\leqq\cdots\leqq t_k\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}} dt_1\cdots dt_k.$$

The arguments of the discrete case are easily adapted to this case.

Theorem (1.3) shows that, in the following, it will be sufficient to study the limits

(6.18)
$$\lim_{N\to\infty}\frac{1}{N^n}E(S_N(b_1)\cdots S_N(b_{2n}))$$

and

(6.19)
$$\lim_{T\to\infty}\frac{1}{T^n}E\left(\int_{[0,T)^{2n}} j_{t_1}(b_1)\cdots j_{t_{2n}}(b_{2n})\right)dt_1\dots dt_{2n}$$

for each $n\in\mathbf{N}$ and $b_1,\dots,b_{2n}\in B$. Our conclusions are stated in Theorems (1.4), (1.5). The proof is rather long therefore we first prove some lemmata.

Applying Lemma (4.2) we find that

(6.20)
$$\frac{1}{N^n}E(S_N(b_1)\cdots S_N(b_{2n}))=$$

$$=\frac{1}{N^n}\sum_{p=1}^{2n}\sum_{m=0}^{n}\sum_{1\leqq p_1<\cdots<p_m\leqq 2n}\sum_{(q_1,\dots,q_m)}{}' \sum_{(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}}$$

$$\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}\ \sum_{t\in I_N(S_1,\dots,S_p,\{p_h,q_h\}_{h=1}^m,\pi)}\ \prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\cdot$$

$$\cdot\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big).$$

Because of Theorem (1.3), the limit of the right hand side of (6.20) is equal to the limit (as $N \to \infty$) of

(6.21)
$$\frac{1}{N^n}\sum_{p=n}^{2n}\sum_{m=0}^{n}\sum_{1\leqq p_1<\dots<p_m\leqq 2n}\sum_{(q_1,\dots,q_m)}{}'\sum_{(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}$$
$$\sum^{(S,p)}_{\substack{1\leqq t_1\leqq\dots\leqq t_{2n}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$
$$\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\,.$$

In the following for each $N\in\mathbf{N}$ and each fixed $p=n,\dots,2n$, $m=$ $=0,1,\dots,n$, $1\leqq p_1<\dots<p_m\leqq 2n$, $1\leqq q_1,\dots,q_m\leqq 2n$ satisfying (3.8a), (3.8b), (3.8c), $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$, $\pi\in\mathcal{S}_{2n}^{(S,p)}$, we denote

(6.22)
$$\Delta_N(2n,(S_1,\dots,S_p),\pi):=$$
$$:=\frac{1}{N^n}\sum^{(S,p)}_{\substack{1\leqq t_1\leqq\dots\leqq t_{2n}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}\prod_{h=1}^{m}\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})$$
$$\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})\cdot E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\,.$$

The first step in dealing with $\Delta_N(2n,(S_1,\dots,S_p),\pi)$ is:

LEMMA (6.1). *For $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$, if there exists an $h'\in$ $\in\{1,\dots,m\}$ satisfying (6.6) (replacing k by $2n$), then*

(6.23)
$$\lim_{N\to\infty}\Delta_N(2n,(S_1,\dots,S_p),\pi)=0.$$

PROOF. Without loss of generality, by the boundedness of the σ-factors, we can neglect the factor $\sigma(t_1,\dots,t_{2n},b_1,\dots,b_{2n})$. We shall also suppose that $h'=1$.

By the same arguments of ii) and iii) in the proof of Theorem (1.3) and in the same notations, we find that

(6.24)
$$\big|\Delta_N(2n,(S_1,\dots,S_p),\pi)\big|\leqq$$

$$\leqq \frac{1}{N^{n-r}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n-2(m-1)} \leqq N \\ t_{\pi^{-1}(p_1)} > t_{\pi^{-1}(q_1)}}}^{(S',p')} M^{m-1} \cdot \left|\varepsilon(t_{\pi^{-1}(p_1)}, t_{\pi^{-1}(q_1)}, b_{p_1}, b_{q_1})\right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|$$

where $r \leqq m-1$. Condition (6.6) implies that in the sum

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n-2(m-1)} \leqq N \\ t_{\pi^{-1}(p_1)} > t_{\pi^{-1}(q_1)}}}^{(S',p')}$$

there are at most $2(n-m)$ free indices, therefore, by relabeling the indices, the right hand side of (6.24) can be rewritten as

$$\text{(6.25)} \qquad \frac{1}{N^{n-r}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n-2m} \leqq N \\ t_{\pi^{-1}(p_1)} > t_{\pi^{-1}(q_1)}}}^{(S',p')} M^{m-1} \cdot \left|\varepsilon(t_{\pi^{-1}(p_1)}, t_{\pi^{-1}(q_1)}, b_{p_1}, b_{q_1})\right|$$

$$\left|E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|.$$

By the boundedness of the factor ε, (6.25) is majorized by

$$\text{(6.26)} \quad \frac{1}{N^{n-r}} \sum_{1 \leqq t_1 \leqq \cdots \leqq t_{2n-2m} \leqq N}^{(S',p')} M^{m} \cdot \left|E\big(j_{t_1}(b_{\pi(r_1)}) \cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|.$$

If $m = n$, then (6.26) is $O(\frac{1}{N})$ which tends to zero; if $m < n$, then since $r \leqq$ $\leqq m-1$, it follows that $n - r \geqq n - m + 1 = \delta(2n-2m)$ with $\frac{1}{2} + \frac{1}{2(n-m)} =:$ $=: \delta > \frac{1}{2}$, therefore the statement follows.

The second step is:

LEMMA (6.2). *For* $(S_1, \ldots, S_p) \in \mathcal{P}_{2n,p}$ *and* $\pi \in \mathcal{S}_{2n}^{(S,p)}$, *if there exists an* $h' \in \{1, \ldots, m\}$ *satisfying* (6.12a) *and* (6.12b) *or* (6.15a) *and* (6.15b), *then*

$$\text{(6.27)} \qquad \lim_{N\to\infty} \Delta_N(2n, (S_1, \ldots, S_p), \pi) = 0.$$

PROOF. By the same arguments as in the proof of Lemma (6.3) and the formula (6.13), one obtains (6.26) again.

Before the third step, we prove the following:

LEMMA (6.3). *For each $k \in \mathbf{N}$, $p = 1, \dots, k$, $b_1, \dots, b_k \in B$, if the limit*

$$\lim_{N \to 0} \frac{1}{N} \sum_{h=1}^{N} E\big(j_h(b \cdot b')\big) \tag{6.28}$$

exists and $E(\cdot, \cdot, b, b')$ is $s - L^1(\mathcal{C}, dn)$ for each $b, b' \in B$, then the quantity

$$W_{k-1} := \frac{1}{N^{ak}} \sum_{t \in I_N(S_1, \dots, S_p)} \left| E\big(j_{t_1}(b_1) \cdots j_{t_k}(b_k)\big) \right| \tag{6.29}$$

with $a \geqq \frac{1}{2}$ is bounded.

PROOF. By Theorem (1.3) we can suppose that $p \geqq n$. Now we prove this lemma by induction. It is clear that one needs only to prove (6.29) in the case of $a = \frac{1}{2}$ and $k \geqq 2$.

If $k = 2$, (6.29) becomes

$$\frac{1}{N} \sum_{h=1}^{N} \left| E\big(j_h(b_1 b_2)\big) \right| + \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \left| E\big(j_h(b_1) j_k(b_2)\big) \right| + \tag{6.29a}$$

$$+ \frac{1}{N} \sum_{1 \leqq k < h \leqq N}^{N} \left| E\big(j_h(b_1) j_k(b_2)\big) \right|.$$

Applying the commutation relation (1.1) to the third term of (6.29a) and by the boundedness of the σ and ε-factors, (6.29a) is dominated by

$$\frac{1}{N} \sum_{h=1}^{N} \left| E\big(j_h(b_1 b_2)\big) \right| + \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \left| E\big(j_h(b_1) j_k(b_2)\big) \right| + \tag{6.30}$$

$$+ \frac{1}{N} \sum_{1 \leqq h < k \leqq N}^{N} \left| E\big(j_h(b_2) j_k(b_1)\big) \right| + M$$

where M is $\frac{1}{N}$ times the sum of N ε-factors. Since the map E is FP mixing, one finds that, for N big enough, (6.30) is majorized by

(6.31)

$$\frac{1}{N} \sum_{\substack{1 \leqq h < k \leqq N \\ k \leqq h + d_N}}^{N} \left| E\big(j_h(b_1) j_k(b_2)\big) \right| + \frac{1}{N} \sum_{\substack{1 \leqq h < k \leqq N \\ h \leqq k + d_N}}^{N} \left| E\big(j_h(b_2) j_k(b_1)\big) \right| + 2M + 1$$

and the right hand side of (6.31) is bounded since $E(\cdot,\cdot,b,b')$ is $s-L^1(\mathcal{C},dn)$ for each $b,b'\in B$. Thus we obtain a bound of W_1, denoted by $|W|_1$.

Suppose that for each $k\leqq n$, we obtain the bound $|W|_k$ on W_k and without loss of generality, assume that $|W|_1\leqq\dots\leqq|W|_{n-1}$. Then

$$W_n\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_{S_p}>t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_{p-1}}}(b_{S_{p-1}})\big)\Big|\cdot\Big|E\big(j_{S_p}(b_{S_p})\big)\Big|+$$

$$+\delta_N\cdot N^p\cdot C$$

where C is a constant. For N large enough, one has $\delta_N\cdot N^p\cdot C\leqq 1$ and so by the induction assumption,

(6.32)

$$W_n\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+|W|_{n-1}\cdot M+1.$$

Repeating the same arguments to (6.32), we find that

(6.33)

$$W_n\leqq\frac{1}{N^{n+1}}\sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_{S_p}\leqq t_{S_{p-1}}+d_N,t_{S_{p-1}}\leqq t_{S_{p-2}}+d_N}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+\big(|W|_{n-1}+|W|_{n-2}\big)\cdot M+2\leqq$$

$$\leqq\dots\leqq$$

$$\leqq\frac{1}{N^{\frac{n+1}{2}}}\sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_{S_h}\leqq t_{S_{h-1}}+d_N,\ h=2,\dots,p}}\Big|E\big(j_{t_{S_1}}(b_{S_1})\cdots j_{t_{S_p}}(b_{S_p})\big)\Big|+$$

$$+(|W|_2+\dots+|W|_{n-1})\cdot M+n-1\leqq(|W|_1+\dots+|W|_{n-1})\cdot M+n.$$

Putting

(6.34) $$|W|_n:=(|W|_1+\dots+|W|_{n-1})\cdot M+n$$

we finish the proof.

Lemmata (6.1) and (6.2) show that in order to consider the limit of $\Delta_N(2n,(S_1,\dots,S_p),\pi)$ we need only to consider those $(S_1,\dots,S_p)$ and $\pi\in$ $\in\mathcal{S}_{2n}^{(S,p)}$ in which

$$\{t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)}\}_{h=1}^{m}\cap\{t_{r_1},\dots,t_{r_{2n-2m}}\}=\emptyset. \tag{6.35}$$

Moreover (third step)

LEMMA (6.4). *For $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$ and $\pi\in\mathcal{S}_{2n}^{(S,p)}$, if the cardinality of the index set $\{t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)}\}_{h=1}^{m}$ is not equal to $2m$, then $\Delta_N(2n,(S_1,\dots,S_p),\pi)$ tends to zero as $N\to\infty$.*

REMARK. Lemma (6.4) shows that one needs only to consider the case when all the $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^{m}$ are singletons of the partition $(S_1,\dots,S_p)$.

PROOF. By the same arguments as in ii) of the proof of Theorem (1.3) and relabeling the indices, one has

$$\left|\Delta_N(2n,(S_1,\dots,S_p),\pi)\right|\leqq \tag{6.36}$$

$$\leqq\frac{1}{N^{n-m}}\sum_{1\leqq t_1\leqq\dots\leqq t_{2n-2m}\leqq N}^{(S',p')}\left|E\big(j_{t_1}(b_{\pi(r_1)})\cdots j_{t_{2n-2m}}(b_{\pi(r_{2n-2m})})\big)\right|$$

$$\cdot\frac{1}{N^m}\sum_{1\leqq h_1<k_1,\dots,h_m<k_m\leqq N}\left|\varepsilon(k_1,h_1,b_1',b_1'')\cdots\varepsilon(k_m,h_m,b_m',b_m'')\right|.$$

Applying Lemma (6.3) we find a majorization of the right hand side of (6.36):

$$\frac{C}{N^m}\cdot\sum_{1\leqq h_1<k_1,\dots,h_m<k_m\leqq N}\left|\varepsilon(k_1,h_1,b_1',b_1'')\cdots\varepsilon(k_m,h_m,b_m',b_m'')\right| \tag{6.37}$$

where C is a constant and $\{b_h',b_h''\}_{h=1}^{m}\subset\{b_h\}_{h=1}^{2n}$. Thus, if the cardinality of the $\{h_j,k_j\}_{j=1}^{m}$ is not $2m$, the statement follows from Lemma (3.4).

REMARK. In the following, for each fixed "good" partition-permutation pair $(S_1,\dots,S_p)$ and $\pi\in\mathcal{S}_{2n}^{(S,p)}$ ("good" means that $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^{m}$ are singletons) we prove that one needs only to consider those terms in which the indices $t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)}$ are consecutive for each $h=1,\dots,m$ (in full analogy with what happens in the weak coupling limit). This is our fourth step. First we prove the following result:

LEMMA (6.5). *For each* $n \in \mathbf{N}$, $1 \leqq p \leqq 2n$, $(S_1, \dots, S_p) \in \mathcal{P}_{2n,p}$ *and for each* $b_1, \dots, b_{2n} \in B$,

$$\lim_{N\to 0} \frac{1}{N^n} \sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N}} E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big) = 0. \tag{6.38}$$

PROOF.

$$\frac{1}{N^n} \sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N}} \Big|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big| = \tag{6.39}$$

$$= \frac{1}{N^n}\Big(\sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N, t_2\leqq d_N+t_1}} + \sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N, t_2> d_N+t_1}} \Big)\Big|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big|.$$

Applying the FP mixing property to the second term of the right hand side of (6.39), one finds that (6.39) is majorized by

$$\frac{1}{N^n} \sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N, t_2\leqq d_N+t_1}} \Big|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\Big| + \tag{6.40}$$

$$+\frac{1}{N^n} \sum_{\substack{1\leqq t_1<\cdots<t_p\leqq N\\ t_1\leqq d_N, t_2> d_N+t_1}} \Big|E\big(j_{t_1}(b_{S_1})\big)\Big|\cdot\Big|E\big(j_{t_2}(b_{S_2})\cdots j_{t_p}(b_{S_p})\big)\Big| + \delta_N\cdot N^{p-n}.$$

Now let us analyze the three terms of (6.40):

— the third term of (6.40) tends clearly to zero;

— if S_1 is a singleton, the second term of (6.40) is equal to zero;

— if S_1 is not singleton then $|S_2| + \cdots + |S_p| \leqq 2n-2$, and by Lemma (6.3),

$$\frac{1}{N^{n-1}} \sum_{1\leqq t_2<\cdots<t_p\leqq N} \Big|E\big(j_{t_2}(b_{S_2})\cdots j_{t_p}(b_{S_p})\big)\Big|$$

is bounded, therefore the second term of (6.40) is dominated by

$$C\cdot\frac{1}{N}\sum_{h=1}^{d_N}\Big|E\big(j_h(b_{S_1})\big)\Big|$$

for some constant C. This clearly tends to zero as $N \to \infty$ because of (1.5a). Thus we have only to prove that the first term of (6.40) tends to zero as $N \to \infty$.

Iterating the above arguments, i.e. splitting the sum with respect to t_3, and then t_4, etc., we find that

(6.41) $$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N}} \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right| =$$

$$= \lim_{N\to\infty} \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 < \cdots < t_p \leqq N \\ t_1 \leqq d_N, t_2 \leqq d_N + t_1, \dots, t_p \leqq d_N + t_{p-1}}} \left| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \right|.$$

And the statement follows again from (1.5a) since the sum on the right hand side of (6.41) is of order less than $O(d_N^p)$.

Now we can make our fourth step:

LEMMA (6.6). *For each $n \in \mathbf{N}$, $1 \leqq p \leqq 2n$, let $(S_1, \dots, S_p) \in \mathcal{P}_{2n,p}$ and $\pi \in \mathcal{S}_{2n}^{(S,p)}$ be good in the sense of the remark preceding Lemma (6.5). Then denoting*

(6.42) $$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\dots,m}}^{0}$$

the subsum of

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\dots,m}}^{(S,p)}$$

extended to all $1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N$ for which there exists an $h = = 1, \dots, m$ such that $t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}$ are not consecutive indices, and denoting

(6.43) $$\Delta_N^0(2n, (S_1, \dots, S_p), \pi) :=$$

$$:= \frac{1}{N^n} \sum_{\substack{1 \leqq t_1 \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\dots,m}}^{0} \prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|,$$

$\Delta_N^0(2n,(S_1,\dots,S_p),\pi)$ *tends to zero as* $N\to\infty$.

PROOF. Let $(S_1,\dots,S_p)\in\mathcal{P}_{2n,p}$ and $\pi\in\mathcal{S}_{2n}^{(S,p)}$ be good. Without loss of generality, we can assume that $t_{\pi^{-1}(p_1)},t_{\pi^{-1}(q_1)}$ are *not* consecutive indices. Denote

$$l:=t_{\pi^{-1}(p_1)},\quad k:=t_{\pi^{-1}(q_1)}$$

and let l_1 be the smallest t_h greater than l.

Since l,k are not consecutive indices, one has $l_1<k$. With these notations (6.43) is majorized by

$$(6.44)\qquad \frac{1}{N^n}\sum\cdots\sum_{l=1}^{N}\sum_{l_1=l+1}^{N}\cdots\sum_{k=l_1+1}^{N}\cdots\sum|\varepsilon(l,k,b_{p_1},b_{q_1})|$$

$$\prod_{h=2}^{m}|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})|\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|.$$

Notice that for each $X(l,\dots,k)$, since both l and k are singletons, we can exchange the l_1-summation with all those preceding the k-summation. That is, by our assumptions on l,k,l_1, one has

$$(6.45)\qquad \sum_{l=1}^{N}\sum_{l_1=l+1}^{N}\cdots\sum_{k=l_1+1}^{N}X(l,\dots,k)=\sum_{l_1=1}^{N}\sum_{l=1}^{l_1-1}\cdots\sum_{k=l_1+1}^{N}X(l,\dots,k)$$

$$=\sum_{l_1=1}^{N}\cdots\sum_{l=1}^{l_1-1}\sum_{k=l_1+1}^{N}X(l,\dots,k).$$

Therefore (6.44) is less than or equal to

$$(6.46)\qquad \frac{1}{N^{n-1}}\sum_{\substack{1\leqq t_1\leqq\cdots\widehat{t_{\pi^{-1}(p_1)}}\leqq\cdots\widehat{t_{\pi^{-1}(q_1)}}\leqq\cdots\leqq t_{2n}\leqq N\\ t_{\pi^{-1}(p_h)}>t_{\pi^{-1}(q_h)},h=2,\dots,m}}\prod_{h=2}^{m}|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})|$$

$$\left|E\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)\right|\frac{1}{N}\sum_{l=1}^{l_1-1}\sum_{k=l_1+1}^{N}|\varepsilon(l,k,b_{p_1},b_{q_1})|.$$

For each $K \in \mathbf{N}$, split the sum

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\dots,m}}$$

into two parts

(6.47)

$$\sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 \leqq K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\dots,m}} + \sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 > K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\dots,m}}$$

Since $\varepsilon(\cdot,\cdot,b_{p_1},b_{q_1})$ is $s - L^1(\mathcal{C}, dn)$, for each $\eta > 0$, we can choose K satisfying

$$\sum_{k=K+1}^{N} \left| \varepsilon(l,k,b_{p_1},b_{q_1}) \right| < \eta. \tag{6.48}$$

By the same arguments as in i), ii) and iii) of the proof of Theorem (1.3) and relabeling the indices, using (6.47) and (6.48), we see that (6.46) is dominated by

$$C \cdot \eta + \frac{1}{N^{n-1}} \sum_{\substack{1 \leqq t_1 \leqq \cdots \widehat{t_{\pi^{-1}(p_1)}} \leqq \cdots \widehat{t_{\pi^{-1}(q_1)}} \leqq \cdots \leqq t_{2n} \leqq N \\ l_1 \leqq K, t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=2,\dots,m}} \tag{6.49}$$

$$\prod_{h=2}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right|$$

$$\frac{1}{N} \sum_{l=1}^{l_1-1} \sum_{k=l_1+1}^{N} \left| \varepsilon(l,k,b_{p_1},b_{q_1}) \right| \leqq C\eta \; + \; C_1 \frac{K}{N}$$

where, C, C_1 are constants. This ends the proof.

In the continuous case, the situation is much easier. In fact, by the same considerations as before (6.18), one needs only to consider the analogue of Lemma (6.6) in the case of $p = 2n$. Replacing again (6.18a) by (6.18b), in this case we are led to estimate the integral

(6.50)

$$\Delta'_T(2n) := \frac{1}{T^k} \int_{\substack{0 \leqq t_1 \leqq \cdots \leqq t_{2n} < T \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h=1,\dots,m}} \prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right|$$

$$\left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right| dt_1 \cdots dt_{2n}$$

which corresponds to the case in which all the blocks of the partitions are singletons. Moreover the partitions $(S_1, \dots, S_{2n})$ are good in the sense of the remark preceding Lemma (6.5). Denoting, in analogy with (6.42)

(6.51) $$\Sigma_{T,\pi,\{p_h,q_h\}_{h=1}^m} := \big\{ (t_1, \dots, t_{2n}) \in [0,T)^{2n} :$$

$$t_1 < t_2 < \cdots < t_{2n};\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)}, h = 1, \dots, m \text{ and there exists a}$$

$$h \in \{1, \dots, m\} \text{ such that } t_{\pi^{-1}(p_h)} \text{ and } t_{\pi^{-1}(q_h)} \text{ are not consecutive}\big\}$$

and

(6.52) $$\Delta^0_T(2n) := \frac{1}{T^k} \int_{\Sigma_{T,\pi,\{p_h,q_h\}_{h=1}^m}} \prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right|$$

$$\left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right| dt_1 \cdots dt_{2n}$$

one has the following

LEMMA (6.7). *$\Delta^0_T(2n)$ defined by (6.52) tends to zero as $T \to \infty$.*

PROOF. The proof is similar to that of Lemma (6.6) (just replace the sums there by integrals).

Now let us come back to the discrete situation. Since the indices

$$\{t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}\}_{h=1}^m$$

are all singletons, we know that $p \geqq 2m$ and the index set $\{r_1, \dots, r_{2n-2m}\}$ is divided into $p - 2m$ sets, hence from Theorem (1.3) we know that $p - 2m$ should be greater than or equal to $n - m$, i.e. $m \leqq p - n$. Now we deal with the property of the partition $(S_1, \dots, S_p)$ and show that for each $h = 1, \dots, m$, S_h has at most 2 elements. This is our fifth step.

As a first generalization of Lemma (2.1) we prove the following result:

LEMMA (6.8). *For each $n \in \mathbf{N}$, $p \geqq n$, $(S_1, \dots, S_p) \in \mathcal{P}_{2n,p}$ and $b_1, \dots, b_{2n} \in B$, if there exists an $h \in \{1, \dots, p\}$ such that the cardinality of the set S_h is greater than or equal to 3, then the limit of the quantity*

$$\Delta_N(n,p) := \frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_p)} \left| E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| \tag{6.53}$$

is zero.

PROOF. We shall prove that $\Delta_N(n,p)$ tends to zero by induction.

For the case of $n = 1$, there is nothing to prove. Suppose that the conclusion of the lemma is true for all integers less than or equal to n, and let us consider the situation for $n+1$. By the FP mixing property of E we have that

$$\Delta_N(n+1,p) = \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_p)\\ t_p\leqq t_{p-1}+d_N}} \left| E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| + \tag{6.54}$$

$$+\frac{1}{N^{n+1}} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \sum_{t_p>t_{p-1}+d_N} \left| E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\right| \cdot$$

$$\cdot \left| E\big(j_{t_p}(b_{S_p})\big)\right| + \delta_N\cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.54) tends to zero as $N\to 0$. Moreover the second term on the right hand side of (6.54) is equal to zero if $|S_p| = 1$. If $|S_p| = 2$ then that term is equal to

(6.55)

$$\frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \left| E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\right| \cdot \frac{1}{N} \sum_{t_p>t_{p-1}+d_N} \left| E\big(j_{t_p}(b_{S_p})\big)\right|$$

$$= \Delta_N(n,p-1)\cdot \frac{1}{N} \sum_{t_p>t_{p-1}+d_N} \left| E\big(j_{t_p}(b_{S_p})\big)\right|.$$

Notice that $\frac{1}{N}\sum_{t_p>t_{p-1}+d_N}\left|E\big(j_{t_p}(b_{S_p})\big)\right|$ is bounded. By assumption there exists an $h\in\{1,\dots,p-1\}$ such that $|S_h|\geqq 3$, so the induction assumption implies that (6.55) tends to zero. If $|S_p|\geqq 3$ then $(S_1,\dots,S_{p-1})$ is a partition of at most $2(n+1)-3$ elements, so we can apply Corollary (2.2) with $k\leqq$ $\leqq 2(n+1)-3$ and $ak = n = \frac{1}{2}\big(2(n+1)-2\big) > \frac{1}{2}k$. Thus we conclude again that (6.55) tends to zero. These arguments show that the limit of

$\Delta_N(n+1,p)$ is equal to the limit of the first term on the right hand side of (6.54). We write that term in the form

$$(6.56)\qquad \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-1})\\ t_p\leqq t_{p-1}+d_N,\, t_{p-1}\leqq t_{p-2}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| +$$

$$+\frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_{p-2})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right| \cdot$$

$$\cdot\frac{1}{N} \sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right| + \delta_N\cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.56) tends to zero as $N\to 0$. Moreover in the second term on the right hand side of (6.56):

— if $|S_{p-1}|=|S_p|=1$, then since $E(j.(b)j.(b'))$ is in $s-L^2(C,dn)$, one has that

$$\frac{1}{N} \sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right|$$

is bounded. By assumption there exists an $h\in\{1,\dots,p-2\}$ such that $|S_h|\geqq 3$, so by the induction assumption this term tends to zero.

— if $|S_{p-1}|+|S_p|>2$, then by Lemma (6.5) with $k\leqq 2n-3$ (so that $n-\frac{1}{2}>\frac{k}{2}$), the quantity

$$(6.57)\qquad \frac{1}{N^{n-\frac{1}{2}}} \sum_{t\in I_N(S_1,\dots,S_{p-2})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right|$$

is bounded and because of (1.5a),

(6.58)
$$\frac{1}{N^{1+\frac{1}{2}}} \sum_{t_{p-1}>t_{p-2}+d_N,\ t_p\leqq t_{p-1}+d_N} \cdot\left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right| = C\cdot\frac{d_N}{N^{1/2}} \longrightarrow 0.$$

Therefore the second term on the right hand side of (6.56) tends to zero as $N\to 0$.

Summing up we have proved that the limit of $\Delta_N(n+1,p)$ is equal to

$$(6.59)\qquad \lim_{N\to\infty} \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-1})\\ t_p\leqq t_{p-1}+d_N,\, t_{p-1}\leqq t_{p-2}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right|.$$

Iterating the above argument one finishes the proof.

Applying Lemma (6.2) to $\Delta_N(2n,(S_1,\ldots,S_p),\pi)$ we conclude that one needs only to consider the case of $|S_h| \leqq 2$ for each $h = 1,\ldots,p$. Moreover Lemma (6.1), Lemma (6.2) and Lemma (6.4) show that we can restrict ourselves to partitions $(S_1,\ldots,S_p)$ and permutations π such that the indices $t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}$ $(h = 1,\ldots,m)$ defined in Lemma (2.6) are singletons of the partition (the other ones give zero contribution). If we omit these singletons, we obtain a subpartition $(S'_1,\ldots,S'_{p'})$ of $(S_1,\ldots,S_p)$. Thus $p' = p - 2m$ and $|S'_h| \leqq 2$ for each $h = 1,\ldots,p' = p-2m$. In the following we shall restrict ourselves to this case.

The sixth step is to show that one needs to consider only partitions with the property that between any two nonsingleton blocks there is an even number of singletons. A corollary of the following lemma is that the total number of singletons is even.

LEMMA (6.9). *With the same notations and assumptions as in Lemma (6.8), if the partition $(S_1,\ldots,S_p)$ is such that either there exist $1 < i < j <$ $< 2n$ with the properties:*
— *S_i and S_j are not singleton,*
— *for each $i < h < j$, S_h is a singleton,*
— *$j - i - 1$ is odd,*
or the partition either begins or ends with an odd number of singletons, then the limit of $\Delta_N(n,p)$, as $N \to \infty$, is zero.

PROOF. The proof is similar to that of Lemma (6.8).

For the case of $n = 1$, nothing has to be proved. Suppose that the conclusion of the lemma is true for all integers less than or equal to n, and consider the situation for $n+1$.

By the FP mixing property of E we have

$$(6.60)\quad \Delta_N(n+1,p) = \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\ldots,S_{p-1})\\ t_p \leqq t_{p-1}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| +$$

$$+\frac{1}{N^{n+1}} \sum_{t\in I_N(S_1,\ldots,S_{p-1})} \sum_{t_p > t_{p-1}+d_N} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\right| \cdot$$

$$\cdot\left|E\big(j_{t_p}(b_{S_p})\big)\right| + \delta_N \cdot N^{p-n-1}.$$

It is clear that the third term on the right hand side of (6.60) tends to zero as $N \to 0$ therefore we can neglect it and consider only the first two terms. We shall distinguish several situations:

The first term of (6.60) is equal to

$$(6.61)\qquad \frac{1}{N^{n+1}} \sum_{\substack{t\in I_N(S_1,\dots,S_{p-2})\\ t_p\leqq t_{p-1}+d_N,\, t_{p-1}\leqq t_{p-2}+d_N}} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_p}(b_{S_p})\big)\right| +$$

$$+\frac{1}{N^{n-\frac{1}{2}}} \sum_{t\in I_N(S_1,\dots,S_{p-2})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-2}}(b_{S_{p-2}})\big)\right| \cdot$$

$$\cdot \frac{1}{N^{1+\frac{1}{2}}} \sum_{t_{p-1}>t_{p-2}+d_N,\, t_p\leqq t_{p-1}+d_N} \left|E\big(j_{t_{p-1}}(b_{S_{p-1}})j_{t_p}(b_{S_p})\big)\right| + \delta_N\cdot N^{p-n-1}.$$

The third term of (6.61) obviously tends to zero and by the same argument as in the proof of Lemma (6.8), the second term of (6.61) also tends to zero. Iterating $(p-1)$-times the above arguments, one reduces the first term of (6.61) to a term of order $d_N^p/N^{n-\frac{1}{2}}$ which tends to zero by (1.5a). So the first term of (6.60) always tends to zero.

If S_p is a singleton, then the second term is equal to zero. If S_p is not a singleton, i.e. $|S_p| = 2$ then the second term of (6.60) is dominated by

$$C\cdot \frac{1}{N^n} \sum_{t\in I_N(S_1,\dots,S_{p-1})} \left|E\big(j_{t_1}(b_{S_1})\cdots j_{t_{p-1}}(b_{S_{p-1}})\big)\right|$$

for some constant C and surely the subpartition $(S_1,\dots,S_{p-1})$ satisfies the condition of this lemma. Therefore the limit is zero by the induction assumption and this ends the proof.

Applying Lemma (6.9) to $\Delta_N(2n,(S_1,\dots,S_p),\pi)$, we conclude that one needs only to consider the situation in which each set of the subpartition $(S'_1,\dots,S'_{p-2m})$ has one or two elements; moreover between any two nonsingleton blocks, there is an even number of singletons (may be zero).

Summing up the results obtained up to now we conclude that for each $p=n,\dots,2n$ (cardinality of the partition), $m=0,1,\dots,p-n$ (number of ε-factors) and $1\leqq p_1<\dots<p_m\leqq 2n$, $1\leqq q_1,\dots,q_m\leqq 2n$ (the indices of those operators that are coupled by an ε-factor) satisfying (3.8a), (3.8b), (3.8c), one needs only to consider those partitions $(S_1,\dots,S_p)$ in which

$$(6.62)\qquad |S_h|\leqq 2, \quad \text{for each}\quad h=1,\dots,p$$

and there exist at least $2m$ singletons; and only those permutations $\pi\in$ $\in \mathcal{S}_{2n}^{(S,p)}$ for which

a) $\{\pi^{-1}(p_h),\pi^{-1}(q_h)\}_{h=1}^m$ are singletons;

b) if we denote $(S'_1, \dots, S'_{p-2m})$ the subpartition obtained by omitting $\{\pi^{-1}(p_h), \pi^{-1}(q_h)\}_{h=1}^m$ from $(S_1, \dots, S_p)$, then in this subpartition, between any two nonsingleton sets there exists an even number of singletons.

REMARK. Notice that if $p = 2n$, the partition $(S_1, \dots, S_{2n})$ satisfies (6.62).

In the following we shall denote all $(S_1, \dots, S_p)$ which satisfy this condition by $\mathcal{P}^0_{2n,p}$ and for each fixed partition $(S_1, \dots, S_p)$, by $\mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$ the set of all $\pi \in \mathcal{S}_{2n}^{(S,p)}$ satisfying a), b) above.

For each fixed $(S_1, \dots, S_p) \in \mathcal{P}^0_{2n,p}$ and $\pi \in \mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$, our seventh step concerns $t \in I_N(S_1, \dots, S_p)$: we shall prove that if S'_i is the $2h-1$-st singleton and S'_j is the $2h$-th singleton and there exists some index l of some ε-factor such that $t_{S'_i} < t_l < t_{S'_j}$, then we get limit zero. More precisely:

LEMMA (6.10). *Denote $\sum_{(S,p,\varepsilon)}$ the sub-sum of*

$$\sum_{\substack{1 \leqq t_1 \leqq \dots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{(S,p)} - \sum_{\substack{1 \leqq t_1 \leqq \dots \leqq t_{2n} \leqq N \\ t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)},\ h=1,\dots,m}}^{0} \tag{6.63}$$

with the property that there exist $1 \leqq i < j \leqq p'$ such that:

— S'_i is the $2h-1$-st singleton and S'_j is the $2h$-th singleton,

— there exists some index t_l of some ε-factor such that $t_{S'_i} < t_l < t_{S'_j}$,

then

$$\frac{1}{N^n} \sum_{(S,p,\varepsilon)} \prod_{h=1}^m \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \tag{6.64}$$

$$\left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right|$$

converges to zero as $N \to \infty$.

PROOF. Since the total number of singletons is even, for each term of (6.64) one can find an integer m' such that there are exactly $2m'$ singletons in the sub-partition $(S'_1, \dots, S'_{p-2m})$, defined in b) above. We label them by $i_1, \dots, i_{2m'}$. By the assumption of this lemma, we know that there exists an $x = 1, \dots, m'$ such that between $t_{S'_{i_{2x-1}}}$ and $t_{S'_{i_{2x}}}$ there is an index of some ε-factor. Since the indices of the ε-factors are consecutive (by Lemma (6.6)), it follows that between $t_{S'_{i_{2x-1}}}$ and $t_{S'_{i_{2x}}}$ there is an even number of indices of ε-factors.

Let us split (6.64) into two parts:

$$\text{(6.65)} \qquad \frac{1}{N^n}\Bigg(\sum_{\substack{(S,p,\varepsilon) \\ t_{S'_{i_{2x-1}}} > t_{S'_{i_{2x-1}-1}} + d_N, \text{ and } t_{S'_{i_{2x}}} + d_N < t_{S'_{i_{2x}+1}}}} + \sum_{\substack{(S,p,\varepsilon) \\ t_{S'_{i_{2x-1}}} \leqq t_{S'_{i_{2x-1}-1}} + d_N, \text{ or } t_{S'_{i_{2x}}} + d_N \leqq t_{S'_{i_{2x}+1}}}} \Bigg)$$

$$\prod_{h=1}^{m} \left| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \right| \cdot \left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})}) \big) \right|.$$

By the same argument as in the proof of Theorem (1.3) it follows that the second term of (6.65) goes to zero as $N \to \infty$. Moreover, the same arguments as in the proof of Lemma (6.1) imply that the first term of (6.65) is majorized by

$$\text{(6.66)} \qquad C \cdot \frac{1}{N^2} \sum_{1 \leqq l_1 < h < k < l_2 \leqq N} \left| E\big(j_{l_1}(b'_1) j_{l_2}(b'_2) \big) \cdot \varepsilon(k, h, b, b') \right|$$

where C is a constant and $b, b', b_1, b_2 \in B$, therefore the statement follows from (4.2a).

In the continuous case, for a fixed partition $(S_1, \dots, S_{2n})$, a fixed permutation π, a fixed m, and a fixed set $p_1 < \dots < p_m$, $q_1 < \dots < q_m$, denote

$$\text{(6.67)} \quad \Omega_c(\pi, \{p_h, q_h\}_{h=1}^m) := \big\{ (t_1, \dots, t_{2n}) \in [0, T)^{2n} : \ \text{(i) } t_1 < \dots < t_{2n};$$

$$\text{(ii) } t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)} \text{ and they are consecutive for any } h = 1, \dots, m;$$

$$\text{(iii) the indices } t_{S'_{2h}},\ t_{S'_{2h+1}} \text{ are consecutive for each } h = 1, \dots, n-m \big\}.$$

Recall that we have already shown that, in the continuous case, we need only to consider the case in which $p = 2n$, i.e. only those partitions which are entirely made up of singletons.

LEMMA (6.11). *As* $T \to \infty$ *the limit of* $\Delta_T(2n, (S_1, \dots, S_{2n}), \pi)$ *is equal to the limit of the quantity*

$$\text{(6.68)} \qquad \frac{1}{T^n} \int_{\Omega(\pi, \{p_h, q_h\}_{h=1}^m)} dt_1 \cdots dt_{2n} \prod_{h=1}^{m} \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right|.$$

PROOF. The same as in Lemma (6.10).

Summing up our conclusions:

— In the discrete case, one needs only to consider the partitions $(S_1, \dots, S_p) \in \mathcal{P}^0_{2n,p}$ and the permutations $\pi \in \mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$ (cf. (6.62) and below). In this situation,

(6.69) $$\lim_{N\to\infty} \Delta_N(2n, (S_1, \dots, S_p), \pi) =$$

$$= \lim_{N\to\infty} \frac{1}{N^n} \sum_{t\in\Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)} \prod_{h=1}^m \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\left| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \right|$$

where

(6.70) $$\Omega_d(\pi, \{p_h, q_h\}_{h=1}^m) :=$$

$$:= \big\{ t \in I_N(S_1, \dots, S_p) : \ \{t_{\pi^{-1}(p_h)}\}, \{t_{\pi^{-1}(q_h)}\} \text{ are singletons, } h = 1, \dots, m;$$
$$t_{\pi^{-1}(p_h)} > t_{\pi^{-1}(q_h)} \text{ and they are consecutive for any } h = 1, \dots, m;$$
$$t_{S'_{2i_l}} \text{ is the consecutive index of } t_{S'_{2i_l-1}},\ l = 1, \dots, m' \text{ and}$$
$$\{S'_{2i_l}, S'_{2i_l-1}\}_{l=1}^{m'} \text{ are singletons}\big\}$$

— In the continuous case, the limit of $\Delta_T(2n, (S_1, \dots, S_{2n}), \pi)$ is given by (6.68).

Moreover for each $p = n, \dots, 2n$ and $(S_1, \dots, S_p)$, the above conclusions show that we need only to consider the term in which there exist $2(m + m')$ singletons and $p - 2(m + m')$ nonsingletons. Since each nonsingleton set must have cardinality 2, we have the following relation

$$2n = 2(m + m') + 2(p - 2(m + m'))$$

i.e.

$$p = n + m + m'.$$

This shows that we have $p - 2(m + m') = p - 2(p - n) = 2n - p$ nonsingletons and each of them is of cardinality 2. Recall that in the product map case one needs only to consider the case $p = n$ and for each partition $(S_1, \dots, S_p)$, $|S_1| = \dots = |S_p| = 2$. Therefore the number of nonsingleton sets is an invariant which depends on the map E and the commutation relations.

Now we pass to the eighth step of the proof:

LEMMA (6.12). *For each* $n \in \mathbf{N}$, $p = n, \dots, 2n$ *and* $(S_1, \dots, S_p) \in \mathcal{P}_{2n,p}$ *if*

— $|S_h| \leqq 2, \quad h = 1, \dots, p$;

— *between each two nonsingletons there exists an even number of singletons.*

Then for $b_1, \dots, b_{2n} \in B$, *the limit of*

$$(6.71)\quad \frac{1}{N^n} \sum_{1 \leqq t_1 < \dots < t_p \leqq N} \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) - E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \Big|$$

is equal to zero, where

(6.72)

$$E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) := E\big(j_{t_1}(b_{S_1})\big) \cdots E\big(j_{t_{i_1}}(b_{S_{i_1}}) j_{t_{i_1+1}}(b_{S_{i_1+1}})\big)$$

$$\cdots E\big(j_{t_{i_{m'}}}(b_{S_{i_{m'}}}) j_{t_{i_{m'}+1}}(b_{S_{i_{m'}+1}}) \cdots E\big(j_{t_p}(b_{S_p})\big)$$

and $\{i_h\}_{h=1}^{m'} \subset \{1, \dots, p\}$ *with* $i_1 < \dots < i_{m'}$, $\{S_{i_h}, S_{i_h+1}\}_{h=1}^{m'}$ *are all singletons.*

PROOF. We apply induction on n. For $n = 1$, (6.71) is identically equal to zero. Supposing that (6.71) tends to zero for each integer less than or equal to n, let us see the situation for $n + 1$.

If $|S_p| = 2$, we split (6.71) into two parts:

$$(6.73)\quad \frac{1}{N^{n+1}} \Bigg(\sum_{\substack{1 \leqq t_1 < \dots < t_p \leqq N \\ t_p > t_{p-1} + d_N}} + \sum_{\substack{1 \leqq t_1 < \dots < t_p \leqq N \\ t_p \leqq t_{p-1} + d_N}} \Bigg)$$

$$\Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) - E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_p}(b_{S_p})\big) \Big|.$$

The same arguments as in the proof of Theorem (1.3) imply that the second term of (6.73) converges to zero as $N \to 0$. From the FP mixing property of E, one knows that the first term of (6.73) is dominated by

$$(6.74)\quad \frac{1}{N^n} \sum_{1 \leqq t_1 < \dots < t_{p-1} \leqq N} \Big| E\big(j_{t_1}(b_{S_1}) \cdots j_{t_{p-1}}(b_{S_{p-1}})\big) -$$

$$- E^c\big(j_{t_1}(b_{S_1}) \cdots j_{t_{p-1}}(b_{S_{p-1}})\big) \Big| \cdot \frac{1}{N} \sum_{1 \leqq t_p \leqq N} \Big| E\big(j_{t_p}(b_{S_p})\big) \Big|$$

where $(n+1)-1=n \leqq p-1$. The second sum of (6.73) is bounded and the induction assumption implies that the first sum of (6.73) goes to zero. Therefore (6.71) tends to zero.

On the other hand, if $|S_p|=1$, then $|S_{p-1}|=1$ and the proof that (6.71) goes to zero is almost the same as before, the only difference being that one has to split (6.71) into two parts between the indices t_{p-2} and t_{p-1}.

As an improvement of Lemma (6.14), one has

LEMMA (6.13). *For each* $n \in \mathbf{N}$, $p=2,\ldots,2n$ *and* $(S_1,\ldots,S_p) \in \in \mathcal{P}^0_{2n,p}$, $\pi \in \mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}$, *the quantity*

$$(6.75) \qquad \frac{1}{N^n} \sum_{t \in \Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)} \prod_{h=1}^m \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$\Big| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) -$$

$$- E^c\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \Big|$$

tends to zero as $N \to \infty$.

PROOF. The lemma is an application of Lemma (6.14) and the property that $\varepsilon(\cdot,\cdot,b_l,b'_l)$ is in $s-L^1(\mathcal{C},dn)$ for each $l=1,\ldots,m$.

The continuous analogue of Lemma (6.13) is

LEMMA (6.14). *For each* $n \in \mathbf{N}$,

$$(6.76) \qquad \frac{1}{T^n} \int_{\Omega_c(\pi,\{p_h,q_h\}_{h=1}^m)} dt_1 \cdots dt_{2n} \prod_{h=1}^m \big| \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h}) \big|$$

$$\Big| E\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) -$$

$$- E^c\big(j_{t_{r_1}}(b_{\pi(r_1)}) \cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big) \Big|$$

tends to zero as $T \to \infty$, *where* E^c *has the same meaning as in* (6.72).

PROOF OF THEOREMS (1.4) AND (1.5). Applying Lemmata (5.2), (5.4) to

$$(6.77) \qquad \frac{1}{N^n} \sum_{t \in \Omega_d(\pi,\{p_h,q_h\}_{h=1}^m)} \prod_{h=1}^m \varepsilon(t_{\pi^{-1}(p_h)}, t_{\pi^{-1}(q_h)}, b_{p_h}, b_{q_h})$$

$$E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)$$

and

$$\frac{1}{T^n}\int_{\Omega_c(\pi,\{p_h,q_h\}_{h=1}^m)} dt_1\cdots dt_{2n}\prod_{h=1}^m\big|\varepsilon(t_{\pi^{-1}(p_h)},t_{\pi^{-1}(q_h)},b_{p_h},b_{q_h})\big| \tag{6.78}$$

$$E^c\big(j_{t_{r_1}}(b_{\pi(r_1)})\cdots j_{t_{r_{2n-2m}}}(b_{\pi(r_{2n-2m})})\big)$$

respectively, one has the following

THEOREM (6.15). *In the discrete case, suppose that for each $b,b'\in B$,*
(i) $\sigma(\cdot,\cdot,b,b')=\sigma(b,b')$,
(ii) $E(\cdot,\cdot,b,b')$, $\varepsilon(\cdot,\cdot,b,b')$ *are in $S-L^1(C,dn)$ in the sense of Definition* (4.1),
(iii) *the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^N E(j_h(b\cdot b')) := C(bb') \tag{6.79}$$

exists.

Then for each $n\in\mathbf{N}$ and $b_1,\ldots,b_{2n}\in B$, the central limit (6.18) *exists and if we denote*

$$f(b,b') := \sum_{k=h+1}^{\infty}\varepsilon(k,h,b,b') \tag{6.80}$$

and

$$F(b,b') := \sum_{k=h+1}^{\infty}E\big(j_h(b)j_k(b')\big) \tag{6.81}$$

then the limit (6.18) *is equal to*

(6.82)

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{m=0}^{p/2\wedge(p-n)}\sum_{1\leqq p_1<\cdots<p_m\leqq 2n}\sideset{}{'}\sum_{(q_1,\ldots,q_m)}\sum_{(S_1,\ldots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p,\{p_h,q_h\}_{h=1}^m)}}$$

$$\sigma(\pi,b_1,\ldots,b_{2n})\cdot\prod_{h=1}^m f(b_{p_h},b_{q_h})C(b_{\pi(S'_1)})\cdots F(b_{\pi(S'_{i_1})},b_{\pi(S'_{i_1+1})})\cdots$$

$$\cdots F(b_{\pi(S'_{i_{p-n-m}})}, b_{\pi(S'_{i_{p-n-m+1}})}) \cdots C(b_{\pi(S'_{p-2m})})$$

where $\{S'_{i_l}, S'_{i_l+1}\}_{l=1}^{p-n-m}$ *are the singletons of* $(S'_1, \dots, S'_{p-2m})$ *and* $\sigma(b_1, \dots, b_{2n})$ *is a product of some* $\sigma(b_k, b_h)$ *with* $1 \leqq k < h \leqq 2n$.

The continuous analogue of Theorem (6.15) is

THEOREM (6.16). *In the continuous case, suppose that for each* $b, b' \in$ $\in B$,

(i) $\sigma(\cdot, \cdot, b, b') = \sigma(b, b')$, *and*

(ii $E(\cdot, \cdot, b, b')$, $\varepsilon(\cdot, \cdot, b, b')$ *are in* $S - L^1(\mathcal{C}, dt)$ *in the sense of Definition* (4.1).

Then for each $n \in \mathbf{N}$ *and* $b_1, \dots, b_{2n} \in B$, *the central limit* (6.19) *exists and moreover if we denote*

$$f(b, b') := \int_{[h,\infty)} ds\varepsilon(s, h, b, b') \tag{6.83}$$

and

$$F(b, b') := \int_{[h,\infty)} dsE\big(j_h(b)j_s(b')\big) \tag{6.84}$$

then limit (6.19) *is equal to*

$$\frac{1}{n!} \sum_{1 \leqq p_1 < \dots < p_m \leqq 2n} \sum_{(q_1, \dots, q_m)}' \sum_{\pi \in \mathcal{S}_{2n}} \sigma(\pi, b_1, \dots, b_{2n}) \cdot \prod_{h=1}^{m} f(b_{p_h}, b_{q_h}) \tag{6.85}$$

$$F(b_{\pi(r_1)}, b_{\pi(r_2)}) \cdots F(b_{\pi(r_{2n-2m-1})}, b_{\pi(r_{2n-2m})})$$

where, $\sigma(b_1, \dots, b_{2n})$ *is a product of some* $\sigma(b_k, b_h)$ *with* $1 \leqq k < h \leqq 2n$.

Now let us consider some special situations in the discrete case.

First of all consider the term with $m = 0$ in (6.82). In this case, we have no ε-factors and (6.82) has the form

$$\frac{1}{n!} \sum_{p=n}^{2n} \sum_{(S_1, \dots, S_p) \in \mathcal{P}^0_{2n,p}} \sum_{\pi \in \mathcal{S}_{2n}^{(S,p)}} \tag{6.86}$$

$$\sigma(\pi, b_1, \dots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots F(b_{\pi(S_{i_1})}, b_{\pi(S_{i_1+1})}) \cdots$$

$$\cdots F(b_{\pi(S_{i_{p-n}})}, b_{\pi(S_{i_{p-n}+1})}) \cdots C(b_{\pi(S_p)})$$

where

(6.87) $\mathcal{P}^0_{2n,p} := \{(S_1, \dots, S_p) \in \mathcal{P}_{2n,p} :$ there exist $2p - 2n$ singletons and

between two nonsingletons there exists an even number of singletons$\}$.

We write (6.86) as a sum of two terms:

$$(6.86\text{a})\quad \frac{1}{n!} \sum_{(S_1,\dots,S_n)\in\mathcal{P}^0_{2n,n}} \sum_{\pi\in\mathcal{S}^{(S,n)}_{2n}} \sigma(\pi, b_1, \dots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots C(b_{\pi(S_n)}) +$$

$$+ \frac{1}{n!} \sum_{p=n+1}^{2n} \sum_{(S_1,\dots,S_p)\in\mathcal{P}^0_{2n,p}} \sum_{\pi\in\mathcal{S}^{(S,p)}_{2n}}$$

$$\sigma(\pi, b_1, \dots, b_{2n}) \cdot C(b_{\pi(S_1)}) \cdots F(b_{\pi(S_{i_1})}, b_{\pi(S_{i_1}+1)}) \cdots$$

$$\cdots F(b_{\pi(S_{i_{p-n}})}, b_{\pi(S_{i_{p-n}}+1)}) \cdots C(b_{\pi(S_p)}).$$

Notice that in the first term of (6.86a), all partitions are pair partitions (without singletons), therefore, each $\pi \in \mathcal{S}^{(S,n)}_{2n}$ exchanges the pairs, i.e. it is equivalent to an n-permutation. More precisely, for each fixed $(S_1, \dots, S_n) \in \mathcal{P}^0_{2n,n}$ with $S_h = \{l_h, k_h\}$ and $l_h < k_h$ for any $h = 1, \dots, n$ and for each $\pi \in \mathcal{S}^{(S,n)}_{2n}$, define a transformation σ by

$$(6.88)\qquad l_{\sigma(h)} := \pi(l_h), \quad k_{\sigma(h)} := \pi(k_h), \quad h = 1, \dots, n.$$

It is easy to check that the first term of (6.86a) is equal to

$$(6.89)\qquad \frac{1}{n!} \sum_{(S_1,\dots,S_n)\in\mathcal{P}^0_{2n,n}} \sum_{\pi\in\mathcal{S}_n} \sigma(\pi, b_1, \dots, b_{2n}) \cdot C(b_{S_{\pi(1)}}) \cdots C(b_{S_{\pi(n)}});$$

this gives the result when E is a product map and the ε factor is zero.

On the other hand, if we introduce the notation

$$(6.90)\qquad C(\pi, S, n, b_i, b_j) := C(b_{\pi(i)} b_{\pi(j)})$$

then we can rewrite (6.89) as

$$(6.91)\quad \frac{1}{n!} \sum_{p.p.} \sum_{\pi\in\mathcal{S}_n} \sigma(\pi, b_1, \dots, b_{2n}) \cdot C(\pi, S, n, b_{l_1}, b_{k_1}) \cdots C(\pi, S, n, b_{l_n}, b_{k_n})$$

where $\sum_{p.p.}$ defined in [1] is the sum over all the pair partitions of $\{1,\dots,2n\}$, i.e. all the $\{l_h,k_h\}_{h=1}^{m}=\{1,\dots,2n\}$ such that

(6.91a) $$l_h<k_h,\quad h=1,\dots,n$$

and

(6.91b) $$l_1<\dots<l_n.$$

More generally, considering also the second term in (6.86a), if we define

(6.92)
$$C(\pi,S,p,b_i,b_j):=\begin{cases}C(b_{\pi(i)}b_{\pi(j)})\,, & \text{if } i,j \text{ are in the same } S_h\\ F(b_{\pi(i)},b_{\pi(j)})\,, & \text{if } i,j \text{ are not in the same } S_h,\end{cases}$$

and notice that in (6.86a), the $\{S_{i_h},S_{i_h+1}\}_{h=1}^{p-n}$ are singletons, then (6.86a) is equal to

(6.93) $$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}\sigma(\pi,b_1,\dots,b_{2n})\cdot C(\sigma,S,p,b_{l_1},b_{k_1})\cdots$$
$$\cdots C(\sigma,S,p,b_{l_n},b_{k_n})$$

where

(6.94) $$\{l_1,k_1\}=S_1,\dots,l_{i_1}=S_{i_1},k_{i_1}=S_{i_1+1},\dots,l_{i_{p-n}}=$$
$$=S_{i_{p-n}},k_{i_{p-n}}=S_{i_{p-n}+1},\dots,\{l_n,k_n\}=S_p$$

In the general case, i.e. when m needs not to be zero, we have

THEOREM (6.17). *Define $C(\pi,S,p,b_i,b_j)$ to be equal to $C(b_{\pi(i)}b_{\pi(j)})$ if i, j are in the same block; equal to $F(b_{\pi(i)},b_{\pi(j)})$ if i, j are in two different blocks and π leaves i and j fixed; equal to $F(b_{\pi(i)},b_{\pi(j)})+f(b_i,b_j)$ if i, j are in two different blocks and π permutes i and j. Then* (6.82) *is equal to*

(6.95) $$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}$$
$$\sigma(\pi,b_1,\dots,b_{2n})\cdot C(\pi,S,p,b_{l_1},b_{k_1})\cdots C(\pi,S,p,b_{l_n},b_{k_n}).$$

PROOF. Since in the product

$$C(\sigma,S,p,b_{l_1},b_{k_1})\cdots C(\sigma,S,p,b_{l_n},b_{k_n}) \tag{6.96}$$

some terms have the form $F+f$, we can expand the product of sums into sums of products. Suppose that in the expansion there are m factors f, then by the definition of $C(\pi,S,p,b_i,b_j)$ we know that m can be equal to $0,1,\dots,p-n$. Moreover, that definition shows that each factor f corresponds to two singletons, therefore, $m \leqq p/2$.

For each fixed $m=0,1,\dots,p/2\wedge(p-n)$, we label the f factors by the indices $\{p_h,q_h\}_{h=1}^m$. Then $\{p_h,q_h\}_{h=1}^m$ are singletons and it follows that we cannot choose all partitions in $\mathcal{P}_{2n,p}$ but only in $\mathcal{P}^0_{2n,p}$; moreover $\pi\in\mathcal{S}_{2n}^{(S,p)}$ should satisfy $\pi^{-1}(p_h)>\pi^{-1}(q_h)$ for all $h=1,\dots,m$. This ends the proof.

Similarly, in continuous case, we have

THEOREM (6.18). *Defining*

$$C(\pi,b_i,b_j):=\begin{cases} F(b_{\pi(i)},b_{\pi(j)})\ , & \text{if } \pi:\ \binom{\cdots i\cdots j\cdots}{\cdots i\cdots j\cdots} \\ F(b_{\pi(i)},b_{\pi(j)})+f(b_i,b_j)\ , & \text{if } \pi:\ \binom{\cdots j\cdots i\cdots}{\cdots i\cdots j\cdots}, \end{cases} \tag{6.97}$$

(6.85) *is equal to*

$$\frac{1}{n!}\sum_{\pi\in\mathcal{S}_{2n}^{(S,p)}}\sigma(b_1,\dots,b_{2n})\cdot C(\pi,b_{l_1},b_{k_1})\cdots C(\pi,b_{l_n},b_{k_n}). \tag{6.98}$$

From Theorem (6.17) (resp. Theorem (6.18)) one can finish the proof of Theorem (1.4) (resp. Theorem (1.5)). In fact, by expanding C_0 to $C+G$, the expression (1.18) is equal to

$$\frac{1}{n!}\sum_{p.p.}\sigma(i_1,j_1,\dots,i_n,j_n)\sum_{l=0}\sum_{1\leqq r_1<\dots<r_l\leqq n}C(b_{i_{\pi(1)}}b_{j_{\pi(1)}})\cdots \tag{6.99}$$

$$\cdots G(b_{i_{\pi(r_1)}}b_{j_{\pi(r_1)}})\cdots G(b_{i_{\pi(r_l)}}b_{j_{\pi(r_l)}})\cdots C(b_{i_{\pi(n)}}b_{j_{\pi(n)}}).$$

Now regard the pair partition

$$\{i_1,j_1,\dots,i_{r_1},j_{r_1},\dots,i_{r_l},j_{r_l},\dots,i_n,j_n\}$$

as partition $(S_1,\dots,S_{n+l})$ with singletons $i_{r_1},j_{r_1},\dots,i_{r_l},j_{r_l}$; and put $n+l=p$. Then (6.99) becomes

$$\frac{1}{n!}\sum_{p=n}^{2n}\sum_{(S_1,\dots,S_p)\in\mathcal{P}^0_{2n,p}}\sum_{\pi\in\mathcal{S}_n}C(b_{i_{\pi(1)}}b_{j_{\pi(1)}})\cdots G(b_{i_{\pi(r_1)}}b_{j_{\pi(r_1)}})\cdots \tag{6.100}$$

$$\cdots G(b_{i_{\pi(r_l)}} b_{j_{\pi(r_l)}}) \cdots C(b_{i_{\pi(n)}} b_{j_{\pi(n)}}).$$

Notice that $G(b_{i_{\pi(r_d)}} b_{j_{\pi(r_d)}})$, $d = 1, \ldots, l$ is a sum of two terms which correspond to an exchange of two singletons. Since the exchanges act on different pairs of singletons, they commute each other. Therefore, combining the exchanges and $\pi \in \mathcal{S}_n$ to a new permutation π', then $\pi' \in \mathcal{S}_{2n}^{(S,p)}$ and which exchanges pairs (non-singleton block or two singletons) and keeps the order of elements *in* each non-singleton block; acts as a 2-exchange on two singletons. This shows that (6.100) is equal to (6.95).

The arguments in the continuous case are similar.

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ON THE NUMBER OF CIRCLES DETERMINED BY n POINTS IN THE EUCLIDEAN PLANE

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1. Let $P = \{p_1, p_2, \dots, p_n\}$ be a set of n points in the Euclidean plane and S be the set of connecting lines, which join these points. The straight line $s \in S$ is called straight line of order i when $|s \cap P| = i$. For a point-set P we denote $S_i(P)$ the set of straight lines of order i and $t_i = |S_i(P)|$. Then

$$t = \sum_{i=2}^{n} t_i \tag{1.1}$$

where $t = |S|$, is true. The connecting line $s \in S$ is called ordinary when $|s \cap P| = 2$.

For a given pointset P let C be a set of circles determined by points of P. The circle $c \in C$ is called circle of order i when $|c \cap P| = i$. The set of circles of order i is denoted by $C_i(P)$ and $k_i = |C_i(P)|$. Then

$$k = \sum_{i=3}^{n} k_i \tag{1.2}$$

where $k = |C|$, is true. A circle $c \in C$ ordinary when $|c \cap P| = 3$.

2. Now we formulate our main result in following two theorems.

THEOREM 2.1. *Let P be a set of $n \geqq 4$ points in the Euclidean plane, not all on a circle or a straight line. Let p_j be an arbitrary point of a set P. Then P determines at least* $15(n-1)/133$ *circles containing exactly three points of P, one of which is p_j.*

Elliott [6] has proved $|C_3(P, p_j)| \geqq 2(n-1)/21$, where $C_i(P, p_j)$ is the set of the circles of order i determined by P and passing through p_j.

THEOREM 2.2. *Let P satisfy the hypotheses of Theorem* 2.1. *Then* $k_3 \geqq 5n(n-1)/133$.

From the above mentioned theorems further results follow. For the upper bound of the number of circles of order m Jucovič [10] has proved the estimate

$$k_m \leqq \frac{1}{\binom{m}{3}} \left\{ \binom{n}{3} - \left\lfloor \frac{n+2}{3} \right\rfloor \right\}. \tag{2.1}$$

There holds the stronger

THEOREM 2.3. *Let P be a set of $n \geqq 4$ points in the Euclidean plane. For every integer $m \geqq 4$*

$$k_m \leqq \frac{1}{\binom{m}{3}} \left\{ \binom{n}{3} - \frac{5n(n-1)}{133} \right\} \tag{2.2}$$

holds.

For the number k of circles determined by n points Elliott [6] has shown the best possible lower bound

$$k \geqq \frac{1}{2}(n-1)(n-3)+1 \quad \text{for} \quad n \geqq 394. \tag{2.3}$$

For $6 \leqq n \leqq 394$ the weaker but quadratic bound gives the following

THEOREM 2.4. *Let P be a set of $n \geqq 6$ points in the Euclidean plane, not all on a circle or a straight line. Then*

$$k \geqq \left(15n(n-1)+1678\right)/266. \tag{2.4}$$

3. For the proof of Theorems 2.1–2.4 some results of Kelly and Moser [13] are needed. It is known that a set of two or more lines in the plane which do not form a pencil realize a subdivision of the plane into two or more regions. The lines of S not passing through the point $p \in P$ dissect the plane into polygonal regions — except in the case to be noted below. If $n-1$ points of P distinct from p are on a line, no division is realized. If exactly $n-1$ of the points, including p, are on a line, then the division is into $n-2$ angular regions, that is, regions bounded by two lines. In all other cases the division is into polygonal regions, bounded by at least three edges. The point p is in the interior of one of these regions, which is called the residence of p. The lines of S containing the edges of the residence are neighbours of p. The number of ordinary lines passing through p is the order of p. The number of neighbours of p which are ordinary lines is the rank of p. The order plus the rank is the index. Kelly and Moser [13] has proved, among others, the following lemmas.

LEMMA 3.1. *The index of each point of P which is not of order two is at least three.*

LEMMA 3.2. *If I_i is the index of the point p_i, then*

$$t_2 \geqq \frac{1}{6}\sum_{i=1}^{n} I_i. \tag{3.1}$$

The main result of Kelly and Moser [13] is the following.

THEOREM 3.3. *Let P be a set of n non-collinear points in the Euclidean plane. For the number of ordinary connecting lines*

$$t_2 \geqq 3n/7 \tag{3.2}$$

is true.

4. PROOF OF THEOREM 2.1. If $n = 4$, then it is sufficient to prove the passing at least one ordinary circle through every point. There are two possibilities.

α) Three points (e.g. p_1, p_2, p_3) lie on a straight line. Then the circles (p_1, p_2, p_4); (p_1, p_3, p_4) and (p_2, p_3, p_4) are ordinary.

β) No three points of p_1, p_2, p_3, p_4 lie on a straight line. Then all four circles determined by them are ordinary.

Let now $n = 5$. Again it is sufficient to show that at least one ordinary circle passes through each point (e.g. p_5). In the following we use $\overline{M}$ to denote the inverse of a set M. Inverting the system with respect to p_5 we distinguish again two cases.

α) If no three points of $\overline{p}_1, \overline{p}_2, \overline{p}_3, \overline{p}_4$ are collinear, then from six lines $\overline{p}_i\overline{p}_j$, $i \neq j$, $i, j \in \{1, 2, 3, 4\}$ at most two are passing through p_5. Those which do not pass through p_5 after inversion, transform into ordinary circles passing through p_5.

β) Now suppose that three points $\overline{p}_1, \overline{p}_2, \overline{p}_3$ lie on a straight line s. The point $\overline{p}_4$ cannot lie on the same line s, because then all points $p_1, p_2, \ldots, p_5$ would lie on a circle $\overline{c}$. Thus at least one of the straight lines $\overline{p}_4\overline{p}_1, \overline{p}_4\overline{p}_2$ is ordinary and does not pass through p_5, and after inversion they transform into ordinary circles passing through p_5.

Let now $n \geqq 6$. Without loss of generality we can take $p_j = p_n$. Let Ω be a circle of inversion with centre p_n. Let m connecting lines of $S(\overline{P})$ and passing through the point p_n have order two. Two cases are possible.

α) $m = 2$. Now the number of ordinary lines of $S(\overline{P})$ which do not pass through p_n is at least $3n/7 - 2$ (see Theorem 3.3). All of them are transformed by inversion with respect Ω into ordinary circles passing through p_n. Thus

$$\left|C_3(P, p_n)\right| \geqq 3n/7 - 2 > 15(n-1)/133. \tag{4.1}$$

β) $m \neq 2$. Now in virtue of Lemma 3.1 for $S(\overline{P})$ the index $I_{p_n} \geqq 3$. We distinguish two cases.

Case 1: $m \leqq 7(n-1)/19$. We denote by r the number of those points of $\overline{P} - p_n$ the order of which is two. Thus the number of those points of $\overline{P}$ which have order distinct from two is $n - r$. Trivially

$$\bar{t}_2 = \left|S_2(\overline{P})\right| \geqq r \tag{4.2}$$

is true. From Lemma 3.2 and Lemma 3.1 for $S(\overline{P})$ we get

$$6\bar{t}_2 \geqq \sum_{i=1}^{n} I_i \geqq 3(n-r-1)+2r+m = 3(n-1)-r+m.$$

This and (4.2) imply

$$\bar{t}_2 \geqq \big(3(n-1)+m\big)/7, \tag{4.3}$$

i.e. the number of those lines from $S(\overline{P})$ which have order two and which do not pass through p_n is at least

$$\bar{t}_2 - m \geqq \frac{3(n-1)+m}{7} - m = \frac{3(n-1-2m)}{7} \geqq$$

$$\geqq \frac{3\left(n-1-\frac{14(n-1)}{19}\right)}{7} = \frac{15(n-1)}{133},$$

and all of them are transformed by the inversion considered into ordinary circles passing through p_n.

Case 2: $m > 7(n-1)/19$. Now as a consequence of Theorem 3.3 we get

$$\left|S_2(\overline{P}-p_n)\right| \geqq 3(n-1)/7. \tag{4.4}$$

In the point-set $\{\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_{n-1}\}$ there are exactly m points, e.g. $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m$ such that the lines $\overline{p}_i p_n$ for $i = 1, 2, \ldots, m$ are ordinary. We can distribute the other $n-1-m$ points into at most $(n-1-m)/2$ pairs such that the lines $\overline{p}_j \overline{p}_s$, $j \neq s$; $j, s \in \{m+1, m+2, \ldots, n-1\}$ have order two and they pass through p_n.

Thus the number of ordinary lines passing through p_n is at most $(n-1-m)/2 \leqq (n-1-7(n-1)/19)/2$. This and (4.4) imply that the number of ordinary lines which do not pass through p_n is at least

$$\frac{3(n-1)}{7} - \frac{n-1-\frac{7(n-1)}{19}}{2} = \frac{15(n-1)}{133}$$

and all of them are transformed by the inversion considered into ordinary circles passing through p_n.

The proof of Theorem 2.1 is complete.

5. Proof of Theorem 2.2. Cases $n=4$, $n=5$ are trivial. Let $n \geqq 6$. Because of Theorem 2.1, at least $15(n-1)/133$ ordinary circles pass through each point of P. Any ordinary circle is in the number $15n(n-1)/133$ included at most three times. Hence $k_3 \geqq 5n(n-1)/133$ and Theorem 2.1 is proved.

PROOF OF THEOREM 2.3. On each circle of order i there are $\binom{i}{3}$ distinct triples of points. For the total number $\binom{n}{3}$ triples of points, which can be made by n points in the plane

$$\binom{n}{3} = \sum_{i=3}^{n} \binom{i}{3} k_i \tag{5.1}$$

is true. This and Theorem 2.2 imply

$$\binom{n}{3} \geqq k_3 + \sum_{i=4}^{n} \binom{i}{3} k_i \geqq \frac{5n(n-1)}{133} + \sum_{i=4}^{n} \binom{i}{3} k_i \geqq \frac{5n(n-1)}{133} + \binom{m}{3} k_m.$$

From this we get the upper bound (2.2) requested. Theorem 2.3 is proved.

6. For the proof of Theorem 2.4 we shall need the following.

LEMMA 6.1 (see Jucovič [10]). *Let $p_1, p_2, \dots, p_6$ be six points in the plane, not all lying on a line or a circle. Then these points determine at least 8 circles.*

PROOF OF THEOREM 2.4. Suppose that $m < n$ is the maximal number of those points $P = \{p_1, p_2, \dots, p_n\}$ which lie on the same line or circle. We denote these points by $p_1, \dots, p_m$. Theorem 2.1 implies that at least $15(n-1)/133$ ordinary circles pass through p_n. If we omit the point p_n from P, then at least $15(n-1)/133$ circles cancel from $C(P)$. The point-set $\{P - p_n\}$ containing $n-1$ points result from that. Again Theorem 2.1 implies that at least $15(n-2)/133$ ordinary circles pass through p_{n-1}. If we omit the point p_{n-1} from $P - p_n$, then from $C(P - p_n)$ at least $15(n-2)/133$ circles cancel. Repeating this procedure and adding the ordinary circles we get

$$\begin{aligned} k = |C(P)| &\geqq \frac{15(n-1)}{133} + |C(P - p_n)| \geqq \\ &\geqq \frac{15}{133}\{(n-1) + (n-2)\} + |C(P - p_n - p_{n-1})| \geqq \dots \\ &\dots \geqq \frac{15}{133}\{(n-1) + (n-2) + \dots + (m+1)\} + \\ &+ |C(P - p_n - p_{n-1} - \dots - p_{m+2})|. \end{aligned} \tag{6.1}$$

Now $m+1$ points $p_1, p_2, \dots, p_{m+1}$ remain. If from these just m lie on the same line, then they determine $\binom{m}{2}$ circles, i.e.

$$k \geqq \frac{15}{133}\{(n-1) + (n-2) + \dots + (m+1)\} + \binom{m}{2} = \tag{6.2}$$

$$= \frac{15n^2 - 15n + 118m^2 - 148m}{266}.$$

If from the points $p_1, p_2, \ldots, p_{m+1}$ just m lie on the same circle, then at least $\frac{m(m-2)}{2} + 1$ circles are determined by them, i.e.

$$(6.3) \qquad k \geqq \frac{15}{133}\{(n-1) + (n-2) + \ldots + (m+1)\} + \frac{m(m-2)}{2} + 1 =$$

$$= \frac{15n^2 - 15n + 118m^2 - 281m + 266}{266}.$$

There are two cases.

1. If $m \geqq 6$, then the inequalities (6.2) and (6.3) imply

$$(6.4) \quad k \geqq \frac{1}{266}\{15n^2 - 15n + 2m(59m - 74)\} \geqq \frac{1}{266}(15n^2 - 15n + 1678)$$

and

$$(6.5) \quad k \geqq \frac{1}{266}\{15n^2 - 15n + 4248 - 1686 + 266\} > \frac{1}{266}(15n^2 - 15n + 1678)$$

respectively.

2. If $m < 6$, then no six points lie on the same line or circle. We shall now do the successive omitting of points up to the point p_7 and we obtain

$$(6.6) \quad k \geqq \frac{15}{133}\{(n-1) + (n-2) + \ldots + 6\} + \left|C(P - p_n - p_{n-1} - \ldots - p_7)\right|.$$

From this and Lemma 6.1 we get

$$(6.7) \qquad k \geqq \frac{15}{133} \cdot \frac{(n-1+6)(n-6)}{2} + 8 = \frac{15n^2 - 15n + 1678}{266}.$$

Theorem 2.4 is proved.

REMARK. As dr. Pach from Budapest informed us, Csima and Sawyer [17] have proved $t_2 \geqq 6n/13$. Using this result one can obtain by analogous arguments $\left|C_3(P, p_j)\right| \geqq 33(n-1)/247$ in Theorem 2.1 and all corollaries, too.

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COMPLETELY CONTRACTIVE HILBERT MODULES AND PARROTT'S EXAMPLE

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1. Introduction

In two earlier papers [9, 10] the present author together with Sastry studied certain finite dimensional Hilbert modules $\mathbf{C}_{\mathbf{N}}^{n+1}$ over the function algebra $A(\Omega)$ for Ω a domain in $\mathbf{C}^m$. This paper is a continuation of that work and provides partial answers to some of the questions raised in [10] for the poly disk algebra. While most of the terminology and notations are from the two papers [9, 10] and will be used without any further apology, we point out in Remark 3.8 that the contractive module $\mathbf{C}_{\mathbf{N}}^{2n}$ (respectively completely contractive) gives rise to a matricially normed $2m$-dimensional vector space and a contractive (respectively completely contractive) linear map on it and conversely.

In the two papers cited above the main result showed that a contractive module $\mathbf{C}^{n+1}$ over the ball algebra $\mathcal{A}(\mathbf{B}^m)$ is completely bounded by $\sqrt{m}$ and examples were given to show that the bound is attained. This, in particular shows that for $m \geqq 2$, contractive modules are not necessarily completely contractive over the ball algebra. However, for the poly disk algebra $\mathcal{A}(\mathbf{D}^m)$, we know via Ando's theorem [2] that every contractive module over $\mathcal{A}\left(\mathbf{D}^2\right)$ is completely contractive while Parrott [11] provides an example of a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ which is not completely contractive. As Paulsen [12] points out, it would be good to know the difference in the internal structure of $\mathcal{A}\left(\mathbf{D}^2\right)$ and $\mathcal{A}\left(\mathbf{D}^3\right)$ that leads to this situation, see 4.4 for a partial answer. Our approach is to actually work out Parrot's example using the notion of complete contractivity rather than dilation, these notions are of course equivalent [cf. 12]. The methods of [9, 10], seem to work well in the context of the ball algebra but the actual computations over the poly disk algebra seem to be very messy. In fact, we are not able to produce an example of a contractive module over $\mathcal{A}\left(\mathbf{D}^3\right)$ which is not completely contractive within the class of the very simple modules considering in [9, 10], however see Remark 4.8. Therefore, we are forced to consider slightly more general module action than those of

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[9, 10]. This necessitates generalising many of the previous results to this new setup. Whenever the proof of a natural generalisation of results from [9, 10] is routine, we omit it. In this more general setup, apart from being able to show that there exists a contractive module over $\mathcal{A}(\mathbf{D}^3)$ which is not even 2-contractive (Theorem 4.7), we show that such phenomenon occurs in dual pairs, that is there also exists a contractive module over $\mathcal{A}(l^1(3)_1)$ which is not completely contractive (Theorem 4.1). Another interesting fact is: In the example of a contractive module over $\mathcal{A}(\mathbf{D}^3)$ in Section 4.6, we use only linear maps in $\mathcal{A}(\mathbf{D}^3) \otimes \mathcal{M}_2$ to detect that it fails to be completely contractive. Lastly following suggestions of Vern Paulsen, we show in Section 4.3, how our methods can be used to answer a question of Loebl [8].

To keep this work as self-contained as possible we have given more details than would seem necessary. In the rest of this section we give basic definitions. In Section 2, we show that most of the results in [10] can be modified to fit into the present context. The main new ideas are contained in Sections 3 and 4. The following definitions and terminology can be found in many places (cf. [5, 9]).

1.1. DEFINITION. A Hilbert module $\mathcal{H}$ over a (not necessarily complete) complex algebra $\mathcal{A}$ consists of a complex Hilbert space $\mathcal{H}$ together with a continuous map $(a, f) \to a \cdot f$ from $\mathcal{A} \times \mathcal{H}$ to $\mathcal{H}$ satisfying the following conditions:

For a, b in $\mathcal{A}$, h, h_i in $\mathcal{H}$ and α, β in $\mathbf{C}$

(i) $1 \cdot h = h$,
(ii) $a \cdot (b \cdot h) = (a \cdot b) \cdot h$,
(iii) $(a + b) \cdot h = a \cdot h + b \cdot h$,
(iv) $a \cdot (\alpha h_1 + \beta h_2) = \alpha (a \cdot h_1) + \beta (a \cdot h_2)$.

The Hilbert module is *bounded* if there exists a constant K such that

$$\|a \cdot h\|_{\mathcal{H}} \leqq K \|a\|_{\mathcal{A}} \|h\|_{\mathcal{H}}$$

for all a in $\mathcal{A}$ and h in $\mathcal{H}$, and is *contractive* if $K \leqq 1$.

1.2. For any region Ω in $\mathbf{C}^m$, let $\mathcal{A}(\Omega)$ denote the closure of the polynomial algebra $\mathcal{P}(\Omega)$ with respect to the supremum norm on $\overline{\Omega}$ the closure of the region Ω. Throughout this paper we will assume that

(i) Ω is a bounded open neighbourhood of the origin in $\mathbf{C}^m$, and
(ii) Ω is convex and balanced.

We note that (i) and (ii) imply that Ω is polynomially convex [6, p.67] and so, by Oka's theorem [6, p.84], $\mathcal{A}(\Omega)$ contains all functions that are holomorphic in a neighbourhood of $\overline{\Omega}$.

The Hilbert $\mathcal{P}(\Omega)$ module structure on the Hilbert space $\mathcal{H}$ determines and is completely determined by a commuting m-tuple $\mathbf{T} = (T_1, \dots, T_m)$ of bounded operators on $\mathcal{H}$ defined by

$$T_i h = z_i h$$

for h in $\mathcal{H}$ and $1 \leqq i \leqq m$. If $\mathcal{H}$ is a bounded (respectively contractive) $\mathcal{P}(\Omega)$ module then the module map extends to $\mathcal{A}(\Omega)$ and we write $\mathcal{H}_{\mathbf{T}}$ for this bounded Hilbert $\mathcal{A}(\Omega)$-module. As explained in [9, Section 1.2] the notion of $\mathbf{T}$ admitting $\overline{\Omega}$ as a k-spectral set is equivalent to $\mathcal{H}_{\mathbf{T}}$ being bounded.

1.3. For any function algebra $\mathcal{A}$ and an integer $k \geqq 1$, let $\mathcal{M}_k(\mathcal{A}) \cong \mathcal{A} \otimes \otimes \mathcal{M}_k(\mathbf{C})$ denote the algebra of $k \times k$ matrices with entries from $\mathcal{A}$. Here for $F = (f_{ij})$ in $\mathcal{M}_k(\mathcal{A})$, the norm $\|F\|$ of F is defined by

$$\|F\| = \sup\{\|(f_{ij}(z))\| : z \in \mathbf{M}\},$$

where $\mathbf{M}$ is the maximal ideal space for $\mathcal{A}$. We note that for $\mathcal{A} = \mathcal{A}(\Omega)$, the maximal ideal space can be identified with Ω [6, Theorem 1.2, p.67] and thus

$$\|F\| = \sup\{\|(f_{ij}(z))\| : z \in \Omega\}.$$

1.4. DEFINITION. If $\mathcal{H}$ is a bounded Hilbert $\mathcal{A}$-module, then $\mathcal{H} \otimes \mathbf{C}^k$ is a bounded $\mathcal{M}_k(\mathcal{A})$-module. For each k let n_k denote the smallest bound for $\mathcal{H} \otimes \mathbf{C}_k$. Then the Hilbert $\mathcal{A}$-module is *completely bounded if*

$$n_\infty = \lim_{k \to \infty} n_k < \infty$$

and is *completely contractive* if $n_\infty \leqq 1$.

1.5. In the following, $l^p(n)$ stands for the vector space $\mathbf{C}^n$ with the usual l^p-norm and $(X)_1$ will denote the open unit ball of the Banach space X. For T a linear operator on $l^2(n)$ and ω any complex number; define the operator $N(T, \omega)$ on $l^2(n) \oplus l^2(n) \cong l^2(2n)$ by

$$N(T, \omega) = \begin{bmatrix} \omega I_n & T \\ \mathbf{0} & \omega I_n \end{bmatrix}.$$

Now, for $\omega = (\omega_1, \dots, \omega_m)$ in Ω, consider the pairwise commuting m-tuple of operators

$$\mathbf{N}(\mathbf{T}, \omega) = \left(N(T_1, \omega_1), \dots, N(T_m, \omega_m)\right).$$

The central object of study is the Hilbert $\mathcal{A}(\Omega)$-module $l^2(2n)_{\mathbf{N}}$ and to determine when it is contractive (respectively completely contractive). We write $\mathbf{N}$ for $\mathbf{N}(\mathbf{T}, \omega)$ when the meaning is clear from the context

2. The functional calculus

In this section we establish that the evaluation map $p \to p(\mathbf{N})$ on $\mathcal{P}(\Omega)$ extends continuously to $H(\omega)$, the algebra of germs of holomorphic functions at ω. This fact will be necessary in proving Lemma 3.3 in the next section.

2.1. LEMMA. *For S, T in $\mathcal{L}\left(l^2(n)\right)$ and λ, μ in* $\mathbf{C}$

(i) $N(S,\lambda)N(T,\mu) = N(\lambda T + \mu S, \lambda\mu)$,

(ii) $\|N(\lambda,\mu)\|^2 = \frac{1}{2}\left\{|\lambda|^2 + 2|\mu|^2 + |\lambda|\sqrt{|\lambda|^2 + 4|\mu|^2}\right\}$,

(iii) $\|N(S,\lambda)\| = \|N(\|S\|, |\lambda|)\|$.

PROOF. (i) and (ii) are straightforward. To prove (iii), note that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det\left(A - BD^{-1}C\right)$$

and

$$N(S,\lambda)N(S,\lambda)^* = |\lambda|^2 I_{2n} + \begin{bmatrix} SS^* & \overline{\lambda}S \\ \lambda S^* & \mathbf{0} \end{bmatrix}.$$

For $x \in \mathbf{C}$, we have

$$\det \begin{bmatrix} SS^* - xI_n & \overline{\lambda}S \\ \lambda S^* & -xI_n \end{bmatrix} = (-x)^n \det\left(SS^* - xI_n + x^{-1}|\lambda|^2 SS^*\right) =$$

$$= (-x)^n \det\left(SS^*\left(1 + |\lambda|^2 x^{-1}\right) - xI_n\right) =$$

$$= (-x)^n \left(1 + |\lambda|^2 x^{-1}\right)^n \det\left(SS^* - \frac{x}{1 + |\lambda|^2 x^{-1}} I_n\right).$$

Thus, the maximum eigenvalue of $\begin{bmatrix} SS^* & \overline{\lambda}S \\ \lambda S^* & \mathbf{0} \end{bmatrix}$ is

$$\frac{\|S\|}{2}\left(\|S\| + \sqrt{\|S\|^2 + 4|\lambda|^2}\right).$$

By the spectral mapping theorem, the maximum eigenvalue of $N(S,\lambda)N(S,\lambda)^*$ is

$$\frac{1}{2}\left\{2|\lambda|^2 + \|S\|^2 + \|S\|\sqrt{\|S\|^2 + 4|\lambda|^2}\right\}.$$

Using (ii) to compute the norm of $N(S,\lambda)$ we verify that (iii) is correct.

2.2. LEMMA. *Let $\mathcal{A}$ be a complex algebra, $\Theta : \mathcal{A} \to \mathbf{C}$ be a continuous algebra homomorphism and $\varphi : \mathcal{A} \to \mathcal{L}\left(l^2(n)\right)$ be a continuous linear map such that*

$$\varphi(ab) = \Theta(a)\varphi(b) + \Theta(b)\varphi(a).$$

Then the map $a \to N(\varphi(a), \Theta(a))$ is a continuous algebra homomorphism from $\mathcal{A}$ to $\mathcal{L}\left(l^2(n)\right)$.

PROOF. The continuity follows from (ii) and (iii) of the previous Lemma.

As in [9] Lemma 2.1 (iii) and Lemma 2.2 yield the following proposition.

2.3. PROPOSITION. *For f in $H(\omega)$ let $\nabla f(\omega)$ be the vector $(a_1, \dots, a_m)$. Then the map $\rho : f \to \mathbf{N}\left(a_1 T_1 + \ldots + a_m T_m, f(\omega)\right)$ is a continuous algebra homomorphism from $H(\omega)$ to $\mathcal{L}\left(l^2(n)\right)$ coinciding with the evaluation map $p \to p(\mathbf{N}(\mathbf{T}, \omega))$ on $\mathcal{P}(\Omega)$.*

2.4. Since the map ρ extends the evaluation map on $\mathcal{P}(\Omega)$ it follows that $\rho \otimes I_k$ is also a continuous algebra homomorphism of $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ coinciding with the evaluation map on $\mathcal{P}(\Omega) \otimes \mathcal{M}_k$.

Let X, Y be finite dimensional normed linear spaces and Ω be an open subset of X. A function $f : \Omega \subseteqq X \to Y$ is said to be holomorphic if the Frechet derivative of f at ω exists as a complex linear map from X to Y. Let $I = (i_1, \dots, i_m)$ denote a multi-index of length $|I| = i_1 + \ldots + i_m$ and let e_k denote the multi-index with a one in the k^{th} position and zero elsewhere. Let $P : \Omega \to \mathcal{M}_k$ be a polynomial matrix valued function, that is, $P(z) = (p_{ij}(z))$, where each p_{ij} is a polynomial function in m-variables. Then we can write

$$P(z) = \sum P_I (z - \omega)^I$$

where each P_I is a scalar $k \times k$ matrix. Now it is easy to verify that the derivative $DP(\omega)$ of P at ω is

$$DP(\omega) = (P_{e_1}, \dots, P_{e_m})$$

which acts on a vector $\mathbf{v} = (v_1, \dots, v_m)$ by

$$DP(\omega) \cdot \mathbf{v} = v_1 P_{e_1} + \ldots + v_m P_{e_m}.$$

However, for notational convenience we always write $DP(\omega)$ for $(P_{e_1}, \dots, P_{e_m})$. We introduce a pairing between two m-tuples of operators $\mathbf{S}$ and $\mathbf{T}$ as follows:

$$\langle \mathbf{S}, \mathbf{T} \rangle = S_1 \otimes T_1 + \ldots + S_m \otimes T_m$$

where the matrix for $A \otimes B$ is just $((a_{ij} b))$. In this notation, we have

$$(\rho \otimes I_k)(F) \sim \begin{bmatrix} I_n \otimes F(\omega) & \langle DF(\omega), \mathbf{N} \rangle \\ \mathbf{0} & I_n \otimes F(\omega) \end{bmatrix},$$

where $\sim$ indicates that the matrix on the right is obtained from the one on the left after elementary row and/or column operations.

3. Characterization of completely contractive modules

The main result in this section says that to determine when $\|\rho \otimes I_k\| \leqq \leqq 1$, it is enough to consider those functions which vanish at a fixed but arbitrary point of Ω. However to prove this we need, as in [10], the following result of Douglas, Muhly and Pearcy [4, Proposition 2.2].

3.1. LEMMA. *For $i = 1, 2$ let T_i be a contraction on a Hilbert space $\mathcal{H}_i$ and let X be an operator mapping $\mathcal{H}_2$ into $\mathcal{H}_i$. A necessary and sufficient condition for the operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by the matrix $\begin{bmatrix} T_1 & X \\ \mathbf{0} & T_2 \end{bmatrix}$ to be a contraction is that there exist a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that*

$$X = (I_{\mathcal{H}_1} - T_1 T_1^*)^{\frac{1}{2}} C (I_{\mathcal{H}_2} - T_2^* T_2)^{\frac{1}{2}}.$$

Again as in [10], we need some results about biholomorphic automorphisms of the unit ball in $\mathcal{M}_k$, which can be found in Harris [7, Theorem 2]. We collect the results we will need in the following.

3.2. LEMMA. *For each B in the unit ball $(\mathcal{M}_k)_1$ of $\mathcal{M}_k$, the Mobius transformation*

$$\varphi_B(A) = (I - BB^*)^{-\frac{1}{2}}(A + B)(I + B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}$$

is a biholomorphic mapping of $(\mathcal{M}_k)_1$ onto itself with $\varphi_B(0) = B$. Moreover,

$$\varphi_B^{-1} = \varphi_{-B}, \quad \varphi_B(A)^* = \varphi_{B^*}(A^*), \quad \|\varphi_B(A)\| \leqq \varphi_{\|B\|}(\|A\|)$$

and

$$D\varphi_B(A)\mathbf{C} = (I - BB^*)^{\frac{1}{2}}(I + AB^*)^{-1} C (I + B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}.$$

Now we are ready to prove the main result of this section. While this lemma is similar to Lemma 3.2 in [9] and the lemma in [10], in the present situation, some extra care is necessary for the proof.

3.3. LEMMA. *If $\|F(\mathbf{N})\| \leqq 1$ for all F in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ with $\|F\|_\infty \leqq 1$ and $F(w) = \mathbf{0}$, then $\|G(\mathbf{N})\| \leqq 1$ for all G in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ with $(\|G)\|_\infty \leqq \leqq 1$.*

PROOF. Any G in $\mathcal{M}_k(\mathrm{Hol}(\overline{\Omega}))$ of norm less than or equal to one, maps Ω into $(\mathcal{M}_k)_1$. In particular, for ω in Ω, $\|G(\omega)\| < 1$ we can form the Mobius transformation $\varphi_{-G(\omega)}$ of $(\mathcal{M}_k)_1$. Consider the map $\varphi_{-G(\omega)} \circ G$, which maps ω onto zero. Thus,

$$1 \geqq \left\| \left(\varphi_{-G(\omega)} \circ G\right)(\mathbf{N}) \right\| = \left\| \begin{bmatrix} \mathbf{0} & \langle D\left(\varphi_{-G(\omega)} \circ G\right)(\omega), \mathbf{T} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|.$$

However,

$$\left\langle D\left(\varphi_{-G(\omega)} \circ G\right), \mathbf{T}\right\rangle = \left\langle D\varphi_{-G(\omega)}(G(\omega)) DG(\omega), \mathbf{T}\right\rangle =$$

$$= (I - G(\omega)G(\omega)^*)^{1/2} \langle DG(\omega), \mathbf{T}\rangle (I - G(\omega)^* G(\omega))^{-1/2}$$

by Lemma 3.2. On the other hand,

$$(I_{nk} - G(\omega)G(\omega)^* \otimes I_n)^{-1/2} \langle DG(\omega), \mathbf{T}\rangle (I_{nk} - G(\omega)^* G(\omega) \otimes I_n)^{-1/2} =$$

$$= \left((I_k - G(\omega)G(\omega)^*)^{-1/2} \otimes I_n\right) \langle DG(\omega), \mathbf{T}\rangle \cdot$$

$$\cdot \left((I_k - G(\omega)^* G(\omega))^{-1/2} \otimes I_n\right) =$$

$$= \left\langle (I_k - G(\omega)G(\omega)^*)^{-1/2} DG(\omega)(I_k - G(\omega)^* G(\omega))^{-1/2}, \mathbf{T}\right\rangle$$

and Lemma 3.1 implies that

$$G(\mathbf{N}) = \begin{bmatrix} G(\omega) \otimes I_n & \langle DG(\omega), \mathbf{T}\rangle \\ 0 & G(\omega) \otimes I_n \end{bmatrix}$$

has norm at most one.

3.4. The hypothesis on Ω guarantees that it can be realized as the unit ball in $\mathbf{C}^m$ with respect to a suitable norm $\|\cdot\|_\Omega$ on $\mathbf{C}^m$. In the following, the norm of a map between two normed linear spaces is always understood to be the usual operator norm. The following definition is an adaptation of Definition 1.2 [10].

3.5. Definition. For ω in Ω, define

$$\mathbf{D}_{\mathcal{M}_k}\Omega(\omega) = \left\{DF(\omega) \in \mathcal{L}\left((\mathbf{C}^m, \|\cdot\|_\Omega), \mathcal{M}_k\right) : F \in \operatorname{Hol}(\overline{\Omega}),\ \|F\|_\infty \leqq 1\right\}.$$

The m-tuple $\mathbf{N}(\mathbf{T}, \omega)$determines a linear map

$$\rho_{\mathbf{N}}^{(k)} : \mathcal{L}\left((C^m, \|\cdot\|_\Omega), \mathcal{M}_k\right) \to \mathcal{L}\left(\mathbf{C}^{kn}, \mathbf{C}^{kn}\right)$$

defined by

$$\rho_{\mathbf{N}}^{(k)}(P_1, \ldots, P_m) = P_1 \otimes T_1 + \ldots + P_m \otimes T_m = \langle \mathbf{P}, \mathbf{T} \rangle$$

We set

$$M_\Omega^k(\mathbf{N}(\mathbf{T}, \omega)) = \sup \left\{ \|\rho_{\mathbf{N}}^{(k)}(\mathbf{P})\| : \mathbf{P} \in \mathbf{D}_{\mathcal{M}_k}\Omega(\omega) \right\}$$

and

$$M_\Omega^c(\mathbf{N}(\mathbf{T}, \omega)) = \sup \left\{ M_\Omega^k(\mathbf{N}(\mathbf{T}, \omega)) : k \in \mathbf{N} \right\}.$$

In what follows, when $k = 1$, we will write $\rho_{\mathbf{N}}$ and $M_\Omega(\mathbf{N}(\mathbf{T}, \omega))$ instead of $\rho_{\mathbf{N}}^{(1)}$ and $M_\Omega^1(\mathbf{N}(\mathbf{T}, \omega))$. The map $\rho_{\mathbf{N}}$ is essentially the map ρ of Proposition 2.3. In view of Lemma 3.3, it is straightforward to prove the following theorem.

3.6. THEOREM. *$l^2(2n)_{\mathbf{N}}$ is a completely contractive $\mathcal{A}(\Omega)$-module if and only if $M_\Omega^c(\mathbf{N}(\mathbf{T}, \omega)) \leqq 1$.*

Parts (a) and (b) of the following theorem are identical to Theorem 1.9 in [10]. However, part (c) and (d) are slightly different in view of the fact that we are using a more general module action than the previous set up. Also note that neither part (a) nor part (c) of the following theorem is very useful unless we assume Ω admits a transitive group of biholomorphic automorphisms.

3.7. THEOREM. *Let ω be in Ω and assume that there exists a biholomorphic automorphism Θ_ω of Ω such that $\Theta_\omega(\omega) = \mathbf{0}$, and let $D\Theta_k$ and $D\Theta^k$ be the k^{th} column and k^{th} row respectively of the derivative $D\Theta_\omega(\omega)$. Then,*

(a) $\mathbf{D}_{\mathcal{M}_k}\Omega(\omega) = \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0}) \cdot D\Theta_\omega(\omega) = \{(DP(0) \cdot D\Theta_1, \ldots, DP(0) \cdot \cdot D\Theta_m) : DP(0) \in \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0})\}$,

(b) $\mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0}) = \{\mathbf{P} \in \mathcal{L}((\mathbf{C}^m, \|\cdot\|_\Omega), \mathcal{M}_k) : \|\mathbf{P}\| \leqq 1\}$,

(c) $M_\Omega^k(\mathbf{N}(\mathbf{T}, \omega)) = M_\Omega^k(\mathbf{N}(D\Theta_\omega(\omega) \cdot \mathbf{T}, \mathbf{0})) = M_\Omega^k(\mathbf{N}(D\Theta^1 \cdot \mathbf{T}, \ldots, D\Theta^m \cdot \mathbf{T};\ \mathbf{0}))$.

(d) $M_\Omega^k(\mathbf{N}(\mathbf{T}, \mathbf{0})) = \sup\{\|\langle \mathbf{P}, \mathbf{T} \rangle\| : \mathbf{P} \in \mathbf{D}_{\mathcal{M}_k}\Omega(\mathbf{0})\}$.

3.8. REMARK. Note that for $k = 1$, $\mathbf{D}_{\mathcal{M}_1}\Omega(\mathbf{0}) = \{\mathbf{P} \in \mathcal{L}((\mathbf{C}^m, \|\cdot\|_\Omega), \mathbf{C}) : \|\mathbf{P}\| \leqq 1\}$. In other words, if $\|\cdot\|_{\Omega^*}$ denotes the norm on $\mathbf{C}^m$ that is dual to $\|\cdot\|_\Omega$ then $\mathbf{D}_{\mathcal{M}_1}\Omega(\mathbf{0})$ can be identified with $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})_1$, which we write as Ω^*. Consequently, $M_\Omega(\mathbf{N}(\mathbf{T}, \mathbf{0}))$ is less than or equal to one if and only if $\|\rho_{\mathbf{N}}\| \leqq 1$, that is, $\|z_1 T_1 + \ldots + z_m T_m\| \leqq 1$ for all $(z_1, \ldots, z_m)$ in Ω^*, that is, $(T_1, \ldots, T_m)$ is in $D_{\mathcal{M}_1}\Omega^*(\mathbf{0})$.

Note that the inclusion $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ in $\mathcal{A}(\Omega)$ via the map $\mathbf{z} \to l_\mathbf{z}$, where for $\mathbf{z}$ in $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ and ω in $(\mathbf{C}^m, \|\cdot\|_{\Omega})$, $l_\mathbf{z}(\omega) = \sum_{j=1}^{n} \omega_j \overline{\mathbf{z}}_j$ is an isometry. We define, for each $[x_{ij}]$ in $(\mathbf{C}^m, \|\cdot\|_{\Omega^*}) \otimes \mathbf{M}_k$ the norm $\|[x_{ij}]\|$ using the inclusion map, which turns $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$ into a matricially normed space. (The definition and other related material is in [3].) Now, it is possible to talk of the *cb*-norm of the map $\rho_\mathbf{N} : (\mathbf{C}^m, \|\cdot\|_{\Omega^*}) \to \mathcal{M}_n$, we again refer the reader to [3] for this definition. It is easy to see, in view of Theorem 3.6, that studying completely bounded modules $\mathbf{C}^{2n}_\mathbf{N}$ over $\mathcal{A}(\Omega)$ is the same as studying completely bounded maps on the matrix normed space $(\mathbf{C}^m, \|\cdot\|_{\Omega^*})$. We will have more to say about this in Section 4.3.

4. Parrott's example and duality

The main theorem in this section is a duality result. As in the previous section, let Ω be the unit ball in $\mathbf{C}^m$ with respect to some norm $\|\cdot\|_\Omega$. Let $\|\cdot\|_{\Omega^*}$ be the dual norm and Ω^* be the unit ball in $\mathbf{C}^m$ with respect to the dual norm $\|\cdot\|_{\Omega^*}$.

4.1. THEOREM. *The following statements are equivalent:*

(i) *If $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ is a contractive module over $\mathcal{A}(\Omega)$, then it is completely contractive.*

(ii) *If $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ is a contractive module over $\mathcal{A}(\Omega^*)$, then it is completely contractive.*

PROOF. We prove (i) implies (ii). Note that by Remark 3.8, $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ is contractive over $\mathcal{A}(\Omega^*)$, if and only if $\mathbf{T}$ is in $\mathbf{D}_{\mathcal{M}_n}(\Omega^*)^*(\mathbf{0})$. But $(\Omega^*)^*$ is equal to Ω so that the module $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ is contractive over $\mathcal{A}(\Omega^*)$ if and only if $\mathbf{T}$ is in $\mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0})$. By Theorem 3.7 (d), to show that $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ is completely contractive, we have to establish for all k in $\mathbf{N}$

$$\|\langle \mathbf{P}, \mathbf{T}\rangle\| \leqq 1, \quad \text{for all} \quad \mathbf{P} \quad \text{in} \quad \mathbf{D}_{\mathcal{M}_k}\Omega^*(\mathbf{0}).$$

However, $\mathbf{P}$ in $\mathbf{D}_{\mathcal{M}_k}\Omega^*(\mathbf{0})$ is equivalent to saying $l^2(2k)_{\mathbf{N}(\mathbf{P},\mathbf{0})}$ is a contractive module over $\mathcal{A}(\Omega)$, again by Remark 3.8. But we are assuming any contractive module over $\mathcal{A}(\Omega)$ is completely contractive, so $l^2(2k)_{\mathbf{N}(\mathbf{P},\mathbf{0})}$ is completely contractive. Or equivalently, via Theorem 3.7(d), for all k in $\mathbf{N}$

$$\|\langle \mathbf{T}, \mathbf{P}\rangle\| \leqq 1, \quad \text{for all} \quad \mathbf{T} \quad \text{in} \quad \mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0}).$$

Using the flip map to change the order of tensor products occurring in $\langle \mathbf{T}, \mathbf{P}\rangle$, we see that for all k and n in $\mathbf{N}$, we have

$$\|\langle \mathbf{P}, \mathbf{T}\rangle\| = \|\langle \mathbf{T}, \mathbf{P}\rangle\| \leqq 1,$$

for all $\mathbf{P}$ in $\mathbf{D}_{\mathcal{M}_n}\Omega^*(\mathbf{0})$ and all $\mathbf{T}$ in $\mathbf{D}_{\mathcal{M}_n}\Omega(\mathbf{0})$. This completes the proof of (i) implies (ii) and the other implication can be verified in a similar manner.

4.2. COROLLARY. *If Ω admits a transitive group of biholomorphic automorphisms and $l^2(2)$ is a contractive module over $\mathcal{A}(\Omega)$ then $l^2(2)$ is completely contractive.*

PROOF. Note that in this case

$$\mathbf{N}(\mathbf{t},\omega) = \left(\begin{bmatrix} \omega_1 & t_1 \\ 0 & \omega_1 \end{bmatrix}, \dots, \begin{bmatrix} \omega_m & t_m \\ 0 & \omega_m \end{bmatrix} \right).$$

If $\omega = 0$ then $\mathbf{N}(\mathbf{t},\omega)$ is contractive if and only if $\mathbf{t} = (t_1, \dots, t_m)$ is in Ω and the proof is obvious. If $\omega \neq 0$ then $\mathbf{N}(\mathbf{t},\omega)$ is contractive if and only if $D\Theta_\omega(\omega) \cdot \mathbf{t}$ is in Ω. To check complete contractivity we have to verify that

$$M_\Omega^k(\mathbf{N}(\mathbf{t},\omega)) = M_\Omega^k(\mathbf{N}(D\Theta_\omega(\omega) \cdot \mathbf{t}, \mathbf{0})) =$$

$$= \sup \{ |\langle D\Theta_\omega(\omega) \cdot \mathbf{t}, \mathbf{P} \rangle| : \mathbf{P} \in \mathbf{D}_{\mathcal{M}_k}\Omega(0) \} \leqq 1,$$

but the last inequality is clearly true since $D\Theta_\omega(\omega) \cdot \mathbf{t} \in \Omega$ and the proof is complete.

We wish to point out in this connection that, while the above corollary is not very hard to prove, J. Agler [1] has shown by using a more refined form of Schwarz lemma that the same statement as in the previous corollary holds for arbitrary convex bounded subsets of $\mathbf{C}^n$. As a consequence he reproves a result from complex geometry, which says that for such domains the Caratheodory and the Kobayashi metric are the same.

4.3. REMARK. Vern Paulsen has shown me that the above theorem can be used to answer the following question.

Note first that $\rho_{\mathbf{N}}$ is a linear map from $(C^m, \| \|_{\Omega^*})$ into the matrix algebra $\mathcal{M}_n$. What we have defined as M_Ω^c is nothing else but the completely bounded norm of $\rho_{\mathbf{N}}$ provided we endow the normed space $(C^m, \| \|_{\Omega^*})$ with matrix norm structure as follows [cf. 3]. If X is a normed space, then the canonical inclusion of X into the continuous functions on the unit ball of its dual allows us to identify X with a subspace of a C^*-algebra and hence endows X with a matrix norm structure such that X is an operator space. Given a matricially normed Banach space X and linear maps $\rho : X \to \mathcal{L}(\mathcal{H})$, define

$$\alpha(X) = \sup \{ \|\rho\|_{cb} : \|\rho\| \leqq 1 \}.$$

One natural question is to determine matricially normed Banach spaces for which $\alpha(X) = 1$. The affirmative answer to this question is equivalent to asserting that any contractive module $l^2(2n)_{\mathbf{N}(\mathbf{T},\mathbf{0})}$ over $\mathcal{A}(\Omega)$ is also completely contractive. This question for $l^1(2)$ was first raised in Loebl [8].

If we look at the finite dimensional vector space $(\mathbf{C}^m, \| \|_{\Omega^*})$, then the matrix norm structure it inherits from $\mathcal{A}(\Omega)$, as discussed in 3.8, is the same as the matrix norm structure defined above. Note that by Ando's theorem [2], $\alpha(l^\infty(2)) \leqq 1$ and examples can be given to show that the bound is attained, thus $\alpha(l^\infty(2)) = 1$. Ando's result together with the previous theorem imply that $\alpha(l^1(2)) = 1$.

4.4. Remark. The example we wish to discuss here is like that of Parrott [11]. Our discussion is computational in nature and shows that there is a contractive Hilbert module over $\mathcal{A}(\mathbf{D}^3)$, which is not even 2-contractive. (For a discussion see [12, p.92]). By Theorem 4.2, it follows that the same is true for $\mathcal{A}\left(\left(l^1(3)\right)_1\right)$, that is, there is a contractive Hilbert module over $\mathcal{A}\left(\left(l^1(3)\right)_1\right)$ which is not 2-contractive. We hope this clarifies some of the mystery surrounding Parrott's example.

4.5. Lemma. *The norm of the map* $V : l^\infty(2) \to l^2(2)$ *is*

$$\|V\| = \left(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + 2\,|\langle \mathbf{v}_1, \mathbf{v}_2\rangle|\right)^{1/2},$$

where $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are the columns of the matrix for* V.

Proof. It is enough to note that

$$\left\| V \cdot \begin{bmatrix} 1 \\ e^{i\vartheta} \end{bmatrix} \right\| = \left\| \mathbf{v}_1 + e^{i\vartheta}\mathbf{v}_2 \right\|_2 =$$

$$= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + e^{-i\vartheta}\langle \mathbf{v}_1, \mathbf{v}_2\rangle + e^{i\vartheta}\langle \mathbf{v}_2, \mathbf{v}_1\rangle.$$

The result follows by choosing $\vartheta = \arg\langle \mathbf{v}_1, \mathbf{v}_2\rangle$.

However, the natural generalisation of this formula does not hold for $V : l^\infty(n) \to l^2(n)$ for $n > 2$. In fact, $\left(\|V\|_\infty^2\right)^2$ is, in general, strictly smaller than $\sum_{i,j} |\langle \mathbf{v}_i, \mathbf{v}_j\rangle|$. This is what we exploit the following.

4.6. Example. Let $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = \frac{1}{2}\left(1, \sqrt{3}\right)$ and $\mathbf{v}_3 = \left(1, -\sqrt{3}\right)$. It is easy to see that the map $\eta_1 : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to \mathbf{v}_1 + e^{i\vartheta}\mathbf{v}_2 + e^{i\varphi}\mathbf{v}_3$ from $l^\infty(3)$ to $l^2(2)$ has norm strictly smaller than $\sqrt{6}$. Similarly for $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = \frac{1}{2}\left(\sqrt{3}, -1\right)$ and $\mathbf{u}_3 = \frac{1}{2}\left(\sqrt{3}, 1\right)$, we see that the norm of the map $\eta_2 : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to \mathbf{u}_1 + e^{i\vartheta}\mathbf{u}_2 + e^{i\varphi}\mathbf{u}_3$ from $l^\infty(3)$ to $l^2(2)$ is strictly less than $\sqrt{6}$. In fact, with a little effort, one can show that each of these norms equals $\frac{3}{\sqrt{2}}$. Let U be the unitary matrix whose rows are $\mathbf{v}_2$ and $\mathbf{u}_2$. Similarly let V be the unitary matrix whose rows are $\mathbf{v}_3$ and $\mathbf{u}_3$. Now, consider the map

$$\rho_{(I,U,V)} : \left(1, e^{i\vartheta}, e^{i\varphi}\right) \to I + e^{i\vartheta}U + e^{i\varphi}V$$

and note that the operator norm of $\left(I + e^{i\vartheta}U + e^{i\varphi}V\right)$ is at most $\sqrt{\|\eta_1\|^2 + \|\eta_2\|^2}$. However if at a fixed ϑ, φ either of the norms $\|\eta_1\|$ or $\|\eta_2\|$ is equal to $3/\sqrt{2}$ then the other one is strictly less than $3/\sqrt{2}$. So that the operator norm of $\left(I + e^{i\vartheta}U + e^{i\varphi}V\right)$ is strictly less than 3. Thus we have shown the form of the map

$$\left(1, e^{i\vartheta}, e^{i\varphi}\right) \to I + e^{i\vartheta}U + e^{i\varphi}V$$

from $l^\infty(3)$ to $\mathcal{L}\left(l^2(2)\right)$ is strictly less than 3.

4.7. THEOREM. *$l^2(4)_{\mathbf{N}((I,U,V),\mathbf{0})}$ is contractive but not completely contractive over $\mathcal{A}(\mathbf{D}^3)$.*

PROOF. To show that $l^2(4)_{\mathbf{N}}$ is contractive we have to establish — by Remark 3.8 — that

$$\|z_1 I + z_2 U + z_3 V\| \leqq 1$$

for all $(z_1, z_2, z_3) \in \left(l^1(3)\right)_1$. But the inequality holds since each of I, U and V is a contraction operator. Note that the above discussion implies that $(I, U, V)/\delta$, for some $\delta < 3$, is in $\mathbf{D}_{\mathcal{M}_2}\mathbf{D}^3(\mathbf{0})$. To show that $l^2_{\mathbf{N}}$ is not completely contractive, we compute

$$\|\langle (I,U,V)/\delta,\ (I,U,V)\rangle\| =$$

$$= \delta^{-1}\|I \otimes I + U \otimes U + V \otimes V\| =$$

$$= \delta^{-1}\left\| \begin{bmatrix} i + \frac{1}{2}u + \frac{1}{2}v & \frac{\sqrt{3}}{2}(U - V) \\ \frac{\sqrt{3}}{2}(U + V) & I - \frac{1}{2}U + \frac{1}{2}V \end{bmatrix} \right\| =$$

$$= \frac{\sqrt{3}}{2}\delta^{-1} \left\| \begin{bmatrix} \sqrt{3} & 0 & 0 & \sqrt{3} \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ \sqrt{3} & 0 & 0 & \sqrt{3} \end{bmatrix} \right\| = 3\delta^{-1} > 1.$$

4.8. REMARK. We have not been able to decide, whether for operators of the form

$$N(V, \mathbf{0}) = \left(N(\mathbf{v}_1, \mathbf{0}), \ldots, N(\mathbf{v}_m, \mathbf{0})\right)$$

as in [9], where $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are vectors in $\mathbf{C}^n$, contractivity implies complete contractivity over $\mathcal{A}(\mathbf{D}^m)$. Vern Paulsen has shown that in this case the complete bound for a contractive map can be at most K_G, the universal constant of Grothendieck.

Note added in proof (November 8, 1993). In the paper "Contractive homomorphisms and tensor product norms", written jointly with B. Bagchi we have obtained many results relating to Remark 4.8.

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THE NUMERICAL RANGE OF NONLINEAR BANACH SPACE OPERATORS

R. U. VERMA (Orlando)

1. Introduction

Zarantonello [8] has introduced the concept of numerical range of nonlinear Hilbert space operators, and proved that the numerical range contains the spectrum. He applied this relation for solving nonlinear functional equations. Furthermore, this technique does work well for solving nonlinear partial differential equations. For more details, see the works of Dolph [2], Minty [5], Browder [1], Williams [7] and Martin, Jr. [4].

The aim of this paper is to generalize the results of Zarantonello [8] on nonlinear Hilbert space operators to the case of nonlinear Banach space operators. The obtained results provide a constructive method for solving nonlinear functional equations similar to that of Zarantonello [8]. Finally, we extend a result of Williams [7] on the spectra of products and numerical ranges to the case of nonlinear operators acting on a Banach space.

Suppose X is a linear space over a field K (real or complex). A mapping $[\cdot,\cdot]\colon X \times X \to K$ is said to be a semi-inner product on X if

i) $[x,x] > 0$ if $x \neq 0$;

ii) $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$ for all $\alpha, \beta \in K$ and $x, y, z \in X$; and

iii) $|[x,y]|^2 \leqq [x,x] \cdot [y,y]$ for all $x, y \in X$.

We call such a space X a semi-inner product space, and define the norm $\|\cdot\|$ on X by

$$\|x\| = [x,x]^{\frac{1}{2}} \quad \text{for all} \quad x \in X.$$

From now on, let X be a complex Banach space, and let $Op(X)$ denote the class of all operators A from its domain $D(A) \subset X$ into X. Let $\|\cdot\|$ denote the norm on X. An operator $A \in Op(X)$ is said to be Lipschitzian if

$$\|Ax - Ay\| \leqq M\|x - y\|$$

for some constant $M > 0$, and $x, y \in D(A)$ with $x \neq y$.

Let Lip(x) denote the class of all Lipschitz operators on X. Suppose $A \in$ Lip(X), and $x, y \in D(A)$ with $x \neq y$. Then a generalized Lipschitz norm of A, denoted $\|A\|_L$, is defined by

$$\|A\|_L = \|A\| + \|A\|_l,$$

where

$$\|A\| = \sup_{\|x\| \leqq 1} \|Ax\|, \quad \text{and} \quad \|A\|_l = \sup_{x \neq y} \frac{\|Ax - Ay\|}{\|x - y\|}.$$

From now on, $G_L(X)$ shall denote the class of all generalized Lipschitz operators. For $A \in Op(X)$, we define the spectrum of A, denoted $\sigma(A)$, as the complement of the resolvent set

$$\rho(A) = \{z: (A - zI)^{-1} \text{ exist and is generalized Lipschitzian}\}.$$

Next we introduce a concept of the numerical range of a nonlinear Banach space operator $A \in Op(X)$. The numerical range of A, denoted $V_L[A]$, is defined by

$$V_L[A] = \left\{ \frac{[Ax, x] + [Ax - Ay, x - y]}{\left(\|x\|^2 + \|x - y\|^2\right)} : x, y \in D(a),\ x \neq y \right\},$$

where $[,]$ is a semi-inner-product on X.

This numerical range has similar properties to that introduced by Zarantonello [8].

2. The numerical range

Let us first consider some properties of the numerical range. Then we establish the main results.

LEMMA 2.1. *Let $A, B \in Op(X)$, and $\lambda \in K$ (a complex field). Then*

(i) $V_L[\lambda A] = \lambda V_L[A]$;

(ii) $V_L[A_z] = V_L[A]$, *where $A_z x = Ax + z$ for all $x \in D(A_z) = D(A)$ and $z \in X$;*

(iii) $V_L[A + B] \subseteqq V_L[A] + V_L[B]$; *and*

(iv) $V_L[A - \lambda I] = V_L[A] - \{\lambda\}$.

PROOF. The assertions (i) and (ii) are obvious. In order to prove (iii), assume $D(A + B) = D(A) \cap D(B)$ and $x, y \in D(A) \cap D(B)$ with $x \neq y$, and $z \in V_L[A + B]$. Then

$$\frac{\left[(A + B)x, x\right] + \left[(A + B)x - (A + B)y, x - y\right]}{\|x\|^2 + \|x - y\|^2} =$$

$$= \frac{[Ax, x] + [Ax - Ay, x - y]}{\|x\|^2 + \|x - y\|^2} + \frac{[Bx, x] + [Bx - By, x - y]}{\|x\|^2 + \|x - y\|^2}.$$

This implies that $z \in V_L[A] + V_L[B]$, and this proves (iii).

To prove assertion (iv), assume $x, y \in D(A)$ with $x \neq y$. Then, for some $z \in V_L[A - \lambda I]$, we have

$$\frac{\big[(A - \lambda I)x, x\big] + \big[(A - \lambda I)x - (A - \lambda I)y, x - y\big]}{\|x\|^2 + \|x - y\|^2} =$$

$$= \frac{[Ax, x] + [Ax - Ay, x - y] - \lambda\big([x, x] + [x - y, x - y]\big)}{\|x\|^2 + \|x - y\|^2} =$$

$$= \frac{[Ax, x] + [Ax - Ay, x - y]}{\|x\|^2 + \|x - y\|^2} - \{\lambda\}.$$

This establishes (iv).

The invertibility of an operator $A \in G_L(X)$ is studied in the following lemma. An $A \in G_L(X)$ is said to be *invertible* if A is injective, surjective, and its inverse map is in $G_L(X)$.

LEMMA 2.2. *Let X be a complex Banach space, $A \in G_L(X)$ and $\|A\|_L < 1$. Then $I - A$ is invertible in $G_L(X)$, and*

$$\|(I - A)^{-1}\|_L \leqq \frac{2 - \|A\|_L}{\big(1 - \|A\|\big)\big(1 - \|A\|_l\big)}.$$

Furthermore, if $B_0 = I$ and $B_n = I - AB_{n-1}$ for $n = 1, 2, \ldots$, then

$$\lim B_n x = (I - A)^{-1}x \quad \textit{for every} \quad x \in X, \quad n \to \infty$$

and

$$\|(I - A)^{-1}x - B_n x\| \leqq \|A\|_l^n \cdot \|Ax\| \cdot \big(1 - \|A\|_l\big)^{-1}$$

for $x \in X$ and $n = 0, 1, \ldots$.

PROOF. Since $A \in G_L(X)$ and hence $A \in \mathrm{Lip}(X)$, for each $x, y \in X$ with $x \neq y$, we have

$$\|(I - A)x - (I - A)y\} \geqq \|z - y\| - \|Ax - Ay\| \geqq \big(1 - \|A\|_l\big)\|x - y\|.$$

This implies that $I - A$ is injective. Next, if $u, v \in R(I - A)$, then

$$\|(I - A)^{-1}u\| \leqq \big(1 - \|A\|\big)^{-1}\|u\|;$$

and

$$\|(I-A)^{-1}u-(I-A)^{-1}v\| \leqq (1-\|A\|_l)^{-1}\|u-v\|.$$

It follows from this that

$$\|(I-A)^{-1}\|_L \leqq \frac{2-\|A\|_L}{(1-\|A\|)(1-\|A\|_l)}.$$

To prove the second part, we show that if $x \in X$, then

$$\|B_{n+1}x - B_n x\| \leqq \|A\|_l^n \|Ax\|$$

for $n = 0, 1, \ldots$. This assertion follows by induction. It is true for $n = 0$. Assume that it holds for $n = k-1$. Then

$$\|B_{k+1}x - B_k x\| = \|AB_k x - AB_{k-1}x\| \leqq$$

$$\leqq \|A\|_l \|B_k x - B_{k-1}x\| \leqq \|A\|_l \|A\|_l^{k-1}\|Ax\|.$$

For a positive integer p,

$$\|B_{n+p}x - B_n x\| = \left\| \sum_{k=0}^{p-1}(B_{n+k+1}x - B_{n+k}x)\right\| \leqq$$

$$\leqq \sum_{k=0}^{p-1}\|B_{n+k+1}x - B_{n+k}x\| \leqq \sum_{k=0}^{p-1}\|A\|_l^{n+k}\|Ax\| =$$

$$= \|A\|_l^n \|Ax\|\{1+\|A\|_l+\ldots+\|A\|_l^{p-1}\} \leqq \|A\|_l^n\|Ax\|(1-\|A\|_l\|)^{-1}.$$

Since, by the hypothesis, $\|A\|_l < 1$ and X is complete, this implies that $\lim_{m\to\infty} B_m x = Ex$ exists for all $x \in X$, and, for $m = n+p$,

$$\|Ex - B_n x\| = \lim_{p\to\infty}\|B_{n+p}x - B_n x\| \leqq \|A\|_l^n\|Ax\|(1-\|A\|_l)^{-1}.$$

Furthermore, A is continuous, and it follows that

$$Ex = \lim_{n\to\infty} B_n x = \lim_{n\to\infty}(I - AB_{n-1})x = (I + AE)x.$$

Thus, $(I-A)E = I$, so E is a right inverse of $I-A$, implying $I-A$ is surjective. This and the first part of the proof imply that

$$E = (I-A)^{-1} \in G_l(X).$$

THEOREM 2.3. *Let $A \in Op(X)$ be closed, and $\lambda \notin V_L[A]$. Then $A - \lambda I$ is injective. If*

$$d(\lambda, V_L[A]) = \inf\{ |\lambda - \mu| : \mu \in V_L[A]\} > 0,$$

then the equation

$$Ax - \lambda x = y$$

has a unique solution for every $y \in R(A - \lambda I)$.

PROOF. Suppose $x, y (x \neq y) \in D(A)$. Then, since $d(\lambda, V_L[A]) > 0$,

$$\big|[(A - \lambda I)x, x] + [(A - \lambda I)x - (A - \lambda I)y, x - y]\big| =$$

$$= \big|[Ax, x] - \lambda \|x\|^2 + [Ax - Ay, x - y] - \lambda \|x - y\|^2\big| =$$

$$= \big|[Ax, x] + [Ax - Ay, x - y] - \lambda(\|x\|^2 + \|x - y\|^2)\big| =$$

$$= \left|\frac{[Ax, x] + [Ax - Ay, x - y]}{\|x\|^2 + \|x - y\|^2} - \lambda\right| (\|x\|^2 + \|x - y\|^2) \geqq$$

$$\geqq d(\lambda, V_L[A]) (\|x\|^2 + \|x - y\|^2).$$

From this, it follows that

$$\|(A - \lambda I)x\| \geqq d(\lambda, V_L[A]) \|x\|;$$

and

$$\|(A - \lambda I)x - (A - \lambda I)y\| \geqq d(\lambda, V_L[A]) \|x - y\|.$$

Consequently,

$$\frac{\|(A - \lambda I)x\|}{\|x\|} + \frac{\|(A - \lambda I)x - (A - \lambda I)y\|}{\|x - y\|} \geqq 2d(\lambda, V_L[A]).$$

Now, from the above inequalities, it is immediate that if $\lambda \notin V_L[A]$, then $A - \lambda I$ is injective, and $(A - \lambda I)^{-1}$ belongs to $G_l(R(A - \lambda I), X)$ with

$$\|(A - \lambda I)^{-1}\|_l \leqq (1/2) d^{-1}(\lambda, V_L[A]).$$

Let $y_k = (A - \lambda I)x_k$ $(k = 1, 2, \dots)$ satisfy $\lim_{k \to \infty} y_k = y$. Then

$$\|x_n - x_m\| = \|(A - \lambda I)^{-1} y_n - (A - \lambda I)^{-1} y_m\| \leqq d^{-1}(\lambda, V_L[A]) \|y_n - y_m\|,$$

so $\lim_{k \to \infty} x_k = x$ exists. Since A is closed and hence $A - \lambda I$ is closed, it is immediate from this that $x \in D(A - \lambda I)$ and $(a - \lambda I)x = y$. Hence, $y \in$ $\in R(A - \lambda I)$ and $R(A - \lambda I)$ is closed in X.

THEOREM 2.4. *Let X be a complex Banach space, and $A \in G_L(X)$. Then*

$$\sigma(A) \subseteqq \overline{\mathrm{co}}(V_L[A]).$$

The proof of Theorem 2.4 follows from an application of the following lemma.

LEMMA 2.5. *Let X be a complex Banach space, and $B \in G_l(X)$. Suppose that there exists a number $\beta > 0$ such that $\operatorname{Re}\gamma \leqq -\beta$ for all $\gamma \in V_L[B]$. Then B is invertible in $G_l(X)$.*

PROOF. If we apply Theorem 2.3, we need only to show that B is surjective. This is equivalent to showing $0 \in R(B)$ since $V_L[B_z] = V_L[B]$ by Lemma 2.1(ii). Suppose, for $\alpha > 0$, $\alpha\|B\| < 1$ so that $(I - \alpha B)^{-1} \in \in G_l(X)$, by Lemma 2.2. Furthermore, since

$$\alpha^{-1}\big((I-\alpha B)^{-1} - I\big) = \alpha^{-1}\big((I-\alpha B)^{-1} - (I-\alpha B)(I-\alpha B)^{-1}\big) =$$

$$= B(I-\alpha B)^{-1},$$

it suffices to show that there exists a $z \in X$ such that $(I - \alpha B)^{-1}z - z = 0$. By Lemma 2.1(i) and (iv), we get $V_L[I - \alpha B] = 1 - \alpha V_L[B]$, and so $\gamma \in \in V_L[I - \alpha B]$ implies that $\operatorname{Re}\gamma \geqq 1 + \alpha\beta$. As a result, for $x, y \in X (x \neq y)$, we have

$$\big\|(I-\alpha B)x\big\|\,\|x\| + \big\|(I-\alpha B)x - (I-\alpha B)y\big\|\,\|x-y\| \geqq$$

$$\geqq \operatorname{Re}\left(\frac{\big[(I-\alpha B)x, x\big] + \big[(I-\alpha B)x - (I-\alpha B)y, x-y\big]}{\|x\|^2 + \|x-y\|^2}\right)\cdot$$

$$\cdot\big(\|x\|^2 + \|x-y\|^2\big) \geqq \big(\|x\|^2 + \|x-y\|^2\big)\operatorname{Re}\big(V_L[I-\alpha B]\big),$$

and it follows that

$$\big\|(I-\alpha B)x\big\|\,\|x\| + \big\|(I-\alpha B)x - (I-\alpha B)y\big\|\,\|x-y\| \geqq$$

$$\geqq (1+\alpha\beta)\big(\|x\|^2 + \|x-y\|^2\big).$$

Consequently, we arrive at

$$\big\|(I-\alpha B)x\big\| \geqq (1+\alpha\beta)\|x\|;$$

and

$$\big\|(I-\alpha B)x - (I-\alpha B)y\big\| \geqq (1+\alpha\beta)\|x-y\|.$$

Hence,

$$\frac{\|(I-\alpha B)x\|}{\|x\|}+\frac{\|(I-\alpha B)x-(I-\alpha B)y\|}{\|x-y\|} \geqq 2(1+\alpha\beta).$$

From this, it is immediate that

$$\|(I-\alpha B)^{-1}\|_l \leqq \frac{1}{2}(1+\alpha\beta)^{-1} < 1.$$

This concludes the proof.

THEOREM 2.6. *Let X be a complex Banach space, let $C(X)$ denote the class of all continuous operators which send bounded sets into bounded sets, let $B \in C(X)$, and $A \in G_L(X)$. If $0 \notin \overline{\mathrm{co}}(V_L[A])$, then*

$$\sigma(A^{-1}B) \subseteqq \overline{\mathrm{co}}(V_L[B]) \,/\, \overline{\mathrm{co}}(V_L[A]).$$

PROOF. If $0 \notin \sigma(B-\lambda A)$, then $(B-\lambda A)^{-1}$ exists, and $(B-\lambda A)^{-1} \in$ $\in G_l(X)$. And, the product $(B-\lambda A)^{-1}A$ also belongs to $G_l(X)$. Since

$$(A^{-1}B-\lambda I)(B-\lambda A)^{-1}A = A^{-1}(B-\lambda A)(B-\lambda A)^{-1}A = I,$$

this implies that $A^{-1}B-\lambda I$ has a generalized Lipschitzian inverse. Hence, $\lambda \notin \sigma(A^{-1}B)$.

This means that $\lambda \in \sigma(A^{-1}B)$ implies $0 \in \sigma(B-\lambda A)$. Then from Theorem 2.4 and Lemma 2.1(iii) it follows that

$$0 \in \overline{\mathrm{co}}(V[B-\lambda A]) \subseteqq \overline{\mathrm{co}}(V[B]) - \lambda\,\overline{\mathrm{co}}(V[A]),$$

and this implies that

$$\lambda \in \overline{\mathrm{co}}(V[B]) \,/\, \overline{\mathrm{co}}(V[A]).$$

This completes the proof.

REMARK 2.7. The obtained results can also be applied to the case of a reflexive Banach space X, and demicontinuous functions A from X to its dual X^*.

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ON EXPONENTIATION OF n-ARY ALGEBRAS

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0. Introduction

For any medial n-ary algebra $\mathbf{G}$ and any n-ary algebra $\mathbf{H}$ the set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$ again forms an n-ary algebra $\mathbf{G}^{\mathbf{H}}$, the so called power of $\mathbf{G}$ and $\mathbf{H}$. These powers are studied in the paper. In particular, we give sufficient conditions for the validity of the law $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \simeq \mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ and for the composition of homomorphisms to be a homomorphism of $\mathbf{G}^{\mathbf{H}} \times \mathbf{K}^{\mathbf{G}}$ into $\mathbf{K}^{\mathbf{G}}$. We also discover a cartesian closed category of n-ary algebras. In the last section we generalize the results attained on universal algebras.

Throughout the article, n denotes a non-negative integer. By an *n-ary algebra* we understand a pair $\mathbf{G} = \langle G, p\rangle$ where G is a nonvoid set (called the *underlying set* of $\mathbf{G}$) and p is an n-ary operation on G. For the fundamental concepts concerning n-ary algebras (as special cases of universal algebras) see e.g. [2]. The direct product of a family of n-ary algebras $\{\mathbf{G}_i \mid i \in I\}$ will be denoted by $\prod_{i\in I} \mathbf{G}_i$. In the case $I = \{i_1, i_2\}$ we shall write $\mathbf{G}_{i_1} \times \times \mathbf{G}_{i_2}$ instead of $\prod_{i\in I} \mathbf{G}_i$. Given two n-ary algebras $\mathbf{G}$ and $\mathbf{H}$, by $\mathrm{Hom}(\mathbf{G},\mathbf{H})$ we denote the set of all homomorphisms of $\mathbf{G}$ into $\mathbf{H}$. If $\mathbf{G}$ and $\mathbf{H}$ are isomorphic, we write $\mathbf{G} \simeq \mathbf{H}$.

Let us recall that an n-ary algebra $\langle G, p\rangle$ is called *idempotent* if $p(x, x, \dots, x) = x$ holds for all $x \in G$, and it is called *medial* if

$$p\big(p(x_{11}, x_{12}, \dots, x_{1n}), p(x_{21}, x_{22}, \dots, x_{2n}), \dots, p(x_{n1}, x_{n2}, \dots, x_{nn})\big) =$$
$$= p\big(p(x_{11}, x_{21}, \dots, x_{n1}), p(x_{12}, x_{22}, \dots, x_{n2}), \dots, p(x_{1n}, x_{2n}, \dots, x_{nn})\big)$$

holds for all $x_{ij} \in G$, $i, j = 1, \dots, n$. For example, any commutative semigroup is a medial binary algebra. It is evident that if $\mathbf{G}_i$ is a medial n-ary algebra for each $i \in I$, then $\prod_{i\in I} \mathbf{G}_i$ is also medial.

1. Mediality and exponentiation of n-ary algebras

As an immediate consequence of the Lemma in [4] we get

LEMMA 1.1. *Let* $\mathbf{G} = \langle G, p\rangle$ *and* $\mathbf{H} = \langle H, q\rangle$ *be two* n*-ary algebras and let* $f_1, \ldots, f_n \in \mathrm{Hom}(\mathbf{G}, \mathbf{H})$. *If* $\mathbf{H}$ *is medial, then the map* $f: G \to H$ *given by* $f(x) = q(f_1(x), \ldots, f_n(x))$ *for each* $x \in G$ *fulfills* $f \in \mathrm{Hom}(\mathbf{G}, \mathbf{H})$.

Lemma 1.1 ensures the correctness of the following definition

DEFINITION 1.2. Let $\mathbf{G} = \langle G, p\rangle$ and $\mathbf{H} = \langle H, q\rangle$ be two n-ary algebras and let $\mathbf{G}$ be medial. Then we put $\mathbf{G}^{\mathbf{H}} = \langle \mathrm{Hom}(\mathbf{H}, \mathbf{G}), r\rangle$ where r is the n-ary operation on $\mathrm{Hom}(\mathbf{H}, \mathbf{G})$ defined by $r(f_1, \ldots, f_n)(x) = p(f_1(x), \ldots, f_n(x))$ for each $f_1, \ldots, f_n \in \mathrm{Hom}(\mathbf{H}, \mathbf{G})$ and each $x \in \mathbf{H}$. The n-ary algebra $\mathbf{G}^{\mathbf{H}}$ is called the power of $\mathbf{G}$ and $\mathbf{H}$.

THEOREM 1.3. *Let* $\mathbf{G}$ *be a medial* n*-ary algebra and let* $\{\mathbf{G}_i \mid i \in I\}$ *be a family of* n*-ary algebras such that* $\mathbf{G}_i \simeq \mathbf{G}$ *for each* $i \in I$. *Let* $\mathbf{H}$ *be an* n*-ary algebra whose underlying set is equipotent with* I. *Then there exists an isomorphic embedding of* $\mathbf{G}^{\mathbf{H}}$ *into* $\prod_{i \in I} \mathbf{G}_i$.

PROOF. The theorem follows immediately from the definition. It is sufficient to consider the evident fact that for any medial n-ary algebra $\mathbf{G}$ and any n-ary algebra $\mathbf{H}$ with the underlying set H the power $\mathbf{G}^{\mathbf{H}}$ is precisely the subalgebra of the direct product $\prod_{i \in I} \mathbf{G}_i$ where $G_i = G$ for each $i \in H$, whose underlying set is $\mathrm{Hom}(\mathbf{H}, \mathbf{G})$.

REMARK. Let the assumptions of Theorem 1.3 be fulfilled. Furthermore, let the underlying set of $\mathbf{G}$ be a singleton, or let $\mathbf{G}$ be idempotent and I be a singleton. Then clearly $\mathbf{G}^{\mathbf{H}} \simeq \prod_{i \in I} \mathbf{G}_i$.

THEOREM 1.4. *Let* $\{\mathbf{G}_i \mid i \in I\}$ *be a family of medial* n*-ary algebras and let* $\mathbf{H}$ *be an* n*-ary algebra. Then*

$$\left(\prod_{i \in I} \mathbf{G}_i\right)^{\mathbf{H}} \simeq \prod_{i \in I} \mathbf{G}_i^{\mathbf{H}}.$$

PROOF. Let $\mathbf{G}_i = \langle G, p_i\rangle$ whenever $i \in I$ and let H be the underlying set of $\mathbf{H}$. Let $p, q, r_i (i \in I), r$ be the n-ary operations of $\prod_{i \in I} \mathbf{G}_i$, $\left(\prod_{i \in I} \mathbf{G}_i\right)^{\mathbf{H}}$, $\mathbf{G}_i^{\mathbf{H}}$, $\prod_{i \in I} \mathbf{G}_i^{\mathbf{H}}$ respectively. By pr_i $(i \in I)$ we denote the i-th projection of $\prod_{i \in I} \mathbf{G}_i$. For any $f \in \mathrm{Hom}\left(\mathbf{H}, \prod_{i \in I} \mathbf{G}_i\right)$ put $\varphi(f) = g$ where $g: I \to \bigcup_{i \in I} \mathrm{Hom}(\mathbf{H}, \mathbf{G}_i)$ is the map given by $g(i) = \mathrm{pr}_i \circ f$ whenever $i \in I$. It can be easily seen that

φ is a bijection of $\operatorname{Hom}\left(\mathbf{H}, \prod_{i\in I} \mathbf{G}_i\right)$ onto $\prod_{i\in I} \operatorname{Hom}(\mathbf{H},\mathbf{G}_i)$. Let $f_1,\dots,f_n \in \operatorname{Hom}\left(\mathbf{H}, \prod_{i\in I} \mathbf{G}_i\right)$. Then for each $i \in I$ and each $x \in H$ we have

$$\begin{aligned}\varphi\big(q(f_1,\dots,f_n)\big)(i)(x) &= \operatorname{pr}_i\big(q(f_1,\dots,f_n)(x)\big) = \\ &= \operatorname{pr}_i\big(p(f_1(x),\dots,f_n(x))\big) = p_i\big(\operatorname{pr}_i(f_1(x)),\dots,\operatorname{pr}_i(f_n(x))\big) = \\ &= r_i(\operatorname{pr}_i\circ f_1,\dots,\operatorname{pr}_i\circ f_n)(x) = r_i\big(\varphi(f_1)(i),\dots,\varphi(f_n)(i)\big)(x) = \\ &= r\big(\varphi(f_1),\dots,\varphi(f_n)\big)(i)(x).\end{aligned}$$

Consequently, φ is a homomorphism of $\left(\prod_{i\in I} \mathbf{G}_i\right)^{\mathbf{H}}$ into $\prod_{i\in I} \mathbf{G}_i^{\mathbf{H}}$.

By the Lemma in [4], the power $\mathbf{G}^{\mathbf{H}}$ of the n-ary algebras $\mathbf{G}$ and $\mathbf{H}$ is medial whenever $\mathbf{G}$ is medial. Consequently, given n-ary algebras $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$, the power $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ is defined whenever $\mathbf{G}$ is medial. Moreover, there holds

THEOREM 1.5. *Let* $\mathbf{G}$ *be a medial n-ary algebra and let* $\mathbf{H}$, $\mathbf{K}$ *be n-ary algebras. Then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \simeq (\mathbf{G}^{\mathbf{K}})^{\mathbf{H}}.$$

PROOF. Let $\mathbf{G} = \langle G,p\rangle$, $\mathbf{H} = \langle H,q\rangle$, $\mathbf{K} = \langle K,r\rangle$. Moreover, let s, t, u, v be the n-ary operations of $\mathbf{G}^{\mathbf{H}}$, $\mathbf{G}^{\mathbf{K}}$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$, $(\mathbf{G}^{\mathbf{K}})^{\mathbf{H}}$ respectively. Let $f \in \operatorname{Hom}(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ be a homomorphism. Given $x \in H$, we put $f_x(y) = f(y)(x)$ for each $y \in K$. Let $y_1,\dots,y_n \in K$. Then

$$\begin{aligned}f_x\big(r(y_1,\dots,y_n)\big) &= f\big(r(y_1,\dots,y_n)\big)(x) = s\big(f(y_1),\dots,f(y_n)\big)(x) = \\ &= p\big(f(y_1)(x),\dots,f(y_n)(x)\big) = p\big(f_x(y_1),\dots,f_x(y_n)\big).\end{aligned}$$

We have shown that $f_x \in \operatorname{Hom}(\mathbf{K},\mathbf{G})$. Now, let $\hat{f}\colon H \to \operatorname{Hom}(\mathbf{K},\mathbf{G})$ be the map defined by $\hat{f}(x) = f_x$ for each $x \in H$. Then $\hat{f}(x)(y) = f(y)(x)$ whenever $x \in H$ and $y \in K$. Let $x_1,\dots,x_n \in H$. Then for any $y \in K$ we have

$$\begin{aligned}\hat{f}\big(q(x_1,\dots,x_n)\big)(y) &= f(y)\big(q(x_1,\dots,x_n)\big) = p\big(f(y)(x_1),\dots,f(y)(x_n)\big) = \\ &= p\big(\hat{f}(x_1)(y),\dots,\hat{f}(x_n)(y)\big) = t\big(\hat{f}(x_1),\dots,\hat{f}(x_n)\big)(y).\end{aligned}$$

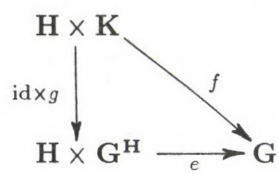
Consequently, $\hat{f} \in \operatorname{Hom}(\mathbf{H},\mathbf{G}^{\mathbf{K}})$. Therefore, putting $\varphi(f) = \hat{f}$ for every $f \in \operatorname{Hom}(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ we get a map $\varphi\colon \operatorname{Hom}(\mathbf{K},\mathbf{G}^{\mathbf{H}}) \to \operatorname{Hom}(\mathbf{H},\mathbf{G}^{\mathbf{K}})$. We shall show that $\varphi \in \operatorname{Hom}\big((\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}, (\mathbf{G}^{\mathbf{K}})^{\mathbf{H}}\big)$. Let us remark that $\varphi(f)(x)(y) = f(y)(x)$. Let $f_1, \ldots, f_n \in \operatorname{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$. Then for each $x \in H$ and each $y \in K$ we have

$$\varphi\big(u(f_1, \ldots, f_n)\big)(x)(y) = u(f_1, \ldots, f_n)(y)(x) =$$
$$= s\big(f_1(y), \ldots, f_n(y)\big)(x) = p\big(f_1(y)(x), \ldots, f_n(y)(x)\big) =$$
$$= p\big(\varphi(f_1)(x)(y), \ldots, \varphi(f_n)(x)(y)\big) = t\big(\varphi(f_1)(x), \ldots, \varphi(f_n)(x)\big)(y) =$$
$$= v\big(\varphi(f_1), \ldots, \varphi(f_n)\big)(x)(y).$$

Hence $\varphi \in \operatorname{Hom}\big((\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}, (\mathbf{G}^{\mathbf{K}})^{\mathbf{H}}\big)$. Since φ is obviously a bijection, we get $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \simeq (\mathbf{G}^{\mathbf{K}})^{\mathbf{H}}$.

Given a medial n-ary algebra $\mathbf{G}$ and an n-ary algebra $\mathbf{H}$, by e we denote the so called *evaluation map* $e\colon \mathbf{H} \times \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ given by $e(x, h) = h(x)$.

LEMMA 1.6. *Let* $\mathbf{G}$ *be a medial* n*-ary algebra, let* $\mathbf{H}$ *and* $\mathbf{K}$ *be idempotent* n*-ary algebras, and let* $f \in \operatorname{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$. *Then there exists a unique homomorphism* $g \in \operatorname{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ *for which the following triangle commutes:*



The unique homomorphism $g \in \operatorname{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ *is given by* $g(y)(x) = f(x, y)$.

PROOF. Let $\mathbf{G} = \langle G, p\rangle$, $\mathbf{H} = \langle H, q\rangle$, $\mathbf{K} = \langle K, r\rangle$ and let s denote the n-ary operation of $\mathbf{G}^{\mathbf{H}}$. For each $y \in K$ put $g(y) = h_y$ where $h_y\colon H \to G$ is the map defined by $h_y(x) = f(x, y)$ whenever $x \in H$. For any $x_1, \ldots, x_n \in H$ and any $y \in K$ there holds

$$h_y\big(q(x_1, \ldots, x_n)\big) = f\big(q(x_1, \ldots, x_n), y\big) =$$
$$= f\big(\big(q(x_1, \ldots, x_n), r(y, \ldots, y)\big)\big) = p\big(f(x_1, y), \ldots, f(x_n, y)\big) =$$
$$= p\big(h_y(x_1), \ldots, h_y(x_n)\big).$$

Therefore $h_y \in \operatorname{Hom}(\mathbf{H}, \mathbf{G})$ for each $y \in K$, i.e. g is a map of K into $\operatorname{Hom}(\mathbf{H}, \mathbf{G})$. Let $y_1, \ldots, y_n \in K$. Then for any $x \in H$ we have

$$g\big(r(y_1, \ldots, y_n)\big)(x) = f\big(x, r(y_1, \ldots, y_n)\big) =$$
$$= f\big(q(x, \ldots, x), r(y_1, \ldots, y_n)\big) = p\big(f(x, y_1), \ldots, f(x, y_n)\big) =$$
$$= p\big(g(y_1)(x), \ldots, g(y_n)(x)\big) = s\big(g(y_1), \ldots, g(y_n)\big)(x).$$

Consequently, $g \in \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$. Evidently, the triangle commutes for g. We prove the uniqueness of g. On that account, let $g' \in \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ be a homomorphism for which the triangle commutes. Suppose that $g' \neq g$, i.e. there exists $y \in K$ with $g'(y) \neq g(y)$. Then there exists $x \in H$ such that $g'(y)(x) \neq g(y)(x)$. Hence $f(x,y) = e(x, g'(y)) = g'(y)(x) \neq g(y)(x) = e(x, g(y)) = f(x,y)$, which is a contradiction. Therefore $g' = g$ and the proof is complete.

THEOREM 1.7. *Let* $\mathbf{G}$ *be a medial* n*-ary algebra and let* $\mathbf{H}$, $\mathbf{K}$ *be idempotent* n*-ary algebras. Then there exists an isomorphic embedding of* $\mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ *into* $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$.

PROOF. Let $\mathbf{G} = \langle G, p\rangle$, $\mathbf{H} = \langle H, q\rangle$, and let r, s, t be the n-ary operations of $\mathbf{G}^{\mathbf{H}}$, $\mathbf{G}^{\mathbf{H}\times\mathbf{K}}$, $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$ respectively. For each $f \in \mathrm{Hom}(\mathbf{H}\times\mathbf{K}, \mathbf{G})$ put $\varphi(f) = g$ where $g\colon \mathbf{K} \to \mathbf{G}^{\mathbf{H}}$ is the homomorphism from Lemma 1.6, i.e. $g(y)(x) = f(x,y)$. We get a map $\varphi\colon \mathrm{Hom}(\mathbf{H}\times\mathbf{K}, \mathbf{G}) \to \mathrm{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ that is evidently injective. Let $f_1, \ldots, f_n \in \mathrm{Hom}(\mathbf{H}\times\mathbf{K}, \mathbf{G})$. Then

$$\varphi\big(s(f_1,\ldots,f_n)\big)(y)(x) = s(f_1,\ldots,f_n)(x,y) =$$
$$= p\big(f_1(x,y)\ldots,f_n(x,y)\big) = p\big(\varphi(f_1)(y)(x),\ldots,\varphi(f_n(y)(x)\big) =$$
$$= r\big(\varphi(f_1)(y),\ldots,\varphi(f_n)(y)\big)(x) = t\big(\varphi(f_1),\ldots,\varphi(f_n)\big)(y)(x)$$

for any $x \in H$ and $y \in K$. Consequently, φ is a homomorphism of $\mathbf{G}^{\mathbf{H}\times\mathbf{K}}$ into $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}$.

PROPOSITION 1.8. *Let* $\mathbf{G}$, $\mathbf{H}$ *be* n*-ary algebras and let* $\mathbf{K}$, $\mathbf{L}$ *be medial* n*-ary algebras. Let* $f \in \mathrm{Hom}(\mathbf{G}, \mathbf{H})$ *and* $g \in \mathrm{Hom}(\mathbf{K}, \mathbf{L})$. *Put* $\varphi(h) = g \circ h \circ f$ *whenever* $h \in \mathrm{Hom}(\mathbf{H}, \mathbf{K})$. *Then* $\varphi \in \mathrm{Hom}(\mathbf{K}^{\mathbf{H}}, \mathbf{L}^{\mathbf{G}})$.

PROOF. Clearly, $\varphi\colon \mathrm{Hom}(\mathbf{H},\mathbf{K}) \to \mathrm{Hom}(\mathbf{G},\mathbf{L})$. Let $\mathbf{G} = \langle G, p\rangle$, $\mathbf{H} = \langle H, q\rangle$, $\mathbf{K} = \langle K, r\rangle$, $\mathbf{L} = \langle L, s\rangle$. Next, let t and u be the n-ary operations of $\mathbf{K}^{\mathbf{H}}$ and $\mathbf{L}^{\mathbf{G}}$ respectively. Let $h_1, \ldots, h_n \in \mathrm{Hom}(\mathbf{H}, \mathbf{K})$. Then for each $x \in G$ we have

$$\varphi\big(t(h_1,\ldots,h_n)\big)(x) = g\big(t(h_1,\ldots,h_n)(f(x))\big) =$$
$$= g\big(r(h_1(f(x)),\ldots,h_n(f(x)))\big) = s\big(g(h_1(f(x))),\ldots,g(h_n(f(x)))\big) =$$
$$= s\big(\varphi(h_1)(x),\ldots,\varphi(h_n)(x)\big) = u\big(\varphi(h_1),\ldots,\varphi(h_n)\big)(x).$$

Therefore $\varphi \in \mathrm{Hom}(\mathbf{K}^{\mathbf{H}}, \mathbf{L}^{\mathbf{G}})$.

2. Diagonality and exponentiation of n-ary algebras

An n-ary algebra $\langle G,p\rangle$ is called *diagonal* if

$$p\big(p(x_{11},x_{12},\dots,x_{1n}),p(x_{21},x_{22},\dots,x_{2n}),\dots,p(x_{n1},x_{n2},\dots,x_{nn})\big) = \\ = p(x_{11},x_{22},\dots,x_{nn})$$

holds for all $x_{ij}\in G$, $i,j=1,\dots,n$. Thus, each diagonal n-ary algebra is medial.

EXAMPLE 2.1. Obviously, if G is a nonvoid set and if $p\colon G^n\to G$ is a constant map or a projection, then $\langle G,p\rangle$ is a diagonal n-ary algebra. Let us describe all the binary operations p on the set $\{1,2,3\}$ for which the grupoid $\langle\{1,2,3\},p\rangle$ is diagonal: these are the three constant operations (i.e. constant maps of $G\times G$ into G), the two projections, the six binary operations given by the following tables

	1	2	3
1	1	1	1
2	2	2	2
3	2	2	2

	1	2	3
1	1	1	1
2	3	3	3
3	3	3	3

	1	2	3
1	1	1	1
2	2	2	2
3	1	1	1

	1	2	3
1	1	1	1
2	1	1	1
3	3	3	3

	1	2	3
1	2	2	2
2	2	2	2
3	3	3	3

	1	2	3
1	3	3	3
2	2	2	2
3	3	3	3

and the six dual operations. (Binary operations p, q on a set G are said to be dual if $p(x,y)=q(y,x)$ for all $x,y\in \mathbf{G}$.)

LEMMA 2.2. *Let* $\mathbf{G}$, $\mathbf{H}$ *be n-ary algebras and let* $\mathbf{G}$ *be diagonal. Then the evaluation map* $e\colon \mathbf{H}\times\mathbf{G}^{\mathbf{H}}\to\mathbf{G}$ *fulfills* $e\in \operatorname{Hom}(\mathbf{H}\times\mathbf{G}^{\mathbf{H}},\mathbf{G})$.

PROOF. Let $\mathbf{G}=\langle G,p\rangle$, $\mathbf{H}=\langle H,q\rangle$, $\mathbf{G}^{\mathbf{H}}=\langle K,r\rangle$, $\mathbf{H}\times\mathbf{G}^{\mathbf{H}}=\langle L,s\rangle$ and let $y_1,\dots,y_n\in L$. Then there exist $x_1,\dots,x_n\in H$ and $f_1,\dots,f_n\in$ $\in\operatorname{Hom}(\mathbf{H},\mathbf{G})$ such that $y_i=(x_i,f_i)$ for each $i=1,\dots,n$. We have

$$e\big(s(y_1,\dots,y_n)\big) = e\big(q(x_1,\dots,x_n),r(f_1,\dots,f_n)\big) = \\ = r(f_1,\dots,f_n)\big(q(x_1,\dots,x_n)\big) = \\ = p\big(f_1(q(x_1,\dots,x_n)),\dots,f_n(q(x_1,\dots,x_n))\big) = \\ = p\big(p(f_1(x_1),\dots,f_1(x_n)),\dots,p(f_n(x_1),\dots,f_n(x_n))\big) = \\ = p\big(f_1(x_1),\dots,f_n(x_n)\big) = p\big(e(x_1,f_1),\dots,e(x_n,f_n)\big) = \\ = p\big(e(y_1),\dots,e(y_n)\big).$$

Therefore $e\in\operatorname{Hom}(\mathbf{H}\times\mathbf{G}^{\mathbf{H}},\mathbf{G})$.

THEOREM 2.3. *Let* $\mathbf{G}$ *be a diagonal* n*-ary algebra and* $\mathbf{H}$, $\mathbf{K}$ *be idempotent* n*-ary algebras. Then*

$$\mathbf{G}^{\mathbf{H}\times\mathbf{K}} \simeq (\mathbf{G}^{\mathbf{H}})^{\mathbf{K}}.$$

PROOF. For any $g \in \operatorname{Hom}(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ put $\psi(g) = e \circ (\mathrm{id} \times g)$ where $e\colon \mathbf{H} \times \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ is the evaluation map and $\mathrm{id}\colon \mathbf{H} \to \mathbf{H}$ is the identity map. By Lemma 2.2, $e \in \operatorname{Hom}(\mathbf{H} \times \mathbf{G}^{\mathbf{H}}, \mathbf{G})$. Consequently, $\psi(g) \in \operatorname{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G})$. Let $\varphi\colon \operatorname{Hom}(\mathbf{H} \times \mathbf{K}, \mathbf{G}) \to \operatorname{Hom}(\mathbf{K}, \mathbf{G}^{\mathbf{H}})$ be the injective homomorphism defined in the proof of Theorem 1.7. It can be easily seen that for any $g \in \operatorname{Hom}(\mathbf{K},\mathbf{G}^{\mathbf{H}})$ there holds $\varphi(\psi(g)) = g$. Hence φ is a surjection.

REMARK. Theorems 1.4, 1.7 and 2.3 can also be obtained as consequences of the results of [6].

THEOREM 2.4. *Let* $\mathbf{G}$ *be an* n*-ary algebra, let* $\mathbf{H}$ *be a medial* n*-ary algebra and let* $\mathbf{K}$ *be a diagonal* n*-ary algebra. Put* $\varphi(f,g) = g \circ f$ *whenever* $f \in \operatorname{Hom}(\mathbf{G},\mathbf{H})$ *and* $g \in \operatorname{Hom}(\mathbf{H},\mathbf{K})$. *Then* $\varphi \in \operatorname{Hom}(\mathbf{H}^{\mathbf{G}} \times \mathbf{K}^{\mathbf{H}}, \mathbf{K}^{\mathbf{G}})$.

PROOF. Clearly, $\varphi\colon \operatorname{Hom}(\mathbf{G},\mathbf{H}) \times \operatorname{Hom}(\mathbf{H},\mathbf{K}) \to \operatorname{Hom}(\mathbf{G},\mathbf{K})$. Let $\mathbf{G} = \langle G,p\rangle$, $\mathbf{H} = \langle H,q\rangle$ and $\mathbf{K} = \langle K,r\rangle$. Let s, t, u, v be the n-ary operations of $\mathbf{H}^{\mathbf{G}}$, $\mathbf{K}^{\mathbf{H}}$, $\mathbf{H}^{\mathbf{G}} \times \mathbf{K}^{\mathbf{H}}$, $\mathbf{K}^{\mathbf{G}}$ respectively. Let $(f_1,g_1),\dots,(f_n,g_n) \in \operatorname{Hom}(\mathbf{G},\mathbf{H}) \times \operatorname{Hom}(\mathbf{H},\mathbf{K})$. Then for any $x \in G$ we have

$$\begin{aligned}
&\varphi\big(u((f_1,g_1),\dots,(f_n,g_n))\big)(x) = \varphi\big(s(f_1,\dots,f_n),t(g_1,\dots,g_n)\big)(x) = \\
&= t(g_1,\dots,g_n)\big(s(f_1,\dots,f_n)(x)\big) = t(g_1,\dots,g_n)\big(q(f_1(x),\dots,f_n(x))\big) = \\
&= r\big(g_1(q(f_1(x),\dots,f_n(x))),\dots,g_n(q(f_1(x),\dots,f_n(x)))\big) = \\
&= r\big(r(g_1(f_1(x)),\dots,g_1(f_n(x))),\dots,r(g_n(f_1(x)),\dots,g_n(f_n(x)))\big) = \\
&= r\big(g_1(f_1(x)),\dots,g_n(f_n(x))\big) = r\big(\varphi(f_1,g_1)(x),\dots,\varphi(f_n,g_n)(x)\big) = \\
&= v\big(\varphi(f_1,g_1),\dots,\varphi(f_n,g_n)\big)(x).
\end{aligned}$$

Therefore $\varphi \in \operatorname{Hom}(\mathbf{H}^{\mathbf{G}} \times \mathbf{K}^{\mathbf{H}}, \mathbf{K}^{\mathbf{G}})$.

We have shown that the diagonality of n-ary algebras gives the important results formulated in Theorems 2.3 and 2.4. Therefore it will be worthwhile to give a criterion for the diagonality of n-ary algebras.

LEMMA 2.5. *Let* $\mathbf{G} = \langle G,p\rangle$ *be an* n*-ary algebra. If* $\mathbf{G}$ *is diagonal, then for any elements* $x_1,\dots,x_n \in G$ *there holds*

$$\begin{aligned}
&p\big(x_1,\dots,x_{n-1},p(x_1,\dots,x_n)\big) = p\big(x_1,\dots,x_{n-2},p(x_1,\dots,x_n),x_n\big) = \\
&= \dots = p\big(p(x_1,\dots,x_n),x_2,\dots,x_n\big) = p(x_1,\dots,x_n).
\end{aligned}$$

PROOF. Let $x_1,\dots,x_n \in G$ and let $p(x_1,\dots,x_n) = t$. Then

$$p(t,\dots,t) = p\big(p(x_1,\dots,x_n),\dots,p(x_1,\dots,x_n)\big) = p(x_1,\dots,x_n) = t.$$

Thus

$$p(x_1,\dots,x_{n-1},t) = p\big(p(x_1,\dots,x_n),\dots,p(x_1,\dots,x_n),p(t,\dots,t)\big) = \\ = p(t,\dots,t).$$

Similarly we can prove the other equalities.

PROPOSITION 2.6. *An n-ary algebra $\langle G,p\rangle$ is diagonal if and only if the following condition is fulfilled:*

$$p\big(x_{11},x_{22},\dots,x_{n-1,n-1},p(x_{n1},\dots,x_{nn})\big) = \\ = p\big(x_{11},x_{22},\dots,x_{n-2,n-2},p(x_{n-1,1},\dots,x_{n-1,n}),x_{nn}\big) = \dots = \\ = p\big(p(x_{11},\dots,x_{1n}),x_{22},x_{33},\dots,x_{nn}\big) = p(x_{11},x_{22},\dots,x_{nn})$$

holds for all $x_{ij} \in G$, $i,j = 1,\dots,n$.

PROOF. Let the condition of Proposition 2.6 be fulfilled. Then for any elements $x_{ij} \in G$, $i,j = 1,\dots,n$, we have

$$p\big(p(x_{11},\dots,x_{1n}),\dots,p(x_{n,1},\dots,x_{nn})\big) = \\ = p\big(p(x_{11},\dots,x_{1n}),\dots,p(x_{n-1,1},\dots,x_{n-1,n}),x_{nn}\big) = \\ = p\big(p(x_{11},\dots,x_{1n}),\dots,p(x_{n-2,1},\dots,x_{n-2,n}),x_{n-1,n-1},x_{nn}\big) = \dots = \\ = p(x_{11},x_{22},\dots,x_{nn}).$$

Hence $\langle G,p\rangle$ is diagonal. Conversely, suppose that $\langle G,p\rangle$ is diagonal. Then, by the help of Lemma 2.5, for any elements $x_{ij} \in G$, $i,j = 1,\dots,n$, we get

$$p\big(x_{11},x_{22},\dots,x_{n-1,n-1}p(x_{n1},\dots,x_{nn})\big) = \\ = p\big(p(x_{11},\dots,x_{1n}),\dots,p(x_{n-1,1},\dots,x_{n-1,n}), \\ p(x_{n1},\dots,x_{n,n-1},p(x_{n1},\dots,x_{nn}))\big) = \\ = p\big(p(x_{11},\dots,x_{1n}),\dots,p(x_{n1},\dots,x_{nn})\big) = p(x_{11},x_{22},\dots,x_{nn}).$$

Similarly we can prove the other equalities of the condition of Proposition 2.6.

REMARK. By Proposition 2.6, a grupoid $\langle G, \cdot \rangle$ is diagonal (i.e. $(x \cdot y) \cdot (z \cdot t) = x \cdot t$ whenever $x, y, z, t \in G$) iff $x \cdot (y \cdot t) = (x \cdot z) \cdot t = x \cdot t$ holds for any $x, y, z, t \in G$. Thus, each diagonal grupoid is a semigroup.

Now we shall consider the n-ary algebras that are both diagonal and idempotent. Such algebras have been studied by J. Płonka in [7].

EXAMPLE 2.7. Obviously, if G is a nonvoid set and if $p: G^n \to G$ is a projection, then $\langle G, p \rangle$ is a diagonal idempotent n-ary algebra. Let us describe all the binary operations p on the set $\{1,2,3,4\}$ for which the grupoid $\langle \{1,2,3,4\}, p \rangle$ is both diagonal and idempotent: these are the two projections, the three binary operations given by the following tables

	1	2	3	4
1	1	2	1	2
2	1	2	1	2
3	3	4	3	4
4	3	4	3	4

	1	2	3	4
1	1	2	2	1
2	1	2	2	1
3	4	3	3	4
4	4	3	3	4

	1	2	3	4
1	1	3	3	1
2	4	2	2	4
3	1	3	3	1
4	4	2	2	4

and the three dual operations.

At the end of the section we formulate a category-theoretical meaning of the introduced exponentiation of n-ary algebras (for the definitions of the concepts used see e.g. [3]). By $\mathcal{A}_n$ we denote the category of diagonal idempotent n-ary algebras with homomorphisms as morphisms. Lemmas 1.6 and 2.2 result directly in

THEOREM 2.8. *Let* $\mathbf{G}$, $\mathbf{H}$ *be objects of* $\mathcal{A}_n$ *and let* $e: \mathbf{H} \times \mathbf{G}^{\mathbf{H}} \to \mathbf{G}$ *be the evaluation map. Then the pair* $(\mathbf{G}^{\mathbf{H}}, e)$ *is a co-universal map for* $\mathbf{G}$ *with respect to the functor* $\mathbf{H} \times \_ : \mathcal{A}_n \to \mathcal{A}_n$. *In other words,* $\mathcal{A}_n$ *is a cartesian closed category in the sense of* [1].

3. Exponentiation of universal algebras

For the basic concepts concerning universal algebras see again [2]. In fact, a universal algebra $\mathbf{G}$ is a pair $\langle G, \{p_\lambda \mid \lambda \in \Omega\} \rangle$ where Ω is a set and $\langle G, p_\lambda \rangle$ is an n_λ-ary algebra for each $\lambda \in \Omega$. The family $\{n_\lambda \mid \lambda \in \Omega\}$ is then the type of $\mathbf{G}$. All the previous results can be easily generalized to universal algebras. As an extension of mediality to universal algebras we must take the commutativity:

A universal algebra $\langle G, \{p_\lambda \mid \lambda \in \Omega\} \rangle$ of type $\{n_\lambda \mid \lambda \in \Omega\}$ is called *commutative* (see [4]) if for any $\lambda, \mu \in \Omega$ and any matrix $(x_{ij})_{n_\lambda \times n_\mu}$ over G there holds

$$p_\lambda\big(p_\mu(x_{11}, \ldots, x_{1n_\mu}), \ldots, p_\mu(x_{n_\lambda 1}, \ldots, x_{n_\lambda n_\mu})\big) =$$

$$= p_\mu\big(p_\lambda(x_{11},\dots,x_{n_\lambda 1}),\dots,p_\lambda(x_{1n_\mu},\dots,x_{n_\lambda n_\mu})\big).$$

If $\mathbf{G} = \big\langle G, \{p_\lambda \mid \lambda \in \Omega\}\big\rangle$, $\mathbf{H} = \big\langle H, \{q_\lambda \mid \lambda \in \Omega\}\big\rangle$ are two universal algebras of the same type $\{n_\lambda \mid \lambda \in \Omega\}$ and if $\mathbf{G}$ is commutative, then we define the power of $\mathbf{G}$ and $\mathbf{H}$ as follows (by $\mathrm{Hom}(\mathbf{H},\mathbf{G})$ we denote the set of all homomorphisms of $\mathbf{H}$ into $\mathbf{G}$): $\mathbf{G}^{\mathbf{H}} = \big\langle \mathrm{Hom}(\mathbf{H},\mathbf{G}), \{r_\lambda \mid \lambda \in \Omega\}\big\rangle$ where, for any $\lambda \in \Omega$, r_λ is the n_λ-ary operation on $\mathrm{Hom}(\mathbf{H},\mathbf{G})$ defined by $r_\lambda(f_1,\dots,f_{n_\lambda})(x) = p_\lambda\big(f_1(x),\dots,f_{n_\lambda}(x)\big)$ for each $f_1,\dots,f_{n_\lambda} \in \mathrm{Hom}(\mathbf{H},\mathbf{G})$ and each $x \in H$. For the case when both $\mathbf{G}$ and $\mathbf{H}$ are commutative this power is dealt with in [5].

We say that a universal algebra $\mathbf{G} = \big\langle G, \{p_\lambda \mid \lambda \in \Omega\}\big\rangle$ is *diagonal* or *idempotent* if the n_λ-ary algebra $\langle G, p_\lambda\rangle$ is diagonal or idempotent respectively for each $\lambda \in \Omega$. All the previous results concerning n-ary algebras are valid also for universal algebras. Note, however, that for universal algebras diagonality does not imply commutativity in general. Therefore, together with diagonality we must suppose commutativity in generalizations of the statements of Section 2 to universal algebras (excluding the statements 2.5 and 2.6).

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ON THE CLASS OF SATURATION IN STRONG APPROXIMATION BY PARTIAL SUMS OF FOURIER SERIES

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Introduction

Let $f(x)$ be a continuous function with period 2π and let its Fourier series be

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x).$$

Further let $s_k(x)$ be the k-th partial sum of the series. The strong approximation of $f(x)$ by partial sums is estimation of the expression

$$h_n(x,p;f) = \left\{\frac{1}{n+1}\sum_{k=0}^{n}|f(x) - s_k(x)|^p\right\}^{\frac{1}{p}}.$$

G. Freud [4] showed that if

$$(1) \qquad h_n(x,p;f) = O\left(n^{-\frac{1}{p}}\right), \qquad 1 < p < \infty,$$

then the saturation occurs, the order of saturation is $n^{-\frac{1}{p}}$ and the class of saturation X_p is a subclass of the Lip $\left(\frac{1}{p}\right)$ class. Furthermore he proved that almost everywhere

$$\lim_{t\to 0} |t|^{-\frac{1}{p}}\{f(x+t) - f(x)\} = 0.$$

In this note we shall improve this. We shall consider the integral modulus of continuity and the fractional differentiability of functions in the class X_p.

1. The integral modulus of continuity

We introduce the L_p modulus of continuity which is

$$\omega_p(t,f) = \sup_{0<h\leqq t} \left\{ \int_0^{2\pi} |f(x+h) - f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

If $\omega_p(t,f) = o(t^\alpha)$, then we say that $f \in \operatorname{lip}(\alpha, p)$.

Furthermore we introduce the L_p-best approximation of f by trigonometric polynomials of order n:

$$E_n(f)_p = \inf_{t_n(x)} \left\{ \int_0^{2\pi} |f(x) - t_n(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

THEOREM 1. *If for an $f \in L_p$*

$$\sum_{n=0}^{\infty} |f(x) - s_n(x)|^p \leqq K, \qquad 1 < p < \infty \tag{2}$$

uniformly, then $f(x) \in \operatorname{lip}\left(\frac{1}{p}, p\right)$.

PROOF. Let $T_n(x)$ be the n-th trigonometric polynomial which attains the L_p best approximation of $f(x)$, that is

$$E_n(f)_p = \left\{ \int_0^{2\pi} |f(x) - T_n(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Then by the M. Riesz theorem, we have

$$\|f - s_n\|_p \leqq A_p \|f - T_n\|_p \leqq A_p \|f - s_n\|_p.$$

Integrating the formula (1) and substituting this we get

$$\sum \left\{ E_n(f)_p \right\}^p \leqq A_p^p \sum \|f - s_n\|_p^p \leqq 2\pi A_p^p K.$$

Since $E_n(f)_p$ tends to zero monotonically, we get

$$n \left\{ E_n(f)_p \right\}^p = o(1),$$

and by [7, p.333],

$$\omega_p(t,f) = o\left(t^{\frac{1}{p}}\right).$$

2. The fractional differentiability

Even if $f(x) \in \operatorname{lip}\left(\frac{1}{p}\right)$, there is a function which has no fractional derivative of degree $1/p$ except in a null set [8, p.67]. We shall investigate fractional differentiability. Without loss of generality, we can suppose that the constant term of the Fourier series is zero.

LEMMA 1. *If for an $f \in L_p$*

$$\sum_{n=1}^{\infty} |f(x) - s_n(x)|^p \leqq K$$

then

$$\sum_{n=1}^{\infty} \frac{|s_n^\alpha(x) - \sigma_n^\alpha(x)|^p}{n} \leqq K', \qquad \alpha = \frac{1}{p}, \tag{3}$$

where

$$s_n^\alpha(x) = \sum_{k=1}^{n} k^\alpha A_k(x), \qquad n = 1, 2, \dots,$$

and $\sigma_n^\alpha(x)$ are the arithmetic means of $s_n^\alpha(x)$.

PROOF. By Hardy's inequality, we have

$$\sum_{n=1}^{\infty} |f(x) - \sigma_n(x)|^p \leqq A_p \sum_{n=1}^{\infty} |f(x) - s_n(x)|^p,$$

and

$$\sum_{n=1}^{\infty} |s_n(x) - \sigma_n(x)|^p \leqq 2A_p K.$$

This is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \left| \sum_{k=1}^{n} k A_k(x) \right|^p \leqq 2A_p K. \tag{4}$$

The formula to prove is also

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left| \sum_{k=1}^{n} k^{1+\alpha} A_k(x) \right|^p \leqq K'. \tag{5}$$

Set $\sum_{k=1}^{n} kA_k(x) = t_n(x)$ and by partial summation, we get

$$\sum_{k=1}^{n} k^{1+\alpha} A_k(x) = \sum_{k=1}^{n-1} t_k(x)\Delta k^{\alpha} + t_n(x)\cdot n^{\alpha} = T_1(x) + T_2(x),$$

say. Then by [5, p.255]

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} |T_1(x)|^p \leqq C \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left| \sum_{k=1}^{n-1} k^{\alpha-1} t_k(x) \right|^p \leqq$$

$$\leqq C' \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left(n^{\alpha} |t_n(x)| \right)^p \leqq C' \sum_{n=1}^{\infty} \frac{1}{n^p} |t_n(x)|^p,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} |T_2(x)|^p \leqq \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} |t_n(x)|^p \cdot n^{\alpha\cdot p} = \sum_{n=1}^{\infty} \frac{1}{n^p} |t_n(x)|^p.$$

Thus we get (5).

By a definition due to Flett [2], this is the same as the following Theorem.

THEOREM 2. *If* (2) *is satisfied, i.e.*

$$\sum_{n=1}^{\infty} |f(x) - s_n(x)|^p \leqq K,$$

then $\sum k^{\alpha} A_k(x)$ *is uniformly* $|C,1|_p$*-summable.*

THEOREM 3. *If* (2) *is satisfied, then for* $1 < p \leqq 2$, $f(x)$ *has the fractional derivative* $f^{\alpha}(x)$ *of order* $\alpha (= \frac{1}{p})$ *almost everywhere, and* f^{α} *belongs to the class* L_{λ} $(1 < \lambda < \infty)$.

PROOF. Integrating (3), we have

$$\int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} |s_n^{\alpha}(x) - \sigma_n^{\alpha}(x)|^p \right\}^{\frac{\lambda}{p}} dx < \infty. \tag{6}$$

By M. Riesz theorem for l_p-valued functions [1, p.1175], we get

$$\int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} |\tilde{s}_n^{\alpha}(x) - \tilde{\sigma}_n^{\alpha}(x)|^p \right\}^{\frac{\lambda}{p}} dx < \infty, \tag{7}$$

where $\tilde{s}_n^\alpha(x)$ are the conjugate of $s_n^\alpha(x)$ etc.

Now we denote the Poisson integral of $f(x)$ by $f(r,x)$ and set

$$\phi(z) = \phi(re^{ix}) = f(r,x) + i\tilde{f}(r,x). \tag{8}$$

Then by (6) and (7) we get for $\phi(z)$ the formula of Littlewood–Paley type

$$\int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} |s_n^\alpha(e^{ix}) - \sigma_n^\alpha(e^{ix})|^p \right\}^{\frac{\lambda}{p}} dx < \infty. \tag{9}$$

Hence, for $\frac{1}{p} = \alpha$ and $1 < p \leqq 2$, we get (see [6, p.528])

$$\left\| \phi^\alpha(e^{ix}) \right\|_\lambda < \infty.$$

For $p > 2$, $\alpha = \frac{1}{p}$, the conclusion of Theorem 3 is not true. The negative example is the following. This example is also a negative answer to a problem associated with Theorem 8 of Flett [2, p.126].

Set

$$f(x) = \sum_{n=1}^{\infty} 2^{-n\alpha} n^{-\frac{1}{2}} \cos 2^n x,$$

where $\alpha = \frac{1}{p}$. If $k = 2^m + 1$, then

$$f(x) - s_k(x) = \sum_{n=m}^{\infty} 2^{-n\alpha} n^{-\frac{1}{2}} \cos 2^n x,$$

and

$$|f(x) - s_k(x) \leqq \sum_{n=m}^{\infty} 2^{-n\alpha} n^{-\frac{1}{2}} \leqq m^{-\frac{1}{2}} \sum_{n=m}^{\infty} 2^{-n\alpha} \leqq Cm^{-\frac{1}{2}} 2^{-m\alpha}.$$

The trigonometric series is lacunary and such 2^m terms as $k = 2^m + 1$, $2^m + 2, \dots, 2^{m+1}$ have the same remainder. Hence

$$\sum_{k=0}^{\infty} |f(x) - s_k(x)|^p \leqq C' \sum_{m=1}^{\infty} 2^m \left(m^{-\frac{1}{2}} 2^{-m\alpha} \right)^p =$$

$$= C' \sum_{m=1}^{\infty} m^{-\frac{p}{2}} < \infty, \qquad \text{for} \quad p > 2.$$

If $f^{\alpha}(x)$ exists at every point of a set of positive measure, then $f_{1-\alpha}(x)$ is derivable [9, p.133] and the series

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \cos 2^n x$$

has to be Abel summable at every point of a set of positive measure. Since the series is lacunary, it is necessary for the series $\sum 1/n$ to converge [9, p.203]. This is evidently a contradiction.

3. Improvement of Theorem 1

We shall prove a slightly better theorem than Theorem 1.

THEOREM 4. *If* (2) *is satisfied, then*

$$\int_0^{2\pi} \left\{ \int_0^{\pi} \frac{|f(x+t)-f(x-t)|^p}{t^2}\, dt \right\}^{\frac{\lambda}{p}} dx < \infty, \qquad p \leqq \lambda < \infty.$$

PROOF. We set

$$\phi^{\alpha}(z) = \sum_{n=1}^{\infty} n^{\alpha} c_n z^n$$

where $\phi(z)$ is defined by (8). Then (9) is true and by [2, p.116] we have

$$(10) \qquad \int_0^{2\pi} \left\{ \int_0^1 (1-r)^{p-1} \left| \{\phi^{\alpha}(re^{ix})\}' \right|^p dr \right\}^{\frac{\lambda}{p}} dx < \infty.$$

Furthermore by Flett [3, p.375], we get

$$\int_0^{2\pi} \left\{ \int_0^{\pi} \frac{\left|\phi\left(e^{i(x+t)}\right) - \phi\left(e^{i(x-t)}\right)\right|^p}{t^2}\, dt \right\}^{\frac{\lambda}{p}} dx < \infty, \quad p \leqq \lambda < \infty,$$

since $\alpha = \frac{1}{p}$. The real part of this gives Theorem 4.

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CANONICAL A-SYSTEMS

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I. Introduction

Let us recall the definition of canonical number systems and C.N.S. rings (cf. [3], [4], [5]).

DEFINITION I.1. Let R be a commutative ring. Let $a \in R$, $N \in \mathbf{N}$. The pair $\{a, N\}$ is a *canonical number system* if every $y \in R$ can be uniquely represented as a sum

$$y = y_0 + y_1 a + \ldots + y_n a^n \tag{I.1}$$

where $y_i \in \{0, 1, \ldots, N-1\}$. The ring R is said to be a C.N.S. *ring* if there is a canonical number system in it.

Kovács gives us many examples of C.N.S. rings (cf. [3], [4]), and a complete description of C.N.S. integral domains, cf. [5]. In this paper, we generalize the notion of C.N.S. rings, taking a polynomial ring $\mathbf{A}$ instead of the ring $\mathbf{Z}$ of integers.

If $\mathbf{A}$ is the ring $k[T]$ of polynomials over the field k, we introduce the notion of canonical $\mathbf{A}$-system, and the notion of C.**A**.S. rings, and we study these rings.

II. Notation and definitions

Let k be a commutative field. Let $\mathbf{A} = k[T]$, $\mathbf{A}^* = \mathbf{A} - \{0\}$, $\mathbf{K} = k(T)$. If $H \in \mathbf{A}$, we denote by $\deg(H)$ the degree of H, and we put $\deg(0) = -\infty$. If $H \in \mathbf{A}^*$, then $\boldsymbol{C}_H$ denotes the additive subgroup of polynomials M of $\mathbf{A}$ such that $\deg(M) < \deg(H)$. If $\mathbf{P}$ is a polynomial in the ring $A[X]$, its degree in the variable X will be denoted by $\deg_X(\mathbf{P})$.

In the sequel, the word ring will be used for a commutative ring which is an $\mathbf{A}$-algebra with unity. If $\mathbf{B}$ is an algebra, the unit element of $\mathbf{B}$ will be denoted by $\mathbf{1_B}$, while the unit element of $\mathbf{A}$ will be denoted by 1.

For every $y \in \mathbf{B}$, for every additive subgroup $\boldsymbol{G}$ of $\mathbf{A}$ we denote by $\boldsymbol{G}[y]$ the set of elements $z \in \mathbf{B}$ for which there exist an integer $m \geqq 0$ and elements $G_0, \ldots, G_m$ in $\boldsymbol{G}$ such that

$$z = G_0 \mathbf{1_B} + G_1 y + \ldots + G_m y^m. \tag{II.1}$$

It is easy to prove that $\boldsymbol{G}[y]$ is an additive subgroup of $\boldsymbol{G}$.

DEFINITION II.1. Let $\mathbf{B}$ be a ring. Let $\beta \in \mathbf{B}$, $H \in \mathbf{A}^*$. We say that (β, H) is a *canonical* $\mathbf{A}$*-system* if, for every $y \in \mathbf{B}$, there exist a unique integer $m \geqq 0$, and a unique sequence $(Y_0, Y_1, \ldots, Y_m)$ in $\boldsymbol{C}_H$ such that

$$y = Y_0 \mathbf{1_B} + Y_1 \beta + \ldots + Y_m \beta^m. \tag{II.2}$$

We say that $\mathbf{B}$ is a **C.A.S.** *ring* if there exists a canonical $\mathbf{A}$-system in it. Choosing a canonical $\mathbf{A}$-system (β, H) in the **C.A.S.** ring $\mathbf{B}$, the representation (II.2) of $y \in \mathbf{B}$ is called a *canonical representation* of y, and the integer m in (II.2) is called the *length of the canonical representation.*

REMARK II.2. If (β, H) is a canonical $\mathbf{A}$-system in $\mathbf{B}$ and $y \in \mathbf{B}$ admits the canonical representation (II.2), then $Y_m \neq 0$ except for $y = 0$.

REMARK II.3. Let $\mathbf{B}$ be a **C.A.S.** ring. Then the map $f: \mathbf{A} \to \mathbf{B}$ defined by $f(a) = a\mathbf{1_B}$ is not zero everywhere and, up to isomorphisms, we have only to consider two cases:

(1) the ring $\mathbf{B}$ contains the ring $\mathbf{A}$,

(2) the ring $\mathbf{B}$ contains a quotient $\mathbf{A}/\mathbf{I}$ where $\mathbf{I}$ is an ideal of $\mathbf{A}$, $\mathbf{I} \neq \{0\}$, $\mathbf{I} \neq \mathbf{A}$.

REMARK II.4. Let (β, H) be a canonical $\mathbf{A}$-system in $\mathbf{B}$. If $\mathbf{B}$ is a ring containing $\mathbf{A}$, then we have the inclusions

$$\mathbf{B} \subset \boldsymbol{C}_H[\beta] \subset \mathbf{A}[\beta] \subset \mathbf{B};$$

if $\mathbf{B}$ is a ring containing $\mathbf{A}/\mathbf{I}$, then

$$\mathbf{B} \subset \boldsymbol{C}_H[\beta] \subset \mathbf{A}[\beta] = (\mathbf{A}/\mathbf{I})[\beta] \subset \mathbf{B}.$$

Hence,

$$\mathbf{B} = \boldsymbol{C}_H[\beta] = \mathbf{A}[\beta], \quad \text{if} \quad \mathbf{B} \quad \text{contains} \quad \mathbf{A}, \tag{II.3}$$

$$\mathbf{B} = \boldsymbol{C}_H[\beta] = (\mathbf{A}/\mathbf{I})[\beta], \quad \text{if} \quad \mathbf{B} \quad \text{contains} \quad \mathbf{A}/\mathbf{I}. \tag{II.4}$$

III. C.A.S. subrings of K

We have the following theorems:

THEOREM III.1. *For every* $(Y, Z) \in \mathbf{A} \times \mathbf{A}^*$, (Y, Z) *is a canonical* **A**-*system in* **A**, *if and only if* $\deg Y = \deg Z$ *and* $\deg Z > 0$.

COROLLARY III.2. *The ring is a* **C.A.S.** *ring.*

THEOREM III.3. *Let* $Y \in \mathbf{A}$, $P \in \mathbf{A}^*$, $Q \in \mathbf{A}^*$ *such that* g.c.d.$(P, Q) =$ $= 1$, *then* $(P/Q, Y)$ *is a canonical* **A**-*system in* $\mathbf{A}[P/Q]$ *if and only if* $\deg P > \deg Q$ *and* $\deg P = \deg Y$.

COROLLARY III.4. *For every* $y \in K$, *the ring* $\mathbf{A}[y]$ *is a* **C.A.S.** *ring.*

We note that Theorem III.3 implies Theorem III.1, and that Corollary III.2 is immediate.

It is not difficult to provide a direct proof of these results. But Theorem III.3 is a particular case of Theorem V.2, and Corollary III.4 is a particular case of Corollary V.3. So, we ask the reader to go forward to Section V.

IV. The non-algebraic case

THEOREM IV.1. *Let* **B** *be a ring containing a quotient* $\mathbf{A}/(Q)$ *where* Q *is a non-constant polynomial in* **A**. *Let* $(\beta, H) \in \mathbf{B} \times \mathbf{A}^*$. *Then* (β, H) *is a canonical* **A**-*system in* **B** *if and only if* $\mathbf{B} = \mathbf{A}/(Q)[\beta]$ *with* β *non-algebraic over* $\mathbf{A}/(Q)$ *and* $\deg H = \deg Q$.

PROOF. Let $a \mapsto s(a)$ be the canonical map from **A** onto $\mathbf{A}/(Q)$. We note that this map is bijective from $\boldsymbol{C}_Q$ to $\mathbf{A}/(Q)$ and that $s(a) = a\mathbf{1}_{\mathbf{B}}$.

Theorem 1, §2, of [1] shows that the subalgebra $\mathbf{A}/(Q)[\beta]$ of **B** is isomorphic to a quotient $\left(\mathbf{A}/(Q)[X]\right)/\mathcal{R}$ of the polynomial ring $\mathbf{A}/(Q)[X]$ by the ideal $\mathcal{R}$ of polynomials $f \in \mathbf{A}/(Q)[X]$ such that $f(\beta) = 0$. If β is non-algebraic, $\mathcal{R} = \{0\}$, $\mathbf{A}/(Q)[\beta]$ is the ring of polynomials over $\mathbf{A}/(Q)$. Every $y \in \mathbf{A}/(Q)[\beta]$ admits a unique representation as a sum

$$y = u_0 + u_1\beta + \ldots + u_m\beta^m,$$

with $u_0, u_1, \ldots, u_m$ in $\mathbf{A}/(Q)$. Using the bijection s, we see that y admits a unique representation

$$y = U_0\mathbf{1}_{\mathbf{B}} + U_1\beta + \ldots + U_m\beta^m,$$

with $U_0, U_1, \ldots, U_m$ belonging to $\boldsymbol{C}_Q$. If $\deg H = \deg Q$, then $\boldsymbol{C}_H = \boldsymbol{C}_Q$ and the "only if" part of theorem is proved.

Now we suppose that (β, H) is a canonical **A**-system in **B**. By (II.4), $\mathbf{B} = \mathbf{A}/(Q)[\beta]$. First we will prove that β is non-algebraic over $\mathbf{A}/(Q)$.

Suppose the contrary. The ideal $\mathcal{R}$ is non-zero. Among the non-zero polynomials in $\mathcal{R}$, we choose one, whose degree n is minimal, say

$$f(X) = s(U_0) + s(U_1)X + \ldots + s(U_n)X^n,\ U_0 \in \boldsymbol{C}_Q,\ U_1 \in \boldsymbol{C}_Q, \ldots, U_n \in \boldsymbol{C}_Q.$$

Let $D = \text{g.c.d.}(U_n, Q)$. We put $U_n = DA_n$ and $Q = DW$. Then, $Wf(\beta) = = 0$, and the polynomial

$$s(WU_0) + s(WU_1)X + \ldots + s(WU_{n-1})X^{n-1}$$

belongs to $\mathcal{R}$. According to the choice of f, it is the zero polynomial, and $Q = DW$ divides every WU_i. We write $U_i = DA_i$. Let U and V in $\mathbf{A}$ be such that $UA_n + VW = 1$. Then

$$0 = UDA_0 + UDA_1\beta + \ldots + UDA_{n-1}\beta^{n-1} + D\beta^n. \tag{1}$$

From (1) we get by induction on the exponent j that for $j \geqq n$ there exists a polynomial $P \in \mathbf{A}[X]$ such that $\deg_X(P) < n$ and $D\beta^j = DP(\beta)$. On the other hand, for $i = 0, 1, \ldots, \deg(Q) - 1$, there exist an integer $r(i)$ and $Y_{0,i}Y_{1,i}, \ldots, Y_{r(i),i}$ in $\boldsymbol{C}_H$ such that

$$T^i = Y_{0,i} + Y_{1,i}\beta + \ldots + Y_{r(i),i}\beta^{r(i)}.$$

Let $r = \max\Big\{r(0), r(1), \ldots, r\big(\deg(Q) - 1\big)\Big\}$. Then for each $Y \in \mathbf{A}/(Q)$, there exists a polynomial $G \in \mathbf{A}[X]$, whose coefficients are in $\boldsymbol{C}_H$ such that $\deg_X(G) \leqq r$ and $Y = G(\beta)$. Hence every Y in $\big(D\mathbf{A}/(Q)\big)[\beta]$ admits a canonical representation (II.2) whose length is less than $r + n$. Since (β, H) is a canonical system in $\mathbf{B}$, this representation would be unique. We write the canonical representation of D as

$$D = D_0\mathbf{1_B} + D_1\beta + \ldots + D_t\beta^t \tag{2}$$

Then for every integer $N > r + n$, $D\mathbf{1_B} + D\beta + \ldots + D\beta^N$ belongs to $\big(D\mathbf{A}/(Q)\big)[\beta]$ and admits the representation

$$D_0\mathbf{1_B} + (D_0 + D_1)\beta + \ldots + (D_{t-1} + D_t)\beta^{t+N}$$

which is canonical and whose length is $t + N \neq r + N$. This is a contradiction; hence β is not algebraic over $\mathbf{A}/(Q)$.

For every $a \in \mathbf{A}$, we have $s(a) = a\mathbf{1_B}$. As $s(0) = Q\mathbf{1_B}$ admits only the trivial representation in the form (II.2), we cannot have $Q \in \boldsymbol{C}_H$, so

$$\deg Q \geqq \deg H. \tag{3}$$

The class $s(H) = H\mathbf{1_B}$ admits a representation in the form (II.2). There exists an integer $r \geqq 0$, $H_0, \ldots, H_r$ in C_H such that

$$s(H) = H_0\mathbf{1_B} + H_1\beta + \ldots + H_r\beta^r, \quad H_r \neq 0.$$

Hence

$$0 = (H_0 - H)\mathbf{1_B} + H_1\beta + \ldots + H_r\beta^r, \quad H_r \neq 0,$$

and the transcendence of β implies that $r = 0$, and that Q divides $H_0 - H$. We have $\deg H_0 < \deg H$, so $H_0 - H \neq 0$ and $\deg(H_0 - H) = \deg(H)$. If Q divides $H_0 - H$, then $\deg(H_0 - H) \geqq \deg(Q)$ and

$$(4) \qquad \deg H \geqq \deg Q.$$

V. The algebraic case

We prove that the ring $\mathbf{A}[\beta]$ is a **C.A.S.** ring if β is algebraic over $\mathbf{K}$. We deduce this fact from Theorem V.2 and Corollary V.3. Theorem V.2 contains a little more information.

First, we make it more precise what we mean by a minimal polynomial.

V.1. Remark and definition. Let β be a non-zero algebraic element over the field $\mathbf{K}$. Let $\mathbf{I}(\beta)$ be the ideal of polynomials of $\mathbf{K}[X]$ whose β is a root. In $\mathbf{I}(\beta)$ there is one and only one polynomial P which satisfies the following conditions:

(1) $\mathbf{I}(\beta)$ is generated by P,

(2) $P \in \mathbf{A}[X]$,

(3) the coefficients of P are coprime in $\mathbf{A}$.

We call P the minimal polynomial of β over $\mathbf{A}$.

THEOREM V.2. *Let* $\mathbf{B}$ *a ring containing* $\mathbf{A}$. *Let* $(\beta, H) \in \mathbf{B} \times \mathbf{A}^*$. *Then* (β, H) *is a canonical* $\mathbf{A}$*-system in* $\mathbf{B}$ *if and only if the following conditions are satisfied:*

(1) $\mathbf{B} = \mathbf{A}[\beta]$ *and* β *is algebraic over* $\mathbf{K}$.

(2) *If* $P = A_0 + A_1X + \ldots + A_nX^n$ *is the minimal polynomial of* β *over* $\mathbf{A}$, *then*

$$\deg A_0 = \deg H,$$

and

$$\deg A_i < \deg A_0 \quad \textit{for every} \quad i = 1, \ldots, n.$$

PROOF. Let (β, H) be a canonical $\mathbf{A}$-system in $\mathbf{B}$. Then, $\beta \neq 0$. With (II.3), we have that $\mathbf{B} = \mathbf{A}(\beta)$, and recalling that $H \in \mathbf{A}$ admits a canonical representation we get that β is algebraic over $\mathbf{K}$, and (1) is proved.

Let

(i) $$P(X) = A_0 + A_1 + \ldots + A_n X^n$$

be the minimal polynomial of β over $\mathbf{A}$. We have

$$0 = A_0 + A_1\beta + \ldots + A_n\beta^n,$$

and the element $A_0\beta^{-1}$ of the field $\mathbf{K}(\beta)$ belongs to the ring $\mathbf{A}[\beta] = \mathbf{B}$. There exist an integer $s \geqq 0$ and $R_0, R_1, \ldots, R_s$ in $\boldsymbol{C}_H$ such that

$$A_0\beta^{-1} = R_0 + R_1\beta + \ldots + R_s\beta^s.$$

Hence

(ii) $$A_0 = R_0\beta + R_1\beta^2 + \ldots + R_s\beta^{s+1}.$$

This is a representation of A_0 in the form (II.2). This representation is the only one. So

$$A_0 = A_0 \mathbf{1_B}$$

is not a representation in the form (II.2). Hence, we have

(iii) $$\deg A_0 \geqq \deg H.$$

As $\mathbf{B}$ is included in $\boldsymbol{C}_H[\beta]$, H is non-zero, it admits a representation

$$H = H_0 + H_1\beta + \ldots + H_m\beta^m,$$

in the form (II.2) with $H_m \neq 0$. In this representation $H_0 \neq H$, so

$$0 = H_0 - H + H_1\beta + \ldots + H_m\beta^m$$

is a non-trivial equation whose β is a root. So the minimal polynomial P divides

$$H_0 - H + H_1X + \ldots + H_mX^m$$

in $\mathbf{K}[X]$. There exist $y_0, y_1, \ldots, y_r$ in $\mathbf{K}$ such that

$$H_0 - H + H_1X + \ldots + H_mX^m = P(X)(y_0 + y_1X + \ldots + y_rX^r).$$

The coefficients of P are pairwise coprime. The product $P(X)(y_0 + y_1X + + \ldots + y_rX^r)$ is in $\mathbf{A}[X]$. By Gauss's lemma, the y_i's are in $\mathbf{A}$. In particular,

$$H_0 - H = A_0y_0, \quad A_0 \in \mathbf{A}, \ y_0 \in \mathbf{A}.$$

As $H_0 - H \neq 0$, $\deg(H_0 - H) \geqq \deg(A_0)$. As $H_0 \in C_H$, $\deg(H_0 - H) = \deg(H)$. Hence, we have

$$\text{(iv)} \qquad \deg H \geqq \deg A_0,$$

and the first part of (2) is proved.

For the same reasons, we deduce from (iii) that there exist $Z_0, Z_1, \dots, Z_t$ in $\mathbf{A}$ such that

$$-A_0 + R_0X + \ldots + R_sX^{s+1} = P(X)(Z_0 + Z_1X + \ldots + Z_tX^t).$$

We put

$$Q(X) = -A_0 + R_0X + \ldots + R_sX^{s+1},$$

$$Z(X) = Z_0 + Z_1X + \ldots + Z_tX^t.$$

For $i = 0, \dots, s$, we have

$$\deg R_i < \deg H = \deg A_0.$$

If $s = 0$, then, $A_0 = R_0\beta$, $\beta \in \mathbf{K}$, $A_0 - R_0X$ is the minimal polynomial of β over $\mathbf{A}$, and the second part of (2) is proved. Now, we suppose $s \geqq 1$. Let ρ be a real number such that

$$\text{(v)} \qquad 0 < \rho < \min_{1 \leqq i \leqq s} \left(\frac{\deg A_0 - \deg R_i}{i} \right).$$

Let v be the valuation on the field $\mathbf{K}$ given by

$$v(A/B) = \deg B - \deg A$$

if A and B are non-zero polynomials in $\mathbf{A}$. There exists one and only one valuation w on the field $\mathbf{K}(X)$ such that for every

$$B = B_0 + \ldots + B_pX^p$$

in $\mathbf{K}[X]$,

$$w(B) = \min_{0 \leqq j \leqq p} \left(v(B_j) - j\rho \right).$$

A proof of this can be found in [2]. The choice of ρ implies that

$$w(Q) = -\deg A_0, \qquad w(Z) \leqq 0,$$

so,

$$w(P) \geqq -\deg A_0,$$

and, for $j = 1, 2, \ldots, n$ we have

$$v(A_j) > v(A_j) - j\rho \geqq w(P) \geqq -\deg A_0,$$

and the second part of (2) is proved.

To prove the converse, we consider β algebraic over $\mathbf{K}$ whose minimal polynomial $P = A_0 + A_1 X + \ldots + A_n X^n$ satisfies the condition

$$\text{(vi)} \qquad \deg A_i < \deg A_0.$$

Then

$$A_0 = -A_1\beta - \ldots - A_n\beta^n$$

is a representation of $H = A_0$ in the form (II.2). Moreover, if $M \in \boldsymbol{C}_H$ then

$$M = M\mathbf{1}_{\mathbf{B}}$$

is a representation of M in the form (II.2). Let $M \in \mathbf{A}$ be such that $\deg M = \deg H$. Let $\operatorname{sgn}(M)$ and $\operatorname{sgn}(H)$ be the leading coefficient in M and in H, resp. Let

$$M_0 = M - \operatorname{sgn}(M)\operatorname{sgn}(H)^{-1}H.$$

Then

$$M = M_0 + \operatorname{sgn}(M)\operatorname{sgn}(H)^{-1}A_1\beta + \ldots + \operatorname{sgn}(M)\operatorname{sgn}(H)^{-1}A_n\beta^n$$

is a representation of M in the form (II.2).

Let $d \geqq \deg H$ be an integer. Suppose that every polynomial of degree less than d admits a representation of the form (II.2). Let $M \in \mathbf{A}$ be a polynomial of degree $d+1$. Write

$$\text{(vii)} \qquad M = \operatorname{sgn}(M)T^{d+1} + M', \quad \deg M' \leqq d.$$

Then by the induction hypothesis, there exist integers $r \geqq 0$, $s \geqq 0$ and polynomials $Y_0, \ldots, Y_r$, $Z_0, \ldots . Z_s$ in $\boldsymbol{C}_H$ such that

$$T^d = Y_0 + Y_1\beta + \ldots + Y_r\beta^r,$$

$$M' = Z_0 + Z_1\beta + \ldots + Z_s\beta^s.$$

We divide the set $\{0, \ldots, r\}$ in two parts $\mathbf{I}$ and $\mathbf{J}$, defined by:

$$i \in \mathbf{I} \Leftrightarrow 1 + \deg Y_i < \deg H,$$

$$i \in \mathbf{J} \Leftrightarrow 1 + \deg Y_i = \deg H.$$

If the set $\mathbf{J}$ is not empty, then for every $i \in \mathbf{J}$ there exist an integer m_i and polynomials $Y_{i,0}, \dots, Y_{i,m_i}$ in $\boldsymbol{C}_H$ such that

$$Y_i T = Y_{i,0} + Y_{i,1}\beta + \dots + Y_{i,m_i}\beta^{m_i}.$$

Finally, (vii) gives

$$M = \sum_{i=0}^{S} Z_i \beta^i + \sum_{i \in \mathbf{I}} \operatorname{sgn}(M) Y_i X \beta^i + \sum_{i \in \mathbf{J}} \sum_{j=0}^{m_i} \operatorname{sgn}(M) Y_{i,j} \beta^{i+j}$$

and this is a representation of M in the form (II.2).

We have proved that every $M \in \mathbf{A}$ admits a representation of the form (II.2). Then every $z \in \mathbf{A}[\beta]$ admits a representation of the form (II.2). It remains to prove the unicity of these representations. To this end, it suffices to prove that 0 admits only the trivial representation. Suppose that there exist an integer $m \geqq 0$ and $Y_0, \dots, Y_m$ in $\boldsymbol{C}_H$, such that

(viii) $$0 = Y_0 + Y_1\beta + \dots + Y_m\beta^m, \quad Y_m \neq 0.$$

Among these non-trivial representations of 0, we choose one for which the length m is minimal. As at the beginning of the proof of this theorem, we can prove that there exists a polynomial $Q(X)$ in $\mathbf{A}[X]$ such that

$$Q(X)P(X) = Y_0 + Y_1 X + \dots + Y_m X^m.$$

This implies that A_0 divides Y_0 in the ring $\mathbf{A}$, but $A_0 = H$ and $Y_0 \in \boldsymbol{C}_H$, so $Y_0 = 0$. Then, (viii) gives us

$$0 = Y_1 + Y_2\beta + \dots + Y_m\beta^{m-1}, \quad Y_m \neq 0,$$

which is non-trivial representation of 0 whose length is $m - 1$. This is in contradiction with the choice of m.

We have proved that (β, A_0) is a canonical $\mathbf{A}$-system in $\mathbf{A}[\beta]$. If the degree of $Y \in \mathbf{A}$ is equal to $\deg A_0$, then $\boldsymbol{C}_Y = \boldsymbol{C}_{A_0}$ and (β, Y) is also a canonical $\mathbf{A}$-system in $\mathbf{A}[\beta]$.

COROLLARY V.3. *Let β be an algebraic element over* $\mathbf{K}$. *Then the ring* $\mathbf{A}[\beta]$ *is a* **C.A.S** *ring.*

PROOF. Let $P(X) = A_0 + A_1 X + \dots + A_n X^n$ be the minimal polynomial of β over $\mathbf{A}$. Let $M \in \mathbf{A}$ and $\alpha = \beta - M$. Then

(i) $$\mathbf{A}[\beta] = \mathbf{A}[\alpha].$$

The polynomial

(ii) $$Q(X) = Q_0 + Q_1 X + \ldots + Q_n X^n$$

where, for $i = 0, \ldots, n$,

$$Q_j = \sum_{i=j}^{n} \binom{i}{j} A_i M^{i-j},$$

admits the root α. The polynomials $Q_0, \ldots, Q_n$ are coprime, therefore, $Q(X)$ is the minimal polynomial of α over $\mathbf{A}$. If we choose M such that for every $i = 0, \ldots, n-1$, $\deg M > \deg A_i$, then, for every $j = 1, \ldots, n$, $\deg Q_j < \deg Q_0$. Theorem V.2 tells us that (α, Q_0) is a canonical $\mathbf{A}$-system in $\mathbf{A}[\alpha]$. We conclude with (i).

COROLLARY V.4. *Let* $\mathbf{B}$ *be the integral closure of* $\mathbf{A}$ *in an algebraic extension* $\mathbf{K}(\Theta)$ *of* $\mathbf{K}$. *Then* $\mathbf{B}$ *is a* **C.A.S.** *ring if and only if there exists* $\beta \in \mathbf{B}$ *such that* $\mathbf{B} = \mathbf{A}[\beta]$.

PROOF. Immediate.

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ON THE LEBESGUE FUNCTION OF HERMITE–FEJÉR INTERPOLATION FOR LAGUERRE ABSCISSAS

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1. Introduction

Let

$$L_n^{(\alpha)}(x) = e^x x^{-\alpha}(e^{-x}x^{n+\alpha})^{(n)}/n!, \qquad n = 1, 2, \dots$$

be the Laguerre polynomial of degree n for $\alpha > -1$ with the usual normalization

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n},$$

and with zeros

$$0 < x_{1n} < x_{2n} < \dots < x_{nn}. \tag{1}$$

For $f(x)$ defined on $[0, \infty)$ the Hermite–Fejér interpolation based on the nodes (1) has the form

$$H_n^{(\alpha)}(f; x) = \sum_{k=1}^{n} f(x_k) v_k(x) l_k^2(x), \tag{2}$$

where

$$v_k(x) = \left[(x - x_k)(\alpha - x_k) + x\right]/x_k, \qquad k = 1, \dots, n,$$

$$l_k(x) = L_n^{(\alpha)}(x)/\left[L_n'^{(\alpha)}(x_k)(x - x_k)\right], \qquad k = 1, \dots, n.$$

Szegö [1], Vértesi [2], and Szabados [3] investigated the convergence of (2). For the convergence of (2), it is well-known that the estimation of the Lebesgue function

$$\Lambda_n^{(\alpha)}(x) = \sum_{k=1}^{n} |v_k(x)| l_k^2(x) \tag{3}$$

and the Lebesgue constant

$$\Lambda_n^{(\alpha)} = \max_{0 \leqq x \leqq A} \Lambda_n^{(\alpha)}(x),$$

where A is an arbitrary but fixed positive number, plays an important role. Szegö [1] and Szabados [3] gave the following estimate

$$\Lambda_n^{(\alpha)} = O\left(n^{1/2(\alpha+|\alpha|)}\right).$$

The purpose of our paper is to establish a two sided uniform estimate of the Lebesgue function (3).

In what follows, the sign "$A_n(x) \sim B_n(x)$" means that there exist two positive constants $C_1 < C_2$ independent of n and x such that

$$C_1 B_n(x) < A_n(x) < C_2 B_n(x).$$

Throughout the paper, C_1, C_2 and C always denote constants independent of n and x, but not the same at each appearance.

2. Main result

Our main result is the following

THEOREM 2.1. *For $\alpha > -1$ and $x \in [0, A]$ $(A > 0)$, the estimate*

$$\left[\Lambda_n^{(\alpha)}(x) - 1\right] \sim n^{-\alpha}\left(L_n^{(\alpha)}(x)\right)^2\left(x^2 - 2(\alpha+2)x + \alpha^2\right)_+ \tag{4}$$

uniformly holds, where

$$\left(f(x)\right)_+ \overset{\text{def}}{=} \begin{cases} f(x), & \text{if} \quad f(x) > 0 \\ 0, & \text{if} \quad f(x) \leqq 0. \end{cases}$$

COROLLARY 2.2. *The following estimates are valid:*

$$\Lambda_n^{(\alpha)} = O(n^\alpha) \qquad (\alpha \geqq 0), \tag{5}$$

$$\Lambda_n^{(\alpha)} = 1 + O(n^{-1/2}) \qquad (-1 < \alpha < 0). \tag{6}$$

3. Lemma and preliminaries

LEMMA 3.1 [1]. *Let $\alpha > -1$. Then the following asymptotic relation holds for the zeros $x_k = x_{kn}$ of $L_n^{(\alpha)}(x)$:*

$$x_k \sim k^2/n, \quad k = 1, \dots, n; \quad n = 1, 2, \dots . \tag{7}$$

We also need the following known results (cf [1]):

$$L_n^{(\alpha)}(x) = \begin{cases} O(n^\alpha), & 0 \leqq x < C/n, \\ x^{-\alpha/2-1/4} O(n^{\alpha/2-1/4}), & C/n \leqq x \leqq A, \end{cases} \tag{8}$$

$$L_n^{\prime(\alpha)}(x_k) \sim x_k^{-\alpha/2-3/4} n^{\alpha/2+1/4} \sim k^{-\alpha-3/2} n^{\alpha+1} \qquad (0 < x_k \leqq 2A). \tag{9}$$

4. Proofs

PROOF OF THEOREM 2.1. If $x = x_k$ $(k = 1, \dots, n)$, (4) holds. Now, suppose $x \neq x_k$ $(k = 1, \dots, n)$. Let

$$a = a(\alpha) := \alpha + 2 - 2\sqrt{\alpha + 1}$$

and

$$b = b(\alpha) := \alpha + 2 + 2\sqrt{\alpha + 1}$$

be the zeros of the quadratic polynomial

$$\Delta = \Delta(x) := x^2 - 2(\alpha + 2)x + \alpha^2.$$

Now we shall distinguish and discuss three cases.

Case (I): $a \leqq x \leqq b$. Then $\Delta = (x - a)(x - b) \leqq 0$. Therefore,

$$(x - x_k)(\alpha - x_k) + x = x_k^2 - (x + \alpha)x_k + (1 + \alpha)x \geqq 0$$

and

$$v_k(x) = \left[(x - x_k)(\alpha - x_k) + x\right]/x_k \geqq 0.$$

Hence, according to the property of interpolation (2), we have

$$\Lambda_n^{(\alpha)}(x) = \sum_{k=1}^{n} \left| v_k(x) \right| l_k^2(x) = 1. \tag{10}$$

If $0 \leqq x < a$ or $b < x \leqq A$, then $\Delta > 0$. Denote by

$$c = c(x,\alpha) := 1/2(x + \alpha - \sqrt{\Delta})$$

and

$$d = d(x,\alpha) := 1/2(x + \alpha + \sqrt{\Delta})$$

the roots of the equation

$$y^2 - (x+\alpha)y + (1+\alpha)x = 0.$$

Therefore, if $0 < x_k \leqq c$ or $x_k \geqq d$, we have

$$v_k(x) = \left[x_k^2 - (x+\alpha)x_k + (1+\alpha)x\right]/x_k \geqq 0.$$

And, if $c < x_k < d$,

$$v_k(x) < 0.$$

Observing the equation

$$\sum_{k=1}^{n} v_k(x) l_k^2(x) = 1$$

and the above results, we obtain

$$\Lambda_n^{(\alpha)}(x) - 1 = \sum_{k=1}^{n} \left| v_k(x) \right| l_k^2(x) - \sum_{k=1}^{n} v_k(x) l_k^2(x) = \tag{11}$$

$$= 2 \sum_{c < x_k < d} l_k^2(x)(x_k - c)(d - x_k)/x_k.$$

Case (II): $b < x \leqq A$. Observing that

$$x_k < d = 1/2\left(x + \alpha + \sqrt{(x-a)(x-b)}\right) \leqq x - 1$$

and

$$x_k > c = 1/2\left(x + \alpha - \sqrt{(x-a)(x-b)}\right) \geqq \alpha + 1 > 0,$$

we have

$$1 \leqq x - x_k < A$$

and

$$\alpha + 1 \leqq x_k \leqq d(A,\alpha) = 1/2\left(A + \alpha + \sqrt{\Delta(A)}\right).$$

That is

$$(12) \qquad |x - x_k| \sim 1, \qquad x_k \sim 1.$$

Therefore, (9), (11)–(12) yield

$$(13) \qquad \left[\Lambda_n^{(\alpha)}(x) - 1\right] \sim n^{-\alpha-1/2}\left(L_n^{(\alpha)}(x)\right)^2 \sum_{c<x_k<d} (x_k - c)(d - x_k).$$

Set

$$\mathbf{P} := \left(c, 1/2\left(x + \alpha - 1/2\sqrt{\Delta}\right)\right),$$

$$\mathbf{Q} := \left[1/2\left(x + \alpha - 1/2\sqrt{\Delta}\right), 1/2\left(x + \alpha + 1/2\sqrt{\Delta}\right)\right]$$

and

$$\mathbf{R} := \left(1/2\left(x + \alpha + 1/2\sqrt{\Delta}\right), d\right).$$

Write

$$(14) \qquad \sum_{c<x_k<d} (x_k - c)(d - x_k) = \sum_{x_k \in \mathbf{P}} + \sum_{x_k \in \mathbf{Q}} + \sum_{x_k \in \mathbf{R}} = R_1 + R_2 + R_3.$$

First, we estimate R_2. Since $x_k \in \mathbf{Q}$,

$$(15) \qquad (x_k - c)(d - x_k) \sim \Delta.$$

Since $x_k \in \mathbf{Q}$ and (7) holds, there exist two constants $0 < C_1 < C_2$ such that

$$\sqrt{(1/2)n\left(x + \alpha - 1/2\sqrt{\Delta}\right)/C_2} \leqq k \leqq \sqrt{(1/2)n\left(x + \alpha + 1/2\sqrt{\Delta}\right)/C_1}.$$

From (15), it follows that

$$(16) \qquad R_2 \sim \Delta\left\{\left\langle\sqrt{(1/2)n\left(x + \alpha + 1/2\sqrt{\Delta}\right)/C_1}\right\rangle - \right.$$

$$\left. - \left\langle\sqrt{(1/2)n\left(x + \alpha - 1/2\sqrt{\Delta}\right)/C_2}\right\rangle + 1\right\},$$

where $\langle\cdot\rangle$ denotes the greatest integer part.

It is easy to prove that

$$(17) \qquad \left\{\left\langle\sqrt{(1/2)n\left(x + \alpha + 1/2\sqrt{\Delta}\right)/C_1}\right\rangle - \right.$$

$$-\left\langle\sqrt{(1/2)n\left(x+\alpha-1/2\sqrt{\Delta}\right)/C_2}\right\rangle+1\Big\}\sim\sqrt{n}.$$

Then, (16)–(17) yield

$$R_2\sim\sqrt{n}\Delta. \tag{18}$$

For R_3, we have

$$R_3\leqq\Delta\sum_{x_k\in\mathbf{R}}1\leqq\Delta\sum_{1\leqq k\leqq C\sqrt{n}}1\leqq C\sqrt{n}\Delta. \tag{19}$$

Similarly,

$$R_2\leqq C\sqrt{n}\Delta. \tag{20}$$

Combining (13)–(14) and (18)–(20), it follows that

$$\left[\Lambda_n^{(\alpha)}(x)-1\right]\sim n^{-\alpha}\Delta\left(L_n^{(\alpha)}(x)\right)^2\qquad(x>b). \tag{21}$$

Case (III): $0\leqq x<a$. If $\alpha=0$, then $a=0$. So this case does not exist. If $\alpha<0$, since

$$0\leqq x<a=\alpha+2-2\sqrt{\alpha+1}<-\alpha,$$

then

$$x+\alpha<0.$$

From this, it is easy to prove that

$$\sqrt{\Delta}\leqq-(x+\alpha).$$

Hence,

$$c<d=1/2\left(x+\alpha+\sqrt{\Delta}\right)\leqq 0.$$

So this case also does not exist.

Now, suppose $\alpha>0$ and let

$$e=e(\alpha):=1+1.5/\alpha,\quad f=f(\alpha):=1+2/\alpha,\quad g=g(\alpha):=\alpha^2/(4.5+3\alpha).$$

First, assume $0\leqq x<g$. It is evident that

$$x\leqq c\leqq ex\leqq fx<d. \tag{22}$$

Write

$$\sum_{c<x_k<d} l_k^2(x)(x_k-c)(d-x_k)= \tag{23}$$

$$=\sum_{c<x_k\leqq fx}+\sum_{fx<x_k<1/2(x+\alpha+1/2\sqrt{\Delta})}+\sum_{1/2(x+\alpha+1/2\sqrt{\Delta})\leqq x_k<d}=r_1+r_2+r_3.$$

When $x=0$, r_1 vanishes. First, we estimate r_2. Since

$$x\leqq fx<x_k, \qquad f>1,$$

then

$$(1-1/f)x_k=x_k-x_k/f<x_k-x\leqq x_k,$$

that is

$$(x_k-x)^2\sim x_k^2. \tag{24}$$

From (22) we obtain

$$x_k\geqq x_k-c\geqq x_k-ex\geqq(1-e/f)x_k.$$

Hence,

$$(x_k-c)\sim x_k \tag{25}$$

and

$$d\geqq d-x_k\geqq 1/4\sqrt{\Delta}\geqq 1/4(a-g)>0. \tag{26}$$

Therefore

$$(d-x_k)\sim 1. \tag{27}$$

Considering (24)–(25) and (27) and using (7) and (9), we obtain

$$r_2\sim\left(L_n^{(\alpha)}(x)\right)^2\sum_{fx<x_k<1/2(x+\alpha+1/2\sqrt{\Delta})}k^{2\alpha+1}n^{-2\alpha-1}.$$

Observing (26), we obtain

$$r_2\sim n^{-\alpha}\Delta\left(L_n^{(\alpha)}(x)\right)^2. \tag{28}$$

For R_3, since

$$d - x_k < 1/4\sqrt{\Delta}$$

and

$$x_k - x \geqq 1/2\left(x + \alpha + 1/2\sqrt{\Delta}\right) - c = 3\sqrt{\Delta}/4 \geqq 3(a-g)/4 > 0,$$

we have

$$r_3 \leqq C\Delta n^{-2\alpha-2}\left(L_n^{(\alpha)}(x)\right)^2 \sum_{0<x_k<d} k^{2\alpha+3} \leqq Cn^{-\alpha}\Delta\left(L_n^{(\alpha)}(x)\right)^2. \tag{29}$$

For r_1, observing (22) and $x > 0$, we have

$$x_k \sim x \tag{30}$$

and

$$(f-1)x \geqq x_k - x \geqq c - x = 2x/\left(\alpha - x + \sqrt{\Delta}\right).$$

Since

$$\alpha - x \geqq \sqrt{\Delta},$$

then

$$(x_k - x) \sim x. \tag{31}$$

Applying (9) and (30)–(31), we obtain

$$r_1 \leqq Cx^{\alpha-3/2}n^{-\alpha-1/2}\left(L_n^{(\alpha)}(x)\right)^2 \sum_{c<x_k<fx} (x_k - c)(d - x_k).$$

It is obvious that

$$x_k - c \leqq (f-1)x$$

and

$$d - x_k \leqq C.$$

Noting the above, we obtain

$$r_1 \leqq Cx^{\alpha-1/2}n^{-\alpha-1/2}\left(L_n^{(\alpha)}(x)\right)^2 \sum_{x<x_k<fx} 1 \leqq \tag{32}$$

$$\leqq Cx^{\alpha}n^{-\alpha}L_n^{(\alpha)}(x) \leqq Cn^{-\alpha}\Delta L_n^{(\alpha)}(x).$$

Combining (11), (23), (28)–(29) and (32) yield (4) for $0 \leqq x < g$.

For $0 < g \leqq x < a$, we can also prove (4) by a similar method used in the proof of Case (II). Q.E.D.

PROOF OF COROLLARY 2.2. (5) follows from (4) and (8). Now suppose $-1 < \alpha < 0$. According to the proof of Theorem 2.1, the case $0 \leqq x < a$ does not exist. So we only have two cases (i) $a \leqq x \leqq b$ and (ii) $x > b$.

For case (i),

$$\left(x^2 - 2(\alpha + 2)x + \alpha^2\right)_+ = 0.$$

Therefore, (6) holds. If $x > b > 1$, using the second estimate of (8), we obtain

$$\Lambda_n^{(\alpha)}(x) - 1 = O(n^{-1/2}).$$

From the above, (6) follows. Q.E.D.

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GEOMETRY OF POSITIVE COMPACT OPERATORS ON l^p

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1. Introduction

In recent years geometric properties of spaces of operators acting on various Banach spaces were intensively investigated.

In this paper [1] we would like to continue this investigation for the space of compact operators acting on l^p. Namely we deal with the positive part of the unit ball. We give the description of its extreme, exposed, strongly exposed and smooth points.

For a Banach space E we denote by $B(E)$ and $S(E)$ the unit ball and the unit sphere of E, respectively. If E is an order space, then the positive part of the unit ball (sphere) is denoted by $B_+(E)$ $\big(S_+(E)\big)$. We denote by l^p $(1 \leqq p \leqq \infty)$ the Banach lattice of p-summable sequences equipped with the standard l^p norm and order. We denote $e_1 = (1,0,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, $\ldots$. The dual space of l^p $(1 \leqq p < \infty)$ is identified with l^q where $1/p + 1/q = 1$. If $1 < p < \infty$ then l^p is strictly convex, hence $\operatorname{ext} B_+(l^p) = \{0\} \cup S_+(l^p)$. Moreover, $\operatorname{ext} B_+(l^1) = \{0\} \cup \{e_i : i \in \mathbf{N}\}$.

If E is a Banach space then we denote by $\mathfrak{K}(E)$ the Banach space of all linear compact operators from E into E equipped with the operator norm. For $z \in E^*$ and $y \in E$ we denote by $z \otimes y$ the one dimensional operator defined by $(z \otimes y)(x) = y\langle z, x\rangle$, $x \in E$. To every operator $T \in \mathfrak{K}(l^p)$ there corresponds a unique matrix (t_{ji}) with scalar entries such that $(Tx)_j = \sum_{i=1}^{\infty} t_{ji}x_i$. The adjoint operator $T^* \in \mathfrak{K}(l^q)$ is determined in the same manner by the transposed matrix. An operator T is said to be positive, denoted by $T \geqq 0$, if $Tf \geqq 0$ whenever $f \geqq 0$. Obviously $T = (t_{ji}) \geqq 0$ if and only if $t_{ji} \geqq 0$ for all $i, j \in \mathbf{N}$. In particular the set of positive operators (contractions) in $\mathfrak{K}(l^p)$ is denoted by $\mathfrak{K}_+(l^p)\big(B_+\big(\mathfrak{K}(l^p)\big)\big)$. We denote the support of $f \in l^p$ by $\operatorname{supp} f = \{i : f_i \neq 0\}$. For a matrix $T = (t_{ji})$ we define the support of T as $\operatorname{supp} T = \{i \in \mathbf{N}$: there exists $j \in \mathbf{N}$ such that $t_{ji} \neq 0\}$. The support of the transposed matrix is denoted by $\operatorname{supp} T^*$.

[1] Written while the author was a research fellow of the Alexander von Humboldt-Stiftung at Mathematisches Institut der Eberhard-Karls-Universität in Tübingen

For $f=(f_i)\in l^p$ $(1\leqq p<\infty)$ we define $f^{p-1}=\left(|f_i|^{p-1}\operatorname{sign} f_i\right)$. Note that if $\|f\|_p=1$ then f^{p-1} is the unique functional such that $\|f^{p-1}\|_q= =\langle f,f^{p-1}\rangle=1$. If an operator $T\in\mathfrak{N}(l^p)$, $p\in(1,\infty)$, attains its norm at $f\in S(l^p)$ then from the strict convexity it follows that $T^*(Tf)^{p-1}=f^{p-1}$. This equality for positive operators gives us additional information about the operator. Namely, if $T\in\mathfrak{N}_+(l^p)$, $1<p<\infty$, and $T^*(Tf)^{p-1}=f^{p-1}$, for some $f\in S_+(l^p)$ with $\operatorname{supp} f=\operatorname{supp} T$, then $\|T\|=1$ (Proposition 1 in [5]).

It is easy to see that for $T=(t_{ji})\in\mathfrak{N}(l^1)$ we have:

$T\in\operatorname{ext} B_+\left(\mathfrak{N}(l^1)\right)$ *if and only if* $t_{ji}=0$ *or* 1 *and in every column of* (t_{ji}) *at most one entry is equal to* 1.

Note that in this case compactness of T implies that only finitely many entries of the extreme T are equal to one. Moreover $\exp B_+\left(\mathfrak{N}(l^1)\right)= =\operatorname{ext} B_+\left(\mathfrak{N}(l^1)\right)$. Indeed, the extreme operator T is exposed by the functional ξ defined by $\xi(R)=\sum_{i,j=1}^{\infty}\alpha_{ji}(2\operatorname{sign} t_{ji}-1)$, where $\alpha_{ji}>0$ is such that $\sum_{i,j=1}^{\infty}\alpha_{ji}<\infty$.

A graph $G(T)$ is associated with each matrix $T=(t_{ji})$ which is defined by the following formula. To the i-th column there corresponds a node u_i, and to the j-th row there corresponds a node v_j. There is an edge joining u_i and v_j if and only if $t_{ji}\neq 0$. We say that the matrix $T\geqq 0$ has no cycle of positive entries provided that every connected component of the graph $G(T)$ is a tree (i.e. the graph $G(T)$ has no cycle). If the graph $G(T)$ is connected we say that the operator T is elementary.

2. Extreme operators

LEMMA 1. *Let* $1\leqq p<\infty$. *If* $0\neq T\in\operatorname{ext} B_+\left(\mathfrak{N}(l^p)\right)$ *then there exists* $f\in S_+(l^p)$ *such that* $\|Tf\|=1$ *and* $\operatorname{supp} f=\operatorname{supp} T$.

PROOF. For $p=1$ it is easy to check it directly. Now assume that $1< <p<\infty$. Let $T=\sum_{k=1}^{k_0}T_k$ (k_0 is finite or infinite) be a decomposition of T into elementary operators $T_k\neq 0$ with $\operatorname{supp} T_k$ pairwise disjoint and $\operatorname{supp} T_k^*$ pairwise disjoint. Then $\operatorname{supp} T=\cup\operatorname{supp} T_k$ and $\operatorname{supp} T^*=\cup\operatorname{supp} T_k^*$. We have $\|T\|=\sup\|T_k\|$, and $T\geqq 0$ if and only if $T_k\geqq 0$ for all k. Therefore $T\in\operatorname{ext} B_+\left(\mathfrak{N}(l^p)\right)$ if and only if $T_k\in\operatorname{ext} B_+\left(\mathfrak{N}(l^p)\right)$ for all k. For every T_k (since T_k is elementary and compact) there exists $f_k\in S_+(l^p)$ such that $\|Tf_k\|=1$ and $\operatorname{supp} f_k=\operatorname{supp} T_k$ (see [5, Theorem 4]. Fix $\alpha_k>0$ such that $\sum_{k=1}^{k_0} \alpha_k^p = 1$. Put $f = \sum_{k=1}^{k_0} \alpha_k f_k$. We have $f_k \in S_+(l^p)$, $\|Tf\| = 1$ and supp $f =$ supp T as required.

We recall some facts about doubly stochastic matrices. Let $\mathbf{r} = (r_j), \mathbf{s} = (s_i) \in S_+(l^1)$. A matrix $P = (p_{ji})$ is called doubly stochastic with respect to $(\mathbf{s}, \mathbf{r})$ if 1^0 $p_{ji} \geqq 0$, 2^0 $\sum_{k=1}^{\infty} p_{ki} = s_i$, 3^0 $\sum_{k=1}^{\infty} p_{jk} = r_j$ for all $i, j \in \mathbf{N}$. We denote by $\mathbf{D}(s, r)$ the set of all doubly stochastic matrices with respect to $(\mathbf{s}, \mathbf{r})$. For $P \in \mathbf{D}(s, r)$ we have $P \in$ ext $\mathbf{D}(s, r)$ if and only if the matrix (p_{ji}) has no cycle of positive entries. We say that a matrix $P \in \mathbf{D}(\mathbf{s}, \mathbf{r})$ is uniquely determined in $\mathbf{D}(\mathbf{s}, \mathbf{r})$ by its graph if for $P, P_1 \in \mathbf{D}(\mathbf{s}, \mathbf{r})$ the condition that $G(P_1)$ is a subgraph of $G(P)$ $\big(G(P_1) \subseteqq G(P)\big)$ implies that $P = P_1$. In fact, the extreme points of $\mathbf{D}(\mathbf{s}, \mathbf{r})$ are those matrices in $\mathbf{D}(\mathbf{s}, \mathbf{r})$ which are uniquely determined in $\mathbf{D}(\mathbf{s}, \mathbf{r})$ by their graphs (see [6]). For other characterizations of extreme doubly stochastic matrices see [15, 14, 3, 2, 1, 16, 8].

THEOREM 1. *Let $1 < p < \infty$ and let $T \in B_+\big(\mathfrak{R}(l^p)\big)$. Then $T = (t_{ji}) \in$ ext $B_+\big(\mathfrak{R}(l^p)\big)$ if and only if*

1^0 $\|T\| = 0$ *or* 1, *and*

2^0 *the matrix has no cycle of positive entries, and*

3^0 *there exists $f \in S_+(l^p)$ with* supp $f =$ supp T *such that* $\|T_f\| = 1$.

PROOF. Obviously the zero operator is an extreme point of $B_+\big(\mathfrak{R}(l^p)\big)$ and every non zero extreme operator has norm one. Let $\|T\| = 1$. In view of Lemma 1, if T is extreme then 3^0 holds, i.e. there exists $f = (f_i) \in S_+(l^p)$ such that $\|Tf\| = 1$. Put $g = (g_j) = (Tf)^{p-1}$. Obviously supp $g =$ supp T^* and $T^*(g) = f^{p-1}$. In particular, $\|f^{p-1}\|_q = \|T^*g\|_q = \|g\|_q = \|T^*\| = 1$.

Put

$$A = \left\{ (r_{ji}) \colon r_{ji} = \begin{cases} p_{ji}/f_i g_j & \text{if } f_i g_j \neq 0 \\ 0 & \text{otherwise} \end{cases}, (p_{ji}) \in \mathbf{D}\big((f_i^p), (g_j^q)\big) \right\}$$

Because $\sum_{i=1}^{\infty} t_{ji} f_i = g_j^{q-1}$ and $\sum_{j=1}^{\infty} t_{ji} g_j = f_i^{p-1}$ we have $(t_{ji} g_j f_i) \in \mathbf{D}\big((f_i^p), (g_j^q)\big)$. Hence $T \in A$. Let $R = (r_{ji}) \in A$. Then $R \geqq 0$, $Rf = g^{q-1}$, $R^* g = f^{p-1}$ with supp $f =$ supp R and $\|f\|_p = 1$. Hence by Proposition 1 in [5], $\|R\| = 1$. Therefore $A \subseteqq B_+\big(\mathfrak{R}(l^p)\big)$. Let $T = \alpha T_1 + (1 - \alpha) T_2$ for some $\alpha \in (0, 1)$ and $T_k = \big(t_{ji}^k\big) \in B_+\big(\mathfrak{R}(l^p)\big)$, $k = 1, 2$. We have supp $T_k \subseteqq$ supp T and supp $T_k^* \subseteqq$ supp T^* since $t_{ji} = 0$ implies $t_{ji}^k = 0$. By the strict convexity of l^p and l^q we have $T_k f = g^{q-1}$ and $T_k^* g = f^{p-1}$. Hence $T_k \in A$. This means that A is a face of $B_+\big(\mathfrak{R}(l^p)\big)$, and $T \in$ ext A if and only if $T \in$ ext $B_+\big(\mathfrak{R}(l^p)\big)$. The correspondence

$$A \ni (r_{ji}) \longrightarrow (r_{ji} f_i g_j) \in \mathbf{D}\big((f_i^p), (g_j^q)\big)$$

is an affine bijection preserving zero entries. We have $(t_{ji}) \in \operatorname{ext} A$ if and only if $(t_{ji} f_i g_j) \in \operatorname{ext} \mathbf{D}\big((f_i^p),(g_j^q)\big)$. Moreover, $(t_{ji} f_i g_j) \in \operatorname{ext} \mathbf{D}\big((f_i^p),(g_j^q)\big)$ if and only if the matrix $(t_{ji} f_i g_j)$ (so also (t_{ji})) has no cycle of positive entries. Conversely, if 1^0, 2^0 and 3^0 hold we can define the face A and because of 2^0 T is an extreme point of A, so is $B_+\big(\mathfrak{R}(l^p)\big)$, which ends the proof.

In view of Theorem 4 in [5], if $T \in \mathfrak{R}(l^p)$ $(1 < p < \infty)$ is elementary then there exists $f \in S_+(l^p)$ such that $\|Tf\| = \|T\|$ and $\operatorname{supp} f = \operatorname{supp} T$. Hence we can easily obtain the following result.

COROLLARY 1. *Let* $1 < p < \infty$ *and let* $T \in S_+\big(\mathfrak{R}(l^p)\big)$ *be elementary. Then* $T = (t_{ji}) \in \operatorname{ext} B_+\big(\mathfrak{R}(l^p)\big)$ *if and only if the matrix has no cycle of positive entries.*

Note that the above theorem extends analogous results for finite dimensional l^p spaces in [4]. The description of extreme positive l^p-contraction in the general case is more complicated and is presented in [7,9].

THEOREM 2. *Let* $1 \leqq p < \infty$. *Then* $\operatorname{conv} \operatorname{ext} B_+\big(\mathfrak{R}(l^p)\big)$ *is norm dense in* $B_+\big(\mathfrak{R}(l^p)\big)$.

PROOF. Let $\mathfrak{R}_n(l^p)$ denote all operators $T \in \mathfrak{R}(l^p)$ concentrated on $[e_1, e_2, \dots, e_n]$ (i.e. $\operatorname{supp} T \cup \operatorname{supp} T^* \subseteqq \{1, 2, \dots, n\}$). It is easy to see that $\operatorname{ext} B_+\big(\mathfrak{R}_n(l^p)\big) \subseteqq \operatorname{ext} B_+\big(\mathfrak{R}(l^p)\big)$ for all $n \in \mathbf{N}$. Since $\mathfrak{R}_n(l^p)$ is finite dimensional $\operatorname{conv} \operatorname{ext} B_+\big(\mathfrak{R}_n(l^p)\big)$ is norm dense in $\mathfrak{R}_n(l^p)$. Since the finite dimensional operators are dense in $\mathfrak{R}(l^p)$ we have

$$B_+\big(\mathfrak{R}(l^p)\big) \subseteqq \overline{\bigcup_{n \in \mathbf{N}} B_+\big(\mathfrak{R}_n(l^p)\big)} \subseteqq \overline{\operatorname{conv}}\Big(\bigcup_{n \in \mathbf{N}} \operatorname{ext} B_+\big(\mathfrak{R}_n(l^p)\big)\Big) \subseteqq$$

$$\subseteqq \overline{\operatorname{conv}} \operatorname{ext} B_+\big(\mathfrak{R}(l^p)\big) \subseteqq B_+\big(\mathfrak{R}(l^p)\big).$$

In contrast to the above theorem recall that $\operatorname{ext} B\big(\mathfrak{R}(l^2)\big) = \emptyset$ and an analogous theorem for the whole unit ball of $B\big(\mathfrak{R}(l^p)\big)$ holds for $p \neq 2$ [11,17].

3. Exposed operators

We recall that a point q_0 in a convex set Q is called *exposed* if there exists a linear functional ξ such that $\xi(q_0) > \xi(q)$ for all $q \in Q \setminus \{q_0\}$. The set of all exposed points is denoted by $\exp Q$. Moreover $q_0 \in \exp Q$ is called *strongly exposed* if $\xi(q_n) \to \xi(q_0)$ implies that $\|q_n - q_0\| \to 0$. In fact $\operatorname{ext} B_+(l^p) = \exp B_+(l^p)$ for $1 \leqq p \leqq \infty$. It is easy to calculate that every exposed point of $B_+(l^1)$ (and $B(l^1)$) is in fact strongly exposed. Moreover because for $1 < p < \infty$ the space l^p is uniformly convex the set of strongly exposed points of $B(l^p)$ coincides with $S(l^p)$, hence also every point of $S_+(l^p)$ is a strongly exposed point of $B_+(l^p)$. We should point out here that 0 is an exposed but not strongly exposed point of $B_+(l^p)$, because 0 is a weak (but not strong) limit of the sequence $\{e_n\}_{n=1}^{\infty}$.

We shall use the fact that $\operatorname{ext} \mathbf{D}(\mathbf{s},\mathbf{r}) = \exp \mathbf{D}(\mathbf{s},\mathbf{r})$ [6, Proposition 2] in the proof of the next theorem.

THEOREM 3. *Let* $1 \leqq p < \infty$. *Then*

$$\exp B_+\big(\mathfrak{N}(l^p)\big) = \operatorname{ext} B_+\big(\mathfrak{N}(l^p)\big).$$

PROOF. For $p = 1$ it is presented in the introduction. Let $1 < p < \infty$. If $T \in \exp B_+\big(\mathfrak{N}(l^p)\big)$, then $T \in \operatorname{ext} B_+\big(\mathfrak{N}(l^p)\big)$. Let now $T \in \operatorname{ext} B_+\big(\mathfrak{N}(l^p)\big)$ with $\|T\| = 1$. Let $\alpha_{ji} > 0$ be such that $\sum_{i,j=1}^{\infty} \alpha_{ji} < \infty$. Let $f \in S_+(l^p)$ be such that $\|Tf\| = 1$ and $\operatorname{supp} f = \operatorname{supp} T$. We claim that a functional ξ exposes $B_+\big(\mathfrak{N}(l^p)\big)$ at T where

$$\xi(R) = \langle Rf, (Tf)^{p-1}\rangle - \sum_{i,j} \alpha_{ji} r_{ji}(1 - \operatorname{sign} t_{ji}), \quad R \in \mathfrak{N}(l^p).$$

Indeed, $\xi(R) \leqq \xi(T)$ for $R \in B_+\big(\mathfrak{N}(l^p)\big)$. Suppose that $\xi(R) = \xi(T)$. Then $Rf = Tf$. Hence $R^*\big((Tf)^{p-1}\big) = f^{p-1}$ and $R \in A$ (where A is defined in the proof of Theorem 1). Moreover $t_{ji} = 0$ implies that $r_{ji} = 0$, so the graph $G(R)$ is a subgraph of $G(T)$. Because the elements of $\operatorname{ext} \mathbf{D}\big((f_p^i)\big)$, $\big((Tf)_j^p)\big)$ are uniquely determined by its graph, so are elements of $\operatorname{ext} A$. Therefore we obtain $T = R$. Now one can easily see that the zero operator is also exposed.

THEOREM 4. *Let* $1 \leqq p < \infty$. *There are no strongly exposed points in* $B_+\big(\mathfrak{N}(l^p)\big)$.

PROOF. To get a contradiction, suppose that a functional ξ strongly exposes some $T \in \operatorname{ext} B_+\big(\mathfrak{N}(l^p)\big)$ with $\|T\| = 1$. In view of the characterization of extreme operators in each (except at most one) column of (t_{ji}) there exists infinitely many zero entries. Therefore there exists a sequence $\big\{(i_n, j_n)\big\}_{n=1}^{\infty}$ such that $i_n < i_{n+1}$, $j < j_{n+1}$ and $t_{j_n i_n} = 0$, $n \in \mathbf{N}$. We define $R_n = (r_{ij}^n)$ by

$$r_{ji}^n = \begin{cases} t_{ji} & \text{if } i = i_n \text{ or } j = j_n \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $0 \leqq \sum_{n=1}^{n_0} R_n \leqq T$ for all $n_0 \in \mathbf{N}$, so $\|\sum_{n=1}^{n_0} R_n\| \leqq 1$. Thus $\sum_{n=1}^{n_0} \xi(R_n) \leqq 1$. We have $\xi(R_n) \geqq 0$, since $T - R_n$ is a positive contraction and $\xi(T) - \xi(R_n) = \xi(T - R_n) \leqq \xi(T)$. Therefore $\xi(R_n)$ tends to 0 as n tends to ∞.

Let N_0 be arbitrary finite subset of $\mathbf{N}$. Because $\sum_{n\in N_0} e_{i_n} \otimes e_{j_n}$ is a positive contraction we have $\sum_{n\in N_0} \xi\left(e_{i_n} \otimes e_{j_n}\right) \leqq 1$. Thus $\xi\left(e_{i_n} \otimes e_{j_n}\right)$ tends to 0. Put $T_n = T - R_n + e_{i_n} \otimes e_{j_n}$. We have $T_n \in B_+\big(\mathfrak{R}(l^p)\big)$ and $\xi(T_n) = \xi(T) - \xi(R_n) + \xi\left(e_{i_n} \otimes e_{j_n}\right) \overset{n\to\infty}{\longrightarrow} 0$, but $\|T_n - T\| = \|e_{i_n} \otimes e_{j_n} - R_n\| \geqq 1$. This contradiction proves that T is not a strongly exposed point of $B_+\big(\mathfrak{R}(l^p)\big)$. Using an argument similar to the one presented above we can see that the zero operator is also not strongly exposed.

4. Quasi interior and smooth points

We say that T is a *smooth* point of $Q\big(Q = \mathfrak{R}_+(l^p)\big)$ or $B_+\big(\mathfrak{R}(l^p)\big)$ if there exists a unique (up to a multiplicative constant) functional $\xi_0 \neq 0$ such that $\xi_0(T) = \sup \xi_0(Q)$. The set of all smooth points of Q is denoted by smooth Q. Recall that a point $T \in Q$ is a support point of Q if there exists $\xi \neq 0$ such that $\xi(T) = \sup \xi(Q)$. An element which is not a support point is called a quasi-interior point of Q. The set of all quasi-interior points of Q is denoted by q-int Q. Obviously smooth $B(l^p) = S(l^p)$ for $1 < p < \infty$ and smooth $B(l^1) = \left\{f \in S(l^1)\colon \operatorname{supp} f = \mathbf{N}\right\}$. Moreover, $f = (f_i) \in$ smooth $B_+(l^p)$ $(1 \leqq p < \infty)$ if and only if 1^0 $f_i > 0$ and $\|f\| = 1$ and 2^0 there exists a unique $n \in \mathbf{N}$ such that $f_i > 0 = f_n$ for all $i \neq n$ and $\|f\| < 1$. Hence $f \in$ smooth l^p_+ $(1 \leqq p < \infty)$ if and only if there exists a unique $n \in \mathbf{N}$ such that $f_i > 0 = f_n$ for all $i \neq n$. We have

$$\text{q-int}\, B_+(l^p) = \{f\colon f_i > 0 \quad \text{for all} \quad i \in \mathbf{N} \quad \text{and} \quad \|f\| < 1\}$$

and

$$\text{q-int}\, l^p_+ = \{f\colon f_i > 0 \quad \text{for all} \quad i \in \mathbf{N}\}.$$

THEOREM 5. *Let* $1 \leqq p < \infty$. *Then* $t \in$ q-int $B_+\big(\mathfrak{R}(l^p)\big)$ *if and only if* $\|T\| < 1$ *and* $t_{ji} > 0$ *for all* $j, i \in \mathbf{N}$.

PROOF. Let $T \in$ q-int $B_+\big(\mathfrak{R}(l^p)\big)$. Then by the Hahn–Banach Theorem, $\|T\| < 1$. We define a functional κ_{ji} by $\kappa_{ji}(R) = -\langle e_j, Re_i\rangle = -r_{ji}$. The functional $\kappa_{j_0 i_0}$ supports $B_+\big(\mathfrak{R}(l^p)\big)$ at every $R = (r_{ji}) \in B_+\big(\mathfrak{R}(l^p)\big)$ with $r_{j_0 i_0} = 0$. Hence all $t_{ji} > 0$. Now let $\|T\| < 1$ and $t_{ji} > 0$ for all $j, i \in \mathbf{N}$. Suppose that $\xi \neq 0$ supports $B_+\big(\mathfrak{R}(l^p)\big)$ at T. Because the set of all one dimensional operators of the form $e_i \otimes e_j$ is total in $\mathfrak{R}(l^p)$, there exists $e_{i_0} \otimes e_{j_0}$ such that $\xi\left(e_{i_0} \otimes e_{j_0}\right) \neq 0$. Let $\varepsilon > 0$ be such that $T \mp \varepsilon e_{i_0} \otimes e_{j_0} \in B_+\big(\mathfrak{R}(l_p)\big)$. We have $\xi\left(T \mp \varepsilon e_{i_0} \otimes e_{j_0}\right) \leqq \sup \xi\big(B_+\big(\mathfrak{R}(l^p)\big)\big) = \xi(T)$, so

$\xi(e_{i_0} \otimes e_{j_0}) = 0$. This contradiction shows that there are no functionals supporting $B_+(\mathfrak{K}(l^p))$ at T i.e. $T \in \text{q-int}\, B_+(\mathfrak{K}(l^p))$.

Using analogous argument as in the proof of the above theorem we get the following result.

THEOREM 6. *Let* $1 \leqq p < \infty$. $T \in \text{q-int}\, \mathfrak{K}_+(l^p)$ *if and only if* $t_{ji} > 0$ *for all* $j, i \in \mathbf{N}$.

The smooth points of the unit ball in the space of compact operators on a Hilbert space are described by Holub [12]. This result was extended by Heinrich [11] to the space of compact operators acting on Banach spaces. Moreover the smooth points of the unit ball of the space of all bounded operators on a Hilbert space (i.e. $\mathfrak{L}(l^2)$) are given in [13].

LEMMA 2. *Let* $1 < p < \infty$. *If* $T \in \text{smooth}\, B_+(\mathfrak{K}(l^p))$ *with* $\|T\| = 1$ *then* T *attains its norm at some unique vector* $f \in S_+(l^p)$.

PROOF. Because T is compact, T is norm attaining operator. To get a contradiction suppose that $\|Tf_1\| = \|Tf_2\| = 1$ for some linearly independent vectors $f_i \in S_+(l^p)$ $(i = 1, 2)$. Let η_i be a functional on l^p such that $\eta_i(Tf_i) = \|\eta_i\| = 1$. We define functionals ξ_i on $\mathfrak{K}(l^p)$ by $\xi_i(R) = \eta_i(Rf_i)$. It is easy to see that ξ_i support $B_+(\mathfrak{K}(l^p))$ at T. Obviously $\xi_1 \neq \xi_2$. Thus $T \notin \text{smooth}\, B_+(\mathfrak{K}(l^p))$. This contradiction ends the proof.

REMARK. Let $1 < p < \infty$. From Heinrich's result and the fact that the unit ball of l^p is smooth we have $T \in \text{smooth}\, B(\mathfrak{K}(l^p))$ if and only if T attains its norm at a unique vector (up to multiplicative constants).

THEOREM 7. *Let* $1 < p < \infty$. *Then* $T \in \text{smooth}\, B_+(\mathfrak{K}(l^p))$ *if and only if*

(i) $t_{ij} > 0$ *for all* $i, j \in \mathbf{N}$ *if* $\|T\| = 1$, *or*

(ii) *there exists a unique pair* (i_0, j_0) *such that* $t_{ji} > 0 = t_{j_0 i_0}$ *for all* $(i, j) \neq (i_0, j_0)$ *if* $\|T\| < 1$.

PROOF. Let $T \in \text{smooth}\, B_+(\mathfrak{K}(l^p))$ with $\|T\| = 1$. T is norm attaining, because T is compact. By Lemma 2, T attains its norm at a unique vector $f \in S_+(l^p)$. Obviously a functional ξ defined by $\xi(R) = \langle (Tf)^{p-1}, Rf \rangle$ supports $B_+(\mathfrak{K}(l^p))$ at T. We have $t_{ji} > 0$ for all $i, j \in \mathbf{N}$. Indeed, if $t_{j_0 i_0}$ were 0 for some j_0, i_0 then there would be a second (different) functional $\kappa_{j_0 i_0}$ (defined as in the proof of Theorem 5) supporting $B_+(\mathfrak{K}(l^p))$ at T. Hence it follows also that for every $T \in \text{smooth}\, B_+(\mathfrak{K}(l^p))$ with $\|T\| \leqq 1$ at most one entry t_{ji} is equal to 0 (in particular $0 \notin \text{smooth}\, B_+(\mathfrak{K}(l^p))$). If $\|T\| < 1$ and $t_{ji} > 0$ for all $j, i \in \mathbf{N}$, then, by Theorem 5, $T \in \text{q-int}\, B_+(\mathfrak{K}(l^p))$, so $T \notin \text{smooth}\, B_+(\mathfrak{K}(l^p))$. Therefore, if $T \in \text{smooth}\, B_+(\mathfrak{K}(l^p))$ with $\|T\| < < 1$ then there exists a unique pair (i_0, j_0) such that $t_{ji} > 0 = t_{j_0 i_0}$ for all $(i, j) \neq (i_0, j_0)$ if $\|T\| < 1$.

Now suppose that (i) holds. Let a functional ξ support $B_+(\mathfrak{N}(l^p))$ at T. Because T is an elementary operator, by [5, Theorem 4] there exists a unique vector $f \in S_+(l^p)$ attaining the norm of T. Obviously $Tf \in \operatorname{smooth} B(l^p)$, then by [10] $T \in \operatorname{smooth} B(\mathfrak{N}(l^p))$. Put $g = (Tf)^{p-1}$. It is easy to check that the functional η defined by $\eta(R) = \langle Rf, g\rangle$ supports $B(\mathfrak{N}(l^p))$ at T. Because η is unique with $\eta(T) = \|\eta\| = \|T\| = 1$, we have

$$\lim_{\lambda\to 0} \frac{\|T+\lambda R\| - \|T\|}{\lambda} = \eta(R) = \langle Rf, g\rangle.$$

Let $u \in l^p$ and $v \in l^p$ be such that 1^0 $\operatorname{supp} u$ and $\operatorname{supp} v$ are finite, and 2^0 $\langle f, u\rangle = 0$ or $\langle v, g\rangle = 0$. Let $\varepsilon > 0$ be such that $T \mp \varepsilon u \otimes v \geqq 0$. We have

$$(T \mp \varepsilon u \otimes v)/\|T \mp \varepsilon u \otimes v\| \in B_+(\mathfrak{N}(l^p)).$$

Hence

$$\xi(T \mp \varepsilon u \otimes v)/\|T \mp \varepsilon u \otimes v\| \leqq \sup \xi\big(B_+(\mathfrak{N}(l^p))\big) = \xi(T),$$

$$\mp\xi(u\otimes v) \leqq \xi(T)\frac{\|T+\lambda u\otimes v\| - \|T\|}{\lambda} \quad \text{for all} \quad \lambda \in (0,\varepsilon).$$

Thus

$$\mp\xi(u\otimes v) \leqq \xi(T)\lim_{\lambda\to 0}\frac{\|T+\lambda u\otimes v\| - \|T\|}{\lambda} = \xi(T)\eta(u\otimes v) = 0.$$

Therefore $\xi(u\otimes v) = 0$. Because finite dimensional operators are dense in $\mathfrak{N}(l^p)$, we have $\xi(S) = 0$ for all $S \in \mathfrak{N}(l^p)$ such that $\langle Sf, g\rangle = 0$. Therefore $\xi(R) = \xi\big(R - \langle Rf, g\rangle f^{p-1} \otimes g^{q-1}\big) + \langle Rf, g\rangle \xi(f^{p-1}\otimes g^{q-1}) = c\langle Rf, g\rangle$, where c is a constant. Hence ξ is unique and $T \in \operatorname{smooth} B_+(\mathfrak{N}(l^p))$.

Finally assume that (ii) holds. Let a functional ξ support $B_+(\mathfrak{N}(l^p))$ at T. Put $\xi_{ji} = \xi(e_i \otimes e_j)$. If $(i,j) \neq (i_0, j_0)$ then there exists $\varepsilon > 0$ such that $T \mp \varepsilon e_i \otimes e_j \in B_+(\mathfrak{N}(l^p))$. Hence

$$(T) \mp \varepsilon\xi_{ji} = \xi(T \mp \varepsilon e_i \otimes e_j) \leqq \sup \xi\big(B_+(\mathfrak{N}(l^p))\big) = \xi(T),$$

so $\xi_{ji} = 0$. Therefore $\xi(R) = \sum_{i,j} \xi_{ji} r_{ji} = \xi_{j_0 i_0}$, i.e. ξ is unique and $T \in$ $\in \operatorname{smooth} B_+(\mathfrak{N}(l^p))$.

THEOREM 8. *Let* $T \in B_+(\mathfrak{N}(l^p))$. *Then* $T \in \operatorname{smooth} B_+(\mathfrak{N}(l^1))$ *if and only if*

(i) *there exists a unique* $n \in \mathbf{N}$ *such that* $\|Te_n\| = 1$ *and* $t_{ji} > 0$ *for all*

i, j if $\|T\| = 1$, or

(ii) *there exists unique pair (i_0, j_0) such that $t_{ji} > 0 = t_{j_0 i_0}$ for all $(i, j) \neq \neq (i_0, j_0)$ if $\|T\| < 1$.*

PROOF. Let $T \in \operatorname{smooth} B_+(\mathfrak{R}(l^1))$ with $\|T\| = 1$. The operator T is compact, so norm attaining. By Lemma 2, T attains its norm at some unique vector $f \in S_+(l^1)$. If T attains its norm at f then T attains its norm at every element of the face F_f generated by f in $B(l^1)$. Because $B(l^1)$ and F_f are $\sigma(l^1, c_0)$-compact, $\operatorname{ext} F_f \neq \emptyset$. Because $\operatorname{ext} F_f \subseteqq \operatorname{ext} B(l^1)$ and $f \in \in S_+(l^1)$ is unique we get $f \in \operatorname{ext} B(l^1)$. Hence $F_f = \{f\}$ and $f = e_n$. Using the functional κ_{ji} and arguments from the proof of Theorem 5 we get that all $t_{ji} > 0$ if $\|T\| = 1$, and (i). Moreover at most one entry of (t_{ji}) is equal to zero if $\|T\| < 1$. Therefore, by Theorem 5, if $T \in \operatorname{smooth} B_+(\mathfrak{R}(l^1))$ with $\|T\| < 1$ then there exists a unique pair (i_0, j_0) such that $t_{ji} > 0 = t_{j_0 i_0}$ for all $(i, j) \neq (i_0, j_0)$.

Now suppose that (i) holds. Let a functional ξ support $B_+(\mathfrak{R}(l^1))$ at T. Fix arbitrary $i \neq n$ and $x \in l^1$. Because there exists $\varepsilon > 0$ such that $T \mp \mp \varepsilon e_i \otimes x \in B_+(\mathfrak{R}(l^1))$, we have $\xi(e_i \otimes x) = 0$. Hence $\xi(R) = \sum_{j=1}^{\infty} \xi_j r_{jn} = = \xi(e_n \otimes Re_n)$, $R \in \mathfrak{R}(l^1)$, where $\xi_j = \xi(e_n \otimes e_j)$. We define a functional η on $(l^1)^*$ by $\eta(x) = \xi(e_n \otimes x)$, $x \in l^1$. We have $\eta(x) = \xi(e_n \otimes x) \leqq \xi(e_n \otimes \otimes Te_n) = \eta(Te_n)$ for all $x \in B_+(l^1)$. Hence η supports $B_+(l^1)$ at Te_n. Put $\eta_j = \eta(e_j)$. We have $\|\eta\| = \sup |\eta_j|$. Since $\operatorname{supp} Te_n = \mathbf{N}$ and $Te_n \in \in \operatorname{smooth} B_+(l^1)$ the functional $\eta/\|\eta\|$ is unique and $\eta_j = \|\eta\| \operatorname{sign}(Te_n)_j = = \|\eta\|$. Thus $\xi_j = \eta_j$ and $\xi(R) = \|\eta\| \sum_{j=1}^{\infty} r_{jn}$, i.e. ξ is unique and $T \in \in \operatorname{smooth} B_+(\mathfrak{R}(l^p))$.

Finally, if (ii) holds, then using the same arguments as in the proof of Theorem 7 we get $T \in \operatorname{smooth} B_+(\mathfrak{R}(l^1))$.

Using similar arguments one can prove the following results.

THEOREM 9. *Let $1 \leqq p < \infty$. Then $T \in \operatorname{smooth} \mathfrak{R}_+(l^p)$ if and only if there exists a unique pair (i_0, j_0) such that $t_{ji} > 0 = t_{j_0 i_0}$ for all $(i, j) \neq \neq (i_0, j_0)$.*

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A NOTE ON THE RELATION BETWEEN ORDINARY AND STRONG APPROXIMATION OF ORTHOGONAL SERIES

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1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{k=0}^{\infty} c_k \varphi_k(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{k=0}^{\infty} c_k^2 < \infty. \tag{1.2}$$

By the Riesz–Fischer theorem, series (1.1) converges in the metric L^2 to a square-integrable function $f(x)$. We denote the n-th partial sum of series (1.1) by $s_n(x)$.

Let $T = (\alpha_{ik})$ $(i, k = 0, 1, \ldots)$ be a double infinite matrix of real numbers. We say that series (1.1) is *T-summable* to $f(x)$ at a point $x \in (a, b)$ if

$$t_i(x) := \sum_{k=0}^{\infty} \alpha_{ik} s_k(x)$$

exists for all i (except perhaps finitely many of them), and

$$\lim_{i\to\infty} t_i(x) = f(x).$$

Let $p > 0$. Series (1.1) will be called *strongly $T(p)$-summable* at x if the relation

$$\lim_{i\to\infty} \sum_{k=0}^{\infty} \alpha_{ik} |s_k(x) - f(x)|^p = 0$$

holds.

Let $\{\gamma_n\}$ be a positive nonincreasing sequence tending to zero. If the relation

$$T_n(x) := T_n(p,f;x) := \left\{ \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x) - f(x)|^p \right\}^{1/p} = o_x(\gamma_n)$$

holds, then we say that the strong $T_n(p,f;x)$-means of (1.1) approximate $f(x)$ in order γ_n at x.

2. At the end of the sixties, it was known that for the most frequently used summability methods, T-summability and strong $T(2)$-summability of series (1.1) coincide under condition (1.2), up to sets of measure zero. These results were proved by various authors and by individial methods. E.g. for the classical $(C,1)$-summation process this was proved by Zygmund [11] (see also Tandori [10]), for $(C,\beta > 0)$- summation by Sunouchi [9], for Riesz summation by Meder [5] and Leindler [2], and for the generalized de la Vallée Poussin summation by the present author ([3]).

Then Móricz [8] raised and answered negatively the following interesting problem: "Does, under condition (1.2), T-summability of series (1.1) almost everywhere (a.e.) imply strong $T(p)$-summability for any permanent T-process?"

Here we recall only Móricz's result.

THEOREM A. *There exist a uniformly bounded orthonormal system $\{\phi_k(x)\}$ on $(0,1)$, a sequence $\{c_k\}$ of coefficients and a permanent T-summation process such that (1.2) is satisfied, the orthogonal series $\sum_{k=0}^{\infty} c_k\phi_k(x)$ is T-summable a.e., but the relation*

$$\varlimsup_{n\to\infty} \sum_{k=0}^{\infty} \alpha_{nk} |s_k(x) - f(x)|^p = \infty$$

holds a.e. in $(0,1)$ for any $p > 0$.

3. The aim of the present note is to prove the analogue of Theorem A for approximation.

Our conjecture that the above mentioned aim can be achieved was born out by the fact (see e.g. [4], Theorem 1.9) that if f and its conjugate function $\tilde{f}$ belong to the class Lip 1 and $(a,b) = (0,2\pi)$, $\{\varphi_n(x)\}$ is the trigonometric system, T is the classical $(C,1)$-summation, then $t_n(x) - f(x) = O(n^{-1})$ everywhere, but $T_n(1,f;0) \geqq Cn^{-1}\log n$ $(C > 0)$, that is, the strong $(C,1)$-means do not approximate as well as the ordinary $(C,1)$-means in this special case; at least not everywhere.

We wanted to prove a similar result for an arbitrary decreasing sequence $\{\gamma_n\}$ instead of $\{1/n\}$, of course giving up the trigonometric system and the

$(C,1)$-summability as hopeless. Unfortunately we were not successful to do this without assuming any additicnal restriction on $\{\gamma_n\}$. Thus, our original problem stays as an interesting open question. Our conjecture, using the notations of Theorem to be proven here, is that our Theorem holds with $\gamma_n = \gamma_n^*$, too.

We are able to establish the following result.

THEOREM. *For any positive nonincreasing sequence* $\{\gamma_n^*\}$ *there exist a positive nonincreasing sequence* $\{\gamma_n\}$, *a uniformly bounded orthonormal system* $\{\Phi_n(x)\}$ *on* $(0,1)$, *a sequence* $\{c_k\}$ *of coefficients and a permanent* T*-summation process such that*

$$0 < \gamma_n \leqq \gamma_n^* \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\gamma_n}{\gamma_n^*} = 1, \tag{3.1}$$

$$\sum_{n=0}^{\infty} c_n^2 \gamma_n^{-2} < \infty, \tag{3.2}$$

and the orthogonal series

$$\sum_{n=0}^{\infty} c_n \Phi_n(x) \tag{3.3}$$

satisfies the following relations:

$$t_n(x) - f(x) = o_x(\gamma_n) \tag{3.4}$$

and

$$\overline{\lim_{n\to\infty}} \gamma_n^{-1} T_n(p,f,x) = \infty \tag{3.5}$$

almost everywhere in $(0,1)$ *for any positive* p, *where* t_n *and* T_n *denote the* n*-th ordinary and strong* T*-means of series* (3.3), *respectively.*

The method of proof follows similar lines as that of Móricz's theorem, which was mainly based on a direct construction of summability T in question using the classical scheme of Menchoff [7].

4. We start the proof of our Theorem by recalling a fundamental lemma of Menchoff [6]. In the sequel, we use $C, C_1, C_2, \ldots$ to denote positive constants, not necessarily the same ones.

LEMMA 1. *Let $\nu > 3$ be a natural number and let $C > 1$. Then there exists in $(-1, C)$ a system $\{\psi_{k\nu}(x)\}$ $(1 \leqq k \leqq \nu^2)$ of orthonormal step functions with the following properties:*

(i) $|\psi_{k\nu}(x)| \leqq C_1$ $(1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C)$;

(ii) *for every point $x \in (1/2, 1)$ there exists an index $\ell(x)$ depending on x such that $1 \leqq \ell(x) \leqq \nu^2$ and*

$$\sum_{k=1}^{\ell(x)} \psi_{k\nu}(x) \geqq C_2 \nu \log \nu.$$

Following the method used by Menchoff [7], we define another system $\{\chi_{k\nu}(x)\}$ $(1 \leqq k \leqq 2\nu^2)$ of orthonormal step functions in $(-2, C)$ as follows:

$$\chi_{k\nu}(x) := \chi_{\nu^2+k,\nu}(x) := \frac{1}{\sqrt{2}}\psi_{k\nu}(x) \quad (1 \leqq k \leqq \nu^2,\ -1 \leqq x \leqq C), \tag{4.1}$$

$$\chi_{k\nu}(x) := \frac{1}{\sqrt{2}} r_k(x+2),\ \chi_{\nu^2+k,\nu}(x) := -\frac{1}{\sqrt{2}} r_k(x+2)$$

$$(1 \leqq k \leqq \nu^2,\ -2 \leqq x < -1),$$

where $r_k(x) := \operatorname{sign} \sin 2^k \pi x$ denotes the k-th Rademacher function. By Lemma 1 it is clear that

$$|\chi_{k\nu}(x)| \leqq C_3 \quad (1 \leqq k \leqq 2\nu^2,\ -2 \leqq x \leqq C);$$

furthermore, for every $x \in (1/2, 1)$, there exists an index $\ell(x)$ $(1 \leqq \ell(x) \leqq \leqq \nu^2)$ such that

$$\sum_{k=1}^{\ell(x)} \chi_{k\nu}(x) = \sum_{k=\nu^2+1}^{\nu^2+\ell(x)} \chi_{k\nu}(x) \geqq C_4 \nu \log \nu. \tag{4.2}$$

In order to construct the required system $\{\Phi_k(x)\}$ we introduce some new notations. Let $g(y)$ be an arbitrary function defined in $(-2, C)$ and let $I := (u, v)$ be an arbitrary finite interval. We proceed from the interval I to the interval $(-2, C)$ by means of the linear transformation

$$y := -2 + \frac{x-u}{v-u}(2+C) \quad (u \leqq x \leqq v,\ -2 \leqq y \leqq C), \tag{4.3}$$

and put

$$g(I; x) := \begin{cases} \sqrt{2+C}\ g(y) & \text{if } u \leqq x \leqq v, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.4}$$

Furthermore, let $E(I)$ denote the image set of an arbitrary set $E \subset (-2, C)$ arising from transformation (4.3).

By (4.4) it is clear that

$$\int_u^v g^2(I;x)dx = \mu(I)\int_{-2}^C g^2(y)dy,$$

where $\mu(I)$ denotes the Lebesgue measure of the interval I.

We may assume that $\gamma_n^* \to 0$, otherwise, with $\gamma_n := \gamma_n^* \geqq C > 0$, the statements of Theorem follow from Theorem A, since in this special case approximation reduces to summability.

First we define four infinite sequences of natural numbers.

Set $\nu_0 = \nu_0' = 4$ and $N_0 = N_0' = 0$. Assume that the numbers $\{\nu_i\}$, $\{\nu_i'\}$, $\{N_i\}$ and $\{N_i'\}$, $i = 0, 1, \ldots, k-1$, are already defined; then we continue the definitions in the following way: Let ν_k' be the smallest natural number such that

$$\gamma^*_{N_{k-1}+(\nu_k')^2} \leqq \frac{1}{2}\gamma^*_{N_{k-1}}, \tag{4.5}$$

furthermore let

$$\nu_k := \max(4, \nu_k', 2^{k^2}), \tag{4.6}$$

$$N_k' := N_{k-1} + \nu_k^2$$

and

$$N_k := N_k' + \nu_k^2 = 2\sum_{i=1}^k \nu_i^2.$$

By this procedure we get four infinite sequences.

Now we set

$$\Phi_k(x) := \Psi_k(x) := r_k(x) \quad (k = 0, 1, \ldots, N_1;\ 0 \leqq x \leqq 1).$$

Let $r > 1$, and assume that the step functions $\Phi_k(x)$, $\Psi_k(x)$ $(k = 0, 1, \ldots, N_{r-1})$ are already defined. Then we divide the interval $(0,1)$ into a finite number of mutually disjoint subintervals $I_1, I_2, \ldots, I_s$, in which every function $\Phi_k(x)$, $\Psi_k(x)$ with $k \leqq N_{r-1}$ is constant. Let I_j', I_j'' denote the two halves of the interval I_j, and set

$$\Phi_k(x) := \begin{cases} \chi_{k-N_{r-1},\nu_r}(I_j'; x) & \text{if} \quad x \in I_j' \\ -\chi_{k-N_{r-1},\nu_r}(I_j''; x) & \text{if} \quad x \in I_j'' \end{cases}$$
$$(j = 1, 2, \ldots, s;\ N_{r-1} < k \leqq N_r);$$

$$\Psi_k(x) := \begin{cases} (2+C)^{-1/2} r_{k-N_{r-1}}(\tilde{I}_j; x) & \text{if} \quad x \in \tilde{I}_j \text{ and } N_{r-1} < k \leqq N'_r, \\ -(2+C)^{-1/2} r_{k-N'_r}(\tilde{I}_j; x) & \text{if} \quad x \in \tilde{I}_j \text{ and } N'_r < k \leqq N_r, \end{cases}$$

where $\tilde{I}_j$ can be either I'_j or I''_j $(j = 1, 2, \ldots, s)$. It is clear that these functions are also step functions.

Set $E_1 := (-2, -1)$, $E_2 := (-1, C)$ and $E_3 := (1/2, 1)$; furthermore, write

$$G'_r(1) := \bigcup_{j=1}^{s} E_1(I'_j), \quad G''_r(1) := \bigcup_{j=1}^{s} E_1(I''_j),$$

and

$$G_r(i) := \bigcup_{j=1}^{s} (E_i(I'_j) \cup E_i(I''_j)) \quad (i = 2, 3).$$

It is obvious that the interval $(0, 1)$ is the union of the mutually disjoint subsets $G'_r(1)$, $G''_r(1)$ and $G_r(2)$, and that

$$\mu\big(G_r(3)\big) = \big(2(2+C)\big)^{-1} \quad (r = 1, 2, \ldots). \tag{4.7}$$

We can easily see that the system $\{\Phi_k(x)\}$ constructed above is orthonormal and uniformly bounded.

In connection with the system $\{\Psi_k(x)\}$, Menchoff [7] proved that $\{\Psi_k(x)\}$ can be divided into two orthonormal *convergence* subsystems. We remind the reader that an orthonormal system $\{\varphi_k(x)\}$ is called a convergence system if every series $\sum c_k \varphi_k(x)$ whose coefficients satisfy condition (1.2) is convergent a.e. The exact result of Menchoff can be written as a lemma.

LEMMA 2. *Let $\{\Psi_k(x)\}$ be the system defined above, and set*

$$S' := \bigcup_{r=1}^{\infty} \{\Psi_k(x) : N_{r-1} < k \leqq N'_r\},$$

$$S'' := \bigcup_{r=1}^{\infty} \{\Psi_k(x) : N'_r < k \leqq N_r\}.$$

Then both S' and S'' are orthonormal convergence systems.

Now, we define the sequences $\{\gamma_n\}$ and $\{c_n\}$, furthermore the required matrix $T = (\alpha_{ik})$. Let

$$\gamma_n := \gamma^*_{N_r} \quad \text{if} \quad N_{r-1} < n \leqq N_r, \quad r = 1, 2, \ldots. \tag{4.8}$$

It is obvious that conditions (3.1) hold.

We set $c_0 := 0$ and

$$(4.9)\qquad c_k = \begin{cases} r^{-1}\nu_r^{-1}\gamma_{N_r} & \text{if } N_{r-1} < k \leqq N_r' \\ -r^{-1}\nu_r^{-1}\gamma_{N_r} & \text{if } N_r' < k \leqq N_r \end{cases} \qquad (r = 1, 2, \dots).$$

It is also clear that (3.2) is satisfied.

Finally, we define $T = (\alpha_{ik})$ $(i, k = 0, 1, \dots)$ as follows:

$$\alpha_{00} = 1 \quad \text{and} \quad \alpha_{0k} = 0 \quad \text{for} \quad k \geqq 1;$$

and, for $i \geqq 1$, we distinguish two cases: if $N_{r-1} < i \leqq N_r'$, $r \geqq 1$, then we set

$$\alpha_{ii} := \alpha_{i,\nu_r^2+i} := \frac{1}{2} \quad \text{and} \quad \alpha_{ik} := 0 \quad \text{otherwise;}$$

if $N_r' < i \leqq N_r$, $r \geqq 1$, then

$$\alpha_{i,N_r} := 1 \quad \text{and} \quad \alpha_{ik} := 0 \quad \text{otherwise.}$$

The permanence of the T-summation process is obvious.

Next, we show that for the N_r-th partial sums of series (3.3) the relation

$$(4.10)\qquad S_{N_r}(x) - f(x) = o_x(\gamma_{N_{r+1}})$$

holds a.e. in $(0,1)$. On account of (4.5), (4.6), (4.8) and (4.9) we have

$$\sum_{r=1}^{\infty} \int_0^1 \gamma_{N_{r+1}}^{-2} \left(S_{N_r}(x) - f(x)\right)^2 dx = \sum_{r=1}^{\infty} \gamma_{N_{r+1}}^{-2} \sum_{k=N_r+1}^{\infty} c_k^2 =$$

$$= \sum_{r=1}^{\infty} \Bigg(\sum_{k=N_r+1}^{N_{r+1}} c_k^2 \Bigg) \sum_{k=1}^{r} \gamma_{N_{k+1}}^{-2} \leqq C \sum_{r=1}^{\infty} \gamma_{N_{r+1}}^{-2} \sum_{k=N_r+1}^{N_{r+1}} c_k^2 < \infty,$$

whence (4.10) follows by B. Levi's theorem.

Using (4.10) we can verify that the relation

$$(4.11)\qquad S_{N_r'}(x) - f(x) = o_x(\gamma_{N_r})$$

holds, too. Namely,

$$\sum_{r=1}^{\infty} \int_0^1 \gamma_{N_r}^{-2} \left(S_{N_r}(x) - S_{N_r'}(x)\right)^2 dx = \sum_{r=1}^{\infty} \gamma_{N_r}^{-2} \sum_{k=N_r'+1}^{N_r} c_k^2 < \infty,$$

thus,

$$S_{N_r}(x) - S_{N_r'}(x) = o_x(\gamma_{N_r}),$$

which, by (4.10), proves (4.11).

Using (4.10) and (4.11) we can prove (3.4). Namely, if $N_r' < i \leqq N_r$, then

$$t_i(x) = S_{N_r}(x);$$

and if $N_{r-1} < i \leqq N_r'$, then

$$t_i(x) = \frac{1}{2}S_i(x) + \frac{1}{2}S_{i+\nu_r^2}(x) = \frac{1}{2}S_{N_{r-1}}(x) + \frac{1}{2}S_{N_r'}(x) +$$

$$+\left\{\sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+\nu_r^2}\right\}c_k\Phi_k(x).$$

Hence, by (4.10) and (4.11), it is easy to see that if we can show that $\gamma_i^{-1}R(r,i;x)$ tends to 0 a.e. in $(0,1)$ as $r \to \infty$, where

$$R(r,i;x) := \left\{\sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_1'+1}^{i+\nu_r^2}\right\}c_k\Phi_k(x),$$

then (3.4) is verified.

Now taking into account (4.1) and the definition of the coefficients c_k and the functions $\Phi_k(x)$, we can see that $R(r,i;x) = 0$ at every point $x \in$ $\in G_r(2)$. If $x \in G_r'(1) \cup G_r''(1)$, then, a simple consideration gives

$$\Phi_k(x) = (1 + C/2)^{1/2}\Psi_k(x) \quad \text{if} \quad x \in G_r'(1),$$

and

$$\Phi_k(x) = -(1 + C/2)^{-1/2}\Psi_k(x) \quad \text{if} \quad x \in G_r''(1)$$

$(N_{r-1} < k \leqq N_r,\ r = 1,2,\dots)$. Hence we obtain

$$\gamma_i^{-1}R(r,i;x) = \pm(1 + C/2)^{1/2}\left\{\sum_{k=N_{r-1}+1}^{i} + \sum_{k=N_r'+1}^{i+\nu_r^2}\right\}r^{-1}\nu_r^{-1}\Psi_k(x)$$

according as $x \in G_r'(1)$ or $x \in G_r''(1)$. Applying Lemma 2, we infer that $\gamma_i^{-1}R(r,i;x)$ tends to 0 a.e. in $(0,1)$ as $r \to \infty$.

Finally we prove (3.5). Let us consider the sets $G_r(3)$ $(r = 1,2,\dots)$. By the definition of the intervals I_j', I_j'' $(j = 1,2,\dots,S)$ and $G_r(3)$, we see that the sets $G_r(3)$ are stochastically independent. Therefore, by (4.7), we can apply the Borel–Cantelli lemma (see e.g. Feller [1], p. 155), whence

$$\mu\big(\overline{\lim_{r\to\infty}} G_r(3)\big) = 1 \tag{4.12}$$

follows.

Let $N_{r-1} < i \leqq N'_r$. Using the inequality

$$|a-b|^p \geqq C(p)|a|^p - |b|^p,$$

where $p>0$ and $C(p)$ denotes a positive constant depending only on p (see e.g. Móricz [8], p. 75) we easily obtain that

$$\gamma_i^{-p}\sum_{k=0}^{\infty}\alpha_{ik}|S_k(x)-f(x)|^p =$$

$$= \frac{1}{2}\big(\gamma_i^{-1}|S_i(x)-f(x)|\big)^p + \frac{1}{2}\big(\gamma_i^{-1}|S_{i+\nu_r^2}(x)-f(x)|\big)^p \geqq$$

$$\geqq \frac{C(p)}{2\gamma_i^p}\left\{\left|\sum_{k=N_{r-1}+1}^{i} c_k\Phi_k(x)\right|^p + \left|\sum_{k=N'_r+1}^{i+\nu_r^2} c_k\Phi_k(x)\right|^p\right\} -$$

$$-\frac{1}{2}\big\{\big(\gamma_i^{-1}|S_{N_{r-1}}(x)-f(x)|\big)^p + \big(\gamma_i^{-1}|S_{N'_r}(x)-f(x)|\big)^p\big\}.$$

Hence, taking into account (4.2), (4.8), (4.9), (4.10) and (4.11), we obtain that there exists an index $i=\ell(x)$ $(N_{r-1} < \ell(x) \leqq N'_r)$ for almost every point $x\in G_r(3)$ such that

$$\gamma_{\ell(x)}^{-p}\sum_{k=0}^{\infty}\alpha_{\ell(x),k}|S_k(x)-f(x)|^p \geqq C\cdot C(\gamma)\left(\frac{1}{r}\log\nu_r\right)^p - o_x(1) \tag{4.13}$$

holds. By (4.12) this estimate holds at almost every point $x\in(0,1)$ for infinitely many values of r. Using (4.6) and (4.13), we can easily see that relation (3.5) is also satisfied a.e.; and this completes the proof.

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A SIMPLE PROOF FOR KÖNIG'S MINIMAX THEOREM

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1. In 1953 Ky Fan [2] proved a minimax theorem without linear structure. Since the appearance of this result, there is a living interest for the axiomatic character of minimax theorems. In 1968 H. König [3] extended Ky Fan's [2] theorem to the case where the constant field for convexity is only a part of $[0,1]$. Applying the ideas of H. Kneser [1] and Ky Fan [2], M. A. Geraghty and B. L. Lin [10] rediscovered König's theorem [3], while S. Simons [11] extended it for two functions. His proof is based on König's version of the Mazur–Orlicz theorem [4].

In the last decade two approaches seemed to be successful for proving minimax theorems: the method of level sets (discovered by I. Joó [5] and applied by L. L. Stachó [6] for quasiconvex-concave functions on interval spaces) and the so called cone method (given in [8] and used by Z. Sebestyén [9, 12], M. Horváth–A. Sövegjártó [13]). Concerning these methods, we notice that by means of them one can prove most of the classical minimax theorems. For instance Ky Fan's theorem can be deduced using the method of level sets (see I. Joó and L. L. Stachó [7] and L. L. Stachó [6]) and also using the cone method [13]. We mention that the function lifting introduced in [7] provides an immediate deduction of König's theorem from Ky Fan's one.

The aim of this note is to give an elementary and simple proof for König's theorem using both methods of Joó [5, 8]. We hope that this proof will be useful also for further generalizations.

2. Let X and Y be nonempty sets and $f: X \times Y \to R$ a given function.

DEFINITION. f is said to be $1/2$ *concave-convex* if the following conditions are fulfilled:

(1) For each $y_1, y_2 \in Y$ there exists $y_3 \in Y$ such that

$$f(x, y_3) \leqq \frac{1}{2}\left[f(x, y_1) + f(x, y_2)\right] \quad \text{for every} \quad x \in X;$$

(2) For every $x_1, x_2 \in X$ there exists $x_3 \in X$ such that

$$f(x_3, y) \geqq \frac{1}{2}\left[f(x_1, y) + f(x_2, y)\right] \quad \text{for every} \quad y \in Y.$$

Denote by D ($\subset [0,1]$) the set of diadic rationals. It is easy to see that if (1) is fulfilled, then

(3) For every $y_1, y_2 \in Y$ and $t \in D$ there exists $y_t \in Y$ such that

$$f(x, y_t) \leqq tf(x, y_1) + (1-t)f(x, y_2) \quad \text{for every} \quad x \in X.$$

A similar statement holds for (2).

THEOREM (H. König [3]). *Suppose X is a compact Hausdorff space and $f(\cdot, y): X \to R$ is upper-semicontinuous for every $y \in Y$, further f is 1/2-concave-convex. Then we have*

$$\sup_x \inf_y f(x,y) = \inf_y \sup_x f(x,y).$$

For the proof we need the following

LEMMA (I. Joó [8]). *Let X and Y be arbitrary sets, $f: X \times Y \to R$ be any function. For $y \in Y$ and $c \in R$ (real) denote*

$$H_y^c = \{x \in X: f(x,y) \geqq c\}, \quad c_* = \sup_x \inf_y f(x,y), \quad c^* = \inf_y \sup_x f(x,y).$$

Then $c_* = c^*$ if and only if for every $c < c^*$ we have

$$\bigcap_{y \in Y} H_y^c \neq \emptyset.$$

PROOF OF THE THEOREM. Since X is compact and f is upper semi-continuous on X for every fixed $y \in Y$, the sets H_y^c are compact. Therefore, it is enough to prove that the family of sets $\{H_y^c: y \in Y\}$ ($c < c^*$) has the finite intersection property. It is obvious that $H_y^c \neq \emptyset$ for every $c < c^*$. First we prove that any two sets of this family have nonempty intersection. Suppose the contrary, i.e. that there exist $c < c^*$ and $y_1, y_2 \in Y$ such that $H_{y_1}^c \cap H_{y_2}^c = \emptyset$ and define the function $h: X \to R^2$ by $h(x) = \big(f(x,y_1) - c, f(x,y_2) - c\big)$; further consider the set $K = \big\{(s,t) \in R^2: s \geqq 0, t \geqq 0\big\}$. According to our assumption, $h(X) \cap K = \emptyset$. Now we show that $\operatorname{Co} h(X) \cap \operatorname{int} K = \emptyset$. For this, suppose that there exist $\lambda_1, \ldots, \lambda_k \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$ and $x_1, \ldots, x_k \in X$ such that $\sum_{i=1}^k \lambda_i h(x_i) \in \operatorname{int} K$. It is easy to see that there exists a dense subset M of $\big\{(t_1, \ldots, t_k) \in R^k: t_i \geqq 0, \sum_{i=1}^k t_i = 1\big\}$ with the following property: for every $(a_1, \ldots, a_k) \in M$ and $x_1, \ldots, x_k \in X$ there exists $x_a \in X$ such that $f(x_a, y) \geqq a_1 f(x_1, y) + \ldots + a_k f(x_k, y)$ for every $y \in Y$. Choose an element $a = (a_1, \ldots, a_k) \in M$

such that $\sum_{i=1}^{k} a_i h(x_i) \in K$. Then $h(x_a) - \sum_{i=1}^{k} a_i h(x_i) \in K$ which contradicts $h(X) \cap K = \emptyset$. By the well-known separation theorem of Hahn–Banach in R^2, there exists a hyperplane (line) which separates the sets $\mathrm{Co}\, h(X)$ and K. That is, there exists $b = (b_1, b_2) \in K$ with $b_1 + b_2 = 1$ such that $b_1 f(x, y_1) + b_2 f(x, y_2) \leqq c$ for every $x \in X$. Let $c_1 \in \mathbf{R}$ be such that $c < c_1 < c^*$ and $d = c_1 - c$. Then we have $b_1[f(x, y_1) - c_1] + b_2[f(x, y_2) - c_1] \leqq -d$ for every $x \in X$, hence the set $h_1(X)$ is separated from K by the line $b_1 s + b_2 t = -d$, where $h_1(x) = \big(f(x, y_1) - c_1, f(x, y_2) - c_1\big)$. Now, since $f(\cdot, y_1)$ and $f(\cdot, y_2)$ are upper-semicontinuous, there exist $p > 0$ and $q > 0$ such that $h_1(X) \subset (-\infty, p] \times (-\infty, q]$. Since $b_1^2 + b_2^2 \neq 0$, the line $b_1 s + b_2 t = -d$ intersects at least one of the lines $s = p$ and $t = q$. Suppose that $b_1 s + b_2 t = -d$ intersects $t = q$. It is clear then, that the line $b_1 q s + (d + b_2 q) t = 0$, which contains the origin and the common point of these lines, separates $h_1(X)$ and K. Let $D \subset [0, 1]$ be the set of diadic rationals. It is clear then one can choose $\alpha \in D$ such that the line $\alpha s + (1 - \alpha) t = 0$ separates $h_1(X)$ and K, or in other words, $\alpha[f(x, y_1) - c_1] + (1 - \alpha)[f(x, y_2) - c_1] \leqq 0$ for every $x \in X$. Consider $y_\alpha \in Y$ such that $f(x, y_\alpha) \leqq \alpha f(x, y_1) + (1 - \alpha) f(x, y_2)$ for every $x \in X$. Then $f(x, y_\alpha) \leqq c_1$ for every $x \in X$ and hence $\sup_x f(x, y_\alpha) \leqq c_1$, consequently $c^* = \inf_y \sup_x f(x, y) \leqq c_1$ which contradicts $c_1 < c^*$.

In order to prove that for any $c < c^*$ and $y_1, \ldots, y_n \in Y$ we have $\bigcap_{i=1} H_{y_i}^c \neq \emptyset$, we use induction. Suppose we know this for $n \leqq N$ and prove it for $N + 1$. To this end denote $C = \bigcap_{i=1}^{N-1} H_{y_i}^c$. This is a nonempty compact subset of X and since the function $\overline{f} = f|_{C \times Y}$ is 1/2-concave-convex, we can repeat the proof above for $\overline{f}$ and for the sets $H_1 = H_{y_N}^c \cap C$, $H_2 = H_{y_{N+1}}^c \cap C$. This completes the proof of the theorem.

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BIFURCATIONS IN A PREDATOR-PREY MODEL WITH MEMORY AND DIFFUSION II: TURING BIFURCATION

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1. Introduction

In the first part of this paper [2] we have introduced the system

$$(1.1)\qquad \begin{cases} \dot{N} = \varepsilon N(1 - N/K) - \beta NP/(\beta + N), \\ \dot{P} = -P(\gamma + \delta P)/(1 + P) + \beta NP/(\beta + N) \end{cases}$$

where $N(t)$ and $P(t)$ are the prey and the predator densities at time t, respectively, $\varepsilon > 0$, $\beta > 0$, $K > 0$ are the specific growth rate of prey, the conversion rate and the carrying capacity with respect to the prey, respectively, $\gamma > 0$ and $\delta > 0$ are the minimal mortality and the limiting mortality of the predator, respectively (the natural assumption is $\gamma < \delta$). We have shown there that the following conditions are reasonable and natural:

$$(1.2)\qquad \gamma < \beta \leqq \delta,$$

$$(1.3)\qquad \beta < K,$$

$$(1.4)\qquad \gamma < \beta K/(\beta + K).$$

Condition (1.2) ensures that predator mortality is increasing with density, and that the predator null-cline has a reasonable concave down shape; (1.3) ensures that for the prey an Allée-effect zone exists where the increase of prey density is favourable to its growth rate; (1.4) is needed to have a positive equilibrium point of system (1.1). Under these assumptions (0,0) and $(K, 0)$ are unstable equilibria of (1.1), and the system has at least one

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equilibrium with positive coordinates. This is the point of intersection of the prey null-cline

$$P = H_1(N) := (K - N)(\beta + N)\varepsilon/(\beta K),$$

and the predator null-cline

$$P = H_2(N) := \big((\beta - \gamma)N - \beta\gamma\big) / \big((\delta - \beta)N + \beta\delta\big).$$

Thus, denoting the coordinates of a positive equilibrium by $(\overline{N}, \overline{P})$, these coordinates satisfy $\overline{P} = H_1(\overline{N}) = H_2(\overline{N})$. (See Fig. 1.) In Part I we have shown that if $(\overline{N}, \overline{P})$ is on the descending branch of the parabola $P = H_1(N)$ then it is asymptotically stable. If it is on the ascending branch in the Allée-effect zone (like on Fig. 1) it may or may not be stable.

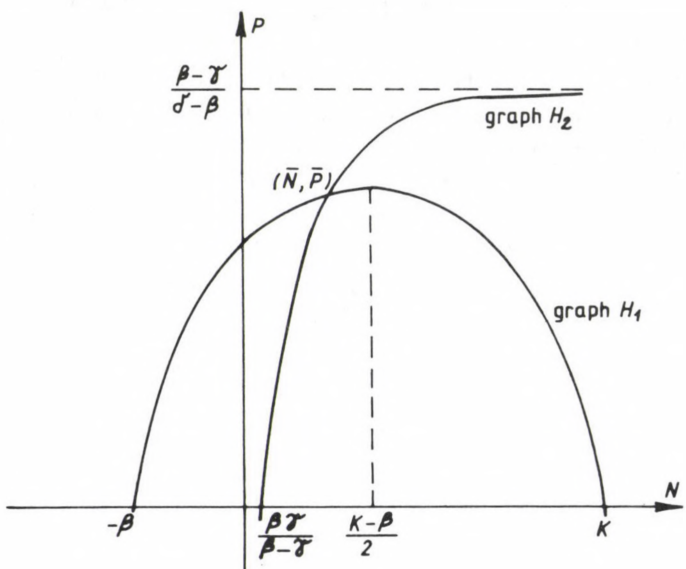

Fig. 1
Prey null-cline $P = H_1(N)$, and predator null-cline $P = H_2(N)$ under assumptions (1.2)–(1.4). The Allée-effect zone is $N \in (0, (K-\beta)/2)$. Here there is a single positive equilibrium $(\overline{N}, \overline{P})$, and it is in the Allée-effect zone.

In Part I we have introduced an infinite distributed delay into the second equation of the system for the prey density, i.e. we replaced N in the second equation by

$$Q(t) = \int_{-\infty}^{t} N(\tau) a \exp\big(- a(t-\tau)\big)\, d\tau, \quad a > 0.$$

We have given conditions for the asymptotic stability of the equilibrium in this case and have shown, among other things, that under some conditions the increase of the delay $1/a$ destabilizes the originally stable equilibrium by an Andronov–Hopf bifurcation.

In Part II we assume that prey and predator are living in a one dimensional, bounded spatial domain where they are diffusing according to Fick's law, i.e. in the direction of the negative gradient of the density, and the diffusion is proportional to the negative divergence of this negative gradient, i.e. to the second derivative of density. We assume zero flux boundary conditions which means that there is no migration across the boundary of the spatial domain.

Conditions will be established for the stability of the spatially homogeneous (constant) stationary solution of the system, and we are going to show that considering one of the diffusion coefficients as a bifurcation parameter, at a certain critical value a Turing instability occurs meaning that the stationary solution stays stable with respect to the original system without diffusion but becomes unstable with respect to the system with diffusion (Section 2). We show in Section 3 that at the critical value of the bifurcation parameter a Turing bifurcation takes place, i.e. a spatially nonhomogeneous (non-constant) stationary solution, in other words, a *pattern* arises. In Section 4 we establish conditions for the stability of the pattern.

2. Turing instability

Let us modify the predator-prey system (1.1) assuming that prey and predator are diffusing according to Fick's law in the interval $x \in [0, l]$, i.e. consider the reaction-diffusion system

$$\text{(2.1)} \qquad \begin{cases} \dfrac{\partial N}{\partial t} = d_N \dfrac{\partial^2 N}{\partial x^2} + \varepsilon N(1 - N/K) - \beta N P/(\beta + N), \\[2mm] \dfrac{\partial P}{\partial t} = d \dfrac{\partial^2 P}{\partial x^2} - P(\gamma + \delta P)/(1 + P) + \beta N P/(\beta + N) \end{cases}$$

on $(t, x) \in \mathbf{R}^+ \times [0, l]$, $l > 0$ where ε, β, K, γ, $\delta > 0$ satisfy conditions (1.2)–(1.4). We are interested in solutions $N : \mathbf{R}^+ \times [0, l] \mapsto \mathbf{R}^+$, $P : \mathbf{R}^+ \times \times [0, l] \mapsto \mathbf{R}^+$ that satisfy the no-flux boundary conditions

$$\text{(2.2)} \qquad N_x(t, 0) = N_x(t, l) = P_x(t, 0) = P_x(t, l) = 0.$$

The diffusion coefficients d_N, d are supposed to be non-negative.

Introducing the two dimensional vector $U = \mathrm{col}(N, P)$, the diagonal matrix $D = \mathrm{diag}(d_N, d)$ and the vector

$$F(U) = \mathrm{col}\big(\varepsilon N(1 - N/K) - \beta NP/(\beta + N),$$

$$-P(\gamma + \delta P)/(1 + P) + \beta NP/(\beta + N)\big)$$

the system and the boundary conditions assume the form

$$U_t = DU_{xx} + F(U), \tag{2.3}$$

$$U_x(t, 0) = U_x(t, l) = 0. \tag{2.4}$$

Clearly, a spatially constant solution $U(t) = \big(N(t),\ P(t)\big)$ of (2.3) satisfies the boundary conditions (2.4) and the *kinetic system*

$$U_t = F(U) \tag{2.5}$$

which is, in fact, system (1.1). The equilibrium of (2.5) $\overline{U} = \big(\overline{N}, \overline{P}\big)$ is at the same time a constant solution of (2.3). We shall be concerned with the positive equilibrium whose existence has been established in Part I and recalled in the Introduction.

DEFINITION 2.1. We say that the equilibrium $\overline{U} = \big(\overline{N}, \overline{P}\big)$ of (2.3) is *Turing (diffusionally) unstable* if it is an asymptotically stable equilibrium of the kinetic system (2.5) but is unstable with respect to solutions of (2.3)–(2.4). (See Okubo [5], Svirezhev, Logofet [8], and Svirezhev [9]).

The latter requirement means that there are solutions of (2.3)–(2.4) that have initial values $U(0, x)$ arbitrarily close to $\overline{U}$ (in the supremum norm) but do not tend to $\overline{U}$ as t tends to infinity. The problem of stability of an equilibrium solution U of (2.3)–(2.4) will be solved via a linearized stability analysis (see Casten, Holland [1], Razzhevaikin [6], Smoller [7], and Conway [3]).

We linearize system (2.3) at the point $\overline{U} = \big(\overline{N}, \overline{P}\big)$. Introducing the new coordinates $V = (V_1, V_2) = \big(N - \overline{N}, P - \overline{P}\big)$, the notations (as in Part I)

$$\eta = \varepsilon/(k\beta), \quad \Theta_1 = \beta\overline{N}/\big(\beta + \overline{N}\big), \quad \Theta_2 = K - \beta - 2\overline{N},$$

$$\Theta_3 = \big(K - \overline{N}\big)/\big(\beta + \overline{N}\big), \quad \Theta_4 = \big((\delta - \beta)\overline{N} + \beta\delta\big)^2/(\delta - \gamma),$$

and

$$A = F_U\big(\overline{U}\big) = \begin{bmatrix} \eta\Theta_1\Theta_2 & -\Theta_1 \\ \beta^2\eta\Theta_3 & -\eta\Theta_3\Theta_4 \end{bmatrix}, \tag{2.6}$$

the linearized system assumes the form

$$V_t = DV_{xx} + AV, \tag{2.7}$$

while the boundary conditions remain

$$V_x(t,0) = V_x(t,l) = 0. \tag{2.8}$$

We solve the linear boundary value problem by Fourier's method. Solutions are assumed in the form $V(t,x) = y(t)\psi(x)$. The functions $y : [0,\infty) \mapsto \mathbf{R}^2$, $\psi : [0,l] \mapsto \mathbf{R}$ are to satisfy

$$\dot{y} = (A - \lambda D)y \tag{2.9}$$

where dot denotes differentiation with respect to time t, and

$$\psi'' = -\lambda\psi, \quad \psi'(0) = \psi'(l) = 0 \tag{2.10}$$

where prime denotes differentiation with respect to the spatial variable x. The eigenvalues of the boundary value problem (2.10) are

$$\lambda_j = (j\pi/l)^2, \quad j = 0,1,2,\dots \tag{2.11}$$

with corresponding eigenfunctions

$$\psi_j(x) = \cos(j\pi x/l). \tag{2.12}$$

Clearly, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$. These eigenvalues are to be substituted into (1.9). Denoting two independent solutions of (2.9) taken with $\lambda = \lambda_j$ by y_{1j}, y_{2j}, the solution of the boundary value problem (2.7)–(2.8) is obtained in the form

$$V(t,x) = \sum_{j=0}^{\infty} \big(a_{1j}y_{1j}(t) + a_{2j}y_{2j}(t)\big) \cos(j\pi x/l) \tag{2.13}$$

where a_{ij} $(i = 1,2;\ j = 0,1,2,\dots)$ are to be determined according to the initial condition $V(0,x)$. If e.g. $y_{1j}(0) = (1,0)$, $y_{2j}(0) = (0,1)$ for $j = 0,1,2,\dots$ then

$$\begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix} = \frac{1}{l}\int_0^l V(0,x)\,dx,$$

$$\begin{bmatrix} a_{1k} \\ a_{2k} \end{bmatrix} = \frac{2}{l}\int_0^l V(0,x)\cos\frac{k\pi x}{l}\,dx \quad (k = 1,2,\dots).$$

The following notations will be used:

$$B(\lambda) = A - \lambda D, \quad B_j = B(\lambda_j) = A - \lambda_j D. \tag{2.14}$$

According to Casten, Holland [1] if for all j both eigenvalues of B_j have negative real parts then the equilibrium $\overline{U}$ of (2.3)–(2.4) is asymptotically stable: if at least one eigenvalue of a matrix B_j has positive real part then $\overline{U}$ is unstable.

The following two lemmata are specializations of results in Conway [3]. The proofs are given here for completeness of the treatment, and because the formulae applied will be used later.

LEMMA 2.1. *If $\overline{U}$ is Turing unstable then it lies in the Allée-effect zone* (see Fig. 1).

PROOF. The equilibrium point $\overline{U} = (\overline{N}, \overline{P})$ lies in the Allée-effect zone if and only if $\Theta_2 = K - \beta - 2\overline{N} > 0$. Now,

$$\mathrm{Tr}\ A = \eta(\Theta_1\Theta_2 - \Theta_3\Theta_4), \tag{2.15}$$

$$\det A = \eta\Theta_1\Theta_3\left(\beta^2 - \eta\Theta_2\Theta_4\right), \tag{2.16}$$

$$\mathrm{Tr}\ B_j = \mathrm{Tr}\ A - (d_N + d)\lambda_j, \tag{2.17}$$

$$\det B_j = \det A + \lambda_j d_N\left(\eta\Theta_3\Theta_4 + \lambda_j d\right) - \lambda_j\eta\Theta_1\Theta_2 d. \tag{2.18}$$

If $\overline{U}$ lies outside the Allée-effect zone then $\Theta_2 \leqq 0$. Since, obviously, Θ_1, Θ_3, $\Theta_4 > 0$ in this case $\mathrm{Tr}\ A < 0$, $\det A > 0$, $\mathrm{Tr}\ B_j < 0$, $\det B_j > 0$, i.e. all eigenvalues of the matrices A, B_j ($j = 0, 1, 2, \ldots$) have negative real parts, so no Turing instability may occur. □

LEMMA 2.2. *Suppose that $\overline{U}$ lies in the Allée-effect zone, and that both eigenvalues of the matrix A have negative real parts. If $\overline{U}$ is Turing unstable then $d > d_N$ must hold.*

PROOF. The condition on the eigenvalues of A imply that $\mathrm{Tr}\ A < 0$, and this, in turn, implies $\mathrm{Tr}\ B_j < 0$. Thus, Turing instability may occur only if for some j we have $\det B_j \leqq 0$. Now, $\det B(0) = \det A > 0$ (see (2.14)), and

$$d \det B(0) / d\lambda = \eta(d_N\Theta_3\Theta_4 - d\Theta_1\Theta_2). \tag{2.19}$$

Because of (2.15) and $\mathrm{Tr}\ A < 0$ we have $0 < \Theta_1\Theta_2 < \Theta_3\Theta_4$ where we have used also the assumption that $\overline{U}$ is in the Allée-effect zone, i.e. $\Theta_2 > 0$. So if $d_N \geqq d$,

$$d \det B(0) / d\lambda > \eta\Theta_1\Theta_2(d_N - d) \geqq 0.$$

Since the quadratic polynomial $\det B(\lambda)$ has a positive leading coefficient $d_N d$, the last inequality means that $\det B(\lambda) > 0$ for $\lambda > 0$, i.e. for all $\lambda = \lambda_j$. □

Note that from the proof one can see that in order to have Turing instability the negativity of expression (2.19) is necessary.

EXAMPLE Set $\beta = 0.1000$, $\gamma = 0.0100$, $\delta = 0.1055$, $\varepsilon = K = 1$. The unique positive equilibrium is $(\overline{N}, \overline{P}) = (0.4486,\ 3.0250)$. It is easy to see that this point is in the Allée-effect zone $(0.4486 < 0.9000/2)$ and it is an asymptotically stable equilibrium of the kinetic system (2.5).

The preceding considerations justify the following

DEFINITION 2.2. The parameters ε, β, K, γ, δ in (2.1) are said to be *Turing-admissible* if (i) the inequalities (1.2)–(1.4) hold, (ii) the kinetic system (2.5) has an equilibrium point $(\overline{N}, \overline{P})$ with positive coordinates in the Allée-effect zone and it is linearly asymptotically stable.

THEOREM 2.1. *Suppose that the parameters ε, β, K, γ, δ are Turing-admissible. Then $0 < d_N < d$ can be chosen such that $\overline{U} = (\overline{N}, \overline{P})$ is Turing unstable. To be sure, for a fixed $d > 0$ a $k \in \mathbf{N}$ can be found such that the right hand side of the following inequality is positive, and if*

$$0 < d_N < \frac{\lambda_k \eta \Theta_1 \Theta_2 d - \det A}{\lambda_k(\eta \Theta_3 \Theta_4 + \lambda_k d)} \tag{2.20}$$

then $\overline{U} = (\overline{N}, \overline{P})$ is Turing unstable.

PROOF. By assumption $\Theta_2 > 0$, and A is a stable matrix, thus, $\det A > 0$. We are going to show that $k \in \mathbf{N}$ and $d_N > 0$ can be chosen such that $\det B_k = \det B(\lambda_k) < 0$. We have

$$\det B(\lambda) = \det A - \lambda \eta \Theta_1 \Theta_2 d + \lambda d_N(\eta \Theta_3 \Theta_4 + \lambda d)$$

(cf. (2.18)). Since $\lambda_j \to \infty$ as $j \to \infty$, for fixed $d > 0$ there exists $k \in \mathbf{N}$ such that $\lambda_k \eta \Theta_1 \Theta_2 d - \det A > 0$. Hence

$$(\lambda_k \eta \Theta_1 \Theta_2 d - \det A) / \big(\lambda_k(\eta \Theta_3 \Theta_4 + \lambda_k d)\big) > 0,$$

and, as a consequence, if d_N is chosen according to (2.20) then $\det B_k < 0$. □

Note that we may proceed the other way around. We may fix $k = 1$, say, i.e. $\lambda_1 = (\pi/l)^2$, and increase d till the numerator of (2.20) will be positive. As d tends to infinity the right hand side of (2.20) is increasing and tends to $\lambda_1 \eta \Theta_1 \Theta_2 / \lambda_1^2$. An easy estimate shows that this is less than $\varepsilon(l/\pi)^2$. This means that irrespective of how large the predator diffusion rate d is, the prey diffusion rate d_N must satisfy

$$d_N < \varepsilon(l/\pi)^2 \tag{2.21}$$

in order to have Turing instability.

Since, clearly, for all $k \in \mathbf{N}$

$$\eta\Theta_1\Theta_2/\lambda_k \leqq \eta\Theta_1\Theta_2/\lambda_1 \leqq \varepsilon(l/\pi)^2$$

the inequality (2.21) is, in fact, a necessary condition of Turing instability.

3. Pattern formation

We start this section with a definition. Consider the reaction-diffusion system

$$\frac{\partial U}{\partial t} = D(\lambda)\frac{\partial^2 U}{\partial x^2} + F(U,\lambda) \tag{3.1}$$

where $U \in \mathbf{R}^n$, D is a non-negative diagonal matrix depending smoothly on the real parameter $\lambda \in [0,\infty)$ and $F : \mathbf{R}^n \times [0,\infty) \mapsto \mathbf{R}^n$ is a smooth function, along with the Neumann boundary conditions

$$\partial U(t,0)/\partial x = \partial U(t,l)/\partial x = 0. \tag{3.2}$$

Assume further that for some $\overline{U} \in \mathbf{R}^n$ we have $F(\overline{U},\lambda) = 0$ for all $\lambda \in$ $\in [0,\infty)$, i.e. $\overline{U}$ is a constant stationary solution of (3.1)–(3.2).

DEFINITION 3.1. We say that $\overline{U}$ undergoes a *Turing bifurcation* at $\lambda_0 \in$ $\in (0,\infty)$ if for $0 < \lambda < \lambda_0$ the solution $\overline{U}$ is asymptotically stable, for $\lambda_0 < \lambda$ it is unstable (or vice versa), and in some neighbourhood of λ_0 the problem (3.1)–(3.2) has non-constant stationary solutions (i.e. solutions which do not depend on time t but are not constant in space, are varying with x).

First we are going to establish the conditions of a Turing bifurcation for the linearized system (2.7)–(2.8). We shall consider the predator diffusion coefficient $d > 0$ as the bifurcation parameter. Keeping the notations of Section 2 the following theorem holds.

THEOREM 3.1. *Suppose that the parameters* ε, β, K, γ, δ *are Turing admissible.*

(i) *If*

$$d_N \geqq \eta\Theta_1\Theta_2/\lambda_1 \tag{3.3}$$

then the zero solution of the linear problem (2.7)–(2.8) *is asymptotically stable for all* $d > 0$.

(ii) *If*

$$\eta\Theta_1\Theta_2/\lambda_1 > d_N \geqq \eta\Theta_1\Theta_2/\lambda_2 \tag{3.4}$$

then at

$$d = d_0 = (\lambda_1 d_N \eta \Theta_3 \Theta_4 + \det A)/(\lambda_1 \eta \Theta_1 \Theta_2 - \lambda_1^2 d_N) \tag{3.5}$$

the zero solution of the linear problem (2.7)–(2.8) *undergoes a Turing bifurcation.*

PROOF. (i) From (2.18) we have

$$\det B_j = (\lambda_j d_N - \eta\Theta_1\Theta_2)(\lambda_j d + \eta\Theta_3\Theta_4) + \eta\beta^2\Theta_1\Theta_3.$$

Since λ_j, $j = 0, 1, 2, \dots$ form a monotone increasing sequence (3.3) implies $\det B_j > 0$ for all $j = 0, 1, 2, \dots$. From (2.17) we see that for Turing admissible parameters $\operatorname{Tr} B_j < 0$ for all $j = 0, 1, 2, \dots$, hence the zero solution of (2.7)–(2.8) is asymptotically stable.

(ii) If d_N satisfies (3.4) and d is chosen according to (3.5) then $\det B_1 = 0$. Clearly, for $0 < d < d_0$ we have $\det B_1 > 0$, and for $d_0 < d$: $\det B_1 < 0$. In all these cases $\det B_j > 0$, $j \neq 1$. Thus, taking into account what has been quoted after formula (2.14), for $0 < d < d_0$ the zero solution is asymptotically stable, for $d_0 < d$ it is unstable. If $d = d_0$ one eigenvalue of B_1 is zero the other is negative. Denote the eigenvector corresponding to the zero eigenvalue by $y_{11} = \operatorname{col}(\eta^1, \eta^2)$, i.e.

$$B_1 y_{11} = (A - \lambda_1 D) y_{11} = 0, \quad y_{11} \neq 0.$$

As we can see from (2.9)–(2.12) the function

$$v_1(x) = y_{11}\psi_1(x) = \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \cos(\pi x / l) \tag{3.6}$$

is a spatially non-constant stationary solution of the linearized problem (2.7)–(2.8).

Note that any scalar multiple of v_1 is also a solution. Since every $y_{ij}(t)$, $(i, j) \neq (1, 1)$ tends to zero exponentially as t tends to infinity, we see from (2.13) that every solution $V(t, x)$ of (2.7)–(2.8) tends to $a_{11} v_1(x)$ where a_{11} is determined by the initial values $V(0, x)$ (also the second, negative eigenvalue and the corresponding eigenvector of B_1 is to be used in the calculation of a_{11}). Thus, in this case the zero solution of (2.7)–(2.8) is Turing unstable, though if we used a finer definition we could say that it is "neutrally stable".

Note also that if instead of (3.4) d_N is fixed between $\eta\Theta_1\Theta_2/\lambda_k$ and $\eta\Theta_1\Theta_2/\lambda_{k+1}$ (the latter being smaller than the former) then a similar result holds; the critical value (3.5) of d is to be replaced by

$$(\lambda_k d_N \eta\Theta_3\Theta_4 + \det A)/(\lambda_k \eta\Theta_1\Theta_2 - \lambda_k^2 d_N).$$

We are going to extend the result about the Turing bifurcation of the zero solution of the linearized system to the non-linear problem (2.3)–(2.4). For easier reference we quote here Theorem 13.5 from Smoller [7] about bifurcation from a simple eigenvalue which is to be used.

THEOREM 3.S. *Let X, Y be Banach spaces, $U = S \times V$ an open subset of $\mathbf{R} \times X$, and $f \in C^2(U,Y)$ such that $f(\lambda,0) = 0$, $\lambda \in S \subset \mathbf{R}$. Denote the linear operators obtained by differentiating f with respect to its second, resp. first and second variables at $\lambda_0 \in S$, $v = 0 \in V$ by $L_0 = f_v(\lambda_0, 0)$ and $L_1 = f_{\lambda v}(\lambda_0, 0)$, resp.; and assume that*

(i) the kernel of L_0, the subspace $\mathcal{N}(L_0)$ of X is one dimensional spanned by $v_1 \in X$;

(ii) the range of L_0, the subspace $\mathcal{R}(L_0)$ of Y has codimension 1, i.e. $\dim[Y/\mathcal{R}(L_0)] = 1$;

(iii) $L_1 v_1 \notin \mathcal{R}(L_0)$.

Let $\mathcal{Z}$ be an arbitrary closed subspace of X such that $X = [\text{Span } v_1] \oplus \oplus \mathcal{Z}$; then there is a $\delta > 0$ and a C^1 curve (λ, ϕ): $(-\delta, \delta) \mapsto S \times \mathcal{Z}$ such that $\lambda(0) = \lambda_0$, $\phi(0) = 0$, and $f\big(\lambda(s), sv_1 + s\phi(s)\big) = 0$ for $|s| < \delta$; furthermore, there is a neighbourhood of $(\lambda_0, 0)$ such that any zero of f either lies on this curve or is of the form $(\lambda, 0)$.

REMARK. In what follows the role of the space X will be played by

$$X = \{V \in C^2\big([0,l], \mathbf{R}^2\big) : V_x(0) = V_x(l) = 0\} \tag{3.7}$$

with the usual supremum norm involving the first and second derivatives, while $Y = C^0([0,l], \mathbf{R}^2)$ with the usual supremum norm. However, in choosing the subspace $\mathcal{Z}$ we shall use the orthogonality induced by the scalar product

$$\langle V, W\rangle = \int_0^l \big(V^1(x)W^1(x) + V^2(x)W^2(x)\big)\, dx \tag{3.8}$$

where $V = (V^1, V^2)$, $W = (W^1, W^2)$.

Now we are able to prove

THEOREM 3.2. *Suppose that the parameters ε, β, K, γ, δ are Turing admissible.*

(i) If (3.3) holds then the constant solution $\overline{U} = \big(\overline{N}, \overline{P}\big)$ of the nonlinear problem (2.3)–(2.4) is asymptotically stable.

(ii) If $[0, \eta^2]$ is not parallel to the second eigenvector y_{21} of B_1 and d_N satisfies (3.4) then at $d = d_0$ the constant solution $\overline{U}$ undergoes a Turing bifurcation.

PROOF. (i) follows immediately from the asymptotic stability of the zero solution of the linear problem (2.7)–(2.8).

(ii) From Theorem 3.1 just like in (i) we have that for $d < d_0$ (given in (3.5)) $\overline{U}$ is asymptotically stable, while for $d_0 < d$ it is unstable. We have to prove yet the existence of a stationary non-constant solution in some neighbourhood of the critical value d_0 of the bifurcation parameter d. Such a stationary solution satisfies the two dimensional system of second order ordinary differential equations with boundary conditions

$$DU_{xx} + F(U) = 0, \quad U_x(0) = U_x(l) = 0. \tag{3.9}$$

We consider (3.9) as an operator equation on the Banach space X given by (3.7), and we apply Theorem 3.S with d as the bifurcation parameter. Introducing the new vector of the variation $V = U - \overline{U}$ (3.9) assumes the equivalent form

$$DV_{xx} + AV + H(V) = 0, \quad V_x(0) = V_x(l) = 0 \tag{3.10}$$

where A is the matrix given by (2.6), and

$$H(V) = F(\overline{U} + V) - AV, \quad H(0) = 0, \quad H_v(0) = 0. \tag{3.11}$$

Denote the left hand side of (3.10) by $T(d, V)$. T is a one-parameter family of operators acting on X and taking its elements into $Y = C^0([0,l], \mathbf{R}^2)$. Clearly, $T \in C^2$. The spectrum of the linear operator $L_0 = T_v(d_0, 0) = \partial T(d_0, 0)/\partial V$ consists of the eigenvalues μ_{ij} ($i = 1, 2$; $j = 0, 1, 2, \ldots$) of the matrices B_j given by (2.14). The corresponding eigenfunctions are $\psi_j(x) y_{ij}$ ($i = 1, 2$; $j = 0, 1, 2, \ldots$) where ψ_j is a given by (2.12) and y_{ij} is the eigenvector of the matrix B_j corresponding to the eigenvalue μ_{ij}. Now, all matrices $B_j = A - \lambda_j D$ are to be taken at $d = d_0$. As it can be seen from the proof of Theorem 3.1 and from (2.17)–(2.18) for $i = 1, 2$; $j = 0, 2, 3, \ldots$ all μ_{ij} have negative real parts. For $j = 1$ one eigenvalue, μ_{11} say, is zero the other one is negative. The eigenfunction corresponding to $\mu_{11} = 0$ is $v_1(x) = y_{11} \cos(\pi x/l)$ (see (3.6)). Thus, the null-space of the operator $L_0 = T_v(d_0, 0)$ is one dimensional spanned by v_1, the range of this operator is, clearly, given by

$$\mathcal{R}(L_0) = \{ W \in C^0([0,l], \mathbf{R}^2) : W\text{'s Fourier expansion does not contain } \cos(\pi x/l) \text{ term} \} \cup \{ y_{21} \cos(\pi x/l) \}$$

because of the orthogonality and completeness of the system (2.12). So the codimension of $\mathcal{R}(L_0)$ is one.

Set $L_1 = \partial T_v(d_0, 0)/\partial d$, then

$$L_1 = D' \frac{\partial^2}{\partial x^2} \quad \text{where} \quad D' = \frac{\partial D}{\partial d} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly (see (3.6)),

$$L_1 v_1 = -D'(\pi/l)^2 y_{11}\cos(\pi x/l) = -(\pi/l)^2 \begin{bmatrix} 0 \\ \eta^2 \end{bmatrix} \cos(\pi x/l).$$

So $L_1 v_1 \nparallel y_{21}\cos(\pi x/l)$ and

$$\langle v_1, L_1, v_1\rangle = -(\pi/l)^2 \int_0^l (\eta^2)^2 \cos^2(\pi x/l)\, dx \neq 0$$

because $\eta^2 \neq 0$. Thus, $L_1 v_1 \notin \mathcal{R}(L_0)$.

Finally, if we define the subspace $\mathcal{Z}$ by

$$\mathcal{Z} = \mathcal{R}(L_0) \tag{3.12}$$

then all the conditions of Theorem 3.S are satisfied. We conclude that $(d_0, 0)$ is a bifurcation point, and there exist a $\delta > 0$, a function $d : (-\delta, \delta) \mapsto \mathbf{R}$ and for $s \in (-\delta, \delta)$ a solution of (3.10) with $d = d(s)$ $\big(|s| < \delta\big)$ substituted

$$V(s,x) = s y_{11}\cos(\pi x/l) + s\phi(s,x)$$

such that $d(0) = d_0$, $\phi(0,x) = 0$, $d \in C^1$, $\phi(\cdot, x) \in C^1$, and $\phi(s,\cdot) \in \mathcal{Z}$. □

Note that the corresponding solution of (3.9), i.e. the non-constant stationary solution of the nonlinear problem (2.3)–(2.4) is

$$U(s,x) = \overline{U} + s y_{11}\cos(\pi x/l) + \mathcal{O}(s^2) \tag{3.13}$$

(corresponding to the choice $d = d(s)$, $|s| < \delta$), i.e.

$$N(x) = \overline{N} + s\eta^1\cos(\pi x/l) + \mathcal{O}(s^2), \quad P(x) = \overline{P} + s\eta^2\cos(\pi x/l) + \mathcal{O}(s^2).$$

Since s is considered to be small here, we call this solution a *small amplitude pattern.*

Note further that Theorem 3.S implies that (2.3)–(2.4) has no other stationary solution apart from $\big(\overline{N}, \overline{P}\big)$ and (3.13) in a neighbourhood of $\big(d_0, \overline{U}\big) \in \mathbf{R} \times X$.

Finally, note that in the linear situation covered by Theorem 3.1 the function $d(s) \equiv d_0$, and the corresponding one parameter family of solutions is $\overline{U} + s v_1(x)$, $s \in \mathbf{R}$ (see (3.6)).

4. The stability of the pattern

In this section we are going to apply the results of Crandall, Rabinowitz [4] (see also [7] Chapter 13) for generating conditions of asymptotic stability of the bifurcating non-constant, stationary solution: the pattern. In the preceding section we have introduced the linearization of the operator $T(d,V)$, the left hand side of (3.10): $L_0 = T_v(d_0,0)$. We saw that all the eigenvalues μ_{ij} of this operator have negative real parts except μ_{11} which is zero. In Theorem 3.2 we have established the existence of the bifurcating pattern $\big(d(s), U(s,x)\big)$, see (3.13). Now, the linearized operator T_v taken along this bifurcating solution $T_v\big(d(s), U(s,x)\big)$ is close to L_0 if s is small. Our aim is to determine the sign of the eigenvalue of this operator bifurcating from $\mu_{11} = 0$. For easier reference we quote here some results from [4,7].

DEFINITION 4.1. Let X and Y be Banach spaces, and L_0, $K \in B(X,Y)$, the set of bounded operators. We say that $\mu \in \mathbf{C}$ is a *K-simple eigenvalue of L_0 with eigenfunction* v_1 if the following conditions hold:
(i) $\dim \mathcal{N}(L_0 - \mu K) = \operatorname{codim} \mathcal{R}(L_0 - \mu K) = 1$,
(ii) v_1 spans $\mathcal{N}(L_0 - \mu K)$,
(iii) $Kv_1 \notin \mathcal{R}(L_0 - \mu K)$,
where $\mathcal{N}$ is the null-space and $\mathcal{R}$ is the range of the operator $L_0 - \mu K$.

LEMMA 4. CR. *Let $L_0, K \in B(X,Y)$, assume that μ_{11} is a K-simple eigenvalue of L_0 with eigenfunction v_1, and let $X = [\text{Span } v_1] \oplus \mathcal{Z}$. Then there is a $\delta > 0$ such that if $L \in B(X,Y)$, $\|L - L_0\| < \delta$ then L has a unique K-simple eigenvalue $\mu(L)$ satisfying $|\mu(L) - \mu_{11}| < \delta$ with eigenfunction $v = v_1 + z$ where $z \in \mathcal{Z}$. The map $L \mapsto \big(\mu(L), v(L)\big)$ is smooth, and $\mu(L_0) = \mu_{11}$, $v(L_0) = v_1$.*

In the following theorem the notations of Theorem 3.S and Lemma 4. CR are used. The assumptions of Theorem 3.S mean that $\mu_{11} = 0$ is an L_1-simple eigenvalue of $L_0 = f_v(\lambda_0, 0)$. For $|\lambda|$ and $|s|$ small the operators $f_v(\lambda,0)$ and $f_v\big(\lambda(s), sv_1 + s\phi(s)\big)$ are close to L_o, thus, by Lemma 4. CR there are unique $\mu_0(\lambda) = \mu\big(f_v(\lambda,0)\big)$, $v_0(\lambda) = v\big(f_v(\lambda,0)\big)$, and unique $\mu(s) = \mu\big(f_v(\lambda(s), sv_1 + s\phi(s))\big)$, $v_b(s) = v\big(f_v(\lambda(s), sv_1 + s\phi(s))\big)$ such that

$$f_v(\lambda,0)v_0(\lambda) = \mu_0(\lambda)v_0(\lambda), \tag{4.1}$$

$$f_v\big(\lambda(s), sv_1 + s\phi(s)\big)\, v_b(s) = \mu(s)v_b(s), \tag{4.2}$$

$\mu_0(\lambda_0) = \mu(0) = \mu_{11} = 0$, and $v_0(\lambda_0) = v_b(0) = v_1$. The functions μ_0 and μ are smooth being compositions of smooth functions.

THEOREM 4. CR. *Let the assumption of Theorem* 3.S *hold, and let the functions* μ_0 *and* μ *defined as above. Then* $\mu_0'(\lambda)0) \neq 0$, *and if* $\mu(s) \neq 0$ *for* $|s|$ *small,* $s \neq 0$, *then*

(4.3) $$\lim_{s \to 0} \left(s\lambda'(s)\mu_0'(\lambda_0)/\mu(s) \right) = -1.$$

We are going to apply formula (4.3) in the situation of Theorem 3.2 (ii), i.e. for the bifurcating non-constant stationary solution of the nonlinear problem (2.3)–(2.4). Our aim is to determine the sign of the bifurcating eigenvalue $\mu(s)$ for small $|s|$. The following calculation aims at the determination of the sign of the other quantities in (4.3). In the context of Theorem 3.2 the role of the bifurcation parameter λ is played by d whose critical value d_0 is given by (3.5), the role of the nonlinear operator f is played by the left hand side of (3.10) denoted by $T(d, V)$.

The determination of $\mu'(d_0)$ is fairly easy. We know that $\mu_0(d)$ satisfies

$$\mu_0^2(d) - \text{Tr } B_1(d)\mu_0(d) + \det B_1(d) \equiv 0.$$

By implicit differentiation

(4.4) $$\mu_0'(d_0) = \lambda_1(\lambda_1 d_N - \eta\Theta_1\Theta_2)/(\text{Tr } A - (d_N + d_0)\lambda_1) > 0$$

where (2.17)–(2.18) were used, and (3.4) was taken into account.

It is more tricky to determine the sign of $d'(s)$ which plays the role of $\lambda'(s)$ in (4.3). First we substitute $d(s)$ and the bifurcating solution

$$V(s, x) = s y_{11} \cos(\pi x/l) + s\phi(s, x)$$

into (3.10). Denoting as before the function $y_{11}\cos(\pi x/l)$ by v_1 we get

$$D(s)\big(- s(\pi/l)^2 v_1 + s\phi_{xx}(s, \cdot)\big) + sAv_1 + sA\phi(s, \cdot) + H\big(sv_1 + s\phi(s, \cdot)\big) \equiv 0$$

where the matrix D depends on s through its element $d(s)$. Dividing the last identity by s, differentiating with respect to s and setting $s = 0$ we obtain

$$D(0)\phi_{xx}(0, \cdot) - D'(0)(\pi/l)^2 v_1 + D'(0)\phi_{xx}(0, \cdot) +$$

$$+ A\phi'(0, \cdot) + H(v_1) = 0$$

where prime denotes differentiation with respect to s and

(4.5) $$D'(0) = \begin{bmatrix} 0 & 0 \\ 0 & d'(0) \end{bmatrix}.$$

Taking into account that $\phi(0,\cdot)=0$ we have finally

$$D(0)\phi'_{xx}(0,\cdot)-(\pi/l)^2D'(0)v_1+A\phi'(0,\cdot)+H(v_1)=0. \tag{4.6}$$

Let us take now the scalar product of the left hand side with v_1:

$$\langle v_1, D(0)\phi'_{xx}(0,\cdot)\rangle-(\pi/l)^2\langle v_1, D'(0)v_1\rangle+$$
$$+\langle v_1, A\phi'(0,\cdot)\rangle+\langle v_1, H(v_1)\rangle=0. \tag{4.7}$$

Applying integration by parts twice in the first term we get that the sum of the first and the third term is

$$\langle v_1, D(0)\phi'_{xx}(0,\cdot)\rangle+\langle v_1, A\phi'(0,\cdot)\rangle=\langle v_1, B_1\phi'(0,\cdot)\rangle \tag{4.8}$$

where $B_1=A-\lambda_1D(0)$ according to (2.14). Let us expand now $\phi'(0,\cdot)\in$ $\in X$ and $H(v_1)\in X$ (see (3.7)) according to $X=[\text{Span } v_1]\oplus\mathcal{Z}$ where $\mathcal{Z}$ is given by (3.12):

$$\phi'(0,\cdot)=a_1v_1+z_1,\quad H(v_1)=av_1+z \tag{4.9}$$

where $a_1,a\in\mathbf{R}$, $z_1,z\in\mathcal{Z}$. Substituting this expansion of $\phi'(0,\cdot)$ into the right hand side of (4.8) we get

$$\langle v_1, B_1(a_1v_1+z_1)\rangle=a_1\mu_{11}\langle v_1,v_1\rangle+\langle v_1,B_1z_1\rangle=\langle v_1,B_1z_1\rangle$$

since $\mu_{11}=0$. Substituting the expansion of $H(v_1)$ into the last term of (4.7) we obtain

$$\langle v_1, av_1+z\rangle=a\langle v_1,v_1\rangle.$$

Finally, for the second term of (4.7) we have, taking into account (3.6) and (4.5), that

$$-(\pi/l)^2\langle v_1, D'(0)v_1\rangle=-\lambda_1(\eta^2)^2d'(0)\int_0^l\cos^2(\pi x/l)\,dx=$$
$$=-\lambda_1(\eta^2)^2d'(0)l/2.$$

Summing up these results (4.7) assumes the form

$$-\lambda_1(\eta^2)^2d'(0)l/2+a\langle v_1,v_1\rangle+\langle v_1,B_1z_1\rangle=0.$$

Hence,

$$\operatorname{sgn} d'(0)=\operatorname{sgn}(a\langle v_1,v_1\rangle+\langle v_1,B_1z_1\rangle). \tag{4.10}$$

The number a can be expressed explicitly in terms of known quantities. We have to fix somehow the coordinates η^1 and η^2 of the vector y_{11} in (3.6). Since now $B_1 y_{11} = 0$ and $\det B_1 = 0$, we have

$$(\eta\Theta_1\Theta_2 - (\pi/l)^2 d_N)\eta^1 - \Theta_1\eta^2 = 0,$$

i.e. we may choose e.g.

$$\eta^1 = \Theta_1, \quad \eta^2 = \eta\Theta_1\Theta_2 - (\pi/l)^2 d_N. \tag{4.11}$$

Then according to the definition (3.11) of $H(V)$

$$H\big(v_1(x)\big) = F\big(\overline{N} + \eta^1\cos(\pi x/l), \overline{P} + \eta^2\cos(\pi x/l)\big) -$$

$$- \begin{bmatrix} \eta\Theta_1\Theta_2 & \Theta_1 \\ \beta^2\eta\Theta_3 & -\eta\Theta_3\Theta_4 \end{bmatrix} \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \cos(\pi x/l).$$

Denoting the first and the second coordinates of the last expression of $H(v_1)$ by $H^1(x)$ and $H^2(x)$, resp., we get according to (4.9) that

$$\begin{bmatrix} H^1(x) \\ H^2(x) \end{bmatrix} = a \begin{bmatrix} \eta^1 \\ \eta^2 \end{bmatrix} \cos(\pi x/l) + \dots$$

where we did not write out the remainder of the Fourier series containing the terms with $\cos(k\pi x/l)$, $k = 2, 3, \dots$. Taking the scalar product of both sides with v_1 we obtain

$$a = \frac{2}{l\big((\eta^1)^2 + (\eta^2)^2\big)} \int_0^l \big(\eta^1 H^1(x) + \eta^2 H^2(x)\big) \cos(\pi x/l)\, dx. \tag{4.12}$$

THEOREM 4.1. *Assume that the conditions of Theorem* 3.2 (ii) *hold, and that the eigenvalue $\mu(s)$ of the non-constant stationary solution bifurcating from the critical eigenvalue $\mu_{11} = \mu(0) = 0$ is non-zero for small $|s| \neq$ $\neq 0$. If* (4.10) *is positive (resp. negative) then the bifurcating non-constant stationary solution of the non-linear problem* (2.3)–(2.4) *is asymptotically stable (resp. unstable) for $s > 0$, and unstable (resp. asymptotically stable) for $s < 0$, $|s|$ small.*

PROOF. If (4.10) is positive then because of continuity $d'(s) > 0$ for $|s|$ small. In our notation (4.3) is

$$\lim_{s\to 0} \big(s d'(s)\mu_0'(d_0)/\mu(s)\big) = -1,$$

and by (4.4) $\mu_0'(d_0) > 0$. Thus, $\mu(s) < 0$ for $s > 0$, $\mu(s) > 0$ for $s < < 0$. Since for small enough $|s|$ the rest of the eigenvalues stay near μ_{ij} ($i = 1, 2$; $j = 0, 2, 3, \ldots$) and to μ_{21} in the left half plane, for $s > 0$ the bifurcating solution is asymptotically stable, for $s < 0$ it is unstable. In case (4.10) is negative the proof is similar. □

COROLLARY 4.2. *Under the assumptions of the previous theorem if* (4.10) *is non-zero then there is a* $\delta > 0$ *such that for* $d \in (d_0 - \delta, d_0 + \delta)$ *the problem* (2.3)–(2.4) *has a non-constant stationary solution which is asymptotically stable for* $d > d_0$ *and unstable for* $d < d_0$.

PROOF. (See Fig. 2.) If (4.10) is positive then $d'(s)$ is positive therefore $d(s) > d_0$ for $s > 0$ and $d(s) < d_0$ for $s < 0$. If (4.10) is negative then $d'(s) < 0$, and $d(s) > d_0$ for $s < 0$, $d(s) < d_0$ for $s > 0$. □

This result, obviously, means that in our system in the generic case as the predator diffusion rate d is increased beyond the critical value d_0, and the spatially constant solution loses its stability, a stable small amplitude pattern arises.

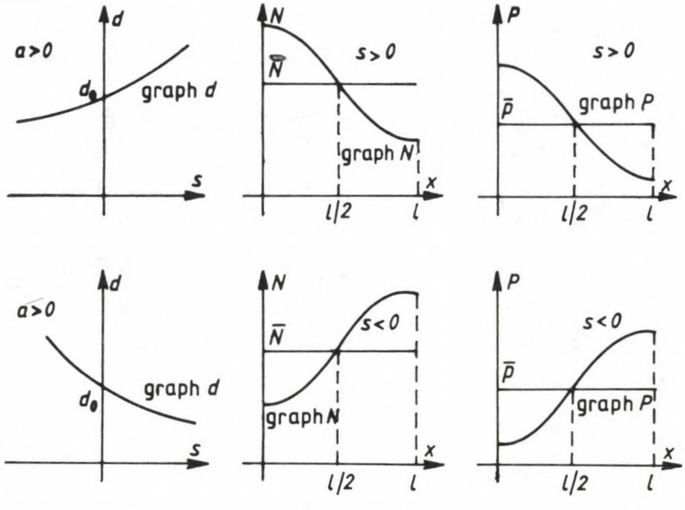

Fig. 2

The graph of $d(s)$ *and of the two coordinates of the stable pattern* (3.13) *choosing* η^1, η^2 *according to* (4.11)

EXAMPLE. We may continue the calculations with the data of the Example of Section 2. If we fix $l = 1$ for the length of the habitat then the interval (3.4) becomes

$$0.000058 \leqq d_N < 0.000232.$$

Fix $d_N = 0.0001$ in this interval. Then the critical value of the predator diffusion coefficient d at which the bifurcation takes place is $d_0 = 0.0385$ by (3.5).

5. Discussion

System (2.1) describes the dynamics of a predator-prey interaction. Prey quantity grows logistically in the absence of predation, predator mortality increases with predator density but is bounded, there is a Holling-type functional response, and both species are subject to Fickian diffusion in a one dimensional spatial habitat from which and into which there is no migration. It is assumed that the system has a positive equilibrium in the Allée-effect zone, i.e. in a neighbourhood of the equilibrium the increase of prey density is beneficial to prey's growth rate.

If the prey diffusion rate is relatively high compared e.g. to the square of the length of the spatial domain (condition (3.3)) then this equilibrium is locally asymptotically, stable. If the prey diffusion rate is lower (condition (3.4)) then one may increase the predator diffusion rate to a value (3.5) higher than the prey diffusion rate at which this equilibrium loses its stability. To be sure, the equilibrium point considered as an equilibrium of the system without diffusion stays stable, i.e. a so called diffusional instability occurs quite naturally, contrary to the general belief that diffusion usually stabilizes systems. At the critical value of the bifurcation parameter (the predator diffusion rate) a *pattern* (a non-constant stationary solution) arises. A first order approximation of this pattern (3.13) is explicitly given. A quantity (4.10) is given; if this quantity is non-zero then for values of the predator diffusion rate higher than the critical one the pattern is stable, for predator diffusion rates lower than the critical the pattern still exists but is unstable.

In a subsequent paper we are going to study what happens if delay is introduced into the system with diffusion which has a stable pattern. We expect to establish an Andronov–Hopf bifurcation superimposed upon the Turing bifurcation, i.e. we expect that the pattern begins to oscillate periodically in time.

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REARRANGEMENT OF FOURIER SERIES AND FOURIER SERIES WHOSE TERMS HAVE RANDOM SIGNS

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1. Introduction

Let us denote by $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ the one dimensional torus, $L^2 := L^2(\mathbf{T})$, and $C := C(\mathbf{T})$ the sets of square (Lebesgue) integrable functions and continuous functions, resp., and $\varepsilon = (\varepsilon_k)$ a Rademacher system on $I = [0, 1]$. In the probability space $(I, \mathcal{L}, P)$ belonging to the Rademacher system ε, $\mathcal{L}$ is the algebra of Lebesgue measurable sets in I and the probability measure P is the Lebesgue measure on I. The expectation with respect to this probability space will be denoted by E throughout the paper.

Following Zygmund [12] we write for the Fourier series of any $f \in L^2$

$$(1) \qquad \begin{cases} f \sim S(f, \cdot) := \sum_{n=0}^{\infty} A_n, & S_n := S_n(f, \cdot) := \sum_{j=0}^{n} A_j, \\ A_j(x) := c_j \cos(jx + \Theta_j) := a_j \cos jx + b_j \sin jx. \end{cases}$$

The series coming from f by giving random signs to its terms is

$$(2) \qquad f_\varepsilon \sim \sum_{n=0}^{\infty} \varepsilon_n A_n,$$

and the Pisier-algebra $\mathcal{P}$ is

$$(3) \qquad \mathcal{P} := \{f \in L^2 : \quad P(f_\varepsilon \in C) = 1\}.$$

The characterization of $\mathcal{P}$ was a long-standing problem of the theory of Fourier series initiated by Payley and Zygmund [5] in 1930. For history and development we refer to [3]. The problem was finally solved by Marcus and Pisier [4] in 1978.

Another old but still open problem is the following. Denote $\nu : \mathbf{N} \leftrightarrow \mathbf{N}$ any permutation of $\mathbf{N}$, and introduce for any f with Fourier series (1) the

ν-rearrangement of the series and the corresponding partial sums as

$$(4)\qquad f \underset{\nu}{\sim} {}_\nu S(f) := \sum_{n=0}^{\infty} A_{\nu(n)}, \quad {}_\nu S_n(f) := \sum_{j=0}^{n} A_{\nu(j)}.$$

The class

$$(5)\qquad U := \{f \in C : \exists \nu : \mathbf{N} \leftrightarrow \mathbf{N}, \quad {}_\nu S_n \to f \quad \text{uniformly on} \quad \mathbf{T}\}$$

is a subspace of C. The problem of deciding if $U = C$ or not, was posed already in 1962 by Ulyanov, cf. [10] pp. 58–59, or [9].

In 1986 the following result was proved [6]. For any $f \in C$ there exist a rearrangement ν and a subsequence (n_k) of $\mathbf{N}$ such that ${}_\nu S_{n_k} \to f$ $(k \to \infty)$ uniformly on $\mathbf{T}$. On the basis of this and other results the conjecture $U = C$ was formulated and an equivalent finite version was given in [6], where the reader may find more about historical background and motivation of the problem.

The aim of the present paper is to study the connection between the classes $\mathcal{P}$ and U.

THEOREM 1 (Pecherskiĭ [13]). $\mathcal{P} \cap C \subset U$.

THEOREM 2. *There exists $f \in U$ with $f \notin \mathcal{P}$.*

As a by-product we obtain several other criteria for an $f \in C$ to belong to U. The results of the paper were obtained in 1987 and form a part (essentially Chapter III.2) of the thesis [7]. The author would like to express his gratitude to Professors G. Halász, B. Kashin and S. Konjagin for useful comments and references. In particular, Professor S. Konjagin called the attention of the author to a recent paper of D. V. Pecherskiĭ [13]. The paper deals with related problems using a key lemma (Lemma 1 in the paper) which is somewhat similar to Chobanjan's Lemma quoted here as Lemma 4. Using his new lemma, Pecherskiĭ proved Theorem 1 of this work as Theorem 2 of [13], cf. p. 25. Hence this result must be attributed to Pecherskiĭ, as his Theorem, and Section 3 of this work describes only a second although independent and different proof for it. Let us mention that this proof was worked out in 1987, before the appearance of [13].

2. Some lemmas

LEMMA 1. *For all $f \in \mathcal{P}$ we have*

$$\delta_n := \delta_n(f) := \sup_{m \geq n} E\|S_m(f_\varepsilon, \cdot) - S_n(f_\varepsilon, \cdot)\|_\infty \to 0 \quad (n \to \infty).$$

PROOF. Well-known, see e.g. [3] Ch. 2, Theorem 4 and Ch. 5, Theorem 3.

Now let us denote the de la Vallée Poussin means of f by

$$(6)\qquad V_n := V_n(f) := \sum_{j=0}^{n} A_j + \sum_{j=n+1}^{2n} \left(2 - \frac{j}{n}\right) A_j.$$

We also introduce for any $f \in L^2$ with Fourier series (1) the usual notation

$$(7)\qquad s_k := \sqrt{\sum_{n=2^k+1}^{2^{k+1}} c_n^2}.$$

LEMMA 2. *For any $f \in L^2$ and $k > 2$ there exists a 0–1 sequence $\omega = (\omega_i)$ with $i = 2^k + 1, \dots, 2^{k+1}$ such that*

$$\left\| S_n(f) + \sum_{i=n+1}^{2n} \omega_i A_i - V_n(f) \right\|_\infty \leqq 8\sqrt{k} \cdot s_k \quad \textit{with} \quad n = 2^k.$$

PROOF. This follows from Lemma 2 of [6].

LEMMA 3. *Let $P(x) = \sum_{i=0}^{N} A_i(x)$ be any trigonometric polynomial of degree not exceeding N. Then we have*

$$E\|P_\varepsilon\|_\infty = E\left\| \sum_{0}^{N} \varepsilon_i A_i \right\|_\infty \leqq 2\sqrt{\log N}\, \|P\|_2.$$

PROOF. See [8], (5.1.2) Lemma, p. 290.

LEMMA 4. *Let X be any normed space and $\mathbf{x}_1, \dots, \mathbf{x}_N$ be elements of X. There exists a permutation σ of $\{1, 2, \dots, N\}$ such that*

$$\max_{M \leqq N} \left\| \sum_{i=1}^{M} \mathbf{x}_{\sigma(i)} \right\|_X \leqq 9 \left\| \sum_{i=1}^{N} \mathbf{x}_i \right\|_X + 9E \left\| \sum_{i=1}^{N} \varepsilon_{j_i} \mathbf{x}_i \right\|_X,$$

where $(\varepsilon_{j_i}) \subset \varepsilon$ with $j_i \neq j_k$ $(i \neq k)$ for $i = 1, 2, \dots, N$.

PROOF. This is Corollary 1 on p. 56 of [1].

LEMMA 5. *Suppose that $f \in L^2$ and s_k is non-increasing. Then $f \in \mathcal{P}$ if and only if*

$$\sum_{k=1}^{\infty} s_k < \infty.$$

PROOF. Necessity is proved in [5], and sufficiency is contained in [3], Ch. 7, Theorem 1.

LEMMA 6. *Let c_n be any sequence satisfying the conditions*

i) $\sum_1^{\infty} c_n^2 < \infty$,

ii) $1/c_n$ *is concave,*

iii) c_n *is monotonically decreasing.*

Then there exists an $f \in C$ with Fourier series (1), *i.e. for some* (Θ_n) *the L^2 series described in* (1) *belongs to a continuous function.*

PROOF. This is a well-known result of Salem, cf. [12] Ch V, (10.1) Theorem. We note that the statement is true even if ii) is not supposed, see [2].

3. Proof of Theorem 1

Let us take any $f \in \mathcal{P}$. Since $f \in L^2$, $\sum s_k^2 = \|f\|_2 < \infty$ and for every $\eta > 0$ one can find a $k \in \mathbf{N}$ with $k \cdot s_k^2 < \eta$. That is, we have some $k_j \to \infty$ with

$$s_{k_j}^2 \leqq \frac{\eta_j}{k_j}, \quad \eta_j \to 0 \quad (j \to \infty). \tag{8}$$

We define ν as the composition of two other permutations,

$$\nu = \sigma \circ \pi \tag{9}$$

First we construct π as the disjoint union of permutations π_k on the blocks $\left[2^k + 1, 2^{k+1}\right]$. For $k = k_j > 2$ we apply Lemma 2 and define π_k with

$$\pi_k(i) := \begin{cases} \min\{p : \omega_p = 1 \text{ and } p \neq \pi_k(l) \text{ for any } l < p\}, & n < i \leqq m_j \\ \min\{p : p \neq \pi_k(l) \text{ for any } m_j < l < p\}, & m_j < i \leqq 2n, \end{cases} \tag{10}$$

where

$$n = 2^k = 2^{k_j} \quad \text{and} \quad m_j = n + \sum_{i=n+1}^{2n} \omega_i. \tag{11}$$

That is, π_k^{-1} places the indices with $\omega_i = 1$ to the beginning of the interval $[n+1, 2n]$, and places the indices with $\omega_i = 0$ to the other end. So for $k = k_j$ we have by the definition of π_k

$$S_{2^k} + \sum_{i=2^k+1}^{2^{k+1}} \omega_i A_i = S_{2^k} + \sum_{i=2^k+1}^{m_j} A_{\pi_k(i)} \quad (k = k_j). \tag{12}$$

Now for $k \neq k_j$ we choose π_k to be identity and take

$$\pi := \bigcup_{k=0}^{\infty} \pi_k. \tag{13}$$

Trivially

$${}_\pi S_{2^k} = S_{2^k} \tag{14}$$

and hence from Lemma 2, (8), (12), (13) and (14) we get

$$\left\|{}_\pi S_{m_j} - V_{2^{k_j}}\right\|_\infty \leqq 8\sqrt{\eta_j}. \tag{15}$$

Next we define σ so that

$$\sigma := \bigcup_{j=0}^{\infty} \sigma_j, \quad \sigma_j : [m_j + 1, m_{j+1}) \leftrightarrow [m_j + 1, m_{j+1}). \tag{16}$$

Here we can take $m_0 := -1$ and σ_0 to be identity on $[0, m_1)$. Consider the polynomial

$$T_j :=_\pi S_{m_{j+1}} -_\pi S_{m_j}. \tag{17}$$

Our goal is to rearrange the order of the terms of T_j by σ_j to ensure small partial sums. We apply Lemma 4 in the Banach space C with the ∞-norm and $N = m_{j+1} - m_j$, $\mathbf{x}_i = A_{\pi(i+m_j)}$, $j_i = \pi(i + m_j)$ $(i = 1, \ldots, N)$. We obtain a certain σ_j with

$$\max_M \left\|\sigma_j(T_j)_M\right\|_\infty \leqq 9\|T_j\|_\infty + 9\left\|(T_j)_\varepsilon\right\|_\infty. \tag{18}$$

Note that $f \in \mathcal{P}$, and Lemma 1 entails

$$E\left\|\left(S_{2^{k_j+1}} - S_{2^{k_j}}\right)\right\|_\infty \to 0 \quad (j \to \infty). \tag{19}$$

Since the left hand side of (12) is exactly ${}_{\pi}S_{m_j}$, we have

$$P_j := \sum_{i=n+1}^{2n} \omega_i A_i = {}_{\pi}S_{m_j} - {}_{\pi}S_n \quad \left(n = 2^{k_j}\right) \tag{20}$$

and also

$$T_j = {}_{\pi}S_{m_{j+1}} - {}_{\pi}S_{m_j} = P_{j+1} - P_j - \left(S_{2^{k_{j+1}}} - S_{2^{k_j}}\right). \tag{21}$$

Hence in view of (19)

$$E\left\|(T_j)_\varepsilon\right\|_\infty \leqq E\left\|(P_j)_\varepsilon\right\|_\infty + E\left\|(P_{j+1})_\varepsilon\right\|_\infty + o(1) \quad (j \to \infty). \tag{22}$$

Now further use of (8), $\|P_j\|_2 \leqq s_{k_j}$, $\deg(P_j) \leqq 2^{k_j}$ and Lemma 3 ensure

$$E\left\|(P_j)_\varepsilon\right\|_\infty \to 0 \quad (j \to \infty), \tag{23}$$

hence from (22) and (23)

$$E\left\|(T_j)_\varepsilon\right\|_\infty \to 0 \quad (j \to \infty). \tag{24}$$

Next we make use of the continuity of f in the form that $V_n(f) \to f$ uniformly on $\mathbf{T}$, cf. [11]. Hence (15) and $f \in C$ entails $\|T_j\|_\infty \to 0$ $(j \to 0)$ and so (18) and (24) imply

$$\max_M \left\|\sigma_j(T_j)_M\right\|_\infty \to 0 \quad (j \to \infty). \tag{25}$$

We obtain from (16), (17) and (25) that

(26)
$$\max_{m_j \leqq M \leqq m_{j+1}} \left\|\sigma_j(T_j)_M\right\|_\infty = \max_{m_j \leqq M \leqq m_{j+1}} \left\|{}_{\sigma\cdot\pi}f_M - {}_{\pi}S_{m_j}\right\|_\infty \to 0 \; (j \to \infty).$$

Since (15) and $f \in C$ entails $\left\|{}_{\pi}S_{m_j} - f\right\|_\infty \to 0$, (26) and (9) concludes the proof of Theorem 1.

4. Further criteria for $f \in U$

THEOREM 3. *If $f \in C$ and s_k is nonincreasing, then $f \in U$.*

To prove this theorem first we note that in view of monotonicity and $f \in C \subset L^2$ we have $s_k = o\left(\frac{1}{\sqrt{k}}\right)$. Hence it suffices to prove the following

THEOREM 4. *If $f \in C$ and $s_k = o\left(\frac{1}{\sqrt{k}}\right)$, then $f \in U$.*

PROOF. The proof is very similar to that of Theorem 1.

If we define $k_j := j$ and $\eta_j := j \cdot s_j^2$, we get (8) with $k_j = j$ according to our assumption on s_k. Now repeating the proof of Theorem 1 with $k_j = j$ the only change is that to prove (19), instead of using Lemma 1, we refer to Lemma 3. Since $f \in \mathcal{P}$ was used only there, this modification proves Theorem 4.

COROLLARY. *If $f \in C$ satisfies the multiplier condition*

$$\sum_{n=1}^{\infty} c_n^2 \cdot \log n < \infty,$$

then $f \in U$.

Professor B. Kashin informed the author that this was conjectured more than ten years ago. It can be compared to the multiplier condition

$$\sum c_n^2 \cdot \log^{1+\varepsilon} n < \infty$$

of Payley and Zygmund to ensure $f \in \mathcal{P}$.

5. Proof of Theorem 2

Let us define

$$c_n := \frac{1}{\sqrt{n} \cdot \log n} \quad (n \geqq 10) \tag{27}$$

and $c_n = c_{10}$ for $n \leqq 10$, say. This sequence satisfies conditions i), ii) and iii) of Lemma 6, hence there exists an $f \in C$ with Fourier series (1), where c_n is defined in (27). Obviously (7) and (27) mean for s_k that

$$\sum s_k^2 < \infty, \quad \sum s_k = \infty \tag{28}$$

and also

(29)
$$s_k^2 = \sum_{2^k+1}^{2^{k+1}} \frac{1}{n\log^2 n} \geqq \sum_{2^k+1}^{2^{k+1}} \left(\frac{1}{(2n)\log^2(2n)} + \frac{1}{(2n-1)\log^2(2n-1)} \right) =$$
$$= s_{k+1}^2,$$

i.e. s_k is monotonic. Now for monotonic s_k (28) and Lemma 5 ensure $f \notin \notin \mathcal{P}$, while Theorem 3 gives $f \in U$. This concludes the proof of Theorem 2.

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