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FURTHER RESULTS ON RETICULATED RINGS

H. AL-EZEH (Amman)

Throughout this paper all rings are commutative with unity and all lattices are distributive with 0 and 1 unless otherwise stated. For a ring R, the reticulation of R is defined as a distributive lattice generated by the symbols $D(a)$, $a \in R$ and satisfying:

$$D(1_R) = 1,\ \ D(0_R) = 0,\ \ D(a \cdot b) = D(a) \wedge D(b),\ \ D(a + b) \leqq D(a) \vee D(b).$$

For more details about the reticulation LR of a ring, see Simmons [9]. For any ideal I of the ring R, let $D(I)$ be the ideal generated by $\{D(a)\colon a \in I\}$ in LR. For any ideal J of the lattice LR, let $D^{-1}(J) = \{a \in R\colon D(a) \in J\}$. Trivially, $D^{-1}(J)$ is an ideal of R.

The reticulation of a ring was investigated by Simmons [9] in order to show that a lot of ring theoretic properties have analogues in lattice theory and vice versa. In this paper we continue this theme and answer a couple of questions raised by Simmons [9]. Then we proceed to prove further results in that direction. Let $\mathrm{Id}(R)$ be the lattice of all ideals of the ring R. It should be noted that this lattice is not necessarily distributive. Let $R\,\mathrm{Id}(R)$ be the distributive lattice of radical ideals of the ring R. For a lattice L, let $\mathrm{Id}(L)$ be the lattice of ideals of L. From now on, LR will denote the reticulation of R.

We start by quoting a theorem that was given by Johnstone [7], p. 194.

THEOREM 1. *Let R be a ring. Then $D^{-1}\colon \mathrm{Id}(LR) \to R\,\mathrm{Id}(R)$ is a lattice isomorphism, moreover for any ideal I of the ring R, $D^{-1}(D(I)) = \sqrt{I}$, the radical of I. Also, D^{-1} defines a bijection from the prime ideals of LR to the prime ideals of R. Hence minimal prime ideals of LR correspond to minimal prime ideals of R.*

Recall that a ring R is called quasi regular if for every $a \in R$, there exists $b \in R$ such that $\mathrm{ann}(\mathrm{ann}(a)) = \mathrm{ann}(b)$, where for any ideal I of R, $\mathrm{ann}(I) = \{x \in R\colon xy = 0 \text{ for all } y \in I\}$. A lattice L is called quasi complemented if for every $a \in L$, there exists $b \in L$ such that $a^{**} = b^*$, where for any ideal I of a lattice L, $I^* = \{x \in L\colon x \wedge y = 0 \text{ for all } y \in I\}$. The first result we want to establish is the following: A semiprime ring (without nontrivial nilpotents) is quasi regular if and only if LR is quasi complemented and for every $a \in LR$, there exists $r \in R$ such that $a^* = D^*(r)$. This result will be proved after the following preliminary lemma.

LEMMA 2. *Let R be a semiprime ring. Then for any ideal I of R, $D(\operatorname{ann}(I)) = D^*(I)$.*

PROOF. The proof is easy if we keep in mind that every $x \in D(I)$ has the form $\bigvee_{i=1}^{n} D(a_i)$ for some $a_1, a_2, \dots, a_n \in I$.

THEOREM 3. *Let R be a semiprime ring. Then R is quasi regular if and only if the lattice LR is quasi complemented and for each $x \in LR$, there exists $a \in R$ such that $x^* = D^*(a)$.*

PROOF. Assume R is quasi regular. Let $x \in LR$, then $x = \bigvee_{i=1}^{n} D(a_i)$ for some $a_1, a_2, \dots, a_n \in R$. Hence

$$x^* = \Big(\bigvee_{i=1}^{n} D(a_i)\Big)^* = \bigwedge_{i=1}^{n} D^*(a_i) = D\Big(\bigcap_{i=1}^{n} \operatorname{ann}(a_i)\Big).$$

Since R is quasi regular, for any $a, b \in R$, there exists $c \in R$ such that $\operatorname{ann}(a) \cap \operatorname{ann}(b) = \operatorname{ann}(c)$, see Henrikson and Jerison [6]. Thus $x^* = D(\operatorname{ann}(r))$ for some $r \in R$. By the previous lemma, $x^* = D^*(r)$, $r \in R$. Now, consider

$$x^{**} = D^{**}(r) = D^*(\operatorname{ann}(r)) = D(\operatorname{ann}(\operatorname{ann}(r))) = D(\operatorname{ann}(s))$$

for some $s \in R$, since R is quasi regular. Consequently, LR is quasi complemented, see Speed [10].

Conversely, let $r \in R$. Consider

$$\begin{aligned} D(\operatorname{ann}(\operatorname{ann}(r)) &= D^*(\operatorname{ann}(r)) \quad \text{by Lemma 2} \\ &= D^{**}(r) = y^* \quad \text{for some } y \in LR, \end{aligned}$$

since LR is quasi complemented. By assumption, $y^* = D^*(s)$ for some $s \in R$. Therefore, $D(\operatorname{ann}(\operatorname{ann}(r))) = D(\operatorname{ann}(s))$. Since R is semiprime, $\operatorname{ann}(\operatorname{ann}(r))$ and $\operatorname{ann}(s)$ are radical ideals. Hence applying Theorem 1, $\operatorname{ann}(s) = \operatorname{ann}(\operatorname{ann}(r))$.

THEOREM 4. *Let R be a quasi regular ring. Then the space of minimal prime ideals of LR with the hull-kernel topology is compact.*

PROOF. By the previous theorem LR is a quasi-complemented lattice. Speed [10] proved that a lattice is quasi complemented if and only if the space of minimal prime ideals is compact. Thus the space of minimal prime ideals of LR is compact.

For any prime ideal P of a semiprime ring R, $O(P) = \{x \in L : \exists y \notin P \ni xy = 0\}$ is an ideal of R contained in P and is contained in any prime ideal contained in P. Also, for any prime ideal P of a lattice L, $O(P) = \{x \in L : \exists y \notin P \ni x \wedge y = 0\}$.

Recall that a ring R is called a PF-ring if every principal ideal aR is a flat R-module. An ideal I of a ring R is called pure if for each $x \in I$, there exists $y \in I$ such that $xy = x$. By now, it is well known that R is a PF-ring if and only if for each $a \in R$, $\mathrm{ann}(a)$ is a pure ideal, see Al-Ezeh [1]. A ring R is called a PP-ring if for each $a \in R$, aR is a projective R-module. In fact, a ring R is a PP-ring if and only if for each $a \in R$, $\mathrm{ann}(a)$ is generated by an idempotent. In the following theorem we give a different characterization of PF-rings.

THEOREM 5. *A semiprime ring R is a PF-ring if and only if each minimal prime ideal is pure.*

PROOF. Assume R is a PF-ring. Let P be a prime ideal, and let $x \in O(P)$, then there exists $y \notin P$ such that $xy = 0$, i.e. $x \in \mathrm{ann}(y)$. Since R is a PF-ring, $\mathrm{ann}(y)$ is pure. So there exists $a \in \mathrm{ann}(y)$ such that $xa = x$. Since $ay = 0$ and $y \notin P$, $a \in O(P)$. Therefore $O(P)$ is pure. For any minimal prime ideal of a semiprime ring, $P = O(P)$, see Henriksen and Jerison [6]. Thus each minimal prime ideal is pure.

Conversely, assume that each minimal prime ideal of R is pure. Let P be a prime ideal in R. Then P contains a minimal prime ideal of R, say Q. Since $Q \subseteqq P$, $O(P) \subseteqq Q$. Now, let $x \in O(Q)$. Since $Q = O(Q)$ is pure, there exists $y \in O(Q)$ such that $xy = x$. Hence $x(1-y) = 0$ and $1 - y \notin P$. So $x \in O(P)$. Therefore $O(Q) \subseteqq O(P)$, and hence $O(P) = O(Q) = Q$. So each prime ideal P contains a unique minimal prime ideal, namely, $O(P)$. Thus R is a PF-ring, see Matlis [8].

Recall that an ideal I of a lattice L is called a σ-ideal if for each $x \in I$, $I \vee x^* = L$. (For more details about σ-ideals see Cornish [3].) A lattice L is called conormal if for any $x, y \in L$, if $x \wedge y = 0$, then there exist $a, b \in L$ satisfying $x \wedge a = y \wedge b = 0$ and $a \vee b = 1$. In fact, Cornish calls these lattices normal ones. Cornish [3] proved the following theorem.

THEOREM 6. *A lattice L is conormal if and only if each minimal prime ideal of L is a σ-ideal.*

Georgescu and Voiculescu [5] showed that for any ring R, $D'\colon \Phi R \to \Phi LR$ defined by $D'(I) = D(\sqrt{I})$ is a lattice isomorphism, where ΦR is the lattice of all pure ideals of R and ΦLR is the lattice of all σ-ideals of LR. If R is a semiprime ring, then for each pure ideal of R, $I = \sqrt{I}$. Hence $D\colon \Phi R \to \Phi LR$ is a lattice isomorphism. So, by Theorems 1, 5, and 6, we get the following theorem that answers a question raised by Simmons [9].

THEOREM 7. *A semiprime ring R is a PF-ring if and only if LR is a conormal lattice.*

As an application of the above results we prove a result that was proved by Simmons [9]. Speed [10] proved that a lattice L is quasi complemented if and only if the space of minimal prime ideals with the hull-kernel topology is compact. Cornish [3] proved that a lattice L is stonian if and only if it is conormal and the space of minimal prime ideals with hull-kernel topology is compact. Endo [4] proved that a ring R is a PP-ring if and only if it is a quasi regular PF-ring. Consequently, we get the following result which was proved by Simmons [9].

THEOREM 8. *A semiprime ring R is a PP-ring if and only if LR is a stonian lattice.*

A ring R is called an almost PP-ring if for each $a \in R$, $\operatorname{ann}(a)$ is generated by idempotents. One can easily show that a ring R is an almost PP-ring if and only if for each $a \in R$, and $x \in \operatorname{ann}(a)$, there exists an idempotent e in $\operatorname{ann}(a)$ such that $xe = x$. Some authors call these rings complementedly normal rings. An element a in a lattice L is called complemented if there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The element b is called the complement of a and will be denoted by a'. For a lattice L, let BL be the set of all complemented elements of L. In fact, BL forms a boolean lattice. An ideal I of a lattice L is called a strongly σ-ideal if for every $a \in I$, there exists $b \in I \cap BL$ such that $a \leqq b$. A lattice L is called almost stonian if for each $a \in L$, a^* is a strongly σ-ideal. Cornish [2] called almost stonian lattices complementedly normal lattices.

Finally, an ideal I of a ring R is called a strongly pure ideal if for each $a \in I$, there exists an idempotent e in I such that $ae = a$. Our aim is to show that a semiprime ring R is an almost PP-ring if and only if LR is an almost stonian lattice.

THEOREM 9. *A semiprime ring R is an almost PP-ring if and only if each minimal prime ideal is a strongly pure one.*

PROOF. Let P be a minimal prime ideal of R. Then $O(P) = P$. Let $x \in O(P)$, then there exists $y \notin P$ such that $xy = 0$. Hence $x \in \operatorname{ann}(y)$. Because R is an almost PP-ring, there exists an idempotent e in $\operatorname{ann}(y)$ such that $xe = x$. Since $y \notin P$ and P is prime, $e \in P$. Therefore, $P = O(P)$ is a strongly pure ideal.

Conversely, assume that each minimal prime ideal of R is a strongly pure one. Therefore, the minimal prime ideals are pure. By Theorem 5, R is a PF-ring. Now, let $y \in R$, and $x \in \operatorname{ann}(y)$. Since R is a PF-ring, $O(P)$, for any prime ideal P of R, is prime (see Matlis [8]). Thus, for any prime ideal P of R, either $x \in O(P)$ or $y \in O(P)$. So we have the following two cases:

(i) if $x \in O(P)$, then by assumption there exists an idempotent $e \in$ $\in O(P) \subseteqq P$ such that $xe = x$. If $f = 1 - e$, then $xf = 0$ and $f \notin P$;

(ii) if $x \notin O(P)$, then $y \in O(P)$. By assumption, there exists an idempotent $q \in O(P)$ such that $yg = y$. If $h = 1 - g$, then $yh = 0$ and $h \in P$. So for each prime ideal P, either there exists an idempotent $f \notin P$ such that $xf = 0$ or there exists an idempotent $h \notin P$ such that $yh = 0$. Let A be the ideal in R generated by all idempotents $f \notin P$ with $xf = 0$ if case (i) holds and those idempotents $h \notin P$ with $yh = 0$ otherwise. Clearly $A = R$ because for each maximal ideal M the ideal A contains an element not in M. Hence

$$1 = r_1 f_1 + \ldots + r_m f_m + s_1 h_1 + \ldots + s_n b_n.$$

Therefore $x = x(s_1 h_1 + \ldots + s_n h_n)$ and $s_1 h_1 + \ldots + s_n h_n \in \operatorname{ann}(y)$. Consequently, there exists an idempotent h in $\operatorname{ann}(y)$ such that $xh = x$. Thus $\operatorname{ann}(y)$ is a strongly pure ideal. Therefore R is an almost stonian ring.

Now, we prove an analogous result for lattices.

THEOREM 10. *A lattice L is almost stonian if and only if each minimal prime ideal of L is a strongly σ-ideal.*

PROOF. Let P be a minimal prime ideal of L. Then $O(P) = P$. Let $x \in O(P)$, then there exists $y \notin P$ such that $x \wedge y = 0$, i.e. $x \in y^*$. Hence there exists $b \in y^* \cap BL$ such that $x \leqq b$. Since $y \notin P$ and $b \wedge y = 0$, $b \in O(P)$. Therefore $P = O(P)$ is a strongly σ-ideal of L.

Conversely, assume that each minimal prime ideal is a strongly σ-ideal. Clearly, every strongly σ-ideal is a σ-ideal. Thus each minimal prime ideal is a σ-ideal. By Theorem 6, L is conormal. Therefore each $O(P)$ is a prime ideal, see Cornish [3]. Now, let $a \in L$ and $x \in a^*$. Then $x \wedge a = 0$. So either $x \in O(P)$ or $a \in O(P)$. So for each prime ideal P of L, we have two cases:

(i) if $x \in O(P)$, then there exists $b_P \in O(P) \cap BL$ such that $x \leqq b_P$. Therefore $x \wedge b'_P = 0$ and $b'_P \notin P$;

(ii) if $x \notin O(P)$, then $a \in O(P)$. So, there exists $b_P \in O(P) \cap BL$ such that $a \leqq b$. Therefore $a \wedge b'_P = 0$ and $b'_P \notin P$.

Let J be the ideal generated by all those b_P's constructed in (i) if it holds and by those b_P's in (ii) otherwise. Clearly, $L = J$ because for each maximal ideal M of L, the ideal J contains an element not in M. Therefore

$$1 = b'_{P_1} \vee \ldots \vee b'_{P_m} \vee b'_{Q_1} \vee \ldots \vee b'_{Q_n}$$

with $b'_{P_i} \wedge x = 0$ and $b'_{Q_i} \cap a = 0$. Thus $x = x \wedge (b'_{Q_1} \vee \ldots \vee b'_{Q_n})$ i.e. $x \leqq b'_{Q_1} \vee \ldots \vee b'_{Q_n}$. Since $b'_{Q_1} \vee \ldots \vee b'_{Q_n} \in a^*$, a^* is a strongly σ-ideal. Therefore L is an almost stonian lattice.

Now we quote a lemma that was proved by Simmons [9].

LEMMA 11. *The mapping $D\colon R \to LR$ restricts to a lattice isomorphism from BR to BLR, where BR is the boolean lattice of idempotents of R.*

THEOREM 12. *For a ring R, D takes strongly pure ideals of R to strongly σ-ideals of LR, and D^{-1} takes strongly σ-ideals of LR to strongly pure ideals of R.*

PROOF. Let I be a strongly pure ideal of R. Let $x \in D(I)$, then $x = \bigvee_{i=1}^{n} D(a_i)$ for some $a_1, a_2, \ldots, a_n \in I$. For each a_i, there exists an idempotent $e_i \in I$ such that $a_i e_i = a_i$. Hence $D(a_i) \leqq D(e_i)$. Thus $x \leqq \bigvee_{i=1}^{n} D(e_i)$.

Since $\bigvee_{i=1}^{n} D(e_i)$ is a complemented element in $D(I)$, $D(I)$ is a strongly σ-ideal of LR.

Now let J be a strongly σ-ideal of LR. Let $a \in D^{-1}(J)$, then $D(a) \in \in J$. Since J is a strongly σ-ideal of LR, there exists $b \in J \cap BLR$ such that $D(a) \leqq b$. By Lemma 11, there exists an idempotent e in R such that $b = D(e)$. Hence $D(a) \leqq D(e)$. Therefore $D(a) \wedge D(1-e) = 0$. Consequently, $D(a(1-e)) = 0$. Thus $a(1-e) = 0$. Moreover $e \in D^{-1}(J)$ since $D(e) = b \in J$. Therefore $D^{-1}(J)$ is a strongly pure ideal of R.

Using Theorems 1, 10, and 11, we get the following theorem.

THEOREM 13. *A semiprime ring R is an almost PP-ring if and only if LR is an almost stonian lattice.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JORDAN
AMMAN
JORDAN

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ON A MESH-INDEPENDENCE PRINCIPLE FOR OPERATOR EQUATIONS AND THE SECANT METHOD

I. K. ARGYROS (Lawton)

1. Introduction

Consider the nonlinear equation

$$F(x)=0 \tag{1}$$

where F is a nonlinear operator between two Banach spaces E, $\hat{E}$. The secant iteration

$$x_{n+1}=x_n-\delta F(x_n,x_{n-1})^{-1}F(x_n),\quad n=0,1,2,\ldots \tag{2}$$

has been used to approximate a solution x^* of equation (1), for some x_{-1}, $x_0\in E$. The linear operator $\delta F(x_n,x_{n-1})$ is assumed to be a consistent approximation of the first Fréchet-derivative $F'(x_n)$, [2], [8], [9], [12].

The iterates $\{x_n\}$, $n=1,2,\ldots$ can rarely be executed in infinite dimensional spaces. In practice we replace equation (1) by a family of discretized equations

$$P_h(x)=0,\quad h>0 \tag{3}$$

where P_h is a nonlinear operator between finite dimensional spaces E_h, $\hat{E}_h$. The discretization on E is defined by the bounded linear operators $L_h\colon E\to$ $\to E_h$.

Let us define the discretized secant iteration for equation (3) by

$$z_0^h=L_h(x_0),\quad z_{-1}^h=L_h(x_{-1}) \tag{4}$$

and

$$z_{n+1}^h=z_n^h-\delta P_h(z_n^h,z_{n-1}^h)^{-1}P_h(z_n^h),\quad n=0,1,2,\ldots. \tag{5}$$

We will show that under certain assumptions the sequence $\{z_n^h\}$ converges to a locally unique solution z_h^* of equation (3) and the following estimates are true:

$$z_h^*=L_h(x^*)+O(h^p),\ \ p>0,\quad z_n^h-z_h^*=L_h(x_n-x^*)+O(h^p),$$

$$z_{n+1}^h - z_n^h + L_n(x_{n+1} - x_n) + O(h^p), \quad P_h(z_n^h) = \hat{L}_n(F(x_n)) + O(h^p)$$

and for any $\varepsilon > 0$ and sufficiently small h

$$|\min\{n \geqq 0, \|x_n - x^*\| < \varepsilon\} - \min\{n \geqq 0, \|z_n^h - z_h^*\| < \varepsilon\} \leqq 1$$

if x_0 belongs to a certain ball centered at x^* and of a specific finite radius.

The above results have been proved when Newton's iteration is used to approximate a solution x^* of equation (1), [2], [3], [4], [6], [9]. Since the iterates for the secant method can be computed easier than the Newton's iterates, both in infinite and finite dimensional spaces, we feel that the results obtained here are useful.

The last recorded result above indicates that there is at most a difference of one between the number of steps required by the iterations (2) and (4) to converge to within a given tolerance $\varepsilon > 0$. This aspect together with the rest of the above results constitute the mesh-independence principle for the secant method.

If we let $\delta F(x_n, x_{n-1}) = F'(x_n)$, $n = 0, 1, 2, \ldots$, our results reduce to the ones in [2], [9]. In this paper we use similar definitions and the proof procedure of [2], [9].

II. Main results

The norms in all spaces E, $\hat{E}$, E_h and $\hat{E}_h$ will be denoted by the same symbol $\|\ \|$ and for any bounded linear operator from E to $\hat{E}$ (or from E_h to $\hat{E}_h$), the induced norm will be used.

We will assume familiarity with the definition of a divided difference operator $\delta F(v, w)$, [10, p. 78].

We can now prove the following theorem (see also [11]).

THEOREM 1. *Let F be a nonlinear operator defined on an open set D of a Banach space E with values in a Banach space $\hat{E}$. Assume:*

(a) *The equation $F(x) = 0$ has a solution $x^* \in D$ at which the Fréchet-derivative $F'(x^*)$ exists and is boundedly invertible with*

$$\|F'(x^*)^{-1}\| \leqq d_1. \tag{6}$$

(b) *The nonlinear operator F has divided differences $\delta F(u, v)$ satisfying the following conditions:*

$$\|\delta F(v, w) - \delta F(u, z)\| \leqq d_0(\|v - u\| + \|w - z\|) \text{ for all } u, v, w, z \in D \tag{7}$$

and

(c) *The open ball $U^* = U(x^*, r^*) = \{x \in E \mid \|x - x^*\| < R^*\} \subset D$ with*

$$r^* = \frac{1}{3d_0 d_1}. \tag{8}$$

Then for any $x_{-1}, x_0 \in U^*$, the secant iteration (2) *remains in U^* and converges to x^* with*

(9)
$$\|x_{n+1}-x^*\| \leqq \frac{d_0 d_1 \|x_{n-1}-x^*\|}{1-d_0 d_1(\|x_n-x^*\|+\|x_{n-1}-x^*\|)}\|x_n-x^*\|, \quad n=0,1,2,\dots .$$

PROOF. Let us denote by L the linear operator given by

$$L = \delta F(v,w) \quad \text{for } v,w \in U^*.$$

It can easily be seen that (7) gives $F'(x^*) = \delta F(x^*,x^*)$. By the choice of r^*, (7) and the identity

$$L = F'(x^*)[I + F'(x^*)^{-1}(L - F'(x^*))]$$

we obtain

$$\left\|F'(x^*)^{-1}(\delta F(v,w)-\delta F(x^*,x^*))\right\| \leqq d_1 d_0(\|v-x^*\|+\|w-x^*\|) \leqq 2d_0 d_1 r^* < 1.$$

By the Banach lemma on invertible operators it follows that L is invertible and

$$\|L^{-1}\| \leqq \frac{d_1}{1-d_0 d_1(\|v-x^*\|+\|w-x^*\|)}. \tag{10}$$

Let us now suppose that $x_{n-1}, x_n \in U^*$. Set $L_n = \delta F(x_n, x_{n-1})$. Then L_n is invertible and we can write by (7) and (10)

$$\|x_{n+1}-x^*\| = \left\|-L_n^{-1}(\delta F(x_n,x^*)-L_n)(x_n-x^*)\right\| \leqq$$
$$\leqq \frac{d_0 d_1 \|x_{n-1}-x^*\|}{1-d_0 d_1\left(\|x_n-x^*\|+\|x_{n-1}-x^*\|\right)}\|x_n-x^*\|, \quad n=0,1,2,\dots .$$

From (9) and the choice of r^* it follows that

$$\|x_{n+1}-x^*\| < \|x_n-x^*\| < r^*, \quad n=0,1,2,\dots \tag{11}$$

and $\lim_{n\to\infty} x_n = x^*$. Therefore, the secant iteration generated by (2) remains in U^* and converges to the solution x^* of equation (1).

That completes the proof of the theorem.

We will now adjust the definition of a Lipschitz uniform discretization given in [2] to fit our purposes.

The solution x^* as well as the secant iterates generated by (2) may have better smoothness properties than the elements of E. That is why we can consider a subset $W^* \subset E$ such that

(12) $x^* \in W^*$, $x_n \in W^*$, $x_n - x^* \in W^*$, $x_{n+1} - x_n \in W^*$, $n = 0,1,2,\dots$.

The discretization methods to be considered here are described by the family of triplets

$$\{P_h, L_h, \hat{L}_h\}, \quad h > 0 \tag{13}$$

where

$$P_h\colon D_h \subset E_h \to \hat{E}_h, \quad h > 0$$

are nonlinear operators and

$$L_h\colon E \to E_h, \quad \hat{L}_h\colon \hat{E} \to \hat{E}_h, \quad h > 0,$$

are bounded linear discretization operators such that

$$L_h(w^* \cap U^*) \subset D_h, \quad h > 0. \tag{14}$$

The discretization (13) is called *Lipschitz uniform* if there exist scalars $R > 0$, $b > 0$ such that

$$\bar{U}^*(L_h(x^*), R) \subset D_h, \quad h > 0 \tag{15}$$

and

$$\begin{cases} \|\delta P_h(\bar{v}, \bar{w}) - \delta P_h(\bar{u}, \bar{z})\| \leqq b(\|\bar{v} - \bar{u}\| + \|\bar{w} - \bar{z}\|), \quad h > 0, \\ \qquad \bar{v}, \bar{u}, \bar{w}, \bar{z} \in \bar{U}^*(L_h(x^*), R). \end{cases} \tag{16}$$

Furthermore, the discretization family (13) is called *bounded* if there is a constant $c_0 > 0$ such that

$$\|L_h(u)\| \leqq c_0\|u\|, \quad u \in W^*, \ h > 0; \tag{17}$$

stable if there is a constant $c_1 > 0$ such that

$$\left\|P_h'(L_h(u))^{-1}\right\| \leqq c_1, \quad u \in W^* \cap U^*, \quad h > 0; \tag{18}$$

and *consistent of order* p if there are two constants c_2, c_3 such that

$$\left\|\hat{L}_h(F(x)) - P_h(L_h(x))\right\| \leqq c_2 h^p, \quad x \in W^* \cap U^*, \quad h > 0, \tag{19}$$

and

$$\left\|\hat{L}_h(F'(u))v - P_h'(L_h(u))L_h(v)\right\| \leqq c_3 h^p, \ u \in W^* \cap U^*, \ v \in W^*, \ h > 0. \tag{20}$$

We can now state and prove the main result.

THEOREM 2. *Let F be a nonlinear operator defined on an open set D of a Banach space E with values in a Banach space $\hat{E}$. Assume:*

(a) *the hypotheses of Theorem* 1 *are true and*

(b) *the Lipschitz uniform discretization* (13) *is bounded, stable and consistent of order p.*

Then

(i) *equation* (3) *has a locally unique solution*

$$z_h^* = L_h(x^*) + O(h^p) \tag{21}$$

for all $h > 0$ satisfying

$$0 < h \leqq h^* = \left[\min\left(\frac{R}{c_4}, \frac{c_2}{c_4^2 b}\right)\right]^{\frac{1}{p}} \quad \textit{with} \quad c_4 = \frac{2c_1c_2}{1 - c_0c_1br^*}. \tag{22}$$

(ii) *There exist constants $h_0 \in (0, h^*]$, $r_0 \in (0, r^*]$ such that the iteration* (4)–(5) *converges to z_h^* such that*

$$z_n^h = L_h(x_n) + O(h^p), \quad n = 0, 1, 2, \dots, \tag{23}$$

$$P_h(z_n^h) = \hat{L}_h(F(x_n)) + O(h^p), \quad n = 0, 1, 2, \dots, \tag{24}$$

$$z_h^h - z_h^* = L_n(x_n - x^*) + O(h^p), \quad n = 0, 1, 2, \dots, \tag{25}$$

for all $h \in (0, h_0]$ and all starting points $x_0 \in U(x^*, r_0)$.

PROOF. Let $x_{-1} \in U^*$. We can assume without loss of generality that $c_0c_1br^* < 1$. Otherwise choose $x_{-1} \in U_1^* = U_1^*(L_h(x^*), r_1^*)$ with $r_1^* < r^*$ and $c_0c_1br_1^* < 1$.

The well established theorem (see, e.g. Theorem 3 in [9]) ensures that when

$$q(h) = c_1b\|L_h(x^* - x_{-1})\| + 2\left[c_1b\|\delta P_h(L_h(x^*), L_h(x_{-1}))^{-1} P_h(L_h(x^*))\|\right]^{1/2} \leqq 1, \tag{26}$$

$$r(h) = \frac{1 - c_1b\|L_h(x^* - x_{-1})\| - d}{2c_1b} \leqq R \tag{27}$$

with

$$d = \Big[(1 - c_1b\|L_h(x^* - x_{-1})\|)^2 - 4c_1b\|\delta P_h(L_h(x^*), L_h(x_{-1}))^{-1} P_h(L_h(x^*))\|\Big]^{1/2}, \tag{28}$$

then (3) has a unique root z_h^* in $\bar{U}(L_h(x^*), r(h))$.

By (18), (19), (17) and (22) we obtain successively

$$q(h) \leqq c_0 c_1 b r^* + 2[c_1 b \| \delta P_h(L_h(x^*), L_h(x_{-1}))^{-1}(P_h(L_h(x^*))- \\ -\hat{P}_h(F(x^*)))\|]^{1/2} \leqq c_0 c_1 b r^* + 2\left[c_1^2 c_2 b h^p\right]^{1/2} \leqq 1 \tag{29}$$

and

$$r(h) \leqq c_4 h^p \leqq R, \quad \text{with } c_4 = \frac{2c_1 c_2}{1 - c_0 c_1 b r^*}, \tag{30}$$

which shows that (26) and (27) hold for all h satisfying (22).

Thus (21) follows from

$$\|z_h^* - L_h(x^*)\| \leqq r(h) \leqq c_4 h^p. \tag{31}$$

By applying Theorem 1 to (3) we see that the secant sequence (4) converges to z_h^* if

$$\|L_h(x_0) - z_h^*\| < \frac{1}{3b\|P_h'(z_h^*)^{-1}\|}, \tag{32}$$

$$\bar{U}(z_h^*, \|L_h(x_0) - z_h^*\|) \subset \bar{U}(L_h(x^*), R). \tag{33}$$

The estimate (33) holds if

$$\|z_h^* - L_h(x^*)\| + \|L_h(x_0) - z_h^*\| \leqq R, \tag{34}$$

and by (17) and (31) we obtain

$$\|L_h(x_0) - z_h^*\| \leqq \|L_h(x_0) - L_h(x^*)\| + \|L_h(x^*) - z_h^*\| \leqq c_0\|x_0 - x^*\| + c_4 h^p. \tag{35}$$

That is, (34) is satisfied if

$$c_0\|x_0 - x^*\| + 2c_4 h^p \leqq R. \tag{36}$$

Using the identity

$$P_h'(z_h^*) = P_h'(L_h(x^*))[I - P_h'(L_h(x^*))^{-1}(P_h'(L_h(x^*) - P_h'(z_h^*))], \tag{37}$$

(16), (18) and (31) we get

$$\|P_h'(z_h^*)^{-1}\| \leqq \frac{\|P_h'(L_h(x^*)^{-1}\|}{1 - b\|P_h'(L_h(x^*))^{-1}\|\|L_h(x^*) - z_h^*\|} \leqq \frac{c_1}{1 - bc_1 c_4 h^p}. \tag{38}$$

The estimate (32) now certainly holds if

$$c_0\|x_0 - x^*\| + 2c_4 h^p \leqq \frac{1 - bc_1 c_4 h^p}{3c_1 b}. \tag{39}$$

It is simple calculus to verify that (36) and (39) are satisfied for all $h \in (0, h_2]$ and $x_0 \in U(x^*, r_2)$ with

$$h_1 = \left[\min\left(\frac{R}{2c_4}, \frac{1}{14c_1c_4b}, \frac{c_2}{c_4^2 b}\right)\right]^{\frac{1}{p}} \tag{40}$$

and

$$r_1 = \min\left(\frac{R}{2c_0}, \frac{1}{6c_0c_1b}\right). \tag{41}$$

That is, with the above values of h and x_0 the sequence (4) converges to z_h^*.

Let us now consider the real quadratic equation

$$As^2 + Bs + C = 0 \tag{42}$$

where

$$A = 5bc_1, \quad B = 8c_0c_1b\|x_0 - x^*\| + c_1c_4bh^p - 1,$$

and

$$C = c_1(c_1 + c_2)h^p.$$

We can now choose

$$h_0 = \left[\min\left(h_1^p, \frac{(8c_0c_1\|x_0 - x^*\| + c_1c_4bh_1^p - 1)^2}{20bc_1^2(c_1 + c_1)}\right)\right]^{\frac{1}{p}} \tag{43}$$

and

$$r_0 = \min\left(r_1, \frac{1}{32c_0c_1b}\right). \tag{44}$$

Then for all $h \in (0, h_0]$ and $x_0 \in U(x^*, r_0)$, equation (42) has a small positive solution s_0 such that

$$s_0 \leqq c_5h^p \tag{45}$$

with $c_5 = 4C$.

We now prove that for $h \in (0, h_1)$ and $x_0 \in U(x^*, r_0)$ and all $n = 0, 1, 2, \ldots$ the estimate

$$\|z_n^h - L_h(x_n)\| \leqq s_0 \tag{46}$$

holds.

For $n = 0$, (46) is trivially true. Suppose that (46) holds for $n = 0, 1, 2, \ldots, i$.

Using (2) and (4) we obtain the identity

$$
\begin{aligned}
(47)\quad z_{i+1}^h - L_h(x_{i+1}) = \delta P_h(z_i^h, z_{i-1}^h)^{-1}\{[\delta P_h(z_i^h, z_{i-1}^h)(z_i^h - L_h(x_i)) -& \\
- P_h(z_i^h) + P_h(L_h(x_i))] + [(\delta P_h(z_i^h, z_{i-1}^h) -& \\
- \delta P_h(L_h(x_i), L_h(x_{i-1}))L_h(\delta F(x_i, x_{i-1})^{-1}F(x_i))] +& \\
+ [\delta P_h(L_h(x_i), L_h(x_{i-1}))L_h(\delta F(x_i, x_{i-1})^{-1}F(x_i)) - \hat{L}_h(F(x_i))] +& \\
+ [\hat{L}_h(F(x_i)) - P_h(L_h(x_i))]\}.&
\end{aligned}
$$

From the identity

$$
\begin{aligned}
(48)\quad \delta P_h(z_i^h, z_{i-1}^h) = \delta P_h(L_h(x^*), L_h(x^*))[I - \delta P_h(L_h(x^*), L_h(x^*))^{-1} \times& \\
\times(\delta P_h(L_h(x^*), L_h(x^*)) - \delta P_h(z_i^h, z_{i-1}^h))],&
\end{aligned}
$$

(16), (17), and (18) we obtain

$$
(49)\qquad \|\delta P_h(z_i^h, z_{i-1}^h)^{-1}\| \leqq \frac{c_1}{1 - c_1 c_4 b h^p - 2bc_1(s_0 + c_0\|x_0 - x^*\|)}.
$$

Let us note that condition (16) implies the following Lipschitz condition for P_h':

$$
(50)\qquad \|P_h'(\bar{u}) - P_h'(\bar{v})\| \leqq 2b\|\bar{u} - \bar{v}\|, \quad \bar{u}, \bar{v} \in \bar{U}^*(L_h(x^*), R).
$$

Using the integral representation

$$
(51)\qquad P_h(\bar{u}) - P_h(\bar{v}) = \left[\int_0^1 P_h'(\bar{v} + t(\bar{u} - \bar{v}))dt\right](\bar{u} - \bar{v})
$$

we deduce that

$$
(52)\qquad \|P_h(\bar{u}) - P_h(\bar{v}) - P_h'(\bar{w})(\bar{u} - \bar{v})\| \leqq b(\|\bar{u} - \bar{w}\| + \|\bar{v} - \bar{w}\|)\|\bar{u} - \bar{v}\|
$$

for all $\bar{u}, \bar{v}, \bar{w} \in \bar{U}(L_h(x^*), R)$.

From (50), (51) and (52) we obtain the estimate

(53)

$$
\|P_h(\bar{u}) - P_h(\bar{v}) - \delta P_h(\bar{w}, \bar{z})(\bar{u} - \bar{v})\| \leqq b(\|\bar{u} - \bar{w}\| + \|\bar{v} - \bar{z}\| + \|\bar{w} - \bar{z}\|)\|\bar{u} - \bar{v}\|.
$$

Using (47), (11) and (53) the norm of the quantity in the first bracket of (47) becomes

$$
\begin{aligned}
(54)\qquad \|P_h(L_h(x_i)) - P_h(z_i^h) - \delta P_h(z_i^h, z_{i-1}^h)(z_i^h - L_h(x_i))\| &\leqq \\
\leqq b(\|L_h(x_i) - z_i^h\| + \|z_i^h - z_{i-1}^h\|)\|z_i^h - L_h(x_i)\| &\leqq \\
\leqq bs(3s + 2c_0\|x_0 - x^*\|),&
\end{aligned}
$$

since

$$z_i^h - z_{i-1}^h = (z_i^h - L_h(x_i)) + (L_h(x_i) - L_h(x_{i-1})) + (L_h(x_{i-1}) - z_{i-1}^h)$$

and

$$\|x_{n+1} - x^*\| \leqq \|x_n - x^*\|.$$

By (11) and (16) we obtain

$$(55)\ \|(\delta P_h(z_i^h, z_{i-1}^h) - \delta P_h(L_h(x_i), L_h(x_{i-1})))L_h(\delta F(x_i, x_{i-1})^{-1}F(x_i))\| \leqq$$
$$\leqq b(\|z_i^h - L_h(x_i)\| + \|z_{i-1}^h - L_h(x_{i-1})\|)c_0\|x_i - x_{i+1}\| \leqq 4c_0bs\|x_0 - x^*\|.$$

Finally, from (19) and (20) we obtain

$$(56)\ \ \|\delta P_h(L_h(x_i), L_h(x_{i-1}))L_h(\delta F(x_i, x_{i-1})^{-1}F(x_i)) - \hat{L}_h(F(x_i))\| \leqq c_1h^p$$

and

$$(57)\qquad \|\hat{L}_h(F(x_i)) - P_h(L_h(x_i))\| \leqq c_2h^p.$$

With these majorizations in (47) we obtain that

$$(58)\qquad \|z_{i+1}^h - L_h(x_{i+1})\| \leqq$$
$$\leqq \frac{c_1[(c_1 + c_2)h^p + 4bc_0\|x_0 - x^*\|s_0 + bs_0(3s_0 + 2c_0\|x_0 - x^*\|)]}{1 - c_1c_4bh^p - 2bc_1(s_0 + c_0\|x_0 - x^*\|)} = s_0$$

by (42).

This completes the introduction and hence the proof of (46) for all $n = 0, 1, 2, \ldots$.

The estimate (23) now follows from (45) and (46) since

$$(59)\qquad \|z_n^h - L_h(x_n)\| \leqq s_0 \leqq c_5h^p.$$

By (50) we deduce that there exists $c_6 > 0$ such that

$$(60)\qquad \|P_h'(\bar{u})\| \leqq c_6, \quad \bar{u} \in U(L_h(x^*), R)$$

and therefore by (17) and (19)

$$(61)\qquad \|P_h(z_n^h) - \hat{L}_h(F(x_n))\| \leqq \|P_h(z_n^h) - P_h(L_h(x_n))\| +$$
$$+\|P_h(L_h(x_n)) - \hat{L}_h(F(x_n))\| \leqq c_6\|z_n^h - L_h(x_n)\| + c_2h^p \leqq (c_6c_5 + c_2)h^p,$$

which shows (24).

Finally by (31) and (59) we obtain

$$(62)\quad \|(z_n^h - z_h^*) - L_h(x_n - x^*)\| \leqq \|z_n^h - L_h(x_n)\| + \|z_h^* - L_h(x^*)\| \leqq$$
$$\leqq c_5h^p + c_4h^p = c_7h^p, \quad \text{with} \quad c_7 = c_5 + c_4$$

which shows (25) and that completes the proof of the theorem.

We now complete the claims made in the introduction concerning the mesh-independence principle as follows:

THEOREM 3. *Let F be a nonlinear operator defined on an open set D of a Banach space E with values in a Banach space $\hat{E}$. Assume:*
(a) *the hypotheses of Theorem 2 are true, and*
(b) *there exists $\delta > 0$ such that*

$$\liminf_{h>0} \|L_h(u)\| \geqq \delta\|u\| \quad \text{for each} \quad u \in W^*. \tag{63}$$

Then for some $r_2 \in (0, r_0]$, and for any fixed $\varepsilon > 0$ and $x_0 \in \bar{U}(x^*, r_2)$ there exists a constant $h_3 = h_3(\varepsilon, x_0) \in (0, h_2]$ such that

$$|\min\{n \geqq 0, \|x_n - x^*\| < \varepsilon\} - \min\{n \geqq 0, \|z_n^h - z_h^*\| < \varepsilon\}| < 1 \tag{64}$$

for all $h \in (0, h_3]$.

PROOF. Let i be the unique integer defined by

$$\|x_{i+1} - x^*\| < \varepsilon \leqq \|x_i - x^*\|. \tag{65}$$

By (63), there exists $h_2 > 0$ (depending on x_0) such that

$$\|L_h(x_i - x^*)\| \geqq \delta\|x_i - x^*\| \quad \text{for all} \quad 0 < h < h_3. \tag{66}$$

We prove the theorem for

$$r_2 = \min\left(r_0, \frac{\beta}{2c_0c_1b(\beta+2)}\right), \tag{67}$$

$$\beta = \min\left(2c_0, \frac{1}{c_0}, \delta\right) \tag{68}$$

and

$$h_3 = \min\left(h_0, h_2, \left(\frac{\beta}{bc_1c_4(\beta+2)}\right)^{\frac{1}{p}}, \left(\frac{\beta\varepsilon}{2c_7}\right)^{\frac{1}{p}}\right). \tag{69}$$

By (62) and (69) we obtain

$$\|z_{i+1}^h - z_h^*\| \leqq \|L_h(x_{i+1} - x^*)\| + c_7h^p \leqq c_0h^p + \frac{\beta\varepsilon}{2} < 2c_0\varepsilon. \tag{70}$$

Using (35), (67), (68) and (69) we get

$$\|z_{i+2}^h - z_h^*\| \leqq \frac{bc_1\|z_i - z_h^*\|}{1 - bc_1(\|z_{i+1} - z^*\| + \|z_i - z^*\|)}\|z_{i+1} - z_h^*\| \leqq \tag{71}$$

$$\leqq \frac{bc_1\|z_0 - z_h^*\|}{1 - 2bc_1\|z_0 - z_h^*\|}\|z_{i+1} - z_h^*\| \leqq \frac{bc_1(c_0r_2 + c_4h^p)}{1 - bc_1(c_0r_2 + c_4h^p)}2c_0\varepsilon \leqq \beta c_0\varepsilon < \varepsilon.$$

By (62) and (66)

$$\varepsilon \leqq \|x_i - x^*\| \leqq \frac{1}{\delta}\|L_h(x_i - x^*)\| \leqq \frac{1}{\delta}(\|z_i^h - z_h^*\| + c_7 h^p)$$

or

(72) $$\|z_i^h - z_h^*\| \geqq \delta\varepsilon - c_7 h^p \geqq \delta\varepsilon - \frac{\delta\varepsilon}{2} = \frac{\delta\varepsilon}{2}.$$

Let us now assume that $\|z_{i-1}^h - z_h^*\| < \varepsilon$, then as in (71) we get

(73) $$\|z_i^h - z_h^*\| < \frac{\beta\varepsilon}{2} \leqq \frac{\delta\varepsilon}{2},$$

contradicting (72). That is, we must have

(74) $$\|z_{i-1}^h - z_h^*\| \geqq \varepsilon.$$

The result (64) now follows from (65), (71) and (74).

That completes the proof of the theorem.

REMARKS. The condition (63) certainly holds if

(75) $$\lim_{h\to 0} \|L_h(u)\| = \|u\| \quad \text{for each} \quad u \in W^*.$$

For some discretizations we have

(76) $$\lim_{h\to 0} \|L_h(u)\| = \|u\| \quad \text{uniformly for} \quad u \in W^*.$$

Both conditions (75) and (76) are almost standard in most discretization studies [1], [2], [8], [10].

A theorem similar to Theorem 3 can now be stated if (63) is replaced by (76) (with h_3 depending only on ε).

III. Applications

EXAMPLE 1. Theorem 1 can be realized for operators F which satisfy an autonomous differential equation of the form

$$F'(x) = G(F(x)), \quad \text{for some given operator} \quad G.$$

As $F'(x^*) = G(0)$, the inverse $F'(x^*)^{-1}$ can be evaluated without knowing the solution x^*. Consider for example the scalar equation

(77) $$F(x) = 0$$

where F is given by

$$F(x) = e^x - q, \quad \text{for some given } q > 0.$$

Note that $F'(x) = F(x) + q$. That is $F'(x^*) = q$.

Let us define the divided difference operator $\delta F(v, w)$ by

$$\delta F(v, w) = \frac{F(v) - F(w)}{v - w}, \quad w \neq v.$$

The linear operator $\delta F(v, w)$ is now a function of two variables v and w. By expanding $\delta F(v, w)$ about (v, w), restricting the domain of it in some ball U^* centered at $x^* = \ell nq$ and using Taylor's theorem in two variables, a number $b \geqq 0$ satisfying (7) can easily be found.

By Theorem 1 if $x_0, x_{-1} \in U^*$, then the iteration (2) can be used to approximate the solution $x^* = \ell nq$ of equation (77).

For further examples on autonomous differential equations one can refer to [7] and the references there.

A more interesting application is given by the following example.

EXAMPLE 2. In this example we study the case of a natural difference approximation for the scalar, second order two point boundary value problem studied in [2]. Consider the operator

$$F\colon D \subset C^2[0,1] \to C[0,1] \times \mathbf{R}^2,$$

$$F(y) = \{y'' - f(x, y, y'); 0 \leqq x \leqq 1, y(0) - \alpha, y(1) - \beta\},$$

where D and f are assumed to be such that equation (1) has a unique solution $x^* \in D$ and

$$f \in C^3(U(x^*, R)),$$

$$U(x^*, R) = \{(x_1, x_2, x_3) \in \mathbf{R}^3; 0 \leqq x_1 < 1, |x_2 - x^*(x_1)|, |x_3 - x^*(x_1)| \leqq R\}.$$

To avoid repetitions we consider the discretization (13) considered in [2, p. 167] with the finite difference operator $\delta P_h(v, w)$ exactly as defined in [10, p. 81, Form (10)].

As in [2] it can easily be seen that all the hypotheses of Theorems 2 and 3 are satisfied and therefore the conclusions apply in this case.

Other examples and applications may be found in the cited articles.

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DEPARTMENT OF MATHEMATICAL SCIENCES
CAMERON UNIVERSITY
LAWTON, OK 73505
U.S.A.

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ON PSEUDO-DIFFERENTIAL OPERATORS WITH AMPLITUDE $Q(x,y,\xi)$

J. TERVO (Kuopio)

1. Introduction

Consider the linear operators Q defined in the Schwartz class S by the requirement

$$(1.1)\qquad (Q\varphi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy\Big)d\xi,$$

for $x \in \mathbf{R}^n$. Here the amplitude $Q(\cdot,\cdot,\cdot)$ is assumed to be a C^∞-function $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ such that

$$(1.2)\qquad \sup_{x,y\in\mathbf{R}^n} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma} k_{N_\gamma+\delta|\alpha+\beta|}(\xi) \text{ for } \xi \in \mathbf{R}^n,$$

with some $C_{\alpha,\beta,\gamma} > 0$, $N_\gamma \in \mathbf{R}$ and $\delta < 1$. The operator Q maps S into S. We establish that $Q = Q_N$ for any $N \in \mathbf{N}$, where

$$(1.3)\qquad Q_N(x,y,\xi) := (1+|x-y|^{2N})^{-1}\Big[\Big(1+\Big(\sum_{l=1}^n D_{\xi_l}^2\Big)^N\Big)Q\Big](x,y,\xi)$$

(cf. Theorem 3.1).

For $N \geqq n$ we define a function $L_N(\cdot,\cdot)$: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ by

$$(1.4)\qquad L_N(x,\xi) := (2\pi)^n \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta-\xi)}dy\Big)d\eta.$$

One obtains that $L_N(\cdot,\cdot) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$,

$$(1.5)\qquad \sup_{x\in\mathbf{R}^n} |(D_x^\alpha D_\xi^\beta L_N)(x,\xi)| \leqq C_{\alpha,\beta} k_{\mu_{\alpha,\beta}}(\xi) \text{ for } \xi \in \mathbf{R}^n$$

with some $C_{\alpha,\beta} > 0$, $\mu_{\alpha,\beta} \in \mathbf{R}$ and that

$$(1.6)\quad (Q\varphi)(x) = (L_N(x,D)\varphi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} L_N(x,\xi)(F\varphi)(\xi)e^{i(\xi,x)}d\xi$$

(cf. Lemma 3.4 and Theorem 3.6). We know some special classes of operators for which the relation (1.6) holds (cf. [3], [4] and [6], for example). Remark that in our considerations, the amplitude $Q(\cdot,\cdot,\cdot)$ is not assumed to be compactly supported in y.

Let $A^{M,m}_{\Phi,\phi}$ be a class of amplitudes so that

(1.7)
$$|(D^\alpha_x D^\beta_y D^\gamma_\xi Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma} \sup_{(u_1,\xi),(u_2,\xi)\in E_{x,y}\times \mathbf{R}^n} \Phi^{M-|\gamma|}(u_1,\xi)\phi^{m-|\alpha+\beta|}(u_2,\xi),$$

where (Φ,ϕ) forms a pair of weight functions in the sense of Beals and Fefferman [1] and where $E_{x,y}$ is the convex hull of $x, y \in \mathbf{R}^n$. Denote by $S^{M,m}_{\Phi,\phi}$ the Beals and Fefferman class of symbols $L(\cdot,\cdot) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ (cf. [1], [2]). As an application we establish that

(1.8)
$$\mathbf{L}^{M,m}_{\Phi,\phi} := \{Q \mid Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}\} = \mathcal{L}^{M,m}_{\Phi,\phi} := \{L(x,D) \mid L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}\},$$

where

$$(L(x,D)\varphi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} L(x,\xi)(F\varphi)(\xi)e^{i(\xi,x)}d\xi$$

(cf. Section 4). Some facts on the decomposition $L(x,D) = Q^\sim + R$, where $Q^\sim$ is properly supported in an open set $G \subset \mathbf{R}^n$ (for the terminology cf. [3]), and where $R \in \bigcap_{N\in\mathbf{N}} \mathbf{L}^{M-N,m}_{\Phi,\phi}$, are also considered.

2. The operator Q with amplitude $Q(x,y,\xi)$

2.1. Denote by S the Schwartz class of all rapidly decreasing smooth functions $\varphi\colon \mathbf{R}^n \to \mathbf{C}$. S is equipped with the standard locally convex topology defined by the seminorms $q_{\alpha,\beta}(\varphi) := \sup_{x\in\mathbf{R}^n} |x^\alpha(D^\beta_x\varphi)(x)|$. Let $Q(\cdot,\cdot,\cdot)$ be a function in $C^\infty(\mathbf{R}^{3n}) := C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n)$. Consider the linear operator Q defined for $\varphi \in S$ by the requirement

$$(2.1)\qquad (Q\varphi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy \Big) d\xi.$$

In the case when $Q(\cdot,\cdot,\cdot)$ together with its derivatives $(D^\alpha_x D^\beta_y D^\gamma_\xi Q)(\cdot,\cdot,\cdot)$ obeys suitable tempered criteria, the operator Q maps S continuously into S. Furthermore, under certain conditions, the formal transpose $Q'\colon S \to S$ exists, that is, there exists a continuous linear operator $Q'\colon S \to S$ such that

$$(2.2)\qquad (Q\varphi)(\psi) = \varphi(Q'\psi) \text{ for all } \varphi,\psi \in S,$$

where we denoted

$$(2.3) \qquad \varphi(\psi) := \int_{\mathbf{R}^n} \varphi(x)\psi(x)dx \quad \text{for} \quad \varphi, \psi \in S$$

(cf. [5]). In the sequel we write $k_s(\xi) = (1 + |\xi|^2)^{s/2}$. We recall

THEOREM 2.1. *Suppose that $Q(\cdot,\cdot,\cdot) \in C^\infty(\mathbf{R}^{3n})$ such that with a constant $\delta < 1$ for all $\alpha, \beta, \gamma \in \mathbf{N}_0^n$ there exist constants $C_{\alpha,\beta,\gamma} > 0$ and $N_\gamma \in \mathbf{R}$ such that*

$$(2.4) \qquad \sup_{x,y\in\mathbf{R}^n} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma} k_{N_\gamma + \delta|\alpha+\beta|}(\xi)$$

for all $\xi \in \mathbf{R}^n$. Then the operator Q defined by (2.1) *maps S continuously into S and the formal transpose Q' of Q exists. The operator Q' is given by*

$$(2.5) \qquad (Q'\psi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q(y,x,-\xi)\psi(y)e^{i(x-y,\xi)}dy \Big) d\xi.$$

For the proof of Theorem 2.1 we refer to [5]. Specifically, in the case when $Q(x,y,\psi)$ is of the form $Q(x,y,\xi) = L(x,\xi)$, where $L(\cdot,\cdot) \in C^\infty(\mathbf{R}^{2n})$ such that

$$(2.6) \qquad \sup_{x\in\mathbf{R}^n} |(D_x^\alpha D_\xi^\beta L)(x,\xi)| \leqq C_{\alpha,\beta} k_{N_\beta + \delta|\alpha|}(\xi) \quad \text{for all} \quad \xi \in \mathbf{R}^n,$$

Theorem 2.1 can be applied and so we obtain: Assume that $L(\cdot,\cdot) \in C^\infty(\mathbf{R}^{2n})$ such that with some $\delta < 1$, the estimate (2.6) holds. Then the operator $L(x,D)$ defined by

$$(2.7) \qquad (L(x,D)\varphi)(x) := (2\pi)^{-n} \int_{\mathbf{R}^n} L(x,\xi)(F\varphi)(\xi)e^{i(x,\xi)}d\xi$$

maps S continuously into S and the formal transpose $L'(x,D)\colon S \to S$ exists. The operator $L'(x,D)$ is given by

$$(2.8) \qquad (L'(x,D)\psi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} L(y,-\xi)\psi(y)e^{i(x-y,\xi)}dy \Big) d\xi$$

(here $F\colon S \to S$ denotes the Fourier transform).

2.2. Suppose that (Φ,ϕ) forms a pair of weight functions in the sense of Beals and Fefferman [1]. Let $E_{x,y}$ be the convex hull of the elements x and $y \in \mathbf{R}^n$, that is,

$$E_{x,y} := \{u \in \mathbf{R}^n \mid u = \alpha_1 x + \alpha_2 y, \text{ where } \alpha_1 \geqq 0,\ \alpha_2 \geqq 0 \text{ and } \alpha_1 + \alpha_2 = 1\}.$$

We say that the amplitude $Q(\cdot,\cdot,\cdot)$ belongs to $A^{M,m}_{\Phi,\phi}$ (where $M, m \in \mathbf{R}$) if the estimate

(2.9)
$$|(D^\alpha_x D^\beta_y D^\gamma_\xi Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma} \sup_{(u_1,\xi),(u_2,\xi)\in E_{x,y}\times\mathbf{R}^n} \Phi^{M-|\gamma|}(u_1,\xi)\phi^{m-|\alpha+\beta|}(u_2,\xi)$$

holds for all $x, y, \xi \in \mathbf{R}^n$. The linear space $\mathbf{L}^{M,m}_{\Phi,\phi}$ of operators Q is defined by

$$\mathbf{L}^{M,m}_{\Phi,\phi} := \{Q \mid Q \text{ is defined by (2.1), where } Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}\}.$$

Furthermore, we define the class $S^{M,m}_{\Phi,\phi}$ of symbols in the following way: The function $L(\cdot,\cdot) \in C^\infty(\mathbf{R}^{2n})$ belongs to $S^{M,m}_{\Phi,\phi}$ if the estimate

$$(2.10) \qquad |(D^\alpha_x D^\beta_\xi L)(x,\xi)| \leqq C_{\alpha,\beta}\Phi^{M-|\beta|}(x,\xi)\phi^{m-|\alpha|}(x,\xi) \text{ for } x,\xi \in \mathbf{R}^n$$

holds. The linear space $\mathcal{L}^{M,m}_{\Phi,\phi}$ of operators is defined by

$$\mathcal{L}^{M,m}_{\Phi,\phi} := \{L(x,D) \mid L(x,D) \text{ is defined by (2.7) where } L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}\}.$$

LEMMA 2.2. *Suppose that* $Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}$. *Then the estimate*

$$(2.11) \quad \sup_{x,y\in\mathbf{R}^n} |(D^\alpha_x D^\beta_y D^\gamma_\xi Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma} k_{|M-|\gamma||+(1-\varepsilon)|m|+(1-\varepsilon)|\alpha+\beta|}(\xi)$$

holds with some $0 < \varepsilon < 1$.

PROOF. In virtue of property (i) of weight functions Φ and ϕ one has with $c > 0$, $C > 0$ and $\varepsilon > 0$:

$$(2.12) \qquad c \leqq \Phi(u,\xi) \leqq C(1+|\xi|)$$

and

$$(2.13) \qquad c(1+|\xi|)^{\varepsilon-1} \leqq \phi(u,\xi) \leqq C$$

for all $(u,\xi) \in E_{x,y} \times \mathbf{R}^n$ (cf. [1]) and so we get

$$\Phi^{M-|\gamma|}(u,\xi) \leqq \max\{C^{|M-|\gamma||}, c^{-|M-|\gamma||}\}(1+|\xi|)^{|M-|\gamma||}$$

and

$$\phi^{m-|\alpha+\beta|}(u,\xi) \leqq c^{-|\alpha+\beta|}\max\{C^{|m|}, c^{-|m|}\}(1+|\xi|)^{(1-\varepsilon)(|m|+|\alpha+\beta|)}.$$

Hence we obtain from (2.9) with some $C_{\alpha,\beta,\gamma} > 0$

$$\sup_{x,y\in\mathbf{R}^n} |(D^\alpha_x D^\beta_y D^\gamma_\xi Q)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma}(1+|\xi|)^{|M-|\gamma||+|m|(1-\varepsilon)}(1+|\xi|)^{(1-\varepsilon)|\alpha+\beta|},$$

which implies (2.11). □

From (2.11) we observe that for any $Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}$ the estimate (2.4) holds (with $N_\gamma := |M-|\gamma|| + (1-\varepsilon)|m|$ and with $\delta := 1-\varepsilon < 1$). Hence any $Q \in \mathbf{L}^{M,m}_{\Phi,\phi}$ maps S continuously into S and the formal transpose $Q' \colon S \to S$ exists.

3. The identities $Q = Q_N$ and $Q = L_N(x, D)$

3.1. Define $Q_N(\cdot,\cdot,\cdot) \in C^\infty(\mathbf{R}^{3n})$, $N \in \mathbf{N}$ by (here $D_{\xi_l} := -i\partial/\partial\xi_l$)

$$Q_N(x,y,\xi) = (1 + |x-y|^{2N})^{-1}((1+\Delta_\xi^N)Q)(x,y,\xi), \tag{3.1}$$

where $\Delta_\xi := \Big(\sum\limits_{l=1}^{n} D_{\xi_l}^2\Big)$.

We begin with

THEOREM 3.1. *Suppose that $Q(\cdot,\cdot,\cdot) \in C^\infty(\mathbf{R}^{3n})$ such that with a constant $\delta < 1$ the estimate* (2.4) *holds. Then one has*

$$Q = Q_N \quad \textit{for any} \quad N \in \mathbf{N}. \tag{3.2}$$

PROOF. A. Let Θ be in C_0^∞ such that $\Theta(x) = 1$ for all $|x| \leqq 1$ and define $\Theta_j(x) := \Theta(x/j)$ for $j \in \mathbf{N}$. Furthermore, define

$$Q_j(x,y,\xi) = Q(x,y,\xi)\Theta_j(\xi). \tag{3.3}$$

Then the Fubini Theorem implies that

$$(Q_j\varphi)(x) = (2\pi)^{-n} \int\limits_{\mathbf{R}^n} \Big(\int\limits_{\mathbf{R}^n} Q_j(x,y,\xi) e^{i(x-y,\xi)} d\xi \Big) \varphi(y) dy. \tag{3.4}$$

B. We obtain for any $\alpha \in \mathbf{N}_0^n$

(3.5)

$$(y-x)^\alpha \Big(\int\limits_{\mathbf{R}^n} Q_j(x,y,\xi) e^{i(x-y,\xi)} d\xi \Big) = \int\limits_{\mathbf{R}^n} Q_j(x,y,\xi)(-D_\xi)^\alpha (e^{i(x-y,\xi)}) d\xi =$$
$$= \int\limits_{\mathbf{R}^n} (D_\xi^\alpha Q_j)(x,y,\xi) e^{i(x-y,\xi)} d\xi,$$

and so by induction (with respect to N)

$$|y-x|^{2N} \Big(\int\limits_{\mathbf{R}^n} Q_j(x,y,\xi) e^{i(x-y,\xi)} d\xi \Big) = \tag{3.6}$$
$$= \Big(\sum_{l=1}^{n} (y_l - x_l)^2 \Big)^N \Big(\int\limits_{\mathbf{R}^n} Q_j(x,y,\xi) e^{i(x-y,\xi)} d\xi \Big) =$$
$$= \int\limits_{\mathbf{R}^n} \bigg(\Big(\sum_{l=1}^{n} D_{\xi_l}^2 \Big)^N Q_j \bigg)(x,y,\xi) e^{i(x-y,\xi)} d\xi$$

(note that $(y-x)^{(0,\dots,\alpha_1,\dots,0)}=(y_1-x_1)^{\alpha_1}$). Thus

(3.7)
$$\int\limits_{\mathbf{R}^n} Q_j(x,y,\xi)e^{i(x-y,\xi)}d\xi=(1+|x-y|^{2N})^{-1}\int\limits_{\mathbf{R}^n}((1+\Delta_\xi^N)Q_j)(x,y,\xi)e^{i(x-y,\xi)}d\xi$$

and so by the Fubini Theorem

$$(3.8)\qquad (Q_j\varphi)(x)=$$

$$=(2\pi)^{-n}\int\limits_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}\left[\int\limits_{\mathbf{R}^n}((1+\Delta_\xi^N)Q_j)(x,y,\xi)e^{i(x-y,\xi)}d\xi\right]\varphi(y)dy=$$
$$=(2\pi)^{-n}\int\limits_{\mathbf{R}^n}\Big(\int\limits_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}((1+\Delta_\xi^N)Q_j)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi.$$

C. We shall show that

$$(3.9)\qquad (Q_j\varphi)(x)\to(Q\varphi)(x)\quad\text{for}\quad x\in\mathbf{R}^n$$

and

$$(3.10)\qquad (Q_j\varphi)(x)\to(Q_N\varphi)(x)\quad\text{for}\quad x\in\mathbf{R}^n,$$

which implies the assertion.

C$_1$. For any $\alpha\in\mathbf{N}_0^n$ one gets

$$(3.11)\qquad \Big|\xi^\alpha\Big(\int\limits_{\mathbf{R}^n}Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy\Big)\Big|=$$
$$=\Big|\sum_{\gamma\leqq\alpha}\int\limits_{\mathbf{R}^n}\binom{\alpha}{\gamma}(D_y^{\alpha-\gamma}Q)(x,y,\xi)(D^\gamma\varphi)(y)e^{i(x-y,\xi)}dy\Big|\leqq$$
$$\leqq\sum_{\gamma\leqq\alpha}\binom{\alpha}{\gamma}C_{0,\alpha-\gamma,0}\|D^\gamma\varphi\|_{L_1}k_{N_0+\delta|\alpha-\gamma|}(\xi)\leqq C_{\alpha,\varphi}k_{N_0+\delta|\alpha|}(\xi),$$

and so with some $C_\varphi>0$

$$(3.12)\qquad \Big|\int\limits_{\mathbf{R}^n}Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy\Big|\leqq C_\varphi k_{-(n+1)}(\xi).$$

Since

$$f_j(\xi) := \Theta_j(\xi)\Big(\int\limits_{\mathbf{R}^n} Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy\Big) \to f(\xi) :=$$
$$:= \int\limits_{\mathbf{R}^n} Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy$$

and since

$$|f_j(\xi)| \leqq (\sup_{\xi} |\Theta(\xi)|)|f(\xi)| \leqq \|\Theta\|_\infty C_\varphi k_{-(n+1)}(\xi),$$

the Dominated Convergence Theorem implies that

$$\int\limits_{\mathbf{R}^n} f_j(\xi)d\xi \to \int\limits_{\mathbf{R}^n} f(\xi)d\xi,$$

and so (3.9) holds.

C_2. Similarly, as in C_1 one gets that

$$\int\limits_{\mathbf{R}^n}\Big(\int\limits_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}Q_j(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi \to \tag{3.13}$$
$$\to \int\limits_{\mathbf{R}^n}\Big(\int\limits_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}Q(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi$$

and then to obtain (3.10) we have to prove that

$$\int\limits_{\mathbf{R}^n}\Big(\int\limits_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}(\Delta_\xi^N Q_j)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi \to \tag{3.14}$$
$$\to \int\limits_{\mathbf{R}^n}\Big(\int\limits_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}(\Delta_\xi^N Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi.$$

Define $P_N(D) := \Delta_\xi^N$. Then by the Leibniz formula we get

$$(P_N(D)Q_j)(x,y,\xi) = \sum_{|\alpha|\leqq 2N} (1/\alpha!)(P_N^{(\alpha)}(D)Q)(x,y,\xi)(D^\alpha\Theta_j)(\xi) =$$
$$= \sum_{|\alpha|\leqq 2N} (1/\alpha!)Q_{j,\alpha}(x,y,\xi),$$

where

$$Q_{j,\alpha}(x,y,\xi) := (P_N^{(\alpha)}(D)Q)(x,y,\xi)(D^\alpha\Theta_j)(\xi).$$

Our task reduces to establish that

$$\text{(3.15)}\quad \int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}Q_{j,\alpha}(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi \to$$

$$\to \begin{cases} \int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}(P_N(D)Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big)d\xi, & \text{for } \alpha=0. \\ 0, & \text{for } \alpha\neq 0. \end{cases}$$

Since the amplitude $(x,y,\xi)\to(1+|x-y|^{2N})^{-1}(P_N(D)Q)(x,y,\xi)$ satisfies the estimate like (2.4) (note that

$$|D_x^\alpha D_y^\beta(1+|x-y|^{2N}))| \leqq C_{\alpha,\beta}(1+|x-y|^2)^{-N-\frac{|\alpha+\beta|}{2}} \leqq C_{\alpha,\beta}),$$

one gets as in C_1 that the convergence (3.15) holds, when $\alpha=0$.

Let $\alpha\neq 0$. Define

$$g_{j,\alpha}(\xi) := \int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}Q_{j,\alpha}(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy.$$

Then

(3.16)

$$g_{j,\alpha}(\xi) = (D^\alpha\Theta_j)(\xi)\int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}(P_N^{(\alpha)}(D)Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy =$$

$$=(1/j^{|\alpha|})(D^\alpha\Theta)(\xi/j)\int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}(P_N^{(\alpha)}(D)Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy \underset{j}{\to} 0$$

for any $\xi\in\mathbf{R}^n$. Furthermore,

$$\Big|\xi^\beta\Big(\int_{\mathbf{R}^n}(1+|x-y|^{2N})^{-1}(P_N^{(\alpha)}(D)Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\Big| \leqq$$

$$\leqq \int_{\mathbf{R}^n}|D_y^\beta[(1+|x-y|^{2N})^{-1}\varphi(y)(P_N^{(\alpha)}(D)Q)(x,y,\xi)]|dy \leqq$$

$$\leqq \sum_{\tau\leqq\beta}\binom{\beta}{\tau}\|D_y^{\beta-\tau}[(1+|x-(\cdot)|^{2N})^{-1}\varphi]\|_{L_1}\sup_{x,y}|(D_y^\tau P_N^{(\alpha)}(D)Q)(x,y,\xi)|.$$

Since $P_N^{(\alpha)}(D)$ is a differential operator of order $2N - |\alpha|$, we get

$$\sup_{x,y} \left|(D_y^\tau P_N^{(\alpha)}(D)Q)(x,y,\xi)\right| \leqq C_{\alpha,\tau} k_{N_\alpha+\delta|\tau|}(\xi),$$

where $C_{\alpha,\tau}$ is a suitable constant and where $N_\alpha := \max_{|\gamma|\leqq 2N-|\alpha|}\{N_\gamma\}$. Hence there exists a constant $C_{\alpha,\varphi} > 0$ such that

$$(3.17)\qquad \left|\int_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}(P_N^\alpha(D)Q)(x,y,\xi)e^{i(x-y,\xi)}\varphi(y)dy\right| \leqq$$
$$\leqq C_{\alpha,\varphi}k_{-(n+1)}(\xi) \quad \text{for all} \quad \xi \in \mathbf{R}^n.$$

This estimate yields from (3.16) that

$$|g_{j,\alpha}(\xi)| \leqq (1/j^{|\alpha|})(\sup_\xi |(D^\alpha\Theta)(\xi)|)C_{\alpha,\varphi}k_{-(n+1)}(\xi) \leqq$$
$$\leqq \|D^\alpha\Theta\|_\infty C_{\alpha,\varphi}k_{-(n+1)}(\xi),$$

and then by (3.16) and by the Dominated Convergence Theorem

$$\int_{\mathbf{R}^n} g_{j,\alpha}(\xi)d\xi \underset{j}{\to} 0 \quad \text{for any} \quad \alpha \neq 0.$$

This completes the proof of (3.15) and so the proof is ready. □

COROLLARY 3.2. *Suppose that $L(\cdot,\cdot) \in C^\infty(\mathbf{R}^{2n})$ such that the estimate* (2.6) *holds with some $\delta < 1$. Then*

$$(3.18)\qquad L(x,D) = L_N,$$

where

$$(3.19)\qquad (L_N\varphi)(x) =$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^n} ((1+\Delta_\xi^N)L)(x,\xi)\int_{\mathbf{R}^n} (1+|x-y|^{2N})^{-1}\varphi(y)e^{i(x-y,\xi)}dyd\xi.$$

3.2. For the amplitude $Q_N(\cdot,\cdot,\cdot)$ the estimate

$$(3.20)\qquad |(D_x^\alpha D_y^\beta D_\xi^\gamma Q_N)(x,y,\xi)| \leqq C_{\alpha,\beta,\gamma}k_{N_{\gamma,N}+\delta|\alpha+\beta|}(\xi)(1+|x-y|^{2N})^{-1}$$

holds, where $N_{\gamma,N} := \max_{w\leqq 2N}\{N_{w+\gamma}\}$. This follows from the fact that $1+\Delta_\xi^N$ is a differential operator of order $2N$ and

$$(3.21)\qquad |D_x^\alpha D_y^\beta((1+|x-y|^{2N})^{-1})| \leqq C'_{\alpha,\beta}(1+|x-y|^{2N})^{-1}.$$

For $N \geqq n$ we define a function $L_N(\cdot,\cdot)\colon \mathbf{R}^{2n} \to \mathbf{C}$ by

$$L_N(x,\xi) := (2\pi)^n \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta) e^{i(x-y,\eta-\xi)} dy \Big) d\eta. \tag{3.22}$$

LEMMA 3.3. *Suppose that $Q(\cdot,\cdot,\cdot) \in C^\infty(\mathbf{R}^{3n})$ such that with a constant $\delta < 1$, the estimate (2.4) holds. Let $Q_N(\cdot,\cdot,\cdot)$ be defined by (3.3), $N \geqq n$ and let $L_N(\cdot,\cdot)$ be defined by (3.22). Then one has for any $R \in \mathbf{N}$*

$$\begin{aligned} & L_N(x,\xi) = \\ &= (2\pi)^n \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} (1+|\eta-\xi|^{2R})^{-1} ((1+\Delta_y^R) Q_N)(x,y,\eta) e^{i(x-y,\eta-\xi)} dy \Big) d\eta =: \\ &=: L_{N,R}(x,\xi). \end{aligned} \tag{3.23}$$

PROOF. For any $\alpha \in \mathbf{N}_0^n$ one has

$$(\eta-\xi)^\alpha \int_{\mathbf{R}^n} Q_N(x,y,\eta) e^{i(x-y,\eta-\xi)} dy = \int_{\mathbf{R}^n} (D_y^\alpha Q_N)(x,y,\eta) e^{i(x-y,\eta-\xi)} dy,$$

and so by induction

$$\begin{aligned} & |\eta-\xi|^{2R} \int_{\mathbf{R}^n} Q_N(x,y,\eta) e^{i(x-y,\eta-\xi)} dy = \\ &= \int_{\mathbf{R}^n} \Big(\Big(\sum_{l=1}^n D_{y_j}^2 \Big)^R Q_N \Big)(x,y,\eta) e^{i(x-y,\eta-\xi)} dy. \end{aligned}$$

This completes the proof. □

Define a function $Q_{N,R}(\cdot,\cdot,\cdot,\cdot)$ by

$$Q_{N,R}(x,y,\xi\eta) := (1+|\eta-\xi|^{2R})^{-1} ((1+\Delta_y^R) Q_N)(x,y,\eta). \tag{3.24}$$

Since the differential operator Δ_y^R is of order $2R$ and since

$$|D_\xi^\alpha D_\eta^\beta ((1+|\eta-\xi|^{2R})^{-1})| \leqq C'_{\alpha,\beta} (1+|\eta-\xi|^{2R})^{-1}$$

one gets the estimate

$$\begin{aligned} & |(D_x^\alpha D_y^\beta D_\xi^\gamma D_\eta^\tau Q_{N,R})(x,y,\xi,\eta)| \leqq \\ & \leqq C_{\alpha,\beta,\gamma,\tau} (1+|\eta-\xi|^{2R})^{-1} (1+|x-y|^{2N})^{-1} k_{N_\tau, N+\delta 2R+\delta|\alpha+\beta|}(\eta) \end{aligned} \tag{3.25}$$

(cf. also (3.20)).

LEMMA 3.4. *The function $L_N(\cdot,\cdot)$ defined by* (3.22) *belongs to $C^\infty(\mathbf{R}^{2n})$ for any $N \geqq n$ and*

$$\sup_x |(D_x^\alpha D_\xi^\beta L_N)(x,\xi)| \leqq C_{\alpha,\beta} K_{\mu_{\alpha,\beta}}(\xi) \quad \textit{for all} \quad \xi \in \mathbf{R}^n, \tag{3.26}$$

with some constants $C_{\alpha,\beta} > 0$ and $\mu_{\alpha,\beta} \in \mathbf{R}$.

PROOF. For any $R \in \mathbf{N}$

$$L_N(x,\xi) = (2\pi)^n \int\limits_{\mathbf{R}^n} \Big(\int\limits_{\mathbf{R}^n} Q_{N,R}(x,y,\xi,\eta) e^{i(x-y,\eta-\xi)} dy \Big) d\eta. \tag{3.27}$$

Let $\alpha, \beta \in \mathbf{N}_0^n$. Choose $R \in \mathbf{N}$ so large that

$$N_{0,N} + (\delta - 1)2R \leqq -(n+1). \tag{3.28}$$

Then

$$|Q_{N,R}(x,y,\xi,\eta)| \leqq C'_{0,0,0,0}(1+|\xi|^2)^R (1+|x-y|^{2N})^{-1} k_{-(n+1)}(\eta)$$

and so the Fubini Theorem implies

$$L_N(x,\xi) = (2\pi)^n \int\limits_{\mathbf{R}^n} \Big(\int\limits_{\mathbf{R}^n} Q_{N,R}(x,y,\xi,\eta) e^{i(x-y,\eta-\xi)} d\eta \Big) dy.$$

Choose $R \in \mathbf{N}$ so that

$$\max_{u \leqq 2|\beta|} \{N_{u,N}\} + (\delta-1)2R + (1+\delta)|\alpha| \leqq -(n+1). \tag{3.29}$$

Similarly as above (cf. the proof of Lemma 3.3) we find that for $r = |\beta|$ the relation

$$L_N(x,\xi) = (2\pi)^n \int\limits_{\mathbf{R}^n} \Big(\int\limits_{\mathbf{R}^n} Q_{N,R,r}(x,y,\xi,\eta) e^{i(x-y,\eta-\xi)} d\eta \Big) dy =: L_{N,R,r}(x,\xi) \tag{3.30}$$

holds, where

$$Q_{N,R,r}(x,y,\xi,\eta) := (1+|x-y|^{2r})^{-1}((1+\Delta_\eta^r)Q_{N,R})(x,y,\xi,\eta).$$

By (3.25)

$$|(D_x^\alpha D_\xi^\beta [Q_{N,R,r})(x,y,\xi,\eta) e^{i(x-y,\eta-\xi)}]| \leqq \tag{3.31}$$

$$\leqq \sum_{u \leqq \alpha} \sum_{v \leqq \beta} \binom{\alpha}{u} \binom{\beta}{v} |(D_x^{\alpha-u} D_\xi^{\beta-v} Q_{N,R,r})(x,y,\xi,\eta)| |x-y|^{|v|} |\xi-\eta|^{|u|} \leqq$$

$$\leqq \sum_{u \leqq \alpha} \sum_{v \leqq \beta} \binom{\alpha}{u} \binom{\beta}{v} C'_{\alpha,\beta,u,v} (1+|x-y|^{2r})^{-1}.$$

$$\cdot \max_{\substack{w\leqq\alpha-u\\ r\leqq\beta-v}} |(D_x^w D_\xi^r(1+\Delta_\eta^r)Q_{N,R})(x,y,\xi,\eta)|(1+|x-y|^r)(1+|\eta-\xi|^2)^{|\alpha|/2}\leqq$$

$$\leqq C'_{\alpha,\beta}(1+|\eta-\xi|^{2R})^{-1}(1+|\xi|^2)^{|\alpha|/2}(1+|x-y|^{2N})^{-1}k_{\max\limits_{u\leqq 2r}\{N_{u,N}\}+\delta 2R+(1+\delta)|\alpha|}(\eta)\leqq$$

$$\leqq C''_{\alpha,\beta}(1+|\xi|^2)^{(2R+|\alpha|)/2}(1+|x-y|^{2N})^{-1}k_{\max\limits_{u\leqq 2r}\{N_{u,N}\}+(\delta-1)2R+(1+\delta)|\alpha|}(\eta)\leqq$$

$$\leqq C''_{\alpha,\beta}(1+|\xi|^2)^{(2R+|\alpha|)/2}(1+|x-y|^{2N})^{-1}k_{-(n+1)}(\eta).$$

Thus by the Dominated Convergence Theorem and by the Mean Value Theorem the function

$$L_N(x,\xi)=(2\pi)^n\int_{\mathbf{R}^{2n}} Q_{N,R,r}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}d\eta dy \tag{3.32}$$

belongs to $C^\infty(\mathbf{R}^{2n})$ and

$$(D_x^\alpha D_\xi^\beta L_N)(x,\xi)=(2\pi)^n\int_{\mathbf{R}^{2n}} D_x^\alpha D_\xi^\beta[Q_{N,R,r}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}]d\eta dy. \tag{3.33}$$

The estimate (3.26) follows immediately from (3.31). This finishes the proof. □

The proof of Lemma 3.4 shows the following fact.

COROLLARY 3.5. *Let $L_N(\cdot,\cdot)$ and $L_{N,R,r}(\cdot,\cdot)$ be defined by* (3.22) *and* (3.30), *resp. Then for any $N\geqq n$ and $r\in\mathbf{N}$ there exists a constant $R\in\mathbf{N}$ so that*

$$L_N(x,\xi)=L_{N,R,r}(x,\xi).$$

REMARK. Since $L_N(\cdot,\cdot)$ obeys the estimate (3.26) we know that the operator $L_N(x,D)$ defined by

$$(L_N(x,D)\varphi)(x)=(2\pi)^{-n}\int_{\mathbf{R}^n} L_N(x,\xi)(F\varphi)(\xi)e^{i(\xi,x)}d\xi$$

maps S into $C^\infty(\mathbf{R}^n)$ (cf. [5]).

In fact, we are able to show

THEOREM 3.6. *Suppose that $Q(\cdot,\cdot,\cdot)\in C^\infty(\mathbf{R}^{3n})$ such that with a constant $\delta<1$ the estimate* (2.4) *holds. Let $L_N(\cdot,\cdot)$ be defined by* (3.22). *Then for any $N\geqq n$ one has*

$$Q_N=L_N(x,D). \tag{3.34}$$

PROOF. In view of (3.20), for $N \geqq n$ the function

$$(y,\xi) \to |Q_N(x,y,\xi)(F\varphi)(\xi)e^{i(y,\xi)}e^{i(x-y,\eta)}|$$

is integrable and so by the Fubini Theorem

$$\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta)}\Big(\int_{\mathbf{R}^n}(F\varphi)(\xi)e^{i(y,\xi)}d\xi\Big)dy =$$

$$= \int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta)+i(y,\xi)}dy\Big)(F\varphi)(\xi)d\xi.$$

For any $\alpha \in \mathbf{N}_0^n$ by (3.20)

$$\left|(\eta-\xi)^\alpha \int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta-\xi)}dy\right| \leqq \tag{3.35}$$

$$\leqq \int_{\mathbf{R}^n} |(D_y^\alpha Q_N)(x,y,\eta)|dy \leqq C_{0,\alpha,0}k_{N_0,N+\delta|\alpha|}(\eta)\int_{\mathbf{R}^n}\left(1+|x-y|^{2N}\right)^{-1}dy$$

and so with a suitable constant $C_m > 0$

$$\left|\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta-\xi)}dy\right| \leqq C_m k_m(\xi)k_{N_0,N+(\delta-1)m}(\xi)$$

for any $m \in \mathbf{N}$. Choosing m large enough one finds that the function

$$(\eta,\xi) \to \left|\Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta-\xi)}dy\Big)(F\varphi)(\xi)e^{i(x,\xi)}\right|$$

is integrable. Thus by the Fubini Theorem

$$(Q_N\varphi)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^n} Q_N(x,y,\eta)\varphi(y)e^{i(x-y,\eta)}dy\Big]d\eta = \tag{3.36}$$

$$= \int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^n} Q_N(x,y,\eta)\Big(\int_{\mathbf{R}^n}(F\varphi)(\xi)e^{i(\xi,y)}d\xi\Big)e^{i(x-y,\eta)}dy\Big]d\eta =$$

$$= \int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta)+i(y,\xi)}dy\Big)(F\varphi)(\xi)d\xi\Big]d\eta =$$

$$= \int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^n}\Big(\int_{\mathbf{R}^n} Q_N(x,y,\eta)e^{i(x-y,\eta-\xi)}dy\Big)(F\varphi)(\xi)e^{i(x,\xi)}d\eta\Big]d\xi =$$

$$= (2\pi)^{-n}\int_{\mathbf{R}^n} L_N(x,\xi)(F\varphi)(\xi)e^{i(x,\xi)}d\xi = (L_N(x,D)\varphi)(x),$$

as required. □

COROLLARY 3.7. *Suppose that $Q(\cdot,\cdot,\cdot) \in C^\infty(\mathbf{R}^{3n})$ such that with a constant $\delta < 1$ the estimate (2.4) holds. Then there exists a symbol $L(\cdot,\cdot) \in$ $\in C^\infty(\mathbf{R}^{2n})$ such that with constants $C_{\alpha,\beta} > 0$, $\mu_{\alpha,\beta} \in \mathbf{R}$ the estimate*

$$\sup_x |(D_x^\alpha D_\xi^\beta L)(x,\xi)| \leqq C_{\alpha,\beta} k_{\mu_{\alpha,\beta}}(\xi) \quad \text{for all} \quad \xi \in \mathbf{R}^n \tag{3.37}$$

holds and

$$Q = L(x, D). \tag{3.38}$$

PROOF. Choose $L(\cdot,\cdot) = L_N(\cdot,\cdot)$, where $N \geqq n$. Then $L(\cdot,\cdot)$ obeys (3.37) (cf. Lemma 3.4) and $Q_N = L(x, D)$ (cf. Theorem 3.6). Due to Theorem 3.1 one has $Q = Q_N$ for any $N \in \mathbf{N}$ and so the assertion follows. □

4. The identity $\mathbf{L}_{\Phi,\phi}^{M,m} = \mathcal{L}_{\Phi,\phi}^{M,m}$

4.1. One sees easily that the inclusion

$$\mathcal{L}_{\Phi,\phi}^{M,m} \subset \mathbf{L}_{\Phi,\phi}^{M,m} \tag{4.1}$$

holds. In the sequel we establish the converse inclusion

$$\mathbf{L}_{\Phi,\phi}^{M,m} \subset \mathcal{L}_{\Phi,\phi}^{M,m}. \tag{4.2}$$

Let $Q(\cdot,\cdot,\cdot)$ be in $A_{\Phi,\phi}^{M,m}$. Then by Corollary 3.7 we know that

$$Q = L_N(x, D) \quad \text{for} \quad N \geqq n, \tag{4.3}$$

and so our task is to verify that $L_N(\cdot,\cdot) \in S_{\Phi,\phi}^{M,m}$. Fix $N \geqq n$ and define

$$b_j(x, y, \xi, \eta) := Q_N(x, y, \eta)\Theta_j(y)\Theta_j(\eta) \tag{4.4}$$

and

$$L_j(x,\xi) := \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b_j(x, y, \xi, \eta) e^{i(x-y,\eta-\xi)} dy d\eta. \tag{4.5}$$

We show that

$$|(D_x^\alpha D_\xi^\beta L_j)(x,\xi)| \leqq C_{\alpha,\beta}\Phi^{M-|\beta|}(x,\xi)\phi^{m-|\alpha|}(x,\xi) \tag{4.6}$$

and

$$(D_x^\alpha D_\xi^\beta L_j)(x,\xi) \to (D_x^\alpha D_\xi^\beta L_N)(x,\xi), \tag{4.7}$$

which implies $L_N(\cdot,\cdot) \in S_{\Phi,\phi}^{M,m}$.

LEMMA 4.1. *Suppose* $Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}$ *and* $L_j(\cdot,\cdot)$ *is defined by* (4.5). *Then*

$$(4.6)\quad |(D^\alpha_x D^\beta_\xi L_j)(x,\xi)| \leqq C_{\alpha,\beta}\Phi^{M-|\beta|}(x,\xi)\phi^{m-|\alpha|}(x,\xi) \quad \textit{for} \quad x,\xi \in \mathbf{R}^n,$$

where $C_{\alpha,\beta}$ *does not depend on* j.

PROOF. A. A direct computations shows that

$$(4.8)\qquad |(D^\alpha_x D^\beta_y D^\gamma_\eta b_j)(x,y,\xi,\eta)| \leqq$$
$$\leqq C_{\alpha,\beta,\gamma} \sup_{(u_1,\eta),(u_2,\eta)\in E_{x,y}\times\mathbf{R}^n} \Phi^{M-|\gamma|}(u_1,\eta)\phi^{m-|\alpha+\beta|}(u_2,\eta),$$

where $C_{\alpha,\beta,\gamma}$ does not depend on j. Here one must note that with constants C_τ and $C_u > 0$

$$(4.9)\qquad |(D^\tau\Theta_j)(\eta)| \leqq (1/j^{|\tau|})C_\tau(1+|\eta/j|)^{-|\tau|} =$$
$$= C_\tau(j^{|\tau|}+|\eta|)^{-|\tau|} \leqq C_\tau(1+|\tau|)^{-|\tau|} \leqq C_\tau C^{|\tau|}\Phi^{-|\tau|}(u_1,\eta)$$

for all $(u_1,\eta) \in E_{x,y}\times\mathbf{R}^n$ and that

$$(4.10)\qquad |(D^u\Theta_j)(y)| \leqq C_u \quad \text{for all} \quad y \in \mathbf{R}^n.$$

B. Since $b_j(x,y,\xi,\eta)$ is compactly supported in (y,η), the derivative $(D^\alpha_x D^\beta_\xi L_j)(x,\xi)$ exists and

$$(4.11)\qquad (D^\alpha_x D^\beta_\xi L_j)(x,\xi) = \int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n} D^\alpha_x D^\beta_\xi(b_j(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)})dyd\eta =$$

$$= \sum_{u\leqq\alpha}\binom{\alpha}{u}\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n}(D^{\alpha-u}_x b_j)(x,y,\xi,\eta)D^u_x D^\beta_\xi(e^{i(x-y,\eta-\xi)})dyd\eta =$$
$$= \sum_{u\leqq\alpha}\binom{\alpha}{u}\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n}(D^{\alpha-u}_x b_j)(x,y,\xi,\eta)(-D_y)^u(-D_\eta)^\beta(e^{i(x-y,\eta-\xi)})dyd\eta =$$
$$= \sum_{u\leqq\alpha}\binom{\alpha}{u}\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n}(D^{\alpha-u}_x D^u_y D^\beta_\eta b_j)(x,u,\xi,\eta)e^{i(x-y,\eta-\xi)}dyd\eta.$$

For any $u \leqq \alpha$, $v,w \in \mathbf{N}^n_0$ we obtain from (4.8)

$$|(D^{\alpha-u}_x D^{u+v}_y D^{\beta+w}_\eta b_j)(x,y,\xi,\eta)| \leqq$$
$$\leqq C_{\alpha-u,u+v,\beta+w} \sup_{(u_1,\eta),(u_2,\eta)\in E_{x,y}\times\mathbf{R}^n} \Phi^{M-|\beta|-|w|}(u_1,\eta)\phi^{m-|\alpha|-|v|}(u_2,\eta).$$

Using Lemma 2 of [1] we see that

$$\left|\int\limits_{\mathbf{R}^n}\int\limits_{\mathbf{R}^n}(D_x^{\alpha-u}D_y^u D_\eta^\beta b_j)(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}dyd\eta\right| \leqq$$
$$\leqq C_{\alpha,\beta,u}\Phi^{M-|\beta|}(x,\xi)\phi^{m-|\alpha|}(x,\xi),$$

which implies the assertion (4.6). $\square$

LEMMA 4.2. *Suppose that $Q(\cdot,\cdot,\cdot)\in A_{\Phi,\phi}^{M,m}$, $L_j(\cdot,\cdot)$ is defined by (4.5) and $L(\cdot,\cdot):=L_N(\cdot,\cdot)$ for some $N\geqq n$. Then*

(4.7) $$(D_x^\alpha D_\xi^\beta L_j)(x,\xi)\to(D_x^\alpha D_\xi^\beta L)(x,\xi)\quad for\quad (x,\xi)\in\mathbf{R}^{2n}.$$

PROOF. Fix $\alpha,\beta\in\mathbf{N}_0^n$ and $N\geqq n$. Choose $R\in\mathbf{N}$ so large that (3.29) holds. Then

(4.12)
$$(D_x^\alpha D_\xi^\beta L)(x,\xi)=(2\pi)^n\int\limits_{\mathbf{R}^{2n}}D_x^\alpha D_\xi^\beta[Q_{N,R,|\beta|}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}]dyd\eta$$

(cf. (3.33)). Similarly as above (cf. the proof of Theorem 3.1 and Lemma 3.3) we get

(4.13) $$L_j(x,\xi)=(2\pi)^n\int\limits_{\mathbf{R}^{2n}}a_{j,N,R,|\beta|}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}dyd\eta.$$

Here we wrote

$$a_{j,N,R,|\beta|}(x,y,\xi,\eta):=\left(1+|x-y|^{2|\beta|}\right)^{-1}(1+\Delta_\eta^{|\beta|})b_{j,R}(x,y,\xi,\eta),$$

where

$$b_{j,R}(x,y,\xi,\eta):=\left(1+|\eta-\xi|^{2R}\right)^{-1}((1+\Delta_y^R)b_j)(x,y,\xi,\eta).$$

A direct computation shows that

(4.14) $$D_x^\alpha D_\xi^\beta[a_{j,N,R,|\beta|}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}]\underset{j}{\rightarrow}$$
$$\underset{j}{\rightarrow}D_x^\alpha D_\xi^\beta[Q_{N,R,|\beta|}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}].$$

Furthermore, one finds by (3.31) that

$$|D_x^\alpha D_\xi^\beta[a_{j,N,R,|\beta|}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}]|\leqq$$
$$\leqq C_{\alpha,\beta}(1+|\xi|^2)^{(R+|\alpha|)/2}\left(1+|x-y|^{2N}\right)^{-1}k_{-(n+1)}(\eta).$$

Here one must note that by (3.31)

$$|D_x^u D_\xi^v [Q_{N,R,r}(x,y,\xi,\eta)e^{i(x-y,\eta-\xi)}]| \leqq$$
$$\leqq C''_{\alpha,\beta}(1+|\xi|^2)^{(R+|\alpha|)/2}(1+|x-y|^{2N})^{-1}k_{-(n+1)}(\eta)$$

for any $u \leqq \alpha$, $v \leqq \beta$ and that

$$\sup_x |(D^\alpha \Theta_j)(x)| \leqq C_\alpha \quad \text{for any} \quad \alpha \in \mathbf{N}_0^n.$$

Hence the Dominated Convergence Theorem implies by (4.12)–(4.14) that the assertion (4.7) holds. □

THEOREM 4.3. *The identity*

$$\mathbf{L}_{\Phi,\phi}^{M,m} = \mathcal{L}_{\Phi,\phi}^{M,m} \tag{4.15}$$

is valid.

PROOF. Let Q be in $\mathbf{L}_{\Phi,\phi}^{M,m}$. Then by (4.3), $Q = L_N(x,D) =: L(x,D)$ where $N \geqq n$. In view of (4.6) and (4.7) one finds that $L(\cdot,\cdot) \in S_{\Phi,\phi}^{M,m}$. Thus $Q = L(x,D) \in \mathcal{L}_{\Phi,\phi}^{M,m}$. □

4.2. Let G be an open set in $\mathbf{R}^n$. Define a class $A_{\Phi,\phi}^{M,m}(G)$ of amplitudes $Q(\cdot,\cdot,\cdot) \in C^\infty(G \times G \times \mathbf{R}^n)$ by the requirement: $Q(\cdot,\cdot,\cdot) \in A_{\Phi,\phi}^{M,m}(G)$ if and only if for all compact sets $K, K' \subset G$ and $\alpha,\beta,\gamma \in \mathbf{N}_0^n$ there exists a constant $C > 0$ such that

$$\sup_{y \in K'} |(D_x^\alpha D_y^\beta D_\xi^\gamma Q)(x,y,\xi)| \leqq C\Phi^{M-|\gamma|}(x,\xi)\phi^{m-|\alpha+\beta|}(x,\xi) \tag{4.16}$$

for all $(x,\xi) \in K \times \mathbf{R}^n$. Then the operator Q defined for $\varphi \in C_0^\infty(G)$ by

$$(Q\varphi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} Q(x,y,\xi)\varphi(y)e^{i(x-y,\xi)}dy \Big) d\xi, \quad x \in G \tag{4.17}$$

maps $C_0^\infty(G)$ into $C^\infty(G)$. Define a class of operators Q by

$$\mathbf{L}_{\Phi,\phi}^{M,m}(G) := \{Q \mid \text{there exists } Q(\cdot,\cdot,\cdot) \in A_{\Phi,\phi}^{M,m}(G)\}.$$

As well-known, there exists a function $h \in C^\infty(G \times G)$ and an open neighbourhood U of the diagonal $D := \{(x,y) \in G \times G \mid x = y\}$ such that

1° $h(x,y) = 1$ for $(x,y) \in U$,

2° for each compact set $K' \subset G$ there exists a compact set $K'' \subset G$ such that

$$\operatorname{supp} h(x,\cdot) \subset K'' \quad \text{for any} \quad x \in K'$$

and

$$\operatorname{supp} h(\cdot, y) \subset K'' \quad \text{for any} \quad y \in K'.$$

Let $Q(\cdot,\cdot,\cdot)$ be in $A^{M,m}_{\Phi,\phi}(G)$. Then a routine evaluation shows that the function $Q^{\sim}(\cdot,\cdot,\cdot)$ defined by

$$Q^{\sim}(x,y,\xi) = h(x,y)Q(x,y,\xi) \tag{4.18}$$

belongs to $A^{M,m}_{\Phi,\phi}(G)$, as well. Furthermore, the function

$$h_\gamma(x,y) := \begin{cases} (1-h(x,y))/(x-y)^\gamma, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

belongs to $C^\infty(G \times G)$ for any $\gamma \in \mathbf{N}_0^n$. Define

$$R(x,y,\xi) := (1-h(x,y))Q(x,y,\xi)$$

and

$$R_\gamma(x,y,\xi) = h_\gamma(x,y)(D_\xi^\gamma Q)(x,y,\xi).$$

Then

LEMMA 4.4. *The functions $R(\cdot,\cdot,\cdot)$ and $R_\gamma(\cdot,\cdot,\cdot)$ belong to $A^{M,m}_{\Phi,\phi}(G)$ and*

$$R = R_\gamma \quad \textit{for any} \quad \gamma \in \mathbf{N}_0^n. \tag{4.19}$$

PROOF. Since $1-h$ and h belong to $C^\infty(G \times G)$, hence $R(\cdot,\cdot,\cdot)$ and $R_\gamma(\cdot,\cdot,\cdot)$ belong to $A^{M,m}_{\Phi,\phi}(G)$. Furthermore, the relation (4.19) is shown as in the proof of Theorem 3.1. □

THEOREM 4.5. *Suppose that $Q(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}(G)$. Then there exist $Q^{\sim} \in \mathbf{L}^{M,m}_{\Phi,\phi}(G)$ and $R \in \bigcap_{N \in \mathbf{N}} \mathbf{L}^{M-N,m}_{\Phi,\phi}(G)$ such that*

$$Q = Q^{\sim} + R, \tag{4.20}$$

where $Q^{\sim}$ is properly supported.

PROOF. Choose $Q^{\sim}(x,y,\xi) := h(x,y)Q(x,y,\xi)$ and $R(x,y,\xi) := (1-h(x,y))Q(x,y,\xi)$. Then (4.20) is valid and $R = R_\gamma$ for any $\gamma \in \mathbf{N}_0^n$. Since $R_\gamma(\cdot,\cdot,\cdot)$ belongs to $A^{M-N,m}_{\Phi,\phi}(G)$ when $|\gamma| = N$, one gets the assertion. □

REMARK 4.6. For example, any symbol $L(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}(G)$ (cf. [2], p. 176) obeys the estimate (4.16) and so $L(x,D)$ can be expressed in the form $L(x,D) = Q^{\sim} + R$ where $Q^{\sim}$ is properly supported and $R \in \bigcap_{N \in \mathbf{N}} \mathbf{L}^{M-N,m}_{\Phi,\phi}(G)$.

Let $\psi \in C_0^\infty(G)$ be such that $\psi(y) = 1$ in some neighbourhood V of supp φ (where $\varphi \in C_0^\infty(G)$ is given). Furthermore, choose Θ from $C_0^\infty(G)$ such that $\Theta(x) = 1$ in some neighbourhood of a given point $x_0 \in G$. Then

$$(Q^\sim \varphi)(x) = (Q^\sim_{\psi,\Theta})(x) \quad \text{for} \quad x \in V,$$

where

$$Q^\sim_{\psi,\Theta}(x,y,\xi) := \begin{cases} \Theta(x) Q^\sim(x,y,\xi)\psi(y) & \text{for } x,y \in G\times G \\ 0 & \text{for } (x,y) \notin G \times G. \end{cases}$$

Since $Q^\sim_{\psi,\Theta}(\cdot,\cdot,\cdot) \in A^{M,m}_{\Phi,\phi}$, we know from Section 3 that

$$Q^\sim_{\psi,\Theta} = L^\sim_{\psi,\Theta}(x,D),$$

where $L^\sim_{\psi,\Theta}(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}$.

Since $L^\sim_{\psi,\Theta}(x,D)\varphi = Q^\sim_{\psi,\Theta}\varphi = \Theta Q^\sim(\psi\varphi) = \Theta Q^\sim \varphi$ and $R_{\psi,\Theta}(x,D)\varphi = = \Theta R\varphi$ for all $\varphi \in C_0^\infty(V)$ we get that for any $G' \Subset G$ there exist $L^\sim_{G'}(\cdot,\cdot) \in \in S^{M,m}_{\Phi,\phi}(G)$ and $R_{G'}(\cdot,\cdot) \in \bigcap\limits_{N \in \mathbf{N}} S^{M-N,m}_{\Phi,\phi}(G)$ so that

$$Q\varphi = L^\sim_{G'}(x,D)\varphi + R_{G'}(x,D)\varphi \quad \text{for all} \quad \varphi \in C_0^\infty(G').$$

Also, for $Q^\sim$ there exists $L^\sim(\cdot,\cdot) \in S^{M,m}_{\Phi,\phi}(G)$ so that

$$Q^\sim \varphi = L^\sim(x,D)\varphi \quad \text{for all} \quad \varphi \in C_0^\infty(G)$$

(one can choose $L^\sim(x,\xi) := (2\pi)^n \int\limits_{\mathbf{R}^n} \Big(\int\limits_G h(x,y) e^{i(x-y,\eta-\xi)} dy\Big) L(x,\eta) d\eta\Big)$.

In Section 4.2 it suffices to assume that (Φ,ϕ) forms a pair of weight functions only locally in G (cf. [2], p. 176).

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UNIVERSITY OF KUOPIO
DEPARTMENT OF APPL. MATH.
70211 KUOPIO
SF–FINLAND

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PROPERTIES OF HYPERCONNECTED SPACES, THEIR MAPPINGS INTO HAUSDORFF SPACES AND EMBEDDINGS INTO HYPERCONNECTED SPACES

N. AJMAL and J. K. KOHLI (Delhi)

1. Introduction

Professor Levine calls a space X a D-space [5] if every nonempty open subset of X is dense in X, or equivalently every pair of nonempty open sets in X intersect. In the literature D-spaces are frequently referred to as hyperconnected spaces (see for example [9], [10]). In this paper we extend the concept of hyperconnectedness to pointwise hyperconnectedness and use it to study the properties of hyperconnected spaces. We shall call a space X *pointwise hyperconnected* at x in X if each open set containing x is dense in X. It is immediate that a space X is hyperconnected if and only if it is pointwise hyperconnected at each of its points.

It is clear from the definition that the property of being a hyperconnected space is open hereditary. In fact every subset of a hyperconnected space having a nonempty interior is hyperconnected in its relative topology. In particular, every β-subset [6] of a hyperconnected space is hyperconnected. However, in the sequel, Example 2.3 shows that hyperconnectedness is not even closed hereditary. This corrects an error in [5] where it is erroneously stated that hyperconnectedness is hereditary (see [5, Theorem 2(1)]). More generally we shall show that every topological space can be realized as a closed subspace of a hyperconnected space (see Theorem 3.1).

Section 2 is devoted to the properties of (pointwise) hyperconnected spaces. We show that hyperconnectedness is preserved under feebly continuous surjections and inversely preserved under feebly open injections while pointwise hyperconnectedness is invariant under continuous surjections. Moreover, we prove that (pointwise) hyperconnectedness is productive and that every subset of a hyperconnected space having a nonempty interior is hyperconnected. Furthermore, we show that a space is hyperconnected if and only if every feebly continuous function from it into a Hausdorff space is constant and that every continuous function from a pointwise hyperconnected space into a Hausdorff space is constant. In the process we improve/generalize certain results of Noiri [7], Pipitone and Russo [8], and Levine [5].

In Section 3, we show that every topological space can be realized as a closed subspace of a hyperconnected space called "hyperconnectification".

Beside discussing basic properties of hyperconnectifications, we also reflect upon their functorial nature.

The closure of a subset A of a topological space X will be denoted either by $\overline{A}$ or $\mathrm{Cl}_X A$ or $\mathrm{Cl}\,A$; and the interior of a subset B of X will be denoted either by B^0 or $\mathrm{int}_X B$ or $\mathrm{int}\,B$.

An open set in a space is said to be *regular open* if it is the interior of its closure.

2. Properties of (pointwise) hyperconnected spaces, characterizations and mappings into Hausdorff spaces

First we give some illustrative examples which either reflect upon the theory or will be referred to in the sequel.

EXAMPLE 2.1. Let X denote $\mathbf{N}$, the set of natural numbers, endowed with the topology generated by taking basic neighbourhoods of each $n \in \mathbf{N}$ the set $\{1, 2, \dots, n\}$. The space X is a T_0 second countable, hyperconnected space.

EXAMPLE 2.2. Let $X = \{a, b, c\}$ and $\mathcal{I} = \{\{a\}, \{b\}, \{a, b\}, X, \emptyset\}$. The space $(X, \mathcal{I})$ is pointwise hyperconnected at c but neither at a nor at b.

EXAMPLE 2.3. Let Y denote the closed unit interval $[0, 1]$ equipped with the usual topology $\mathcal{U}$. Let $X = Y \cup \{w\}$, where $w \notin Y$. A topology on X is defined by declaring $V \subset X$ to be open if either V is empty or $V = U \cup \{w\}$, for some $U \in \mathcal{U}$. The space X is a hyperconnected space. The relative topology of $\left[0, \frac{1}{2}\right]$ which it inherits as a subspace of X coincides with the Euclidean topology. Thus the closed set $\left[0, \frac{1}{2}\right]$ is not hyperconnected in its relative topology. This example shows that hyperconnectedness is not a closed hereditary property.

PROPOSITION 2.1. *For a topological space X, the set of all points where X is pointwise hyperconnected is a closed subset of X.*

PROOF. Let F denote the set of all points where X is pointwise hyperconnected. To show that F is closed, we shall show that $X - F$ is open. To this end, let $x \in X - F$. Then there is an open set U containing x such that $\overline{U} \neq X$ and so $U \subset X - F$. Thus $X - F$ being the union of open sets is open.

THEOREM 2.2. *For a topological space X the following statements are equivalent:*

(1) *X is pointwise hyperconnected at x.*

(2) *Every nonempty open set intersects every open set containing x.*

(3) *Every open set containing x is connected.*

(4) *Every closed subset of X not containing x is nowhere dense in X.*

(5) *X is the only regular open set containing x.*

PROOF. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are easy. To see that (3) $\Rightarrow$ (4), let F be a nonempty closed set not containing x. Then $X - F$ is an open set containing x. Now, if F is not nowhere dense, then F^0 is nonempty and so $F^0 \cup (X - F) = U$ is an open set containing x which is not connected.

To show that (4) $\Rightarrow$ (1), let U be an open set containing x. Now, if U is not dense in X, then there exists a nonempty open set V disjoint from U and so $X - U$ is a nonempty closed set not containing x which fails to be nowhere dense. Hence U is dense in X and thus X is pointwise hyperconnected at x.

Equivalence of the conditions (1) and (5) is straightforward.

COROLLARY 2.3. *If X is pointwise hyperconnected at x, then X is a connected space which is locally connected at x.*

COROLLARY 2.4 [10]. *A hyperconnected space is connected and locally connected.*

THEOREM 2.5. *If $f\colon X \to Y$ is a continuous surjection and X is pointwise hyperconnected at x, then Y is pointwise hyperconnected at $f(x)$.*

PROOF. Let $V \subset Y$ be any open set containing $f(x)$. Then $f^{-1}(V)$ is an open set containing x and so $f^{-1}(V)$ is dense in X. Since continuous surjections preserve dense sets, $V = f(f^{-1}(V))$ is dense in Y and thus Y is pointwise hyperconnected at $f(x)$.

DEFINITION 2.1 [2]. A function $f\colon X \to Y$ from a topological space X into a topological space Y is said to be *feebly continuous* if for every open set V of Y, $f^{-1}(V) \neq \emptyset$ implies that int $f^{-1}(V) \neq \emptyset$.

In the above definition a feebly continuous function is not necessarily assumed to be surjective as assumed in Frolík's original definition [2].

DEFINITION 2.2 [4]. A set H in a topological space X is said to be *semi-open* if there exists an open set U in X such that $U \subset H \subset \overline{U}$. A function $f\colon X \to Y$ from a topological space X into a topological space Y is said to be *semi-continuous* if the inverse image of every open subset of Y is semi-open.

PROPOSITION 2.6. *Every semi-continuous function is feebly continuous.*

PROOF. Suppose $f\colon X \to Y$ is semi-continuous and let V be any open subset of Y such that $f^{-1}(V) \neq \emptyset$. In view of semi-continuity of f, $f^{-1}(V)$ is a nonempty semi-open set in X and hence there exists a nonempty open set U in X such that $U \subset f^{-1}(V) \subset \overline{U}$. Clearly $\emptyset \neq U \subset$ int $f^{-1}(V)$ and thus int $f^{-1}(V) \neq \emptyset$. So f is feebly continuous.

The following example shows that the converse of Proposition 2.6 is false.

EXAMPLE 2.4. Let $X = Y = \{1,2,3,4\}$ and let X be endowed with the topology $\mathcal{I} = \{\emptyset, \{3\}, \{1,4\}, \{1,3,4\}, X\}$ and Y be endowed with the topology $\mathcal{V} = \{\emptyset, \{1,2,3\}, Y\}$. Let f denote the identity mapping of X onto Y. Then f is feebly continuous but not semi-continuous.

THEOREM 2.7. *If $f: X \to Y$ is a feebly continuous surjection from a hyperconnected space X onto Y, then Y is hyperconnected.*

PROOF. Let U and V be any nonempty open sets in Y. Since f is a feebly continuous surjection, int $f^{-1}(U)$ and int $f^{-1}(V)$ are nonempty open sets in X. By hyperconnectedness of X, the sets int $f^{-1}(U)$ and int $f^{-1}(V)$ have a nonempty intersection. Let $x \in \operatorname{int} f^{-1}(U) \cap \operatorname{int} f^{-1}(V) \subset f^{-1}(U) \cap$ $\cap f^{-1}(V)$. Then $f(x) \in U \cap V$. Thus any pair of nonempty open sets in Y have a nonempty intersection and so Y is hyperconnected.

COROLLARY 2.8 (Noiri [7]). *If X is hyperconnected and $f: X \to Y$ is a semi-continuous surjection, then Y is hyperconnected.*

PROOF. It is immediate in view of Theorem 2.7 and Proposition 2.6.

COROLLARY 2.9 (Levine [5]). *A continuous image of a hyperconnected space is hyperconnected.*

PROOF. It is immediate from Theorem 2.5 or Theorem 2.7.

COROLLARY 2.10 (Pipitone and Russo [12]). *If X is hyperconnected and $f: X \to Y$ is a semi-continuous surjection, then Y is connected.*

PROOF. It is immediate from Corollary 2.8.

REMARK 2.1. In view of Theorem 2.7, at a first glance, one might conjecture that Theorem 2.5 remains true if the term "continuous function" is replaced by "feebly continuous function". However, this conjecture is immediately put to rest by the following example.

EXAMPLE 2.5. Let X be the space of Example 2.2 and let Y be the two point discrete space $\{0,1\}$. Let $f: X \to Y$ be defined by $f(a) = f(c) = 0$ and $f(b) = 1$. Then f is a semi-continuous function and X is pointwise hyperconnected at c but Y is nowhere pointwise hyperconnected. Thus pointwise hyperconnectedness is not preserved even under semi-continuous surjections.

DEFINITION 2.3 [2]. A function $f: X \to Y$ is said to be *feebly open* if for every nonempty open set U in X, there exists a nonempty open set V in Y such that $V \subset f(U)$.

DEFINITION 2.4 [1]. A function $f: X \to Y$ is said to be *semi-open* if $f(U)$ is semi-open for every open set U in X.

It is easily seen that every semi-open function is feebly open. However, the converse is not true as is shown by the following example.

EXAMPLE 2.6. Let $X = Y = \{a,b,c,d\}$. Let $\mathcal{I} = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, Y\}$ and let $\mathcal{V} = \{\emptyset, \{a,c,d\}, X\}$. Let X be endowed with the topology $\mathcal{V}$ and Y be equipped with the topology $\mathcal{I}$ and let f denote the identity mapping of X onto Y. Then f is a feebly open surjection which is not semi-open.

THEOREM 2.11. *If Y is hyperconnected and $f: X \to Y$ is a feebly open injection, then X is hyperconnected.*

PROOF. Let U and V be any two nonempty open sets in X. Then since f is feebly open, $\operatorname{int} f(U) \neq \emptyset \neq \operatorname{int} f(V)$. Since Y is hyperconnected, $\operatorname{int} f(U) \cap \operatorname{int} f(V) \neq \emptyset$ and hence $f(U) \cap f(V) \neq \emptyset$. Since f is one-one, $f(U) \cap f(V) = f(U \cap V)$ and so $U \cap V \neq \emptyset$. Thus X is hyperconnected.

COROLLARY 2.12 (Noiri [7], Theorem 3.3). *If Y is hyperconnected and $f: X \to Y$ is a semi-open injection, then X is hyperconnected.*

COROLLARY 2.13 (Levine [5], Theorem 4). *If Y is hyperconnected and $f: X \to Y$ is an open injection, then X is hyperconnected.*

DEFINITION 2.5 [6]. Let X be a topological space and let $A \subset X$. Then A is said to be

(a) an *α-set* if $A \subset \operatorname{int}(\operatorname{Cl}(\operatorname{int} A))$,

and

(b) a *β-set* if $A \subset \operatorname{Cl}(\operatorname{int} A)$.

Every open set is an α-set, every α-set is semi-open, every semi-open set is a β-set and every nonempty β-set has a nonempty interior. However, none of these implications can be reversed (for details see [4], [6]).

THEOREM 2.14. *Let X be a hyperconnected space and let $A \subset X$. If A has nonempty interior, then A is hyperconnected in its relative topology. In particular, every β-subset of a hyperconnected space is hyperconnected.*

PROOF. Let V be any nonempty open set in A. Then $V = U \cap A$, where U is a nonempty open set in X. Since $\operatorname{int}_X A \neq \emptyset$ and since X is hyperconnected, $U \cap \operatorname{int}_X A$ is a nonempty open set in X and hence it is dense in X. Therefore,

$$X = \operatorname{Cl}_X(U \cap \operatorname{int}_X A) \subset \operatorname{Cl}_X(U \cap A) = X.$$

Since $\operatorname{Cl}_A V = \operatorname{Cl}_X(U \cap A) \cap A$, it follows that V is dense in A and thus A is hyperconnected.

COROLLARY 2.15 (Pipitone and Russo [8, Theorem 6.3]). *Every semi-open subset of a hyperconnected space is hyperconnected.*

Mappings into Hausdorff spaces

THEOREM 2.16. *For a topological space X the following statements are equivalent.*

(a) *X is hyperconnected.*

(b) *Every feebly continuous function from X into a Hausdorff space is constant.*

(c) *Every feebly continuous function from X into the two point discrete space $\{0, 1\}$ is constant.*

(d) *Every semi-continuous function from X into the two point discrete space $\{0,1\}$ is constant.*

(e) *Every semi-continuous function from X into a Hausdorff space is constant.*

PROOF. (a) $\Rightarrow$ (b). Let $f\colon X \to Y$ be a feebly continuous function from X into a Hausdorff space Y. Suppose f is not constant. Then there exist $x, y \in X$ such that $f(x) \neq f(y)$. By Hausdorfness of Y, there are disjoint open sets U and V containing $f(x)$ and $f(y)$, respectively and so $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint sets. Since f is feebly continuous, it follows that int $f^{-1}(U)$ and int $f^{-1}(V)$ are nonempty disjoint open sets contradicting the fact that X is hyperconnected.

The implication (b) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (d) is immediate in view of the fact that every semi-continuous function is feebly continuous.

(d) $\Rightarrow$ (e). Let $g\colon X \to Y$ be a semi-continuous function from X into a Hausdorff space Y. Suppose g is not constant. Then there exist $x, y \in X$ such that $g(x) \neq g(y)$. In view of Hausdorffness of Y, there are disjoint open sets U_1 and U_2 containing $g(x)$ and $g(y)$, respectively. Since g is semi-continuous, $g^{-1}(U_1)$ and $g^{-1}(U_2)$ are nonempty disjoint semi-open sets in X and so there exist nonempty open sets U and V in X such that

$$U \subset g^{-1}(U_1) \subset \overline{U}, \quad V \subset g^{-1}(U_2) \subset \overline{V} \quad \text{and} \quad \overline{U} \cap V = \emptyset.$$

Let $f\colon X \to \{0,1\}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \overline{U}, \\ 1, & \text{if } x \in X - \overline{U}. \end{cases}$$

Clearly, f is a semi-continuous surjection onto the two point discrete space $\{0,1\}$.

(e) $\Rightarrow$ (a). Suppose X is not hyperconnected. Then there exists a nonempty open set U in X such that $\overline{U} \neq X$. Define $f\colon X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \overline{U}, \\ 1, & \text{if } x \in X - \overline{U}. \end{cases}$$

Then f is a non-constant semi-continuous function from X onto the Hausdorff space $\{0,1\}$. This contradiction to (e) completes the proof of the theorem.

REMARK 2.2. Equivalence of the assertions (a) and (d) in the above theorem is due to Noiri [7].

A continuous function into a Hausdorff space is completely determined by its values on a dense set. However, in the case of pointwise hyperconnected spaces, the following stronger result holds.

THEOREM 2.17. *Let X be a pointwise hyperconnected space at a point $x \in X$. Then every continuous function from X into a Hausdorff space is constant. In particular, every continuous real-valued function on X is constant.*

PROOF. Let $f\colon X \to Y$ be a continuous function from X into a Hausdorff space Y. Suppose f is not constant. Then there exists $p \in X$ such that $f(x) \neq f(p)$. Since Y is Hausdorff, there are disjoint open sets U and V containing $f(x)$ and $f(p)$, respectively. By continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint open sets such that $x \in f^{-1}(U)$. In view of Theorem 2.2 this contradicts the fact that X is pointwise hyperconnected at x.

COROLLARY 2.18. *If X is pointwise hyperconnected at a point, then X is ultra-pseudo-compact.*

REMARK 2.3. The above corollary improves an observation of [10], p. 30.

PROBLEM 2.1. Give an example of a space which is not pointwise hyperconnected at any of its points and such that every continuous function from X into every Hausdorff space is constant.

We may point out that it was proved in [3] that for every T_1-space Y a regular T_1-space X exists such that every continuous function from X to Y is necessarily constant. Thus, if $Y = \{0, 1\}$, the two point discrete space, there exists a nondegenerate regular T_1-space X such that every continuous function from X into Y is constant; clearly, such a space X cannot be pointwise hyperconnected at any of its points. However, Problem 2.1 remains open.

Products

THEOREM 2.19. *A product space πX_α is pointwise hyperconnected at a point $x = (x_\alpha)$ if and only if each factor space X_α is pointwise hyperconnected at x_α.*

PROOF. Since a projection onto a factor space is a continuous surjection, necessity is immediate in view of Theorem 2.5. To prove sufficiency, let $X = \pi X_\alpha$ and suppose that each X_α is pointwise hyperconnected at x_α. It suffices to show that each basic open set containing $x = (x_\alpha)$ is dense in X. To this end, let U be a basic open set in X containing x. Then $U = \pi U_\alpha$, where $U_\alpha = X_\alpha$ for all but finitely many α. Since $\overline{U} = \pi \overline{U}_\alpha$ and since each X_α is pointwise hyperconnected at x_α, $\overline{U}_\alpha = X_\alpha$ for all α and so $\overline{U} = X$. Thus X is pointwise hyperconnected at x.

COROLLARY 2.20 (Levine [5]). *A product space is hyperconnected if and only if each factor space is hyperconnected.*

3. Embeddings into hyperconnected spaces

THEOREM 3.1. *Every topological space can be embedded as a nowhere dense closed subspace into a hyperconnected space.*

PROOF. Let $(X,\mathcal{I})$ be a topological space. Let $X^* = X \cup \{\infty\}$, where $\infty \notin X$ and let $\mathcal{I}^* = \{U \cup \{\infty\} : U \in \mathcal{I}\} \cup \{\emptyset\}$. The collection $\mathcal{I}^*$ is a topology for X^*. Since the closure of every nonempty open set in X^* is X^*, the space X^* is hyperconnected. Moreover, the inclusion map $i: X \to X^*$ is a closed embedding and hence we may consider X as a closed subspace of X^*. Clearly X is nowhere dense in X^*.

The space X^* constructed in Theorem 3.1 is called one point hyperconnectification of X. In the sequel, the notation X^* will always have the same meaning as in Theorem 3.1.

COROLLARY 3.2. *Every topological space can be embedded as a closed subspace into a connected and locally connected space.*

PROOF. It is immediate in view of Theorem 3.1 and Corollary 2.4.

THEOREM 3.3. *If $f: X \to Y$ is a continuous function from a space X into a space Y, then there exists a continuous extension $f^*: X^* \to Y^*$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_x"'] & Y \arrow[d, "i_y"] \\ X^* \arrow[r, "f^*"] & Y^* \end{tikzcd}$$

Moreover, if f has any one of the following properties, so does f^*:

(a) *open,* (b) *closed surjection,* (c) *homeomorphism,* (d) *quotient.*

PROOF. Suppose $X^* = X \cup \{\infty_x\}$ and $Y^* = Y \cup \{\infty_y\}$. Let $f^*: X^* \to Y^*$ be the function whose restriction to X is f and $f^*(\infty_x) = \infty_y$. Now, commutativity of the given diagram is obvious. To show that f^* is continuous, let W be an open set in Y^*. Then $W = V \cup \{\infty_y\}$, where V is open in Y. Thus, $f^{*-1}(W) = f^{-1}(V) \cup \{\infty_x\}$ and since f is continuous, $f^{-1}(V)$ is open in X and so $f^{*-1}(W)$ is open in X^*. This proves that f^* is continuous.

(a) Suppose f is an open function and let W be an open set in X^*. Then $W = U \cup \{\infty_x\}$, where U is open in X, and so $f(U)$ is open in Y. Thus, $f^*(W) = f(U) \cup \{\infty_y\}$ is open in Y^*.

(b) Suppose f is a closed surjection and let F be a closed set in X^*. Obviously, f^* is a surjection. If $F = X^*$, then $f^*(F) = Y^*$ which is closed. Assume $F \neq X^*$. Then $\infty_x \notin F$ and $F = F \cap X$ is closed in X. Since f is a closed function, $f(F) = f^*(F)$ is closed in Y. Now, $\infty_y \notin f^*(F)$ and so $Y^* \setminus f^*(F) = (Y \setminus f(F)) \cup \{\infty_y\}$ is open in Y^*. Hence $f^*(F)$ is closed in Y^*.

(c) In case f is a homeomorphism, the result follows as in the case (a) or (b) together with the fact that f^* is a bijection whenever f is.

(d) We omit the easy proof in this case.

Let $\mathcal{C}$ denote the category of topological spaces and continuous functions and let $\mathcal{S}$ be its full subcategory consisting of hyperconnected spaces. Define $F\colon \mathcal{C} \to \mathcal{S}$ by $F(X) = X^*$ for each object X in $\mathcal{C}$ and $F(f) = f^*$ for each morphism f in $\mathcal{C}$. We point out that F is a functor and leave the simple verification to the reader.

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DEPARTMENT OF MATHEMATICS
HINDU COLLEGE
UNIVERSITY OF DELHI
DELHI–110007, INDIA

DEPARTMENT OF MATHEMATICS
ZAKIR HUSSAIN COLLEGE
UNIVERSITY OF DELHI
DELHI–110006, INDIA

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ON THE SIMPLICITY OF SANDWICH NEAR-RINGS

R. D. HOFER (Plattsburgh) and K. D. MAGILL, Jr. (Buffalo)

1. Introduction

It will be assumed throughout this paper that all topological groups are Hausdorff which means, of course that they are also completely regular. We will also assume, without further comment, that the groups discussed here have more than one element. Let X be a topological space, H an additive topological group and α a continuous function from H into X. Denote by $N(X, H, \alpha)$ the near-ring of all continuous functions from X into H where addition of functions is pointwise and the product fg of two functions $f, g \in \in N(X, H, \alpha)$ is defined by $fg = f \circ \alpha \circ g$. Denote by $N_0(X, H, \alpha)$ the subnear-ring of $N(X, H, \alpha)$ which consists of all those continuous functions f with the property that $f(\alpha(0)) = 0$. Then $\langle 0 \rangle$, the constant function which maps all of X into $0 \in H$, is the additive identity of $N(X, H, \alpha)$ and $N_0(X, H, \alpha)$ is the largest subnear-ring of $N(X, H, \alpha)$ for which $\langle 0 \rangle$ is a two-sided multiplicative zero. We refer to a near-ring of the form $N(X, H, \alpha)$ as a sandwich near-ring and we use the term Z-sandwich near-ring for one of the form $N_0(X, H, \alpha)$. In this paper, we investigate the simplicity of both sandwich near-rings and Z-sandwich near-rings.

The first results concerning the simplicity of near-rings of functions were obtained a long time ago. If $X = H$ and one takes α to be the identity map then $N(X, H, \alpha)$ is simply $N(H)$, the near-ring of all continuous selfmaps of the topological group H and $N_0(X, H, \alpha)$ is just $N_0(H)$ the near-ring of all continuous selfmaps which fix 0. These early results were obtained by Berman and Silverman [1] and Nobauer and Philipp [5] and dealt with $N(H)$ and $N_0(H)$. Specifically, it was shown that if H is discrete, then $N_0(H)$ is simple and so is $N(H)$ except in the one case where H is of order two. The first author of this paper then showed that within a huge class of topological groups, the converse holds for $N_0(H)$ [2]. That is, if H is any one of these groups, then $N_0(H)$ is simple if and only if H is discrete. He also showed in [2] that there are many nondiscrete groups H for which $N(H)$ is simple. The second author of this paper modified some of the techniques used in [2] and produced in [3] still another class of groups such that $N(H)$ is simple if H is any one of these groups. In particular, $N(R^N)$ is simple where R^N is the additive topological group of Euclidean N-space.

The results obtained to date indicate that while it is rare for $N_0(H)$ to be simple, it is not so rare for $N(H)$ to be simple. In this paper we look at particular classes of sandwich near-rings and Z-sandwich near-rings. We show that such a sandwich near-ring is simple if and only if it is isomorphic to some $N(H)$ where, of course, H is one of the groups such that $N(H)$ is simple. Various other equivalent conditions are also given for one of these sandwich near-rings to be simple. For example, one such condition is that the near-ring have a multiplicative identity. Analogous results are obtained for Z-sandwich near-rings.

2. Sandwich near-rings

Throughout the paper the symbol $\langle x\rangle$ will be used to denote a constant function which maps everything into the point x. The domain of $\langle x\rangle$ will vary but will be evident from context.

LEMMA 2.1. *Let X be a completely regular Hausdorff space, H a topological group and suppose that either H contains an arc or X is 0-dimensional. Let α be a continuous function from H into X and suppose $N(X,H,\alpha)$ is simple. Then* Ran α *(the range of α) is a dense subspace of X.*

PROOF. Suppose $\operatorname{Ran}\alpha$ is not dense in X and define a map φ from $N(X,H,\alpha)$ to $N(H)$ by $\varphi(f) = f\circ\alpha$. One verifies in a straightforward manner that φ is a homomorphism from $N(X,H,\alpha)$ into $N(H)$. Moreover, it is nontrivial since $\varphi[N(X,H,\alpha)]$ contains all the constant functions mapping H into H. Specifically, if $y\in H$ and $\langle y\rangle$ denotes the constant function mapping all of X into y, then $\varphi(\langle y\rangle) = \langle y\rangle\circ\alpha$ is the constant function which maps all of H into y. Since Ran α is not dense in X, we can choose $p\in X$-clRan α. If H contains an arc, then it also contains an arc A with one endpoint 0 and the other endpoint $y\neq 0$. Since X is completely regular and Hausdorff, there exists a continuous function f from X into H such that $f(x) = 0$ for $x\in$ clRan α and $f(p) = y$. If H does not contain an arc, then X is 0-dimensional and there exists a clopen (simultaneously closed and open) set V such that $p\in V\subsetneqq$ clRan α. In this case, choose any $y\in$ $\in H$, $y\neq 0$ and define $f(x) = y$ for $y\in V$ and $f(x) = 0$ for $x\in X-V$. In either event we have a function $f\in N(X,H,\alpha)$ such that $f\neq\langle 0\rangle$ but $\varphi(f) = f\circ\alpha = \langle 0\rangle = \varphi(\langle 0\rangle)$. Thus $\varphi[N(X,H,\alpha)]$ is a proper homomorphic image of $N(X,H,\alpha)$. This is, of course, a contradiction and we conclude that Ran α is, indeed, dense in X.

COROLLARY 2.2. *Let G be any abstract group, let H be a proper subgroup and let $N(G,H)$ denote the near-ring of all functions from G into H under pointwise addition and composition. Then $N(G,H)$ is not simple.*

PROOF. Let G and H have the discrete topologies and define $\alpha(y) = y$ for all $y\in H$. Then $N(G,H)$ is just the sandwich near-ring $N(G,H,\alpha)$ and since Ran α is not dense in G, the conclusion follows from Lemma 2.1.

In much of what follows X will also be a topological group and when this is the case we will emphasize it by replacing X by the symbol G. Nevertheless, $N(G,H,\alpha)$ will still be the near-ring of all continuous functions from G into H with pointwise addition and multiplication defined by $fg = f \circ \alpha \circ g$. When we speak of a homomorphism α from the topological group H into the topological group G it will be assumed, as is customary, that α is also continuous and an isomorphism from H into G will be assumed to also be a homeomorphism into G. When α is not necessarily a homeomorphism we will refer to it as an algebraic isomorphism. Finally, we will use the symbol 0 to denote the identity of G as well as that of H.

DEFINITION 2.3. Let G and H be topological groups and let α be a homomorphism from H into G. We will say that the triple (G,H,α) is *compatible* if the following two conditions are satisfied.

(2.3.1) Either H contains an arc or G is 0-dimensional.

(2.3.2) Either $\operatorname{Ker}\alpha$ is pathwise connected or G is 0-dimensional.

LEMMA 2.4. *Let (G,H,α) be a compatible triple and suppose $N(G,H,\alpha)$ is simple. Then α is an algebraic isomorphism from H onto a dense subgroup of G.*

PROOF. $\alpha[H]$ is dense in view of Lemma (2.1). We need only show that α is injective. Define a map φ from $N(G,H,\alpha)$ into $N(G)$ by $\varphi(f) = \alpha \circ f$. Since α is a group homomorphism, it readily follows that φ is a homomorphism from the near-ring $N(G,H,\alpha)$ into the near-ring $N(G)$. Since, by assumption, our groups contain more than one point and since $\alpha[H] = \operatorname{Ran}\alpha$ is dense in G, it follows that $\operatorname{Ran}\alpha$ contains more than one point. For each $x \in \operatorname{Ran}\alpha$ choose $y \in H$ such that $\alpha(y) = x$. Then $\langle y\rangle \in N(G,H,\alpha)$ and $\varphi(\langle y\rangle) = \alpha \circ \langle y\rangle = \langle \alpha(y)\rangle = \langle x\rangle$ where the domain of $\langle x\rangle$ is G. Thus, $\varphi[N(G,H,\alpha)]$ contains all $\langle x\rangle$ such that $x \in \operatorname{Ran}\alpha$ and is therefore a nontrivial near-ring. Now suppose α is not injective. Then $\alpha(y) = 0$ for some $y \in H$, $y \neq 0$. Choose two distinct points $a, b \in G$. If $\operatorname{Ker}\alpha$ is pathwise connected then there is an arc $A \subseteqq \operatorname{Ker}\alpha$ with endpoints 0 and y respectively. Since G is completely regular and Hausdorff there exists a continuous function f from G into A such that $f(a) = 0$ and $f(b) = y$. If $\operatorname{Ker}\alpha$ is not pathwise connected then G is 0-dimensional and there exists a clopen set V containing a but not b. In this case, define $f(x) = 0$ for $x \in V$ and $f(x) = y$ for $x \in G - V$. In either event $f \in N(G,H,\alpha)$, $f \neq \langle 0\rangle$ and $\operatorname{Ran} f \subseteqq \operatorname{Ker}\alpha$. Thus $\varphi(f) = \alpha \circ f = \langle 0\rangle$. That is, φ is a proper homomorphism from $N(G,H,\alpha)$ onto a nontrivial subnear-ring of $N(G)$. Because of this contradiction, we conclude that α is injective.

At this point we actually have all the preliminary results we need in order to get the main result of this section. However, at one point in the proof of that result we would, as things now stand, find it necessary to appeal to Corollary (3.3) of [4], a result about semigroups which applies to sandwich near-rings. The problem is that it does not apply to Z-sandwich near-rings and so is of no use to us in Section 3 when we deal with these near-rings. What we need are generalizations of some results in [4] and it seems appropriate to get them now as they apply to both sandwich near-rings and Z-sandwich near-rings.

Let X and Y be topological spaces and let α be a continuous function from Y into X. $S(X,Y,\alpha)$ denotes the semigroup of all continuous functions from X into Y where the product fg is defined by $fg = f \circ \alpha \circ g$. The semigroup $S(X,Y,\alpha)$ is referred to as a *sandwich semigroup*.

DEFINITION 2.5. A subsemigroup $T(X,Y,\alpha)$ of $S(X,Y,\alpha)$ is said to be an *adequate semigroup* if whenever $\alpha(y) = x$ then $f(x) = y$ for some $f \in$ $\in T(X,Y,\alpha)$.

LEMMA 2.6. *Let $T(X,Y,\alpha)$ be an adequate semigroup which has a left identity. Then α maps Y homeomorphically onto a retract of X.*

PROOF. Let l be a left identity of $T(X,Y,\alpha)$. Then $\alpha \circ l$ is a continuous selfmap of X and since

$$(\alpha \circ l) \circ (\alpha \circ l) = \alpha \circ (l \circ \alpha \circ l) = \alpha \circ (ll) = \alpha \circ l$$

we see that $\alpha \circ l$ is an idempotent continuous selfmap of X with respect to composition. Thus Ran $\alpha \circ l$ is a retract of X. Suppose $\alpha(y) = x$ and let f be any function in $T(X,Y,\alpha)$ such that $f(x) = y$. We then have

$$\begin{gathered}(\alpha \circ l)(x) = \alpha \circ l \circ \alpha(y) = \alpha \circ l \circ \alpha \circ f(x) = \\ = \alpha \circ (lf)(x) = \alpha \circ f(x) = \alpha(y) = x.\end{gathered}$$

It follows that Ran $\alpha \subseteqq$ Ran $\alpha \circ l$ and $\alpha \circ l$ is the identity on Ran α. Evidently, Ran $\alpha \circ l \subseteqq$ Ran α so, in fact, we have Ran $\alpha =$ Ran $\alpha \circ l$ and we conclude that Ran α is a retract of X. Now choose any $y \in Y$ and let g be any function in $T(X,Y,\alpha)$ such that $g(\alpha(y)) = y$ and we have

$$(l \circ \alpha)(y) = l \circ \alpha \circ g \circ \alpha(y) = (lg)(\alpha(y)) = g(\alpha(y)) = y.$$

Thus, we have shown that not only is $\alpha \circ l$ the identity on Ran α but $l \circ \alpha$ is the identity on Y and it follows that α is a homeomorphism from Y onto the retract Ran α.

We recall that a collection $\mathcal{F}$ of functions from X into Y is said to *separate points* if for distinct points $a, b \in X$, $f(a) \neq f(b)$ for some $f \in \mathcal{F}$. The proof of our next result is identical to the proof of Theorem (3.2) of [4] but since it is short we include it here for the sake of completeness.

LEMMA 2.7. *Let $T(X,Y,\alpha)$ be any (not necessarily adequate) subsemigroup of $S(X,Y,\alpha)$ which separates points and suppose that $T(X,Y,\alpha)$ has a right identity. Then α maps some retract of Y homeomorphically onto X.*

PROOF. Let r be a right identity of $T(X,Y,\alpha)$. Then $r\circ\alpha$ is a continuous selfmap of Y and we have

$$(r\circ\alpha)\circ(r\circ\alpha)=(r\circ\alpha\circ r)\circ\alpha=(rr)\circ\alpha=r\circ\alpha.$$

Thus $r\circ\alpha$ is an idempotent continuous selfmap of Y and therefore Ran $r\circ\alpha$ is a retract of Y. Let any $y\in$ Ran r be given and choose any $x\in X$ such that $r(x)=y$. We then have

$$(r\circ\alpha)(y)=(r\circ\alpha\circ r)(x)=(rr)(x)=r(x)=y.$$

Hence, $r\circ\alpha$ is the identity on Ran r and Ran $r\subseteqq$ Ran $r\circ\alpha$. Since it is evident that Ran $r\circ\alpha\subseteqq$ Ran r, we actually have Ran $r=$ Ran $r\circ\alpha$ so that Ran r is a retract of Y. Next, let $f\in T(X,Y,\alpha)$ and note that

$$f(\alpha\circ r(x))=(f\circ\alpha\circ r)(x)=(fr)(x)=f(x).$$

Since $T(X,Y,\alpha)$ separates points it follows that $\alpha\circ r$ is the identity map on X. Because of this and the fact that $r\circ\alpha$ is the identity on Ran r, we conclude that α is a homeomorphism from the retract Ran r onto X.

LEMMA 2.8. *Let $T(X,Y,\alpha)$ be an adequate semigroup which separates points and has an identity. Then α is a homeomorphism from Y onto X.*

PROOF. This is an immediate consequence of the previous two lemmas.

REMARKS. Let $T(X,Y,\alpha)$ be any adequate semigroup which separates points. We have just seen that if $T(X,Y,\alpha)$ has an identity then α must be a homeomorphism from Y onto X. However, the converse does not, in general, hold. For example, let $X=Y=R$ be the space of real numbers, let α be the identity map and take $T(X,Y,\alpha)$ to be the collection of all bounded continuous selfmaps of R. Then $T(X,Y,\alpha)$ is an adequate semigroup which separates points and α is certainly a homeomorphism from Y onto X. Suppose some function $f\in T(X,Y,\alpha)$ is an identity. Since the binary operation here is simple composition, we would have

$$\langle f(x)\rangle=f\circ\langle x\rangle=\langle x\rangle$$

for all $x\in X=R$. In other words, f would have to be the identity map. But $T(X,Y,\alpha)$ contains no functions with unbounded range so we see that $T(X,Y,\alpha)$ does not have an identity.

We next give a name to some groups we introduced in [3], p. 41.

DEFINITION 2.9. A topological group is referred to as a TR-*group* if it is the additive group of some Hausdorff topological ring T which satisfies the following conditions:

(2.9.1) T contains a multiplicative identity 1.

(2.9.2) The element $2 = 1 + 1$ has a multiplicative inverse.

(2.9.3) Either the elements 1 and 2 are connected by an arc or T is 0-dimensional.

Now we need only put together the results we have obtained here with some results of previous papers in order to get the main result of this section.

THEOREM 2.10. *Let (G, H, α) be a compatible triple, let H be a TR-group and suppose, in addition, that α is topologically a quotient map and* Ran α *is a closed subgroup of G. Then the following statements are equivalent:*

(2.10.1) *$N(G, H, \alpha)$ is simple.*

(2.10.2) *$N(G, H, \alpha)$ has a multiplicative identity.*

(2.10.3) *$N(G, H, \alpha)$ is isomorphic to $N(G)$.*

(2.10.4) *$N(G, H, \alpha)$ is isomorphic to $N(H)$.*

(2.10.5) *α is an isomorphism from H onto G.*

PROOF. We first observe that (2.10.1) implies (2.10.5). The mapping α is an algebraic isomorphism from H onto a dense subgroup of G by Lemma 2.4. Since α is a quotient map and Ran α is closed, it follows that α is an isomorphism from H onto G. If α is an isomorphism from the group H onto the group G then one can show that the mapping which sends f into $f \circ \alpha$ is an isomorphism from the near-ring $N(G, H, \alpha)$ onto the near-ring $N(H)$ so that (2.10.5) implies (2.10.2). In view of Theorem (4.3) of [3], $N(H)$ is simple so that (2.10.4) implies (2.10.1) and we have shown thus far that (2.10.1), (2.10.4) and (2.10.5) are all equivalent. The mapping which sends f into $\alpha \circ f$ is a near-ring isomorphism from $N(G, H, \alpha)$ onto $N(G)$ whenever α is an isomorphism from H onto G so that (2.10.5) implies (2.10.3). It is immediate that (2.10.3) implies (2.10.2). Suppose (2.10.2) holds. Now $N(G, H, \alpha)$ separates points and its multiplicative semigroup is adequate since it contains all constant functions from G into H. Thus, it follows from Lemma (2.8) that α is a homeomorphism from H onto G since $N(G, H, \alpha)$ has a multiplicative identity. Since, by assumption α is a group homomorphism, we conclude that it must be an isomorphism from H onto G so that (2.10.2) implies (2.10.5) and we have now shown that (2.10.2), (2.10.3) and (2.10.5) are also equivalent. This completes the proof.

The results of this section indicate that while there are a number of groups G for which $N(G)$ is simple, a lot of sandwich near-rings are not simple.

3. Z-sandwich near-rings

LEMMA 3.1. *Let X be a completely regular Hausdorff space, H a topological group and suppose that either H contains an arc or X is 0-dimensional. Let α be a nonconstant continuous function from H into X and suppose $N_0(X,H,\alpha)$ is simple. Then* Ran α *is a dense subspace of X.*

PROOF. The proof of this lemma is the same as the proof of Lemma 2.1 with the exception of the portion where we must show that $\varphi[N_0(X,H,\alpha)]$ has a nonzero element. In Lemma 2.1 we had all the constant functions at our disposal but the only constant function in $N_0(X,H,\alpha)$ is $\langle 0\rangle$. We remedy the situation in what follows. Since α is nonconstant there exists a point $y \in H$ such that $\alpha(y) \neq \alpha(0)$. Now we argue as before. Either H contains an arc, in which case there exists an arc with endpoints 0 and z or X is 0-dimensional. In either event, there exists a continuous function f from X into H such that $f(\alpha(0)) = 0$ and $f(\alpha(y)) = z \neq 0$. Evidently, $f \in N_0(X,H,\alpha)$ and $\varphi(f) = f \circ \alpha \neq \langle 0\rangle$. This concludes the proof.

LEMMA 3.2. *Let X be a nondegenerate completely regular first countable Hausdorff space, let H be a first countable group, let α be a nonconstant continuous map from H into X and suppose the following two conditions are satisfied:*

(3.2.1) *Either H contains an arc or X is 0-dimensional.*

(3.2.2) *If H is discrete then so is X.*

Then if $N_0(X,H,\alpha)$ is simple, the following two conditions must hold:

(3.2.3) Ran α *is dense in X.*

(3.2.4) *$\alpha(0)$ is an isolated point of X.*

PROOF. It follows immediately from Lemma 3.1 that (3.2.3) must hold. We deny (3.2.4) and obtain a contradiction. Let

$$J = \{f \in N_0(X,H,\alpha)\colon f \text{ vanishes in a neighborhood of } \alpha(0)\}.$$

It is immediate that J is a normal subgroup of $(N_0(X,H,\alpha),+)$. Let $g_1, g_2 \in N_0(X,H,\alpha)$ and $f \in J$. Then $f(x) = 0$ for $x \in V$ where V is some neighborhood of $\alpha(0)$. Now $(\alpha \circ g_1)^{-1}[V]$ is a neighborhood of $\alpha(0)$ on which fg_1 vanishes so that $fg_1 \in J$. Moreover, for any $x \in V$,

$$\begin{aligned}(g_1(g_2+f) - g_1g_2)(x) &= g_1 \circ \alpha(g_2(x) + f(x)) - g_1 \circ \alpha \circ g_2(x) = \\ &= g_1 \circ \alpha(g_2(x)) - g_1 \circ \alpha \circ g_2(x) = 0.\end{aligned}$$

That is, $g_1(g_2 + f) - g_1g_2$ vanishes on V and therefore belongs to J. This establishes the fact that J is an ideal of $N_0(X, H, \alpha)$. We next show that $J \neq \{\langle 0\rangle\}$. Choose $p \in X - \{\alpha(0)\}$ and let W be a closed neighborhood of $\alpha(0)$ not containing the point p. Suppose first that H contains an arc. Then it must necessarily contain an arc A with endpoints 0 and y and there exists a continuous function f from X into A such that $f(x) = 0$ for $x \in W$ and $f(p) = y$. On the other hand, if H contains no arcs then X is 0-dimensional by (3.2.1) and there exists a clopen set V such that $p \in V \subseteqq X - W$. In this case simply choose $y \neq 0$ and define $f(x) = y$ for $x \in V$ and $f(x) = 0$ for $x \in X - V$. In either event f vanishes on W so that $f \in J$ and $f \neq \langle 0\rangle$.

Next, we show that $J \neq N_0(X, H, \alpha)$. Again, we first consider the case where H contains an arc and hence contains an arc with endpoints 0 and y. Since $\alpha(0)$ is a G_δ and X is completely regular and Hausdorff, it is well known that there exists a continuous function f mapping X into A which vanishes at $\alpha(0)$ and nowhere else. Evidently, $f \in N_0(X, H, \alpha) - J$.

Now consider the case where H contains no arcs. Then X is 0-dimensional and since X is first countable, there exists a countable collection $\{V_n\}_{n=1}^{\infty}$ of clopen sets such that $V_1 \neq X$, $V_{n+1} \subseteqq V_n$ and $V_{n+1} \neq V_n$ for all n and $\cap\{V_n\}_{n=1}^{\infty} = \{\alpha(0)\}$. For each V_n, define a continuous function f_n from X into the closed unit interval $I = [0, 1]$ by $f_n(x) = 0$ for $x \in V_n$ and $f_n(x) = 1$ for $x \in X - V_n$ and then define

$$f(x) = \sum_{n=1}^{\infty}(f_n(x))/2^n \quad \text{for each} \quad x \in X.$$

Then f is a continuous function from X into I with the property that $f(\alpha(0)) = 0$, $f(x) \neq 0$ for $x \neq \alpha(0)$ and $\operatorname{Ran} f = \{0\} \cup \{1/2^n\}_{n=1}^{\infty}$. Since, by assumption, $\alpha(0)$ is not isolated, H is not discrete by (3.2.2) and since H is a group, it follows that 0 is not isolated. Thus there exists a sequence of points $\{y_n\}_{n=1}^{\infty}$ in H distinct from 0 and from each other which converges to 0. Define a map g from $\operatorname{Ran} f$ into H by

$$g(0) = 0 \quad \text{and} \quad g(1/2^n) = y_n \quad \text{for each} \quad n.$$

Then g is continuous and it follows that $g \circ f$ is a continuous map from X into H which vanishes at $\alpha(0)$ and only at $\alpha(0)$. Thus $g \circ f \in N_0(X, H, \alpha) - J$. All this contradicts the simplicity of $N_0(X, H, \alpha)$ and we conclude finally that $\alpha(0)$ is isolated.

We are now in a position to present an analogue of Theorem 2.10 for near-rings of the form $N_0(G, H, \alpha)$.

THEOREM 3.4. *Let G and H be first countable topological groups, let α be a nonzero homomorphism from H into G and suppose the following two conditions are satisfied:*

(3.4.1) *Either H contains an arc or G is 0-dimensional.*

(3.4.2) *If H is discrete then so is G.*

Then the following statements are equivalent:

(3.4.3) $N_0(G,H,\alpha)$ *is simple.*

(3.4.4) $N_0(G,H,\alpha)$ *has a multiplicative identity and G is discrete.*

(3.4.5) $N_0(G,H,\alpha)$ *is isomorphic to* $N_0(G)$ *and G is discrete.*

(3.4.6) $N_0(G,H,\alpha)$ *is isomorphic to* $N_0(H)$ *and H is discrete.*

(3.4.7) α *is an isomorphism from H onto G and H is discrete.*

PROOF. We show first that (3.4.3) implies (3.4.7). Lemma 3.2 tells us that $\alpha(0)$ is isolated and that $\operatorname{Ran}\alpha$ is dense in G. It follows immediately that G is discrete and $\operatorname{Ran}\alpha = G$. To complete the verification that (3.4.3) implies (3.4.7), we need only show that α is injective. To this end, we define a homomorphism φ from the near-ring $N_0(G,H,\alpha)$ into the near-ring $N_0(G)$ by $\varphi(f) = \alpha \circ f$. Since α is a group endomorphism, it readily follows that φ is a near-ring homomorphism. We want to show that $\varphi[N_0(G,H,\alpha)] \neq \{\langle 0\rangle\}$. Choose $y \in H$ such that $\alpha(y) \neq 0$ and define $f(\alpha(y)) = y$ and $f(x) = 0$ for $x \neq \alpha(y)$. Then f is continuous since G is discrete and $\varphi(f) = \alpha \circ f \neq \langle 0\rangle$.

Now suppose α is not injective. Then $\alpha(y) = 0$ for some $y \neq 0$. Choose $a \in G - \{0\}$ and define $f(a) = y$ and $f(x) = 0$ for $x \neq y$. Again, f is continuous since G is discrete so that $f \in N_0(G,H,\alpha)$. But $\varphi(f) = \alpha \circ f = \langle 0\rangle$ and φ is a proper homomorphism from $N_0(G,H,\alpha)$ onto $\varphi[N_0(G,H,\alpha)] \neq \{\langle 0\rangle\}$. This contradicts the simplicity of $N_0(G,H,\alpha)$ and we conclude that α is injective. It is now immediate that H is also discrete and we have shown that (3.4.3) implies (3.4.7). When α is an isomorphism from H onto G, the map which sends f into $f \circ \alpha$ is a near-ring isomorphism from $N_0(G,H,\alpha)$ onto $N_0(H)$ so that (3.4.7) implies (3.4.6) and it is well known that $N_0(H)$ is simple whenever H is discrete [1], [5] so that (3.4.6) implies (3.4.3). At this point, we have shown that (3.4.3), (3.4.6) and (3.4.7) are all equivalent.

If (3.4.7) holds, G is discrete as well as H and since α is a group isomorphism from H onto G, the map which sends f onto $\alpha \circ f$ is a near-ring isomorphism from $N_0(G,H,\alpha)$ onto $N_0(G)$. Thus, (3.4.7) implies (3.4.5) which immediately implies (3.4.4). If (3.4.4) holds, then the multiplicative semigroup of $N_0(G,H,\alpha)$ is adequate and separates points since G is discrete and it follows from Lemma 2.8 that α is a homeomorphism from H onto G. Thus H is discrete and α is an isomorphism from H onto G. That is, (3.4.4) implies (3.4.7). It follows that the statements (3.4.3) to (3.4.7) inclusive are all equivalent and the proof is complete.

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SUNY COLLEGE AT PLATTSBURGH
PLATTSBURGH, NEW YORK 12901
U.S.A.

106 DIEFENDORF HALL
SUNY AT BUFFALO
BUFFALO, NEW YORK 14214–3093
U.S.A.

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LOCAL SOLUTION OF A MIXED PROBLEM FOR A DEGENERATED HYPERBOLIC EQUATION

D. C. PEREIRA (Belém), M. C. DE CAMPOS VIEIRA
and T. N. RABELLO (São José dos Campos)

1. Introduction

In this paper we study the existence of a local solution for the mixed problem associated to the equation

$$(*)\qquad u'' + (1 + M_0(|A^{1/2}u|^2))Au + M_1(|A^{1/2}u|^2)Au'' = 0$$

where A is a selfadjoint operator defined in a Hilbert space H with norm $|\cdot|$; M_0, M_1 are real functions with $M_0(\lambda) \geqq 0$ and $M_1(\lambda) \geqq 0$ for $\lambda \geqq 0$.

When $M_0(\lambda) \equiv 0$ for $\lambda \geqq 0$, equation $(*)$ is an abstract model for the equation of vibrations of thin rods (cf. Love [8]) and it was studied, for example, by one of the authors in [9], [10] and [11].

When $M_1(\lambda) \equiv 0$ for $\lambda \geqq 0$, equation $(*)$ has its motivation in the mathematical description of the vibrations of an elastic stretched string and it was studied by Bernstein [2], Dickey [3], Pohozaev [12], Arosio–Spagnolo [1], Lions [7], Ebihara–Medeiros–Milla Míranda [5], Yamada [13] among others.

We use the penalty method as in Ebihara [4] combined with Faedo–Galerkin's method and compactness arguments (see Lions [6]).

2. Notations, assumptions and main results

Let V, H be two real Hilbert spaces whose scalar product and norm are $((\cdot,\cdot))$, $\|\cdot\|$ and $(\cdot,\cdot)$, $|\cdot|$ respectively. We suppose that V is continuously embedded in H and dense. We identify H to its dual so that we have $V \subset H \subset V'$.

Let A be a given operator such that $A \in \mathcal{L}(V, V')$, $A^* = A$, $\langle Au, v\rangle = ((u,v))$ for all $u, v \in V$.

Suppose:

$$(2.1)\qquad M_0 \in C^1[0,\infty),\quad M_0(\lambda) \geqq 0 \quad \text{for} \quad \lambda \geqq 0,$$

$$(2.2)\qquad \begin{gathered} M_1 \in C^1[0,\infty),\quad M_1(\lambda) \geqq 0 \quad \text{for} \quad \lambda \geqq 0 \quad \text{and} \\ |M_1'(\lambda)\lambda| \leqq CM_1(\lambda) \quad \text{for} \quad \lambda \geqq 0, \end{gathered}$$

where C is a positive constant.

(2.3) The injection of V in H is compact.

Then, the spectral resolution of A is given by

$$Aw_\nu = \lambda_\nu w_\nu, \qquad \nu = 1, 2, \dots, \tag{2.4}$$

where $\{\lambda_\nu\}$ and $\{w_\nu\}$ are the eigenvalues and eigenvectors of A, resp. It is known that $0 < \lambda_1 \leqq \lambda_2 \leqq \dots$ and $\lambda_j \to +\infty$. Therefore the power $A^{k/2}$ of A is well defined for all positive integers k. We denote by V_k the domain of $A^{k/2}$, that is, $V_k = D(A^{k/2})$ equipped with the scalar product and norm

$$(u, v)_k = (A^{k/2}u, A^{k/2}v) \quad \text{and} \quad |v|_k^2 = |A^{k/2}v|^2.$$

We have $V_1 = V$, $V_0 = H$ and

(2.5) the embedding of V_{k+1} in V_k is compact for all k.

Then we have the following result:

THEOREM 1. *Assume* (2.1), (2.2), (2.3), $u_0, u_1 \in V_{k+3}$ *with* $u_0 \neq 0$ *and* $k \geqq 2$. *Then there exist a real number* $T_0 > 0$ *and a vector valued function* $u\colon [0, T_0] \to H$ *such that*

$$u \in L^\infty(0, T_0; V_{k+3}), \tag{2.6}$$

$$u' \in L^\infty(0, T_0; V_{k+2}), \tag{2.7}$$

$$u'' \in L^\infty(0, T_0; V_k), \tag{2.8}$$

and u *satisfies*

$$(u'' + (1 + M_0(|A^{1/2}u|^2))Au + M_1(|A^{1/2}u|^2)Au'', v) = 0, \tag{2.9}$$

for all $v \in V$ *in* $L^2(0, T_0)$, *and*

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{2.10}$$

3. Proof of the theorem

The proof will follow by using the penalty method as in Ebihara [4]. Let $F\colon (0, \infty) \to \mathbf{R}$ satisfying the conditions

$$\left\{\begin{array}{l} F \in C^1(0, \infty), \\ \text{there exist numbers } \alpha_0 > 0,\ \beta_0 \geqq 1,\ \delta > 0 \text{ such that} \\ F(\xi) \geqq \frac{\alpha_0}{\xi^{\beta_0}} \text{ for all } \xi \in (0, \delta], \\ F'(\xi) \leqq 0 \text{ for all } \xi > 0, \\ F(\xi) = 1 \text{ for all } \xi \geqq 1. \end{array}\right. \tag{3.1}$$

For u_1 given in V_k, let $K > 0$ be a real number such that

$$|u_1|_k^2 < K. \tag{3.2}$$

The penalized problem associated to (2.4), (2.5) is the following: For each $\varepsilon > 0$ find $u_\varepsilon(t)$ such that

$$\begin{aligned} u_\varepsilon''(t) + (1 + M_0(|u_\varepsilon(t)|_1^2))Au_\varepsilon(t) + M_1(|u_\varepsilon(t)|_1^2)Au_\varepsilon''(t) + \\ +\varepsilon F\left(\frac{K - |u_\varepsilon'(t)|_k^2}{\varepsilon}\right) u_\varepsilon'(t) = 0, \end{aligned} \tag{3.3}$$

$$u_\varepsilon(0) = u_0, \qquad u_\varepsilon'(0) = u_1 \tag{3.4}$$

where K satisfies the condition (3.2).

We shall prove that there exists a solution u_ε of the above problem which converges to u when $\varepsilon \to 0$. Furthermore u is defined on an interval $[0, T_0]$ and it satisfies (2.6)–(2.10).

a) *Approximate scheme for the penalized problem.* Let us consider the eigenvectors w_ν as in (2.4), let $[w_1, \dots, w_m]$ be the subspace of V, generated by the first m eigenvectors of $\{w_\nu\}$. Let

$$u_{\varepsilon m}(t) = \sum_{\nu=1}^{m} g_{\varepsilon\nu m}(t) w_\nu \in [w_1, \dots, w_m]$$

be defined by

$$\begin{aligned} (u_\varepsilon''m(t), v) + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))(Au_{\varepsilon m}(t), v) + \\ +M_1(|u_{\varepsilon m}(t)|_1^2)(Au_{\varepsilon m}''(t), v) + \varepsilon F\left(\frac{K - |u_{\varepsilon m}'(t)|_k^2}{\varepsilon}\right)(u_{\varepsilon m}'(t), v) = 0 \end{aligned} \tag{3.5}$$

for all $v \in [w_1, \dots, w_m]$,

$$u_{\varepsilon m}(0) = u_{0m} \to u_0 \quad \text{strongly in} \quad V_{k+3}, \tag{3.6}$$

$$u_{\varepsilon m}'(0) = u_{1m} \to u_1 \quad \text{strongly in} \quad V_{k+3}. \tag{3.7}$$

By (3.2) it follows that there exists a solution $u_{\varepsilon m}(t)$ of the system (3.5)–(3.7) defined on the interval $[0, T_{\varepsilon m})$, and from $F \in C^1(0, \infty)$ we have $u_{\varepsilon m} \in$ $\in C^2[0, T_{\varepsilon m})$.

b) *Estimate I.* This is given by the following result:

LEMMA. *For all $\varepsilon > 0$ and $m \geqq m_0$ we have*

$$|u'_{\varepsilon m}(t)|_k^2 < K \quad \textit{for all} \quad t \in [0,\infty). \tag{3.8}$$

PROOF. We fix $\varepsilon > 0$ and $m \geqq m_0$ such that $[0, T_{\varepsilon m})$ is an interval of the solution and

$$0 < \varepsilon < 1, \qquad \frac{K - |u_{1m}|_k^2}{\varepsilon} > 1. \tag{3.9}$$

We note that (3.8) is satisfied for all $t \in [0, T_{\varepsilon m})$. It is sufficient to show that

$$\lim_{t \to T_{\varepsilon m}} |u'_{\varepsilon m}(t)|_k^2 \quad \text{exists and} \quad \lim_{t \to T_{\varepsilon m}} |u'_{\varepsilon m}(t)|_k^2 < K \tag{3.10}$$

because then the conclusion of the lemma will follow by Zorn's lemma.

Let $t = 0$ in (3.5) and $v = A^{k+1} u''_{\varepsilon m}(0)$. Using (2.1), (3.1), (3.6) and (3.7) we have

$$|u''_{\varepsilon m}(0)|_{k+1} \leqq C \tag{3.11}$$

where C is a positive constant independent of $\varepsilon > 0$ and m.

Now taking the derivate with respect to t of the approximated equation (3.5) we obtain:

$$\begin{gathered}(u'''_{\varepsilon m}(t), v) + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))(Au'_{\varepsilon m}(t), v) + \\ +2M'_0(|u_{\varepsilon m}(t)|_1^2)(u_{\varepsilon m}(t), u'_{\varepsilon m}(t))_1 (Au_{\varepsilon m}(t), v) + \\ +M_1(|u_{\varepsilon m}(t)|_1^2)(Au'''_{\varepsilon m}(t), v) + \varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)(u''_{\varepsilon m}(t), v) + \\ +2M'_1(|u_{\varepsilon m}(t)|_1^2)(u_{\varepsilon m}(t), u'_{\varepsilon m}(t))_1 (Au''_{\varepsilon m}(t), v) - \\ -2F'\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)(u'_{\varepsilon m}(t), u''_{\varepsilon m}(t))_k (u'_{\varepsilon m}(t), v) = 0.\end{gathered} \tag{3.12}$$

If we consider $v = A^k u''_{\varepsilon m}(t)$ in (3.12) we have from (3.1)

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}[|u''_{\varepsilon m}(t)|_k^2 + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))|u'_{\varepsilon m}(t)|_{k+1}^2 + \\ +M_1(|u_{\varepsilon m}(t)|_1^2)|u''_{\varepsilon m}(t)|_{k+1}^2] \leqq \\ \leqq |M'_0(|u_{\varepsilon m}(t)|_1^2)||u_{\varepsilon m}(t)|_1 |u'_{\varepsilon m}(t)|_1 |u'_{\varepsilon m}(t)|_{k+1}^2 + \\ +2|M'_0(|u_{\varepsilon m}(t)|_1^2)||u_{\varepsilon m}(t)|_1 |u'_{\varepsilon m}(t)|_1 |u_{\varepsilon m}(t)|_{k+2} |u''_{\varepsilon m}(t)|_k + \\ +2|M'_1(|u_{\varepsilon m}(t)|_1^2)||u_{\varepsilon m}(t)|_1 |u'_{\varepsilon m}(t)|_1 |u''_{\varepsilon m}(t)|_{k+1}^2 .\end{gathered}$$

But in $[w_1, \ldots, w_m]$ the norms are equivalent, thus from (3.8), (2.1) and (2.2) we obtain in $[0, T_{\varepsilon m})$

$$\frac{d}{dt}[|u''_{\varepsilon m}(t)|_k^2 + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))|u'_{\varepsilon m}(t)|_{k+1}^2 + $$
$$+ M_1(|u_{\varepsilon m}(t)|_1^2)|u''_{\varepsilon m}(t)|_{k+1}^2] \leqq C_1|u'_{\varepsilon m}(t)|_{k+1}^2 + C_2|u''_{\varepsilon m}(t)|_k^2 + C_3.$$

Integrating from 0 to $t < T_{\varepsilon m}$, using (3.6), (3.7) and (3.11) we have from Gronwall's inequality

$$(3.13) \qquad |u''_{\varepsilon m}(t)|_k \leqq C(\varepsilon, m) \quad \text{for all} \quad t \in [0, T_{\varepsilon m}),$$

where $C(\varepsilon, m)$ is a constant depending on $\varepsilon > 0$ and m. It follows from (3.13) that $\lim\limits_{t \to T_{\varepsilon m}} |u'_{\varepsilon m}(t)|_k^2$ exists.

Let us suppose that

$$(3.14) \qquad \lim_{t \to T_{\varepsilon m}} |u'_{\varepsilon m}(t)|_k^2 = K.$$

Let $v = A^k u'_{\varepsilon m}(t)$ in (3.5). Then we have

$$\frac{1}{2}\frac{d}{dt}[|u'_{\varepsilon m}(t)|_k^2 + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))|u_{\varepsilon m}(t)|_{k+1}^2 + $$
$$+ M_1(|u_{\varepsilon m}(t)|_1^2)|u'_{\varepsilon m}(t)|_{k+1}^2] + \varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)|u'_{\varepsilon m}(t)|_k^2 \leqq$$
$$\leqq 2|M_0'(|u_{\varepsilon m}(t)|_1^2)||u_{\varepsilon m}(t)|_1|u'_{\varepsilon m}(t)|_1|u_{\varepsilon m}(t)|_{k+1}^2 + $$
$$+ 2|M_1'(|u_{\varepsilon m}(t)|_1^2)||u_{\varepsilon m}(t)|_1|u'_{\varepsilon m}(t)|_1|u'_{\varepsilon m}(t)|_{k+1}^2.$$

Integrating both sides from 0 to t, $t < T_{\varepsilon m}$, we obtain from (2.1), (2.2), (3.6) and (3.7)

$$(3.15) \qquad \varepsilon \int_0^t F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)|u'_{\varepsilon m}(s)|_k^2 ds \leqq C(\varepsilon, m),$$

where $C(\varepsilon, m)$ is a constant independent of t.

Let $E(t) = |u'_{\varepsilon m}(t)|_k^2$. We shall prove that under the hypothesis (3.14), there exists an interval $(\tau_0, T_{\varepsilon m})$ such that $E'(t) > 0$ for all $t \in (\tau_0, T_{\varepsilon m})$. In fact, let us suppose the contrary. Then for each interval $(\tau_\nu, T_{\varepsilon m})$ with $\tau_\nu \to T_{\varepsilon m}$, there exist $t_\nu \in (\tau_\nu, T_{\varepsilon m})$ such that

$$(3.16) \qquad E'(t_\nu) = 2(u'_{\varepsilon m}(t_\nu), u''_{\varepsilon m}(t_\nu))_k = 0.$$

By the continuity of $E(t)$ and properties of F we have

$$(3.17) \qquad E(t_\nu) \to K \quad \text{as} \quad \nu \to \infty,$$

and

$$F\left(\frac{K-E(t_\nu)}{\varepsilon}\right) \to \infty \quad \text{as} \quad \nu \to \infty. \tag{3.18}$$

Let $v = A^k u'_{\varepsilon m}(t)$ in (3.5) and observing (3.16) we obtain

$$\varepsilon F\left(\frac{K-|u'_{\varepsilon m}(t_\nu)|_k^2}{\varepsilon}\right)|u'_{\varepsilon m}(t_\nu)|_k^2 = \tag{3.19}$$

$$= -(1+M_0(|u_{\varepsilon m}(t_\nu)|_1^2))\frac{1}{2}\frac{d}{dt}|u_{\varepsilon m}(t_\nu)|_{k+1}^2 - M_1(|u_{\varepsilon m}(t_\nu)|_1^2)\frac{1}{2}\frac{d}{dt}|u'_{\varepsilon m}(t_\nu)|_{k+1}^2.$$

But by the continuity of M_0 and M_1 and the estimates for $u_{\varepsilon m}$, $u'_{\varepsilon m}$ and $u''_{\varepsilon m}$ in $[0, T_{\varepsilon m})$ we have that the right hand side of (3.19) is bounded. On the other hand the left hand side of (3.19) approaches infinity when $\nu \to \infty$. This is a contradiction which proves our claim.

We have $0 < E'(t) \leqq C(\varepsilon, m)$ for all $t \in (\tau_0, T_{\varepsilon m})$, whence

$$\xi(s) = \frac{K-E(s)}{\varepsilon}$$

is strictly decreasing in $(\tau_0, T_{\varepsilon m})$. Let η be the inverse function of ξ. By a change of variable we obtain

$$\int_{\tau_0}^{t} F\Big(\frac{K-E(s)}{\varepsilon}\Big)ds = \int_{\xi(\tau_0)}^{\xi(t)} \frac{-\varepsilon F(\xi)}{E'(\eta(\xi))}d\xi = \tag{3.20}$$

$$= \varepsilon \int_{\xi(t)}^{\xi(t_0)} \frac{F(\xi)}{E'(\eta(\xi))}d\xi \geqq \frac{\varepsilon}{C(\varepsilon,m)} \int_{\xi(t)}^{\xi(t_0)} F(\xi)d\xi.$$

Let $\varrho > 0$ be a positive real number such that $K - \varrho > 0$. Then by (3.14) there exists $\delta_1 > 0$ satisfying $\tau_0 < T_{\varepsilon m} - \delta_1$ and

$$E(t) > K - \varrho \quad \text{for all} \quad t \in (T_{\varepsilon m} - \delta_1, T_{\varepsilon m}). \tag{3.21}$$

By using (3.21) in (3.15), it follows that

$$C(\varepsilon, m) \geqq K - \varrho \int_{T_{\varepsilon m}-\delta_1}^{t} F\Big(\frac{K-E(s)}{\varepsilon}\Big)ds.$$

Choosing τ_0 such that $\xi(\tau_0) \leqq \delta$ (δ as in (3.1)) and observing (3.20) we obtain

$$\frac{\varepsilon\alpha_0}{C(\varepsilon,m)} \int\limits_{\xi(t)}^{\xi(T_{\varepsilon m}-\delta_1)} \frac{1}{\xi^{\beta_0}} d\xi \leqq \frac{\varepsilon}{C(\varepsilon,m)} \int\limits_{\xi(t)}^{\xi(T_{\varepsilon m}-\delta_1)} F(\xi) d\xi \leqq C(\varepsilon,m)$$

which is a contradiction, because the first integral is divergent. Therefore the lemma is proved.

Before obtaining more estimates, let us first prove that $|u_{\varepsilon m}(t)|_1 > 0$ in a neighborhood of $t=0$. In fact, by the lemma we have

$$\left| |u_{\varepsilon m}(t)|_1 - |u_{0m}|_1 \right| \leqq C_k \sqrt{K} t,$$

or

$$|u_{0m}|_1 - C_k\sqrt{K}t \leqq |u_{\varepsilon m}(t)|_1. \tag{3.22}$$

Let $\delta_2 > 0$ be such that $0 < 2\delta_2 < |u_0|_1$. This is possible because $u_0 \neq 0$. Let

$$T_1 = \frac{|u_0|_1 - 2\delta_2}{C_k\sqrt{K}};$$

then we have

$$C_k\sqrt{K}t + \delta_2 \leqq |u_0|_1 - \delta_2 \leqq |u_{0m}|_1 \quad \text{for all} \quad t \in [0, T_1]. \tag{3.23}$$

From (3.22) and (3.23) it follows that

$$|u_{\varepsilon m}(t)|_1 \geqq \delta_2 \quad \text{for all} \quad t \in [0, T_1]. \tag{3.24}$$

c) *Estimate II.* Let $v = A^{k+2}u'_{\varepsilon m}(t)$ in (3.5). Using $F(\xi) \geqq 0$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[|u'_{\varepsilon m}(t)|^2_{k+2} + (1 + M_0(|u_{\varepsilon m}(t)|^2_1))|u_{\varepsilon m}(t)|^2_{k+3} + \\
&\qquad + M_1(|u_{\varepsilon m}(t)|^2_1)|u'_{\varepsilon m}(t)|^2_{k+3}] \leqq \\
&\leqq |M'_0(|u_{\varepsilon m}(t)|^2_1)||u_{\varepsilon m}(t)|_1|u'_{\varepsilon m}(t)|_1|u_{\varepsilon m}(t)|_{k+3} + \\
&\quad + |M'_1(|u_{\varepsilon m}(t)|^2_1)||u_{\varepsilon m}(t)|_1|u'_{\varepsilon m}(t)|^2_{k+3}|u'_{\varepsilon m}(t)|_1.
\end{aligned}$$

From $|u_{\varepsilon m}(t)|_k \leqq \overline{K}$, $|u'_{\varepsilon m}(t)|_k \leqq K$ for all $t \in [0,T]$ and $V_k \subset V_1$ with compact embedding, we obtain by using (2.1), (2.2) and (3.24) in the above inequality:

$$\begin{aligned}
&\frac{d}{dt}[|u'_{\varepsilon m}(t)|^2_{k+2} + (1 + M_0(|u_{\varepsilon m}(t)|^2_1))|u_{\varepsilon m}(t)|^2_{k+3} + \\
&\quad + M_1(|u_{\varepsilon m}(t)|^2_1)|u'_{\varepsilon m}(t)|^2_{k+3}] \leqq C_4|u_{\varepsilon m}(t)|^2_{k+3} + \\
&\qquad + \frac{C_5}{\delta_2} M_1(|u_{\varepsilon m}(t)|^2_1)|u'_{\varepsilon m}(t)|^2_{k+3} + C_6.
\end{aligned}$$

Integrating from 0 to $t \leqq T_1$, using (2.5), (3.6), (3.7) and Gronwall's inequality we obtain:

$$(3.26)\qquad \begin{cases} |u'_{\varepsilon m}(t)|_{k+2} \leqq C & \text{for all} \quad t \in [0,T_1] \\ |u_{\varepsilon m}(t)|_{k+3} \leqq C & \text{for all} \quad t \in [0,T_1] \\ M_1(|u_{\varepsilon m}(t)|_1^2)|u'_{\varepsilon m}(t)|_{k+3}^2 \leqq C & \text{for all} \quad t \in [0,T_1] \end{cases}$$

where C is a positive constant independent of $\varepsilon > 0$ and m.

d) *Estimate III.* In (3.12) let $v = A^k u''_{\varepsilon m}(t)$, then we obtain

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}|u''_{\varepsilon m}(t)|_k^2 + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))\frac{1}{2}\frac{d}{dt}|u'_{\varepsilon m}(t)|_{k+1}^2 + \\ +M_1(|u_{\varepsilon m}(t)|_1^2)\frac{1}{2}\frac{d}{dt}|u''_{\varepsilon m}(t)|_{k+1}^2 + \\ +2M_0'(|u_{\varepsilon m}(t)|_1^2)(u_{\varepsilon m}(t)), u'_{\varepsilon m}(t))_1(Au_{\varepsilon m}(t), A^k u''_{\varepsilon m}(t)) + \\ +2M_1'(|u_{\varepsilon m}(t)|_1^2)(u_{\varepsilon m}(t), u'_{\varepsilon m}(t))_1|u''_{\varepsilon m}(t)|_{k+1}^2 + \\ +\varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)|u''_{\varepsilon m}(t)|_k^2 - 2F'\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)(u''_{\varepsilon m}(t), u'_{\varepsilon m}(t))^2 = 0 \end{gathered}$$

because of the assumptions on $F(\xi)$ and $F'(\xi)$ together with our previous estimates. Thus we get

$$\begin{gathered} \frac{d}{dt}[|u''_{\varepsilon m}(t)|_k^2 + (1 + M_0(|u_{\varepsilon m}(t)|_1^2))|u'_{\varepsilon m}(t)|_{k+1}^2 + \\ +M_1(|u_{\varepsilon m}(t)|_1^2)|u''_{\varepsilon m}(t)|_{k+1}^2] \leqq C_7 + C_8|u''_{\varepsilon m}(t)|_k^2 + \\ +\frac{c_9}{\delta_2}M_1(|u_{\varepsilon m}(t)|_1^2)|u''_{\varepsilon m}(t)|_{k+1}^2. \end{gathered}$$

Integration from 0 to $t \leqq T_1$, using the convergence of the initial data, the continuity of M_0, M_1 and the estimates (3.11) it follows from the Gronwall's inequality that

$$(3.27)\qquad \begin{cases} |u''_{\varepsilon m}(t)|_k \leqq C & \text{for all} \quad t \in [0,T_1] \\ M_1(|u_{\varepsilon m}(t)|_1^2)|u''_{\varepsilon m}(t)|_{k+1}^2 \leqq C & \text{for all} \quad t \in [0,T_1] \end{cases}$$

where C is a positive constant independent of $\varepsilon > 0$ and m.

e) *Estimate IV.* Our objective now is to prove that

$$(3.28)\qquad \varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \leqq C \quad \text{for all} \quad t \in [0,T_1],$$

where C is a positive constant independent of $\varepsilon > 0$ and m, ε being small enough.

In fact, if $|u'_{\varepsilon m}(t)|_k^2 \leqq \frac{K}{2}$ for some t, then

$$\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon} \geqq \frac{K}{2\varepsilon}$$

and for sufficiently small $\varepsilon > 0$, we have

$$F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \leqq 1.$$

Suppose $|u'_{\varepsilon m}(t)|_k^2 > \frac{K}{2}$. Let $v = A^k u'_{\varepsilon m}(t)$ in (3.5). Then we have

$$\frac{1}{2}\frac{d}{dt}|u'_{\varepsilon m}(t)|_k^2 + \frac{1}{2}(1 + M_0(|u_{\varepsilon m}(t)|_k^2))\frac{d}{dt}|u_{\varepsilon m}(t)|_{k+1}^2 +$$
$$+\frac{1}{2}M_1(|u_{\varepsilon m}(t)|_1^2)\frac{d}{dt}|u'_{\varepsilon m}(t)|_{k+1}^2 + \varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right)|u'_{\varepsilon m}(t)|_{k+1}^2 = 0$$

or

$$\frac{K}{2}\varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \leqq |u'_{\varepsilon m}(t)|_k |u''_{\varepsilon m}(t)|_k +$$
$$+(1 + M_0(|u_{\varepsilon m}(t)|_1^2))|u_{\varepsilon m}(t)|_{k+1}|u'_{\varepsilon m}(t)|_{k+1} +$$
$$+M_1(|u_{\varepsilon m}(t)|_1^2)|u'_{\varepsilon m}(t)|_{k+2}|u''_{\varepsilon m}(t)|_k.$$

Estimate (3.28) follows from (3.26) and (3.27).

Because of the estimates (3.26), (3.27) and (3.20), there exist two functions $u(t)$, $\chi(t)$ and subsequences still represented by $(u_{\varepsilon m})$, extracted from $(u_{\varepsilon m})$, such that

$$(3.29)\qquad \begin{cases} u_{\varepsilon m} \to u \quad \text{weak star in} \quad L^\infty(0, T_1; V_{k+3}) \\ u'_{\varepsilon m} \to u' \quad \text{weak star in} \quad L^\infty(0, T_1; V_{k+2}) \\ u''_{\varepsilon m} \to u'' \quad \text{weak star in} \quad L^\infty(0, T_1; V_k) \\ \varepsilon F\left(\frac{K - |u'_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \to \chi \quad \text{weak star in} \quad L^\infty(0, T_1). \end{cases}$$

Since the embedding of V_{k+1} in V_k is compact, it follows from Aubin–Lions' theorem [6] that

$$u_{\varepsilon m} \to u \quad \text{strongly in} \quad L^\infty(0, T_1; V_{k+2}),$$

and

$$u'_{\varepsilon m} \to u' \quad \text{strongly in} \quad L^\infty(0, T_1; V_{k+1}).$$

Since $M_0, M_1 \in C^1[0, \infty)$, it follows that

$$M_0(|u_{\varepsilon m}|_1^2) \to M_0(|u|_1^2) \quad \text{strongly in} \quad L^2(0, T_1)$$

and

$$M_0(|u_{\varepsilon m}|_1^2)Au_{\varepsilon m} \to M_0(|u|_1^2)Au \quad \text{weak in} \quad L^2(0,T_1;V_k).$$

Also, we have

$$M_1(|u_{\varepsilon m}|_1^2)Au''_{\varepsilon m} \to M_1(|u|_1^2)Au'' \quad \text{weak in} \quad L^2(0,T_1;V_{k-2}).$$

The above convergences permit us to pass to the limit in the approximated equation (3.5) when $\varepsilon \to 0$ and $m \to \infty$. Thus we obtain

$$\begin{aligned}(3.30)\qquad &(u''(t),v)+(1+M_0(|u(t)|_1^2))(Au(t),v)+\\ &+M_1(|u(t)|_1^2)(Au''(t),v)+\chi(t)(u'(t),v)=0,\end{aligned}$$

for each $v \in V$ in $L^2(0,T_1)$ and $u(0)=u_0$, $u'(0)=u_1$.

It is sufficient to show that $\chi \equiv 0$ in some interval $[0,T_0]$. In fact, from $u' \in C^0([0,T_1],V_k)$, $u'(0)=u_1$, $|u_1|_k^2 < K$, there exists an interval $[0,T_0]$ where $|u'(t)|_k^2 < K$ for all $t \in [0,T_0]$. If $\beta_1 = \max\{|u'(t)|_k^2; 0 \leqq t \leqq T_0\}$, then $\varrho_1 = K - \beta_1 > 0$ and

$$(3.31)\qquad |u'(t)|_k^2 < K - \frac{\varrho_1}{2} \quad \text{for all} \quad t \in [0,T_0].$$

By (3.27) we have

$$|u'_{\varepsilon m}(t) - u'_{\varepsilon m}(s)|_k \leqq C|t-s|.$$

By (3.26) and the compactness of the embedding of V_{k+1} in V_k, we can use Arzelá–Ascoli's theorem to conclude that

$$u'_{\varepsilon m} \to u' \quad \text{in} \quad C^\circ([0,T_0],V_k).$$

Then for $\frac{\varrho_1}{4}$ given, there exist ε_0 and m_1, such that

$$\big||u'_{\varepsilon m}(t)|_k^2 - |u'(t)|_k^2\big| \leqq \frac{\varrho_1}{4}, \quad 0 < \varepsilon \leqq \varepsilon_0, \quad m \geqq m_1.$$

From (3.31) it follows that

$$K - |u'_{\varepsilon m}(t)|_k^2 \geqq \frac{\varrho_1}{4} \quad \text{for all} \quad t \in [0,T_0], \quad 0 < \varepsilon \leqq \varepsilon_0, \quad m \geqq m_1,$$

and from the definition of $F(\xi)$ we find that

$$\varepsilon F\left(\frac{K - |u_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \leqq \varepsilon$$

for small $\varepsilon > 0$. Then

$$\varepsilon F\left(\frac{K - |u_{\varepsilon m}(t)|_k^2}{\varepsilon}\right) \to \chi(t) \equiv 0 \quad \text{in} \quad L^\infty(0,T_0)$$

since $\chi(t) \equiv 0$. Thus (3.30) implies that u is a solution and Theorem 1 is proved.

REMARK. If we consider the equation

$$u'' + M_1(|A^{1/2}u|^2)Au + M_2(|A^{1/2}u|^2)Au'' = 0$$

with $M_i(\lambda) \geqq 0$, for all $\lambda \geqq 0$, and $|M_i'(\lambda)\lambda| \leqq CM_i(\lambda)$, for all $\lambda \geqq 0$ $(i = 1, 2)$, when C is a positive constant, then we obtain a similar result as in Theorem 1 because the same method applies.

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DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO PARÁ
66.000 – BELÉM – PA – BRASIL

CENTRO TÉCNICO AEROESPACIAL
INSTITUTO TECNOLÓGICO DE AERONÁUTICA
DIVISÃO FUNDAMENTAL – DEPTO. DE MATEMÁTICA
12225 – SÃO JOSÉ DOS CAMPOS – SP – BRASIL

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ON MOMENTS OF THE SUPREMUM OF NORMED PARTIAL SUMS OF RANDOM VARIABLES INDEXED BY $\mathbf{N}^k$

N. V. GIANG (Hanoi)

1. Introduction and preliminaries

Let $k \geqq 1$ be an integer and let $\mathbf{N}^k$ denote the positive integer k-dimensional lattice points with coordinate-wise partial ordering, $\leqq$, i.e. for every $\mathbf{m} = (m_1, \dots, m_k)$, $\mathbf{n} = (n_1, \dots, n_k) \in \mathbf{N}^k$ $\mathbf{m} \leqq \mathbf{n}$ if and only if $m_i \leqq n_i$, $i = 1, 2, \dots, k$. Further, $|\mathbf{n}|$ is used to denote the product $n_1 n_2 \cdots n_k$, and $\mathbf{n} \to \infty$ means that $|\mathbf{n}| \to \infty$.

Let $\{X(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ be independent and identically distributed (i.i.d.) random variables and let $S(\mathbf{n}) = \sum_{\mathbf{k} \leqq \mathbf{n}} X(\mathbf{k})$ denote their partial sums. Let X be a random variable which has the same distribution as X_1 and which is independent of all other random variables.

Let $0 < t_1, \dots, t_k < 2$ and $b(\mathbf{n})$ be positive real numbers of the type

$$b(\mathbf{n}) = n_1 \dots n_k. \tag{1.1}$$

Denote $t = \max\{t_i\}$.

It is well-known that in some special cases of $b(\mathbf{n})$ the problem of relating moments of X to moments of $\sup_{\mathbf{n}} |b^{-1}(\mathbf{n}) X(\mathbf{n})|^p$ and $\sup_{\mathbf{n}} |b^{-1}(\mathbf{n}) S(\mathbf{n})|^p$, $p \geqq t$, has been studied by several authors.

The simplest case $k = 1$, $b(\mathbf{n}) = \mathbf{n}$ has been investigated first by Marcinkiewicz, Zygmund and Burkholder. Namely, Marcinkiewicz and Zygmund [10] proved that

$$E \sup_n |S(n)/n|^p < \infty, \quad p \geqq 1 \tag{1.2}$$

provided

$$E|X| \log^+ |X| < \infty \quad \text{if} \quad p = 1 \quad \text{and} \quad E|X|^p < \infty \quad \text{if} \quad p > 1. \tag{1.3}$$

Here and in the following $\log^+ x = \max\{1, \log x\}$.

Burkholder [1] proved that (1.2), (1.3) and

$$E \sup_n |X(n)/n|^p < \infty, \quad p \geq 1 \tag{1.4}$$

are equivalent. Furthermore, when $p = 1$, McCabe and Shepp [11], by a different way, proved the equivalence of (1.2)–(1.4).

For the case $k > 1$, $b(\mathbf{n}) = n_1 \cdots n_k$, $p = 1$, replacing condition (1.3) by

$$E|X|(\log^+ |X|)^k < \infty, \tag{1.3'}$$

Gabriel [3] has shown the equivalence of (1.2), (1.3) and (1.4).

Generalizing the last result of Gabriel, Gut [5] has obtained the following result.

THEOREM 1.1(Gut [5]). *Let $\{X, X(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ be i.i.d. random variables and suppose that $EX = 0$ whenever it is finite. Let $0 < r < 2$ and $p \geqq r$. The following statements are equivalent:*

$$E|X|^p(\log^+ |X|)^k < \infty \ \text{ if } p = r \ \text{ and } E|X|^p < \infty \ \text{ if } p > r; \tag{1.5}$$

$$E \sup_n \left| \frac{X(\mathbf{n})}{|\mathbf{n}|^{1/r}} \right|^p < \infty, \tag{1.6}$$

$$E \sup_n \left| \frac{S(\mathbf{n})}{|\mathbf{n}|^{1/r}} \right|^p < \infty. \tag{1.7}$$

In this note we wish to generalize the above result of Gut to the general case when the normalizing sequence $\{b(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ is of the type (1.1).

The proof of the main result of this paper will be based on the following lemmas.

LEMMA 1.1. *Let $\alpha_1, \dots, \alpha_k$ be real numbers, $\alpha = \max\{\alpha_i, i = 1, \dots, k\}$; $t_1, \dots, t_k > 0$. Put $p = \max\{\alpha_i t_i,\ i = 1, \dots, k\}$, $q = \operatorname{card}\{i\colon 1 \leqq i \leqq k,\ \alpha_i t_i = p\}$, $r = \operatorname{card}\{i\colon 1 \leqq i \leqq k,\ \alpha_i = 0\}$. For each $x > 0$, put*

$$f(x) = \sum_{n_1^{1/t_1} \dots n_k^{1/t_k} \leqq x} n_1^{\alpha_1 - 1} \cdots n_k^{\alpha_k - 1}.$$

Then we have the following conclusions:

a) *If $\alpha \leqq 0$, then*[1]

$$f(x) \asymp (\log^+ x)^r. \tag{1.8}$$

[1] Here the notation $f(x) \asymp g(x)$ means that there exist constants C_1 and C_2 such that for every x greater than some x_0

$$C_1 g(x) \leqq f(x) \leqq C_2 g(x).$$

b) *If* $\alpha > 0$, *then*

$$f(x) \asymp x^p (\log^+ x)^{q+r-1}. \tag{1.9}$$

PROOF. The proof of this lemma proceeds by induction on the dimension k.

a) Consider first the case $\alpha < 0$. In this case $r = 0$ and it is easy to see that $f(x) < \infty$ for every x, thus the conclusion is evident.

Assume now $\alpha = 0$. The proof proceeds as follows. For $k = 1$, (1.8) is obvious. Now suppose the conclusion holds with $k-1$-dimensions. Obviously,

$$f(x) = \sum_{n_1 \leqq x^{t_1}} n_1^{\alpha_1 - 1} \sum_{n_2^{1/t_2} \dots n_k^{1/t_k} \leqq x / n_1^{1/t_1}} n_2^{\alpha_2 - 1} \cdots n_k^{\alpha_k - 1}.$$

If $\alpha_1 < 0$, then by the induction hypothesis

$$f(x) \asymp \sum n_1^{\alpha_1 - 1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^r. \tag{1.10}$$

Choosing only one term on the right-hand side of (1.10) corresponding to $n_1 = 1$ we get

$$f(x) \geqq C_1 (\log^+ x)^r.$$

On the other hand, for every n_1

$$n_1^{\alpha_1 - 1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^r \leqq n_1^{\alpha_1 - 1} (\log^+ x)^r,$$

so we obtain

$$f(x) \leqq C_2 (\log^+ x)^r \sum_{n_1 \leqq x^{t_1}} n_1^{\alpha_1 - 1} \leqq C_2' (\log^+ x)^r$$

(with $\alpha_1 < 0$ the series $\sum n_1^{\alpha_1 - 1}$ converges). If $\alpha_1 = 0$, then (1.10) becomes

$$f(x) \asymp \sum_{n_1 \leqq x^{t_1}} n_1^{-1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{r-1}.$$

It is easy to verify that

$$\sum_{n_1 \leqq x^{t_1}} n_1^{-1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{r-1} \asymp \int_1^{x^{t_1}} u \left(\log^+ \frac{x}{u^{1/t_1}} \right)^{r-1} du \asymp (\log^+ x)^r.$$

This terminates the first step of the proof.

b) As above the conclusion is true when $k = 1$. Suppose that it holds for $k-1$. We may assume, without loss of generality, that $\alpha_i > 0$, $i = 1, \dots, q$. Then if $\alpha_i t_i = p$ for all $1 \leqq i \leqq q$, it follows from the induction hypothesis that

$$f(x) \asymp x^p \sum_{n_1 \leqq x^{t_1}} n_1^{-1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{q+r-2} \asymp x^p (\log^+ x)^{q+r-1}.$$

If $\alpha_i t_i = p$ does not hold for all $i = 1, \dots, q$, then we may assume that $\alpha_1 t_1 < p$. By induction

$$f(x) \asymp \sum_{n_1 \leqq x^{t_1}} n_1^{\alpha_1 - 1} \left(\frac{x}{n_1^{1/t_1}} \right)^p \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{q+r-1} =$$

$$= x^p \sum_{n_1 \leqq x^{t_1}} n_1^{\alpha_1 - 1 - p/t_1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{q+r-1}.$$

Choosing only one term corresponding to $n_1 = 1$ on the right-hand side of the last expression we obtain

$$f(x) \geqq C_1 x^p (\log^+ x)^{q+r-1}.$$

On the other hand, the sum

$$\sum_{n_1 \leqq x^{t_1}} n_1^{\alpha_1 - 1 - p/t_1} \left(\log^+ \frac{x}{n_1^{1/t_1}} \right)^{q+r-1}$$

is majorized by

$$\sum_1^{\infty} n_1^{-1-(p-\alpha_1 t_1)/t_1} (\log^+ x)^{q+r-1} \leqq C (\log^+ x)^{q+r-1}.$$

The lemma is completely proved.

REMARK. In the case $\alpha_1 \geqq 1, \dots, \alpha_k \geqq 1$ the lemma has been stated and proved by Klesov (Lemma 2, [9], p. 923).

The following result is an extension of Lemma 2.2 of [5].

LEMMA 1.2. *Let E be a Banach space with norm $\| \cdot \|$ and let $\{Y(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ be independent E-valued random variables. Further let $\{a(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ be a set of positive real numbers such that $a(\mathbf{m}) \leqq a(\mathbf{n})$ if $\mathbf{m} \leqq \mathbf{n}$. Set*

$$U(\mathbf{n}) = a^{-1}(\mathbf{n}) \sum_{\mathbf{m} \leqq \mathbf{n}} Y(\mathbf{m}), \quad V = \sup_{\mathbf{n}} \|U(\mathbf{n})\|, \quad W = \sup_{\mathbf{n}} \|a^{-1}(\mathbf{n}) Y(\mathbf{n})\|$$

and suppose that $V < \infty$ a.s. Then $W < \infty$ a.s. and if $EW^p < \infty$ for some $0 < p < \infty$, then $EV^p < \infty$.

REMARK. For $k = 1$, $a_{(n)} \equiv 1$, this is Theorem 3.1 of [8]; for $k = 1$ and general nondecreasing sequences $\{a(n), n = 1, 2, 3, \dots\}$ it is Corollary 3.4 of [8]. For the case $k > 1$ and $a(\mathbf{n})$ there are functions of $|\mathbf{n}|$ such that $a(\mathbf{m}) \leqq a(\mathbf{n})$ if $|\mathbf{m}| \leqq |\mathbf{n}|$, this lemma is Lemma 2.2 of Gut [5].

PROOF. The first statement of the lemma is obvious because of $W \leqq 2^k V$.

For the case $a(n) \equiv 1$ the lemma has been proved in [5].

For the general case of $a(\mathbf{n})$ we proceed roughly as in [5], p. 208. The details are omitted.

2. Results

In this section we state and prove the main results of this note.

THEOREM 2.1. *Let $\{X, X(\mathbf{n}), \mathbf{n} \in \mathbf{N}^k\}$ be i.i.d. random variables and suppose that $EX = 0$ whenever it is finite. Let $0 < t_1, \dots, t_k < 2$, $t = \max\{t_i\}$, $r = \operatorname{card}\{i \colon t_i = t\}$. Let $b(\mathbf{n})$ be of the type (1.1) and $p \geqq t$. The following statements are equivalent:*

$$(2.1)\qquad E|X|^p(\log^+|X|)^r < \infty \ \text{ if } p = t \ \text{ and } E|X|^p < \infty \ \text{ if } p > t,$$

$$(2.2)\qquad E\sup_n \left|\frac{X(\mathbf{n})}{b(\mathbf{n})}\right|^p < \infty,$$

$$(2.3)\qquad E\sup_n \left|\frac{S(\mathbf{n})}{b(\mathbf{n})}\right|^p < \infty.$$

PROOF. (2.1) $\Rightarrow$ (2.2). Define

$$X'(\mathbf{n}) = X(\mathbf{n})I(|X(\mathbf{n})| < b(\mathbf{n})), \quad X''(\mathbf{n}) = X(\mathbf{n}) - X'(\mathbf{n}).$$

Here $I\{\cdot\}$ denotes the indicator function of the set in the braces.

It is evident that

$$(2.4)\quad E\sup_{\mathbf{n}}|b^{-1}(\mathbf{n})X(\mathbf{n})|^p \leqq E\sup_{\mathbf{n}}|b^{-1}(\mathbf{n})X'(\mathbf{n})|^p + E\sup_{\mathbf{n}}|b^{-1}(\mathbf{n})X''(\mathbf{n})|^p.$$

Obviously,

$$E\sup_{\mathbf{n}}|b^{-1}(\mathbf{n})X'(\mathbf{n})|^p \leqq 1.$$

We estimate now the second term on the right-hand side of (2.4). We have by Lemma 1.1

$$E \sup_{\mathbf{n}} |b^{-1}(\mathbf{n})X''(\mathbf{n})|^p \leqq \sum_{\mathbf{n}\in\mathbf{N}^k} b^{-p}(\mathbf{n})E|X''(\mathbf{n})|^p =$$

$$= \sum_{\mathbf{n}\in\mathbf{N}^k} b^{-p}(\mathbf{n}) \int_{b(\mathbf{n})}^{\infty} x^p dP(|X(\mathbf{n})| < x) = \sum_{\mathbf{n}\in\mathbf{N}^k} b^{-p}(\mathbf{n}) \int_{b(\mathbf{n})}^{\infty} x^p (dP(|X| < x) =$$

$$= \sum_{0<b(\mathbf{n})\leqq 1} b^{-p}(\mathbf{n}) \int_{b(\mathbf{n})}^{\infty} x^p dP(|X| < x) + \sum_{1<b(\mathbf{n})\leqq 2} b^{-p}(\mathbf{n}) \int_{b(\mathbf{n})}^{\infty} x^P dP(|x| < x) + \ldots \leqq$$

$$\leqq \sum_{j=1}^{\infty} j^{-p} a_j \int_{j}^{\infty} x^p dP(|X| < x) = \sum_{j=1}^{\infty} j^{-p} a_j \sum_{i=j}^{\infty} \int_{i}^{i+1} x^p dP(|X| < x) =$$

$$= \sum_{i=1}^{\infty} \int_{i}^{i+1} x^p dP(|X| < x) \sum_{j\leqq i} a_j j^{-p} \leqq$$

$$\leqq C \sum_{i=1}^{\infty} i^p P(i \leqq |X| < i+1) \sum_{b(\mathbf{n})\leqq i} b^{-p}(\mathbf{n}) \leqq$$

$$\leqq \begin{cases} C_1 \sum\limits_{i=1}^{\infty} i^p P(i \leqq |X| < i+1) & \text{if } p > t \\ C_2 \sum\limits_{i=1}^{\infty} i^p (\log^+ i)^r P(i \leqq |X| < i+1) & \text{if } p = t. \end{cases}$$

$$\leqq \begin{cases} C_1\, E\,|X|^p & \text{if } p > t, \\ C_2\, E\,|X|^p (\log^+ |X|)^r & \text{if } p = t \end{cases}$$

(where $a_j = \operatorname{card}\{\mathbf{n} : j < b(\mathbf{n}) \leqq j+1\}$). This terminates the first step of the proof.

(2.2) $\Rightarrow$ (2.1). Since

$$\sup_{\mathbf{n}} |X(\mathbf{n})/b(\mathbf{n})|^p \geqq |X(\mathbf{1})| \tag{2.2}$$

implies that $E\,|X|^p < \infty$ and therefore (2.2) is trivially necessary for (2.1) to hold if $p > t$.

Assume now that $p = t$. When $r = 0$, proceeding as in the first step of the proof above we find that the conclusion is true. Suppose now that $r > 0$. It is in fact no loss of generality to assume that $t_1 = \ldots = t_r = t = p$. In this case it is easy to see that

$$(2.5) \qquad E\sup_{\mathbf{n}} |b^{-1}(\mathbf{n})X(\mathbf{n})|^p \geqq E\sup_{\mathbf{n}} \left| \frac{X(\mathbf{n}')}{n_1^{1/p}\cdots n_r^{1/p}} \right|^p = \\ = E \sup_{\mathbf{n}\in\mathbf{N}^r} |\mathbf{n}'|^{-1}|X(\mathbf{n}')|^p.$$

Here $\mathbf{n}' = (n_1, \dots, n_r, 1, \dots, 1) \in \mathbf{N}^r \times \{1\} \times \dots \times \{1\} \simeq \mathbf{N}^r$.

According to Theorem 1.1, (2.5) is equivalent to $E|X|^p(\log^+|X|)^r < \infty$. Thus (2.1) follows.

The proof of (2.2) $\Leftrightarrow$ (2.3) is an appropriate modification of that given in [5] for Theorem 3.2.

(2.2) $\Rightarrow$ (2.3). Because of (2.1) the Marcinkiewicz–Zygmund law is true (see [9], Corollary 1, p. 918). It follows that $V = \sup_{\mathbf{n}} |b^{-1}(\mathbf{n})S(\mathbf{n})| < \infty$ a.s. Since (2.2) holds, that is $E\,W^p < \infty$, where $W = \sup_{\mathbf{n}} |b^{-1}(\mathbf{n})X(\mathbf{n})|$, an application of Lemma 1.2 yields $E\,V^p < \infty$, i.e. (2.3).

(2.3) $\Rightarrow$ (2.2). Immediate, because of $W \leqq 2^k V$. The theorem is completely proved.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HANOI
THUONG DINH, DONG DA
HANOI, VIETNAM

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ON ENUMERATION OF SPANNING SUBGRAPHS WITH A PREASSIGNED CYCLOMATIC NUMBER IN A GRAPH

Y. CHOW (Milwaukee)

1. Introduction

For a given connected (labelled) *planar* graph, the enumeration of its connected spanning subgraphs containing the same number of "cycles" was carried out recently [1] by a method using the real vector space of some formal sums and certain operators defined on the space. By the number of *cycles* of a connected graph (or subgraph) G here we mean the cyclomatic number $\chi(G)$ defined by

$$\chi(G) = |e(G)| - |v(G)| + 1$$

with e and v being the edge set and the vertex set, respectively. $|(\ )|$ denotes the cardinal number of a set. The purpose of this article is to treat the case of a graph whether planar or *non-planar*.

It is interesting for reasons of exposition to mention the following fact. Shortly after the completion of the work which appeared in [1], we were led by the *duality* concept to a solution of enumeration of spanning (m-tree) *forests* for any given (labelled) graph with no planarity restriction by an approach similar to that of [1] used in dealing with *cycles*, thus achieved a generalization of the celebrated Kirchhoff's matrix-tree theorem to the case of m-forests, to be referred to hereafter simply as the matrix-forest theorem (Theorem I of [2]; also [4] in a different form). That the solution of the "cycle" problem for *planar* graphs could lead to the solution of the "forest" problem for *any* graph (planar or non-planar) is due to the very fortunate and crucial fact: even though the concept of "duality" is well-defined for planar graphs only, i.e. a technique used on a graph can sometimes be passed over to its *dual* graph under certain correspondence, yet in the particular case of the matrix-forest theorem one realizes that the proof in fact involves *no planarity arguments.* Hence the theorem stands for the general case of any graph. However, it is not possible to use duality on the result of a matrix-forest theorem to attack the cycle enumeration problem for *non-planar* graphs, as *dual* graphs are no longer defined for these graphs. Our approach here is to use some subgraphs (i.e. "deletion") and also the graphs obtained by identifying certain vertices ("contractions") of the given graph, usually in several consecutive steps. The procedure allows repeated iterations so the final computation may involve comparatively simple graphs since the number of edges and the number of vertices are reduced. In fact, for a non-planar graph, one may stop this process at the stage when all the relevant graphs (subgraphs or contracted graphs) involved become *planar*. Then the operator method of [1] (cf. also [3] for computational details) can take over. Thanks are due to Professors C. J. Liu and R. McQuistan for helpful comments. I would like to especially thank Professor Bennett Chow for his continued interest and encouragement.

Main results

We first fix some terminologies. By the "contraction" of the edges $e_{ij}^{(m)}$ (m = multiplicity label for the different edges), joining the pair of vertices v_i and v_j we mean that all edges $e_{ij}^{(m)}$ are first deleted, then the two vertices v_i and v_j are identified. For convenience, we shall label the new vertex v_i if $i < j$. We shall refer to such an operation as the "contraction" of the edges e_{ij}. Either a "deletion" or a "contraction", as just defined, will be called a "reduction". We shall use the following notation:

$$(1)\quad \begin{cases} \dot{e}_{ij} & \equiv \text{the contraction of all edges } e_{ij} \text{ (regardless of multiplicity of edges)} \\ \hat{e}_{ij} & \equiv \text{the deletion of all edges } e_{ij} \text{ (regardless of multiplicity of edges)} \\ S_{ij} & \equiv \text{the set of all edges joining vertices } v_i \text{ and } v_j \\ m_{ij} & \equiv \text{the multiplicity of } e_{ij}, \text{ i.e. the number of edges joining directly vertices } v_i \text{ and } v_j. \end{cases}$$

The graphs considered here are connected, labelled, undirected and without self-loops. We shall first analyze the enumeration of connected spanning subgraphs with a fixed cyclomatic number, i.e. all the subgraphs to be enumerated have a fixed χ. Denote by $\sigma_\chi(G)$ the total number of such subgraphs of a graph G (G can be either planar or non-planar). Consider first $\sigma_1(G)$ where the given graph G is connected and labelled but is in general a *multigraph.* Then we have the following basic property:

For any pair of vertices i and j,

$$(2)\qquad \sigma_1(G) = \sigma_1(\hat{e}_{ij}G) + m_{ij}\sigma_1(\dot{e}_{ij}G) + \binom{m_{ij}}{2}\sigma_0(\dot{e}_{ij}G),$$

provided that $\hat{e}_{ij}G$ remains connected.

The proof of (2) is straightforward. Let us check the terms on the right-hand side of (2). The first term is the contribution of those subgraphs (in G) that contain *none* of the edges from S_{ij}. The second and third terms count, respectively, those subgraphs containing *one* and *two* of the edges from S_{ij}. One of the important features of expression (2) is its capability for iteration, i.e. (2) can be repeatedly applied to the reduced graphs which appeared on its right-hand side. As it stands the double subscripts i, j which appeared in (2) are too tedious to write after several iterations, therefore we simplify now the notation by using a single subscript. First we write e_N in place of $e_{N,N-1}$: the rule is that we shall hereafter always label the vertices in G in the reversed order according to which we proceed to reduce the edges. So the first pair of vertices to be reduced will be the pair $N, N-1$ and we shall label the coalesced vertex $N-1$ if the reduction is a *contraction*. We shall next reduce the pair $N-1, N-2$ and shall label the coalesced vertex $N-2$ if it is a *contraction*, and so forth. This simplification of notation also helps us to keep track of the number of iterations involved with ease. It is also important to realize that in general the operations of "deletion" and "contraction" do not commute, i.e. for instance

$$\dot{e}_{N-1}\hat{e}_N \neq \hat{e}_{N-1}\dot{e}_N$$

as depicted in Fig. 1.

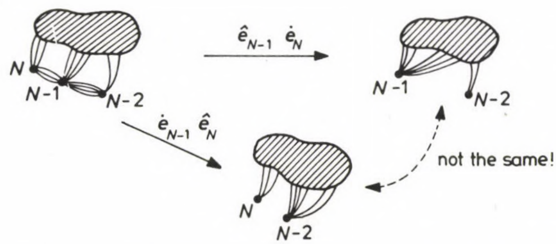

Fig. 1. Non-commutativity of "deletion" and "contraction"

Following these conventions, (2) can be written into

$$\sigma_1(G) = \sigma_1(\hat{e}_N G) + \binom{m_N}{1}\sigma_1(\dot{e}_N G) + \binom{m_N}{2}\sigma_0(\dot{e}_N G) \tag{3}$$

or simply

$$\sigma_1(G) = \sigma_1(\hat{e}_N G) + m_N\sigma_1(\dot{e}_N G) + \frac{1}{2}m_N(m_N - 1)\sigma_0(\dot{e}_N G). \tag{4}$$

Before we can carry out the iteration of (3) we must look into the effect of $\hat{e}_N G$ and $\dot{e}_N G$ on the *multiplicities of edges*. In other words, what will be the *new* multiplicities in $\hat{e}_N G$ and also in $\dot{e}_N G$; they will be different from those of G where the symbol m_N is used. First, the only change in $\hat{e}_N G$, from G, is that the *new* m_N, denoted by $\hat{m}_N$, is simply $\hat{m}_N = 0$ (i.e. there is no edge linking directly v_N and v_{N-1} as a result of the deletion $\hat{e}_N$). Next, the only changes in $\dot{e}_N G$, from G, are the absence of the vertex N and the change of multiplicity from the original $m_{N-1,i}$ to the *new* multiplicity $m_{N-1,i}$ given by

$$m^c_{N-1,i} - m_{N-1,i} = m_{N,i}, \qquad i = 1, 2, \ldots, N-2 \tag{5}$$

where the superscript c in (5) denotes a single "contraction" and where the subscripts conform to that in (1). Therefore, we have

$$\sigma_1(\hat{e}_N G) = \tag{6}$$
$$= \sigma_1(\hat{e}_{N-1}\hat{e}_N G) + m_{N-1}\sigma_1(\dot{e}_{N-1}\hat{e}_N G) + \frac{1}{2}m_{N-1}(m_{N-1}-1)\sigma_0(\dot{e}_{N-1}\hat{e}_N G)$$

and

$$\sigma_1(\dot{e}_N G) = \tag{7}$$
$$= \sigma_1(\hat{e}_{N-1}\dot{e}_N G) + m^c_{N-1}\sigma_1(\dot{e}_{N-1}\dot{e}_N G) + \frac{1}{2}m^c_{N-1}(m^c_{N-1}-1)\sigma_0(\dot{e}_{N-1}\dot{e}_N G)$$

where m^c_{N-1} is given by (5). We may now do the *first iteration* of (3); a substitution of (6) and (7) into (3) yields

$$\begin{aligned} \sigma_1(G) = {} & \sigma_1(\hat{e}_{N-1}\hat{e}_N G) + m_{N-1}\sigma_1(\dot{e}_{N-1}\hat{e}_N G) + m_N\sigma_1(\hat{e}_{N-1}\dot{e}_N G) + \\ & + m_N m^c_{N-1}\sigma_1(\dot{e}_{N-1}\dot{e}_N G) + \frac{1}{2}m_{N-1}(m_{N-1}-1)\sigma_0(\dot{e}_{N-1}\hat{e}_N G) + \\ = {} & \frac{1}{2}m_N m^c_{N-1}(m^c_{N-1}-1)\sigma_0(\dot{e}_{N-1}\dot{e}_N G) + \frac{1}{2}m_N(m_N-1)\sigma_0(\dot{e}_N G). \end{aligned} \tag{8}$$

By carrying out the *second* iteration one finds that all the multiplicity factors are due to *contractions* ($\dot{e}$); their positions among the reduction operators decide the coefficients, e.g. if $\dot{e}$ is in the rightest position it contributes a factor m_N for σ_1 and a factor $\binom{m_N}{2}$ for σ_0 when $\dot{e}$ is the only operator acting on G. Some details of the second iteration of (3) are given in the Appendix.

The general expression of σ_n is

$$\sigma_n(G) = \sigma_n(\hat{e}_N G) + \sum_{i=1}^{n+1} \binom{m_N}{n+2-i} \sigma_{i-1}(\dot{e}_N G) \tag{9}$$

following a similar proof as that of (3). Iterations can be carried out in the same way as for σ_1.

Since, by the algebraic method of [1], we can make direct calculations once the graphs involved are reduced to *planar* ones, it is useful to define the "planarity reduction index" ζ. We say that $\zeta = \kappa$ if κ is the minimum number of reductions such that all the graphs resulted from these κ consecutive reductions (i.e. either "deletion" or "contraction") are *planar* and still *connected*. Obviously a planar graph has $\zeta = 0$ by definition. As an example, $\zeta(G) = 3$ if all the following eight graphs are *connected* and *planar*: $\dot{e}\dot{e}\dot{e}G$, $\dot{e}\dot{e}\hat{e}G$, $\dot{e}\hat{e}\dot{e}G$, $\hat{e}\dot{e}\dot{e}G$, $\dot{e}\hat{e}\hat{e}G$, $\hat{e}\dot{e}\hat{e}G$, $\hat{e}\hat{e}\dot{e}G$, $\hat{e}\hat{e}\hat{e}G$ where $\dot{e}\dot{e}\dot{e}G$ denotes

$\dot{e}_{N-2}\dot{e}_{N-1}\dot{e}_N G$ and so on. Therefore, it is only necessary to go through ζ steps of iterations to yield an expression that can be readily computed by the algebraic method of [1]. It is also clear that there can be different series of reductions with the same ζ, in general. It is important to note that since the purpose of the present method is to treat a *non-planar* graph, by means of suitable reductions, in terms of its planar reduced graphs we will always choose a minimum number of reductions (which defines ζ) to do the job. Hence the situation of having *disconnected* reduced graphs will not happen under this situation. In other words, in the above example of $\zeta(G) = 3$ the situation of e.g. $\hat{e}\hat{e}\hat{e}G$ being disconnected will not arise.

The examples of non-planar graphs $K_{3,3}$ and K_5

Consider first the non-planar graph $K_{3,3}$. We have $\zeta(K_{3,3}) = 1$, as it is easily seen that both $\dot{e}K_{3,3}$ and $\hat{e}K_{3,3}$ are planar, in this case for *any* edge e (Fig. 5). We shall compute only σ_1 here since σ_2, etc. can be computed by the same method. By (2), we have

$$\sigma_1(K_{3,3}) = \sigma_1(\hat{e}_6 K_{3,3}) + \sigma_1(\dot{e}_6 K_{3,3}). \tag{10}$$

First, we compute $\sigma_1(\hat{e}_6 K_{3,3})$ following the method of [1]. The so-called cycle-adjacency matrix can be written into, according to the cycle assignments of Fig. 6(a),

$$\hat{\mathbf{E}} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -2 \\ -1 & -2 & 4 \end{bmatrix}. \tag{11}$$

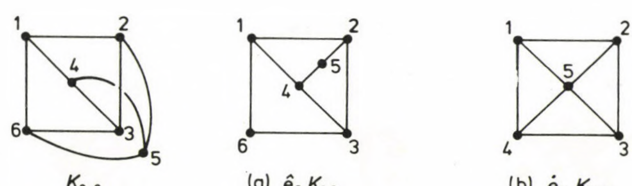

Fig. 5. The graph $K_{3,3}$ and some reduced graphs

The computation can be carried out by the formula (cf. (67) of [3])

$$\sigma_1(\hat{e}_6 K_{3,3}) = \sum_{i=1}^{3} \det \hat{\mathbf{E}}(i) + \sum_{1 \leqq i < j \leqq 3} \hat{E}_{ij} \det \hat{\mathbf{E}}(i,j). \tag{12}$$

We find, by (12),

$$\sigma_1(\hat{e}_6 K_{3,3}) = 26. \tag{13}$$

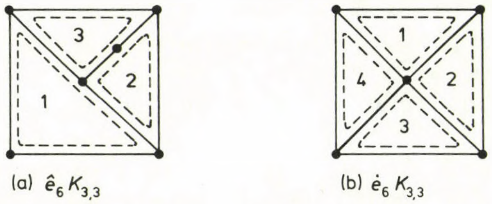

Fig. 6. Cycle labellings of the reduced graphs

To compute $\sigma_1(\dot{e}_6K_{3,3})$, we assign cycle labels as shown in Fig. 6(b). The corresponding cycle-adjacency matrix is

$$\dot{\mathbf{E}} = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}. \tag{14}$$

The computational formula to be used is

$$\sigma_1(\dot{e}_6K_{3,3}) = \sum_{i=1}^{4} \det \dot{\mathbf{E}}(i) + \sum_{1 \leqq i < j \leqq 4} \dot{E}_{ij} \det \dot{\mathbf{E}}(i,j) \tag{15}$$

from which we find

$$\sigma_1(\dot{e}_6K_{3,3}) = 52. \tag{16}$$

Therefore, by (10),

$$\sigma_1(K_{3,3}) = 26 + 52 = 78, \tag{17}$$

i.e. $K_{3,3}$ has 78 connected spanning subgraphs each of which has exactly *one* cycle. This can be easily checked graphically.

Consider next the non-planar graph K_5. Again, it is easy to see that $\zeta(K_5) = 1$. According to the labellings of Fig. 8(a) we have, for $\hat{e}_5K_5$, the cycle-adjacency matrix:

$$\hat{\mathbf{E}} = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix}. \tag{18}$$

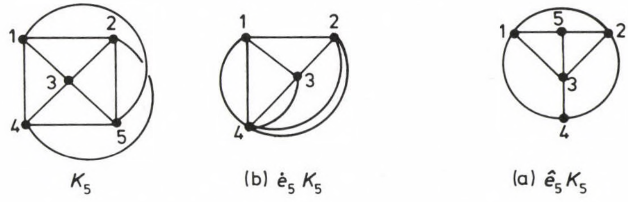

Fig. 7. The graph K_5 and some reduced graphs

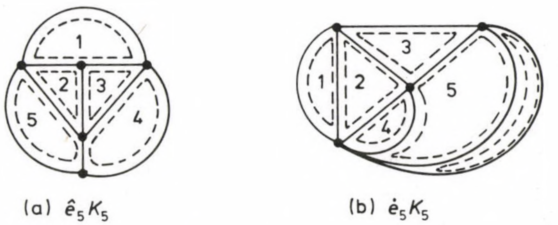

Fig. 8. Cycle labellings of two reduced graphs of K_5

The computational formula will be, similar to (12),

$$\sigma_1(\hat{e}_5 K_5) = \sum_{i=1}^{5} \det \hat{\mathbf{E}}(i) + \sum_{1 \leqq i < j \leqq 5} \hat{E}_{ij} \det \hat{\mathbf{E}}(i,j) \tag{19}$$

from which we get

$$\sigma_1(\hat{e}_5 K_5) = (45+55+55+40+40)-(21+21+24+21+16+21) = 111. \tag{20}$$

Next, we compute $\sigma_1(\dot{e}_5 K_5)$ in a similar manner. The cycle-adjacency matrix corresponding to Fig. 8(b) is

$$\dot{E} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}. \tag{21}$$

The computational formula will be

$$\sigma_1(\dot{e}_5 K_5) = \sum_{i=1}^{6} \det \dot{\mathbf{E}}(i) + \sum_{1 \leqq i < j \leqq 6} \dot{E}_{ij} \det \dot{\mathbf{E}}(i,j). \tag{22}$$

We find

(23)
$$\sigma_1(\dot{e}_5K_5) = (35+40+30+55+40+35)-(20+16+26+20+16+26) = 111.$$

Combining (20) and (23) according to

$$\sigma_1(K_5) = \sigma_1(\hat{e}_5K_5) + \sigma_1(\dot{e}_5K_5), \tag{28}$$

we get

$$\sigma_1(K_5) = 111 + 111 = 222, \tag{29}$$

i.e. K_5 has 222 connected spanning subgraphs each of which has exactly *one* cycle. This has been checked graphically. Computations for σ_2 and so on are entirely similar and we shall not carry them out here any further.

An important remark is in order: *further* iterations can be carried out in general *even after* the associated reduced graphs become *planar* as long as they are still *connected* graphs. In this way, the computations may sometimes be reduced to the extremely simple ones. However, one must realize that the reduction operations are "geometrical" (i.e. "graphical") procedures. Until one can formulate edge-deletions as well as contractions in a simple *algebraic* fashion, there is often some competition between "algebraic" and "geometric" procedures in this kind of mixed computations. Their relative rule is, therefore, decided by the over-all computational efficiency.

Appendix

It is very instructive to carry out the *second iteration* of (3). We need to expand all the four σ_1 terms in (8) since σ_0, counting trees, can be computed directly by Kirchhoff's matrix-tree theorem. First, we have

$$\begin{aligned}\sigma_1(\hat{e}_{N-1}\hat{e}_NG) &= \sigma_1(\hat{e}_{N-2}\hat{e}_{N-1}\hat{e}_NG) + m_{N-2}\sigma_1(\dot{e}_{N-2}\hat{e}_{N-1}\hat{e}_NG)+\\ &+\frac{1}{2}m_{N-2}(m_{N-2}-1)\sigma_0(\dot{e}_{N-2}\hat{e}_{N-1}\hat{e}_NG).\end{aligned} \tag{A.1}$$

Next, in order to expand $\sigma_1(\dot{e}_{N-1}\hat{e}_NG)$ by means of (3) we have first to figure out the numerical factor which appeared before $\sigma_1(\dot{e}_{N-2}\dot{e}_{N-1}\hat{e}_NG)$, i.e. the multiplicity of edges directly connecting $N-2$ and $N-3$ in the reduced graph $\dot{e}_{N-1}\hat{e}_NG$. This is just (see Fig. 2)

$$m_{N-2} + m_{N-1,N-3} \equiv m^{cd}_{N-2} \tag{A.2}$$

where the superscripts *cd* mean "contraction"-"deletion" in that order.

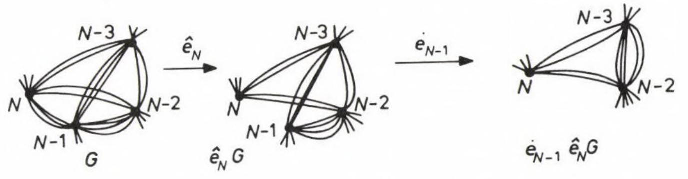

Fig. 2. Expression (A.2)

Hence we have, by (3),

$$\sigma_1(\dot{e}_{N-1}\hat{e}_N G) = \sigma_1(\hat{e}_{N-2}\dot{e}_{N-1}\hat{e}_N G) + m_{N-2}^{cd}\sigma_1(\dot{e}_{N-2}\dot{e}_{N-1}\hat{e}_N G) + \tag{A.3}$$
$$+\frac{1}{2}m_{N-2}^{cd}(m_{N-2}^{cd}-1)\sigma_0(\dot{e}_{N-2}\dot{e}_{N-1}\hat{e}_N G).$$

Next, we have (see Fig. 3)

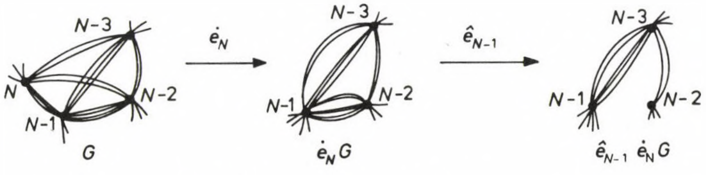

Fig. 3. Expression (A.5)

$$\sigma_1(\hat{e}_{N-1}\dot{e}_N G) = \sigma_1(\hat{e}_{N-2}\hat{e}_{N-1}\dot{e}_N G) + m_{N-2}^{dc}\sigma_1(\dot{e}_{N-2}\hat{e}_{N-1}\dot{e}_N G) + \tag{A.4}$$
$$+\frac{1}{2}m_{N-2}^{dc}(m_{N-2}^{dc}-1)\sigma_0(\dot{e}_{N-2}\hat{e}_{N-1}\dot{e}_N G)$$

with

$$m_{N-2}^{dc} \equiv m_{N-2}. \tag{A.5}$$

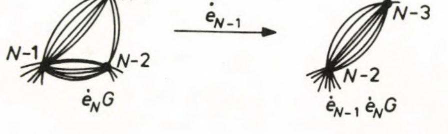

Fig. 4. Expression (A.7)

Then finally (see Fig. 4),

$$\text{(A.6)}\qquad \sigma_1(\dot e_{N-1}\dot e_N G)=\sigma_1(\hat e_{N-2}\dot e_{N-1}\dot e_N G)+m^{cc}_{N-2}\sigma_1(\dot e_{N-2}\dot e_{N-1}\dot e_N G)+ \\ +\frac{1}{2}m^{cc}_{N-2}(m^{cc}_{N-2}-1)\sigma_0(\dot e_{N-2}\dot e_{N-1}\dot e_N G)$$

with

$$\text{(A.7)}\qquad m^{cc}_{N-2}\equiv m_{N-2}+m_{N,N-3}+m_{N-1,N-3}.$$

Before we put (A.1), (A.3), (A.4), (A.6) into (8) we need to simplify our notation. We shall write hereafter

$$\text{(A.8)}\qquad \dot e\dot e\dot e G\equiv \dot e_{N-2}\dot e_{N-1}\dot e_N G,\ \ \dot e\hat e\dot e G\equiv \dot e_{N-2}\hat e_{N-1}\dot e_N G, \\ \hat e\hat e\dot e G\equiv \hat e_{N-2}\hat e_{N-1}\dot e_N G,\ \ \text{etc.}$$

i.e. all the subscripts of e will be omitted. Define next the "direct sum" $\oplus$. For any real numbers a, b, c, d we have

$$\text{(A.9)}\qquad \sigma_k\{(a\dot e\hat e\dot e\oplus b\dot e\hat e\dot e\dot e\oplus c\dot e\hat e\oplus d)G\}\equiv \\ \equiv a\sigma_k(\dot e\hat e\dot e G)+b\sigma_k(\dot e\hat e\dot e\dot e G)+c\sigma_k(\dot e\hat e G)+d\sigma_k(G)$$

and also, for instance,

$$\dot e(a\hat e\oplus b\dot e\hat e\dot e)=a\dot e\hat e\oplus b\dot e\dot e\hat e\dot e,\quad \text{etc.}$$

for real numbers a and b. (8) becomes, after the second iteration,

$$\text{(A.11)}\ \sigma_1(G)=\sigma_1\{\hat e[\hat e\hat e\oplus m_{N-1}\dot e\hat e\oplus m_N\hat e\dot e\oplus(m_{N,N-2}+m_{N-1})m_N\dot e\dot e]G\}+ \\ +\sigma_1\{\dot e[m_{N-2}\hat e\hat e\oplus(m_{N-1,N-3}+m_{N-2})m_{N-1}\dot e\hat e\oplus m_{N-2}m_N\hat e\dot e\oplus \\ \oplus 4(m_{N,N-3}+m_{N-1,N-3}+m_{N-2})(m_{N,N-2}+m_{N-1})m_N\dot e\dot e]G\}+ \\ +\frac{1}{2}\sigma_0\Big\{\dot e[m_{N-2}(m_{N-2}-1)\hat e\hat e\oplus \\ \oplus(m_{N-1,N-3}+m_{N-2})(m_{N-1,N-3}+m_{N-2}-1)m_{N-1}\dot e\hat e\oplus \\ \oplus m_{N-2}(m_{N-2}-1)m_N\hat e\dot e\oplus(m_{N,N-3}+m_{N-1,N-3}+m_{N-2})\cdot \\ \cdot(m_{N,N-3}+m_{N-1,N-3}+m_{N-2}-1)(m_{N,N-2}+m_{N-1})m_N\dot e\dot e\oplus \\ \oplus m_{N-1}(m_{N-1}-1)\hat e\oplus(m_{N,N-2}+m_{N-1})(m_{N,N-2}+m_{N-1}-1)m_N\dot e\oplus \\ \oplus m_N(m_N-1)]G\Big\},$$

where $m_i\equiv m_{i,i-1}$ as defined before. We emphasize that the "direct sum" $\oplus$ used here is a computational device that has *no* clear graphic meaning.

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DEPARTMENT OF PHYSICS
UNIVERSITY OF WISCONSIN
MILWAUKEE, WI 53201
U.S.A.

[1] In spite of the actual publication dates, reference [1] was submitted before reference [2].

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QUASIIDEALS IN ALTERNATIVE RINGS[1]

J. A. LALIENA (Logroño)

1. Introduction

The notion of quasiideal is a generalization of the notion of one-sided ideal. It was introduced by O. Steinfeld in 1953 [4] for associative rings and semigroups. Quasiideals have been the theme of several papers (Clifford [1], Lajos and Szász [2], Lhu [3], Weinert [8], etc.) and an interesting monograph, [7], has been published containing the principal results. Quasiideals are useful in associative rings, for example, in the characterization of division rings, in the characterization of von Neumann regular rings and elements, and also for some decomposition theorems of semiprime rings.

Here we are going to extend the notion of quasiideal to alternative rings (§1). Also we study the minimal quasiideals in general alternative rings (§2) and then in semiprime alternative rings (§3). In §4, as in associative rings, we will give for certain semiprime alternative rings some decomposition theorems based on quasiideals.

An alternative ring is a ring not necessarily associative such that it satisfies the identities $x(xy) = x^2y$ and $(yx)x = yx^2$. We denote by A an alternative ring. If $X, Y \subset A$, then XY means the subgroup of $(A, +)$ spanned by the elements xy with $x \in X$ and $y \in Y$, and $XA^{\#}$ means $XA + X$. If $Z \subset A$, we understand by $[X, Y, Z]$ the abelian subgroup spanned by the associators $[x, y, z] = (xy)z - x(yz)$ with $x \in X$, $y \in Y$, $z \in Z$. The least ideal of A containing the set of all associators of A is $[A, A, A]A^{\#} = A^{\#}[A, A, A]$. This ideal is called the *associator ideal* and it is denoted by $D(A)$ or D. Also for an alternative ring the nucleus is defined by $N(A) = \{x \in A | [x, a, b] = 0\ \forall a, b \in A\}$. The biggest ideal of A contained in $N(A)$ is called the maximal nuclear ideal of A and it is denoted by $U(A)$ or U. Let C, D be subrings of A, then $C \oplus D$ means the direct sum of the subrings C and D.

We recall also that simple nonassociative alternative rings with divisors of zero have a basis over their center: $\{x_i\} \cup \{y_i\}$ with $i = 0, 1, 2, 3$ such that

$$x_iy_0 = x_i,\quad x_0x_i = x_i,\quad y_ix_0 = y_i,\quad y_0y_i = y_i,\quad x_0x_0 = x_0,\quad y_0y_0 = y_0,$$
$$y_ix_i = -y_0,\quad x_iy_i = -x_0,\quad -x_{i+1}x_i = x_ix_{i+1} = y_{i+2},$$
$$-y_{i+1}y_i = y_iy_{i+1} = x_{i+2}\quad \text{with } i = 1, 2, 3$$

[1] This paper is a part of the author's doctoral dissertation directed by Professor Santos González at University of Zaragoza.

where the indices are taken modulo 3, and the other products are zero.

§1. Definition and first properties

Let A be an alternative ring and Q a subgroup of $(A,+)$, then Q is quasiideal of A, abbreviated Q q.i. A, if

$$(QA+(QA)A+\dots)\cap(AQ+A(AQ)+\dots)\subset Q.$$

A more useful characterization of this notion is given by the following result.

PROPOSITION 1.1. *Let A be an alternative ring and Q a subgroup of $(A,+)$. Then Q is quasiideal of A if and only if $[A,A,Q]\subset Q$ and $(AQ+$ $+[A,A,Q])\cap(QA+[Q,A,A])\subset Q$.*

PROOF. We suppose first Q is a quasiideal of A. Then $[A,A,Q]\subset AQ+$ $+A(AQ)$. From the alternation of the associator $[A,A,Q]=[Q,A,A]\subset$ $\subset QA+(QA)A$. Thus $[A,A,Q]\subset Q$ and also $(AQ+[A,A,Q])\cap(QA+$ $+[Q,A,A])\subset Q$.

Conversely, let Q be a subgroup of $(A,+)$ such that $[A,A,Q]\subset Q$ and $(AQ+[A,A,Q])\cap(QA+[Q,A,A])\subset Q$. To see that Q is a quasiideal it is sufficient to prove $A(A\dots(AQ))\subset AQ+[A,A,Q]$ and also $((QA)\dots A)A\subset$ $\subset QA+[A,A,Q]$. We only prove the first one, and we do this by induction on n, the number of factors in $A(A\dots(AQ))$. For $n=1$ this is clear. For $n=2$, $A(AQ)\subset AQ+[A,A,Q]$. Suppose it is true for $n-1$. Then by the induction hypothesis $A(A\dots(AQ))\subset A(AQ+[A,A,Q])\subset(AA)Q+[A,A,Q]+AQ\subset$ $\subset AQ+[A,A,Q]$, because $[A,A,Q]\subset Q$ and case $n=2$.

OBSERVATIONS. 1. Another characterization of quasiideals for a subgroup Q of $(A,+)$ in an alternative ring A is the following: Q is quasiideal of A if and only if $[A,A,Q]\subset Q$ and $(AQ+(A(AQ))\cap(QA+(QA)A)\subset Q$.

2. Each one-sided ideal of an alternative ring is a quasiideal. However, not all quasiideals are one-sided ideals. For example: Let A be the subring of a split Cayley–Dickson algebra over F spanned as F-subspace by the elements $\{x_0,x_1,x_2,y_0,y_1,y_3\}$. Let Q be spanned over F by $\{x_0,x_2,y_3\}$. Then Q is a quasiideal but not a one-sided ideal.

3. Every quasiideal of an alternative ring is a subring, but not every subring is quasiideal. There are many examples of this.

If $B\leqq A$ and Q q.i. A, then $Q\cap B$ q.i. B.

4. The intersection of every set of quasiideals of A is also a quasiideal. Thus if $X\subset A$, the q.i. of A spanned by X is

$$\cap\{Q\mid Q\ \text{ q.i. } A \text{ such that } \ X\subset Q\}.$$

5. The sum of quasiideals need not be a quasiideal. An example is given in [7].

Properties:

(P.1) Let Q be a q.i. of A. If X is a subring of A such that $(AQ + [A,A,Q]) \cap ((QA + [Q,A,A]) \leqq X \leqq Q$, then X is a q.i. A.

(P.2) i) The intersection of a left and a right ideal is q.i. ii) Every q.i. is a quadratic ideal ($q \in Q$ implies $qAq \leqq Q$).

(P.3) Let A be an alternative ring and let Q be a q.i. of A such that $Q \leqq QA$ or $Q \leqq AQ$. Then Q is the intersection of $Q + AQ$ and $Q + QA$.

(P.4) If A is a unitary ring, every q.i. of A is the intersection of a left and a right ideal of A.

(P.5) Let Q be q.i. A such that Q is not one-sided ideal of A. Then $AQ + [A,A,Q]$ is a proper left ideal of A and $QA + [Q,A,A]$ is a proper right ideal of A.

PROPOSITION 1.2. *Let A be an alternative ring such that $A^2 \neq 0$. Then A is a division associative ring or a Cayley–Dickson algebra over its center if and only if A does not have proper quasiideals.*

PROOF. If A is a division associative ring, it is clear that A does not have proper quasiideals. If A is a Cayley–Dickson algebra over its center, A is a unitary ring and from (P.4) every q.i. of A is the intersection of a left ideal and a right ideal. But from the Corollary to Lemma 10.2 in [9], a Cayley–Dickson algebra has not got proper one-sided ideals. Therefore A does not have proper quasiideals.

Conversely, suppose A does not have proper quasiideals. In particular A does not have one-sided ideals. Thus A is simple, and therefore A will be a simple associative ring or a Cayley–Dickson algebra. If A is a simple associative ring without one-sided ideals and $A^2 \neq 0$ it is known that A is a division ring.

§2. Minimal quasiideals of alternative rings

Often, if we are going to study some special subset of a ring, it is interesting to know the minimal elements among these subsets.

We will say that $0 \neq Q$ q.i. A is a minimal quasiideal (q.i.m.) if Q is contained properly in no one quasiideal of A.

PROPOSITION 2.1. *Let L be a minimal left ideal of A and let R be a minimal right ideal of A. Then $R \cap L$ is either* 0 *or a q.i.m. A.*

PROOF. We denote $Q = R \cap L$. We know Q is q.i. Now we suppose $Q \neq 0$ and we will show Q is minimal. Let $0 \neq Q' \neq Q$ and $Q' \subset Q$ q.i. A. Q' is not a one-sided ideal of A, since R and L are minimal. From (P.5) $AQ' + [A,A,Q']$ is a proper left ideal of A and it is contained in L. From L being minimal, $AQ' + [A,A,Q'] = 0$ or $AQ' + [A,A,Q'] = L$. If $AQ' + [A,A,Q'] = 0$, then Q' would be a left ideal, but this is not true. Therefore $AQ' + [A,A,Q'] = L$. Similarly $Q'A + [A,A,Q'] = R$. Thus

$$Q' \subset Q = L \cap R = (AQ' + [A,A,Q']) \cap (Q'A + [A,A,Q']) \subset Q'.$$

Therefore $Q = Q'$. Contradiction.

PROPOSITION 2.2. *Let Q be q.i.m. A. Then either* 1. $Q \subset N(A)$ *and if* $Q^2 \neq 0$, Q *is a division associative ring such that* $Q = eAe$ *with e a nuclear idempotent of A; or* 2. $Q \subset D(A)$.

PROOF. We consider $A^{\#}[A, A, Q]$. This is a left ideal: $A(A[A, A, Q]) \subset \subset [A, A, [A, A, Q]] + (A^2[A, A, Q]) \subset [A, A, Q] + A^2[A, A, Q] \subset A^{\#}[A, A, Q]$.

Similarly $[A, A, Q]A^{\#}$ is a right ideal. Therefore $A^{\#}[A, A, Q] \cap [A, A, Q]A^{\#}$ is a q.i. contained in Q. From Q minimal either $A^{\#}[A, A, Q] \cap [A, A, Q]A^{\#} = 0$ or $A^{\#}[A, A, Q] \cap [A, A, Q]A^{\#} = Q$. In the first case $[A, A, Q] = 0$ and thus $Q \subset N(A)$. If we suppose $Q^2 \neq 0$, from Theorem 6.5 in [7], Q is a division ring and $Q = eAe$, with e a nuclear idempotent of A. In the second case $Q \subset D(A)$.

Now from the proof of Proposition 2.2 and the proof of Theorem 7.2 in [7] we have the following corollary:

COROLLARY 2.3. *Let Q be q.i.m. of A such that $Q^2 \neq 0$, then Q is the intersection of a right and a left ideal of A.*

There are minimal quasiideals of A contained in $D(A)$ with square zero. For example, take the subalgebra of a split Cayley–Dickson algebra with basis over the center: $\{x_0, x_1, x_2, y_0, y_1, y_3\}$ and let Q be spanned over the center by $\{x_2, y_3\}$.

PROPOSITION 2.4. *Let Q be q.i. A such that Q is a division subring of A or Q is a Cayley–Dickson algebra over its center. Then Q is q.i.m. A.*

PROOF. It follows from Proposition 1.2.

PROPOSITION 2.5. *The product RL of a minimal right ideal R of A and a minimal left ideal L of A is either* 0 *or a minimal quasiideal of A.*

PROOF. We will suppose $RL \neq 0$. We denote $A(RL) + A(A(RL)) + \dots$ by $\Sigma A^{(n}(RL)$ and $(RL)A + ((RL)A)A + \dots$ by $\Sigma(RL)A^{(n}$. If $\Sigma A^{(n}(RL) = 0$ or $\Sigma(RL)A^{(n} = 0$, then $\Sigma(RL)A^{(n} \cap \Sigma A^{(n}(RL) = 0 \subset RL$ and thus RL is q.i. Since $0 \neq RL \subset R \cap L$ and $R \cap L$ is q.i.m., then $RL = R \cap L$. If $\Sigma A^{(n}(RL) \neq 0$ and $\Sigma(RL)A^{(n} \neq 0$, from L and R are minimal, we have $\Sigma A^{(n}(RL) = L$ and $\Sigma(RL)A^{(n} = R$. Now we are going to show that $R^2 \neq \neq 0$. We have $0 \neq RL = R(\Sigma A^{(n)}(RL))$. Let $0 \neq x \in R(\Sigma A^{(n)}(RL))$ be $x = r(a_n(\dots(a_1(r_0y_0))))$ with $r, r_0 \in R$, $a_i \in A$ for $i = 1, \dots, n$ and $y_0 \in L$. We proceed by induction on n, and we show that $x = r(a_n(\dots(a_1(r_0y_0)))) \in \in \Sigma R^2A^{(m}$ and $y = [r, a_n, a_{n-1}(\dots(a_1(r_0y_0)))] \in \Sigma R^2A^{(m}$ with $m \geqq 0$. If $x = r(a_1(r_0y_0))$ then $x = (ra_1)(r_0y_0) - [r, a_1, r_0y_0] = (ra_1)(r_0y_0) + + [r, r_0y_0, a_1] \in \Sigma R^2A^{(m}$. Suppose $x \in \Sigma R^2A^{(m}$ and $y \in \Sigma R^2A^{(m}$ for

$k \leqq n-1$. Then for n, from the Zorn–Moufang identities, we have

$$[r,a_n,(a_{n-1}(\dots(a_1(r_0y_0))))] = [r,a_n,a_{n-2}(\dots(a_1(r_0y_0)))]a_{n-1}+$$
$$+[a_{n-1},a_n,(a_{n-2}(\dots(a_1(r_0y_0))))]r - [a_{n-1},a_n,r(a_{n-2}(\dots(a_1(r_0y_0)))]$$

where

$$[a_{n-1},a_n,a_{n-2}(\dots(a_1(r_0y_0)))]r=[ra_{n-1},a_n,a_{n-2}(\dots(a_1(r_0y_0)))]+$$
$$+[a_na_{n-1},r,a_{n-2}(\dots(a_1(r_0y_0)))]-[a_{n-1},r,a_{n-2}(\dots(a_1(r_0y_0)))]a_n\in\Sigma R^2A^{(m}$$

because of the induction hypothesis and the fact that R is a right ideal. Thus

$$[r,a_n,(a_{n-1}(\dots(a_1(r_0y_0))))] \in (\Sigma R^2A^{(m})A + \Sigma R^2A^{(m}.$$

Therefore

$$x = (ra_n)(a_{n-1}(\dots(a_1(r_0y_0)))) - [r,a_n(a_{n-1}(\dots(a_1(r_0y_0))))] \in \Sigma R^2A^{(m}$$

from the induction hypothesis. Thus $R^2 \neq 0$, because $RL \neq 0$. In the same way we can prove that L is a minimal left ideal of A such that $L^2 \neq 0$.

From [5] we have one of the following situations:

(i) $R \subset U$,

(ii) $R \subset D$, and in this case R is an ideal and a Cayley–Dickson algebra and $A = R \oplus A'$;

and the same about L. If R and L satisfy (i), from the associative case, $RL = R \cap L$ and thus RL is q.i.m. If $R \subset U$ and $L \subset D$, from $UD = 0$, then we have $RL = 0$, contradiction. The same if $R \subset D$ and $L \subset U$, from $DU = 0$. If $R \subset D$ and $L \subset D$, then R and L are ideals of A and Cayley–Dickson algebras. Since RL is an ideal, $RL \subset R, L$ and then $RL = R = L$.

PROPOSITION 2.6. *Let L be a minimal left ideal of A which is not a split Cayley–Dickson algebra over its center. Let $0 \neq e$ be an idempotent in L, then eL is q.i.m. of A and a division subring of A.*

PROOF. From $e \in L$ and $e^2 = e$, we have $L^2 \neq 0$. Thus L is a minimal left ideal of A such that $L^2 \neq 0$, and from [5] $L \subset U$ or $L \subset D$ (and then L is a Cayley–Dickson algebra). If $L \subset U$ and $0 \neq e \in L$ is an idempotent as in the associative case eL is a division subring of A and a q.i. If $L \subset D$, from the hypothesis L will be a Cayley–Dickson division algebra. Then $e = 1_L$ and thus $eL = L$.

NOTE. This result is not true if L is a split Cayley–Dickson algebra. From the Cayley–Dickson process [7] we can suppose $e = x_0$. But then x_0L has a basis over the center of the Cayley–Dickson algebra, namely $\{x_0, x_1, x_2, x_3\}$ that is not a quasiideal.

§3. Minimal quasiideals of semiprime alternative rings

We recall a known result.

LEMMA 3.1. *Let M be an ideal of A. The following conditions are equivalent:*

(i) *If B is an ideal of A such that $B^2 \subset M$, then $B \subset M$.*
(ii) *If L is a left ideal of A such that $L^2 \subset M$, then $L \subset M$.*
(iii) *If R is a right ideal of A such that $R^2 \subset M$, then $R \subset M$.*

If M is an ideal of A satisfying (i) or (ii) or (iii), we say M is a semiprime ideal. If 0 is a semiprime ideal of A, then we say A is a semiprime ring. Let M be an ideal of A, A is M-semiprime if M does not contain one-sided ideals (or ideals) with product zero.

PROPOSITION 3.2. *Let Q be a minimal quasiideal of A such that $[A,A,Q] \neq 0$. If A is $A^{\#}[A,A,Q]$-semiprime or $[A,A,Q]A^{\#}$-semiprime, then Q is a Cayley–Dickson algebra, $Q \leqq D(A)$ and Q is an ideal of A.*

PROOF. From Proposition 2.2 and its proof $Q = A^{\#}[A,A,Q] \cap [Q,A,A]A^{\#}$ with $A^{\#}[A,A,Q] = AQ + [A,A,Q]$ and $[A,A,Q]A^{\#} = QA + [A,A,Q]$. We are going to see that $A^{\#}[A,A,Q]$ is a minimal left ideal of A. We suppose L is a left ideal of A such that $L \subset A^{\#}[A,A,Q]$. We will show $L^2 = 0$. We consider the q.i. $AL \cap (QA + [Q,A,A])$, contained in $(AQ + [A,A,Q]) \cap \cap (QA + [Q,A,A]) \subset Q$. Since Q is q.i.m., $AL \cap (QA + [Q,A,A]) = 0$ or $AL \cap (QA + [Q,A,A]) = Q$. In the second case $Q \subset AL \subset L \subset A^{\#}[A,A,Q]$ and thus $A^{\#}[A,A,Q] = AQ + [A,A,Q] = L$, because Q is q.i. and L is a left ideal. But this is a contradiction. Therefore $AL \cap (QA + [Q,A,A]) = 0$. Thus $QL \subset AL \cap (QA + [Q,A,A]) = 0$ and $[Q,A,L] \subset AL \cap (QA + [Q,A,A]) = 0$. Therefore $L^2 \subset A^{\#}[A,A,Q]$. $L \subset (AQ)L + QL \subset A(QL) + [A,Q,L] + QL \subset \subset [Q,A,L] = 0$. Now from [5] and the fact that A is $A^{\#}[A,A,Q]$-semiprime, $L = 0$ and thus $A^{\#}[A,A,Q]$ is an ideal and a Cayley–Dickson algebra over its center. But $Q \subset A^{\#}[A,A,Q]$, thus from Proposition 1.2 we have $Q = A^{\#}[A,A,Q]$.

Similarly if A is $[A,A,Q]A^{\#}$-semiprime, we will obtain that $[A,A,Q]A^{\#}$ is a minimal right ideal and then Q is an ideal and also a Cayley–Dickson algebra.

COROLLARY 3.3. *Let Q be a minimal quasiideal of a semiprime ring A. Then Q is the intersection of a minimal left ideal and a minimal right ideal.*

PROOF. It follows from Proposition 2.2, Theorem 7.2 and its proof in [7] and Proposition 3.2.

THEOREM 3.4. *Every minimal quasiideal Q of a semiprime ring A is such that either*

(a) *$Q \subset D(A)$, and then Q is an ideal and a Cayley–Dickson algebra with $A = Q \oplus A'$; or*

(b) *$Q \subset U(A)$.*

Moreover $Q = fAe$ with f and e nuclear idempotents of A.

PROOF. If $[Q,A,A] \neq 0$ we know from Proposition 3.2 that $Q \leqq D(A)$, Q is ideal and Q is a Cayley–Dickson algebra. Therefore Q is a minimal ideal and thus, from Theorem D in [5], $Q = eAe$ with e a central idempotent and $A = Q \oplus A'$.

If $[Q, A, A] = 0$, then $Q \leqq N(A)$ and from the associative case (see [7]) $Q = AQ \cap QA$ with AQ, QA minimal one-sided ideals. From Theorem C in [5] we know $AQ \leqq U(A)$ or $AQ \leqq D(A)$, and if $AQ \leqq D(A)$ then AQ is an ideal. Therefore in the last case $AQ = QA = Q \leqq U(A) \cap D(A)$, that is, since $UD = 0$, $Q^2 = 0$ and this contradicts A is semiprime. Hence $AQ \leqq U(A)$ and thus $Q \leqq U(A)$. Now from [5] $AQ = QA$ and $QA = fA$ with e and f nuclear idempotents. Therefore $Q = fAe$.

Observation. If A is semiprime and Q is a non-associative q.i.m. of A, then $Q^2 \neq 0$. This is curious because for associative rings there are semiprime rings with minimal quasiideals with square zero (in matrix algebras, for example). Moreover we remark that in the non-associative alternative case we determine quasiideals better than in the associative case. This happens also for minimal ideals and minimal one-sided ideals.

Proposition 3.5. *Let e be a nonzero idempotent of a semiprime ring A. The following conditions are equivalent:*

(i) *Ae is a minimal left ideal of A,*

(ii) *eAe is a minimal quasiideal of A,*

(iii) *eA is a minimal right ideal of A.*

Proof. (i) implies (ii). We suppose Ae is a minimal left ideal of A. From [5] we have $Ae \subset U$ or $Ae \subset D$. If $Ae \subset U$, $eAe = eA \cap Ae$ with eA a minimal right ideal [7, Proposition 7.4]. Thus eAe is q.i.m. If $Ae \subset D$, then Ae is a Cayley–Dickson algebra and it is clear that $e = 1_{Ae}$. Thus $eAe = Ae$ is q.i.m.

(ii) implies (iii). We suppose eAe is q.i.m. of A. From Corollary 3.3 $eAe = L \cap R$ with L and R minimal one-sided ideals. If $R \subset U$, then $e \in U$ and thus eA is a right ideal such that $0 \neq eA \subset R$. From minimality of R, $eA = R$. If $R \subset D$ we have $eAe \subset D$ is a Cayley–Dickson algebra and an ideal. Therefore $eAe = eA$.

(iii) implies (i). Let eA be a minimal right ideal. If $eA \subset U$, then from Proposition 2.6, eAe is q.i.m. and we can apply "(ii) implies (iii)" to obtain that Ae is a minimal left ideal. If $eA \subset D$ then eA is a Cayley–Dickson algebra and an ideal with $e = 1_{eA}$ a nuclear idempotent of A. Therefore $Ae = eA$.

Corollary 3.6. *If A is a semiprime alternative ring, the following conditions are equivalent:*

(i) *A has a minimal quasiideal,*

(ii) *A has a minimal left ideal,*

(iii) *A has a minimal right ideal.*

Theorem 3.7. *Let A be a semiprime alternative ring. The product of two minimal quasiideals of A is 0 or a minimal quasiideal of A.*

Proof. Let Q, Q' be q.i.m.

(i) If $Q \subset U$ and $Q' \subset D$, from $UD = 0$, we get $QQ' = 0$.

(ii) If $Q \subset D$ and $Q' \subset U$, from $DU = 0$, we get $QQ' = 0$.

(iii) If $Q \subset D$ and $Q' \subset D$, then Q and Q' are minimal ideals and since QQ' is an ideal it follows that $QQ' = 0$ or $QQ' = Q = Q'$.

(iv) If $Q \subset U$ and $Q' \subset U$, from the associative case, QQ' is q.i.m.

PROPOSITION 3.8. *Let R be a minimal right ideal such that $R^2 \neq 0$. Then R is a union of minimal quasiideals such that the intersection of each two different of them is zero.*

PROOF. From [5] we have $R \subset D$ and then it is a Cayley–Dickson algebra and minimal quasiideal, or $R \subset U$ and then we only need to apply the associative case (see [7]).

§4. Some decomposition theorems for semiprime rings based on quasiideals

It is known that semiprime rings which are the sums of their minimal left ideals have a decomposition into direct sums of minimal ideals. In the following we are going to study semiprime rings that are sums of their minimal quasiideals.

DEFINITION. We say that the quasiideals $Q_{\gamma\delta}$ ($\gamma \in \Gamma$, $\delta \in \Delta$) of a ring A are a complete system, K, if the following three conditions hold:

(1) either $Q_{\gamma\delta} = 0$ or $Q_{\gamma\delta}$ is a minimal quasiideal of A,

(2) for every $Q_{\gamma\delta} \neq 0$ there are nuclear idempotents e_γ, $f_\delta \in A$ such that $Q_{\gamma\delta} = e_\gamma A f_\delta$,

(3) for every finite subset $Q_{11}, \dots, Q_{1e}, Q_{21}, \dots, Q_{2e}, \dots, Q_{k1}, \dots, Q_{ke}$ of K there are nuclear idempotents g_i, h_j with $i = 1, \dots, r \leqq k$ and $j = 1, \dots, s \leqq e$ such that

$$\sum_{x=1}^{k}\sum_{y=1}^{e} Q_{xy} \leqq \sum_{i=1}^{r}\sum_{j=1}^{s} g_i A h_j$$

where $g_i A h_j$ are 0 or q.i.m. such that $g_i A h_j \cdot h_j A g_i = g_i A g_i$.

THEOREM 4.1. *For an alternative ring A the following conditions are equivalent:*

(i) *A is a semiprime ring and the sum of its minimal left ideals.*

(ii) *$A = (\oplus B_i) \oplus (\oplus A_j)$ with $\{B_i\}_{i \in I}$ simple subrings of A containing at least one minimal left ideal and with $\{A_j\}_{j \in J}$ Cayley–Dickson algebras. Moreover $\oplus B_i = U(A)$ and $\oplus A_j = D(A)$.*

(iii) *A is a sum of quasiideals which are a complete system.*

(iv) *A is semiprime and the sum of its minimal quasiideals.*

PROOF. It follows from Proposition 4.15 in [5], the associative case and Corollary 3.3.

DEFINITION. The quasiideals $Q_{11}, Q_{12}, \dots, Q_{mm}$ of a ring A are a finite complete system if there are nuclear idempotents $e_1, \dots, e_m \in A$ such that $Q_{ik} = 0$ or $Q_{ik} = e_i A e_k$ $(i \leqq m,\ k \leqq m)$ is q.i.m. of A with the property that $e_i A e_k \cdot e_k A e_i = e_i A e_i$.

Now from the associative case and Theorem 4.1 it follows:

THEOREM 4.2. *The following conditions on an alternative ring A are equivalent:*

(i) *A is a semiprime ring and a sum of a finite number of its minimal left ideals.*

(ii) *$A = A_1 \oplus \dots \oplus A_s \oplus B_1 \oplus \dots \oplus B_r$ such that $A_1, \dots, A_s$ are Cayley-Dickson algebras and $B_1, \dots, B_r$ are simple associative rings which are sums of a finite number of their minimal left ideals. Moreover $A_1 \oplus \dots \oplus A_s = U(A)$ and $B_1 \oplus \dots \oplus B_r = D(A)$.*

(iii) *A is a sum of quasiideals which are a finite complete system.*

(iv) *A is semiprime and a sum of a finite number of its minimal quasi-ideals.*

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COLEGIO UNIVERSITARIO DE LA RIOJA (UNIVERSITY OF ZARAGOZA)
C/ OBISPO BUSTAMANTE, NO. 3
26001 LOGROÑO
SPAIN

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d-ISOMORPHIC SEMIGROUPS OF CONTINUOUS FUNCTIONS IN LOCALLY COMPACT SPACES

D. T. THÀNH (Hânôi)

To Professor Á. Császár on his 65^{th} birthday

In this paper, we use the following concepts introduced in [1]:

Let S be a semigroup. For $f, g \in S$, $f <_d g$ iff there is $h \in S$ such that $g = hf$.

A subset D of S is said to be a d-ideal iff $\emptyset \neq D \neq S$ and

(1) $f \in D$, $f <_d g$ implies $g \in D$,

(2) For $f, g \in D$, there exists $h \in D$ such that

$$h <_d f, \quad h <_d g.$$

D is a maximal d-ideal iff it is a d-ideal and coincides with any d-ideal in which it is contained.

Let S_1, S_2 be semigroups. S_1, S_2 are said to be d-isomorphic iff there is a bijective map $\varphi \colon S_1 \to S_2$ such that $f <_d g$ iff $\varphi(f) <_d \varphi(g)$.

REMARK. If S_1, S_2 are (semigroup) isomorphic then they are clearly d-isomorphic bu the converse is false: e.g. two groups of the same cardinality are always d-isomorphic.

Let X be a locally compact Hausdorff space. We denote $Z(f) = \{x \in X \colon f(x) = 0\}$, $Z^c(f) = \{x \in X \colon f(x) \neq 0\}$ for $f \colon X \to \mathbf{R}$; $C_c(X) = \{f \colon f \colon X \to \mathbf{R}$ is continuous and $\overline{Z^c(f)}$ is compact$\}$.

We know [4] that $C_c(X)$, $C_c(Y)$ are linear lattice isomorphic (so ring isomorphic) iff X, Y are homeomorphic, provided X, Y are locally compact Hausdorff spaces.

In [3], it has been shown that two locally compact Hausdorff spaces X, Y are homeomorphic iff $C_c(X)$, $C_c(Y)$ are semigroup isomorphic. A generalization of this theorem is contained in [2]. Our purpose is to prove another generalization.

Given a locally compact Hausdorff space X, we consider the semigroup $C_c(X)$. Then, if $f, g \in C_c(X)$, $f <_d g$, there is $h \in C_c(X)$ satisfying $g = hf$, consequently $Z(f) \subset Z(g)$.

LEMMA 1. *Ever d-ideal D in $C_c(X)$ is fixed (i.e. $\bigcap_{f \in D} Z(f) \neq \emptyset$).*

PROOF. There is $f \in C_c(X)$ not belonging to D. Suppose $g \in C_c(X)$ and $Z(g) \cap \overline{Z^c(f)} = \emptyset$. Then, by the compactness of $\overline{Z^c(f)}$, there is $\varepsilon > 0$

such that $|g| > \varepsilon$ on $\overline{Z^c(f)}$. Let

$$h(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0. \end{cases}$$

It is easy to prove that h is continuous on X. In fact, h is continuous at $x \in Z^c(f)$ because f, g are continuous and $g(x) \neq 0$. If $x_0 \in Z(f)$, then there exists a neighbourhood V of x_0 such that $|f| < \varepsilon\delta$ on V for a given $\delta > 0$. For $x \in V$ either $|h(x)| = \left|\frac{f(x)}{g(x)}\right| < \frac{\varepsilon\delta}{\varepsilon} = \delta$ or $h(x) = 0$. So h is continuous at $x_0 \in Z(f)$.

$\overline{Z^c(h)} = \overline{Z^c(f)}$ implies $h \in C_c(X)$.

Now $g <_d f$ by $f = hg$. So $g \in D$ would imply $f \in D$ contrary to the choice of f, hence $Z(g) \cap \overline{Z^c(f)} \neq \emptyset$ whenever $g \in D$.

For $g, g' \in D$, there is $h \in D$ such that $h <_d g$, $h <_d g'$. Then $Z(h) \subset \subset Z(g) \cap Z(g')$. Thus the collection $\{Z(g) \cap \overline{Z^c(f)}: g \in D\}$ of closed sets in the compact space $\overline{Z^c(f)}$ has the finite intersection property and therefore a non-empty intersection. □

Now let $x_0 \in X$, $D_{x_0} = \{f \in C_c(X): f(x_0) = 0\}$. As X is locally compact and Hausdorff, $\bigcap\limits_{f \in D_{x_0}} Z(f) = \{x_0\}$.

LEMMA 2. *D_{x_0} is a maximal d-ideal.*

PROOF. First of all, we prove D_{x_0} is a d-ideal. If $g = hf$, $f \in D_{x_0}$, $h \in \in C_c(X)$, then clearly $g \in D_{x_0}$. For $f, g \in D_{x_0}$, define $h(x) = \sqrt{|f(x)| + |g(x)|}$,

$$f^*(x) = \begin{cases} \frac{f(x)}{h(x)} & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0. \end{cases}$$

f^* is continuous at $x \in Z^c(f)$ because f and h are continuous and $h(x) \neq 0$. If $x \in Z(f)$, there is a neighbourhood V of x such that $|f| < \varepsilon^2$ on V. For $y \in V$ either

$$|f^*(y)| = \frac{|f(y)|}{\sqrt{|f(y)| + |g(y)|}} \leqq \sqrt{|f(y)|} < \varepsilon$$

or $f^*(y) = 0$. So f^* is continuous and $Z(f) = Z(f^*)$ implies $f^* \in C_c(X)$. From $f = f^*h$ we obtain $h <_d f$. Similarly $h <_d g$.

A d-ideal containing D_{x_0} must be fixed by Lemma 1 at some point $x_1 \in X$. Then D_{x_0} is fixed at x_1, consequently $x_1 = x_0$, and the d-ideal in question is contained in D_{x_0}. So, the latter is a maximal d-ideal. □

By the above, $\psi(x) = D_x$ is a bijection from X onto the set of all maximal d-ideals in $C_c(X)$. Clearly $x \in Z(f)$ iff $f \in D_x$ so that ψ determines the subsets $Z(f)$ of X which constitute a base for the closed set in the locally compact Hausdorff space X.

Thus we have proved:

THEOREM. *Two locally compact Hausdorff spaces X and Y are homeomorphic iff the semigroups $C_c(X)$ and $C_c(Y)$ are d-isomorphic.*

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BUONG 328, C4, PHUONG TAN MAI
KHU HAI BA
HANOI
VIETNAM

Acta Math. Hung.
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RINGS WITH LEFT SELF DISTRIBUTIVE MULTIPLICATION

G. F. BIRKENMEIER, H. HEATHERLY (Lafayette) and T. KEPKA (Prague)

Introduction

Throughout this note all rings are associative, but not necessarily commutative or with unity. A ring is *left self distributive* (an *LD-ring*) if it satisfies the identity: $xyz = xyxz$. Similarly one defines a *right self distributive ring* (an *RD-ring*). Petrich [7] classified all rings which are both LD and RD-rings as those rings which are the direct sum of a Boolean ring and a nilpotent ring of index at most three. As shown by several examples given herein, this does not hold for all LD-rings. These examples illustrate how rich the variety of LD-rings is.

If R is an LD-ring and N is the set of nilpotent elements of R, then N is an ideal, $N^3 = 0$, and R/N is Boolean. If R/N contains unity, then $R = A \oplus N$ as a direct sum of left ideals, with A a Boolean ring with unity. This condition is implied by several others; e.g., R has d.c.c. on ideals or a.c.c. on ideals. Without any finiteness condition we are still able to find an ideal $B = A + N$, where A is Boolean and is a left ideal, such that B is completely semiprime and is left and right essential in R.

Somewhat surprising from the viewpoint of semigroup theory [5] is the result that every LD-ring is left permutable (satisfies $abc = bac$ identically). Other useful identities are developed.

A complete classification of subdirectly irreducible LD-rings is given. Such a ring is either nilpotent of index at most three, $\mathbf{Z}_2$, or a certain four element ring.

1. Preliminaries

Let R be a ring and M a non-empty subset of R. Then $\mathrm{Id} = \mathrm{Id}(R)$ is the set of idempotents of R, $\mathbf{l}(M) = \{x \in R; xM = 0\}$, $\mathbf{r}(M) = \{x \in R; Mx = 0\}$, and $\langle M \rangle$ is the ideal of R generated by M. The ideal generated by $[R, R] = \{ab - ba; a, b \in R\}$ will be denoted $\langle R, R \rangle$, the *commutator ideal* or *Lie ideal* of R. The set M is said to be *reduced* if $M \cap N = 0$.

An ideal I of R is called *right (left) essential* if $I \cap K \neq 0$ for each right (left) ideal of R; *completely semiprime* if $x \in I$ whenever $x^n \in I$ (equivalently if R/I is reduced); and *completely prime* if $xy \in I$ implies $x \in I$ or $y \in I$.

A ring R is called *left (right) permutable* if the identity $xyz = yxz$ (respectively $xyz = xzy$) holds and *medial* if the identity $xyzv = xzyv$ holds.

LEMMA 1.1. *Let R be an LD-ring and $a, b, c, d \in R$.*

(i) $a^3 = a^n \in \mathrm{Id}$ *for every* $n \geqq 4$.
(ii) $2abc = 0$.
(iii) $(abc - acbc)^2 = 0$.
(iv) *If* $b, c \in \mathrm{Id}$ *and* $bc = 0$, *then* $cb = 0$.
(v) $R(a - a^2)R = R(a - a^3)R = 0$.
(vi) $(a - a^2)^3 = (a - a^3)^3 = 0$.
(vii) $abcd = -cbad = cbad$.
(viii) *If* $a \in N$, *then* $abc = bac = 0$.

PROOF. The proofs of (i), ...,(iv) are straightforward calculations.

(v) $b(a-a^2)c = bac-ba^2c = b(abc)-ba^2c = babac-ba^2c = ba^2c-ba^2c = 0$, so that $R(a - a^2)R = 0$. Similarly, $R(a - a^3)R = 0$.

(vi) This is an immediate consequence of (v).

(vii) Consider the equality $0 = (a+c)b(a+c)d-(a+c)bd = abcd+cbad$. By (ii), $cbad = -cbad$.

(viii) We have $0 = aba^3c = abc = babc = bac$.

PROPOSITION 1.2. *Let R be an LD-ring.*
(i) *R is left permutable, and hence medial.*
(ii) *If* $x \in \langle R^2\rangle$, *then* $x^2 = x^n \in \mathrm{Id}$ *for every* $n \geqq 3$.

PROOF. (i) Using Lemma 1.1 (vii) and left distributivity, $abc - bac = abac-babc = aba^2c-babac = a^2bac-ba^2c = a^2bc-ba^2c = ba^2c-ba^2c = 0$ for all $a, b, c \in R$.

(ii) Let $x = \Sigma a_ib_i$. Using (i), $x^2 = x\Sigma a_ib_i = x\Sigma a_ixb_i = x\Sigma xa_ib_i = x^3 \in \mathrm{Id}$.

We note that an important question in semigroup theory is to determine those semigroups which are the multiplicative semigroups of rings [6]. By Proposition 1.2, a left distributive semigroup that is not left permutable cannot be the multiplicative semigroup of a ring.

2. Subdirectly irreducible rings

In this section R will be a subdirectly irreducible LD-ring with heart H (i.e., H is the smallest non-zero ideal of R). It is well known (see e.g. [3], Lemma 75) that the heart is either square zero or is a simple ring. Thus either $H^2 = 0$ or H is isomorphic to the ring $\mathbf{Z}_2$ (integers modulo 2); in the latter case $R = H$ (if the heart has a unity, the ring is simple).

LEMMA 2.1. *If* $R^3 \neq 0$, *then:*
(i) $\mathbf{r}(R) = 0$, $RH = H$ *and* $\mathrm{Id} \neq 0$.
(ii) $x^2 \in \mathrm{Id}$, $xy = x^2y$ *and* $2xy = 0$ *for all* $x, y \in R$.

PROOF. (i) We have $abc \neq 0$ for some $a, b, c \in R$. But $abc = ab^3c$, which yields a non-zero idempotent $e = b^3$. Then eR is a non-zero ideal (use Proposition 1.2 (i)), $H \subseteqq eR$ and $0 \neq ed \in H$ for some $d \in R$. Since $e(ed) = ed$, H is not contained in $\mathbf{r}(R)$, and therefore $\mathbf{r}(R) = 0$.

(ii) For any $x, y, z \in R$, we have $zxy = zx^2y$, $xy - x^2y \in \mathbf{r}(R) = 0$ and $xy = x^2y$. The rest follows from Lemma 1.1.

LEMMA 2.2. *If $R^3 \neq 0$ and $H^2 = 0$, then:*
(i) $H \subseteqq \langle R, R \rangle$ *and* $H \subseteqq \mathrm{l}(R)$.
(ii) $H = \{0, w\}$ *is a two-element set and* $xw = w$ *for every* $x \in R - \mathrm{l}(R)$.

PROOF. (i) If R is commutative, then $H \cap \mathrm{Id} \neq 0$ (see the proof of Lemma 2.1 (i)) and $H^2 \neq 0$. Hence R is not commutative and $H \subseteqq \langle R, R \rangle$. From this, $HR \subseteqq \langle R, R \rangle R = 0$ (the last equality follows from the left permutability of R).

(ii) Since $RH = H$, $Rw = H$ for some $w \in H$. If $xw \neq 0$ for an $x \in R$, then xR is a non-zero ideal and $H \subseteqq xR$. Thus $w = xy$ for some $y \in R$ and we have $xw = x^2y = xy = w$. Consequently, $H = \{0, w\}$. Further, if $xw = 0$, then H is not contained in xR, and so $xR = 0$ and $x \in \mathrm{l}(R)$.

LEMMA 2.3. *Let $R^3 \neq 0$ and $H^2 = 0$. Then $H = \mathrm{l}(R)$ and $R = H \cup \mathrm{Id}$.*

PROOF. Let $x \in R - (\mathrm{l}(R) \cup \mathrm{Id})$. Then $y = x - x^2 = x + x^2$ is a non-zero element of $\mathrm{l}(R)$, and so $H \subseteqq Ry$, Ry being a non-zero ideal (we have $\mathbf{r}(R) = 0$). From this $w = zy$ for suitable $z \in R$ and $w = zx + zx^2 = zx + z^2x^2 = zx + (zx)^2$. The supposition $zx \notin \mathrm{l}(R)$ yields $w = zxw = (zx)^2 + (zx)^3 = 0$, a contradiction. Thus $zx \in \mathrm{l}(R)$, $w = zx$ and $0 = w^2 = zxw = zw = w$, a contradiction. We have proved $R = \mathrm{l}(R) \cup \mathrm{Id}$.

Now, if $0 \neq a \in \mathrm{Id}$ and $b \in \mathrm{l}(R)$, then $a+b \notin \mathrm{l}(R)$, and so $a+b \in \mathrm{Id}$. Then $a+b = (a+b)^2 = a+ab$, which implies $b = ab$. If $b \neq 0$, we have $w = cb$ for some $c \in \mathrm{Id}$, and hence $w = cb = b$ (as proved above). Thus $H = \mathrm{l}(R)$.

LEMMA 2.4. *Let $R^3 \neq 0$ and $H^2 = 0$. Then* $\mathrm{card}(R) = 4$.

PROOF. The ring R must contain 0, w and a non-zero idempotent e (see Lemmas 2.1 and 2.2). By Lemma 2.3, $e + w \in \mathrm{Id}$ and $e + w$ is a fourth distinct element. Now, let $x \in R - H$, $x \neq e, e+w$. Then, by Lemma 2.3, x, $e+x$ are non-zero idempotents and $w + xe + e = w + (x+e)e = (x+e)w + (x+e)e = (x+e)(w+e) = x(w+e) + e(w+e) = w + xe + w + e = xe + e$ implies $w = 0$, a contradiction.

EXAMPLE 2.5. There is (up to isomorphism) exactly one non-nilpotent subdirectly irreducible LD-ring of order 4. This ring can be formed by taking w and e as the generators for the group $C(2) \oplus C(2)$ and defining multiplication via the following table:

	0	w	e	$w+e$
0	0	0	0	0
w	0	0	0	0
e	0	w	e	$w+e$
$w+e$	0	w	e	$w+e$

This example and the preceding results give us the main classification theorem of this section.

THEOREM 2.6. *Every subdirectly irreducible LD-ring is either nilpotent of index at most three, isomorphic to $\mathbf{Z}_2$, or to the four element ring from Example* 2.5.

REMARK 2.7. Let $R \neq 0$ be a subdirectly irreducible ring with $R^3 = 0$. Then either

(i) $R^2 = 0$ and $(R, +)$ is isomorphic to $C(p^n)$, $p \geqq 2$ a prime, $1 \leqq n \leqq \infty$, or

(ii) $0 \neq R^2 \subseteqq \mathbf{l}(R) \cap \mathbf{r}(R) \cong C(p^n)$ and $p^n R = 0$, $1 \leqq n < \infty$, or

(iii) $0 \neq R^2 \subseteqq \mathbf{l}(R) \cap \mathbf{r}(R) \cong C(p^\infty)$.

PROPOSITION 2.8. *Let T be an LD-ring.*

(i) *If T is simple, then either $T \cong \mathbf{Z}_2$ or $T^2 = 0$ and $(T, +)$ is isomorphic to $C(p)$, p prime.*

(ii) *If I is a minimal ideal of T, then either $I^2 = 0$ or $I \cong \mathbf{Z}_2$.*

PROOF. (i) If $T^2 \neq 0$, then $N = 0$ and T is a simple Boolean ring.

(ii) A minimal ideal in a ring is either square zero or simple [3], p. 135.

3. Direct decompositions of LD-rings

PROPOSITION 3.1. *Let R be an LD-ring.*

(i) *N is an ideal of R, $N^3 = 0$ and R/N is a Boolean ring. Hence $N = \langle K \rangle$, where $K = \{a - a^2; a \in R\}$.*

(ii) *If R has a right unity element, then R is a Boolean ring with unity.*

(iii) *If I is a semiprime ideal of R, then R/I is a Boolean ring. Hence I is a completely semiprime ideal.*

(iv) *If I is a prime ideal of R, then I is a completely prime ideal, a maximal ideal and R/I is isomorphic to $\mathbf{Z}_2$.*

PROOF. (i) $N^3 = 0$ by Lemma 1.1 (viii). Hence, if $a, b \in N$, then $(a + b)^3 = 0$ and it is clear that N is an ideal of R. By Lemma 1.1 (vi), R/N is idempotent, and hence Boolean.

(ii) Let e be a right unity for R. If $x \in R$, then $x = xe = xee = xexe = x^2$.

(iii) and (iv) These proofs are immediate.

LEMMA 3.2. *Let R be an LD-ring and $C \subseteqq R$ a maximal set of pairwise commuting idempotents. Then C is a maximal reduced left ideal.*

PROOF. Let $a, b \in C$. Then $(a+b)^2 = a + 2ab + b = a + b$ by Lemma 1.1 (ii). Clearly, $a + b$ commutes with every element of C, and therefore $a + b \in C$. Further, let $x \in R$. Then $xa \in \mathrm{Id}$ and again, by left permutability, xa commutes with every element of C. Thus C is a maximal reduced left ideal of R.

THEOREM 3.3. *Let R be an LD-ring and A a maximal reduced left ideal of R.*

(i) *$\mathbf{l}(A) = \mathbf{l}(\mathrm{Id}) = N$ and $\mathbf{r}(\mathrm{Id}) \subseteqq \mathbf{r}(A) \subseteqq N$.*

(ii) *A contains every reduced right ideal of R.*

(iii) *$B = A \oplus N$ is an ideal of R which is both left and right essential in R.*
(iv) *B is a completely semiprime ideal.*
(v) *If $\mathrm{Id} \subseteqq B$, then $R = B$.*

PROOF. (i) By Lemma 1.1 (viii), $N \subseteqq \mathbf{l}(\mathrm{Id})$ and we have $\mathbf{l}(\mathrm{Id}) \subseteqq \mathbf{l}(A)$, since A is Boolean. Now, let $x \in \mathbf{l}(A) - N$. Then $x^3 = e \neq 0$, $e \in \mathrm{Id}$ and $eA = 0$, so that $Re \cap A = 0$. By the maximality of A, $0 \neq y = ze + a$ and $y^2 = 0$ for some $z \in R$ and $a \in A$. Thus $0 = y^2 = ze + aze + a$, so $a \in Re \cap A = 0$. Then $y \in Re \cap N = 0$, a contradiction. We have proved that $\mathbf{l}(A) = \mathbf{l}(\mathrm{Id}) = N$. Similarly, $\mathbf{r}(\mathrm{Id}) \subseteqq \mathbf{r}(A) \subseteqq N$.

(ii) Let $0 \neq I$ be a reduced right ideal of R. Since every element of I is idempotent and R is left permutable, I is an ideal. If $I \not\subseteqq A$, then $(I + A) \cap$ $\cap N \neq 0$. Hence, let $0 \neq x + a$, $x \in I$, $a \in A$, be such that $(x + a)^2 = 0$. Now $x(x + a) = x + xa \in I \cap N = 0$, so that $x + a = -xa + a \in A \cap N = 0$, a contradiction.

(iii) Let $b \in B$ and $x \in R$. Since R is left permutable, $bx - xb \in N$. From this conclude that B is an ideal. Further, let $0 \neq y \in R$ be such that $y \notin B$. By (i), $yB \neq 0$. Hence B is right essential in R and similarly we can show that B is left essential.

(iv) By Proposition 3.1, R/B is Boolean, and hence reduced. Thus B is completely semiprime.

(v) This is an easy consequence of (iv).

As a corollary of the preceding theorem, we have the following result of M. Petrich [7], Theorem 2.

COROLLARY 3.4. *A ring R is both left and right self distributive if and only if $R = A \oplus N$ (ring direct sum) where A is a Boolean ring and $N^3 = 0$.*

PROOF. Let R be both left and right self distributive. We can assume that $R \neq N$. Then there exists $x \in R - N$, so Rx^3 is a non-zero reduced left ideal of R. By Zorn's Lemma there exists a non-zero left ideal A which is maximal among reduced left ideals of R. Let $e \in \mathrm{Id}$. Since R is right self distributive, $eR \cap N = 0$. By Theorem 3.3, $eR \subseteqq A$ and $R = A \oplus N$. Since R is right permutable, A is an ideal. The converse is obvious.

PROPOSITION 3.5. *Let R be an LD-ring. Then R is an RD-ring if and only if $x^2y = xy^2$ for all $x, y \in R$.*

PROOF. Let R be an RD-ring. By Corollary 3.4, $x^2y = xy^2$ for all $x, y \in R$. Conversely, assume $x^2y = xy^2$ for all $x, y \in R$. Then the ring of Example 2.5 cannot be a homomorphic image of R. By Theorem 2.6, R is an RD-ring.

LEMMA 3.6. *Let R be an LD-ring, $x \in R$ and $e = x^3$.*
(i) *$Rx = Re \oplus R(x + e)$ (left ideal decomposition), Re is a Boolean ring with unity and $R(x + e) = \mathbf{l}(R) \cap Rx$ (hence $R(x + e)$ is an ideal).*
(ii) *$xR = eR \oplus (x + e)R$ (ideal decomposition) and $xR = Re \oplus Y \oplus (x + e)R$ (left ideal decomposition) where $Y = \mathbf{l}(R) \cap eR$ and $Y \oplus (x + e)R = \mathbf{l}(R) \cap xR$.*

PROOF. (i) Let $y \in R$. Then $yx = ye + y(x+e)$ and $R(x+e) \subseteqq \mathrm{l}(R) \cap Rx$ by Lemma 1.1 (v). The inclusion $\mathrm{l}(R) \cap Rx \subseteqq R(x+e)$ is easily seen.

(ii) Since R is left permutable, xR is an ideal, $Re \subseteqq eR$ and $(x+e)R \subseteqq \subseteqq \mathrm{l}(R) \cap \mathbf{r}(R)$. Now let $y \in R$. Then $ey = ye + ey + ye$, $ey + ye \in \mathrm{l}(R) \cap eR$. Hence $eR = Re \oplus Y$, $Y = \mathrm{l}(R) \cap eR$, and $xR = Re \oplus Y \oplus (x+e)R$.

THEOREM 3.7. *Let R be an LD-ring such that the ring R/N is finitely generated as an ideal. Then $R = A \oplus N$, where A is a left ideal of R which is a Boolean ring with unity. If, in addition, $R = Rx_1 + \ldots + Rx_n$ or $R = x_1R + \ldots + x_nR$ for some $1 \leqq n$ and $x_1, \ldots, x_n \in R$, then $N = \mathrm{l}(R)$.*

PROOF. Since R/N is Boolean and idempotents lift modulo N, we have $R = Re_1 + \ldots + Re_m + N$ for some $1 \leqq m$ and $e_1, \ldots, e_m \in \mathrm{Id}(R)$. Now assume $m = 2$. Note that $e_2 = e_2e_1 + a$, $a = e_2 + e_2e_1$, $R = Re_1 + Ra + N$. By Lemma 3.6 (i), $Ra = Re \oplus R(a+e)$, $e = a^3$. Now, $0 = ae_1 = ee_1 = e_1e$. Let $b = e_1 + e_2$, so that $b = b^2$. Then we have $R = Rb \oplus N$ and Rb is a Boolean ring with unity. The general result follows by induction. The equality $N = \mathrm{l}(R)$ may be proved similarly.

Note: If R is a Boolean ring then R is finitely generated as an ideal if and only if R has a unity.

COROLLARY 3.8. *Let R be an LD-ring. If R satisfies any of the following conditions, then $R = A \oplus N$, where A is a left ideal of R which is a finite Boolean ring.*

(i) *R/N is finite.*

(ii) *R has no infinite set of orthogonal idempotents.*

(iii) *R has a.c.c. or d.c.c. on ideals.*

PROOF. In all cases R/N is finite and the result follows from Theorem 3.7.

REMARK 3.9. Example 2.5 shows that the direct decomposition $R = A \oplus N$ cannot be improved to a direct sum of ideals.

Every left permutable ring (and hence every LD-ring) is medial. Medial rings are investigated in [2]. The next result is a partial converse to Theorem 3.7.

THEOREM 3.10. *Let R be a medial ring.*

(i) *N is an ideal.*

(ii) *If $R = A \oplus N$ where A is a left ideal of R which is a Boolean ring and $N^3 = 0$, then R is an LD-ring.*

PROOF. (i) For all $x, y \in R$, $(xy - yx)^3 = 0$. Now, N is an ideal by [3], Theorem 54.

(ii) First, notice that $NA = 0$. Let $a_1, a_2, a_3 \in A$ and $x_1, x_2, x_3 \in N$. Then

$$(a_1 + x_1)(a_2 + x_2)(a_1 + x_1)(a_3 + x_3) - (a_1 + x_1)(a_2 + x_2)(a_3 + x_3) =$$

$$= a_1a_2a_3 + a_1a_2x_1x_3 + a_1a_2a_1x_3 - a_1a_2a_3 - a_1a_2x_3 - a_1x_2x_3 =$$
$$= a_1a_2x_1x_3 - a_1x_2x_3 = a_1x_1a_2x_3 - a_1x_2a_1x_3 = 0.$$

LEMMA 3.11. *Let R be an LD-ring, A a maximal reduced left ideal and $B = A \oplus N$. Let $e \in \mathrm{Id}$ be such that $e \notin B$. Then there exists $b \in R - B$ and $0 \neq a \in A$ such that $ea = a$, $0 \neq ae \in B$, $e = b + ae$, and b and ae are non-zero orthogonal idempotents.*

PROOF. By Theorem 3.3 (i), $ea = a \neq 0$ for some $a \in A$. We have $0 \neq ae \in B$ by Theorem 3.3 (iii). Put $b = e + ae$. Since $e \notin B$, also $b \notin B$. But R is left permutable and $b^2 = e + eae = e + ae = b$. Also $aeb = 0 = bae$.

PROPOSITION 3.12. *Let R be an LD-ring, A a maximal reduced left ideal and $B = A \oplus N$. If $B \neq R$, then there are infinite sets $\{b_0, b_1, b_2, \dots\}$ and $\{a_1, a_2, a_3, \dots\}$ of non-zero idempotents such that:*

(i) *$b_i \in R - B$ for every $i = 0, 1, 2, \dots$.*

(ii) *$\{a_1, a_2, a_3, \dots\} \subseteqq A$ is a set of orthogonal idempotents.*

(iii) *$b_ia_{i+1} = a_{i+1}$ and $a_{i+1}b_i \in B$ for every $i = 0, 1, 2, \dots$.*

(iv) *$b_i = b_{i+1} + a_{i+1}b_i$ and $b_{i+1}, a_{i+1}b_i$ are non-zero orthogonal idempotents for every $i = 0, 1, 2, \dots$.*

(v) *$Rb_0 \supset Rb_1 \supset Rb_2 \supset \dots$.*

(vi) *$\mathbf{r}(b_0) \subset \mathbf{r}(b_1) \subset \mathbf{r}(b_2) \subset \dots$.*

PROOF. Let $x \in R$ be such that $x \notin B$. Put $e = x^3$. Since R/B is a Boolean ring, $e \notin B$. Now, the existence of the sets of idempotents as well as the assertions (i), ..., (iv) follow from Lemma 3.11 by induction. Moreover, (v) is clear, since $b_i \notin Rb_{i+1}$. For (vi), $\mathbf{r}(b_i) \subseteqq \mathbf{r}(b_{i+1})$. However, $a_{i+1}b_i \in \mathbf{r}(b_{i+1})$ and $a_{i+1}b_i \notin \mathbf{r}(b_i)$.

4. Examples

In this section, three construction schemes are given for building LD-rings. Here examples show that the variety of LD-rings contains much more than just direct sums or products of Boolean rings and nilpotent class three rings. We close the section with an open problem.

EXAMPLE 4.1. Let T be a ring, M a left T-module and $f\colon M \to T$, $h\colon M \to M$ be T-homomorphisms satisfying $fh = f$ and $h^2 = h$. Define $x * y = f(x)h(y)$ for each $x, y \in M$. Then $(M, +, *)$ is a ring. If $f(x)f(y) = f(x)f(y)f(x)$ for each $x, y \in N$, then $(M, +, *)$ is an LD-ring.

One concrete realization of this is given by taking M to be the full set of n by n matrices, $n > 1$, over a Boolean ring T and use $f = \text{trace}$, $h = 1_M$. The ring so formed will always contain nonzero nilpotents and idempotents. Another realization arises by using the same M, T, f but taking $h(a_{ij}) = (a'_{ij})$ where $a'_{ii} = a_{ii}$ and $a'_{ij} = 0$ for $i \neq j$.

Other similar constructions come readily to mind.

EXAMPLE 4.2. Let S be a ring and f, h endomorphisms of the additive group $(S, +)$ such that:

(i) for each $y \in f(S)$, $x \in h(S)$, $f(yx) = yf(x)$ and $h(yx) = yh(x)$;

(ii) $h^2 = h$, $f^2 = f$ and $fh = f$.

Define $a * b = f(a)h(b)$ for each $a, b \in S$. Then $(S, +, *)$ is a ring. If $f(a)f(b)f(a)h(c) = f(a)f(b)h(c)$ for all $a, b, c \in S$ and if $f(S)$ is a subsemigroup of the multiplicative semigroup of S, then $(S, +, *)$ is an LD-ring.

As a particular example of this take R to be an LD-ring, S the full ring of n by n matrices over R, $n > 1$, and define f and h via $f(a_{ij}) = (b_{ij})$, where $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = a_{ii}$, $h(a_{ij}) = (c_{ij})$, where $c_{ij} = 0$ for $i > j$ and $c_{ij} = a_{ij}$ otherwise. Then $(S, +, *)$ is an LD-ring which is not right permutable.

EXAMPLE 4.3. Let F be a free ring and let I be the ideal of F generated by the set $\{abac - abc;\ a, b, c \in F\}$. Then F/I is an LD-ring which is neither Boolean nor nilpotent.

The construction schemes given in Examples 4.1 and 4.2 are special cases of a general algebraic construction developed in [1].

The results of Section 3 indicate that for a large class of LD-rings, the decomposition $R = A \oplus N$, where A is a left ideal and a Boolean ring, is valid. So far we have been unable to prove or disprove this for the whole class of LD-rings and we leave it as an open problem.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHWESTERN LOUISIANA
BOX 41010
LAFAYETTE, LOUISIANA 70504
U.S.A.

DEPARTMENT OF MATHEMATICS
CHARLES UNIVERSITY
SOKOLOVSKÁ 83
PRAGUE 8
CZECHOSLOVAKIA

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QUASI-P RADICALS OF ASSOCIATIVE RINGS

XI OUGEN (Ningbo)

§1. Quasi-P rings and quasi-P radicals

In order to study the radicals of associative rings in a unified way, we introduce the notions of quasi-P rings and quasi-P ideals where P means an arbitrary property of rings.

DEFINITION 1.1. Let R be an associative ring, R° an extensive ring of R, and $\overline{R^\circ}$ an aribtrary homomorphic image of R°: $R^\circ \overset{f}{\sim} \overline{R^\circ}$. If the images $\overline{R} = f(R)$ of subrings R of R° under the arbitrary homomorphism f are all $\{0\}$, or, if some $f(R) = \overline{R} \neq \{0\}$, then, it must contain a non-zero P ideal of $\overline{R}^\circ$. We call R a quasi-P subring of R°, or briefly, a quasi-P-ring, where P is an arbitrary property of rings.

DEFINITION 1.2. Let I be an ideal of an associative ring R (one-sided or two-sided). When I is regarded as a ring, which is a quasi-P subring of R, we call I a quasi-P ideal of R.

LEMMA 1.1. *An arbitrary quasi-P one-sided ideal of an associative ring R is contained in a quasi-P two-sided ideal of R.*

LEMMA 1.2. *The sum I of all quasi-P two-sided ideals of an associative ring R is also a quasi-P two-sided ideal of R. Therefore, it is a unique maximal quasi-P ideal of R.*

PROOF. Suppose that I is not a quasi-P ideal of R, then R has at least a homomorphic image $\overline{R}$ ($R \overset{\varrho}{\sim} \overline{R} \cong R/N$), $\varrho(I) = \overline{I} \cong (I+N)/N$ in $\overline{R}$ does not contain non-zero P ideals of $\overline{R}$. Thus, we have a quasi-P ideal $I_\alpha \nsubseteqq N$ in I at least. (Otherwise, the sum I of all I_α must be $I \subseteqq N$, i.e. $\overline{I} = \{0\}$, whence I would be quasi-P ideal.) Since I_α are quasi-P ideals, $\overline{I}_\alpha$ must contain a non-zero P ideal of $\overline{R}$, but it is also a non-zero P ideal of $\overline{R}$ in $\overline{I}$, a contradiction. □

LEMMA 1.3. *Let I be a quasi-P two-sided ideal of R, $\overline{H}$ a non-zero quasi-P ideal of R/I. Thus the inverse image H of $\overline{H}$ under the natural homomorphism ϱ ($R \overset{\varrho}{\sim} R/I$) is also a non-zero quasi-P ideal of R.*

LEMMA 1.4. *Let I be a unique maximal quasi-P ideal, then R/I does not contain any non-zero quasi-P ideals.*

Proof. If R/I has a quasi-P ideal $\overline{H} \neq \{0\}$ then, by Lemma 1.3, the complete inverse image H of $\overline{H}$ under the homomorphism f $(R \overset{f}{\sim} R/I)$ is a non-zero quasi-P ideal of R. Furthermore, as I is a maximal quasi-P ideal, so $H \subseteqq I$. Hence $\overline{H} = \{0\}$, a contradiction to $\overline{H} \neq \{0\}$. □

Definition 1.3. The unique maximal quasi-P ideal of an associative ring R is called quasi-P radical of R, and an associative ring having no non-zero quasi-P ideals is called quasi-P semi-simple ring.

According to the above lemmas and Definition 1.3, we have

Theorem 1.1. *In an arbitrary associative ring there must exist a quasi-P radical I, which is the sum of all quasi-P two-sided ideals, and R/I is quasi-P semi-simple.*

Therefore, if P is a concrete property of rings, for instance, nilpotent property, local nilpotent property, nil property, b-property, left quasi-regular property, and g-regular property, we obtain the quasi-nilpotent radical, quasi-local nilpotent radical, quasi-nil radical, quasi-b-radical, quasi-Jacobson radical, and quasi-Brown–McCoy radical etc. respectively. Moreover, for an arbitrary ring R, it need not have a P-radical, but it must have a quasi-P radical.

§2. A quasi-P radical is an Amitsur–Kurosh radical

Lemma 2.1. *Let I be the subring of an associative ring R, and let $\overline{I}$ be the homomorphic image of I: $I \overset{\varphi}{\sim} \overline{I}$. Then, we have a ring $\overline{R}$ such that:* (1) $R \overset{f}{\sim} \overline{R}$, (2) *$\overline{I}$ is a subring of $\overline{R}$,* (3) *the homomorphism f restricted to I equals φ, namely $f|_1 = \varphi$.*

Proof. Let $\overline{R} = \overline{I} \cup R/I$, and consider the mapping

$$\begin{aligned} f\colon\ & R \to \overline{R}, \\ & a \mapsto \overline{a} = a \quad (\text{when } a \in R/I), \\ & a \mapsto \overline{a} = \varphi(a) \quad (\text{when } a \in I). \end{aligned}$$

Obviously, f is a surjection. We define an addition in $\overline{R}$: let $\overline{a}, \overline{b} \in \overline{R}$, and a, b be inverse images of $\overline{a}$, $\overline{b}$, respectively, in R. If $a + b = c$ in R, and $c \overset{f}{\mapsto} \overline{c}$, then we define $\overline{a} \oplus \overline{b} = \overline{c}$. We can similarly define multiplication $\odot$. Obviously, $R \overset{f}{\sim} \overline{R}$, thus $(\overline{R}, \oplus, \odot)$ is an associative ring. From the definition of f, it follows $f|_I = \varphi$, and by $I \overset{\varphi}{\sim} \overline{I}$, we know $\overline{I}$ is a subring of $\overline{R}$. □

Theorem 2.1. *A homomorphic image of a quasi-P ring is also a quasi-P ring.*

Proof. Let R be a quasi-P ring, then it is the quasi-P subring of a certain extensive ring R°. Let $\overline{R}$ be a homomorphic image of R: $R \overset{\varphi}{\sim} \overline{R}$.

By Lemma 2.1, there is an extensive ring $\overline{R}^\circ$ of $\overline{R}$ such that $R^\circ \overset{f}{\sim} \overline{R}^\circ$ and $f\,|_R = \varphi$. We can show that $\overline{R}$ is a quasi-P subring of $\overline{R}^\circ$.

Let $\overline{\overline{R}}^\circ$ be an arbitrary homomorphic image of $\overline{R}^\circ$: $\overline{R}^\circ \overset{f'}{\sim} \overline{\overline{R}}^\circ$ and let $\overline{\overline{R}}$ be the image of $\overline{R}$ under f'. Thus $\overline{\overline{R}} = \{0\}$ or $\overline{\overline{R}} \neq \{0\}$ alternatively. If $\overline{\overline{R}} \neq \{0\}$, there is a non-zero P ideal of $\overline{\overline{R}}^\circ$ in $\overline{\overline{R}}$. In fact, since $R^\circ \overset{f'f}{\sim} \overline{\overline{R}}^\circ$ and $\overline{\overline{R}}$ is the image of R under $f'f$, as R is the quasi-P subring of R°, therefore, there is a non-zero P ideal of $\overline{\overline{R}}^\circ$ in $\overline{\overline{R}}$. Hence $\overline{R}$ is the quasi-P subring of $\overline{R}^\circ$, namely $\overline{R}$ is a quasi-P ring. □

By Theorems 2.1 and 1.1, we have the following result immediately:

THEOREM 2.2. *Quasi-P radicals are Amitsur–Kurosh radicals.* □

According to Theorem 2.2, as soon as we give an arbitrary property P of rings we get an Amitsur–Kurosh radical — quasi-P radical. For instance, a ring (or an ideal) is called an idempotent element ring (or an iderl) if its every element is an idempotent element. Obviously, the idempotent element property is not a radical property. However, by Theorems 1.1 and 2.2, because every ring R has a quasi-idempotent element radical, and a quasi-idempotent element radical is an Amitsur–Kurosh radical, therefore, the quasi-idempotent element property is a radical property. So, if we give an arbitrary property of rings, we do get a radical property.

§3. Quasi-P radical and P radical

LEMMA 3.1. *If P is a property of rings, then a P semi-simple ring must be a quasi-P semi-simple ring.*

PROOF. Let R be P semi-simple, i.e. let R have no non-zero P ideal. If R contains a non-zero quasi-P ideal I, by definition of quasi-P ideals, the image $\overline{I}$ $(= f(I) = I)$ of I under the homomorphism f $(R \overset{f}{\sim} \overline{R} = R/\{0\} = R)$ has a non-zero P ideal S of $\overline{R}$, i.e. R contains a non-zero P ideal S. This is a contradiction. Therefore R is quasi-P semi-simple. □

LEMMA 3.2. *Let P be a radical property, then a quasi-P semi-simple ring R must be a P semi-simple ring.*

PROOF. If the quasi-P semi-simple ring R is not P semi-simple, i.e., R has a non-zero P ideal I, then it can be shown that I is also a quasi-P ideal. In fact, since P is of radical property, the homomorphic images of P rings are still P rings. This means that property P is homomorphically closed. Hence, when $f(I) = \overline{I} \neq \{0\}$, under any arbitrary natural homomorphism f $(R \overset{f}{\sim} R/H)$, it is also an ideal having property P, namely, $\overline{I}$ contains a non-zero P ideal $\overline{I}$ of $\overline{R}$ $(\cong R/H)$. Thus I is a quasi-P ideal. This contradicts the quasi-P semi-simple property of R. □

THEOREM 3.1. *Let P be a radical property. Then an arbitrary associative ring R:*

$$\text{Quasi-}P \text{ radical} = P \text{ radical.}$$

PROOF. Let I, I_Q be expressed P radical and quasi-P radical of R, respectively. Since R/I is P semi-simple, by Lemma 3.1, it is also quasi-P semi-simple, thus $I_Q \subsetneqq I$, and by Lemma 3.2, the quasi-P semi-simple ring R/I_Q is also P semi-simple, whence $I \subsetneqq I_Q$, i.e. $I = I_Q$. □

Since local nilpotent property, nil property, b-property, left quasi-regular property, g-regular property and so on, in the associative rings are of radical property, we obtain, by Theorem 3.1, the following results in an associative ring R: quasi-L-radical $=$ L-radical, quasi-K-radical $=$ K-radical, quasi-b-radical $=$ b-radical $=$ Baer radical, quasi-J-radical $=$ J-radical, quasi-BM-radical $=$ BM-radical, etc.

Therefore, a lot of concrete Amitsur–Kurosh radicals can be unified into a quasi-P-radical. So, while studying the problems of concrete Amitsur–Kurosh radicals and semi-simple property etc., we may study, in a unified way, the quasi-P radical and quasi-P semi-simple rings.

Moreover, the nilpotent property is not a radical property. However, if R has a nilpotent radical, the quasi-nilpotent radical of R equals nilpotent radical, too. Since,

COROLLARY 3.1. *If P is a property of rings other than radical property, and property P is homomorphically closed, with an arbitrary associative ring R having P-radical I (I is a maximal P ideal of R, and R/I is P semi-simple), we have:*

$$\text{Quasi-}P \text{ radical of } R = P \text{ radical of } R.$$

Since the nilpotent property is homomorphically closed, so, by Corollary 3.1, if R has a nilpotent radical, we have:

$$\text{Quasi-nilpotent radical} = \text{nilpotent radical.}$$

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DEPARTMENT OF MATHEMATICS
NINGBO UNIVERSITY
NINGBO, ZHEJIANG
P. R. OF CHINA

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SOME REMARKS ON GENERAL RADICAL THEORY AND DISTRIBUTIVE NEAR-RINGS

S. VELDSMAN (Port Elizabeth)

§1. Introduction

The Kurosh–Amitsur radical theory of associative rings is a well developed theory; largely due to the well known ADS-theorem which is a useful substitute for the non-transitivity of the relation of being an ideal. To prove this theorem one uses, in concerto, the commutativity of the underlying group, the associativity of the multiplication and the (right and left) distributivity of the multiplication over the addition. It is already known that the last two axioms (or at least suitable substitutes) are necessary for the ADS-theorem to hold. In this note we collect together these results and investigate the influence of the commutativity of the underlying group on the validity of the ADS-theorem. In fact, we show that in the variety of distributive near-rings (or equivalently, associative rings with not necessarily abelian addition), the ADS-theorem is no longer valid. Contrary to the varieties of all near-rings or all 0-symmetric near-rings, this variety is an Andrunakievič variety — hence all hypersolvable radicals have the ADS-property.

§2. Background

Let $\mathcal{W}$ be a universal class of Ω-groups (i.e. $\mathcal{W}$ is homomorphically closed and hereditary on ideals). A *(Kurosh–Amitsur) radical class* $\mathbf{R}$ in $\mathcal{W}$ is a subclass $\mathbf{R}$ of $\mathcal{W}$ which is homomorphically closed, closed under extensions (i.e. if $I \triangleleft A \in \mathcal{W}$ and both I and A/I are in $\mathbf{R}$, then $A \in \mathbf{R}$) and for all $A \in \mathcal{W}$, $\mathbf{R}(A) := \sum(I \triangleleft A \mid I \in \mathbf{R}) \in \mathbf{R}$. With $\mathbf{R}$ we associate its *semisimple class* $\mathcal{S}\mathbf{R}$ which is defined by

$$\mathcal{S}\mathbf{R} := \{A \in \mathcal{W} \mid \mathbf{R}(A) = 0\}.$$

$\mathbf{R}$ has the *ADS-property* if $\mathbf{R}(I) \triangleleft A$ holds for all $I \triangleleft A \in \mathcal{W}$. Radicals with this property always have a hereditary semisimple class (i.e. $I \triangleleft A \in \mathcal{S}\mathbf{R}$ implies $I \in \mathcal{S}\mathbf{R}$; or equivalently $\mathbf{R}(I) \subseteqq \mathbf{R}(A)$ for all $I \triangleleft A \in \mathcal{W}$). For the basics on the general radical theory of associative rings, Wiegandt [15] can be consulted.

An Ω-group A is a *zero Ω-group* if $\Omega A = 0$ (i.e. $a_1 a_2 \dots a_n \omega = 0$ for all $a_i \in A$, $\omega \in \Omega$). A radical class $\mathbf{R}$ is *hypersolvable* if $\mathbf{R}$ contains all zero Ω-groups in $\mathcal{W}$. For $A \in \mathcal{W}$, A^+ denotes the underlying group and A^0 denotes the zero Ω-group built on A^+.

For the purpose of comparison, we recall some known results:

2.1. THEOREM (Anderson, Divinsky and Suliński [1]). *Let $\mathcal{W}$ be the variety of all associative rings. Then every radical class in $\mathcal{W}$ has the ADS-property and (consequently) all semisimple classes are hereditary.*

Remark that this result remains true if the associativity is replaced by weaker conditions, e.g. the universal class of alternative rings (Anderson, Divinsky and Suliński [1]), the universal class of Jordan R-algebras, R associative and commutative with $\frac{1}{2} \in R$, the universal class of (γ, δ)-algebras with $\frac{1}{6} \in R$ and the Andrunakievič s-varieties (cf. Beidar [2]). However, with no assumption on the associativity, we have the degenerate results of Gardner:

2.2. THEOREM (Gardner [6] and [7]). *Let $\mathcal{W}$ be the variety of all rings and let $\mathbf{R} \subseteqq \mathcal{W}$ be a radical class. Then the following are equivalent:*

(1) $\mathbf{R}$ *has the ADS-property.*

(2) $\mathcal{S}\mathbf{R}$ *is hereditary.*

(3) $\mathbf{R}$ *is an A-radical (these are radical classes $\mathbf{R}$ for which $A \in \mathbf{R}$ and $B \in \mathcal{W}$ with $A^+ \cong B^+$ implies $B \in \mathbf{R}$).*

The above equivalences are well-known, except maybe (3) $\Rightarrow$ (1): Let $\mathbf{R}$ be an A-radical. From Gardner [5] we know that $\mathbf{R}_G := \{A^+ \mid A \in \mathbf{R}\}$ is a radical class in the variety of all abelian groups and $\mathbf{R}_G(A^+) = (\mathbf{R}(A))^+$ for all $A \in \mathcal{W}$. Consider $\mathbf{R}(I) \triangleleft I \triangleleft A \in \mathcal{W}$ and let $a \in A$. Since $(aI)^+$ is a subgroup of I^+ and a acts homomorphically on I^+, we get

$$a\mathbf{R}(I) = a\mathbf{R}_G(I^+) \subseteqq \mathbf{R}_G(aI^+) \subseteqq \mathbf{R}_G(I^+) = \mathbf{R}(I).$$

Likewise, $\mathbf{R}(I)a \subseteqq \mathbf{R}(I)$ and $\mathbf{R}(I) \triangleleft A$ follows.

Concerning the influence of the distributivity, firstly recall that a *near-ring* is an associative ring for which the underlying group is not necessarily abelian and only one distributive law is required (in our case, we assume the right distributive law holds). The folklore of near-ring theory can be found in Pilz [10]. If N is a near-ring, then $0x = 0$ for all $x \in N$, but $x0$ need not be 0. Near-rings N for which $x0 = 0$ for all $x \in N$ are called *0-symmetric.* A *constant* near-ring N is one for which $x0 = x$ for all $x \in N$.

We will also need

2.3. PROPOSITION (Mlitz and Oswald [9]). *Let $\mathcal{W}$ be a universal class of near-rings. If $\mathbf{R} \subseteqq \mathcal{W}$ is a hypersolvable radical class, then $\mathbf{R}$ contains all nilpotent near-rings in $\mathcal{W}$. The hypersolvable radicals $\mathbf{R}$ in $\mathcal{W}$ can be characterized as all those radical classes $\mathbf{R}$ for which*

$$\mathbf{R}(N) = \{x \in N \mid xN \subseteqq \mathbf{R}(N)\}$$

holds for all $N \in \mathcal{W}$.

The presence of only one distributive law also causes a degeneracy — however, in this case it is good.

2.4. THEOREM (Betsch and Kaarli [3], Veldsman [14]). *Let* $\mathcal{W}$ *be the variety of all near-rings or all abelian near-rings (or only the* 0*-symmetric ones in each case). Let* $\mathbf{R} \subseteqq \mathcal{W}$ *be a radical class with a hereditary semisimple class. Then* $\mathbf{R}$ *contains all the nilpotent near-rings and all the constant near-rings.*

2.5. COROLLARY. *If* $\mathcal{W}$ *is as in Theorem* 2.4, *then a semisimple class* $\mathcal{S}\mathbf{R}$ *is hereditary if and only if* $\mathbf{R}$ *is hypersolvable and* $\mathbf{R}(I)A \subseteqq \mathbf{R}(A)$ *for all* $I \triangleleft A \in \mathcal{W}$.

The proof follows readily from Proposition 2.3 and Theorem 2.4.

§3. Distributive near-rings

A distributive near-ring is an associative ring with not necessarily abelian addition. If N is such a near-ring, then $ab + cd = cd + ab$ for all $a, b, c, d \in N$ and if N' is the commutator subgroup of N^+, then N' is an ideal of N and $NN' = 0 = N'N$ (cf. [4], [10] and [11]). In particular, $(N')^2 = 0$. Remark that here, as is customary in near-ring theory, for subsets $X, Y \subseteqq N$, $XY = \{xy \mid x \in X, y \in Y\}$.

3.1. PROPOSITION. *The variety* $\mathcal{W}$ *of all distributive near-rings is an Andrunakievič variety (i.e. if* $J \triangleleft I \triangleleft N \in \mathcal{W}$, *then* $\overline{J}^3 \subseteqq J$ *where* $\overline{J}$ *is the ideal in* N *generated by* J*).*

PROOF. Let $X = J + NJ + JN + NJN$. Then

$$X^3 \subseteqq IXI = I(J + NJ + JN + NJN)I \subseteqq$$
$$\subseteqq IJI + (IN)JI + IJ(NI) + (IN)J(NI) \subseteqq J$$

since $J \triangleleft I \triangleleft N$. Let $\langle X \rangle$ be the normal subgroup in N generated by X, i.e.

$$\langle X \rangle = \left\{ \sum_i^{\text{finite}} \sigma_i(a_i + x_i - a_i) \mid \sigma_i = \pm 1,\ a_i \in N,\ x_i \in X \right\}.$$

Then $J \subseteqq \langle X \rangle$ and $\langle X \rangle$ is in fact an ideal in N (using the fact that N^2 is abelian). Hence $\overline{J} = \langle X \rangle$. Using $X^3 \subseteqq J$ and once again the fact that N^2 is abelian, it follows that $\overline{J}^3 \subseteqq J$.

3.2. THEOREM. *Let* $\mathbf{R}$ *be a hypersolvable radical class in the variety* $\mathcal{W}$ *of all distributive near-rings. Then* $\mathbf{R}$ *has the ADS-property and every near-ring in* $\mathcal{S}\mathbf{R}$ *is in fact a ring.*

PROOF. Consider $\mathbf{R}(I) \triangleleft I \triangleleft N \in \mathcal{W}$. Then

$$\overline{\mathbf{R}(I)}/\mathbf{R}(I) \triangleleft I/\mathbf{R}(I) \in \mathcal{S}\mathbf{R}$$

and $\overline{\mathbf{R}(I)}/\mathbf{R}(I)$ is nilpotent by Proposition 3.1. Since $\mathbf{R}$ is hypersolvable, $\mathbf{R}(I) = \overline{\mathbf{R}(I)}$ holds, i.e. $\mathbf{R}(I) \triangleleft N$. Lastly, if $A \in \mathcal{S}\mathbf{R}$, then the commutator A' of A^+ is a nilpotent ideal of A. Since $\mathbf{R}$ is hypersolvable, $A' = 0$; hence A^+ is abelian which means that A is a ring.

Next we give a characterization of the radicals with hereditary semisimple classes (compare this with Corollary 2.5):

3.3. THEOREM. *Let* $\mathbf{R}$ *be a radical class in the variety of all distributive near-rings. Then* $\mathcal{S}\mathbf{R}$ *is hereditary if and only if*

$$\mathbf{R}(I)A + A\mathbf{R}(I) \subseteqq \mathbf{R}(A) \quad \textit{for all} \quad I \triangleleft A \in \mathcal{W}.$$

PROOF. If $\mathcal{S}\mathbf{R}$ is hereditary, then $\mathbf{R}(I) \subseteqq \mathbf{R}(A)$ for all $I \triangleleft A \in \mathcal{W}$ and the result follows since $\mathbf{R}(A) \triangleleft A$. Conversely, let $I \triangleleft A \in \mathcal{S}\mathbf{R}$. We show $I \in \mathcal{S}\mathbf{R}$, i.e. $\mathbf{R}(I) = 0$. This follows if we can prove that $\mathbf{R}(I) \triangleleft A$. Since $a\mathbf{R}(I) + \mathbf{R}(I)a \subseteqq \mathbf{R}(A) = 0$ for all $a \in A$, $\mathbf{R}(I)$ is both left and right invariant in A. Thus it is sufficient to show that $\mathbf{R}(I)$ is a normal subgroup of A. Suppose this is not the case. Then there is an $a_0 \in A$ such that $a_0 + \mathbf{R}(I) - a_0 \nsubseteqq \mathbf{R}(I)$. Let $J := a_0 + \mathbf{R}(I) - a_0 + \mathbf{R}(I)$. Then straightforward calculations show that $J/\mathbf{R}(I) \triangleleft I/\mathbf{R}(I)$ and $J/\mathbf{R}(I)$ is a homomorphic image of $\mathbf{R}(I)$ by $x \to a_0 + x - a_0 + \mathbf{R}(I)$. Since $\mathbf{R}$ is homomorphically closed, $J/\mathbf{R}(I) \in \mathbf{R}$. But $I/\mathbf{R}(I) \in \mathcal{S}\mathbf{R}$; hence $J/\mathbf{R}(I) = 0$ which contradicts $a_0 + \mathbf{R}(I) - a_0 \nsubseteqq \mathbf{R}(I)$ and proves the theorem.

A radical class $\mathbf{R}$ is an *invariantly strong radical* if S is an invariant subgroup of A and $S \in \mathbf{R}$, then $S \subseteqq \mathbf{R}(A)$.

3.4. THEOREM. *Let* $\mathbf{R}$ *be an invariantly strong radical class in the variety of distributive near-rings. Then* $\mathbf{R}$ *has the ADS-property.*

PROOF. Let $I \triangleleft A$. As in the proof of Theorem 3.3 it can be shown that $\mathbf{R}(I)$ is a normal subgroup of A. Let $a \in A$ and let $J = a\mathbf{R}(I) + \mathbf{R}(I)$. Then $J/\mathbf{R}(I)$ is a homomorphic image of $\mathbf{R}(I)$ and it is an invariant subgroup of $I/\mathbf{R}(I) \in \mathcal{S}\mathbf{R}$. By the assumption on $\mathbf{R}$, we get $J/\mathbf{R}(I) = 0$, i.e. $a\mathbf{R}(I) \subseteqq \mathbf{R}(I)$. A symmetrical argument yields $\mathbf{R}(I)a \subseteqq \mathbf{R}(I)$ for all $a \in A$ and hence $\mathbf{R}(I) \triangleleft A$.

Invariantly strong radicals are plentiful in view of Theorem 3.2 and the next result:

3.5. THEOREM. *Let* $\mathbf{R}$ *be a radical class for which* $\mathcal{S}\mathbf{R}$ *consists of rings. Then* $\mathbf{R}$ *is invariantly strong.*

PROOF. Let S be an invariant subgroup of A with $S \in \mathbf{R}$. Then

$$S/S \cap \mathbf{R}(A) \cong S + \mathbf{R}(A)/\mathbf{R}(A)$$

and the latter is an invariant subgroup of the ring $A/\mathbf{R}(A)$; hence an ideal. Since $\mathcal{S}\mathbf{R}$ consists of rings, it is hereditary and consequently $S+\mathbf{R}(A)/\mathbf{R}(A) = 0$. Whence $S \subseteqq \mathbf{R}(A)$.

Standard radical theoretic arguments will show that if $\mathbf{R}$ is a radical class in the variety of distributive near-rings, then $\mathbf{R}$ is invariantly strong if and only if $\mathcal{S}\mathbf{R}$ is hereditary on invariant subgroups.

We now give an example which shows that not all the semisimple classes in the variety of distributive near-rings are hereditary; hence not all radical classes have the ADS-property.

3.6. EXAMPLE. Let D_8 be the dihedral group with 8 elements generated by a and b subject to $4a = 0 = 2b$ and $ia + b = b + (4 - i)a$ for $i = 1, 2, 3$. Since $U := \{0, a, 2a, 3a\}$ is a normal subgroup of D_8 for which D_8/U is a finite cyclic group, we can use Theorem 2.1 in Heatherly [8] to obtain a distributive near-ring M on D_8 for which the multiplication is given by

$\cdot$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0
$2a$	0	0	0	0	0	0	0	0
$3a$	0	0	0	0	0	0	0	0
b	0	0	0	0	b	b	b	b
$a+b$	0	0	0	0	b	b	b	b
$2a+b$	0	0	0	0	b	b	b	b
$3a+b$	0	0	0	0	b	b	b	b

The only ideals of M are $\{0\}$, M, $T := \{0, 2a\}$, $U = \{0, a, 2a, 3a\}$ and $V := \{0, 2a, b, 2a + b\}$. Apart from the trivial ideals, V has ideals T and $Y := \{0, b\}$. Note that Y is not an ideal of M since it is not a normal subgroup.

Let $\mathbf{R} := \{N \in \mathcal{W} \mid N^2 = N\}$ where N^2 is the ideal in N generated by N^2. As is well-known, $\mathbf{R}$ is a radical class in $\mathcal{W}$, in fact the upper radical class determined by the class of all distributive near-rings with zero multiplication. M has no non-zero ideals which are in $\mathbf{R}$; hence $M \in \mathcal{S}\mathbf{R}$. However, $\mathbf{R}(V) = Y \neq 0$, i.e. $V \notin \mathcal{S}\mathbf{R}$.

Finally, remark that if we have a suitable substitute for the lack of commutativity of the underlying group, then some of the results of the associative ring case can be recovered, cf. [13], Section 5.6.

Specifically, if $\mathcal{V}$ is a universal class of distributive near-rings for which aI is a normal subgroup of A for all $I \triangleleft A \in \mathcal{V}$, $a \in A$, then every radical class in $\mathcal{V}$ has the ADS-property.

In the present paper we only considered the omission of one of the ring axioms mentioned in the introduction. Of course, omitting two or more may yield further degeneracy, see for example [12].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PORT ELIZABETH
P.O.BOX 1600
6000 PORT ELIZABETH
SOUTH AFRICA

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ON THE CONVERGENCE OF EIGENFUNCTION EXPANSIONS

I. JOÓ (Budapest)

Let $\Omega \subset \mathbf{R}^N$ ($N \geqq 1$) be any bounded domain. A function $0 \neq u \in C^2(\Omega)$ is said to be an eigenfunction of the Laplace operator with (complex) eigenvalue λ, if $-\Delta u = \lambda u$ in Ω. Consider any complete orthonormal system $(u_i) \subset L^2(\Omega)$ of eigenfunctions of the Laplace operator with arbitrary (complex) eigenvalues $(\lambda_i) \subset \mathbf{C}$, i.e.

$$-\Delta u_i = \lambda_i u_i. \tag{1}$$

Denote μ_i the square root of λ_i with Re $\mu_i \geqq 0$; further use the notations $\varrho_i = \mathrm{Re}\,\mu_i \geqq 0$, $\nu_i = \mathrm{Im}\,\mu_i$. Introduce the Bessel–MacDonald function [1]

$$v_\alpha(r) = v_\alpha^N(r) = \frac{2^{(2-\alpha)/2}}{(2\pi)^{N/2}\Gamma\left(\frac{\alpha}{2}\right)} \cdot \frac{K_{(N-\alpha)/2}(r)}{r^{(N-\alpha)/2}}, \qquad \alpha > 0 \tag{2}$$

(K_ν denotes the ν-th MacDonald function). The function space $L_p^\alpha(\mathbf{R}^N)$ consists of the complex-valued functions on $\mathbf{R}^N$ representable in the form $f(x) = \int\limits_{\mathbf{R}^N} v_\alpha(|x-y|)h(y)dy$ with some $h \in L^p(\mathbf{R}^N)$ and the norm of f is defined by $\|f\|_{L_p^\alpha} = \|h\|_{L^p}$. The function space $L_p^\alpha(\mathbf{R}^N)$ is a Banach space with this norm and is called Liouville space. It is a natural and simple generalization of the Soboleff spaces W_p^α for nonintegral α. We shall prove in this paper the following three theorems.

THEOREM 1 (uniform convergence). *Let $f \in L_p^\alpha(\mathbf{R}^3)$, $\alpha \geqq 1$, $\alpha p > 3$, $p \geqq 1$. Suppose f has compact support* supp f *in Ω. Then the partial sums*

$$S_\mu(f,x) := \sum_{\varrho_i<\mu} {}_i f_i u_i(x), \quad f_i : = \int\limits_\Omega f\overline{u_i}, \quad \mu > 0 \tag{3}$$

of the expansion of f with respect to the eigenfunctions (u_i) converge to f uniformly on every compact subset of Ω as $\mu \to \infty$.

THEOREM 2 (localization principle). *Let $f \in L_2^\alpha(\mathbf{R}^3)$, $\alpha \geqq 1$. Suppose $f = 0$ on some domain $\Omega_0 \subset \Omega$. Then $S_\mu(f,x) \to f(x) = 0$ uniformly on every compact subset of Ω_0 as $\mu \to \infty$.*

THEOREM 3 (absolute convergence). *Let $f \in L_p^\alpha(\mathbf{R}^3)$, $\alpha > 3/2$, $\alpha p > 3$, $p \geqq 1$. Suppose f has compact support in Ω. Then the Fourier series of f*

with respect to the system (u_i) *converges absolutely and uniformly on every compact subset of* Ω.

The case $N \geqq 1$, $\lambda_i \geqq 0$ was proved by Titchmarsh and V. A. Il'in, and these results are classical (see e.g. in [1]). The case $N = 1$ was proved for arbitrary complex eigenvalues in [3] for the Schrödinger operator and the case $N = 3$ was first investigated in [4], [5] for any $\lambda_i \in \mathbf{C}$.

For the proof we need a series of lemmas. We give these in the subsequent sections.

§1. Upper and lower estimate for the square sums of eigenfunctions

In this section $\Omega \subset \mathbf{R}^3$ is an arbitrary (not necessarily bounded) domain. Suppose $(u_i) \subset L^2(\Omega)$ is an arbitrary system of eigenfunctions: $-\Delta u_i = \lambda_i u_i$ $(\lambda_i \in \mathbf{C})$. The system (u_i) is said to be a Bessel system if

$$\sum_{i=1}^{\infty} |\langle f, u_i\rangle|^2 \leqq C\|f\|^2_{L^2(\Omega)}, \quad f \in L^2(\Omega) \tag{4}$$

holds with a constant C independent of f. (u_i) is called a Hilbert system if

$$\sum_{i=1}^{\infty} |\langle f, u_i\rangle|^2 \geqq C\|f\|^2_{L^2(\Omega)}, \qquad f \in L^2(\Omega) \tag{5}$$

for some constant $C > 0$. In (4) and (5) $\langle f, u_i\rangle := \int_\Omega f\overline{u_i}$. Finally (u_i) is called a Riesz system if it is a Bessel and a Hilbert system and in this case we have

$$c_1\|f\|^2_{L^2(\Omega)} \leqq \sum_{i=1}^{\infty} |\langle f, u_i\rangle|^2 \leqq c_2\|f\|^2_{L^2(\Omega)}.$$

For this case we shall use the shorter notation

$$\sum_{i=1}^{\infty} |\langle f, u_i\rangle|^2 \asymp \|f\|^2_{L^2(\Omega)}. \tag{6}$$

First we prove

LEMMA 1. *Let* $(u_i) \subset L^2(\Omega)$ *be any Bessel system of eigenfunctions of the Laplace operator, let* $K \subset \Omega$ *be any fixed compact set; further let* $0 < R < \min(\pi/4, 0.5 \operatorname{dist}(K, \partial\Omega))$ *be an arbitrary fixed number. Then there exists a constant* $c = c(K, R)$ *independent of* x *and* μ *such that*

$$\sum_{|\mu-\varrho_i|\leqq 1} \left(|u_i(x)|e^{2|\nu_i|R}\right)^2 \leqq c\mu^2 \qquad (x \in K,\ \mu \geqq 1). \tag{7}$$

Proof. This is proved in [5]. For the sake of completeness we give a short proof here. According to the well-known mean-value formula we have

$$\int_\theta u_i(x+r\theta)d\theta = 4\pi\frac{\sin r\mu_i}{r\mu_i}u_i(x) \quad \text{if } \mu_i \neq 0, \tag{8}$$

$$\int_\theta u_i(x+r\theta)d\theta = 4\pi u_i(x) \quad \text{if } \mu_i = 0. \tag{8'}$$

For any $0 < R < 0.5\,\mathrm{dist}(K, \partial\Omega)$ define

$$d(r,\mu) := \begin{cases} \mu\dfrac{\sin r\mu}{r} & \text{if } R < r < 2R, \\ 0 & \text{otherwise.} \end{cases}$$

According to (8), for any fixed $x \in K$ the Fourier coefficients of $d(|x-y|,\mu)$ are

$$\overline{d_i} = \int_\Omega d(|x-y|,\mu)u_i(y)dy = \int_R^{2R} d(r,\mu)\int_\theta u_i(x+r\theta)d\theta r^2 dr = \tag{9}$$
$$= 4\pi u_i(x)\frac{\mu}{\mu_i}\int_R^{2R}\sin r\mu\,\sin r\mu_i dr.$$

Our aim is to get (7) from the Bessel inequality $\sum\limits_{i=1}^{\infty}|d_i|^2 \leqq c\|d(|x-y|,\mu)\|^2_{L^2(\Omega)}$. Hence we have to give a lower estimate for

$$\int_R^{2R}\sin r\mu\,\sin r\mu_i dr = \frac{1}{2}\int_R^{2R}[\cos r(\mu-\mu_i) - \cos r(\mu+\mu_i)] =$$
$$= \left[\frac{\sin r(\mu-\mu_i)}{2(\mu-\mu_i)} - \frac{\sin r(\mu+\mu_i)}{2(\mu+\mu_i)}\right]_R^{2R}.$$

We give the lower estimate for four cases separately:

Case 1: $\varrho_i \geqq B|\nu_i| \geqq B^2$, where $B \gg 1$. In this case we have

$$\left|\frac{\sin r(\mu-\mu_i)}{\mu-\mu_i}\right| \asymp \frac{e^{r|\nu_i|}}{|\nu_i|} \quad \text{and} \quad \left|\frac{\sin r(\mu+\mu_i)}{\mu+\mu_i}\right| \asymp \frac{e^{r|\nu_i|}}{r|\varrho_i|}$$

with absolute constants. Hence

$$\left|\int_R^{2R} \sin r\mu \, \sin r\mu_i dr\right| \geqq c\frac{e^{2R|\nu_i|}}{|\nu_i|}$$

and we get

$$\sum_{\substack{|\varrho_i-\mu|\leqq 1\\ \varrho_i\geqq B|\nu_i|\geqq B^2}} |u_i(x)|^2\frac{e^{4R|\nu_i|}}{|\nu_i|^2} \leqq c\sum_{i=1}^{\infty}|d_i|^2 \leqq \tag{10}$$

$$\leqq c\|d(|x-y|,\mu)\|^2_{L^2(\Omega)} = c\int_R^{2R} r^2\mu^2\frac{\sin^2 r\mu}{r^2}dr \leqq c\mu^2.$$

Case 2: $\mu \geqq \mu_0$, $|\nu_i| \leqq B$, and $\mu_0 \gg B$. Taking into account the identity $\sin r\mu_i = \sin r\varrho_i \operatorname{ch} r\nu_i + i \cos r\varrho_i \operatorname{sh} r\nu_i$ we have

$$\left|\left[\frac{\sin r(\mu-\mu_i)}{\mu-\mu_i}\right]_R^{2R}\right| \geqq \left|\frac{\sin 2R(\mu-\varrho_i)\operatorname{ch} 2R\nu_i - \sin R(\mu-\varrho_i)\operatorname{ch} R\nu_i}{\mu-\mu_i}\right| =$$

$$= \frac{|\sin R|\mu-\varrho_i||}{|\mu-\varrho_i|}\cdot|2\cos R(\mu-\varrho_i)\operatorname{ch} 2R\nu_i - \operatorname{ch} R\nu_i|.$$

Hence for $R < \pi/4$ we get $\left|\frac{\sin r(\mu-\mu_i)}{\mu-\mu_i}\right| \geqq C(R) > 0$. On the other hand

$$\left|\left[\frac{\sin r(\mu+\mu_i)}{\mu+\mu_i}\right]_R^{2R}\right| \leqq \frac{e^{2R|\nu_i|}}{|\mu+\mu_i|} \leqq \frac{e^{2R|\nu_i|}}{2\mu-1},$$

i.e. if μ_0 is large enough, then the term containing $\mu-\mu_i$ is dominant, and we have

$$\sum_{\substack{|\varrho_i-\mu|\leqq 1\\ |\nu_i|\leqq B}} |u_i(x)|^2 \leqq c\mu^2 \qquad (\mu \geqq \mu_0). \tag{11}$$

Case 3: $\varrho_i \leqq B|\nu_i|$, $\mu \geqq \mu_0$, $B \leqq |\nu_i|$. In this case the magnitude of the terms considered may be equal, hence expand now the function

$$d^1(r,\mu) := \begin{cases} \mu/r & \text{if } R < r < 2R\\ 0 & \text{otherwise.}\end{cases}$$

Obviously

$$\|d^1(|x-y|,\mu)\|^2_{L^2(\Omega)} = c\int_R^{2R} r^2\frac{\mu^2}{r^2}dr \leqq c\mu^2,$$

$$d_i^1 = cu_i(x)\frac{\mu}{\mu_i}\int\limits_R^{2R} \sin r\mu_i dr = cu_i(x)\frac{\mu}{\mu_i}\frac{\cos R\mu_i - \cos 2R\mu_i}{\mu_i}$$

hence $|d_i^1| \geqq c|u_i(x)|\dfrac{e^{2R|\nu_i|}}{|\nu_i|}$, consequently

$$\sum_{\substack{|\varrho_i-\mu|\leqq 1\\ \varrho_i\leqq B|\nu_i|\\ B\leqq|\nu_i|}} |u_i(x)|^2\frac{e^{4R|\nu_i|}}{|\nu_i|^2} \leqq c\sum_{i=1}^{\infty}|d_i^1|^2 \leqq c\mu^2 \qquad (\mu \geqq \mu_0). \tag{12}$$

Case 4: $\mu \leqq \mu_0$. Now expand the function

$$d^2(r,\mu) := \begin{cases} 1/r & \text{if } < r < 2R,\\ 0 & \text{otherwise.}\end{cases}$$

Obviously,

$$d_i^2 = cu_i(x)\int\limits_0^{2R}\mu_i^{-1}\sin r\mu_i dr = cu_i(x)\frac{1-\cos 2R\mu_i}{\mu_i^2}.$$

a) If $2R|\nu_i| > 2$ then

$$\left|\int\limits_0^R \frac{\sin r\mu_i}{\mu_i}dr\right| \geqq c\frac{e^{2R|\nu_i|}}{|\nu_i|^2}$$

hence

$$\sum_{\substack{\varrho_i\leqq\mu_0+1\\ 2R|\nu_i|>2}} |u_i(x)|^2\frac{e^{4R|\nu_i|}}{|\nu_i|^2} \leqq c\sum_{i=1}^{\infty}|d_i|^2 \leqq c\|d^2(|x-y|,\mu)\|^2_{L^2(\Omega)} \leqq c. \tag{13}$$

b) If $2R|\nu_i| < 2$ then denote $T := \{\varrho_i \leqq \mu_0+1,\ |\nu_i| < 1/R\}$. Pick $R' < R$ such that $k\pi/R \neq \ell\pi/R'$, then

$$\min_{z\in T}\max\left\{\left|\frac{1-\cos 2Rz}{z^2}\right|, \left|\frac{1-\cos 2R'z}{z^2}\right|\right\} > 0$$

i.e. we can divide the set $\{\mu_i : \mu_i \in T\}$ into two parts, so that for one we can use R for the other R' and we get

$$\sum_{\substack{\varrho_i\leqq\mu_0+1\\ 2R|\nu_i|\leqq 2}} |u_i(x)|^2 \leqq c. \tag{14}$$

From (10), (11), (12), (13), (14) the desired (7) follows. □

LEMMA 2. *Let $(u_i) \subset L^2(\Omega)$ be any Riesz system of eigenfunctions of the Laplace operator. Then for any compact set $K \subset \Omega$ and $R > 0$ there exists $M > 0$ and $c > 0$ such that*

$$\sum_{|\mu-\varrho_i|\leqq M} |u_i(x)|^2 e^{4R|\nu_i|} > c\mu^2 \quad (x \in K,\ \mu \geqq 1). \tag{15}$$

PROOF. Obviously we can suppose that $R < \min(\pi/4,\ 0.5\operatorname{dist}(K, \partial G))$ and in this case we can apply (7), i.e. for any $\delta > 0$ we have

$$\sum_{i=1}^{\infty} \frac{|u_i(x)|^2 e^{4R|\nu_i|}}{(1+\varrho_i)^{3+\delta}} \leqq c(\delta) < \infty \qquad (x \in K). \tag{16}$$

According to (9) we have

$$c\mu^2 \leqq \|d(|x-y|,\mu)\|^2_{L^2(\Omega)} \leqq c\sum_{i=1}^{\infty} |d_i|^2 \leqq$$
$$\leqq c\sum_{i=1}^{\infty} \frac{\mu^2}{|\mu_i|^2} \left| \int_R^{2R} \sin r\mu \ \sin r\mu_i dr \right|^2 |u_i(x)|^2,$$

further

$$\left| \int_R^{2R} \sin\ r\mu \sin r\mu_i dr \right| = \left| \left[\frac{\sin r(\mu-\mu_i)}{2(\mu-\mu_i)} - \frac{\sin r(\mu+\mu_i)}{2(\mu+\mu_i)} \right]_R^{2R} \right| \leqq c\frac{e^{2R|\nu_i|}}{1+|\mu-\varrho_i|},$$

hence

$$\sum_{i=1}^{\infty} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1+|\mu-\varrho_i|^2} |u_i(x)|^2 > c\mu^2 \qquad (x \in K). \tag{17}$$

We will choose $M > 0$ later, now suppose only that $\mu \geqq 2M$, then it follows from (17)

$$c\sum_{|\mu-\varrho_i|\leqq M} |u_i(x)|^2 e^{4R|\nu_i|} > c\mu^2 - \sum_{|\mu-\varrho_i|>M} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1+|\mu-\varrho_i|^2} |u_i(x)|^2. \tag{18}$$

We have to estimate the sum on the right hand side of (18). For this use the partition

$$S = \sum_{|\mu-\varrho_i|>M} = \sum_{M<|\mu-\varrho_i|\leqq\frac{\mu}{2}} + \sum_{\frac{3}{2}\mu\leqq\varrho_i} + \sum_{1\leqq\varrho_i\leqq\frac{1}{2}\mu} + \sum_{\varrho_i\leqq 1} = \tag{19}$$
$$= S_1 + S_2 + S_3 + S_4.$$

If $R < \min(\pi/4,\ \mathrm{dist}(K, \partial\Omega))$ then (7) holds and hence

$$\sum_{\substack{i=1\\ \varrho_i \geqq 1}}^{\infty} \frac{|u_i(x)|^2 e^{4R|\nu_i|}}{|\varrho_i|^{3+\delta}} \leqq c(\delta), \tag{20}$$

consequently

$$S_2 \leqq c\frac{\mu^2}{\mu^{1/2}} \sum_{\frac{3}{2}\mu \leqq \varrho_i} \frac{e^{4R|\nu_i|}}{\varrho_i^{3+\frac{1}{2}}} |u_i(x)|^2 \leqq c\mu^{3/2}, \tag{21}$$

$$S_3 \leqq c\frac{\mu^2}{\mu^{1/2}} \sum_{1 \leqq \varrho_i \leqq \frac{\mu}{2}} \frac{e^{4R|\nu_i|}}{\varrho_i^{3+\frac{1}{2}}} |u_i(x)|^2 \leqq c\mu^{3/2}. \tag{22}$$

It follows from (7) that

$$\sum_{\substack{\varrho_i \leqq 1\\ |\mu_i| \geqq 1}} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1 + |\mu - \varrho_i|^2} |u_i(x)|^2 \leqq c \sum_{\varrho_i \leqq 1} |u_i(x)|^2 e^{4R|\nu_i|} \leqq c\mu^2. \tag{23}$$

If $|\mu_i| \leqq 1$, then we have to estimate as follows:

$$\int_R^{2R} \sin r\mu \, \sin r\mu_i dr = \left[-\frac{\cos r\mu}{\mu} \sin r\mu_i\right]_R^{2R} + \frac{\mu_i}{\mu} \int_R^{2R} \cos r\mu \, \cos r\mu_i dr,$$

hence

$$\frac{\mu^2}{|\mu_i|^2} \left| \int_R^{2R} \sin r\mu \, \sin r\mu_i dr \right| \leqq c.$$

So we get a more exact estimate than that of (17), namely

$$\sum_{|\mu_i| \geqq 1} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1 + |\mu - \varrho_i|^2} |u_i(x)|^2 + \sum_{|\mu_i| \leqq 1} |u_i(x)|^2 \geqq c\mu^2, \tag{17'}$$

hence

(18′)

$$\sum_{|\mu - \varrho_i| \leqq M} |u_i(x)|^2 e^{4R|\nu_i|} \geqq c\mu^2 - c \sum_{\substack{|\mu - \varrho_i| > M\\ |\mu_i| \leqq 1}} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1 + |\mu - \varrho_i|^2} |u_i(x)|^2 - c \sum_{|\mu_i| \leqq 1} |u_i(x)|^2.$$

Obviously

$$S_4 \leqq \sum_{|\mu_i| \leqq 1} |u_i(x)|^2 \leqq c. \tag{24}$$

We have to estimate only S_1 in (19). For this look for M in the form $M = 2^m$ and let p be a natural number such that $2^{p-1} < \mu/2 \leqq 2^p$. Then for $m+1 \leqq \leqq k \leqq p$ we have

$$\sum_{2^{k-1} < |\mu - \varrho_i| \leqq 2^k} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1 + |\mu - \varrho_i|^2} |u_i(x)|^2 \leqq$$

$$\leqq c \sum_{2^{k-1} < |\mu - \varrho_i| \leqq 2^k} 2^{-2k} e^{4R|\nu_i|} |u_i(x)|^2 \leqq$$

$$\leqq c 2^{-2k} \sum_{j=2^{k-1}+1}^{2^k} \sum_{j-1 < |\mu - \varrho_i| \leqq j} |u_i(x)|^2 e^{4R|\nu_i|} \leqq c 2^{-2k} \sum_{j=2^{k-1}+1}^{2^k} c\mu^2 = c 2^{-2k} \mu^2,$$

i.e.

$$S_1 \leqq \sum_{k=m+1}^{p} \sum_{2^{k-1} < |\mu - \varrho_i| \leqq 2^k} \frac{\mu^2}{|\mu_i|^2} \frac{e^{4R|\nu_i|}}{1 + |\mu - \varrho_i|^2} |u_i(x)|^2. \tag{25}$$

It is clear from (21), (22), (24), (25) and (18′) (because the constants in these estimates do not depend on M), that (15) is fulfilled for sufficiently large M if $\mu \geqq 2M$. Choose $2M$ instead of M then we get (15) for every $\mu \geqq 1$, but the constant c depends on μ.

§2. Estimates for the spectral function

Let $\Omega \subset \mathbf{R}^3$ be any not necessarily bounded domain and $(u_i)_1^\infty \subset L^2(\Omega)$ any complete orthonormal system of eigenfunctions of the Laplace operator with arbitrary complex eigenvalues $(\lambda_i) \subset \mathbf{C}$. The spectral function of the system (u_i, λ_i) is defined as

$$\theta(x, y, \mu) := \sum_{\varrho_i < \mu} u_i(x) u_i(y).$$

This may be a sum with infinitely many terms, but according to Lemma 1 this sum converges absolutely for every $x, y \in \Omega$, $\mu > 0$.

Define the function $V_R(r, \mu)$ as follows:

$$V_R(r, \mu) = \begin{cases} \dfrac{1}{2\pi^2} \dfrac{\mu}{r^2} \Big(-\cos r\mu + \dfrac{\sin r\mu}{r\mu} \Big) & \text{if } 0 < r \leqq R, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3. *For any complete orthonormal system of eigenfunctions of the Laplace operator with any complex eigenvalues we have*

$$\theta(x,y,\mu) = V_R(|x-y|,\mu) + \hat{\theta}(x,y,\mu) \tag{26}$$
$$\left(x \in K, \quad y \in \Omega, \quad \mu \geqq 1, \quad 0 < R < \min(\operatorname{dist}(K,\partial\Omega), \frac{\pi}{4})\right)$$

where $x \in K$, K is any fixed compact subset of Ω, $\mu \geqq 1$ and for the function $\hat{\theta}$ we have the estimate

$$\|\hat{\theta}(x,\cdot,\mu)\|_{L^2(\Omega)} \leqq C(R,K)\mu \qquad (x \in K, \ \mu \geqq 1). \tag{27}$$

PROOF. Calculate the Fourier coefficients of the function V_R at arbitrarily fixed $x \in K$ with respect to the system (u_i). Taking into account (8) we obtain

$$V_i = \int_\Omega V_R(|x-y|,\mu)u_i(y)dy = \int_0^R r^2 V_R(r,\mu) \int_\theta u_i(x+r\theta)d\theta dr =$$
$$= \int_0^R r^2 \frac{1}{2\pi^2}\frac{\mu}{r^2}\left(-\cos r\mu + \frac{\sin r\mu}{r\mu}\right)\cdot 4\pi \frac{\sin r\mu_i}{r\mu_i} u_i(x)dr =$$
$$= \frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^R \left(-\cos r\mu + \frac{\sin r\mu}{r\mu}\right)\frac{\sin r\mu_i}{r}dr \cdot u_i(x);$$

further, according to orthogonality and completeness of (u_i) we get

$$\sum_{i=1}^\infty V_i\overline{u_i(y)} = \sum_{i=1}^\infty \langle u_i, V_R\rangle\, \overline{u_i(y)} = \overline{V_R(x-y,\mu)} = V_R. \tag{29}$$

Define the function $\delta(\mu,\varrho_i)$ as follows:

$$\delta(\mu,\delta_i) := \begin{cases} 1, & \varrho_i < \mu \\ 1/2, & \varrho_i = \mu \\ 0, & \varrho_i > \mu. \end{cases}$$

It is easy to check (see [7]) that

$$\frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^R \left(-\cos r\mu + \frac{\sin r\mu}{r\mu}\right)\frac{\sin r\mu_i}{r}dr = \frac{\mu^{\frac{3}{2}}}{\mu_i^{1/2}}\int_0^R J_{\frac{3}{2}}(r\mu)J_{\frac{1}{2}}(r\mu_i)dr.$$

First of all we shall prove the following estimate:

$$\left|\frac{\mu^{\frac{3}{2}}}{\mu_i^{1/2}}\int_0^R J_{\frac{3}{2}}(r\mu)J_{\frac{1}{2}}(r\mu_i)dr-\delta(\mu,\varrho_i)\cdot\frac{\varrho_i}{\mu_i}\right|\leqq \tag{30}$$

$$\leqq c\frac{e^{R|\nu_i|}}{1+|\mu-\varrho_i|}\cdot\frac{\mu}{1+|\mu_i|}.$$

According to the identity

$$\sin r\mu_i=\sin r\varrho_i+\sin r\varrho_i(\operatorname{ch} r\nu_i-1)+i\cos r\varrho_i\operatorname{sh} r\nu_i$$

we have

(31)

$$\frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^R\left(-\cos r\mu+\frac{\sin r\mu}{r\mu}\right)\frac{\sin r\mu_i}{r}dr=\frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^\infty\left(-\cos r\mu+\frac{\sin r\mu}{r\mu}\right)\frac{\sin r\mu_i}{r}dr-$$

$$-\frac{2}{\pi}\frac{\mu}{\mu_i}\int_R^\infty\left(-\cos r\mu+\frac{\sin r\mu}{r\mu}\right)\cdot\frac{\sin r\varrho_i}{r}dr+$$

$$+\frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^R\left(-\cos r\mu+\frac{\sin r\mu}{r\mu}\right)\cdot\sin r\varrho_i\frac{\operatorname{ch} r\nu_i-1}{r}dr+$$

$$+i\frac{2}{\pi}\frac{\mu}{\mu_i}\int_0^R\left(-\cos r\mu+\frac{\sin r\mu}{r\mu}\right)\cos r\varrho_i\frac{\operatorname{sh} r\nu_i}{r}dr=$$

$$=I_1+I_2+I_3+I_4.$$

Integrating by parts we get

$$I_1=\frac{2}{\pi}\frac{\mu}{\mu_i}\cdot\frac{\varrho_i}{\mu}\int_0^\infty\sin r\mu\frac{\cos r\varrho_i}{r}dr;$$

further, taking into account the well-known identities

$$\int_0^\infty\frac{\sin t}{t}dt=\frac{\pi}{2}=\int_0^\infty\frac{\sin\alpha t}{t}dt\qquad(\alpha>0),$$

we have

$$\delta(\mu,\varrho_i)=\frac{2}{\pi}\int_0^\infty\sin r\mu\frac{\cos r\varrho_i}{r}dr. \tag{32}$$

Consequently

$$I_1 = \frac{\mu^{\frac{3}{2}}\varrho_i^{\frac{1}{2}}}{\mu_i}\int\limits_0^\infty J_{\frac{3}{2}}(r\mu)J_{\frac{1}{2}}(r\varrho_i)dr = \delta(\mu\varrho_i)\frac{\varrho_i}{\mu_i}. \tag{33}$$

Estimate the two parts of I_2 separately. One of them is

$$\int\limits_R^\infty \cos r\mu\frac{\sin r\varrho_i}{r}dr = \int\limits_R^\infty \frac{\sin r(\varrho_i-\mu)+\sin r(\varrho_i+\mu)}{2r}dr.$$

On the other hand we have, for any $a>0$,

$$\int\limits_R^\infty \frac{\sin ra}{r}dr = \int\limits_{aR}^\infty \frac{\sin t}{t}dt = \left[-\frac{\cos t}{t}\right]_{aR}^\infty - \int\limits_{aR}^\infty \frac{\cos t}{t^2}dt$$

i.e.

$$\left|\int\limits_R^\infty \frac{\sin ra}{r}dr\right| \leqq \frac{c}{1+aR}.$$

Hence in our case we have

$$\left|\int\limits_R^\infty \cos r\mu\frac{\sin r\varrho_i}{r}dr\right| \leqq \frac{c}{1+|\mu-\varrho_i|}.$$

For the other term of I_2 we obtain obviously

$$\int\limits_R^\infty \frac{\sin r\mu}{r\mu}\cdot\frac{\sin r\varrho_i}{r}dr = \int\limits_R^\infty \frac{\cos r(\mu-\varrho_i)-\cos r(\mu+\varrho_i)}{2r^2\mu}dr =$$

$$= \left[\left(\frac{\sin r(\mu-\varrho_i)}{\mu-\varrho_i}-\frac{\sin r(\mu+\varrho_i)}{\mu+\varrho_i}\right)\frac{1}{2r^2\mu}\right]_R^\infty +$$

$$+\int\limits_R^\infty \frac{1}{r^3\mu}\left(\frac{\sin r(\mu-\varrho_i)}{\mu-\varrho_i}-\frac{\sin r(\mu+\varrho_i)}{\mu+\varrho_i}\right)dr$$

hence

$$\left|\int\limits_R^\infty \frac{\sin r\mu}{r\mu}\cdot\frac{\sin r\varrho_i}{r}dr\right| \leqq \frac{c}{\mu}\cdot\frac{1}{1+|\mu-\varrho_i|},$$

and the desired estimate follows for the case $|\mu_i| \geqq 1$. But in the case of $|\mu_i| \leqq 1$ the estimate (30) is obvious, namely in this case, taking into account $\cos r\mu_i = 1 + O(|r\mu_i|^2)$, $\sin r\mu_i = r\mu_i + O(|r\mu_i|^3)$ and $\int\limits_0^R \frac{\sin r\mu}{r} dr = O(1)$ we obtain from the identities

$$\frac{\mu}{\mu_i}\int_0^R \cos r\mu \frac{\sin r\mu_i}{r} dr = \left[\sin r\mu \frac{\sin r\mu_i}{r\mu_i}\right]_0^R -$$

$$-\int_0^R \sin r\mu \frac{\cos r\mu_i}{r} dr + \int_0^R \sin r\mu \frac{\sin r\mu_i}{r} dr,$$

$$\frac{\mu}{\mu_i}\int_0^R \frac{\sin r\mu}{r\mu}\cdot\frac{\sin r\mu_i}{r} dr = \int_0^R \frac{\sin r\mu}{r} dr + \int_0^R \sin r\mu \frac{\sin(r\mu_i) - r\mu_i}{r^2\mu_i}$$

the estimate

$$\left|\frac{\mu}{\mu_i}\int_0^R \left(-\cos r\mu + \frac{\sin r\mu}{r\mu}\right)\frac{\sin r\mu_i}{r} dr\right| \leqq c$$

and (30) follows in the case $|\mu_i| \leqq 1$. Hence from now on we may suppose that $|\mu_i| \geqq 1$.

For the main term of I_3 we have obviously

$$\int_0^R \cos r\mu \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r} dr = \left[\left(-\frac{\cos r(\varrho_i - \mu)}{2(\varrho_i - \mu)} - \frac{\cos r(\varrho_i + \mu)}{2(\varrho_i + \mu)}\right)\frac{\operatorname{ch} r\nu_i - 1}{r}\right]_0^R +$$

$$+\int_0^R \left(\frac{\cos r(\varrho_i - \mu)}{2(\varrho_i - \mu)} + \frac{\cos r(\varrho_i + \mu)}{2(\varrho_i + \mu)}\right)\cdot\left(\frac{\operatorname{ch} r\nu_i - 1}{r}\right)' dr,$$

hence

$$\left|\int_0^R \cos r\mu \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r} dr\right| \leqq \frac{c}{|\mu - \varrho_i|} e^{R|\nu_i|}.$$

Further, as easy to see, we can estimate this integral by $ce^{R|\nu_i|}$, i.e.

$$\left|\int_0^R \cos r\mu \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r} dr\right| \leqq \frac{c}{1 + |\mu - \varrho_i|} e^{R|\nu_i|}.$$

For the smaller term in I_3 we can write

$$\int_0^R \frac{\sin r\mu}{r\mu} \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r} dr =$$

$$= \int_0^R \frac{1}{2}(\cos r(\mu - \varrho_i) - \cos r(\mu + \varrho_i)) \frac{\operatorname{ch} r\nu_i - 1}{r^2\mu} dr =$$

$$= \left[\left(\frac{\sin r(\mu - \varrho_i)}{2(\mu - \varrho_i)} - \frac{\sin r(\mu + \varrho_i)}{2(\mu + \varrho_i)} \right) \frac{\operatorname{ch} r\nu_i - 1}{r} \right]_0^R -$$

$$- \int_0^R \left(\frac{\sin r(\mu - \varrho_i)}{2(\mu - \varrho_i)} - \frac{\sin r(\mu + \varrho_i)}{2(\mu + \varrho_i)} \right) \left(\frac{\operatorname{ch} r\nu_i - 1}{2r^2\mu} \right)' dr,$$

hence

$$\left| \int_0^R \frac{\sin r\mu}{r\mu} \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r} dr \right| \leqq \frac{c}{\mu} \frac{e^{R|\nu_i|}}{1 + |\mu - \varrho_i|}$$

i.e. we have proved the estimate

$$|I_3| \leqq c \frac{e^{R|\nu_i|}}{1 + |\mu - \varrho_i|} \cdot \frac{\mu}{1 + |\mu_i|}.$$

(We used the interesting fact that $\left(\frac{\operatorname{ch} r\nu_i - 1}{r^2} \right)' > 0$, $r > 0$.)

For the main term of I_4 we have obviously

$$\int_0^R \cos r\mu \cos r\varrho_i \frac{\operatorname{sh} r\nu_i}{r} dr = \left[\left(\frac{\sin r(\mu - \varrho_i)}{\mu - \varrho_i} - \frac{\sin r(\mu + \varrho_i)}{\mu + \varrho_i} \right) \frac{\operatorname{sh} r\nu_i}{2r} \right]_0^R -$$

$$- \int_0^R \left(\frac{\sin r(\mu - \varrho_i)}{\mu - \varrho_i} - \frac{\sin r(\mu + \varrho_i)}{\mu + \varrho_i} \right) \left(\frac{\operatorname{sh} r\nu_i}{2r} \right)' dr,$$

hence

$$\left| \int_0^R \cos r\mu \cos r\varrho_i \frac{\operatorname{sh} r\nu_i}{r} dr \right| \leqq \frac{c}{1 + |\mu - \varrho_i|} e^{R|\nu_i|}.$$

Write the remainder part of I_4 in the form

$$\int_0^R \frac{\sin r\mu}{r\mu}\cos r\varrho_i \frac{\operatorname{sh} r\nu_i - r\nu_i}{r}dr + \int_0^R \frac{\sin r\mu}{r\mu}\cos r\varrho_i \frac{r\nu_i}{r}dr = I_4' + I_4'',$$

and estimate these terms separately. Obviously, we can write

$$I_4' = \left[-\left(\frac{\cos r(\mu-\varrho_i)}{\mu-\varrho_i} + \frac{\cos r(\mu+\varrho_i)}{\mu+\varrho_i}\right)\left(\frac{\operatorname{sh} r\nu_i - r\nu_i}{2r}\right)' + \right.$$
$$+ \int_0^R \left(\frac{\cos r(\mu-\varrho_i)}{\mu-\varrho_i} + \frac{\cos r(\mu+\varrho_i)}{\mu+\varrho_i}\right)\left(\frac{\operatorname{sh} r\nu_i - r\nu_i}{2r^2\mu}\right)' dr,$$

hence

$$\left|\int_0^R \frac{\sin r\mu}{r\mu}\cos r\varrho_i \frac{\operatorname{sh} r\nu_i - r\nu_i}{r}dr\right| \leqq \frac{c}{1+|\mu-\varrho_i|}e^{R|\nu_i|}\cdot\frac{1}{\mu}$$

(because we can estimate by $\leqq ce^{R|\nu_i|}\mu^{-1}$ according to the obvious estimate

$$\left|\int_0^R \frac{\operatorname{sh} r\nu_i - r\nu_i}{r^2}dr\right| = \int_0^R \sum_{k=1}^{\infty} \frac{(r|\nu_i|)^{1+2k}}{(1+2k)!r^2}dr =$$
$$= \sum_{k=1}^{\infty} \frac{|\nu_i|^{1+2k}}{(1+2r)!}\cdot\frac{R^{2k}}{2k} \leqq c\sum_{k=0}^{\infty}\frac{(R|\nu_i|)^{2k+1}}{(2k+1)!} = c \operatorname{sh} R|\nu_i| \leqq ce^{R|\nu_i|}.\Bigg)$$

On the other hand,

$$I_4'' = \frac{\nu_i}{\mu}\int_0^R \frac{\sin r\mu \cos r\varrho_i}{r}dr = \frac{\nu_i}{2\mu}\int_0^R \frac{\sin r(\mu-\varrho_i) + \sin r(\mu+\varrho_i)}{r}dr.$$

If $\mu-\varrho_i$ and $\mu+\varrho_i$ are positive, say $\varrho_i \leqq \mu - 1$, then the trivial estimate

$$\left|\frac{\nu_i}{2\mu}\int_0^R \frac{\sin r(\mu-\varrho_i)+\sin r(\mu+\varrho_i)}{r}dr\right| \leqq c\frac{\nu_i}{\mu} \leqq c\frac{e^{R|\nu_i|}}{1+|\mu-\varrho_i|}$$

is satisfactory. If $\mu - 1 \leqq \varrho_i \leqq \mu + 1$, then the same estimate works. If $\varrho_i \geqq \mu + 1$, i.e. $\mu - \varrho_i$ is negative and $\mu + \varrho_i$ is positive, then using

$$\int_0^R \frac{\sin r(\mu - \varrho_i)}{r} dr = \frac{\pi}{2} - \int_R^\infty \frac{\sin r(\mu - \varrho_i)}{r} dr,$$
$$\int_0^R \frac{\sin r(\mu + \varrho_i)}{r} dr = \frac{\pi}{2} - \int_R^\infty \frac{\sin r(\mu + \varrho_i)}{r} dr,$$

we obtain

$$\left| \frac{\nu_i}{2\mu} \int_0^R \frac{\sin r(\mu - \varrho_i) + \sin r(\mu + \varrho_i)}{r} dr \right| \leqq c \frac{|\nu_i|}{\mu} \frac{1}{1 + |\mu - \varrho_i|}$$

i.e. we have proved

$$|I_4| \leqq c \frac{e^{R|\nu_i|}}{1 + |\mu - \varrho_i|} \cdot \frac{\mu}{1 + |\mu_i|}.$$

The case $\mu_i = 0$ is obvious, and summarizing our estimates (30) follows. Now we return to the proof of Lemma 3.

According to (28) and (30) we have for $R < \min(\pi/4, \operatorname{dist}(K, \partial\Omega))$

$$(34) \qquad \left\| V_R(|x-y|, \mu) - \theta(x, y, \mu) - 0.5 \sum_{\varrho_i = \mu} u_i(x) u_i(y) \right\|^2_{L^2_y(\Omega)} \leqq$$
$$= \sum_{i=1}^\infty |V_i - \delta(\mu, \varrho_i) u_i(x)|^2 \leqq c \sum_{i=1}^\infty \frac{\mu^2}{1 + |\mu_i|^2} \frac{e^{2R|\nu_i|}}{(1 + |\mu - \varrho_i|)^2} |u_i(x)|^2 \leqq$$
$$\leqq c \sum_{k=0}^\infty \frac{\mu^2}{(1+k)^2} \cdot \frac{1}{(1 + |\mu - k|)^2} \sum_{k \leqq \varrho_i \leqq k+1} e^{2R|\nu_i|} |u_i(x)|^2 \leqq$$
$$\leqq c \sum_{k=0}^\infty \frac{\mu^2}{(1 + |\mu - k|)^2} \leqq c\mu^2,$$

further

$$\left\| \sum_{\varrho_i = \mu} u_i(x) \cdot u_i(y) \right\|^2_{L^2_y(\Omega)} = \sum_{\varrho_i = \mu} |u_i(x)|^2 \leqq c\mu^2.$$

Hence (26) and (27) follow. □

In the proof we have used the fact that (30) is equivalent to

$$(30')\qquad \left|\frac{\mu^{3/2}}{\mu_i^{1/2}}\int_0^R J_{\frac{3}{2}}(r\mu)\cdot J_{\frac{1}{2}}(r\mu_i)dr-\delta(\mu,\varrho_i)\right|\leqq c\frac{e^{R|\nu_i|}}{1+|\mu-\varrho_i|}\cdot\frac{\mu}{1+|\mu_i|}$$

because it is easy to see that

$$\left|\delta(\mu,\varrho_i)\Big(1-\frac{\varrho_i}{\mu_i}\Big)\right|\leqq\left|\delta(\mu,\varrho_i)\frac{\nu_i}{\mu_i}\right|\leqq c\frac{e^{R|\nu_i|}}{1+|\mu-\varrho_i|}\cdot\frac{\mu}{1+|\mu_i|}.$$

§3. Estimates of the kernel of fractional order

Let $\Omega\subset\mathbf{R}^3$ be any domain, (u_i) any complete in $L^2(\Omega)$ orthonormal system of eigenfunctions of the Laplace operator with arbitrary (complex) eigenvalues $(\lambda_i)\subset\mathbf{C}$, i.e. $-\Delta u_i=\lambda_i u_i$ in Ω, further $\sqrt{\lambda_i}=:\mu_i=\varrho_i+i\nu_i$, $\varrho_i\geqq 0$.

For any $0<\alpha<6$ introduce the kernel $T_\alpha(x,y)$ of order α as follows: this is a function defined for $x,y\in\Omega$ whose coefficients for all fixed $x\in\Omega$ are

$$(35)\qquad \int_\Omega T_\alpha(x,y)\overline{u_i(y)}dy=\frac{\overline{u_i(x)}}{(1+\varrho_i^2)^{\alpha/2}}\qquad (i=1,2,\dots).$$

It is convenient to use the form

$$T_\alpha(x,y)\sim\sum_{i=1}^\infty\frac{\overline{u_i(x)}u_i(y)}{(1+\varrho_i^2)^{\alpha/2}},$$

if it exists. For the investigation of this question introduce the polynomial

$$w_\alpha(r)=\sum_{k=0}^n a_k r^{2k},$$

by

$$(36)\quad v_\alpha(R)=w_\alpha(R),\ \ v_\alpha'(R)=w_\alpha'(R),\dots,v_\alpha^{(n)}(R)=w_\alpha^{(n)}(R)\qquad(n\geqq 0).$$

Obviously, (36) determine uniquely the polynomial w_α.

$$v_\alpha^R(r):=\begin{cases}v_\alpha(r), & r\leqq R\\ 0, & r>R\end{cases},\qquad w_\alpha^R(r):=\begin{cases}w_\alpha(r), & r\leqq R\\ 0, & r>R.\end{cases}$$

Calculate the i-th Fourier coefficient of the function $v_\alpha^R(|x-y|)-w_\alpha^R(|x-y|)$ with respect to the system (u_i). Taking into account (8) we get for any $0<R<\operatorname{dist}(x,\partial,\Omega)$:

$$\overline{\varphi_i(x)} := \langle u_i, v_\alpha^R - w_\alpha^R\rangle = \int\limits_\Omega u_i(y)[v_\alpha^R(|x-y|)-w_\alpha^R(|x-y|)]dy =$$

$$=\int\limits_0^R r^2[v_\alpha(r)-w_\alpha(r)]\int\limits_\omega u_i(x+r\theta)d\theta dr = \int\limits_0^R r^2[v_\alpha(r)-w_\alpha(r)]\cdot 4\pi\frac{\sin r\mu_i}{r\mu_i}dr\cdot u_i(x) =$$

$$= \frac{4\pi}{\varrho_i}u_i(x)\int\limits_0^\infty rv_\alpha(r)\sin r\varrho_i dr - \frac{4\pi}{\varrho_i}u_i(x)\int\limits_R^\infty rv_\alpha(r)\sin r\varrho_i dr +$$

$$+4\pi u_i(x)\int\limits_0^R rv_\alpha(r)\left[\frac{\sin r\mu_i}{\mu_i}-\frac{\sin r\varrho_i}{\varrho_i}\right]dr -$$

$$-\frac{4\pi}{\mu_i}u_i(x)\int\limits_0^R rw_\alpha(r)\sin r\mu_i dr =: I_1+I_2+I_3+I_4.$$

We know that

$$\frac{1}{u_i(x)}I_1 = \frac{2^{(3-\alpha)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{\varrho_i}\int\limits_0^\infty r^{(\alpha-1)/2}K_{(3-\alpha)/2}(r)\sin r\varrho_i dr =$$

$$= \frac{2^{(2-\alpha)/2}}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{\sqrt{\varrho_i}}{\varrho_i}\int\limits_0^\infty r^{\alpha/2}K_{(3-\alpha)/2}(r)J_{1/2}(r)dr = \frac{1}{(1+\varrho_i^2)^{\alpha/2}};$$

on the other hand,

$$I_2 = -\frac{(2\pi)^{3/2}}{\sqrt{\varrho_i}}u_i(x)\int\limits_R^\infty r^{3/2}v_\alpha(r)J_{1/2}(r)dr,$$

$$I_3 = (2\pi)^{3/2}\cdot u_i(x)\int\limits_0^R r^{3/2}v_\alpha(r)\left[\frac{J_{1/2}(r\mu_i)}{\sqrt{\mu_i}}-\frac{J_{1/2}(r\varrho_i)}{\sqrt{\varrho_i}}\right]dr,$$

$$I_4 = -\frac{(2\pi)^{3/2}}{\sqrt{\mu_i}}u_i(x)\int\limits_0^R r^{3/2}w_\alpha(r)J_{1/2}(r)dr.$$

We know that

$$\int r^{\nu+1} J_\nu(r\mu)dr = \frac{1}{\mu} r^{\nu+1} J_{\nu+1}(r).$$

Denote $Df(r) := r^{-1} f'(r)$. Integrating by parts we obtain

$$I_2 = -(2\pi)^{3/2} u_i(x) \left[r^{3/2} v_\alpha(r) \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} \right]_{r=R}^{\infty} +$$

$$+(2\pi)^{3/2} u_i(x) \int_R^\infty Dv_\alpha(r) r^{5/2} \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} dr,$$

$$I_3 = (2\pi)^{3/2} u_i(x) \left[r^{3/2} v_\alpha(r) \left(\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} \right) \right]_{r=0}^{R} -$$

$$-(2\pi)^{3/2} u_i(x) \int_0^R Dv_\alpha(r) r^{5/2} \left(\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} \right) dr.$$

Because $|K_\nu(x)| \leqq cx^{-|\nu|}$ $(|x| < 1)$, hence $v_\alpha(r) \leqq cr^{-|3-\alpha|} \leqq cr^{-3}$, consequently the integrated part of I_3 vanishes at $r = 0$. Because $v_\alpha(r)$ decreases exponentially, hence the integrated part of I_2 vanishes at $r = \infty$. Finally,

$$I_4 = -(2\pi)^{3/2} u_i(x) \left[r^{3/2} w_\alpha(r) \frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} \right]_{r=0}^{R} +$$

$$+(2\pi)^{3/2} u_i(x) \int_0^R Dw_\alpha(r) r^{5/2} \frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} dr,$$

where the integrated part vanishes at $r = 0$. I.e. the sum of integrated parts of I_2, I_3 and I_4 is:

$$(2\pi)^{3/2} u_i(x) R^{3/2} \Bigg\{ v_\alpha(R) \left[\frac{J_{3/2}(R\varrho_i)}{\varrho_i^{3/2}} + \right.$$

$$\left. + \left(\frac{J_{3/2}(R\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(R\varrho_i)}{\varrho_i^{3/2}} \right) \right] - w_\alpha(R) \frac{J_{3/2} R\mu_i}{\mu_i^{3/2}} \Bigg\} = 0$$

because $v_\alpha(R) = w_\alpha(R)$, consequently

$$I_2 + I_3 + I_4 = -(2\pi)^{3/2} u_i(x) \left\{ - \int_R^\infty D v_\alpha(r) r^{5/2} \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} dr + \right.$$

$$\left. + \int_0^R D v_\alpha(r) r^{5/2} \left[\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} \right] dr \right\} =$$

$$= c_\alpha u_i(x) \left\{ - \int_R^\infty K_{1+\frac{3-\alpha}{2}}(r) \cdot r^{\alpha/2} \cdot \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} dr + \right.$$

$$\left. + \int_0^R K_{1+\frac{3-\alpha}{2}}(r) r^{\alpha/2} \left[\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} \right] dr \right\}.$$

According to $|K_\nu(r)| \leqq c(\nu) e^{-r}/\sqrt{r}$ and $|J_\nu(r)| \leqq c(\nu)/\sqrt{r}$, we have the estimate

$$\left| \int_R^\infty K_{1+\frac{3-\alpha}{2}}(r) r^{\alpha/2} \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} dr \right| \leqq c/\varrho_i^2. \tag{37}$$

We shall prove

LEMMA 4. *We have the following estimate for the Fourier coefficients of $v_\alpha(|x-y|) - w_\alpha(|x-y|)$ with respect to the system (u_i):*

$$\left| \varphi_i(x) - \overline{u_i(x)} \frac{1}{(1+\varrho_i^2)^{\alpha/2}} \right| \leqq \begin{cases} c_R \dfrac{|u_i(x)| e^{R|\nu_i|}}{(1+\varrho_i^2)^{\alpha/2}}, & 0 < \alpha \leqq 1, \\ c_R \dfrac{|u_i(x)| e^{R|\nu_i|}}{1+\varrho_i^2}, & 1 \leqq \alpha. \end{cases} \tag{38}$$

PROOF. If $\varrho_i \leqq 1$, then $\left| \dfrac{\sin r\mu_i}{r\mu_i} \right| \leqq c e^{R|\nu_i|}$, hence

$$\left| \frac{\varphi_i(x)}{u_i(x)} \right| = \left| \int_0^R r^2 [v_\alpha(r) - w_\alpha(r)] \cdot 4\pi \frac{\sin r\mu_i}{r\mu_i} dr \right| \leqq$$

$$\leqq c e^{R|\nu_i|} \int_0^R r^2 \cdot r^{\alpha-3} dr \leqq c e^{R|\nu_i|} \quad \text{if} \quad \alpha < 3,$$

$$\leqq c e^{R|\nu_i|} \int_0^R r^2 \ln r \, dr \leqq c e^{R|\nu_i|} \quad \text{if} \quad \alpha \geqq 3,$$

i.e. from now on we may suppose that $\varrho_i \geqq 1$.

Taking into account (37) and

$$\left|\int_0^R K_{\frac{5-\alpha}{2}}(r)r^{\alpha/2}\left[\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}}\right] dr\right| \leqq$$

$$\leqq c\int_0^R r^{\frac{5-\alpha}{2}} \ln r r^{\alpha/2}\frac{1}{r^{1/2}\varrho_i^2} dr \leqq c/\varrho_i^2 \qquad (\alpha \geqq 5),$$

we may suppose $\alpha < 5$. Consider the following partition:

$$\text{(39)} \qquad \sqrt{\frac{\pi'}{2}}r^{1/2}\left[\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}}\right] = \frac{-\cos r\mu_i + \dfrac{\sin r\mu_i}{r\mu_i}}{\mu_i^2} -$$

$$-\frac{-\cos r\varrho_i + \dfrac{\sin r\varrho_i}{r\varrho_i}}{\varrho_i^2} = \frac{-\cos r\varrho_i(\text{ch } r\nu_i - 1)}{\mu_i^2} + \frac{i\ \sin r\varrho_i \text{sh } r\nu_i}{\mu_i^2} +$$

$$+\frac{\sin r\varrho_i(\text{ch } r\nu_i - 1)}{r\mu_i^3} + \frac{i\ \cos r\varrho_i(\text{sh } r\nu_i - r\nu_i)}{r\mu_i^3} +$$

$$+\left[\frac{-\cos r\varrho_i}{\mu_i^2} + \frac{\sin r\varrho_i}{r\mu_i^3} + \frac{i\nu_i \cos r\varrho_i}{\mu_i^3} - \frac{-\cos r\varrho_i + \dfrac{\sin r\varrho_i}{r\varrho_i}}{\varrho_i^2}\right] =$$

$$= \frac{-\cos r\varrho_i(\text{ch } r\nu_i - 1)}{\mu_i^2} + i\frac{\sin r\varrho_i \text{sh } r\nu_i}{\mu_i^2} + \frac{\sin r\varrho_i(\text{ch } r\nu_i - 1)}{r\mu_i^3} +$$

$$+i\frac{\cos r\varrho_i(\text{sh } r\nu_i - r\nu_i)}{r\mu_i^3} + \frac{\varrho_i(\varrho_i^3 - \mu_i^3)}{\varrho_i^3\mu_i^3}\left(-\cos r\varrho_i + \frac{\sin r\varrho_i}{r\varrho_i}\right),$$

and estimate term by term. We saw before Lemma 4 that

$$\text{(40)} \qquad \varphi_i(x) - u_i(x)\frac{1}{(1+\varrho_i^2)^{\alpha/2}} = cu_i(x)\left\{-\int_R^\infty K_{(5-\alpha)/2}(r)r^{\alpha/2}\cdot\right.$$

$$\left.\cdot\frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}} dr + \int_0^R K_{(5-\alpha)/2}(r)r^{\alpha/2}\left[\frac{J_{3/2}(r\mu_i)}{\mu_i^{3/2}} - \frac{J_{3/2}(r\varrho_i)}{\varrho_i^{3/2}}\right] dr\right\}.$$

According to (39) and (40) it is enough to apply the estimate

$$\left|\int_0^R K_{(5-\alpha)/2}(r)r^{\frac{\alpha-1}{2}}\frac{\cos r\varrho_i(\operatorname{ch} r\nu_i-1)}{\mu_i^2}dr\right| =$$

$$= \frac{1}{|\mu_i|^2}\left|\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i\frac{\operatorname{ch} r\nu_i-1}{r^2}dr\right| \leqq$$

$$\leqq \frac{1}{|\mu_i|^2}\left|\frac{\operatorname{ch} R\nu_i-1}{R^2}\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i dr\right| +$$

$$+\frac{1}{|\mu_i|^2}\int_0^R\left|\int_0^t K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i dr\right|\left(\frac{\operatorname{ch} t\nu_i-1}{t}\right)'dt \leqq c\frac{e^{R|\nu_i|}}{|\mu_i|^2},$$

because

$$\left|\int_0^t K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i dr\right| \leqq c\int_0^t r^{\frac{\alpha-5}{2}}r^{\frac{\alpha+3}{2}}dr = c\int_0^t r^{\alpha-1}dr \leqq c$$

if $t \leqq R$.

$$\left|\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha-1)/2}\sin r\varrho_i\frac{\operatorname{ch} r\nu_i}{\mu_i^2}dr\right| \leqq$$

$$\leqq \frac{1}{|\mu_i|^2}\left|\frac{\operatorname{sh} R\nu_i}{R}\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha+1)/2}\sin r\varrho_i dr\right| +$$

$$+\frac{1}{|\mu_i|^2}\int_0^R\left|\int_0^t K_{(5-\alpha)/2}(r)r^{(\alpha+1)/2}\sin r\varrho_i dr\right|\left(\frac{\operatorname{sh} t\nu_i}{t}\right)'dt \leqq$$

$$\leqq \begin{cases} \dfrac{c}{|\mu_i|^2}e^{R|\nu_i|} & \text{if} \quad \alpha \geqq 1, \\[2mm] \dfrac{c}{\varrho_i^{\alpha+1}}e^{R|\nu_i|} & \text{if} \quad \alpha < 1, \end{cases}$$

because

$$\int_0^{1/\varrho_i} \left| K_{(5-\alpha)/2}(r) r^{(\alpha+1)/2} \sin r\varrho_i \right| dr \leqq c \int_0^{1/\varrho_i} r^{(\alpha-5)/2} r^{(\alpha+1)/2} r\varrho_i dr =$$

$$= c\varrho_i \int_0^{1/\varrho_i} r^{\alpha-1} dr \leqq c/\varrho_i^{\alpha-1}$$

and for $R > t > 1/\varrho_i$

$$\left| \int_{\frac{1}{\varrho_i}}^{t} K_{(5-\alpha)/2}(r) r^{(\alpha+1)/2} \sin r\varrho_i dr \right| \leqq c \int_{\frac{1}{\varrho_i}}^{t} r^{\alpha-2} dr \leqq \begin{cases} \frac{c}{\varrho_i^{\alpha-1}} & \text{if } \alpha < 1, \\ c & \text{if } \alpha > 1. \end{cases}$$

Further in the case $\alpha = 1$ we have

$$\left| \int_{1/\varrho_i}^{t} K_2(r) r \sin r\varrho_i dr \right| = \left| \int_{1/\varrho_i}^{t} r^3 \frac{K_2(r)}{r^2} \sin r\varrho_i dr \right| \leqq$$

$$\leqq \left| \left[r^3 \frac{K_2(r)}{r} \frac{\cos r\varrho_i}{\varrho_i} \right]_{\frac{1}{\varrho_i}}^{t} \right| + \left| \int_{1/\varrho_i}^{t} 3r^2 \frac{K_2(r)}{r} \frac{\cos r\varrho_i}{\varrho_i} dr \right| +$$

$$+ \left| \int_{1/\varrho_i}^{t} r^3 \frac{K_2(r)}{r^2} \frac{\cos r\varrho_i}{\varrho_i} dr \right| \leqq c + \int_{1/\varrho_i}^{t} \frac{c}{r^2 \varrho_i} dr \leqq c.$$

We have also

$$\left| \int_0^R K_{(5-\alpha)/2}(r) r^{(\alpha-1)/2} \sin r\varrho_i \frac{\operatorname{ch} r\nu_i - 1}{r\mu_i^3} dr \right| \leqq$$

$$\leqq \frac{1}{|\mu_i|^3} \left| \frac{\operatorname{ch} R\nu_i - 1}{R^2} \int_0^R K_{(5-\alpha)/2}(r) r^{(\alpha+1)/2} \sin r\varrho_i dr \right| +$$

$$+ \frac{1}{|\mu_i|^3} \int_0^R \left| \int_0^t K_{(5-\alpha)/2}(r) r^{(\alpha+1)/2} \sin r\varrho_i dr \right| \left(\frac{\operatorname{ch} t\nu_i - 1}{t^2} \right)' dt \leqq$$

$$\begin{cases} \dfrac{c}{|\mu_i|^3}\dfrac{e^{R|\nu_i|}}{\varrho_i^{\alpha-1}} & \text{if } \alpha \leqq 1, \\ \dfrac{c}{|\mu_i|^3}e^{R|\nu_i|} \leqq c\dfrac{e^{R|\nu_i|}}{\varrho_i^3} & \text{if } \alpha \geqq 1, \end{cases}$$

further

$$\left|\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha-1)/2}\cos r\varrho_i \frac{\operatorname{sh} r\nu_i - r\nu_i}{r\mu_i^3}dr\right| \leqq$$

$$\leqq \frac{1}{|\mu_i|^3}\left|\frac{\operatorname{sh} R\nu_i - R\nu_i}{R}\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i dr\right| +$$

$$+\frac{1}{|\mu_i|^3}\int_0^R\left|\int_0^t K_{(5-\alpha)/2}(r)r^{(\alpha+3)/2}\cos r\varrho_i dr\right|\left(\frac{\operatorname{sh} t\nu_i - t\nu_i}{t^3}\right)' dt \leqq c\frac{e^{R|\nu_i|}}{|\mu_i|^3}$$

and finally

$$\left|\frac{\varrho_i^3-\mu_i^3}{\varrho_i^3}\cdot\frac{\varrho_i}{\mu_i^3}\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha-1)/2}\left(-\cos r\varrho_i + \frac{\sin r\varrho_i}{r\varrho_i}\right)dr\right| =$$

$$= c\left|\frac{\varrho_i^3-\mu_i^3}{\varrho_i^3}\cdot\frac{\varrho_i^3}{\mu_i^3}\int_0^R K_{(5-\alpha)/2}(r)r^{\alpha/2}J_{3/2}(r\varrho_i)dr\right| \leqq$$

$$\leqq c\frac{(1+|\nu_i|)^3}{\varrho_i}\cdot\frac{\varrho_i^{3/2}}{|\mu_i|^3}\left|\int_0^R K_{(5-\alpha)/2}(r)r^{\alpha/2}J_{3/2}(r\varrho_i)dr\right|.$$

We know (see [7] p. 410) that

$$\int_0^\infty K_{(5-\alpha)/2}(r)r^{\alpha/2}J_{3/2}(r\varrho_i)dr = c\frac{\varrho_i^{3/2}}{(1+\varrho_i^2)^{\alpha/2}}$$

and taking into account also (37) we obtain

$$\left|\frac{\varrho_i^3-\mu_i^3}{\varrho_i^3}\cdot\frac{\varrho_i}{\mu_i^3}\int_0^R K_{(5-\alpha)/2}(r)r^{(\alpha-1)/2}\left(-\cos r\varrho_i + \frac{\sin r\varrho_i}{r\varrho_i}\right)dr\right| \leqq$$

$$\leqq c\frac{\varrho_i^{1/2}}{|\mu_i|^3}(1+|\nu_i|)^3\left(\varrho_i^{\frac{3}{2}-\alpha}+\varrho_i^{\frac{3}{2}-2}\right)\leqq\begin{cases} c\dfrac{(1+|\nu_i|)^3}{\varrho_i^{5/2}}\varrho_i^{\frac{3}{2}-\alpha}\leqq c\dfrac{e^{R|\nu_i|}}{\varrho_i^{\alpha+1}} & \text{if } \alpha\leqq 2,\\ c\dfrac{e^{R|\nu_i|}}{\varrho_i^3} & \text{if } \alpha\geqq 2.\end{cases}$$

Summarizing our estimates, (38) follows. □

Define the partial sums of T_α as follows:

$$E_\mu T_\alpha(x,y):=\sum_{\varrho_i<\mu}\frac{\overline{u_i(x)}u_i(y)}{(1+\varrho_i^2)^{\alpha/2}}.$$

LEMMA 5. *Let $\Omega\subset\mathbf{R}^3$ be any bounded domain and $K\subset\Omega$ any compact subset, then there exists $c=c(K,q,\Omega)>0$ such that*

$$\|E_\mu T_1(x,\cdot)\|_{L^q(\Omega)}\leqq c;\quad \mu\geqq 1,\quad 1<q<3/2,\quad x\in K \tag{41}$$

further

$$\|E_\mu T_1(x,\cdot)\|_{L^2(\Omega\setminus\Omega_1)}\leqq c;\quad \mu\geqq 1,\quad x\in K, \tag{42}$$

if $K\subset\Omega_1\subset\Omega$ is another domain.

PROOF. By integration by parts

$$E_\mu T_1(x,y)=\int_0^\mu\frac{1}{(1+t^2)^{1/2}}d_t\theta(x,y,t)=\frac{\theta(x,y,\mu)}{(1+\mu^2)^{1/2}}+$$

$$+\frac{1}{2}\int_0^\mu\theta(x,y,t)\frac{2t}{(1+t^2)^{3/2}}dt=:I_1+I_2.$$

Further, obviously,

$$\left\|u^{\frac{3}{2}}V_R(|x-y|,\mu)\right\|_{L^q(\Omega)}=\mu^{\frac{3}{2}}\left(\int_0^R r^2\left(\frac{J_{3/2}(r\mu)}{r^{3/2}}\right)^q dr\right)^{\frac{1}{q}}\leqq$$

$$\leqq c\cdot\mu\left(\int_0^R r^2\frac{1}{r^{2q}}dr\right)^{\frac{1}{q}}\leqq c\mu$$

and hence

$$\|I_1\|_{L^q_y(\Omega)}\leqq c,\quad x\in K,\quad 1<q<3/2.$$

It is enough to consider the integral $\int\limits_1^\mu$ in I_2.

According to the partition (26) first consider the main part. Using the notation $r = |x-y|$ we obtain for $r \leqq 1/\mu$

$$\left|\int\limits_1^\mu V_R(r,t)\frac{t}{(1+t^2)^{3/2}}dt\right| = c\left|\int\limits_1^\mu \left(\frac{t}{r}\right)^{\frac{3}{2}} J_{3/2}(rt)\frac{t}{(1+t^2)^{3/2}}dt\right| \leqq$$

$$\leqq c\int\limits_1^\mu \left(\frac{t}{r}\right)^{\frac{3}{2}} (rt)^{\frac{3}{2}}\frac{t}{(1+t^2)^{3/2}}dt \leqq c\int\limits_1^\mu t^2 dt \leqq c\mu^2$$

and

$$\|\mu^2\|_{L^q(|x-y|<1/\mu)} \leqq c\mu^{2q-3} \leqq c.$$

In the case $r \geqq 1/\mu$ we have obviously

$$\int\limits_1^{1/r} V_R(r,t)\cdot\frac{t}{(1+t^2)^{3/2}}dt \leqq c\int\limits_1^{1/r} tdt \leqq c/r^2,$$

further, taking into account the asymptotical expansion

$$J_{3/2}(rt) = c\frac{\cos(rt+\gamma)}{(rt)^{1/2}} + O\left((rt)^{-3/2}\right)$$

we obtain for the remainder term

$$\int\limits_{1/r}^\mu \left(\frac{t}{r}\right)^{\frac{3}{2}} \frac{1}{(rt)^{3/2}}\cdot\frac{t}{(1+t^2)^{3/2}}dt \leqq \frac{c}{r^3}\int\limits_{1/r}^\mu \frac{t}{(1+t^2)^{3/2}}dt \leqq c/r^2$$

and for the main term

$$\left|\int\limits_{1/r}^\mu \left(\frac{t}{r}\right)^{\frac{3}{2}} \frac{\cos(rt+\gamma)}{(rt)^{1/2}}\frac{t}{(1+t^2)^{3/2}}dt\right| = \frac{1}{r^2}\left|\int\limits_{1/r}^\mu \frac{\cos(rt+\gamma)}{t}\frac{t^3}{(1+t^2)^{3/2}}dt\right| \leqq$$

$$\leqq \frac{1}{r^2}\left\{\left(\frac{\mu^2}{1+\mu^2}\right)^{\frac{3}{2}}\left|\int\limits_{1/r}^\mu \frac{\cos(rt+\gamma)}{t}dt\right| + \right.$$

$$\left. + \int\limits_{1/r}^\mu \left|\int\limits_{1/r}^u \frac{\cos(rt+\gamma)}{t}dt\right| \left[\left(\frac{u^2}{1+u^2}\right)^{\frac{3}{2}}\right]' du\right\} \leqq c/r^2,$$

because $\left(\frac{u^2}{1+u^2}\right)' > 0$, $\left|\int\limits_{1/r}^{u} \frac{\cos(rt+\gamma)}{t} dt\right| \leqq c$; consequently

$$\left\| \int\limits_1^\mu V_R(r,t) \frac{t}{(1+t^2)^{3/2}} dt \right\|_{L^q(|x-y|<1/\mu)} \leqq c \int\limits_{1/\mu}^{R} r^{2-2q} dr \leqq c.$$

Now estimate the remainder term in I_2, i.e. the term containing $\hat{\theta}$ (we use the partition (26)). Let $0 < R < \min\left(\operatorname{dist}(K, \partial\Omega), \frac{\pi}{4}\right)$, apply (30′) and

$$\hat{\theta}(x,y,\mu) = \sum_{\varrho_i<\mu} (\overline{u_i(x)} - V_i) u_i(y) - \sum_{\varrho_i \geqq \mu} V_i u_i(y)$$

to get

$$\hat{\theta}(x,y,t) = -\frac{1}{2} \sum_{\varrho_i = t} \overline{u_i(x)} u_i(y) +$$

$$+ \sum_{i=1}^{\infty} \overline{u_i(x)} u_i(y) \cdot \frac{e^{R|\nu_i|}}{1+|t-\varrho_i|} \cdot \frac{t}{1+\varrho_i} \cdot R_i, \qquad |R_i| \leqq c$$

where c does not depend on x, y, t and i; hence

$$\int\limits_1^\mu \hat{\theta}(x,y,t) \frac{t}{(1+t^2)^{3/2}} dt = \overline{u_i(x)} u_i(y) \frac{e^{R|\nu_i|}}{1+\varrho_i^2} R_i I,$$

where

$$I = \int\limits_1^\mu \frac{t}{1+|t-\varrho_i|} \cdot \frac{t}{(1+t^2)^{3/2}} dt \leqq \int\limits_1^\mu \frac{1}{1+|t-\varrho_i|} \frac{t}{1+t^2} dt \leqq$$

$$\leqq c \frac{\ln(1+\varrho_i)}{1+\varrho_i} \leqq c \,/\, (1+\varrho_i)^{2/3},$$

i.e.

$$\left\| \int\limits_1^\mu \hat{\theta}(x,\cdot,t) \frac{t}{(1+t^2)^{3/2}} dt \right\|_{L^q(\Omega)} \leqq c \left\| \sum_{i=1}^{\infty} |u_i(x) u_i(y)| \frac{e^{R|\nu_i|}}{(1+\varrho_i)^{4/3}} \right\|^2_{L^2_y(\Omega)} =$$

$$= c \sum_{i=1}^{\infty} \frac{|u_i(x)|^2 e^{2R|\nu_i|}}{(1+\varrho_i)^{4/3}} \leqq c,$$

so we have proved (41). To prove (42) choose R such that $0 < R < \min(\operatorname{dist}(K,\Omega), \pi/4)$; then the main term in the partition (26) vanishes: $V_R r = 0$ since $r > R$, so $\theta(x,y,t) = \hat\theta(x,y,t)$, consequently

$$\|I_1\|^2_{L^2(\Omega\setminus\Omega_1)} \leqq c\mu^{-2}\|\hat\theta(x,\cdot,\mu)\|^2_{L^2(\Omega)} =$$

$$= c\mu^{-2}\left(\sum_{\varrho_i=\mu}|u_i(x)|^2 + \sum_{i=1}^{\infty}\frac{|u_i(x)|^2e^{2R|\nu_i|}}{1+|\mu-\varrho_i|}\left(\frac{\mu}{1+\varrho_i}\right)^2\right) \leqq c\sum_{k=1}^{\infty}\frac{1}{1+|\mu-k|^2} \leqq c,$$

$$\|I_2\|^2_{L^2(\Omega\setminus\Omega_1)} \leqq c + c\left\|\sum_{i=1}^{\infty}|u_i(x)u_i(y)|\frac{e^{R|\nu_i|}}{(1+\varrho_i)^{7/4}}\right\|^2_{L^2(\Omega)} \leqq$$

$$\leqq c + c\sum_{i=1}^{\infty}\frac{|u_i(x)|^2e^{2R|\nu_i|}}{(1+\varrho_i)^{4/3}} \leqq c,$$

and Lemma 5 is proved. □

LEMMA 6. *The expansion of any $f \in C_0^\infty(\Omega)$ with respect to the system (u_i) converges absolutely and locally uniformly in Ω (i.e. uniformly on every compact subset of Ω).*

PROOF. For the case $\varrho_i \geqq 1$ calculate the coefficients f_i of f by repeated application of Green's formula:

$$f_i = \int_\Omega f(x)\overline{u_i(x)}dx = -\frac{1}{\lambda_i}\int_\Omega f(x)\Delta\overline{u_i(x)}dx =$$

$$= -\frac{1}{\overline{\lambda}_i}\int_\Omega \Delta f(x)u_i(x)dx = \frac{1}{\overline{\lambda}_i^2}\int_\Omega \Delta f\cdot\Delta u_i = \frac{1}{\overline{\lambda}_i^2}\int_\Omega \Delta^2 f\cdot u_i = \ldots = \frac{-1}{\overline{\lambda}_i^n}\int_\Omega \Delta^n f u_i,$$

consequently

$$\sum_{i=k}^{k+m}|f_i\cdot u_i(x)| \leqq c\int_\Omega|\Delta^n f(y)|\sum_{i=k}^{k+m}\frac{|u_i(x)u_i(y)|}{(1+\varrho_i)^{2n}}dy \leqq$$

$$\leqq c\int_\Omega|\Delta^n f(y)|\left(\sum_{i=k}^{k+m}\frac{|u_i(x)|^2}{(1+\varrho_i)^{2n}}\right)^{\frac12}\left(\sum_{i=k}^{k+m}\frac{|u_i(y)|^2}{(1+\varrho_i)^{2n}}\right)^{\frac12}dy \leqq$$

$$\leqq c\|\Delta^n f\|_{L^2(\Omega)}\left(\int_{\operatorname{supp} f}\sum_{i=k}^{k+m}\frac{|u_i(y)|^2}{(1+\varrho_i)^{2n}}dy\right)^{\frac12} \quad \text{if} \quad n \gg 1. \qquad \square$$

§4. Proof of Theorem 1

In our case $\alpha \geqq 1$, $\alpha p > 3$, $p \geqq 1$. According to a known embedding theorem (see [2], p. 315) for such α and p

$$L_p^\alpha \subset L_{p'}^1 = W_{p'}^1 \quad \text{with some} \quad p' > 3. \tag{43}$$

Indeed, according to the mentioned theorem of [2], $L_p^\alpha(\mathbf{R}^N) \subset L_q^\beta(\mathbf{R}^N)$ if $\beta = \alpha - N\left(\frac{1}{p} - \frac{1}{q}\right) \geqq 0$ and $1 < p < q < \infty$. (43) is obvious if $\alpha = 1$; for $\alpha > 1$ we can determine p' from $\alpha - 1 = 3\left(\frac{1}{p} - \frac{1}{p'}\right)$: $\frac{1}{p'} = \frac{1}{p} - \frac{\alpha-1}{3} = \frac{3-\alpha p+p}{3p} < 1/3$, i.e. we may suppose that $f \in W_p^1(\mathbf{R}^3)$ for some $p > 3$. We know further ([2], p. 288–289) that

$$W_p^1 = \left\{F\colon F(x) = \int\limits_{\mathbf{R}^3} v_1(|x-y|)f(y)dy\colon f \in L^p(\mathbf{R}^3)\right\}$$

further

$$\|F\|_{W_p^1(\mathbf{R}^3)} \asymp \|f\|_{L^p(\mathbf{R}^3)}.$$

Define $\Psi_1(x,y)$ by

$$T_1(x,y) = v_1^R(|x-y|) - w_1^R(|x-y|) + \Psi_1(x,y),$$

then by Lemma 4 we have

$$\Psi_1(x,y) = \sum_{i=1}^{\infty} \gamma_i \overline{u_i(x)} u_i(y),$$

where $|\gamma_i| \leqq ce^{R|\nu_i|}/(1+\varrho_i)^2$. Let $0 < R < \min(\operatorname{dist}(K,\partial\Omega), \pi/4)$, then (7) holds, hence

$$\sup_{x\in K} \frac{|u_i(x)|^2 e^{2R|\nu_i|}}{(1+\varrho_i)^{3/2+\varepsilon}} < \infty$$

and so

$$\|\Psi_1(x,\cdot)\|_{L^2(\Omega)} \leqq c \sum_{i=1}^{\infty} \frac{|u_i(x)|^2 \cdot e^{2R|\nu_i|}}{(1+\varrho_i^2)^2} \leqq c \qquad (x \in K).$$

It is known (see [2]) that if $f \in W_p^1(\mathbf{R}^3)$, $p > 3$, and f has compact support in Ω, then there exists $h \in L^p(\mathbf{R}^3)$ with compact support in Ω such that

$$f(x) = \int\limits_{\Omega} (v_1^R - w_1^R)(|x-y|)h(y)dy,$$

further $\|f\|_{W_p^1} \asymp \|h\|_{L^p}$ (here $p > 3$). It is easy to prove that similar statements hold for the transformation $\int\limits_\Omega T_1(x,y)h(y)dy$.

Now let $g(x) := \int\limits_\Omega \Psi_1(x,y)\cdot h(y)dy$. Because $p > 3$ hence $p \geqq 2$ and h has compact support, consequently $h \in L^2(\Omega)$,

$$\|h\|_{L^2(\Omega)} = \sum_{i=1}^{\infty} |h_i|^2$$

where h_i denotes the i-th Fourier coefficient of h with respect to the system (u_i), further $g(x) = \sum_{i=1}^{\infty} u_i(x)\gamma_i h_i$ is the Fourier series of g. Indeed,

$$\int\limits_\Omega g(x)\overline{u_i(x)}dx = \int\limits_\Omega h(y)\int\limits_\Omega \Psi_1(x,y)\cdot\overline{u_i(x)}dxdy =$$
$$= \int\limits_\Omega h(y)\overline{u_i(y)}\gamma_i dy = \gamma_i h_i.$$

Obviously,

$$\sum |u_i(x)\gamma_i h_i| \leqq \left(\sum |u_i(x)\gamma_i|^2\right)^{\frac{1}{2}}\left(\sum |h_i|^2\right)^{\frac{1}{2}} \leqq$$
$$\leqq c\|h\|_{L^2(\Omega)}\left(\sum \frac{|u_i(x)|^2 e^{2R|\nu_i|}}{(1+\varrho_i^2)^2}\right) \leqq c\|h\|_{L^2(\Omega)},$$

i.e. the Fourier series of g converges absolutely and locally uniformly in Ω. It is enough to show that for any $h \in L^p(\mathbf{R}^3)$ $(p > 3)$ with compact support in Ω the Fourier series of the function $F(x) := \int\limits_\Omega T_1(x,y)h(y)dy$ converges locally uniformly in Ω. If F_i denotes the i-th Fourier coefficient of F, then according to (41) we have

$$\sum_{\varrho_i<\mu} F_i u_i(x) = \sum_{\varrho_i<\mu} u_i(x)\int\limits_\Omega \overline{u_i(x)}\int\limits_\Omega T_1(x,y)h(y)dy =$$
$$= \sum_{\varrho_i<\mu} u_i(x)\int\limits_\Omega h(y)\Big(\int\limits_\Omega T_1(x,y)\overline{u_i(x)}dx\Big)dy =$$
$$= \sum_{\varrho_i<\mu} u_i(x)\int\limits_\Omega h(y)\overline{u_i(y)}\frac{1}{(1+\varrho_i^2)^{\alpha/2}}dy =$$
$$= \int\limits_\Omega h(y)\overline{\sum_{\varrho_i<\mu}\overline{u_i(x)}u_i(y)\frac{1}{(1+\varrho_i^2)^{\frac{\alpha}{2}}}}dy = \int\limits_\Omega h(y)\overline{E_\mu T_1(x,y)}dy,$$

$$\Big|\sum_{\varrho_i<\mu} F_i u_i(x)\Big| \leqq \|E_\mu T_1(x,y)\|_{L^q(\Omega)} \cdot \|h\|_{L^p(\Omega)} \leqq c\|h\|_{L^p(\Omega)}.$$

Let $(F_n) \subset C_0^\infty(\Omega)$ such that $F_n \underset{n\to\infty}{\overset{W_p^1}{\longrightarrow}} F$. $F_n \in W_p^1$ and supp F_n is compact in Ω, hence according to the statement above, there exists $h_n \in L^p$ with compact support in Ω such that

$$F_n(x) = \int_\Omega T_1(x,y)h_n(y)dy \text{ and } \|h_n - h\|_{L^p} \to 0 \text{ as } n \to \infty.$$

We know from Lemma 6 that $\sum_{\varrho_i<\mu} (F_n)_i \cdot u_i(x)$ tends to $F_n(x)$ locally uniformly in Ω as $\mu \to \infty$, further

$$\Big|F(x) - \sum_{\varrho_i<\mu} F_i u_i(x)\Big| \leqq \Big|F_n(x) - \sum_{\varrho_i<\mu} (F_n)_i u_i(x)\Big| +$$

$$+|F(x) - F_n(x)| + \Big|\sum_{\varrho_i<\mu} [F_i - (F_n)_i]u_i(x)\Big| \leqq c\|F - F_n\|_{W_p^1} < \varepsilon,$$

$$\Big|\sum_{\varrho_i<\mu} [F_i - (F_n)_i]u_i(x)\Big| \leqq c\|h - h_n\|_{L^p(\Omega)} < \varepsilon$$

if n is large enough, and hence Theorem 1 follows. □

§5. Proof of Theorem 2

Obviously $L_2^\alpha \subset L_2^1 = W_2^1(\mathbf{R}^3)$. Suppose f satisfies the conditions of Theorem 2, then there exists $h \in L^2(\mathbf{R}^3)$ with compact support in Ω such that

$$f(x) = \int_\Omega (v_1^R - w_1^R)(|x-y|)h(y)dy$$

and $h = 0$ on $(\Omega_0)_{-R} := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geqq R\}$. Let $g(x) := \int_\Omega \Psi_1(x,y) \cdot h(y)dy$; then as we have seen in Section 4, the Fourier series of g converges to g locally uniformly in Ω. Let $F = f + g = \int_\Omega T_1(x,y)h(y)dy$. It is enough to show that the Fourier series of F converges locally uniformly in Ω. We know that

$$\Big|\sum_{\varrho_i<\mu} F_i u_i(x)\Big| = \Big|\int_\Omega h(y)\overline{E_\mu T_1(x,y)}dy\Big| \leqq$$

$$\leqq \|E\mu T_1(x,y)\|_{L_y^2(\Omega\setminus(\Omega_0)_{-R})} \cdot \|h\|_{L^2(\mathbf{R}^3)} \leqq c\|h\|_{L^2}.$$

Hence we can finish the proof of the local uniform convergence as in the proof of Theorem 1. Because the Fourier series of f tends to f locally uniformly in Ω, hence it tends to 0 locally uniformly in Ω_0.

§6. Proof of Theorem 3

If f satisfies the conditions of Theorem 3, then there exists $\beta > 3/2$ such that $f \in L_2^{\beta}(\mathbf{R}^3)$. Namely, if $p \geqq 2$, then because supp f is compact in Ω, we have $f \in L_2^{\alpha}$. If $p < 2$, then applying the embedding theorem given in the proof of Theorem 1, we obtain

$$\beta = \alpha - 3\left(\frac{1}{p} - \frac{1}{2}\right) = \frac{\alpha p - 3}{p} + \frac{3}{2} > 3/2.$$

Consequently we may suppose $f \in L_2^{\alpha}(\mathbf{R}^3)$, supp f is compact in Ω and $\alpha > 3/2$. Choose an $h \in L^2(\mathbf{R}^3)$ with compact support in Ω and such that

$$f(x) = \int\limits_{\Omega} [v_{\alpha}^R(|x-y|) - w_{\alpha}^R(|x-y|)]h(y)dy$$

and let

$$g(x) := \int\limits_{\Omega} \Psi_{\alpha}(x,y) \cdot h(y)dy.$$

We have seen that the Fourier series of g converges absolutely and locally uniformly in Ω, hence it is enough to show that the Fourier series of

$$F(x) := f(x) + g(x) = \int\limits_{\Omega} T_{\alpha}(x,y)h(y)dy$$

converges absolutely (the local uniform convergence is proved in Theorem 1). Obviously,

$$F_i = \int\limits_{\Omega} F(x)\overline{u_i(x)}dx = \int\limits_{\Omega} h(y) \int\limits_{\Omega} T_{\alpha}(x,y)\overline{u_i(x)}dxdy =$$
$$= \int\limits_{\Omega} h(y)\overline{u_i(y)}\frac{1}{(1+\varrho_i^2)^{\alpha/2}} = \frac{h_i}{(1+\varrho_i^2)^{\alpha/2}},$$

hence

$$\sum_{i=1}^{\infty} |F_i u_i(x)| \leqq \left\{\sum_{i=1}^{\infty} |F_i|^2 (1+\varrho_i^2)^{\alpha}\right\}^{\frac{1}{2}} \left\{\sum_{i=1}^{\infty} \frac{|u_i(x)|^2}{(1+\varrho_i^2)^{\alpha}}\right\}^{\frac{1}{2}} \leqq c\|h\|_{L^2},$$

because $\alpha > 3/2$, i.e. the Fourier series of f is absolutely convergent. Theorem 3 is proved.

REMARKS. 1. The following is true for $N = 3, 5, 7$ and probably it holds for every natural number $N \geqq 1$:

CONJECTURE.

$$\left| \int_0^R r^{\frac{\alpha}{2}} K_{(N-\alpha)/2}(r) \left[\frac{J_{N/2}(r\mu_i)}{\mu_i^{N/2}} - \frac{J_{N/2}(r\varrho_i)}{\varrho_i^{N/2}} \right] dr \right| \leqq$$

$$\leqq \begin{cases} \dfrac{c}{(1+\varrho_i)^{\frac{N-1}{2}+\alpha}} e^{R|\nu_i|} & \text{if } \alpha \leqq 1, \\ \dfrac{c}{(1+\varrho_i)^{\frac{N+1}{2}}} e^{R|\nu_i|} & \text{if } \alpha \geqq 1. \end{cases}$$

In the proofs of the theorems we have used only the estimate $|\ldots| \leqq \leqq \dfrac{c}{(1+\varrho_i)^{1+\alpha}} e^{R|\nu_i|}$ because we used the case $\alpha = \frac{N-1}{2}$ and for this case these estimates coincide. The conjectured estimate is stronger than that of the proved, if α is small.

2. In this paper we had to develop some ideas of the papers [8], [9] of V. Komornik, too.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6–8
H–1088, HUNGARY

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INVERSE AND SYMMETRIC RELATORS

Á. SZÁZ (Debrecen)

Introduction

Relators are simply nonvoid collections of reflexive relations on sets. They are straightforward generalizations of the various uniformities [24], and are essentially identical to the generalized uniformities of Konishi [12] and Krishnan [13] and to the connector systems of Nakano–Nakano [23].

Relator spaces were proposed in our former paper [28] as the most suitable basic terms which topology and analysis should be based on. In [28], we introduced and studied the most important basic tools in relator spaces and mild continuities of relations from one relator space into another.

In a subsequent paper [29], to provide a primary classification for relator spaces, which is necessary to formulate and prove generalized forms of many of the important theorems of topology and analysis, we introduced and studied various directedness, topologicalness and transitiveness properties of relators.

In the present paper, to start a similar investigation in connection with symmetries, we establish the most important basic properties of inverse relators, and introduce and study six fundamental and two supplementary symmetry properties of relators. The results obtained are mainly illustrated with the help of some particular Davis–Pervin relators [29, p. 195].

0. Notations and terminology

A nonvoid family $\mathcal{R}$ of reflexive relations on a set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space.

If (x_α) and (y_α) are nets, A and B are sets, and x and y are points in $X(\mathcal{R})$, then we write

(i) $(y_\alpha) \in \mathrm{Lim}_{\mathcal{R}}(x_\alpha)$ $((y_\alpha) \in \mathrm{Adh}_{\mathcal{R}}(x_\alpha))$ if $((y_\alpha, x_\alpha))$ is eventually (frequently) in each $R \in \mathcal{R}$;

(ii) $x \in \lim_{\mathcal{R}}(x_\alpha)$ $(x \in \mathrm{adh}_{\mathcal{R}}(x_\alpha))$ if $(x) \in \mathrm{Lim}_{\mathcal{R}}(x_\alpha)$ $((x) \in \mathrm{Adh}_{\mathcal{R}}(x_\alpha))$;

(iii) $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ $(B \in \mathrm{Int}_{\mathcal{R}}(A))$ if $R(B) \cap A \neq \emptyset$ $(R(B) \subset A)$ for all (some) $R \in \mathcal{R}$;

(iv) $x \in \mathrm{cl}_{\mathcal{R}}(A)$ ($x \in \mathrm{int}_{\mathcal{R}}(A)$) if $\{x\} \in \mathrm{Cl}_{\mathcal{R}}(A)$ ($\{x\} \in \mathrm{Int}_{\mathcal{R}}(A)$);
(v) $A \in \mathcal{F}_{\mathcal{R}}$ ($A \in \mathcal{T}_{\mathcal{R}}$) if $\mathrm{cl}_{\mathcal{R}}(A) = A$ ($\mathrm{int}_{\mathcal{R}}(A) = A$);
(vi) $u \in \varrho_{\mathcal{R}}(x)$ if $y \in \mathrm{cl}_{\mathcal{R}}(\{x\})$.

If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^* = \{S \subset X \times X : \exists R \in \mathcal{R} : R \subset S\},$$

$$\mathcal{R}^\# = \{S \subset X \times X : \forall A \subset X : \exists R \in \mathcal{R} : R(A) \subset S(A)\}$$

and

$$\mathcal{R}^\wedge = \{S \subset X \times X : \forall x \in X : \exists R \in \mathcal{R} : R(x) \subset S(x)\}$$

are called the uniform, proximal and topological refinements of $\mathcal{R}$, respectively.

Namely, $\mathcal{R}^*$, $\mathcal{R}^\#$ and $\mathcal{R}^\wedge$ are the largest relators on X such that $\mathrm{Lim}_{\mathcal{R}^*} = \mathrm{Lim}_{\mathcal{R}}$ ($\mathrm{Adh}_{\mathcal{R}^*} = \mathrm{Adh}_{\mathcal{R}}$), $\mathrm{Cl}_{\mathcal{R}^\#} = \mathrm{Cl}_{\mathcal{R}}$ ($\mathrm{Int}_{\mathcal{R}^\#} = \mathrm{Int}_{\mathcal{R}}$) and $\mathrm{lim}_{\mathcal{R}^\wedge} = \mathrm{lim}_{\mathcal{R}}$ ($\mathrm{adh}_{\mathcal{R}^\wedge} = \mathrm{adh}_{\mathcal{R}}$) and $\mathrm{cl}_{\mathcal{R}^\wedge} = \mathrm{cl}_{\mathcal{R}}$ ($\mathrm{int}_{\mathcal{R}^\wedge} = \mathrm{int}_{\mathcal{R}}$), respectively.

Two relators $\mathcal{R}$ and $\mathcal{S}$ on the same set are called uniformly, proximally and topologically (weakly topologically) equivalent if $\mathcal{R}^* = \mathcal{S}^*$, $\mathcal{R}^\# = \mathcal{S}^\#$ and $\mathcal{R}^\wedge = \mathcal{S}^\wedge$ ($\varrho_{\mathcal{R}} = \varrho_{\mathcal{S}}$), respectively.

Moreover, a relator $\mathcal{R}$ is called uniformly, proximally and topologically fine if $\mathcal{R}^* = \mathcal{R}$, $\mathcal{R}^\# = \mathcal{R}$ and $\mathcal{R}^\wedge = \mathcal{R}$, respectively.

A relator $\mathcal{R}$ on X is called
(i) weakly transitive if $\cap\mathcal{R}$ is transitive;
(ii) weakly topological if $\varrho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$;
(iii) topological if $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ for all $A \subset X$.

Moreover, a relator $\mathcal{R}$ is called inversely topological (bitopological) if $\mathcal{R}^{-1}$ is topological (both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topological).

Finally, we remark that if $\mathcal{A}$ is a nonvoid family of subsets of X and

$$R_A = A \times A \cup (X \setminus A) \times X$$

for all $A \in \mathcal{A}$, then the relator

$$\mathcal{R}_{\mathcal{A}} = \{R_A : A \in \mathcal{A}\}$$

is called the Davis–Pervin relator on X generated by $\mathcal{A}$.

Note that to be more precise the ground set X should also be indicated in the above notations.

1. Inverse relators

DEFINITION 1.1. If $\mathcal{R}$ is a relator, then the relator

$$\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$$

is called the inverse of $\mathcal{R}$.

The most important basic properties of the inverse relators are contained in the next obvious

THEOREM 1.2. *If $\mathcal{R}$ is a relator on X, then*
(i) $\mathrm{Lim}_{\mathcal{R}^{-1}} = (\mathrm{Lim}_{\mathcal{R}})^{-1}$; (ii) $\mathrm{Adh}_{\mathcal{R}^{-1}} = (\mathrm{Adh}_{\mathcal{R}})^{-1}$;
(iii) $\mathrm{Cl}_{\mathcal{R}^{-1}} = (\mathrm{Cl}_{\mathcal{R}})^{-1}$; (iv) $\mathrm{Int}_{\mathcal{R}^{-1}} = \mathcal{C} \circ (\mathrm{Int}_{\mathcal{R}})^{-1} \circ \mathcal{C}$,
where $\mathcal{C}$ means the complementation operator with respect to X.

As an easy consequence of this theorem, we can at once state

COROLLARY 1.3. *If $\mathcal{R}$ is a relator, then*

$$(\mathcal{R}^{-1})^{\sim} = (\mathcal{R}^{\sim})^{-1}.$$

PROOF. By [28, Definition 4.7] and Theorem 1.2, it is clear that

$$(R^{-1})^{\sim} = (\mathrm{Cl}_{\{R^{-1}\}})^{-1} = \mathrm{Cl}_{\{R\}} = (R^{\sim})^{-1}$$

for all $R \in \mathcal{R}$, which is apparently a little more than stated.

However, at present, it is more important to point out that Theorem 1.2 can also be used to prove easily

THEOREM 1.4. *If $\mathcal{R}$ is a relator, then*
(i) $(\mathcal{R}^{-1})^{*} = (\mathcal{R}^{*})^{-1}$;
(ii) $(\mathcal{R}^{-1})^{\#} = (\mathcal{R}^{\#})^{-1}$.

PROOF. By Theorem 1.2 and [28, Corollary 5.5], it is clear that

$$\mathrm{Lim}_{(\mathcal{R}^{*})^{-1}} = (\mathrm{Lim}_{\mathcal{R}^{*}})^{-1} = (\mathrm{Lim}_{\mathcal{R}})^{-1} = \mathrm{Lim}_{\mathcal{R}^{-1}}$$

and hence $(\mathcal{R}^{*})^{-1} \subset (\mathcal{R}^{-1})^{*}$.

Moreover, a quite similar application of Theorem 1.2 and [28, Corollary 5.9] shows that $(\mathcal{R}^{\#})^{-1} \subset (\mathcal{R}^{-1})^{\#}$.

Thus, since the reverse inclusions are immediate consequences of the former ones, the proof is complete.

As a useful consequence of this theorem, we can at once state

COROLLARY 1.5. *A relator $\mathcal{R}$ is uniformly (proximally) fine if and only if its inverse $\mathcal{R}^{-1}$ is uniformly (proximally) fine.*

Unfortunately, the corresponding assertions do not hold for the topological refinements. Namely, for instance, we have

EXAMPLE 1.6. If X is the set of all real numbers and $\mathcal{A}$ is the family of all half-open intervals $[a, b[$ in X with $-\infty < a < b < +\infty$, then the Davis–Pervin relator $\mathcal{R} = \mathcal{R}_{\mathcal{A}}$ has the following properties:

(i) $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$; (ii) $\mathcal{R} \cap (\mathcal{R}^{-1})^{\wedge} = \emptyset$;
(iii) $(\mathcal{R}^{\wedge})^{-1} \not\subset (\mathcal{R}^{-1})^{\wedge}$; (iv) $(\mathcal{R}^{-1})^{\wedge} \not\subset (\mathcal{R}^{\wedge})^{-1}$.

Namely, if $A \in \mathcal{A}$, then we clearly have $R_A^{-1}(x) = X \setminus A$ if $x \in X \setminus A$ and $R_A^{-1}(x) = X$ if $x \in A$. Hence, since $R_A(x) = A$ if $x \in A$ and $R_A(x) = X$ if $x \in X \setminus A$, it is clear that

$$R_A^{-1} \in \mathcal{R}^\wedge, \quad \text{but} \quad R_A \notin (\mathcal{R}^{-1})^\wedge.$$

Therefore, the properties (i)–(iii) are now quite obvious.

On the other hand, if $S \subset X \times X$ such that

$$S(x) = X \setminus [x-1, x[$$

for all $x \in X$, then it is clear that $S \in (\mathcal{R}^{-1})^\wedge$. Moreover, since

$$S^{-1}(y) = \{x \in X : y \in S(x)\} = X \setminus]y, y+1]$$

for all $y \in X$, it is also clear that $S^{-1} \notin \mathcal{R}^\wedge$, i.e., $S \notin (\mathcal{R}^\wedge)^{-1}$. Therefore, now property (iv) is also quite obvious.

REMARK 1.7. Note that, because of the properties (iv) and $\mathcal{R} \subset \mathcal{R}^\wedge$, the inverse of the topologically fine relator $\hat{R}$ cannot be topologically fine.

In addition to Theorems 1.2 and 1.4, we can also state

THEOREM 1.8. *If $\mathcal{R}$ is a relator, then*
(i) $\varrho_{\mathcal{R}^{-1}} = \varrho_{\mathcal{R}}^{-1}$; (ii) $(\mathcal{R}^{-1})' = (\mathcal{R}')^{-1}$.

PROOF. To check this, note that by [28, Theorem 2.22], we have $\varrho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = (\bigcap \mathcal{R})^{-1}$.

Moreover, recall that $\mathcal{R}'$ is the family of all finite intersections of members of $\mathcal{R}$ [29, p. 181].

2. Symmetries of relators

DEFINITION 2.1. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, will be called

(i) strongly symmetric if each $R \in \mathcal{R}$ is symmetric;
(ii) properly symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}$;
(iii) uniformly symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^*$;
(iv) proximally symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^\#$;
(v) topologically symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^\wedge$;
(vi) weakly symmetric if $\bigcap \mathcal{R}$ is symmetric.

REMARK 2.2. Clearly, each of the properties (i) through (iv) implies its successor.

Moreover, if (v) holds, then it is clear that

$$\bigcap_{R \in \mathcal{R}} R(x) \subset \bigcap_{R \in \mathcal{R}} R^{-1}(x)$$

for all $x \in X$. Therefore, we also have $\cap\mathcal{R} \subset (\cap\mathcal{R})^{-1}$, and hence (vi).

The fact that the converse implications are not, in general, true is apparent from the subsequent examples.

EXAMPLE 2.3. If R is a reflexive relation on X such that R is not symmetric, then $\mathcal{R} = \{R, R^{-1}\}$ is a properly symmetric relator on X such that $\mathcal{R}$ is not strongly symmetric.

EXAMPLE 2.4. If R is as in Example 2.3 and Δ_X is the identity relation on X, then $\mathcal{R} = \{\Delta_X, R\}$ is a uniformly symmetric relator on X such that $\mathcal{R}$ is not properly symmetric.

To check the uniform symmetry of $\mathcal{R}$, note that now $\mathcal{R}^* = \{\Delta_X\}^*$ is the largest relator on X.

EXAMPLE 2.5. If $\operatorname{card}(X) \geqq 3$ and $R_x = \Delta_X \cup \{x\} \times X$ for all $x \in X$, then $\mathcal{R} = \{R_x\}_{x \in X}$ is a proximally symmetric relator on X such that $\mathcal{R}$ is not uniformly symmetric.

To check this, note that

$$R_x(z) = X \quad \text{if} \quad z = x \quad \text{and} \quad R_x(z) = \{z\} \quad \text{if} \quad z \in X \setminus \{x\}$$

and

$$R_x^{-1}(z) = \{x\} \quad \text{if} \quad z = x \quad \text{and} \quad R_x^{-1}(z) = \{x, z\} \quad \text{if} \quad z \in X \setminus \{x\}.$$

Therefore, we have

$$R_x(A) = \Delta_X(A) \quad \text{for all} \quad A \subset X \quad \text{and} \quad x \in X \setminus A$$

and

$$R_z(z) \not\subset R_x^{-1}(z) \quad \text{for all} \quad x, z \in X.$$

Whence, it is clear that

$$\mathcal{R}^{\#} = \{\Delta_X\}^*, \quad \text{but} \quad \mathcal{R}^{-1} \cap \mathcal{R}^* = \emptyset.$$

EXAMPLE 2.6. If $\operatorname{card}(X) \geqq 3$ and $\mathcal{A} = \{\{x\}\}_{x \in X}$, then the Davis–Pervin relator $\mathcal{R} = \mathcal{R}_{\mathcal{A}}$ is a topologically symmetric relator on X such that $\mathcal{R}$ is not proximally symmetric.

To check this, note that if now

$$R_x = R_{\{x\}} = \{x\} \times \{x\} \cup (X \setminus \{x\}) \times X,$$

then

$$R_x(z) = \{x\} \quad \text{if} \quad z = x \quad \text{and} \quad R_x(z) = X \quad \text{if} \quad z \in X \setminus \{x\}$$

and

$$R_x^{-1}(z) = X \quad \text{if} \quad z = x \quad \text{and} \quad R_x^{-1}(z) = X \setminus \{x\} \quad \text{if} \quad z \in X \setminus \{x\}.$$

Therefore, we have

$$R_x(x) = \Delta_X(x) \quad \text{for all} \quad x \in X$$

and

$$R_y(\{z, w\}) \not\subset R_x^{-1}(\{z, w\}) \text{ for all } x, y \in X \text{ and } z, w \in X \setminus \{x\} \text{ with } z \neq w.$$

Whence, it is clear that

$$\mathcal{R}^{\wedge} = \{\Delta_X\}^*, \quad \text{but} \quad \mathcal{R}^{-1} \cap \mathcal{R}^{\#} = \emptyset.$$

EXAMPLE 2.7. If X and $\mathcal{R}$ are as in Example 2.6, then $\mathcal{R}^{-1}$ is a weakly symmetric Davis–Pervin relator on X such that $\mathcal{R}^{-1}$ is not topologically symmetric.

To check this, note that under the notations of Example 2.6, we have

$$\bigcap_{x \in X} R_x^{-1}(z) = \Delta_X(z) \quad \text{for all} \quad z \in X$$

and

$$R_y^{-1}(x) \not\subset R_x(x) \quad \text{for all} \quad x, y \in X.$$

Whence, it is clear that

$$\cap \mathcal{R}^{-1} = \Delta_X, \quad \text{but} \quad \mathcal{R} \cap (\mathcal{R}^{-1})^{\wedge} = \emptyset.$$

REMARK 2.8. Note that the relator $\mathcal{R}$ considered in Example 1.6 is also a topologically symmetric relator such that $\mathcal{R}^{-1}$ is not topologically symmetric.

Thus, Examples 2.6 and 2.7 are somewhat superfluous. They have only been stated here because of their close analogy with Example 2.5.

3. Proximally symmetric relators

Because of the corresponding definitions and Corollary 1.3, we obviously have

THEOREM 3.1. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is strongly symmetric;*
(ii) *$\mathcal{R}^{-1}$ is strongly symmetric;*
(iii) *$\mathcal{R}'$ is strongly symmetric;*
(iv) *$\mathcal{R}^{\sim}$ is strongly symmetric.*

THEOREM 3.2. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is properly symmetric;*
(ii) *$\mathcal{R}^{-1}$ is properly symmetric;*
(iii) *$\mathcal{R}^{-1}$ is equal to $\mathcal{R}$;*
(iv) *$\mathcal{R}^{\sim}$ is properly symmetric.*

To prove the implication (iv) $\Rightarrow$ (i), note that $\mathcal{R}^{\sim} = \mathcal{S}^{\sim}$ also implies $\mathcal{R} = \mathcal{S}$.

Now as an immediate consequence of Theorems 3.2 and 1.8, we can also state

COROLLARY 3.3. *If $\mathcal{R}$ is a properly symmetric relator, then the relator $\mathcal{R}'$ is also properly symmetric.*

The fact that the converse statement need not be true is apparent from the next

EXAMPLE 3.4. If X is the set of all real numbers and $R_i \subset X \times X$ for $i = 1, 2$ such that

$$R_i(x) = [x - i, +\infty[$$

for all $x \in X$, then

$$\mathcal{R} = \{R_1, R_2, R_1^{-1}, R_2^{-1}, R_1^{-1} \cap R_2\}$$

is a relator on X such that $\mathcal{R}'$ is properly symmetric, but $\mathcal{R}$ is not even topologically symmetric.

To check this, note that

$$\mathcal{R}' = \mathcal{R} \cup \{R_1 \cap R_1^{-1}, R_2 \cap R_2^{-1}, R_1 \cap R_2^{-1}\},$$

and moreover

$$R(0) \not\subset (R_1^{-1} \cap R_2)^{-1}(0) = [-1, 2]$$

for all $R \in \mathcal{R}$.

REMARK 3.5. As an immediate consequence of Theorems 3.2, 1.4 and 1.2, we can also state:

If $\mathcal{R}$ is a properly symmetric relator, then the relators $\mathcal{R}^*$ and $\mathcal{R}^{\#}$ are properly symmetric and the relations $\mathrm{Lim}_{\mathcal{R}}$ and $\mathrm{Cl}_{\mathcal{R}}$ are symmetric.

However, this fact is of no importance for us since by using Theorems 1.4 and 1.2, we can prove much more.

THEOREM 3.6. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is uniformly symmetric;*
(ii) *$\mathcal{R}^{-1}$ is uniformly symmetric;*
(iii) *$\mathcal{R}^{-1}$ is uniformly equivalent to $\mathcal{R}$;*
(iv) *$\mathcal{R}^*$ is properly symmetric.*

PROOF. If (i) holds, then by Theorem 1.4, it is clear that $(\mathcal{R}^{-1})^{-1} \subset \subset (\mathcal{R}^*)^{-1} = (\mathcal{R}^{-1})^*$, and thus (ii) also holds. Hence, since $\mathcal{R} = (\mathcal{R}^{-1})^{-1}$, it is clear that (ii) also implies (i).

Therefore, if (ii) holds, then we have not only $\mathcal{R} \subset (\mathcal{R}^{-1})^*$, but also $\mathcal{R}^{-1} \subset \mathcal{R}^*$. Hence, it follows that $\mathcal{R}^* \subset (\mathcal{R}^{-1})^*$ and $(\mathcal{R}^{-1})^* \subset \mathcal{R}^*$. Consequently, we have $(\mathcal{R}^{-1})^* = \mathcal{R}^*$, and thus (iii) also holds.

Finally, to complete the proof, note that if (iii) holds, then again by Theorem 1.4, $(\mathcal{R}^*)^{-1} = (\mathcal{R}^{-1})^* = \mathcal{R}^*$. Moreover, if (iv) holds, then because of $\mathcal{R} \subset \mathcal{R}^*$, we also have $\mathcal{R}^{-1} \subset \mathcal{R}^*$.

Because of this theorem, we obviously have

COROLLARY 3.7. *If $\mathcal{R}$ and $\mathcal{S}$ are uniformly equivalent relators, then $\mathcal{R}$ is uniformly symmetric if and only if $\mathcal{S}$ is uniformly symmetric.*

COROLLARY 3.8. *A relator $\mathcal{R}$ is uniformly symmetric if and only if $\mathcal{R}$ is uniformly equivalent to a properly symmetric relator $\mathcal{S}$.*

Moreover, from Theorem 3.6, by [28, Corollary 5.5] and Theorem 1.2, it is clear that we also have

THEOREM 3.9. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is uniformly symmetric;*

(ii) $\mathrm{Lim}_{\mathcal{R}^{-1}} = \mathrm{Lim}_{\mathcal{R}}$ $(\mathrm{Adh}_{\mathcal{R}^{-1}} = \mathrm{Adh}_{\mathcal{R}})$;

(iii) $\mathrm{Lim}_{\mathcal{R}}$ $(\mathrm{Adh}_{\mathcal{R}})$ *is symmetric.*

On the other hand, by using a quite similar argument as in the proof of Theorem 3.6, we can also easily prove

THEOREM 3.10. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is proximally symmetric;*

(ii) *$\mathcal{R}^{-1}$ is proximally symmetric;*

(iii) *$\mathcal{R}^{-1}$ is proximally equivalent to $\mathcal{R}$;*

(iv) *$\mathcal{R}^{\#}$ is properly symmetric.*

Because of this theorem, we obviously have

COROLLARY 3.11. *If $\mathcal{R}$ and $\mathcal{S}$ are proximally equivalent relators, then $\mathcal{R}$ is proximally symmetric if and only if $\mathcal{S}$ is proximally symmetric.*

COROLLARY 3.12. *A relator $\mathcal{R}$ is proximally symmetric if and only if $\mathcal{R}$ is proximally equivalent to a properly symmetric relator $\mathcal{S}$.*

Moreover, from Theorem 3.10, by [28, Corollary 5.9] and Theorem 1.2, it is clear that we also have

THEOREM 3.13. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is proximally symmetric;*

(ii) $\mathrm{Cl}_{\mathcal{R}^{-1}} = \mathrm{Cl}_{\mathcal{R}}$ $(\mathrm{Int}_{\mathcal{R}^{-1}} = \mathrm{Int}_{\mathcal{R}})$;

(iii) $\mathrm{Cl}_{\mathcal{R}}$ *is symmetric* $((\mathrm{Int}_{\mathcal{R}})^{-1} = \mathcal{C} \circ \mathrm{Int}_{\mathcal{R}} \circ \mathcal{C})$.

REMARK 3.14. This latter theorem, together with [28, Theorem 6.7], will allow us to easily prove that $\mathcal{R}^{\wedge}$ is properly symmetric if and only if $B \cap \mathrm{cl}_{\mathcal{R}}(A) \neq \emptyset$ implies $A \cap \mathrm{cl}_{\mathcal{R}}(B) \neq \emptyset$.

4. Topologically symmetric relators

From Examples 2.6 and 2.7, it is clear that an analogue of Theorems 3.6 and 3.10 cannot be true for topological symmetry.

However, as some immediate consequences of [28, Corollaries 5.13, 5.16 and 5.19], we still have the next two theorems.

THEOREM 4.1. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is topologically symmetric;*
(ii) $\lim_{\mathcal{R}} \subset \lim_{\mathcal{R}^{-1}}$ $(\mathrm{adh}_{\mathcal{R}} \subset \mathrm{adh}_{\mathcal{R}^{-1}})$*;*
(iii) $\mathrm{cl}_{\mathcal{R}} \subset \mathrm{cl}_{\mathcal{R}^{-1}}$ $(\mathrm{int}_{\mathcal{R}^{-1}} \subset \mathrm{int}_{\mathcal{R}})$.

THEOREM 4.2. *If $\mathcal{R}$ is an inversely topological relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is topologically symmetric;*
(ii) $\mathcal{F}_{\mathcal{R}^{-1}} \subset \mathcal{F}_{\mathcal{R}}$ $(\mathcal{T}_{\mathcal{R}^{-1}} \subset \mathcal{T}_{\mathcal{R}})$.

REMARK 4.3. By Theorem 4.1, it is clear that the implication (i) $\Rightarrow$ (ii) is always true.

On the other hand the next simple example shows that the converse implication is not, in general, true.

EXAMPLE 4.4. If $X = \{1, 2, 3, 4\}$ and $R \subset X \times X$ such that

$$R(1) = R(3) = X \quad \text{and} \quad R(2) = R(4) = X \setminus \{1\},$$

then $\mathcal{R} = \{R\}$ is a relator on X such that

$$\mathcal{T}_{\mathcal{R}^{-1}} = \mathcal{T}_{\mathcal{R}} = \{\emptyset, X\},$$

but $\mathcal{R}$ is still not even weakly symmetric.

Because of the definition of R, it is clear that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$. Moreover, it is also clear that

$$R^{-1}(1) = \{1, 2\} \quad \text{and} \quad R^{-1}(i) = X \quad \text{if} \quad i \in X \setminus \{1\}.$$

Thus, we also have $\mathcal{T}_{\mathcal{R}^{-1}} = \{\emptyset, X\}$. Moreover, since $R(1) \not\subset R^{-1}(1)$, it is clear that $\mathcal{R}$ is not weakly symmetric.

REMARK 4.5. Note that for a singleton relator $\mathcal{R} = \{R\}$ all the possible symmetry properties coincide.

Since the inverse of a topologically symmetric relator need not be topologically symmetric, we must also have

DEFINITION 4.6. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, is called
(i) inversely topologically symmetric if $\mathcal{R}^{-1}$ is topologically symmetric;
(ii) topologically bisymmetric if $\mathcal{R}$ is both topologically symmetric and inversely topologically symmetric.

REMARK 4.7. By Theorem 3.10 and Remark 2.2, it is clear that a proximally symmetric relator is also topologically bisymmetric.

On the other hand, the following example shows that the converse statement is not, in general, true.

EXAMPLE 4.8. If X and $\mathcal{A}$ are as in Example 1.6, and $\mathcal{B} = \mathcal{A} \setminus \{I\}$, where $I = [0,1[$, then

$$\mathcal{R} = \mathcal{R}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{B}}^{-1}$$

is a topologically bisymmetric Davis–Pervin relator on X such that $\mathcal{R}$ is not proximally symmetric.

The topological bisymmetry of $\mathcal{R}$ is immediate from the facts that

$$\mathcal{R}^{-1} = \mathcal{R}_{\mathcal{A}}^{-1} \cup \mathcal{R}_{\mathcal{B}}$$

and

$$\mathcal{R}_{\mathcal{A}}^{-1} \subset \mathcal{R}_{\mathcal{A}}^{\wedge} \quad \text{and} \quad \mathcal{R}_{\mathcal{A}} \subset \mathcal{R}_{\mathcal{B}}^{\wedge}.$$

Note that the first inclusion has been established in Example 1.6, while the second inclusion is apparent from the facts that $\mathcal{B} = \mathcal{A} \setminus \{I\}$ and

$$R_{[x,2^{-1}(x+1)[}(x) \subset R_I(x)$$

for all $x \in I$, and $R_I(x) = X$ for all $x \in X \setminus I$.

To check that $\mathcal{R}$ is not proximally symmetric, note that

$$R_I^{-1} \in \mathcal{R}_{\mathcal{A}}^{-1}, \quad \text{but} \quad R_I^{-1} \notin \mathcal{R}^{\#},$$

since

$$R_A(X \setminus I) \not\subset R_I^{-1}(X \setminus I) \quad \text{and} \quad R_B^{-1}(X \setminus I) \not\subset R_I^{-1}(X \setminus I)$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Namely, if $A \in \mathcal{A}$, then we obviously have

$$R_A(X \setminus I) = X \quad \text{and} \quad R_I^{-1}(X \setminus I) = R_{X \setminus I}(X \setminus I) = X \setminus I.$$

Moreover, if $B \in \mathcal{B}$ such that

$$R_{X \setminus B}(X \setminus I) = R_B^{-1}(X \setminus I) \subset R_I^{-1}(X \setminus I) = X \setminus I,$$

then since

$$R_{X\setminus B}(X\setminus I)=X \text{ if } X\setminus I \not\subset X\setminus B \text{ and } R_{X\setminus B}(X\setminus I)=X\setminus B \text{ if } X\setminus I\subset X\setminus B,$$

we also have

$$X\setminus I\subset X\setminus B\subset X\setminus I,$$

i.e., $B=I$, which is a contradiction.

REMARK 4.9. Note that under the above notations, we actually have

$$\mathcal{R}^{-1}\subset\mathcal{R}_{\mathcal{A}}^{\wedge} \quad\text{and}\quad \mathcal{R}\subset\mathcal{R}_{\mathcal{A}}^{-1}\cup\mathcal{R}_{\mathcal{B}}^{\wedge},$$

which is a little more than the topological bisymmetry of $\mathcal{R}$.

Now, as some immediate consequences of Theorems 4.1 and 4.2 and Definition 4.6, we can also state the next useful theorems.

THEOREM 4.10. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is topologically bisymmetric;*
(ii) $\lim_{\mathcal{R}^{-1}}=\lim_{\mathcal{R}}$ $(\mathrm{adh}_{\mathcal{R}^{-1}}=\mathrm{adh}_{\mathcal{R}})$;
(iii) $\mathrm{cl}_{\mathcal{R}^{-1}}=\mathrm{cl}_{\mathcal{R}}$ $(\mathrm{int}_{\mathcal{R}^{-1}}=\mathrm{int}_{\mathcal{R}})$.

THEOREM 4.11. *If $\mathcal{R}$ is a bitopological relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is topologically bisymmetric;*
(ii) $\mathcal{F}_{\mathcal{R}^{-1}}=\mathcal{F}_{\mathcal{R}}$ $(\mathcal{T}_{\mathcal{R}^{-1}}=\mathcal{T}_{\mathcal{R}})$.

REMARK 4.12. By Theorem 4.10, it is clear that the implication (i) $\Rightarrow$ (ii) is always true.

On the other hand, Example 4.4 shows also that the converse implication is not, in general, true.

5. Weakly symmetric relators

As an immediate consequence of [28, Theorem 2.22] and Theorem 1.8, we can at once state

THEOREM 5.1. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) $\varrho_{\mathcal{R}}=\bigcap\mathcal{R}$;
(iii) $\varrho_{\mathcal{R}^{-1}}=\varrho_{\mathcal{R}}$;
(iv) *$\varrho_{\mathcal{R}}$ is symmetric.*

Hence

Corollary 5.2. *If $\mathcal{R}$ and $\mathcal{S}$ are weakly topologically equivalent relators, then $\mathcal{R}$ is weakly symmetric if and only if $\mathcal{S}$ is weakly symmetric.*

Moreover, from Theorems 5.1 and 3.6, it is clear that now we also have

Theorem 5.3. *If $\mathcal{R}$ is a relator, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$\mathcal{R}^{-1}$ is weakly symmetric;*
(iii) *$\mathcal{R}^{-1}$ is weakly topologically equivalent to $\mathcal{R}$;*
(iv) *$\{\cap\mathcal{R}\}^*$ is properly symmetric.*

However, at present, it is more important to point out that by using Theorem 5.1, now we can also easily prove

Theorem 5.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$x \in \operatorname{int}_{\mathcal{R}}(A)$ implies $\varrho_{\mathcal{R}}(x) \subset A$ for all $A \subset X$;*
(iii) *$A \cap \varrho_{\mathcal{R}}(x) \neq \emptyset$ implies $x \in \operatorname{cl}_{\mathcal{R}}(A)$ for all $x \in X$ and $A \subset X$.*

Proof. If $x \in \operatorname{int}_{\mathcal{R}}(A)$, then there exists an $R \in \mathcal{R}$ such that $R(x) \subset A$. Therefore, if (i) holds, then by Theorem 5.1, we also have $\varrho_{\mathcal{R}}(x) = (\cap\mathcal{R})(x) \subset \subset R(x) \subset A$. Consequently, (ii) also holds.

On the other hand, if $x \notin \operatorname{cl}_{\mathcal{R}}(A)$, then by [28, Theorem 2.13], $x \in \in \operatorname{int}_{\mathcal{R}}(X \setminus A)$. Therefore, if (ii) holds, then we also have $\varrho_{\mathcal{R}}(x) \subset X \setminus A$, i.e., $A \cap \varrho_{\mathcal{R}}(x) = \emptyset$. Consequently, (iii) also holds.

Finally, to complete the proof, note that if (iii) holds, then $y \in \varrho_{\mathcal{R}}(x)$, i.e., $\{y\} \cap \varrho_{\mathcal{R}}(x) \neq \emptyset$ implies $x \in \operatorname{cl}_{\mathcal{R}}(\{y\}) = \varrho_{\mathcal{R}}(y)$ for all $x, y \in X$. And thus, again by Theorem 5.1, (i) also holds.

Remark 5.5. In the light of the above theorem and Remark 3.14, it is clear that the weak symmetry of $\mathcal{R}$ is a natural localization of the proper symmetry of $\mathcal{R}^\wedge$.

Despite this, it is still rather surprising that Theorem 5.4 allows us to prove easily

Theorem 5.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$\mathcal{R}^\wedge$ is inversely topologically symmetric.*

Proof. If $x \in X$ and $A \subset X$, then by [28, Theorem 6.7] and Theorem 1.2, it is clear that

$$A \cap \varrho_{\mathcal{R}}(x) \neq \emptyset \Leftrightarrow A \in \operatorname{Cl}_{\mathcal{R}^\wedge}(\{x\}) \Leftrightarrow \{x\} \in \operatorname{Cl}_{(\mathcal{R}^\wedge)^{-1}}(A) \Leftrightarrow$$
$$\Leftrightarrow x \in \operatorname{cl}_{(\mathcal{R}^\wedge)^{-1}}(A).$$

On the other hand, from [28, Corollary 5.7], we know that $x \in \mathrm{cl}_{\mathcal{R}}(A)$ if and only if $x \in \mathrm{cl}_{\mathcal{R}^\wedge}(A)$.

Therefore, by Theorem 5.4, the assertion (i) is equivalent to the inclusion

$$\mathrm{cl}_{(\mathcal{R}^\wedge)^{-1}} \subset \mathrm{cl}_{\mathcal{R}^\wedge},$$

which is, by Theorem 4.1, equivalent to the assertion (ii).

REMARK 5.7. Note that the implication (ii) $\Rightarrow$ (i) is already an immediate consequence of Remark 2.2, Theorem 5.3 and Corollary 5.2.

Moreover, note also that the topological symmetry of $\mathcal{R}^\wedge$ is nothing else but the proper symmetry of $\mathcal{R}^\wedge$ which is a rather restrictive property.

6. Weakly symmetric weakly transitive relators

As an immediate consequence of Theorem 5.1, [29, Theorem 3.21] and [30, Corollary 4.13] we can also state

THEOREM 6.1. *If $\mathcal{R}$ is a weakly transitive relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$\varrho_{\mathcal{R}}$ is an equivalence;*
(iii) *$\varrho_{\mathcal{R}}$ is nonmingled-valued.*

REMARK 6.2. Note that the implications (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii) are always true.

Moreover, note that if (ii) or (iii) holds, then $\mathcal{R}$ is necessarily weakly transitive.

Now, by considering the Davis–Pervin relator $\mathcal{R}_{X/\varrho_{\mathcal{R}}}$, we can also easily prove

THEOREM 6.3. *If $\mathcal{R}$ is a weakly transitive relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$\mathcal{R}_{X/\varrho_{\mathcal{R}}}$ is topologically symmetric.*

PROOF. If (i) holds, then by Theorem 6.1, we have

$$R_{\varrho_{\mathcal{R}}(y)}(y) = \varrho_{\mathcal{R}}(y) \subset \left(R_{X\setminus\varrho_{\mathcal{R}}(x)}\right)(y) = \left(R_{\varrho_{\mathcal{R}}(x)}\right)^{-1}(y)$$

for all $x, y \in X$, which is apparently a little more than (ii).

Conversely, if (ii) holds and $x, y \in X$ such that $y \notin \varrho_{\mathcal{R}}(x)$, then there exists a $z \in X$ such that

$$R_{\varrho_{\mathcal{R}}(z)}(y) \subset \left(R_{\varrho_{\mathcal{R}}(x)}\right)^{-1}(y) = \left(R_{X\setminus\varrho_{\mathcal{R}}(x)}\right)(y) = X \setminus \varrho_{\mathcal{R}}(x).$$

Hence, it follows that

$$y \in \varrho_{\mathcal{R}}(z) \subset X \setminus \varrho_{\mathcal{R}}(x).$$

Because of [29, Theorem 3.21], this implies now that

$$\varrho_{\mathcal{R}}(y) \subset X \setminus \varrho_{\mathcal{R}}(x).$$

Therefore, we also have $x \notin \varrho_{\mathcal{R}}(y)$. Consequently, $\varrho_{\mathcal{R}}$ is symmetric and thus by Theorem 5.1, (i) also holds.

REMARK 6.4. Note that this theorem extends the first assertion of Example 2.6, and is very similar to Theorem 5.6.

But, in contrast to Theorem 5.6, neither of the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) in Theorem 6.3 is true without supposing the weak transitivity of $\mathcal{R}$.

EXAMPLE 6.5. If $X = \{1,2,3\}$ and $R \subset X \times X$ such that

$$R(1) = \{1,2\}, \quad R(2) = X, \quad R(3) = \{2,3\},$$

then $\mathcal{R} = \{R\}$ is a strongly symmetric relator on X such that $\mathcal{R}_{X/\varrho_{\mathcal{R}}}$ is not even weakly symmetric.

To check this note that $\varrho_{\mathcal{R}} = R^{-1} = R$, but

$$\bigcap \mathcal{R}_{X/\varrho_{\mathcal{R}}} = \Delta_X \cup \{(1,2),(3,2)\}$$

is not symmetric.

EXAMPLE 6.6. If $X = \{1,2,3\}$ and $R \subset X \times X$ such that

$$R(1) = \{1,2\} \quad \text{and} \quad R(2) = R(3) = \{2,3\},$$

then $\mathcal{R} = \{R\}$ is a relator on X such that $\mathcal{R}_{X/\varrho_{\mathcal{R}}}$ is properly symmetric despite that $\mathcal{R}$ is not weakly symmetric.

To check this, note that $\varrho_{\mathcal{R}} = R^{-1}$ and

$$R^{-1}(1) = \{1\}, \quad R^{-1}(2) = X, \quad R^{-1}(3) = \{2,3\}.$$

Therefore

$$\left(R_{\varrho_{\mathcal{R}}(1)}\right)^{-1} = R_{X \setminus \varrho_{\mathcal{R}}(1)} = R_{\varrho_{\mathcal{R}}(3)} \quad \text{and} \quad \left(R_{\varrho_{\mathcal{R}}(2)}\right)^{-1} = R_{\varrho_{\mathcal{R}}(2)}.$$

Analogously to Theorem 6.3, we can also easily prove

THEOREM 6.7. *If $\mathcal{R}$ is a weakly transitive relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*
(ii) *$\mathcal{R}_{X/\varrho_{\mathcal{R}}}$ is topologically finer than $\mathcal{R}$.*

PROOF. If (i) holds, then by Theorem 5.1, we have

$$R_{\varrho_{\mathcal{R}}(x)}(x) = \varrho_{\mathcal{R}}(x) = (\cap \mathcal{R})(x) \subset R(x)$$

for all $x \in X$, which is apparently a little more than (ii).

Conversely, if (ii) holds, then for any $x \in X$ and $R \in \mathcal{R}$ there exists a $y \in X$ such that

$$R_{\varrho_{\mathcal{R}}(y)}(x) \subset R(x).$$

Hence, if $R(x) \neq X$, it follows that

$$x \in \varrho_{\mathcal{R}}(y) \subset R(x).$$

Because of [29, Theorem 3.21], this implies now that

$$\varrho_{\mathcal{R}}(x) \subset R(x).$$

Consequently, by [28, Theorem 2.22], we have

$$(\cap\mathcal{R})^{-1}(x) = \varrho_{\mathcal{R}}(x) \subset (\cap\mathcal{R})(x)$$

for any $x \in X$, whence (i) is immediate.

REMARK 6.8. From the first part of this proof, it is clear that the implication (i) $\Rightarrow$ (ii) is always true.

On the other hand, Example 6.6 shows also that the converse implication is not, in general, true.

Namely, if $\mathcal{R}$ is as in Example 6.6, then we have

$$R_{\varrho_{\mathcal{R}}(1)}(1) \subset R(1) \quad \text{and} \quad R_{\varrho_{\mathcal{R}}(3)}(i) = R(i) \quad \text{if} \quad i = 2,3.$$

Consequently, $\mathcal{R}_{X/\varrho_{\mathcal{R}}}$ is topologically finer than $\mathcal{R}$ despite that $\mathcal{R}$ is not weakly symmetric.

7. Weakly symmetric weakly topological relators

From Theorem 5.4, by [28, Theorem 2.20] and Theorem 5.1, it is clear that we also have

THEOREM 7.1. *If $\mathcal{R}$ is a weakly topological relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*

(ii) *$x \in G$ implies $\varrho_{\mathcal{R}}(x) \subset G$ for all $G \in \mathcal{T}_{\mathcal{R}}$;*

(iii) *$F \cap \varrho_{\mathcal{R}}(x) \neq \emptyset$ implies $x \in F$ for all $x \in X$ and $F \in \mathcal{F}_{\mathcal{R}}$.*

To check the implication (iii) $\Rightarrow$ (i), note that if (iii) holds, then because of $y \in \varrho_{\mathcal{R}}(y) \in \mathcal{F}_{\mathcal{R}}$, the condition $y \in \varrho_{\mathcal{R}}(x)$ implies $x \in \varrho_{\mathcal{R}}(y)$. And thus by Theorem 5.1, (i) also holds.

REMARK 7.2. By Theorem 5.4 and [28, Theorem 2.20], it is clear that the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are always true.

On the other hand, our former Example 4.4 shows also that the implication (ii) $\Rightarrow$ (i) is not, in general, true.

Now, as an immediate consequence of Theorem 7.1 and [28, Theorem 2.20], we can also state

THEOREM 7.3. *If $\mathcal{R}$ is a weakly topological relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*

(ii) *for each $F \in \mathcal{F}_\mathcal{R}$ and $x \in X \setminus F$ there exists a $G \in \mathcal{T}_\mathcal{R}$ such that $F \subset G$, but $x \notin G$;*

(iii) *each member of $\mathcal{F}_\mathcal{R}$ ($\mathcal{T}_\mathcal{R}$) is the intersection (union) of certain members of $\mathcal{T}_\mathcal{R}$ ($\mathcal{F}_\mathcal{R}$).*

To check the implication (i) $\Rightarrow$ (ii), note that if $F \in \mathcal{F}_\mathcal{R}$, then $X \setminus F \in \mathcal{T}_\mathcal{R}$. Therefore, if $x \in X \setminus F$ and (i) holds, then by Theorem 7.1, $\varrho_\mathcal{R}(x) \subset X \setminus F$. Hence, since $x \in \varrho_\mathcal{R}(x) \in \mathcal{F}_\mathcal{R}$, it is clear that $G = X \setminus \varrho_\mathcal{R}(x) \in \mathcal{T}_\mathcal{R}$ such that $F \subset G$, but $x \notin G$.

REMARK 7.4. It is clear that the equivalence of (ii) and (iii) is always true.

On the other hand, again Example 4.4 shows that the implication (ii) $\Rightarrow$ (i) is not, in general, true.

Thus, for the sake of completeness, we need only show that the implication (i) $\Rightarrow$ (ii) is not also true in general.

EXAMPLE 7.5. If $X = \{0, 1, 2\}$ and $R_i \subset X \times X$ for $i = 1, 2$ such that

$$R_i(0) = \{0, i\} \quad \text{and} \quad R_i(k) = \{1, 2\} \quad \text{if} \quad k \in \{1, 2\},$$

then $\mathcal{R} = \{R_1, R_2\}$ is a weakly symmetric and weakly transitive relator on X such that the assertion (ii) of Theorem 7.3 does not hold.

To check this latter statement, note that

$$\mathcal{T}_\mathcal{R} = \{\emptyset, \{1, 2\}, X\} \quad \text{and} \quad \mathcal{F}_\mathcal{R} = \{\emptyset, \{0\}, X\}$$

and thus the assertion (iii) of Theorem 7.3 cannot hold.

REMARK 7.6. The above relations R_1 and R_2 have also been used in our former papers ([28, Example 5.23] and [29, Example 2.9]).

From Theorems 7.1 and 5.1, it is clear that we also have

THEOREM 7.7. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is weakly symmetric;*

(ii) $\varrho_\mathcal{R}(x) = \bigcap_{R \in \mathcal{R}} \operatorname{int}_\mathcal{R}(R(x))$ *for all $x \in X$;*

(iii) $\bigcap\{F : x \in F \in \mathcal{F}_\mathcal{R}\} = \bigcap\{G : x \in G \in \mathcal{T}_\mathcal{R}\}$ *for all $x \in X$.*

To check the implication (i) $\Rightarrow$ (ii), note that if $x \in X$ and $R \in \mathcal{R}$, then because of the topologicalness of $\mathcal{R}$, we have $\operatorname{int}_\mathcal{R}(R(x)) \in \mathcal{T}_\mathcal{R}$. Therefore, if (i) holds, then by Theorem 7.1, $\varrho_\mathcal{R}(x) \subset \operatorname{int}_\mathcal{R}(R(x))$ also holds. Thus, we have $\varrho_\mathcal{R}(x) \subset \bigcap_{R \in \mathcal{R}} \operatorname{int}_\mathcal{R}(R(x))$.

Moreover, if (i) holds, then by Theorem 5.1, it is clear that we also have

$$\bigcap_{R \in \mathcal{R}} \operatorname{int}_\mathcal{R}(R(x)) \subset \bigcap_{R \in \mathcal{R}} R(x) = \varrho_\mathcal{R}(x),$$

even if $\mathcal{R}$ is not supposed to be topological. Consequently, if (i) holds, then (ii) also holds.

REMARK 7.8. By [28, Theorem 2.22], it is clear that the implication (ii) $\Rightarrow$ (i) is always true.

On the other hand, Examples 4.4 and 7.5 show that the implications (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are not, in general, true.

Thus, for the sake of completeness, we need only show that the implication (i) $\Rightarrow$ (ii) is also not true in general.

EXAMPLE 7.9. If R is a reflexive and symmetric relation on X such that R is not transitive, then $\mathcal{R} = \{R\}$ is a strongly symmetric relator on X such that the assertion (ii) of Theorem 7.7 does not hold.

To check this latter statement, note that for a singleton relator $\mathcal{R} = \{R\}$, the assertion (ii) of Theorem 7.7 means only that $R \circ R^{-1} \subset R$, i.e., $R^{-1} \subset R$ and $R \circ R \subset R$.

REMARK 7.10. By using some convenient definitions the assertions (ii) and (iii) of Theorems 7.3 and 7.7 can be stated in more suggestive forms.

For instance, under the notations of [29, Theorem 2.10], the assertion (ii) of Theorem 7.7 means only that the relators $\mathcal{R}$ and $(\mathcal{R}^{\circ})^{-1}$ are weakly topologically equivalent.

Notes and comments

The inverse $\mathcal{U}^{-1}$ of a quasi-uniformity $\mathcal{U}$ was called the dual and the conjugate of $\mathcal{U}$ by Nachbin [21, p. 57] and Murdeshwar–Naimpally [20, p. 16], respectively. The assertion (ii) of Theorem 1.4 is essentially a particular case of Theorem 2 of Kenyon [11].

The useful example 1.6 has mainly been suggested by the half-open interval space of Kelley [10, p. 59] and the generalized uniformization techniques of Davis [6] and Pervin [25]. Another important relator constructed from half-open intervals was utilized in [27].

A strongly symmetric (uniformly symmetric) generalized uniformity was called symmetric (feeble symmetric) by Krisnan [13], while an inversely topologically symmetric quasi-uniformity was called point-symmetric by Fletcher–Lindgren [8, p. 36]. The local symmetry of [8] is an important hybrid property.

Most of the results of Examples 1.6, 2.6, 2.7 and 4.8 can be easily extended to more general Davis–Pervin relators. Moreover, there is a natural generalization of the Davis–Pervin relators whose symmetry properties can also be nicely described [31].

The definitions of uniform and proximal symmetries are completely justified by Theorems 3.6 and 3.10 and their consequences. However, in the light of Theorems 4.1 and 4.2 and Examples 2.6 and 2.7, the topological symmetry should be rather called a topological semisymmetry.

In [18], a relator $\mathcal{R}$ will be called properly topologically symmetric if its topological refinement $\mathcal{R}^\wedge$ is properly symmetric, and it is shown that a relator $\mathcal{R}$ is properly topologically symmetric if and only if $\mathcal{R}$ is topologically equivalent to a symmetric singleton relator $\mathcal{S} = \{S\}$.

Thus, a properly topologically symmetric relator is always topologically symmetric, but not necessarily topologically bisymmetric. Moreover, even a strongly symmetric relator need not be properly topologically symmetric. Thus, neither the topological bisymmetry nor the proper topological symmetry can be naturally called the topological symmetry.

Our weak symmetry corresponds to the famous regularity axiom R_0. This axiom was first introduced by Shanin [26], and later rediscovered by Davis [6]. Since then a lot of work has been done on the R_0-axiom. See, for instance, [17], [1, p. 402], [20, p. 37], [22], [5], [9], [16], [7], [2, p. 93], [14], [4], [33] and [3].

In particular, the equivalence of the assertions (iii) of Theorem 6.1 and (ii) of Theorems 7.1 and 7.3 in topological spaces, and the equivalence of the assertions (i) and (ii) of Theorem 5.1 in quasi-uniform spaces have also been observed by Davis [6, Theorem 2] and Murdeshwar–Naimpally [20, Theorems 3.8 and 3.10], respectively.

These latter two authors, in their paper [20, Theorem 3.6], have also proved a close analogue of Theorem 5.6 which shows that a topological relator is weakly symmetric if and only if it is topologically equivalent to an inversely topologically symmetric quasi-uniformity. A few theorems of this kind will also be proved in [19] and [32].

Added in proof (April 11, 1991). The author is deeply indebted to Jenő Deák who has pointed out several small errors in the text and suggested the following more valuable examples in place of Examples 2.5, 4.8 and 7.9.

EXAMPLE 1. If X is an infinite set, $\mathcal{E}$ is the family of all equivalences on X having only finitely many equivalence classes, and L is a linear ordering on X, then the family

$$\mathcal{R} = \{E \cap L \colon E \in \mathcal{E}\}$$

is a base for a non-symmetric transitive quasi-uniformity $\mathcal{U}$ on X which induces the discrete proximity on X.

The curious fact that a non-symmetric quasi-uniformity may induce a proximity is certainly well-known, but we could not find it in the existing literature.

EXAMPLE 2. If X is an uncountable set, then the family $\mathcal{R}$ of all relations

$$R_{(A,B)} = A \times B \cup (X \setminus A) \times X,$$

where $A \subset B \subset X$ such that A is finite or $X \setminus B$ is countable, is a subbase for a topologically bisymmetric totally bounded quasi-uniformity $\mathcal{U}$ on X which induces a non-symmetric quasi-proximity on X.

For the necessary prerequisites on quasi-uniformities and quasi-proximities the reader is referred to Chapter 1 of Fletcher–Lindgren [8].

EXAMPLE 3. If X is the set of all positive integers and for each $n \in X$, we set $R_n \subset X \times X$ such that

$$R_n(1) = \{1,2\}, \quad R_n(2) = \{1,2\} \cup \{k\}_{k=n}^{\infty}$$

and

$$R_n(k) = \{k\} \quad \text{for} \quad k = 3,4\dots,$$

then the family $\mathcal{R} = \{R_n\}_{n=1}^{\infty}$ is a weakly topological and weakly symmetric relator on X such that the assertion (ii) of Theorem 7.7 does not hold.

Moreover, Jenő Deák has also observed that the assertions (i) and (iii) still remain equivalent under the weaker assumption that $\mathcal{R}$ is only weakly topological.

In this respect it is also worth mentioning that most of the results of our present and former papers remain true if arbitrary relation systems are called relators.

Added in proof (April 16, 1992). Meantime, we learned that various symmetry properties of quasi-uniformities have also been studied by P. Fletcher and W. Hunsaker [Symmetry conditions in terms of open sets, to appear in *Top. and its Appl.*] and J. Deák [A note on weak symmetry properties of quasi-uniformities, to appear in *Studia Sci. Math. Hungar.*].

For some closely related results, see also H. P. A. Künzi, M. Mršević, I. L. Reilly and M. K. Vamanamurthy [Convergence, precompactness and symmetry in quasi-uniform spaces, to appear].

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INSTITUTE OF MATHEMATICS
LAJOS KOSSUTH UNIVERSITY
H-4010 DEBRECEN, PF. 12
HUNGARY

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EXACT NORM ESTIMATES FOR THE SINGULAR SCHRÖDINGER OPERATOR

M. HORVÁTH (Budapest)

In this paper we consider the manifolds $S_k \subset \mathbf{R}^N$, $k = 1, \dots K$ defined by the rules

$$S_k := \{(\xi, \eta) : \xi \in \mathbf{R}^{m_k},\ \eta \in \mathbf{R}^{N-m_k},\ \xi = \varphi_k(\eta)\}$$

where the partial derivatives of the functions $\varphi_k \colon \mathbf{R}^{N-m_k} \to \mathbf{R}^{m_k}$ are uniformly bounded:

$$|\nabla \varphi_k(\eta)| \leqq c, \quad \eta \in \mathbf{R}^{N-m_k}, \quad k = 1, \dots, K.$$

We define

$$S := \bigcup_{k=1}^{K} S_k.$$

Let further $0 \leqq \tau < 2$, $m := \min\limits_k m_k$ and let

(i) $q \in C^\infty(\mathbf{R}^N \setminus S)$ be a real valued function satisfying

(ii) $|D^\alpha q(x)| \leqq c_0[\operatorname{dist}(x, S)]^{-\tau-|\alpha|}, \quad x \in \mathbf{R}^N, 0 \leqq |\alpha| < m + 2 - \tau.$

Introduce the Schrödinger operator

$$\mathbf{L}_\mu f := -\Delta f + qf + \mu f.$$

Recall the definition of the Liouville classes $L_p^t(\mathbf{R}^N)$, $1 \leqq p \leqq \infty$, $t \geqq 0$. L_p^t is the space of all functions

$$F^{-1}((1+|x|^2)^{-\frac{t}{2}} Ff), \qquad f \in L_p(\mathbf{R}^N)$$

where F is the Fourier transform defined on the Schwartz distributions. The corresponding norm is defined by

$$\left\| F^{-1}((1+|x|^2)^{-\frac{t}{2}} Ff) \right\|_{L_p^t(\mathbf{R}^N)} := \|f\|_{L_p(\mathbf{R}^N)}.$$

We shall prove that in case $1 < p < \frac{m}{\tau}$, $\mathbf{L}_\mu$ makes an isomorphism between L_p^2 and L_p. Let ℓ_0 be the integer defined by

$$\frac{m}{\tau + 2\ell_0} \leqq 1 < \frac{m}{\tau + 2(\ell_0 - 1)}.$$

Then we have the following:

THEOREM. a) *Let* $1 < p < \frac{m}{\tau}$, $0 \leqq s \leqq 2\ell_0$, $s < \frac{m}{p} + 2 - \tau$. *Then*

$$\mathbf{L}_\mu^{\frac{s}{2}} : L_p^s \to L_p$$

is an isomorphism of L_p^s *onto* L_p.

b) *Suppose that the conclusion of* a) *holds and consider the bounded linear operators*

$$A_n : L_p \to L_p, \qquad L_p^s \to L_p^s.$$

Suppose further that A_n *is changeable with* $\mathbf{L}_\mu$ *and*

$$A_n f \xrightarrow{L_p} f, \qquad n \to \infty, \quad f \in L_p.$$

Then we have

$$A_n f \xrightarrow{L_p^s} f, \qquad n \to \infty, \quad f \in L_p^s.$$

c) *Let* $1 < p < \frac{m}{\tau}$, $s = \frac{m}{p} + 2 - \tau$. *There exist a potential* q *satisfying conditions* (i), (ii) *and a function*

$$f \in C_0^\infty(\mathbf{R}^N), \qquad f \notin D(\mathbf{L}_\mu^{\frac{s}{2}}).$$

REMARK. a) and b) generalize some results of Nikolskii, Lions, Lisorkin [8] and Joó [6] and Joó [7]. c) shows that the condition $s < \frac{m}{p} + 2 - \tau$ can not be improved in a); this answers a problem raised in Joó [7].

We recall the following result of Marcinkiewicz:

THEOREM A [3]. *Suppose that the function*

$$\lambda : \mathbf{R}^N \to \mathbf{R}$$

satisfies the following property: if

$$1 \leqq k_1 < k_2 < \ldots < k_r \leqq N, \qquad r \leqq N$$

is an arbitrary index sequence, then the derivative

$$D^{\mathbf{k}}\lambda := D_{x_{k_1}} \ldots D_{x_{k_r}} \lambda(x)$$

exists and is continuous at the points $x \in \mathbf{R}^N$, $x_{k_1} \neq 0, \ldots, x_{k_r} \neq 0$; *further*

$$|x^{\mathbf{k}} D^{\mathbf{k}} \lambda| = |x_{k_1} \ldots x_{k_r} D_{x_{k_1}} \ldots D_{x_{k_r}} \lambda(x)| \leqq M, \qquad x \in \mathbf{R}^N.$$

In this case for any $1 < p < \infty$ *there exists a constant* c_p, *independent of* f *and* M *such that*

$$\|F^{-1}(\lambda F f)\|_{L_p} \leqq c_p M \|f\|_{L_p}, \qquad f \in L_p(\mathbf{R}^N). \tag{1}$$

Such a function $\lambda(x)$ is called multiplicator.

Our first statement is analogous to Lemma 1 of Joó [6].

LEMMA 1 (Joó [7]). *Let* $m \geqq 2$, $\alpha \geqq 0$, $1 < p < \frac{m}{\alpha}$. *Then*

$$\|\varrho_k^\alpha f\|_{L_p} \leqq c\|f\|_{L_p^\alpha}, \quad f \in L_p^\alpha(\mathbf{R}^N), \quad \varrho_k(x) := |\xi - \varphi_k(\eta)|^{-1}, \ x = (\xi, \eta)$$

where $c = c(\alpha, m, N, p)$ *is independent of* f.

Remark that in [7] a slightly different notation is used, namely

$$S_k\{(\xi,\eta) \in \mathbf{R}^{m_k} \times \mathbf{R}^{N-m_k} : \eta = \varphi_k(\xi)\}, \quad \dim S_k = m_k,$$
$$m = \min(N - m_k), \qquad \varrho_k(x) = |\eta - \varphi_k(\xi)|^{-1}.$$

Taking these modifications into account, Lemma 1 is transformed into Lemma 1 of [7].

Introduce the notation

$$a(f) \asymp b(f), \qquad f \in \mathcal{F}; \tag{2}$$

this means that there exist constants $0 < c_1 \leqq c_2 < \infty$, independent of f such that

$$c_1 a(f) \leqq b(f) \leqq c_2 a(f), \qquad f \in \mathcal{F}. \tag{3}$$

LEMMA 2. *Let* $1 < p < \infty$, $\mu \geqq 0$ *and* $\ell \in \mathbf{N} = \{1, 2, \dots\}$. *Then*

$$\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} \asymp \|(\mu + 1 - \Delta)f\|_{L_p^{2\ell-2}}, \qquad f \in L_p^{2\ell}, \tag{4}$$

$$\|(\mu+1-\Delta)^\ell f\|_{L_p} \asymp \|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} + \mu^2\|f\|_{L_p^{2\ell-4}} + \dots + \mu^\ell\|f\|_{L_p}, \quad f \in L_p^{2\ell} \tag{5}$$

where the implicit constants do not depend on f *and* μ.

PROOF. First verify (4). The estimate

$$\|(\mu + 1 - \Delta)f\|_{L_p^{2\ell-2}} \leqq c\Big(\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}\Big)$$

being trivial, it remains to show the converse. The estimate

$$\|a\| + \|b\| \leqq \|a + b\| + \|a - b\|$$

is true in any normed space, hence

$$\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} = \|F^{-1}((1 + |x|^2)^\ell Ff)\|_{L_p} + \|F^{-1}((1 + |x|^2)^{\ell-1}\mu Ff)\|_{L_p} \leqq \tag{6}$$
$$\leqq \|F^{-1}((1 + |x|^2)^{\ell-1}(1+|x|^2+\mu)Ff)\|_{L_p} + \|F^{-1}((1 + |x|^2)^{\ell-1}(1+|x|^2-\mu)Ff)\|_{L_p}.$$

Consider the function

$$\lambda(x) = \frac{1 + |x|^2 - \mu}{1 + |x|^2 + \mu} = 1 - 2\mu\frac{1}{1 + |x|^2 + \mu}.$$

It is easy to see that $|\lambda(x)| \leqq 3$, λ is a multiplicator and

$$|x_{i_1}\dots x_{i_k} D_{x_{i_1}}\dots D_{x_{i_k}}\lambda| \leqq c(k)\frac{\mu|x_{i_1}\dots x_{i_k}|^2}{(1+|x|^2+\mu)^{k+1}} \leqq c(k).$$

Hence Theorem A implies that

$$\|F^{-1}((1+|x|^2)^{\ell-1}(1+|x|^2-\mu)Ff)\|_{L_p} \leqq$$
$$\leqq c(p,N)\|F^{-1}((1+|x|^2+\mu)(1+|x|^2)^{\ell-1}Ff)\|_{L_p}, \qquad f\in L_p^{2\ell}.$$

Now we can continue the estimate (6):

$$\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} \leqq c\|F^{-1}((1+|x|^2)^{\ell-1}(1+|x|^2+\mu)Ff)\|_{L_p} =$$
$$= c\|F^{-1}((1+|x|^2)^{\ell-1}F((\mu+1-\Delta)f))\|_{L_p} = c\|(\mu+1-\Delta)f\|_{L_p^{2\ell-2}}$$

which proves (4). The estimate (5) is proved in (4) for $\ell = 1$ and for larger ℓ we use a simple induction: if (5) holds for ℓ, then

$$\|(\mu+1-\Delta)^{\ell+1}f\|_{L_p} \asymp \|(\mu+1-\Delta)f\|_{L_p^{2\ell}} + \mu\|(\mu+1-\Delta)f\|_{L_p^{2\ell-2}} + \dots$$
$$\dots + \mu^{\ell}\|(\mu+1-\Delta)f\|_{L_p} \asymp \|f\|_{L_p^{2\ell+2}} + \mu\|f\|_{L_p^{2\ell}} + \mu^2\|f\|_{L_p^{2\ell-2}} + \dots$$
$$\dots + \mu^{\ell}\|f\|_{L_p^2} + \mu^{\ell+1}\|f\|_{L_p}.$$

Lemma 2 is proved.

LEMMA 3. *Let $m \geqq 2$, $0 \leqq \tau < 2$, $\ell \in \mathbf{N}$, $1 < p < \frac{m}{\tau}$, $2\ell < \frac{m}{p} + 2 - \tau$ and let $\mu_0(N,p,\tau)$ be large enough. Then*

$$\|\mathbf{L}_\mu f\|_{L_p^{2\ell-2}} \asymp \|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}, \tag{7}$$

$$\|\mathbf{L}_\mu^{\ell} f\|_{L_p} \asymp \|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} + \dots + \mu^{\ell}\|f\|_{L_p}, \tag{8}$$

$$\|\mathbf{L}_\mu^{\ell} f - (\mu-\Delta)^{\ell} f\|_{L_p} \leqq \tag{9}$$
$$\leqq c\Big[\|f\|_{L_p^{\tau+2(\ell-1)}} + \mu\|f\|_{L_p^{\tau+2(\ell-2)}} + \dots + \mu^{\ell-2}\|f\|_{L_p^{\tau+2}} + \mu^{\ell-1}\|f\|_{L_p^{\tau}}\Big]$$

hold for $f \in L_p^{2\ell}$, $\mu \geqq \mu_0$, with constants independent of μ.

PROOF. We know that

$$\|qf\|_{L_p^{2\ell-2}} \leqq c \sum_{0\leqq|\beta_1|+|\beta_2|\leqq 2\ell-2} \|D^{\beta_1}q\cdot D^{\beta_2}f\|_{L_p} \leqq \tag{10}$$
$$\leqq c\sum\|D^{\beta_2}f\|_{L_p^{\tau+|\beta_1|}} \leqq c\|f\|_{L_p^{\tau+2\ell-2}}$$

(we have used that $\lambda(x) = \frac{x^{\beta_2}}{(1+|x|^2)^{\frac{|\beta_2|}{2}}}$ is a multiplicator, see [3], 1.5.5). Consequently

$$\|\mathbf{L}_\mu f\|_{L_p^{2\ell-2}} \leqq \|(\mu-\Delta)f\|_{L_p^{2\ell-2}} + \|qf\|_{L_p^{2\ell-2}} \leqq c(\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}).$$

Conversely, we have for $\mu \geqq 1$

$$\|\mathbf{L}_\mu f\|_{L_p^{2\ell-2}} \geqq \|(\mu-\Delta)f\|_{L_p^{2\ell-2}} - \|qf\|_{L_p^{2\ell-2}} \geqq$$
$$\geqq c_1(\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}) - c_2\|f\|_{L_p^{\tau+2\ell-2}}.$$

Consider the function

$$\lambda(x) := \frac{(1+|x|^2)^{\frac{\tau}{2}}}{\varepsilon(1+|x|^2)+c(\varepsilon)}$$

where $\varepsilon > 0$ is arbitrary and $c(\varepsilon) > 0$ is a constant to be specified later. We shall show that λ is a multiplicator; we have to prove the estimates

$$|x_{i_1}\dots x_{i_k} D_{x_{i_1}}\dots D_{x_{i_k}}\lambda| \leqq M, \qquad x \in \mathbf{R}^N.$$

Up to permutations of indices, $x_{i_1}\dots x_{i_k} D_{x_{i_1}}\dots D_{x_{i_k}}\lambda$ is the sum of expressions of type

$$x_{i_1}\dots x_{i_k} D_{x_{i_1}}\dots D_{x_{i_r}}(1+|x|^2)^{\frac{\tau}{2}} \cdot D_{x_{i_{r+1}}}\dots D_{x_{i_k}}[\varepsilon(1+|x|^2)+c(\varepsilon)]^{-1}.$$

Expanding the derivatives we get

$$\left|x_{i_1}\dots x_{i_k} D_{x_{i_1}}\dots D_{x_{i_r}}(1+|x|^2)^{\frac{\tau}{2}} \cdot D_{x_{i_{r+1}}}\dots D_{x_{i_k}}[\varepsilon(1+|x|^2)+c(\varepsilon)]^{-1}\right| =$$

$$= c\frac{|x_{i_1}\dots x_{i_r}|^2}{(1+|x|^2)^r} \cdot \frac{|x_{i_{r+1}}\dots x_{i_k}|^2}{[\varepsilon(1+|x|^2)+c(\varepsilon)]^{k-r}} \cdot \frac{(1+|x|^2)^{\frac{\tau}{2}}}{\varepsilon(1+|x|^2)+c(\varepsilon)} \leqq$$
$$\leqq \begin{cases} c\cdot 1\cdot \varepsilon^{r-k}\cdot\varepsilon^{-1}(1+|x|^2)^{\frac{\tau}{2}-1} \\ c\cdot 1\cdot\varepsilon^{r-k}\cdot\frac{1+|x|^2}{c(\varepsilon)}. \end{cases}$$

Now if $(1+|x|^2)^{\frac{\tau}{2}-1} \leqq \varepsilon^{k+1-r}$ i.e. $1+|x|^2 \geqq \left(\frac{1}{\varepsilon}\right)^{\frac{k+1-r}{1-\frac{\tau}{2}}}$, then $\varepsilon^{r-k}\cdot$ $\cdot\varepsilon^{-1}(1+|x|^2)^{\frac{\tau}{2}-1} \leqq 1$. If $1+|x|^2 \leqq \left(\frac{1}{\varepsilon}\right)^{\frac{k+1-r}{1-\frac{\tau}{2}}}$, then

$$\varepsilon^{r-k}\frac{1+|x|^2}{c(\varepsilon)} \leqq \frac{1}{c(\varepsilon)}\cdot\varepsilon^{r-k-\frac{k+1-r}{1-\frac{\tau}{2}}} \leqq 1$$

if we suppose that

$$c(\varepsilon) \geqq \varepsilon^{r-k-\frac{k+1-r}{1-\frac{\tau}{2}}}$$

for all $0 \leqq r \leqq k \leqq N$ i.e. if

$$c(\varepsilon) \geqq \varepsilon^{-N-\frac{N+1}{1-\frac{\tau}{2}}}.$$

Therefore we have proved by Theorem A the following statement:

Let $0 \leqq \tau < 2$, $1 < p < \infty$ and $\varepsilon > 0$. Then there exists $c(\varepsilon) = c(\varepsilon, \tau, N) > 0$ such that for any $\ell \geqq 1$

$$\|f\|_{L_p^{\tau+2\ell-2}} \leqq \varepsilon\|f\|_{L_p^{2\ell}} + c(\varepsilon)\|f\|_{L_p^{2\ell-2}}, \qquad f \in L_p^{2\ell}(\mathbf{R}^N). \tag{11}$$

Consequently

$$\begin{aligned} \|\mathbf{L}_\mu f\|_{L_p^{2\ell-2}} &\geqq c_1\Big(\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}\Big) - c_2\Big(\varepsilon\|f\|_{L_p^{2\ell}} + c(\varepsilon)\|f\|_{L_p^{2\ell-2}}\Big) = \\ &= (c_1 - c_2\varepsilon)\|f\|_{L_p^{2\ell}} + (c_1\mu - c_2 c(\varepsilon))\|f\|_{L_p^{2\ell-2}}. \end{aligned}$$

Let $\varepsilon > 0$ be chosen so small that $c_1 - c_2\varepsilon \geqq \frac{c_1}{2}$ and then for fixed ε let μ be large enough to ensure $c_1\mu - c_2 c(\varepsilon) \geqq \frac{c_1}{2}\mu$. Then

$$\|\mathbf{L}_\mu f\|_{L_p^{2\ell-2}} \geqq \frac{c_1}{2}\Big(\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}}\Big).$$

Hence (7) is proved. Statement (8) follows from (7) by induction on ℓ as in Lemma 2. Now we show (9). Expanding the brackets in $((\mu - \Delta) + q)^\ell f$ we get that $(\mu - s + q)^\ell f - (\mu - \Delta)^\ell f$ is the sum of expressions of the form

$$c\mu^\alpha D^{\beta_1} q \cdot \ldots \cdot D^{\beta_r} q \cdot D^\beta f$$

where

$$2\alpha + 2r + |\beta_1| + \ldots + |\beta_r| + |\beta| = 2\ell, \qquad r \geqq 1.$$

By Lemma 1 we obtain

$$\begin{aligned} \mu^\alpha\|D^{\beta_1} q \cdot \ldots \cdot D^{\beta_r} q \cdot D^\beta f\|_{L_p} &\leqq c\mu^\alpha \sum_k \|\varrho_k^{|\beta_1|+\ldots+|\beta_r|+r\tau} D^\beta f\|_{L_p} \leqq \\ &\leqq c\mu^\alpha\|D^\beta f\|_{L_p^{|\beta_1|+\ldots+|\beta_r|+r\tau}} = c\mu^\alpha\|D^\beta f\|_{L_p^{2\ell-|\beta|-2\alpha+r(\tau-2)}} \leqq \\ &\leqq c\mu^\alpha\|f\|_{L_p^{2\ell-2\alpha+r(\tau-2)}} \leqq c\mu^\alpha\|f\|_{L_p^{\tau+2(\ell-1-\alpha)}}. \end{aligned}$$

Lemma 3 is proved.

LEMMA 4 [8]. *Let $\ell \in \mathbf{N}$. Then for any $\mu \geqq 1$*

a) $(-\Delta + \mu)^\ell : L_p^{2\ell} \to L_p$ *is an (onto) isomorphism.*

b) $-\Delta + \mu : L_p^{2\ell} \to L_p^{2\ell-2}$ *is an (onto) isomorphism.*

LEMMA 5. *Let $m \geqq 2$, $0 \leqq \tau < 2$, $1 < p < \frac{m}{\tau}$, $2\ell < \frac{m}{p} + 2 - \tau$, $\mu \geqq \mu_0$. Then*

a) $\mathbf{L}_\mu^\ell : L_p^{2\ell} \to L_p$ *is an (onto) isomorphism, and*
b) $\mathbf{L}_\mu : L_p^{2\ell} \to L_p^{2\ell-2}$ *is an (onto) isomorphism.*

PROOF. Since b) follows from a) by induction on ℓ, we show only a). That $\mathbf{L}_\mu^\ell$ is an isomorphic embedding has been proved in Lemma 3. Suppose indirectly that there exists

$$0 \neq g \in L_{p^*}, \qquad \frac{1}{p} + \frac{1}{p^*} = 1$$

satisfying

$$\int\limits_{\mathbf{R}^N} g\mathbf{L}_\mu^\ell f = 0, \qquad f \in L_p^{2\ell}. \tag{12}$$

There exists $h \in L_p$ such that

$$\|h\|_{L_p} = 1, \qquad \int\limits_{\mathbf{R}^N} gh = \|g\|_{L_{p^*}}.$$

By Lemma 4 there exists f such that

$$f \in L_p^{2\ell}, \qquad (\mu - \Delta)^\ell f = h.$$

Consequently, using also (9), we obtain

$$\|g\|_{L_{p^*}} = \Big|\int\limits_{\mathbf{R}^N} gh\Big| = \Big|\int\limits_{\mathbf{R}^N} g(\mu - \Delta)^\ell f\Big| = \Big|\int\limits_{\mathbf{R}^N} g[(\mu - \Delta)^\ell - \mathbf{L}_\mu^\ell]f\Big| \leqq$$

$$\leqq \|g\|_{L_{p^*}} \|[(\mu-\Delta)^\ell - \mathbf{L}_\mu^\ell]f\|_{L_p} \leqq c_0\|g\|_{L_{p^*}}\Big[\|f\|_{L_p^{\tau+2(\ell-1)}} + \ldots + \mu^{\ell-1}\|f\|_{L_p^\tau}\Big].$$

By (11) we get

$$\begin{aligned} \frac{1}{c_0} &\leqq \varepsilon\Big[\|f\|_{L_p^{2\ell}} + \mu\|f\|_{L_p^{2\ell-2}} + \ldots + \mu^{\ell-1}\|f\|_{L_p^2}\Big] + \\ &+c(\varepsilon)\Big[\|f\|_{L_p^{2\ell-2}} + \mu\|f\|_{L_p^{2\ell-4}} + \ldots + \mu^{\ell-1}\|f\|_{L_p}\Big] \leqq \\ &\leqq 2\varepsilon\Big[\|f\|_{L_p^{2\ell}} + \ldots + \mu^\ell\|f\|_{L_p}\Big] \end{aligned} \tag{13}$$

if

$$c(\varepsilon) \leqq \varepsilon\mu. \tag{14}$$

On the other hand (8) implies that

$$1 = \|h\|_{L_p} \geqq c_1\Big[\|f\|_{L_p^{2\ell}} + \ldots + \mu^\ell\|f\|_{L_p}\Big]. \tag{15}$$

Now if $0 < \varepsilon < \frac{c_1}{2c_0}$ and for fixed ε, μ is chosen satisfying (14), the relations (13) and (15) stand in contradiction. This proves Lemma 5.

REMARK. Using (4), resp. (7) and Lemmas 4, 5 we see that

$$\mathbf{L}_\mu^{-1} : L_p \to L_p, \qquad (\mu + 1 - \Delta)^{-1} : L_p \to L_p$$

are continuous and

$$(16) \qquad \|\mathbf{L}_\mu^{-1}\| \leqq \frac{c}{1+\mu}, \qquad \|(\mu+1-\Delta)^{-1}\| \leqq \frac{c}{1+\mu}.$$

Consequently the operators $\mathbf{L}_{\mu_0}$, $1 - \Delta$ with domains

$$D(\mathbf{L}_{\mu_0}) = D(1-\Delta) = L_p^2(\mathbf{R}^N)$$

are positive operators in the Banach space $L_p(\mathbf{R}^N)$ in the sense of Triebel [1, 1.14.1]. We can define the (nonintegral) power Λ^α of a positive operator $\Lambda\colon A \hookleftarrow\!\!\!\rightarrow A$ in the Banach space A as follows. Let $\alpha \in \mathbf{C}$, $n, m_1 \in \mathbf{Z}$, $\sigma \in \mathbf{R}$ be arbitrary satisfying

$$(17) \qquad n \geqq 0, \quad 0 < \sigma < m_1, \quad -n < \operatorname{Re}\alpha \leqq \sigma - n.$$

Then for any

$$a \in (A, D(\Lambda^{m_1}))_{\frac{\sigma}{m_1},1}$$

the integral

$$(18) \qquad \Lambda_\sigma^\alpha a := \frac{\Gamma(m_1)}{\Gamma(\alpha+n)\Gamma(m_1-n-\alpha)} \int_0^\infty t^{\alpha+n-1}\Lambda^{m_1-n}(\Lambda+t)^{-m_1} a\,dt$$

converges in the norm of A and does not depend on n, m_1. Also, the closure of Λ_σ^α does not depend on σ. This closure will be defined as Λ^α, the α-th power of Λ. If α is entire, $\alpha \in \mathbf{Z}$, then Λ^α is the usual α-th iteration of Λ (for $\alpha \geqq 0$) or the inverse of the usual meaning of $\Lambda^{-\alpha}$ (for $\alpha < 0$).

Further we know that if A is a Hilbert space and Λ is a strongly positive selfadjoint operator of A, then

$$\Lambda^\alpha = \int \lambda^\alpha dE_\lambda$$

where dE_λ is the spectral measure associated with Λ. We shall use

THEOREM B (Triebel [1], 1.15.2). a) *Let m_1 be large enough,* $\operatorname{Re}\alpha, \operatorname{Re}\beta <$ $< m_1$, *then*

$$\Lambda^\alpha\Lambda^\beta a = \Lambda^{\alpha+\beta} a, \qquad a \in D(\Lambda^{2m_1}).$$

b) *If* $\operatorname{Re}\alpha < 0$ *then* Λ^α *is continuous and* $\Lambda^{-\alpha}\Lambda^\alpha = 1$.

c) *If* $\operatorname{Re}\alpha \cdot \operatorname{Re}\beta > 0$, *then* $\Lambda^\alpha\Lambda^\beta = \Lambda^{\alpha+\beta}$.

d) *If* $\operatorname{Re}\alpha > 0$ *and* $\mu > 0$ *then* Λ^α *maps isomorphically* $D(\Lambda^\alpha)$ *onto* A, $D(\Lambda^{\alpha+\mu})$ *onto* $D(\Lambda^\mu)$ *(the space* $D(\Lambda^\alpha)$ *is endowed with the norm* $\|a\|_{D(\Lambda^\alpha)} := \|\Lambda^\alpha a\|_A$).

The explicit description of $D(\Lambda^\alpha)$ for nonintegral α was studied by many authors. J.-L. Lions [4], [5], [8] discovered the nice complex interpolation property (20) (below) between the domains $D(\Lambda^\alpha)$ for a large class of operators. We shall apply a variant further developed by Triebel:

THEOREM C (Triebel [1], 1.15.3). *Let* Λ *be a positive operator. Suppose that there exists* $\varepsilon > 0$ *such that for* $-\varepsilon \leqq t \leqq \varepsilon$ *the operator* Λ^{it} *is continuous and*

$$\|\Lambda^{it}\| \leqq c, \qquad |t| \leqq \varepsilon. \tag{19}$$

Then for any complex numbers α, β *with* $0 \leqq \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ *we have*

$$(D(\Lambda^\alpha), D(\Lambda^\beta))_{[\Theta]} = D(\Lambda^{(1-\Theta)\alpha+\Theta\beta}), \qquad 0 < \theta < 1 \tag{20}$$

with equivalent norms.

The following three lemmas are devoted to the verification of (19) for our operators $1 - \Delta$ and $\mathbf{L}_{\mu_0}$. In (17) we can define

$$\alpha = it, \qquad n = 1 = \sigma < 2 = m_1. \tag{21}$$

LEMMA 6. *Consider* $\Lambda = 1 - \Delta$ *and let* $1 < p < \infty$, $0 < T < \infty$. *Then we have*

$$\|(1-\Delta)^{it} f\|_{L_p} \leqq c(p, N, T)\|f\|_{L_p}, \qquad f \in L_p(\mathbf{R}^N), \quad |t| < T \tag{22}$$

and $(1-\Delta)^{it} f$ *can be realized by the corresponding integral (18) with (21).*

PROOF. By the substitution $r = (1 + |x|^2)u$ we get

$$\int_0^\infty r^{it} \frac{1 + |x|^2}{(1 + |x|^2 + r)} dr = (1 + |x|^2)^{it} \int \frac{u^{it}}{(1+u)^2} du = c(t)(1 + |x|^2)^{it}$$

where

$$|c(t)| \leqq \int (1+u)^{-2} du < \infty.$$

The function $(1 + |x|^2)^{it}$ is a multiplicator, since

$$|x_{i_1} \dots x_{i_k} D_{x_{i_1}} \dots D_{x_{i_k}} (1 + |x|^2)^{it}| = c|x_{i_1} \dots x_{i_k}|^2 (1 + |x|^2)^{it-k} \leqq c(N, T).$$

It follows from Theorem A that

$$\left\| F^{-1}\left[\left(\int_0^\infty r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right)Ff\right]\right\|_{L_p} \leqq c\|f\|_{L_p}, \quad f\in L_p. \tag{23}$$

We shall show that for any fixed $\omega<\infty$

$$\int_0^\omega r^{it}(1-\Delta)(1+r-\Delta)^{-2}f\,dr = F^{-1}\left[\left(\int_0^\omega r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right)Ff\right] \tag{24}$$

holds for $f\in L_p$. Take any sequence of partitions $\varphi_n=(r_k)$, $r_k=r_{k,n}$ of the segment $[0,\omega]$ with

$$\sup_k(r_{k+1,n}-r_{k,n})\to 0, \qquad n\to\infty.$$

We shall see that the functions

$$\lambda_n(x):=\int_0^\omega r^{it}\frac{1+|x|^2}{(1+|x|^2+r_k)^2}\Delta r_k-\sum_k r_k^{it}\frac{1+|x|^2}{(1+|x|^2+r_k)^2}\Delta r_k$$

are multiplicators and

$$|x_{i_1}\dots x_{i_s}D_{x_{i_1}}\dots D_{x_{i_s}}\lambda_n(x)|\leqq M_n\to 0, \qquad n\to\infty \tag{25}$$

i.e.

$$\left|x_{i_1}\dots x_{i_s}\left[\int_0^\omega r^{it}D_{x_{i_1}}\dots D_{x_{i_s}}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr-\right.\right.$$

$$\left.\left.-\sum_k r_k^{it}D_{x_{i_1}}\dots D_{x_{i_s}}\frac{1+|x|^2}{(1+|x|^2+r_k)^2}\Delta r_k\right]\right|\leqq M_n\to 0.$$

Let $0<\varepsilon<1$ be fixed. Using the simple estimate

$$\left|x_{i_1}\dots x_{i_s}D_{x_{i_1}}\dots D_{x_{i_s}}\frac{1+|x|^2}{(1+|x|^2+r)^2}\right|\leqq\frac{c(N)}{1+|x|^2+r}\leqq c(N)$$

we see that

$$\left|x_{i_1}\dots x_{i_s}\int_0^\varepsilon r^{it}D_{x_{i_1}}\dots D_{x_{i_s}}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right|<\varepsilon\cdot c(N),$$

$$\left| x_{i_1} \dots x_{i_s} \cdot \sum_{r_k<\varepsilon} r_k^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r_k)^2} \Delta r_k \right| < \varepsilon \cdot c(N).$$

Suppose that

$$\sup_k \Delta r_k < \varepsilon^2.$$

There exist intermediate points $r_{k,1} \in [r_k, r_{k+1}]$ such that

$$\left| x_{i_1} \dots x_{i_s} \left[\int_\varepsilon^\omega r^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r)^2} dr - \right.\right.$$

$$\left.\left. - \sum_{r_k \geqq \varepsilon} r_{k,1}^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r_{k,1})^2} \Delta r_k \right] \right| \leqq c(N)\varepsilon^2.$$

Finally we have

$$\left| x_{i_1} \dots x_{i_s} \sum_{r_k \geqq \varepsilon} \left[r_k^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r_k)^2} - \right.\right.$$

$$\left.\left. - r_{k,1}^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r_{k,1})^2} \right] \cdot \Delta r_k \right| \leqq$$

$$\leqq |x_{i_1} \dots x_{i_s}| \sum_{r_k \geqq \varepsilon} (\Delta r_k)^2 \max_{r \in [r_k, r_{k+1}]} \left| \frac{d}{dr} r^{it} D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r)^2} \right| \leqq$$

$$\leqq |x_{i_1} \dots x_{i_s}| \sum_{r_k \geqq \varepsilon} (\Delta r_k)^2 \frac{|t|}{r_k} \max_{[r_k, r_{k+1}]} \left| D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r)^2} \right| +$$

$$+ 2|x_{i_1} \dots x_{i_s}| \sum_{r_k \geqq \varepsilon} (\Delta r_k)^2 \max_{[r_k, r_{k+1}]} \left| D_{x_{i_1}} \dots D_{x_{i_s}} \frac{1+|x|^2}{(1+|x|^2+r)^3} \right| \leqq$$

$$\leqq c(N)\left(2+\frac{|t|}{\varepsilon}\right) \sum_{r_k \geqq \varepsilon} (\Delta r_k)^2 \leqq c(N,T)\varepsilon \sum_{r_k \geqq \varepsilon} \Delta r_k \leqq c(N,T) \cdot \omega \cdot \varepsilon.$$

The above estimates show that (25) holds indeed. By Theorem A this implies that

(26)

$$F^{-1}\left[\left(\sum_k r_k^{it} \frac{1+|x|^2}{(1+|x|^2+r_k)^2} \Delta r_k\right) Ff\right] \overset{L_p}{\to} F^{-1}\left[\left(\int_0^\omega r^{it} \frac{1+|x|^2}{(1+|x|^2+r)^2} dr\right) Ff\right],$$

$n \to \infty$. Using the identity

$$\frac{1+|x|^2}{(1+|x|^2+r_k)^2} Ff = F[(1-\Delta)(1+r_k-\Delta)^{-2} f]$$

we see that the left hand side of (26) is an approximating sum of the integral

$$\int_0^\omega r^{it}(1-\Delta)(1+r-\Delta)^{-2} f dr$$

and this proves (24). Consider now the $L_p \to L_p$ operators

$$A_\omega f := F^{-1}\left[\left(\int_0^\omega r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right)Ff\right]$$

and

$$A_\infty f := F^{-1}\left[\left(\int_0^\infty r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right)Ff\right].$$

Since the functions

$$\int_0^\omega r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr$$

are multiplicators with estimating constants independent of ω, there exists $c>0$ satisfying

$$\|A_\omega f\|_{L_p},\ \|A_\infty f\|_{L_p} \leqq c\|f\|_{L_p}, \qquad f\in L_p,\ \omega<\infty. \tag{27}$$

On the other hand the functions

$$\int_\omega^\infty r^{it}(1+|x|^2+r)^{-2}dr$$

are multiplicators and

$$\left| x_{i_1}\dots x_{i_s} D_{x_{i_1}}\dots D_{x_{i_s}} \int_\omega^\infty r^{it}(1+|x|^2+r)^{-2}dr\right| \leqq$$

$$\leqq c(N)\int_\omega^\infty (1+r)^2 dr =: M_\omega \to 0, \qquad \omega\to\infty.$$

Consequently for $f \in L_p^2$ we have

$$\|A_\omega f - A_\infty f\|_{L_p} = \left\| F^{-1}\left[\left(\int_\omega^\infty \frac{r^{it}\,dr}{(1+|x|^2+r)^2}\right)(1+|x|^2)Ff\right]\right\|_{L_p} =$$

$$= \left\| F^{-1}\left[\int_\omega^\infty \frac{r^{it}\,dr}{(1+|x|^2+r)^2} F((1-\Delta)f)\right]\right\|_{L_p} \leqq c(p)M_\omega \|(1-\Delta)f\|_{L_p}.$$

This means that if f is taken from a dense subset L_p^2 of L_p, then $A_\omega f$ converge in L_p to $A_\infty f$. Taking into account the uniform boundedness (27) this means that $A_\omega f \xrightarrow{L_p} A_\infty f$ for all $f \in L_p$. But then the integral

$$\int_0^\infty r^{it}(1-\Delta)(1+r-\Delta)^{-2} f\,dr$$

converges in L_p for any $f \in L_p$ and equals

$$F^{-1}\left[\left(\int_0^\infty r^{it}\frac{1+|x|^2}{(1+|x|^2+r)^2}dr\right)Ff\right].$$

Now (23) implies (22), hence Lemma 6 is proved.

REMARK. From the identities

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad \Gamma(z+1) = z\Gamma(z)$$

we can express the coefficient of the integral (18):

$$\frac{1}{\Gamma(1+it)\Gamma(1-it)} = \frac{\operatorname{sh} \pi t}{\pi t}.$$

LEMMA 7. *Let $m \geqq 2$, $0 \leqq \tau < 2$, $1 < p < \frac{m}{\tau}$. Then for large μ_0*

$$\|\mathbf{L}_{\mu_0+r}^{-1} f\|_{L_p^s} \leqq \frac{c}{(1+r)^{1-\frac{s}{2}}}\|f\|_{L_p} \quad f \in L_p, \quad 0 \leqq s \leqq 2.$$

PROOF. The case $s = 0$ is contained in (16). For $f \in L_p$ there exists $f_1 \in L_p^2$ with $\mathbf{L}_{\mu_0+r} f_1 = f$ and by (7) we get

$$\|f\|_{L_p} = \|\mathbf{L}_{\mu_0+r} f_1\|_{L_p} \geqq c\|f_1\|_{L_p^2}$$

which is the case $s = 2$. Hence the operator $\mathbf{L}_{\mu_0+r}^{-1} : L_p \to L_p, \qquad L_p \to L_p^2$ is continuous with norms $\leqq \frac{c}{1+r}$ and $\leqq c$ respectively. Since the complex interpolation is exact, we get that $\mathbf{L}_{\mu_0+r}^{-1} : L_p \to L_p^s$ is also continuous with norm $\leqq \frac{c}{(1+r)^{1-\frac{s}{2}}}$ as we asserted.

LEMMA 8. *Let $m \geqq 2$, $0 \leqq \tau < 2$, $1 < p < \frac{m}{\tau}$ and $0 < T < \infty$. Then for $\mu \geqq \mu_0$ we have*

$$\|\mathbf{L}_\mu^{it} f\|_{L_p} \leqq c(p, N, T, \tau)\|f\|_{L_p}, \quad f \in L_p, \ |t| < T$$

and $\mathbf{L}_\mu^{it} f$ can be expressed by the corresponding integral for all $f \in L_p$.

PROOF. In the integral defining $(\mu - \Delta)^{it} f$ we shall change the operators $\mu - \Delta$ by $\mathbf{L}_\mu$ step by step. First let

$$I_1 := \int_0^\infty r^{it}(\mu - \Delta)(\mu + r - \Delta)^{-1}[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f dr =$$

$$= \int_0^\infty r^{it}[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f dr - \int_0^\infty r^{it+1}(\mu + r - \Delta)^{-1}[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f dr.$$

Since

$$\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1} = (\mu + r - \Delta)^{-1}[(\mu + r - \Delta) - \mathbf{L}_{\mu+r}]\mathbf{L}_{\mu+r}^{-1} = -(\mu + r - \Delta)^{-1} q \mathbf{L}_{\mu+r}^{-1},$$

hence

$$\left\| \int_0^\infty r^{it}[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f dr \right\|_{L_p} \leqq \int_0^\infty \|(\mu + r - \Delta)^{-1} q \mathbf{L}_{\mu+r}^{-1} f\|_{L_p} dr \leqq$$

$$\leqq c \int_0^\infty \frac{1}{1+r} \|q \mathbf{L}_{\mu+r}^{-1} f\|_{L_p} dr \leqq c \int_0^\infty \frac{1}{1+r} \|\mathbf{L}_{\mu+r}^{-1} f\|_{L_p^\tau} dr \leqq$$

$$\leqq c \int_0^\infty (1 + r)^{\frac{\tau}{2} - 2} dr \cdot \|f\|_{L_p} = c\|f\|_{L_p},$$

$$\left\| \int_0^\infty r^{it+1}(\mu + r - \Delta)^{-1}[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f dr \right\|_{L_p} \leqq$$

$$\leqq c \int_0^\infty \frac{r}{1+r} \|[\mathbf{L}_{\mu+r}^{-1} - (\mu + r - \Delta)^{-1}] f\|_{L_p} dr \leqq c\|f\|_{L_p}$$

therefore

$$\|I_1\|_{L_p} \leqq c\|f\|_{L_p}.$$

Let now

$$I_2 := \int_0^\infty r^{it}(\mu-\Delta)[(\mu+r-\Delta)^{-1} - \mathbf{L}_{\mu+r}^{-1}]\mathbf{L}_{\mu+r}^{-1} f\,dr =$$

$$= \int_0^\infty r^{it}(\mu-\Delta)(\mu+r-\Delta)^{-1} q\mathbf{L}_{\mu+r}^{-2} f\,dr =$$

$$= \int_0^\infty r^{it} q\mathbf{L}_{\mu+r}^{-2} f\,dr + \int_0^\infty r^{it+1}(\mu+r-\Delta)^{-1} q\mathbf{L}_{\mu+r}^{-2} f\,dr.$$

Then we have

$$\left\| \int_0^\infty r^{it} q\mathbf{L}_{\mu+r}^{-2} f\,dr \right\|_{L_p} \leqq c \int_0^\infty \|q\mathbf{L}_{\mu+r}^{-2} f\|_{L_p} dr \leqq$$

$$\leqq c \int_0^\infty \|\mathbf{L}_{\mu+r}^{-2} f\|_{L_p^\tau} dr \leqq c \int_0^\infty (1+r)^{\frac{\tau}{2}-1} \|\mathbf{L}_{\mu+r}^{-1} f\|_{L_p} dr \leqq$$

$$\leqq c \int_0^\infty (1+r)^{\frac{\tau}{2}-2} dr \cdot \|f\|_{L_p} = c\|f\|_{L_p}$$

and similarly

$$\left\| \int_0^\infty r^{it+1}(\mu+r-\Delta)^{-1} q\mathbf{L}_{\mu+r}^{-2} f\,dr \right\|_{L_p} \leqq c \int_0^\infty \frac{r}{1+r} \|q\mathbf{L}_{\mu+r}^{-2} f\|_{L_p} dr \leqq c\|f\|_{L_p}$$

i.e.

$$\|I_2\|_{L_p} \leqq c\|f\|_{L_p}.$$

Finally let

$$I_3 := \int_0^\infty r^{it}[\mathbf{L}_\mu - (\mu-\Delta)]\mathbf{L}_{\mu+r}^{-2} f\,dr = \int_0^\infty r^{it} q\mathbf{L}_{\mu+r}^{-2} f\,dr,$$

then

$$\|I_3\|_{L_p} \leqq \int_0^\infty \|q\mathbf{L}_{\mu+r}^{-2} f\|_{L_p} dr \leqq c\|f\|_{L_p}.$$

Our estimations show that

$$\int_0^\infty \|(\mu-\Delta)(\mu+r-\Delta)^{-2}f - \mathbf{L}_\mu \mathbf{L}_{\mu+r}^{-2} f\|_{L_p} dr \leqq c\|f\|_{L_p}.$$

This implies that the integral

$$\int_0^\infty r^{it}\mathbf{L}_\mu \mathbf{L}_{\mu+r}^{-2} f\, dr$$

converges in L_p-norm and

$$\left\| \int_0^\infty r^{it}(\mu-\Delta)(\mu+r-\Delta)^{-2} f\, dr - \int_0^\infty r^{it}\mathbf{L}_\mu \mathbf{L}_{\mu+r}^{-2} f\, dr \right\|_{L_p} \leqq c\|f\|_{L_p}.$$

Since we can prove Lemma 6 for $\mu-\Delta$ instead of $1-\Delta$, Lemma 8 follows from Lemma 6.

Proof of the Theorem. a) Let $1<p<\frac{m}{\tau}$, $\frac{m}{\tau+2\ell_0} \leqq 1 < \frac{m}{\tau+2(\ell_0-1)}$, $0<s\leqq 2\ell_0$, $s<\frac{m}{p}+2-\tau$. Suppose first that there exists $\ell \leqq \ell_0$ with

$$s \leqq 2\ell < \frac{m}{p}+2-\tau.$$

By Lemma 5 we have $D(\mathbf{L}_\mu^\ell) = L_p^{2\ell}$, hence

$$D(\mathbf{L}_\mu^s) = (L_p, D(\mathbf{L}_\mu^\ell))_{[\Theta]} = L_p^s, \quad \Theta = \frac{s}{2\ell}.$$

If there is no integer ℓ with $s \leqq 2\ell < \frac{m}{p}+2-\tau$, then take $1 \leqq \ell \leqq \ell_0 - 1$ satisfying $2\ell < s < 2\ell+2$. Let further

$$1<p_0<\frac{m}{\tau+2(\ell-1)}, \qquad 1<p_1<\frac{m}{\tau+2\ell},$$

then

$$\mathbf{L}_\mu \colon L_{p_0}^{2\ell} \to L_{p_0}^{2\ell-2},\ \ L_{p_1}^{2\ell+2} \to L_{p_1}^{2\ell}$$

are (onto) isomorphisms. Now take $\Theta := \frac{s-2\ell}{2}$. Applying the complex interpolation for $\mathbf{L}_\mu$ and $\mathbf{L}_\mu^{-1}$ we get that

$$\mathbf{L}_\mu \colon L_{p^*}^s \to L_{p^*}^{s-2} \quad \frac{1}{p^*} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$$

is isomorphism. For fixed Θ,

$$1 > \frac{1}{p^*} > \frac{(1-\Theta)(\tau+2(\ell-1))}{m} + \frac{\Theta(\tau+2\ell)}{m} = \frac{s+\tau-2}{m},$$

hence for appropriate p_0 and p_1 we get $p^* = p$. Now $s < \frac{m}{p} + 2 - \tau$ implies $s-2 < 2\ell < \frac{m}{p} + 2 - \tau$ therefore $L_p^{s-2} = D(\mathbf{L}_\mu^{\frac{s}{2}-1})$. Since $\mathbf{L}_\mu \colon D(\mathbf{L}_\mu^{\frac{s}{2}}) \to D(\mathbf{L}_\mu^{\frac{s}{2}-1})$ is isomorphism, it follows that in general $D(\mathbf{L}_\mu^{\frac{s}{2}}) = L_p^s$ which proves part a).

b) This is an easy consequence of a). Indeed, let $f \in L_p^s$, then

$$\|A_n f - f\|_{L_p^s} \leqq c\|\mathbf{L}_\mu^{\frac{s}{2}}(A_n f - f)\|_{L_p} = c\left\|A_n(\mathbf{L}_\mu^{\frac{s}{2}} f) - \mathbf{L}_\mu^{\frac{s}{2}} f\right\|_{L_p} \to 0$$

since $\mathbf{L}_\mu^{\frac{s}{2}} f \in L_p$.

c) Let $s = \frac{m}{p} + 2 - \tau$, $1 < p < \frac{m}{\tau}$ and suppose indirectly that $C_0^\infty \subset \subset D(\mathbf{L}_\mu^{\frac{s}{2}})$. Since $\mathbf{L}_\mu \colon D(\mathbf{L}_\mu^{\frac{s}{2}}) \to D(\mathbf{L}_\mu^{\frac{s}{2}-1})$ is isomorphism and $D(\mathbf{L}_\mu^{\frac{s}{2}-1}) = L_p^{s-2}$ by a), we get $\mathbf{L}_\mu(C_0^\infty) \subset L_p^{s-2}$. Obviously $(-\Delta + \mu)(C_0^\infty) \subset C_0^\infty \subset L_p^{s-2}$, hence $q\varphi \in L_p^{s-2}$ if $\varphi \in C_0^\infty$. Now suppose that the singularity surface is a hyperplane as follows:

$$S = \{\xi, \eta) : \xi = 0\}$$

and suppose that

$$q(x) = q(\xi), \qquad x = (\xi, \eta).$$

Then we have dist $(x, S) = |\xi|$; hence the condition on the singularity of q is

$$|D^\alpha q(\xi)| \leqq c|\xi|^{-\tau-|\alpha|}, \qquad 0 \leqq |\alpha| \leqq 2\ell_0 - 2. \tag{28}$$

Consider the function

$$\varphi(x) = \varphi_1(\xi)\varphi_2(\eta), \qquad \varphi_1 \in C_0^\infty(\mathbf{R}^m), \quad \varphi_2 \in C_0^\infty(\mathbf{R}^{N-m}).$$

The condition $q\varphi \in L_p^{s-2}$ can be rewritten as

$$F^{-1}\Big((1+|x|^2)^{\frac{s}{2}-1} F(q\varphi_1\varphi_2)\Big) \in L_p(\mathbf{R}^N).$$

It is easy to check that

$$\lambda(x) := \left(\frac{1+|\xi|^2}{1+|x|^2}\right)^{\frac{s}{2}-1}$$

is a multiplicator, consequently

$$F^{-1}((1+|\xi|^2)^{\frac{s}{2}-1} F(q\varphi_1\varphi_2)) \in L^p(\mathbf{R}^N).$$

Denote F_m the Fourier transform on $\mathbf{R}^m$. Then

$$F(q\varphi_1\varphi_2)(x) = (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbf{R}^N} e^{-i\langle x,x'\rangle} q(\xi')\varphi_1(\xi')\varphi_2(\eta')d\eta' d\xi' =$$

$$= (2\pi)^{-\frac{m}{2}} \int\limits_{\mathbf{R}^m} e^{-i\langle \xi,\xi'\rangle} q(\xi')\varphi_1(\xi')d\xi' \cdot (2\pi)^{\frac{m-N}{2}} \int\limits_{\mathbf{R}^{N-m}} e^{-i\langle \eta,\eta'\rangle}\varphi_2(\eta')d\eta' =$$

$$= F_m(q\varphi_1)(\xi)\cdot F_{N-m}(\varphi_2)(\eta),$$

consequently

$$F^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}-1}F(q\varphi_1\varphi_2)\Big) = F_m^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}-1}F_m(q\varphi_1)\Big)(\xi)\cdot\varphi_2(\eta) \in L_p(\mathbf{R}^N)$$

and then

$$F_m^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}-1}F_m(q\varphi_1)\Big) \in L_p(\mathbf{R}^m)$$

i.e.

$$q\varphi_1 \in L_p^{s-2}(\mathbf{R}^m) \quad \text{for} \quad \varphi_1 \in C_0^\infty(\mathbf{R}^m).$$

We use the following embedding theorem of Triebel [1]:

$$L_p^s \subset L_{p^*}^t \quad \text{if} \quad 1 < p \leqq p^* < \infty, \quad s - \frac{N}{p} \geqq t - \frac{N}{p^*}.$$

In particular this gives

$$L_p^{s-2}(\mathbf{R}^m) \subset L_{\frac{m}{\tau}}(\mathbf{R}^m)$$

and then

$$q\varphi_1 \in L_{\frac{m}{\tau}}(\mathbf{R}^m).$$

If $\varphi_1 = 1$ in a sufficiently large ball with centre at the origin, we get

$$q \in L_{\frac{m}{\tau}}^{\mathrm{loc}}(\mathbf{R}^m). \tag{29}$$

But we can define $q(\xi) := |\xi|^{-\tau}$; this satisfies (28) but (29) does not hold, since

$$\int\limits_{S(0,R)} |q(\xi)|^{\frac{m}{\tau}} d\xi = \int\limits_0^R c r^{m-1}(r^{-\tau})^{\frac{m}{\tau}} dr = c\int\limits_0^R \frac{1}{r}dr = \infty.$$

The contradiction proves c).

The proof of the Theorem is complete.

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TECHNICAL UNIVERSITY
FACULTY OF ELECTRICAL ENGINEERING
DEPARTMENT OF MATHEMATICS
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STOCZEK U. 2–4
HUNGARY

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ALMOST SURE CONVERGENCE OF SET-VALUED MARTINGALES AND SUBMARTINGALES

B. K. DAM (Budapest)

1. Introduction. The almost sure convergence of a class of set-valued martingales in the Hausdorff metric has been discussed by F. Hiai and H. Umegaki [4], [5]. Later on, the authors of [1] have treated the almost sure convergence of a larger class than that of set-valued martingales, the so called set-valued amarts. In this paper we present the Mosco-type convergence (see [7], [6]) for martingales and submartingales taking values in $\mathcal{P}_c(\mathcal{X})$, where $\mathcal{X}$ is a reflexive separable Banach space and $\mathcal{P}_c(\mathcal{X})$ is the class of all convex, closed, bounded and non-empty subsets of $\mathcal{X}$. N. Papageorgiou [8] has given some sufficient conditions for convergence of set-valued martingales and submartingales in Mosco sense. However, his conditions are rather strong. We shall show here that every set-valued martingale (or submartingale), which is "L^1-bounded", converges almost surely in Mosco sense. This result is a generalization of Doob's convergence theorem (for real-valued martingales) and of that of Chatterji (for vector-valued martingales).

2. Preliminaries. Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{X}$ a real separable Banach space with the dual space $\mathcal{X}^*$. For each $X \subset \mathcal{X}$, $\mathrm{cl}\, X$, $\overline{\mathrm{co}}\, X$ will denote the norm-closure and the closed, convex hull of X, respectively. Let $\mathcal{P}(\mathcal{X})$ (resp. $\mathcal{P}_c(\mathcal{X})$) denote the family of all nonempty, closed, bounded (resp. non-empty, closed, convex, bounded) subsets of $\mathcal{X}$.

The convergence in the Mosco sense is the following (see [7]). Let $\{X_n\}_{n \geqq 1}$ be a sequence in $\mathcal{P}_c(\mathcal{X})$. Denote

$$s\text{-}\liminf X_n = \Big\{ x \in \mathcal{X} : \exists x_n \in X_n,\ \lim_{n\to\infty} \|x_n - x\| = 0 \Big\},$$

$$w\text{-}\limsup X_n = \Big\{ x \in \mathcal{X} : \exists x_k \in X_{n_k}, x_k \xrightarrow{w} x \text{ (converges weakly)} \Big\}.$$

We say that X_n converges to X in the Mosco sense and write $X_n \to X$ if

$$s\text{-}\liminf X_n = w\text{-}\limsup X_n = X.$$

For every $X \subset \mathcal{X}$, denote $\sigma_X(x^*) = \sup\limits_{x^* \in \mathcal{X}} \langle x, x^* \rangle$, $\|X\| = \sup\limits_{x \in X} \|x\|$.

The set-valued function $F \colon \Omega \to \mathcal{P}(\mathcal{X})$ is called to be $\mathcal{A}$-measurable if for every open subset 0 of $\mathcal{X}$ we have

$$F^{-1}(0) = \{\omega \in \Omega \colon F(\omega) \cap 0 \neq \emptyset\} \in \mathcal{A}$$

(see [2]).

We denote by $\mathcal{M}[\Omega, \mathcal{A}, P; \mathcal{X}] = \mathcal{M}[\Omega; \mathcal{X}]$ the family of all measurable set-valued functions $F: \Omega \to \mathcal{P}(\mathcal{X})$ and we set

$$\mathcal{L}^1[\Omega; \mathcal{X}] = \{F \in \mathcal{M}[\Omega; \mathcal{X}]: \int_\Omega \|F(\omega)\| dP < +\infty\},$$
$$\mathcal{L}_c^1[\Omega; \mathcal{X}] = \{F \in \mathcal{L}^1[\Omega; \mathcal{X}]: F(\omega) \in \mathcal{P}_c(\mathcal{X}) \text{ a.s.}\}.$$

We denote by $L^1(\Omega, \mathcal{S}, P; \mathcal{X}) = L^1(\Omega; \mathcal{X})$ the Banach space of all measurable functions $f: \Omega \to \mathcal{X}$ such that the norm

$$\|f\|_1 = \int_\Omega \|f(\omega)\| dP$$

is finite.

For $F \in \mathcal{M}[\Omega; \mathcal{X}]$, let

$$S_F^1 = \{f \in L^1(\Omega; \mathcal{X}): f(\omega) \in F(\omega) \text{ a.s.}\}.$$

Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and besides S_F^1 defined on $(\Omega, \mathcal{A}, P)$, we take on $(\Omega, \mathcal{B}, P)$ the family

$$S_F^1(\mathcal{B}) = \{f \in L^1(\Omega, \mathcal{B}, P; \mathcal{X}): f(\omega) \in F(\omega) \text{ a.s.}\}.$$

Recall that for $f \in L^1(\Omega; \mathcal{X})$ the conditional expectation of f relative to $\mathcal{B}$ is given as a function $E(f \mid \mathcal{B}) \in L^1(\Omega, \mathcal{B}, P; \mathcal{X})$ such that

$$\int_B E(f \mid \mathcal{B}) dP = \int_B f dP$$

for all $B \in \mathcal{B}$. If $F \in \mathcal{M}[\Omega; \mathcal{X}]$ with $S_F^1 \neq \emptyset$, then by virtue of [4, Theorem 5-1] there exists a unique (in the a.s. sense) $\mathcal{B}$-measurable function $E[F \mid \mathcal{B}]$ satisfying

$$S^1_{E[F|\mathcal{B}]}(\mathcal{B}) = \mathrm{cl}\{E(f \mid \mathcal{B}): f \in S_F^1\},$$

where on the right hand side we have taken the closure in the norm topology of $L^1(\Omega; \mathcal{X})$.

We call $E[F \mid \mathcal{B}]$ the (set-valued) conditional expectation of F relative to $\mathcal{B}$.

Let $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ be an increasing sequence of sub-σ-fields of $\mathcal{A}$ and let $F_1, F_2, \dots$ be a sequence of set-valued functions in $\mathcal{L}^1[\Omega; \mathcal{X}]$. We say that $(F_n, \mathcal{A}_n)_{n \geqq 1}$ is a martingale (submartingale) if $F_n = E[F_{n+1} \mid \mathcal{A}_n]$ a.s. ($F_n \subset E[F_{n+1} | \mathcal{A}_n)$ a.s.) for all $n \geqq 1$.

3. Results. We show here that every submartingale $(F_n, \mathcal{A}_n)_{n \geqq 1}$ belonging to $\mathcal{L}^1_c[\Omega; \mathcal{X}]$, where $\mathcal{X}$ is a separable and reflexive Banach space and

$$\sup_{n \geqq 1} \int_\Omega \|F_n(\omega)\| dP < +\infty$$

is satisfied, converges almost surely to some set-valued function belonging to $\mathcal{L}^1_c[\Omega; \mathcal{X}]$. We begin with the case of regular martingales.

THEOREM 3.1. *Let $\mathcal{X}$ be a reflexive and separable Banach space and let F be an element of $\mathcal{L}^1_c[\Omega, \mathcal{A}, P; \mathcal{X}]$. Let $(\mathcal{A}_n)_{n \geqq 1}$ be a sequence of sub-σ-fields of $\mathcal{A}$ such that $\mathcal{A} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$ and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$. Then the martingale $F_n = E[F \mid \mathcal{A}_n]$ converges a.s. to F in the Mosco sense.*

PROOF. Since $\mathcal{X}$ is a Banach space and F is bounded and integrable, we can use the following representation of F due to Castaing (see [2]):

$$F(\omega) = \mathrm{cl}\{f_k(\omega)\}, \qquad \forall \omega \in \Omega, \ \forall k \geqq 1$$

where

$$\int_\Omega \|f_k(\omega)\| dP < +\infty.$$

By virtue of Theorem 5.1 of [4] for every $k \geqq 1$ and $n \geqq 1$ we have $E[f_k|\mathcal{A}_n] \in$ $\in F_n$ a.s. Fix $k \geqq 1$. Then $\{E(f_k \mid \mathcal{A}_n)\}_{n \geqq 1}$ is a vector-valued regular martingale. Consequently (see [3, Theorem 2])

$$\lim_{n \to \infty} \|f_k(\omega) - E(f_k|\mathcal{A}_n)\| = 0 \quad \text{a.s.}$$

This means that $f_k(\omega) \in s\text{-}\liminf F_n(\omega)$ a.s. for every $k \geqq 1$. Therefore, $F(\omega) \subset s\text{-}\liminf F_n(\omega)$ a.s.

Now let $(x_j^*)_{j \geqq 1}$ be a dense sequence in $\mathcal{X}^*$, $\|x_j^*\| \leqq 1$ ($\mathcal{X}^*$ is separable since $\mathcal{X}$ is reflexive and separable.) Then on the basis of the paper by Valadier [9] we have for every $j \geqq 1$

$$E(\sigma_F(x_j^*) \mid \mathcal{A}_n) = \sigma_{E[F|\mathcal{A}_n]}(x_j^*) = \sigma_{F_n}(x_j^*) \qquad \text{a.s.}$$

Thus $\{\sigma_{F_n}(x_j^*), \mathcal{A}_n\}_{n \geqq 1}$ is a real regular martingale. Hence

$$\lim_{n \to \infty} \sigma_{F_n(\omega)}(x_j^*) = \sigma_{F(u)}(x_j^*) \qquad \text{a.s.}$$

This implies that there exists a set $N \in \mathcal{A}$ with $P(N) = 0$ such that for arbitrary $\omega \in \Omega \setminus N$ and for every $j \geqq 1$ we have

$$\lim_{n \to \infty} \sigma_{F_n(\omega)}(x_j^*) = \sigma_{F(\omega)}(x_j^*).$$

Let us suppose that $x \in w\text{-}\limsup F_n(\omega)$ for $\omega \notin N$. Then there exists a sequence $x_k \in F_{n_k}(\omega)$ such that $x_k \xrightarrow{w} x$ as $k \to \infty$. Therefore we have

$$(3.1) \qquad \langle x, x_j^* \rangle = \lim_{k\to\infty} \langle x_k, x_j^* \rangle \leqq \lim_{k\to\infty} \sigma_{F_{n_k(\omega)}}(x_j^*) = \sigma_{F(\omega)}(x_j^*).$$

Note that $F(\omega)$ is weakly compact since $\mathcal{X}$ is reflexive and $F(\omega)$ is closed and bounded. By the properties of the support function, for each $j \geqq 1$ there exists an $x_j \in F(\omega)$ such that

$$\langle x_j, x_j^* \rangle = \sigma_{F(\omega)}(x_j^*) \qquad \Big(= \sup_{x\in F(\omega)} \langle x, x_j^* \rangle\Big)$$

holds. So we can write (3.1) in the form

$$(3.2) \qquad \langle x, x_j^* \rangle \leqq \langle x_j, x_j^* \rangle.$$

Again, we deduce the existence of a $z \in F(\omega)$ such that $x_{j_i} \xrightarrow{w} z$ as $i \to \infty$, since $x_j \in F(\omega)$ for all $j \geqq 1$ and $F(\omega)$ is weakly compact. Without loss of generality we can suppose that for $x^* \in \mathcal{X}^*$, $\|x^*\| \leqq 1$ we have

$$\lim_{j\to\infty} \|x_j^* - x^*\| = 0$$

because in general we can pick out a subsequence of $(x_j^*)_{j\geqq 1}$ which converges to x^*. Thus inequality (3.2) implies that

$$\langle x, x^* \rangle \leqq \langle z, x^* \rangle \leqq \sup_{y\in F(\omega)} \langle y, x^* \rangle = \sigma_{F(\omega)}(x^*)$$

since

$$\lim_{j\to\infty} \langle x, x_j^* \rangle = \langle x, x^* \rangle \quad \text{and} \quad \lim_{i\to\infty} \left\langle x_{j_i}, x_{j_i}^* \right\rangle = \langle x, x^* \rangle.$$

Finally, note that $F(\omega)$ is convex, closed and bounded, so by the separation theorem (see [10]) we deduce $x \in F(\omega)$. Consequently, $w\text{-}\limsup F_n(\omega) \subset \subset F(\omega)$ a.s. This completes the proof.

Now we present a general convergence theorem for L_1-bounded set-valued submartingales.

THEOREM 3.2. *Let $\mathcal{X}$ be a reflexive, separable Banach space and let $(F_n, \mathcal{A}_n)_{n\geqq 1}$ be a multivalued submartingale in $\mathcal{L}_c^1[\Omega; \mathcal{X}]$ such that*

$$\sup_{n\geqq 1} \int_\Omega \|F_n(\omega)\| dP < +\infty$$

holds. Then there exists a random element $F \in \mathcal{L}_c^1[\Omega, \mathcal{A}, P; \mathcal{X}]$ such that $F_n \to F$ a.s. where $\mathcal{A} = \sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{A}_n\Big)$.

For the proof of this theorem we establish the following

LEMMA 3.3. *Let $\mathcal{X}$ be a Banach space whose dual $\mathcal{X}^*$ is separable. Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Suppose that*

$$S_F^1 = \{f \in L^1(\Omega, \mathcal{A}, P; \mathcal{X}) \colon f(\omega) \in F(\omega) \text{ a.s.}\}, \quad F \in \mathcal{L}_c^1[\Omega, \mathcal{A}, P; \mathcal{X}]$$

is a weakly compact subset in $L_1(\Omega, \mathcal{A}, P; \mathcal{X})$, i.e. S_F^1 is compact with respect to the topology $\sigma(L_1(\mathcal{X}), L_\infty(\mathcal{X}^*))$. Then the set

$$G = \{E(f \mid \mathcal{B}) \colon f \in S_F^1\}$$

is a closed subset of $L_1(\Omega, \mathcal{B}, P; \mathcal{X})$ with respect to the norm-topology in $L_1(\Omega, \mathcal{B}, P; \mathcal{X})$.

PROOF. Let $g_n \in G$, $n \geqq 1$ and suppose that

$$\lim_{n\to\infty} \int_\Omega \|g(\omega) - g_n(\omega)\| dP = 0.$$

Since $g_n \in G$ we have $g_n = E(f_n \mid \mathcal{B})$ for some $f_n \in S_F^1$. But S_F^1 is compact with respect to the topology $\sigma(L_1(\mathcal{X}), L_\infty(\mathcal{X}^*))$. Therefore, there exists an $f \in S_F^1$ and a subsequence (f_{n_k}) of (f_n) such that for every $A \in \mathcal{B}$ and $x^* \in \mathcal{X}^*$ we have

$$\lim_{k\to\infty} \int_\Omega \langle f_{n_k}(\omega), \chi_A(\omega) x^* \rangle \, dP = \int_\Omega \langle f(\omega), \chi_A(\omega) x^* \rangle \, dP$$

where

$$\chi_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in \Omega \setminus A. \end{cases}$$

This means that

$$\lim_{k\to\infty} \left\langle \int_A f_{n_k}(\omega) dP, x^* \right\rangle = \left\langle \int_A f(\omega) dP, x^* \right\rangle \tag{3.3}$$

holds. However,

$$\int_A f_{n_k}(\omega) dP = \int_A g_{n_k}(\omega) dP$$

and

$$\lim_{k\to\infty} \left\| \int_A g_{n_k}(\omega) dP - \int_A g(\omega) dP \right\| = 0.$$

Consequently, we have

$$\lim_{k\to\infty} \left\langle \int_A f_{n_k}(\omega) dP, x^* \right\rangle = \left\langle \int_A g(\omega) dP, x^* \right\rangle. \tag{3.4}$$

Combining (3.3) and (3.4) we deduce

$$\left\langle \int_A f(\omega)dP, x^* \right\rangle = \left\langle \int_A g(\omega)dP, x^* \right\rangle \tag{3.5}$$

for every $x^* \in \mathcal{X}^*$ and $A \in \mathcal{B}$. This means that

$$g(\omega) = E(f \mid \mathcal{B})(\omega) \quad \text{a.s.}$$

and so $g \in G$. The proof is thus complete.

PROOF OF THEOREM 3.2. Since $(F_n, \mathcal{A}_n)_{n \geqq 1}$ is a submartingale, for every $n \geqq 1$ we have

$$S^1_{F_n}(\mathcal{A}_n) \subset S^1_{E[F_{n+1}|\mathcal{A}_n]}(\mathcal{A}_n).$$

By Theorem 5.1 of [4] for arbitrary $n \geqq 1$ we get

$$S^1_{E[F_{n+1}|\mathcal{A}_n]}(\mathcal{A}_n) = \mathrm{cl}\{E(f \mid \mathcal{A}_n) : f \in S^1_{F_{n+1}}\}. \tag{3.6}$$

Note that $\mathcal{X}$ is reflexive. Thus by Theorem 3.7 (iii) of [4], $S^1_{F_{n+1}}$ is weakly compact for every $n \geqq 1$ and, consequently, by Lemma 3.3 $\{E(f \mid \mathcal{A}_n) : f \in \in S^1_{F_{n+1}}\}$ is closed. On the basis of this, (3.6) can be written in the form

$$S^1_{E[F_{n+1}|\mathcal{A}_n]}(\mathcal{A}_n) = \{E(f \mid \mathcal{A}_n) : f \in S^1_{F_{n+1}}\} \tag{3.7}$$

for every $n \geqq 1$.

First assume that $\sup\limits_{n \geqq 1} \|F_n(\omega)\| \in L_1(\mathbf{R})$. Let

$$F(\omega) = w\text{-}\limsup F_n(\omega), \quad G(\omega) = s\text{-}\liminf F_n(\omega).$$

By Theorem 2.2 of [6], $F(\omega)$ and $G(\omega)$ are measurable (we show below that $G(\omega) \neq \emptyset$ a.s.). Using the Castaing representation of F_n (see [2]) we have for arbitrary $n \geqq 1$

$$F_n(\omega) = \mathrm{cl}\{f_i^{(n)}(\omega) \colon i = 1, 2, \dots ; f_i^{(n)}(\omega) \in S^1_{F_n}\}. \tag{3.8}$$

Let $n, i \geqq 1$ be fixed. By (3.7) there exists a sequence $f^{(n)}_{(i)_j} \in S^1_{F_{n+j}}$, $j \geqq 1$ such that $\{f^{(n)}_{(i)_j}, \mathcal{A}_{n+j}\}_{j \geqq 0}$ is a vector-valued martingale. But

$$\sup_{j \geqq 0} \|f^{(n)}_{(i)_j}(\omega)\| \leqq \sup_{j \geqq 0} \|F_{n+j}(\omega)\| \in L_1(\mathbf{R})$$

and $\mathcal{X}$ is a reflexive Banach space (so $\mathcal{X}$ has the Radon–Nikodym property). Therefore, there exists $g_i^{(n)} \in L_1(\Omega, \mathcal{A}, P; \mathcal{X})$ such that

$$f_{(i)_j}^{(n)}(\omega) = E(g_i^{(n)} \mid \mathcal{A}_{n+j})(\omega) \quad \text{a.s.} \tag{3.9}$$

holds. By the convergence theorem for regular martingales (see [3, Theorem 1]) we have

$$\lim_{j\to\infty} \|f_{(i)_j}^{(n)}(\omega) - g_i^{(n)}(\omega)\| = 0 \quad \text{a.s.}$$

At the same time

$$f_{(i)_j}^{(n)}(\omega) \in F_{n+j}(\omega) \quad \text{a.s.}$$

Therefore

$$g_i^{(n)}(\omega) \in G(u) = s\text{-}\liminf F_n(\omega) \quad \text{a.s.}$$

This means that $G(\omega) \neq \emptyset$ a.s. Moreover, $g_i^{(n)} \in S_G^1$ since

$$\int_\Omega \|g_i^{(n)}(\omega)\| dP < +\infty.$$

Now suppose that

$$x \in F(\omega) = w\text{-}\limsup F_n(\omega).$$

Then by (3.8), there exists a sequence $f_{(i_k)}^{(n_k)}(\omega)$ such that $f_{(i_k)}^{(n_k)}(\omega) \xrightarrow{w} X$. But (3.9) implies that $f_{i_k}^{(n_k)} = E(g_{i_k}^{(n_k)} \mid \mathcal{A}_{n_k})$ for $g_{i_k}^{(n_k)} \in S_G^1$. So, $f_{i_k}^{(n_k)} \in$ $\in E[G \mid \mathcal{A}_{n_k}]$. This means that

$$x \in w\text{-}\limsup E[G \mid \mathcal{A}_{n_k}].$$

But $(E[G \mid \mathcal{A}_{n_k}])_{k\geq 1}$ is a regular martingale, whence by Theorem 3.1 we get

$$w\text{-}\limsup E[G \mid \mathcal{A}_n] = G = s\text{-}\liminf E[G \mid \mathcal{A}_n] \quad \text{a.s.}$$

Therefore,

$$x \in G(\omega) = s\text{-}\liminf F_n(\omega).$$

In other words,

$$w\text{-}\limsup F_n(\omega) \subset s\text{-}\liminf F_n(\omega) \quad \text{a.s.}$$

In the general case, when $\sup_{n\geq 1} \|F_n(\omega)\| \notin L_1(\mathbf{R})$ by using the maximal lemma (see [1, Lemma 2.2]) we can reduce the problem to the above one as follows.

Fix a positive constant $a > 0$ and define a stopping time σ in the following way:

$$\sigma(\omega) = \begin{cases} \infty, & \text{if } \sup \|F_n(\omega)\| \leqq a \\ \inf(n\colon \|F_n(\omega)\| > a), & \text{otherwise.} \end{cases}$$

Consider the sequence $H_n(\omega) = F_{n\wedge\sigma(\omega)}(\omega)$. It is easy to see that $(H_n, \mathcal{A}_{n\wedge\sigma})_{n\geqq 1}$ is also a submartingale. Let

$$H(\omega) = \sup_{n\geqq 1} \|H_n(\omega)\|.$$

We note that if we put

$$\overline{F}_\sigma(\omega) = \begin{cases} a, & \text{if } \sigma = +\infty \\ F_\sigma, & \text{if } \sigma < +\infty \end{cases}$$

then $\|H_n(\omega)\| \leqq \|\overline{F}_\sigma(\omega)\|$, $\{\sigma = +\infty\}$ and $\|H_n(\omega)\| \uparrow \|\overline{F}_\sigma(\omega)\|$ on the set $\{\sigma < +\infty\}$. Using Fatou's lemma we see that

$$\int_\Omega \|H(\omega)\| dP = \int_{\{\sigma=+\infty\}} \|H(\omega)\| dP + \int_{\{\sigma<+\infty\}} \|H(\omega)\| dP \leqq$$

$$\leqq a + \int_{\{\sigma<+\infty\}} \|H(\omega)\| dP \leqq a + \liminf \int_{\{\sigma<+\infty\}} \|F_{n\wedge\sigma}\| dP \leqq a + \sup_{n\geqq 1} \int_\Omega \|F_n\| dP < +\infty.$$

By the maximal lemma we have (see [1])

$$P(\sup \|F_n(\omega)\| \geqq a) \leqq \frac{1}{a} \sup_{n\geqq 1} \int_\Omega \|f_n(\omega)\| dP.$$

This means that $(F_{n\wedge\sigma})$ coincides with F_n except on a set of measure arbitrary small if a is large enough. Therefore, we can assume without loss of generality that (F_n) itself has the property that

$$\sup_{n\geqq 1} \|F_n(\omega)\| \in L_1(\mathbf{R}).$$

This completes the proof.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF PROBABILITY
BUDAPEST, MÚZEUM KRT. 6–8
H-1088
HUNGARY

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CARTESIAN PRODUCTS OF FRÉCHET TOPOLOGICAL GROUPS AND FUNCTION SPACES

V. I. MALYHIN and D. B. SHAKHMATOV (Moscow)

1. Introduction

It is well-known that there exists a Tychonoff Fréchet space X whose square $X \times X$ is not Fréchet and has even uncountable tightness ([2], [3, Proposition 3.14]; see also Example 1 in [9]). At the same time there is a lot of examples serving to show that in topological groups many convergence properties tend to improvements. For example, any Fréchet topological group is strongly Fréchet [23, Theorem 4] whereas there are Fréchet spaces which are not strongly Fréchet [3], [21]. A bisequential topological group is metrizable [1] while there are even non-first-countable bisequential topological spaces ([21, Example 10.4], [13, Theorem 5]). A topological group satisfying the weak first axiom of countability [30] is metrizable ([22], [23, Theorem 3]) but for general topological spaces this does not hold [3], [30]. So one may hope that some convergence properties which are not multiplicative in the class of all topological spaces, would become multiplicative in the class of topological groups. However, it is clear now that this is not the case. Indeed, V. I. Malyhin constructed under CH a hereditarily separable (countably compact, hereditarily normal) topological group G such that $G \times G$ has uncountable tightness (and is neither countably compact nor normal) [15]. He also constructed via forcing a countable Fréchet group the square of which is not Fréchet [18].

It is worth calling the reader's attention to the difference between cases of distinct groups G_1 and G_2 and of a single group G. For general topological spaces this difference does not matter. Indeed, having a pair $\langle X_1, X_2\rangle$ of topological spaces with any of the considered convergence properties and forming their topological sum $X = X_1 \oplus X_2$, one obtains the pair $\langle X, X\rangle$ with the same properties. Further, if $X_1 \times X_2$ does not have some convergence property, the same remains valid for $X \times X$. For topological groups there is no analogue of topological sum, and here is the main difficulty in constructing pairs of the form $\langle G, G\rangle$. Moreover, the following problem of A. V. Arhangel'skii is still open. Let G_1 and G_2 be Fréchet (sequential, of countable tightness) topological groups. Is there a Fréchet (sequential, with countable tightness) topological group G which contains G_1 and G_2 as (closed, normal) subgroups or even as subspaces?

Now let us turn to function spaces. We denote by $C_p(X)$ the space of all real-valued continuous functions defined on a Tychonoff space X equipped with the topology of pointwise convergence. Spaces X and Y such that both $C_p(X)$ and $C_p(Y)$ have countable tightness but $C_p(X) \times C_p(Y)$ has not, had been constructed under ZFC by joint efforts of A. V. Arhangel'skii, E. G. Pytkeev and T. Przymusiński. Lately, under Jensen's $\diamondsuit$, S. Todorčević, F. Galvin and A. Miller constructed spaces X and Y for which $C_p(X)$ and $C_p(Y)$ are Fréchet while $C_p(X) \times C_p(Y)$ is not Fréchet but nevertheless has countable tightness.[2] In both cases the spaces X and Y are different from each other. This is not by accident. Unlike topological groups, to demonstrate that convergence properties are not multiplicative for function spaces, one *has* to consider $C_p(X)$ and $C_p(Y)$ with different X and Y. Of course, if $C_p(X)$ has countable tightness, then so does the space $C_p(X)^\omega$. This immediately follows from Fact 3 below, as soon as one observes that $C_p(X)^\omega$ is homeomorphic to $C_p(X \times D)$, where D is a countable discrete space. If $C_p(X)$ is Fréchet, then so is $C_p(X)^\omega$ [31]. Recall also that $C_p(X)$ is sequential iff it is Fréchet [25], [8].

The aim of this paper is to strengthen all the above results by showing via forcing that there exist: a) a Hausdorff hereditarily separable Fréchet topological group G whose square $G \times G$ has uncountable tightness (Corollary 1); b) Tychonoff spaces X and Y such that $C_p(X)^\omega$ and $C_p(Y)^\omega$ are hereditarily separable and Fréchet but $C_p(X) \times C_p(Y)$ has uncountable tightness (Corollary 2). Incidentally, we expand the general technique for constructing Fréchet spaces in Cohen generic extensions (see Clue Lemma).[3] The results of this paper were announced in [20], [14], [16].

2. Notations and terminology

Our topological notations and terminology follow [6]. All spaces are assumed to be Tychonoff. $X \oplus Y$ denotes the topological sum of spaces X

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[1]Here is the history of this example. E. K. van Douwen observed in a letter to A. V. Arhangel'skii that T. Przymusiński's technique [24] allows to construct spaces X and Y with the following properties: (i) all finite powers X^n and Y^n are Lindelöf, and (ii) $X \times Y$ is not Lindelöf. In fact, if in the definition of the space X from Remark 4.7 of [24] one sets $k = \omega$ and $m = 1$, then the spaces X and $Y = D_{\omega+1}$ will be as required. Here the set $D_{\omega+1} \subset \mathbf{R}$ is equipped with the subspace topology. The same arguments as in Theorem 1.5 of [24] show that $X \times Y$ contains a closed copy of the non-normal space $(D_{\omega+1}, \mathcal{T})$. If we observe that $C_p(X) \times C_p(Y)$ is homeomorphic to $C_p(X \oplus Y)$, then from (ii) and Fact 4 below it follows that $C_p(X) \times C_p(Y)$ has uncountable tightness, whereas (i) and Fact 4 imply countable tightness of both $C_p(X)$ and $C_p(Y)$.

[2]To construct the spaces X and Y, one needs to combine Theorem 5 of [7] with Theorem 2 of [10]. Both spaces X and Y are subspaces of the real line $\mathbf{R}$ and thus $X \oplus Y$ has a countable base. It follows from Fact 4 that $C_p(X) \times C_p(Y)$ has countable tightness.

[3]Using this technique the first author succeeded in constructing a Hausdorff Fréchet compact space without points of countable character [17].

and Y [6, 2.2], ω denotes the first infinite ordinal. We use $|X|$, $t(X)$ and $\ell(X)$ for denoting *cardinality, tightness* and *Lindelöf number* of a space X, respectively (see [6, pp. 87, 248]). If $Y \subset X$, then $\overline{Y}^X$ and $\overline{Y}$ denote the closure of Y in X. We fix the symbol $C_p(X)$ for the space of all real-valued continuous functions defined on a space X equipped with the topology of pointwise convergence [4], [10]. CH, MA, MA (σ-centered) and $P(\mathfrak{c})$ stand for Continuum Hypothesis, Martin's Axiom, Martin's Axiom restricted to σ-centered posets and combinatorial principle equivalent to Booth's Lemma respectively (see [33] for details). A space X is *Fréchet* (has *countable tightness*) iff $x \in X$, $Y \subset X$ and $x \in \overline{Y}$ imply that there is a sequence of points of Y converging to x (there is a countable set $Z \subset Y$ with $x \in \overline{Z}$ respectively). The reader interested in convergence properties is referred to [21], [3], [9]. A space X is a *strong L-space* (*strong S-space*) iff X^ω is hereditarily Lindelöf (hereditarily separable) but X is not hereditarily separable (is not hereditarily Lindelöf) [11], [27]. For basic forcing facts we refer the reader to [12], [29]. Our set-theoretic notations are standard and follow [12]. As usual, an ordinal is identified with the set of all its predecessors. For a function f, dom f and rng f denote its domain and range, respectively. $H(A)$ is the set of all functions from A to 2 having finite domains. If τ is a cardinal and A is a set, then $[A]^\tau = \{B \subset A \colon |B| = \tau\}$, $[A]^{<\tau} = \{B \subset A \colon |B| < \tau\}$.

3. A Fréchet hereditarily separable topological group G whose square has uncountable tightness

THEOREM 1. *Suppose M is a model of ZFC and M' is a model obtained from M via adding a single Cohen real. Then, in M', the group 2^{ω_1} contains a hereditarily separable subgroup G the square $G \times G$ of which has uncountable tightness.*

PROOF. Consider the poset $\mathcal{P}$ consisting of all functions p from $\omega \times 2$ to 2 such that dom $p = e \times 2$, $e \in [\omega]^{<\omega}$, and for every $j \in e$, we have either $p(j,0) = 1$ or $p(j,1) = 1$. The partial ordering $\leqq$ of $\mathcal{P}$ is the reverse inclusion: $p \leqq q$ iff $p \supset q$. Let H be a generic subset of $\mathcal{P}$ over M and $M[H]$ be a generic extension of M. According to [29, Chapter I, Theorem 5.6] the forcing notion $\mathcal{P}$ is equivalent to the forcing notion $Fn(\omega, 2)$ which adds a single Cohen real, hence without loss of generality we will identify $M[H]$ with M'. In M', set $r = \cup H$. Then r is a function from $\omega \times 2$ to 2 with the following property:

(*) for all $k \in \omega$, either $r(k,0) = 1$ or $r(k,1) = 1$.

In M, for every $\alpha \in \omega_1 \setminus \omega$ fix an injection $\theta_\alpha \colon \alpha \to \omega$ such that the family $\{\text{rng}\, \theta_\alpha : \alpha \in \omega_1 \setminus \omega\}$ is almost disjoint. In M', for any $\beta \in \omega_1 \setminus \omega$ and $i \in 2$ we define a function $y_{\beta,i} \in 2^{\omega_1}$ by letting

$$y_{\beta,i}(\gamma)=\begin{cases}0 & \text{if } \gamma\leqq\beta,\\ r(\theta_\gamma(\beta),i) & \text{if } \gamma>\beta.\end{cases}$$

For the sake of simplicity we will denote the set $(\omega_1\setminus\omega)\times 2$ by Ω.

Now, set $Y=\{y_{\beta,i}\colon(\beta,i)\in\Omega\}\subset 2^{\omega_1}$ and let G be the subgroup of 2^{ω_1} generated by Y. For every $a\in[\Omega]^{<\omega}$ set $g_a=\sum\{y_{\beta,i}\colon(\beta,i)\in a\}\in G$. If $h\in G$, then we define $\mathrm{Ord}\,(h)$ to be any element of the set $[\Omega]^{<\omega}$ with $h=g_{\mathrm{Ord}\,(h)}$. For $X\subset G$ we set $\mathrm{Ord}\,(X)=\{\mathrm{Ord}(x)\colon x\in X\}\subset[\Omega]^{<\omega}$.

Claim 1. *$G\times G$ has uncountable tightness.*

Proof. Let $\mathbf{0}\in 2^{\omega_1}$ be the function with $\mathbf{0}(\alpha)=0$ for all $\alpha\in\omega_1$. For $\alpha\in\omega_1+1$ let $Z_\alpha=\{(y_{\beta,0},y_{\beta,1})\colon\beta\in\alpha\setminus\omega\}\subset G\times G$. To prove that $G\times G$ has uncountable tightness, it suffices to show that $\langle\mathbf{0},\mathbf{0}\rangle\in\overline{Z_{\omega_1}}^{G\times G}$ but $\langle\mathbf{0},\mathbf{0}\rangle\notin\overline{Z_\alpha}^{G\times G}$ whenever $\alpha\in\omega_1$. The first inclusion follows from the definition of Y. Let us establish the second one. From the definition of $y_{\beta,0}$, $y_{\beta,1}$'s and $(*)$ it follows that $\beta\in\alpha\setminus\omega$ implies either $Y_{\beta,0}(\alpha)=1$ or $y_{\beta,1}(\alpha)=1$. Thus, letting $U=\{g\in G\colon g(\alpha)=0\}$, we would have the open set $U\times U$ disjoint from Z_α, and $\langle\mathbf{0},\mathbf{0}\rangle\in U\times U$. Therefore, $\langle\mathbf{0},\mathbf{0}\rangle\notin\overline{Z_\alpha}^{G\times G}$

Claim 2. *G is hereditarily separable.*

Proof. Recall that a set $S\subset 2^{\omega_1}$ is called to be finally dense in 2^{ω_1} if there is an $\alpha_S\in\omega_1$ such that for any $\varepsilon\in H(\omega_1\setminus\alpha_S)$ one can find an $s\in S$ with $s\supset\varepsilon$. A set $X\subset 2^{\omega_1}$ is said to be *weakly HFD-set* if for every $Z\in\in[X]^{\omega_1}$ there is an $S\in[Z]^\omega$ which is finally dense in 2^{ω_1} [11, Definition 1.9]. If $X\subset 2^{\omega_1}$ is a weakly HFD-set, then the space X is hereditarily separable [11, Theorem 1.10]. So to verify Claim 2 it suffices to prove the following

Lemma 1. *G is a weakly HFD-set.*

Proof. Fix an $X\in[G]^{\omega_1}$. Then $\mathrm{Ord}(X)\in[[\Omega]^{<\omega}]^{\omega_1}$. Now we need the following

Fact 1 [26]. *Suppose that a model M' is obtained by adding a single Cohen real to a model M. Let γ be an ordinal. If in M', $E'\subset[[\gamma]^{<\omega}]^{\omega_1}$, then there is a set $E\in[E']^{\omega_1}$ with $E\in M$.*

Identifying Ω with the ordinal $(\omega_1-\omega)+(\omega_1-\omega)$ and applying Fact 1 to the family $\mathrm{Ord}(X)\in[[\Omega]^{<\omega}]^{\omega_1}$, we obtain a set $A'\in[\mathrm{Ord}(X)]^{\omega_1}$ with $A'\in\in M$. Choose an $A\in[A']^\omega$ with $A\in M$. From our constructions it follows that for $A^*=\{g_a\colon a\in A\}$ we have $A^*\in[X]^\omega$. We will show that A^* is finally dense in 2^{ω_1}. To do this let us define $\delta\in\omega_1$ by $\delta=\sup\{\alpha\in\omega_1\colon\langle\alpha,i\rangle\in a$ for some $a\in A$ and $i\in 2\}$ and show that for every $\varepsilon\in H(\omega_1\setminus\delta)$, there is an $a\in A$ with $g_a\supset\varepsilon$. Assume that $\varepsilon=\{\langle\alpha_0,i_0\rangle,\langle\alpha_1,i_1\rangle,\dots,\langle\alpha_n,i_n\rangle\}$ where $n\in\omega$, $\alpha_0,\alpha_1,\dots,\alpha_n\in\omega_1\setminus\delta$ and $i_0,i_1,\dots,i_n\in 2$. Fix a $p\in\mathcal{P}$.

Everything will be done if we find a $q \in \mathcal{P}$ and an $a \in A$ such that $q \leqq p$ and $q \Vdash \dot{g}_a \supset \check{\varepsilon}$.

Since $\{\operatorname{rng} \theta_\alpha : \alpha \in \omega_1 \setminus \omega\}$ is an almost disjoint family, and dom p is a finite set, there is a $b \in [\omega]^{<\omega}$ such that $\operatorname{dom} p \subset b \times 2$, and $\operatorname{rng} \theta_{\alpha_i} \cap \cap \operatorname{rng} \theta_{\alpha_j} \subset b$ provided that $i, j \in n+1$ and $i \neq j$. Choose a condition $p' \leqq p$ for which $b \times 2 \subset \operatorname{dom} p' = e' \times 2$, and let $c = \cup\{\theta_{\alpha_i}^{-1}(e') : i \in n+1\}$. Obviously $c \in [\omega_1]^{<\omega}$. Hence one can find an $a \in A$ such that $a' \setminus c \neq \emptyset$ where $a' = \{\alpha \in \omega_1 : \langle \alpha, i\rangle \in a \text{ for some } i \in 2\}$. Choose an $\alpha \in a' \setminus c$. The definition of δ implies that $\alpha \in \alpha_i$ for every $i \in n+1$. Let $n_j = \theta_{\alpha_j}(\alpha)$, $\operatorname{dom} t_j = \operatorname{dom} p' \cup (e_j \times 2)$ where $e_j = \{\theta_{\alpha_j}(\gamma) : \gamma \in a'\}$. Choose a condition $s_j \leqq p'$ with $\operatorname{dom} s_j = \operatorname{dom} t_j \setminus \{(n_j, 0), (n_j, 1)\}$. If $a_\alpha = a \setminus \{(\alpha, 0), (\alpha, 1)\}$, then $s_j \Vdash \dot{g}_{a_\alpha}(\check{\alpha}_j) = \check{k}_j$ for some $k_j \in 2$. Now we have to consider three cases.

Case 1: $(\alpha, 0), (\alpha, 1) \in a$. We set

$$t_j = \begin{cases} s_j \cup \{\langle (n_j, 0), 1\rangle, \langle (n_j, 1), 1\rangle\} & \text{if } k_j = i_j, \\ s_j \cup \{\langle (n_j, 0), 1\rangle, \langle (n_j, 1), 0\rangle\} & \text{if } k_j \neq i_j. \end{cases}$$

Case 2: $(\alpha, 0) \in a$, $(\alpha, 1) \notin a$. We set

$$t_j = \begin{cases} s_j \cup \{\langle (n_j, 0), 0\rangle, \langle (n_j, 1), 1\rangle\} & \text{if } k_j = i_j, \\ s_j \cup \{\langle (n_j, 0), 1\rangle, \langle (n_j, 1), 0\rangle\} & \text{if } k_j \neq i_j. \end{cases}$$

Case 3: $(\alpha, 0) \notin a$, $(\alpha, 1) \in a$. We set

$$t_j = \begin{cases} s_j \cup \{\langle (n_j, 0), 1\rangle, \langle (n_j, 1), 0\rangle\} & \text{if } k_j = i_j, \\ s_j \cup \{\langle (n_j, 0), 0\rangle, \langle (n_j, 1), 1\rangle\} & \text{if } k_j \neq i_j. \end{cases}$$

One can easily verify that in all cases, $t_j \leqq s_j \leqq p'$ and

$$t_j \Vdash \dot{g}_a(\check{\alpha}_j) = \check{i}_j.$$

Repeat the above construction consecutively to obtain $t_0, t_1, \dots, t_n$, and note that, since $\operatorname{dom} t_k \cap \operatorname{dom} t_\ell = \operatorname{dom} p'$ provided that $k, \ell \in n+1$ and $k \neq \ell$, and $t_k \leqq p'$ whenever $k \in n+1$, it follows that $q = t_0 \cup t_1 \cup \ldots \cup t_n$ is a condition, i.e. $q \in \mathcal{P}$. Finally, $q \leqq p$ and $q \Vdash \dot{g}_a \supset \check{\varepsilon}$, so q does the job.

FACT 2 [19]. *Assume $P(\mathfrak{c})$. Then a space of countable tightness and of character $< \mathfrak{c}$ is Fréchet.*

FACT 3 (J. Roitman, M. Bell). *Adding a single Cohen real does not destroy $P(\mathfrak{c})$.*

In fact, J. Roitman [26] proved that adding a single Cohen real does not destroy MA (σ-centered), and M. Bell [5] showed that MA (σ-centered) and $P(\mathfrak{c})$ are equivalent.

Facts 2 and 3 yield

Clue Lemma. *Add a single Cohen real r to a model M of $MA + \neg CH$. Then, in the generic extension $M[r]$, spaces of countable tightness and of character $\leqq \omega_1$ are Fréchet.*

Proof. Adding a single Cohen real does not change the power function, hence in $M[r]$, $\omega_1 < \mathfrak{c}$. Since $P(\mathfrak{c})$ follows from $MA + \neg CH$, Fact 3 implies that $P(\mathfrak{c})$ holds in $M[r]$. The conclusion of Clue Lemma follows from Fact 2.

Clue Lemma immediately yields the first main result of this paper.

Corollary 1. *Add a single Cohen real to a model M of $MA + \neg CH$. Then, in the generic extension M', the group G constructed in Theorem 1 is Fréchet. Therefore, in M', there exists a hereditarily separable Fréchet topological group G the square $G \times G$ of which has uncountable tightness.*

4. Cartesian products of strong S- and L-spaces in Cohen generic extensions

Theorem 2. *Suppose that M' is a generic extension of a model M by adding a single Cohen real. Then, in M', there exist the following spaces:*

(i) *strong L-spaces $L_0, L_1 \subset 2^{\omega_1}$ (of cardinality ω_1) the Cartesian product $L_0 \times L_1$ of which is not Lindelöf;*

(ii) *strong S-spaces $S_0, S_1 \subset 2^{\omega_1}$ (of cardinality ω_1) the Cartesian product $S_0 \times S_1$ of which has uncountable tightness.*

Proof. In M, for every $\alpha \in \omega_1 \setminus \omega$ fix an injection $\theta_\alpha : \alpha \to \omega$ such that the family $\{\operatorname{rng} \theta_\alpha : \alpha \in \omega_1 \setminus \omega\}$ is almost disjoint. Let $r : \omega \to 2$ be a Cohen real. In $M' = M[r]$, for every $\alpha \in \omega_1 \setminus \omega$, define $A^0_\alpha = \{\beta \in \alpha : r(\theta_\alpha(\beta)) = 0\}$, $A^1_\alpha = \{\beta \in \alpha : r(\theta_\alpha(\beta)) = 1\}$, and let $\ell^0_\alpha \in 2^{\omega_1}$ and $\ell^1_\alpha \in 2^{\omega_1}$ be the characteristic functions of the sets $A^0_\alpha \subset \omega_1$ and $A^1_\alpha \subset \omega_1$ respectively. For $\alpha \in \omega$, let $\ell^0_\alpha = \ell^1_\alpha = \mathbf{0}$. Let $L_0 = \{\ell^0_\alpha : \alpha \in \omega_1\} \subset 2^{\omega_1}$ and $L_1 = \{\ell^1_\alpha : \alpha \in \omega_1\} \subset 2^{\omega_1}$.

Claim 3. *The space $L_0 \times L_1$ is not Lindelöf.*

Proof. Let $V_\alpha = \{x \in 2^{\omega_1} : x(\alpha) = 0\}$. It is an easy exercise on Cohen forcing to show, applying the almost disjointness of $\{\operatorname{rng} \theta_\alpha : \alpha \in \omega_1 \setminus \omega\}$, that the family $\sigma = \{V_\alpha \times V_\alpha : \alpha \in \omega_1\}$ constitutes an open covering of $L_0 \times L_1$. If $\sigma' \in [\sigma]^\omega$, then $\sigma' \subset \sigma_\beta = \{V_\alpha \times V_\alpha : \alpha \in \beta\}$ for some $\beta \in \omega_1$. Further, $(\ell^0_\beta, \ell^1_\beta) \notin \cup \sigma_\beta$, since $\alpha \in \beta$ implies that either $\ell^0_\beta(\alpha) = 1$ or $\ell^1_\beta(\alpha) = 1$. Therefore, no countable $\sigma' \subset \sigma$ covers $L_0 \times L_1$.

Claim 4. *Both L_0 and L_1 are strong L-spaces.*

Proof. By symmetry of the definitions of L_i's, it suffices to prove that L_0 is a strong L-space. But this is just what J. Roitman has shown in [26] (her space $X_{\mathcal{F},r}$ is exactly our space L_0).

If $L = \{\ell_\alpha : \alpha \in \omega_1\} \subset 2^{\omega_1}$ is a strong L-space, then the space $S = \{s_\beta : \beta \in \omega_1\} \subset 2^{\omega_1}$, where $s_\beta \in 2^{\omega_1}$ is defined by $s_\beta(\alpha) = \ell_\alpha(\beta)$ for every $\alpha \in \omega_1$, is a strong S-space [26], [27], [28]. Apply this construction to $L_i = \{\ell^i_\alpha : \alpha \in \omega_1\}$ to obtain $S'_i = \{s^i_\beta : \beta \in \omega_1\}$ and let $S_i = S'_i \cup \{\mathbf{0}\}$ $(i = 0, 1)$. Then both S_0 and S_1 are strong S-spaces.

Claim 5. *The space $S_0 \times S_1$ has uncountable tightness.*

Proof. Obviously, $(\mathbf{0}, \mathbf{0}) \in \overline{\{(s^0_\alpha, s^1_\alpha) : \alpha \in \omega_1\}}^{S_0 \times S_1}$, but $(\mathbf{0}, \mathbf{0}) \notin \overline{\{(s^0_\alpha, s^1_\alpha) : \alpha \in \beta\}}^{S_0 \times S_1}$ whenever $\beta \in \omega_1$.

5. Products of convergence properties in function spaces

Fact 4 (A. V. Arhangel'skii, E. G. Pytkeev, cited in [4, Theorem 4.1.2]). *For every space X, $t(C_p(X)) = \sup\{\ell(X^n) : n \in \omega\}$.*

Fact 5 [34], [32]. *For every space X the following conditions are equivalent:*

(i) *X^ω is hereditarily Lindelöf,*

(ii) *$C_p(X)^\omega$ is hereditarily separable.*

Now we are ready to prove the second main result of our paper.

Corollary 2. *Add a single Cohen real to a model of $MA + \neg CH$. In the generic extension, let L_0 and L_1 be the spaces constructed in item* (i) *of Theorem* 2. *Then:*

(i) *$C_p(L_0)^\omega$ and $C_p(L_1)^\omega$ are hereditarily separable and Fréchet, while*

(ii) *$C_p(L_0) \times C_p(L_1)$ has uncountable tightness.*

Proof. The first part of (i) follows from Fact 5. To obtain the second, observe that, since $C_p(L_i)^\omega$ is dense in $\mathbf{R}^{L_i \times \omega}$, the character of $C_p(L_i)^\omega$ is equal to ω_1, and then apply Clue Lemma. To show (ii), remark that $C_p(L_0) \times C_p(L_1)$ is homeomorphic to $C_p(L_0 \oplus L_1)$, the space $(L_0 \oplus L_1)^2$ is not Lindelöf, and then apply Fact 4.

Remark. Theorem 1 and Corollary 1 were obtained at about the same time by both authors independently from each other. Theorem 2 and Corollary 2 are due to V. I. Malyhin.

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CHAIR OF APPLIED MATHEMATICS
DEPARTMENT OF ECONOMICS AND COMPUTER SCIENCES
MOSCOW INSTITUTE OF CONTROL AND MANAGEMENT
RJAZANSKIĬ PROSPEKT 99, 109542 MOSCOW
RUSSIA

CHAIR OF GENERAL TOPOLOGY AND GEOMETRY
DEPARTMENT OF MECHANICS AND MATHEMATICS
MOSCOW STATE UNIVERSITY
119899 MOSCOW
RUSSIA

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CONTINUOUS FUNCTIONS WHOSE LEVEL SETS ARE ORTHOGONAL TO ALL POLYNOMIALS OF A GIVEN DEGREE

H. DEHLING (Groningen) and M. S. TAQQU* (Boston)

1. Introduction

Let $w(t)$ be a continuous, non-negative weight function on $\mathbf{R}$ with $\int_{\mathbf{R}} w(t)dt = 1$ and with finite absolute moment of order k, i.e., $\int_{\mathbf{R}} |t|^k w(t)dt < \infty$. Denote by $\mathcal{P}_k$ the subspace of $L_2(\mathbf{R}, \mathcal{B}, w(t)dt)$ spanned by the polynomials $p(t)$ of degree less than or equal to k. (The degree of a polynomial $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ is $\deg(P(t)) = \max\{j : a_j \neq 0\}$.) Let $W_0 \equiv 1$, $W_1(t), \ldots, W_k(t)$ be an orthogonal basis for $\mathcal{P}_k$.

In this paper we investigate the class of functions $G(t)$ satisfying

$$(1.1) \qquad \int_{\{G(\cdot)\leqq s\}} W_j(t)w(t)dt = 0 \quad \text{for} \quad j = 1, \ldots, k \quad \text{and for all} \quad s \in \mathbf{R}.$$

A condition equivalent to (1.1) is that $1\{t : G(t) \leqq s\} - \int_{\mathbf{R}} 1\{G(t) \leqq s\} w(t)dt$ lies in $\mathcal{P}_k^{\perp}$, the orthogonal complement of $\mathcal{P}_k$. Alternatively,

$$\int_{\{G(\cdot)\leqq s\}} p(t)w(t)dt = \int_{\mathbf{R}} p(t)w(t)dt \int_{\{G\{\cdot\}\leqq s\}} w(t)dt$$

for all polynomials $p(t)$ in $\mathcal{P}_k$.

The existence of a non-trivial function G satisfying (1.1) can be obtained as a corollary to Liapunov's theorem (see [3], Th. 5.5) on the convexity of the range of a vector measure. Namely, let μ_j be the signed measure given by $\frac{d\mu_j}{dt} = W_j(t)w(t)$ and let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^T$ be the corresponding vector measure. (T denotes transpose.) We have $\boldsymbol{\mu}(\mathbf{R}) = 0$ since $W_j \perp W_0$ for $j \geqq 1$. We want to show first the existence of a non-trivial set $A \subset \mathbf{R}$, i.e., $\lambda(A) > 0$ and $\lambda(\mathbf{R} \setminus A) > 0$, with $\boldsymbol{\mu}(A) = 0$. (Here λ is the Lebesgue

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measure.) If $\boldsymbol{\mu}(\mathbf{R}_+) = 0$, simply take $A = \mathbf{R}_+$. Otherwise, apply Liapunov's theorem to $\boldsymbol{\mu}$, restricted to $\mathbf{R}_+$ and $\mathbf{R}_-$ respectively, to get sets A_+ and A_- with $\boldsymbol{\mu}(A_+) = \frac{1}{2}\boldsymbol{\mu}(\mathbf{R}_+)$ and $\boldsymbol{\mu}(A_-) = \frac{1}{2}\boldsymbol{\mu}(\mathbf{R}_-)$. Then take $A = A_+ \cup \cup A_-$. Obviously $\boldsymbol{\mu}(A) = 0$. Moreover, this set A is non-trivial. Indeed, $\lambda(A_+) > 0$, because $\boldsymbol{\mu}(A_+) \neq 0$ and $\boldsymbol{\mu} \ll \lambda$. Also $\lambda(\mathbf{R}_+ \setminus A_+) > 0$ so that finally $\lambda(A) > \lambda(A_+) > 0$ as well as $\lambda(\mathbf{R} \setminus A) > \lambda(\mathbf{R}_+ \setminus A_+) > 0$. This set A yields a non-trivial function $G(t) = 1_A(t)$ whose level sets $1\{t : G(t) \leqq \leqq s\}$, $-\infty < s < \infty$, are $\emptyset$, A and $\mathbf{R}$. Therefore $G(t) = 1_A(t)$ satisfies (1.1). Although Liapunov's theorem establishes the existence of such a function G, it does not offer a way of constructing it.

Our goal is two-fold. We want first to give an explicit construction of a family of sets A_s, $-\varepsilon \leqq s \leqq \varepsilon$, satisfying

$$\int_{A_s} W_j(t)w(t)dt = 0 \quad \text{for} \quad j = 1, \dots, k \quad \text{and for all} \quad s. \tag{1.2}$$

We will find that A_s can be taken to be a union of $k+1$ intervals, whose endpoints depend differentiably on s. With the help of this explicit construction we will be able to achieve our second goal, namely the construction of a continuous function $G(t)$ satisfying (1.1).

THEOREM. *Let $w(t)$ be a continuous, non-negative weight function on* $\mathbf{R}$ *with* $\int_{\mathbf{R}} w(t)dt = 1$ *and* $\int_{\mathbf{R}} |t|^k w(t)dt < \infty$ *for some integer k. Let* $W_0 \equiv 1$, $W_1(t), \dots, W_k(t)$ *be orthogonal polynomials of degree less than or equal to k. Then:*

(a) *There exist an $\varepsilon > 0$ and distinct points $t_1, \dots, t_{k+1}$ and k functions*

$$f_j : [-\varepsilon, \varepsilon] \to \mathbf{R}, \qquad j = 1, \dots, k$$

with the properties:

(i) *f_j are strictly increasing, differentiable and $f_j(0) = 0$,*

(ii) $A_s = \bigcup_{j=1}^{k} [t_j + f_j(-s), t_j + f_j(s)] \cup [t_{k+1} - s, t_{k+1} + s]$ *satisfies* (1.2), $\forall s \in [0, \varepsilon]$.

(b) *There exists a continuous function $G(t)$ satisfying* (1.1).

In the light of recent results of the authors, this theorem has remarkable consequences for the large sample behavior of the empirical process of long-range dependent observations. To be more specific, choose as weight functions $w(t)$ the $N(0,1)$ density $\phi(t) = (2\pi)^{-1/2}e^{-t^2/2}$. The corresponding orthogonal polynomials are the Hermite polynomials $H_q(t)$, $q = 0, 1, \dots$. Let $(X_j)_{j=1}^{\infty}$ be a mean-zero unit variance stationary Gaussian sequence with covariance $r(k) = k^{-D}L(k)$ where $0 < D < 1$ and L slowly varying at infinity.

Let $G\colon \mathbf{R} \to \mathbf{R}$ be any measurable function and let $Y_j = G(X_j)$. The empirical distribution function of Y_j is given by $F_N(x) = \frac{1}{N} \sum_{j=1}^{N} 1\{Y_j \leqq x\}$. Clearly $F_N(x) \to F(x) = P(Y_1 \leqq x)$ uniformly in x almost surely. In Dehling and Taqqu [1] we proved that $(F_N - F)$ can be normalized in such a way that a non-degenerate limit is obtained. It turns out that the normalizing constants as well as the type of process that arises in the limit depend heavily on the Hermite rank of the family of functions $\Delta_x(t) = 1\{G(t) \leqq x\} - F(x)$, namely, on $m = \inf\{q : \exists x \text{ with } \int_{\mathbf{R}} \Delta_x(t) H_q(t)\phi(t)dt \neq 0\}$. The higher the Hermite rank, the more complex the behavior. The theorem obtained in this paper shows that there are processes of the type $Y_j = G(X_j)$ with G continuous, for which $\Delta_x(t)$ has an arbitrarily high Hermite rank.

2. An example

To illustrate the main ideas of the proof, we consider the special case of Hermite polynomials and $k = 2$. We have $H_0(t) = 1$, $H_1(t) = t$ and $H_2(t) = t^2 - 1$. We want to find a set $A = A(2)$ which is a union of N disjoint intervals, i.e., $A = \bigcup_{i=1}^{N} [t_i, t_i + a_i]$ with $a_i \geqq 0$. In order to satisfy (1.2), we must have

$$\sum_{i=1}^{N} \int_{[t_i, t_i + a_i]} H_j(t)\phi(t)dt = 0, \qquad j = 1, 2, \tag{2.1}$$

or, since the a_i are non-negative,

$$\sum_{i=1}^{N} \int_{t_i}^{t_i + a_i} H_j(t)\phi(t)dt = 0, \qquad j = 1, 2. \tag{2.2}$$

We wiew (2.2) as a system of non-linear equations in the a_i's, $i = 1, \dots, N$. When $N = 2$, the trivial solution $a_1 = a_2 = 0$ may be the only one. Hence we take $N = 3$, so that (2.2) becomes a system of 2 equations with 3 unknowns. It still admits the trivial solution $a_1 = a_2 = a_3 = 0$. The implicit function theorem however, yields the existence of a continuous family of solutions $a_1 = a_1(a_3), a_2 = a_2(a_3), a_3$, for a_3 in a small neighborhood of zero, if the Jacobian

$$\begin{vmatrix} H_1(t_1)\phi(t_1) & H_1(t_2)\phi(t_2) \\ H_2(t_1)\phi(t_1) & H_2(t_2)\phi(t_2) \end{vmatrix} \neq 0,$$

or equivalently, if the determinant

$$\begin{vmatrix} H_1(t_1) & H_1(t_2) \\ H_2(t_1) & H_2(t_2) \end{vmatrix} \neq 0. \tag{2.3}$$

The derivatives $\frac{da_1}{da_3}$ and $\frac{da_2}{da_3}$ may be obtained by implicit differentiation in (2.2); they satisfy

$$\begin{cases} \frac{da_1}{da_3} H_1(t_1)\phi(t_1) + \frac{da_2}{da_3} H_1(t_2)\phi(t_2) + H_1(t_3)\phi(t_3) = 0 \\ \frac{da_1}{da_3} H_2(t_1)\phi(t_1) + \frac{da_2}{da_3} H_2(t_2)\phi(t_2) + H_2(t_3)\phi(t_3) = 0. \end{cases} \tag{2.4}$$

Since $a_1 = a_2 = a_3 = 0$ is a solution of (2.2), we can get a continuum of solutions with $a_i > 0$, $i = 1,2,3$ by requiring that $\frac{da_1}{da_3}$ and $\frac{da_2}{da_3}$ be positive around zero. Hence, we have to find points t_1, t_2, t_3 such that (2.3) holds and (2.4) has a positive solution. This is equivalent geometrically to requiring that the origin $\mathbf{0} = (0,0)^T$ be in the interior of the convex hull of the three points $(H_1(t_i), H_2(t_i))^T$, $i = 1,2,3$. It is easy to find such points t_i by drawing the graph of the curve $(t, t^2 - 1)$.

3. The basic proposition

When $k > 2$, graphical inspection is no longer available. In this section, we prove the following proposition:

PROPOSITION 3.1. *Let $w(t)$ be as in the theorem and let $p_1(t), \dots, p_k(t)$ be polynomials satisfying* $\deg(p_j) = j$ *and*

$$\int_{\mathbf{R}} p_j(t) w(t) dt = 0, \qquad j = 1, \dots, k. \tag{3.1}$$

Then we can find points $t_1, \dots, t_{k+1} \in S = \{t : w(t) > 0\}$ such that

$$\begin{pmatrix} p_1(t_1) \\ \vdots \\ p_k(t_1) \end{pmatrix}, \dots, \begin{pmatrix} p_1(t_k) \\ \vdots \\ p_k(t_k) \end{pmatrix} \tag{3.2}$$

are linearly independent vectors in $\mathbf{R}^k$, and such that the system of equations

$$\sum_{i=1}^{k+1} p_j(t_i) a_i = 0, \qquad j = 1, \dots, k, \tag{3.3}$$

has a solution $(a_1, \dots, a_{k+1})$ with all $a_i > 0$.

For abbreviation, we let

$$\mathbf{P}(t) = (p_1(t), \dots, p_k(t))^T$$

denote the vector of polynomials. The convex hull of the set H will be denoted by $\mathrm{conv}(H)$ and the interior of $\mathrm{conv}(H)$ by $\mathrm{conv}^\circ(H)$. Let $\mathbf{0}$ denote the zero vector in $\mathbf{R}^k$.

LEMMA 3.1. $\mathbf{0} \in \mathrm{conv}^\circ(\{\mathbf{P}(t) : t \in S\}) \neq \emptyset$.

PROOF. If $\mathrm{conv}^\circ(\{\mathbf{P}(t) : t \in S\}) = \emptyset$ then $\{\mathbf{P}(t) : t \in S\}$ is contained in a hyperplane $\{\mathbf{x} : \mathbf{c}^T\mathbf{x} = a\}$ where $\mathbf{c} = (c_1, \dots, c_k)^T$ is a non-zero vector and $a \in \mathbf{R}$. Therefore $\sum_{i=1}^{k} c_i p_i(t) = a$ for every $t \in S$. Since S is open and non-empty, this implies that the polynomial $q = \sum_{i=1}^{k} c_i p_i$ is constant. However, this contradicts $\mathbf{c} \neq 0$ and $\deg(p_i) = i$, $i = 1, \dots, k$.

Thus $\mathrm{conv}^\circ(\{\mathbf{P}(t) : t \in S\}) \neq \emptyset$ and hence $\{\mathbf{P}(t) : t \in S\}$ is contained in the closure of $\mathrm{conv}^\circ(\{\mathbf{P}(t) : t \in S\})$.

Now, suppose, ad absurdum, that $\mathbf{0} \notin \mathrm{conv}^\circ\{\mathbf{P}(t) : t \in S\}$. Then, by the separating hyperplane theorem, there exists a non-zero vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)^T$ such that $\boldsymbol{\alpha}^T\mathbf{x} \geqq 0$ for all $\mathbf{x} \in \mathrm{conv}^\circ\{\mathbf{P}(t) : t \in S\}$. Hence, by continuity, $\boldsymbol{\alpha}^T\mathbf{P}(t) \geqq 0$, i.e., $\sum_{q=1}^{k} \alpha_q p_q(t) \geqq 0$ for all $t \in S$. By (3.1), we have $\int_{\mathbf{R}} \sum_{q=1}^{k} \alpha_q p_q(t) w(t)\,dt = 0$ and thus $\sum_{q=1}^{k} \alpha_q p_q = 0$ on S. Since S is an open set, this implies $\sum_{q=1}^{k} \alpha_q p_q \equiv 0$. The linear independence of the polynomials $p_1, \dots, p_k$ implies $\alpha_1 = \dots = \alpha_k = 0$, contradicting our assumption on $\boldsymbol{\alpha}$. □

LEMMA 3.2. *Let* $N = 2^k(k+1)$, *and let*

$$A = \{(t_1, \dots, t_N) \colon t_i \in S \ \ (i = 1, \dots, N) \text{ and } \mathbf{0} \in \mathrm{conv}^\circ(\{\mathbf{P}(t_1), \dots, \mathbf{P}(t_N)\})\}.$$

Then A is a non-empty open subset of $\mathbf{R}^N$.

PROOF. By Lemma 3.1 we can find $\varepsilon > 0$ such that the ball of radius ε centered at the origin is contained in $\mathrm{conv}\{\mathbf{P}(t) : t \in S\}$. In particular, the 2^k vectors $\mathbf{x}_{\boldsymbol{\sigma}} = (\sigma_1\varepsilon, \dots, \sigma_k\varepsilon)^T$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k) \in \{+1, -1\}^k$ are in $\mathrm{conv}\{\mathbf{P}(t) \colon t \in S\}$. By Caratheodory's theorem ([2], Th. 17.1), each vector $\mathbf{x}_{\boldsymbol{\sigma}}$ is in the convex hull of $k+1$ points of $\{\mathbf{P}(t) \colon t \in S\}$. Hence there are numbers $t_i \in S$ $(i = 1, \dots, 2^k(k+1))$ such that the convex hull of the corresponding $\mathbf{P}(t_i)$ contains all 2^k vectors $\mathbf{x}_{\boldsymbol{\sigma}}$ and thus has $\mathbf{0}$ in its interior. Therefore, $A \neq \emptyset$.

Next we show that A is open, or equivalently that its complement is closed. Let $t_n = (t_1^n, \dots, t_N^n)$ be a sequence in $\mathbf{R}^N \setminus A$ converging to some point $\mathbf{t} = (t_1, \dots t_N)$. Then

$$\mathbf{0} \notin \operatorname{conv}^\circ(\{\mathbf{P}(t_1^n), \dots, \mathbf{P}(t_N^n)\}) = T_n$$

and hence there are unit vectors $\mathbf{e}_n$ such that the hyperplane $\{x : \mathbf{e}_n^T \mathbf{x} = 0\}$ does not meet the set T_n. By selecting a subsequence we may assume that the vectors $\mathbf{e}_n$ converge to a unit vector $\mathbf{e}$. It follows from the continuity of the map $\mathbf{P}$ that the hyperplane $\{\mathbf{x} : \mathbf{e}^T \mathbf{x} = 0\}$ does not meet the set $\operatorname{conv}^\circ\{\mathbf{P}(t_1), \dots, \mathbf{P}(t_N)\}$, and hence $\mathbf{t} \notin A$. □

LEMMA 3.3. *There are $N = 2^k(k+1)$ distinct points $t_1, \dots, t_N \in S$ such that*

(i) $\mathbf{0} \in \operatorname{conv}^\circ(\{\mathbf{P}(t_1), \dots, \mathbf{P}(t_N)\})$ *and*

(ii) *the vectors $\mathbf{P}(t_{i_1}), \dots \mathbf{P}(t_{i_k})$ are linearly independent for every k-subset $\{t_{i_1}, \dots, t_{i_k}\}$ of $\{t_1, \dots, t_N\}$.*

PROOF. Let A be as in Lemma 3.2 and let B denote the set of those points $t = (t_1, \dots, t_N)$ for which Condition (ii) of Lemma 3.3 does not hold. Since every element of $A \setminus B$ satisfies the requirements of Lemma 3.3, it is sufficient to prove that $A \setminus B \neq \emptyset$.

Observe first that B is a nowhere dense set of $\mathbf{R}^N$. Indeed, if the vectors $\mathbf{P}(t_{i_1}), \dots, \mathbf{P}(t_{i_k})$ are linearly dependent, then the determinant

$$d(t) = \det(p_j(t_{i_n}) : \ j, n = 1, \dots, k)$$

is zero. Now $d(\mathbf{t})$ is a non-vanishing polynomial in the variables $t_1, \dots, t_N$ (whose functional form depends on the subset $\{t_{i_1}, \dots, t_{i_k}\}$), and hence the set $\{\mathbf{t} \in \mathbf{R}^N : d(t) = 0\}$ is nowhere dense. Since B is the union of finitely many sets of this form, B is nowhere dense.

By Lemma 3.2, the set A is non-empty and open. Hence $A \setminus B \neq \emptyset$. □

PROOF OF PROPOSITION 3.1. Let $\{t_1, \dots, t_N\}$ be the set obtained in Lemma 3.3. By Caratheodory's theorem ([2], Th. 17.1) there exists a subset $t_{i_1}, \dots, t_{i_{k+1}}$ such that

$$\mathbf{0} \in \operatorname{conv}\{\mathbf{P}(t_{i_1}), \dots \mathbf{P}(t_{i_{k+1}})\}.$$

Thus (3.3) has a solution with all $a_i \geqq 0$. If one of the a_i's were equal to zero, a k-subset of $\mathbf{P}_k(t_{i_1}), \dots, \mathbf{P}_k(t_{i_{k+1}})$ would be linearly dependent, contradicting Lemma 3.3. Thus (3.3) is satisfied, and also (3.2). □

4. Proof of the Theorem

To establish the first part, we apply Proposition 3.1 to obtain points $t_1, \dots, t_{k+1} \in S$ satisfying (3.2) and (3.3) (with p_i replaced by W_i). Consider the following system of k non-linear equations in $k+1$ variables $a_1, \dots, a_{k+1}$:

$$\sum_{i=1}^{k+1} \int_{t_i}^{t_i+a_i} W_j(t)w(t)dt = 0, \qquad j = 1, \dots, k. \tag{4.1}$$

(Here we follow the usual convention for definite integrals, defining $\int\limits_{t_i}^{t_i+a_i} = -\int\limits_{t_i+a_i}^{t_i}$ when $a_i < 0$.)

This system has a trivial solution $a_1 = a_2 = \dots = a_{k+1} = 0$. The matrix of partial derivatives is

$$\begin{pmatrix} W_1(t_1)w(t_1) & \dots & W_1(t_k)w(t_k) & W_1(t_{k+1})w(t_{k+1}) \\ \vdots & & \vdots & \vdots \\ W_k(t_1)w(t_1) & \dots & W_k(t_k)w(t_k) & W_k(t_{k+1})w(t_{k+1}) \end{pmatrix}.$$

Since the $k \times k$ sub-matrix consisting of the first k columns is non-singular, we may apply the implicit function theorem. There exists then an $\varepsilon' > 0$ and differentiable functions $f_1, \dots, f_k \colon [-\varepsilon', \varepsilon'] \to \mathbf{R}$ such that $a_1 = f_1(a)$, $a_2 = f_2(a), \dots, a_k = f_k(a), a_{k+1} = a$, solves (4.1). Moreover, using the chain rule we have the following system of linear equations for $f_i'(0) = \frac{df_i}{da}(0)$:

$$f_1'(0)W_j(t_1)w(t_1) + f_2'(0)W_j(t_2)w(t_2) + \dots + f_k'(0)W_j(t_k)w(t_k) + \\ + W_j(t_{k+1})w(t_{k+1}) = 0.$$

By the choice of the t_i's this system has a unique solution $f_i'(0) > 0$, $i = 1, \dots, k$. Thus all f_i are strictly increasing in some neighborhood $[-\varepsilon, \varepsilon]$ of 0 with $\varepsilon < \varepsilon'$. Hence (4.1) holds with $a_i = f_i(s)$, $i = 1, \dots, k$, $a_{k+1} = s$ for any $0 < s \leqq \varepsilon$. It also holds with $a_i = f_i(-s) < 0$, $i = 1, \dots, k$, $a_{k+1} = -s$. Therefore

$$\sum_{i=1}^{k} \int_{t_i+f_i(-s)}^{t_i+f_i(s)} W_j(t)w(t)dt + \int_{t_{k+1}-s}^{t_{k+1}+s} W_j(t)w(t)dt = 0$$

for $j = 1, \dots, k$ and for all $0 \leqq s \leqq \varepsilon$.

We now turn to the proof of the second part of the theorem. Assume w.l.o.g. (by suitably decreasing ε) that the intervals $[t_j + f_j(-\varepsilon), t_j + f_j(\varepsilon)]$, $1 \leqq j \leqq k$ and $[t_{k+1} - \varepsilon, t_{k+1} + \varepsilon]$ are disjoint. For $0 \leqq s \leqq \varepsilon$ define $A_s = \bigcup\limits_{j=1}^{k} [t_j - f_j(-s), t_j + f_j(s)] \cup [t_{k+1} - s, t_{k+1} + s]$. Now let $G \colon \mathbf{R} \to \mathbf{R}$ be given by

$$G(x) = \begin{cases} \varepsilon & \text{for } x \notin A_\varepsilon \\ s & \text{for } x \in \partial A_s, \text{ i.e., for } x = t_j + f_j(\pm s),\ 1 \leqq j \leqq k \text{ and } x = t_{k+1} \pm s. \end{cases}$$

Locally, in $[t_j, t_j + f_j(\varepsilon)]$, G is the inverse function of $t_j + f_j(s)$; in $[t_j + + f_j(-\varepsilon); t]$, it is the inverse function of $t_j + f_j(-s)$. Thus G is continuous, actually even differentiable except for finitely many points. □

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RIJKSUNIVERSITEIT GRONINGEN
WISKUNDE
POSTBUS 800
9700 AV GRONINGEN
THE NETHERLANDS

DEPARTMENT OF MATHEMATICS
BOSTON UNIVERSITY
111 CUMMINGTON ST.
BOSTON, MA 02215
U.S.A.

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HERMITE INTERPOLATION AT THE ZEROS OF CERTAIN FREUD-TYPE ORTHOGONAL POLYNOMIALS

H. N. MHASKAR* (Los Angeles) and Y. XU (Austin)

1. Introduction. Let

$$(1.1)\qquad A := \{x_{kn},\ k = 1,\dots,n,\ n = 1,2\dots\},$$

$$(1.2)\qquad B := \{y_{kn},\ k = 1,\dots,n,\ n = 1,2\dots\}$$

be triangular schemes of real numbers and $f\colon \mathbf{R} \to \mathbf{R}$. The Hermite interpolation polynomial $H_n(f,x) := H_n(A,B,f,x)$ is the unique polynomial of degree at most $2n-1$ which satisfies

$$(1.3)\quad H_n(f,x_{kn}) = f(x_{kn}),\ \ H_n'(f,x_{kn}) = y_{kn},\ \ k = 1,\dots,n,\ \ n = 1,2\dots.$$

In 1930, L. Fejér proved the following

THEOREM 1.1 [24]. *Let f be continuous on $[-1,1]$, $x_{kn} = \cos\left(\frac{2k-1}{2n}\pi\right)$, $k = 1,\dots,n$, $n = 1,2,\dots$ and*

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq n} \left\{ \frac{\log n}{n} \sqrt{1 - x_{kn}^2}\,|y_{kn}| \right\} = 0.$$

Then the sequence $\{H_n(A,B,f)\}$ converges to f uniformly on $[-1,1]$ as $n \to \infty$.

There is a great deal of literature concerning the behavior of $\{H_n(A,B,f)\}$ for various choices of A,B and the function classes to which f belongs. In particular, it is known (cf. [30]) that when $\{x_{kn}\}$ are the zeros of the Jacobi polynomials and the y_{kn}'s are uniformly bounded, then, for every $f\colon [-1,1] \to \mathbf{R}$, continuous on $[-1,1]$, the sequence $\{H_n(A,B,f)\}$ converges to f uniformly on compact subintervals of $(-1,1)$ as $n \to \infty$. The interpolation polynomials $\{H_n(A,B,f)\}$ are analyzed in an even greater detail when all y_{kn}'s are equal to zero, in which case, $H_n(A,B,f)$ is called the Hermite–Fejér interpolation polynomial. In many cases of interest, such as when the nodes x_{kn} are as in Theorem 1.1, H_n is a positive operator. We do not intend to give a complete survey here, but quote the following theorem, which is relevant to the present work.

* Part of this author's work was done during his visit to the Center for Approximation Theory, Texas A & M University, College Station, Texas, during the Fall of 1988.

THEOREM 1.2 [3]. *Let $f\colon [-1,1] \to \mathbf{R}$ be continuous on $[-1,1]$ and for $\delta > 0$, we put*

$$(1.4) \qquad \omega(f,\delta) := \max\{|f(x)-f(t)|, \quad |x-t| \leqq \delta, \quad x,t \in [-1,1]\}.$$

Let $\{x_{kn}\}$ be as in Theorem 1.1 and y_{kn}'s be all equal to zero. Then for $x \in [-1,1]$,

$$(1.5) \qquad |H_n(f,x) - f(x)| \leqq \frac{c}{n}\sum_{k=1}^{n}\omega\Big(f,\frac{1}{k}\Big), \qquad n = 1,2,\dots$$

where c is an absolute constant.

In the case when $\{x_{kn}\}$ are the zeros of orthogonal polynomials with respect to weight functions whose support is the whole real line, relatively less is known about the polynomials $H_n(A,B,f)$. There is an extensive literature when the nodes are the zeros of either Laguerre or Hermite polynomials. We will not venture to give a complete list, but quote ([1], [11], [26], [28], [29], [31]) as examples. When the weight function is a *general* one, we are aware of only [12], where processes closely related to Hermite–Fejér interpolation are studied. The results in [12] are stated, however, only when f satisfies certain restrictive growth conditions.

With the aid of the recent research in the theory of orthogonal polynomials, it has now become possible for us to obtain the analogues of Theorems 1.1 and 1.2 in the case when the x_{kn}'s are the zeros of polynomials orthogonal on the whole real line with respect to a weight function of the form $w_Q^2(x) := \exp(-2Q(x))$ which satisfies various technical conditions. Thus, in this paper, we assume only that $w_Q^p f \in C_0(\mathbf{R})$ for some p, $0 < p < 2$ and show that under certain conditions on B, $\{H_n(A,B,f)\}$ converges uniformly on compact subsets of $\mathbf{R}$ when x_{kn}'s are as described. Moreover, we use a modulus of continuity which is "natural" for weighted polynomial approximation, to give an estimate analogous to (1.5) for the rate of convergence. We observe that, unlike the "classical" case, the Hermite–Fejér interpolation operator in our case is not a positive operator. Thus, just as in the case of the general Jacobi nodes, we do not expect uniform convergence on the whole real line, even after multiplication by w_Q. The novelty of our paper lies in the fact that we assume very little of the function being approximated, and in our use of the modified modulus of continuity. The proofs use the ideas in [21] and [13] as well as many recent estimates on the orthogonal polynomials, including a differential equation which they satisfy.

In Section 2, we discuss our main results. In Section 3, we discuss certain preliminary facts concerning the interpolation process and the modulus of continuity, as well as review the known estimates on quantities related to orthogonal polynomials which will be needed. The proofs are completed in Section 4 except for a technical estimate which is proved in an Appendix.

The authors would like to thank Professor J. Szabados for his keen interest and suggestions for improvement in the presentation of this work.

2. Main results. Throughout this paper, we shall adopt the following convention concerning constants. The lower case letters $c, c_1, c_2, \ldots$ etc. will denote constants depending only upon the weight function in question, unless otherwise indicated. Their value may be different at different occurrences, even within a single formula. The constants denoted by capital letters will retain their values.

We consider weight functions of the form $w_Q(x) := \exp(-Q(x))$ where Q satisfies each of the following properties.

(W1) *Q is an even, convex function in $C^2(0,\infty)$ and Q'' is nondecreasing on $(0,\infty)$.*

(W2) *There exist constants $c_1, c_2 > 0$ such that*

$$1 < c_1 \leqq \frac{xQ''(x)}{Q'(x)} \leqq c_2 < \infty, \qquad x \in \mathbf{R}. \tag{2.2}$$

(W3) *Let, for $x \in \mathbf{R}$, $\delta > 0$,*

$$\operatorname{osc}(Q'', x, \delta) := \max_{|t-x| \leqq \delta} |Q''(t) - Q''(x)| \tag{2.2}$$

and denote q_n the least positive solution of the equation

$$q_n Q'(q_n) = n, \qquad n = 1, 2, \ldots. \tag{2.3}$$

Then, for any $c > 0$

$$\int_0^1 \frac{\operatorname{osc}(Q'', x, t)}{t} dt \leqq c_1 \frac{n}{q_n}, \qquad |x| \leqq c q_n. \tag{2.4}$$

The prototypical weights which satisfy all of these conditions are $\exp(-|x|^\alpha)$, $\alpha \geqq 2$.

If $n \geqq 0$ is an integer, let Π_n denote the class of all polynomials of degree at most n. We denote by $\{p_n\}_{n=0}^\infty$ the system of polynomials orthonormal on $\mathbf{R}$ with respect to w_Q^2, i.e.

$$\int p_n(t) p_m(t) w_Q^2(t) dt = \delta_{mn}, \qquad p_n \in \Pi_n \tag{2.5}$$

and introduce the notation

$$p_n(x) =: \gamma_n \prod_{k=1}^n (x - x_{kn}) \tag{2.6a}$$

where

$$\gamma_n > 0, \qquad -X_n := x_{nn} < x_{n-1,n} < \ldots < x_{1n} = X_n. \tag{2.6b}$$

Let $B := \{y_{kn}\}$ be a triangular scheme of real numbers, which will be fixed throughout this paper. We let

$$(2.7)\qquad Y_n := \max_{1 \leqq k \leqq n} |w_Q(x_{kn}) y_{kn}|.$$

If $f \colon \mathbf{R} \to \mathbf{R}$, we shall denote for the sake of brevity, the Hermite interpolation polynomial $H_n(A, B, f, x)$ by $H_n(f, x)$, where $A = \{x_{kn}\}$, $B = \{y_{kn}\}$.

In order to state our results concerning the convergence properties of $H_n(f, x)$, we need to introduce a modified modulus of continuity (equation (2.12)) introduced in [9], [10] and found useful in the study of weighted polynomial approximation (cf. [18], [19]). If $g \in C_0(\mathbf{R})$, we set

$$(2.8)\qquad \|g\| := \max_{x \in \mathbf{R}} |g(x)|$$

and define the difference operator by

$$(2.9)\qquad \Delta_t g(x) := g(x+t) - g(x), \qquad x, t \in \mathbf{R}.$$

Let $w_Q f \in C_0(\mathbf{R})$ and $\delta > 0$. We set

$$(2.10)\qquad \Omega(Q, f, \delta) := \sup_{|t| \leqq \delta} \|\Delta_t(w_Q f)\| + \delta \|Q'_\delta w_Q f\|$$

where

$$(2.11)\qquad Q'_\delta(x) := \min\{\delta^{-1}, [1 + Q'^2(x)]^{1/2}\}.$$

The modified modulus of continuity is then defined by the formula

$$(2.12)\qquad \Omega^*(Q, f, \delta) := \inf_{a \in \mathbf{R}} \Omega(Q, f - a, \delta).$$

We wish to point out that there is a different, equivalent expression for the modulus of continuity which is even more useful in certain applications ([5]). For our purposes, however, it is more convenient to use the expression (2.12).

Our main result can now be formulated as follows.

THEOREM 2.1. *Let w_Q satisfy the conditions* (W1), (W2), (W3) *and q_n be as defined in* (2.3). *We assume further that*

$$(2.13)\qquad |w_Q(t) p_n(t)| \leqq c q_n^{-1/2}, \qquad |t| \leqq A^* q_n, \qquad n = 1, 2, \ldots .$$

Let $p \in (0, 2)$, $w_Q^p f \in C_0(\mathbf{R})$ and $n \geqq 1$ be an integer. Then there exist positive constants D and c depending only on Q such that for every real x with $|x| \leqq D q_n$,

$$(2.14)\qquad |H_n(f, x) - f(x)| \leqq c\left(\frac{q_n}{n} \log n\right) Y_n + c(1 + |x|)(1 + p|Q'(x)|) w_Q^{-2-p}(x) \times$$
$$\times \left\{ \Omega^*\left(pQ, f, \frac{q_n}{n}\right) + \frac{q_n p_n^2(x) w_Q^2(x)}{n} \sum_{k=1}^{n} \Omega^*\left(pQ, f, \frac{q_n}{k}\right) \right\}.$$

We note that an argument similar to the one in [21] shows that the second term on the right hand side of (2.14) tends to 0 as $n \to \infty$. Hence, in particular, the Hermite–Fejér process converges uniformly on compact subsets of the real line. We also note the following corollary concerning the convergence of the Hermite process.

COROLLARY 2.2. *Let the conditions of Theorem* 2.1 *be satisfied and in addition, we assume that f is continuously differentiable and that $w_Q^p f' \in \in C_0(\mathbf{R})$. For $k = 1, \dots, n$, let $y_{kn} = f'(x_{kn})$. Then, for any compact interval $I \subset \mathbf{R}$ and $x \in I$,*

$$|H_n(f,x) - f(x)| \leqq c\left(\frac{q_n}{n}\log n\right)\varepsilon_n(pQ, f'), \tag{2.15}$$

where c is a positive constant depending only on Q, p, and I and

$$\varepsilon_n(pQ, f') := \min_{P\in\Pi_n} \|w_Q^p(f' - P)\|. \tag{2.16}$$

(2.13) seems to be an important condition, occurring as a hypothesis in many theorems (cf. [12], [13], [14], [21]). It is now known to be true for a fairly general class of weight functions including $\exp(-|x|^\alpha)$ when $\alpha > 3$ ([16]). It is announced in [15], without proof, that it is satisfied for the weight functions $\exp(-|x|^\alpha)$ when $\alpha > 0$ as conjectured in [25]. When the results in [15] are proved, our theorem will be valid for the weight functions $\exp(-|x|^\alpha)$ when $\alpha \geqq 2$. Currently, it is valid for these weights when $\alpha = 2$ and when $\alpha > 3$.

3. Preliminaries. In this section, we review certain known facts about orthogonal polynomials, the Hermite interpolation polynomials and the modulus of continuity (2.12). We adopt the following notation. $A \sim B$ will mean that $c_1 A \leqq B \leqq c_2 A$.

We begin with certain facts concerning orthogonal polynomials.

PROPOSITION 3.1 ([6], [13]). (a) *(Recurrence relation.)*

$$xp_{n-1}(x) = \varrho_n p_n(x) + \beta_n p_{n-1}(x) + \varrho_{n-1}p_{n-2}(x), \quad x \in \mathbf{R}, \quad n = 2, 3, \dots, \tag{3.1}$$

where

$$\varrho_n := \gamma_{n-1}/\gamma_n, \tag{3.2a}$$

$$\beta_n := \int t p_{n-1}^2(t) w_Q^2(t)\,dt. \tag{3.2b}$$

When Q is even, then $\beta_n = 0$.

(b) *(Christoffel–Darboux formula.)*

$$K_n(x,t) := \sum_{k=0}^{n-1} p_k(x)p_k(t) = \varrho_n \frac{p_n(x)p_{n-1}(t) - p_n(t)p_{n-1}(x)}{x - t}. \tag{3.3}$$

(c) *If Φ is a linear functional on Π_{n-1}, then we define the Φ-Christoffel function by the formula*

$$\Lambda_n(\Phi) := \min_{P\in\Pi_{n-1}} (\Phi(P))^{-2} \int |P(t)|^2 w_Q^2(t)\,dt. \tag{3.4}$$

Then

$$\Lambda_n(\Phi) = \left\{\sum_{k=0}^{n-1}(\Phi(p_k))^2\right\}^{-1}. \tag{3.5}$$

In particular, when $\Phi_x(P) := P(x)$, *then*

$$\lambda_n(x) := \Lambda_n(\Phi_x) = (K_n(x,x))^{-1}, \tag{3.6}$$

$$\lambda_n^{-1}(x) = \varrho_n[p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)]. \tag{3.7}$$

(d) *(Quadrature formula.) For every* $P \in \Pi_{2n-1}$,

$$\sum_{k=1}^{n} \lambda_{kn} P(x_{kn}) = \int P(t) w_Q^2(t)dt \tag{3.8}$$

where x_{kn} *are as in* (2.6b) *and*

$$\lambda_{kn} := \lambda_n(x_{kn}). \tag{3.9}$$

Next, we develop a differential equation satisfied by p_n.

PROPOSITION 3.2 ([4], [2], [20]). *Let Q be twice continuously differentiable. Then, for $n = 1,2,\dots$ and $x \in \mathbf{R}$,*

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) \tag{3.10}$$

where, with

$$\overline{Q}(x,t) := \frac{Q'(t) - Q'(x)}{t - x}, \tag{3.11}$$

we have,

$$A_n(x) := 2\varrho_n \int p_n^2(t)w_Q^2(t)\overline{Q}(x,t)dt, \tag{3.12a}$$

$$B_n(x) := 2\varrho_n \int p_n(t)p_{n-1}(t)w_Q^2(t)\overline{Q}(x,t)dt. \tag{3.12b}$$

Moreover, for $n = 2,3,\dots$ and $x \in \mathbf{R}$,

$$p_n''(x) + M_n(x)p_n'(x) + N_n(x)p_n(x) = 0 \tag{3.13}$$

where,

$$M_n(x) := -2Q'(x) - A_n'(x)/A_n(x) \tag{3.14}$$

and

$$N_n(x) := \frac{A_n(x)A_{n-1}(x)\varrho_n}{\varrho_{n-1}} - \frac{A_{n-1}(x)B_n(x)(x-\beta_n)}{\varrho_{n-1}} + \\ + B_n(x)B_{n-1}(x) + B_n'(x) - \frac{A_n'(x)}{A_n(x)}B_n(x). \tag{3.15}$$

Next, we state a representation for $H_n(f,x)$ in terms of the *fundamental polynomials.*

PROPOSITION 3.3 ([30]). *We have, for $n = 1,2,\dots,$*

$$(3.16)\qquad H_n(f,x)=\sum_{k=1}^{n} f(x_{kn})v_{kn}(x)l_{kn}^2(x)+\sum_{k=1}^{n} y_{kn}(x-x_{kn})l_{kn}^2(x)$$

where

(3.17)

$$l_{kn}(x):=\frac{p_n(x)}{p_n'(x_{kn})(x-x_{kn})}=\varrho_n\lambda_{kn}p_{n-1}(x_{kn})\frac{p_n(x)}{x-x_{kn}}=\lambda_{kn}K_n(x,x_{kn}),$$

and

$$(3.18)\qquad v_{kn}(x)=1-2l_{kn}'(x_{kn})(x-x_{kn})=1-\frac{p_n''(x_{kn})}{p_n'(x_{kn})}(x-x_{kn})=$$
$$=1+\lambda_{kn}^{-1}(\lambda_n'(x_{kn}))(x-x_{kn}).$$

In view of (3.13) and (3.14), we obtain an alternative expression for v_{kn} when Q is twice continuously differentiable. We observe that in this case,

$$(3.19)\qquad v_{kn}(x)=1-\left[2Q'(x_{kn})+\frac{A_n'(x_{kn})}{A_n(x_{kn})}\right](x-x_{kn}).$$

This observation plays a crucial role in our proof of Theorem 2.1.

Next, we state certain estimates concerning the various quantities related to the system $\{p_n\}$. However, we note that some of the statements in the next proposition are valid under conditions weaker than those stated below.

PROPOSITION 3.4. *Let w_Q satisfy the conditions* (W1), (W2), (W3) *stated near the beginning of Section 2. Then, there exist positive constants $c, c_1, c_2, \ldots$ and $D_1, D_2 \ldots$ such that each of the following statements hold.*

(a) ([7])

$$(3.20)\qquad cq_n \leqq X_n \leqq D_{1q_n},\qquad \varrho\sim q_n,\qquad q_n\sim q_{2n}.$$

(b) ([7], [8]) *For* $x\in\mathbf{R}$,

$$(3.21)\qquad \sum_{k=0}^{n-1} p_k^2(x)\leqq c(n/q_n)w_Q^{-2}(x),$$

$$(3.22)\qquad \sum_{k=0}^{n-1} p_k'^2(x)\leqq c(n/q_n)^3w_Q^{-2}(x).$$

(c) ([7]) *For* $\max\{|x|,|x_{kn}|,|x_{k+1,n}|\}\leqq D_2q_n$,

$$(3.23)\qquad \lambda_n(x)\leqq c(q_n/n)w_Q^2(x),$$

$$(3.24)\qquad D_3\frac{q_n}{n}\leqq x_{kn}-x_{k+1,n}\leqq D_4\frac{q_n}{n}.$$

(d)

$$(3.25)\qquad w_Q(x)\sim w_Q(y)\quad \textit{if}\quad |x-y|\leqq c(q_n/n).$$

(e) [(20)] *For every* $c > 0$ *and* $|x| \leqq cq_n$,

$$A_n(x) \sim n/q_n. \tag{3.26}$$

(e) ([13]) *Let* $0 < p < 2$. *Then for integer* k, $1 \leqq k \leqq n$, *we have,*

$$\lambda_{kn} w_Q^{-p}(x_{kn}) \leqq \int_{x_{k+1,n}}^{x_{k-1,n}} w_Q^{2-p}(t)dt \tag{3.27}$$

where

$$x_{n+1,n} := \infty, \qquad x_{0,n} := -\infty. \tag{3.28}$$

We observe that (3.26) is stated in [20] under more restrictive conditions on the weight function, but the proof reveals that the conditions (W1), (W2), (W3) are enough. Also, (3.27) is not stated explicitly in [13]. However, it is an easy consequence of Lemma 3.2 in [13] if we observe that there exists an even entire function G with positive Taylor coefficients such that

$$G(x) \sim w_Q^{-p}(x), \qquad x \in \mathbf{R} \tag{3.29}$$

(cf. [14]). In fact, Lemma 3.2 of [13] is used in [13] in this way.

Finally, we state a known result about the modulus of continuity $\Omega^*(Q, f, \delta)$ defined in (2.12), which is true for a far more general class of weight functions. Let $w_Q f \in C_0(\mathbf{R})$, $\delta > 0$. We define a K-functional by the formula

$$K(Q, f, \delta) = \inf[\|w_Q(f-g)\| + \delta\|w_Q g'\|] \tag{3.30}$$

where the inf is taken over all continuously differentiable functions g such that $w_Q g' \in C_0(\mathbf{R})$.

PROPOSITION 3.5 ([9], [10]). *Let Q satisfy the conditions* (W1), (W2), (W3). *Then for* $w_Q f \in C_0(\mathbf{R})$ *and* $0 < \delta \leqq 1$,

$$\Omega^*(Q, f, \delta) \sim K(Q, f, \delta). \tag{3.31}$$

4. Proofs. In order to prove Theorem 2.1, we observe that

$$\sum_{k=1}^{n} v_{kn}(x) l_{kn}^2(x) = 1 \quad \text{for every} \quad x \in \mathbf{R}. \tag{4.1}$$

Hence, for $x \in \mathbf{R}$,

(4.2)

$$H_n(f, x) - f(x) = \sum_{k=1}^{n} (f(x_{kn}) - f(x)) v_{kn}(x) l_{kn}^2(x) + \sum_{k=1}^{n} y_{kn}(x - x_{kn}) l_{kn}^2(x).$$

We start by replacing the differences $f(x_{kn}) - f(x)$ in the above sum by $\Omega^*(pQ, f, |x - x_{kn}|)$, where we recall that $0 < p < 2$, and $w_Q^{-p} f \in C_0(\mathbf{R})$. In the sequel, we shall write w_{kn} instead of $w_Q(x_{kn})$, w instead of w_Q.

LEMMA 4.1. *We have,*

$$|f(x_{kn}) - f(x)| \leqq c(1 + p|Q'(x)|)w^{-p}(x)w_{kn}^{-p}\Omega^*(pQ, f, |x - x_{kn}|). \tag{4.3}$$

PROOF. Without loss of generality, we may assume that Q is nonnegative on $\mathbf{R}$ and take $p = 1$. Let $h := x - x_{kn}$. Then,

$$|f(x_{kn}) - f(x)| \leqq w^{-1}(x)|\Delta_h(wf)(x_{kn})| + |w_{kn}f(x_{kn})||w_{kn}^{-1} - w^{-1}(x)|. \tag{4.4}$$

Now, since $Q'w^{-1}$ is an increasing function on $[0, \infty)$, the mean value theorem yields

$$\begin{aligned} |w_{kn}^{-1} - w^{-1}(x)| &= |w^{-1}(|x_{kn}|) - w^{-1}(|x|)| \leqq \\ &\leqq \max\{|Q'(x_{kn})|w_{kn}^{-1}, |Q'(x)|w^{-1}(x)\}||x| - |x_{kn}|| \leqq \\ &\leqq [1 + |Q'(x)|]w^{-1}(x)w_{kn}^{-1}[1 + |Q'(x_{kn})|]|x - x_{kn}| \leqq \\ &\leqq \sqrt{2}[1 + |Q'(x)|]w^{-1}(x)w_{kn}^{-1}[1 + Q'^2(x_{kn})]^{1/2}|x - x_{kn}|. \end{aligned} \tag{4.5}$$

Also,

$$|w_{kn}^{-1} - w^{-1}(x)| \leqq w_{kn}^{-1} + w^{-1}(x) \leqq 2w^{-1}(x)w_{kn}^{-1}. \tag{4.6}$$

With the notation of (2.11), the estimates (4.5), (4.6) yield

$$|w_{kn}^{-1} - w^{-1}(x)| \leqq c[1 + |Q'(x)|]w^{-1}(x)Q'_{|h|}(x_{kn})|x - x_{kn}|w_{kn}^{-1}. \tag{4.7}$$

Substituting from (4.7) into (4.4), we get

$$\begin{aligned} |f(x_{kn}) - f(x)| &\leqq c[1 + |Q'(x)|]w^{-1}(x)w_{kn}^{-1}\{|\Delta_h(wf)(x_{kn})| + \\ &+ |h|Q'_{|h|}(x_{kn})w_{kn}|f(x_{kn})|\} \leqq c[1 + |Q'(x)|]w^{-1}(x)w_{kn}^{-1}\Omega(Q, f, |h|). \end{aligned} \tag{4.8}$$

The estimate (4.3) follows from (4.8). □

Using the estimates (4.3) and (2.7) in (4.2), we arrive at

$$\begin{aligned} &|H_n(f, x) - f(x)| \leqq \\ &\leqq c[1 + p|Q'(x)|]w^{-p}(x)\Big\{\sum_{k=1}^{n} w_{kn}^{-p}\Omega^*(pQ, f, |x - x_{kn}|)|v_{kn}(x)|l_{kn}^2(x)\Big\} + \\ &\qquad + Y_n \sum_{k=1}^{n} w_{kn}^{-p}|x - x_{kn}|l_{kn}^2(x). \end{aligned} \tag{4.9}$$

Thus, in order to prove Theorem 2.1, we need to estimate the two sums on the right hand side of (4.9). The details of this estimation are organized in Lemmas 4.2, 4.3 and 4.4.

We assume that

$$(4.10)\qquad |x| \leqq Dq_n, \qquad D := \frac{1}{4}\min\Big\{D_1, D_2, \frac{1}{2}A^*\Big\}$$

where D_1, D_2 are defined in Proposition 3.4 and A^* is the constant appearing in (2.13). With D_4 as in (3.24), we define three sets of indices

$$(4.11a)\qquad N := \{k\colon k \text{ integer}, \ 1 \leqq k \leqq n, \ |x - x_{kn}| \leqq 2D_4 q_n/n\},$$

$$(4.11b)\qquad M := \{k\colon k \text{ integer}, \ 1 \leqq k \leqq n, \ |x - x_{kn}| > 2D_4 q_n/n, \ |x_{kn}| \leqq 2Dq_n\},$$

$$(4.11c)\qquad F := \{k\colon k \text{ integer}, \ 1 \leqq k \leqq n, \ |x_{kn}| > 2Dq_n\}.$$

We note that the set N can contain at most $4D_4/D_3$ members, where D_3 is defined in (3.24).

LEMMA 4.2. *Let ϕ be a nondecreasing, nonnegative function on $[0,\infty)$ such that $\phi(2u) \sim \phi(u)$ for $u \in [0,\infty)$. Then,*

$$(4.12)\qquad \sum_{k\in N} \lambda_{kn}^{-1}\phi(|x - x_{kn}|)l_{kn}^2(x) \leqq c w_Q^{-2}(x)\frac{n}{q_n}\phi\Big(\frac{q_n}{n}\Big),$$

$$(4.13)\qquad \sum_{k\in M} \lambda_{kn}^{-1}\phi(|x - x_{kn}|)l_{kn}^2(x) \leqq c q_n p_n^2(x)\int_{c_1 q_n/n}^{c_2 q_n} \frac{\phi(u)}{u^2}\,du,$$

$$(4.14)\qquad \sum_{k\in F} \lambda_{kn}^{-1}\phi(|x - x_{kn}|)l_{kn}^2(x) \leqq c p_n^2(x)\phi(q_n).$$

PROOF. In view of (3.17) and the Schwarz inequality,

$$(4.15)\qquad l_{kn}^2(x) = \lambda_{kn}^2 K_n^2(x, x_{kn}) \leqq \lambda_{kn} K_n(x,x).$$

The estimate (4.12) now follows from (3.21), the conditions on ϕ and the fact that N contains at most $4D_4/D_3$ members. Next, using (3.17), (3.20), (2.13), (3.23) and (3.24), we see that

$$(4.16)\qquad l_{kn}^2(x) = \varrho_n^2\lambda_{kn}^2 p_{n-1}^2(x_{kn})\frac{p_n^2(x)}{(x - x_{kn})^2} \leqq c q_n^2 \lambda_{kn}^2 q_n^{-1} w_{kn}^{-2}\frac{p_n^2(x)}{(x - x_{kn})^2} \leqq$$
$$\leqq c q_n p_n^2(x)\lambda_{kn}\frac{q_n}{n}\frac{1}{(x - x_{kn})^2} \leqq c q_n p_n^2(x)\lambda_{kn}\frac{x_{kn} - x_{k+1,n}}{(x - x_{kn})^2}.$$

Now, using (3.24) and the definition of M (equation (4.11b)), it is easy to verify that

$$(4.17)\qquad |x - t| \sim |x - x_{kn}|, \qquad t \in [x_{k+1,n}, x_{kn}], \qquad k \in M.$$

So using (4.16) and the properties of ϕ we get

$$\sum_{k\in M}\lambda_{kn}^{-1}\phi(|x-x_{kn}|)l_{kn}^2(x) \leqq \tag{4.18}$$

$$\leqq cq_np_n^2(x)\sum_{k\in M}\frac{\phi(|x-x_{kn}|)}{(x-x_{kn})^2}(x_{kn}-x_{k+1,n}) \leqq cq_np_n^2(x)\int\limits_{c_1q_n/n}^{c_2q_n}\frac{\phi(u)}{u^2}du.$$

The estimate (4.13) is thus proved.

When $k\in F$, we have $|x-x_{kn}|\sim q_n$. So, using (3.17), (3.20), the properties of ϕ and the quadrature formula (3.8) with $P=p_{n-1}^2\in\Pi_{2n-2}$, we get

(4.19)

$$\sum_{k\in F}\lambda_{kn}^{-1}\phi(|x-x_{kn}|)l_{kn}^2(x) \leqq c\phi(q_n)\sum_{k\in F}\varrho_n^2p_n^2(x)\lambda_{kn}p_{n-1}^2(x_{kn})(x-x_{kn})^{-2} \leqq$$

$$\leqq c\phi(q_n)p_n^2(x)\sum_{k\in F}\lambda_{kn}p_{n-1}^2(x_{kn}) \leqq c\phi(q_n)p_n^2(x).$$

This proves (4.14). $\square$

LEMMA 4.3. *We have*

$$\lambda_{kn}w_{kn}^{-p}|v_{kn}(x)| \leqq c\frac{q_n}{n}(1+|x|), \qquad k\in N\cup M, \tag{4.20}$$

$$\lambda_{kn}w_{kn}^{-p}|v_{kn}(x)| \leqq c\exp(-c_1n), \qquad k\in F. \tag{4.21}$$

PROOF. If $k\in N\cup M$, then $|x_{kn}|\leqq (A^*/2)q_n$. Consequently, in view of Proposition A.1 in the Appendix, there exists $c>0$ such that

$$\left|\frac{A_n'(x_{kn})}{A_n(x_{kn})}\right| \leqq c. \tag{4.22}$$

Hence, (3.19) yields

(4.23)

$$|v_{kn}(x)| \leqq c[1+(2|Q'(x_{kn})|+1)(|x|+|x_{kn}|)] \leqq c(1+|x|)(1+|x_{kn}|)(1+|Q'(x_{kn})|).$$

Since the function $w^{2-p}(y)(1+|y|)(1+|Q'(y)|)$ is bounded from above on $\mathbf{R}$, (4.23) gives

$$w_{kn}^{2-p}|v_{kn}(x)| \leqq c(1+|x|).$$

The estimate (4.20) now follows from (3.23).

To prove (4.21), we first obtain a general estimate. Let $y \in \mathbf{R}$. Using Schwarz inequality and (3.22), we get

$$(4.24)\quad |\lambda_n'(y)| = \left|\frac{d}{dy}\left(\sum p_k^2(y)\right)^{-1}\right| = 2\left(\sum p_k^2(y)\right)^{-2}\left|\sum p_k'(y)p_k(y)\right| \leqq$$
$$\leqq 2\lambda_n^{3/2}(y)\left(\sum p_k'^2(y)\right)^{1/2} \leqq c\left(\frac{n}{q_n}\right)^{3/2} w^{-1}(y)\lambda_n^{3/2}(y).$$

In particular,

$$(4.25)\qquad |\lambda_{kn}'/\lambda_{kn}| \leqq c\left(\frac{n}{q_n}\right)^{3/2} w_{kn}^{-1}\lambda_{kn}^{1/2}.$$

Thus, in view of (3.18), (3.20) and (3.21),

$$(4.26)\qquad |v_{kn}(x)| \leqq 1 + c\left(\frac{n}{q_n}\right)^{3/2} w_{kn}^{-1}\lambda_{kn}^{1/2} q_n \leqq c\left(\frac{n}{q_n}\right)^{3/2} w_{kn}^{-1}\lambda_{kn}^{1/2} q_n.$$

Now, let $k \in F$. Since $\frac{2}{3}(p+1) < 2$, it is easy to see using Proposition 3.4(f) that

$$(4.27)\quad \lambda_{kn} w_{kn}^{-\frac{2}{3}(p+1)} \leqq \int_{x_{k+1,n}}^{x_{k-1,n}} [w(t)]^{2-\frac{2}{3}(p+1)}dt \leqq \int_{cq_n}^{\infty} [w(t)]^{2-\frac{2}{3}(p+1)}dt \leqq$$
$$\leqq c\frac{q_n}{n}\int_{cq_n}^{\infty} Q'(t)[w(t)]^{2-\frac{2}{3}(p+1)}dt \leqq c\frac{q_n}{n}w(cq_n)^{2-\frac{2}{3}(p+1)}.$$

In view of (4.26) and (4.27), if $k \in F$, then

$$(4.28)\quad \lambda_{kn} w_{kn}^{-p}|v_{kn}(x)| \leqq c(n/q_n)^{3/2} q_n w_{kn}^{-p-1}\lambda_{kn}^{3/2} \leqq c q_n w(cq_n)^{2-p} \leqq c\ \exp(-c_1 n).$$

This proves (4.21). □

LEMMA 4.4. *We have*

$$(4.29)\qquad \lambda_{kn} w_{kn}^{-p} \leqq c q_n/n, \qquad k \in N \cup M$$

$$(4.30)\qquad \lambda_{kn} w_{kn}^{-p} \leqq c\ \exp(-c_1 n), \qquad k \in F.$$

PROOF. The estimate (4.29) follows from (3.23) since w^{2-p} is bounded from above. The estimate (4.30) follows from Proposition 3.4 exactly as in (4.27). □

COMPLETION OF THE PROOF OF THEOREM 2.1. We shall denote $\Omega^*(pQ, f, \delta)$ by $\Omega^*(\delta)$. Then Ω^* is a nondecreasing, nonnegative function on $[0, \infty)$, and satisfies the condition $\Omega^*(2\delta) \sim \Omega^*(\delta)$, $\delta > 0$. Hence, (4.20), (4.12) imply that

$$(4.31)\qquad \sum_{k\in N} w_{kn}^{-p}\Omega^*(|x-x_{kn}|)|v_{kn}(x)|l_{kn}^2(x) \leqq$$

$$\leqq c(1+|x|)\frac{q_n}{n}\sum_{k\in N}\lambda_{kn}^{-1}\Omega^*(|x-x_{kn}|)l_{kn}^2(x) \leqq c(1+|x|)w^{-2}(x)\Omega^*(q_n/n).$$

Next, (4.20) and (4.13) imply that

$$(4.32)\qquad \sum_{k\in M} w_{kn}^{-p}\Omega^*(|x-x_{kn}|)v_{kn}(x)|l_{kn}^2(x) \leqq$$

$$\leqq c(1+|x|)\frac{q_n}{n}\sum_{k\in M}\lambda_{kn}^{-1}\Omega^*(|x-x_{kn}|)l_{kn}^2(x) \leqq c(1+|x|)q_np_n^2(x)\frac{q_n}{n}\int_{cq_n/n}^{c_1q_n}\frac{\Omega^*(u)}{u^2}du \leqq$$

$$\leqq c(1+|x|)q_np_n^2(x)\frac{1}{n}\int_{c_1}^{c_2n}\Omega^*(q_n/v)dv \leqq c(1+|x|)q_np_n^2(x)\frac{1}{n}\sum_{k=1}^{n}\Omega^*(q_n/k).$$

Next, (4.21) and (4.14) imply that

$$(4.33)\qquad \sum_{k\in F} w_{kn}^{-p}\Omega^*(|x-x_{kn}|)|v_{kn}(x)|l_{kn}^2(x) \leqq$$

$$\leqq c\exp(-c_1n)\sum_{k\in F}\lambda_{kn}^{-1}\Omega^*(|x-x_{kn}|)l_{kn}^2(x) \leqq cp_n^2(x)\exp(-c_1n)\Omega^*(q_n).$$

We get from (4.31), (4.32), (4.33) that

$$(4.34)\qquad \sum_{k=1}^{n} w_{kn}^{-p}\Omega^*(|x-x_{kn}|)|v_{kn}(x)|l_{kn}^2(x) \leqq$$

$$\leqq c(1+|x|)\Big\{w^{-2}(x)\Omega^*(q_n/n) + q_np_n^2(x)\frac{1}{n}\sum_{k=1}^{n}\Omega^*(q_n/k)\Big\}.$$

Next, we let $\phi(u)=u$. Then (4.29) and (4.12) give

$$(4.35)\quad \sum_{k\in N} w_{kn}^{-p}|x-x_{kn}|l_{kn}^2(x) \leqq c\frac{q_n}{n}\sum_{k\in N}\lambda_{kn}^{-1}|x-x_{kn}|l_{kn}^2(x) \leqq cw^{-2}(x)\frac{q_n}{n}.$$

Next, (4.29), (4.13) and (2.13) give

$$(4.36)\qquad \sum_{k\in M} w_{kn}^{-p}|x-x_{kn}|l_{kn}^2(x) \leqq c\frac{q_n}{n}\sum_{k\in M}\lambda_{kn}^{-1}|x-x_{kn}|l_{kn}^2(x) \leqq$$

$$\leqq cq_np_n^2(x)\frac{q_n}{n}\int_{c_2q_n/n}^{c_2q_n}\frac{du}{u} \leqq cw^{-2}(x)\frac{q_n}{n}\log n.$$

Next, (4.30), (4.14) and (2.13) give

$$(4.37) \sum_{k\in F} w_{kn}^{-p}|x-x_{kn}|l_{kn}^2(x) \leqq c\exp(-c_1 n)\sum_{k\in F}\lambda_{kn}^{-1}|x-x_{kn}|l_{kn}^2(x) \leqq$$
$$\leqq c\exp(-c_1 n)p_n^2(x)q_n \leqq c\exp(-c_1 n)w^{-2}(x).$$

In view of (4.35), (4.36), (4.37), we have proved

$$(4.38) \qquad \sum_{k=1}^{n} w_{kn}^{-p}|x-x_{kn}|l_{kn}^2(x) \leqq c\frac{q_n}{n}\log n w^{-2}(x).$$

Theorem 2.1 now follows from (4.9), (4.34) and (4.38). □

Appendix

We wish to show the following.

PROPOSITION A.1. *Let w_Q be a weight function which satisfies conditions* (W1), (W2), (W3) *and let* (2.13) *hold. Then, there exists a constant $c>0$ such that*

$$(A.1) \qquad |A_n'(x)/A_n(x)| \leqq c, \qquad |x| \leqq \frac{1}{2}A^*q_n$$

where A_n is defined in (3.12a).

PROOF. Using an argument based on the infinite-finite range inequalities of [22], [23], [17], we obtain, exactly as in [20] that

$$(A.2) \qquad 2\varrho_n \int_{|t|\geqq A_1^* q_n} p_n^2(t)w_Q^2(t)\left|\frac{\partial}{\partial x}\overline{Q}(x,t)\right| dt \leqq c\exp(-c_1 n),$$

for a suitable constant $A_1^* \geqq A^*+1$. Next, we observe that

$$(A.3) \quad \left|\frac{\partial}{\partial x}\overline{Q}(x,t)\right| = \frac{|Q'(t)-Q'(x)-Q''(x)(t-x)|}{(t-x)^2} \leqq \frac{\operatorname{osc}(Q'',x,|x-t|)}{|x-t|}.$$

Hence, using (2.13), we see that when $|x| \leqq (1/2)A^*q_n$,

$$(A.4) \quad 2\varrho_n \int_{|t-x|\leqq 1} p_n^2(t)w_Q^2(t)\left|\frac{\partial}{\partial x}\overline{Q}(x,t)\right| dt \leqq c\int_0^1 \frac{\operatorname{osc}(Q'',x,u)}{u}du \leqq c\frac{n}{q_n}.$$

Further,

(A.5)

$$2\varrho_n \int_{\substack{|t-x|\geqq 1\\ |t|\leqq A^* q_n}} p_n^2(t)w_Q^2(t)\left|\frac{\partial}{\partial x}\overline{Q}(x,t)\right| dt \leqq c\int_{\substack{|t-x|\geqq 1\\ |t|\leqq A^* q_n}} \frac{\operatorname{osc}(Q'',x,|x-t|)}{|x-t|}dt \leqq$$
$$\leqq cq_nQ''(c_1q_n) \leqq cQ'(q_n) \leqq cn/q_n.$$

If $|t| \geqq A^*q_n$ then necessarily $|x-t| \geqq (1/2)A^*q_n \geqq 1$. We then have

$$2\varrho_n \int\limits_{A^* q_n \leqq |t| \leqq A_1^* q_n} p_n^2(t) w_Q^2(t) \left| \frac{\partial}{\partial x} \overline{Q}(x,t) \right| dt \leqq \tag{A.6}$$

$$\leqq \int\limits_{A^* q_n \leqq |t| \leqq A_1^* q_n} p_n^2(t) w_Q^2(t) \operatorname{osc}(Q'', x, cq_n) dt \leqq cn/q_n^2.$$

The estimate (A.1) follows from (A.2), (A.4), (A.6) and (3.26). □

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DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY
LOS ANGELES, CA 90032
U.S.A.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS
AUSTIN, TX 78712
U.S.A.

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ERGODIC PROPERTIES OF SOME INVERSE POLYNOMIAL SERIES EXPANSIONS OF LAURENT SERIES

J. KNOPFMACHER (Johannesburg)

1. Introduction

Recently A. Knopfmacher and the present author [8] introduced and studied some properties of various unique expansions of formal Laurent series over a field F, as the sums of reciprocals of polynomials, involving "digits" $a_1, a_2, \ldots$ lying in a polynomial ring $F[X]$ over F. In particular, one of these expansions (described below) turned out to be analogous to the so-called Lüroth expansion of a real number, discussed in Perron [13] Chapter 4.

In a partly parallel way, Artin [1] and Magnus [9, 10] had earlier studied a Laurent series analogue of simple *continued fractions* of real numbers, involving "digits" $x_1, x_2, \ldots$ in a polynomial ring as above. In addition to sketching elementary properties of an n-dimensional "Jacobi–Perron" variant of this, Paysant–Leroux and Dubois [11, 12] also briefly outlined certain "metric" theorems analogous to some of Khintchine [7] for real continued fractions, in the case when F is a finite field. The main aim of this paper is to derive some similar metric or ergodic results for the Laurent series Lüroth-type expansion referred to above. (For analogous results concerning Lüroth expansions of *real* numbers, see Jager and de Vroedt [5] and Salát [14], and also [16, 17, 18].)

In order to explain the conclusions, we first fix some notation and describe the inverse-polynomial Lüroth-type representation to be considered:

Let $\mathcal{L} = F((z))$ denote the field of all *formal Laurent series* $A = \sum_{n=v}^{\infty} c_n z^n$ in an indeterminate z, with coefficients c_n all lying in a given field F. Although the main case of importance usually occurs when F is the field $\mathbf{C}$ of complex numbers, certain interest also attaches to other ground fields F and most of the results of [8] hold for arbitrary F. It will be convenient to write $X = z^{-1}$ and also consider the ring $F[X]$ of *polynomials* in X, and the field $F(X)$ of *rational functions* in X, with coefficients in F.

If $c_v \neq 0$, we call $v = v(A)$ the *order* of A above, and define the *norm* (or *valuation*) of A to be $\|A\| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant, but *later will be chosen* as $q = \operatorname{card}(F)$, if F is finite. Letting $v(0) = +\infty$, $\|0\| = 0$, one then has (cf. Jones and Thron [6] Chapter 5):

$$(1.1)\qquad \begin{cases} \|A\| \geqq 0 \text{ with } \|A\| = 0 \text{ iff } A = 0, \\ \|AB\| = \|A\| \cdot \|B\|, \text{ and} \\ \|\alpha A + \beta B\| \leqq \max(\|A\|, \|B\|) \text{ for non-zero } \alpha, \beta \in F, \\ \text{with equality when } \|A\| \neq \|B\|. \end{cases}$$

By (1.1), the norm $\| \ \|$ is non-Archimedean, and it is well known that $\mathcal{L}$ forms a *complete* metric space relative to the metric ϱ such that $\varrho(A, B) = \|A - B\|$.

In terms of the notation $X = z^{-1}$ above, we shall make frequent use of the polynomial $[A] = \sum_{v \leqq n \leqq 0} c_n X^{-n} \in F[X]$, and refer to $[A]$ as the *integral part* of $A \in \mathcal{L}$. Then $v = v(A)$ is the *degree* $\deg[A]$ of $[A]$ relative to X, and the same function $[\ ]$ was used by Artin [1] and Magnus [9, 10] for their continued fractions. (For a recent application of Artin's algorithm, F finite, see Hayes [4].)

Given $A \in \mathcal{L}$, now note that $[A] = a_0 \in F[X]$ iff $v(A_1) \geqq 1$ where $A_1 = A - a_0$. As in [8], if $A_n \neq 0$ $(n > 0)$ is already defined, we then let $a_n = \left[\frac{1}{A_n}\right]$ and put $A_{n+1} = (a_n - 1)(a_n A_n - 1)$. If some $A_m = 0$ or $a_n = 0$, this recursive process stops. It was shown in [5] that this algorithm leads to a finite or convergent (relative to ϱ) *Lüroth-type* series expansion

$$(1.2)\qquad A = a_0 + \frac{1}{a_1} + \sum_{r \geqq 2} \frac{1}{a_1(a_1 - 1) \dots a_{r-1}(a_{r-1} - 1) a_r},$$

where $a_r \in F[X]$, $a_0 = [A]$, and $\deg(a_r) \geqq 1$ for $r \geqq 1$. Furthermore this expansion is *unique* for A subject to the preceding conditions on the "digits" a_r.

If I denotes the ideal in the power series ring $F[[z]]$, consisting of all power series x such that $x(0) = 0$, then another way of looking at this expansion algorithm is in terms of operators $a\colon I - \{0\} \to F[X]$, $T\colon I \to I$ such that $a(x) = \left[\frac{1}{x}\right]$, $T(0) = 0$ and otherwise $T(x) = (a(x) - 1)(xa(x) - 1)$. Then, for $x = A_1 \in I$, $a_1 = a_1(x) = a(x)$, and more generally $a_n = a_n(x) = a_1(T^{n-1}x)$ if $0 \neq T^{n-1}x \in I$. It will be shown below that $x \in I \Rightarrow T(x) \in I$.

From now on, unless otherwise stated, it will be assumed that $F = \mathbf{F}_q$ is a *finite* field with exactly q elements. Then it will be shown below that $T\colon I \to I$ is *ergodic* relative to the Haar measure μ on I such that $\mu(I) = 1$. This fact will then be used to deduce in particular:

THEOREM 1. (i) *For any given polynomial $k \in \mathbf{F}_q[X]$, $\deg(k) \geqq 1$, and all $x \in I$ outside a set of Haar measure 0, the digit value k has asymptotic frequency*

$$\lim_{n \to \infty} \frac{1}{n} \#\{r \leqq n : a_r(x) = k\} = \|k\|^{-2} = q^{-2\deg(k)}.$$

(ii) *For all $x \in I$ outside a set of Haar measure 0 there exists a single asymptotic mean-value*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \deg(a_r(x)) = \frac{q}{q-1}.$$

(iii) *For all $x \in I$ outside a set of Haar measure* 0,

$$\|x-\omega_n\| = q^{\left(-\frac{2q}{q-1}+o(1)\right)n} \quad as \quad n\to\infty,$$

where

$$\omega_n = \omega_n(x) = \sum_{r=1}^{n} \frac{\lambda_{r-1}}{a_r}, \quad \lambda_0 = 1, \quad \lambda_r = \frac{1}{a_1(a_1-1)\dots a_r(a_r-1)}.$$

Before turning to proofs, it is interesting to note that the limit in (ii) above coincides with an analogous one quoted by Paysant–Leroux and Dubois [12] for their very different Jacobi–Perron expansion. Regarding (iii) above, it may also be noted that it was shown in [8] that

$$\|x-\omega_n\| \leqq q^{-2n-1} \quad \text{for all} \quad x,$$

a similar but weaker algebraic result.

2. Ergodic properties of the basic operator

In order to show that $x \in I \Rightarrow T(x) \in I$, first note that $0 \neq x \in I \Rightarrow \Rightarrow v\left(\frac{1}{x}\right) \leqq -1 \Rightarrow a(x) = \left[\frac{1}{x}\right] \neq 0$ and $v(a(x)-1) = v(a(x)) = v\left(\frac{1}{x}\right) \leqq -1$. Further

$$0 \neq x \in I \Rightarrow \frac{1}{x} = a(x) + \sum_{r\geqq 1} c_r' z^r, \quad \text{say} \quad \Rightarrow 1 = xa(x) + x\sum_{r\geqq 1} c_r' z^r \Rightarrow$$
$$\Rightarrow v(xa(x)-1) = v\Big(x\sum_{r\geqq 1} c_r' z^r\Big) \Rightarrow v(xa(x)-1) \geqq v(x)+1.$$

Thus (even if F were an infinite field)

$$0\neq x\in I \Rightarrow v(T(x)) = v(a(x)-1)+v(xa(x)-1) \geqq -v(x)+(v(x)+1) \Longrightarrow T(x)\in I.$$

LEMMA 1. *The operator $T\colon I \to I$ is ergodic relative to the Haar measure μ on I with $\mu(I) = 1$.*

PROOF. A convenient description of the Haar measure μ on the ideal I of power series x in $\mathbf{F}_q[[z]]$ with $x(0) = 0$ is given in Sprindžuk [15], pages 67–70. In particular $\mu(C) = q^{-r}$ for any *circle* ("disc", "ball")

$$C = C(x, q^{-r-1}) := \{y \in \mathcal{L} : \|x-y\| \leqq q^{-r-1}\}$$

of radius q^{-r-1}. So $\mu(I) = 1$, since $I = C(0, q^{-1})$.

Now note that every "digit" $a(x)$ lies in $\mathcal{F}_1 := \{k \in \mathbf{F}_q[X] : \deg(k) \geqq 1\}$. For any given digits $k_1, \dots, k_n \in \mathcal{F}_1$, let

$$I_n = I_n(k_1, \dots, k_n) := \{x \in I : a_1(x) = k_1, \dots, a_n(x) = k_n\}$$

and call I_n *a basic (Lüroth) cylinder* of *rank* n. Also let $I_0 = I$.

The Lüroth-type expansion (1.2) of any $x \in I_n$ then has the form

$$x = \omega_n + \lambda_n \sum_{r>n} \frac{1}{a_{n+1}(a_{n+1}-1)\dots a_{r-1}(a_{r-1}-1)a_r},$$

where $\lambda_0 = 1$, $\lambda_r = \frac{1}{k_1(k_1-1)\dots k_r(k_r-1)}$ for $1 \leqq r \leqq n$, and $\omega_n = \sum_{r=1}^{n} \frac{\lambda_{r-1}}{k_r}$. Thus $x = \omega_n + \lambda_n T^n(x) = \psi_n(T^n(x))$, if $\psi_n = \psi_n(k_1, \dots, k_n) : I \to I_n$ is defined by $\psi_n(y) = \omega_n + \lambda_n y$ ($y \in I$). The "linear-type" map ψ_n is then 1-1 onto, with inverse map $T^n \colon I_n \to I$. In particular $I_n = \operatorname{Im}(\psi_n) = \omega_n + \lambda_n I$. Since $I = C(0, q^{-1})$, it then follows that $I_n = C(\omega_n, q^{-1}\|\lambda_n\|)$ and has Haar measure $\mu(I_n) = q^{-v(\lambda_n)} = \|\lambda_n\|$. Hence

$$\mu(I_n) = \frac{1}{\|k_1(k_1-1)\dots k_n(k_n-1)\|} = \frac{1}{\|k_1k_2\dots k_n\|^2}, \tag{2.1}$$

since $\deg(k) = \deg(k-1)$ for $\deg(k) \geqq 1$.

Using (2.1), we readily deduce that the digit functions $a_r \colon I - \{0\} \to \mathcal{F}_1$ are *identically distributed independent random variables* relative to μ. In a standard way, quite similar to that followed by Jager and de Vroedt [5] for real series, one may then conclude that the operator T is measure-preserving and ergodic.

Knowing Lemma 1 and the fact that the a_r are identically-distributed random variables, various deductions may now be made with the aid of standard results like the Ergodic Theorem or the laws of large numbers. However, here we shall merely sketch a few arguments which include ones leading to Theorem 1 above:

By special choices of f in the ergodic formula

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n-1} f(T^{r-1}x) = \int_I f\,d\mu \quad \text{a.e.},$$

we obtain:

(i) If f is the characteristic function of a basic cylinder $I_1(k)$, we get

$$\lim_{n\to\infty} \frac{1}{n} \#\{r \leqq n : a_r(x) = k\} = \mu(I_1(k)) = \|k\|^{-2} \quad \text{a.e.}$$

(ii) If instead $f(x) = \deg(a_1(x))$, then for almost all x $\Big($since $I - \{0\} =$ $= \bigcup_{k\in\mathcal{F}_1} I_1(k)\Big)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\deg(a_r(x)) = \int\limits_{I-\{0\}} f d\mu = \sum_{k\in\mathcal{F}_1}\int\limits_{I_1(k)}\deg(a_1(x))d\mu =$$

$$= \sum_{k\in\mathcal{F}_1}\mu(I_1(k))\deg(k) = \sum_{r=1}^{\infty} rq^{-2r}\cdot(q-1)q^r = \frac{q}{q-1}.$$

(iii) It follows from (ii) that there exists a *Khintchine-type* constant

$$\lim_{n\to\infty}\|a_1(x)a_2(x)\dots a_n(x)\|^{1/n} = q^{q/(q-1)} \quad \text{a.e.}$$

(iv) If $f(x) = \|a_1(x)\|$, then $\int\limits_I f d\mu = +\infty$ but a well-known truncation argument shows that, in contrast with the finite asymptotic geometric-mean, we have $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\|a_r(x)\| = +\infty$ a.e.

(v) By choosing f to be the characteristic function of any given circle $C \subseteqq I$, it may be deduced that the values $x, Tx, T^2x, \dots$ are *uniformly distributed* in I, for almost all x.

(vi) Every non-rational $x \in I$ can be written as $x = \lim\limits_{n\to\infty}\omega_n$ in $\mathcal{L}$, where

$$\omega_n = \omega_n(x) = \sum_{r=1}^{n}\frac{\lambda_{r-1}}{a_r}, \qquad \lambda_0 = 1, \qquad \lambda_r = \frac{1}{a_1(a_1-1)\dots a_r(a_r-1)}.$$

So $x \in I_n = I_n(a_1, \dots, a_n) = C(\omega_n, q^{-1}\|\lambda_n\|)$, and $x = \omega_n + \lambda_n T^n(x)$. Thus

$$\|x-\omega_n\| \leqq q^{-1}\|\lambda_n\| = q^{-1}\mu(I_n),$$

and

$$x = \omega_{n+1} + \lambda_{n+1}T^{n+1}(x) = \omega_n + \lambda_{n+1}(a_{n+1} - 1 + T^{n+1}(x)).$$

Since $\|a_{n+1}\| = \|a_{n+1}-1\| \geqq q > \|T^{n+1}(x)\|$, it follows also that

$$\|x-\omega_n\| = \|\lambda_{n+1}\|\cdot\|(a_{n+1}-1)+T^{n+1}(x)\| = \|\lambda_{n+1}\|\cdot\|a_{n+1}\| \geqq q\mu(I_{n+1}).$$

Thus $q\mu(I_{n+1}) \leqq \|x-\omega_n\| \leqq q^{-1}\mu(I_n)$, giving

$$1 - \sum_{r=1}^{n+1}\log_q\|a_r\|^2 \leqq \log_q\|x-\omega_n\| \leqq -1 - \sum_{r=1}^{n}\log_q\|a_r\|^2.$$

Choosing f as in (ii) above, these inequalities then imply that

$$\lim_{n\to\infty}\frac{1}{n}\log_q\|x-\omega_n\| = -\frac{2q}{q-1} \quad \text{a.e.},$$

which yields part (iii) of Theorem 1.

(vii) With reference to Billingsley [2] Section 13, say, the preceding argument also now shows that the basic operator T above has *entropy*

$$h(T) = -\lim_{n\to\infty} \frac{1}{n} \log_e \mu(I_n) = \frac{2q \log_e q}{q-1}.$$

Various other and stronger results may be derived in the present context, from the laws of large numbers (and iterated logarithms), but details are omitted here.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE WITWATERSRAND
JOHANNESBURG, 2050
SOUTH AFRICA

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A CLASS OF OPERATORS ASSOCIATED WITH L_h^2

N. S. FAOUR (Kuwait)

1. Introduction

The space L_h^2 is the space of all harmonic functions f defined on the open unit disk D such that they are square integrable with respect to the area measure $dA = \frac{1}{\pi}dydx$. Following similar arguments to that used by Conway [2, p. 175] it can be shown that L_h^2 is a closed subspace of L^2 with orthonormal basis $\dots, \sqrt{3}\bar{z}^2, \sqrt{2}\bar{z}, 1, \sqrt{2}z, \sqrt{3}z^2, \dots$. The Bergman space A^2 is the space of all analytic functions f defined on D and square integrable with respect to the area measure. It is known that A^2 is a closed subspace of L^2 with orthonormal basis $\{\sqrt{n+1}z^n\}_{n \geqq 0}$. The space $L_h^2 = A^2 \oplus \overline{A_0^2}$, where $\overline{A_0^2}$ is the space of all complex conjugates of functions in A^2 which vanish at the origin.

Let $\varphi \in L^\infty(D)$ and Q be the orthogonal projection of L^2 onto L_h^2. Define the operator C_φ on L_h^2 by $C_\varphi(f) = Q(\varphi \cdot f)$. The Toeplitz operator T_φ is defined on A^2 by $T_\varphi(f) = P(\varphi \cdot f)$, where P is the orthogonal projection of L^2 onto A^2. Let $B_\varphi = PC_\varphi$. Note that B_φ restricted to A^2 is T_φ. It can be easily established that

$$C_{\alpha\psi_1+\beta\psi_2} = \alpha C_{\psi_1} + \beta C_{\psi_2}, \qquad \psi_1, \psi_2 \in L^\infty(D), \qquad \alpha, \beta \in \mathbf{C},$$

$C_\psi^* = C_{\overline{\psi}}$ where C_ψ^* is the adjoint of C_ψ, and $\psi \in L^\infty(D)$. Similar results are true for Toeplitz operators defined on A^2.

In this paper it is shown that if $\varphi \in L^\infty(D)$, $\psi \in C(\overline{D})$, then $C_{\varphi\psi} - C_\varphi C_\psi$ and $C_\varphi C_\psi - C_\psi C_\varphi$ are compact. Also a result related to Toeplitz operators defined on the Bergman space A^2 is proved. Finally, the spectrum of C_φ is studied.

2. Results

THEOREM 1. *Let $\varphi \in L^\infty(D)$, $\psi \in C(\overline{D})$. Then $C_{\varphi\psi} - C_\varphi C_\psi$ and $C_\varphi C_\psi - C_\psi C_\varphi$ are compact operators on L_h^2.*

To prove Theorem 1, a definition and two lemmas are needed.

DEFINITION 1. For $f \in L^\infty(D)$, define the operator $H_f\colon L_h^2 \to (L_h^2)^\perp$ by $H_f g = (I-Q)fg$ and the operator $S_f\colon (L_h^2)^\perp \to (L_h^2)^\perp$ by $S_f g = (I-Q)fg$.

LEMMA 1. (i) *The adjoint operator* $H_f^*\colon (L_h^2)^\perp \to L_h^2$ *is given by* $H_f^* g = Q(\overline{f}g)$.

(ii) $C_{fg} - C_f C_g = H_{\overline{f}}^* H_g,\ f,g \in L^\infty(D)$.

(iii) $H_{fg} = S_f H_g + H_f C_g$.

Lemma 1 is easily established by straightforward computation.

To state and prove Lemma 2, we use the fact that $L^2 = L_h^2 \oplus (L_h^2)^\perp$. Thus using this decomposition of L^2, one can easily see that for $n \geqq 1$

$$(1)\qquad (\overline{z})^{n-1}|z|^2 = \frac{n}{n+1}(\overline{z})^{n-1} \oplus \left((\overline{z})^{n-1}|z|^2 - \frac{n}{n+1}(\overline{z})^{n-1}\right),$$

and if i, m are non-negative integers, then

$$(2)\qquad z^i|z|^{2m} = \frac{i+1}{m+i+1}z^i \oplus \left(z^i|z|^{2m} - \frac{i+1}{m+i+1}z^i\right).$$

From (1) and (2) it follows that

$$C_{\overline{z}}C_z\overline{z} = \frac{1}{2}\overline{z}, \quad \text{and} \quad C_{|z|^2}\overline{z} = \frac{2}{3}\overline{z}.$$

LEMMA 2. (i) *The self-commutator of* C_z, $C_{\overline{z}}C_z - C_zC_{\overline{z}}$ *is of trace class.*
(ii) $C_{|z|^2} - C_zC_{\overline{z}}$ *and* $C_{|z|^2} - C_{\overline{z}}C_z$ *are of trace class.*

PROOF. By using (1) and (2), it is easily verified that for $n \geqq 0$,

$$(3)\qquad C_{\overline{z}}C_z z^n = \frac{n+1}{n+2}z^n.$$

Note that for $n = 0$, $C_{\overline{z}}C_z 1 = C_zC_{\overline{z}}1 = \frac{1}{2}$. Moreover, for $n \geqq 1$,

$$(4)\qquad C_zC_{\overline{z}}z^n = \frac{n}{n+1}z^n,$$

$$(5)\qquad C_{\overline{z}}C_z\overline{z}^n = \frac{n}{n+1}\overline{z}^n,$$

and

$$(6)\qquad C_zC_{\overline{z}}\overline{z}^n = \frac{n+1}{n+2}\overline{z}^n.$$

For simplicity of notation, let $e_n = \sqrt{n+1}z^n$, $\overline{e}_n = \sqrt{n+1}\overline{z}^n$. Then

$$\sum_{n=1}^\infty \langle (C_{\overline{z}}C_z - C_zC_{\overline{z}})e_n, e_n\rangle = -\sum_{n=1}^\infty \langle (C_{\overline{z}}C_z - C_zC_{\overline{z}})\overline{e}_n, \overline{e}_n\rangle = \sum_{n=1}^\infty \frac{1}{(n+1)(n+2)} < \infty.$$

This proves (i) due to the fact that a compact self-adjoint operator is of trace class if the sum of the absolute values of its eigenvalues is finite. A similar calculation yields (ii).

PROOF OF THEOREM 1. Let $A = \{f \in C(\overline{D}) : H_f \text{ is compact}\}$. Note that $H_1 = 0$, thus $1 \in A$. It is easily established that A is a closed subspace of $C(\overline{D})$. In addition, by Lemma 1, A is an algebra. Further $C_{\overline{z}z} - C_{\overline{z}}C_z = H_z^* H_z$, and $C_{z\overline{z}} - C_z C_{\overline{z}} = H_{\overline{z}}^* H_{\overline{z}}$; thus according to Lemma 2, $H_z^* H_z$, and $H_{\overline{z}}^* H_{\overline{z}}$ are compact operators on L_h^2. Consequently, H_z and $H_{\overline{z}}$ are compact, and therefore $z, \overline{z}$ are elements in A. The Stone–Weierstrass Theorem implies that $A = C(\overline{D})$, and thus H_ψ is compact. It follows that the operator $C_{\varphi\psi} - C_\varphi C_\psi = H_{\overline{\varphi}}^* H_\psi$ is compact. Since $C_\psi C_\varphi - C_\varphi C_\psi = (C_\psi C_\varphi - C_{\psi\varphi}) + (C_{\varphi\psi} - C_\varphi C_\psi)$ and $(C_\psi C_\varphi - C_{\psi\varphi})^* = C_{\overline{\varphi}} C_{\overline{\psi}} - C_{\overline{\psi\varphi}}$, then the property just established implies that $C_\psi C_\varphi - C_\varphi C_\psi$ is compact. The proof of the theorem is complete.

REMARK 1. The technique used in the proof of Theorem 1 is similar to that used in [1, Proposition 8].

THEOREM 2. *Let* $\overline{\varphi} = \varphi_1 + \varphi_2$, $\psi = \psi_1 + \psi_2$ *be elements in* $L_h^\infty(D) \subset L_h^2 = A^2 \oplus \overline{A_0^2}$, *and let* $T_\varphi T_\psi = T_{\varphi\psi}$. *If* C_{φ_2} *or* C_{ψ_2} *has dense range in* L_h^2 *then* $\psi \in H^\infty$ *or* $\overline{\varphi} \in H^\infty$ *respectively.*

PROOF. For all non-negative integers n and m we have

$$\langle T_\varphi T_\psi z^n, z^m \rangle = \langle T_{\varphi\psi} z^n, z^m \rangle .$$

Thus,

$$\langle T_\psi z^n, \overline{\varphi} z^m \rangle = \langle \psi z^n, \overline{\varphi} z^m \rangle .$$

Therefore,

$$\langle \psi z^n - T_\psi z^n, \overline{\varphi} z^m \rangle = 0,$$

and hence,

$$\bigl\langle \psi z^n - T_\psi z^n, \overline{\varphi} z^m - T_{\overline{\varphi}} z^m \bigr\rangle = 0. \tag{7}$$

For $n = 0$, $m = 0$, it follows from (7) that $\langle \psi_2, \varphi_2 \rangle = 0$. Moreover, if $n = 0$, $m = 1$, we get $\langle \psi_2, \varphi_2 z - P(\varphi_2 z) \rangle = 0$. Using the fact that $\psi_2 \in \overline{A_0^2}$, it follows that $\langle \psi_2, \varphi_2 z \rangle = 0$. Continuing this process for $n = 0$, $m \geqq 0$, we get

$$\langle \psi_2, \varphi_2 z^m \rangle = 0. \tag{8}$$

By the same techniques it is easily established that for $n \geqq 0$,

$$\langle \psi_2, \varphi_2 \overline{z}^n \rangle = 0. \tag{9}$$

If C_{φ_2} has dense range, (8) and (9) will imply that $\psi_2 = 0$, and hence $\psi \in H^\infty$. By similar argument, it can be concluded that if C_{ψ_2} has dense range, then $\varphi_2 = 0$, and hence $\overline{\varphi} \in H^\infty$.

From Theorem 2, the following corollary is easily established.

COROLLARY 1. *Let* $\overline{\varphi} = \varphi_1 + \varphi_2$, $\psi = \psi_1 + \psi_2$ *be elements in* $L_h^\infty(D)$, ψ_2 *or* $\varphi_2 \in L^\infty(D)$. *If* $T_\varphi T_\psi = T_{\varphi\psi}$, *then* $\psi_2\overline{\varphi_2} \in (L_h^2)^\perp$.

It is well-known that the spectrum of the Toeplitz operator T_z, $\sigma(T_z)$ is $\overline{D}$. Moreover, it is easily established that the operator C_z is a bilateral weighted shift, $\sigma(C_z)$ is ∂D, and the closed convex hull of the essential range of z, $\overline{\text{co}}(R(z))$ is $\overline{D}$. Consequently, $\sigma(C_z)$ is different from $\overline{\text{co}}(R(z))$. However, we have the following:

PROPOSITION 1. *If* φ *is a function in* $L^\infty(D)$, *then*

$$\sigma(C_\varphi) \subset \overline{\text{co}}(R(\varphi)).$$

PROOF. Suppose that $\lambda \in \mathbf{C}$, $\lambda \notin \overline{\text{co}}(R(\varphi))$. Since $R(\varphi)$ is a compact subset of the complex plane, there is a disk D with center z_0 containing $R(\varphi)$ but not the point λ. Thus

$$|\lambda - z_0| > \text{ess sup } |\varphi(z) - z_0| = \|\varphi - z_0\|_\infty \geqq \|C_{\varphi - z_0}\| .$$

From this it follows that $\lambda - z_0 \notin \sigma(C_{\varphi - z_0})$ and hence $\lambda \notin \sigma(C_\varphi)$, and hence $\lambda \notin \sigma(C_\varphi)$, completing the proof.

REMARK 2. The Toeplitz operator T_z is hyponormal. However, the operator C_z is not hyponormal; this is due to the fact that every hyponormal operator whose spectrum has zero area is normal.

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KUWAIT UNIVERSITY
DEPARTMENT OF MATHEMATICS
P.O. BOX 5969
KUWAIT

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CONGRUENCE LATTICES OF PLANAR LATTICES

G. GRÄTZER and H. LAKSER (Winnipeg)*

1. Introduction. Let L be a lattice. It was proved in N. Funayama and T. Nakayama [5] that the congruence lattice of L is distributive. For a finite lattice L, the converse of this result was proved by R. P. Dilworth: Every finite distributive lattice D can be represented as the lattice of congruence relations of a suitable finite lattice L. The first published proof of this result is in G. Grätzer and E. T. Schmidt [15]. For another proof of this result by the present authors, see [6, pp. 81–84]. See also [1], [2], [9], [10], [18], [21].

In all these proofs, we construct a lattice L starting from a finite distributive lattice D with n nonzero join-irreducible elements. This lattice L turns out to be rather large: it has $O(4^n)$ (or more) elements; it is also rather complex: it is of order-dimension $O(n)$ (or higher).

There are results in the literature providing stronger forms of Dilworth's result by constructing finite lattices L representing D as the congruence lattice and having additional properties:

(i) L is sectionally complemented and the length of L is $2n-1$. See G. Grätzer and E. T. Schmidt [15].

(ii) The length of L is $5m$, where m is the number of dual atoms of D. See S.-K. Teo [20]; for $m=1$ and for the conjecture solved by Teo, see E. T. Schmidt [17].

We add to this list with the following result:

THEOREM. *Let D be a finite distributive lattice with more than one element. Then there exists a finite planar lattice L with no proper automorphism such that the congruence lattice of L is isomorphic to D. The lattice L can be chosen to have $O(|J(D)|^3)$ elements, where $J(D)$ is the set of nonzero join-irreducible elements in D.*

There are many other related results relaxing the condition that D or L be finite. These go beyond the scope of this paper. For a brief review, see G. Grätzer [9].

Now consider the automorphism group of L. Obviously, it is a group. The characterization theorem of the automorphism group of a finite lattice is due to R. Frucht [4]: Every finite group G can be represented as the automorphism group of a suitable finite lattice L. In fact, Frucht's construction yields a simple lattice of length three.

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As an application of the Theorem, we prove a result of V. A. Baranskiĭ and A. Urquhart (see [1], [2], [21]) that the congruence lattice and the automorphism group of a finite lattice are *independent*:

COROLLARY 1. *Let D be a finite distributive lattice with more than one element, and let G be a finite group. Then there exists a finite lattice L such that the congruence lattice of L is isomorphic to D, and the automorphism group of L is isomorphic to G.*

Again, the lattice L we construct is much smaller than the lattices in [2], [21].

Combining Frucht's result with the result of G. Sabidussi [16], the automorphism group of a lattice is characterized as an arbitrary group. As another application of the Theorem, we prove a result of V. A. Baranskiĭ [1] and [2]:

COROLLARY 2. *Let D be a finite distributive lattice with more than one element, and let G be an arbitrary group. Then there exists a lattice L such that the congruence lattice of L is isomorphic to D, and the automorphism group of L is isomorphic to G.*

The Theorem of this paper is closely connected to several other results in the literature: the independence of the congruence lattice, the automorphism group, and the subalgebra lattice of a (universal) algebra, finitary or infinitary (G. Grätzer and W. A. Lampe, see Appendix 7 of G. Grätzer [7] for a complete discussion); the independence of the complete congruence lattice and the automorphism group of a complete lattice (G. Grätzer [8], G. Grätzer and H. Lakser [11], and G. Grätzer and H. Lakser [12]).

The basic notation is explained in §2. In §3, we introduce the coloring of a chain, which originated in S.-K. Teo [19], and investigate the congruences of the associated extension. We discuss in §4 a generalization of this construction introduced in G. Grätzer and H. Lakser [13]. This is then applied in §5 to construct the finite lattice L representing D. In §6, we show how to modify the construction to make L planar, proving the Theorem. Finally, in §7, we augment L to additionally represent G as an automorphism group, proving the corollaries. Some concluding comments are collected in §8.

2. Notation. D is the finite distributive lattice we want to represent in the Theorem. $J(D)$ is the partially ordered set of (nonzero) join-irreducible elements of D. $\mathfrak{M}_3$ denotes the five-element modular nondistributive lattice.

For a lattice A, let $\mathrm{Ip}\, A$ denote the set of *prime intervals* in A, i.e., the set of all intervals $\mathfrak{p} = [u, v]$, where $u \prec v$ (u is *covered by* v) in A. If $\mathfrak{p} = [u, v]$ is an interval of A, then for any lattice B and $b \in B$, we use the notation $\mathfrak{p} \times \{b\}$ for the interval $[\langle u, b\rangle, \langle v, b\rangle]$ of $A \times B$. Note that if $\mathfrak{p}$ is prime, then $\mathfrak{p} \times \{b\} \in \mathrm{Ip}(A \times B)$.

Let $\mathfrak{p}_0 = [x_0, y_0]$ be a (prime) interval of A_0, and let $\mathfrak{p}_1 = [x_1, y_1]$ be a (prime) interval of A_1. It will be convenient to refer to the elements of the sublattice of $A \times B$ generated by $\mathfrak{p}_0 \times \{x_1\}$ and $\{x_0\} \times \mathfrak{p}_1$ as follows (see Fig. 1):

$$o(\mathfrak{p}_0, \mathfrak{p}_1) = \langle x_0, x_1 \rangle, \qquad a(\mathfrak{p}_0, \mathfrak{p}_1) = \langle x_0, y_1 \rangle,$$
$$b(\mathfrak{p}_0, \mathfrak{p}_1) = \langle y_0, x_1 \rangle, \qquad i(\mathfrak{p}_0, \mathfrak{p}_1) = \langle y_0, y_1 \rangle.$$

$\mathfrak{p}_0 \times \mathfrak{p}_1$ shall refer to the interval $[o(\mathfrak{p}_0, \mathfrak{p}_1), i(\mathfrak{p}_0, \mathfrak{p}_1)]$.

For an interval $\mathfrak{p} = [u, v]$ in the lattice A, we shall denote by $\Theta_A(\mathfrak{p})$ or $\Theta_A(u, v)$ the congruence relation generated by the interval $\mathfrak{p}$. If A is understood, we use the notation $\Theta(\mathfrak{p})$ or $\Theta(u, v)$. Note that $u \equiv v$ (Θ) is equivalent to $\Theta(\mathfrak{p}) \leqq \Theta$.

We refer the reader to G. Grätzer [6] for the standard notation in lattice theory.

3. Coloring. A *coloring* of a chain C is a surjective (onto) map

$$\varphi\colon \mathrm{Ip}\ C \to J(D).$$

If $\mathfrak{p} \in \mathrm{Ip}\ C$ and $\mathfrak{p}\varphi = a$, one should think of $\Theta_K(\mathfrak{p})$ as the congruence representing $a \in J(D)$ in some extension K of C.

Following S.-K. Teo [19], for the chains C_0 and C_1 and colorings φ_0 and φ_1, respectively, we define the lattice $K = C_0 \times_\varphi C_1$, as follows: the lattice K is $C_0 \times C_1$ augmented with the elements $m(\mathfrak{p}_0, \mathfrak{p}_1)$ whenever $\mathfrak{p}_0 \in \mathrm{Ip}\ C_0$, $\mathfrak{p}_1 \in \mathrm{Ip}\ C_1$, and $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$; we require that the elements

$$o(\mathfrak{p}_0, \mathfrak{p}_1),\ a(\mathfrak{p}_0, \mathfrak{p}_1),\ b(\mathfrak{p}_0, \mathfrak{p}_1),\ i(\mathfrak{p}_0, \mathfrak{p}_1),\ m(\mathfrak{p}_0, \mathfrak{p}_1) \tag{3.1}$$

form a sublattice of K isomorphic to $\mathfrak{M}_3$, as illustrated by Fig. 1.

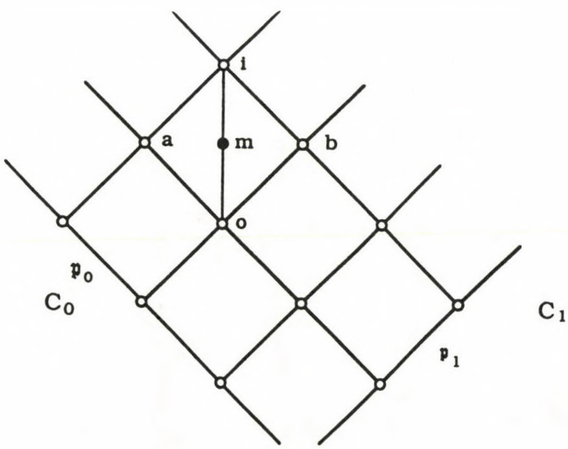

Fig. 1

In Teo's paper, $C_0 = C_1$ and $\varphi_0 = \varphi_1$, but the idea is the same.

As an illustration, let D be the distributive lattice of Fig. 2; the join-irreducible elements are marked with $\bullet$. Then $J(D)$ has four elements, as illustrated. Let C_0 and C_1 be the chains of Fig. 3; the color of a prime interval appears to the right of the edge. Fig. 4 illustrates $K = C_0 \times_\varphi C_1$.

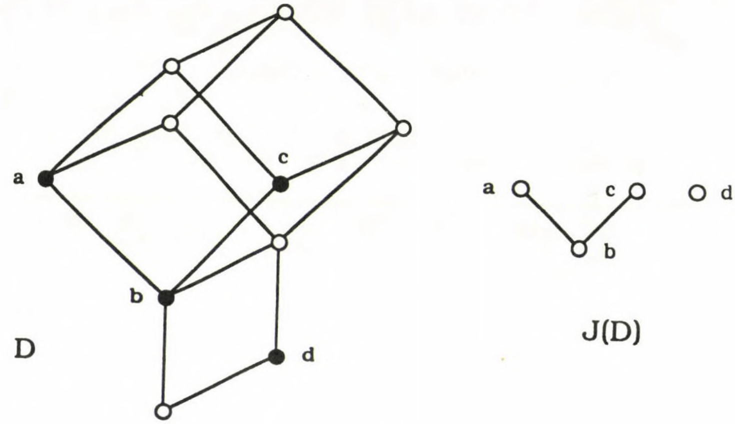

Fig. 2

The congruences of $C_0 \times C_1$ are of the form $\Theta_0 \times \Theta_1$, where Θ_0 is a congruence of C_0 and Θ_1 is a congruence of C_1. Now take only Θ_0 and Θ_1 with the following property:

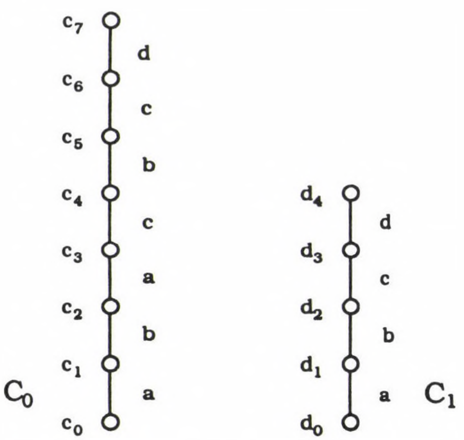

Fig. 3

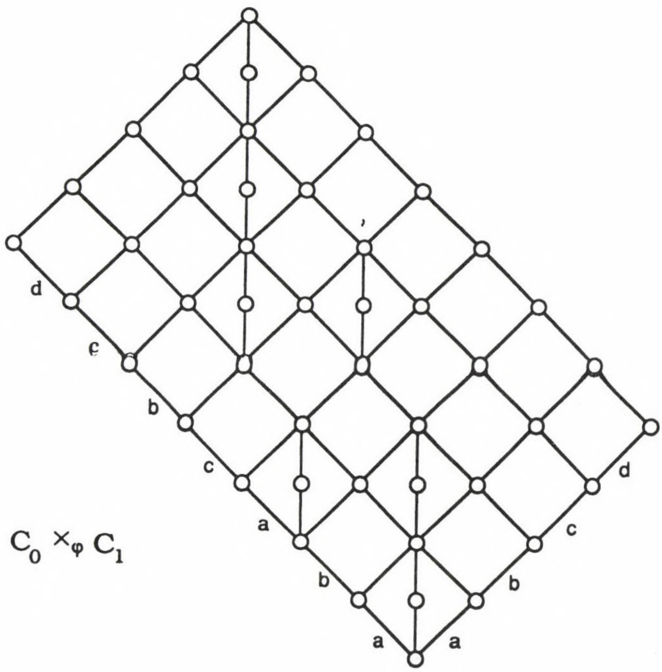

Fig. 4

(3.2) If $\mathfrak{p}_0 \in \mathrm{Ip}\ C_0$, $\mathfrak{p}_1 \in \mathrm{Ip}\ C_1$, and $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$, then

$$\Theta(\mathfrak{p}_0) \leqq \Theta_0 \quad \text{iff} \quad \Theta(\mathfrak{p}_1) \leqq \Theta_1.$$

Now, we extend the congruence $\Theta_0 \times \Theta_1$ on $C_0 \times C_1$ to a congruence $\Theta_0 \times_\varphi \Theta_1$ on K as follows:

Let $\mathfrak{p}_0 \in \mathrm{Ip}\ C_0$ and $\mathfrak{p}_1 \in \mathrm{Ip}\ C_1$. If $\mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1$, then let the elements in (3.1) be in one congruence class modulo $\Theta_0 \times_\varphi \Theta_1$.

It is easy to compute (see also §4) that the congruences of K are exactly these $\Theta_0 \times_\varphi \Theta_1$.

As an example, take the congruence Θ_0 of the chain C_0 and the congruence Θ_1 of the chain C_1 of Fig. 5. Then $\Theta_0 \times_\varphi \Theta_1$ is the congruence of K as illustrated by Fig. 6.

Thus the congruences of $C_0 \times_\varphi C_1$ are in one-to-one correspondence with subsets of $J(D)$; hence the congruence lattice of $C_0 \times_\varphi C_1$ is a finite Boolean lattice.

4. Generalized coloring. In [13], we generalized the construction $C_0 \times_\varphi C_1$ of §3 as follows.

Let L be a lattice and let Λ be a set of *proper intervals* in L, i.e., intervals with more than one element. We define a lattice $L[\Lambda]$ by adjoining the family of new pairwise distinct elements $\{m_I \mid I \in \Lambda\}$ to L, and requiring that $u \prec m_I \prec v$ for each $I = [u, v] \in \Lambda$.

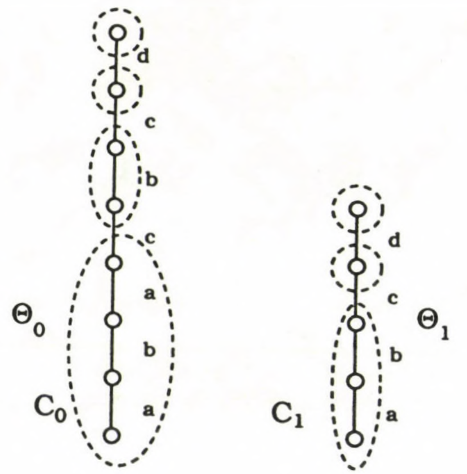

Fig. 5

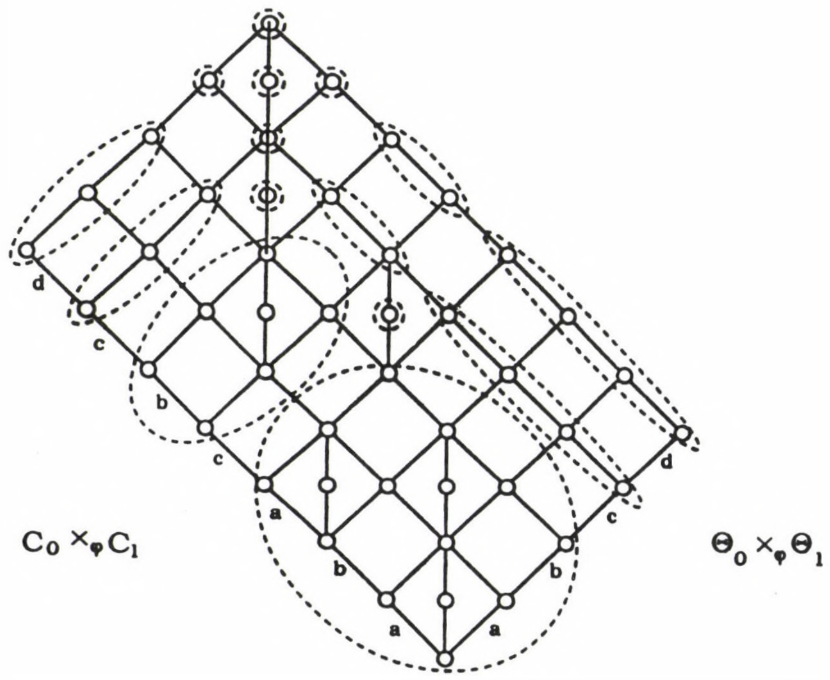

Fig. 6

We associate with $x \in L[\Lambda]$ the elements $\underline{x}$ and $\overline{x}$ of L: for $x \in L$, set

$\underline{x} = \overline{x} = x$; for $I = [u,v] \in \Lambda$, set $\underline{m}_I = u$ and $\overline{m}_I = v$. We then, more formally, define the relation $\leqq$ on the set $L[\Lambda]$ as follows:

$$x \leqq y \quad \text{if and only if} \quad x = y \quad \text{or} \quad \overline{x} \leqq_L \underline{y},$$

where $\leqq_L$ denotes the partial ordering in L.

Then $\langle L[\Lambda], \leqq \rangle$ is a lattice extending L. If X is a subset of $L[\Lambda]$, then $\bigvee X$ exists in $L[\Lambda]$ if and only if either there is an $x \in X$ such that, for all $y \in X$, we have $x \geqq y$, in which case $\bigvee X = x$; or there is no such x and $\bigvee_L(\overline{x} \mid x \in X)$ exists, in which case

$$\bigvee X = \bigvee_L (\overline{x} \mid x \in X),$$

where $\bigvee_L$ is the complete join in L; and dually for $\bigwedge$.

Let C_0 and C_1 be finite chains with colorings φ_0 and φ_1, respectively, as in §3. Let $A = C_0 \times C_1$. Observe that $C_0 \times_\varphi C_1$ can be obtained as $A[\Lambda]$ in the obvious way with

$$\Lambda = \{\mathfrak{p}_0 \times \mathfrak{p}_1 \mid \mathfrak{p}_0 \in \mathrm{Ip}\, C_0,\ \mathfrak{p}_1 \in \mathrm{Ip}\, C_1,\ \mathfrak{p}_0\varphi_0 = \mathfrak{p}_1\varphi_1\}.$$

The following result describes which congruences extend from L to $L[\Lambda]$:

One Point Extension Theorem [13]. *Let Λ be a set of nontrivial, nonprime intervals in the lattice L, and let Θ be a congruence relation on L. Then Θ has an extension $\Theta[\Lambda]$ to $L[\Lambda]$ if and only if Θ satisfies the following conditions and their duals (see Fig. 7):*

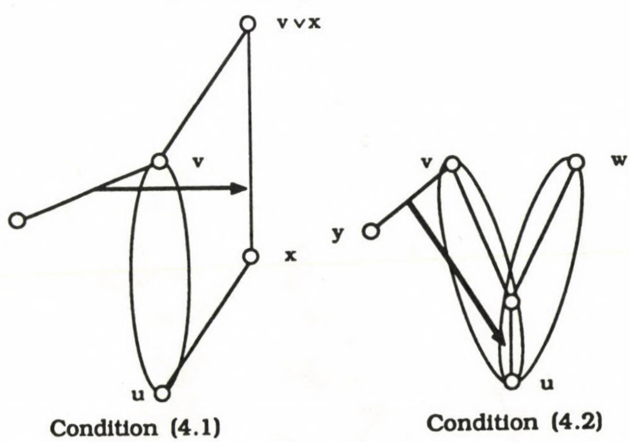

Fig. 7

(O1) *For $[u,v] \in \Lambda$, $x, y \in L$ with $y < v$ and $u < x$,*

$$y \equiv v \quad (\Theta) \qquad \textit{implies that} \qquad x \equiv v \vee x \quad (\Theta).$$

(O2) *For* $[u, v], [u, w] \in \Lambda$, *with* $v \neq w$ *and* $y \in L$ *with* $y < v$,

$$y \equiv v \quad (\Theta) \qquad \textit{implies that} \qquad u \equiv v \wedge w \quad (\Theta).$$

The extension $\Theta[\Lambda]$ *of* Θ *to* $L[\Lambda]$ *is unique. It can be described as follows: For all* $a \in L[\Lambda]$, *set* $a \equiv a$ $(\Theta[\Lambda])$. *For all* $a, b \in L[\Lambda]$, *with* $a \neq b$, *set*

$$a \equiv b \quad (\Theta[\Lambda])$$

if and only if the following three conditions hold:

(O3) $\overline{a \wedge b} \equiv \underline{a \vee b}$ (Θ).

(O4) $a \wedge b \in L$ *or* $a \wedge b \notin L$ *and there is an* $x_{a \wedge b} \in L$ *with*

$$\underline{a \wedge b} < x_{a \wedge b} \qquad \textit{and} \qquad \underline{a \wedge b} \equiv x_{a \wedge b} \quad (\Theta).$$

(O5) $a \vee b \in L$ *or* $a \vee b \notin L$ *and there is a* $y_{a \vee b} \in L$ *with*

$$y_{a \vee b} < \overline{a \vee b} \qquad \textit{and} \qquad y_{a \vee b} \equiv \overline{a \vee b} \quad (\Theta).$$

An interesting special case can be developed by generalizing the concept of coloring from §3. Let P be a set of nontrivial intervals in A. A *(generalized) coloring* φ *of a lattice* A *by a set* X is a surjective map $\varphi \colon P \to X$.

For each $i = 0, 1$, let A_i be a lattice with a coloring $\varphi_i \colon P_i \to X$. We consider the set Λ of all intervals in $A_0 \times A_1$ defined by setting

$$\Lambda = \{ I_0 \times I_1 \mid I_0 \in P_0, \ I_1 \in P_1, \quad \text{and} \quad I_0\varphi_0 = I_1\varphi_1 \}.$$

Let us denote the lattice $(A_0 \times A_1)[\Lambda]$ by $A_0 \times_\varphi A_1$, and the element $m_{I_0 \times I_1}$ by $m(I_0, I_1)$.

The next result is an application of the One Point Extension Theorem to determine the congruence relations on $A_0 \times A_1$ that extend to $A_0 \times_\varphi A_1$. Recall that any congruence relation Θ on the lattice $A = A_0 \times A_1$ is of the form $\Theta_0 \times \Theta_1$, where, for $i = 0, 1$, Θ_i is a congruence relation on A_i.

COLORED PRODUCT EXTENSION THEOREM [13]. *The congruence relation* $\Theta = \Theta_0 \times \Theta_1$ *on* $A_0 \times A_1$ *extends to* $A_0 \times_\varphi A_1$ *if and only if the following two conditions and the dual of the second condition hold:*

(C1) *For* $I_0 \in P_0$, $I_1 \in P_1$, *if* $I_0\varphi_0 = I_1\varphi_1$, *then*

$$\Theta(I_0) \leqq \Theta_0 \quad \textit{is equivalent to} \quad \Theta(I_1) \leqq \Theta_1.$$

(C2) *For* $i = 0, 1$, *if* $I = [u, v] \in P_i$ *and there is a* $y < v$ *with* $y \equiv v$ (Θ_i), *then* $\Theta(I) \equiv \Theta_i$.

In that event, the extension is unique.

The reader should find it evident that the last statement of §3 follows from the Colored Product Extension Theorem.

5. Constructing L for D. Now let D be given. In order to construct a lattice with no nontrivial automorphisms in an efficient manner, we restrict the construction outlined in §3 to those D where each join-irreducible element is comparable to some other join-irreducible element.

Specifically, we can set

$$D \cong D' \times B,$$

where B is Boolean and each element of $J(D')$ is comparable to some other element of $J(D')$ — see Fig. 8 for the D' associated with D of Fig. 2. We note that B can be represented as the congruence lattice of a chain.

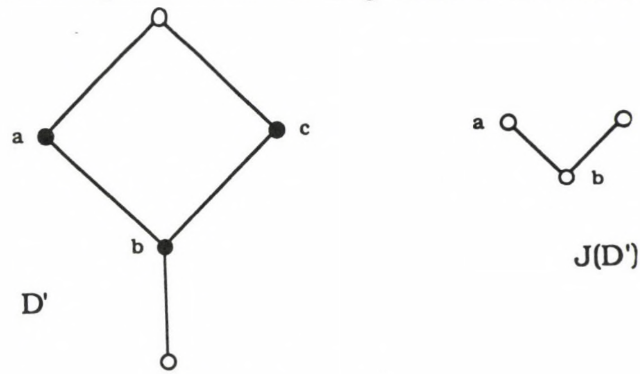

Fig. 8

We now construct a lattice L' whose congruence lattice is isomorphic to D'.

For every $a, b \in J(D')$ with $a \prec b$ (note: $a \prec b$ in $J(D')$, not in D'), we construct a four-element chain $C(a,b)$, see Fig. 9, with elements $o(a,b)$, $m(a,b)$, $n(a,b)$, $i(a,b)$, satisfying the relations:

$$o(a,b) \prec m(a,b) \prec n(a,b) \prec i(a,b).$$

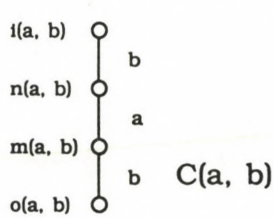

Fig. 9

We define a map $\varphi_{a,b}$ of Ip $C(a,b)$ into $J(D')$ by

$$\begin{aligned}[o(a,b), m(a,b)]\varphi_{a,b} &= b,\\ [m(a,b), n(a,b)]\varphi_{a,b} &= a,\\ [n(a,b), i(a,b)]\varphi_{a,b} &= b.\end{aligned}$$

We list all covering pairs in $J(D')$:

$$a_0 \prec b_0, \quad a_1 \prec b_1, \dots, a_{n-1} \prec b_{n-1}.$$

We construct two chains:

$$C_0\colon c_0 \prec c_1 \prec \dots \prec c_{3n-1}, \quad \text{and} \quad C_1\colon d_0 \prec d_1 \prec \dots \prec d_j,$$

where $j = |J(D')|$. Observe that $n = O(j^2)$, and so $|C_0| = O(j^2)$.

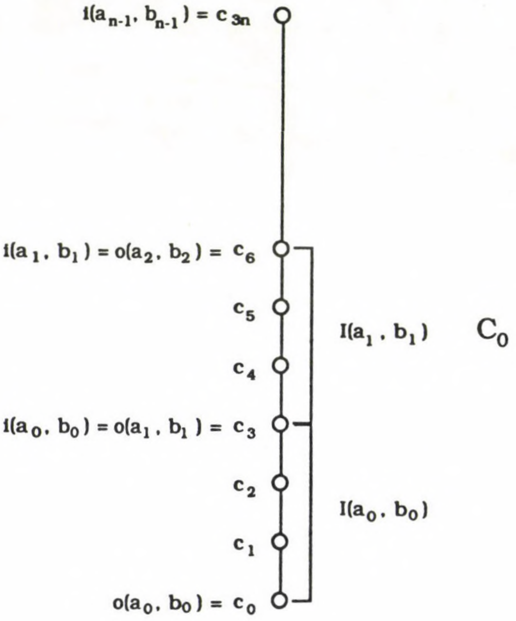

Fig. 10

We regard C_0, see Fig. 10, as the ordinal sum of

$$C(a_0, b_0), C(a_1, b_1), \dots, C(a_{n-1}, b_{n-1}),$$

with $i(a_0, b_0)$ identified with $o(a_1, b_1)$, $i(a_1, b_1)$ identified with $o(a_2, b_2)$, and so on.

In C_0, let $I(a, b)$ denote the interval $[o(a, b), i(a, b)]$ for $a, b \in J(D')$ with $a \prec b$.

We define a coloring φ_0 of C_0. First of all, we define the set of intervals

$$P_0 = \mathrm{Ip}\ C_0 \cup \{I(a, b) \mid a, b \in J(D'), a \prec b\}.$$

Now if $\mathfrak{p} \in \mathrm{Ip}\ C_0$ with $\mathfrak{p} \in I(a, b)$, we define $\mathfrak{p}\varphi_0 = \mathfrak{p}\varphi_{a,b}$. For all $a, b \in J(D)$ with $a \prec b$, we set $I(a, b)\varphi_0 = b$.

Set $P_1 = \mathrm{Ip}\ C_1$, and choose φ_1 as an arbitrary surjective map. Note that φ_1 is a bijection, and so in C_1, for every $a \in J(D')$, there is a unique $\mathfrak{p}_a \in \mathrm{Ip}\ C_1$ with $\mathfrak{p}_a\varphi_1 = a$. Set $\mathfrak{p}_a = [(a)_0, (a)_1]$.

We define L' by setting

$$L' = C_0 \times_\varphi C_1.$$

The lattice L' for the lattice D' of Fig. 8 can be obtained by omitting the unit element of the lattice depicted in Fig. 11. Note that $|L'| = O(j^3)$ where $j = |J(D')|$.

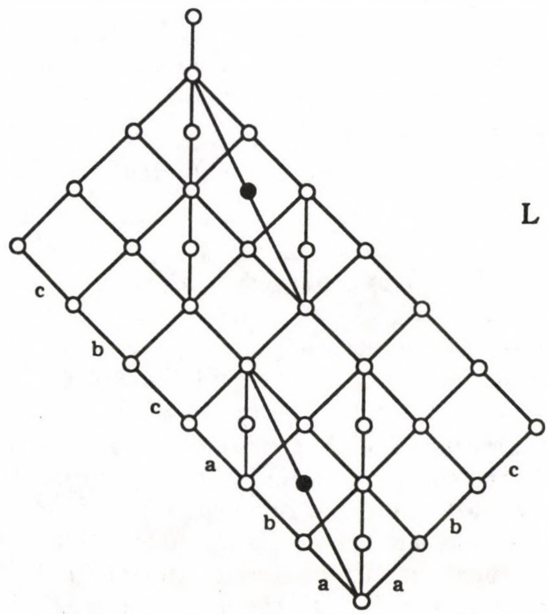

Fig. 11

Now we prove that the congruence lattice of L' is isomorphic to D'. With every hereditary subset H of $J(D')$, we associate a congruence relation Θ^H as follows: for $i = 0, 1$, we define on the chain C_i the relation Θ_i^H ($x, y \in C_i$, $x \leqq y$):

$$x \equiv y \quad (\Theta_i^H) \qquad \text{iff} \qquad \mathfrak{p}\varphi_i \in H \quad \text{for any} \quad \mathfrak{p} \in \mathrm{Ip}[x, y].$$

Let us verify that Conditions (C1), (C2), and the dual of (C2) hold for Θ_0^H and Θ_1^H. Indeed, (C1) holds by definition if I_0 is prime. Let $I_0 = I(a, b)$ for some $a, b \in J(D')$ with $a \prec b$. Then

$$\Theta(I_0) = \Theta(I(a,b)) = \Theta(o(a,b), m(a,b)) \vee \Theta(m(a,b), n(a,b)) =$$
$$= \Theta(o(a,b), m(a,b)),$$

since $\Theta(m(a,b), n(a,b)) \leqq \Theta(o(a,b), m(a,b))$ by virtue of $a \prec b$. Now this case is reduced to the case of the prime interval $[o(a,b), m(a,b)]$, already considered.

To verify (C2), observe that it obviously holds for prime intervals in chains. Hence we are left with the case $i = 0$ and $I_0 = I(a,b)$ for some $a, b \in \in J(D')$ with $a \prec b$. Since $y < i(a,b)$ implies that $y \leqq n(a,b)$, it is obvious (again utilizing that $a \prec b$) that any congruence collapsing y and $i(a,b)$ also collapses all of $I(a,b)$, concluding (C2). The dual of (C2) follows similarly. By the Colored Product Product Extension Theorem, Θ_0^H and Θ_1^H uniquely determine a congruence Θ^H of L'.

Conversely, let Θ be a congruence of L'. By the Colored Product Extension Theorem, Θ is uniquely determined by its restrictions Θ_0 and Θ_1 to C_0 and C_1 respectively, which satisfy Conditions (C1), (C2), and the dual of Condition (C2). For $i = 0, 1$, define

$$H_i = \{\mathfrak{p}\varphi_i \mid \mathfrak{p} \in \operatorname{Ip} C_i,\ \ \Theta(\mathfrak{p}) \leqq \Theta_i\}.$$

Then Condition (C1) yields that $H_0 = H_1$; set $H = H_0 = H_1$. Obviously, H is a subset of $J(D')$. It is hereditary. Indeed, if $a \prec b$ in $J(D')$, $b \in \in H$, then $[n(a,b), i(a,b)]$ is collapsed by Θ_0 since it has color b. Applying Condition (C2) with $i = 0$, $I = I(a,b)$, and $y = n(a,b)$, we obtain that $I(a,b)$ is collapsed by Θ_0. Thus $[m(a,b), n(a,b)]$ is also collapsed by Θ_0. Since $[m(a,b), n(a,b)]\varphi_0 = a$, we conclude that $a \in H$ by the definition of $H_0 = H$. Thus H is a hereditary subset of $J(D')$.

It is now straightforward that $H \mapsto \Theta^H$ is an isomorphism between the lattice of hereditary subsets of $J(D')$ and the congruence lattice of L', and so the congruence lattice of L' is isomorphic to D'.

Finally, if the Boolean lattice B has t atoms, let C be a chain of length t. Then the congruence lattice of C is isomorphic to B. Let the lattice L be the ordinal sum of L' and C with the unit element of L' identified with the zero element of C — see Fig. 11 for the lattice L constructed for the lattice D of Fig. 2. Then

$$\operatorname{Con} L \cong \operatorname{Con} L' \times \operatorname{Con} C \cong D' \times B \cong D,$$

where $\operatorname{Con} A$ denotes the congruence lattice of the lattice A.

6. Planar lattices. The lattice L constructed in §5 is close to being planar; it is in fact of order-dimension 3. It is not planar because of the elements of the form $m(I(a,b), \mathfrak{p}_b)$ where $a \prec b$ in $J(D')$ (recall that $I(a,b)$ is the interval of C_0 defined in §5, and $\mathfrak{p}_b$ is the unique prime interval of C_1 of color b). There are two such elements in Fig. 11; they are black-filled.

To transform L into a planar lattice without changing its congruence lattice requires a few steps.

For the first step, let

$$e_0, e_1, \ldots, e_{k-1}$$

list all the nonminimal elements of $J(D')$; these are the elements that occur as the element b in a pair $a, b \in J(D')$ with $a \prec b$. We rearrange the list of all covering pairs in $J(D')$:

$$a_0 \prec b_0,\ \ a_1 \prec b_1, \ldots,\ \ a_{n-1} \prec b_{n-1}$$

so that we start with all the pairs of the form x, e_0 followed by all the pairs of the form x, e_1, and so on.

In the second step, we redefine φ_1 so that the bottom prime interval of C_1 is colored by e_0, the next with e_1, and so on. Past e_{k-1} we do not care how the coloring is done except that φ_1 be onto.

As the third step, we define a subset L'_1 of L'. Let $\langle x, d_0\rangle$ belong to L'_1 iff $x \in I(a, e_0)$ for some $a \in J(D')$; let $\langle x, d_1\rangle$ belong to L'_1 iff $x \in I(a, e_0)$ or $x \in I(a, e_1)$ for some $a \in J(D')$; in general, let $\langle x, e_s\rangle$ belong to L'_1 iff $x \in I(a, e_i)$ for some $a \in J(D')$ and $s \leqq i$. All $\langle x, d_t\rangle$ are in L'_1 for $k \leqq t < j$. We retain all the elements of L' of the form of $m(I, J)$.

Observe that we only threw away elements that play no role in determining the congruence structure of L', so that the congruence lattice of L'_1 is still isomorphic to D'. To be more precise, any prime interval of L' is projective to a prime interval of L'_1, and any two prime intervals of L'_1 that are projective in L' are already projective in L'_1.

L'_1 is still not planar; however, all the elements that cause problems (that is, the elements of the form $m(I(a, b), \mathfrak{p}_b)$ where $a \prec b$ in $J(D')$) are in intervals $I(a, b) \times \mathfrak{p}_b$ where the "left-side" of the direct product is also the "left-side" of the lattice L'_1.

As the fourth, and final, step, observe that, by the One Point Extension Theorem, the role of the element $m(I(a, b), \mathfrak{p}_b)$ can be taken over by the element

$$m(I(a, b), [(b)_0, (b)_1]), \quad \text{where} \quad \mathfrak{p}_b = [(b)_0, (b)_1].$$

After these replacements, the resulting lattice L'_2 is planar. Let L_2 be the ordinal sum of L'_2 with the chain C, with the unit element of L'_2 and the zero element of C identified; the lattice L_2 we obtain for the lattice D of Fig. 2 is shown in Fig. 12. Although the lattice L_2 is smaller than L, we have not improved the order of $|L_2|$; we still have $|L_2| = O(|J(D)|^3)$.

To conclude the proof of the Theorem, we need only prove that L_2 has no proper automorphisms. Clearly, we need only show that L'_2 has no proper automorphisms. If D' is trivial, then so is L'_2, and we are done. Otherwise, let α be an automorphism of L'_2. If $j = 1$, then D' would be Boolean; hence $j > 1$, and so $[d_{j-1}, d_j] \neq [d_0, d_1]$. Since φ_1 is bijective, this implies that

$$[d_{j-1}, d_j]\varphi_1 \neq [d_0, d_1]\varphi_1 = [c_0, c_1]\varphi_0.$$

It follows that $\langle c_0, d_j\rangle$ is the only doubly-irreducible element of L'_2 that lies in an interval that is a four-element Boolean lattice. Thus $\langle c_0, d_j\rangle\alpha = \langle c_0, d_j\rangle$, and consequently α is the identity mapping on the chain $\{c_0\} \times C_1$. Since those elements of L'_2 that are not doubly-irreducible are precisely the remaining elements of $C_0 \times C_1$ in L'_2, it follows that α is the identity mapping on $(C_0 \times C_1) \cap L'_2$. It is then immediate that α is the identity mapping, concluding the proof of the Theorem.

7. Automorphism groups. R. Frucht [3] proved that we can represent the group G as the automorphism group of a connected undirected graph

$\mathfrak{G} = \langle V, E\rangle$ with more than one edge and without loops, where V is the set of vertices and E is the set of edges.

Next, we represent G by a bounded lattice and lattice automorphisms. As in R. Frucht [4], from $\mathfrak{G}$, we form the lattice:

$$H = V \dot{\cup} E \dot{\cup} \{0, 1\},$$

where, for all $v \in V$ and $e \in E$, the relations $0 < v < 1$ and $0 < e < 1$ hold; let $v < e$ in H iff $v \in e$. Note that H is of length three.

The graphs constructed in R. Frucht [3] and G. Sabidussi [16] have the following property:

(7.1) For $v \in V$, there are $e_0, e_1 \in E$ with $v \nleq e_0$, e_1 and $e_0 \cap e_1 = \emptyset$.

It is easy to prove that if the graph $\mathfrak{G}$ has Property (7.1), then the associated lattice is simple. Hence, H is a simple lattice.

Let L be the lattice we obtained at the end of §6 (see Fig. 12) with o and i as the zero element and unit element of L, respectively.

If L is a chain, let the lattice K be defined by replacing the bottom prime interval of L by H — see Fig. 13. Then, since H is simple, the congruence lattice of K is isomorphic to D. Clearly, the automorphism group of K is isomorphic to G.

Attach H to L by identifying 1 with o. Set $v = \langle c_1, d_1\rangle$, in the notation of §5. We add a relative complement q of o in $[0, v]$, and obtain the lattice K — see Fig. 14.

It is easy to see that any automorphism of K keeps o fixed. Therefore, any automorphism takes L into L and H into H. Since, by the Theorem, L has no proper automorphism, any automorphism of K is an automorphism of H trivially extended to K. It follows that the automorphism group of K is isomorphic to G.

A congruence Θ of $K - \{q\}$ is formed from a congruence of H and a congruence of L. Since H is simple, we only have the two trivial choices on H. That the congruence lattice of K is isomorphic to that of L follows from the following lemma, concluding the proofs of Corollaries 1 and 2.

LEMMA. *A congruence Θ on $K - \{q\}$ extends to K if and only if either*

$$\Theta_H = \omega_h \quad \text{and} \quad o \not\equiv v \quad (\Theta_L) \tag{1}$$

or

$$\Theta_H = \iota_H \quad \text{and} \quad o \equiv v \quad (\Theta_L). \tag{2}$$

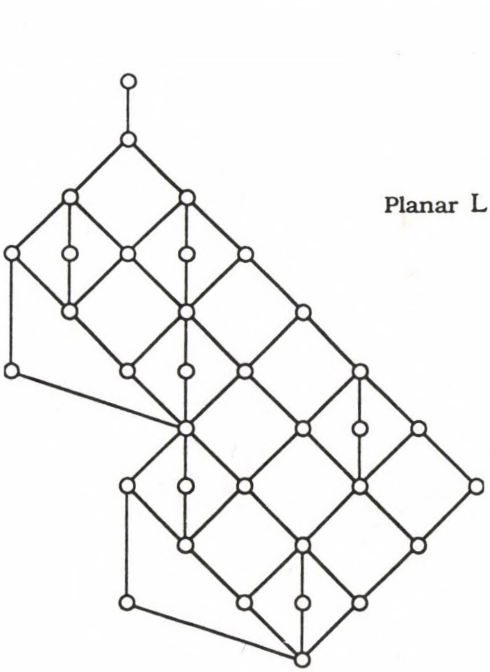

Fig. 12

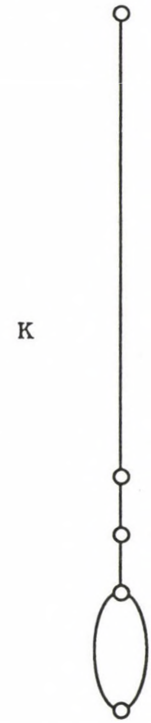

Fig. 13

PROOF. The One Point Extension Theorem applies to extending Θ to K. Since $\Lambda = \{[0, v]\}$ is a singleton, Condition (O2) holds vacuously.

Assume first that Θ extends to K. Then Condition (O1) and its dual hold. We show that either Condition (1) or Condition (2) of the Lemma holds.

Let $\Theta_H = \omega_H$. If $o \equiv v\ (\Theta_L)$, then, in Condition (O1), set $y = o$ and let $x \in H$ with $0 < x < 1$. Then $y \equiv v\ (\Theta)$ and so $x \equiv v\ (\Theta)$, that is, $x \equiv 1\ (\Theta_H)$, contradicting $\Theta_H = \omega_H$. Thus, if $\Theta_H = \omega_H$, then $o \not\equiv v\ (\Theta_L)$.

On the other hand, if $\Theta \neq \omega_H$, then $\Theta_H = \iota_H$. But then set $y = o$ and let x be any lower cover of v in L. Applying the dual of Condition (O1), we conclude that $x \equiv 0\ (\Theta)$, and so that $o \equiv v\ (\Theta_L)$.

Thus, if Θ extends to K, then either Condition (1) or Condition (2) holds.

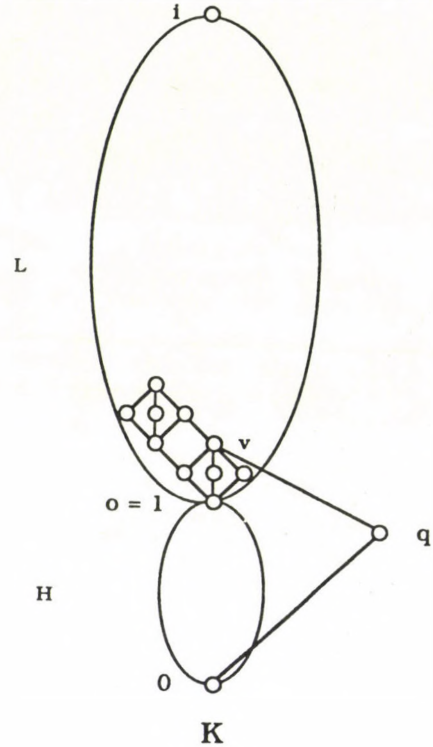

Fig. 14

Now let one of Condition (1) and Condition (2) hold. We show that Θ extends to K by establishing Condition (O1) and its dual.

We establish Condition (O1). Let $y < v$ and let $y \equiv v$ (Θ). Then $o \equiv v$ (Θ_L). Thus Condition (2) holds, and so $\Theta_H = \iota_H$, that is, $0 \equiv o$ (Θ). Consequently $0 \equiv v$ (Θ), and Condition (O1) follows immediately.

Next, we establish the dual of Condition (O1). Let $y > 0$ with $0 \equiv y$ (Θ). Then $\Theta_H = \iota_H$, and so Condition (2) holds, that is, $o \equiv v$ (Θ_L). Thus, $0 \equiv v$ (Θ), whereby the dual of Condition (O1) follows immediately.

8. Concluding comments. We can make the lattice L of the Theorem smaller by making the intervals $I(a_i, b_i)$ and $I(a_{i+1}, b_{i+1})$ overlap in C_0 by two elements provided that $b_i = b_{i+1}$. Note that the Colored Product Extension Theorem permits the intervals to overlap. While this can reduce the size of L by up to a third, it does not affect $O(|L|)$.

In [10], we prove a generalization of the theorem of Dilworth: Given two finite distributive lattices D_0 and D_1, and a $\{0,1\}$-homomorphism φ of D_0 into D_1, we show that there exist a finite lattice L and an ideal I of L such that the congruence lattice of L is isomorphic to D_0, the congruence lattice of I is isomorphic to D_1, and the restriction of a congruence from L to I

induces the homomorphism φ. See also E. T. Schmidt [18] for a different proof.

Using the construction developed in this paper, we can improve on this result by requiring either that L be planar, or alternatively, that L and I have given finite automorphism groups. The details will appear in [14].

The main problem, originally raised in [6], see Problem II.18, remains unresolved: Is the congruence lattice of a lattice always independent of the automorphism group?

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MAN. R3T 2N2
CANADA

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BERICHTIGUNG ZU MEINER ARBEIT: KONSTRUKTION DES REGULÄREN SIEBZEHNECKS MIT LINEAL UND STRECKENÜBERTRAGER[1]

J. STROMMER (Budapest)

In die Arbeit ist ein Fehler unterlaufen. Um dies zu verbessern, sind folgende Veränderungen vorzunehmen:

1) Auf S. 222 ist der vorletzte Absatz durch das folgende zu ersetzen:

Da ferner

$$(x_2 + x_6)(x_8 - x_7) = -1$$

ist, so sind die Größen

$$\text{(9)} \qquad z_1 = x_2 + x_6 \qquad (> 0),$$

$$\text{(10)} \qquad z_2 = x_8 - x_7 \qquad (< 0)$$

Wurzeln der quadratischen Gleichung:

$$x^2 - (u_1 + v_1)x - 1 = 0;$$

also ist

$$z_1 = \frac{1}{2}(u_1 + v_1) + \frac{1}{2}\sqrt{(u_1 + v_1)^2 + 4},$$

$$z_2 = \frac{1}{2}(u_1 + v_1) - \frac{1}{2}\sqrt{(u_1 + v_1)^2 + 4}.$$

Die Größen w_1, w_2, u_1, u_2, v_1, v_2, z_1, z_2 können wir als Strecken mit unseren beschränkten Hilfsmitteln leicht konstruieren, und so auf der Zahlengeraden OA, deren Anfangs- und Einheitspunkt O, bzw. A ist, auch die Punkte von der Abszisse w_1, w_2, u_1, u_2, v_1, v_2, z_1, z_2 bestimmen.

2) Auf S. 223 ist statt der beiden Absätze nach der Formel (8) folgendes zu setzen:

Aus den Gleichungen (6), (9) und (2) folgt, daß

$$x_1 = -\frac{2u_2 + z_1}{2 + v_2}$$

ist. Mittels dieser Formel kann man die Größe x_1 mit Hilfe der vorgelegten beschränkten Hilfsmittel leicht konstruieren und dann aus den Gleichungen

---

[1] In diesem *Acta*, **59** (1992), S. 217–226.

(2), (5), (7), (1), (3), (8) und (4) der Reihe nach auch die Größen x_4, x_6, x_2, x_7, x_5 und x_3 als Strecken geometrisch bestimmen.

(Eingegangen am 24. Juni 1992.)

TECHNISCHE UNIVERSITÄT
LEHRSTUHL FÜR GEOMETRIE
H-1111 BUDAPEST
STOCZEK UTCA 2-4

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ON A PROBLEM OF P. ERDŐS

G. N. SÁRKÖZY (Budapest)

1. Throughout the paper the following notations will be used: $\|x\|$ denotes the distance of x from the nearest integer. We write $e^{2\pi ix} = e(x)$. The cardinality of the finite set X is denoted by $|X|$, $\Lambda(n)$ is the Mangoldt function and $\pi(x)$ is the number of prime numbers not exceeding x.

2. Divisibility properties of sums of integers have been studied by many authors (see, e.g., [1], [2], [3] and [10]). P. Erdős asked the following related question: If $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$, $\mathcal{M} \subset \{1, \dots, [\sqrt{N}]\}$, $|\mathcal{M}| = k$, the elements of $\mathcal{M}$ are pairwise coprime and $m \nmid a+b$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $m \in \mathcal{M}$, then how large can $|\mathcal{A}||\mathcal{B}|$ be? In this paper we will study this problem. We will give the nearly best possible bound for $\max |\mathcal{A}||\mathcal{B}|$ under these assumptions. One may ask the question why the condition that the elements of $\mathcal{M}$ should be less than or equal to $\sqrt{N}$ is needed. We will return to this question.

In Section 3 we will give the upper bound $|\mathcal{A}||\mathcal{B}| \leqq \frac{4N^2}{k^2}$ for $|\mathcal{A}||\mathcal{B}|$ and the proof will use the large sieve. An application of this result will be shown. It is easy to see that this upper bound is the best possible apart from constants and a factor $(\log k)^2$. To see this let us take $\mathcal{A} = \mathcal{B} = \left\{n \mid 1 \leqq n < \frac{\sqrt{N}}{4},\ n \text{ integer}\right\}$ and $\mathcal{M} = \left\{p \mid \frac{\sqrt{N}}{2} \leqq p \leqq \sqrt{N},\ p \text{ prime}\right\}$, then $m \nmid a+b$ for every $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $|\mathcal{A}||\mathcal{B}| \geqq c\frac{N^2}{k^2(\log k)^2}$. In Section 5 we will show that the upper bound is best possible not only for this special, large k but for all k-s. Finally in Section 6 we will estimate $|\mathcal{A}||\mathcal{B}|$ for fixed $|\mathcal{M}|$ and we will derive a corollary in the case of prime powers.

3. In this section we will give an upper bound for $|\mathcal{A}||\mathcal{B}|$ if $m \nmid a+b$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $m \in \mathcal{M}$.

THEOREM 1. *Let N be a positive integer, assume that $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$, $\mathcal{M} \subset \{1, \dots, [\sqrt{N}]\}$, the elements of $\mathcal{M}$ are pairwise coprime and $m \nmid a+b$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$, $m \in \mathcal{M}$. Then*

$$|\mathcal{A}||\mathcal{B}| \leqq \frac{4N^2}{k^2}.$$

For the proof we need the following lemma:

LEMMA 1. *Assume that $\mathcal{M} \subset \{1, \dots, N\}$ and the elements of $\mathcal{M}$ are pairwise coprime. For each $m \in \mathcal{M}$ remove $f(m)$ different residue classes* mod m. *Then the number of positive integers* $n \leqq N$ *which remain is at most*

$$\frac{N+Q^2}{\sum\limits_{q\leqq Q}{}' \prod\limits_{\substack{m|q \\ m\in\mathcal{M}}} \frac{f(m)}{m-f(m)}}$$

where the dash indicates that q *is the product of distinct elements of* $\mathcal{M}$.

PROOF. Gallagher writes in [5] on page 492, that this can be shown in a similar way as Montgomery proves it in [8] for primes instead of $\mathcal{M}$. For the sake of completeness we give here the sketch of a proof.

Let us define $(a,b)_{\mathcal{M}}$ in this way: it is a divisor of both a and b and a product of elements of $\mathcal{M}$, and this is the largest number with these two properties. Write

$$\mu_{\mathcal{M}}(n)=\begin{cases} 1 & \text{if } n=1 \\ (-1)^r & \text{if } n=m_1m_2\dots m_r \text{ where } m_1,m_2,\dots m_r \\ & \text{are distinct elements of } \mathcal{M} \\ 0 & \text{otherwise.}\end{cases}$$

We prove the lemma in the following form:

Let a_n $(M+1\leqq n\leqq M+N)$ be arbitrary real or complex numbers. For each $m\in\mathcal{M}$ let $f(m)$ be the number of residue classes h mod m for which $a_n=0$ whenever $n\equiv h \pmod m$. Then for any $Q\geqq 1$

$$\left|\sum_n a_n\right|^2 \leqq \frac{N+Q^2}{\sum\limits_{q\leqq Q}\mu^2_{\mathcal{M}}(q)\prod\limits_{\substack{m|q \\ m\in\mathcal{M}}}\frac{f(m)}{m-f(m)}}\sum_n |a_n|^2.$$

Our lemma follows from this, if the a_n's assume only the values 0 and 1. Let us write $S(x)=\sum\limits_n a_n e(nx)$. First we prove:

$$\left|\sum_n a_n\right|^2 \mu^2_{\mathcal{M}}(q)\prod_{\substack{m|q \\ m\in\mathcal{M}}}\frac{f(m)}{m-f(m)} \leqq \sum_{\substack{a=1 \\ (a,q)_{\mathcal{M}}=1}}^{q}\left|S\left(\frac{a}{q}\right)\right|^2. \tag{1}$$

If q is not the product of distinct elements of $\mathcal{M}$, (1) is trivial. Thus we may say that q is the product of distinct elements of $\mathcal{M}$. Let $R(q)$ consist of those numbers r, $1\leqq r\leqq q$, for which $(n-r,q)_{\mathcal{M}}=1$ whenever $a_n\neq 0$. From the definition of $f(m)$ and the Chinese remainder theorem we see that $R(q)$ contains precisely $\prod\limits_{\substack{m|q \\ m\in\mathcal{M}}} f(m)$ numbers. We replace the Jensen–Ramanujan identity

$$\mu(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ah}{q}\right),\qquad (h,q)=1$$

by the following generalization of it:

$$\mu_{\mathcal{M}}(q)=\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{q} e\left(\frac{ah}{q}\right)\quad\text{where}\quad (h,q)_{\mathcal{M}}=1.$$

To see this we have to generalize the Moebius inversion for $\mathcal{M}$-divisibility, but this can be done in the same way.

Hence for every $r\in R(q)$ we have

$$a_n\mu_{\mathcal{M}}(q)=\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{q} a_n e\left(\frac{(n-r)a}{q}\right).$$

We sum this over n and all $r\in R(q)$ to obtain

$$\Big(\sum_n a_n\Big)\mu_{\mathcal{M}}(q)\prod_{\substack{m|q\\ m\in\mathcal{M}}} f(m)=\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{q}\Big(\sum_N a_n e\Big(\frac{an}{q}\Big)\Big)\Big(\sum_{r\in R(q)} e\Big(\frac{-ar}{q}\Big)\Big)=$$

$$=\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{q} S\Big(\frac{a}{q}\Big)\sum_{r\in R(q)} e\Big(\frac{-ar}{q}\Big).$$

By Cauchy's inequality we see that

(2)

$$\Big|\sum_n a_n\Big|^2\mu_{\mathcal{M}}^2(q)\prod_{\substack{m|q\\ m\in\mathcal{M}}} f(m)^2\leqq\Big(\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{q}\Big|S\Big(\frac{a}{q}\Big)\Big|^2\Big)\Big(\sum_{\substack{a=1\\(a,q)_{\mathcal{M}}=1}}^{2}\Big|\sum_{r\in R(q)} e\Big(\frac{-ar}{q}\Big)\Big|^2\Big).$$

Now a little consideration shows that the second factor is multiplicative. Hence it is

$$=\prod_{\substack{m|q\\ m\in\mathcal{M}}}\Big(\sum_{a=1}^{m-1}\Big|\sum_{r\in R(m)} e\Big(\frac{-ar}{m}\Big)\Big|^2\Big)=\prod_{\substack{m|q\\ m\in\mathcal{M}}}\Big(\sum_{r_1\in R(m)}\sum_{r_2\in R(m)}\sum_{a=1}^{m-1} e\Big(\frac{a(r_2-r_1)}{m}\Big)\Big).$$

The value of the innermost sum is $m-1$ or -1 according as $m\mid r_2-r_1$ or not, so the above is

$$=\prod_{\substack{m|q\\ m\in\mathcal{M}}}((m-1)f(m)-f(m)(f(m)-1))=\prod_{\substack{m|q\\ m\in\mathcal{M}}} f(m)(m-f(m)).$$

This together with (2) proves (1).

Now we use the analytical form of the large sieve:

LEMMA 2. *Let a_n $(M+1 \leqq n \leqq M+N)$ be arbitrary real or complex numbers and put*

$$S(x) = \sum_n a_n e(nx).$$

Let $\mathcal{X}$ be a set of real numbers for which $\|x - x'\| \geqq \delta > 0$ whenever x and x' are distinct members of $\mathcal{X}$. Then

$$\sum_{x \in \mathcal{X}} |S(x)|^2 \leqq (\delta^{-1} + N) \sum_n |a_n|^2.$$

The proof is in [9].

We now derive Lemma 1 from this lemma. We choose $\mathcal{X}$ as the set of the fractions $\frac{a}{q}$, $q \leqq Q$, where q is the product of elements of $\mathcal{M}$ and $(a,q)_{\mathcal{M}} = 1$. It is easy to see that in this case $\delta \geqq Q^{-2}$, thus we have

$$\sum_{q \leqq Q}{}' \sum_{\substack{a=1 \\ (a,q)_{\mathcal{M}}=1}}^{q} \left| S\left(\frac{a}{q}\right)\right|^2 \leqq (N + Q^2) \sum_n |a_n|^2$$

and this with (1) proves our lemma.

PROOF OF THEOREM 1. Let us assume that for $m \in \mathcal{M}$ there are $f(m)$ residue classes mod m which contain no element of $\mathcal{A}$ and there are $g(m)$ residue classes which contain no element of $\mathcal{B}$. If $m \nmid a + b$, then it is clear that $g(m) \geqq m - f(m)$, i.e., $g(m) + f(m) \geqq m$. Using the previous Lemma 1 with $Q = \sqrt{N}$ we get

$$|\mathcal{A}| \leqq \frac{N + Q^2}{\sum\limits_{q \leqq Q} \mu^2_{\mathcal{M}}(q) \prod\limits_{\substack{m|q \\ m \in \mathcal{M}}} \frac{f(m)}{m - f(m)}} \leqq \frac{2N}{\sum\limits_{m \in \mathcal{M}} \frac{f(m)}{m - f(m)}}, \tag{3}$$

$$|\mathcal{B}| \leqq \frac{N + Q^2}{\sum\limits_{q \leqq Q} \mu^2_{\mathcal{M}}(q) \prod\limits_{\substack{m|q \\ m \in \mathcal{M}}} \frac{g(m)}{m - g(m)}} \leqq \frac{N + Q^2}{\sum\limits_{q \leqq Q} \mu^2_{\mathcal{M}}(q) \prod\limits_{\substack{m|q \\ m \in \mathcal{M}}} \frac{m - f(m)}{f(m)}} \leqq \frac{2N}{\sum\limits_{m \in \mathcal{M}} \frac{m - f(m)}{f(m)}}.$$

By Cauchy's inequality we see that

$$|\mathcal{A}||\mathcal{B}| \leqq \frac{(2N)^2}{\left(\sum\limits_{m \in \mathcal{M}} \frac{f(m)}{m - f(m)}\right)\left(\sum\limits_{m \in \mathcal{M}} \frac{m - f(m)}{f(m)}\right)} \leqq \frac{4N^2}{\left(\sum\limits_{m \in \mathcal{M}} \left(\frac{f(m)}{m - f(m)}\right)^{\frac{1}{2}} \left(\frac{m - f(m)}{f(m)}\right)^{\frac{1}{2}}\right)^2} =$$

$$= \frac{4N^2}{\left(\sum\limits_{m \in \mathcal{M}} 1\right)^2} = \frac{4N^2}{k^2}.$$

4. Step (3) plays an important role in the proof. In certain cases, when many products of the m's are less than $\sqrt{N}$, then it gives a very "rough" estimate and it can be improved by the missing factor $(\log k)^2$. But if no product of the m's is less than $\sqrt{N}$, (3) is an equality and we cannot save in this way. An example for this case is

COROLLARY 1. *Let ℓ and $N \geqq N_0(\ell)$ be positive integers, $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$. If*

$$|\mathcal{A}||\mathcal{B}| > \frac{4}{\ell^2(1-2^{-\frac{1}{\ell}})^2} N^{2-\frac{1}{\ell}} (\log N)^2, \tag{4}$$

then there exists a prime p such that $\frac{\sqrt{N}}{2} \leqq p^\ell \leqq \sqrt{N}$ and $p^\ell \mid a+b$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$.

PROOF. We use Theorem 1 choosing

$$\mathcal{M} = \left\{ p^\ell \mid p \text{ prime}, \frac{\sqrt{N}}{2} \leqq p^\ell \leqq \sqrt{N} \right\}.$$

If N is large enough in terms of ℓ, then by the prime number theorem we have

$$|\mathcal{M}| = k \geqq \frac{N^{\frac{1}{2\ell}}(1-2^{-\frac{1}{\ell}})}{2 \log N^{\frac{1}{2\ell}}} = \frac{\ell N^{\frac{1}{2\ell}}(1-2^{-\frac{1}{\ell}})}{\log N}.$$

For this sufficiently large N by (4) we have

$$\frac{4N^2}{k^2} \leqq 4N^2 \left(\frac{\ell N^{\frac{1}{2\ell}}(1-2^{-\frac{1}{\ell}})}{\log N} \right)^{-2} = 4\ell^{-2}(1-2^{-\frac{1}{\ell}})^{-2} N^{2-\frac{1}{\ell}} (\log N)^2 < |\mathcal{A}||\mathcal{B}|.$$

Hence by Theorem 1 there exist a prime p and integers a, b such that $\frac{\sqrt{N}}{2} \leqq$ $\leqq p^\ell \leqq \sqrt{N}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $p^\ell \mid a+b$ and this completes the proof of Corollary 1.

If we drop the condition $p^\ell \geqq \frac{\sqrt{N}}{2}$, then many products of the m's are less than $\sqrt{N}$, and using this fact we get similarly

COROLLARY 2. *Let ℓ and $N > N_0(\ell)$ be positive integers, $\mathcal{A}, \mathcal{B} \subset$ $\subset \{1, \dots, N\}$. If*

$$|\mathcal{A}||\mathcal{B}| > c(\ell) N^{2-\frac{1}{\ell}},$$

then there eixsts a prime p such that $p^\ell \leqq \sqrt{N}$ and $p^\ell \mid a+b$ for some $a \in \mathcal{A}$, $b \in \mathcal{B}$.

We remark that in the special case $\ell = 2$ this corollary improves a result of [4] by N^ε.

5. In this section we give a lower bound. First we give a proof of existence for all k and then for "small" k's we construct $\mathcal{M}$ and $\mathcal{A}, \mathcal{B}$.

THEOREM 2. *If $N \geqq N_0$, then there exists a constant $c_1 > 0$ such that for every positive integer k with $k \leqq \pi(\sqrt{N}) - \pi(\frac{\sqrt{N}}{2})$ there exist $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$, $\mathcal{M} \subset \{1, \dots, [\sqrt{N}]\}$, $|\mathcal{M}| = k$, the elements of $\mathcal{M}$ are pairwise coprime,*

$$|\mathcal{A}||\mathcal{B}| > c_1 \frac{N^2}{k^2(\log k)^2} \tag{5}$$

and $m \nmid a + b$ for every $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$.

PROOF. First we are going to show that there exist constants $c_2, c_3 > 0$ such that for all $1 \leqq k \leqq c_2(\pi(\sqrt{N}) - \pi(\frac{\sqrt{N}}{2}))$ there exist k pairwise coprime numbers belonging to a subinterval of length at most $c_3 k \log k \leqq \frac{\sqrt{N}}{6}$ of the interval $[\frac{\sqrt{N}}{2}, \sqrt{N}]$.

If $c_2(\pi(\sqrt{N}) - \pi(\frac{\sqrt{N}}{2})) \leqq k \leqq \pi(\sqrt{N}) - \pi(\frac{\sqrt{N}}{2})$ then let $\mathcal{M}$ be the set of any k primes from the interval $[\frac{\sqrt{N}}{2}, \sqrt{N}]$, and let $\mathcal{A} = \mathcal{B} = \left\{n \mid 1 \leqq n < \frac{\sqrt{N}}{4},\ n \text{ integer}\right\}$. It is easy to see that (5) holds with a constant c_1' for all such k's.

LEMMA 3. *If $N \geqq N_1$ and $z \leqq \frac{\sqrt{N}}{6}$, then there exists a constant $c_4 > 0$ such that*

$$\left|\left\{n \,\middle|\, \frac{\sqrt{N}}{2} \leqq n \leqq \sqrt{N},\ p \mid n \Rightarrow p > z\right\}\right| \geqq c_4 \frac{\sqrt{N}}{\log z}.$$

PROOF. There exists a number $\delta > 0$ such that for $z \leqq \left(\frac{\sqrt{N}}{2}\right)^{\delta}$ the assertion follows from Brun's sieve ([6], page 82, Theorem 2.5), while for greater z's it suffices to take the primes between $\frac{\sqrt{N}}{2}$ and $\sqrt{N}$ to see the inequality to be proved.

It is clear by this lemma that there is a constant $c_5 > 0$ such that for all $z \leqq \frac{\sqrt{N}}{6}$ there exists an interval of length at most z in the interval $[\frac{\sqrt{N}}{2}, \sqrt{N}]$, which contains at least $c_5 \frac{z}{\log z}$ numbers whose prime factors are greater than z. But then these numbers are pairwise coprime. So we have found $c_2, c_3 > 0$ such that for all natural numbers $1 \leqq k \leqq c_2(\pi(\sqrt{N}) - \pi(\frac{\sqrt{N}}{2}))$ there exist k pairwise coprime numbers belonging to a subinterval of length at most $c_3 k \log k \leqq \frac{\sqrt{N}}{6}$ of the interval $[\frac{\sqrt{N}}{2}, \sqrt{N}]$ and this was to be proved. Let $\mathcal{M}$ consist of these numbers and let us denote the right end of this interval by L. Then $\mathcal{M} \subset [L - c_3 k \log k, L]$ and here $L > \frac{\sqrt{N}}{2}$. Let us multiply the elements of $\mathcal{M}$ by i. These products belong to the interval $[iL - ic_3 k \log k, iL]$ and therefore the intervals $(iL, iL + \frac{L}{2})$ do not contain multiples of elements of $\mathcal{M}$ if $0 \leqq i \leqq \frac{L}{2(c_3 k \log k)} - 1$.

Then the construction is the following:

$$\mathcal{A} = \mathcal{B} = \bigcup_i \left\{ n \mid iL < n < iL + \frac{L}{4},\ n \text{ integer} \right\} \text{ where } 0 \leqq i \leqq \frac{L}{4(c_3 k \log k)} - \frac{1}{2}.$$

This set is not empty because $c_3 k \log k \leqq \frac{\sqrt{N}}{6}$. For these $\mathcal{A}, \mathcal{B}$, for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ we have $a + b \in (iL, iL + \frac{L}{2})$ for some $0 \leqq i \leqq \frac{L}{2(c_3 k \log k)} - 1$ thus $m \nmid a + b$ for every $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$. Furthermore, if $N > N_2$, then

$$|\mathcal{A}| = |\mathcal{B}| \geqq \left[\frac{L}{4(c_3 k \log k)} + \frac{1}{2} \right] \left[\frac{L}{4} - 2 \right] > \frac{L}{5(c_3 k \log k)} \frac{L}{5} > \frac{N}{100 c_3 k \log k}.$$

Hence

$$|\mathcal{A}||\mathcal{B}| > c_1'' \frac{N^2}{k^2 (\log k)^2}$$

so that choosing $c_1 = \min(c_1', c_1'')$, (5) holds for all k.

We remark that this construction cannot be improved apart from a constant factor. Namely, it is easy to see that an interval of length H can contain only at most $c \frac{H}{\log H}$ pairwise coprime numbers. In fact the number of primes not exceeding $\sqrt{H}$ is at most $c_1 \frac{\sqrt{H}}{\log H}$ and each of them may have one multiple among our numbers, and the number of those integers, all of whose prime factors are greater than or equal to $\sqrt{H}$, is less than or equal to $c_2 \frac{H}{\log H}$ by Brun's sieve ([6], p. 72, Corollary 2.3.1).

Now I would like to return to the question, why we need the assumption that the elements of $\mathcal{M}$ be less than or equal to $\sqrt{N}$. If the m's may exceed $\sqrt{N}$, then the same construction can be given with $L > \sqrt{N k \log k}$ and then $\mathcal{A}, \mathcal{B}$ may contain the positive percentage of the integers up to N, so that $m \mid a + b$ never holds since $\frac{L}{2(ck \log k)} L > c'N$. Thus for "small" k going a little over $\sqrt{N}$, there are $\mathcal{M}, \mathcal{A}, \mathcal{B}$ such that $\mathcal{A}, \mathcal{B}$ are dense and $m \nmid a + b$.

At the end of this section we give a constructive proof of the lower bound for "small" k's:

THEOREM 3. *Let $N > N_0$ be a positive integer. For every positive integer $k_0 \leqq k \leqq \frac{\log N}{4(\log \log N)^2}$ we can construct $\mathcal{M}, \mathcal{A}, \mathcal{B}$ such that $|\mathcal{M}| = k$, $\mathcal{M} \subset \{1, \dots, [\sqrt{N}]\}$, the elements of $\mathcal{M}$ are pairwise coprime, $\mathcal{A}, \mathcal{B} \subset \subset \{1, \dots, N\}$,*

$$|\mathcal{A}||\mathcal{B}| > \frac{N^2}{32400 k^2 (\log k)^2}$$

and $m \nmid a + b$ for every $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$.

PROOF. Let us fix a number $k_0 \leqq k \leqq \frac{\log N}{4(\log\log N)^2}$. Let us define the number x by the inequality

$$x([2k\log k]!) \leqq \sqrt{N} < (x+1)([2k\log k]!). \tag{6}$$

Let $L = x([2k\log k]!)$ and let us denote the i-th prime number by p_i. Then by the prime number theorem, for $k > k_1$ we have $p_k < 2k\log k$. Define $\mathcal{M}$ in the following way:

$$\mathcal{M} = \{L - p_i \mid i = 1, \dots, k\}.$$

We are going to show that $\mathcal{M}$ satisfies our conditions. Clearly $|\mathcal{M}| = k$ and by (6) every element of $\mathcal{M}$ is less than $\sqrt{N}$. The elements of $\mathcal{M}$ must be pairwise coprime. Let us assume indirectly that there are two elements of $\mathcal{M}$ having a common prime factor. In fact, let $p \mid L - p_i$ and $p \mid L - p_j$ where $i < j \leqq k$. Then $p \mid p_j - p_i$, consequently, $p \leqq p_j < 2k\log k$ but then $p \mid L$ thus $p \mid p_i$ and $p \mid p_j$, i.e., $p = p_j = p_i$ which is a contradiction. So we have $\mathcal{M} \subset [L - 2k\log k, L]$ where $L > \frac{\sqrt{N}}{2}$ and from here the proof is the same as in the previous theorem.

6. In this section we will study that for fixed $\mathcal{M}$ how large $|\mathcal{A}||\mathcal{B}|$ can be under the condition that $m \mid a + b$ never holds. First we will discuss the case when $\mathcal{M}$ consists of "small" numbers where we drop the conditions that the elements of $\mathcal{M}$ must be less than $\sqrt{N}$ and pairwise coprime and an application will be shown.

THEOREM 4. *Let $N > N_0$ be a positive integer. Assume that $\mathcal{M} = \{m_1, \dots, m_k\} \subset \{2, \dots, N\}$ where $m_1 < m_2 < \dots < m_k$. Let $r \leqq k$, $P := m_1 \dots m_r$ and assume that $P < \frac{N}{8}$. Let*

$$\sum_{i=1}^{r} \frac{1}{m_i} \leqq \gamma \tag{7}$$

and

$$\sum_{i=r+1}^{k} \frac{1}{m_i} \leqq \delta \tag{8}$$

where assume that $\delta e^{2\gamma} \leqq \frac{1}{8}$. Then there exist $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ such that $m \nmid a + b$ if $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and

$$|\mathcal{A}||\mathcal{B}| > \frac{N}{8e^{2\gamma}} \min\left(\left[\frac{N}{4P}\right], \left[\frac{1}{8\delta e^{2\gamma}}\right]\right) \quad \textit{where} \quad \frac{1}{0} = \infty.$$

PROOF. Let us take all the positive integers n such that $n \not\equiv 0 \pmod{m_i}$ for $1 \leqq i \leqq r$. Then by a well-known lemma of Behrend ([7], Lemma 5, p. 263) these numbers form at least

$$\prod_{i=1}^{r}(m_i - 1) = P\prod_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) = P\,\exp\left(\sum_{i=1}^{r}\log\left(1 - \frac{1}{m_i}\right)\right) \tag{9}$$

residue classes mod P. Let us use the trivial inequality $\log(1-x) \geqq -2x$ if $0 < x \leqq \frac{1}{2}$. Here $1 \notin \mathcal{M}$ implies $\frac{1}{2} \geqq \frac{1}{m_i} > 0$. Therefore by (7) and (9) we get

$$\prod_{i=1}^{r}(m_i - 1) > P \exp\Big(-\sum_{i=1}^{r} 2\frac{1}{m_i}\Big) > \frac{P}{e^{2\gamma}}.$$

Thus those integers not exceeding N, which are not divisible by any of $m_1, m_2, \dots, m_r$, form at least $\frac{P}{e^{2\gamma}}$ arithmetic progressions of difference P.

Now let us give an upper estimate for the number of those "bad" integers not exceeding N, which are divisible by at least one of $m_{r+1}, m_{r+2}, \dots, m_k$. By (8)

$$\sum_{i=r+1}^{k} \left[\frac{N}{m_i}\right] \leqq N \sum_{i=r+1}^{k} \frac{1}{m_i} \leqq N\delta.$$

Hence the arithmetic progressions above contain at most $\frac{N\delta e^{2\gamma}}{P}$ "bad" numbers on the average, thus at least $\frac{P}{2e^{2\gamma}}$ arithmetic progressions contain at most $\frac{2N\delta e^{2\gamma}}{P}$ "bad" numbers. Let us denote the set of these arithmetic progressions by $\mathbf{V}$, then

$$|\mathbf{V}| \geqq \frac{P}{2e^{2\gamma}}. \tag{10}$$

Let us consider an arithmetic progression $\mathcal{V} = \{h, h+P, \dots, h+KP\} \in \mathbf{V}$, where $0 < h \leqq P$, $h + KP \leqq N < h + (K+1)P$. Then $K = [\frac{N-h}{P}]$ whence $\frac{N}{P} - 2 < K \leqq \frac{N}{P}$. Let the "bad" elements of this arithmetic progression be $h + k_1P$, $h + k_2P, \dots, h + k_TP$ where by the discussion above $T \leqq \frac{2N\delta e^{2\gamma}}{P}$. Let $k_{T+1} = K+1$ and $Z := \min([\frac{N}{4P}], [\frac{1}{8\delta e^{2\gamma}}])$. Let us define the subset $\mathcal{B}(\mathcal{V})$ of $\mathcal{V}$ in the following way: for $0 \leqq j < K$ let $h + jP \in \mathcal{B}(\mathcal{V})$ if and only if for every $1 \leqq i \leqq Z$ we have $h + jP + iP \notin \{h + k_1P, h + k_2P, \dots h + k_{T+1}P\}$.

Now we will give a lower estimate for $|\mathcal{B}(\mathcal{V})|$. Obviously, for one of the k j's, satisfying the condition $0 \leqq j < k$, $h + jP \in \mathcal{B}(\mathcal{V})$ does not hold if and only if for some $1 \leqq i \leqq Z$ and for some $1 \leqq \ell \leqq T+1$ we have $h + jP + iP = h + k_\ell P$, i.e., $j = k_\ell - i$. Here ℓ and i can be chosen in $T+1$ and Z many ways, respectively, thus $h + jP \notin \mathcal{B}(\mathcal{V})$ holds for at most $(T+1)Z$ j's. Therefore (by $P < \frac{N}{8}$) we have

(11)
$$|\mathcal{B}|(\mathcal{V})| \geqq k - (T+1)Z > \frac{N}{P} - 2 - \Big(\frac{2N\delta e^{2\gamma}}{P} + 1\Big)\min\Big(\frac{N}{4P}, \frac{1}{8\delta e^{2\gamma}}\Big) >$$
$$> \frac{N}{P} - \frac{N}{4P} - \frac{N}{4P} - \frac{N}{4P} = \frac{N}{4P}.$$

Now the construction is the following:

$$\mathcal{B} = \bigcup_{\mathcal{V} \in \mathbf{V}} \mathcal{B}(\mathcal{V}), \qquad \mathcal{A} = \{P, 2P, \dots, ZP\}.$$

It follows from the construction that $m \nmid a + b$ for $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and by (10) and (11) we have

$$|\mathcal{A}||\mathcal{B}| \geqq Z|\mathbf{V}| \min_{\mathcal{V} \in \mathbf{V}} |\mathcal{B}(\mathcal{V})| > \frac{N}{8e^{2\gamma}} \min\left(\left[\frac{N}{4P}\right], \left[\frac{1}{8\delta e^{2\gamma}}\right]\right).$$

This completes the proof of the theorem.

COROLLARY 3. *Let ℓ and N be positive integers, $\ell \geqq 2$, $N > N_0(\ell)$. Then there exist sequences $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ such that $p^\ell \nmid a + b$ for every prime p, $a \in \mathcal{A}$, $b \in \mathcal{B}$, and*

$$|\mathcal{A}||\mathcal{B}| > \frac{(\ell - 1)N(\log N)^{\ell - 1}}{65e^4(3\ell)^{\ell - 1}}.$$

PROOF. We apply the previous theorem with

$$\mathcal{M} = \{p^\ell \mid p \text{ prime}, p^\ell \leqq 2N\} \quad \text{and} \quad p_1^\ell p_2^\ell \dots p_{r-1}^\ell < \sqrt{N} \leqq p_1^\ell p_2^\ell \dots p_r^\ell := P.$$

Then γ can be chosen in the following way:

$$\sum_{i=1}^{r} \frac{1}{p_i^\ell} < \sum_{i=1}^{\infty} \frac{1}{p_i^\ell} < \sum_{n=1}^{\infty} \frac{1}{n^\ell} \leqq \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} < 2 := \gamma.$$

Furthermore

$$\sum_{i=r+1}^{k} \frac{1}{p_i^\ell} < \sum_{i=r+1}^{\infty} \frac{1}{p_i^\ell} < \sum_{n=p_r+1}^{\infty} \frac{1}{n^\ell} < \int_{p_r}^{\infty} \frac{1}{x^\ell} dx = \frac{1}{(\ell - 1)p_r^{\ell - 1}}.$$

By the prime number theorem

$$\log P = \ell \sum_{i=1}^{r} \log p_i \sim \ell \sum_{n \leqq p_r} \Lambda(n) \sim \ell p_r,$$

i.e., $p_r \sim \frac{\log P}{\ell}$. Thus

$$\frac{\ell^\ell P}{(\log P)^\ell} \sim \frac{P}{p_r^\ell} < \sqrt{N} \leqq P,$$

whence $\log P \sim \frac{1}{2} \log N$ (so that $P < \frac{N}{8}$ holds for $N > N_1$). Hence

$$p_r \sim \frac{\log N}{2\ell} > \frac{\log N}{3\ell} \quad \text{for} \quad N > N_2(\ell)$$

and

$$\sum_{i=r+1}^{k} \frac{1}{p_i^\ell} < \frac{(3\ell)^{\ell-1}}{(\ell-1)(\log N)^{\ell-1}} := \delta.$$

Then for $N > N_3(\ell)$, $\delta e^{2\gamma} < \frac{1}{8}$ holds trivially. Applying Theorem 4 there exist sequences $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ such that $p^\ell \nmid a + b$ and

$$|\mathcal{A}||\mathcal{B}| > N \frac{1}{8e^4} \left[\frac{(\ell-1)(\log N)^{\ell-1}}{(3\ell)^{\ell-1} 8e^4} \right] > \frac{(\ell-1)(\log N)^{\ell-1}}{65(3\ell)^{\ell-1} e^8} N.$$

Namely we have to take the second term of the minimum in Theorem 4, since the orders of magnitude $[\frac{N}{4P}]$ and $[\frac{1}{8\delta e^{2\gamma}}]$ are $\sqrt{N}$ and a power of $\log N$, respectively.

P. Erdős and A. Sárközy remark that in the special case $\ell = 2$ there exist sequences $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ such that $\frac{|\mathcal{A}||\mathcal{B}|}{N} \to \infty$ and $p^2 \nmid a + b$.

In Theorem 4 $\mathcal{M}$ is chosen in a special way. One may ask what can be said for an arbitrary set $\mathcal{M}$ (e.g., if $\mathcal{M}$ is the set of primes not exceeding $\sqrt{N}$). Then the following lemma can be used:

LEMMA 3. *There is an effectively computable constant c_1 such that if $N > c_1$ is a positive integer and $t < \log N$, then there exist sequences $\mathcal{A}, \mathcal{B} \subset \subset \{1, \dots, N\}$ such that $|\mathcal{B}| = t$,*

$$|\mathcal{A}| > \frac{N}{t(2 \log N)^t},$$

and $\mathcal{A} + \mathcal{B}$ consists of primes between $\frac{N}{2}$ and N.

PROOF. This can be found in [10].

This implies that for $N > c_1$ and for an arbitrary set $\mathcal{M}$ there exist sequences $\mathcal{A}, \mathcal{B} \subset \{1, \dots, N\}$ such that $|\mathcal{B}| = t < \log N$,

$$|\mathcal{A}||\mathcal{B}| > \frac{N}{(2 \log N)^t}$$

and $m \nmid a + b$ for every $m \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Finally I would like to thank Professor Erdős for his valuable remarks.

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RUTGERS UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
NEW BRUNSWICK, NJ 08903
U.S.A.

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NEW UNIFIED RADON INVERSION FORMULAS

Á. KURUSA (Szeged)

1. Introduction

Let f be a real function on $\mathbf{R}^n$ and assume that it is integrable on each hyperplane. Let $\mathbf{P}^n$ denote the space of all hyperplanes in $\mathbf{R}^n$. The Radon transform Rf of f is defined by

$$Rf(\xi) = \int_{\xi} f(x)dx,$$

where dx is the natural measure on the hyperplane ξ. Each hyperplane $\xi \in \in \mathbf{P}^n$ can be written as $\xi = \{x \in \mathbf{R}^n : \langle x, \omega\rangle = p\}$, where $\omega \in S^{n-1}$ is a unit vector and $\langle .,.\rangle$ is the usual inner product on $\mathbf{R}^n$. In what follows we identify the continuous functions ϕ on $\mathbf{P}^n$ with continuous functions ϕ on $S^{n-1} \times \mathbf{R}$ satisfying $\phi(\omega, p) = \phi(-\omega, -p)$.

We introduce also the dual transform R_t which maps a continuous function $\phi \in C^0(\mathbf{P}^n)$ to the function $R_t\phi \in C^0(\mathbf{R}^n)$ defined by

$$R_t\phi(x) = \int_{S^{n-1}} \phi(\omega, \langle\omega, x\rangle)d\omega.$$

First Radon [10] and John [8] proved that any C^∞ function f of compact support can be reconstructed from Rf. More precisely, if L denotes the Laplacian on $\mathbf{R}^n$ and $d\omega$ is the area element on S^{n-1} then

$$f(x) = 2(2\pi)^{1-n}(-L)^{(n-1)/2} \int_{S^{n-1}} Rf(\omega, \langle\omega, x\rangle)d\omega \qquad \text{if } n \text{ is odd,} \tag{1}$$

(2)

$$f(x) = -(2\pi)^{-n}(-L)^{(n-2)/2} \int_{S^{n-1}} \int_{-\infty}^{\infty} \partial_2 Rf(\omega, p)\frac{dp}{\langle\omega, x\rangle - p}d\omega \qquad \text{if } n \text{ is even.}$$

In formula (2), the Cauchy principal value is taken. Later these formulas were proved under many different assumptions [4, 6, 7, 9, 11]. These proofs are based on advanced potential analysis and the inversion formulas are different in the odd and even dimensional cases. Deans [3] gave a unified inversion formula which covered both cases but his formula was not so explicit as (1) and (2).

In this paper we prove two explicit unified inversion formulas, given in the next theorem, using elementary geometry and analysis rather than the potential theory employed by previous authors. In the following theorem, $S(\mathbf{R}^n)$ denotes the Schwartz-space of smooth rapidly decreasing functions on $\mathbf{R}^n$.

THEOREM. *If* $f \in S(\mathbf{R}^n)$, $2 \leqq n \in \mathbf{N}$ *and*

(1.1)

$$h_A(\omega,p)=C\lim_{\varepsilon\to 0}\int_1^\infty (r^2-1)^{\frac{n-3}{2}}\left(\left(\frac{d}{dr}\right)^{n-1}Rf(\omega,p+r\varepsilon)+\left(\frac{d}{dr}\right)^{n-1}Rf(\omega,p-r\varepsilon)\right)dr,$$

$$h_B(\omega,p) = C\lim_{\varepsilon\to 0}\int_{|r|>\varepsilon} r^{n-2}\left(\frac{d}{rdr}\right)^{n-1}(Rf(\omega,p-r))dr, \tag{1.2}$$

where $C = (-1)^{n-1}\Gamma(n/2)\pi^{1/2}/\Gamma((n-1)/2)(2\pi)^n$, *then* $f = R_t h_A = R_t h_B$.

It is well known that the dual transform R_t has non-trivial kernel. So, for any function f, the above functions h_A and h_B are in the preimage of f at R_t, i.e. $h_A, h_B \in R_t^{-1} f$. For a clearer formulation, we introduce the operators $\square$ and Ξ by

$$\square f(\omega,p) = C\lim_{\varepsilon\to 0}\int_1^\infty (r^2-1)^{\frac{n-3}{2}}\left(\left(\frac{d}{dr}\right)^{n-1}f(\omega,p+r\varepsilon)+\left(\frac{d}{dr}\right)^{n-1}f(\omega,p-r\varepsilon)\right)dr,$$

$$\Xi f(\omega,p) = C\lim_{\varepsilon\to 0}\int_{|r|>\varepsilon} r^{n-2}\left(\frac{d}{rdr}\right)^{n-1}(f(\omega,p-r))dr.$$

Then our inversion formulas appear in the form $f = R_t\square Rf$ and $f = R_t\Xi Rf$. These formulas are very similar to the Radon formulas $f = cR_t\Lambda^{n-1}Rf$, where Λ is the Calderon–Zygmund operator in one dimension [12]. Straightforward but lengthy calculations on the Taylor expansion of $(r^2-1)^{(n-3)/2}$ show that $\square = \Lambda^{n-1}$. Also $\Xi = \Lambda^{n-1}$ can be proved by integration by parts (see (12) on p. 11 of [5]). We do not go into details in this paper.

The dual transform notion R_t appears in the previously mentioned form in the literature [6]. Now we slightly modify this notion because this (equivalent!) version is more treatable in our considerations. To avoid misunderstanding, this version is said to be boomerang transform and it is denoted by B [13]. The function space $C^0B(\mathbf{R}^n\backslash 0)$ consists of such continuous real functions on $\mathbf{R}^n\backslash 0$ which can be extended into continuous functions also at the origin 0 along any line lying on 0. The boomerang transform B: $C^0B(\mathbf{R}^n\backslash 0) \to C^0(\mathbf{R}^n)$ is defined by

$$Bf(x) = \frac{1}{2} \int_{S^{n-1}} f(\omega\langle\omega, x\rangle)d\omega.$$

The simple connection between R_t and B can be described as follows. For a real function f on $\mathbf{R}^n$ let Pf be the function on $\mathbf{P}^n$ defined by $Pf(\xi) = f(x_\xi)$, where x_ξ is the orthogonal projection of the origin 0 on the hyperplane ξ. Then $Bf = R_tPf$.

Also a useful geometric interpretation of the boomerang transform can be given as follows [13]. Let $f_\omega(t)$ be a continuous function defined on the line $l_\omega = \{t\omega : t \in \mathbf{R}\}$. Then the function $f_\omega^W \in C^0(\mathbf{R}^n)$, defined by $f_\omega^W(x) := f_\omega(\langle x, \omega\rangle)$, is a so called 'plane wave' with the axis ω. The function f_ω^W is constant along the hyperplanes which intersect the line l_ω orthogonally. Now take a function $f \in C^0B(\mathbf{R}^n\backslash 0)$ and for any $\omega \in S^{n-1}$ consider the function $f_\omega(t) := f(t\omega)$ on l_ω. Then the map $\omega \to f_\omega^W$ is a function-valued (plane wave-valued) function defined on S^{n-1}. The integral of this function is just Bf i.e.

$$Bf = \frac{1}{2} \int_{S^{n-1}} f_\omega^W d\omega.$$

Finally we sketch the main ideas of the paper. We start by the investigation of the radial function; this is the main point of our approach. First we show that the transform B is one to one on the space G_0 of smooth radial functions and prove three inversion formulas on this space G_0. (It is worth to note here that a different consideration of the boomerang transform on this space G_0 can be found also in [13].) In the next step, we prove inversion formulas for the radial functions which are defined around an arbitrary point $P \in \mathbf{R}^n$. Using Dirac sequences and convolution, we prove our general inversion formulas from these special ones.

2. Inversion formulas on radial functions

A function $f(x) \in C^0(\mathbf{R}^n)$ is said to be radial at $P \in \mathbf{R}^n$, if there exists a function $\overline{f}\colon \mathbf{R}_+ \to \mathbf{R}$ such that $f(x) = \overline{f}(|x - P|)$. If f is radial at 0 then

$$Bf(x) = |S^{n-2}| \int_0^{\pi/2} \cos^{n-2}(\alpha)\overline{f}(|x| \sin\alpha)d\alpha. \tag{2.1}$$

LEMMA 2.1. *If h is a continuous radial function then*

$$(2.2)\qquad Bh(x) = |S^{n-2}| \int_0^1 h(p|x|)(1-p^2)^{(n-3)/2}dp$$

and so if $f_i(x) = |x|^i$ $(i \in \mathbf{N})$ *then*

$$(2.3)\qquad Bf_i(x) = f_i(x)\pi^{(n-1)/2}\frac{\Gamma((i+1)/2)}{\Gamma((n+i)/2)}.$$

The proof is a simple calculation which is left to the reader.

COROLLARY 2.2. *If f is a continuous radial function, then*
(i) $f_{n-1}B(f_{n-1}Bf) = Q^{n-1}(f)(2\pi)^{n-1}$,
(ii) $f_{n-2}B(f_1Bf) = I^{n-1}(f)(2\pi)^{n-1}$,
(iii) $f_{n-1}Bf = Q^{(n-1)/2}(f)(2\pi)^{(n-1)/2}$ *if n is odd, where*

$$Qf(x) = |x|\int_0^{|x|} \overline{f}(t)dt \quad \text{and} \quad If(x) = \int_0^{|x|} \overline{f}(t)dt.$$

PROOF. If $f = f_i$ then the formulas follow directly from Lemma 2.1. Since B, Q and I are linear operators, the formulas are valid for polynomials as well. As these integral operators are continuous with respect to the uniform convergence, the proof can be finished by the Weierstrass theorem.

Let G_p denote the space of C^∞ radial functions at the point $P \in \mathbf{R}^n$. The following theorem gives our inversion formulas for the radial functions $f \in G_0$.

THEOREM 2.3. *The boomerang transform is an injection on G_0 onto G_0. If $f \in G_0$ then*
(i) $B^{-1}f = h_1 = (2\pi)^{1-n}\left(\frac{d}{drr}\right)^{n-1}(f_{n-1}B(f_{n-1}f))$,
(ii) $B^{-1}f = h_2 = (2\pi)^{1-n}\left(\frac{d}{dr}\right)^{n-1}(f_{n-2}B(f_1f))$,
(iii) $B^{-1}f = h_3 = (2\pi)^{(1-n)/2}\left(\frac{d}{drr}\right)^{(n-1)/2}(f_{n-1}f)$ *if n is odd, where $\frac{d}{dr}$ is the radial differentiation.*

PROOF. Suppose that h is a continuous radial function and $Bh = 0$. Then by Corollary 2.2 we get $I^{n-1}(h) = 0$. Using differentiation $(n-1)$-times, we have $h = 0$, i.e. the boomerang transform B is one-to-one.

Since the three cases are very similar we deal only with the second one. $h_2 \in G_0$ follows immediately from

$$(2.4)\qquad |x|^{n-2}B(f_1f)(x) = |S^{n-2}|\int_0^{|x|} \overline{f}(p)p(|x|^2-p^2)^{(n-3)/2}dp.$$

To see $f = Bh_2$, integrate (ii) $(n-1)$-times. Since h_2 is zero in order $n-1$ at the origin, we get

$$f_{n-2}B(f_1 f) = I^{n-1}(h_2)(2\pi)^{n-1}.$$

This implies $f = Bh_2$ by (ii) of Corollary 2.2.

The following statement easily follows from $h_\omega^W(x+y) = h_\omega(\langle x,\omega\rangle + {}$ $+ \langle y,\omega\rangle)$ and from

$$Bh = \frac{1}{2}\int\limits_{S^{n-1}} h_\omega^W \, d\omega. \tag{2.5}$$

LEMMA 2.4. *If $f = Bh$, then*

$$f_y = B\Big(h\Big(x + x\frac{\langle x,y\rangle}{\langle x,x\rangle}\Big)\Big), \tag{2.6}$$

where $f_y(x) = f(x+y)$.

Notice that by this lemma and by Theorem 2.3, inversion formulas can be introduced for the radial functions at an arbitrary point P. Using radial Dirac-sequences and convolution, the procedure leads to the general inversion formulas. We follow this way in our proof. A sequence of functions $\{v_k\}$ is called delta-convergent if it tends to the Dirac distribution in the dual space of continuous bounded functions.

PROPOSITION 2.5. *Let $f \in S(\mathbf{R}^n)$ and let $\{v_k\} \subset G_0$ be delta-convergent. If the sequence*

$$h_k(x) = \int\limits_{\mathbf{R}} Rf(e_x, |x| - r)\overline{B^{-1}v_k}(|r|)dr, \qquad x \in \mathbf{R}^n \setminus 0 \tag{2.7}$$

where $e_x = x/|x|$, has limit function h, then $f = Bh$.

PROOF. By the substitution $r = |x| - s$ and by the Fubini theorem we get

$$h_k(x) = \int\limits_{\mathbf{R}} Rf(e_x, s)B^{-1}v_k(x - se_x)ds = \int\limits_{\mathbf{R}^n} f(y)B^{-1}v_k\Big(x - x\frac{\langle x,y\rangle}{\langle x,x\rangle}\Big)dy. \tag{2.8}$$

From Lemma 2.4 we obtain

$$Bh_k(x) = \int\limits_{\mathbf{R}^n} f(y)v_k(x-y)dy, \qquad x \in \mathbf{R}^n, \tag{2.9}$$

which proves the proposition completely.

3. Proof of the main theorem

We need two technical lemmas. The first statement immediately follows by integrating in polar coordinates.

LEMMA 3.1. *If $\{v_k\} \subset G_0$ is a delta-convergent sequence, then*

$$w_k : \mathbf{R} \to \mathbf{R} \quad \left(r \mapsto |r|^{n-1}\overline{v}_k(|r|)\,|S^{n-1}|\,/2\right) \tag{3.1}$$

is also delta-convergent.

LEMMA 3.2. *If $\gamma \in C^\infty(\mathbf{R})$ and $f(r) = \gamma(r) - \gamma(-r)$, then $\frac{1}{r}\left(\frac{d}{drr}\right)^k f \in$ $\in C^\infty$ and*

$$\lim_{r\to 0} \frac{1}{r}\Big(\frac{d}{drr}\Big)^k f(r) = f^{(2k+1)}(0)\frac{1}{(2k+1)!!}.$$

PROOF. By induction we get

$$\Big(\frac{d}{drr}\Big)^k f(r) = \gamma_k(r) - \gamma_k(-r), \tag{3.2}$$

where $\gamma_k \in C^\infty$ and $\gamma_k(0) = 0$. This proves the first statement. The second assertion follows immediately using the Taylor expansion of γ.

Now we prove our main theorem. The function $Rf(e_x, |x| - r)$ is denoted by $\varphi(r)$. When the x-dependence is important, we denote it by $\varphi_x(r)$.

PROOF OF (1.1). By (ii) of Theorem 2.3 and by Proposition 2.5, we have

$$h_k(x) = (2\pi)^{1-n}\int_0^\infty (\varphi(r)+\varphi(-r))\Big(\frac{d}{dr}\Big)^{n-1}(f_{n-2}B(f_1 v_k))\Big|_{re_x} dr. \tag{3.3}$$

By Theorem 2.3 there exists a function $U_k \in G_0$ such that $v_k = BU_k$ and so

$$I^m U_k = (2\pi)^{1-n}\Big(\frac{d}{dr}\Big)^{n-1-m}(f_{n-2}B(f_1 v_k)). \tag{3.4}$$

Therefore by integration by parts in (3.3) we get

$$h_k(x) = (-2\pi)^{1-n}\int_0^\infty \Big(\frac{d}{dr}\Big)^{n-1}(\varphi(r)+\varphi(-r))r^{n-2}B(f_1 v_k)(re_x)dr, \tag{3.5}$$

where the remainders vanish at 0 by (3.4) and at ∞ by $Rf \in S(S^{n-1} \times \mathbf{R})$ [7]. Use (2.4) and Lemma 3.1 furthermore reverse the order of integrations to see

$$h_k(x) = (-2\pi)^{1-n}\frac{2|S^{n-2}|}{|S^{n-1}|}\int_0^\infty w_k(t)\int_t^\infty g(r)\frac{(r^2-t^2)^{(n-3)/2}}{t^{n-2}}drdt, \tag{3.6}$$

where $g(r) = (\frac{d}{dr})^{n-1}\varphi(r) + (\frac{-d}{dr})^{n-1}\varphi(-r)$ and w_k comes from Lemma 3.1. Making use of the substitution $r = st$ in the inner integral results in

$$(3.7)\quad h(x) = \lim_{k\to\infty} h_k(x) = (-2\pi)^{1-n}\frac{|S^{n-2}|}{|S^{n-1}|}\lim_{t\to 0}\int_t^\infty g(st)(s^2-1)^{(n-3)/2}ds,$$

which completes the proof.

Proof of (1.2). By (i) of Theorem 2.3 and by Proposition 2.5 we have

$$(3.8)\quad h_k(x) = (2\pi)^{1-n}\int_0^\infty (\varphi(r)+\varphi(-r))\Big(\frac{d}{drr}\Big)^{n-1}(f_{n-1}B(f_{n-1}v_k))\Big|_{re_x}dr.$$

As in the previous proof, use integrations by parts to get

$$(3.9)\ h_k(x) = (-2\pi)^{1-n}\int_0^\infty \Big(\frac{d}{rdr}\Big)^{n-1}(\varphi(r)+\varphi(-r))r^{n-1}B(f_{n-1}v_k)(re_x)dr.$$

The function $g(r) = (\frac{d}{rdr})^{n-1}(\varphi(r)+\varphi(-r))r^{n-1}$ is of class C^∞ by Lemma 3.2. Therefore use Lemma 3.1, Lemma 2.1 and the Fubini theorem to get

$$(3.10)\quad h_k(x) = (-2\pi)^{1-n}\frac{2|S^{n-2}|}{|S^{n-1}|}\int_0^\infty w_k(x)\int_s^\infty \frac{g(r)}{r}(1-s^2/r^2)^{(n-3)/2}drds,$$

where w_k comes from Lemma 3.1. Thus we have
(3.11)

$$h(x) = \lim_{k\to\infty} h_k(x) = (-2\pi)^{1-n}\frac{|S^{n-2}|}{|S^{n-1}|}\lim_{s\to 0}\int_s^\infty \frac{g(r)}{r}(1-s^2/r^2)^{(n-3)/2}dr.$$

To obtain the theorem, we have to prove that

$$(3.12)\qquad 0 = \lim_{s\to 0}\int_s^\infty \frac{g(r)}{r}(1-(1-s^2/r^2)^{(n-3)/2})dr.$$

For this purpose break up the integral into two parts as $[s, 2s]$ and $(2s, \infty)$ and transform the first part into an integral on $[1, 2]$ to see that it tends to zero. The other integral on $(2s, \infty)$ converges to zero simply by the Lebesgue dominated convergence theorem. This completes the proof.

It should be mentioned that the odd dimensional inversion formula (1) can be proved easily using (iii) of Theorem 2.3, Proposition 2.5 and Lemma 3.1.

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
H–6720 SZEGED
HUNGARY

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RELATORS GENERATING THE SAME GENERALIZED TOPOLOGY

J. MALA (Nyíregyháza)

Introduction

A nonvoid family $\mathcal{R}$ of reflexive relations on a set X is called a relator on X [8]. If $A \subset X$, then the set

$$\mathrm{int}_{\mathcal{R}}(A) = \{x \in X : \exists R \in \mathcal{R} : R(x) \subset A\}$$

is called the $\mathcal{R}$-interior of A. The members of the family

$$\mathcal{T}_{\mathcal{R}} = \{V \subset X : V \subset \mathrm{int}_{\mathcal{R}}(V)\}$$

are called the $\mathcal{R}$-open sets.

If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{\wedge} = \{S \subset X \times X : \forall x \in X : x \in \mathrm{int}_{\mathcal{R}}(S(x))\}$$

is called the topological refinement of $\mathcal{R}$. Namely, if $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then by [8, Corollary 5.16], we have $\mathrm{int}_{\mathcal{S}} \subset \mathrm{int}_{\mathcal{R}}$ if and only if $\mathcal{S} \subset \mathcal{R}^{\wedge}$ or equivalently $\mathcal{S}^{\wedge} \subset \mathcal{R}^{\wedge}$. Hence it is clear that $\mathcal{R}^{\wedge}$ is the largest relator on X such that $\mathrm{int}_{\mathcal{R}^{\wedge}} \subset \mathrm{int}_{\mathcal{R}}$. Moreover, $\mathrm{int}_{\mathcal{R}^{\wedge}} = \mathrm{int}_{\mathcal{R}}$.

Since the $\mathcal{R}$-interiors of subsets of X need not be $\mathcal{R}$-open, the corresponding assertions do not, in general, hold for $\mathcal{T}_{\mathcal{R}}$ instead of $\mathrm{int}_{\mathcal{R}}$. Therefore, it is of some interest to point out that by introducing the transitive modification

$$\mathcal{R}^{-} = \left\{ \bigcup_{n=1}^{\infty} R^n : R \in \mathcal{R} \right\}$$

of a relator $\mathcal{R}$, we can still prove some similar equivalents of the inclusion $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$.

For this, it is convenient to consider first a singleton relator $\{R\}$ which can often be identified with the relation R without any danger of confusion.

The author is indebted to Árpád Száz who suggested many improvements in the notation and the formulation as well as the proofs of theorems.

1. Topologies generated by singleton relators

Because of the definition of open sets, we clearly have

THEOREM 1.1. *If R is a reflexive relation on X and $V \subset X$, then the following assertions are equivalent:*

(i) $V \in \mathcal{T}_R$;
(ii) $R(V) \subset V$;
(iii) $R(V) = V$.

Using this obvious theorem, we can easily prove

THEOREM 1.2. *If R is a reflexive relation on X, then $\mathcal{T}_R$ is closed under arbitrary unions and intersections.*

PROOF. If $(V_i)_{i\in I}$ is a family in $\mathcal{T}_R$, then by Theorem 1.1, we evidently have

$$R\Big(\bigcup_{i\in I} V_i\Big) = \bigcup_{i\in I} R(V_i) = \bigcup_{i\in I} V_i$$

and

$$R\Big(\bigcap_{i\in I} V_i\Big) \subset \bigcap_{i\in I} R(V_i) = \bigcap_{i\in I} V_i,$$

and hence

$$\bigcup_{i\in I} V_i \in \mathcal{T}_R \qquad \text{and} \qquad \bigcap_{i\in I} V_i \in \mathcal{T}_R.$$

Moreover, simple applications of the definitions of $\mathcal{T}_R$ and $\mathcal{R}^\wedge$ give

THEOREM 1.3. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{T}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}^\wedge} \mathcal{T}_R.$$

PROOF. If $V \in \mathcal{T}_{\mathcal{R}}$, then for each $x \in V$, there exists an $R_x \in \mathcal{R}$ such that $R_x(x) \subset V$. Thus, by defining $R \subset X \times X$ such that $R(x) = R_x(x)$ if $x \in V$ and $R(x) = X$ if $x \in X \setminus V$, we can at once state that $R \in \mathcal{R}^\wedge$ such that $R(V) \subset V$, i.e., $V \in \mathcal{T}_R$.

Conversely, if $V \in \mathcal{T}_R$ for some $R \in \mathcal{R}^\wedge$, then $R(x) \subset V$ for all $x \in V$. Hence, since $x \in \operatorname{int}_{\mathcal{R}}(R(x))$ for all $x \in X$, it is clear that $x \in \operatorname{int}_{\mathcal{R}}(V)$ for all $x \in V$. That is, $V \subset \operatorname{int}_{\mathcal{R}}(V)$, and thus $V \in \mathcal{T}_{\mathcal{R}}$.

2. Singleton relators generated by topologies

Now, as a certain converse to Theorem 1.2, we can also prove

THEOREM 2.1. *If $\mathcal{A}$ is a family of subsets of X and $R \subset X \times X$ such that*

$$R(x) = \bigcap\{A \in \mathcal{A} : x \in A\}$$

for all $x \in X$, then

(i) *R is the largest reflexive relation on X such that $\mathcal{A} \subset \mathcal{T}_R$;*
and
(ii) *$\mathcal{T}_R = \mathcal{A}$ if and only if $\mathcal{A}$ is closed under arbitrary unions and intersections.*

PROOF. If $A \in \mathcal{A}$ and $x \in A$, then by the definition of R it is clear that $R(x) \subset A$. Therefore, $A \in \mathcal{T}_R$. On the other hand, if S is a reflexive relation on X such that $\mathcal{A} \subset \mathcal{T}_S$ and $x \in X$, then $x \in A \in \mathcal{A}$ implies that $S(x) \subset A$. Therefore, $S(x) \subset R(x)$.

From Theorem 1.2 we know that if $\mathcal{A} = \mathcal{T}_R$, then $\mathcal{A}$ is closed under arbitrary unions and intersections. On the other hand, if $V \in \mathcal{T}_R$, then by Theorem 1.1 and the definition of R, it is clear that

$$V = \bigcup_{x \in V} \bigcap \{A \in \mathcal{A} : x \in A\}.$$

Therefore, if $\mathcal{A}$ is closed under arbitrary unions and intersections, then $V \in \in \mathcal{A}$ also holds.

REMARK 2.2. The relation R defined in Theorem 2.1 is always transitive. Namely, if $y \in R(x)$ then $x \in A \in \mathcal{A}$ implies $y \in A$. Therefore, we also have $R(y) \subset R(x)$.

A simple application of Theorem 2.1 and Remark 2.2 gives

THEOREM 2.3. *If R is a reflexive relation on X, then*

$$R^- = \bigcup_{n=1}^{\infty} R^n$$

is the largest reflexive relation on X such that $\mathcal{T}_R \subset \mathcal{T}_{R^-}$. Moreover, $\mathcal{T}_R = = \mathcal{T}_{R^-}$.

PROOF. Because of Theorems 2.1 and 1.2, we need only show that if $S \subset X \times X$ such that

$$S(x) = \bigcap \{V \in \mathcal{T}_R : x \in V\},$$

for all $x \in X$, then $S = R^-$. From Theorem 2.1 and Remark 2.2 we know that $R \subset S$ and $S^2 \subset S$. Hence by the definition of R^- it is clear that $R^- \subset S$. Moreover, if $x \in X$, then also by the definition of R^- it is clear that

$$R(R^-(x)) = \bigcup_{n=1}^{\infty} R^{n+1}(x) \subset R^-(x).$$

Thus, we have not only $x \in R^-(x)$, but also $R^-(x) \in \mathcal{T}_R$. Whence, by the definition of S, it is evident that $S(x) \subset R^-(x)$ also holds.

REMARK 2.4. Note that R^- is the smallest transitive relation such that $R \subset R^-$.

3. Relators generating the same generalized topology

Now, combining Theorems 1.3 and 2.3 and using the obvious notation

$$\mathcal{R}^- = \{R^- : R \in \mathcal{R}\},$$

we can easily establish the following improvement of [8, Corollary 5.19].

THEOREM 3.1. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(i) $\mathcal{T}_\mathcal{S} \subset \mathcal{T}_\mathcal{R}$;
(ii) $\mathcal{S}^{\wedge -} \subset \mathcal{R}^\wedge$;
(iii) $\mathcal{S}^{\wedge - \wedge} \subset \mathcal{R}^\wedge$;
(iv) $\mathcal{S}^{\wedge -} \subset \mathcal{R}^{\wedge -}$.

PROOF. If (ii) holds, then by Theorems 1.3 and 2.3 it is clear that

$$\mathcal{T}_\mathcal{S} = \bigcup_{S \in \mathcal{S}^\wedge} \mathcal{T}_S = \bigcup_{S \in \mathcal{S}^\wedge} \mathcal{T}_{S^-} = \bigcup_{T \in \mathcal{S}^{\wedge -}} \mathcal{T}_T \subset \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_R = \mathcal{T}_\mathcal{R}.$$

That is, (i) also holds.

On the other hand, if $T \in \mathcal{S}^{\wedge -}$, then there exists an $S \in \mathcal{S}^\wedge$ such that $T = S^-$. Hence, by Remark 2.4 and Theorems 1.1, 2.3 and 1.3, it is clear that

$$T(x) \in \mathcal{T}_T = \mathcal{T}_S \subset \mathcal{T}_\mathcal{S}$$

for all $x \in X$. Therefore, if (i) holds then we also have $T(x) \in \mathcal{T}_\mathcal{R}$ for all $x \in X$, which implies $T \in \mathcal{R}^\wedge$. Consequently, if (i) holds then (ii) also holds.

Finally, the equivalence of (iii) and (iv) to (ii) is an immediate consequence of the fact that the mappings

$$\mathcal{R} \to \mathcal{R}^\wedge \quad \text{and} \quad \mathcal{R} \to \mathcal{R}^-$$

are increasing idempotent operations on relators such that $\mathcal{R} \subset \mathcal{R}^\wedge$ and $\mathcal{R}^- \subset \mathcal{R}^\wedge$.

Now, as an immediate consequence of Theorem 3.1, we can also state

COROLLARY 3.2. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X then $\mathcal{T}_\mathcal{R} = \mathcal{T}_\mathcal{S}$ if and only if $\mathcal{R}^{\wedge -} = \mathcal{S}^{\wedge -}$.*

Hence by noticing that

$$\mathcal{T}_{X \times X} = \{\emptyset, X\} \quad \text{and} \quad \mathcal{T}_{\Delta X} = \mathcal{P}(X),$$

we can also easily get

THEOREM 3.3. *If $\mathcal{R}$ is a relator on X, then*

(i) $\mathcal{T}_\mathcal{R} = \{\emptyset, X\}$ *if and only if* $\mathcal{R}^{\wedge -} = \{X \times X\}$;
(ii) $\mathcal{T}_\mathcal{R} = \mathcal{P}(X)$ *if and only if* $\mathcal{R}^{\wedge -} = \{\Delta_X\}^{\wedge -}$.

REMARK 3.4. The first statement of this latter theorem is identical to [4, Theorem 3.5].

4. A new characterization of topological relators

To show that [8, Corollary 5.19] can also be derived from Theorem 3.1, we have only to prove the next striking analogue of [9, Theorems 2.10 and 3.4].

THEOREM 4.1. *If $\mathcal{R}$ is a relator on X then the following assertions are equivalent:*

(i) *$\mathcal{R}$ is topological;*

(ii) *$\mathcal{R}^{\wedge -}$ is topologically equivalent to $\mathcal{R}$.*

PROOF. By Remark 2.4 and [9, Theorem 3.3], it is clear that $\mathcal{R}^{\wedge -}$ is a topological relator on X. Therefore the implication (ii) $\Rightarrow$ (i) is an immediate consequence of [9, Corollary 2.4].

On the other hand, by Theorem 3.1, it is clear that $\mathcal{R}^{\wedge - \wedge} \subset \mathcal{R}^{\wedge}$ is always true. Therefore, to prove the converse implication (i) $\Rightarrow$ (ii), we need only show that if (i) is true then $\mathcal{R}^{\wedge} \subset \mathcal{R}^{\wedge - \wedge}$, i.e., $\mathcal{R} \subset \mathcal{R}^{\wedge - \wedge}$ is also true.

For this, note that if (i) holds, and $R \in \mathcal{R}$ and $x \in X$, then by [9, Theorem 2.3] and Theorem 3.1 we necessarily have

$$\operatorname{int}_{\mathcal{R}}(R(x)) \in \mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{R}^{\wedge -}}.$$

Therefore we also have

$$x \in \operatorname{int}_{\mathcal{R}^{\wedge -}}(\operatorname{int}_{\mathcal{R}}(R(x))) \subset \operatorname{int}_{\mathcal{R}^{\wedge -}}(R(x)),$$

which shows that $R \in \mathcal{R}^{\wedge - \wedge}$.

Now, as an immediate consequence of Theorems 3.1 and 4.1 we can also state

THEOREM 4.2. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{S}$ is topological, then the following assertions are equivalent:*

(i) $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$;

(ii) $\mathcal{S}^{\wedge} \subset \mathcal{R}^{\wedge}$;

(iii) $\mathcal{S} \subset \mathcal{R}^{\wedge}$.

Hence, by Theorems 1.3 and 4.1, it is clear that we also have

COROLLARY 4.3. *If $\mathcal{R}$ is a topological relator on X, then $\mathcal{R}^{\wedge}$ is the largest topological relator on X such that $\mathcal{T}_{\mathcal{R}^{\wedge}} \subset \mathcal{T}_{\mathcal{R}}$. Moreover $\mathcal{T}_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$.*

Combining Theorems 3.3 and 4.1, now we can also state

THEOREM 4.4. *If $\mathcal{R}$ is a topological relator on X, then*

(i) *$\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$ if and only if $\mathcal{R} = \{X \times X\}$;*

(ii) *$\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X)$ if and only if $\mathcal{R}^{\wedge} = \{\Delta_X\}^{\wedge}$.*

5. A few useful counterexamples

The fact that the conditions of topologicalness cannot be omitted from the above assertions can also be at once seen from the next simple

EXAMPLE 5.1. If $X = \{1, 2, 3\}$ and $R \subset X \times X$ such that

$$R(1) = \{1, 2\}, \quad R(2) = \{2, 3\}, \quad R(3) = \{1, 3\},$$

then $\mathcal{R} = \{R\}$ is a relator on X such that

$$\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\} \quad \text{and} \quad \mathcal{T}_{\mathcal{R}^{-1}} = \{\emptyset, X\}$$

but $\mathcal{R}^{\wedge}$ and $(\mathcal{R}^{-1})^{\wedge}$ are still incomparable.

REMARK 5.2. Since

$$R^{-1}(1) = \{1, 3\}; \quad R^{-1}(2) = \{1, 2\}; \quad R^{-1}(3) = \{2, 3\}$$

now we also have

$$\mathcal{R}^{\wedge} \cap (\mathcal{R}^{-1})^{\wedge} = \{X \times X\}.$$

Moreover, in addition to the above example, now we can also easily prove

EXAMPLE 5.3. If X is a set with $\operatorname{card}(X) \geqq 3$, then there is no largest relator $\mathcal{R}$ on X such that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$.

PROOF. If $B \subset X$, then for each proper subset A of B

$$R_A = A \times B \cup (X \setminus A) \times X$$

is a reflexive relation on X such that $\mathcal{T}_{R_A} = \{\emptyset, X\}$. Therefore, if $\mathcal{R}$ is the largest relator on X such that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, then we necessarily have $R_A \in \mathcal{R}$ for all proper subset A of B. Hence, it is clear that $B \in \mathcal{T}_{\mathcal{R}}$ if $\operatorname{card}(B) \geqq 2$, which is a contradiction if $B \neq X$. Consequently, the assertion of the example is true.

REMARK 5.4. Clearly if $\operatorname{card}(X) \leqq 2$ then $\mathcal{R} = \{X \times X\}$ is the only relator on X such that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$.

The above proof also yields a remarkable example to Theorem 1.3 and [4, Theorems 3.1 and 3.5].

EXAMPLE 5.5. If X is a set with $\operatorname{card}(X) \geqq 3$, then the family $\mathcal{R}$ of all relations

$$R_{(A,B)} = A \times B \cup (X \setminus A) \times X$$

with $A \subset B \subset X$, $A \neq B$ and $\operatorname{card}(B) \geqq 2$ is a relator on X such that $\mathcal{T}_R = \{\emptyset, X\}$ for all $R \in \mathcal{R}$, but $\mathcal{T}_{\mathcal{R}} = \{V \subset X : \operatorname{card}(V) \neq 1\}$.

REMARK 5.6. Note that in contrast to the transitivity of the Davis–Pervin relations

$$R_A = R_{(A,A)} = A \times A \cup (X \setminus A) \times X,$$

where $A \subset X$, now we have

$$(R_{(A,B)})^{-} = (R_{(A,B)})^2 = X \times X,$$

whenever $A \subset B \subset X$ such that $A \neq B$.

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INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF AGRICULTURAL SCIENCES
H–2103 GÖDÖLLŐ
PÁTER KÁROLY U. 1
HUNGARY

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ON A MULTIPLICATIVE PROPERTY OF SEQUENCES OF INTEGERS[1]

I. Z. RUZSA and A. SÁRKÖZY (Budapest)

1. Throughout this paper, we use the following notations: $\mathcal{A}, \mathcal{A}_1, \mathcal{B}, \mathcal{M}, \ldots$ denote (finite or infinite) sequences of positive integers. $c_1, c_2, \ldots$ denote positive constants. We write $\log\log x = l_2(x)$, $\log\log\log x = l_3(x)$. The least common multiple of the integers a, b is denoted by $[a,b]$. p_i denotes the i^{th} prime number (so that $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$). $\omega(n)$ denotes the number of distinct prime factors of n: $\omega(n) = \sum_{p|n} 1$ while $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity: $\Omega(n) = \sum_{p^\alpha \| n} \alpha$, and we write $\mathcal{D}(x,u) = \{n : n \leqq x, \Omega(n) \geqq u\}$. $\mu(n)$ denotes the Möbius function.

2. Erdős and Graham [1, p. 88] raised the following question: "Is it true that if $a_1 < a_2 < \ldots$ is a sequence of integers satisfying

$$\frac{1}{l_2(x)} \sum_{a_i < x} \frac{1}{a_i} \to +\infty \tag{1}$$

then

$$\left(\sum_{a_i < x} \frac{1}{a_i} \right)^{-2} \sum_{1 < a_i < a_j \leqq x} \frac{1}{[a_i, a_j]} \to +\infty \quad ?\text{"} \tag{2}$$

(By $(a_i, a_j) = \frac{a_i a_j}{[a_i, a_j]}$, (2) says that $(a_i, a_j) \to +\infty$ holds "on average".) This paper is devoted to the study of this problem. However, it will be shown that to ensure (2), $\sum_{a_i < x} \frac{1}{a_i}$ must grow much faster: one must have

$$\sum_{a_i < x} \frac{1}{a_i} > \exp\left(f(x) \left(l_2(x)\right)^{1/2} l_3(x) \right)$$

where $f(x) \to +\infty$ and this is the best possible.

It is well-known that there is a constant c_1 such that

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$$\sum_{p \leqq x} \frac{1}{p} < l_2(x) + c_1 \quad \text{for} \quad x \geqq 3. \tag{3}$$

THEOREM 1. *For all $\varepsilon > 0$, there are constants $c_2, c_3, c_4, x_0, \delta$ (all depending on ε) such that if $x > x_0$, $\mathcal{A} \subset \{1, 2, \dots, [x]\}$ and $\sum_{a \in \mathcal{A}} \frac{1}{a} > c_2$, then, defining the unique integer k by*

$$\frac{(l_2(x) + c_1)^k}{(k-1)!} \leqq \frac{1}{8} \sum_{a \in \mathcal{A}} \frac{1}{a} < \frac{(l_2(x) + c_1)^{k+1}}{k!} \tag{4}$$

(where c_1 is defined by (3)), we have

$$\sum_{a, a' \in \mathcal{A}} \frac{1}{[a, a']} > c_3 L(x, k) \left(\sum_{a \in \mathcal{A}} \frac{1}{a} \right)^2 \tag{5}$$

where

$$L(x, k) = \begin{cases} \frac{k \log x}{l_2(x)} \frac{(k + [l_2(x)])!}{(l_2(x) + c_4)^{k + [l_2(x)]}} & \text{for } k \leqq (1 - \delta) l_2(x) \\ (\log x)^{\log 4 - 1 - \varepsilon} & \text{for } k > (1 - \delta) l_2(x). \end{cases}$$

An easy computation shows that

$$\left(\log \sum_{a \in \mathcal{A}} \frac{1}{a} \right) \left((l_2(x))^{1/2} l_3(x) \right)^{-1} \to +\infty$$

implies $k (l_2(x))^{-\frac{1}{2}} \to +\infty$, whence $L(x, k) \to +\infty$, and for all $\varepsilon > 0$ there is an $\eta (> 0)$ such that $c_3 L(x, k) > (\log x)^{\log 4 - 1 - \varepsilon}$ for $x > x_0(\varepsilon)$, $\sum_{a \in \mathcal{A}} \frac{1}{a} > (\log x)^{1 - \eta}$. Thus we obtain

COROLLARY 1. *For every $L > 0$ there are numbers K, x_0 such that $x > x_0$ and*

$$\sum_{a \in \mathcal{A}} \frac{1}{a} > \exp \left(K (l_2(x))^{1/2} l_3(x) \right)$$

imply

$$\sum_{a, a' \in \mathcal{A}} \frac{1}{[a, a']} > L \left(\sum_{a \in \mathcal{A}} \frac{1}{a} \right)^2.$$

COROLLARY 2. *For every $\varepsilon > 0$ there are numbers $x_0, \eta (> 0)$ such that $x > x_0$ and*

$$\sum_{a\in\mathcal{A}} \frac{1}{a} > (\log x)^{1-\eta}$$

imply

$$\sum_{a,a'\in\mathcal{A}} \frac{1}{[a,a']} > (\log x)^{\log 4-1-\varepsilon}\left(\sum_{a\in\mathcal{A}}\frac{1}{a}\right)^2 .$$

As the following theorem shows, Corollary 1 is the best possible.

THEOREM 2. *For every $K>0$ there are numbers x_0, H and an infinite set $\mathcal{A}$ of integers such that for $x>x_0$ we have*

$$\sum_{a\in\mathcal{A},\, a\leqq x} \frac{1}{a} > \exp\left(K\,(l_2(x))^{1/2}\, l_3(x)\right)$$

and

(6)
$$\sum_{a,a'\in\mathcal{A},\, a,a'\leqq x} \frac{1}{[a,a']} < H\left(\sum_{a\in\mathcal{A},\, a\leqq x}\frac{1}{a}\right)^2 .$$

3. To prove Theorem 1, we need three lemmas.

LEMMA 1. *If $x\geqq 3$ and t is a positive integer with $t\leqq l_2(x)$, then*

$$\sum_{n\leqq x,\Omega(n)\leqq t}\frac{1}{n} < 4\frac{(l_2(x)+c_1)^t}{(t-1)!} .$$

PROOF. If $n\leqq x$ and $\Omega(n)\leqq t$, then, writing n in the form $n=u^2v$ where $|\mu(v)|=1$, we have $v\leqq n\leqq x$ and $\omega(v)=\Omega(v)\leqq\Omega(n)\leqq t$. Thus by (3) we have

$$\sum_{n\leqq x,\Omega(n)\leqq t}\frac{1}{n} \leqq \sum_{u=1}^{+\infty}\sum_{i=0}^{t}\sum_{\substack{v\leqq x,|\mu(v)|=1\\ \Omega(v)=i}}\frac{1}{u^2v} = \left(\sum_{u=1}^{+\infty}\frac{1}{u^2}\right)\left(\sum_{i=0}^{t}\sum_{\substack{v\leqq x,|\mu(v)|=1\\ \Omega(v)=i}}\frac{1}{v}\right) <$$

$$< 2\sum_{i=0}^{t}\frac{1}{i!}\left(\sum_{p\leqq x}\frac{1}{p}\right)^i < 2\sum_{i=0}^{t}\frac{(l_2(x)+c_1)^i}{i!} < 2(t+1)\frac{(l_2(x)+c_1)^t}{t!} \leqq$$

$$\leqq 4\frac{(l_2(x)+c_1)^t}{(t-1)!} .$$

LEMMA 2. *There is a constant $c_6(>0)$ such that if $\varepsilon>0$ and $x>x_0(\varepsilon)$, then for every integer u with $l_2(x)+c_6<u<(2-\varepsilon)l_2(x)$ we have*

$$|\mathcal{D}(x,u)| < c_7 \frac{x}{\log x} \frac{(l_2(x))^{u-1}}{(u-1)!} \frac{u}{u - l_2(x)}$$

where $c_7 = c_7(\varepsilon)$ depends on ε (but it is independent of x and u).

PROOF. This is a slightly modified form of the first half of Corollary 1 in [4] and, in fact, it follows from this corollary and a result of Sathe [5] and Selberg [6].

LEMMA 3. *For $x > x_0$ we have*

$$|\mathcal{D}(x, 1 + [l_2(x)])| = |\{n : n \leqq x, \Omega(n) > l_2(x)\}| > \frac{1}{3}x .$$

PROOF. This follows from a theorem of Erdős and Kac [2], [3].

4. COMPLETION OF THE PROOF OF THEOREM 1. Let $\mathcal{A}_1$ denote the set of integers a with $a \in \mathcal{A}$, $\Omega(a) > k$. Then by (4) and Lemma 1 we have

$$\sum_{a\in\mathcal{A}_1} \frac{1}{a} = \sum_{a\in\mathcal{A}} \frac{1}{a} - \sum_{a\in\mathcal{A},\Omega(a)\leqq k} \frac{1}{a} > \tag{9}$$

$$> \sum_{a\in\mathcal{A}} \frac{1}{a} - 4\frac{(l_2(x)+c_1)^k}{(k-1)!} \geqq \sum_{a\in\mathcal{A}} \frac{1}{a} - \frac{1}{2}\sum_{a\in\mathcal{A}} \frac{1}{a} = \frac{1}{2}\sum_{a\in\mathcal{A}} \frac{1}{a} .$$

For $n = 1, 2, \ldots$, let $g(n)$ denote the number of pairs a, m with $a \in \mathcal{A}_1$, $\Omega(m) > [l_2(x)]$ and $am = n$, and write $\mathcal{M} = \{n : n \leqq x^2, g(n) > 0\}$. If $n \in \mathcal{M}$, then we have

$$\Omega(n) = \Omega(am) = \Omega(a) + \Omega(m) > k + [l_2(x)]$$

so that, by Lemma 2,

$$|\mathcal{M}| \leqq |\mathcal{D}(x^2, k + [l_2(x)] + 1)| < \tag{10}$$

$$< c_7 \frac{x^2}{\log x^2} \frac{(l_2(x^2))^{k+[l_2(x)]}}{(k+[l_2(x)])!} \frac{k + [l_2(x)] + 1}{k + [l_2(x)] + 1 - l_2(x^2)} <$$

$$< c_8 \frac{x^2}{\log x} \frac{(l_2(x)+c_9)^{k+[l_2(x)]}}{(k+[l_2(x)])!} \frac{l_2(x)}{k} \quad \text{for} \quad k \leqq (1-\delta) l_2(x)$$

and

$$|\mathcal{M}| \leqq |\mathcal{D}(x^2, k + [l_2(x)] + 1)| \leqq |\mathcal{D}(x^2, [(2-\delta)l_2(x)])| < \tag{11}$$

$$< c_7 \frac{x^2}{\log x^2} \frac{(l_2(x^2))^{[(2-\delta)l_2(x)]-1}}{([(2-\delta)l_2(x)]-1)!} \frac{[(2-\delta)l_2(x)]}{[(2-\delta)l_2(x)] - l_2(x^2)} <$$

$$< x^2(\log x)^{1-\log 4+\varepsilon/2} \quad \text{for} \quad k > (1-\delta)l_2(x)$$

if δ is small enough in terms of ε.

By the definition of the function $g(n)$, clearly we have

$$\sum_{n=1}^{x^2} g^2(n) \leqq \sum_{n=1}^{x^2} \left(\sum_{a\in\mathcal{A}, a|n} 1 \right)^2 = \tag{12}$$

$$= \sum_{n=1}^{x^2} \left(\sum_{a\in\mathcal{A}, a|n} \sum_{a'\in\mathcal{A}, a'|n} 1 \right) = \sum_{a\in\mathcal{A}} \sum_{a'\in\mathcal{A}} \sum_{n\leqq x^2, [a,a']|n} 1 =$$

$$= \sum_{a\in\mathcal{A}} \sum_{a'\in\mathcal{A}} \left[\frac{x^2}{[a,a']} \right] \leqq x^2 \sum_{a\in\mathcal{A}} \sum_{a'\in\mathcal{A}} \frac{1}{[a,a']}.$$

On the other hand, by Cauchy's inequality and Lemma 3, and in view of (9), we have

(13)

$$\sum_{n=1}^{x^2} g^2(n) \geqq \frac{1}{|\mathcal{M}|} \left(\sum_{n\in\mathcal{M}} g(n) \right)^2 = \frac{1}{|\mathcal{M}|} \left(\sum_{a\in\mathcal{A}_1} \sum_{m\leqq x^2/a, \Omega(m)>[l_2(x)]} 1 \right)^2 \geqq$$

$$\geqq \frac{1}{|\mathcal{M}|} \left(\sum_{a\in\mathcal{A}_1} \sum_{m\leqq x^2/a, \Omega(m)>[l_2(x^2/a)]} 1 \right)^2 >$$

$$> \frac{1}{|\mathcal{M}|} \left(\sum_{a\in\mathcal{A}_1} \frac{1}{3}\frac{x^2}{a} \right)^2 = \frac{x^4}{9|\mathcal{M}|} \left(\sum_{a\in\mathcal{A}_1} \frac{1}{a} \right)^2 > \frac{x^4}{36|\mathcal{M}|} \left(\sum_{a\in\mathcal{A}} \frac{1}{a} \right)^2.$$

(5) follows from (10), (11), (12) and (13), and this completes the proof of Theorem 1.

5. For a finite set $\mathcal{B}$ of integers we introduce the notations

$$|\mathcal{B}| = \sum_{b\in\mathcal{B}} 1/b \quad \text{and} \quad \|\mathcal{B}\| = \left(\sum_{b,b'\in\mathcal{B}} \frac{1}{[b,b']} \right)^{1/2}.$$

We establish some properties of these "norms" that will be used to prove Theorem 2.

LEMMA 4. *If* $\mathcal{A} \cap \mathcal{B} = \emptyset$ *, then*

$$|\mathcal{A} \cup \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}| \tag{14}$$

and

$$\|\mathcal{A} \cup \mathcal{B}\| \leqq \|\mathcal{A}\| + \|\mathcal{B}\|. \tag{15}$$

PROOF. (14) is obvious. To prove (15), let D be the least common multiple of all elements of $\mathcal{A} \cup \mathcal{B}$. For any set $\mathcal{M}$ of divisors of D and $1 \leqq i \leqq D$ let

$$x_i = \sum_{d \in \mathcal{M}, d|i} 1.$$

This defines a vector $(x_i) = \overline{x}_{\mathcal{M}}$. We easily find (now $|\overline{x}|$ denotes the usual Euclidean norm) that

$$|\overline{x}_{\mathcal{M}}| = \sum_i x_i^2 = \sum_{d,d' \in \mathcal{M}} \sum_{i \leqq D, d, d'|i} 1 = \sum_{d,d' \in \mathcal{M}} \frac{D}{[d,d']} = D\|\mathcal{M}\|^2.$$

Hence (15) follows from the triangle inequality.

We define the product of two sets by

$$\mathcal{AB} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}. \tag{16}$$

We call the sets $\mathcal{A}$ and $\mathcal{B}$ *coprime*, if $(a,b) = 1$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

LEMMA 5. *If $\mathcal{A}$ and $\mathcal{B}$ are coprime sets, then we have*

$$|\mathcal{AB}| = |\mathcal{A}||\mathcal{B}|, \tag{17}$$

$$\|\mathcal{AB}\| = \|\mathcal{A}\|\|\mathcal{B}\|. \tag{18}$$

PROOF. To show (17), it is enough to observe that if these sets are coprime, then the products ab, $a \in \mathcal{A}$, $b \in \mathcal{B}$ are all distinct. To prove (18) we add that $[ab, a'b'] = [a,a'][b,b']$ if $(a,b) = (a,b') = (a',b) = (a',b') = 1$.

6. PROOF OF THEOREM 2. Select a $c > 0$ (we shall specify it in terms of K, and it will determine H). We define a sequence $(x_0, x_1, \dots)$ of integers by recursion. If x_{k-1} is given, let x_k be the smallest integer such that

$$\alpha_k = \sum_{x_{k-1} < p \leqq x_k} 1/p > ck. \tag{19}$$

These numbers x_k are obviously primes and by the minimality we have

$$ck < \alpha_k \leqq ck + 1/x_k. \tag{20}$$

(19) implies that

$$x_k = x_{k-1}^{(\exp ck)+o(1)} \tag{21}$$

and that

$$\sum_{p \leqq x_k} 1/p = \alpha_1 + \ldots + \alpha_k = ck(k+1)/2 + O(1),$$

hence

$$x_k = e^{\exp(ck(k+1)/2+O(1))}. \tag{22}$$

From (21) we also infer that

$$x_1 x_2 \ldots x_{k-1} < x_k \tag{23}$$

for $k > k_0$.

Let $\mathcal{P}_k$ be the set of primes in $(x_{k-1}, x_k]$ and put

$$\mathcal{A}_k = \mathcal{P}_k \mathcal{P}_{k+1} \ldots \mathcal{P}_{2k},$$

(product in the sense of (16)),

$$\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i.$$

This will be the set $\mathcal{A}$ of the theorem.

Take an integer x and put $\mathcal{B} = \mathcal{A} \cap [1, x]$. Our aim is to estimate $|\mathcal{B}|$ from below and $\|\mathcal{B}\|/|\mathcal{B}|$ from above.

The maximal element of $\mathcal{A}_j$ is

$$y_j = x_j x_{j+1} \ldots x_{2j}.$$

Now define k by $y_{k-1} \leqq x < y_k$. This means that $\mathcal{A}_1, \ldots, \mathcal{A}_{k-1}$ are completely contained in $\mathcal{B}$, $\mathcal{A}_k$ may be partially contained or disjoint. Since the minimal element of $\mathcal{A}_{k+i}$ is larger than

$$x_{k+i-1} \ldots x_{2k+2i-1},$$

the sets $\mathcal{A}_{k+1}, \ldots$ are all disjoint to $\mathcal{B}$. Write

$$\mathcal{B} = \bigcup_{i=1}^{k-1} \mathcal{A}_i \cup \mathcal{C},$$

where $\mathcal{C} = \mathcal{A}_k \cap [1, x]$.

First we estimate $|\mathcal{B}|$. Since

$$x < y_k = x_k \ldots x_{2k} < x_{2k+1},$$

by (22) we infer

$$l_2(x) < ck^2/2 + O(k),$$

that is,

$$k \geqq (2l_2(x)/c)^{1/2} - O(1).$$

Consequently we have

$$|\mathcal{B}| \geqq |\mathcal{A}_{k-1}| = \alpha_{k-1}...\alpha_{2k-2} \geqq (k-1)...(2k-2) \geqq$$
$$\geqq (k-1)^k \gg \exp\left(K(l_2(x))^{1/2} l_3(x)\right)$$

as wanted, if $c < 1/K$.

Now we begin to estimate $\|\mathcal{B}\|/|\mathcal{B}|$.

If $\mathcal{P}$ is any set of primes with

$$|\mathcal{P}| = \sum_{p\in\mathcal{P}} 1/p = \alpha,$$

then we have

$$\|\mathcal{P}\|^2 = \sum_{p,q\in\mathcal{P}} \frac{1}{[p,q]} = \alpha^2 + \sum_{p\in\mathcal{P}}\left(\frac{1}{p} - \frac{1}{p^2}\right) \leqq \alpha^2 + \alpha,$$

thus

$$\frac{\|\mathcal{P}\|}{|\mathcal{P}|} \leqq \sqrt{1 + \frac{1}{\alpha}}.$$

Consequently by Lemma 5 we have

$$\frac{\|\mathcal{A}_j\|}{|\mathcal{A}_j|} \leqq \prod_{i=j}^{2j} \frac{\|\mathcal{P}_i\|}{|\mathcal{P}_i|} \leqq \prod_{i=j}^{2j}\left(1 + \frac{1}{\alpha_i}\right)^{1/2} \leqq \prod_{i=j}^{2j}\left(1 + \frac{1}{ci}\right)^{1/2} \leqq \tag{24}$$
$$\leqq \left(1 + \frac{1}{cj}\right)^{(j+1)/2} \leqq \exp\frac{j+1}{2cj} \leq H'$$

with $H' = \exp 1/c$.

If $\mathcal{C} = \emptyset$, then by (24) Lemma 4 implies

$$\|\mathcal{B}\| \leqq \|\mathcal{A}_1\| + ... + \|\mathcal{A}_{k-1}\| \leqq H'(|\mathcal{A}_1| + ... + |\mathcal{A}_{k-1}|) = H'|\mathcal{B}|,$$

thus the conclusion of the theorem holds with $H = H'$. If $\mathcal{C} \neq \emptyset$, we need to estimate also $\|\mathcal{C}\|$.

Put

$$\mathcal{U} = \mathcal{P}_k...\mathcal{P}_{2k-1}.$$

The elements of $\mathcal{C}$ are of the form up, $u \in \mathcal{U}$, $p \in \mathcal{P}_{2k}$, $p \leqq x/u$. Write

$$\gamma_u = \sum_{p \in \mathcal{P}_{2k}, p \leqq x/u} 1/p = \sum_{x_{2k-1} < p \leq \min(x_{2k}, x/u)} 1/p.$$

Let $\gamma = \min_{u \in \mathcal{U}} \gamma_u$, $\Gamma = \max_{u \in \mathcal{U}} \gamma_u$. With this notation we have

$$|\mathcal{C}| = \sum_{u \in \mathcal{U}} \gamma_u/u \geqq \gamma|\mathcal{U}|. \tag{25}$$

We have also

$$\|\mathcal{C}\|^2 = \sum_{up, u'p' \in \mathcal{C}} \frac{1}{[up, u'p']} \leqq \sum_{u, u' \in \mathcal{U}} \frac{1}{[u, u']}(\gamma_u \gamma_{u'} + \gamma_u),$$

where the first summand corresponds to the terms with $p \neq p'$ and the second to those with $p = p'$. Consequently we have

$$\|\mathcal{C}\|^2 \leqq (\Gamma^2 + \Gamma)\|\mathcal{U}\|^2. \tag{26}$$

To make use of these estimates we need to show that Γ is not much larger than γ.

Let $\underline{u}$ and $\overline{u}$ denote the minimal and maximal elements of $\mathcal{U}$. We have

$$\overline{u} = x_k \dots x_{2k-1} < x_{2k-1}^2, \qquad \underline{u} > x_{k-1} \dots x_{2k-2},$$

hence

$$\overline{u}/\underline{u} < x_{2k-1}/x_{k-1} < x_{2k-1}.$$

If $x \geqq \overline{u} x_{2k-1}$, then we have $x/\overline{u} \geqq x_{2k-1} > \overline{u}/\underline{u}$, hence $x/\underline{u} = (x/\overline{u})(\overline{u}/\underline{u}) \leqq$ $\leqq (x/\overline{u})^2$, thus

$$\Gamma - \gamma \leqq \sum_{x/\overline{u} \leqq p \leqq x/\underline{u}} 1/p < 1.$$

If $x < \overline{u} x_{2k-1}$, then $x/\underline{u} < x_{2k-1}(\overline{u}/\underline{u}) < x_{2k-1}^2$, hence

$$\Gamma = \sum_{x_{2k-1} < p \leqq x/\underline{u}} 1/p < \sum_{x_{2k-1} < p \leqq x_{2k-1}^2} 1/p < 1 \leqq \gamma + 1$$

again.

By view of $\Gamma \leqq \gamma + 1$, we have $\Gamma^2 + \Gamma < (\gamma + 2)^2$ and thus (26) yields

$$\|\mathcal{C}\| \leqq (\gamma + 2)\|\mathcal{U}\|. \tag{27}$$

We have $|\mathcal{U}| = \alpha_k \dots \alpha_{2k-1}$, hence

$$|\mathcal{U}|/|\mathcal{A}_{k-1}| = \alpha_{2k-1}/\alpha_{k-1} \sim (2k-1)/(k-1) \to 2,$$

thus $|\mathcal{U}| \leqq 3|\mathcal{A}_{k-1}|$ for large k. Moreover, similarly to (24) we can deduce

$$\|\mathcal{U}\|/|\mathcal{U}| \leqq H'.$$

Substituting these estimates into (27), by (25) we obtain

$$\|\mathcal{C}\| \leqq (\gamma+2)\|\mathcal{U}\| \leqq H'(\gamma+2)|\mathcal{U}| \leqq H'|\mathcal{C}| + 6H'|\mathcal{A}_{k-1}|.$$

By Lemma 4 this implies

$$\|\mathcal{B}\| \leqq \|\mathcal{A}_1\| + \ldots + \|\mathcal{A}_{k-1}\| + \|\mathcal{C}\| \leqq H'|\mathcal{A}_1| + \ldots + H'|\mathcal{A}_{k-1}| + \\ + H'|\mathcal{C}| + 6H'|\mathcal{A}_{k-1}| \leqq 7H'(|\mathcal{A}_1| + \ldots + |\mathcal{A}_{k-1}| + |\mathcal{C}|) = 7H'|\mathcal{B}|,$$

thus the theorem is proved with $H = 7H'$.

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MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, P.O.BOX 127
H–1364 HUNGARY

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EIGENVALUES AND EIGENVECTORS OF SOME TRIDIAGONAL MATRICES

L. LOSONCZI (Debrecen)[1]

1. Introduction

Let n, k be fixed natural numbers, $1 \leqq k \leqq n$, and denote by $M_{n,k} = M_{n,k}(v, a, b, s, t)$ the $(n+1) \times (n+1)$ matrix

$$(1) \qquad \left(\begin{array}{ccc|ccccccc}
a+v & & & t & & & & & & \\
& \ddots & & & \ddots & & & & & \\
& & a+v & & & \ddots & & & & \\
\hline
s & & & v & & & \ddots & & & \\
& \ddots & & & \ddots & & & \ddots & & \\
& & \ddots & & & v & & & & t \\
\cline{7-10}
& & & \ddots & & & \multicolumn{1}{|c}{v+b} & & & \\
& & & & \ddots & & \multicolumn{1}{|c}{} & \ddots & & \\
& & & & & s & \multicolumn{1}{|c}{} & & & v+b
\end{array}\right)$$

where $v, a, b, s, t \in \mathbf{C}$. The entries $m_{j\ell}$ $(j, \ell = 0, \ldots, n)$ of $M_{n,k}$ are

$$m_{jj} = \begin{cases} a+v & \text{if } \ j = 0, \ldots, k-1 \\ v & \text{if } \ j = k, \ldots, n-k \\ b+v & \text{if } \ j = n-k+1, \ldots, n \end{cases} \qquad (n+1-2k \geqq 0),$$

$$m_{jj} = \begin{cases} a+v & \text{if } \ j = 0, \ldots, n-k \\ a+b+v & \text{if } \ j = n-k+1, \ldots, k-1 \\ b+v & \text{if } \ j = k, \ldots, n \end{cases} \quad (n+1-2k < 0)$$

and

$$m_{j\ell} = \begin{cases} s & \text{if } \ j = \ell + k, \ \ell = 0, \ldots, n-k \\ 0 & \text{if } \ 0 < |j - \ell| \neq k \\ t & \text{if } \ \ell = j + k, \ j = 0, \ldots, n-k. \end{cases}$$

In [7] we factorized $\det M_{n,k}$ if $s = t = 1$ and used this result to find the best constants in some quadratic inequalities. Here we apply another

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method to determine the eigenvalues of $M_{n,k}$. This method enables us to find the eigenvectors of $M_{n,k}$ too. In possession of the eigenvectors we can complement Theorem 4 of [6] by giving the cases of equality. Moreover, we can find some new discrete quadratic inequalities of Wirtinger type.

We remark that multidiagonal matrices (i.e. matrices which have the same entries in the diagonals — except possibly the main diagonal) appear in different areas of mathematics [2], [3], [4], [9]. Thus the investigation of $M_{n,k}$ is of interest in itself too. Throughout the paper $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$ denote the set of complex, real, and integer numbers respectively.

2. Evaluation of $\det M_{n,k}$

Denote by $D_{n,k} = D_{n,k}(v,a,b,s,t)$ the determinant of $M_{n,k}(v,a,b,s,t)$, and let $D_{0,1}(v,a,b,s,t) = a+b+v$.

THEOREM 1. *Let $n+1 = kq+r$ $(0 \leqq r < k)$. Then we have*

$$D_{n,k}(v,a,b,s,t) = D_{q,1}(v,a,b,s,t)^r D_{q-1,1}(v,a,b,s,t)^{k-r}. \tag{2}$$

PROOF. Let us rearrange both the rows and columns of $M_{n,k}$ in the order of indices

$$0,k,\ldots,qk;1,k+1,\ldots,qk+1;\ldots;r-1,k+r-1,\ldots,qk+r-1; \tag{3}$$

$$\begin{gathered} r,k+r,\ldots,(q-1)k+r;r+1,k+r+1,\ldots,(q-1)k+r+1;\ldots \\ \ldots;k-1,2k-1,\ldots,(q-1)k+k-1. \end{gathered} \tag{4}$$

We remark that throughout the paper $\{0,1,\ldots,n\}$ is used as the index set of rows and columns of $M_{n,k}$. This index set is more convenient for our purpose than the usual set $\{1,2,\ldots,n+1\}$.

(3) contains r groups of $q+1$ indices while (4) has $k-r$ groups of q indices. If $r=0$, (3) is empty and all indices are contained in (4) .

The rearranged matrix has r blocks of $M_{q,1}(v,a,b,s,t)$ and $k-r$ blocks of $M_{q-1,1}(v,a,b,s,t)$ in the "diagonal". Hence (2) follows. □

A similar rearrangement has been applied by Egerváry and Szász [2] to find $M_{n,k}(-\lambda,0,0,1,1)$. Next we find $D_{q,1}(v,a,b,s,t)$. We may suppose that $st \neq 0$ since if $s=0$ or $t=0$ then $D_{q,1}$ is a triangular determinant whose value is the product of its diagonal elements.

THEOREM 2. *Let* $q = 0, 1, \ldots; v, a, b, s, t \in \mathbf{C}$, $st \neq 0$, $\sigma = \sqrt{st}$. *If* $v \neq \pm 2\sigma$ *then*

$$(5)\quad D_{q,1}(v,a,b,s,t) = \frac{\sigma^{q+1}}{\sin\vartheta}\left[\sin(q+2)\vartheta + \frac{a+b}{\sigma}\sin(q+1)\vartheta + \frac{ab}{\sigma^2}\sin q\vartheta\right]$$

where $\vartheta \in \mathbf{C}$ *is such that* $v = 2\sigma\cos\vartheta$ $(\sin\vartheta \neq 0)$. *If* $v = \pm 2\sigma$ *then*

(6)

$$D_{q,1}(v,a,b,s,t) = \sigma^{q+1}\left[(\pm 1)^{q+2}(q+2) + \frac{a+b}{\sigma}(\pm 1)^{q+1}(q+1) + \frac{ab}{\sigma^2}(\pm 1)^q q\right]$$

i.e. in this case $D_{q,1}$ *can be obtained as the limit of the right hand side of* (5) *as* $\vartheta \to m\pi$ *where* $v = (-1)^m 2\sigma$, $m \in \mathbf{Z}$.

$D_{q,1}$ *can also be expressed by help of the Chebychev polynomials* U_j *of the second kind as*

$$(7)\quad D_{q,1}(v,a,b,s,t) = \sigma^{q+1}\left[U_{q+1}\left(\frac{v}{2\sigma}\right) + \frac{a+b}{\sigma}U_q\left(\frac{v}{2\sigma}\right) + \frac{ab}{\sigma^2}U_{q-1}\left(\frac{v}{2\sigma}\right)\right]$$

where U_j *is defined as the extension of the polynomial*

$$(8)\qquad U_j(\cos\vartheta) = \frac{\sin(j+1)\vartheta}{\sin\vartheta}\quad (j = -1, 0, 1\ldots).$$

PROOF. Expanding $D_{q,1}(v,a,b,s,t)$ by the zeroth row and also expanding the cofactor of t by the zeroth column we get

$$(9)\qquad D_{q,1}(v,a,b,s,t) = (a+v)D_{q-1,1}(v,0,b,s,t) - tsD_{q-2,1}(v,0,b,s,t)$$

if $q \geqq 2$. (9) shows that

$$(10)\qquad d_q := D_{q,1}(v,0,b,s,t)$$

satisfies the linear homogeneous difference equation

$$(11)\qquad d_{\ell+2} - vd_{\ell+1} + tsd_\ell = 0\quad (\ell = 0, 1, \ldots)$$

with the initial conditions

$$(12)\qquad d_0 = b + v,\quad d_1 = (b+v)v - ts.$$

Since $\sigma \neq 0$ we can always find $\vartheta \in \mathbf{C}$ such that $v = 2\sigma\cos\vartheta$ (ϑ is unique if we require that $-\pi \leqq \operatorname{Re}\vartheta < \pi$, $\operatorname{Im}\vartheta > 0$ or $-\pi \leqq \operatorname{Re}\vartheta \leqq 0$, $\operatorname{Im}\vartheta = 0$ hold).

Then the roots of the characteristic equation

$$\lambda^2 - (2\sigma\cos\vartheta)\lambda + \sigma^2 = 0$$

of (11) are $\lambda_{1,2} = \sigma e^{\pm i\vartheta}$. $\lambda_1 \neq \lambda_2$ if and only if $\vartheta \neq m\pi$, $m \in \mathbf{Z}$. Hence

$$d_\ell = \begin{cases} \sigma^\ell(c_1 e^{i\ell\vartheta} + c_2 e^{-i\ell\vartheta}) & \text{if} \quad \vartheta \neq m\pi \ (\lambda_1 \neq \lambda_2) \\ \sigma^\ell(c_1 e^{i\ell\vartheta} + c_2 \ell e^{i\ell\vartheta}) & \text{if} \quad \vartheta = m\pi \ (\lambda_1 = \lambda_2) \end{cases}$$

for $\ell = 0, 1, \ldots$ where c_1, c_2 are constants to be determined from (12). Calculating c_1, c_2 we obtain that

$$(13) \quad d_\ell = \begin{cases} \dfrac{\sigma^\ell}{\sin\vartheta}[\sigma \sin(\ell+2)\vartheta + b\sin(\ell+1)\vartheta] & \text{if} \quad \vartheta \neq m\pi \\ \sigma^\ell\left[\sigma(\ell+2)(-1)^{m(\ell+2)} + b(\ell+1)(-1)^{m(\ell+1)}\right] & \text{if} \quad \vartheta = m\pi. \end{cases}$$

From (9), (10), (13) by applying the formula $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$ several times we get exactly (5) and (6). (7) follows from (5) and (8) taking into consideration that $D_{q,1}$ is a polynomial of its variables. □

REMARKS. In (5)–(13) any fixed value of the square root $\sqrt{st} = \sigma$ can be used.

$D_{q,1}(v, a, b, 1, 1)$ has been evaluated by Rutherford [10].

3. Eigenvalues and eigenvectors of $M_{q,1}$

Let λ be an eigenvalue of $M_{q,1}(v, a, b, s, t)$ and consider the system of equations

$$(14) \qquad M_{q,1}(v-\lambda, a, b, s, t)y = 0$$

where $y = (y_0, \ldots, y_q)^T$ and T denotes transposition. (14) can be written as

$$(15) \qquad \begin{cases} (a+v-\lambda)y_0 + ty_1 = 0 \\ sy_0 + (v-\lambda)y_1 + ty_2 = 0 \\ \qquad \vdots \\ sy_{q-2} + (v-\lambda)y_{q-1} + ty_q = 0 \\ sy_{q-1} + (b+v-\lambda)y_q = 0. \end{cases}$$

Let $\sigma = \sqrt{st} \neq 0$ and assuming $v - \lambda \neq \pm 2\sigma$ substitute $v - \lambda = 2\sigma\cos\vartheta$, $\vartheta \in \mathbf{C}$. From the first equation of (15)

$$y_1 = -\frac{1}{t}(a + 2\sigma\cos\vartheta)y_0 = -\frac{\sigma\sin 2\vartheta + a\sin\vartheta}{t\sin\vartheta}y_0.$$

We easily obtain by induction that

$$(16) \qquad y_\ell = (-1)^\ell \frac{\sigma^\ell \sin(\ell+1)\vartheta + a\sigma^{\ell-1}\sin\ell\vartheta}{t^\ell \sin\vartheta} y_0 \qquad (\ell = 1, 2, \ldots, q).$$

Substituting y_q and y_{q-1} into the last equation of (15) we get after some calculations that

$$\frac{(-1)^q}{t^q} D_{q,1}(v-\lambda, a, b, s, t) y_0 = 0.$$

Since $D_{q,1}(v-\lambda, a, b, s, t) = 0$, y_0 is arbitrary and $y_1, \ldots, y_q$ are given by (16). If $v - \lambda = 2(-1)^m \sigma$, $m \in \mathbf{Z}$, then in (16) the limit of the right hand side has to be taken as $\vartheta \to m\pi$.

It can be seen that the dimension of the subspace spanned by the eigenvectors corresponding to the eigenvalue λ (i.e. the geometric multiplicity of λ, see [5], § 50) is one. $\square$

Hence we have proved

THEOREM 3. *Let λ be an eigenvalue of $M_{q,1}(v,a,b,s,t)$ ($st \neq 0$, $v,a,b,s,t \in \mathbf{C}$). Then the eigenvectors $y = (y_0, y_1, \ldots, y_q)^T$ corresponding to λ are given by*

$$(17) \qquad y_\ell = \left(-\frac{\sigma}{t}\right)^\ell \left[\sin(\ell+1)\vartheta + \frac{a}{b}\sin \ell\vartheta\right] C \qquad (\ell = 0,1,\ldots,q)$$

if $v - \lambda = 2\sigma\cos\vartheta$ ($\vartheta \neq m\pi$, $m \in \mathbf{Z}$) $\sigma = \sqrt{st}$ and by

(18)
$$y_\ell = \left(-\frac{\sigma}{t}\right)^\ell \left[(\ell+1)(-1)^{m\ell} + \frac{a}{\sigma} - \ell(-1)^{m(\ell-1)}\right] C \qquad (\ell = 0,1,\ldots,q)$$

if $v - \lambda = 2\sigma\cos m\pi$, $m \in \mathbf{Z}$ where $C \neq 0$ is an arbitrary constant. Each eigenvalue is of geometric multiplicity 1.

If $\frac{a}{\sigma}, \frac{b}{\sigma} \in \{0, 1, -1\}$ then the eigenvalues of $M_{q,1}$ can be explicitly given.

THEOREM 4. *The following identities hold:*

$$(19) \qquad D_{q,1}(v-\lambda, 0, 0, s, t) = \prod_{j=0}^{q}\left(v - \lambda - 2\sigma\cos\frac{(j+1)\pi}{q+2}\right),$$

(20)
$$D_{q,1}(v-\lambda, 0, \sigma, s, t) = D_{q,1}(v-\lambda, \sigma, 0, s, t) = \prod_{j=0}^{q}\left(v - \lambda - 2\sigma\cos\frac{2(j+1)\pi}{2q+3}\right),$$

$$(21) \qquad D_{q,1}(v-\lambda, 0, -\sigma, s, t) = D_{q,1}(v-\lambda, -\sigma, 0, s, t) =$$

$$= \prod_{j=0}^{q}\left(v - \lambda - 2\sigma\cos\frac{(2j+1)\pi}{2q+3}\right),$$

$$D_{q,1}(v-\lambda,\sigma,\sigma,s,t)=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{(j+1)\pi}{q+1}\right), \tag{22}$$

$$\begin{aligned} D_{q,1}(v-\lambda,\sigma,-\sigma,s,t) &= D_{q,1}(v-\lambda,-\sigma,\sigma,s,t)= \\ &=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{(2j+1)\pi}{2q+2}\right), \end{aligned} \tag{23}$$

$$D_{q,1}(v-\lambda,-\sigma,-\sigma,s,t)=\prod_{j=0}^{q}\left(v-\lambda-2\sigma\cos\frac{j\pi}{q+1}\right), \tag{22}$$

where $\sigma=\sqrt{st}$.

PROOF. The proof is analogous to that of [7] hence it is omitted (we remark that in [7] from (26) a factor -1 and from (28) a factor $\sin\vartheta$ is missing). □

From the above formulae one can see that the eigenvalues of e.g. $M_{q,1}(v,\sigma,\sigma,s,t)$ are $\lambda_j=v-2\sigma\cos\frac{(j+1)\pi}{q+1}$ $(j=0,\dots,q)$.

4. Eigenvalues and eigenvectors of $M_{n,k}$

By Theorem 1 the eigenvalues of $M_{n,k}(v,a,b,s,t)$ are the zeros λ of the polynomials $D_{q,1}(v-\lambda,a,b,s,t)$ and $D_{q-1,1}(v-\lambda,a,b,s,t)$. First we consider the possibility of these polynomials to have common zeros.

THEOREM 5. *The polynomials* $D_{q,1}(v-\lambda,a,b,s,t)$ *and* $D_{q-1,1}(v-\lambda,a,b,s,t)$ *with* $st\neq 0$ *have a common zero* λ *if and only if*

$$ab=st \tag{25}$$

holds. In this case the common zero is $\lambda=v+a+b$.

In particular (25) *holds if*

$$\frac{a}{\sigma}=\frac{b}{\sigma}=1 \quad or \quad \frac{a}{\sigma}=\frac{b}{\sigma}=-1 \tag{26}$$

where $\sigma=\sqrt{st}$.

PROOF. By Theorem 2, $D_{q,1}(v-\lambda,a,b,s,t)=D_{q-1,1}(v-\lambda,a,b,s,t)=0$ holds if and only if *either* $\sin\vartheta\neq 0$ and $v-\lambda=2\sigma\cos\vartheta$ satisfies the equations

$$\sin(q+2)\vartheta+\frac{a+b}{\sigma}\sin(q+1)\vartheta+\frac{ab}{\sigma^2}\sin q\vartheta=0, \tag{27}$$

$$\sin(q+1)\vartheta + \frac{a+b}{\sigma}\sin q\vartheta + \frac{ab}{\sigma^2}\sin(q-1)\vartheta = 0, \tag{28}$$

or if $\sin\vartheta = 0$, $\vartheta = m\pi$, $m \in \mathbf{Z}$, $v - \lambda = 2(-1)^m\sigma$ and the derivatives of the left hand sides of (27), (28) vanish at $\vartheta = m\pi$.

In the first case by the addition theorem of the sine function and by (28) we can rewrite (27) as

$$\cos(q+1)\vartheta + \frac{a+b}{\sigma}\cos q\vartheta + \frac{ab}{\sigma^2}\cos(q-1)\vartheta = 0. \tag{29}$$

Multiplying (28) by i and adding it to (29) we get

$$e^{i(q-1)\vartheta}\left(e^{i\vartheta} + \frac{a}{\sigma}\right)\left(e^{i\vartheta} + \frac{b}{\sigma}\right) = 0 \tag{30}$$

thus

$$\frac{a}{\sigma} = -e^{i\vartheta} \quad \text{or} \quad \frac{b}{\sigma} = -e^{i\vartheta}. \tag{31}$$

If e.g. $\frac{a}{\sigma} = -e^{i\vartheta}$ then

$$\sin \ell\vartheta = \frac{1}{2i}\left(e^{i\ell\vartheta} - e^{-i\ell\vartheta}\right) = \frac{i}{2}(-1)^{\ell+1}\left[\left(\frac{a}{\sigma}\right)^{\ell} - \left(\frac{\sigma}{a}\right)^{\ell}\right] \qquad (\ell = 1, 2, \dots).$$

Substituting this into (28) we obtain after some calculations that (28) holds if and only if

$$(\sigma^2 - a^2)(\sigma^2 - ab) = 0$$

i.e. if

$$a^2 = \sigma^2 \quad \text{or} \quad ab = \sigma^2. \tag{32}$$

$a^2 = \sigma^2$ implies $e^{2i\vartheta} = 1$, $\sin\vartheta = 0$ which was excluded. Thus $ab = \sigma^2 = st$ is necessary for $D_{q,1}$ and $D_{q-1,1}$ to have common zero. It is sufficient too since (25) ensures that (28) holds while by (31) the equation (29) and hence (27), too, hold.

$$\cos\vartheta = \frac{1}{2}\left(e^{i\vartheta} + e^{-i\vartheta}\right) = -\left(\frac{a}{\sigma} + \frac{\sigma}{a}\right) = -\frac{a+b}{\sigma}$$

thus $\lambda = v - 2\sigma\cos\vartheta = v + a + b$ is the common zero.

Starting with the root $\frac{b}{\sigma} = -e^{i\vartheta}$ of (31) we obtain the same result.

In the second case $\vartheta = m\pi$, $m \in \mathbf{Z}$ and the derivatives of (27), (28) give

$$(q+2)(-1)^{m(q+2)} + \frac{a+b}{\sigma}(q+1)(-1)^{m(q+1)} + \frac{ab}{\sigma^2}q(-1)^{mq} = 0, \tag{33}$$

$$(34)\qquad (q+1)(-1)^{m(q+1)} + \frac{a+b}{\sigma}q(-1)^{mq} + \frac{ab}{\sigma^2}(q-1)(-1)^{m(q-1)} = 0.$$

If m is even then the subtraction of (34) from (33) leads to

$$\left(1+\frac{a}{\sigma}\right)\left(1+\frac{b}{\sigma}\right) = 0$$

hence $\frac{a}{\sigma} = -1$ or $\frac{b}{\sigma} = -1$. If $\frac{a}{\sigma} = -1$ then from (33) we get $\frac{b}{\sigma} = -1$ and conversely $\frac{b}{\sigma} = -1$ and (33) imply $\frac{a}{\sigma} = -1$. Thus

$$(35)\qquad \frac{a}{\sigma} = \frac{b}{\sigma} = -1, \quad \lambda = v - 2\sigma\cos\vartheta = v - 2\sigma = v + a + b.$$

If m is odd then the difference of (33) and (34) can be written as

$$(-1)^q\left(1-\frac{a}{\sigma}\right)\left(1-\frac{b}{\sigma}\right) = 0.$$

Thus $\frac{a}{\sigma} = 1$ or $\frac{a}{\sigma} = 1$. If e.g. $\frac{a}{\sigma} = 1$ then from (33) we get $\frac{b}{\sigma} = 1$ and conversely. Hence

$$(36)\qquad \frac{a}{\sigma} = \frac{b}{\sigma} = 1, \quad \lambda = v - 2\sigma\cos\vartheta = v + 2\sigma = v + a + b.$$

Since (35), (36) are particular cases of (25) we proved that (25) is a necessary condition. It is sufficient too since with $\vartheta = m\pi$ and with (35) or (36) both (27), (28) and (33), (34) are satisfied. □

THEOREM 6. *The eigenvectors $x = (x_0, x_1, \dots, x_n)^T$ of $M_{n,k}(v,a,b,s,t)$ corresponding to the eigenvalue $\lambda = v - 2\sigma\cos\vartheta$ are given by*

$$(37)\qquad x_{u+hk} = \left(-\frac{\sigma}{t}\right)^h \left[\sin(h+1)\vartheta + \frac{a}{\sigma}\sin h\vartheta\right] C_u$$

if $\vartheta \neq m\pi$, $m \in \mathbf{Z}$, and by

$$(38)\qquad x_{u+hk} = \left(-\frac{\sigma}{t}\right)^h \left[(h+1)(-1)^{mh} + \frac{a}{\sigma}h(-1)^{m(h-1)}\right] C_u$$

if $\vartheta = m\pi$, $m \in \mathbf{Z}$. In (37), (38) $u = 0,1,\dots,r-1$; $h = 0,1,\dots,q$ and $u = r, r+1, \dots, k-1$; $h = 0,1,\dots,q-1$. $C_0, C_1, \dots, C_{k-1} \in \mathbf{C}$ are constants such that

(i) *$C_0, C_1, \dots C_{r-1}$ are arbitrary constants not all zero, $C_r = C_{r+1} = \dots \cdots = C_{k-1} = 0$ if $D_{q,1}(v-\lambda,a,b,s,t) = 0 \neq D_{q-1,1}(v-\lambda,a,b,s,t)$,*

(ii) *$C_0 = C_1 = \cdots = C_{r-1} = 0$ and $C_r, C_{r+1}, \dots, C_{k-1}$ are arbitrary constants not all zero if $D_{q,1}(v-\lambda,a,b,s,t) \neq 0 = D_{q-1,1}(v-\lambda,a,b,s,t)$,*

(iii) $C_0, C_1, \ldots, C_{k-1}$ *are arbitrary constants not all zero if* $D_{q,1}(v-\lambda, a, b, s, t) = 0 = D_{q-1,1}(v-\lambda, a, b, s, t)$ *(by Theorem* 5 *this case occurs if and only if* $ab = st$*).*

The geometric multiplicity of the eigenvalue λ *in the cases* (i), (ii), (iii) *is* r, $k-r$, k *respectively.*

PROOF. We have to solve the system of equations

$$M_{n,k}(v-\lambda, a, b, s, t)x = 0. \tag{39}$$

Rearranging the equations of (39) according to (3), (4) one can recognize that (39) decomposes to the following systems:

$$M_{q,1}(v-\lambda, a, b, s, t)z_u = 0 \tag{40}$$

where $z_u = (x_u, x_{u+k}, \ldots, x_{u+qk})^T$, $u = 0, 1, \ldots, r-1$ and

$$M_{q-1,1}(v-\lambda, a, b, s, t)w_u = 0 \tag{41}$$

where $w_u = (x_u, x_{u+k}, \ldots, x_{u+(q-1)k})^T$, $u = r, r+1, \ldots, k-1$.

In the case (i) Theorem 3 gives the solution (40) for each $u = 0, 1, \ldots, r-1$. This justifies (37) and (38) for $u = 0, 1, \ldots, r-1$; $h = 0, 1, \ldots, q$. (41) has trivial solution only for each $u = r, r+1, \ldots, k-1$ since the determinant of the system $D_{q-1,1}(v-\lambda, a, b, s, t)$ is not zero. These trivial solutions are included in (37) and (38) by requiring $C_r = C_{r+1} = \cdots = C_{k-1} = 0$.

In the case (ii) (40) has trivial solutions only (therefore $C_0 = C_1 = \cdots = C_{r-1} = 0$) and the solutions of (41) are (by Theorem 3) given by (37), (38) for $u = r, r+1, \ldots, k-1$; $h = 0, 1, \ldots, q-1$.

In the case (iii) both systems (40), (41) have nontrivial solutions which are given by (37), (38).

The statement concerning the geometric multiplicity of the eigenvalues is obvious. □

5. Applications

Here we apply our results to study some discrete quadratic inequalities of Wirtinger type. Let A be an Hermitian matrix of order $n+1$ with eigenvalues $\lambda_0 \geqq \lambda_1 \geqq \ldots \geqq \lambda_n$ and let $x^{(0)}, x^{(1)}, \ldots, x^{(n)}$ be the corresponding linearly independent eigenvectors. Then

$$\lambda_n\langle x, x\rangle \leqq \langle Ax, x\rangle \leqq \lambda_0\langle x, x\rangle \tag{42}$$

holds for every vector $x \in \mathbf{C}^{n+1}$ where $\langle\cdot,\cdot\rangle$ is the usual inner product

$$\langle x, y\rangle = \sum_{j=0}^{n} x_j \bar{y}_j$$

for $x = (x_0, \dots, x_n)^T$, $y = (y_0, \dots, y_n)^T$. The equality $\lambda_n\langle x, x\rangle = \langle Ax, x\rangle$ holds if and only if $x = 0$ or x is an eigenvector corresponding to λ_n (if $\lambda_n < \lambda_{n-1}$, then x is a scalar multiple of $x^{(1)}$). Similarly $\langle Ax, x\rangle = \lambda_0\langle x, x\rangle$ holds if and only if $x = 0$ or x is an eigenvector corresponding to λ_i (see e.g. [1]).

Let now $v, a, b \in \mathbf{R}$, $\alpha, \beta \in \mathbf{C} \setminus \{0\}$, $t = \alpha\overline{\beta}$, $s = \overline{t} = \overline{\alpha}\beta$, then $M_{n,k}(v, a, b, \alpha\overline{\beta}, \overline{\alpha}\beta)$ is an Hermitian matrix and

$$\text{(43)} \qquad \langle M_{n,k}(v, a, b, \overline{\alpha}\beta, \alpha\overline{\beta})x, x\rangle = \\ = \sum_{j=0}^{n-k} \left[|\alpha x_{j+k} + \beta x_j|^2 - (a + |\alpha|^2)|x_{j+k}|^2 - \right. \\ \left. -(b + |\beta|^2)|x_j|^2\right] + (v + a + b)\sum_{j=0}^{n} |x_j|^2.$$

We are going to formulate inequalities of the form (42) in the cases when the best constants (the least and greatest eigenvalues) can explicitly be given.

We have seen that this is the case if $\frac{a}{\sigma} = \varepsilon$, $\frac{b}{\sigma} = \rho$, $\varepsilon, \rho = 0, \pm 1$. The possible values of ε, ρ are listed in the next table.

Table 1

ℓ	ε_ℓ	ρ_ℓ
1	0	0
2	0	1
3	0	−1
4	1	−1
5	1	1
6	−1	−1
7	1	0
8	−1	0
9	−1	1

With $a = \varepsilon_\ell\sigma = \varepsilon_\ell|\alpha\beta|$, $b = \rho_\ell\sigma = \rho_\ell|\alpha\beta|$, $v = -a - b = -|\alpha\beta|(\varepsilon_\ell + \rho_\ell)$ in (43) we have

$$\text{(44)} \qquad \langle A_{n,k}^{(\ell)}x, x\rangle = \sum_{j=0}^{n-k} \left[|\alpha x_{j+k} + \beta x_j|^2 - (\varepsilon_\ell|\alpha\beta| + |\alpha|^2)|x_{j+k}|^2 - \right. \\ \left. -(\rho_\ell|\alpha\beta| + |\beta|^2)|x_j|^2\right]$$

where

$$\text{(45)} \qquad A_{n,k}^{(\ell)} = M_{n,k}(-|\alpha\beta|(\varepsilon_\ell + \rho_\ell), \varepsilon_\ell|\alpha\beta|, \rho_\ell|\alpha\beta|, \overline{\alpha}\beta, \alpha\overline{\beta})$$

for $\ell = 1, \ldots, 9$.

The eigenvalues of $A_{n,k}^{(\ell)}$ are of the form

$$\lambda = v - 2\sigma \cos \vartheta = -|\alpha\beta|(\varepsilon_\ell + \rho_\ell + 2\cos \vartheta). \tag{46}$$

Based on Theorem 4 it is easy to find the eigenvalues of $A_{n,k}^{(\ell)}$. By (46) Table 2 gives the values of ϑ corresponding to the eigenvalues λ of $A_{n,k}^{(\ell)}$ and also gives $m(\lambda)$, the algebraic multiplicity of λ (which is now equal to the geometric multiplicity).

Table 2

ℓ	ϑ	$(m(\lambda) = r)$	ϑ	$(m(\lambda) = k - r)$	ϑ $(m(\lambda) = k)$
1	$\frac{(j+1)\pi}{q+2}$	$(j = 0, \ldots, q)$	$\frac{(j+1)\pi}{q+1}$	$(j = 0, \ldots, q-1)$	–
2	$\frac{2(j+1)\pi}{q+3}$	$(j = 0, \ldots, q)$	$\frac{2(j+1)\pi}{2q+1}$	$(j = 0, \ldots, q-1)$	–
3	$\frac{(2j+1)\pi}{2q+3}$	$(j = 0, \ldots, q)$	$\frac{(2j+1)\pi}{2q+1}$	$(j = 0, \ldots, q-1)$	–
4	$\frac{(2j+1)\pi}{2q+2}$	$(j = 0, \ldots, q)$	$\frac{(2j+1)\pi}{2q}$	$(j = 0, \ldots, q-1)$	–
5	$\frac{(j+1)\pi}{q+1}$	$(j = 0, \ldots, q-1)$	$\frac{(j+1)\pi}{q}$	$(j = 0, \ldots, q-2)$	π
6	$\frac{j\pi}{q+1}$	$(j = 1, \ldots, q)$	$\frac{j\pi}{q}$	$(j = 1, \ldots, q-1)$	0
7		same	as	$\ell = 2$	
8		same	as	$\ell = 3$	
9		same	as	$\ell = 4$	

In Table 3 we collected $\vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell)}$ such that for each $\ell = 1, \ldots, 6$, (46) gives the minimal and maximal eigenvalue of $A_{n,k}^{(\ell)}$ if $\vartheta = \vartheta_{\min}^{(\ell)}$ and $\vartheta = \vartheta_{\max}^{(\ell)}$ respectively. Here we used the relation

$$\left[\frac{n}{k}\right] = \begin{cases} q & \text{if } r > 0 \\ q - 1 & \text{if } r = 0 \end{cases}$$

where $[x]$ denotes the greatest integer not exceeding x. We omitted $\ell = 7, 8, 9$ since by Table 2 $\vartheta_{\min}^{(\ell+5)} = \vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell+5)} = \vartheta_{\max}^{(\ell)}$ for $\ell = 2, 3, 4$.

Table 3

ℓ	$\vartheta_{\min}^{(\ell)}$	$\vartheta_{\max}^{(\ell)}$
1	$\dfrac{\pi}{\left[\frac{n}{k}\right]+2}$	$\pi-\vartheta_{\min}^{(1)}$
2	$\dfrac{2\pi}{2\left[\frac{n}{k}\right]+3}$	$\pi-\frac{1}{2}\vartheta_{\min}^{(2)}$
3	$\dfrac{\pi}{2\left[\frac{n}{k}\right]+3}$	$\pi-2\vartheta_{\min}^{(3)}$
4	$\dfrac{\pi}{2\left[\frac{n}{k}\right]+2}$	$\pi-\vartheta_{\min}^{(4)}$
5	$\dfrac{\pi}{\left[\frac{n}{k}\right]+2}$	π
6	0	$\pi-\dfrac{\pi}{\left[\frac{n}{k}\right]+1}$

Finally we formulate the main result of this section.

THEOREM 7. *Let n,k be fixed natural numbers, $1\leqq k\leqq n$, $n+1=kq+r$ $(0\leqq r<k)$. For every $x=(x_0,\dots,x_n)^T\in\mathbf{C}^{n+1}$ and $\ell=1,\dots,6$ the inequality*

$$(47)\qquad -|\alpha\beta|\left(\varepsilon_\ell+\rho_\ell+2\cos\vartheta_{\min}^{(\ell)}\right)\langle x,x\rangle\leqq\langle A_{n,k}^{(\ell)}x,x\rangle\leqq$$

$$\leqq-|\alpha\beta|\left(\varepsilon_\ell+\rho_\ell+2\cos\vartheta_{\max}^{(\ell)}\right)\langle x,x\rangle$$

holds where

$$\langle A_{n,k}^{(\ell)}x,x\rangle=\sum_{j=0}^{n-k}\left[|\alpha x_{j+k}+\beta x_j|^2-(\varepsilon_\ell|\alpha\beta|+|\alpha|^2)|x_{j+k}|^2-\right.$$

$$\left.-(\rho_\ell|\alpha\beta|+|\beta|^2)|x_j|^2\right],$$

$\alpha,\beta\in\mathbf{C}\setminus\{0\}$ and ε_ℓ, ρ_ℓ; $\vartheta_{\min}^{(\ell)}$, $\vartheta_{\max}^{(\ell)}$ are given by Tables 1 and 3 respectively.

Equality on the left hand side of (47) occurs for $\ell=1,2,3,4,5$ if and only if

$$(48)\qquad x_{u+hk}=\left(-\frac{|\alpha\beta|}{\alpha\bar{b}}\right)^h\left[\sin(h+1)\vartheta_{\min}^{(\ell)}+\varepsilon_\ell\sin h\vartheta_{\min}^{(\ell)}\right]C_u$$

holds for

$$u = 0, 1, \ldots, r-1; \quad h = 0, 1, \ldots, q \quad \text{and} \tag{49}$$

$$u = r, r+1, \ldots, k-1; \quad h = 0, 1, \ldots, q-1$$

while for $\ell = 6$ *if and only if*

$$x_{u+hk} = \left(-\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right)^h D_u \tag{50}$$

holds for the subscripts (49).

Equality on the right hand side of (47) *is valid for* $\ell = 1, 2, 3, 4, 6$ *if and only if*

$$x_{u+hk} = \left(-\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right)^h \left[\sin(h+1)\vartheta_{\max}^{(\ell)} + \varepsilon_\ell \sin h\vartheta_{\max}^{(\ell)}\right] C_u \tag{51}$$

holds for the indices (49) *while for* $\ell = 5$ *if and only if*

$$x_{u+hk} = \left(\frac{|\alpha\beta|}{\alpha\overline{\beta}}\right) D_u \tag{52}$$

is fulfilled for the subscripts (49).

Here $C_0, \ldots, C_{r-l}$ *are arbitrary constants,* $C_r = C_{r+1} = \cdots = C_{k-1} = 0$ *(if* $r = 0$ *then all* C_u*'s are arbitrary) and* $D_0, \ldots, D_{k-1}$ *are arbitrary constants.*

PROOF. The statement of Theorem 7 follows from the general result (42) taking into consideration Theorem 6 and the calculations of Section 5 concerning $A_{n,k}^{(\ell)}$ and its "parameters". □

The cases $\ell = 7, 8, 9$ can be obtained from $\ell = 2, 3, 4$ by exchanging ε_ℓ, α to ρ_ℓ, β respectively.

Several special cases of (47) are known. If $k = 1$, $\alpha = \beta = 1$, (47) has been proved by Fan, Taussky and Todd [3]. Their inequalities are discrete analogues of Wirtinger's inequality; see e.g. Hardy, Littlewood and Pólya [6], p. 184. For $\beta = 1$, $\alpha = \pm 1$, $\ell = 1, 3, 6$, Theorem 7 (without the equality clause) has been proved by the author [7] (the cases (i), (ii), (iii), (iv) of [7] can easily be rewritten to the cases $\ell = 6, 3, 3, 1$ respectively). Concerning inequalities related to (47) we refer to [2], [8].

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L. KOSSUTH UNIVERSITY
DEPARTMENT OF MSTHEMATICS
4010 DEBRECEN, EGYETEM TÉR 1
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