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UNIQUENESS THEOREMS FOR WALSH SERIES UNDER A STRONG CONDITION

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1. Introduction

Let

$$\mu = \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x)$$

be a Walsh series and $\mathcal{A}$ a certain class of Walsh series. When E is a subset of the dyadic group, it is called a set of uniqueness for $\mathcal{A}$, if $\mu \in \mathcal{A}$ and

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{everywhere except on } E$$

imply that $\hat{\mu}(k) = 0$ for $k = 0, 1, 2, \ldots$. When E is not a set of uniqueness for $\mathcal{A}$, it is called a set of multiplicity for $\mathcal{A}$.

It is easy to see that a subset of the dyadic group is a set of uniqueness for the class of all Walsh series μ such that

$$\sum_{k=0}^{\infty} |\hat{\mu}(k)|^2 < \infty,$$

if and only if it is of measure zero. This class of Walsh series coincides with L^2-space.

In this paper we shall consider the uniqueness problem for the class of all Walsh series μ such that

$$\left| \sum_{k=2^n}^{2^{n+1}-1} \hat{\mu}(k) w_k(x) \right| \equiv 2^n |\Delta m_\mu(I_n(x))| = o(1) \quad \text{uniformly in } x \text{ as } n \to \infty.$$

Let $\mathcal{B}$ be the class of these Walsh series. It is easy to see that $\mathcal{B}$ and L^2 are not subsets of each other.

We shall prove the following three theorems.

THEOREM 1. *Assume that E is a subset of the dyadic group, and there exists a couple of sequences of integers*

$$\{N_n\}_n \equiv \{N_n(x)\}_n \quad \textit{and} \quad \{k_n\}_n \equiv \{k_n(x)\}_n$$

for each $x \in E$ such that

(i) $N_n \uparrow \infty$ *as* $n \to \infty$*;*

(ii) $\liminf_{n\to\infty} k_n < \infty$;

(iii) *there exists a dyadic interval* $I_{N_n+k_n}$ *of rank* $N_n + k_n$ *such that*

$$I_{N_n+k_n} \subsetneqq I_{N_n}(x) \quad \text{and} \quad I_{N_n+k_n} \cap E = \emptyset \quad \text{for} \quad n = 1, 2, \ldots .$$

Then E *is a set of uniqueness for* $\mathcal{B}$.

An $H^{(1)}$-set on the dyadic group satisfies the above condition (see [3]).

THEOREM 2. *There exists a set of multiplicity for* $\mathcal{B}$ *which is of measure zero.*

THEOREM 3. *A set of positive measure is a set of multiplicity for* $\mathcal{B}$.

Let m_μ be the dyadic measure associated with a Walsh series μ, that is,

$$m_\mu(I_n^p) = \lim_{N\to\infty} \int_{I_n^p} \sum_{k=0}^{N} \hat{\mu}(k) w_k(x)dx = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k\Big(\frac{p}{2^n}\Big)$$

where I_n^p is the set of all $0-1$ sequences, $x = (t_1, t_2, \ldots)$ such that

$$\sum_{k=1}^{n} \frac{t_k}{2^k} = \frac{p}{2^n}$$

for $p = 0, 1, \ldots, 2^n - 1$. I_n^p is called a dyadic interval of rank n. Throughout this paper, $I, I', \ldots$ denote dyadic intervals. $I_n(x)$ denotes the dyadic interval of rank n which contains x.

We identify $x = (t_1, t_2, \ldots)$ with the number $\sum_{k=1}^{\infty} \frac{t_k}{2^k}$ if $\lim_{k\to\infty} t_k \neq 1$ and write

$$x = \Big(\sum_{k=1}^{\infty} \frac{t_k}{2^k}\Big)^-$$

if $\lim_{k\to\infty} t_k = 1$. For details of the dyadic group and dyadic measures we refer the reader to [2], [4], and [5].

2. Proof of Theorem 1

If $x \in E$, then there exists a finite set of dyadic intervals $\{I'_{N_n+j}\}_j$ such that

$$I_{N_n}(x) \supsetneqq I'_{N_n+1} \supsetneqq \cdots \supsetneqq I'_{N_n+k_n} = I_{N_n+k_n}^{p_n}$$

for $n = 1, 2, \ldots$. If $\mu \in \mathcal{B}$ and

$$\lim_{j\to\infty} \sum_{k=0}^{2^j-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{everywhere on} \quad I_{N_n+k_n}^{p_n},$$

then by Corollary 1 of [6] we have $m_\mu(I) = 0$ for each $I \subset I_{N_n+k_n}^{p_n}$. Since

$$\Delta m_\mu(I_n^p) = m_\mu(I_{n+1}^{2p}) - m_\mu(I_{n+1}^{2p+1}),$$

we have

$$m_\mu(I_{N_n}(x)) = m_\mu(I_{N_n}(x) \setminus I'_{N_n+1}) + m_\mu(I'_{N_n+1}) =$$
$$= \pm\Delta m_\mu(I_{N_n}(x)) + 2_\mu(I'_{N_n+1}) = o\Big(\frac{1}{2^{N_n}}\Big) + 2m_\mu(I'_{N_n+1}) =$$
$$= o\Big(\frac{1}{2^{N_n}}\Big) + 2\{m_\mu(I'_{N_n+1} \setminus I'_{N_n+2}) + m_\mu(I'_{N_n+2})\} =$$
$$= o\Big(\frac{1}{2^{N_n}}\Big) + 2\{\pm\Delta m_\mu(I'_{N_n+1}) + 2m_\mu(I'_{N_n+2})\} =$$
$$= o\Big(\frac{1}{2^{N_n}}\Big) + 2\cdot o\Big(\frac{1}{2^{N_n+1}}\Big) + 4m_\mu(I'_{N_n+2}) = \ldots =$$
$$= o\Big(\frac{1}{2^{N_n}}\Big) + \ldots + o\Big(\frac{1}{2^{N_n}}\Big) + 2^{k_n} m_\mu(I'_{N_n+k_n}) = o\Big(\frac{k_n}{2^{N_n}}\Big) \text{ as } n \to \infty.$$

Thus we proved that

$$2^{N_n} m_\mu(I_{N_n}(x)) = o(k_n) \quad \text{as} \quad n \to \infty.$$

From the assumption we have

$$\liminf_{n\to\infty} 2^n |m_\mu(I_n(x))| = \liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x) \right| = 0 \text{ everywhere on } E.$$

Hence we proved that

$$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x) \right| = 0 \quad \text{everywhere.}$$

By Lemma 1 of [6], we have $\hat\mu(k) = 0$ for $k = 0, 1, \ldots$.

From Theorem 3, the set introduced in Theorem 1 is of measure zero.

3. Proof of Theorem 2

To prove Theorem 2, we need the following Lemma.

LEMMA 4. *When N is a positive integer, there exists a nonnegative dyadic measure m_μ which satisfies the following conditions:*

(i) $m_\mu(I_0^0) = 1$;

(ii) $\lim_{n\to\infty} 2^n m_\mu(I_n(x)) = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat\mu(k) w_k(x) = 0$ *except some perfect set of measure zero;*

(iii) $|\Delta m_\mu(I_n(x))| = \frac{1}{2^n} \left| \sum_{k=2^n}^{2^{n+1}-1} \hat\mu(k) w_k(x) \right| \leqq \frac{1}{N 2^n}$ *for all x and n.*

PROOF. Suppose that $N \geqq 2$. Put $m_\mu(I_0^0) = 1$,

$$m_\mu(I_1^p) = \begin{cases} \frac{1}{2} + \frac{1}{4N} & \text{for } p = 0; \\ \frac{1}{2} - \frac{1}{4N} & \text{for } p = 1 \end{cases}$$

and

$$m_\mu(I_2^p) = \begin{cases} \frac{1}{4} + \frac{1}{8N} + \frac{1}{8N} & \text{for } p = 0; \\ \frac{1}{4} + \frac{1}{8N} - \frac{1}{8N} & \text{for } p = 1; \\ \frac{1}{4} - \frac{1}{8N} + \frac{1}{8N} & \text{for } p = 2; \\ \frac{1}{4} - \frac{1}{8N} - \frac{1}{8N} & \text{for } p = 3. \end{cases}$$

Continuing in this way, we have

$$m_\mu(I_{2N}^p) \begin{cases} > 0 & \text{for } 0 \leqq p < 2^{2N} - 1 \\ = 0 & \text{for } p = 2^{2N} - 1. \end{cases}$$

If $I \subset I_{2N}^{2^{2N}-1}$, then put $m_\mu(I) = 0$. It is easy to see that the number of dyadic intervals of rank $2N$ satisfying

$$m_\mu(I_{2N}^p) = \frac{2k}{N 2^{2N}}$$

is $\binom{2N}{k}$.

If $m_\mu(I_{2N}^p) > 0$, continue in this way on each I_{2N}^p. The number of dyadic intervals of rank $(2N + 2k)$ satisfying

$$m_\mu(I_{2N+2k}^p) = \frac{2k}{N 2^{2N}} \cdot \frac{2k'}{k 2^{2k}}$$

is $\binom{2N}{k} \cdot \binom{2k}{k'}$.

Continuing in this way, we can construct the dyadic measure m_μ.

Let S be the sum of dyadic intervals I such that $m_\mu(I) = 0$. Thus the measure of S is given by the following equation:

$$|S| = \frac{1}{4^N} + \sum_{k_1=1}^{2N} \binom{2N}{k_1} \frac{1}{4^{N+k_1}} + \sum_{k_1=1}^{2N} \binom{2N}{k_1} \sum_{k_2=1}^{2k_1} \binom{2k_1}{k_2} \frac{1}{4^{N+k_1+k_2}} +$$

$$+ \ldots + \sum_{k_1=1}^{2N} \sum_{k_2=1}^{2k_1} \cdots \sum_{k_s=1}^{2k_{s-1}} \binom{2N}{k_1} \binom{2k_1}{k_2} \cdots \binom{2k_{s-1}}{k_s} \frac{1}{4^{N+k_1+\ldots+k_s}} + \ldots .$$

Since

$$\sum_{k_s=1}^{2k_{s-1}} \binom{2k_{s-1}}{k_s} \frac{1}{4^{k_s}} = \sum_{k_2=1}^{2k_{s-1}} \binom{2k_{s-1}}{k_s} \left(\frac{1}{2}\right)^{2k_{s-1}} - 1 = \left(1 + \frac{1}{2}\right)^{2k_{s-1}} - 1,$$

we have

$$\sum_{k_{s-1}=1}^{2k_{s-2}} \binom{2k_{s-2}}{k_{s-1}} \cdot \frac{1}{4^{k_{s-1}}} \left\{ \left(1+\frac{1}{2}\right)^{2k_{s-1}} - 1 \right\} = \sum_{k_{s-1}=1}^{2k_{s-2}} \binom{2k_{s-2}}{k_{s-1}} \left\{ \frac{1}{2}\left(1+\frac{1}{2}\right) \right\}^{2k_{s-1}} -$$

$$- \sum_{k_{s-1}=1}^{2k_{s-2}} \binom{2k_{s-2}}{k_{s-1}} \cdot \frac{1}{4^{r_{s-1}}} = \left\{ 1 + \left(1+\frac{1}{2}\right) \right\}^{2k_{s-2}} - \left(1+\frac{1}{2}\right)^{2k_{s-2}}$$

and so on. Thus we have

$$\sum_{k_1=1}^{2N} \binom{2N}{k_1} \frac{1}{k_1} \cdot \sum_{k_2=1}^{2k_1} \binom{2k_1}{k_2} \cdot \frac{1}{4^{k_2}} \cdots \sum_{k_s=1}^{2k_{s-1}} \binom{2k_{s-1}}{k_s} \cdot \frac{1}{4^{k_s}} \equiv (A_s)^{2N} - (A_{s-1})^{2N},$$

where $A_0 = 0$, $A_1 = 1 + \frac{A_0}{2}, \dots, A_{k+1} = 1 + \frac{A_k}{2}, \dots$. It is obvious that

$$A_0 < A_1 < A_2 \dots .$$

Put $\lim_{k\to\infty} A_k = A$. Then A satisfies $1 + \frac{A}{2} = A$, from which $A = 2$ follows immediately. Hence we have

$$|S| = \lim_{k\to\infty} \frac{1}{4^N} \left\{ 1 + \left(1+\frac{1}{2}\right)^{2N} - 1 + \dots + \left(1 + \frac{A_k}{2}\right)^{2N} - \left(1 + \frac{A_{k-1}}{2}\right)^{2N} \right\} =$$

$$= \lim_{k\to\infty} \frac{1}{4^N} \left(1+\frac{A_k}{2}\right)^{2N} = \left(\frac{1}{4^N}\right) \cdot 2^{2N} = 1.$$

The dyadic measure m_μ is nonnegative and satisfies

$$|\Delta m_\mu(I_n(x))| = \begin{cases} 0 & \text{if } I_n(x) \subset S \\ \frac{1}{N2^n} & \text{otherwise.} \end{cases}$$

The proof is complete.

Proof of Theorem 2. Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers such that $\varepsilon_n \downarrow 0$ as $n \to \infty$. We shall construct a dyadic measure m_μ. First put $m_\mu(I_0^0) = 1$. When $N = 2$, by Lemma 4, there exist an integer n_1 and a set of measure $1, S_2$ such that

$$\Big| \bigcup_{s_2 \supset I_{n_1}^p} I_{n_1}^p \Big| > 1 - \varepsilon_1$$

and m_μ satisfies

$$\begin{cases} \lim_{n\to\infty} 2^n m_\mu(I_n(x)) = 0 & \text{if } x \in \bigcup_{S_2 \supset I_{n_1}^p} I_{n_1}^p; \\ |\Delta m_\mu(I_n(x))| \leqq \frac{1}{2^{n+1}} & \text{for all } x \text{ and } n < n_1. \end{cases}$$

Put

$$\bigcup_{x \in S_2^c} I_{n_1}(x) \equiv \sum_{k=1}^{L} I_{n_1}^{p_k} \qquad \text{(disjoint sum)}.$$

On each $I_{n_1}^{p_k}$, there exist an open subset $S^{(k)} \subset I_{n_1}^{p_k}$ and an integer $n_2(> n_1)$ such that $m_\mu(I) = 0$ if $I \subset S^{(k)}$ and

$$\Big| \bigcup_{S^{(k)} \supset I_{n_2}^p} I_{n_2}^p \Big| > \frac{1}{2^{n_1}}(1 - \varepsilon_2).$$

By Lemma 4, for a sufficiently large N, we can construct a dyadic measure m_μ on each $I_{n_1}^{p_k}$ such that

$$|\Delta m_\mu(I_n(x))| \leqq \frac{1}{3 \cdot 2^n} \quad \text{for all} \quad x \quad \text{and} \quad n_1 \leqq n < n_2.$$

Continuing in this way, we can construct m_μ on the dyadic group. Hence there exist an increasing sequence of integers $\{n_k\}_k$ and an open set S of measure 1 such that

$$\begin{cases} m_\mu(I_0^0) = 1; \\ |\Delta m_\mu(I_n(x))| \leqq \frac{1}{(k+2)2^n} \text{ for all } x \text{ and } n_k \leqq n < n_{k+1}; \\ m_\mu(I) = 0 \text{ for all } I \subset S \ \ (k = 1, 2, \dots). \end{cases}$$

Therefore S^c is a set of measure zero and a set of multiplicity for $\mathcal{B}$.

4. Proof of Theorem 3

To prove Theorem 3, we need the following lemma.

LEMMA 5. *Let* $E = \sum_{p \in \Lambda} I_N^p$ *(disjoint sum) and* $s = s^N - \#\Lambda$ *where* $\#\Lambda$ *is the number of all elements of* Λ. *For each positive integer* j *such that*

$$0 \leqq s < 2^{N+1-j},$$

there exists a nonnegative dyadic measure m_μ *which satisfies the following conditions:*

$$(1) \quad \begin{cases} \text{(i)} & m_\mu(I_0^0) = \frac{j+1}{2} - \frac{s}{2^{N+1-j}}; \\ \text{(ii)} & |\Delta m_\mu(I_n^p)| \leqq \frac{1}{2^{n+1}} \textit{ for } 0 \leqq n < N \textit{ and } p = 0, 1, \dots, 2^n - 1; \\ \text{(iii)} & m_\mu(I) = 0 \textit{ if } I \cap E = \emptyset; \\ \text{(iv)} & m_\mu(I_N^p) \leqq \frac{j+1}{2^{N+1}} \textit{ for } p = 0, 1, \dots, 2^N - 1. \end{cases}$$

PROOF. When a nonnegative dyadic measure m_μ satisfies the above conditions for some j, N and s, we call it a dyadic measure of (j, N, s)-type.

We shall prove Lemma 5 by induction. When $j = 1$, put $m_\mu(I) = |I \cap E|$. Obviously m_μ satisfies the condition (1). Thus m_μ is a dyadic measure of $(1, N, s)$-type. Lemma 5 is valid for $j = 1$.

When $j = k - 1$, $N = 0, 1, \dots$ and $0 \leqq s < 2^{N+2-k}$, assume that there exists a nonnegative dyadic measure of $(k-1, N, s)$-type for each E. We shall prove that there exists a nonnegative dyadic measure of (k, N, s)-type.

When $0 \leqq N < k$, we have $s = 0$. Put

$$(2) \qquad m_\mu(I_n^p) = \frac{(k+1)}{2^{N+1}} \quad \text{for all} \quad p.$$

It is easy to see that m_μ satisfies (1).

When $N = k$, two cases arise; $s = 0$ and $s = 1$.

When $s = 0$, define m_μ by (2).

When $s = 1$, we can assume without loss of generality that

$$I_k^{p_1} \cap E = \emptyset \quad \text{and} \quad I_k^{p_1} \subset I_1^1.$$

Let m' be a nonnegative dyadic measure of type $(k, k, 0)$ and m'' be of type $(k-1, k-1, 1)$. Put

$$(3) \qquad m_\mu(I_n^p) = \begin{cases} \frac{1}{2} m'(I_{n-1}^p) & \text{if } I_n^p \subset I_1^0; \\ \frac{1}{2} m''\left(\frac{1}{2} \dot{+} I_{n-1}^p\right) & \text{if } I_n^p \subset I_1^1. \end{cases}$$

Obviously m_μ satisfies the conditions, (ii) of (1) for $0 < n < k$ and (iii) and (iv) of (1). Since

$$\begin{cases} m_\mu(I_1^0) = \frac{1}{2} m'(I_0^0) = \frac{k+1}{4}; \\ m_\mu(I_1^1) = \frac{1}{2} m''(I_0^0) = \frac{1}{2}\left(\frac{k}{2} - \frac{1}{2}\right), \end{cases}$$

we have

$$\begin{cases} m_\mu(I_0^0) = \frac{1}{2}\left\{\frac{k+1}{2} + \frac{k}{2} - \frac{1}{2}\right\} = \frac{k+1}{2} - \frac{1}{2}; \\ |\Delta m_\mu(I_0^0)| = \frac{1}{2}\left\{\frac{k+1}{2} - \frac{k}{2} + \frac{1}{2}\right\} = \frac{1}{2}. \end{cases}$$

Hence m_μ satisfies (i) and (ii) of (1) for $n = 0$. We proved that m_μ is a dyadic measure of type $(k, k, 1)$.

Assume that Lemma 5 holds for $N = 0, 1, \dots, k, \dots q$; we shall prove it for $N = q + 1$. We can take $s = 0, 1, \dots, 2^{q+2-k}$. When $s = 2s'$, put $s_1 = s' - r$ and $s_2 = s' + r$, and assume that

$$(I_{q+1}^{p_1} \cup \dots \cup I_{q+1}^{p_{s_1}}) \cap E = \emptyset; \quad (I_{q+1}^{p_1} \cup \dots \cup I_{q+1}^{p_{s_1}}) \subset I_1^0;$$
$$(I_{q+1}^{p'_1} \cup \dots \cup I_{q+1}^{p'_{s_2}}) \cap E = \emptyset; \quad (I_{q+1}^{p'_1} \cup \dots \cup I_{q+1}^{p'_{s_2}}) \subset I_1^1.$$

When $s' < 2^{q-k}$, let m' and m'' be nonnegative dyadic measures of type (k,q,s_1) and type (k,q,s_2), respectively. Let m_μ be the dyadic measure defined by (3). Thus m_μ satisfies

$$m_\mu(I_0^0)=\frac{1}{2}\{m'(I_0^0)+m''(I_0^0)\}=\frac{1}{2}\Big\{\frac{k+1}{2}-\frac{s'-r}{2^{q+1-k}}+\frac{k+1}{2}-\frac{s'+r}{2^{q+1-k}}\Big\}=\frac{k+1}{2}-\frac{s'}{2^{q+1-k}}=\frac{k+1}{2}-\frac{s}{2^{(q+1)+1-k}}$$

and

$$|\Delta m_\mu(I_0^0)|=\frac{1}{2}\left\{\frac{k+1}{2}-\frac{s'-r}{2^{q+1-k}}-\frac{k+1}{2}+\frac{s'+r}{2^{q+1-k}}\right\}=$$
$$=\frac{1}{2}\cdot\frac{2r}{2^{q+1-k}}\leqq\frac{s'}{2^{q+1-k}}<\frac{2^{q-k}}{2^{q+1-k}}=\frac{1}{2}.$$

It is easy to see that m_μ satisfies (ii) of (1) for $0<n<q+1$, (iii) of (1) and (iv) of (1). Therefore m_μ is a dyadic measure of type $(k,q+1,s)$. When $2^{q-k}\leqq s'<2^{q+1-k}$, two cases arise: $s'+r<2^{q+1-k}$ and $2^{q+1-k}\leqq s'+r\leqq 2s'=s<2^{q+2-k}$.

In the first case, we define m_μ similarly as we did in the preceding case and we can easily prove that m_μ is a dyadic measure of type $(k,q+1,s)$.

We shall consider the second case. Put $d=s'+r-2^{q+1-k}$. When d is an even number, let m' be a nonnegative dyadic measure of type $(k,q,2s'-2^{q+1-k}-\frac{d}{2})$ and satisfy

$$m'(I_q^{p_1})=m'(I_q^{p_2})=\ldots=m'(I_q^{p_{s_1}})=0.$$

When d is an odd number, let m^* be a nonnegative dyadic measure of type $\left(k,q,2s'-2^{q+1-k}-\frac{d+1}{2}\right)$ and satisfy

$$m^*(I_q^{p_1})=m^*(I_q^{p_2})=\ldots=m^*(I_q^{p_{s_1}})=0.$$

Moreover let m^{**} be a nonnegative dyadic measure of type $\left(k,q,2s'-2^{q+1-k}-\frac{d-1}{2}\right)$ and satisfy $m^{**}(I)=0$ if $m^*(I)=0$. Put

$$m'(I)=\frac{1}{2}\{m^*(I)+m^{**}(I)\}\quad\text{for all}\quad I.$$

Thus m' satisfies

$$m'(I_0^0) = \frac{1}{2}\left\{\frac{k+1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d+1}{2}\right)+\right.$$
$$\left.+\frac{k+1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d-1}{2}\right)\right\} =$$
$$= \frac{k+1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d}{2}\right),$$
$$|\Delta m'(I_0^0)| = \left|\frac{1}{2}\left\{\frac{k+1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d+1}{2}\right)-\right.\right.$$
$$\left.\left.-\frac{k+1}{2} + \frac{1}{2^{q+1-k}}\left(2s' - s^{q+1-k} - \frac{d-1}{2}\right)\right\}\right| = \frac{1}{2}\cdot\frac{1}{2^{q+1-k}}\cdot\left(\frac{1}{2}+\frac{1}{2}\right) \leqq \frac{1}{2}$$

and

$$m'(I_q^p) = \frac{1}{2}\{m^*(I_q^p) + m^{**}(I_q^p)\} \leqq$$
$$\leqq \frac{1}{2}\cdot\frac{1}{2^q}\left\{\frac{k+1}{2} + \frac{k+1}{2}\right\} = \frac{k+1}{2^{q+1}} \quad \text{for all} \quad p.$$

Next, let m'' be a nonnegative dyadic measure of type $(k-1, q, s'+r)$ and satisfy

$$m''\left(\frac{1}{2}\dot{+}I_q^{p_1'}\right) = m''\left(\frac{1}{2}\dot{+}I_q^{p_2'}\right) = \ldots = m''\left(\frac{1}{2}\dot{+}I^{p'_{s_2}}\right) = 0.$$

Thus m'' satisfies

$$\begin{cases} m''(I_0^0) = \frac{k}{2} - \frac{s'+r}{2^{q+1-(k-1)}}; \\ m''(I_q^p) \leqq \frac{k}{2^{q+1}} \quad \text{for all} \quad p. \end{cases}$$

Let m_μ be the dyadic measure defined by (3). Then we have

$$m_\mu(I_0^0) = \frac{1}{2}\left\{\frac{k+1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d}{2}\right) + \frac{k}{2} - \frac{1}{2^{q+1-k}}(s'+r)\right\} =$$
$$= \frac{1}{2}\left\{k + \frac{1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - 2^{q+1-k} - \frac{d}{2} + \frac{1}{2}(s'+d+2^{q+1-k}-s')\right)\right\} =$$
$$= \frac{1}{2}\left\{k + \frac{1}{2} - \frac{1}{2^{q+1-k}}\left(2s' - \frac{2^{q-k+1}}{2}\right)\right\} =$$
$$= \frac{1}{2}\left\{k + \frac{1}{2} - \frac{2s'}{2^{q+1-k}} + \frac{1}{2}\right\} = \frac{k+1}{2} - \frac{s}{2^{(q+1)+1-k}},$$

$$0 \leqq \Delta m_\mu(I_0^0) = \frac{1}{2}\Big[\Big\{\frac{k+1}{2} - \frac{1}{2^{q+1-k}}\Big(2s' - 2^{q+1-k} - \frac{d}{s}\Big)\Big\} -$$
$$-\Big\{\frac{k}{2} - \frac{1}{2^{q+2-k}}(s' + d + 2^{q+1-k} - s')\Big\}\Big] =$$
$$= \frac{1}{2}\Big\{\frac{1}{2} - \frac{1}{2^{q+2-k}}(4s' - 2^{q+2-k} - 2d - 2^{q-k+1})\Big\} =$$
$$= \frac{1}{2}\Big\{\frac{1}{2} - \frac{s'}{2^{q-k}} + 1 + \frac{d}{2^{q+1-k}} + \frac{1}{2}\Big\} = \frac{1}{2}\Big\{2 - \frac{s'}{2^{q-k}} + \frac{d}{2^{q+1-k}}\Big\} =$$
$$= \frac{1}{2}\Big\{2 - \frac{1}{2^{q+1-k}}(s' + 2^{q+1-k} - r)\Big\} \leqq \frac{1}{2}\Big\{2 - \frac{1}{2^{q+1-k}}(s' - s' + 2^{q+1-k})\Big\} = \frac{1}{2}$$

and

$$\max_p m_\mu(I_{q+1}^p) = \frac{1}{2}\max_p m'(I_q^p) \leqq \frac{1}{2^{q+1}}\cdot\frac{k+1}{2}.$$

It is easy to see that m_μ satisfies (1). Thus m_μ is a dyadic measure of type $(k, q+1, s)$.

Finally, we shall consider the case $s = 2s' + 1$. Without loss of generality we can write $s_1 = s' - r$ and $s_2 = s' + r + 1$ and

$$\begin{cases} (I_{q+1}^{p_1} \cup \ldots \cup I_{q+1}^{p_{s_1}}) \cap E = \emptyset; & (I_{q+1}^{p_1} \cup \ldots \cup I_{q+1}^{p_{s_1}}) \subset I_1^0; \\ (I_{q+1}^{p'_1} \cup \ldots \cup I_{q+1}^{p'_{s_2}}) \cap E = \emptyset; & (I_{q+1}^{p'_1} \cup \ldots \cup I_{q+1}^{p'_{s_2}}) \subset I_1^1. \end{cases}$$

When $s' + 1 + r < 2^{q+1-k}$, we can proceed similarly to the case $s = 2s'$ and $s' + r < 2^{q+1-k}$. When $2^{q+1-k} \leqq s' + 1 + r \leqq 2s' + 1$, define m' and m'' similarly to the case $2s' = s$ and $2^{q+1-k} \leqq s' + r < 2^{q+2-k}$. Since

$$m'(I_0^0) = \frac{k}{2} + \frac{1}{2^{q+1-k}}(2^{q+1-k} - s' - 1 + 1 + 2^{q+1-k} - s' - 1) + \frac{d}{2^{q+2-k}} =$$
$$= \frac{k}{2} + \frac{1}{2^{q+2-k}}(2^{q+2-k} - 2s' - 1) + \frac{d}{2^{q+2-k}}$$

and

$$m''(I_0^0) = \frac{k}{2} - \frac{d}{2^{q+2-k}},$$

we have

$$\left|\Delta m_\mu(I_0^0)\right| = \frac{1}{2}\left|m'(I_0^0) - m''(I_0^0)\right| =$$
$$= \frac{1}{2}\Big|\frac{k}{2} + \frac{1}{2^{q+1-k}}(2^{q+2-k} - 2s' - 1) + \frac{d}{2^{q+2-k}} - \frac{k}{2} + \frac{d}{2^{q+2-k}}\Big| =$$
$$= \frac{1}{2}\Big|2 - \frac{1}{2^{q+1-k}}(2s' + 1 - d)\Big| \leqq 1 - \frac{2^{q+1-k}}{2^{q+2-k}} = \frac{1}{2}$$

and

$$m_\mu(I_0^0) = \frac{1}{2}\{m'(I_0^0) + m''(I_0^0)\} =$$
$$= \frac{1}{2}\left\{\frac{k}{2} + \frac{1}{2^{q+1-k}}(2^{q+2-k} - 2s' + 1) + \frac{d}{2^{q+2-k}} + \frac{k}{2} - \frac{d}{2^{q+2-k}}\right\} =$$
$$= \frac{k+1}{2} - \frac{2s'+1}{2^{q+2-k}}.$$

It is easy to see that m_μ satisfies (ii) of (1) for $0 < n < q+1$ and (iii) and (iv) of (1).

Hence m_μ is a dyadic measure of type $(k, q+1, 2s'+1)$. The proof is complete.

PROOF OF THEOREM 3. Assume that E is a closed subset of the dyadic group and

$$\frac{1}{2^{k_1}} \leqq 1 - |E| < \frac{2}{2^{k_1}}.$$

By the Lebesgue theorem, we have

$$\lim_{n\to\infty} 2^n |E \cap I_n(x)| = 1 \quad \text{a.e. on} \quad E.$$

Thus by Egoroff theorem, for sufficiently small $\varepsilon_1 > 0$, there exist a perfect set $E_1 \subset E$ and positive integer N_1 such that

$$1 - 2^{N_1}|E \cap I_{N_1}(x)| < \varepsilon_1 \quad \text{on} \quad E_1,$$

that is,

$$|E \cap I_n(x)| > \frac{1}{2^{N_1}}(1 - \varepsilon_1) \quad \text{on} \quad E_1.$$

Put

$$F_1 = \bigcup_{x\in E_1} I_{N_1}(x) \equiv \sum_{p\in\Lambda^{(0)}} I_{N_1}^p \qquad \text{(disjoint sum)}.$$

Moreover we can make E_1 satisfy

$$|F_1| \geqq |E_1| > 1 - \frac{2}{2^{k_1}}.$$

Put $s = 2^{N_1} - \#\Lambda^{(0)} \equiv \#\Lambda^{(0)c}$ where $\Lambda^{(0)c} \equiv \{0, 1, 2, \dots, 2^{N_1} - 1\} \setminus \Lambda^{(0)}$. Thus we have

$$\left|\bigcup_{p\in\Lambda^{(0)c}} I_{N_1}^p\right| = \frac{s_1}{2^{N_1}} < \frac{2}{2^{k_1}},$$

from which $s < 2^{N_1+1-k_1}$.

By Lemma 5 we can find a nonnegative dyadic measure which satisfies the following:

(i) $m_1(I_0^0) = \frac{k_1+1}{2} - \frac{s_1}{2^{N_1+1-k_1}}$;

(ii) $|\Delta m_1(I_n^p)| \leqq \frac{1}{2^{n+1}}$ for $0 \leqq n < N_1$ and $0 \leqq p < 2^n$;

(iii) $m_1(I) = 0$ if $I \cap F_1 = \emptyset$;

(iv) $m_1(I_{N_1}^p) \leqq \frac{k_1+1}{2^{N_1+1}}$ for $0 \leqq p < 2^{N_1}$.

When $p_1 \in \Lambda^{(0)}$, we shall discuss on $I_{N_1}^{p_1}$ similarly to the preceding case. Let k_2 be a number such that

$$\frac{1}{2^{N_1}} - \left|E \cap I_{N_1}^{p_1}\right| < \frac{1}{2^{N_1}} \cdot \frac{1}{2^{k_2}}.$$

By the Lebesgue theorem, we have

$$\lim_{n\to\infty} 2^{N_1+n}|E \cap I_{N_1+n}(x)| = 1 \quad \text{a.e. on} \quad E \cap I_{N_1}^{p_1}.$$

By the Egoroff theorem, there exist a set $E_2^{(p_1)} \subset I_{N_1}^{p_1} \cap E$ and a positive integer N_2, which does not depend on $p_1 \in \Lambda^{(0)}$, such that

$$1 - 2^{N_1+N_2}|E \cap I_{N_1+N_2}(x)| < \varepsilon_2 \quad \text{on} \quad E_2^{(p_1)},$$

where ε_2 is a sufficiently small positive number. Thus we have

$$|E \cap I_{N_1+N_2}(x)| > \frac{1}{2^{N_1+N_2}}(1-\varepsilon_2) \quad \text{on} \quad E_2^{(p_1)}.$$

Put

$$F_2^{(p_1)} = \bigcup_{x\in E_2^{(p_1)}} I_{N_1+N_2}(x) \equiv \sum_{p\in\Lambda_{p_1}^{(1)}} I_{N_1+N_2}^p \quad \text{(disjoint sum)}.$$

We can make $E_2^{(p_1)}$ satisfy

$$\left|F_2^{(p_1)}\right| \geqq \left|E_2^{(p_1)}\right| > \frac{1}{2^{N_1}}\left(1 - \frac{2}{2^{k_2}}\right).$$

Put $s_2 = 2^{N_2} - \#\Lambda_{p_1}^{(1)} = \#\Lambda_{p_1}^{(1)^c}$ where $\Lambda_{p_1}^{(1)^c} = \{p_1 2^{N_2}, p_1^{N_2}+1, \dots, (p_1 + {} + 1)2^{N_2} - 1\} \setminus \Lambda_{p_1}^{(1)}$. Thus we have

$$\left|\bigcup_{P_2\in\Lambda_{p_1}^{(1)^c}} I_{N_1+N_2}^{p_2}\right| = \frac{s_2}{2^{N_1+N_2}} < \frac{1}{2^{N_1}} \cdot \frac{1}{2^{k_1}}.$$

By Lemma 5 we can find a nonnegative dyadic measure $m_2^{(p_1)}$ which satisfies the following:

(i) $m_2^{(p_1)}(I_{N_1}^{p_1}) = \frac{1}{2^{N_1}}\left(\frac{k_2+1}{2} - \frac{s_2}{2^{N_2+1-k_2}}\right)$;

(ii) $\left|\Delta m_2^{(p_1)}(I_{N_1+n}^p)\right| \leqq \frac{1}{2^{N_1+n}}$ for $0 \leqq n < N_2$ and $I_{N_1+n}^p \subset I_{N_1}^{p_1}$;

(iii) $m_2^{(p_1)}(I) = 0$ if $I \cap F_2^{(p_1)} = \emptyset$;

(iv) $m_2^{(p_1)}(I_{N_1+N_2}^p) \leqq \frac{k_2+1}{2^{N_1+N_2+1}}$ for all $p \in \{p_1 2^{N_2}, \dots, (p_1+1)2^{N_2} - 1\}$.

Continuing in this process, we can find three sequences of numbers, a sequence of sets of integers and a sequence of dyadic measures such that

(i) $N_1 < N_2 < \dots$ and $\lim_{j\to\infty} N_j = \infty$;

(ii) $k_1 < k_2 < \dots$, $\lim_{j\to\infty} k_j = \infty$ and $\prod_{j=1}^{\infty} \frac{k_j+1}{k_j-1} < \infty$;

(iii) $s_1, s_2, \dots$ such that $s_j < 2^{N_j+1-k_j}$;

(iv) $\Lambda^{(0)}, \Lambda^{(1)}_{p_1}, \Lambda^{(2)}_{p_1,p_2}, \dots \Lambda^{(j)}_{p_1\dots p_j}, \dots$;

(v) $m_1, m_2^{(p_1)}, m_3^{(p_1,p_2)}, \dots m_j^{(p_1\dots p_{j-1})} \dots$.

where $m_j^{(p_1\dots p_{j-1})}$ satisfies the following:

(i)′ $m_j^{(p_1\dots p_{j-1})}(I^{p_{j-1}}_{N_1+\dots+N_{j-1}}) = \frac{1}{2^{N_1+\dots+N_{j-1}}}\left(\frac{k_j+1}{2} - \frac{s_j}{2^{N_j+1-k_k}}\right)$;

(ii)′ $\left|\Delta m_j^{(p_1\dots p_{j-1})}(I^p_{N_1+\dots+N_{j-1}+n})\right| \leqq \frac{1}{2^{N_1+\dots+N_{j-1}+n+1}}$ for $0 \leqq n < N_j$ and $I^p_{N_1+\dots+N_{j-1}+n} \subset I^{p_{j-1}}_{N_1+\dots+N_{j-1}}$;

(iii)′ $m_j^{(p_1\dots p_{j-1})}(I) = 0$ if $I \cap \left(\bigcup_{p\in\Lambda^{(j)c}_{p_1\dots p_j}} I^p_{N_1+\dots+N_j}\right) = \emptyset$;

(iv)′ $m_j^{(p_1\dots p_{j-1})}(I^p_{N_1+\dots+N_j}) \leqq \frac{k_j+1}{2^{N_1+\dots+N_j+1}}$ for $I^p_{N_1+\dots+N_j} \subset I^{p_{j-1}}_{N_1+\dots+N_{j-1}}$.

Put

$$A(k,N,s) = \frac{k+1}{2} - \frac{s}{2^{N+1-k}}.$$

We shall construct a dyadic measure m_μ as follows. When $0 \leqq n \leqq N_1$, put

$$m_\mu(I^p_n) = \frac{1}{A(k_1,n_1,s_1)} m_1(I^p_n).$$

When $0 \leqq n \leqq N_2$, put

$$m_\mu(I^p_{N_1+n}) = \begin{cases} \frac{1}{A(k_1,N_1,s_1)} m_1(I^{p_1}_{N_1}) \frac{2^{N_1}}{A(k_2,N_2,s_2)} m_2^{(p_1)}(I^p_{N_1+n}) \\ \qquad \text{if } I^p_{N_1+n} \subset I^{p_1}_{N_1} \text{ where } p_1 \in \Lambda^{(0)}; \\ 0 \qquad \text{if } I^p_{N_1+n} \subset \bigcup_{p\in\Lambda^{(0)c}} I^{p_1}_{N_1}. \end{cases}$$

Continuing in this process, when $0 \leqq n < N_j$, we can define $m_\mu(I^p_{N_1+\dots+N_{j-1}+n})$ as follows:

$$m_\mu(I^p_{N_1+\dots+N_{j-1}+n}) = \begin{cases} \frac{1}{A(k_1,N_1,s_1)} m_1(I^{p_1}_{N_1}) \cdot \frac{2^{N_1}}{A(k_2,N_2,s_2)} m_2^{(p_1)}(I^{p_2}_{N_1+N_2}) \dots \\ \qquad \frac{2^{N_1+\dots+N_{j-1}}}{A(k_j,N_j,s_j)} m_j^{(p_1\dots p_{j-1})}(I^p_{N_1+\dots+N_{j-1}+n}) \\ \text{if } I^p_{N_1+\dots+N_{j-1}+n} \subset I^{p_{j-1}}_{N_1+\dots+N_{j-1}} \text{ for } p_{j-1} \in \Lambda^{(j-2)}_{p_1\dots p_{j-2}}; \\ 0 \text{ otherwise.} \end{cases}$$

Since for $0 \leqq n < N_j$,

$$\left|\Delta m_\mu(I^p_{N_1} + \ldots + N_{j-1} + n)\right| \leqq \frac{1}{A(k_1, N_1, S_1)} m_1(I^{p_1}_{N_1}) \cdot \frac{2^{N_1}}{A(k_2, N_2, S_2)} \cdots$$

$$\cdots \frac{1}{2^{N_1+\ldots+N_{j-1}+n+1}} \leqq \frac{1}{\frac{k_1+1}{2} - 1} \cdot \frac{k_1+1}{2^{N_1+1}} \cdot \frac{2^{N_1}}{\frac{k_2+1}{2} - 1} \cdot \frac{k_2+1}{2^{N_2+1}} \cdots$$

$$\cdots \frac{2^{N_1+\ldots+N_{j-1}}}{\frac{k_j+1}{2} - 1} \cdot \frac{1}{2^{N_1+\ldots+N_{j-1}+n+1}} = \prod_{\ell=1}^{j-1} \frac{k_\ell + 1}{k_\ell - 1} \cdot \frac{1}{k_j - 1} \cdot \frac{1}{2^{N_1+\ldots+N_{j-1}+n}},$$

we have

$$2^{N_1+\ldots+N_{j-1}+n} \left|\Delta m_\mu(I^p_{N_1+\ldots+N_{j-1}+n})\right| \leqq \prod_{\ell=1}^{j-1} \frac{k_\ell+1}{k_\ell-1} \cdot \frac{1}{k_j-1} = o(1)$$

as $j \to \infty$, from which

$$2^n \left|\Delta m_\mu(I_n(x))\right| = \left|\sum_{k=2^n}^{2^{n+1}-1} \hat{\mu}(k) w_k(x)\right| = o(1) \quad \text{uniformly as } j \to \infty.$$

Thus $\mu \in \mathcal{B}$. On the other hand, if

$$I^{p_1}_{N_1} \supset I^{p_2}_{N_1+N_2} \supset \ldots \supset I^{p_k}_{N_1+\ldots+N_k} \supset \ldots \supset \{x\}$$

and $I^{p_k}_{N_1+\ldots+N_k} \cap E \neq \emptyset$ for all k, then there exist $x_k \in E \cap I^{p_k}_{N_1+\ldots+N_k}$ for all k and $x_k \to x$ as $k \to \infty$. Since E is a closed set, we have $x \in E$. Conversely if $x' \notin E$ then there exists $I^{p'}_{N_1+\ldots+N_{k'}}$ such that

$$I^{p'}_{N_1+\ldots+N_{k'}} \ni x' \quad \text{and} \quad I^{p'}_{N_1+\ldots+N_{k'}} \cap E = \emptyset.$$

Hence if $x' \notin E$, then $2^n m_\mu(I_n(x)) = 0$ for sufficiently large n, that is

$$\lim_{n\to\infty} 2^n m_\mu(I_n(x)) = 0 \quad \text{except on} \quad E.$$

Moreover $m_\mu(I^0_0) = 1$. Thus we proved that E is a set of multiplicity for $\mathcal{B}$.

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UNIFORM CONTINUITY IN SEQUENTIALLY UNIFORM SPACES

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Introduction. For a metric space (X,d) the following three levels of uc-ness (continuity of some set of functions is uniform) coincide: (a) any real-valued continuous and bounded function is uniformly continuous, i.e. $C^*(X,R) = U^*(X,R)$; (b) any real-valued continuous function is uniformly continuous, i.e. $C(X,R) = U(X,R)$; (c) any continuous function from X to any (metrizable) uniform space is uniformly continuous, i.e. $C(X,Y) = U(X,Y)$ for any uniform space Y. The equivalence is due essentially to the following sequential characterization of uniform continuity in the metric case. A function $f\colon (X,d) \to (Y,d')$ is uniformly continuous iff for each pair of sequences $(x_n), (y_n)$ of X if $\lim d(x_n, y_n) = 0$ then $\lim d'(f(x_n), f(y_n)) = 0$. The uniform version of the previous sequential characterization of uniform continuity has been used to define sequential uniform spaces. The category of sequential uniform spaces introduced by Hušek [4] is a wide class of uniform spaces including metric spaces, closed under sums and quotients. A sequentially uniform space is a remarkable example of uniform space in which if any real-valued continuous and bounded function is uniformly continuous then any continuity is uniform. We will obtain this result proving that a sequential uniformity is the largest member of its proximity class. Further, after generalizing in a natural way the notion of pseudo-Cauchy sequence, we will show that a normal sequential uniformity is fine (any continuity is uniform) iff any pseudo-Cauchy sequence with distinct points has a cluster point.

1. Sequential uniformities

Let $(X,\mathcal{U})$ be a uniform space. Two sequences $(x_n), (y_n)$ of X are called *adjacent* iff for any diagonal neighborhood $V \in \mathcal{U}$ there exists $n_0 \in N$ such that $(x_n, y_n) \in V$ for any $n > n_0$, [4].

A uniform space $(X,\mathcal{U})$ is called *sequentially uniform* iff any function from X to any (metrizable) uniform space which preserves adjacent sequences is uniformly continuous.

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The category of sequentially uniform spaces is a wide class including metric spaces closed under sums and quotients.

Any uniformity $\mathcal{U}$ induces a proximity δ in the following way:

$$A\delta B \Leftrightarrow (\forall V \in \mathcal{U} \Rightarrow V[A] \cap B \neq \emptyset).$$

Usually the family of all uniformities inducing a fixed proximity has no maximum, [5].

PROPOSITION 1.1. *A sequential uniformity $\mathcal{U}$ is the largest member of its proximity class.*

PROOF. Let $\mathcal{V}$ be a uniformity proximally equivalent to $\mathcal{U}$. We have to show that the identity $f: (X, \mathcal{U}) \to (X, \mathcal{V})$ is uniformly continuous. Suppose not. Then there would exist a pair of sequences $(x_n), (y_n)$ adjacent with respect to $\mathcal{U}$ and a diagonal neighborhood $V \in \mathcal{V}$ such that $(x_n, y_n) \notin V$ for each $n \in N$. By Efremovich lemma if $W \in \mathcal{V}$ and $W^4 \subset V$, then one can find subsequences $(x_{n_k}), (y_{n_k})$ such that $(x_{n_k}, y_{n_l}) \notin W$ for each $k, l \in N$. So $A = \{x_{n_k}: k \in N\}$ and $B = \{y_{n_k} : k \in N\}$ are far in contrast with adjacency.

Let $C^*(X, \mathbf{R})$ $(U^*(X, \mathbf{R}))$ be the set of all real-valued continuous (uniformly continuous) and bounded functions on X.

PROPOSITION 1.2. *If $\mathcal{U}$ is sequential and $C^*(X, \mathbf{R}) = U^*(X, \mathbf{R})$, then any continuous function from X to any uniform space Y is uniformly continuous.*

PROOF. If continuity of real-valued bounded functions is uniform, then $\mathcal{U}$ must be finer than the uniformity induced on X by its Stone–Čech compactification. Thus $\mathcal{U}$ induces the largest compatible proximity δ_F, which is called the *functionally indistinguishable proximity* [5] ($A \not\delta_F B \Leftrightarrow$ there exists a continuous function $f: X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$). From Proposition 1.1 it follows that $\mathcal{U}$ is the finest compatible uniformity. It is well-known that the fine uniformity is the only one for which any continuity is uniform.

2. Sequential characterization of uc-ness

We can give a characterization of uc-ness in sequentially uniform spaces in terms of sequences as in the metric case by generalizing the concept of a pseudo-Cauchy sequence in a natural way [2]. In a metric space a sequence is pseudo-Cauchy iff the pairs of terms are frequently arbitrarily close.

A sequence (x_n) is called *pseudo-Cauchy* iff for each $n_0 \in N$ there exist $A, B \subset N$ such that $A \cap B = \emptyset$, $n_0 < A$, $n_0 < B$ and $\{x_n: n \in A\}\delta\{x_n: n \in B\}$. In the metric case the two definitions agree.

Remark that for a normal space the largest compatible proximity is $\delta_0: A\delta_0 B \Leftrightarrow A^- \cap B^- \neq \emptyset$.

PROPOSITION 2.3. *Let $(X,\mathcal{U})$ be a normal sequentially uniform space. The following are equivalent:*

(1) *Any continuous function from X to any (metrizable) uniform space is uniformly continuous.*

(2) *Any real-valued bounded continuous function on X is uniformly continuous.*

(3) $\delta = \delta_0$.

(4) *Every pseudo-Cauchy sequence with distinct points has a cluster point.*

PROOF. It is easy to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (1). Suppose there exist a uniform space Y, a continuous function $f\colon X \to Y$ and two adjacent sequences (x_n), (y_n) in X with no adjacent images $(f(x_n))$, $(f(y_n))$ in Y. By their adjacency, by the non-adjacency of their images and by continuity of f, (x_n), (y_n) both do not cluster in X. Further, working with subsequences, we can suppose that (x_n), (y_n) have distinct points and $x_n \neq y_m$ for each $n \neq m$. Finally, putting $z_{2h} = y_h$ and $z_{2h-1} = x_h$ we obtain the sequence (z_n) which is pseudo-Cauchy with distinct points but with no cluster point.

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HAAR SYSTEMS FOR COMPACT GEOMETRIES

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§1. **Introduction.** Haar functions have many applications to information theory. Because the diameter of their supports shrink monotonically to zero they are especially useful for "problems where a better representation of certain sections" of the information is required (Harmuth [4]). Problems of this type include pattern recognition and image enhancement.

In the literature, Haar analysis of two dimensional problems has been carried out primarily by the Kronecker product system, i.e., the system $\{X_m(x)X_n(y)\}$ where x, y belong to the unit interval, the X_j's are the classical Haar functions, and $m, n = 0, 1, 2, \ldots$.

This hybrid system has at least two drawbacks. Without drastic rearrangement the diameter of its supports do not shrink monotonically to zero, e.g., compare $\mathrm{spt}(X_{2^n} \cdot X_{2^n}) = [0, 2^{-n}] \times [0, 2^{-n}]$ with $\mathrm{spt}(X_1 \cdot X_{2^{n+1}}) = [0,1] \times [0, 2^{-n-1}]$. And, the union of these supports fills the unit square. In particular, when analysis of the unit disc is undertaken there is considerable waste storing information from corners which are both unwanted and unneeded.

To introduce systems which lack these drawbacks, recall that the classical Haar system is defined in dyadic blocks $\{X_j \colon 2^\ell \leqq j < 2^{\ell+1}\}$. Indeed, $X_0(x) \equiv 1$, $x \in [0,1)$, the functions X_j in the ℓ-th dyadic block, $\ell \geqq 0$, are defined as follows. Divide the unit interval $\ell + 1$ times generating $2^{\ell+1}$ even subintervals. Using pairs of these subintervals as supports and sweeping left to right, define

$$X_j(x) = \begin{cases} 2^{\ell/2} & \text{if } \frac{p}{2^\ell} \leqq x < \frac{p+\frac{1}{2}}{2^\ell} \\ -2^{\ell/2} & \text{if } \frac{p+\frac{1}{2}}{2^\ell} \leqq x < \frac{p+1}{2^\ell} \\ 0 & \text{otherwise} \end{cases}$$

where $j = 2^\ell + p$ and $0 \leqq p < 2^\ell$. In particular, to define a Haar-like system on some region in n-dimensional Euclidean space we need only specify a

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method of division and an ordering on the subregions generated, and define the functions block by block.

For example, for the unit square set $f_0(x,y) \equiv 1$. To define f_j for $2^\ell \leqq j < 2^{\ell+1}$ where $\ell \geqq 0$ divide the unit square $\ell + 1$ times, alternating horizontal cuts with vertical cuts, generating $2^{\ell+1}$ even subrectangles. Use pairs of these subrectangles as supports, sweeping left to right from the bottom left corner to the top right corner. Thus for $\ell = 0$ define

$$f_1(x,y) = \begin{cases} +1 & \text{if } 0 \leqq x < 1, \quad 0 \leqq y < \frac{1}{2} \\ -1 & \text{if } 0 \leqq x < 1, \quad \frac{1}{2} \leqq y < 1. \end{cases}$$

For ℓ odd, say $\ell = 2m + 1$ for some $m \geqq 0$ and $2^\ell \leqq j < 2^{\ell+1}$, write j uniquely as

$$j = 2^{2m+1} + q2^m + p$$

where $0 \leqq p < 2^m$ and $0 \leqq q < 2^{m+1}$. Set

$$f_j(x,y) = \begin{cases} (-1)^p 2^{\ell/2} & \text{if } \frac{p}{2^m} \leqq x < \frac{p+\frac{1}{2}}{2^m}, \quad \frac{q}{2^{m+1}} \leqq y < \frac{q+1}{2^{m+1}} \\ (-1)^{p+1} 2^{\ell/2} & \text{if } \frac{p+\frac{1}{2}}{2^m} \leqq x < \frac{p+1}{2^m}, \quad \frac{q}{2^{m+1}} \leqq y < \frac{q+1}{2^{m+1}} \\ 0 & \text{otherwise.} \end{cases}$$

For ℓ even, say $\ell = 2m$ for some $m \geqq 1$ and $2^\ell \leqq j < 2^{\ell+1}$ write j uniquely as

$$j = 2^{2m} + q2^m + p$$

where $0 \leqq p < 2^m$, $0 \leqq q < 2^m$. Set

$$f_j(x,y) = \begin{cases} (-1)^q 2^{\ell/2} & \text{if } \frac{p}{2^m} \leqq x < \frac{p+1}{2^m}, \quad \frac{q}{2^m} \leqq y < \frac{q+\frac{1}{2}}{2^m} \\ (-1)^{q+1} 2^{\ell/2} & \text{if } \frac{p}{2^m} \leqq x < \frac{p+1}{2^m}, \quad \frac{q+\frac{1}{2}}{2^m} \leqq y < \frac{q+1}{2^m} \\ 0 & \text{otherwise.} \end{cases}$$

Extend each f_j to the closed unit square $[0,1] \times [0,1]$ by

$$f_j(1,y) = \lim_{x \to 1-} f_j(x,y), \qquad y \in [0,1)$$

and

$$f_j(x,1) = \lim_{y \to 1-} f_j(x,y), \qquad x \in [0,1].$$

Call the collection $\{f_0, f_1, \ldots\}$ the square Haar functions. The supports of the first five dyadic blocks of square Haar functions are represented in Figures 1 through 5. (The black subrectangles represent a region where some f_j from that block takes on the value $+2^{\ell/2}$, the white subrectangles represent the value $-2^{\ell/2}$.) Clearly, the diameter of the support of f_j tends

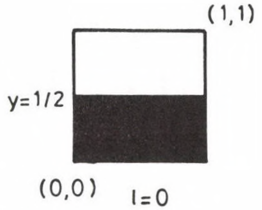

Fig. 1

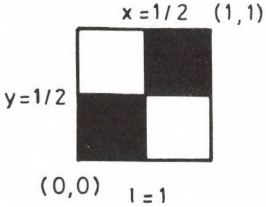

Fig. 2

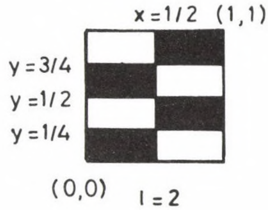

Fig. 3

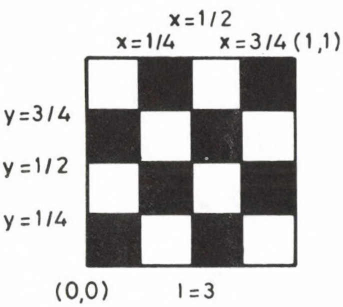

Fig. 4

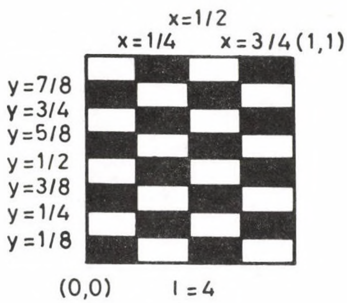

Fig. 5

monotonically to zero as $j \to \infty$. We shall see that the square Haar functions form a complete orthonormal system with at least as good convergence and uniqueness properties as the Kronecker product system.

For the unit disc set $g_0(r,\theta) \equiv 1$. To define g_j for $2^\ell \leqq j < 2^{\ell+1}$ where $\ell \geqq 0$ divide the unit disc $\ell + 1$ times alternating cuts along diameters with cuts along concentric circles centered at the origin, generating $2^{\ell+1}$ even subregions. Use pairs of these subregions as supports, sweeping counterclockwise spiraling outward from the origin to the boundary. Thus for $\ell = 0$ set

$$g_1(r,\theta) = \begin{cases} +1 & \text{if } 0 \leqq \theta < \pi \\ -1 & \text{if } \pi \leqq \theta < 1. \end{cases}$$

For ℓ odd, say $\ell = 2m+1$ for some $m \geqq 0$ and $2^\ell \leqq j < 2^{\ell+1}$ write j uniquely as

$$j = 2^{2m+1} + p2^{m+1} + q$$

where $0 \leqq p < 2^m$ and $0 \leqq q < 2^{m+1}$. Set

$$g_j(r,\theta) = \begin{cases} (-1)^q 2^{\ell/2} & \text{if } \sqrt{\frac{p}{2^m}} \leqq r < \sqrt{\frac{p+\frac{1}{2}}{2^m}}, \quad \frac{q\pi}{2^m} \leqq \theta < \frac{(q+1)\pi}{2^m} \\ (-1)^{q+1} 2^{\ell/2} & \text{if } \sqrt{\frac{p+\frac{1}{2}}{2^m}} \leqq r < \sqrt{\frac{p+1}{2^m}}, \quad \frac{q\pi}{2^m} \leqq \theta < \frac{(q+1)\pi}{2^m} \\ 0 & \text{otherwise.} \end{cases}$$

For ℓ even, say $\ell = 2m$ for some $m \geqq 1$ and $2^\ell \leqq j < 2^{\ell+1}$ write j uniquely as

$$j = 2^{2m} + p2^m + q$$

where $0 \leqq p < 2^m$ and $0 \leqq q < 2^m$. Set

$$g_j(r,\theta) = \begin{cases} (-1)^p 2^{\ell/2} & \text{if } \sqrt{\frac{p}{2^m}} \leqq r < \sqrt{\frac{p+1}{2^m}}, \quad \frac{q\pi}{2^{m-1}} \leqq \theta < \frac{(q+\frac{1}{2})\pi}{2^{m-1}} \\ (-1)^{p+1} 2^{\ell/2} & \text{if } \sqrt{\frac{p}{2^m}} \leqq r < \sqrt{\frac{p+1}{2^m}}, \quad \frac{(q+\frac{1}{2})\pi}{2^{m-1}} \leqq \theta < \frac{(q+1)\pi}{2^{m-1}} \\ 0 & \text{otherwise.} \end{cases}$$

Extend each g_j to the closed unit disc by radial continuity, i.e.,

$$g_j(1,\theta) = \lim_{r\to 1-} g_j(r,\theta), \qquad \theta \in [0, 2\pi).$$

Call the collection $\{g_0, g_1, \dots\}$ the polar Haar functions. The supports of the first five dyadic blocks of polar Haar functions are represented in Figures 6 through 10. Again, a black subregion indicates a value of $+2^{\ell/2}$, a white subregion a value $-2^{\ell/2}$. Clearly, the diameter of the support of g_j tends monotonically to zero, and the union of the supports of polar Haar functions fill the unit disc (but not the unit square). We shall see that the polar Haar functions form a complete orthonormal system with Haar-like convergence and uniqueness properties. Hence this system offers an improvement for circular geometries over the Kronecker product system.

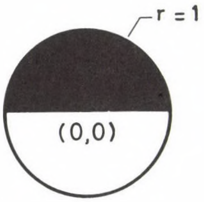

Fig. 6

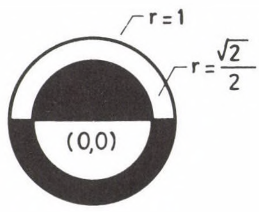

Fig. 7

Fig. 8

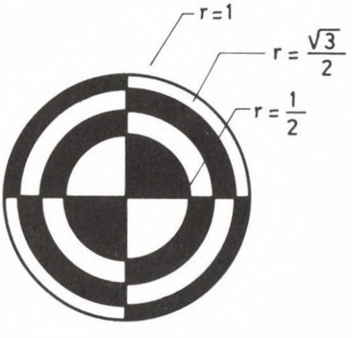

Fig. 9

Fig. 10

It is clear that the method above can be applied to any region in the plane, indeed to any region in n-dimensional Euclidean space. In order to be useful, the systems so generated must have good convergence properties.

Gundy [3] introduced H-systems which behave wonderfully in this regard. Hence a general construction conforming to the procedure set forth above would be easily analyzed if it generated H-systems.

In §2 we shall give a general procedure for constructing a system $\mathcal{H}$ on any compact, metrizable measure space (Δ, μ), hence on any compact region of Euclidean space. We will show $\mathcal{H}$ is complete and orthonormal in $L^2_\mu(\Delta)$ and that it includes the square and polar Haar systems generated above, after a dyadic block rearrangement.

In §3 we verify that $\mathcal{H}$ is an H-system. We show that $\mathcal{H}$ has the desired convergence properties. Among them, the $\mathcal{H}$-Fourier series of a function f converges uniformly when f is continuous on Δ and converges in L^p_μ norm when $f \in L^p_\mu(\Delta)$ for some $1 \leqq p < \infty$.

In §4 we obtain sufficient conditions for uniqueness to hold for $\mathcal{H}$-series. Our conditions improve earlier work by Movsisjan [5], which applied only to the Kronecker product system. In particular, we show that for uniqueness not only countable subsets, but any countable union of embedded arcs can be disregarded.

§2. **The general procedure.** An explicit description of the square and polar Haar systems appears in §1. This was done to indicate how they might be programmed. Fortunately, such explicitness is unnecessary for the general case.

Let Δ be any compact metric space and μ be a non-negative regular Borel measure on Δ of total variation 1 (see Rudin [6] for terminology). Let

$\{\Delta_i^{(n)}: 1 \leqq i \leqq n\}$, $n = 1, 2, \ldots$, be a sequence of partitions of Δ such that for each integer $n \geqq 1$,

$$\Delta = \Delta_1^{(n)} \cup \Delta_2^{(n)} \cup \cdots \cup \Delta_n^{(n)}, \tag{1}$$

$$\Delta_i^{(n)} \cap \Delta_j^{(n)} = \emptyset \qquad \text{for} \quad i \neq j, \tag{2}$$

$$\Delta_1^{(n)} = \Delta_n^{(n+1)} \cup \Delta_{n+1}^{(n+1)}, \tag{3}$$

$$\Delta_i^{(n)} = \Delta_{i-1}^{(n+1)} \qquad \text{for} \quad i = 2, 3, \ldots, n, \tag{4}$$

$$\mu\left(\Delta_n^{(n+1)}\right) = \mu\left(\Delta_{n+1}^{(n+1)}\right) \tag{5}$$

and such that

$$\lim_{n\to\infty}\left(\max_{1\leqq i\leqq n} \operatorname{diam}(\Delta_i^{(n)})\right) = 0. \tag{6}$$

Such partitions can be generated recursively by setting $\Delta_1^{(1)} = \Delta$ and successive division using (2) through (6). Notice that the partitions $\{\Delta_i^{(n)}: 1 \leqq i \leqq \leqq n\}$ are even only when $n = 2^\ell$ for some $\ell = 0, 1, \ldots$. It is these partitions which correspond to the dyadic blocks illustrated in Figures 1 through 10. The partitions corresponding to n for $2^\ell < n < 2^{\ell+1}$ are hybrid ones bridging the gap from one dyadic block to another. We have allowed this redundancy to simplify the theory below.

For each integer $n \geqq 1$ let $[n]$ represent the greatest integral power of 2 in n, i.e., $[n] = 2^\ell$ where ℓ is the unique integer determined by $2^\ell \leqq n < 2^{\ell+1}$. Define the system $\mathcal{H} \equiv \{h_0, h_1, \ldots\}$ on Δ as follows. Set $h_0(x) \equiv 1$ and

$$h_1(x) = \begin{cases} +1 & \text{if } x \in \Delta_1^{(2)} \\ -1 & \text{if } x \in \Delta_2^{(2)}. \end{cases}$$

For $n \geqq 2$ and $x \in \Delta$ set

$$h_n(x) = \begin{cases} [n]^{\frac{1}{2}} & \text{if } x \in \Delta_n^{(n+1)} \\ -[n]^{\frac{1}{2}} & \text{if } x \in \Delta_{n+1}^{(n+1)} \\ 0 & \text{otherwise.} \end{cases}$$

Notice for a suitable choice of the partitions $\{\Delta_i^{(n)}: 1 \leqq i \leqq n\}$ in the unit square, respectively the unit disc, that the system $\mathcal{H}$ is equivalent to the square Haar system, respectively the polar Haar system. Indeed, the system $\mathcal{H}$ so generated is a dyadic block rearrangement of the square Haar system, respectively the polar Haar system.

To see this let Δ represent the unit disc and examine the polar Haar system $\{g_0, g_1, \dots\}$. Set

$$\Delta_1^{(2)} = \{(r,\theta) \colon 0 \leqq \theta < \pi,\ 0 \leqq r \leqq 1\},$$
$$\Delta_2^{(2)} = \{(r,\theta) \colon \pi \leqq \theta < 2\pi,\ 0 \leqq r \leqq 1\}.$$

Clearly $g_0 = h_0$ and $g_1 = h_1$. Fix $\ell \geqq 1$. We are out to show that the partitions $\{\Delta_i^{(n)} \colon 1 \leqq i \leqq n\}$ can be chosen so that

$$\{g_j \colon 2^\ell \leqq j < 2^{\ell+1}\} = \{h_j \colon 2^\ell \leqq j < 2^{\ell+1}\}. \tag{7}$$

Notice that the ℓ-th dyadic block of polar Haar functions is completely determined by wedges of the form

$$\left\{(r,\theta) \colon \sqrt{\frac{p}{2^m}} \leqq r < \sqrt{\frac{p+1}{2^m}},\ \frac{q\pi}{2^m} \leqq \theta < \frac{(q+1)\pi}{2^m}\right\}$$

where $\ell = 2m$ or $2m+1$ and $0 \leqq p < 2^m$, $0 \leqq q < 2^{m+1}$. Moreover, notice that the support of any polar Haar function from the ℓ-th dyadic block is contained in the closures of a region of constancy of some polar Haar function from the $\ell - 1$-st dyadic block. Hence by successive division of the sets $\Delta_1^{(2)}, \Delta_2^{(2)}$ using properties (2) through (5) it is possible to generate partitions so that $\{\Delta_i^{(2^\ell)} \colon 1 \leqq i \leqq 2^\ell\}$ coincide with these wedges. In particular, (7) holds. Notice that $g_j \neq h_j$ for many j's after $\ell = 3$. Indeed, instead of spiraling counterclockwise from the origin to the boundary, condition (4) forces the wedges $\Delta_i^{(2^\ell)}$ to flipflop from origin toward boundary and back again, as $i = 1, 2, \dots, 2^\ell$.

The fact that special cases of $\mathcal{H}$ turn out to be dyadic block rearrangements of the square and polar Haar systems means that any theorem about $\mathcal{H}$ is at once a theorem about both these systems. For example, if $\mathcal{H}$ is complete and orthonormal, then so are these systems. If $\mathcal{H}$-Fourier series converge, then 2^n-th partial sums of series in both these systems converge. And, if uniqueness holds for 2^n-th partial sums of $\mathcal{H}$-series then uniqueness holds for square and polar Haar series. Thus we shall analyze $\mathcal{H}$ in general.

LEMMA 1. *The system $\mathcal{H}$ is orthonormal in $L^2_\mu(\Delta)$ whether (6) holds or not.*

PROOF. We must show

$$\int\limits_\Delta h_n h_m \, d\mu = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

For simplicity we suppose $n \geqq 1$.

If $m = n$ we have by construction that

$$\int_{\Delta} h_n h_m \, d\mu = \int_{\Delta_1^{(n)}} |h_n|^2 d\mu.$$

Since $|h_n|^2 \equiv [n]$ is constant on $\Delta_1^{(n)}$, and since (5) and (2) imply that

$$\mu\left(\Delta_1^{(n)}\right) = [n]^{-1}, \qquad n \geqq 1, \tag{8}$$

it follows that

$$\int_{\Delta} |h_n|^2 d\mu = 1.$$

If $m \neq n$ we may assume by symmetry that $m > n$. Hence $\Delta_1^{(m)}$ intersects $\Delta_1^{(n)}$ only when $\Delta_1^{(m)} \subset \Delta_1^{(n)}$. Hence by construction,

$$\int_{\Delta} h_n h_m \, d\mu \equiv \int_{\Delta_1^{(n)} \cap \Delta_1^{(m)}} h_n h_m \, d\mu = \pm [n]^{\frac{1}{2}} \int_{\Delta_1^{(m)}} h_m \, d\mu.$$

Since (5) and (8) imply

$$\int_{\Delta_1^{(m)}} h_m \, d\mu = 0$$

we conclude that $\mathcal{H}$ is orthonormal in $L_\mu^2(\Delta)$. □

In the next section we shall see that property (6) implies $\mathcal{H}$ is complete.

§3. Convergence of $\mathcal{H}$-Fourier series. A function f on Δ is called an $\mathcal{H}$-polynomial of order n if there exist constants a_k such that

$$f = \sum_{k=0}^{n-1} a_k h_k.$$

A function f on Δ is called an $\mathcal{H}$-step function of order n if there exist constants c_i such that

$$f = \sum_{i=1}^{n} c_i I_{\Delta_i^{(n)}}$$

where for each measurable $E \subseteqq \Delta$,

$$I_E(x) \equiv \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in \Delta - E \end{cases}$$

represents the indicator function of E.

It is clear that every $\mathcal{H}$-polynomial of order n is an $\mathcal{H}$-step function of order n. The converse of this statement also holds:

LEMMA 2. *Let i, n be integers with $1 \leqq i \leqq n$. There exist constants $a_k(i,n)$ such that*

$$I_{\Delta_i^{(n)}} = \sum_{k=0}^{n-1} a_k(i,n) h_k. \tag{9}$$

PROOF. We proceed by induction on n.

For $n = 1$ we have

$$I_{\Delta_1^{(1)}} \equiv h_0$$

and for $n = 2$ we have by construction that

$$I_{\Delta_1^{(2)}} = \frac{1}{2}h_0 + \frac{1}{2}h_1, \qquad I_{\Delta_2^{(2)}} = \frac{1}{2}h_0 - \frac{1}{2}h_1.$$

Suppose the lemma holds for some $n \geqq 2$ and all $1 \leqq i \leqq n$. Observe by (3) that

$$I_{\Delta_n^{(n+1)}} = \frac{1}{2} I_{\Delta_1^{(n)}} + \frac{1}{2}\frac{h_n}{[n]^{1/2}}, \quad I_{\Delta_{n+1}^{(n+1)}} = \frac{1}{2} I_{\Delta_1^{(n)}} - \frac{1}{2}\frac{h_n}{[n]^{1/2}},$$

and by (4) that

$$I_{\Delta_{i-1}^{(n+1)}} = I_{\Delta_i^{(n)}}, \qquad i = 2, 3, \dots, n.$$

Consequently, it follows from the inductive hypothesis that the lemma holds for $n+1$ and all $1 \leqq j \leqq n+1$. □

For $f \in L^1_\mu(\Delta)$ define $\mathcal{H}$-Fourier coefficients by

$$\hat{f}(n) = \int_\Delta f \cdot h_n d\mu, \qquad n = 0, 1, \dots .$$

Define the $\mathcal{H}$-Fourier series of f by

$$Sf = \sum_{k=0}^{\infty} \hat{f}(k) h_k$$

and the n-th order partial sums of Sf by

$$S_n f = \sum_{k=0}^{n-1} \hat{f}(k) h_k, \qquad n = 0, 1, \dots .$$

The $\mathcal{H}$-Dirichlet kernel is defined by

$$D_n(x,t) = \sum_{k=0}^{n-1} h_k(x) h_k(t)$$

for $n = 1, 2, \dots,$ and $x, t \in \Delta$. Notice by Fubini's theorem and definition that

$$(S_n f)(x) = \int_\Delta f(t) D_n(x,t) d\mu(t) \tag{10}$$

for $x \in \Delta$, $f \in L^1_\mu(\Delta)$, and $n = 1, 2, \dots .$

LEMMA 3. *If P is a step function of order n then*

$$(S_nP)(x) = P(x)$$

for every $x \in \Delta$.

PROOF. By Lemma 2 there exist constants a_k such that

$$P = \sum_{k=0}^{n-1} a_k h_k.$$

Therefore by (10) we have

$$(S_nP)(x) = \sum_{k=0}^{n-1} a_k \int_\Delta h_k(t) D_n(x,t) d\mu(t).$$

Since Lemma 1 implies

$$\int_\Delta h_k(t) D_n(x,t) d\mu(t) = \sum_{j=0}^{n-1} h_j(x) \int_\Delta h_k(t) h_j(t) d\mu(t) \equiv h_k(x),$$

we conclude that

$$(S_nP)(x) = \sum_{k=0}^{n-1} a_k h_k(x) \equiv P(x). \qquad \square$$

This result allows us to obtain a closed form for the $\mathcal{H}$-Dirichlet kernel.

LEMMA 4. *Let i, n be integers with $1 \leqq i \leqq n$. If $t \in \Delta_i^{(n)}$ then*

$$D_n(x,t) = \begin{cases} \frac{1}{\mu(\Delta_i^{(n)})} & \textit{if } x \in \Delta_i^{(n)} \\ 0 & \textit{if } x \in \Delta - \Delta_i^{(n)}. \end{cases}$$

PROOF. Fix $x \in \Delta$ and observe that $D_n(x,t)$ is a step function in t of order n. Thus there exist constants $c_k(x)$ such that

$$D_n(x,t) = \sum_{k=1}^{n} c_k(x) I_{\Delta_k^{(n)}}(t). \tag{11}$$

These constants can be explicitly computed.

Indeed, use (2) to write

$$c_k(x) = \sum_{\ell=1}^{n} c_\ell(x) I_{\Delta_\ell^{(n)}}(t) I_{\Delta_k^{(n)}}(t) \equiv I_{\Delta_k^{(n)}}(t) D_n(x,t),$$

for each $t \in \Delta_k^{(n)}$. Integrate this identity $d\mu(t)$ over $\Delta_k^{(n)}$ to obtain

$$c_k(x)\mu(\Delta_k^{(n)}) = \int_{\Delta_k^{(n)}} I_{\Delta_k^{(n)}}(t) D_n(x,t) d\mu(t) \equiv S_n P(x)$$

where $P = I_{\Delta_k^{(n)}}$. Consequently Lemma 3 implies

$$c_k(x) = \frac{I_{\Delta_k^{(n)}}(x)}{\mu(\Delta_k^{(n)})}.$$

Putting these values into (11) we conclude that

$$D_n(x,t) = \sum_{k=1}^{n} \frac{I_{\Delta_k^{(n)}}(x) I_{\Delta_k^{(n)}}(t)}{\mu(\Delta_k^{(n)})}.$$

In particular, since $t \in \Delta_i^{(n)}$ implies $I_{\Delta_k^{(n)}}(t) \neq 0$ only when $k = i$, it follows that

$$D_n(x,t) = \frac{I_{\Delta_i^{(n)}}(x)}{\mu(\Delta_i^{(n)})}$$

as required. □

Since the measure μ is a Borel measure and Δ is compact, it is clear that each continuous function on Δ is measurable and bounded, hence integrable. Therefore, such a function has an $\mathcal{H}$-Fourier series and we may ask whether this series converges.

THEOREM 1. *If f is continuous on Δ then $S_n f$ converges to f, as $n \to \infty$, uniformly on Δ.*

PROOF. By (10) and Lemma 4,

$$S_n f(x) = \int_{\Delta} f(t) D_n(x,t) d\mu(t) = \sum_{i=1}^{n} \int_{\Delta_i^{(n)}} f(t) D_n(x,t) d\mu(t) =$$

$$= \sum_{i=1}^{m} I_{\Delta_i^{(n)}}(x) \frac{1}{\mu(\Delta_i^{(n)})} \int_{\Delta_i^{(n)}} f(t) d\mu(t).$$

Since this sum has at most one non-zero term for each fixed x, and since

$$f(x) = \frac{1}{\mu(\Delta_i^{(n)})} \int_{\Delta_i^{(n)}} f(x) d\mu(t),$$

it follows that

$$(12) \qquad S_nf(x) - f(x) = \sum_{i=1}^{n} I_{\Delta_i^{(n)}}(x)\frac{1}{\mu(\Delta_i^{(n)})}\int_{\Delta_i^{(n)}} (f(t) - f(x))d\mu(t).$$

Since f is continuous and Δ is a compact metric space, f is uniformly continuous on Δ. Thus given $\varepsilon > 0$ we can choose by (6) an integer N so large that $|f(t) - f(x)| < \varepsilon$ for $x, t \in \Delta_i^{(n)}$, $1 \leqq i \leqq n$, and for $n = N, N+1, \ldots$. In particular, it follows from (2) and (12) that

$$|S_nf(x) - f(x)| < \varepsilon$$

for $n \geqq N$. □

For each integer $n \geqq 1$ let $\mathcal{B}_n$ denote the σ-algebra generated by $\{\Delta_i^{(n)} : 1 \leqq \leqq i \leqq n\}$. Recall given $f \in L^1_\mu(\Delta)$ that the conditional expectation of f given $\mathcal{B}_n$, written $g \equiv E(f \mid \mathcal{B}_n)$, is the $\mathcal{B}_n$-measurable function g (guaranteed to exist by the Radon–Nikodym theorem) which satisfies

$$(13) \qquad \int_E g\,d\mu = \int_E f\,d\mu$$

for all $e \in \mathcal{B}_n$.

Lemma 5. *If $f \in L^1_\mu(\Delta)$ and $n \geqq 1$ then $S_nf = E(f \mid \mathcal{B}_n)$.*

Proof. Clearly, a function is $\mathcal{B}_n$-measurable if and only if it is a step function of order n. Hence the function

$$g \equiv S_nf$$

is $\mathcal{B}_n$-measurable and it suffices to verify (13) for $E = \Delta_k^{(n)}$ and $1 \leqq k \leqq n$ fixed.

A trivial integration and Lemma 4 leads to

$$\int_{\Delta_k^{(n)}} f(t)d\mu(t) = \frac{1}{\mu(\Delta_k^{(n)})}\int_{\Delta_k^{(n)}}\int_{\Delta_k^{(n)}} f(t)d\mu(x)d\mu(t) =$$

$$= \sum_{i=1}^{n}\int_{\Delta_i^{(n)}}\int_{\Delta_k^{(n)}} f(t)D_n(x,t)d\mu(x)d\mu(t) \equiv \int_{\Delta}\int_{\Delta_k^{(n)}} f(t)D_n(x,t)d\mu(x)d\mu(t).$$

Consequently it follows from Fubini's theorem and (10) that

$$\int_{\Delta_k^{(n)}} f\,d\mu = \int_{\Delta_k^{(n)}} S_nf(x)d\mu(x) \equiv \int_{\Delta_k^{(n)}} g\,d\mu.$$

Thus (13) holds for $E = \Delta_k^{(n)}$ and the proof is complete. □

Thus we see that the partial sums of any $\mathcal{H}$-Fourier series form a martingale.

THEOREM 2. *If $f \in L_\mu^p(\Delta)$, $1 \leqq p < \infty$, then $S_n f \to f$, as $n \to \infty$, a.e. on Δ and in L_μ^p norm.*

PROOF. Fix $1 \leqq p < \infty$. Let $\mathcal{B}_\infty$ denote the smallest σ-algebra containing the σ-algebras $\mathcal{B}_1, \mathcal{B}_2, \dots$. A classical martingale convergence theorem of Lévy (see Doob [2]) states that if $f \in L_\mu^p(\Delta)$ is $\mathcal{B}_\infty$-measurable and $f_n = E(f|\mathcal{B}_n)$, then $f_n \to f$ a.e. and in $L_\mu^p(\Delta)$ norm. But μ is a regular Borel measure. Hence, by Lusin's theorem the space of continuous functions $C(\Delta)$ is dense in $L_\mu^p(\Delta)$. Since by Theorem 1 the collection of $\mathcal{B}_\infty$-measurable functions is dense in $C(\Delta)$ it follows that each $L_\mu^p(\Delta)$ function is $\mathcal{B}_\infty$-measurable. We conclude from Lemma 5 and Lévy's theorem that $S_n f$ converges to f a.e. and in L_μ^p norm. □

COROLLARY 1. *The system $\mathcal{H}$ is complete.*

PROOF. If $f \in L_\mu^1(\Delta)$ and $\hat{f}(n) = 0$ for $n = 0, 1, \dots$, then the $\mathcal{H}$-Fourier series of f is identically zero. Hence $f = 0$ a.e. by Theorem 2. □

We have shown, therefore, that $\mathcal{H}$ is a complete H-system. In particular, by Gundy [3] we have the following two results.

COROLLARY 2. *If f is measurable and a.e. finite then there is an $\mathcal{H}$-series*

$$S = \sum_{n=0}^{\infty} a_n h_n \tag{14}$$

which converges a.e. to f.

COROLLARY 3. *An $\mathcal{H}$-series* (14) *converges a.e. on a set $E \subsetneqq \Delta$ if and only if*

$$\sum_{n=0}^{\infty} (a_n h_n)^2 < \infty$$

a.e. on E.

§4. Uniqueness. A set $E \subsetneqq \Delta$ is called distinguished if $E = \Delta_i^{(n)}$ for some integer n and some choice of $i = n - 1$ or $i = n$. Notice for each integer $m \geqq 0$ that the collection of distinguished sets of measure 2^{-m} forms a partition of Δ; indeed,

$$\Delta = \bigcup_{k=2^m+1}^{2^{m+1}} \left(\Delta_{k-1}^{(k)} \cup \Delta_k^{(k)} \right).$$

Also notice for each distinguished set $E = \Delta_i^{(n)}$ that h_ℓ is constant on E for $0 \leqq \ell < n$. In particular, given any $\mathcal{H}$-series and any distinguished set $E = \Delta_i^{(n)}$, the partial sum S_n is constant on E.

In this section we consider the following question. When is an a.e. convergent $\mathcal{H}$-series S an $\mathcal{H}$-Fourier series? We begin by showing that if S is not an $\mathcal{H}$-Fourier series then some partial sum of S is as large as we wish.

LEMMA 6. *Let $f \in L^1_\mu(\Delta)$, let $E_0 \equiv \Delta_{i_0}^{(n_0)}$ be a distinguished set and suppose $S_n \to f$ as $n \to \infty$, a.e. on E_0. If $S_{n_0} \neq S_{n_0} f$ on E_0 and M is any positive real number then there is a distinguished set $\Delta_i^{(m)} \subseteqq E_0$ such that $|S_m| > M$ on $\Delta_i^{(m)}$.*

PROOF. Suppose to the contrary that $|S_m| \leqq M$ on $\Delta_i^{(m)}$ for every distinguished $\Delta_i^{(m)}$ contained in E_0. Fix such a $\Delta_i^{(m)}$, let $\Delta_{i'}^{(m')}$ be distinguished and suppose $\Delta_{i'}^{(m')} \subset \Delta_i^{(m)}$ and

$$\mu\left(\Delta_{i'}^{(m')}\right) = \frac{1}{2}\mu\left(\Delta_i^{(m)}\right).$$

By definition $h_\ell(x) \equiv 0$ for $x \in \Delta_i^{(m)}$ and $m \leqq \ell \leqq m' - 2$. Consequently, $S_N \equiv S_m$ on $\Delta_i^{(m)}$ for all $m \leqq N < m'$. Since the collection of distinguished sets forms a partition of E_0, it follows from our assumption that

$$|S_N| \leqq M \quad \text{on} \quad E_0 \tag{15}$$

for $N = n_0, n_0 + 1, \ldots$.

Let $T = S - Sf$ and choose by hypothesis a non-zero constant d such that $T_{n_0} \equiv d$ on E_0. Since

$$\int_{E_0} h_j d\mu = 0$$

for $j = n_0, n_0 + 1, \ldots$ it is clear that

$$\lim_{N\to\infty} \int_{E_0} T_N d\mu = d \cdot \mu(E_0) \neq 0.$$

On the other hand, by (15), hypothesis, and the bounded convergence theorem, the series S converges to f in the $L^1_\mu(E_0)$ norm. Hence it follows from Theorem 2 that $T_N \to 0$ in $L^1_\mu(E_0)$ norm. In particular,

$$\lim_{N\to\infty} \int_{E_0} T_N d\mu = 0,$$

a contradiction. □

THEOREM 3. *Let $f \in L^1_\mu(\Delta)$ and S be an $\mathcal{H}$-series which converges a.e. to f. If for every $x_0 \in \Delta$*

$$\limsup_{n\to\infty}\left(\limsup_{x\to x_0}|S_n(x)|\right) < \infty,$$

then S is the $\mathcal{H}$-Fourier series of f.

PROOF. Suppose S is not the $\mathcal{H}$-Fourier series of f. Let $a_0, a_1, \ldots$ represent the coefficients of S and choose $n_0 \geqq 1$ to be the smallest integer which satisfies $a_{n_0-1} \neq \hat{f}(n_0-1)$. Then $S_{n_0} \neq S_{n_0}f$ on $E_0 \equiv \Delta^{(n_0)}_{n_0}$ and by repeated applications of Lemma 6 we can choose distinguished sets $\Delta^{(n_k)}_{i_k}$ such that $\Delta^{(n_{k-1})}_{i_{k-1}} \supset \Delta^{(n_k)}_{i_k}$ and

$$|S_{n_k}| > k \quad \text{on} \quad \Delta^{(n_k)}_{i_k}, \qquad k = 1, 2, \ldots. \tag{16}$$

Since Δ is a compact metric space there is an x_0 which belongs to the closures $\overline{\Delta^{(n_k)}_{i_k}}$ for $k = 0, 1, \ldots$. By (16) we have

$$\limsup_{x\to x_0}|S_{n_k}(x)| \geqq k$$

for $k = 1, 2, \ldots$. It follows, therefore, that

$$\limsup_{n\to\infty}\limsup_{x\to x_0}|S_n(x)| = +\infty.$$

Since this contradicts the hypothesis, we conclude that S is the $\mathcal{H}$-Fourier series of f. □

Since the one-dimensional Haar system is an $\mathcal{H}$-system, this result contains a new uniqueness theorem in the classical case.

COROLLARY. *If f is integrable on the interval $[0,1]$, if S is a Haar series which converges a.e. to f and if*

$$\limsup_{n\to\infty}|S_n(x \pm 0)| < \infty$$

for all $x \in [0,1]$, then S is the Haar–Fourier series of f.

The hypotheses of Theorem 3 might prove difficult to verify in practice. A more easily verifiable hypothesis would be some growth condition on the coefficients of S.

Since such a growth condition must be satisfied by all $\mathcal{H}$-Fourier series we begin with the following.

LEMMA 7. *Let $x_0 \in \Delta$ and $k_1 < k_2 < \dots$ be the collection of integers ℓ which satisfy $h_\ell(x_0) \neq 0$. If f is integrable on Δ then*

$$\lim_{j\to\infty} \frac{\hat{f}(k_j)}{h_{k_j}(x_0)} = 0.$$

PROOF. By definition, $h_\ell(x_0) \neq 0$ if and only if $x_0 \in \Delta_1^{(\ell)}$. Moreover, each h_ℓ has constant absolute value on $\Delta_1^{(\ell)}$. Consequently, for each $j \geqq 1$ we have

$$|\hat{f}(k_j)| = \left| \int\limits_{\Delta_1^{(k_j)}} f \cdot h_{k_j} \, d\mu \right| \leqq |h_{k_j}(x_0)| \int\limits_{\Delta_1^{(k_j)}} |f| \, d\mu.$$

Since the indefinite integral of f is absolutely continuous with respect to μ and $\mu(\Delta_1^{(k_j)}) \to 0$ as $j \to \infty$, it is now clear that

$$\hat{f}(k_j)/h_{k_j}(x_0) \to 0$$

as $j \to \infty$. □

This growth condition was first identified for the classical Haar system by Arutunjan and Talaljan [1]. Accordingly, we say that an $\mathcal{H}$-series

$$S = \sum_{k=0}^{\infty} a_k h_k$$

satisfies the A–T condition at a point $x_0 \in \Delta$ if

$$\lim_{j\to\infty} \frac{a_{k_j}}{h_{k_j}(x_0)} = 0$$

where $k_1 < k_2 < \dots$ are all integers ℓ which satisfy $h_\ell(x_0) \neq 0$.

By Lemma 7 every $\mathcal{H}$-Fourier series satisfies the A–T condition at every point in Δ. On the other hand, if an $\mathcal{H}$-series fails to satisfy the A–T condition at even one point it may not be an $\mathcal{H}$-Fourier series.

To construct such an example, let $\Delta = \Delta_{i_1}^{(n_1)} \supset \Delta_{i_2}^{(n_2)} \supset \dots$ be distinguished sets such that

$$\mu\left(\Delta_{i_j}^{(n_j)}\right) = \frac{1}{2}\mu\left(\Delta_{i_{j-1}}^{(n_{j-1})}\right)$$

for $j = 2, 3, \dots$ and

$$\{x_0\} = \bigcap_{j=1}^{\infty} \Delta_{i_j}^{(n_j)}.$$

Define coefficients a_m by $a_m = 0$ when $m \neq k_j$ and

$$a_{k_j} = (-1)^{k_j - i_j}[k_j]^{1/2},$$

where $k_j = n_j - 1$ for $j = 1, 2, \ldots$. Consider the $\mathcal{H}$-series

$$S = \sum_{m=0}^{\infty} a_m h_m \equiv \sum_{j=1}^{\infty} a_{k_j} h_{k_j}.$$

An easy induction argument verifies

$$S_n(x) = \begin{cases} 2^{j-1} & \text{if } x \in \Delta_{i_j}^{(n_j)} \\ 0 & \text{if } x \in \Delta \sim \Delta_{i_j}^{(n_j)} \end{cases}$$

for $n_j \leqq n < n_{j+1}$ and $j = 1, 2, \ldots$. Since the $\Delta_{i_j}^{(n_j)}$'s shrink to the point x_0, it follows that S converges to zero everywhere on $\Delta \sim \{x_0\}$. In view of Theorem 2, S cannot be an $\mathcal{H}$-Fourier series. On the other hand, since $a_{k_j}/h_{k_j} = \pm 1$ it is clear that S fails to satisfy the A–T condition at x_0. But given $x \neq x_0$ and $x \in \Delta_i^{(m)}$, a distinguished set, we have that $m \neq k_j$ for m large. Consequently, $a_m = 0$ for m large and S satisfies the A–T condition at every point in Δ except x_0.

This example used a sequence of distinguished sets with non-empty intersection to force the series S to have partial sums identically zero on larger and larger subsets of Δ. The following result clarifies the role of the distinguished sets by showing such cancellation cannot subsist for $\mathcal{H}$-series which satisfy the A–T condition.

LEMMA 8. *Let $E_j = \Delta_{i_j}^{(n_j)}$ be a sequence of distinguished sets which satisfy $E_0 \supset E_1 \supset \ldots$ and $\bigcap_{j=0}^{\infty} E_j \neq \emptyset$. Suppose further for each $j \geqq 1$ that*

$$\mu(E_j) = \frac{1}{2}\mu(E_{j-1}) \tag{17}$$

and

$$F_j = E_{j-1} \setminus E_j. \tag{18}$$

If S is an $\mathcal{H}$-series which satisfies the A–T condition everywhere on Δ and

$$S_{n_0} \neq 0 \quad \text{on} \quad E_0, \tag{19}$$

then

$$S_{n_j} \neq 0 \quad \text{on} \quad F_j \tag{20}$$

for infinitely many integers $j \geqq 1$.

PROOF. By iteration it suffices to show (20) holds for at least one integer $j \geqq 1$. Suppose to the contrary that

$$S_{n_j} \equiv 0 \quad \text{on} \quad F_j \tag{21}$$

for $j = 1, 2, \ldots$. For each $j \geqq 0$ set $k_j = n_j - 1$. Since each h_m vanishes off $\Delta_1^{(m)}$ and for each m the sets $\Delta_\ell^{(m)}$ are pairwise disjoint for $\ell = 1, 2, \ldots, m$, it is clear by (17) that

$$S_{n_j} = S_{n_{j-1}} + a_{k_j} h_{k_j} \quad \text{on} \quad E_j \cup F_j, \tag{22}$$

that h_{k_j} and S_{n_j} are constant on E_j and on F_j, and that h_{k_j} changes signs from F_j to E_j for each integer $j \geqq 1$.

Choose by (19) a non-zero constant d such that $S_{n_0} \equiv d$ on E_0, and suppose for some $j \geqq 1$ that

$$S_{n_{j-1}} = 2^{j-1} d \quad \text{on} \quad E_{j-1}. \tag{23}$$

Since $F_j \subset E_{j-1}$ it is clear by (21), (22), and (23) that

$$a_{k_j} h_{k_j} = -2^{j-1} d \quad \text{on} \quad F_j.$$

Since h_{k_j} changes signs from F_j to E_j, it follows that

$$a_{k_j} h_{k_j} = +2^{j-1} d \quad \text{on} \quad E_j. \tag{24}$$

Let

$$x_0 \in \bigcap_{j=0}^{\infty} E_j.$$

Notice for each integer $\ell \geqq k_0$ that $h_\ell(x_0) \neq 0$ if and only if $\ell = k_j$ for some integer $j \geqq 0$. Since

$$\left| h_{k_j}(x_0) \right|^2 = [k_j] = 2^j [k_0]$$

we conclude by (24) that

$$\frac{a_{k_j}}{h_{k_j}(x_0)} = \frac{1}{2} \frac{d}{[k_0]} \neq 0$$

for $j = 1, 2, \ldots$. Contrary to hypothesis, therefore, S cannot satisfy the A–T condition at x_0. □

An $\mathcal{H}$-system is called weakly nested if each distinguished set E contains a distinguished set F whose closure satisfies $\overline{F} \subsetneqq E$. Notice that the square, polar and classical Haar systems are all weakly nested. In fact, if Δ and the distinguished subsets of Δ are simply connected it is always possible to divide the boundaries of the distinguished sets in such a way that the resulting $\mathcal{H}$-system is weakly nested.

A subset $w \subseteqq \Delta$ is called $\mathcal{H}$-negligible if given any distinguished set E there is a distinguished set $F \subseteqq E$ such that $w \cap F = \emptyset$. Recall that an embedded arc in $\mathbf{R}^2$ is a $1-1$ bicontinuous image of an interval. It is clear that embedded arcs are $\mathcal{H}$-negligible for both the square and polar Haar systems.

THEOREM 4. *Let $f \in L^1_\mu(\Delta)$, $p_1 < p_2 < \ldots$ be any sequence of positive integers, $\mathcal{H}$ be weakly nested, S be an $\mathcal{H}$-series which satisfies the A–T condition everywhere on Δ, and $w_1, w_2, \ldots$ be a countable collection of $\mathcal{H}$-negligible sets. If*

$$\lim_{k\to\infty} S_{p_k} = f \quad \text{a.e. on} \quad \Delta \tag{25}$$

and

$$\limsup_{k\to\infty} |S_{p_k}(x)| < \infty \quad \text{for} \quad x \notin \bigcup_{j=1}^{\infty} w_j \tag{26}$$

then S is the $\mathcal{H}$-Fourier series of f.

PROOF. Assume for simplicity that $p_k = k$ for $k = 1, 2, \ldots$.

Suppose the theorem is false, i.e., that S is not the $\mathcal{H}$-Fourier series of f. Choose an integer n_0 and a distinguished set $\Delta_{i_0}^{(n_0)}$ such that

$$S_{n_0} - S_{n_0} f \neq 0 \quad \text{on} \quad \Delta_{i_0}^{(n_0)}.$$

Since w_1 is $\mathcal{H}$-negligible there exists a distinguished set $V_0 \subseteqq \Delta_{i_0}^{(n_0)}$ such that $w_1 \cap V_0 = \emptyset$. Moreover, since $\mathcal{H}$ is weakly nested there exist distinguished sets $V_1, V_2, \ldots$ such that

$$\overline{V}_k \subset V_{k-1}, \qquad k = 1, 2, \ldots. \tag{27}$$

Set $E_0 = \Delta_{i_0}^{(n_0)}$. By successive division choose distinguished sets $E_j = \Delta_{i_j}^{(n_j)}$ which satisfy $\mu(E_j) = \frac{1}{2}\mu(E_{j-1})$, $E_j \subset E_{j-1}$ for $j = 1, 2, \ldots$ and

$$E_{j_k} = V_k \tag{28}$$

for some integers j_k, $k = 0, 1, \ldots$.

Set $F_j = E_{j-1} \setminus E_j$, $j = 1, 2, \ldots$, and $T = S - Sf$. Notice by (27) and (28) that $\cap E_j \neq \emptyset$. Also notice by Lemma 7 and hypothesis that T satisfies the A–T condition everywhere on Δ. Thus by Lemma 8 we can choose an integer j so large that $E_{j-1} \subseteqq V_1$

$$S_{n_j} - S_{n_j} f \neq 0 \quad \text{on} \quad F_j.$$

Applying Lemma 6 to the distinguished set F_j we can choose a distinguished set $\Delta_\ell^{(m)} \subseteqq F_j$ such that $|S_m| > 1$ on $\Delta_\ell^{(m)}$.

Set $m_0 = n_0$, $\ell_0 = i_0$, $m_1 = m$, $\ell_1 = \ell$ and notice by construction that

$$w_1 \cap \Delta_{\ell_1}^{(m_1)} = \emptyset, \qquad \overline{\Delta_{\ell_1}^{(m_1)}} \subset \Delta_{\ell_0}^{(m_0)},$$

and

$$|S_{m_1}(x)| > 1 \quad \text{for} \quad x \in \Delta_{\ell_1}^{(m_1)}.$$

Continuing this process we generate distinguished sets $\Delta_{\ell_k}^{(m_k)}$, $k = 0, 1, \dots$ such that

$$w_k \cap \Delta_{\ell_k}^{(m_k)} = \emptyset, \tag{29}$$

$$\overline{\Delta_{\ell_k}^{(m_k)}} \subset \Delta_{\ell_{k-1}}^{(m_{k-1})}, \tag{30}$$

and

$$|S_{m_k}(x)| > k \quad \text{for} \quad x \in \Delta_{\ell_k}^{(m_k)}. \tag{31}$$

By (30) choose $x_0 \in \bigcap_{k=1}^{\infty} \Delta_{\ell_k}^{(m_k)}$. We have by (31) that

$$\limsup_{m \to \infty} |S_m(x_0)| = +\infty.$$

Consequently, it follows from hypothesis that $x_0 \in w_j$ for some $j \geqq 1$. However, (29) implies $x_0 \notin w_k$ for any k. This contradiction proves that $S_m \equiv S_m f$ for all integers m and thus S is the $\mathcal{H}$-Fourier series of f. □

COROLLARY. *Let f be a finite-valued, integrable function on Δ, $\mathcal{H}$ be weakly nested, and S be an $\mathcal{H}$-series which satisfies the A–T condition everywhere on Δ. If S converges to f except perhaps on a countable collection of $\mathcal{H}$-negligible sets, then S is the $\mathcal{H}$-Fourier series of f.*

Movsisjan [5] has shown for the Kronecker product Haar system on the d-dimensional unit cube that uniqueness holds for a series satisfying the A–T condition everywhere which converges to a finite-valued integrable f off a countable set. Our techniques show that the countable set can be replaced by a countable union of $1-1$ bicontinuous image of compact sets of dimension less than d, because such sets are negligible for the collection of d-dimensional rectangles which form the partitions $\{\Delta_j^n : 1 \leqq i \leqq n\}$. In particular, in two dimensions any countable union of embedded arcs can be discarded and uniqueness will still hold.

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TRANSITION LAYER PHENOMENA OF THE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DISCONTINUITY

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1. Introduction

In this paper we consider the interior transition layer phenomena of solutions for the nonlinear boundary value problems with discontinuity. M. Slatkin [4] has considered the following differential equation with discontinuity:

$$A'' = \begin{cases} (1-A^2)/2 & \text{if } 0 < \xi \leqq \ell_0 \\ -(1-A^2)/2 & \text{if } \ell_0 \leqq \xi < \infty \end{cases}$$

(ℓ_0 is a constant). On the other hand P. Fife [2] (p. 68) considered the equation with discontinuity:

$$\frac{1}{2}\frac{d^2\hat{\varrho}}{dy^2} = -G(\hat{\varrho}) = \begin{cases} G^+(\hat{\varrho}) & \text{if } \hat{\varrho} > \nu \\ G^-(\hat{\varrho}) & \text{if } \hat{\varrho} \leqq \nu \end{cases}$$

(ν is a constant). In this paper we consider the singular perturbation problem for the following differential equation:

$$(0.1) \qquad \varepsilon^2 u'' = \hat{F}(u,t) = \begin{cases} \hat{f}_0(u,t) & \text{if } ((u,t) \in (-\infty,\infty) \times [0,t_0]) \\ \hat{f}_1(u,t) & \text{if } ((u,t) \in (-\infty,\infty) \times [t_0,1]), \end{cases}$$

$$(0.2) \qquad u(0) = a_0, \quad u(1) = a_1$$

($\varepsilon > 0$, t_0 is given later). This problem is generalized to the following singular perturbation problem for a system of differential equations:

$$(1.1) \qquad \varepsilon^2 u'' = F(u,v) = \begin{cases} f_0(u,v) & \text{if } ((u,v) \in (-\infty,\infty) \times [b_0,z_0]) \\ f_1(u,v) & \text{if } ((u,v) \in (-\infty,\infty) \times [z_0,b_1]), \end{cases}$$

$$(1.2) \qquad v'' = g(u,v),$$

$$(1.3) \qquad u(0) = a_0, \quad u(1) = a_1, \quad v(1) = b_0, \quad v(1) = b_1.$$

In this paper we consider the existence of the solution of (1) when ε is small enough, as well as the asymptotic behavior of this solution as $\varepsilon \to 0$ in Theorems 4 and 5. This result is a generalization of Fife [1]. The boundary value problem (0) is a special case of (1) and the existence and asymptotic behavior of the solution of (0) are given as results of Theorems 4 and 5. We illustrate the properties of the solutions on some examples of type (0) or (1).

2. Preliminaries

Let $f_0(u,v)$ be sufficiently many times continuously differentiable in $(-\infty,\infty)\times[b_0,w_0]=(-\infty,\infty)\times I_0$ and let $f_0(u,v)=0$ have twice continuously differentiable solutions $h_{00}(v)$, $r(v)$, $h_{01}(v)$ in I_0. Let $f_1(u,v)$ be sufficiently many times continuously differentiable in $(-\infty,\infty)\times[w_1,b_1]$ $(w_0>w_1)$ and let there be solutions $h_{10}(v)$, $r(v)$, $h_{11}(v)$ for $f_1(u,v)=0$ in I_1. We suppose $h_{10}(v)<h_{00}(v)<r(v)<h_{01}(v)<h_{11}(v)$ in the interval $I=I_0\cap I_1$.

We investigate the existence and asymptotic properties of the solutions of (1) under some assumptions. We assume that the following conditions hold:

$$\text{(I)}\qquad \begin{cases} \frac{\partial}{\partial u}f_0(h_{00}(v),v)\geqq 2\beta>0 & (v\in I_0),\\ \frac{\partial}{\partial u}f_1(h_{11}(v),v)\geqq 2\beta>0 & (v\in I_1),\end{cases}$$

where β is a constant.

$$\text{(II)}\qquad \begin{cases} \int_{h_{00}(b_0)}^{k} f_0(u,b_0)du>0 \quad (k\in[a_0,h_{00}(b_0)) \quad\text{or}\quad k\in(h_{00}(b_0),a_0]),\\ \int_{h_{11}(b_1)}^{k} f_1(u,b_1)du>0 \quad (k\in[a_1,h_{11}(b_1)) \quad\text{or}\quad k\in(h_{11}(b_1),a_1]).\end{cases}$$

Let us introduce the function $J(z_0)$ as follows:

$$J(z_0)=\int_{h_{00}(z_0)}^{r(z_0)} f_0(u,z_0)du-\int_{h_{11}(z_0)}^{r(z_0)} f_1(u,z_0)du,\quad\text{where}\quad z_0\in I.$$

(III) There is a zero of $J(z_0)$ at $z_0=z_0^*$ and $J(z_0)$ changes sign as z_0 passes through z_0^*.

$$\text{(IV)}\qquad \begin{cases} \int_{h_{00}(z_0^*)}^{k} f_0(u,z_0^*)du>0 & (k\in(h_{00}(z_0^*),r(z_0^*))),\\ \int_{h_{11}(z_0^*)}^{k} f_1(u,z_0^*)du>0 & (k\in(r(z_0^*),h_{11}(z_0^*))).\end{cases}$$

Let us define $G(v)$ by the following equation:

$$G(v)=\begin{cases} g(h_{00}(v),v) & \text{if } b_0\leqq v\leqq z_0^*\\ g(h_{11}(v),v) & \text{if } z_0^*<v\leqq b_1.\end{cases}$$

(V) There is a solution $V=V(t)$ $(V'(t)\neq 0)$ of the following equation:

$$V''=G(V),\qquad V(0)=b_0,\qquad V(1)=b_1,$$

and there is a $t=t_0^*$ $(t_0^*\in(0,1))$ such that $V(t_0^*)=z_0^*$.

LEMMA 1. *When $|\omega|$ is small enough there is a solution $V_0 = V_0(t)$ ($V_0'(t) \neq 0$) of the boundary value problem*

$$V_0'' = g(h_{00}(V_0), V_0), \quad V_0(0) = b_0, \quad V_0(t_0^* + \delta) = V(t_0^*) + \omega,$$

and there is a solution $V_1 = V_1(t)$ ($V_1'(t) \neq 0$) of the boundary value problem

$$V_1'' = g(h_{11}(V_1), V_1), \quad V_1(t_0^* + \delta) = V(t_0^*) + \omega, \quad V_1(0) = b_1,$$

where $|\delta|$ is enough small.

PROOF. Lemma 1 follows from condition (V). The details are omitted (see Fife [1]). □

We consider the following boundary value problems:

(2.1) $$\varepsilon^2 u_0'' = f_0(u_0, v_0),$$

(2.2) $$v_0'' = g(u_0, v_0),$$

(2.3) $$u_0(0) = a_0, \ u_0(t_0^* + \delta) = r(V(t_0^*) + \omega), \ v_0(0) = b_0, \ v_0(t_0^* + \delta) = V(t_0^*) + \omega,$$

(2.1′) $$\varepsilon^2 u_1'' = f_1(u_1, v_1),$$

(2.2′) $$v_1'' = g(u_1, v_1),$$

(2.3′) $$u_1(t_0^* + \delta) = r(V(t_0^*) + \omega), \ u_1(1) = a_1, \ v_1(t_0^* + \delta) = V(t_0^*) + \omega, \ v_1(1) = b_1.$$

LEMMA 2. *Assume the conditions* (I), (II), (IV) *and* (V). *If* $\max|g_u(h_{00}(v), v)| + \max|g_u(h_{11}(v), v)| = \pi_1$ *is small enough, then there is a solution* $(u_0(t, \delta, \varepsilon, \omega), v_0(t, \delta, \varepsilon, \omega))$ *for the problem* (2) *and a solution* $(u_1(t, \delta, \varepsilon, \omega), v_1(t, \delta, \varepsilon, \omega))$ *for the problem* (2′) *respectively as* ε *is small enough.*

PROOF. See Fife [1]. □

LEMMA 3. *The asymptotic properties of these solutions are given by the following relations as* $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} u_0(t, \delta, \omega, \varepsilon) = h_{00}(V(t)) \quad \textit{uniformly in} \quad [\lambda, t_0^* + \delta - \lambda] \qquad (\lambda > 0),$$

$$\lim_{\varepsilon \to 0} u_1(t, \delta, \omega, \varepsilon) = h_{11}(V(t)) \quad \textit{uniformly in} \quad [t_0^* + \delta + \lambda, 1 - \lambda],$$

$$\lim_{\varepsilon \to 0} \|V_0(t) - v_0(t, \delta, \omega, \varepsilon)\|_{C^1[0, t_0^* + \delta]} = 0,$$

$$\lim_{\varepsilon \to 0} \|V_1(t) - v_1(t, \delta, \omega, \varepsilon)\|_{C^1[t_0^* + \delta, 1]} = 0.$$

PROOF. See Fife [1] (p. 504). □

3. Existence and asymptotic behavior of the solutions

In this section we construct the solution of the problem (1) and study its asymptotic behavior in Theorems 4 and 5. We consider the existence and asymptotic properties of the solution of (0).

THEOREM 4. *Assume* (I), (II), (III), (IV) *and* (V). *If* π_1 *is small enough, then there is a solution* $(u,v)=(u(t,\varepsilon),v(t,\varepsilon))$ *of the boundary value problem* (1).

PROOF. Put

$$\Phi(\delta,\omega,\varepsilon)=\left(\varepsilon u_0'(t_0^*+\delta,\omega,\varepsilon)\right)^2-\left(\varepsilon u_1'(t_0^*+\delta,\omega,\varepsilon)\right)^2,$$
$$\Psi(\delta,\omega,\varepsilon)=v_0'(t_0^*+\delta,\omega,\varepsilon)-v_1'(t_0^*+\delta,\omega,\varepsilon).$$

By the conditions (III) and (V) there is a solution $(\delta,\omega)=(\delta(\varepsilon),\omega(\varepsilon))$ of the equation $\Phi(\delta,\omega,\varepsilon)=0$, $\Psi(\delta,\omega,\varepsilon)=0$ such that $|\delta(\varepsilon)|+|\omega(\varepsilon)|\to 0$ as $\varepsilon\to 0$. Thus

$$u(t,\varepsilon)=\begin{cases} u_0(t,\delta(\varepsilon),\omega(\varepsilon),\varepsilon) & \text{if } 0\leqq t\leqq t_0^*+\delta(\varepsilon)\\ u_1(t,\delta(\varepsilon),\omega(\varepsilon),\varepsilon) & \text{if } t_0^*+\delta(\varepsilon)<t\leqq 1\end{cases}$$

is continuously differentiable at $t=t_0^*+\delta(\varepsilon)$. Define

$$v(t,\varepsilon)=\begin{cases} v_0(t,\delta(\varepsilon),\omega(\varepsilon),\varepsilon) & \text{if } 0\leqq t\leqq t_0^*+\delta(\varepsilon)\\ v_1(t,\delta(\varepsilon),\omega(\varepsilon),\varepsilon) & \text{if } t_0^*+\delta(\varepsilon)<t\leqq 1,\end{cases}$$

then $v(t,\varepsilon)$ is continuously differentiable at $t=t_0^*+\delta(\varepsilon)$.

Due to the structure of $(u(t,\varepsilon),v(t,\varepsilon))$, it is twice continuously differentiable in $(0,1)$ and satisfies the differential equations (1.1), (1.2) and the boundary condition (1.3). This $(u,v)=(u(t,\varepsilon),v(t,\varepsilon))$ is the solution which we were looking for. □

THEOREM 5. *The asymptotic properties of this solution are given by following:*

$$\lim_{\varepsilon\to 0}u(t,\varepsilon)=\begin{cases} h_{00}(v(t)) & \text{uniformly in } [\lambda,t_0^*-\lambda]\quad(\lambda>0),\\ h_{11}(V(t)) & \text{uniformly in } [t_0^*+\lambda,1-\lambda],\end{cases}$$
$$\lim_{\varepsilon\to 0}\|V(t)-v(t,\varepsilon)\|_{C^1[0,1]}=0.$$

PROOF. This is proved in Lemma 3. □

$t=t_0^*$ is the transition point for this problem and $z_0=V(t_0^*)+\omega(\varepsilon)$. We consider the problem (0). Put $t=v$, then (0) can be rewritten in the type of (1):

$$\varepsilon^2u''=\hat{F}(u,v),\tag{0.1}$$
$$v''=0,\tag{0.2}$$
$$u(0)=a_0,\quad u(1)=a_1,\quad v(0)=0,\quad v(1)=1.\tag{0.3}$$

We assume that the conditions on $f_i(u,v)$ $(i=0,1)$ and $h_{ij}(v)$ $(i,j=0,1)$ are satisfied by $\hat{f}_i(u,v)$ $(i=0,1)$ and $\hat{h}_{ij}(v)$ (the solutions of $\hat{f}_i(u,v)=0$ $(i,j=0,1)$).

THEOREM. *Suppose the conditions* (I), (II), (III) *and* (IV), *then there is a solution* $u = u(t, \varepsilon)$ *of the boundary value problem* (0) *such that it has an interior transition point in* $\hat{I}$ *(a sub-interval of* $(0, 1)$*).*

PROOF. By Theorems 4 and 5, this theorem is proved. □

4. Examples

In this section we study the properties of the solutions to problems of type (0) and (1).

EXAMPLE 1. Consider the following problem:

$$\varepsilon^2 u'' = F(u, v), \tag{3.1}$$

$$v'' = -au = g(u, v) \qquad (a > 0), \tag{3.2}$$

$$u(0) = u(1) = 2 - \sqrt{3/2}, \qquad v(0) = v(1) = 2 - \sqrt{3/2} \tag{3.3}$$

where

$$F(u, v) = \begin{cases} (u^2 - 1)(u - v) & (v \leqq z_0) \\ (u^2 - 4)(u - v) & (v > z_0). \end{cases}$$

We show the existence of the solutions $(u(t, \varepsilon), v(t, \varepsilon))$, with the transition points from -1 to 2 or from 2 to -1, of (3). In this problem $h_{00} = -1$, $h_{01} = 1$, $h_{10} = -2$, $h_{11} = 2$ and $r = v$. We define $J(z_0)$ as follows:

$$J(z_0) = \int_{-1}^{z_0} (u^2 - 4)(u - z_0)du - \int_{2}^{z_0} (u^2 - 4)(u - z_0)du = -\frac{3}{2}\left(z_0^2 - 4z_0 + \frac{5}{2}\right).$$

We can find the root $z_0^* = 2 - \sqrt{3/2}$ of $J(z_0)$. Consider the following problem:

$$V'' = G(V) = \begin{cases} g(-1, V) = a & (V \leqq z_0^*) \\ g(2, V) = -2a & (V > z_0^*), \end{cases} \tag{4}$$

$$V(0) = V(1) = z_0^* = 2 - \sqrt{3/2}. \tag{5}$$

Then there is a solution $V_1(t)$ of (4), (5):

$$V_1(t) = \begin{cases} -a\left(t - \frac{1}{6}\right)^2 - \frac{a}{6}\left(t - \frac{1}{6}\right) + z_0^*, & \text{if } 0 \leqq t \leqq \frac{1}{6} \\ \frac{a}{2}\left(t - \frac{1}{6}\right)^2 - \frac{a}{6}\left(t - \frac{1}{6}\right) + z_0^*, & \text{if } \frac{1}{6} < t \leqq \frac{1}{2} \\ -a\left(t - \frac{2}{3}\right)^2 - \frac{a}{6}\left(t - \frac{2}{3}\right) + z_0^*, & \text{if } \frac{1}{2} < t \leqq \frac{2}{3} \\ \frac{a}{2}\left(t - \frac{2}{3}\right)^2 - \frac{a}{6}\left(t - \frac{2}{3}\right) + z_0^*, & \text{if } \frac{2}{3} < t \leqq 1. \end{cases}$$

$V_1(t)$ is a periodic function with period $\frac{1}{2}$. Similarly there are solutions $V_n(t)$ with period $\frac{1}{2^n}$ for every integer $n \geqq 1$. By the use of $V_n(t)$ we can construct the solution $(u_n(t,\varepsilon), v_n(t,\varepsilon))$ of (3). Notice that

$$\max V_n(t) = z_0^* + a\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{2}\right)^{2n+2} \quad \text{and} \quad \min V_n(t) = z_0^* - a\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{2}\right)^{2n+1}.$$

For all n, $-1 < V_n(t) < 1$ must be satisfied. Thus $a < 36(\sqrt{3/2} - 1)$. By the use of this condition $0 < V_n(t) < 1$ for all n. Thus every transition layer phenomena of $(u_n(t,\varepsilon), v_n(t,\varepsilon))$ are caused by the discontinuity of the differential equation (3.1). There are $2^{n+1} - 1$ transition points for this solution. We give the asymptotic properties of $(u_1(t,\varepsilon), v_1(t,\varepsilon))$ as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} u_1(t,\varepsilon) = \begin{cases} 2 & (\text{uniformly in } \lambda \leqq t \leqq \frac{1}{6} - \lambda) \\ -1 & (\text{uniformly in } \frac{1}{6} + \lambda \leqq t \leqq \frac{1}{2} - \lambda) \\ 2 & (\text{uniformly in } \frac{1}{2} + \lambda \leqq t \leqq \frac{2}{3} - \lambda) \\ -1 & (\text{uniformly in } \frac{2}{3} + \lambda \leqq t \leqq 1 - \lambda), \end{cases}$$

$$\lim_{\varepsilon \to 0} v_1(t,\varepsilon) = V_1(t) \quad (\text{uniformly in } 0 \leqq t \leqq 1).$$

Similarly the asymptotic properties of $(u_n(t,\varepsilon), v_n(t,\varepsilon))$ can be given. In this problem z_0 is determined as a function of ε and it is near z_0^*.

EXAMPLE 2. Consider the following boundary value problem:

$$\varepsilon^2 u'' = F(u,t), \tag{6}$$

$$u(0) = -\frac{1}{2}, \qquad u(1) = \frac{3}{2}, \tag{7}$$

where

$$F(u,t) = \begin{cases} (u^2-1)(u-t) & (t \leqq t_0) \\ (u^2-4)(u-t) & (t > t_0). \end{cases}$$

By the theorem there is a solution $u = u(t,\varepsilon)$ with only one transition point $t_0^* = 2 - \sqrt{3/2}$. The asymptotic properties are given as follows:

$$\lim_{\varepsilon \to 0} u(t,\varepsilon) = \begin{cases} -1 & (\text{uniformly in } \lambda \leqq t \leqq t_0^* - \lambda) \\ 2 & (\text{uniformly in } t_0^* + \lambda \leqq t \leqq 1 - \lambda). \end{cases}$$

In this problem t_0 is determined as a function of ε and it is near t_0^*.

EXAMPLE 3. Consider the following boundary value problem:

$$\varepsilon^2 u'' = F(u,t), \tag{8}$$

$$u(-b) = \sqrt{3/2} - 2, \quad u(b) = 2 - \sqrt{3/2} \quad (0 < b < 1), \tag{9}$$

$$F(u,t) = \begin{cases} (u^2-4)(u-t) & (t \leqq t_{-1}) \\ (u^2-1)(u-t) & (t_{-1} < t < t_1) \\ (u^2-4)(u-t) & (t_1 \leqq t). \end{cases}$$

(a) $2-\sqrt{3/2} < b < 1$. In this case there are three transition points: $t^*_{-1} = \sqrt{3/2}-2$, $t^*_1 = 2-\sqrt{3/2}$ and $t^*_0 = 0$. The first two are caused by discontinuity of the differential equation. The asymptotic properties are as follows.

$$\lim_{\varepsilon\to 0} u(t,\varepsilon) = \begin{cases} -2 & \text{uniformly in } -b+\lambda \leqq t \leqq t^*_{-1}-\lambda \\ 1 & \text{uniformly in } t^*_{-1}+\lambda \leqq t \leqq t^*_0-\lambda \\ -1 & \text{uniformly in } t^*_0+\lambda \leqq t \leqq t^*_1-\lambda \\ 2 & \text{uniformly in } t^*_1+\lambda \leqq t \leqq b-\lambda. \end{cases}$$

(b) $0 < b < 2-\sqrt{3/2}$. In this case there is a solution with a transition point $t^*_0 = 0$ which is not caused by discontinuity of the differential equation.

The asymptotic property is given by the following formula:

$$\lim_{\varepsilon\to 0} u(t,\varepsilon) = \begin{cases} 1 & \text{uniformly in } -b+\lambda \leqq t \leqq -\lambda \\ -1 & \text{uniformly in } \lambda \leqq t \leqq b-\lambda. \end{cases}$$

By considering the feature of the solutions of the above three examples we see that there are different cases. Under some conditions there are solutions for the boundary value problems for systems of differential equations with arbitrarily many interior transition points. But for the boundary value problems for one differential equation with discontinuity we got one and three transition points.

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AN ERGODIC MAXIMAL EQUALITY FOR NONSINGULAR FLOWS AND APPLICATIONS

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Dedicated to Professor Shigeru Tsurumi on his retirement from Tokyo Metropolitan University

Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space and $\{T_t\}$ a measurable flow of nonsingular automorphisms of $(X, \mathcal{B}, \mu)$. For an f in $L_1(\mu)$ we define the *maximal ergodic function* f^* by

$$f^*(x) = \sup_{b>0} \frac{\int_0^b f(T_t x) \frac{d\mu \circ T_t}{d\mu}(x)dt}{\int_0^b \frac{d\mu \circ T_t}{d\mu}(x)dt}. \tag{1}$$

In this paper we assume that the flow $\{T_t\}$ is *conservative* and *ergodic*, and prove that

$$\alpha\mu\{f^* > \alpha\} = \int_{\{f^*>\alpha\}} f d\mu \quad \text{for all} \quad \alpha > \int f d\mu. \tag{2}$$

This equality implies that a nonnegative f in $L_1(\mu)$ satisfies $\int_{\{f>1\}} f \log f d\mu < \infty$ if and only if $\int f^* d\mu < \infty$. (For a related topic, see [1], [9], [11] and [14].)

About ten years ago, Marcus and Petersen [8] proved equality (2), under the assumption that all the T_t are measure preserving. Thus our result may be regarded as a generalization of their theorem to noninvariant measures. Further we shall apply the proof of (2) to investigate an integrability problem for the *maximal function* Mf defined by

$$Mf(x) = \max\{f^*(x), (-f)^*(x)\}.$$

To do this, we introduce the decreasing function $\hat{f}$ on the interval $[0, 1)$ which is equidistributed with $f \in L_1(\mu)$. Extending $\hat{f}$ to the real line by $\hat{f}(t+1) = \hat{f}(t)$, we define

$$H(f) = \int_0^{1/2} \frac{1}{t} \left| \int_{-t}^{t} \hat{f}(s)\, ds \right| dt.$$

Clearly, $H(f) = 0$ if and only if f and $-f$ are equidistributed. Further, it will be seen that if $f \geqq 0$ then $H(f) < \infty$ if and only if $\int Mf d\mu < \infty$. However, if the nonnegativity of f is not assumed, then, as is easily seen by a simple example, $H(f) < \infty$ does not necessarily imply $\int Mf d\mu < \infty$. (It will be proved below that $\int Mf d\mu < \infty$ implies $H(f) < \infty$.) Therefore it would be of interest to know what condition on f is necessary (and sufficient) for $H(f) < \infty$. Already, this problem was discussed by the author in [13], under the assumption that all the T_t are measure preserving. In this paper we shall prove that the results in [13] hold without the assumption.

1. Preliminaries and the maximal equality

Let $\{T_t\} = \{T_t\colon -\infty < t < \infty\}$ be a measurable flow of nonsingular automorphisms of $(X, \mathcal{B}, \mu)$, where we may and will assume without loss of generality that the probability space $(X, \mathcal{B}, \mu)$ is nonatomic and complete. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero. Since each T_t is nonsingular, the Radon–Nikodym theorem can be applied to define a function $w_t = d\mu \circ T_t / d\mu$ in $L_1(\mu)$ satisfying

$$\int_A w_t d\mu = \mu(T_t A) \quad \text{for all} \quad A \in B, \tag{3}$$

and let us put

$$U_t f(x) = f(T_t x) w_t(x) \quad \text{for} \quad f \in L_1(\mu). \tag{4}$$

As is easily seen, $\{U_t\} = \{U_t\colon -\infty < t < \infty\}$ becomes a one-parameter group of positive linear isometries on $L_1(\mu)$. It follows from Krengel [3] (see also Sato [12]) that strong-$\lim_{t \to 0} U_t = I$, I being the identity operator. Thus we may suppose without loss of generality that the function $(t, x) \to w_t(x)$ is measurable on $(-\infty, \infty) \times X$. This was used implicitly in the definition of f^*.

The flow $\{T_t\}$ is called *conservative* if each T_t is conservative. Recall that a nonsingular automorphism T is conservative if and only if $A \subset TA$ implies $A = TA$. It is known (see e.g. Krengel [6], §3.1) that T is conservative if and only if

$$\sum_{n=0}^{\infty} \frac{d\mu \circ T^n}{d\mu}(x) = \infty \quad \text{for almost all} \quad x \in X.$$

It is easily seen that $\{T_t\}$ is conservative if and only if

$$\int_0^{\infty} U_t 1(x) dt = \int_0^{\infty} w_t(x) dt = \infty \quad \text{for almost all} \quad x \in X.$$

$\{T_t\}$ is called *ergodic* if $A = T_t A$ for all real t implies $\mu A = 0$ or $\mu(X \setminus A) = 0$.

THEOREM 1. *Let $\{T_t\}$ be an ergodic, conservative, measurable flow of nonsingular automorphisms of $(X,\mathcal{B},\mu)$ with $\mu X = 1$. Let f be a function in $L_1(\mu)$ and α be a constant such that $\alpha > \int f d\mu$. Then*

$$\alpha\mu\{f^* > \alpha\} = \int\limits_{\{f^*>\alpha\}} f d\mu. \tag{5}$$

PROOF. It follows that

$$\{f^* > \alpha\} = \left\{x\colon \sup_{b>0} \int_0^b U_t(f-\alpha)(x)dt > 0\right\},$$

and hence a continuous time version of Hopf's maximal inequality yields

$$\alpha\mu\{f^* > \alpha\} \leqq \int\limits_{\{f^*>\alpha\}} f d\mu. \tag{6}$$

Therefore it remains to prove the converse inequality. To do this, we first notice that:

If T is a conservative, nonsingular automorphism of $(X,\mathcal{B},\mu)$, then for any f and g in $L_1(\mu)$ with $g > 0$ a.e. on X

$$\int\limits_{(A\cup T^{-1}A)\cap I(X\setminus A)} f d\mu \leqq \alpha \int\limits_{(A\cup T^{-1}A)\cap I(X\setminus A)} g d\mu, \tag{7}$$

where we set

$$A = \left\{x\colon \sup_{n\geqq 1} \sum_{i=0}^{n-1} f(T^i x)\frac{d\mu\circ T^i}{d\mu}(x) \Big/ \sum_{i=0}^{n-1} g(T^i x)\frac{d\mu\circ T^i}{d\mu}(x) > \alpha\right\} \tag{8}$$

and $I(X\setminus A)$ denotes the smallest T-invariant set containing $X\setminus A$.

In fact, since T is conservative, it follows that

$$I(X\setminus A) = \bigcup_{n=0}^{\infty} T^{+n}(X\setminus A),$$

$$A\cap I(X\setminus A) = \bigcup_{n=1}^{\infty}\left(T^{+n}(X\setminus A)\cap\bigcap_{i=0}^{n-1} T^{+i}A\right)$$

and

$$T^{-n}A\cap\bigcap_{i=n+1}^{\infty} T^{-i}(X\setminus A) = \emptyset \quad \text{for all} \quad n\geqq 0.$$

Thus we get

$$\int_{A\cap I(X\setminus A)} f d\mu = \sum_{n=1}^{\infty} \int_{T^{+n}(X\setminus A)\cap\bigcap_{i=0}^{n-1} T^{+i}A} f d\mu =$$

$$= \sum_{n=1}^{\infty} \int_{(X\setminus A)\cap\bigcap_{i=1}^{n} T^{-i}A} f(T^n x)\cdot \frac{d\mu\circ T^n}{d\mu}(x) d\mu(x) =$$

$$= \sum_{n=1}^{\infty} \int_{(X\setminus A)\cap\left(\bigcap_{i=1}^{n} T^{-i}A\right)\cap T^{-n-1}(X\setminus A)} \left(\sum_{i=1}^{n} f(T^i x)\cdot \frac{d\mu\circ T^i}{d\mu}(x)\right) d\mu(x)$$

and therefore by (8)

$$\int_{(A\cup T^{-1}A)\cap I(X\setminus A)} f d\mu = \int_{A\cap I(X\setminus A)} f d\mu + \int_{(X\setminus A)\cap T^{-1}A} f d\mu =$$

$$= \sum_{n=1}^{\infty} \int_{(X\setminus A)\cap\left(\bigcap_{i=1}^{n} T^{-i}A\right)\cap T^{-n-1}(X\setminus A)} \left(\sum_{i=0}^{n} f(T^i x)\cdot \frac{d\mu\circ T^i}{d\mu}(x)\right) d\mu(x) \leqq$$

$$\leqq \sum_{n=1}^{\infty} \int_{(X\setminus A)\cap\left(\bigcap_{i=1}^{n} T^{-i}A\right)\cap T^{-n-1}(X\setminus A)} \alpha\left(\sum_{i=0}^{n} g(T^i x)\cdot \frac{d\mu\circ T^i}{d\mu}(x)\right) d\mu(x) =$$

$$= \alpha \int_{(A\cup T^{-1}A)\cap I(X\setminus A)} g d\mu,$$

establishing inequality (7). (This is an adaptation of Derriennic's argument in [2].)

We now proceed as follows. For an integer $n \geqq 1$, write

$$f_n(x) = 2^n \int_0^{2^{-n}} f(T_t x) w_t(x) dt$$

and

$$g_n(x) = 2^n \int_0^{2^{-n}} w_t(x)dt.$$

It follows that

$$\lim_n \|f_n - f\|_1 = 0 = \lim_n \|g_n - 1\|_1.$$

Further if we set

$$\tilde{f}_n(x) = \sup_{k \geqq 1} \frac{\sum_{i=0}^{k-1} f_n\left(T_{i/2^n}x\right) w_{i/2^n}(x)}{\sum_{i=0}^{k-1} g_n\left(T_{i/2^n}x\right) w_{i/2^n}(x)}$$

and $A_n = \{\tilde{f}_n > \alpha\}$, and if $I_n(X \setminus A_n)$ denotes the smallest $T_{2^{-n}}$-invariant set containing $X \setminus A_n$, then

$$\tilde{f}_n(x) \uparrow f^*(x) \quad \text{a.e. on} \quad X \tag{9}$$

and

$$\int_{\left(A_n \cup T_{2^{-n}}^{-1} A_n\right) \cap I_n(X \setminus A_n)} f_n d\mu \leqq \alpha \int_{\left(A_n \cup T_{2^{-n}}^{-1} A_n\right) \cap I_n(X \setminus A_n)} g_n d\mu. \tag{10}$$

Since $\alpha > \int f d\mu$ implies $\mu(X \setminus \{f^* > \alpha\}) > 0$ by (6), the ergodicity of $\{T_t\}$ and (9) yield

$$I_n(X \setminus \{f^* > \alpha\}) \uparrow X.$$

Since $A_n \uparrow \{f^* > \alpha\}$, it is then enough to show that

$$\lim_n \mu\left(\left(T_{2^{-n}}^{-1} A\right) \setminus A\right) = 0 \quad \text{for each} \quad A \in \mathcal{B}. \tag{11}$$

For this purpose we use the strong continuity of $\{U_t\}$. It follows that

$$\lim_{t \to 0} \|w_t - 1\|_1 = \lim_{t \to 0} \|U_t 1 - 1\|_1 = 0$$

and

$$\lim_{t \to 0} \left\| w_t \cdot 1_{T_t^{-1} A} - 1_A \right\|_1 = 0.$$

Hence given an $\varepsilon > 0$ we can choose a real number $t_0 > 0$ so that if $|t| < t_0$ then

$$\mu\{|w_t - 1| > \varepsilon\} < \varepsilon, \qquad \int_{\{|w_t - 1| > \varepsilon\}} w_t d\mu < \varepsilon$$

and

$$\int\limits_{(T_t^{-1}A)\setminus A} w_t d\mu < \varepsilon.$$

Therefore

$$\begin{aligned}\mu((T_t^{-1}A)\setminus A) = \mu((T_t^{-1}A)\setminus A) \cap \{|w_t - 1| > \varepsilon\})+\\ +\mu((T_t^{-1}A)\setminus A) \cap \{|w_t - 1| \leqq \varepsilon\}) <\\ < \varepsilon + \frac{1}{1-\varepsilon} \int\limits_{(T_t^{-1}A)\setminus A} w_t d\mu < \varepsilon + \frac{\varepsilon}{1-\varepsilon}\end{aligned}$$

for $|t| < t_0$. This establishes (11), and the proof is complete.

2. Applications

Theorem 1 can be applied to obtain the following

THEOREM 2. *Let $\{T_t\}$ be as in Theorem 1. Let f be a nonnegative function in $L_1(\mu)$ and r a nonnegative constant. Then*

$$\int\limits_{\{f>1\}} f(\log f)^{r+1} d\mu < \infty \text{ if and only if } \int\limits_{\{f^*>1\}} f^*(\log f^*)^r d\mu < \infty.$$

PROOF. See the proof of Theorem 2 in Petersen [10].

THEOREM 3. *Let $\{T_t\}$ be as in Theorem 1. Then there exists an absolute constant $c > 0$ such that*

$$\int Mf d\mu \geqq cH(f) \quad \text{for all} \quad f \in L_1(\mu).$$

PROOF. By a slight modification of the proof of Theorem 1 it follows easily that if $A = \{Mf \geqq \alpha\}$ and $\mu A < 1$ then

$$\int\limits_A f d\mu \leqq \alpha\mu A.$$

Using this, Theorem 3 can be proved as in Theorem 1 of [13]; the details are omitted.

THEOREM 4. *Let $\{T_t\}$ be as in Theorem 1. Then there exists an absolute constant $C > 0$ such that to each $f \in L_1(\mu)$ there corresponds an $f' \in L_1(\mu)$, which is equidistributed with f, such that*

$$\int M(f')d\mu \leqq C[H(f) + \int |f|d\mu].$$

PROOF. By a theorem of Kubo [7] and Krengel [4] (cf. also [5]), $\{T_t\}$ can be represented as a nonsingular flow $\{S_t\}$ under a function. More precisely, $\{S_t\}$ has the following structure. Let $(Y, \mathcal{F}, m)$ be a complete finite measure space, T a nonsingular automorphism of $(Y, \mathcal{F}, m)$, and h a real valued measurable function on Y with $h(y) \geqq d$ for all $y \in Y$, where $d > 0$ is a constant. Put

$$\overline{Y} = \{(y, u) : y \in Y, 0 \leqq u < h(y)\},$$

and let $\overline{m}$ denote the restriction of the completed product measure of μ and the Lebesgue measure on the real line to $\overline{Y}$. $\overline{\mathcal{F}}$ denotes the σ-field of all $\overline{m}$-measurable subsets of $\overline{Y}$. Lastly, put

$$\overline{\mu} = e(y, u)d\overline{m}(y, u),$$

where $e(y, u)$ is a positive measurable function on $\overline{Y}$ such that $\int e d\overline{m} = 1$. Then $\{S_t\}$ is a flow on $(\overline{Y}, \overline{\mathcal{F}}, \overline{\mu})$ defined by

$$S_t(y,u) = \begin{cases} (y, u+t), & \text{if } 0 \leqq u+t < h(y), \\ \left(T^n y, u+t-\sum\limits_{i=0}^{n-1} h(T^i y)\right), & \text{if } \sum\limits_{i=0}^{n-1} h(T^i y) \leqq \\ & \leqq u+t < \sum\limits_{i=0}^{n} h(T^i y), \quad n \geqq 1, \\ \left(T^{-n} y, u+t+\sum\limits_{i=1}^{n} h(T^{-i} y)\right), & \text{if } -\sum\limits_{i=1}^{n} h(T^{-i} y) \leqq \\ & \leqq u+t < -\sum\limits_{i=1}^{n-1} h(T^{-i} y), \quad n \geqq 1. \end{cases}$$

Since we may assume that $(X, \mathcal{B}, \mu) = (\overline{Y}, \overline{\mathcal{F}}, \overline{\mu})$ and $\{T_t\} = \{S_t\}$, Theorem 4 can be proved as in Theorem 2 of [13]; the details are omitted.

THEOREM 5. *Let $\{T_t\}$ be as in Theorem 1. Suppose $f \in L_1(\mu)$ satisfies $\int_{\{|f|>1\}} |f| \log |f| d\mu = \infty$. Then there exists an $f' \in L_1(\mu)$, which is equidistributed with f, such that*

$$\int M(f')d\mu = \infty.$$

PROOF. See the proof of Theorem 3 in [13].

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ON FISSIBLE MODULES

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The additive groups of rings have been studied by several authors, [1], [2], and have played a role in solving several purely ring theoretical problems. In contrast, very little attention has been paid to the additive groups of modules. In this note several elementary facts concerning additive groups of modules will be obtained. The notion of a fissible module will be defined in a manner which parallels the definition of a fissible ring. A complete description will be given of rings all of whose modules are fissible.

All rings R are assumed to be associative, and with unity. An R-module is always meant to be a unital left R-module.

Notation:

R	a ring, and the left R-module ${}_RR$
1	the unity of R
M	an R-module
M^+	the additive group of M
M_t	the torsion part of M^+
M_p	the p-primary component of M^+, p a prime
$Z(n)$	a cyclic group of order n
ω	the first infinite ordinal
h_p	the p-height function, p a prime.

LEMMA 1. *If R^+ is a torsion group with $|1| = n$, then $nM = 0$ for every R-module M.*

PROOF. Let M be an R-module, and let $m \in M$. Then $nm = (n1)m = 0m = 0$.

LEMMA 2. *Let p be a prime. If R^+ is p-divisible then every R-module M satisfies:*

1) *M^+ is p-divisible, and*
2) $M_p = 0$.

PROOF. 1) Let $m \in M$. There exists $a \in R$ such that $1 = pa$. Hence $m = p(am)$ and so $pM = M$.

2) Suppose there exists $m \in M_p$, $m \neq 0$, such that $|m| = p^k$ with $k > 0$. Since $1 = p^k a$ for some $a \in R$, it follows that $m = p^k am = a(p^k m) = 0$, a contradiction.

An immediate consequence of Lemma 2 is:

COROLLARY 3. *If R^+ is divisible then M^+ is a divisible torsion free group for every R-module M. In particular R^+ is torsion free.*

The following observation is obvious.

OBSERVATION 4. *The class of additive groups of R-modules is closed with respect to direct sums, and direct products.*

THEOREM 5. *Let R be a torsion ring with $|1| = n$. A group G is the additive group of an R-module if and only if $nG = 0$.*

PROOF. If G is the additive group of an R-module, then $nG = 0$ by Lemma 1.

Let $n = p_1^{k_1} \dots p_t^{k_t}$ be the prime decomposition of n. To prove the converse it suffices, by Observation 4, to show that $Z(p^k)$ is the additive group of an R-module where $p = p_j$ for any $1 \leqq j \leqq t$, and k is an arbitrary integer satisfying $1 \leqq k \leqq k_j$. Now $1 = a_1 + \dots + a_t$ with $|a_i| = p_i^{k_1}$, $1 < i < t$. By [3, Proposition 27.1], $R^+ = \bigoplus_{i=1}^{t}(a_i) \oplus H$. There exist integers s_i, $1 \leqq i \leqq t$ such that $\sum_{i=1}^{t} s_i p_i^{k_i} = 1$. Let $Z(p^k) = (b)$. The products $a_i b = s_i p^{k_i} b$ for $i = 1, \dots, t$, and $hb = 0$ for all $h \in H$, induce a unital R-module structure on $Z(p^k)$.

LEMMA 6. 1) *Let M be an R-module, $m \in M$, and p a prime. Then $h_p(1) \leqq h_p(m)$.*

2) *$h_p(1) = 0$ or ∞.*

PROOF. 1) For every positive integer n, if $1 = p^n a$ for some $a \in R^+$, then $m = p^n(am)$.

2) Follows immediately from the fact that $1^2 = 1$.

LEMMA 7. *Let M be an injective R-module, and let p be a prime. If $M_p = 0$ then M^+ is p-divisible.*

PROOF. Suppose that $M_p = 0$. Then the map $f\colon pM \to M$ defined by $f(pm) = m$ for all $m \in M$ is well defined. Since M is injective there is an R-homomorphism $g\colon M \to M$ whose restriction to pM is f. Therefore $m = f(pm) = g(pm) = p \cdot g(m)$ for all $m \in M$ and so $pM = M$.

A consequence of Lemma 7 is:

COROLLARY 8. *If M is an injective R-module with M^+ torsion free, then M^+ is divisible.*

LEMMA 9. *A torsion group G is the additive group of a ring with unity if and only if G is bounded.*

PROOF. Suppose that G is bounded. Since a finite direct sum of rings with unity is a ring with unity, and since G is a finite direct sum of groups of the form $G(p,n) = \bigoplus_{\alpha} Z(p^n)$ with p a prime, and n a positive integer, it suffices to construct a ring with unity R such that $R^+ = G(p,n)$. Such a ring R exists by [4, Lemma 122.3].

The converse is an immediate consequence of Lemma 1.

Recall, [2, p. 9], that a ring R is fissible if R is a direct sum $R = R_t \oplus S$. Similarly, one has the following:

DEFINITION. An R-module M is fissible if M is an R-module direct sum $M = M_t \oplus N$.

A simple consequence of Lemma 9 is:

COROLLARY 10. *If R is a fissible ring with unity, then R_t is bounded.*

LEMMA 11. *Every R-module is fissible if and only if R is a ring direct sum $R = R_t \oplus S$ such that every S-module is fissible.*

PROOF. Suppose that every R-module is fissible. Then R is an R-module direct sum $R = R_t \oplus S$. Let $1 = e_t + e_s$ with $e_t \in R_t$ and $e_s \in S$. It is readily seen that e_t and e_s are left unities for R_t and S respectively, that $R_t^2 \subseteqq R_t$ and that $S^2 \subseteqq S$. Let $r \in R_t$. Then $r = re_t + re_s$, and so $re_s = r - re_t$. Since S is a left R-module, it follows that $re_s \in S$, while on the other hand $r - re_t \in R_t$. Therefore $re_s \in R_t \cap S = 0$. This yields that $re_t = r$ and that $re_s = 0$, i.e., e_t is a unity for R_t and $R_t e_s = 0$. A similar argument shows that e_s is a unity for S, and that $Se_t = 0$. Hence $R_t S = R_t e_s S = 0$, and $SR_t = Se_t R_t = 0$. This implies that $R = R_t \oplus S$ is a ring direct sum. Let M be an S-module. For $r \in R_t$, $s \in S$, and $m \in M$, define $(r+s)m = sm$ where the product on the right hand side is determined by the action of S on M. These products induce an R-module structure on M. Hence M is an R-module direct sum $M = M_t \oplus N$. However this decomposition is also an S-module direct sum, and so M is a fissible S-module.

Conversely, suppose that R is a ring direct sum $R = R_t \oplus S$, and that every S-module is fissible. Let $1 = e_t + e_s$ with $e_t \in R_t$ and $e_s \in S$. Clearly e_t and e_s are unities for R_t and S respectively. Let M be an R-module, $M_0 = e_t M$, and $N = e_s M$. Both M_0 and N are R-modules, and $M = M_0 + N$. Let $m \in M_0 \cap N$. Then there exist $m_1, m_2 \in M$ such that $m = e_t m_1 = e_s m_2$. However $m = e_t m_1 = e_t^2 m_1 = e_t e_s m_2 = 0$, i.e., $M_0 \cap N = 0$. Therefore $M = M_0 \oplus N$ is an R-module direct sum. One may view N as an S-module. Therefore N is an S-module direct sum $N = N_t \oplus K$. Since R_t annihilates N, this is an R-module decomposition. Clearly $M_t = M_0 \oplus N_t$, and $M = M_t \oplus K$, i.e., M is a fissible R-module.

THEOREM 12. *Every R-module is fissible if and only if R is a ring direct sum $R = R_t \oplus D$ with D^+ a torsion-free divisible group.*

PROOF. By Lemma 11, it may be assumed that R^+ is torsion-free. If R^+ is divisible then every R-module is trivially fissible by Corollary 3. Conversely, suppose that every R-module is fissible, but that R^+ is not divisible.

Let p be a prime for which $pR \neq R$. Then $M = \prod_{n<\omega} R/p^n R$ is an R-module, but M_t is not a direct summand of M, a contradiction.

A ring R is p-fissible, p a prime, if R is a ring direct sum $R = R_p \oplus S$, [2, p. 28]. The analogous concept for modules is the following:

DEFINITION. An R-module M is p-fissible, p a prime, if M is an R-module direct sum $M = M_p \oplus N$.

Arguments similar to those used in proving Lemma 11 and Theorem 12 yield the following:

LEMMA 13. *Let p be a prime. Every R-module is p-fissible if and only if R is a ring direct sum $R = R_p \oplus S$ such that every S-module is p-fissible.*

THEOREM 14. *Let p be a prime. Every R-module is p-fissible if and only if R is a direct sum $R = R_p \oplus S$ with S^+ a p-divisible group.*

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INTERPOLATION AND SIMULTANEOUS MEAN CONVERGENCE OF DERIVATIVES

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Introduction

We will discuss simultaneous mean convergence to a function $f \in C^r[-1,1]$ and to its derivatives $f^{(0)}, \dots, f^{(r)}$ by polynomial interpolation. We will use an arbitrary array of distinct nodes $\{X_n\}$ where for each n, $X_n = \{x_{1,n}, \dots, x_{n,n}\} \subsetneqq (-1,1)$. We will also use an *admissible point system* $\{T_n\}$ introduced in Balázs and Kilgore [1], defined by

$$T_n = \{t_{0,n}, \dots, t_{a-1,n}, s_{0,n}, \dots, s_{a-1,n}\},$$

where the index a is fixed by the relationship $a = \left[\frac{r+1}{2}\right]$. The points in T_n must satisfy for some $C > 0$, $K \geqq [C]$, and $n = 1, 2, \dots$

$$(1) \qquad -1 \leqq t_{0,n} \leqq \dots \leqq t_{a-1,n} \leqq -1 + \frac{C}{(n+K)^2} < 1 - \frac{C}{(n+K)^2} \leqq \leqq s_{a-1,n} \leqq \dots \leqq s_{0,n} \leqq 1.$$

On X_n we will define $L_n f$ for each $f \in C^r[-1,1]$ to be the Lagrange interpolant to f among the polynomials of degree $n-1$ or less, and we will define $P_n f$ to be the polynomial of degree $n - 1 + 2a$ which interpolates f on the set $X_n \cup T_n$. For any given n, the operator P_n is ordinary Lagrange interpolation if $X_n \cap T_n = \emptyset$ and if all of the inequalities in (1) are strict. If however for a given n these conditions are not met, then $P_n f$ will interpolate $f^{(0)}, \dots, f^{(k-1)}$ on any point which is listed k times in $X_n \cup T_n$. In such a case, $k \leqq a + 1$ follows from the definitions of X_n and T_n.

In our previous contribution [1], the error in $\|f^{(k)} - P_n^{(k)} f\|_\infty$, $k = 0, \dots, r$ was estimated in terms of $\|L_n\|_\infty$ and $\|L_n^*\|_\infty$ where

$$L_n f(x) := \sum_{j=1}^{n} f(x_{j,n}) \ell_{j,n}(x)$$

and

$$(2) \qquad L_n^* f(x) := \sum_{j=1}^{n} f(x_{j,n}) \left(\frac{1-x^2}{1-x_{j,n}^2} \right)^{1/2} \ell_{j,n}(x),$$

the polynomials $\ell_{j,n}(x)$ being the fundamental polynomials satisfying $\ell_{j,n}(x_j) = \delta_{jk}$ (Kronecker delta), for $1 \leqq j,\ k \leqq n$. Here, we will make similar estimates in weighted L^p spaces. Among other things, we will give simultaneous approximation extensions of the Erdős–Turán Theorem [5] and of the result of Erdős–Feldheim [4] and answer affirmatively a conjecture of Szabados and Varma [9].

Notation: For $1 \leqq p \leqq \infty$, we will say that $f \in L^p$ if

$$\int_{-1}^{1} |f|^p dx < \infty$$

and $f \in (L \log^+ L)^p$ if $f \log^+ |f| \in L^p$. When we wish to speak of the "weighted norm" with weight $w(x)$, we will assume that $w(x) \in L^1$, $w(x) \geqq 0$, with $w(x) > 0$ on a subset of positive measure in $[-1, 1]$ and that

$$\|f\|_{p,w} := \left(\int_{-1}^{1} |f(x)|^p w(x) dx\right)^{1/p} \qquad (p < \infty). \tag{3}$$

However, a weighted norm is sometimes represented as

$$\left(\int_{-1}^{1} |f(x)u(x)|^p dx\right)^{1/p} = \|fu\|_p \qquad (p < \infty). \tag{4}$$

We will also agree that $\|fu\|_\infty$ is a "weighted norm", in the sense of (4). We have an actual norm only when the weight function is positive a.e. Otherwise, we have only defined a seminorm. And we do not have a norm if $p < 1$. Nevertheless, we continue to use the notation $\|\cdot\|_p$ in these cases. We define

$$\|uL_n\|_p = \sup_{\|f\|_\infty \leqq 1} \|uL_n f\|_p \quad \text{and} \quad \|uL_n^*\|_p = \sup_{\|f\|_\infty \leqq 1} \|uL_n^* f\|_p.$$

We will say that the weight function v is of *Markov–Bernstein class* for a given p, $0 < p \leqq \infty$, if for any non-negative integer s the weight $u(x) = v(x)(1 - x^2)^{s/2}$ satisfies for p_n any polynomial of degree n or less the inequalities

$$\left\|u(x)(1 - x^2)^{r/2} p_n^{(r)}(x)\right\|_p \leqq M_1 n^r \|u(x)p_n(x)\|_p, \tag{5}$$

where r is any positive integer, and

$$\|u(x)p_n(x)\|_p \leqq M_2 \|u(x)p_n(x)\|_{p,I_n}, \tag{6}$$

in which the norm $\|\cdot\|_{p,I_n}$ is taken on $[-1+c/n^2, 1-c/n^2]$ for some $c>0$, and M_1 and M_2 are constants independent of n and p_n. An interesting problem would be to characterize this "Markov–Bernstein" class of weight functions with non-trivial necessary and sufficient conditions. We do not enter into this question here but rather point out two large and slightly different classes of weights which share the requisite properties.

The class of *generalized Jacobi* weights (GJ) is defined for example in Nevai [7]. The weight $w(x)$ is GJ if

$$w(x)=\psi(x)(1-x)^{\Gamma_0}\prod_{k=1}^{m}|t_k-x|^{\Gamma_k}(1+x)^{\Gamma_{m+1}} \tag{7}$$

for $-1 \leqq x \leqq 1$, where $-1<t_m<t_{m-1}<\ldots<t_1<1$, $\Gamma_k>-1$ for $k=0,\ldots,m+1$, and both $\psi(x)$ and $(\psi(x))^{-1}$ are in $L^\infty[-1,1]$. The definition may be modified if we want the representation (4). If ψ is continuous with its modulus of continuity satisfying

$$\int_0^1 \frac{\omega(\psi;t)}{t}dt<\infty,$$

then we say that w is *generalized smooth Jacobi* (GSJ). The weight class GJ is contained in the Markov–Bernstein class for $0<p<\infty$ (Lubinsky and Nevai [6]).

The class J_p^*, $1 \leqq p \leqq \infty$, of Ditzian and Totik [3, Definition 8.1.1], is similar to the GJ class, except that $\psi(x)$ need not be bounded at ± 1 but can have, for example, logarithmic growth near these points. The weight class J_p^* is contained in the Markov–Bernstein class for $1 \leqq p \leqq \infty$ (Ditzian and Totik [3], (8.1.3) and (8.1.4)).

A Jacobi weight w may be described by

$$w(x)=(1-x)^\alpha(1+x)^\beta, \quad \text{where} \quad \alpha,\beta>-1,$$

and is in both of the classes GSJ and J_p^*.

Finally, we denote by $E_n(g)$ the error in the best uniform approximation to a function g by polynomials of degree n or less, and we denote by $Q_n(x)$ the monic polynomial of degree $2a$ which is zero on the points of T_n:

$$Q_n(x)=(x-t_{0,n})\ldots(x-t_{a-1,n})(x-s_{0,n})\ldots(x-s_{a-1,n}). \tag{8}$$

Results

Our main result has already been established for the case $p=\infty$ and $u(x)=1$ in Balázs and Kilgore [1]:

THEOREM. *Let $f \in C^r[-1,1]$ and for $n = 1,2,\dots$ let X_n be a set of nodes in $(-1,1)$ and T_n an admissible point system. Let L_n be the Lagrange interpolation on X_n and L_n^* the operator defined in (2). Let P_n be the interpolation operator defined on $X_n \cup T_n$. Then for $0 < p \leqq \infty$, and for a weight function u of Markov–Bernstein class and for $k = 0,\dots,r$*

(a) *for r even*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(1)n^{k-r} E_{n-1}(f^{(r)})\|uL_n\|_p,$$

(b) *for r odd*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(1)n^{k-r+1} E_{n-1}(f^{(r)}) \left\|u\sqrt{1-x^2}L_n\right\|_p,$$

(c) *for r odd*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(1)n^{k-r} E_{n-1}(f^{(r)}) \|uL_n^*\|_p.$$

REMARK. The constants denoted by $O(1)$ may be seen to depend on the ratio C/K in (1), but not *per se* on the choice of T_n.

Our theorem links the simultaneous approximation properties of P_n to the more fundamental approximation properties of L_n. In particular, it states that $P_n^{(k)} f$ converges to $f^{(k)}$ and gives a rate of convergence for each $k = 0,\dots,r$ *provided only that the norms of L_n or respectively of L_n^* remain uniformly bounded.* We make this statement explicit in the following:

COROLLARY 1. *If in Theorem 1 we have $\|uL_n\|_p \leqq M$ for all n, then*

(a) *for r even*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(n^{k-r})E_{n-1}(f^{(r)}),$$

(b) *for r odd*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(n^{k-r+1})E_{n-1}(f^{(r)}).$$

If on the other hand $\|uL_n^*\|_p \leqq M$ for all n, then

(c) *for r odd*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(n^{k-r})E_{n-1}(f^{(r)}).$$

The hypotheses of this corollary are, of course, not satisfied if $p = \infty$. But for $p < \infty$ nodes can exist which do satisfy the hypotheses. Our next result also uses Nevai [7, Theorem 1] and provides methods for satisfying the hypotheses of Corollary 1. Nevai and Vértesi [8] have recently stated a similar result to Corollary 2 for weights $u \in \mathrm{GSJ}$, with more restrictive conditions on the added node set T_n.

COROLLARY 2. *Let $0 < p < \infty$. Let $w \in$ GSJ. Let u be a weight function of Markov–Bernstein class such that $u \in (L \log^+ L)^p$ and such that $u \Big/ \sqrt{w\sqrt{1-x^2}} \in L^p$. For $n = 1,2,\dots$ let X_n be the zero set of the n-th degree orthogonal polynomial associated with w, and let L_n be the Lagrange interpolation based on X_n. Then there exists M such that for $n = 1,2,\dots, \|uL_n\|_p \leqq M$ and $\|uL_n^*\|_p \leqq M$. Therefore for any integer $r \geqq 0$ and for $f \in C^r[-1,1]$, and for T_n any admissible point system, the sequence of interpolation operators P_n defined on $X_n \cup T_n$ for $n = 1,2,\dots$ yields for $k = 0,\dots,r$ the rates of simultaneous convergence given in*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p = O(1) n^{k-r} E_{n-1}(f^{(r)}).$$

REMARKS. 1. The hypotheses of Corollary 2 regarding u are met if $u \in$ GJ for some $p, 0 < p < \infty$ (representation 4). They are also met if $u \in J_p^*$ for $1 \leqq p < \infty$. Of course, one must provide a suitable weight $w \in$ GSJ. This will be done in Corollary 4.

2. The Erdős–Turán Theorem [5] asserts the following:

For each $f \in C[a,b]$, let $L_n f$ interpolate f on the zeroes of the n-th degree orthogonal polynomial defined for $w \in L^2$. Then $\|w(f - L_n f)\|_2 \to 0$.

3. It is of interest to see when Corollary 2 allows the choice $u = w^{1/p}$. Since $w^{1/p} \Big/ \sqrt{w\sqrt{1-x^2}} \in L^p$ we have $\frac{\Gamma_0}{p} - \frac{\Gamma_0}{2} - \frac{1}{4} > -\frac{1}{p}$. If this inequality should hold for all p, $0 < p < \infty$, it necessarily follows that $\Gamma_0 \leqq -\frac{1}{2}$. Similar bounds exist for Γ_{m+1}. In particular an augmented version of the Erdős–Turán theorem follows with any weight function $w(x)$ in GSJ, with weighted mean convergence of the derivatives up to the r-th as well.

More generally, one may state the following:

COROLLARY 3. *Let u^2 be a weight function of Markov–Bernstein class. Let X_n, $n = 1,2,\dots$ be the set of zeroes of the n-th degree orthogonal polynomial associated with u^2. Let $f \in C^r[-1,1]$, and let $a = \left[\frac{r+1}{2}\right]$. Let T_n be an admissible point system. Then for $k = 0,\dots,r$, with $q = r$ if r is even and $q = r-1$ if r is odd*

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_2 \leqq \text{const} \cdot n^{k-q} E_{n-1}(f^{(r)}).$$

The result of Erdős–Feldheim [3] is that for all functions $f \in C[-1,1]$ and for $1 \leqq p < \infty$, $\|w(f - L_n f)\|_p \to 0$, where w is the Chebyshev weight $(1-t^2)^{-1/2}$, and L_n interpolates on the zero set of the n-th degree Chebyshev polynomial. Szabados and Varma [9] have shown that a simultaneous interpolation version of this result must hold for $p = 2$, when the nodes T_n are added at the locations $\pm \cos \frac{k}{n}\pi$, $k = 0,\dots,r$, leading to their conjecture that the same is true for all other values of $p < \infty$. We affirm this conjecture here, showing that a weight function u and added nodes T_n can be chosen in the general setting. On the other hand, the choice of the Chebyshev nodes is nearly unique.

COROLLARY 4. *Let $0 < p < \infty$, and let u be a weight of Markov–Bernstein class such that $u \in (L \log^+ L)^p$. Let X_n be the set of zeroes of the n-th degree Chebyshev polynomial, and let L_n be the interpolation defined on X_n for $n = 1, 2, \dots$. Then*

(a) *$\|uL_n\|_p$ and $\|uL_n^*\|_p$ are uniformly bounded, whence $\|u(f - L_n f)\|_p \to \to 0$, all $f \in C[-1,1]$.*

(b) *Let $f \in C^r[-1,1]$. If T_n is any admissible point system and if P_n, $n = 1, 2, \dots$ is the sequence of interpolation operators based on $X_n \cup T_n$, then for $k = 0, \dots, r$*

$$\left\| u(f^{(k)} - P_n^{(k)} f) \right\|_p = O(n^{k-r}) E_{n-1}(f^{(r)}).$$

(c) *More generally, part* (a) *holds for all p and for all $u \in (L \log^+ L)^p$, with L_n interpolating on nodes generated by $w \in$ GSJ, if and only if $\Gamma_0 = = -1/2$, $\Gamma_{m+1} = -1/2$, and $0 = \Gamma_1 = \dots = \Gamma_m$* (cf. (7)).

In view of Corollary 4, it is interesting to speculate exactly which weights $u \in (L \log^+ L)^p$ are of Markov–Bernstein class. However, we do get convergence for all weights $u \in L^{p+\varepsilon}$, for any $\varepsilon > 0$ and near-convergence for all weights $u \in L^p$, with no special assumptions about u.

COROLLARY 5. *Let $0 < p < \infty$; let X_n be the set of zeroes of the n-th degree Chebyshev polynomial, and let L_n be defined on X_n. Then*

a) *If $u \in L^{p+\varepsilon}$ for some $\varepsilon > 0$ we have for $f \in C^r[-1,1]$, T_n any admissible point system, and P_n based on $X_n \cup T_n$*

$$\left\| u(f^{(k)} - p_n^{(k)} f) \right\|_p = O(n^{k-r}) E_{n-1}(f^{(r)}).$$

b) *If merely $u \in L^p$, then*

$$\left\| u(f^{(k)} - p_n^{(k)}) \right\|_p = O(n^{k-r}) E_{n-1}(f^{(r)}).$$

Existing results

For Theorem 1, there are two basic components for the proof. The first component is the "Timan theorem with interpolation":

THEOREM A (Balázs–Kilgore–Vértesi [2]). *Let $f \in C^r[-1,1]$. Let T_n be an admissible point system in $[-1,1]$. Then there exists a sequence $\{p_n\}$ of* *polynomials of degree* $m = n - 1 + 2a$ *or less such that* p_n *interpolates* f *on* T_n *and for* $k = 0, \dots, r$ *and for* $|x| \leqq 1$

$$\left| f^{(k)}(x) - p_n^{(k)}(x) \right| = O(1) \cdot \left(\frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n} \right)^{q-k} E_{m-r}(f^{(r)}).$$

COROLLARY OF THEOREM A (Balázs–Kilgore [1]). *Let* Q_n *be the sequence of monic polynomials of degree* $2a$ *such that* Q_n *is zero on* T_n. *Then the following statements hold for* $x \in [-1, 1]$ *and in particular by continuity at the zeroes of* Q_n:

(a) *for* r *even*

$$\left| \frac{f(x) - p_n(x)}{Q_n(x)} \right| = O(n^{a-r}) E_{n-1}(f^{(r)}), \tag{9}$$

(b) *for* r *odd*

$$\left| \frac{f(x) - p_n(x)}{Q_n(x)} \right| = O(n^{a-r+1}) E_n(f^{(r)}) \tag{10}$$

and

$$\left| \frac{(f(x) - p_n(x))\sqrt{1-x^2}}{Q_n(x)} \right| = O(n^{a-r}) E_n(f^{(r)}). \tag{11}$$

The second component of our proof is an inequality of Dzyadyk type, which follows easily from the inequalities (5) and (6):

LEMMA. *For* $a = \left[\frac{r+1}{2}\right]$ *we have for* p_n *a polynomial of degree* n, $k = 0, \dots, r$, *for* w *of Markov–Bernstein class, and for* $Q_n(x)$ *as in* (5)

$$\left\| w(x)[Q_n(x)p_n(x)]^{(k)} \right\|_p = O(1) n^k \left\| w(x) p_n(x) \right\|_p \quad \textit{for } r \textit{ even} \tag{12}$$

and

$$\left\| w(x)[Q_n(x)p_n(x)]^{(k)} \right\|_p = O(1) n^k \left\| w(x)\sqrt{1-x^2} p_n(x) \right\|_p \quad \textit{for } r \textit{ odd.} \tag{13}$$

Corollary 2 is central to the further results stated, and it depends upon our theorem and upon Nevai [7, Theorem 1]:

THEOREM B. *Let* $w \in \mathrm{GSJ}$ *and* $0 < p < \infty$. *Let* v *be a not necessarily integrable Jacobi weight function and let* u *be a nonnegative function defined in* $[-1, 1]$ *such that* $u \in L^p$, $uv \in (L \log^+ L)^p$, $\frac{u}{\sqrt{w\sqrt{1-x^2}}} \in L^p$ *and* $v\sqrt{w\sqrt{1-x^2}} \in L^1$. *Then for every bounded function* f

$$\sup_{n \geqq 1} \| L_n(w, vf) u \|_p \leqq \mathrm{const} \cdot \| f \|_\infty. \tag{14}$$

In formula (14), $L_n(w, vf)u$ signifies the expression

$$u(x)\sum_{j=1}^{n} v(x_j)f(x_j)\ell_j(x),$$

which interpolates vf on the set X_n consisting of the zeroes of the n-th degree orthogonal polynomial associated with the weight w.

Proofs

All of our results may be seen easily to follow from the Theorem, whose proof we will defer until the end. Corollary 1 is obvious. We turn to Corollaries 2, 3, 4 and 5:

PROOF OF COROLLARY 2. Invoking Corollary 1, we use in Theorem B $v(x) = 1$ when r is even. When r is odd, we apply Theorem B with $u(x)\sqrt{1-x^2}$ in place of $u(x)$ and using $v(x) = (1-x^2)^{-1/2}$. We then have

$$u(x)L_n^* f(x) = u(x)\sqrt{1-x^2}L_n(vf)(x).$$

It follows automatically from $w \in L^1$ that

$$v\sqrt{w\sqrt{1-x^2}} \in L^1 \quad \text{for} \quad v = 1 \quad \text{or} \quad v = (1-x^2)^{-1/2}.$$

PROOF OF COROLLARY 3. From the Erdős–Turán theorem, $\|uL_n\|_2$ is bounded. The result follows then from Corollary 1.

PROOF OF COROLLARY 4. In Corollary 2, we use $w(x) = (1-x^2)^{-1/2}$. Thus

$$\frac{u}{\sqrt{w\sqrt{1-x^2}}} \in L^p \quad \text{if and only if} \quad u \in L^p,$$

and for $v(x) \equiv 1$ (r even) or for $v(x) = (1-x^2)^{-1/2}$ (r odd) we also have

$$v(x)\sqrt{w(x)\sqrt{1-x^2}} = v(x) \in L^1.$$

PROOF OF COROLLARY 5. Applying Hölder's inequality we see that

$$\|u(f - P_n f)\|_p \leqq \|(f - P_n f)\|_{pr}\|u\|_{ps}$$

where $1/r + 1/n = 1$. If $s = 1$, we obtain part (b) of the corollary, applying the theorem of Balázs and Kilgore [1] (uniform norm version of the Theorem). If $u \in L^{p+\varepsilon}$ for some $\varepsilon > 0$, then let $p < ps < p + \varepsilon$, and r is finite. Part (a) follows from Corollary 4, with p replaced by pr and u replaced by 1.

Finally, we give the proof of the Theorem. We will henceforth abbreviate the cumbersome double subscript representation for the nodes.

PROOF OF THEOREM. We begin by invoking Theorem A to write

$$\left\|u(f^{(k)} - P_n^{(k)} f)\right\|_p \leqq K_p \left(\left\|u(f^{(k)} - p_n^{(k)})\right\|_p + \left\|uP_n^{(k)}(f - p_n)\right\|_p \right), \tag{15}$$

in which $\{p_n\}$ is the sequence of polynomials guaranteed by Theorem A, and

$K_p = 1$ if $p \geqq 1$, $K_p = 2^{1/p}$ if $0 < p < 1$.

From Theorem A we may obtain the uniform estimate

$$\left\| f^{(k)} - p_n^{(k)} \right\|_\infty = O(1) n^{k-r} E_{m-r}(f^{(r)}) \qquad (m = n - 1 + 2a)$$

which, combined with the fact that

$$\left\| u(f^{(k)} - p_n^{(k)}) \right\|_p \leqq \|u\|_p \left\| f^{(k)} - p_n^{(k)} \right\|_\infty ,$$

establishes that the first term on the right of (15) satisfies the conclusions of the Theorem. Therefore, we consider only the second term.

From Theorem A, the polynomials p_n precisely interpolate f on the points in T_n for each n (including the interpolation of derivatives on any points listed with multiplicity), and thus we have

$$(16) \quad \left\| u P_n^{(k)}(f - p_n) \right\|_p = \left\| u(x) \sum_{j=1}^{n} \left(\frac{f(x_j) - p_n(x_j)}{Q_n(x_j)} \right) [Q_n(x) \ell_j(x)]^{(k)} \right\|_p ,$$

in which $\frac{f(x_j) - p_n(x_j)}{Q_n(x_j)}$ may be defined as a limit, if needed.

We now proceed in various ways from (16) to prove parts (a), (b), and (c) of the Theorem:

(a) From (16) we may note using (12) and then (9) that

$$\left\| u(x) \sum_{j=1}^{n} \left(\frac{f(x_j) - p_n(x_j)}{Q_n(x_j)} \right) [Q_n(x) \ell_j(x)]^{(k)} \right\|_p \leqq$$

$$\leqq M n^k \max_{x_j \in X_n} \left\| \frac{f(x_j) - p_n(x_j)}{Q_n(x_j)} \right\| \cdot \sup_{|\sigma_j| \leqq 1} \left\| u(x) \sum_{j=1}^{n} \sigma_j \ell_j(x) \right\|_p =$$

$$= O(1) n^{k-r} E_{n-1}(f^{(r)}) \|u L_n\|_p .$$

(b) From (16) we may carry out analogous steps to those performed in part (a), only using (13) in place of (12) to reach

$$\left\| u(x) \sum_{j=1}^{n} \left(\frac{f(x_j) - p_n(x_j)}{Q_n(x_j)} \right) [Q_n(x) \ell_j(x)]^{(k)} \right\|_p \leqq$$

$$\leqq M n^{k+1-r} E_n(f^{(r)}) \left\| u \sqrt{1 - x^2} L_n \right\|_p .$$

(c) We merely note that

$$\left\| u(x) \sum_{j=1}^{n} \left(\frac{f(x_j) - p_n(x_j)}{Q_n(x_j)} \right) [Q_n(x) \ell_j(x)]^{(k)} \right\|_p =$$

$$= \left\| u(x) \sum_{j=1}^{n} \left(\frac{(f(x_j) - p_n(x_j))\sqrt{1-x_j^2}}{Q_n(x_j)} \right) \left[\frac{Q_n(x)}{\sqrt{1-x_j^2}} \ell_j(x) \right]^{(k)} \right\|_p \leqq$$

$$\leqq Mn^k \left\| u(x) \sum_{j=1}^{n} \left(\frac{(f(x_j) - p_n(x_j))\sqrt{1-x_j^2}}{Q_n(x_j)} \right) \frac{\sqrt{1-x^2}}{\sqrt{1-x_j^2}} \ell_j(x) \right\|_p ,$$

using along the way (13). We now may use (11) and an argument similar to that in (a) to reach the conclusion that

$$\left\| u(f^{(k)} - P_n^{(k)} f) \right\|_p = O(1) n^{k-r} E_n(f^{(r)}) \left\| u L_n^* \right\|_p .$$

This concludes the proof of the Theorem.

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SOME CONDITIONS FOR THE COMMUTATIVITY OF RINGS

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1.

Throughout, R will represent an associative ring (may be without unity 1) with $Z(R)$, $N(R)$ and $C(R)$ denoting its centre, the set of nilpotent elements and the commutator ideal of R respectively. For any $a, b \in R$ as usual $[a, b] = ab - ba$. In his paper [2] Bell proved that a ring R generated by n^{th} power of its elements and satisfying the polynomial identity $[x^n, y] = [x, y^n]$ for all x, y in R and a fixed positive integer $n > 1$, is commutative. Motivated by the above observation Harmanci [4] proved that a ring R with unity 1 satisfying $[x^k, y] = [x, y^k]$ $(k = n, n+1)$, is commutative. Recently, Gupta [3] generalized the above result as follows:

THEOREM G. *Let R be a semiprime ring with unity* 1 *satisfying*

$$[x^n, y] - [x, y^n] \in Z(R), \tag{1}$$

$$[x^{n+1}, y] - [x, y^{n+1}] \in Z(R) \tag{2}$$

for all $x, y \in R$ and a fixed integer $n > 1$. Then R is commutative.

Gupta also remarked that in view of an example [9, Example 2], Theorem G can not be extended to primary rings. However, the question remains open whether the existence of unity 1 in the ring of the above theorem is essential and both the identities are necessary for the result. In this direction we generalize the mentioned result to a great extent as follows:

THEOREM 1. *Let R be a semiprime ring. Then the following statements are equivalent:*

(i) *R is commutative.*

(ii) *There exists a positive integer $n > 1$ such that $[x, [x^n, y] - [x, y^n]] = 0$ for all x, y in R.*

(iii) *There exists a positive integer $n > 1$ such that $[y, [x^n, y] - [x, y^n]] = 0$ for all x, y in R.*

Before beginning the proof, we state the following result due to Bell [2] which will be used in the subsequent text of our paper.

LEMMA 1. *Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of noncommuting indeterminates, its coefficients being integers with highest common factor* 1. *If there exists no prime*

p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then R has a nil commutator ideal and the nilpotent elements of R form an ideal.

LEMMA 2. *Let R be a prime ring satisfying any one of the conditions* (ii) *or* (iii) *of our theorem, then R has no nonzero nilpotent elements.*

PROOF. Let a be an element of R with the property that $a^2 = 0 \neq a$.

Assume that R satisfies (ii). Replace ax for x and axa for y and use the fact that $a^2 = 0$, to get $ax(ax)^n axa = 0$ i.e. $(ax)^{n+3} = 0$ for all x in R. Again if R satisfies (iii), then by putting axa for x and xa for y in the identity and using $a^2 = 0$, we get $(axa)(xa)^n(xa) = 0$ i.e. $(ax)^{n+3} = 0$ for all x in R. Hence, in both cases $(ax)^{n+3} = 0$ for all x in R. If $aR \neq 0$, then the above shows that aR is a nonzero nil right ideal satisfying $z^{n+3} = 0$ for all z in aR. But a well-known result of Levitzki [6, Lemma 1.1] rules this out and hence $aR = 0$. Thus, the primeness of R forces $a = 0$.

PROOF OF THEOREM 1. Since R is semiprime, it is isomorphic to a subdirect sum of prime rings R_α, each of which as a homomorphic image of R satisfies the hypothesis placed on R. Hence, we may assume that the ring R is prime satisfying the conditions of the theorem.

Every commutative ring R satisfies (ii) as well as (iii) and so (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

Next we show that (ii) $\Rightarrow$ (i):

Let R satisfy the condition $f(x,y) = [x,[x^n,y] - [x,y^n]] = 0$ which is a polynomial identity with coprime integral coefficients. Consideration of $x = E_{11} + E_{12}$ and $y = E_{21}$ assures that no ring of 2×2 matrices over $GF(p)$, p a prime, satisfies the identity $f(x,y) = 0$ and hence by Lemma 1, the commutator ideal of R is nil. But in view of Lemma 2, R has no nonzero nilpotent elements and, therefore, R is commutative.

Arguing on the same lines, we can prove that (iii) implies (i).

2. Another commutativity condition

THEOREM 2. *Let R be a ring with unity 1 in which for every x, y in R, $[x, x^n y - xy^m] = 0$, where $m > 1$ and $n \geqq 1$ are fixed positive integers. Then R is commutative.*

In preparation for the proof of the above theorem we begin with the following lemmas, whose proofs can be looked in [5], [8, p. 221] and [10] respectively.

LEMMA 3. *If for any x, y in R we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $x^2 p_{x,y}(x) - x$ commutes with y, then R is commutative.*

LEMMA 4. *If $[x,[x,y]] = 0$, then $[x^k, y] = kx^{k-1}[x,y]$ for all positive integers k.*

LEMMA 5. *Let R be a ring with unity 1 and $f\colon R \to R$ be a function such that $f(x) = f(x+1)$ holds for all x in R. If for some positive integer k, $x^k f(x) = 0$ for all x in R, then necessarily $f(x) = 0$.*

PROOF OF THEOREM 2. Our identity of the theorem gives

$$(*) \qquad x^n[x,y] = x[x,y^m] \quad \text{for all } x,y \text{ in } R.$$

If $n = 1$, then we have $x[x,y] = x[x,y^m]$. On replacing x by $x+1$ and simplifying with the help of $x[x,y] = x[x,y^m]$ we get $[x, y - y^m] = 0$ for all x in R. Thus R is commutative by Lemma 3.

Let $n > 1$. Now we show that $C(R) \subseteqq Z(R)$. In view of Lemma 1, $C(R) \subseteqq N(R)$, since $x = E_{11} + E_{12}$ and $y = E_{12}$ fail to satisfy $(*)$. By making repeated use of $(*)$, we see that for any positive integer t

$$x^{tn}[x,y] = x^{(t-1)n}x[x,y^m] = xx^{(t-2)n}x\left[x,y^{m^2}\right] = \ldots = x^t\left[x,y^{m^t}\right].$$

Now if $a \in N(R)$, then for sufficiently large t, we get

$$(1) \qquad x^{tn}[x,a] = 0 \quad \text{for all } x \text{ in } R.$$

In view of Lemma 5, this yields $[x,a] = 0$ for all x in R. Hence $a \in Z(R)$, which gives that

$$(2) \qquad C(R) \subseteqq Z(R).$$

Replace x by $2x$ in $(*)$ to get $2^{n+1}x^n[x,y] = 2^2x[x,y^m]$. Combining this with $(*)$, we obtain $(2^{n+1} - 2^2)x^n[x,y] = 0$. If $q = 2^{n+1} - 2^2$, then $qx^n[x,y] = 0$. By using Lemma 5 we have $q[x,y] = 0$. Since commutators are central, hence by Lemma 4, $[x^q, y] = qx^{q-1}[x,y] = 0$, i.e.

$$(3) \qquad x^q \in Z(R) \quad \text{for all } x \text{ in } R.$$

Further by replacing y with y^m in $(*)$, we get $x^n[x,y^m] = x\,[x,(y^m)^m]$. In view of (2) and Lemma 4 this yields $x^n[x,y^m] = my^{m(m-1)}x[x,y^m]$. Combining the last identity with $(*)$, we get $x^n[x,y^m] = my^{m(m-1)}x^n[x,y]$. Again in view of (2) and Lemma 4, this implies that $x^n[x,y^m] = y^{(m-1)^2}[x,y^m]x^n$, i.e. $(1 - y^{(m-1)^2})[x,y^m]x^n = 0$. With the help of (2) and $(*)$ we have $(1 - y^{(m-1)^2})[x,y]x^{2n-1} = 0$ and by Lemma 5, we conclude that $(1 - y^{(m-1)^2})[x,y] = 0$. But in view of (3), this yields $[x, y - y^{q(m-1)^2+1}] = (1 - y^{q(m-1)^2})[x,y] = 0$ and, therefore, R is commutative by Lemma 3.

The following example shows that the existence of unity 1 in the hypothesis of the above result is not superfluous.

EXAMPLE 1. Let D_k be the ring of $k \times k$ matrices over a division ring D and $A_k = \{(a_{ij}) \in D_k \mid a_{ij} = 0 \ (i \geqq j)\}$. If $k > 2$ then A_k is a noncommutative nilpotent ring of index k. For any positive integer m and n, A_3 satisfies $[x, x^n y - xy^m] = 0$.

However, the above theorem can be extended to a wider class of rings called s-unital. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for all x in R and R is called s-unital if $x \in Rx \cap xR$ for each x in R.

Let R be a left s-unital ring satisfying the identity $[x, x^n y - xy^m] = 0$. Using the same arguments as used to get (1) in case of Theorem 2, this yields that $x^{tn}[x, a] = 0$ for $a \in N(R)$ and $x \in R$. Since R is left s-unital, choose $e \in R$ with $ea = a$, then by above we can easily see that $a - ae = e^{tn}[e, a] = 0$.

Now let x be an arbitrary element of R and choose $e' \in R$ with $e'x = x$, then there exists $e'' \in R$ such that $e''x = x$ and $e''e' = e'$. Now $(x - xe'')^2 = 0$ i.e. $x - xe''$ is nilpotent and $e'(x - xe'') = x - xe''$. Thus the fact just claimed above implies that $x - xe'' = (x - xe'')e' = 0$, which shows that $xe'' = x$. Hence, R is s-unital. Now in view of [7, Proposition 1] we may assume that R has unity 1 and the commutativity of R follows from Theorem 2, which establish the following result:

THEOREM 3. *Let R be a left s-unital ring in which for every x, y in R, $[x, x^n y - xy^m] = 0$ where $m > 1$, $n \geqq 1$ are fixed positive integers. Then R is commutative.*

One might conjecture that the above theorem should hold for a right s-unital ring as well. But the following example suggests that a right s-unital ring satisfying $[x, x^n y - xy^m] = 0$ need not be commutative.

EXAMPLE 2. Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ which is a right s-unital subring of the ring of all 2×2 matrices over $GF(2)$. It can be easily seen that $[x, x^2 y - xy^2] = 0$ for all x, y in S. However, S is not commutative.

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ESTIMATES OF THE SHEPARD INTERPOLATORY PROCEDURE

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1. Introduction

In this paper we consider a linear positive operator V_m which generates a rational F-stable interpolatory procedure [8, 13]; some properties of V_m are shown and a convergence theorem of Korovkin type is given.

An interesting particular case of V_m is the Shepard operator S_m introduced in [14] and studied by numerous authors; we recall for example [3, 4, 8, 10]. The principal aim of the present paper is to estimate the convergence order of the operator S_m with respect to different matrices of knots.

2. The operator $V_m(A;\varphi)$

Let $A := \{x_{m,i}, i = 1, 2, \dots, m, m \in N\}$ be an infinite matrix of different knots belonging to $I := [-1, 1]$. We denote by $l_{m,k}$ the k-th fundamental Lagrange polynomial of degree $m-1$ corresponding to the matrix A and defined by

$$l_{m,k}(x) = \frac{p_m(x)}{p'_m(x_{m,k})(x - x_{m,k})}, \qquad k = 1, 2, \dots, m,$$

where

$$p_m(x) = \prod_{i=1}^{m}(x - x_{m,i}).$$

Let

$$\Phi_m(x) = \left[\sum_{k=1}^{m} \frac{l^2_{m,k}(x)}{\varphi^2_m(x_{m,k})}\right]^{-1},$$

where $\{\varphi_m\}$ is a sequence of functions such that $\varphi_m(x_{m,i}) \neq 0$, $i = 1, 2, \dots, m$. Thus, for every function f defined on I we introduce the operator V_m by

$$\text{(2.1)} \qquad V_m(A;\varphi;f;x) = (V_m f)(x) = V_m(f;x) :=$$
$$:= \Phi_m(x) \sum_{k=1}^{m} \frac{l^2_{m,k}(x)}{\varphi^2_m(x_{m,k})} f(x_{m,k}), \qquad x \in I.$$

Obviously, V_m is a linear positive operator corresponding to the matrix A and to the sequence $\{\varphi_m\}$; by (2.1) it follows also that if $f = e_0$ with $e_k(x) = x^k$,

$k \in N$, then $V_m f = f$. Further, $V_m(f;x)$ is a rational function of degree $(2m-2, 2m-2)$ and being $\Phi_m(x_{m,k}) = \varphi_m^2(x_{m,k})$, it can be written in the form

$$V_m(f;x) = \Phi_m(x) \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\Phi_m(x_{m,k})} f(x_{m,k}). \tag{2.2}$$

In particular, by (2.2) we have $V_m(f;x_{m,i}) = f(x_{m,i})$, $i = 1, 2, \dots, m$. Moreover, since

$$\min_{x \in I} f(x) \leqq V_m(f;x) \leqq \max_{x \in I} f(x),$$

we deduce that V_m is an F-stable interpolatory procedure, i.e. stable in the Fejér sense (see, for example [12, 13]).

Denoting by $H_m f$ the Hermite–Fejér polynomial defined by $(H_m f)(x_{m,k}) = f(x_{m,k})$, $(H_m f)'(x_{m,k}) = 0$, we prove the following

PROPOSITION 2.1. *For every matrix A and for every sequence $\{\varphi_m\}$, we have*

$$(V_m f)'(x_{m,i}) = 0, \qquad i = 1, 2, \dots, m, \tag{2.3}$$

$$V_m(f;x) = H_m(f;x) + R_m(f;x), \qquad x \in I, \tag{2.4}$$

where $H_m f$ is the Hermite–Fejér polynomial and

$$R_m(f;x) = \frac{1}{2} \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\Phi_m(x_{m,k})} (x - x_{m,k})^2 f(x_{m,k}) \times$$
$$\times \int_0^1 \Phi_m''(x_{m,k} + (x - x_{m,k})t)t \, dt.$$

PROOF. Since

$$\frac{d}{dx} \Phi_m^{-1}(x) = 2 \sum_{k=1}^{m} l_{m,k}(x) \frac{l_{m,k}'(x)}{\Phi_m(x_{m,k})},$$

we have

$$-\frac{\Phi_m'(x)}{\Phi_m^2(x)} = 2 \sum_{k=1}^{m} l_{m,k}(x) \frac{l_{m,k}'(x)}{\Phi_m(x_{m,k})};$$

thus,

$$\frac{\Phi_m'(x_{m,k})}{\Phi_m(x_{m,k})} = -2 l_{m,k}'(x_{m,k}). \tag{2.5}$$

Differentiating (2.2), in view of (2.5) we deduce (2.3).

On the other hand, relation (2.4) follows by (2.2), (2.5) and recalling that

$$\Phi_m(x) = \Phi_m(x_{m,k}) \left[1 + \frac{\Phi'_m(x_{m,k})}{\Phi_m(x_{m,k})}(x - x_{m,k})\right] +$$

$$+\frac{(x - x_{m,k})^2}{2} \int_0^1 \Phi''_m(x_{m,k} + (x - x_{m,k})t)t\,dt. \qquad \square$$

We remark that by Proposition 2.1 it follows that the positive operator V_m is a rational non-integer extension of the Hermite–Fejér operator H_m, which in general is not positive. It follows also that $R_m(e_0; x) = 0$, $R_m(f; x_{m,k}) = 0$.

Particularly interesting cases of the operator V_m have been introduced and studied separately in previous papers. For instance, if $\{p_m(w)\}$ is a sequence of orthonormal polynomials in I with respect to a weight function w and we set

$$A = \{x_{m,i} \,/\, p_m(w; x_{m,i}) = 0,\ i = 1, 2, \dots, m,\ m \in N\},$$

$$\varphi_m(x) = \lambda_m(w; x) = \left[\sum_{k=0}^{m-1} p_k^2(w; x)\right]^{-1}$$

(m-th Cristoffel function), then since

$$\lambda_m^{-1}(w; x) = \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\lambda_{m,k}(w)},$$

where $\lambda_{m,k}(w) = \lambda_m(w; x_{m,k})$, we obtain the operator

$$F_m(f; x) = \lambda_m(w; x) \sum_{k=1}^{m} \frac{l_{m,k}^2(x)}{\lambda_{m,k}(w)} f(x_{m,k}), \qquad x \in I,$$

introduced by Nevai in [11] and studied in [6].

If, for any matrix A of knots, we set $\varphi_m(x) \equiv 1$, then we obtain the operator

$$L_m(f; x) = \frac{\sum\limits_{k=1}^{m} l_{m,k}^2(x) f(x_{m,k})}{\sum\limits_{k=1}^{m} l_{m,k}^2(x)},$$

introduced by Hermann and Vértesi in [9].

Finally, if for any matrix $A = \{x_{m,i},\ i = 1,2,\dots,m,\ m \in N\}$ we set

$$\varphi_m(x) = \frac{1}{p'_m(x)}, \qquad p_m(x) = \prod_{k=1}^{m}(x - x_{m,k}),$$

then we have the Shepard operator

$$S_m(A;f;x) = S_m(f;x) = \frac{\sum_{k=1}^{m}(x - x_{m,k})^{-2} f(x_{m,k})}{\sum_{k=1}^{m}(x - x_{m,k})^{-2}}, \tag{2.6}$$

see [14]. This last operator is important in approximation theory and it is used for graphic representation of surfaces; for this reason it has interested numerous authors; among all we recall [3, 4, 8, 10].

In the next section we shall give pointwise estimates for the remainder term $f(x) - S_m(f;x)$ in the cases of different distributions of the knots. For the general operator V_m we state a necessary and sufficient condition for the uniform convergence of $V_m f$ to f.

Let $\omega(f;\delta) = \max\limits_{|x-y| \leqq \delta} |f(x) - f(y)|$ be the ordinary modulus of continuity of the function f. Then, the following theorem holds.

THEOREM 2.2. *Let $T_{m,2}(x) = V_m(h_x;x)$ with $h_x(t) = (x-t)^2$; then $V_m f$ converges to f on I if and only if*

$$\lim_{m\to\infty} \sup_{x\in I} T_{m,2}(x) = 0.$$

Moreover, we have

$$|f(x) - V_m(f;x)| \leqq 2\omega\left(f; \sqrt{T_{m,2}(x)}\right). \tag{2.7}$$

PROOF. Since $V_m(e_0;x) = 1$, the first statement of the theorem is an equivalent formulation of the celebrated Korovkin theorem. Inequality (2.7) can be obtained by standard computations. □

3. Convergence estimates for the Shepard operator

In the present section we consider the Shepard operator defined by (2.6) corresponding to two different matrices of knots and we give pointwise estimates of the convergence order.

The case of the equidistant knots.

THEOREM 3.1. *If* $X = \left\{-1+\frac{2k}{m},\ k=0,1,\dots,m,\ m \in N\right\}$, *then we have*

$$(3.1)\qquad |f(x)-S_m(X;f;x)| \leqq \frac{9}{m}\int_{m^{-1}}^{1}\frac{\omega(f;t)}{t^2}dt,\quad f\in C^0(I),\ x\in I.$$

PROOF. Let $x_{m,i}=-1+\frac{2i}{m}$, $i=0,1,\dots,m$, $m\in N$. Being $S_m(X;f;x_{m,i})= = f(x_{m,i})$, in the following we assume that $x \neq x_{m,i}$. Let

$$x_{m,c-1} < x_{m,c} < x < x_{m,c+1},$$

and let $x_{m,c}$ be the knot closest to x, i.e. $x - x_{m,c} \leqq x_{m,c+1} - x$. Thus, $x - x_{m,c} < m^{-1}$ and $x_{m,c+1} - x > m^{-1}$. Moreover, if $x_{m,k} < x_{m,c}$ then $2(c-k)m^{-1} < x - x_{m,k} < [2(c-k)+1]m^{-1}$ and if $x_{m,k} \geqq x_{m,c+1}$ then $[2(k-c)-1]m^{-1} < x_{m,k} - x < 2(k-c)m^{-1}$. Thus for every $x \in I$ with $x \neq x_{m,c}$, we have

$$(3.2)\qquad |f(x)-S_m(X;f;x)| \leqq \sum_{k=0}^{m}\frac{(x-x_{m,c})^2}{(x-x_{m,k})^2}\omega(f;|x-x_{m,k}|) \leqq$$

$$\leqq \omega(f;|x-x_{m,c}|)+\sum_{k=0}^{c-1}\frac{\omega(f;|x-x_{m,k}|)}{m^2(x-x_{m,k})^2}+\sum_{k=c+1}^{m}\frac{\omega(f;|x-x_{m,k}|)}{m^2(x-x_{m,k})^2} \leqq$$

$$\leqq \omega(f;m^{-1})+\frac{1}{4}\sum_{k=0}^{c-1}\frac{\omega(f;m^{-1}[2(c-k)+1])}{(c-k)^2}+\sum_{k=c+1}^{m}\frac{\omega(f;2(k-c)m^{-1})}{[2(k-c)-1]^2} \leqq$$

$$\leqq \omega(f;m^{-1})+\frac{3}{4}\sum_{k=0}^{c-1}\frac{\omega(f;(c-k)m^{-1})}{(c-k)^2}+\sum_{k=c+1}^{m}\frac{\omega(f;(k-c)m^{-1})}{(k-c)^2} \leqq$$

$$\leqq \frac{11}{4}\omega(f;m^{-1})+\frac{7}{4}\sum_{i=2}^{m}\frac{\omega(f;im^{-1})}{i^2}.$$

Now

$$\frac{1}{i^2} \leqq m^{-1}\int_{(i-1)m^{-1}}^{im^{-1}}\frac{dt}{t^2},\qquad i \geqq 2,$$

and

$$\omega(f;im^{-1}) \leqq 2\omega(f;t),\qquad (i-1)m^{-1} \leqq t \leqq im^{-1}.$$

Thus

$$(3.3)\qquad \sum_{i=2}^{m} \frac{\omega(f; im^{-1})}{i^2} \leqq \frac{2}{m} \sum_{i=2}^{m} \int\limits_{(i-1)m^{-1}}^{im^{-1}} \omega(f;t)t^{-2}dt = \frac{2}{m} \int\limits_{m^{-1}}^{1} \omega(f;t)t^{-2}dt.$$

Since

$$\omega(f;m^{-1}) \leqq \frac{2}{m} \int\limits_{m^{-1}}^{1} \omega(f;t)t^{-2}dt,$$

by (3.2) and (3.3), inequality (3.1) follows. □

By a proof similar to that of Theorem 3.1 (cf. Bojanic [5]), one can prove also the following

COROLLARY 3.2. *If the knots $x_{m,i}$, $i = 1, 2, \dots, m$, $m \in N$ of the matrix X satisfy the condition*

$$c_1 m^{-1} \leqq |x_{m,i+1} - x_{m,i}| \leqq c_2 m^{-1}, \quad i = 1, 2, \dots, m-1, \quad m \in N,$$

with constants c_1 and c_2 independent of i and m, then we have

$$|f(x) - S_m(X; f; x)| \leqq C m^{-1} \int\limits_{m^{-1}}^{1} \frac{\omega(f;t)}{t^2} dt, \qquad f \in C^0(I),\ x \in I,$$

where C is a constant depending only on c_1 and c_2.

We remark that in general the Hermite–Fejér interpolation procedure for continuous functions and with respect to equidistant knots is not convergent.

Furthermore, in general the estimate (3.1) is more precise than the following

$$(3.4)\qquad \|f - S_m(X;f)\| \leqq 8\omega(f; m^{-1}\log m), \quad m \geqq 2, \ f \in C^0(I),$$

obtained by Newman and Rivlin in [12] (cf. also [1]).

Indeed, if $f \in \mathrm{Lip}_M \alpha$, $0 < \alpha < 1$ then by (3.1) we deduce

$$\|f - S_m(X;f)\| \leqq \mathrm{const.} m^{-\alpha},$$

whereas the inequality (3.4) implies

$$\|f - S_m(X;f)\| \leqq \mathrm{const.} \Big(\frac{\log m}{m}\Big)^{\alpha}.$$

In the case $f \in \mathrm{Lip}_M 1$, both inequalities (3.1) and (3.5) give

$$\|f - S_m(X;f)\| \leqq \mathrm{const.} \frac{\log m}{m}.$$

Moreover, since for any matrix of knots X there exists an $f \in \mathrm{Lip}_M 1$ such that

$$\|f - S_m(X;f)\| \geqq \frac{1}{300}\frac{\log m}{m},$$

(see [12]), we can deduce that

$$\sup_{f \in \mathrm{Lip}_M 1} \|f - S_m(X;f)\| \geqq \text{const.}\frac{\log m}{m}.$$

Finally, the error $f - S_m(X;f)$ does not improve by assuming only higher smoothness of the function f. Indeed, the following theorem holds.

THEOREM 3.3. *The asymptotic relation*

$$\frac{m}{\log m}\left[S_m(X;f;x) - f(x)\right] = o(1), \qquad m \to \infty, \tag{3.5}$$

is not valid for every $x \in I$ and for every non-constant function $f \in C^1$.

PROOF. Let $f \in C^1(I)$ with $f'(1) > 0$ and $\int_0^1 \omega(f';t)t^{-1}dt < \infty$; it results

$$f(x_{m,i}) - f(x) = (x_{m,i} - x)f'(x) + G_x(x_{m,i}), \tag{3.6}$$

with

$$|G_x(x_{m,i})| \leqq |x - x_{m,i}|\omega\left(f'; |x - x_{m,i}|\right). \tag{3.7}$$

Thus

$$S_m(X;f;x) - f(x) = f'(x)S_m(X;g_x;x) + S_m(X;G_x;x), \tag{3.8}$$

where $g_x(t) = t - x$. In view of (3.7) and proceeding as in the proof of Theorem 3.1, we deduce

$$|S_m(X;G_x;x)| \leqq \frac{35}{2m}\int_{m^{-1}}^{1} \omega(f';t)t^{-1}dt.$$

So by the assumption $\int_0^1 \omega(f';t)t^{-1}dt < \infty$, we have

$$\lim_{m\to\infty} \frac{m}{\log m} S_m(X;G_x;x) = 0,$$

uniformly on I. Thus by (3.8)

$$\lim_{m\to\infty} \frac{m}{\log m}\left[S_m(X;f;x) - f(x)\right] = \lim_{m\to\infty} \frac{m}{\log m} S_m(X;g_x;x)f'(x).$$

Finally, let $x \in \left(1-\frac{2p}{m}, 1\right]$ for some fixed $p < m$; by some calculation

$$S_m(X; g_x; x) \geqq \text{const.} \frac{\log m}{m},$$

whence we deduce that

$$\lim_{m\to\infty} \frac{m}{\log m} S_m(X; g_x; x) f'(x) > 0. \qquad \square$$

As for the saturation of the Shepard operator S_m, it was proved in [7] that

$$\|f - S_m(X; f)\| = o\left(\frac{1}{m}\right) \Leftrightarrow f \quad \text{is a constant.}$$

However, the saturation class is an open problem. J. Szabados [15] proved that

$$f'(0) = f'(1) = 0 \text{ and } \int_0^1 \frac{\omega(f'; t)}{t} dt < \infty \Rightarrow \|f - S_m(X; f)\| = O\left(\frac{1}{m}\right),$$

and he conjectured that the converse implication is also true. At present, in the converse direction only $f \in \bigcap_{\alpha<1} \operatorname{Lip} \alpha$ is known (see [7]).

The case of the zeros of orthogonal polynomials

Let $\{p_m(w)\}$ be the sequence of the orthonormal polynomials with respect to the weight function $w \in \text{GSJ}$ defined by

$$w(x) = \psi(x) \prod_{k=0}^{s+1} |x - t_k|^{\gamma_k}, \qquad x \in I,$$

where $-1 = t_0 < t_1 < \ldots < t_s < t_{s+1} = 1$, $\gamma_k > -1$, $k = 0, 1, \ldots, s+1$ and the function $\psi > 0$ is such that $\int_0^1 \omega(\psi, \delta)\delta^{-1} d\delta < \infty$. The zeros of $p_m(w; x) = \alpha_m(w)x^m +$ lower degree terms, $\alpha_m(w) > 0$, are denoted by $x_{m,i} = x_{m,i}(w)$ and they are ordered so that $x_{m,1} < x_{m,2} < \ldots < x_{m,m}$. Set $x_{m,i} = \cos\theta_{m,i}$ for $0 \leqq i \leqq m+1$ where $x_{m,0} = -1$, $x_{m,m+1} = 1$ and $0 \leqq \theta_{m,i} \leqq \pi$. Then[1]

$$\theta_{m,i} - \theta_{m,i+1} \sim m^{-1}, \tag{3.9}$$

[1] If A and B are two expressions depending on some variables then we write

$$A \sim B \quad \text{iff} \quad |AB^{-1}| \leqq \text{const. and } |A^{-1}B| \leqq \text{const.}$$

uniformly for the variables in consideration.

uniformly for $0 \leqq i \leqq m$, $m \in N$. (See [11, Theorem 9.22, p. 166].) Now we denote by Y the matrix having as knots the zeros of $q_m(x) = (1-x^2)p_m(w;x)$ and consider the Shepard operator $S_m(Y)$ corresponding to the matrix Y. Let $g_x(t) = t - x$; in order to study the convergence of $S_m(Y)$, the following lemma is needed.

LEMMA. *Let Y be the matrix of knots corresponding to a weight $w \in$ $\in$ GSJ. Then*

$$|S_m(Y; g_x; x) \leqq \text{const.} \frac{\sqrt{1-x^2}}{m} \tag{3.10}$$

holds with a constant independent of x and m.

PROOF. By (2.6) we can write

$$S_m(Y; g_x; x) = -\frac{\sum\limits_{i=0}^{m+1} (x - x_{m,i})^{-1}}{\sum\limits_{i=0}^{m+1} (x - x_{m,i})^{-2}} = -\frac{q_m(x)q'_m(x)}{\sum\limits_{i=0}^{m+1} l^2_{m,i}(x)[q'_m(x_{m,i})]^2}.$$

For every $x \in I$, we denote by c the index corresponding to the knot $x_{m,c}$ closest to x. Recalling that $l_{m,c}(x) \sim 1$, uniformly for $x \in I$, $m \in N$ (see [11, p. 171]), we deduce

$$|S_m(Y; g_x; x)| \leqq \frac{|q_m(x)q'_m(x)|}{l^2_{m,c}(x)[q'_m(x_{m,c})]^2} \sim \frac{|q_m(x)q'_m(x)|}{[q'_m(x_{m,c})]^2}. \tag{3.11}$$

Since $q'_m(x) = (1-x^2)p'_m(w;x) - 2xp_m(w;x)$ we obtain

$$|q_m(x)q'_m(x)| \leqq (1-x^2)\left[2p^2_m(w;x) + (1-x^2)|p_m(w;x)||p'_m(w;x)|\right]. \tag{3.12}$$

Now, define w_m by

$$w_m(x) = \left(\sqrt{1+x} + m^{-1}\right)^{2\gamma_0+1} \prod_{k=1}^{s} (|x-t_k| + m^{-1})^{\gamma_k} \left(\sqrt{1-x} + m^{-1}\right)^{2\gamma_{s+1}+1}.$$

Then

$$|p_m(w;x)| \leqq \text{const.}[w_m(x)]^{-1/2}, \tag{3.13}$$

uniformly in $x \in I$ and $m \in N$ (see [2, Theorem 1.1, p. 226]). Moreover

$$\max_{x\in I}\left\{|p'_m(w;x)|\sqrt{w_m(x)}\left(\sqrt{1-x} + m^{-1}\right)\left(\sqrt{1+x} + m^{-1}\right)\right\} \leqq \tag{3.14}$$
$$\leqq \text{const.}\, m \max_{x\in I}\left\{|p_m(w;x)|\sqrt{w_m(x)}\right\}$$

(see [11, Theorem 19, p. 164]). Being by (3.12)

$$|q_m(x)q_m'(x)| \leqq (1-x^2)w_m^{-1}(x)\Big\{2w_m(x)p_m^2(w;x)+ \\ +\left[\sqrt{1-x^2}\sqrt{w_m(x)}|p_m(w;x)|\right]\times \\ \times\left[\left(\sqrt{1-x}+m^{-1}\right)\left(\sqrt{1+x}+m^{-1}\right)\sqrt{w_m(x)}|p_m'(w;x)|\right]\Big\},$$

the inequalities (3.13) and (3.14) give

$$|q_m(x)q_m'(x)| \leqq \text{const.}(1-x^2)w_m^{-1}(x)\left\{1+m\sqrt{1-x^2}\right\}. \tag{3.15}$$

On the other hand,

$$[p_m'(w;x_{m,i})]^{-1} = \frac{\alpha_{m-1}(w)}{\alpha_m(w)}\lambda_{m,i}(w)p_{m-1}(w;x_{m,i}),$$

where $\lambda_{m,i}(w)$, $i=1,2\dots,m$ denote the Christoffel constants defined by $\lambda_{m,i}(w)=\lambda_m(w;x_{m,i})$ with $\lambda_m(w;x)=\left[\sum_{k=0}^{m-1}p_k^2(w;x)\right]^{-1}$ (m-th Christoffel function). Consequently

$$[q_m'(x_{m,c})]^{-2} = \frac{1}{(1-x_{m,c}^2)\,[p_m'(w;x_{m,c})]^2} = \\ = \left(\frac{\alpha_{m-1}(w)}{\alpha_m(w)}\right)^2\frac{\lambda_{m,c}^2(w)p_{m-1}^2(w;x_{m,c})}{(1-x_{m,c}^2)^2}.$$

Since $\alpha_{m-1}(w)/\alpha_m(w)\sim 1$ and $\lambda_{m,c}(w)p_{m-1}^2(w;x_{m,c})\sim m^{-1}(1-x_{m,c}^2)$ (see [11, Theorem 6.3.28, p. 120 and Theorem 9.31, p. 170]), we deduce

$$[q_m'(x_{m,c})]^{-2}\sim\frac{\lambda_{m,c}(w)}{m(1-x_{m,c}^2)}.$$

Furthermore, by (3.9) $\lambda_{m,c}(w)\sim\lambda_m(w;x)$ and recalling that $\lambda_m(w;x)\sim$ $\sim m^{-1}w_m(x)$, uniformly in $x\in I$, $m\in N$, we obtain

$$[q_m'(x_{m,c})]^{-2}\sim\frac{w_m(x)}{m^2\left[\sqrt{1-x^2}+m^{-1}\right]^2}. \tag{3.16}$$

Combining (3.15) and (3.16) with (3.11), inequality (3.10) follows. □

Now we are able to prove the following

THEOREM 3.4. *Let Y be the matrix of knots corresponding to a weight $w \in$ GSJ. Then we have*

(3.17)
$$|f(x)-S_m(Y;f;x)| \leqq \text{const.} m^{-1} \int_{m^{-1}}^{1} \omega\left(f; t\sqrt{1-x^2}\right)\frac{dt}{t^2}, \quad f \in C^0(I),\ x \in I,$$

(3.18)
$$|f(x)-S_m(Y;f;x)| \leqq \text{const.} \frac{\sqrt{1-x^2}}{m}\left\{|f'(x)| + \int_{m^{-1}}^{1} \omega\left(f'; t\sqrt{1-x^2}\right)\frac{dt}{t}\right\},$$
$$f \in C^1(I), \quad x \in I.$$

PROOF. Assume that $x \neq x_{m,i}$ and $x_{m,c} < x < x_{m,c+1}$ with $x_{m,c}$ the closest knot to x. Taking (3.9) into account, it is easy to prove that

$$|x-x_{m,c}| \leqq \text{const.} m^{-1}\sqrt{1-x^2}, \quad |x-x_{m,k}| \geqq \text{const.} m^{-1}|k-c|\sqrt{1-x^2}, \quad k \neq c,$$

and for sake of brevity we omit the computations. Thus,

$$|f(x)-S_m(Y;f;x)| \leqq \sum_{k=0}^{m} \frac{(x-x_{m,c})^2}{(x-x_{m,k})^2}\omega(f;|x-x_{m,k}|).$$

Since
$$\frac{\omega(f;\delta_2)}{\delta_2} \leqq 2\frac{\omega(f;\delta_1)}{\delta_1}, \qquad \delta_2 \geqq \delta_1,$$
we have

$$|f(x)-S_m(Y;f;x)| \leqq$$
$$\leqq \text{const.}\left\{\omega\left(f; \frac{\sqrt{1-x^2}}{m}\right) + \sum_{\substack{k=0\\k\neq c}}^{m} \frac{1}{(k-c)^2}\omega\left(f; \frac{|k-c|}{m}\sqrt{1-x^2}\right)\right\} \leqq$$
$$\leqq \text{const.}\left\{\omega\left(f; \frac{\sqrt{1-x^2}}{m}\right) + \sum_{i=2}^{m} \frac{1}{i^2}\omega\left(f; \frac{i}{m}\sqrt{1-x^2}\right)\right\} \leqq$$
$$\leqq \text{const.}\left\{\omega\left(f; \frac{\sqrt{1-x^2}}{m}\right) + m^{-1}\int_{m^{-1}}^{1} \omega\left(f; t\sqrt{1-x^2}\right)t^{-2}dt\right\},$$

and observing that $\omega(f; m^{-1}\sqrt{1-x^2}) \leqq \text{const.} m^{-1} \int_{m^{-1}}^{1} \omega(f; t\sqrt{1-x^2})t^{-2}dt$, inequality (3.17) follows.

In order to prove (3.18) we recall (3.6) whence

$$S_m(Y;f;x) - f(x) = f'(x)S_m(Y;g_x;x) + S_m(Y;G_x;x), \tag{3.19}$$

with $g_x(t) = t - x$. In view of (3.7) we deduce as above:

$$|S_m(Y;G_x;x)| \leqq \text{const.}\frac{\sqrt{1-x^2}}{m}\int_{m^{-1}}^{1} \omega\left(f';t\sqrt{1-x^2}\right)t^{-1}dt.$$

Finally, taking into account (3.19) by the Lemma we deduce also (3.18). □

A particular case of the last theorem interesting in the applications corresponds to $w(x) = \sqrt{1-x^2}$, that is to say that $x_{m,i} = \cos\frac{\pi i}{m+1}$, $i = 0,1,\dots,m+1$.

We remark that the estimate (3.17) for the presence of the term $\sqrt{1-x^2}$, is better than the previous estimate (3.1) near to the end points ± 1 corresponding to a thicker mesh in their neighbours. However, we suspect that the Shepard procedure corresponding to another distribution of knots can have a better behaviour near those points corresponding to a thicker mesh and they may not necessarily be the end points of the interval. In particular, by Theorem 3.3 it follows that in the case of equidistant knots it is not possible to establish an inequality of the same kind as (3.18).

An estimate similar to (3.17) is also valid for the operator L_m, whereas for the operator F_m both estimates (3.17) and (3.18) are true; see [6] and [4] respectively.

Moreover, we recall that for the Hermite–Fejér operator corresponding to the zeros of the Jacobi polynomials $p_m^{\alpha,\beta}$, Vértesi [16], generalizing a result of Bojanic [3], has proved the estimate

$$|f(x)-H_m(f;x)| = O(1)\sum_{i=1}^{m}\left[\omega\left(f;\frac{i\sqrt{1-x^2}}{m}\right)+\omega\left(f;\frac{i^2|x|}{m^2}\right)\right]i^{2\gamma-1},\quad x\in I,$$

where the O sign depends on α and $\beta, \gamma = \max(\alpha,\beta,-1/2)$. This estimate is of the same kind as (3.17); however this last one gives a good estimation only when $\alpha,\beta \leqq -1/2$.

Finally, we remark that the problem of saturation for the operator F_m has been partially resolved [6]; whereas for the operator $S_m(Y)$ it is still an open problem.

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A DECOMPOSITION OF CONTINUITY AND α-CONTINUITY

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Let B be a subset of a topological space (X,T). The closure and the interior of B are denoted by $\mathrm{Cl}(B)$ and $\mathrm{Int}(B)$, respectively.

A subset $B \subset X$ is said to be semi-open [5] (resp. an α-set [8], pre-open [6]) if $B \subset \mathrm{Cl}(\mathrm{Int}(B))$ (resp. $B \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$, $B \subset \mathrm{Int}(\mathrm{Cl}(B))$). The collection of all subsets of a space (X,T) which are semi-open (resp. α-sets, pre-open) is denoted by $\mathrm{SO}(X,T)$ (resp. T^α, $\mathrm{PO}(X,T)$). It was observed in [8] that T^α is a topology on X and that $T \subset T^\alpha \subset \mathrm{SO}(X,T)$. Moreover, $T^\alpha = \mathrm{SO}(X,T) \cap \mathrm{PO}(X,T)$ [11].

The union of all semi-open (resp. pre-open) sets contained in B is called the semi-interior of B [2] (resp. pre-interior of B [6]) and is denoted by $\mathrm{sInt}(B)$ (resp. $\mathrm{pInt}(B)$). The interior of a subset B of the space (X,T^α) is denoted by $\alpha\mathrm{Int}(B)$.

The following result will be useful in the sequel.

LEMMA 1 [1]. *If B is a subset of a space (X,T), then*

(i) $\mathrm{sInt}(B) = B \cap \mathrm{Cl}(\mathrm{Int}(B))$,

(ii) $\mathrm{pInt}(B) = B \cap \mathrm{Int}(\mathrm{Cl}(B))$,

(iii) $\alpha\mathrm{Int}(B) = B \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$.

Let X and Y be topological spaces. A map $f\colon X \to Y$ is called:

— semi-continuous if for every open set V of Y, $f^{-1}(V)$ is semi-open [5];

— pre-continuous if for every open set V of Y, $f^{-1}(V)$ is pre-open [6];

— α-continuous if for every open set V of Y, $f^{-1}(V)$ is an α-set [11].

Let $\ddot{A}$ and $\ddot{U}$ be collections of subsets of a topological space. We use the notion $\ddot{A} \wedge \ddot{U} = \{A \cap U\colon A \in \ddot{A}, U \in \ddot{U}\}$.

In this paper we introduce the following types of maps:

DEFINITION 1. Let (X,T) and (Y,T') be topological spaces and let $\ddot{A}$ be a collection of subsets of X. A map $f\colon (X,T) \to (Y,T')$ is said to be locally $\ddot{A}$-continuous if for every open set V of (Y,T'), the set $f^{-1}(V)$ belongs to $(\ddot{A} \wedge T) \cup T$.

It is evident that every continuous map is locally $\ddot{A}$-continuous for any collection $\ddot{A}$ of subsets of X.

We say that a map $f\colon (X,T) \to (Y,T')$ is $\ddot{A}$-continuous if for every open set V of (Y,T'), the set $f^{-1}(V)$ belongs to $\ddot{A}$, where $\ddot{A}$ is a collection of subsets of X. Obviously every $\ddot{A}$-continuous map is locally $\ddot{A}$-continuous since $\ddot{A} \subset (\ddot{A} \wedge T) \cup T$.

For a topological space (X,T) we denote:

$$D(c,\alpha) = \{B \subset X \colon \mathrm{Int}(B) = \alpha\mathrm{Int}(B)\},$$
$$D(c,p) = \{B \subset X \colon \mathrm{Int}(B) = \mathrm{pInt}(B)\},$$
$$D(\alpha,p) = \{B \subset X \colon \mathrm{Int}(B) = \mathrm{pInt}(B)\}.$$

The most significant result in this note is the following triplet of theorems.

THEOREM 1. *For a topological space (X,T) we have:*

(i) *A map $f\colon (X,T) \to (Y,T')$ is continuous if and only if it is both α-continuous and $D(c,\alpha)$-continuous.*

(ii) *Let $\ddot{A}$ be a collection of subsets of X satisfying the following:*

(c,α) *A map $f\colon (X,T) \to (Y,T')$ is continuous if and only if it is both α-continuous and locally $\ddot{A}$-continuous.*

Then $\ddot{A} \subset D(c,\alpha)$.

THEOREM 2. *For a topological space (X,T) we have:*

(i) *A map $f\colon (X,T) \to (Y,T')$ is continuous if and only if it is both pre-continuous and $D(c,p)$-continuous.*

(ii) *Let $\ddot{A}$ be a collection of subsets of X satisfying the following:*

(c,p) *A map $f\colon (X,T) \to (Y,T')$ is continuous if and only if it is both pre-continuous and locally $\ddot{A}$-continuous.*

Then $\ddot{A} \subset D(c,p)$.

THEOREM 3. *For a topological space (X,T) we have:*

(i) *A map $f\colon (X,T) \to (Y,T')$ is α-continuous if and only if it is both pre-continuous and $D(\alpha,p)$-continuous.*

(ii) *Let $\ddot{A}$ be a collection of subsets of X satisfying the following:*

(α,p) *A map $f\colon (X,T) \to (Y,T')$ is α-continuous if and only if it is both pre-continuous and locally $\ddot{A}$-continuous.*

Then $\ddot{A} \subset D(\alpha,p)$.

Before proving the above results we need the following three lemmas.

LEMMA 2. *For a topological space (X,T) the following hold.*

(i) *$T^\alpha \cap D(c,\alpha) = T$.*

(ii) *If $\ddot{A}$ is a collection of subsets of X such that $T^\alpha \cap ((\ddot{A} \wedge T) \cup T) = T$, then $\ddot{A} \subset D(c,\alpha)$.*

PROOF. (i) The conditions $B \in T^\alpha$ and $B \in D(c,\alpha)$ imply $B = \alpha\mathrm{Int}(B)$ and $\mathrm{Int}(B) = \alpha\mathrm{Int}(B)$ and consequently $B \in T$. Conversely, if $B \in T$, then $B = \mathrm{Int}(B)$ and $B = \alpha\mathrm{Int}(B)$ since $T \subset T^\alpha$. Thus $B \in T^\alpha \cap D(c,\alpha)$ which finishes the proof of (i).

(ii) Let $A \in \ddot{A}$. According to Lemma 1 (iii) we have $\alpha\mathrm{Int}(A) = A \cap \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. Clearly, the set $\alpha\mathrm{Int}(A)$ belongs to T^α. We see also that it is of the form $A \cap U$, where $A \in \ddot{A}$ and $U \in T$. From this follows $\alpha\text{Int}(A) \in T^\alpha \cap ((\ddot{A} \wedge T) \cup T)$ and consequently $\alpha\text{Int}(A) \in T$. Thus $\alpha\text{Int}(A) = \text{Int}(A)$, which means $A \in D(c, \alpha)$ and finishes the proof.

LEMMA 3. *For a topological space (X, T) the following hold.*

(i) $\text{PO}(X,T) \cap D(c,p) = T$.

(ii) *If $\ddot{A}$ is a collection of subsets of X such that* $\text{PO}(X,T) \cap ((\ddot{A} \wedge T) \cup T) = T$, *then* $\ddot{A} \subset D(c,p)$.

PROOF. Since the proof is analogous to that in Lemma 2, it is omitted.

LEMMA 4. *For a topological space (X, T) the following hold.*

(i) $\text{PO}(X,T) \cap D(\alpha,p) = T^\alpha$.

(ii) *If $\ddot{A}$ is a collection of subsets of X such that* $\text{PO}(X,T) \cap ((\ddot{A} \wedge T) \cup T) = T^\alpha$, *then* $\ddot{A} \subset D(\alpha,p)$.

PROOF. If $B \in \text{PO}(X,T) \cap D(\alpha,p)$, then $B = \text{pInt}(B)$ and $\alpha\text{Int}(B) = \text{pInt}(B)$ and consequently, $B = \alpha\text{Int}(B)$, which means $B \in T^\alpha$. Conversely if $B \in T^\alpha$, then $B = \alpha\text{Int}(B)$ and consequently $B = \text{pInt}(B)$ since $\alpha\text{Int}(B) \subset \text{pInt}(B)$. Thus we have $\alpha\text{Int}(B) = \text{pInt}(B) = B$, so $B \in \text{PO}(X,T) \cap D(\alpha,p)$, which finishes the proof of (i).

(ii) If $A \in \ddot{A}$, then $A \cap \text{Int}(\text{Cl}(A)) = \text{pInt}(A)$ by Lemma 1 (ii). It is obvious that $\text{pInt}(A) \in \text{PO}(X,T) \cap ((\ddot{A} \wedge T) \cup T)$. It implies $\text{pInt}(A) \in T^\alpha$. Hence $\text{pInt}(A) \subset \alpha\text{Int}(A)$ and we have shown $\ddot{A} \subset D(\alpha,p)$.

PROOF OF THEOREMS 1, 2 AND 3. The proofs are all similar. Statement (i) of Theorem 1 (resp. Theorem 2, Theorem 3) follows easily from (i) of Lemma 2 (resp. Lemma 3, Lemma 4).

Now we will prove (ii) of Theorem 1. Let us assume that $\ddot{A} \not\subset D(c,\alpha)$. Then there exists $A \in \ddot{A}$ for which $\text{Int}(A) \neq \alpha\text{Int}(A)$ holds, which implies $\alpha\text{Int}(A) \notin T$. Let $Y = X$ and $T' = \{Y, \emptyset, \alpha\text{Int}(A)\}$. The identity map $f\colon (X,T) \to (Y,T')$ is clearly α-continuous, but not continuous. Since the set $\alpha\text{Int}(A)$ is of the form $A \cap U$, $A \in \ddot{A}$, $U \in T$, the map f is locally $\ddot{A}$-continuous. This means that $\ddot{A}$ does not satisfy the condition (c,α) which finishes the proof.

The proof of (ii) of Theorem 2 (resp. Theorem 3) is analogous to that of Theorem 1, it is sufficient to observe that the condition $\text{Int}(A) \neq \text{pInt}(A)$ (resp. $\alpha\text{Int}(A) \neq \text{pInt}(A)$) is equivalent to the condition $\text{pInt}(A) \notin T$ (resp. $\text{pInt}(A) \notin T^\alpha$).

Let us observe that the condition $T^\alpha \cap \ddot{A} = T$ (resp. $\text{PO}(X,T) \cap \ddot{A} = T$, $\text{PO}(X,T) \cap \ddot{A} = T^\alpha$) implies $\ddot{A} \subset T \cup \{B \subset X\colon B \notin T^\alpha\}$ (resp. $\ddot{A} \subset T \cup \{B \subset X\colon B \notin \text{PO}(X,T)\}$, $\ddot{A} \subset T^\alpha \cup \{B \subset X\colon B \notin \text{PO}(X,T)\}$).

The following example shows that $D(c,\alpha) \neq T \cup \{B \subset X\colon B \notin T^\alpha\}$, $D(c,p) \neq T \cup \{B \subset X\colon B \notin \text{PO}(X,T)\}$ and $D(\alpha,p) \neq T^\alpha \cup \{B \subset X\colon B \notin \text{PO}(X,T)\}$. At first we observe that $D(c,p) = D(c,\alpha) \cap D(\alpha,p)$. Thus, it is sufficient to prove the existence of a set $B \subset X$ for which $B \notin D(c,\alpha)$, $B \notin D(\alpha,p)$ and $B \notin \text{PO}(X,T)$ hold.

EXAMPLE 1. Let $B=[-1,1]\setminus(\{\frac{1}{n}: n=2,3,\dots\}\cup\{-\frac{1}{n}: n=2,3,\dots\})\cup\cup((1,2)\setminus Q)$, where Q denotes the set of rational numbers. We consider B as a subset of the space of real numbers with the natural topology. Then we have $\mathrm{Int}(\mathrm{Cl}(B))=(-1,2)$, $\mathrm{Int}(B)=\bigcup\left\{\left(-\frac{1}{n},-\frac{1}{n+1}\right): n=1,2,\dots\right\}\cup\cup\bigcup\left\{\left(\frac{1}{n+1},\frac{1}{n}\right): n=1,2,\dots\right\}$ and $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))=(-1,1)$. Hence it follows $B\not\subset\mathrm{Int}(\mathrm{Cl}(B))$, $\mathrm{Int}(B)\neq B\cap\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$ and $B\cap\mathrm{Int}(\mathrm{Cl}(B))\neq B\cap\cap\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$. This means $B\notin\mathrm{PO}(X,T)$, $B\notin D(c,\alpha)$ and $B\notin D(\alpha,p)$.

In [12], Jingchen Tong defined an $\mathcal{A}$-set in a topological space as a set B such that $B=U\cap C$, where U is an open set and C is a regular closed set.

A map $f\colon(X,T)\to(Y,T')$ is said to be $\mathcal{A}$-continuous [12] if, for every open set V of Y, $f^{-1}(V)$ is an $\mathcal{A}$-set.

It is easy to verify that every $\mathcal{A}$-set belongs to $D(c,p)$. Thus, the following corollaries are immediate consequences of Theorems 1, 2, resp. 3.

COROLLARY 1 [12, Theorem 4.1]. *A map $f\colon(X,T)\to(Y,T')$ is continuous if and only if it is both α-continuous and $\mathcal{A}$-continuous.*

COROLLARY 2. *A map $f\colon(X,T)\to(Y,T')$ is continuous if and only if it is both pre-continuous and $\mathcal{A}$-continuous.*

COROLLARY 3. *If a map $f\colon(X,T)\to(Y,T')$ is both pre-continuous and $\mathcal{A}$-continuous, then it is α-continuous.*

It is clear that Corollary 2 is a generalization of Theorem 4.1 in [12]. We see also that Corollary 3 follows trivially from Corollary 2. Moreover, we observe that Corollary 3 follows from the fact that $\mathcal{A}$-continuous maps are semi-continuous ([12], Theorem 5.1) and, a map is α-continuous if and only if it is pre-continuous and semi-continuous (see [9], Theorem 3.2, where pre-continuous maps are called almost continuous).

It is known that α-continuous maps into regular spaces are continuous. Then from Theorem 3 (i) follows

COROLLARY 4. *If (Y,T') is a regular space, then a map $f\colon(X,T)\to\to(Y,T')$ is continuous if and only if it is both pre-continuous and $D(\alpha,p)$-continuous.*

Let (X,T) be a topological space and let I be an ideal of subsets of X. For a subset $A\subset X$, $D_I(A)=\{x\in X: U\cap A\notin I$ for each neighbourhood U of $x\}$. Assume that I satisfies the following: $A\in I$ if and only if $D_I(A)=\emptyset$. Then the operation $A\to A\cup D_I(A)$ is the closure operation [4]; the topology defined by this way is denoted by $H(T,I)$. A subset $B\subset X$ is open in $(X,H(T,I))$ if and only if B is the difference of an open set in (X,T) and a set belonging to I (see [4], Theorem 1).

When $\ddot{A}$ is a family of subsets of X we use the notation $\ddot{A}^-$ for the family $\{X\setminus A: A\in\ddot{A}\}$. It is clear that $H(T,I)=T\wedge I^-$. Then the continuity of $f\colon(X,H(T,I))\to(Y,T')$ is equivalent to the local I^--continuity of $f\colon(X,T)\to(Y,T')$.

We will need the following lemma.

LEMMA 5. *For a topological space (X,T) the following hold:*
(i) $D(c,\alpha) \wedge D(c,\alpha) = D(c,\alpha)$,
(ii) $D(c,p) \wedge D(c,p) = D(c,p)$,
(iii) $D(\alpha,p) \wedge D(\alpha,p) = D(\alpha,p)$.

The proof is obvious and thus omitted.

Let $f\colon X \to Y$ be a map and I an ideal with the above property of subsets of X. We consider the following properties for f.

(1) The map $f\colon (X,T) \to (Y,T')$ is continuous.

(2) The map $f\colon (X,H(T,I)) \to (Y,T')$ is continuous and the map $f\colon (X,T) \to (Y,T')$ is α-continuous.

(3) The map $f\colon (X,H(T,I)) \to (Y,T')$ is continuous and the map $f(X,T) \to (Y,T')$ is pre-continuous.

(4) The map $f\colon (X,T) \to (Y,T')$ is α-continuous.

THEOREM 4. *Let (X,T), (Y,T') be topological spaces and I an ideal of subsets of X. For a map $f\colon X \to Y$ we have:*
(i) *Statements* (1) *and* (2) *are equivalent if and only if $I^- \subset D(c,\alpha)$.*
(ii) *Statements* (1) *and* (3) *are equivalent if and only if $I^- \subset D(c,p)$.*
(iii) *Statements* (4) *and* (3) *are equivalent if and only if $I^- \subset D(\alpha,p)$.*

PROOF. (i) If Statements (1) and (2) are equivalent, then the family I^- satisfies the condition (c,α) of Theorem 1 and consequently $I^- \subset D(c,\alpha)$. Conversely, let $I^- \subset D(c,\alpha)$. Since $H(T,I) = T \wedge I^-$ and $T \subset D(c,\alpha)$ we obtain $H(T,I) \subset D(c,\alpha) \wedge D(c,\alpha) = D(c,\alpha)$ by Lemma 5 (i). Thus from Theorem 1 (i) it follows that conditions (1) and (2) are equivalent.

The proof of (ii) (resp. (iii)) is analogous to that of (i); it follows from Theorem 2 (resp. Theorem 3) and from Lemma 5 (ii) (resp. Lemma 5 (iii)).

A subset $B \subset X$ is said to be simply-open if $B = U \cup K$, where U is an open set and K is nowhere dense [7]. A map $f\colon (X,T) \to (Y,T')$ is said to be simply-continuous if for every open set V of Y, $f^{-1}(V)$ is simply-open [3, 7].

We have the following theorem on simply-continuity:

THEOREM 5. *A map $f\colon (X,T) \to (Y,T')$ is α-continuous if and only if it is both simply-continuous and pre-continuous.*

PROOF. Evidently, by Theorem 3, it is sufficient to prove that every simply-open set belongs to $D(\alpha,p)$. At first we shall show that $B \in D(\alpha,p)$ if and only if $X \setminus B \in D(\alpha,p)$:

If $B \in D(\alpha,p)$, then $B \cap \mathrm{Int}(\mathrm{Cl}(B)) = B \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$. Thus we obtain $\mathrm{Cl}(\mathrm{Int}(B)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))) = \mathrm{Cl}(\mathrm{Cl}(B) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))) =$ $= \mathrm{Cl}(B \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))) = \mathrm{Cl}(B \cap \mathrm{Int}(\mathrm{Cl}(B))) = \mathrm{Cl}(\mathrm{Cl}(B) \cap \mathrm{Int}(\mathrm{Cl}(B)) =$ $= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))$; consequently $\mathrm{Int}(\mathrm{Cl}(B)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))$. Now let us observe that $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B))) = X \setminus \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(X \setminus B)))$ and $\mathrm{Int}(\mathrm{Cl}(B)) = X \setminus \mathrm{Cl}(\mathrm{Int}(X \setminus B)))$. This implies $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(X \setminus B))) =$

$= \mathrm{Cl}(\mathrm{Int}(X \setminus B))$ and consequently $\mathrm{Int}(\mathrm{Cl}(X \setminus B)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(X \setminus B)))) =$ $= \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(X \setminus B)))$. So $(X \setminus B) \cap \mathrm{Int}(\mathrm{Cl}(X \setminus B)) = (X \setminus B) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(X \setminus B)))$, which means $X \setminus B \in D(\alpha, p)$.

Secondly, we observe that every open set belongs to $D(\alpha, p)$ and every nowhere dense set belongs to $D(\alpha, p)$. Therefore, by the above fact, every closed set belongs to $D(\alpha, p)$ and every set of the form $X \setminus K$, where K is nowhere dense, also belongs to $D(\alpha, p)$. Then every simply-open set $U \cup K$ is of the form $X \setminus ((X \setminus U) \cap (X \setminus K))$, where $(X \setminus U) \cap (X \setminus K)$ belongs to $D(\alpha, p)$ by Lemma 5 (iii). Thus the set $U \cup K$ belongs to $D(\alpha, p)$, which finishes the proof of the theorem.

We see by an argument similar to that in Corollary 4, that the above result implies:

COROLLARY 5. *If* (Y, T') *is a regular space, then a map* $f\colon (X, T) \to$ $\to (Y, T')$ *is continuous if and only if it is both simply-continuous and pre-continuous.*

Since every semi-open set is simply-open, the last corollary implies

COROLLARY 6 [10, Lemma 5]. *If* (Y, T') *is a regular space, then a map* $f\colon (X, T) \to (Y, T')$ *is continuous if and only if it is both semi-continuous and pre-continuous.*

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ON THE CONTROL OF A NET OF STRINGS

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In what follows we consider a net of homogeneous strings connected at the endpoints. This system can be represented as a graph. It is controlled at the vertices. We consider the problem of relaxing the system in a fixed finite time from any initial conditions. We shall show that this is possible if and only if the graph is a tree. Then we give necessary and sufficient conditions for the approximate controllability and also for the case of non-fixed finite relaxation time. In this paper we continue the investigations initiated by S. Rolewicz [1] and give the correction of a mistake stated there.

1. Consider a connected graph whose edges are strings connected at the knots of the graph. The strings and knots are indexed by $s = 1, \dots, M$ and $p = 1, \dots, N$, resp. We apply controls $u_p(t) \in L^2(0,T)$ at each vertex p. Suppose that the s-th string has endpoints p and p'; then its movement is described by the equation

$$\varrho_s \frac{\partial^2 y_s(x,t)}{\partial t^2} = \frac{\partial^2 y_s(x,t)}{\partial x^2}, \quad 0 < x < \ell_s, \quad 0 < t < T \tag{1}$$

where $\varrho_s > 0$ is the mass density and ℓ_s is the length of the s-th string. We investigate the following boundary and initial conditions:

$$y_s(0,t) = u_p(t), \qquad y_s(\ell_s, t) = u_{p'}(t), \tag{2}$$

$$y_s(x,0) = y_s^0(x), \qquad \frac{d}{dt} y_s(x,0) = y_s^1(x) \tag{3}$$

where

$$y_s^0 \in L^2(0,\ell_s), \qquad y_s^1 \in H^{-1}(0,\ell_s).$$

Here H^{-1} is the dual space of H_0^1 with respect to L^2, see [2]. Using the Fourier method we ask for y_s in the form

$$y_s(x,t) = \sqrt{\frac{2}{\ell_s \varrho_s}} \sum_{n=1}^{\infty} c_{n,s}(t) \cdot \sin \frac{n\pi}{\ell_s} x. \tag{4}$$

Let

$$y_s^0(x) = \sqrt{\frac{2}{\ell_s \varrho_s}} \sum_{n=1}^{\infty} c_{n,s}^0 \sin \frac{n\pi}{\ell_s} x, \qquad y_s^1(x) = \sqrt{\frac{2}{\ell_s \varrho_s}} \sum_{n=1}^{\infty} c_{n,s}^1 \sin \frac{n\pi}{\ell_s} x,$$

then we get by Fourier's method that

$$c_{n,s}(t)=c^0_{n,s}\cos\sqrt{\lambda_{n,s}}t+c^1_{n,s}\frac{\sin\sqrt{\lambda_{n,s}}t}{\sqrt{\lambda_{n,s}}}+ \tag{5}$$

$$+\sqrt{\frac{2}{\ell_s}}\int_0^t[u_p(\tau)-(-1)^n u_{p'}(\tau)]\sin\sqrt{\lambda_{n,s}}(t-\tau)d\tau,$$

$$\lambda_{n,s}:=\left(\frac{n\pi}{\ell_s\varrho_s}\right)^2,\qquad n=1,2,\dots .$$

Indeed, multiply both sides of (1) by $z(x,t)\in C^2([0,\ell_s]\times[0,T])$ satisfying

$$z(x,T)=z_t(x,T)=0,\qquad z(0,t)=z(\ell_s,t)=0.$$

Then integration by parts gives

$$\int_0^{\ell_s}\int_0^T\varrho_s(y_s)_{tt}z=\int_0^{\ell_s}\varrho_s[(y_s)_t(x,t)z(x,t)-y_s(x,t)z_t(x,t)]_{t=0}^T dx+\int_0^{\ell_s}\int_0^T\varrho_s y_s z_{tt}=$$

$$=\int_0^{\ell_s}\varrho_s[y^0_s(x)z_t(x,0)-y^1_s(x)z(x,0)]dx+\int_0^{\ell_s}\int_0^T\varrho_s y_s z_{tt};$$

$$\int_0^T\int_0^{\ell_s}(y_s)_{xx}z=\int_0^T[(y_s)_x(x,t)z(x,t)-y_s(x,t)z_x(x,t)]_{x=0}^{\ell_s}dt+\int_0^T\int_0^{\ell_s}y_s z_{xx}=$$

$$=\int_0^T[u_p(t)z_x(0,t)-u_{p'}(t)z_x(\ell_s,t)]dt+\int_0^T\int_0^{\ell_s}y_s z_{xx}.$$

We can summarize this as

$$\int_0^{\ell_s}\int_0^T y_s[\varrho_s z_{tt}-z_{xx}]dtdx=\int_0^T[u_p(t)z_x(0,t)-u_{p'}(t)z_x(\ell_s,t)]dt- \tag{6}$$

$$-\int_0^{\ell_s}\varrho_s[y^0_s(x)z_t(x,0)-y^1_s(x)z(x,0)]dx.$$

Let

$$z(x,t)=\sqrt{\frac{2}{\ell_s\varrho_s}}b(t)\sin\frac{n\pi}{\ell_s}x$$

where $b\in C^2[0,T]$, $b(T)=b'(T)=0$ is arbitrary. Then

$$c_{n,s}(t) = \int_0^{\ell_s} y_s(x,t)\varrho_s\sqrt{\frac{2}{\ell_s\varrho_s}}\sin\frac{n\pi}{\ell_s}x\,dx$$

and hence and from (6) we obtain the relation .

$$\int_0^T [c_{n,s}(t)b''(t) + \lambda_{n,s}c_{n,s}(t)b(t)]dt =$$

$$= \sqrt{\frac{2}{\ell_s}}\sqrt{\lambda_{n,s}}\int_0^T [u_p(t) - (-1)^n u_{p'}(t)]b(t)dt - c_{n,s}^0 b'(0) + c_{n,s}^1 b(0).$$

This holds for all $b(t)$ with the above restrictions, hence
(7)

$$c''_{n,s} + \lambda_{n,s}c_{n,s} = \sqrt{\frac{2}{\ell_s}}\sqrt{\lambda_{n,s}}[u_p - (-1)^n u_{p'}], \quad c_{n,s}(0) = c_{n,s}^0, \quad c'_{n,s}(0) = c_{n,s}^1$$

and

(8)
$$c_{n,s}(t) + i\frac{c'_{n,s}(t)}{\sqrt{\lambda_{n,s}}} =$$

$$= e^{-i\sqrt{\lambda_{n,s}}t}\left\{\left[c_{n,s}^0 + i\frac{c_{n,s}^1}{\sqrt{\lambda_{n,s}}}\right] + i\sqrt{\frac{2}{\ell_s}}\int_0^t [u_p(\tau) - (-1)^n u_{p'}(\tau)]e^{i\sqrt{\lambda_{n,s}}\tau}d\tau\right\}.$$

DEFINITION 1. The system of strings is controllable in finite time T if for any initial conditions

$$(y_s^0, y_s^1) \in L^2(0,\ell_s) \oplus H^{-1}(0,\ell_s) =: W_s, \qquad s = 1,\dots,M$$

there exists a control

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \in L^2(0,T;\mathbf{R}^N) =: H(T)$$

relaxing the system in time T:

$$y_s(x,T) = (y_s)_t(x,T) = 0 \qquad (s = 1,\dots,M).$$

By (4) this is equivalent to

$$c_{n,s}(T) = c'_{n,s}(T) = 0.$$

Using the well-known properties of the space H^{-1} we see that the sequences

$$\left\{c_{n,s}^0 + i\frac{c_{n,s}^1}{\sqrt{\lambda_{n,s}}} : \quad n \geqq 1, \quad s = 1, \dots, M\right\}$$

run over the complex ℓ_2 space when the initial conditions run over the sets W_s. Hence (8) implies

$$\left\{c_{n,s}(t) + i\frac{c_{n,s}'(t)}{\sqrt{\lambda_{n,s}}}\right\} \in \ell_2$$

i.e. in every time T the movement state of y_s remains in the function space W_s of the initial conditions. Let $\alpha_{ns} \in \mathbf{R}^N$ be the vector, whose p-th coordinate is 1, the p'-th one is $(-1)^{n+1}$ and the other coordinates are zero. By (8) the system can be controlled in time T if and only if the sequence

$$\left\{\int_0^T \left\langle \alpha_{n,s} e^{i\sqrt{\lambda_{n,s}}t}, u(t)\right\rangle dt : \quad n \geqq 1, \quad s = 1, \dots, M\right\}$$

runs over the complex valued ℓ_2 while u runs over $H(T)$; in other words, if the sequences

$$\left\{\int_0^T \left\langle \alpha_{n,s} e^{\pm i\sqrt{\lambda_{n,s}}t}, u(t)\right\rangle dt : \quad n \geqq 1, \quad s = 1, \dots, M\right\}$$

run over the complex ℓ_2 space when $u \in L^2(0, T; \mathbf{C}^N)$ (we identify here $\ell_2 \times \ell_2$ and ℓ_2).

By a fundamental theorem of N. K. Bari [3] this means exactly that the system

$$\Phi := \left\{\alpha_{n,s} e^{\pm i\sqrt{\lambda_{n,s}}t} : \quad n \geqq 1, \quad s = 1, \dots, M\right\}$$

is a Riesz basis in its closed linear hull. We shall say that Φ is an L-basis. The above arguments give the following

LEMMA 1. *The system is controllable in time T if and only if Φ is an L-basis in $L^2(0, T; \mathbf{C}^n)$.*

We shall show the following

THEOREM 1. *The system is controllable if and only if its graph is a tree.*

PROOF. First we recall

$$\sqrt{\lambda_{n,s}} = \frac{n\pi}{L_s}, \qquad L_s := \ell_s\sqrt{\varrho_s}.$$

The quantity L_s is called the optical length of the s-th string; for inhomogeneous mass distribution $\varrho_s(x)$ it is defined by the formula

$$L_s := \int_0^{\ell_s} \sqrt{\varrho_s}.$$

The system Φ is the union of the following $2M$ systems:

$$\left\{\alpha_{1,s}e^{\pm i(2n+1)\pi x/L_s}\right\}_{n=0}^{\infty}, \quad \left\{\alpha_{2,s}e^{\pm i2n\pi x/L_s}\right\} \qquad (s=1,\dots,M).$$

Each of them is an L-basis in $L^2(0,T;\mathbf{C}^N)$ for $T \geqq L_s$. Hence we have to show that for large T the $2M$ generated subspaces are independent in the sense that any of them and the linear hull of the others have a positive angle.

a) Let first $M < N$. In this case $M+1 = N$ and the graph is a tree. A single string ($M=1$) is controllable also with the assumption that in one of the endpoints the control is zero. This can be seen e.g. from (8). Now we choose in the graph a vertex p of degree 1 and its neighbour vertices p'. The strings between p and p'-s can be relaxed by appropriate controls $u_p \equiv 0$, $u_{p'} \in L^2(0,T_1)$. Then take the neighbours p'' of some p'. Now if $u_{p'}(t) = 0$ for $t > T_1$ and $u_{p''} \in L^2(T_1,T_2)$ are appropriate then the strings (p',p'') are also relaxed. Since the graph is connected, the iteration of the above processes relaxes all the strings.

b) Let $M = N$. Then the graph is a circle, whose vertices are possibly joined by a tree. Since the trees can be controlled by a), we restrict ourselves to the case of a single circle. Since in $\alpha_{2,s}$ there is one coordinate $+1$ and -1 hence in this case

$$\sum_{s=1}^{N} \alpha_{2,s} = 0. \tag{9}$$

On the other hand for every $\varepsilon > 0$ there exist integers $n_s = 0$, $k = 0$ satisfying

$$\left|\frac{2n_s\pi}{L_s} - k\right| < \varepsilon \qquad (s=1,\dots,M), \tag{10}$$

or with another notation

$$\left|\left|\left|k\frac{L_s}{2\pi}\right|\right|\right| < \varepsilon \qquad (s=1,\dots,M);$$

(see the Kronecker theorem on simultaneous diophantine approximation [4]). Now by (9) and (10) we obtain

$$\left\|\sum_{s=1}^{N} \alpha_{2,s}e^{i2n_s\pi x/L_s}\right\|_{L^2(0,T;\mathbf{C}^N)} < c\cdot\varepsilon.$$

Since the L^2-norm of the above sum is bounded below, hence Φ can not be an L-basis and then the system is not controllable..

c) Finally let $M > N$. Then (10) can be proved in the same way as in b). Since $M > N$, the vectors $\alpha_{2,s}$, $s = 1, \dots, M$ are linearly dependent. This can be put instead of (9) and the proof can be finished as in b).

Theorem 1 is proved.

REMARK 1. S. Rolewicz [1] investigated the same problem with the difference that the initial conditions are taken from the spaces

$$y_s^0 \in L^2(0, \ell_s), \qquad y_s^1 \in L^2(0, \ell_s). \tag{11}$$

By (8), the controllability in this case is equivalent to the statement that the sequences

$$\left\{ \left\langle u, \alpha_{n,s} e^{i\sqrt{\lambda_{n,s}}t} \right\rangle_{H(T)} \right\}_{n,s}$$

run over the space $h_2 \oplus i\ell_2$ where we use the real valued ℓ_2 space and h_2 is the space of real sequences $\{c_n\}$ satisfying $\sum (nc_n)^2 < \infty$. In particular, the moment space of the sequence $\{\alpha_{n,s} \sin \sqrt{\lambda_{n,s}} t\}$ with respect to the real space $H(T)$ is the real ℓ_2. Consequently the moment operator $F \colon H(T) \to \ell_2$,

$$Fu := \left\{ \left\langle u, \alpha_{n,s} \sin \sqrt{\lambda_{n,s}} t \right\rangle_{H(T)} \right\}_{n,s}$$

is continuous and onto, further it is isomorphic restricted to the orthocomplement of Ker F, which is the closed linear hull of all $\alpha_{n,s} \sin \sqrt{\lambda_{n,s}} t$. In case $M \geqq N$ this is in contradiction with the estimates of the form

$$\left\| \sum_{s=1}^{M} c_s \alpha_{2,s} e^{i2n_s\pi x/L_s} \right\|_{H(T)} < c \cdot \varepsilon, \qquad \sum_{s=1}^{M} c_s^2 = 1$$

proved in Theorem 1, hence for $M \geqq N$ the system is not controllable. If $M < N$ the controllability can be proved quickly as in Theorem 1a). Remark that the movement of the strings does not remain in the class (11). Formula (8) shows only that

$$c_{n,s} + i \frac{c'_{n,s}}{\sqrt{\lambda_{n,s}}}$$

is in the complex valued ℓ_2 space, but if we relax successively the strings, we can control also this larger class of movement states.

COROLLARY 1. *The system* (1)–(3) *with the modification* (11) *is controllable only in the case of trees.*

REMARK. Rolewicz allowed in [1] that the relaxation time $T < \infty$ may depend on the initial conditions. We continue our investigations in this direction.

Introduce some notions connected with the above statements.

DEFINITON 2. The system of strings is

a) controllable in (unbounded) finite time, if for any initial conditions (y_s^0, y_s^1) there exists $T < \infty$ and a control $u \in H(T)$ relaxing the system in time T, i.e. $y_s(T,\cdot) = \frac{d}{dt}y_s(T,\cdot)$ for $s = 1,\dots,M$;

b) approximately controllable in time T if given any initial conditions (y_s^0, y_s^1) from a fixed dense subset of $\bigoplus_{s=1}^{M} W_s$ we can find $u \in H(T)$ relaxing the system in time T;

c) approximately controllable in (unbounded) finite time if for any initial conditions (y_s^0, y_s^1) from a dense subset of $\bigoplus_{s=1}^{M} W_s$ we can find $T < \infty$ and $u \in H(T)$ relaxing the system in time T.

THEOREM 2. 1) *If the graph is not a tree, then the system of strings is never controllable in the sense of* 2a).

2) *The approximate controllability property of the system in the sense of* 2b) *and* 2c) *are equivalent and it does not hold if and only if we can find equal values* $\lambda_{n,s}$, *for which the corresponding vectors* $\alpha_{n,s}$ *are dependent.*

In the proof we return to the moment space

$$R(T) := \left\{ \left(\int_0^T \left\langle \alpha_{n,s} e^{i\sqrt{\lambda_{n,s}}t}, u(t) \right\rangle dt \right)_{n=1,s=1}^{\infty\ \ M} : u \in H(T) \right\}.$$

From (8) we see that the controllability properties can be described as above:

2a) $\Leftrightarrow \bigcup_{T<\infty} R(T) = \ell_2$,

2b) $\Leftrightarrow R(T)$ is dense in ℓ_2,

2c) $\Leftrightarrow \bigcup_{T<\infty} R(T)$ is dense in ℓ_2.

Introduce the notation

$$c(u) := \left(\int_0^T \left\langle \alpha_{n,s} e^{i\sqrt{\lambda_{n,s}}t}, u(t) \right\rangle dt \right)_{n=1,s=1}^{\infty\ \ M}$$

and consider the isomorphism

$$(c(u_1), c(u_2)) \to (c(u_1) + ic(u_2), \overline{c(u_1) - ic(u_2)})$$

in the space $\ell_2 \oplus \ell_2$ which we consider as the space ℓ_2 with twice as many complex coordinates. Since we have

$$c(u_1)+ic(u_2)=\left(\int_0^T\left\langle \alpha_{n,s}e^{i\sqrt{\lambda_{n,s}}t},u_1(t)+iu_2(t)\right\rangle dt\right),$$

$$\overline{c(u_1)-ic(u_2)}=\left(\int_0^T\left\langle \alpha_{n,s}e^{-i\sqrt{\lambda_{n,s}}t},u_1(t)+iu_2(t)\right\rangle dt\right),$$

hence the properties 2a), 2b), 2c) are equivalent to the statements: $\bigcup_{T<\infty} R_1(T)=\ell_2$, $R_1(T)$ is dense in ℓ_2, and $\bigcup_{T<\infty} R_1(T)$ is dense in ℓ_2, resp., where

$$R_1(T):=\left\{\left(\int_0^T\left\langle \alpha_{n,s}e^{\pm i\sqrt{\lambda_{n,s}}t},u(t)\right\rangle dt\right)_{n=1,s=1}^{\infty\ \ M}:u\in L^2(0,T;\mathbf{C}^N)\right\}.$$

We show that the sets $R_1(T)$ stop growing in a finite time, i.e. we have

LEMMA 2. *There exists $T_0<\infty$ satisfying $R_1(T_0)=R_1(T)$ for all $T>T_0$.*

PROOF. Let

$$\omega_{n,s}:=\sqrt{\lambda_{n,s}}+i,\qquad \sigma:=\{\omega_{n,s}:n=1,2,\dots,\ s=1,\dots,M\}.$$

We shall prove that for large $T_0<\infty$ the operator

$$P_{T_0}:L^2(0,\infty;\mathbf{C}^N)\to L^2(0,T_0;\mathbf{C}^N)$$

restricting the functions to $(0,T_0)$, maps isomorphically the system

$$\bigvee_{(0,\infty)}\left(\alpha_{n,s}e^{\pm i\omega_{n,s}t}:\omega_{n,s}\in\sigma\right)$$

into $L^2(0,T_0;\mathbf{C}^N)$; in fact we show that for large T_0 it maps isomorphically the larger subspace

$$\bigvee_{(0,\infty)}\left(\mathbf{C}^N e^{\pm i\omega_{n,s}t}:\omega_{n,s}\in\sigma\right).$$

This last statement can be reformulated as follows. The one-dimensional operator

$$P_{T_0}:L^2(0,\infty)\to L^2(0,T_0)$$

maps isomorphically the set

$$\bigvee_{(0,\infty)}\left(e^{\pm i\omega_{n,s}t}:\omega_{n,s}\in\sigma\right)$$

into $L^2(0,T_0)$. We shall prove the existence of an exponential type entire function $F(z)$ whose zeros are the values $\omega_{n,s}$ and finitely many further zeros $\mu_1,\dots,\mu_R$ and $|F(x)|\asymp 1$, $x\in\mathbf{R}$. As it is well-known from the theory of scalar exponential bases [3, 6], the operator P_{T_0} maps isomorphically

$$\bigvee_{(0,\infty)}\left(e^{i\mu_1 t},\dots,e^{i\mu_R t},e^{\pm i\omega_{n,s}t}:\omega_{n,s}\in\sigma\right)$$

onto the whole $L^2(0,T_0)$ where $T_0<\infty$ is defined as the length of the indicator diagram of F ([5]). Consider the partition

$$\{1,2,\dots,M\}=\bigcup_{j=1}^{R}S_j$$

induced by the equivalence relation

$$s_1\sim s_2\Leftrightarrow L_{s_1}/L_{s_2}\in\mathbf{Q}.$$

Denote further

$$\sigma^{(s)}:=\left\{\frac{\pi r}{L_s}+i:r\in\mathbf{Z}\setminus\{0\}\right\},\qquad \sigma_j:=\{\sigma^{(s)}:s\in S_j\}.$$

Then we have

$$\sigma=\bigcup_{j=1}^{R}\sigma_j.$$

We can easily find an entire function whose zero set is $\sigma^{(s)}$. For $s\in S_j$ the sets $\sigma^{(s)}$ are not disjoint. Let $s_1,s_2,\dots,s_k\in S_j$, then $\sigma^{(s_1)}\cap\dots\cap\sigma^{(s_k)}$ is an arithmetic progression without the number i. Therefore for appropriate $\mu\in\mathbf{R}$ the zero set of the function

$$f_{(s_1,\dots,s_k)}(z):=\frac{\sin\mu(z-i)}{z-i}$$

is $\sigma^{(s_1)}\cap\dots\cap\sigma^{(s_k)}$ without multiple zeros. Let

$$f_k:=\prod_{\substack{s_1,\dots,s_k\\ s_1<\dots<s_k}}f_{(s_1,\dots,s_k)}$$

then the function

$$F_j:=\frac{f_1f_3f_5\dots}{f_2f_4f_6\dots}$$

will be of exponential type with zero set σ_j and with simple zeros. From the estimate

$$|f_k(x)| \asymp (1+|x|)^{-\binom{n}{k}}$$

we obtain by

$$1 = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \dots$$

that

$$|F_j(x)| \asymp \frac{1}{1+|x|} \qquad (x \in \mathbf{R}).$$

Let $\mu_1, \dots, \mu_k \in \mathbf{C}_+ \setminus \sigma$ be arbitrary different numbers. Then the exponential type entire function

$$F(z) := \prod_{j=1}^{R} (z - \lambda_j) F_j(z)$$

satisfies the above formulated conditions, consequently P_{T_0} maps isomorphically $\bigvee_{(0,\infty)} \left(\alpha_{n,s} e^{\pm i\omega_{n,s} t} : \omega_{n,s} \in \sigma\right)$ into $L^2(0, T_0; \mathbf{C}^N)$. This implies

$$R_1(T_0) = R_1(\infty) =$$

$$= \left\{ \left(\int_0^\infty \left\langle \alpha_{n,s} e^{\pm i\omega_{n,s} t}, u(t) \right\rangle dt \right) : \omega_{n,s} \in \sigma,\ \ u \in L^2(0, \infty; \mathbf{C}^N) \right\}.$$

Now take the function

$$\hat{F}(z) := F(z) \sin \mu(z - i + 1)$$

instead of $F(z)$, where $\mu > 0$ is not commensurable with any of the numbers L_s and among the zeros of $\sin \mu(z-i+1)$ the values $\mu_1, \dots, \mu_R$ do not occur. The length of the indicator diagram of $\hat{F}$ is $T_0 + \mu$, hence repeating the above proof we obtain

$$(12) \qquad R_1(T) = R_1(\infty) \quad \text{for every} \quad T \geqq T_0,$$

which completes the proof of Lemma 2.

Returning to the proof of Theorem 2, the controllability in the sense 2a) would mean that $\bigcup_{T<\infty} R_1(T) = \ell_2$ i.e. $R_1(T_0) = \ell_2$; but this is true only for systems whose graph is a tree. We see also easily that 2b) and 2c) are equivalent and both mean exactly the density of $R_1(\infty)$ in ℓ_2. Clearly, if we have linear dependence among some vectors $\alpha_{n,s}$ where the corresponding values $\lambda_{n,s}$ are identical, then this implies dependence among the corresponding coordinates in $R_1(\infty)$ and hence $R_1(\infty)$ can not be dense. Conversely, suppose that we have no such dependence among the vectors $\alpha_{n,s}$. Fix an $\varepsilon > 0$ and take disks with center $\omega_{n,s}$ and radius ε. Take the topological connectedness components of the union of these disks. The centers of the disks belonging to the same components give a partition

$$\sigma = \cup\sigma_{(k)}.$$

The "theorem of making blocks" [6] states that for sufficiently small $\varepsilon > 0$ the subspaces

$$H_k := \text{Lin}\left\{e^{i\omega_{n,s}t} : \omega_{n,s} \in \sigma_{(k)}\right\}$$

form an L-basis in $L^2(0,\infty)$. Consequently the system $\{\mathbf{C}^N H_k : k \in \mathbf{Z}\}$ is L-basis in $L^2(0,\infty;\mathbf{C}^N)$ and then à fortiori the system

$$\hat{H}_k := \text{Lin}\left\{\alpha_{n,s}e^{i\omega_{n,s}t} : \omega_{n,s} \in \sigma_{(k)}\right\}$$

is also L-basis. Therefore there exists a system $\{Y_k\}$ of subspaces biorthogonal to $\{H_n\}$ in

$$\bigvee_{(0,\infty)} \left(\alpha_{n,s}e^{i\omega_{n,s}t} : \omega_{n,s} \in \sigma\right).$$

This means that $H_k \perp Y_\ell$ $(k \neq \ell)$, $\dim Y_k = \dim H_k$. Take the elements $\alpha_{n,s}e^{i\omega_{n,s}t}$ generating H_k. Their orthogonal projection onto Y_k give a basis in Y_k hence there exists a system $\{y_{n,s}\} \subset Y_k$ biorthogonal to them. We unify the systems $\{y_{n,s}\}$ so constructed for all $\sigma_{(k)}$ to see that

$$\int_0^T \left\langle \alpha_{n,s}e^{i\omega_{n,s}t}, y_{n',s'}(t)\right\rangle dt = \delta_{n,n'} \cdot \delta_{s,s'},$$

hence we gave a system biorthogonal to Φ. But this implies at once the density of $R_1(\infty)$ in ℓ_2 since if $u(t)$ runs over the finite linear combinations of the $y_{n,s}$, the moment sequences run over all finite sequences. Theorem 2 is proved. Remark, that in this paper we have used the ideas of the works [7–11].

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$*$-INDEPENDENT SUBSETS IN MODULAR LATTICES OF BREADTH TWO

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As a weaker version of weak independence (see G. Czédli, A. P. Huhn and E. T. Schmidt [1]) the notion of $*$-independence was introduced by G. Czédli in [2]. Let L be a lattice and let H be a subset of L. H is called $*$-independent if whenever $h, h_1, \ldots, h_k \in H$ and $h_1 \vee \ldots \vee h_k = h$ then $h = h_i$ for some $1 \leqq i \leqq k$. A maximal $*$-independent subset is called a $*$-basis of L. If L is of finite length then any maximal chain and $J(L)$, the set of all join-irreducible elements are $*$-bases of L. It was proved in [2] that if L is a finite distributive lattice and H is a $*$-basis of L then H has at least as many elements as any maximal chain, i.e., $|H| \geqq \ell(L) + 1$ where $\ell(L)$ is the length of L. The authors also asked whether the same was true for modular lattices. Here we give a partial answer for modular lattices of breadth two. Let us recall that the breadth $b(L)$ of a lattice L is the least natural number n with the property that for each $X \subseteq L$ there is a subset X' of X such that $\vee X = \vee X'$ and $|X'| \leqq n$. Thus $b(L) \leqq 2$ means that any join $a_1 \vee \ldots \vee a_k$ equals $a_i \vee a_j$ for some $1 \leqq i, j \leqq k$.

THEOREM. *Let L be a modular lattice of finite length and of breadth at most two. Then for any $*$-basis H of L we have $|H| \geqq \ell(L) + 1$.*

PROOF. If H is infinite then we are done. Suppose H is finite and let C be a maximal chain in L. Since $H \cup C$ is finite from C. Herrmann [4] we know that L', the sublattice generated by $H \cup C$ is finite as well. Clearly, H is a $*$-basis in L', moreover, $\ell(L') = \ell(L)$ whence it is enough to consider finite lattices. We will proceed by induction on $|L|$.

OBSERVATION 1. If for the pairwise distinct $a, b_1, b_2, b_3 \in L$ we have $a \prec b_1, b_2, b_3$ then the sublattice generated by $\{b_1, b_2, b_3\}$ is isomorphic to the five-element non-distributive modular lattice M_3 (the notation $a \prec b$ means a is covered by b i.e. $a < b$ and if $a \leqq c \leqq b$ then $a = c$ or $b = c$).

Indeed, the well-known Interval Isomorphism Theorem (see Grätzer [3]) implies $b_i \vee b_j \succ b_i, b_j$ if $i \neq j$. If $b_1 \vee b_2$, $b_1 \vee b_3$, $b_2 \vee b_3$ are pairwise incomparable then $b_1 \vee b_2$, $b_1 \vee b_3$, $b_2 \vee b_3$ generate an 8-element Boolean lattice (cf. Grätzer [3]) which is of breadth 3 whence this case is impossible.

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If e.g. $b_1 \vee b_2 \geqq b_2 \vee b_3$ then by the above covering relations we must have $b_1 \vee b_2 = b_2 \vee b_3$ but then the sublattice generated by $\{b_1, b_2, b_3\}$ is M_3.

OBSERVATION 2. If $c \in J(L)$ and $c \prec b, b'$ with $b \neq b'$ then $b \in J(L)$ or $b' \in J(L)$.

Indeed, if $b, b' \notin J(L)$ then there exist $x, x' \neq c$ such that $x \prec b$ and $x' \prec b'$. The Interval Isomorphism Theorem gives $x \wedge c \prec x, c$ and $x' \wedge c \prec x', c$. As $c \in J(L)$ we must have $x \wedge c = x' \wedge c = e$ and $e \prec x, x', c$ where e is the unique lower cover of c. Now by Observation 1 we get $b = b'$, a contradiction.

Let us consider

$$0 = c_0 \prec c_1 \prec \ldots \prec c_k,$$

a chain of join-irreducible elements which is maximal in the sense that if $c_k \prec c$ for some c then c is not join-irreducible. Notice that c_k must be meet-irreducible by Observation 2. We distinguish four cases the first three of which are the trivial ones.

Case 1: $c_k = 1$, the greatest element of L, i.e. c_k has no upper cover at all. Now $L - \{c_k\}$ is a sublattice of L and $H - \{c_k\}$ is a $*$-basis of $L - \{c_k\}$. Also, $\ell(L - \{c_k\}) = \ell(L) - 1$ and $|H - \{c_k\}| = |H| - 1$ (obviously, $c_k \in H$) and the assertion follows by induction.

Case 2: c_k has a unique upper cover and $c_k \notin H$. Since $c_k \notin H$ and c_k is both join- and meet-irreducible there must be a maximal chain not containing c_k. This means the length of the sublattice $L - \{c_k\}$ is $\ell(L)$ and since H is a $*$-basis in $L - \{c_k\}$ as well the assertion follows by induction.

Case 3: c_k has a unique upper cover b and $c_k \in H$ but $b \notin H$. We have two possibilities:

(i) $b = h_1 \vee h_2$ for some $h_1, h_2 \in H - \{c_k\}$ and $h_1, h_2 < b$.

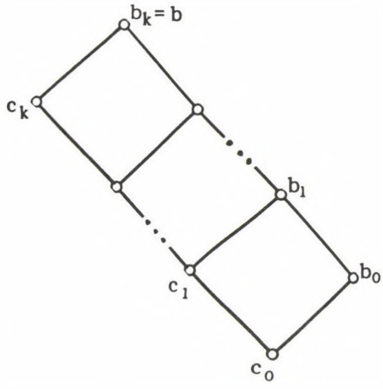
(ii) The above h_1 and h_2 do not exist.

In case (i) $H - \{c_k\}$ is a $*$-basis of $L - \{c_k\}$. This can be shown easily by observing that in any non-trivial join (i.e. when the elements to be joined are strictly less than their join) containing c_k we can replace c_k by $h_1 \vee h_2$.

In case (ii) $(H - \{c_k\}) \cup \{b\}$ is a $*$-basis of $L - \{c_k\}$. Similarly to case (i), this can be seen by observing that in non-trivial joins c_k and b can be replaced by each other.

Since $b \notin J(L)$ some maximal chain of L avoids c_k whence $\ell(L - \{c_k\}) = \ell(L)$. Then the assertion follows by induction as before.

Case 4: Both c_k and b belong to H. Let $b_0 \neq c_1$ such that $0 \prec b_0$ (if b_0 does not exist then $0 \in H$ is meet-irreducible and we can induct with $L - \{0\}$ and $H - \{0\}$). Further define $b_i = c_i \vee b_{i-1}$ for $i = 1, \ldots, k$. Using again the Interval Isomorphism Theorem and join-irreducibility of c_i we get the following diagram where, as usual, the line segments denote covering relations:



Suppose now that $H' \supseteqq H - \{c_k\}$ is a $*$-basis in $L - \{c_k\}$ and let $h' \in \in H' - H$. We claim that this can happen in two ways: either $h' = b_i$ for some i or $c_{i-1} \prec h' \prec b_i$ for some i. To see this assume $h' \neq b_i$ $(i = 0, \dots, k)$. Since $h' \notin H$ and H was a maximal $*$-independent subset there must be some $g \in H$ with $h' = c_k \vee g$ and $c_k, g < h'$ or $g = c_k \vee h'$ and $c_k, h' < g$. Since c_k is meet-irreducible if we replace c_k by b_k the equations remain valid. Then $*$-independence of H' gives $h' = b_k$ or g and $g = b_k$ or h', respectively. Taking into account $h' \notin H$ we are left with the only possibility $g = b_k$. Let i be the minimal index such that $h' \leqq b_i$ and let j be the maximal index such that $h' \geqq c_j$. We show $c_i \vee h' = b_i$ and $c_i \wedge h' = c_j$. That $c_i \leqq c_i \vee h' \leqq b_i$ is clear and $c_i = c_i \vee h'$ would imply $h' \leqq c_i \leqq c_k$ however we have $b_k = c_k \vee h'$. Thus the first equation follows from $c_i \prec b_i$.

Any meet involving any of $c_0, \dots, c_k$ equals one of $c_0, \dots, c_k$ since these elements cover each other and are join-irreducible. Hence, the second equation is immediate.

The Interval Isomorphism Theorem gives $c_j \prec h'$ and Observation 1 says $\{c_j, c_{j+1}, h', b_j\}$ generates M_3. This means $i = j + 1$ and $c_j \prec h' \prec b_{j+1}$.

Notice that we also have $h' \in J(L)$. This is trivial if $j = 0$. If $j > 0$ then $b_j \notin J(L)$ and $c_j \prec h', b_j$ which give $h' \in J(L)$ by Observation 2.

On the other hand, if $h' = b_i$ for some i then $c_i \in H$. Suppose not. Then there must be $g, g' \in H$ such that $g' = c_i \vee g$ and $c_i, g < g'$. Let C be a maximal chain between c_i and g'. Suppose that x is the least element in C which is different from all c_j and y is the least element in C with $y \notin J(L)$. Further, let $z \in C$ with $z \prec y$ and let $c_j \in C$ such that $c_j \prec x$.

First observe that $x = b_j$ cannot occur. If $x = b_j$ then $c_i \leqq h' = b_i \leqq b_j \leqq g'$ would give $g \vee h' = g'$ which in turn would give $g = g'$ or $h' = g'$ by $*$-independence of H'. However both are impossible since $g < g'$ and $h' \notin H$. Our Observations also give $c_{j+1} \vee x = b_{j+1}$ and $x \in J(L)$. One consequence is $z \geqq x$. Since all $z' \in C$ with $z' \leqq z$ are join-irreducible we have $z \wedge c_{j+1} = c_j$. The Interval Isomorphism Theorem gives now $z \prec c_{j+1} \vee z$. By Observation 2, the only join-reducible cover of z is y whence $c_{j+1} \vee z = y$.

This together with $x \leqq z$ and $c_{j+1} \vee x = b_{j+1}$ yield $b_{j+1} \leqq y$. Then by $b_i < b_{j+1}$ and $y \leqq g'$ we get $b_i < g'$. This implies $g' = b_i \vee g$ which contradicts the $*$-independence of H'.

If $|H' - H| \leqq 1$ then the assertion follows by induction. Let us suppose now $h' \neq h''$ and $h', h'' \in H' - H$. Using the information derived above we distinguish four subcases:

(i) $h' = b_i$, $h'' = b_j$ and $i < j$.

Then $c_j \in H'$ and $c_j \vee b_i = b_j$ contradicts the $*$-independence of H'.

(ii) $c_i \prec h' \prec b_{i+1}$ and $c_j \prec h'' \prec b_{j+1}$ with $i \leqq j$.

Then let $c_0 \prec \ldots \prec c_j \prec h'' \prec d_0 \prec \ldots \prec d_n$ be a maximal chain of join-irreducibles. We may suppose d_n has a unique upper cover e with $d_n, e \in H$ otherwise we could induct as in case 1 or 2 or 3. But then a similar argument to those above shows $b_{j+1} \wedge d_n = h''$ whence $b_{j+1} \vee d_n = e$ which gives $h' \leqq e$. This and $h' \nleqq d_n$ yield $h' \vee d_n = e$, contradicting the $*$-independence of H'.

(iii) $h' = b_i$, $c_j \prec h' \prec b_{j+1}$ with $i \leqq j$.

This case can be handled as (ii).

(iv) $h' = b_i$, $c_j \prec h' \prec b_{j+1}$ with $i > j$.

This case is essentially the same as (i).

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A CONSISTENT EDGE PARTITION THEOREM FOR INFINITE GRAPHS

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0. Introduction

The fundamental problem of partition theory of infinite graphs is if for every graph Y and cardinal μ there exists a graph X such that if the vertices (or edges) of X are colored with μ colors then there is a copy of Y with all the vertices (edges) getting the same color. This is denoted as $X \to (Y)^1_\mu$ and $X \to (Y)^2_\mu$; if these statements fail, then, of course, the arrow is crossed. Let $K(\alpha)$ denote the complete graph on α vertices, and let $K(\alpha) \leqq X$ denote that the graph contains $K(\alpha)$ as subgraph. If κ is an infinite cardinal, then obviously $K(\kappa^+) \to K(\kappa^+)^1_\kappa$, and by the Erdős–Rado theorem [6], $K((2^\kappa)^+) \to (K(\kappa^+))^2_\kappa$, and this result gives the existence of X for any Y, μ.

To make the problem harder, one might require the copy to be *induced*. This relation is denoted as $\rightarrowtail$. Though the vertex problem is still fairly easy, the edge case even for finite X, μ was only solved around 1973 by Deuber, Nesetril–Rödl, and Erdős–Hajnal–Pósa [1, 11, 5]. The latter authors even showed that for μ finite, Y countable, there is an appropriate X. Hajnal and Komjáth [9] proved that it is consistent that there exists a Y of size $\aleph_1$ such that no X (of any size) has $X \rightarrowtail (Y)^2_2$. Shelah [14] proved that it is consistent that for any Y, μ there is an X with $X \rightarrowtail (Y)^2_\mu$. Hajnal recently proved [8] that if Y is finite, μ is infinite, an appropriate X exists, in ZFC.

Another way of making the problem harder is to pose restrictions on X. We may require that if $K(\alpha) \nleqq Y$, then $K(\alpha) \nleqq X$, either. This excludes the possibility of getting an easy solution by using the above-mentioned Erdős–Rado theorem. For finite X, μ, Folkman showed the existence of such an X with $X \to (Y)^1_\mu$ and also, for finite α, μ the existence of a finite X with $K(\alpha+1) \nleqq X \to (\alpha)^2_\mu$. [7]. Nesetril and Rödl solved the edge case, for finite Y, μ [12]. The infinite case for vertices, but if α is finite, was solved by Komjáth and Rödl [10]. The case of general α is given by Hajnal and Komjáth [9]. As for the edge coloring, Hajnal and Komjáth proved in [9] that it is consistent that there is a Y of size $\aleph_1$, with $K(3) \nleqq Y$ and if $X \to (Y)^2_\omega$ then $K(\omega) \leqq X$. It was an old problem of Erdős and Hajnal if a graph Y

with $K(4) \nleqq Y \to K(3)^2_\omega$ exists. S. Shelah in [14] proved that such a Y may consistently exist. Another old Erdős–Hajnal question was if a Y with $K(\omega_1) \nleqq Y \to (K(\omega))^2_\omega$ may exist. Here we solve (at least consistently) this problem by showing the consistency of the statement that if Y is a graph, μ a cardinal, then there exists a graph X with $X \rightarrowtail (Y)^2_\mu$ and if $K(\alpha) \nleqq Y$ then $K(\alpha) \nleqq X$, either.

We first show that if $2^\mu = \mu^+$, $\kappa > \mu$ is measurable, Y is a graph on μ, then there is a $\leqq \mu^+$-closed poset of size κ, adding a graph X on κ as above. From this, we can get the general result, if we assume that $\{\kappa_\alpha : \alpha \text{ ordinal}\}$ is a class of measurable cardinals, and take the iteration $\{P_\alpha, Q_\alpha : \alpha \text{ ordinal}\}$ of posets, where Q_α is the poset of Theorem 1 with $\mu = \kappa_\alpha^+$, $\kappa = \kappa_{\alpha+1}$, and Y is some graph on μ. We take inverse limits at singular ordinals, direct limits otherwise. This will guarantee enough closure properties for getting a model of ZFC, and for that the graphs preserve their partition property at later iterations.

1. The consistency proof

THEOREM 1. *If $2^\mu = \mu^+$, Y is a graph on μ, $\kappa > \mu$ is a measurable cardinal, then there exists a $\leqq \mu^+$-closed partial order P, $|P| = \kappa$, adding a graph X such that $X \rightarrowtail (Y)^2_\mu$, and whenever $K(\alpha) \leqq X$, then $K(\alpha) \leqq Y$.*

PROOF. The vertex set of X will be $[\kappa]^2$. We define a partial ordering $<$ on it by putting $\{\beta_0, \alpha_0\}_< < \{\beta_1, \alpha_1\}_<$ iff $\beta_0 < \beta_1$ and $\alpha_0 < \alpha_1$. A condition is of the form $p = (s, g, \varphi)$ where $s \subseteqq [\kappa]^2$, $|s| \leqq \mu^+$, $g \subseteqq [s]^2$. If $\{\{\beta_0, \alpha_0\}_<, \{\beta_1, \alpha_1\}_<\} \in g$, then either $\beta_0 < \beta_1 < \alpha_0 < \alpha_1$ or $\beta_1 < \beta_0 < \alpha_1 < \alpha_0$. φ is a function with $\mathrm{Dom}(\varphi) = \{A \leqq s : |A| > 2, [A]^2 \subseteqq g\}$. For $A \in \mathrm{Dom}(\varphi)$, $\varphi(A) \leqq \mu$ spans a complete graph in Y, $|\varphi(A)| = |A|$. We also require that if B properly end-extends A, then $\varphi(B)$ should properly end-extend $\varphi(A)$.

Condition $p' = (s', g', \varphi')$ extends $p = (s, g, \varphi)$ if $s' \supseteqq s$, $g = [s]^2 \cap g'$, $\varphi' \supseteqq \varphi$, and if $A \subseteqq s$, $|A| > 2$ spans a complete graph in g, $x \in s' - s$, and $A \cup \{x\}$ is complete in g', then $A < x$, i.e. $y < x$ holds for every $y \in A$.

Let P be the set of conditions defined so far.

LEMMA 1. *$(P, \leqq)$ is transitive.*

PROOF. Straightforward.

LEMMA 2. *If $p = (s, g, \varphi) \in P$, $A \subseteqq \kappa$, then $p \mid A \in P$. If $A \cap s$ is an initial segment in s, then $p \leqq p \mid A$.*

PROOF. Immediate from the definitions.

LEMMA 3. *$(P, \leqq)$ is $\leqq \mu^+$-closed.*

PROOF. Assume that $p_\xi = (s_\xi, g_\xi, \varphi_\xi)$ is a decreasing, continuous sequence of conditions ($\xi < \xi_0 \leqq \mu^+$). Take $p = (s, g, \varphi)$, where $s = \bigcup\{s_\xi : \xi < < \xi_0\}$, $g = \bigcup\{g_\xi : \xi < \xi_0\}$, and whenever $A \subseteqq s$ spans a complete subgraph in

g, $|A| > 2$, then $\varphi(A) = \cup\{\varphi_\xi(A \cap s_\xi): \xi < \xi_0, |A \cap s_\xi| > 2\}$. For $\xi < \zeta < \xi_0$, $\varphi_\zeta(A \cap s_\zeta)$ end-extends $\varphi_\xi(A \cap s_\xi)$, so $\varphi(A)$ induces a complete subgraph in Y, and $|\varphi(A)| = |A|$. If B end-extends A, then select $\xi < \xi_0$ with $s_\xi \cap \cap (B - A) \neq \emptyset$. By the definition of order on P, $A \subseteqq s_\xi$, so $\varphi(A) = \varphi_\xi(A)$ and $\varphi(B)$ end-extends $\varphi_\xi(B \cap s_\xi)$ which in turn end-extends $\varphi(A)$. To check $p \leqq p_\xi$ $(\xi < \xi_0)$, the only nontrivial thing is the clause on $A \cup \{x\}$. If $A \in \in \mathrm{Dom}(\varphi_\xi)$, $x \in s - s_\xi$, we can assume that $x \in s_{\xi+1} - s_\xi$, so $A < x$, and we are done.

LEMMA 4. *If $p_i = (s_i, g_i, \varphi_i)$ are conditions for $i < 2$, they agree on $s_0 \cap s_1$, then $q = (s_0 \cup s_1, g_0 \cup g_1, \varphi_0 \cup \varphi_1)$ is a condition. If $s_0 \cap s_1 < < (s_0 - s_1) \cup (s_1 - s_0)$, then $q \leqq p_0, p_1$.*

PROOF. Straightforward.

If $G \subseteqq P$ is a generic subset, we let $X = \cup\{g: (s, g, \varphi) \in G\}$.

LEMMA 5. *If $K(\alpha) \leqq X$ for some α, then $K(\alpha) \leqq Y$.*

PROOF. $K(\mu + 1) \nleqq X$, as if $A \subseteqq [\kappa]^2$ spans a complete graph of type $\mu + 1$, pick $p = (s, g, \varphi) \in G$ fixing A. This is possible by Lemma 3. But then, $\varphi(A)$ would give a $K(\mu+1)$ in Y, a contradiction. If $K(\alpha) \leqq X$, $\alpha \leqq \mu$, argue similarly.

In order to finish the proof of Theorem 1, assume without loss of generality that $1 \Vdash F: X \to \mu$. By Fact 2.4 in [14] there is a set A of measure one, $\{N_s: s \in [A]^{<\omega}\}$ such that

(1) $N_s \prec (H(2^\kappa); \in, F, \Vdash, \dots)$;
(2) $[N_s]^{\mu^+} \subseteqq N_s$;
(3) $|N_s| = 2^{\mu^+}$;
(4) $N_{s_0} \cap N_{s_1} = N_{s_0 \cap s_1}$;
(5) there is an isomorphism $H(N_{s_0}, N_{s_1})$ between N_{s_0} and N_{s_1} for $|s_0| = |s_1|$, mapping s_0 onto s_1;
(6) $N_s \cap A = s$;
(7) if s_0 is end-extended to s_1, then N_{s_0} is end-extended by N_{s_1}.

Let $A' \subseteqq A$ be a set of indiscernibles for $\{N_s: s \in [A]^{<\omega}\}$. Enumerate the first $\mu 2$ elements of A' in increasing order as $\{\beta(i) : i < \mu\} \cup \{\alpha(i) : i < \mu\}$. Put $t(i) = \{\beta(i), \alpha(i)\}$, $M_i = N_{t(i)}$ for $i < \mu$.

DEFINITION. For $p, q \in P$, $p \sim q$ denotes that $p \mid N_\emptyset = q \mid N_\emptyset$.

LEMMA 6. *If $p(i) \in M_i$, $p(j) \in M_j$, $p(i) \sim p(j)$, then $p(i)$, $p(j)$ are compatible.*

PROOF. By (4), the non-edge amalgamation works.

We next show that one-edge amalgamation can also be constructed.

DEFINITION. If $i < j < \mu$, $p(i) = (s(i), g(i), \varphi(i)) \in M_i$, $p(j) = = (s(j), g(j), \varphi(j)) \in M_j$, $p(i) \sim p(j)$, then put $p(i) + p(j) = (s, g, \varphi)$ with $s = s(i) \cup s(j)$, $g = g(i) \cup g(j) \cup \{\{t(i), t(j)\}\}$, $\varphi = \varphi(i) \cup \varphi(j)$.

LEMMA 7. *$p(i)+p(j)$ is a condition, extending both $p(i)$ and $p(j)$.*

PROOF. As $\beta(i) < \beta(j) < \alpha(i) < \alpha(j)$, it is possible to join $t(i)$ and $t(j)$. As $\sup(N_\emptyset) < \beta(i) < \beta(j)$, $t(i)$ and $t(j)$ are not joined into $N_\emptyset$, so no new complete subgraph with more than two elements is formed.

DEFINITION. If $i < j < \mu$, $p(i) \in M_i$, $p(j) \in M_j$, $\xi < \mu$, we call the pair $(p(i), p(j))$ ξ-good, if $p(i) \sim p(j)$, and for every selection of $p'(i) \leqq p(i)$, $p'(j) \leqq p(j)$ with $p'(i) \in M_i$, $p'(j) \in M_j$, $p'(i) \sim p'(j)$, there is a $q \leqq p'(i)+p'(j)$ such that $q \Vdash F(\{t(i),t(j)\}) = \xi$.

LEMMA 8. *If $i < j < \mu$, $p(i) \in M_i$, $p(j) \in M_j$, $p(i) \sim p(j)$, then there exist $\xi < \mu$, $p'(i) \leqq p(i)$, $p'(j) \leqq p(j)$, $p'(i) \in M_i$, $p'(j) \in M_j$ such that $(p'(i), p'(j))$ is ξ-good.*

PROOF. Assume that the statement is false. Put $p(i,0)=p(i)$, $p(j,0)=$ $=p(j)$, and we are going to construct decreasing, continuous sequences $p(i,\xi)$, $p(j,\xi)$ for $\xi \leqq \mu$. If $p(i,\xi)$, $p(j,\xi)$ are defined, let $p(i,\xi+1) \sim p(j,\xi+1)$ be such that no $q \leqq p(i,\xi+1)+p(j,\xi+1)$ can force $F(\{t(i),t(j)\}) = \xi$. If $q \leqq p(i,\mu)+p(j,\mu)$ determines $F(\{t(i),t(j)\})$, then we get a contradiction.

By transfinite recursion on $\alpha < \mu^+$, we select, for every $f\colon \alpha \to 2$, a condition $p(i,f) \in M_i$, and an ordinal $\xi(f) < \mu$ such that

(8) $H(M_i,M_j)(p(i,f)) = p(j,f) \quad (i<j<\mu)$;

(9) $(p(i,f\hat{\ }0), p(j,f\hat{\ }1))$ is $\xi(f)$-good $(i<j)$;

(10) $p(i,f') \leqq p(i,f)$ when $f' \supseteqq f$;

(11) $p(i,f) \sim p(j,g)$ when $f,g\colon \alpha \to 2$, $i<j$.

For α limit, we can take unions. Given $\{p(i,f)\colon f\colon \alpha \to 2,\ i<\mu\}$ we select $p(i,f\hat{\ }0)$, $p(i,f\hat{\ }1)$ by a transfinite recursion of length $|2^\alpha| \leqq \mu^+$, using Lemma 8. To insure (11), we must keep extending $p(i,f) \mid N_\emptyset$, this can be done by Lemmas 3 and 4.

By the Baire category theorem, there exist $\xi < \mu$, and increasing $\tau_i < \mu^+$ $f_i\colon \alpha \to 2$ $(i<\mu)$ for some $\alpha < \mu^+$, such that

(12) $f_i(\tau_i) = 0$, $f_j(\tau_i) = 1$, $f_i \mid \tau_i \subseteqq f_j \mid \tau_j$ $(i<j)$;

(13) $\xi(f_i \mid \tau_i) = \xi$.

Put $Y = \{\{\delta(i),\varepsilon(i)\} : i<\mu\}$.

We are going to construct $q(\gamma,i)$ for $\gamma \leqq \mu$, $i<\mu$. Put $q(0,i) = p(i,f_i)$, for γ limit, $q(\gamma,i) = \bigcup\{q(\gamma',i)\colon \gamma' < \gamma\}$. If the construction is given, up to the γth level, let $u(\gamma) \in N_{t(i)\cup t(j)}$ be such that

$$u(\gamma) \leqq q(\gamma,\delta(\gamma)) + q(\gamma,\varepsilon(\gamma))$$

and $u(\gamma) \Vdash F(\{t(\delta(\gamma)), t(\varepsilon(\gamma))\}) = \xi$. We then take $q(\gamma+1,i) = q(\gamma,f) \cup$ $\cup\, u(\gamma) \mid M_i$.

LEMMA 9. *$u(\gamma)$ exists.*

PROOF. By Lemma 8 and by $q(\gamma,i) \sim q(\gamma,j)$. This latter property holds for γ limit by continuity, for $\gamma = 0$ by definition and (11), and for $\gamma+1$ by definition.

LEMMA 10. $q(\gamma+1,i) \leqq q(\gamma,i)$.

PROOF. By Lemma 4.

If $u(\gamma) = (s(\gamma), g(\gamma), \varphi(\gamma))$ for $\gamma < \mu$, then put $u = (s, g, \varphi)$ where $s = \cup \cup\{s(\gamma) : \gamma < \mu\}$, $g = \cup\{g(\gamma) : \gamma < \mu\}$, and φ is such that it extends all $\varphi(\gamma)$, and $\varphi(\{t(i) : i \in A\}) = A$, when $|A| > 2$, and A spans a complete subgraph in Y.

LEMMA 11. $u \in P$.

PROOF. It suffices to show that if $B \subseteqq s$, $|B| > 2$, spans a complete subgraph then it is either in the domain of some $\varphi(\gamma)$ or it is of the form $B = \{t(i) : i \in A\}$ for some $A \subseteqq \mu$.

If two M_i-s cover B, then one of them covers, too, or else $\{t(i), t(j)\} \subseteqq B$, but then $B \cap N_\emptyset = \emptyset$, so $B = \{t(i), t(j)\}$. If no two M_i-s cover B, then $B \subseteqq \{t(i) : i < \mu\}$, and we are done, again.

LEMMA 12. $u \leqq u(\gamma)$.

PROOF. There is no complete subgraph in u which is extended the wrong way. The only candidate for this is a set of type $\{t(i) : i \in A\}$ of which only two vertices are in $u(\gamma)$.

LEMMA 13. $u \Vdash \{t(i) : i < \mu\}$ *span a monocolored copy of* Y.

PROOF. Obvious.

Clearly, Lemma 13 concludes the proof of Theorem 1.

THEOREM 2. *If the existence of class many measurable cardinals is consistent, then it is consistent that for every* Y, μ *there exists an* X *with* $X \rightarrowtail \rightarrowtail (Y)^2_\mu$ *such that if* $K(\alpha) \leqq X$, *then* $K(\alpha) \leqq Y$.

PROOF. By iterating the poset in Theorem 1.

The assumption on the existence of measurables can be eliminated, see [14] Sections 3, 4.

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ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH SMALL DAMPING

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1. Introduction

In this paper we consider the linear differential equation

$$\ddot{x} + a(t)\dot{x} + x = 0, \tag{1}$$

where the function $a(t)$ is supposed to be nonnegative and piecewise continuous. The classical problem of finding conditions guaranteeing that every solution of (1) tends to zero as $t \to \infty$, has been the subject of a great number of publications [3]–[8] for equation (1) and recently also for more general nonlinear equations. There are rather sharp sufficient conditions, but the sharpness problem, that is to find necessary and sufficient conditions, has not been solved even for the linear equation (1). This problem is the subject of our paper.

It is known that if $a(t)$ is too small, i.e. $\int^{\infty} a < \infty$, the solutions are oscillatory and do not tend to zero as $t \to \infty$ [8]. If the condition

$$\int_0^{\infty} a = \infty \tag{2}$$

holds, then there exists a solution tending to zero [2], but examples show [4, 5] that there may be also solutions not converging to the equilibrium position. We note that if $a(t)$ is bounded, this solution is oscillatory, but if $a(t)$ is too large, such as t^2, it is monotone. To exclude the existence of monotone solutions not reaching zero, the condition

$$\int_0^{\infty} \frac{1}{A(t)} \int_0^t A(s)ds\, dt = \infty, \qquad \text{where} \qquad A(t) := \exp\left\{\int_0^t a(s)ds\right\}$$

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is necessary and sufficient [3, 6, 7, 8] provided $a(t)$ is bounded below by a positive constant. However, this is not the case for the equations with small damping, in other words, there is no necessary and sufficient condition to exclude the situation in which oscillatory solutions tending and not tending to zero exist simultaneously.

In this paper we consider the bounded damping case ($a(t)$ is bounded).

After general statements we give some types of criteria for the case of step-function damping and then we extend these results to the case of general damping. Finally, we show that condition (2) is "almost sufficient" in the following sense. If $\limsup_{t\to\infty} a(t) < 2$ and (2) holds then every solution of the equation

$$\ddot{x} + \frac{1}{2}(a(t) + a(t-\sigma))\dot{x} + x = 0$$

tends to zero as $t \to \infty$ with every sufficiently small σ.

2. General results

Using the polar transformation

$$x = R\cos\varphi, \qquad \dot{x} = -R\sin\varphi$$

we transform equation (1) into the system

$$\dot{\varphi} = 1 - \frac{1}{2}a(t)\sin 2\varphi, \qquad \dot{R} = -Ra(t)\sin^2\varphi. \tag{3}$$

Since $x^2(t) + \dot{x}^2(t) = R^2(t)$, we may prove $R(t) \to 0$ instead of $x(t) \to 0$ ($t \to \infty$). It is easy to see that

$$R(t) = R(0)\exp\left\{-\int_0^t a(s)\sin^2\varphi(s)ds\right\} > \frac{R(0)}{A(t)}. \tag{4}$$

Inequality (4) is of extreme importance. From (4) we can derive that the solutions can tend to zero not faster than the function $1/A(t)$. We may ask whether the solutions can reach this rate. The following theorem gives the answer.

THEOREM 1. *Equation* (1) *has a solution not tending to zero* ($t \to \infty$) *if and only if it has a solution for which* $R(t) = O(1/A(t))$ ($t \to \infty$).

The proof of this result is analogous to that of Atkinson's theorem [1] for the equation $\ddot{x} + q(t)x = 0$.

This theorem suggests that the case when solutions tending and not tending to zero exist together is only a "singular" situation. In the next part we try to characterize the nature of this "singularity" investigating the case of step-function damping.

3. The step-function damping

Let the functions a_n (steps) be defined by

$$a_n(t) = \begin{cases} \alpha_n & \text{if } t \in [0, i_n] \\ 0 & \text{if } t < 0, \text{ or } t > i_n \end{cases} \tag{5}$$

$(\alpha_n, i_n > 0)$ for $n = 1, 2, \ldots$. Let the sequence $\{s_n\}$ be given such that $s_0 \geqq 0$, $s_{n+1} \geqq s_n + i_n$. Now define

$$a(t) := \sum_{n=1}^{\infty} a_n(t - s_n) \tag{6}$$

and suppose

$$\sum_{n=1}^{\infty} \alpha_n i_n = \infty. \tag{7}$$

As in general, further conditions on $\{\alpha_n\}, \{i_n\}, \{s_n\}$ are needed making all solutions tend to zero.

In the following theorem, published earlier, the conditions will be independent of the distribution of the steps, i.e. of the sequence $\{s_n\}$.

THEOREM 2 [4]. *Suppose that $a(t)$ is bounded, and*

$$\sum_{n=1}^{\infty} \alpha_n i_n \min(1, i_n^2) = \infty \tag{8}$$

holds. Then every solution of (1) *tends to zero as $t \to \infty$.*

On the other hand, if

$$\sum_{n=1}^{\infty} \alpha_n i_n^3 < \infty, \tag{9}$$

then there exists a sequence $\{s_n\}$, $s_{n+1} > s_n + i_n$, $s_0 \geqq 0$, such that (1) *has at least one solution not tending to zero ($t \to \infty$).*

The first part of the theorem is valid for nonlinear equations as well. The proof of the second part gives a way to construct counterexamples ([4, 5]) and expresses the fact that if (9) holds, it is impossible to give distribution independent results. For example, if $\alpha_n = 1$, $s_n = n$, $i_n = 1/n$, the first part of the previous theorem cannot guarantee that the solutions tend to zero as $t \to \infty$.

In the following theorems conditions for the distribution of the steps are essential. By Theorem 2, if $\liminf_{n\to\infty} i_n > 0$ then (7) suffices the solutions to tend to zero. First we consider the case, when this condition is not satisfied.

THEOREM 3. *Suppose that there is an increasing sequence* $\{n_j\}$ *of natural numbers such that* $\lim_{j\to\infty}(i_{n_j}+i_{n_j+1})=0$ *is satisfied. If the sequence* $\{\max(\alpha_{n_j},\alpha_{n_j+1})\}$ *is bounded and the conditions*

$$\liminf_{j\to\infty}((s_{n_j+1}-s_{n_j}) \bmod \pi)>0, \tag{10}$$

$$\limsup_{j\to\infty}((s_{n_j+1}-s_{n_j}) \bmod \pi)<\pi, \tag{11}$$

$$\sum_{j=1}^{\infty}\min(\alpha_{n_j}i_{n_j},\alpha_{n_j+1}i_{n_j+1})=\infty \tag{12}$$

are satisfied, then every solution of (1) *tends to zero as* $t\to\infty$.

PROOF. Let us consider system (3). Since $R(t)$ is constant on the intervals (s_n+i_n,s_{n+1}), the right-hand side of the first equation is periodic in φ and $d\varphi/dt\equiv 1$, we can suppose that $s_{n+1}-s_n-i_n<\pi$. Let us consider the sequence $\{n_j\}$. If $k=n_j$ for some j large enough, we have

$$\delta \leqq s_{k+1}-s_k-i_k \leqq \varphi(s_{k+1})-\varphi(s_k+i_k),$$

and also

$$\varphi(s_{k+1}+i_{k+1})-\varphi(s_k) \leqq s_{k+1}-s_k-i_k+\frac{M}{2}(i_k+i_{k+1})<\pi-\delta,$$

with some positive δ where M is a bound for the sequence $\{\max(\alpha_{n_j},\alpha_{n_j+1})\}$. So the function $\sin^2\varphi$ has at most one zero on $[s_k,s_{k+1}+i_{k+1}]$ and there is a number $\beta>0$ for which

$$\min\Big(\min_{[s_k,s_k+i_k]}\sin^2\varphi(t),\min_{[s_{k+1},s_{k+1}+i_{k+1}]}\sin^2\varphi(t)\Big)>\beta>0,$$

for every large enough $k=n_j$ $(j>J)$, i.e. $\sin^2\varphi(t)$ is uniformly strictly positive at least on one of the associated neighbouring intervals. Now we get the following estimate for $R(t)$:

$$R(s_{n_L+1}+i_{n_L+1}) \leqq R(0)\exp\Big\{-\frac{1}{2}\sum_{j=J}^{L}\Big(\alpha_{n_j}\int_{s_{n_j}}^{s_{n_j}+i_{n_j}}\sin^2\varphi(u)du+$$

$$+\alpha_{n_j+1}\int_{s_{n_j+1}}^{s_{n_j+1}+i_{n_j+1}}\sin^2\varphi(u)du\Big)\Big\} \leqq K\exp\Big\{-\frac{1}{2}\beta\sum_{j=J}^{L}\min(\alpha_{n_j}i_{n_j},\alpha_{n_j+1}i_{n_j+1})\Big\}$$

where the right hand side tends to zero ($L \to \infty$). The theorem is proved. □

Conditions (10) and (11) say that the accumulation points of the sequence $\{s_{n_j+1} - s_{n_j}\}$ cannot be equal to any multiples of π. In the special case when the sequence $\{(s_{n+1} - s_n) \bmod \pi\}$ is convergent, we obtain the following result.

COROLLARY 4. *Suppose that* (7) *holds,* $\limsup_{n\to\infty} \alpha_n < \infty$, $\lim_{n\to\infty} i_n = 0$. *If*

$$(13) \qquad 0 < \lim_{n\to\infty} ((s_{n+1} - s_n) \bmod \pi) < \pi$$

then every solution of equation (1) *tends to zero as* $t \to \infty$.

Theorems 2 and 3 are independent. For the above mentioned example $s_n = n$, $i_n = 1/n$, $\alpha_n = 1$ condition (8) is not satisfied but Theorem 3 is applicable. On the other hand, if $s_n = n\pi$, $\alpha_n = 1$, $i_n = 1/n^{1/4}$, condition (8) holds but Corollary 4 is not applicable.

Theorem 3 has the following interesting consequence, which shows that a slight modification of the equation kills the solutions which do not tend to zero.

COROLLARY 5. *Suppose that* $a(t)$ *satisfies the conditions of Corollary* 4, *except* (13). *Then every solution of the equation*

$$\ddot{x} + \frac{1}{2}(a(t) + a(t-\sigma))\dot{x} + x = 0$$

tends to zero ($t \to \infty$) *for every* $0 < \sigma \neq j\pi$ ($j = 1, 2, \dots$).

Without assuming $i_{n_j} \to 0$ ($j \to \infty$) we can state the following

THEOREM 6. *Suppose that there is an increasing sequence* $\{n_j\}$ *of natural numbers such that the conditions*

$$(14) \qquad \liminf_{j\to\infty} ((s_{n_j+1} - s_{n_j} - i_{n_j}) \bmod \pi) > 0,$$

$$(15) \qquad \limsup_{j\to\infty} ((s_{n_j+1} - s_{n_j} - i_{n_j}) \bmod \pi) + \frac{1}{2}(\alpha_{n_j} i_{n_j} + \alpha_{n_j+1} i_{n_j+1}) < \pi$$

and (12) *hold. Then every solution of equation* (1) *tends to zero as* $t \to \infty$.

4. General damping

In this section we consider the more general equation

$$(16) \qquad \ddot{x} + h(t, x, \dot{x})\dot{x} + x = 0,$$

$h(t, x, y)$ is continuous and satisfies

$$(17) \qquad 0 \leqq a(t) \leqq h(t, x, y) \leqq b(t),$$

where $a(t)$, $b(t)$ are piecewise continuous on $[0, \infty)$.

Theorem 6 can be easily generalized to (16). The following result holds.

THEOREM 7. *Suppose that* (17) *holds and there exist sequences of intervals* $\{|s_n, s_n + i_n]\}, \{t_n, t_n + l_n]\}$ *such that* $0 \leqq s_0$, $s_n < s_n + i_n < t_n < < t_n + l_n < s_{n+1}$. *If*

$$\limsup_{n\to\infty} \{b(t) : t \in [s_n, t_n + l_n]\} < 2, \tag{18}$$

$$\liminf_{n\to\infty} (t_n - s_n - i_n) > 0, \tag{19}$$

$$\limsup_{n\to\infty} (t_n + l_n - s_n) \leqq \pi/2, \tag{20}$$

$$\sum_{n=1}^{\infty} \min\left(\int_{s_n}^{s_n+i_n} a, \int_{t_n}^{t_n+l_n} a \right) = \infty, \tag{21}$$

then every solution of (16) *tends to zero as* $t \to \infty$.

PROOF. Now we have the following estimates on the variations of $\varphi(t)$ if n is large enough:

$$0 < \delta_1 < \delta(t_n - s_n - i_n) < \varphi(t_n) - \varphi(s_n + i_n),$$

and

$$\varphi(t_n + l_n) - \varphi(s_n) < (2 - \delta)(t_n + l_n - s_n) < \pi - \delta_1$$

with some δ, δ_1. From here the proof can be finished similarly to that of Theorem 3. □

A straightforward application of the previous theorem gives the following generalization of Corollary 5, which especially shows that (2) is an "almost sufficient" condition for $R(t) \to 0$ $(t \to \infty)$ for every solution of equation (1).

THEOREM 8. *Suppose that* $\int^{\infty} a = \infty$, $\limsup_{t\to\infty} b(t) < 2$ *and* (17) *holds. Then every solution of the equation*

$$\ddot{x} + \frac{1}{2}(h(t, x, \dot{x}) + h(t - \sigma, x, \dot{x}))\dot{x} + x = 0 \tag{22}$$

tends to zero as $t \to \infty$ *for each* $0 < \sigma < \pi/2$.

PROOF. Let $0<\sigma<\pi/2$ be given. Let N be a natural number for which $\sigma/N<\pi/2-\sigma$. Let us now define the sets

$$K_k := \bigcup_{n=0}^{\infty} [(n\sigma+k)\sigma/N, (n\sigma+k+1)\sigma/N] \quad (k = 0, 1, \dots, N-1).$$

It is obvious that $\bigcup_{k=0}^{N-1} K_k = [0, \infty)$, and there is a K for which $\int_{K_k} a = \infty$. Now we can apply Theorem 7 to equation (22) with that K_k $(i_n, l_n = \sigma/N$, $s_n = (2nN + k)\sigma/N$, $t_n = ((2n + 1)N + k)\sigma/N)$. The theorem is proved. □

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ON ADDITIVE FUNCTIONS

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A classical theorem of P. Erdős [1] states that if a real-valued additive function is non-decreasing or $f(n+1)-f(n)\to 0$ as $n\to\infty$, then it must have the form $c\log n$ for some constant c. Both statements were generalized and strengthened by many authors; see e.g. the book [2] of P. D. T. A. Elliott and references there.

The aim of the present note is to give an elementary and straightforward proof for the following theorem, which is a corollary of the theorems proved in the papers [3], [4] and [5].

THEOREM. *Let f be an additive real-valued arithmetical function; A, B positive integers with $(A,B)=1$. Suppose that f is non-decreasing on the set $\{An+B : n=0,1,\dots\}$. Then $f(n)=c\log n$ if $(n,A)=1$, where $c>0$ is a suitable constant.*

PROOF. Our idea is to prove that f is completely additive on the set $\{An+B : n=0,1,\dots\}$, from which the statement follows at once.

1. First we show that if $h\equiv 1 \pmod A$ then

$$f(h)=\alpha\log h. \tag{1}$$

To this end let us denote $c_h := f(h)/\log h$ and $d_h(n) := f(An+B)-c_h\log n$ for any fixed $h\in\mathbf{N}$, $h\equiv 1 \pmod A$, $h\neq 1$. We shall show that $d_h(n)$ is bounded, which implies the statement (1) taking into account that

$$d_{h_1}(n)-d_{h_2}(n)=\left(\frac{f(h_2)}{\log h_2}-\frac{f(h_1)}{\log h_1}\right)\log n = O(1)$$

(for any pair of h_1, h_2).

First we show that $d_h(n)$ is bounded above, i.e. if $n>n_0=n_0(c_h,h)$ then there exists $m_h<n$ for which

$$f(An+B)-c_h\log n < f(Am_h+B)-c_h\log m_h. \tag{2}$$

We are looking for an M such that

$$h(AM+B) > An+B \tag{3}$$

and

$$AM+B\equiv 1 \pmod h. \tag{4}$$

Let us denote m_h the minimal solution of (4) which satisfies (3). It is easy to see that $m_h = \frac{n}{h} + O(1)$. f is non-decreasing on $\{An + B\}$ thus we have

$$(5)\qquad f(An + B) \leqq f(h(Am_h + B)) = f(h) + f(Am_h + B),$$

hence

$$(6)\quad f(An+B) - c_h \log n \leqq f(h) - c_h \log \frac{n}{m_h} + (f(Am_h + B) - c_h \log m_h) =$$
$$= f(Am_h + B) - c_h \log m_h + O(\tfrac{h}{n}).$$

By iteration (repeating the above calculation with m_h in place of n) we get that $d_h(n)$ is bounded above. It can be proved similarly that $d_h(n)$ is bounded below. Hence (1) is proved.

2. Now let c_i ($i = 1$ or 2) be a natural number which is coprime to A and denote $c_i^* > 0$ one of the multiplicative inverses of c_i mod A. By Dirichlet's theorem there exists a prime p_i such that $p_i = An_i + c_i$ with suitable n_i and $(p_1, c_2) = (p_2, c_1) = (p_1p_2, c_1^*c_2^*) = 1$. Then $p_ic_i^* = Ac_i^*n_i + c_ic_i^* = Am + 1$, so using the result of Part 1 we have

$$(7)\qquad f(p_i) + f(c_i^*) = \alpha \log p_i + \alpha \log c_i^*$$

and

$$(8)\qquad f(p_1) + f(p_2) + f(c_1^*c_2^*) = \alpha \log p_1 + \alpha \log p_2 + \alpha \log c_1^* + \alpha \log c_2^*.$$

From (7) and (8) we obtain that $f(ab) = f(a) + f(b)$ for every pair a, b coprime to A. Thus

$$\alpha \log p^{\varphi(A)} = f(p^{\varphi(A)}) = \varphi(A) f(p)$$

for all primes $p \nmid A$ by (1), which yields the theorem.

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INVESTIGATIONS OF CERTAIN OPERATORS WITH RESPECT TO THE VILENKIN SYSTEM

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Introduction and results

We introduce some notations and definitions. Let $m = (m_0, m_1, \dots, m_k, \dots)$ be a sequence of natural numbers, each of them not less than 2. Denote by Z_{m_k} ($k \in \mathbf{N} := \{0, 1, \dots\}$) the m_k-th discrete cyclic group, i.e. $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ ($k \in \mathbf{N}$). If we define the group G_m as the direct product of the groups Z_{m_k}, then G_m is a compact Abelian group with Haar measure 1. The elements of G_m are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in Z_{m_k}$ ($k \in \mathbf{N}$). The sum of $x, y \in G_m$, $x \dot{+} y$ is obtained by adding the n-th coordinates of x and y modulo m_n ($n \in \mathbf{N}$). (Let $\dot{-}$ be the inverse of operator $\dot{+}$.) The topology of G_m is completely determined by the following subgroups of G_m:

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbf{N}).$$

For a fixed $x \in G_m$ and for $n \in \mathbf{N}$ let

$$I_n(x, k) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}, y_n = k\} \qquad (k \in Z_{m_n}).$$

It is obvious that $I_n(x) = \bigcup_{k \in Z_{m_n}} I_n(x, k)$ and this decomposition contains disjoint sets, furthermore $|I_n(x, k)| = m_n^{-1} |I_n(x)|$ ($k \in Z_{m_n}$ and $|A|$ denotes the measure of the Haar measurable set $A \subset G_m$).

It is well-known [1] that the characters of G_m form a complete orthonormal system $\hat{G}_m$ in $L^1(G_m)$. The elements of $\hat{G}$ can be obtained as follows. Define the sequence (M_k, $k \in \mathbf{N}$) as $M_0 := 1$, $M_{k+1} := m_k M_k$ ($k \in \mathbf{N}$), then all $n \in \mathbf{N}$ have a unique representation of the form $n = \sum_{k=0}^{\infty} n_k M_k$ ($n_k \in Z_{m_k}$, $k \in \mathbf{N}$). If

$$r_n(x) := \exp(2\pi i x_n / m_n) (=: r_n(x_n))$$
$$(n \in \mathbf{N}, \quad x = (x_0, x_1, \dots) \in G_m, \quad i = (-1)^{1/2}),$$

then the elements of $\hat{G}_m$ are nothing but the functions $\psi_n := \prod_{k=1}^{\infty} r_k^{n_k}$ ($n \in \mathbf{N}$) (cf. [1]). (ψ_n, $n \in \mathbf{N}$) is the so-called Vilenkin system. The Fourier coefficients of a function $f \in L^1(G_m)$ with respect to $\hat{G}_m$ are denoted by $\hat{f}(k)$ $(k \in \mathbf{N})$ and let

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k \quad (n \in \mathbf{N}), \quad \sigma_n = n^{-1} \sum_{k=1}^{n} S_k f \quad (n \in \mathbf{N} \setminus \{0\}).$$

The kernels of Dirichlet type are of the form $D_n := \sum\limits_{k=0}^{n-1} \psi_k$ $(n \in \mathbf{N})$. It is known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } (x \in I_n) \\ 0 & \text{if } (x \notin I_n) \quad (n \in \mathbf{N}) \end{cases}$$

and also

$$D_n = \psi_n \sum_{k=0}^{\infty} \sum_{j=m_k-n_k}^{m_k-1} r_k^j D_{M_k}$$

(cf. [1]). We define the maximal Hardy space $H(G_m)$ as follows ([4, 10]). Let $f \in L^1(G_m)$ belong to the maximal Hardy space $H(G_m)$ iff the maximal function $f^* := \sup\limits_n |S_{M_n} f|$ is an element of $L^1(G_m)$. $\|f\|_H := \|f^*\|_1$. The concept of the so-called atomic Hardy space $H^1(G_m)$ is as follows.

First we define the set of intervals. If the sequence m is bounded, then this set is $\{I_n(x) \mid x \in G_m,\ n \in \mathbf{N}\}$, [10]. If m is an arbitrary sequence, then a set $I \subset G_m$ is called an interval if for some $x \in G_m$ and $n \in \mathbf{N}$, I is of the form $I = \bigcup\limits_{k \in U} I(x,k)$, where U is obtained from Z_{m_n} by dyadic partition. (The sets $U_1, U_2, \ldots \subset Z_{M_n}$ are obtained by means of such a partition if

$$U_1 = \{0, \ldots, [m_n/2]-1\}, \qquad U_2 = \{[m_n/2], \ldots, m_n - 1\},$$
$$U_3 = \{0, \ldots, [([m_n/2]-1)/2]-1\}, \quad U_4 = \{[([m_n/2]-1)/2], \ldots [m_n/2]-1\}, \ldots$$

and so on, [ ] denotes the entire part.)

We define the atoms as follows: the function $a \in L^\infty(G_m)$ is called an atom if either $a = 1$ or there exists an interval $I_a := I$ for which $\operatorname{supp} a \subset I$, $|a| \leqq |I|^{-1}$ and $\int\limits_I a = 0$ hold. The space $H^1(G_m)$ is the set of all functions $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$, where $a_i \in A(G_m)$ are atoms and for the coefficients we have $\sum\limits_{i=0}^{\infty} |\lambda_i| < \infty$. $H^1(G_m)$ is a Banach space with respect to the norm $\|f\|_{H^1} :=$ $:= \inf \sum\limits_{i=0}^{\infty} |\lambda_i|$. (The infimum is taken over all decompositions $f = \sum \lambda_i a_i$.) It is known that for bounded m the spaces $H^1(G_m)$ and $H(G_m)$ coincide and the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_H$ are equivalent, [2].

It is known that the Walsh system is not a Schauder basis in $L^1(G_m)$ (where each m_j is 2). Moreover there exists a function $f \in H(G_m)$ such that $\limsup \|S_n f\|_1 = \infty$. The following theorem for Walsh system was proved by P. Simon [8]. The trigonometric analogue was verified by B. Smith [11].

THEOREM 1. *If $f \in H^1(G_m)$ (m is arbitrary), then*

$$\lim_{n\to\infty} \log^{-1} n \sum_{k=1}^{n} k^{-1} \|S_k f\|_1 = \|f\|_1.$$

Define the Sunouchi operator T as follows [9, 12, 13]:

$$Tf := \Big(\sum_{n=0}^{\infty} |S_{M_n} f - \sigma_{M_n} f|^2\Big)^{1/2}.$$

G. I. Sunouchi proved [12–13] that in the case of Walsh system ($m_j = 2$, $M_j = 2^j$ for all $j \in \mathbf{N}$) this operator T as mapping from $L^r(G_m)$ into $L^r(G_m)$ is bounded, if $1 < r < \infty$. The analogous statement for $r = 1$ does not hold thus it is of interest to study what happens in this case. P. Simon [9] proved that in the case of Walsh system ($m_j = 2, j \in \mathbf{N}$), T is bounded as mapping from H^1 into L^1.

For arbitrary sequence m there are some different cases. First we prove

THEOREM 3. *If $\limsup_k m_k = \infty$, then there exists a function $f \in H(G_m)$ such that $\|Tf\|_1 = \infty$.*

That is, the theorem of Simon does not hold in the case of $H(G_m)$ and $\limsup m_k = \infty$. Define the sequence m' in the following way:

$$m'_k := M_{k+1}^{-1} \sum_{j=0}^{k-1} M_{j+1} \log m_j \qquad (k = 1, 2, \dots).$$

If m is bounded, then so is m' and H coincides with H^1.

THEOREM 4. *If $\limsup_k m'_k < \infty$, then there exists a constant $c > 0$ such that $\|Tf\|_1 \leqq c\|f\|_{H^1}$ for all $f \in H^1(G_m)$.*

THEOREM 5. *If $\limsup_k m'_k = \infty$, then there exists a function $f \in H^1(G_m)$ such that $\|Tf\|_1 = \infty$.*

Thus for bounded m the theorem of Simon and Sunouchi holds, and also for unbounded m this theorem is true if and only if m' is bounded. To prove Theorems 3, 4, 5 the following lemma is often used.

LEMMA 2. *If $f \in L^1(G_m)$, $n \in \mathbf{N}$, then*

$$S_{M_n}f(x)-\sigma_{M_n}f(x)=(M_n-1)/2\int\limits_{I_n(x)} f-$$

$$-\sum_{k=0}^{n-1}\sum_{y=1}^{m_k-1}(M_k/(1-e^{-2\pi i y/m_k}))\int\limits_{I_n(x)\dot{+}e_k y} f \quad (x\in G_m,\ e_k:=(0,\dots,0,\overset{k}{\breve{1}},0,\dots)).$$

Lemma 2 can be proved by the method of Pál and Simon [7]. Throughout this paper c denotes an absolute constant, which may vary from line to line.

Proofs

PROOF OF THEOREM 1. We prove that

$$\sup_{a\in A(G_m)}\sup_{N\in\mathbf{N}}\log^{-1}N\sum_{k=1}^{N}\|S_k a\|_1 k^{-1}<\infty. \tag{1}$$

We complete the proof of Theorem 1 by (1) in a similar way as P. Simon [8] did in the case of $m_i=2$ $(i\in\mathbf{N})$. Let a be an atom for which $I_a:=$ $:=\bigcup_{k=\alpha}^{\beta}I_n(y,k)$. Since case $a=1$ is trivial, we suppose that $a\neq 1$.

(2) If $q<M_n$, then ψ_q is constant on I_a, thus $S_q a=0$. So let

$$q\geqq M_n \qquad (q=q_sM_s+\dots+q_nM_n+\dots+q_0M_0).$$

(3) If $x\in I_i(y)\setminus I_{i+1}(y)$, $i=0,\dots,n-1$, then

$$S_q a(x)=\int\limits_{I_a}a(t)\psi_q(x-t)\left[\sum_{k=0}^{i-1}q_kM_k+M_i\sum_{j=m_i-q_i}^{m_i-1}\exp(2\pi(-1)^{1/2}j(x_i\dot{-}t_i)/m_i\right]=$$

$$=\left\{\int\limits_{I_a}a(t)\psi_q(x-t)\right\}\left\{\sum_{0}^{i-1}q_kM_k+M_i\sum_{j=m_i-q_i}^{m_i-1}\exp(2\pi(-1)^{1/2}j(x_i\dot{-}t_i)/m_i)\right\}$$

$(t_i=y_i)$. Consequently

$$|S_q a(x)|=|\hat{a}(q)|\left|\sum_{0}^{i-1}q_kM_k+M_i\sum_{j=m_i-q_i}^{m_i-1}\exp(2\pi(-1)^{1/2}j(x_i\dot{-}t_i)/m_i)\right|.$$

From this it follows that

$$\int\limits_{I_i(y)\setminus I_{i+1}(y)} |S_q a| = \sum_{l=0,l\neq y_i}^{m_i-1} \int\limits_{\{(y_0,\dots,y_{i-1},l,\dots)\}} |S_q a(x)|dx \leqq$$

$$\leqq |\hat{a}(q)|\left(1 + m_i^{-1} \sum_{\substack{x_i=0,\\ x_i\neq y_i}}^{m_i-1} \left| \sum_{j=m_i-q_i}^{m_i-1} \exp(2\pi(-1)^{1/2} j(x_i \dot{-} y_i)/m_i)\right|\right) =$$

$$= |\hat{a}(q)|\left(1 + m_i^{-1} \sum_{k=1}^{m_i-1} |\sin(\pi q_i k/m_i)/\sin(\pi k/m_i)|\right) \leqq c|\hat{a}(q)|\log q_i.$$

Hence

$$(4) \qquad \int\limits_{G_m\setminus I_n(y)} |S_q a(x)| \leqq c|\hat{a}(q)|\log(m_0\dots m_{n-1}) = c|\hat{a}(q)|\log M_n.$$

$$(5) \qquad x \in I_n(y) \setminus \bigcup_{k=\alpha}^{\beta} I_n(y,k) =: B.$$

(5) gives $x - t \in I_n \setminus I_{n+1}$.

$$S_q a(x) = \int\limits_{I_a} a(t)\psi_q(x-t)\left\{\sum_{k=0}^{n-1} q_k M_k + M_n \sum_{j=m_n-q_n}^{m_n-1} r_n^j(x \dot{-} t)\right\}dt =$$

$$=: \int\limits_{I_a} a(t)\psi_q(x-t)(D+E)dt.$$

It is easy to get

$$(6) \qquad \int\limits_B \left|\int\limits_{I_a} a(t)\psi_q(x-t)D\right|dx \leqq |\hat{a}(q)|.$$

Define A_k ($k \in J := \{\alpha, \alpha+1, \dots, \beta\}$) in the following way:

$$A_k := (\beta-\alpha) \int\limits_{I_n(y,k)} a(z)\bar{\psi}_{q_{n+1}M_{n+1}+\dots+q_s M_s}(z)dz \qquad (k \in J).$$

Thus $|A_k| \leqq 1$ ($k \in J$) and in the case of $q < M_{n+1}$, $\sum_{k\in J} A_k = 0$ holds.

$$(7) \qquad \int\limits_B \left|\int\limits_{I_a} a(t)\psi_q(x-t)E\right|dx = m_n^{-1} \sum_{x_n\in J}\left|\sum_{j=1}^{q_n} \hat{a}(q-jM_n) r_n^{q_n-j}(x_n)\right| =$$

$$= 1/((\beta-\alpha+1)m_n)\sum_{x\in J}\left|\sum_{j=0}^{q_n-1}\sum_{t\in J} A_t r_n^j(x-t)\right|.$$

From now on we assume that $q < M_{n+1}$, and let

$$t_0 := \begin{cases} (\alpha+\beta)/2 & (\alpha \equiv \beta \pmod 2) \\ (\alpha+\beta-1)/2 & (\alpha \not\equiv \beta \pmod 2). \end{cases}$$

(7) is equivalent to

$$(8) \qquad ((\beta-\alpha+1)m_n)^{-1} \sum_{x\notin J} \Big| \sum_{t\in J} A_t (r_n^{q_n}(x-t)-1) \, / \, (r_n(x-t)-1) \Big| .$$

We give an upper bound for (8) in a way that we also give some upper bounds for the following two sums:

$$(9) \qquad \frac{1}{(\beta-\alpha+1)m_n} \sum_{x\notin J} \Big| \sum_{t\in J} A_t r_n^{q_n}(x-t)/(r_n(x-t)-1) \Big| =: \sum_2^1 ,$$

$$(10) \qquad \frac{1}{(\beta-\alpha+1)m_n} \sum_{x\notin J} \Big| \sum_{t\in J} A_t \frac{1}{r_n(x-t)-1} \Big| =: \sum_1^1 .$$

First we deal with (10). Let $\Delta J := J \cup \{\beta\dot{+}1, \beta\dot{+}2, \dots, \beta\dot{+}(\beta-\alpha+1)\} \cup \cup \{\alpha\dot{-}(\beta-\alpha+1), \alpha\dot{-}(\beta-\alpha), \dots \alpha\dot{-}1\}$. Then

$$\sum_1^1 \leqq \frac{1}{(\beta-\alpha+1)m_n} \sum_{x\notin \Delta J} \Big| \sum_{t\in J} A_t \frac{1}{r_n(x-t)-1} \Big| +$$
$$+ \frac{1}{(\beta-\alpha+1)m_n} \sum_{x\in \Delta J} \Big| \sum_{t\in J} A_t \frac{1}{r_n(x-t)-1} \Big| =: \sum_1^2 + \sum_1^3 .$$

(11)

$$\sum_1^2 \leqq \frac{c}{(\beta-\alpha+1)m_n} \sum_{x\notin\Delta J} \Big| \sum_{t\in J} A_t \{\cot(\pi(x-t)/m_n) - \cot(\pi(x-t_0)/m_n)\} \Big| \leqq$$
$$\leqq \frac{c}{(\beta-\alpha+1)} \sum_{t\in J} |A_t||t-t_0| \sum_{x\notin\Delta J} \frac{1}{|x-t||x-t_0|} \leqq c(\beta-\alpha+1) \sum_{k\geqq\beta-\alpha+1} k^{1/2} \leqq c.$$

Since the discrete Hilbert transformation [15] is of type (2, 2) we get the boundedness of $\sum_1^3$:

$$\sum_1^3 \leqq c m_n^{-1} (\beta-\alpha+1)^{-1/2} \Big\{ \sum_{x\in\Delta J/J} \Big| \sum_{t\in J} A_t \cot(\pi(x-t)/m_n) \Big|^2 \Big\}^{1/2} \leqq$$
$$\leqq c(\beta-\alpha+1)^{-1/2} \Big\{ \sum_{t\in J} |A_t|^2 \Big\}^{1/2} \leqq c.$$

This and (11) give $\sum_1^1 \leqq c$. Now we deal with (9) that is construct some upper bound for $\sum_2^1$:

$$\sum_2^1 \leqq \frac{1}{(\beta-\alpha+1)m_n}\sum_{x\notin J}\Big|\sum_{t\in J} A_t r_n^{q_n}(x-t)\times$$

$$\times\{\cot(\pi(x-t)/m_n)-\cot((\pi(x-t_0)/m_n)\}\Big| + |\hat{a}(q_nM_n)|\sum_{x\notin J}\frac{1}{|x-t_0|}.$$

$$|\hat{a}(q_nM_n)|\sum_{x\notin J}|x-t_0|^{-1} < c|\hat{a}(q_nM_n)|\log\Big(\frac{m_n}{\beta-\alpha+1}\Big).$$

Then in the same way as we get $\sum_1^1 \leqq c$ we also get that

$$\sum_2^1 \leqq c\Big(1+|\hat{a}(q_nM_n)|\log\Big(\frac{m_n}{\beta-\alpha+1}\Big)\Big). \tag{12}$$

Let now $q \geqq M_{n+1}$. Then

$$\int_B\Big|\int_{I_a} a(t)\psi_q(x-t)(D+E)\Big|dx \leqq |\hat{a}(q)|+c\Big(\frac{1}{(\beta-\alpha+1)m_n}\sum_{x\notin J}\Big|\sum_{t\in J}A_t\frac{r_n^{q_n}(x-t)}{r_n(x-t)-1}\Big|+$$

$$+\frac{1}{(\beta-\alpha+1)m_n}\sum_{x\notin J}\Big|\sum_{t\in J}A_t\frac{1}{r_n(x-t)-1}\Big|\Big) =: |\hat{a}(q)|+c\Big(\sum^5+\sum^6\Big),$$

$$\sum^6 \leqq c+\frac{c}{(\beta-\alpha+1)m_n}\sum_{x\notin J}\Big|\sum_{t\in J}A_t(\cot(\pi(x-t)/m_n)-\cot(\pi(x-t_0)/m_n)\Big|+$$

$$+c|\hat{a}(q_sM_s+\ldots+q_{n+1}M_{n+1})|\log\Big(\frac{m_n}{\beta-\alpha+1}\Big) \leqq$$

$$\leqq c\Big(1+|\hat{a}(q_sM_s+\ldots+q_{n+1}M_{n+1})|\log\Big(\frac{m_n}{\beta-\alpha+1}\Big)\Big)$$

as it comes from the techniques used in this proof before. We also get that

$$\sum^5 \leqq c\Big(1+|\hat{a}(q)|\log\Big(\frac{m_n}{(\beta-\alpha+1)}\Big)\Big).$$

Next we define an atom $b \in A(G_m)$. Let $y_n = 1$, and if $z \notin I_{n+1}(y)$, then $b(z) = 0$,

$$b(y_0, \dots, y_{n-1}, y_n, t_{n+1}, \dots, t_s, \dots) :=$$
$$:= (1/2) \sum_{k=\alpha}^{\beta} a(y_0, \dots, y_{n-1}, k, t_{n+1}, \dots, t_s, \dots).$$

Then

$$|b| \leqq M_{n+1}, \quad \operatorname{supp} b = I_{n+1}(y), \quad \int_{G_m} b = \int_{I_{n+1}(y)} b = 0,$$

that is b is an atom. If $q \geqq M_{n+1}$, then $2|b^\wedge(q)| = |a^\wedge(q_s M_s + \dots + q_{n+1} M_{n+1})|$. By the application of a theorem of S. Fridli and P. Simon [4], we get

$$(13) \qquad (1/\log N) \sum_{q=M_{n+1}}^{N} \frac{|a^\wedge(q_s M_s + \dots + q_{n+1} M_{n+1})| \log \frac{m_n}{\beta-\alpha+1}}{q} \leqq$$
$$\leqq \frac{\log m_n}{\log M_{n+1}} \sum_{q=M_{n+1}}^{N} \frac{|b^\wedge(q)|}{q} < c.$$

The upper bound for $\sum^{5}$, (6), (4), the application of the Fridli–Simon theorem [4] and (13) imply

$$(14) \qquad (1/\log N) \sum_{q=M_{n+1}}^{N} \frac{\|S_q a\|_1}{q} < c.$$

(14) shows that inequality $q < M_{n+1}$ can be supposed (forever from now). Summarizing our achievements the rest in order to prove (1) is to get

(15)

$$A =: (1/\log N) \sum_{q=M_n}^{N} \frac{|a^\wedge(q)| \log(m_n/(\beta - \alpha + 1))}{q} < c, \quad \text{where } N < M_{n+1}.$$

Let $L := N_n$.

$$(16) \qquad A \leqq c(1/(\log L M_n)) \log(m_n/(\beta - \alpha + 1)) \sum_{j=1}^{L} \frac{|a^\wedge(j M_n)|}{j} =: B.$$

By the method of the paper of Fridli and Simon $|a^\wedge(j M_n)| \leqq c \frac{(\beta-\alpha+1)j}{m_n}$. If $L \geqq m_n/(\beta - \alpha + 1)$, then

$$B \leqq c \frac{1}{\log \frac{m_n}{\beta-\alpha+1} + \log M_n} \log \frac{m_n}{\beta - \alpha + 1} < c.$$

If $L < m_n/(\beta - \alpha + 1)$, then

$$B < c\frac{1}{n+\log L}\log(m_n/(\beta-\alpha+1))\frac{L(\beta-\alpha+1)}{m_n} =: C. \tag{17}$$

Now

$$1 \leqq \frac{m_n}{\beta-\alpha+1} =: v \leqq m_n. \qquad \frac{L\,\log v}{(n+\log L)v} =: u(L,v), \qquad 1 \leqq L < v.$$

$$u'_L = \frac{\log v}{v(n+\log L)} - \frac{L\,\log v}{v(n+\log L)^2}\frac{1}{L} = \frac{\log v}{v(n+\log L)}\Big(1-\frac{1}{n+\log L}\Big) > 0,$$

$$u(v,v) = \frac{v\,\log v}{(n+\log L)v} < 1,$$

hence $C < c$. This gives that $A < c$. Thus (1) is proved.

Let $f \in H^1(G_m)$. Then $f = \sum\limits_{i=0}^{\infty} \lambda_i a_i$, where $\sum |\lambda_i a_i| < \infty$ and $a_i \in A(G_m)$ $(i \in \mathbf{N})$. Now

$$\Big|1/(\log N)\sum_{k=1}^{N} k^{-1}\|S_k f\|_1 - \|f\|_1\Big| = \Big|\frac{1}{\log N}\sum_{k=1}^{N} k^{-1}(\|S_k f\|_1 - \|f\|_1) +$$

$$+\|f\|_1\Big(\frac{1}{\log N}\sum_{k=1}^{N} k^{-1} - 1\Big)\Big| \leqq \frac{1}{\log N}\sum_{k=1}^{N} k^{-1}\|S_k f - f\|_1 + o(1) = R_N f + o(1).$$

Let $\varepsilon > 0$ and define q as $\sum\limits_{i=q+1}^{\infty} |\lambda_i| < \varepsilon$. We have

$$R_N f \leqq \sum_{i=0}^{\infty} |\lambda_i| R_N a_i = \sum_{i=q+1}^{\infty} |\lambda_i| R_N a_i + \sum_{i=0}^{q} |\lambda_i| R_N a_i =: \sum^{7} + \sum^{8}.$$

(1) implies that $\sum\limits^{7} < c\varepsilon$. We prove that for a fixed atom $a \in A(G_m)$, $R_N a \to 0$ $(N \to \infty)$. That is $\sum\limits^{8} < \varepsilon$ if N is large enough. This would complete the proof of Theorem 1. By standard argument (Simon [8, 9]) $\int\limits_{I_a} |S_k a - a| \to 0$ $(k \to +\infty$, and not greater than 2 for all $k \in \mathbf{N})$. Using the notations and results achieved for the atom a in the proof of (1),

$$\int\limits_{G_m \setminus I_n(y)} |S_k a| \leqq c|a^{\wedge}(k)| \log M_n = c|a^{\wedge}(k)|,$$

because $a \in A(G_m)$ is fixed. We proved that for $k \geqq M_{n+1}$

$$\int\limits_{I_a \setminus I_n(y)} |S_k a| \leqq c(|a^\wedge(k)| + |b^\wedge(k)|) \log \frac{m_n}{\beta - \alpha + 1},$$

where the atom b corresponding to the atom a is also fixed. Since $\log m_n/(\beta - -\alpha + 1)$ is constant, for $N \geqq M_{n+1}$ by the theorem of Fridli and Simon [4] we get

$$\log^{-1} N \sum_{k=1}^{N} k^{-1} \int\limits_{G_m \setminus I_a} |S_k a| \leqq c \log^{-1} N \sum_{k=1}^{N} \frac{|a^\wedge(k)| + |b^\wedge(k)|}{k} \leqq c/\log N \to 0.$$

The proof of Theorem 1 is complete.

Proof of Theorem 3. Denote $\Delta_k := \left[\frac{m_k}{2}\right] + 1$ if $m_k \geqq 3$, and $\Delta_k := 1$ if $m_k = 2$. Let

$$f_k(x) := \begin{cases} M_{k+1} & \text{if } x \in I_k(0,1),\ x_{k+1} \neq 0, \\ -M_{k+1} & \text{if } x \in I_k(0,\Delta_k),\ x_{k+1} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In a joint paper of S. Fridli and P. Simon [4] it is proved that for each $\sum |\lambda_i| < \infty$ the function $f := \sum f_{2k} \lambda_k$ is an element of the maximal Hardy space $H(G_m)$. We give the construction of a series λ_i ($i \in \mathbf{N}$) of the above type such that $\|Tf\|_1 = \infty$. Denote $P_k := \{x \in G_m | x_0 = \ldots = x_{k-1} = 0, x_k \neq 0\}$. It is easy to get

$$\|Tf\|_1 \geqq \sum_{k=0}^{\infty} \int\limits_{P_k} |Tf| \geqq \sum_{k=0}^{\infty} |S_{M_{k+1}} f - \sigma_{M_{k+1}} f|. \tag{31}$$

Therefore we have to give a lower bound for $\int\limits_{P_k} |S_{M_{k+1}} f - \sigma_{M_{k+1}} f|$. Lemma 2 states that

$$S_{M_{k+1}} f - \sigma_{M_{k+1}} f = (M_{k+1} - 1)/2 \int\limits_{I_{k+1}(x)} f - \tag{32}$$

$$- \sum_{n=0}^{k} \sum_{y=1}^{m_n - 1} \frac{M_n}{1 - e^{-2\pi i y/m_n}} \int\limits_{I_{k+1}(y) + e_n y} f =: A_1 - A_2.$$

Since $x \in P_k$, $I_{k+1}(x) \cap \operatorname{supp} f_j = \emptyset$ for $j = 0, 1, \ldots, k-1$ and $I_{k+1}(x) \cap \cap \operatorname{supp} f_j = \emptyset$ for $j = k+1, \ldots$. This implies that $A_1 = \frac{M_{k+1} - 1}{2} \int\limits_{I_{k+1}(x)} f_k \lambda_{k/2}$

if $2 \mid k$, and if $2 \nmid k$, then $A_1 = 0$. That is we deal only with case $2 \mid k$. Then

$$\int_{P_k} |A_1| = \int_{I_k(0,1)} \left| \frac{M_{k+1}-1}{2} \int_{I_{k+1}(x)} f_k \lambda_{k/2} \right| +$$

$$+ \int_{I_k(0,\Delta_k)} \left| (M_{k+1}-1)/2 \int_{I_{k+1}(x)} f_k \lambda_{k/2} \right| \leqq |\lambda_{k/2}|.$$

Now we give a lower bound for $\int_{P_k} |A_2|$. If $n \leqq k-2$, then $I_{k+1}(x) + e_n z \cap$ $\cap \operatorname{supp} f_j = \emptyset$ for each $j \in \mathbf{N}$, $z \in Z_{m_n} \setminus \{0\}$. That is

(33)

$$A_2 = \sum_{y=1}^{m_{k-1}-1} \frac{M_{k-1}}{1-e^{-2\pi i y/m_{k-1}}} \int_{I_{k+1}(x)+e_{k-1}y} f + \sum_{y=1}^{m_k-1} \frac{M_k}{1-e^{-2\pi i y/m_k}} \int_{I_{k+1}(x)+e_k y} f.$$

$I_{k+1}(x) + e_{k-1}y \cap \operatorname{supp} f_j \neq \emptyset$ may hold only for $j = k-1$, that is if $2 \mid k$, then the first term of (33) equals zero. If $2 \nmid k$, then the first term of (33) is as follows:

$$\left(\frac{M_{k-1}}{1-e^{-2\pi i/m_{k-1}}} \frac{M_k}{M_{k+1}} - \frac{M_{k-1}}{1-e^{-2\pi i \Delta_{k-1}/m_{k-1}}} \frac{M_k}{M_{k+1}} \right) \lambda_{\frac{k-1}{2}},$$

meanwhile the second term (as it comes from the discussion of $I_{k+1}(x) + e_k y$ and the definition of f_j's) equals zero. Thus if $2 \nmid k$, then

$$\int_{P_k} |A_2| \leqq |\lambda_{(k-1)/2}| c m_k^{-1} \leqq c |\lambda_{(k-1)/2}|,$$

where c is an absolute constant. If $2 \mid k$, then the second term of (33) is as follows:

$$A_2 = \sum_{y=1}^{m_k-1} \frac{M_k}{1-e^{-2\pi i y/m_k}} \int_{I_{k+1}(x)+e_k y} f. \tag{34}$$

If $x_k \dot{+} y = 0$, then either $I_{k+1}(x) + e_k y \supset \operatorname{supp} f_j$ or $I_{k+1}(x) + e_k y \cap \operatorname{supp} f_j = \emptyset$ for each $j \in \mathbf{N}$, thus $\int_{I_{k+1}(x)+e_k y} f = 0$. If $x_k \dot{+} y \neq 0$, then

$$\int_{I_{k+1}(x)+e_k y} f = \int_{I_{k+1}(x)+e_k y} f_k \lambda_{k/2}.$$

As a consequence of this in the case of $2 \mid k$ by (34) we get

$$\int_{P_k} |A_2| = \sum_{j=1}^{m_k-1} \int_{\{x|x_0=\ldots=x_{k-1}=0,x_k=j\}} |A_2| =$$

$$= |\lambda_{k/2}|/m_k \sum_{\substack{j=2\\ j\neq\Delta_k}}^{m_k-1} \left| \frac{1}{1-e^{2\pi i(j-1)/m_k}} - \frac{1}{1-e^{2\pi i(j-\Delta_k)/m_k}} \right| +$$

$$+|\lambda_{k/2}|/m_k^{-1} 2|\sin\pi(1-\Delta_k)/m_k|^{-1} \geqq$$

$$\geqq c|\lambda_{k/2}| \frac{1}{m_k} \sum_{\substack{j=2\\ j\neq\Delta_k}}^{m_k-1} \frac{1}{\left|\sin\frac{\pi(j-1)}{m_k}\sin\frac{\pi(j-\Delta_k)}{m_k}\right|} \geqq$$

$$\geqq c|\lambda_{k/2}| \frac{m_k}{\Delta_k} \sum_{j=1}^{\Delta_k} \frac{1}{j} \geqq c_0|\lambda_{k/2}| \log m_k,$$

where $0 < c_0$ is a fixed constant. Summarizing our achievements we get

$$\|Tf\|_1 \geqq \sum_{k=0}^{\infty} \int_{P_k} |S_{M_{k+1}}f - \sigma_{M_{k+1}}f| \geqq c_0 \sum_{s=0}^{\infty} \log m_{2s}|\lambda_s| - c\sum_{j=0}^{\infty}|\lambda_j|.$$

If $\limsup_j m_{2j} = \infty$, then there exists an index series $j_\nu \nearrow \infty$ such that $\sum_{\nu=0}^{\infty} (\log m_{2j_\nu})^{-1} < \infty$. In this case let $\lambda_{j_\nu} := (\log m_{2j_\nu})^{-1}$, and the rest of λ_j equals zero. Thus $\sum_{j=0}^{\infty} |\lambda_j| < \infty$, and $\|Tf\|_1 = +\infty$. If $\limsup_j m_{2j} < \infty$, then $\limsup_j m_j = \infty$ gives that $\limsup_j m_{2j+1} = +\infty$. In this case the proof of Theorem 3 is the same, but now instead of $f := \sum_{k=0}^{\infty} \lambda_k f_{2k}$ we define $f := \sum_{k=0}^{\infty} \lambda_k f_{2k+1}$. Repeating the whole procedure of the proof of Theorem 3 we get that the statement holds in this case too. The proof of Theorem 3 is complete.

Proof of Theorem 4. We prove that for every atom $a \in A(G_m)$, $\|Ta\|_1 \leqq c$ for some absolute constant c. By standard arguments this gives the proof of the theorem (see e.g. F. Schipp and P. Simon [5, 6]). Case $a = 1$ is trivial and that is why from now $a \neq 1$ is supposed. Let $I_a = \bigcup_{j=\alpha}^{\beta} I_k(y,j)$, where $\{\alpha, \alpha+1, \ldots, \beta\} \subset Z_{m_k}$, $k, y \in G_m$ fixed. Then

$|a| \leqq M_{k+1}(\beta-\alpha+1)^{-1}$. First we construct an upper bound for the value of the integral $\int\limits_{I_j(y)\setminus I_{j+1}(y)} |Ta|$, where $j=0,1,\dots,k-1$. Then we do the same for $\int\limits_{I_k(y)\setminus I_a} |Ta|$ and at last for the value of $\int\limits_{I_a} |Ta|$. In the first case by Lemma 2 we have

$$A := S_{M_N}a(x) - \sigma_{M_N}a(x) = \frac{M_N-1}{2} \int\limits_{I_N(x)} a -$$

$$-\sum_{n=0}^{N-1}\sum_{z=1}^{m_n-1} \frac{M_n}{1-e^{-2\pi i z/m_n}} \int\limits_{I_N(x)+e_n z} a =: A_1 - A_2$$

$$(N = 0, 1 \dots \text{ and } x \in I_j(y)\setminus I_{j+1}(y)).$$

If $N \leqq j$, then $I_N(x) \supset I_a$ and $I_N(x) \cap I_a = \emptyset$ for $N > j$, hence $A_1 = 0$. If $N \leqq j$, then for $n = 0, 1, \dots, N-1$ $I_N(x) + e_n z \cap I_a = \emptyset$ and this implies that $A_2 = 0$. If $N-1 \geqq j$, then $I_N(x) = \{(y_0, \dots, y_{j-1}, x_j, \dots x_{N-1}, \dots)\}$ $(x_j \neq y_j)$. $I_N(x) + e_n z \cap I_a \neq \emptyset$ implies that $n = j$ and $z = y_j \dot{-} x_j$. In this case we get $I_N(x) + e_n z = \{(y_0, \dots, y_j, x_{j+1}, \dots, x_{N-1}, \dots)\}$. If $N-1 = j$, then $I_N(x) + e_n z \supset I_a$ and this is why A_2 equals zero. That is $N-1 \geqq j+1$ can be supposed. If $N \leqq k$, then for $I := I_N(x) + e_n z$ we get either $I \supset I_a$ or $I \cap I_a = \emptyset$. In both cases $A_2 = 0$. Hence the only case to be dealt with is $N \geqq k+1$. Suppose that $N \geqq k+1$. Since I is a "complete" interval, $I \cap I_a \neq \emptyset$ implies $x_{j+1} = y_{j+1}, \dots x_{k-1} = y_{k-1}$, $x_k \in [\alpha, \beta]$. Thus in this case $I \subset I_a$, hence $|\int\limits_I a| \leqq M_{k+1}/(\beta-\alpha+1)M_N$. This gives

$$|A_2| \leqq \frac{M_j M_{k+1}}{|\sin \pi(x_j - y_j)/m_j|(\beta-\alpha+1)M_N}.$$

$$\Big\{\sum_{N=k+2}^{\infty} |A_2|^2\Big\}^{1/2} \leqq c \frac{M_j}{(\beta-\alpha+1)|\sin \pi(x_j-y_j)/m_j| m_{k+1}}.$$

As a consequence of this we get

$$\int\limits_{I_j(y)\setminus I_{j+1}(y)} |T_1 a| \leqq M_{k+1}^{-1} \sum_{x_k=\alpha}^{\beta} \sum_{\substack{x_j=0\\ x_j\neq y_j}}^{m_j-1} \frac{M_j}{(\beta-\alpha+1)|\sin\pi(x_j-y_j)/m_j|} \leqq$$

$$\leqq c M_{k+1}^{-1} M_{j+1} \log m_j,$$

where

$$T_1 a := \Big\{\sum_{\substack{N=0\\ N\neq k+1}}^{\infty} |S_{M_N}a - \sigma_{M_N}a|^2\Big\}^{1/2}.$$

This immediately gives

$$\int\limits_{G_m \setminus I_k(y)} |T_1 a| \leqq c M_{k+1}^{-1} \sum_{j=0}^{k-1} M_{j+1} \log m_j = c m_k' < c.$$

Case $N = k+1$ is discussed later in a separate way. Now we give an upper bound for the value of $\int\limits_{I_k(y) \setminus I_a} |Ta|$. If $N \leqq k$, then $I_N(x) \supset I_a$ and if $N > k$, then (since $x_k \notin [\alpha, \beta]$) we get $I_N(x) \cap I_a = \emptyset$. That is in both cases $A_1 = 0$. Hence we discuss only A_2. If $N \leqq k$, then $I_N(x) \supset I_a$, thus $I := I_N(x) + e_n z \cap \cap I_a = \emptyset$, that is $A_2 = 0$. Suppose that $N \geqq k+2$. In a simple way we get that

$$|A| = |A_2| \leqq \sum_{z \in J_{x_k}} \frac{M_k M_{k+1}}{|\sin \pi z / m_k| (\beta - \alpha + 1) M_N} \quad (J_{x_k} := \{\alpha \dot{-} x_k, \dots, \beta \dot{-} x_k\}).$$

This implies the following inequality

$$\left\{ \sum_{N=k+2}^{\infty} |A|^2 \right\}^{1/2} \leqq 2 M_k \sum_{y_k \in [\alpha, \beta]} |\sin \pi (x_k - y_k)/m_k|^{-1} (\beta - \alpha + 1)^{-1} m_{k+1}^{-1}.$$

Hence

$$\int\limits_{I_k(y) \setminus I_a} |T_1 a| \leqq 2 M_{k+1}^{-1} \sum_{x_k \notin [\alpha, \beta]} \sum_{y_k \in [\alpha, \beta]} |\sin \pi (x_k - y_k)/m_k|^{-1} M_k (\beta - \alpha + 1)^{-1} m_{k+1}^{-1}.$$

That is we have to give an upper bound for the sum

$$A_3 := \frac{1}{(\beta - \alpha + 1) m_k m_{k+1}} \sum_{x_k = \beta+1}^{m_k - 1} \sum_{y_k = \alpha}^{\beta} |\sin \pi (x_k - y_k)/m_k|^{-1} +$$

$$+ \frac{1}{(\beta - \alpha + 1) m_k m_{k+1}} \sum_{x_k = 0}^{\alpha - 1} \sum_{y_k = \alpha}^{\beta} |\sin \pi (x_k - y_k)/m_k|^{-1} = A_{31} + A_{32}.$$

$$A_{31} \leqq \frac{c m_k}{(\beta - \alpha + 1) m_k m_{k+1}} \sum_{x = \beta+1}^{m_k - 1} \sum_{y = \alpha}^{\beta} \frac{1}{x - y} \leqq$$

$$\leqq \frac{c m_k}{(\beta - \alpha + 1) m_k m_{k+1}} \sum_{x = \beta+1}^{m_k - 1} \log\left(\frac{x - \alpha}{x - \beta}\right) =$$

$$= \frac{c m_k}{(\beta - \alpha + 1) m_k m_{k+1}} \log\left(\frac{(m_k - 1 - \alpha)!}{(\beta - \alpha + 1)! (m_k - 1 - \beta)!}\right) =$$

$$= \frac{cm_k}{(\beta-\alpha+1)m_k m_{k+1}} \log\left(\binom{m_k-1-\alpha}{\beta-\alpha+1}\right) \leqq \frac{m_k c \log(m_k-1-\alpha)}{m_k m_{k+1}} \leqq$$

$$\leqq \frac{c \log m_k}{m_{k+1}} = \frac{cM_{k+1}\log m_k}{M_{k+2}} \leqq cm'_{k+1} \leqq c,$$

and similarly $A_{32} \leqq c$. Thus we proved that $\int\limits_{I_k(y)\setminus I_a} |T_1 a| \leqq c$. Since the operator T is of type $(2,2)$ (this can be proved in a simple way), we get $\int\limits_{I_a} |Ta| \leqq c$. Hence all we have to prove is

$$\text{(41)} \qquad \int\limits_{G_m\setminus I_k(y)} |S_{M_{k+1}}a - \sigma_{M_{k+1}}a| \leqq c$$

and

$$\text{(42)} \qquad \int\limits_{I_k(y)\setminus I_a} |S_{M_{k+1}}a - \sigma_{M_{k+1}}a| \leqq c.$$

First we prove (42). If $x \in I_k(y) \setminus I_a$, then $x = (y_0, \ldots, y_{k-1}, x_k, \ldots)$, where $x_k \notin [\alpha, \beta]$. Lemma 2 shows that $I_{k+1}(x) \cap I_a = \emptyset$, hence $A_1 = 0$. $I := I_{k+1}(x) + e_n z \cap I_a \neq \emptyset$ is possible only in the case when $n = k$. Thus

$$|A_2| = \left|\sum_{z=1}^{m_k-1} \frac{M_k}{1-e^{-2\pi iz/m_k}} \int\limits_{I_{k+1}(x)+e_k z} a\right|.$$

Let

$$B_t := (\beta-\alpha+1) \int\limits_{(y_0,\ldots,y_{k-1},t,\ldots)+I_{k+1}} a$$

where $t \in Z_{m_k}$. $|B_t| \leqq 1$ $(t \in Z_{m_k})$, $B_t = 0$ $(t \notin [\alpha,\beta])$ and $\sum\limits_{t=\alpha}^{\beta} B_t = 0$ hold. We have

$$\text{(43)} \quad |A_2| = \frac{M_k}{\beta-\alpha+1}\left|\sum_{z=1}^{m_k-1} \frac{B_{x_k+z}}{1-e^{-2\pi iz/m_k}}\right| = \frac{M_k}{\beta-\alpha+1}\left|\sum_{t=\alpha}^{\beta} \frac{B_t}{e^{-2\pi i(t-x_k)/m_k}}\right|.$$

(43) gives

$$\text{(44)} \qquad \int\limits_{I_k(y)\setminus I_a} |A_2| = M_{k+1}^{-1} \sum_{\substack{x_k=0 \\ x_k\notin[\alpha,\beta]}}^{m_k-1} \frac{M_k}{\beta-\alpha+1}\left|\sum_{t=\alpha}^{\beta} B_t \frac{1}{r_k(x_k-t)-1}\right|.$$

Then we can use the method of the proof of Theorem 1 to show that (42) holds.

Now we prove (41). If $x \in G_m \setminus I_k(y)$, then the already known technics give that $A_1 = 0$. Thus only A_2 is to be discussed. Let $x \in I_j(y) \setminus I_{j+1}(y)$ $(j \in 0,1,\dots,k-1)$, $x = (y_0,\dots,y_{j-1},x_j,\dots)$. Since $N = k+1$, $I_N(x)+e_n z \cap$ $\cap\, I_a \neq \emptyset$ implies $n = j$, $z = y_j - x_j$ and also $x_{j+1} = y_{j+1},\dots,x_{k-1} = y_{k-1}$, $x_k \in [\alpha,\beta]$. Thus

$$\int\limits_{I_j(y)\setminus I_{j+1}(y)} |A_2| = M_{k+1}^{-1} \sum_{x_k\in[\alpha,\beta]} \sum_{\substack{x_j=0\\ x_j\neq y_j}}^{m_j-1} \left| \frac{M_j}{1-e^{-2\pi i(y_j-x_j)/m_j}} B_{x_k}(\beta-\alpha+1)^{-1} \right| \leqq$$

$$\leqq \frac{M_j}{M_{k+1}(\beta-\alpha+1)} \sum_{x_k=\alpha}^{\beta} \sum_{\substack{x_j=0\\ x_j\neq y_j}}^{m_j-1} \frac{1}{|\sin \pi(x_j-y_j)/m_j|} \leqq$$

$$\leqq \frac{cM_j}{(\beta-\alpha+1)M_{k+1}} \sum_{x_k=\alpha}^{\beta} m_j \log m_j = \frac{cM_{j+1}\log m_j}{M_{k+1}}.$$

This immediately gives that the left side of (41) is not greater than

$$cM_{k+1}^{-1} \sum_{j=0}^{k-1} M_{j+1} \log m_j = cm_k' \leqq c.$$

Thus (41) is verified. Summarizing our results,

$$\left\| \left\{ \sum_{\substack{n=0\\ n\neq k+1}}^{\infty} |S_{M_n} a - \sigma_{M_n} a|^2 \right\}^{1/2} \right\|_1 \leqq c \quad \text{and} \quad \|S_{M_{k+1}} a - \sigma_{M_{k+1}} a\|_1 \leqq c.$$

By these last inequalities we get that $\|Ta\|_1 \leqq c$. The proof of Theorem 4 is complete.

Proof of Theorem 5. Let $a \in A(G_m)$, $I_a := \bigcup_{j=\alpha}^{\beta} I_k(y,j)$ and $|a| :=$ $:= \mu^{-1}(I_a)$ everywhere, where $\alpha,\beta \in Z_{m_k}$, $\alpha \leqq \beta$, $y \in G_m$ and $k \in \mathrm{N}$ fixed. We estimate the value of integral

$$D := \int\limits_{G_m\setminus I_k(y)} |S_{M_{k+1}} a - \sigma_{M_{k+1}} a|$$

with the technics used in the proof of Theorem 4:

$$D = \sum_{j=0}^{k-1} \int\limits_{I_j(y)\setminus I_{j+1}(y)} |S_{M_{k+1}}a - \sigma_{M_{k+1}}a| =$$

$$= \sum_{j=0}^{k-1} M_{k+1}^{-1} \sum_{x_k=\alpha}^{\beta} \sum_{\substack{x_j=0 \\ x_j \neq y_j}}^{m_j-1} \left| \frac{M_j}{1-r_j(x-y)} B_{x_k} (\beta-\alpha+1)^{-1} \right| =$$

$$= M_{k+1}^{-1}(\beta-\alpha+1)^{-1} \sum_{j=0}^{k-1} M_j \sum_{x_k=\alpha}^{\beta} \sum_{\substack{x_j=0 \\ x_j \neq y_j}}^{m_j-1} |\sin \pi(x_j-y_j)/m_j|^{-1} \geqq$$

$$\geqq M_{k+1}^{-1}(\beta-\alpha+1)^{-1} \sum_{j=0}^{k-1} M_j \sum_{x_k=\alpha}^{\beta} c_0 m_j \log m_j = c_0 m_k'$$

for some fixed absolute positive constant c_0. Then define $f := \sum \lambda_i a_i$ where the sum of the absolute values of the complex numbers λ_i is finite and for the atoms $a_i \in A(G_m)$ $I_{a_i} := I_{k_i}(y)$ $(y \in G_m$ fixed), where the index series $k_\nu \nearrow +\infty$ is defined later. Let $|a_i| = M_{k_i}$. Consider the left side of the following inequality:

$$\int\limits_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} f - \sigma_{M_{k_\nu+1}} f \right| \leqq \|Tf\|_1. \tag{51}$$

By the above proved lower bound for the atom a_ν we get

$$\int\limits_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} a_\nu - \sigma_{M_{k_\nu+1}} a_\nu \right| \geqq c_0 m_{k_\nu}. \tag{52}$$

Let $n > \nu$ and denote

$$F_j := \int\limits_{I_j(y)\setminus I_{j+1}(y)} \left| S_{M_{k_\nu+1}} a_\nu - \sigma_{M_{k_\nu+1}} a_\nu \right|, \qquad j = 0, 1, \dots, k_\nu - 1.$$

If $x \in I_j(y) \setminus I_{j+1}(y)$, then $I_{k_\nu+1}(x) \cap I_{a_n} = \emptyset$, hence

$$A_1 = (M_{k_\nu+1} - 1)/2 \int\limits_{I_{k_\nu+1}(x)} a_n = 0.$$

Now consider A_2 corresponding to the atom a_n and $M_{k_\nu+1}$.

$$I := I_{k_\nu+1}(x) + e_s z \cap I_{a_n} \neq \emptyset$$

is possible only in the case when $s = j$ and $z = y_j \dot{-} x_j$. Let s and z be fixed in this way. Hence $I \cap I_a \neq 0$ also implies that $I \supset I_{a_n}$ $(\nu + 1 \leqq n)$ which yields $A_2 = 0$. So we proved that $F_j = 0$ for each atom a_n $(n \geqq \nu + 1)$ and $j \in \{0, 1, \dots, k_\nu - 1\}$, i.e.

$$(53) \qquad \int_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} a_n - \sigma_{M_{k_\nu+1}} a_n \right| = 0, \qquad n \geqq \nu + 1.$$

Now we consider the left side of (53) for the atoms a_n $(n \leqq \nu - 1)$. Let now $n \leqq \nu - 1$. By the technics of the proof of Theorem 4 one can prove that $\|Ta_n\|_1 \leqq c\left(m'_{k_n} + m'_{k_n+1}\right)$ for some absolute constant c. Assume that a series $\lambda_i \in \mathbf{C}$ $(i \in \mathbf{N})$, $\sum_{i=0}^{\infty} |\lambda_i| < \infty$, $|\lambda_i| > 0$ is given and also that $k_0, k_1, \dots, k_{\nu-1}$ are given. Since $\limsup_k m'_k = \infty$, there exists an index k_ν such that

$$c_0 |\lambda_\nu| m'_{k_\nu} \geqq 2^\nu + c \sum_{i=0}^{\nu-1} |\lambda_i| \left(m'_{k_i} + m'_{k_i+1}\right).$$

Define $f := \sum_{i=0}^{\infty} \lambda_i a_i$. Then $\sum_{i=0}^{\infty} |\lambda_i| < \infty$, $f \in H^1(G_m)$,

$$\|Tf\|_1 \geqq \int_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} f - \sigma_{M_{k_\nu+1}} f \right| \geqq \int_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} a_\nu - \sigma_{M_{k_\nu+1}} a_\nu \right| |\lambda_\nu| -$$

$$- \sum_{i=0}^{\nu-1} \int_{G_m \setminus I_{k_\nu}(y)} \left| S_{M_{k_\nu+1}} a_i - \sigma_{M_{k_\nu+1}} a_i \right| |\lambda_i| \geqq$$

$$\geqq c_0 |\lambda_\nu| m'_{k_\nu} - c \sum_{i=0}^{\nu-1} |\lambda_i| (m'_{k_i} + m'_{k_i+1}) \geqq 2^\nu.$$

Since this holds for all natural numbers ν, we proved that $\|Tf\|_1 = \infty$. Thus the proof of Theorem 5 is complete.

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ON PSEUDOMANIFOLDS WITH BOUNDARY. II

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In this paper we shall prove that each orientable (nonclosed) n-dimensional pseudomanifold with boundary and without homologically singular interior points — i.e., without interior points having noncyclic n-dimensional local Betti group with respect to the coefficient group Z — is absolutely nonlinked.

We shall use the definitions and notations of [8] without any comment.

1. c-regular domains in (n,p)-cells

Let p be a prime number and n a positive integer. Let Z_p be the cyclic group of integers $\bmod p$ and H the Čech homology theory defined on the category of compact pairs over the coefficient group Z_p. Let (X,A) be an (n,p)-cell (see [8] 1.2).

1.1. Definition. Let U be a domain i.e., a nonempty connected open set in $X \setminus A$. We say that U is a *c-regular domain of* (X,A) if

$$H_n(X, X \setminus U) \approx Z_p.$$

1.2. Remark. Let U be a c-regular domain of (X,A). Then the homomorphism $j_{1*}: H_n(X,A)|! \to H_n(X,X \setminus U)$ induced by the inclusion $j_1: (X,A) \subset (X,X \setminus U)$ is an isomorphism.

Indeed, let U_1 be a nonempty open subset in U such that the homomorphism $j_*: H_n(X,A) \to H_n(X,X \setminus U_1)$ induced by the inclusion $j: (X,A) \subset \subset (X,X \setminus U_1)$ is a monomorphism. 1.2(d) of [8] shows the existence of such a U_1. However $j_* = j_{2*}j_{1*}$ where $j_{2*}: H_n(X,X \setminus U) \to H_n(X,X \setminus U_1)$ is induced by the inclusion $j_2: (X,X \setminus U) \subset (X,X \setminus U_1)$, and thus j_{1*} is a monomorphism as well. Taking also $H_n(X,A) \approx H_n(X,X \setminus U) \approx Z_p$ (cf. [8] 1.2(b)) into account we obtain that j_{1*} is an isomorphism as required.

1.3. Remark. Let U be a domain in $X \setminus A$. If $(X,X \setminus U)$ is an (n,p)-cell then $H_n(X,X \setminus U) \approx Z_p$ and thus U is a c-regular domain of (X,A).

Conversely, if U is a c-regular domain of (X,A) then $(X,X \setminus U)$ is an (n,p)-cell.

Indeed, since $X \setminus A$ is a nonempty connected locally connected space with countable base (see [8] 1.2(a)) so is $U = X \setminus (X \setminus U)$.

Since $H_n(X,X \setminus U) \approx Z_p$ (see 1.1) the compact pair $(X,X \setminus U)$ satisfies condition 1.2(b) of [8] as well.

Let $i_1\colon (X,\emptyset)\subset(X,X\setminus U)$, $i\colon (X,\emptyset)\subset(X,A)$ and $j_1\colon (X,A)\subset(X,X\setminus U)$ be inclusions. Then by $i_{1*} = j_{1*}i_*\colon H_n(X) \to H_n(X,X\setminus U)$ and by the triviality of the homomorphism i_* (see [8] 1.2(c)) we obtain that i_{1*} is a trivial homomorphism, $i_{1*}(H_n(X)) = 0$.

To see that 1.2(d) of [8] is satisfied, let V_1 be a domain in $X\setminus(X\setminus U) = U$. Then $V_1 \subset X\setminus A$ and thus by [8] 1.2(d) there is a nonempty open subset $U_1 \subset V_1$ such that the homomorphism $j_{2*}\colon H_n(X,A) \to H_n(X,X\setminus U_1)$ induced by the inclusion $j_2\colon (X,A)\subset(X,X\setminus U_1)$ is a monomorphism. Let $j_{3*}\colon H_n(X,X\setminus U) \to H_n(X,X\setminus U_1)$ be the homomorphism induced by the inclusion $j_3\colon (X,X\setminus U)\subset(X,X\setminus U_1)$ and let j_{1*} be the same as above. Then $j_{2*} = j_{3*}j_{1*}$ and since j_{1*} is an isomorphism (see 1.2) it follows that j_{3*} is a monomorphism as required.

1.4. REMARK. Let U be a c-regular domain of (X,A). Let U_1 be a domain in U. Then obviously U_1 is a c-regular domain of the (n,p)-cell (X,A) if and only if U_1 is a c-regular domain of the (n,p)-cell $(X,X\setminus U)$ (cf. 1.3).

1.5. DEFINITION. Let q be a point of $X\setminus A$. q is said to be a *c-regular point of* (X,A) if q has a base of neighbourhoods in $X\setminus A$ consisting of c-regular domains of (X,A).

1.6. DEFINITION. We say that the (n,p)-cell (X,A) *has no c-singularity* or it is *without c-singularity* if each point of $X\setminus A$ is c-regular.

1.7. REMARK. The (n,p)-cell (X,A) is obviously without c-singularity if and only if there exists a base of $X\setminus A$ consisting of c-regular domains of (X,A).

2. (n,p)-cells without c-singularity in R^{n+1}

Let p, n, Z_p and H be the same as in Section 1. Let (X,A) be an (n,p)-cell in R^{n+1} i.e., X is a subspace of the $(n+1)$-euclidean space R^{n+1}. Suppose that (X,A) has no c-singularity (see 1.6).

2.1. DEFINITION. A domain G of $R^{n+1}\setminus A$ is said to be a *$c-e$-regular domain of* (X,A) if it is e-regular (see [8] 2.7) and if $G\cap X$ is a c-regular domain of (X,A).

2.2. REMARK. Since each domain of R^{n+1} lying in a ball disjoint to A is e-regular (see [8] 2.7) taking also 1.7 into account we obtain that each $q\in X\setminus A$ has a base of neighbourhoods consisting of $c-e$-regular domains of (X,A).

2.3. LEMMA. *Let G be a $c-e$-regular domain of (X,A). Then $G\setminus X$ has two components and the closure of each component contains $G\cap X$.*

PROOF. Let $U = G\cap X$. Since $H_n(X,X\setminus U)\approx Z_p$ Theorem 2 of [5] shows that $G\setminus X$ has at most two components. On the other hand by Theorem 2.15 of [8] $G\setminus X$ has at least two components. Consequently $G\setminus X$ has exactly two components — say G_1 and G_2.

Let $q \in U$. We have to show that q is a limit point of both domains G_1 and G_2, i.e., that for each open ball G' in R^{n+1} around q, G' meets both components G_1 and G_2.

Let G' be such a ball and U' a c-regular domain of (X, A) such that $q \in U' \subset U \cap G'$. By 1.6 such a U' exists. 1.3 shows that $(X, X \setminus U)$ is an (n,p)-cell and by 1.4, U' is a c-regular domain of $(X, X \setminus U)$. Hence the homomorphism $j_*\colon H_n(X, X \setminus U) \to H_n(X, X \setminus U')$ induced by the inclusion $j\colon (X, X \setminus U) \subset (X, X \setminus U')$ is an isomorphism (see 1.2). Moreover by [8] 2.16, $U' = X \setminus (X \setminus U')$ is nowhere dense in R^{n+1} and thus according to [5] Theorem 3, $H_{n+1}(X, X \setminus U') = 0$. Consider the segment

$$H_n(X, X \setminus U') \xleftarrow{j_*} H_n(X, X \setminus U) \longleftarrow H_n(X \setminus U', X \setminus U) \longleftarrow H_{n+1}(X, X \setminus U')$$

of the exact homology sequences of the triple $(X, X \setminus U', X \setminus U)$. Since j_* is a monomorphism and $H_{n+1}(X, X \setminus U') = 0$ we get $H_n(X \setminus U', X \setminus U) = 0$ and thus $G \setminus ((X \setminus U') \setminus (X \setminus U)) = G \setminus (X \setminus U')$ is connected (see the consequence of Theorem 2 in [5]).

Let $Q = G \setminus (X \setminus U')$. Q is clearly a connected open subset of R^{n+1} and $U' = Q \cap X$ is a closed subset of Q. Moreover we clearly have $Q \setminus U' = Q \setminus X = G \setminus X$ and thus G_1 and G_2 are the components of $Q \setminus U'$. However the open subset $P = G' \cap Q$ of Q contains U' and thus $P = G' \cap Q$ meets both G_1 and G_2 (see [7] 3.2). Consequently $G' \cap G_1 \neq \emptyset$ and $G' \cap G_2 \neq \emptyset$ as required.

We now recall some definitions of [7] concerning k-manifolds.

Let R be a T_2-space and (Y, B) a compact pair in R.

2.4. Definition. Let V be a domain in R. We say that V is a *regularly intersecting domain of* (Y, B) if

(a) $V \cap B = \emptyset$.

(b) $V \cap Y$ is a domain of $Y \setminus B$.

If V is a regularly intersecting domain of (Y, B) and $U = V \cap Y$ we then say that V *regularly intersects the compact pair* (Y, B) *in* U.

2.5. Definition. A domain V of R is said to be *k-regular* mod (Y, B) if the following conditions are fulfilled:

(a) V is a regularly intersecting domain of (Y, B).

(b) $V \setminus Y$ consists of two components.

(c) The closure of each component of $V \setminus Y$ contains $V \cap Y$.

2.6. Definition. The compact pair (Y, B) itself is called a *k-manifold in R* if it satisfies the following two conditions:

(a) $Y \setminus B$ is a nonempty connected space,

(b) for every $q \in Y \setminus B$ the k-regular domains that contain the point q form a base for the neighbourhood system of the point q in R.

We now return to the (n,p)-cell (X, A) in R^{n+1}.

2.7. A domain V of R^{n+1} is said to be *small* if it is contained in an open ball disjoint to A. Let Σ be the set of mod (X, A) k-regular small domains. Each member of Σ is clearly an e-regular domain of (X, A).

We show that each $q \in X \setminus A$ has a base of neighbourhoods in R^{n+1} consisting of members of Σ.

Indeed, let V' be a neighbourhood of q in R^{n+1}. Let G be an open ball around q disjoint to A. Since the $c-e$-regular domains of (X, A) containing q form a base of neighbourhoods of q (see 2.2) it follows the existence of a $c-e$-regular domain V of (X, A) such that $q \in V \subset V' \cap G$. However according to 2.3, V is a $\bmod (X, A)$ k-regular domain (see also 2.5). Thus $V \in \Sigma$ as required.

Now taking also 1.2(a) of [8] and 2.6 into account we can state that (X, A) is a k-manifold in R^{n+1}.

2.8. Definitions. Let $V, V' \in \Sigma$ (cf. 2.7). V and V' are said to be *compatible* if either $V \subset V'$ or $V' \subset V$.

By a ***Σ-chain*** we mean a sequence $\alpha = (V_1, \ldots, V_k)$ in Σ such that for $i = 1, \ldots, k-1$, V_i and V_{i+1} are compatible members of Σ. α is said to be *closed* if $V_1 = V_k$.

Let $K: q \to q'$ be a continuous path in $X \setminus A$ (see [8] 2.4) and let $K = K_1 K_2 \ldots K_m$ (cf. [8] 2.4) be a subdivision of K into factors where $K_i = K_i: q_{i+1} \to q_i$ for $i = 1, \ldots, m$. Then we clearly have $q_1 = q'$ and $q_{m+1} = q$.

Now the Σ-chain $\alpha = (V_1, \ldots, V_{m+1})$ is said to be ***associated to the subdivision*** $K = K_1 K_2 \ldots K_m$ ***of*** K if $q_i \in V_i$ for $i = 1, \ldots, m+1$ and $\tilde{K}_i \subset \subset V_i \cup V_{i+1}$ i.e., $\tilde{K}_i \subset V_i$ or $\tilde{K}_i \subset V_{i+1}$ for $i = 1, \ldots, m$ (cf. [8] 2.4). In this case we also say that ***the subdivision*** $K = K_1 K_2 \ldots K_m$ ***of*** K ***is associated to the*** Σ***-chain*** $\alpha = (V_1, \ldots, V_{m+1})$.

Let K be a continuous path in $X \setminus A$ and $\alpha = (V_1, \ldots, V_{m+1})$ a Σ-chain. We say that ***K and α are associated*** or ***K is associated to α*** or ***α is associated to K*** if there exists a subdivision $K = K_1 K_2 \ldots K_m$ of K associated to α.

2.9. Observe that for each continuous path $K: q \to q'$ in $X \setminus A$ and for each $V, V' \in \Sigma$ with $q \in V$ and $q' \in V'$ there is a Σ-chain $\alpha = (V_1, \ldots, V_{m+1})$ associated to K such that $V_1 = V'$ and $V_{m+1} = V$ (see [3] 6.8).

Consequently to each closed path K of $X \setminus A$ there is a closed Σ-chain $\alpha = (V_1, \ldots, V_{m+1})$ associated to K.

Observe that in this case the closed path $K \cdot K$ and the closed Σ-chain $\beta = (V_1, \ldots, V_{m+1} = V_1, V_2, \ldots, V_{m+1})$ are clearly associated to each other.

2.10. Let $V \in \Sigma$. Then by the ***banks of*** V we mean the components of $V \setminus X$ and we design them by $P^1(V)$ and $P^2(V)$. Obviously the numeration here is arbitrary.

2.11. Let $\alpha = (V_1, \ldots, V_m)$ be a Σ-chain and let $P^1(V_1)$ and $P^2(V_1)$ be the banks of V_1. Then there exists a numeration P_i^1 and P_i^2 of the banks of V_i such that

(a) $P_1^1 = P^1(V_1)$ and $P_1^2 = P^2(V_1)$,

(b) $P_i^1 \cap P_{i+1}^1 \neq \emptyset$ and $P_i^2 \cap P_{i+1}^2 \neq \emptyset$ for $i = 1, \ldots, m+1$ and this numeration is unique (see [4] 2.10).

Hence two sequences $\alpha_P(1) = (P_1^1, \dots, P_m^1)$ and $\alpha_P(2) = (P_1^2, \dots, P_m^2)$ of the banks belong to the Σ-chain α. Moreover for $j = 1, 2$ we have $P_i^j \subset \subset P_{i+1}^j$ in the case $V_i \subset V_{i+1}$ and $P_{i+1}^j \subset P_i^j$ in the case $V_{i+1} \subset V_i$. The sequences $\alpha_P(1)$ and $\alpha_P(2)$ are called *the chains of banks associated to the Σ-chain α.*

If α is closed i.e., if $V_m = V_1$ then two cases are possible:

(i) $P_m^1 = P_1^1$ and $P_m^2 = P_1^2$,

(ii) $P_m^1 = P_1^2$ and $P_m^2 = P_1^1$.

In the first case we say that α *preserves its banks* and in the second case that α *changes its banks.*

Observe that for each closed Σ-chain $\alpha = (V_1, \dots, V_m)$ the closed Σ-chain $\beta = (V_1, \dots, V_m = V_1, V_2, \dots, V_m)$ clearly preserves its banks.

2.12. THEOREM. *Let K be a continuous closed path in $X \setminus A$ and α a closed Σ-chain associated to K (see 2.9). Suppose that α preserves its banks. Then K is nonlinked to (X, A) (see [8] 2.6).*

PROOF. Let $\alpha = (V_1, \dots, V_{m+1} = V_1)$ and let $K = K_1 K_2 \dots K_m$ be a subdivision of K into factors associated to α. For $i = 1, \dots, m$ let $K_i = K_i \colon q_{i+1} \to q_i$. Then $q_1 = q_{m+1}$. Moreover for $i = 1, \dots, m+1$ we have $q_i \in V_i$ and for $i = 1, \dots, m$ $\tilde{K}_i \subset V_i \cup V_{i+1}$ (see 2.8). Let $(P_1^1, \dots, P_{m+1}^1)$ be a chain of banks associated to the chain α. Since α preserves its banks we have $P_{m+1}^1 = P_1^1$. For $i = 1, \dots, m$ take a point q_i' from P_i^1 and let $q_{m+1}' = q_1' \in \in P_{m+1}^1 = P_1^1$. For $i = 1, \dots, m$ let $K_i' \colon q_i' \to q_{i+1}'$ be a continuous path in the connected and in R^{n+1} open $P_i^1 \cup P_{i+1}^1$ which is either P_i^1 or P_{i+1}^1 and for $i = 1, \dots, m$ let $K_i'' \colon q_i' \to q_i$ be a continuous path in the domain V_i. Let $K_{m+1}'' = K_1''$. Now for $i = 1, \dots, m$ the closed path $\overline{K_i} = (K_i'')^{\cdot} K_i K_{i+1}'' K_i'$ is lying in $V_i \cup V_{i+1}$ which is either V_i or V_{i+1} and since both domains V_i and V_{i+1} are e-regular (see 2.7), it follows that $\overline{K_i}$ is nonlinked to (X, A) (see [8] 2.7). On the other hand $K' = K_m' \dots K_2' K_1'$ is a continuous closed path in $R^{n+1} \setminus X$. Hence K' is nonlinked to (X, A), too (see [8] 2.6). Consequently by [8] 2.13, K is nonlinked to (X, A) as required.

2.13. THEOREM. *Let K be a continuous closed path in $X \setminus A$ and α a closed Σ-chain associated to K. Suppose that α changes its banks. Then K is linked to (X, A).*

PROOF. Let $\alpha = (V_1, \dots, V_{m+1} = V_1)$. V_1 is an e-regular domain of (X, A) that meets X (see 2.7 and 2.5). According to [8] 2.15, there are points $q', q'' \in V_1 \setminus X$ and continuous paths $K^4 \colon q'' \to q'$ and $K^3 \colon q' \to q''$ in $R^{n+1} \setminus A$ such that

(a) q' and q'' are in distinct components of $V_1 \setminus X$,

(b) $\tilde{K}^4 \subset V_1$,

(c) $\tilde{K}^3 \subset R^{n+1} \setminus X$,

(d) $K^3 K^4$ is linked to (X, A).

Let $K = K_m \dots K_2 K_1$ be a subdivision of K into factors associated to α where $K_i = K_i : q_i \to q_{i+1}$ for $i = 1, \dots, m$ and thus $q_1 = q_{m+1}$. As we have seen in 2.8 for $i = 1, \dots, m+1$ one has $q_i \in V_{m+2-i}$ and for $i = 1, \dots, m$ $\tilde{K}_i \subset V_{m+2-i} \cup V_{m+1-i}$.

Let $K_{m+1} : q_1 \to q_1$ be a degenerated path. Then $K = K_{m+1} K_m \dots K_1$.

Since $V_1 \in \Sigma$ it follows that V_1 is a k-regular domain $\operatorname{mod}(X, A)$ (see 2.7). Select the numeration of the banks of V_1 so that $q' \in P^1(V_1)$ and $q'' \in P^2(V_1)$ (cf. 2.10). Let

$$\alpha_P(1) = (P_1^1, \dots, P_{m+1}^1)$$

be the chain of banks associated to the chain α so that $P_1^1 = P^1(V_1)$. Since the chain α changes its banks it follows $P_{m+1}^1 = P^2(V_1)$ (see 2.11).

For $i = 1, \dots, m+1$ take a point q_i' in P_{m+2-i}^1. Hence $q_1' \in P^2(V_1)$. For $i = 1, \dots, m$ let $K_i' : q_{i+1}' \to q_i'$ be a continuous path in $P_{m+2-i}^1 \cup P_{m+1-i}^1$ which is either P_{m+2-i}^1 or P_{m+1-i}^1 and for $i = 1, \dots, m+1$ let $K_i'' : q_i \to q_i'$ be a continuous path in the domain V_{m+2-i}. Let $K'^1 : q_1' \to q''$ and $K'^2 : q' \to q_{m+1}'$ be continuous paths in $P^2(V_1) = P_{m+1}^1$ and $P_1(V_1) = P_1^1$ respectively. Let

$$K^2 = K'^1 K_1' K_2' \dots K_m' K'^2 : q' \to q''$$

and

$$K' = K^2 K^4 : q'' \to q''.$$

We first show that the closed path K' is linked to (X, A).

Indeed, let $K_1^0 : q' \to q'$ and $K_2^0 : q'' \to q''$ be degenerated paths and let $K^1 = (K^4)^{\cdot}$. The closed path $(K^2)^{\cdot} K^3 = (K^2)^{\cdot} K_2^0 K^3 K_1^0$ is lying in $R^{n+1} \setminus X$ and thus it is nonlinked to (X, A) (see [8] 2.6). Hence if $K' = K^2 K^4 = K^2 (K^1)^{\cdot}$ were nonlinked to (X, A) then by [8] 2.9, [8] 2.11 and [8] 2.9 again the closed paths $(K^1)^{\cdot} K^2 = (K^1)^{\cdot} K_2^0 K^2 K_1^0$, $(K^1)^{\cdot} K_2^0 K_2^0 K^3 K_1^0 K_1^0 = (K^1)^{\cdot} K^3 = K^4 K^3$ and $K^3 K^4$ would be nonlinked to (X, A) as well, contradicting the assumption (d).

Let $K_{m+1}' = K'^2 K^4 K'^1 : q_1' \to q_{m+1}'$. Since the closed path

$$K' = K^2 K^4 = K'^1 K_1' K_2' \dots K_m' K'^2 K^4$$

is linked to (X, A) it follows by [8] 2.9 that the closed path

$$K_1' K_2' \dots K_m' K'^2 K^4 K'^1 = K_1' \dots K_m' K_{m+1}'$$

is linked to (X, A) as well.

Let

$$\overline{K_{m+1}} = (K_{m+1}'')^{\cdot} K_{m+1}' K_1'' K_{m+1}.$$

$\overline{K_{m+1}}$ is a closed path in the e-regular domain $V_1 = V_{m+1}$ of (X, A) and thus $\overline{K_{m+1}}$ is nonlinked to (X, A). For $i = 1, \dots, m$ let $\overline{K_i} = (K_i'')^{\cdot} K_i' K_{i+1}'' K_i$.

$\overline{K_i}$ is a closed path in $V_{m+2-1} \cup V_{m+1-i}$ which is either V_{m+2-i} or V_{m+1-i}. However each member of Σ is an e-regular domain of (X, A) and thus $\overline{K_i}$ is a nonlinked closed path of (X, A). The closed paths $\overline{K_1}, \dots, \overline{K_{m+1}}$ are all nonlinked to (X, A) while $K'_1 \dots K'_m K'_{m+1}$ is a linked closed path of (X, A). Thus by [8] 2.13 $K = K_{m+1} K_m \dots K_1$ is a linked closed path of (X, A) as required.

The proof of the theorem is complete.

2.14. THEOREM. *If $p \neq 2$ then (X, A) is a nonlinked (n, p)-cell* (cf. [8] 1.11).

PROOF. Let $f\colon [a, b] \to X \setminus A$ be a closed continuous line in $X \setminus A$ (see [8] 1.6) and let K be the equivalence class of f, i.e., the closed path with the representative f (see [8] 2.4). According to [8] 2.6, we only need to show that K is nonlinked to (X, A).

Let $\alpha = (V_1, \dots, V_{m+1})$ be a closed Σ-chain associated to K (see 2.9) and

$$\beta = (V_1, \dots, V_{m+1} = V_1, V_2, \dots, V_{m+1}).$$

The closed Σ-chain β is associated to the closed path $K \cdot K$ (see 2.9) and β preserves its banks (see 2.11). Hence according to 2.12 the closed path $K \cdot K$ is nonlinked to (X, A).

Let $\mathfrak{V} = \mathfrak{V}_{p,n-1,1}$ be a nondegenerated theory of linking in R^{n+1} (cf. [8] 1.8). Since $K \cdot K$ is nonlinked to (X, A) it follows

$$\mathfrak{v}_{A,\widetilde{KK}}(A_*, (KK)_*) = 0 \tag{1}$$

(see [8] 2.6, [8] 1.4 and [8] 2.5). However

$$\mathfrak{v}_{A,\widetilde{KK}}(A_*, (KK)_*) = \mathfrak{v}_{A,\tilde{K}}(A_*, K_*) + \mathfrak{v}_{A,\tilde{K}}(A_*, K_*) = 2 \cdot \mathfrak{v}_{A,\tilde{K}}(A_*, K_*) \tag{2}$$

(see [8] 2.10) and since $\mathfrak{v}_{A,\tilde{K}}(A_*, K_*) \in Z_p$ (see [8] 1.8), $p \neq 2$ and the relations (1) and (2) imply $\mathfrak{v}_{A,\tilde{K}}(A_*, K_*) = 0$. Consequently K is nonlinked to (X, A) as required.

2.15. THEOREM. *If $p \neq 2$ and if (X, A) is simultaneously an $(n, 2)$-cell which has no c-singularity then (X, A) is a nonlinked $(n, 2)$-cell.*

PROOF. Let f be a closed line in $X \setminus A$ and let K be the equivalence class of f i.e., the closed path with the representative f. We only need to show that K is nonlinked to the $(n, 2)$-cell (X, A).

Let Σ be the same as in 2.7. Let α be a closed Σ-chain associated to K (see 2.9). According to 2.14, K is nonlinked to the (n, p)-cell (X, A) hence by 2.13 the chain α preserves its banks. Consequently by 2.12, K is nonlinked to the $(n, 2)$-cell (X, A) as required.

3. Pseudomanifolds with boundary and without homologically singular interior points

3.1. A partially ordered set $(E, \leqq)$ is said to be *directed* if for any $e_1, e_2 \in \in E$ there is an $e \in E$ with $e_1 \leqq e$ and $e_2 \leqq e$.

In the sequel all groups are abelian. Accordingly the group operation will be referred to as addition.

3.2. A *directed system* $D = (G_e, \varphi_{e,e'}, (E, \leqq))$ *of abelian groups* consists of the following: A directed partially ordered set $(E, \leqq)$; for each $e \in E$ an abelian group G_e; for each pair $e \leqq e'$ from E a homomorphism $\varphi_{e,e'} : G_e \to \to G_{e'}$ satisfying the following two conditions:

$$\varphi_{e,e} = \mathrm{id}_{G_e} \quad \text{for} \quad e \in E,$$

$$\varphi_{e_2,e_3} \circ \varphi_{e_1,e_2} = \varphi_{e_1,e_3} \quad \text{for} \quad e_1 \leqq e_2 \leqq e_3.$$

Let $D = (G_e, \varphi_{e,e'}, (E, \leqq))$ be a directed system of groups and let $\overline{D} = = \bigcup_{e \in E} G_e \times \{e\}$. For (g_1, e_1) and (g_2, e_2) in $\overline{D}$ let $(g_1, e_1) \sim (g_2, e_2)$ if there is an $e_3 \geqq e_1, e_2$ so that

$$\varphi_{e_1,e_3}(g_1) = \varphi_{e_2,e_3}(g_2).$$

$\sim$ is clearly an equivalence on $\overline{D}$.

Let $\overline{D}/\sim$ be the family of equivalence classes of $\sim$. For each $(g, e) \in \in \overline{D}$ let $\widetilde{(g, e)}$ denote the equivalence class of (g, e). (g, e) is said to be a *representative* of $\widetilde{(g, e)}$.

The addition in $\overline{D}/\sim$ is defined as follows: Let $(g_1, e_1), (g_2, e_2) \in \overline{D}$ and select e_3 so that $e_1 \leqq e_3$ and $e_2 \leqq e_3$. Let

$$\widetilde{(g_1, e_1)} + \widetilde{(g_2, e_2)} = \widetilde{(\varphi_{e_1,e_3}(g_1) + \varphi_{e_2,e_3}(g_2), e_3)}.$$

This addition is clearly well defined and $\overline{D}/\sim$ equipped with this addition becomes an abelian group called the *limit group* of the directed system of groups D. We use $\varinjlim D$ to denote this limit group.

3.3. We should mention that if E^* is a cofinal subset of E and for each $e, e' \in E^*$ with $e \leqq e'$ $\varphi_{e,e'}$ is an isomorphism then the groups G_e $(e \in E^*)$ are clearly isomorphic to each other and these groups are isomorphic to the group $\varinjlim D$.

3.4. Let $\mathbf{N}_0$ be the set of nonnegative integers, i.e., $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For $k \in \mathbf{N}$ let Z_k be the cyclic group of integers mod k and let $Z_0 = Z$. Moreover for $k \in \mathbf{N}_0$ let H^k be the Čech homology theory defined on the category of compact pairs with the coefficient group Z_k.

3.5. Let $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$. Let (X, A) be a compact pair and $q \in X \setminus A$.

Let E be the family of all open neighbourhoods of q in $X \setminus A$ and for $U, U' \in E$ let $U \leqq U'$ if $U' \subset U$. Thus $(E, \leqq)$ is clearly a directed set.

For each $U \in E$ let $G_U = H_n^k(X, X \setminus U)$ and for $U \leqq U'$ $(U, U' \in E)$ let $\varphi_{U,U'}: G_U \to G_{U'} = i_{U,U'*}: H_n^k(X, X \setminus U) \to H_n^k(X, X \setminus U')$ be the homomorphism induced by the inclusion $i_{U,U'}: (X, X \setminus U) \subset (X, X \setminus U')$.

Thus we obtain a directed system of groups

$$D_n^k(q) = (G_U, \varphi_{U,U'}, (E, \leqq)).$$

The limit group $\varinjlim D_n^k(q)$ of this directed system is called the *n-dimensional local Betti group of the compact pair* (X, A) *at the point* q *with respect to the coefficient group* Z_k. We use $H_n^k(X, A, q)$ to denote this local Betti group.

3.6. PROPOSITION. *Let* K *be a triangulation situated in some euclidean space* R^s. *Let* L *be a closed subcomplex of* K (see [1] p. 126). *Let* Y *and* F *be the body of* K *and* L *respectively* (see [1] p. 136), *i.e.* $(Y, F) = (\|K\|, \|L\|)$. *Let* $q \in Y \setminus F$. *Let* $O_K(q)$ *be the set of all simplexes* $T \in K$ *with* $q \in \overline{T}$ *where* $\overline{T}$ *is the closure of* T. $O_K(q)$ is an open subcomplex of K. *Let* $r \in \mathbf{N}_0$ *and* $n \in \mathbf{N}$. *Then*

$$H_n^r(Y, F, q) \approx \Delta_r^n(O_k(q)) \qquad \text{(cf. [2] p. 50)}.$$

PROOF. Consider R^s as a hyperplane of the euclidean $(s+1)$-space R^{s+1}. Let c be a point in $R^{s+1} \setminus R^s$.

Let $O'_K(q)$ be the subcomplex of K consisting of all simplexes of $O_K(q)$ and of all faces of such simplexes. Let $B_K(q) = O'_K(q) \setminus O_K(q)$. $O'_K(q)$ and $B_K(q)$ are closed subcomplexes of K.

Let $M^*(q)$ be the set of all open cones with the vertex c where the base of the cones runs over all open simplexes of $B_K(q)$ (see [1] p. 214). Let

$$M(q) = B_K(q) \cup M^*(q) \cup \{c\},$$

where $\{c\}$ is the 0-simplex with the vertex c. $M(q)$ and $M(q) \cup O'_K(q) = M(q) \cup O_K(q)$ are clearly triangulations in R^{s+1} and $M(q) \cap O'_K(q) = B_K(q)$.

An easy computation shows that

$$\Delta_r^n(O_K(q)) \approx \Delta_r^n(M(q) \cup O'_K(q)).$$

Also, observe that the group

$$\Delta_r^n(M(q) \cup O'_K(q)) = \Delta^n(M(q) \cup O'_K(q), Z_r)$$

is isomorphic to the group

$$\Delta^n(\|M(q) \cup O'_K(q)\|, Z_r)$$

(see [2] p. 166 and [2] p. 159) and this latter group is isomorphic to $H_n^r(\|M(q) \cup O'_K(q)\|)$ (see [6], 24). However $\|M(q)\|$ is contractible to a point over itself and thus it is homologically trivial. Moreover the compact pair $(\|M(q)\cup \cup O'_K(q)\|, \|M(q)\|)$ can be triangulated and thus its Čech homology sequence is exact (see IX.9.4 of [9] p. 251). Consequently the groups $H^r_n(\|M(q)\cup \cup O'_K(q)\|)$ and $H^r_n(\|M(q)\cup O'_K(q)\|, \|M(q)\|)$ are isomorphic:

$$H^r_n(\|M(q)\cup O'_K(q)\|) \approx H^r_n(\|M(q)\cup O'_K(q)\|, \|M(q)\|)$$

(see [9] p. 23).

Now let $V = \|M(q)\| \setminus \|B_K(q)\| = \|M(q)\cup O'_K(q)\| \setminus \|O'_K(q)\|$. The subset V of $\|M(q)\|$ is open in $\|M(q)\cup O'_K(q)\|$ and we have

$$\|M(q)\cup O'_K(q)\| \setminus V = \|O'_K(q)\|, \qquad \|M(q)\| \setminus V = \|B_K(q)\|.$$

Hence the inclusion map

$$j\colon (\|O'_K(q)\|, \|B_K(q)\|) \subset (\|M(q)\cup O'_K(q)\|, \|M(q)\|)$$

induces an isomorphism in each dimension (see [9] X.5.4, pp. 266, 267) and thus

$$H^r_n(\|M(q)\cup O'_K(q)\|, \|M(q)\|) \approx H^r_n(\|O'_K(q)\|, \|B_K(q)\|).$$

Hence

$$\Delta^n_r(O_K(q)) \approx H^r_n(\|O'_K(q)\|, \|B_K(q)\|). \tag{3}$$

We now construct a suitable base of neighbourhoods of the point q in $Y \setminus F$.

First observe that $\|O'_K(q)\|$ is a closed cone over $\|B_K(q)\|$ with vertex q.

Now for $m \in \mathbf{N}$ let ψ_m be the positive dilatation of R^s with the invariant point q and with the ratio of minification $\frac{1}{m}$, i.e.

$$\overrightarrow{q\psi_m(q')} = \frac{1}{m}\overrightarrow{qq'}$$

for each $q' \in R^s$ and let

$$U_m = \psi_m(\|O'_K(q)\| \setminus \|B_K(q)\|).$$

U_m is clearly an open subset of $\|O'_K(q)\| \setminus \|B_K(q)\|$ and we have

$$\|O'_K(q)\| \setminus \|B_K(q)\| = U_1 \supset U_2 \supset \ldots \supset U_m \supset \ldots .$$

Moreover $\{U_1, \ldots, U_m, \ldots\}$ is a base of neighbourhoods of q in $Y \setminus F$.

Let

$$\overline{\psi_m} = \psi_m \,|_{\|O'_K(q)\|}\colon (\|O'_K(q)\|, \|B_K(q)\|) \to (\psi_m(\|O'_K(q)\|), \psi_m(\|B_K(q)\|))$$

and let $P_m = \|O'_K(q)\| \setminus U_m$. Thus $P_1 = \|B_K(q)\|$. $\overline{\psi_m}$ is a homeomorphism and thus the induced

$$\overline{\psi_m}_*: H^r_n(\|O'_K(q)\|, \|B_K(q)\|) \to H^r_n(\psi_m(\|O'_K(q)\|), \psi_m(\|B_K(q)\|))$$

is an isomorphism. Let

$$i_m: (\|O'_K(q)\|, \|B_K(q)\|) \subset (\|O'_K(q)\|, P_m)$$

and

$$j_m: (\psi_m(\|O'_K(q)\|), \psi_m(\|B_K(q)\|)) \subset (\|O'_K(q)\|, P_m)$$

be inclusion maps. Then $j_m\overline{\psi_m}$ is clearly homotopic to i_m and thus $j_{m*}\overline{\psi_m}_* = i_{m*}$ where j_{m*} and i_{m*} are the homomorphisms induced by the inclusions j_m and i_m respectively. However by

$$\psi_m(\|O'_K(q)\|) \setminus \psi_m(\|B_K(q)\|) = \|O'_K(q)\| \setminus P_m = U_m,$$

j_m is an excision and thus j_{m*} is an isomorphism. Consequently i_{m*} is an isomorphism as well.

Let $h_m: (\|O'_K(q)\|, P_m) \subset (Y, Y \setminus U_m)$ be an inclusion map. By

$$\|O'_K(q)\| \setminus P_m = Y \setminus (Y \setminus U_m) = U_m,$$

h_m is an excision and thus the induced

$$h_{m*}: H^r_n(\|O'_K(q)\|, P_m) \to H^r_n(Y, Y \setminus U_m)$$

is an isomorphism, too.

Hence for $m \in \mathbf{N}$

$$H^r_n(\|O'_K(q)\|, \|B_K(q)\|) \approx H^r_n(Y, Y \setminus U_m). \tag{4}$$

For $m, m' \in \mathbf{N}$ with $m \leqq m'$ let

$$\varphi_{U_m,U_{m'}}: H^r_n(Y, Y \setminus U_m) \to H^r_n(Y, Y \setminus U_{m'})$$

be the homomorphism induced by the inclusion $i_{U_m,U_{m'}}: (Y, Y \setminus U_m) \to (Y, Y \setminus U_{m'})$ and for $m \in \mathbf{N}$ consider the commutative diagram

$$\begin{array}{ccc} H^r_n(Y, Y \setminus U_1) & \xrightarrow{\varphi_{U_1,U_m}} & H^r_n(Y, Y \setminus U_m) \\ \uparrow h_{1*} & & \uparrow h_{m*} \\ H^r_n(\|O'_K(q)\|, P_1) & \xrightarrow{i_{m*}} & H^r_n(\|O'_K(q)\|, P_m). \end{array}$$

Since i_{m*}, h_{m*} and h_{1*} are isomorphisms it follows that φ_{U_1,U_m} is an isomorphism as well. Moreover since for $m, m' \in \mathbf{N}$ with $m \leqq m'$ one has

$$\varphi_{U_m,U_{m'}}\varphi_{U_1,U_m} = \varphi_{U_1,U_{m'}}$$

we obtain that $\varphi_{U_m,U_{m'}}$ is an isomorphism, too. Now according to 3.5, 3.3 and (4) $H^r_n(Y,F,q)$ is isomorphic to $H^r_n(\|O'_K(q)\|,\|B_K(q)\|)$ and thus by (3)

$$H^r_n(Y,F,q) \approx \Delta^n_r(O_K(q))$$

as required.

The proof is complete.

3.7. REMARK. As we have seen in the proof of 3.6, the assumptions of 3.6 imply the existence of a countable base of neighbourhoods $\{U_1,\dots,U_m,\dots\}$ of q in $Y\setminus F$ such that for each $m\in \mathbf{N}$ and $r\in \mathbf{N}_0$

$$H^r_n(Y,Y\setminus U_m) \approx \Delta^n_r(O_K(q)).$$

3.8. Let K and L be the same as in 3.6 and suppose that K is an orientable n-dimensional (combinatorial) pseudomanifold with boundary L and $L\neq\emptyset$ (see [2] pp. 72, 74). Then according to [8] 3.4 $(\|K\|,\|L\|)=(Y,F)$ is an (n,p)-cell for each prime p.

Let $q\in\|K\|\setminus\|L\|$. Then $O_K(q)$ can be clearly uniquely represented in the form

$$O_K(q)=E_1\cup\dots\cup E_{t(q)}$$

where for $j=1,\dots,t(q)$ E_j is a closed subcomplex of $O_K(q)$, it is an orientable n-pseudomanifold and for $j\neq j'$ $(j,j'\in\{1,\dots,t(q)\})$ the dimension of the subcomplex $E_j\cap E_{j'}$ of K is less than $n-1$.

Let $r\in\mathbf{N}_0$. Then a light computation shows that

$$\Delta^n_r(O_K(q))\approx\Delta^n_r(E_1)\oplus\dots\oplus\Delta^n_r(E_{t(q)})\approx \underbrace{Z_r}_{1}\oplus\dots\oplus\underbrace{Z_r}_{t(q)}.$$

Hence according to 3.6, $H^0_n(Y,F,q)$ is isomorphic to a cyclic group for each $q\in Y\setminus F$ if and only if $t(q)=1$ for each $q\in Y\setminus F$ and in this case by 3.7 each $q\in Y\setminus F$ has a countable base of neighbourhoods $\{U_1(q),\dots,U_m(q),\dots\}$ in $Y\setminus F$ so that $H^p_n(Y,Y\setminus U_m(q))\approx Z_p$ for each $m\in\mathbf{N}$ and each prime p.

Consequently taking also 1.5 and 1.6 into account we can state the following theorem:

If (Y,F) is an orientable n-dimensional (topological) pseudomanifold with boundary and without singular interior points i.e., *without interior points having noncyclic n-dimensional local Betti group with respect to the coefficient group $Z=Z_0$ then for each prime p, (Y,F) is an (n,p)-cell without c-singularity.*

3.9. DEFINITION. The (n,p)-cell (Y,B) is said to be an *absolutely nonlinked cell* if for each topological embedding $\varphi\colon Y\to R^{n+1}$ $(\varphi(Y),\varphi(B))$ is a nonlinked (n,p)-cell in R^{n+1}.

Now we can state the following theorem.

3.10. THEOREM. *Let (Y, B) be an orientable n-dimensional pseudomanifold with boundary and without singular interior points i.e., without interior points having noncyclic n-dimensional local Betti group with respect to the coefficient group Z then for each prime p, (Y, B) is an absolutely nonlinked (n, p)-cell.*

PROOF. Let $\varphi: Y \to R^{n+1}$ be a topological embedding of Y into R^{n+1}. Let $X = \varphi(Y)$ and $A = \varphi(B)$. Then by the theorem of 3.8 for each prime p, (X, A) is an (n, p)-cell without c-singularity in R^{n+1}. Consequently according to 2.14 and 2.15 for each prime p, (X, A) is a nonlinked (n, p)-cell. Hence for each prime p, (Y, B) is an absolutely nonlinked (n, p)-cell as required.

Our program is finished.

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MEASURABLE SOLUTIONS OF FUNCTIONAL EQUATIONS OF SUM FORM

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1. Introduction

Let $n \geqq 2$ be an integer and denote by Γ_n the set of all complete n-ary probability distributions, that is

$$\Gamma_n = \left\{ x = (x_1, \dots, x_n) \mid x_i \geqq 0, \ \sum_{i=1}^{n} x_i = 1 \right\}$$

and let Γ_n^0 be the same set but with positive probabilities:

$$\Gamma_n^0 = \left\{ x = (x_1, \dots, x_n) \mid x_i > 0, \ \sum_{i=1}^{n} x_i = 1 \right\}.$$

I and Δ_n will denote either $[0,1]$ and Γ_n or $]0,1[$ and Γ_n^0. Let $f_{ij}, g_{is}, h_{js}: I \to \to \mathbf{C}$ $(i = 1, \dots, k;\ j = 1, \dots, l,\ s = 1, \dots, N;\ k, l \geqq 2)$ be given or unknown functions. By a *functional equation of sum form* we shall mean an equation of the form

$$(1) \qquad \sum_{i=1}^{k} \sum_{j=1}^{l} \left[f_{ij}(x_i y_j) - \sum_{s=1}^{N} g_{is}(x_i) h_{js}(y_j) \right] = 0 \qquad (x \in \Delta_k,\ y \in \Delta_l)$$

(see [19]). The pair (k,l) will be called *the type of* (1) while we shall refer to N as the *index of equation* (1).

Functional equations of sum form have important applications in characterization problems of entropies having the sum property (see Aczél–Daróczy [2]). During the past thirty years several special cases of (1) have been solved by various authors: Chaundy–McLeod [5], Aczél–Daróczy [3], Behara–Nath [4], Kannappan [10]–[15], Daróczy [6], Losonczi [17]–[21], Losonczi–Maksa [22], [23], Maksa [24], [25], Daróczy–Járai [7], Mittal [26], Sahoo [28]. Although nowadays functional equations of sum form cannot give much new in entropy characterizations, there are several interesting unsolved equations of sum form. General results are known only if $k, l \geqq 3$ [18]. The most difficult equations are the equations of type (2,2). In this direction we mention the paper of Daróczy and Járai [7] who determined the measurable solutions of

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$$\sum_{i=1}^{2}\sum_{j=1}^{2}[f(x_iy_j) - x_if(y_j) - y_jf(x_i)] = 0 \qquad (x, y \in \Gamma_2^0). \tag{2}$$

The same equation with several unknown functions was solved by Kannappan and Ng [16]. The author determined the $C_3[0,1]$ solutions of

$$\sum_{i=1}^{2}\sum_{j=1}^{2}[f(x_iy_j) - f(x_i)f(y_j)] = 0 \qquad (x, y \in \Gamma_2) \tag{3}$$

(see [19]). The aim of this paper is to study the measurable solutions of (1) if $k = l = 2$.

If in (1) $I =]0,1[$, $\Delta_n = \Gamma_n^0$ we refer to (1) as *equation of sum form on the open domain* while in case $I = [0,1]$, $\Delta_n = \Gamma_n$ we call (1) an *equation of sum form on the closed domain.* It is clear that the first case is the more complicated; we shall deal with this case (open domain). Our results support the view that the true domain of definition for (1) is the open one.

2. The differentiability of measurable solutions

If $k = l = 2$ it will be more convenient to write (1) in the form

$$f_1(xy)+f_2(x(1-y))+f_3((1-x)y)+f_4((1-x)(1-y))=\sum_{s=1}^{N} g_s(x)h_s(y) \;\; (x,y\in I).$$

This is obtained from (1) by writing f_1, f_2, f_3, f_4 for $f_{11}, f_{12}, f_{21}, f_{22}$ and by $g_s(x) = g_{1s}(x) + g_{2s}(1-x)$, $h_s(y) = h_{1s}(y) + h_{2s}(1-y)$ $(s = 1, \dots, N)$.

The aim of this section is to prove

THEOREM 1. *Suppose that the functions $f_i, g_s, h_s\colon]0,1[\to \mathbf{C}$ $(i=1,2,3,4;$ $s = 1, \dots, N)$*

(i) *satisfy the functional equation*

$$\begin{aligned} f_1(xy) + f_2(x(1-y)) + f_3((1-x)y) + f_4((1-x)(1-y)) &= \\ = \sum_{s=1}^{N} g_s(x)h_s(y) \qquad (x, y \in]0,1[) \end{aligned} \tag{4}$$

(ii) *f_1, f_2, f_3, f_4 are measurable on $]0,1[$,*

(iii) *the functions $g_1, \dots, g_N$ and $h_1, \dots, h_N$ are linearly independent on $]0,1[$.*

Then f_i, g_s, h_s $(i = 1,2,3,4;\ s = 1, \dots, N)$ are infinitely many times differentiable functions on $]0,1[$.

PROOF. We need some lemmas for the proof.

LEMMA 1. *Suppose that* (i) *and* (iii) *hold. If* f_1, f_2, f_3, f_4 *are measurable (continuous or* n *times differentiable) then* g_s, h_s $(s = 1, \dots, N)$ *are measurable (continuous or* n *times differentiable) too.*

PROOF OF LEMMA 1. By the linear independence of $h_1, \dots, h_N$ there exist points $y_1, \dots, y_N \in]0,1[$ such that $\det(h_s(y_k))_{s,k=1}^N \neq 0$ (see [1]). Substituting $y = y_k$ $(k = 1, \dots, N)$ in (4) and solving the obtained linear system for $g_1, \dots, g_N$ we get

$$(5) \qquad g_s(x) = \sum_{t=1}^{N} \alpha_{st} \mathcal{L}(x, y_t) \qquad (s = 1, \dots, N)$$

where $\mathcal{L}(x, y)$ denotes the left hand side of (4) and α_{st} $(s, t = 1, \dots, N)$ are constants. If f_1, f_2, f_3, f_4 are measurable (continuous or n times differentiable) then so are $\mathcal{L}(x, y_t)$ $(t = 1, \dots, N)$ and by (5) g_s $(s = 1, \dots, N)$ too. The statement for h_s $(s = 1, \dots, N)$ can be proved similarly. □

LEMMA 2. *If* (i), (ii), (iii) *hold then* f_i, g_s, h_s $(i = 1, 2, 3, 4;\ s = 1, \dots, N)$ *are continuous functions on* $]0,1[$.

PROOF OF LEMMA 2. We need the following result of A. Járai ([9], Theorem 2.7.2; we slightly changed the notations).

THEOREM J. *Let* T *be a locally compact metric space, let* Z_1 *be a metric space and let* Z_i $(i = 2, \dots, n)$ *be separable metric spaces. Suppose that* D *is an open subset of* $T \times \mathbf{R}^k$ *and* $X_i \subset \mathbf{R}^k$ *for* $i = 2, \dots, n$. *Let* $F_1 : T \to Z_1$, $F_i : X_i \to Z_i$, $G_i : D \to X_i$, $H : D \times Z_2 \times \dots \times Z_n \to Z_1$ *be functions. Suppose that the following conditions hold:*

(I) *For every* $(t, y) \in D$

$$F_1(t) = H(t, y, F_2(G_2(t, y)), \dots, F_n(G_n(t, y))).$$

(II) F_i *are Lebesgue measurable over* X_i *for* $i = 2, \dots, n$.

(III) H *is continuous on compact sets.*

(IV) *For* $i = 2, \dots, n$ G_i *is continuous and for every fixed* $t \in T$ *the mappings* $y \to G_i(t, y)$ *are differentiable with derivative* $D_2G_i(t, y)$ *and with the Jacobian* $J_2G_i(t, x)$, *moreover the mapping* $(t, y) \to D_2G_i(t, y)$ *is continuous on* D *and for every* $t \in T$ *there exist a* $(t, y) \in D$ *so that*

$$J_2G_i(t, y) \neq 0 \quad \text{for} \quad i = 2, \dots, n.$$

Then F_1 *is continuous on* T. □

First we prove the continuity of f_1. We transform equation (4) into the form given by condition (I) of the above theorem. From (4) with $t = xy$ we obtain

$$(6) \quad f_1(t) = -f_2\left(\frac{t}{y} - t\right) - f_3(y - t) - f_4\left(1 - \frac{t}{y} - y + t\right) + \sum_{s=1}^{N} g_s\left(\frac{t}{y}\right) h_s(y)$$

for $0 < t < y < 1$.

Let $T =]0,1[$, $n = 2N+4$, $Z_1 = Z_2 = \ldots = Z_n = \mathbf{C}$, $X_2 = \ldots = X_n =]0,1[$, $D = \{(t,y) \subset R^2 \mid 0 < t < y < 1\}$. Define the functions G_i on D by

$$(7)\quad \begin{cases} G_2(t,y) = \frac{t}{y} - t, \quad G_3(t,y) = y - t,\ G_4(t,y) = 1 - \frac{t}{y} - y + t, \\ G_5(t,y) = \ldots = G_{4+N}(t,y) = \frac{t}{y},\ G_{5+N}(t,y) = \ldots = G_{4+2N}(t,y) = y \end{cases}$$

and let

$$H(t,y,z_2,\ldots,z_n) = -z_2 - z_3 - z_4 + \sum_{k=5}^{N+4} z_k z_{N+k}.$$

It follows from (6) that the functions F_i $(i = 1,\ldots,n)$ given by

$$F_1 = f_1,\ F_2 = f_2,\ F_3 = f_3,\ F_4 = f_4,\ F_{4+s} = g_s,\ F_{4+N+s} = h_s\ (s = 1,\ldots,N)$$

satisfy the equation in (I) for all $(t,y) \in D$. F_i $(i = 1,\ldots,n)$ are measurable by (ii) and Lemma 1. H is clearly continuous and condition (IV) of Theorem J holds too since calculating D_2G_i one can see that for every $t \in T =]0,1[$

$$D_2G_i(t,y) \neq 0 \quad \text{for} \quad i = 2,\ldots,n \quad \text{if} \quad y \neq \sqrt{t}.$$

Thus by Theorem J, $f_1 = F_1$ is continuous on $]0,1[$. The continuity of f_2, f_3 and f_4 can be proved by making the substitutions $x \to 1-x$; $y \to 1-y$ and $x \to 1-x$, $y \to 1-y$ in (4) respectively and repeating the above argument. The continuity of g_s, h_s $(s = 1,\ldots,N)$ follows from Lemma 1. □

Let $C^{(n)}]0,1[$ be the space of all functions $f\colon]0,1[\to \mathbf{C}$ such that $f^{(n)}$ is continuous on $]0,1[$. $C^{(0)}]0,1[= C]0,1[$ is the space of all continuous functions on $]0,1[$.

LEMMA 3. *Suppose that* (i) *and* (iii) *hold. If* $f_1, f_2, f_3, f_4 \in C^{(n)}]0,1[$ *then* $g_s, h_s \in C^{(n+1)}]0,1[$ $(s = 1,\ldots,N;\ n = 0,1,2,\ldots)$.

PROOF OF LEMMA 3. By Lemma 1, $f_1, f_2, f_3, f_4 \in C^{(n)}]0,1[$ imply $g_s, h_s \in C^{(n)}]0,1[$ $(s = 1,\ldots,N)$. We show that together with g_s, h_s the functions $x \to \int_{1/2}^{x} g_s(u)du$, $x \to \int_{1/2}^{x} h_s(u)du$ $(s = 1,\ldots,N)$ are also linearly independent. Namely if

$$\sum_{s=1}^{N} c_s \int_{1/2}^{x} g_s(u)du = 0 \qquad (x \in]0,1[)$$

holds with some constants c_s $(s = 1, \dots, N)$ then by differentiation we get $\sum_{s=1}^{N} c_s g_s(x) = 0$ $(x \in]0,1[)$ thus by (iii) $c_s = 0$ $(s = 1, \dots, N)$. Hence we can find a system $a_1, \dots, a_N \in]0,1[$ such that

$$\det\left(\int_{1/2}^{a_t} g_s(u)du\right)_{s,t=1}^{N} \neq 0. \tag{8}$$

Integrating (4) with respect to x from 1/2 to a_t we obtain, after suitable transformations in the integrals that

$$\frac{1}{y}\int_{y/2}^{a_t y} f_1(u)du - \frac{1}{y}\int_{y/2}^{(1-a_t)y} f_2(u)du + \frac{1}{1-y}\int_{(1-y)/2}^{a_t(1-y)} f_3(u)du - \tag{9}$$

$$-\frac{1}{1-y}\int_{(1-y)/2}^{(1-a_t)(1-y)} f_4(u)du = \sum_{s=1}^{N}\left(\int_{1/2}^{a_t} g_s(u)du\right)h_s(y)$$

for $t = 1, \dots, N$. It is well known (see e.g. [8]) that if $f_1, f_2, f_3, f_4 \in C^{(n)}]0,1[$ then $l_t(y)$, the left hand side of (9), is in $C^{(n+1)}]0,1[$. Solving (9) as a linear system for the unknowns $h_s(y)$ we get

$$h_s(y) = \sum_{t=1}^{N} \beta_{st} l_t(y) \qquad (s = 1, \dots, N;\ y \in]0,1[)$$

with suitable constants β_{st}. Hence $h_s \in C^{(n+1)}]0,1[$ and by symmetry reasons the same holds for g_s $(s = 1, \dots, N)$ too. □

LEMMA 4. *If* (i) *and* (iii) *hold and* $f_1, f_2, f_3, f_4 \in C^{(n)}]0,1[$, $h_s, g_s \in C^{(n+1)}]0,1[$ $(s = 1, \dots, N)$ *then* $f_1, f_2, f_3, f_4 \in C^{(n+1)}]0,1[$ *for* $n = 0, 1, \dots$.

PROOF OF LEMMA 4. Write (4) in the form (6). Let $0 < \alpha < \beta < 1$ and choose the interval $[\lambda, \mu]$ such that $\sqrt{\beta} < \lambda < \mu < 1$ (then $[\alpha, \beta] \times [\lambda, \mu] \subset D$ holds). Integrating (6) with respect to y on $[\lambda, \mu]$ we obtain

$$(\lambda - \mu)f_1(t) = -\int_{\lambda}^{\mu} f_2\left(\frac{t}{y} - t\right)dy - \int_{\lambda}^{\mu} f_3(y - t)dy - \int_{\lambda}^{\mu} f_4\left(1 - \frac{t}{y} - y + t\right)dy +$$

$$+\sum_{s=1}^{N}\int_{\lambda}^{\mu} g_s\left(\frac{t}{y}\right)h_s(y)dy.$$

We use the substitution $G_i(t,y) = u$ $(i = 2,3,4)$ in the first three integrals where G_i are defined by (7). It is easy to check that the equations $G_i(t,y) = u$ $(i = 2,3,4)$ can uniquely be solved for y if $t \in [\alpha,\beta]$. In the case of $i = 4$ this uniqueness is ensured by the assumption $\sqrt{\beta} < \lambda$, namely, by this condition, the derivative of the function $y \to G_4(t,y)$:

$$D_2G_4(t,y) = \frac{t}{y^2} - 1$$

is negative on $[\alpha,\beta] \times [\lambda,\mu]$ hence our function is strictly decreasing. The solutions $y = \gamma_i(t,u)$ of $G_i(t,y) = u$ $(i = 2,3,4)$ are infinitely many times differentiable functions of t and u. Performing the substitutions we have for $t \in [\alpha,\beta]$

$$f_1(t) = \frac{1}{\mu-\lambda}\sum_{i=2}^{4}\int\limits_{G_i(t,\lambda)}^{G_i(t,\mu)} f_i(u)D_2\gamma_i(t,u)\,du + \frac{1}{\lambda-\mu}\sum_{s=1}^{N}\int\limits_{\lambda}^{\mu} g_s\Big(\frac{t}{y}\Big)h_s(y)\,dy.$$

If $g_s, h_s \in C^{(n+1)}]0,1[$ then the second sum is in $C^{(n+1)}]0,1[$ too. In the first sum the functions f_2, f_3, f_4 are at least continuous hence by repeated application of the theorem concerning the differentiation of parametric integrals (see e.g. Dieudonné [8]) the first sum is differentiable infinitely many times on $[\alpha,\beta]$. Since $[\alpha,\beta]$ is an arbitrary subinterval of $]0,1[$ we have $f_1 \in C^{(n+1)}]0,1[$ and similarly $f_2, f_3, f_4 \in C^{(n+1)}]0,1[$. □

Now we return to the proof of Theorem 1. Let

$$\mathcal{F} = \{f_i, g_s, h_s \mid i = 1,2,3,4;\ s = 1,\dots,N\}.$$

By Lemma 2 $\mathcal{F} \subset C]0,1[$. If $\mathcal{F} \subset C^{(n)}]0,1[$ then by Lemmas 3 and 4 $\mathcal{F} \subset$ $\subset C^{(n+1)}]0,1[$ $(n = 0,1,\dots)$ hence $\mathcal{F} \subset \bigcap_{n=0}^{\infty} C^{(n)}]0,1[$. □

3. Differential equations for f_i

A well known method of solving functional equations is their reduction to differential equations. From the solutions of the differential equation one can select the solutions of the functional equation by substitution or by other means (see Aczél [1] pp. 186–201). Usually one can obtain differential equations from a given functional equation in many ways. For example differentiating (4) (with $f_1 = f_2 = f_3 = f_4 = f$) with respect to y n times and substituting $y = 1/2$ we obtain

$$[1+(-1)^n]\Big[x^n f^{(n)}\Big(\frac{x}{2}\Big) + (1-x)^n f^{(n)}\Big(\frac{1-x}{2}\Big)\Big] = \sum_{s=1}^{n} g_s(x)h_s^{(n)}\Big(\frac{1}{2}\Big).$$

For odd n the left hand side is zero hence the right hand side must be zero too. Under suitable assumptions on $h_s^{(2k+1)}\left(\frac{1}{2}\right)$ this may determine the functions g_s. For even n we get

$$(10)_k \qquad x^{2k} f^{(2k)}\left(\frac{x}{2}\right) + (1-x)^{2k} f^{(2k)}\left(\frac{1-x}{2}\right) = \sum_{s=1}^{N} g_s(x) h_s^{(2k)}\left(\frac{1}{2}\right).$$

Combining equation $(10)_k$, its first and second derivative: $\frac{d}{dx}(10)_k$, $\frac{d^2}{dx^2}(10)_k$ and $(10)_{k+1}$ by coefficients $2k(2k+1)$, $4k(1-x)$, $(1-x)^2$ and $-1/4$ respectively we can get rid of the derivatives $f^{(j)}\left(\frac{1-x}{2}\right)$. We have

$$\frac{x^{2k}(1-2x)}{4} f^{(2k+2)}\left(\frac{x}{2}\right) + 2kx^{2k-1}(1-x) f^{(2k+1)}\left(\frac{x}{2}\right) +$$
$$+2kx^{2k-2}(2x+2k-1) f^{(2k)}\left(\frac{x}{2}\right) = \sum_{s=1}^{N} [(1-x)^2 g_s''(x) + 4k(1-x) g_s'(x) +$$
$$+2k(2k+1) g_s(x)] h_s^{(2k)}\left(\frac{1}{2}\right) - \frac{1}{4} \sum_{s=1}^{N} g_s(x) h_s^{(2k+2)}\left(\frac{1}{2}\right).$$

This equation is valid for $k = 0, 1, \ldots$; $x \in]0,1[$ provided that the functions involved are differentiable sufficiently many times. Let $k = 0, 1, \ldots, 3N$ here. Eliminating $(1-x)^2 g_s''(x)$, $(1-x) g_s'(x)$, $g_s(x)$ $(s = 1, \ldots, N)$ from this system, we obtain a linear homogeneous differential equation of degree $\leqq 2 \cdot 3N + 2 = 6N + 2$ with polynomial coefficients. Unfortunately this differential equation cannot help much in solving (4) since the structure of its solutions is quite difficult.

Next we shall deduce a differential equation of Euler type for $f_1 (f_2, f_3, f_4)$.

THEOREM 2. *Suppose that conditions* (i), (ii), (iii) *of Theorem* 1 *are satisfied. Then there exists functions* $\gamma_k, \tilde{\gamma}_k :]0,1[\to \mathbf{C}$ $(k = 3, \ldots, 15N+5)$ *such that choosing any* $y \in]0,1[$ *not all of the functions* γ_k *and* $\tilde{\gamma}_k$ *vanish at* y *and* f_1, f_3 *satisfy the Euler differential equation*

$$(11) \qquad \sum_{k=3}^{15N+5} \gamma_k(y) u^k f_i^{(k)}(u) = 0 \qquad (i = 1, 3;\ 0 < u < y < 1)$$

further f_2, f_4 *satisfy the equation*

$$(12) \qquad \sum_{k=3}^{15N+5} \tilde{\gamma}_k(y) u^k f_i^{(k)}(u) = 0 \qquad (i = 2, 4;\ 0 < u < y < 1).$$

PROOF. First we define some basic differential operators L_i $(i = 1,2,3,4)$ by

$$L_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad L_2 = x\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y},$$
$$L_3 = (1-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad L_4 = (1-x)\frac{\partial}{\partial x} - (1-y)\frac{\partial}{\partial y}.$$

These and all other differential operators to be defined later will be applied on the equation (4) in which by Theorem 1 all functions are differentiable arbitrary many times.

The effect of L_i on the left hand side of (4) will be the disappearance of f_i $(i = 1,2,3,4)$. Let I be the identical operator and let

$$D_2 = \frac{\partial}{\partial y},$$

$$\begin{aligned}
A_n^{(1)} &= L_3 + nI = (1-x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + nI,\\
A_n^{(2)} &= L_4 + (n+1)I = (1-x)\frac{\partial}{\partial x} - (1-y)\frac{\partial}{\partial y} + (n+1)I,\\
A_n^{(3)} &= L_2 - (n-2)I = x\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y} - (n-2)I,\\
A_n^{(4)} &= L_2 - (n-1)I = x\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y} - (n-1)I,\\
A_n^{(5)} &= L_2 - nI = x\frac{\partial}{\partial x} + (1-y)\frac{\partial}{\partial y} - nI,\\
A_n^{(6)} &= L_1 - (n+1)I = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - (n+1)I,\\
A_n^{(7)} &= L_1 - (n+2)I = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - (n+2)I,\\
A_n^{(8)} &= L_1 - (n+3)I = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - (n+3)I.
\end{aligned}$$

Further define $B_n^{(j)}$ by

$$B_n^{(j)} = A_n^{(j)}A_n^{(j-1)}\dots A_n^{(1)}D_2^n \qquad (j = 1,\dots,8).$$

It is easy to check that $A_n^{(j)}D_2^n = D_2^n A_0^{(j)}$ hence $B_n^{(j)} = D_2^n A_0^{(j)}A_0^{(j-1)}\dots A_0^{(1)}$ holds too. It is clear that $A_n^{(3)}$, $A_n^{(4)}$, $A_n^{(5)}$ and $A_n^{(6)}$, $A_n^{(7)}$, $A_n^{(8)}$ pairwise commute. Using the relations

$$L_2 = L_1 + \frac{\partial}{\partial y}, \qquad L_3 = -L_1 + \frac{\partial}{\partial x}, \qquad L_4 = -L_1 + \frac{\partial}{\partial x} - \frac{\partial}{\partial y},$$
$$L_1\frac{\partial}{\partial x} - \frac{\partial}{\partial x}L_1 = -\frac{\partial}{\partial x}, \qquad L_1\frac{\partial}{\partial y} - \frac{\partial}{\partial y}L_1 = \frac{\partial}{\partial y}$$

we easily get

$$L_1L_2 - L_2L_1 = \frac{\partial}{\partial y}, \quad L_2L_3 - L_3L_2 = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad L_1L_3 - L_3L_1 = -\frac{\partial}{\partial y},$$
$$L_2L_4 - L_4L_2 = -\frac{\partial}{\partial x}, \quad L_1L_4 - L_4L_1 = -\frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad L_3L_4 - L_4L_3 = \frac{\partial}{\partial y}.$$

By the above relations $B_n^{(j)}$ can be factorized in many ways. For example

$$B_n^{(1)} = A_n^{(1)}D_2^n = D_2^nA_0^{(1)} = D_2^nL_3,$$
$$B_n^{(2)} = A_n^{(2)}A_n^{(1)}D_2^n = D_2^nA_0^{(2)}A_0^{(1)} = D_2^n(L_4 + I)L_3 = D_2^n(L_3 + I)L_4,$$

etc. Denote by $\mathcal{L}(x, y)$ and $\mathcal{R}(x, y)$ the left and right hand side of (4), resp. The factorizations above show that $B_n^{(1)}\mathcal{L}(x, y)$ does not contain f_3 (and its derivatives), $B_n^{(2)}\mathcal{L}(x, y)$ contains only f_1, f_2. $B_n^{(5)}\mathcal{L}(x, y)$ contains only f_1 (and its derivatives) and finally $B_n^{(8)}\mathcal{L}(x, y) = 0$. In addition from $j = 6$ on $B_n^{(j)}\mathcal{L}(x, y)$ is a differential operator of Euler type in the variable x (at fixed y). Next we give the more detailed form of the equations

$$(13)_j \qquad B_n^{(j)}\mathcal{L}(x, y) = B_n^{(j)}\mathcal{R}(x, y).$$

For $j \geqq 2$ $B_n^{(j)}\mathcal{R}(x, y)$ gets quite complicated, we shall leave it as it is.

$$(13)_1 \qquad nx^{n-1}f_1^{(n)}(xy) + x^nyf_1^{(n+1)}(xy) + (-1)^nnx^{n-1}f_2^{(n)}(x(1-y)) +$$
$$+(-1)^nx^n(1-y-x)f_2^{(n+1)}(x(1-y)) -$$
$$-(-1)^n(1-x)^{n+1}f_4^{(n+1)}((1-x)(1-y)) =$$
$$= \sum_{s=1}^{N}\left[(1-x)g_s'(x)h_s^{(n)}(y) + g_s(x)(yh_s^{(n+1)}(y) + nh_s^{(n)}(y))\right],$$

$$(13)_2 \quad (2nx^{n-1} + n(n-1)x^{n-2})f_1^{(n)}(xy) + ((2y-n-1)x^n + 2nyx^{n-1})f_1^{(n+1)}(xy) +$$
$$+(-yx^{n+1} + y^2x^n)f_1^{(n+2)}(xy) + (-1)^n(2nx^{n-1} + n(n-1)x^{n-2})f_2^{(n)}(x(1-y)) +$$

$$+(-1)^n((-2y-n+1)x^n+2n(1-y)x^{n-1})f_2^{(n+1)}(x(1-y))+$$
$$+(-1)^n(-(1-y)x^{n+1}+(1-y)^2x^n)f_2^{(n+2)}(x(1-y))=$$
$$=\sum_{s=1}^{N}\Big[(1-x)^2g_s''(x)h_s^{(n)}(y)+(1-x)g_s'(x)((2y-1)h_s^{(n+1)}(y)+2nh_s^{(n)}(y))+$$
$$+g_s(x)(-y(1-y)h_s^{(n+2)}(y)+(n+1)(2y-1)h_s^{(n+1)}(y)+n(n+1)h_s^{(n)}(y))\Big],$$

$(13)_3$
$$2nx^{n-1}f_1^{(n)}(xy)+(2yx^n+n(n+1)x^{n-1})f_1^{(n+1)}(xy)+$$
$$+(-(n+2)x^{n+1}+2(n+1)yx^n)f_1^{(n+2)}(xy)+(-yx^{n+2}+y^2x^{(n+1)})f_1^{(n+3)}(xy)+$$
$$+(-1)^n2nx^{n-1}f_2^{(n)}(x(1-y))+(-1)^n(-2(y+n)x^n)f_2^{(n+1)}(x(1-y))+$$
$$+(-1)^n(-2(1-y)x^{n+1})f_2^{(n+2)}(x(1-y))=B_n^{(3)}\mathcal{R}(x,y),$$

$(13)_4$
$$2(n+1)f_1^{(n+1)}(xy)+(2(y-n-2)x^{n+1}+(n+1)(n+2)x^n)f_1^{(n+2)}(xy)+$$
$$+((-2y-n-3)x^{n+2}+2(n+2)yx^{n+1})f_1^{(n+3)}(xy)+$$
$$+(-yx^{n+3}+y^2x^{n+2})f_1^{(n+4)}(xy)-2(-1)^n(n+1)x^nf_2^{(n+1)}(x(1-y))-$$
$$-2(-1)^n(1-y)x^{n+1}f_2^{(n+2)}(x(1-y))=B_n^{(4)}\mathcal{R}(x,y),$$

$(13)_5$
$$(-4(n+3)x^{n+2}+(n+2)(n+3)x^{n+1})f_1^{(n+3)}(xy)+$$
$$+(-(4y+n+4)x^{n+3}+2(n+3)yx^{n+2})f_1^{(n+4)}(xy)+$$
$$+(-yx^{n+4}+y^2x^{n+3})f_1^{(n+5)}(xy)=B_n^{(5)}\mathcal{R}(x,y),$$

$(13)_6$
$$-4(n+3)x^{n+2}f_1^{(n+3)}(xy)-(4y+2n+8)x^{n+3}f_1^{(n+4)}(xy)-$$
$$-2yx^{n+4}f_1^{(n+5)}(xy)=B_n^{(6)}\mathcal{R}(x,y),$$

$(13)_7$
$$-2(n+4)x^{n+3}f_1^{(n+4)}(xy)-2yx^{n+4}f_1^{(n+5)}(xy)=B_n^{(7)}\mathcal{R}(x,y),$$

$(13)_8$
$$0=B_n^{(8)}\mathcal{R}(x,y).$$

We can get differential equation for f_1 from $(13)_5$, $(13)_6$ or $(13)_7$. Using $(13)_5$ the equation obtained will not be of Euler type while in case of $(13)_7$

we obtain an Euler equation but its order will be too high. Hence we use $(13)_6$:

$$(13)_6 \qquad -2yx^{n+4}f_1^{(n+5)}(xy)-(4y+2n+8)x^{n+3}f_1^{(n+4)}(xy)- -4(n+3)x^{n+2}f_1^{(n+3)}(xy)=B_n^{(6)}\mathcal{R}(x,y).$$

It is essential that $B_n^{(6)}\mathcal{R}(x,y)$ has the following decomposition

$$(14) \qquad B_n^{(6)}\mathcal{R}(x,y)=\sum_{s=1}^{N}\left[\sum_{k=1}^{15}G_{s,k}(x)H_{s,k}(y,n)\right]$$

where

$$(15) \qquad \begin{cases} G_{s,1}(x)=g_s(x), \quad G_{s,2}(x)=g_s'(x), \quad G_{s,3}(x)=xg_s'(x), \\ G_{s,4}(x)=g_s''(x), \quad G_{s,5}(x)=xg_s''(x), \quad G_{s,6}(x)=x^2g_s''(x), \\ G_{s,7}(x)=xg_s'''(x), \quad G_{s,8}(x)=x^2g_s'''(x), \quad G_{s,9}(x)=x^3g_s'''(x), \\ G_{s,10}(x)=x^2g_s^{(4)}(x), \quad G_{s,11}(x)=x^3g_s^{(4)}(x), \\ G_{s,12}(x)=x^4g_s^{(4)}(x), \quad G_{s,13}(x)=x^3(x-1)g_s^{(5)}(x), \\ G_{s,14}(x)=x^4(x-1)g_s^{(5)}(x), \quad G_{s,15}(x)=x^4(x-1)^2g_s^{(6)}(x). \end{cases}$$

The functions $H_{s,k}$ have the form

$$(16) \qquad H_{s,k}(y,n)=\sum_{p=0}^{6}P_{k,p}(y,n)h_s^{(n+p)}(y)$$

where $P_{k,p}$ are suitable polynomials of n and y. For example

$$H_{s,15}(y,n)=h_s^{(n)}(y),$$

$$H_{s,14}(y,n)=(-6n+16)h_s^{(n)}(y)+(4-6y)h_s^{(n+1)}(y),$$

$$H_{s,13}(y,n)=(4n-8)h_s^{(n)}(y)+(4y-3)h_s^{(n+1)}(y),$$

$$H_{s,12}(y,n)=(15n^2-65n+72)h_s^{(n)}(y)+$$
$$+[(30y-20)n+(-50y+32)]h_s^{(n+1)}(y)+(15y^2-20y+6)h_s^{(n+2)}(y),$$

$$H_{s,11}(y,n)=(-20n^2+76n-72)h_s^{(n)}(y)+$$
$$+[(-40y+28)n+(56y-38)]h_s^{(n+1)}(y)+(-20y^2+28y-9)h_s^{(n+2)}(y),$$

$$H_{s,10}(y,n)=(6n^2-18n+12)h_s^{(n)}(y)+$$
$$+[(12y-9)n+(-12y+9)]h_s^{(n+1)}(y)+(6y^2-9y+3)h_s^{(n+2)}(y).$$

The remaining functions $H_{s,k}$ are too complicated to be reproduced here.

REMARK. We found $B_n^{(6)}\mathcal{R}(x,y)$ by using personal computer. Due to software problems we calculated

$$A_n^{(6)}A_n^{(5)}\dots A_n^{(1)}(b_s^n e^{a_s x}e^{b_s y})$$

and factorized the variables a_s, b_s, x, y. Replacing $a_s^k e^{a_s x}$ by $g_s^{(k)}(x)$ and $b_s^l e^{b_s y}$ by $h_s^{(l)}(y)$ we obtain the general term (the bracket) of (14).

Denote by $-F_n(x,y)$ the left hand side of $(13)_6$ then it can be written as

$$(17)\qquad 1\cdot xF_n(x,y)+\sum_{s=1}^{N}\sum_{k=1}^{15}xG_{s,k}(x)H_{s,k}(y,n)=0 \qquad (x,y\in]0,1[).$$

For every fixed $y\in]0,1[$, (17) with $n=0,1,\dots,15N$ is a linear homogeneous system of equations for the unknowns 1, $xG_{1,1}(x),\dots,xG_{1,15}(x),\dots,$ $xG_{N,15}(x)$. It has nontrivial solutions hence its determinant is zero:

$$(18)\qquad \begin{vmatrix} xF_0(x,y) & H_{1,1}(y,0) & H_{1,2}(y,0) & \dots & H_{1,15}(y,0) & \dots & H_{N,15}(y,0)\\ xF_1(x,y) & H_{1,1}(y,1) & H_{1,2}(y,1) & \dots & H_{1,15}(y,1) & \dots & H_{N,15}(y,1)\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ xF_{15N}(x,y) & H_{1,1}(y,15N) & H_{1,2}(y,15N) & \dots & H_{1,15}(y,15N) & \dots & H_{N,15}(y,15N) \end{vmatrix}=0.$$

This implies (see e.g. [27]) that the rows of the determinant in (18) form a linearly dependent vector-system. Writing out the linear dependence for the first coordinate we have

$$(19)\qquad \sum_{l=0}^{15N}\alpha_l(y)xF_l(x,y)=0$$

where for any fixed $y\in]0,1[$ not all coefficients $\alpha_l(y)$ are zero. Using the definition of F_l we can rewrite (19) into the form

$$\sum_{k=3}^{15N+5}\beta_k(y)x^k f_1^{(k)}(xy)=0 \qquad (x,y\in]0,1[)$$

where not all $\beta_k(y)$'s are zero (at any fixed $y\in]0,1[$). Here we needed the fact that in $F_l(x,y)$ the coefficients of $f_1^{(l+j)}(xy)$ $(j=3,4,5)$ are positive.

Substituting $u=xy$, $\gamma_k(y)=\beta_k(y)y^{-k}$ we have

$$\sum_{k=3}^{15N+5}\gamma_k(y)u^k f_1^{(k)}(u)=0 \qquad (0<u<y<1)$$

and again not all the coefficients $\gamma_k(y)$ are zero at any fixed $y \in]0,1[$.

This is exactly the *Euler differential equation* (11) for f_1 we intended to derive. If we replace x by $1-x$ in (4), f_1 goes over into f_3 and $g_s(x)$ goes over into $g_s(1-x)$. This transformation leaves $H_{s,k}(y,n)$ unchanged while in $F_n(x,y)$, f_1 has to be replaced by f_3. Thus repeating the above argument we obtain that f_3 satisfies the same differential equation (11) as f_1. By similar reasoning we obtain that f_2, f_4 *satisfy another Euler differential equation*

$$\sum_{k=3}^{15N+5} \tilde{\gamma}_k(y)u^k f_j^{(k)}(u) = 0 \qquad (0 < u < y < 1;\ j = 2,4) \tag{12}$$

where not all the coefficients $\tilde{\gamma}_k(y)$ are zero at any fixed $y \in]0,1[$. □

4. The structure of solutions

Our main result is

THEOREM 3. *Suppose that the functions* $f_i, g_s, h_s\colon]0,1[\mapsto \mathbf{C}$ $(i=1,2,3,4;$ $s = 1,\dots,N)$ *satisfy conditions* (i), (ii), (iii) *of Theorem* 1. *Then there exist distinct complex numbers* $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4,\dots,\lambda_M$ *and natural numbers* $m_1,\dots,m_M$ *with* $\sum_{j=1}^{M} m_j \leqq 30N+7$ *such that*

$$f_i \in \mathcal{E}\,(\lambda_1,\dots,\lambda_M; m_1,\dots,m_M) \qquad (i = 1,2,3,4) \tag{20}$$

$\mathcal{E}\,(\lambda_1,\dots,\lambda_M; m_1,\dots,m_M)$ *being the vector space of all functions*

$$x \to \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} c_{jk} x^{\lambda_j} \log^k x, \qquad x \in]0,1[$$

where c_{jk} *are complex constants.*

Further g_s, h_s $(s = 1,\dots,N)$ *can be written as*

$$g_s(x) = \gamma_{s1}(x) + \gamma_{s2}(1-x), \qquad h_s(x) = \chi_{s,1}(x) + \chi_{s2}(1-x) \tag{21}$$

with

$$\gamma_{s1}, \gamma_{s2}, \chi_{s1}, \chi_{s2} \in \mathcal{E}\,(\lambda_1,\dots,\lambda_M; m_1,\dots,m_M). \tag{22}$$

Moreover the index set $J = \{(j,k) \mid j = 1,\dots,M; k = 0,\dots,m_j-1\}$ *has three subsets* I_0, I_1, I_2 *each containing at most* $15N+5$ *elements, among them the elements* $(1,0)$, $(2,0)$, $(3,0)$ *such that*

$$f_i(x) = \sum_{(j,k)\in I_1} c_{jk}^{(i)} x^{\lambda_j} \log^k x \qquad \textit{if}\quad i = 1,3, \tag{23}$$

$$f_i(x) = \sum_{(j,k)\in I_2} c_{jk}^{(i)} x^{\lambda_j} \log^k x \qquad \textit{if}\quad i = 2,4 \tag{24}$$

and

$$(25)\qquad \sum_{i=1}^{4} f_i(x) = \sum_{(j,k)\in I_0} \Big(\sum_{i=1}^{4} c_{jk}^{(i)}\Big) x^{\lambda_j} \log^k x$$

hold where $c_{jk}^{(i)}$ $(j = 1, \dots, M;\ k = 0, \dots, m_j - 1;\ i = 1,2,3,4)$ *are complex constants.*

PROOF. We have seen that f_1 satisfies the Euler equation

$$(11)\qquad \sum_{k=3}^{15N+5} \gamma_k(y) u^k f_1^{(k)}(u) = 0 \qquad (0 < u < y < 1)$$

of order $\leqq 15N+5$. For a fixed $y = y_1 \in]0,1[$ let $\mu_1^{(1)}, \dots, \mu_{P_1}^{(1)}$ be the distinct roots of the characteristic equation of (11) and let $n_1^{(1)}, \dots, n_{P_1}^{(1)}$ be their multiplicities. Then we have $\sum_{k=1}^{P_1} n_k^{(1)} \leqq 15N+5$ and we may suppose $\mu_1^{(1)} = 0$, $\mu_2^{(1)} = 1$, $\mu_3^{(1)} = 2$. Since the functions $u \to u^{\mu_j^{(1)}} \log^k u$ $(j = 1, \dots, P_1;$ $k = 0, \dots, n_j^{(1)} - 1)$ form a linear independent set of solutions we have

$$(26)\qquad f_1(u) = \sum_{j=1}^{P_1} \sum_{k=0}^{n_j^{(1)}-1} a_{jk} u^{\mu_j^{(1)}} \log^k u \qquad (0 < u < y_1)$$

where a_{jk} are constants. Fixing another value $y_2 \in]0,1[$ we similarly obtain that

$$(27)\qquad f_1(u) = \sum_{j=1}^{P_2} \sum_{k=0}^{n_j^{(2)}-1} b_{jk} u^{\mu_j^{(2)}} \log^k u \qquad (0 < u < y_2)$$

where $\mu_1^{(2)} = 0$, $\mu_2^{(2)} = 1$, $\mu_3^{(2)} = 2, \dots, \mu_{P_2}^{(2)}$ are distinct complex numbers, $n_1^{(2)}, \dots, n_{P_2}^{(2)}$ are natural numbers and b_{jk} are constants.

Since the functions (26), (27) are identical on the interval $]0, \min\{y_1, y_2\}[$ by the linear independence of the functions involved the terms on the right hand side of (26), (27) must be the same.

More precisely if a term $u^{\mu_j^{(1)}} \log^k u$ of (26) appears in (27) (i.e. $\mu_j^{(1)} = \mu_l^{(2)}$ for some $l = 1, 2, \dots, P_2$) and $0 \leqq k \leqq \min\{n_j^{(1)}, n_l^{(2)}\} - 1$) then $a_{jk} = b_{lk}$ while if $u^{\mu_j^{(1)}} \log^k u$ does not occur in (27) then $a_{jk} = 0$ and conversely: terms of (27) occurring in (26) have the same coefficient, those terms of (27) which are not present in (26) have zero coefficient.

Keeping only the terms with nonzero coefficients and introducing new notation for the exponents and multiplicities we have

$$f_1(u) = \sum_{j=1}^{P} \sum_{k=0}^{n_j-1} c_{jk} u^{\mu_j} \log^k u \tag{28}$$

for all $u \in]0,1[$ where $A = \{\mu_1, \dots, \mu_P\}$ has distinct complex numbers $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 2$; $n_1, \dots, n_P$ are natural numbers with $\sum_{j=1}^{P} n_j \leqq 15N + 5$ and c_{jk} are constants.

Since f_3 satisfies the same Euler equation we conclude that

$$f_3(u) = \sum_{j=1}^{P} \sum_{k=0}^{n_j-1} d_{jk} u^{\mu_j} \log^k u, \qquad u \in]0,1[\tag{29}$$

where d_{jk} are constants.

Similarly, from (12) we deduce that

$$f_2(u) = \sum_{j=1}^{Q} \sum_{k=0}^{p_j-1} e_{jk} u^{\nu_j} \log^k u, \qquad u \in]0,1[, \tag{30}$$

$$f_4(u) = \sum_{j=1}^{Q} \sum_{k=0}^{p_j-1} h_{jk} u^{\nu_j} \log^k u, \qquad u \in]0,1[\tag{31}$$

where $B = \{\nu_1, \dots, \nu_Q\}$ has distinct complex numbers, $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$; $p_1, \dots, p_Q$ are natural numbers, $\sum_{j=1}^{Q} p_j \leqq 15N + 5$ and e_{jk}, h_{jk} are constants.

Let $\lambda_1, \dots, \lambda_M$ be the distinct elements of $A \cup B$ and define m_j ($j = 1, \dots, M$) by

$$m_j = \begin{cases} \max\{n_k, p_l\} & \text{if } \lambda_j \in A \cap B \text{ and } \lambda_j = \mu_k = \nu_l \\ n_k & \text{if } \lambda_j \in A \setminus B \text{ and } \lambda_j = \mu_k \\ p_l & \text{if } \lambda_j \in B \setminus A \text{ and } \lambda_j = p_l. \end{cases}$$

With this notation (28)–(31) can be written as

$$f_i(u) = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} c_{jk}^{(i)} u^{\lambda_j} \log^k u \qquad (u \in]0,1[, \ i = 1,2,3,4) \tag{32}$$

where $c_{jk}^{(i)} \in \mathbf{C}$ are constants. This proves (20). (23), (24) follow from (28)–(31).

Let $f(u) = \sum_{i=1}^{4} f_i(u)$ then from (4) it easily follows that

$$f(xy) + f(x(1-y)) + f((1-x)y) + f((1-x)(1-y)) = \\ = \sum_{s=1}^{N} [g_s(x) + g_s(1-x)][h_s(y) + h_s(1-y)].$$

Applying e.g. (23) for this equation we obtain that (25) holds.

To complete the proof we show that (21), (22) are valid. In the proof of Lemma 1 we have seen that

$$g_s(x) = \sum_{t=1}^{N} \alpha_{st} \mathcal{L}(x, y_t) \qquad (s = 1, \dots, N; \ x \in]0,1[).$$

Using (32) let us calculate $\mathcal{L}(x, y_t)$. We have

$$\mathcal{L}(x, y_t) = \sum_{j=1}^{M} \sum_{k=0}^{m_j-1} \Big[c_{jk}^{(1)} (xy_t)^{\lambda_j} \log^k xy_t + c_{jk}^{(2)} (x(1-y_t))^{\lambda_j} \log^k x(1-y_t) + \\ + c_{jk}^{(3)} ((1-x)y_t)^{\lambda_j} \log^k (1-x)y_t + c_{jk}^{(4)} ((1-x)(1-y_t))^{\lambda_j} \log^k (1-x)(1-y_t) \Big].$$

Expanding the logarithmic factors by the binomial theorem, applying the identity

$$\sum_{k=0}^{M_j-1} \sum_{l=0}^{k} u_{kl} = \sum_{l=0}^{m_j-1} \sum_{k=l}^{m_j-1} u_{kl}$$

we obtain

$$g_s(x) = \sum_{j=1}^{M} \sum_{l=0}^{m_j-1} \big[A_{jls} x^{\lambda_j} \log^l x + B_{jls} (1-x)^{\lambda_j} \log^l (1-x) \big] \tag{33}$$

where

$$A_{jls} = \sum_{k=l}^{m_j-1} \sum_{t=1}^{N} \binom{k}{l} \alpha_{st} \Big[c_{jk}^{(1)} y_t^{\lambda_j} \log^{k-l} u_t + c_{jk}^{(2)} (1-y_t)^{\lambda_j} \log^{k-l} (1-y_t) \Big],$$

$$B_{jls} = \sum_{k=l}^{m_j-1} \sum_{t=1}^{N} \binom{k}{l} \alpha_{st} \Big[c_{jk}^{(3)} y_t^{\lambda_j} \log^{k-l} u_t + c_{jk}^{(4)} (1-y_t)^{\lambda_j} \log^{k-l} (1-y_t) \Big].$$

(33) justifies the statements (21), (22) concerning the representation of the functions g_s. A similar calculation shows that the statement is valid for h_s too. □

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SOME METHODS FOR FINDING ERROR BOUNDS FOR NEWTON-LIKE METHODS UNDER MILD DIFFERENTIABILITY CONDITIONS

I. K. ARGYROS (Lawton)

I. Introduction

Let E and $\hat{E}$ be Banach spaces and consider a nonlinear operator $F\colon D \subseteqq E \to \hat{E}$ which is Fréchet-differentiable on an open convex set $D_0 \subset D$.

The most popular method for approximating a solution $x^* \in D_0$ of the equation

$$F(x) = 0 \tag{1}$$

are the so-called Newton-like methods of the form

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n), \quad x_0 \in D_0 \text{ prechosen}, \quad n = 0, 1, 2, \ldots. \tag{2}$$

The linear operator $A(x)$ is a conscious approximation to the Fréchet-derivative $F'(x)$ of F at $x \in D_0$. For $A(x) = F'(x)$ and $A(x) = F'(x_0)$ one obtains the Newton–Kantorovich and the modified Newton–Kantorovich method.

Several authors including Balazs and Goldner [2], [3], Janko [5], Schmidt [13], [14], Rheinboldt [12], Dennis [4], Miel [9], Moret [10], Potra and Ptak [11] have proved convergence theorems for (2) or special cases of it providing several error bounds on the distances $\|x_n - x^*\|$ and $\|x_n - x_{n+1}\|$, $n = 0, 1, 2, \ldots$. The latter authors above have improved the results of the former using Kantorovich type hypotheses.

Recently, Yamamoto in an excellent paper has unified and improved these results in [16].

The main hypothesis of all the above authors is that the Fréchet-derivative $F'(x)$ of F satisfies a Lipschitz condition. However, there are many interesting differential equations or singular integral equations that can be written in the form (1) where $F'(x)$ is only (K,p), $0 \leqq p \leqq 1$ Hölder continuous (to be precised later) (see also [1]).

Here we extend the results mentioned above in this new setting. Our results reduce to the ones obtained by Yamamoto [16] and the others for $p = 1$.

Finally, we provide an example of a two-point boundary value problem on which our results apply whereas the results obtained by the above authors do not.

II. Main convergence results

We will need a definition:

DEFINITION. Let F be a nonlinear operator and L a boundedly invertible operator defined on a convex set $D_0 \subset E$ with values in a Banach space $\hat{E}$. We say that the Fréchet-derivative $F'(x)$ of F is (c,p)-Hölder continuous on $D_0 \subset E$ if for some $c > 0$, $p \in [0,1]$

$$\|L^{-1}(F'(x) - F'(y))\| \leqq c\|x-y\|^p \quad \text{for all} \quad x,y \in D_0. \tag{3}$$

We then say that $F'(\cdot) \in H_{D_0}(c,p)$.

It is well established [5, p. 142] that

$$\|L^{-1}(F(x) - F(y) - F'(x)(x-y))\| \leqq \frac{c}{1+p}\|x-y\|^{1+p} \text{ for all } x,y \in D_0. \tag{4}$$

We can now prove the following convergence theorem for (2).

THEOREM 1. *Let $D \subseteqq E$ and $F\colon D \to \hat{E}$ and assume $F'(\cdot) \in H_{D_0}(K,p)$ on a convex set $D_0 \subseteqq D$. Let $A\colon D_0 \to L(E,\hat{E})$ and a point x_0 be such that $A(x_0)^{-1}$ exists and*

$$\|A(x_0)^{-1}(F'(x) - F'(y))\| \leqq K\|x-y\|^p, \quad x,y \in D_0, \quad K > 0, \quad p \in (0,1], \tag{5}$$

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leqq L\|x - x_0\|^p + \ell, \quad x \in D_0, \quad L \geqq 0, \quad \ell \geqq 0, \quad p \in (0,1], \tag{6}$$

$$\|A(x_0)^{-1}(F'(x) - A(x))\| \leqq M\|x - x_0\|^p + m, \quad x \in D_0, \quad M \geqq 0, \quad m \geqq 0, \quad p \in (0,1], \tag{7}$$

and

$$\eta \geqq \|A(x_0)^{-1}F(x_0)\| > 0. \tag{8}$$

Assume:

(a) *The real function g defined by*

$$g(t) = (M+L)t^{p+1} - L\eta t^p + \left(m - 1 + \ell + \frac{K\eta^p}{1+p}\right)t + \eta(1-\ell) \tag{9}$$

has a smallest positive zero $r^* > \eta$.

(b) *The following inequalities are satisfied:*

$$L(r^*)^p + \ell < 1 \tag{10}$$

and

$$h_0 = \frac{1}{1 - \ell - L(r^*)^p}\left[\frac{K}{1+p}\eta^p + m + M(r^*)^p\right] < 1. \tag{11}$$

Then

(i) *If* $\overline{U}(x_0, r^*) \subseteqq D_0$ *then the sequence* $\{x_n\}$, $n = 0, 1, 2, \ldots$ *generated by* (2) *is well defined, remains in* $U(x_0, r^*)$ *and converges to a solution* $x^* \in \overline{U}(x_0, r^*)$ *of equation* (1).

(ii) *Moreover, if*

$$h^* = \frac{1}{1 - \ell - L(r^*)^p}\Big[\frac{2^p K(r^*)^p}{1+p} + m + M(r^*)^p\Big] < 1 \tag{12}$$

then x^* *is the unique solution of equation* (1) *in*

$$\overline{U}(x_1, r^* - \eta) \subseteqq \overline{U}(x_0, r^*). \tag{13}$$

(iii) *Furthermore, the following estimates are true:*

$$\|x_{n+1} - x_n\| \leqq t_{n+1} - t_n, \quad n = 0, 1, 2, \ldots \tag{14}$$

and

$$\|x_n - x^*\| \leqq t^* - t_n, \quad n = 0, 1, 2, \ldots \tag{15}$$

where the real sequence $\{t_n\}$ *is nonnegative, increasingly converging to some* $t^* \geqq 0$ *and is given by*

(16)

$$t_{n+1} - t_n = \frac{1}{1 - \ell - Lt_n^p}\Big[\frac{K}{1+p}(t_n - t_{n-1})^{1+p} + (m + Mt_{n-1}^p)(t_n - t_{n-1})\Big], \quad n = 1, 2, \ldots$$

with $t_0 = 0$ *and* $t_1 = \eta$.

PROOF. (i) Let $x \in U(x_0, r^*)$. Then we can write

$$A(x) = A(x_0)(I + A(x_0)^{-1}(A(x) - A(x_0))).$$

Using (6) and (10) we get

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leqq L\|x - x_0\|^p + \ell \leqq L(r^*)^p + \ell < 1$$

and by the Banach lemma on invertible operators the linear operator $A(x)$ is invertible for all $x \in U(x_0, r^*)$ and

$$\|A(x)^{-1}A(x_0)\| \leqq \frac{1}{1 - \ell - L(r^*)^p}. \tag{17}$$

Therefore $T(x) = x - A^{-1}(x)F(x)$ is defined on $U(x_0, r^*)$ and if x, $T(x) \in U(x_0, r^*)$, using (4), (17) and (7) we obtain

$$\begin{aligned} \|T(T(x)) - T(x)\| &= \| - A^{-1}(T(x))F(T(x))\| \leqq \\ &\leqq \frac{1}{1 - \ell - L\|T(x) - x_0\|^p}[\|F(T(x)) - F(x) - F'(x)(T(x) - x)) + \\ &\qquad + (F'(x) - A(x))(T(x) - x)\|] \leqq \\ &\leqq \frac{1}{1 - \ell - L\|T(x) - x_0\|^p}\Big\{\frac{K}{1+p}\|T(x) - x\|^{1+p} + (m + M\|x - x_0\|^p)\|T(x) - x\|\Big\} = \\ &= \overline{g}(\|T(x) - x\|, \|T(x) - x_0\|, \|x - x_0\|) \end{aligned} \tag{18}$$

where

$$\overline{g}(u,v,w) = \frac{1}{1-\ell-LV^p}\Big[\frac{Ku^{1+p}}{1+p}+(m+Mw^p)u\Big]. \tag{19}$$

The difference equation given by (16) is such that $t_2 - t_1 \leqq h_0\eta$ and

$$t_2 \leqq t_1 + h_0\eta = \eta(1+h_0) < \frac{\eta}{1-h_0}.$$

Using (9) and $g(r^*) = 0$ we get $\eta = (1-h_0)r^*$. That is $t_2 < r^*$. We can easily show using induction on n that

$$t_{k+1} - t_k \leqq h_0(t_k - t_{k-1}), \quad t_{k+1} - t_k \leqq \eta, \quad \text{and} \quad t_{k+1} \leqq r^*.$$

It now follows

$$\lim_{k\to\infty} t_k = t^*, \quad \frac{\eta}{1-h_0} = r^*.$$

Therefore, we have shown

$$\|x_{n+1} - x_n\| \leqq t_{n+1} - t_n, \quad n = 0,1,2,\dots .$$

That is by a well known lemma on majorizing sequences, there exists an element $x^* \in \overline{U}(x_0, r^*)$ such that $T(x^*) = x^*$.

We can now get

$$\|F(x_k)\| \leqq \|A(x_0)A(x_0)^{-1}(A(x_k)) - A(x_0))(x_{k+1} - x_k)\| +$$
$$+\|A(x_0)(x_{k+1} - x_k)\| \leqq [L\|x_k - x_0\|^p + \ell + 1]\|A(x_0)\|\|x_{k+1} - x_k\| \leqq$$
$$\leqq [L(r^*)^p + \ell + 1]\|A(x_0)\|\|x_{k+1} - x_k\| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence $F(x^*) = 0$.

(ii) Let us assume that there exists another solution $z^* \in \overline{U}(x_0, r^*)$ of equation (1). By (2), (7), (17) and (12) we get

$$x_{n+1} - z^* =$$
$$= A(x_n)^{-1}[(A(x_n) - F'(x_n))(x_n - z^*) + F'(x_n)(x_n - z^*) - (F(x_n) - F(z^*))]$$

and

$$\|x_{n+1} - z^*\| \leqq \frac{1}{1-\ell-L(r^*)^p}\Big[m + M\|x_n - x_0\|^p +$$
$$+\frac{K}{1+p}(\|x_n - x_0\| + \|x_0 - z^*\|)^p\Big]\|x_n - z^*\| \leqq$$
$$\leqq h_1\|x_n - z^*\| \leqq \cdots \leqq h_1^{n+1}\|x_0 - z^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

That is, $x^* = z^*$. The rest follows from the observation that

$$x^* \in \overline{U}(x_1, r^* - \eta) \subset \overline{U}(x_0, r^*), \tag{20}$$

since $\|x_0 - x_1\| + r^* - \eta \leqq r^*$.

(iii) The results in this part follow immediately from (i) and (20).

That completes the proof of the theorem.

We now give some sufficient conditions for the existence of a minimum solution of the real equation $g(t) = 0$.

PROPOSITION. *Assume that the following conditions are satisfied:*

(21) $$B = -m + 1 - \ell + L\eta - \frac{K\eta^p}{1+p} \geqq 0,$$

(22) $$B^2 > 4\eta(1-\ell)(M+L),$$

and

(23) $$g(t_q) \leqq 0$$

with

(24) $$t_q = \frac{B}{2(M+L)}.$$

Then the equation

(25) $$g(t) = 0$$

where g is given by (9) has a smallest positive zero r^*.

PROOF. By (21), (22) it follows that the quadratic equation

$$(M+L)t^2 + \Big(m - 1 + \ell + \frac{K\eta^p}{1+p} - L\eta\Big)t + \eta(1-\ell) = 0$$

has two positive zeroes and a minimum at $t_q = \frac{B}{2(M+L)}$. We can easily see that $g(t)$ is continuous, $g(0) > 0$ and $g(t) > 0$ for t sufficiently large and since $g(t_q) \leqq 0$, by (23) it follows that $g(t)$ has two positive zeroes r^* and r_1^* with $r^* \leqq r_1^*$ of which we can choose the minimum to be r^* and r_1^* to be a second minimum zero. That is if $\overline{r}_1^*$ is a zero also with $r^* \neq \overline{r}_1^* \leqq r_1^*$ then $\overline{r}_1^* = r_1^*$.

That completes the proof of the proposition.

THEOREM 2. *Let F, D_0, x_0 be as in the introduction and assume:*

(a) *the equation* (1) *has a solution $x^* \in D_0$;*

(b) *there exist nonnegative numbers a_n^0, a_n^1, a_n^2 such that*

(26)
$$\|x_{n+1} - x^*\| \leqq \frac{1}{1+p} a^0 + n\|x_n - x^*\|^{1+p} + a_n^1\|x_n - x^*\| + a_n^2 \textit{ for all } n = 0, 1, 2, \ldots$$

with

(27) $$a_n^0 > 0 \quad \textit{and} \quad a_n^1 \leqq 1;$$

(c) *the real function*

(28) $$q(t) = \frac{1}{1+p} a_n^3 t^{1+p} - (1 - a_n^4)t + a_n^5 + d_n, \quad d_n = \|x_{n+1} - x_n\|$$

with

$$a_n^3 \geqq a_n^0,\ 1 \geqq a_n^4 \geqq a_n^1 \quad \textit{and} \quad a_n^5 \geqq a_n^2, \quad \textit{for all} \quad n = 0, 1, 2, \ldots \tag{29}$$

has two positive zeroes r^* and r_1^*, $r^* \leqq r_1^*$ with r^* being the smallest positive zero and r_1^* is such that if $\overline{r}_1^* \neq r^*$ is a zero also with $\overline{r}_1^* \leqq r_1^*$ then $\overline{r}_1^* = r_1^*$.

Then

(i) *The real function*

$$q_0(t) = \frac{1}{1+p} a_n^0 t^{1+p} - (1 - a_n^1)t + a_n^2 + d_n \tag{30}$$

has two positive zeroes r_2^* and r_3^* such that

$$r_2^* \leqq r^* \leqq r_1^* \leqq r_3^*. \tag{31}$$

(ii) *Moreover, if*

$$\|x_n - x^*\| \leqq r^* \tag{32}$$

then

$$\|x_n - x^*\| \leqq r_2^*. \tag{33}$$

PROOF. The result in (i) follows immediately from the easy observation that

$$q(t) \geqq q_0(t) \quad \text{for all} \quad t > 0. \tag{34}$$

Using (26) we get

$$\|x_n - x^*\| - d_n \leqq \|x_{n+1} - x^*\| \leqq \frac{1}{1+p} a_n^0 \|x_n - x^*\|^{1+p} + a_n^1 \|x_n - x^*\| + a_n^2. \tag{35}$$

By hypothesis we must have $q_0(\|x_n - x^*\|) > 0$ and either

$$\|x_n - x^*\| \leqq r_2^* \quad \text{or} \quad \|x_n - x^*\| \geqq r_3^*$$

the latter is excluded, however, and the result in (ii) follows.

REMARKS. (a) Note that for $p = 1$, Theorem 2 reduces to Theorem 3.1 in [16].

(b) The result in (ii) above constitutes an improved error estimate on the distances $\|x_n - x^*\|$, $n = 0, 1, 2, \ldots$.

According to the proposition, the equation

$$g_{1n}(t) = 0, \quad n = 0, 1, 2, \ldots \tag{36}$$

where

(37)

$$\begin{cases} g_{1n}(t) = \frac{1}{1+p}\sigma K t^{1+p} + (m + \ell - 1 + \sigma K t_n^p)t + (1 - \ell - Lt_n^p)(t_{n+1} - t_n), \\ \sigma = \max\left(1, \frac{L+M}{K}\right) \end{cases}$$

has a minimal solution s_n^5 and a second minimal solution s_n^6 with $s_n^5 \leqq s_n^6$ if the following set of conditions is satisfied for all $n = 0, 1, 2, \ldots$:

(c)

$$m + \ell - 1 + \sigma K t_n^p \leqq 0, \tag{38}$$

$$(m + \ell - 1 + \sigma K t_n^p)^2 \geqq \frac{4\sigma K}{1+p}(1 - \ell - Lt_n^p)(t_{n+1} - t_n) \tag{39}$$

and

$$g_{1n}(t_{1n}) \leqq 0 \tag{40}$$

with

$$t_{1n} = -\frac{(m + \ell - 1 + \sigma K t_n^p)(1+p)}{2\sigma K}. \tag{41}$$

It can easily be seen that the conditions (c) are certainly satisfied if the following are true:

(c_1)

$$m + \ell - 1 + 2\sigma K (r^*)^p \leqq 0, \tag{42}$$

$$(m + \ell - 1)^2 \geqq \frac{4\sigma K}{1+p}\left[\frac{K}{1+p}(r^*)^p + (m + M(r^*)^p)\right]r^* \tag{43}$$

and

$$g_2(tr^*) \leqq 0 \tag{44}$$

with

(45)

$$g_2(t) = \frac{1}{1+p}\sigma K t^{1+p} + (m + \ell - 1 + \sigma K (r^*)^p)t + \frac{K}{1+p}(r^*)^{1+p}(m + M(r^*)^p)r^*$$

and

$$tr^* = -\frac{(m + \ell - 1 + \sigma K (r^*)^p)(1+p)}{2\sigma K}. \tag{46}$$

Indeed, (38) follows from (42) since $t_n \leqq r^*$, $n = 0,1,2,\ldots$. We also have that for all $n = 0,1,2,\ldots$ (39) follows from (43) since

$$\begin{aligned}(m+\ell-1+\sigma K t_n^p)^2 &\geqq (m+\ell-1)^2 \geqq \\ \geqq \frac{4\sigma K}{1+p}\Big[\frac{K}{1+p}(r^*)^P &+ (m+M(r^*)^p)\Big]r^* \geqq \\ \geqq \frac{4\sigma K}{1+p}\Big[\frac{K}{1+p}(t_n-t_{n-1})^{1+p} &+ (m+Mt_{n-1}^p)(t_n-t_{n-1})\Big] \geqq \\ \geqq \frac{4\sigma K}{1+p}(1-\ell-Lt_n^p)&(t_{n+1}-t_n).\end{aligned}$$

Moreover,

$$tr^* \leqq t_{1n}, \qquad g_{1n}(t) \leqq g_2(t), \quad t \geqq 0$$

and the function g_2 is decreasing on $[0, r^*]$ by (42).

That is, (40) follows from (44) since for all $n = 0,1,2,\ldots$

$$g_{1n}(t_{1n}) \leqq g_2(t_{1n}) \leqq g_n(tr^*) \leqq 0.$$

We can now prove the main result.

THEOREM 3. *Let $D \subseteqq E$ and $F\colon D \to \hat{E}$ and $F'(\cdot) \in H_{D_0}(K,p)$ on a convex set $D_0 \subset D$. Assume:*

(a) *the hypotheses of Theorem 1 are satisfied;*

and

(b) *the set of conditions* (c) *or* (c_1) *are satisfied.*

Then:

(i) *The sequence $\{x_n\}$, $n = 0,1,2,\ldots$ generated by* (2) *is well defined, remains in $U(x_0,r^*)$ and converges to a unique solution x^* of equation* (1) *in $\overline{U}(x_1, r^*-\eta) \subset \overline{U}(x_0,r^*)$.*

(ii) *Let $\overline{U}_0 = \overline{U}(x_0,r^*)$, $\overline{U}_n = \overline{U}(x_n, r^*-t_n)$, $n = 1,2,\ldots$, $K_0 = L_0 = K$,*

$$K_n = \sup_{\substack{x,y\in\overline{U}_n\\ x\neq y}} \frac{\|A(x_n)^{-1}(F'(x)-F'(y))\|}{\|x-y\|^p},$$

$$L_n = \sup_{\substack{x,y\in\overline{U}_0\\ x\neq y}} \frac{\|A(x_n)^{-1}(F'(x)-F'(y))\|}{\|x-y\|^p}.$$

Then we have

(47) $$x^* \in \overline{U}_n \subset \overline{U}_{n-1} \subset \cdots \subset \overline{U}_0,$$

(48) $$\|x_n - x^*\| \leqq s_n^1 \leqq s_n^2 \leqq s_n^3 \leqq s_n^4 \leqq t^* - t_n \text{ for all } n = 0,1,2,\ldots,$$

where s_n^1, s_n^2, s_n^3 *and* s_n^4 *are the least solutions of the equations*

$$P_n(t) = p_n(t) - t + d_n, \qquad V_n(t) = v_n(t) - t + d_n,$$
$$W_n(t) = w_n(t) - t + d_n, \qquad Y_n(t) = y_n(t) - t + d_n,$$

with

$$p_n(t) = \frac{1}{1+p} K_n t^{1+p} + \frac{(m + M\|x_n - x_0\|^p)t}{1 - \ell - L\|x_n - x_0\|^p},$$

$$v_n(t) = \frac{1}{1+p} L_n t^{1+p} + \frac{(m + M\|x_n - x_0\|^p)t}{1 - \ell - L\|x_n - x_0\|^p},$$

$$w_n(t) = (1 - \ell - L\|x_n - x_0\|^p)^{-1} \Big[\frac{K}{1+p} t^{1+p} + (m + M\|x_n - x_0\|^p)t\Big]$$

and

$$y_n(t) = (1 - \ell - Lt_n^p)^{-1} \Big[\frac{K}{1+p} t^{1+p} + (m + Mt_n^p)t\Big].$$

PROOF. Part (i) follows immediately from Theorem 1. It is easy to see that $x^* \in \overline{U}_n \subset \overline{U}_{n-1}$. Let us define the real functions

$$Z_n(t) = z_n(t) - t + d_n$$

and

$$z_n(t) = (1 - \ell - Lt_n^p)^{-1} \Big[\frac{\sigma K}{1+p} t^{1+p} + ((m + (\sigma K - L)t_n^p)t\Big] + (t_{n+1} - t_n) - d_n$$

for all $n = 0, 1, 2, \ldots$.

It can easily be seen that condition (c) or (c_1) imply that the equation $Z_n(t) = 0$ has a minimal solution s_n^5 and a second minimal solution s_n^6 with $s_n^5 \leqq s_n^6$.

We also have that

$$\|x_{n+1} - x^*\| \leqq p_n(\|x_n - x^*\|) \leqq v_n(\|x_n - x^*\|) \leqq w_n(\|x_n - x^*\|) \leqq$$

$$\leqq y_n(\|x_n - x^*\|) \leqq z_n(\|x_n - x^*\|)$$

for all $n = 0, 1, 2, \ldots$.

With the above results it can easily be seen that the hypotheses of Theorem 2 are satisfied. Therefore we can apply (33) to obtain (48) for all $n = 0, 1, 2, \ldots$.

Note that for $p = 1$ our results can reduce to the ones obtained in [16].

We now complete this paper with some applications where our results apply but the ones in [2]–[16] do not.

III. Applications

Consider the differential equation

$$y'' + y^{1+p} = 0, \quad p \in (0,1], \quad y(0) = y(1) = 0. \tag{49}$$

We divide the interval $[0,1]$ into n subintervals and we set $h = \frac{1}{n}$. Let $\{v_k\}$ be the points of subdivision with

$$0 \leqq v_0 < v_1 < \cdots < v_n = 1.$$

A standard approximation for the second derivative is given by

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \quad y_i = y(v_i), \quad i = 1, 2, \dots, n-1.$$

Take $y_0 = y_n = 0$ and define the operator $F\colon \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ by

$$F(y) = H(y) + h^2\varphi(y),$$

$$H = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ \vdots & \ddots & \ddots & -1 \\ 0 & -1 & & 2 \end{bmatrix}, \qquad \varphi(y) = \begin{bmatrix} y_1^{1+p} \\ y_2^{1+p} \\ \vdots \\ y_{n-1}^{1+p} \end{bmatrix}$$

and

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}.$$

Then

$$F'(y) = H + h^2(p+1) \begin{bmatrix} y_1^p & & & 0 \\ & y_2^p & & \\ & & \ddots & \\ 0 & & & y_{n-1}^p \end{bmatrix}. \tag{50}$$

The Newton–Kantorovich hypotheses on which the work in [2]–[16] is based for the solution of the equation

$$F(y) = 0 \tag{51}$$

may not be satisfied.

We may not be able to evaluate the second Fréchet-derivative since it would involve the evaluation of quantities y_i^{-p} and they may not exist.

Let $y \in \mathbf{R}^{n-1}$, $M \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ and define the norms of y and M by

$$\|y\| = \max_{1 \leqq j \leqq n-1} |y_j|, \qquad \|M\| = \max_{1 \leqq j \leqq n-1} \sum_{k=1}^{n-1} |m_{jk}|.$$

For all y, $z \in \mathbf{R}^{n-1}$ for which $|y_i| > 0$, $|z_i| > 0$, $i = 0, 1, 2, \dots, n-1$ we obtain for $p = \frac{1}{2}$, say

$$\|F'(y) - F'(z)\| = \left\|\operatorname{diag}\left\{\frac{3}{2}h^2(y_j^{1/2} - z_j^{1/2})\right\}\right\| = \frac{3}{2}h^2 \max_{1 \leqq j \leqq n-1} |y_j^{1/2} - z_j^{1/2}| \leqq$$
$$\leqq \frac{3}{2}h^2[\max|y_j - z_j|]^{1/2} = \frac{3}{2}h^2\|y - z\|^{1/2}.$$

That is, $K = \frac{3}{2}h^2$ and $p = \frac{1}{2}$. Therefore, the results in [2]–[13] cannot be applied here. Let us assume that $A(x) = F'(x_0)^{-1}$ for all $x \in D_0$.

We can choose $n = 10$ which gives (9) equations for iteration (2). Since a solution would vanish at the end points and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin \pi x$. This gives us the following vector:

$$z_0 = \begin{bmatrix} 4.01524E+01 \\ 7.63785E+01 \\ 1.05135E+02 \\ 1.23611E+02 \\ 1.29999E+02 \\ 1.23675E+02 \\ 1.05257E+02 \\ 7.65462E+01 \\ 4.03495E+01 \end{bmatrix}.$$

Using the iterative algorithm (2), after seven iterations we get

$$z_7 = \begin{bmatrix} 3.35740E+01 \\ 6.52027E+01 \\ 9.15664E+01 \\ 1.09168E+02 \\ 1.15363E+02 \\ 1.09168E+02 \\ 9.15664E+01 \\ 6.52027E+01 \\ 3.35740E+01 \end{bmatrix}.$$

We choose z_7 as our x_0 for our Theorem 1. We get the following results:

$$L = \ell = m = 0, \quad M = K = .383823, \quad \eta = 9.15311 \cdot 10^{-5}, \quad p = \frac{1}{2}.$$

The function g given by (9) becomes

$$g(t) = .383823t^{3/2} - .99362187t + 9.15311 \cdot 10^{-5}.$$

This function has a minimal zero $r^* = 9.211864469 \cdot 10^{-5} > \eta$. The rest of the hypotheses of Theorem 1 are satisfied with $h^* = 7.15706368 \cdot 10^{-3} < 1$ and $r^* - \eta = 5.8754469 \cdot 10^{-7}$. Hence, by Theorem 1, the sequence generated by (2) is well defined, remains in $\overline{U}(x_0, r^*)$ and converges to a unique solution x^* of equation (51) in $\overline{U}(x_1, r^* - \eta)$.

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EPIS IN CATEGORIES OF CONVERGENCE SPACES

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1. In what follows **X** will denote a topological category in the sense of Herrlich [10]. All subcategories of **X** considered in the paper are full and isomorphism-closed. We do not distinguish between **X** and Ob **X**, a class (always non empty) of spaces (**X**-objects) and the corresponding full and isomorphism-closed subcategory, a space (or a subspace = extremal subobject) and its underlying set (subset), an **X**-morphism and the corresponding set-function.

The categorical terminology is that of [11].

A *closure operator* C of **X** is an assignment to each subset M of (the underlying set of) any object X of a subset $c_X M$ of X such that:

a) $M \subseteqq c_X M$;

b) $c_X N \subseteqq c_X M$ whenever $N \subseteqq M$;

c) (*continuity condition*). For each $f\colon X \to Y$ in **X** and M subset of X, $f(c_X M) \subseteqq c_Y(fM)$.

Furthermore C is called *idempotent* if $c_X(c_X M) = c_X M$.

In case $c_X\emptyset = \emptyset$ is always true, this coincides with the notion of closure operator given in [8].

A subset $M \subseteqq X$ is called C*-closed* (respectively C*-dense*) in X if $c_X M = M$ (respectively $c_X M = X$). A **X**-morphism $f\colon X \to Y$ is called C*-dense* if $f(X)$ is C-dense in Y.

Notice that many classical operators in **Top** are closure operators in the previous sense, e.g., θ-closure, sequential closure, compact closure, z-closure (defined as the intersection of all zero-sets containing the given subset). However the semiregularization operator does not satisfy the continuity condition (cf. [3]).

The conglomerate of all closure operators of **X** is endowed with the 'pointwise' preorder defined by $C \leqq C'$ iff $c_X M \subseteqq c'_X M$ for each $X \in \mathbf{X}$ and $M \subseteqq X$.

For each ordinal α we define the α-th iteration of C as the closure operator C^α given recursively by $C^1 = C$, $C^{\alpha+1} = CC^\alpha$ and $C^\alpha = \sup\{C^\beta\colon \beta < \alpha\}$ for limit α and for $\alpha = \infty$ (with $\beta < \infty$ for all small β). For each $X \in \mathbf{X}$ there exists α such that $(c_X)^\alpha M = (c_X)^{(\alpha+1)} M$ for each $M \subseteqq X$, so C^∞ is

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idempotent and it is the least idempotent closure operator coarser than C (the *idempotent hull* of C, cf. [6, Section 4]).

For a given closure operator C of $\mathbf{X}$ set

$\mathbf{X}_{0C} = \{X \in \mathbf{X}\colon x \in c_X(\{y\}) \text{ and } y \in c_X(\{x\}) \Rightarrow x = y\}$.

$\mathbf{X}_{1C} = \{X \in \mathbf{X}\colon \{x\} = c_X(\{x\}), \text{ for each } x \in X\}$.

$\mathbf{X}_{2C} = \{X \in \mathbf{X}\colon c_{X\times X}(\Delta_X) = \Delta_X\}$.

Let us note that the first two categories are defined by conditions on the points of the space, while the third one is defined by a global property of the space. Since in this paper we are going to deal mainly with the first two categories, for $i = 0, 1$ we say that two points x and y of a space X are (i, C)*-separated* if they satisfy the condition given in the respective definition.

For $\mathbf{X} = \mathbf{Top}$ and C the ordinary closure we obtain the class of T_0-spaces, T_1-spaces and T_2-spaces respectively. It is also easy to show that $\mathbf{X}_{iC}$, $i = 0, 1, 2$, are quotient-reflective subcategories of $\mathbf{X}$.

Since $C \leqq C^2 \leqq \dots$, then trivially $\mathbf{X}_{0C} \supseteqq \mathbf{X}_{0C^2} \supseteqq \dots$, and by definition $\mathbf{X}_{0C^\alpha} \supseteqq \mathbf{X}_{1C}$ for each ordinal α so we obtain a chain

$$\mathbf{X}_{0C} \supseteqq \mathbf{X}_{0C^2} \supseteqq \dots \supseteqq \mathbf{X}_{0C^\alpha} \supseteqq \dots \supseteqq \mathbf{X}_{1C}.$$

Every class of $\mathbf{X}$-objects $\mathbf{S}$ defines an idempotent closure operator in the following way: $F \subseteqq X$ is called $\mathbf{S}$-closed in X iff for each $x \in (X \setminus F)$ there exist $S \in \mathbf{S}$ and $f, g\colon X \to S$ such that the restrictions $f|F$ and $g|F$ coincide and $f(x) \neq g(x)$. A pair (f, g) as above is said to be a $\mathbf{S}$*-separating pair* for (x, F) in $\mathbf{X}$. The $\mathbf{S}$*-closure* of $M \subseteqq X$ is defined as the intersection of all $\mathbf{S}$-closed subsets of X containing M and it is denoted by $[M]_{\mathbf{S}}$.

The closure operator defined above characterizes the epimorphisms of the full subcategory $\mathbf{S}$ of $\mathbf{X}$ as the $[\ ]_{\mathbf{S}}$-dense maps (cf. [8, Theorem 2.8]).

For $X \in \mathbf{X}$ and $M \subseteqq X$, $X \coprod_M X$ will denote the quotient of the coproduct $X \coprod X = X \times \{0, 1\}$ obtained by identifying each $(m, 0)$, $m \in M$, with $(m, 1)$. Let $q\colon X \coprod X \to X \coprod_M X$ be the quotient map. The maps

$$k_i\colon X \to X \coprod\nolimits_M X, \quad s\colon X \coprod\nolimits_M X \to X \coprod\nolimits_M X \text{ and } p\colon X \coprod\nolimits_M X \to X$$

are respectively defined by $k_i(x) = q(x, i)$, $s(q(x, 0)) = q(x, 1)$, $s(q(x, 1)) = q(x, 0)$ and $p(q(x, i)) = x$, for $x \in X$ and $i = 0, 1$.

LEMMA 1.1 ([8, Proposition 2.6]). *Let* $\mathbf{S}$ *be a quotient-reflective subcategory of* $\mathbf{X}$ *containing a space with at least two points and let* $X \in \mathbf{S}$. *A subset* $M \subseteqq X$ *is* $\mathbf{S}$*-closed iff* $X \coprod_M X$ *belongs to* $\mathbf{S}$.

Let C be a closure operator of $\mathbf{X}$. For $X \in \mathbf{X}$ and $M \subseteqq X$ set

$$\mathrm{cl}_C M = \left\{x \in X\colon q(x, 0) \in c_{X \coprod_M X}\{q(x, 1)\}\right\}.$$

It is easy to show that cl_C is an extensive, monotone and continuous operator, i.e. cl_C is a closure operator of $\mathbf{X}$.

Let $\mathrm{Eq}(f, g)$ denote the equalizer of f and g and, for each reflective subcategory $\mathbf{A}$ of $\mathbf{X}$ let $R\colon \mathbf{X} \to \mathbf{A}$ denote the $\mathbf{A}$-reflection functor and $r\colon X \to RX$ the $\mathbf{A}$-reflection of $X \in \mathbf{X}$.

LEMMA 1.2. *If* $\mathbf{A}$ *is a reflective subcategory of* $\mathbf{X}$ *then, for each* $X \in \mathbf{X}$ *and* $M \subseteqq X$ *the following holds*

$$[M]_{\mathbf{A}} = \mathrm{Eq}(rk_0, rk_1)$$

where $r\colon X \coprod_M X \to R(X \coprod_M X)$.

PROOF. Since $M \subseteqq \mathrm{Eq}(k_0, k_1) \subseteqq \mathrm{Eq}(rk_0, rk_1)$ and $R(X \coprod_M X) \in \mathbf{A}$, then $[M]_{\mathbf{A}} \subseteqq \mathrm{Eq}(rk_0, rk_1)$. On the other hand, if $x \notin [M]_{\mathbf{A}}$ and $f, g\colon X \to A$, $A \in \mathbf{A}$ is an $\mathbf{A}$-separating pair for (x, M), then $(f \coprod_M g)(q(x,0)) \neq$ $\neq (f \coprod_M g)(q(x,1))$ so, using the universal property of reflections which says that $(f \coprod_M g)$ admits a factorization $(f \coprod_M g) = hr$, we obtain that $r(q(x,0)) \neq r(q(x,1))$. Since $r(q(x,i)) = (rk_i)(x)$, $i = 0,1$, then we deduce that $x \notin \mathrm{Eq}(rk_0, rk_1)$.

THEOREM 1.3. *Let* C *be a closure operator of* $\mathbf{X}$, $i = 0,1$, $X \in \mathbf{X}$ *and* $M \subseteqq X$. *Then*

(a) $(\mathrm{cl}_C)^\infty M \subseteqq [M]_{\mathbf{X}_{iC}}$;

(b) $(\mathrm{cl}_C)^\infty M = [M]_{\mathbf{X}_{iC}}$ *if* $X \in \mathbf{X}_{iC}$.

PROOF. (a) Since $\mathbf{X}_{iC}$-closure is idempotent it is enough to show that $\mathrm{cl}_C M \subseteqq [M]_{\mathbf{X}_{iC}}$. If $x \in \mathrm{cl}_C M$ then $q(x,0) \in c_{X \coprod_M X}\{q(x,1)\}$ so, applying the symmetry s and the property of continuity, also $q(x,1) \in c_{X \coprod_M X}\{q(x,0)\}$. If $f\colon X \coprod_M X \to Y$ with $Y \in \mathbf{X}_{iC}$, $i = 0,1$, then $f(q(x,0)) \in c_Y(f(\{q(x,1)\}))$ and $f(q(x,1)) \in c_Y(f(\{q(x,0)\}))$ by continuity. Now $Y \in \mathbf{X}_{iC}$ gives $f(q(x,0)) = f(q(x,1))$. In particular, for the $\mathbf{X}_{iC}$-reflection $r\colon X \coprod_M X \to$ $\to R(X \coprod_M X)$, $(rk_0)x = r(q(x,0)) = r(q(x,1)) = (rk_1)(x)$, so $x \in$ $\in \mathrm{Eq}(rk_0, rk_1)$. Consequently, by Lemma 1.2, $x \in [M]_{\mathbf{X}_{ic}}$.

(b) Since $\mathbf{X}_{ic}$-closure is idempotent, it is enough to show that cl_C-closed sets are $\mathbf{X}_{iC}$-closed in $X \in \mathbf{X}_{iC}$. Let $M \subseteqq X$, $X \in \mathbf{X}_{iC}$, and assume $M =$ $\mathrm{cl}_C M$. In virtue of Lemma 1.1, to prove that $\mathbf{M} = [M]_{\mathbf{X}_{iC}}$ it suffices to show that $X \coprod_M X \in \mathbf{X}_{iC}$. Let $z_1 = q(x, \ell)$, $z_2 = q(y, j)$ be two distinct points in $X \coprod_M X$. If $x \neq y$, then applying the projection $p\colon X \coprod_M X \to X$ we get $p(z_1) \neq p(z_2)$. Then by $X \in \mathbf{X}_{iC}$ it follows that $x = p(z_1)$ and $y = p(z_2)$ can be (i, C)-separated $(\ell = 0,1)$. So, by continuity, z_1 and z_2 are (i, C)-separated too. If $x = y$, then $i = 0$, $j = 1$, say, and $x \notin M$. By $M = \mathrm{cl}_C M$, $x \notin \mathrm{cl}_C M$, so $z_1 = q(x,0) \notin c_{X \coprod_M X}\{q(x,1)\}$ and $z_2 =$ $q(x,1) \notin c_{X \coprod_M X}\{q(x,0)\}$ (by the definition of $\mathrm{cl}_C M$). Thus $X \coprod_M X \in$ $\in \mathbf{X}_{iC}$, $i = 0,1$.

COROLLARY 1.4. *Let* C *be a closure operator of* $\mathbf{X}$. *Then for* $i = 0,1$, *the epimorphisms in the category* $\mathbf{X}_{iC}$ *are the* $(\mathrm{cl}_C)^\infty$*-dense maps.*

2. We recall that a *filter convergence structure* on a set X simply consists of a function $f_X\colon X \to \mathcal{P}F(X)$ where $F(X)$ is the family of filters on $\mathbf{X}$. A filter Φ f_X-converges to $x \in X$, and we write $\Phi \to x$, iff $\Phi \in f_X(x)$. Let us consider the following conditions on f_X:

a) For each $x \in X$, the filter generated by $\{x\}$ converges to x;

b) if $\Phi \to x$ and $\Phi \subseteqq \Psi$ then $\Psi \to x$;

c) $\Phi \cap \Psi \to x$ whenever $\Phi \to x$ and $\Psi \to x$;

d) $\Phi \to x$ iff every ultrafilter containing Φ converges to x;

e) the intersection of all filters converging to x is a filter converging to x (called the neighbourhood filter of x).

Fil, **Lim**, **PsT**, **PrT** will denote the category of all filter convergence spaces satisfying a) and b), resp. a), b) and c), resp. a), b) and d), resp. a), b) and e). The morphisms in **Fil** are the maps $f\colon (X, f_X) \to (Y, f_Y)$ such that $\Phi \to x$ in X implies $f(\Phi) \to f(x)$ in Y, where $f(\Phi)$ is the filter in Y generated by the family $\{f(F)\colon F \in \Phi\}$. **Lim**, **PsT**, **PrT** are considered as full subcategories of **Fil**.

The following inclusions hold: **Fil** $\supseteqq$ **Lim** $\supseteqq$ **PsT** $\supseteqq$ **PrT** $\supseteqq$ **Top** (= the category of topological spaces), and every category in this chain is bireflective in the previous one.

Every f_X defines an operator $k_{(X,f_X)} = k_X\colon \mathcal{P}X \to \mathcal{P}X$ in X by setting, for each $M \subseteqq X$,

$$k_X M = \{x \in X\colon \text{ there is } \Phi \to x \text{ and } M \cap F \neq \emptyset, \text{ for each } F \in \Phi\}.$$

It is easy to see that this is a closure operator of **X**. It will be denoted by K.

More on filter convergence spaces can be found e.g. in [9], [14] and [1].

Recall that a (Fréchet–Kuratowski) *sequential structure* on a set X consists, for each $x \in X$, of a family of sequences in X (called the sequences *converging* to x) such that

a) the constant sequence $(x, x, \dots)$ converges to x;

b) if a sequence converges to x then every subsequence converges to x;

c) if every subsequence of a given sequence (x_n) has a subsequence converging to x, then (x_n) converges to x.

FK will denote the category of sequential spaces. The morphisms in **FK** are the maps f such that if (x_n) converges to x in the domain of f then the sequence $(f(x_n))$ converges to $f(x)$ in the codomain of f. The closure operator K in **KF** is defined, for each $X \in$ **FK** and $M \subseteqq X$, by

$$k_X M = \{x \in X\colon \text{ there is a sequence in } M \text{ converging to } x\}.$$

A good reference for these spaces is [4].

PROPOSITION 2.1. *Let* **X** *be one of the categories* **Fil**, **Lim**, **PsT**, **PrT**, **FK** *and let K be the closure operator of* **X** *defined above. Then, for $X \in$ **X**, $M \subseteqq X$, $x \in X \setminus M$ and $A \subseteqq X$, the following holds: $q(x,0) \in k_{X \coprod_M X}(q(A \times \{1\}))$ iff $x \in k_X(M \cap A)$.*

PROOF. If **X** = **Fil**, **Lim**, **PsT**, **PrT**, then for $x \in X \setminus M$ and Φ a filter on $X \coprod_M X$ converging to $q(x,0)$, one has $q(X \times \{0\}) \in \Phi$. In fact in such a case there exists a filter Ψ in $X \coprod X$ converging to $(x,0)$ such that $q(\Psi) \subseteqq \Phi$. Since, by definition of $X \coprod X$, $X \times \{0\} \in \Psi$ thus $q(X \times \{0\}) \in \Phi$. So we can also assume that $F \subseteqq q(X \times \{0\})$ for each $F \in \Phi$. By $q(x,0) \in$ $\in k_{X \coprod_M X}(q(A \times \{1\}))$ it follows that $q(X \times \{0\}) \cap q(A \times \{1\}) \neq \emptyset$ which is equivalent to $M \cap A \neq \emptyset$. Since $q(x,0) \in k_{X \coprod_M X}(q(A \times \{1\}))$ or equivalently $q(x,0) \in k_{X \coprod_M X}(q(X \times \{0\}) \cap q(A \times \{1\})) = k_{X \coprod_M X}(q((M \cap A) \times \{0\}))$, then $x \in k_X(M \cap A)$.

For $\mathbf{X} = \mathbf{FK}$, if $q(x_n, i_n)$ converges to $q(x,0)$ then both (x_n) converges to x and (i_n) converges to 0, which shows the "only if" part. The "if" part of the Proposition is obvious.

COROLLARY 2.2. *For* $\mathbf{X}$ *as above,* $X \in \mathbf{X}$ *and* $M \subseteqq X$, $q(x,0) \in$ $\in k_{X \coprod_M X}\{q(x,1)\})$ *iff* $x \in M$.

COROLLARY 2.3. *For* $\mathbf{X}$ *as above, and* $i = 0, 1$, *the* $\mathbf{X}_{iK}$*-closure is discrete in* $\mathbf{X}_{iK}$*-spaces.*

COROLLARY 2.4. *For* $\mathbf{X}$ *as above the epimorphisms in* $\mathbf{X}_{iK}$, $i = 0, 1$, *are onto. In particular* $\mathbf{X}_{0K}$ *and* $\mathbf{X}_{1K}$ *are co-wellpowered categories.*

Notice that for $\mathbf{X} = \mathbf{Top}$ Proposition 2.1 is not true. In contrast with $\mathbf{PrT}_0$ the epimorphisms in $\mathbf{Top}_0$ need not be onto (cf. the proof of Corollary 3.3), so the inclusion $\mathbf{Top}_0 \hookrightarrow \mathbf{PrT}_0$ does not preserve epimorphisms. Another notable difference between $\mathbf{Top}_0$ and $\mathbf{PrT}_0$ is that $\mathbf{Top}_0$ is simply cogenerated while $\mathbf{PrT}_0$ is not (as conjectured by the authors in 1987 and recently shown in [13]).

3. In what follows we will show that, for each ordinal $\alpha \geqq 2$, $\mathbf{PrT}_{0K^\alpha}$-epimorphisms need not be onto.

LEMMA 3.1. *Let* X *be a pretopological space and let* $M \subseteqq X$. *Then a point* $x \in X$ *belongs to* $\mathrm{cl}_{K^2}(M)$ *iff* x *belongs to* $k_X(M \cap k_X(\{x\}))$.

PROOF. We may suppose $x \in X \setminus M$. In $X \coprod_M X$, $q(y,i) \in$ $\in k_{X \coprod_M X}\{q(x,i)\}$, $i = 0, 1$, iff every neighbourhood of $q(y,i)$ contains $q(x,i)$ and this is true iff every neighbourhood of (y,i) contains (x,i) in $X \times \{i\}$. This proves that $k_{X \coprod_M X}\{q(x,i)\} = k_i(k_X\{x\})$. Since $k_i(X)$ is a neighbourhood of $q(x,i)$ for $i = 0, 1$, then clearly $q(x,0) \in k^2_{X \coprod_M X}\{q(x,1)\}$ iff $q(x,0) \in k_{X \coprod_M X}(k_0(X) \cap k_1(k_X\{x\})) = k_{X \coprod_M X}(k_0(M \cap (k_X\{x\})))$ and this is equivalent to $x \in k_X(M \cap k_X\{x\})$ according to Proposition 2.1.

THEOREM 3.2. *The inclusion* $\mathbf{Top}_0 \hookrightarrow \mathbf{Pr\ T}_{0K^\alpha}$ *preserves epimorphisms for each ordinal* $\alpha \geqq 2$.

PROOF. First observe that the $\mathbf{Top}_0$-closure (which coincides with b-closure (cf. [15], [5]) is precisely the cl_{K^2}-closure, in virtue of Lemma 3.1.

So the inclusion $\mathbf{Top}_0 \hookrightarrow \mathbf{PrT}_{0K^2}$ preserves epimorphisms. Consequently, for each ordinal $\alpha \geqq 2$, the inclusion $\mathbf{Top}_0 \hookrightarrow \mathbf{PrT}_{0K^\alpha}$ preserves epimorphisms. In fact, if f is a $\mathbf{Top}_0$-epimorphism, then it is also a $\mathbf{PrT}_{0K^2}$-epimorphism, being at the same time a $\mathbf{PrT}_{0K^\alpha}$-morphism, thus f is also a $\mathbf{PrT}_{0K^\alpha}$-epimorphism.

COROLLARY 3.3. *For each ordinal $\alpha \geqq 2$ there exist $\mathbf{PrT}_{0K^\alpha}$-epimorphisms which are not onto.*

PROOF. There exist $\mathbf{Top}_0$-epimorphisms which are not onto. Take any infinite set X and a point x_0 of X. The topology in X contains as closed sets X and all finite subsets not containing x_0. Then X is a T_0-space and the inclusion map $(X \setminus \{x_0\}) \to X$ is a $\mathbf{Top}_0$-epimorphism.

4. Let us denote by $\Theta\colon \mathbf{PrT} \to \mathbf{PrT}$ the Θ-closure functor (Θ is concrete, and $\theta_X M = \{x \in X : M \cap k_X U \neq \emptyset$ for each neighbourhood U of $x\}$). It is not difficult to see that a pretopological space X belongs to $\mathbf{PrT}_{0\Theta}$ iff every convergent filter admits a unique limit point, i.e. it is a Hausdorff pretopological space. So $\mathbf{PrT}_{0\Theta} = \mathbf{PrT}_{2K}$.

LEMMA 4.1. *For each pretopological space X and $M \subseteqq X$, $\mathrm{cl}_\Theta M = k_X M$.*

PROOF. Let $x \in X \setminus M$. Notice that $q(x,0) \in \theta_{X \coprod_m X}\{q(x,1)\}$ iff for each W_0 and W_1 neighbourhoods of $q(x,0)$ and $q(x,1)$ respectively, $W_0 \cap W_1 \neq \emptyset$. Since the intersection of two neighbourhoods is again a neighbourhood we can assume without loss of generality that $W_i = q(U \times \{i\})$, where U is a neighbourhood of x in X. Clearly $W_0 \cap W_1 \neq \emptyset$ iff $U \cap M \neq \emptyset$. This proves that $q(x,0) \in \theta_{X \coprod_M X}\{q(x,1)\}$ iff $x \in k_X M$ (note that this is evident for $x \in M$).

THEOREM 4.2. *The $\mathbf{PrT}_{2K}$-epimorphisms are precisely the K^∞-dense maps.*

PROOF. We have observed that $\mathbf{PrT}_{0\Theta} = \mathbf{PrT}_{2K}$ so, by Corollary 1.4 the $\mathbf{PrT}_{2K}$-epimorphisms are the $(\mathrm{cl}_\Theta)^\infty$-dense maps. Now the statement follows from Lemma 4.1.

Let us denote by **Ury** the category of all topological spaces in which different points can be separated by disjoint closed neighbourhoods. It is well known (cf. [17]) that the epimorphisms in **Ury** are the Θ^∞-dense maps. Consequently the functor $\Theta\colon \mathbf{Ury} \to \mathbf{PrT}_{2K}$ preserves epimorphisms. On the other hand it is shown in [16] that the category **Ury** is not co-well-powered, so $\mathbf{PrT}_{2K}$ also is not co-well-powered. The non-co-well-poweredness of $\mathbf{PrT}_{2K}$ was first shown by Kneis [12] who produced an example. The fact that the non co-well-poweredness of $\mathbf{PrT}_{2K}$ can be deduced by the non co-well-poweredness of the category **Ury** was observed in [7]. In contrast with the previous result the category $\mathbf{Top}_2$ of all topological Hausdorff spaces is a co-well-powered category.

A notable difference between topological spaces and pseudotopological spaces is that the compact Hausdorff topological spaces form an epireflective subcategory of $\mathbf{Top}_2$, while the class of all compact pseudotopological Hausdorff spaces is not reflective in $\mathbf{PsT}_{2K}$, as it is shown in [2].

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SUM FORM EQUATIONS OF MULTIPLICATIVE TYPE

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1. Introduction

Let $\Gamma_n^o = \left\{P = (p_1, p_2, \dots, p_n) | 0 < p_k < 1, \sum_{k=1}^{n} p_k = 1\right\}$ be the set of all n-ary complete discrete probability distributions. Let $\mathbf{R}$ be the set of reals and I_o be the unit open interval $]0,1[$. Let $D^o = \{(x,y)|x,y,x+y \in \in I_o\}$. A real valued function $A: I_o \to \mathbf{R}$ is said to be *additive* if, and only if, $A(x+y) = A(x)+A(y)$. An additive map $A: I_o \to \mathbf{R}$ has a unique extension $\bar{A}: \mathbf{R} \to \mathbf{R}$. A map $L: I_o \to \mathbf{R}$ is called *logarithmic* provided $L(xy) = = L(x) + L(y)$ holds in I_o. A function $M: I_o \to \mathbf{R}$ is called *multiplicative* if $M(xy) = M(x)M(y)$. A multiplicative function $M: I_o \to \mathbf{R}$ can be uniquely extended to $\bar{M}: \mathbf{R}_+ \to \mathbf{R}$, where $\mathbf{R}_+ = \{x \in \mathbf{R} | x > 0\}$.

In 1948, Shannon [9] introduced the following measure of information

$$H_n(P) = -\sum_{i=1}^{n} p_i \log p_i,$$

which is now known as *Shannon's entropy.* This has been generalized to entropy of type (α, β) [5]

$$H_n^{(\alpha,\beta)}(P) = (2^{1-\alpha} - 2^{1-\beta})^{-1} \sum_{i=1}^{n} (p_i^\alpha - p_i^\beta), \tag{1.1}$$

where $P \in \Gamma_n^o$ and α, β are real nonzero parameters. While characterizing the entropy of type (α, β) we come across the following functional equation

$$\sum_{i=1}^{2}\sum_{j=1}^{3} f(p_i q_j) = \sum_{i=1}^{2} p_i^\alpha \sum_{j=1}^{3} f(q_j) + \sum_{j=1}^{3} q_j^\beta \sum_{i=1}^{2} f(p_i),$$

where $P \in \Gamma_2^o$, $Q \in \Gamma_3^o$, $\alpha, \beta \in \mathbf{R} \setminus \{0\}$. The above functional equation can be solved (see [4]) with the aid of the following functional equation:

$$f(pq) + f((1-p)q) = f(q)\{M(p) + M(1-p)\} + M(q)\{f(p) + f(1-p)\}, \tag{1.2}$$

where $p, q \in I_o$ and $M\colon I_o \to \mathbf{R}$ is a multiplicative function.

The objective of this paper is to find all solutions f of the *sum form equation of multiplicative type*, that is, of the functional equation

$$(1.3)\quad f(pq)+f((1-p)q) = f(q)\{m(p)+m(1-p)\}+M(q)\{f(p)+f(1-p)\},$$

where $p, q \in I_o$ and $m, M\colon I_o \to \mathbf{R}$ are multiplicative functions. Notice that (1.3) generalizes (1.2). Also, we will see later that if $m\colon I_o \to \mathbf{R}$ is additive in addition to being multiplicative, then the sum form equation of multiplicative type is connected to the *fundamental equation of information.* An account of the history of results related to the fundamental equation together with an extensive list of references, can be found in [8].

2. Auxiliary results

In this section we prepare a series of auxiliary results following methods from [6] to prove our main theorem. Let $f, M\colon I_o \to \mathbf{R}$ satisfy

$$(2.1\text{a})\qquad f(pq) + f((1-p)q) = f(q)[M(p) + M(1-p)]$$

and

$$(2.1\text{b})\qquad M(pq) = M(p)M(q).$$

Suppose f and M are nonconstant solutions of (2.1). We define the set

$$\Omega = \{x \in I_o \mid f(tx) = M(x)f(t) \text{ for all } t \in I_o\}.$$

LEMMA 1. *Suppose $M, f\colon I_o \to \mathbf{R}$ are solutions of* (2.1); *and f and M are not identically constant. Then Ω has the following properties:*

(a) $\frac{1}{2} \in \Omega$,

(b) *if $x \in \Omega$, then $1 - x \in \Omega$,*

(c) *if $x \in \Omega \cap \left]0, \frac{1}{2}\right[$, then $2x \in \Omega$,*

(d) *if $x, y \in \Omega$, then $xy \in \Omega$,*

(e) *if $x, y \in \Omega$ and $\frac{x}{y} \in I_o$, then $\frac{x}{y} \in \Omega$,*

(f) *if $x, y \in \Omega$ and $y > x$, then $y - x \in \Omega$,*

(g) *if $x, y \in \Omega$ and $x + y \in I_o$, then $x + y \in \Omega$.*

PROOF. $\frac{1}{2} \in \Omega$ follows from (2.1a) with $p = \frac{1}{2}$. Part (b) is an obvious consequence of (2.1a). Since $\frac{1}{2} \in \Omega$, for $x \in \Omega \cap \left]0, \frac{1}{2}\right[$ and $t \in I_o$,

$$M\Big(\frac{1}{2}\Big)f(t \cdot 2x) = f(tx) = f(t)M(x) = f(t)M\Big(2x \cdot \frac{1}{2}\Big) = f(t)M(2x)M\Big(\frac{1}{2}\Big).$$

Hence $f(t \cdot 2x) = f(t)M(2x)$. Thus if $x \in \Omega \cap \left]0, \frac{1}{2}\right[$, then $2x \in \Omega$. Next we prove (d). Let $x, y \in \Omega$. Then

$$f(txy) = f(tx)M(y) = f(t)M(x)M(y) = f(t)M(xy), \quad \text{for} \quad t \in I_o.$$

Hence $xy \in \Omega$. To prove (e), let $x, y \in \Omega$ and $\frac{x}{y} \in I_o$. Consider

$$f(t)M(x) = f(tx) = f\Big(t\frac{x}{y}y\Big) = f\Big(t\,\frac{x}{y}\Big)M(y).$$

Hence

$$f(t)\frac{M(x)}{M(y)} = f\Big(t\,\frac{x}{y}\Big).$$

Thus

$$f\Big(t\,\frac{x}{y}\Big) = f(t)M\Big(\frac{x}{y}\Big).$$

Hence $\frac{x}{y} \in \Omega$. Next we prove (f). Since $y - x = y(1 - \frac{x}{y})$ if $y > x$, it is easy to see that $y - x \in \Omega$ by (b), (e) and (d). Consider $x, y \in \Omega$ and $x + y \in I_o$. Then

$$x + y = 2\begin{cases} y - \frac{1}{2}(y - x) & \text{if } y \geqq x, \\ x - \frac{1}{2}(x - y) & \text{if } x > y. \end{cases}$$

Thus $x + y \in \Omega$. This completes the proof of Lemma 1.

LEMMA 2. *If f and M are nonconstant solutions of* (2.1) *and if there exist $u, v \in \Omega$, $u < v$ such that $]u, v[\subset \Omega$, then f is a solution of $f(pq) = M(p)f(q)$, that is, $f(p) = cM(p)$, where c is an arbitrary constant.*

PROOF. Let $u, v \in \Omega$ with $u < v$ such that $]u, v[\subset \Omega$. It is enough to show that $]0, 1[\subseteqq \Omega$. Since $u, v \in \Omega$, $\frac{u}{v} \in \Omega$ by Lemma 1(e). First we show that $\left]\frac{u}{v}, 1\right[\subset \Omega$. Let $x \in \left]\frac{u}{v}, 1\right[$. Then $\frac{u}{v} < x < 1$. Thus $u < xv < v$. Since $]u, v[\subset \Omega$, $xv \in \Omega$. We know that $v \in \Omega$. Hence by Lemma 1, $\frac{xv}{v} \in \Omega$. That is $x \in \Omega$. This implies $\left]\frac{u}{v}, 1\right[\subset \Omega$. Again by Lemma 1(d), it can be shown by induction that $\left(\frac{u}{v}\right)^k \in \Omega$ for all natural numbers k. Hence $\left]\left(\frac{u}{v}\right)^k, 1\right[\subset \Omega$. This shows that $]0, 1[\subseteqq \Omega$. This completes the proof.

LEMMA 3. *If f and M are not identically constants and satisfy* (2.1a,b) *and Ω does not contain an interval, then f is additive and $M(x) = x$.*

PROOF. Define $\phi\colon I_o^2 \to \mathbf{R}$ by setting

$$\phi(t, x) := f(tx) - M(x)f(t). \tag{2.3}$$

Then

$$\phi(t, x) + \phi(t, 1 - x) = f(tx) + f(t(1 - x)) - f(t)[M(x) + M(1 - x)] = 0 \tag{2.4}$$

for all t and x in I_o. In particular,

$$\phi\Big(t, \frac{1}{2}\Big) = 0 \tag{2.5}$$

for all $t \in I_o$. Consider (2.1) and let $p = t(x+y)$ and $q = \frac{x}{(x+y)}$. Then (2.1) becomes

$$(2.6) \qquad f(tx) + f(ty) = f(tx+ty)\Big[M\Big(\frac{x}{x+y}\Big) + M\Big(\frac{y}{x+y}\Big)\Big].$$

Now using (2.6) we compute

$$(2.7) \quad M(x+y)[\phi(t,x)+\phi(t,y)] = M(x+y)[f(tx)-M(x)f(t)+f(ty)-M(y)f(t)] =$$
$$= M(x+y)[f(tx)+f(ty)] - M(x+y)[M(x)+M(y)]f(t) =$$
$$= M(x+y)f(tx+ty)\Big[M\Big(\frac{x}{x+y}\Big)+M\Big(\frac{y}{x+y}\Big)\Big] - M(x+y)[M(x)+M(y)]f(t) =$$
$$= f(tx+ty)[M(x)+M(y)] - M(x+y)[M(x)+M(y)]f(t) =$$
$$= [M(x)+M(y)][f(tx+ty) - M(x+y)f(t)] = [M(x)+M(y)]\phi(t,x+y).$$

Let $t \in I_o$, $(x,y) \in D^o$ and write $1-x-y$ instead of y in (2.7). Then

$$(2.8) \quad M(1-y)[\phi(t,x)+\phi(t,1-x-y)] = \phi(t,1-y)[M(x)+M(1-x-y)].$$

Since $\phi(t,x) = -\phi(t,1-x)$, from (2.8) we get

$$(2.9) \quad M(1-y)[\phi(t,x)-\phi(t,x+y)] = -\phi(t,y)[M(x)+M(1-x-y)].$$

We eliminate $\phi(t,x+y)$ from (2.9) by (2.7). Then

$$(2.10) \qquad M(1-y)[M(x+y)-M(x)-M(y)]\phi(t,x) =$$
$$= \phi(t,y)[(M(x)+M(1-x-y))(M(x)+M(y)) - M(1-y)M(x+y)]$$

holds for all $t \in I_o$ and $(x,y) \in D^o$.

Let $y \in \Omega$ and $x \in I_o \setminus \Omega$ such that $(x,y) \in D^o$. Then by definition of Ω there exists $t_o \in I_o$ such that $\phi(t_o,x) \neq 0$ and $\phi(t_o,y) = 0$. Hence from (2.10) with $t = t_o$, we obtain

$$(2.11) \qquad M(1-y)[M(x+y)-M(x)-M(y)] = 0.$$

Since $M\colon I_o \to \mathbf{R}$ is a multiplicative function and not identically constant, (2.11) reduces to

$$(2.12) \qquad M(x+y) = M(x)+M(y)$$

for all $x \in I_o \setminus \Omega$, $y \in \Omega$ and $(x,y) \in D^o$.

Let $x,y \in \Omega$ and $x+y \in I_o$. Then there exists $x_o \in]0,1-x-y[$ such that $x+x_o \in I_o \setminus \Omega$ since in the opposite case Ω would contain the interval

$[x,1-y]$. By Lemma 1(g) we see that $x_o \in I_o \setminus \Omega$ and $x+y \in \Omega$. Hence $x+x_o+y \in I_o \setminus \Omega$. Thus by (2.12) we get

$$M(x+y)=M(x+y+x_o)-M(x_o)=M(x)+M(y),$$

for all $x,y \in \Omega$. We have shown so far that $M(x+y)=M(x)+M(y)$ for all $(x,y) \in D^o$ and $y \in \Omega$. In particular $M\left(x+\frac{1}{2}\right)=M(x)+M\left(\frac{1}{2}\right)$ holds for all $x \in \left]0,\frac{1}{2}\right[$. Letting $y=\frac{1}{2}$ in (2.9) for all $x \in \left]0,\frac{1}{2}\right[$, we get

$$(2.13)\quad M\Big(\frac{1}{2}\Big)\Big[\phi(t,x)-\phi\Big(t,\frac{1}{2}+x\Big)\Big]=-\phi\Big(t,\frac{1}{2}\Big)\Big[M(x)+M\Big(1-x-\frac{1}{2}\Big)\Big].$$

Since $\phi\left(t,\frac{1}{2}\right)=0$, (2.13) reduces to $\phi(t,x)=\phi\left(t,\frac{1}{2}+x\right)$, that is by (2.3),

$$(2.14)\qquad f\Big(tx+\frac{t}{2}\Big)=M\Big(x+\frac{1}{2}\Big)f(t)+f(tx)-M(x)f(t)$$

for all $t \in I_o$ and $x \in \left]0,\frac{1}{2}\right]$. Again, since $f\left(\frac{t}{2}\right)=M\left(\frac{1}{2}\right)f(t)$, (2.14) becomes

$$(2.15)\qquad f\Big(tx+\frac{t}{2}\Big)=f(tx)+f(t)\Big[M\Big(x+\frac{1}{2}\Big)-M(x)\Big]=$$
$$=f(tx)+M\Big(\frac{1}{2}\Big)f(t)=f(tx)+f\Big(\frac{t}{2}\Big).$$

Hence we have shown that $f(x+y)=f(x)+f(y)$ for all $x,y \in \left]0,\frac{1}{2}\right]$. Next let $(x,y) \in D^o$ then $\frac{x}{2},\frac{y}{2} \in \left]0,\frac{1}{2}\right]$ and

$$(2.16)\quad M\Big(\frac{1}{2}\Big)f(x+y)=f\Big(\frac{x}{2}+\frac{y}{2}\Big)=f\Big(\frac{x}{2}\Big)+f\Big(\frac{y}{2}\Big)=M\Big(\frac{1}{2}\Big)[f(x)+f(y)].$$

Thus

$$(2.17)\qquad f(x+y)=f(x)+f(y)$$

for all $x,y,x+y \in I_o$. This in (2.1a) yields

$$(2.18)\qquad M(p)+M(1-p)=1$$

for all $p \in]0,1[$. Thus from Rathie and Kannappan [7, p. 157], M is monotone increasing. Then from [1, p. 33, p. 41], we conclude that $M(x)=x$ for all $x \in I_o$. This completes the proof of the Lemma.

By Lemma 1, 2 and 3, we have the following theorem.

THEOREM 4. *Let $f,M\colon I_o \to \mathbf{R}$ satisfy* (2.1a) *and* (2.1b). *Suppose f and M are not identically constant. Then either f is additive and $M(x)=x$ (i.e. M is additive also) or $f(x)=cM(x)$, where c is an arbitrary constant.*

THEOREM 5. *Let $f,M\colon I_o \to \mathbf{R}$ satisfy*

$$(2.19a)\quad f(pq)+f((1-p)q)=f(q)[M(p)+M(1-p)]+M(q)[f(p)+f(1-p)]$$

and

$$M(pq) = M(p)M(q). \tag{2.19b}$$

Suppose f and M are not identically constant. Further we suppose that M is not additive. Then

$$f(p) = M(p)L(p) \tag{2.20}$$

where $L\colon I_o \to \mathbf{R}$ is a logarithmic function.

PROOF. Letting $p = \frac{u}{u+v}$, $q = u+v$ such that $u, v, u+v \in I_o$ into (2.19a), we get
(2.21)

$$f(u)+f(v)-\left[M\left(\frac{u}{u+v}\right)+M\left(\frac{v}{u+v}\right)\right]f(u+v)=\left[f\left(\frac{u}{u+v}\right)+f\left(\frac{v}{u+v}\right)\right]M(u+v).$$

For $t \in I_o$, we get from (2.21)

$$f(ut) + f(vt) - \left[M\left(\frac{u}{u+v}\right) + M\left(\frac{v}{u+v}\right)\right]f(ut+vt) = \tag{2.22}$$
$$= \left[f\left(\frac{u}{u+v}\right) + f\left(\frac{v}{u+v}\right)\right]M(u+v)M(t).$$

From (2.21) and (2.22) we get

$$f(ut) + f(vt) - \frac{M(u)+M(v)}{M(u+v)}f(ut+vt) = \tag{2.23}$$
$$= M(t)[f(u)+f(v)] - \frac{M(u)+M(v)}{M(u+v)}f(u+v)M(t).$$

We temporarily fix $t \in I_o$ and define

$$F(u) := f(ut) - M(t)f(u) - M(u)f(t). \tag{2.24}$$

Using (2.24) in (2.23), we obtain

$$F(u) + F(v) = \frac{M(u)+M(v)}{M(u+v)}F(u+v). \tag{2.25}$$

Suppose $F(u) = \text{constant}$ for all $u \in I_o$. Then by (2.24), we have

$$f(ut) = M(t)f(u) + M(u)f(t) + k, \tag{2.26}$$

where k is a constant. Since M is not identically constant, we get from (2.26) a Pexider type functional equation

$$\frac{f(ut)-k}{M(ut)} = \frac{f(u)}{M(u)} + \frac{f(t)}{M(t)}. \tag{2.27}$$

Thus,

$$f(x) = M(x)L(x) + 2bM(x) + k \tag{2.28}$$

and also

$$f(x) = M(x)L(x) + bM(x), \tag{2.29}$$

where b is an arbitrary constant. Hence $k = 0 = b$ and f has the form as asserted in (2.20).

Next suppose that $F(u)$ is not identically constant. The functional equation (2.25) can be transformed to

$$F(pq) + F((1-p)q) = F(q)[M(p) + M(1-p)], \tag{2.30}$$

with $p = \frac{u}{u+v}$, and $q = u + v$. Then from Theorem 4, since M is not additive, we have

$$F(p) = cM(p), \tag{2.31}$$

where c is an arbitrary real constant. From (2.31) and (2.24), we obtain

$$f(ut) = M(t)f(u) - M(u)f(t) = c(t)M(u), \tag{2.32}$$

where $c\colon I_o \to \mathbf{R}$ is an arbitrary function. Notice that the left side of (2.32) is symmetric in u and t. Thus by symmetry of the left side of (2.32), we get

$$c(t)M(u) = c(u)M(t).$$

Hence $c(t) = c_o M(t)$, where c_o is an arbitrary constant. Now (2.32) becomes

$$f(ut) = M(t)f(u) + M(u)f(t) + c_o M(ut). \tag{2.33}$$

The above equation can be reduced to a Pexider type functional equation as above and from that

$$f(x) = L(x)M(x) + c_o M(x) + 2b, \tag{2.34}$$

and also

$$f(x) = L(x)M(x) + b \tag{2.35}$$

for all $x \in I_o$. Hence from (2.34) and (2.35) we get $c_o = 0 = b$. Thus again f has the form as claimed in (2.20). This completes the proof of Theorem 5.

3. Solution of the sum form equation of multiplicative type

Now we proceed to find the general solution of the functional equation (1.3). Let $f\colon I_o \to \mathbf{R}$ be a nonconstant real valued function and $m, M\colon I_o \to \mathbf{R}$ be multiplicative functions satisfying the functional equation (1.3) for all $p, q \in I_o$. While finding the general solutions of (1.3), we consider the following cases as illustrated in the tree diagram.

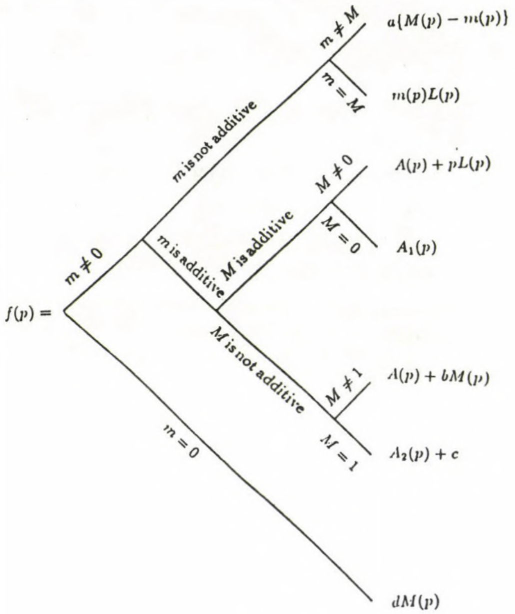

Fig. 1

For $(x, y) \in D^o$, let

$$(3.1)\qquad p = \frac{x}{x+y} \quad \text{and} \quad q = x + y.$$

Letting (3.1) into (1.3), we get

(3.2)
$$f(x)+f(y)=f(x+y)\Big\{m\Big(\frac{x}{x+y}\Big)+m\Big(\frac{y}{x+y}\Big)\Big\}+M(x+y)\Big\{f\Big(\frac{x}{x+y}\Big)+f\Big(\frac{y}{x+y}\Big)\Big\}.$$

Case 1. Suppose $m = 0$ on I_o. Then (3.2) reduces to

$$(3.3)\qquad f(x) + f(y) = M(x + y)\Big\{f\Big(\frac{x}{x+y}\Big) + f\Big(\frac{y}{x+y}\Big)\Big\}.$$

Replacing x by xt and y by yt, for $t \in I_o$, we get from (3.3)

$$(3.4)\qquad f(xt) + f(yt) = M(t)M(x + y)\Big\{f\Big(\frac{x}{x+y}\Big) + f\Big(\frac{y}{x+y}\Big)\Big\}.$$

Hence (3.3) and (3.4) imply

$$(3.5)\qquad f(xt) - M(t)f(x) = -\{f(yt) - M(t)f(y)\}$$

for all $(x, y) \in D^o$ and $t \in I_o$. Hence

$$(3.6)\qquad K(t) = f(xt) - M(t)f(x),$$

where $K\colon I_o \to \mathbf{R}$. But by (3.5), $K(t) = 0$ on I_o and

$$f(xt) = M(t)f(x) \tag{3.7}$$

for all $x, t \in I_o$. Interchanging x and t in (3.7) and using (3.7), we get

$$f(p) = dM(p), \tag{3.8}$$

where d is an arbitrary constant since f is nonconstant.

Case 2. Suppose $m \neq 0$ on I_o. Now we split this case into two subcases based on whether, on I_o, m is additive or not.

Subcase 2.1. Suppose m is not additive on I_o. As before, for $t \in I_o$, we replace x by xt and y by yt in (3.2) to obtain

$$\begin{aligned} f(xt) + f(yt) = f(xt + yt)\Big\{m\Big(\frac{x}{x+y}\Big) + m\Big(\frac{y}{x+y}\Big)\Big\} + \\ + M(x+y)M(t)\Big\{f\Big(\frac{x}{x+y}\Big) + f\Big(\frac{y}{x+y}\Big)\Big\}. \end{aligned} \tag{3.9}$$

Then from (3.2) and (3.9), we get

$$\begin{aligned} f(xt) + f(yt) - \frac{m(x) + m(y)}{m(x+y)} f(xt + yt) = \\ = M(t)f(x) + M(t)f(y) - \frac{m(x) + m(y)}{m(x+y)} f(x+y)M(t). \end{aligned} \tag{3.10}$$

For fixed $t \in I_o$, we define $F\colon I_o \to \mathbf{R}$ by

$$F(x) := f(xt) - M(t)f(x) - m(x)f(t). \tag{3.11}$$

Then from (3.10), using (3.11) we get

$$F(x) + F(y) = \frac{m(x) + m(y)}{m(x+y)} F(x+y), \tag{3.12}$$

where $(x, y) \in D^o$. First we suppose that F is a constant on I_o, say k. Then (3.11) yields

$$f(xt) = M(t)f(x) + m(x)f(t) + k \tag{3.13}$$

$$\text{also} = M(x)f(t) + m(t)f(x) + k. \tag{3.14}$$

Hence

$$f(x)\{M(t) - m(t)\} = f(t)\{M(x) - m(x)\}. \tag{3.15}$$

Thus if $m \neq M$ on I_o, then (3.15) implies

$$f(p) = a\{M(p) - m(p)\} \tag{3.16}$$

where a is a constant.

If $m = M$, then from (3.13), we get

$$f(xt) = m(t)f(x) + m(x)f(t) + k. \tag{3.17}$$

Since $m = M$ and $m \neq 0$, we get from (3.17)

$$\frac{f(xt) - k}{M(xt)} = \frac{f(x)}{M(x)} + \frac{f(t)}{M(t)} \tag{3.18}$$

for all $x, t \in I_o$. Equation (3.18) is a Pexider type functional equation and

$$f(p) = M(p)\{L(p) + c_1\} = M(p)\{L(p) + 2c_1\} + k, \tag{3.19}$$

where $L\colon I_o \to \mathbf{R}$ is a logarithmic function. Hence $k = c_1 = 0$, and we have

$$f(p) = M(p)L(p). \tag{3.20}$$

Suppose F is not identically constant on I_o. Equation (3.12) was treated in Theorem 4 and its solution can be obtained from Theorem 4 as

$$F(p) = cm(p) \tag{3.21}$$

where c is a constant. Hence (3.11) and (3.21) yield

$$f(xt) = M(t)f(x) + m(x)f(t) + c(t)m(x), \tag{3.22}$$

where $c\colon I_o \to \mathbf{R}$. Interchanging x with t, we get from (3.22)

$$M(t)f(x) + m(x)f(t) + c(t)m(x) = M(x)f(t) + m(t)f(x) + c(x)m(t),$$

which is

$$\{M(t) - m(t)\}f(x) = \{M(x) - m(x)\}f(t) + c(x)m(t) - c(t)m(x). \tag{3.23}$$

Now we suppose that $m \neq M$ on I_o. Then using (3.22) we compute $f(xtu)$ in two different ways, first as $f(xt \cdot u)$ and then as $f(x \cdot tu)$ to obtain

$$\begin{aligned} f(xt \cdot u) = M(u)M(t)f(x) + M(u)m(x)f(t) + \\ + M(u)m(x)c(t) + m(x)m(t)f(u) + c(u)m(x)m(t) \end{aligned} \tag{3.24}$$

$$\begin{aligned} f(x \cdot tu) = M(t)M(u)f(x) + m(x)M(u)f(t) + \\ + m(x)m(t)f(u) + c(u)m(t)m(x) + c(tu)m(x). \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) we get

$$M(u)c(t)m(x) = c(tu)m(x). \tag{3.26}$$

As $m \neq 0$ on I_o, (3.26) yields

$$c(tu) = c(t)M(u) \tag{3.27}$$

for all $u, t \in I_o$. From (3.27), we get

$$c(x) = c_o M(x), \quad x \in I_o, \tag{3.28}$$

where c_o is a constant. Letting (3.28) into (3.23) and then fixing t in the resulting expression, we get

$$f(x) = a_o\{M(x) - m(x)\} + b_1 M(x) + b_2 m(x) \tag{3.29}$$

where a_o, b_1, b_2 are real constants. Putting (3.29) into (1.3), we obtain

$$(b_1 + b_2)\{m(p) + m(1-p)\}M(q) = 0. \tag{3.30}$$

If $M \neq 0$, then (3.30) implies that $b_1 = -b_2$ and hence from (3.29) we get

$$f(p) = a\{M(p) - m(p)\}, \tag{3.31}$$

where $a(= a_o + b_1)$ is an arbitrary constant. If $M = 0$ on I_o, then (1.3) yields

$$f(pq) + f((1-p)q) = f(q)\{m(p) + m(1-p)\}. \tag{3.32}$$

The functional equation (3.32) was investigated in Theorem 4. Hence from Theorem 4, we get the general solution of (3.32) as

$$f(p) = -am(p). \tag{3.33}$$

Thus, since $M = 0$, the solution f of (1.3) in this case is again of the form of (3.31).

Now we treat the subcase when $m = M$ on I_o. Then from (3.23), we get

$$c(x)m(t) = c(t)m(x).$$

Thus from the above equation we get

$$c(x) = c_o m(x), \tag{3.34}$$

where c_o is a constant. Letting (3.34) into (3.22), we obtain

$$f(xt) = M(t)f(x) + M(x)f(t) + c_o M(t)M(x). \tag{3.35}$$

Rearranging (3.35), we obtain

$$\frac{f(xt) - c_o M(xt)}{M(xt)} = \frac{f(x)}{M(x)} + \frac{f(t)}{M(t)}. \tag{3.36}$$

The Pexider equation (3.36) yields

$$f(x) = M(x)\{L(x) + c_2\} \tag{3.37}$$

and also

$$f(x) = M(x)\{L(x) + 2c_2\} + c_o M(x),$$

where $L\colon I_o \to \mathbf{R}$ is a logarithmic function. Thus $c_2 = -c_o$. Hence (3.37) reduces to

$$f(x) = M(x)L(x) - c_o M(x). \tag{3.38}$$

Letting (3.38) into (1.3), we get $c_o\{m(pq) + m((1-p)q)\} = 0$. Hence letting $p = q = \frac{1}{2}$ in the above equation we get $c_o = 0$. Thus

$$f(p) = M(p)L(p), \quad p \in I_o \tag{3.39}$$

where $L\colon I_o \to \mathbf{R}$ is logarithmic on I_o.

Subcase 2.2. Next, we consider the subcase when m is additive. (Notice that m is not identically 0). Then (1.3) becomes

$$f(pq) + f((1-p)q) - f(q) = M(q)\{f(p) + f(1-p)\}. \tag{3.40}$$

We first define

$$G(x) := f(x) + f(1-x), \quad x \in I_o \tag{3.41}$$

and for $(x,y) \in D^o$, we compute using (3.40)

$$G(x) + M(1-x)G\Big(\frac{y}{1-x}\Big) = \tag{3.42}$$

$$= f(x) + f(1-x) + M(1-x)\Big\{f\Big(\frac{y}{1-x}\Big) + f\Big(\frac{1-x-y}{1-x}\Big)\Big\} =$$

$$= f(x) + f(1-x) + f(y) + f(1-x-y) - f(1-x) = f(x) + f(y) - f(1-x-y).$$

On the right side, the expression (3.42) is symmetric in x and y. Thus, we have

$$G(x) + M(1-x)G\Big(\frac{y}{1-x}\Big) = G(y) + M(1-y)G\Big(\frac{x}{1-y}\Big), \tag{3.43}$$

for all $(x,y) \in D^o$. The functional equation (3.43) is known as the *fundamental equation of information of multiplicative type.* The solutions of (3.43) can be obtained from [2] or [3] as:

(A) If M is additive with $M \neq 0$ on I_o, then

$$G(x) = M(x)L(x) + M(1-x)L(1-x) + b_1 M(x)$$

where $L\colon I_o \to \mathbf{R}$ is logarithmic and b_1 is a constant.

(B) If M is additive with $M = 0$ on I_o, then $G(x) = b_2$ where b_2 is a constant.

(C) If M is not additive with $M \neq 1$ on I_o, then $G(x) = b_3M(x) + b_4M(1-x) - b_4$ where b_3 and b_4 are constants.

(D) If M is not additive with $M = 1$ on I_o, then $G(x) = L(1-x) + c$ where c is a constant.

Now using definition of G in (3.41) and the form of G in (A), (B), (C) and (D), we determine the solution f of (3.40).

Subcase 2.2.A. Suppose M is additive with $M \neq 0$ on I_o. Then from (A) and (3.41), we get

$$f(x) + f(1-x) = xL(x) + (1-x)L(1-x) + b_1x. \tag{3.44}$$

Interchanging x with $1-x$ in (3.44), we obtain $b_1 = 0$. Letting (3.44) into (3.40) with $b_1 = 0$, we obtain

$$f(pq) + f((1-p)q) - f(q) = q\{pL(p) + (1-p)L(1-p)\}.$$

Defining

$$A(x) := f(x) - xL(x) \tag{3.45}$$

for all $x \in I_o$, we obtain from the above

$$A(pq) + A((1-p)q) = A(q) \tag{3.46}$$

for all $p, q \in I_o$. That is A is an additive function. Thus, by (3.45), we get

$$f(p) = A(p) + pL(p), \quad p \in I_o. \tag{3.47}$$

Now (3.47) is a solution of (3.40) provided $A(1) = 0$.

Subcase 2.2.B. Suppose M is additive with $M = 0$ on I_o. Then (3.40) reduces to

$$f(pq) + f((1-p)q) = f(q) \tag{3.48}$$

that is f is additive on I_o and

$$f(p) = A_1(p) \tag{3.49}$$

where $A_1\colon \mathbf{R} \to \mathbf{R}$ is an additive function.

Subcase 2.2.C. Suppose M is not additive with $M \neq 1$ on I_o. Then from (C) and (2.40), we get

(3.50)
$$f(x)+f(y)-f(x+y)=M(x+y)\Big\{b_3M\Big(\frac{x}{x+y}\Big)+b_4M\Big(\frac{y}{x+y}\Big)-b_4\Big\}=$$
$$= b_3M(x) + b_4M(y) - b_4M(x+y).$$

Symmetry of the left side of (3.50) implies $b_3 = b_4 = b$ (say). Hence we get

$$f(x) + f(y) - f(x+y) = b\{M(x) + M(y) - M(x+y)\}$$

for all $(x, y) \in D^o$. Thus if

$$A(x) := f(x) - bM(x) \tag{3.51}$$

then the above functional equation reduces to $A(x)+A(y) = A(x+y)$. Hence

$$f(p) = A(p) + bM(p) \tag{3.52}$$

where A is additive, b is an arbitrary constant and $A(1) = -b$.

Subcase 2.2.D. Next we consider the subcase when M is not additive with $M = 1$ on I_o. Then from (3.40) and (D) with (3.1), we get

$$f(x) + f(y) - f(x+y) = L\Big(\frac{x}{x+y}\Big) + c.$$

The symmetry of the left side of the above equation implies that

$$L(x) = L(y) \tag{3.53}$$

for all $(x, y) \in D^o$. But $L\colon I_o \to \mathbf{R}$ is logarithmic, hence in view of (3.53) L must be identically 0. Thus we get $f(x) + f(y) - f(x+y) = c$ which in fact implies

$$f(p) = A_2(p) + c, \quad p \in I_o \tag{3.54}$$

where $A_2\colon \mathbf{R} \to \mathbf{R}$ is an additive function and d is an arbitrary constant with $A_2(1) + d = 0$.

Thus we have proved the following theorem.

THEOREM 6. *Let $f\colon I_o \to \mathbf{R}$ be a nonconstant function and $m, M\colon I_o \to \mathbf{R}$ be multiplicative functions. If f, m, M satisfy the functional equation* (1.3), *then the general solution of* (1.3) *is given by*

$$f(p) = \begin{cases} a\{M(p) - m(p)\} & \text{if } m \neq 0,\ m \text{ is not additive and } m \neq M \\ M(p)L(p) & \text{if } m \neq 0,\ m \text{ is not additive and } m = M \\ A(p) + pL(p) & \text{if } m \neq 0,\ m \text{ is additive, } M \text{ is additive and } M \neq 0 \\ A_1(p) & \text{if } m \neq 0,\ m \text{ is additive, } M \text{ is additive and } M = 0 \\ A(p) + bM(p) & \text{if } m \neq 0,\ m \text{ is additive, } M \text{ is not additive} \\ & \quad \text{and } M \neq 1 \\ A_2(p) + c & \text{if } m \neq 0,\ m \text{ is additive, } M \text{ is not additive} \\ & \quad \text{and } M = 1 \\ dM(p) & \text{if } m = 0 \end{cases}$$

where $A, A_1, A_2\colon \mathbf{R} \to \mathbf{R}$ are additive functions, $L\colon I_o \to \mathbf{R}$ is logarithmic, a, b, c, d are constants with $A(1) = 0$ and $A_2(1) + c = 0$.

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ON A GENERALIZED EULER SPLINE AND ITS APPLICATIONS TO THE STUDY OF CONVERGENCE IN CARDINAL INTERPOLATION AND SOLUTIONS OF CERTAIN EXTREMAL PROBLEMS*

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1. Introduction

It is well known that the Euler spline is instrumental to deriving the integral representation of the error function in cardinal interpolation of a function f from a spline space $\mathcal{S}$ in terms of $\mathcal{L}f$, where $\mathcal{L}$ is the differential operator governing the spline space $\mathcal{S}$. The classical polynomial spline results are well documented in Schoenberg's CBMS monograph [8] and its follow-up paper [1], and some of its analogous results for the more general differential operator

$$\mathcal{L} = \prod_{j=0}^{n}(D - \gamma_j),$$

where D denotes the differentiation operator and $\gamma_0, \ldots, \gamma_n$ are real numbers, were obtained in [2], [3], [4], [7], and [9]. It should be remarked that the mathematical analysis in the above mentioned literature depends very much on the property of translation invariance of the spline space $\mathcal{S}$, which, in turn, requires the linear differential operator $\mathcal{L}$ to have constant coefficients. The objective of this paper is to initiate a study of this problem where non-constant coefficients are allowed. In particular, a generalized Euler spline will be introduced, its intrinsic properties carefully studied, and from it, the kernel function of the integral representation of the error function in cardinal interpolation will be constructed, and its sign pattern will be given. The importance of this integral representation formula is that it provides a very important tool for estimating the error of convergence and solving certain extremal problems. We will also give an example of such applications.

The operator $\mathcal{L}$ we will study in this paper can be described as follows:

$$(1.1)\qquad \begin{cases} \mathcal{L} = \mathcal{L}_{n+1}, \text{ where} \\ \mathcal{L}_j = D_j D_{j-1} \ldots D_1, \quad j = 1, \ldots, n+1, \text{ and} \\ (D_j g)(t) = \alpha_{j-1}(t) D(\beta_{j-1}(t) g(t)), \end{cases}$$

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with $\alpha_j \in C^{n-j}(\mathbf{R}^1)$ and $\beta_j \in C^{n+1-j}(\mathbf{R}^1)$, α_j and β_j are positive functions such that

$$(1.2) \qquad \begin{cases} \alpha_j(t+h) = c_j\alpha_j(t) \\ \beta_j(t+h) = c_j^{-1}\beta_j(t), \end{cases}$$

for all $t \in \mathbf{R}^1$ and $j = 0, \dots, n+1$. Here, $h > 0$ is some fixed constant. Hence, from (1.2), we have $c_j = \beta_j(0)/\beta_j(h)$. It is worthwhile to mention that the linear differential operators $\mathcal{L}_j$ defined above extend those with constant coefficients and real roots. Indeed, by setting $\alpha(t) = e^{\gamma_j t}$ and $\beta_k(t) = e^{-\gamma_j t}$, γ_j real, we have

$$\mathcal{L} = \mathcal{L}_{n+1} = \prod_{j=0}^{n}(D - \gamma_j).$$

In addition, by setting $\alpha_k(t) \equiv 1$ and β_j h-periodic functions, we have

$$(\mathcal{L}_j g)(t) = D(\beta_{j-1}(t)g(t)).$$

To simplify our presentation, the additional assumption

$$(1.3) \qquad \alpha_{-1}(t) = \beta_{n+1}(t) \equiv 1$$

will be made throughout the paper. Let

$$(1.4) \qquad w_j(t) = \begin{cases} \frac{1}{\beta_0(t)} & \text{for } j = 0 \\ \frac{1}{\alpha_{j-1}(t)\beta_j(t)} & \text{for } j = 1, \dots, n+1, \end{cases}$$

and consider the functions

(1.5)

$$u_j(t) = \begin{cases} w_0(t) & \text{for } j = 0 \\ w_0(t)\int_0^1 w_1(t_1)\int_0^{t_1} w_2(t_2)\dots\int_0^{t_{j-1}} w_j(t_j)dt_1\dots dt_j & \text{for } j = 1, \dots, n+1. \end{cases}$$

Then it follows that

$$(1.5') \qquad U_{n+1} := \operatorname{span}\{u_0, \dots, u_n\}$$

is the null space of the operator $\mathcal{L} = \mathcal{L}_{n+1}$. Now, let $\Delta = \{x_j : j \in \mathbf{Z}\}$ be a bi-infinite knot sequence. We will consider the spline space

$$\mathcal{S} = \mathcal{S}(\mathcal{L}_{n+1}; \Delta) = \{f \in C^{n-1}(\mathbf{R}^1) : (\mathcal{L}_{n+1}f)(t) = 0 \text{ for all } t \neq x_j, j \in \mathbf{Z}\}.$$

Let us introduce the function

$$\omega_n(x,t) = \frac{1}{\alpha_n(x)\beta_n(t)}$$

and fix a negative real number λ.

We will show that for any fixed x, the problem

(1.6)

$$\mathcal{L}_j y(t)|_{t=x+h-0} = \lambda \mathcal{L}_j y(t)|_{t=x+0} + \omega_n(x,x)\delta_{jn}, \quad j = 1, \dots, n, \text{ and } y \in U_{n+1}$$

has a unique solution $A_n(t,x,\lambda)$, which is considered as a function of t in the interval $(x, x+h)$. Here, δ_{jn} denotes, as usual, the Kronecker delta. We will also show that $A_n(\cdot, x, \lambda)$ can be extended to all of $\mathbf{R}^1$ in an elegant manner. Its extension will be called the generalized Euler spline relative to the operator $\mathcal{L}$. This topic will be discussed in the next section. A detailed study of the intrinsic properties of this function will be given in Section 3.

To discuss the integral representation of the error of approximation by cardinal interpolation from $\mathcal{S}$, we need the following notation. Let α be any fixed real number. The interpolation will be taken at the nodes $\alpha + h\mathbf{Z}$, $0 \leqq \alpha < h$. Let $L_\alpha \in \mathcal{S} = \mathcal{S}(\mathcal{L}; \Delta)$ denote the fundamental spline function; that is, L_α satisfies the interpolation condition

$$L_\alpha(\alpha + jh) = \delta_{j0}, \quad j \in \mathbf{Z},$$

where $\Delta = \{x_j \colon x_j = jh, j \in \mathbf{Z}\}$. Hence, the spline function in $\mathcal{S}$ that interpolates a sufficiently well-behaved function f at the nodes $\alpha + h\mathbf{Z}$ is given by

$$\sum_{j=-\infty}^{\infty} f(\alpha + jh) L_\alpha(\cdot - jh).$$

In Section 4, we will show that the function

$$H(t,x,\alpha) = \frac{1}{2\pi i} \int\limits_{|z|=1} \left[A_n(t,x,z) - A_n(t,0,z) \frac{A_n(\alpha,x,z)}{A_n(\alpha,0,z)} \right] dz$$

provides the kernel of the integral representation of the error of interpolation from $\mathcal{S}$, namely:

$$f(t) = \sum_{j=-\infty}^{\infty} f(\alpha + jh) L_\alpha(t - jh) - \int\limits_{-\infty}^{\infty} H(t,x,\alpha)(\mathcal{L}_{n+1} f)(x)\,dx.$$

As mentioned above, this formula is important not only for estimating the error of approximation, but also in providing a useful tool for solving certain extremal problems. For this purpose, we need to have some knowledge of the sign pattern of the kernel function $H(t,x,\alpha)$. In Section 5, we will give an exact formulation of $\operatorname{sgn} H(t,x,\alpha)$. Hence, the L^1 norm of $H(t,x,\alpha)$ can be evaluated, and as an application, it gives rise to the solution of certain extremal problems. This application will be given in Section 6.

2. The generalized Euler spline

We first introduce the "conjugate" $U^*_{n+1} = \text{span}\{u^*_0, \dots, u^*_n\}$ of the Haar space U_{n+1} defined in (1.5)–(1.5′) by setting

$$(2.1)\quad u^*_j(x) = \begin{cases} w^*_0(x) \equiv \frac{1}{\beta^*_0(x)} & \text{for } j=0 \\ w^*_0(x)\int\limits_0^x w^*_1(t_1)\int\limits_0^{t_1} w^*_2(t_2)\dots\int\limits_0^{t_{j-1}} w^*_j(t_j)dt_j\dots dt_1 & \text{for } 1 \leqq j \leqq n, \end{cases}$$

where $w^*_j := w_{n+1-j}$ and $\beta^*_j := \alpha_{n-j}$. To complete the definition of full conjugation, we let $\alpha^*_j := \beta_{n-j}$, $D^*_j f := \alpha^*_{j-1} D(\beta^*_{j-1} f)$ and

$$(2.2)\qquad \mathcal{L}^* = \mathcal{L}^*_{n+1}, \quad \mathcal{L}^*_j = D^*_j \dots D^*_1.$$

It can be shown that both $\mathcal{L}_j$ defined in (1.1) and $\mathcal{L}^*_j$ in (2.2) satisfy:

$$(2.3)\qquad \begin{cases} [\mathcal{L}_j f(t)]_{t=\tau+\nu h} = \mathcal{L}_j f(\tau+\nu h) \\ [\mathcal{L}^*_j f(t)]_{t=\tau+\nu h} = \mathcal{L}^*_j f(\tau+\nu h). \end{cases}$$

The bridge between the two Haar spaces U_{n+1} and U^*_{n+1} is the function

$$(2.4)\qquad u_n(t,x) := w_0(t)w_{n+1}(x)\int\limits_x^t w_1(t_1)\dots\int\limits_x^{t_{n-1}} w(t_n)dt_n\dots dt_1;$$

namely, the following formulation is obtained.

LEMMA 1.

$$(2.5)\qquad u_n(t,x) \equiv \sum_{k=0}^{n} (-1)^{n-k} u_k(t) u^*_{n-k}(x).$$

PROOF. For each fixed value of x, since $u_n(\cdot,x)$ is in U_{n+1} there are constants $c_j(x)$, $j=0,\dots,n$, such that

$$u_n(\cdot,x) = \sum_{j=0}^{n} c_j(x) u_j(\cdot).$$

Now, by applying the operator $\alpha_j^{-1}\mathcal{L}_j$ on both sides, setting $t=0$, and using the identity

$$\int\limits_t^x w_n(t_n)\int\limits_t^{t_n} w_{n-1}(t_{n-1})\dots\int\limits_t^{t_{j+2}} w_{j+1}(t_{j+1})dt_n\dots dt_{j+1} \equiv$$

$$\equiv \int\limits_t^x w_{j+1}(t_{j+1})\dots\int\limits_{t_{n-1}}^x w_n(t_n)dt_{j+1}\dots dt_n,$$

we have

$$c_j(x) = \int_x^0 w_{j+1}(t_{j+1}) \int_x^{t_{j+1}} w_{j+2}(t_{j+2}) \dots \int_x^{t_{n-1}} w_n(t_n)dt_n \dots dt_{j+1} = \\ = (-1)^{n-j} u^*_{n-j}(x). \qquad \square$$

Next, since the space U_{n+1} has dimension $n+1$, the function $u_n(\cdot - (n+1)h, x)$ is a linear combination of $u_n(\cdot - jh, x)$ where $j = 0, \dots, n$; that is, for any fixed value of x there exist constants $b_0, \dots, b_{n+1}$, with $b_{n+1} \neq 0$, $\operatorname{sgn} b_{n+1} = (-1)^{n+1}$, such that

$$\sum_{j=0}^{n+1} b_j u_n(t - jh, x) = 0 \tag{2.6}$$

for all t. Following Schoenberg [9], we define the so-called "B-spline" function:

$$B_n(t, x) = \sum_{j=0}^{n+1} b_j u_n(t - jh, x)[t - jh - x]^0_+ \tag{2.7}$$

which, by the definition of $[\cdot]^0_+$ and (2.6), clearly has support in $[x, x + (n+1)h]$. It is also easy to verify that

$$\mathcal{L}_n B_n(t, x) = \sum_{j=0}^{\ell} b_j c_n^{-j} (\alpha_n(x)\beta_n(t))^{-1}, \tag{2.8}$$

for $t \in (x + \ell h, x + \ell h + h)$ and $\ell = 0, \dots, n$, so that

$$\sum_{j=0}^{n+1} b_j c_n^{-j} = 0. \tag{2.9}$$

The Euler spline corresponding to the differential operators $\mathcal{L}_j$, $j = 1, \dots, n+1$, can now be defined by

$$A_n(t, x, \lambda) = \sum_{\nu=-\infty}^{\infty} \lambda^\nu \frac{B_n(t - \nu h, x)}{-\lambda T(\lambda^{-1})}, \tag{2.10}$$

where

$$T(\lambda^{-1}) = \sum_{j=0}^{n+1} b_j \lambda^{-j}. \tag{2.11}$$

The roots of the polynomial T can be shown to be

$$x_k := \frac{\beta_k(h)}{\beta_h(0)} = c_k^{-1}, \quad k = 0, 1, \dots, n. \tag{2.12}$$

Indeed, from (2.7) and an application of Lemma 1, we have

$$\sum_{j=0}^{n+1} b_j u_k(t - jh) = 0$$

for all $k = 0, \dots, n$; so that by applying the operator $\mathcal{L}_k$, it follows that

$$\sum_{j=0}^{n+1} b_j x_k^j = 0,$$

and hence

$$T(\lambda^{-1}) = b_{n+1} \prod_{k=0}^{n} (\lambda^{-1} - x_k). \tag{2.13}$$

3. Intrinsic properties of the Euler spline

In this section, we will derive some of the important properties of the Euler spline $A_n(t, x, \lambda)$.

LEMMA 2. *For any fixed value of x, the function $A_n(\cdot, x, \lambda)$ satisfies the following:*

(i) $A_n(\cdot + h, x, \lambda) = \lambda A_n(\cdot, x, \lambda)$,

(ii) $A_n(\cdot, x, \lambda) \in \mathcal{S}(\mathcal{L}_{n+1}, \Delta_x)$, *where* $\Delta_x = \{x + \nu h\}_{\nu \in \mathbf{Z}}$, *and*

(iii) $A_n(\cdot, x, \lambda)$ *is the unique solution of problem* (1.6) *on* $(x, x + h)$.

PROOF. The first two properties of $A_n(t, x, \lambda)$ are simple consequences of the definition of the Euler spline itself and that of the B-spline function $B_n(t, x)$ in (2.7). To verify (iii), we first note that

$$\mathcal{L}_n A_n(x + h - 0, x, \lambda) - \lambda \mathcal{L}_n A_n(x + 0, x, \lambda) =$$

$$= \mathcal{L}_n A_n(x + h - 0, x, \lambda) - \mathcal{L}_n A_n(x + h + 0, x, \lambda) =$$

$$= (\alpha_n(x)\beta_n(x))^{-1} \Big[\sum_{\nu=0}^{n} \lambda^{-\nu} \Big(\sum_{j=0}^{\nu} b_j c_n^{-j+\nu+1} \Big) -$$

$$- \sum_{\nu=-1}^{n-1} \lambda^{-\nu} \Big(\sum_{j=0}^{\nu+1} b_j c_n^{-j+\nu+1} \Big) \Big] \Big/ (-\lambda T(\lambda^{-1})) =$$

$$= (\alpha_n(x)\beta_n(x))^{-1}\Big[-b_{n+1}\lambda^{-n} - b_0\lambda - \sum_{\nu=0}^{n-1} b_{\nu+1}\lambda^{-\nu}\Big] \Big/ (-\lambda T(\lambda^{-1})) =$$

$$= (\alpha_n(x)\beta_n(x))^{-1};$$

and since $A_n(\cdot, x, \lambda)$ is in $\mathcal{S}(\mathcal{L}_{n+1}, \Delta_x)$, we have

$$\mathcal{L}_k A_n(x+h-0, x, \lambda) - \lambda \mathcal{L}_k A_n(x+0, x, \lambda) =$$
$$= \mathcal{L}_k A_n(x+h-0, x, \lambda) - \mathcal{L}_k A_n(x+h-0, x, \lambda) = 0$$

for $k = 1, \dots, n-1$. Hence, $A(\cdot, x, \lambda)$ solves (1.6). The uniqueness of this solution can be shown by performing the operations $\mathcal{L}_n, \dots, \mathcal{L}_1$ consecutively to the difference of two solutions, showing that this difference, written as a linear combination of $u_0(t), \dots, u_n(t)$, must be identically zero. □

Now let us fix t instead. Then as a function of x, $A_n(t, x, \lambda)$ also has analogous properties as follows.

LEMMA 2′. *Let $\lambda < 0$ and t be fixed. Then $A_n(t, \cdot, \lambda)$ satisfies the following:*

(i) $A_n(t, \cdot + h, \lambda) = \lambda^{-1} A_n(t, \cdot, \lambda)$,

(ii) $A_n(t, \cdot, \lambda) \in \mathcal{S}(\mathcal{L}^*_{n+1}, \Delta_x)$, *where* $\Delta_t = \{t + \nu h\}_{\nu \in \mathbf{Z}}$, *and*

(iii) $A_n(t, \cdot, \lambda)$ *is the unique solution of the problem*

$$(3.1) \qquad \begin{cases} \mathcal{L}^*_j y(x)|_{x=t-h+0} = \lambda \mathcal{L}^*_j y(x)|_{x=t-0} + (-1)^n \omega_0(t,t)\delta_{jn} \\ \text{for } j = 1, \dots, n \text{ and } y \in U^*_{n+1} \end{cases}$$

where $\omega_0(x,t) := (\alpha_0(x)\beta_0(t))^{-1}$ *and* $y \in U^*_{n+1}$.

PROOF. Again (i) and (ii) follow directly from the definition of $A_n(t, x, \lambda)$, and the proof of (iii) is similar to that of Lemma 2. □

From its definition (2.7), we have

$$B_n(t, x) = b_0 u_n(t, x) > 0$$

for $t \in (x, x+h)$. By using the generalized Budan–Fourier theorem (cf. [2] and [7]), we may even conclude that

$$(3.2) \qquad B_n(t, x) > 0, \quad t \in (x, x + nh + h),$$

and that for $t \in (x, x+h)$, $\{B_n(t+jh, x)\}$, $j = 0, \dots, n$, is a Pólya frequency sequence [6], and for fixed values of t and x, $A(t, x, \cdot)$ has μ_0 zeros $\lambda = \lambda_j(t, x)$ where

$$\mu_0 = \begin{cases} n & \text{if } t \in (x, x+h) \\ n-1 & \text{if } t = x \text{ or } t = x+h. \end{cases}$$

Let us consider the case $x = 0$. Suppose that ξ_n is a real zero of $A_n(\cdot, 0, -1)$ and $0 \leqq \alpha < h$. Then for $t = \alpha \neq \xi_n$, we may label the zeros of $A_n(\alpha, 0, \cdot)$ by $\lambda_j = \lambda_j(\alpha, \cdot)$ such that

$$(3.3) \qquad \lambda_1 < \lambda_2 < \dots < \lambda_{m-1} < -1 < \lambda_m < \dots < \lambda_{\mu_0} < 0.$$

In what follows, we will assume, without loss of generality, that $\alpha \neq 0$, so that $x = 0 < t = \alpha < h = x + h$ and $\mu_0 = n$.

4. The error formula for cardinal interpolation

Consider the knot sequence

$$\Delta_\alpha := \{\alpha + jh\colon j \in \mathbf{Z}\} \cup \{t\}$$

where $0 < \alpha < h$, and let

$$\mathcal{S}(\mathcal{L}^*_{n+1}, \Delta_\alpha) = \{f \in C^{n-1}(\mathbf{R}^1)\colon (\mathcal{L}^*_{n+1} f)(x) = 0,\ x \notin \Delta_\alpha\}$$

be the spline space corresponding to the linear operator $\mathcal{L}^*_{n+1}$. Then for any constant $\eta_0 \geqq 0$, it can be shown that there exists a unique function $H_t(\cdot)$ in $\mathcal{S}(\mathcal{L}^*_{n+1}, \Delta_\alpha)$ that satisfies the conditions

$$\text{(A)} \qquad \begin{cases} H_t(\nu h) = 0, & \nu \in \mathbf{Z} \\ \mathcal{L}^*_n H_t(t+0) - \mathcal{L}^*_n H_t(t-0) = (-1)^n \omega_0(t,t) \\ H_t(x) = O(e^{-\eta_0 |x|}) & \text{as} \quad |x| \to \infty. \end{cases}$$

Indeed, if y is the difference of two solutions of (A), then it is a null spline, with knot sequence $\{\alpha + \nu\mathbf{Z}\}$, having zeros at $h\mathbf{Z}$, so that

$$\text{(4.1)} \qquad y(x) = \sum_{j=1}^{n} k_j S_j(x), \quad S_j(\cdot) := A_n(\alpha, \cdot, \lambda_j)$$

where $\lambda_j = \lambda_j(\alpha, 0)$; but the asymptotic condition in (A) then forces y to be identically zero.

REMARK. The asymptotic condition in (A) can be replaced by $O(|x|^\beta)$, $\beta > 0$, since any nontrivial $y(x)$ in (4.1) must necessarily be of exponential growth in $|x|$, namely:

$$|S_j(x + \nu h)| = |A_n(\alpha, x + \nu h, \lambda_j)| = |\lambda_j^{-\nu h} A_n(\alpha, x, \lambda_j)|$$

tends to infinity exponentially as $\nu \to +\infty$ for $|\lambda_j| < 1$ or $\nu \to -\infty$ for $|\lambda_j| > 1$.

For $\eta_0 = 0$, the solution $y = H_t(\cdot)$ has the following expression:

$$\text{(B}_1\text{)} \qquad y(x) := \begin{cases} \sum\limits_{j=m}^{n} k_j S_j(x) & \text{for} \quad x \leqq \alpha - h \\ \sum\limits_{j=0}^{n} e_j u_j^*(x) & \text{for} \quad \alpha - h \leqq x \leqq t \\ \sum\limits_{j=0}^{n} d_j u_j^*(x) & \text{for} \quad t \leqq x \leqq \alpha \\ \sum\limits_{j=1}^{m-1} k_j S_j(x) & \text{for} \quad x \geqq \alpha \end{cases}$$

and satisfies:

$$(\mathrm{B}_2)\qquad \begin{cases} y \in C^{n-1}(\mathbf{R}^1), \quad y(0) = 0, \quad \text{and} \\ \mathcal{L}_j^* y(t+0) - \mathcal{L}_j^*(t-0) = (-1)^n \omega_0(t,t)\delta_{jn}, \quad j = 1, \dots, n. \end{cases}$$

Of course, for any fixed t, and $\alpha - h < 0 < t < \alpha$, (B_1) and (B_2) determine $y = H_t(\cdot)$ uniquely. In other words, (B_1) and (B_2) characterize the unique solution of (A) for $\eta_0 = 0$.

Now, we may give the integral representation of $y = H_t(\cdot)$ as follows:

$$H_t(x) = \frac{1}{2\pi} \int_\Gamma \Big[A_n(t,x,z) - A_n(t,0,z) \frac{A_n(\alpha,x,z)}{A_n(\alpha,0,z)} \Big] dz, \tag{4.2}$$

where $\Gamma = \{z \colon |z| = 1\}$ is the unit circle oriented in the counterclockwise direction. Indeed, it is clear that the integral (4.2) satisfies all three conditions in (A) with $\eta_0 = 0$. Using this formulation, we are now able to determine all the coefficients in (B_1), as follows:

(i) Let $t = \tau + \nu h$ and $x = \ell h + y$, $0 \leqq \tau < y < h$. Then

$$H_t(x) = \begin{cases} -\sum\limits_{k=m}^{n} \lambda_k^{\nu-\ell} A_n(\tau,0,\lambda_n) \dfrac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial \lambda_k}} & \text{for } \nu - \ell \geqq 1 \\ \sum\limits_{k=1}^{m-1} \lambda_k^{\nu-\ell} A_n(\tau,0,\lambda_k) \dfrac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial \lambda_k}} & \text{for } \nu - \ell \leqq 1 \\ -\sum\limits_{k=m}^{n} A_n(\tau,0,\lambda_k) \dfrac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial \lambda_k}} + \eta & \text{for } \nu - \ell = 0 \end{cases} \tag{4.3}$$

where η is some bounded function of y and τ.

(ii) Let $t = \tau + \nu h$, $x = \ell h + y$ and $0 \leqq y < \tau < h$. Then $H_t(\cdot)$ has the same expression as the first two in (4.3), but the function η in the third expression may be a different bounded function of y and τ.

For $0 < \xi < \xi_0$, $\xi_0 := \min\{\ln|\lambda_{m-1}|, \ln|\lambda_m|^{-1}\}$ as given in (3.3), let $\Phi(\xi) = \{f | f^{(n)}$ absolutely continuous on any bounded interval, $f^{(n+1)}(x) = O(e^{\xi|x|})$, $|x| \to \infty\}$. We will give an integral representation for the functions in $\Phi(\xi)$. The representation formula here extends the results in [3].

Let $f \in \Phi(\xi)$ and N be a sufficiently large number, such that

$$[\alpha_\ell(x)\beta_\ell(x)(\mathcal{L}_{n-\ell}^* H_t(x))\mathcal{L}_\ell f(x)]_{-Nh+\alpha}^{Nh+\alpha} = o(1) \tag{4.4}$$

as $N \to \infty$. It can be shown that for each $\nu \in \mathbf{Z}$,

$$\delta \mathcal{L}_n^* H_t(x)|_{x=\alpha+\nu h} := \mathcal{L}_n^* H_t(\alpha + \nu h + 0) - \mathcal{L}_n^* H_t(\alpha + \nu h - 0) = \tag{4.5}$$

$$= (-1)^{n+1}\omega_0(\alpha,\alpha) \frac{1}{2\pi i} \int_\Gamma \frac{A_n(t - \nu h - h, 0, z)}{A_n(\alpha, 0, z)} dz = (-1)^{n+1}\omega_0(\alpha,\alpha) L_\alpha(t - \nu h),$$

and

$$L_\nu(t-\nu h)=\begin{cases}1 & \text{for } t=\alpha+\nu h,\ \nu\in\mathbf{Z}\\ 0 & \text{for } t=\alpha+\mu h,\ \mu\neq\nu,\ \mu\in\mathbf{Z}.\end{cases}$$

Here, $L_\alpha(t-\nu h)\in\mathcal{S}(\mathcal{L}_{n+1},\Delta)$, where $\Delta=\{jh\}_{j\in\mathbf{Z}}$, is so-called the fundamental cardinal spline function in $\mathcal{S}(\mathcal{L}_{n+1},\Delta)$.

We also have

$$\delta\mathcal{L}_n^*H_t(x)|_{x=t+\nu h}=\begin{cases}(-1)^n\omega_0(t,t), & \nu=0\\ 0, & \nu\neq0,\ \nu\in\mathbf{Z}.\end{cases}\tag{4.6}$$

Note that (4.5) and (4.6) follow directly from Lemma 2′(iii). Next, set

$$I_N=\int\limits_{\alpha-Nh}^{\alpha+Nh}H_t(x)\mathcal{L}_{n+1}f(x)dx,\quad f\in\Phi(\xi).$$

Then we have

$$I_N=o(1)+(-1)^j\int\limits_{\alpha-Nh}^{\alpha+Nh}(\mathcal{L}_j^*H_t(x))\mathcal{L}_{n+1-j}f(x)dx=$$

$$=o(1)+(-1)^n\int\limits_{\alpha-Nh}^{\alpha+Nh}[\alpha_0(x)\mathcal{L}_n^*H_t(x)]\frac{d}{dx}(\beta_0(x)f(x))dx=$$

$$=o(1)+(-1)^n\sum_{\nu=1-N}^{N}(g(\alpha+\nu h-0)-g(\alpha+\nu h-h+0))+$$

$$+(-1)^n(g(t-0)-g(t+0))+(-1)^{n+1}\int\limits_{\alpha-Nh}^{\alpha+Nh}f(x)\mathcal{L}_{n+1}^*H_t(x)dx,$$

where $g(x)=\alpha_0(x)\beta_0(x)f(x)\mathcal{L}_n^*H_t(x)$. Hence, since $\mathcal{L}_{n+1}^*H_t(x)\equiv0$, we have

$$I_N=o(1)+(-1)^{n+1}\Big\{\sum_{j=1-N}^{N}[g(\alpha+jh+0)-g(\alpha+jh-0)]+g(t+0)-g(t-0)\Big\}=\tag{4.7}$$

$$=o(1)+(-1)^n\Big\{(-1)^n\omega_0(\alpha,\alpha)\sum_{\nu=1-N}^{N-1}\alpha_0(\alpha)\beta_0(\alpha)f(\alpha+\nu h)L_\alpha(t-\nu h)+(-1)^{n+1}f(t)\Big\},$$

and we obtain the following.

THEOREM 1. *Let $\alpha \in [0,h)$, $\alpha \neq \xi_n$, where ξ_n is the root of the equation $A_n(x,0,-1)=0$. Then for any $f \in \Phi(\xi)$,*

$$f(t)=\sum_{\nu=-\infty}^{\infty} f(\nu h+\alpha)L_\alpha(t-\nu h)-\int_{-\infty}^{\infty} H_t(x)\mathcal{L}_{n+1}f(x)dx. \tag{4.8}$$

5. The sign pattern of the kernel function

In the following, we will study the sign pattern of the function $H_x(t):=$ $:= H(t,x,\alpha)$, where x is fixed.

Let $x=\ell h+y$ where $y\in[0,h)$, $\ell\in\mathbf{Z}$. Then the function $H_x(\cdot)$ is a spline function; and in fact, $H_x(\cdot)\in\mathcal{S}(\mathcal{L},\Delta_1)$ where $\Delta_1=\{x\}\cup\{\nu h\}_{\nu\in\mathbf{Z}}$.

Let $t=\nu h+\tau$, $\nu\in\mathbf{Z}$, $\tau\in[0,h)$. From the residue theorem, we have

$$H_x(t)=\begin{cases} -\sum\limits_{k=m}^{n}\lambda_k^{\nu-\ell}A_n(\tau,0,\lambda_k)\dfrac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial\lambda_k}}, & \nu-\ell\geqq 1,\\ \sum\limits_{k=1}^{m-1}\lambda_k^{\nu-\ell}A_n(\tau,0,\lambda_k)\dfrac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial\lambda_k}}, & \nu-\ell\leqq -1.\end{cases} \tag{5.1}$$

REMARK. In this section, we only consider $H_x(t)=H(t,x,\alpha)$, where x is fixed and t is a variable. If t is fixed and x is the variable, then $\tilde H_t(x):=$ $:= H(t,x,\alpha)$ has similar properties.

In the following, we set $S_j(t):=A_n(t,0,\lambda_j)$, $\lambda_j=\lambda_j(\alpha)$. Then we have

$$H_x(t)=\begin{cases}\sum\limits_{j=1}^{m-1}c_jS_j(t), & \nu\leqq\ell-1\\ \sum\limits_{j=m}^{n}c_jS_j(t), & \nu\geqq\ell+1\end{cases}$$

where $t=\nu h+\tau$, $\tau\in[0,h)$ for some $c_1,\dots,c_m$.

Let μ_0,μ_1 be two integers such that $\mu_0\leqq\ell-1$ and $\mu_1\geqq\ell+1$. Then from Gantmacher's theorem (cf. [5]), we have

$$\begin{aligned}S^+((-1)^i\mathcal{L}_iH_x(\mu_0h+0))_0^n &\geqq n-1-S^-(\mathcal{L}_iH_x(\mu_0h+0))_0^{n-1}\geqq\\ &\geqq n-1-S^+(\mathcal{L}_iH_x(\mu_0h+0))_0^{n-1}\geqq n-1-(m-2)=n+1-m,\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}S^+(\mathcal{L}_iH_x(\mu_1h-0))_0^n &\geqq S^+(\mathcal{L}_iH_x(\mu_1h-0))_0^{n-1}\geqq\\ &\geqq S^-(\mathcal{L}_iH_x(\mu_1h-0))_0^n\geqq m-1,\end{aligned} \tag{5.3}$$

where the notations of S^+, S^- may be found in [6]. Since $H_x(jh+\alpha)=0$, $j\in\mathbf{Z}$, there are $\mu_1+|\mu_0|$ zeros in the interval $I=[\mu_0 h,\mu_1 h]$. The number of knots in the function $H_x(t)$ on I is also $\mu_1+|\mu_0|$ (which agrees with the cardinality of the set $N:=\{jh\}_{j=\mu_0+1}^{\mu_1-1}\cup\{x\}$). Now, let

$$(5.4)\qquad W(H_x,\omega)=S^+(-\mathcal{L}_n H_x(\omega-0),\mathcal{L}_{n-1}H_x(\omega),\mathcal{L}_n H_x(\omega+0)).$$

Then the number of zeros of H_x on (μ_0,μ_1) is given by

$$(5.5)\qquad \begin{aligned} Z(H_x,(\mu_0 h,\mu_1 h))=n-S^+((-1)^i\mathcal{L}_i H_x(\mu_0 h+0))_0^n-S^+(\mathcal{L}_i H_x(\mu_1 h-0))_0^n+\\ +\sum_{\omega\in N}(W(H_x,\omega)-1))-\text{ a non-negative even integer}\end{aligned}$$

(see [2]). It is easily seen that $W(H_x,\omega)-1\leqq 1$, and thus, from (5.2), (5.3), (5.4), we have $Z(H_x(\mu_0,\mu_1))=\mu_1+|\mu_0|$, so that the equalities in (5.2) and (5.3) must hold. Therefore $S^+(\mathcal{L}_i H_x(\mu_0 h+0))_0^{n-1}=S^-(\mathcal{L}_i H_x(\mu_0 h+0))_0^{n-1}=$ $=m-2$, and

$$(5.6)\qquad \operatorname{sgn} H_x(\mu_0 h)=(-1)^{m-2}\operatorname{sgn}\mathcal{L}_{n-1}H_x(\mu_0 h).$$

Again, from (5.2), (5.6), and the fact $W(H_x,\mu_0 h)=2$, we also have

$$(5.7)\quad \operatorname{sgn} H_x(\mu_0 h)=(-1)^m\operatorname{sgn}\mathcal{L}_{n-1}H_x(\mu_0 h)=(-1)^{m+1}\operatorname{sgn}\mathcal{L}_n H_x(\mu_0 h+0);$$

and from (5.1), we have

$$(5.8)\qquad \mathcal{L}_n H_x(t)=\begin{cases}-\sum\limits_{j=m}^{n}\frac{\lambda_j^{\nu-\ell}}{1-\lambda_j}\frac{A_n(\alpha,y,\lambda_j)}{\frac{\partial A_n(\alpha,0,\lambda_j)}{\partial\lambda_j}}, & \nu-\ell\geqq 1\\ \sum\limits_{k=1}^{m-1}\frac{\lambda_k^{\nu-\ell}}{1-\lambda_k}\frac{A_n(\alpha,y,\lambda_k)}{\frac{\partial A_n(\alpha,0,\lambda_k)}{\partial\lambda_k}}, & \nu-\ell\leqq -1\end{cases}$$

where $t\in[\nu h,\nu h+h)$, $x\in[\ell h,\ell h+h)$. It can be shown as in [3] that

$$(5.9)\qquad \operatorname{sgn}\left(\frac{A_n(\alpha,y,\lambda_j)}{\frac{\partial A_n(\alpha,0,\lambda_j)}{\partial\lambda_j}}\right)=1$$

since $\lambda_1<\lambda_2<\ldots<\lambda_{m-1}<-1<\lambda_m<\ldots<\lambda_n<0$. Then from (5.8), we have, for sufficiently large $|\nu-\ell|$,

$$(5.10)\qquad \operatorname{sgn}\mathcal{L}_n H_x(t)=(-1)^{\nu-\ell},\quad \nu-\ell\leqq -1.$$

However, $W(H,\omega) = 2$, where ω is any point in Δ_1. Hence, considering the first formula in (5.8), we conclude that (5.10) holds for any $\nu - \ell \in \mathbf{Z}$. Combining this with (5.7), we have

$$\operatorname{sgn} H_x(\mu_0 h) = (-1)^{m+1+\mu_0-\ell}, \quad x \in [\ell h, \ell h + h).$$

From the above discussion we have shown that $H_x(\cdot)$ has only simple zeros on $\mathbf{R}^1$, which are $\{\nu h + \alpha\}_{\nu \in \mathbf{Z}}$. Since $\mu_0 h \in (\mu_0 h - h + \alpha, \mu_0 h + \alpha)$, it follows that

$$(5.11)\quad \operatorname{sgn} H_x(t) = (-1)^{m+\nu-\ell} \text{ for } x \in [\ell h, \ell h + h) \text{ and } t \in [\nu h + \alpha, \nu h + h + \alpha).$$

That is, we have proved the following.

THEOREM 2. *The sign pattern of the kernel function* $H_x(t) := H(t, x, \alpha)$ *is given by* (5.11).

We shall apply Theorem 2 to solve an extremal problem in the following section.

6. An extremal problem

Let $\mathcal{L}_{n+2} = D\mathcal{L}_{n+1}$, where $\mathcal{L}_{n+1}$ is the differential operator defined in (1.1). Then analogous to Lemma 2, we establish the following.

LEMMA 3. *There exists a function* $E_{n+1}(t,x)$ *that satisfies the following conditions:*

(i) $\mathcal{L}_{n+2}(D)E_{n+1}(\cdot, x) = 0$ *on* $(x + \ell h, x + h + \ell h)$, $\ell \in \mathbf{Z}$,

(ii) $\mathcal{L}_{n+1}(D)E_{n+1}(\cdot, x) = (-1)^\ell$ *on* $(x + \ell h, x + h + \ell h)$, $\ell \in \mathbf{Z}$,

(iii) $E_{n+1}(t + h, x) = -E_{n+1}(t, x)$ *for all* t,

(iv) $E_{n+1}(\cdot, x) \in C^n(R^1)$, *and*

(v) $E_{n+1}(\cdot, x)$ *has exactly one simple zero* $\xi_{n+1}^\nu(x)$ *in the interval* $[x + \nu h, x + h + \nu h)$ *for each* $\nu \in \mathbf{Z}$.

Next, consider the function class

$$(6.1)\quad F(\xi) := \{f \mid f \in \Phi(\xi), f(\nu h + \alpha) = 0,\ \nu \in \mathbf{Z}, |\mathcal{L}_{n+1}(D)f(t)| \leqq 1,\ t \in \mathbf{R}^1\}$$

where $\Phi(\xi)$ was defined in §4. Set $\xi_n(x) = \xi_n^0(x)$, and recall that $\xi_n(0)$ and $\xi_{n+1}(0)$ are roots of the equations $A_n(t, 0, -1) = 0$ and $E_{n+1}(t, 0) = 0$, respectively. In the following discussion, we will assume that $\xi_n(0) \neq \xi_{n+1}(0)$.

REMARK. For the constant coefficient differential operator $\mathcal{L}_j = \prod_{k=0}^{j-1}(D - t_k)$, $j = 1, \ldots, n+2$, it is known that $\xi_n(0) \neq \xi_{n+1}(0)$ (see [3]).

Set $\alpha = \xi_{n+1}(0)$. Since $E_{n+1}(\xi_{n+1}(0)+\nu h, 0) = 0$ for all $\nu \in \mathbf{Z}$, it follows from Lemma 3(ii) and (5.11) that

$$E_{n+1}(t,0) = -\int_{-\infty}^{\infty} H(t,x,\alpha)\mathcal{L}_{n+1}E_{n+1}(x,0)dx = \tag{6.2}$$

$$= (-1)^{m+\nu+1}\int_{-\infty}^{\infty} |H(t,x,\alpha)|dx, \quad t \in I_{\nu,\alpha}.$$

The following result extends Theorem 3 in [3].

THEOREM 3. *Let ξ be a positive number such that $0 < \xi < \xi_0$ Then*

$$\sup_{f\in F(\xi)} |f(t)| = \int_{-\infty}^{\infty} |H(t,x,\alpha)|dx. \tag{6.3}$$

Moreover, if there is a function g in $F(\xi)$ and a point $t_1 \in I_{\nu,\alpha}$ such that

$$|g(t_1)| = \int_{-\infty}^{\infty} |H(t_1,x,\alpha)|dx, \tag{6.4}$$

then

$$g(t) = \varepsilon E_{n+1}(t+\alpha, 0) \tag{6.5}$$

where $\alpha = \xi_{n+1}(0)$, and $\varepsilon = +1$ or -1.

PROOF. From (4.8), it is clear that

$$|f(t)| \leqq \int_{-\infty}^{\infty} |H(t,x,\alpha)|dx, \quad f \in F(\xi).$$

Hence, since $E_{n+1}(t,0) \in F(\xi)$, we have (6.3), and $E_{n+1}(t,0)$ is an extremal function.

To verify the second assertion, let g be a function in $F(\xi)$ that satisfies (6.4). Then

$$g(t_1) = -\int_{-\infty}^{\infty} H(t_1,x,\alpha)\mathcal{L}_{n+1}g(x)dx. \tag{6.6}$$

Suppose that $\varepsilon_1 = +1$ or -1 is so chosen that $\varepsilon_1 g(t_1) < 0$. Then multiplying ε_1 to both sides of (6.6) and adding to (6.4), we have

$$0 = \int_{-\infty}^{\infty} \{|H(t_1, x, \alpha)| - \varepsilon_1 H(t_1, x, \alpha)\mathcal{L}_{n+1} g(x)\} dx \tag{6.7}$$

so that, since $g \in F(\xi)$, $|\mathcal{L}_{n+1} g(x)| = 1$ a.e. From (6.7), we also have $\varepsilon_1 \operatorname{sgn} H(t_1, x, \alpha) = \operatorname{sgn} \mathcal{L}_{n+1} g(x)$. Thus $\operatorname{sgn} \mathcal{L}_{n+1} g(x) = \varepsilon_1 (-1)^{m+\nu+\ell}$ for $t_1 \in I_{\nu,\alpha}$ and $x \in [\ell h, \ell h + h)$. Now let $\varepsilon = \varepsilon_1 (-1)^{m+\nu}$. Then $\mathcal{L}_{n+1}[\varepsilon E_{n+1}(x, 0) - g(x)] = 0$ for $x \in \mathbf{R}^1$, since the function $J(x) = \varepsilon E_{n+1}(x, 0) - g(x)$ is in $F(\xi)$. Then from (4.8) we may conclude that $J(x) = 0$ and Theorem 3 immediately follows.

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ON SOME TOPOLOGICAL VECTOR SPACES RELATED TO THE GENERAL OPEN MAPPING THEOREM

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1. Introduction

The spaces V_r, W_r, V and W were introduced by Valdivia [4], showing they were maximal classes of locally convex spaces for the isomorphism and homomorphism theorem, respectively, when in the range a class of locally convex spaces containing barrelled spaces was desired.

Mas [2] gave new characterizations of these classes of spaces without using the dual spaces, showing that a locally convex space $E(\mathcal{T})$ is a V_r-space if, and only if, there is no locally convex and barrelled topology on E, strictly coarser than $\mathcal{T}$; and $E(\mathcal{T})$ is a V-space, if, and only if, every separated quotient of $E(\mathcal{T})$ is a V_r-space. This allowed him to find easy proofs of Valdivia's isomorphism and homomorphism theorem and generalizations to classes of topological spaces. In this way we shall say that a topological vector space $E(\mathcal{T})$ is an $\mathcal{L} - V_r$-space if there is no Hausdorff $\mathcal{L}$-barrelled linear topology on E, strictly coarser than $\mathcal{T}$. Moreover, a topological vector space $E(\mathcal{T})$ is an $\mathcal{L}-V$-space if every separated quotient of $E(\mathcal{T})$ is an $\mathcal{L}-V_r$-space.

In this paper we introduce the $\mathcal{L}$-strongly almost open mappings and characterize $\mathcal{L} - V_r$- and $\mathcal{L} - V$-spaces in an analogous way as Valdivia [5] characterizes the V_r- and V-spaces by means of the strongly almost open mappings. We also obtain some results concerning $\mathcal{L}$- B_r-complete and $\mathcal{L}-B$-complete spaces, analogous to those obtained by Valdivia [6] in the locally convex case though with different proofs since we are not able to use the dual spaces. Besides, we prove that if a topological vector space E contains an $\mathcal{L} - V_r$-subspace ($\mathcal{L} - V$-subspace) of finite codimension, then E is an $\mathcal{L} - V_r$-space ($\mathcal{L} - V$-space) and obtain two open mapping theorems.

Topological vector space will stand for Hausdorff topological vector space and our notation follows [1]. However, let us recall that if $\mathcal{U}$ and $\mathcal{V}$ are the sets of all the neighbourhoods of the origin of the topological vector spaces E and F, respectively, and f is a linear mapping of E in F, then f is called weakly singular if $\overline{N(f)} = \cap\{\overline{f^{-1}(V)}: V \in \mathcal{V}\}$, where $N(f)$ denotes the kernel of f. f is called almost continuous if for each $V \in \mathcal{V}$, $\overline{f^{-1}(V)} \in \in \mathcal{U}$. Being f onto, f is called almost open if for each $U \in \mathcal{U}$, $\overline{f(U)} \in \mathcal{V}$. A topological vector space F is called $\mathcal{L}-B_r$-complete if each almost continuous linear mapping with closed graph of each topological vector space E in F is continuous. A topological vector space E is called $\mathcal{L}-B$-complete if each weakly singular almost open linear mapping of E onto each topological vector space F is open and this holds if, and only if, every separated quotient of E is $\mathcal{L}-B_r$-complete. If $E(\mathcal{T}_0)$ is a topological vector space and $\mathcal{T}$ is a further linear topology on E, $\overline{\mathcal{T}_0}^{\mathcal{T}}$ will denote the linear topology on E, whose neighbourhoods of the origin are the $\mathcal{T}$-closures, $\overline{U}^{\mathcal{T}}$, of the $\mathcal{T}_0$-neighbourhoods, U, of the origin.

2. $\mathcal{L}$-strongly almost continuous linear mappings

Definition 1. Let E and F be two topological vector spaces and f a linear mapping of E in F. We shall say that f is $\mathcal{L}$-strongly almost open if for each closed string $\mathcal{U}=(U_n)_{n=1}^{\infty}$ in E, $\overline{f(\mathcal{U})}^{f(E)}=\left(\overline{f(U_n)}^{f(E)}\right)_{n=1}^{\infty}$ is a topological string in $f(E)$.

Theorem 1. *A topological vector space $E(\mathcal{T})$ is an $\mathcal{L}-V_r$-space if, and only if, each continuous one-to-one $\mathcal{L}$-strongly almost open linear mapping of E onto each topological vector space is an isomorphism.*

Proof. Let f be a continuous one-to-one $\mathcal{L}$-strongly almost open linear mapping of E onto the topological vector space F and suppose f is not an isomorphism. Let $\mathcal{V}=(V_n)_{n=1}^{\infty}$ be a closed string in F. As $f^{-1}(\mathcal{V})=$ $=(f^{-1}(V_n))_{n=1}^{\infty}$ is a closed string in E, $\overline{f(f^{-1}(V))}^{F}=\mathcal{V}$ is a topological string in F. Hence F is $\mathcal{L}$-barrelled and E is not an $\mathcal{L}-V_r$-space.

Conversely, if E is not an $\mathcal{L}-V_r$-space then there exists a Hausdorff linear topology $\mathcal{T}^*$ on E, strictly coarser than $\mathcal{T}$, such that $E(\mathcal{T}^*)$ is $\mathcal{L}$-barrelled. Now the identity $i\colon E(\mathcal{T})\to E(\mathcal{T}^*)$ is a bijective continuous linear mapping but it is not an isomorphism. However, i is $\mathcal{L}$-strongly almost open since if $\mathcal{U}=(U_n)_{n=1}^{\infty}$ is a closed string in $E(\mathcal{T})$, then $\overline{i(\mathcal{U})}^{\mathcal{T}^*}=\overline{\mathcal{U}}^{\mathcal{T}^*}$ is a closed string and, consequently, topological in $E(\mathcal{T}^*)$. □

Theorem 2. *A topological vector space $E(\mathcal{T})$ is an $\mathcal{L}-V$-space if, and only if, each continuous $\mathcal{L}$-strongly almost open linear mapping of E in each topological vector space is a homomorphism.*

Proof. Let f be a continuous $\mathcal{L}$-strongly almost open linear mapping of E in the topological vector space F. Without loss of generality we may assume $f(E)=F$. The linear mapping $\hat{f}$ of the quotient space $E/f^{-1}(0)$ onto F defined by passing to the quotient is one-to-one, continuous and $\mathcal{L}$-strongly almost open. Hence $\hat{f}$ is an isomorphism and f is a homomorphism.

Conversely, if E is not an $\mathcal{L}-V$-space then there exists a Hausdorff linear topology $\mathcal{T}^*$ on E/F, strictly coarser than the quotient topology such that $E/F(\mathcal{T}^*)$ is $\mathcal{L}$-barrelled. Hence the identity $i\colon E/F \to E/F(\mathcal{T}^*)$ is not a homomorphism. However, the linear mapping $g = i \circ k$, where k denotes the canonical mapping of E onto E/F is continuous and $\mathcal{L}$-strongly almost open. So, g is a homomorphism and, consequently, i is a homomorphism, too. Contradiction. □

COROLLARY 1. *If $E(\mathcal{T})$ is an $\mathcal{L}$-barrelled $\mathcal{L} - V_r$-space then $E(\mathcal{T})$ is $\mathcal{L} - B_r$-complete.*

PROOF. Let $\mathcal{T}^*$ be a Hausdorff linear topology on E strictly coarser than $\mathcal{T}$, such that $\overline{\mathcal{T}}^{\mathcal{T}^*} \subset \mathcal{T}^*$. We just have to show $\mathcal{T} = \mathcal{T}^*$ [1, S10(3)]. The identity $i\colon E(\mathcal{T}) \to E(\mathcal{T}^*)$ fulfils the condition of Theorem 1. So, i is an isomorphism and $\mathcal{T} = \mathcal{T}^*$. □

COROLLARY 2. *If $E(\mathcal{T})$ is an $\mathcal{L}$-barrelled $\mathcal{L} - V$-space then $E(\mathcal{T})$ is $\mathcal{L} - B$-complete.*

COROLLARY 3. *If $E(\mathcal{T})$ is an $\mathcal{L}$-barrelled topological vector space which is not $\mathcal{L} - B_r$-complete, then there exists an $\mathcal{L}$-barrelled space F and a continuous one-to-one linear mapping of E onto F which is not an isomorphism.*

PROOF. By Corollary 1 there is a Hausdorff linear topology $\mathcal{T}^*$, on E, strictly coarser than $\mathcal{T}$ in such a way that $E(\mathcal{T}^*)$ is $\mathcal{L}$-barrelled. Now the identity i of $E(\mathcal{T})$ onto $E(\mathcal{T}^*)$ is one-to-one and continuous but it is not an isomorphism. □

COROLLARY 4. *If $E(\mathcal{T})$ is an $\mathcal{L}$-barrelled topological vector space which is not $\mathcal{L} - B$-complete, then there exists an $\mathcal{L}$-barrelled space F and a continuous linear mapping of E onto F which is not a homomorphism.*

PROOF. If E is not $\mathcal{L} - B$-complete then there is a separated quotient, E/G, of E which is not $\mathcal{L} - B_r$-complete. Then, by Corollary 3, there exists an $\mathcal{L}$-barrelled space F and a continuous one-to-one linear mapping v of E/G onto F, which is not an isomorphism. Let k be the canonical mapping of E onto E/G, then the linear mapping $u = v \circ k$ of E onto F is continuous and is not a homomorphism. □

3. Some results concerning subspaces of finite codimension

Valdivia [7] shows that if E is a locally convex space and contains a subspace of countable codimension which is a V_r-space then E is a V_r-space. We shall show this to be true for subspaces of finite codimension in the topological vector case, without convexity conditions using the following result (cf. [3]):

PROPOSITION 1. *Let $E(\mathcal{T})$ be a topological vector space and $\mathcal{T}^*$ a Hausdorff linear topology on E, coarser than $\mathcal{T}$ and F a subspace of E. If the respective induced topologies coincide on F, as well as the respective quotient topologies on E/F, then $\mathcal{T}$ and $\mathcal{T}^*$ coincide on E.*

PROPOSITION 2. *Let $E(\mathcal{T})$ be a topological vector space. If F is a subspace of finite codimension, which is an $\mathcal{L}-V_r$-space, then E is an $\mathcal{L}-V_r$-space.*

PROOF. We shall assume F is a hyperplane. Two cases are likely to happen:

i) F is closed in $E(\mathcal{T})$. Then if $x \in E \setminus F$, $E(\mathcal{T}) = F \oplus_t [x]$ and if $\mathcal{T}^*$ is a Hausdorff linear topology on E, strictly coarser than $\mathcal{T}$, $F(\mathcal{T}^*)$ is not $\mathcal{L}$-barrelled since $\mathcal{T}^* \subset \mathcal{T}$ but $\mathcal{T}^* \neq \mathcal{T}$. Hence $E(\mathcal{T}^*)$ is not $\mathcal{L}$-barrelled.

ii) F is dense in $E(\mathcal{T})$. Let $\mathcal{T}^*$ be a Hausdorff linear topology on E, strictly coarser than $\mathcal{T}$. If $\mathcal{T}^*$ and $\mathcal{T}$ do not coincide on F, neither $F(\mathcal{T}^*)$ nor $E(\mathcal{T}^*)$ would be $\mathcal{L}$-barrelled. In the other case, i.e. if $\mathcal{T}^*$ and $\mathcal{T}$ coincide on F, by Proposition 1, $\mathcal{T}^*$ and $\mathcal{T}$ also coincide on E. Contradiction.

Therefore, in any case, there is no Hausdorff linear topology $\mathcal{T}^*$ on E, strictly coarser than $\mathcal{T}$, such that $E(\mathcal{T}^*)$ is $\mathcal{L}$-barrelled, and $E(\mathcal{T})$ is an $\mathcal{L}-V_r$-space. □

PROPOSITION 3. *Let $E(\mathcal{T})$ be a topological vector space. If F is a subspace of finite codimension, which is an $\mathcal{L}-V$-space, then E is an $\mathcal{L}-V$-space.*

PROOF. Let η be the linear topology on E which has as a fundamental set of strings to the set of all the strings in E whose intersection with F is a topological string. Then $\mathcal{T} \subset \eta$ and F is a closed subspace of $E(\eta)$. Let us see now that $E(\eta)$ is an $\mathcal{L}-V$-space. Let $E(\eta)/G$ be a separated quotient of $E(\eta)$. As $F/G \cap F$ is a separated quotient of the $\mathcal{L}-V_r$-space F and of finite codimension in E/G, $E(\eta)/G$ is an $\mathcal{L}-V_r$-space by Proposition 2, so $E(\eta)$ is an $\mathcal{L}-V$-space. Hence $E(\mathcal{T})$ is an $\mathcal{L}-V$-space since $\mathcal{T}$ is coarser than η. □

4. Two open mapping theorems

Valdivia [8] proves that the B_r-complete and B-complete spaces have analogous properties to those shown in Proposition 2 and uses it to obtain two open mapping theorems. We do not know whether $\mathcal{L}-B_r$-complete and $\mathcal{L}-B$-complete spaces share that property. However we have been able to obtain the analogous version of Valdivia's two open mapping theorems in the context of topological vector spaces, without convexity conditions.

THEOREM 3. *Let $E(\mathcal{T})$ be a topological vector space, covered by a sequence of linear subspaces $\{E_n : n \in \mathbf{N}\}$, such that for each $n \in \mathbf{N}$ there* *exists a topology $\mathcal{T}_n$ on E_n finer than the induced topology by E, in such a way that $E_n(\mathcal{T}_n)$ is an $\mathcal{L} - V$-space. If f is a continuous linear mapping of E in the Baire space F, such that $F(E)$ is of countable codimension in F, then f is open of E in $f(E)$ and $f(E)$ is $\mathcal{L} - B$-complete and of finite codimension in F.*

PROOF. Let $\{x_n : n \in \mathbf{N}\}$ be a cobasis of $f(E)$ in F and $F_n := [f(E_n) \cup \cup \{x_1, x_2, \dots, x_n\}]$ for each $n \in \mathbf{N}$. As $F = \cup\{F_n : n \in \mathbf{N}\}$ there exists a $p \in \mathbf{N}$ such that F_p is of second category in F and, consequently, $\mathcal{L}$-barrelled. $f(E_p)$ is also $\mathcal{L}$-barrelled since it is a subspace of F_p of finite codimension. The restriction of f to E_p, f_p, is continuous of $E_p(\mathcal{T}_p)$ onto $f(E_p)$ and, therefore, open ([2]). Hence $E_p(\mathcal{T}_p)/f_p^{-1}(0)$ is topologically isomorphic to $f(E_p)$, which will be $\mathcal{L}$-barrelled and an $\mathcal{L} - V$-space and, by Corollary 2, $\mathcal{L} - B$-complete. So $f(E_p)$ is complete and, therefore, closed in F and, since $f(E_p)$ is of finite codimension in F_p, F_p is closed in F, which implies that F_p coincides with F. Now $f(E)$ is an $\mathcal{L} - V$-space since $f(E_p) \subset f(E) \subset F$ and $f(E_p)$ is an $\mathcal{L} - V$-space of finite codimension in $f(E)$. Moreover, $f(E)$ is of finite codimension in the Baire space F, so $f(E)$ is $\mathcal{L}$-barrelled and, by Corollary 2, $f(E)$ is $\mathcal{L} - B$-complete.

Finally, if U is a neighbourhood of the origin in E, $f(U) \cap f(E_p)$ is a neighbourhood of the origin in $f(E_p)$ since $f(U) \cap f(E_p) \supset f(U \cap E_p) = f_p(U \cap E_p)$, f_p is an open mapping and $U \cap E_p$ is a neighbourhood of the origin in $E_p(\mathcal{T}_p)$. □

In the same way, using properties of the $\mathcal{L} - V_r$-spaces instead of the $\mathcal{L} - V$-spaces, we obtain:

THEOREM 4. *Let $E(\mathcal{T})$ be a topological vector space, covered by a sequence of linear subspaces $\{E_n : n \in \mathbf{N}\}$, such that, for each $n \in \mathbf{N}$, there exists a topology $\mathcal{T}_n$ on E_n finer than the induced topology by E, in such a way that $E_n(\mathcal{T}_n)$ is an $\mathcal{L} - V_r$-space. If f is a continuous one-to-one linear mapping of E in the Baire space F, such that $f(E)$ is of countable codimension in F, then f is open on E in $f(E)$ and $f(E)$ is $\mathcal{L} - B_r$-complete and of finite codimension in F.*

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SOME APPLICATIONS TO ZERO DENSITY THEOREMS FOR L-FUNCTIONS

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1. Introduction

For several problems with primes like Goldbach's or the twin problem one has to study the sum

$$S(x,\alpha)=\sum_{n\leqq x}\Lambda(n)e(n\alpha) \tag{1.1}$$

($x\geqq 2$, $\alpha\in\mathbf{R}$, $e(\beta)=e^{2\pi i\beta}$, $\Lambda=$ von-Mangoldt's function). The asymptotic behaviour is well known in the neighbourhood of rational numbers $\alpha=a/q$ with 'small' denominator q:

$$S\Big(x,\frac{a}{q}+\beta\Big)=\frac{\mu(q)}{\varphi(q)}\sum_{n\leqq x}e(n\beta)+O(x\exp(-c_1(\ln x)^{1/2})) \tag{1.2}$$

($q\leqq(\ln x)^A$, $|\beta|\leqq\exp(c_2(\ln x)^{1/2})x^{-1}$; $c_1,c_2>0$, sufficiently small. See Prachar [9], Ch. 6). For $q\geqq(\ln x)^A$ one has upper bounds for $|S|$ by the methods of Vinogradov (Prachar [9], Ch. 6), Vaughan (Davenport [3], § 25), or Cudakov–Montgomery (Montgomery [7], Ch. 16).

It is the first aim of this paper to prove a mean value result for $S(x,\alpha)$ similar to the Bombieri–Vinogradov prime number theorem.

THEOREM 1. *Let $x\geqq 2$, $A>0$, $1\leqq Q\leqq x^{1/4}$, $\vartheta=\min(Q^{-4},(\ln x)^{-8(A+21)})$. Then*

$$\sum_{q\leqq Q}\max_{(a,q)=1}\max_{y\leqq x}\max_{|\beta|\leqq\vartheta}\Big|S\Big(y,\frac{a}{q}+\beta\Big)-\frac{\mu(q)}{\varphi(q)}\sum_{n\leqq y}e(n\beta)\Big|\ll\frac{x}{(\ln x)^A}.$$

REMARKS. 1. All constants implied by the symbols $O(\ )$ and $\ll$ depend at most on A and ε (which occur later). The constants $c_3,c_4,\dots$ will be positive and absolute.

2. For x sufficiently large we have $x^{-1}\leqq\vartheta\leqq 1$.

3. The bound $x^{1/4}$ for Q can be reduced slightly by more careful calculations. The Riemann hypothesis on the zeros of the functions $L(s,\chi)$ gives the asymptotic behaviour of $S(x,a/q)$ for $q=O(x^{1/3}(\ln x)^{-4/3-\varepsilon})$. The density

hypothesis, i.e. an upper bound $(U^2T)^{2(1-\sigma)+\varepsilon}$ in (2.5), leads to $Q \leqq x^{1/3-\varepsilon}$ in Theorem 1. It seems to be impossible to reach this by means of zero density results which are available at present.

Several authors considered the ternary Goldbach problem in a localized form (Haselgrove [5], Pan [8], Chen [2]), for example, for $N \equiv 1\ (2)$,

$$(1.3) \qquad N = p_1 + p_2 + p_3, \quad p_v = \frac{N}{3} + O(N^{2/3+\varepsilon})$$

(Chen [2]).

A similar question is, how to find thin subsets $\mathbf{P}'$ of the set $\mathbf{P}$ of primes such that every $n \geqq n_0$, $n \equiv 1\ (2)$ can be written as

$$n = p_1' + p_2' + p_3' \qquad (p_v' \in \mathbf{P}').$$

(1.3) does not lead to such sets. By probabilistic arguments Wirsing [12] proved the existence of a set $\mathbf{P}'$, where

$$P'(x) = \#\{p' \leqq x, p' \in \mathbf{P}'\} \ll (x \ln x)^{1/3}.$$

Apart from the factor $(\ln x)^{1/3}$ this bound is optimal. It is however a disadvantage of the method that it does not show how such sets $\mathbf{P}'$ really look like. It is the second aim of the present paper to describe how a set $\mathbf{P}'$ (which will of course be much bigger than Wirsing's) can be constructed explicitly. This set will consist of primes in certain residue classes to certain prime moduli. It seems to be a very hard problem to describe explicitly sets $\mathbf{P}'$ which are nearly as thin as Wirsing's.

THEOREM 2. *One can construct a subset $\mathbf{P}'$ of the set $\mathbf{P}$ of primes such that*

(i) $P'(x) = \#\{p' \leqq x, p' \in \mathbf{P}\} = O(x^{15/16})$, *and*

(ii) *every sufficiently large $n \equiv 1\ (2)$ can be written as*

$$n = p_1' + p_2' + p_3' \qquad (p_j' \in \mathbf{P}').$$

It is highly probable that the Pjatecki-Shapiro sets

$$\{p, p = [n^c]\} \quad \left(c \in \left(1, \frac{755}{662}\right)\right)$$

(Pjatecki-Shapiro [10], Heath–Brown [6]) can serve as $\mathbf{P}'$-s.*

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* *Added in proof (February 10, 1993).* This was recently shown for $1 < c < \frac{21}{20}$ by A. Balog and J. Friedlander (*Pacific J. Math.*, **156** (1992), 45–62).

2. Proof of Theorem 1

2.1. Let $q \leqq Q$, $(a,q)=1$. For a character $\chi \bmod q$, $\chi^* \bmod q^*$ (q^*/q) will denote the primitive character that induces χ. The principal character $\chi_0 \bmod 1$ will be regarded as primitive.

$$\tau(\chi)=\sum_{c=1}^{q}{}^{*}\chi(c)e\Big(\frac{c}{q}\Big)$$

is the well known Gaussian sum ($\sum^*$ means summation over a reduced residue system $\bmod q$). In particular, $\tau(\chi_0 \bmod q)=\mu(q)$. We have the inequality $\tau(\chi)\ll q^{*1/2}$ (Davenport [3], §9). For $z \geqq 1$, put

$$\psi(z,\chi)=\sum_{n\leqq z}\Lambda(n)\chi(n),\quad \psi(z)=\psi(z,\chi_0 \bmod 1)=\sum_{n\leqq z}\Lambda(n).$$

For $1\leqq z\leqq x$ one easily sees

$$S\Big(z,\frac{a}{q}\Big)=\frac{\mu(q)}{\varphi(q)}\psi(z)+ \tag{2.1}$$
$$+\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0(q)}\chi(a)\psi(z,\chi)\tau(\overline{\chi})+O(q^{1/2}(\ln x)^2).$$

2.2. Some facts about the zeros of L-functions will be used. Let $3\leqq U\leqq$ $\leqq Q$, $q\leqq U$, $\chi \bmod q$, χ primitive, $c_3>0$ (sufficiently small),

$$M(U,t)=\max(\ln U,(\ln(|t|+3))^{3/4}(\ln\ln(|t|+3))^{3/4}).$$

Then $L(\sigma+it,\chi)\neq 0$ for

$$\sigma\geqq 1-\frac{c_3}{M(U,t)} \tag{2.2}$$

with the possible exception of a real, simple zero β_1 of $L(s,\chi_1)$ with a real, primitive character $\chi_1 \bmod q_1$. If β_1 occurs, then q_1 and χ_1 are uniquely determined by U and c_3 (Prachar [9], VIII, Satz 6.2; Davenport [3], §1). For the exceptional — or Siegel — zero β_1 the inequality

$$\beta_1\leqq 1-c(\varepsilon)q_1^{-\varepsilon} \tag{2.3}$$

holds ($\varepsilon>0$, arbitrary. Prachar [9], IV, Satz 8.2).

In particular, (2.2) implies that, except the possible β_1, $L(s,\chi)$ ($\chi=\chi^* \bmod q$, $q\leqq 2U$) has no zero in the set

$$\begin{cases} s=\sigma+it, \quad |t|\leqq x,\\ \sigma\geqq 1-\delta_U=\begin{cases}1-c_3(\ln x)^{-4/5}, & \text{if } U\leqq Q_1=\exp((\ln x)^{4/5})),\\ 1-c_3(\ln x)^{-1}, & \text{if } Q_1\leqq U\leqq Q.\end{cases}\end{cases} \tag{2.4}$$

As usual, for $T \geqq 2$, $0 \leqq \sigma \leqq 1$, $N(\sigma, T, \chi)$ will denote the number of zeros $\varrho = \xi + i\eta$ (counted with multiplicity) of $L(s, \chi)$ in the rectangle $|\eta| \leqq T$, $\sigma \leqq \xi \leqq 1$. Then

$$\sum_{q \leqq U} \sum_{\chi(q)}{}^{*} N(\sigma, T, \chi) \ll \tag{2.5}$$

$$\ll \begin{cases} (U^2T)^{\frac{3(1-\sigma)}{2-\sigma}} (\ln(UT))^9, & \text{if } \frac{1}{2} \leqq \sigma \leqq \frac{4}{5}, \\ (U^2T)^{\frac{2(1-\sigma)}{\sigma}} (\ln(UT))^{14}, & \text{if } \frac{4}{5} \leqq \sigma \leqq 1 \end{cases}$$

(Montgomery [7], Theorem 12.2).

2.3. We have the well known explicit formula

$$\psi(z, \chi) = E_0(\chi) z - \sum_{\varrho, |\eta| \leqq x} \frac{z^\varrho}{\varrho} - \tag{2.6}$$

$$-(1 - a(\chi)) \ln z - b(\chi) + \sum_{m=1}^{\infty} \frac{z^{a-2m}}{2m - a} + O((\ln z)^2),$$

where

$$E_o(\chi) = \begin{cases} 1, & \text{if } \chi = \chi_0 \bmod q, \\ 0 & \text{otherwise,} \end{cases}$$

$\varrho = \xi + i\eta$ denotes non-trivial zeros of $L(s, \chi)$, $a = a(\chi) \in \{0, 1\}$,

$$b(\chi) = - \sum_{\varrho, |\eta| < 1} \varrho^{-1} + O(\ln q) \tag{2.7}$$

(Davenport [3], §19).

If χ is induced by the exceptional character $\chi_1 \bmod q_1$ ($q_1 \mid q, q_1 \leqq Q$), then, because of (2.3), the zero $1 - \beta_1$ gives a contribution $\frac{1}{1-\beta_1} \ll_\varepsilon q_1^\varepsilon$ in (2.7). Because of $\sum_{\varrho, |\eta| < 1} 1 \ll \ln q$ this, together with (2.4), implies

$$b(\chi) \ll (\ln x)^2 + q_1^\varepsilon. \tag{2.8}$$

In all other cases (2.8) remains valid without the term q_1^ε.

Put

$$E_1(\chi) = \begin{cases} 1, & \text{if } \chi_1 \text{ exists for } Q \text{ and } \chi^* = \chi_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $z \leqq x$, (2.6) and (2.8) (with $\varepsilon = \frac{1}{2}$) give

$$\psi(z, \chi) = E_0(\chi) z - \sum_{\varrho(\chi), |\eta| \leqq x} \frac{z^\varrho}{\varrho} + O\left((\ln x)^2 + E_1(\chi) q_1^{1/2}\right).$$

(2.1) leads to

$$S\Big(z,\frac{a}{q}\Big)=\frac{\mu(q)}{\varphi(q)}z-\frac{1}{\varphi(q)}\sum_{\chi(q)}\tau(\overline{\chi})\chi(a)\sum_{\varrho,|\eta|\leqq x}\frac{z^{\varrho}}{\varrho}+O\left(q^{1/2}(\ln x)^2\right).$$

Partial summation gives, for $y\leqq x$,

$$S\Big(y,\frac{a}{q}+\beta\Big)=\frac{\mu(q)}{\varphi(q)}\sum_{n\leqq y}e(n\beta)-$$

$$-\frac{1}{\varphi(q)}\sum_{\chi(q)}\tau(\overline{\chi})\chi(a)\sum_{\substack{\varrho(\chi^*)\\|\eta|\leqq x}}\int_1^y dzz^{\varrho-1}e(z\beta)+O\left((1+|\beta|x)q^{1/2}(\ln x)^2\right).$$

If $\Delta=\Delta(x,Q,\vartheta)$ denotes the expression to be estimated in Theorem 1, then

$$\Delta\ll\sum_{q\leqq Q}\frac{1}{\varphi(q)}\sum_{\chi(q)}q^{*1/2}\sum_{\varrho(\chi^*),|\eta|\leqq x}\max_{y\leqq x}\max_{|\beta|\leqq\vartheta}\left|\int_1^y dzz^{\varrho-1}e(z\beta)\right|+$$
$$+x\vartheta(\ln x)^4\cdot Q^{3/2}.$$

Let $U_v=2^{v-1}$, $1\leqq v\leqq v_0$, $v_0=\left[\frac{\ln Q}{\ln 2}\right]$. Then the last inequality, together with the choice of ϑ, implies

$$\Delta\ll(\ln x)^2\sum_{1\leqq v\leqq v_0}U_v^{-1/2}\sum_{U_v\leqq q\leqq U_{v+1}}\cdot \tag{2.9}$$

$$\cdot\sum_{\chi(q)}{}^{*}\sum_{\varrho(\chi),|\eta|\leqq x}\max_{y\leqq x}\max_{|\beta|\leqq\vartheta}\cdot\left|\int_1^y dzz^{\varrho-1}e(z\beta)\right|+x(\ln x)^{-A}.$$

2.4. For $\varepsilon=\frac{2}{5(A+3)}$ and $C'=C'(\varepsilon)$ sufficiently large we have

$$1-\frac{C(\varepsilon)}{q_1^{\varepsilon}}\leqq 1-\frac{C_3}{(\ln x)^{4/5}},\quad\text{if}\quad q_1\leqq C'(\ln x)^{\frac{4}{5\varepsilon}}.$$

Exceptional zeros β_1 with $\beta_1\geqq 1-\delta_U$ therefore only occur in the case $U\geqq C'(\ln x)^{4(5\varepsilon)^{-1}}$. Hence the contribution of these β_1 to (2.9) is

$$\ll_{\varepsilon}x(\ln x)^{3-\frac{2}{5\varepsilon}}=x(\ln x)^{-A}.$$

(In general, the constants depending on ε or A cannot be calculated effectively.)

(2.9) therefore remains true if for all $U = U_v$ the ϱ-summation is restricted to those ϱ for which

$$(2.10)\qquad |\eta| \leqq x, \quad \xi \leqq 1 - \delta_U.$$

This will be indicated by $\sum\limits_{\varrho}'$.

2.5. With Lemma 4.3 and 4.5 of Titchmarsh [11] one gets the following inequalities. For $1 \leqq M \leqq M' \leqq 2M$ we have

$$(2.10)\qquad \int\limits_M^{M'} dz z^{\varrho-1} e(z\beta) \ll \begin{cases} \frac{M^\xi}{|\xi|}, & \text{if } |\eta| \geqq 8|\beta|M, \\ \frac{M^{\xi-1}}{|\beta|}, & \text{if } |\eta| \leqq \pi|\beta|M, \\ \frac{M^\xi}{|\eta|^{1/2}+1} & \text{in any case.} \end{cases}$$

Because of $\xi = \operatorname{Re}\varrho \geqq C_3(\ln x)^{-1}$ (see (2.4)) $\frac{1}{\varrho+\kappa} \ll \frac{\ln x}{|\eta|+1}$ is true for $\kappa \geqq 0$. Hence, by expanding $e(z\beta)$, for $1 \leqq y \leqq x$ and $|\beta| \leqq y^{-1}$,

$$(2.11)\qquad \int\limits_1^y dz z^{\varrho-1} e(z\beta) \ll \frac{x^\xi}{|\eta|+1}.$$

Let (M_μ) be a sequence of $\mu_0 \ll \ln x$ numbers with

$$M_1 = 1, \quad M_\mu \leqq M_{\mu+1} \leqq 2M_\mu \leqq x, \quad 2M_{\mu_0} \geqq x.$$

Then (2.9), 2.4, (2.11), and (2.10) give

$$\Delta \ll \frac{x}{(\ln x)^A} + (\ln x)^4 \sum_{1 \leqq \nu \leqq \nu_0} U_v^{-1/2} \sum_{U_v \leqq q \leqq U_{v+1}} \cdot$$

$$\cdot \Bigg\{ \sum_{\chi(q)}{}^{*} \sum_{\varrho(\chi), |\eta| \leqq x}{}' \frac{x^\xi}{|\eta|+1} + \max_{x^{-1} \leqq |\beta| \leqq \vartheta} \max_{M_\mu \leqq |\beta|^{-1}} \sum_{\chi(q)}{}^{*} \sum_{\substack{\varrho(\xi) \\ |\eta| \leqq x}}{}' \frac{M_\mu^\xi}{|\eta|+1} +$$

$$+ \max_{x^{-1} \leqq |\beta| \leqq \vartheta} \max_{|\beta|^{-1} \leqq M_\mu \leqq x} \sum_{\chi(q)}{}^{*} \cdot$$

$$\cdot \Bigg(\sum_{\substack{\varrho(\chi) \\ |\eta| \leqq \pi|\beta|M_\mu}}{}' \frac{M_\mu^{\xi-1}}{|\beta|} + \sum_{\substack{\varrho(\chi) \\ \pi|\beta|M_\mu \leqq |\eta| \leqq 8\pi|\beta|M_\mu}}{}' \frac{M_\mu^\xi}{|\eta|^{1/2}} + \sum_{\substack{\varrho(\chi) \\ 8\pi|\beta|M_\mu \leqq |\eta| \leqq x}}{}' \frac{M_\mu^\xi}{|\eta|} \Bigg) \Bigg\} \ll$$

$$\ll \frac{x}{(\ln x)^A} + (\ln x)^5 \sum_{1 \leqq \nu \leqq \nu_0} U_\nu^{-1/2} \cdot \Bigg\{ x^{1/2} \sum_{q,\chi}{}_{\nu} \sum_{\substack{\varrho(\chi) \\ \xi \leqq 1/2,\, 0 \leqq \eta \leqq x}} \frac{1}{\eta+1} +$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\beta^{-1}\max_{\beta^{-1}\leqq M_\mu\leqq x}M_\mu^{-1/2}\sum_{q,\chi}{}_\nu\sum_{\substack{\varrho(\chi)\\ \xi\leqq 1/2,\,0\leqq\eta\leqq\pi\beta M_\mu}}1+$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\max_{\beta^{-1}\leqq M_\mu\leqq x}M_\mu^{1/2}\sum_{q,\chi}{}_\nu\cdot$$

$$\cdot\Bigg(\sum_{\substack{\varrho(\chi)\\ \xi\leqq 1/2,\,\pi\beta M_\mu\leqq\eta\leqq 8\pi\beta M_\mu}}\eta^{-1/2}+\sum_{\substack{\varrho(\chi)\\ \xi\leqq 1/2,\,8\pi\beta M_\mu\leqq\eta\leqq x}}\eta^{-1}\Bigg)+$$

$$+\int_{1/2}^{1-\delta_\nu}d\sigma\int_0^x d\tau\frac{x^\sigma}{(\tau+1)^2}\sum_{q,\chi}{}_\nu N(\sigma,\tau,\chi)+$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\max_{\beta^{-1}\leqq M_\mu\leqq x}\beta^{-1}\cdot\int_{1/2}^{1-\delta_\nu}d\sigma\int_0^{\pi\beta M_\mu}d\tau\frac{M_\mu^{\sigma-1}}{\tau+1}\sum_{q,\chi}{}_\nu N(\sigma,\tau,\chi)+$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\max_{\beta^{-1}\leqq M_\mu\leqq x}\int_{1/2}^{1-\delta_\nu}d\sigma\cdot$$

$$\cdot\Bigg(\int_{\pi\beta M_\mu}^{8\pi\beta M_\mu}d\tau\frac{M_\mu^\sigma}{\tau^{3/2}}+\int_{8\pi\beta M_\mu}^{x}d\tau\frac{M_\mu^\sigma}{\tau^2}\Bigg)\cdot\sum_{q,\chi}{}_\nu N(\sigma,\tau,\chi)\Bigg\}.$$

Here $\sum_{q,\chi}{}_\nu$ means $\sum_{U_\nu\leqq q\leqq U_{\nu+1}}\ \sum_{\chi(q)}{}^*$, $\delta_\nu=\delta_{U\nu}$. The contribution of the zeros ϱ with $\xi\leqq\frac12$ is, because of $\sum_{\varrho(\chi),\,|\eta|\leqq T}1\ll T\ln(qT)$,

$$\ll(\ln x)^5\sum_{1\leqq\nu\leqq\nu_0}U_\nu^{-1/2}U_\nu^2\Big\{x^{1/2}(\ln x)^2+$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\beta^{-1}\max_{\beta^{-1}\leqq M_\mu\leqq x}M_\mu^{-1/2}\beta M_\mu\ln x+$$

$$+\max_{x^{-1}\leqq\beta\leqq\vartheta}\max_{\beta^{-1}\leqq M_\mu\leqq x}M_\mu^{1/2}\Big((\beta M_\mu)^{1/2}\ln x+(\ln x)^2\Big)\Big\}\ll$$

$$\ll(\ln x)^7Q^{3/2}(x^{1/2}+\vartheta^{1/2}x)\ll\frac{x}{(\ln x)^A}.$$

Inserting (2.5), one gets

$$\Delta\ll\frac{x}{(\ln x)^A}+(\ln x)^{20}\sum_{1\leqq\nu\leqq\nu_0}U_\nu^{-1/2}.$$

$$\cdot\left\{\int_{1/2}^{4/5} d\sigma U_\nu^{\frac{6(1-\sigma)}{2-\sigma}} x^\sigma + \int_{4/5}^{1-\delta_\nu} d\sigma U_\nu^{\frac{2(1-\sigma)}{\sigma}} x^\sigma + \right.$$

$$+ \max_{x^{-1}\leqq\beta\leqq\delta} \beta^{-1} \max_{\beta^{-1}\leqq M_\mu\leqq x} \left(\int_{1/2}^{4/5} d\sigma M_\mu^{\sigma-1} U_\nu^{\frac{6(1-\sigma)}{2-\sigma}} (\beta M_\mu)^{\frac{3(1-\sigma)}{2-\sigma}} + \right.$$

$$\left. + \int_{4/5}^{1-\delta_\nu} d\sigma M_\mu^{\sigma-1} U_\nu^{\frac{4(1-\sigma)}{\sigma}} (\beta M_\mu)^{\frac{2(1-\sigma)}{\sigma}} \right) + \max_{x^{-1}\leqq\beta\leqq\delta} \max_{\beta^{-1}\leqq M_\mu\leqq x} \cdot$$

$$\left.\cdot\left(\int_{1/2}^{4/5} d\sigma M_\mu^\sigma U^{\frac{6(1-\sigma)}{2-\sigma}} (\beta M_\mu)^{\frac{3(1-\sigma)}{2-\sigma}-\frac{1}{2}} + \int_{4/5}^{1-\delta_\nu} d\sigma M_\mu^\sigma U_\nu^{\frac{4(1-\sigma)}{\sigma}} (\beta M_\mu)^{\frac{2(1-\sigma)}{\sigma}-\frac{1}{2}} \right)\right\}.$$

The exponent of M_μ is $\geqq 0$ in all integrals, so $\max M_\mu$ can be taken as x. In the third and fourth integral the exponent of β is negative. So these integrals are of the same order of magnitude as the first and the second. In the fifth integral the exponent of β is positive, in the last it is negative. Therefore

$$\Delta \ll \frac{x}{(\ln x)^A} + x(\ln x)^{20} \sum_{1\leqq\nu\leqq\nu_0} U_\nu^{-1/2}. \tag{2.12}$$

$$\cdot\left\{ \int_{1/2}^{4/5} d\sigma \left(U_\nu^{\frac{6}{2-\sigma}} x^{-1} \right)^{1-\sigma} + \int_{4/5}^{1-\delta_\nu} (U_\nu^{4/\sigma} x^{-1})^{1-\sigma} + \right.$$

$$\left. +(x\vartheta)^{-1/2} \int_{1/2}^{4/5} d\sigma \left(U_\nu^{\frac{6}{2-\sigma}} x^{\frac{3}{2-\sigma}-1} \vartheta^{\frac{3}{2-\sigma}} \right)^{1-\sigma} \right\}.$$

Because of $\frac{3}{2-\sigma}(1-\sigma) - \frac{1}{2} \geqq 0$ for $\sigma \in [1/2, 4/5]$ the first integral is of lower order than the third. The exponent of U_ν is $\geqq 1/6$ in the third integral. In the second integral the exponent of U_ν is

$$\geqq 0 \text{ in } [4/5, 8/9] \quad \text{and} \quad \leqq 0 \text{ in } [8/9, 1-\delta_\nu].$$

(2.12) and (2.4) therefore give

$$\Delta \ll \frac{x}{(\ln x)^A} + x(\ln x)^{21}. \tag{2.13}$$

$$\cdot\Big\{Q^{-1/2}(x\vartheta)^{-1/2}\int\limits_{1/2}^{4/5} d\sigma\left(Q^{\frac{6}{2-\sigma}}(x\vartheta)^{\frac{3}{2-\sigma}}x^{-1}\right)^{1-\sigma}+Q^{-1/2}\int\limits_{4/5}^{8/9} d\sigma(Q^{4/\sigma}x^{-1})^{1-\sigma}+$$

$$+\int\limits_{8/9}^{1-c_3(\ln x)^{-4/5}} d\sigma x^{\sigma-1}+Q_1^{-1/2}\int\limits_{8/9}^{1-c_3(\ln x)^{-1}} d\sigma(Q_1^{4\sigma}x^{-1})^{1-\sigma}\Big\}.$$

The third integral is

$$\ll x^{-c_3(\ln x)^{-4/5}}\ll(\ln x)^{-21-A}.$$

In the last term in (2.13) the exponent of Q_1 is $\leqq 0$ in $[8/9, 9/10]$, and $\leqq -1/18$ in $[9/10, 1-c_3(\ln x)^{-1}]$. So this term can be estimated by

$$\ll\int\limits_{8/9}^{9/10} d\sigma x^{\sigma-1}+\int\limits_{9/10}^{1-c_3(\ln x)^{-1}} d\sigma Q_1^{-1/18}\ll(\ln x)^{-21-A}.$$

It is therefore sufficient to consider the first integral in (2.13).

2.6 Let $Q=x^{\alpha}$, $0\leqq\alpha\leqq 1/4$, and, in the first case

$$Q^{-4}\leqq(\ln x)^{-8(A+21)},\quad\text{i.e. }\vartheta=Q^{-4}. \tag{2.14}$$

Here we have

$$\Delta\ll\frac{x}{(\ln x)^A}+x(\ln x)^{21}\cdot x^{\frac{3}{2}\alpha-\frac{1}{2}}\int\limits_{1/2}^{4/5} d\sigma\left(x^{\frac{3-6\alpha}{2-\sigma}-1}\right)^{1-\sigma}. \tag{2.15}$$

The function

$$f(\sigma)=\frac{3}{2}\alpha-\frac{1}{2}+\Big(\frac{3-6\sigma}{2-\sigma}-1\Big)(1-\sigma)$$

is maximal at

$$\sigma_0=\sigma_0(\alpha)=2-(3(1-2\alpha))^{1/2}.$$

For $0\leqq\alpha\leqq 1/8$ one sees $\sigma_0\leqq 1/2$. Because of $f(1/2)=-\alpha/2$ in this sub-case (2.14) and (2.15) lead to

$$\Delta\ll x(\ln x)^{-A}+x(\ln x)^{21}\cdot x^{-\alpha/2}\ll x(\ln x)^{-A}.$$

In the case $1/8\leqq\alpha\leqq 1/4$ we have $f(\sigma)\leqq f(\sigma_0)\leqq -1/16$, which also implies $\Delta\ll x(\ln x^{-A}$.

Assume now $Q \leqq (\ln x)^{2(A+21)}$, $\vartheta = (\ln x)^{-8(A+21)}$. Here (2.15) yields

$$\Delta \ll x(\ln x)^{-A} + x(\ln x)^{21}(Qx\vartheta)^{-1/2} \cdot \int_{1/2}^{4/5} d\sigma \left(Q^{\frac{6}{2-\sigma}} \vartheta^{\frac{3}{2-\sigma}} x^{\frac{3}{2-\sigma}-1} \right)^{1-\sigma}.$$

The exponents of Q, x, and ϑ are decreasing in $1/2 \leqq \sigma \leqq 4/5$. Hence

$$\Delta \ll \frac{x}{(\ln x)^A} + x(\ln x)^{21} Q^{3/2} \vartheta^{-\frac{1}{2}} \int_{1/2}^{4/5} d\sigma \vartheta^{\frac{3(1-\sigma)}{2-\sigma}} \ll$$

$$\ll x(\ln x)^{-A} + x(\ln x)^{21} Q^{3/2} \vartheta^{1/2} \ll x(\ln x)^{-A}.$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

3.1. Let

(3.1) $$M \geqq M_0 \quad \text{and} \quad R = M^{1/11}.$$

With a prime $p = p_M \in (R/2, R]$ (to be chosen later) and numbers

$$b_1, \dots, b_\nu \quad (\nu \ll R^{1/3},\ 0 < b_1 < \dots < b_\nu < p)$$

put

$$P'_M = \bigcup_{\mu=1}^{\nu} \{r \text{ prime},\ r \leqq M, r \equiv b_\mu(p)\}.$$

At the end it will be shown that

$$N = p'_1 + p'_2 + p'_3 \quad (p'_1, \dots, p'_3 \in P'_M) \quad \text{for} \quad M/2 < N \leqq M,\ N \equiv 1\,(2).$$

3.2. The next lemma is important for the choice of P_M.

LEMMA 1. *For* $p \in \mathbf{P}$, $(a,q) = 1$, $b \not\equiv 0(p)$, *put*

$$f(p,q,a,b) = \begin{cases} \frac{\mu(q)}{\varphi(pq)}, & \text{if } p \nmid q \\ \frac{\mu(q_1)}{\varphi(q)} e\left(\frac{abq_1^*}{p}\right), & \text{if } q = pq_1,\ p \nmid q_1,\ q_1 q_1^* \equiv 1(p), \\ 0, & \text{if } p^2/q. \end{cases}$$

Then, for $\varepsilon = \frac{1}{200}$,

$$\sum_{R/2<p\leqq R}\Big(\sum_{\substack{q\leqq R^{4/3+\varepsilon}\\ p\nmid q}} + \sum_{\substack{q\leqq R^{2+\varepsilon}\\ p\mid q}}\Big)\max_{(b,p)=1}\max_{(a,q)=1}\max_{|\beta|\leqq R^{2+\varepsilon}q^{-1}M^{-1}}\cdot$$

$$\cdot\Big|\sum_{\substack{n\leqq M\\ n\equiv b(p)}}\Lambda(n)e\Big(\Big(\frac{a}{q}+\beta\Big)n\Big)-f(p,q,a,b)\sum_{n\leqq M}e(n\beta)\Big| \ll M(\ln M)^{-3}.$$

PROOF. 1. Put, for $(a,q)=1$ and $p\nmid b$,

$$T=T\Big(p,b,\frac{a}{q},\beta\Big)=\sum_{\substack{n\leqq M\\ n\equiv b(p)}}\Lambda(n)e\Big(\Big(\frac{a}{q}+\beta\Big)n\Big),$$

and, for a character χ, $U(\chi,\beta)=\sum_{\substack{\varrho(\chi)=\xi+i\eta\\ |\eta|\leqq M}}\Big|\int_1^M dz z^{\varrho-1}e(z\beta)\Big|$.

2. In the case $p\nmid q$ one gets

$$T=\sum_{c=1}^{q}{}^{*}e\Big(\frac{ac}{q}\Big)\frac{1}{\varphi(pq)}\sum_{\chi(p)}\sum_{\chi'(q)}\overline{\chi}(b)\overline{\chi}'(c)\cdot$$

$$\cdot\sum_{n\leqq M}\chi\chi'(n)\Lambda(n)e(n\beta)+O((\ln M)^2)=f(p,q,a,b)\sum_{n\leqq M}\Lambda(n)e(n\beta)+$$

$$+O\Big(\frac{1}{p\varphi(q)}\sum_{\chi(q)\neq\chi_0}|\tau(\overline{\chi})|\Big|\sum_{n\leqq M}\chi(n)\Lambda(n)e(n\beta)\Big|\Big)+O(p^{-1}q^{1/2}(\ln M)^2)+$$

$$+O\Big(\frac{1}{p\varphi(q)}\sum_{\substack{\chi(pq)\\ \chi=\chi_1\chi_2,\chi_1(p),\chi_2(q)\\ \chi_1\neq\chi_0}}|\tau(\overline{\chi}_2)|\Big|\sum_{n\leqq M}\chi(n)\Lambda(n)e(n\beta)\Big|\Big)+O((\ln M)^2).$$

Because of $\tau(\chi_1)\tau(\chi_2)=\overline{\chi}_1(q)\overline{\chi}_2(p)\tau(\chi_1\chi_2)$ and $\chi=\chi^*(p)$ one gets $|\tau(\overline{\chi}_2)|=$ $=p^{-1/2}|\tau(\overline{\chi})|$ in the last sum. Treating $\sum_{n\leqq M}\chi(n)\Lambda(n)e(n\beta)$ in the same manner as in Section 2.3, one arrives at

$$T=f(p,q,a,b)\sum_{n\leqq M}e(n\beta)+O\Big(\frac{1}{p\varphi(q)}\sum_{\chi(q)}|\tau(\overline{\chi})|U(\chi,\beta)\Big)+ \tag{3.2}$$

$$+O\Big(p^{-3/2}\frac{1}{\varphi(q)}\sum_{\chi(pq)}|\tau(\overline{\chi})|U(\chi,\beta)\Big)+O(q^{1/2}M^{\varepsilon}(1+|\beta|M))$$

in the first case.

3. If $q = pq_1$, $p \nmid q_1$, then

$$T = \sum_{c=1}^{q_1}{}^{*} \sum_{d=1}^{p}{}^{*} e\Big(\frac{ac}{q_1}\Big) \sum_{\substack{n \leqq M \\ n \equiv bq_1 q_1^* + cp(q)}} \Lambda(n) e(n\beta) + O((\ln M)^2).$$

Introducing characters, one sees, as in the first case,

$$T = f(p,q,a,b) \sum_{n \leqq M} \Lambda(n) e(n\beta) + \tag{3.3}$$

$$+ O\Big(\frac{1}{\varphi(q)} \sum_{\chi(q_1), \chi \neq \chi_0} |\tau(\overline{\chi})| \Big| \sum_{n \leqq M} \Lambda(n)\chi(n) e(n\beta)\Big|\Big) +$$

$$+ O\Big(p^{-3/2} \frac{1}{\varphi(q_1)} \sum_{\chi(q), \chi \neq \chi_0} |\tau(\overline{\chi})| \Big| \sum_{n \leqq M} \cdots \Big|\Big) + O((\ln M)^2) =$$

$$= f(p,q,a,b) \sum_{n \leqq M} e(n\beta) + O\Big(\frac{1}{\varphi(q)} \sum_{\chi(q_1)} |\tau(\overline{\chi})| U(\chi,\beta)\Big) +$$

$$+ O\Big(p^{-1/2} \frac{1}{\varphi(q)} \sum_{\chi(q)} |\tau(\overline{\chi})| U(\chi,\beta)\Big) + O(q^{1/2} M^{\varepsilon}(1 + |\beta| M)).$$

4. The last case one has to treat is $q = p^2 q_2$, $p \nmid q_2$. If $q_2 q_2^* \equiv 1(p)$, $q_2 \tilde{q}_2 \equiv 1(p^2)$, then

$$T = \sum_{c=1}^{q_2}{}^{*} \sum_{y=1}^{p^2}{}^{*} e\Big(\frac{a(cp^2 + yq_2)}{q}\Big) \sum_{t=1}^{p} \sum_{\substack{n \leqq M \\ n \equiv b + tp(p^2), yq_2(p^2), cp^2(q_2)}} \Lambda(n) e(\beta n) + O((\ln M)^2) =$$

$$= \sum_{c=1}^{q_2}{}^{*} \sum_{t=1}^{p} e\Big(\frac{ab\tilde{q}_2}{p_2} + \frac{at\tilde{q}_2}{p} + \frac{ac}{q_2}\Big) \sum_{\substack{n \leqq M \\ n \equiv b + tp(p^2), cp^2(q_2)}} \Lambda(n) e(\beta n) + O((\ln M)^2).$$

One now introduces characters $\bmod p^2$ and q_2. If a character $\chi \bmod p^2$ can be defined $\bmod p$, then $\chi(b + tp) = \chi(b)$, and the sum over t vanishes. Hence

$$T = \frac{1}{\varphi(p^2)\varphi(q_2)} \sum_{t=1}^{p} e\Big(\frac{ab\tilde{q}_2}{p^2} + \frac{at\tilde{q}_2}{p}\Big) \cdot$$

$$\cdot \sum_{\chi(p^2)}{}^{*} \overline{\chi}(b + tp) \sum_{\chi_2(q_2)} \chi_2(a) \tau(\overline{\chi}_2) \sum_{n \leqq M} \chi\chi_2(n) \Lambda(n) e(\beta n) + O((\ln M)^2).$$

Therefore, in this case,

$$T \ll \frac{1}{\varphi(q)} \sum_{\chi(q)} |\tau(\overline{\chi})| U(\chi,\beta) + q^{1/2} M^{\varepsilon}(1+|\beta|M). \tag{3.4}$$

5. If Δ denotes the expression to be estimated in the lemma, then by (3.2), ..., (3.4),

$$\Delta \ll \sum_{\frac{R}{2}<p\leqq R} \Bigg\{ \sum_{\substack{q\leqq R^{4/3+\varepsilon}\\ p\nmid q}} \max_{|\beta|\leqq \frac{R^{2+\varepsilon}}{qM}} \cdot$$

$$\cdot\Big(\frac{1}{p\varphi(q)} \sum_{\chi(q)} |\tau(\overline{\chi})| U(\chi,b) + \frac{1}{p^{3/2}\varphi(q)} \sum_{\chi(pq)} |\tau(\overline{\chi})| U(\chi,\beta) + q^{-1/2} M^{\varepsilon} R^{2+\varepsilon}\Big) +$$

$$+ \sum_{\substack{q_1\leqq R^{1+\varepsilon}\\ p\nmid q_1}} \max_{|\beta|\leqq \frac{R^{2+\varepsilon}}{pq_1 M}} \Big(\frac{1}{p\varphi(q_1)} \sum_{\chi(q_1)} |\tau(\overline{\chi})| U(\chi_1\beta) +$$

$$+ \frac{1}{p^{3/2}\varphi(q_1)} \sum_{\chi(pq_1)} |\tau(\overline{\chi})| U(\chi,\beta) + q_1^{-1/2} p^{-1} M^{\varepsilon} R^{2+\varepsilon}\Big) +$$

$$+ \sum_{\substack{q_2\leqq R^{\varepsilon}\\ p\nmid q_2}} \max_{|\beta|\leqq \frac{R^{2+\varepsilon}}{p^2 q_2 M}} \Big(\frac{1}{p^2\varphi(q_2)} \sum_{\chi(p^2 q_2)} |\tau(\overline{\chi})| U(\chi,\beta) + p^{-1} q_2^{-1/2} R^{2+\varepsilon} M^{\varepsilon}\Big)\Bigg\} \ll$$

$$\ll \Big(\sum_{q\leqq R^{2+\varepsilon}} \max_{|\beta|\leqq \frac{R^{2+\varepsilon}}{qM}} + R^{-1/2} \sum_{q\leqq R^{\frac{7}{3}+\varepsilon}} \max_{|\beta|\leqq \frac{R^{3+\varepsilon}}{qM}} \Big) \cdot$$

$$\cdot \frac{1}{\varphi(q)} \sum_{\chi(q)} q^{*1/2} U(\chi,\beta) + M^{\varepsilon} R^{\frac{11}{3}+\varepsilon} \ll \ln M \sum_{U_\nu \leqq R^{2+\varepsilon}} U_\nu^{-1/2} \sum_{U_\nu\leqq q\leqq U_{\nu+1}} \sideset{}{^*}\sum_{\chi(q)} \cdot$$

$$\cdot \sum_{\varrho(\chi),|\eta|\leqq M} \max_{|\beta|\leqq \frac{R^{2+\varepsilon}}{U_\nu M}} \left| \int_1^M dz z^{\varrho-1} e(z\beta) \right| + \ln M \cdot R^{-1/2} \sum_{U_\nu\leqq R^{\frac{7}{3}+\varepsilon}} U_\nu^{-1/2} \sum_{U_\nu\leqq q\leqq U_{\nu+1}} \cdot$$

$$\cdot \sideset{}{^*}\sum_{\chi(q)} \sum_{\varrho(\chi),|\eta|\leqq M} \max_{|\beta|\leqq \frac{R^{3+\varepsilon}}{U_\nu M}} \left| \int_1^M dz z^{\varrho-1} e(z\beta) \right| + M^{\varepsilon} R^{\frac{11}{3}+\varepsilon}$$

(U_ν like in Section 2.3).

It is easy to see that in the last sums $U_{\nu+1} \leqq M^{1/4}$ and

$$\frac{R^{2+\varepsilon}}{U_\nu M}, \frac{R^{3+\varepsilon}}{U_\nu M} \leqq \min(U_\nu^{-4}, (\ln M)^{-8(3+21)}),$$

if ε, M, and R fulfil (3.1). Hence the last expression is of the type (2.9), and can be estimated by $\ll M(\ln M)^{-3}$. This proves Lemma 1.

3.3. By Lemma 1 we know that there exist primes $p \in (R/2, R]$, such that

$$(3.5) \qquad \Bigg(\sum_{\substack{q \leqq R^{\frac{4}{3}+\varepsilon} \\ p \nmid q}} + \sum_{\substack{q \leqq R^{2+\varepsilon} \\ p \mid q}} \Bigg) \Delta(q) \ll \frac{M}{R(\ln M)^2},$$

where

$$(3.6) \qquad \Delta(q) = \max_{(b,p)=1} \max_{(a,q)=1} \max_{|\beta| \leqq \frac{R^{2+\varepsilon}}{qM}} \cdot$$

$$\cdot \Bigg| \sum_{\substack{n \leqq M \\ n \equiv b(p)}} \Lambda(n) e\Big(\Big(\frac{a}{q} + \beta\Big) n\Big) - f(p,q,a,b) \sum_{n \leqq M} e(n\beta) \Bigg|.$$

Such a prime $p = p_M$ will be fixed for the rest of the proof.

3.4. LEMMA 2. *There is a set $B \subseteqq \mathbf{N}_0$ (depending on $p = p_M$) with the properties*

(i) $B \subseteqq \{1, \ldots, p-1\}$, $|B| \ll R^{1/3}$, *and*

(ii) *for every $a \in \{0, \ldots, p-1\}$ the congruence $a \equiv b_1 + b_2 + b_3 \bmod p$ is solvable in $b_1, b_2, b_3 \in B$.*

PROOF. Following Stöhr and Raikov (see Halberstam–Roth [4], p. 36) one can construct a basis of third order $B' \subseteqq \mathbf{N}_0$ of $\mathbf{N}_0$ with

$$B'(x) = \#\{b' \in B', b' \leqq x\} \ll x^{1/3} \qquad (x \geqq 1).$$

Put $B'_p = \{b' \in B', b' < p\}$. Hence, for $0 \leqq a < p$, there are $b'_1, b'_2, b'_3 \in B'_p$ with

$$(3.7) \qquad a = b'_1 + b'_2 + b'_3.$$

Change B'_p into B as follows.

a) Omit $b = 0$ from B'_p.

b) Add (if necessary) to B'_p the numbers $1, 2, p-1, p-2, b-1, b-2$ (for $b \in B'_p$, $b \geqq 4$).

Then B obviously fulfils (i). Going through all cases in which $b'_\nu = 0$ occurs in (3.7), one sees that each $a \in \{3, \ldots, p-1\} \cup \{p+1, p+2\}$ can be represented as $a = b_1 + b_2 + b_3$ ($b_\nu \in B$). This proves Lemma 2.

By means of Lemma 2 the set P'_M can now be defined as

$$(3.8) \qquad P'_M = \bigcup_{b \in B} \{r \in \mathbf{P}, r \equiv b(p), r \leqq M\}.$$

The Brun–Titchmarsh Theorem or even a trivial bound, and (3.1) imply

$$|P'_M| \ll M^{15/16}. \tag{3.9}$$

3.5. The next lemma gives a bound for exponential sums on "minor arcs".

LEMMA 3. *For* $\varepsilon = 1/200$, $(b,p) = 1$ $(p = p_M)$, $(a,q) = 1$, $|\beta| \leqq R^{2+\varepsilon}q^{-1}M^{-1}$, *and*

a) $R^{\frac{4}{3}+\varepsilon} < q \leqq MR^{-2-\varepsilon}$, *if* $p \nmid q$,

b) $R^{2+\varepsilon} < q \leqq MR^{-2-\varepsilon}$, *if* $p \mid q$

we have

$$\sum_{\substack{n \leqq M \\ n \equiv b(p)}} \Lambda(n) e\Big(\Big(\frac{a}{q} + \beta\Big)n\Big) \ll \frac{M}{R(\ln M)^2}.$$

PROOF. According to Balog and Perelli [1] one has, for $M' \leqq M$ and $h = (p,q)$, the inequality

$$\sum_{\substack{r \leqq M', r \text{ prime} \\ r \equiv b(p)}} e\Big(\frac{a}{q}r\Big) \ll (\ln M)^3 \Big(\frac{hM}{pq^{1/2}} + \frac{q^{1/2}M^{1/2}}{h^{1/2}} + \frac{M^{4/5}}{p^{2/5}}\Big).$$

Distinguishing the cases a) and b), and using partial summation, one easily gets the statement above. This lemma is the reason for introducing the factor R^ε. One could have used $(\ln M)^C$ instead.

3.6. For $p = p_M$, $b \in B$ put $S(b,\alpha) = \sum\limits_{\substack{n \leqq M \\ n \equiv b(p)}} \Lambda(n) e(n\alpha)$, further, with $\varepsilon = 1/200$,

$$Q = \frac{M}{R^{2+\varepsilon}}, \quad Q_1 = R^{4/3+\varepsilon}, \quad Q_2 = R^{2+\varepsilon},$$

and for $M/2 < N \leqq M$, $N \equiv 1(2)$,

$$B_3(N) = \sum_{\substack{b_1, b_2, b_3 \in B \\ b_1+b_2+b_3 \equiv N(p)}} 1 \quad (>0), \text{ and } H_3(N) = \sum_{\substack{n_1, n_2, n_3 \\ n_1+n_2+n_3 = N}} 1 \gg N^2.$$

For

a) $q \leqq Q_1$, $p \nmid q$, or

b) $q \leqq Q_2$, $p \mid q$

and $(a,q) = 1$ let $I_{a/q} = \Big[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\Big]$. Then the intervals $I_{a/q}$ are pairwise disjoint, and, because of Lemma 3, for all $b \in B$ we have

$$S(b,\alpha) \ll \frac{M}{R(\ln M)^2}, \quad \text{if} \quad \alpha \in m = [Q^{-1}, 1 - Q^{-1}] \setminus \cup I_{a/q}. \tag{3.10}$$

Then

$$D(N) = \sum_{\substack{r_1, r_2, r_3 \in P'_M \\ r_1+r_2+r_3=N}} \ln r_1 \ln r_2 \ln r_3 = \tag{3.11}$$

$$= \sum_{\substack{b_1, b_2, b_3 \in B \\ b_1+b_2+b_3 \equiv N(p)}} \int\limits_{Q^{-1}}^{1-Q^{-1}} d\alpha \prod_{\nu=1}^{3} S(b_\nu, \alpha) e(-N\alpha) + O\Big(\frac{M^{3/2}}{R} B_3(N)\Big).$$

The last error is $\ll B_3(N)\frac{M^2}{R^2 \ln M}$. Because of (3.10) the contribution of the minor arcs to (3.11) is

$$\ll \sum_{b_1+b_2+b_3 \equiv N(p)} \prod_{\nu=1}^{2} \Big(\int\limits_0^1 d\alpha |S(b_\nu, \alpha)|^2 \Big)^{1/2} \cdot \frac{M}{R(\ln M)^3} \ll \frac{B_3(N) M^2}{R^2 (\ln M)^3},$$

hence

$$D(N) = \sum_{\substack{b_1, b_2, b_3 \in B \\ b_1+b_2+b_3 \equiv N(p)}} \Big(\sum_{\substack{q \leqq Q_1 \\ p \nmid q}} + \sum_{\substack{q \leqq Q_2 \\ p \mid q}} \Big) \cdot \tag{3.12}$$

$$\cdot \sum_{a=1}^{q}{}^{*} e\Big(-\frac{a}{q} N\Big) \int\limits_{-\frac{1}{qQ}}^{\frac{1}{qQ}} d\beta \prod_{\nu=1}^{3} S\Big(b_\nu, \frac{a}{q} + \beta\Big) e(-n\beta) + O\Big(\frac{B_3(N) M^2}{R^2 \ln M}\Big).$$

For $q \leqq Q_2$, $p^2 | q$, $|\beta| \leqq \frac{1}{qQ}$ (3.6) gives $\left|S\left(b_\nu, \frac{a}{q} + \beta\right)\right| \leqq \Delta(q)$. Therefore the contribution of these q to (3.12) is, by (3.5),

$$\ll \sum_{b_1+b_2+b_3 \equiv N(p)} \sum_{\substack{q \leqq Q_2 \\ p \mid q}} \Delta(q) \cdot \prod_{\nu=1}^{2} \Big(\int\limits_0^1 d\alpha |S(b_\nu, \alpha)|^2 \Big)^{1/2} \ll B_3(N) \frac{M^2}{R^2 (\ln M)^2}.$$

Consider next $q \leqq Q_1$, $p \nmid q$. Here we have

$$S\Big(b_\nu, \frac{a}{q} + \beta\Big) = \frac{\mu(q)}{\varphi(pq)} \sum_{n \leqq M} e(n\beta) + O(\Delta(q)).$$

Hence the contribution of these q to (3.12) is

$$\ll \sum_{b_1+b_2+b_3 \equiv O(N)} \sum_{\substack{q \leqq Q_1 \\ p \nmid q_1}} \sum_{a=1}^{q}{}^{*} \Big(\frac{1}{p^3 \varphi^3(q)} \int\limits_{|\beta| \leqq (qQ)^{-1}} d\beta \Big| \sum_{n \leqq M} e(n\beta) \Big|^3 +$$

$$+\Delta(q)\frac{1}{(p\varphi(q))^2}\int\limits_{|\beta|\leqq(qQ)^{-1}} d\beta\Big|\sum_{n\leqq M} e(n\beta)\Big|^2+$$

$$+\frac{\Delta(q)}{p\varphi(q)}\Bigg(\int\limits_{|\beta|\leqq(qQ)^{-1}} d\beta\Big|\sum_{n\leqq M} e(n\beta)\Big|^2\Bigg)^{1/2}\Bigg(\int\limits_{\alpha\in I_{a|q}} d\alpha|S(b_3,\alpha)|^2\Bigg)^{1/2}+$$

$$+\Delta(q)\prod_{\nu=1}^{2}\Bigg(\int\limits_{\alpha\in I_{a|q}} d\alpha|S(b_\nu,\alpha)|^2\Bigg)^{1/2}\Bigg).$$

Again, with (3.5), this turns out to be $\ll B_3(N)\frac{M^2}{R^2(\ln M)^2}$.

In the same manner one sees that for $q\leqq Q_2$, $q=pq_1$, $p\nmid q_1$, the sum $S(b_\nu,\alpha)$ can be replaced by $\frac{\mu(q_1)}{\varphi(q)}e\left(\frac{abq_1^*}{p}\right)\sum\limits_{n\leqq M}e(n\beta)$. Therefore

(3.13)

$$D(N)=\sum_{\substack{b_1,b_2,b_3\in B\\ b_1+b_2+b_3\equiv N(p)}}\ \sum_{\substack{q=pq_1\leqq Q_2\\ p\nmid q_1}}\ \sum_{a=1}^{q}{}^{*}e\Big(-\frac{a}{q}N\Big)\frac{\mu(q_1)}{(p-1)^3\varphi^3(q_1)}e\Big(\frac{aq_1^*}{p}(b_1+b_2+b_3)\Big)\cdot$$

$$\cdot\int\limits_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\Big(\sum_{n\leqq M}e(n\beta)\Big)^3 e(-N\beta)d\beta+O\Big(\frac{B_3(N)M^2}{R^2\ln M}\Big).$$

Changing $\int\limits_{-(qQ)^{-1}}^{(qQ)^{-1}}$ into $\int\limits_{-1/2}^{1/2}$ one gets an error

$$\ll B_3(N)\sum_{\substack{q=pq_1\leqq Q_2\\ p\nmid q_1}}\frac{\varphi(q)}{R^3\varphi^3(q_1)}(qQ)^2\ll\frac{B_3(N)M^2}{R^2\ln M}.$$

This, with (3.13), implies

$$D(N)=\frac{H_3(N)}{(p-1)^3}\sum_{\substack{b_1,b_2,b_3\in B\\ b_1+b_2+b_3\equiv N(p)}}\ \sum_{\substack{q_1\leqq Q_2/p\\ p\nmid q_1}}\frac{\mu(q_1)}{\varphi^3(q_1)}\cdot$$

$$\cdot\sum_{d=1}^{p-1}\sum_{f=1}^{q_1}{}^{*}e\Big(\frac{d}{p}(b_1+b_2+b_3-N)\Big)e\Big(-\frac{fN}{q_1}\Big)+O\Big(B_3(N)\frac{M^2}{R^2\ln M}\Big)=$$

$$= \frac{B_3(N)H_3(N)}{(p-1)^2} \sum_{\substack{q_1 \leqq Q_2/p \\ p \nmid q_1}} \frac{\mu(q_1)c_{q_1}(-N)}{\varphi^3(q_1)} + O\left(B_3(N)\frac{M^2}{R^2 \ln M}\right) =$$

$$= \frac{B_3(N)H_3(N)}{p^2} \prod_{\substack{r \in \mathbf{P} \\ r \neq p}} \left(1 - \frac{c_r(-N)}{(r-1)^3}\right) + O\left(B_3(N)\frac{M^2}{R^2 \ln M}\right).$$

The last product is $\gg 1$, consequently

(3.14) $$D(N) = \#\{p_1, p_2, p_3 \in P'_M,\ p_1 + p_2 + p_3 = N\} \geqq 1$$

for $\frac{M}{2} < N \leqq M$, $N \equiv 1(2)$. If one puts $M_\nu = 2^\nu$ and

$$\mathbf{P}' = \bigcup_{\nu \geqq \nu_0} P'_{M_\nu},$$

then (3.9) and (3.14) imply the statement of Theorem 2.

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ORDER OF STRONG UNIQUENESS IN BEST L_∞-APPROXIMATION BY SPLINE SPACES

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I. Introduction

Let X be a normed linear space and let U_n denote an n-dimensional subspace of X. A function $u_0 \in U_n$ is called a *best approximation of a function $f \in X$ from U_n* if $\|f - u_0\| \leqq \|f - u\|$ for each $u \in U_n$. If f has a unique best approximation $u(f)$ from U_n, then one is interested in the behavior of $u - u(f)$ for every function $u \in U_n$ satisfying

$$\|f - u\| \leqq \|f - u(f)\| + \delta \tag{1.1}$$

as $\delta \to 0$. This is the so-called strong uniqueness problem. If for a function $f \in X$ and any $u \in U_n$ satisfying (1.1) with some $0 < \delta \leqq 1$, the relation

$$\|u - u(f)\| \leqq K(f)\delta^\gamma$$

holds where $0 < \gamma \leqq 1$ and $K(f) > 0$ is independent of u and δ, then γ is called the *order of strong uniqueness at f with respect to U_n*. Moreover, $u(f)$ is called the *strongly unique best approximation of f from U_n of order γ*. In the literature the following equivalent statement of the above property is often used: For any $u \in U_n$ such that $\|f - u\| \leqq \|f - u(f)\| + 1$,

$$\|f - u\| \geqq \|f - u(f)\| + \tilde{K}(f)\|u - u(f)\|^{1/\gamma}. \tag{1.2}$$

(The reason why we consider only such functions $u \in U_n$ satisfying $\|f-u\| \leqq \leqq \|f - u(f)\| + 1$ consists in the fact that (1.2) does not hold if $\gamma < 1$ and $\|u\| \to \infty$.)

Kroó [1] studied the problem in the case when $X = C^r[a,b]$ endowed with the supremum norm, and U_n denotes an n-dimensional Haar subspace of X.

In this paper we consider the case when $X = C[a,b]$, the space of all real-valued continuous functions on the compact interval $[a,b]$ endowed with the supremum norm $\|f\| = \max\{|f(x)|: x \in [a,b]\}$ $(f \in C[a,b])$, and $U_n = = S_m(\Delta)$, the subspace of spline functions of degree m with k fixed knots at Δ. Nürnberger [2] gave a characterization of those functions in $C[a,b]$ at which the order γ of strong uniqueness with respect to $S_m(\Delta)$ is equal to one. We are now interested in those functions $f \in C[a,b]$ which have a unique best approximation $u(f)$ from $S_m(\Delta)$ but $u(f)$ is not strongly unique of order one. Using a characterization of unique best approximations from $S_m(\Delta)$ given by Nürnberger and Singer [3] we are able to determine the order of strong uniqueness at f for all f in a wide subclass of $C[a,b]$ (Theorem 3.3). Moreover, we show that the order given in Theorem 3.3 is sharp (Proposition 3.5).

II. Characterization of uniqueness and strong uniqueness of order one

Let $\Delta = \{x_1, \dots, x_k\}$ with $a = x_0 < x_1 < \dots < x_{k+1} = b$ be a partition of $[a,b]$. Then the *space of polynomial spline functions of degree* m $(m \geqq 1)$ *with* k *fixed simple knots at* Δ is defined by

$$S_m(\Delta) = \{s\colon s \text{ is } (m-1)\text{-times continuously differentiable and } s|_{[x_i,x_{i+1}]} \text{ is a polynomial of degree at most } m,\ 0 \leqq i \leqq k\}.$$

In order to define the B-spline basis of $S_m(\Delta)$ we introduce additional knots

$$x_{-m} < x_{-m+1} < \dots < x_{-1} < a, \quad b < x_{k+2} < \dots < x_{m+k+1}.$$

It is wellknown (see e.g. Schumaker [4]) that $\dim S_m(\Delta) = m + k + 1$ and there exists a basis $\{B_0, \dots, B_{m+k}\}$ of $S_m(\Delta)$ consisting of the so-called B-splines which satisfy the following properties:

$$B_l(x) > 0 \text{ for every } x \in (x_{l-m}, x_{l+1}),$$
$$B_l(x) = 0 \text{ for every } x \notin (x_{l-m}, x_{l+1}), \quad 0 \leqq l \leqq m+k.$$

Moreover, it is wellknown that there exist functions in $C[a,b]$ which have at least two best spline approximations.

We call the points $a \leqq t_1 < \dots < t_r \leqq b$ *alternating extreme points* of a function $f \in [a,b]$ if $\sigma(-1)^l f(t_l) = \|f\|$, $2 \leqq l \leqq r$, $\sigma \in \{-1,1\}$. We count the number of alternating extreme points of f in a subinterval I of $[a,b]$ by

$$A_I(f) = \max\{r\colon \text{ there exist } r \text{ alternating extreme points of } f \text{ in } I\}.$$

The following characterization of strongly unique best spline approximations of order one was given in [2].

THEOREM 2.1. *For* $f \in C[a,b]$ *and* $s_0 \in S_m(\Delta)$ *the following conditions* (2.1) *and* (2.2) *are equivalent:*

(2.1) s_0 *is the strongly unique best approximation of* f *from* $S_m(\Delta)$ *of order one.*

(2.2) $A_{[a,b]}(f - s_0) \geqq m + k + 2$ *and* $A_{[a,x_l)}(f - s_0) \geqq l + 1$,

$A_{(x_{k+1-l},b]}(f - s_0) \geqq l + 1,\ \ 1 \leqq l \leqq k,$ *and* $A_{(x_r,x_{r+m+l})}(f - s_0) \geqq l + 1$,

if $(x_r, x_{r+l+m}) \subset [a, b]$, $l \geqq 1$.

The alternation properties in statement (2.2) were firstly introduced in [5] ensuring uniqueness of best spline approximations. In that paper it turned out that alternation properties do not suffice to characterize uniqueness. One has also to consider the behavior of the error function in a neighborhood of certain knots.

For this reason the following notation was introduced in [3].

DEFINITION 2.2. A function $f \in C[a, b]$ is called *flat of order* m *from the right* (resp. *from the left*) at $t_0 \in (a, b)$ if there exists a sequence $(\lambda_l) \subset \subset (0, b - t_0)$ (resp. $(\lambda_l) \subset (a - t_0, 0)$) converging to zero such that $\lim\limits_{l\to\infty}(|f(t_0+\lambda_l) - - f(t_0)|/|\lambda_l|^m) = 0$.

Using this definition the following characterization of uniqueness of best spline approximations was obtained in [3].

THEOREM 2.3. *For* $f \in C[a, b]$ *and* $s_0 \in S_m(\Delta)$ *the following conditions* (2.3) *and* (2.4) *are equivalent:*

(2.3) s_0 *is the unique best approximation of* f *from* $S_m(\Delta)$.

(2.4) (a) $A_{[a,b]}(f - s_0) \geqq m + k + 2$ *and* $A_{[a,x_l]}(f - s_0) \geqq l + 1$, $A_{[x_{k+1-l},b]}(f - s_0) \geqq l + 1$, $1 \leqq l \leqq k$, *and* $A_{[x_r,x_{r+m+l}]}(f - s_0) \geqq l + 1$, *if* $[x_r, x_{r+m+l}] \subset [a, b]$, $l \geqq 1$.

(b) *If* $A_{[a,x_l)}(f - s_0) = l$ *resp.* $A_{[x_r,x_{r+m+l})}(f - s_0) = 1$ *(resp.* $A_{(x_{k+1-l},b]}(f - s_0) = l$ *resp.* $A_{(x_r,x_{r+m+l}]}(f - s_0) = l$*), then* $f - s_0$ *is flat of order* m *from the left at* x_l *resp.* x_{r+m+l} *(resp. flat of order* m *from the right at* x_{k+1-l} *resp.* x_r*).*

III. Main results

At first we extend the property of flatness of order m to order $m + \alpha$ where $\alpha > 0$.

DEFINITION 3.1. A function $f \in C[a, b]$ is called *flat of order at least* $m + \alpha$ *from the right* (resp. *from the left*) *at* $t_0 \in (a, b)$ where $\alpha > 0$ if there exist a positive constant M and a sequence $(\lambda_l) \subset (0, b - t_0)$ (resp. $(\lambda_l) \subset (a - t_0, 0)$) converging to zero such that

$$|f(t_0 + \lambda_l) - f(t_0)| \leqq M|\lambda_l|^{m+\alpha} \text{ for all } l \in \mathbf{N}.$$

Using this property we will now determine the order of strong uniqueness at f for certain functions $f \in C[a, b]$. To do this assume that $f \in C[a, b]$

has a unique best spline approximation $s(f)$ which is not strongly unique of order one. Then by Theorems 2.1 and 2.3, $f - s(f)$ is flat of order m at certain knots. Let

$$Z_1(f) = \{x_i \in \Delta\colon f - s(f) \text{ is flat of order } m \text{ from the left at } x_i \text{ and } A_{[a,x_i)}(f - s(f)) = i \text{ or } A_{[x_{i-m-r},x_i)}(f - s(f)) = r, \text{ if } [x_{i-m-r}, x_i) \subset [a,b] \ (r \geqq 1)\}$$

and

$$Z_2(f) = \{x_i \in \Delta\colon f - s(f) \text{ is flat of order } m \text{ from the right at } x_i \text{ and } A_{(x_i,b]}(f - s(f)) = k + 1 - i \text{ or } A_{(x_i,x_{i+m+r}]}(f - s(f)) = r, \text{ if } (x_i, x_{i+m+r}] \subset [a,b] \ (r \geqq 1)\}.$$

We will show in Lemma 4.1 that there exist closed knot subintervals $I_l = [x_{1,l}, x_{2,l}]$, $1 \leqq l \leqq r$, of $[a,b]$ such that

$$(3.1)\quad \begin{cases} I_1 < \ldots < I_r, \text{ i.e. if } x \in I_l,\ y \in I_{l+1}, \text{ then } x < y,\ 1 \leqq l \leqq r-1, \\ s(f)|_{I_l} \text{ is the strongly unique best approximation of } f|_{I_l} \\ \text{from } S_m(\Delta)|_{I_l} \text{ of order one,} \\ I_l \text{ is maximal, i.e. if } [x_u, x_v] \cap I_l \neq \emptyset \text{ and } [x_u, x_v] \cap ([a,b] \setminus I_l) \neq \emptyset, \\ \text{then } s(f)|_{[x_u,x_v]} \text{ is not strongly unique or order one, } 1 \leqq l \leqq r, \\ \text{there do not exist } r+1 \text{ intervals with the above properties.} \end{cases}$$

Now let

$$(3.2)\quad \begin{cases} J_l = [x_{2,l}, x_{1,l+1}], & 1 \leqq l \leqq r-1, \\ J_0 = [a, x_{1,1}], & \text{if } x_{1,1} > a, \\ J_r = [x_{2,r}, b], & \text{if } x_{2,r} < b. \end{cases}$$

Without loss of generality we may assume that $x_{1,1} > a$ and $x_{2,r} < b$.

We now introduce a subclass K of $C[a,b]$ and determine the order of strong uniqueness at f for every $f \in K$. Define

$$K = \{f \in C[a,b]\colon f \text{ has a unique best approximation } s(f) \text{ from } S_m(\Delta) \text{ and for every } x \in Z_1(f) \cup Z_2(f) \text{ there exists a strictly monotone sequence } (\lambda_l) \text{ as in Definition 2.2 resp. in Definition 3.1 and a positive constant } \Lambda \text{ such that } |\lambda_l/\lambda_{l+1}| \leqq \Lambda \text{ for each } l \in \mathbf{N}\}.$$

REMARK 3.2. The class K contains at least all such functions which have a "good behavior" near the points $x \in Z_1(f) \cup Z_2(f)$: Let $f \in C[a,b]$ have a unique best spline approximation $s(f)$ and assume that for each $x \in$ $\in Z_1(f) \cup Z_2(f)$ there exist real positive numbers M, ε and α such that

$$|(f(x + \sigma\lambda) - s(f)(x + \sigma\lambda)) - (f(x) - s(f)(x))| \leqq M\lambda^{m+\alpha}$$

for all $\lambda \in (0,\varepsilon]$ where $\sigma = 1$, if $x \in Z_2(f)$, and $\sigma = -1$, if $x \in Z_1(f)$. Then $f \in K$.

More generally, if $f \in \mathrm{Lip}(m+\alpha)$ in a right-hand neighborhood of x, if $x \in Z_2(f)$, and $f \in \mathrm{Lip}(m+\alpha)$ in a left-hand neighborhood of x, if $x \in Z_1(f)$, then also $f \in K$.

Now let intervals $J_0, \dots, J_r$ be given as in (3.2). If for some $i \in \{0,\dots,r\}$, $|J_i \cap Z_1(f)| = \mu_i$, the number of elements of $J_i \cap Z_1(f)$, then there exist knots $\{z_{i1},\dots,z_{i\mu_i}\} = \Delta \cap J_i \cap Z_1(f)$ such that $f - s(f)$ is flat from the left at z_{il} of order at least $m + \alpha_{il}$ for some $\alpha_{il} \geqq 0$, $1 \leqq l \leqq \mu_i$. Set $P_i = \prod_{l=1}^{\mu_i} \alpha_{il}/(m+\alpha_{il})$. Analogously define $Q_i = \prod_{l=1}^{\nu_i} \tilde{\alpha}_{il}/(m+\tilde{\alpha}_{il})$ where $\Delta \cap J_i \cap Z_2(f) = \{\tilde{z}_{i1},\dots,\tilde{z}_{i\nu_i}\}$ and $f - s(f)$ is flat from the right at $\tilde{z}_{il}$ of order at least $m + \tilde{\alpha}_{il}$ for some $\tilde{\alpha}_{il} \geqq 0$, $1 \leqq l \leqq \nu_i$. If for some $i \in \{0,\dots,r\}$ $\mu_i = 0$ (resp. $\nu_i = 0$), we set $P_i = 1$ (resp. $Q_i = 1$). Then we define

$$\gamma = \min\{P_i, Q_i \colon 0 \leqq i \leqq r\}. \tag{3.3}$$

We are now in the position to state the main result of this paper.

THEOREM 3.3. *Let $f \in K$. Then for every $s \in S_m(\Delta)$ such that*

$$\|f - s\| \leqq \|f - s(f)\| + \delta \tag{3.4}$$

where $0 < \delta \leqq 1$, we have

$$\|s - s(f)\| \leqq K(f)\delta^\gamma \tag{3.5}$$

where γ is the integer from (3.3) *and the constant $K(f) > 0$ is independent of s and δ.*

(Hence the order of strong uniqueness at f with respect to $S_m(\Delta)$ depends on the order of flatness and the number of flatness points.)

REMARK 3.4. At the end of this section we will present a function $f \in C[a,b]$ such that $f - s(f)$ is flat of order m at a certain knot $x \in \Delta$ and it is not flat of order $m+\alpha$ at x for any $\alpha > 0$. Hence by (3.3) $\gamma = 0$ and then by Theorem 3.3, zero is the order of strong uniqueness at f with respect to $S_m(\Delta)$.

Moreover, every f which has a unique best approximation from $S_m(\Delta)$ but it is not contained in K has at least order $\gamma = 0$, because it follows from $\|f - s\| \leqq \|f - s(f)\| + \delta$ where $s \in S_m(\Delta)$ and $0 < \delta \leqq 1$ that $\|s - s(f)\| \leqq 2\|f - s(f)\| + 1 = K(f)$.

The following proposition will show that the order of strong uniqueness given in Theorem 3.3 is sharp.

PROPOSITION 3.5. *For every set of positive real numbers $\alpha_1,\dots,\alpha_P$ and every $1 \leqq P \leqq k$ there exists a function $f \in K$ and P knots $\{x_{i_1},\dots,x_{i_P}\}$ such that f is flat from the right of order $m+\alpha_j$ at x_{i_j}, $1 \leqq j \leqq P$. Moreover,* *there exists* $0 < \tilde{\delta} \leqq 1$ *such that for every* $0 < \delta \leqq \tilde{\delta}$ *there is a function* $s_\delta \in$ $\in S_m(\Delta)$ *such that* $\|s(f) - s_\delta\| \geqq M\delta^{\gamma_P}$ *and*

$$\|f - s_\delta\| \leqq \|f - s(f)\| + \delta \qquad (0 < \delta \leqq \tilde{\delta}) \tag{3.6}$$

where $\gamma_P = \prod_{j=1}^{P} \alpha_j/(m + \alpha_j)$ *and* M *is independent of* f *and* δ.

We now present a function which is flat of order m at a knot and not flat of order $m + \alpha$ at this knot for any $\alpha > 0$.

EXAMPLE 3.6. Let $a = x_0 = -1, x_1 = 0, b = x_2 = 1$ and let $f \in C[-1, 1]$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x = -1 \\ -1, & \text{if } x \in \{-1/2, 1\} \\ 1 - x^{m+|\ln x|^{-1/2}}, & \text{if } 0 \leqq x \leqq 1/2 \\ \text{linear}, & \text{elsewhere.} \end{cases}$$

Then an easy calculation shows that $f(0) = 1$ and f is flat of order m at x_1. Moreover, it follows that $f \in K$, $\|f\| = 1$ and $s(f) = 0$, the unique best approximation of f from $S_m(\Delta)$ where $\Delta = \{0\}$. Now we show that f is not flat of order $m + \alpha$ for any $\alpha > 0$. If $\alpha > 0$, then $x^{m+|\ln x|^{-1/2}} \,/\, x^{m+\alpha} =$ $= x^{-\alpha+|\ln x|^{-1/2}}$. Since $|\ln x|^{-1/2} \to 0$ as $x \searrow 0$, there exists an $\varepsilon \in (0, 1)$ such that $0 < |\ln x|^{-1/2} < \alpha/2$ for every $x \in (0, \varepsilon]$. Then $x^{|\ln x|^{-1/2}} > x^{\alpha/2}$ for every $x \in (0, \varepsilon]$. Hence $x^{-\alpha+|\ln x|^{-1/2}} > x^{-\alpha/2}$ and therefore, $x^{-\alpha+|\ln x|^{-1/2}} \to \infty$, if $x \searrow 0$.

IV. Proof of the main results

In this section we prove the statements given in Section III. Throughout this section let $f \in C[a, b]$ and assume that f has the unique best approximation $s(f)$ from $S_m(\Delta)$ which is not strongly unique of order one. Without loss of generality we may assume that $s(f) = 0$.

LEMMA 4.1. *The conditions* (3.1) *are satisfied.*

PROOF. It follows from Theorem 2.3 that $A_{[a,b]}(f) \geqq m + k + 2$. Then by a simple combinatorial argument there exists a subinterval $[x_i, x_{i+j}]$ of $[a, b]$ such that $A_{[x_i,x_{i+j}]}(f) \geqq m + j + 1$ and $A_{[x_p,x_{p+q}]}(f) \leqq m + q$ in every proper subinterval $[x_p, x_{p+q}]$ of $[x_i, x_{i+j}]$.

We will show that $I = [x_i, x_{i+j}]$ satisfies (2.2). Let $T = \{t_0, \dots, t_{m+j}\}$ where $x_i \leqq t_0 < \dots < t_{m+j} \leqq x_{i+j}$ are alternating extreme points of f. Then by the choice of T, $[x_{i+l}, x_{i+j}]$ contains at most $m + j - 1$ points of T, $1 \leqq l \leqq j - 1$. Hence, $t_l < x_{i+l}$, $1 \leqq l \leqq j - 1$. Analogously, $t_{m+j-l} > x_{i+j-l}$, $1 \leqq l \leqq j - 1$.

Now assume that for some $l \geqq i$ and $r \geqq 1$, $(x_l, x_{l+m+r}) \subset [x_i, x_{i+j}]$. Since $[x_i, x_l]$ contains at most $m+l-i$ points of T and $[x_{l+m+r}, x_{i+j}]$ at most $i+j-l-r$ points of T, the interval (x_l, x_{l+m+r}) contains at least $m+j+1-(m+l-i)-(i+j-l-r) = r+1$ points of T. Thus, we have shown that f satisfies (2.2) on I. Then by Theorem 2.1, 0 is the strongly unique best approximation of f from $S_m(\Delta)$ of order one on I. We may assume that I is maximal, i.e. if $[x_u, x_v] \supsetneqq I$, then 0 is not strongly unique of order one on $[x_u, x_v]$, i.e. (2.2) is not satisfied on $[x_u, x_v]$.

Now suppose that $\tilde{I} = [x_p, x_{p+q}]$ is also such a maximal strong uniqueness subinterval of $[a,b]$ and assume that $\tilde{I} \neq I$ and $\tilde{I} \cap I \neq \emptyset$. Without loss of generality we may assume that $x_p < x_i$. Then $x_{p+q} \in [x_i, x_{i+j})$. We show that f satisfies (2.2) even on $[x_p, x_{i+j}]$ which yields a contradiction to the maximality of I and $\tilde{I}$. Let $\tilde{T} = \{\tilde{t}_0, \ldots, \tilde{t}_{m+q}\}$ where $x_p \leqq \tilde{t}_0 < \ldots < \tilde{t}_{m+q} \leqq \leqq x_{p+q}$ are alternating extreme points of f. Since f satisfies (2.2) on $\tilde{I}$, $[x_p, x_i)$ contains at least $i-p+1$ points of $\tilde{T}$. Then $A_{[x_p, x_{i+j}]}(f) \geqq m+i+j-p+1$. Moreover, since $A_{[x_p, x_{p+l})}(f) \geqq l+1$, $1 \leqq l \leqq i-p$, and $A_{[x_i, x_{i+l})}(f) \geqq l+1$, $1 \leqq l \leqq j$, it follows that $A_{[x_p, x_{p+l})}(f) \geqq l+1$, $1 \leqq l \leqq i+j-p$. Analogously, $A_{(x_{i+j-l}, x_{i+j}]}(f) \geqq l+1$, $1 \leqq l \leqq i+j-p$.

Now let $(x_r, x_{r+m+l}) \subset [x_p, x_{i+j}]$ $(l \geqq 1,\ r \geqq 1)$. If $x_r \geqq x_i$ or $x_{r+m+l} \leqq \leqq x_{p+q}$, then by the properties of I and $\tilde{I}$, $A_{(x_r, x_{r+m+l})}(f) \geqq l+1$. Hence assume that $x_r < x_i$ and $x_{r+m+l} > x_{p+q}$. If $x_{r+m+u} = x_i$ for some $u \in \in \{1, \ldots, l-1\}$, then $A_{(x_r, x_{r+m+u})}(f) \geqq u+1$. Then, since $A_{[x_i, x_{r+m+l})}(f) \geqq \geqq r+m+l-i+1$, it follows that $A_{(x_r, x_{r+m+l})}(f) \geqq u+r+m+l-i+1 = l+1$. If $x_{r+m+1} > x_i$, then $A_{(x_r, x_{r+m+l})}(f) \geqq A_{[x_i, x_{r+m+l})}(f) \geqq r+m+l-i+1 \geqq l+1$. Thus, we have shown that f satisfies (2.2) on $[x_p, x_{i+j}]$, a contradiction. Hence for every two maximal strong uniqueness intervals I_1, I_2 with $I_1 \neq \neq I_2$ we have $I_1 < I_2$ or $I_2 < I_1$. Now considering the intervals $[x_0, x_i]$ and $[x_{i+j}, x_{k+1}]$ we can conclude as above and find all other maximal knot intervals on which 0 is the strongly unique best approximation of order one. □

In the following statement we study the relationship between strong uniqueness intervals as in (3.1) and the points of flatness of f.

LEMMA 4.2. *Let $x_p \in Z_1(f)$ (resp. $x_p \in Z_2(f)$). Then the following conditions hold:*

(4.1) *There exists a subinterval $I = [x_i, x_{i+j}]$ of $[x_p, x_{k+1}]$ (resp. of $[x_0, x_p]$) such that f satisfies* (2.2) *on I.*

(4.2) *If x_i is minimal under condition* (4.1) *(resp. x_{i+j} maximal under conditions* (4.1)*), i.e. there does not exist any other interval $\tilde{I} = [x_\mu, x_{\mu+\nu}]$ satisfying* (4.1) *such that $x_p \leqq x_\mu < x_i$ (resp. $x_{i+j} < x_{\mu+\nu} \leqq x_p$), then $x_1 \notin Z_2(f)$, $p \leqq l \leqq i-1$ (resp. $x_l \notin Z_1(f)$, $i+j+1 \leqq l \leqq p$).*

PROOF. We only treat the case when $x_p \in Z_1(f)$. (The other case will follow analogously.) By definition of $Z_1(f)$ we distinguish two cases:

(i) Let $A_{[a,x_p)}(f) = p$. Since by Theorem 2.3, $A_{[a,b]}(f) \geqq m + k + 2$ and $A_{[a,x_p]}(f) = p + 1$, it follows that $A_{[x_p,b]}(f) \geqq m + k - p + 2$. Then using the arguments in the proof of Lemma 4.1 we can find a subinterval $I = [x_i, x_{i+j}] \subset [x_p, b]$ such that f satisfies (2.2) on I and x_i is minimal under (4.1).

(ii) Let $A_{[x_{p-m-r},x_p)}(f) = r$ where $[x_{p-m-r}, x_p) \subset [a, x_p)$ $(r \geqq 1)$. Since by Theorem 2.3, $A_{[x_{p-m-r},b]}(f) \geqq k-p+m+r+2$ and $A_{[x_{p-m-r},x_p]}(f) = r+1$, it follows that $A_{[x_p,b]}(f) \geqq k - p + m + 2$. Now arguing as in (i) we find a subinterval $I = [x_i, x_{i+j}]$ with the desired properties. This proves statement (4.1).

To prove (4.2) assume that $x_l \in Z_2(f)$ for some $l \in \{p, \dots, i-1\}$. We distinguish once more.

(α) Let $A_{(x_l,b]}(f) = k + 1 - l$. We have shown in (i) and (ii) that $A_{[x_p,b]}(f) \geqq m + k - p + 2$. This implies that $A_{[x_p,x_l]}(f) \geqq m + l - p + 1$. Then arguing as in the proof of Lemma 4.1 we find a knot subinterval $\tilde{I}$ of $[x_p, x_l]$ such that f satisfies (2.2) on $\tilde{I}$. This contradicts the minimality of x_i in $[x_p, b]$.

(β) Let $A_{(x_l,x_{l+m+r}]}(f) = r$ where $(x_l, x_{l+m+r}] \subset (x_l, b]$ $(r \geqq 1)$. It follows from (2.4) that $A_{[a,x_{l+m+r}]}(f) \geqq l + m + r + 1$. Now we use again the fact that $x_p \in Z_1(f)$. If $A_{[a,x_p)}(f) = p$, then $A_{[x_p,x_{l+m+r}]}(f) \geqq l + m + r - p + 1$ and therefore, $A_{[x_p,x_l]}(f) \geqq l + m - p + 1$. As in (i) we get a contradiction to the minimality of x_i.

Finally assume that $A_{[x_{p-m-\varrho},x_p)}(f) = \varrho$ where $[x_{p-m-\varrho}, x_p) \subset [a, x_p)$ $(\varrho \geqq 1)$. Then, since by (2.4), $A_{[x_{p-m-\varrho},x_{l+m+r}]}(f) \geqq \varrho - p + l + m + r + 1$, it follows that $A_{[x_p,x_l]}(f) \geqq \varrho - p + l + m + r + 1 - \varrho - r = l + m - p + 1$. As above this yields a contradiction. □

In the following three lemmas we consider certain subspaces of $S_m(\Delta)$ and determine the order of strong uniqueness with respect to these subspaces.

LEMMA 4.3. *Let $f \in K$ and assume that there exists a knot subinterval $[x_p, x_{p+m+r}]$ of $[x_1, x_k]$ $(r \geqq 1)$ such that f has alternating extreme points $x_p = t_0 < t_1 < \ldots < t_r < x_{p+m+r}$ (resp. $x_p < t_0 < t_1 < \ldots < t_r = x_{p+m+r}$) satisfying*

(4.3)
$$t_l \in (x_{p+l}, x_{p+l+m}),\ 1 \leqq l \leqq r\ \text{(resp. } t_l \in (x_{p+l}, x_{p+l+m}),\ 0 \leqq l \leqq r-1).$$

Moreover, assume that $x_p \in Z_2(f)$ (resp. $x_{p+m+r} \in Z_1(f)$) and f is flat of order at least $m+\alpha$ $(\alpha \geqq 0)$ at x_p (resp. at x_{p+m+r}). Let $S = \operatorname{span}\{B_{m+p}, \dots, B_{m+p+r-1}\}$ and $0 < \delta \leqq 1$. If $s \in S$ satisfies $\|f - s\| \leqq \|f\| + K_1\delta^\beta$ where $K_1 > 0$ is independent of f and δ, and $\beta \geqq 0$, then

$$\|s\| \leqq K_2\delta^{\alpha\beta/(m+\alpha)}$$

where $K_2 > 0$ is independent of s and δ.

PROOF. Recall that $s(f) = 0$. We will only treat the case when $t_0 = x_p$ and $x_p \in Z_2(f)$ and $\alpha > 0$. (The other cases will follow analogously.)

Let $0 < \delta \leqq 1$ and $s = \sum_{l=m+p}^{m+p+r-1} a_l B_l$ such that $\|f - s\| \leqq \|f\| + K_1\delta^\beta$. Without loss of generality we may assume that $f(x_p) = \|f\|$. It first follows that $\|s\| \leqq 2\|f\| + K_1\delta^\beta \leqq 2\|f\| + K_1$. If $x \in [x_p, x_{p+1}]$, then by the properties of the B-splines $s(x) = a_{m+p}c(x - x_p)^m$ where $c > 0$ is independent of f, s and δ. Hence

$$|a_{m+p}| \leqq (2\|f\| + K_1)/(c(x_{p+1} - x_p)^m) = C,$$

i.e. a_{m+p} is bounded. We distinguish two cases.

(i) Let $a_{m+p} \geqq 0$. Then there exists an $\varepsilon \in (0, x_{p+1} - x_p)$ such that $f(x) \geqq 2/3\|f\|$ and $0 \leqq s(x) \leqq 1/3\|f\|$ for every $x \in [x_p, x_p + \varepsilon]$. Define $\tilde{f} \in C[a,b]$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \notin [x_p, x_p + \varepsilon], \\ \|f\|, & \text{if } x \in [x_p, x_p + \varepsilon/2], \\ \|f\| + (2/\varepsilon)(x - x_p - \varepsilon/2)(f(x_p+\varepsilon) - \|f\|), & \text{if } x \in (x_p + \varepsilon/2, x_p + \varepsilon). \end{cases}$$

Then $\|\tilde{f}\| = \|f\|$, $\tilde{f}(x) \geqq 2/3\|f\|$ for every $x \in [x_p, x_p + \varepsilon]$ and $\tilde{f}$ has alternating extreme points $x_p < \tilde{t}_0 < \ldots < \tilde{t}_r < x_{p+m+r}$ where $\tilde{t}_0 = t_0 + \varepsilon/2$, $\tilde{t}_l = t_l$, $1 \leqq l \leqq r$. Hence $\tilde{t}_l \in (x_{p+l}, x_{p+l+m})$, $0 \leqq l \leqq r$. Moreover, it follows that $\|\tilde{f} - s\| \leqq \|\tilde{f}\| + K_1\delta^\beta$. Let $q \in \{0, \ldots, r\}$ be arbitrary and set $\hat{t}_l = \tilde{t}_l$, $0 \leqq l < q$, $\hat{t}_l = \tilde{t}_{l+1}$, $q \leqq l \leqq r-1$. Then $\hat{t}_l \in (x_{p+l}, x_{p+l+m+1})$, $0 \leqq l \leqq r-1$. From this and Nürnberger [2, Corollary 1.6] it follows that 0 is the strongly unique best approximation of $\tilde{f}$ from S of order one. Therefore $\|s\| \leqq \hat{K}\delta^\beta$ where $\hat{K} > 0$ is independent of s and δ.

(ii) Let $a_{m+p} < 0$. Since $f \in K$ and $x_p \in Z_2(f)$, there exists a strictly decreasing sequence (λ_l) converging to zero such that $f(x_p) - f(x_p + \lambda_l) \leqq$ $\leqq M\lambda_l^{m+\alpha}$ and $\lambda_l/\lambda_{l+1} \leqq \Lambda$ for each $l \in \mathbf{N}$. Without loss of generality we may assume that $x_p + \lambda_1 < x_{p+1}$ and $f(x_p + \lambda_l) > \|f\|/2$ for each $l \in \mathbf{N}$. Let $\lambda \in (\lambda_l)$. Then

$$(f(x_p+\lambda) - s(x_p+\lambda))^2 = (f(x_p+\lambda))^2 - 2f(x_p+\lambda)s(x_p+\lambda) + (s(x_p+\lambda))^2 \leqq$$

$$\leqq \|f - s\|^2 \leqq (\|f\| + K_1\delta^\beta)^2 \leqq \|f\|^2 + c_1\delta^\beta$$

where $c_1 > 0$ is independent of s and δ. Since $x_p + \lambda < x_{p+1}$, it follows that $s(x_p + \lambda) = a_{m+p}c\lambda^m$. Then it follows from the above inequality that

$$a_{m+p}c\lambda^m \geqq -(\|f\|^2 - (f(x_p+\lambda))^2 + c_1\delta^\beta)/(2f(x_p+\lambda)) \geqq$$

$$\geqq -((\|f\| - f(x_p + \lambda))(\|f\| + f(x_p + \lambda)) + c_1\delta^\beta)/\|f\| \geqq$$
$$\geqq -(2\|f\|M\lambda^{m+\alpha} + c_1\delta^\beta)/\|f\| \geqq -(c_2\lambda^{m+\alpha} + c_1\delta^\beta)/\|f\|.$$

This implies $|a_{m+p}| \leqq c_3\lambda^\alpha + c_4\delta^\beta\lambda^{-m}$ for each $\lambda \in (\lambda_l)$ where c_3 and c_4 are positive real numbers independent of s and δ.

We distinguish two cases. If $\delta^{\beta/(m+\alpha)} > \lambda_1$, then setting $\lambda = \lambda_1$ we obtain

$$|a_{m+p}| \leqq c_3\delta^{\alpha\beta/(m+\alpha)} + c_4\delta^\beta\lambda_1^{-m} \leqq c_5\delta^{\alpha\beta/(m+\alpha)}$$

where $c_5 > 0$ is independent of s and δ. If $\lambda_{l+1} < \delta^{\beta/(m+\alpha)} \leqq \lambda_l$ for some $l \in \mathbf{N}$, then setting $\lambda = \lambda_{l+1}$ we obtain

$$|a_{m+p}| \leqq c_3\lambda_{l+1}^\alpha + c_4\delta^\beta\lambda_{l+1}^{-m} \leqq c_3\delta^{\alpha\beta/(m+\alpha)} + c_4\delta^\beta\lambda_l^{-m}\Lambda^m \leqq$$
$$\leqq c_3\delta^{\alpha\beta/(m+\alpha)} + c_4\delta^{\beta-m\beta/(m+\alpha)}\Lambda^m \leqq c_5\delta^{\alpha\beta/(m+\alpha)}$$

where $c_5 > 0$ is independent of s and δ.

Now we have to estimate the coefficients $a_{m+p+1}, \dots, a_{m+p+r-1}$ of s. Let $\tilde{s} = \sum_{l=m+p+1}^{m+p+r-1} a_l B_l$. Then $\|f - \tilde{s}\| - \|a_{m+p}B_{m+p}\| \leqq \|f - s\| \leqq \|f\| + K_1\delta^\beta$ which implies that $\|f - \tilde{s}\| \leqq \|f\| + \overline{K}\delta^{\alpha\beta/(m+\alpha)}$ where $\overline{K} > 0$ is independent of s and δ. Then, since (4.3) is satisfied, applying the arguments in case (i) to the space $\operatorname{span}\{B_{m+p+1}, \dots, B_{m+p+r-1}\}$ we get that $\|\tilde{s}\| \leqq \overline{\overline{K}}\delta^{\alpha\beta/(m+\alpha)}$ and

$$\|s\| \leqq \|\tilde{s}\| + |a_{m+p}|\|B_{m+p}\| \leqq \tilde{\tilde{K}}\delta^{\alpha\beta/(m+\alpha)}$$

where $\tilde{\tilde{K}} > 0$ is independent of s and δ. □

The following statement can be proved analogously as Lemma 4.3.

LEMMA 4.4. *Let $f \in K$ and assume that there exists a knot subinterval $[x_p, x_q]$ of $[x_1, b]$ (resp. of $[a, x_k]$) such that f has alternating extreme points $x_p = t_0 < t_1 < \dots < t_{q-p} < x_q$ (resp. $x_p < t_0 < t_1 < \dots < t_{q-p} = x_q$) satisfying $t_l \in (x_{p+l}, x_{p+l+m})$, $1 \leqq l \leqq q-p-1$. Moreover, assume that $x_p \in Z_2(f)$ (resp. $x_q \in Z_1(f)$) and f is flat of order at least $m + \alpha$ ($\alpha \geqq 0$) at x_p (resp. at x_q). Let $S = \operatorname{span}\{B_{m+p}, \dots, B_{m+q-1}\}$ (resp. $S = \operatorname{span}\{B_p, \dots, B_{q-1}\}$) and $0 < \delta \leqq 1$. If $s \in S$ satisfies $\|f - s\| \leqq \|f\| + K_1\delta^\beta$ where $K_1 > 0$ is independent of s and δ and $\beta \geqq 0$, then*

$$\|s\| \leqq K_2\delta^{\alpha\beta/(m+\alpha)}$$

where $K_2 > 0$ is independent of s and δ.

LEMMA 4.5. *Let $f \in K$ and assume that there exists a knot subinterval $[x_p, x_{p+m+r}]$ of $[x_1, x_k]$ ($r \geqq 1$) such that f has alternating extreme points $x_p = t_0 < t_1 < \dots < t_r = x_{p+m+r}$ satisfying*

$$t_l \in (x_{p+l}, x_{p+l+m}), \quad 1 \leqq l \leqq r-1. \tag{4.4}$$

Moreover, assume that $x_p \in Z_2(f)$ *and* $x_{p+m+r} \in Z_1(f)$ *and* f *is flat of order at least* $m+\alpha$ $(\alpha \geqq 0)$ *at* x_p *and of order at least* $m+\tilde{\alpha}$ $(\tilde{\alpha} \geqq 0)$ *at* x_{p+m+r}. *Let* $\alpha \leqq \tilde{\alpha}$ *and* $S = \operatorname{span}\{B_{m+p}, \dots, B_{m+p+r-1}\}$ *and* $0 < \delta \leqq 1$. *If* $s \in S$ *satisfies* $\|f-s\| \leqq \|f\| + K_1\delta^\beta$ *where* $K_1 > 0$ *is independent of* s *and* δ *and* $\beta \geqq 0$, *then*

$$\|s\| \leqq K_2 \delta^{\alpha\beta/(m+\alpha)}$$

where $K_2 > 0$ *is independent of* s *and* δ.

PROOF. Let $0 < \delta \leqq 1$ and $s = \sum_{l=m+p}^{m+p+r-1} a_l B_l$ such that $\|f-s\| \leqq \leqq \|f\| + K_1\delta^\beta$. Then $s(x) = a_{m+p}c(x-x_p)^m$, if $x \in [x_p, x_{p+1}]$, and $s(x) = a_{m+p+r-1}\tilde{c}(x_{m+p+r}-x)^m$, if $x \in [x_{m+p+r-1}, x_{m+p+r}]$ where $c > 0$ and $\tilde{c} > 0$ are independent of s and δ. As in the proof of Lemma 4.3 we show that

$$|a_{m+p}| \leqq C, \quad |a_{m+p+r-1}| \leqq C$$

for some $C > 0$. Without loss of generality we may assume that $f(x_p) = \|f\|$.

We now distinguish three cases.

(i) Let $a_{m+p} \geqq 0$. Then analogously as in the proof of Lemma 4.3 we define a function $\tilde{f} \in C[a,b]$. Then $\|\tilde{f}\| = \|f\|$, $\|\tilde{f}-s\| \leqq \|\tilde{f}\| + K_1\delta^\beta$ and $\tilde{f}$ has alternating extreme points $\tilde{t}_0 = x_p + \lambda < \tilde{t}_1 < \ldots < \tilde{t}_r = x_{p+m+r}$ such that $\tilde{t}_l \in (x_{p+l}, x_{p+l+m})$, $0 \leqq l \leqq r-1$. Then by Lemma 4.3 it follows that $\|s\| \leqq \tilde{K}\delta^{\tilde{\alpha}\beta/(m+\tilde{\alpha})}$ where $\tilde{K} > 0$ is independent of s and δ.

(ii) Let $a_{m+p+r-1}f(x_{p+m+r}) \geqq 0$. As in case (i) we show that $\|s\| \leqq \leqq \tilde{K}\delta^{\alpha\beta/(m+\alpha)}$.

(iii) Let $a_{m+p} < 0$ and $a_{m+p+r-1}f(x_{p+m+r}) < 0$. Assume that $r = 1$. Then $f(x_{p+m+r}) = -\|f\|$ and $a_{m+p+r-1} = a_{m+p} < 0$. This implies $a_{m+p+r-1}f(x_{p+m+r}) > 0$, a contradiction. Hence $r \geqq 2$. As in the proof of Lemma 4.3, case (ii) we show that

$$|a_{m+p}| \leqq \hat{K}\delta^{\alpha\beta/(m+\alpha)}, \qquad |a_{m+p+r-1}| \leqq \hat{K}\delta^{\tilde{\alpha}\beta/(m+\tilde{\alpha})}.$$

To estimate the coefficients $a_{m+p+1}, \dots, a_{m+p+r-2}$ we set $\tilde{s} = \sum_{l=m+p+1}^{m+p+r-2} a_l B_l$.

Then

$$\|f-\tilde{s}\| - \|a_{m+p}B_{m+p}\| - \|a_{m+p+r-1}B_{m+p+r-1}\| \leqq \|f-s\| \leqq \|f\| + K_1\delta^\beta$$

which implies that $\|f-\tilde{s}\| \leqq \|f\| + \overline{K}\delta^{\alpha\beta/(m+\alpha)}$ where $\overline{K} > 0$ is independent of s and δ. Then, since (4.4) is satisfied, we conclude analogously as in Lemma 4.3, case (ii) (with respect to $\operatorname{span}\{B_{m+p+1}, \dots, B_{m+p+r-2}\}$) and obtain $\|\tilde{s}\| \leqq \overline{K}\delta^{\alpha\beta/(m+\alpha)}$. Then it follows that $\|s\| \leqq \|\tilde{s}\| + |a_{m+p}|\|B_{m+p}\| + |a_{m+p+r-1}|\|B_{m+p+r-1}\| \leqq \approx{K}\delta^{\alpha\beta/(m+\alpha)}$ where $\approx{K} > 0$ is independent of s and δ. □

Now we are able to prove the main result.

PROOF OF THEOREM 3.3. Let $f \in K$ and assume that $s(f) = 0$. Moreover, let $0 < \delta \leqq 1$ and $s = \sum_{l=0}^{m+k} a_l B_l$ such that $\|f - s\| \leqq \|f\| + \delta$. Assume that $\{I_l\}_{l=1}^{r}$ are subintervals of $[a, b]$ such that the conditions (3.1) are satisfied. In particular, it follows that $I_1 < \ldots < I_r$. Set $I_1 = [x_i, x_{i+j}]$. Since 0 is the strongly unique best approximation of $f|_{I_1}$ from $S_m(\Delta)|_{I_1}$ of order one, it follows that

$$\|s|_{I_1}\| = \left\| \sum_{l=i}^{m+i+j-1} a_l B_l \right\| \leqq K_1(f)\delta \tag{4.5}$$

where $K_1(f)$ is independent of s and δ.

Now we want to estimate s on $[a, x_i]$. If $x_i = a$, then we are finished. Hence assume that $x_i > a$. Then we show that

$$\{x_1, \ldots, x_{i-1}\} \cap Z_2(f) = \emptyset. \tag{4.6}$$

Assume to the contrary that $x_p \in Z_2(f)$ for some $p \in \{1, \ldots, i-1\}$. Then by Lemma 4.2 there exists a subinterval $I = [x_u, x_{u+v}]$ of $[a, x_p]$ such that f satisfies (2.2) on I. Then by (2.1) 0 is the strongly unique best approximation of $f|_I$ from $\dot{S}_m(\Delta)|_I$ of order one. This is a contradiction to the assumption that I_1 is maximal and also the number r of the intervals $I_1, \ldots, I_r$. This proves (4.6).

Now we show that

$$x_i \in Z_1(f). \tag{4.7}$$

Assume that (4.7) does not hold. Then we have to consider the following two cases: There exists some $p \in \{1, \ldots, i-1\}$ such that $x_p \in Z_1(f)$ and $x_l \notin Z_1(f)$, $p+1 \leqq l \leqq i$, or $\{x_1, \ldots, x_i\} \cap Z_1(f) = \emptyset$. We will only treat the first case. (The other case will follow similarly.) By Theorem 2.3 we have to distinguish:

(4.7.1) We first assume that $A_{[a,x_p)}(f) = p$. Since $x_l \notin Z_1(f)$, $p+1 \leqq l \leqq i$, it follows from (4.6) and Theorem 2.3 that $A_{[a,x_l)} \geqq l+1$, $p+1 \leqq l \leqq i$, and $A_{(x_l,x_{l+m+q})}(f) \geqq q+1$ where $(x_l, x_{l+m+q}) \subset [x_p, x_i]$ $(q \geqq 1)$. This implies that $A_{[x_p,x_l)}(f) \geqq l+1-p$, $p+1 \leqq l \leqq i$. We show that condition (2.2) is satisfied after replacing $[a, b]$ by the interval $[x_p, x_{i+j}]$. Since by (3.1), 0 is the strongly unique best approximation of $f|_{I_1}$ from $S_m(\Delta)|_{I_1}$ of order one, it follows from Theorem 2.1 that $A_{I_1}(f) \geqq m+j+1$, $A_{[x_i,x_{i+l})}(f) \geqq l+1$, $A_{(x_{i+j-l},x_{i+j}]}(f) \geqq l+1$, $1 \leqq l \leqq j$, and $A_{(x_l,x_{l+m+q})}(f) \geqq q+1$ $(q \geqq 1)$ where $(x_l, x_{l+m+q}) \subset I_1$. Combining the above statements we obtain $A_{[x_p,x_{i+j}]}(f) \geqq m+i+j-p+1$ and $A_{[x_p,x_{p+l})}(f) \geqq l+1$, $1 \leqq l \leqq i+j-p$.

Now we show that $A_{(x_{i+j-l},x_{i+j}]}(f) \geqq l+1$, $1 \leqq l \leqq i+j-p$. This is true by the above statements, if $1 \leqq l \leqq j$. Moreover, if $j < l \leqq m+j$ then $A_{(x_{i+j-l},x_{i+j}]}(f) \geqq A_{[x_i,x_{i+j}]}(f) \geqq m+j+1 \geqq l+1$. Hence let $m+j<l \leqq$ $\leqq i+j-p$. Then $i+j-l < i-m$. Since $x_l \notin Z_1(f)$, $p+1 \leqq l \leqq i$, and $x_l \notin Z_2(f)$, $1 \leqq l \leqq i-1$, it follows from Theorem 2.3 that $A_{(x_{i+j-l},x_i)}(f) \geqq$ $\geqq l-j-m+1$, if $m+j<l \leqq i+j-p$. This implies that $A_{(x_{i+j-l},x_{i+j}]}(f) \geqq$ $\geqq l-j-m+m+j+1 = l+1$. Now we show that $A_{(x_l,x_{l+m+q})}(f) \geqq q+1$, if $(x_l, x_{l+m+q}) \subset [x_p, x_{i+j}]$ $(q \geqq 1)$. By the above arguments it suffices to consider the case when $x_p \leqq x_l < x_i < x_{l+m+q} \leqq x_{i+j}$. As above it follows that $A_{[x_i,x_{l+m+q})}(f) \geqq l+m+q-i+1$. Then $A_{(x_l,x_{l+m+q})}(f) \geqq q+1$, if $l \geqq i-m$. But if $l < i-m$, then $A_{(x_l,x_i)}(f) \geqq i-l-m+1$ and therefore $A_{(x_l,x_{l+m+q})}(f) \geqq i-l-m+l+m+q-i+1 = q+1$. Thus, we have shown that $[x_p, x_{i+j}]$ satisfies (2.2) and therefore, 0 is the strongly unique best approximation of $f|_{[x_p,x_{i+j}]}$ from $S_m(\Delta)|_{[x_p,x_{i+j}]}$ of order one. This is a contradiction to the maximality of I_1 and proves (4.7.1).

(4.7.2) We now assume that $A_{[x_{p-m-q},x_p)}(f) = q$ where $[x_{p-m-q}, x_p) \subset$ $\subset [a, x_p]$ $(q \geqq 1)$. As in (4.7.1) it follows that $A_{[a,x_l)}(f) \geqq l+1$, $p+1 \leqq l \leqq i$, and $A_{(x_l,x_{l+m+q})}(f) \geqq q+1$ where $(x_l, x_{l+m+q}) \subset [x_p, x_i]$ $(q \geqq 1)$. Moreover, since $x_l \notin Z_1(f)$, $p+1 \leqq l \leqq i$, it follows that $A_{[x_{p-m-q},x_l)}(f) \geqq l+q-p+1$. Hence it follows that $A_{[x_p,x_l)}(f) \geqq l+1-p$, $p+1 \leqq l \leqq i$. Thus, we have the same hypotheses as in (4.7.1). Then concluding as in that case we obtain a contradiction. This proves (4.7).

Now assume that for some $q \in \{1, \dots, i-1\}$, $x_q \in Z_1(f)$ and $x_l \notin Z_1(f)$, $q+1 \leqq l \leqq i$. (If no such integer q exists, set $x_q = a$.) We show that

$$A_{[x_q,x_i]}(f) \geqq i-q+1. \tag{4.8}$$

Since by (2.4) $A_{[a,x_i]}(f) \geqq i+1$, (4.8) follows, if $x_q = a$. If not, then $x_q \in$ $\in Z_1(f)$. Then using similar arguments as in the proof of (4.7) statement (4.8) easily follows. Now we show that

$$\begin{cases} \text{there exists an } x_\mu \in [x_q, x_{i-1}] \text{ such that } f \text{ has alternating} \\ \text{extreme points } x_\mu \leqq \tilde{t}_0 < \ldots < \tilde{t}_{i-\mu} \leqq x_i \text{ and } \tilde{t}_l \in (x_{\mu+l-m}, x_{\mu+l}), \\ 1 \leqq l \leqq i-\mu-1. \end{cases} \tag{4.9}$$

By (4.8) there exist alternating extreme points $x_q \leqq t_0 < \ldots < t_{i-q} \leqq x_i$ of f. Then (4.9) is trivially satisfied, if $q = i-1$. Hence let $q < i-1$. Since $x_i \in$ $\in Z_1(f)$, it follows that x_i is an extreme point of f. Then without loss of generality we may assume that $t_{i-q} = x_i$ and $t_0, \ldots, t_{i-q-1}$ are maximal, i.e. if $x_q \leqq$ $\leqq \hat{t}_0 < \ldots < \hat{t}_{i-q} = x_i$ are alternating extreme points of f, then $\hat{t}_l \leqq t_l$, $0 \leqq l \leqq$ $\leqq i-q-1$.

Now we show that for some $\mu \in \{q, \dots, i-1\}$ the set $\{t_{\mu-q}, \dots, t_{i-q}\}$ satisfies (4.9). We distinguish two cases:

(4.9.1) Let $i - q \leqq m + 1$. Then for all $\mu \in \{q, \dots, i-1\}$ and all $1 \leqq l \leqq i - \mu - 1$ we have

$$t_{\mu-q+1} \geqq t_{\mu-q+1} > t_{\mu-q} \geqq t_0 \geqq x_q \geqq x_{i-m-1} \geqq x_{\mu+l-m}.$$

Hence it follows that $t_{\mu-q+l} > x_{\mu+l-m}$, $1 \leqq l \leqq i-\mu-1$, $q \leqq \mu \leqq i-1$.

Now assume that there exists no $x_\mu \in \{x_{q+1}, \dots, x_{i-1}\}$ such that $t_{\mu-q+l} < x_{\mu+l}$, $1 \leqq l \leqq i-\mu-1$, and $t_{\mu-q} \geqq x_\mu$. Hence $t_{i-q-1} < x_{i-1}$, if $\mu = i-1$, $t_{i-q-2} < x_{i-2}$, if $\mu = i-2, \dots, t_1 < x_{q+1}$, if $\mu = q+1$. But this implies that $t_l < x_{q+l}$, $1 \leqq l \leqq i-q-1$, and therefore (4.9) is satisfied, if $\mu = q$.

(4.9.2) Let $i-q > m+1$. If $t_{i-q-l} \geqq x_{i-l}$ for some $l \in \{1, \dots, m+1\}$, then applying case (4.9.1) to the interval $[x_{i-l}, x_i]$ we can verify the existence of an $x_\mu \in [x_{i-l}, x_{i-1}]$ such that (4.9) holds. Hence assume that $t_{i-q-l} < x_{i-l}$, if $1 \leqq l \leqq m+1$. Let $r \in \{m+2, \dots, i-q\}$ be the maximal integer such that $t_{i-q-r} \geqq x_{i-r}$. Then it follows that $t_{i-q-l} > x_{i-m-l}$, if $r-m \leqq l \leqq r-1$. Since by (4.6) $x_{i-m-l} \notin Z_2(f)$, $1 \leqq l \leqq r-m-1$, it follows from Theorem 2.3 that $A_{(x_{i-m-l}, x_i]}(f) \geqq l+1$, $1 \leqq l \leqq r-m-1$. Then by the choice of $\{t_0, \dots, t_{i-q}\}$, $t_{i-q-l} > x_{i-m-l}$, $1 \leqq l \leqq r-m-1$. Summarizing we have $t_{i-q-l} > x_{i-m-l}$, $1 \leqq l \leqq r-1$. Moreover by the choice of r we have $t_{i-q-l} < x_{i-l}$, $1 \leqq l \leqq r-1$. Then setting $\mu = i-r$ and $\tilde{t}_l = t_{i-q-r+l}$, $0 \leqq l \leqq i-\mu$, we can conclude from the above inequalities that $\tilde{t}_l \in (x_{\mu+l-m}, x_{\mu+l})$, $1 \leqq l \leqq r-1$. This proves (4.9).

Now we are able to estimate s on $[x_\mu, x_i]$ where x_μ is chosen as in (4.9). To do this let $S = \operatorname{span}\{B_\mu, \dots, B_{i-1}\}$. Recall that $s = \sum_{l=0}^{m+k} a_l B_l$ such that $\|f - s\| \leqq \|f\| + \delta$. Then for $x \in [x_\mu, x_{i+j}]$,

$$s(x) = \tilde{s}(x) + \sum_{l=i}^{m+i+j-1} a_l B_l(x)$$

where $\tilde{s} = \sum_{l=\mu}^{i-1} a_l B_l \in S$. Hence it follows that

$$\|(f - \tilde{s})|_{[x_\mu, x_{i+j}]}\| - \|(s - \tilde{s})|_{[x_\mu, x_{i+j}]}\| \leqq \|f - s\| \leqq \|f\| + \delta.$$

Using this inequality together with (4.5) we obtain

$$\|(f - \tilde{s})|_{[x_\mu, x_i]}\| \leqq \|(f - \tilde{s})|_{[x_\mu, x_{i+j}]}\| \leqq \tag{4.10}$$

$$\leqq \|f\| + \delta + \|(s - \tilde{s})|_{[x_\mu, x_{i+j}]}\| \leqq \|f\| + \delta + \Big\| \sum_{l=i}^{m+i+j-1} a_l B_l \Big\| \leqq \|f\| + (1 + K_1(f))\delta.$$

Since $x_i \in Z_1(f)$ and $Z_1(f) \cap [a, x_i] = \{z_{01}, \dots, z_{0\mu_0}\}$, it follows that $z_{0\mu_0} = x_i$. Moreover, recall that f is flat at z_{0l} of order at least $m + \alpha_{0l}$ for some $\alpha_{0l} \geqq 0$, $1 \leqq l \leqq \mu_0$. Then by Lemma 4.4,

$$\|\tilde{s}\| = \Big\|\sum_{l=\mu}^{i-1} a_l B_l\Big\| \leqq K_2(f)\delta^{\alpha_{0\mu_0}/(m+\alpha_{0\mu_0})} \tag{4.11}$$

where $K_2(f)$ is independent of s and δ. Thus, we have estimated the function s on $[x_\mu, x_{i+j}]$.

Now we want to estimate s on $[x_q, x_\mu]$. Since $x_q = a$ or $x_q \in Z_1(f)$ and $x_l \notin Z_1(f)$, $q+1 \leqq l \leqq \mu$, we can conclude as in the proof of (4.7) and verify that $A_{[x_q, x_\mu)}(f) \geqq \mu - q + 1$. Then arguing similarly as in the proof of (4.9) we can show that

$$\begin{cases} \text{there exists an } x_\nu \in [x_q, x_{\mu-1}] \text{ such that } f \text{ has alternating} \\ \text{extreme points } x_\nu \leqq t_0 < \dots < t_{\mu-\nu} < x_\mu \\ \text{and } t_l \in (x_{\nu+l-m}, x_{\nu+l}),\ 1 \leqq l \leqq \mu - \nu. \end{cases} \tag{4.12}$$

Now using (4.10) and (4.11) we get that

$$\Big\|\Big(f - \sum_{l=\nu}^{\mu-1} a_l B_l\Big)\Big|_{[x_\nu, x_\mu]}\Big\| \leqq \|f\| + \delta + K_2(f)\delta^{\alpha_{0\mu_0}/(m+\alpha_{0\mu_0})} \leqq \tag{4.13}$$

$$\leqq \|f\| + K_3(f)\delta^{\alpha_{0\mu_0}/(m+\alpha_{0\mu_0})}$$

where $K_3(f)$ is independent of s and δ. Since by (4.12) and [2, Corollary 1.6], 0 is the strongly unique best approximation of f from span$\{B_\nu, \dots, B_{\mu-1}\}$ of order one on $[x_\nu, x_\mu]$, it follows from (4.13) that

$$\Big\|\sum_{l=\nu}^{\mu-1} a_l B_l\Big\| \leqq K_4(f)\delta^{\alpha_{0\mu_0}/(m+\alpha_{0\mu_0})} \tag{4.14}$$

where $K_4(f)$ is independent of s and δ. By a repeated application of the above statements we can estimate s on $[x_q, x_\nu]$ and combining this together with (4.11) and (4.14) we obtain that

$$\Big\|\sum_{l=q}^{m+i+j-1} a_l B_l\Big\| \leqq K_5(f)\delta^{\alpha_{0\mu_0}/(m+\alpha_{0\mu_0})} \tag{4.15}$$

where $K_5(f)$ is independent of s and δ. If $x_q = a$, then we are finished in the interval $[a, x_{i+j}]$. Hence assume that $x_q = z_{0,\mu_0-1}$. Then replacing the interval $[x_q, x_i]$ in (4.9) by the intervals $[z_{0,l-1}, z_{0l}]$, $1 \leqq l \leqq \mu_0 - 1$, where $z_{00} = a$, and arguing as in (4.9)–(4.15) we get the estimate

$$\Big\| \sum_{l=0}^{m+i+j-1} a_l B_l \Big\| \leqq K_6(f)\delta^{P_0}$$

where as in (3.3) $P_0 = \prod_{l=1}^{\mu_0} \alpha_{0l}/(m + \alpha_{0l})$ and $K_6(f)$ is independent of s and δ. Thus, we have obtained an estimate of s on the interval $[a, x_{i+j}]$. Now we proceed by estimating s on $[x_{i+j}, b]$. Let $I_2 = [x_p, x_{p+q}]$ such that $I_2 > I_1$ and the conditions (3.1) are satisfied. Then as in (4.5) we obtain

$$\Big\| \sum_{l=p}^{m+p+q-1} a_l B_l \Big\| \leqq K_7(f)\delta \tag{4.16}$$

where $K_7(f)$ is independent of s and δ.

Now we want to estimate s on $[x_{i+j}, x_p]$. If $p - i - j = m - \varrho$ for some $\varrho \geqq 0$, then the estimate of s on $[x_{i+j}, x_p]$ has already been given by (4.5) and (4.16). Hence assume that $p - i - j \geqq m + 1$. We first show that

$$\{x_{i+j}, \dots x_p\} \cap (Z_1(f) \cup Z_2(f)) \neq \emptyset. \tag{4.17}$$

Assume that (4.17) does not hold. Since by (2.4) $A_{[x_{i+j}, x_p]}(f) \geqq p - i - j - m + 1$ and by the choice of I_1 and I_2 $A_{I_1}(f) \geqq m + j + 1$ and $A_{I_2}(f) \geqq m + q + 1$, it follows that

$$A_{[x_i, x_{p+q}]}(f) \geqq m + j + p - i - j - m + m + q + 1 = m + p + q - i + 1.$$

Moreover, it follows from Theorem 2.1 that $A_{[x_i, x_{i+l})}(f) \geqq l + 1$, $1 \leqq l \leqq j$. Since $A_{I_1}(f) \geqq m + j + 1$, we obtain $A_{[x_i, x_{i+l})}(f) \geqq m + j + 1 \geqq l + 1$, $j + 1 \leqq l \leqq j + m$. Thus, we have shown that $A_{[x_i, x_{i+l})}(f) \geqq l + 1$, $1 \leqq l \leqq j + m$. Since (4.17) is not true, it follows from (2.2) that $A_{(x_{i+j}, x_{i+j+m+l})}(f) \geqq l + 1$, $1 \leqq l \leqq p + q - i - j - m$, and $A_{I_1}(f) \geqq m + j + 1$. This implies that $A_{[x_i, x_{i+j+m+l})}(f) \geqq m + j + l + 1$, $1 \leqq l \leqq p + q - i - j - m$. Analogously we show that $A_{(x_{p+q-l}, x_{p+q}]}(f) \geqq l + 1$, $1 \leqq l \leqq p + q - i$. Moreover, it follows from (2.2) that $A_{(x_u, x_{u+m+l})}(f) \geqq l + 1$, if $(x_u, x_{u+m+l}) \subset [x_i, x_{p+q}]$ $(l \geqq 1)$. Then Theorem 2.1 implies that 0 is the strongly unique best approximation of $f|_{[x_i, x_{p+q}]}$ from $S_m(\Delta)|_{[x_i, x_{p+q}]}$ of order one, a contradiction to the maximality of I_1 and I_2. This proves (4.17).

Now we show that

$$x_{i+j} \in Z_2(f) \quad \text{or} \quad x_p \in Z_1(f). \tag{4.18}$$

Assume that $x_{i+j} \in Z_2(f)$. Then by (4.17) there exists an $x_\mu \in Z_1(f) \cup \cup Z_2(f)$ where $x_{i+j} < x_\mu \leqq x_p$ and $x_l \notin Z_1(f) \cup Z_2(f)$, if $i+j+1 \leqq l \leqq \mu-1$. If $x_\mu \in Z_2(f)$, then arguing as in the proof of (4.7) we show that $[x_i, x_\mu]$ satisfies (2.2), a contradiction to the maximality of I_1. Hence $x_\mu \in Z_1(f)$. Then, since I_2 satisfies the conditions (3.1), it follows from Lemma 4.2 that $x_l \notin Z_2(f)$, $\mu \leqq l \leqq p-1$. Now assume that $x_p \notin Z_1(f)$. Let $x_\mu \leqq x_\nu < x_p$ such that $x_\nu \in Z_1(f) \cup Z_2(f)$ and $x_l \notin Z_1(f) \cup Z_2(f)$, if $\nu+1 \leqq l \leqq p-1$. Then it follows from the above argument that $x_\nu \in Z_1(f)$. Now using again the proof of (4.7) we can conclude that $[x_\nu, x_{p+q}]$ satisfies (2.2), a contradiction to the maximality of I_2. This proves (4.18).

Without loss of generality we may assume that

$$x_{i+j} \in Z_2(f). \tag{4.19}$$

Recall that $Z_2(f) \cap [x_{i+j}, x_p] = \{\tilde{z}_{11}, \dots, \tilde{z}_{1\nu_1}\}$ and f is flat of order at least $\tilde{\alpha}_{1l}$ at $\tilde{z}_{1l}$, $1 \leqq l \leqq \nu_1$. Then $\tilde{z}_{11} = x_{i+j}$. Using the maximality of I_2 and Lemma 4.2 we can conclude as in the proof of (4.18) in order to show that $[x_{i+j}, \tilde{z}_{1\nu_1}] \cap Z_1(f) = \emptyset$. Set $x_u = \tilde{z}_{1\nu_1}$. Since for $x \in [x_{i+j}, x_p]$, $s(x) = \sum_{l=i+j}^{p-1+m} a_l B_l(x)$ and therefore the estimate of s on $[x_{i+j}, x_p]$ depends only on the given estimates (4.5) on I_1 resp. (4.16) on I_2, we can argue as in (4.8)–(4.15) and obtain

$$\Big\| \sum_{l=m+i+j}^{u-1+m} a_l B_l \Big\| \leqq K_8(f)\delta^{\tilde{Q}_1} \tag{4.20}$$

where $\tilde{Q}_1 = \prod_{l=1}^{\nu_1-1} \tilde{\alpha}_{1l}/(m+\tilde{\alpha}_{1l})$ and $K_8(f)$ is independent of s and δ.

Now we want to estimate s on $[x_u, x_p]$. We distinguish two cases: We first assume that

$$(x_u, x_p] \cap Z_1(f) \neq \emptyset. \tag{4.21}$$

Then by the notation in (3.3), $(x_u, x_p] \cap Z_1(f) = \{z_{11}, \dots, z_{1\mu_1}\}$ and f is flat of order at least α_{1l} at z_{1l}, $1 \leqq l \leqq \mu_1$. We set $x_v = z_{11}$. Now arguing as in (4.7)–(4.16) we obtain that $z_{1\mu_1} = x_p$ and

$$\Big\| \sum_{l=v}^{p-1} a_l B_l \Big\| \leqq K_9(f)\delta^{\tilde{P}_1} \tag{4.22}$$

where $\tilde{P}_1 = \prod_{l=2}^{\mu_1} \alpha_{1l}/(m+\alpha_{1l})$ and $K_9(f)$ is independent of s and δ. Hence we have still to estimate s on $[x_u, x_v]$. Recall that $x_u = \tilde{z}_{1\nu_1} < z_{11} = x_v$.

If $v-u \leqq m$, then the estimate of s on $[x_u, x_v]$ has already been given by (4.20) and (4.22). Hence assume that

$$v-u=m+\varrho \tag{4.23}$$

for some $\varrho \geqq 1$. Then it follows from (2.4) that $A_{[x_u,x_v]}(f) \geqq \varrho+1$. Now we distinguish the cases (4.24) and (4.28):

$$\left\{\begin{array}{l}\text{There does not exist any } x_\mu \in [x_{u+1}, x_{v-1}] \\ \text{such that } A_{[x_u,x_\mu]}(f) \geqq \mu-u+1.\end{array}\right. \tag{4.24}$$

Then we show that

$$\left\{\begin{array}{l} f \text{ has alternating extreme points } x_u=t_0<\ldots<t_\varrho \leqq x_v \\ \text{such that } t_l \in (x_{u+l}, x_{u+l+m}), \quad 1 \leqq l \leqq \varrho-1.\end{array}\right. \tag{4.25}$$

Since by hypothesis $A_{[x_u,x_v]}(f) \geqq \varrho+1$ and $x_u \in Z_2(f)$, without loss of generality we may assume that $t_0=x_u$. Moreover, let $t_0<t_1<\ldots<t_\varrho \leqq x_v$ be alternating extreme points of f that are minimal, i.e. if $x_u=\hat{t}_0<\ldots<$ $<\hat{t}_\varrho \leqq x_v$ are also alternating extreme points of f, then $t_l \leqq \hat{t}_l$, $1 \leqq l \leqq \varrho$. We show that $\{t_0,\ldots,t_\varrho\}$ satisfies (4.25). It first follows from (4.24) that $t_l > x_{u+l}$, $1 \leqq l \leqq \varrho-1$. Moreover, it is obvious that $t_l < x_{u+l+m}$, if $x_{u+l+m} \geqq x_v$ for some $l \in \{1,\ldots,\varrho-1\}$. Hence assume that $x_{u+l+m}<x_v$. Then it follows from (2.4) and the fact that $x_l \notin Z_1(f)$, $u+1 \leqq l \leqq v-1$, that $A_{[x_u,x_{u+l+m})}(f) \geqq l+1$. Then the minimality of $\{t_0,\ldots,t_\varrho\}$ implies that $t_l<x_{u+l+m}$, $1 \leqq l \leqq \varrho-1$. This proves (4.25). If $t_\varrho<x_v$, then by Lemma 4.3, (4.15) and (4.20) we obtain the estimate

$$\left\| \sum_{l=u+m}^{v-1} a_l B_l \right\| \leqq K_{10}(f)\delta^{Q_1} \tag{4.26}$$

where Q_1 is defined as in (3.3) and $K_{10}(f)$ is independent of s and δ. If $t_\varrho=x_v$, then by Lemma 4.5, (4.15), (4.16), (4.20) and (4.22) we obtain the estimate

$$\left\| \sum_{l=u+m}^{v-1} a_l B_l \right\| \leqq K_{11}(f)\delta^{R_1} \tag{4.27}$$

where $R_1=\min\{P_1,Q_1\}$ and P_1 is defined as in (3.3). Moreover, $K_{11}(f)$ is independent of s and δ. By (4.15), (4.16), (4.20), (4.22), (4.26) and (4.27) we are finished in $[x_{i+j}, x_p]$.

Now we assume that (4.24) does not hold. Hence

(4.28) there exists an $x_\mu \in [x_{u+1}, x_{v-1}]$ such that $A_{[x_u,x_\mu]}(f) \geqq \mu-u+1$.

Let $x_u \leqq t_0 < \ldots < t_{\mu-u} \leqq x_\mu$ be alternating extreme points of f. Since $x_u \in Z_2(f)$, without loss of generality we may assume that $t_0 = x_u$. Moreover, assume that x_μ is minimal with respect to (4.28) and $t_0, \ldots, t_{\mu-u}$ are minimal, i.e. if $x_u = \hat{t}_0 < \ldots < \hat{t}_{\mu-u} \leqq x_\mu$ are alternating extreme points of f, then $t_l \leqq \hat{t}_l$, $1 \leqq l \leqq \mu - u$. Then as in (4.25) we show that $t_l \in \in (x_{u+l}, x_{u+l+m})$, $1 \leqq l \leqq \mu - u - 1$. Now using Lemma 4.4, (4.15) and (4.20) we obtain

$$\left\| \sum_{l=u+m}^{\mu-1+m} a_l B_l \right\| \leqq K_{12}(f)\delta^{Q_1} \tag{4.29}$$

where $K_{12}(f)$ is independent of s and δ. If $v - \mu \leqq m$, then by (4.15), (4.16), (4.20), (4.22) and (4.29) we get an estimate of s on $[x_u, x_v]$ and we are finished in $[x_{i+j}, x_p]$. If $v - \mu = m + \sigma$ for some $\sigma \geqq 1$, then from (2.4) and the fact that $x_\mu \notin Z_2(f)$ it follows that $A_{(x_\mu, x_v]}(f) \geqq \sigma + 1$. Then the following cases can occur:

$$\left\{ \begin{array}{l} \text{There does not exist any } x_\nu \in [x_{\mu+1}, x_{v-1}] \\ \text{such that } A_{[x_\nu, x_v]}(f) \geqq v - \nu + 1. \end{array} \right. \tag{4.30}$$

Then similarly as in (4.25) we show that f has alternating extreme points $x_\mu < t_0 < \ldots < t_\sigma = x_v$ such that $t_l \in (x_{\mu+l}, x_{\mu+l+m})$, $1 \leqq l \leqq \sigma - 1$. Hence by Lemma 4.3, (4.16) and (4.22) we obtain

$$\left\| \sum_{l=\mu+m}^{v-1} a_l B_l \right\| \leqq K_{13}(f)\delta^{P_1} \tag{4.31}$$

where $K_{13}(f)$ is independent of s and δ and we are finished in $[x_{i+j}, x_p]$.

If (4.30) does not hold, then

(4.32) there exists an $x_\nu \in [x_{\mu+1}, x_{v-1}]$ such that $A_{[x_\nu, x_v]}(f) \geqq v - \nu + 1$.

Then arguing as in (4.28) and using Lemma 4.4 we obtain

$$\left\| \sum_{l=\nu}^{v-1} a_l B_l \right\| \leqq K_{14}(f)\delta^{P_1} \tag{4.33}$$

where $K_{14}(f)$ is independent of s and δ. If $\nu - \mu \leqq m$, then by (4.15), (4.16), (4.20), (4.22), (4.29) and (4.33) we get an estimate of s on $[x_u, x_v]$ and we are finished in $[x_{i+j}, x_p]$. If $\nu - \mu = m + \tau$ for some $\tau \geqq 1$, then from (2.4) it follows that $A_{(x_\mu, x_\nu)}(f) \geqq \tau + 1$. By a repeated application of the above arguments and [2, Corollary 1.6] we obtain

$$\left\| \sum_{l=\mu+m}^{\nu-1} a_l B_l \right\| \leqq K_{15}(f)\delta^{R_1} \tag{4.34}$$

where $R_1 = \min\{P_1, Q_1\}$, $K_{15}(f)$ is independent of s and δ, and we are finished in $[x_{i+j}, x_p]$. This completes the case when (4.21) holds.

Now assume the converse of (4.21), i.e.

$$(4.35) \qquad (x_u, x_p] \cap Z_1(f) = \emptyset.$$

As in (4.23) we may assume that $p - u = m + \chi$ for some $\chi \geqq 1$. Then from (2.4) and (4.35) it follows that $A_{[x_u,x_p)}(f) \geqq \chi + 1$. Arguing as in (4.24)–(4.34) we obtain

$$(4.36) \qquad \Big\| \sum_{l=u+m}^{p-1} a_l B_l \Big\| \leqq K_{16}(f)\delta^{Q_1}$$

where $K_{16}(f)$ is independent of s and δ.

Summarizing all the above estimates of s and using the hypothesis that $s = \sum_{l=0}^{m+k} a_l B_l$ we obtain for every $x \in [a, x_{p+q}]$,

$$(4.37) \qquad |s(x)| \leqq \tilde{K}(f)\delta^{\tilde{\gamma}}$$

where $\tilde{\gamma} = \min\{P_0, P_1, Q_1\}$ and $\tilde{K}(f)$ is independent of s and δ. Now arguing in $J_2, \dots, J_{r-1}$ as in J_1 (for the notations see (3.2)) and in J_r as in J_0 we finally obtain the desired statement (3.4). This completes the proof of Theorem 3.3. □

PROOF OF PROPOSITION 3.5. Without loss of generality assume that $P = k$ and let some positive real numbers $\alpha_1, \dots, \alpha_k$ be given. Set $x_i = i$, $0 \leqq i \leqq k+1$, and $\Delta = \{x_i\}_{i=1}^k$. Let $f \in C[0, k+1]$ be defined by

$$f(x) = \begin{cases} (-1)^{i+m}(2(m+1)x - (1+2i)), & \text{if } x \in [i/(m+1), (i+1)/(m+1)], \\ & 0 \leqq i \leqq m \\ (-1)^{i+1}(1 - 2(x-i)^{m+\alpha_i}), & \text{if } x \in [i, i+1],\ 1 \leqq i \leqq k. \end{cases}$$

Set $\beta_i = \alpha_i/(m+\alpha_i)$ and $\gamma_i = \prod_{j=1}^{i} \beta_j$, $1 \leqq i \leqq k$, and define $s_\delta \in S_m(\Delta)$ by

$$s_\delta(x) = \sum_{i=1}^{k} (-1)^i \delta^{\gamma_i}(x-i)_+^m$$

for some $0 < \delta \leqq 1$. It is easily verified that 0 is a best approximation of f from $S_m(\Delta)$ and f is flat of order $m + \alpha_i$ from the right at each x_i, $1 \leqq i \leqq k$. Hence it follows from Theorem 2.3 that 0 is the unique best approximation of f from $S_m(\Delta)$. Since $0 < \gamma_i < 1$, $1 \leqq i \leqq k$, and $\gamma_i > \gamma_j$, if $i < j$, and

$0 < \delta \leqq 1$, there exist a positive constant M independent of δ and a constant $0 < \delta_0 \leqq 1$ such that

$$\|s_\delta\| \geqq M\delta^{\gamma_k} \quad \text{and} \quad \|s_\delta\| \leqq 1$$

for all $0 < \delta \leqq \delta_0$. Furthermore, there exists a constant $0 < \delta_1 \leqq \delta_0$ such that $\operatorname{sgn} s_\delta(i) = (-1)^{i+1}$, $2 \leqq i \leqq k+1$, for all $0 < \delta \leqq \delta_1$.

Now we show that there exists a constant $0 < \delta_2 \leqq \delta_1$ such that $|f(x) - s_\delta(x)| \leqq \|f\| = 1$ for all $x \in [2, k+1]$ and all $0 < \delta \leqq \delta_2$. Without loss of generality assume that i is odd. (The case when i is even follows analogously.) Let $x \in [i, i+1]$. Then

$$f(x) - s_\delta(x) = 1 - 2(x-i)^{m+\alpha_i} + \sum_{j=1}^{i} (-1)^{j+1}\delta^{\gamma_j}(x-j)^m.$$

This implies that

$$f'(x) - s'_\delta(x) = -2(m+\alpha_i)(x-i)^{m+\alpha_i-1} + m\sum_{j=1}^{i}(-1)^{j+1}\delta^{\gamma_j}(x-j)^{m-1}$$

for all $x \in [i, i+1]$.

Let us first assume that $f' - s'_\delta$ has no zero on $(i, i+1)$. Then $f - s_\delta$ can have an extremal value only in i or $i+1$. It follows from $\|s_\delta\| \leqq 1$, $\operatorname{sgn} s_\delta(i) = (-1)^{i+1}$ and the definition of f that $|f(x) - s_\delta(x)| \leqq 1$, if $x = i$ or $x = i+1$. Therefore we assume that there is an $\tilde{x} \in (i, i+1)$ such that $f'(\tilde{x}) - s'_\delta(\tilde{x}) = 0$. Hence

$$2(m+\alpha_i)(\tilde{x}-i)^{m+\alpha_i-1} = m\sum_{j=1}^{i}(-1)^{j+1}\delta^{\gamma_j}(\tilde{x}-j)^{m-1}.$$

For some $0 < \delta_3 \leqq \delta_2$ the inequality

$$2(m+\alpha_i)(\tilde{x}-i)^{m+\alpha_i-1} < m\delta^{\gamma_i}(\tilde{x}-i)^{m-1}$$

is valid for all $0 < \delta \leqq \delta_3$, since $\gamma_i > \gamma_j$ for all $j = 1, \dots, i-1$. Hence

$$(\tilde{x}-i)^{\alpha_i} < (m/(2(m+\alpha_i)))\delta^{\gamma_i}$$

for $0 < \delta \leqq \delta_3$. It follows from $\tilde{x} > i$ and the last inequality that

$$f(\tilde{x}) - s_\delta(\tilde{x}) < 1 + \sum_{j=1}^{i}(-1)^{j+1}\delta^{\gamma_j}(\tilde{x}-j)^m <$$

$$< 1 + \sum_{j=1}^{i-1}(-1)^{j+1}\delta^{\gamma_j}(\tilde{x}-j)^m + (m/(2(m+\alpha_i)))^{m/\alpha_i}\delta^{\gamma_{i-1}}.$$

Therefore,

$$f(\tilde{x})-s_\delta(\tilde{x})<1+\sum_{j=1}^{i-2}(-1)^{j+1}\delta^{\gamma_j}(\tilde{x}-j)^m-\delta^{\gamma_{i-1}}\left((\tilde{x}-i+1)^m-(m/(2(m+\alpha_i)))^{m/\alpha_i}\right).$$

Since $\tilde{x}-i+1>1$ and $m/(2(m+\alpha_i))<1/2$, it follows that

$$(\tilde{x}-i+1)^m-(m/(2(m+\alpha_i)))^{m/\alpha_i}>0.$$

Hence for some $0<\delta_4\leqq\delta_3$ we have

$$0<f(\tilde{x})-s_\delta(\tilde{x})<1 \quad \text{for all} \quad 0<\delta\leqq\delta_4.$$

$f'-s'_\delta$ can have at most m zeros on $[i,i+1]$. This follows from Rolle's theorem and the fact that $(f^{(m)}-s_\delta^{(m)})\,|_{[i,i+1]}=a_i(x-i)^{\alpha_i}+b_i$, $a_i,b_i\in\mathbf{R}$, has at most one zero. Therefore, $f'-s'_\delta$ has only a finite number of zeros on $[2,k+1]$. Now it follows from the above results that for some $0<\delta_5\leqq\delta_4$

$$|f(x)-s_\delta(x)|\leqq 1, \qquad x\in[2,k+1],$$

for all $0<\delta\leqq\delta_5$.

Finally we consider $f-s_\delta$ on $[1,2]$. Since $f'-s'_\delta$ has the zero

$$\tilde{x}=1+((m/(2(m+\alpha_1)))\delta^{\gamma_1})^{1/\alpha_1}$$

on $[1,2]$, and

$$f(\tilde{x})-s_\delta(\tilde{x})=1+\gamma_1(m/(2(m+\alpha_1)))^{m/\alpha_1}\delta\leqq\|f\|+\delta,$$

it follows that $|f(x)-s_\delta(x)|\leqq\|f\|+\delta$ for all $x\in[1,2]$. Summarizing the above results we obtain that

$$\|f-s_\delta\|\leqq\|f\|+\delta \quad \text{for all} \quad 0<\delta\leqq\tilde{\delta}=\delta_5.$$

This completes the proof of Proposition 3.5. □

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ON APPROXIMATION OF UNBOUNDED FUNCTIONS BY LINEAR COMBINATIONS OF MODIFIED SZÁSZ–MIRAKIAN OPERATORS

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1. S. M. Mazhar and V. Totik [6] have proposed the integral modification of Szász–Mirakian operators to approximate Lebesgue integrable functions defined on $[0,\infty)$ as

$$L_n(f,x)=n\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\int_0^{\infty}p_{n,\nu}(t)f(t)dt, \tag{1.1}$$

where

$$p_{n,\nu}(x)=e^{-nx}\frac{(nx)^{\nu}}{\nu!},\qquad f\in L_1[0,\infty).$$

The operator (1.1) was studied for *simultaneous approximation* by Singh [8] and moreover, similar results have been derived by Sahai and Prasad [7] for modified Lupas operators.

However, we consider the class P of all measurable functions defined on $[0,\infty)$ such that

$$P[0,\infty)=\left\{f\colon\int_0^{\infty}e^{-nt}f(t)dt<\infty,\ \ n>n_0(f)\right\}.$$

Obviously, $L_1[0,\infty)\subsetneqq P[0,\infty)$ and hence L_n may be utilised for studying a larger class of functions. For $m\in\mathbf{N}^0$ (set of nonnegative integers), the mth order moment of L_n is defined as

$$T_{n,m}(x)=n\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\int_0^{\infty}p_{n,\nu}(t)(t-x)^m dt. \tag{1.2}$$

Further, we define

$$P_\alpha[0,\infty)=\{f\in P[0,\infty)\colon f(t)=O(e^{\alpha t}),\ t\to\infty,\ \alpha>0\}.$$

In the sequel we assume $0<a<a_1<b_1<b<\infty$ and $\|\cdot\|_{[a,b]}$ means the sup norm on the space $C[a,b]$. Moreover, $\|\cdot\|_\alpha$ denotes a norm on the space $C_\alpha[0,\infty)=C[0,\infty)\cap P_\alpha[0,\infty)$ such that for $f\in C_\alpha[0,\infty)$,

$$\|f\|_\alpha=\sup_{0\leqq t<\infty}|f(t)|e^{-\alpha t}.$$

Let $d_0, d_1, \dots, d_k$ be arbitrary but fixed distinct positive integers. Then, following Kasana and Agrawal [4], the *linear combinations* $L_n(f,k,x)$ of $L_{d_j n}(f,x)$, $j = 0,1,\dots,k$ are introduced as

$$(1.3)\qquad L_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} L_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \dots & d_0^{-k} \\ L_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \dots & d_1^{-k} \\ \dots\dots & \dots & \dots & \dots & \dots \\ \dots\dots & \dots & \dots & \dots & \dots \\ L_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \dots & d_k^{-k} \end{vmatrix},$$

where Δ is the *Vandermonde determinant* obtained by replacing the operator column of the determinant by the entries 1. On simplification (1.3) is reduced to

$$(1.4)\qquad L_n(f,k,x) = \sum_{j=0}^{k} c(j,k) L_{d_j n}(f,x),$$

where

$$(1.5)\qquad c(j,k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \quad \text{and} \quad c(0,0) = 1;$$

and (1.4) is the form of linear combinations considered by May [5].

For $f \in C_\alpha[0,\infty)$, $\delta > 0$ and $m \in \mathrm{N}$, the *Steklov mean* $f_{2m,\delta}$ is defined by (1.6)

$$f_{2m,\delta}(x) = \frac{1}{\binom{2m}{m}\delta^{2m}} \left(\int_0^{\delta}\right)^{2m} \left(\binom{2m}{m} f(x) + (-1)^{m-1}\overline{\Delta}_h^{2m} f(x)\right) dt_1 dt_2 \dots dt_{2m},$$

where

$$h = \sum_{i=1}^{2m} t_i \quad \text{and} \quad \overline{\Delta}_h^{2m} f(x) = \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} f(x + (m-i)h).$$

This paper contains *Woronowskaja type asymptotic formula* and an error estimate in terms of *higher order modulus of continuity* for unbounded functions on the semi-real axis.

2. In this section we introduce certain auxiliary results which will be utilised in Section 3.

LEMMA 1 [8]. *For $T_{n,m}(x)$ there holds the recurrence relation:*

$$(2.1)\quad nT_{n,m+1}(x) = xT'_{n,m}(x) + (m+1)T_{n,m}(x) + 2mxT_{n,m-1}(x), \quad m \geqq 1.$$

By direct computation from (1.2), *we have* $T_{n,0}(x) = 1$ *and* $T_{n,1}(x) = \frac{1}{n}$ *and further, using the recurrence relation* (2.1) *it can be verified that*

(i) $T_{n,m}(x)$ *is a polynomial in* x *of degree* $[m/2]$ *and in* n^{-1} *of degree* m.

(ii) *For every* $x \in [0, \infty)$, $T_{n,m}(x) = O(n^{-[(m+1)/2]})$.

(iii) *The coefficients of* $n^{-(k+1)}$ *in* $T_{n,2k+2}(x)$ *and* $T_{n,2k+1}(x)$ *are* $(2k+2)!x^{k+1}/(k+1)!$ *and* $(2k+2)!x^k/2k!$, *respectively.*

LEMMA 2. *If* $c(j,k)$, $j = 0, 1, \dots, k$ *are defined as in* (1.5), *then*

$$\sum_{j=0}^{k} c(j,k) d_j^{-m} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, 2, \dots, k. \end{cases} \tag{2.2}$$

PROOF. If the operator column of the determinant in (1.3) is replaced by the entries $d_0^{-m} d_1^{-m}, \dots, d_k^{-m}$ and then observing the determinant consecutively, for $m = 0, 1, \dots, k$ this lemma follows. However, May [5] has given a different proof by using Lagrange polynomials.

LEMMA 3 [3]. *Let* δ *be a positive number. Then, for every* $m \in \mathbf{N}$ *and* $x \in [0, \infty)$ *there exists a positive constant* $K_{m,x}$ *such that*

$$\int\limits_{|t-x|>\delta} W(n,x,t) e^{\alpha t} dt \leqq K_{m,x} n^{-m}, \tag{2.3}$$

where $K_{m,x}$ *is a positive constant depending on* m *and* x *and*

$$W(n,x,t) = n \sum_{\nu=0}^{\infty} p_{n,\nu}(x) p_{n,\nu}(t).$$

LEMMA 4. *The function* $f_{2m,\delta}$ *defined in* (1.6) *has the properties:*

(a) $\|f_{2m,\delta} - f\|_{[a_1,b_1]} \leqq M_1 \omega_{2m}(f;\delta,a,b)$;

(b) $\|f_{2m,\delta}\|_{[a_1,b_1]} \leqq M_2 \|f\|_{[a,b]} \leqq M_2' \|f\|_\alpha$;

(c) $\|f_{2m,\delta}^{(2m)}\|_{[a_1,b_1]} \leqq M_3 \delta^{-2m} \omega_{2m}(f;\delta,a,b)$,

where $M_2' = M_2 e^{\alpha b}$, M_i*'s are positive constants depending on* m *only and* $\omega_{2m}(f;\delta,a,b)$ *is the modulus of continuity of order* $2m$ *corresponding to* f:

$$\omega_{2m}(f;\delta,a,b) = \sup\{|\Delta_h^{2m} f(x)|;\ |h| \leqq \delta,\ x + 2mh \in [a,b]\}$$

such that

$$\Delta_h^{2m} f(x) = \sum_{i=0}^{2m} (-1)^{2m-i} \binom{2m}{i} f(x+ih).$$

The properties (a) to (c) are extensions of the calculations found in a paper by Freud and Popov [1]. However, for conciseness the proof is given as follows:

PROOF. From the definition of $f_{2m,\delta}$, we have

$$|f_{2m,\delta}(x) - f(x)| \leqq \frac{1}{\binom{2m}{m}\delta^{2m}} \left(\int_0^\delta\right)^{2m} |\overline{\Delta}_k^{2m} f(x)| dt_1 dt_2 \dots dt_{2m} \leqq$$

$$\leqq M\omega_{2m}(f;\delta m, a, b) \leqq Mm\omega_{2m}(f;\delta, a, b) \leqq M_1\omega_{2m}(f;\delta, a, b), \quad x \in [a_1, b_1].$$

Hence (a) is obtained. The proof of (b) is trivial and to prove (c) we observe that

$$(2.4) \quad \frac{d^{2m}}{dx^{2m}} \left(\int_0^\delta\right)^{2m} \left(f\left(x + \sum_{i=0}^{2m} t_i\right) + f\left(x - \sum_{i=0}^{2m} t_i\right)\right) dt_1 dt_2 \dots dt_{2m} =$$

$$= (\Delta_\delta^{2m} + \Delta_{-\delta}^{2m}) f(x),$$

and

$$(2.5) \qquad \omega_{2m}(f;|m-i|\delta, a, b) \leqq |m-i|\omega_{2m}(f;\delta, a, b) = K_i\omega_{2m}(f;\delta, a, b).$$

Using (2.4), (2.5) and (1.6) we have

$$\|f_{2m,\delta}^{(2m)}\|_{[a_1,b_1]} =$$

$$= \frac{1}{\binom{2m}{m}\delta^{2m}} \left\| \sum_{i=0}^{m-1} (-1)^{i+m-1} \binom{2m}{i} \left(\Delta_{(m-i)\delta}^{2m} + \Delta_{-(m-i)\delta}^{2m}\right) f(x) \right\|_{[a_1,b_1]} \leqq$$

$$\leqq \frac{1}{\binom{2m}{m}\delta^{2m}} \left(\sum_{i=0}^{m-1} \binom{2m}{i} (m-i) \right) \omega_{2m}(f;\delta, a, b)$$

and thus (c) is immediate.

3. Here we prove our main results.

THEOREM 1. *Let $f \in P_\alpha[0,\infty)$ and let $f^{(2k+2)}$ exist at a point $x \in [0,\infty)$. Then*

$$(3.1) \qquad \lim_{n\to\infty} n^{k+1}[L_n(f,k,x) - f(x)] = \sum_{m=k+1}^{2k+2} \frac{f^{(m)}(x)}{m!} Q(m,k,x)$$

and

$$(3.2) \qquad \lim_{n\to\infty} n^{k+1}[L_n(f,k+1,x) - f(x)] = 0,$$

where $Q(m,k,x)$ are polynomials in x of degree at most $[m/2]$ such that

$$Q(2k+1,k,x)=\frac{(-1)^k(2k+2)!}{2k!\prod_{j=0}^{k} d_j}x^k; \quad Q(2k+2,k,x)+\frac{(-1)^k(2k+2)!}{(k+1)!\prod_{j=0}^{k} d_j}x^{k+1}.$$

Moreover, if $f^{(2k+2)}$ *exists and is continuous on* $[a_1,b_1]$, *then* (3.1) *and* (3.2) *hold uniformly on* $[a_1,b_1]$.

PROOF. Since $f^{(2k+2)}$ exists at $x\in[0,\infty)$, it follows that

$$f(t)=\sum_{m=0}^{2k+2}\frac{f^{(m)}(x)}{m!}(t-x)^m+\varepsilon(t,x)(t-x)^{2k+2},$$

where $\varepsilon(t,x)\to 0$ as $t\to x$ and is contained in $P_\alpha[0,\infty)$. Writing

$$n^{k+1}[L_n(f,k,x)-f(x)]=n^{k+1}\left\{\sum_{m=1}^{2k+2}\frac{f^{(m)}(x)}{m!}L_n((t-x)^m,k,x)+\right.$$

$$\left.+\sum_{j=0}^{k}c(j,k)L_{d_jn}(\varepsilon(t,x)(t-x)^{2k+2},x)\right\}=I_1+I_2$$

(say). Hence

$$I_1=n^{k+1}\sum_{m=1}^{2k+2}\frac{f^{(m)}(x)}{m!}\sum_{j=0}^{k}c(j,k)T_{d_jn,m}(x).$$

Using Lemma 1, we have

$$T_{d_jn,m}(x)=\frac{P_0(x)}{(d_jn)^{[\frac{m+1}{2}]}}+\frac{P_1(x)}{(d_jn)^{[\frac{m+1}{2}]+1}}+\ldots+\frac{P_{[m/2]}(x)}{(d_jn)^m}$$

for certain polynomials P_i, $i=0,1,\ldots,[m/2]$ in x of degree at most $[m/2]$. Clearly,

$$\sum_{j=0}^{k}c(j,k)T_{d_jn,m}(x)= \tag{3.3}$$

$$=\frac{1}{\Delta}\begin{vmatrix}\frac{P_0(x)}{(d_0n)^{[\frac{m+1}{2}]}}+\frac{P_1(x)}{(d_0n)^{[\frac{m+1}{2}]+1}}+\ldots+\frac{P_{[m/2]}(x)}{(d_0n)^m} & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k}\\ \frac{P_0(x)}{(d_1n)^{[\frac{m+1}{2}]}}+\frac{P_1(x)}{(d_1n)^{[\frac{m+1}{2}]+1}}+\ldots+\frac{P_{[m/2]}(x)}{(d_1n)^m} & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k}\\ \cdots\cdots\cdots & \ldots & \ldots & \ldots & \ldots\\ \cdots\cdots\cdots & \ldots & \ldots & \ldots & \ldots\\ \frac{P_0(x)}{(d_kn)^{[\frac{m+1}{2}]}}+\frac{P_1(x)}{(d_kn)^{[\frac{m+1}{2}]+1}}+\ldots+\frac{P_{[m/2]}(x)}{(d_kn)^m} & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k}\end{vmatrix}=$$

$$=n^{-(k+1)}\{Q(m,k,x)+o(1)\},\qquad m=k+1,k+2,\ldots,2k+2.$$

So I_1 is determined by

$$\sum_{m=k+1}^{2k+2} \frac{f^{(m)}(x)}{m!} Q(m,k,x) + o(1).$$

The expression for $Q(2k+1,k,x)$ and $Q(2k+2,k,x)$ can be obtained by application of Lemma 1 in (3.3).

Now it suffices to show that $I_2 \to 0$ as $n \to \infty$. For a given $\varepsilon > 0$, there exists a $\delta \equiv \delta(\varepsilon, x) > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, $0 < |t-x| \leqq \delta$, and for $|t-x| > \delta$, we notice that $\varepsilon(t,x) = O(e^{\alpha t})$. Treat I_2 as

$$I_2 = n^{k+1} \sum_{j=0}^{k} c(j,k) \Bigg(\int\limits_{|t-x| \leqq \delta} + \int\limits_{|t-x|>\delta} \Bigg) W(d_j n, x, t) \varepsilon(t,x)(t-x)^{2k+2} dt = I_3 + I_4.$$

Again using Lemma 1 we get

$$|I_3| \leqq \varepsilon n^{k+1} \sum_{j=0}^{k} |c(j,k)| \max_{0 \leqq j \leqq k} L_{d_j n}((t-x)^{2k+2}, x) < K\varepsilon.$$

Similarly (in view of (2.3)),

$$|I_4| \leqq n^{k+1} \sum_{j=0}^{k} |c(j,k)| \int\limits_{|t-x|>\delta} W(d_j n, x, t)(t-x)^{2k+2} e^{\alpha t} dt \leqq$$

$$\leqq n^{k+1} \sum_{j=0}^{k} |c(j,k)| \Bigg(\int_0^\infty W(d_j n, x, t)(t-x)^{4(k+1)} dt \int\limits_{|t-x|>\delta} W(d_j n, x, t) e^{2\alpha t} dt \Bigg)^{1/2} \leqq$$

$$\leqq K_{m,x} n^{-\frac{m+2(k+1)}{2}} = o(1), \qquad \frac{m}{2} > k+1.$$

Since $\varepsilon > 0$ is arbitrary, combining I_3 and I_4 we conclude $I_2 \to 0$.

The assertion (3.2) can be proved along similar lines by noting the fact:

$$L_n((t-x)^m, k+1, x) = O(n^{-(k+2)}).$$

The limits in (3.1) and (3.2) hold uniformly due to the uniform continuity of $f^{(2k+2)}$ on $[a,b]$ and uniformness of the term $o(1)$ occurring in the estimates of I_1. This completes the proof.

THEOREM 2. *Let $f \in C_\alpha[0,\infty)$. Then, for all n sufficiently large,*

$$\text{(3.4)} \quad \|L_n(f,k,\cdot) - f\|_{[a_1,b_1]} \leqq M_{k,b} \left\{ \omega_{2k+2}(f; n^{-1/2}, a, b) + n^{-(k+1)} \|f\|_\alpha \right\},$$

where $M_{k,b}$ is a positive constant dependent on k and b but independent of f and n.

Proof. Let $f_{2k+2,\delta}$ be the Steklov mean of $(2k+2)$-th order corresponding to f. In view of linearity of $L_n(\cdot,k,x)$ we write

$$L_n(f,k,x)-f(x)=L_n(f-f_{2k+2,\delta},k,x)+(f_{2k+2,\delta}(x)-f(x))+$$
$$+L_n(f_{2k+2,\delta},k,x)-f_{2k+2,\delta}(x)=J_1(x)+J_2(x)+J_3(x).$$

The estimate of $J_1(x)$ follows from Lemma 3; indeed, we have

$$|J_1(x)|\leqq\sum_{j=0}^{k}|c(j,k)|\int_0^\infty W(d_jn,x,t)|f_{2k+2,\delta}(t)-f(t)|dt$$

and

$$\int_0^\infty W(d_jn,x,t)|f_{2k+2,\delta}(t)-f(t)|dt=$$
$$=\int_{|t-x|\leqq\delta}+\int_{|t-x|>\delta}\leqq\|f_{2k+2,\delta}-f\|_{[a_1-\delta,b_1+\delta]}+K_mn^{-m}\|f\|_\alpha,$$

where $\delta\leqq\min\{a_1-a,b-b_1\}$. Hence, using Lemma 4(a)

$$\|J_1\|_{[a_1,b_1]}\leqq M_1\omega_{2k+2}(f;\delta,a,b)+K_mn^{-m}\|f\|_\alpha.$$

In a similar manner,

$$\|J_2\|_{[a_1,b_1]}\leqq M_1\omega_{2k+2}(f;\delta,a,b).$$

It remains to estimate $J_3(x)$. Expanding $f_{2k+2,\delta}$ by Taylor's formula,

$$f_{2k+2,\delta}(t)=\sum_{i=0}^{2k+1}\frac{f^{(i)}_{2k+2,\delta}(x)}{i!}(t-x)^i+\frac{f^{(2k+2)}_{2k+2,\delta}(\xi)}{(2k+2)!}(t-x)^{2k+2},\tag{3.5}$$

where ξ lies between t and x. Operating $L_n(\cdot,k,x)$ on (3.5) and separating the integral into two parts as in the estimation of $J_1(x)$, we obtain (in view of Lemma 1 and (2.2))

$$\|L_n(f_{2k+2,\delta},k,\cdot)-f_{2k+2,\delta}\|_{[a_1,b_1]}\leqq$$
$$\leqq Mn^{-(k+1)}\sum_{i=k+1}^{2k+2}\|f^{(i)}_{2k+2,\delta}\|_{[a_1,b_1]}+K_mn^{-m}\|f_{2k+2,\delta}\|_\alpha.$$

Using the interpolation property

$$\|f^{(i)}_{2k+2,\delta}\|_{[a_1,b_1]}\leqq M'(\|f_{2k+2,\delta}\|_{[a_1,b_1]}+\|f^{(2k+2)}_{2k+2,\delta}\|_{[a_1,b_1]})$$

due to Goldberg and Meir [2] in the above inequality, we further have, for $m\geqq k+1$,

(3.6)
$$\|L_n(f_{2k+2,\delta},k,\cdot)-f_{2k+2,\delta}\|_{[a_1,b_1]}\leqq M_k n^{-(k+1)}\left(\|f^{(2k+2)}_{2k+2,\delta}\|_{[a_1,b_1]}+\|f_{2k+2,\delta}\|_\alpha\right).$$

Consequently, J_3 is estimated by application of inequalities (b) and (c) of Lemma 4 in (3.6) as

$$\|J_3\|_{[a,b]}\leqq M_k n^{-(k+1)}\left(M_3\delta^{-(2k+2)}\omega_{2k+2}(f;\delta,a,b)+M_2'\|f\|_\alpha\right).$$

Choosing $\delta=n^{-1/2}$ and combining the estimates of $J_1(x)$, $J_2(x)$ and $J_3(x)$, the required result follows.

REMARKS. (i) The concept of linear combination was developed to increase the order of approximation of functions with higher smoothness. We preferred form (1.3) to (1.4), since it is not possible to calculate $Q(2k+1,k,x)$ and $Q(2k+2,k,x)$ by using the form (1.4) due to May [5].

(ii) An analogous result to (3.4) was recently obtained by Wood [9] for Bernstein polynomials using the technique of Peetre's K-functionals.

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EXPANSIONS IN LEGENDRE POLYNOMIALS AND LAGRANGE INTERPOLATION

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This paper is divided into four sections.

In the first section we consider the problem of the convergence and divergence of Fourier series with respect to Legendre polynomials of functions in Lorentz spaces. The critical index for the convergence of Legendre–Fourier series of functions in L^p spaces is $p = 4/3$. We prove that the Legendre–Fourier series of functions in the Lorentz space $L^{4/3,1}$ converge almost everywhere and in the norm of $L^{4/3,\infty}$, while there exist functions in $L^{4/3,r}$, $r > 1$, with Legendre–Fourier series diverging in measure and pointwise everywhere. We also give precise estimates for the norms of the partial sum operators.

In the second section we consider the problem of the convergence in Lorentz spaces of the Lagrange interpolation taken at the zeros of Legendre polynomials. Here the critical index is $p = 4$. We prove that the Lagrange polynomials which interpolate a continuous function at the zeros of Legendre polynomials converge in the norm of $L^{4,\infty}$ and may diverge in the norm of $L^{4,s}$ if $s < +\infty$. Again we give precise estimates for the rate of divergence.

The third section contains the proof of the theorems.

The fourth section is devoted to some concluding remarks. In particular we briefly consider the problem of the convergence of Legendre–Fourier series with an arbitrary reordering or grouping of the terms, and the problem of the convergence of Lagrange interpolation polynomials of functions in some Lipschitz classes.

1. Fourier series with respect to Legendre polynomials

Let $\{p_n\}_{n=0}^{+\infty}$ be the orthonormal system of Legendre polynomials,

$$p_n(x) = \frac{\sqrt{n+\frac{1}{2}}}{2^n n!} \frac{d^n}{dx^n} \left\{ (x^2-1)^n \right\}.$$

The n-th partial sums of the Fourier–Legendre expansion of a function f integrable on $[-1,1]$ are defined by

$$S_n f(x) = \sum_{k=0}^{n} \hat{f}(k) p_k(x),$$

where $\hat{f}(k) = \int_{-1}^{1} f(y)p_k(y)dy$.

H. Pollard has shown that if f is in $L^p[-1,1]$, $4/3 < p < 4$, then $\{S_n f\}$ converges to f in the L^p-norm. Also, if $p > 4/3$, $\{S_n f(x)\}$ converges for almost every x in $[-1,1]$. These results are essentially the best possible for L^p-spaces, because J. Newman, W. Rudin, and C. Meaney, have constructed functions in $L^{4/3}[-1,1]$ with Legendre expansions diverging both in norm and almost everywhere.

At the critical indexes $p = 4/3$ and $p = 4$ one can obtain positive convergence results only for functions in suitable subspaces of $L^p[-1,1]$. In particular S. Chanillo has shown that the operators $\{S_n\}$ are not uniformly of weak type $(4,4)$, but they are of restricted weak type $(4,4)$, i.e. for the characteristic functions of measurable sets E,

$$|\{x \in [-1,1] \colon |S_n \chi_E(x)| > t\}| \leqq ct^{-4}|E|.$$

Chanillo's result has a natural interpretation in terms of Lorentz spaces.

The Lorentz space $L^{p,r}[-1,1]$, $1 < p < +\infty$ and $1 \leqq r \leqq +\infty$, is the Banach space of functions f integrable on $[-1,1]$, with

$$\|f\|_{p,r} = \left\{ \frac{r}{p} \int_0^2 \left[t^{1/p} f^*(t) \right]^r \frac{dt}{t} \right\}^{1/r} < +\infty,$$

where f^* is the decreasing rearrangement of $|f|$.

In particular $L^{p,p}[-1,1] = L^p[-1,1]$, $L^{p,\infty}[-1,1] = \text{Weak-}L^p[-1,1]$, and $L^{p,r}[-1,1] \subset L^{p,s}[-1,1]$ if $r < s$. The operators of weak and restricted weak type (p,p) are precisely those bounded from $L^{p,p}[-1,1]$ into $L^{p,\infty}[-1,1]$, and from $L^{p,1}[-1,1]$ into $L^{p,\infty}[-1,1]$ respectively.

As a general reference on Lorentz spaces see Hunt [10]. For the norm convergence of Legendre expansions see Pollard [15, 16, 17], Newman–Rudin [14], Dreseler–Soardi [6], and Cartwright [4]. For the pointwise convergence see Pollard [18] and Meaney [12]. For the weak behaviour of these expansions see Chanillo [5] and Guadalupe–Pérez–Varona [9]. See also the survey Badkov [3].

Our aim is to provide a fairly simple approach to the problem of convergence and divergence of Fourier–Legendre expansions for functions in these Lorentz spaces. In particular we shall prove the following results.

THEOREM I. *Let f be a function integrable on $[-1,1]$. Then if $(1-x^2)^{-1/4} f(x)$ is also integrable, the partial sums $\{S_n f\}$ of the Fourier–Legendre series of f converge to f in measure, while if the Fourier–Legendre transform $\{\hat{f}(n)\}$ of f is an unbounded sequence, the partial sums $\{S_n f\}$ diverge in measure and pointwise almost everywhere.*

THEOREM II. *The partial sums* $\{S_n f(x)\}$ *of the Fourier–Legendre series of a function* f *in* $L^{4/3,1}[-1,1]$ *converge pointwise to* $f(x)$ *for almost every* x *in* $[-1,1]$.

THEOREM III. *There exist functions* f *in* $L^{4/3,r}[-1,1]$, $1 < r \leqq +\infty$, *with* $\{S_n f(x)\}$ *unbounded for every* x *in* $[-1,1]$.

THEOREM IV. *Let* $p = 4/3$ *or* $p = 4$, *and let* $1 \leqq r$, $s \leqq +\infty$. *Then*

$$\sup\left\{\frac{\|S_n f\|_{p,s}}{\|f\|_{p,r}}\right\} \simeq c[\log(2+n)]^{1-\frac{1}{r}+\frac{1}{s}}.$$

In particular, if $1 < r \leqq +\infty$ *there exist functions* f *in* $L^{p,r}[-1,1]$ *with* $\left\{\|S_n f\|_{p,\infty}\right\}$ *unbounded, while for every function* f *in* $L^{p,1}[-1,1]$, $\{S_n f\}$ *converges to* f *in the norm of* $L^{p,\infty}[-1,1]$.

Theorem I is a simple corollary of the Haar–Szegő equiconvergence theorem between Legendre and cosine expansions, and perhaps it is already known. Anyhow, in our view this result explains why Theorems II, III, and IV, are natural. Indeed, Theorem I naturally leads to consider functions f which are integrable against $(1-x^2)^{-1/4}$. Since this function is the typical representative of $L^{4,\infty}[-1,1]$ and this space is the dual of $L^{4/3,1}[-1,1]$, the connection between Legendre expansions and the Lorentz space $L^{4/3,1}[-1,1]$ is clear.

Theorem II extends Pollard's result on the almost everywhere convergence of Legendre series, while Theorem III is the analogue for Legendre series of Kolmogorov's construction of an everywhere divergent trigonometric Fourier series. Theorem IV is related to the work of D. I. Cartwright on the Lebesgue constants associated to Jacobi series. This last theorem also gives a different and, we believe, very simple proof of Chanillo's result on the restricted weak behaviour of Legendre expansions at the critical indexes.

We remark that since the Lorentz spaces are interpolation spaces between L^p-spaces, and vice versa, Theorems I, II, III, IV, imply some of the main results on the convergence and divergence of Legendre expansions on L^p-spaces. Also, it is easy to extend these theorems to more general expansions in Jacobi polynomials.

2. Lagrange interpolation at the zeros of Legendre polynomials

Let $\{x_{k,n}\}$ be the zeros of $p_n(x)$ ordered by $1 > x_{1,n} > x_{2,n} > \ldots > > x_{n,n} > -1$, and let $\{\ell_{k,n}\}$ be the fundamental polynomials of the Lagrange interpolation at the points $\{x_{k,n}\}$,

$$\ell_{k,n}(x) = \frac{p_n(x)}{p_n'(x_{k,n})(x-x_{k,n})}.$$

The n-th Lagrange polynomial which interpolates a function f on $[-1,1]$ at the points $\{x_{k,n}\}$ is defined by

$$L_n f(x) = \sum_{k=1}^{n} f(x_{k,n})\ell_{k,n}(x).$$

We note that a natural domain for the definition of the operators $\{L_n\}$ is the space $C[-1,1]$ of continuous functions on $[-1,1]$ with the norm $\|\cdot\|_\infty$.

S. Bernstein, G. Grünwald and J. Marcinkiewicz have constructed continuous functions f on $[-1,1]$ with sequences of interpolating polynomials diverging everywhere. However R. Askey, extending previous results of P. Erdős and P. Turán, has shown that if f is continuous on $[-1,1]$ and $p < 4$, then

$$\lim_{n\to+\infty} \int_{-1}^{1} |L_n f(x) - f(x)|^p dx = 0.$$

P. Nevai has shown that this fails if $p = 4$. See Askey [1], Nevai [13], Erdős–Vértesi [7], and also Szegő [19] and Zygmund [23] for general references.

Our contribution is a proof of the fact that at the critical index $p = 4$ the operators $\{L_n\}$ map $C[-1,1]$ into $L^{4,\infty}[-1,1]$ uniformly. More precisely we have the following result.

THEOREM V. *Let* $1 \leqq s \leqq +\infty$. *Then*

$$\sup\left\{\frac{\|L_n f\|_{4,s}}{\|f\|_\infty}\right\} \simeq c[\log(2+n)]^{1/s}.$$

In particular, for every function f continuous on $[-1,1]$, $\{L_n f\}$ converges to f in the norm of $L^{p,\infty}[-1,1]$, while if $1 \leqq s < +\infty$ there exist functions f continuous on $[-1,1]$ with $\left\{\|L_n f\|_{4,s}\right\}$ unbounded.

Again, it is possible to prove a similar theorem for Lagrange interpolation at the zeros of Jacobi polynomials. In this paper we choose to consider only the case of Legendre polynomials in order to simplify some of the formulas and make the reading easier.

3. Proof of the theorems

By the inequalities 7.21.1 and 7.3.8 of Szegő [19], if $-1 < x < 1$,

$$(1) \qquad |p_n(x)| \leqq \begin{cases} \sqrt{n+\frac{1}{2}}, \\ \sqrt{\frac{2}{\pi}+\frac{1}{\pi n}}(1-x^2)^{-1/4}, \end{cases}$$

and also, by the asymptotic formula 8.21.18 of Szegő [19],

$$(2)\quad p_n(\cos\vartheta) = \sqrt{\frac{2}{\pi}}|\sin\vartheta|^{-1/2}\cos\Big(\Big(n+\frac{1}{2}\Big)\vartheta - \frac{\pi}{4}\Big) + O(n^{-1}|\sin\vartheta|^{-3/2}).$$

This asymptotic formula has, as a consequence, the following equiconvergence theorem between Legendre and cosine expansions (Szegő [19], Theorem 9.1.12):

Let the function $(1-\cos^2\vartheta)^{1/4} f(\cos\vartheta)$ be integrable on $[0,\pi]$, and denote by $s_n f(\cos\vartheta)$ the n-th partial sum of its Fourier cosine expansion. Then, for $-1 < x < 1$,

$$\lim_{n\to+\infty}\left\{S_n f(x) - (1-x^2)^{-1/4} s_n f(x)\right\} = 0,$$

and, if $0 < \varepsilon < 1$, the convergence is uniform in every interval $\varepsilon - 1 \leqq x \leqq \leqq 1 - \varepsilon$.

Proof of Theorem I. Since

$$\int_{-1}^{1}(1-x^2)^{-1/4}|f(x)|dx = \int_0^{\pi}|1-\cos^2\vartheta|^{1/4}|f(\cos\vartheta)|d\vartheta,$$

the convergence in measure of $\{S_n f\}$ is an easy consequence of the Haar–Szegő equiconvergence theorem and Kolmogorov result on the weak behaviour of cosine expansions.

If the sequence $\{\hat{f}(n)\}$ is unbounded, the divergence in measure of $\{S_n f\}$ follows easily from the asymptotic formula (2) for Legendre polynomials, and the almost everywhere divergence follows from the asymptotic formula for Legendre polynomials and the Cantor–Lebesgue theorem. See 9.1.2 of Zygmund [23], and Meaney [12]. □

To apply the Haar–Szegő equiconvergence theorem to the problem of convergence of Legendre expansions of functions in Lorentz spaces we need the following lemma which is an immediate consequence of the duality between the spaces $L^{4/3,1}[-1,1]$ and $L^{4,\infty}[-1,1]$.

Lemma 1. *Let f be a function in $L^{4/3,1}[-1,1]$. Then the function $|1-\cos^2\vartheta|^{1/4} f(\cos\vartheta)$ is in $L^1[0,\pi]$, and, if $0 < \varepsilon < 1$,*

$$\chi_{[\varepsilon-1,1-\varepsilon]}(\cos\vartheta)|1-\cos^2\vartheta|^{1/4} f(\cos\vartheta) \quad \textit{is in} \quad L^{4/3}[0,\pi].$$

Proof of Theorem II. By the Haar–Szegő equiconvergence theorem $\{S_n f(x)\}$ converges provided the n-th partial sums of the cosine Fourier series of $|\sin\vartheta|^{1/2} f(\cos\vartheta)$ converge in $\cos\vartheta = x$. By the Riemann localization principle, if $|\cos\vartheta| < 1-\varepsilon$ the cosine Fourier series of $|\sin\vartheta|^{1/2} f(\cos\vartheta)$ is equiconvergent with the cosine Fourier series of

$$\chi_{[\varepsilon-1,1-\varepsilon]}(\cos\vartheta)|\sin\vartheta|^{1/2}\cdot f(\cos\vartheta).$$

But by the previous lemma this function is in $L^{4/3}[0,\pi]$ if f is in $L^{4/3,1}[-1,1]$, and by the Carleson–Hunt theorem its cosine Fourier series converges almost everywhere. □

The following lemmas are easy consequences of the asymptotic formulas for Legendre polynomials, and of the duality between the Lorentz spaces $L^{p,r}[-1,1]$ and $L^{q,s}[-1,1]$ if $1/p+1/q = 1/r+1/s = 1$.

LEMMA 2. $\|p_n\|_{4,s} \simeq c[\log(2+n)]^{1/s}$.

LEMMA 3. $|\hat f(n)| \leqq c[\log(2+n)]^{1-\frac{1}{r}}\|f\|_{4/3,r}$. *In particular, the Fourier–Legendre transform* $\{\hat f(n)\}$ *of a function* f *in* $L^{4/3,1}[-1,1]$ *converges to* 0, *while, if* $r>1$ *there exist functions* f *in* $L^{4/3,r}[-1,1]$ *with* $\{\hat f(n)\}$ *unbounded.*

Let $\{p_n^{(1,1)}\}$ be the system of Jacobi polynomials orthonormal on $[-1,1]$ with respect to the measure $(1-x^2)dx$. For this system we have estimates analogous to (1) and (2). In particular, if $-1<x<1$, then

$$p_n^{(1,1)}(x) = \begin{cases} O(n^{3/2}), \\ A_n(x)(1-x^2)^{-3/4}, \end{cases} \tag{3}$$

with A_n bounded and oscillating. See 7.32.5 and 8.21.18 of Szegő [19].

Define $q_n(x) = (1-x^2)p_n^{(1,1)}(x)$. Then, by formula 4.5.5 of Szegő [19],

$$q_n(x) = \sqrt{\frac{(n+1)(n+2)}{(2n+1)(2n+3)}}p_n(x) - \sqrt{\frac{(n+1)(n+2)}{(2n+3)(2n+5)}}p_{n+2}(x).$$

LEMMA 4. $|\hat f(n+2) - \hat f(n)| \leqq c\|f\|_{4/3,r}$.

PROOF. It is enough to observe that

$$\int_{-1}^{1} f(x)q_n(x)dx = \sqrt{\frac{(n+1)(n+2)}{(2n+1)(2n+3)}}\hat f(n) - \sqrt{\frac{(n+1)(n+2)}{(2n+3)(2n+5)}}\hat f(n+2),$$

and that the functions $\{q_n\}$ are uniformly bounded. □

LEMMA 5. *Let* $0<\varepsilon<\pi/4$. *Then for every* n *and every* ϑ *with* $\varepsilon<\vartheta< <\frac{\pi}{2}-\varepsilon$ *one has*

$$|p_n(\cos\vartheta)| + |p_{n+2}(\cos\vartheta)| > c > 0.$$

PROOF OF THEOREM IV. We shall prove the theorem for functions f in $L^{4,r}[-1,1]$ and the proof for functions in the dual space will follow. By Pollard's formula,

$$S_nf(x)=\alpha_n\int_{-1}^{1}p_{n+1}(x)q_n(y)\frac{1-\left[\frac{1-x^2}{1-y^2}\right]^{1/4}}{x-y}f(y)dy+$$

$$+\alpha_n\int_{-1}^{1}\frac{(1-x^2)^{1/4}p_{n+1}(x)(1-y^2)^{-1/4}q_n(y)f(y)}{x-y}dy-$$

$$-\alpha_n\int_{-1}^{1}\frac{q_n(x)p_{n+1}(y)f(y)}{x-y}dy+\beta_n\int_{-1}^{1}p_{n+1}(x)p_{n+1}(y)f(y)dy=$$

$$=\alpha_nK_nf(x)+\alpha_n(1-x^2)^{1/4}p_{n+1}(x)H[(1-y^2)^{-1/4}q_nf](x)-$$
$$-\alpha_nq_n(x)H[p_{n+1}f](x)+\beta_n\hat{f}(n+1)p_{n+1}(x),$$

where H denotes the Hilbert transform and K_n is the operator with kernel

$$K_n(x,y)=p_{n+1}(x)q_n(y)\frac{1-\left[\frac{1-x^2}{1-y^2}\right]^{1/4}}{x-y}=\frac{(x+y)p_{n+1}(x)(1-y^2)^{-1}q_n(y)}{\left(\left[\frac{1-x^2}{1-y^2}\right]^{1/4}+1\right)\left(\left[\frac{1-x^2}{1-y^2}\right]^{1/2}+1\right)}.$$

To obtain this expression for K_n multiply and divide by $(x+y)$ and factor $(1-y^2)$ from the denominator.

By Lemma 2,

$$\|\hat{f}(n+1)p_{n+1}\|_{4,s}\leqq c[\log(2+n)]^{1/s}\|f\|_{4,r},$$

and since by (1) and (3) the functions $(1-x^2)^{1/4}p_{n+1}(x)$ and $(1-y^2)^{-1/4}q_n(y)$ are bounded, by the boundedness of the Hilbert transform on $L^{4,s}[-1,1]$ we have

$$\left\|(1-x^2)^{1/4}p_{n+1}H[(1-y^2)^{-1/4}q_nf]\right\|_{4,s}\leqq c\|f\|_{4,s}.$$

LEMMA 6. *The operator* $f(x)\to(1-x^2)^{1/4}H[(1-y^2)^{-1/4}f](x)$ *is bounded on every* $L^{p,s}[-1,1]$, $4/3<p<+\infty$, $1\leqq s\leqq+\infty$.

PROOF. The result for the spaces $L^p[-1,1]$ is an extension of the M. Riesz theorem due to G. H. Hardy and J. E. Littlewood, but it also follows from the fact that for $p/4<p-1$ the weight $(1-x^2)^{p/4}$ is in the Muckenhoupt class A_p (see Torchinsky [20], 9.4.4). The result for Lorentz spaces follows by interpolation. □

The constant c depends on ε but not on n and ϑ.

PROOF. Denote by $\vartheta_{k,n}$ the k-th zero of $p_n(\cos\vartheta)$, and assume t $\varepsilon/2 < \vartheta_{k,n} < \pi/2 - \varepsilon/2$ and $\varepsilon < \vartheta < \pi/2 - \varepsilon$. From the asymptotic form for $p_n(\cos\vartheta)$ it follows that $\vartheta_{k,n} = \frac{k-1/4}{n+1/2}\pi + O(n^{-2})$. Also, for some consta a and b, $|p_n(\cos\vartheta)| > a\varepsilon$ if $|\vartheta - \vartheta_{k,n}| > b\varepsilon/n$ for every k. (See also Trico [21].) To prove the lemma it is thus enough to check that every ϑ in interval $[\varepsilon, \pi/2 - \varepsilon]$ is at a distance at least $c\varepsilon/n$ from the set $\{\vartheta_{k,n}\}$ or fr the set $\{\vartheta_{k,n+2}\}$. □

PROOF OF THEOREM III. Let $n_k = 2^{2^k}$. By Lemmas 3 and 4, if r there exist functions g_k with $\|g_k\|_{4/3,r} \leqq c$ and with $\hat{g}_k(n_k) > (\log n_k)^{1-}$ and $\hat{g}_k(n_k+2) > (\log n_k)^{1-1/r}$. By cutting the Fourier–Legendre transfo of g_k with suitable smooth multipliers we may also assume that $\hat{g}_k(j) =$ $0 \leqq j \leqq 2^{-1}n_k$ or $j \geqq 2n_k$.

Define $f(x) = \sum_{k=1}^{+\infty} k^{-2} g_k(x)$. This function is in $L^{4/3,r}[-1,1]$ and b sequences $\{\hat{f}(n_k)\}$ and $\{\hat{f}(n_k+2)\}$ diverge to $+\infty$. Using Lemma 5 it is t easy to see that for every x with $0 < x < 1$ the sequence $\{|\hat{f}(n_k)p_{n_k}(x)$ $+ |\hat{f}(n_k+2)p_{n_k+2}(x)|\}$ is unbounded. Since the polynomials p_{n_k} and p_n are even, the same is true for every x with $-1 < x < 0$. Since $p_{n_k}(1)$ $= p_{n_k}(-1) = (n_k + 1/2)^{1/2}$ and $p_{n_k}(0) = (2/\pi)^{1/2}$, the above sequenc unbounded even for $x = -1, 0, 1$.

We have thus proved that for every x in $[-1,1]$ the sequence $\{\hat{f}(n)p_n($ is unbounded, so that the series $\sum_{n=0}^{+\infty} \hat{f}(n)p_n(x)$ cannot converge. □

The point of departure for the proof of Theorem IV is the follow rearrangement of the Christoffel–Darboux formula, which is due to H. Poll (§9 of Pollard [15] or §2 of Pollard [16]):

$$S_nf(x) = \frac{n+1}{\sqrt{2n+1}\sqrt{2n+3}} \int_{-1}^{1} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} f(y)dy =$$

$$= \alpha_n p_{n+1}(x) \int_{-1}^{1} \frac{q_n(y)f(y)}{x-y}dy - \alpha_n q_n(x) \int_{-1}^{1} \frac{p_{n+1}(y)f(y)}{x-y}dy +$$

$$+\beta_n p_{n+1}(x) \int_{-1}^{1} p_{n+1}(y)f(y)dy,$$

with $\{\alpha_n\} \to 1/2$, $\{\beta_n\} \to -1/2$, and as before $q_n(x) = (1-x^2)p_n^{(1,1)}(x)$.

From (1), (3), and this lemma, we immediately obtain

$$\|q_n H[p_{n+1} f]\|_{4,s} \leqq c\|f\|_{4,s}.$$

Finally, by the duality between $L^{4/3,t}[-1,1]$ and $L^{4,r}[-1,1]$ if $1/t+1/r=1$, and since $|K_n(x,y)| \leqq 2|p_{n+1}(x)||(1-y^2)^{-1} q_n(y)|$,

$$\left|\int_{-1}^{1} K_n(x,y) f(y)dy\right| \leqq c|p_{n+1}(x)| \left\|(1-y^2)^{-1} q_n\right\|_{4/3,t} \|f\|_{4,r}.$$

Then, by Lemma 2 and the analogous statement for the system $\{p_n^{(1,1)}\}$,

$$\|K_n f\|_{4,s} \leqq c\|p_{n+1}\|_{4,s}\|(1-y^2)^{-1} q_n\|_{4/3,t}\|f\|_{4,r} \leqq$$

$$\leqq c[\log(2+n)]^{1-\frac{1}{r}+\frac{1}{s}}\|f\|_{4,r}.$$

If $r \leqq s$, then $\|f\|_{4,s} \leqq c\|f\|_{4,r}$, so that collecting all these estimates we obtain the desired upper bound for the norm of S_n as operator from $L^{4,r}[-1,1]$ into $L^{4,s}[-1,1]$ in the case $1 \leqq r \leqq s \leqq +\infty$.

In the case $1 \leqq s < r \leqq +\infty$ we argue as follows. A moment's reflection shows that it is enough to prove the inequality

$$\|S_n g_{2n}\|_{4,s} \leqq c[\log(2+n)]^{1-\frac{1}{r}+\frac{1}{s}}\|g_{2n}\|_{4,r}$$

for every polynomial g_{2n} of degree $2n$. Indeed, using a variant of the delayed means of de la Vallée-Poussin, for every function f in $L^{4,r}[-1,1]$ it is possible to construct polynomials g_{2n} of degree at most $2n$ with $\hat{f}(k) = \hat{g}_{2n}(k)$ if $0 \leqq k \leqq n$, and $\|g_{2n}\|_{4,r} \leqq c\|f\|_{4,r}$. (See e.g. Zygmund [22] or Askey [2].)

LEMMA 7. *Let $1 \leqq s < r \leqq +\infty$. Then for every polynomial g_n of degree n we have*

$$\|g_n\|_{4,s} \leqq c[\log(2+n)]^{\frac{1}{s}-\frac{1}{r}}\|g_n\|_{4,r}.$$

PROOF. $|g_n(x)| \leqq \sum_{k=0}^{n} |\hat{g}_n(k)|\sqrt{k+\frac{1}{2}} \leqq \frac{n+1}{\sqrt{2}}\left\{\sum_{k=0}^{n} |\hat{g}_n(k)|^2\right\}^{1/2}$. Hence

$$\|g_n\|_{4,s} = \left\{\frac{s}{4}\int_0^2 \left[t^{1/4} g_n^*(t)\right]^s \frac{dt}{t}\right\}^{1/s} \leqq c\left\{(1+n)^s \int_0^{(1+n)^{-4}} t^{s/4}\frac{dt}{t}\right\}^{1/s} \|g_n\|_{4,r} +$$

$$+c\left\{\int_{(1+n)^{-4}}^{2} \frac{dt}{t}\right\}^{1/s-1/r}\left\{\int_{(1+n)^{-4}}^{2} [t^{1/4} g_n^*(t)]^r \frac{dt}{t}\right\}^{1/r} \leqq$$

$$\leqq c[\log(2+n)]^{\frac{1}{s}-\frac{1}{r}}\|g_n\|_{4,r}. \qquad \square$$

We have already proved that the norm of S_n on $L^{4,s}[-1,1]$ is at most $c\log(2+n)$, so that if $s < r$ using the above lemma we obtain

$$\|S_n g_{2n}\|_{4,s} \leqq c\log(2+n)\|g_{2n}\|_{4,s} \leqq c[\log(2+n)]^{1-\frac{1}{r}+\frac{1}{s}}\|g_{2n}\|_{4,r}.$$

To obtain a lower bound for the norm of S_n as operator from $L^{4,r}[-1,1]$ into $L^{4,s}[-1,1]$ we use as a test function

$$f(y) = \begin{cases} q_n(y)(1-y)^{-1/2} & \text{if } 0 < y < 1, \\ 0 & \text{if } -1 < y \leqq 0. \end{cases}$$

It is easy to see that $\|f\|_{4,s} \simeq c[\log(2+n)]^{1/s}$, and

$$\|\hat{f}(n+1)p_{n+1}\|_{4,s} + \|q_n H[p_{n+1}f]\|_{4,s} +$$

$$+\left\|(1-x^2)^{1/4}p_{n+1}H[(1-y^2)^{-1/4}q_n f]\right\|_{4,s} \leqq c[\log(2+n)]^{1/s}.$$

Also, if $1/2 < x < 1$,

$$|K_n f(x)| \geqq c|p_{n+1}(x)|\int_0^x (1-y)^{-1}dy \geqq c\log(1-x)^{-1}|p_{n+1}(x)|,$$

so that $\|K_n f\|_{4,s} \geqq c[\log(2+n)]^{1+1/s}$.

Since $\|f\|_{4,r} \simeq c[\log(2+n)]^{1/r}$, these estimates imply that

$$\|S_n f\|_{4,s} \geqq c[\log(2+n)]^{1-\frac{1}{r}+\frac{1}{s}}\|f\|_{4,r}. \qquad \square$$

Let $\{\lambda_{k,n}\}$ be the Christoffel numbers of the Gauss quadrature at the points $\{x_{k,n}\}$. These are positive numbers defined by

$$\int_{-1}^{1} g_{2n-1}(x)dx = \sum_{k=1}^{n} g_{2n-1}(x_{k,n})\lambda_{k,n}$$

for every polynomial g_{2n-1} of degree at most $2n-1$.

Using the positivity of the Cesàro means of order two and a variant of the delayed means of de la Vallée-Poussin, A. Zygmund has shown that if $1 \leqq p \leqq +\infty$ and if g_n is a polynomial of degree n, then

$$\left\{\sum_{k=1}^{n} |g_n(x_{k,n})|^p \lambda_{k,n}\right\}^{1/p} \leqq c\left\{\int_{-1}^{1} |g_n(x)|^p dx\right\}^{1/p},$$

where c is an absolute constant. See Zygmund [22] and Askey [1,2].

Theorem V is an easy consequence of this result and Theorem IV.

PROOF OF THEOREM V. Let f be in $C[-1,1]$ and g be in $L^{4/3,1}[-1,1]$. Then,

$$\left|\int_{-1}^{1} L_n f(x) g(x) dx\right| = \left|\int_{-1}^{1} L_n f(x) S_{n-1} g(x) dx\right| =$$

$$= \left|\sum_{k=1}^{n} f(x_{k,n}) S_{n-1} g(x_{k,n}) \lambda_{k,n}\right| \leqq \|f\|_\infty \sum_{k=1}^{n} |S_{n-1} g(x_{k,n})| \lambda_{k,n} \leqq$$

$$\leqq c\|f\|_\infty \int_{-1}^{1} |S_{n-1} g(y)| dy \leqq c\|f\|_\infty \|S_{n-1} g\|_{4/3,\infty} \leqq c\|f\|_\infty \|g\|_{4/3,1}.$$

We have thus proved that $\|L_n f\|_{4,\infty} \leqq c\|f\|_\infty$. Since $L_n f$ is a polynomial of degree at most $n-1$, by Lemma 7 we also have

$$\|L_n f\|_{4,s} \leqq c[\log(2+n)]^{1/s} \|L_n f\|_{4,\infty} \leqq c[\log(2+n)]^{1/s} \|f\|_\infty.$$

To estimate the norm of L_n from below we need a continuous function f with norm one, and such that

$$f(x_{k,n}) = \begin{cases} \frac{p_n'(x_{k,n})}{|p_n'(x_{k,n})|} & \text{if } 1 \leqq k \leqq [n/2], \\ 0 & \text{if } [n/2] < k \leqq n. \end{cases}$$

Then, if $-1 \leqq x \leqq -1/2$,

$$|L_n f(x)| = \left|\sum_{k=1}^{[n/2]} \frac{p_n(x)}{|p_n'(x_{k,n})|(x - x_{k,n})}\right| \geqq \frac{1}{2} |p_n(x)| \sum_{k=1}^{[n/2]} |p_n'(x_{k,n})|^{-1}.$$

But, by Szegő [19] 8.9.2, $|p_n'(x_{k,n})|^{-1} \simeq ck^{3/2} n^{-5/2}$, so that, if $-1 \leqq x \leqq -1/2$, then $|L_n f(x)| \geqq c|p_n(x)|$. This estimate and the asymptotic formula for Legendre polynomials imply that

$$\|L_n f\|_{4,s} \geqq c[\log(2+n)]^{1/s} \|f\|_\infty. \qquad \square$$

4. Concluding remarks

The first observation is that if a function f in $L^p[-1,1]$, $p > 1$, is supported away from the points $+1$ and -1, then its Legendre–Fourier series converges almost everywhere. However it is possible to construct functions f in $L^{4/3,r}[-1,1]$, $r > 1$, which are supported in an interval $[1-\varepsilon, 1]$ with ε arbitrarily small, but with Legendre–Fourier series diverging everywhere. Therefore for the operators $\{S_n\}$ there is no localization. For the operators $\{L_n\}$ a sort of localization holds only away from the points $+1$ and -1.

The second observation is that using the family $\{q_n\}$ as test functions it is possible to prove a stronger version of the divergence results in Theorem IV.

THEOREM VI. *Let $E_0 \subset E_1 \subset E_2 \dots$ be an increasing sequence of subsets of the integers, and define*

$$S_{E_n}f(x) = \sum_{k \in E_n} \hat{f}(k)p_k(x).$$

Then there exist functions f in $L^{4/3,r}[-1,1]$, $1 < r \leqq +\infty$, with $\left\{\|S_{E_n}f\|_{4/3,\infty}\right\}$ unbounded. Similarly, if $1 \leqq s < +\infty$ then there exist functions f in $L^{4,1}[-1,1]$ with $\left\{\|S_{E_n}f\|_{4,s}\right\}$ unbounded.

PROOF. If $k \in E_n$ but $k+2 \notin E_n$, then

$$\|S_{E_n}q_k\|_{4,s} = \left\| \sqrt{\frac{(k+1)(k+2)}{(2k+1)(2k+3)}}p_k \right\|_{4,s} \geqq c[\log(2+k)]^{1/s}\|q_k\|_{4,r}. \quad \square$$

The above theorem is essentially contained in Giulini–Soardi–Travaglini [8] or Dreseler–Soardi [6]. Note that we make no assumptions on the ordering and the cardinality of the sets $\{E_n\}$. Since the system of Legendre polynomials is an unconditional basis only for the space $L^2[-1,1]$, it is clear that without extra assumptions on these sets we cannot expect the operators $\{S_{E_n}\}$ to be of restricted weak type.

The last observation is that for functions with some smoothness we can improve the convergence result in Theorem V.

Let $C^\alpha[-1,1]$, $0 < \alpha < 1$, be the subspace of $C[-1,1]$ defined by the norm

$$\|f\|_{C^\alpha} = \|f\|_\infty + \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : -1 \leqq x,\ y \leqq 1\right\},$$

and let $C_0^\alpha[-1,1]$ be the closure in $C^\alpha[-1,1]$ of the set of polynomials.

The following theorem holds.

THEOREM VII. *For every function f in the Lipschitz space $C_0^\alpha[-1,1]$, $0 < \alpha \leqq 1/4$, we have*

$$\lim_{n \to +\infty} \|L_nf - f\|_{4/(1-2\alpha)} = 0.$$

If $0 < \alpha < 1/4$ and $p > 4/(1-2\alpha)$ then there exist functions f in $C_0^\alpha[-1,1]$ with $\{\|L_nf\|_p\}$ unbounded.

PROOF. By Jackson's theorem if f is in $C_0^\alpha[-1,1]$ then there exist polynomials g_n of degree at most n such that

$$\lim_{n \to +\infty} n^\alpha \|f - g_n\|_\infty = 0.$$

(See e.g. Lorentz [11].) On the other hand, by computations similar to those in the proof of Theorems IV and V, if f is a bounded function and $4 < p \leqq +\infty$, then

$$\|L_n f\|_p \leqq cn^{1/2-2/p}\|f\|_\infty.$$

Hence

$$\lim_{n\to+\infty} \|L_n f - f\|_{4/(1-2\alpha)} \leqq$$

$$\leqq \limsup_{n\to+\infty} \left(\|L_n[f-g_{n-1}]\|_{4/(1-2\alpha)} + \|g_{n-1} - f\|_{4/(1-2\alpha)}\right) \leqq$$

$$\leqq c \limsup_{n\to+\infty} n^\alpha \|f - g_{n-1}\|_\infty = 0.$$

Test functions similar to those used in the proof of Theorem V show that the norms of the operators $\{L_n\}$ from $C_0^\alpha[-1,1]$ into $L^p[-1,1]$ are not uniformly bounded if $p > 4/(1-2\alpha)$. □

The case $\alpha = 1/2$ and $p = +\infty$ of the above theorem is contained in Theorem 14.4 of Szegő [19].

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ON THE CONTROL OF A CIRCULAR MEMBRANE. I

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Let $\Omega := \{(x,y) : x^2 + y^2 < 1\} \subset \mathbf{R}^2$ be the unit circle, and take some (different) points $P_1, \dots, P_N \in \Omega$, $S_{N+1}, \dots S_M \in \partial\Omega$. Consider the following system for $u = u(t,x,y)$:

$$u_{tt} = \Delta u + \sum_{j=1}^{N} \delta\left((x,y) - P_j\right) v_j, \tag{1}$$

$$\left.\frac{\partial u}{\partial r}\right|_{\partial\Omega\times(0,T)} = \sum_{j=N+1}^{M} \delta(s - S_j) v_j, \qquad s \in \partial\Omega, \tag{2}$$

$$u(0,.,.) = u_t(0,.,.) = 0 \tag{3}$$

with controls $v_j(t) \in L^2(0,T)$.

We shall investigate the approximative controllability of the system (1)–(3) describing the control of the circular membrane in the points P_j ($j = 1, \dots, N$), S_j ($j = N+1, \dots, M$). First we give an outline of the related results (known for us). The analogous control problem when a rectangular membrane is controlled in one side of the rectangle, is investigated in [4]. The independence of the movement of a membrane in finitely many different points is proved for the rectangular membrane in [5] and [6]. In [6] the author of the present paper used Lemma 7 below which was unnecessarily strong because we need for that problem only the completeness of the exponentials. The "real" application of Lemma 7 would be a control problem; this motivated the author in finding the problem of the present note. For the circular membrane the problem is solved partly in [7], the complete solution is given in [8]. We shall use some ideas of [4] and [8] here.

Introduce the eigenfunctions

$$-\Delta\varphi = \lambda^2\varphi, \qquad \frac{\partial\varphi}{\partial r} = 0 \quad \text{on} \quad \partial\Omega \tag{4}$$

corresponding to (1)–(3). For fixed $m = 1, 2, \dots$ denote $0 < \lambda_1^{(m)} < \lambda_2^{(m)} < \dots$ the zeros of J'_m, where J_m is the Bessel function of order m. It is known [1] that if we take polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, then [4] has the solutions

$$\varphi^{+}_{m,n}(x,y) := J_m\left(\lambda^{(m)}_n r\right)\cos m\varphi, \quad \varphi^{-}_{m,n}(x,y) := J_m\left(\lambda^{(m)}_n r\right)\sin m\varphi,$$
$$(m = 0,1,\dots; \quad n = 1,2,\dots)$$

and $\lambda^{\pm}_{m,n} := \lambda^{(m)}_n$.

It is known [1] that the eigenfunctions $1, \varphi^{+}_{0,n}, \varphi^{\pm}_{m,n}$ give a complete orthogonal system in the weighted space $L^2_r((0,1)\times(0,2\pi))$ and

$$(4') \qquad \left\|\varphi^{\pm}_{m,n}\right\|^2 = \pi\int_0^1 \left|J_m\left(\lambda^{(m)}_n r\right)\right|^2 r\,dr = \pi\frac{\left|J_m(\lambda^{(m)}_n)\right|^2}{2}\left(1-\frac{m^2}{\lambda^{(m)2}_n}\right);$$

$$\left\|\varphi^{\pm}_{0,n}\right\|^2 = 2\pi\int_0^1 \left|J_0(\lambda^{(0)}_n r)\right|^2 r\,dr = \pi\left|J_0(\lambda^{(0)}_n r)\right|^2;$$

this follows from [10] 7.14.1 (10) and 7.2.8 (56)–(57). Next we refine the meaning of (1)–(3). Consider a function $z(t,x,y)$ with the properties

$$z(T,\cdot,\cdot) = z_t(T,\cdot,\cdot) = 0, \qquad \left.\frac{\partial z}{\partial r}\right|_{\partial\Omega\times(0,T)} = 0.$$

Taking a formal twofold integration by parts in $\int_0^T\int_\Omega uz_{tt}$ and a formal application of Green's formula in $\int_0^T\int_\Omega u\Delta z$ we get

$$(5) \qquad \int_0^T\int_\Omega u(z_{tt}-\Delta z) = \int_0^T\int_\Omega z(u_{tt}-\Delta u) + \int_0^T\sum_{j=N+1}^{M} z(.,S_j)v_j =$$
$$= \int_0^T\left(\sum_{j=1}^{N} z(\cdot,P_j)v_j + \sum_{j=N+1}^{M} z(\cdot,S_j)v_j\right).$$

We ask for the solution u in the form

$$u(t,x,y) = c_0(t) + \sum_{n=1}^{\infty} c^{+}_{0,n}(t)\varphi^{+}_{0,n}(x,y) + \sum c^{\pm}_{m,n}(t)\varphi^{\pm}_{m,n}(x,y);$$

then we have

$$c_0(t) = \frac{1}{\pi}\int_\Omega u, \quad c^{\pm}_{m,n}(t) = \frac{1}{\gamma_{m,n}}\int_\Omega u\varphi^{\pm}_{m,n}, \quad \gamma_{m,n} = \left\|\varphi^{\pm}_{m,n}\right\|^2.$$

Apply (5) to the function

$$z(t,x,y)=\varphi^{\pm}_{m,n}(x,y)b(t)$$

where $b\in C^2[0,T]$, $b(T)=b'(T)=0$. We get

$$\gamma_{m,n}\int_0^T\left(b''+\left(\lambda_n^{(m)}\right)^2 b\right)c^{\pm}_{m,n}=\int_0^T bh^{\pm}_{m,n},$$

$$h^{\pm}_{m,n}(t):=\sum_{j=1}^{N}\varphi^{\pm}_{m,n}(P_j)v_j(t)+\sum_{j=N+1}^{M}\varphi^{\pm}_{m,n}(S_j)v_j(t)$$

i.e.

$$c^{\pm}_{m,n}(t):=\frac{1}{\gamma_{m,n}}\int_0^t\frac{\sin\lambda_n^{(m)}(t-\tau)}{\lambda_n^{(m)}}h^{\pm}_{m,n}(\tau)\,d\tau \tag{6}$$

and analogously $c_0(t)=\frac{1}{\pi}\int_0^t(t-\tau)\sum_{j=1}^M v_j(\tau)d\tau$. Introduce the notation

$$e^{\pm}_{m,n}:=\begin{pmatrix}\varphi^{\pm}_{m,n}(P_1)\\ \vdots\\ \varphi^{\pm}_{m,n}(P_N)\\ \varphi^{\pm}_{m,n}(S_{N+1})\\ \vdots\\ \varphi^{\pm}_{m,n}(S_M)\end{pmatrix},\qquad v(t):=\begin{pmatrix}v_1(t)\\ \vdots\\ v_M(t)\end{pmatrix}.$$

Then $h^{\pm}_{m,n}=\langle v,e^{\pm}_{m,n}\rangle$. Define the spaces

$$W_r:=\Bigg\{f=c_0+\sum_{n=1}^{\infty}c^{+}_{0,n}\varphi^{+}_{0,n}+\sum c^{\pm}_{m,n}\varphi^{\pm}_{m,n}:\|f\|^2_{W_r}:=$$
$$=|c_0|^2+\sideset{}{^*}\sum_{\substack{m\geqq 0\\ n\geqq 1}}\left|\frac{\gamma_{m,n}}{e^{\pm}_{m,n}}c^{\pm}_{m,n}\right|^2\left|\lambda_n^{(m)}\right|^{2r}<\infty\Bigg\}$$

and let $\mathcal{H}_r:=W_{r+1}\oplus W_r$. Here $\sum^*$ is the sum over all indices m,n satisfying $|e^{\pm}_{m,n}|\neq 0$. Denote further

$$\xi^{\pm}_{m,n}(t):=i\lambda_n^{(m)}c^{\pm}_{m,n}(t)+c^{\pm\prime}_{m,n}(t),\quad \xi^{\pm}_{m,-n}(t):=-i\lambda_n^{(m)}c^{\pm}_{m,n}(t)+c^{\pm\prime}_{m,n}(t),$$

then

$$\xi^{\pm}_{m,n}(t) = \frac{1}{\gamma_{m,n}} \int_0^t e^{i\lambda_n^{(m)}(t-\tau)} \left\langle v(\tau), e^{\pm}_{m,n} \right\rangle d\tau,$$

$$\xi^{\pm}_{m,-n}(t) = \frac{1}{\gamma_{m,n}} \int_0^t e^{-i\lambda_n^{(m)}(t-\tau)} \left\langle v(\tau), e^{\pm}_{m,n} \right\rangle d\tau.$$

If we introduce the notations

$$\mathbf{K} := \mathbf{Z} \setminus \{0\}, \quad \omega_{m,k} := \lambda^m_{|k|} \operatorname{sgn} k \qquad (m \in \mathbf{N},\ k \in \mathbf{K}),$$

we can give a unified form

$$\xi^{\pm}_{m,k}(t) := \frac{\left|e^{\pm}_{m,n}\right|}{\gamma_{m,n}} \int_0^t e^{i\omega_{m,k}(t-\tau)} \left\langle v(\tau), \frac{e^{\pm}_{m,n}}{\left|e^{\pm}_{m,n}\right|} \right\rangle d\tau \tag{7}$$

$$(m \in \mathbf{N}, \quad k \in \mathbf{K}, \quad n = |k|).$$

We need the following simple

LEMMA 1. *Let*

$$f = a_0 + \sum_{n=1}^{\infty} a^{+}_{0,n} \varphi^{+}_{0,n} + \sum a^{\pm}_{m,n} \varphi^{\pm}_{m,n},$$

$$g = b_0 + \sum_{n=1}^{\infty} b^{+}_{0,n} \varphi^{+}_{0,n} + \sum b^{\pm}_{m,n} \varphi^{\pm}_{m,n},$$

$$\xi^{\pm}_{m,k} := i\omega_{m,k} a^{\pm}_{m,n} + b^{\pm}_{m,n}, \qquad n := |k|.$$

Then the mapping

$$\{f, g\} \to \left\{ a_0, b_0, |\omega_{0,k}|^r \xi^{+}_{0,k} \frac{\gamma_{0,n}}{|e^{+}_{0,n}|}, |\omega_{m,k}|^r \xi^{\pm}_{m,k} \frac{\gamma_{m,n}}{|e^{\pm}_{m,n}|} : m \in \mathbf{N},\ k \in \mathbf{K},\ |e^{\pm}_{m,n}| \neq 0 \right\}$$

establishes an isomorphism between $\mathcal{H}_r$ and ℓ_2.

Now we can prove

THEOREM 1. *For $r < -1/2$ and for any control $v \in L^2(0,T;\mathbf{C}^N)$ we have*

$$\{u, u_t\} \in C([0,T], \mathcal{H}_r).$$

PROOF. The statement $\{u, u_t\} \in \mathcal{H}_r$ means by (7) and Lemma 1 that

$$\left\{ |\omega_{m,k}|^r \int_0^t \left\langle v(\tau), \frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|} e^{i\omega_{m,k}\tau} \right\rangle d\tau \right\} \in \ell_2. \tag{8}$$

But we know ([1]) that the distance between the consecutive zeros of J_m tends to π decreasingly and that the zeros of J_m and J'_m are interlacing. So the zeros of J'_m can be divided into two separate sequences, i.e. for which $\mu_{n+1} - \mu_n \geqq \pi$. Therefore the Bessel inequality holds for them (see in [11]), and then

$$\sum_k \left| \int_0^t \left\langle v(\tau), \frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|} e^{i\omega_{m,k}\tau} \right\rangle d\tau \right|^2 \leqq c \int_0^t |v(\tau)|^2 d\tau, \tag{9}$$

where $c < \infty$ is independent of m. From the estimate $|\omega_{m,k}|^r \leqq (m+1)^r$ we obtain (8). The continuity of $\{u(t,\cdot,\cdot), u_t(t,\cdot,\cdot)\} \in \mathcal{H}_r$ in $t \in [0,T]$ can be proved similarly as in [14]. □

REMARK. Analogous investigations concerning vibrating strings are given in [14].

DEFINITION. The system (1)–(3) is approximately controllable in a finite time T if the set of all reachable movement states

$$R(T) := \{\{u(T,\cdot,\cdot), u_t(T,\cdot,\cdot)\}; v \in L^2(0,T : \mathbf{C}^N)\}$$

is dense in $\mathcal{H}_r$. We shall prove

THEOREM 2. *The system (1)–(3) is not approximately controllable, i.e. $R(T)$ is not dense in $\mathcal{H}_r$, for any $T > 0$ and $r < -1/2$.*

PROOF. Suppose that $P_j \neq 0$ for all j. Then for any $T > 0$ the system

$$e(\Lambda) := \left\{ \frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|} e^{i\omega_{m,k}\tau} : m \in \mathbf{N},\ k \in \mathbf{K},\ n = |k|,\ |e^{\pm}_{m,n}| \neq 0 \right\}$$

contains a Riesz basis in $L^2(0,T;\mathbf{C}^N)$.

A similar statement was proved in [8]. We shall transform some ideas applied there to our case.

LEMMA 2. *If we arrange the positive zeros $\lambda_n^{(m)}$ of J'_m increasingly into a sequence* $0 < \mu_1 < \mu_2 < \dots$ *then*

$$\mu_{n+1} - \mu_n \to 0 \quad (n \to \infty).$$

PROOF. We start with the asymptotical formula [1]

$$J_m(x) = \frac{1}{\sqrt{3}}\sqrt{1 - \frac{\arctan w}{w}}\,\left[J_{1/3}(z) + J_{-1/3}(z)\right] + O(m^{-4/3}),$$
$$x > m, \quad w := \sqrt{\frac{x^2}{m^2} - 1}, \quad z := m(w - \arctan w),$$

where the O-term is uniform in x. Since

$$2J'_m(x) = J_{m-1}(x) - J_{m+1}(x)$$

we also need the notations

$$w_1 := \sqrt{\frac{x^2}{(m-1)^2} - 1}, \qquad z_1 := (m-1)(w_1 - \arctan w_1),$$

$$w_2 := \sqrt{\frac{x^2}{(m+1)^2} - 1}, \qquad z_2 := (m+1)(w_2 - \arctan w_2).$$

We shall consider for fixed $c \geqq 1$ the values

$$x = m\sqrt{c^2+1} + t, \qquad t = O(m^{1/3})$$

with large m. In this case

$$(m-1)w_1 - mw = \sqrt{x^2 - (m-1)^2} - \sqrt{x^2 - m^2} =$$
$$= \sqrt{x^2 - m^2}\left(\sqrt{1 + \frac{2m+1}{x^2 - m^2}} - 1\right) =$$
$$= \sqrt{x^2 - m^2}\left(\frac{m}{x^2 - m^2} + O(m^{-2})\right) = \frac{m}{\sqrt{x^2 - m^2}} + O\left(\frac{1}{m}\right),$$

hence

$$m(w_1 - w) = w_1 + \frac{m}{\sqrt{x^2 - m^2}} + O\left(\frac{1}{m}\right).$$

Now

$$\sqrt{x^2 - m^2} = \sqrt{c^2t^2 + 2tm\sqrt{c^2+1} + t^2} = cm\left(1 + \frac{t}{m}\frac{\sqrt{c^2+1}}{c^2} + O(m^{-1/3})\right)$$

hence

$$\frac{1}{\sqrt{x^2-m^2}} = \frac{1}{mc} + O\Big(\frac{t}{m^2}\Big) = \frac{1}{mc} + O(m^{-5/3}).$$

Further we have

$$w_1 = \frac{1}{m-1}\sqrt{x^2-m^2+2m-1} = \frac{1}{m-1}cm\sqrt{1+O\Big(\frac{t}{m}\Big)} = c + O(m^{-2/3})$$

and then

$$m(w_1 - w) = c + \frac{1}{c} + O(m^{-2/3}).$$

Now we can calculate

$$z_1 - z = \frac{m}{\sqrt{x^2-m^2}} + \arctan w_1 + m(\arctan w - \arctan w_1) + O\Big(\frac{1}{m}\Big).$$

Since

$$\arctan w_1 = \arctan c + O(m^{-2/3}),$$
$$\arctan w - \arctan w_1 = \frac{w-w_1}{1+w_1^2} + O(|w-w_1|^2),$$

hence

$$z_1 - z = \frac{1}{c} + \arctan c - \frac{c+\frac{1}{c}}{1+w_1^2} + O(m^{-2/3}) = \frac{1}{c} + \arctan c - \frac{c+\frac{1}{c}}{1+c^2} + O(m^{-2/3})$$

i.e.

(10) $$z_1 - z = \arctan c + O(m^{-2/3}).$$

Analogously

$$mw - (m+1)w_2 = \sqrt{x^2-m^2} - \sqrt{x^2-(m+1)^2} = \frac{m}{\sqrt{x^2-m^2}} + O\Big(\frac{1}{m}\Big) =$$
$$= \frac{1}{c} + O(m^{-2/3}),$$
$$w_2 = c + O(m^{-2/3}),$$
$$mw - w_2 = w_2 + \frac{m}{\sqrt{x^2-m^2}} + O\Big(\frac{1}{m}\Big) = c + \frac{1}{c} + O(m^{-2/3}),$$

and then

(10′) $$z - z_1 = \arctan c + O(m^{-2/3}).$$

Using asymptotical formulae ([1]) for $J_{\pm 1/3}$, we obtain for $x = m\sqrt{c^2+1}+t$

(11)

$$J_m(x) = \sqrt{\frac{2}{3\pi}}\sqrt{1-\frac{\arctan w}{w}}\frac{1}{\sqrt{z}}\left[\cos\left(z-\frac{\pi}{2}\left(\frac{1}{3}+\frac{1}{2}\right)\right)+\cos\left(z-\frac{\pi}{2}\left(-\frac{1}{3}+\frac{1}{2}\right)\right)+\right.$$

$$\left.+O\left(\frac{1}{z}\right)\right]+O(m^{-4/3}) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{mw}}\left[\cos\left(z-\frac{\pi}{4}\right)+O(m^{-5/6})\right],$$

hence

$$2J_m'(x) = J_{m-1}(x)-J_{m+1}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(m-1)w_1}}\left[\cos\left(z_1-\frac{\pi}{4}\right)+O(m^{-5/6})\right]-$$

$$-\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{(m+1)w_2}}\left[\cos\left(z_2-\frac{\pi}{4}\right)+O(m^{-5/6})\right] =$$

$$= O(m^{-3/2})+\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{mw}}\left[\cos\left(z_1-\frac{\pi}{4}\right)-\cos\left(z_2-\frac{\pi}{4}\right)+O(m^{-5/6})\right] =$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{mw}}\left[-2\sin\left(\frac{z_1+z_2}{2}-\frac{\pi}{4}\right)\sin\frac{z_1-z_2}{2}+O(m^{-5/6})\right].$$

Applying (10) and (10′) we obtain

$$\frac{z_1+z_2}{2} = z+O(m^{-2/3}), \qquad \frac{z_1-z_2}{2} = \arctan c+O(m^{-2/3}).$$

If we denote $y := \sin\arctan c$, then $0 < \arctan c < \frac{\pi}{2}$ implies that $c = \frac{y}{\sqrt{1-y^2}}$ i.e. $\sin(\arctan c) = \frac{c}{\sqrt{c^2+1}}$. Consequently

$$(12) \qquad J_m'(x) = -\sqrt{\frac{2}{\pi}}\frac{c}{\sqrt{c^2+1}}\frac{1}{\sqrt{mw}}\left[\sin\left(z-\frac{\pi}{4}\right)+O(m^{-2/3})\right].$$

Since z is a monotone increasing function of x, we see from (12) that $J_m'(x)$ must have a zero $x = \mu_k^{(m)}$ if $z = \frac{\pi}{4}+k\pi$ and if the value x corresponding to z has the form

$$z = m\sqrt{c^2+1}+t, \quad t = O(m^{1/3}).$$

Denote

$$\mu_k^{(m)} = m\sqrt{c^2+1}+t_k^{(m)};$$

then

$$w = \sqrt{\frac{[\mu_k^{(m)}]^2}{m^2}-1} = \frac{1}{m}\sqrt{m^2c^2+2\sqrt{c^2+1}mt_k^{(m)}+[t_k^{(m)}]^2} =$$

$$= c\left(1+\frac{\sqrt{c^2+1}}{c^2}\frac{t_k^{(m)}}{m}+O(m^{-4/3})\right)$$

and

$$\arctan w = \arctan c + \frac{1}{c\sqrt{c^2+1}}\frac{t_k^{(m)}}{m} + O(m^{-4/3}).$$

Consequently

$$z = m(w - \arctan w) = m(c - \arctan c) + \frac{c}{\sqrt{c^2+1}}t_k^{(m)} + O(m^{-1/3}),$$

hence

$$m(c - \arctan c) + \frac{c}{\sqrt{c^2+1}}t_k^{(m)} + O(m^{-1/3}) = \frac{\pi}{4} + k\pi.$$

From here

$$(13)\qquad t_k^{(m)} = \frac{\sqrt{c^2+1}}{c}\Big[\frac{\pi}{4} + k\pi - m(c - \arctan c)\Big] + O(m^{-1/3})$$

i.e.

$$(14)\quad \mu_k^{(m)} = m\sqrt{c^2+1} + t_k^{(m)} = \pi\frac{\sqrt{c^2+1}}{c}\Big[k + \frac{1}{4} + m\frac{\arctan c}{\pi}\Big] + O(m^{-1/3}).$$

Now we can easily prove Lemma 2. Let $x > 0$ be a large number. Choose m_0 with $x = m_0\sqrt{c^2+1} + O(1)$ and k_0 with $t_{k_0}^{(m_0)} = O(1)$. Then if we choose any $k = k_0 + O(m^{1/3})$, $m = m_0 + O(m^{1/3})$, we have $t_k^{(m)} = O(m^{1/3})$. Suppose that $\frac{\arctan c}{\pi}$ is irrational. Then the mod 1 distribution of its multiples is uniform and hence we can choose $m = m_0 + O(m^{1/3})$ such that

$$(15)\qquad \left\|\frac{xc}{\pi\sqrt{c^2+1}} - \frac{1}{4} - m\frac{\arctan c}{\pi}\right\| = o_m(1) \qquad (m \to \infty).$$

where $\|y\|$ denotes the distance of y from the set $\mathbf{Z}$ of integers. (15) means that for some $k \in \mathbf{Z}$

$$x = \pi\frac{\sqrt{c^2+1}}{c}\Big[k + \frac{1}{4} + m\frac{\arctan c}{\pi}\Big] + o_x(1) = \mu^{(m)})_k + o_x(1).$$

From $x = \mu_{k_0}^{(m)} + O(1)$ we obtain that $k = k_0 + O(m^{1/3})$. Lemma 2 is proved.

Next we need

LEMMA 3 [8]. *If $0 \leqq \varphi_1 < \ldots < \varphi_M < 2\pi$, then the vectors*

$$\begin{pmatrix} \sin m\varphi_1 \\ \vdots \\ \sin m\varphi_M \end{pmatrix}, \begin{pmatrix} \cos m\varphi_1 \\ \vdots \\ \cos m\varphi_M \end{pmatrix}, \qquad m = 0, 1, \ldots, M-1$$

span the space $\mathbf{C}^M$.

Consider the points $P_j = (r_j, \varphi_j)$, $S_j = (1, \varphi_j)$ given by its polar coordinates. Arrange the indices such that $r_1, r_2, \dots, r_{N_0}, r_{N_0+1}$ are all different, $r_{N_0+1} = 1$ and the other r_j do not give new values. Introduce further the sets $\sigma_j = \{\ell\colon r_j = r_\ell\}$ for $j = 1, \dots, N_0 + 1$; and the notation

$$d_j := \sqrt{r_j^2(c^2+1) - 1}.$$

Suppose that P_j is not the origin for all j and $r_j^2(c^2+1) - 1 > 0$. We shall use the following fact.

LEMMA 4 [8]. *There exists a residual set* $D \subset [1, \infty)$ *such that for* $c \in D$ *any equation of the form*

$$0 = n\pi + n_0 \arctan c + \sum_{j=1}^{N_0} n_j \arctan d_j + \sum_{j=1}^{M} n_j' \varphi_j$$

with entire coefficients n, n_j, n_j' *implies*

$$n_0 = n_1 = \ldots = n_{N_0} = 0.$$

Next we define a basis $e_1, \dots, e_M$ of $\mathbf{R}^M$ as follows. For fixed j the values φ_ℓ, $\ell \in \sigma_j$ must be different, so by Lemma 3 we get a basis in the coordinates $\ell \in \sigma_j$. We define the other coordinates to be zeros and consider these vectors together for $1 \leqq j \leqq N_0 + 1$. Then multiply the j-th coordinate by $\frac{1}{\sqrt{d_j}}$ for $1 \leqq j \leqq M$. The resulting set of vectors will be denoted by $e_1, \dots, e_M$; they form obviously a basis in $\mathbf{R}^M$.

LEMMA 5. *Suppose that* P_j *is not the origin for any* j. *Let* $\varepsilon, T > 0$ *be arbitrary. The set* $e(\Lambda)$ *contains a subsystem*

$$\Phi := \left\{ e_n^j e^{\pm i\lambda_{n,j} t} : j = 1, \dots, M;\ n = 1, 2, \dots \right\} \cup \left\{ e_0^j e^{i\lambda_{0,j} t} : j = 1, \dots, M \right\}$$

such that

(a) $\left| e_n^j - \frac{e_j}{\|e_j\|} \right| < \varepsilon, \qquad j = 1, \dots, M;\ n = 0, 1, \dots,$

(b) $\left| \lambda_{n,j} - 2\pi \frac{n}{T} \right| < \varepsilon, \qquad j = 1, \dots, M; \quad n \geqq n_0$

for some large n_0.

PROOF. The construction of e_n^j and $\lambda_{n,j}$ will be given only for $j = 1$; for other j it goes in a similar manner.

Choose $c \in D$ (see Lemma 4) satisfying

$$3 \geqq \min_j r_j \sqrt{c^2+1} \geqq 2.$$

Let first n be large and approximate $x = 2\pi \frac{n}{T}$ by the $\mu_k^{(m)}$ as in Lemma 2. We can choose c such that the further condition

$$\left\| \frac{n}{T} \frac{c}{\sqrt{c^2+1}} - \frac{1}{8} \right\| < \varepsilon^2 \tag{16}$$

holds; if the construction goes with $j \in \sigma_{N_0+1}$, (16) must be substituted by

$$\left\| \frac{n}{T} \frac{c}{\sqrt{c^2+1}} + \frac{1}{8} \right\| < \varepsilon^2. \tag{16'}$$

Take m_0 with $2\pi \frac{n}{T} = m_0\sqrt{c^2+1} + O(1)$ and k_0 with $t_{k_0}^{(m_0)} = O(1)$; then $2\pi \frac{n}{T} = \mu_{k_0}^{(n_0)} + O(1)$. The estimate

$$\varepsilon > \left| 2\pi \frac{n}{T} - \mu_k^{(m)} \right| = \pi \frac{\sqrt{c^2+1}}{c} \left| 2 \frac{n}{T} \frac{c}{\sqrt{c^2+1}} - \frac{1}{4} - m \frac{\arctan c}{\pi} - k \right|$$

can be ensured if we require

$$\left\| m \frac{\arctan c}{\pi} \right\| < \varepsilon^2. \tag{17}$$

Indeed, if we take $m = m_0 + O(1)$ satisfying (17) then we get $k \in \mathbf{Z}$ such that $\left| 2\pi \frac{n}{T} - \mu_k^{(m)} \right| < 2\varepsilon^2$ and then $k = k_0 + O(1)$ follows for small ε. Define the vectors

$$e_{mk0}^{+} := \begin{pmatrix} \cos m\varphi_1 J_m(\lambda_k^{(m)} r_1) \\ \vdots \\ \cos m\varphi_M J_m(\lambda_k^{(m)} r_M) \end{pmatrix}, \qquad e_{mk0}^{-} := \begin{pmatrix} \sin m\varphi_1 J_m(\lambda_k^{(m)} r_1) \\ \vdots \\ \sin m\varphi_M J_m(\lambda_k^{(m)} r_M) \end{pmatrix}.$$

By (11)

$$J_m(\mu_k^{(m)} r_j) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{m w_j}} \left[\cos\left(z_j - \frac{\pi}{4} \right) + O(m^{-5/6}) \right] =$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{m d_j}} \left[\cos\left(z_j - \frac{\pi}{4} \right) + O(m^{-5/6}) \right];$$

where we used that

$$m w_j = \sqrt{r_j^2 [\mu_k^{(m)}]^2 - m^2} = \sqrt{m^2 (r_j^2 (c^2+1) - 1) + r_j^2 2\sqrt{c^2+1}\, m t_k^{(m)} + [t_k^{(m)}]^2} =$$

$$= m d_j \sqrt{1 + \frac{2 r_j^2 \sqrt{c^2+1}}{d_j^2} \frac{t_k^{(m)}}{m} + O(m^{-4/3})} =$$

$$= m d_j \left(1 + \frac{r_j^2 \sqrt{c^2+1}}{d_j^2} \frac{t_k^{(m)}}{m} + O(m^{-4/3}) \right).$$

Further we have by (13)

$$z_j = m(w_j - \arctan w_j) = m(d_j - \arctan d_j) + \frac{r_j^2\sqrt{c^2+1}}{d_j}\Big(1 - \frac{1}{1+d_j^2}\Big) t_k^{(m)} +$$

$$+ O(m^{-1/3}) = m(d_j - \arctan d_j) + t_k^{(m)} \frac{d_j}{\sqrt{c^2+1}} + O(m^{-1/3}) =$$

$$= \frac{d_j}{c}\Big(\frac{\pi}{4} + k\pi\Big) + m\Big(\frac{d_j}{c}\arctan c - \arctan d_j\Big) + O(m^{-1/3}).$$

If we have an $m = m_0 + O(1)$ satisfying (17), for the corresponding k we obtain

$$k = -m\frac{\arctan c}{\pi} + 2\frac{n}{T}\frac{c}{\sqrt{c^2+1}} - \frac{1}{4} + 2\varrho\varepsilon^2, \quad |\varrho| < 1.$$

Consequently for large m

$$z_j = 2\pi\frac{n}{T}\frac{d_j}{\sqrt{c^2+1}} - m\arctan d_j + 8\varrho_1\varepsilon^2, \qquad |\varrho_1| < 1,$$

hence

$$J_m(\mu_k^{(m)} r_j) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{md_j}}[\cos(m \arctan d_j - \gamma_j) + 8\varrho_2\varepsilon^2], \tag{18}$$

with some $|\varrho_2| < 1$. Here $\gamma_j := 2\pi\frac{n}{T}\frac{d_j}{\sqrt{c^2+1}} + \frac{\pi}{4}$. Define the vectors

$$e_{mk1}^{+} := \sqrt{\frac{2}{\pi m}}\begin{pmatrix} \frac{1}{\sqrt{d_1}}\cos m\varphi_1 \cos(m \arctan d_1 - \gamma_1) \\ \vdots \\ \frac{1}{\sqrt{d_M}}\cos m\varphi_M \cos(m \arctan d_M - \gamma_M) \end{pmatrix},$$

$$e_{mk1}^{-} := \sqrt{\frac{2}{\pi m}}\begin{pmatrix} \frac{1}{\sqrt{d_1}}\sin m\varphi_1 \cos(m \arctan d_1 - \gamma_1) \\ \vdots \\ \frac{1}{\sqrt{d_M}}\sin m\varphi_M \cos(m \arctan d_M - \gamma_M) \end{pmatrix}.$$

The estimate (18) implies

$$\left|e_{mk0}^{\pm} - e_{mk1}^{\pm}\right| \leqq \sqrt{\frac{2}{\pi m}} 8m\varrho_3\varepsilon^2, \qquad |\varrho_3| < 1. \tag{19}$$

Take the number $0 \leqq m' \leqq |\sigma_1| - 1$ such that the ℓ-th coordinates of e_1 are $\frac{1}{\sqrt{d_\ell}}\cos m'\varphi_\ell$ $\Big($or $\frac{1}{\sqrt{d_\ell}}\sin m'\varphi_\ell\Big)$. Consider for $m = m_0 + O(1)$ the following problem of simultaneous diophantine approximation:

$$(20)\qquad \begin{cases} \left\| m\frac{\arctan c}{2\pi} \right\| < \varepsilon^2, \\ \left\| m\frac{\arctan d_1}{2\pi} - \frac{\gamma_1}{2\pi} \right\| < \varepsilon^2, \\ \left\| m\frac{\arctan d_j}{2\pi} - \frac{\gamma_j + \frac{\pi}{2}}{2\pi} \right\| < \varepsilon^2, \quad 2 \leqq j \leqq N_0, \\ \left\| m\frac{\varphi_\ell}{2\pi} - m'\frac{\varphi_\ell}{2\pi} \right\| < \varepsilon^2, \quad \ell \in \sigma_1. \end{cases}$$

This system implies

$$\left\| m\frac{\arctan d_j}{2\pi} - \frac{\gamma_j}{2\pi} + \frac{1}{4} \right\| < 2\varepsilon^2$$

also for $j = N_0 + 1$, since in this case

$$d_{N_0+1} = c, \qquad N_0 + 1 = 2\pi\frac{n}{T}\frac{c}{\sqrt{c^2+1}} + \frac{\pi}{4},$$

and

$$\left\| \frac{\gamma_{N_0+1} - \frac{\pi}{2}}{2\pi} \right\| = \left\| \frac{n}{T}\frac{c}{\sqrt{c^2+1}} - \frac{1}{8} \right\| < \varepsilon^2.$$

The system (20) has a solution $m = m_0 + O(1)$ if and only if for any integers n_j, n'_j for which

$$n_0\frac{\arctan c}{\pi} + \sum_{j=1}^{N_0} n_j\frac{\arctan d_j}{2\pi} + \sum_{\ell\in\sigma_1} n'_\ell\frac{\varphi_\ell}{2\pi}$$

is entire, the expression

$$n_1\frac{\gamma_1}{2\pi} + \sum_{j=1}^{N_0} n_j\frac{\gamma_j - \frac{\pi}{2}}{2\pi} + \sum_{\ell\in\sigma_1} n'_\ell m'\frac{\varphi_\ell}{2\pi}$$

is also entire ([9]). This property holds by Lemma 4, hence (20) has a solution. For this solution we get

$$\sqrt{\frac{\pi m}{2}}e^+_{mk1} - e_1 = O(\varepsilon^2).$$

Taking into account (19) this means that

$$\left| \frac{e^+_{mk0}}{|e^+_{mk0}|} - \frac{e_1}{|e_1|} \right| = O(\varepsilon^2) < \varepsilon$$

$\left(\text{or } \left|\frac{e^{-}_{mk0}}{|e^{-}_{mk0}|} - \frac{e_1}{|e_1|}\right| = O(\varepsilon^2) < \varepsilon\right)$ if $\varepsilon > 0$ is small enough. Now we define

$$e_n^1 := \frac{e^{+(-)}_{mk0}}{|e^{+(-)}_{mk0}|}, \qquad \lambda_{n,1} := \mu_k^{(m)}$$

then (a) and (b) hold for large n, say, for $n \geqq n_0$. For $j \in \sigma_{N_0+1}$ the construction is similar, only (16) is substituted by (16′) and (20) by

$$(20') \qquad \begin{cases} \left\|m\frac{\arctan c}{2\pi}\right\| < \varepsilon^2, \\ \left\|m\frac{\arctan d_j}{2\pi} - \frac{\gamma_j + \frac{\pi}{2}}{2\pi}\right\| < \varepsilon^2, \quad j = 1, \dots, N_0, \\ \left\|m\frac{\varphi_\ell}{2\pi} - m'\frac{\varphi_\ell}{2\pi}\right\| < \varepsilon^2, \quad \ell \in \sigma_{N_0+1}. \end{cases}$$

Then

$$\left\|m\frac{\arctan d_{N_0+1}}{2\pi} - \frac{\gamma_{N_0+1}}{2\pi}\right\| = \left\|m\frac{\arctan c}{2\pi} - \left(\frac{n}{T}\frac{c}{\sqrt{c^2+1}} + \frac{1}{8}\right)\right\| < 2\varepsilon^2$$

and the above arguments apply. In case $0 \leqq n < n_0$ we repeat the above process with

$$x = \frac{(2n_0+1)\pi}{T}, \quad x = \frac{(2n_0+3)\pi}{T}, \dots, \quad x = \frac{(4n_0-1)\pi}{T}$$

instead of $x = \frac{2n\pi}{T}$. If we choose $\varepsilon < \frac{\pi}{2T}$, the values $\mu_n^{(m)}$ will be different from the values $\mu_n^{(m)}$ constructed in the earlier steps. Lemma 5 is proved.

LEMMA 6 [5, 6]. *Suppose that the system $e_n e^{i\lambda_n t} : n \in \mathbf{Z}$ is a Riesz basis in $L^2\ (0, T; \mathbf{C}^M)$. Then there exists an $\varepsilon > 0$ such that*

$$\lambda_n' \in \mathbf{C}, \qquad |\lambda_n - \lambda_n'| < \varepsilon$$

implies that $\{e_n e^{i\lambda_n' t}\}$ is also a Riesz basis in $L^2(0, T; \mathbf{C}^M)$. The constant ε depends only on T and $0 < c \leqq C < \infty$, where

$$c\sum|\alpha_n|^2 \leqq \left\|\sum \alpha_n e_n e^{i\lambda_n t}\right\|^2 \leqq C\sum|\alpha_n|^2,$$

namely $e^{T\varepsilon} - 1 < \sqrt{c/C}$ is sufficient.

LEMMA 7. *Let $e_n e^{i\lambda_n t} : n \in \mathbf{Z}$ be a Riesz basis in $L^2(0, T; \mathbf{C}^M)$ and let $\lambda_0' \in \mathbf{C}$, $\lambda_0' \neq \lambda_n$, $n \in \mathbf{Z}$. Then the new system*

$$e_0 e^{i\lambda_0' t}, \qquad e_n e^{i\lambda_n t} \quad (n \in \mathbf{Z} \setminus \{0\})$$

forms also a Riesz basis in $L^2(0,T;\mathbf{C}^M)$.

The proof requires some notions and theorems of the theory of vector exponentials. The matrix valued function

$$F\colon \mathbf{C}_+ \to \mathbf{C}^{M\times M}$$

is called strong H^2_+-function ([3]) if $\frac{f(z)e}{z+\lambda} \in H^2_+(\mathbf{C}^M)$ for every $e \in \mathbf{C}^M$, $\lambda \in \mathbf{C}_+$; here H^2_+ is the ordinary Hardy space on the upper half plane. The strong H^2_+-function F is called strong outer H^2_+-function if we have also

$$\bigvee\left\{\left(\frac{z-i}{z+i}\right)^n \frac{F(z)e}{z+i} : e \in \mathbf{C}^M,\ n \in \mathbf{N}\right\} = H^2_+(\mathbf{C}^M),$$

where $\bigvee$ denotes the closed linear hull in $H^2_+(\mathbf{C}^M)$. The strong (outer) H^2_--functions are defined similarly, only H^2_+ is replaced by H^2_-, $\mathbf{C}_+$ by $\mathbf{C}_-$ and $\pm i$ by $\mp i$. If $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$ is a Riesz basis in $L^2(0,T;\mathbf{C}^M)$ then there exists an entire function $F\colon \mathbf{C} \to \mathbf{C}^{M\times M}$ (the so-called generating function) having the factorizations

$$\text{(21)} \qquad \begin{cases} F(z) = \Pi(z)\cdot F_e^+(z) & (z \in \mathbf{C}_+), \\ F(z) = e^{iTz} F_e^-(z) & (z \in \mathbf{C}_-), \end{cases}$$

where $F_e^\pm$ are strong outer $H^2_\pm$-functions and $\Pi(z)$ is the Blaschke–Potapov product corresponding to

$$\text{(22)} \qquad \operatorname{Ker} \Pi^*(\lambda_n) = \bigvee\nolimits_{\mathbf{C}^M} \{e_k; \lambda_k = \lambda_n\} = E_n.$$

Now consider a system $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$, where $|e_n| \asymp 1$ and $0 < \delta \leqq |\operatorname{Im} \lambda_n| \leqq H < \infty$, $n \in \mathbf{Z}$. Suppose that there exists a generating function $F(z)$ satisfying (21), where Π is defined by (22). Then

THEOREM A [3]. *The following statements are equivalent:*

1) $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$ *is a Riesz basis in* $L^2(0,T;\mathbf{C}^M)$,

2) *The same system is a Riesz basis in its closed linear hull in* $L^2(0,\infty;\mathbf{C}^M)$ *and*

$$\text{(23)} \qquad \|F(x)Hu(x)\|_{L^2(\mathbf{R},\mathbf{C}^M)} \leqq c\|F(x)u(x)\|_{L^2(\mathbf{R},\mathbf{C}^M)}$$

holds for all functions u *with* $F(x)u(x) \in L^2(\mathbf{R},\mathbf{C}^M)$, *where* H *denotes the Hilbert transform*

$$Hu(x) := \frac{1}{2\pi}\ \text{p.v.} \int_{\mathbf{R}} \frac{u(t)}{x-t}\,dt.$$

Now we are able to prove Lemma 7. Since $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$ is a Riesz basis, there exists a generating function $F(z)$. We can assume that the Blaschke–Potapov product $\Pi(z)$ has the form

$$\Pi(z) = b_0(z)b_1(z)b_{-1}(z)b_2(z)b_{-2}(z)\ldots$$

where

$$b_0(z) = U^{-1}\begin{pmatrix} \frac{1-\frac{z}{\lambda_0}}{1-\frac{z}{\overline{\lambda}_0}} I_r & 0 \\ 0 & I_{M-r} \end{pmatrix} U$$

with some unitary matrix U transforming E_0 onto the subset of $\mathbf{C}^M$ consisting of the vectors vanishing in the last $M - r$ coordinates, $r = \dim E_0$ and I_r is the $r \times r$ identity matrix. We can also suppose that

$$U : e_0 \mapsto \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now define

$$F_0(z) = U^{-1}\begin{pmatrix} \frac{1-\frac{z}{\lambda'_0}}{1-\frac{z}{\overline{\lambda}'_0}} & 0 \\ 0 & I_{M-1} \end{pmatrix} U$$

and let

$$F_1(z) := F_0(z)F(z).$$

Then $F_1(z)$ is entire and it is a strong H^2_+-function. Consequently ([2]) there exists a factorization $F_1 = \Theta F^+_{e,1}$, where Θ is an operator valued inner function and $F^+_{e,1}$ is a strong outer H^2_+-function. Potapov proved in [12] that Θ can be further factorized in the form $\Theta = \Pi_1 S$, where Π_1 is the Blaschke–Potapov product and S is a singular inner function, i.e. $\det S(z) \neq 0$ for $z \in \mathbf{C}_+$. By the Helson–Lowdenslager theorem ([2]) in the factorization

$$\det F_1 = \det \Pi_1 \det S \det F^+_{e,1}$$

$\det F^+_{e,1}$ is outer, $\det \Pi_1$ is a Blaschke product and $\det S$ is a singular inner function. On the other hand by $F_1 = F_0 F$ we obtain

$$\det F_1 = \frac{1-\frac{z}{\lambda'_0}}{1-\frac{z}{\lambda_0}} \det F =$$

$$= \left[\frac{1-\frac{z}{\lambda'_0}}{1-\frac{z}{\overline{\lambda}'_0}} \cdot \frac{1-\frac{z}{\overline{\lambda}_0}}{1-\frac{z}{\lambda_0}} \det \Pi\right] \left[\frac{1-\frac{z}{\overline{\lambda}'_0}}{1-\frac{z}{\overline{\lambda}_0}} \det F^+_e\right].$$

In the first brackets stands a Blaschke product and since

$$\left|\frac{1-\frac{z}{\overline{\lambda_0'}}}{1-\frac{z}{\overline{\lambda_0}}}\right| \asymp 1 \qquad (z \in \mathbf{C}_+),$$

hence in the second brackets we have an outer function. Since the factorization of scalar functions is unique up to a multiplicative constant, we must have

$$\det S(z) = c, \quad |c| = 1 \quad (z \in \mathbf{C}_+).$$

Since $S(z)$ is contractive, we have $I - S^*(z)S(z) \geqq 0$ hence the eigenvalues of $S^*(z)S(z)$ are between 0 and 1 and their product equals $\det S^*(z)S(z) = 1$. Consequently $I = S^*(z)S(z)$, i.e. $S(z)$ is unitary on $\mathbf{C}_+$. But then $S(z)$ and $S^*(z) = S^{-1}(z)$ are both analytic in $\mathbf{C}_+$ and this means that $S(z)$ is a constant unitary matrix. It can be put into the outer factor, so we obtained the factorization

$$F_1(z) = \Pi_1(z) F_{e,1}^+(z), \qquad z \in \mathbf{C}_+.$$

In the lower halfplane we have

$$F_1(z) = e^{iTz} F_{e,1}^-(z), \qquad F_{e,1}^-(z) := F_0(z) F_e^-(z).$$

Since

$$\left|\frac{1-\frac{z}{\lambda_0'}}{1-\frac{z}{\lambda_0}}\right| \asymp 1 \qquad (z \in \mathbf{C}_-),$$

hence $\|F_0(z)\| \leqq c$, $\|F_0^{-1}(z)\| \leqq c$, so F_0 gives an isomorphism of $H^2_-(\mathbf{C}^M)$ onto itself and then maps complete sets onto complete sets. In particular we get that $F_{e,1}^-$ is a strong outer H^2_--function. Thus we showed that $F_1(z)$ is a generating function of the new system $e_0 e^{i\lambda_0' t}$, $e_n e^{i\lambda_n t}$, $n \in \mathbf{Z} \setminus \{0\}$. We have to verify (23) with F_1 instead of F. But $\|F_0(x)\|, \|F_0^{-1}(x)\| \leqq c$ implies that

$$\|F_0(x) f(x)\|_{L^2(\mathbf{R},\mathbf{C}^M)} \asymp \|f(x)\|_{L^2(\mathbf{R},\mathbf{C}^M)} \qquad \left(f \in L^2(\mathbf{R}, \mathbf{C}^M)\right),$$

and since (23) holds with F it holds also with F_1. Lemma 7 is proved.

PROOF OF THEOREM 3. We shall show that the system $\Phi \subset e(\Lambda)$ constructed in Lemma 5 is in fact a Riesz basis in $L^2(0,T;\mathbf{C}^M)$. Let

$$f \in L^2(0,T;\mathbf{C}^M), \quad f(t) = \sum_{n\in\mathbf{Z}} f_n e^{i2\pi\frac{n}{T}t}, \quad f_n = \sum_{j=1}^{M} \alpha_{n,j} e_{|n|}^j.$$

If $\varepsilon > 0$ is small enough then

$$\|f\|_{L^2(0,T;\mathbf{C}^M)} \asymp \sum_{n\in\mathbf{Z}} |f_n|^2 \asymp \sum_{n\in\mathbf{Z}} \sum_{j=1}^{M} |\alpha_{n,j}|^2$$

which proves that the system

$$\Phi_0 := \left\{ e^j_{|n|} e^{i2\pi \frac{n}{T} t} : n \in \mathbf{Z},\ j = 1, \dots, M \right\}$$

is a Riesz basis in $L^2(0,T;\mathbf{C}^M)$. By Lemma 5 (b) and Lemma 6 we obtain that for large n_0 the system

$$\hat{\Phi}_0 := \left\{ e^j_{|n|} e^{i2\pi \frac{n}{T} t} : |n| \leqq n_0;\ j = 1, \dots, M \right\} \cup$$
$$\cup \left\{ e^j_{|n|} e^{i \operatorname{sgn} n \cdot \lambda_{n,j} \cdot t} : |n| \geqq n_0 + 1,\ j = 1, \dots, M \right\}$$

is a Riesz basis in $L^2(0,T;\mathbf{C}^M)$. The last step is to apply Lemma 7. It may happen that for some $n \leqq n_0 : \lambda_{n,j} = 2\pi \frac{k}{T}$ (here $k \leqq n_0$ if n_0 is large enough). Therefore define new exponents $\hat{\lambda}_{n,j}$, $0 \leqq n \leqq n_0$, $1 \leqq j \leqq M$ which are different and do not contain elements from the set

$$\left\{ \pm\lambda_{n,j} : n \in \mathbf{N},\ j = 1, \dots, M \right\} \cup \left\{ 2\pi \frac{n}{T} : n \in \mathbf{Z} \right\}.$$

By Lemma 7 we get that

$$\{ e^j_n e^{\pm i \hat{\lambda}_{n,j} t} : n \leqq n_0, j = 1, \dots, M \} \cup \{ e^j_n e^{\pm i \lambda_{n,j} t} : n \geqq n_0 + 1, j = 1, \dots, M \}$$

is a Riesz basis; then again by Lemma 7, Φ is a Riesz basis in $L^2(0,T;\mathbf{C}^M)$. Theorem 3 is proved.

For the proof of Theorem 2 we recall first the following result of D. L. Russell.

PROPOSITION 1 [3]. *Let* $\{e^{i\lambda_n t} : n \in \mathbf{Z}\}$ *be a Riesz basis in* $L^2(0,T)$ *and let* $\mu_1, \dots, \mu_s$ *be different (complex) values different also from the numbers* λ_n. *Then the system* $\{e^{i\mu_j t}\}_{1 \leqq j \leqq s} \cup \{e^{i\lambda_n t}\}_{n \in \mathbf{Z}}$ *forms a Riesz basis in the Hilbert space* $H^s(0,T)$.

We generalize this to the case of vector exponentials as follows.

PROPOSITION 2. *Let* $\sigma_n \subset \mathbf{C}$ *be finite sets and*

$$H_n := \bigvee\nolimits_{L^2(0,T;\mathbf{C}^M)} \{ e_\lambda e^{i\lambda t} : \lambda \in \sigma_n \}, \quad n \in \mathbf{Z}$$

with some $e_\lambda \in \mathbf{C}^M$. *Suppose that* H_n $n \in \mathbf{Z}$ *is a Riesz basis in* $L^2(0,T;\mathbf{C}^M)$ *(see this notion in* [2]*). Let* $s \geqq 1$ *be entire and*

$$H_{(0)} := \bigvee\nolimits_{L^2(0,T;\mathbf{C}^M)} \{ e_j e^{i\mu_j t} : j = 1, \dots, M_s \},$$

where

a) *the system* $\{e_j e^{i\mu_j t}\}$ *is linearly independent,*

b) $H_{(0)} \cap H_n = \{0\}$, $n \in \mathbf{Z}$,

c) *from the vectors* e_j *we can form* s *bases in* $\mathbf{C}^M$, *e.g. let* $\{e_1, \ldots, e_M\}$, $\{e_{M+1}, \ldots, e_{2M}\}$, $\ldots$, $\{e_{(s-1)M+1}, \ldots, e_{sM}\}$ *be all bases in* $\mathbf{C}^M$. *Then* $H_{(0)}$, H_n $(n \in \mathbf{Z})$ *forms a Riesz basis in* $L^2(0,T;\mathbf{C}^M)$.

We shall prove the following statement which implies trivially Proposition 2.

PROPOSITION 2′. *Let* $s \geqq 1$ *be entire and* $H_n := \bigvee\{e_\lambda e^{i\lambda t} : \lambda \in \sigma_n\}$ *be a Riesz basis in* $H^{s-1}(0,T;\mathbf{C}^M)$. *Let further the sets* σ_n *be finite and define*

$$H_{(0)} := \bigvee\{e_j e^{i\mu_j t} : j = 1, \ldots, M\}$$

where

a) $H_{(0)} \cap H_n = \{0\}$ *for all* n,

b) $e_1, \ldots, e_M$ *is a basis in* $\mathbf{C}^M$.

Then $H_{(0)}$, H_n $(n \in \mathbf{Z})$ *is a Riesz basis in* $H^s(0,T;\mathbf{C}^M)$.

REMARK. The case $N = 1$ of Proposition 2 is asserted in [4] without proof.

For the proof of Proposition 2′ we introduce first the closed subspace

$${}_0H^s := \{f \in H^s(0,T;\mathbf{C}^M) : f(0) = 0\}.$$

Let further $T\colon \mathbf{C}^M \to \mathbf{C}^M$ be the linear mapping defined by

$$Te_j := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(the j-th coordinate is 1), i.e. T maps the basis e_j to the standard orthonormal basis. Finally define

$$A(z) := T^{-1} \begin{pmatrix} z - i\mu_1 & & 0 \\ & \ddots & \\ 0 & & z - i\mu_N \end{pmatrix} T = zI + C$$

and the operator $A(\frac{d}{dt})$ by

$$A\Big(\frac{d}{dt}\Big) f = Af := f' + Cf.$$

LEMMA 8. 1) *$A\colon H^s \to H^{s-1}$ is continuous and its kernel is* $\operatorname{Ker} A = H_{(0)}$.

2) *The mapping $f \mapsto f(0)$ establishes an isomorphism between $H_{(0)}$ and* $\mathbf{C}^M$.

3) *$A\colon {}_0H^s \to H^{s-1}$ is an (onto) isomorphism.*

PROOF. Statement 2) is trivial. To prove 1) expand any function $f \in H^s$ in the form

$$f = \sum_{j=1}^{N} e_j f_j$$

where $f_j(t) \in H^s(0,T)$ are scalar-valued functions. Then we have

$$Af = \sum e_j \left(\frac{d}{dt} - i\mu_j\right) f_j(t). \tag{24}$$

From this we see at once that $Af = 0$ if and only if $f_j = \alpha_j e^{i\mu_j t}$ for all j, so $\operatorname{Ker} A = H_{(0)}$. On the other hand A is continuous since

$$\|Af\|^2_{H^{s-1}} \asymp \sum_{j=1}^{M} \left\| \left(\frac{d}{dt} - i\mu_j\right) f_j \right\|^2_{H^{s-1}} \leqq c \sum \|f_j\|^2_{H^s} \asymp \|f\|^2_{H^s},$$

so 1) is proved. To show 3), we prove that A maps ${}_0H^s$ onto H^{s-1}. By (24) it means that for arbitrary $\mu \in \mathbf{C}$

$$\left\{ \left(\frac{d}{dt} - i\mu\right) f : f \in H^s(0,T) \right\} = H^{s-1}(0,T)$$

i.e. for any $g \in H^{s-1}$ we have to find $y \in H^s$ satisfying

$$y' - i\mu y = g.$$

Introduce the function $Y(t)$ by

$$Y(t) \cdot e^{i\mu t} = y(t).$$

Then we have

$$y' - i\mu y = Y' e^{i\mu t} = g \quad \text{i.e.} \quad Y = \int g(t) e^{-i\mu t} dt.$$

This is a correct solution since $g \in H^{s-1}$ implies $ge^{-i\mu t} \in H^{s-1}$ hence $Y \in$ $\in H^s$ and so $y \in H^s$. Hence the mapping $A\colon {}_0H^s \to H^{s-1}$ is onto indeed and then by 1) it is a one-to-one continuous linear mapping between Banach spaces, hence is isomorphic. Lemma 8 is proved.

PROOF OF PROPOSITION 2. Consider a finite sum $\sum_n f_n$ where $f_n \in H_n$. Since $f_n' \in H_n$ and H_n is a Riesz basis in H^{s-1}, we have

$$\Big\|\sum f_n\Big\|_{H^s}^2 \asymp \Big\|\sum f_n\Big\|_{H^{s-1}}^2 + \Big\|\sum f_n'\Big\|_{H^{s-1}}^2 \asymp$$

$$\asymp \sum \left(\|f_n\|_{H^{s-1}}^2 + \|f_n'\|_{H^{s-1}}^2\right) \asymp \sum \|f_n\|_{H^s}^2,$$

therefore H_n $(n \in \mathbf{Z})$ forms an L-basis in H^s (i.e. a Riesz basis in its closed linear hull). On the other hand we have

$$H_{(0)} \cap \bigvee_{H^s} \{H_n : n \in \mathbf{Z}\} = \{0\}. \tag{25}$$

To prove this, consider the projection $P\colon H^s \to {}_0H^s$ of H^s onto ${}_0H^s$ parallel to $H_{(0)}$. By Lemma 8, 1) and 2), we have ${}_0H^s \cap H_{(0)} = \{0\}$, ${}_0H^s \dot{+} H_{(0)} = H^s$, hence P is uniquely defined. Since $H_{(0)}$ is finite dimensional and ${}_0H^s$ is closed, the angle of these subspaces must be positive, thus P is continuous (in H^s). We know that $AH_n = H_n$ because $AH_n \subset H_n$ is obvious and $H_n \cap \cap \operatorname{Ker} A = \{0\}$ implies the equality of the dimensions. Define $H_n^* := PH_n$. Since P changes only the component lying in $\operatorname{Ker} A$, we have $AH_n^* = H_n$. By Lemma 8, 3), this means that the system $\{H_n : n \in \mathbf{Z}\}$ is a Riesz basis in ${}_0H^s$ and then $H_{(0)}, H_n^* : n \in \mathbf{Z}$ is a Riesz basis in H^s. To prove (25) suppose that

$$f_{(0)} = \sum f_n,$$

the sum being convergent in H^s. Applying P we get $0 = \sum Pf_n$ and then the L-basis property of H_n implies $0 = Pf_n$ for all n. But $0 = Pf_n$ implies $0 = f_n$, otherwise $H_n \cap H_{(0)} \neq \{0\}$ would follow. This implies $f_{(0)} = 0$, so (25) is proved. This yields that $H_{(0)}, H_n$, $n \in \mathbf{Z}$ form an L-basis in H^s. The completeness of this system follows from the completeness of $H_{(0)}, H_n^*$, $n \in \mathbf{Z}$. Proposition 2′ and thus Proposition 2 are proved.

PROOF OF THEOREM 2. Suppose first that the origin does not occur among the points P_j. By Theorem 3 the system $\phi \subset e(\Lambda)$ constructed in Lemma 5 is a Riesz basis in $L^2(0,T;\mathbf{C}^M)$. Taking any values $x = \frac{(2n+1)\pi}{T}$ with large n, the method of construction of Lemma 5 give us some exponentials $e_x^j e^{i\lambda_{xj}t} \in e(\Lambda)$ with

$$\left| e_x^j - \frac{e_j}{|e_j|} \right| < \varepsilon, \quad \left| \lambda_{n,j} - \frac{(2n+1)\pi}{T} \right| < \varepsilon, \quad j = 1, \ldots, M.$$

Now by Proposition 2 we see that for a given $s \geqq 1$ there exists a new system $\Phi \subset \hat{\Phi} \subset e(\Lambda)$ which is a Riesz basis in $H^s(0,T;\mathbf{C}^M)$. Of course, $e(\Lambda) \setminus \hat{\Phi}$

contains infinitely many exponentials. Expand one of them in H^s:

$$e_0 e^{i\lambda_0 t} = \sum_{r=1}^{M_s} \alpha_r e_r e^{i\mu_r t} + \sum_{n\in\mathbf{Z}} \beta_n e^j_{|n|} e^{i\cdot\operatorname{sgn} n\cdot\lambda_{n,j}\cdot t}$$

if we say that $\operatorname{sgn} 0 = 1$. This sum can be differentiated s times; the resulting series will converge in L^2 and

$$\sum_{n\in\mathbf{Z}} |\beta_n|^2(|n|+1)^s < \infty.$$

Consequently among the coordinates of the moment sequence

$$|\omega_{m,k}|^r \xi^{\pm}_{m,k}(T)\frac{\gamma_{m,n}}{|e^{\pm}_{m,n}|} = |\omega_{m,k}|^r \int_0^T \left\langle v(t), e^{i\omega_{m,k}(t-T)}\frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|}\right\rangle dt,$$

there exists a linear connection with some coefficient sequence from ℓ_2 if s is large enough. Indeed, if we have

$$\sum\nolimits^{*} d_{m,k} e^{i\omega_{m,k}t}\frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|} = 0, \tag{26}$$

the sum being convergent in ℓ_2, then

$$0 = \sum\nolimits^{*} \frac{d_{m,k}}{|\omega_{m,k}|^r}|\omega_{m,k}|^r \int_0^T \left\langle v(t), e^{i\omega_{m,k}(t-T)}\frac{e^{\pm}_{m,n}}{|e^{\pm}_{m,n}|}\right\rangle dt$$

and here $\left\{\frac{d_{m,k}}{|\omega_{m,k}|^r}\right\} \in \ell_2$ if s is large enough. This completes the proof for the case when $P_j \neq 0$. If the origin occurs, for example $P_1 = 0$, then for $m > 0$ the first coordinates of $e^{\pm}_{m,n}$ are zeros. In this case we consider only the last $M-1$ coordinates of the vectors; in these coordinates we get a Riesz basis in $H^s(0,T;\mathbf{C}^{M-1})$ by the above argument if the point P_1 is supposed to be omitted. By the above way we get in these coordinates a relation of type (26) which leads to the noncompleteness of the moment sequence as above. Theorem 2 is completely proved.

REMARK. The method given in [6] shows that $R(T)$ does not give up growing for large T, contrary to the one-dimensional case of vibrating strings (see e.g. [14]). We investigate the space $\mathcal{H}_r$ and give controllability results in a forthcoming paper.

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UNIFORM ESTIMATIONS OF THE GREEN FUNCTION FOR THE SINGULAR SCHRÖDINGER OPERATOR

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Consider the manifolds $S_1, \dots, S_{\ell_0} \subset \mathbf{R}^N$, $\dim S_k = N - m_k$ defined by the rules

$$S_k = \{x = (\xi, \eta) \in \mathbf{R}^{m_k} \times \mathbf{R}^{N-m_k} : \varphi_k(\eta) = \xi\}$$

where the partial derivatives of order 1 of the functions $\varphi_k \colon \mathbf{R}^{N-m_k} \to \mathbf{R}^{m_k}$ are uniformly bounded:

$$|\nabla \varphi_k(\eta)| \leqq C, \qquad \eta \in \mathbf{R}^{N-m_k}, \quad k = 1, \dots, \ell_0.$$

Define further $S := \bigcup_{k=1}^{\ell_0} S_k$ and choose arbitrarily $0 \leqq \tau < 2$. Take a function $q \in C^\infty(\mathbf{R}^N \setminus S)$ satisfying

$$|q(x)| \leqq c_0[\operatorname{dist}(x, S)]^{-\tau}.$$

Fix a number $0 < \varepsilon_0 < \pi$ and define the sector

$$Z_0 = \{z \in \mathbf{C} : \varepsilon_0 \leqq \arg z \leqq \pi - \varepsilon_0\}.$$

We consider the Green function of the operator $-\Delta + q - \mu^2$, $\mu \in Z_0$. We shall prove the same uniform estimate for the Green function which is known in case $q = 0$ (for the Laplace operator); see Theorem 1 below. Analogous results were obtained in Joó [6] for the case of a bounded domain Ω and spherically symmetrical potential $q(x)$. In [4] Joó investigated potentials where the spherical symmetry of the main term of q is perturbed, where the perturbation diminishes quickly if we approximate the origin of symmetry. The second main result of the present paper is stated in Theorem 2 below. It is an analogue of the square sum estimate for a complete system of eigenfunctions (if the spectrum is discrete); see e.g. [7]. In our case with domain $\Omega = \mathbf{R}^N$ the spectrum is continuous, hence we have a square integral estimate of a complete system of generalized eigenfunctions with respect to an appropriate measure. This is based on the Neumann spectral theorem and on some ideas of Maurin [9].

We shall prove the estimate of the Green function. In case $q = 0$ the Green function is explicitly given in Titchmarsh [1], 13.7.2:

$$(1) \qquad \begin{cases} E_0(x,y,\mu) = c_N\left(\frac{\mu_2}{r}\right)^{\frac{N}{2}-1} H^{(1)}_{\frac{N}{2}-1}(\mu_2 r), \quad r = |x-y|,\ x, y \in \mathbf{R}^N, \\ c_N = i2^{-\frac{N}{2}-1}\pi^{-\frac{N}{2}+1},\ \mu_2 = \sqrt{\mu^2 - \mu_1},\ \operatorname{Im}\mu_2 > 0. \end{cases}$$

LEMMA 1. *There exists $\alpha > 0$ such that*

$$|E_0(x,y,\mu)| \leqq cr^{2-N}e^{-2\alpha|\mu_2|r}, \quad x,y \in \mathbf{R}^N,\ \mu \in Z_0,\ \mu_1 > 0 \tag{2}$$

and the estimate is uniform in x, y, μ, μ_1.

PROOF. Using the asymptotical formula [2], 7.13.1 we get that

$$\left|H^{(1)}_{\frac{N}{2}-1}(\mu_2 r)\right| \leqq \frac{c}{\sqrt{|\mu_2|r}}e^{-\operatorname{Im}\mu_2 r}, \qquad |\mu_2 r| \geqq 1.$$

From $\mu_1 > 0$ and $\mu \in Z_0$ it follows that $\mu_2 \in Z_0$, consequently $-\operatorname{Im}\mu_2 r \leqq -c(\varepsilon_0)|\mu_2 r| \leqq -c(\varepsilon_0) < 0$, hence

$$e^{-\operatorname{Im}\mu_2 r} \leqq e^{-c(\varepsilon_0)|\mu_2|r} \leqq c(|\mu_2|r)^{\frac{3-N}{2}}e^{-\frac{c(\varepsilon_0)}{2}|\mu_2|r}$$

which proves (2) for the case $|\mu_2|r \geqq 1$. From the estimate

$$|H^{(1)}_\nu(z)| \leqq c_\nu|z|^{-\nu}, \qquad |z| \leqq 1,\ \nu > 0$$

we get easily (2) in case $|\mu_2|r \leqq 1$. Lemma 1 is proved.

LEMMA 2. *Let $\alpha > 0$, $\ell, s \geqq 0$, $\ell + s < N - \tau$. Then there exists $c = c(\alpha, \ell, N, s, \tau) < \infty$ such that*

$$\int_{\mathbf{R}^N} e^{-\alpha|x-u||\mu|}|x-u|^{-\ell}|y-u|^{-s}|q(u)|du \leqq c|\mu|^{s+\ell+\tau-N}$$

uniformly in $x, y \in \mathbf{R}^N$, $\mu \in Z_0$.

PROOF. Use the decomposition

$$\int_{\mathbf{R}^N} = \int_{|x-u|\leqq|u-y|} + \int_{|x-u|>|u-y|} =: I_1 + I_2.$$

Clearly we have

$$I_1 \leqq \int_{\mathbf{R}^N} e^{-\alpha|x-u||\mu|}|x-u|^{-\ell-s}|q(u)|du,$$

$$I_2 \leqq \int_{\mathbf{R}^N} e^{-\alpha|u-y||\mu|}|u-y|^{-\ell-s}|q(u)|du.$$

By symmetry it is enough to estimate I_1. It is known [4] that $\operatorname{dist}(u, S_k) \geqq c|\xi - \varphi_k(\eta)|$. Consequently

$$|q(u)| \leqq c[\operatorname{dist}(u, S)]^{-\tau} \leqq c\sum_{k=1}^{\ell_0}[\operatorname{dist}(u, S_k)]^{-\tau} \leqq$$

$$\leqq c\sum_{k=1}^{\ell_0}|\xi_k - \varphi_k(\eta_k)|^{-\tau}, \qquad u = (\xi_k, \eta_k) \in \mathbf{R}^{m_k} \times \mathbf{R}^{N-m_k}.$$

To simplify the notations, we shall omit the indices in ξ_k and η_k. We have

$$I_1 \leqq c\sum_k \int_{\mathbf{R}^n} e^{-\alpha|x-u||\mu|}|x - u|^{-\ell-s}|\xi - \varphi_k(\eta)|^{-\tau}d\xi d\eta =$$

$$= c\sum_k \int_{\mathbf{R}^N} e^{-\alpha|x-(z+\varphi_k(\eta),\eta)||\mu|}|x - (z + \varphi_k(\eta), \eta)|^{-\ell-s}|x|^{-\tau}dzd\eta.$$

We know that

$$|x - (z + \varphi_k(\eta), \eta)| \geqq \begin{cases} |x^1 - \eta| \\ |x^0 - (z + \varphi_k(\eta))| \end{cases}, \qquad x = (x^0, x^1).$$

Using the known identity

$$\int_0^\infty e^{-\beta t}t^{\gamma-1}dt = \Gamma(\gamma)\beta^{-\gamma}, \qquad \beta > 0,\ \gamma > 0$$

we obtain for any $0 \leqq p_k \leqq \ell + s$ that

$$I_1 \leqq c\sum_k \int_{\mathbf{R}^N} \left(e^{-\frac{\alpha}{2}|x^1-\eta||\mu|}|x^1 - \eta|^{-\ell-s+p_k}\right) \cdot$$

$$\cdot \left(e^{-\frac{\alpha}{2}|x^0-z-\varphi_k(\eta)||\mu|}|x^0 - z - \varphi_k(\eta)|^{-p_k}|z|^{-\tau}\right) d\eta dz =$$

$$= c\sum_k \int_{\mathbf{R}^{N-m_k}} e^{-\frac{\alpha}{2}|x^1-\eta||\mu|}|x^1 - \eta|^{-\ell-s+p_k} \cdot$$

$$\cdot\left(\int_{\mathbf{R}^{m_k}} e^{-\frac{\alpha}{2}|x^0-z-\varphi_k(\eta)||\mu|}|x^0 - z - \varphi_k(\eta)|^{-p_k}|z|^{-\tau}dz\right)d\eta.$$

We estimate first the inner integral:

$$\int_{\mathbf{R}^{m_k}} e^{-\frac{\alpha}{2}|x^0-z-\varphi_k(\eta)||\mu|}|x^0-z-\varphi_k(\eta)|^{-p_k}|z|^{-\tau}dz = \int_{|z|\leqq|x^0-z-\varphi_k(\eta)|} + \int_{|z|\geqq|x^0-z-\varphi_k(\eta)|} \leqq$$

$$\leqq \int_{\mathbf{R}^{m_k}} e^{-\frac{\alpha}{2}|z\mu|}|z|^{-p_k-\tau}dz + \int_{\mathbf{R}^{m_k}} e^{-\frac{\alpha}{2}|x^0-z-\varphi_k(\eta)||\mu|}|x^0 - z - \varphi_k(\eta)|^{-p_k-\tau}dz =$$

$$= 2\int_{\mathbf{R}^{m_k}} e^{-\frac{\alpha}{2}|z\mu|}|z|^{-p_k-\tau}dz = c\int_0^\infty r^{m_k-1-p_k-\tau}e^{-\frac{\alpha}{2}|\mu|r}dr = c|\mu|^{p_k+\tau-m_k}$$

provided $m_k > p_k + \tau$. Then we have

$$I_1 \leqq c\sum_k |\mu|^{p_k+\tau-m_k} \int_{\mathbf{R}^{N-m_k}} e^{-\frac{\alpha}{2}|x^1-\eta||\mu|}|x^1 - \eta|^{-\ell-s+p_k}d\eta \leqq$$

$$\leqq c\sum_k |\mu|^{p_k+\tau-m_k} \int_0^\infty r^{N-m_k-1}e^{-\frac{\alpha}{2}|\mu|r}r^{-\ell-s+p_k}dr \leqq$$

$$\leqq c\sum_k |\mu|^{p_k+\tau-m_k}|\mu|^{m_k+\ell+s-p_k-N} = c|\mu|^{\ell+s+\tau-N}$$

provided $N + p_k - m_k - \ell - s > 0$. All the above conditions assumed for the numbers p_k are

$$0 \leqq p_k \leqq \ell + s, \qquad m_k + \ell + s - N < p_k < m_k - \tau.$$

These can be simultaneously satisfied, hence the proof is complete.

Define the functions $F_n(x, y, \mu)$ by

$$\tag{3} \begin{cases} F_0(x,y,\mu) := E_0(x,y,\mu), \\ F_n(x,y,\mu) := -\int_{\mathbf{R}^N} E_0(x,u,\mu)F_{n-1}(u,y,\mu)q(u)du. \end{cases}$$

LEMMA 3. *Let* $n = 0, 1, \dots, \left[\frac{N-2}{2-\tau}\right]$ *and* $0 < \delta < 2-\tau$ *be arbitrary. Then*

$$\tag{4} |F_n(x,y,\mu)| \leqq c|x-y|^{2-N+n\delta}e^{-\alpha|x-y||\mu_2|}|\mu_2|^{-n(2-\delta-\tau)}$$

holds uniformly in $x, y \in \mathbf{R}^N$, $\mu \in Z_0$, $\mu_1 > 0$.

PROOF. In case $n = 0$, (2) is even stronger than (4). For larger values of n we use induction:

$$|F_n(x,y,\mu)| \leqq \int_{\mathbf{R}^N} |E_0(x,u,\mu)||F_{n-1}(u,y,\mu)||q(u)|du \leqq$$

$$\leqq c|\mu_2|^{(1-n)(2-\delta-\tau)}\int_{\mathbf{R}^n} e^{-2\alpha|x-u||\mu_2|-\alpha|y-u||\mu_2|}|x-u|^{2-N}|y-u|^{2-N+(n-1)\delta}|q(u)|du.$$

Using $|x-u|+|u-y| \geqq |x-y|$ we get by Lemma 2 that

$$|F_n(x,y,\mu)| \leqq c|\mu_2|^{(1-n)(2-\delta-\tau)} e^{-\alpha|x-y||\mu_2|}\cdot$$
$$\cdot \int\limits_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{2-N}|y-u|^{2-N+(n-1)\delta}|q(u)|du =$$
$$= c|\mu_2|^{(1-n)(2-\delta-\tau)} e^{-\alpha|x-y||\mu_2|}\Bigg(\int\limits_{|u-y|>\frac{|x-y|}{2}} + \int\limits_{|u-y|\leqq\frac{|x-y|}{2}} \Bigg) \leqq$$
$$\leqq c|\mu_2|^{(1-n)(2-\delta-\tau)} e^{-\alpha|x-y||\mu_2|}|x-y|^{2-N+n\delta}\cdot$$
$$\cdot\Bigg(\int\limits_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{2-N}|y-u|^{-\delta}|q(u)|du+$$
$$+ \int\limits_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{-n\delta}|y-u|^{2-N+(n-1)\delta}|q(u)|du\Bigg) \leqq$$
$$\leqq c|\mu_2|^{(1-n)(2-\delta-\tau)} e^{-\alpha|x-y||\mu_2|}|x-y|^{2-N+n\delta}|\mu_2|^{\delta+\tau-2} =$$
$$= c|x-y|^{2-N+n\delta} e^{-\alpha|x-y||\mu_2|}|\mu_2|^{-n(2-\tau-\delta)}$$

as we asserted.

LEMMA 4. *There exist constants* $c_0, c_1 > 0$ *such that for* $n \geqq \left[\frac{N-2}{2-\tau}\right]+1$

$$|F_n(x,y,\mu)| \leqq c_1 e^{-\alpha|x-y||\mu_2|}(c_0|\mu_2|)^{N-2-n(2-\tau)} \tag{5}$$

uniformly in $x, y \in \mathbf{R}^N$, $\mu \in Z_0$, $\mu_1 \geqq 0$.

PROOF. Consider first $n = \left[\frac{N-2}{2-\tau}\right]+1$ and apply the previous Lemma:

$$|F_n(x,y,\mu)| \leqq c|\mu_2|^{(1-n)(2-\delta-\tau)} \int\limits_{\mathbf{R}^N} e^{-2\alpha|x-u||\mu_2|-\alpha|y-u||\mu_2|}|x-u|^{2-N}\cdot$$
$$\cdot|u-y|^{2-N+(n-1)\delta}|q(u)|du \leqq c|\mu_2|^{(1-n)(2-\delta-\tau)} e^{-\alpha|x-y||\mu_2|}\cdot$$
$$\cdot \int\limits_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{2-N}|u-y|^{2-N+(n-1)\delta}|q(u)|du.$$

We know that

$$(n-1)(2-\tau) = (2-\tau)\left[\frac{N-2}{2-\tau}\right] > N-2-(2-\tau) = N-4+\tau$$

hence there exists δ, $\frac{N-4+\tau}{n-1} < \delta < 2-\tau$ and then Lemma 2 applies:

$$|F_n(x,y,\mu)| \leqq c_2|\mu_2|^{(1-n)(2-\delta-\tau)}e^{-\alpha|x-y||\mu_2|}|\mu_2|^{2N-4-(n-1)\delta+\tau-N} =$$
$$= c_2|\mu_2|^{-\left[\frac{N-2}{2-\tau}\right](2-\tau)+N-4+\tau}e^{-\alpha|x-y||\mu_2|}.$$

For larger values of n we use induction. Denote c_3 and c_4 the constants in Lemmas 1 and 2, resp. and define

$$c_0 := (c_3c_4)^{\frac{1}{\tau-2}}, \qquad c_1 := c_2c_0^{\left[\frac{N-2}{2-\tau}\right](2-\tau)-N+4-\tau}.$$

Then (5) holds for $n = \left[\frac{N-2}{2-\tau}\right]+1$ and for larger n we have

$$|F_n(x,y,\mu)| \leqq c_1(c_0|\mu_2|)^{N-2-(n-1)(2-\tau)}\cdot$$
$$\cdot c_3\int_{\mathbf{R}^N} e^{-2\alpha|x-u||\mu_2|-\alpha|y-u||\mu_2|}|x-u|^{2-N}|q(u)|du \leqq$$
$$\leqq c_1c_3(c_0|\mu_2|)^{N-2-(n-1)(2-\tau)}e^{-\alpha|x-y||\mu_2|}c_4|\mu_2|^{\tau-2} =$$
$$= c_1(c_0|\mu_2|)^{N-2-n(2-\tau)}e^{-\alpha|x-y||\mu_2|}$$

which proves Lemma 4.

In what follows we choose $\mu_1 > 0$ large enough to satisfy $c_0|\mu_2| \geqq 2$. Consider the function

$$E_\mu(x,y) := \sum_{n=0}^{\infty} F_n(x,y,\mu); \tag{6}$$

the series is uniformly convergent in $x, y \in \mathbf{R}^N$ if $\mu \in Z_0$ is fixed. Further we have

THEOREM 1. *We have*

$$|E_\mu(x,y)| \leqq c|x-y|^{2-N}e^{-\frac{\alpha}{2}|x-y||\mu_2|} \tag{7}$$

uniformly in $x, y \in \mathbf{R}^N$, $\mu \in Z_0$; *moreover*

$$E_\mu(y,x) = E_\mu(x,y). \tag{8}$$

PROOF. By Lemma 3

$$\sum_{n=0}^{\left[\frac{N-2}{2-\tau}\right]} |F_n(x,y,\mu)| \leqq \begin{cases} c|x-y|^{2-N}e^{-\alpha|x-y||\mu_2|} & \text{if } |x-y| \leqq 1 \\ ce^{-\alpha|x-y||\mu_2|} & \text{if } |x-y| \geqq 1. \end{cases}$$

From the estimate

$$e^{-\frac{\alpha}{2}|x-y||\mu_2|} \leqq c|x-y|^{2-N}, \qquad |x-y| \geqq 1$$

we obtain

$$\sum_{n=0}^{[\frac{N-2}{2-\tau}]} |F_n(x,y,\mu)| \leqq c|x-y|^{2-N} e^{-\frac{\alpha}{2}|x-y||\mu_2|}, \quad x,y \in \mathbf{R}^N, \ \mu \in Z_0.$$

On the other hand (5) implies that

$$\sum_{n=[\frac{N-2}{2-\tau}]+1}^{\infty} |F_n(x,y,\mu)| \leqq ce^{-\alpha|x-y||\mu_2|}$$

which proves (7). To see (8) consider the statements

$$(\mathrm{A}_n) \ F_n(x,y,\mu) = \int_{\mathbf{R}^N} F_i(x,u,\mu)F_{n-1-i}(u,y,\mu)q(u)du, \quad 0 \leqq i \leqq n-1$$

$$(\mathrm{B}_n) \qquad F_n(y,x,\mu) = F_n(x,y,\mu).$$

Now (B_0), (A_1) are trivial and (A_n), (B_{n-1}) imply (B_n) since

$$F_n(x,y,\mu) = -\int_{\mathbf{R}^N} F_{n-1}(x,u,\mu)E_0(u,y,\mu)q(u)du =$$

$$= -\int_{\mathbf{R}^N} E_0(y,u,\mu)F_{n-1}(u,x,\mu)q(u)du = F_n(y,x,\mu).$$

On the other hand $(\mathrm{B}_0), \dots, (\mathrm{B}_{n-1})$ imply (A_n). Indeed, (A_n) holds for $i = 0$, and if it holds for some $0 \leqq i \leqq n-2$ then it also holds for $i+1$, since

$$F_n(x,y,\mu) = \int_{\mathbf{R}^N} F_i(x,u,\mu)q(u)\left(\int_{\mathbf{R}^N} E_0(u,v,\mu)F_{n-i-2}(v,y,\mu)q(v)dv\right)du =$$

$$= \int_{\mathbf{R}^N} F_{n-i-2}(v,y,\mu)q(v)\left(\int_{\mathbf{R}^N} E_0(v,u,\mu)F_i(u,x,\mu)q(u)du\right)dv =$$

$$= -\int_{\mathbf{R}^N} F_{n-i-1}(v,y,\mu)q(v)F_{i+1}(v,x,\mu)dv =$$

$$= -\int_{\mathbf{R}^N} F_{i+1}(x,u,\mu)F_{n-i-2}(u,y,\mu)q(u)du.$$

Consequently (A_n) and (B_n) hold for all n and then (8) follows. Theorem 1 is proved.

LEMMA 5. a) *$E_\mu(x,y)$ is an (exponentially decreasing) solution of the integral equation*

$$E_\mu(x,y) = E_0(x,y,\mu) - \int_{\mathbf{R}^N} E_0(x,u,\mu)E_\mu(u,y)q(u)du. \tag{9}$$

b) *E_μ is the Green function of $-\Delta + q - \mu^2 + \mu_1$ i.e.*

$$\hat{E}_\mu\left((-\Delta + q - \mu^2 + \mu_1)\varphi\right) = \varphi, \qquad \varphi \in C_0^\infty(\mathbf{R}^N) \tag{10}$$

where $\hat{E}_\mu$ denotes the transformation with kernel E_μ:

$$\hat{E}_\mu f(x) := \int_{\mathbf{R}^N} E_\mu(x,y)f(y)dy.$$

PROOF. Applying Lemmas 1, 2, 4 we see that

$$\sum_{n=N_1}^{\infty} \int_{\mathbf{R}^N} |E_0(x,u,\mu)F_n(u,y,\mu)q(u)|du \leqq$$

$$\leqq c \int_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{2-N} \sum_{n=N_1}^{\infty} (c_0|\mu_2|)^{N-2-n(2-\tau)}|q(u)|du \leqq$$

$$\leqq c \sum_{n=N_1}^{\infty} 2^{N-2-n(2-\tau)} \int_{\mathbf{R}^N} e^{-\alpha|x-u||\mu_2|}|x-u|^{2-N}|q(u)|du < \infty.$$

Consequently

$$E_\mu(x,y) = \sum_{n=0}^{\infty} F_n(x,y,\mu) =$$

$$= E_0(x,y,\mu) - \sum_{n=0}^{\infty} \int_{\mathbf{R}^N} E_0(x,u,\mu)F_n(u,y,\mu)q(u)du =$$

$$= E_0(x,y,\mu) - \int_{\mathbf{R}^N} E_0(x,u,\mu)E_\mu(u,y)q(u)du$$

follows from Lebesgue's dominated convergence theorem. Thus (9) is proved.

The statement (10) is known in case $q = 0$. If the potential exists, we have

$$\hat{E}_\mu((-\Delta + q - \mu^2 + \mu_1)\varphi)(x) = \int_{\mathbf{R}^N} q(y)\varphi(y)E_\mu(x,y)dy+$$

$$+\int_{\mathbf{R}^N} (-\Delta_y - \mu^2 + \mu_1)\varphi(y)\cdot E_\mu(x,y)dy = \int_{\mathbf{R}^N} q(y)\varphi(y)E_\mu(x,y)dy+$$

$$+\int_{\mathbf{R}^N} (-\Delta_y - \mu^2 + \mu_1)\varphi(y)\cdot E_0(x,y,\mu)dy - \int_{\mathbf{R}^N} (-\Delta_y - \mu^2 + \mu_1)\varphi(y)\cdot$$

$$\cdot\Big(\int_{\mathbf{R}^N} E_0(x,u,\mu)E_\mu(u,y)q(u)du\Big)dy =: I_1 + I_2 + I_3.$$

First we observe that $I_2 = \varphi(x)$. (8) implies that in I_3

$$\int_{\mathbf{R}^N} E_0(x,u,\mu)E_\mu(u,y)q(u)du = \int_{\mathbf{R}^N} E_0(y,u,\mu)E_\mu(u,x)q(u)du.$$

Now we shall show that for fixed x,

(11) $\quad (\Delta_y + \mu^2 - \mu_1)\varphi(y)\cdot E_0(y,u,\mu)E_\mu(u,x)q(u) \in L_1(\mathbf{R}^N \times \mathbf{R}^N).$

Indeed,

$$\int_{\mathbf{R}^N} |E_0(y,u,\mu)||E_\mu(u,x)q(u)|du \leqq$$

$$\leqq \int_{\mathbf{R}^N} ce^{-2\alpha|y-u||\mu_2|-\frac{\alpha}{2}|x-u||\mu_2|}|y-u|^{2-N}|x-u|^{2-N}|q(u)|du \leqq$$

$$\leqq \int_{|x-u|\geqq\frac{|x-y|}{2}} + \int_{|y-u|\geqq\frac{|x-y|}{2}} \leqq c|x-y|^{2-N}\int_{\mathbf{R}^N} e^{-2\alpha|y-u||\mu_2|}|y-u|^{2-N}|q(u)|du+$$

$$+c|x-y|^{2-N}\int_{\mathbf{R}^N} e^{-\frac{\alpha}{2}|x-u||\mu_2|}|x-u|^{2-N}|q(u)|du \leqq c|x-y|^{2-N}$$

and for fixed x

$$\int_{\operatorname{supp}\varphi} |x-y|^{2-N}dy \leqq c\int_0^R r^{N-1}r^{2-N}dr < \infty$$

which proves (11). Consequently the Fubini theorem can be applied in I_3:

$$I_3 = -\int\limits_{\mathbf{R}^N} E_\mu(x,u)q(u)\Bigg(\int\limits_{\mathbf{R}^N} E_0(u,y,\mu)(-\Delta_y - \mu^2 + \mu_1)\varphi(y)dy\Bigg)du =$$

$$= -\int\limits_{\mathbf{R}^N} E_\mu(x,u)q(u)\varphi(u)du = -I_1$$

which implies (10).

Introduce the functions

$$G_\mu^{(1)}(x,y) := E_\mu(x,y),$$

$$G_\mu^{(\nu)}(x,y) := \int\limits_{\mathbf{R}^N} E_\mu(x,u)G_\mu^{(\nu-1)}(u,y)du, \quad \nu = 2,3,\dots.$$

LEMMA 6. *Let $1 \leqq \nu < \frac{N}{2}$ be an integer, then*

$$\left|G_\mu^{(\nu)}(x,y)\right| \leqq c|x-y|^{2\nu-N}e^{-\frac{\alpha}{4}|x-y||\mu_2|} \tag{12}$$

uniformly in $x, y \in \mathbf{R}^N$, $\mu \in Z_0$.

PROOF. The case $\nu = 1$ is proved. We apply induction on ν:

$$|G_\mu^{(\nu)}(x,y)| \leqq c\int\limits_{\mathbf{R}^N} e^{-\frac{\alpha}{2}|x-u||\mu_2|-\frac{\alpha}{4}|u-y||\mu_2|}|x-u|^{2-N}|u-y|^{2\nu-2-N}du \leqq$$

$$\leqq ce^{-\frac{\alpha}{4}|x-y||\mu_2|}\int\limits_{\mathbf{R}^N} e^{-\frac{\alpha}{4}|x-u||\mu_2|}|x-u|^{2-N}|u-y|^{2\nu-2-N}du.$$

Here we set

$$\int\limits_{\mathbf{R}^N} = \int\limits_{|x-u|\leqq\frac{|x-y|}{2}} + \int\limits_{|y-u|\leqq\frac{|x-y|}{2}} + \int\limits_{\begin{cases}|x-u|\geqq\frac{|x-y|}{2}\\ |u-y|\geqq\frac{|x-y|}{2}\end{cases}}.$$

Now

$$\int\limits_{|x-u|\leqq\frac{|x-y|}{2}} \leqq c|x-y|^{2\nu-2-N}\int\limits_{|x-u|\leqq\frac{|x-y|}{2}} |x-u|^{2-N}du =$$

$$= c|x-y|^{2\nu-2-N}\int\limits_0^{\frac{|x-y|}{2}} r^{N-1}r^{2-N}dr = c|x-y|^{2\nu-N},$$

$$\int\limits_{|y-u|\leqq\frac{|x-y|}{2}} \leqq c|x-y|^{2-N} \int\limits_{|y-u|\leqq\frac{|x-y|}{2}} |u-y|^{2\nu-2-N}du =$$

$$= c|x-y|^{2-N} \int\limits_0^{\frac{|x-y|}{2}} r^{N-1}r^{2\nu-2-N}dr = c|x-y|^{2\nu-N},$$

$$\int\limits_{\left\{\begin{array}{l}|x-u|\geqq\frac{|x-y|}{2}\\ |u-y|\geqq\frac{|x-y|}{2}\end{array}\right.} \leqq c \int\limits_{\frac{|x-y|}{2}\leqq|x-u|} e^{-\frac{\alpha}{4}|x-u||\mu_2|}|x-u|^{2\nu-2N}du =$$

$$= c\int\limits_{\frac{|x-y|}{2}}^{\infty} r^{2\nu-N-1}e^{-\frac{\alpha}{4}r|\mu_2|}dr \leqq$$

$$\leqq \begin{cases} c\int\limits_{\frac{|x-y|}{2}}^{1} r^{2\nu-N-1}dr + c\int\limits_1^{\infty} r^{2\nu-N-1}e^{-\frac{\alpha}{4}r|\mu_2|}dr \leqq c|x-y|^{2\nu-N} & \text{if } |x-y|\leqq 2,\\ c|x-y|^{2\nu-N}\int\limits_1^{\infty} e^{-\frac{\alpha}{4}r|\mu_2|}\frac{1}{r}dr \leqq c|x-y|^{2\nu-N} & \text{if } |x-y|\geqq 2.\end{cases}$$

The above estimates imply Lemma 6.

REMARK. If N is even and $\nu = \frac{N}{2}$, the above calculation shows that

$$(13)\qquad |G_\mu^{(\nu)}(x,y)| \leqq \begin{cases} c\ln\frac{2}{|x-y|}\cdot e^{-\frac{\alpha}{4}|x-y||\mu_2|} & \text{if } |x-y|\leqq 2,\\ ce^{-\frac{\alpha}{4}|x-y||\mu_2|} & \text{if } |x-y|\geqq 2.\end{cases}$$

Now introduce the notation $\nu_0 := \left[\frac{N}{4}\right]$, and take any value σ, $\frac{N}{4} < \sigma < \nu_0+1$. Consider the function

$$G^{(\sigma)}(x,y) := \frac{\Gamma(\nu_0+1)}{\Gamma(\nu_0+1-\sigma)\Gamma(\sigma)}\int\limits_0^{\infty} r^{-\sigma+\nu_0}G_{i\sqrt{r}}^{(\nu_0+1)}(x,y)dr.$$

LEMMA 7. *If $N\neq 4$ (and $N\geqq 3$) then*

$$(14)\qquad |G^{(\sigma)}(x,y)| \leqq c|x-y|^{2\sigma-N}e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}}.$$

In case $N=4$ we have

$$(14')\qquad |G^{\sigma}(x,y)| \leqq \begin{cases} c|x-y|^{2\sigma-N}e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}} & \textit{if } |x-y|\geqq 2,\\ c\ln\frac{2}{|x-y|}\cdot|x-y|^{2\sigma-N}e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}} & \textit{if } |x-y|\leqq 2.\end{cases}$$

PROOF. In case $N \neq 4$ we have $\nu_0 + 1 < \frac{N}{2}$, hence by Lemma 6

$$|G^\sigma(x,y)| \leqq c \int_0^\infty r^{\nu_0-\sigma} \left| G^{(\nu_0+1)}_{i\sqrt{r}}(x,y) \right| dr \leqq$$

$$\leqq c \int_0^\infty r^{\nu_0-\sigma} |x-y|^{2\nu_0+2-N} e^{-\frac{\alpha}{4}|x-y|\sqrt{r+\mu_1}} dr \leqq$$

$$\leqq c|x-y|^{2\nu_0+2-N} e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}} \int_0^\infty r^{\nu_0-\sigma} e^{-\frac{\alpha}{8}|x-y|\sqrt{r}} dr.$$

Use the decomposition $\int_0^\infty = \int_0^{|x-y|^{-2}} + \int_{|x-y|^{-2}}^\infty$. Then

$$\int_0^{|x-y|^{-2}} \leqq \int_0^{|x-y|^{-2}} r^{\nu_0-\sigma} dr = c|x-y|^{2(\sigma-\nu_0-1)},$$

and substituting $\frac{\alpha}{8}|x-y|\sqrt{r} = u$ we get

$$\int_{|x-y|^{-2}}^\infty \leqq |x-y|^{2(\sigma-\nu_0)} \int_{|x-y|^{-2}}^\infty e^{-\frac{\alpha}{8}|x-y|\sqrt{r}} dr =$$

$$= c|x-y|^{2(\sigma-\nu_0)} \int_{\frac{\alpha}{8}}^\infty e^{-u} \frac{u}{|x-y|^2} du = c|x-y|^{2(\sigma-\nu_0-1)}$$

which implies (14). In case $N = 4$ and $|x-y| \geqq 2$ the estimate (13) for $G^{(\nu_0+1)}_\mu = G^{(2)}_\mu$ is identical with (12) hence the same proof works. If $|x-y| \leqq 2$, then

$$|G^{(\sigma)}(x,y)| \leqq c \int_0^\infty r^{1-\sigma} \ln \frac{2}{|x-y|} \cdot e^{-\frac{\alpha}{4}|x-y|\sqrt{r+\mu_1}} dr \leqq$$

$$\leqq c \ln \frac{2}{|x-y|} \cdot e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}} \int_0^\infty r^{1-\sigma} e^{-\frac{\alpha}{8}|x-y|\sqrt{r}} dr \leqq$$

$$\leqq c \ln \frac{2}{|x-y|} \cdot e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}} |x-y|^{2(\sigma-2)}.$$

Lemma 7 is proved.

LEMMA 8. *Let* $\tau < \frac{3}{2}$ *in case* $m = 3$ *and* $\tau < 2$ *in case* $m \geqq 4$. *Then*

a) $\hat{G}_\mu^{(\nu)} = (-\Delta + q - \mu^2 + \mu_1)^{-\nu}\colon L_2(\mathbf{R}^N) \to L_2(\mathbf{R}^N)$, $1 \leqq \nu \leqq \nu_0 + 1$, $\mu = it$, $t > 0$,

b) $\hat{G}^{(\sigma)} = (-\Delta + q + \mu_1)^{-\sigma}\colon L_2(\mathbf{R}^N) \to L_2(\mathbf{R}^N)$

are continuous mappings.

PROOF. The above conditions imply that

$$-\Delta + q - \mu^2 + \mu_1 \colon L_2^2(\mathbf{R}^N) \to L_2(\mathbf{R}^N)$$

is an isomorphism onto L_2 (see [4], [7]), consequently $(-\Delta + q - \mu^2 + \mu_1)^{-1}$ is a bounded operator of $L_2(\mathbf{R}^N)$. From (12) and (13) we see that

$$|G_\mu^{(\nu)}(x,y)| \leqq K(x-y), \qquad 1 \leqq \nu \leqq \nu_0 + 1$$

where $K(x) \in L_1(\mathbf{R}^N)$. Consequently $\hat{G}_\mu^{(\nu)}$ is a bounded operator of $L_2(\mathbf{R}^N)$. If $\varphi \in C_0^\infty$, then

$$\int_{\mathbf{R}^N} G_\mu(x,y)\Big(\int_{\mathbf{R}^N} G_\mu^{(\nu)}(y,z)\varphi(z)dz\Big)dy =$$

$$= \int_{\mathbf{R}^N} \varphi(z)\Big(\int_{\mathbf{R}^N} G_\mu(x,y)G_\mu^{(\nu)}(y,z)dy\Big)dz = \int_{\mathbf{R}^N} G_\mu^{(\nu+1)}(x,z)\varphi(z)dz$$

i.e. $\hat{G}_\mu(\hat{G}_\mu^{(\nu)}\varphi) = \hat{G}_\mu^{(\nu+1)}\varphi$. Since the operators $\hat{G}_\mu^{(\nu)}$ are bounded, it follows that $\hat{G}_\mu^{(\nu)}$ is the ν-th iteration of the operator $\hat{G}_\mu$. The set C_0^∞ is dense in $L_2^2(\mathbf{R}^N)$, hence $(-\Delta + q - \mu^2 + \mu_1)(C_0^\infty)$ is dense in $L_2(\mathbf{R}^N)$. By (10) the continuous mappings $(-\Delta+q-\mu^2+\mu_1)^{-1}$ and $\hat{G}_\mu$ are identical on this dense set, hence $(-\Delta+q-\mu^2+\mu_1)^{-1} = \hat{G}_\mu$ and then $(-\Delta+q-\mu^2+\mu_1)^{-\nu} = \hat{G}_\mu^{(\nu)}$. This proves a). To see b), argue as follows. The operator $-\Delta + q + \mu_1$ is positive for sufficiently large $\mu_1 > 0$ (see [7]). In Triebel [5], 1.15.1 a formula is given for the fractional powers of positive operators. In our case this means the following statement: There exists a dense linear subset $D \subset L_2(\mathbf{R}^N)$ such that

$$(-\Delta + q + \mu_1)^{-\sigma} f = \frac{\Gamma(\nu_0+1)}{\Gamma(\nu_0+1-\sigma)\Gamma(\sigma)}\cdot$$

$$\cdot \int_0^\infty r^{\nu_0-\sigma}(-\Delta + q + \mu_1 + r)^{-\nu_0-1} f dr, \qquad f \in D$$

the integral being convergent in $L_2(\mathbf{R}^N)$. Consequently

$$(-\Delta+q+\mu_1)^{-\sigma}f=$$
$$=\frac{\Gamma(\nu_0+1)}{\Gamma(\nu_0+1-\sigma)\Gamma(\sigma)}\cdot\int_0^\infty r^{\nu_0-\sigma}\left[\int_{\mathbf{R}^N} G^{(\nu_0+1)}_{i\sqrt{r}}(x,y)f(y)dy\right]dr=$$
$$=\int_{\mathbf{R}^N} f(y)\left[\frac{\Gamma(\nu_0+1)}{\Gamma(\nu_0+1-\sigma)\Gamma(\sigma)}\int_0^\infty r^{\nu_0-\sigma}G^{(\nu_0+1)}_{i\sqrt{r}}(x,y)dr\right]dy=$$
$$=\int_{\mathbf{R}^N} G^{(\sigma)}(x,y)f(y)dy,\quad f\in D.$$

The application of the Fubini theorem was correct, since

$$\int_0^\infty\int_{\mathbf{R}^N} r^{\nu_0-\sigma}\left|G^{(\nu_0+1)}_{i\sqrt{r}}(x,y)f(y)\right|dydr\leqq$$
$$\leqq c\int_{\mathbf{R}^N}|x-y|^{2\nu_0+2-N}\left(1+\left|\ln\frac{2}{|x-y|}\right|\right)\cdot$$
$$\cdot\int_0^\infty r^{\nu_0-\sigma}e^{-\frac{\alpha}{16}|x-y|\sqrt{r}}dr\cdot e^{-\frac{\alpha}{16}|x-y|\sqrt{\mu_1}}|f(y)|dy\leqq$$
$$\leqq c\int_{\mathbf{R}^N}|x-y|^{2\sigma-N}\left(1+\left|\ln\frac{2}{|x-y|}\right|\right)e^{-\frac{\alpha}{16}|x-y|\sqrt{\mu_1}}|f(y)|dy\leqq$$
$$\leqq c\|f\|_{L_2}\cdot\left(\int_{\mathbf{R}^N}|x-y|^{4\sigma-2N}\left(1+\left|\ln\frac{2}{|x-y|}\right|\right)^2e^{-\frac{\alpha}{8}|x-y|\sqrt{\mu_1}}dy\right)^{\frac{1}{2}}=$$
$$=c\|f\|_{L_2}\left(\int_0^\infty r^{4\sigma-N-1}\left(1+\left|\ln\frac{2}{|x-y|}\right|\right)^2e^{-\frac{\alpha}{8}r\sqrt{\mu_1}}dr\right)^{\frac{1}{2}}=c\|f\|_{L_2}<\infty.$$

The continuous mappings $\hat{G}^{(\sigma)}$ and $(-\Delta+q+\mu_1)^{-\sigma}$ are identical on the dense set D, hence they are equal on L_2. Lemma 8 is proved.

Before stating Theorem 2 we need the Neumann spectral theorem in the following form:

THEOREM A (Maurin [9], IX.§4). *Let A be a selfadjoint operator in a* *separable Hilbert space H. Then there exists a unitary isomorphism*

$$U\colon H \to \int_{\Lambda} \oplus \hat{H}_\lambda d\mu(\lambda) \tag{15}$$

where $\hat{H}_\lambda$ are complex ℓ_2-spaces of sequences with $n(\lambda) \leqq \infty$ coordinates, $\Lambda \subset \mathbf{R}$ is the spectrum of A, μ is a nonnegative Borel measure on Λ and the direct integral space on the right of (15) is the space of all vectors $h(\lambda) = (h_k(\lambda))_{k=1}^{n(\lambda)}$ for which

$$\|h\|^2 := \int_{\Lambda} \sum_{k=1}^{n(\lambda)} |h_k(\lambda)|^2 d\mu(\lambda) < \infty. \tag{16}$$

Finally, if $\varphi(\lambda)$ is an a.e. finite Borel measurable function, then for the domain $D(\varphi(A))$ of the operator $\varphi(A)$

$$U(D(\varphi(A))) = \left\{ h : \int_{\Lambda} \sum_{k=1}^{n(\lambda)} |h_k(\lambda)|^2 |\varphi(\lambda)|^2 d\mu(\lambda) < \infty \right\} \tag{17}$$

and

$$U\varphi(A)U^{-1}\colon h(\lambda) \mapsto \varphi(\lambda)h(\lambda) \quad \textit{for} \quad h \in U(D(\varphi(A))). \tag{18}$$

As we have seen, the kernel function $G^{(\sigma)}$ of the operator $\mathbf{L}_{\mu_1}^{-\sigma}$ satisfies for $\frac{N}{4} < \sigma < \nu_0 + 1$ the estimate

$$\sup_{x \in \mathbf{R}^N} \int_{\mathbf{R}^N} |G^{(\sigma)}(x, y)|^2 dy < \infty. \tag{19}$$

On the other hand $\mathbf{L}_{\mu_1} : L_2^2 \to L_2$ is an isomorphism (see [7]) and $D(\mathbf{L}_{\mu_1}) = L_2^2$ is dense in L_2. This implies that $\mathbf{L}_{\mu_1}$ is selfadjoint, and then Theorem A applies. Define the functions

$$u(x, \lambda) = (u_k(x, \lambda))_{k=1}^{n(\lambda)} := (\lambda + \mu_1)^\sigma U(G^{(\sigma)}(x, \cdot))(\lambda). \tag{20}$$

Since U is norm preserving, we have

THEOREM 2. *The following square sum estimate holds:*

$$\sup_{x \in \mathbf{R}^N} \int_{\mu_0}^{\infty} \sum_{k=1}^{n(\lambda)} |u_k(x, \lambda)|^2 (\lambda + \mu_1)^{-2\sigma} d\mu(\lambda) < \infty.$$

The functions $u(\cdot, \lambda)$ are generalized eigenfunctions of $\mathbf{L}_{\mu_1}$ with eigenvalue λ; for their properties see [9], XVII.

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ON THE DISTRIBUTION OF TRANSLATES OF ADDITIVE FUNCTIONS

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0. Introduction

Our main purpose is to give necessary and sufficient conditions for the existence of the limit distribution of $P(E)f(n)-\alpha(x)$, where f is an additive function and $P(E)f(n)=a_0f(n)+a_1f(n+1)+\ldots+a_kf(n+k)$.

A sharp theorem can be derived from a recent result due to A. Hildebrand [5], if $P(z)=\sum a_jz^j$ satisfies the condition $P(1)\neq 0$ or $P'(1)\neq 0$. We consider the case $(z-1)^2 \mid P(z)$ (see Theorem 1). In Section 2 we shall investigate conditions under which

$$\sum_{n\leqq x}\phi(P(E)f(n)-\alpha(x))\ll x$$

holds true.

In Section 3 we give a necessary and sufficient condition for the existence of the limit distributions of $g(n+1)-f(n)$, where f and g are additive functions.

Theorem 2 is a special case of a theorem due to P. D. T. A. Elliott [6] (see 1.13). We derive this theorem from Hildebrand's result (see 1.13) very simply.

1.1. An infinite sequence $\{u_n\}_{n\in\mathbf{N}}$ of real (or complex) numbers is called a tight sequence if for every $\delta>0$ there exists a number $c<\infty$, such that

$$\sup_{n\geqq 1} x^{-1}\#\{n\leqq x \mid |u_n|>c\}<\delta. \tag{1.1}$$

Let $\mathcal{T}$ be the set of tight sequences.

1.2. Let $\mathcal{T}'$ denote the set of those sequences $\{u_n\}_{n\in\mathbf{N}}$ for which the relation

$$\sup_{x\geqq 1} x^{-1}\#\{n\leqq x \mid |u_n-\alpha(x)|>c\}<\delta \tag{1.2}$$

holds for every $\delta>0$ with a suitable constant $c=c(\delta)$ and with a suitable function $\alpha(x)$.

1.3. Let E denote the shifting operator, $Eu_n = u_{n+1}$. Let $Iu_n = u_n$, $\Delta = E - I$, and for an arbitrary polynomial $P(z) \in C[z]$, $P(z) = a_0 + a_1 z + \ldots + a_k z^k$ let $P(E)u_n := a_0 u_n + a_1 u_{n+1} + \ldots + a_k u_{n+k}$.

It is clear that $\{u_n\} \in \mathcal{T}'$ implies that $\{P(E)u_n\} \in \mathcal{T}'$, $\{\Delta u_n\} \in \mathcal{T}$, and that $\{P(E)u_n\} \in \mathcal{T}$ if $P(1) = 0$.

1.4. Let $\mathcal{A}$ be the class of real valued additive functions. Let $\mathcal{P} = \{p\}$ denote the set of all prime numbers, and let $\mathcal{P}^*$ be the set of all prime powers.

1.5. We say that a sequence $\{u_n\}_{n \in \mathbf{N}}$ of real numbers has a limit distribution if there exists a distribution function F such that

$$\lim_{x \to \infty} x^{-1} \#\{n \leqq x \mid u_n < y\} = F(y) \tag{1.3}$$

holds for all continuity points y of F.

Let $\mathcal{D}$ denote the set of the sequences having a limit distribution.

1.6. We say that the sequence $\{u_n\}_{n \in \mathbf{N}}$ of real numbers has a limit distribution with the centering function $\alpha(x)$ if there exists a distribution $G(x)$ such that

$$\lim x^{-1} \#\{n \leqq x \mid u_n - \alpha(x) < y\} = G(y) \tag{1.4}$$

holds for all continuity points of G.

Let $\mathcal{D}_\alpha$ denote the set of those sequences which have limit distributions with the centering function $\alpha(x)$.

One can easily see that $\{u_n\} \in \mathcal{D}_{\alpha_1} \cap \mathcal{D}_{\alpha_2}$ if and only if $\{u_n\} \in \mathcal{D}_{\alpha_1}$ and $\alpha_1(x) - \alpha_2(x) \to c$ where c is an arbitrary (finite) number.

1.7. For an arbitrary $f \colon \mathcal{P} \to \mathbf{R}$ let (E_f), (E_f^*), (B_f) denote the conditions:

(E_f): $\displaystyle\sum_{|f(p)| \leqq 1} \frac{f(p)}{p}$ is convergent,

(E_f^*): $\displaystyle\sum_{|f(p)| > 1} \frac{1}{p}$ is convergent,

(B_f): $\displaystyle\sum_{\substack{|f(p)| \leqq 1 \\ p \leqq x}} \frac{1}{p}$ is bounded in x.

1.8. A classical theorem due to P. Erdős and A. Wintner [1] asserts that $f \in A \cap \mathcal{D}$ if and only if (E_f), (E_{f^2}), (E_f^*) hold true.

A function $f \in \mathcal{A}$ belongs to $\mathcal{D}_\alpha$ with a suitable centering function $\alpha(x)$ if and only if f can be written as $f(n) = \lambda \log n + h(n)$ with some $\lambda \in \mathbf{R}$, where h satisfies the conditions (E_{h^2}), (E_h^*). Furthermore, if $f \in \mathcal{D}_\alpha$, then $\alpha(x) = \lambda \log x + A(h, x) + O(1)$, where

$$A(h; x) := \sum_{\substack{|h(p)| \leqq 1 \\ p \leqq x}} \frac{h(p)}{p} \tag{1.5}$$

(cf. Theorem 7.1 in [2]).

1.9. We say after P. Erdős that $f \in \mathcal{A}$ is finitely distributed if there exists a sequence $x_1 < x_2 < \dots$ of real numbers, $x_\nu \to \infty$, some positive numbers δ and c, and for every ν at least $k \geqq \delta x_\nu$ integers $a_1 < a_2 < \dots < a_k \leqq x_\nu$ such that $|f(a_i) - f(a_j)| \leqq c$ for every $i \neq j$.

Erdős proved [1] that f is finitely distributed if and only if $f = \lambda \log + h$, with some $\lambda \in \mathbf{R}$ and with some h for which (E_{h^2}), (E_h^*) hold true.

Let $\mathcal{A}_{fi}$ denote the class of finitely distributed (additive) functions.

It is easy to characterize all those $g \in \mathcal{A}_{fi}$ for which $P(E)g :=$ $:= \{P(E)g(n)\}_{n \in \mathbf{N}} \in \mathcal{D}_\alpha$ or $\mathcal{D}$.

Let $g = \lambda \log + h$, such that (E_{h^2}), (E_h^*) hold. Then $P(E)g \in \mathcal{D}_\alpha$ and $P(E)g \in \mathcal{D}$ if $P(1) = 0$. If $P(1) \neq 0$ then $P(E)g \in \mathcal{D}$ if and only if $\lambda = 0$ and (E_f) holds.

These assertions have been known for a long time.

It seems probable that if $f \in \mathcal{A}$ and $P \in \mathbf{R}[z]$, P is not identically zero, then $P(E)f \in \mathcal{D}_\alpha$ only if $f \in \mathcal{A}_{fi}$. This assertion was proved for $P = \Delta$ only recently by A. Hildebrand [5]. Hildebrand proved somewhat more in his paper [5], namely that $f \in \mathcal{A}$, $\Delta f \in \mathcal{T}$ imply that f is finitely distributed (see [5], proof of Theorem 1; necessity).

It is quite natural to formulate the following

CONJECTURE. Let $f \in \mathcal{A}$ and $P \in \mathbf{C}[z]$, P not identically zero. Assume that $P(E)f \in \mathcal{T}'$. Then $f \in \mathcal{A}_{fi}$.

1.10. LEMMA A. *Let $f \in \mathcal{A}$, $|f(p^\alpha)| \leqq 1$ if $p^\alpha \in P^*$. Assume that (E_{f^2}) holds. Then for every $\gamma > 0$,*

$$\sum_{n \leqq x} |f(n) - A_x|^\gamma \ll x, \tag{1.6}$$

where

$$A_x = \sum_{p \leqq x} \frac{f(p)}{p}, \tag{1.7}$$

and the constant implied by $\ll$ may depend on f and γ.

(1.6) is a special case of Elliott's moment inequality [7].

1.11. The following assertion is an immediate consequence of Hildebrand's theorem and of Erdős' theorem on finitely distributed functions.

Let $\phi\colon \mathbf{R} \to \mathbf{R}_{\geqq 0}$ be an arbitrary function such that $\phi(x) \to \infty$ as $|x| \to \infty$. Let $f \in \mathcal{A}$ and assume that

$$x^{-1} \sum_{n \leqq x} \phi(f(n) - \alpha(x)) = O(1) \qquad \text{as } x \to \infty$$

or

$$x^{-1}\sum_{n\leqq x}\phi(\Delta f(n)) = O(1) \qquad \text{as } x\to\infty$$

holds. Then $f\in\mathcal{A}_{fi}$, $f=\lambda\log+h$, (E_{h^2}), (E_h^*) are satisfied, $\alpha(x) = \lambda\log x + A(h;x)+O(1)$.

2. On the distribution of $P(E)f(n)$

2.1. Let $b(H)$ be a monotonically increasing function defined on $[1,\infty)$ such that $b(1)=1$, $b(H)\to\infty$ as $H\to\infty$. Let $\mathcal{T}_b$ denote the set of those sequences $u:=\{u_n\}_{n\in\mathbf{N}}$ of complex numbers for which

$$\sup_{x\geqq 1} x^{-1}\#\{n\leqq x \mid |u_n|>H\} < \frac{c_1}{b(H)}, \tag{2.1}$$

where c_1 is a suitable constant that may depend on u.

In what follows we shall assume that $b(H)$ is slowly growing, namely that

$$b(H^2)\leqq c_2 b(H) \tag{2.2}$$

holds with some suitable constant c_2.

2.2. Let us fix a sequence $\mathbf{u}$, and denote by $I(\mathbf{u},b)$ the set of those polynomials $P\in\mathbf{C}[z]$ for which $P(E)\mathbf{u}:=\{P(E)u_n\}_{n\in\mathbf{N}}\in\mathcal{T}_b$. One can see easily that $I(\mathbf{u},b)$ is an ideal.

2.3. STATEMENT. *Let $f\in\mathcal{A}$, $P\in\mathbf{C}[z]$ be a non-zero polynomial for which $P(E)f\in\mathcal{T}_b$. Then the generating polynomial $Q(z)$ of $I(f,b)$ has the form $(z-1)^k$ where $k\geqq 0$.*

PROOF. The assertion can be proved by a method which was worked out by Elliott and Kátai, independently.

Let $\deg P = M$, $\deg Q = k$, $Q(z)=\prod_{j=1}^{k}(z-\theta_j)$. The assertion is clear if $k=0$ or if $M=0$. Assume that $k\geqq 1$.

Let $m\in\mathbf{N}$, $Q_m(z)=\prod_{j=1}^{k}(z-\theta_j^m)$. Since $Q(z)$ divides $Q_m(z^m)$, therefore $Q_m(E^m)f\in I(f,b)$. Let $Q_m(z)=\beta_0+\beta_1 z+\ldots+\beta_k z^k$, $\beta_k=1$. It is clear that $\beta_0\neq 0$. Let $u_n:=Q_m(E^m)f(mn)$, i.e. u_n is the value of $Q_m(E^m)f$ at place mn. Let $v_n:=Q_m(E)f(n)$, $\Delta(m,n):=u_n-v_n$. Then $\mathbf{u}\in\mathcal{T}_b$, consequently $Q(E)\mathbf{u}\in\mathcal{T}_b$. Furthermore $Q(E)\mathbf{v}=Q(E)Q_m(E)\mathbf{f}=Q_m(E)Q(E)\mathbf{f}\in\mathcal{T}_b$, and so $Q(E)\Delta(m,n)\in\mathcal{T}_b$. It has the form

$$Q(E)\Delta(m,n)=\sum_{j=0}^{k}\alpha_j\Delta(m,n+j)=\sum_{h=0}^{2k}\gamma_h\{f(m(n+h)-f(n+h)\}. \tag{2.3}$$

Here $\gamma_0 \neq 0$, $\gamma_{2k} = 1$. Let $m = p \in \mathcal{P}$.

Thus we have

$$(2.4)\qquad \sum_{\alpha=0}^{\infty} x^{-1}\#\left\{\nu \leqq \frac{x}{p^\alpha}\;\middle|\;(\nu,p)=1, |Q(e)\Delta(p,p^\alpha\nu)| \geqq H\right\} < \frac{c_2}{b(H)}.$$

Observe that $Q(E)\Delta(p,p^\alpha\nu) = \gamma_0(f(p^{\alpha+1}) - f(p^\alpha)) + O(1)$ as $\alpha \to \infty$, uniformly for $(\nu,p)=1$. Consequently, for large H we have

$$(2.5)\qquad \sum_{\substack{\alpha=0\\ |f(p^{\alpha+1})-f(p^\alpha)|\geqq 2H}}^{\infty} p^{-\alpha} < \frac{c_3}{b(H)}.$$

Let $g_p \in \mathcal{A}$ be so defined that

$$(2.6)\qquad g_p(q^\beta) = \begin{cases} f(p^\alpha) & \text{if } q^\beta = p^\alpha \text{ for some } \alpha\\ 0 & \text{otherwise, if } q^\beta \in \mathcal{P}^*.\end{cases}$$

From (2.2) and (2.5) we get readily that $g_p \in \mathcal{T}_b$. To provide this, it is enough to show that

$$\sum := \sum_{|f(p^\alpha)|\geqq H} p^{-\alpha} < \frac{c_4}{b(H)}.$$

(2.2) implies that $b(H) = O((\log H)^c)$ with some positive constant c. Let $H \geqq 20$, and let γ be the smallest integer for which $|f(p^\gamma)| \geqq H$. We may assume that $\gamma \leqq \frac{\log H}{\log p}$, since in the opposite case

$$\sum \leqq \sum_{l=0}^{\infty} \frac{1}{p^{\gamma+l}} \leqq \frac{2}{p^\gamma} \leqq \frac{1}{H} < \frac{c_4}{b(H)}$$

clearly holds. Let $l_\beta = (\gamma - \beta + 2)^{-2}$. Since $H \leqq |f(p^\gamma)| \leqq \sum_{\beta=1}^{\gamma} |f(p^\beta) - f(p^{\beta-1})|$, $\sum l_\beta \leqq 1$, therefore $|f(p^\beta) - f(p^{\beta-1})| \geqq l_\beta H$ holds for at least one $\beta \leqq \gamma$, which by (2.5) implies that

$$\frac{1}{p^\gamma} \leqq \frac{1}{p^\beta} < \frac{c_3}{b\left(\frac{l_\beta H}{2}\right)}.$$

Since $l_\beta \geqq \frac{1}{(\gamma+2)^2} > \frac{1}{\sqrt{H}}$, the right hand side of the inequality is less than $< \frac{c}{b(H)}$. Thus $g_p \in \mathcal{T}_\beta$. Consequently

$$\Delta(p,n) = \sum_{j=0}^{k} \beta_j\{f(p(n+j)) - f(n+j)\}$$

belongs to $\mathcal{T}_b$ as well. Thus $\mathbf{v} = Q_p(E)f \in \mathcal{T}_b$, $Q_p(z) \in I(f,b)$. Hence we have that $Q_p(z) = Q(z)$, $\{\theta_1^p, \dots, \theta_k^p\} = \{\theta_1, \dots, \theta_k\}$ for evey $p \in \mathcal{P}$. This can occur only if $\theta_1 = \dots = \theta_k = 1$, which completes the proof of our assertion.

REMARK. We proved that $g_p \in \mathcal{T}_b$. Hence it follows immediately that $I(f,b) = I(f_p, b)$, where $f_p = f - g_p$.

2.4. Let $b_k(H) = (\log eH)^{k-1}[\log(e \log eH)]^{1+\delta}$, where $\delta > 0$ is a constant.

LEMMA 1. *If $f \in \mathcal{A}$ and $\Delta^k f \in \mathcal{T}_{b_k}$ for some $k \geqq 2$, then $\Delta^{k-1} f \in \mathcal{T}_{b_{k-1}}$, and so $\Delta f \in \mathcal{T}_{b_1} \subseteqq \mathcal{T}$.*

PROOF. By the Remark in 2.3 we may assume that $f(2^\alpha) = \alpha f(2)$ holds for $\alpha \in \mathbf{N}$.

Let $\Delta_2 = E^2 - I$. Then $\Delta_2^{k-1} = (E+I)^{k-1}\Delta^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} \Delta^{k-1} E^j$, whence

$$\Delta_2^{k-1} f(2n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \Delta^{k-1} f(2n+j).$$

Furthermore, $\Delta_2^{k-1} f(2n) = \Delta^{k-1} f(n)$. Let $y_n := \Delta^{k-1} f(n)$,

$$d_{2n+l} = \left| y_n - 2^{k-1} y_{2n+l} \right| \qquad (l = 0,1). \tag{2.7}$$

By the above relations we get

$$0 \leqq d_{2n+l} \leqq c \sum_{h=0}^{k} |\Delta^k f(2n+h)|$$

and so $\{d_n\}_{n\in\mathbf{N}} \in \mathcal{T}_{b_k}$.

For an arbitrary $M \in \mathbf{N}$ let $M_1 = \left[\frac{M}{2}\right]$, $M = 2M_1 + e_0$, $e_0 \in \{0,1\}$, and in general $M_{s+1} = \left[\frac{M_s}{2}\right]$, $M_s = 2M_{s+1} + e_s$ $(s = 0,1,2,\dots)$. If $2^t \leqq M_s < 2^{t+2}$, then $2^{t+1} \leqq M_{s-1} < 2^{t+2}, \dots, 2^{t+s} \leqq M < 2^{t+s+1}$. Furthermore, for every $n \in \mathbf{N}$ there exist exactly 2^s of M for which $M_s = n$.

Let $M \in [2^\nu, 2^{\nu+1})$, $\theta := 2^{-(k-1)}$. From (2.7) we get

$$|y_M| \leqq d_M \theta + d_{M_1}\theta^2 + \dots + d_{M_\nu}\theta^{\nu+1} \tag{2.8}$$

Let $H_l = \theta^{-(l+1)} H/(l+1)(\log e(l+1))^{1+\delta}$, $0 \leqq l \leqq \nu$. The number of distinct integers M_l for which $d_{M_l} \geqq H_l$ is less than

$$\frac{c2^{\nu-l}}{b_k(H_l)} < \frac{c_1 2^{\nu-l}}{(l+1+\log H)^{k-1}[\log e \log(l+1+H)]^{1+\delta}}. \tag{2.9}$$

Let T_ν be the number of integers $M \in [2^\nu, 2^{\nu+1}]$ for which $M_l > H_l$ occurs for some l.

From (2.9) we have

$$(2.10)\quad T_\nu \leqq c_2 2^\nu \sum_{l=0}^{\infty} (l + \log H)^{-(k-1)} (\log(l + \log H))^{-1-\delta} \leqq c_2 2^\nu / b_{k-1}(H).$$

On the other hand, if $d_{M_l} < H_l$ holds for every l ($\leqq \nu$), then

$$|y_M| \leqq H \sum_{l=0}^{\infty} (l+1)^{-1} [\log e(l+1)]^{-1-\delta} < c_4 H.$$

Hence our assertion, namely that $\{y_n\} \in \mathcal{T}_{b_{k-1}}$, readily follows.

2.5. Let $f \in \mathcal{A}$ and $P \in C[z]$ be a non-zero polynomial for which $P(E)f \in \mathcal{T}$. Let I_f be the set of those polynomials $Q \in \mathbf{C}[z]$ for which $Q(E)f \in \mathcal{T}$. Repeating the argument used in Sections 2.1–2.4 one can see that I_f is an ideal the generating polynomial of which has the form $(z-1)^k$ where $k \geqq 0$. Since $(z-1)^k | P(z)$, we have that $P(z)$ has a root at $z = 1$ with multiplicity at least k. If $P'(1) \neq 0$, then $k \leqq 1$, and so $\Delta f \in \mathcal{T}$, if $P(1) \neq 0$, then $k = 0$ and so $f \in \mathcal{T}$.

2.6. THEOREM 1. *Let $f \in \mathcal{A}$, $P \in \mathbf{R}[z]$ be a non-zero polynomial.*

(1) *Assume that $P(1) \neq 0$.*

(a) *Then $P(E)f \in \mathcal{D}$ if and only if (E_f), (E_{f^2}), (E_f^*) hold.*

(b) *$P(E)f \in \mathcal{D}_\alpha$ if and only if $f \in \mathcal{A}_{fi}$. If $f \in \mathcal{A}_{fi}$, and $\lambda \in \mathbf{R}$, $h \in \mathcal{A}$ be so defined that (E_{h^2}), (E_h^*) hold, then $\alpha(x) = (\lambda \log x + A(h,x))P(1) + O(1)$.*

(2) *Assume that $P(1) = 0$ and $P'(1) \neq 0$. Then $P(E) \in \mathcal{D}$ if and only if $\Delta f \in \mathcal{D}$, i.e. if $f \in \mathcal{A}_{fi}$.*

(3) *Assume that $P(z) = (z-1)^k T(z)$, $T(1) \neq 0$, $k \geqq 2$. If $P(E)f \in \mathcal{T}_{b_k}$, then $\Delta f \in \mathcal{T}_{b_1} \subseteqq \mathcal{T}$, consequently $f \in \mathcal{F}_{fi}$, $P(E)f \in \mathcal{D}$.*

PROOF. The sufficiency of the conditions is known.

Necessity. (1a) If $P(E)f \in \mathcal{D}$, then $P(E)f \in \mathcal{T}$, and so by 2.5 we have $f \in \mathcal{T} \subset \mathcal{A}_{fi}$. Hence one gets easily that (E_{f^2}), (E_f^*), (B_f) hold. These conditions imply that $P(E)f \in \mathcal{D}_\alpha$ with $\alpha(x) = P(1)A(f,x)$. Hence we obtain that $\lim A(f,x)$ exists, (see [2]) i.e. (E_f) holds.

(1b) If $P(E)f \in \mathcal{D}_\alpha$ then $P(E)\Delta f \in \mathcal{T}$, consequently by 2.5, $\Delta f \in \mathcal{T}$, and so by Hildebrand's theorem, $f \in \mathcal{A}_{fi}$.

(2) If $P(E)f \in \mathcal{D}$, then $P(E)f \in \mathcal{T}$ and by 2.5 $\Delta f \in \mathcal{T}$, consequently by Hildebrand's theorem $f \in \mathcal{A}_{fi}$.

(3) This follows immediately from Lemma 1 and from Hildebrand's theorem.

3. On the distribution of $g(n+1)-f(n)$

Let $f,g \in \mathcal{A}$, $H(n) := g(n) - f(n)$, $\varrho(n) := g(n+1) - f(n)$.

LEMMA 2. *Assume that* $\varrho := \{\varrho(n)\}_{n\in\mathbf{N}} \in \mathcal{T}$. *Then* $H \in \mathcal{T}$, $\Delta g \in \mathcal{T}$, $\Delta f \in \mathcal{T}$. *Consequently* $g \in \mathcal{A}_{fi}$, $g = \lambda \log + t$, (E_{t^2}), (E_t^*), (E_{H^2}), (E_H^*), (B_H) *hold true.*

PROOF. The density of $n \in \mathbf{N}$ for which $2^\alpha \mid n(n+1)$ is $1/2^{\alpha-1}$ for $\alpha \geqq 1$. Let $\lambda_n := g(2n+2) - f(2n)$. Observe that

$$\lambda_n - \varrho(n) = (g(2n+2) - g(n+1)) - (f(2n) - f(n)).$$

Since $\{f(2n)-f(n)\}, \{g(2n+2)-g(n+1)\} \in \mathcal{T}$, therefore $\{\lambda_n - \varrho(n)\} \in \mathcal{T}$, whence by $\varrho \in \mathcal{T}$ we get $\{\lambda_n\} \in \mathcal{T}$. Since

$$\lambda_n = \varrho(2n+1) - H(2n+1) + \varrho(2n),$$

we get immediately that $\{H(2n+1)\} \in \mathcal{T}$. Let $H_0 \in \mathcal{A}$ be defined by $H_0(2^\alpha) = H(2^\alpha)$, $H_0(2n+1) = 0$ $(n = 0,1,2,\ldots)$. Since $\{H_0(n)\} \in \mathcal{T}$ therefore $\{H(n)\}_{n\in\mathbf{N}} \in \mathcal{T}$. Now $\Delta g(n) = \varrho(n) - H(n)$, $\varrho, H \in \mathcal{T}$, therefore $\Delta g \in \mathcal{T}$. Similarly we have that $\Delta f \in \mathcal{T}$.

The further assertions are immediate consequences of the results stated in Section 1.

THEOREM 2. *In order that $\varrho(n)$ possess a limiting distribution, it is both necessary and sufficient that the conditions* (E_H), (E_H^*), (E_{H^2}), $g \in \mathcal{A}_{fi}$ *would hold.*

PROOF. *Sufficiency.* It can be proved in a routine way.

Necessity. Taking into consideration Lemma 1, we have to prove only that (E_H) holds. Let $F(x)$ be the limit law of the sequence $\varrho(n) = \Delta g(n) + H(n)$. Let $\lambda \in \mathbf{R}$ and $t \in \mathcal{A}$ be so defined that $g = \lambda \log + t$, where (E_{t^2}), (E_t^*) hold. The fulfilment of these conditions and that of (E_{H^2}), (E_H^*) follow from Lemma 2. One can prove easily that under these conditions the frequencies

$$\lim_{N\to\infty} N^{-1} \#\{n \leqq N \mid \Delta g(n) + H(n) - A(H,N) < y\}$$

converge to a proper distribution function $G(y)$. By 1.6 we obtain that $\lim A(H,N)$ exists, i.e. (E_H) holds true.

4. We say that a function $\phi(x)$ defined for $x \geqq 0$ is subadditive if it is monotonically increasing, $0 \leqq \phi(0)$, $\phi(x) \to \infty$, and $\phi(x_1+x_2) \leqq c_1(\phi(x_1) + \phi(x_2))$ for $x_1, x_2 \geqq 1$. We extend the domain of ϕ to the whole complex plane so that $\phi(z) = \phi(|z|)$.

For some $f \in \mathcal{A}$ let $\mathcal{P}^*(f) \subseteqq \mathcal{P}^*$ denote the set of those p^m for which $f(p^m) \neq 0$. Let furthermore

$$\pi(f) := \sum_{p^m \in \mathcal{P}^*(f)} 1/p^m.$$

In this section we assume that ϕ is a subadditive function, $f \in \mathcal{A}$, $P \in \mathbf{C}[z]$ is a nonzero polynomial.

LEMMA 3. *Assume that* $\pi(f) < \infty$. *If*

$$x^{-1} \sum_{n \leqq x} \phi(P(E)f(n)) = O(1), \tag{4.1}$$

then

$$x^{-1} \sum_{n \leqq x} \phi(f(n)) = O(1). \tag{4.2}$$

PROOF. Let $P(z) = a_0 + a_1 z + \ldots + a_k z^k$. We may assume that $a_0 \neq 0$. Let γ_p be the least exponent for which $p^{\gamma_p} > k+1$. Let $\mathcal{I}$ be the set of those integers d all prime factors of which are greater than d and which can be written as products of mutually coprime factors from $\mathcal{P}^*(f)$.

For some $d \in \mathcal{I}$ let $\mathcal{M}_d$ be the set of those integers n which can be written as

$$n = d\nu \prod_{p \leqq k+1} p^{\gamma_p},$$

and for which $a_n = \nu(n+1)\ldots(n+k)$ satisfies the conditions:
(a) if $p > k+1$, then $p^2 \nmid a_n$,
(b) if $p > k+1$ and $p \in \mathcal{P}^*(f)$, then $p \nmid a_n$.
It is clear that, with some constant $c_1 > 0$,

$$x^{-1} \# \{ n \leqq x \mid n \in \mathcal{M}_d \} > \frac{c_1}{d}$$

uniformly for $1 \leqq d \leqq \log x$, say. This can be proved by the Eratosthenian sieve.

Observe that $P(E)f(n) = f(d) + O(1)$, if $n \in \mathcal{M}_d$. This, by (4.1) implies that

$$\sum_{d \in \mathcal{I}} \frac{\phi(f(d))}{d} < \infty. \tag{4.3}$$

Let $f = f_1 + f_2$, $f_1, f_2 \in \mathcal{A}$ such that $f_1(p^\beta) = f(p^\beta)$ if $p \leqq k+1$ and $= 0$ if $p > k+1$. From (4.3) we get

$$\sum_{n \leqq x} \phi(f_2(n)) \leqq \sum_{d \in \mathcal{I}} \phi(f(d)) \left[\frac{x}{d} \right] \ll x$$

whence $\sum_{n\leqq x}\phi(P(E)f_2(n))=O(x)$, and so

$$\sum_{n\leqq x}\phi(P(E)f_1(n))=O(x).$$

The last expression implies readily that

$$\sum_{p\leqq k+1}\sum_{\beta=1}^{\infty}\frac{\phi(f(p^\beta))}{p^\beta}<\infty.$$

Hence

$$\sum_{n\leqq x}\phi(f_1(n))\ll x,$$

consequently

$$\sum_{n\leqq x}\phi(f(n))\ll\sum\phi(f_1(n))+\sum\phi(f_2(n))\ll x.$$

This completes the proof of our lemma.

LEMMA 4. *Assume that* (E_{f^2}), (E_f^*) *hold, and that*

$$U(P,f):\quad U(P,f,x):=\sum_{n\leqq x}\phi(P(E)f(n))\ll x$$

is valid. Then

$$\sum_{\substack{|f(p^\alpha)|\geqq 1\\ p^\alpha\in\mathcal{P}^*}}p^{-\alpha}\phi(f(p^\alpha))<\infty. \tag{4.4}$$

If $P(1)\neq 0$, *then* (B_f) *holds as well.*

If (E_{f^2}), (E_f^*), (4.4) *and in addition in the case* $P(1)\neq 0$ *the condition* (B_f) *hold, then* $U(P,f)$ *is satisfied.*

PROOF. Let $f=f_1+f_2$, $f_1,f_2\in\mathcal{A}$, where

$$f_1(p^\alpha)=\begin{cases}0 & \text{if } \alpha\geqq 2 \text{ or if } |f(p)|\geqq 1\\ f(p) & \text{if } \alpha=1 \text{ and } |f(p)|<1.\end{cases}$$

Let S^* denote the set of those integers d for which $p\mid d$ implies that $p^2\mid d$ or $|f(p)|\geqq 1$.

Assume that (E_{f^2}), (E_f^*) and $U(P,f)$ hold. The fulfilment of (E_f^*) implies that

$$\sum_{d\in S^*}d^{-1}<\infty.$$

Observing that $\phi(z) \ll z^c$, and that f_1 is bounded on $\mathcal{P}^*$, by Lemma A we get that

$$U(P, f_1 - A(f_1, x), x) \ll U(1, f_1 - A(f_1, x), x) \ll x.$$

Since $|f_2 - A| \leqq |f| + |f_1 - A|$, assuming the validity of $U(P, f)$, we obtain that

$$\sum_{n \leqq x} \phi(c_n) \ll x, \qquad c_n = P(E) f_2(n) + P(1) A(f_1, x). \tag{4.5}$$

Hence

$$x \gg \sum_{n \leqq x} \phi(\Delta c_n) = \sum_{n \leqq x} \phi(P(E) \Delta f_2(n)),$$

and so by Lemma 3 the inequality (4.4) is true. Assume now that $P(1) \neq 0$. From Lemma 3 and (4.4) we have that $\sum_{n \leqq x} \phi(f_2(n)) \ll x$, from which by (4.5) we derive that

$$\sum_{n \leqq x} \phi(P(1) A(f_1, x)) \ll \sum \phi(c_n) + \sum \phi(f_2(n)) \ll x$$

i.e. $A(f_1, x) = O(1)$ follows.

It is enough to prove the second assertion for $P = 1$, since $U(P, f, x) \ll \ll U(1, f, 2x)$. This would follow from

$$\sum_{d \in S^*} \frac{\phi(|f_2(d)|)}{d} < \infty. \tag{4.6}$$

The subadditivity of ϕ implies that $\phi(x_1 + x_2) \leqq c_1[\phi(x_1) + \phi(x_2) + c_2]$ holds with some constants $c_1 \geqq c_2 > 0$, for all complex numbers x_1, x_2. By iterating this inequality, we can derive that

$$\phi(x_1 + \ldots + x_s) \leqq \sum_{l=1}^{s} c_1^{s-1}(c_2 + \phi(x_l)).$$

By using this and (4.4) we get that

$$\sum_{d \in S^*} \frac{\phi(|f_2(d)|)}{d} \ll c_2 \sum \frac{c_1^{\omega(d)}}{d} + \sum \frac{c_1^{\omega(d)}}{d} \sum_{p^\alpha \| d} \phi(|f_2(p^\alpha)|),$$

and the right hand side is finite.

Let $(V(P, f))$, $(M(f))$ denote the conditions:

$(V(P, f))$ $\quad \sum_{n \leqq x} \phi(P(E) f(n)) \ll x,$

$(M(f))$ $\quad \sum_{|f(p^\alpha)| \geqq 1} \frac{\phi(f(p^\alpha))}{p^\alpha} < \infty.$

THEOREM 3. *Let* $G \in C[z]$, $P(z) = (z-1)^k G(z)$, $G(1) \neq 0$.

(1) *Case* $k = 0$. *The relation* $(V(P,f))$ *holds if and only if* (B_f), (E_{f^2}), (E_f^*) *and* $(M(f))$ *are satisfied.*

(2) Case $k = 1$. The relation $V(P,f)$ holds if and only if $f \in \mathcal{A}_{fi}$, $f = \lambda \log + h$, with some suitable $\lambda \in \mathbf{R}$ and $h \in \mathcal{A}$ such that (E_{h^2}), (E_h^*), $(M(h))$ are satisfied.

(3) Case $k \geqq 2$. Assume that $\phi(H) \gg b_k(H)$. Then the relation $(V(P,f))$ holds if and only if $f = \lambda \log + h$ with some $\lambda \in \mathbf{R}$ and $h \in \mathcal{A}$ such that (E_{h^2}), (E_h^*), $(M(h))$ are satisfied.

PROOF. (1) *Case* $k \geqq 1$. *Necessity.* We have from $V(P,f)$ that $P(E)f \in \in \mathcal{T}$, and in the case $k \geqq 2$ that $P(E)f \in \mathcal{T}_{b_k}$. Then, by Theorem 1 we conclude that $f = \lambda \log + h$. It is clear that the conditions $(V(P,f))$ and $(V(P,h))$ are equivalent. Then, by Lemma 3, (E_{h^2}), (E_h^*), $(M(h))$ are satisfied.

Sufficiency. By Lemma 3 we have$(U(P,h))$ under the conditions $(V(P,h))$, (E_{h^2}), (E_h^*), which by $P(E)f = O(1) + P(E)h$ implies $(V(P,f))$.

(2) *Case* $k = 0$. The sufficiency is clear. To prove the necessity, we observe that $(V(P,f))$ implies the fulfilment of $(V(P\Delta, f))$. Thus we can apply the already proved part of the theorem, $P(z)(z-1)$ leads to the case $k = 1$, whence we have that $f = \lambda \log + h$, (E_{h^2}), (E_h^*) are satisfied. Furthermore we have

$$\sum_{n \leqq x} \phi(P(E)h(n) + \lambda P(1) \log n) \ll x.$$

We shall prove that $\lambda = 0$. This is clear, since $P(E)h_2(n)$ is bounded on a set K of integers n having a positive density, $A(h,x) = O(\log\log x)$,

$$\sum_{n \leqq x} \phi(P(E)(h_1(n) - A(h,x))) \ll \sum_{n \leqq 2x} |h_1(n) - A(h,x)|^c \ll x, \tag{4.7}$$

and so

$$\sum_{\substack{n \leqq x \\ n \in K}} \phi(\lambda P(1) \log n + P(1)A(h,x)) \ll \sum_{\substack{n \leqq x \\ n \in K}} \phi(P(E)h_2(n)) +$$
$$+ \sum_{n \leqq x} \phi(P(E)(h_1(n) - A(h,x))) \ll x.$$

Assuming that $\lambda \neq 0$, the left hand side is larger than

$$\gg x\phi\left(\frac{|\lambda||P(1)|}{2} \log x\right).$$

This is a contradiction to $\phi(y) \to \infty$ for $|y| \to \infty$. From Lemma 4 we obtain that (B_f) holds.

5. Let $f, g \in \mathcal{A}$, $H(n) := g(n) - f(n)$, $\varrho(n) := g(n+1) - f(n) = \Delta g(n) + H(n)$, ϕ *be a subadditive function.*

Let $(C(g,H))$, $(\Delta(g))$, $\mathcal{D}(H)$ denote the conditions

$$(C(g,H)) \qquad \sum_{n \leqq x} \phi(\Delta g(n) + H(n)) \ll x,$$

$$(\Delta(g)) \qquad \sum_{n \leqq x} \phi(\Delta g(n)) \ll x,$$

$$(\mathcal{D}(H)) \qquad \sum_{n \leqq x} \phi(H(n)) \ll x.$$

THEOREM 4. $(C(g,H))$ *holds if and only if* $(\Delta(g))$ *and* $(\mathcal{D}(H))$ *are valid.*

PROOF. It is clear that $(\Delta(g))$ and $(\mathcal{D}(H))$ imply the fulfilment of $(C(g,H))$. To prove the necessity of these conditions, we observe that $(C(g,H))$ implies $\varrho \in \mathcal{T}$, and so by Lemma 2 we have that $g = \lambda \log + t$, (E_{t^2}), (E_t^*), (B_H), (E_{H^2}), (E_H^*) hold. Let $t = t_1 + t_2$, $H = H_1 + H_2$, $t_1, t_2, H_1, H_2 \in \mathcal{A}$,

$$t_1(p^\alpha) = \begin{cases} 0 & \text{if } |t(p^\alpha)| < 1 \\ t(p^\alpha) & \text{if } |t(p^\alpha)| \geqq 1, \end{cases} \qquad H_1(p^\alpha) = \begin{cases} 0 & \text{if } |H(p^\alpha)| < 1 \\ H(p^\alpha) & \text{if } |H(p^\alpha)| \geqq 1. \end{cases}$$

It is clear that $\pi(t_1) < \infty$, $\pi(H_1) < \infty$. Let K_1 be the set of those integers d which can be written as products of mutually coprime prime powers belonging to $\mathcal{P}^*(t_1)$ and let K_2 be the set composed similarly from $\mathcal{P}^*$ (H_1) instead of t_1. We have

$$\sum_{d \in K_i} \frac{1}{d} < \infty.$$

By Lemma A and from the fulfilment of (E_{f^2}), (B_H), (E_{H^2}) we obtain $(\Delta(t_2))$ and $\mathcal{D}(H_2)$, consequently that $C(t_1, H_2)$ hold. Since $\Delta g(n) = \Delta t(n) + O(1)$, and

$$\phi(\Delta g_1(n) + H_1(n)) \ll 1 + \phi(\Delta g(n) + H(n)) + \phi(\Delta g_2(n) + H_2(n)),$$

therefore

$$\sum_{n \leqq x} \phi(\Delta t_1(n) + H_1(n)) \ll x, \tag{5.1}$$

i.e. $C(t_1, H_2)$ holds as well.

Let $d \leqq \log x$, $d \in K_1$. The number of integers $n \leqq x$ for which $n + 1 = 2d\nu$, $(\nu, \mathcal{P}^*(t_1)) = 1$ and $(n, \mathcal{P}^*(t_1) \cup \mathcal{P}^*(H_1)) = 1$ is greater than $c_1 x/d$. For such an n we have $\Delta t_1(n) + H_1(n) = t_1(2d)$, and so from (5.1) we get

$$\sum_{\substack{d \in K_1 \\ (d,2)=1}} \frac{\phi(t(d))}{d} < \infty. \tag{5.2}$$

Let us now choose d to run over the set 2^α $(\alpha = 1, 2, \dots)$. Just as above we get

$$\sum_{\alpha=0}^{\infty} \frac{\phi(t_1(2^\alpha))}{2^\alpha} < \infty. \tag{5.3}$$

This, by Lemma 4 implies $\mathcal{D}(H_1)$, consequently from (5.1) we get $\Delta(t_1)$. The fulfilment of $\mathcal{D}(H_2)$ and $\Delta(t_2)$ was proved earlier. Thus $\Delta(t)$ as well as $\Delta(g)$ and $\mathcal{D}(H)$ readily follow.

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ON THE ORDER OF MAGNITUDE OF FUNDAMENTAL POLYNOMIALS OF HERMITE INTERPOLATION

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Let

$$(-1 \leqq) x_{nn} < x_{n-1,n} < \cdots < x_{1n} (\leqq 1) \tag{1}$$

be an arbitrary system of nodes of interpolation, and let $m \geqq 1$ be an arbitrary integer. (We shall often abbreviate x_{kn} as x_k.) For an arbitrary $m-1$ times differentiable function $f(x)$ in the interval $[-1,1]$, consider the Hermite interpolation polynomial

$$H_{mn}(f,x) := \sum_{k=1}^{n} \sum_{j=0}^{m-1} f^{(j)}(x_k) A_{jk}(x) \tag{2}$$

where the polynomials $A_{jk}(x)$ (more precisely, $A_{jkmn}(x)$) of degree at most $mn-1$ satisfy the conditions

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq} \qquad (j,p = 0,\dots,m-1; \quad k,q = 1,\dots,n) \tag{3}$$

(δ is the Kronecker delta). Thus the operator (2) has the interpolatory properties

$$H_{mn}^{(p)}(f,x_q) = f^{(p)}(x_q) \qquad (p = 0,\dots,m-1; \quad q = 1,\dots,n).$$

In this paper we determine the exact lower bounds for the quantities

$$L_{jmn} := \left\| \sum_{k=1}^{n} |A_{jk}(x)| \right\| \qquad (j = 0,\dots,m-1) \tag{4}$$

where $\|\cdot\|$ means supremum norm of the corresponding function over the interval $[-1,1]$. The motivation for investigating these quantities is obvious: L_{0mn} is the so-called Lebesgue constant of the Hermite–Fejér operator

$$\overline{H}_{mn}(f,x) := \sum_{k=1}^{n} f(x_k) A_{0k}(x), \tag{5}$$

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and the other quantities L_{jmn} $(j = 1, \dots, m-1)$ play also an important role in investigating the convergence behavior of (5). Namely, when estimating the error of approximation by (5), one usually needs an estimate for this error when the operator is applied to the best approximating polynomial $p(x)$ of $f(x)$ of degree at most $mn - 1$, i.e.

$$p(x) - \overline{H}_{mn}(p, x) = \sum_{k=1}^{n} \sum_{j=1}^{m-1} p^{(j)}(x_k) A_{jk}(x).$$

In case we know structural properties of $f(x)$, we also have some information on $p^{(j)}(x_k)$, and this leads to the quantities L_{jmn}.

Our main result is the following:*

THEOREM 1. *For an arbitrary system of nodes* (1) *we have*

$$L_{jmn} \geqq \begin{cases} c_1 \frac{\log n}{n^j} & \text{if } m - j \text{ is odd,} \\ \frac{c_2}{n^j} & \text{if } m - j \text{ is even} \end{cases} \qquad (j = 0, \dots, m-1).$$

This is a very general result whose particular cases had been known. (Of course, if m is even then $L_{0mn} \geqq c_2$ is obvious.) For Lagrange interpolation $(m = 1)$ we get

$$L_{0,1,n} \geqq c_1 \log n$$

(this is G. Faber's classical result [3]); for Hermite–Fejér interpolation $(m = 2)$

$$L_{1,2,n} \geqq c_1 \frac{\log n}{n}$$

was proved by P. Erdős and P. Turán [2]; finally

$$L_{0,3,n} \geqq c_1 \log n \tag{6}$$

has been shown recently by J. Szabados and A. K. Varma [5]. (Actually, P. Vértesi [7] proved more than (6) by showing that the corresponding Lebesgue function is $> c \log n$ on a large set in [-1,1].)

The most interesting special case of Theorem 1 is

$$L_{0,m,n} \geqq c_1 \log n \qquad (m \text{ odd})$$

which was conjectured in [5].

Since the proof of Theorem 1 is long, we break it into a series of lemmas. In what follows, let

$$\omega_n(x) := \prod_{k=1}^{n} (x - x_k), \quad l_k(x) := l_{kn}(x) := \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)} \quad (k = 1, \dots, n).$$

* In what follows $c_1, c_2, \dots$ will denote positive constants depending only on j and m.

LEMMA 1. *We have*

$$A_{jk}(x)=\frac{l_k(x)^m}{j!}\sum_{i=0}^{m-j-1}\frac{[l_k(x)^{-m}]^{(i)}_{x=x_k}}{i!}(x-x_k)^{i+j} \tag{7}$$
$$(j=0,\dots,m-1;\ k=1,\dots,n).$$

PROOF. Since the degree of (7) is at most $mn-1$, we only have to prove (3). The polynomial (7) contains the factor $(x-x_i)^m$ $(i\neq k)$, whence

$$A^{(p)}_{jk}(x_i)=0 \qquad (p=0,\dots,m-1,\ i\neq k).$$

Also, if $0\leqq p<j$, then it contains a factor $(x-x_k)^{p+1}$, i.e.

$$A^{(p)}_{jk}(x_k)=0 \qquad (p=0,\dots,j-1).$$

Finally, if $j\leqq p\leqq m-1$ then applying the Newton–Leibniz rule twice we obtain

$$A^{(p)}_{jk}(x_k)=\frac{1}{j!}\sum_{i=0}^{p-j}\frac{[l_k(x)^{-m}]^{(i)}_{x=x_k}}{i!}\binom{p}{i+j}(i+j)![l_k(x)^m]^{(p-i-j)}_{x=x_k}=$$
$$=\binom{p}{j}\sum_{i=0}^{p-j}\binom{p-j}{i}[l_k(x)^{-m}]^{(i)}_{x=x_k}[l_k(x)^m]^{(p-i-j)}_{x=x_k}=\binom{p}{j}[1]^{(p-j)}_{x=x_k}=\delta_{pj}$$
$$(p=j,j+1,\dots,m-1). \qquad \square$$

Now we introduce the notations

$$a_{ik}:=a_{ikmn}:=m\sum_{\substack{\nu=1\\ \nu\neq k}}^{n}\frac{1}{(x_\nu-x_k)^i} \qquad (k=1,\dots,n;\ i=1,2,\dots) \tag{8}$$

and

$$b_{ik}:=b_{ikmn}:=\frac{[l_k(x)^{-m}]^{(i)}_{x=x_k}}{i!} \qquad (k=1,\dots,n;\ i=0,1,\dots). \tag{9}$$

LEMMA 2. *We have*

$$b_{ik}=\frac{1}{i}\sum_{\nu=1}^{i}a_{\nu k}b_{i-\nu,k} \qquad (k=1,\dots,n;\ i=1,2,\dots). \tag{10}$$

PROOF. Since by (8)

$$a_{ik}=\frac{m}{(i-1)!}\left[\frac{1}{x-x_k}-\frac{\omega_n'(x)}{\omega_n(x)}\right]^{(i-1)}_{x=x_k} \qquad (k=1,\dots,n;\ i=1,2,\dots),$$

we obtain by (9) and the Newton–Leibniz formula again

$$\frac{1}{i}\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu,k} = \frac{m}{i}\sum_{\nu=1}^{i}\frac{1}{(\nu-1)!}\left[\frac{1}{x-x_k}-\frac{\omega_n'(x)}{\omega_n(x)}\right]_{x=x_k}^{(\nu-1)}\cdot\frac{[l_k(x)^{-m}]_{x=x_k}^{(i-\nu)}}{(i-\nu)!}=$$

$$=\frac{m}{i!}\left[\left(\frac{1}{x-x_k}-\frac{\omega_n'(x)}{\omega_n(x)}\right)l_k(x)^{-m}\right]_{x=x_k}^{(i-1)}=\frac{1}{i!}\left[-\frac{l_k'(x)}{l_k(x)}ml_k(x)^{-m}\right]_{x=x_k}^{(i-1)}=$$

$$=\frac{1}{i!}[l_k(x)^{-m}]_{x=x_k}^{(i)}=b_{ik}\qquad(k=1,\dots,n;\ i=1,2,\dots).\qquad\square$$

Now let

(11)
$$B_{jk}(x):=B_{jkmn}(x):=\sum_{i=0}^{m-j-1}b_{ik}(x-x_k)^i\quad(j=0\dots,m-1;\ k=1,\dots,n).$$

LEMMA 3. *We have*

(12)
$$B_{jk}(x)\geqq c_3\left(\frac{x-x_k}{x_k-x_{k\pm1}}\right)^{m-j-1}$$
$$(-\infty<x<\infty,\ m-j\ \textit{odd},\ 0\leqq j\leqq m-1,\ 1\leqq k\leqq n)$$

with one of the signs in $x_{k\pm1}$.

PROOF. For $j=m-1$ we have $B_{m-1,k}(x)\equiv b_{0k}=1$ (see (9) and (11)), whence we may assume that $j\leqq m-3$ (i.e. $m\geqq 3$) (the case $j=m-2$ does not arise, since by assumption $m-j$ is odd.) By (9) and (11), $B_{jk}(x)$ is nothing else but an *even order* partial sum of the Taylor expansion of the rational function $l_k(x)^{-m}$ about $x=x_k$. Since $l_k(x)^{-m}$ is the reciprocal of a polynomial having only real zeros and taking the value 1 at $x=x_k$, by a theorem of Laguerre (see G. Pólya and G. Szegö [4], Problem 50 on p. 43)

(13)
$$B_{jk}(x)>0\qquad(-\infty<x<\infty).$$

Hence $b_{m-j-1,k}\geqq 0$. First we prove that deg $B_{jk}=m-j-1$, i.e. $b_{m-j-1,k}>0$. Namely, if $b_{m-j-1,k}=0$ then by (13) $b_{m-j-2,k}=0$, and deg $B_{jk}\leqq m-j-3$, deg $A_{jk}^{(j+1)}(x)\leqq mn-j-4$. Hence we will deduce a contradiction by counting the number of roots of $A_{jk}^{(j+1)}(x)$. The x_i's are all roots of multiplicity at least $m-j-1\,(i=1,\dots,n)$. Applying Rolle's theorem to the intervals $(x_{i+1},x_i)\,(i=1,\dots,n-1)$ and then successively to subintervals we get j roots for $A_{jk}^{(j)}(x)$ in each of the intervals $(x_{i+1},x_i)\,(i=1,\dots,n-1)$, and then for $A_{jk}^{(j+1)}(x)$ we get $j+1$ roots in each $(x_{i+1},x_i)\,(i=1,\dots,k-1,k+$ $+1,\dots,n-1)$, while in (x_{k+1},x_k) and (x_k,x_{k-1}) we get only j roots (since

$A_{jk}^{(j)}(x) = 1$). Altogether we obtain

$$n(m-j-1)+(n-3)(j+1)+2j = mn-j-3$$

roots for $A_{jk}^{(j+1)}(x)$, i.e. $A_{jk}^{(j+1)}(x) \equiv 0$, a contradiction, since $A_{jk}^{(j)}(x)$ is not a constant by (3).

Thus if we let

$$c_{jk} = \sup\{c \mid B_{jk}(x) \geqq c(x-x_k)^{m-j-1} \quad (-\infty < x < \infty)\}. \tag{14}$$

then $c_{jk} > 0$, and evidently

$$C_{jk}(x) := B_{jk}(x) - c_{jk}(x-x_k)^{m-j-1} \geqq 0 \qquad (-\infty < x < \infty). \tag{15}$$

Being $b_{m-j-1,k} > 0$ the leading coefficient of $B_{jk}(x)$, (15) implies that $c_{jk} \leqq \leqq b_{m-j-1,k}$. Using Taylor expansion about $x = x_k$, we obtain by (15), (11), (7) and (3)

$$C_{jk}(x) l_k(x)^m \frac{(x-x_k)^j}{j!} = A_{jk}(x) - \frac{c_{jk}}{j!} l_k(x)^m (x-x_k)^{m-1} =$$
$$= \frac{(x-x_k)^j}{j!} - \frac{c_{jk}}{j!}(x-x_k)^{m-1} + (x-x_k)^m D_{jk}(x)$$

where $D_{jk}(x)$ is a polynomial. Hence

$$E_{jk}(x) := C_{jk}(x) l_k(x)^m = 1 - c_{jk}(x-x_k)^{m-j-1} + j!(x-x_k)^{m-j} D_{jk}(x). \tag{16}$$

We now distinguish two cases.

Case 1: $c_{jk} = b_{m-j-1,k}$. Then $C_{jk}(x)$ is of degree at most $m-j-3$ (since $C_{jk}(x)$ must be of even degree), and $E_{jk}(x)$ is of degree at most $m-j-3+m(n-1) = mn-j-3$. We determine the roots of $E'_{jk}(x)$. The x_i's ($i \neq k$) are roots of multiplicity at least $m-1$, while x_k is a root of multiplicity $m-j-2 \geqq 1$. Further $n-3$ roots are obtained by Rolle's theorem applied to the $n-3$ intervals determined by the roots $x_1, \dots, x_{k-1}$ and $x_{k+1}, \dots, x_n$ of $E_{jk}(x)$. This is altogether

$$(m-1)(n-1)+(m-j-2)+(n-3) = mn-j-4,$$

which is the degree of $E'_{jk}(x)$; i.e. there are no other roots.

Case 2: $c_{jk} < b_{m-j-1,k}$. Then there exists an $\alpha_{jk} \neq x_k$ such that

$$C_{jk}(\alpha_{jk}) = C'_{jk}(\alpha_{jk}) = 0. \tag{17}$$

Namely, if we had $C_{jk}(x) > 0\,(-\infty < x < \infty)$ then this would contradict the definition of c_{jk} in (14). Again, we determine the roots of $E'_{jk}(x)$. Besides the roots listed above, α_{jk} is an additional root, and since it is a root of $E_{jk}(x)$ as well, either the application of Rolle's theorem yields yet another root, or, if $\alpha_{jk} = x_i\,(i \neq k)$, say, then this is a root of multiplicity 2 higher than in Case 1. So the total number of real roots of $E'_{jk}(x)$ is now $mn-j-2$, which is the degree of $E'_{jk}(x)$.

Thus we have a complete description of the roots of $E'_{jk}(x)$, and in both cases we can write by (16)

$$E'_{jk}(x) = (x - x_k)^{m-j-2} F_{jk}(x)$$

where

$$\text{(18)}\quad F_{jk}(x) = -(m-j-1)c_{jk} + (x-x_k)j![(m-j)D_{jk}(x) + (x-x_k)D'_{jk}(x)].$$

Without loss of generality we may assume that in Case 2 $x_k < \alpha_{jk}$. (The case $x_k > \alpha_{jk}$ leads to entirely analogous, symmetric considerations.) Again, we distinguish two cases.

Case 1: $2 \leqq k \leqq n$. With the notation

$$\beta_{jk} = \begin{cases} x_{k-1} & \text{if } c_{jk} = b_{m-j-1,k} \\ \min(\alpha_{jk}, x_{k-1}) & \text{if } c_{jk} < b_{m-j-1,k} \end{cases}$$

we may say that the polynomial $F_{jk}(x)$ has no roots in the interval (x_{k+1}, β_{jk}). (If $k = n$ then this interval is $(-\infty, \beta_{jn})$.) Since $E'_{jk}(x)$, and hence $F_{jk}(x)$, has only real roots, it follows that $F_{jk}(x)$ is monotone in at least one of the intervals (x_{k+1}, x_k) and (x_k, β_{jk}). (If $k = n$ then this interval is $(-\infty, \beta_{jn}) \supset$ $\supset (x_n, \beta_{jn})$.) Again, without loss of generality we may assume the first possibility, i.e. by (18)

$$\text{(19)}\qquad 0 \geqq F_{jk}(x) \geqq F_{jk}(x_k) = -(m-j-1)c_{jk} \qquad (x_{k+1} \leqq x \leqq x_k).$$

By (16), $E_{jk}(x_{k+1}) = 0$, $E_{jk}(x_k) = 1$, and thus by (19)

$$\text{(20)}\qquad 1 = \int_{x_{k+1}}^{x_k} E'_{jk}(x)\,dx = \int_{x_{k+1}}^{x_k} (x-x_k)^{m-j-2} F_{jk}(x)\,dx \leqq$$

$$\leqq -(x_k - x_{k+1})^{m-j-2} \int_{x_{k+1}}^{x_k} F_{jk}(x)\,dx \leqq (x_k - x_{k+1})^{m-j-1}(m-j-1)c_{jk}.$$

Hence and from (14) we obtain (12). (If the monotonicity interval is (x_k, β_{jk}) then, instead of $x_k - x_{k+1}$ in (20), we end up with $\beta_{jk} - x_k \leqq x_{k-1} - x_k$.)

Case 2: $k = 1$. This case requires a slightly different argument only if $c_{j1} < b_{m-j-1,1}$; namely, otherwise $F_{j1}(x)$ will be monotone in the interval (x_2, x_1). Now according to the previous argument, $F_{j1}(x) \neq 0$ in the interval (x_2, α_{j1}). If $F_{j1}(x)$ is monotone in (x_2, x_1), then the same argument works as before. It remains to settle the case when $F_{j1}(x)$ attains its minimum in (x_2, α_{j1}) at $y \in (x_2, x_1)$. Again, we distinguish two cases.

Case 2.1: $F_{j1}(y) \leqq 2F_{j1}(x_1)$. Then it follows from the convexity from below of $F_{j1}(x)$ in (y, ∞) that

$$0 < \alpha_{j1} - x_1 \leqq x_1 - y \leqq x_1 - x_2.$$

Thus as before

$$1 = -\int_{x_1}^{\alpha_{j1}} E'_{j1}(x)\, dx = -\int_{x_1}^{\alpha_{j1}} (x - x_1)^{m-j-2} F_{j1}(x)\, dx \leqq$$
$$\leqq -(\alpha_{j1} - x_1)^{m-j-1} F_{j1}(x_1) \leqq (x_1 - x_2)^{m-j-1}(m - j - 1)c_{j1},$$

which is (20) with $k = 1$.

Case 2.2: $F_{j1}(y) \geqq 2F_{j1}(x_1)$. Then

$$1 = \int_{x_2}^{x_1} E'_{j1}(x)\, dx \leqq -(x_1 - x_2)^{m-j-1} F_{j1}(y) \leqq$$
$$\leqq -2(x_1 - x_2)^{m-j-1} F_{j1}(x_1) = 2(x_1 - x_2)^{m-j-1}(m - j - 1)c_{j1},$$

which is again (20) apart from a factor 2. □

Lemmas 1 and 3 imply

COROLLARY. *We have*

$$|A_{jk}(x)| \geqq \frac{c_3}{j!} \left| \frac{\omega_n(x)}{\omega'_n(x_k)} \right|^m \frac{1}{|x - x_k| \cdot |x_k - x_{k\pm1}|^{m-j-1}}$$

$$(-\infty < x < \infty,\ m - j \textit{ odd},\ 0 \leqq j \leqq m - 1;\ k = 1, \ldots, n)$$

with one of the signs in $x_{k\pm1}$.

LEMMA 4 (see Erdős–Turán [2]). *With the notations*

$$I_n = \left[-\frac{1}{\log n}, \frac{1}{\log n} \right], \quad I'_n = \left[-\left(1 - \frac{\log^2 n}{\sqrt{n}}\right) \frac{1}{\log n}, \left(1 - \frac{\log^2 n}{\sqrt{n}}\right) \frac{1}{\log n} \right],$$
$$M_n = \max_{|x| \leqq 1} |\omega_n(x)|, \quad \overline{M}_n = \max_{x \in I_n} |\omega_n(x)|$$

we have

$$\max_{x \in I'_n} |\omega'_n(x)| \leqq c_4 n \left(\frac{M_n}{\log^2 n} + \overline{M}_n \right)$$

with an absolute constant $c_4 > 0$.

PROOF OF THEOREM 1. *Case 1:* $m - j$ is odd.*

* We note that this part of the proof in the particular case $m = 3$, $j = 0$ has been done in [5].

Case 1.1: there is a $1 \leqq k_0 \leqq n$ such that $|l_{k_0}(y)| = \|l_{k_0}(x)\| \geqq n^2$. Then by Markov's inequality

$$|l_{k_0}(x)| \geqq \frac{1}{2}n^2 \qquad \left(|x-y| \leqq \frac{1}{2n^2},\ |x| \leqq 1\right).$$

Hence there exists $z \in [-1,1]$ such that

$$|z-y| \leqq \frac{1}{2n^2} \leqq |z - x_{k_0}|,$$

whence by the Corollary

$$|A_{jk_0}(z)| \geqq \frac{c_3}{j!}\frac{\left(\frac{1}{2n^2}\right)^{m-1}\left(\frac{1}{2}n^2\right)^m}{2^{m-j-1}} \geqq c_5 n^2$$

which is more than we stated.

Case 1.2: $\|l_k(x)\| = O(n^2)$ $(k = 1,\dots,n)$. Then, according to a result of P. Erdős [1], the system of nodes (1) is asymptotically uniformly distributed, in the sense that with $x_k = \cos\theta_k$ $(k = 1,\dots,n)$ we have

$$\left|\sum_{\theta_k \in I} 1 - \frac{|I|}{\pi}n\right| \leq \log^2 n \qquad (I \subseteqq [0,\pi])$$

where $|I|$ denotes the length of the interval I. Hence

$$\sum_{x_k \in I} 1 \geqq \frac{|I|}{15}n \quad \text{if} \quad I \subset [-1,1], \quad |I| \geqq 4\frac{\log^2 n}{n} \quad \text{and} \quad n \geqq n_0.$$

Using the harmonic-geometric-arithmetic means inequalities we get for $p \geqq 1$

$$\text{(21)} \qquad \sum_{x_k \in I} \frac{1}{|x_k - x_{k\pm1}|^p} \geqq \frac{\sum_{x_k \in I} 1}{\left(\prod_{x_k \in I} |x_k - x_{k\pm1}|\right)^{p/\sum_{x_k \in I} 1}} \geqq$$

$$\geqq \frac{\left(\sum_{x_k \in I} 1\right)^{p+1}}{\left(\sum_{x_k \in I} |x_k - x_{k\pm1}|\right)^p} \geqq \frac{\left(\frac{|I|}{15}n\right)^{p+1}}{\left(2|I| + 8\frac{\log^2 n}{n}\right)^p} \geqq \frac{|I| n^{p+1}}{4^{3p+2}}$$

if $|I| \geqq 4\frac{\log^2 n}{n}$, $n \geqq n_0$, no matter which sign is taken in $x_{k\pm1}$.

Case 1.2.1: $\overline{M}_n \leqq \frac{M_n}{\log^2 n}$. Then by Lemma 4

$$\max_{x \in I_n'} |\omega_n'(x)| \leqq \frac{2c_4 M_n n}{\log^2 n}$$

and hence with $|\omega_n(y)| = M_n$ we get by the Corollary and by (21) with $p = m - j - 1$ and $I = I'_n$

$$L_{jmn} \geqq \sum_{x_k \in I'_n} |A_{jk}(y)| \geqq \frac{c_3}{j!} \sum_{x_k \in I'_n} \left| \frac{\omega_n(y)}{\omega'_n(x_k)} \right|^m \frac{1}{|y - x_k| \cdot (x_k - x_{k\pm 1})^{m-j-1}} \geqq$$

$$\geqq c_6 \left(\frac{\log^2 n}{n} \right)^m \sum_{x_k \in I'_n} \frac{1}{(x_k - x_{k\pm 1})^{m-j-1}} \geqq c_7 \frac{\log^{2m-1} n}{n^j}$$

which is more than stated in the theorem.

Case 1.2.2: $M_n \leqq \overline{M}_n \log^2 n$. Then by Lemma 4

$$\max_{x \in I'_n} |\omega'_n(x)| \leqq 2c_4 n \overline{M}_n,$$

and hence with $|\omega_n(z)| = \overline{M}_n$, $-\frac{1}{\log n} \leqq z \leqq 0$ (say) and

$$I_{n,\lambda} := \left[z + (2\lambda + 1) \frac{\log n}{\sqrt{n}},\ z + (2\lambda + 3) \frac{\log n}{\sqrt{n}} \right] \subset I'_n$$

$$\left(\lambda = 0, 1, \ldots, \left[\frac{\sqrt{n}}{\log^3 n} \right] := q \right)$$

we get

$$L_{jmn} \geqq \sum_{x_k \in I'_n} |A_{jk}(z)| \geqq \frac{c_3}{j!} \sum_{x_k \in I'_n} \left| \frac{\omega_n(z)}{\omega'_n(x_k)} \right|^m \frac{1}{|z - x_k| \cdot (x_k - x_{k\pm 1})^{m-j-1}} \geqq$$

$$\geqq \frac{c_3}{j!(2c_5 n)^m} \sum_{\lambda=1}^{q} \sum_{x_k \in I_{n,\lambda}} \frac{1}{|z - x_k| \cdot (x_k - x_{k\pm 1})^{m-j-1}} \geqq$$

$$\geqq \frac{c_8}{n^{m-1/2} \log n} \sum_{\lambda=1}^{q} \frac{1}{2\lambda + 3} \sum_{x_k \in I_{n,\lambda}} \frac{1}{(x_k - x_{k\pm 1})^{m-j-1}} \geqq$$

$$\geqq \frac{c_9}{n^{m-1/2} \log n} \cdot \log n \cdot \frac{n^{m-j} \log n}{\sqrt{n}} = c_9 \frac{\log n}{n^j}.$$

Case 2: $m - j$ is even. Then by (4),

$$\|A_{jk}(x)\| \leqq L_{jmn} \qquad (k = 1, \ldots, n),$$

whence by Bernstein's inequality and by (3)

$$1 = A_{jk}^{(j)}(x_k) \leqq \frac{(mn-1)^j \|A_{jk}\|}{(1 - x_k^2)^{j/2}} = O\left(\frac{n^j L_{jmn}}{(1 - x_k^2)^{j/2}} \right)$$

$$(k = 2, \ldots, n-1;\ j = 0, \ldots, m-1).$$

Thus

$$L_{jmn} \geqq c_{10} \frac{(1-x_k^2)^{j/2}}{n^j} \qquad (k=2,\dots,n-1;\ j=0,\dots,m-1). \tag{22}$$

Now if

$$\sup_n \min_{1 \leqq k \leqq n} |x_{kn}| < 1 \tag{23}$$

then the statement of Theorem 1 follows from (22).

Finally, assume that (23) does not hold, i.e. there exists an infinite sequence of indices $n_1 < n_2 < \dots$ such that

$$\lim_{s\to\infty} \min_{1 \leqq k \leqq n_s} |x_{kn_s}| = 1.$$

Then let k_s, $0 \leqq k_s \leqq n_s$ be a sequence such that (with the notation $x_{n_s+1,n_s} = -1$, $x_{0,n_s} = 1$)

$$\lim_{s\to\infty} x_{k_s,n_s} = 1, \quad \lim_{s\to\infty} x_{k_s+1,n_s} = -1. \tag{24}$$

Assume that $n_s/2 \leqq k_s \leqq n_s$ (the case $0 \leqq k_s < n_s/2$ is even simpler). Then by (8) and (10) (since $b_{01} = 1$)

$$(-1)^i a_{i1} > 0, \quad (-1)^i b_{i1} > 0 \qquad (i = 1, 2, \dots). \tag{25}$$

On the other hand, by (24)

$$|l_{1,n_s}(0)| \geqq \left(\frac{x_{k_s,n_s}}{1-x_{k_s,n_s}}\right)^{k_s-1} \left(\frac{-x_{k_s+1,n_s}}{2}\right)^{n_s-k_s} \geqq$$
$$\geqq \left(\frac{2x_{k_s,n_s}}{1-x_{k_s,n_s}}\right)^{k_s-1} \cdot \frac{1}{3^{n_s}} \geqq \frac{9^{n_s/2}}{3^{n_s}} = 1$$

for sufficiently large s. Thus (7), (9) and (25) yield

$$|A_{j1}(0)| \geqq \frac{|l_{1,n_s}(0)|^m}{j!}(-x_{1,n_s})^j \geqq c_{11} \qquad (j = 0, \dots, m-2),$$

which is more than we had to prove. □

Now we prove that the lower estimates given in Theorem 1 are sharp. In fact, we prove slightly more:

THEOREM 2. *For the Chebyshev nodes*

$$x_{kn} = \cos\frac{2k-1}{2n}\pi \qquad (k = 1, \dots, n;\ n = 1, 2, \dots) \tag{26}$$

we have

$$L_{jmn} \leqq L^{\star}_{jmn} := \left\| \sum_{k=1}^{n} \frac{|A_{jk}(x)|}{(1-x_k^2)^{j/2}} \right\| = \begin{cases} O\left(\frac{\log n}{n^j}\right) & \textit{if } m-j \textit{ is odd,} \\ O\left(\frac{1}{n^j}\right) & \textit{if } m-j \textit{ is even} \end{cases}$$
$$(j = 0, \ldots, m-1).$$

PROOF. For the Chebyshev nodes we have

$$b_{ik} = \begin{cases} O\left(\left(\frac{n}{\sin\theta_k}\right)^i\right) & \text{if } i \text{ is even} \\ O\left(\frac{n^{i-1}}{\sin^{i+1}\theta_k}\right) & \text{if } i \text{ is odd,} \end{cases}$$

where $\theta_k = \frac{2k-1}{2n}\pi \quad (k = 1, \ldots, n)$ (see P. Vértesi [6], Lemma 3.11). Also, if for a fixed $x \in [-1, 1]$,

$$|x - x_s| = \min_{1 \leqq k \leqq n} |x - x_k|$$

then

$$|l_k(x)| = \begin{cases} O\left(\frac{\sin\theta_k}{n|x-x_k|}\right) & \text{if } k \neq s, \\ O(1) & \text{if } k = s. \end{cases}$$

Thus, if $m - j$ is odd we obtain from (7)

$$\text{(27)} \qquad |A_{jk}(x)| = O\left(\frac{\sin^m\theta_k}{n^m|x-x_k|^m}\right) \sum_{i=0}^{m-j-1} \left(\frac{n}{\sin\theta_k}\right)^i |x - x_k|^{i+j} =$$
$$= O(n^{-m}) \sum_{i=0}^{m-j-1} \frac{n^i \sin^{m-i}\theta_k}{|x-x_k|^{m-j-i}} = O\left(\frac{\sin^j\theta_k}{n^m}\right) \sum_{i=0}^{m-j-i} \frac{n^i}{\sin^{m-j-i}\frac{|\theta-\theta_k|}{2}} =$$
$$= O\left(\frac{\sin^j\theta_k}{n^j|s-k|}\right) \qquad (k \neq s),$$

and

$$\text{(28)} \qquad |A_{js}(x)| = O(1) \sum_{i=0}^{m-j-1} \left(\frac{n}{\sin\theta_k}\right)^i |x - x_s|^{i+j} = O\left(\left(\frac{\sin\theta_s}{n}\right)^j\right)$$

(of course, this holds for $m - j$ even, too), whence

$$L^{\star}_{jmn} = \left\| \sum_{k=1}^{n} \frac{|A_{jk}(x)|}{(1-x_k^2)^{j/2}} \right\| = O\left(\frac{\log n}{n^j}\right) \qquad (m - j \text{ odd}).$$

If $m-j \geqq 2$ is even, then in (27), for $i = m-j-1$ we use the second relation to get

$$|A_{jk}(x)| = O\left(\frac{\sin^m \theta_k}{n^m|x-x_k|^m}\right)\left\{\sum_{i=0}^{m-j-2}\left(\frac{n}{\sin\theta_k}\right)^i |x-x_k|^{i+j}+ + \frac{n^{m-j-2}}{\sin^{m-j}\theta_k}|x-x_k|^{m-1}\right\} =$$

$$= O\left(\frac{\sin^j \theta_k}{n^m}\right)\sum_{i=0}^{m-j-2}\frac{n^i}{\sin^{m-j-i}\frac{|\theta-\theta_k|}{2}} + O\left(\frac{\sin^j\theta_k}{n^{j+2}|x-x_k|}\right) =$$

$$= O\left(\frac{\sin^j\theta_k}{n^{j+2}\sin^2\frac{\theta-\theta_k}{2}} + \frac{\sin^j\theta_k}{n^{j+2}\sin\frac{|\theta-\theta_k|}{2}\sin\frac{\theta+\theta_k}{2}}\right) = O\left(\frac{\sin^j\theta_k}{n^j(s-k)^2}\right) \qquad (k \neq s)$$

whence and from (28)

$$L^\star_{jmn} = \left\|\sum_{k=1}^{n}\frac{|A_{jk}(x)|}{(1-x_k^2)^{j/2}}\right\| = O\left(\frac{1}{n^j}\right) \qquad (m-j \text{ even}).$$

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APPROXIMATE HIGH ORDER SMOOTHNESS[1]

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A substantial amount is known concerning the properties of smooth functions. (For example, see [15], [8], and [9].) Recently, Dutta [4] introduced the notion of high order smoothness and established a number of interesting results analogous to those for smooth functions. (See also [5] and [6].) Also, the notion of approximate smoothness generalizes that of smoothness and has been found to have a number of similarities. (For example, see [16] and [13].) Here we combine these latter two concepts in the obvious manner to arrive at the notion of approximate high order smoothness, and the purpose of this paper is to show that results analogous to those of Dutta carry over to this setting.

We begin with some definitions and notation. All functions considered here are assumed to be real-valued, Lebesgue measurable functions defined at least on an open interval of the real line. First, such a function f, defined on an open interval I, is said to have an approximate kth Peano derivative at $x_0 \in I$ if there is a polynomial $Q_{x_0,k}(h)$ of degree at most k such that $Q_{x_0,k}(0) = f(x_0)$ and

$$f(x_0 + h) - Q_{x_0,k}(h) = o_{\mathrm{ap}}(h^k) \quad (h \to 0),$$

where the $o_{\mathrm{ap}}(h^k)$ notation is used to indicate that the left hand side, divided by h^k, tends to zero as h tends to zero through a set having density one at zero. The value of the kth approximate Peano derivative of f at x_0 is denoted $f_{(k)}(x_0)$, where $f_{(k)}(x_0)/k!$ is the coefficient of h^k in $Q_{x_0,k}(h)$. (As is customary, we use $f^{(k)}(x_0)$ to denote the value of the ordinary kth derivative of f at x_0, if it exists.) The basic properties of approximate Peano derivatives are described in [2].

More generally, if there is a polynomial $P_{x_0,k}(h)$ of degree at most k for which

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$$\frac{f(x_0+h)+(-1)^k f(x_0-h)}{2} - P_{x_0,k}(h) = o_{\mathrm{ap}}(h^k) \quad (h \to 0),$$

then f is said to have a kth approximate symmetric derivative at x_0 and the value of this derivative at x_0 is denoted by $D^k_{\mathrm{ap}} f(x_0)$, where $D^k_{\mathrm{ap}} f(x_0)/k!$ is the coefficient of h^k in $P_{x_0,k}(h)$. If k is even, we further require that $P_{x_0,k}(0) = f(x_0)$. It is well known that if k is even, then $P_{x_0,k}(h)$ has only even powers of h, and if k is odd, only odd powers. Clearly, if f has a kth approximate Peano derivative at x_0, then it has a kth approximate symmetric derivative at x_0 and the values of the two derivatives are equal.

If we suppress the word "approximate" everywhere in the previous paragraph, we have the concept of a kth symmetric derivative at x_0, which we shall denote by $D^k f(x_0)$.

Next, suppose that m is a natural number greater than or equal to 2 and that f has an $m-2$ approximate symmetric derivative at x_0. If, further, it happens that

$$\frac{f(x_0+h)+(-1)^m f(x_0-h)}{2} - P_{x_0,m-2}(h) = o_{\mathrm{ap}}(h^{m-1}) \quad (h \to 0),$$

then f is said to be approximately m-smooth at x_0. If f is approximately m-smooth at each point of an interval, we shall say that f is approximately m-smooth on the interval, and if we simply say that f is approximately m-smooth, we mean that it is approximately m-smooth on $(-\infty,\infty)$. Thus, approximate 2-smoothness is the notion of approximate smoothness explored in [16], [13], and [12]. If the word "approximate" is suppressed from the above definitions and discussion, then we arrive at Dutta's [4] concept of m-smoothness, again with 2-smoothness being what is more commonly called smoothness.

Throughout we shall use $\lambda(S)$ to denote the Lebesgue measure of a measurable set S. We shall also use what is becoming standard notation by saying that a function f is a Baire* 1 function if every non-empty perfect set contains a portion such that the restriction of f to that portion is continuous.

We are now ready to begin our investigation into the continuity and differentiability of approximately m-smooth functions.

THEOREM 1. *If f is approximately continuous and approximately m-smooth, then $D^{m-2}_{\mathrm{ap}} f$ is a Baire* 1 function.*

PROOF. We shall deal with odd and even m's separately, providing an inductive proof in each instance.

Dealing with the odd numbers first, consider $m=3$, i.e., suppose that f is approximately continuous and approximately 3-smooth. For the sake of reaching a contradiction, assume that $D^1_{\mathrm{ap}} f$ is not Baire* 1. Then there exists a non-empty perfect set Q such that on any portion of Q the restriction of $D^1_{\mathrm{ap}}f$ is not continuous.

By the approximate 3-smoothness of f we can choose for each number x a set $H_x(f) \subset [-1,1]$ symmetric about 0 such that 0 is a density point of $H_x(f)$, and if $h \in H_x(f)$ then

$$|f(x+h) - f(x-h) - D^1_{\mathrm{ap}}f(x) \cdot 2h| < h^2.$$

For each natural number q, put

$$A_q = \{x: \text{ if } 0 < t < 2/q \text{ then } \lambda(H_x \cap [-t,t]) > 0.9 \cdot 2t\}.$$

By the Baire Category Theorem there is a natural number q and a portion Q' of Q such that A_q is dense in Q'. By our assumption $D^1_{\mathrm{ap}}f$ is not continuous on Q' and hence, using the density of A_q in Q', it is a straightforward matter to show that there is a point $y \in Q'$, an $\varepsilon > 0$, and a sequence of points $p_k \in Q' \cap A_q$, such that $p_k \to y$, and $|D^1_{\mathrm{ap}}f(y) - D^1_{\mathrm{ap}}f(p_k)| > \varepsilon$ for each natural number k.

Let g be the function defined by $g(x) = f(x) - D^1_{\mathrm{ap}}f(y) \cdot (x-y)$. Then $D^1_{\mathrm{ap}}g(y) = 0$ and for each k we have $|D^1_{\mathrm{ap}}g(p_k)| > \varepsilon$. Also note that for each x we have $H_x(f) = H_x(g)$. For the sake of brevity we shall denote the sets $H_{p_k}(g)$ and $H_y(g)$ by H_k and H, respectively.

Without loss of generality we shall assume that the p_k's decrease monotonically to y.

Choose $h_0 > 0$ such that $h_0 < \min\{1/q,\ \varepsilon/21\}$ and such that for all $0 < t < h_0$ we have $\lambda(H \cap [-t,t]) > 0.9 \cdot 2t$. Next, choose K so large that for all $k > K$, $p_k - y < 0.01h_0$. For $k > K$ let T_k denote the set of all t in $(0.1h_0,\ h_0)$ such that $y - t \in (y+H) \cap (p_k + H_k)$. Then for $t \in T_k$ we have

$$|g(y+t) - g(y-t)| < t^2,$$

and

$$|g(p_k+(p_k-(y-t)))-g(p_k-(p_k-(y-t)))-D^1_{\mathrm{ap}}g(p_k)2(p_k-(y-t))| < (p_k-(y-t))^2,$$

and hence, by adding the previous two inequalities we obtain

$$|g(y+t)-g(y+t+2(p_k-y)) + D^1_{\mathrm{ap}}g(p_k)2(p_k-y+t)| < t^2 + (p_k-y+t)^2 <$$
$$< h_0^2 + 1.01^2h_0^2 < 2.1h_0^2 < 2.1h_0\frac{\varepsilon}{21} = 0.1\varepsilon h_0.$$

Therefore

$$|g(y+t) - g(y+t+2(p_k-y))| > |D^1_{\mathrm{ap}}g(p_k) \cdot 2(p_k-y+t)| - 0.1\varepsilon h_0 >$$
$$> \varepsilon \cdot 2(p_k-y+t) - 0.1\varepsilon h_0 > \varepsilon 2t - 0.1\varepsilon h_0 > 0.2\varepsilon h_0 - 0.1\varepsilon h_0 = 0.1\varepsilon h_0.$$

Next, we claim that $\lambda(T_k) > 0.6h_0$. To see this, first note that by the choice of h_0 and the symmetry of H we have $\lambda([y-h_0, y]\backslash(y+H)) < 0.1h_0$. Next, since $p_k - y < 0.01h_0 < 0.1h_0$ and H_k is symmetric about 0, we have that $\lambda([p_k - 1.1h_0,\ p_k]\backslash(p_k + H_k)) < 0.1 \cdot 1.1h_0$, and hence by using that $p_k - 1.1h_0 < y - h_0 < y < p_k$, we obtain that $\lambda([y-h_0, y]\backslash(p_k+H_k)) < 0.11h_0$. Therefore $\lambda([y-h_0, y] \cap (y+H) \cap (p_k+H_k)) > (1-0.21)h_0$ and so $\lambda(T_k) = \lambda(y-T_k) = \lambda([y-h_0, y-0.1h_0] \cap (y+H) \cap (p_k+H)) > (1-0.31)h_0 > 0.6h_0$.

Next, put $C_k = y + T_k$. Then $C_k \subset [y+0.1h_0, y+h_0]$, $\lambda(C_k) > 0.6h_0$, and if $x \in C_k$ then

$$|g(x) - g(x + 2(p_k - y))| > 0.1\varepsilon h_0.$$

Put $h_k = 0.9h_0/[0.9h_0/(p_k - y)]$, where $[x]$ denotes the integer part of x. Since $p_k - y < 0.01h_0$ we clearly have $2(p_k - y) > h_k > p_k - y$. Denote by Φ_k the set of integers ℓ for which

$$\lambda([y + 0.1h_0 + (\ell-1)h_k,\ y + 0.1h_0 + \ell h_k] \cap C_k) > 0.1h_k.$$

Set

$$D_k = \bigcup_{\ell \in \Phi_k} [y + 0.1h_0 + (\ell-1)h_k,\ y + 0.1h_0 + \ell h_k].$$

Thus,

$$\lambda(C_k \cap ([y+0.1h_0, y+h_0]\backslash D_k)) < 0.1\lambda([y+0.1h_0, y+h_0]) < 0.09h_0.$$

If $\lambda(D_k) < 0.5h_0$ then $\lambda(C_k) \leq \lambda(D_k) + \lambda(C_k \cap ([y+0.1h_0, y+h_0]\backslash D_k) < 0.59h_0$, contrary to $\lambda(C_k) > 0.6h_0$. Consequently, we have $\lambda(D_k) \geqq 0.5h_0$ for every $k > K$. It follows that there is a set U of measure at least $0.5h_0$ such that if $x \in U$ then x belongs to infinitely many D_k. Choose an $x \in U$ and denote the set of corresponding indices k by Ψ.

If $k \in \Psi$ then $x \in D_k$ and hence there exists an $\ell \in \Phi_k$ such that $x \in [y + 0.1h_0 + (\ell-1)h_k,\ y + 0.1h_0 + \ell h_k]$ and $\lambda([y+0.1h_0+(\ell-1)h_k,\ y+0.1h_0+\ell h_k] \cap C_k) > 0.1h_k$. Put $W_k = [y + 0.1h_0 + (\ell-1)h_k,\ y + 0.1h_0 + \ell h_k] \cap C_k$. If $w \in W_k$ then

$$|g(w) - g(w + 2(p_k - y))| > 0.1\varepsilon h_0.$$

Since $p_k - y < h_k < 2(p_k - y)$, we have $W_k \cup (W_k + 2(p_k - y)) \subset [x - 4(p_k - y),\ x + 4(p_k - y)]$. Since g is approximately continuous at x, there is a set E and a $\delta > 0$ such that if $|t| < \delta$ then $\lambda([x-t, x+t] \cap E) > 2 \cdot 0.99t$, and for $w \in [x-\delta, x+\delta] \cap E$ we have $|g(w) - g(x)| < 0.05\varepsilon h_0$. Choose $k \in \Psi$ such that $4(p_k - y) < \delta$; this is possible since $p_k \to y$ and Ψ is infinite. If $w \in W_k$ then $|g(w) - g(w + 2(p_k - y))| > 0.1\varepsilon h_0$ and hence either w or $w + 2(p_k - y)$ must fail to belong to $[x - \delta, x + \delta] \cap E$. Since $\lambda(W_k) > 0.1h_k > 0.1(p_k - y)$, this would imply that $\lambda([x - 4(p_k - y), x + 4(p_k - y)] \cap E) < 8(p_k - y) - 0.1(p_k - y)$, contrary to $\lambda([x - 4(p_k - y), x + 4(p_k - y)] \cap E) > 2 \cdot 0.99 \cdot 4(p_k - y)$. This contradiction completes the proof for the case $m = 3$.

Now, suppose the theorem is true for $m = 3, 5, \dots, n-2$, and suppose that f is an approximately continuous, approximately n-smooth function. Assume that $D^{n-2}_{\mathrm{ap}} f$ is not Baire* 1. Then there is a non-empty perfect set Q such that for each portion of Q, the restriction of $D^{n-2}_{\mathrm{ap}} f$ to that portion fails to be continuous. By the inductive hypothesis, there is a nonempty set Q' of the form $[c,d] \cap Q$ on which each of the functions $D^1_{\mathrm{ap}} f$, $D^3_{\mathrm{ap}} f, \dots, D^{n-4}_{\mathrm{ap}} f$ is continuous.

By the approximate n-smoothness of f we can choose for each number x a set $H_x(f) \subset [-1,1]$ symmetric about 0 such that 0 is a density point of $H_x(f)$, and if $h \in H_x(f)$ then

$$|f(x+h) - f(x-h) - 2P_{x,n-4}(h) - 2\frac{D^{n-2}_{\mathrm{ap}} f(x)}{(n-2)!} h^{n-2}| < h^{n-1}.$$

For each natural number q, put

$$A_q = \{x : \text{ if } 0 < t < 2/q \text{ then } \lambda(H_x \cap [-t,t]) > 0.9 \cdot 2t\}.$$

By the Baire Category Theorem there is a natural number q and a portion Q'' of Q' such that A_q is dense in Q''. Since $D^{n-2}_{\mathrm{ap}} f$ is not continuous on Q'', there exists a $y \in Q''$, and $\varepsilon > 0$, and a sequence of points $p_k \in Q'' \cap A_q$ such that $|D^{n-2}_{\mathrm{ap}} f(y) - D^{n-2}_{\mathrm{ap}} f(p_k)| > \varepsilon(n-2)!$ for each natural number k.

Let g be the function given by $g(x) = f(x) - P_{y,n-2}(x-y)$. Then $D^1_{\mathrm{ap}} g(y) = D^3_{\mathrm{ap}} g(y) = \dots = D^{n-2}_{\mathrm{ap}} g(y) = 0$ and for each k, $|D^{n-2}_{\mathrm{ap}} g(p_k)| > \varepsilon(n-2)!$. Also, note that for each x, $H_x(f) = H_x(g)$. For the sake of brevity we shall denote the sets $H_{p_k}(g)$ and $H_y(g)$ by H_k and H, respectively.

Without loss of generality we assume that the p_k's decrease monotonically to y.

Choose $h_0 > 0$ such that $h_0 < \min\{1/q, \varepsilon/(20.1)^{n-1}\}$ and such that for all $0 < t \leqq h_0$ we have $\lambda(H \cap [-t,t]) > 0.9 \cdot 2t$. Next, choose K so large that for all $k > K$ and all $0 \leqq h \leqq 1.1h_0$, we have $|P_{p_k,n-4}(h)| < \frac{\varepsilon}{4}(0.1h_0)^{n-2}$, and $p_k - y < 0.01h_0$. Now let $k > K$ and let T_k denote the set of all t in $(0.1h_0, h_0)$ such that $y - t \in (y+H) \cap (p_k + H_k)$. Then for $t \in T_k$ we have

$$|g(y+t) - g(y-t)| < t^{n-1},$$

and

$$|g(p_k + (p_k - (y-t))) - g(p_k - (p_k - (y-t))) - 2P_{p_k,n-4}(p_k - y + t) - \\ -2D^{n-2}_{\mathrm{ap}} g(p_k)(n-2)!(p_k - y + t)^{n-2}| < (p_k - y + t)^{n-1},$$

and, consequently,

(1)

$$|g(y+t) - g(2p_k - y + t) + 2P_{p_k,n-4}(p_k - y + t) + 2\frac{D^{n-2}_{\mathrm{ap}} g(p_k)}{(n-2)!}(p_k - y + t)^{n-2}| < \\ < t^{n-1} + (p_k - y + t)^{n-1}.$$

Hence,

$$|g(y+t)-g(2p_k-y+t)|>$$

$$>\left|2\frac{D_{\mathrm{ap}}^{n-2}g(p_k)}{(n-2)!}(p_k-y+t)^{n-2}\right|-t^{n-1}-(p_k-y+t)^{n-1}-|2P_{p_k,n-4}(p_k-y+t)|>$$

$$>2\varepsilon(p_k-y+t)^{n-2}-t^{n-1}-(p_k-y+t)^{n-1}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}>$$

$$>2\varepsilon t^{n-2}-h_0^{n-1}-(1.01h_0)^{n-1}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}>$$

$$>2\varepsilon t^{n-2}-(2.01h_0)^{n-1}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}=$$

$$=2\varepsilon(0.1h_0)^{n-2}-(2.01h_0)(2.01h_0)^{n-2}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}>$$

$$>2\varepsilon(0.1h_0)^{n-2}-(2.01\cdot\frac{\varepsilon}{(20.1)^{n-1}})(2.01h_0)^{n-2}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}=$$

$$=2\varepsilon(0.1h_0)^{n-2}-\frac{\varepsilon}{10}(0.1h_0)^{n-2}-\frac{\varepsilon}{2}(0.1h_0)^{n-2}>$$

$$>2\varepsilon(0.1h_0)^{n-2}-\varepsilon(0.1h_0)^{n-2}=\varepsilon(0.1h_0)^{n-2}.$$

In summary, for $t\in T_k$ we have

$$|g(y+t)-g(y+t+2(p_k-y))|>\varepsilon(0.1h_0)^{n-2}.$$

Next we claim that $\lambda(T_k)>0.6h_0$. Indeed, this is established in exactly the same manner as in the $m=3$ case and the rest of the proof is completed in precisely the same manner as before with the sole exception being that the positive number δ is chosen so small this time that for $w\in[x-\delta,x+\delta]\cap E$ we have $|g(w)-g(x)|<\varepsilon(0.1h_0)^{n-2}/2$. This again will produce a contradiction and complete the proof for the odd numbers m.

We now turn our attention to the even numbers m. The case $m=2$ was proved by Larson [12]; i.e., he showed that if an approximately continuous function f is approximately 2-smooth, then f is a Baire* 1 function.

Proceeding to the inductive step, assume that the theorem holds for the even integers $m=2,4,\dots,n-2$, and suppose that the approximately continuous function f is approximately n-smooth. Assume that $D_{\mathrm{ap}}^{n-2}f$ is not Baire* 1. Then there is a non-empty perfect set Q such that for each portion of Q, the restriction of $D_{\mathrm{ap}}^{n-2}f$ to that portion fails to be continuous. By the inductive hypothesis, there is a nonempty set Q' of the form $[c,d]\cap Q$ on which each of the functions, $f, D_{\mathrm{ap}}^2f, D_{\mathrm{ap}}^4f,\dots,D_{\mathrm{ap}}^{n-4}f$ is continuous.

By the approximate n-smoothness of f we can choose for each number x a set $H_x(f)\subset[-1,1]$ symmetric about 0, such that 0 is a density point of $H_x(f)$, and if $h\in H_x(f)$ then

$$|f(x+h)+f(x-h)-2P_{x,n-4}(h)-2\frac{D_{\mathrm{ap}}^{n-2}f(x)}{(n-2)!}h^{n-2}|<h^{n-1}.$$

For each natural number q, put

$$A_q = \{x : \text{ if } 0 < t < 2/q \text{ then } \lambda(H_x \cap [-t,t]) > 0.9 \cdot 2t\}.$$

By the Baire Category Theorem there is a natural number q and a portion Q'' of Q' such that A_q is dense in Q''. Since $D_{\text{ap}}^{n-2}f$ is not continuous on Q'', there exists a $y \in Q''$, an $\varepsilon > 0$, and a sequence of points $p_k \in Q'' \cap A_q$ such that $|D_{\text{ap}}^{n-2}f(y) - D_{\text{ap}}^{n-2}f(p_k)| > \varepsilon(n-2)!$ for each natural number k.

Let g be the function given by $g(x) = f(x) - P_{y,n-2}(x-y)$. Then $g(y) = D_{\text{ap}}^2 g(y) = D_{\text{ap}}^4 g(y) = \ldots = D_{\text{ap}}^{n-2} g(y) = 0$ and for each k we have $|D_{\text{ap}}^{n-2}g(p_k)| > \varepsilon(n-2)!$. Also, note that for each x, $H_x(f) = H_x(g)$. For the sake of brevity we shall denote the sets $H_{p_k}(g)$ and $H_y(g)$ by H_k and H, respectively.

Without loss of generality we assume that the p_k's decrease monotonically to y.

Choose $h_0 > 0$ such that $h_0 < \min\{1/q, \varepsilon/(20.1)^{n-1}\}$ and such that for all $0 < t \leqq h_0$ we have $\lambda(H \cap [-t,t]) > 0.9 \cdot 2t$. Next, choose K so large that for all $k > K$ and all $0 \leqq h \leqq 1.1h_0$, we have $|P_{p_k,n-4}(h)| < \frac{\varepsilon}{4}(0.1h_0)^{n-2}$, and $p_k - y < 0.01h_0$. Now let $k > K$ and let T_k denote the set of all t in $(0.1h_0, h_0)$ such that $y - t \in (y + H) \cap (p_k + H_k)$. Whereas in the odd case we had for $t \in T_k$, $|g(y+t) - g(y-t)| < t^{n-1}$, here for $t \in T_k$ we have

$$|g(y+t) + g(y-t)| < t^{n-1},$$

and

$$\begin{gathered}|g(p_k + (p_k - (y-t))) + g(p_k - (p_k - (y-t)))- \\ -2P_{p_k,n-4}(p_k - y + t) - 2D_{\text{ap}}^{n-2}g(p_k)(n-2)!(p_k - y + t)^{n-2}| < (p_k - y + t)^{n-1},\end{gathered}$$

and, consequently,

$$\begin{gathered}|g(y+t) - g(2p_k - y + t) + 2P_{p_k,n-4}(p_k - y + t) + \\ +2\frac{D_{\text{ap}}^{n-2}g(p_k)}{(n-2)!}(p_k - y + t)^{n-2}| < t^{n-1} + (p_k - y + t)^{n-1}.\end{gathered} \tag{2}$$

This inequality (2) is analogous to inequality (1) in the proof for the odd integers m, and the proof from here on is identical to that one. Thus our theorem is proved.

Two observations on this result may be appropriate here. First, we do not know if the assumption of approximate continuity is necessary. Furthermore, notice that in the even cases, the original function f turns out to be a Baire* 1 function. In the odd cases this need not be true as the following example shows.

EXAMPLE 1. There is an approximately continuous function f which is approximately m-smooth for every odd natural number $m \geqq 3$, but f is not a Baire* 1 function.

PROOF. We begin by defining several sequences in $[-1,1]$. For $n \in \mathbf{N}$, the set of natural numbers, we define $a_{n,0} = 1$ and then for $n, k \in \mathbf{N}$ we let

$$a_{n,k} = \frac{a_{n,k-1}}{2^{n+k}}, \text{ and } b_{n,k} = a_{n,k-1} - 2a_{n,k}.$$

The 'n' should be thought of as fixed and the a's and b's defining a sequence of intervals $\{(a_{n,k}, b_{n,k})\}$ converging to 0. We also need central sequences of intervals $[c_{n,k}, d_{n,k}]$ which are defined by

$$c_{n,k} = a_{n,k} + \frac{(n+2)a_{n,k+1}}{2^{n+2}}, \quad d_{n,k} = b_{n,k} - \frac{(n+2)a_{n,k}}{2^{n+2}}.$$

We extend to the left half of $[-1,1]$ by defining $a_{n,-k} = -a_{n,k}$, and so on; $a_{n,0}$ will be used to designate either $+1$ or -1 as the situation dictates. Now, fix n and k and define $f_{n,k}$ as follows:

$$f_{n,k}(x) = \begin{cases} 0 & \text{for } x \notin (a_{n,k}, b_{n,k}); \\ 1/n & \text{for } x \in [c_{n,k}, d_{n,k}]; \\ \text{a } C^\infty & \text{spline on each of } [a_{n,k}, c_{n,k}] \text{ and } [d_{n,k}, b_{n,k}]. \end{cases}$$

It is easy to see that $f_{n,k}$ is C^∞. For fixed n we define:

$$f_n(x) = \sum_{k \in \mathbf{Z}} f_{n,k}(x) + \frac{\chi_{\{0\}}(x)}{n},$$

where $\mathbf{Z}$ denotes the set of integers. This function is continuous except at 0 and is actually C^∞ at points of $\bigcup_{k=-\infty}^{\infty} (a_{n,k}, b_{n,k})$. It is also clear that $E_{0,h} = \{h : f_n(h) = f_n(-h)\}$ has full density at 0, and consequently, f_n is $2m-1$ approximately smooth at 0 for every m. The functions f_n serve as building blocks, and to take full advantage of their properties, it is necessary to define them on arbitrary compact intervals. If I is such an interval, $f_{n,I}$ denotes the composition of f_n with the affine map from I to $[-1,1]$; points of I corresponding to the $a_{n,k}$'s will be denoted by $a_{n,k}(I)$. Other corresponding points are similarly denoted.

Easy computations show that:

$$b_{n,k}(I) - a_{n,k}(I) = \left(1 - \frac{3}{2^{n+k}}\right) a_{n,k-1} \frac{|I|}{2},$$

$$d_{n,k}(I) - c_{n,k}(I) \geqq \left(1 - \frac{4}{2^{n+k}}\right) a_{n,k-1} \frac{|I|}{2},$$

$$a_{n,k-1}(I) - b_{n,k}(I) = a_{n,k}|I|,$$

$$c_{n,k-1}(I) - d_{n,k}(I) = \left(\frac{n+2}{2^{n+2}} + 1\right) a_{n,k}|I|.$$

First define $g_1(x) = f_1(x)$. Then the complement of the support of g_1 consists of countably many compact intervals, the two extreme of which are set aside. The function f_2 is then inserted into each of the remaining intervals and the process repeated. More specifically, set $T_1 = \{[-1, b_{1,-1}], [b_{1,1}, 1]\}$, and define

$$g_2(x) = \sum f_{2,I}(x)$$

where the sum is taken over all compact components of the complement of the support of g_1 which are not in T_1. Suppose g_n has been defined and a set, T_n, has been designated. Then,

$$g_{n+1}(x) = \sum f_{n+1,I}(x)$$

where the sum is taken over all compact components of the complement of the support of $\sum_{k=1}^{n} g_k$ which are not in T_n. The set T_{n+1} consists of T_n together with all intervals of the form $[a_{n+1,0}(I), b_{n+1,-1}(I)]$ and $[b_{n+1,1}(I), a_{n+1,0}(I)]$ where I is as in the last sentence. Finally, let

$$G_n(x) = \sum_{k=1}^{n} g_k(x), \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} g_k(x).$$

Let $P = G^{-1}(0)$. Then $M \equiv \mathrm{cl}(P)\backslash P$ (cl $\equiv$ closure) is a countable set corresponding to midpoints of contiguous intervals. Several properties of G are easily deduced.

1. The set $S = (-1, 1)\backslash \mathrm{cl}(P)$ is an open subset of $(-1, 1)$, and G is C^∞ on that set.

2. G is continuous at each point of P.

3. $\mathrm{cl}(M)\backslash M \subset P$, yet $M \cap P = \emptyset$.

4. $\mathrm{cl}(M)$ is perfect.

It follows from the third and fourth remarks that G is not Baire*1. What remains is to show that G is approximately $2m - 1$ smooth at each point of $\mathrm{cl}(M)$. This is obvious at points of M because the intervals $[c_{n,k}, d_{n,k}] : k = 0, \pm 1, \ldots$ have full density at 0. The remainder of this section is devoted to verifying that if $x \in \mathrm{cl}(M)\backslash M$, then $E_x \equiv \{h : G(x - h) \neq G(x + h)\}$ has density 0 at 0. If $x \in \mathrm{cl}(M)\backslash M$, then

$$x = \bigcap_{n=0}^{\infty} I_n(x)$$

where $I_0 = [-1, 1]$ and $I_n(x)$ is the unique interval contiguous with the support of G_n which contains x. There is a corresponding sequence of integers, k_n, such that

$$I_n = [b_{n,k_n+1}(I_{n-1}), a_{n,k_n}(I_{n-1})], \quad n = 1, 2, \ldots .$$

Note that $|I_n| = 2a_{n,k_n+1}|I_{n-1}|$, where $|I|$ denotes the length of the interval I. We show that if

$$\frac{|I_n|}{2} = a_{n,k_n+1}|I_{n-1}| \leqq h \leq a_{n-1,k_{n-1}+1}|I_{n-2}| = \frac{|I_{n-1}|}{2},$$

then $\Delta(E_{x,h}) < \frac{1}{n}$, where

$$E_{x,h} = \{t \in (0,h] : G(x+t) \neq G(x-t)\} \text{ and } \Delta(E_{x,h}) = \lambda(E_{x,h})/h.$$

If $S \subseteq \mathbf{R}$ and I is an interval we let $\Delta(S,I) = \frac{\lambda(S\cap I)}{|I|}$. The computations used below make use of the following remark.

REMARK. If S is a measurable set and both $\Delta(S^c,[x,x+h]) < \varepsilon$ and $\Delta(S^c,[x-h,x]) < \varepsilon$ then $\Delta(T,[0,h]) < 2\varepsilon$ where $T = \{t : \text{either } x-t \notin S \text{ or } x+t \notin S\}$.

The proof that $\Delta(E_{x,h}) < 1/n$ for $h \in [|I_n|/2, |I_{n-1}|/2]$ is carried out in three parts. The first part verifies the result for $h \in [|I_n|/2, x - c_{n,-k_n}(I_{n-1})]$. This is accomplished by considering five critical h values in this interval and applying Remark 1.

For notational convenience, we suppose $k_{n+1} > 0$ so that x lies in the right half of I_n which we suppose to be centered at 0.

Part 1

Define $h_0 = |I_n|/2$, $h_1 = c_{n,k_n}(I_{n-1}) + 2a_{n+1,1}|I_n| - x$, $h_2 = x - c_{n,k_n+1}(I_{n-1})$, $h_3 = x - c_{n,-k_n-1}(I_{n-1})$, $h_4 = x - d_{n,-k_n-1}(I_{n-1})$ and $h_5 = x - c_{n,-k_n}(I_{n-1})$. Other than h_0 and h_1 these critical values determine symmetric intervals centered at x which are determined by a point labelled on Diagram 1. The value of h_1 is chosen to insure that both $x + h_1 \in (c_{n,k_n}(I_{n-1}), d_{n,k_n}(I_{n-1}))$ and $x - h_1 \in (c_{n,k_n+1}(I_{n-1}), d_{n,k_n+1}(I_{n-1}))$. Roughly speaking, h_1 is the smallest convenient value we can safely say has these two properties. The entirety of Diagram 1 is contained in I_{n-1} and each of the displayed points has been labeled without the suffix (I_{n-1}), i.e., c_{n,k_n} denotes $c_{n,k_n}(I_{n-1})$.

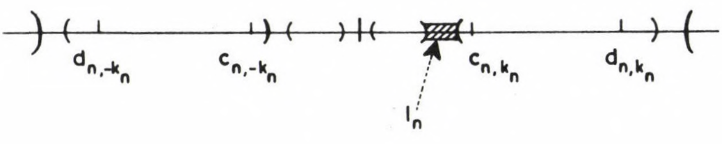

Diagram 1

a. Suppose $h \in [h_0, h_1]$ and let $S = G^{-1}(\frac{1}{n+1})$. Then

$$(d_{n+1,-1}(I_n), c_{n+1,-1}(I_n)) \subseteq S \cap [x-h, x]$$

so that

$$\Delta(S^c, [x-h, x]) \leq \frac{h - (c_{n+1,-1}(I_{n-1}) - d_{n+1,-1}(I_{n-1}))}{h} <$$

$$< \frac{h_1 - (c_{n+1,-1}(I_{n-1}) - d_{n+1,-1}(I_{n-1}))}{h_0} =$$

$$= \frac{(c_{n,k_n}(I_{n-1}) + 2a_{n+1,1}|I_n| - x) - (c_{n+1,-1}(I_{n-1}) - d_{n+1,-1}(I_{n-1}))}{|I_n|/2} <$$

$$< \frac{c_{n,k_n}(I_{n-1}) - (c_{n+1,-1}(I_n) - d_{n+1,-1}(I_n) + 2a_{n+1,1}|I_n|)}{|I_n|/2} =$$

$$= \frac{(c_{n,k_n}(I_{n-1}) - a_{n,k_n}(I_{n-1})) + (a_{n+1,0}(I_n) - b_{n+1,1}(I_n))}{|I_n|/2} +$$

$$+ \frac{(b_{n+1,1}(I_n) - d_{n+1,1}(I_n)) + (c_{n+1,1}(I_n) - a_{n+1,1}(I_n)) + 3a_{n+1,1}|I_n|}{|I_n|/2} \leqq$$

$$\leqq \frac{(c_{n,k_n}(I_{n-1}) - a_{n,k_n}(I_{n-1})) + a_{n+1,1}|I_n| + \frac{n+3}{2^{n+3}}\, a_{n+1,1}|I_n|}{|I_n|/2} +$$

$$+ \frac{\frac{n+3}{2^{n+3}}\, a_{n+1,2}|I_n| + 3a_{n+1,1}|I_n|}{|I_n|/2} =$$

$$= \frac{(c_{n,k_n}(I_{n-1}) - a_{n,k_n}(I_{n-1})) + \frac{n+3}{2^{n+3}}\, a_{n+1,1}|I_n|}{|I_n|/2} +$$

$$+ \frac{\frac{n+3}{2^{n+3}} \frac{1}{2^{n+3}}\, a_{n+1,1}|I_n| + 4a_{n+1,1}|I_n|}{|I_n|/2} <$$

$$< \frac{(c_{n,k_n}(I_{n-1}) - a_{n,k_n}(I_{n-1})) + 5a_{n+1,1}|I_n|}{|I_n|/2} =$$

$$= \frac{\frac{n+2}{2^{n+2}}\, a_{n,k_n+1}\, |I_{n-1}| \;+\; 5a_{n+1,1}|I_n|}{|I_n|/2} = \frac{n+2}{2^{n+1}} \frac{a_{n,k_n}}{2^{n+k_n}} + \frac{5}{2^{n+2}}$$

(see Diagram 2), which is $o(1/n)$. Also, $(c_{n+1,1}(I_n), d_{n+1,1}(I_n)) \subseteq S \cap [x, x+h]$ and a similar computation shows that $\Delta(S^c, [x, x+h])$ is also $o(1/n)$. Hence, for sufficiently large n, $\Delta(E_{x,h}) < 1/n$ for each $h \in [h_0, h_1]$, uniformly in h.

b. At this point two remarks are relevant. First, note that the interval $(c_{n,k_n}(I_{n-1}), d_{n,k_n}(I_{n-1}))$ is extraordinarily long compared to the interval $(c_{n,k_{n+1}}(I_{n-1}), d_{n,k_{n+1}}(I_{n-1}))$ and second, on each of these intervals

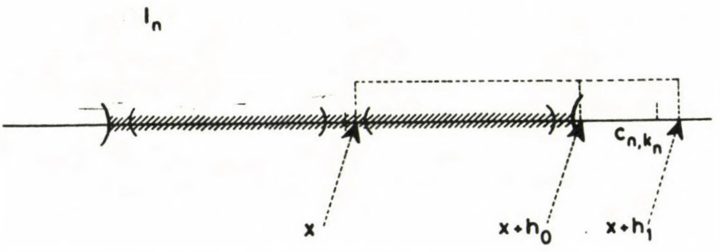

Diagram 2

G is identically $\frac{1}{n}$. The choice of h_1 then entails that for $h \in [h_1, h_2]$, $G(x+h) = G(x-h) = \frac{1}{n}$ and hence, $\Delta(E_{x,h}) \leqq \Delta(E_{x,h_1})$. By observing that $\Delta(E_{x,h_2}) \leqq \Delta(E_{x,h_1})$ and using the fact, proved in Section a, that $\Delta(E_{x,h_1}) = o(1/n)$, it follows that $\Delta(E_{x,h_2}) = o(1/n)$.

c. The ratio $\frac{h_3-h_2}{h_2}$ is also $o(1/n)$ and as a consequence for $h \in [h_2, h_3]$, we have $\Delta(E_{x,h}) = o(1/n)$ (see Diagram 3).

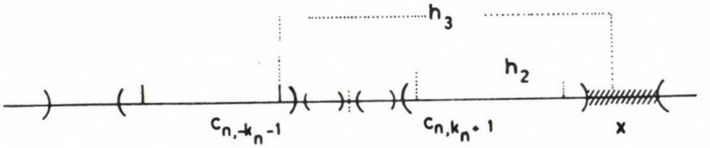

Diagram 3

d. If $h \in [h_3, h_4]$, then $x-h \in (d_{n,-k_n-1}(I_{n-1}), c_{n,-k_n-1}(I_{n-1}))$, $x+h \in$ $\in (c_{n,k_n}(I_{n-1}), d_{n,k_n}(I_{n-1}))$, and each of these intervals is contained in $G^{-1}(\frac{1}{n})$. Hence, for such h, $\Delta(E_{x,h}) \leqq \Delta(E_{x,h_3})$. In particular, if $h = h_4$ we have $\Delta(E_{x,h_4}) = o(1/n)$.

e. The ratio $\frac{h_5-h_4}{h_4}$ is also $o(1/n)$ and hence for $h \in [h_4, h_5]$, $\Delta(E_{x,h}) =$ $= o(1/n)$. This completes the first part of the argument.

Part 2

In this part we show that if $t \in [x - c_{n,-k_n}(I_{n-1}), x - c_{n,-1}(I_{n-1})]$ then $\Delta(E_{x,t}) < \frac{1}{n}$ for all sufficently large n and uniformly in t. The technique of proof is similar to that of Part 1 except that the critical values are more regular. Suppose $m = 0, 1, \ldots, k_n - 2$, let $t_{0,m} = x - c_{n,-k_n+m}(I_{n-1})$, and $t_{1,m} = d_{n,k_n-m}(I_{n-1}) - x$. If $t \in [t_{0,m}, t_{1,m}]$, then

$$x - t \in (d_{n,-k_n+m}(I_{n-1}), c_{n,-k_n+m}(I_{n-1}))$$

and

$$x+t \in (c_{n,k_n-m}(I_{n-1})), d_{n,k_n-m}(I_{n-1}))$$

and each of these intervals is contained in $G^{-1}(\frac{1}{n})$. Hence, for such t, we have $\Delta(E_{x,t}) \leqq \Delta(E_{x,t_{0,m}})$. See Diagram 4. Note that $t_{0,0} = h_5$, and therefore $\Delta(E_{x,t_{0,0}}) = o(1/n)$. Using the above it now follows that $\Delta(E_{x,t_{1,0}}) = o(1/n)$. Let $t_{2,m} = t_{0,m+1} = x - c_{n,-k_n+m+1}(I_{n-1})$. Then it is easy to see that the ratio $\frac{t_{2,m}-t_{1,m}}{t_{1,m}}$ is $o(1/n)$. Hence it follows that if $t \in [t_{1,0}, t_{2,0}]$, then $\Delta(E_{x,t}) = o(1/n)$. As $t_{2,0} = t_{0,1}$, we can continue this process to conclude that $\Delta(E_{x,t}) < \frac{1}{n}$ for sufficiently large n and uniformly for all $t \in \in [x - c_{n,-k_n}(I_{n-1}), x - c_{n,-1}(I_{n-1})]$. This completes the second part of the argument.

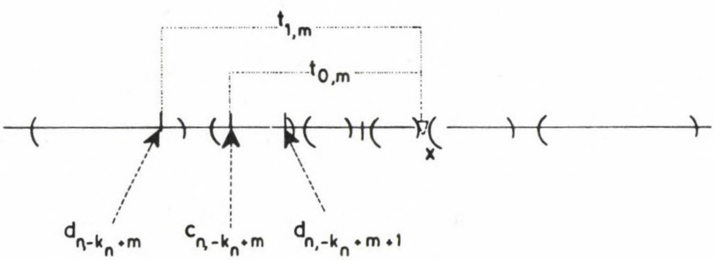

Diagram 4

Part 3

In this part we complete the proof that there is an N sufficiently large that if $n \geqq N$, $\Delta(E_{x,h}) < \frac{1}{n}$ for all $h \in \left[\frac{|I_n|}{2}, \frac{|I_{n-1}|}{2}\right]$. It is sufficient to show that $\Delta(E_{x,s}) = o(1/n)$ uniformly for $s \in [x - c_{n,-1}(I_{n-1}), |I_{n-1}|/2]$. Let $s_0 = x - c_{n,-1}(I_{n-1})$ and $s_1 = d_{n,1}(I_{n-1}) - x$. For $s \in (s_0, s_1)$, $x - s \in \in (d_{n,-1}(I_{n-1}), c_{n,-1}(I_{n-1}))$ and $x+s \in (c_{n,1}(I_{n-1}), d_{n,1}(I_{n-1}))$ both of which intervals are contained in S. Consequently, $\Delta(E_{x,s}) \leqq \Delta(E_{x,s_0})$ which we saw in Part 2 is $o(1/n)$. Further, a proof entirely analogous to the first portion of Part 2 shows that $\Delta(E_{x,s_1}) = o(1/n)$. To complete the proof we show that if $s_2 = |I_{n-1}|/2$ then the ratio $\frac{s_2-s_1}{s_1}$ is also $o(1/n)$.

$$\frac{s_2-s_1}{s_1} = \frac{\frac{|I_{n-1}|}{2} - (d_{n,1}(I_{n-1}) - x)}{d_{n,1}(I_{n-1}) - x} \leqq \frac{\frac{|I_{n-1}|}{2} - (d_{n,1}(I_{n-1}) - a_{n,1}(I_{n-1}))}{d_{n,1}(I_{n-1}) - c_{n,1}(I_{n-1})} \leqq$$

$$\leqq \frac{(a_{n,0}(I_{n-1}) - b_{n,1}(I_{n-1})) + (b_{n,1}(I_{n-1}) - a_{n,1}(I_{n-1}))}{d_{n,1}(I_{n-1}) - c_{n,1}(I_{n-1})} +$$

$$+\frac{a_{n,1}|I_{n-1}|-(d_{n,1}(I_{n-1})-a_{n,1}(I_{n-1}))}{d_{n,1}(I_{n-1})-c_{n,1}(I_{n-1})}=$$

$$=\frac{(a_{n,0}(I_{n-1})-b_{n,1}(I_{n-1}))+(b_{n,1}(I_{n-1})-d_{n,1}(I_{n-1}))+a_{n,1}|I_{n-1}|}{d_{n,1}(I_{n-1})-c_{n,1}(I_{n-1})}\leqq$$

$$\leqq\frac{a_{n,1}|I_{n-1}|+\frac{n+2}{2^{n+2}}a_{n,1}|I_{n-1}|+a_{n,1}|I_{n-1}|}{(1-\frac{4}{2^{n+1}})a_{n,0}\frac{|I_{n-1}|}{2}}=\frac{\frac{1}{2^n}\frac{n+2}{2^{n+2}}+\frac{1}{2^{n-1}}}{1-\frac{4}{2^{n+1}}}$$

This then completes the proof of Part 3.

If $h\in[\frac{|I_n|}{2},\frac{|I_{n-1}|}{2}]$, then $\Delta(E_{x,h})$ is bounded by the maximum of the densities $\Delta(E_{x,t})$ where t is one of the critical values from Part 1, 2 or 3, i.e.,

$$t\in\{h_0,h_1,h_2,h_3,h_4,h_5,t_1,t_2,s_1,s_2\}.$$

Hence, there is an N such that for $n\geqq N$, $\Delta(E_{x,h})<\frac{1}{n}$ uniformly for all $h\in[|I_n|/2,|I_{n-1}|/2]$. That is, for every $x\in\mathrm{cl}(M)\setminus M$, $E_x=\{h:G(x+h)\neq\neq G(x-h)\}$ has density 0 at x. This then completes the proof that G has all the desired attributes.

However, even though an approximately continuous, approximately m-smooth function (m odd) can fail to be a Baire*1 function, it will have to be continuous on a dense open set. This is our next Theorem 2. First we need the following lemma, which is based on a recent result of Freiling and Rinne [10].

LEMMA 1. *Let f be an approximately continuous function which is approximately symmetrically differentiable on an interval I.*

a) *If $D^1_{\mathrm{ap}}f$ is bounded either above or below on I, then f is continuous and symmetrically differentiable on I.*

b) *If $D^1_{\mathrm{ap}}f$ is continuous at $x_0\in I$, then f is differentiable at x_0.*

PROOF (of a)). Assume that $D^1_{\mathrm{ap}}f$ is bounded below by M on I. Then the function g, given by $g(x)=f(x)-Mx$, is approximately continuous and approximately symmetrically differentiable on I with $D^1_{\mathrm{ap}}g(x)==D^1_{\mathrm{ap}}f(x)-M\geqq 0$ for all $x\in I$. Hence, according to [10], g is nondecreasing on I. As noted in [7], it is easy to see that if a monotone function is approximately symmetrically differentiable, then it must be symmetrically differentiable. Thus, g is symmetrically differentiable on I. Since a symmetrically differentiable function cannot have a jump discontinuity, g must be continuous on I. Clearly, then, f is also both continuous and symmetrically differentiable on I. The analogous proof holds if $D^1_{\mathrm{ap}}f$ is assumed to be bounded above.

PROOF (of b)). Since $D^1_{\text{ap}}f$ is continuous at x_0, it is bounded on a neighborhood J of x_0 and hence equals $D^1 f$ on J by part a) of this lemma. Applying the quasi-mean-value theorem of [1], we conclude that f is differentiable at x_0.

Neither a) nor b) remains true if the assumption of approximate continuity in the previous lemma is weakened to Baire one, Darboux. For example, Croft [3] has constructed a Baire class one, Darboux function which is almost everywhere zero, but nonzero on a dense set.

THEOREM 2. *If f is approximately continuous and approximately m-smooth, then f is continuous on a dense open set.*

PROOF. If m is even then Theorem 1 guarantees that f is a Baire*1 function and is hence continuous on a dense open set. Suppose that m is odd. Then f is approximately 3-smooth. Let I be any open interval. According to [11], $D^1_{\text{ap}}f$ is a function of Baire class one and is, therefore, bounded on some subinterval J of I. Consequently, from part a) of Lemma 2, it follows that f is continuous on J.

We now turn to differentiability properties of approximately m-smooth functions. The following lemma is useful.

LEMMA 2. *Let f be approximately continuous on an open interval I.*

a) *If $D^{2k-1}_{\text{ap}}f$, $k = 1, 2, \ldots, m$, exist and are continuous on I, then the ordinary derivative $f^{(2m-1)}$ exists and is continuous on I.*

b) *If $D^{2k}_{\text{ap}}f$, $k = 1, 2, \ldots, m$, exist and are continuous on I, then the ordinary derivative $f^{(2m)}$ exists and is continuous on I.*

PROOF (of a)). If $m = 1$, the result is immediate from Lemma 1, part b). Suppose that the result holds for $m = n$ and let us show that it must also hold for $m = n + 1$. The proof we shall give for this inductive step is virtually identical to that given by Dutta in [4] for his Lemma 2 and we include it here solely for the sake of completeness. We are assuming that $D^{2k-1}_{\text{ap}}f$, $k = 1, 2, \ldots, n+1$, exist and are continuous on I, and that, via the inductive hypothesis, $f^{(2n-1)}$ exists and is continuous on I. Let $[a, b]$ be contained in I. For each $x \in I$ and each h such that $x \pm h \in I$, there is, by the mean value theorem, a $\theta, 0 < \theta < 1$, such that

$$(2n+1)! \left[(f(x+h) - f(x-h))/2 - \sum_{k=1}^{n} \frac{h^{2k-1}}{(2k-1)!} f^{(2k-1)}(x) \right] \Big/ h^{2n+1} = \\ = \frac{\Delta_2(f^{(2n-1)}; x, \theta h)}{(\theta h)^2},$$

where $\Delta_2(\phi; x, t)$ is the usual second symmetric difference for a function ϕ:

$$\Delta_2(\phi; x, t) = \phi(x+t) + \phi(x-t) - 2\phi(x).$$

Hence, letting $\underline{D}^2\phi(x) = \liminf_{h\to 0} \Delta_2(\phi; x, h)$ and $\overline{D}^2\phi(x) = \limsup_{h\to 0} \Delta_2(\phi; x, h)$, we have

$$\underline{D}^2 f^{(2n-1)}(x) \leqq D_{\mathrm{ap}}^{2n+1} f(x) \leqq \overline{D}^2 f^{(2n-1)}(x)$$

for all $x \in I$. Now, since $D_{\mathrm{ap}}^{2n+1} f$ is continuous on $[a, b]$ and satisfies the above inequality, Lemma (3.13) on p. 327 of [17, Vol. I] indicates that the function

$$f^{(2n-1)}(x) - \int_a^x \int_a^t D_{\mathrm{ap}}^{2n+1} f(u)du$$

is linear in $[a, b]$ and, hence, $f^{(2n+1)} = D_{\mathrm{ap}}^{2n+1} f$ on (a, b). Since $[a, b]$ is arbitrary in I, we conclude that $f^{(2n+1)}(x) = D_{\mathrm{ap}}^{2n+1} f(x)$ for all $x \in I$, completing the proof of a).

PROOF (of b)). First consider the case $m = 1$. Here we are assuming that f is approximately continuous on I and that $D_{\mathrm{ap}}^2 f$ is continuous on I, and wish to conclude that $f'' = D_{\mathrm{ap}}^2 f$ on I. Let $[a, b] \subset I$. From Theorem 3 in [13] we may first conclude that f is actually continuous on $[a, b]$. Next, since f and $D_{\mathrm{ap}}^2 f$ are both continuous on $[a, b]$ and

$$\underline{D}^2 f(x) \leqq D_{\mathrm{ap}}^2 f(x) \leqq \overline{D}^2 f(x)$$

for all $x \in [a, b]$ we conclude, again via Lemma (3.13) on p. 327 [17, Vol. I] that the function

$$f(x) - \int_a^x \int_a^t D_{\mathrm{ap}}^2 f(u)du$$

is linear in $[a, b]$ and, hence, $f'' = D_{\mathrm{ap}}^2 f$ on (a, b). Since $[a, b]$ is arbitrary in I, we conclude that $f'' = D_{\mathrm{ap}}^2 f$ on I, completing the proof for $m = 1$. The inductive step may now be justified in a manner entirely analogous to that utilized in the odd case above.

From Theorem 1 and Lemma 2, we immediately obtain the following:

THEOREM 3. *If f is approximately continuous and approximately m-smooth, then $f^{(m-2)}$ exists and is continuous on an open dense set.*

Here again, the example in [3] shows that the assumption of approximate continuity in Theorem 1 cannot be replaced by Baire one, Darboux.

LEMMA 3. *Let f be approximately m-smooth and suppose that $f_{(m-2)}$ exists on an open interval I. If $f_{(m-2)}$ attains a local maximum or minimum at x_0 in I, then $f_{(m-1)}(x_0)$ exists and equals* 0.

PROOF. Note that the $m = 2$ case reduces to the assumption that f is approximately smooth and approximately continuous. That the approximate derivative $f_{(1)}$ or f'_{ap} exists at a point where a local extremum is attained was proved in [16]. Consequently, we need only consider the situation where $m > 2$.

Suppose that $f_{(m-2)}$ has a local minimum at x_0. Since f is approximately m-smooth at x_0, there is a set E_{x_0} of density one at zero such that for $h \in \in E_{x_0}$, we have

$$(3) \qquad [f(x_0+h)+(-1)^m f(x_0-h)]/2 - P_{x_0,m-2}(h) = o(h^{m-1}).$$

Since $f_{(m-2)}$ has a local minimum at x_0, it is bounded below on a neighborhood J of x_0 and, hence, from [2] we conclude that $f^{(m-2)}$ exists on J. Thus we may rewrite the left-hand side of (1) as

(4)

$$\frac{1}{2}\left[f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(x_0)\right]+\frac{(-1)^m}{2}\left[f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f^{(k)}(x_0)\right].$$

Next, we shall show that for $h > 0$ and sufficiently small, both of the expressions $f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(x_0)$ and $f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f^{(k)}(x_0)$ are nonnegative. Once this is established, it will follow from (3) and (4) that for sufficiently small $h \in E_{x_0}$

$$f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(x_0) = o(h^{m-1}),$$

implying that $f_{(k-1)}(x_0)$ exists and equals 0.

To verify the claim mentioned above, let $h > 0$ be small enough so that both $x_0 - h$ and $x_0 + h$ are in J. From Taylor's Theorem we know that there is a $0 < \delta_1 < 1$ and a $0 < \delta_2 < 1$ such that

$$(5)\ f(x_0+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(x_0) = \frac{h^{m-2}}{(m-2)!}[f^{(m-2)}(x_0+\delta_1 h)-f^{(m-2)}(x_0)]$$

and

$$f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f^{(k)}(x_0) = \frac{(-h)^{m-2}}{(m-2)!}[f^{(m-2)}(x_0-\delta_2 h)-f^{(m-2)}(x_0)].$$

We may rewrite the latter equality as

$$(6) \qquad (-1)^m\left[f(x_0-h)-\sum_{k=0}^{m-2}\frac{(-h)^k}{k!}f^{(k)}(x_0)\right] =$$

$$= \frac{h^{m-2}}{(m-2)!}[f^{(m-2)}(x_0-\delta_2 h)-f^{(m-2)}(x_0)].$$

Since $f^{(m-2)}$ has a local minimum at x_0 we know that for $h > 0$ and sufficiently small, the right hand sides of both (5) and (6) are nonnegative, completing the proof for the case when f has a local minimum at x_0.

The remaining case may be disposed of by considering $-f$.

THEOREM 4. *Let f be approximately continuous and approximately m-smooth. Then $f_{(m-1)}(x)$ exists and is finite at each point of a set having the power of the continuum in every interval.*

PROOF. From Theorem 3 we know that $f^{(m-2)}$ exists and is continuous on a dense open set. Let I be an interval on which $f^{(m-2)}$ is continuous. We may now proceed exactly as in the proof of Theorem 2 in [4] to show that $f_{(m-1)}(x)$ will exist for all points x in a set having the power of the continuum in I. The only modification to be made in that proof is to use Lemma 3 of this paper in place of Lemma 4 in [4].

THEOREM 5. *Let f be an approximately m-smooth function for which $f_{(m-2)}$ exists on an open interval I. Then $f_{(m-1)}$ has the Darboux property on $E = \{x \in I : f_{(m-1)}(x)$ exists$\}$.*

PROOF. First note that for the $m = 2$ case the assumption that $f_{(m-2)}$ exist on I merely means that f is approximately continuous on I. With this observation we see that the conclusion follows from Theorem 2 in [13]. Thus we need only consider $m > 2$.

For $m > 2$ the assumption that $f_{(m-2)}$ exists on I guarantees that f is approximately continuous on I. Hence, Theorem 4 asserts that E is of the power of the continuum in I. Let a and b be two points in E where $a < b$ and $f_{(m-1)}(a) \neq f_{(m-1)}(b)$. For each c between $f_{(m-1)}(a)$ and $f_{(m-1)}(b)$ we must produce an $x_0 \in E \cap (a, b)$ such that $f_{(m-1)}(x_0) = c$. Clearly, it suffices to verify this for the situation where $c = 0$ and $f_{(m-1)}(a) < 0 < f_{(m-1)}(b)$.

Being an approximate Peano derivative, $f_{(m-2)}$ has the Darboux property on I [2]. Hence, it is a Baire*1, Darboux function on I. According to Theorem 1 in [14], $f_{(m-2)}$ is either monotone (and hence continuous) on $[a, b]$, or there is a subinterval $[\alpha, \beta]$ of (a, b) on which $f_{(m-2)}$ is continuous but not monotone. If the latter situation holds, then $f_{(m-2)}$ must obviously have a local extremum at some point $x_0 \in (\alpha, \beta)$, and Lemma 3 guarantees that $x_0 \in E$ and $f_{(m-1)}(x_0) = 0$, exactly the situation we are seeking. We shall complete the proof by showing that it is impossible for $f_{(m-2)}$ to be monotone on all of $[\alpha, \beta]$.

So suppose that $f_{(m-2)}$ is nondecreasing on $[\alpha, \beta]$. Then $f_{(m-2)} = f^{(m-2)}$ on $[\alpha, \beta]$ [2]. Consequently, if $\alpha + h \in (\alpha, \beta)$, then Taylor's Theorem assures us of a $0 < \delta < 1$ such that

$$(7)\quad f(\alpha+h) - \sum_{k=0}^{m-2} \frac{h^k}{k!} f^{(k)}(\alpha) = \frac{h^{m-2}}{(m-2)!}[f^{(m-2)}(\alpha+\delta h) - f^{(m-2)}(\alpha)] \geqq 0.$$

Also, we know that there is a set H having density one at 0 such that for $h \in H$ we have

$$(8) \qquad f(\alpha+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(\alpha)-\frac{h^{m-1}}{(m-1)!}f_{(m-1)}(\alpha)=o(h^{m-1}).$$

From (7), (8), and the fact that $f_{(m-1)}(\alpha)<0$, we conclude that for positive h's in H we have

$$f(\alpha+h)-\sum_{k=0}^{m-2}\frac{h^k}{k!}f^{(k)}(\alpha)=o(h^{m-1}),$$

which is a contradiction because $f_{(m-1)}(\alpha)\neq 0$.

Similarly, a contradiction is obtained if $f_{(m-2)}$ is assumed to be nonincreasing on $[\alpha,\beta]$, completing the proof.

Utilizing a very similar line of proof we also obtain the following monotonicity result.

THEOREM 6. *Let f be an approximately m-smooth function for which $f_{(m-2)}$ exists on an open interval I. If $f_{(m-1)}(x)\geqq 0$ for all $x\in E=\{x\in$ $\in I: f_{(m-1)}(x)$ exists$\}$, then $f_{(m-2)}$ is nondecreasing on I.*

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