Acta Mathematica Hungarica

VOLUME 59, NUMBERS 1–2, 1992

Akadémiai Kiadó
Budapest

Kluwer Academic Publishers
Dordrecht / Boston / London

ACTA MATH. HU ISSN 0236-5294

ACTA MATHEMATICA
HUNGARICA

Acta Mathematica publishes papers on mathematics in English, German, French and Russian.

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For Albania, Bulgaria, China, Cuba, Czech and Slovak Federal Republic, Hungary, Korean People's Republic, Mongolia, Poland, Romania, successor states of the U.S.S.R., successor states of Yugoslavia, Vietnam

AKADÉMIAI KIADÓ

P.O. Box 254, 1519 Budapest, Hungary

For all other countries with

KLUWER ACADEMIC PUBLISHERS

P.O. Box 17, 3300 AA Dordrecht, Holland

Publication programme: 1992: Volumes 59–60 (eight issues)

Subscription price per volume: Dfl 206,– / US $ 105 (incl. postage)

Acta Mathematica Hungarica is abstracted/indexed in Current Contents — Physical, Chemical and Earth Sciences, Mathematical Reviews.

ACTA MATHEMATICA HUNGARICA

VOLUME 59

AKADÉMIAI KIADÓ, BUDAPEST

1992

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VOLUME 59

Acta Math. Hung.
59 (1–2) (1992), 1–10.

ϕ-ORTHOGONALLY ADDITIVE MAPPINGS. II

Gy. SZABÓ (Debrecen)

1. Introduction

This is the second part of a series of papers describing the properties of ϕ-orthogonally additive mappings for a sesquilinear form ϕ. While in the first part [3] the symmetric orthogonality has been studied, here we examine the cases of non-symmetric and totally isotropic orthogonalities showing essentially the general solution to be additive (see Sections 2 and 3 below). Furthermore, we illustrate our theory by some examples and summarize the results obtained in the different possible cases (see Section 4 below). These results turn out to be proper generalizations of those of Sundaresan–Kapoor [2].

We shall use the same notation and terminology as in the first part [3]. Namely, throughout the paper, Φ will denote a field of char $\Phi \neq 2$, X a Φ-vector space with $\dim_\Phi X \geqq 2$ and $(Y, +)$ an abelian group. Furthermore, consider a sesquilinear functional $\phi\colon X \times X \to \Phi$ with respect to an automorphism $\bar{\cdot}\colon \Phi \to \Phi$ and define the *ϕ-orthogonality* relation $\perp^\phi$ on X by

$$\perp^\phi = \{(x, y) \in X \times X \mid \phi(x, y) = 0\}.$$

A vector $x \in X$ is said to be *isotropic,* if $\phi(x, x) = 0$. For $x \in X$ define the linear functional $\phi_x\colon X \to \Phi$ by $\phi_x(t) = \phi(t, x)$ and consider the linear subspace $X^*_\phi = \{\phi_x \mid x \in X\}$ in the algebraic conjugate X^* of X. Denoting by $\mathcal{P}$ the family of all 2-dimensional linear subspaces in X, for $P \in \mathcal{P}$ let $\perp^\phi{}_P$ be the set of all $(u, v) \in \perp^\phi$ such that $\operatorname{lin}\{u, v\} = P$ (the set of all *ϕ-orthogonal bases* in P), where $\operatorname{lin} V$ stands for the linear hull of a subset $V \subset X$.

The vector $x \in X$ is said to be a *τ_0-element,* if $x \in \operatorname{lin}\{u, v\}$ for some $u, v \in X$ such that ($x \perp^\phi u$ or $u \perp^\phi x$) and ($x \perp^\phi v$ or $v \perp^\phi x$). Let X_0 denote the set of all τ_0-elements in X and define the subfamily $\mathcal{P}_0 = \{P \in \mathcal{P} \mid \perp^\phi{}_P \cap (X_0 \times X_0) \neq \emptyset\}$ in $\mathcal{P}$.

We say $P \in \mathcal{P}$ to be a *$\perp^\phi$-normal plane,* if there are $(u_i, v_i) \in \perp^\phi{}_P$ $(i = 1, 2)$ with

$$\bigcap_{i=1}^{2} (\operatorname{lin}\{u_i\} \cup \operatorname{lin}\{v_i\}) = \{0\}.$$

The subfamily of all $\perp^\phi$-normal planes in $\mathcal{P}$ will be denoted by $\mathcal{P}_n$. A vector $x \in X$ is called a τ_1-*element,* if it is contained in a $\perp^\phi$-normal plane: $x \in P \in \mathcal{P}_n$.

The mappings A, Q and $F\colon X \to Y$ are said to be *additive, quadratic* or *orthogonally additive* ($\perp^\phi$*additive*), if they satisfy the equations:

$$A(x+y) = A(x) + A(y), \qquad x, y \in X,$$
$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \qquad x, y \in X, \text{ or}$$
$$F(x+y) = F(x) + F(y), \qquad x, y \in X,\ x \perp^\phi y,$$

respectively. For any $t \in X$ let $F_t\colon \Phi \to Y$ be defined by $F_t(\lambda) = F(\lambda t)$. Furthermore, we shall use the notations:

$$\mathrm{Hom}(X,Y) = \{A\colon X \to Y | A \text{ is additive}\},$$
$$\mathrm{Quad}(X,Y) = \{Q\colon X \to Y | Q \text{ is quadratic}\},$$
$$\mathrm{Hom}_{\perp^\phi}(X,Y) = \{F\colon X \to Y | F \text{ is } \perp^\phi\text{-additive}\}.$$

Finally, $\mathbf{R}$ is the real line, 0 denotes the scalar zero, the zero vector as well as the identity element of the group Y. The actual meaning of 0 always will be clear from the context. The sign $\mathbf{0}$ stands for the constant zero mapping.

2. The non-symmetric ϕ-orthogonality

LEMMA 2.1. *Suppose that* $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ *and*
(i) *either* $P \in \mathcal{P}_0$,
(ii) *or* $P \in \mathcal{P}_n$ *such that* F *is additive on* $\mathrm{lin}\{v\}$ *for some* $(u,v) \in \perp^\phi{}_P$. *Then* F *is additive on the whole* P.

PROOF. (i) We can choose $(u,v) \in \perp^\phi{}_P \cap (X_0 \times X_0)$, i.e., a ϕ-orthogonal base in P consisting of τ_0-elements. Then by [1], Lemma 2.4, F is additive on $\mathrm{lin}\{u\}$ and $\mathrm{lin}\{v\}$, whence [1], Lemma 2.1 yields the additivity of F on $\mathrm{lin}\{u,v\} = P$.

(ii) Since P is a normal plane, there exists $(x,y) \in \perp^\phi{}_P$ with $x, y \notin \notin \mathrm{lin}\{v\}$. By the homogeneity, we may assume that $x = (u+\xi v) \perp^\phi (u+\eta v) = = y$ and so formula (2.5) in [1] turns into

$$F_u(\lambda+\mu) = F_u(\lambda) + F_u(\mu), \qquad \lambda, \mu \in \Phi,$$

i.e., F is additive on $\mathrm{lin}\{u\}$. Finally, [1], Lemma 2.1 completes the proof.

LEMMA 2.2. *If* $P \in \mathcal{P}$ *is such that the* ϕ*-orthogonality is not symmetric on* P, *then there is a non-isotropic vector in* P *and any* $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ *is additive on* P.

PROOF. Choose vectors $u, v \in P$ with $\phi(u,v) \neq 0 = \phi(v,u)$. Then obviously $\mathrm{lin}\{u,v\} = P$ and one of the vectors u, v and $u+v$ is not isotropic. Now defining ζ to satisfy $\bar{\zeta} = \phi(u,u)/\phi(u,v)$, we have for $z = u - \zeta v$ that

$$\phi(u,z) = 0 \quad \text{and} \quad \phi(u-z,u) = 0,$$

i.e., u is a τ_0-element. On the other hand, if $w = [\phi(v,v)/\phi(u,v)]u$, then

$$\phi(v,w) = 0 \quad \text{and} \quad \phi(v-w,v) = 0,$$

i.e., v is also a τ_0-element. These mean that $P \in \mathcal{P}_0$ and so the previous lemma implies the additivity of F on P.

LEMMA 2.3. *Suppose that $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ and there is a non-isotropic vector $z \in X$ such that F is additive on* $\mathrm{lin}\{z\}$. *Then F is additive on each* $\mathrm{lin}\{t\}$, $t \in X \setminus \{0\}$, *separately.*

PROOF. Let $t \in X \setminus \{0\}$ be arbitrarily fixed. We may and do assume that t is not isotropic and t, z are linearly independent, i.e., $\mathrm{lin}\{t,z\} = P \in \mathcal{P}$. Now there may occur exactly one of the following possibilities:

a) $\perp^\phi$ *is non-symmetric on P*: Then Lemma 2.2 yields the additivity of F on $P \supset \mathrm{lin}\{t\}$.

b) $\perp^\phi$ *is symmetric on P*: Then we have to deal with the cases below:

b/1) $\phi(t,z) = 0$: Then $(t+\xi z) \perp^\phi (t-z)$ holds for $\xi = \phi(t,t)/\phi(z,z)$, and so formula (2.5) in [1] yields the desired additivity:

$$F_t(\lambda + \mu) = F_t(\lambda) + F_t(\mu), \qquad \lambda, \mu \in \Phi.$$

b/2) $\phi(t,z) \neq 0$: If $x = [\phi(z,t)/\phi(t,t)]t - z$, $y = t - [\phi(t,z)/\phi(z,z)]z$, then clearly $(x,t), (y,z) \in \perp^\phi{}_P$ and $\phi(t,z) \neq 0 \neq \phi(z,t)$ implies that $\mathrm{lin}\{t,y\} = P = \mathrm{lin}\{x,z\}$. Furthermore, we have

— *either* $y \in \mathrm{lin}\{x\}$, i.e., $y = \lambda x$ for some $\lambda \in \Phi$. Then $\phi(y,t) = \lambda\phi(x,t) = 0$, and so y is a τ_0-element showing by [1], Lemma 2.4, that F is additive on $\mathrm{lin}\{y\}$. Since $(y,z) \in \perp^\phi{}_P$, [1], Lemma 2.1, completes the proof.

— *or* $\mathrm{lin}\{x,y\} = P$, i.e., P is a normal plane with

$$(u_1, v_1) = (x,t) \quad \text{and} \quad (u_2, v_2) = (y,z).$$

Thus Lemma 2.1 implies that F is additive on $\mathrm{lin}\{y,z\} = P \supset \mathrm{lin}\{t\}$.

LEMMA 2.4. *Let $P \in \mathcal{P}$ and suppose that each $x \in P$ is isotropic, but ϕ is not identically zero on $P \times P$. Then $\perp^\phi$ is symmetric on P and there exist vectors $u, v \in P$ such that $\mathrm{lin}\{u,v\} = P$ and $\phi(u,v) = 1 = -\phi(v,u)$. Moreover, $\bar{\alpha} = \alpha$ for all $\alpha \in \Phi$.*

PROOF. The symmetry of $\perp^\phi$ is a direct consequence of Lemma 2.2. Now choose an arbitrary base x, y for P: $\mathrm{lin}\{x,y\} = P$. Then $\phi(x,y) + \phi(y,x) = \phi(x+y, x+y) = 0$. Here $\phi(x,y) \neq 0$, since otherwise ϕ would vanish identically on $P \times P$. Thus $u = x/\phi(x,y)$ and $v = y$ satisfy all the required properties:

$$\phi(u,v) = 1 \quad \text{and} \quad \phi(v,u) = \phi(u+v, u+v) - \phi(u,v) = -1.$$

Finally, for any $\alpha \in \Phi$, we have

$$\alpha - \bar{\alpha} = \phi(\alpha u + v, \alpha u + v) = 0.$$

LEMMA 2.5. *Suppose that $P \in \mathcal{P}$ and each $x \in P$ is isotropic. If $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ and there is a non-isotropic vector $z \in X$ such that F is additive on* $\mathrm{lin}\{z\}$, *then F is additive on the whole P, too.*

PROOF. If $\phi(x,y) = 0$ for all $x, y \in P$, then there is nothing to prove. Otherwise, by Lemma 2.4, $\bar{\alpha} = \alpha$ whenever $\alpha \in \Phi$ and we can choose $u, v \in P$ such that $\mathrm{lin}\{u,v\} = P$ and $\phi(u,v) = 1 - \phi(v,u)$. Next define

$$x = \phi(z,v)u - \phi(z,u)v \in P,$$

for which $\phi(x,u) = \phi(z,u)$ and $\phi(x,v) = \phi(z,v)$. Thus setting $z_0 = z - x$, we get $\phi(z_0,u) = 0 = \phi(z_0,v)$ and so $\phi(z_0,x) = 0$. Now we can proceed in different ways depending on the cases as follows:

a) $\phi(u,z_0) \neq 0$: Let $\beta = -1/\phi(u,z_0)$ for which $\phi(u, \beta z_0 + v) = 0$ holds.

b) $\phi(v,z_0) \neq 0$: Let $\alpha = 1/\phi(v,z_0)$ for which $\phi(v, \alpha z_0 + u) = 0$ holds.

c) $\phi(u,z_0) = 0 = \phi(v,z_0)$: Then $\phi(x,z_0) = 0$ and

$$\begin{aligned}\phi(z_0,z_0) &= \phi(z-x,z_0) = \phi(z,z_0) = \phi(z,z-x) = \\ &= \phi(z,z) - \phi(z_0+x,x) = \phi(z,z) \neq 0.\end{aligned}$$

Thus for $\alpha = -1/\phi(z_0,z_0)$, we have $\phi(\alpha z_0 + u, z_0 + v) = 0$. These mean that in any case there are scalars $\alpha, \beta \in \Phi$ with

$$(\alpha z_0 + u) \perp^\phi (\beta z_0 + v) \quad \text{or} \quad (\beta z_0 + v) \perp^\phi (\alpha z_0 + u).$$

Now using repeatedly the orthogonal additivity of F, we obtain for all $\lambda, \mu \in \Phi$ that

$$\begin{aligned}F(\lambda\alpha z_0 + \mu\beta z_0) + F(\lambda u + \mu v) &= F(\lambda\alpha z_0 + \mu\beta z_0 + \lambda u + \mu v) = \\ = F(\lambda[\alpha z_0 + u] + \mu[\beta z_0 + v]) &= F(\lambda\alpha z_0 + \lambda u) + F(\mu\beta z_0 + \mu v) = \\ &= F(\lambda\alpha z_0) + F(\lambda u) + F(\mu\beta z_0) + F(\mu v).\end{aligned}$$

Due to Lemma 2.3, F is additive on $\mathrm{lin}\{z_0\}$, thus we gain

$$F(\lambda u + \mu v) = F(\lambda u) + F(\mu v), \qquad \lambda, \mu \in \Phi. \tag{2.1}$$

Finally, because of the same Lemma 2.3 (or simply by the isotropy of u, v), F is additive on $\mathrm{lin}\{u\}$ and $\mathrm{lin}\{v\}$, and so (2.1) yields the additivity of F on $P = \mathrm{lin}\{u,v\}$.

THEOREM 2.6. *If the ϕ-orthogonality is not symmetric on X, then*

$$\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y).$$

PROOF. Let $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ and choose $u_0, v_0 \in X$ such that $\phi(u_0,v_0) \neq 0 = \phi(v_0,u_0)$. Then u_0, v_0 are linearly independent, and by Lemma 2.2, there exists a non-isotropic vector $z_0 \in P_0 = \mathrm{lin}\{u_0,v_0\}$ and F is additive on $P_0 \supset \operatorname{lin}\{z_0\}$. Since F is additive on $\operatorname{lin}\{z_0\}$, Lemma 2.3 ensures the additivity of F on each $\operatorname{lin}\{t\}$, $t \in X$. Now it suffices to prove the additivity of F on any $P \in \mathcal{P}$. There are two possibilities:

a) $\phi(x,x) = 0$ *for all* $x \in P$: Then Lemma 2.5 gives the desired result.

b) $\phi(v,v) \neq 0$ *for some* $v \in P$: Then choosing an $x \in P \setminus \operatorname{lin}\{v\}$, for $u = x - [\phi(x,v)/\phi(v,v)]v$, we have $(u,v) \in \perp^{\phi}{}_P$. Finally, regarding that F is additive on $\operatorname{lin}\{u\}$ and $\operatorname{lin}\{v\}$, [1], Lemma 2.1, completes the proof.

3. The totally isotropic ϕ-orthogonality

PROPOSITION 3.1. *Let $P \in \mathcal{P}$ and suppose that each $x \in P$ is isotropic. If $\phi_z \notin \{\phi_x | x \in P\}$ for some $z \in X$, then every $F \in \operatorname{Hom}_{\perp^\phi}(X,Y)$ is additive on P.*

PROOF. By Theorem 2.6, it suffices to deal with symmetric $\perp^{\phi}$. If $\phi(x,y) = 0$ for all $x,y \in P$, then there is nothing to prove. Otherwise, by Lemma 2.4, $\bar{\alpha} = \alpha$ whenever $\alpha \in \Phi$ and we can choose $u,v \in P$ such that $\operatorname{lin}\{u,v\} = P$ and $\phi(u,v) = 1 = -\phi(v,u)$. Next define

$$x = \phi(z,v)u - \phi(z,u)v \in P,$$

for which $\phi(z,u) = \phi(x,u)$ and $\phi(z,v) = \phi(x,v)$. With respect to our hypothesis, $\phi_z \neq \phi_x$ and so there exists $t \in X$ such that $\phi(t,z) \neq \phi(t,x)$. Let further

$$y = \phi(t,v)u - \phi(t,u)v \in P.$$

Then $\phi(t,u) = \phi(y,u)$ and $\phi(t,v) = \phi(y,v)$, and for $z_0 = z - x$, $t_0 = t - y$, we have

$$\phi(z_0,u) = 0 = \phi(z_0,v) \quad \text{and} \quad \phi(t_0,u) = 0 = \phi(t_0,v).$$

The symmetry of $\perp^{\phi}$ yields also

$$\phi(u,z_0) = 0 = \phi(v,z_0) \quad \text{and} \quad \phi(u,t_0) = 0 = \phi(v,t_0),$$

whence by the biadditivity of ϕ, we obtain

$$\phi(z_0,x) = 0 = \phi(z_0,y) \quad \text{and} \quad \phi(t_0,x) = 0 = \phi(t_0,y),$$

$$\phi(x,z_0) = 0 = \phi(y,z_0) \quad \text{and} \quad \phi(x,t_0) = 0 = \phi(y,t_0).$$

On the other hand,

$$\phi(t_0,z_0) = \phi(t_0 + y, z_0 + x) - \phi(y,x) =$$

$$= \phi(t,z) - \phi(t_0 + y, x) = \phi(t,z) - \phi(t,x) \neq 0.$$

Thus we may assume that $\phi(t_0,z_0) = 1$. Then for any $\lambda, \mu \in \Phi$, we have

$$\phi(\pm\lambda u + \mu t_0, \pm\mu v - \lambda z_0) = \lambda\mu - \mu\lambda = 0.$$

Now we can compute according to the orthogonal additivity of F:

$$F(\pm\lambda u \pm \mu v) + F(\mu t_0 - \lambda z_0) = F([\pm\lambda u + \mu t_0] + [\pm\mu v - \lambda z_0]) =$$

$$= F(\pm\lambda u + \mu t_0) + F(\pm\mu v - \lambda z_0) = F(\pm\lambda u) + F(\mu t_0) + F(\pm\mu v) + F(-\lambda z_0).$$

Since F is obviously odd on P, therefore subtraction of the above equalities from each other yields

$$2F(\lambda u + \mu v) = 2F(\lambda u) + 2F(\mu v), \qquad \lambda, \mu \in \Phi.$$

Hence, taking into account the additivity of F on $\operatorname{lin}\{u\}$, $\operatorname{lin}\{v\}$ and $\operatorname{lin}\{\lambda u + \mu v\}$, we obtain the required result: F is additive on $P = \operatorname{lin}\{u, v\}$.

COROLLARY 3.2. *If all $x \in X$ are isotropic and $\dim X_\phi^* \geqq 3$, then*

$$\operatorname{Hom}_{\perp\phi}(X, Y) = \operatorname{Hom}(X, Y).$$

PROOF. Let $F \in \operatorname{Hom}_{\perp\phi}(X, Y)$ and $P \in \mathcal{P}$. Since $\dim\{\phi_x | x \in P\} \leqq 2$, there exists $z \in X$ such that $\phi_z \notin \{\phi_x | x \in P\}$. Therefore Proposition 3.1 implies the additivity of F on P, and because of the arbitrary choice of P, F is additive on the whole X.

LEMMA 3.3. *Let $P \in \mathcal{P}$ and suppose that each $x \in P$ is isotropic but ϕ does not vanish identically on $P \times P$. If the ϕ-orthogonality on X is symmetric and $X_\phi^* = \{\phi_x | x \in P\}$, then there exists a unique linear projection $L\colon X \to P$ such that $F \in \operatorname{Hom}_{\perp\phi}(X, Y)$ if and only if it has the form*

$$F(z) = H(L(z)) + A(z - L(z)), \qquad z \in X, \tag{3.1}$$

where $H\colon P \to Y$ is a mapping, additive on any straight line through the origin and $A\colon N = \{z \in X | L(z) = 0\} \to Y$ is an additive mapping.

PROOF. Lemma 2.4 ensures the existence of vectors $u, v \in P$ such that $\phi(u, v) = 1 = -\phi(v, u)$ and by the same Lemma, $\bar{\alpha} = \alpha$ for all $\alpha \in \Phi$. This implies that

$$\phi(\lambda_1 u + \mu_1 v, \lambda_2 u + \mu_2 v) = \lambda_1\mu_2 - \lambda_2\mu_1, \qquad \lambda_1, \mu_1, \lambda_2, \mu_2 \in \Phi,$$

and so

$$\phi(t, x) = 0 \iff x \in \operatorname{lin}\{t\}, \quad t, x \in P, \ t \neq 0. \tag{3.2}$$

Hence for $x_1, x_2 \in P$, we have

$$\phi(t, x_1) = \phi(t, x_2), \qquad t \in X \iff x_1 = x_2. \tag{3.3}$$

By our assumption, if $z \in X$, then $\phi_z = \phi_x$ for some $x \in P$, i.e. there is a mapping $L\colon X \to P$ such that

$$\phi(t, z) = \phi(t, L(z)), \qquad t, z \in X.$$

The uniqueness of L is clear from (3.3). Next we show that L is a linear projection. Indeed, for any $z \in X$, $L(z)$ and $L(L(z))$ are in P and

$$\phi(t, L(z)) = \phi(t, L(L(z))), \qquad t \in X,$$

whence by (3.3), $L(z) = L(L(z))$. On the other hand, if $z_1, z_2 \in X$, $\zeta_1, \zeta_2 \in \Phi$, then simple computations show that

$$\phi(t, L(\zeta_1 z_1 + \zeta_2 z_2) - \zeta_1 L(z_1) - \zeta_2 L(z_2)) = 0 = \phi(t, 0), \qquad t \in X.$$

This together with (3.3) give the linearity of L. Now introducing the null space $N = \{z \in X | L(z) = 0\}$, any vector $z \in X$ can uniquely be written in the form $z = x + w$ with $x = L(z) \in P$ and $w = z - L(z) \in N$, i.e. $X = P \oplus N$. Furthermore, for any $z \in X$ and $w \in N$, we have $\phi(z, w) = = \phi(z, L(w)) = \phi(z, 0) = 0$. Thus taking into account the symmetry of $\perp^\phi$, we obtain

$$\phi(z, w) = \phi(w, z) = 0, \qquad z \in X, \ w \in N. \tag{3.4}$$

This enables us to represent the bilinear form ϕ as follows: If $z_i \in X$, $L(z_i) = \lambda_i u + \mu_i v \in P$ and $w_i = z_i - L(z_i) \in N$ $(i = 1, 2)$, then

$$\phi(z_1, z_2) = \phi(\lambda_1 u + \mu_1 v + w_1, \lambda_2 u + \mu_2 v + w_2) = \lambda_1 \mu_2 - \lambda_2 \mu_1. \tag{3.5}$$

Finally, let $F \in \mathrm{Hom}_{\perp^\phi}(X, Y)$. Then for any $z \in X$, we have $P \ni L(z) \perp^\phi (z - L(z)) \in N$ and so the orthogonal additivity of F yields

$$F(z) = F(L(z)) + F(z - L(z)).$$

Now define $H \colon P \to Y$ by $H(x) = F(x)$ for all $x \in P$ and $A \colon N \to Y$ by $A(w) = F(w)$ for all $w \in N$. Since every vector in P is isotropic, H is additive on each one dimensional subspace of P and by (3.4), $A \in \mathrm{Hom}(N, Y)$ giving formula (3.1).

Conversely, consider an arbitrary mapping F of the form (3.1). Then for $(z_1, z_2) \in \perp^\phi$, formulas (3.5) and (3.2) imply that $L(z_1)$ and $L(z_2)$ are linearly dependent and so

$$F(z_1 + z_2) = H(L(z_1 + z_2)) + A([z_1 + z_2] - L(z_1 + z_2)) =$$

$$H(L(z_1) + L(z_2)) + A([z_1 - L(z_1)] + [z_2 - L(z_2)]) =$$

$$= [H(L(z_1)) + A(z_1 - L(z_1))] + [H(L(z_2)) + A(z_2 - L(z_2))] = F(z_1) + F(z_2),$$

i.e. $F \in \mathrm{Hom}_{\perp^\phi}(X, Y)$.

4. Supplementary results

EXAMPLE 4.1. The conditions of Lemmas 2.1 and 2.2 are not necessary for the additivity of an arbitrary $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ on some $P \in \mathcal{P}$. Let, e.g., $X = \mathbf{R}^4$ and $\phi\colon X \times X \to \mathbf{R}$ be defined by

$$\phi((\xi_1;\xi_2;\xi_3;\xi_4),(\eta_1;\eta_2;\eta_3;\eta_4)) = \xi_1\eta_2 - \xi_2\eta_1 + \xi_3\eta_4 - \xi_4\eta_3.$$

Then $\perp^\phi$ is symmetric on the plane $P = \mathrm{lin}\{(1;0;0;0),(0;1;0;0)\}$ (in fact on the whole X) and $\perp^\phi{}_P = \emptyset$, however every vector in X is isotropic and for $z = (0;0;1;0)$, we have

$$\phi_z \notin \{\phi_x | x \in P\}.$$

Thus by Proposition 3.1, F is additive on P.

EXAMPLE 4.2. The condition of Proposition 3.1 is not necessary for the additivity of an arbitrary $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ on some $P \in \mathcal{P}$. Indeed, e.g. consider on $X = \mathbf{R}^3$ the bilinear form $\phi\colon X \times X \to \mathbf{R}$ defined by

$$\phi((\xi_1;\xi_2;\xi_3),(\eta_1;\eta_2;\eta_3)) = \xi_1\eta_2 - (\xi_2 - \xi_3)\eta_1.$$

Then every $x \in P = \mathrm{lin}\{1;0;0),(0;1;0)\}$ is isotropic and for any $t = (\tau_1;\tau_2;\tau_3)$, $z = (\zeta_1;\zeta_2;\zeta_3) \in X$, we have

$$\phi(t,z) = \phi((\tau_1;\tau_2;\tau_3),(\zeta_1;\zeta_2;0)),$$

i.e., $\phi_z \in \{\phi_x | x \in P\}$. However

$$\phi((1;1;0),(1;2;1)) \neq 0 = \phi((1;2;1),(1;1;0)),$$

showing that $\perp^\phi$ is not symmetric and so by Theorem 2.6, F is additive.

However, in this context we can state the following

PROPOSITION 4.3. *Suppose that* $\mathrm{Hom}(\Phi,Y) \neq \{\mathbf{0}\}$ *and* $P \in \mathcal{P}$. *Then every* $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ *is additive on* P *if and only if one of the conditions below is satisfied:*

(i) $\phi(x,y) = 0$ *for all* $x, y \in P$;

(ii) $\perp^\phi$ *is not symmetric;*

(iii) *Every* $x \in P$ *is isotropic and* $\phi_z \notin \{\phi_x | x \in P\}$ *for some* $z \in X$.

PROOF. *Sufficiency.* It is clear from Lemma 2.1, Theorem 2.6 and Proposition 3.1 above.

Necessity. If none of the conditions (i), (ii) and (iii) holds, then $\perp^\phi$ is symmetric and

— *either* $\phi(v,v) \neq 0$ *for some* $v \in P$, when $F\colon X \to Y$ defined with the aid of some $a \in \mathrm{Hom}(\Phi,Y) \setminus \{\mathbf{0}\}$ by

$$F(x) = a(\phi(x,x)), \qquad x \in X,$$

is not additive on P, but anyway $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$.

— *or each $x \in P$ is isotropic,* when conditions of Lemma 3.3 are satisfied and so one can easily define a mapping $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ of form (3.1) which is not additive on P.

REMARK 4.4. In the rest of the paper we are going to summarize our results concerning ϕ-orthogonally additive mappings. There may occur exactly the following possibilities:

a) *$\perp^\phi$ is not symmetric:* Then

$$\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y).$$

b) *$\perp^\phi$ is symmetric:* Now

b/1) *either each $x \in X$ is isotropic:* Then there may happen that

b/1) (i) $\dim X^*_\phi = 0$: Then

$$\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y).$$

b/1) (ii) $\dim X^*_\phi = 1$: This is impossible, since then $X^*_\phi = \mathrm{lin}\{\phi_x\}$ and $\phi(t,x) \neq 0$ for some $t, x \in X$, and so we could choose $\lambda \in \Phi$ such that $\lambda\phi(x,x) = \lambda\phi_x(x) = \phi_t(x) = \phi(x,t) \neq 0$, i.e., x would not be isotropic.

b/1) (iii) $\dim X^*_\phi = 2$: Then $X^*_\phi = \mathrm{lin}\{\phi_x, \phi_y\}$ for some $x, y \in X$. Obviously x and y are linearly independent and so $\mathrm{lin}\{x,y\} = P \in \mathcal{P}$. Then $X^*_\phi = \{\phi_p | p \in P\}$. Since $\phi_x \neq \mathbf{0}$, there exists $t \in X$ with $\phi(t,x) \neq 0$ and so by the isotropy, $0 \neq \phi(t,x) = \phi(t+x, t+x) - \phi(x,t) = \phi(x,p)$ for some $p \in P$, i.e., ϕ does not vanish on $P \times P$. Thus Lemma 3.3 provides a unique linear projection $L\colon X \to P$ such that $F \in \mathrm{Hom}_{\perp^\phi}(X,Y)$ if and only if

$$F(z) = H(L(z)) + A(z - L(z)), \qquad z \in X,$$

where $H\colon P \to Y$ is a mapping, additive on any straight line through the origin and $A\colon N = \{z \in X | L(z) = 0\} \to Y$ is an additive mapping.

b/1) (iv) $\dim X^*_\phi \geqq 3$: Then

$$\mathrm{Hom}_{\perp^\phi}(X,Y) = \mathrm{Hom}(X,Y).$$

b/2) *or there is a non-isotropic vector $v \in X$:* Then we have to deal with the cases:

b/2) (i) $\dim X^*_\phi = 1$: Then $X^*_\phi = \mathrm{lin}\{\phi_v\}$. This means that for any $y \in X$ there is a unique $\mu(y) \in \Phi$ with $\phi(x,y) = \phi_y(x) = \mu(y)\phi_v(x) = \mu(y)\phi(x,v)$ whenever $x \in X$. For $x = v$ this yields $\phi(v,y) = \mu(y)\phi(v,v)$, i.e., $\mu(y) = \phi(v,y)/\phi(v,v)$. Now consider the linear functional $\lambda\colon X \to \Phi$ defined by $\lambda(x) = \phi(x,v)/\phi(v,v)$, $x \in X$. It is clear that for any $x \in X$, $\phi(x - \lambda(x)v, v) = 0$, whence using the symmetry of $\perp^\phi$, we have $\phi(v,x) - \overline{\lambda(x)}\phi(v,v) = \phi(v, x - \lambda(x)v) = 0$, i.e., $\overline{\lambda(x)} = \mu(x)$. Thus $\phi(x,y) =$

$= \mu(y)\phi(x,v) = \lambda(x)\overline{\lambda(y)}\phi(v,v)$ whenever $x, y \in X$. Hence one can see easily that $F \in \mathrm{Hom}_{\perp\phi}(X,Y)$ if and only if

$$F(x) = f(\lambda(x)), \qquad x \in X,$$

with arbitrary $f\colon \Phi \to Y$ satisfying $f(0) = 0$.

b/2) (ii) dim $X_\phi^* \geqq 2$: If $\Phi \neq GF(3)$, then by [3], Corollary 2.9, the odd mappings in $\mathrm{Hom}_{\perp\phi}(X,Y)$ are additive, while the even ones are quadratic. Moreover, using the notations $\Omega = \{\alpha \in \Phi | \alpha = \bar{\alpha}\}$, $\Omega_+ = \{\mu\bar{\mu} | \mu \in \Phi\}$, $\Omega_- = -\Omega_+$ and assuming that

$$\Omega_+ + \Omega_+ \subset \Omega_+; \quad \Omega = \Omega_- \cup \Omega_+; \quad \Omega_+ = \{\omega^2 | \omega \in \Omega_+\},$$

for 2-torsion free Y, Corollary 3.5 in [3] implies that $F \in \mathrm{Hom}_{\perp\phi}(X,Y)$ if and only if

$$F(x) = A(x) + a(\phi(x,x)), \qquad x \in X,$$

with some $A \in \mathrm{Hom}(X,Y)$ and $a \in \mathrm{Hom}(\Phi,Y)$.

REMARK 4.5. It is worth mentioning that the planes P defined in the Examples 4.1, 4.2, contain no ϕ-orthogonal base at all. This is the reason why the theory of abstract orthogonality spaces developed in [1] can not be applied for non-symmetric or totally isotropic ϕ-orthogonality. Anyway, the results of Sundaresan–Kapoor [2] are completely covered now.

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(Received July 24, 1989; revised November 2, 1989)

INSTITUTE OF MATHEMATICS
L. KOSSUTH UNIVERSITY
EGYETEM TÉR 1
H–4010 DEBRECEN, PF. 12
HUNGARY

Acta Math. Hung.
59 (1–2) (1992), 11–24.

ON SOME PROBLEMS OF P. TURÁN CONCERNING POWER SUMS OF COMPLEX NUMBERS

L. ERDŐS (Budapest)

1. Introduction

Let $z_1, z_2, \dots, z_n$ be complex numbers satisfying

$$\max_{j=1,2,\dots,n} |z_j| = |z_1| = 1, \tag{1.1}$$

and, for arbitrary positive integer ν, let $s_\nu = z_1^\nu + z_2^\nu + \dots + z_n^\nu$. We are interested in the asymptotic behaviour of the expression

$$\min_{z_1,\dots,z_n} \max_{\nu\in H} |s_\nu| \tag{1.2}$$

under the condition (1.1), where H denotes a fixed finite subset of the positive integers. The minimum in (1.2) exists because of the compactness of the domain (1.1) in C^n. One can easily prove by rotation and homothetic transformation that the condition (1.1) can be exchanged for

$$z_1 = 1 \tag{1.3}$$

without changing the value of (1.2).

Several papers deal with the case $H = \{1, 2, \dots, n\}$ (see P. Turán's book [6]). F. V. Atkinson [1] has proved that

$$\alpha_n := \min_{z_1=1} \max_{\nu=1,2,\dots,n} |s_\nu|$$

is greater than $1/6$. On the other hand, a suitably chosen set $\{z_1, z_2, \dots, z_n\}$ proves that $\alpha_n < 1$ (for the best known result see [3]).

The situation essentially changes omitting the number 1 from H and replacing it by integers greater than n. Let us denote

$$\beta_n := \min_{z_1=1} \max_{\nu=2,3,\dots,n+1} |s_\nu|. \tag{1.4}$$

P. Erdős has proved (see [6], Theorem 4.1) that, for large enough n,

$$\beta_n \leqq 2(n+1)^2 e^{-\vartheta n} < \frac{1}{1.32^n},$$

where ϑ is the only positive root of the equation

$$1 + \vartheta + \log\vartheta = 0 \tag{1.5}$$

($\vartheta = 0.2784\dots$), i.e., β_n is exponentially small.

P. Turán ([6], Problem 6) posed the problem of the exact value of β_n. In our Theorem 1 we prove the following estimations for β_n.

THEOREM 1. *Under the assumption* (1.1) *and with the notations* (1.4) *and* (1.5), *we have*

$$n^{-0.7823}\exp(-2\vartheta n) \leqq \beta_n \leqq n^{4.5}\exp(-2\vartheta n) \tag{1.6}$$

for sufficiently large n.

COROLLARY. *For* $n > n_0$, *we have* $\frac{1}{1.746^n} < \beta_n < \frac{1}{1.745^n}$.

V. T. Sós and P. Turán asked for the "smallest" numerical positive value A_1 for which the inequality

$$\max_{\substack{m+1\leqq\nu\leqq m+n\\ \nu \text{ integer}}} |s_\nu| \geqq \left(\frac{n}{A_1(m+n)}\right)^n \tag{1.7}$$

holds for arbitrary nonnegative integer m and positive integer n under the assumption (1.1) (see [7]).

Using the result of P. Erdős quoted above it is trivial that $A_1 > 1.321$. The authors of [7] mentioned that the case $m = 2$ can yield a better lower estimation for A_1. So denoting

$$\gamma_n := \min_{z_1=1}\max_{\nu=3,4,\dots,n+2} |s_\nu| \tag{1.8}$$

they have shown in Section 11 of [7] that a good estimation of the minimal absolute value of the zeros of a certain equation involving Hermite polynomials (see formula (4.8) in [7]) can give an upper estimation for γ_n, and that can yield a better estimation for A_1.

Fulfilling this programme E. Makai proved that, for sufficiently large n,

$$\gamma_n \leqq \frac{1}{1.473^n},$$

which implies that $A_1 \geqq 1.473$ (see [4]).

The idea of the proof of our Theorem 1 can also be applied for the estimation of γ_n. Similar but longer calculations show the following

THEOREM 1′. *We have*

$$\gamma_n \leqq \min_{z_1=1}\max_{\nu=2,3,\dots,n+2} |s_\nu| \leqq \frac{1}{1.745^n}. \tag{1.9}$$

The main difference between our idea and that of V. T. Sós, P. Turán and E. Makai is that we do not stick to choose $s_3, s_4, \dots, s_{n+1}$ all to be zero as they did but we exploit the possibility of choosing certain s_i's to be small but not zero.

In Section 3 we answer another question of P. Turán. Problem 7 of his book [6] asked whether the expression (1.2) is exponentially small under the assumption (1.1) if $H = \{2, 3, \dots, n+2\}$. Theorem 1′ shows that the answer is yes. Following the idea of P. Erdős' proof of $\beta_n \leqq 1/1.32^n$ we give a positive answer in the following *general* form.

THEOREM 2. *Under the assumption* (1.1) *if* $k \leqq 0.2783\frac{n}{\log n}$, *then*

$$\min_{z_1=1}\ \max_{\nu=2,3,\dots,n+k} |s_\nu|$$

is exponentially small.

2. Proof of Theorem 1

The lower estimation. Let us denote $\varepsilon(n) = \exp(-2\vartheta n - c\log n)$ where the number c will be chosen later. We are going to prove that $\beta_n \geqq \varepsilon(n)$. On the contrary let us assume that $|s_2|, |s_3|, \dots, |s_{n+1}|$ are smaller than $\varepsilon(n)$.

By Waring's formulae (see L. Rédei [5]) the coefficients a_k of the polynomial $(z-z_1)(z-z_2)\dots(z-z_n) = z^n + a_1 z^{n-1} + \dots + a_n$ can be expressed by $s_1, s_2, \dots, s_n$ in the following way:

$$(2.1)\qquad a_k = \sum_{\substack{i_1,i_2,\dots,i_n \geqq 0\\ i_1+2i_2+\dots+ni_n=k}} \frac{1}{i_1!i_2!\dots i_n!}\left(-\frac{s_1}{1}\right)^{i_1}\left(-\frac{s_2}{2}\right)^{i_2}\dots\left(-\frac{s_n}{n}\right)^{i_n}$$

$$(k = 1, 2, \dots, n).$$

For $k = 1$ we have $a_1 = (-s_1)$. For $k \geqq 2$ separating the summands corresponding to $i_1 = k$, $i_2 + i_3 + \dots + i_n = 1$ and $i_2 + \dots + i_n \geqq 2$ we obtain that

$$(2.2)\qquad a_k = \frac{(-s_1)^k}{k!} + \sum_{i_1=0}^{k-2}\frac{(-s_1)^{i_1}}{i_1!}\left(\frac{-s_{k-i_1}}{k-i_1}\right) +$$

$$+\sum_{i_1=0}^{k-2}\frac{(-s_1)^{i_1}}{i_1!}\sum_{\substack{i_2,\dots,i_n\geqq 0\\ i_2+\dots+i_n\geqq 2\\ 2i_2+\dots+ni_n=k-i_1}}\frac{1}{i_2!\dots!i_n!}\left(\frac{-s_2}{2}\right)^{i_2}\dots\left(\frac{-s_n}{n}\right)^{i_n}.$$

(It is trivial that i_1 cannot be $k-1$ in (2.1) because $2i_2 + \dots + ni_n$ cannot be 1.)

Using the roughest estimation for the third sum we get

$$(2.3)\qquad \left|a_k - \frac{(-s_1)^k}{k!}\right| \leqq \sum_{i=0}^{k-2}\frac{|s_1|^i}{i!}\varepsilon(n) + \sum_{i=0}^{k-2}\frac{|s_1|^i}{i!}\sum_{\substack{i_2,\dots,i_n\geqq 0\\ i_2+\dots+i_n\geqq 2\\ 2i_2+\dots+ni_n=k-i}}\varepsilon(n)^{i_2+\dots+i_n}.$$

The number of summands in the third sum is equal to the number of partitions of $k-i$ not containing 1 as a summand and containing at least two terms. This number is smaller than that of all partitions of n which can be estimated by $\exp(d\sqrt{n})$ where d is a fixed positive number (see [2]).

Thus from (2.3)

$$\left|a_k - \frac{(-s_1)^k}{k!}\right| \leqq e^{|s_1|}\varepsilon(n) + e^{|s_1|}\varepsilon(n)^2 e^{d\sqrt{n}} = e^{|s_1|}\varepsilon(n)t_n, \tag{2.4}$$

where t_n converges to 1 exponentially fast.

By the remark in Section 1 about the conditions (1.1) and (1.3) we may assume that $z_1 = 1$, consequently $z_1 = 1$ is a root of the equation $z^n + a_1 z^{n-1} + \ldots + a_n = 0$, thus

$$a_n = -(1 + a_1 + \ldots + a_{n-1}) = -\sum_{i=0}^{n-1} \frac{(-s_1)^i}{i!} + e^{|s_1|}\varepsilon(n)nr_n, \tag{2.5}$$

where $|r_n| \leqq 1 + o(1)$.

Since $|s_1| \leqq |z_1| + |z_2| + \ldots + |z_n| \leqq n$, an easy computation shows that

(2.6)
$$\left|e^{-s_1} - \sum_{i=0}^{n-1} \frac{(-s_1)^i}{i!}\right| \leqq \sum_{i=n}^{\infty} \frac{|s_1|^i}{i!} \leqq \frac{|s_1|^n}{n!}\left(1 + \frac{|s_1|}{n+1} + \frac{|s_1|^2}{(n+1)^2} + \ldots\right) =$$
$$= \frac{|s_1|^n(n+1)}{n!(n+1-|s_1|)}.$$

Thus by (2.5)–(2.6)

$$|a_n| \geqq |e^{-s_1}| - \frac{|s_1|^n(n+1)}{n!(n+1-|s_1|)} - e^{|s_1|}\varepsilon(n)n|r_n|. \tag{2.7}$$

Using (2.4) and Newton–Girard's formula

$$s_{n+1} + a_1 s_n + \ldots + a_n s_1 = 0$$

(see L. Rédei [5]) we get

$$|a_n||s_1| \leqq |s_{n+1}| + |a_1||s_n| + \ldots + |a_{n-1}||s_2| \leqq (1 + |a_1| + \ldots + |a_{n-1}|)\varepsilon(n) \leqq$$
$$\leqq \varepsilon(n)\Big(\sum_{i=0}^{n-1} \frac{|s_1|^i}{i!} + e^{|s_1|}\varepsilon(n)nt_n\Big).$$

Comparing this inequality with (2.7) and using $|e^{-s_1}| \geqq e^{-|s_1|}$, we obtain that

(2.8)
$$|s_1|\left(e^{-|s_1|}-\frac{|s_1|^n(n+1)}{n!(n+1-|s_1|)}-e^{|s_1|}\varepsilon(n)n|r_n|\right)<\varepsilon(n)\Big(e^{|s_1|}+e^{|s_1|}\varepsilon(n)nt_n\Big).$$

First case: $|s_1|\leqq \vartheta n-c_1\log n$ ($c_1>0$ will be chosen later). It can be assumed that $|s_1|>1/6$ since Atkinson's result [1] claims that $\max\limits_{\nu=1,\dots,n}|s_\nu|>$ $>1/6$ and $|s_2|,|s_3|,\dots,|s_n|$ are small. (Instead of Atkinson's deep result, one can use (2.7) and (2.4) with $k=n>n_0$.)

In this case for large enough n

$$e^{|s_1|}\varepsilon(n)n|r_n|\leqq \exp\{-\vartheta n+(1-c_1-c)\log n+\log(1+o(1))\} \tag{2.9}$$

and by Stirling's formula

(2.10)
$$\frac{|s_1|^n(n+1)}{n!(n+1-|s_1|)}\leqq \exp\Big\{n\Big(\log\Big(\vartheta-c_1\frac{\log n}{n}\Big)+1\Big)+\log\frac{1}{1-\vartheta}\Big\}\leqq$$
$$\leqq \exp\Big\{\log\frac{1}{1-\vartheta}-n\vartheta\Big\}$$

since $\vartheta+\log\Big(\vartheta-c_1\frac{\log n}{n}\Big)+1\leqq 0$ by the definition of ϑ.

Using these estimations in (2.8)

$$\frac{1}{6}\Big\{\exp(-\vartheta n+c_1\log n)-\exp\Big(\log\frac{1}{1-\vartheta}-n\vartheta\Big)- \tag{2.11}$$
$$-\exp(-\vartheta n+(1-c_1-c)\log n+\log(1+o(1)))\Big\}<$$

$$<\exp(-2\vartheta n-c\,\log n)\{\exp(\vartheta n-c_1\log n)+\exp(-\vartheta n+(1-c_1-c)\log n+\log t_n)\}.$$

Since $\log(1+o(1))\to 0$ and $\log t_n\to 0$, the main term on the left side is $(1/6)\exp(-\vartheta n+c_1\log n)$ if

$$c_1>1-c_1-c \tag{2.12}$$

and the main term on the right side is $\exp(-\vartheta n-(c+c_1)\log n)$. Thus we get a contradiction if

$$c_1>-(c+c_1) \tag{2.13}$$

but (2.12) guarantees (2.13).

Second case: $|s_1|>\vartheta n-c_1\log n$. The estimation (2.4) yields that

$$|a_n|\geqq \frac{|s_1|^n}{n!}-e^{|s_1|}\varepsilon(n)t_n. \tag{2.14}$$

We repeat the calculation of the first case replacing the estimation of $|a_n|$ in (2.7) by (2.14). Similarly to (2.8) we get

$$|s_1|\Big(\frac{|s_1|^n}{n!} - e^{|s_1|}\varepsilon(n)t_n\Big) < \varepsilon(n)\Big(e^{|s_1|} + e^{|s_1|}\varepsilon(n)nt_n\Big) \tag{2.15}$$

and

$$\frac{1}{6}\frac{|s_1|^n}{n!} < e^{|s_1|}\varepsilon(n)\Big(1 + \frac{t_n}{6} + \varepsilon(n)nt_n\Big). \tag{2.16}$$

By Stirling's formula, we have

$$\exp\Big(-|s_1| + n\log|s_1| - n\log n + n - \frac{1}{12n} - \log\sqrt{2\pi n}+ \tag{2.17}$$

$$+2\vartheta n + c\log n\Big) \leqq 6\Big(\frac{7}{6} + o(1)\Big).$$

Simple differentiation shows that in the interval $[\vartheta n - c_1\log n, n]$ the function $f(x) = -x + n\log x$ is increasing, so the minimal value is attained at $x = \vartheta n - c_1\log n$. Since $|s_1|$ is in this interval by (1.1), we have

$$\exp\Big\{-\vartheta n + c_1\log n + n\log(\vartheta n - c_1\log n) - n\log n + n - \frac{1}{12n}-$$

$$-\log\sqrt{2\pi n} + 2\vartheta n + c\log n\Big\} \leqq 7 + o(1),$$

or

$$\exp\Big\{n\Big(\vartheta + \log\Big(\vartheta - c_1\frac{\log n}{n}\Big) + 1\Big) + \Big(c_1 + c - \frac{1}{2}\Big)\log n- \tag{2.18}$$

$$-\frac{1}{12n} - \log\sqrt{2\pi}\Big\} \leqq 7 + o(1).$$

But

$$n\Big(\vartheta + \log\Big(\vartheta - c_1\frac{\log n}{n}\Big) + 1\Big) = n\Big(\log\Big(\vartheta - c_1\frac{\log n}{n}\Big) - \log\vartheta\Big) \geqq$$

$$\geqq n\Big(-c_1\frac{\log n}{n}\Big)\max_{\vartheta - c_1\frac{\log n}{n}\leqq x\leqq\vartheta}(\log x)' = -c_1\log n\frac{1}{\vartheta - c_1\frac{\log n}{n}}.$$

Thus from (2.18) we get a contradiction if

$$c_1 + c - \frac{1}{2} > \frac{c_1}{\vartheta}. \tag{2.19}$$

Now we can choose the constants c_1 and c on the grounds of (2.12) and (2.19). These inequalities will be satisfied if

$$c_1 = \frac{\vartheta}{2(\vartheta+1)} \quad \text{and} \quad c = 0.7823 > \frac{1}{\vartheta+1}.$$

The upper estimation. We are going to prove the existence of complex numbers $z_1, z_2, \dots, z_n$ such that $z_1 = 1$ and $|s_2|, |s_3|, \dots, |s_{n+1}| \leqq \exp(-2\vartheta n + 4.5 \log n)$ for large enough n (by the remark in Section 1 about (1.1) and (1.3) it is enough to guarantee (1.3) instead of (1.1)).

From Waring's formulae it follows that for an arbitrary system of the sums of the first n powers there uniquely exists a system of numbers $z_1, z_2, \dots, z_n$ resulting the given sums of powers. So we shall give explicitly the numbers $s_1, s_2, \dots, s_n$ instead of $z_1, z_2, \dots, z_n$.

Let $\varepsilon(n) = \exp(-2\vartheta n + 4.5 \log n)$; we are going to choose $s_2, s_3, \dots, s_{n-1}$ to be smaller than $\varepsilon(n)$ in absolute value. For the moment we do not care about s_n because it plays no role in (2.2) for $k = 1, 2, \dots, n-1$. Assuming this choice of $s_2, s_3, \dots, s_{n-1}$ we have an estimation similar to (2.3) using the partitions. For $2 \leqq k \leqq n-1$,

$$a_k = \frac{(-s_1)^k}{k!} + \sum_{i=0}^{k-2} \frac{(-s_1)^i}{i!} \frac{-s_{k-i}}{k-i} + O\left(e^{|s_1|}\varepsilon^2(n) e^{d\sqrt{n}}\right). \tag{2.20}$$

Let us choose $s_1 = [\vartheta n]$. Then $e^{|s_1|}\varepsilon^2(n) e^{d\sqrt{n}} n \leqq e^{-2.5\vartheta n}$. Now we choose the value of a_n to guarantee $z_1 = 1$ to be a root:

$$a_n := -(1 + a_1 + \ldots + a_{n-1}) = \tag{2.21}$$

$$= -\Big\{\sum_{k=0}^{n-1} \frac{(-s_1)^k}{k!} + \sum_{k=2}^{n-1}\sum_{i=0}^{k-2} \frac{(-s_1)^i}{i!}\Big(\frac{-s_{k-i}}{k-i}\Big)\Big\} + O(e^{-2.5\vartheta n}) =$$

$$= -\Big\{\sum_{k=0}^{n-1} \frac{(-s_1)^k}{k!} + \sum_{i=0}^{n-3} \frac{(-s_1)^i}{i!} \sum_{j=2}^{n-1-i} \frac{-s_j}{j}\Big\} + O(e^{-2.5\vartheta n}).$$

By the Newton–Girard's formulae and (2.20)

$$-s_n = a_1 s_{n-1} + \ldots + a_{n-1}s_1 + n a_n = \sum_{k=1}^{n-1} \frac{(-s_1)^k}{k!} s_{n-k} +$$

$$+ \sum_{k=2}^{n-1}\sum_{i=0}^{k-2} \frac{(-s_1)^i}{i!}\Big(\frac{-s_{k-i}}{k-i}\Big) s_{n-k} + O(e^{-2.5\vartheta n}) + n a_n,$$

consequently,

$$(2.22)\qquad -s_n=\sum_{k=1}^{n-1}\frac{(-s_1)^k}{k!}s_{n-k}+\sum_{i=0}^{n-3}\frac{(-s_1)^i}{i!}\left(\frac{-s_{n-1-i}}{n-1-i}\right)s_1+$$

$$+\sum_{k=2}^{n-2}\sum_{i=0}^{k-2}\frac{(-s_1)^i}{i!}\left(\frac{-s_{k-i}}{k-i}\right)s_{n-k}+O(e^{-2.5\vartheta n})+na_n.$$

The double sum on the right side can be estimated by $e^{|s_1|}\varepsilon^2(n)n$, which can be endorsed to the error term. Expressing s_{n+1} we get

$$(2.23)\qquad -s_{n+1}=a_1s_n+\ldots+a_ns_1=\sum_{k=2}^{n-1}\frac{(-s_1)^k}{k!}s_{n+1-k}+$$

$$+\sum_{k=2}^{n-1}\sum_{i=0}^{k-2}\frac{(-s_1)^i}{i!}\left(\frac{-s_{k-i}}{k-i}\right)s_{n+1-k}+O(e^{-2.5\vartheta n})+s_1a_n-s_1s_n.$$

The double sum on the right side can be treated as in (2.22). So it is enough to guarantee that

$$(\mathrm{I})\qquad \sum_{k=0}^{n-1}\frac{(-s_1)^k}{k!}+\sum_{i=0}^{n-3}\frac{(-s_1)^i}{i!}\sum_{j=2}^{n-1-i}\left(\frac{-s_j}{j}\right)=0,$$

$$(\mathrm{II})\qquad \sum_{k=1}^{n-1}\frac{(-s_1)^k}{k!}s_{n-k}+\sum_{i=0}^{n-3}\frac{(-s_1)^i}{i!}\left(\frac{-s_{n-i-1}}{n-i-1}\right)s_1=0,$$

$$(\mathrm{III})\qquad \sum_{k=2}^{n-1}\frac{(-s_1)^k}{k!}s_{n+1-k}=0.$$

Indeed, (I) implies that $|a_n|=O(e^{-2.5\vartheta n})$, then this and (II) imply that $|s_n|=O(ne^{-2.5\vartheta n})<\varepsilon(n)$, finally, these both and (III) imply that $|s_{n+1}|=$ $=O(n^2e^{-2.5\vartheta n})<\varepsilon(n)$.

All we have to do is to prove the existence of a "small" solution $s_2,s_3,\ldots,$ s_{n-1} of the equations (I)-(III). The number A will be chosen as a suitable integer not far from $n-s_1$. All the $s_2,s_3,\ldots,s_{n-1}$ are to be zero except s_A, s_{A+1}, s_{A+2}. Thus (I)–(III) take the following form:

$$(\mathrm{I}')\qquad \sum_{i=0}^{n-A-3}\frac{(-s_1)^i}{i!}\left(\frac{s_A}{A}+\frac{s_{A+1}}{A+1}+\frac{s_{A+2}}{A+2}\right)+\frac{(-s_1)^{n-A-2}}{(n-A-2)!}\left(\frac{s_A}{A}+\frac{s_{A+1}}{A+1}\right)+$$

$$+\frac{(-s_1)^{n-A-1}}{(n-A-1)!}\frac{s_A}{A}=\sum_{k=0}^{n-1}\frac{(-s_1)^k}{k!},$$

$$\text{(II}'\text{)}\quad \frac{s_A}{A}\Big(\frac{-s_1}{n-A}\Big)+\frac{s_{A+1}}{A+1}+\frac{s_{A+2}}{A+2}\Big(\frac{n-A-1}{-s_1}\Big)=\frac{(-s_1)^{A+1}(n-A-1)!}{n!},$$

$$\text{(III}'\text{)}\qquad \Big(\frac{-s_1}{n-A+1}\Big)s_A+s_{A+1}+\Big(\frac{n-A}{-s_1}\Big)s_{A+2}=0.$$

Anticipating that for suitable A, this system of linear equations has a nonzero determinant (cf. (2.33)) we obtain that

$$\text{(2.24)}\qquad s_A=A\Big\{\sum_{k=0}^{n-1}\frac{(-s_1)^k}{k!}-\frac{(-s_1)^{n-1}(n-A)(n-A-1)(A+2)}{(n+1)!}+$$

$$+\sum_{i=0}^{n-A-3}\frac{(-s_1)^i}{i!}\frac{(-s_1)^{A+1}(n-A-1)!}{(n+1)!}(-s_1(A+1)-(n-A)(A+2))\Big\}\times$$

$$\times\Big\{\sum_{i=0}^{n-A-3}\frac{(-s_1)^i}{i!}\Big(1+\frac{2s_1}{n-A+1}+\frac{s_1^2}{(n-A)(n-A+1)}\Big)+$$

$$+\frac{(-s_1)^{n-A-2}}{(n-A-2)!}\Big(1+\frac{2s_1}{n-A+1}\Big)+\frac{(-s_1)^{n-A-1}}{(n-A-1)!}\Big\}^{-1},$$

$$\text{(2.25)}\qquad s_{A+1}=s_1s_A\frac{2(A+1)}{A(n-A+1)}+\frac{(-s_1)^{A+1}(n-A)!(A+1)(A+2)}{(n+1)!},$$

(2.26)

$$s_{A+2}=s_1^2s_A\frac{A+2}{A(n-A)(n-A+1)}-\frac{(-s_1)^{A+2}(n-A-1)!(A+1)(A+2)}{(n+1)!}.$$

The second terms in (2.25) and (2.26) can be estimated easily, anticipating that A will be chosen $n-s_1$ or $n-s_1+1$. The second term in (2.26) is not smaller in absolute value than that of (2.25), so we are dealing with it:

$$\Big|\frac{(-s_1)^{A+2}(n-A-1)!(A+1)(A+2)}{(n+1)!}\Big|\leqq$$

$$\leqq\frac{s_1^{n-s_1+2}(s_1-1)!(n-s_1)^2(1+o(1))}{(n+1)!}\leqq$$

$$\leqq\frac{(\vartheta n)^n(\vartheta n)^{2.5}n^2(1-\vartheta)^2e^{n(1-\vartheta)}(1+o(1))}{(n+1)\sqrt{n}n^n}=$$

$$=\vartheta^{2.5}(1-\vartheta)^2n^3e^{-2\vartheta n}(1+o(1))\leqq\frac{1}{2}\exp(-2\vartheta n+3\log n).$$

Thus from (2.25) and (2.26) (for $A = n - s_1$ and $A = n - s_1 + 1$) we get

$$(2.27) \qquad |s_{A+1}| \leqq (2 + o(1))|s_A| + \frac{1}{2}\exp(-2\vartheta n + 3\log n),$$

$$(2.28) \qquad |s_{A+2}| \leqq (1 + o(1))|s_A| + \frac{1}{2}\exp(-2\vartheta n + 3\log n).$$

Now we estimate s_A from (2.24). We start with the first $\{\dots\}$ consisting of three terms, T_1, T_2, T_3. The first term T_1 can be treated similarly to (2.6):

$$|T_1| = \left|\sum_{k=0}^{n-1} \frac{(-s_1)^k}{k!}\right| \leqq e^{-s_1} + \frac{{s_1}^n}{n!}\frac{n}{n - s_1} \leqq \exp(-\vartheta n + 1) +$$

$$+ \exp(n(\log \vartheta + 1))\frac{1}{1 - \vartheta}.$$

Since $-\vartheta = \log \vartheta + 1$,

$$(2.29) \qquad |T_1| \leqq \exp(-\vartheta n)\Big(e + \frac{1}{1 - \vartheta}\Big).$$

The second term T_2 gives:

(2.30)

$$|T_2| = \left|\frac{(-s_1)^{n-1}(n - A)(n - A - 1)(A + 2)}{(n + 1)!}\right| \leqq (e\vartheta)^n \frac{(1 - \vartheta)\vartheta}{\sqrt{2\pi}}\sqrt{n}(1+o(1)) \leqq$$

$$\leqq \frac{1}{2}\exp\Big(-\vartheta n + \frac{1}{2}\log n\Big).$$

The estimation of the third term T_3 yields:

$$|T_3| = \left|\sum_{i=0}^{n-A-3} \frac{(-s_1)^i}{i!}\frac{(-s_1)^{A+1}(n - A - 1)!}{(n + 1)!}(-s_1(A + 1) - (n - A)(A + 2))\right| \leqq$$

$$\leqq e^{s_1}\frac{s_1^{n-s_1+1}(s_1 - 1)!}{(n + 1)!}2\vartheta(1 - \vartheta)n^2(1 + o(1))$$

since $s_1 = (1 + o(1))\vartheta n$, $A = (1 + o(1))(1 - \vartheta)n$. Thus we get

$$(2.31) \quad |T_3| \leqq \exp\Big\{n(1 + \log \vartheta) + \log n + \frac{1}{2}\log \vartheta + \log(2\vartheta(1 - \vartheta)) + o(1)\Big\} <$$

$$< 0.213\exp(-\vartheta n + \log n).$$

Thus the first $\{\dots\}$ in (2.24) is smaller than the sum of the right sides of (2.29), (2.30) and (2.31), i.e.

$$\text{the first } \{\dots\} \text{ in (2.24) } \leqq 0.22\exp(-\vartheta n+\log n). \tag{2.32}$$

The second $\{\dots\}$ of (2.24) cannot be estimated from below in a direct way but we have a certain freedom in the choice of A. We are going to prove that $A=n-s_1$ *or* $A=n-s_1+1$ suits. Let

$$f(A):=\sum_{i=0}^{n-A-3}\frac{(-s_1)^i}{i!}\Big(1+\frac{2s_1}{n-A+1}+\frac{s_1^2}{(n-A)(n-A+1)}\Big)+$$
$$+\frac{(-s_1)^{n-A-2}}{(n-A-2)!}\Big(1+\frac{2s_1}{n-A+1}\Big)+\frac{(-s_1)^{n-A-1}}{(n-A-1)!}.$$

LEMMA. *We have*

$$\max\{|f(n-s_1)|,|f(n-s_1+1)|\}\geqq\frac{1}{2}\exp(\vartheta n-2.5\log n). \tag{2.33}$$

PROOF. Indirectly, let us suppose that

$$\begin{cases} |f(n-s_1)|<\frac{1}{2}\exp(\vartheta n-2.5\log n), \\ |f(n-s_1+1)|<\frac{1}{2}\exp(\vartheta n-2.5\log n). \end{cases} \tag{2.34}$$

In order to eliminate the annoying $\sum\limits_{i=0}^{n-A-3}\frac{(-s_1)^i}{i!}$, let us consider the following linear combination:

$$L:=\frac{4s_1-3}{s_1-1}f(n-s_1)-\frac{4s_1+1}{s_1+1}f(n-s_1+1). \tag{2.35}$$

From (2.34) it follows that

$$|L|\leqq 5\exp(\vartheta n-2.5\log n). \tag{2.36}$$

A simple calculation shows that

$$L=\frac{(-s_1)^{s_1}}{s_1!}\frac{-2}{(s_1-1)(s_1+1)}. \tag{2.37}$$

Thus

$$|L|\geqq\exp(\vartheta n-2.5\log n)\frac{(2+o(1))}{\sqrt{2\pi\vartheta}e\vartheta^2}>7\exp(\vartheta n-2.5\log n)$$

which contradicts (2.36).

Now let us choose A so that $|f(a)| \geqq \frac{1}{2}\exp(\vartheta n - 2.5\log n)$. Then from (2.24) and (2.32) it follows that

$$|s_A| \leqq (n(1-\vartheta)+2)\cdot 0.22\exp(-\vartheta n+\log n)\cdot 2\exp(-\vartheta n + 2.5\log n) \leqq$$
$$\leqq 0.4\exp(-2\vartheta n + 4.5\log n),$$

and from (2.27) and (2.28) we get

$$|s_{A+1}| \leqq \exp(-2\vartheta n + 4.5\log n), \quad |s_{A+2}| \leqq \exp(-2\vartheta n + 4.5\log n).$$

REMARK. The proof of Theorem 1′ is essentially the same but we must choose s_A, s_{A+1}, s_{A+2}, s_{A+3} so that they satisfy a system of linear equations similar to (I)–(III) consisting of four equations. The index A can be chosen either $n - s_1$ or $n - s_1 + 1$ so that the absolute values of the solutions s_A, s_{A+1}, s_{A+2}, s_{A+3} are "small". The calculation is elementary but rather long so we omit the details.

3. Proof of Theorem 2

Let $k_0 = 0.2789\frac{n}{\log n}$. We prove the existence of numbers $z_1 = 1$, $z_2, \dots, z_n$ so that $s_2, s_3, \dots, s_{n+k_0}$ are exponentially small. Instead of $z_1, z_2, \dots, z_n$ we prescribe the values of the sums of powers. Following the idea of P. Erdős let $s_1 = \vartheta n$, $s_2 = s_3 = \dots = s_{n-1} = 0$. From Newton–Girard's formulae one can easily get that $a_j = \frac{(-s_1)^j}{j!}$ $(j = 1, 2, \dots, n-1)$ where the a_j's are the same as at the beginning of Section 2. To guarantee $z_1 = 1$ let us choose

$$a_n := -(1 + a_1 + \dots + a_{n-1}) = -\sum_{j=0}^{n-1}\frac{(-s_1)^j}{j!}.$$

The coefficients $a_1, a_2, \dots, a_n$ being given, all the sums of powers are determined by Newton–Girard's formulae. We are going to prove the following estimation by induction on k:

$$|s_{n+k}| \leqq 2ns_1^k e^{-s_1}\left(\frac{1}{\log 2}\right)^k \quad \text{if} \quad k = 0, 1, \dots, n-1. \tag{3.1}$$

Case $k = 0$. By $s_n + a_1 s_{n-1} + \dots + a_{n-1}s_1 + na_n = 0$ we get

$$s_n = n\sum_{j=0}^{n}\frac{(-s_1)^j}{j!} = n\Big(e^{-s_1} - \sum_{j=n+1}^{\infty}\frac{(-s_1)^j}{j!}\Big).$$

Since $\sum\limits_{j=n+1}^{\infty} \frac{(-s_1)^j}{j!}$ is an alternating series with terms decreasing in absolute value, we obtain that

$$|s_n| \leqq n\Big(e^{-s_1} + \frac{s_1^{n+1}}{(n+1)!}\Big) \leqq ne^{-s_1}\Big(1 + \exp(n(\vartheta + \log\vartheta + 1))\Big) = 2ne^{-s_1}.$$

Case $k = 1$. Expressing s_{n+1} from Newton–Girard's formulae we get

$$|s_{n+1}| = s_1\Big|n\sum_{j=0}^{n}\frac{(-s_1)^j}{j!} + \sum_{j=0}^{n-1}\frac{(-s_1)^j}{j!}\Big| \leqq$$

$$\leqq s_1e^{-s_1}(n+1)\Big(1 + e^{s_1}\frac{s_1^n}{(n+1)!}(s_1+1)\Big) \leqq 2ns_1e^{-s_1}\Big(\frac{1}{\log 2}\Big).$$

Case $k \geqq 2$.

$$|s_{n+k}| = \Big|-\sum_{j=1}^{k}\frac{(-s_1)^j}{j!}s_{n+k-j}\Big| \leqq \sum_{j=1}^{k}\frac{s_1^j}{j!}|s_{n+k-j}| \leqq$$

$$\leqq 2ne^{-s_1}\sum_{j=1}^{k}\frac{s_1^j}{j!}s_1^{k-j}\Big(\frac{1}{\log 2}\Big)^{k-j} = 2ne^{-s_1}s_1^k\Big(\frac{1}{\log 2}\Big)^k\sum_{j=1}^{k}\frac{(\log 2)^j}{j!} \leqq$$

$$\leqq 2ne^{-s_1}s_1^k\Big(\frac{1}{\log 2}\Big)^k.$$

By (3.1) easy calculation shows that s_{n+k} is exponentially small if $k \leqq k_0$ since

$$2ne^{-s_1}s_1^k\Big(\frac{1}{\log 2}\Big)^k = 2n\exp\Big\{-n\Big(\vartheta - \frac{k}{n}\log n + \frac{k}{n}(\log\log 2 - \log\vartheta)\Big)\Big\} \leqq$$

$$\leqq 2n\exp(-n(\vartheta - 0.2783)).$$

REMARK. A slightly more accurate calculation proves that s_{n+k} is exponentially small if $k = O\Big(\frac{n}{\log n}\Big)$. The proof is based on the following inequalities which can be proved by induction:

$$(1-\varepsilon_k)\Big(\frac{n}{k!} + \frac{1}{(k-1)!}\Big)s_1^ke^{-s_1} \leqq s_{n+k} \leqq (1+\varepsilon_k)\Big(\frac{n}{k!} + \frac{1}{(k-1)!}\Big)s_1^ke^{-s_1},$$

where $\varepsilon_0 = \exp\{n(\gamma + \log\gamma + 1)\}$, $\varepsilon_k = \varepsilon_0 \cdot 3^{k-1}k!$, $s_1 = \gamma n$, $s_2 = s_3 = \ldots = s_{n-1} = 0$, $z_1 = 1$ and γ denotes a positive constant smaller than $\exp\Big\{-\frac{k\log n}{n} - 2\Big\}$ (the term $\frac{1}{(k-1)!}$ is considered 0 for $k = 0$).

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(Received August 22, 1989)

Acta Math. Hung.
59 (1–2) (1992), 25–32.

ON THE CONTROL OF A RECTANGULAR MEMBRANE

I. JOÓ (Budapest)

1. Let $\Omega = (0,a)\times(0,b)$ be a rectangle and consider the following control problem:

$$u_{tt} = \Delta u + \sum_{j=1}^{N} \delta((x,y)-P_j)v_j, \quad (x,y)\in\Omega \tag{1}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega\times(0,T)} = \sum_{j=N+1}^{M} \delta(s-S_j)v_j, \quad s\in\partial\Omega. \tag{2}$$

Here $u(t,x,y)$ gives the height of the point $(x,y)\in\Omega$ in time t, $P_1,\dots,P_N\in$ $\in\Omega$, $S_{N+1},\dots,S_M\in\partial\Omega$ and

$$v_j(t)\in L^2(0,T) \quad (j=1,\dots,M)$$

are the controls. We suppose further that the initial state is relaxed, i.e.

$$u(0,x,y) = u_t(0,x,y) = 0 \quad \text{on} \quad \Omega. \tag{3}$$

Define further the reachability set

$$R(T) := \{(u(T,.,.),\, u_t(T,.,.)) : v_j\in L^2(0,T); j=1,\dots,M\}$$

and consider the following system of eigenfunctions:

$$-\Delta\varphi_{mn} = \lambda_{mn}\varphi_{mn}, \quad \frac{\partial\varphi_{mn}}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega.$$

It has solutions

$$\varphi_{mn}(x,y) = \gamma_{mn}\cos m\frac{\pi}{a}x\cos n\frac{\pi}{b}y, \quad \lambda_{mn} = \left(m\frac{\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)^2,$$
$$(m,n = 0,1,2,\dots),$$

forming a complete orthonormal system in $L^2(\Omega)$ if

$$\gamma_{mn} := \begin{cases} \dfrac{2}{\sqrt{ab}} & \text{if } m,n\neq 0 \\ \dfrac{1}{\sqrt{ab}} & \text{if } m=n=0 \\ \sqrt{\dfrac{2}{ab}} & \text{if one of } m,n \text{ is } 0. \end{cases}$$

Introduce the spaces

$$W_r := \left\{ f = \sum_{m,n} c_{mn}\varphi_{mn} : \| f \|^2_{W_r} := |c_{00}|^2 + \sum_{(m,n)\neq(0,0)} |c_{mn}|^2 \lambda^r_{mn} < \infty \right\},$$

$$\mathcal{H}_r := W_{r+1} \oplus W_r.$$

First we prove

THEOREM 1. *For any control the solution of* (1)–(3) *satisfies* $(u, u_t) \in C([0,T], \mathcal{H}_r)$ *for* $r < -3/4$.

We recall that the system (1)–(3) is approximately controllable in time T if for all $r < -3/4$ $R(T) \subset \mathcal{H}_r$ is dense in $\mathcal{H}_r$. The system is approximately controllable in nonbounded finite time if $\bigcup_{T<\infty} R(T)$ is dense in $\mathcal{H}_r$ for all $r < -3/4$.

THEOREM 2. (a) *The system* (1)–(3) *is not approximately controllable in any time* $T < \infty$ *(i.e.* $R(T) \cap \mathcal{H}_r$ *is not dense in* $\mathcal{H}_r$ *for any* $r \in \mathbf{R}$*).*

(b) *The system* (1)–(3) *is approximately controllable in nonbounded finite time i.e.* $\bigcup_{T<\infty} R(T)$ *is dense in* $\mathcal{H}_r$ *if and only if all* λ_{mn} *are different (i.e.* a^2/b^2 *is not rational) and if all* e_{mn} *are nonzero.*

As a corollary we get that, contrary to the onedimensional case of vibrating strings ([1], [2]) the reachability set $R(T)$ does not give up growing for large T.

We give to (1)–(3) the following interpretation. The function $u(t,x,y)$ satisfies (1)–(3) if for every function $z(t,x,y) \in C^2$ with

$$z(T,.,.) = z_t(T,.,.) = 0, \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega\times(0,T)} = 0,$$

the following relation holds:

$$\int_0^T \int_\Omega u(z_{tt} - \Delta z) = \int_0^T \left(\sum_{j=1}^N z(.,P_j)v_j + \sum_{j=N+1}^M z(.,S_j)v_j \right). \tag{4}$$

We ask for the solution u in the form

$$u(t,x,y) = \sum_{m,n\in\mathbf{N}} c_{mn}(t)\varphi_{mn}(x,y);$$

then

$$c_{mn} = \int_\Omega u\varphi_{mn}.$$

Applying (4) to the function $z(t,x,y) = b(t)\varphi_{mn}(x,y)$ with $b \in C^2[0,T]$, $b(T) = b'(T) = 0$ we obtain

$$c_{mn}(t) = \int_0^t \frac{\sin\sqrt{\lambda_{mn}}(t-\tau)}{\sqrt{\lambda_{mn}}} h(\tau)\,d\tau, \quad (m,n) \neq (0,0), \tag{5}$$

$$c_{00}(t) = \int_0^t (t-\tau)h(\tau)d\tau,$$

$$h(t) := \sum_{j=1}^{N} \varphi_{mn}(P_j)v_j(t) + \sum_{j=N+1}^{M} \varphi_{mn}(S_j)v_j(t).$$

Denote

$$\xi_{m+1,\pm(n+1)}(t) := \pm i\sqrt{\lambda_{mn}}c_{mn}(t) + c'_{mn}(t) =$$

$$= \int_0^t e^{\pm i\sqrt{\lambda_{mn}}(t-\tau)}h(\tau)d\tau = \int_0^t \langle e_{mn}e^{\pm i\sqrt{\lambda_{mn}}\tau}, \quad v(\tau)\rangle d\tau,$$

$$\xi_{1,1}(t) := c_{00}(t) \int_0^t (t-\tau)h(\tau)d\tau = \int_0^t \langle e_{00}\tau, v_(\tau)\rangle d\tau,$$

$$\xi_{1,-1}(t) := c'_{00}(t) = \int_0^t h(\tau)d\tau = \int_0^t \langle e_{00}, v_{(\tau)}\rangle d\tau,$$

$$e_{mn} := \begin{pmatrix} \varphi_{mn}(P_1) \\ \vdots \\ \varphi_{mn}(P_N) \\ \varphi_{mn}(S_{N+1}) \\ \vdots \\ \varphi_{mn}(S_M) \end{pmatrix}, \quad v(\tau) := \begin{pmatrix} \overline{v_1(t-\tau)} \\ \vdots \\ \overline{v_m(t-\tau)} \end{pmatrix}.$$

For given functions

$$f = \sum_{m,n} f_{mn}\varphi_{mn}, \quad g = \sum_{m,n} g_{mn}\varphi_{mn}$$

it is easy to check

LEMMA 1. *The mapping*

$$A:(g,f)\to\left\{f_{00},g_{00},(\lambda_{m|k|})^{r/2}\xi_{mk}:m,|k|=1,2,\dots;m+|k|>2\right\}$$

establishes an isomorphism between $\mathcal{H}_r$ and ℓ_2.

This means that in proving Theorems 1 and 2 we can investigate $AR(T)$ in ℓ_2. It is not hard to see that for fixed m the sequence $\sqrt{\lambda_{mn}}$ has a partition into $c\sqrt{m}$ sequences in which the distance of the neighbouring elements is not smaller than 1 (see [13]). Consequently

$$\sum_n\left|\int_0^t\langle e_{mn}e^{i\sqrt{\lambda_{mn}}\tau},v(\tau)\rangle d\tau\right|^2\leqq c\sqrt{m}\parallel v\parallel^2_{L^2(0,t;C^N)}$$

which implies that $AR(t)\subset\ell_2$. The fact that for fixed v $A(u(t,.),u_t(t,.))$ is continuous in t can be proved similarly.

Return to the proof of Theorem 2/a. It is easy to see that:

LEMMA 2 ([3], [4]). *The vectors e_{mn} $(0\leqq m,k\leqq 2M-2)$ span the space C^M.*

Using this, we obtain a basis $e_1,\dots,e_M$. We have

LEMMA 3 ([3], [4]). *Let $T,\varepsilon>0$. Then the system*

$$e(\Lambda):=\left\{e_{mn}e^{\pm i\sqrt{\lambda_{mn}}t}:m+n>0;\quad m,n\geqq 0\right\}$$

contains a subsystem

$$\Phi=\left\{e_n^je^{i\lambda_n^{(j)}t}:j=1,\dots,M;n=1,2,\dots\right\}\cup\left\{e_o^je^{i\lambda_o^{(j)}t}:j=1,\dots,M\right\}$$

such that

(a) $|e_n^j-e_j|<\varepsilon$ $(j=1,\dots,M;n=0,1,\dots)$,

(b) $\lambda_n^{(j)}=2\pi\frac{n}{T}+O(1/\sqrt{n})$.

If $e_j=e_{mk}$ then by simultaneous diophantine approximation we get that for any pair (m',k') there exist $m=m'+O(1)$, $k=k'+O(1)$ satisfying $|e_j-e_{mk}|<\varepsilon$. Now approximate $2\pi\frac{n}{T}$ by $\lambda_{m'k'_0}$ with $k'_0=O(1)$. Then take appropriate $m=m'+O(1)$, further $k'=O(m^{1/2})$ such that $\lambda_{mk'}=$ $=2\pi\frac{n}{T}+O(m^{-1/2})$. Finally take $k=k'+O(1)$, then $\lambda_{mk}=2\pi\frac{n}{T}+O(m^{-1/2})$; the proof of Lemma 3 is based on these ideas.

We see from (a) that the system

$$\Phi_0=\left\{e^j_{|n|}e^{i2\pi(n/T)t}:j=1,\dots,M;n\in Z\right\}$$

is a Riesz basis in $L^2(0,T;C^M)$. We have

LEMMA 4 ([5] in case $M = 1$). *If $\{e_n e^{i\lambda_n t} : n \in Z\}$ is a Riesz basis in $L^2(0,T;C^M)$ with*

$$c_1 \sum |\alpha_n|^2 \leqq \left\| \sum \alpha_n e_n e^{i\lambda_n t} \right\|_{L^2}^2 \leqq c_2 \sum |\alpha_n|^2$$

and if $\varepsilon > 0$ satisfies $e^{T\varepsilon} - 1 \leqq \sqrt{c_1/c_2}$, then for any shifted exponents $\lambda'_n \in C$, $|\lambda_n - \lambda'_n| < \varepsilon$ the new system $\left\{e_n e^{i\lambda'_n t} : n \in Z\right\}$ remains a Riesz basis in $L^2(0,T;C^M)$.

The proof is an easy application of the method of Duffin and Eachus [5]. As a consequence we get that if in Φ we substitute all exponentials $2\pi\frac{n}{T}$ (with finitely many exceptions) by $\lambda_n^{(j)}$, we get a Riesz basis. That the remaining finitely many exponents can also be shifted (to obtain that Φ is a Riesz basis) is proved in

LEMMA 5. *Let $\{e_n e^{i\lambda_n t} : n \in Z\}$ be a Riesz basis in $L^2(0,T;C^M)$ and let $\lambda'_0 \neq \lambda_n$ $(n \in Z)$. Then the new system $\left\{e_0 e^{i\lambda'_0 t}, e_n e^{i\lambda_n t} : n \in Z \setminus \{0\}\right\}$ is also a Riesz basis in $L^2(0,T;C^M)$.*

The proof is based on the Riesz basis criterion given in [6]; namely the Hilbert transform must be bounded in the L^2 space weighted by the generating function. Now we shall join some elements to Φ to obtain a basis in $H^s(0,T;C^M)$. We use the following generalization of a theorem of Russell [7].

LEMMA 6. *Let*

$$H_n = \bigvee_{L^2(0,T;C^M)} \left\{e_\lambda e^{i\lambda t} : \lambda \in \sigma_n\right\}, \quad n \in Z,$$

where $\bigvee_H \psi$ denotes the closed linear hull of ψ in the space H, $\sigma_n \subset C$ are finite sets and $e_\lambda \in C^M$. Suppose that $\{H_n : n \in Z\}$ is a Riesz basis in $L^2(0,T;C^M)$. Let $s \geqq 1$ be an integer and

$$H_{(0)} := \bigvee_{L^2(0,T;C^M)} \{e_j e^{i\mu_j t} : j = 1, \ldots, Ms\}$$

where

(a) *the system $\{e_j e^{i\mu_j t}\}$ is linearly independent,*

(b) *$H_{(0)} \cap H_n = \{0\}$ for every n,*

(c) *the vector systems $\{e_1, \ldots, e_M\}, \{e_{M+1}, \ldots, e_{2M}\}, \ldots, \{e_{(s-1)M+1}, \ldots, e_{sM}\}$ are bases in C^M.*

Then $\{H_{(0)}, H_n : n \in Z\}$ *forms a Riesz basis in* $H^s(0,T;C^M)$.

This can be proved by induction on s. If we introduce the operator

$$B : H^s(0,T;C^M) \to H^{s-1}(0,T;C^M), \quad Bf = f' + Cf$$

where C is a constant matrix of the form

$$C = T^{-1}\begin{pmatrix} -i\mu_1 & & 0 \\ & \ddots & \\ 0 & & -i\mu_M \end{pmatrix} T \cdot$$

then B maps ${}_0H^s := \{f \in H^s(0,T;C^M) : f(0) = 0\}$ isomorphically onto $H^{s-1}(0,T;C^M)$. Using this, the proof of Lemma 6 is easy. So we can get a system Φ_1, $\Phi \subset \Phi_1 \subset e(\Lambda)$ forming a Riesz basis in H^s. Expanding an element of $e(\Lambda) \setminus \Phi_1$ in H^s we get a relation

$$0 = \sum_{m,n} d_{mn} e_{mn} e^{\pm i\sqrt{\lambda_{mn}}t}$$

which converges in $L^2(0,T;C^M)$ with its derivatives up to the order s, i.e. the series

$$0 = \sum_{m,n} d_{mn}(\lambda_{mn})^{s/2} e_{mn} e^{\pm i\sqrt{\lambda_{mn}}t}$$

is convergent in L^2. But then

$$0 = \sum_{m,n} d_{mn}(\lambda_{mn})^{-r/2}(\lambda_{mn})^{r/2}\xi_{mn}(T)$$

which proves that if s is large, then $R(T)$ is not dense in $\mathcal{H}_r$ as we asserted.

Finally we prove Theorem 2/b.

Let $\nu_{mn} := \sqrt{\lambda_{mn}} + i$, $\nu_{m,-n} := -\sqrt{\lambda_{mn}} + i$ and

$$\tilde{R}(\infty) := \Big\{\{\langle e_{00}\tau e^{-\tau}, v\rangle_{L^2}, \langle e_{00}e^{-\tau}, v\rangle_{L^2},$$

$$(\lambda_{m|k|})^{\frac{r}{2}}\langle e^{i\nu_{mk}\tau} e_{m|k|}, v\rangle_{L^2} : m,\ |k| = 1, 2, \dots; m + |k| > 2\}; v \in L^2(0.\infty, C^M)\Big\}.$$

LEMMA 7. *We have*

$$\bigcup_{T<\infty} AR(T) \subset \tilde{R}(\infty) \subset \overline{\bigcup_{T<\infty} AR(T)}.$$

The proof is a standard limiting argument hence we omit it.

By Lemma 7 we have to investigate the density of $\tilde{R}(\infty)$ in ℓ_2. Suppose that for some $a_{mn} \in \ell_2$ we have

$$0 = \sum_{\substack{m,|k|=0 \\ m+|k|>0}}^{\infty} a_{mk}(\lambda_{m|k|})^{r/2} \int_0^{\infty} \langle e_{mk} e^{i\nu_{mk}\tau}, v(\tau)\rangle d\tau + \tag{6}$$

$$+a_0 \int_0^{\infty} \langle e_{00} e^{-\tau}\tau, v(\tau)\rangle d\tau + a_1 \int_0^{\infty} \langle e_{00} e^{-\tau}, v(\tau)\rangle d\tau$$

for all $v \in L^2$. Using the notation $f(z) := \int_0^{\infty} v(t)e^{izt}dt$ we can reformulate (6) as follows:

(7)

$$0 = \sum_{\substack{m,|k|=0 \\ m+|k|>0}}^{\infty} a_{mk}(\lambda_{m|k|})^{r/2} \langle e_{mk}, f(-\overline{\nu_{mk}})\rangle + a_0\langle e_{00}, i^{-1}f'(i)\rangle + a_1\langle e_{00}, f(i)\rangle.$$

Let $(m_0, k_0) \neq (0,0)$ be fixed and define for $p \in N$

$$f_p(z) := \left(\frac{2i}{z+\nu_{m_0k_0}}\right)^p \cdot e_{m_0k_0}.$$

Then $f_p \in H^2_+(C^M)$ and

$$f_p(-\overline{\nu_{m_0k_0}}) = e_{m_0k_0}, \ |f_p(-\overline{\nu_{mk}})| \leqq (1+\delta)^{-p}, \quad (m,k) \neq (m_0,k_0),$$

further $|f_p'(i)| \leqq (1+\delta)^{-p}$, where

$$\delta = \min\left\{ \left|\sqrt{\lambda_{m|k|}} - \sqrt{\lambda_{m_0|k_0|}}\right| : (m,k) \neq (m_0,k_0)\right\}.$$

Consequently we get by (7) that

$$|e_{m_0k_0}|^2(\lambda_{m_0|k_0|})^{p/2}|a_{m_0k_0}| \leqq (1+\delta)^{-p}(|a_0|+|a_1|) +$$
$$+ \sum_{\substack{m+|k|>0 \\ (m,k)\neq(m_0,k_0) \\ \lambda_{m|k|}\leqq 9\lambda_{m_0|k_0|}}} |a_{mk}|(\lambda_{m|k|})^{r/2}(1+\delta)^{-p} +$$
$$+ \sum_{\lambda_{m|k|}>9\lambda_{m_0|k_0|}} |a_{mk}|(\lambda_{m|k|})^{r/2}(2\lambda_{m|k|})^{-\frac{p}{2}} \leqq \frac{c}{(1+\delta)^p} + \frac{c}{2^{p/2}} \to 0$$

as $p \to \infty$ and hence all $a_{mk} = 0$ and (7) is reduced to

$$0 = a_0 \langle e_{00}, i^{-1} f'(i) \rangle + a_1 \langle e_{00}, f(i) \rangle \quad \text{for all} \quad f \in H^2_+$$

and hence $a_0 = a_1 = 0$. The proof is complete. □

Added in proof (June 28, 1991). S. A. Avdonin informed the author that they obtained independently Lemma 6 in their book [14], p. 107 by different proof.

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(Received September 5, 1989)

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13–15
H–1053 BUDAPEST
HUNGARY

Acta Math. Hung.
59 (1–2) (1992), 33–38.

ON TOTAL ADDITIVE SOLUTION OF SOME EQUATIONS

KATALIN KOVÁCS (**Budapest**)

In 1946 P. Erdős proved the following theorem [1]:

THEOREM 1. *Let f be additive function. If*

$$f(n+1)-f(n)\to 0,$$

then $f(n)=c\log n$.

E. Wirsing gave a more general result for total additive functions [4]:

THEOREM 2. *Let f be a total additive function. If*

$$f(n+1)-f(n)=o(\log n),$$

then $f(n)=c\log n$.

Using the above result I. Kátai proved [2]

THEOREM 3. *Let f be a total additive function and $f_i=c_i f$. If*

$$\sum_{i=1}^{m} f_i(n+a_i)=o(\log n),$$

then $f(n)=c\log n$ or $f\equiv 0$.

In these results the arguments are linear. In this article we examine some equations which contain quadratic arguments or arguments n^k+s too. These are only special cases of the

CONJECTURE. Let $\chi_i(n)$ be polynomials. For the total additive solution of the equations

$$f(\chi_i(n))=o(\log n)\quad (i\in\{1,\dots,k\})$$

$f(p)=c\log p$ or $f(p)=0$ for the primes p, which divides at least one of the $\chi_i(n)$ for some n.

REMARK. We cannot write $f(n)=c\log n$ or $f\equiv 0$ in the conjecture. For example $f(n^2+1)=o(\log n)$ allows arbitrary choice of the $f(p)$'s if $p\equiv -1 \mod 4$.

DEFINITION. Let $\chi_i(n)$ be the polynomials in the arguments of our equations. We call a prime p *relevant*, if there exists a pair (n,i), such that $p|\chi_i(n)$.

In [3] we got the following assertions for total additive f:

THEOREM 4. (i) *$f(n^2+a)=0$ for $|a|\leqq 20$ implies $f(p)=0$ for the relevant primes.*

(ii) $f(n^2+k)=f(n^2)+o(\log n)\Rightarrow f(n)=c\log n$.

(iii) $f(n^3-1)=t\,f(n^3)+o(\log n)$ $(t\neq 0)\Rightarrow f(n)=c\log n$ *or* $f\equiv 0$.

Here we show some other equations where we can use the same methods.

THEOREM 5. *Let f be total additive.*

(i) *If $f(n^2+n+a)=0$ for some $|a|\leqq 20$, then $f(p)=0$ for the relevant primes.*

(ii) *Let $\varepsilon\in\{1,2,-2\}$ or $\{p_1,-p_2,p_1p_2\}$ (p_j are odd primes), $a_{i\varepsilon}\in Z$, $i\in\{1,\dots,s_\varepsilon\}$, $M_\varepsilon=\max\{\{\max_i a_{i\varepsilon}\},0\}$, $m_\varepsilon=\min\{\{\min_i a_{i\varepsilon}\},0\}$,*

$$T=\max_{\varepsilon}\left\{2,|m_\varepsilon|,m_\varepsilon^2+\varepsilon,M_\varepsilon+\max_{\varepsilon<0}\sqrt{2|\varepsilon|},M_\varepsilon^2+\varepsilon,\left|\frac{M_\varepsilon^2}{4}+\varepsilon\right|\right\}.$$

If for $p\leqq T$ $f(p)=0$ and $f(n^2+\varepsilon)=\sum_{i=1}^{s_\varepsilon}c_{i\varepsilon}f(n+a_{i\varepsilon})$, then $f\equiv 0$.

REMARKS. (i) If Theorem 5 (i) and Theorem 4 (i) were valid for all $a\in Z$, then $f(n^2+bn+a)=0$ in the case $b\leqq 1$ would imply $f(p)=0$ for the relevant primes. (Replacing n by $n-\left[\frac{b}{2}\right]$ we get this result.) For a fixed pair (a,b) it is enough to know $f(p)=0$ for the relevant primes not greater than $\max\left\{\frac{a-b+1}{-b+2},\frac{-b+\sqrt{b^2-4a}}{2},|a-0.25b^2|\right\}$ for the prime–divisors of a and for the first relevant prime. (The proof is similar to that of Theorem 5 (i).)

(ii) We show two examples, which imply $f\equiv 0$.

1. $f(n^2+2)=f(n+7)+f(n-2)$, $f(n^2-2)=f(n)$, $f(n^2+1)=f(n)$. (We must control $f(p)=0$ for $p\leqq 51$.)

2. $f(n^2+5)=f(n-4)+f(n)$, $f(n^2-3)=f(n-2)+f(n-3)$, $f(n^2+15)=f(n+1)+f(n+3)$. (Here $T=23$.)

THEOREM 6. *Let f be a total additive function. The following conditions imply $f(n)=c\log n$ or $f\equiv 0$.*

1. $f(n^2+1)=s_1f(n)+s_2f(n-1)+o(\log n)$ $(s_1,s_2\neq(0,0))$.

2. (i) $f(n^2+2k)=f(n-k)+o(\log n)$.

(ii) $f(n^2+k)=\sum_{i=1}^{m}c_if(n+a_i)+o(\log n)$, *if $a_i=-s_i^2-s_i-k$ with some $s_i\in Z$.*

3. (i) $f(n^2+n+k)=f(n+1)+o(\log n)$, *if $k=-s^2-s-1$ with some $s\in Z$.*

(ii) $f(n^2+n+k)=\sum_{i=1}^{m}c_if(n+a_i)+o(\log n)$, *if $a_i=1-k-s_i^2$ with some $s_i\in Z$.*

4. *Let $f(n^3+1)=o(\log n)$ and*

(i) $f(n^{3\cdot 2^k}-1)=o(\log n)$

or

(ii) $f(n^2+1)=\sum\limits_{i=1}^{m} c_i f(n+a_i)(\neq -f(n+1)-f(n+2))+o(\log n)$

or

(iii) $f(n^4-1)=o(\log n)$

or

(iv) $f(n^4+\varepsilon)=o(\log n)$, $\varepsilon=1$ *or* 2

and

$f(n^8-1)=o(\log n)$.

PROOF OF THEOREM 5. (i) For each relevant prime $p \nmid a$ there exists an $n\in\{0,\dots,p-1\}$ such that $p|n^2+n+a$ and in the case $a>0$

$$0<\frac{n^2+n+a}{p}\leqq\frac{(p-1)^2+p-1+a}{p}=p-1+\frac{a}{p}<p \quad \text{if} \quad p>a$$

resp. in the case $a<0$

$$0<\frac{n^2+n+a}{p}<p-1 \quad \text{if} \quad p>\frac{-1+\sqrt{1-4a}}{2}.$$

If $f(p)=0$ for the primes $p\leqq a$ (if $a>0$) resp. for the primes $p\leqq\sqrt{|a|}$ (if $a<0$), then $f(p)=0$ for all relevant primes by induction, using that

$$0=f(n^2+n+a)=f(p)+f(s) \quad \text{with} \quad s\leqq p-1.$$

So we must control only the relevant primes $p\leqq a$ resp. $p\leqq\sqrt{|a|}$, whether $f(p)=0$ or not. Let us see two cases:

$a=-14$: $p\in\{2,3,7\}$. For $n\in\{5,4,7\}$ we have $f(2^4)=f(2\cdot 3)=$ $=f(2\cdot 3\cdot 7)=0$, which imply $f(2)=f(3)=f(7)=0$.

$a=19$: $p\in\{3,5,7,13,19\}$. For $n\in\{1,2,4,5,19\}$ we get

$$f(3\cdot 7)=f(5^2)=f(3\cdot 13)=f(7^2)=f(19\cdot 3\cdot 7)=0.$$

(ii) Replacing n by $n-M_\varepsilon$ in

$$f(n^2+\varepsilon)=\sum_{i=1}^{s_\varepsilon} c_{i\varepsilon} f(n+a_{i\varepsilon})$$

we get

$$f\left((n-M_\varepsilon)^2+\varepsilon\right)=\sum_{i=1}^{s_\varepsilon} c_{i\varepsilon} f(n+a_{i\varepsilon}-M_\varepsilon). \tag{1}$$

For the two sets of ε's every prime is relevant. So for each p there exists an ε and an $n_p \in \{0, \dots, p-1\}$ such that $p|(n_p - M)^2 + \varepsilon$. For $n \leqq |m_\varepsilon| + M_\varepsilon$

$$(n - M_\varepsilon)^2 + \varepsilon \leqq \max\left\{M_\varepsilon^2 + \varepsilon, m_\varepsilon^2 + \varepsilon, \left|\left(\frac{M}{2}\right)^2 + \varepsilon\right|\right\} \leqq T.$$

So for $p > T$ we have $n > |m_\varepsilon| + M_\varepsilon$. Replacing these n's in (1), the arguments of the right hand side are all in the interval $[1, p-1]$. Using that $p > 1 + M_\varepsilon + \max_{\varepsilon<0} \sqrt{2|\varepsilon|}$, we get

$$0 < \frac{(n_p - M_\varepsilon)^2 + \varepsilon}{p} \leqq \max\left\{\frac{\left|\left(\frac{M_\varepsilon}{2}\right)^2 + \varepsilon\right|}{p}, \frac{(p - 1 - M_\varepsilon)^2 + \varepsilon}{p}\right\} < p.$$

If we know $f(m) = 0$ for all $m < p$, then we have

$$f((n_p - M)^2 + \varepsilon) = f(p) = 0. \tag{2}$$

For $p < T$ $f(p) = 0$ is assumed and for the other p's we get it by induction after (2).

PROOF OF THEOREM 6. We use the following equality: If $\Theta(n) = n^2 + tn + k$, then

$$\Theta(\Theta(n) + n) = \Theta(n) - \Theta(n+1),$$

i.e. $f(\Theta(\Theta(n) + n)) = f(\Theta(n)) + f(\Theta(n+1))$.

1. By the choice $\Theta(n) = n^2 + 1$ in the equation

$$s_1 f(n^2+n+1) + s_2 f(n^2+n) = (s_1+s_2)f(n) + s_2 f(n-1) + s_1 f(n+1) + o(\log n),$$

i.e.

$$s_1 f(n^2 + n + 1) = s_1 f(n) + s_2 f(n-1) + (s_1 - s_2) f(n+1) + o(\log n).$$

Choosing $\Theta_2(n) = n^2 + n + 1$ we have

$$\begin{aligned} 2s_1 f(n+1) + s_2 f(n^2 + 2n) + (s_1 - s_2) f((n+1)^2 + 1) = \\ = (s_1 + s_2) f(n) + s_2 f(n-1) + (2s_1 - s_2) f(n+1) + \\ + (s_1 - s_2) f(n+2) + o(\log n). \end{aligned}$$

Using our basic equation we can replace

$$f((n+1)^2 + 1) = s_1 f(n+1) + s_2 f(n) + o(\log n).$$

So Theorem 3 is applicable.

2. (i) By $\Theta_1(n) = n^2 + 2k$

$$f(n^2 + n + k) = f(n - k) + f(n - k + 1) + o(\log n).$$

Here the choice $\Theta_2(n) = n^2 + n + k$ implies

$$f(n^2+2n)+f(n^2+2n+1) = f(n-k)+2f(n-k+1)+f(n-k+2)+o(\log n).$$

We can apply Theorem 3 again.

(ii) Let us take $\Theta(n) = n^2 + k$ and apply Theorem 3.

3. (i) By the choice $\Theta_1(n) = n^2 + n + k$

$$f((n + 1)^2 + k) = f(n + 1) + f(n + 2) + o(\log n).$$

Replacing n by $n - 1$ and considering $\Theta_2(n) = n^2 + k$

$$f(n^2 + n + k) = f(n^2 + n + k + 1) = f(n) + 2f(n + 1) + f(n + 2) + o(\log n).$$

Our basic equation and the choice $k = -s^2-s-1$ give only linear arguments. Theorem 3 is applicable.

(ii) Let us take $\Theta(n) = n^2 + n + k$ and apply Theorem 3.

4. (i)

$$f(n^{3\cdot 2^k} - 1) = f\left(n^{3\cdot 2^{k-1}} - 1\right) + f\left(\left(n^{2^{k-1}}\right)^3 + 1\right) =$$

$$= f\left(n^{3\cdot 2^{k-1}} - 1\right) + o(\log n) = \cdots = f(n^3 - 1) + o_k(\log n).$$

We can prove Theorem 4 (iii) for $t = 0$ too: In $f(n^2 + n + 1) = -f(n - 1)+$ $+o(\log n)$ by the choice $\Theta(n) = n^2 + n + 1$ we get

$$-f(n^2 + 2n) = -f(n - 1) - f(n) + o(\log n).$$

Applying Theorem 3 we get $f \equiv 0$.

(ii) By $\Theta(n) = n^2 - n + 1$ in

$$f(n^3 + 1) = f(n^2 - n + 1) + f(n + 1) = o(\log n)$$

we get

$$f(n^2 + 1) = -f(n - 1) - f(n + 2) + o(\log n). \tag{3}$$

Considering the other equation Theorem 3 is applicable.

(iii) $f(n^3 + 1) = o(\log n)$ gives (3), which implies

$$f(n^4 - 1) = f(n^2 + 1) + f(n^2 - 1) = f(n - 1) - f(n + 2) + o(\log n).$$

Let us apply Theorem 3 again.

(iv) The case $\varepsilon = 1$: $f(n^8 - 1) = f(n^4 - 1) + f(n^4 + 1)$ gives $f(n^4 - 1) =$ $= o(\log n)$ which we examined in (iii).

The case $\varepsilon = 2$: $f(n^{16} - 1) = f(n^8 + 1) + f(n^8 - 1)$ gives $f(n^8 + 1) =$ $= o(\log n)$. Replacing n by n^4 in (3) we have

$$f(n^4+2)+f(n^4+1) = f(n^4+1)+o(\log n) = -f(n^8+1)+o(\log n) = o(\log n),$$

i.e. the case $\varepsilon = 1$. □

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(Received September 26, 1989)

DEPARTMENT OF ALGEBRA AND NUMBER THEORY
LORÁND EÖTVÖS UNIVERSITY
H–1088, BUDAPEST, MÚZEUM KRT. 6–8

Acta Math. Hung.
59 (1–2) (1992), 39–48.

THE NUMERICAL SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS BY SPLINE FUNCTIONS

J. GYŐRVÁRI and Cs. MIHÁLYKÓ (Veszprém)

1. Introduction

In this paper we are going to approximate the solution of the nonlinear differential equation

$$(1)\qquad \begin{cases} y^{(m)}(x) = f\left[x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right], & x \in [0;1], \\ y^{(j)}(0) = y_0^{(j)} \;\; (j = 0, 1, \ldots, m-1;\; m \geqq 2) \end{cases}$$

supposing

$$(1.1)\qquad f\left[x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right] \in C^{(r)}([0;1])$$

where r is a fixed integer,

$$(1.2)\qquad \left| f^{(q)}\left[x, y_1, y_1', y_1'', \ldots, y_1^{(m-1)}\right] - f^{(q)}\left[x, y_2, y_2', y_2'', \ldots, y_2^{(m-1)}\right] \right| \leqq$$

$$\leqq L \cdot \sum_{i=0}^{m-1} \left| y_1^{(i)} - y_2^{(i)} \right| \quad (q = 0, 1, \ldots, r)$$

(Lipschitz condition), where

$$f^{(0)} = f,\; f^{(q+1)} = f_x^{(q)} + f_y^{(q)} \cdot y' + f_{y'}^{(q)} \cdot y'' + \cdots + f_{y^{(m-2)}}^{(q)} \cdot y^{(m-1)} + f_{y^{(m-1)}}^{(q)} \cdot f$$
$$(q = 0, 1, 2, \ldots, r-1).$$

The problem of approximating the solution of nonlinear differential equations has been always of special interest. By spline functions the Cauchy problem $y' = f(x, y)$ was discussed by F. R. Loscalzo and T. D. Talbot [1], [2] and Th. Fawzy [3]. Some problems of a second order differential equation was solved by Gh. Micula [4], [5] by Th. Fawzy [6], [7] and by J. Győrvári [8], [9], [10].

The main point of our method is that approximated values of $y^{(q)}(x_k)$ are constructed by the help of $y_0^{(j)}$ $(j = 0, 1, \ldots, m-1)$ and the function f, then using these approximate values $\overline{y}_k^{(q)}$ $(k = 0, 1, 2, \ldots, n;\; q = 0, 1, 2)$ the solution of (1) and its derivatives are approximated up to the $(m+r)$–th order by the spline function of type $(0;1;2)$ defined in [6]. In the approximation theorems we use the average moduli defined in [11].

2. The first approximation process

2.1. Definition of the approximate values $\overline{y}_k^{(q)}$. Let

$$x_k = \frac{k}{n}; \quad h = \frac{1}{n}; \quad x_{k+1/2} = x_k + \frac{h}{2} \quad (k = 0, 1, \dots, n);$$

$$\omega(f^{(r)}, x, h) = \sup_{t_1, t_2 \in \left[x - \frac{h}{2}; x + \frac{h}{2}\right] \cap [0,1]} \left| f^{(r)}(t_1) - f^{(r)}(t_2) \right|,$$

$$\tau\left(f^{(r)}, h\right) = \int_0^1 \omega\left(f^{(r)}, x, h\right) dx.$$

DEFINITION.

$$\overline{y}_0^{(j)} := y_0^{(j)} \quad (j = 0, 1, \dots, m-1),$$
$$\overline{y}_0^{(m+q)} := f^{(q)}\left[x_0, y_0, y_0', \dots, y_0^{(m-1)}\right] \quad (q = 0, 1, \dots, r),$$
$$\overline{y}_k^{(q)} := G_k^{(q)}(x_k) \quad (k = 1, 2, \dots, n-1;\ q = 0, 1, \dots, m+r),$$
$$\overline{y}_n^{(q)} := G_{n-1}^{(q)}(x_n) \quad (q = 0, 1, \dots, m+r),$$

where

(2.1.1)
$$G_0(x) := \sum_{j=0}^{m-1} \frac{y_0^{(j)}}{j!}(x - x_0)^j + \sum_{q=0}^{r} \frac{f^{(q)}\left[x_0, y_0, y_0', \dots, y_0^{(m-1)}\right]}{(m+q)!}(x - x_0)^{m+q}$$

if $x_0 \leqq x \leqq x_1$;

$$G_k(x) := \sum_{j=0}^{m-1} \frac{G_{k-1}^{(j)}(x_k)}{j!}(x - x_k)^j + \tag{2.1.2}$$
$$+ \sum_{q=0}^{r} \frac{f^{(q)}\left[x_k, G_{k-1}(x_k), G_{k-1}'(x_k), \dots, G_{k-1}^{(m-1)}(x_k)\right]}{(m+q)!}(x - x_k)^{m+q}$$

if $x_k \leqq x \leqq x_{k+1}$ $(k = 1, 2, \dots, n-1)$.

One can see that

$$G_{k-1}^{(j)}(x_k) = G_k^{(j)}(x_k) \quad (k = 1, 2, \dots, n-1;\ j = 0, 1, \dots, m-1).$$

2.2. The convergence process. In this section we prove a theorem which shows how the approximate values $\overline{y}_k^{(j)}$ converge to the exact values of $y_k^{(j)} = y^{(j)}(x_k)$ $(k = 1, 2, \ldots, n;\ j = 0, 1, \ldots, m + r)$.

DEFINITION. We shall denote the estimated errors of $\overline{y}_k^{(j)}$ and $y_k^{(j)}$ $(k = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m + r)$ by

$$e_k^{(j)} = \left| y_k^{(j)} - \overline{y}_k^{(j)} \right| \quad (k = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m + r).$$

THEOREM 2.2.1. *We have*

$$e_{k+1}^{(j)} \leqq C_{j,k+1} h^r \cdot \tau \left(f^{(r)}, h \right) \quad (k = 0, 1, \ldots, n - 1;\ j = 0, 1, \ldots, m + r)$$

where the constants $C_{j,k+1}$ are independent of n.

PROOF. We shall often use the following Taylor–formulas:

$$y^{(\nu)}(x) = \sum_{j=\nu}^{m-1} \frac{y_k^{(j)}}{(j-\nu)!} (x - x_k)^{j-\nu} + \tag{2.2.1}$$
$$+ \sum_{q=0}^{r-1} \frac{f^{(q)} \left[x_k, y_k, \ldots, y_k^{(m-1)} \right]}{(m + q - \nu)!} (x - x_k)^{m+q-\nu} +$$
$$+ \frac{1}{(m + r - 1 - \nu)!} \int_{x_k}^{x} (x - t)^{m+r-1-\nu} f^{(r)} \left[t, y(t), \ldots, y^{(m-1)}(t) \right] dt$$
$$(k = 0, 1, \ldots, n - 1;\ \nu = 0, 1, \ldots, m - 1),$$

$$y^{(m+q)}(x) = \sum_{j=q}^{r-1} \frac{f^{(j)} \left[x_k, y_k, \ldots, y_k^{(m-1)} \right]}{(j - q)!} (x - x_k)^{j-q} + \tag{2.2.2}$$
$$+ \frac{1}{(r - 1 - q)!} \int_{x_k}^{x} (x - t)^{r-1-q} f^{(r)} \left[t, y(t), \ldots, y^{(m-1)}(t) \right] dt$$
$$(k = 0, 1, \ldots, n - 1;\ q = 0, 1, \ldots, r - 1).$$

Let $x_k \leqq x \leqq x_{k+1}$ $(k = 0, 1, \ldots, n - 1)$. Using equations (2.1.2) and (2.2.2) and the Lipschitz condition (1.2) we get

$$|y(x) - G_k(x)| \leqq$$
$$\leqq \sum_{j=0}^{m-1} \frac{e_k^{(j)}}{j!} h^j + L \sum_{q=0}^{r-1} \left\{ e_k + e_k' + \cdots + e_k^{(m-1)} \right\} \frac{h^{m+q}}{(m+q)!} + \frac{1}{(m + r - 1)!}.$$

$$\cdot\int_{x_k}^{x}(x-t)^{m+r-1}\left\{f^{(r)}\left[t,y(t),\dots,y^{(m-1)}(t)\right]-f^{(r)}\left[x_k,y_k,\dots,y_k^{(m-1)}\right]\right\}dt+$$

$$+\left|f^{(r)}\left[x_k,y_k,\dots,y_k^{(m-1)}\right]-f^{(r)}\left[x_k,G_{k-1}(x_k),\dots,G_{k-1}^{(m-1)}(x_k)\right]\right|\frac{h^{m+r}}{(m+r)!}.$$

At $x=x_{k+1}$ from this inequality we get

$$e_{k+1}\leqq e_k\left\{1+L\sum_{q=0}^{r}\frac{h^{m+q}}{(m+q)!}\right\}+e_k'\left\{h+L\sum_{q=0}^{r}\frac{h^{m+q}}{(m+q)!}\right\}+$$

$$+e_k''\left\{\frac{h^2}{2!}+L\sum_{q=0}^{r}\frac{h^{m+q}}{(m+q)!}\right\}+\dots+e_k^{(m-1)}\left\{\frac{h^{m-1}}{(m-1)!}+L\sum_{q=0}^{r}\frac{h^{m+q}}{(m+q)!}\right\}+$$

$$+\frac{h^{m+r}}{(m+r)!}\,\omega\left(f^{(r)},x_{k+\frac{1}{2}},h\right).$$

Using the inequality $\sum\limits_{q=0}^{r}\frac{h^q}{(m+q)!}\leqq\frac{1}{m!}\sum\limits_{q=0}^{r}h^q=\text{const.}$ we get

$$e_{k+1}\leqq e_k\left\{1+a_{0,0}h^m\right\}+e_k'a_{0,1}h+e_k''a_{0,2}h^2+\dots+e_k^{(m-1)}a_{0,m-1}h^{m-1}+$$

$$+\frac{h^{m+r}}{(m+r)!}\omega\left(f^{(r)},x_{k+\frac{1}{2}},h\right).$$

Similarly we get

$$e_{k+1}'\leqq e_ka_{1,0}h^{m-1}+e_k'\left\{1+a_{1,1}h^{m-1}\right\}+e_k''a_{1,2}h+$$

$$+\dots+e_k^{(m-1)}a_{1,m-1}h^{m-2}+\frac{h^{m+r-1}}{(m+r-1)!}\omega\left(f^{(r)},x_{k+\frac{1}{2}},h\right),$$

$$e_{k+1}''\leqq e_ka_{2,0}h^{m-2}+e_k'a_{2,1}h^{m-2}+e_k''\left\{1+a_{2,2}h^{m-2}\right\}+\dots+$$

$$+e_k^{(m-1)}a_{2,m-1}h^{m-3}+\frac{h^{m+r-2}}{(m+r-2)!}\omega\left(f^{(r)},x_{k+\frac{1}{2}},h\right)$$

$$\vdots$$

$$e_{k+1}^{(m-1)}\leqq e_ka_{m-1,0}h+e_k'a_{m-1,1}h+e_k''a_{m-1,2}h+\dots+$$

$$+e_k^{(m-1)}\left\{1+a_{m-1,m-1}h\right\}+\frac{h^{r+1}}{(r+1)!}\omega\left(f^{(r)},x_{k+\frac{1}{2}},h\right)$$

where

$$a_{i,0} = L \sum_{q=0}^{r} \frac{h^q}{(m+q)!} \quad (i = 0, 1, \dots, m-1),$$

$$a_{i,0} = a_{i,1} = \dots = a_{i,i} \quad (i = 0, 1, \dots, m-1),$$

$$a_{i,j} = \frac{1}{(j-i)!} + h^{m-j} a_{i,0} \quad (i = 0, 1, \dots, m-1;\ j = i+1, i+2, \dots, m-1).$$

These inequalities are true for $k = 0$, too. Hence

$$E_{k+1} \leqq \{I + Ah\} E_k + C\,\omega\left(f^{(r)}, x_{k+\frac{1}{2}}, h\right) \quad (k = 0, 1, \dots, m-1)$$

where

$$E_k = \left[e_k, e_k', \dots, e_k^{(m-1)}\right]^T,$$

$$C = \left[\frac{h^{m+r}}{(m+r)!}, \frac{h^{m+r-1}}{(m+r-1)!}, \dots, \frac{h^{r+1}}{(r+1)!}\right]^T,$$

and I is the identity matrix;

$$A = \begin{bmatrix} a_{0,0}h^{m-1} & a_{0,1} & a_{0,2}h & \dots & a_{0,m-1}h^{m-2} \\ a_{1,0}h^{m-2} & a_{1,1}h^{m-2} & a_{1,2} & \dots & a_{1,m-1}h^{m-3} \\ a_{2,0}h^{m-3} & a_{2,1}h^{m-3} & a_{2,2}h^{m-3} & \dots & a_{2,m-1}h^{m-4} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m-1,0} & a_{m-1,1} & a_{m-1,2} & \dots & a_{m-1,m-1} \end{bmatrix}$$

By the repeated use of this inequality we get

$$E_{k+1} \leqq \{I + Ah\} E_k + C\omega\left(f^{(r)}, x_{k+\frac{1}{2}}, h\right),$$

$$E_{k+1} \leqq \{I + Ah\} \left[\{I + Ah\} E_{k-1} + C\omega\left(f^{(r)}, x_{k-1+\frac{1}{2}}, h\right)\right] + \\ + C\omega\left(f^{(r)}, x_{k+\frac{1}{2}}, h\right),$$

$$\vdots$$

$$E_{k+1} \leqq \{I + Ah\}^{k+1} E_0 + \sum_{j=0}^{k} \{I + Ah\}^j C\omega\left(f^{(r)}, x_{k-j+\frac{1}{2}}, h\right),$$

$$E_{k+1} \leqq \sum_{j=0}^{k} \{I + Ah\}^j C\omega\left(f^{(r)}, x_{k-j+\frac{1}{2}}, h\right).$$

From this

$$\|E_{k+1}\|_\infty \leqq C_1 \frac{h^{r+1}}{(r+1)!} \sum_{j=0}^{k} \{1 + \|A\|_\infty h\}^j \cdot \omega\left(f^{(r)}, x_{k-j+\frac{1}{2}}, h\right) \leqq$$

$$\leqq C_2 h^r \sum_{j=0}^{k} \int_{x_j}^{x_{j+1}} \omega\left(f^{(r)}, x_{k-j+\frac{1}{2}}, h\right) dx \leqq C_3 h^r \tau\left(f^{(r)}, h\right)$$

that is

$$e_{k+1}^{(j)} \leqq C_{j,k+1} h^r \tau(f^{(r)}, h) \quad (k = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots, m-1).$$

Using this inequality we get

$$e_k^{(m+q)} = \left| y^{(m+q)}(x_k) - G_k^{(m+q)}(x_k) \right| =$$

$$= \left| f^{(q)}\left[x_k, y_k, \ldots, y_k^{(m-1)}\right] - f\left[x_k, G_{k-1}(x), \ldots, G_{k-1}^{(m-1)}(x_k)\right] \right| \leqq$$

$$\leqq L\left\{e_k + e_k' + \cdots + e_k^{(m-1)}\right\} \leqq C_{m+q,k} h^r \tau\left(f^{(r)}, h\right).$$

3. The second approximation process

As we have seen before, we have the set of the approximate values

$$\overline{y}_0^{(q)}, \overline{y}_1^{(q)}, \ldots, \overline{y}_n^{(q)} \quad (q = 0, 1, 2)$$

which are the approximate values of the exact solution $y(x)$ of (1) and its derivatives at the points $x_0, x_1, \ldots, x_n$.

Using these approximate values $\overline{y}_k^{(q)}$ $(k = 0, 1, \ldots n;\ q = 0, 1, 2)$ following the idea of Th. Fawzy in [6] we are going to construct a spline function $\overline{S}_\Delta(x)$ interpolating the set $\overline{y}_k^{(q)}$ and approximating the solution of (1) and its derivatives. Also, we shall discuss the convergence of this function to $y(x)$.

3.1. The construction of the spline function. Similarly as Fawzy did in [6] we construct the spline function $\overline{S}_\Delta(x)$ $(\overline{S}_\Delta(x) = \overline{S}_k(x)$ if $x_k \leqq x \leqq x_{k+1})$ in the following theorem:

THEOREM 3.1.1. *Let* $\overline{y}_k^{(q)}$ $(k = 0, 1, \dots n;\ q = 0, 1, 2)$ *be the approximate values defined above. Then there exists a unique spline function* $\overline{S}_\Delta(x)$ *satisfying:*

$$\overline{S}_\Delta(x) \in C^2([0,1]), \tag{3.1.1}$$

$$\overline{S}_k^{(q)}(x_k) := \overline{y}_k^{(q)} \quad (k = 0, 1, \dots n-1;\ q = 0, 1, 2), \tag{3.1.2}$$

$$\overline{S}_{n-1}^{(q)}(x_n) := \overline{y}_n^{(q)} \quad (q = 0, 1, 2), \tag{3.1.3}$$

$$\overline{S}_\Delta(x) \quad \textit{is a polynomial of minimal degree on } [x_k, x_{k+1}]. \tag{3.1.4}$$

We get the following equalities:

$$\overline{S}_k(x) = \overline{y}_k + \overline{y}'_k(x - x_k) + \frac{\overline{y}''_k}{2!}(x - x_k)^2 + \sum_{i=1}^{3} \overline{a}_{i,k}(x - x_k)^{2+i}, \tag{3.1.5}$$

where

$$\overline{a}_{1,k} = 10\overline{F}_k - 4\overline{F}'_k + \frac{1}{2}\overline{F}''_k, \tag{3.1.6}$$

$$\overline{a}_{2,k} = \frac{1}{h}(-15\overline{F}_k + 14\overline{F}'_k - 2\overline{F}''_k), \tag{3.1.7}$$

$$\overline{a}_{3,k} = \frac{1}{h^2}(6\overline{F}_k - 3\overline{F}'_k + \frac{1}{2}\overline{F}''_k), \tag{3.1.8}$$

$$\overline{F}_k = \frac{1}{h^3}(\overline{y}_{k+1} - \overline{y}_k - \overline{y}'_k h - \frac{1}{2}\overline{y}''_k h^2), \tag{3.1.9}$$

$$\overline{F}'_k = \frac{1}{h^2}(\overline{y}'_{k+1} - \overline{y}'_k - \overline{y}''_k h), \tag{3.1.10}$$

$$\overline{F}''_k = \frac{1}{h}(\overline{y}''_{k+1} - \overline{y}''_k). \tag{3.1.11}$$

3.2. A general convergence process. In this section we prove the essential theorem concerned with the convergence of our spline function constructed in Theorem 3.1.1 to the exact solution of (1) and also we prove that this function satisfies this differential equation as $n \to \infty$, if $2 \leqq m$ and $m + r \leqq 5$.

THEOREM 3.2.1. *If $y(x) \in C^{m+r}([0,1])$ is the exact solution of* (1) *and $\overline{S}_\Delta(x)$ is the spline function mentioned in Theorem* 3.1.1 *then the following inequalities hold:*

$$\left|y^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq B_{q,k} h^{r-q} \tau(y^{(m+r)}, h)$$

$(k = 0, 1, \dots, n-1;\ q = 0, 1, \dots, m+r;\ 2 \leqq m + r \leqq 5)$ *where the constants $B_{q,k}$ are independent of n.*

THEOREM 3.2.2. *If the function f in* (1) *satisfies conditions* (1.1) *and* (1.2), *then the following inequalities hold:*

$$\left|\overline{S}_\Delta^{(m)}(x) - f[x, \overline{S}_\Delta(x), \dots, \overline{S}_\Delta^{(m-1)}(x)]\right| \leqq K\, h^{r-m} \tau(y^{(m+r)}, h),$$

where the constant K is independent of n.

To prove these theorems we need the following lemmas, which can be easily seen using formulas (3.1.5–3.1.11) and Theorem 2.2.1.

LEMMA 3.2.1. *Let $a_{j,k}$ $(j = 1, 2, 3;\ k = 0, 1, \dots, n-1)$ and $\overline{a}_{j,k}$ denote the coefficients of the spline function $S_\Delta(x)$ constructed with the exact values $y^{(j)}(x_k)$ and the ones in Theorem* 3.1.1. *Then we have*

$$|a_{j,k} - \overline{a}_{j,k}| \leqq A_{j,k} h^{r-3-j} \tau(y^{(m+r)}, h)\ (j = 1, 2, 3;\ k = 0, 1, \dots, n-1),$$

where the constants $A_{j,k}$ are independent of n.

LEMMA 3.2.2. *Let $S_\Delta(x)$ and $\overline{S}_\Delta(x)$ denote the spline function constructed with the exact values and the one constructed with the approximate values* (*Theorem* 3.1.1), *respectively. Then we have*

$$\left|S_k^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq D_{q,k} h^{r-q} \tau(y^{(m+r)}, h)$$
$$(k = 0, 1, \dots, n-1; \quad q = 0, 1, \dots, m+r),$$

where the constants $D_{q,k}$ are independent of n.

PROOF OF THEOREM 3.2.1. Lemma 3.2.2, Theorem 1.1.2 of [10] and the inequality

$$\left|y^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq \left|y^{(q)}(x) - S_k^{(q)}(x)\right| + \left|S_k^{(q)}(x) - \overline{S}_k^{(q)}(x)\right|$$
$$(q = 0, 1, \dots, m+r)$$

imply Theorem 3.2.1.

PROOF OF THEOREM 3.2.2. Using conditions (1), (1.1) and (1.2) we obtain

$$\left|\overline{S}_\Delta^{(m)}(x) - f\left[x, \overline{S}_\Delta(x), \dots, \overline{S}_\Delta^{(m-1)}(x)\right]\right| \leqq$$

$$\leqq \left|\overline{S}_\Delta^{(m)}(x) - y^{(m)}(x)\right| + \left|y^{(m)}(x) - f\left[x, \overline{S}_\Delta(x), \dots, \overline{S}_\Delta^{(m-1)}(x)\right]\right| \leqq$$

$$\leqq \left|\overline{S}_\Delta^{(m)}(x) - y^{(m)}(x)\right| + \left|f\left[x, y(x), \dots, y^{(m-1)}(x)\right] - \right.$$

$$\left. - f\left[x, \overline{S}_\Delta(x), \dots, \overline{S}_\Delta^{(m-1)}(x)\right]\right| \leqq$$

$$\leqq \left|\overline{S}_\Delta^{(m)}(x) - y^{(m)}(x)\right| + L\sum_{j=0}^{m-1}\left|y^{(j)}(x) - \overline{S}_\Delta^{(j)}(x)\right|$$

which implies Theorem 3.2.2 by the help of Theorem 3.2.1.

REMARK. If $f^{(r)}$ has finite variation then $\tau\left(f^{(r)}, h\right) = O(h)$, (see in [11]). Using this equation we get the following theorems from Theorems 2.2.1, 3.2.1 and 3.2.2:

THEOREM 2.2.1.a. *We have*

$$e_{k+1}^{(j)} \leqq C_{j,k+1}^* h^{r+1} \quad (k = 0, 1, \dots, n-1;\ j = 0, 1, \dots, m+r)$$

where the constants $C_{j,k+1}^*$ *are independent of* n.

THEOREM 3.2.1.a. *We have*

$$\left|y^{(q)}(x) - \overline{S}_k^{(q)}(x)\right| \leqq B_{q,k}^* h^{r+1-q} \quad (k = 0, 1, \dots, n;\ q = 0, 1, \dots, m+r)$$

where the constants $B_{q,k}^*$ *are independent of* n.

THEOREM 3.2.2.a. *We have*

$$\left|\overline{S}_\Delta^{(m)}(x) - f\left[x, \overline{S}_\Delta(x), \dots, \overline{S}_\Delta^{(m-1)}(x)\right]\right| \leqq K^* h^{r+1-m},$$

where the constant K^* *is independent of* n.

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(Received November 28, 1989)

UNIVERSITY OF VESZPRÉM
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
P.O.BOX 158
8201 VESZPRÉM HUNGARY

Acta Math. Hung.
59(1-2) (1992), 49–58.

ON APPROXIMATION OF FUNCTIONS BY POLYNOMIALS WITH WEIGHT

N. X. KY (Budapest)

Introduction

Let

$$u_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta} \qquad (\alpha,\beta>-1,\ x\in(-1,1)).$$

Denote $L^p_{\alpha,\beta}$ the Banach space of all measurable functions defined on the interval $[-1,1]$ for which

$$\|f\|_{p,\alpha,\beta}=\left(\int_{-1}^{1}|f(x)|^p u_{\alpha,\beta}(x)\,dx\right)^{1/p}<\infty \qquad (1\leqq p<\infty).$$

$L^{\infty}_{\alpha,\beta}=L^{\infty}$ is the space of all measurable, essentially bounded functions on $[-1,1]$ with the usual norm

$$\|f\|_{\infty}=\operatorname*{ess\,sup}_{x\in(-1,1)}|f(x)|.$$

For every $f\in L^p_{\alpha,\beta}$ $(1\leqq p\leqq\infty)$, we define the best approximation

(1) $$E_n(p,\alpha,\beta,f)=\inf_{p_n\in\Pi_n}\|f-p_n\|_{p,\alpha,\beta} \qquad (n\in\mathbf{N}),$$

where Π_n denotes the set of all algebraic polynomials of degree at most n, and $\mathbf{N}$ is the set of non-negative integers.

The aim of the present paper is to consider some fundamental problems concerning the approximation (1): an inequality of Jackson-type (in §.1), equivalent theorems (in §.2), direct and converse theorems (in §.3 and 4).

§.1. A Jackson type inequality

One of the most important results of the theory of trigonometric approximation is Jackson's inequality (see e.g. [8]). We are going to consider an inequality of similar type for the approximation (1).

Denote by $\mathbf{N}^+$ the set of all positive integers. For $k \in \mathbf{N}^+$, $1 \leqq p \leqq \infty$; $\alpha, \beta > -1$, let $W^{(k)}(p, \alpha, \beta)$ be the set of all measurable functions f defined on the interval $[-1, 1]$, whose derivatives of order at most $k-1$ are absolutely continuous on every subinterval $[-1+\varepsilon, 1-\varepsilon]$ $(0 < \varepsilon < 1/2)$, $(f^{(0)} := f)$ and

$$\Delta^\ell(x) f^{(\ell)}(x) \in L^p_{\alpha,\beta} \qquad (\ell = 0, 1, \ldots, k),$$

where

$$\Delta(x) = \sqrt{1-x^2} \qquad (x \in (-1, 1)).$$

THEOREM 1. *Let $k \in \mathbf{N}^+$, $1 \leqq p < \infty$, $\alpha, \beta > -1$. For every $f \in W^{(k)}(p, \alpha, \beta)$ we have*

$$E_n(p, \alpha, \beta, f) \leqq c(p, \alpha, \beta) n^{-k} \|\Delta^k f^{(k)}\|_{p,\alpha,\beta} \quad (n \in \mathbf{N}). \tag{2}$$

Here and later on $c(., \ldots)$ denotes a constant depending only on variables in the parenthesis.

Inequality (2) is best possible in the sense that the factors n^{-k} on the right-hand side cannot be replaced by $o(n^{-k})$.

For the proof of the theorem we need some lemmas.

Let $P_n(\alpha, \beta, x)$ be the n-th orthonormal Jacobi polynomial with respect to the parameters α and β. Every function $f \in L^p_{\alpha,\beta}$ has the Jacobi–Forier series

$$f(x) \sim \sum_{k=0}^{\infty} a_k(f) P_k(\alpha, \beta, x) \tag{3}$$

where

$$a_k(f) = \int_{-1}^{1} f(x) P_k(\alpha, \beta, x) u_{\alpha,\beta}(x) dx \qquad (k \in \mathbf{N}).$$

Let

$$\sigma_n^\delta(f, x) = \sum_{k=0}^{n} \frac{\binom{n-k+\delta}{n-k}}{\binom{n+\delta}{n}} a_k(f) P_k(\alpha, \beta, x) \qquad (\delta > 0, \ n \in \mathbf{N}).$$

Pollard [10] proved the inequality

$$\|\sigma_n^\delta(f)\|_{p,\alpha,\beta} \leqq c(p, \alpha, \beta) \|f\|_{p,\alpha,\beta} \tag{4}$$

$$(f \in L^p_{\alpha,\beta}, \quad 1 \leqq p \leqq \infty, \quad \delta > \max(\alpha, \beta), \quad n \in \mathbf{N}^+).$$

We shall apply the so-called generalization de la Vallée-Poussin means

$$V_n^\delta(f,x)=\frac{\sum_{j=0}^{\delta}(-1)^{j+1}\binom{\delta}{j}\binom{(j+1)n+\delta}{\delta}\sigma_{(j+1)n}^\delta(f,x)}{\delta! n^\delta}\qquad(\delta,n\in\mathbf{N}^+)$$

of the series (3), which was introduced in this form by G. Freud and J. Szabados [2].

LEMMA 1. *Let* $1\leqq p\leqq\infty$, $\alpha,\beta>-1$, $n\in\mathbf{N}^+$ *and let* $\delta>\max(\alpha,\beta)$ *be an integer. Then* V_n^δ *has the following properties:*

$$V_n^\delta(f)\in\Pi_{(\delta+1)n}\qquad(f\in L^p_{\alpha,\beta}),\tag{5}$$

$$V_n^\delta(p_n)=p_n\qquad(p_n\in\Pi_n),\tag{6}$$

$$\|V_n^\delta(f)\|_{p,\alpha,\beta}\leqq c(p,\alpha,\beta)\|f\|_{p,\alpha,\beta}\qquad(f\in L^p_{\alpha,\beta}),\tag{7}$$

and

$$\|V_n^\delta(f)-f\|_{p,\alpha,\beta}\leqq c(p,\alpha,\beta)E_n(p,\alpha,\beta,f)\qquad(f\in L^p_{\alpha,\beta}).\tag{8}$$

PROOF. (5) and (6) were proved in [2]. (7) follows from (4) and the definition of V_n^δ (see also [2]). Finally, we get (8) by (6) and (7).

Let now $\Pi_n^\perp(\alpha,\beta)$ $(n\in\mathbf{N}^+)$ be the set of all functions $g\in L^\infty$ satisfying

$$\int_{-1}^{1} g p_n u_{\alpha,\beta}\,dx=0$$

for any $p_n\in\Pi_n$.

LEMMA 2 ([7], Theorem 1). *For every* $g\in\Pi_n^\perp(\alpha,\beta)$, $n\in\mathbf{N}^+$ *we have*

$$\left|\int_{-1}^{x} g(t)u_{\alpha,\beta}(t)\,dt\right|\leqq\frac{c(\alpha,\beta)}{n}u_{\alpha,\beta}(x)\Delta(x)\|g\|_\infty\qquad(-1<x<1).\tag{9}$$

PROOF OF THEOREM 1. First let us prove the theorem for the case $k=1$. Our proof consists of three steps.

1. The case $p=1$. Let $f\in W^{(1)}(1,\alpha,\beta)$. By the duality principle of Nikolskiĭ we have (see e.g. [11], p.22)

$$E_n(1,\alpha,\beta,f)=\sup_{\substack{g\in\Pi_n^\perp(\alpha,\beta)\\ \|g\|_\infty\leqq1}}\int_{-1}^{1} g(x)f(x)u_{\alpha,\beta}(x)\,dx.\tag{10}$$

Now for any $g \in \Pi_n^{\perp}(\alpha, \beta)$, $\|g\|_\infty \leqq 1$ let

$$G(x) := \int_{-1}^{x} g(t) u_{\alpha,\beta}(t) dt \qquad (x \in (-1,1)).$$

Then by (9)

$$|G(x)| \leqq \frac{c(\alpha,\beta)}{n} u_{\alpha,\beta}(x) \Delta(x) \qquad (x \in (-1,1)). \tag{11}$$

Furthermore it is easy to see that

$$|G(x)| = O[(1-|x|) u_{\alpha,\beta}(x)] \qquad (|x| \to 1). \tag{12}$$

Using these estimates we show that

$$|f(x)G(x)| = o(1) \qquad (|x| \to 1). \tag{13}$$

Let for example $-1 \leqq x \leqq 0$. Then by (12) we get

$$\begin{aligned}
|f(x)G(x)| &= O\left\{ (1+x) u_{\alpha,\beta}(x) \left[\int_0^x f'(t)dt + f(0) \right] \right\} = \\
&= O\{(1+x)u_{\alpha,\beta}(x)\} + O\left\{ (1+x) u_{\alpha,\beta}(x) \int_0^x f'(t)dt \right\} = \\
&= o(1) + O\left\{ (1+x) u_{\alpha,\beta}(x) \int_0^x f'(t)dt \right\} \qquad (x \to -1).
\end{aligned} \tag{14}$$

Now if $\beta \geqq -1/2$, then

$$\begin{aligned}
|(1+x)u_{\alpha,\beta}(x) \int_0^x f'(t)dt| &= (1+x)^{1/2} \left| \int_0^x u_{\alpha,\beta}(x)\Delta(x) f'(t)dt \right| \leqq \\
&\leqq c(1+x)^{1/2} \left| \int_0^x u_{\alpha,\beta}(t)\Delta(t)|f'(t)|dt \right| \leqq \\
&\leqq c(1+x)^{1/2} \|\Delta f'\|_{1,\alpha,\beta} \to 0 \qquad \text{as} \quad x \to -1.
\end{aligned}$$

In the case when $-1<\beta<-1/2$ from the assumption $f\in W^{(1)}(1,\alpha,\beta)$ it follows that f' is integrable over the interval $[-1,0]$. Thus

$$\left|(1+x)u_{\alpha,\beta}(x)\int_0^x f'(t)dt\right|=o(1)\qquad (x\to -1).$$

This together with (14) implies (13) in the case $x\to -1$. The proof of the case $x\to 1$ of (13) is the same.

Now, using (11) and (13), by integration by parts we get

$$E_n(1,\alpha,\beta,f)=\sup_{\substack{g\in\Pi_n^{\perp}(\alpha,\beta)\\ \|g\|_\infty\leqq 1}}\int_{-1}^{1} f(x)g(x)u_{\alpha,\beta}(x)dx\leqq$$

$$\leqq \sup_{\substack{g\in\Pi_n^{\perp}(\alpha,\beta)\\ \|g\|_\infty\leqq 1}}\left|\int_{-1}^{1} f'(x)G(x)dx\right|\leqq\frac{c(\alpha,\beta)}{n}\|\Delta f'\|_{1,\alpha,\beta}$$

and so the case $p=-1$, $k=1$ of (2) is proved.

2. In the case $p=\infty$, $k=1$, (2) indeed follows from the well known Jackson inequality concerning trigonometric approximation (see e.g. [9], p. 177).

3. We obtain (2) in the case $1<p<\infty$, $k=1$ by application of Riesz–Thorin interpolation theorem and the just proved border line cases $p=1$, ∞ of it (see [13], p. 95). Here the linear operator we are applying for the interpolation is

$$T(f):=V_n^\delta(f)-f.$$

So our theorem is proved for $1\leqq p\leqq\infty$, $k=1$. The case $k>1$ of (2) follows from the case $k=1$ by induction.

It remains to prove that inequality (2) is best possible. It is enough to see this fact in the case $k=1$.

If $p=\infty$, then this statement follows from the same concerning trigonometric approximation (see e.g. [9], p. 177).

Let now $p=2$. By the formulas (see [12], (4.3.4), (4.5.5))

$$P'_{n+2}(\alpha,\beta,x)=\chi_{n+2}P_{n+1}(\alpha+1,\beta+1,x)\qquad (n\in\mathbf{N})$$

with $\chi_{n+2}\sim n$ we obtain for $f_0(x):=\chi_{n+2}^{-1}P_{n+2}(\alpha,\beta,x)$ that

$$E_{n+1}(2,\alpha,\beta,f_0)\sim(n+2)^{-1}$$

and

$$\|\Delta f_0'\|_{2,\alpha,\beta}\sim 1.$$

This fact proves that (2) is the best possible in the case $k=1$, $p=2$.

Now, suppose that in some case $1 \leqq p_0 < 2$, (2) is not the best possible in the above sense. Then by interpolating between p_0 and ∞ we obtain that the same is true for $p=2$ too, which contradicts the just proved case $p=2$.

By analogous argument we can prove that (2) is the best possible in the case $2<p<\infty$.

The proof of our theorem is now complete.

§.2. An equivalence theorem

We shall apply the concept of K-functional

$$K_k(p,\alpha,\beta,f,t) := \inf_{g\in W^{(k)}(p,\alpha,\beta)} \left\{ \|f-g\|_{p,\alpha,\beta} + t\|\Delta^k g^{(k)}\|_{p,\alpha,\beta} \right\}$$

$$(f\in L^p_{\alpha,\beta},\quad 1\leqq p<\infty,\quad \alpha,\beta>-1,\quad k\in \mathbf{N}^+,\quad 0<t<\infty)$$

due to J. Peetre.

Let furthermore $p_n(f)$ be the n-th polynomial of best approximation of f in $L^p_{\alpha,\beta}$, i.e.

$$E_n(p,\alpha,\beta,f) = \|f-p_n(f)\|_{p,\alpha,\beta}.$$

The following theorem is true.

THEOREM 2. *Let $1\leqq p<\infty$, $\alpha,\beta>-1$, k, j, $r\in\mathbf{N}^+$, $\lambda>0$, $1\leqq s\leqq\infty$. For every $f\in L^p_{\alpha,\beta}$ the following conditions are equivalent:*

a) $$\left\{ \sum_{n=1}^{\infty} [n^{r+\lambda} E_n(p,\alpha,\beta,f)]^s \frac{1}{n} \right\}^{1/s} < \infty.$$

b) $f\in W^{(k)}(p,\alpha,\beta)$ *and*

$$\left\{ \sum_{n=1}^{\infty} (n^{r+\lambda-k} \|\Delta^k [f-p_n(f)]^{(k)}\|_{p,\alpha,\beta})^s \frac{1}{n} \right\}^{1/s} < \infty \quad (k<r+\lambda).$$

c) $$\left\{ \sum_{n=1}^{\infty} (n^{r+\lambda-j} \|\Delta^j p_n^{(j)}(f)\|_{p,\alpha,\beta})^s \frac{1}{n} \right\}^{1/s} < \infty \qquad (r+\lambda<j).$$

d) $$\left\{ \int_0^{\infty} [t^{-(r+\lambda)/j} K_j(p,\alpha,\beta,f,t)]^s \frac{dt}{t} \right\}^{1/s} < \infty \qquad (r+\lambda<j).$$

PROOF. V. I. Ivanov [4] proved the inequality

$$\|\Delta^k p_n^{(k)}\|_{p,\alpha,\beta} \leqq c(k,p,\alpha,\beta) n^k \|p_n\|_{p,\alpha,\beta} \tag{15}$$

$$(1\leqq p<\infty,\quad \alpha,\beta>-1,\quad k,n\in\mathbf{N}^+,\quad p_n\in\Pi_n).$$

We obtain Theorem 2 by applying Theorem 3 in [1], and inequalities (2), (15).

§.3. Direct and converse theorems

For a function $f(x)$ defined on $[-1,1]$, let

$$f^*(\theta) := f(\cos\theta) \qquad (0 \leqq \theta \leqq \pi).$$

Let $0 < a < b < \pi$ be two arbitrary constants, which will be fixed in the sequel.

We define the modulus of continuity of a function $f \in L^p_{\alpha,\beta}$ $(\alpha, \beta \geqq -1/2,$ $1 \leqq p \leqq \infty)$ as follows:

$$(16) \qquad \omega(f,\delta)_p = \omega(a,b,\alpha,\beta,f,\delta)_p :=$$

$$= \sup_{0<h<\delta} \left\{ \int_0^b |f^*(\theta+h) - f^*(\theta)|^p \Delta^*(\theta) u^*_{\alpha,\beta}(\theta) d\theta \right\}^{1/p} +$$

$$+ \sup_{0<h<\delta} \left\{ \int_a^\pi |f^*(\theta-h) - f^*(\theta)|^p \Delta^*(\theta) u^*_{\alpha,\beta}(\theta) d\theta \right\}^{1/p}$$

$$(0 < \delta < I = \min(a, \pi - b)).$$

The existence of this modulus of continuity follows from the following inequalities:

$$(17) \qquad \begin{cases} u^*_{\alpha,\beta}(\theta) \leqq c(a,b,\delta) u^*_{\alpha,\beta}(\theta+h) & (a < \theta \leqq b) \\ u^*_{\alpha,\beta}(\theta) \leqq c(a,b,\delta) u^*_{\alpha,\beta}(\theta-h) & (a < \theta \leqq \pi) \end{cases}$$

$$(\alpha, \beta \geqq -1/2, \quad 0 < h \leqq d < \min(a, \pi - b)).$$

It is easy to see that $\omega(f,\delta)_p$ has the properties of modulus of continuity, i.e. it is a continuous increasing function of δ on $[0, I]$ tending to 0 with δ, and

$$\omega(f, \delta_1 + \delta_2) \leqq \omega(f, \delta_1) + \omega(f, \delta_2) \qquad (0 < \delta_1,\ \delta_2 < \delta_1 + \delta_2 < I).$$

Before we state direct and converse theorems, we remark that although (16) depends on the given constants a and b, it has the same order while $\delta \to 0$ for every pair a, b. Our modulus of continuity is equivalent to the K-functional defined above; more precisely we have

$$(18) \qquad c(p,\alpha,\beta,a,b) K_1(p,\alpha,\beta,f,t) \leqq$$

$$\omega(a,b,\alpha,\beta,f,\delta)_p \leqq c(p,\alpha,\beta,a,b) K_1(p,\alpha,\beta,f,t)$$

$$(0 < t < I).$$

The proof of this fact is essentially contained in [5] and [6]. The definition of this type can be applied to many other weights and in many cases it is useful for computation. According to this fact, in [8] we had applied a similar modulus of continuity for consideration of Uljanov type embedding problems, too.

THEOREM 3. *Let $1 \leqq p < \infty$, $\alpha, \beta \geqq -1/2$. For every $f \in L^p_{\alpha,\beta}$ we have*

$$(19)\qquad E_n(p,\alpha,\beta,f) \leqq c(p,\alpha,\beta,a,b)\omega\left(a,b,\alpha,\beta,f,\frac{1}{n}\right)_p \qquad (n \in \mathbf{N}^+),$$

$$(20)\qquad \omega(a,b,\alpha,\beta,f,\delta)_p \leqq c(p,\alpha,\beta,a,b)\delta \sum_{k=0}^{[\delta^{-1}]} E_k(p,\alpha,\beta,f) \qquad (0<\delta<I).$$

PROOF. The case $\alpha=\beta=0$, $1 \leqq p < \infty$ of the theorem was proved in [5], while the case $\alpha, \beta \geqq -1/2$, $p=2$ in [6]. The proof of the general case is the same, therefore we do not repeat it.

§.4. Direct and converse theorems with the moduli of Ditzian–Totik

Direct and converse theorems concerning the best approximation (1) were also given recently by Z. Ditzian and V. Totik (see [2] Theorems 7.2.1 and 8.2.1). Using new moduli of continuity of functions introduced by them, the authors proved direct and converse theorems for the best approximation by algebraic polynomials in $L^p[-1,1]$ with more general weights than $u_{\alpha,\beta}$. Their theorems imply the characterization of many classes of functions having a given order of best approximation, for example the order $n^{-\alpha}$ $(\alpha>0)$. However, in case of the weight $u_{\alpha,\beta}$ with $\alpha,\beta \geqq 0$, $\alpha+\beta>0$ their result (Theorem 8.2.1) can still be made sharp by using the results of this paper. Let us detail this fact.

Let $1 \leqq p < \infty$, $\alpha, \beta \geqq 0$, $r \in \mathbf{N}^+$. Let

$$(21)\qquad \begin{cases} w(x) := u_{\alpha/p,\beta/p}(x), \\ \varphi(x) := \sqrt{1-x^2}. \end{cases}$$

Denote $\omega^r_\varphi(f,t)_{w,p}$ the modulus defined by (B.1) or (B.2) in [2], p.218, with the weights w, φ defined by (21). Then by Theorem 6.1.1 in [2]

$$(22)\qquad \omega^r_\varphi(f,t)_{w,p} \sim K_r(p,\alpha,\beta,f,t).$$

On the other hand, using inequalities (2) and (15) by a usual method of approximation theory we can prove that for $f \in L_{\alpha,\beta}$, $r \in \mathbf{N}^+$, $n>r$

$$(23)\qquad E_n(p,\alpha,\beta,f) \leqq c(p,\alpha,\beta,r)K_r\left(p,\alpha,\beta,f,\frac{1}{n}\right),$$

and

$$K_r\left(p,\alpha,\beta,f,\frac{1}{n}\right)\leqq c(p,\alpha,\beta,r)\frac{1}{n^r}\sum_{k=0}^{n}(k+1)^{r-1}E_n(p,\alpha,\beta,f). \tag{24}$$

Combining these inequalities with (22) we get

THEOREM 4. *Let* $1\leqq p<\infty$; $\alpha,\beta\geqq 0$, $r\in\mathbf{N}^+$, $f\in L^p_{\alpha,\beta}$. *Let* $\omega^r_\varphi(f,t)_{w,p}$ *be the modulus defined above. Then*

$$E_n(p,\alpha,\beta,f)\leqq c(p,\alpha,\beta,r)\omega^r_\varphi\left(f,\frac{1}{n}\right)_{w,p} \qquad (n>r), \tag{25}$$

$$\omega^r_\varphi(f,h)_{w,p}\leqq c(p,\alpha,\beta,r)h^r\sum_{0\leqq k<1/h}(k+1)^{r-1}E_k(p,\alpha,\beta,f). \tag{26}$$

Inequality (26) indeed was proved in [2] (Theorem 8.2.4), while (25) is sharper than (8.2.1) in [2]; this fact follows from Theorem 6.2.2 in the same work. In the case $\alpha=\beta=0$, (25) was proved in [2] (Theorem 7.2.1). The authors proved that case by another method not applying inequality (2).

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(Received May 4, 1984; revised September 8, 1989)

L. EÖTVÖS UNIVERSITY
COMPUTING CENTRE
H–1117 BUDAPEST
BOGDÁNFY ÚT 10/B

Acta Math. Hung.
59(1-2) (1992), 59–74.

VARIETIES OF AUTOMATA AND TRANSFORMATION SEMIGROUPS

Z. ÉSIK (Szeged)

Introduction

The best formulation of the Krohn–Rhodes Decomposition Theorem in automata theory can perhaps be given in terms of certain 'generalized' α_0-products (equivalently cascade products) possessing feedback functions that permit encoding input signs with arbitrary nonempty input words of the component automata. Here, such generalized α_0-products are referred to as α_0^+-products as opposed to the α_0-product that only admits strict sign to sign encoding facility. It is apparent that there is natural bijective correspondence between α_0^+-varieties of automata, i.e. varieties of automata closed under the α_0^+-product, and closed classes of (complete) transformation semigroups in the sense of [2]. The latter are those classes of transformation semigroups closed under division (or covering) and wreath product. As regards α_0-varieties, i.e. varieties of automata closed under the α_0-product, no such natural bijective correspondence exists in general. The aim of this paper is to show that 'large' α_0-varieties are α_0^+-varieties, henceforth establishing a natural bijective correspondence between large α_0-varieties and large closed classes of transformation semigroups. An α_0-variety is large if it contains the automaton $\mathbf{D}_0=(\{1,2\},\{x,y\},\delta)$ with $\delta(i,x)=1$, $\delta(i,y)=2$, $i=1,2$, as well as all the automata $\mathbf{C}_p^1=(\{1,\dots,p\},\{x_0,x\},\delta)$ with $\delta(i,x_0)=i$, $\delta(i,x)\equiv i+ +1 \mod p$, $i=1,\dots,p$, where p is any prime. In other words, an α_0-variety is large if it contains all definite automata and all commutative permutation automata.

The paper is organized as follows. The first section contains all necessary definitions. The second section is devoted to recalling some results on definite and commutative automata. The third section exhibits an auxiliary construction that will be our fundamental tool in proving the above mentioned main result. A similar construction was used in [3] and then in a number of papers, e.g., [1], [5], [4]. The proof of the main result is given in the fourth section. The paper ends with a few remarks.

1. Preliminaries

An *automaton* is a system $\mathbf{A}=(A,X,\delta)$ consisting of finite nonempty sets A and X (the *state set* and the set of *input signs*, respectively), and *transition function* $\delta : A\times X \to A$. The transition is also used in the extended sense, i.e. as a mapping $\delta : A\times X^* \to A$, where X^* is the free monoid of all words generated by X. Thus, X^* includes the empty word λ. We set $X^+ = X^* - \{\lambda\}$.

The *characteristic monoid* $S_1(A)$ of the automaton $\mathbf{A}$ is the semigroup of all transformations $A\to A$ induced by words in X^*, i.e. $S_1(\mathbf{A})=\{\delta_u : u\in X^*\}$ where $\delta_u : A\to A$ is given by $\delta_u(a)=\delta(a,u)$. We let $S(\mathbf{A})$ denote the subsemigroup of $S_1(\mathbf{A})$ consisting of all transformations δ_u with $u\in X^+$. $S(A)$ is called the *characteristic semigroup* of $\mathbf{A}$.

Our fundamental concept is that of the α_0-product (see [6]). The α_0-product is known to be equivalent to the cascade composition ([12]) and the loop-free product ([8]). Let $\mathbf{A}_t=(A_t,X_t,\delta_t)$, $t\in[n]=\{1,\ldots,n\}$, $n\geqq 0$, be automata. Suppose we are given a family of *feedback functions*

$$\phi_t : A_1\times\ldots\times A_{t-1}\times X\to X_t, t\in[n],$$

where X is a new finite nonempty set. The *α_0-product* $\mathbf{A}_1\times\ldots\times\mathbf{A}_n(X,\phi)$ is then defined to be the automaton $\mathbf{A}=(A,X,\delta)$ with

$$A=A_1\times\ldots\times A_n$$

and

$$\delta((a_1,\ldots,a_n),x)=(\delta_1(a_1,x_1),\ldots,\delta_n(a_n,x_n))$$
$$x_t=\phi_t(a_1,\ldots,a_{t-1},x)$$

for all $(a_1,\ldots,a_n)\in A$ and $x\in X$.

Besides α_0-products, we shall be interested in the following generalizations. Suppose that each feedback function ϕ_t maps to the set X_t^* rather than X_t. Then, in exact analogue with the definition of the α_0-product, we obtain the concept of the *α_0^*-product* (see [7]). If only nonempty words occur in the ranges of the feedback functions, we speak about an *α_0^+-product.* That is, in case of an α_0^+-product, the feedback functions are of the form $\phi_t : A_1\times\ldots\times A_{t-1}\times X\to X_t^+$.

An *α_0^λ-product* is an α_0^*-product such that for each t, the feedback function ϕ_t can be treated as a mapping $A_1\times\ldots\times A_{t-1}\times X\to X_t^\lambda = X_t\cup\{\lambda\}$.

A special case of the α_0-product is the *quasidirect product* or *q-product* for short. An α_0-product as above is called a q-product if each feedback function ϕ_t is independent of any state variable a_s $(1\leqq s<t)$. Thus, in a q-product, we can alternatively think of a feedback function ϕ_t as a map

$X \to X_t$. In the same way as for the α_0-product, the q-product generalizes to the following concepts: *q^*-product, q^+-product* and *q^λ-product.*

Let $\mathcal{K}$ be a (possibly empty) class of automata. We define:

$\mathbf{P}_{\alpha_0}(\mathcal{K})$:= all α_0-products of automata from $\mathcal{K}$,

$\mathbf{P}^+_{\alpha_0}(\mathcal{K})$:= all α_0^+-products of automata from $\mathcal{K}$,

$\mathbf{P}^\lambda_{\alpha_0}(\mathcal{K})$:= all α_0^λ-products of automata from $\mathcal{K}$,

$\mathbf{P}^*_{\alpha_0}(\mathcal{K})$:= all α_0^*-products of automata from $\mathcal{K}$,

$\mathbf{P}_q(\mathcal{K})$:= all q-products of automata from $\mathcal{K}$,

$\mathbf{P}^+_q(\mathcal{K})$:= all q^+-products of automata from $\mathcal{K}$,

$\mathbf{P}^\lambda_q(\mathcal{K})$:= all q^λ-products of automata from $\mathcal{K}$,

$\mathbf{P}^*_q(\mathcal{K})$:= all q^*-products of automata from $\mathcal{K}$,

$\mathbf{H}(\mathcal{K})$:= all homomorphic images of automata from $\mathcal{K}$,

$\mathbf{S}(\mathcal{K})$:= all subautomata of automata from $\mathcal{K}$,

Given an automaton $\mathbf{A} = (A, X, \delta)$, we define the automata $\mathbf{A}^* = (A, S_1(\mathbf{A}), \delta^*)$, $\mathbf{A}^+ = (A, S(\mathbf{A}), \delta^+)$ and $\mathbf{A}^\lambda = (A, X \cup \{x_0\}, \delta^\lambda)$ by $\delta^*(a, \delta_u) = \delta(a, u)$, $\delta^+(a, \delta_v) = \delta(a, v)$ and $\delta^\lambda(a, x) = \delta(a, x)$, $\delta^\lambda(a, x_0) = a$, where $a \in A$, $u \in X^*$, $v \in X^+$ and $x \in X$. It is assumed that $x_0 \notin X$. For a class $\mathcal{K}$ we set $\mathcal{K}^* = \{\mathbf{A}^* : \mathbf{A} \in \mathcal{K}\}$, $\mathcal{K}^+ = \{\mathbf{A}^+ : \mathbf{A} \in \mathcal{K}\}$ and $\mathcal{K}^\lambda = \{\mathbf{A}^\lambda : \mathbf{A} \in \mathcal{K}\}$. Obviously, $\mathbf{P}^*_{\alpha_0}(\mathcal{K}) = \mathbf{P}_{\alpha_0}(\mathcal{K}^*)$, $\mathbf{P}^+_{\alpha_0}(\mathcal{K}) = \mathbf{P}_{\alpha_0}(\mathcal{K}^+)$ and $\mathbf{P}^\lambda_{\alpha_0}(\mathcal{K}) = \mathbf{P}_{\alpha_0}(\mathcal{K}^\lambda)$.

Our main concern will be α_0-varieties. By definition, a class $\mathcal{K}$ of automata is an *α_0-variety* if and only if $\mathbf{H}(\mathcal{K}) \subseteqq (\mathcal{K})$, $\mathbf{S}(\mathcal{K}) \subseteqq \mathcal{K}$ and $\mathbf{P}_{\alpha_0}(\mathcal{K}) \subseteqq \mathcal{K}$. Requiring closure under $\mathbf{P}^+_{\alpha_0}$ $(\mathbf{P}^\lambda_{\alpha_0}, \mathbf{P}^*_{\alpha_0})$ instead of closure under $\mathbf{P}_{\alpha_0}$, we get the notion of the *α_0^+-variety (α_0^λ-variety, α_0^*-variety)*. The concept of a *q-variety, (q^+-variety, q^λ-variety, q^*-variety)* is similarly defined. It is known that $\mathbf{HSP}_{\alpha_0}(\mathcal{K})$ is the smallest α_0-variety containing a given class $\mathcal{K}$, i.e. the α_0-variety generated by $\mathcal{K}$. Analogous statements are true for any other variety concept defined above. The easy proofs of the following two propositions are omitted.

PROPOSITION 1.1. *Let z be any of the modifiers $+$, λ and $*$. The following are equivalent for an α_0-variety (q-variety) $\mathcal{K}$:*

(i) *$\mathcal{K}$ is an α_0^z-variety (q^z-variety),*

(ii) *$\mathcal{K}^z \subseteqq \mathcal{K}$,*

(iii) *$\mathcal{K} = \mathbf{HSP}^z_{\alpha_0}(\mathcal{K}_0)$ $(\mathcal{K} = \mathbf{HSP}^z_q(\mathcal{K}_0))$ for a class $\mathcal{K}_0$.*

PROPOSITION 1.2. *Let $\mathcal{K}$ be an α_0^λ-variety (q^λ-variety). Then $\mathcal{K}$ is an α_0^*-variety if and only if $\mathcal{K}$ is an α_0^+-variety (q^+-variety).*

There is an obvious connection between α_0^+-varieties and classes of transformation semigroups closed under *division* (or *covering*) and *wreath product.*

According to [2], such classes of transformation semigroups will simply be referred to as *closed classes.* (More exactly, partial transformation semigroups are also treated in the book of Eilenberg [2], but we will not consider this possibility.)

Let $\mathbf{A}$ be an automaton (A, X, δ). We can form the transformation semigroup $(A, S(\mathbf{A}))$, which is just the automaton $\mathbf{A}^+$ considered as a transformation semigroup. For a class $\mathcal{K}$ of automata, let $\mathbf{T}(\mathcal{K})$ denote the class of all transformation semigroups so obtained from automata in $\mathcal{K}$. Now, the obvious connection is this: The operation $\mathcal{K} \to \mathbf{T}(\mathcal{K})$ is a bijective correspondence between α_0^+-varietes and (nonempty) closed classes of transformation semigroups. Further, when restricted to α_0^*-varieties, this operation is a bijective correspondence between α_0^*-varieties and closed classes of transformation semigroups that are generated by transformation monoids, see [2] for the definition of a transformation monoid. The latter observation follows from the following statement whose easy proof is omitted.

PROPOSITION 1.3. *An α_0^+-variety $\mathcal{K}$ is an α_0^*-variety if and only if $\mathbf{T}(\mathcal{K})$ is generated by transformation monoids.*

We note that a similar bijective correspondence can be established between q^+-varieties and so-called *weakly closed classes* of transformation semigroups.

2. Commutative and definite automata

An automaton $\mathbf{A} = (A, X, \delta)$ is called *commutative* if $\delta(a, xy) = \delta(a, yx)$ $(a \in A, x, y \in X)$ holds universally, i.e. when $S(\mathbf{A})$, or $S_1(\mathbf{A})$, is commutative. If in addition δ_x is a permutation of the state set for each input sign x, we call $\mathbf{A}$ a *commutative permutation automaton.* Clearly, $\mathbf{A}$ is a commutative permutation automaton if and only if $S_1(\mathbf{A})$ is an Abelian group. Note that $S_1(\mathbf{A}) = S(\mathbf{A})$ in this case. Commutative automata form a q^*-variety as do commutative permutation automata. Examples of commutative permutation automata are the following.

Counters. A counter is an automaton $\mathbf{C}_n = ([n], \{x\}, \delta)$ with $\delta(i, x) \equiv i + 1 \mod n$.

Counters with identity. These are the automata $\mathbf{C}_n^1 = ([n], \{x_0, x\}, \delta)$ with $\delta(i, x_0) = i$, $\delta(i, x) \equiv i + 1 \mod n$.

The automata $\mathbf{Z}_n$. We set $\mathbf{Z}_n = ([n], \{x_1, \ldots, x_n\}, \delta)$ with $\delta(i, x_j) \equiv i + j \mod n$.

The following result for which we give a simple proof is implicit in [6].

THEOREM 2.1. *Every q^λ-variety $\mathcal{K}$ of commutative automata is a q^*-variety.*

PROOF. We must show that $\mathcal{K}^+ \subseteq \mathcal{K}$. This will be obvious once we have proved that if $\mathbf{A} = (A, X, \delta) \in \mathcal{K}$ and $x_1, x_2 \in X$ then the automaton $\mathbf{A}' =$

$=(A, X\cup\{y\}, \delta')$ is also in $\mathcal{K}$, where $y\notin X$, $\delta'_y=\delta_{x_1x_2}$ and $\delta'_x=\delta_x$ for all $x\in X$.

Suppose first that $\mathbf{A}'$ (and thus $\mathbf{A}$ too) is generated by a single state a_0. Take the q^λ-product $\mathbf{B}=\mathbf{A}\times\mathbf{A}(X,\phi)$ with $\phi_1(x)=x$, $\phi_2(x)=\lambda$, all $x\in X$, and $\phi_1(y)=x_1$, $\phi_2(y)=x_2$. A straightforward computation proves that the map

$$(\delta(a_0,u),\delta(a_0,v))\to\delta'(a_0,uv),$$

where u, $v\in X^*$, is a homomorphism of $\mathbf{B}$ onto $\mathbf{A}'$, so that $\mathbf{A}'\in\mathcal{K}$. If $\mathbf{A}'$ is not generated by a single state then let $\mathbf{A}'_1,\dots,\mathbf{A}'_k$ ($k\geqq 2$) be all maximal singly generated subautomata of $\mathbf{A}'$. The first part of the proof gives that the automata $\mathbf{A}'_i$ are in $\mathcal{K}$. Let $\mathbf{C}$ be the disjoint sum of the $\mathbf{A}'_i$. Since $\mathcal{K}$ clearly contains the automaton $(\{0,1\},\{y\},\overline{\delta})$ with $\overline{\delta}(i,y)=i$, $i=0,1$, it follows that $\mathcal{K}$ is closed under disjoint sums. So we have $\mathbf{C}\in\mathcal{K}$. Since $\mathbf{A}'$ is a homomorphic image of $\mathbf{C}$, we obtain $\mathbf{A}'\in\mathcal{K}$ as claimed.

The Krohn–Rhodes Decomposition Theorem, see [2], and the above result jointly give the following.

COROLLARY 2.2. *For every integer $n\geqq 1$ we have $\mathbf{Z}_n\in\mathbf{HSP}_{\alpha_0}(\{\mathbf{C}^1_p : p$ is a prime divisor of $n\})$.*

Besides commutative permutation automata, we shall make use of definite automata. An automaton $\mathbf{A}=(A,X,\delta)$ is called *definite* with degree $n\geqq 0$ if $\delta(a,x_1\dots x_n)=\delta(b,x_1\dots x_n)$ holds for every $a,b\in A$ and $x_1,\dots,x_n\in$ $\in X$. Further, an automaton is definite if it is definite with degree n for some n. The proof of the following result (stated in an other form) can be found in [2], p. 95 and p. 141. Recall the definition of $\mathbf{D}_0$ from the Introduction.

THEOREM 2.3. *$\mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0\})$ is exactly the class of all definite automata.*

As shown in [9], even $\mathbf{ISP}_{\alpha_0}(\{\mathbf{D}_0\})$ contains all the definite automata where $\mathbf{I}$ denotes the formation of isomorphic images. The class of all definite automata is an α_0^+-variety.

Important examples of definite automata are the shiftregisters. Let X be a finite nonempty set. The *shiftregister* of degree $n\geqq 1$ over X is the automaton $\mathbf{R}_n(X)=(R_n(X),X,\delta)$ with $R_n(X)=\{x_1\dots x_n : x_1,\dots,x_n\in X\}$ and $\delta(x_1\dots x_n,x)=x_2\dots x_n x$, $x_1\dots x_n\in R_n(X)$, $x\in X$.

3. An auxiliary result

In this section we present a modified version of a construction used [4] and provide a lemma that will be our fundamental tool. But first some more notation.

Let X be a finite nonempty set and n a nonnegative integer. We denote by $X^{(n)}$ the set of all words $u=x_1\dots x_n\in X^*$, $x_1,\dots,x_n\in X$. Thus, $X^{(n)}$

is the set of all words $u \in X^*$ having length $\|u\| = n$. Let u, v, $w \in X^*$ with $w = uv$ and $\|u\| = n$. We write $u = \mathrm{pre}_n w$ and $v = w/n$.

Now let A, X and Y be finite nonempty sets, m, n, $k \geqq 1$ integers with $m \geqq n$, and let

$$\varrho : Y^{(k)} \to X^+,$$
$$\tau_1 : Y^{(m)} \to X^{(m-n+1)} \cup \ldots \cup X^{(m)},$$
$$\tau_2 : Y^{(m)} \to X^{(m+1)} \cup \ldots \cup X^{(m+n)}$$

be mappings subject to the following two conditions:

$$m < \|\varrho(v)\| - k \leqq m + n, \text{ for all } v \in Y^{(k)}, \tag{1}$$

$$\|\tau_1(v)\| \equiv \|\tau_2(v)\| \mod n, \text{ for all } v \in Y^{(m)}. \tag{2}$$

To each such system we correspond an automaton $\mathbf{R}_{A,\varrho,\tau_1,\tau_2} = (R, Y, \delta)$. Set $R = R_1 \cup R_2$,

$$R_1 = \{(a, v) : a \in A, v \in Y^*, \|v\| \leqq k\},$$
$$R_2 = \{(v, u) : v \in Y^+, u \in X^+, \|v\| \leqq m,\ m + 1 \leqq \|v\| + \|u\| \leqq m + n\}.$$

Define for each $y \in Y$,

$$\text{(i)} \qquad \delta((a, v), y) = \begin{cases} (a, vy) & \text{if } \|v\| < k, \\ (y, \varrho(v)/k+1) & \text{if } \|v\| = k, \end{cases}$$

all $(a, v) \in R_1$,

$$\text{(ii)} \qquad \delta((v, u), y) = \begin{cases} (vy, u/1) & \text{if } \|v\| < m, \\ (y, w) & \text{if } \|v\| = m, \end{cases}$$

all $(v, u) \in R_2$, where $w = (u/1)\tau_1(v)$ or $w = (u/1)\tau_2(v)$ depending on whether or not $j - 1 + r \geqq n$ with $j, r \in [n]$ given by $j \equiv \|u\| \mod n$ and $r \equiv \|\tau_1(v)\| - m \mod n$. (Note that the second subcase never applies in (ii) above with $\|\tau_2(v)\| = m + n$. Thus, if $n = 1$, the definition of $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$ is independent of τ_2.)

Some computation is needed in order to check that our definition of $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$ works properly. As regards (i) and the first subcase of (ii), this is obvious. For the second subcase of (ii) suppose $u \in X^+$ with $1 \leqq \|u\| \leqq n$, and let $j, r \in [n]$ be determined by $j \equiv \|u\| \mod n$ and $r \equiv \|\tau_1(v)\| - m \mod n$, respectively, which is to say that $\|u\| = j$, $\|\tau_1(v)\| = m - n + r$ and $\|\tau_2(v)\| = m + r$. Assume $j - 1 + r \geqq n$ and put $w = u/1\tau_1(v)$. We obtain $m = n + m - n \leqq j - 1 + r + m - n = j - 1 + m - n + r = \|u/1\| + \|\tau_1(v)\| = \|w\|$ and $\|w\| = \|u/1\| + \|\tau_1(v)\| = j - 1 + m - n + r = m + j + r - n - 1 \leqq m + 2n - n - 1 < m + n$. Thus, $m \leqq \|w\| < m + n$, as was to be seen. Next let $j - 1 + r < n$, $w = u/1\tau_2(v)$. From $\|\tau_2(v)\| = m + r$ and $r \in [n]$ we obtain $m \leqq \|w\|$. On the other hand, $\|w\| = \|u/1\| + \|\tau_2(v)\| = j - 1 + m + r = m + j - 1 + r < m + n$, proving $m \leqq \|w\| < m + n$.

LEMMA 3.1. $\mathbf{R}_{A,\varrho,\tau_1,\tau_2} \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m, \mathbf{C}_n^1\})$.

PROOF. We define an α_0-product

$$\mathbf{B} = (B, Y, \delta') = \mathbf{Q} \times \mathbf{C}_m \times \mathbf{R}_1 \times \mathbf{Z}_n \times \mathbf{R}_2(Y, \phi)$$

where $\mathbf{Q}, \mathbf{R}_1$ and $\mathbf{R}_2$ are definite automata, and show that $\mathbf{R}_{A,\varrho,\tau_1,\tau_2} \in \in \mathbf{HS}(\{\mathbf{B}\})$. Thus the result follows by Corollary 2.2 and Theorem 2.3. The symbol $*$ will be supposed not to be in $X * \cup Y *$. We define

$$\mathbf{Q} = (\{(a, v) : a \in A,\ v \in Y^*,\ \|v\| \leqq k\} \cup \{*\}, Y, \delta_1),$$
$$\delta_1((a, v), y) = \begin{cases} (a, vy) & \text{if } \|v\| < k, \\ * & \text{if } \|v\| = k, \end{cases}$$
$$\delta_1(*, y) = *,$$

all $y \in Y$, $a \in A$, $v \in Y^*$ with $\|v\| \leqq k$;

$$\mathbf{R}_1 = \mathbf{R}_m(Y),$$
$$\mathbf{R}_2 = ((X \cup \{*\})^{(m+n-1)}, [m+n-1] \times (X \cup \dots \cup X^{(m+n-1)}) \cup \{*\}, \delta_2),$$
$$\delta_2(z_1 \dots z_{m+n-1}, *) = z_2 \dots z_{m+n-1} *,$$
$$\delta_2(z_1 \dots z_{m+n-1}, (t, x_1 \dots x_s)) = \begin{cases} z_2 \dots z_t x_1 \dots x_{m+n-t} & \text{if } m+n-t \leqq s, \\ z_2 \dots z_t x_1 \dots x_s * \dots * & \text{if } m+n-t > s, \end{cases}$$

where $z_1, \dots, z_{m+n-1} \in X \cup \{*\}$, $t, s \in [m+n-1]$, $x_1, \dots, x_s \in X$. Notice that $\mathbf{R}_2$ is definite of degree $m+n-1$, and $\mathbf{Q}$ is definite of degree $k+1$. ($\mathbf{Q}$ is even nilpotent.)

The feedback functions are defined as follows. Let $y \in Y$, $a \in A$, $v_0 \in Y^*$, $\|v_0\| \leqq k$, $i \in [m]$, $v \in Y^{(m)}$ and $j \in [n]$. We define

(α) $\phi_1(y) = y$,

(β) $\phi_2((a, v_0), y) = x, \quad \phi_2(*, y) = x$,

(γ) $\phi_3((a, v_0), i, y) = y, \quad \phi_3(*, i, y) = y$,

(δ) $$\phi_4((a, v_0), i, v, y) = \begin{cases} x_t & \text{where } t = \|\varrho(v_0)\| - (m+k) \text{ if } \|v_0\| = k, \\ x_n & \text{if } \|v_0\| < k, \end{cases}$$

$$\phi_4(*, i, v, y) = \begin{cases} x_r & \text{where } r \in [n] \text{ with } \|\tau_1(v)\| - m \equiv r \mod n, \\ & \qquad \text{if } i \equiv k \mod m, \\ x_n & \text{if } i \not\equiv k \mod m, \end{cases}$$

(η) $$\phi_5((a, v_0), i, v, j, y) = \begin{cases} * & \text{if } \|v_0\| < k, \\ (1, \varrho(v_0)/k+1) & \text{if } \|v_0\| = k, \end{cases}$$

$$\phi_5(*, i, v, j, y) = \begin{cases} * & \text{if } i \not\equiv k \mod m, \\ (j, w) & \text{if } i \equiv k \mod m), \end{cases}$$

where $w=\tau_1(v)$ or $w=\tau_2(v)$ depending on whether or not $j-1+r\geqq n$ with $r\in[n]$ given by $r\equiv\|\tau_1(v)\|-m \mod n$.

Set

$$B_1=\{((a,v_0),i,v,n,*\ldots *):a\in A, v_0\in Y^*, \|v_0\|\leqq k, i\in[m],$$
$$i\equiv\|v_0\| \mod m, v\in Y^{(m)}\},$$
$$B_2=\{(*,i,v,j,u*\ldots *):i\in[m], v\in Y^{(m)}, j\in[n], u\in X^+,$$
$$\nu(i,k)+\|u\|=m+j\},$$

where, for each r, $s\geqq 0$, $\nu(r,s)=t$ if and only if $t\in[m]$ and $r-s\equiv t \mod m$. It is obvious that B_1, $B_2\subseteqq B$. Put $B'=B_1\cup B_2$. We claim that $\mathbf{B}'=(B',Y',\delta')$ is a subautomaton of $\mathbf{B}$ that can be mapped homomorphically onto $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$.

Let $h:B'\to R$ be the mapping given by

$$h((a,v_0),i,v,n,*\ldots *)=(a,v_0),$$

all $((a,v_0),i,v,n,*\ldots *)\in B_1$,

$$h(*,i,vv_0,j,u*\ldots *)=(v_0,u),$$

all $(*,i,vv_0,j,u*\ldots *)\in B_2$ with $\|v_0\|=\nu(i,k)$, $u\in X^+$. Obviously, h maps B' onto R. Let $b\in B'$ and $y\in Y$. There are two cases.

(1) $b=((a,v_0),i,v,n,*\ldots *)\in B_1$ so that $i\equiv\|v_0\| \mod m$. Define $i'\in[m]$, $i'\equiv i+1 \mod m$. Obviously, $i'\equiv\|v_0y\| \mod m$.

(1.1) $\|v_0\|<k$. Then $\delta'(b,y)=((a,v_0y),i',v/1y,n,*\ldots *)$. We see that $\delta'(b,y)\in B_1$ and $h(\delta'(b,y))=(a,v_0y)=\delta((a,v_0),y)=\delta(h(b),y)$.

(1.2) $\|v_0\|=k$. Put $t=\|\varrho(v_0)\|-(m+k)$. By our assumptions on ϱ, we have $t\in[n]$. Obviously, $\|\varrho(v_0)\|>k+1$ so that $\varrho(v_0)/k+1\in X^+$. Since $i'\equiv\|v_0y\| \mod m$ and $\|v_0\|=k$, $\nu(i',k)=1$. We have $\nu(i',k)+\|\varrho(v_0)/k+1\|=1+m+t-1=m+t$. Thus $\delta'(b,y)=(*,i',v/1y,t,\varrho(v_0)/k+1*\ldots *)$ is in B_2. Further, $h(\delta'(b,y))=(y,\varrho(v_0)/k+1)=\delta((a,v_0),y)=\delta(h(b),y)$.

(2) $b=(*,i,vv_0,j,u*\ldots *)\in B_2$ with $\|v_0\|=\nu(i,k)$ and $u\in X^+$ so that $\nu(i,k)+\|u\|=m+j$. Again let $i'\in[m]$ with $i'\equiv i+1 \mod m$.

(2.1) $i\not\equiv k \mod m$, i.e. $1\leqq\nu(i,k)<m$. We see that $\|u\|>1$, $\|v\|\geqq 1$, $\delta'(b,y)=(*,i',v/1v_0y,j,u/1*\ldots *)$. Since $\nu(i',k)+\|u/1\|=\nu(i,k)+1+\|u\|-1=\nu(i,k)+\|u\|=m+j$, it follows that $\delta'(b,y)\in B_2$. Moreover, $h(\delta'(b,y))=(v_0y,u/1)=\delta((v_0,u),y)=\delta(h(b),y)$.

(2.2) $i\equiv k \mod m$, i.e. $\nu(i,k)=m$. Obviously, $v=\lambda$, $\|u\|=j$. Define $r\in[n]$, $r\equiv\|\tau_1(v)\|-m \mod n$, $j'\in[n]$, $j+r\equiv j' \mod n$. We have $\|\tau_1(v)\|=m-n+r$, $\|\tau_2(v)\|=m+r$. Since $\nu(i,k)=m$, then $\nu(i',k)=1$.

(2.2.1) $j-1+r\geqq n$. Compute $\nu(i',k)+\|u/1\tau_1(v)\|=1+\|u\|-1+\|\tau_1(v)\|=j+m-n+r=m-n+(j+r)=m-n+(n+j')=m+j'$. (The use of $u/1$ is legitimate because $\|u\| = j \in [n]$.) It follows that $\delta'(b,y) = (*, i', v_0/1y, j', u/1\tau_1(v)* \ldots *) \in B_2$. Further, $h(\delta'(b,y)) = (y, u/1\tau_1(v)) = \delta((v_0,u),y) = \delta(h(b),y)$.

(2.2.2) $j-1+r < n$. We have $\nu(i',k) + \|u/1\tau_2(v)\| = 1 + \|u\| - 1 + \|\tau_2(v)\| = j+m+r = m+(j+r) = m+j'$. Thus, $\delta(b,y) = (*, i', v_0/1y, j', u/1\tau_2(v)* \ldots *) \in B_2$, and $h(\delta'(b,y)) = (y, u/1\tau_2(v)) = \delta((v_0,u),y) = \delta(h(b),y)$.

The proof of Lemma 3.1 is complete.

As we have already noted, $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$ does not depend on τ_2 if $n=1$. This observation gives rise to the following definition.

Let $k, m \geqq 1$ and let A, X, Y, ϱ and $\tau = \tau_1$ be as in the definition of $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$ when $n=1$, i.e.,

$$\varrho : Y^{(k)} \to X^{(m+k+1)},$$
$$\tau : Y^{(m)} \to X^{(m)}.$$

We define $\mathbf{R}_{A,\varrho,\tau} = (R, Y, \delta)$ with

$$R = R_1 \cup R_2,$$
$$R_1 = \{(a,v) : a \in A, v \in Y^*, \|v\| \leqq k\},$$
$$R_2 = \{(v,u) : v \in Y^+, u \in X^+, \|v\| + \|u\| = m+1\},$$
$$\delta((a,v),y) = \begin{cases} (a, vy) & \text{if } \|v\| < k, \\ (y, \varrho(v)/k+1) & \text{if } \|v\| = k, \end{cases}$$

all $(a,v) \in R_1$, $y \in Y$,

$$\delta((v,u),y) = \begin{cases} (vy, u/1) & \text{if } \|v\| < m, \\ (y, \tau(v)) & \text{if } \|v\| = m, \end{cases}$$

all $(v,u) \in R_2$, $y \in Y$.

COROLLARY 3.2. $\mathbf{R}_{A,\varrho,\tau} \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m\})$.

PROOF. By Lemma 3.1, $\mathbf{R}_{A,\varrho,\tau} \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m, \mathbf{C}_n^1\})$. Since $\mathbf{C}_n^1$ is trivial, we have $\mathbf{R}_{A,\varrho,\tau} \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m\})$.

4. The main result

In this section we prove the following result.

THEOREM 4.1. *Let $\mathcal{K}$ be an α_0-variety of automata containing $\mathbf{D}_0$ and all of the automata $\mathbf{C}_p^1$ where p is any prime. Then $\mathcal{K}$ is an α_0^+-variety.*

PROOF. It suffices to prove that $\mathcal{K}^+ \subseteq \mathcal{K}$. Take an arbitrary automaton $\mathbf{A} = (A, X, \delta) \in \mathcal{K}$ and set $S = S(\mathbf{A})$. We claim that there exist integers m,

$n,\ k \geqq 1$ with $m \geqq n$ and mappings

$$\gamma : S \cup \ldots \cup S^{(k)} \to X^+,$$
$$\tau_1 : S^{(m)} \to X^{(m-n+1)} \cup \ldots \cup X^{(m)},$$
$$\tau_2 : S^{(m)} \to X^{(m+1)} \cup \ldots \cup X^{(m+n)}$$

satisfying the following conditions:

(1) $\|\gamma(v)\| \geqq \|v\|$ for all $v \in S^+,\ \|v\| \leqq k$,

(2) $\mathrm{pre}_{\|v\|}\gamma(vs) = \mathrm{pre}_{\|v\|}\gamma(v)$, for all $v \in S^+,\ s \in S$ with $\|vs\| \leqq k$,

(3) $m < \|\gamma(v)\| - k \leqq m + n$, for all $v \in S^{(k)}$,

(4) $\|\tau_1(v)\| \equiv \|\tau_2(v)\| \mod n$, for all $v \in S^{(m)}$,

(5) $\overline{v} = \delta_{\gamma(v)}$, for all $v \in S^{(k)}$,

(6) $\overline{v} = \delta_{\tau_1(v)} = \delta_{\tau_2(v)}$, for all $v \in S^{(m)}$,

where, for any $u = s_1 \ldots s_i \in S^+$ with $s_1, \ldots, s_i \in S$, $\overline{u}$ denotes the product $s_1 \cdots s_i$ in S. For later use we define $\overline{\lambda} = \delta_\lambda$, i.e. $\overline{\lambda}$ is the identity map $A \to A$.

Let $k = \mathrm{card}(S)$ and denote by E the set of all idempotents in S. Obviously, there exists an integer $n \geqq 1$ with $E \subseteqq \{\delta_u : u \in X^{(n)}\}$. Let $m_0 \geqq n$ be any integer with $S \subseteqq \{\delta_u : u \in X^+,\ \|u\| \leqq m_0\}$. To each $s \in S$ let us assign a fixed word $u(s) \in X^+$ such that $s = \delta_{u(s)}$ and $\|u(s)\| \leqq m_0$. If $s = e \in E$ choose $u(e)$ with $\|u(e)\| = n$. Finally, let m be an integer with

$$m \geqq \max\{k, (m_0 - 1)k + (k-2)n, 2m_0 + n\}.$$

Denote by Z the collection of all words $v = s_1 \ldots s_i$ $(1 \leqq i \leqq k,\ s_i \in S)$ for which the following are valid:

(i) there is an $e \in E$ with $\overline{v}e = \overline{v}$,

(ii) $\overline{v}_j e \neq \overline{v}_j$ for all $v_j = s_1 \ldots s_j$ with $1 \leqq j < i$ and $e \in E$. Since $k = \mathrm{card}(S)$, by Proposition III.9.1 in [2], for every $u \in S^{(k)}$ there is a $v \in Z$ such that v is a prefix of u. Obviously, this v is unique. To define the mapping γ, take an $s_1 \ldots s_k \in S^{(k)}$ with $s_1, \ldots, s_k \in S$. There is a unique i $(1 \leqq i \leqq k)$ with $s_1 \ldots s_i \in Z$. Let e be an idempotent, fixed to $s_1 \ldots s_i \in Z$, such that $\overline{s_1 \ldots s_i}e = \overline{s_1 \ldots s_i}$. Suppose first that $i < k$. Define

$$\gamma(s_1) = u(s_1), \ldots, \gamma(s_1 \ldots s_{i-1}) = u(s_1) \ldots u(s_{i-1}),$$
$$\gamma(s_1 \ldots s_i) = u(s_1) \ldots u(s_i)u(e)^{k-1},$$
$$\gamma(s_1 \ldots s_{i+1}) = u(s_1) \ldots u(s_i)u(e)^{k-1}u(s_{i+1}), \ldots,$$
$$\gamma(s_1 \ldots s_{k-1}) = u(s_1) \ldots u(s_i)u(e)^{k-1}u(s_{i+1}) \ldots u(s_{k-1}),$$

finally,

$$\gamma(s_1 \dots s_k) = u(s_1) \dots u(s_i) u(e)^{k-1+t} u(s_{i+1}) \dots u(s_k),$$

where $t \geqq 0$ is the integer with

$$m + k < \|u(s_1)\| + \ldots + \|u(s_k)\| + (k-1)n + tn \leqq m + k + n.$$

Since $m_0 k + (k-1)m \leqq m + k + n$, there exists such a t. Next suppose $i = k$. Define

$$\gamma(s_1) = u(s_1), \ldots, \gamma(s_1 \dots s_{k-1}) = u(s_1) \dots u(s_{k-1}),$$
$$\gamma(s_1 \dots s_k) = u(s_1) \dots u(s_k) u(e)^t,$$

where $t \geqq 0$ is determined by the condition

$$m + k < \|u(s_1)\| + \ldots + \|u(s_k)\| + tn \leqq m + k + n.$$

It is now obvious that γ is well-defined (use the fact that Z is prefix free) and satisfies (1), (2), (3) and (5). Indeed, $\overline{v} = \delta_{\gamma(v)}$ for all $v \in S \cup \ldots \cup S^{(k)}$. For notational convenience, we extend the domain of definition of γ to include the empty word by setting $\gamma(\lambda) = \lambda$.

As regards τ_1 and τ_2, recall that by $m \geqq k$, $S^m = SES$ (Proposition III.9.2 in [2]). Let $v \in S^{(m)}$. There are $s_1, s_2 \in S$ and $e \in E$ with $\overline{v} = s_1 e s_2$. Since $2m_0 + n \leqq m$, there is a (unique) $t \geqq 1$ with

$$m - n < \|u(s_1)\| + tn + \|u(s_2)\| \leqq m.$$

Set $\tau_1(v) = u(s_1) u(e)^t u(s_2)$, $\tau_2(v) = u(s_1) u(e)^{t+1} u(s_2)$. We have $\|\tau_1(v)\| \equiv \equiv \|\tau_2(v)\| \mod n$ and $\overline{v} = \delta_{\tau_1(v)} = \delta_{\tau_2(v)}$.

Define $\varrho : S^{(k)} \to X^+$ by $\varrho(v) = \gamma(v)$, $v \in S^{(k)}$. To complete the proof we show that $\mathbf{A}^+$ is in $\mathbf{HSP}_{\alpha_0}(\{\mathbf{R}_{A,\varrho,\tau_1,\tau_2}, \mathbf{A}\})$. Thus the result follows by Lemma 3.1. Define the α_0-product

$$\mathbf{B} = (B, S, \delta') = \mathbf{R}_{A,\varrho,\tau_1,\tau_2} \times \mathbf{A}(S, \phi)$$

by

$$\phi_1(s) = s,$$
$$\phi_2((a,v),s) = \begin{cases} \mathrm{pre}_1 \gamma(vs)/\|v\| & \text{if } \|v\| < k, \\ \mathrm{pre}_1 \gamma(v)/\|v\| & \text{if } \|v\| = k, \end{cases}$$

for all $s \in S$, $a \in A$, $v \in S^*$ with $\|v\| \leqq k$,

$$\phi_2((v,u),s) = \mathrm{pre}_1 u,$$

for all $s \in S$, $v \in S^+$, $u \in X^+$ with $\|v\| \leqq m$, $m+1 \leqq \|v\| + \|u\| \leqq m+n$.

Set $B' = B_1 \cup B_2$, where

$$B_1 = \{((a,v),b) : a,b \in A,\ v \in S^*,\ \|v\| \leqq k,\ b = \delta(a, \mathrm{pre}_{\|v\|}\gamma(v))\},$$
$$B_2 = \{((v,u),a) : a \in A,\ v \in S^+,\ u \in X^+,\ \|v\| \leqq m,\ m+1 \leqq \|v\| + \|u\| \leqq m+n\}.$$

Clearly, $B' \subseteq B$. Let $\mathbf{B}' = (B', S, \delta')$, and define

$$h((a,v),b) = \overline{v}(a),$$

for all $((a,v),b) \in B_1$,

$$h((v,u),a) = \overline{v}(\delta(a,u)),$$

for all $((v,u),a) \in B_2$.

We are going to show that $\mathbf{B}'$ is a subautomaton of $\mathbf{B}$ and h is a homomorphism of $\mathbf{B}'$ onto $\mathbf{A}^+$. It is clear that h maps B' onto A: for any $a \in A$, we have $((a,\lambda),a) \in B'$ and $h((a,\lambda),a) = a$. Let $c \in B'$ and $s \in S$. We distinguish four cases according to the definition of the transition in $\mathbf{R}_{A,\varrho,\tau_1,\tau_2}$.

Case 1: $c = ((a,v),b) \in B_1$ with $\|v\| < k$. We have $b = \delta(a, \mathrm{pre}_{\|v\|}\gamma(v))$ and

$$\begin{aligned}\delta'(c,s) &= ((a,vs), \delta(b, \mathrm{pre}_1\gamma(vs)/\|v\|)) = \\ &= ((a,vs), \delta(a, (\mathrm{pre}_{\|v\|}\gamma(v))(\mathrm{pre}_1\gamma(vs)/\|v\|))) = \\ &= ((a,vs), \delta(a, (\mathrm{pre}_{\|v\|}\gamma(vs))(\mathrm{pre}_1\gamma(vs)/\|v\|))) = \\ &= ((a,vs), \delta(a, \mathrm{pre}_{\|vs\|}\gamma(vs))) \in B_1.\end{aligned}$$

Further, $h(\delta'(c,s)) = \overline{vs}(a) = \overline{s}(\overline{v}(a)) = \overline{s}(h(c)) = \delta^+(h(c),s)$.

Case 2: $c = ((a,v),b) \in B_1$ with $\|v\| = k$. Again, $b = \delta(a, \mathrm{pre}_{\|v\|}\gamma(v)) = \delta(a, \mathrm{pre}_k\varrho(v))$. We have

$$\begin{aligned}\delta'(c,s) &= ((s, \varrho(v)/k+1), \delta(b, \mathrm{pre}_1\gamma(v)/\|v\|)) = \\ &= ((s, \varrho(v)/k+1), \delta(b, \mathrm{pre}_1\varrho(v)/k)) = \\ &= ((s, \varrho(v)/k+1), \delta(a, (\mathrm{pre}_k\varrho(v))(\mathrm{pre}_1\varrho(v)/k))) = \\ &= ((s, \varrho(v)/k+1), \delta(a, \mathrm{pre}_{k+1}\varrho(v))) \in B_2.\end{aligned}$$

Moreover, $h(\delta'(c,s)) = \overline{s}(\delta(a, (\mathrm{pre}_{k+1}\varrho(v))(\varrho(v)/k+1))) = \overline{s}(\delta(a,\varrho(v)) = \overline{s}(\overline{v}(a)) = \overline{s}(h(c)) = \delta^+(h(c),s)$.

Case 3: $c = ((v,u),a) \in B_2$ with $\|v\| < m$. We have

$$\delta'(c,s) = ((vs, u/1), \delta(a, \mathrm{pre}_1 u)) \in B_2.$$

But then $h(\delta'(c,s)) = \overline{vs}(\delta(a, (\mathrm{pre}_1 u)(u/1))) = \overline{vs}(\delta(a,u)) = \overline{s}(\overline{v}(\delta(a,u))) = \overline{s}(h(c)) = \delta^+(h(c),s)$.

Case 4: $c=((v,u),a)\in B_2$ with $\|v\|=m$. Now

$$\delta'(c,s)=((s,u/1\tau_i(v)),\delta(a,\mathrm{pre}_1u))\in B_2 \text{ for } i=1 \text{ or } i=2.$$

We obtain $h(\delta'(c,s)) = \overline{s}(\delta(a,(\mathrm{pre}_1u)(u/1\tau_i(v)))) = \overline{s}(\delta(a,u\tau_i(v))) = = \overline{s}(\overline{v}(\delta(a,u))) = \overline{s}(h(c)) = \delta^+(h(c),s)$.

The proof of Theorem 4.1 is complete.

Let us denote by $\mathbf{A}_0$ the automaton $\mathbf{D}_0^\lambda$ which is essentially the automaton $\mathbf{D}_0^*$. Thus, $\mathbf{A}_0$ is the identity-reset automaton that plays fundamental role in the Krohn–Rhodes Decomposition Theorem, cf. [2].

COROLLARY 4.2. *Every α_0-variety containing $\mathbf{A}_0$ and all the automata $\mathbf{C}_p^1$ where p is any prime is an α_0^*-variety.*

PROOF. We know that every such α_0-variety $\mathcal{K}$ is an α_0^+-variety. To see that $\mathcal{K}$ is an α_0^*-variety, by Proposition 1.3, it suffices to show that $\mathbf{T}(\mathcal{K})$ is generated by transformation monoids. However, this follows from the Krohn–Rhodes Decomposition Theorem.

We note that if $n=1$ in the proof of Theorem 4.1 then, by Corollary 3.2, the counters suffice instead of the counters with identity. The situation $n=1$ can always be achieved if $\mathbf{A}$ has an input sign inducing the identity state transformation. On the other hand, the integer m can always be taken to be a power of a fixed prime p. These observations lead to the following result proved in [5] by using a much simpler argument:

THEOREM 4.3. *Every α_0^λ-variety containing $\mathbf{A}_0$ and an automaton $\mathbf{C}_p^1$ for a prime number p is an α_0^*-variety.*

Note that an α_0^λ-variety $\mathcal{K}$ contains $\mathbf{A}_0$ if and only if it contains $\mathbf{D}_0$, and $\mathbf{C}_p^1\in\mathcal{K}$ if and only if $\mathbf{C}_p\in\mathcal{K}$.

Let us denote by $\mathcal{K}_{ls}$ the α_0-variety generated by the automata $\mathbf{D}_0$ and $\mathbf{C}_p^1$ for all prime numbers p, i.e. $\mathcal{K}_{ls}=\mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0,\mathbf{C}_p^1 : p \text{ is a prime}\})$. Moreover, let $\mathcal{K}_s=\mathbf{HSP}_{\alpha_0}(\{\mathbf{A}_0,\mathbf{C}_p^1 : p \text{ is a prime}\})$. Now, Theorem 4.1 can be rephrased as follows: Every α_0-variety containing $\mathcal{K}_{ls}$ is an α_0^+-variety. Similarly, Corollary 4.2 says that every α_0-variety containing $\mathcal{K}_s$ is an α_0^*-variety. According to results proved in [10,11], $\mathcal{K}_{ls}$ can be called the class of all *locally solvable* automata, i.e. the class of all automata $\mathbf{A}$ such that $eS(\mathbf{A})e$ is a solvable group for each idempotent e in $S(\mathbf{A})$. The Krohn–Rhodes Decomposition Theorem implies that $\mathcal{K}_s$ consists of all automata $\mathbf{A}$ such that every subgroup of $S(\mathbf{A})$ is solvable. The latter automata can be called *solvable*, explaining the notation $\mathcal{K}_s$.

Now let $\mathcal{L}$ be the lattice of all α_0-varieties $\mathcal{K}$ with $\mathcal{K}_{ls}\subseteq\mathcal{K}$ and $\mathcal{L}_1$ the lattice of all α_0-varieties containing $\mathcal{K}_s$. Denote by $\mathcal{L}'$ the lattice of those closed classes of transformation semigroups containing $\mathbf{T}(\mathcal{K}_{ls})$ and by $\mathcal{L}_1'$ the lattice of closed classes containing $\mathbf{T}(\mathcal{L}_1)$. Each member of $\mathcal{L}_1'$ is generated by transformation monoids.

COROLLARY 4.4. *$\mathcal{L}$ is isomorphic to $\mathcal{L}'$ and $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_1'$ under the correspondence $\mathcal{K} \mapsto \mathbf{T}(\mathcal{K})$.*

Given a prime number p, the class

$$\mathcal{K}_p = \mathbf{HSP}^{\lambda}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_p\}) = \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}_0, \mathbf{C}_p^1\})$$

consists of those automata $\mathbf{A}$ such that each subgroup of $S(\mathbf{A})$ is a p-group. This observation is again a consequence of the Krohn–Rhodes Decomposition Theorem. By Theorem 4.3, every α_0^{λ}-variety containing the class $\mathcal{K}_p$ for some prime number is an α_0^*-variety, and a bijective correspondence as in Corollary 4.4 can be established. By the Krohn–Rhodes Decomposition Theorem, every such α_0^*-variety $\mathcal{K}$ is uniquely described by those simple groups G for which the automaton obtained by letting G act on itself belongs to $\mathcal{K}$.

Comparing Theorems 4.1 and 4.3, it can be asked if each of the automata $\mathbf{C}_p^1$ is strictly needed in Theorem 4.1. Below we give two examples showing that none of them can be omitted.

EXAMPLE 4.5. Let Q be any set of prime numbers and define $\mathcal{K}_0$ to be the class of all automata $\mathbf{A} = (A, X, \delta)$ such that for each $u \in X^+$, the length of the period of the subsemigroup of $S(\mathbf{A})$ generated by δ_u is not divisible by any prime $q \in Q$. Thus $\mathbf{A} \in \mathcal{K}_0$ if and only if no subgroup of $S(\mathbf{A})$ has order a multiple of a prime $q \in Q$. Let $\mathcal{K}_1$ be the class of all counters and $\mathcal{K} = \mathbf{HSP}_{\alpha_0}(\mathcal{K}_0 \cup \mathcal{K}_1)$, so that $\mathcal{K}$ is an α_0-variety. $\mathcal{K}$ contains the automaton $\mathbf{A}_0$ and thus $\mathcal{K}$ contains $\mathbf{D}_0$ and all aperiodic automata. Moreover, for every prime p we have $\mathbf{C}_p^1 \in \mathcal{K}$ if and only if $p \notin Q$. On the other hand, $\mathbf{C}_p^1 \in \mathbf{HSP}^+_{\alpha_0}$ $+(\mathcal{K}_0 \cup \mathcal{K}_1)$ for all prime numbers p. Thus $\mathcal{K}$ is an α_0^+-variety if and only if Q is empty.

EXAMPLE 4.6. Again let Q be a set of primes. Define $\mathcal{K}$ to be the class of those automata $\mathbf{A} = (A, X, \delta)$ satisfying the following property: If the period of the subsemigroup generated by δ_u in $S(\mathbf{A})$ is divisible by a prime $q \in Q$ then $\|u\|$ too is divisible by some prime number in Q. It can be seen that $\mathbf{C}_p^1 \in \mathcal{K}$ whenever p is a prime not in Q, but $\mathbf{C}_q^1 \notin \mathcal{K}$ for all $q \in Q$. Further, $\mathcal{K}$ contains all aperiodic automata. On the other hand, $\mathbf{HSP}^+_{\alpha_0}(\mathcal{K})$ is the class of all automata whenever there is a prime not in Q. Thus, $\mathcal{K}$ is an α_0^+-variety if and only if Q is empty.

5. Some remarks

The construction that we used in the proof of Theorem 4.1 can be further generalized to some extent. A few possible extensions are given below.

Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, Y, \delta')$ be automata. We write $\mathbf{B}|\mathbf{A}$ if there exist $C \subseteq A$, $h: C \to B$ and $\gamma': Y \to X^+$ such that

(1) h is onto,

(2) $\delta(c,\gamma'(y)) \in C$, $h(\delta(c,\gamma'(y))) = \delta'(h(c),y)$, for all $a \in C$ and $y \in Y$. (Note that this just means that $(B, S(\mathbf{B}))$ is covered by $(A, S(\mathbf{A}))$, see [2].)

Now let $m \geqq n$ be positive integers. We shall write $\mathbf{B}|^{(m,n)}\mathbf{A}$ if and only if, there exist $k \geqq 1$, $C \subseteq A$, and mappings

$$h : C \to B,$$
$$\gamma : Y \cup \cdots \cup Y^{(k)} \to X^{+},$$
$$\tau_1 : Y^{(m)} \to X^{(m-n+1)} \cup \cdots \cup X^{(m)},$$
$$\tau_2 : Y^{(m)} \to X^{(m+1)} \cup \cdots \cup X^{(m+n)}$$

such that the following hold:

(1) h is onto,

(2) $\|\gamma(v)\| \geqq \|v\|$, for all $v \in Y^{+}$, $\|v\| \leqq k$,

(3) $\mathrm{pre}_{\|v\|}\gamma(vy) = \mathrm{pre}_{\|v\|}\gamma(v)$, for all $v \in Y^{*}$, $y \in Y$ with $\|vy\| \leqq k$,

(4) $m < \|\gamma(v)\| - k \leqq m + n$, for all $v \in Y^{(k)}$,

(5) $\|\tau_1(v)\| \equiv \|\tau_2(v)\| \mod n$, for all $c \in C$, $v \in Y^{(k)}$,

(6) $\delta(c,\gamma(v)) \in C$, $h(\delta(c,\gamma(v))) = \delta'(h(c),v)$, for all $c \in C$, $v \in Y^{(k)}$,

(7) $\delta(c,\tau_i(v)) \in C$, $h(\delta(c,\tau_i(v))) = \delta'(h(c),\tau_i(v))$, for all $c \in C$, $v \in Y^{(m)}$ and $i = 1, 2$.

Similarly, we write $\mathbf{B}|^{(m)}\mathbf{A}$ if the above conditions, except for (5) and (7) when $i = 2$, hold with $n = 1$, for some k, C, h, γ and $\tau = \tau_1$.

It can be seen that $\mathbf{B}|\mathbf{A}$ implies $\mathbf{B}|^{(m,n)}\mathbf{A}$ for some m and n. Namely, n can be any positive integer with $E \subseteq \{\delta_u : \|u\| = n\}$, where E is the set of all idempotents in $S(\mathbf{A})$. Another choice can be the following. Suppose $\mathbf{B}|\mathbf{A}$ via C, h and γ'. We can extend γ' to a morphism $Y^{+} \to X^{+}$. Now, for every idempotent $e \in S(\mathbf{B})$, choose a word $v(e) \in Y^{+}$ inducing e. Then let $u(e) = \gamma'(v(e))$, and $n = $ l.c.m.$\{\|u(e)\| : e \in S(\mathbf{B})$ is an idempotent$\}$. If in either case we have $n = 1$, then $\mathbf{B}|^{(m)}\mathbf{A}$ for some m.

PROPOSITION 5.1. *If* $\mathbf{B}|^{(m,n)}\mathbf{A}$ *then* $\mathbf{B}$ *is in* $\mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m, \mathbf{C}_n^1, \mathbf{A}\})$. *If* $\mathbf{B}|^{(m)}\mathbf{A}$ *then* $\mathbf{B}$ *is in* $\mathbf{HSP}_{\alpha_0}(\{\mathbf{D}_0, \mathbf{C}_m, \mathbf{A}\})$.

The proof carries over with a slight modification of the argument applied in proving Theorem 4.1. If $\mathbf{B}|^{(m)}\mathbf{A}$ is assumed, one uses Corollary 3.2.

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(Received January 18, 1988)

A. JÓZSEF UNIVERSITY
BOLYAI INSTITUTE
SZEGED, ARADI V. TERE 1
6720 HUNGARY

Acta Math. Hung.
59(1-2) (1992), 75–83.

ЭКСТРЕМАЛЬНЫЕ ЗАДАЧИ ТЕОРИИ ПОЛИНОМИАЛЬНЫХ ОПЕРАТОРОВ

Д. Л. БЕРМАН (Ленинград)

1. Пусть n и m заданные натуральные числа и C — пространство 2π-периодических непрерывных функций $f(x)$ с нормой $\|f(x)\| = \max|f(x)|$. Через Π_n обозначим множество всех тригонометрических полиномов порядка не выше n. Пусть Ω некоторое множество линейных операторов из C в C, обладаюших тем свойством, что если $A \in \Omega$ и $f \in \Pi_n$, то $A(f,x) = f(x)$. Пусть $E_n(f)$ — наилучшее приближение функции $f(x)$ с помощью полиномов из Π_n. Тогда

$$\|A(f,x) - f(x)\| \leqq (1 + \|A\|)E_n(f) \tag{*}$$

Следовательно, для того чтобы погрешность $\|A(f,x) - f(x)\|$ была наименьшей нужно, чтобы норма оператора A была минимальна. Стало быть, возникает вопрос о вычислении $\varrho = \inf\limits_{A\in\Omega} \|A\|$ и о нахождении оператора $A \in \Omega$ для которого $\|A\| = \varrho$. Обозначим через L множество всех суммируемых 2π-периодических функций. Введем линейное нормипованное функциональное пространство E, обладающее свойствами: 1) элементы E суть функции из L; 2) если $f \in E$, то смещенная функция $f_t(x) = f(x+t)$ при любом $-\infty < t < \infty$ также из E, причем $\|f_t\| \leqq \|f\|$; 3) E содержит множество всех тригонометрических полиномов. Важнейшим частным случаем пространства E является пространство C. Через $\Omega_{n,n+m}(E)$ обозначим множество всех линейных операторов $U_{n,n+m}(f,x)$ из E в E, обладающих следующими свойствами: 1) для любой $f \in E$ $U_{n,n+m}(f,x) \in \Pi_{n+m}$; 2) если $f \in \Pi_n$, то $U_{n,n+m}(f,x) = f(x)$. Важнейшей операцией вида $U_{n,n+m}(f,x)$ является частная сумма Валле–Пуссена

$$\sigma_{n,n+m}(f,x) = \frac{1}{m+1} \sum_{k=n}^{n+m} S_k(f,x), \tag{1}$$

где $S_k(f,x)$ — частная сумма порядка k ряда Фурье функции $f(x)$. К операциям вида $U_{n,n+m}(f,x)$ относятся также известные интерполяционные процессы С. Н. Бернштейна [1]. Операторы из [2] также типа $U_{n,n+m}(f,x)$. Положим $\varrho_{n,n+m} = \inf\limits_{U_{n,n+m}\in\Omega_{n,n+m}} \|U_{n,n+m}\|$. Пусть $\overline{U} \in \Omega_{n,n+m}$. Будем говорить, что оператор $\overline{U}$ экстремальный в классе $\Omega_{n,n+m}$, если $\varrho_{n,n+m} = \|\overline{U}\|$. Возникает естественный вопрос о нахождении в множестве операторов $\Omega_{n,n+m}(E)$ оператора с наименьшей нормой и о вычислении $\varrho_{n,n+m}(E)$. Эта задача была поставлена в [3], [4]. Видимо при произвольных натуральных n и m её трудно решить. Вместе с тем в статье указываются частные случаи, когда её решения весьма простые. Это когда $m = n-1$ или $m = 2n-1$, где n — произвольное натуральное число. Класс операторов $\Omega_{n,n+m}(E)$ допускает следующее обобщение. Обозначим через $\Omega^{(r)}_{n,n+m}$ множество всевозможных линейных операторов $U_{n,n+m}(f,x)$ из E в E, обладающих свойствами: 1) для любой $f \in E$ $U_{n,n+m}(f,x) \in \Pi_{n+m}$; 2) если $f \in \Pi_n$, то $U_{n,n+m}(f,x) = f^{(r)}(x)$. При этом $f^{(r)}(x)$ — производная порядка r от $f(x)$. Положим $\varrho^{(r)}_{n,n+m}(E) = \inf\limits_{U_{n,n+m} \in \Omega^{(r)}_{n,n+m}(E)} \|U_{n,n+m}\|$. Возникает вопрос о нахождении в множестве $\Omega^{(r)}_{n,n+m}$ экстремального оператора и о вычислении $\varrho^{(r)}_{n,n+m}$. Эта задача также была поставлена в [3], [[4]. Она там была решена для $r=1$ и $m=n-1$. В настоящей статье формулируется полное её решение для любого натурального r и $m=n-1$. При этом n — любое натуральное число. Доказательство упомянутого утверждения находится в [8].

2. Рассмотрим сперва случай, когда $m = 2n-1$, где n — любое натуральное число. Обозначим через $\phi_n(t)$ ядро Фейера. Справедлива

ТЕОРЕМА 1. *При любом n оператор*

$$(2)\qquad A(f,x) = \int_0^{2\pi} f(x+t)\cos nt\,\phi_{2n}(t)dt, \quad \phi_n(t) = \frac{1}{2\pi n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2,$$

является экстремальным в классе $\Omega_{n,3n-1}(C)$. При этом $\varrho_{n,3n-1} = \frac{4}{\pi}$.

Для доказательства этой теоремы нужны следующие теоремы из [4].

ТЕОРЕМА 2. *Среди всех линейных операторов из $\Omega_{n,n+m}(C)$ оператор*

$$(3)\quad \overline{U} = \overline{U}_{n,n+m}(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\left[D_n(t) + \sum_{j=1}^{m}\tilde{\alpha}_j\cos(n+j)t\right]dt,$$

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}},$$

где числа $\{\tilde{\alpha}_j\}_{j=1}^m$ определяются из условия минимума интеграла

$$Y=Y(\alpha_1,\alpha_2,\dots,\alpha_m)=\int_0^{\pi}\left|D_n(t)+\sum_{j=1}^{m}\alpha_j\cos(n+j)t\right|dt,$$

обладает наименьшей нормой. При этом

$$\varrho_{n,n+m}(C)=\frac{2}{\pi}\int_0^{\pi}\left|D_n(t)+\sum_{j=1}^{m}\tilde{\alpha}_j\cos(n+j)t\right|dt. \tag{3'}$$

Если воспользоваться известным фактами теории наилучших приближений в метрике L [5], то теорема 2 можно сформулировать в следующей эквивалентной форме.

Теорема 3. *Для того чтобы оператор* (3) *обладал наименьшей нормой в классе операторов* $\Omega_{n,n+m}(C)$ *необходимо и достаточно, чтобы числа* $\{\tilde{\alpha}_j\}_{j=1}^m$ *из формул* (3) *удовлетворяли условиям:*

$$\int_0^{\pi}\cos(n+i)t\,\mathrm{sign}\left[D_n(t)+\sum_{j=1}^{m}\tilde{\alpha}_j\cos(n+j)t\right]dt=0,\quad i=1,2,\dots,m. \tag{4}$$

Теперь можно перейти к доказательству теоремы 1. Докажем, что частная сумма Валле-Пуссена $\sigma_{n,3n-1}(f,x)$ совпадает с оператором (2) и является экстремальным оператором в классе $\Omega_{n,3n-1}(C)$. Из (1) видно, что $\sigma_{n,3n-1}\in\Omega_{n,3n-1}$. Хорошо известно что

$$\sigma_{n,n+m}=\int_0^{2\pi}f(x+t)V_{n,m}(t)dt, \tag{5}$$

где

$$V_{n,m}(t)=\frac{1}{2\pi(m+1)}\sin(n+\frac{m+1}{2})t\sin\frac{m+1}{2}t\sin^{-2}\frac{t}{2}.$$

Поэтому при $m=2n-1$ получим, что

$$\sigma_{n,3n-1}(f,x)=\frac{1}{4\pi n}\int_0^{2\pi}f(x+t)\sin 2nt\sin nt\sin^{-2}\frac{t}{2}dt,$$

или

$$\sigma_{n,3n-1}(f,x)=\frac{1}{2\pi n}\int_0^{2\pi}f(x+t)\cos nt\sin^2 nt\sin^{-2}\frac{t}{2}dt.$$

Итак, $\sigma_{n,3n-1}$ совпадает с оператором (2). Так как $\phi_n(t) \geqq 0$, $t \in$ $\in (-\infty, \infty)$, то для оператора (2) равенства (4) принимают вид:

$$\int_0^\pi \cos(n+j)t \operatorname{sign} \cos nt dt = 0, \quad 1 \leqq j \leqq 2n-1. \tag{6}$$

Известно [5], что

$$\operatorname{sign} \cos t = \frac{4}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\cos(2k+1)t}{2k+1}.$$

Поэтому очевидно, что равенства (6) выполняются. Их выполнение можно также получить с помощью утверждения из [5], стр. 99–100. Итак, оператор (2) является экстремальным в классе $\Omega_{n,3n-1}$. Из (2) следует, что

$$\varrho_{n,3n-1}(C) = \int_0^{2\pi} |\cos nt| \phi_{2n}(t) dt. \tag{7}$$

Известно [6], стр. 183--184, что

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2-1}. \tag{7'}$$

Поэтому $|\cos nt| = \frac{2}{\pi} - \frac{4}{\pi} \sum\limits_{i=1}^{\infty} (-1)^i \frac{\cos 2nit}{4i^2-1}$. Известно, что

$$\phi_n(t) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \cos kt \right). \tag{8}$$

Поэтому из (7), (8) и равенства Парсеваля получим, что $\varrho_{n,3n-1}(C) = \frac{4}{\pi}$. Обозначим через $\mathscr{E}$ множество всех функциональных пространств типа E.

Теорема 4. *Имеет место равенство*

$$\sup_{E \in \mathscr{E}} \varrho_{n,3n-1}(E) = \frac{4}{\pi}. \tag{9}$$

Доказательство. Будем рассматривать оператор (2), как оператор из E в E. Обозначим его через $A(f, x, E)$. Из доказательства теоремы 1 следует что он принадлежит классу $\Omega_{n,3n-1}(E)$. Известно

[7], что для интеграла и нормы имеет место неравенство $\left\|\int f d\mu\right\| \leqq \leqq \int \|f\| d\mu$. Поэтому из (2) получим, что

$$(10)\qquad \|A(f,x,E)\| \leqq \int_0^{2\pi} \|f(x+t)\| \; |\cos nt| \phi_{2n}(t)dtt.$$

Согласно определению пространства типа E $\|f(x+t)\| \leqq \|f\|$. Стало быть, из (10) выводим, что

$$(11)\qquad \|A(f,x,E)\| \leqq \|f\| \int_0^{2\pi} |\cos nt| \phi_{2n}(t)dt.$$

В ходе доказательства теоремы 1 было установлено, что интеграл из правой части неравенства (11) равен $\frac{2}{\pi}$. Итак, из (11) заключаем, что $\|A(E)\| \leqq \frac{4}{\pi}$. Отсюда и из очевидного неравенства $\varrho_{n,3n-1}(E) \leqq \leqq \|A(E)\|$ получаем, что $\varrho_{n,3n-1}(E) \leqq \frac{4}{\pi}$. Следовательно,

$$(12)\qquad \sup_{E\in\mathscr{E}} \varrho_{n,3n-1}(E) \leqq \frac{4}{\pi}.$$

С другой стороны, поскольку C пространство типа E, то $\sup\limits_{E\in\mathscr{E}} \varrho_{n,3n-1}(e) \geqq \varrho_{n,3n-1}(C)$. Согласно теореме 1 $\varrho_{n,3n-1}(C) = \frac{4}{\pi}$. Стало быть,

$$\sup_{E\in\mathscr{E}} \varrho_{n,3n-1}(E) \geqq \frac{4}{\pi}.$$

Отсюда и из (12) выводим (9).

Обозначим через L_p пространство интегрируемых с p-ой степенью, $p \geqq 1$, 2π-периодических функций с нормой $\|f(x)\|_{L_p} = = (\int_0^{2\pi} |f(x)|^p dx)^{1/p}$. Очевидыо, что L_p — пространство типа E. Поэтому справедливо следствие

Следствие 1. *Справедливо неравенство* $\varrho_{n,3n-1}(L_p) \leqq \frac{4}{\pi}$.

Следствие 2. *Для любой* $f \in C$ *выполняется неравенство*

$$\|\sigma_{n,3n-1}(f,x) - f(x)\| \leqq (1+\frac{4}{\pi})E_n(f).$$

Константа $1+\frac{4}{\pi}$ *на всем классе* $\Omega_{n,3n-1}(C)$ *неулучшаемая.*

Доказательство. Следует воспользоваться теоремой 1 и неравенством $(*)$ из пункта 1.

3. Рассмотрим теперь случай $m = n-1$.

Теорема 5. *При любом n оператор*

$$B(f,x)=\int_0^{2\pi} f(x+t)(1+2\cos nt)\phi_n(t)dt \tag{13}$$

является экстремальным в классе $\Omega_{n,2n-1}(C)$. При этом $\varrho_{n,2n-1}(C)=$ $=\frac{1}{3}+\frac{2\sqrt{3}}{\pi}$.

Доказательство. Докажем, что частная сумма Валле-Пуссена $\sigma_{n,2n-1}(f,x)$ совпадает с оператором (13) и является экстремальным оператором в классе $\Omega_{n,2n-1}(C)$. Из (5) вытекает, что

$$\sigma_{n,2n-1}(f,x)=\frac{1}{2\pi n}\int_0^{2\pi} f(x+t)\sin\frac{3nt}{2}\sin\frac{nt}{2}\sin^{-2}\frac{t}{2}dt. \tag{14}$$

Отметим тождество $\sin\frac{3\alpha}{2}\sin\frac{\alpha}{2}=\sin^2\alpha-\sin^2\frac{\alpha}{2}$. Поэтому из (14) выводим

$$\sigma_{n,2n-1}(f,x)=\frac{1}{2\pi n}\int_0^{2\pi} f(x+t)(\sin^2 nt-\sin^2\frac{nt}{2})\sin^{-2}\frac{t}{2}dt.$$

Отсюда после простых преобразований получим, что $\sigma_{n,2n-1}(f,x)$ совпадает с оператором (13). Поэтому из (1) следует, что оператор (13) из класса $\Omega_{n,2n-1}(C)$. Докажем, что он экстремальный в $\Omega_{n,2n-1}(C)$. Для этого воспользуемя теоремой 3. Так как ядро Фейера $\phi_n(t)\geqq 0$, $t\in(-\infty,\infty)$, то равенства (4) из теоремы 3 принимают вид:

$$\int_0^{2\pi} \operatorname{sign}\varphi(t)\cos(n+i)tdt=0, \quad i=1,2,\dots,n-1, \tag{15}$$

где $\varphi(t)=1+2\cos nt$. Разложим в ряд Фурье функцию $\Psi(x)=\operatorname{sign}(\frac{1}{2}+$ $+\cos x)$. Так как она четная, то можно полагать, что

$$\Psi(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx,$$

где $a_0=\frac{2}{\pi}\int_0^\pi \operatorname{sign}(\frac{1}{2}+\cos x)dx$. Очевидно, что $\frac{1}{2}+\cos x\geqq 0$ при $0\leqq x\leqq\frac{2\pi}{3}$ и $\frac{1}{2}+\cos x<0$ при $\frac{2\pi}{3}<x\leqq\pi$. Поэтому $a_0=\frac{1}{3}$. Очевидно, что $\operatorname{sign}\varphi(t)=$ $=\Psi(nt)$. Стало быть, разложение в ряд Фурье функции $\operatorname{sign}\varphi(t)$ имеет вид

$$\operatorname{sign}\varphi(t)=\frac{1}{3}+\sum_{k=1}^{\infty} a_k\cos knt. \tag{16}$$

Из (16) и равенства Парсеваля следует, что равенства (15) выполняются. Поэтому $\sigma_{n,2n-1}$ экстремальный оператор в $\Omega_{n,2n-1}(C)$. В силу (13) имеем, что[1]

$$\varrho_{n,2n-1}(C)=\int_0^{2\pi}|\varphi(t)|\phi_n(t)dt. \tag{17}$$

Вычислим интеграл (17) с помощью равенства Парсеваля. Для этого составим разложение в ряд Фурье функции $|\varphi(t)|$. Пусть $\{\gamma_k\}_{k=0}^{\infty}$ коэффициенты Фурье $|\varphi(t)|$. Имеем, что

$$\gamma_0=\frac{1}{\pi}\int_0^{2\pi}|1+2\cos nt|dt=\frac{2}{\pi n}\int_0^{2\pi n}\left|\frac{1}{2}+\cos t\right|dt. \tag{18}$$

Очевидно, что $\int_0^{2\pi n}|\frac{1}{2}+\cos x|dx=n\int_0^{2\pi}|\frac{1}{2}+\cos x|dx$. Поэтому из (18) вывидим, что

$$\gamma_0=\frac{4}{\pi}\int_0^{\pi}\left|\frac{1}{2}+\cos t\right|dt.$$

Как раньше, учтем распределение знаков функции $f(x)=\frac{1}{2}+\cos x$ в $[0,\pi]$. Получим, что $\gamma_0=\frac{2}{3}+\frac{4\sqrt{3}}{\pi}$. Поэтому разложение $|\varphi(t)|$ имеет вид:

$$|\varphi(t)|=\frac{1}{3}+\frac{2\sqrt{3}}{\pi}+\gamma_1\cos nt+\gamma_2\cos 2nt+\dots \tag{19}$$

Из (8), (19) и равенства Парсеваля вытекает, что

$$\int_0^{2\pi}|\varphi(t)|\phi_n(t)dt=\frac{1}{3}+\frac{2\sqrt{3}}{\pi}. \tag{20}$$

Из (17) и (20) вытекает, что $\varrho_{n,2n-1}(C)=\frac{1}{3}+\frac{2\sqrt{3}}{\pi}$.

Следствие 3. *Для любой $f\in C$ выполняется неравенство*

$$\|\sigma_{n,2n-1}(f,x)-f(x)\|\leqq(\frac{4}{3}+\frac{2\sqrt{3}}{\pi})E_n(f).$$

Константа $\frac{4}{3}+\frac{2\sqrt{3}}{\pi}$ на всем классе $\Omega_{n,2n-1}(C)$ неулучшаемая.

Доказательство. Следует воспользоваться теоремой 5 и неравенством $(*)$ из пункта 1.

Замечание 1. Оценкка $\|\sigma_{n,2n-1}(f,x)-f(x)\|\leqq 4E_n(f)$ принадлежит Валле-Пуссену [63].

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[1] $\varrho_{n,2n-1}(C)$ можно также вычислить по формулам $(3')$, (16).

Замечание 2. Оператор (13) другим методом изучался в [9]. При вычислении $\varrho_{n,2n-1}(C)$ была в [9] допущена ошибка. См. [9] стр. 7. Для её исправления следует в равенстве (20) из [9] правую часть считать равным $(n+1)\sqrt{3}$.

Замечание 3. Экстремальность операторов (2) и (13) следует из нашей общей теоремы [3], [4]. Изложенные здесь доказательства проще чем доказательство упомянутой общей теоремы. Кроме того, здесь найдены $\varrho_{n,2n-1}$ и $\varrho_{n,3n-1}$, а в [4] этого нет.

Теорема 6. *Имеет место равенство*

$$\sup_{E\in\mathcal{E}} \varrho_{n,2n-1}(E) = \frac{1}{3} + \frac{2\sqrt{3}}{\pi}. \tag{21}$$

Метод доказательства равенства (21) не отличается от метода доказательства равенства (9).

4. Для класса операторов $\Omega^{(r)}_{n,2n-1}(C)$ имеет место

Теорема 7. *Оператор*

$$C(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\cos\left(nt+\frac{r\pi}{2}\right)\left[n^r + 2\sum_{k=1}^{n-1} k^2\cos(n-k)t\right]dt$$

принадлежит классу $\Omega^{(r)}_{n,2n-1}(C)$ и является экстремальным в этом классе. При этом

$$\varrho^{(r)}_{n,2n-1}(C) = \frac{4}{\pi}n^r, \quad r = 1,2,\dots$$

Доказательство теоремы 7 находится в [8].

Теорема 8. *Имеет место равенство*

$$\sup_{E\in\mathcal{E}} \varrho^{(r)}_{n,2n-1}(E) = \frac{4}{\pi}n^r, \quad r = 1,2,\dots$$

Доказательство теоремы 8 не отличается от доказательства теоремы 4. Нужно только учесть, что

$$\int_0^{\pi} \left|\cos\left(nt+\frac{r\pi}{2}\right)\right| n^r + 2\sum_{k=1}^{n-1} k^r\cos(n-k)t| dt = 4n^r. \tag{22}$$

Для доказательства равенства (22) ннужно воспользоваться неравенством

$$n^r + 2\sum_{k=1}^{\infty} k^2 \cos(n-k)t \geqq 0,\ t \in (-\infty, \infty),$$

разложением в ряд Фурье функции $|\cos(nt + \frac{r\pi}{2})|$ и равенством Парсеваля. При разложении в ряд Фурье функции $|\cos(nt + \frac{r\pi}{2})|$ можно использовать равенство (7′).

Примечание при корректуре (6.10.1991). В [3] и [4] было установлено, что для того чтобы оператор (1) был экстремальным в классе $\Omega_{n,n+m}(C)$ необходимо и достаточно, чтобы число $v = \frac{2n}{m+1}$ было челым. Добавим, что если v — четное чусло, то

$$\varrho_{n,n+m} = \frac{1}{1+v} + \frac{2}{\pi}\sum_{k=1}^{v/2} \frac{1}{k} \operatorname{tg} \frac{\pi k}{1+v},$$

если v — нечетное число, то

$$\varrho_{n,n+m} = \frac{4}{\pi} \sum_{k=1}^{(1+v)/2} \frac{1}{2k-1} \operatorname{tg} \frac{\pi(2k-1)}{2(1+v)}.$$

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(Поступило 25. 4. 1988.)

1357 WEST ESTES NO. Q2
CHICAGO, IL 60626
U.S.A.

Acta Math. Hung.
59(1-2) (1992), 85–89.

TAUBERIAN THEOREMS CONCERNING POWER SERIES WITH NON-NEGATIVE COEFFICIENTS

D. BORWEIN* (London, Ontario)

1. Introduction

Suppose throughout that $\{a_n\}$ is a sequence of non-negative number, that

$$s_n := \sum_{k=0}^{n} a_k$$

and that

$$0 < f(x) := \sum_{k=0}^{\infty} a_k x^k < \infty \quad \text{for} \quad 0 < x < 1.$$

Hardy and Littlewood [4, Theorem 10] have proved the following theorem.

THEOREM H–L. *If*

$$f(x) \sim (1-x)^{-\rho} L(x) \quad \textit{as} \quad x \to 1-,$$

where $\rho \geqq 0$ *and* $L(1-\frac{1}{u})$ *is a logarithmico-exponential function such that*

$$u^{-\delta} \prec L\left(1-\frac{1}{u}\right) \prec u^{\delta},$$

then

$$s_n \sim \frac{n^{\rho}}{\Gamma(\rho+1)} L\left(1-\frac{1}{n}\right).$$

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* This research was supported in part by Natural Sciences and Engineering Research Council of Canada.

See [3] for definitions and properties of logarithmico-exponential functions. Examples of logarithmico-exponential functions satisfying the above condition are given by

$$L\left(1-\frac{1}{u}\right) := (\log u)^{c_1}(\log\log u)^{c_2}\ldots,$$

where c_1, c_2, ... are real numbers. Theorem H–L is Tauberian in nature in that it yields information about the asymptotic behavior of s_n from the asymptotic behavior of $f(x)$.

The primary object of this note is to supply a simple and straightforward proof of the following generalization of Theorem H–L.

THEOREM 1. (i) *Suppose*

$$\lim_{x\to 1-}\frac{f(x^m)}{f(x)}=\lambda_m>0 \quad \textit{for} \quad m=2 \quad \textit{and} \quad m=3. \tag{1}$$

Then

$$f(x)=(1-x)^{-\rho}\phi(x)$$

where $\rho=-\log_2\lambda_2 \geqq 0$ *and, for all* $t\geqq 1$,

$$\lim_{x\to 1-}\frac{\phi(x^t)}{\phi(x)}=1.$$

Moreover

$$s_n\sim\frac{n^\rho}{\Gamma(\rho+1)}\phi\left(1-\frac{1}{n}\right)=\frac{1}{\Gamma(\rho+1)}f\left(1-\frac{1}{n}\right)$$

and

$$s_{n+1}\sim s_n \quad \textit{and} \quad \lim_{n\to\infty}\frac{s_n}{s_{mn}}=\lambda_m>0 \quad \textit{for} \quad m=2 \quad \textit{and} \quad m=3. \tag{2}$$

(ii) *Conversely,* (2) *implies* (1).

If follows from Theorem 1.8 in [5] that the integers 2, 3 in (1) can be replaced by any two positive numbers p, $q\neq 1$ such that $\log_q p$ is irrational. It was proved in [2] that

$$\lim_{x\to 1-}\frac{f(x^2)}{f(x)}=\lambda>0 \tag{3}$$

alone does not imply (1) when $\lambda<1$, though (1) and (3) are equivalent when $\lambda=1$. Part (i) of Theorem 1 can be deduced from Karamata's Tauberian theorem and a known result about regularly varying functions (see Theorems 2.3 and 1.8 in [5]). We give an alternate proof which is more direct and more elementary, not involving, in particular, the extended continuity theorem for Laplace-Stieltjes transforms on which the proof of Karamata's theorem is based. Part (ii) of Theorem 1 is interesting in that it shows that (1) and (2) are in fact equivalent.

2. Preliminary results

THEOREM 2. *Suppose* $b_n \geqq 0$ *for* $n=0,1,\dots,$

$$t_n := \sum_{k=0}^{n} b_k, \quad \textit{and} \quad g(x) := \sum_{n=0}^{\infty} b_n x^n < \infty \quad \textit{for} \quad 0<x<1.$$

If (1) *holds and* $\frac{g(x)}{f(x)} \to \lambda$ *as* $x \to 1-$, *then* $\frac{t_n}{s_n} \to \lambda$.

PROOF. The result is evidently true if $f(x)$ tends to a finite limit as $x \to 1-$. Suppose therefore that $f(x) \to \infty$ as $x \to 1-$.

Case (i) : $a_n > 0$ for $n=0,1,\dots$. This case follows immediately from the theorem in [2].

Case (ii) : $a_n \geqq 0$ for $n=0,1,\dots$. Let

$$f^\star(x) := f(x) + e^x, \quad g^\star(x) := g(x) + e^x$$

and define $a_n^\star$, $s_n^\star$, $b_n^\star$, $t_n^\star$ in the obvious way. Then $a_n^\star > 0$ for $n=0,1,\dots,$ and, since $f(x) \to \infty$ as $x \to 1-$, (1) is satisfied with $f^\star$ in place of f. Further

$$\frac{g^\star(x)}{f^\star(x)} \to \lambda \quad \text{as} \quad x \to 1- \quad \text{if and only if} \quad \frac{g(x)}{f(x)} \to \lambda \quad \text{as} \quad x \to 1-,$$

and

$$\frac{t_n^\star}{s_n^\star} \to \lambda \quad \text{if and only if} \quad \frac{t_n}{s_n} \to \lambda.$$

Case (ii) now follows from Case (i). □

LEMMA 1. *If* (1) *holds, then, for* $m=1,2,\dots$ *and* $\rho = -\log_2 \lambda_2 \geqq 0$,

$$\lim_{x \to 1-} \frac{f(x^m)}{f(x)} = m^{-\rho}$$

and, for every $c \in (0,1)$,

$$\lim_{n \to \infty} \frac{s_n}{f(c^{1/n})} = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}.$$

PROOF. The result is evidently true with $\rho = 0$ if $f(x)$ tends to a finite limit as $x \to 1-$. Suppose therefore that $f(x) \to \infty$ as $x \to 1-$. It has been shown in [2] that this together with (1) implies the first conclusion. Further, when $\rho > 0$,

$$(m+1)^{-\rho} = \int_0^1 t^m d\chi(t) \quad \text{with} \quad \chi(t) := \frac{1}{\Gamma(\rho)} \int_0^t (-\log u)^{\rho-1} du.$$

It was proved in [1] that the above implies that, when $\rho = 0$,

$$\lim_{n\to\infty} \frac{s_n}{f(c^{1/n})} = 1,$$

and, when $\rho > 0$,

$$\lim_{n\to\infty} \frac{s_n}{f(c^{1/n})} = \int_c^1 t^{-1} d\chi(t) = \frac{1}{\Gamma(\rho)} \int_c^1 t^{-1}(-\log t)^{\rho-1} dt = \frac{(-\log c)^\rho}{\Gamma(\rho+1)}. \ \square$$

The next lemma has been proved in essence in [1].

LEMMA 2. *If $s_{n+1} \sim s_n$ and $\lim_{n\to\infty} \frac{s_n}{s_{mn}} = \lambda > 0$ where m is a positive integer, then*

$$\lim_{x\to 1-} \frac{f(x^m)}{f(x)} = \lambda.$$

3. Proof of Theorem 1

(i) The first conclusion has been proved in [2]. To establish the asymptotic expression for s_n observe that, given $\gamma > 1$,

$$e^{-\gamma/n} < 1 - \frac{1}{n} < e^{-1/n}$$

for n sufficiently large. Hence for such n

$$\frac{s_n}{f(e^{-\gamma/n})} \geq \frac{s_n}{f(1-1/n)} \geq \frac{s_n}{f(e^{-1/n})}$$

and so, by Lemma 1,

$$\frac{\gamma^\rho}{\Gamma(\rho+1)} \geq \limsup_{n\to\infty} \frac{s_n}{f(1-1/n)} \geq \liminf_{n\to\infty} \frac{s_n}{f(1-1/n)} \geq \frac{1}{\Gamma(\rho+1)}.$$

Since $\gamma^\rho \to 1$ as $\gamma \to 1-$, it follows that

$$\lim_{n\to\infty} \frac{s_n}{f(1-1/n)} = \frac{1}{\Gamma(\rho+1)},$$

i.e.,

$$s_n \sim \frac{n^\rho}{\Gamma(\rho+1)}\phi\left(1-\frac{1}{n}\right) = \frac{1}{\Gamma(\rho+1)}f\left(1-\frac{1}{n}\right).$$

To establish (2) we first observe that, by Lemma 1,

$$\lim_{n\to\infty} \frac{s_n}{s_{mn}} = \lim_{n\to\infty} \frac{s_n}{f(e^{-1/n})} \cdot \frac{f(e^{-1/mn})}{s_{mn}} \cdot \frac{f(e^{-1/n})}{f(e^{-1/nm})} = m^{-\rho}.$$

Next we suppose without loss of generality that $s_n \to \infty$. Then, by Theorem 2 with $b_n = a_{n+1}$, we see that

$$\frac{g(x)}{f(x)} = \frac{f(x)-a_0}{f(x)} \to 1 \quad \text{as} \quad x \to 1-, \quad \text{and hence} \quad \frac{t_n}{s_n} = \frac{s_{n+1}-s_0}{s_n} \to 1$$

so that $s_{n+1} \sim s_n$.

(ii) This follows immediately from Lemma 2. □

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(Received May 2, 1988)

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA N6A 5B7

Acta Math. Hung.
59(1-2) (1992), 91–102.

RATE OF CONVERGENCE OF HERMITE–FEJÉR POLYNOMIALS FOR FUNCTIONS WITH DERIVATIVES OF BOUNDED VARIATION

R. BOJANIC (Columbus, Ohio) and F. CHENG (Lexington, Kentucky)

1. Introduction

Let f be a function defined on $[-1,1]$. The Hermite–Fejér interpolation polynomial $\mathbf{H}_n(f,x)$ of f, based on the zeros

$$x_{kn}=\cos\left(\frac{2k-1}{2n}\pi\right),\quad k=1,2,\ldots,n \tag{1.1}$$

of the Chebysev polynomial $T_n(x)=\cos(n\cos^{-1}x)$, is defined by

$$\mathbf{H}_n(f,x)=\sum_{k=1}^{n} f(x_{kn})(1-x_{kn}x)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2. \tag{1.2}$$

It was proved by L. Fejér [1] that $\mathbf{H}_n(f,x)$ converges uniformly to $f(x)$ if $f(x)$ is a continuous function on $[-1,1]$. The rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ when $f(x)$ is a continuous function has been extensively studied before ([2]--[9]). A survey of various quantitative estimates of the rate of convergence can be found in [9], where it was proved that

$$|\mathbf{H}_n(f,x)-f(x)|\leqq \frac{C_1}{n}T_n^2(x)\sum_{k=1}^{n}\left[W_f\left(\frac{(1-x^2)^{1/2}}{k}\right)+W_f\left(\frac{1}{k^2}\right)\right]+ \\ +C_2W_f\left(\frac{|T_n(x)|}{n}\right).$$

Here C_1 and C_2 are positive constants and W_f is the *modulus of continuity* of f.

The behavior of $\mathbf{H}_n(f,x)$ when $f\in\mathbf{BV}[-1,1]$ (i.e., f is of *bounded variation* on $[-1,1]$) was studied by Bojanic and Cheng [10]. It was proved that if $f\in\mathbf{BV}[-1,1]$ and continuous at $x\in(-1,1)$ then $\mathbf{H}_n(f,x)$ converges to $f(x)$ when n tends to $+\infty$ and the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ satisfies the following inequality

$$|\mathbf{H}_n(f,x)-f(x)|\leqq\frac{64T_n^2(x)}{n}\sum_{k=1}^{n}\mathbf{V}_{x-\pi/k}^{x+\pi/k}(f)+2\mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(f) \tag{1.3}$$

where $\mathbf{V}_a^b(f)$ is the *total variation* of f on $[a,b]$. (1.3) can not be improved asymptotically.

However, if x is a point of discontinuity of f where $f(x+)\neq f(x-)$, the sequence $(\mathbf{H}_n(f,x))$ is no longer convergent. This follows from the following observation

$$\text{(1.4)}\quad \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} H_n(f,x)=\frac{1}{2}(f(x+)+f(x-))\pm\frac{1}{2}|f(x+)-f(x-)|\beta(x)$$

where $\beta(x)=1$ if $x=\cos(\alpha\pi)$ and α is irrational, and

$$\beta(x)=\left(\frac{\sin(\pi/2q)}{\pi/2q}\right)^2\left(1-\sum_{k=1}^{\infty}\frac{8qk}{(4q^2k^2-1)^2}\right)$$

if $x=\cos(p\pi/q)$.

(1.4) shows that, unlike Fourier series of 2π-periodic functions of bounded variation or Bernstein polynomials of functions of bounded variation which all converge to $(f(x+)+f(x-))/2$ ([11], [12]), the Hermite–Fejér interpolation polynomials of a function of bounded variation converge only if $f(x+)=f(x-)$.

Although smoother functions find more applications in various fields such as computer aided geometric design, computer vision, graphics and image processing, the asymptotic behavior of Hermite–Fejér interpolation polynomials for functions smoother than continuous functions has been studied only for functions with continuous derivatives.

In this paper we shall investigate the asymptotic behavior of Hermite–Fejér polynomials for functions defined the following way

$$\text{(1.5)}\qquad f(x)=f(-1)+\int_{-1}^{x}\varphi(t)dt,\quad x\in[-1,1]$$

where φ is a function of bounded variation on $[-1,1]$. This class of functions can be described as the class of differentiable functions whose derivatives are of bounded variation and will be denoted by $\mathbf{DBV}[-1,1]$. It is clear that this class of functions is much more general than functions with continuous derivatives. However, as it will be seen in the next section, the asymptotic behavior of Hermite–Fejér interpolation polynomials for functions in this category is also much better than the asymptotic behavior of Hermite–Fejér interpolation polynomials for continuous functions. Results for Bernstein polynomials for functions of this type can be found in [13].

2. Results

Let f be a function in $\mathbf{DBV}[-1,1]$ and $\varphi\in\mathbf{BV}[-1,1]$ so that (1.5) is satisfied. For any $x\in(-1,1)$ such that $x\neq x_{kn}$ for $k=1,2,\dots,n$ we have, from (1.2) and (1.5),

$$\text{(2.1)} \qquad \mathbf{H}_n(f,x)-f(x)=\sum_{k=1}^{n}\left(\int_x^{x_{kn}}\varphi(t)dt\right)\mathbf{H}_{kn}(x)=$$

$$=-\sum_{x_{kn}<x}\left(\int_{x_{kn}}^{x}\varphi(t)dt\right)\mathbf{H}_{kn}(x)+\sum_{x_{kn}>x}\left(\int_x^{x_{kn}}\varphi(t)dt\right)\mathbf{H}_{kn}(x)$$

where

$$\mathbf{H}_{kn}(x)=(1-xx_{kn})\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2, \quad k=1,2,\ldots,n.$$

If we define $\varphi_x(t)$ the following way

$$\varphi_x(t)=\begin{cases}\varphi(t)-\varphi(x-), & t<x\\ 0, & t=x\\ \varphi(t)-\varphi(x+), & t>x\end{cases}$$

then (2.1) can be expressed as

$$\text{(2.2)} \qquad \mathbf{H}_n(f,x)-f(x)=$$

$$=-\sum_{x_{kn}<x}\left(\int_{x_{kn}}^{x}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x)+\sum_{x_{kn}>x}\left(\int_x^{x_{kn}}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x)$$

$$-\varphi(x-)\sum_{x_{kn}<x}(x-x_{kn})\mathbf{H}_{kn}(x)+\varphi(x+)\sum_{x_{kn}>x}(x_{kn}-x)\mathbf{H}_{kn}(x).$$

Since

$$\varphi(x+)\sum_{x_{kn}>x}(x_{kn}-x)\mathbf{H}_{kn}(x)-\varphi(x-)\sum_{x_{kn}<x}(x-x_{kn})\mathbf{H}_{kn}(x)=$$

$$=\frac{\varphi(x+)-\varphi(x-)}{2}\sum_{k=1}^{n}|x_{kn}-x|\mathbf{H}_{kn}(x)+\frac{\varphi(x+)+\varphi(x-)}{2}\sum_{k=1}^{n}(x_{kn}-x)\mathbf{H}_{kn}(x)$$

and the two summations on the right-hand side of the equation are the Hermite–Fejér interpolation polynomials of $f_x(t)=|t-x|$ and $g_x(t)=t-x$, $-1\leqq t\leqq 1$, respectively, we can further convert (2.2) as follows

$$\text{(2.3)} \qquad \mathbf{H}_n(f,x)-f(x)=\frac{\sigma}{2}\mathbf{H}_n(f_x,x)+\frac{\lambda}{2}\mathbf{H}_n(g_x,x)+\mathbf{P}_n(f,x)$$

where

$$\sigma=\varphi(x+)-\varphi(x-);\quad \lambda=\varphi(x+)+\varphi(x-) \tag{2.4}$$

and

$$\mathbf{P}_n(f,x)= \tag{2.5}$$
$$=-\sum_{x_{kn}<x}\left(\int_{x_{kn}}^{x}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x)+\sum_{x_{kn}>x}\left(\int_{x}^{x_{kn}}\varphi_x(t)dt\right)\mathbf{H}_{kn}(x).$$

Therefore, evaluation of the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ is simply a matter of evaluating $\mathbf{H}_n(f_x,x)$, $\mathbf{H}_n(g_x,x)$ and $\mathbf{P}_n(f,x)$.

We shall give estimates for $\mathbf{H}_n(f_x,x)$ and $\mathbf{H}_n(g_x,x)$ first and then use these estimates to get an estimate for the rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$. Since the estimates we get for $\mathbf{H}_n(f_x,x)$ and $\mathbf{H}_n(g_x,x)$ are of some interest, we shall state them as an independent theorem.

THEOREM 1. *If $x\in(-1,1)$ and $x\neq x_{kn}$ for any $k=1,2,\dots,n$ then*

$$\left|\sum_{k=1}^{n}|x_{kn}-x|\mathbf{H}_{kn}(x)-\frac{2(1-x^2)^{1/2}T_n^2(x)}{\pi}\frac{\log n}{n}\right|\leqq C\frac{|T_n(x)|}{n}, \tag{2.6}$$

$$\left|\sum_{k=1}^{n}(x_{kn}-x)\mathbf{H}_{kn}(x)\right|\leqq C\frac{|T_n(x)|}{n} \tag{2.7}$$

where $C=7+\pi$.

The rate of convergence of $\mathbf{H}_n(f,x)$ to $f(x)$ can be estimated as follows.

THEOREM 2. *Let f be a function in* $\mathbf{DBV}[-1,1]$ *and $\varphi\in\mathbf{BV}[-1,1]$ so that* (1.5) *is satisfied. Then for any $x\in(-1,1)$ such that $x\neq x_{kn}$ for $k=1,2,\dots,n$ we have*

$$\left|\mathbf{H}_n(f,x)-f(x)-\frac{\sigma(1-x^2)^{1/2}T_n^2(x)}{\pi}\frac{\log n}{n}\right|\leqq\frac{C(|\sigma|+|\lambda|)}{2}\frac{|T_n(x)|}{n}+ \tag{2.8}$$
$$+\frac{\pi|T_n(x)|}{n}\mathbf{V}_{x-\pi|T_n(x)|/n}^{x+\pi|T_n(x)|/n}(\varphi_x)+\frac{12T_n^2(x)}{n}\sum_{k=1}^{n}\frac{\mathbf{V}_{x-\pi/k}^{x+\pi/k}(\varphi_x)}{k}$$

where $\mathbf{V}_a^b(\varphi_x)$ is the total variation of φ_x on $[a,b]$, σ and λ are defined in* (2.4), *and C is defined in Theorem* 1.

*We assume here and in the rest of the paper that f is extended to the entire real line by $f(x)=f(1)$ for $x>1$ and $f(x)=f(-1)$ for $x<-1$.

If f' is continuous at x, i.e., $\sigma = 0$ and $\lambda = 2f'(x)$, then (2.8) can be simplified as follows.

$$(2.9)\quad |\mathbf{H}_n(f,x) - f(x)| \leqq \frac{C|f'(x)T_n(x)|}{n} + \frac{\pi|T_n(x)|}{n}\mathbf{V}_{x-\pi|T_n(x)|/n}^{x+\pi|T_n(x)|/n}(f') + \frac{12T_n^2(x)}{n}\sum_{k=1}^{n}\frac{\mathbf{V}_{x-\pi/k}^{x+\pi/k}(f')}{k}.$$

The right-hand side of (2.8) converges to zero as $n \to \infty$ since continuity of φ_x at x implies that

$$\mathbf{V}_{x-\beta}^{x+\alpha}(\varphi_x) \to 0 \quad (\alpha, \beta \to 0+).$$

Actually, the last term of the right-hand side of (2.8) is $o(\log n/n)$-convergent. Therefore (2.8) can also be expressed as

$$\mathbf{H}_n(f,x) = f(x) + \frac{\sigma(1-x^2)^{1/2}T_n^2(x)}{\pi}\frac{\log n}{n} + o\left(\frac{\log n}{n}\right).$$

Note that all the estimates mentioned in Section 1 for continuous functions or functions of bounded variation are $o(1)$-convergent only.

As far as the precision of (2.9) is concerned, consider the Hermite–Fejér polynomials of the function $f(x) = x^2$ at $x = 0$ for even n. Since $T_n(0) = 1$ if n is an even integer, we have

$$\mathbf{H}_n(f,0) - f(0) = \sum_{k=1}^{n}\frac{T_n^2(0)}{n^2} = \frac{1}{n}.$$

On the other hand, since $\sigma = \lambda = 0$ at $x = 0$, it follows from (2.9) that

$$|\mathbf{H}_n(f,0) - f(0)| \leqq \frac{\pi}{n}\mathbf{V}_{-\pi/n}^{+\pi/n}(\varphi_0) + \frac{12}{n}\sum_{k=1}^{n}\frac{\mathbf{V}_{-\pi/k}^{+\pi/k}(\varphi_0)}{k}.$$

since $\varphi(t) = 2t$ and $\varphi_0(t) = \varphi(t)$, we have

$$|\mathbf{H}_n(f,0) - f(0)| \leqq \frac{4\pi^2}{n^2} + \frac{12}{n}\sum_{k=1}^{n}\frac{4\pi}{k^2} < \frac{C}{n}$$

for some $C > 0$. Hence for the function $f(x) = x^2$ when n is an even integer we have

$$\frac{1}{n} \leqq |\mathbf{H}_n(f,0) - f(0)| \leqq \frac{C}{n}$$

for some positive constant $C > 0$. Therefore (2.9) can not be improved asymptotically.

3. Proofs

3.1. PROOF OF THEOREM 1. To prove (2.6), observe that

$$\left|\mathbf{H}_n(f_x,x)-\sum_{k=1}^{n}|x_{kn}-x|(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2\right| \leqq \frac{|x|T_n^2(x)}{n}. \tag{3.1}$$

Therefore, it is sufficient to study the asymptotic behavior of the second term on the left-hand side of (3.1) only.

Let $x=\cos\vartheta$, $0<\vartheta<\pi$, $x_{kn}=\cos\vartheta_{kn}$, $\vartheta_{kn}=(2k-1)\pi/(2n)$, $k=1,2,\dots,$ n, and define

$$\delta_\vartheta(\alpha)=\begin{cases}1, & \text{if } 0<\alpha<\vartheta\\ -1, & \text{if } \vartheta<\alpha<\pi.\end{cases}$$

Then

(3.2)
$$\sum_{k=1}^{n}|x_{kn}-x|(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2=\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}.$$

Since $\cos\vartheta_{kn}-\cos\vartheta=(\vartheta-\vartheta_{kn})\sin\overline{\vartheta}_{kn}$ for some $\overline{\vartheta}_{kn}$ between ϑ and ϑ_{kn}, it follows that

$$\left|\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}-\sum_{k=1}^{n}\delta_\vartheta(\vartheta_{kn})\frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})}\right|\leqq$$

$$\leqq\frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\left(\frac{\sin\vartheta}{\cos\vartheta_{kn}-\cos\vartheta}-\frac{1}{\vartheta-\vartheta_{kn}}\right)\right|\leqq$$

$$\leqq\frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\left(\frac{\sin\vartheta}{\sin\overline{\vartheta}_{kn}(\vartheta-\vartheta_{kn})}-\frac{1}{\vartheta-\vartheta_{kn}}\right)\right|\leqq$$

$$\leqq\frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\vartheta-\vartheta_{kn}}\left(\frac{\sin\vartheta-\sin\overline{\vartheta}_{kn}}{\sin\overline{\vartheta}_{kn}}\right)\right|\leqq$$

$$\leqq\frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\vartheta-\vartheta_{kn}}\,\frac{\vartheta-\overline{\vartheta}_{kn}}{\sin\overline{\vartheta}_{kn}}\right|\leqq\frac{\sin\vartheta\cos^2 n\vartheta}{n^2}\sum_{k=1}^{n}\left|\frac{1}{\sin\overline{\vartheta}_{kn}}\right|$$

Furthermore, since $|\sin\frac{\vartheta}{2}|\geqq|\vartheta|/\pi$ if $|\vartheta|\leqq\pi$, we have

$$\sin \overline{\vartheta}_{kn} = \left| \frac{\cos \vartheta - \cos \vartheta_{kn}}{\vartheta - \vartheta_{kn}} \right| = \left| \frac{2 \sin \frac{\vartheta+\vartheta_{kn}}{2} \sin \frac{\vartheta-\vartheta_{kn}}{2}}{\vartheta - \vartheta_{kn}} \right| \geqq$$

$$\geqq \left| \frac{2 \sin \frac{\vartheta+\vartheta_{kn}}{2} \frac{\vartheta-\vartheta_{kn}}{\pi}}{\vartheta - \vartheta_{kn}} \right| = \frac{2}{\pi} \left| \sin\left(\frac{\vartheta+\vartheta_{kn}}{2} \right) \right| \geqq \frac{2}{\pi} M(\vartheta)$$

where $M(\vartheta) = \min(\sin \frac{\vartheta}{2}, \sin \frac{\pi-\vartheta}{2})$. Therefore,

(3.3)
$$\left| \sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin^2 \vartheta \cos^2(n\vartheta)}{n^2(\cos \vartheta_{kn} - \cos \vartheta)} - \sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin \vartheta \cos^2(n\vartheta)}{n^2(\vartheta - \vartheta_{kn})} \right| \leqq \frac{\pi \cos^2 n\vartheta}{n}$$

by noticing that $M(\vartheta) \geqq \sin \vartheta/2$.

Let j be the integer such that $\vartheta_{jn} < \vartheta < \vartheta_{j+1,n}$. It is easy to see that

$$j = \left[\frac{n\vartheta}{\pi} + \frac{1}{2} \right].$$

Since $n(\vartheta - \vartheta_{kn}) = \pi(\frac{n\vartheta}{\pi} + \frac{1}{2} - k)$, we have

(3.4)
$$\sum_{k=1}^{n} \delta_\vartheta(\vartheta_{kn}) \frac{\sin \vartheta \cos^2(n, \vartheta)}{n^2(\vartheta - \vartheta_{kn})} =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=1}^{j} \frac{1}{(\frac{n\vartheta}{\pi} + \frac{1}{2} - k)} - \sum_{k=j+1}^{n} \frac{1}{(\frac{n\vartheta}{\pi} + \frac{1}{2} - k)} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j + k} - \sum_{k=0}^{n-j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j - 1 - k} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\frac{n\vartheta}{\pi} + \frac{1}{2} - j + k} + \sum_{k=0}^{n-j-1} \frac{1}{k + 1 - (\frac{n\vartheta}{\pi} + \frac{1}{2}) + j} \right) =$$

$$= \frac{\sin \vartheta \cos^2 n\vartheta}{n\pi} \left(\sum_{k=0}^{j-1} \frac{1}{\rho(n\vartheta) + k} + \sum_{k=0}^{n-j-1} \frac{1}{1 - \rho(n\vartheta) + k} \right) = \Delta_1 + \Delta_2$$

where

(3.5)
$$\rho(x) = \frac{x}{\pi} + \frac{1}{2} - \left[\frac{x}{\pi} + \frac{1}{2} \right].$$

We shall prove that Δ_1 and Δ_2 are both asymptotically equal to $\sin \vartheta \cos^2 n\vartheta \log n/(n\pi)$.

First, observe that

$$\Delta_1 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi} = \delta_{1,1} + \delta_{1,2} \tag{3.6}$$

where

$$\delta_{1,1} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\frac{1}{\rho(n\vartheta)}, \quad \delta_{1,2} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\left(\sum_{k=1}^{j-1}\frac{1}{\rho(n\vartheta)+k} - \log n\right).$$

Since $\cos n\vartheta = (-1)^j \sin(\rho(n\vartheta)\pi)$ where $j = [n\vartheta/\pi + 1/2]$, if follows that

$$|\delta_{1,1}| \leqq \frac{\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.7}$$

On the other hand, it is easy to see that

$$\left|\sum_{k=1}^{j-1}\frac{1}{\rho(n\vartheta)+k} - \log n\right| \leqq 5.$$

Hence,

$$|\delta_{1,2}| \leqq \frac{5\sin\vartheta\cos^2 n\vartheta}{n\pi}. \tag{3.8}$$

Therefore, from (3.6), (3.7) and (3.8) we have

$$\left|\Delta_1 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi}\right| \leqq \frac{3\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.9}$$

The evaluation of the asymptotic behavior of Δ_2 can be carried out in a similar way. First, observe that

$$\Delta_2 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi} = \delta_{2,1} + \delta_{2,2} \tag{3.10}$$

where

$$\delta_{2,1} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\frac{1}{1-\rho(n\vartheta)},$$

$$\delta_{2,2} = \frac{\sin\vartheta\cos^2 n\vartheta}{n\pi}\left(\sum_{k=1}^{n-j-1}\frac{1}{1-\rho(n\vartheta)+k} - \log n\right).$$

Since $\cos n\vartheta = (-1)^j \sin(1-\rho(n\vartheta)\pi)$ where $j = [n\vartheta/\pi + 1/2]$, it follows that

$$|\delta_{2,1}| \leqq \frac{\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.11}$$

On the other hand, it is also easy to see that

$$\left|\sum_{k=1}^{n-j-1} \frac{1}{1-\rho(n\vartheta)+k} - \log n\right| \leqq 5.$$

Hence,

$$|\delta_{2,2}| \leqq \frac{5\sin\vartheta\cos^2 n\vartheta}{n\pi}. \tag{3.12}$$

Therefore, from (3.10), (3.11) and (3.12) we have

$$\left|\Delta_2 - \frac{\sin\vartheta\cos^2 n\vartheta\log n}{n\pi}\right| \leqq \frac{3\sin\vartheta|\cos n\vartheta|}{n}, \tag{3.13}$$

and the estimate (2.6) follows from (3.1), (3.2), (3.3), (3.4), (3.6), (3.9) and (3.13).

To prove (2.7), observe that by using a similar technique we can show that

$$\left|\mathbf{H}_n(g_x, x) - \sum_{k=1}^{n}(x_{kn}-x)(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2\right| \leqq \frac{|x|T_n^2(x)}{n}, \tag{3.14}$$

$$\sum_{k=1}^{n}(x_{kn}-x)(1-x^2)\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2 = \sum_{k=1}^{n} \frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)}, \tag{3.15}$$

$$\left|\sum_{k=1}^{n} \frac{\sin^2\vartheta\cos^2(n\vartheta)}{n^2(\cos\vartheta_{kn}-\cos\vartheta)} - \sum_{k=1}^{n} \frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})}\right| \leqq \frac{\pi\cos^2 n\vartheta}{n}, \tag{3.16}$$

where $x=\cos\vartheta$, $0<\vartheta<\pi$, $x_{kn}=\cos\vartheta_{kn}$, $\vartheta_{kn}=(2k-1)\pi/2n$, $k=1,2,\dots,n$, and

$$\sum_{k=1}^{n} \frac{\sin\vartheta\cos^2(n\vartheta)}{n^2(\vartheta-\vartheta_{kn})} = \Delta_1 - \Delta_2 \tag{3.17}$$

where Δ_1 and Δ_2 are defined in (3.4). It follows immediately from (3.9) and (3.13) that

$$|\Delta_1 - \Delta_2| \leqq \frac{6\sin\vartheta|\cos n\vartheta|}{n}. \tag{3.18}$$

Therefore, (2.7) follows from (3.14), (3.15), (3.16), (3.17) and (3.18).

3.2. Proof of Theorem 2. Since the evaluation of $\mathbf{H}_n(f_x, x)$ and $\mathbf{H}_n(g_x, x)$ is already done in Theorem 1, the only thing we have to do now is the evaluation of $\mathbf{P}_n(f, x)$. The technique used here is similar to the one used in [10].

For any $x \in (-1,1)$ such that $x \neq x_{kn}$ for $k = 1, 2, \ldots, n$ we have

$$|P_n(f,x)| \leqq \sum_{k=1}^{n} \left| \int_{x_{kn}}^{x} \varphi_x(t)dt \right| \mathbf{H}_{kn}(x) \leqq$$

$$\leqq \sum_{k=1}^{n} \left| \int_{x_{kn}}^{x} \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)dt \right| \mathbf{H}_{kn}(x) \leqq \sum_{k=1}^{n} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)\mathbf{H}_{kn}(x)$$

where $t_{kn} = |x - x_{kn}|$ and $\mathbf{V}_a^b(\varphi_x)$ is the total variation of φ_x on $[a, b]$. Let $x = \cos\vartheta$, $0 < \vartheta < \pi$, $x_{kn} = \cos\vartheta_{kn}$, $\vartheta_{kn} = (2k-1)\pi/(2n)$, $k = 1, 2, \ldots, n$, and define

$$E_r(n,\vartheta) = \left\{ k : \frac{r\pi}{2n} < |\vartheta - \vartheta_{kn}| \leqq \frac{(r+1)\pi}{2n} \right\}, \quad r = 0, 1, \ldots, 2n-1.$$

Then we have

$$|\mathbf{P}_n(f,x)| \leqq \sum_{r=0}^{2n-1} \sum_{k \in E_r(n,\vartheta)} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)\mathbf{H}_{kn}(x).$$

Since $t_{kn} = |x - x_{kn}| \leqq |\vartheta - \vartheta_{kn}| \leqq \pi|T_n(x)|/2n$ if $k \in E_0(n,\vartheta)$ (see [9], p.257) and $E_0(n,\vartheta)$ has at most two elements, it follows that

(3.19)
$$\sum_{k \in E_0(n,\vartheta)} |x - x_{kn}| \mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)\mathbf{H}_{kn}(x) \leqq \frac{\pi|T_n(x)|}{n} \mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(\varphi_x).$$

On the other hand, since $E_r(n,\vartheta)$ has at most two elements,

$$|x - x_{kn}|\mathbf{H}_{kn}(x) \leqq (1 - \cos\vartheta\cos\vartheta_{kn} + \sin\vartheta\sin\vartheta_{kn}) \frac{T_n^2(x)}{n^2|\cos\vartheta - \cos\vartheta_{kn}|} \leqq$$

$$\leqq (1 - \cos(\vartheta_{kn} + \vartheta)) \frac{T_n^2(x)}{2n^2 \sin\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \sin\left|\frac{\vartheta-\vartheta_{kn}}{2}\right|} \leqq$$

$$\leqq 2\sin^2\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \frac{T_n^2(x)}{2n^2 \sin\left(\frac{\vartheta+\vartheta_{kn}}{2}\right) \sin\left|\frac{\vartheta-\vartheta_{kn}}{2}\right|} \leqq$$

$$\leqq \frac{T_n^2(x)\pi}{n^2|\vartheta - \vartheta_{kn}|} \leqq \frac{2T_n^2(x)}{nr}$$

and $t_{kn} \leqq (r+1)\pi/2n$ if $k \in E_r(n,\vartheta)$, we have

$$(3.20) \qquad \sum_{k\in E_r(n,\vartheta)} |x-x_{kn}|\mathbf{V}_{x-t_{kn}}^{x+t_{kn}}(\varphi_x)\mathbf{H}_{kn}(x) \leqq \frac{4T_n^2(x)}{nr}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x)$$

for $r=1,2,\dots,2n-1$. Therefore, by (3.19) and (3.20),

$$(3.21) \qquad |\mathbf{P}_n(f,x)| \leqq \frac{\pi|T_n(x)|}{n}\mathbf{V}_{x-\pi|T_n(x)|/2n}^{x+\pi|T_n(x)|/2n}(\varphi_x)+ \\ +\frac{4T_n^2(x)}{n}\sum_{r=1}^{2n-1}\frac{1}{r}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x).$$

Let $Q(t)=\mathbf{V}_{x-t}^{x+t}(\varphi_x)$. Then

$$(3.22) \qquad \sum_{r=1}^{2n-1}\frac{1}{r}\mathbf{V}_{x-(r+1)\pi/2n}^{x+(r+1)\pi/2n}(\varphi_x)= \\ =\sum_{r=2}^{2n}\frac{1}{r-1}Q\left(\frac{r\pi}{2n}\right) \leqq 2\sum_{r=2}^{2}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right).$$

By virtue of the fact that $Q(t)$ is a non-decreasing function, we have

$$\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{Q(t)}{t}dt \geqq Q\left(\frac{r\pi}{2n}\right)\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{dt}{t} \geqq Q\left(\frac{r\pi}{2n}\right)\log\left(1+\frac{1}{r}\right)$$

or

$$\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq \frac{3}{2}\int_{r\pi/2n}^{(r+1)\pi/2n}\frac{Q(t)}{t}dt.$$

Hence

$$\sum_{r=2}^{2n}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq \frac{3}{2}\int_{\pi/n}^{\pi(2n+1)/2n}\frac{Q(t)}{t}dt.$$

Since $Q(\pi/t)$ is non-decreasing and $Q(\pi/t)=Q(\pi)$ for $0<t\leqq 1$, we have

$$\int_{2n/(2n+1)}^{n}\frac{Q(\pi/t)}{t}dt \leqq \int_{2n/(2n+1)}^{1}\frac{Q(\pi/t)}{t}dt+\int_{1}^{n}\frac{Q(\pi/t)}{t}dt \leqq \\ \leqq \frac{Q(\pi)}{2n+1}+\sum_{k=1}^{n}\frac{Q(\pi/k)}{k} \leqq 2\sum_{k=1}^{n}\frac{Q(\pi/k)}{k}$$

and therefore

$$(3.23) \qquad \sum_{r=2}^{2n}\frac{1}{r}Q\left(\frac{r\pi}{2n}\right) \leqq 3\sum_{k=1}^{n}\frac{Q(\pi/k)}{k}.$$

The proof of Theorem 2 follows now from (3.21) and (3.23).

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(Received May 26, 1988)

DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210
USA

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY 40506
USA

Acta Math. Hung.
59(1-2) (1992), 103–106.

NOTE ON DECOMPOSITION OF BOUNDED FUNCTIONS INTO THE SUM OF PERIODIC TERMS

Z. GAJDA (Katowice)

A motivation for the present note arises from the paper [2] written jointly by M. Laczkovich and Sz. Révész. The following two definitions are due to the authors of the paper just mentioned.

DEFINITION 1. Let A be a non-empty set and let T be a transformation of A into itself. A real-valued function g defined on A is called T-periodic if and only if $\Delta_T g = 0$, where $\Delta_T g := g \circ T - g$.

DEFINITION 2. Let $\mathcal{F}$ be a class of real-valued functions defined on a set A and let $T_1, \ldots, T_k$ map A into itself. The class $\mathcal{F}$ is said to have the decomposition property (d. pr.) with respect to $T_1, \ldots, T_k$ if and only if for every function $f \in \mathcal{F}$ satisfying the condition

$$\Delta_{T_1} \ldots \Delta_{T_k} f = 0$$

there exist functions $f_1, \ldots, f_k \in \mathcal{F}$ such that f_i is T_i-periodic $(i = 1, \ldots, k)$ and $f = f_1 + \ldots + f_k$.

It was proved in [2] that numerous important classes of real functions have the d. pr. with respect to reasonable families of transformations. Among others the following result holds true:

THEOREM 1. *Suppose $\{T_1, \ldots, T_k\}$ to be a family of commuting transformations of a set A into itself (i.e. $T_i \circ T_j = T_j \circ T_i$ for $i, j = 1, \ldots, k$). Then the class $B(A)$ of all bounded real-valued functions defined on A has the d. pr. with respect to $T_1, \ldots, T_k$.*

The proof of this fact presented in [2] is based on a theorem describing kernels of superpositions of bounded linear operators in some Banach spaces. The main purpose of the present note is to give another quite simple and straightforward proof of the same result (it is mentioned in [2], Remark 4.8). Our method relies on the well known fact (cf. [1], II.4, Exercise 22) that the space ℓ^∞ of all bounded real sequences admits a linear functional $\mathrm{LIM} : \ell^\infty \to \mathbf{R}$ with the following two properties:

$$\liminf_{n\to\infty} x_n \leqq \operatorname*{LIM}_{n\to\infty} x_n \leqq \limsup_{n\to\infty} x_n, \tag{1}$$

$$\operatorname*{LIM}_{n\to\infty} x_{n+1} = \operatorname*{LIM}_{n\to\infty} x_n \tag{2}$$

for all sequences $\{x_n\}_{n\in\mathbf{N}} \in \ell^\infty$. Here and in the sequel we write $\underset{n\to\infty}{\mathrm{LIM}}\, x_n$ instead of $\mathrm{LIM}(\{x_n\}_{n\in\mathbf{N}})$. In particular, (1) implies that $\underset{n\to\infty}{\mathrm{LIM}}\, 1 = 1$ and $\|\mathrm{LIM}\| = 1$. Any linear functional on ℓ^∞ satisfying (1) and (2) is called a Banach limit.

PROOF OF THEOREM 1. For the sake of simplicity we restrict ourselves to the case $k = 2$. The proof of the general case is similar.

Assume that a function $f \in B(A)$ fulfils the condition

$$\Delta_{T_1}\Delta_{T_2} f = 0.$$

Denoting by I the identity operator and putting $B_i f := f \circ T_i$ $(i = 1, 2)$ we infer that for each positive integer j the following formula holds:

$$(I - B_i^j) = (I - (\Delta_{T_i} + I)^j) = \Delta_{T_i} C_i,$$

where B_i^j stands for the j-th iterate of the operator B_i and C_i is a polynomial of Δ_{T_i}. Hence and from the fact that the operators T_1 and T_2 commute it follows that

$$f - B_1^n f - B_2^m f + B_1^n B_2^m f = (I - B_1^n)(I - B_2^m) f = 0$$

for all $n, m \in \mathbf{N}$, or more explicitly,

$$f(x) - f(T_1^n(x)) - f(T_2^m(x)) + f(T_1^n(T_2^m(x))) = 0 \tag{3}$$

for all $x \in A$ and $n, m \in \mathbf{N}$.

Let LIM denote a Banach limit defined for all sequences in ℓ^∞. For each $x \in A$ the sequence $\{f(T_1^n(x))\}_{n\in\mathbf{N}}$ is bounded and, therefore, we may correctly define

$$g(x) := \underset{n\to\infty}{\mathrm{LIM}}\, f(T_1^n(x)), \quad x \in A.$$

Since $\|\mathrm{LIM}\| = 1$, we obtain

$$|g(x)| \leqq \sup_{n\in\mathbf{N}} |f(T_1^n(x))| \leqq \sup_{y\in A} |f(y)| = \|f\|_\infty, \quad x \in A,$$

which ensures that $g \in B(A)$. Now fix an $m \in \mathbf{N}$ in equation (3) and let the functionanl LIM act on both sides of (3) considered as sequences corresponding to the index n running over $\mathbf{N}$. It is clear that a Banach limit of a constant sequence is just the same constant. Thus, equation (3) and linearity of the functional LIM yield

$$f(x) - g(x) - f(T_2^m(x)) + g(T_2^m(x)) = 0$$

for all $x \in A$ and $m \in \mathbf{N}$.

Similarly, setting

$$h(x) := \operatorname*{LIM}_{m\to\infty} f(T_2^m(x)) - \operatorname*{LIM}_{m\to\infty} g(T_2^m(x)), \quad x \in A$$

we deduce that $h \in B(A)$ and

$$f(x) - g(x) - h(x) = 0, \quad x \in A.$$

Moreover,

$$g(T_1(x)) = \operatorname*{LIM}_{n\to\infty} f(T_1^{n+1}(x)) = \operatorname*{LIM}_{n\to\infty} f(T_1^n(x)) = g(x)$$

and

$$h(T_2(x)) = \operatorname*{LIM}_{m\to\infty} f(T_2^{m+1}(x)) - \operatorname*{LIM}_{m\to\infty} g(T_2^{m+1}(x)) =$$
$$= \operatorname*{LIM}_{m\to\infty} f(T_2^m(x)) - \operatorname*{LIM}_{m\to\infty} g(T_2^m(x)) = h(x),$$

which completes the proof.

From now on we assume that the set A is endowed with a uniformity $\mathcal{U}$; in other words $(A, \mathcal{U})$ is a uniform space.

DEFINITION 3. A family $\{T_s : s \in S\}$ of transformations of A into itself is called equicontinuous if and only if for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ with the following property:

$$(x, y) \in V \text{ implies that } (T_s(x), T_s(y)) \in U \text{ for all } s \in S.$$

In what follows we apply this general definition to special families consisting of iterates of given transformations.

THEOREM 2. *Let $T_1, \ldots, T_k$ be commuting transformations of A into itself such that for each $i = 1, \ldots, k$ the family $\{T_i^n : n \in \mathbf{N}\}$ is equicontinuous. Then the class $UCB(A)$ of all real-valued uniiformly continuous bounded functions defined on A has the d. pr. with respect to $T_1, \ldots, T_k$.*

PROOF. A direct inspection of the proof of Theorem 1 shows that the following fact is sufficient to derive the assertion of Theorem 2: if $f \in UCB(A)$ and $T : A \to A$ is a transformation with the equicontinuous family of its iterates, then the function $g : A \to \mathbf{R}$ defined by

$$g(x) := \operatorname*{LIM}_{n\to\infty} f(T^n(x)), \quad x \in A$$

is also an element of the class $UCB(A)$. To check this let us fix an $\varepsilon > 0$. From the uniform continuity of f we infer the existence of such a $U \in \mathcal{U}$ that

$$|f(u) - f(v)| \leqq \varepsilon \quad \text{whenever} \quad (u, v) \in U.$$

On the other hand, by virtue of the equicontinuity of the family $\{T^n : n \in \mathbf{N}\}$, there exists a $V \in \mathcal{U}$ such that $(x, y) \in V$ implies that $(T^n(x), T^n(y)) \in U$ for all $n \in \mathbf{N}$. Consequently,

$$\sup_{n \in \mathbf{N}} |f(T^n(x)) - f(T^n(y))| \leqq \varepsilon,$$

provided $(x, y) \in V$. Since $\| \mathrm{LIM} \| = 1$, we have

$$|g(x) - g(y)| = | \operatorname*{LIM}_{n \to \infty} (f(T^n(x)) - f(T^n(y)))| \leqq \varepsilon$$

whenever $(x, y) \in V$, which proves that g is uniformly continuous. Its boundedness can be verified by the same argument as in the proof of Theorem 1.

COROLLARY. *If G is an Abelian topological group, then the class $UCB(G)$ has the d. pr. with respect to any finite system of translations of G.*

PROOF. Since G is Abelian, any two translations of G commute. Moreover, let us note that if T is a translation of G, then for each $n \in \mathbf{N}$ and all $(x, y) \in G$ we have

$$T^n(x) - T^n(y) = x - y.$$

This guarantees that the family $\{T^n : n \in \mathbf{N}\}$ is equcontinuous with respect to the standard uniformity generated by the topology of G. Finally, it suffices to apply Theorem 1.

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(Received May 19, 1988)

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
40-007 KATOWICE
POLAND

Acta Math. Hung.
59(1-2) (1992), 107–110.

INFLUENCE OF NORMALITY ON MAXIMAL SUBGROUPS OF SYLOW SUBGROUPS OF A FINITE GROUP

M. RAMADAN (Cairo)

1. Introduction

Throughout this paper the term group always means a group of finite order. In [5] Srinivasan showed that groups in which maximal subgroups of Sylow subgroups are normal are supersolvable. In Section 3, we prove that if G is a solvable group and all maximal subgroups of Sylow subgroups of $\mathrm{Fit}(G)$ are normal in G, then G is supersolvable. We also prove that if all maximal subgroups of Sylow subgroups of G, except possibly for the largest prime dividing $|G|$, are normal in G, then G possesses an ordered Sylow tower and $G/O_p(G)$ is supersolvable. Furthermore, using the above results, we also obtain some results concerning the supersolvability of a finite group G.

Our notation is standard and taken mainly from [2].

2. Preliminaries

Let G be a finite group of order $p_1^{e_1}p_2^{e_2}\dots p_n^{e_n}$, where $p_1 > p_2 > \dots > p_n$. We say that G has an ordered Sylow tower if there exists a series $1 = G_0 < G_1 < \dots < G_n = G$ of normal subgroups of G such that for each $i = 1, 2, \dots, n$, G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G. If G is supersolvable, then G has an ordered Sylow tower [6].

We shall need the following result in the next section.

2.1 (Baer [4]; see also [1, p. 720]). *Let G be a solvable group. Suppose that* $\mathrm{Fit}(G)$ *possesses a normal series:*

$$\Phi(G) = K_0 \leqq K_1 \leqq \dots \leqq K_n = \mathrm{Fit}(G)$$

such that K_i are normal subgroups of G and $|K_j/K_{j-1}| =$ prime, $1 \leqq i \leqq n$. Then G is supersolvable.

3. Main results

In this section, we prove the following theorems.

THEOREM 3.1. *Assume that G is solvable and every maximal subgroup of the Sylow subgroups of* $\mathrm{Fit}(G)$ *is normal in G. Then G is supersolvable.*

PROOF. We prove the theorem by induction on $|G|$. Suppose that $\Phi(G) \neq 1$. Then there exists a prime p such that $p \,\Big|\, |\Phi(G)|$. Since $\Phi(G) \leqq \leqq \mathrm{Fit}(G)$, it follows that $p \,\Big|\, |\mathrm{Fit}(G)|$. Let P_1 be a Sylow p-subgroup of $\Phi(G)$. Since $P_1 \mathrm{char} \Phi(G) \lhd G$, we have that $P_1 \lhd G$. Set $\mathrm{Fit}(G/P_1) = L/P_1$. Since L/P_1 is a normal nilpotent subgroup of G/P_1 and $P_1 \leqq \Phi(G)$, by [6, p. 168], it follows that L is a normal nilpotent subgroup of G, and so $L \leqq \mathrm{Fit}(G)$. Hence $\mathrm{Fit}(G/P_1) = \mathrm{Fit}(G)/P_1 = L/P_1$. Let P_2/P_1 be a maximal subgroup of the Sylow p-subgroup of $\mathrm{Fit}(G)/P_1$. Then $P_2/P_1 \lhd G/P_1$ as $P_2 \lhd G$ by hypothesis. Also, if Q_1P_1/P_1 is a maximal subgroup of the Sylow q-subgroup of $\mathrm{Fit}(G/P_1) = \mathrm{Fit}(G)/P_1$, then by hypothesis, $Q_1 \lhd G$ and so $Q_1P_1/P_1 \lhd G/P_1$. Hence G/P_1 is supersolvable by induction on the order of G.

Since $G/P_1 \Big/ \Phi(G)/P_1 \cong G/\Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable. By [1, p. 713], we have that G is supersolvable. Thus $\Phi(G) = 1$. Let P be a Sylow subgroup of $\mathrm{Fit}(G)$. Since $P \mathrm{char} \mathrm{Fit}(G) \lhd G$, we have that $P \lhd G$. By [6, p. 162], $\Phi(P) \leqq \Phi(G) = 1$. Hence $\Phi(P) = 1$, for every Sylow subgroup P of $\mathrm{Fit}(G)$.

Since G is solvable and $\Phi(G) = 1$, by [1, p. 279], we have that $\mathrm{Fit}(G) = = R_1 \times R_2 \times \ldots R_m$, where R_i are (elementary abelian) minimal normal subgroups of G. Clearly, we may assume that $R_1 \leqq P$, where P is a Sylow p-subgroup of $\mathrm{Fit}(G)$. If $R_1 = P$, then there exists a maximal subgroup P_1 of $P = R_1$. By hypothesis, $P_1 \lhd G$. If $P_1 \neq 1$, then we have a contradiction as R_1 is a minimal normal subgroup of G. Thus $P_1 = 1$, and so $|R_1| = = \text{prime}$. Now, we may assume that $R_1 < P$. Since $\Phi(P) = 1$, there exists a maximal subgroup P_2 of P such that $R_1 \not\leqq P_2$. By hypothesis, $P_2 \lhd G$, and so $P_2 \cap R_1 \lhd G$. Since R_1 is a minimal normal subgroup of G, we have that $P_2 \cap R_1 = 1$. Since $P = P_2R_1$, it follows that $p = |P : P_2| = |R_1 : P_2 \cap \cap R_1| = |R_1|$. Set $K_i = R_1 \times R_2 \times \cdots \times R_i$, where $i = 1, 2, \ldots, m$. Consider the chain $1 = \Phi(G) \leqq K_1 \leqq \ldots \leqq K_m = \mathrm{Fit}(G)$. Clearly $K_i \lhd G$, for each i, and $|K_i/K_{i-1}| = \text{prime}$. Applying Baer's theorem (2.1), it follows that G is supersolvable.

The following example shows that the solvability of G in Theorem 3.1 can not be omitted.

EXAMPLE 3.2. Set $G = H \times K$, where H is nilpotent and K is a non--abelian simple group. Clearly, G is not solvable. We notice that $\mathrm{Fit}(G) = = \mathrm{Fit}(H) = H$ and every maximal subgroup of the Sylow subgroups of $\mathrm{Fit}(G)$ is normal in G.

Now, we give an example showing that the converse of our Theorem 3.1 is not true:

EXAMPLE 3.3. Set $G = Z/3Z \times S_3$; G is supersolvable, but there exists a maximal subgroup of a Sylow subgroup of $\mathrm{Fit}(G)$ which is not normal in G.

As an immediate consequence of theorem 3.1 we have:

COROLLARY 3.4 (Srinivasan [5]). *Let G be a finite group. If every maximal subgroup of every Sylow subgroup of G is normal, then G is supersolvable.*

PROOF. Every Sylow subgroup of G is either normal or cyclic. Clearly, G is solvable. Now, it follows easily that the maximal subgroups of the Sylow subgroups of $\mathrm{Fit}(G)$ are normal in G. Applying Theorem 3.1, G is supersolvable.

THEOREM 3.5. *Assume that G/H is supersolvable and all maximal subgroups of the Sylow subgroups of H are normal in G. Then G is supersolvable.*

PROOF. We consider two cases.

Case 1: $P = H$, where P is a p-subgroup of G. Let $\mathbf{m}(P) = \{P_1, P_2, \ldots, P_n\}$, where $n \geqq 1$, be the set of all maximal subgroups of P. By hypothesis, $P_i \lhd G$ for all $i = 1, 2, \ldots, n$. Since $G/P_i \big/ P/P_i \cong G/P$, we have that $G/P_i \big/ P/P_i$ is supersolvable. But $|P/P_i| = p$, and so by [6, p. 158], G/P_i is supersolvable, $1 \leqq i \leqq n$. Hence by [6, p. 159], $G/P_1 \times G/P_2 \times \cdots \times G/P_n$ is supersolvable. Since $G/(P_1 \cap P_2 \cap \ldots \cap P_n) \mathrel{\tilde{\subset}} G/P_1 \times G/P_2 \times \cdots \times G/P_n$, then $G \big/ \bigcap_{i=1}^{n} P_i$ is supersolvable. By [1, p. 713] G is supersolvable.

Case 2: $P < H$. By Corollary 3.4, it follows that H is supersolvable. Hence by [6, p. 158], H possesses an ordered Sylow tower. So, P_1 is normal in H, where P_1 is a Sylow p_1-subgroup of H and p_1 is the largest prime dividing the order of H. By induction G/P_1 is supersolvable. By Case 1, it follows that G is supersolvable.

We are now in the position to prove the following result:

THEOREM 3.6. *Assume that p is the largest prime dividing the order of G and that every maximal subgroup of the Sylow q-subgroups of G is normal for all $q \in \Pi(G) - \{p\}$. Then G possesses an ordered Sylow tower and $G/O_p(G)$ is supersolvable. In particular, G is solvable.*

PROOF. Let L be the product of the maximal subgroups of the Sylow p_i-subgroups of G for all $p_i \in \Pi(G) - \{p\}$. Then G/L is a group of order dividing $p_1 p_2 \ldots p_r \cdot |P|$, where $p_1 < p_2 < \ldots < p_r < p$. By [2, p. 257], it follows that G/L possesses an ordered Sylow tower and so G/L is solvable. Clearly L is nilpotent. Hence G is solvable. By Hall's theorem (see [2, p. 232]), there exists a set $\{P_1, P_2, \ldots, P_n = P\}$ of Sylow subgroups of G, such that $P_i P$, $1 \leqq i \leqq n-1$, are subgroups of G. Set $G_i = P_i P$, $1 \leqq i \leqq n-1$. Let $\mathbf{m}(P_i) = \{P_{i1}, P_{i2}, \ldots, P_{is}\}$ be the set of all maximal subgroups of P_i. By hypothesis $P_{ij} \lhd G$, $1 \leqq j \leqq s$, and so $|G_i/P_{ij}| = p_i |P|$, $p_i < p$. By [2, p. 257], G_i/P_{ij} has a normal p_i-complement and so $G_i \big/ \bigcap_{j=1}^{s} P_{ij}$ has a normal p_i-complement, that is $G_i/\Phi(P_i)$ also has one. If P_i is not cyclic, then there exist two distinct maximal subgroups P_{i1}, P_{i2} of P_i. Since $P_{i1} \lhd G$; $P_{i2} \lhd G$ by hypothesis, then $P_i \lhd G$, so $P_i \lhd G_i$. Hence $\Phi(P_i) \leqq \Phi(G_i)$ by [6, p. 162]. So $G_i/\Phi(G_i)$ has a normal p_i-complement. Hence by [1, p. 689] G_i has one as well. Now suppose that P_i is cyclic, by [2, p. 257], G_i has a normal p_i-complement. Since $P \lhd P_iP$, we have that $P_iP \leqq N_G(P)$, $1 \leqq i \leqq n-1$, and so $P \lhd G$, $P = O_p(G)$. Now $G/O_p(G) \cong K$, where K is a p'-Hall subgroup of G. Since every maximal subgroup of the Sylow subgroups of K is normal in K, by Corollary 3.4, K is supersolvable. Hence G possesses an ordered Sylow tower and $G/O_p(G)$ is supersolvable.

As an immediate consequence of Theorem 3.6 we have:

COROLLARY 3.7. *Assume that $p \geqq q$ for every prime q in $\Pi(G)$, $O_p(G) = 1$ and every maximal subgroup of the Sylow q-subgroups of G, for all $q \neq p$ is normal in G. Then G is a supersolvable p'-group.*

PROOF. By Theorem 3.6, G possesses an ordered Sylow tower, and $G/O_r(G)$ is supersolvable, where r is the largest prime in $\Pi(G)$.

Since $O_p(G) = 1$, it follows easily that $p \nmid |G|$ and so G is a p'-group. Applying Theorem 3.6, it follows that G is supersolvable.

ACKNOWLEDGEMENT. The author wishes to thank Professor Dr. M. Asaad, Math. Dept., Faculty of Science, Cairo University, for his work in reviewing the paper and for a number of valuable suggestions and comments.

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(Received June 14, 1988, revised September 30, 1988)

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CAIRO UNIVERSITY
GIZA, EGYPT

Acta Math. Hung.
59(1-2) (1992), 111–114.

CERTAIN UNBOUNDED HERMITE–FEJÉR INTERPOLATORY POLYNOMIAL OPERATORS

R. SAKAI (Aichi Nishikamo)

1. Introduction

Let $S_n = \{x_k = x_{kn};\ k = 1, 2, \ldots, n\}$ denote the zeros of the Chebyshev polynomial $T_n(x) = \cos(nt)$, $x = \cos t$, $n = 1, 2, 3, \ldots$ in decreasing order; $-1 < x_n < x_{n-1} < \ldots < x_1 < 1$. We consider the generalized Hermite–Fejér interpolatory polynomial $L_{m,n}[f, x]$ corresponding to the abscissas S_n as follows:

$$L_{m,n}[f, x_k] = f(x_k), \quad L^{(j)}_{m,n}[f, x_k] = 0, \quad k = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m-1,$$

for $f \in C[-1, 1]$.

In [1] we showed that for every even m

$$\|L_{m,n}[f] - f\|_{C[-1,1]} = O(1) w(f, n^{-1} \log n),$$

where $w(f, h)$ is the modulus of continuity of f, and for odd $m \leqq 17$ the operator $L_{m,n}[f]$ is unbounded. We conjectured that for all odd numbers m the operators $L_{m,n}[f]$ are unbounded. In this paper we prove that this is correct.

2. Preliminaries and theorems

The polynomial

$$(1) \qquad \ell_k(x) = \ell_{k,n}(x) = \omega_n(x) / \{(x - x_k)\omega'_n(x_k)\}, \qquad k = 1, 2, \ldots, n$$

where $\omega_n(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$ are the fundamental polynomials of Lagrange interpolation corresponding to the set S_n. We define the fundamental polynomials of the generalized Hermite–Fejér interpolation by

$$(2) \qquad H_k(x) = {}_mH_{k,n}(x) = \ell_{k,n}(x)^m \sum_{s=0}^{n-1} A_k(n, s)(x - x_k)^s$$

such that

$$(3) \quad H_k(x_s) = \delta_{s,k}, \quad H_k^{(j)}(x_s) = 0, \quad s, k = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m-1.$$

Our main purpose is tp show the following theorem.

THEOREM 1. *For each odd number m we have*

$$\sum_{k=1}^{n} |{}_m H_{k,n}(x_0)| \sim \log n, \tag{4}$$

where $-1 < x_0 < 1$. *Consequently, the operator* $L_{m,n}[f]$ *is unbounded.*

The polynomial $y = T_n(x)$ satisfies the following linear homogeneous differential equations:

$$(1-x^2)y^{(j+2)} - (2j+1)xy^{(j+1)} + (n^2 - j^2)y^{(j)} = 0, \qquad j = 0, 1, 2, \ldots .$$

Using this for $|x_k| \leqq 1 - \varepsilon$, $0 < \varepsilon < 1$, we have

$$\begin{cases} T_n^{(2j)}(x_k)/T_n'(x_k) = (-1)^{j+1} j(2j-1)x_k X_k^{-2j} n^{2j-2} + O(n^{2j-4}), \\ T_n^{(2j+1)}(x_k)/T_n'(x_k) = (-1)^j X_k^{-2j} n^{2j} + O(n^{2j-2}), \end{cases} \tag{5}$$

where $X_k = \sin(\cos^{-1} x_k)$. From (5) for $|x_k| \leqq 1 - \varepsilon$ we obtain

$$\begin{aligned} \ell_k^{(t)}(x_k) &= T_n^{(t+1)}(x_k)/\{(t+1)T_n'(x_k)\} = \\ &= \begin{cases} (-1)^j (2j+1)^{-1} X_k^{-2j} n^{2j} + O(n^{2j-2}) & (t = 2j), \\ (-1)^j \{(2j+1)/2\} x_k X_k^{-2j-2} n^{2j} + O(n^{2j-2}) & (t = 2j+1), \end{cases} \end{aligned} \tag{6}$$

In what follows, ε denotes a fixed number with $0 < \varepsilon < 1/2$, and we consider k's such that $-1 + \varepsilon \leqq x_k \leqq 1 - \varepsilon$. Then we write the above estimation (6) as follows:

$$\ell^{(t)}(x_k) \approx \begin{cases} (-1)^j (2j+1)^{-1} X_k^{-2j} n^{2j} & (t = 2j), \\ (-1)^j \{(2j+1)/2\} x_k X_k^{-2j-2} n^{2j} & (t = 2j+1), \end{cases} \tag{7}$$

If we set

$$K_{m,j}(x) = [\{\ell_k(x)\}^m]^{(j)}, \tag{8}$$

from (7) we have

$$K_{m,2j}(x_k) \approx (-1)^j b(2j, m)(n/X_k)^{2j}, \tag{9}$$

where $b(2j, m)$ is a polynomial in m of degree j. In [1,§4] we showed that if $b(2j, m)$ has the form

$$b(2j,m)=\sum_{s=1}^{j}(-1)^{j-s}b_{s,j}m^{s}, \tag{10}$$

where $b_{sj}>0$, the estimation (4) is correct, so we have to prove

THEOREM 2. *In formula* (10),

$$b_{sj}>0, \quad s=1,2,\dots,j,\ j=1,2,\dots,(m-1)/2.$$

Let us remark that (10) is true in more general cases, too.

PROOF. First we investigate $\{(d/dm)K_{m,2j}(x_k)\}_{m=0}$. Since we consider a neighbourhood of $x=x_k$ we may assume $\ell_k(x)>0$. It can be proved that

$$\{(d/dm)K_{m,2j}(x_k)\}_{m=0}=[(d^{2j}/dx^{2j})\{(\ell_k(x))^m\log(\ell_k(x))_{m=0}\}_{x=x_k}=$$
$$=[(d^{2j}/dx^{2j})\log(\ell_k(x))]_{x=x_k}=-\sum_{s\neq k}(2j-1)!(x_s-x_k)^{-2j}<0,$$

thus, by (9) and (10), $b_{1j}>0$.

Now we consider the other coefficients. We see that

$$K_{2m,2(j+1)}(x_k)=\sum_{s=0}^{2j+2}\binom{2j+2}{s}K_{m,s}(x_k)K_{m,2j-s+2}(x_k),$$

thus

$$K_{2m,2(j+1)}(x_k)-2K_{m,2(j+1)}(x_k)K_{m,0}(x_k)=$$
$$=\sum_{s=1}^{2j+1}\binom{2j+2}{s}K_{m,s}(x_k)K_{m,2j-s+2}(x_k).$$

Since $K_{m,2t+1}(x_k)=o(n^{2t+1})$ and by [1, (4.10)] $b(2j,m)>0$,

$$K_{2m,2(j+1)}(x_k)-2K_{m,2(j+1)}(x_k)\approx\sum_{s=1}^{j}\binom{2j+2}{2s}K_{m,2s}(x_k)K_{m,2j-2s+2}(x_k),$$

that is

$$b(2(j+1),2m)-2b(2(j+1),m)=\sum_{s=1}^{j}\binom{2j+2}{2s}b(2s,m)b(2j-2s+2,m),$$

whence comparing the coefficients and by induction we get Theorem 2.

ACKNOWLEDGEMENT. The author would like to thank the referee and Professor P. Vértesi for their helpful suggestions.

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(Received June 17, 1988; revised April 12, 1989)

DEPT. OF MATH.
AICHI PREFECTURAL
KAMOGAOKA SENIOR HIGH SCHOOL
OTASHIRO 1137, FUJIOKA-CHO INO
AICHI NISHIKAMO
JAPAN

Acta Math. Hung.
59(1-2) (1992), 115–120.

THE MODULUS OF CONTINUITY AND THE BEST APPROXIMATION OVER THE DYADIC GROUP

J. TATEOKA (Akita)

1. Introduction

The connection between the modulus of continuity and the best approximation of functions by Walsh polynomials was studied by Watari [5] for L^p space, $1 \leqq p < \infty$. A similar result for $0 < p < 1$ was obtained by Stroženko, Krotov and Oswal'd [3]. On the other hand, direct and converse theorems for the Hardy space H^p, $0 < p < \infty$, over the n-dimensional torus were proved by Colzani [1]. In this paper we shall prove these results for H^p space, $0 < p \leqq 1$, and VMO space over the dyadic group.

The dyadic group G is the set of the sequences $x = (x_n, n = 1, 2, \dots)$, consisting of 0's and 1's, with termwise addition modulo 2, denoted by $+$. The topology of G is defined by the neighborhoods $V_n = \{x \in G : x_1 = \dots = x_n = 0\}$, $(n = 0, 1, \dots)$ (with a convenience $V_0 = G$), of the identity element. The Walsh functions are given by

$$w_0(x) \equiv 1, \; w_n(x) = r_{n_1}(x) \dots r_{n_k}(x) \quad \text{for} \quad n = 2^{n_1} + \dots + 2^{n_k} (n_1 > \dots n_k \geqq 0),$$

where $r_n(x) = (-1)^{x_{n+1}} (n = 0, 1, \dots)$. A Walsh polynomial of degree n is a linear combination $\sum_{k=0}^{n-1} c_k w_k$ with $c_{n-1} \neq 0$; the totality of polynomials of degree not exceeding n is denoted by $\mathfrak{S}(n)$ and the union of the $\mathfrak{S}(n)$'s by $\mathfrak{S}$. We denote by $\mathfrak{S}'$ the set of all formal Walsh series with complex coefficients.

For $\hat{f} = \sum_{k=0}^{\infty} \hat{f}(k) w_k \in \mathfrak{S}'$,

$$S_n \hat{f}(x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x), \quad f^*(x) = \sup\{|S_{2^n} \hat{f}(x)| : n \geqq 0\}$$

will denote the n-th partial sum and the maximal function, respectively. The spaces considered in this paper are the following ones.

$H^p(G)$ $(0 < p \leqq 1)$ is the space of all $\hat{f} \in \mathfrak{S}'$ such that $f^* \in L^p(G)$; the quasi-norm in H^p is defined by $\|\hat{f}\|_{H^p} = \|f^*\|_{L^p}$. BMO$(G)$ is the space of integrable functions f on G such that $\|f\|_{\mathrm{BMO}} = \sup\{|f - f_I|_I\} < \infty$, where $f_I = |I|^{-1} \int_I f(x)dx$ and the sup is taken over all dyadic intervals I in G. VMO(G) is the space of all BMO functions f on G such that $\lim\limits_{|I|\to 0} |f - f_I|_I = 0$.

It is known that for all $\hat{f} \in H^p(G)$ $(0 < p \leqq 1)$ the limit $f(x) = \lim\limits_{n\to\infty} S_{2^n}\hat{f}(x)$ exists a.e. and $f \in L^p(G)$. Moreover, if $p \geqq 1$ then $\hat{f}(n)$, $n \geqq 0$, are the Walsh–-Fourier coefficients of f.

We shall use the common notation X for the spaces $H^p(G)$ and BMO(G).

Finally let us write, for the modulus of continuity and the best approximation

$$\omega^X(\delta; \hat{f}) = \sup\{\|\hat{f}(\cdot + h) - \hat{f}(\cdot)\|_X : |h| \leqq \delta\},$$

and

$$E_n^X(\hat{f}) = \inf\{\|\hat{f} - P\|_X : P \in \mathfrak{S}(n)\},$$

respectively, where $\hat{f}(\cdot + h)$ has the coefficients $\hat{f}(n)w_n(h)$, $n \geqq 0$.

The letter C denotes positive constant which may have different values in each occasion.

2. We begin with some lemmas:

LEMMA 1. *For any $f \in X$, there is a polynomial of best approximation $P \in \mathfrak{S}(n)$; that is,*

$$E_n^X(f) = \|f - P\|_X.$$

The assertion follows from standard compactness argument. We omit the proof.

LEMMA 2. $E_{2^n}^X(\hat{f}) \leqq \|\hat{f} - S_{2^n} f\|_X \leqq C\ E_{2^n}^X(\hat{f})$.

PROOF. The first inequality is trivial. By Lemma 1 and $\|S_{2^n}\hat{f}\|_{H^p} \leqq$ $\leqq \|\hat{f}\|_{H^p}$, we have the second inequality for $X = H^p$. The BMO case is deduced from the $H^1 -$BMO duality.

LEMMA 3. *If $\lim\limits_{k\to\infty} \|\hat{f}_k - \hat{f}\|_X = 0$, then $\lim\limits_{k\to\infty} E_n^X(\hat{f}_k) = E_n^X(\hat{f})$ and $\lim\limits_{k\to\infty} \omega^X(\delta; \hat{f}_k) = \omega^X(\delta; \hat{f})$ uniformly with respect to δ.*

PROOF. The first statement of the Lemma follows from Lemma 2 and the two inequalities:

$$|E_n^X(\hat{f}_1 + \hat{f}_2)|^p \leqq |E_n^X(\hat{f}_1)|^p + |E_n^X(\hat{f}_2)|^p, \ \hat{f}_1, \hat{f}_2 \in H^p,$$
$$|E_n^X(f_1 + f_2)| \leqq |E_n^X(f_1)| + |E_n^X(f_2)|, \ f_1, f_2 \in \mathrm{BMO}.$$

The second is similar.

LEMMA 4. *If $H \in \mathfrak{S}(2^n)$, then $E^X_{2^k}(H) \leqq \omega^X(2^{-k}; H)$ for any integer $k \geqq 0$.*

PROOF. Our proof proceeds along the line of Strożenko, Krotov and Oswal'd [3]. Firstly we consider the H^p case.

Let $H(x) = \alpha_i$ for $x \in x_i + V_n$ $(i = 0, 1, \dots, 2^n - 1)$, where $G = \bigcup_{i=0}^{2^n-1} (x_i + V_n)$. Also let $Q(x) = \beta_j$ for $x \in y_j + V_k$ $(j = 0, 1, \dots, 2^k - 1)$. Then $Q \in \mathfrak{S}(2^k)$. We set $G(x) = H(x) - Q(x)$ and $K_h(x) = H(x+h) - H(x)$. The lemma easily follows from the inequality

$$\inf\{\|G\|^p_{H^p}; \{\beta_0, \dots, \beta_{2^k-1}\} \subset \mathbf{C}\} \leqq 2^k \int_{V_k} \|K_h\|^p_{H^p} dh.$$

This is obvious for $k \geqq n$. We may assume that $k < n$. Let $x \in x_i + V_n$ be fixed. Let us evaluate $G_I(x) = |I|^{-1} \int_{x \in I} G(t)dt$ and $(K_h)_I(x)$ for $I = z + V_m$. We have three cases:

(a) $x \in I \subset x_i + V_n \subset y_j + V_k$ $(n \leqq m)$,

(b) $x \in x_i + V_n \subset I \subset y_j + V_k$ $(k \leqq m \leqq n)$,

(c) $x \in x_i + V_n \subset y_j + V_k \subset I$ $(0 \leqq m \leqq k)$.

For the case (a), we have $G_I(x) = \alpha_i - \beta_j = G(x)$ and $(K_h)_I(x) = \alpha_{i(h)} - \alpha_i = K_h(x)$, where $x + h \in x_{i(h)} + V_n$ for some integer $i(h)$ depending on i and h. For the case (b), we can write $I = \bigcup_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (x_{i'} + V_n)$ for some integer $\varepsilon(i)$ depending on i. Then we have

$$G_I(x) = \sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (\alpha_{i'} - \beta_j) \quad \text{and} \quad (K_h)_I(x) = 2^{m-n} \sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (\alpha_{i'(h)} - \alpha_{i'}).$$

For the case (c), we can write $I = \bigcup_{j'=\varepsilon(j)}^{\varepsilon(j)+2^{k-m}-1} \bigcup_{i'=j'2^{n-k}}^{(j'+1)2^{n-k}-1} (x_{i'} + V_n)$ for some integer $\varepsilon(j)$ depending on j. Then we have $G_I(x) = 2^{m-n} \sum_{j'} \sum_{i'} (\alpha_{i'} - \beta_{j'})$ and $(K_h)_I(x)$ is the same as in case (b).

Since $G(x)$ is locally constant, there exist $I = z + V_m$ in some case such that $G^*(x) = |G_I(x)|$. (Note that $G^*(x) = \sup\{|G_I(x)| : x \in I\}$.) If $G^*(x) = |G_I(x)|$ as in case (b), then

$$G_{(\mathrm{b})} \equiv \inf\left\{\|G\|^p_{H^p} : (\beta_j) \subset \mathbf{C}\right\} = \inf\left\{\|G^*\|^p_{L^p} : (\beta_j) \subset \mathbf{C}\right\} =$$
$$= \inf\left\{ \sum_{j=0}^{2^k-1} \sum_{i=j2^{n-k}}^{(j+1)2^{n-k}-1} 2^{-n} \left| 2^{m-n} \sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (\alpha_{i'} - \beta_j) \right|^p : (\beta_j) \right\}.$$

Similarly if $K_h^*(x) = |(K_h)_I(x)|$ in case (b), then

$$K_{(\mathrm{b})} \equiv 2^k \int\limits_{V_k} \|K_h\|_{H^p}^p dh = 2^k \sum_{a=0}^{2^{n-k}-1} \int\limits_{h_a+V_n} \|K_h^*\|_{L^p}^p dh \geqq$$

$$\geqq 2^k \sum_{a=0}^{2^{n-k}-1} \sum_{j=0}^{2^k-1} \sum_{i=j2^{n-k}}^{(j+1)2^{n-k}-1} 2^{-2n} \left| 2^{m-n} \sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (\alpha_{i'(h_a)} - \alpha_{i'}) \right|^p .$$

Since

$$\inf \left\{ \sum_{i=j2^{n-k}}^{(j+1)2^{n-k}-1} 2^{-n} \left| 2^{m-n} \sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} (\alpha_{i'} - \beta_j) \right|^p : (\beta_j) \right\} \leqq$$

$$\leqq \sum_{i=j2^{n-k}}^{(j+1)2^{n-k}-1} 2^{-n} \left| 2^{m-n} \left(\sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} \alpha_{i'} - \sum_{k'=\varepsilon(s)}^{\varepsilon(s)+2^{n-m}-1} \alpha_{k'} \right) \right|^p$$

for any $j = 0, \ldots, 2^k - 1$ and $s = j2^{n-k}, \ldots, (j+1)2^{n-k} - 1$, where $\varepsilon(s)$ is an integer depending on s, we have

$$G_{(\mathrm{b})} \leqq$$

$$\leqq \sum_{j=0}^{2^k-1} \sum_{i=j2^{n-k}}^{(j+1)2^{n-k}-1} 2^{-2n+k} \sum_{s=j2^{n-k}}^{(j+1)2^{n-k}-1} \left| 2^{m-n} \left(\sum_{i'=\varepsilon(i)}^{\varepsilon(i)+2^{n-m}-1} \alpha_{i'} - \sum_{k'=\varepsilon(s)}^{\varepsilon(s)+2^{n-m}-1} \alpha_{k'} \right) \right|^p .$$

It is seen that the right is less than $K_{(\mathrm{b})}$.

If the maximal function is attained in other case, similar arguments show that $G_{(\mathrm{a})} \leqq K_{(\mathrm{a})}$ and $G_{(\mathrm{c})} \leqq K_{(\mathrm{b})}$. Thus we have the desired inequality.

By an argument similar to the H^p case, we can see that

$$\inf \{ \|G\|_{\mathrm{BMO}} : (\beta_j) \subset \mathbf{C} \} \leqq 2^m \int\limits_{V_m} \|K_h\|_{\mathrm{BMO}} dh.$$

Therefore the lemma is proved.

3. Now we are in position to state our theorems.

THEOREM 1 (direct theorem). *Let Y be $H^p(G)$ or* VMO(G). *Then we have*

$$E_{2^k}^Y(f) \leqq \omega^Y(2^{-k}; f) \text{ for any } f \in Y.$$

PROOF. The Theorem follows from Lemma 3, Lemma 4 and the following relations:

$$\lim_{n\to\infty} \|f - S_{2^n} f\|_{H^p} = 0 \text{ for } f \in H^p,$$

$$\lim_{n\to\infty} \|f - S_{2^n} f\|_{\mathrm{BMO}} = 0 \text{ for } f \in \mathrm{VMO}.$$

The first equality is easily deduced from the definition of H^p. In order to show the second equality, it is sufficient to note that

$$\|f - S_{2^n} f\|_{\mathrm{BMO}} \leqq \int_G D_{2^n}(t) \|f(\cdot + t) - f(\cdot)\|_{\mathrm{BMO}} dt,$$

where D_{2^n} is the Dirichlet kernel of order 2^n, since

$$\|f(\cdot + t) - f(\cdot)\|_{\mathrm{BMO}} \to 0 \quad \text{as} \quad |t| \to 0 \quad \text{for} \quad f \in \mathrm{VMO}.$$

THEOREM 2 (converse theorem). *If $f \in X$, then*

$$\omega^X(2^{-k}; f) \leqq C E_{2^k}^X(f).$$

PROOF. Let P be the polynomial of order 2^k of best approximation to f. Since $P(x+h) = P(x)$ for $h \in V_k$, we obtain

$$\|f(\cdot + h) - f(\cdot)\|_{H^p}^p \leqq 2\|f - P\|_{H^p}^p, \quad f \in H^p,$$
$$\|f(\cdot + h) - f(\cdot)\|_{\mathrm{BMO}} \leqq 2\|f - P\|_{\mathrm{BMO}}, \quad f \in \mathrm{BMO},$$

which prove the Theorem.

THEOREM 3. *For $f \in Y$ ($Y = H^p$ or VMO), and a positive number α, the following four statements are equivalent:*

(a) $\omega^Y(2^{-k}; f) = O(2^{-k\alpha})$,

(b) $E_{2^k}^Y(f) = O(2^{-k\alpha})$,

(c) $\|f - S_{2^k} f\|_Y = O(2^{-k\alpha})$,

(d) $\|S_{2^k} f - S_{2^{k-1}} f\|_Y = O(2^{-k\alpha})$.

PROOF. Theorem 2, Lemma 2 and Theorem 1 give the equivalence of (a), (b) and (c). Or it is clear that (d) implies (c), the only one to be verified is the implication (c)→(d). This is easily deduced once we use the inequality

$$\sup_{k \geqq 0} 2^{k\alpha} \sum_{n=k}^{\infty} a_n \leqq C \sup_{k \geqq 0} 2^{k\alpha} a_k$$

for any $\alpha > 0$ and any $a_k \geqq 0$. (See Taibleson [4], p. 179.)

REMARK. We can find results analogous to those presented here in the context of H^p space and VMO space associated with local fields. Furthermore, in the local fields, we have that for any function from X, the second modulus of continuity is estimated from below and above by the first modulus of continuity, that is, $\sup\{\|f(\cdot + h) - 2f(\cdot) + f(\cdot - h)\|_X : |h| \leqq q^{-k}\}$ is equivalent to $\sup\{\|f(\cdot + h) - f(\cdot)\|_X : |h| \leqq q^{-k}\}$. In fact, by the argument of Lemma 4, we obtain

$$\sup\{\|f(\cdot + h) - 2f(\cdot) + f(\cdot - h)\|_X : |h| \leqq q^{-k}\} \geqq C E_{q^k}^X(f).$$

Combining this inequality with Theorem 2, we obtain the equivalence. In the case of functions in $L^2(G)$ or $C(G)$ defined on a zero-dimensional compact abelian group G, this fact was proved by Rubinshtein [2].

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(Received June 28, 1988; revised March 1, 1989)

DEPARTMENT OF MATHEMATICS
AKITA UNIVERSITY
TEGATA, AKITA, 010
JAPAN

Acta Math. Hung.
59(1-2) (1992), 121–146.

COMPLETE QUASI-PSEUDO-METRIC SPACES

H. P. KÜNZI (Berne)

1. Introduction

We will use the terminology of [8]. The letter **N** will denote the set of positive integers. First let us recall some definitions.

A *quasi-pseudo-metric* space is a pair (X,p) where X is a set and p is a mapping from $X\times X$ into the set of real numbers satisfying for all x, y, $z\in X$:

(i) $p(x,y)\geqq 0$; (ii) $p(x,x)=0$; (iii) $p(x,y)\leqq p(x,z)+p(z,y)$.

If p satisfies the additional condition (iv) that $p(x,y)=0$ implies that $x=y$, then p is called a *quasi-metric* on X. Note that in this case p^{-1} defined by $p^{-1}(x,y)=p(y,x)$ for all x, $y\in X$ is also a quasi-metric on X and p^* defined by $p^*(x,y)=\max\{p(x,y),p(y,x)\}$ for all x, $y\in X$ is a metric on X.

For a quasi-pseudo-metric space (X,p) the set $B^p_n(x)=\{y\in X: p(x,y)<$ $<2^{-n}\}$ is said to be the *p-ball with center* $x\in X$ *and radius* 2^{-n} (where $n\in\mathbf{N}$). The *topology* $\mathcal{T}(p)$ *induced by* p *on* X is the topology generated by the base $\{B^p_n(x): x\in X, n\in\mathbf{N}\}$ on X. A quasi-pseudo-metric space (X,p) is called *precompact* if for every $n\in\mathbf{N}$ there is a finite subset F of X such that $\bigcup\{B^p_n(f): f\in F\}=X$. A sequence $(x_n)_{n\in\mathbf{N}}$ in a quasi-pseudo-metric space (X,p) is said to be a *left p-Cauchy sequence* [16, p. 128] if for each real number $\delta>0$ there are a point $x\in X$ and a positive integer k such that $p(x,x_n)<\delta$ whenever $n\in\mathbf{N}$ and $n\geqq k$. A quasi-pseudo-metric space (X,p) is called *left p-sequentially complete* [16, p. 132] if every left p-Cauchy sequence in X converges to some point in X (with respect to the topology induced by p on X). In this paper it seems helpful to change this well-established terminology slightly. Since in the following left p-Cauchy sequences are the only Cauchy sequences in quasi-pseudo-metric spaces that we consider, it will suffice to call them *Cauchy sequences*. Furthermore, in order to obtain a clearer terminology we will use the expression *"sequentially convergence complete"* instead of "left p-sequentially complete". Accordingly, we will call a quasi-pseudo-metric space X *sequentially complete* if every Cauchy sequence in X has a cluster point in X. Of course, each sequentially convergence complete quasi-pseudo-metric space is sequentially complete. It is well known that the converse is wrong (see e.g. Example 1 below).

By definition, the *quasi-uniformity* $\mathcal{U}(p)$ *induced by a quasi-pseudo-metric* p on a set X is the quasi-uniformity on X that is generated by the base $\{U_n : n \in \mathbf{N}\}$ where $U_n = \{(x,y) \in X \times X : p(x,y) < 2^{-n}\}$ whenever $n \in \mathbf{N}$. We recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a *$\mathcal{U}$-Cauchy filter* if for each $V \in \mathcal{U}$ there is an $x \in X$ such that $V(x) = \{y \in X : (x,y) \in \in V\} \in \mathcal{F}$ [8, p. 47]. A quasi-uniform space $(X,\mathcal{U})$ is called *convergence complete* if each $\mathcal{U}$-Cauchy filter on X converges in X and it is said to be *complete* if each $\mathcal{U}$-Cauchy filter on X has a cluster point in X [8, p. 50]. Clearly each convergence complete quasi-uniform space is complete. It is well known that the converse does not hold (see e.g. Example 1 below). In this paper we will say that a quasi-pseudo-metric space (X,p) is *(convergence) complete* if the quasi-uniformity $\mathcal{U}(p)$ is (convergence) complete.

It is easy to see that each (convergence) complete quasi-pseudo-metric space is sequentially (convergence) complete. Correcting a statement in the literature, we show in this paper that a sequentially (convergence) complete quasi-metric space need not be complete. We use our example to construct a complete and sequentially convergence complete quasi-metric space that is not convergence complete. We also observe that while a Tychonoff sequentially complete quasi-metric space need not be complete, a Hausdorff sequentially convergence complete quasi-metric space is convergence complete and a normal sequentially complete quasi-metric space is complete. Furthermore we give an example of a Tychonoff complete quasi-metric space that does not have a G_δ-diagonal. This example contradicts another statement in the literature. It follows from our results that an orthocompact Tychonoff space admits a convergence complete quasi-metric if and only if it is a Čech complete space with a G_δ-diagonal.

In the second part of this note we obtain several results related to Problem U of [8, p. 179]. This problem asks for a characterization of the class of quasi-metric spaces having a quasi-metric completion. In this paper we present a solution to this problem. Furthermore we show that a quasi-metric space has a quasi-metric sequential convergence completion if and only if it has a quasi-metric convergence completion. Similarly we prove that a quasi-metric space of non-measurable cardinality has a quasi-metric sequential completion if and only if it has a quasi-metric completion. Finally we present an example showing that the condition "of non-measurable cardinality" cannot be omitted in this result.

We would like to thank the referee for his valuable observations. In particular by suggesting an improved version of Lemma 7 he helped to strengthen several results in the section on quasi-metric completions.

In this paper we will use the following well-known method to construct quasi-metrics.

LEMMA 1. *Let* X *be a set and for each* $n \in \mathbf{N}$ *and each* $x \in X$ *let* $T_n(x)$ *be* *a subset of X. For each $n \in \mathbf{N}$ set $T_n = \bigcup\{\{x\} \times T_n(x) : x \in X\}$. If $(T_n)_{n\in\mathbf{N}}$ is a decreasing sequence of transitive relations on X such that $\bigcap\{T_n : n \in \in \mathbf{N}\} = \{(x,x) : x \in X\}$, then there exists a quasi-metric p on X such that $T_n(x) = B_n^p(x)$ for each $n \in \mathbf{N}$ and each $x \in X$.*

PROOF. We sketch the proof. For each $x \in X$ set $T_0(x) = X$. For each $x, y \in X$ define $p(x,y)$ as follows: $p(x,y) = 0$ if $x = y$, and $p(x,y) = 2^{-n}$ if $y \in T_{n-1}(x) \setminus T_n(x)$ and $n \in \mathbf{N}$. We verify that p is a quasi-metric on X. Clearly conditions (i), (ii) and (iv) are satisfied. It remains to check that condition (iii) is fulfilled. Obviously it suffices to consider the case that $p(x,z) = 2^{-k}$ and $p(z,y) = 2^{-s}$ where k and s are positive integers. Then $z \in T_{k-1}(x)$ and $y \in T_{s-1}(z)$. Thus $y \in T_{f-1}(x)$ where $f = \min\{k,s\}$. Hence $p(x,y) \leqq \max\{p(x,z), p(z,y)\}$, and condition (iii) is satisfied. Let $x \in X$ and $n \in \mathbf{N}$. Clearly the conditions $p(x,y) < 2^{-n}$ and $y \in T_n(x)$ are equivalent by the construction of the quasi-metric p on X. Thus $B_n^p(x) = T_n(x)$ for each $n \in \mathbf{N}$ and each $x \in X$.

We conclude this introduction with an easy example of a complete quasi-metric space that is not sequentially convergence complete.

EXAMPLE 1. A complete quasi-metric space that is not sequentially convergence complete (see also Example 5 and Example 7).

Let $X = (\mathbf{N} \times \{1,2\}) \cup \{-\infty, \infty\}$. For each $n \in \mathbf{N}$ and each $x \in X$ define a subset $T_n(x)$ of X as follows:

$$T_n(x) = \begin{cases} \{x\} & \text{if } x \in \mathbf{N} \times \{1\}; \\ \{\infty\} \cup \{(k,1) : k \in \mathbf{N},\ k \text{ even},\ k \geqq n\} & \text{if } x = \infty; \\ \{-\infty\} \cup \{(k,1) : k \in \mathbf{N},\ k \text{ odd},\ k \geqq n\} & \text{if } x = -\infty; \\ \{(a,2)\} \cup \{(k,1) : k \in \mathbf{N},\ k \geqq n\} & \text{if } x = (a,2),\ a \in \mathbf{N},\ a \geqq n; \\ \{(a,2)\} & \text{if } x = (a,2),\ a \in \mathbf{N},\ a < n. \end{cases}$$

It is easy to check that the sets $T_n(x)$ satisfy the conditions of Lemma 1. Hence there exists a quasi-metric p on X such that $T_n(x) = B_n^p(x)$ for each $n \in \mathbf{N}$ and each $x \in X$. Since $\{(k,1) : k \in \mathbf{N},\ k \geqq n\} \subseteq B_n^p((n,2))$ for each $n \in \mathbf{N}$, the sequence $(x_n)_{n\in\mathbf{N}}$ in X defined by $x_n = (n,1)$ for each $n \in \mathbf{N}$ is a Cauchy sequence in (X,p). Clearly $(x_n)_{n\in\mathbf{N}}$ does not converge in X. Hence (X,p) is not sequentially convergence complete.

We want to show that (X,p) is complete. Let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy ultrafilter on X. Assume that $\mathcal{F}$ does not have a cluster point in X. Since $-\infty$ and ∞ are not cluster points of $\mathcal{F}$ and $\mathcal{F}$ is a free ultrafilter on X, we conclude that $\mathbf{N} \times \{2\} \in \mathcal{F}$. Furthermore since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, for each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B_n^p(x_n) \in \mathcal{F}$. Since $\mathbf{N} \times \{2\} \in \in \mathcal{F}$, this means that there is an $x \in \mathbf{N} \times \{2\}$ such that $\{x\} \in \mathcal{F}$. Hence $\mathcal{F}$ converges — a contradiction. We conclude that each $\mathcal{U}(p)$-Cauchy filter on X has a cluster point in X. Hence (X,p) is complete.

Let us note that the topology induced by the quasi-metric p on X is metrizable.

2. Completeness in quasi-pseudo-metric spaces

Using the terminology of section 1 we can speak of sequentially complete, sequentially convergence complete, complete and convergence complete quasi-pseudo-metric spaces. In this section we study some relations among these four classes of quasi-pseudo-metric spaces. Our investigations are mainly motivated by an error contained in [8] (see Example 2 below). At the end of this section we will use our methods to answer a question of S. Romaguera and A. Gutiérrez about countably compact quasi-pseudo-metric spaces.

The following obvious result is mentioned in [8, § 7.33]. For the sake of completeness we give its elementary proof.

LEMMA 2. *Each (convergence) complete quasi-pseudo-metric space is sequentially (convergence) complete.*

PROOF. Let (X,p) be a complete quasi-pseudo-metric space and let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in X. Then the filter generated by $\{\{x_n : n \in \in \mathbf{N},\ n \geqq k\} | k \in \mathbf{N}\}$ on X is a $\mathcal{U}(p)$-Cauchy filter on X. Thus it has a cluster point in X. Clearly this cluster point is a cluster point of the sequence $(x_n)_{n\in\mathbf{N}}$. Hence (X,p) is sequentially complete.

It remains to show that a convergence complete quasi-pseudo-metric space is sequentially convergence complete. Let (X,p) be a convergence complete quasi-pseudo-metric space and let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in X. For each $n \in \mathbf{N}$ there are $y_n \in X$ and $k_n \in \mathbf{N}$ such that $k \in \mathbf{N}$ and $k \geqq k_n$ imply that $x_k \in B^p_n(y_n)$. Since X is convergence complete, the filter generated by $\{B^p_n(y_n) : n \in \mathbf{N}\}$ on X converges to some point x in X. Obviously the sequence $(x_n)_{n\in\mathbf{N}}$ converges to x, too. Hence (X,p) is sequentially convergence complete.

In [8, Theorem 7.33] it is stated that each sequentially complete quasi-metric space is complete. Unfortunately the proof of this statement is not correct, as our next example shows. (The error seems to occur in line 2 on page 177.)

EXAMPLE 2. We construct a sequentially convergence complete quasi-metric space that is not complete.

Let $Z = Y \cup \mathcal{A} \cup \mathbf{N}$ where the sets Y, $\mathcal{A}$ and $\mathbf{N}$ are supposed to be pairwise disjoint. Here, as usual, $\mathbf{N}$ denotes the set of positive integers. The set Y is the union of a countable collection $\{X_n : n \in \mathbf{N}\}$ of pairwise disjoint uncountable sets and $\mathcal{A} = \{x : x$ is a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of Y so that there is a strictly increasing sequence $(k_n(x))_{n\in\mathbf{N}}$ of positive integers such that for all $n \in \mathbf{N}$ and all $k \in \mathbf{N}$ with $k \geqq k_n(x)$ we have that $x_k \in \bigcup\{X_s : s \in \mathbf{N},\ s \geqq n\}\}$. For each $n \in \mathbf{N}$ and each $x \in Z$ we define a subset $T_n(x)$ of Z

as follows:

$$T_n(x)=\begin{cases} \{x\} & \text{if } x\in Y;\\ \{x\}\cup\{x_i : i\in \mathbf{N},\ i\geqq k_n(x)\} & \text{if } x=(x_i)_{i\in\mathbf{N}}\in\mathcal{A};\\ \{x\}\cup(\bigcup\{X_s : s\in\mathbf{N},\ s\geqq n\}) & \text{if } x\in\mathbf{N} \text{ and } x\geqq n;\\ \{x\} & \text{if } x\in\mathbf{N} \text{ and } x<n.\end{cases}$$

Note that $\bigcup\{\{x\}\times T_n(x) : x\in Z\}$ is a transitive relation on Z whenever $n\in\mathbf{N}$. Furthermore $T_{n+1}\subseteq T_n$ for each $n\in\mathbf{N}$ and $\bigcap\{T_n : n\in\mathbf{N}\}$ is equal to the diagonal of Z. According to Lemma 1 we can define a quasi-metric p on Z such that $B_n^p(x)=T_n(x)$ for each $n\in\mathbf{N}$ and each $x\in Z$.

Observe that $\mathcal{M}=\{\bigcup\{X_s : s\in\mathbf{N},\ s\geqq n\}\mid n\in\mathbf{N}\}\cup\{Z\setminus\{x_i : i\in\mathbf{N}\}\mid (x_i)_{i\in\mathbf{N}}\in\mathcal{A}\}$ has the finite intersection property, because X_n is uncountable for each $n\in\mathbf{N}$. Since $B_n^p(n)=\{n\}\cup(\bigcup\{X_s : s\in\mathbf{N},\ s\geqq n\})$ whenever $n\in\mathbf{N}$, the filter generated by $\mathcal{M}$ on Z is a $\mathcal{U}(p)$-Cauchy filter on Z. Obviously it does not have a cluster point in Z. Hence (Z,p) is not complete.

Let us prove that (Z,p) is sequentially convergence complete. To this end let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (Z,p). We have to show that $(x_n)_{n\in\mathbf{N}}$ converges in Z. Since $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in Z, there are a strictly increasing sequence $(k_n)_{n\in\mathbf{N}}$ of positive integers and a sequence $(y_n)_{n\in\mathbf{N}}$ of points of Z such that $x_k\in T_n(y_n)$ whenever $k\in\mathbf{N}$ and $k\geqq k_n$. If there is an $n\in\mathbf{N}$ such that $y_n\in Y$, then $x_k=y_n$ for each $k\in\mathbf{N}$ such that $k\geqq k_n$. Hence $(x_n)_{n\in\mathbf{N}}$ converges in Z. Thus it suffices to consider the case that $y_n\in\mathcal{A}\cup\mathbf{N}$ for each $n\in\mathbf{N}$.

Assume first that for infinitely many $n\in\mathbf{N}$ we have that $x_n\in(\mathcal{A}\cup\mathbf{N})$. We conclude that $(\mathcal{A}\cup\mathbf{N})\cap T_n(y_n)\cap T_1(y_1)\neq\emptyset$ whenever $n\in\mathbf{N}$. Hence by the definition of the sets $T_n(y_n)$ (where $y_n\in\mathcal{A}\cup\mathbf{N}$) we see that $y_n=y_1$ for each $n\in\mathbf{N}$. Therefore $(x_n)_{n\in\mathbf{N}}$ converges to y_1 in this case.

It remains to consider the case that there is a $t\in\mathbf{N}$ such that $x_k\in Y$ whenever $k\in\mathbf{N}$ and $k\geqq t$. Since we assume that $y_n\in\mathcal{A}\cup\mathbf{N}$ for each $n\in\mathbf{N}$, we have that $x_k\in(T_n(y_n)\cap Y)\subseteq\bigcup\{X_s : s\in\mathbf{N},\ s\geqq n\}$ whenever $n,k\in\mathbf{N}$ and $k\geqq\max\{t,k_n\}$. Hence the sequence b defined by $b=(x_k)_{k\geqq t}$ is an element of $\mathcal{A}$. (We can set $k_n(b)=k_{n+t}$ for each $n\in\mathbf{N}$). Thus we conclude in this case that the sequence $(x_k)_{k\in\mathbf{N}}$ in Z converges to $b\in Z$. This completes the proof.

It is interesting to note that a slight modification of Example 2 yields a complete and sequentially convergence complete quasi-metric space that is not convergence complete.

Example 3. A complete and sequentially convergence complete quasi-metric space that is not convergence complete.

Let $E=Z\cup\{\infty,-\infty\}$ where Z denotes the set defined in Example 2 and ∞ and $-\infty$ are two different elements not belonging to Z. For each $n\in\mathbf{N}$

and each $x \in E$ define a subset $S_n(x)$ of E as follows:

$$S_n(x) = \begin{cases} T_n(x) \text{ (as defined in Example 2)} & \text{if } x \in Z; \\ \{-\infty\} \cup (\bigcup\{X_s : s \in \mathbf{N},\ s \text{ odd},\ s \geqq n\}) & \text{if } x = -\infty; \\ \{\infty\} \cup (\bigcup\{X_s : s \in \mathbf{N},\ s \text{ even},\ s \geqq n\}) & \text{if } x = \infty. \end{cases}$$

Again it is easy to check that the conditions of Lemma 1 are satisfied. Hence there is a quasi-metric q on E such that $B_n^q(x) = S_n(x)$ whenever $n \in \mathbf{N}$ and $x \in E$.

Note that the filter generated by $\{\bigcup\{X_s : s \in \mathbf{N},\ s \geqq n\} | n \in \mathbf{N}\}$ on E is a $\mathcal{U}(q)$-Cauchy filter on E that does not converge in E. Hence (E, q) is not convergence complete.

Next we want to show that (E, q) is sequentially convergence complete. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (E, q). By definition, for each $n \in \mathbf{N}$ there are $y_n \in E$ and $k_n \in \mathbf{N}$ such that $x_k \in B_n^q(y_n)$ whenever $k \in \mathbf{N}$ and $k \geqq k_n$. If $y_n \in \{-\infty, \infty\}$ for infinitely many $n \in \mathbf{N}$, then, obviously, $(x_n)_{n\in\mathbf{N}}$ converges in E. Hence it remains to consider the case that there is an $h \in \mathbf{N}$ such that $y_n \in Z$ whenever $n \in \mathbf{N}$ and $n \geqq h$. In this case we conclude that $x_k \in Z$ whenever $k \in \mathbf{N}$ and $k \geqq k_h$. Thus the sequence $(x_n)_{n \geqq k_h}$ is a Cauchy sequence in the subspace Z of E. Therefore the sequence $(x_n)_{n\in\mathbf{N}}$ in E converges to a point in Z, because the subspace Z of E is sequentially convergence complete according to Example 2. We have shown that (E, q) is sequentially convergence complete.

It remains to show that (E, q) is complete. Let $\mathcal{F}$ be a $\mathcal{U}(q)$-Cauchy ultrafilter on E. Assume that $\mathcal{F}$ does not have a cluster point in E. Since $-\infty$ and ∞ are not cluster points of $\mathcal{F}$ and $\mathcal{F}$ is an ultrafilter on E, we have three possibilities to consider: There is an $s \in \mathbf{N}$ such that $X_s \in \mathcal{F}$, or $\mathcal{A} \in \mathcal{F}$, or $\mathbf{N} \in \mathcal{F}$.

We want to see that in each of these three cases the filter $\mathcal{F}$ on E contains a singleton. Since $\mathcal{F}$ is a $\mathcal{U}(q)$-Cauchy filter on E, for each $n \in \mathbf{N}$ there is an $x_n \in E$ such that $B_n^q(x_n) \in \mathcal{F}$. Hence by the definition of the sets $S_n(x)$ it is immediately clear that $\mathcal{F}$ is fixed if $\mathcal{A} \in \mathcal{F}$ or $\mathbf{N} \in \mathcal{F}$. However, it is also easy to see that $\mathcal{F}$ is fixed if $X_s \in \mathcal{F}$ for some $s \in \mathbf{N}$, because $B_n^q(y) \cap X_s = \emptyset$ whenever $y \in E \setminus X_s$ and $n \in \mathbf{N}$ such that $n > s$. We have shown that in each of the three cases under consideration $\mathcal{F}$ contains a singleton. Thus $\mathcal{F}$ converges in E — a contradiction. We conclude that each $\mathcal{U}(q)$-Cauchy filter on E has a cluster point in E. Hence (E, q) is complete.

It is obvious that Example 2 (and thus Example 3) are not Hausdorff spaces. Our first proposition shows that this fact is not an accident. On the other hand, Example 4 below will show that a sequentially complete quasi-metric Tychonoff space need not be complete.

PROPOSITION 1. *A quasi-metric Hausdorff space is sequentially convergence complete if and only if it is convergence complete.*

The nontrivial implication of this proposition is an immediate consequence of the following two auxiliary results, which seem to be of independent interest and are therefore given in a form more general than needed here.

LEMMA 3. *Let (X,p) be a sequentially convergence complete quasi-metric Hausdorff space and let $(x_k)_{k\in\mathbf{N}}$ be a sequence in X such that $\{B^p_k(x_k): k\in\mathbf{N}\}$ has the finite intersection property. Then $\bigcap\{\overline{\cap\{B^p_k(x_k): k=1,\dots,n\}}\mid n\in\mathbf{N}\}$ is a singleton.*

PROOF. Let $(x_k)_{k\in\mathbf{N}}$ be a sequence in a sequentially convergence complete quasi-metric Hausdorff space (X,p) such that $\{B^p_k(x_k): k\in\mathbf{N}\}$ has the finite intersection property. Set $K=\bigcap\{\overline{\cap\{B^p_k(x_k): k=1,\dots,n\}}\mid n\in\mathbf{N}\}$. For each $n\in\mathbf{N}$ choose $y_n\in\cap\{B^p_k(x_k): k=1,\dots,n\}$. Then $(y_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p) and thus has a cluster point x in X, because X is sequentially complete. Clearly $x\in K$. Assume that there is a $y\in K$ such that $y\neq x$. For each $n\in\mathbf{N}$ there are points a_n and b_n in X such that $a_n\in B^p_n(x)\bigcap\bigcap(\cap\{B^p_k(x_k): k=1,\dots,n\})$ and $b_n\in B^p_n(y)\bigcap(\cap\{B^p_k(x_k): k=1,\dots,n\})$. Set $z_n=a_n$ if $n\in\mathbf{N}$ and n is even, and set $z_n=b_n$ if $n\in\mathbf{N}$ and n is odd. Obviously $(z_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in X. Since X is sequentially convergence complete, $(z_n)_{n\in\mathbf{N}}$ converges in X. Furthermore x and y are two different cluster points of the sequence $(z_n)_{n\in\mathbf{N}}$ in X. Since X is a Hausdorff space, we have obtained a contradiction. Hence $\bigcap\{\overline{\cap\{B^p_k(x_k): k=1,\dots,n\}}\mid n\in\mathbf{N}\}$ is a singleton.

LEMMA 4. *Let (X,p) be a sequentially (convergence) complete quasi-pseudo-metric space. Assume that for each sequence $(x_k)_{k\in\mathbf{N}}$ in X such that the collection $\{B^p_k(x_k): k\in\mathbf{N}\}$ has the finite intersection property the set $\bigcap\{\overline{\cap\{B^p_k(x_k): k=1,\dots,n\}}\mid n\in\mathbf{N}\}$ is a Lindelöf subspace of X. Then (X,p) is (convergence) complete.*

PROOF. Let (X,p) be a sequentially complete quasi-pseudo-metric space that satisfies the condition mentioned in the lemma and let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy filter on X. For each $n\in\mathbf{N}$ there is an $x_n\in X$ such that $B^p_n(x_n)\in\in\mathcal{F}$. Set $K=\bigcap\{\overline{\cap\{B^p_k(x_k): k=1,\dots,n\}}\mid n\in\mathbf{N}\}$. By our assumption, K is a Lindelöf subspace of X. We want to show that $\mathcal{F}$ has a cluster point in X. Assume the contrary. Then for each $x\in K$ there is an open neighborhood G_x of x in X such that $(X\setminus G_x)\in\mathcal{F}$. Since K is a Lindelöf subspace of X, there is a sequence $(y_n)_{n\in\mathbf{N}}$ of elements of K such that $K\subseteq\bigcup\{G_{y_n}: n\in\mathbf{N}\}$. For each $n\in\mathbf{N}$ set $H_n=\left(\bigcap\{B^p_k(x_k): k=1,\dots,n\}\right)\setminus\left(\bigcup\{G_{y_i}: i=1,\dots,n\}\right)$. Note that $H_n\in\mathcal{F}$ whenever $n\in\mathbf{N}$. For each $n\in\mathbf{N}$ choose $y_n\in H_n$. Since (X,p) is sequentially complete, the Cauchy sequence $(y_n)_{n\in\mathbf{N}}$ in (X,p) has a cluster point y in X. Clearly $y\in K$, but $y\notin\bigcup\{G_{y_i}: i\in\mathbf{N}\}$ — a contradiction. We conclude that $\mathcal{F}$ has a cluster point in X. Hence (X,p) is complete.

In order to prove the second assertion let (X,p) be a sequentially convergence complete quasi-pseudo-metric space that satisfies the condition mentioned in the lemma and let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy filter on X. For each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B_n^p(x_n) \in \mathcal{F}$. Set $K = \bigcap\{\overline{\cap\{B_k^p(x_k) : k=1,\dots,n\}} \mid n \in \mathbf{N}\}$. By our assumption K is a Lindelöf subspace of X. We want to show that $\mathcal{F}$ converges in X. Assume the contrary. Then for each $x \in K$ there is an open neighborhood G_x of x in X such that $G_x \notin \mathcal{F}$. Since K is a Lindelöf subspace of X, there is a sequence $(y_n)_{n\in\mathbf{N}}$ of elements of K such that $K \subseteq \subseteq \bigcup\{G_{y_i} : i \in \mathbf{N}\}$. For each $n \in \mathbf{N}$ and each $i \in \mathbf{N}$ choose $z_{n,i} \in (\cap\{B_k^p(x_k) : k = = 1,\dots,n\}) \setminus G_{y_i}$. Construct a sequence $(a_n)_{n\in\mathbf{N}}$ in X by enumerating the elements $z_{n,i}$ (where $n, i \in \mathbf{N}$, $i \leqq n$) according to the lexicographic order on the indexing set $\mathbf{N} \times \mathbf{N}$. Obviously $(a_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p). Since (X,p) is sequentially convergence complete, $(a_n)_{n\in\mathbf{N}}$ has a limit point z in X. Clearly $z \in K$. Hence there is a $k \in \mathbf{N}$ such that $z \in G_{y_k}$. However by the construction of the sequence $(a_n)_{n\in\mathbf{N}}$ we have that $a_n \notin G_{y_k}$ for infinitely many $n \in \mathbf{N}$. We have reached a contradiction. We conclude that $\mathcal{F}$ converges in X. Thus (X,p) is convergence complete.

COROLLARY 1. *A quasi-metric space in which each Cauchy sequence has exactly one cluster point is convergence complete.*

PROOF. Let (X,p) be a quasi-metric space in which each Cauchy sequence has exactly one cluster point. Since convergent sequences are Cauchy sequences, it is clear that X is a Hausdorff space. Obviously X is sequentially complete. Since each subsequence of a Cauchy sequence is a Cauchy sequence, and thus has a cluster point in a sequentially complete quasi-metric space, a nonconvergent Cauchy sequence in X would have at least two different cluster points. We conclude that each Cauchy sequence in X is convergent. Hence X is sequentially convergence complete. The result follows from Proposition 1.

EXAMPLE 4. A zero-dimensional sequentially complete quasi-metric space that is not complete.

We show that the well-known zero-dimensional pseudocompact space Ψ (see e.g. [11, 5I]) can be equipped with a compatible quasi-metric p that is sequentially complete, but not complete. Let $\mathcal{A}$ be an infinite maximal almost disjoint family of infinite subsets of $\mathbf{N}$. Let $(a_n)_{n\in\mathbf{N}}$ be an injective sequence of elements of $\mathcal{A}$ and let $\mathcal{H} = \mathcal{A} \setminus \{a_n : n \in \mathbf{N}\}$. As usual set $\Psi = \mathcal{A} \cup \cup \mathbf{N}$. For each $n \in \mathbf{N}$ and each $x \in \Psi$ define a subset $T_n(x)$ of Ψ as follows:

$$T_n(x) = \begin{cases} \{x\} & \text{if } x \in \mathbf{N}; \\ \{x\} \cup (x \setminus \{1,\dots,n\}) & \text{if } x \in \mathcal{H}; \\ \{x\} \cup (\mathbf{N} \setminus \{1,\dots,n\}) & \text{if } x = a_p,\ p \in \mathbf{N},\ p \geqq n; \\ \{x\} \cup (x \setminus \{1,\dots,n\}) & \text{if } x = a_p,\ p \in \mathbf{N},\ p < n. \end{cases}$$

Considering Lemma 1 we see that there is a quasi-metric p on Ψ such that $B_n^p(x) = T_n(x)$ for each $n \in \mathbf{N}$ and each $x \in \Psi$. Clearly p induces the usual topology on Ψ. Set $\mathcal{M}=\{\mathbf{N}\setminus\{n\}:n\in\mathbf{N}\}\cup\{\mathbf{N}\setminus a:a\in\mathcal{A}\}$. Since $\mathcal{A}$ is infinite, $\mathcal{M}$ has the finite intersection property. Since for each $n\in\mathbf{N}$ the set $T_n(a_n)$ is a member of the filter $\mathcal{F}$ generated by $\mathcal{M}$ on Ψ, $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X. Clearly it does not have a cluster point in Ψ. We conclude that $\mathcal{U}(p)$ is not complete.

We want to show that (Ψ,p) is sequentially complete. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (Ψ,p). For each $n\in\mathbf{N}$ there are $k_n\in\mathbf{N}$ and $y_n\in\Psi$ such that $x_k\in T_n(y_n)$ whenever $k\in\mathbf{N}$ and $k\geqq k_n$. Recall the first three arguments given in that part of the proof of Example 2 where we show that the space Z is sequentially convergence complete. We see by arguments similar to the arguments given there (note that the subspace Y of Z corresponds to the subspace $\mathbf{N}$ of Ψ in an obvious way) that the sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point in Ψ, except, maybe, if the following three conditions are satisfied simultaneously:

(i) $y_n\in\mathcal{A}$ whenever $n\in\mathbf{N}$;

(ii) there is a $k_0\in\mathbf{N}$ such that $x_k\in\mathbf{N}$ for each positive integer k such that $k\geqq k_0$;

(iii) $x_k>n$ whenever n, $k\in\mathbf{N}$ and $k\geqq\max\{k_0,k_n\}$.

However, in this case it is clear that the sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point in the subspace $\mathcal{A}$ of Ψ, because $\{x_n:n\in\mathbf{N}\}$ has infinitely many elements in $\mathbf{N}$ and $\mathcal{A}$ is maximal. This completes the proof.

Let us remark that in order to show that the quasi-metric space in Example 4 is not complete we could also use the following technical result related to Lemma 4. (Consider the sets $B^p_n(a_n)$ where $n\in\mathbf{N}$.)

REMARK 1. *Let (X,p) be a regular complete quasi-metric space and let $(x_k)_{k\in\mathbf{N}}$ be a sequence in X such that $\{B^p_k(x_k):k\in\mathbf{N}\}$ has the finite intersection property. Then $\bigcap\{\overline{\cap\{B^p_k(x_k):k=1,\dots,n\}}|n\in\mathbf{N}\}$ is compact.*

PROOF. Let (X,p) be a regular complete quasi-metric space and let $(x_k)_{k\in\mathbf{N}}$ be a sequence in X such that $\{B^p_k(x_k):k\in\mathbf{N}\}$ has the finite intersection property. Set $K=\bigcap\{\overline{\cap\{B^p_k(x_k):k=1,\dots,n\}}|n\in\mathbf{N}\}$. Let $\mathcal{C}$ be a collection of X-open sets that covers K. We want to show that there is a finite subcollection of $\mathcal{C}$ covering K. For each $x\in K$ choose an X-open neighborhood G_x of X such that $\overline{G_x}\subseteqq D$ for some $D\in\mathcal{C}$. Let $\mathcal{M}=\{B^p_k(x_k):k\in\mathbf{N}\}\cup$ $\cup\{X\setminus G_x:x\in K\}$. If $\mathcal{M}$ has the finite intersection property, then the filter generated by $\mathcal{M}$ on X is a $\mathcal{U}(p)$-Cauchy filter on X without a cluster point in X — a contradiction to the completeness of (X,p). Hence there are an $n\in\mathbf{N}$ and a finite subset P of K such that $\bigcap\{B^p_k(x_k):k=1,\dots,n\}\subseteqq\bigcup\{G_f:f\in$ $\in P\}$. Thus $K\subseteqq\overline{\bigcap\{B^p_k(x_k):k=1,\dots,n\}}\subseteqq\bigcup\{\overline{G_f}:f\in P\}\subseteqq\bigcup\{D:D\in\mathcal{H}\}$ for some finite subcollection $\mathcal{H}$ of $\mathcal{C}$. We conclude that K is compact.

Next we want to study some conditions under which sequentially complete quasi-metric spaces are complete (compare Lemma 4).

Recall that a family of subsets of a topological space is called *discrete* if each point of the space has a neighborhood that intersects at most one member of the family. A subset B of a topological space X is called a *discrete set* in X [18] if the family $\{\{b\} : b \in B\}$ is discrete. A topological space X is said to have *property wD* [18, p. 238] if for each countably infinite discrete set B in X there are a countably infinite subset C of B and a discrete collection $\{G_c : c \in C\}$ of open subsets of X such that $G_c \cap C = \{c\}$ for each $c \in C$. It is well known that a Tychonoff space that is realcompact, or countably paracompact, or normal has this property [18, p. 240 and p. 252].

PROPOSITION 2. *A quasi-metric space that has property wD is sequentially (convergence) complete if and only if it is (convergence) complete.*

PROOF. Let (X,p) be a sequentially (convergence) complete quasi-metric space that has property wD. In order to prove Proposition 2 it suffices by Lemma 2 to show that (X,p) is (convergence) complete. Let $(x_k)_{k\in\mathbf{N}}$ be a sequence in X such that $\{B^p_k(x_k) : k \in \mathbf{N}\}$ has the finite intersection property. Set $K = \bigcap\{\overline{\cap B^p_k(x_k) : k = 1,\dots,n\}}|n \in \mathbf{N}\}$. We want to prove that K is countably compact. Then the assertion will follow from Lemma 4, because a countably compact quasi-metric space is compact [8, Corollary 2.29].

Assume that K is not countably compact. Let $(x_n)_{n\in\mathbf{N}}$ be an injective sequence without a cluster point in K. Since K is a closed subspace of the T_1 space X, $\{x_n : n \in \mathbf{N}\}$ is a discrete set in X. Since X has property wD, there are an (injective) subsequence $(y_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ and a discrete collection $\{G_n : n \in \mathbf{N}\}$ of open subsets of X such that $G_n \cap \{y_k : k \in \in \mathbf{N}\} = \{y_n\}$ for each $n \in \mathbf{N}$. Since $y_n \in G_n \cap K$ whenever $n \in \mathbf{N}$, we have that $G_n \cap \left(\bigcap\{B^p_k(x_k) : k = 1,\dots,n\}\right) \neq \emptyset$ for each $n \in \mathbf{N}$. Choose $z_n \in G_n \cap \cap \left(\bigcap\{B^p_k(x_k) : k = 1,\dots,n\}\right)$ for each $n \in \mathbf{N}$. Since $(z_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p) and (X,p) is sequentially (convergence) complete, $(z_n)_{n\in\mathbf{N}}$ has a cluster point x in X. Hence for each neighborhood G of x we have that $z_n \in G$ for infinitely many $n \in \mathbf{N}$ and, thus, G intersects infinitely many different members of the collection $\{G_n : n \in \mathbf{N}\}$. Therefore the collection $\{G_n : n \in \mathbf{N}\}$ is not even locally finite at x — a contradiction. We conclude that K is countably compact. Hence we have shown that (X,p) is (convergence) complete.

It follows from Proposition 2 that each Tychonoff realcompact sequentially complete quasi-metric space is complete. By using the following generalization of realcompactness introduced in [9] we can still strengthen this result further. Recall that a topological space X is called *almost realcompact* [9] if for each maximal filter $\mathcal{U}$ of open sets in X such that $\{\overline{U} : U \in \mathcal{U}\}$ has the countable intersection property it is the case that $\mathcal{U}$ has a cluster point in X.

PROPOSITION 3. *Each regular almost realcompact sequentially complete quasi-metric space is complete.*

PROOF. Let (X,p) be a regular almost realcompact sequentially complete quasi-metric space. Let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy filter on X and let $\mathcal{H}$ be a maximal open filter on X such that $\{G \in \mathcal{F} : G \text{ is open in } X\} \subseteq \mathcal{H}$. We want to show that $\{\overline{G} : G \in \mathcal{H}\}$ has the countable intersection property. For each $n \in \mathbf{N}$ let $G_n \in \mathcal{H}$. Since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, for each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B^p_n(x_n) \in \mathcal{F}$. Thus $B^p_n(x_n) \in \mathcal{H}$ whenever $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ choose $y_n \in \bigcap\{G_k \cap B^p_k(x_k) : k = 1, \ldots, n\}$. Since X is sequentially complete, we conclude that the Cauchy sequence $(y_n)_{n \in \mathbf{N}}$ in X has a cluster point in $\bigcap\{\overline{G_n} : n \in \mathbf{N}\}$. Hence $\{\overline{G} : G \in \mathcal{H}\}$ has the countable intersection property.

Since X is almost realcompact, there is an $x \in X$ such that $x \in \bigcap\{\overline{G} : G \in \in \mathcal{H}\}$. Assume that there is a $U \in \mathcal{F}$ such that $x \notin \overline{U}$. Since X is regular, there are disjoint open sets V and W in X such that $x \in V$ and $\overline{U} \subseteq W$. Therefore $W \in \mathcal{H}$ and $x \notin \overline{W}$ — a contradiction. Thus x is a cluster point of $\mathcal{F}$. We conclude that $\mathcal{U}(p)$ is a complete quasi-uniformity. Hence (X, p) is complete.

We note that it is shown in [13] that the Dieudonné plank is almost realcompact. On the other hand it is easy to see that it does not have property wD. (Consider the countably infinite closed discrete subspace $\{(\omega_1, n) : n \in \in \mathbf{N}\}$ in this space.)

Finally let us point out that, essentially, the correct part in the argument of Theorem 7.33 in [8] shows that each Tychonoff sequentially complete quasi-metric space is Čech complete. (Recall that a Tychonoff space X is *Čech complete* if and only if there exists a countable family $\{\mathcal{G}_n : n \in \mathbf{N}\}$ of open covers of X such that whenever $\mathcal{F}$ is a family of closed subsets of X that has the finite intersection property and that contains for each $n \in \mathbf{N}$ a member F_n with $F_n \subseteq G_n$ for some $G_n \in \mathcal{G}_n$, then $\mathcal{F}$ has nonempty intersection [3, Theorem 3.9.2].

PROPOSITION 4. *Each Tychonoff sequentially complete quasi-metric space is Čech complete.*

PROOF. We sketch the proof. It is easy to see that in order to prove Proposition 4 it suffices to show that each sequentially complete quasi-metric space (X, p) has the following property: Whenever $\mathcal{F}$ is a filter on X such that for each $n \in \mathbf{N}$ there is a closed set $F_n \in \mathcal{F}$ that is contained in some member of the open cover $\mathcal{G}_n = \{B^p_n(x) : x \in X\}$ of X, then $\mathcal{F}$ has a cluster point in X.

However this is clear by the following two observations.

First note that in this case $\{\overline{F} : F \in \mathcal{F}\}$ has the countable intersection property: Let $H_n \in \mathcal{F}$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ choose $y_n \in \bigcap\{F_k \cap \cap H_k : k = 1, \ldots, n\}$. Then the Cauchy sequence $(y_n)_{n \in \mathbf{N}}$ in (X, p) has a cluster point y in $\bigcap\{\overline{H_k} : k \in \mathbf{N}\}$, because X is sequentially complete. Hence $\{\overline{F} : F \in \mathcal{F}\}$ has the countable intersection property.

Second we conclude that $\bigcap\{F_n : n \in \mathbf{N}\}$ is countably compact, because each sequence in the closed subspace $\bigcap\{F_n : n \in \mathbf{N}\}$ of X is a Cauchy sequence in the sequentially complete quasi-metric space (X,p). Hence $\bigcap\{F_n : n \in \in \mathbf{N}\}$ is compact, because a countably compact quasi-metric space is compact [8, Corollary 2.29]. Using that $\overline{F} \cap (\bigcap\{F_n : n \in \mathbf{N}\}) \neq \emptyset$ for each $F \in \mathcal{F}$, we deduce that $\mathcal{F}$ has a cluster point in $\bigcap\{F_n : n \in \mathbf{N}\}$.

We conclude this section with some remarks on countably compact quasi-pseudo-metric spaces. In [17, p. 194] Romaguera and Gutiérrez ask whether a precompact left-p-sequentially complete quasi-pseudo-metric space must be compact. As we are going to show now, this question has an affirmative answer. We obtain the answer as Corollary 3 to the following proposition.

PROPOSITION 5. *Each countably compact quasi-pseudo-metric space is convergence complete.*

PROOF. Let (X,p) be a countably compact quasi-pseudo-metric space and let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy filter on X. Then for each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B^p_n(x_n) \in \mathcal{F}$. Since X is countably compact, the sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point $y \in X$. Let $m \in \mathbf{N}$. There is a $k \in \mathbf{N}$ such that $k > m$ and such that $x_k \in B^p_{m+1}(y)$. Hence $B^p_k(x_k) \subseteq B^p_m(y)$. Thus $B^p_m(y) \in \in \mathcal{F}$. We conclude that $\mathcal{F}$ converges to y. Therefore (X,p) is convergence complete.

COROLLARY 2. *A precompact countably compact quasi-pseudo-metric space is compact.*

PROOF. Let (X,p) be a precompact countably compact quasi-pseudo-metric space. By Proposition 5 we know that $(X,\mathcal{U}(p))$ is a precompact (convergence) complete quasi-uniform space. The usual argument shows that X is compact: Let $\mathcal{H}$ be an ultrafilter on X. Since (X,p) is precompact, $\mathcal{H}$ is a $\mathcal{U}(p)$-Cauchy filter on X. Hence $\mathcal{H}$ has a cluster point in X, because $\mathcal{U}(p)$ is complete. We conclude that X is compact.

COROLLARY 3. *A precompact sequentially complete quasi-pseudo-metric space is compact.*

PROOF. Let (X,p) be a precompact sequentially complete quasi-pseudo-metric space. It is observed in [17, p. 194] that in a precompact quasi-pseudo-metric space each sequence has a (left p-)Cauchy subsequence. Thus each sequence in X has a cluster point, because X is sequentially complete. Hence X is countably compact. The result follows from Corollary 2 given above.

Note that a countably compact quasi-pseudo-metric space need not to be compact [5,14].

Recall that a quasi-pseudo-metric space (X,p) is called *equinormal*, if $p(A,B) = \inf\{p(a,b) : a \in A,\ b \in B\} > 0$ whenever A and B are two disjoint nonempty closed subsets of X (see e.g. [8, p. 37]). It is easy to see that each countably compact quasi-pseudo-metric space is equinormal [8, Proposition 2.24].

EXAMPLE 5. An equinormal quasi-metric space that is not sequentially convergence complete.

Let $Y = \mathbf{N}\times(\mathbf{N}\cup\{0\})$ and set $P_n = \{(i,j)\in \mathbf{N}\times\mathbf{N} : 1 \leqq i \leqq n,\ 1 \leqq j \leqq n\}$ for each $n\in\mathbf{N}$. For each $n\in\mathbf{N}$ and each $x\in Y$ define a subset $T_n(x)$ of Y as follows:

$$T_n(x) = \begin{cases} \{x\} & \text{if } x\in \mathbf{N}\times\mathbf{N};\\ \{(m,0)\}\cup[(\mathbf{N}\times\mathbf{N})\setminus P_n] & \text{if } x=(m,0),\ m\in\mathbf{N},\ m\geqq n;\\ \{(m,0)\}\cup[(\mathbf{N}\times(\mathbf{N}\setminus\{m\}))\setminus P_n] & \text{if } x=(m,0),\ m\in\mathbf{N},\ m<n.\end{cases}$$

By Lemma 1 there is a quasi-metric p on Y such that $B_n^p(x) = T_n(x)$ for each $n\in\mathbf{N}$ and each $x\in Y$. Construct a sequence $(a_n)_{n\in\mathbf{N}}$ in Y by enumerating the points of the set $\{(i,j)\in\mathbf{N}\times\mathbf{N} : i\geqq j\}$ according to the lexicographic order on $\mathbf{N}\times\mathbf{N}$. Since $T_n((n,0)) = \{(n,0)\}\cup[(\mathbf{N}\times\mathbf{N})\setminus P_n]$ for each $n\in\mathbf{N}$, $(a_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (Y,p). Clearly $(a_n)_{n\in\mathbf{N}}$ is not convergent. Thus (Y,p) is not sequentially convergence complete. It remains to show that (Y,p) is equinormal. Let F_1 and F_2 be disjoint nonempty closed subsets of Y. We want to show that there is an $n\in\mathbf{N}$ such that $F_1\cap T_n(F_2) = \emptyset$. First assume that there are two different points in $\mathbf{N}\times\{0\}$ not belonging to F_1. Then $F_1\cap(\mathbf{N}\times\mathbf{N})$ is finite, because F_1 is closed. Hence there is an $n\in\mathbf{N}$ such that $F_1\cap T_n(F_2\cap(\mathbf{N}\times\{0\}))=\emptyset$. Then $F_1\cap T_n(F_2)=\emptyset$. Second consider the case that $(\mathbf{N}\times\{0\})\setminus F_1$ contains at most one point. Then $F_2\cap(\mathbf{N}\times\{0\})$ contains at most one point. Since F_1 is closed, there is an $n\in\mathbf{N}$ such that $F_1\cap T_n(F_2\cap(N\times\{0\}))=\emptyset$. Thus $F_1\cap T_n(F_2)=\emptyset$. We conclude that (Y,p) is equinormal.

REMARK 2. The author discovered Example 5 in July 1988. Independently the referee obtained essentially the same example when studying the first version of this paper. The referee also observes that each Hausdorff equinormal quasi-metric space (X,p) is convergence complete. (PROOF. The space X is regular, because each closed subset of the Hausdorff space X is of countable character [7, p. 112]. (In fact X is metrizable and the set of nonisolated points of X is compact [7, Proposition 4.1].) Hence the quasi-uniformity $\mathcal{U}(p)$ on X is locally symmetric [8, Proposition 2.26], because p is equinormal. Thus each cluster point of a $\mathcal{U}(p)$-Cauchy filter $\mathcal{F}$ on X is a limit point of $\mathcal{F}$ [8, Proposition 3.9]. The assertion is now a consequence of the following proposition.)

The last proposition in this section shows that each equinormal quasi-metric space is complete. Note that this result is related to Theorem 7.34 of [8]. In fact the proof of Theorem 7.34 given in [8] shows that an equinormal quasi-metric space is sequentially complete. Our Proposition 6 needs a proof however, because, as we have seen above, there are sequentially complete quasi-metric spaces that are not complete.

PROPOSITION 6. *Each equinormal quasi-metric space is complete.*

PROOF. Let (X,p) be an equinormal quasi-metric space and let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy ultrafilter on X. We have to show that $\mathcal{F}$ converges in X.

Since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, for each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B^p_n(x_n) \in \mathcal{F}$.

If the sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point y in X, then, as the proof of Proposition 5 shows, $\mathcal{F}$ converges to y. Hence in the following it suffices to consider the case that $\{x_n : n \in \mathbf{N}\}$ is an infinite closed discrete subspace of X. Let us assume that $\mathcal{F}$ does not converge in X. Since for each $n \in \mathbf{N}$ the filter $\mathcal{F}$ does not converge to x_n, there is an $s_n \in \mathbf{N}$ such that $B^p_{s_n}(x_n) \notin \mathcal{F}$. For each $n \in \mathbf{N}$ set $H_n = B^p_n(x_n) \setminus (\bigcup\{B^p_{s_i}(x_i) : i = 1, \ldots, n\})$. Then $H_n \in \mathcal{F}$ whenever $n \in \mathbf{N}$. Define inductively a sequence $(a_n)_{n\in\mathbf{N}}$ of points of X and a strictly increasing sequence $(k_n)_{n\in\mathbf{N}}$ of positive integers as follows. Choose $a_1 \in H_1$, and choose $k_1 \in \mathbf{N}$ such that $x_{k_1} \neq a_1$. Let $n \in \mathbf{N}\setminus\{1\}$. Assume that a_{n-1} and k_{n-1} are defined. Since $\mathcal{F}$ is a free ultrafilter, we can choose an $a_n \in H_{k_{n-1}} \setminus \{a_1, a_2, \ldots, a_{n-1}, x_1, x_2, \ldots, x_{k_{n-1}}\}$. Moreover, since $\{x_n : n \in \in \mathbf{N}\}$ has infinitely many elements, we can find a $k_n \in \mathbf{N}$ such that $k_n > k_{n-1}$ and such that $x_{k_n} \notin \{a_1, a_2, \ldots, a_n, x_1, x_2, \ldots, x_{k_{n-1}}\}$. This completes the construction of the sequences $(a_n)_{n\in\mathbf{N}}$ and $(k_n)_{n\in\mathbf{N}}$.

It is easy to check that $\overline{\{a_n : n \in \mathbf{N}\}} \cap \{x_{k_n} : n \in \mathbf{N}\} = \emptyset$. In fact, $\{a_n : n \in \mathbf{N}\} \cap \{x_{k_n} : n \in \mathbf{N}\} = \emptyset$, because if $n \in \mathbf{N}$ and $r \in \mathbf{N}$ such that $r > > n$, then $a_r \notin B^p_{s_{k_n}}(x_{k_n})$, since $a_r \in H_{k_{r-1}}$. Because (X, p) is equinormal, there is an $f \in \mathbf{N}$ such that $a_n \notin B^p_f(x_{k_i})$ for each n, $i \in \mathbf{N}$. Then there is an $m \in \mathbf{N}$ such that $k_m \geqq f$. However $a_{m+1} \in H_{k_m} \subseteq B^p_{k_m}(x_{k_m}) \subseteq B^p_f(x_{k_m})$ — a contradiction. We conclude that $\mathcal{F}$ converges. Hence the quasi-uniformity $\mathcal{U}(p)$ is complete.

3. Quasi-metric completions

Let $(X, \mathcal{U})$ be a T_1 quasi-uniform space. In [8, § 3.39] a quasi-uniform space $(Y, \mathcal{V})$ is called a *completion* of $(X, \mathcal{U})$ if $(Y, \mathcal{V})$ is a complete T_1 quasi-uniform space that has a dense subspace quasi-unimorphic (relative to $\mathcal{U}$ and $\mathcal{V}$) to $(X, \mathcal{U})$. Similarly we will call a quasi-uniform space $(Y, \mathcal{V})$ a *convergence completion* of $(X, \mathcal{U})$ if $(Y, \mathcal{V})$ is a convergence complete T_1 quasi-uniform space that has a dense subspace quasi-unimorphic (relative to $\mathcal{U}$ and $\mathcal{V}$) to $(X, \mathcal{U})$.

In [8, Theorem 3.43] it is shown that a T_1 quasi-uniform space $(X, \mathcal{U})$ has a completion if and only if whenever $\mathcal{F}$ is a $\mathcal{U}$-Cauchy filter on X and $x \in X$ is a $\mathcal{T}(\mathcal{U}^{-1})$-cluster point of $\mathcal{F}$, then x is a $\mathcal{T}(\mathcal{U})$-cluster point of $\mathcal{F}$. It is also known that a T_1 quasi-uniform space $(X, \mathcal{U})$ has a convergence completion if and only if every fixed $\mathcal{U}$-Cauchy filter on X is convergent in $(X, \mathcal{U})$ [2, essentially Theorem 3.9 and Lemma 3.8].

In this section we want to study several variants of Problem U posed by P. Fletcher and W. F. Lindgren in [8, p. 179]. As mentioned above, Problem U asks for a characterization of the class of quasi-metric spaces having quasi-metric completions. In order to discuss this problem we introduce the following terminology.

Let (X,p) be a quasi-metric subspace of a quasi-metric space (Y,q). We say that (Y,q) is a *quasi-metric sequential (convergence) completion* of (X,p) if X is dense in Y and Y is sequentially (convergence) complete.

Similarly we say that (Y,q) is a *quasi-metric (convergence) completion* of (X,p) if X is dense in Y and Y is (convergence) complete. Note that in this case the quasi-uniform space $(Y,\mathcal{U}(q))$ is a (convergence) completion of the quasi-uniform space $(X,\mathcal{U}(p))$.

3.1. Preliminary results. Let us first consider several topological properties that are of interest in this context. We note that Lemma 5 and Lemma 6 below are related to [8, Theorem 7.37] and [6, Proposition 2.10], respectively.

Recall that a T_1 space X has a *base of countable order* if and only if there is a sequence $(\mathcal{B}_n)_{n\in\mathbf{N}}$ of bases for X that satisfies the following condition: Whenever $x\in X$ and $(b_n)_{n\in\mathbf{N}}$ is a decreasing sequence of subsets of X such that $x\in b_n\in\mathcal{B}_n$ for each $n\in\mathbf{N}$, then $\{b_n : n\in\mathbf{N}\}$ is a neighborhood base at x [19, Theorem 2].

LEMMA 5. *Each sequentially complete quasi-metric space has a base of countable order.*

PROOF. Let (X,p) be a sequentially complete quasi-metric space. For each $n\in\mathbf{N}$ set $\mathcal{B}_n=\{B^p_k(x): x\in X,\ k\in\mathbf{N},\ k\geqq n\}$. Clearly $\mathcal{B}_n$ is a base for X whenever $n\in\mathbf{N}$. Let $x\in X$ and let $(b_n)_{n\in\mathbf{N}}$ be a decreasing sequence of subsets of X such that $x\in b_n\in\mathcal{B}_n$ for each $n\in\mathbf{N}$. Then for each $n\in\mathbf{N}$ there are $y_n\in X$ and $k_n\in\mathbf{N}$ such that $b_n=B^p_{k_n}(y_n)$ and $k_n\geqq n$. Since $(b_n)_{n\in\mathbf{N}}$ is decreasing, it is clear that $(y_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p). Since (X,p) is sequentially complete, $(y_n)_{n\in\mathbf{N}}$ has a cluster point y in X. Let $m\in\mathbf{N}$. Then there is an $s\in\mathbf{N}$ such that $s>m$ and such that $y_s\in B^p_{m+1}(y)$. Hence $x\in b_s=B^p_{k_s}(y_s)\subseteq B^p_m(y)$. Thus $x\in B^p_m(y)$ for each $m\in\mathbf{N}$. Since X is a T_1 space, we conclude that $x=y$. Hence $\{b_n : n\in\mathbf{N}\}$ is a neighborhood base at x. We have shown that X has a base of countable order.

In the light of Proposition 4 it is interesting to note that by a similar proof one can show that each Tychonoff Čech complete quasi-metric space (X,p) has a base of countable order. (Let $(\mathcal{G}_n)_{n\in\mathbf{N}}$ be a sequence of open covers of X witnessing Čech completeness of X. Set $\mathcal{B}_n=\{B^p_k(x): x\in X,$ $k\in\mathbf{N}$, $k\geqq n$ and $\overline{B^p_k(x)}\subseteq G$ for some $G\in\mathcal{G}_n\}$ for each $n\in\mathbf{N}$.)

COROLLARY 4. *Each θ-refinable sequentially complete quasi-metric space is developable.*

PROOF. Each θ-refinable T_1 space with a base of countable order is developable [19, Theorem 3].

COROLLARY 5. *If a quasi-metric space has a quasi-metric sequential completion, then it has a base of countable order.*

PROOF. A subspace of a topological space that has a base of countable order has a base of countable order [19, Theorem 1].

COROLLARY 6. *Each generalized ordered topological space admitting a sequentially complete quasi-metric is completely metrizable.*

PROOF. Let X be a generalized ordered topological space that admits a sequentially complete quasi-metric. It is known that each generalized ordered quasi-metrizable topological space is paracompact [4, Theorem 4.1]. Hence X is metrizable, because each paracompact Hausdorff space that has a base of countable order is metrizable (see e.g. [19]). Since by Proposition 4 the space X is Čech complete, it is completely metrizable [3, Theorem 4.3.26].

EXAMPLE 6. The Michael line and the Sorgenfrey line (see e.g. [8]) are quasi-metrizable generalized ordered topological spaces that do not admit quasi-metrics that have a quasi-metric sequential completion, because these spaces do not have a base of countable order. [A related result is Corollary 7.39 of [8]. Note, however, that this result is based on the incorrect Proposition 7.39 in [8] (see below)].

Let us recall that a topological space X is said to be *subdevelopable* [8, p. 169] if there exists a sequence $(\mathcal{G}_n)_{n\in\mathbf{N}}$ of open covers of X such that whenever $x \in X$ and $x \in H_n \in \mathcal{G}_n$ for each $n \in \mathbf{N}$, then $\{H_n : n \in \mathbf{N}\}$ is a local subbase at x. In this case the sequence $(\mathcal{G}_n)_{n\in\mathbf{N}}$ is called a *subdevelopment* for X (see also [12, Theorem 3.1, 1; 10]). Note that each subdevelopable T_1 space has a G_δ-diagonal. In fact it is known that a Tychonoff space is subdevelopable if and only if it is a p-space having a G_δ-diagonal [10, Theorem 1]. (The characterization given on p. 169 of [8] is not correct.)

LEMMA 6. *Each sequentially convergence complete quasi-metric space is subdevelopable.*

PROOF. Let (X,p) be a sequentially convergence complete quasi-metric space. For each $n \in \mathbf{N}$ set $\mathcal{B}_n = \{B^p_n(x) : x \in X\}$. We wish to show that $(\mathcal{B}_n)_{n\in\mathbf{N}}$ is a subdevelopment for X. Assume that $x \in X$ and that for each $n \in \mathbf{N}$ we have that $x \in B^p_n(x_n)$ where $x_n \in X$. We argue that $\{B^p_n(x_n) : n \in \mathbf{N}\}$ is a local subbase at x. Assume the contrary. Then there is an $m \in \mathbf{N}$ and for each $n \in \mathbf{N}$ there is a $y_n \in X$ such that $y_n \in \left(\bigcap\{B^p_k(x_k) : k = 1, \dots, n\}\right) \setminus B^p_m(x)$. Set $z_n = y_n$ if $n \in \mathbf{N}$ and n is even, and set $z_n = x$ if $n \in \mathbf{N}$ and n is odd. The Cauchy sequence $(z_n)_{n\in\mathbf{N}}$ in (X,p) converges to some point z in X, because X is sequentially convergence complete. Since X is a T_1 space and $(z_n)_{n\in\mathbf{N}}$ has a constant subsequence, we conclude that $z = x$, although $z_n \notin B^p_m(x)$ for infinitely many $n \in \mathbf{N}$. We have reached a contradiction. We conclude that X is subdevelopable.

COROLLARY 7. *A quasi-metric space that has a quasi-metric sequential convergence completion is subdevelopable.*

PROOF. It is clear that a subspace of a subdevelopable space is subdevelopable (see [12, Theorem 3.2]).

In [8, Proposition 7.39] it is stated that if a quasi-metrizable quasi-uniform space has a completion, then it has a subdevelopment. However as we are going to show now there is an example of a zero-dimensional complete quasi-metric space that does not have a G_δ-diagonal. Of course this example contradicts Proposition 7.39 of [8]. (The error in [8] seems to occur in line 6 on page 180.)

EXAMPLE 7. A zero-dimensional complete quasi-metric space without a G_δ-diagonal.

Our construction combines an example of J. Gerlits [10, p. 345] with some ideas contained e.g. in [15].

Denote by I the interval $[0,1]$ of the reals and by $\{(x_n(\alpha))_{n\in\mathbf{N}} : \alpha \in \mathcal{A}\}$ a family of sequences in I with the following properties:

a) For each $\alpha \in \mathcal{A}$ we have that $(x_n(\alpha))_{n\in\mathbf{N}}$ is a convergent injective sequence in I without its limit point.

b) For each $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq \beta$ we have that $\{x_n(\alpha) : n \in \mathbf{N}\}$ and $\{x_n(\beta) : n \in \mathbf{N}\}$ have a finite intersection.

c) $\{(x_n(\alpha))_{n\in\mathbf{N}} : \alpha \in \mathcal{A}\}$ is maximal with respect to the properties a) and b).

Now we put $X = (I \times \{1,2\}) \cup \mathcal{A}$. For each z, $y \in X$ define $p(z,y)$ as follows:

$$p(z,y) = \begin{cases} 0 & \text{if } z = y; \\ 2^{-n} & \text{if } z \in \mathcal{A},\ n \in \mathbf{N},\ y = (x_n(z), j) \text{ and } j \in \{1,2\}; \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that p is a quasi-metric on X. Note that for each $n \in \mathbf{N}$ and each $x \in X$ the set $B_n^p(x)$ is closed. Thus X is zero-dimensional.

Let us show that the quasi-uniformity $\mathcal{U}(p)$ is complete. Let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy ultrafilter on X. We have to show that $\mathcal{F}$ has a cluster point in X. Assume the contrary. Since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, for each $n \in \mathbf{N}$ there is an $x_n \in X$ such that $B_n^p(x_n) \in \mathcal{F}$. Since $\mathcal{F}$ does not converge in X, the ultrafilter $\mathcal{F}$ on X cannot contain a finite set. Thus $x_n \in \mathcal{A}$ and $x_n = x_1$ for each $n \in \mathbf{N}$. Hence $\mathcal{F}$ converges to x_1 — a contradiction. We conclude that each $\mathcal{U}(p)$-Cauchy filter on X has a cluster point in X. Therefore $\mathcal{U}(p)$ is complete.

It remains to show that X does not have a G_δ-diagonal. Assume the contrary. Then there exists a sequence $(\mathcal{G}_n)_{n\in\mathbf{N}}$ of open covers of X so that whenever x, $y \in X$ and $x \neq y$ there is a $k \in \mathbf{N}$ such that $x \notin \operatorname{st}(y, \mathcal{G}_k)$. For each $n \in \mathbf{N}$ set $S_n = \{i \in I : (i,2) \notin \operatorname{st}((i,1), \mathcal{G}_n)\}$. Then $I = \bigcup\{S_n : n \in \mathbf{N}\}$. Since I is uncountable, there exists a $k \in \mathbf{N}$ such that S_k is infinite. Therefore, since the collection $\mathcal{A}$ is maximal and since each infinite subset of I has a subset that is the range of an injective and in I (with respect to the usual topology) convergent sequence, there is an $\alpha \in \mathcal{A}$ such that $x_n(\alpha) \in S_k$ for infinitely many $n \in \mathbf{N}$. There is also a $C \in \mathcal{G}_k$ such that $\alpha \in C$. Since C is open, there is an $m \in \mathbf{N}$ such that $(x_n(\alpha), j)$ belongs to C whenever $n \in \mathbf{N}$,

$n \geqq m$ and $j \in \{1,2\}$. Hence there is an $s \in \mathbf{N}$ such that $x_s(\alpha) \in S_k$, and $\{(x_s(\alpha),1),(x_s(\alpha),2)\} \subseteq C$. However, this contradicts the definition of S_k. We conclude that X does not have a G_δ-diagonal. In particular X is not sequentially convergence complete.

Our last result in this section is related to Theorem 4.4 of [6]. Note that it shows that the Niemytzki plane admits a convergence complete quasi-metric [8, 5.17; 3, 3.9.D].

PROPOSITION 7. *Each Tychonoff orthocompact Čech complete space with a G_δ-diagonal admits a convergence complete quasi-metric.*

PROOF. Let X be a Tychonoff orthocompact Čech complete space with a G_δ-diagonal. Since X is a Čech complete Tychonoff space, there is a countable collection $\{\mathcal{G}_n : n \in \mathbf{N}\}$ of open covers of X such that whenever $\mathcal{F}$ is a collection of closed subsets of X that has the finite intersection property and that has the property that for each $n \in \mathbf{N}$ there is an $F_n \in \mathcal{F}$ such that $F_n \subseteq G_n$ for some $G_n \in \mathcal{G}_n$, then $\mathcal{F}$ has nonempty intersection. Since X has a G_δ-diagonal, there is a countable collection $\{\mathcal{H}_n : n \in \mathbf{N}\}$ of open covers of X so that whenever x, $y \in X$ and $x \neq y$ there is an $n \in \mathbf{N}$ such that $y \notin \mathrm{st}(x, \mathcal{H}_n)$. For each $n \in \mathbf{N}$ set $\mathcal{R}_n = \{R$ is open in $X : \overline{R} \subseteq (H \cap G)$ for some $H \in \mathcal{H}_n$ and some $G \in \mathcal{G}_n\}$. Since X is orthocompact (see e.g. [8, p. 100] for the definition of the notion of orthocompactness), for each $n \in \mathbf{N}$ there is an interior-preserving open refinement $\mathcal{P}_n$ of $\mathcal{R}_n$. For each $n \in \mathbf{N}$ and each $x \in X$ set $T_n(x) = \bigcap\{S \in \mathcal{P}_k : x \in S,\ k \in \mathbf{N}$ and $k \leqq n\}$.

Let $x \in X$. First let us show that $\{T_n(x) : n \in \mathbf{N}\}$ is a neighborhood base at x. Assume the contrary. Then there is an open neighborhood G of x such that $\mathcal{D} = \{\overline{T_n(x)} \setminus G : n \in \mathbf{N}\}$ is a closed filterbase on X. By the property of the collection $\{\mathcal{G}_n : n \in \mathbf{N}\}$ the filterbase $\mathcal{D}$ on X has a cluster point y in $X \setminus G$. By the property of the collection $\{\mathcal{H}_n : n \in \mathbf{N}\}$ we have that $\bigcap\{\overline{T_n(x)} : n \in \mathbf{N}\} = \{x\}$. Hence $y = x$ — a contradiction. Thus $\{T_n(x) : n \in \in \mathbf{N}\}$ is neighborhood base at x.

By Lemma 1 there is a quasi-metric p on X such that for each $n \in \mathbf{N}$ and each $x \in X$ we have that $B_n^p(x) = T_n(x)$. In particular p induces the topology on X.

Let us show that (X,p) is convergence complete. Because of Corollary 1 it suffices to show that each Cauchy sequence in (X,p) has exactly one cluster point in X. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (X,p). Then for each $n \in \in \mathbf{N}$ there are $y_n \in \mathbf{N}$ and $k_n \in \mathbf{N}$ such that $\{x_k : k \in \mathbf{N},\ k \geqq k_n\} \subseteq T_n(y_n)$. Clearly by the property of the collection $\{\mathcal{G}_n : n \in \mathbf{N}\}$ the sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point y in X. By the property of the collection $\{\mathcal{H}_n : n \in \mathbf{N}\}$ we get that $\bigcap\{\overline{T_n(y_n)} : n \in \mathbf{N}\} = \{y\}$. This shows that y is the only cluster point of the sequence $(x_n)_{n\in\mathbf{N}}$ in X. Hence (X,p) is convergence complete.

3.2. Construction of completions. Now we are ready to answer Problem U of [8]. We will give necessary and sufficent conditions in order that a quasi-metric space (X,p) has a quasi-metric sequential completion (quasi-metric sequential convergence completion, quasi-metric completion, quasi-metric convergence completion). Our characterizations of these spaces should be compared with the characterization (given in the beginning of section 3) of the T_1 quasi-uniform spaces having a (convergence) completion.

PROPOSITION 8. *Let p be a quasi-metric on a set X. Then the following conditions are equivalent:*

a) *The quasi-metric space (X,p) has a quasi-metric sequential completion (Y,q).*

b) *Whenever $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p) and $x\in X$ is a $\mathcal{T}(p^{-1})$-cluster point of the sequence $(x_n)_{n\in\mathbf{N}}$, then x is a $\mathcal{T}(p)$-cluster point of the sequence $(x_n)_{n\in\mathbf{N}}$.*

c) *The quasi-uniform space $(X,\mathcal{U}(p))$ has a completion.*

The proof of Proposition 8 is based on following technical lemma. For later use we formulate it in a fairly general form.

LEMMA 7. *Let p be a quasi-metric on a set X and let $Y=X\cup\mathcal{A}$ where X and $\mathcal{A}$ are disjoint. For each $x\in\mathcal{A}$ let be given a point $s(x)\in X$ and a decreasing sequence $(S_n(x))_{n\in\mathbf{N}}$ of nonempty subsets of X such that $S_1(x)=$ $=B_n^p(s(x))$. For each x, $y\in Y$ define $q(x,y)$ as follows:*

$$q(x,y)=\begin{cases} p(x,y) & \text{if } x,\ y\in X;\\ p(s(x),s(y))+2 & \text{if } x,\ y\in\mathcal{A},\ x\neq y;\\ 0 & \text{if } x,\ y\in\mathcal{A},\ x=y;\\ p(x,s(y))+3 & \text{if } x\in X,\ y\in\mathcal{A};\\ \inf\left\{\max\left\{p(S_n(x),y),\frac{1}{n}\right\}:n\in\mathbf{N}\right\} & \text{if } x\in\mathcal{A},\ y\in X.\end{cases}$$

Then q is a quasi-pseudo-metric on Y. Furthermore

$$B_m^q(x)=\left(\bigcup\{B_m^p(b):b\in S_{2^m+1}(x)\}\right)\cup\{x\}$$

for each $x\in\mathcal{A}$ and $m\in\mathbf{N}$.

If q satisfies the condition

(v) *$q(x,y)\neq 0$ whenever $x\in\mathcal{A}$ and $y\in X$,*

then q is a quasi-metric on Y.

PROOF. First we prove that q satisfies the following condition:

$$|q(x,y)-p(s(x),y)|\leqq 1 \text{ whenever } x\in\mathcal{A} \text{ and } y\in X.$$

In order to see this, let $x\in\mathcal{A}$ and $y\in X$. Note that for each $n\in\mathbf{N}$ and each $z\in S_n(x)$ we have that $p(s(x),y)\leqq p(s(x),z)+p(z,y)\leqq 1+p(z,y)$. Thus $p(s(x),y)-1\leqq\inf\left\{\max\left\{p(S_n(x),y),\frac{1}{n}\right\}:n\in\mathbf{N}\right\}=q(x,y)$. Furthermore observe that $q(x,y)\leqq\max\{p(B_1^p(s(x)),y),1\}\leqq p(s(x),y)+1$. Hence $|q(x,y)-p(s(x),y)|\leqq 1$.

It is obvious that q is a quasi-pseudo-metric on Y provided that q satisfies the triangle inequality. Let us verify that q fulfills this condition. To this end let x, y and z be elements of Y such that $x \neq z$ and $z \neq y$. It is straightforward to check that the inequality $q(x,y) \leqq q(x,z)+q(z,y)$ follows immediately from the definition of q or from the inequality proved above whenever z or y belong to $\mathcal{A}$. In order to give an example consider the case that $x \in \mathcal{A}$, $z \in \mathcal{A}$ and $y \in X$. Then $q(x,y) \leqq p(s(x),y)+1 \leqq p(s(x),s(z))+2+p(s(z),y)-1 \leqq \leqq q(x,z)+q(z,y)$. The remaining five cases are similar.

Therefore it remains to consider the case that z and y belong to X. This case is trivial if $x \in X$. Hence it suffices to study the case that $x \in \in \mathcal{A}$, $z \in X$ and $y \in X$. For convenience set $h = q(x,z)$. Let s be a real number such that $s > h$. By the definition of $q(x,z)$ there are an $n \in \mathbf{N}$ and a $b \in S_n(x)$ such that $1/n < s$ and $p(b,z) < s$. Hence $1/n < s+p(z,y)$ and $p(b,y) \leqq p(b,z)+p(z,y) < s+p(z,y)$. Thus by the definition of $q(x,y)$ we deduce that $q(x,y) \leqq h+p(z,y) = q(x,z)+q(z,y)$. We have shown that (Y,q) is a quasi-pseudo-metric space.

Since the last assertion made in Lemma 7 is obvious, only the set-theoretic equality remains to be verified. Suppose that $x \in \mathcal{A}$ and $m \in \mathbf{N}$. Let $y \in B^p_m(b)$ for some $b \in S_{2^m+1}(x)$. Set $n = 2^m+1$. Then $\max\left\{\frac{1}{n}, p(b,y)\right\} < < 2^{-m}$ and $b \in S_n(x)$. Therefore $y \in B^q_m(x)$. On the other hand let $y \in B^q_m(x)$ such that $y \neq x$. Then $q(x,y) < 2^{-m}$. There are a positive integer n and $b \in \in S_n(x)$ such that $1/n < 2^{-m}$ and $p(b,y) < 2^{-m}$. Hence $y \in B^p_m(b)$ and $n > 2^m$. We conclude that $y \in B^p_m(b)$ for some $b \in S_n(x) \subseteq S_{2^m+1}(x)$.

Proof of Proposition 8. We recall that a T_1 quasi-uniform space $(X,\mathcal{U})$ has a completion if and only if whenever $\mathcal{F}$ is a $\mathcal{U}$-Cauchy filter on X and $x \in X$ is a $\mathcal{T}(\mathcal{U}^{-1})$-cluster point of $\mathcal{F}$, then x is a $\mathcal{T}(\mathcal{U})$-cluster point of $\mathcal{F}$ [8, Theorem 3.43]. Using this result we wish to see first that conditions b) and c) are equivalent.

c)$\to$b): This is an obvious consequence of the result just cited.

b)$\to$c): Let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy filter on X and let $x \in X$ be a $\mathcal{T}(p^{-1})$-cluster point of $\mathcal{F}$. By the result mentioned above it suffices to show that x is a $\mathcal{T}(p)$-cluster point of $\mathcal{F}$. Assume the contrary. Then there exists a $\mathcal{T}(p)$-closed subset F of X belonging to $\mathcal{F}$ such that $x \notin F$. Since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ of points of X such that $B^p_n(x_n) \in \mathcal{F}$ whenever $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ choose $y_n \in B^{p^{-1}}_n(x) \cap F \cap \cap \left(\bigcap\{B^p_k(x_k) : k = 1,\dots,n\}\right)$. By condition b) the point x is a $\mathcal{T}(p)$-cluster point of the Cauchy sequence $(y_n)_{n\in\mathbf{N}}$ in (X,p). Hence x belongs to F — a contradiction. We conclude that x is a $\mathcal{T}(p)$-cluster point of $\mathcal{F}$. Therefore condition c) is satisfied.

We finish the proof by showing that conditions a) and b) are equivalent.

a)$\to$b): Assume that $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p) and that $x \in X$ is a $\mathcal{T}(p^{-1})$-cluster point of $(x_n)_{n\in\mathbf{N}}$ in X. Then there is a subsequence $(z_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that $x \in B^p_n(z_n)$ whenever $n \in \mathbf{N}$. Since

$(z_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in the sequentially complete quasi-metric space (Y,q), the sequence $(z_n)_{n\in\mathbf{N}}$ has a $\mathcal{T}(q)$-cluster point r in Y. Then we have that $\lim\limits_{n\to\infty} q(r,y_n)=0$ for some subsequence $(y_n)_{n\in\mathbf{N}}$ of $(z_n)_{n\in\mathbf{N}}$. Moreover $\lim\limits_{n\to\infty} q(y_n,x)=0$. Since q is a quasi-metric on Y, $r=x\in X$. Hence x is a cluster point of $(x_n)_{n\in\mathbf{N}}$ in (X,p). Therefore (X,p) satisfies condition b).

b)$\to$a): In order to prove this implication we assume that (X,p) is a quasi-metric space that satisfies condition b). We will construct a quasi-metric sequential completion of (X,p). Let $\mathcal{A}=\{x: x$ is a Cauchy sequence in (X,p) without $\mathcal{T}(p)$-cluster point in $X\}$.

Set $Y=X\cup\mathcal{A}$. Let $x=(x_n)_{n\in\mathbf{N}}\in\mathcal{A}$. Choose a sequence $([x](n))_{n\in\mathbf{N}}$ of points in X and a strictly increasing sequence $(k_n(x))_{n\in\mathbf{N}}$ of positive integers such that $\{x_k: k\in\mathbf{N},\ k\geqq k_n(x)\}\subseteqq B_n^p([x](n))$. Set $S_n(x)=\{x_k: k\in\mathbf{N},\ k\geqq k_n(x)\}$ whenever $n\in\mathbf{N}\setminus\{1\}$, $S_1(x)=B_1^p([x](1))$ and $s(x)=[x](1)$. Clearly, $(S_n(x))_{n\in\mathbf{N}}$ is a decreasing sequence of nonempty subsets of X. Hence we can use Lemma 7 to construct a quasi-metric q on Y, provided that q satisfies condition (v):

Assume that $y\in X$ and $x=(x_n)_{n\in\mathbf{N}}\in\mathcal{A}$ such that $q(x,y)=0$. We have that $\lim\limits_{n\to\infty} p(y_n,y)=0$ for some subsequence $(y_n)_{n\in\mathbf{N}}$ of the Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ in X. Because of condition b) the point y is a cluster point of the Cauchy sequence $(x_n)_{n\in\mathbf{N}}$ in (X,p) — a contradiction, since $x\in\mathcal{A}$. Hence (Y,q) is a quasi-metric space.

Note that the set-theoretic equality proved in Lemma 7 shows that for each $x=(x_n)_{n\in\mathbf{N}}\in\mathcal{A}$ the sequence $(x_n)_{n\in\mathbf{N}}$ of points of Y converges to the point x in Y. In particular X is dense in Y.

Finally we want to show that (Y,q) is sequentially complete. Let $(z_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (Y,q). Consider the $\mathcal{U}(q)$-Cauchy filter $\mathcal{F}$ generated by $\{\{z_k: k\in\mathbf{N},\ k\geqq n\}|n\in\mathbf{N}\}$ on Y. For each $n\in\mathbf{N}$ there is a $y_n\in Y$ such that $B_n^q(y_n)\in\mathcal{F}$. If $X\notin\mathcal{F}$, then $\bigcap\{B_k^q(y_k): k=1,\dots,n\}\setminus X\neq\emptyset$ for each $n\in\mathbf{N}$. Hence $y_n\in\mathcal{A}$ and $y_n=y_1$ whenever $n\in\mathbf{N}$. Thus $\mathcal{F}$ (and, therefore, $(z_n)_{n\in\mathbf{N}}$) converge to y_1.

It remains to consider the case that $X\in\mathcal{F}$. In this case we want to show that the filter $\{F\cap X: F\in\mathcal{F}\}$ is a $\mathcal{U}(p)$-Cauchy filter on the subspace X of Y. Assume the contrary. Then clearly there is a $k\in\mathbf{N}$ such that $y_n\in\mathcal{A}$ whenever $n\in\mathbf{N}$ and $n\geqq k$. Let $s\in\mathbf{N}$ and let $h\in\mathbf{N}$ such that $h>\max\{k,s\}$. By Lemma 7 $(X\cap B_h^q(y_h))\subseteqq\bigcup\{B_h^p(z): z\in S_h(y_h)\}\subseteqq B_{h-1}^p([y_h](h))\subseteqq B_s^p([y_h](h))$, because $S_h(y_h)\subseteqq B_h^p([y_h](h))$. Since $X\cap B_h^q(h_y)\in\mathcal{F}$, we have that $B_s^p([y_h](h))\in\mathcal{F}$. Hence for each $s\in\mathbf{N}$ there is a $w_s\in X$ such that $B_s^p(w_s)\in\mathcal{F}$ — a contradiction. We conclude that, in the case under consideration, $\{F\cap X: F\in\mathcal{F}\}$ is a $\mathcal{U}(p)$-Cauchy filter on the subspace X of Y.

Hence there is a $j\in\mathbf{N}$ such that $(z_n)_{n\geqq j}$ is a Cauchy sequence in the subspace X of Y. If the sequence $(z_n)_{n\geqq j}$ does not have a cluster point in (X,p), then by definition $z:=(z_n)_{n\geqq j}\in\mathcal{A}$ and, thus, the sequence $(z_n)_{n\in\mathbf{N}}$ in Y has the limit point $z\in Y$ by an observation made above. Otherwise, obviously, $(z_n)_{n\in\mathbf{N}}$ has a cluster point in Y.

We conclude that each Cauchy sequence in (Y,q) has a cluster point in (Y,q). Hence (Y,q) is sequentially complete.

Next we wish to characterize the quasi-metric spaces that have a quasi-metric sequential convergence completion.

PROPOSITION 9. *Let p be a quasi-metric on a set X. For each $n\in\mathbf{N}$ set $\mathcal{B}_n=\{B^p_n(x):x\in X\}$. Then the following conditions are equivalent:*

a) *(X,p) has a quasi-metric sequential convergence completion.*

b) *The collection $\{\mathcal{B}_n:n\in\mathbf{N}\}$ is a subdevelopment for X.*

c) *(X,p) has a quasi-metric convergence completion.*

d) *The quasi-uniform space $(X,\mathcal{U}(p))$ has a convergence completion.*

PROOF. Again one equivalence is nearly obvious.

d)$\to$b) and b)$\to$d): It is straightforward to check that the condition "Each nonconvergent $\mathcal{U}(p)$-Cauchy filter on X is free" and condition b) are equivalent. Hence the two implications follow from the result [2, Theorem 3.9 and Lemma 3.8] (cited in the introduction to section 3) that a T_1 quasi-uniform space $(X,\mathcal{U})$ has a convergence completion if and only if each fixed $\mathcal{U}$-Cauchy filter on X converges in $(X,\mathcal{U})$.

We are going to show now that the remaining conditions are equivalent to condition b).

a)$\to$b): Let (X,p) be a quasi-metric space that has a quasi-metric sequential convergence completion (Y,q). For each $n\in\mathbf{N}$ set $\mathcal{C}_n=\{B^q_n(y):y\in Y\}$. Then $\{\mathcal{C}_n:n\in\mathbf{N}\}$ is a subdevelopment for Y by the proof of Lemma 6. Since (X,p) is a subspace of (Y,q), it is clear that $\{\mathcal{B}_n:n\in\mathbf{N}\}$ is a subdevelopment for X.

b)$\to$c): Let (X,p) be a quasi-metric space such that $\{\mathcal{B}_n:n\in\mathbf{N}\}$ is a subdevelopment for X. We construct a quasi-metric convergence completion of (X,p). Set $\mathcal{A}=\{x:x \text{ is a nonconvergent } \mathcal{U}(p)\text{-Cauchy filter on } X\}$ and $Y=X\cup\mathcal{A}$. Let $x\in\mathcal{A}$. For each $n\in\mathbf{N}$ choose $[x](n)\in X$ such that $B^p_n([x](n))\in x$. Set $s(x)=[x](1)$ and for each $n\in\mathbf{N}$ set $S_n(x)=\bigcap\{B^p_k([x](k)):k=1,\dots,n\}$. We wish to construct a quasi-metric q on Y according to Lemma 7. It remains to check that q satisfies condition (v).

Assume the contrary. Hence $q(x,y)=0$ for some $y\in X$ and some $x\in\mathcal{A}$. By the definition of q there is a sequence $(z_n)_{n\in\mathbf{N}}$ in X such that $z_n\in\bigcap\{B^p_k([x](k)):k=1,\dots,n\}$ and such that $y\in B^p_n(z_n)$ whenever $n\in\mathbf{N}$. Hence $y\in B^p_{n-1}([x](n))$ for each $n\in\mathbf{N}\setminus\{1\}$. Since $\{\mathcal{B}_n:n\in\mathbf{N}\}$ is a subdevelopment for X, we conclude that the filter x converges to y. Since $x\in\mathcal{A}$, we have reached a contradiction. Hence (Y,q) defined as in Lemma 7 is a quasi-metric space.

It remains to show that (Y,q) is a convergence completion of (X,p). Let $x\in\mathcal{A}$. Since, according to Lemma 7, $S_{2^m+1}(x)\subseteq B^q_m(x)$ for each $m\in\mathbf{N}$, the filterbase x on Y converges to the point x in Y. In particular X is dense in Y.

Let $\mathcal{F}$ be a $\mathcal{U}(q)$-Cauchy filter on Y. Assume that $\mathcal{F}$ does not converge in Y. The argument given in the corresponding part of the proof of the last proposition shows that $X \in \mathcal{F}$ and that $\mathcal{H} := \{F \cap X : F \in \mathcal{F}\}$ is a $\mathcal{U}(p)$-Cauchy filter on the subspace X of Y. Since $\mathcal{F}$ does not converge in Y, we have that $\mathcal{H} \in \mathcal{A}$. Then, clearly $\mathcal{F}$ converges to $\mathcal{H} \in Y$ — a contradiction. Hence we conclude that each $\mathcal{U}(q)$-Cauchy filter on Y converges in Y. Thus (Y, q) is a quasi-metric convergence completion of (X, p).

c)$\to$a): By Lemma 2 each convergence complete quasi-pseudo-metric space is sequentially convergence complete.

Proposition 9 naturally suggests the question whether a quasi-metric space (X, p) has a quasi-metric completion whenever it has a quasi-metric sequential completion. The final paragraphs of this paper are used to point out the set-theoretic nature of this question.

PROPOSITION 10. *Let p be a quasi-metric on a set X. Then the following conditions are equivalent:*

a) *The quasi-metric space (X, p) has a quasi-metric completion.*

b) *The quasi-uniformity $\mathcal{U}(p)$ has a quasi-uniform completion with a countable base.*

c) *Each $\mathcal{U}(p)$-Cauchy ultrafilter $\mathcal{F}$ on X without limit point in (X, p) has a countable subcollection of $\mathcal{T}(p^{-1})$-closed subsets of X with empty intersection.*

PROOF. a)$\to$b): Let (X, p) be a quasi-metric space that has a quasi-metric completion (Y, q). The quasi-uniformity $\mathcal{U}(q)$ satisfies the necessary conditions.

b)$\to$c): Let $(Y, \mathcal{V})$ be a quasi-uniform completion of $(X, \mathcal{U}(p))$ (where we suppose that $(X, \mathcal{U}(p))$ is a dense subspace of $(Y, \mathcal{V})$) with a countable base $\{V_n : n \in \mathbf{N}\}$ and let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy ultrafilter on X that does not have a limit point in (X, p). Since $(Y, \mathcal{V})$ is complete, the $\mathcal{V}$-Cauchy filterbase $\mathcal{F}$ on Y has a cluster point y in $(Y, \mathcal{V})$. For each $n \in \mathbf{N}$ set $F_n = \mathrm{cl}_{\mathcal{T}(p^{-1})}(V_n(y) \cap X)$. Then $(V_n(y) \cap X) \subseteqq F_n \subseteqq (V_n^2(y) \cap X)$ whenever $n \in \mathbf{N}$. Thus $\bigcap\{F_n : n \in \mathbf{N}\} = \emptyset$, because $(Y, \mathcal{T}(\mathcal{V}))$ is a T_1 space. Since y is a cluster point of the filterbase $\mathcal{F}$ on $(Y, \mathcal{V})$, it is clear that $\{F_n : n \in \mathbf{N}\}$ is a subcollection of the ultrafilter $\mathcal{F}$ on X.

c)$\to$a): We want to contruct a quasi-metric completion (Y, q) of (X, p). Set $\mathcal{A} = \{x : x \text{ is a } \mathcal{U}(p)\text{-Cauchy ultrafilter on } X \text{ without limit point in } (X, p)\}$ and $Y = X \cup \mathcal{A}$. Let $x \in \mathcal{A}$. Then for each $n \in \mathbf{N}$ there is an $[x](n) \in X$ such that $B_n^p([x](n)) \in x$. By our assumption, x has a countable subcollection $\{F_k(x) : k \in \mathbf{N}\}$ of $\mathcal{T}(p^{-1})$-closed subsets of X with empty intersection. Set $s(x) = [x](1)$, $S_1(x) = B_1^p([x](1))$ and $S_n(x) = \bigcap\{F_k(x) \cap B_k^p([x](k)) : k = 1, \ldots, n\}$ whenever $n \in \mathbf{N} \setminus \{1\}$. Then $(S_n(x))_{n \in \mathbf{N}}$ is a decreasing sequence of nonempty subsets of X. We define a distance function q on Y as in Lemma 7. Let us check that q satisfies condition (v):

Assume that $q(x, y) = 0$ for some $x \in \mathcal{A}$ and some $y \in X$. Then there is a sequence $(z_n)_{n\in\mathbf{N}}$ in X such that $\lim_{n\to\infty} p(z_n, y) = 0$ and such that $z_n \in \in \bigcap\{F_k(x) : k = 1, \ldots, n\}$ for each $n \in \mathbf{N}$. Since $F_n(x)$ is $\mathcal{T}(p^{-1})$-closed whenever $n \in \mathbf{N}$, we have that $y \in \bigcap\{F_n(x) : n \in \mathbf{N}\}$ — a contradiction. Thus (Y, q) is a quasi-metric space.

Let $x \in \mathcal{A}$. Since by Lemma 7 we have that $S_{2^m+1}(x) \subseteq B^q_m(x)$ for each $m \in \mathbf{N}$, the filterbase x on Y converges to the point x in Y. In particular X is dense in Y.

It remains to show that (Y, q) is complete. Let $\mathcal{F}$ be a $\mathcal{U}(q)$-Cauchy ultrafilter on Y. We assume that $\mathcal{F}$ does not converge in (Y, q). The same arguments as in the corresponding part of the proof of Proposition 9 show that this assumption is incorrect. (In particular note that in our case the filter $\mathcal{H}$ on X is an ultrafilter.) Hence each $\mathcal{U}(q)$-Cauchy filter on Y has a cluster point in (Y, q). Thus (Y, q) is complete.

Recall that a metric space X is realcompact if and only if every closed discrete subspace of X is of non-measurable cardinality [e.g. 3, 5.5.10 and 11, 12.2].

PROPOSITION 11. *Let p be a quasi-metric on a set X such that the metric space (X, p^*) is realcompact. Then the quasi-metric space (X, p) has a quasi-metric sequential completion if and only if it has a quasi-metric completion.*

PROOF. Let p be a quasi-metric on a set X such that the metric space (X, p^*) is realcompact and such that (X, p) has a quasi-metric sequential completion. It suffices to show that (X, p) has a quasi-metric completion. Let $\mathcal{F}$ be a $\mathcal{U}(p)$-Cauchy ultrafilter on X without a $\mathcal{T}(p)$-limit point in X. The ultrafilter $\mathcal{F}$ on X has a countable subcollection $\{Z_n : n \in \mathbf{N}\}$ of $\mathcal{T}(p^*)$-zero-sets such that $\bigcap\{Z_n : n \in \mathbf{N}\} = \emptyset$, because $\mathcal{F}$ does not have a $\mathcal{T}(p^*)$-limit point in X and $\mathcal{T}(p^*)$ is realcompact [11, 10M.2].

Since $\mathcal{F}$ is a $\mathcal{U}(p)$-Cauchy filter on X, for each $n \in \mathbf{N}$ there is a $y_n \in X$ such that $B^p_n(y_n) \in \mathcal{F}$. For each $n \in \mathbf{N}$ set $H_n = \bigcap\{Z_k \cap B^p_k(y_k) : k = 1, \ldots, n\}$. Then $H_n \in \mathcal{F}$ whenever $n \in \mathbf{N}$.

Assume that there is an $x \in \bigcap\{\mathrm{cl}_{\mathcal{T}(p^{-1})} H_n : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$ choose $z_n \in B^{p^{-1}}_n(x) \cap H_n$. Then $x \in B^p_n(z_n)$ for each $n \in \mathbf{N}$. Thus x is a $\mathcal{T}(p^{-1})$-cluster point of $(z_n)_{n\in\mathbf{N}}$. Moreover $(z_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X, p). Hence according to Proposition 8 we get that x is a cluster point of the sequence $(z_n)_{n\in\mathbf{N}}$ in (X, p), because (X, p) has a quasi-metric sequential completion. We deduce that $x \in \mathrm{cl}_{\mathcal{T}(p^*)} H_n$ for each $n \in \mathbf{N}$. Thus $x \in \mathrm{cl}_{\mathcal{T}(p^*)} H_n \subseteq \subseteq Z_n$ whenever $n \in \mathbf{N}$ — a contradiction. Hence $\bigcap\{\mathrm{cl}_{\mathcal{T}(p^{-1})} H_n : n \in \mathbf{N}\} = \emptyset$. Furthermore $\{\mathrm{cl}_{\mathcal{T}(p^{-1})} H_n : n \in \mathbf{N}\}$ is a subcollection of $\mathcal{F}$. By Proposition 10 we see that (X, p) has a quasi-metric completion.

Our last example shows that we cannot omit the cardinality assumption implicitly contained in Proposition 11.

EXAMPLE 8. A quasi-metric space (X,p) that has a quasi-metric sequential completion, but does not have a quasi-metric completion.

Let X be a set of measurable cardinality. Then there exists a free ultrafilter $\mathcal{P}$ on X with the countable intersection property (see e.g. [11, 12.2]). Let $(x_n)_{n\in\mathbf{N}}$ be an injective sequence of points of X and let $(U_n)_{n\in\mathbf{N}}$ be a strictly decreasing sequence of subsets of X belonging to $\mathcal{P}$ such that $x_n \notin U_1$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ and each $x \in X$ define $T_n(x)$ as follows:

$$T_n(x) = \begin{cases} \{x\} & \text{if } x \in (X \setminus \{x_k, k \in \mathbf{N}\}); \\ U_n \cup \{x\} & \text{if } x = x_k,\ k \in \mathbf{N},\ k \geqq n; \\ \{x\} & \text{if } x = x_k,\ k \in \mathbf{N},\ k < n. \end{cases}$$

By Lemma 1 there is a quasi-metric p on X such that $B_n^p(x) = T_n(x)$ whenever $n \in \mathbf{N}$ and $x \in X$. Clearly $\mathcal{P}$ is a $\mathcal{U}(p)$-Cauchy filter on X without cluster point in X. Since $\mathcal{P}$ has the countable intersection property, (X,p) does not have a quasi-metric completion by Proposition 10.

We want to show that (X,p) has a quasi-metric sequential completion. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in (X,p) such that $x \in X$ is a $\mathcal{T}(p^{-1})$-cluster point of $(x_n)_{n\in\mathbf{N}}$. Then there is a subsequence $(a_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that $x \in B_n^p(a_n)$ for each $n \in \mathbf{N}$. By Proposition 8 in order to show that (X,p) has a quasi-metric sequential completion it will suffice to prove that $(a_n)_{n\in\mathbf{N}}$ is eventually constant, because then, of course, x is a $\mathcal{T}(p)$-cluster point of $(a_n)_{n\in\mathbf{N}}$ and, thus, of $(x_n)_{n\in\mathbf{N}}$. Assume that $(a_n)_{n\in\mathbf{N}}$ is not eventually constant. Since $(a_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,p), for each $n \in \mathbf{N}$ there are $y_n \in X$ and $h_n \in \mathbf{N}$ such that $\{a_k : k \in \mathbf{N},\ k \geqq h_n\} \subseteqq B_n^p(y_n)$. Then for each $n \in \mathbf{N}$ there is a $k_n \in \mathbf{N}$ such that $k_n \geqq n$ and $y_n = x_{k_n}$. In particular the sequence $(y_n)_{n\in\mathbf{N}}$ cannot be constant. Hence there is an $s \in \mathbf{N}$ such that $\{a_n : n \in \mathbf{N},\ n \geqq s\} \subseteqq (X \setminus \{x_n : n \in \mathbf{N}\})$. Since $x \in B_n^p(a_n)$ for each $n \in \mathbf{N}$, we conclude by the definition of the sets $T_n(a_n)$ that, nevertheless, $(a_n)_{n\in\mathbf{N}}$ is eventually constant — a contradiction.

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(Received July 27, 1988; revised November 18, 1988)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BERNE
SIDLERSTRASSE 5
3012 BERNE
SWITZERLAND

Acta Math. Hung.
59(1-2) (1992), 147–149.

A NONSTANDARD RESULT ABOUT PATH CONTINUITY

J. MASTERSON (East Lansing)

The purpose of this note is to show that the nonstandard characterization of approximate continuity given in [4] by Wattenberg holds more generally in the context of "continuity paths" described by Bruckner, O'Malley and Thomson [3].

It is assumed that the reader is familiar with the principal elementary tools of nonstandard analysis: the *transfer principle, internality* and *concurrence.* [1] is a readily understandable reference. Beyond these, the more powerful extension of concurrence called *saturation* is of critical importance. [2] (page 27) illucidates this sufficiently.

Let $\mathcal{M}$ denote a higher order structure including $\mathbf{R}$ and its Lebesgue measure space, and ${}^*\mathcal{M}$ a $\varkappa$-saturated nonstandard extension of $\mathcal{M}$. With $\varkappa$ assumed to be sufficiently large Wattenberg made the following definition.

DEFINITION 1. $x \in {}^*[0,1]$ is *negligible* in case there is a standard set $A \subset [0,1]$ so that

(i) $\mathrm{st}(x)$ is a point of dispersion for A,

(ii) $x \in {}^*A$.

Let $\mathcal{N}$ denote the set of negligible points of ${}^*[0,1]$. Then, Wattenberg proved the following:

THEOREM. *Suppose $f:[0,1]\to\mathbf{R}$ and $x\in[0,1]$. Then f is approximately continuous at x if and only if for each $t\in\mu(x)\setminus\mathcal{N}$, ${}^*f(t)\sim{}^*f(x)$.*

In some sense, then, the set $\mathcal{N}$ universally selects from each monad the points eliminated from consideration by the local selection of a set of density 1 at each of the points of $[0,1]$.

A proof is not supplied since an examination of the result in [4] indicates that it holds more generally in the setting of continuity *paths* described by Bruckner, O'Malley and Thomson in [3].

DEFINITION 2. Let $x\in\mathbf{R}$

(1) A *path leading to* x is a set $E_x\subset\mathbf{R}$ such that $x\in E_x$ and x is an accumulation point of E_x.

(2) A *system of paths at* x, ε_x, is a collection of paths leading to x.

(3) A *system of paths* is a collection $\varepsilon=\bigcup\{\varepsilon_x : x\in\mathbf{R}\}$.

The properties listed below are easily shown to hold for several of the well-known systems of paths studied in [3], including the ordinary type, the $(1,1)$-density type and the qualitative type. They are sufficient to give us the generalization referred to above. The (1, 1)-density type is the result in [4].

DEFINITION 3. A *system of paths* will be called *locally thick* in case the following conditions are satisfied.

(i) If $E_{x^{(1)}}$ and $E_{x^{(2)}}$ are in ε_x, there is $E_{x^{(3)}}$ in ε_x so that $E_{x^{(3)}} \subset E_{x^{(1)}} \cap \cap E_{x^{(2)}}$.

(ii) If $E_x \in \varepsilon_x$, then for $\eta > 0$, $E_x \cap (x-\eta, x+\eta) \in \varepsilon_\chi$.

(iii) If $E_{x^n} \in \varepsilon_x$ for $n \in \mathbf{N}$, there is decreasing sequence $\langle c_n \rangle \to 0$ so that $\bigcup_{n=1}^{\infty} (J_n \cap E_{x^n}) \cup \{x\} \in \varepsilon_x$ where $J_n = (x - c_n, x - c_{n+1}] \cup [x + c_{n+1}, x + c_n)$.

DEFINITION 4. A function $f: \mathbf{R} \to \mathbf{R}$ is said to be ε-continuous at x in case $\exists E_x \in \varepsilon_x$ so that $\lim_{\substack{y \to x \\ y \in E_x}} f(y) = f(x)$.

For the remainder of this section, the family ε will be assumed to be locally thick. Also, D^c denotes the complement of D in $\mathbf{R}$.

THEOREM 1. *$f: \mathbf{R} \to \mathbf{R}$ is ε-continuous at x if and only if for each $\varepsilon > 0$, $\{y : |f(y) - f(x)| \geqq \varepsilon\} \subset (E_x)^c$ for some $E_x \in \varepsilon_x$.*

PROOF. Suppose f is ε-continuous at x. Then, there is $E_x \in \varepsilon_x$ so that: for each $\varepsilon > 0$ there is $\delta > 0$ so that $\{y : |f(y) - f(x)| \geqq \varepsilon\} \subset [E_x \cup (x - \delta, x + + \delta)]^c$. Property (ii) of Definition 3 produces $F_x \in \varepsilon_x$ so $\{y : |f(y) - f(x)| \geqq \geqq \varepsilon\} \subset (F_x)^c$.

Suppose for $\varepsilon > 0$, $\{y : |f(y) - f(x)| \geqq \varepsilon\} \subset (E_x)^c$. Given $n \in \mathbf{N}$, there is $E_{x^n} \in \varepsilon_x$ so $E_{x^n} \subset \{y : |f(y) - f(x)| < 1/n\}$. By Property (iii) of Definition 4, there is $\langle c_n \rangle \to 0$ so $F_x = (E_{x^n} \cap J_n) \cup \{x\} \in \varepsilon_x$. It is easily computed that $\lim_{\substack{y \to x \\ y \in F_x}} f(y) = f(x)$.

For the system of paths ε we define the nonstandard set $\mathcal{N}(\varepsilon)$ in ${}^*\mathbf{R}$.

DEFINITION 5. $\mathcal{N}(\varepsilon) = \{z \in {}^*\mathbf{R}$ and $E_x \in \varepsilon_x$ so that $x = st(z)$ and $z \in \in {}^*(E_x)^c\}$. $\mathcal{N}(\varepsilon)$ is called the ε-*negligible* elements of ${}^*\mathbf{R}$.

THEOREM 2. *$f: \mathbf{R} \to \mathbf{R}$ is ε-continuous at x if and only if for every $y \in \mu(x) \setminus \mathcal{N}(\varepsilon)$, ${}^*f(y) \sim {}^*f(x)$.*

The proof is by way of the following lemma.

LEMMA. *Suppose $A \subset \mathbf{R}$ and $x \in \mathbf{R}$. Then $A \subset (E_x)^c$ for some $E_x \in \varepsilon_x \Leftrightarrow \Leftrightarrow \mu(x) \cap {}^*A \subset \mathcal{N}(\varepsilon)$.*

PROOF. Suppose first that A is not a subset of $(E_x)^c$ for any $E_x \in \varepsilon_x$. By Property (ii), for $\eta > 0$ and any $E_x \in \varepsilon_x$, $A \cap E_x \cap (x - \eta, x + \eta) \neq \emptyset$. The transfer principle gives us ${}^*A \cap {}^*E_x \cap {}^*(x - \varrho, x + \varrho) \neq \emptyset$ for ϱ infinitesimal, and hence ${}^*A \cap {}^*E_x \cap \mu(x) \neq \emptyset$. Let r be the relation defined on $\varepsilon_x \times \mu(x)$ by $(E_x, y) \in r$ in case $E_x \in \varepsilon_x$ and $y \in {}^*E_x \cap {}^*A \cap \mu(x)$. We show that r is a concurrent, nonstandard relation. If $\{E_{x^1}, \ldots, E_{x^n}\}$ is a finite subset of ε_x, there is by Property (i) an $E_x \subset \bigcap_{k=1}^{n} E_{x^k}$. Pick $y \in {}^*E_x \cap {}^*A \cap \mu(x)$. Then $(E_{x^k}, y) \in r$ for $k = 1, \ldots, n$ and the concurrence of r is established. The saturation of our model now gives a $z \in \mu(x)$ so that $(E_x, z) \in r$ for all $E_x \in \in \varepsilon_x$. But then $z \in \mu(x) \cap {}^*A$ but $z \notin (E_x)^c$ for any $E_x \in \varepsilon_x$. So, $z \notin \mathcal{N}(\varepsilon)$ and the conclusion is negated.

Assume now $A \subset (E_x)^c$ for some $E_x \in \varepsilon_x$. Let $z \in \mu(x) \cap {}^*A$. Then $x = = \mathrm{st}(z)$ and $z \in {}^*(E_x)^c$, hence $z \in \mathcal{N}(\varepsilon)$. So $\mu(x) \cap {}^*A \subset \mathcal{N}(\varepsilon)$ and the proof of the lemma is complete.

Proof of Theorem. By Theorem 1, f is ε-continuous at $x \Leftrightarrow \forall \varepsilon > 0$, $\{y : |f(y) - f(x)| \geqq \varepsilon\} \subset (E_x)^c$ for some $E_x \in \varepsilon_x \Leftrightarrow \forall \varepsilon > 0$, $\mu(x) \cap \{y : |{}^*f(y) - - {}^*f(x)| \geqq \varepsilon\} \subset \mathcal{N}(\varepsilon)$ (by above lemma). We need only show this latter statement is equivalent to the conclusion of the theorem.

Assume first the latter statement and let $y \in \mu(x) \sim \mathcal{N}(\varepsilon)$. Then for each real $\varepsilon > 0$, $|{}^*f(y) - {}^*f(x)| < \varepsilon$, hence ${}^*f(y) \sim {}^*f(x)$.

On the other hand, assume the conclusion of the theorem. For any $\varepsilon > 0$, let $z \in \mu(x)) \cap \{y : |{}^*f(y) - {}^*f(x)| \geqq \varepsilon\}$. If $z \notin \mathcal{N}(\varepsilon)$ then ${}^*f(z) \sim {}^*f(x)$, contradicting the choice of z. So $z \in \mathcal{N}(\varepsilon)$, verifying the latter statement.

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(Received August 2, 1988)

MICHIGAN STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
EAST LANSING, MICHIGAN 48824--1027
USA

Acta Math. Hung.
59(1-2) (1992), 151–157.

REMARKS ON THE OPTIMAL CONTROL PROBLEM FOR A STRONGLY NON LINEAR HYPERBOLIC SYSTEM

J. E. MUÑOZ RIVERA (Rio de Janeiro)

§ 1. Introduction

Let Ω be an open bounded set of $\mathbf{R}^n$ with smooth boundary Γ; set $Q = \Omega\times]0,T[$ and $\Sigma=\Gamma\times]0,T[$. In [1], the existence of a singular system of optimality for the optimal control problem was treated, when the state is defined by

$$(1.1)\qquad \begin{cases} z''-\Delta z-f(z)=v & \text{in } D'(Q)\\ z(x,t)=0 & \text{on } \Sigma\\ z(0)=y_0,\ z'(0)=y_1 & \text{in } \Omega \end{cases}$$

and the cost function is given by

$$(1.2)\qquad J(v,z)=\|f(z)-z_d\|_Q^2+N\|v\|_Q^2,$$

where $\|.\|_Q$ is the $L^2(Q)$-norm, and $N>0$. We consider (1.1) as a set of restrictions for the couple (v,z). Hence the admissible set is

$$(1.3)\qquad X_{\mathrm{ad}}=\{(v,z)\in L^2(Q)\times W;(v,z) \text{ satisfy (1.1) and } f(z)\in L^2(Q)\}$$

where $W=\{z\in L^\infty(0,T;H_0^1(\Omega));\ z'\in L^\infty(0,T;L^2(\Omega))\}$. Let us denote by $\|.\|_W$ the norm of W defined by $\|z\|_W=\|z\|_{L^\infty(0,T;H_0^1)}+\|z'\|_{L^\infty(0,T;L^2)}$. Komornik and Tiba showed that there exists at least one solution for

$$(1.4)\qquad J(u,y)=\inf\{J(v,z);(v,z)\in X_{\mathrm{ad}}\},\quad (v,z)\in X_{\mathrm{ad}}$$

and in order to prove the existence of a singular system of optimality they imposed the two following conditions to $f\in\mathbf{C}^1$

$$(1.5)\qquad c_0=\sup_{s\in\mathbf{R}}\frac{\|f'(s)\|}{1+\|f(s)\|}<\infty,$$

$$(1.6)\qquad \sup_{\substack{s,t\in\mathbf{R}\\ \|s-t\|<1}}\frac{\|f(s)\|}{1+\|f(t)\|}<\infty.$$

The main result of this paper is to prove that assumption (1.6) is not necessary, that is, we will prove that there exists an optimality system when $f\in C^1$ satisfies only condition (1.5).

§ 2. Approximated problem

Let us consider the following approximated η-system for the sequence of continuous functions $(F_\eta)_{\eta\in\mathbf{N}}$:

$$(2.1)\qquad \begin{cases} z'' - \Delta z - F_\eta(z) = v & \text{in } Q \\ z(x,t) = 0 & \text{on } \Sigma \\ z(0) = y_0,\, z'(0) = y_1 & \text{in } \Omega \end{cases}$$

and let us define the η-functional for the sequence $(F_\eta)_{\eta\in\mathbf{N}}$ as

$$(2.3)\qquad J_\eta(v,z) = \|F_\eta(z) - z_d\|_Q^2 + N\|v\|_Q^2 + \|z-y\|_Q^2 + \|v-u\|_Q^2.$$

The couple (u,y) is the solution of problem (1.4). Let us denote by $X_{\mathrm{ad}}^\eta = \{(v,z)\in L^2(Q)\times W;\ (v,z)$ satisfies (2.1) and $f(z)\in L^2(Q)\}$. Then the η-problem for the sequence $(F_\eta)_{\eta\in\mathbf{N}}$ is given by

$$(2.3)\qquad J_\eta(u_\eta, y_\eta) = \inf\{J_\eta(v,z); (v,z)\in X_{\mathrm{ad}}\},\quad (u_\eta,y_\eta)\in X_{\mathrm{ad}}^\eta.$$

In this conditions we have

LEMMA 2.1. *If $F_\eta \to f$ uniformly on bounded sets of $\mathbf{R}$ and*

$$(2.4)\qquad \|F_\eta(s)\| \leqq c_1 + c_1\|f(s)\| \quad \forall s\in\mathbf{R},\ \forall\eta\in\mathbf{R}$$

then

$$(2.5)\qquad (u_\eta, y_\eta, F_\eta(y_\eta)) \to (u,y,f(y)) \textit{ strongly in } L^2(Q)\times W\times L^2(Q).$$

PROOF. For the existence of the solution of problem (2.3) see [1]. Let us define $v_\eta = y'' - \Delta y - F_\eta(y)$; certainly the couple $(v_\eta, y)\in X_{\mathrm{ad}}^\eta$ and of course $J_\eta(u_\eta, y_\eta)$ is bounded. From (2.1) and (2.4) we conclude that $\exists c_3 > 0$ such that

$$(2.6)\qquad \|F_\eta(y_\eta)\|_Q^2 + \|u_\eta\|_Q^2 + \|y_\eta\|_W \leqq c_3.$$

Since $W \hookrightarrow L^2(Q)$ compactly we obtain a subsequence such that

$$u_\eta \to v \text{ weakly in } L^2(Q),$$
$$y_\eta \to \theta \text{ strongly in } L^2(Q),$$
$$y_\eta \to \theta \text{ a.e. in } Q.$$

Then $F_\eta(y_\eta)\to f(\theta)$ a.e. in Q. By (2.7) and Lions' Lemma (see [2]) we have: $F_\eta(y_\eta)\to f(\theta)$ weakly in $L^2(Q)$. By (2.6) it follows

$$\|f(\theta) - z_d\|_Q^2 + N\|v\|_Q^2 + \|y-\theta\|_Q^2 + \|u-v\|_Q^2 \leqq \liminf J_\eta(u_\eta, y_\eta) \leqq J(u,y).$$

Since $(v,\theta)\in X_{\mathrm{ad}}$, by (1.4) $(u,y) = (v,\theta)$ and the result follows. □

We will prove the existence of sequences $(s_\nu)_{\nu\in\mathbf{N}}$ a $(s_{-\nu})_{\nu\in\mathbf{N}}$ such that

$$(2.7)\qquad s_\nu \geqq \nu \quad \forall\nu\in\mathbf{N} \text{ and } \|f(s_\nu)\| \leqq c_4 + \|f(s)\| \quad \forall s\geqq\nu,$$

$$(2.8)\qquad s_{-\nu} \leqq -\nu \quad \forall\nu\in\mathbf{N} \text{ and } \|f(s_{-\nu})\| \leqq c_4 + \|f(s)\| \quad \forall s\leqq -\nu.$$

These sequences are going to play an important role in the sequel:

LEMMA 2.2. *Let f be a continuous function. Then there exists a sequence of numbers $(s_\nu)_{\nu\in\mathbf{N}}, (s_{-\nu})_{\nu\in\mathbf{N}}$ satisfying* (2.7) *and* (2.8).

PROOF. Put $I_\nu = \inf\{\|f(s)\|; s \geqq \nu\}$. If for all $\nu \in \mathbf{N}$, there exist $s_\nu \geqq \geqq \nu$ such that $f(s_\nu) = I_\nu$, then the sequence $(s_\nu)_{\nu\in\mathbf{N}}$, constructed this way satisfies condition (2.8). We can suppose that there exist a natural number, say ν_0 such that $\|f(s)\| > I = \inf\{\|f(s)\|; s \geqq \nu_0\}$, $\forall s \geqq \nu_0$. Denote by $(T_\gamma)_{\gamma\in\mathbf{N}}$ the minimizing sequence. Clearly, this sequence is not bounded; hence there exists a subsequence satisfying $t_{\gamma_\nu} \geqq \nu$, $\forall \nu \in \mathbf{N}$. Taking $s_\nu = t_{\gamma_\nu}$ we have that condition (2.8) is valid also in this case. To construct $(s_{-\nu})_{\nu\in\mathbf{N}}$ consider $I_{-\nu} = \inf\{\|f(s)\|; s \leqq -\nu\}$ and use the same argument. □

Let us construct the sequence $(f_\nu)_{\nu\in\mathbf{N}}$ in the following way:

$$f_\nu(s) = \begin{cases} f(s) & s_{-\nu} \leqq s \leqq s_\nu \\ f(s_\nu) & s \geqq s_\nu \\ f(s_{-\nu}) & s \leqq s_{-\nu}. \end{cases}$$

Then f_ν is a bounded and lipschitzian function satisfying

$$\text{(2.9)} \qquad \|f_\nu(s)\| \leqq c + \|f(s)\|, \ \|f'_\nu(s)\| \leqq c + c\|f_\nu(s)\|, \ \forall \nu \in \mathbf{N},$$

$$\text{(2.10)} \qquad f_\nu \to f, f'_\nu \to f' \text{ uniformly on bounded sets of } \mathbf{R},$$

where $c = \max\{c_0, c_4\}$. Now denote by $(\varrho_\mu)_{\mu\in\mathbf{N}}$ the regularising sequence, and put $f_{\nu\mu} = f_\nu * \varrho_\mu$ and $C_\nu = \sup\{\|f_\nu(s)\|, s \in \mathbf{R}\}$. For each ν the sequence $(f_{\nu\mu})_{\mu\in\mathbf{N}}$ satisfies

$$\text{(2.11)} \qquad f_{\nu\mu} \to f_\nu \text{ uniformly on bounded sets of } \mathbf{R},$$

(2.12)

$$\|f_{\nu\mu}(s)\| \leqq \frac{1}{\mu} C_\nu + \|f_\nu(s)\|, \ \|f'_{\nu\mu}(s)\| \leqq c + c(\|f_\nu\| * \varrho_\mu)(s), \ \forall \mu \in \mathbf{N}.$$

Let us define the state for a fixed $\nu \in \mathbf{N}$:

$$\text{(2.13)} \qquad \begin{cases} z'' - \Delta z - f_{\nu\mu}(z) = v & \text{in } Q \\ z(x,t) = 0 & \text{on } \Sigma \\ z(0) = y_0, z'(0) = y_1 & \text{in } \Omega, \end{cases}$$

and the cost function

$$J_{\nu\mu}(v,z) =$$
$$= \|f_{\nu\mu}(z) - z_d\|_Q^2 + N\|v\|_Q^2 + \|z - y_\nu\|_Q^2 + \|v - u_\nu\|_Q^2 + \|z - y\|_Q^2 + \|v - u\|_Q^2$$

where (u_ν, y_ν) is the solution of the ν-problem for the sequence $(f_\nu)_{\nu\in\mathbf{N}}$, and (u, y) solution of problem (1.4). Consider the problem

$$\text{(2.14)} \qquad J_{\nu\mu}(u_{\nu\mu}, y_{\nu\mu}) = \inf\{J_{\nu\mu}(v,z); (v,z) \in X_{\text{ad}}^{\nu\mu}\}; \ (u_{\nu\mu}, y_{\nu\mu}) \in X_{\text{ad}}^{\nu\mu}$$

where $X_{\mathrm{ad}}^{\nu\mu} = \{(v,z) \in L^2(q) \times W;\ (v,z)$ satisfying $(2.19)\}$. In order to obtain a characterization for $(u_{\nu\mu}, y_{\nu\mu})$ define
(2.15)

$$J_{\nu\mu\varepsilon}(v,z) = J_{\nu\mu}(v,z) + \frac{1}{\varepsilon}\|z'' - \Delta z - f_{\nu\mu}(z)\|_Q^2 + \|z - y_{\nu\mu}\|_Q^2 + \|v - u_{\nu\mu}\|_Q^2.$$

If we put $D_{\mathrm{ad}} = \{(v,z) \in L^2(Q) \times W;\ z'' - \Delta z \in L^2(Q), z(Q) = y_0, z'(0) = y_1\}$ then the penalised problem is given by

$$(2.16)\qquad J_{\nu\mu\varepsilon}(u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon}) = \inf\{J_{\nu\mu\varepsilon}(v,z) \in D_{\mathrm{ad}}\},\ (u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon}) \in D_{\mathrm{ad}}.$$

$$\varrho_{\nu\mu\varepsilon} = -\frac{1}{\varepsilon}\{y''_{\nu\mu\varepsilon} - \Delta y_{\nu\mu\varepsilon} - f_{\nu\mu\varepsilon}(y_{\nu\mu\varepsilon}) - u_{\nu\mu\varepsilon}\}.$$

Under these conditions we have:

LEMMA 2.3. *$\forall \varepsilon > 0$ and $\forall \nu, \mu \in \mathbf{N}$, the couple $(\varrho_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon})$ satisfies*

$$(\varrho_{\nu\mu\varepsilon}, \xi'' - \Delta\xi - f'_{\nu\mu\varepsilon}(y_{\nu\mu\varepsilon})\xi)_Q = (f_{\nu\mu\varepsilon}(y_{\nu\mu\varepsilon}) -$$
$$- z_d, f'_{\nu\mu\varepsilon}(y_{\nu\mu\varepsilon})\xi)_Q + (y_{\nu\mu\varepsilon} - y_{\nu\mu}, \xi)_Q + (y_{\nu\mu\varepsilon} - y_\nu, \xi)_Q + (y_{\nu\mu\varepsilon} - y, \xi)_Q$$

whenever $\xi \in C^2(\overline{Q})$, $\xi(0) = \xi'(0) = 0$ in Ω and $\xi = 0$ on Σ. Also

$$\varrho_{\nu\mu\varepsilon} + N u_{\nu\mu\varepsilon} + 3u_{\nu\mu\varepsilon} - u_{\nu\mu} - u_\nu - u = 0.$$

PROOF. Since $(u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon})$ is the solution of problem (2.16) we have

$$\frac{d}{d\lambda} J_{\nu\mu\varepsilon}(u_{\nu\mu\mu\varepsilon}, y_{\nu\mu\varepsilon} + \lambda\xi)|_{\lambda=0} = 0.$$

Since for each ν, μ, $f_{\nu\mu}$ is a Lipschitz function of class C^1, we can pass the derivative into the integral. Then we obtain the first part of the lemma. Finally from (2.21) we have

$$\frac{1}{\lambda}\{J_{\nu\mu\varepsilon}(u_{\nu\mu\varepsilon} + \lambda(v - u_{\nu\mu\varepsilon}), y_{\nu\mu\varepsilon}) - J_{\nu\mu\varepsilon}(u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon})\} \geqq 0,\ \forall\lambda \in]0,1[.$$

Letting $\lambda \to 0$ it follows

$$\int_Q (\varrho_{\nu\mu\varepsilon} + N u_{\nu\mu\varepsilon} + 3u_{\nu\mu\varepsilon} - u_{\nu\mu} - u_\nu - u)(v - u_{\nu\mu\varepsilon})\,dx\,dt \geqq 0,\ \forall v \in L^2(Q)$$

and from this expression the proof of the lemma is complete. □

LEMMA 2.4. *When $\varepsilon \to 0$, the solution of problem* (2.16) *satisfies*

$$(u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon}, f_{\nu\mu\varepsilon}(y_{\nu\mu\varepsilon})) \to (u_{\nu\mu}, y_{\nu\mu}, f_{\nu\mu}(y_{\nu\mu})) \text{ strongly in } L^2(Q)\times W \times L^2(Q),$$
$$J_{\nu\mu\varepsilon}(u_{\nu\mu\varepsilon}, y_{\nu\mu\varepsilon}) \to J_{\nu\mu}(u_{\nu\mu}, y_{\nu\mu}),$$
$$\varrho_{\nu\mu\varepsilon} \to \varrho_{\nu\mu} = -N u_\nu - (u_\nu - u) \text{ weakly in } L^2(Q).$$

PROOF. See [1]. □

From Lemma 2.3 and 2.4 it follows

$$(2.17)\qquad (\varrho_{\nu\mu}, \xi'' - \Delta\xi - f'_{\nu\mu}(y_{\nu\mu})\xi)_Q = \\ = (f_{\nu\mu}(y_{\nu\mu}) - z_d, f'_{\nu\mu}(y_{\nu\mu})\xi)_Q + (2y_{\nu\mu} - y_\nu - y, \xi)_Q.$$

COROLLARY 2.5. *The solution of problem* (2.14) *satisfies*

$$(u_{\nu\mu}, y_{\nu\mu}, f_{\nu\mu}(y_{\nu\mu})) \to (u_\nu, y_\nu, f_\nu(y_\nu)) \text{ strongly in } L^2(Q)\times W\times L^2(Q),$$
$$\varrho_{\nu\mu} \to \varrho_\nu = -N u_\nu - (u_\nu - u) \text{ strongly in } L^2(Q).$$

PROOF. Put $f = f_\nu$, and $J_\eta = J_{\nu\mu}$. From (2.11), (2.12) and Lemma 2.1 the result follows. □

Since for all μ, $(f'_{\nu\mu}(y_{\nu\mu}))_{\mu\in\mathbf{N}}$ is uniformly bounded by a constant, say k_ν, we can obtain a subsequence and an element χ_ν in $L^2(Q)$ such that

$$(2.18)\qquad f'_{\nu\mu}(y_{\nu\mu}) \to \chi_\nu \text{ weakly in } L^2(Q).$$

By Lemma 2.4, Corollary 2.5 and from this last convergence we have

$$(2.19)\ (\varrho_\nu, \xi'' - \Delta\xi - \chi_\nu\xi)_Q = (f_\nu(y_\nu) - z_d, \chi_\nu\xi)_Q + (y_\nu - y, \xi)_Q,\ \forall \nu\in\mathbf{N}.$$

COROLLARY 2.6. *The solution of the ν-problem for the sequence $(f_\nu)_{\nu\in\mathbf{N}}$ satisfies*

$$(u_\nu, y_\nu, f_\nu(y_\nu)) \to (u, y, f(y)) \text{ strongly in } L^2(Q)\times W\times L^2(Q).$$

PROOF. The result follows from Lemma 2.1, (2.9) and (2.10). □

LEMMA 2.7. *Let O be an open and bounded set of $\mathbf{R}^n$ and let $(G_\nu)_{\nu\in\mathbf{N}}$ be a sequence in $L^p(O)$, $1 < p < \infty$, weakly convergent to G. Put*

$$L(x) = \liminf_{\nu\to\infty} G_\nu(x) \quad \text{and} \quad U(x) = \limsup_{\nu\to\infty} G_\nu(x);$$

then we have $L(x) \leqq G(x) \leqq U(x)$.

PROOF. Let us define $O_m = \{x\in O; L(x) - \varepsilon \leqq G_\nu(x) \leqq U(x) + \varepsilon,\ \forall\nu \geqq m\}$, for an arbitrary but fixed $\varepsilon > 0$, and denote by D_m the set of all the

$L^q(O)$-functions which have support in O_m. Put $D = \bigcup\limits_m^\infty D_m$. Take $\varphi \in D$ such that $\varphi \geqq 0$, then $\exists m' \in \mathbf{N}$ such that

$$[L(x)-\varepsilon]\varphi \leqq G_\nu(x)\varphi \leqq [U(x)+\varepsilon]\varphi, \ \forall \nu \geqq m'$$

integrating the last expression and letting $\nu \to \infty$ we have

$$\int\limits_O [L(x)-\varepsilon]\varphi(x)dx \leqq \int\limits_O G(x)\varphi(x)dx \leqq \int\limits_O [U(x)+\varepsilon]\varphi(x)dx.$$

Since this last expression does not depend on m' the inequalities are valid for all $\varphi \geqq 0$ in D. Since the measure of O_m tends to measure of O we have that D is dense in $L^q(O)$ where $\frac{1}{p}+\frac{1}{p}=1$. Hence

$$\int\limits_O [L(x)-\varepsilon]w(x)dx \leqq \int\limits_O G(x)w(x)dx \leqq \int\limits_O [U(x)+\varepsilon]w(x)dx, \ \forall w \geqq 0 \text{ in } L^q(O)$$

and from which we conclude $L(x)-\varepsilon \leqq G(x) \leqq U(x)+\varepsilon$. Since ε is arbitrary, the result follows. □

THEOREM 2.8. *Let f be a function of class $\mathbf{C}^1$ satisfying condition* (1.5). *Then there exists a subsequence of $(\chi_\nu)_{\nu\in\mathbf{N}}$ satisfying*

$$\chi_\nu \to f'(y) \textit{ strongly in } L^2(Q).$$

PROOF. From Corollaries 2.5 and 2.6 there exists a subsequence of $(y_{\nu\mu})_{\mu\in\mathbf{N}}$ such that

$$y_{\nu\mu} \to y_\nu \text{ a.e. in } Q, \tag{2.21}$$

$$y_\nu \to y \text{ a.e. in } Q. \tag{2.22}$$

If we put N_ν and M the subsets of Q where the first and the second convergence fails, respectively, then we have that $N = \bigcup\limits_\nu^\infty N_\nu \cup M$ has null measure. Take (x,t) in $Q-N$, then $\exists \nu_0$ such that $\|y_\nu(x,t)\| < \nu_0$, $\forall \nu \geqq \nu_0$, and also $\exists \mu_0$ such that $\|y_{\nu\mu}(x,t)\| < \nu_0$, $\forall \mu \geqq \mu_0$. Since $f'_{\nu\mu}$ is uniformly convergent on $[s_{-\nu_0}, s_{\nu_0}]$, we have that

$$f'_{\nu\mu}(y_{\nu\mu}(x,t)) \to f'_\nu(y_\nu(x,t)), \quad \forall \nu \geqq \nu_0 \ (\nu_0 = \nu_0(x,t)).$$

On the other hand by (2.18) and Lemma 2.7 we conclude that

$$\chi_\nu(x,t) = f'_\nu(y_\nu(x,t)), \quad \forall \nu \geqq \nu_0 \ (\nu_0 = \nu_0(x,t)).$$

By (2.10) and (2.21) we have that $\chi_\nu(x,t) \to f'(y(x,t))$ a.e. in Q and finally from (2.12), (2.20) and Lemma 2.7 we have that

$$\|\chi_\nu(x,t)\| \leqq c + c\|f_\nu(y_\nu(x,t))\|.$$

Then by Corollary 2.6 and Lebesgue's dominated convergence theorem the result follows. □

Finally as a Corollary we have

THEOREM 2.9. *If (u,y) is the solution of problem (1.4), then there exists ϱ in $L^2(Q)$ such that the following optimality system holds:*

$$\begin{gathered}
y'' - \Delta y - f(y) = u \text{ in } D'(Q),\\
\varrho'' - \Delta\varrho - f'(y)\varrho = [f(y) - z_d]f'(y) \text{ in } D'(Q),\\
y(0) = y_0, \quad y'(0) = y_1 \text{ in } \Omega,\\
y \in L^\infty(0,T; H_0^1(\Omega)), \quad y' \in L^\infty(0,T; L^2(\Omega)),\\
\varrho(T) = \varrho'(T) = 0, \quad \varrho = 0 \text{ on } \Sigma,\\
\varrho = -Nu, \quad u \in L^2(Q).
\end{gathered}$$

PROOF. This follows immediately from (2.20), Corollary 2.6 and Theorem (2.11). □

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(Received August 9, 1988)

NATIONAL LABORATORY OF CIENTIFIC COMPUTATION (LNCC)
RUA LAURO MULLER 455 (LNCC–CNPQ)
BOTAFOGO 22290. RIO DE JANEIRO. BRASIL
IMUFRJ P. P. BOX 68530, RIO DE JANEIRO, R. J. BRASIL

Acta Math. Hung.
59(1-2) (1992), 159–174.

NUMERICAL SOLUTION BY SPLINE METHOD FOR AN ELASTIC PROBLEM

A. S. MOHAMED (Budapest)

1. Introduction

In recent papers [1], [2] we have introduced a spline function method approximating the solution of the partial differential equation

$$(1.1) \qquad Au := -\frac{\partial}{\partial r}\left[\frac{1}{r^3}\frac{\partial u}{\partial r}\right] - \frac{\partial}{\partial z}\left[\frac{1}{r^3}\frac{\partial u}{\partial z}\right] = 0$$

for an elastic circular cylinder domain. Spline approximation methods for an ordinary differential equation are discussed in the papers of T. Fawzy [9], [10], and J. Győrvári [11], [12].

In the present paper we shall propagate the method introduced inn [1], [2] for a complicated elastic axisymmetric cylinder domain (see Fig. 1.1). Moreover it is known (see [3]) that if the torqaue T increases then it will appear the plastic region Ω_p in the domain of the cylinder. We shall suppose that the problem remains an axisymmetric one, and the consideration will be taken only to the elastic problem in a given domain Ω_e separately (see case 2) because it is an important part in finding the solution of the elastic-plastic free boundary problem which will be considered in a subsequent work.

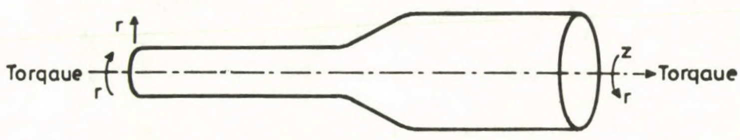

Fig. 1.1. An axisymmetric cylinder

The formulation of the problem [3] can be given in a two dimensional domain Ω or Ω_e (see Fig. 2.1, Fig. 2.2) in the rz-plane, and the problem is summarized to find the function u which must satisfy (1.1) in Ω or Ω_e with the boundary conditions

$$u = 0, \text{ on } \Gamma_0, \tag{1.2}$$

$$u = F(z), \text{ on } \Gamma_1, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial z} = \varphi_1(r), \text{ on } \Gamma_2, \tag{1.4}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \varphi_2(r), \text{ on } \Gamma_3, \tag{1.5}$$

where Γ_i are as show in Fig. 2.1 and 2.2.

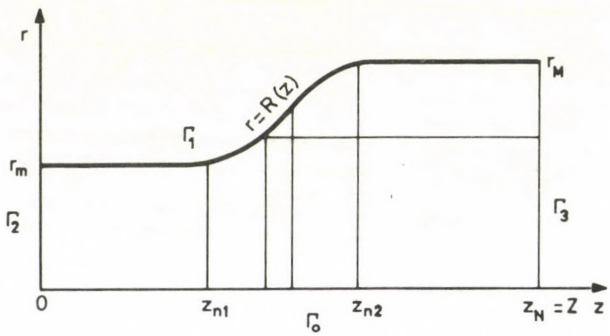

Fig. 2.1. The elastic domain Ω

2. The construction of the spline function

Case 1: the elastic problem in the domain $\Omega_e \equiv \Omega$ (Fig. 2.1).

The intervals of the rz-plane are defined by the partitions

$$0 = z_0 < z_1 < \cdots < z_N = z, \tag{2.1}$$

$$0 = r_0 < r_1 < \cdots < r_M, \tag{2.2}$$

and we suppose that the function $r = R(Z)$, which represents the boundary Γ_1, is constant for $z \in [0, z_{n_1}]$ and $z \in [z_{n_2}, z]$, and increasing in the interval (z_{n_1}, z_{n_2}). Furthermore, we suppose that (z_{n_1}, r_m) and (z_{n_2}, r_M) are grid-points, and

$$r_{m+i} = R(z_{n_1+i}), \quad i = 0, \dots, n_2 - n_1. \tag{2.3}$$

Hence

$$n_2 - n_1 = M - m. \tag{2.4}$$

The fourth power of the function $r = R(z)$ (which we shall use later on) is approximated by the pieciewise linear function

$$(2.5)\quad R^4(z) \approx \tilde{R}^4(z) := r_j^4 + \frac{r_{j+1}^4 - r_j^4}{z_{i+1} - z_i}(z - z_i), \quad z_i < z < z_{i+1},\ n_1 \leqq i < n_2,$$

and the boundary value function $u = F(z)$ on Γ_1 is approximated by the piecewise linear function

$$(2.6)\quad F(z) \approx \tilde{F}(z) := F(z_i) + \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}(z - z_i), \quad i = 0, \ldots, N-1.$$

Let Ω be partitioned into the elements Ω_{ij} which are rectangulars into the domain Ω and triangulars with one curved side (2.5) at the neighborhood of Γ_1 in the interval (z_{n_1}, z_{n_2}), such that

$$(2.7)\qquad \Omega \cup \left(\bigcup_{i=0}^{3} \Gamma_i \right) = \overline{\Omega} = \bigcup_{i,j} \overline{\Omega}_{ij},$$

where $\overline{\Omega}_{ij} = \{z, r | z_i \leqq z_{i+1},\ r_j \leqq r \leqq {}_{j+1},\ z_i = ih_z,\ r_j = jh_r\}$ and the total number of the elements $\overline{\Omega}_{ij}$ is

$$(2.8)\qquad N \cdot m + (N - n_2)(M - m) + \frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2}.$$

The form of the spline function inside Ω, which was constructed in [1] is

$$(2.9)\qquad S_{ij}(z, r) = a_{ij}(r^4 - r_j^4) + b_{ij}(r^4 - r_j^4)(z - z_i) + c_{ij}(z - z_i) + d_{ij},$$

This will have the following forms at the neighborhood of the boundaries Γ_0, Γ_1:

$$(2.10)\qquad S_{i,0}(z, r) = a_{i,0} r^4 + b_{i,0} r^4 (z - z_i), \quad i = 0, \ldots, N-1,$$

$$(2.11)\quad S_{i,m-1}(z, r) = a_{i,m-1}(r^4 - r_m^4) + b_{i,m-1}(r^4 - r_m^4)(z - z_i) + \tilde{F}(z),\quad i = 0, \ldots, n_1 - 1,$$

$$(2.12)\quad S_{i,j}(z, r) = a_{i,j}(r^4 - \tilde{R}^4(z)) + \tilde{F}(z), \quad i = n_1 + \ell, j = m_1 + \ell,\quad \ell = 0, \ldots, n_2 - n_1 - 1,$$

$$(2.13)\ S_{i,M-1}(z, r) = a_{i,M-1}(r^4 - r_M^4) + b_{i,M-1}(r^4 - r_M^4)(z - z_i) + \tilde{F}(z),\quad i = n_2, \ldots, N-1.$$

From [1], and from Theorem 1 in [2] we can show that there exist continuous spline functions (2.9–2.13) in $\overline{\Omega}$ and a system of

$$(2.14)\qquad (2(n_2 - n_1)^2 + (1 - M)(2 - 4N) + 3(n_2 - n_1) - 4n_2(M - m))$$

equations with

$$(2.15)\qquad (4(n_2-n_1)^2+2N(3+2M)-n_2(M-m)-n_2-n_1).$$

The coefficients are obtained from the supposition that the spline functions are continuous in $\overline{\Omega}$. The analysis of the system shows that among these equations there are

$$(2.16)\qquad \left((n_2-n_1)^2-N(1-M)-n_2(M-m)-m-\frac{(n_2-n_1)(n_2-n_1-1)}{2}+1\right)$$

dependent. Deleting the dependent equations from the system we get the independent system of the equations

(2.17)

$$a_{i,j}+b_{i,j}(z_{i+1}-z_i)=a_{i+1,j},\qquad i=0,\dots,N-2,j=0,\dots,m-1.$$

$$c_{i,j}(z_{i+1}-z_i)+d_{i,j}=d_{i+1,j},\qquad i=0,\dots,N-2,j=1,\dots,m-2.$$

$$c_{i,m-1}(z_{i+1}-z_i)+d_{i,m-1}=d_{i+1,m-1},\qquad i=n_1,\dots,N-2.$$

$$a_{i,j}=a_{i+1,j},\qquad i=n_1+\ell,j=m+\ell,\quad \ell=0,\dots,n_2-n_1-1.$$

$$a_{i,j}+b_{i,j}(z_{i+1}-z_i)=a_{i+1,j},\qquad i=n_1+\ell+1,j=m+k,\quad \ell=0,\dots,n_2-n_1-k-2,\quad k=0,\dots,n_2-n_1-2.$$

$$c_{i,j}(z_{i+1}-z_i)+d_{i,j}=d_{i+1,j},\qquad i=n_1+\ell+1,j=m+k,\quad \ell=0,\dots,n_2-n_1-k-2,\quad k=0,\dots,n_2-n_1-2.$$

$$a_{i,j}+b_{i,j}(z_{i+1}-z_i)=a_{i+1,j},\qquad i=n_2,\dots,N-2,j=m,\dots,M-1.$$

$$c_{i,j}(z_{i+1}-z_i)+d_{i,j}=d_{i+1,j},\qquad i=n_2,\dots,N-2,j=m,\dots,M-2.$$

$$a_{0,j}(r_{j+1}^4-r_j^4)+d_{0,j}=d_{0,j+1},\qquad j=0,\dots,m-3,$$

$$b_{i,j}(r_{j+1}^4-r_j^4)+c_{i,j}=c_{i,j+1},\qquad i=0,\dots,N-1,j=0,\dots,m-3.$$

$$a_{0,m-1}(r_{m-1}^4-r_{m-2}^4)+d_{0,m-2}=$$
$$=a_{0,m-1}(r_{m-1}^4-r_m^4)+F(0),$$

$$b_{i,m-2}(r_{m-1}^4-r_{m-2}^4)+c_{i,m-2}=$$
$$=b_{i,m-1}(r_{m-1}^4-r_m^4)+\frac{F(z_{i+1})-F(z_i)}{z_{i+1}-z_i},\quad i=0,\dots,n_1-1.$$

$$b_{i,j}(r_{j+1}^4-r_j^4)+c_{i,j}=c_{i,j+1},\qquad i=n_2,\dots,N-1,\quad j=m-2,\dots,M-3.$$

$$b_{i,M-2}(r_{M-1}^4-r_{M-2}^4)+c_{i,m-2}=$$

$$=b_{i,M-1}(r_{M-1}^4-r_M^4)+\frac{F(z_{i+1})-F(z_i)}{z_{i+1}-z_i},\ i=n_2,\dots,N-1.$$

$$b_{i,j}(r_{j+1}^4-r_j^4)+c_{i,j}=c_{i,j+1},\qquad i=n_1+\ell+k+1,\ j=m+k-1,$$
$$\ell=0,\dots,n_2-n_1-k-2,$$
$$k=0,\dots,n_2-n_1-2.$$

$$a_{i,j}(r_{j+1}^4-r_j^4)+d_{i,j}=F(z_i),\qquad i=n_1+\ell,\ j=m+\ell-1,$$
$$\ell=0,\dots,n_2-n_1-1.$$

$$b_{i,j}(r_{j+1}^4-r_j^4)+c_{i,j}=$$
$$=-a_{i,j+1}\frac{(r_{j+2}^4-r_{j+1}^4)}{z_{i+1}-z_i}+\frac{F(z_{i+1})-F(z_i)}{z_{i+1}-z_i},\quad i=n_1+\ell,\ j=m+\ell-1,$$
$$\ell=0,\dots,n_2-n_1-1.$$

$$b_{i,m-2}(r_{m-2}^4-r_{m-1}^4)+c_{i,m-2}=c_{i,m-1},\qquad i=n_1,\dots,n_2-1.$$

The number of the equations in (2.17) is

$$(2.18)\ (n_2-n_1)^2+3(n_2-n_1)-3N(1-M)+m(1+3n_2)-M(2+3n_2)+ +\frac{(n_2-n_1)(n_2-n_1-1)}{2}+1,$$

with the number of coefficients (2.15).

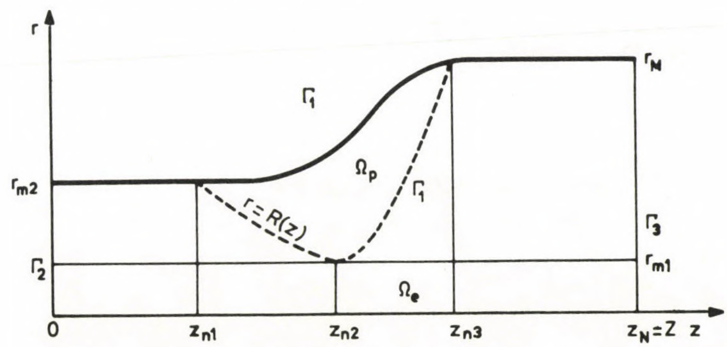

Fig. 2.2. The elastic domain Ω_e

Case 2: the elastic problem in domain Ω_e (Fig. 2.2). Now we suppose that the plastic region Ω_p exists in Ω. For z_i and r_j we have

$$(2.19)\qquad 0=z_0<z_1<\dots<z_N=Z,\quad 0<n_1<n_2<n_3<N,$$
$$(2.20)\qquad 0=r_0<r_1<\dots<r_M,\quad 0<m_1<m_2<M.$$

We have supposed that the function $r=R(z)$ is constant for $z\in[0,z_{n_1}]$ and $z\in[z_{n_3},Z]$, decreasing in the interval (z_{n_1},z_{n_2}) and increasing in the interval (z_{n_2}, z_{n_3}), where its minimum is attained at the point $z_{n_2}, r_{m_1})$. Furthermore, (z_{n_1}, r_{m_2}), (z_{n_2}, r_{m_1}) and (z_{n_3}, r_M) are grid-points, and we suppose that

$$r_{m_2-i} = R(z_{n_1+i}), \quad i = 1, \dots, n_2 - n_1, \tag{2.21}$$

$$r_{m_1+i} = R(z_{n_2+i}), \quad i = 1, \dots, n_3 - n_2, \tag{2.22}$$

therefore,

$$n_2 - n_1 = m_2 - m_1, \quad n_3 - n_2 = M - m_1 \tag{2.23}$$

The fourth power of the function $r = R(z)$ is approximated by the piecewise linear functions

$$R^4(z) \approx \tilde{R}^4(z) := r_{j+1}^4 + \frac{r_j^4 - r_{j+1}^4}{z_{i+1} - z_i}(z - z_i), \quad n_1 \leqq i < n_2, \tag{2.24}$$

$$R^4(z) \approx \tilde{R}^4(z) := r_j^4 + \frac{r_{j+1}^4 - r_j^4}{z_{i+1} - z_i}(z - z_i), \quad n_2 \leqq i < n_3, \tag{2.25}$$

the function $u = F(z)$ on Γ_1 has been approximated as in (2.6).

Similarly as in Case 1, the domain Ω_e is divided into the elements Ω_{ij}, and the total number of the elements is

$$\begin{aligned} &N m_1 + n_1(m_2 - m_1) + (N - n_3)(M - m_1) + \\ &+ \frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2} + \frac{(n_3 - n_2)(n_3 - n_2 + 1)}{2}. \end{aligned} \tag{2.26}$$

The spline function inside Ω_e will have the form (2.9) and it will have the following forms at the neighborhood of the boundaries Γ_0, Γ_1:

$$S_{i,0}(z, r) = a_{i,0} r^4 + b_{i,0} r^4 (z - z_i), \quad i = 0, \dots, N - 1. \tag{2.27}$$

$$\begin{gathered} S_{i,m_2-1}(z, r) = a_{i,m_2-1}(r^4 - r_{m_2}^4) + b_{i,m_2-1}(r^4 - r_{m_2}^4)(z - z_i) + \tilde{F}(z), \\ i = 0, \dots, n_1 - 1. \end{gathered} \tag{2.28}$$

$$\begin{gathered} S_{i,j}(z, r) = a_{i,j}(r^4 - \tilde{R}^4(z)) + \tilde{F}(z), \\ i = n_1 - \ell_1 - 1, \quad j = m_1 + \ell_1, \quad \ell_1 = 0, \dots, n_2 - n_1 - 1. \end{gathered} \tag{2.29}$$

$$\begin{gathered} S_{i,j}(z, r) = a_{i,j}(r^4 - \tilde{R}^4(z)) + \tilde{F}(z), \\ i = n_2 + \ell_2, \quad j = m_1 + \ell_2, \quad \ell_2 = 0, \dots, n_3 - n_2 - 1. \end{gathered} \tag{2.30}$$

$$\begin{gathered} S_{i,M-1}(z, r) = a_{i,M-1}(r^4 - r_M^4) + b_{i,M-1}(r^4 - r_M^4)(z - z_i) + \tilde{F}(z), \\ i = n_3, \dots, N - 1. \end{gathered} \tag{2.31}$$

From [1] or Theorem 1 in [2] we can show that there exist continuous spline functions (2.27 – 2.31) and (2.9) in $\overline{\Omega}_e$ and a system of

$$(2.32) \qquad 2(n_2-n_1)^2+2(n_3-n_2)^2+4n_1(m_2-m_1)-N(1-4M)- \\ -4n_3(M-m_1)-M-m_1-n_1,$$

equations with

$$(2.33) \qquad 2(n_2-n_1)^2+2(n_3-n_2)^2+4N(M-1)+4n_1(m_2-m_1)+ \\ +4n_3(m_1-M)-n_1+n_3$$

coefficients is obtained from the supposition that the spline functions are continuous in Ω_e. The analysis of the system shows that among these equations

$$(2.34)\ (n_2-n_1)^2+(n_3-n_2)^2+n_1(m_2-m_1)+M(N+1)-n_3M-m_1- \\ -m_1+2n_1-n_3-\frac{(n_2-n_1)(n_2-n_1-1)}{2}-\frac{(n_3-n_2)(n_3-n_2-1)}{2}$$

are dependent. By deleting the dependent equations from the system we get an independent system (similar to (2.17) and the difference is obvious) of

$$(n_2-n_1)^2+(n_3-n_2)^2+3n_1(m_2-m_1)-N(1-3M)-3n_3(M-m_1)-3n_1+ \\ +n_3-2M+\frac{(n_2-n_1)(n_2-n_1-1)}{2}+\frac{(n_3-n_2)(n_3-n_2-1)}{2}$$

equations with the number of coefficients (2.33).

3. The spline function properties

Let the spline functions given in Section 2 have similar properties to those described in Section 3 in [2]. In this section we are going to prove a theorem for the density of the spline functions in Ω_e.

Let

$$[u,v]=\int_\Omega \frac{1}{r^3}\left[\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right]dzdr,$$

$$H_A=\left\{u\ \middle|\ \frac{1}{r^{3/2}}\left[\left[\frac{\partial u}{\partial r}\right]^2+\left[\frac{\partial u}{\partial z}\right]^2\right]^{1/2}\in L_2(\Omega),u(z,0)=0\right\},$$

$$[u]=[u,u]^{1/2}.$$

THEOREM 1. *The set of the spline functions* $S_\Delta(z,r)$ *in* (2.9) *are dense in* H_A, *that is for every* $f(z,r)\in W_2^2\cap H_A$ *there exists a spline function* $S_\Delta(z,r)$ *such that*

$$[f(z,r)-S_\Delta(z,r)]=O(h_z+h_r)$$

and for $f(z,r)$ *the following estimates are true*

$$\|f(z,r)-S_\Delta(z,r)\|_C=O(h_z^2+h_r^2),$$
$$\|f(z,r)-S_\Delta(z,r)\|_{W_2^1}=O(h_z+h_r).$$

PROOF. We note that because $f\in W_2^2$, the grid-points (z_i,r_j) can be chosen in such a way that the second derivates $(\frac{\partial^2 f}{\partial z^2},\frac{\partial^2 f}{\partial z\partial r},\frac{\partial^2 f}{\partial r^2})$ are bounded at these grid-points.

For the spline functions in the rectangular elements we let

$$S_\Delta(z,r)=S_{i,j}(z,r):=\frac{f_{i,j+1}-f_{i,j}}{r_{j+1}^4-r_j^4}(r^4-r_j^4)+ \tag{3.1}$$
$$+\frac{1}{h_z(r_{j+1}^4-r_j^4)}(f_{i+1,j+1}-f_{i+1,j}-f_{i,j+1}+f_{i,j})(r^4-r_j^4)(z-z_i)+$$
$$+\frac{f_{i+1,j}-f_{i,j}}{h_z}(z-z_i)+f_{i,j}.$$

The estimates of the approximation by (3.1) in the rectangular elements are introduced in Theorem 4 in [1].

For the spline functions in the left-side triangular elements with the points (z_i,r_j), (z_{i+1},r_j), (z_{i+1},r_{j+1}) we let

$$S_\Delta(z,r)=S_{i,j}(z,r):= \tag{3.2}$$
$$:=\frac{f_{i+1,j+1}-f_{i+1,j}}{r_{j+1}^4-r_j^4}\left(r^4-r_j^4-\frac{r_{j+1}^4-r_j^4}{z_{i+1}-z_i}(z-z_i)\right)+$$
$$+F(z_i)+\frac{F(z_{i+1})-F(z_i)}{z_{i+1}-z_i}(z-z_i)$$

where

$$f_{i,j}:=f(z_i,r_j)\text{ and }F(z):=(z,R(z)).$$

It is obvious that the spline functions given by (3.1) and (3.2) are continuous in Ω_e.

The Taylor expansion for the mentioned triangular elements is

$$(3.3)\qquad f(z,r)=f_{i+1,j}+\lambda_r h_r\frac{\partial f}{\partial r}\bigg|_{i+1,j}+\lambda_z h_z\frac{\partial f}{\partial z}\bigg|_{i+1,j}+Q_2(z,r),$$

where

$$(3.4)\qquad \begin{aligned} z&=z_{i+1}+\lambda_z h_z \quad (-1\leqq\lambda_z\leqq 0),\\ r&=r_j+\lambda_r h_r \quad (0\leqq\lambda_r\leqq 1).\end{aligned}$$

The Peano form of the remainder in (3.3) is [4]

$$(3.6)\qquad \begin{aligned} Q_2(z,r)=\lambda_r^2h_r^2\,\frac{\partial^2 f}{\partial r^2}\bigg|_{i+1,j}&+2\lambda_r\lambda_z h_r h_z\,\frac{\partial^2 f}{\partial z\partial r}\bigg|_{i+1,j}+\\ &+\lambda_z^2h_z^2\frac{\partial^2 f}{\partial z^2}\bigg|_{i+1,j}+o(h_r^2+h_z^2),\end{aligned}$$

and from (3.6) we have

$$(3.7)\qquad |Q_2(z,r)|\leqq c(h_r^2+h_z^2).$$

(3.3) can be used to get the functions $f_{i+1,j+1}, f_{i+1,j}$ in (3.2). By subtracting (3.2) from (3.3) we get that

$$(3.8)\qquad \begin{aligned} f(z,r)-S_{i,j}(z,r)=Q_2(z,r)&-\lambda_r Q_2(z_{i+1},r_{j+1})+\\ &+\lambda_z Q_2(z_i,r_j)+O(h_r^2+h_z^2),\end{aligned}$$

therefore

$$(3.9)\qquad |f(z,r)-S_{i,j}(z,r)|=O(h_z^2+h_r^2).$$

Because $f(z,r)-S_\Delta(z,r)$ is continuous in Ω_e we get

$$(3.10)\qquad \|f(z,r)-S_\Delta(z,r)\|_c=O(h_z^2+h_r^2).$$

The first partial derivatives of the Taylor expansion (3.3) are

$$(3.11)\qquad \frac{\partial f}{\partial r}(z,r)=\frac{\partial f}{\partial r}\bigg|_{i+1,j}+Q_1^r(z,r),$$

$$(3.12)\qquad \frac{\partial f}{\partial z}(z,r)=\frac{\partial f}{\partial z}\bigg|_{i+1,j}+Q_1^z(z,r),$$

where the Peano remainders in (3.11) and (3.12) are

$$(3.13)\qquad Q_1^r(z,r)=(z-z_{i+1})\left.\frac{\partial^2 f}{\partial r\partial z}\right|_{i+1,j}+(r-r_j)\left.\frac{\partial^2 f}{\partial r^2}\right|_{i+1,j}+$$
$$+o(|z-z_{i+1}|)+o(|r-r_j|),$$
$$(3.14)\qquad Q_1^z(z,r)=(z-z_{i+1})\left.\frac{\partial^2 f}{\partial z^2}\right|_{i+1,j}+(r-r_j)\left.\frac{\partial^2 f}{\partial z\partial r}\right|_{i+1,j}+$$
$$+o(|z-z_{i+1}|)+o(|r-r_j|),$$

and so

$$(3.15)\qquad |Q_1^r|\leqq c_1(|z-z_{i+1}|+|r-r_j|),$$
$$(3.16)\qquad |Q_1^z|\leqq c_2(|z-z_{i+1}|+|r-r_j|).$$

The first partial derivatives of the spline functions (3.2) are

$$(3.17)\qquad \frac{\partial S_\Delta}{\partial r}(z,r)=(1+O(h_r))h_r\left(\left.\frac{\partial f}{\partial r}\right|_{i+1,j}+\frac{1}{h_r}Q_2(z_{i+1},r_{j+1})\right),$$
$$(3.18)\qquad \frac{\partial S_\Delta}{\partial r}(z,r)=(1+O(h_r))h_r\left(\left.\frac{\partial f}{\partial r}\right|_{i+1,j}+Q_2(z_{i+1},r_j)\right).$$

From the formulas (3.11 – 3.12) and (3.17 – 3.18) we get

(3.19)
$$\left|\frac{\partial f}{\partial r}-\frac{\partial S_\Delta}{\partial r}\right|=\left|Q_1^r(z,r)-\frac{1}{h_r}Q_2(z_{i+1},r_{j+1})+O(h_r)\right|=O(h_z+h_r),$$
$$(3.20)\qquad \left|\frac{\partial f}{\partial z}-\frac{\partial S_\Delta}{\partial z}\right|=\left|Q_1^z(z,r)+\frac{1}{h_z}Q_2(z_{i+1},r_j)\right|=O(h_z+h_r).$$

Similar results can be obtained for the right-side triangular elements too.

The first derivatives of the spline function S_Δ are bounded in Ω_e, therefore, from (3.10), (3.19 – 3.20) and the results obtained from Theorem 4 in [1] we have

$$(3.21)\qquad \|f(z,r)-S_\Delta(z,r)\|_{W_2^1}=O(h_z+h_r).$$

For the triangular elements in Ω_{ij} we have that

$$(3.22)\qquad \iint_{\Omega_{ij}}\frac{1}{r^3}\left(\left(\frac{\partial}{\partial r}(f-s_{ij})\right)^2+\left(\frac{\partial}{\partial z}(f-s_{ij})\right)^2\right)dzdr=O(h_r^2+h_z^2).$$

Therefore, from theorem 4 in [1] and (3.22) we can write the norm in all the domain Ω_e as follows

$$(3.23) \qquad [f(z,r)-S_\Delta(z,r)]^2 =$$

$$= \sum_{i=0}^{N-1} \sum_{j=1}^{M_i} \iint_{\Omega_{ij}} \frac{1}{r^3}\left(\left(\frac{\partial}{\partial r}(f-s_{ij})\right)^2 + \left(\frac{\partial}{\partial z}(f-s_{ij})\right)^2\right) dzdr +$$

$$+ \sum_{i=0}^{N-1} \int_0^{r_1} \int_{z_i}^{z_{i+1}} \frac{1}{r^3}\left(\left(\frac{\partial}{\partial r}(f-s_{ij})\right)^2 + \left(\frac{\partial}{\partial z}(f-s_{ij})\right)^2\right) dzdr =$$

$$= O(h_z^2 + h_r^2), \qquad M_i := \left[\frac{R(z_i)}{h_r}\right] + 1.$$

Finally, from (3.23) the norm in H_A space will have the following estimate:

$$(3.24) \qquad [f(z,r)-S_\Delta(z,r)] = O(h_z + h_r).$$

4. Convergence of the approximate solution

In this section we introduce a general definition for the partition of the domain Ω_e, from which we can define the domains used in cases 1 and 2 precisely, because we have to integrate the spline function over the domain Ω_e.

Let

$$(4.1) \qquad (z^k, r^k) \in \Gamma_1: \quad R'(z^k) = 0, \quad R''(z^k) > 0, \quad k = 1, \dots, n$$

where

$$(4.2) \qquad 0 = z^0 < z^1 < \dots < z^n < z^{n+1} = Z$$

and

$$(4.3) \qquad (\overline{z}^k, \overline{r}^k) \in \Gamma_1: \quad R'(\overline{z}^k) = 0, \quad R''(\overline{z}^k) < 0, \quad k = 1, \dots, n$$

where

$$(4.4) \qquad z^k < \overline{z}^k < z^{k+1}.$$

Consider the following facts:

a) If $R'(z) \geqq 0$ (or $R'(z) \leqq 0$) for $z \in [0, Z]$ then $n = 1$ and $\overline{z}^0 = 0$ (or $\overline{z}^0 = Z$).

b) If $R'(0) < 0$ then $\overline{z}^0 = 0$.
c) If $R'(Z) > 0$ then $\overline{z}^n = Z$.
d) If $R'(0) = 0$ and $\max R(z) = R_1$ for $z \in [0, z^1]$ then $\overline{z}^0 = 0$.
e) If $R'(Z) = 0$ and $\max R(z) = R_2$ for $z \in [z^n, Z]$ then $\overline{z}^n = Z$.
Let the inverse of the boundary functions as follows

$$\text{inv} R^k(r) = \text{inv}(R(z)), \quad z^k \leqq z < \overline{z}^k. \tag{4.5}$$

$$\text{inv} \tilde{R}^k(r) = \text{inv}(R(z)), \quad \overline{z}^k \leqq z < z^{k+1}. \tag{4.6}$$

Let us introduce the following functions:

$$Z^0(r) = \begin{cases} 0, & 0 \leqq r \leqq r^0 \\ \text{inv} R^0(r), & r^0 < r \leqq \overline{r}^0, \end{cases} \tag{4.7}$$

$$\overline{Z}^0(r) = \begin{cases} \text{inv} \tilde{R}^0(r), & r^0 > r \geqq r^1 \\ \text{inv} \tilde{R}^{m_{i-1}}(r), & r^{m_{i-1}} > r \geqq r^{m_i} \\ Z, & \min\limits_j r^j > r \geqq 0 \end{cases} \tag{4.8}$$

where

$$m_0 = 1 \quad \text{and} \quad m_i = \min\{j | j > m_{i-1}, r^j < r^{m_{i-1}}\}. \tag{4.9}$$

Furthermore, let

$$Z^k(r) = \text{inv} R^k(r), \qquad r^k \leqq r < \overline{r}^k, \tag{4.10}$$

$$\overline{Z}^k(r) = \begin{cases} \text{inv} \tilde{R}^k(r), & \overline{r}^k \geqq r \geqq \max\{r^k, r^{k+1}\} \\ \text{inv} \tilde{R}^{m^k_{i-1}}(r), & r^{m^k_{i-1}} > r \geqq \max\{r^k, r^{m^k_i}\} \\ Z, & r^{k+1} > r \geqq r^k \text{ if } r^k = \min\{r^j | j \geqq k\} \end{cases}, \tag{4.11}$$

where

$$m^k_0 = k + 1, \quad m^k_i = \min\{j | j > m^k_{k-1}, r^j < r^{m^k_{i-1}}\}. \tag{4.12}$$

We have $\Omega_e = \bigcup\limits_{k=0}^{n} \Omega_e^k$, where the elements Ω_e^k are defined as follows:

$$\Omega_e^0 = [0, \overline{r}^0] \times [Z^0(r), \overline{Z}^0(r)], \tag{4.13}$$

$$\Omega_e^k = [r^k, \overline{r}^k] \times [Z^k(r), \overline{Z}^k(r)], \quad k = 1, \dots, n. \tag{4.14}$$

It can be seen that if $(z^*, r^*) \in \Omega_e^k$, then for every $z < z^*$,

$$(z, r^*) \in \Omega_e^k, \quad k = 0, \dots, n. \tag{4.15}$$

LEMMA 1. *In problem* (1.1), *for every* $u(z,r)\in H_A$ *there exists a constant* c_0 *such that*

$$\|u\|_{W_2^1} \leqq c_0[u].$$

PROOF. For the L_2-norm of the first derivatives of the function $u(z,r)$ in Ω_e we get the same result obtained in Lemma 2 in [1], i.e.

$$\left\|\frac{\partial u}{\partial r}\right\|_{L_2}^2 + \left\|\frac{\partial u}{\partial z}\right\|_{L_2}^2 \leqq \overline{R}^3[u,u], \tag{4.16}$$

where $\overline{R} = \max\limits_z R(z)$.

To get the L_2-norm of the function $u(z,r)$ in Ω_e let

$$u(z,r) = \int\limits_0^r \frac{\partial u(z,x)}{\partial x}dx = \int\limits_0^r x^{3/2}\cdot\frac{1}{x^{3/2}}\frac{\partial u(z,x)}{\partial x}dx. \tag{4.17}$$

By using Cauchy–Bunyakovsky inequality to the square of the integral (4.17) we get

(4.18)

$$|u(z,r)|^2 \leqq \int\limits_0^r x^3 dx \cdot \int\limits_0^r \frac{1}{x^3}\left|\frac{\partial u(z,x)}{\partial x}\right|^2 dx \leqq \frac{r^4}{4}\int\limits_0^{R(z)} \frac{1}{x^3}\left|\frac{\partial u(z,x)}{\partial x}\right|^2 dx.$$

Integrating both sides of (4.18) over the domain Ω_e

(4.19)

$$\int\limits_{\Omega_e} |u(z,r)|^2 drdz = \|u\|_{L_2}^2 \leqq \int\limits_0^Z \left(\int\limits_0^{R(z)} \frac{r^4}{4}dr\cdot\int\limits_0^{R(z)} \frac{1}{x^3}\left|\frac{\partial u(z,x)}{\partial x}\right|^2 dx\right)dz \leqq$$

$$\leqq \int\limits_0^{\overline{R}} \frac{r^4}{4}dr\cdot\int\limits_0^Z\left(\int\limits_0^{R(z)} \frac{1}{r^3}\left|\frac{\partial u(z,x)}{\partial x}\right|^2 dr\right)dz = \frac{\overline{R}^5}{20}\int\limits_{\Omega_e}\frac{1}{r^3}\left|\frac{\partial u(z,r)}{\partial r}\right|^2 drdz.$$

Now let

$$u(z,r) = \int\limits_{Z^k(r)}^z \frac{\partial u(y,r)}{\partial y}dy, \quad (z,r)\in\Omega_e^k, \tag{4.20}$$

$$u(z,r) = 0 \ \text{ if } \ (z,r)\in\Gamma_1\cup\Gamma_2. \tag{4.21}$$

By using Cauchy–Bunyakovsky inequality to the square of the integral (4.20) we get

$$|u(z,r)|^2 \leqq \int_{Z^k(r)}^{z} dy \cdot \int_{Z^k(r)}^{z} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy \leqq \tag{4.22}$$

$$\leqq (z - Z^k(r)) \cdot \int_{Z^k(r)}^{\overline{Z}^k(r)} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy.$$

Integration of the left hand side of (4.22) over the domain Ω_e yields

$$\int_{\Omega_e} |u(z,r)|^2 dzdr = \sum_{k=0}^{n} \int_{\Omega_e^k} |u(z,r)|^2 dzdr. \tag{4.23}$$

Integrating both sides of (4.22) over Ω_e^k we obtain

(4.24)

$$\int_{\Omega_e^k} |u(z,r)|^2 dzdr \leqq \int_{r^k}^{\overline{r}^k} \left(\int_{Z^k(r)}^{\overline{Z}^k(r)} (z - Z^k(r))dz \cdot \int_{Z^k(r)}^{\overline{Z}^k(r)} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy \right) dr =$$

$$= \int_{r^k}^{\overline{r}^k} \left(\frac{(\overline{Z}^k(r) - Z^k(r))^2}{2} \cdot r^3 \int_{Z^k(r)}^{\overline{Z}^k(r)} \frac{1}{r^3} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy \right) dr \leqq$$

$$\leqq \frac{Z^2 \cdot \overline{R}^3}{2} \cdot \int_{r^k}^{\overline{r}^k} \int_{Z^k(r)}^{\overline{Z}^k(r)} \frac{1}{r^3} \left|\frac{\partial u(z,r)}{\partial z}\right|^2 dzdr = \frac{Z^2 \overline{R}^3}{2} \cdot \int_{\Omega_e^k} \frac{1}{r^3} \left|\frac{\partial u(z,r)}{\partial z}\right|^2 dzdr.$$

Hence

$$\|u\|_{L_2}^2 \leqq \frac{Z^2 \overline{R}^3}{2} \sum_{k=0}^{n} \int_{\Omega_e^k} \frac{1}{k^3} \left|\frac{\partial u(z,r)}{\partial z}\right|^2 dzdr. \tag{4.25}$$

Now from (4.19) and (4.25) we get the L_2-norm of the function $u(z,r)$:

(4.26)

$$\|u\|_{L_2}^2 \leqq \frac{1}{2} \max\left\{ \frac{\overline{R}^5}{2}, \frac{Z^2 \overline{R}^3}{2} \right\} \int_{\Omega_e} \frac{1}{r^3} \left(\left|\frac{\partial u(z,r)}{\partial r}\right|^2 + \left|\frac{\partial u(z,r)}{\partial z}\right|^2 \right) dzdr.$$

Finally, from (4.16) and (4.26) we get the W_2^1-norm of the function $u(z,r)$:

$$(4.27)\qquad \|u\|^2_{W_2^1} \leqq \left(\overline{R}^3 + \max\left\{\frac{\overline{R}^3}{40}, \frac{Z^2\overline{R}^3}{4}\right\}\right)[u]^2.$$

So there exists $c_0 > 0$ such that

$$(4.28)\qquad \|u\|_{W_2^1} \leqq c_0[u]$$

where

$$(4.29)\qquad c_0 = \sqrt{\left(R^3 + \max\left\{\frac{\overline{R}^3}{40}, \frac{Z^2\overline{R}^3}{4}\right\}\right)}.$$

THEOREM 2. *The approximate solutions $\tilde{S}$, obtained by the minimization of the functional* (3.8) *in* [2] *over a family of splines (see cases* 1 *and* 2*), are convergent to the generalized solutions in the W_2^1-norm:*

$$\|u_0 - \tilde{S}\|_{W_2^1} = O(h_z + h_r),$$

where u_0 denotes the generalized solutions of the original differential problems.

PROOF. In Theorem 5 in [2] we proved the following inequality:

$$(4.30)\qquad [u_o - \tilde{S}] \leqq \inf_s [u_0 - S_\Delta].$$

Therefore, from Theorem 1, Lemma 1 and (4.30) we get that

$$(4.31)\qquad \|u_0 - \tilde{S}\| \leqq c_3[u_0 - \tilde{S}] \leqq c_3 \inf_s [u_0 - S_\Delta] = O(h_z + h_r).$$

Thus the proof is complete.

Acknowledgement. The author express his sincere thanks to Dr. R. Farzan for the valuable advice and comments.

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(Received August 30, 1988)

COMPUTING CENTER OF L. EÖTVÖS UNIVERSITY
1117 BUDAPEST, BOGDÁNFFY U, 10/B
HUNGARY

Acta Math. Hung.
59(1-2) (1992), 175–216.

POLYHEDRON COMPLEXES WITH SIMPLY TRANSITIVE GROUP ACTIONS AND THEIR REALIZATIONS

E. MOLNÁR (Budapest)*

The purpose of this paper to extend algorithmically Poincaré's polyhedron theorem (Theorem 8.1) as far as reasonable so that we could treat the theory of discontinuous transformation groups in d-dimensional spaces of constant curvature by a unified method. For $d \geqq 3$ this problem is very actual now.

We start with a d-dimensional polyhedron $\mathcal{P}$ given by its finite combinatorial flag structure $\mathcal{F}$.

We describe a computer algorithm for finding all combinatorially different pairings $\mathcal{I}$ of $(d-1)$-facets of $\mathcal{P}$ to generate all possible groups $\mathcal{G}$ for which $\mathcal{P}$ is a fundamental polyhedron (Algorithm 2.2).

In each pairing $\mathcal{I}$ each identifying generator g of $\mathcal{G}$ maps a facet $\mathrm{f}_{\mathrm{g}^{-1}}$ of $\mathcal{P}$ onto the corresponding facet f_{g} and g maps the polyhedron $\mathcal{P}$ onto its g-image $\mathcal{P}^{\mathrm{g}}$. These two polyhedra are adjacent along the identified facets $\mathrm{f}_{\mathrm{g}} = (\mathrm{f}_{\mathrm{g}^{-1}})^{\mathrm{g}}$. So we shall have a polyhedron complex $\mathcal{P}^{\mathcal{G}}$ on which $\mathcal{G}$ acts simply transitively and the factor space $\mathcal{P}^{\mathcal{G}}/\mathcal{G}$ (orbit space) is the same as $\mathcal{P}$ equipped with the identifications of $\mathcal{I}$.

The pairing $\mathcal{I}$, called also identifications on $\mathcal{P}$, induces the $\mathcal{G}$-equivalence of n-faces of $\mathcal{P}$ for every n $(0 \leqq n \leqq d-1)$. For a fixed n-face x^n the stabilizer subgroup $\mathcal{G}_{\mathrm{x}^n}$ can be determined algorithmically (Algorithm 5.1) by choosing a fundamental domain $\mathcal{P}_{\mathrm{x}^n}$ for $\mathcal{G}_{\mathrm{x}^n}$ locally in $\mathcal{P}^{\mathcal{G}}$.

The basic algorithm is due to Poincaré in cases $d=2$, 3. For a fixed $(d-2)$-face e (edge if $d=3$) this stabilizer $\mathcal{G}_{\mathrm{e}}$ will be determined by the $\mathcal{G}$-equivalence class of e up to a free choice of the order ν_{e} for the "rotation" subgroup $\mathcal{R}_{\mathrm{e}}$ of $\mathcal{G}_{\mathrm{e}}$ (Algorithm 4.1). This natural number ν_{e} can be chosen arbitrarily: either $1 \leqq \nu_{\mathrm{e}}$ if $\mathcal{R}_{\mathrm{e}}$ consists of finitely many "proper" rotations fixing e and its subflags; or $\nu_{\mathrm{e}} = 0$ (instead of ∞) if the order of $\mathcal{R}_{\mathrm{e}}$ is infinite (e.g. in the hyperbolic space).

A collection of ν_{e}'s, chosen for equivalence classes of $(d-2)$-faces of $\mathcal{P}$, fixes the group $\mathcal{G}$ by a presentation. In such a presentation of $\mathcal{G}$ we give the pairing $\mathcal{I}$, which provides the identifying generators of $\mathcal{G}$, and the defining relations, specifying the order of proper rotations according to edge cycles.

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* Supported by Hungarian National Foundation for Scientific Research (OTKA) No. 424 (86) and 1238 (86).

For a given polyhedron $\mathcal{P}$ and a fixed corresponding group $\mathcal{G}$ the basic problem is whether the polyhedron complex $\mathcal{P}^{\mathcal{G}}$ is realizable in the projective model of a space of constant curvature (Sections 7–9). A number of important results, in the theory of discontinuous isometry groups, acting in spaces of constant curvature, can be reformulated in this manner.

In Section 6 we introduce a topology on the identified polyhedron $\mathcal{P}$. We show how to construct an atlas $\widetilde{\mathbf{P}}$ which contains for every point of the identified $\mathcal{P}$ a connected neighbourhood (Theorem 6.1). Our algorithm provides the atlas by its flag structure, and we do not know apriori whether $\widetilde{\mathbf{P}}$ is realizable, e.g. metrically, in a space of constant curvature. Our algorithm prepares this realization, giving necessary conditions based on the knowledge of lower dimensional situations. Since the planar discontinuous groups have already been classified, so we are able to determine the $\mathcal{G}$-stabilizer for each typical point of the identified $\mathcal{P}$ in dimension 3. So we have an effective method to describe the 3-dimensional discontinuous groups for a concrete $\mathcal{P}$. In Section 7 we indicate the tools of the constructions by sketching the vector models of the classical spaces of constant curvature.

In Theorem 8.1 we shall prove that the metric realization of the atlas $\widetilde{\mathbf{P}}$ involves that of the global polyhedron complex $\mathcal{P}^{\mathcal{G}}$. This can be considered as a generalized Poincaré theorem, since it gives "local criteria" to guarantee a global discontinuous group action on a space of constant curvature. In contrast to this investigations so far, we permit for $\mathcal{P}$ concave realizations as well as other more general properties. In the proof we use the covering space argument applied by A. D. Aleksandrov in a different situation.

Of course, our method is also restricted because of the algorithmic unsolvability of the basic problems in the combinatorial theory of groups. However, for polyhedra with few flags (also in higher dimensions) our method seems to be useful as the examples show in Section 9. We suggest the reader to examine the figures to study these examples.

Extremely important are the cases when the group $\mathcal{G}$ acts freely on the complex $\mathcal{P}^{\mathcal{G}}$. Then the identified polyhedron $\mathcal{P}$ may have a manifold structure and we can attack the space form problem which is very actual in the Bolyai–Lobachevskian hyperbolic 3-space $\mathcal{H}^3$. For this reason we can prescribe additional requirements for the pairing $\mathcal{I}$ of $\mathcal{P}$ to guarantee the torsion free action of the corresponding group $\mathcal{G}$ on $\mathcal{P}^{\mathcal{G}}$.

I shall mention some results obtained by computer in joint works with I. Prok and illustrate the algorithm and the occurring phenomena by examples in 2 and 3 dimensional spaces of constant curvature.

1. Polyhedra and flag structures, isomorphisms

Suppose we are given a d-dimensional polyhedron $\mathcal{P}$ by a finite incidence structure. This means, we are given the sets of the n-faces of $\mathcal{P}$, denoted by F^n, $n=0,1,\ldots,d-2,d-1$ (and the special cases V, E, F in dimension 3), which are identified with the index sets $\{1,\ldots,f^n\}$ (and $\{1,\ldots,v\}$, $\{1,\ldots,e\}$, $\{1,\ldots,f\}$, called vertices, edges, facets, respectively). We suppose that all incidences are given, satisfying some requirements as listed e.g. in [8, 36]. In Fig. 1 we have described a scheme about the incidence structure of $\mathcal{P}$ (a trigonal prisma) where on different levels the faces, ..., the vertices appear as (numbered) nodes, and chains of arrows indicate the incidences. The facets are groupped into isomorphism classes for the reason to be explained later on.

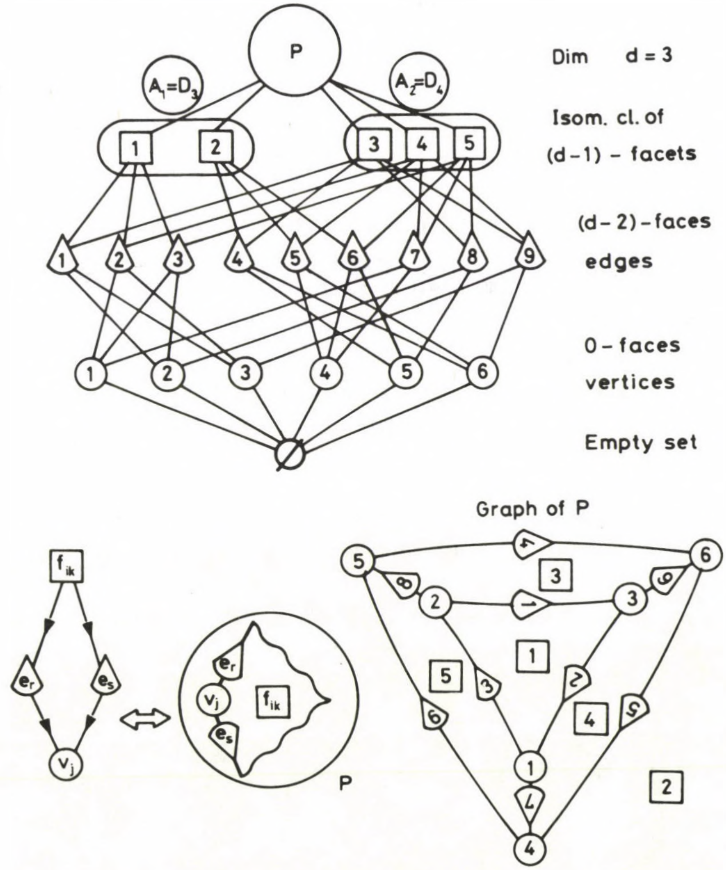


Fig. 1

Now, we formulate the requirements for the flags of $\mathcal{P}$ which reflect the incidence structure, too.

We define a flag as an ordered d-tuple of incident 0-face, ..., m-face, ..., $(d-1)$-facet. A non-empty subset $\mathcal{F}$ of the direct d-product $\mathrm{V}\times\cdots\times\mathrm{E}\times\mathrm{F}$ is defined as the *flag structure* of the polyhedron $\mathcal{P}$ iff the following properties F. 1–2 are fulfilled:

F.1. *Each flag in $\mathcal{P}$ has exactly one n-adjacent flag for each dimension n, $0 \leqq n \leqq d-1$, i.e. only their n-faces are different, the other components coincide.*

Imagine $\mathcal{F}$ like a graph whose points are the flags, the n-adjacent flags are connected with an edge marked by the n-th colour [11, 22].

F.2. *$\mathcal{F}$ is strictly connected, i.e. if f and g are different flags of $\mathcal{F}$, then there is a finite number of flags $f = f_1, f_2, f_3, \dots, f_r = g$, each of them has the common components of f and g, such that f_{s+1} is n-adjacent to f_s for certain n $(1 \leqq s \leqq r-1)$.*

We see that an n-face x^n of $\mathcal{P}$ can be characterized by that subset $\mathcal{F}_{\mathrm{x}^n}$ of $\mathcal{F}$ where in the flags of $\mathcal{F}_{\mathrm{x}^n}$, x^n is just the common n-th component $(0 \leqq n \leqq \leqq d-1)$. The subflags of an n-face as corresponding n-tuples can be defined obviously.

The n-face x^n and the m-face y^m are incident iff there is a flag whose n-th and m-th component is just x^n and y^m, respectively; i.e. $\mathcal{F}_{\mathrm{x}^n} \cap \mathcal{F}_{\mathrm{y}^m} \neq \emptyset$.

We easily conclude that the incidence structure of $\mathcal{P}$ and its flag structure $\mathcal{F}$ determine each other.

Fig. 1 shows two 1-adjacent flags ${}^1f_{ik}(\mathrm{v}_j, \mathrm{e}_r, \mathrm{f}_{ik})$ and ${}^2f_{ik}(\mathrm{v}_j, \mathrm{e}_s, \mathrm{f}_{ik})$, $\mathrm{e}_r \neq \neq \mathrm{e}_s$, symbolically and with picture as well $(d = 3)$.

Note that the polyhedron $\mathcal{P}$ can be given to a computer as the set $\mathcal{F}$ of its flags, i.e. ordered d-tuples of natural numbers. For instance a 3-dimensional tetrahedron is represented by 24 flags, a cube by 48 flags. The one-to-one correspondence between an n-face x^n and the flag subset $\mathcal{F}_{\mathrm{x}^n}$, defined above, makes possible the convenient translation of geometric facts into flag language, i.e. computer one, and vice versa.

The main purpose of this section is to describe an algorithm for determining the *automorphism group* of $\mathcal{P}$, denoted by $\mathrm{Aut}\,\mathcal{P}$. This automatically describes the *automorphism group* $\mathrm{Aut}\,\mathcal{F}$ *of the flag structure* $\mathcal{F}$.

$\mathrm{Aut}\,\mathcal{P}$ is a permutation group. Each permutation in $\mathrm{Aut}\,\mathcal{P}$ consists of d component permutations. The n-th component permutes the n-faces of $\mathcal{P}$, $0 \leqq n \leqq d-1$. Every elements of $\mathrm{Aut}\,\mathcal{P}$ preserves the incidence structure of $\mathcal{P}$, i.e. if an n-face x^n and an m-face y^m are incident then for each permutation p from $\mathrm{Aut}\,\mathcal{P}$ the n-th and m-th components of p maps x^n onto $(\mathrm{x}^n)^{\mathrm{p}}$, y^m onto $(\mathrm{y}^m)^{\mathrm{p}}$so that the images are also incident.

$\mathrm{Aut}\,\mathcal{F}$ consists of all self-bijections of $\mathcal{F}$ which preserve n-adjacency relations for every n $(0 \leqq n \leqq d-1)$. These bijections form a group by the composition as group operation.

LEMMA 1.1. *An automorphism* a *from* $\mathrm{Aut}\,\mathcal{F}$*, resp.* $\mathrm{Aut}\,\mathcal{P}$*, is uniquely determined (if it exists) by a flag f and its* a*-image* f^{a}.

The proof is a trivial consequence of requirements F.1–2. A bijection $\mathrm{a}: f \mapsto f^{\mathrm{a}}$ extends uniquely, if the extension exists, to the 0-adjacent flags of f and of f^{a}, respectively, then to the 1-adjacent ones, and so on. Since $\mathcal{F}$ is connected by F.2 and finite thus the process ends, or it turns out that the extension does not exist. Q.e.d.

The method in the preceding proof suggests to construct the

ALGORITHM 1.1 *enumerating all flags in* $\mathcal{F}$ *from* 1 *up to cardinality* $|\mathcal{F}|$, *uniquely if the starting flag* ${}^1f$ *is presribed:*

We choose the 0-adjacent flag of ${}^1f$ to obtain ${}^2f$. Suppose that we have already listed the flags ${}^1f, {}^2f, \ldots, {}^rf$. To ${}^rf$ we consider its 0-adjacent, 1-adjacent, ..., $(d-1)$-adjacent flags in this order. The first flag among them, not listed yet, will be ${}^{r+1}f$, if such one exists. If every adjacent flag of ${}^rf$ has already been listed then we return to ${}^{r-1}f$ and repeat the previous process with ${}^{r-1}f$.

Since $\mathcal{F}$ is (strictly) connected and finite, the enumerating algorithm ends at ${}^{|\mathcal{F}|}f$. □

We remark that the selection rule of the adjacency relations depends on us, we could start first, e.g., with $(d-1)$-adjacent, flag, then $(d-2)$-adjacent one, etc., in choosing ${}^{r+1}f$ to ${}^rf$.

Now we formulate

ALGORITHM 1.2 *for finding* Aut $\mathcal{F}$ *and* Aut $\mathcal{P}$.

Fix the numbering of flags ${}^1f, {}^2f, \ldots, {}^{|\mathcal{F}|}f$ by Algorithm 1.1. Choose another starting flag from among them (e.g. first ${}^2f$) and denote it by ${}^1f' =: {}^1f^{\mathrm{a}}$. We start with constructing an automorphism $\mathrm{a} \in \mathrm{Aut}\,\mathcal{F}$ and a permutation a in Aut $\mathcal{P}$, if it is possible. The m-th component of a begins with permuting m-faces: we associate the m-th component of ${}^1f$ with the m-th component of ${}^1f'$ for any m, $0 \leqq m \leqq d-1$. The 0-adjacent flag of ${}^1f'$ is ${}^2f' =: {}^2f^{\mathrm{a}}$. This continues also the construction of $\mathrm{a} \in \mathrm{Aut}\,\mathcal{P}$ with building its 0-th component associating the 0-face in ${}^2f$ with the 0-face in ${}^2f'$. Having had the list

$$ {}^1f' =: {}^1f^{\mathrm{a}},\ {}^2f' = {}^2f^{\mathrm{a}},\ \ldots,\ {}^rf' =: {}^rf^{\mathrm{a}} $$

by Algorithm 1.1 and constructed the components of $\mathrm{a} \in \mathrm{Aut}\,\mathcal{P}$ without contradiction, we turn to ${}^{r+1}f' = {}^{r+1}f^{\mathrm{a}}$ and continue building components of permutation $\mathrm{a} \in \mathrm{Aut}\,\mathcal{P}$.

If, for some m $(0 \leqq m \leqq d-1)$, the m-th component (the m-face) of ${}^{r+1}f'$, ordered to that of ${}^{r+1}f$, is different as defined in a previous step of the process, then this a does not extend to an element of Aut $\mathcal{P}$, respectively of Aut $\mathcal{F}$, so we drop this ${}^1f'$ and the involved constructions and turn to another starting flag (to the next one in the fixed list).

If such a contradiction does not occur, then we take ${}^{r+2}f' =: {}^{r+2}f^{\mathrm{a}}$ and continue constructing $\mathrm{a} \in \mathrm{Aut}\,\mathcal{P}$ till ${}^{|\mathcal{F}|}f' =: {}^{|\mathcal{F}|}f^{\mathrm{a}}$. Then a will be taken into the list of Aut $\mathcal{P}$, and we turn to the next starting flag till ${}^{1}f' := {}^{|\mathcal{F}|}f$. □

Note that every element of Aut $\mathcal{P}$ acts on the flags in $\mathcal{F}$ component-wise, equivalently as Aut $\mathcal{F}$ acts on $\mathcal{F}$. In the following we prefer Aut $\mathcal{P}$, it need less memory capacity.

To summarize, we generally formulate

THEOREM 1.2. *Algorithm* 1.2 *leads to all elements of* Aut $\mathcal{P}$ *(and* Aut $\mathcal{F}$*). If we take the flags of fixed* $(d-1)$*-th component, i.e. fixed* $(d-1)$*-facet* f, *and modify the selection rule in Algorithm* 1.1 *so that we prefer* 0*-adjacency up to* $(d-2)$*-adjacency in choosing the next flag in* $\mathcal{F}_{\mathrm{f}}$ *(the flag structure of facet* f*), then Algorithm* 1.2 *leads to the automorphism group* Aut f *of the facet* f *(respectively* Aut $\mathcal{F}_{\mathrm{f}}$*). The flags with fixed* m*-th component, i.e. fixed* m*-face* x^m, *and with fixed components from* $(m+1)$*-th up to the* $(d-1)$*-th one, provide us the automorphism group* Aut x^m *by Algorithm* 1.2, *if we prefer* 0*-adjacency up to* $(m-1)$*-adjacency.*

PROOF. 1) Algorithm 1.2 leads to bijections, preserving incidences by construction. Lemma 1.1 says that we obtain all automorphisms in Aut $\mathcal{P}$, respectively in Aut $\mathcal{F}$.

2) Fixing the last component, we also obtain a flag structure of dimension $d-1$.

3) Fixing the m-th component x^m, the flags in $\mathcal{F}_{\mathrm{x}^m}$ distribute into as many "congruent" classes as many different choices we have at the last $d-m-1$ components. Fixing any choice of them we obtain a subflag structure of dimension m in the first m components. Q.e.d.

EXAMPLES. 1) The 3-tetrahedron has 24 automorphisms, any face of it has 6 ones, any edge has 2 ones, any vertex has only the trivial automorphism.

2) The trigonal prisma (Fig. 1) has 12 automorphisms, one of its quadrangle faces has 8, one of its triangle faces has 6, and so on.

3) The Archimedian 3-solid of face symbol (8,8,3) (truncated 3-cube) has 48 automorphisms, as many as the 3-cube (Section 9, Fig. 10).

4) The 4-cube has $2\cdot 4\cdot 6\cdot 8 = 384$ automorphisms, any cube facet of it has 48, any 2-face of it, from the 24, has 8 automorphisms.

2. Involutive pairings on polyhedron $\mathcal{P}$

In this section our aim is to find all combinatorially different involutive facet pairings of the polyhedron $\mathcal{P}$.

First we formulate

ALGORITHM 2.1 *determining the isomorphism classes of facets of* $\mathcal{P}$.

For any (i-th) class we start with a flag with fixed last facet component f_{i1} and denote it by ${}^{1}f_{i1}$. We look for another facet f_{ik} of $\mathcal{P}$ and a flag ${}^{1}f_{ik}$ to it (with f_{ik} as last component). Then we proceed with Algorithm 1.2 with ${}^1\mathcal{f}_{ik} = {}^1\mathcal{f}_{i1}^{\varphi_{ik}}$ to get an isomorphism φ_{ik} between $(d-1)$-dimensional facets. If it exists, φ_{ik} maps the corresponding faces of f_{i1} onto those of f_{ik} by a map of $d-2$ components (vertex to vertex, m-face to m-face). If not, then we choose the next flag to f_{ik} as starting flag. Else we turn to the next facet of $\mathcal{P}$.

Suppose, we have found k_i facets in the i-th isomorphism class, and all the assigning maps φ_{ik}, $1 < k \leqq k_i$. Moreover, let $\mathcal{A}_i$ denote the automorphism group of the face f_{i1} (see also Theorem 1.2). Then every isomorphism

$$(1) \qquad g_{ikm} : f_{ik} \mapsto f_{im}, \qquad 1 \leqq k,\ m \leqq k_i,$$

can be expressed in the form

$$(2) \qquad g_{ikm} = \varphi_{ik}^{-1} a_{ikm} \varphi_{im}$$

where a_{ikm} runs over the element of $\mathcal{A}_i$. Note that $m = k$ is also allowed; even more, if e.g. $k = 1$ then $\varphi_{ik} = \varphi_{i1} = \mathrm{id}\, f_{i1}$, the identity of f_{i1}, holds by agreement.

We form the next isomorphism class by choosing a new facet and a new starting flag to it. The number I of isomorphism classes is finite. □

We remark that in a real computer program it is worthwhile to determine first the isomorphism classes of facets and the typical automorphism groups $\mathcal{A}_i$ and, only after that, Aut $\mathcal{P}$. For a particular polyhedron $\mathcal{P}$ these can be given to the computer in advance (e.g. for $d = 3$).

Second we define an involutive pairing $\mathcal{I}$, called also identifications of $\mathcal{P}$, as a collection of fixed isomorphisms between facets of $\mathcal{P}$ such that

a) every facet is paired in $\mathcal{I}$;

b) if g_{ikm}, by formulas (1) and (2), occurs in $\mathcal{I}$ according to i-th facet class in Algorithm 2.1 then $g_{imk} = g_{ikm}^{-1} : f_{im} \mapsto f_{ik}$ also occurs in $\mathcal{I}$ in the form $g_{imk} = \varphi_{im}^{-1} a_{imk} \varphi_{ik}$ where $a_{imk} = a_{ikm}^{-1}$ holds in $\mathcal{A}_i$;

c) if a face f_{ik} paired with itself, i.e. $g_{ikk} = \varphi_{ik}^{-1} a_{ikk} \varphi_{ik} : f_{ik} \mapsto f_{ik}$, then $g_{ikk} = g_{ikk}^{-1}$ and consequently $a_{ikk} = a_{ikk}^{-1}$.

Requirement c) is a particular case of b), they say that $\mathcal{I}$ is involutive. To summarize we formulate

THEOREM 2.1. *Each involutive pairing $\mathcal{I}$ of $\mathcal{P}$ can be given by a collection of involutive permutations*

$$(3) \qquad \begin{pmatrix} \dots, k, \dots, m, \dots \\ \dots, m, \dots, k, \dots \end{pmatrix}_i, \qquad 1 \leqq i \leqq \mathrm{I},$$

and, for the i-th permutation, by an ordered set of automorphisms from $\mathcal{A}_i$

$$(4) \qquad (\dots, a_{ikm}, \dots, a_{imk}, \dots) \ \textit{with}\ \ a_{imk} = a_{ikm}^{-1} \ \ \textit{in}\ \ \mathcal{A}_i.$$

These prescribe the isomorphisms $\mathrm{g}_{ikm}: \mathrm{f}_{ik} \mapsto \mathrm{f}_{im}$ *in* $\mathcal{I}$, *according to the* i*-th isomorphism class of facets of* $\mathcal{P}$, *by* $\mathrm{g}_{ikm} = \varphi_{ik}^{-1} \mathrm{a}_{ikm} \varphi_{im}$ *where, e.g.* $\varphi_{ik}: \mathrm{f}_{i1} \mapsto \mathrm{f}_{ik}$ *the fixed assigning isomorphism.*

PROOF is obvious by requirements of involutive pairing.

Note that each pairing $\mathcal{I}$ (the adjective "involutive" will be omitted later on) can be considered as an involutive self-bijection of the flag set $\mathcal{F}$ of $\mathcal{P}$, which preserves the incidence structure of facets as $(d-1)$-polyhedra. The automorphisms of $\mathcal{P}$ are also such self-bijections but they preserve the complete incidence structure of $\mathcal{P}$. So $\operatorname{Aut} \mathcal{P}$ acts on the set of involutive pairings by conjugations.

Two involutive pairings $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *of* $\mathcal{P}$ *are said to be equivalent if an automorphism* a *from* $\operatorname{Aut} \mathcal{P}$ *exists so that* $\mathcal{I}_2 = \mathrm{a}^{-1} \mathcal{I}_1 \mathrm{a}$.

Third, by Theorem 2.1, we can formulate

ALGORITHM 2.2 *determining all non-equivalent involutive facet pairings of* $\mathcal{P}$.

To the i-th facet isomorphism class of $\mathcal{P}$, for each i, we prepare all involutive permutations according to formula (3). These permute k_i elements of the i-th facet class.

0) The identity pairing, of $\mathcal{P}$ is the first in the list of pairings.

1) For a fixed i-permutation we choose a k_i-tuple a_{ikm} of $\mathcal{A}_i$ by (4), considering the requirements of involutivity. We take this for each i, i.e. for each isomorphism class. So we obtain a pairing $\mathcal{I}$.

2) We prepare the conjugates $\mathrm{a}^{-1} \mathcal{I} \mathrm{a}$ with the elements a of $\operatorname{Aut} \mathcal{P}$ and compare them with the former pairings listed already.

3) If no conjugate has been listed formerly then we take $\mathcal{I}$ into the list of pairings. If a conjugate of $\mathcal{I}$ has been already listed then we drop $\mathcal{I}$. We turn to a new pairing by a new choice of the k_i-tuple a_{ikm} from $\mathcal{A}_i$ to the i-permutation.

4) If we have exhausted the k_i-tuples from $\mathcal{A}_i$, then we turn to the next i-permutation with the starting k_i-tuple from $\mathcal{A}_i$.

5) If we have exhausted the i-permutation, then we return to the starting state of i-permutation and of k_i-tuple from $\mathcal{A}_i$, and we take the next k_{i+1}-tuple from $\mathcal{A}_{i+1}$ to the $(i+1)$-permutation.

6) Then we vary the $(i+1)$-permutation, and so on.

7) The process ends and we have got the list of different involutive facet pairings of $\mathcal{P}$. $\square$

Observe that this is a very long algorithm; in general, it is of exponential complexity. Now we do not deal with questions to spare place and time. For particular graph algorithms we refer to [10].

EXAMPLES. 1) In [28] we reported that a 3-dimensional tetrahedron has 64 different involutive facet pairings. We listed 15 of them which determine 26 Euclidean space tilings, realized by 21 crystallographic space groups.

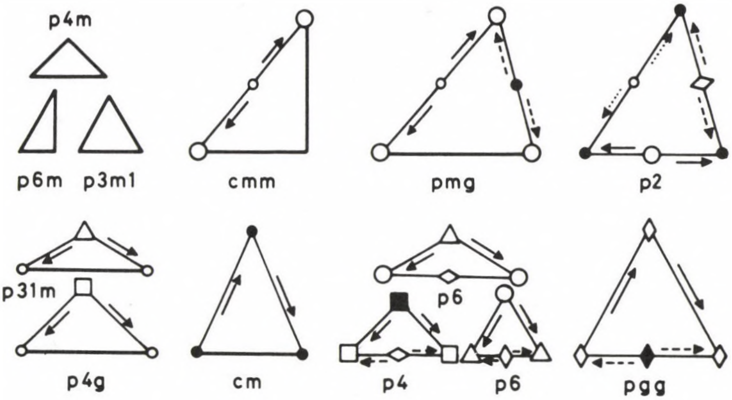


Fig. 2

2) The plane triangle has 8 involutive side pairings determining 13 Euclidean tilings and 12 crystallographic groups (Fig. 2). Taking also quadrangle, pentagon, hexagon we obtain the 46 fundamental tilings for the 17 plane crystallographic groups [16, 17].

3. A group generated by a pairing of $\mathcal{P}$

We generalize a usual construction, see e.g. in [5, 20, 33, 34] for $d=2,3$. A fixed facet pairing $\mathcal{I}$ of the polyhedron $\mathcal{P}$ is given by a collection

$$\ldots,\quad \mathrm{g}_{ikm}:\mathrm{f}_{ik}\mapsto \mathrm{f}_{im},\quad \mathrm{g}_{imk}=\mathrm{g}_{ikm}^{-1}:\mathrm{f}_{im}\mapsto \mathrm{f}_{ik},\quad \ldots$$

of isomorphisms defined in Section 2. Each $\mathrm{g}_{ikm}=:\mathrm{g}$ as a generator of a group $\mathcal{G}$ maps the facet $\mathrm{f}_{ik}=:\mathrm{f}_{\mathrm{g}^{-1}}$ of $\mathcal{P}$ onto the corresponding facet $\mathrm{f}_{im}= =:\mathrm{f}_{\mathrm{g}}$ and we define g to map the polyhedron $\mathcal{P}=:(\mathcal{P},1)$ onto its g-image $(\mathcal{P},\mathrm{g})$. These two polyhedra are said to be adjacent (neighbouring) along the identified facet $(\mathrm{f}_{\mathrm{g}},1)=(\mathrm{f}_{\mathrm{g}^{-1}},\mathrm{g})$. Similarly, the inverse $\mathrm{g}_{imk}=:\mathrm{g}^{-1}$ maps f_{g} onto $\mathrm{f}_{\mathrm{g}^{-1}}$ and we say that g^{-1} maps $\mathcal{P}=:(\mathcal{P},1)$ onto $(\mathcal{P},\mathrm{g}^{-1})$ which are adjacent along the identified facet $(\mathrm{f}_{\mathrm{g}^{-1}},1)=(\mathrm{f}_{\mathrm{g}},\mathrm{g}^{-1})$. Note that $\mathrm{f}_{\mathrm{g}}=\mathrm{f}_{\mathrm{g}^{-1}}$ is also allowed, then $(\mathcal{P},\mathrm{g})=(\mathcal{P},\mathrm{g}^{-1})$ and $\mathrm{g}=\mathrm{g}^{-1}$ is an involutive generator of $\mathcal{G}$ ($\mathrm{g}^2=1$ is a relation in $\mathcal{G}$) (Fig. 3).

We define a combinatorial space tiling by $\mathcal{P}$ under $\mathcal{G}$, i.e. a polyhedron complex as a collection

$$(\mathcal{P},\mathcal{G}):=\{(\mathcal{P},\mathrm{h}):h\in\mathcal{G}\}$$

where h is a product of the generators or their inverses from $\mathcal{I}$, and some identification in $(\mathcal{P},\mathcal{G})$ will be introduced.

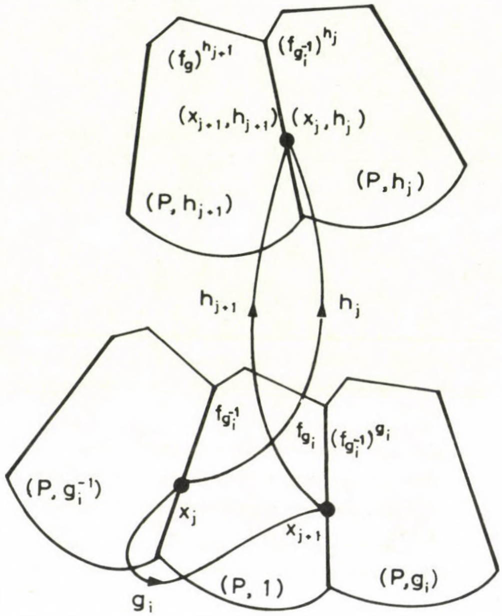


Fig. 3

REMARK. We mention *that a flag structure of* $d+1$ *components will also be defined.*

$$(\mathcal{F}, \mathcal{G}) := \{(\mathscr{f}, \mathrm{h}) : \mathscr{f} \in \mathcal{F},\ \mathrm{h} \in \mathcal{G}\}.$$

Every flag receives a group element h as a d-dimensional component or solid component. We say that each flag $(\mathscr{f}, 1)$, with facet component $\mathrm{f_g}$ and solid component 1 (the identity), is d-adjacent (solid-adjacent) with the flag $(\mathscr{f}^{\mathrm{g}^{-1}}, \mathrm{g})$, whose components from 0-th to $(d-1)$-th one are the same as those of $\mathscr{f}^{\mathrm{g}^{-1}}$, as a flag of $\mathcal{F}$, and the d-th component is the generator g of $\mathcal{G}$ $(\mathrm{g} \in \mathcal{I})$. □

In general, *we define the identification of m-faces of* $(\mathcal{P}, \mathcal{G})$. Two m-faces (x, g) and (y, h) in $(\mathcal{P}, \mathcal{G})$ are identified (glued), iff for m-faces x, y of $\mathcal{P}$ and for g, $\mathrm{h} \in \mathcal{G}$ either

1) $(\mathrm{x}, \mathrm{g}) = (\mathrm{y}, \mathrm{h})$

holds, i.e. $\mathrm{x} = \mathrm{y}$ in $\mathcal{P}$ and $\mathrm{g} = \mathrm{h}$ in the group $\mathcal{G}$ (i.e. they are the same words, may be empty, of the identifying generators g_j and g_j^{-1} of $\mathcal{G}$), or

2) for some $(\mathrm{x}_i, \mathrm{h}_i)$ we have

$$(\mathrm{x}, \mathrm{g}) = (\mathrm{x}_1, \mathrm{h}_1) \sim (\mathrm{x}_2, \mathrm{h}_2) \sim \cdots \sim (\mathrm{x}_r, \mathrm{h}_r) = (\mathrm{y}, \mathrm{h}),$$

where $(\mathrm{x}_i, \mathrm{h}_i) \sim (\mathrm{x}_{i+1}, \mathrm{h}_{i+1})$ holds iff an identifying generator (or an inverse) g_j exists such that x_i is an m-face of the facet $\mathrm{f}_{\mathrm{g}_j^{-1}}$ of $\mathcal{P}$, $\mathrm{x}_{i+1} = \mathrm{x}_i^{\mathrm{g}_j}$ is the image m-face of the facet $\mathrm{f}_{\mathrm{g}_j}$ of $\mathcal{P}$, and $\mathrm{h}_i = \mathrm{g}_j \mathrm{h}_{i+1}$ (Fig. 3).

We write $\langle \mathrm{x}, \mathrm{g}\rangle = \langle \mathrm{y}, \mathrm{h}\rangle$ or $\mathrm{x}^{\mathrm{g}} = \mathrm{y}^{\mathrm{h}}$ defined by the above equivalence relation. This generally defines the quotient space $\langle \mathcal{P}, \mathcal{G}\rangle$. It will be denoted also by $\mathcal{P}^{\mathcal{G}}$.

Note that if $\langle \mathrm{x}, \mathrm{g}\rangle = \langle \mathrm{y}, \mathrm{h}\rangle$ *then* $\langle \mathrm{x}, \mathrm{gk}\rangle = \langle \mathrm{y}, \mathrm{hk}\rangle$ also holds *for every* $\mathrm{g}, \mathrm{h}, \mathrm{k} \in \mathcal{G}$, *and every m-faces* x, y of $\mathcal{P}$.

Each $\mathrm{k} \in \mathcal{G}$ defines an action $\langle \mathrm{k}\rangle$ on $\langle \mathcal{P}, \mathcal{G}\rangle$ just by $\langle \mathrm{k}\rangle : \langle \mathcal{P}, \mathcal{G}\rangle \to \langle \mathcal{P}, \mathcal{G}\rangle$, $\langle \mathrm{x}, \mathrm{h}\rangle \mapsto \langle \mathrm{x}, \mathrm{hk}\rangle$ for any m-face x of $\mathcal{P}$, $0 \leqq m \leqq d-1$.

We can easily see that

$$\langle \mathrm{k}^{-1}\rangle = \langle \mathrm{k}\rangle^{-1} \quad \text{and} \quad \langle \mathrm{kl}\rangle = \langle \mathrm{k}\rangle\langle \mathrm{l}\rangle$$

so we can define a group $\langle \mathcal{G}\rangle$ of such $\langle \mathrm{k}\rangle$'s as self-bijections of $\langle \mathcal{P}, \mathcal{G}\rangle$, moreover,

$$\mathcal{G} \to \langle \mathcal{G}\rangle, \qquad \mathrm{k} \mapsto \langle \mathrm{k}\rangle$$

is just an isomorphism. So we shall obtain the polyhedron complex $\langle \mathcal{P}, \mathcal{G}\rangle = {=:}\, \mathcal{P}^{\mathcal{G}}$ on which $\mathcal{G}$ acts simply transitively, preserving all the incidences. We formulate these important facts in the language of flags.

THEOREM 3.1. *Let $\langle \mathcal{P}, \mathrm{g}\rangle$ and $\langle \mathcal{P}, \mathrm{h}\rangle$ be any two polyhedra in the polyhedron complex $\langle \mathcal{P}, \mathcal{G}\rangle$. Then to any flag $(\mathfrak{f}, \mathrm{g})$ of $(\mathcal{F}, \mathrm{g})$ for $(\mathcal{P}, \mathrm{g})$ there exists uniquely the flag $(\mathfrak{f}, \mathrm{h})$ of $(\mathcal{F}, \mathrm{h})$ for $\langle \mathcal{P}, \mathrm{h}\rangle$ such that $\mathrm{g}^{-1}\mathrm{h} \in \mathcal{G}$ maps $(\mathfrak{f}, \mathrm{g})$ onto $(\mathfrak{f}, \mathrm{h})$, $\langle \mathcal{P}, \mathrm{g}\rangle$ onto $\langle \mathcal{P}, \mathrm{h}\rangle$ and $\langle \mathcal{P}, \mathcal{G}\rangle$ onto itself. Moreover $\mathrm{g}^{-1}\mathrm{h}$ preserves all incidences of $\langle \mathcal{P}, \mathcal{G}\rangle$.*

PROOF. The elements g, h of $\mathcal{G}$ and so $\mathrm{g}^{-1}\mathrm{h}$ are products of generators from $\mathcal{I}$. So we reduce the procedure of construction by elementary observations.

First, we restrict ourselves to the polyhedra $\langle \mathcal{P}, \mathrm{h}\rangle$ and $\langle \mathcal{P}, \mathrm{g}_i\mathrm{h}\rangle$ where $\mathrm{h} \in \mathcal{G}$, $\mathrm{g}_i \in \mathcal{I}$.

These are mapped by

$$\langle \mathrm{h}^{-1}\mathrm{g}_i\mathrm{h}\rangle : \langle \mathcal{P}, \mathrm{h}\rangle \mapsto \langle \mathcal{P}, \mathrm{g}_i\mathrm{h}\rangle : \mathrm{f}_{\mathrm{g}_i}^{\mathrm{g}_i^{-1}\mathrm{h}} = \mathrm{f}_{\mathrm{g}_i^{-1}}^{\mathrm{h}} \mapsto \mathrm{f}_{\mathrm{g}_i^{-1}}^{\mathrm{g}_i\mathrm{h}} = \mathrm{f}_{\mathrm{g}_i}^{\mathrm{h}}$$

each onto another. We know that any flag $\mathfrak{z} \in \mathcal{F}$ with facet component $\mathrm{f}_{\mathrm{g}_i^{-1}}$ and its g_i-image $\mathfrak{z}^{\mathrm{g}_i} \in \mathcal{F}$, prescribed by $\mathcal{I}$, determines this mapping uniquely, so that it preserves every m-adjacency of flags $0 \leqq m \leqq d-1$.

We can uniquely extend this mapping to the generalized flags (see Remark of this section)

$$\langle \mathrm{h}^{-1}\mathrm{g}_i\mathrm{h}\rangle : (\mathcal{F}, \mathrm{h}) \mapsto (\mathcal{F}, \mathrm{g}_i\mathrm{h}), \qquad (\mathfrak{f}, \mathrm{h}) \mapsto (\mathfrak{f}, \mathrm{g}_i\mathrm{h}), \qquad \mathfrak{f} \in \mathcal{F}$$

with the requirements that $\langle \mathrm{h}^{-1}\mathrm{g}_i\mathrm{h}\rangle$ shall preserve the m-adjacency structure of flags for every dimension m, $0 \leqq m \leqq d$.

Second, we define for any generator $\mathrm{g}_i \in \mathcal{I}$ the mapping

$$\langle \mathrm{g}_i\rangle : \langle \mathcal{P}, \mathrm{k}\rangle \mapsto \langle \mathcal{P}, \mathrm{kg}_i\rangle; \qquad (\mathfrak{f}, \mathrm{k}) \mapsto (\mathfrak{f}, \mathrm{kg}_i)$$

of the polyhedron complex $\langle\mathcal{P},\mathcal{G}\rangle$ and of the generalized flag structure $(\mathcal{F},\mathcal{G})$ inductively.

Let $\mathrm{k}=\mathrm{g_{k1}g_{k2}\dots g_{kr}}$ be the expression of $\mathrm{k}\in\mathcal{G}$ by the $\mathrm{g}_{\mathrm{k}j}$'s from $\mathcal{I}$. Then we apply the previous procedure step by step.

$$(f,\mathrm{k})\mapsto(f,\mathrm{g_{k1}^{-1}k})\mapsto(f,\mathrm{g_{k2}^{-1}g_{k1}^{-1}k})\mapsto\dots\mapsto(f,1)\mapsto(f,\mathrm{g}_i)\mapsto$$
$$\mapsto(f,\mathrm{g}_{\mathrm{k}r}\mathrm{g}_i)\mapsto\dots\mapsto(f,\mathrm{g_{k2}}\dots\mathrm{g}_{kr}\mathrm{g}_i)\mapsto(f,\mathrm{kg}_i).$$

Third, we define for any element $\mathrm{h}=\mathrm{g_{h1}g_{h2}}\dots\mathrm{g}_{\mathrm{h}s}$ $(\mathrm{g}_{\mathrm{h}j}\in\mathcal{I})$ the mapping

$$\langle\mathrm{h}\rangle:\langle\mathcal{P},\mathrm{k}\rangle\mapsto\langle\mathcal{P}.\mathrm{kh}\rangle;\qquad(f,\mathrm{k})\mapsto(f,\mathrm{kh})$$

inductively again. We can check that $\langle\mathrm{h}\rangle$ is uniquely defined by the requirements that it shall preserve the m-adjacency structure of generalized flags for every dimension m, $0\leqq m\leqq d$. Q.e.d.

We remark that the automorphism group of the flag structure $(\mathcal{F},\mathcal{G})$ is a supergroup of $\mathcal{G}$. Investigation of $\mathrm{Aut}(\mathcal{F},\mathcal{G})$ and its subgroups, e.g. the normalizer of $\mathcal{G}$ in $\mathrm{Aut}(\mathcal{F},\mathcal{G})$ seems to be an important problem, e.g. in 3-dimensional crystallography [9, 26].

We see that the factor space $\langle\mathcal{P},\mathcal{G}\rangle/\langle\mathcal{G}\rangle$ as $\mathcal{G}$-orbits of m-faces of $\mathcal{P}$ $(0\leqq\leqq m\leqq d)$ is the same as m-faces of $\mathcal{P}$ factored by the equivalence induced by $\mathcal{I}$.

4. Barycentric subdivision of $\mathcal{P}$, stabilizers of $(d-2)$-faces, presentation of $\mathcal{G}$

The pairing $\mathcal{I}$ induces the $\mathcal{G}$-equivalence of n-faces of $\mathcal{P}$ for every n $(0\leqq\leqq n\leqq d-1)$. Such an equivalence class of an n-face x^n of $\mathcal{P}$ is in strict connections with the stabilizer subgroup $\mathcal{G}_{\mathrm{x}^n}$ of x^n in $\mathcal{G}$, and this can be described by those images of $\mathcal{P}$ under $\mathcal{G}$ which join at x^n in the complex $\mathcal{P}^{\mathcal{G}}$.

We turn to new notations: $\mathcal{P}^{\mathcal{G}}:=\langle\mathcal{P},\mathcal{G}\rangle$; $(\mathrm{x}^n)^{\mathrm{g}}:=\langle\mathrm{x}^n,\mathrm{g}\rangle$. A facet $\mathrm{f_g}$ of $\mathcal{P}$ which is not self-paired by the generator g in $\mathcal{I}$ has trivial stabilizer in $\mathcal{G}$. If $\mathrm{f_g}=\mathrm{f_{g^{-1}}}$, i.e. g is involutive generator of $\mathcal{G}$, then the stabilizer $\mathcal{G}_{\mathrm{f_g}}$ contains the identity 1 and $\mathrm{g}=\mathrm{g}^{-1}$, only. We know that any involutive transformation g may have geometrically different realizations, e.g. in the 3-space g is either a plane reflection or a half-turn about an axis or central inversion in a point. These transformations differ in their fixed objects (points, lines, etc.). Other stabilizers also need finer distinctions.

We have not introduced the points and the topology of the polyhedron yet, since we can neglect them in combinatorial algorithms. We formally introduce the *barycentric subdivision* of $\mathcal{P}$ and $\mathcal{P}^{\mathcal{G}}$ into d-simplices:

To any m-face of $\mathcal{P}$ and of $\mathcal{P}^{\mathcal{G}}$, $0\leqq m\leqq d$ we order a formal midpoint. Any flag (f,h) in $(\mathcal{F},\mathcal{G})$, with its vertex component, ..., facet one and solid *one, determines a simplex in* $\mathcal{P}^{\mathcal{G}}$, *whose (vertex) points are just the formal midpoints of the vertex (it is itself), ..., of the edge, the facet, the solid (this is element of* $\mathcal{G}$*), respectively. Any* $m+1$ *vertices of a simplex defined by* (ℓ, h) *determine its* m*-face. This* m*-face is characterized by the corresponding* $m+1$ *components of the flag* (ℓ, h).

This definition gives more visuality to our flag structures $\mathcal{F}$ and $(\mathcal{F}, \mathcal{G})$ in general. For instance, flags with common solid component and facet one, describing a facet of a polyhedron considered in $\mathcal{P}^{\mathcal{G}}$, are characterized by the simplices with the corresponding two common midpoints. The $(d-1)$-facets of these simplices, opposite to the common solid midpoint, amount the $(d-1)$-facet of the polyhedron considered in $\mathcal{P}^{\mathcal{G}}$.

Briefly and intuitively the *stabilizer of any* n*-face* $(\mathrm{x}^n)^{\mathrm{h}}$ *will be described by a fundamental polyhedron for it.* This will be a connected union of those simplices in $\mathcal{P}^{\mathcal{G}}$ which have the common n-midpoint according to $(\mathrm{x}^n)^{\mathrm{h}}$ and have different $\mathcal{G}$-images in $\mathcal{P} := \mathcal{P}^1$. We glue the simplices at their $\mathcal{G}$-equivalent $(d-1)$-facets as dictated by $\mathcal{P}^{\mathcal{G}}$. All these simplex facets contain the n-midpoint ordered to $(\mathrm{x}^n)^{\mathrm{h}}$ and we take identifications of these facets induced by $\mathcal{G}$. So we obtain a new d-dimensional polyhedron $\mathcal{P}_{(\mathrm{x}^n)^{\mathrm{h}}}$ whose flag structure is $(d-1)$-dimensional because of the fixed n-midpoint in common.

It is clear that n-faces in $\mathcal{P}$ which are $\mathcal{G}$-equivalent to $(\mathrm{x}^n)^{\mathrm{h}}$ have conjugate stabilizers. Thus any of them provides us the typical stabilizer.

We turn to the most important case. This will be the *generalized Poincaré algorithm* for describing the stabilizer of $(d-2)$-faces (see [5, 20, 26] for $d = 2, 3$).

ALGORITHM 4.1 *for finding stabilizer* $\mathcal{G}_{\mathrm{e}}$ *of a fixed* $(d-2)$*-face* e *of* $\mathcal{P}$ *by a fundamental polyhedron* $\mathcal{P}_{\mathrm{e}}$.

1) Consider all the flags in $\mathcal{F}$ whose $(d-2)$-component is the edge $\mathrm{e} = \mathrm{e}_0$ itself, i.e. consider all the simplices in $\mathcal{P}$ whose common $(d-2)$-midpoint is e_0. Their union is denoted by $\mathcal{D}_{\mathrm{e}_0} =: \mathcal{D}^1$; this is an edge-domain ($(d-2)$-domain) in $\mathcal{P}$.

2) Assume that $\mathrm{f}_{\mathrm{g}_1^{-1}}$ is a facet incident to e_0 such that $\mathrm{e}_0^{\mathrm{g}_1} =: \mathrm{e}_1 \neq \mathrm{e}_0$ holds for a generator g_1 from $\mathcal{I}$. Take the flags in $\mathcal{F}$ whose $(d-2)$-component is e_1. Some of them have the facet component $\mathrm{f}_{\mathrm{g}_1}$, the others have another facet component, say $\mathrm{f}_{\mathrm{g}_2^{-1}}$. The corresponding simplices form an edge-domain $\mathcal{D}_{\mathrm{e}_1}$ at e_1 in $\mathcal{P}$. The g_1^{-1}-image if $\mathcal{D}_{\mathrm{e}_1}$ is simply denoted by $\mathcal{D}^{\mathrm{g}_1^{-1}} \subset \mathcal{P}^{\mathrm{g}_1^{-1}}$ which joins $\mathcal{D}^1$ at the facet $\mathrm{f}_{\mathrm{g}_1^{-1}}$ and e represents the common $(d-2)$-midpoint of the simplices forming $\mathcal{D}^1$ and $\mathcal{D}^{\mathrm{g}_1^{-1}}$ in $\mathcal{P}^{\mathcal{G}}$.

Then we introduce $\mathrm{e}_2 := \mathrm{e}_1^{\mathrm{g}_2}$ if it is different from e_1 and e_0. The flags in $\mathcal{F}$ with $(d-2)$-component e_2, the corresponding simplices and the edge-domain $\mathcal{D}_{\mathrm{e}_2}$ are analogously defined. Now $\mathcal{D}_{\mathrm{e}_2}^{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}} =: \mathcal{D}^{\mathrm{g}_2^{-1}\mathrm{g}_1^{-1}}$ joins $\mathcal{D}^{\mathrm{g}_1^{-1}}$ at

$(\mathrm{f}_{g_2^{-1}})^{g_1^{-1}}$ so that e represents the common $(d-2)$-midpoint of the simplices forming $\mathcal{D}^1, \mathcal{D}^{g_1^{-1}}, \mathcal{D}^{g_2^{-1}g_1^{-1}}$.

Assume that we have already defined the different edges

(1) $$e = e_0,\ e_1 := e_0^{g_1},\ e_2 := e_1^{g_2},\ \dots,\ e_{r-1} := e_{r-2}^{g_{r-1}} = e_0^{g_1 g_2 \cdots g_{r-1}}$$

and the corresponding edge domains

(2) $$\mathcal{D}^1,\ \mathcal{D}^{g_1^{-1}},\ \mathcal{D}^{g_2^{-1}g_1^{-1}},\ \dots,\ \mathcal{D}^{g_{r-1}^{-1}\cdots g_2^{-1}g_1^{-1}},$$

meeting at e in $\mathcal{P}^{\mathcal{G}}$ and having the facets

(3) $$(\mathrm{f}_{g_1^{-1}}),\ (\mathrm{f}_{g_2^{-1}})^{g_1^{-1}} =: \mathrm{f}^*_{g_2^{-1}},\ \dots,\ (\mathrm{f}_{g_{r-1}^{-1}})^{g_{r-2}^{-1}\cdots g_1^{-1}} =: \mathrm{f}^*_{g_{r-1}^{-1}},$$

after each other, respectively (Fig. 4).

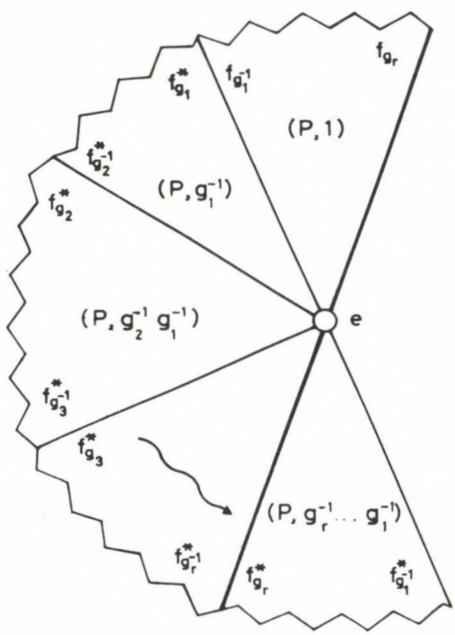


Fig. 4

Then one of the following cases in 3) and 4) occurs.

3) The flags in $\mathcal{F}$ and the corresponding edge domain $\mathcal{D}_{e_{r-1}}$ in $\mathcal{P}$ have the facets $f_{g_{r-1}}$ and $f_{g_r^{-1}}$ such that $e_{r-1}^{g_r} = e_0^{g_1 g_2 \dots g_r} = e_0$ and f_{g_r} is the second facet containing the edge $e = e_0$ (the first was $f_{g_1^{-1}}$).

i) In addition, suppose that

$$c_e := g_1 g_2 \dots g_r \qquad \text{and so} \qquad c_e^{-1} := g_r^{-1} g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}$$

fixes any n-face of e $(0 \leqq n \leqq d-2)$.

Therefore, c_e (and its inverse c_e^{-1}) is called *cycle rotation* to the edge equivalence cycle of e. Now, the union of edge-domains (2) in $\mathcal{P}^{\mathcal{G}}$ determine a "dihedral" fundamental polyhedron $\mathcal{P}_e$ for the stabilizer $\mathcal{G}_e = \mathcal{R}_e$ (Fig. 5.a, $d = 3$), where the facet f_{g_r} of $\mathcal{D}^1$ and the facet $f^*_{g_r^{-1}}$ of $\mathcal{D}^{g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}}$ are paired by the generating "rotation" c_e^{-1} of $\mathcal{G}_e = \mathcal{R}_e$.

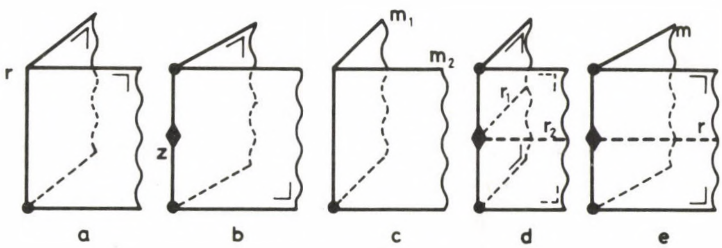


Fig. 5

We can arbitrarily prescribe the order ν_e of rotation c_e belonging to the e-cycle.

If $2 \leqq \nu_e$ then c_e is a proper rotation. if $\nu_e = 1$ then $c_e = 1$ the identity, the "dihedral $\mathcal{P}_e$ entirely surrounds" the $(d-2)$-midpoint of e. In these cases the relation

$$(4) \qquad 1 = c_e^{\nu_e} = (g_1 g_2 \dots g_r)^{\nu_e}$$

holds in $\mathcal{G}$ which determines the stabilizer $\mathcal{G}_e$. If we prescribe the order of c_e to be infinite then we write $\nu_e = 0$, since (4) trivially holds and the equivalence class of e does not serve a proper relation for $\mathcal{G}$.

ii) Now we suppose that $d_e := g_1 g_2 \dots g_r$ is a non-trivial self-transformation of the edge $e = e_0$ (Fig. 5.b, $d = 3$). Then there exists a natural number $n_e \geqq 2$, determined by the pairing $\mathcal{I}$, such that

$$c_e := d_e^{n_e} = (g_1 g_2 \dots g_r)^{n_e} \quad \text{and so} \quad c_e^{-1} := d_e^{-n_e}$$

already fixes any n-face of e $(0 \leqq n \leqq d-2)$. Such n_e is determined by an algorithm as follows. We pick out a $(d-2)$-subflag of $e = e_1$ and proceed the transformation d_e so many times until the subflag maps onto itself.

Again, we arbitrarily prescribe the order ν_e of "rotation" c_e. Now the rotational subgroup $\mathcal{R}_e$ is of index n_e in $\mathcal{G}_e$. The relation

$$1 = d_e^{n_e \nu_e} = (g_1 g_2 \dots g_r)^{n_e \nu_e} \tag{5}$$

holds in general to the equivalence class of e. In case $\nu_e = 0$ this is no proper relation to the edge class of e.

The edge domains (2) in $\mathcal{P}^{\mathcal{G}}$ form a "dihedral" fundamental polyhedron $\mathcal{P}_e$ where the facet f_{g_r} of $\mathcal{D}^1$ and the facet $f^*_{g_r^{-1}}$ of $\mathcal{D}^{g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}}$ are paired by the unique generator $d_e^{-1} = g_r^{-1} g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}$ for the stabilizer $\mathcal{G}_e$.

We remark that i) is a particular case of ii) when $n_e = 1$. In both cases in 3) also $r = 1$ is permitted.

4) Assume that in the procedure described in 2) we have already defined the edges by (1), the edge domains by (2) which join at the facets listed in (3). Moreover, assume that the flags and the corresponding edge domain $\mathcal{D}_{e_{r-1}}$ in $\mathcal{P}$ have the facets $f_{g_{r-1}}$ and $f_{g_r^{-1}}$ such that $e_{r-1}^{g_r} = e_{r-1}$. Then the flag requirement F.1 in Section 1 implies that $f_{g_r^{-1}}$ is paired with itself by $g_r : f_{g_r^{-1}} \mapsto f_{g_r} = f_{g_r^{-1}}$ as involutive generator from $\mathcal{I}$ ($g_r^{-1} = g_r$). That means the facet $(f_{g_r^{-1}})^{g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}} =: f^*_{g_r^{-1}}$ of the edge domain $\mathcal{D}^{g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}}$ is paired with itself by the involutive generator u_r for the stabilizer $\mathcal{G}_e$ where

$$u_r := g_1 g_2 \dots g_{r-1} g_r g_{r-1}^{-1} \dots g_2^{-1} g_1^{-1}. \tag{6}$$

Now we turn back to the starting edge $e = e_0$. Denote by $f_{g_{-1}^{-1}}$ the second facet incident to e_0 such that $e_0^{g_{-1}} =: e_{-1} \neq e_0$ holds for a generator g_{-1} from $\mathcal{I}$. Take the flags in $\mathcal{F}$ whose $(d-1)$-component is e_{-1}. Some of them have the facet component $f_{g_{-1}}$ the others have another facet component, say $f_{g_{-2}^{-1}}$. The corresponding simplices form the edge domain $\mathcal{D}_{e_{-1}}$ at e_{-1} in $\mathcal{P}$. The g_{-1}^{-1}-image of $\mathcal{D}_{e_{-1}}$ is denoted by $\mathcal{D}^{g_{-1}^{-1}}$ which joins $\mathcal{D}^1$ at the facet $f_{g_{-1}^{-1}}$, and so on.

Assume that we have already defined the different edges

$$e_0, \quad e_{-1} := e_0^{g_{-1}}, \quad \dots, \quad e_{-s+1} = e_{-s+2}^{g_{-s+1}} \tag{7}$$

and the edge domains to e

$$\mathcal{D}^1, \quad \mathcal{D}^{g_{-1}^{-1}}, \quad \mathcal{D}^{g_2^{-1} g_1^{-1}}, \quad \dots, \quad \mathcal{D}^{g_{-s+1}^{-1} \dots g_{-2}^{-1} g_{-1}^{-1}} \tag{8}$$

joining in $\mathcal{P}^{\mathcal{G}}$ at the facets

$$f_{g_{-1}^{-1}}, \quad (f_{g_{-2}^{-1}})^{g_{-1}^{-1}} =: f^*_{g_{-2}^{-1}}, \quad \dots, \quad (f_{g_{-s+1}^{-1}})^{g_{-s+2}^{-1} \dots g_{-1}^{-1}} =: f^*_{g_{-s+1}^{-1}} \tag{9}$$

after each other.

Finally, $e_{-s+1}^{g_{-s}} = e_{-s+1}$ shall hold with an involutive generator $g_{-s} : f_{g_{-s}^{-1}} \mapsto$ $\mapsto f_{g_{-s}}$ ($g_{-s}^{-1} = g_{-s}$ from $\mathcal{I}$). That means, the facet $f^*_{g_{-s}^{-1}}$ of the edge domain $\mathcal{D}^{g_{-s+1}^{-1} \cdots g_{-2}^{-1} g_{-1}^{-1}}$ is paired with itself by the second involutive generator u_{-s} for the stabilizer $\mathcal{G}_e$

$$u_{-s} := g_{-1} g_{-2} \cdots g_{-s+1} g_{-s} g_{-s+1}^{-1} \cdots g_{-2}^{-1} g_{-1}^{-1}. \tag{6'}$$

We have analogous cases as in 3). To be short, we define the transformation

$$d_e := u_r u_{-s} \tag{10}$$

which maps the staring edge e onto itself. By means of a $(d-2)$-subflag of e we algorithmically determine the exponent n_e for which $d_e^{n_e}$ fixes any n-face of e $(0 \leqq n \leqq d-2)$. We arbitrarily prescribe the order ν_e of the "rotation" $c_e = d_e^{n_e}$. The relation

$$1 = d_e^{n_e \nu_e} = (u_r u_{-s})^{n_e \nu_e} = c_e^{\nu_e}, \tag{11}$$

with u_r by (6) and u_{-s} by (6'), holds in general to the equivalence class of e. In case $\nu_e = 0$ this is no proper relation for $\mathcal{G}$.

We remark again that $r = 1$ and $-s = -1$ are allowed in (6) and (6'). In Fig. 5.c–e we illustrate all possibilities for $d = 3$, m's denote plane reflections, r's are half-turns. The edge domains in (2) and (8) form a "dihedral" fundamental domain $\mathcal{P}_e$ for the stabilizer $\mathcal{G}_e$. Both facets of $\mathcal{P}_e$ determine involutive generators u_r, u_{-s}, respectively.

The procedure ends, since we have finitely many flags in $(\mathcal{F}, 1)$ or simplices in $\mathcal{P}$ to the equivalence class of the edge e. Moreover, we have finitely many edge equivalence classes. □

PRESENTATION OF $\mathcal{G}$. *Let $\mathcal{I}$ be a pairing of the polyhedron $\mathcal{P}$ which induces the equivalence of edges ($(d-2)$-faces) of $\mathcal{P}$. For each edge equivalence class* e *let ν_e be an arbitrarily prescribed natural number (zero is in), moreover let $\{\nu_e\}$ be a fixed collection of them. Then $\mathcal{P}$, $\mathcal{I}$ and $\{\nu_e\}$ define a group $\mathcal{G} := (\mathcal{P}, \mathcal{I}, \{\nu_e\})$ which is generated by $\mathcal{I}$, and the occasional reflection relations $g_u^2 = 1$ for each involutive generator g_u from $\mathcal{I}$, furthermore the cycle relations $c_e^{\nu_e} = 1$ provide the defining relations of $\mathcal{G}$. If $\nu_e = 0$ for some* e, *then this trivial relation has only a geometric importance.*

Now the polyhedron complex $\mathcal{P}^{\mathcal{G}}$ is defined by $\mathcal{P}$, $\mathcal{I}$, $\{\nu_e\}$ and we have to describe all the stabilizers of m-faces of $\mathcal{P}^{\mathcal{G}}$ for each m $(0 \leqq m \leqq d-3,\ d \geqq 3)$.

This will be the main goal of the next section.

5. Stabilizer of an m-face

Let x be an m-face of $\mathcal{P}$ $(0 \leqq m \leqq d-3)$. We construct a fundamental polyhedron $\mathcal{P}_x$ for the stabilizer $\mathcal{G}_x$. $\mathcal{P}_x$ will have a $(d-1)$-dimensional flag structure.

ALGORITHM 5.1 *for constructing* $\mathcal{P}_x$ *and* $\mathcal{G}_x$.

1) Consider all the flags in $\mathcal{F}$ whose m-component is x itself, i.e. consider all the simplices in $\mathcal{P}$ whose common m-points are x. Their union is denoted by $\mathcal{D}_x =: \mathcal{D}^1$.

2) Take the first flag of $\mathcal{D}_x$ with a facet component, say, $f_{g_1^{-1}}$ such that $x^{g_1} \neq x$ holds for a generator g_1 from $\mathcal{I}$. Then we take all the flags in $\mathcal{F}$ with m-component x^{g_1} and so all the simplices in $\mathcal{P}$ whose common m-points are x^{g_1}. These form the m-domain $\mathcal{D}_{x^{g_1}}$ in $\mathcal{P}$ whose g_1^{-1}-image is simply denoted by $\mathcal{D}^{g_1^{-1}}$ (it lies in $\mathcal{P}^{g_1^{-1}}$). $\mathcal{D}^1$ and $\mathcal{D}^{g_1^{-1}}$ meet at the facet $f_{g_1^{-1}} = (f_{g_1})^{g_1^{-1}} =: f^*_{g_1}$ and they have the m-face x in common in $\mathcal{P}^{\mathcal{G}}$.

3) Then we take the next flag of $\mathcal{D}_x$ or of $\mathcal{D}_{x^{g_1}}$, not taken yet, with a facet component, say, $f_{g_2^{-1}}$ such that x^{g_2} or $x^{g_1g_2}$ is different from the m-faces already listed. Again, we take all the flags and simplices with the considered m-component. These form the corresponding m-domain $\mathcal{D}_{x^{g_2}}$ or $\mathcal{D}_{x^{g_1g_2}}$ is $\mathcal{P}$ whose g_2^{-1}-image or $g_2^{-1}g_1^{-1}$-image is denoted by $\mathcal{D}^{g_2^{-1}}$ or $\mathcal{D}^{g_2^{-1}g_1^{-1}}$, respectively. This latter m-domain joins the previous one at the facet $f^*_{g_2}$ where the exponent $*$ means g_2^{-1} or $g_2^{-1}g_1^{-1}$, respectively. So we have a new m-domain in $\mathcal{P}^{\mathcal{G}}$ to the m-face x.

4) The procedure is already clear. We use Algorithm 1.1 to list the flags with fixed m-component, first for x, then for $x^{g_1}, \ldots, x^h$. Then we pick out new image x^{hg_j} on $\mathcal{P}$ and new m-domains $\mathcal{D}_{x^{hg_j}}$ in $\mathcal{P}$ so that its $g_j^{-1}h^{-1}$-image $\mathcal{D}^{g_j^{-1}h^{-1}}$ (this lies in $\mathcal{P}^{g_j^{-1}h^{-1}}$) joins the previous $\mathcal{D}^{h^{-1}}$ at the facet $f^*_{g_j^{-1}} := (f_{g_j^{-1}})^{h^{-1}} = (f_{g_j})^{g_j^{-1}h^{-1}} =: f^*_{g_j}$. Here g_j is a corresponding generator from $\mathcal{I}$. So we have a new m-domain in $\mathcal{P}^{\mathcal{G}}$ to the m-face x.

5) The procedure ends if we have exhausted the $\mathcal{G}$-equivalents of x on $\mathcal{P}$. Then we have the glued union of the m-domains

$$\mathcal{D}_x =: \mathcal{D}^1, \quad \mathcal{D}^{g_1^{-1}}, \quad \ldots, \quad \mathcal{D}^{h^{-1}}, \quad \mathcal{D}^{g_j^{-1}h^{-1}}, \quad \ldots \tag{1}$$

in $\mathcal{P}^{\mathcal{G}}$ which form a fundamental polyhedron $\mathcal{P}_x$ for the stabilizer $\mathcal{G}_x$. $\mathcal{P}_x$ will have a $(d-1)$-dimensional flag structure $\mathcal{F}_x$ with facet identifications induced by the procedure.

Let $f^{h^{-1}}_{g_i^{-1}}$ be any "free" (not yet glued) facet of $\mathcal{D}^{h^{-1}}$ in (1) and let $f^{k^{-1}}_{g_i}$ be the corresponding free facet of $\mathcal{D}^{k^{-1}}$ in (1). That means, the flags of "this part" of $f_{g_i^{-1}}$ (having the prescribed m-component) are associated with the corresponding flags of f_{g_i} in the original polyhedron $\mathcal{P}$ by the generator g_i of $\mathcal{I}$. Then

$$hg_ik^{-1} : f^{h^{-1}}_{g_i^{-1}} \mapsto f^{k^{-1}}_{g_i}, \qquad g_i := hg_ik^{-1} \in \mathcal{I}_x \tag{2}$$

will be a facet identifying generator of $\mathcal{G}_{\mathrm{x}}$ denoted by $\overline{\mathrm{g}}_i \in \mathcal{I}_{\mathrm{x}}$.

6) So we can reduce our problem to investigate the $(d-1)$-dimensional polyhedron $\mathcal{P}_{\mathrm{x}}$ equipped by facet identifications $\mathcal{I}_{\mathrm{x}}$ which generate the stabilizer $\mathcal{G}_{\mathrm{x}}$. Those $(d-2)$-face classes which contain the $\mathcal{G}$-equivalents of x provide the relations of $\mathcal{G}_{\mathrm{x}}$ determined by the presentation of $\mathcal{G}$. □

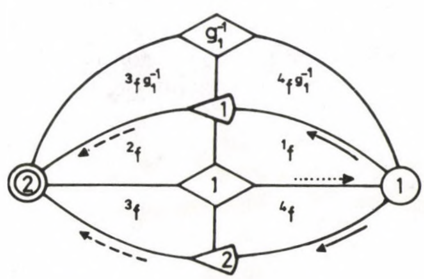
REMARKS. 1. Each $(d-2)$-face class of $\mathcal{P}_{\mathrm{x}}$ which provide a relation of $\mathcal{G}_{\mathrm{x}}$ in a form

$$\overline{\mathrm{g}}_i\overline{\mathrm{g}}_j^{-1}=1 \quad \text{with} \quad \overline{\mathrm{g}}_i,\ \overline{\mathrm{g}}_j \in \mathcal{I}_{\mathrm{x}} \tag{3}$$

reduces the number of generators in $\mathcal{I}_{\mathrm{x}}$, since we have the consequence $\overline{\mathrm{g}}_i=\overline{\mathrm{g}}_j$ in $\mathcal{I}_{\mathrm{x}}$. Such reductions may also yield that the stabilizer $\mathcal{G}_{\mathrm{x}}$ is trivial.

2. We cannot exclude apriori that the starting polyhedron $\mathcal{P}$ with identifications $\mathcal{I}$ and exponents $\{\nu_{\mathrm{e}}\}$ involve that the group $\mathcal{G}$ is trivial and so is the polyhedron complex $\mathcal{P}^{\mathcal{G}}$.

This is an accordance with the combinatorial theory of groups [7, 18] and indicates also the difficulties of the problem.



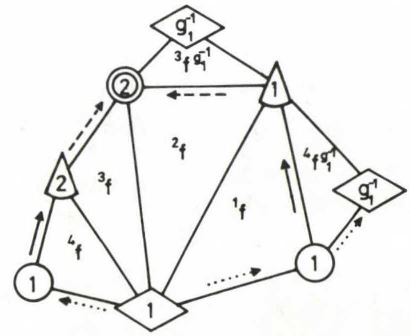


Fig. 6

For instance a 2-dimensional digon with 4 flags (Fig. 6)

$$(1,1), \quad (1,2), \quad ,(2,1), \quad (2,2)$$

allows the identification

$$\mathrm{g}_1:(1,1)\mapsto(1,2), \qquad (2,1)\mapsto(2,2)$$

to generate a free group. We have two non-equivalent $(d-2)$-faces now, the vertices 1 and 2, we may choose the exponents ν_1 and ν_2 independently. If we choose $\nu_1=\nu_2=:\nu$, then the relation $\mathrm{g}_1^{\nu}=1$ leads to the cyclic group $\mathcal{G}=\mathcal{C}_\nu$ and this is realized metrically on the sphere $\mathcal{S}^2$ with a digonal fundamental domain. If $\nu_1\neq\nu_2$, then the greatest common divisor $\nu=(\nu_1,\nu_2)$ gives the order of the corresponding cyclic group $\mathcal{G}=\mathcal{C}_\nu$, especially $\nu=1$ leads to the trivial group. These latter cases are not realizable in the classical sense.

6. Topology on identified polyhedron

In Section 4 we have already introduced the formal barycentric subdivision of $\mathcal{P}^{\mathcal{G}}$ into d-simplices. Any flag (ℓ, h) in $(\mathcal{F}, \mathcal{G})$ with its vertex component, ..., facet one and solid one determines a simplex first in $\mathcal{P}^{\mathcal{G}}$, whose vertices are just the formal midpoints of the m-components of (ℓ, h), $0 \leqq m \leqq d$.

Now we take a $(d+1)$-dimensional real vector space $\mathbf{V}$ and its dual $\mathbf{V}^*$, and define the points of the projective d-sphere S^d as rays of $\mathbf{V}$. The relation $\mathbf{x} \sim \mathbf{y}$ stands for $\mathbf{x}$, $\mathbf{y} \in \mathbf{V} - \{\mathbf{0}\}$ iff there is a positive c $(\in \mathbf{R}^+)$ with $\mathbf{y} = c\mathbf{x}$. The equivalence class of $\mathbf{x}$ will be denoted by $(\mathbf{x})$ and called the ray of $\mathbf{x}$. Identifying opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$, we define the point $[\mathbf{x}]$ of the projective d-space P^d. The d-subspaces of $\mathbf{V}$ can be described by the non-zero elements of $\mathbf{V}^*$. A $\boldsymbol{u} \in \mathbf{V}^*$ gives us a $(d-1)$-subsphere of S^d by

$$\text{(1)} \qquad \mathsf{S}^{d-1}(\boldsymbol{u}) = \{(\mathbf{x}) \subset \mathbf{V} : \mathbf{x}\boldsymbol{u} = 0, \quad \boldsymbol{u} \in \mathbf{V}^* - \{o\}\}.$$

A ray $(\boldsymbol{u})$ in $\mathbf{V}^*$ characterizes a half-space in $\mathbf{V}$ or the positive half-sphere

$$\text{(2)} \qquad \mathsf{S}^+(\boldsymbol{u}) = \{(\mathbf{x}) \subset \mathbf{V} : \mathbf{x}\boldsymbol{u} \geqq 0\}.$$

Identifying opposite rays $(\boldsymbol{u})$ and $(-\boldsymbol{u})$, we define the $(d-1)$-plane $[\boldsymbol{u}]$ of the projective d-space P^d. A point $[\mathbf{x}]$ and a $(d-1)$-plane $[\boldsymbol{u}]$ of P^d are incident, i.e.

$$\text{(3)} \qquad [\mathbf{x}] \; \mathrm{I} \; [\boldsymbol{u}] \quad \text{iff} \quad \mathbf{x}\boldsymbol{u} = 0.$$

A set $\mathbf{C}$ of the projective sphere S^d is convex iff it does not contain opposite rays, and with any two rays $(\mathbf{x})$, $(\mathbf{y}) \in \mathbf{C}$ all the rays $(\mathrm{t}\mathbf{x} + (1-\mathrm{t})\mathbf{y})$, for $0 < \mathrm{t} < 1$, belong to $\mathbf{C}$. Clearly, any convex set of S^d is contained in a half-sphere.

A half-sphere in (2) has also an affine structure, if we take in $\mathbf{V}$ the d-plane (hyperplane)

$$\text{(4)} \qquad \mathsf{A}^d(\boldsymbol{u}) = \{(\mathbf{x}) \subset \mathbf{V} : \mathbf{x}\boldsymbol{u} = c \quad \text{fixed}, \quad \mathrm{c} \in \mathbf{R}^+\},$$

parallel to the d-plane $(\boldsymbol{u})$. The rays $(\mathbf{y})$, $(-\mathbf{y})$ with $\mathbf{y}\boldsymbol{u} = 0$ assign one ideal point to $\mathsf{A}^d(\boldsymbol{u})$. The d-plane $[\boldsymbol{u}]$ represents the ideal $(d-1)$-plane of $\mathsf{A}^d(\boldsymbol{u})$.

Fix a basis $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d$ in $\mathbf{V}$. We define the d-simplex

$$\text{(5)} \qquad \mathbf{S}^d = \operatorname*{Span}_i(\mathbf{a}_i) :=$$

$$:= \{\mathbf{x} = x^0\mathbf{a}_0 + x^1\mathbf{a}_1 + \ldots + \mathrm{x}^d\mathbf{a}_d : x^0, x^1, \ldots, x^d \geqq 0, x^0 + x^1 + \ldots + x^d = 1\}$$

in $\mathbf{V}$ first. But we can consider a model of $\mathbf{S}^d$ also in the projective sphere S^d of $\mathbf{V}$, that means, any vector $\mathbf{x} \in \mathbf{S}^d$ will be represented by its ray $(\mathbf{x})$.

To a basis $\mathbf{a}_0, \ldots, \mathbf{a}_d$ we can uniquely introduce its dual basis $\boldsymbol{b}^0, \ldots, \boldsymbol{b}^d$ in $\mathbf{V}^*$ by requiring

$$\mathbf{a}_i \boldsymbol{b}^j = \delta_i^j \qquad \text{(the Kronecker symbol).} \tag{6}$$

Then the simplex $\mathbf{S}^d$ by (5) will be described in $\mathcal{S}^d$ by

$$\mathbf{S}^d\{(\mathbf{x}) \subset \mathbf{V} : \mathbf{x}\boldsymbol{b}^i \geqq 0 \text{ for } i = 0, 1, \ldots, d\} \tag{7}$$

We say $\mathbf{S}^d$ is the intersection of its positive half-spheres.

By (4), (5) we can also say that $\mathbf{S}^d$ lies in the affine d-space $\mathsf{A}^d(\boldsymbol{b}^0 + \boldsymbol{b}^1 + + \ldots + \boldsymbol{b}^d)$ with $c = 1$.

ALGORITHM 6.1 *for introducing $\mathcal{P}$ as a simplicial complex.*

We know from Algorithm 1.1 how to enumerate the flags in $\mathcal{F}$ on the base of their adjacencies. The first flag ${}^1f$ will be associated with $\mathbf{S}^d =: {}^1\mathbf{S}$ so that the formal midpoint of the m-component maps onto $\mathbf{a}_m$ for each m $(0 \leqq m \leqq d)$. The second flag ${}^2f$ is 0-adjacent to ${}^1f$, so we can choose ${}^2\mathbf{a}_0$ in the interior of the simplex $\mathbf{S}_0^d$ spanned by $-\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_d$ just arbitrarily and define the simplex ${}^2\mathbf{S}$ just by (5) taking ${}^2\mathbf{a}_0, {}^2\mathbf{a}_1 = \mathbf{a}_1, \ldots, {}^2\mathbf{a}_d = \mathbf{a}_d$ as vertices. The union ${}^1\mathbf{S} \cup {}^2\mathbf{S}$ will be a convex polyhedron on the projective sphere $\mathcal{S}^d$ of $\mathbf{V}$, whose facets are $(d-1)$-simplices.

Suppose that we have already listed the simplices ${}^1\mathbf{S}, {}^2\mathbf{S}, \ldots, {}^r\mathbf{S}$ according to the flags ${}^1f, {}^2f, \ldots, {}^rf$ in Algorithm 1.1, so that

$$\mathcal{P}^r := \bigcup_{i=1}^{r} {}^i\mathbf{S} \tag{8}$$

is convex on $\mathcal{S}^d$ with simplicial facets. Take ${}^{r+1}f$ which is m-adjacent, say, with ${}^sf$ $(s \leqq r)$. Then we define ${}^{r+1}\mathbf{S}$ to ${}^s\mathbf{S}$ by taking

$$\operatorname*{Span}_j({}^{r+1}\mathbf{a}_j) =: {}^{r+1}\mathbf{S} \tag{9}$$

with ${}^{r+1}\mathbf{a}_j = {}^s\mathbf{a}_j$ if $j \in \{0, \ldots, d\} - \{m\}$ and ${}^{r+1}\mathbf{a}_m \in \operatorname{Int}({}^s\mathbf{P}^r - \mathbf{P}^r)$.

Here ${}^s\mathbf{P}^r$ is a convex hull of the positive half-spheres of all facets of $\mathbf{P}^r$ except that of facet $\operatorname*{Span}_{j \neq m}({}^s\mathbf{a}_j)$, and we have taken the interior of the difference set.

Since the flag structure $\mathcal{F}$ satisfies the requirements F.1–2 of Section 1 and $\mathcal{F}$ is finite, the convex polyhedron

$$\mathbf{P} := \bigcup_{i=1}^{|\mathcal{F}|} {}^i\mathbf{S} \qquad \text{is well-defined.} \tag{10}$$

The adjacency structure of $\mathcal{F}$ and so the incidence structure of $\mathcal{P}$ will be realized on S^d, if we define the rest of adjacencies by identifications of corresponding simplicial facet pairs of $\mathbf{P}$.

It follows by the construction that $\mathbf{P}$ lies in an open half-sphere bounded by a d-subspace of $\mathbf{V}$. Introducing a parallel d-plane, called δ (a hyperplane in $\mathbf{V}$), all the simplices of $\mathbf{P}$ can be assumed to lie in δ which is a real affine d-space A^d. The corresponding identifications of simplicial facets of $\mathbf{P}$ are uniquely defined by affinities. Recall that any two m-simplices spanned in $\mathbf{V}$ by $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_m$ and $\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_m$, $0 \leqq m \leqq d$ can be mapped one onto another by the unique affinity

$$(11) \qquad x^0\mathbf{a}_0 + x^1\mathbf{a}_1 + \ldots + x^m\mathbf{a}_m \mapsto x^0\mathbf{b}_0 + x^1\mathbf{b}_1 + \ldots + x^m\mathbf{b}_m,$$

with any $x^0, x^1, \dots, x^m \geqq 0$, $x^0 + x^1 + \ldots + x^m = 1$.

So we define the points of $\mathcal{P}$ and the topology on $\mathcal{P}$ by means of $\mathbf{P}$, endowed by facet identifications, and by the affine topology of $\mathbf{P}$. Clearly, the construction of $\mathbf{P}$ and its topology is determined up to homeomorphisms defined by piece-wise affinities. We can say $\mathcal{P}$ is a simplicial complex realized by $\mathbf{P}$ in the affine d-space A^d [21, 23]. □

THEOREM 6.1. *Let $\mathcal{P}$ be a d-polyhedron whose flag structure $\mathcal{F}$ satisfies the flag requirements* 1.F1–2. *Furthermore, let $\mathcal{I}$ be a pairing of the facets of $\mathcal{P}$ which generate a group $\mathcal{G}$ by identifications from $\mathcal{I}$. There is a finite simplicial complex $\tilde{\mathbf{P}}$ in the affine d-space which provides a convex atlas containing connected neighbourhoods for every point of the identified polyhedron $\mathcal{P}$.*

PROOF. For $\mathcal{P}$ we have already glued a convex affine polyhedron $\mathbf{P}$ consisting of d-simplices. Some facets of $\mathbf{P}$ are identified (by 1 of $\mathcal{G}$) as dictated by $\mathcal{F}$, or by the incidence structure of $\mathcal{P}$.

The $\mathcal{G}$ -equivalence defined on $\mathcal{P}$ induces also the $\mathcal{G}$-equivalence on $\mathbf{P}$. Any point X of $\mathbf{P}$ lies on an m-face x of a d-simplex from $\mathbf{P}$, $0 \leqq m \leqq d$. We have finitely many such faces (of any dimension). The $\mathcal{G}$-equivalents of m-face x, containing all the $\mathcal{G}$-equivalents of X, are distributed in finitely many d-simplices from $\mathbf{P}$.

Now consider the construction of the fundamental polyhedron $\mathcal{P}_{\mathrm{x}}$ for the stabilizer $\mathcal{G}_{\mathrm{x}}$ of the m-face x in Algorithm 5.1. This requires gluing finitely many d-simplices to the common m-face x along $(d-1)$-facets of simplices. $\mathcal{G}$-equivalent free facets also occur after the procedure by Algorithm 6.1.

So, after finitely many steps, we shall have all "typical" neighbourhoods in the connected union of d-simplices glued to $\mathbf{P}$. So we get a convex affine polyhedron $\tilde{\mathbf{P}}$ desired in the theorem. Q.e.d.

In Fig. 6 we illustrate the situation in dimension $d=2$ for a digonal $\mathcal{P}$ whose sides are identified by "rotation" g_1. A glued collection of 6 simplices constitutes $\tilde{\mathbf{P}}$.

In most cases the atlas $\tilde{\mathbf{P}}$ can be simplified by gluing along neighbouring identified faces (as we intuitively imagine), but such a gluing cannot be guaranteed in general. We do not know apriori whether $\mathcal{P}$ is realizable in the affine d-space or not.

We shall construct a 3-polyhedron whose flag structure satisfies the requirements F.1–2, but it has no realization whatsoever in the affine 3-space, since its surface is homeomorphic to a non-orientable 2-manifold (e.g. to the projective plane, see Section 9, Fig. 9).

7. Spaces of constant curvature modelled on projective sphere

To prepare the promised generalization of Poincaré's theorem, recall that each simply connected d-space ($d \geqq 2$) of constant sectional curvature can be modelled on the projective d-sphere S^d [25, 29, 39]. We fix a basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_d$ in $\mathbf{V}$ and the dual basis $e^0, e^1, \ldots, e^d$ in $\mathbf{V}^*$ with $\mathbf{e}_i e^j = \delta_i^j$ (the Kronecker symbol).

A *linear polarity* $(_*)$ or, equivalently, a *symmetric bilinear form* or *scalar product* $\langle ; \rangle$ will be introduced by the following notations:

$$(1) \qquad (_*) : \mathbf{V}^* \to \mathbf{V}, \qquad u \mapsto u_* =: \mathbf{u};$$

$$(2) \qquad \langle ; \rangle : \mathbf{V}^* \times \mathbf{V}^* \to \mathbf{R}, \qquad \langle u; v \rangle = \mathbf{u} v = \mathbf{v} u.$$

To define the spherical d-space $\mathcal{S}^d$ we make the basis of $\mathbf{V}^*$ orthonormal by

$$(3) \qquad \langle e^i; e^j \rangle = \delta^{ij} \quad \text{for} \quad i,\ j = 0, 1, \ldots, d$$

which extends to a bilinear form of signature $\langle +, \ldots, +, + \rangle$.

For the *hyperbolic space* $\mathcal{H}^d$ we define

$$(4) \qquad \begin{cases} \langle e^0; e^0 \rangle = -1; \quad \langle e^i; e^i \rangle = 1 \quad \text{for} \quad i = 1, \ldots, d; \\ \text{and} \quad \langle e^i; e^j \rangle = 0 \quad \text{for} \quad i \neq j = 0, 1, \ldots, d. \end{cases}$$

These determine a bilinear form of signature $\langle +, \ldots, +, - \rangle$. For $\mathcal{S}^d$ and $\mathcal{H}^d$ the polarity $(_*)$ is regular, the inverse map is

$$(^*) : \mathbf{V} \to \mathbf{V}^*, \quad \mathbf{x} \mapsto \mathbf{x}^* = x \quad \text{so that} \quad x_* = \mathbf{x}.$$

The *induced symmetric bilinear form on* $\mathbf{V}$ can also be introduced by

$$(5) \qquad \langle ; \rangle : \mathbf{V} \times \mathbf{V} \to \mathbf{R}, \qquad \langle \mathbf{x}; \mathbf{y} \rangle := \mathbf{x} y = \mathbf{y} x = \langle x; y \rangle.$$

So $\mathbf{V}$ and its dual $\mathbf{V}^*$ give us an alternative description for $\mathcal{S}^d$ and $\mathcal{H}^d$. A polar $(d-1)$-plane (u) and its pole $(\mathbf{u})$ can be replaced to get dual statements.

The points of $\mathcal{S}^d$ are modelled also on the unit d-sphere

$$(6) \qquad \{\mathbf{x} \in \mathbf{V} : \langle \mathbf{x}; \mathbf{x} \rangle = 1\}.$$

Clearly, every ray of $\mathbf{V}$ meets exactly one point of this sphere. From $\mathcal{S}^d$ we turn to elliptic d-space if we replace the projective d-sphere S^d with the projective d-space P^d, keeping the metric (3) on $\mathbf{V}^*$ and the induced metric on $\mathbf{V}$.

The *proper points* and *proper* $(d-1)$*-planes of* $\mathcal{H}^d$ are described in the *projective d-space* P^d where opposite rays of S^d are identified:

$$(7)\qquad \mathcal{C}_- := \{[\mathbf{x}] : \langle \mathbf{x};\mathbf{x}\rangle < 0\}, \text{ resp. } \mathcal{C}_+{}^* := \{[\boldsymbol{u}] : \langle \boldsymbol{u};\boldsymbol{u}\rangle > 0\}.$$

Their polars and poles

$$(8)\qquad \mathcal{C}_-^* := \{[\boldsymbol{x}] : \langle \boldsymbol{x};\boldsymbol{x}\rangle < 0\} \text{ and } \mathcal{C}_+ := \{[\mathbf{u}] : \langle \mathbf{u};\mathbf{u}\rangle > 0\},$$

respectively, define improper $(d-1)$-planes and improper points of $\mathcal{H}^d$. The incident poles and polars are defined by

$$(9)\qquad \mathcal{C}_0 := \{[\mathbf{x}] : \langle \mathbf{x};\mathbf{x}\rangle = 0\}, \text{ resp. } \mathcal{C}_0^* := \{[\boldsymbol{u}] : \langle \boldsymbol{u};\boldsymbol{u}\rangle = 0\}.$$

These provide a *conic* and its dual which describe the *points at infinity* (points on the *absolute* or *ends*) and the *boundary* $(d-1)$*-planes*, respectively, as improper elements of $\mathcal{H}^d$.

For the proper points of $\mathcal{H}^d$ we can take the "imaginary sphere" in $\mathbf{V}$ with equation

$$(10)\qquad \langle \mathbf{x};\mathbf{x}\rangle := -x^0x^0 + x^1x^1 + \ldots + x^dx^d = -1 \quad (= \mathrm{i}^2),$$

where $\mathbf{x} = x^0\mathbf{e}_0 + x^1\mathbf{e}_1 + \ldots + x^d\mathbf{e}_d$. Any ray $(\mathbf{y})$, with $\langle \mathbf{y};\mathbf{y}\rangle < 0$, meets the imaginary d-sphere exactly at $\mathbf{y}/\sqrt{-\langle \mathbf{y};\mathbf{y}\rangle}$. The opposite points of (10) are identified for $\mathcal{H}^d$.

For the *Euclidean d-space* $\mathcal{E}^d$ to the basis of $\mathbf{V}^*$ we define

$$(11)\qquad \langle \boldsymbol{e}^0;\boldsymbol{e}^j\rangle = 0 \text{ for } j = 0,1,\ldots,d;\ \langle \boldsymbol{e}^i;\boldsymbol{e}^j\rangle = \delta^{ij} \text{ for } i, j = 1,\ldots,d.$$

This extends to a bilinear form of signature $\langle +,\ldots,+,0\rangle$. Now

$$(12)\qquad (_*) : \boldsymbol{e}^i + \boldsymbol{e}^0\cdot c \mapsto \mathbf{e}_i \quad \text{for} \quad i = 1,\ldots,d; \quad c \in \mathbf{R}.$$

This induces a scalar product in the d-subspace $(\mathbf{e}_1,\ldots,\mathbf{e}_d)$ of $\mathbf{V}$ which is defined by

$$(13)\qquad \langle \mathbf{e}_i;\mathbf{e}_j\rangle := \langle \boldsymbol{e}^i + \boldsymbol{e}^0\cdot c; \boldsymbol{e}^j + \boldsymbol{e}^0\cdot d\rangle = \delta^{ij}.$$

In the affine space

$$(14)\qquad \mathsf{A}^d(\boldsymbol{e}^0) = \{\mathbf{x} \in \mathbf{V}; \mathbf{x}\boldsymbol{e}^0 =: \mathrm{x}^0 = 1\}$$

we get the classical Euclidean d-space.

The *reflection in the* $(d-1)$*-space* $(\boldsymbol{u})$ with $\langle \boldsymbol{u}; \boldsymbol{u}\rangle \neq 0$ is defined by

$$(15) \qquad (\boldsymbol{v}) \mapsto (\boldsymbol{w}), \quad \boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} \cdot \frac{2\langle \boldsymbol{u}; \boldsymbol{v}\rangle}{\langle \boldsymbol{u}; \boldsymbol{u}\rangle}$$

uniformly for $(d-1)$-planes $(\boldsymbol{v})$ of $\mathcal{S}^d$, $\mathcal{H}^d$ and $\mathcal{E}^d$. For Euclidean points $(\mathbf{x})$ the reflection formula is

$$(16) \qquad (\mathbf{x}) \mapsto (\mathbf{y}), \quad \mathbf{y} = \mathbf{x} - \frac{2\mathbf{x}\boldsymbol{u}}{\langle \boldsymbol{u}; \boldsymbol{u}\rangle}\mathbf{u}.$$

For spherical and hyperbolic points $(\mathbf{x})$ this formula can be written in the form

$$(16') \qquad (\mathbf{x}) \mapsto (\mathbf{y}), \quad \mathbf{y} = \mathbf{x} - \frac{2\langle \mathbf{x}; \mathbf{u}\rangle}{\langle \mathbf{u}; \mathbf{u}\rangle}\mathbf{u}.$$

In (16) and (16') we have denoted by $(\mathbf{u})$ the pole of the $(d-1)$-plane $(\boldsymbol{u})$.

The *isometries of our spaces* $\mathcal{S}^d$, $\mathcal{H}^d$ and $\mathcal{E}^d$ are uniformly defined as compositions (products) of finitely many reflections. The groups $\mathrm{Iso}\,\mathcal{S}^d$ and $\mathrm{Iso}\,\mathcal{H}^d$ can also be defined by those linear self-bijections of $\mathbf{V}$ (and $\mathbf{V}^*$, respectively) which preserve the corresponding bilinear forms. But it is not true in $\mathcal{E}^d$, although any reflection of the form (16) leaves the affine space (14) and the scalar product (13) in it invariant.

The group of all self-bijections of $\mathcal{E}^d$ which preserve the bilinear form of $\mathbf{V}^*$ is called similarities of $\mathcal{E}^d$. This contains $\mathrm{Iso}\,\mathcal{E}^d$ as a subgroup. But, in addition, there are typical similarities, called dilatations. These are defined by

$$(17) \qquad \begin{array}{l} \mathbf{x} = 1 \cdot \mathbf{e}_0 + x^1\mathbf{e}_1 + \ldots + x^d\mathbf{e}_d \mapsto \mathbf{y} = c \cdot \mathbf{e}_0 + x^1\mathbf{e}_1 + \ldots + x^d\mathbf{e}_d \\ \text{in } \mathbf{V} \text{ with any fixed } c \in \mathbf{R}, \end{array}$$

or by

$$(17') \quad \boldsymbol{u} = \boldsymbol{e}^0 \cdot u_0 + \boldsymbol{e}^1 u_1 + \ldots + \boldsymbol{e}^d u_d \mapsto \boldsymbol{v} = \boldsymbol{e}^0 \frac{u_0}{c} + \boldsymbol{e}^1 u_1 + \ldots + \boldsymbol{e}^d u_d \ \text{ in } \mathbf{V}^*,$$

so that $\mathbf{x}\boldsymbol{u} = \mathbf{y}\boldsymbol{v} = u_0 + x^1 u_1 + \ldots + x^d u_d$ holds for any $\mathbf{x}$ and $\boldsymbol{u}$ and their images $\mathbf{y}$ and $\mathbf{v}$, respectively. We look at (17') that $\langle ; \rangle$ on $\mathbf{V}^*$ is invariant for a dilatation, since $\boldsymbol{e}^0$ is orthogonal to any other forms in $\mathbf{V}^*$.

We could define the *angles of* $(d-1)$*-planes* and the *distances of points* in general [29, 39]. Now we mention only some hyperbolic cases.

The proper $(d-1)$-planes $[\boldsymbol{u}]$ and $[\boldsymbol{v}]$ intersect in proper $(d-2)$-planes, iff

$$(18) \qquad \langle \boldsymbol{u}; \boldsymbol{u}\rangle\langle \boldsymbol{v}; \boldsymbol{v}\rangle - \langle \boldsymbol{u}; \boldsymbol{v}\rangle^2 > 0 \quad \text{with} \quad \langle \boldsymbol{u}; \boldsymbol{u}\rangle,\ \langle \boldsymbol{v}; \boldsymbol{v}\rangle > 0.$$

We define the angle α of the positive domain, with the inequalities

$$(19)\qquad \mathbf{x}\boldsymbol{u} \geqq 0, \quad \mathbf{x}\boldsymbol{v} \geqq 0, \quad \text{by} \quad \cos\alpha = -\langle \boldsymbol{u}, \boldsymbol{v}\rangle / \sqrt{\langle \boldsymbol{u}; \boldsymbol{u}\rangle \langle \boldsymbol{v}; \boldsymbol{v}\rangle}.$$

If the proper points $[\mathbf{x}]$ and $[\mathbf{y}]$ are taken so that

$$(20)\qquad \langle \mathbf{x}; \mathbf{y}\rangle < 0 \quad \text{with} \quad \langle \mathbf{x}; \mathbf{x}\rangle, \quad \langle \mathbf{y}; \mathbf{y}\rangle < 0,$$

then their distance s is defined by

$$(21)\qquad \operatorname{ch} s = -\langle \mathbf{x}, \mathbf{y}\rangle / \sqrt{\langle \mathbf{x}; \mathbf{x}\rangle \langle \mathbf{y}; \mathbf{y}\rangle} > 1.$$

By (19) α lies in the open interval $(0, \pi)$, and by (21) $s > 0$ holds. But these formulas can be extended to other $(d-1)$-planes and other points if we think of complex angles and distances, (including ∞), too [25, 29]. The isometries of $\mathcal{E}^d$, $\mathcal{S}^d$, $\mathcal{H}^d$ preserve distances and angles.

We mention a characteristic theorem for d-simplices in $\mathcal{S}^d$, $\mathcal{H}^d$, $\mathcal{E}^d$.

THEOREM 7.1. *Let (b^{ij}) be a symmetric $(d+1) \times (d+1)$ matrix so that the bilinear form*

$$(22)\qquad \langle ; \rangle : \mathbf{R}^*_{d+1} \times \mathbf{R}^*_{d+1} \to \mathbf{R}, \qquad \langle \boldsymbol{u}; \boldsymbol{v}\rangle := \sum_{i,j=0}^{d} u_i b^{ij} v_j$$

has the signature $\langle +, \dots, +, +\rangle$, $\langle +, \dots, +, -\rangle$ or $\langle +, \dots, +, 0\rangle$. Then there exists a simplex

$$(23)\qquad \mathbf{S} = \{(\mathbf{x}) : \mathbf{x}\boldsymbol{b}^i \geqq 0, \ \mathbf{x} \in \mathbf{V}, \quad \textit{for each} \quad \boldsymbol{b}^0, \boldsymbol{b}^1, \dots, \boldsymbol{b}^d \in \mathbf{V}^*\}$$

whose vertices are $(\mathbf{a}_i)$ with $\mathbf{a}_i \boldsymbol{b}^j = \delta_i^j$ on the projective metric d-sphere S^d $(\mathbf{V}, \mathbf{V}^*)$, which has spherical, hyperbolic or Euclidean metric, respectively, such that the facet forms $\boldsymbol{b}^0, \boldsymbol{b}^1, \dots, \boldsymbol{b}^d$ of $\mathbf{S}$ satisfy the relations

$$(24)\qquad \langle \boldsymbol{b}^i, \boldsymbol{b}^j\rangle = b^{ij} \quad \textit{for} \quad i, \ j = 0, 1, \dots, d.$$

For any two such simplices ${}^1\mathbf{S}$, ${}^2\mathbf{S}$ there exists either exactly one isometry for $\mathcal{S}^d$ resp. $\mathcal{H}^d$, or exactly one similarity for $\mathcal{E}^d$ which maps ${}^1\mathbf{S}$ onto ${}^2\mathbf{S}$.

The matrix (b^{ij}) describes the generalized angles α^{ij} of corresponding facets of $\mathbf{S}$ by

$$(25)\qquad \cos\alpha^{ij} = -b^{ij} / \sqrt{b^{ii} \cdot b^{jj}}.$$

For $\mathcal{S}^d$ and $\mathcal{H}^d$ the inverse of (b^{ij}) is just $(a_{ij}) = (\langle \mathbf{a}_i; \mathbf{a}_j\rangle)$ and it describes the generalized distances s_{ij} between vertices $(\mathbf{a}_i)$ and $(\mathbf{a}_j)$ of the simplex $\mathbf{S}$ by

$$(26)\qquad \cos s_{ij} = a_{ij} / \sqrt{a_{ii} \cdot a_{jj}} \quad \textit{for} \quad \mathcal{S}^d,$$

$$(27)\qquad \operatorname{ch} s_{ij} = -a_{ij} / \sqrt{a_{ii} \cdot a_{jj}} \quad \textit{for} \quad \mathcal{H}^d,$$

For $\mathcal{E}^d$ let B_{ij} be the corresponding signed minor determinants of (b^{ij}), and assume that $B_{00} > 0$. Then the vertex $(\mathbf{a}_0)$ of $\mathbf{S}$ is a proper point of $\mathcal{E}^d$. If B_{0i}, $B_{0j} \neq 0$ and $B_{00}^{(ij)}$ denotes the corresponding signed $(d-1)$-minor determinants of B_{00}, then $\frac{B_{00}}{B_{0i}}\mathbf{a}_i$ and $\frac{B_{00}}{B_{0j}}\mathbf{a}_j$ are also proper vertices of $\mathbf{S}$ and the following distance formulas hold:

$$s_{0i}^2 = B_{00} \cdot B_{00}^{(ii)} / (B_{0i})^2, \tag{28}$$

$$s_{ij}^2 = B_{00}\left[B_{00}^{(ii)}\frac{1}{(B_{0i})^2} - 2B_{00}^{(ij)}\frac{1}{B_{0i}B_{0j}} + B_{00}^{(jj)}\frac{1}{(B_{0j})^2}\right]. \tag{28'}$$

If $B_{ii} = 0$ then $(\mathbf{a}_i)$ is an ideal point of $\mathbf{S}$.

The proof is a simple consequence of the preceding facts. For $\mathcal{S}^d$ and $\mathcal{H}^d$ see e.g. [29]. We only prove the Euclidean distance formulas. To the bilinear form of signature $\langle +, \ldots, +, 0\rangle$ there is a polarity $(_*) : \mathscr{b}^i \mapsto \mathbf{b}^i = b^{ij}\mathbf{a}_j$ with a (radical) form $\mathscr{b} = \mathscr{b}^j b_j \mapsto \mathbf{0}$, $0 = \langle \mathscr{b}^i; \mathscr{b}\rangle = b^{ij} b_j$ for $i = 0, 1, \ldots, d$. (We apply Einstein–Schouten's summation convention.) It follows

$$b_j = B_{0j} \quad \text{for} \quad j = 0, 1, \ldots, d \tag{29}$$

up to a factor of proportionality.

A vertex $\frac{1}{b_j}\mathbf{a}_j$ lies in the d-space $\mathbf{x}\mathscr{b} = 1$ iff

$$0 \neq \mathbf{a}_j \mathscr{b} = \mathbf{a}_j \mathscr{b}^i b_i = \delta_j^i b_i = b_j = B_{0j} \quad \text{for} \quad j = 0, 1, \ldots, d. \tag{30}$$

Then the following vector lies in the image d-space at $(_*)$:

$$\frac{1}{b_i}\mathbf{a}_i - \frac{1}{b_j}\mathbf{a}_j = c_r \mathbf{b}^r = c_r b^{rs}\mathbf{a}_s \quad \text{for} \quad r = 1, 2, \ldots, d; \tag{31}$$

and the distance formula for $\mathcal{E}^d$ is defined by

$$s_{ij}^2 = \langle c_u \mathbf{b}^u; c_v \mathbf{b}^v\rangle = c_u b^{uv} c_v \quad \text{for} \quad u,\ v = 1, 2, \ldots, d. \tag{32}$$

After having solved equation (31) for c_r, we have essentially different cases in (32) for $i = 0$ and $0 \neq i$, j as indicated in (28) and (28'). Q.e.d.

We remark that less explicit criteria for polyhedra are mentioned in [39, 40].

8. The algorithmic extension of Poincaré's polyhedron theorem

In Section 6 we have defined the simplicial complex $\tilde{\mathbf{P}}$ on the projective d-sphere. $\tilde{\mathbf{P}}$ has piece-wise affine topology and provides an atlas containing connected neighbourhoods for every point of the identified d-polyhedron $\mathcal{P}$ by Theorem 6.1. We define the *metric realization of* $\tilde{\mathbf{P}}$, *i.e. the local metric realization of* $[\mathcal{P}^{\mathcal{G}}; \mathcal{G}(\mathcal{I}, \{\nu_e\})]$, which will be denoted by $\tilde{\mathsf{P}}$, with the following properties.

MR.1. $\tilde{\mathsf{P}}$ *is a simplicial complex on the projective metric d-sphere* S^d *with spherical, hyperbolic or Euclidean metric.* $\tilde{\mathsf{P}}$ *has only proper facets.*

MR.2. *A simplicial homeomorphism* $\varphi : \tilde{\mathbf{P}} \to \tilde{\mathsf{P}}$ *is required such that* φ *is compatible with the actions of* $\mathcal{G}$ *on* $\tilde{\mathbf{P}}$ *and*

$$\varphi^{-1}\mathrm{h}\varphi =: \mathsf{h} : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}}$$

is a "simplex-wise" isometry in S^d *for any occurring action* $\mathrm{h} \in \mathcal{G}$ *on* $\tilde{\mathbf{P}}$. *We say that* $\mathrm{h} \in \mathcal{G}$ *is* φ*-associated with* $\mathsf{h} \in \mathrm{Iso}\,\mathsf{S}^d$; $\mathrm{h} = 1$ *also allowed, then* h *is a restriction of the identity of* S^d.

MR.3. *We require for each edge class* e *that the simplices in* $\tilde{\mathbf{P}}$, *describing the fundamental polyhedron* $\mathcal{P}_\mathrm{e}$ *of the stabilizer* $\mathcal{G}_\mathrm{e}$, *will have images* P_e *in* S^d *such that the generators of* $\mathcal{G}_\mathrm{e}$ *are* φ*-associated with the generators identifying the corresponding facets of* P_e *by isometries. Furthermore, we require that any cycle relation, with exponent* ν_e (0 is included), *shall be isometrically satisfied for these generators in* S^d.

REMARK. Instead of metric realization we can obviously define topologic, affine, conform, etc. realization of $\tilde{\mathbf{P}}$ as well. The existence of these realizations is always questionable as we indicated after Theorem 6.1. It may depend on $\mathcal{P}$, $\mathcal{I}$ and $\{\nu_\mathrm{e}\}$, too. Examples show these facts in Section 9. □

The main problem is *how to realize the global polyhedron complex* $[\mathcal{P}^{\mathcal{G}}; \mathcal{G}(\mathcal{I}, \{\nu_\mathrm{e}\})]$ *on a projective metric d-sphere* S^d, particularly in $\mathcal{S}^d$, $\mathcal{H}^d$, $\mathcal{E}^d$. *The metric realization* $[\mathsf{P}^\mathsf{G}, \mathsf{G}(\mathsf{I}, \{\nu_\mathrm{e}\})]$ exists on S^d, modelling $\mathcal{S}^d$, $\mathcal{H}^d$ or $\mathcal{E}^d$, if there is an *equivariant bijection* Φ of $\mathcal{P}^{\mathcal{G}}$ onto P^G on S^d, i.e. there is an isomorphism f of $\mathcal{G}$ onto an isometry group $\mathsf{G} \subset \mathrm{Iso}\,\mathsf{S}^d$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathcal{G} & : & \mathcal{P}^{\mathcal{G}} & \longrightarrow & \mathcal{P}^{\mathcal{G}} \\
\Big\downarrow f & & \Big\downarrow \Phi & & \Big\downarrow \Phi \\
\mathsf{G} & : & \mathsf{P}^\mathsf{G} & \longrightarrow & \mathsf{P}^\mathsf{G}
\end{array}$$

that means $\mathrm{g}^f = \mathbf{g} = \Phi^{-1}\mathrm{g}\Phi$ for any $\mathrm{g} \in \mathcal{G}$.

Poincaré made the first steps [33, 34] to solve particular cases in dimensions $d = 2$, 3, by formulating angle conditions for a fundamental polygon, resp. a polyhedron. The d-dimensional formulation is indicated in [20]. A. D. Aleksandrov [1] gave a principal generalization for filling the d-space of constant curvature by face-to-face compact polyhedra from finitely many congruence classes. His criterion says that it is satisfactory to guarantee the (unique) continuation of the filling around $(d-2)$-faces. His topological method, using covering space arguments, is very effective. (For non-compact hyperbolic polyhedra see also [3, 4].) In [40] there is a formulation of the

Poincaré theorem in this sense. Applications can be found in many papers, see e.g. [6, 13, 14, 19, 24, 25, 26, 30, 31, 32, 35, 40, 41, 43].

Our formulation will also reduce the problem to checking finitely many cases. The combinatorial algorithms, described in Sections 1–6, can be extended by geometric considerations which need to examine $(d-1)$-dimensional cases, namely the stabilizers of faces of $\mathcal{P}$. Thus we get a hopeful inductive procedure.

Consider a local metric realization $\tilde{\mathsf{P}}$ which may have curved faces, too [26, 27, 30]. It contains the φ-images of those simplices which amount $\mathbf{P} \subset \tilde{\mathbf{P}}$ corresponding to $\mathcal{P}$ by its flags in $\mathcal{F}$. We denote $\mathbf{P}^{\varphi} =: \mathsf{P} := \mathcal{P}^{\Phi}$. Each face identification of $\mathcal{P}$ is also described by simplices in $\tilde{\mathbf{P}}$ and so in $\tilde{\mathsf{P}}$ as defined in Algorithm 6.1. Imagine that these simplices are glued to the corresponding facet by the isometry $\varphi^{-1}\mathrm{g}_i^{-1}\varphi =: \mathbf{g}_i^{-1} \in \mathrm{Iso}\,\mathsf{S}^d$. So the $\mathbf{g}_i^{-1}$-image of P can be defined as an extension of the simplex-wise isometry in S^d. Hence, for any element $\mathrm{h} = \mathrm{g}_1\mathrm{g}_2 \dots \mathrm{g}_{s-1}\mathrm{g}_s$ of $\mathcal{G}$ with generators $\mathrm{g}_1 \dots \mathrm{g}_s \in \mathcal{I}$ we can define the corresponding Φ-image of $\mathcal{P}^{\mathrm{h}}$, i.e. the polyhedron P^{h}, passing through $\mathsf{P}^{\mathbf{g}_s}, \mathsf{P}^{\mathbf{g}_{s-1}\mathbf{g}_s}, \dots, \mathsf{P}^{\mathbf{g}_2 \dots \mathbf{g}_{s-1}\mathbf{g}_s}$ as described in Theorem 3.1.

For a while we do not know, whether Φ extends uniquely or not, and the Φ-image of $\mathcal{P}^{\mathcal{G}}$ cover the proper points of S^d or not.

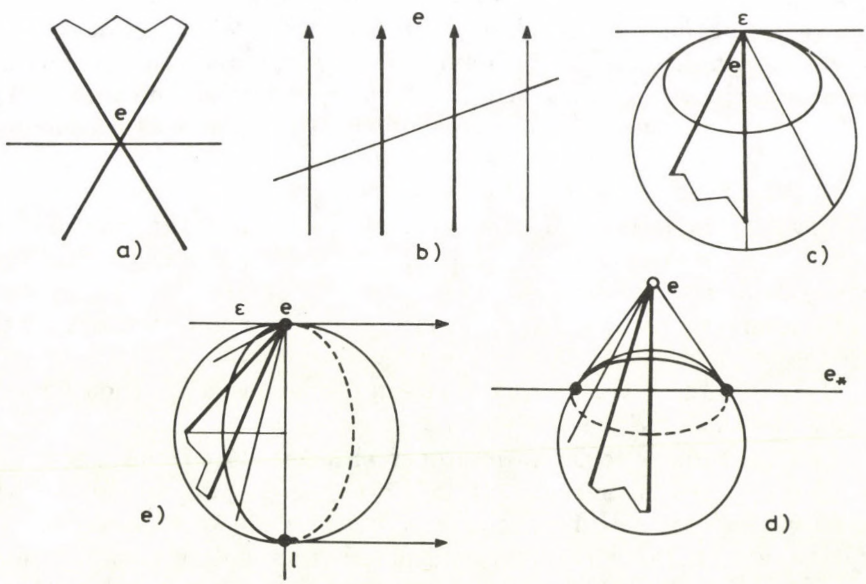


Fig. 7

Now consider the simplices in $\tilde{\mathsf{P}}$ which constitute the fundamental domain P_e for the stabilizer G_e of a $(d-2)$-face e. We have the cases, described in Fig. 7.a–e.

If $1 \leqq \nu_e$ is finite then P_e has proper $(d-2)$-face e (Fig. 7.a). The stabilizer G_e has the rotation subgroup R_e of order ν_e in $\mathcal{S}^d$, $\mathcal{H}^d$ or $\mathcal{E}^d$.

If $\nu_e = 0$, i.e. R_e has infinite order, then we may have 4 cases:

Fig. 7.b shows when R_e consists of Euclidean translations, since e is an ideal $(d-2)$-face. Fig. 7.c is the case where R_e is generated by a "horocyclic translation" in $\mathcal{H}^d$ about the end of e.

In Fig. 7.d the $(d-2)$-face e lies out of the absolute of $\mathcal{H}^d$, its 1-dimensional polar is the proper line e_*, R_e is generated by a hypercyclic translation along e_* which happens not to intersect P.

In the last cases, when e is not proper, the metric data of the corresponding generators of G_e are not unique in general.

Fig. 7.e shows such an important 2-dimensional hyperbolic case, where G_e is generated by a hyperbolic translation along a line ℓ. One end of ℓ is the unique end of e. The line ℓ does not meet any G_e-images of P_e. In this way we cannot get a fundamental domain for the starting group G.

The d-dimensional generalization of this case $(d \geqq 3)$ yields also a hyperbolic translation along a line ℓ, where the $(d-2)$-face e is fixed but not point-wise.

On the $(d-1)$-plane ε, tangent to the absolute of $\mathcal{H}^d$ at the end of e, we have a Euclidean projective metric structure, i.e. a bilinear form of signature $\langle +, +, \ldots, 0 \rangle$, so that the vector $\mathbf{e}$, assigning the end of e, is orthogonal to each vector of the corresponding d-subspace $\mathbf{E}^d$ of $\mathbf{V}^{d+1}$ (see Section 7). The hyperbolic translation along ℓ induces a Euclidean dilatation in ε with centre at the end of e, the axis (as ideal $(d-2)$-plane is cut off ε by the $(d-1)$-plane tangent to the absolute at the second end of ℓ. Such a case cannot yield a fundamental domain for G, since ℓ will not be intersected by P_e-images under G_e, but any neighbourhood of any point on ℓ will.

Now let us consider the simplices in $\tilde{P}$ which constitute the fundamental domain P_x for the stabilizer G_x of an m-face x $(0 \leqq m \leqq d-3)$. We have analogous cases as before.

G_x is finite iff x is a proper m-face. The midpoint of x is a centre of a $(d-1)$-sphere on which G_x will act discontinuously.

If G_x is infinite, then x shall be an improper m-face without or with end on the absolute of $\mathcal{E}^d$ or $\mathcal{H}^d$.

First, assume x to be an improper m-face without end, then the polar $(d-m-1)$-plane x_* is invariant in $\mathcal{H}^d$ under G_x and so are the $(d-m-1)$-hyperspheres with base x_*. Then we can define an appropriate $(d-1)$-hypersphere (distance surface) with pole in the midpoint of x, intersecting only those $(d-1)$-facets of P_x which contain x. On this hypersphere G_x acts as a $(d-1)$-dimensional hyperbolic isometry group with a fundamental domain cut off by P_x off the $(d-1)$-hypersphere.

Assume that x has one end. Then G_x acts either as a Euclidean motion group with invariant $(d-m-1)$-planes, respectively with invariant $(d-m-1)$-horospheres (in analogy with Fig. 7.b–c), or as a Euclidean (proper) similarity group (in analogy with Fig. 7.e).

In this latter case there exists a line ℓ (at least one) which will not be intersected by the $\mathsf{P_x}$-images under the stabilizer $\mathsf{G_x}$, but any neighbourhood of any point of ℓ will. Thurston [37] developed a method (called Dehn's surgery) which may lead to a discontinuous transformation group also in this case, but then we have additional defining relation and a modified fundamental domain. I only *remark* that the famous Thurston's manifold of volume 0.98 and the hyperbolic manifold of volume 0.94 discovered by Matveev–Fomenko [12] can also be constructed in our manner by a compact fundamental polyhedron.

Summarizing, if the cases Fig. 7.e occur for the local realization $\tilde{\mathsf{P}}$ in $\mathcal{H}^d$, then this does not extend to a global realization of $[\mathcal{P}^{\mathcal{G}}, \mathcal{G}(\mathcal{I}, \{\nu_e\})]$ in general.

THEOREM 8.1. *The d-dimensional polyhedron complex* $[\mathcal{P}^{\mathcal{G}}; \mathcal{G}(\mathcal{I}, \{\nu_e\})]$ *can be realized on the projective metric d-sphere* S^d, *if the corresponding finite simplicial complex* $\tilde{\mathbf{P}}$ *has a metric realization* $\tilde{\mathsf{P}}$ *on* S^d *by* MR.1–3 *such that any stabilizer of any face with exactly one end has only horospherical generators by* $\tilde{\mathsf{P}}$.

PROOF. We follow A. D. Aleksandrov [1] with some modifications and prove by induction on dimension. We consider the local realization $\tilde{\mathsf{P}}$ as a finite collection of simplices on a projective metric d-sphere S^d. Our conditions guarantee that we can uniquely tile S^d with G-images of P around $(d-1)$-facets and $(d-2)$-faces of P.

Now we prepare all the typical neighbourhoods for the point equivalence classes on m-faces of P in the simplices of $\tilde{\mathsf{P}}$ for any m, $0 \leqq m \leqq d$.

For any proper point X on an m-face we consider a ball $\mathsf{B_X}$ with centre in X, intersecting only those facets of $\tilde{\mathsf{P}}$ which contain X. The $(d-1)$-dimensional surface $\mathsf{S_X}$ of $\mathsf{B_X}$ cut off $\tilde{\mathsf{P}}$ a $(d-1)$-dimensional complex $\mathsf{P_X}$ which satisfies the conditions of our theorem for dimension $d-1$. By induction we can uniquely tile the whole sphere $\mathsf{S_X}$ and so the whole ball $\mathsf{B_X}$, since on $\mathsf{S_X}$ we have a spherical metric $(d-1)$-space.

For any improper m-face with exactly one end X we consider the corresponding horoball $\mathsf{B_X}$ whose surface $\mathsf{S_X}$ intersects (in proper points) only those facets of $\tilde{\mathsf{P}}$ which contain the end X. The horosphere $\mathsf{S_X}$ cut off $\tilde{\mathsf{P}}$ a $(d-1)$-complex $\mathsf{P_X}$ which satisfies our conditions in less dimension. Therefore, by induction, we can uniquely tile the whole $\mathsf{S_X}$ and so the whole $\mathsf{B_X}$, since on the horosphere $\mathsf{S_X}$ we have a Euclidean metric $(d-1)$-space. This arguments hold also for $\mathcal{E}^d$ where a horoball means a half-space.

Consider any improper m-face in $\mathcal{H}^d$ with an outer midpoint X. We take such a hypersphere $\mathsf{S_X}$ with base $(d-1)$-plane polar to the centre X, and that proper part of $\mathcal{H}^d$, the hyperball $\mathsf{B_X}$, which intersect only the facets of $\tilde{\mathsf{P}}$ having X as common point. This hyperball $\mathsf{B_X}$ does not contain any midpoints of the proper m-faces of $\tilde{\mathsf{P}}$, $0 \leqq m \leqq d$. The hypersphere S_X cut off $\tilde{\mathsf{P}}$ a $(d-1)$-complex P_X which satisfies our conditions in less dimension. Again by induction, we can uniquely tile the whole S_X and so the whole B_X, since on S_X we have a hyperbolic metric $(d-1)$-space.

Now we determine a universal distance δ for $\tilde{\mathsf{P}}$ (Lebesgue-distance). Cutting off $\tilde{\mathsf{P}}$ all the open horoballs and hyperballs defined before, we get a compact subset $\tilde{\mathsf{P}}^c$ of $\tilde{\mathsf{P}}$. Any point A of $\tilde{\mathsf{P}}^c$ has a typical ball-like neighbourhood $\mathsf{B}(\mathrm{A}, r_\mathrm{A})$ with radius r_A. Take the open balls $\mathsf{B}\left(\mathrm{A}, \frac{1}{2}r_\mathrm{A}\right)$ at every point A of $\tilde{\mathsf{P}}^c$. These balls cover $\tilde{\mathsf{P}}^c$. By the Borel lemma, from these balls we can choose a finite covering. The minimal radius of these finitely many balls will be a universal distance δ with the property that any point X of $\tilde{\mathsf{P}}$ has a ball-like neighbourhood of radius δ which is contained in a typical generalized neighbourhood. Here "generalized" means that we count also the finitely many horoballs and hyperballs to the typical ball-like neighbourhoods.

Consider any proper point A of the projective metric sphere S^d. Connect it with a point C in $\tilde{\mathsf{P}}$ by a geodesic line. We shall prove that A is uniquely covered by a G-iamge of P.

We cover the segment CA by finitely many subsegments $\mathrm{CC}_1, \mathrm{C}_1\mathrm{C}_2, \dots,$ $\mathrm{C}_n\mathrm{A}$ whose length is smaller than δ. We go from C to A step-by-step using the facts that the typical generalized neighbourhoods have uniquely been covered by the induction assumption. We easily see as well that the segment CA is covered by finitely many G-images of P joining each other along adjacent facets.

The covering of the proper part of S^d is unique, since it is simply connected.

The theorem is true for $d=2$, since the conditions for facets and $(d-2)$-faces, i.e. vertices, guarantee the (unique) tiling of the typical generalized neighbourhoods in $\tilde{\mathsf{P}}$.

We do not claim that the improper points of $\mathsf{S}^d = \mathcal{H}^d$ will be covered by G-images of P if P has also outer points. This is not true in general. But the covering is unique, since each outer point can be characterized by a bundle of proper lines running inside of a G-image of P. Q.e.d.

9. Examples

In this section we illustrate our method by further examples.

1. Fig. 8 shows a 3-simplex $A_0A_1A_2A_3 =: \mathcal{P}$ with a pairing $\mathcal{I}$ of its facets from among the possible 64 different pairings [28, 43]. Now, the generators of $\mathcal{G}(a, b, ; c, d)$ are:

$$(1) \qquad \mathrm{z}_1 : A_1A_2A_3 \to A_0A_3A_2 : \quad \mathrm{z}_2 : A_0A_1A_2 \to A_1A_0A_3.$$

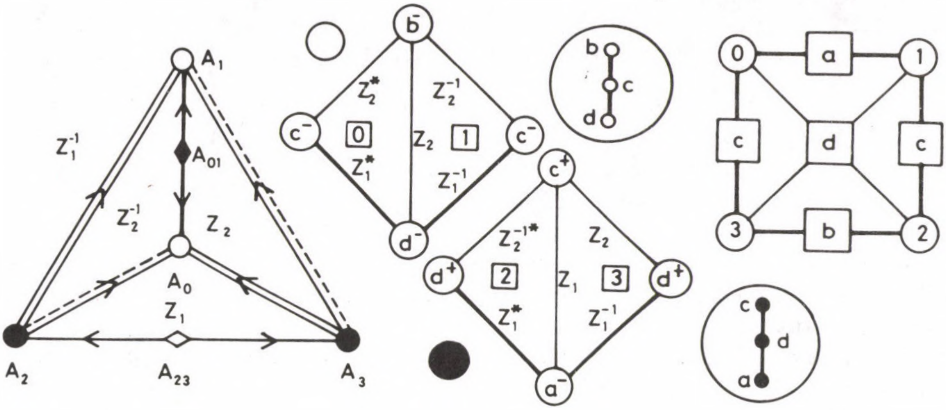


Fig. 8

The edge classes and the cycle relations are read off Fig. 8, by Section 4.

(2)
$$\mathrm{a} \rightarrow : (z_1 z_1)^a = 1; \quad \mathrm{b} \rightarrow : (z_2 z_2)^b = 1;$$
$$\mathrm{c} \Rightarrow : (z_1 z_2^{-1})^c = 1; \quad \mathrm{d} \dashrightarrow : (z_1 z_2)^d = 1.$$

The stabilizers of the edges are: $\mathcal{G}_a := (z_1 - z_1^{2a})$; $\mathcal{G}_b := (z_2 - z_2^{2b})$; $\mathcal{G}_c := (r_1 :=$ $:= z_1 z_2^{-1} - r_1^c)$; $\mathcal{G}_d := (r_2 := z_1 z_2 - r_2^d)$.

The stabilizers of the vertices are also illustrated in Fig. 8 (see [16] for the surface diagram of 2-dimensional groups):

(3)
$$\circ\{A_0, A_1\} : \mathcal{G}_{A_1} := (r_2 := z_1 z_2, r_3 := z_2^2 - r_2^d, r_3^b, (r_2 r_3^{-1})^c);$$
$$\bullet\{A_2, A_3\} : \mathcal{G}_{A_3} := (r_4 := z_1^2, r_5 := z_1^{-1} z_2 - r_4^a, r_5^c, (r_4 r_5)^d).$$

We applied the considerations from Section 5.

Now we examine the possible collections of exponents $\{\nu_e\} = \{a, b; c, d\}$ and the realizations of $\mathcal{P}^{\mathcal{G}}$.

$\{a, b; c, d\} = \{1, 1; 1, 1\}$ involves the group $\mathcal{G} := \{z_1 - z_1^2 = 1\}$ of order two. $\mathcal{P}^{\mathcal{G}}$ has a realization in the spherical space $\mathcal{S}^3$, z_1 will be an antiinversion changing P as a ball with the complementary ball P^{z_1}.

$\{a, b; c, d\} = \{1, 1; 1, 2\}$ involves also $\mathcal{G} := \{z_1 - z_1^2\}$. But $\mathcal{P}^{\mathcal{G}}$ has no metric realization in our sense.

For the general situation we consider a tetrahedron P on a projective metric sphere S^3 with the congruent facets indicated in Fig. 8. It follows that P has a half-turn with axis $A_{01}A_{23}$ as a self-symmetry. This implies the angles of P to be $\frac{\pi}{a}, \frac{\pi}{b}; \frac{\pi}{c}, \frac{\pi}{d}$ and $a, b; c, d \geqq 2$, thus we have the following

Schläfli–Coxeter matrix [7]:
(4)
$$(b^{ij}) = (\langle \ell^i; \ell^j \rangle) = \begin{pmatrix} 1 & -\cos\frac{\pi}{a} & -\cos\frac{\pi}{d} & -\cos\frac{\pi}{c} \\ -\cos\frac{\pi}{a} & 1 & -\cos\frac{\pi}{c} & -\cos\frac{\pi}{d} \\ -\cos\frac{\pi}{d} & -\cos\frac{\pi}{c} & 1 & -\cos\frac{\pi}{b} \\ -\cos\frac{\pi}{c} & -\cos\frac{\pi}{d} & -\cos\frac{\pi}{b} & 1 \end{pmatrix}, \quad i,\ j = 0,1,2,3$$

in the sense of Theorem 7.1. The change $a \leftrightarrow b$ or $c \leftrightarrow d$ does not make any difference. These simplices have the Coxeter diagram in Fig. 8 which has an involutive automorphism without fixed nodes, since P has a half-turn as a self-symmetry. Theorem 7.1 provides a complete description of the metric realizations of these simplices.

By (4), the bilinear form

$$(5) \qquad \langle u, v \rangle = \sum_{i,j=0}^{3} u_i b^{ij} v_j$$

on $\mathbf{R}_4^*$ with the matrix (b^{ij}) is of signature $\langle +,+,+,+ \rangle$ iff

$$\begin{aligned}(6)\quad \{a,b;c,d\} &= \{m,n;2,2\} \qquad m,n \geqq 2;\\ &= \{2,2;m,2\} \text{ (this has the same simplex as } \{m,m;2,2\});\\ &\qquad (m \geqq 3)\\ &= \{3,2;3,2\}.\end{aligned}$$

So we have got all the metric realizations of $\mathcal{P}^{\mathcal{G}}$ in the spherical space $\mathcal{S}^3$.

The signature of (5) is $\langle +,+,+,0 \rangle$ iff (see [28])

$$\begin{aligned}(7)\qquad \{a,b;c,d\} &= \{3,3;3,2\}, \quad 203.\ \mathbf{Fd3} \\ &= \{2,2;3,3\}, \quad 219.\ \mathbf{F43c} \\ &= \{3,2;4,2\}, \quad 222.\ \mathbf{Pn3n}.\end{aligned} \quad \Big\}\ \text{these have the same simplex.}$$

These are Euclidean crystallographic groups, realizing $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{E}^3$.

The signature of (5) is $\langle +,+,+,- \rangle$ in the following cases (8)–(10):

$$\begin{aligned}(8)\quad \{a,b;c,d\} &= \{5,2;3,2\},\ \{3,2;5,2\};\\ &= \{4,3;3,2\},\ \{5,3;3,2\},\ \{5,4;3,2\};\\ &= \{4,4;3,2\} - \{3,3;4,2\} - \{2,2;3,4\} \text{ for the same simplex;}\\ &= \{5,5;3,2\} - \{3,3;5,2\} - \{2,2;5,3\} \text{ for the same simplex.}\end{aligned}$$

These are so-called Lanner simplices [40], when P is compact in $\mathcal{H}^3$. They realize 11 non-isomorphic $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{H}^3$.

$$\begin{aligned}(9)\quad \{a,b;c,d\} &= \{3,2;6,2\},\ \{6,2;3,2\},\ \{4,2;4,2\};\\ &= \{3,2;3,3\},\ \{6,3;3,2\},\ \{6,4;3,2\},\ \{6,5;3,2\},\ \{4,3;4,2\}.\end{aligned}$$

These are non-compact simplices in $\mathcal{H}^3$ with 2 proper vertices and 2 ends. They realize 8 non-isomorphic $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{H}^3$.

$$\begin{aligned}(10)\qquad \{a,b;c,d\} &= \{6,6;3,2\} - \{3,3;6,2\} - \{2,2;6,3\};\\ &= \{4,4;4,2\} - \{2,2;4,4\};\\ &= \{3,3;3,3\}.\end{aligned}$$

So we have 3 simplices in $\mathcal{H}^3$ with 4 ends as vertices. They realize 6 non-isomorphic $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{H}^3$.

Of course, we may have also non-proper edges for P.

For instance, the case $\{0,2;2,2\}$ is realized by the matrix

$$(11)\qquad (b^{ij}) = \begin{pmatrix} 1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

of signature $\langle +,+,+,0\rangle$ (see Theorem 7.1). Then we have a Euclidean simplex P. A_0, A_1 are proper vertices; A_2, A_3 are ideal points. $\mathcal{P}^{\mathcal{G}}$ is realized by a Euclidean "rod group" G, generated by the glide-reflection $\mathbf{z}_1$ and the rotatory reflection $\mathbf{z}_2$ about the line A_0A_1.

The extreme case $\{0,0;0,0\}$ is realized, e.g., by the matrix

$$(12)\qquad (b^{ij}) = \begin{pmatrix} 1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\end{pmatrix}$$

of signature $\langle +,+,+,-\rangle$, since its eigenvalues are $\langle 2,2,2,-2\rangle$. Then we have a hyperbolic simplex P with proper facets. Every edge of P has exactly one end as midpoint. All the vertices lie out of the absolute. So we realize $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{H}^d$, G is generated by two horospherical glide-reflections $\mathbf{z}_1$ and $\mathbf{z}_2$ so that $\mathbf{z}_1\mathbf{z}_2^{-1}$ and $\mathbf{z}_1\mathbf{z}_2$ are horospherical translations. The stabilizers of vertices are plane hyperbolic groups.

The earlier case $\{0,2;2,2\}$ also allows us to define the matrix

$$(11)\qquad (b^{ij}) = \begin{pmatrix} 1 & -\operatorname{ch} t & 0 & 0\\ -\operatorname{ch} t & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

of signature $\langle +,+,+,-\rangle$, since its eigenvalues are $(1,1,1+\operatorname{ch} t,1-\operatorname{ch} t)$. That means, the planes $(b^0) = A_1A_2A_3$ and $(b^1) = A_0A_2A_3$ have the distance t along the common perpendicular A_0A_1, polar to the improper edge A_2A_3. The generator $\mathbf{z}_1$ is a glide reflection along A_0A_1, $\mathbf{z}_2$ is a rotatory reflection

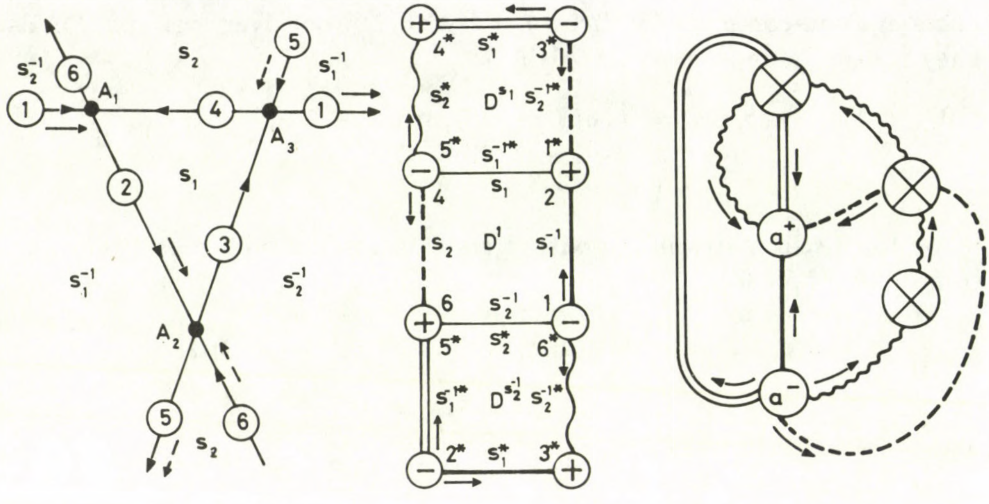


Fig. 9

about A_0A_1. We have got isomorphic groups by (11) and (13) but different realizations.

2. Fig. 9 shows a polyhedron $\mathcal{P}$ whose surface is homeomorphic to the projective plane and this is while $\mathcal{P}$ cannot be realized in $\mathcal{E}^3$ or in $\mathcal{H}^3$, $\mathcal{S}^3$. We start with a division of the projective plane by three lines into 4 triangles. We introduce the formal midpoint if $\mathcal{P}$ which is connected with the points of the projective plane to form a cone. We shall pair the triangles by associating their flags. The vertices A_1, A_2, A_3 divide the lines into 6 numbered segments. All these will be equivalent at the pairing $\mathcal{I}$, whose generators are

$$\mathrm{s}_1 : \mathit{125} \to \mathit{234}; \qquad \mathrm{s}_2 : \mathit{631} \to \mathit{546}.$$

This yields the cycle rotation $\mathrm{s}_1^2\mathrm{s}_2\mathrm{s}_1^{-1}\mathrm{s}_2^{-2}$ by Section 4. Taking an exponent a, we get a representation of the group

$$\mathcal{G} := (\mathrm{s}_1, \mathrm{s}_2 - (\mathrm{s}_1^2\mathrm{s}_2\mathrm{s}_1^{-1}\mathrm{s}_2^{-2})^a = 1). \tag{14}$$

The stabilizer $\mathcal{G}_{A_1}$ for the equivalence class of vertices A_1, A_2, A_3 will be determined by constructing $\mathcal{P}_{A_1}$ as in Section 5. $\mathcal{P}_{A_1}$ consists of 3 quadrangular vertex domains glued at A_1. Fig. 9 shows this 2-dimensional structure. $\mathcal{D}^1$ is the vertex domain of A_1 with 4 facets, denoted by $\mathrm{f}_{\mathrm{s}_1}$, $\mathrm{f}_{\mathrm{s}_1^{-1}}$, $\mathrm{f}_{\mathrm{s}_2^{-1}}$, $\mathrm{f}_{\mathrm{s}_2}$ (after the generators, the f's are omitted in Fig. 9), and with 4 oriented edges *1*, *2*, *4*, *6*. Along the facet $\mathrm{f}_{\mathrm{s}_1}$ we glue the s^1-image of the vertex domain A_2 to $\mathcal{D}^1$ so that $\mathrm{f}_{\mathrm{s}_1} = (\mathrm{f}_{\mathrm{s}_1^{-1}})^{\mathrm{s}_1}$ (in Fig. 9 for brevity $(\mathrm{f}_{\mathrm{s}_1^{-1}})^{\mathrm{s}_1} =: (\mathrm{s}_1^{-1})^*$), $\mathit{2} = \mathit{1}^{\mathrm{s}_1} =: \mathit{1}^*$, $\mathit{4} = \mathit{5}^*$. This domain is denoted by $\mathcal{D}^{\mathrm{s}_1}$. Along $\mathrm{f}_{\mathrm{s}_2^{-1}}$ we glue the s_2^{-1}-image of the vertex domain A_3 to $\mathcal{D}^1$, so that $\mathrm{f}_{\mathrm{s}_2^{-1}} = (\mathrm{f}_{\mathrm{s}_2})^{\mathrm{s}_2^{-1}} =: \mathrm{s}_2^*$, $\mathit{6} = \mathit{5}^*$, $\mathit{1} = \mathit{6}^*$.

This domain is denoted by $\mathcal{D}^{s_2^{-1}}$. Now the facets of $\mathcal{P}_{A_1} = \mathcal{D}^1 \cup \mathcal{D}^{s_1} \cup \mathcal{D}^{s_2^{-1}}$ are paired as indicated in Fig. 9 by special lines. So we get the generators of $\mathcal{G}_{A_1}$ as follows:

$$
(15)\qquad
\begin{array}{ll}
\text{———} & f_{s_1^{-1}} \mapsto (f_{s_1})^{s_2^{-1}} : g_1 := s_1 s_2^{-1};\\
\sim\sim\sim & (f_{s_2^{-1}})^{s_2^{-1}} \mapsto (f_{s_2})^{s_1} : g_2 := s_2^2 s_1;\\
\text{- - - -} & (f_{s_2^{-1}})^{s_1} \mapsto f_{s_2} : p_1 := s_1^{-1} s_2;\\
=\!= & (f_{s_1^{-1}})^{s_2^{-1}} \mapsto (f_{s_1})^{s_1} : p_2 := s_2 s_1^2.
\end{array}
$$

The relations for $\mathcal{G}_{A_1}$ are

$$
(16)\qquad \ominus : (p_2 p_1 g_2^{-1} g_1)^a = 1; \qquad \oplus : (g_1 g_2 p_2^{-1} p_1^{-1})^a = 1.
$$

Both relations are consequences of (14–15). We have described also the surface diagram of $\mathcal{G}_{A_1}$ in Fig. 9 in the sense of [16]. This is a graph on the non-oriented surface of genus 3 (a sphere with 3 cross-graphs) endowed with two centres of a-fold rotation. If we cut the surface along this graph, then we get the fundamental polygon of $\mathcal{G}_{A_1}$ described above. The case $a = 1$ seems to be very important. Then $\mathcal{P}$ with the above identifications is "almost a manifold" with two exceptional points, the centre of $\mathcal{P}$ and the equivalence class of the vertices.

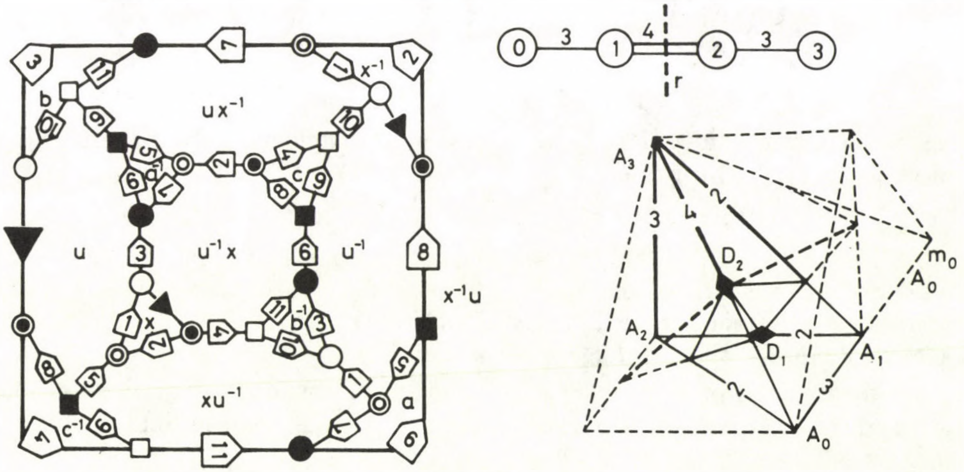


Fig. 10

3. Now we shall construct a Threlfall's manifold of constant positive curvature by means of the method applied first in [31]. In Fig. 10 we have described a truncated cube as $\mathcal{P}$ and a pairing $\mathcal{I}$ on it, so that each equivalence class of the edges has exactly 3 edges at the joint of a triangle–octagon,

octagon--octagon, octagon--triangle, respectively. All the exponents will be 1. Consequently, each equivalence class of the vertices has exactly 4 vertices starting or ending 4 edge classes. We define the generators step-by-step and get the relations by the scheme as follows.

First we assign the edge class $-\!\!\blacktriangleright\!\!-$ and the facets, joining at these edges, together with two generators:

$$\mathrm{u}: \mathrm{f}_{\mathrm{u}^{-1}} \mapsto \mathrm{f}_{\mathrm{u}}; \qquad \mathrm{x}: \mathrm{f}_{\mathrm{x}^{-1}} \mapsto \mathrm{f}_{\mathrm{x}}. \tag{17}$$

This determines the generator $\mathrm{x}^{-1}\mathrm{u}: \mathrm{f}_{\mathrm{u}^{-1}\mathrm{x}} \mapsto \mathrm{f}_{\mathrm{x}^{-1}\mathrm{u}}$ as a product. In Fig. 10 the f's are omitted. Surprisingly, our previous requirements will determine the whole construction. All the generators are expressed by u and x, and we also get the defining relations.

At the joint of f_{u} and f_{x} we choose the edge *1*. This has two images on $\mathrm{f}_{\mathrm{u}^{-1}}$ and on $\mathrm{f}_{\mathrm{x}^{-1}}$. The joining facets at the previous edge images shall be paired, so the generator $\mathrm{ux}^{-1}: \mathrm{f}_{\mathrm{xu}^{-1}} \mapsto \mathrm{f}_{\mathrm{ux}^{-1}}$ shall be introduced.

At the joint of already marked facets we define a newer edge class containing 3 edges and we get a new generator, a trivial relation, a non-trivial relation or a consequence relation by the following scheme:

$$\begin{aligned}
&-\!\!\blacktriangleright\!\!-\mathrm{x}^{-1}\mathrm{u}; \quad \mathit{1}: \mathrm{ux}^{-1}; \quad \mathit{2}: \mathrm{x}^{-1}(\mathrm{u}^{-1}\mathrm{x})(\mathrm{xu}^{-1}) = 1;\\
&\mathit{3}: \mathrm{b} := \mathrm{u}(\mathrm{x}^{-1}\mathrm{u}); \quad \mathit{4}: \mathrm{c} := (\mathrm{u}^{-1}\mathrm{x})(\mathrm{ux}^{-1});\\
&\mathit{5}: \mathrm{a} := (\mathrm{xu}^{-1})\mathrm{u}^{-1}; \quad \mathit{6}: \mathrm{a}(\mathrm{u}^{-1}\mathrm{x})\mathrm{u} \overset{5}{=} \mathrm{xu}^{-3}\mathrm{xu} = 1;\\
&\mathit{7}: \mathrm{a}(\mathrm{ux}^{-1})(\mathrm{u}^{-1}\mathrm{x}) \overset{5}{=} \mathrm{xu}^{-1}\mathrm{x}^{-1}\mathrm{u}^{-1}\mathrm{x} \overset{2}{=} 1;\\
&\mathit{8}: \mathrm{c}(\mathrm{x}^{-1}\mathrm{u})\mathrm{u} \overset{4}{=} \mathrm{u}^{-1}\mathrm{xux}^{-2}\mathrm{u}^{2} \overset{2}{=} 1;\\
&\mathit{9}: \mathrm{cu}(\mathrm{xu}^{-1}) \overset{4}{=} \mathrm{u}^{-1}\mathrm{xu}(\mathrm{x}^{-1}\mathrm{ux})\mathrm{u}^{-1} \overset{2}{=} \mathrm{u}^{-1}\mathrm{xu}(\mathrm{xu}^{-1})\mathrm{u}^{-1} \overset{6}{=} 1;\\
&\mathit{10}: \mathrm{bu}^{-1}\mathrm{xu}^{-1} \overset{3}{=} 1;\\
&\mathit{11}: \mathrm{b}(\mathrm{xu}^{-1})(\mathrm{u}^{-1}\mathrm{x}) \overset{3}{=} \mathrm{ux}^{-1}(\mathrm{uxu}^{-2})\mathrm{x} \overset{6}{=} \mathrm{ux}^{-1}(\mathrm{x}^{-1}\mathrm{u})\mathrm{x} \overset{2}{=} 1.
\end{aligned} \tag{18}$$

As we see, from the relations at 2 and 6 we can derive the others. So we have for $\mathcal{G}(\mathcal{P},\mathcal{I})$ the presentation

$$\mathcal{G} := (\mathrm{u}, \mathrm{x} - \mathrm{x}^{2}\mathrm{u}^{-1}\mathrm{x}^{-1}\mathrm{u}^{-1} = 1 = \mathrm{u}^{3}\mathrm{x}^{-1}\mathrm{u}^{-1}\mathrm{x}^{-1}). \tag{19}$$

This is Threlfall's binary octahedral group [7] of symbol $\langle 2,3,4\rangle$ and of order 48 mentioned as $\mathcal{G}_{43}$ in [30] and described there by a spherical tetrahedron. It is shown first by Threlfall that this $\mathcal{G}$, as a fundamental group. defines a *spherical space form* $\mathcal{S}^3/\mathcal{G}$.

Now we shall prove that $\mathcal{P}^{\mathcal{G}}$ has a metric realization in the spherical space $\mathcal{S}^3$, indeed. Let us consider the simplex $A_0A_1A_2A_3$ whose Schläfli–Coxeter matrix is

(20)

$$(b^{ij}) = (\langle \boldsymbol{b}^i; \boldsymbol{b}^j \rangle) = \begin{pmatrix} 1 & -\cos\frac{\pi}{3} & 0 & 0 \\ -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{4} & 0 \\ 0 & -\cos\frac{\pi}{4} & 1 & -\cos\frac{\pi}{3} \\ 0 & 0 & -\cos\frac{\pi}{3} & 1 \end{pmatrix}, \quad i,\ j = 0,1,2,3$$

of signature $\langle +,+,+,+\rangle$. Hence $A_0A_1A_2A_3$ can be realized in $\mathcal{S}^3$. Its Coxeter-diagram, described in Fig. 10, has a symmetry which induces a half-turn about the axis D_1D_2. Place a plane through D_1D_2 perpendicularly to the edge A_0A_3. This plane divides $A_0A_1A_2A_3$ into two halves. Take the part with A_3 and the side planes $(\mathcal{b}^0)=m_0$, $(\mathcal{b}^1)=m_1$, $(\mathcal{b}^2)=m_2$. The reflections in m_0, m_1, m_2 would generate the Coxeter group C_3 as the symmetry group of an octahedron $\mathcal{O}$ centred in A_3. A_2 would be a facet centre, A_1 an edge one, and A_0 a vertex of $\mathcal{O}$.

Now we reflect the half simplex around A_3 under the above group C_3, then we just get the truncated cube as an Archimedian solid $(8,8,3)$. To see this we need the fact that the half-turn r changes the planes $m_1 \leftrightarrow m_2$ and $m_0 \leftrightarrow m_3$. So around D_2 we get a group of order 16, generated by m_1 and r, thus we have an octagon in the symmetry plane of A_0A_3.

Summarizing, the group $\mathcal{N}$, generated by m_0, m_1, r, forms the orbit of A_3. And a Dirichlet–Voronoi cell of this orbit is just the truncated cube considered. In this D–V–tiling 3 domains meet at each edge where a triangle and two octagons join. At every point of the tiling 4 domains meet in 4 edges. Thus we have explained the requirements of the construction and checked the local realization of $\mathcal{P}^{\mathcal{G}}$ in $\mathcal{S}^3$. Theorem 8.1 guarantees the global realization.

We remark that $A_0A_1A_2A_3$, as a characteristic simplex of the spherical octahedron $\mathcal{O}$, allows us to define another spherical manifold with fundamental domain $\mathcal{O}$ and with fundamental group $\mathcal{G}_{33}$ (binary tetrahedral group of Threlfall) as described in [30]. $\mathcal{S}^3/\mathcal{G}_{33}$ is a twofold covering of $\mathcal{S}^3/\mathcal{G}_{43}$. In [31] we have constructed "Archimedian manifolds" of similar structure in $\mathcal{H}^3$.

The method makes it possible to construct manifolds, where all the proper points of $\mathcal{P}$ have trivial stabilizer under $\mathcal{G}$. Given $\mathcal{P}$, we can look for those manifolds which are derived from $\mathcal{P}$ by identifications. We are working on this program using computers (see e.g. [28, 32]).

In [28] one can also find non-isomorphic Euclidean space tilings $\mathcal{P}^{\mathcal{G}}$ by simplices where the corresponding groups $\mathcal{G}$ are isomorphic. Such difficulties naturally arise. Although we know that there does not exist a universal algorithm for the isomorphism problem of groups, we expect some partial results also in this field.

Added in proof. After having completed this paper, I got the possibility to study the paper of Takeshi Morokuma, A characterization of fundamental domains of discontinuous groups acting on real hyperbolic spaces, *Journal of the Faculty of Science, the University of Tokyo, Section 1A: Mathematics,* **25** (1978), 157–183. (*MR* 80d:32.027, *Zbl* 418. 57.017.) This paper intends to extend Maskit's result [20] on the 3-dimensional Poincaré theorem to dimension $n \geqq 3$. Maskit gives necessary and sufficient conditions for a polyhedron D of finite volume to be a fundamental domain for an isometry group G acting discontinuously on the hyperbolic space $\mathcal{H}^3$ (and on any 3-space of constant curvature). But he restricts himself only to cycle transformations which preserve the orientation of edges of D (condition (h)). So he excludes the involutive line reflections from among the generators of G and, in general, he excludes the orientation reversing isometries which could cause troubles at the cycle transformations (see our Algorithm 4.1 and Figure 5).

Morokuma introduces further restrictions for his discrete group Γ and for the identified polyhedron F. In Definition 1.ii) he requires that Γ *has no reflection* (involutive transformation?). In Definition 2.ii–3) he requires cycle transformation for any m-face class which fixes the starting m-face point-wise ($1 \leqq m \leqq n-2$? this is understandable for $m = n-2$, see our Section 5).

These are too strong restrictions and exclude the most interesting cases. In his main theorem Morokuma *considers the second part to be elementary and proves the first part by restricted assumptions* in Definition 2. *The second part, however, seems to be not true*, even he excludes any involutive transformations from Γ in Definition 1, or considers only orientation preserving isometries in Γ. For dimensions $n \geqq 4$ an $(n-2)$-dimensional face of F may have such stabilizer which does not fix the $(n-2)$-face pointwise (see our Section 4).

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(Received March 13, 1989)

TECHNICAL UNIVERSITY
DEPARTMENT OF GEOMETRY
BUDAPEST, MŰEGYETEM RKP. 3–9
H–1521

Acta Math. Hung.
58(1-2) (1992), 217–226.

KONSRUKTION DES REGULÄREN SIEBZEHNECKS MIT LINEAL UND STRECKENÜBERTRAGER

J. STROMMER (Budapest)

Herrn em. o. Prof. Karl Strubecker zum 85. Geburtstag gewidmet

D. Hilbert hat im Jahre 1899 in seiner berühmten Festschrift "Grundlagen der Geometrie" bewiesen, daß diejenigen regulären Vielecke, die mit Zirkel und Lineal konstruierbar sind, notwändig mit Lineal und Streckenübertrager, d. h. mit einem Instrument konstruierbar sind, welches das Abtragen von Strecken ermöglicht.* Im folgenden geben wir die Konnstruktion des regulären Siebzehnecks mit diesen Zeichenhilfsmitteln an.

Zu diesem Zweck tragen wir die Einheitsstrecke von dem Scheitel eines Winkels, der gleich $\pi/34$ ist, an einem Schenkel desselben ab (Fig. 1); dann schlagen wir um den so erhaltenen Punkt durch den Scheitel des Winkels einen Kreis, der den anderen Schenkel noch in einem Punkt schneidet; um diesen Punkt schlagen wir durch den vorher konstruierten Punkt einen Kreis, der den ersten Schenkel noch in einem Punkt schneidet usw. Wenn wir die nacheinander konstruierten Punkte miteinander verbinden, erhalten wir sechzehn gleichschenklige Dreiecke, deren Schenkel alle gleich 1 sind. In der Figur sind die Maßzahlen der einzelnen Winkel in bezug auf $\pi/34$ als Winkeleinheit angegeben. Wir bezeichnen diejenigen Basen der einzelnen Dreiecke, die auf demselben Schenkel wie die Einheitsstrecke liegen, mit $x_1, x_2, \dots, x_8$ und die Basen, die auf dem anderen Schenkel liegen, mit $y_1, y_2, \dots, y_8$. Unter den Größen x und y besteht eine Reihe von einfachen Relationen:

$$1:\frac{1}{2}y_1 = y_1:\left(1+\frac{1}{2}x_1\right),$$

oder:

$$y_1^2 = 2 + x_1;$$

ebenso:

$$1:\frac{1}{2}y_1 = y_2:\frac{1}{2}(x_1+x_2)$$

*Mit Lineal und Streckenübertrager kann man durch einen gegebenen Punkt zu einer Geraden die Parallele ziehen, auf einer Geraden eine Senkrechte errichten, und alle Ausdrücke konstruieren, welche durch Addieren, Subtrahieren, Multiplizieren, Dividieren und aus einer Summe von Streckenquadraten durch Quadratwurzelziehen hervorgehen.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	$2+x_2$	x_1+x_3	x_2+x_4	x_3+x_5	x_4+x_6	x_5+x_7	x_6+x_8	x_7-x_8	y_1+y_2	y_1+y_3	y_2+y_4	y_3+y_5	y_4+y_6	y_5+y_7	y_6+y_8	y_7
x_2	x_1+x_3	$2+x_4$	x_1+x_5	x_2+x_6	x_3+x_7	x_4+x_8	x_5-x_8	x_6-x_7	y_2+y_3	y_1+y_4	y_1+y_5	y_2+y_6	y_3+y_7	y_4+y_8	y_5	y_6-y_8
x_3	x_2+x_4	x_1+x_5	$2+x_6$	x_1+x_7	x_2+x_8	x_3-x_8	x_4-x_7	x_5-x_6	y_3+y_4	y_2+y_5	y_1+y_6	y_1+y_7	y_2+y_8	y_3	y_4-y_8	y_5-y_7
x_4	x_3+x_5	x_2+x_6	x_1+x_7	$2+x_8$	x_1-x_8	x_2-x_7	x_3-x_6	x_4-x_5	y_4+y_5	y_3+y_6	y_2+y_7	y_1+y_8	y_1	y_2-y_8	y_3-y_7	y_4-y_6
x_5	x_4+x_6	x_3+x_7	x_2+x_8	x_1-x_8	$2-x_7$	x_1-x_6	x_2-x_5	x_3-x_4	y_5+y_6	y_4+y_7	y_3+y_8	y_2	y_1-y_8	y_1-y_7	y_2-y_6	y_3-y_5
x_6	x_5+x_7	x_4+x_8	x_3-x_8	x_2-x_7	x_1-x_6	$2-x_5$	x_1-x_4	x_2-x_3	y_6+y_7	y_5+y_8	y_4	y_3-y_8	y_2-y_7	y_1-y_6	y_1-y_5	y_2-y_4
x_7	x_6+x_8	x_5-x_8	x_4-x_7	x_3-x_6	x_2-x_5	x_1-x_4	$2-x_3$	x_1-x_2	y_7+y_8	y_6	y_5-y_8	y_4-y_7	y_3-y_6	y_2-y_5	y_1-y_4	y_1-y_3
x_8	x_7-x_8	x_6-x_7	x_5-x_6	x_4-x_5	x_3-x_4	x_2-x_3	x_1-x_2	$2-x_1$	y_8	y_7-y_8	y_6-y_7	y_5-y_6	y_4-y_5	y_3-y_4	y_2-y_3	y_1-y_2
y_1	y_1+y_2	y_2+y_3	y_3+y_4	y_4+y_5	y_5+y_6	y_6+y_7	y_7+y_8	y_8	$2+x_1$	x_1+x_2	x_2+x_3	x_3+x_4	x_4+x_5	x_5+x_6	x_6+x_7	x_7+x_8
y_2	y_1+y_3	y_1+y_4	y_2+y_5	y_3+y_6	y_4+y_7	y_5+y_8	y_6	y_7-y_8	x_1+x_2	$2+x_3$	x_1+x_4	x_2+x_5	x_3+x_6	x_4+x_7	x_5+x_8	x_6-x_8
y_3	y_2+y_4	y_1+y_5	y_1+y_6	y_2+y_7	y_3+y_8	y_4	y_5-y_8	y_6-y_7	x_2+x_3	x_1+x_4	$2+x_5$	x_1+x_6	x_2+x_7	x_3+x_8	x_4-x_8	x_5-x_4
y_4	y_3+y_5	y_2+y_6	y_1+y_7	y_1+y_8	y_2	y_3-y_8	y_4-y_7	y_5-y_6	x_3+x_4	x_2+x_5	x_1+x_6	$2+x_7$	x_1+x_8	x_2-x_8	x_3-x_7	x_4-x_6
y_5	y_4+y_6	y_3+y_7	y_2+y_8	y_1	y_1-y_8	y_2-y_7	y_3-y_6	y_4-y_5	x_4+x_5	x_3+x_6	x_2+x_7	x_1+x_8	$2-x_8$	x_1-x_7	x_2-x_6	x_3-x_5
y_6	y_5+y_7	y_4+y_8	y_3	y_2-y_8	y_1-y_7	y_1-y_6	y_2-y_5	y_3-y_4	x_5+x_6	x_4+x_7	x_3+x_8	x_2-x_8	x_1-x_7	$2-x_6$	x_1-x_5	x_2-x_4
y_7	y_6+y_8	y_5	y_4-y_8	y_3-y_7	y_2-y_6	y_1-y_5	y_1-y_4	y_2-y_3	x_6+x_7	x_5+x_8	x_4-x_8	x_3-x_7	x_2-x_6	x_1-x_5	$2-x_4$	x_1-x_3
y_8	y_7	y_6-y_8	y_5-y_7	y_4-y_6	y_3-y_5	y_2-y_4	y_1-y_3	y_1-y_2	x_7+x_8	x_6-x_8	x_5-x_7	x_4-x_6	x_3-x_5	x_2-x_4	x_1-x_3	$2-x_2$

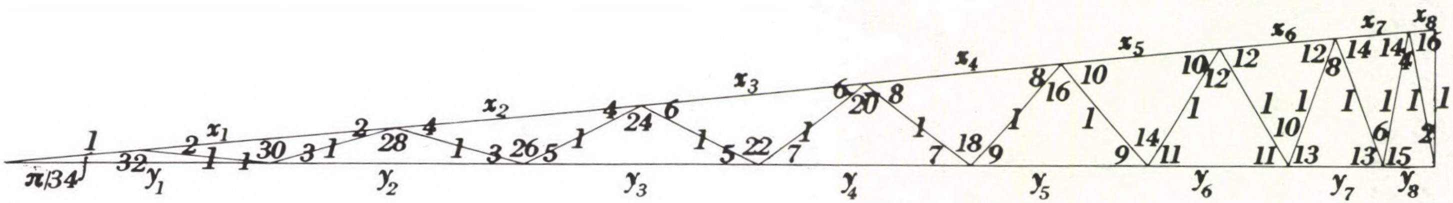

Fig. 1

$$y_1 y_2 = x_1 + x_2;$$

$$1 : \frac{1}{2} y_1 = y_3 : \frac{1}{2}(x_2 + x_3)$$

$$y_1 y_3 = x_2 + x_3;$$

$$\cdot \quad \cdot \quad \cdot$$

$$y_1 : \left(1 + \frac{1}{2} x_1\right) = x_1 : \frac{1}{2}(y_1 + y_2)$$

$$x_1^2 = y_1^2 + y_1 y_2 - 2x_1$$
$$= 2 + x_2;$$

$$y_1 : \left(1 + \frac{1}{2} x_1\right) = x_2 : \frac{1}{2}(y_2 + y_3)$$

$$x_1 x_2 = y_1 y_2 + y_1 y_3 - 2x_2$$
$$= x_1 + x_2;$$

$$\cdot \quad \cdot \quad \cdot$$

$$1 : \frac{1}{2} y_1 = x_1 : \frac{1}{2}(y_1 + y_2)$$

$$x_1 y_1 = y_1 + y_2;$$

$$\cdot \quad \cdot \quad \cdot$$

$$y_2 : \frac{1}{2}(x_1 + x_2) = x_1 : \frac{1}{2}(y_1 + y_2)$$

$$y_2^2 = x_1^2 + x_1 x_2 - y_1 y_2$$
$$= 2 + x_3;$$

$$\cdot \quad \cdot \quad \cdot$$

Die so erhaltenen Produkte von je zwei der Größen x und y stellen wir in der auf S. 218 befindlichen Multiplikationstabelle zusammen.

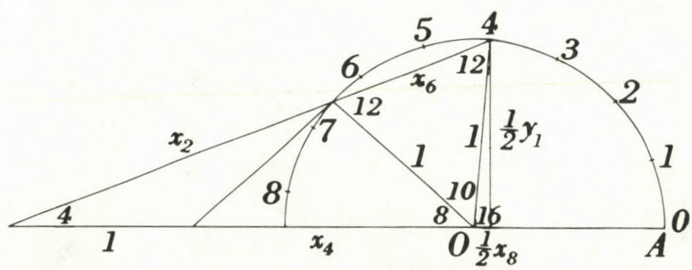

Fig. 2

Auch die Höhen der einzelnen Dreiecke können wir aus der Figur entnehmen. Z. B. die Höhenlinien der Dreiecke mit den Basen x_8 und y_1 teilen dieselbe in je zwei rechtwinklige Dreiecke mit den Katheten $\frac{1}{2}x_8$ und $\frac{1}{2}y_1$. So ist die Höhe des Dreiecks, dessen Base x_8 ist, gleich der Hälfte von y_1.

Aus den Dreiecken, deren Base x_2, x_4 und x_6 sind, sowie aus einer der beiden Hälften des Dreiecks, dessen Base x_8 ist, kann man ein rechtwinkliges Dreieck zusammenstellen (Fig. 2). Schlägt man nun um die Spitze O des Dreiecks, dessen Base x_6 ist, einen Kreis mit dem Radius 1, und teilt die Peripherie desselben von dem Punkt A angefangen, der auf der Verlängerung der größeren Kathete des rechtwinkligen Dreiecks liegt, in 17 gleiche Teile, so fällt einer der Eckpunkte dieses Dreiecks mit dem vierten Teilpunkt zusammen. Das von diesem Punkt auf OA gefällte Lot ist gleich $\frac{1}{2}y_1$ und die Entfernung des Fußpunktes desselben von O ist gleich $\frac{1}{2}x_8$. Unsere Aufgabe wird also gelöst sein, wenn es gelungen ist, die Größen x_8 und y_1 als Strecken mit Lineal und Streckenübertrager zu konstruieren, da man ja dann durch wiederholte Winkelhalbierung den Winkel $2\pi/17$ herstellen kann.

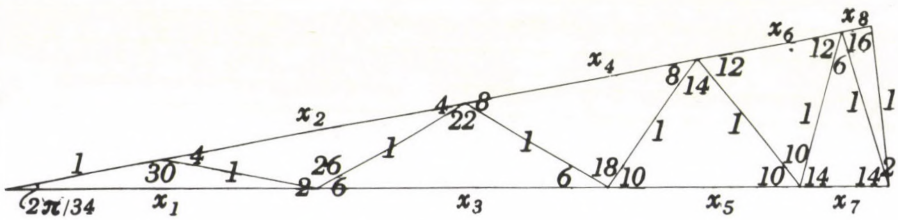

Fig. 3

Aus denjenigen Dreiecken der ersten Figur, deren Base einer der x-Strecken ist, können wir ein gleichschenkliges Dreieck zusammenstellen (Fig. 3), aus dem die Gleichung:

$$x_1+x_3+x_5+x_7-x_2-x_4-x_6-x_8=1$$

folgt.

Ferner ist:

$$\begin{aligned}(x_2+x_8+x_4-x_1)(x_3+x_5-x_6+x_7) &= x_2x_3+x_3x_8+x_3x_4-x_1x_3\\ &\quad+x_2x_5+x_5x_8+x_4x_5-x_1x_5\\ &\quad-x_2x_6-x_6x_8-x_4x_6+x_1x_6\\ &\quad+x_2x_7+x_7x_8+x_4x_7-x_1x_7\\ &= x_1+x_5-x_6+x_5+x_1+x_7-x_4-x_2\\ &\quad+x_3+x_7+x_3-x_4+x_1-x_8-x_6-x_4\\ &\quad-x_4-x_8-x_2+x_3-x_2+x_7+x_5+x_7\\ &\quad+x_5-x_8+x_1-x_2-x_6+x_3-x_6-x_8.\end{aligned}$$

Nach obigem ist also:

$$(x_2+x_8+x_4-x_1)(x_3+x_5-x_6+x_7)=4.$$

So sind die Größen

$$w_1 = x_2 + x_8 + x_4 - x_1,$$
$$w_2 = -x_3 - x_5 + x_6 - x_7$$

Wurzeln der quadratischen Gleichung:

$$w^2 + w - 4 = 0.$$

Nun is $w_2 < 0$, da ja bereits $x_3 > x_6$ ist, und so ist $w_1 > 0$, da ja $w_1 w_2 = -4$ ist; also ist:

$$w_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{17},$$
$$w_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{17}.$$

Ferner ist:

$$(x_2 + x_8)(x_4 - x_1) = x_2x_4 + x_4x_8 - x_1x_2 - x_1x_8$$
$$= x_2 + x_6 + x_4 - x_5 - x_1 - x_3 - x_7 + x_8 = -1,$$

und so sind die Größen

(1) $$u_1 = x_2 + x_8, \qquad (>0)$$
(2) $$u_2 = x_4 - x_1 \qquad (<0)$$

Wurzeln der quadratischen Gleichung:

$$u^2 - w_1 u - 1 = 0;$$

also ist:

$$u_1 = \frac{1}{2}w_1 + \frac{1}{2}\sqrt{w_1^2 + 4},$$
$$u_2 = \frac{1}{2}w_1 - \frac{1}{2}\sqrt{w_1^2 + 4}.$$

Ebenso erhält man, daß

$$(x_7 - x_6)(x_3 + x_5) = -1$$

ist, und so sind die Größen

(3) $$v_1 = x_6 - x_7, \qquad (>0)$$

(4) $$v_2 = -x_3 - x_5 \qquad (<0)$$

Wurzeln der quadratischen Gleichung:

$$v^2 - w_2 v - 1 = 0;$$

also ist:

$$v_1 = \frac{1}{2}w_2 + \frac{1}{2}\sqrt{w_2^2+4},$$
$$v_2 = \frac{1}{2}w_2 - \frac{1}{2}\sqrt{w_2^2+4}.$$

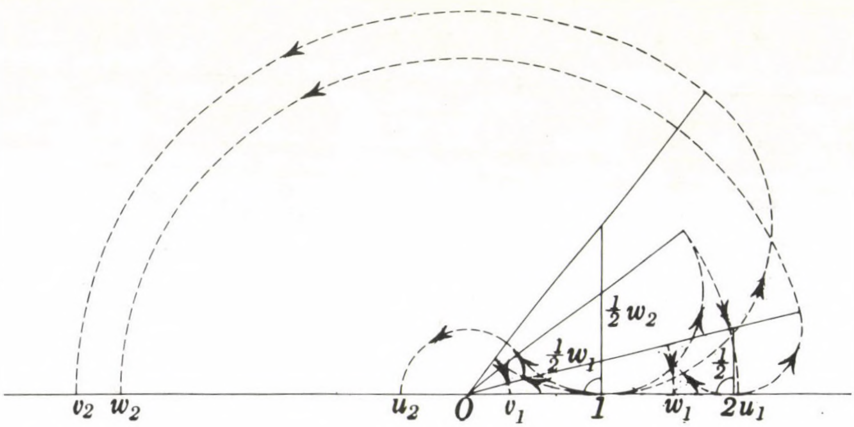

Fig. 4

Die Größen w_1, w_2, u_1, u_2, v_1, v_2 können wir als Strecken leicht konstruieren (s. Fig. 4), und so auf der Zahlengerade OA, deren Anfangs- und Einheitspunkt O, bzw. A ist, auch die Punkte von der Abszisse w_1, w_2, u_1, u_2, v_1, v_2.

Aus den Gleichungen (1), (2), (3), (4) können wir weitere Gleichungen für die Größen x ableiten, indem wir dieselben der Reiche nach mit $x_1, x_2, \dots, x_8$ multiplizieren und die Produkte von je zwei Größen x nach der Multiplikationstabelle in eine Summe von denselben umwandeln. So erhalten wir z. B. folgende Gleichungen:

$$x_4 - x_1 = u_2,$$
$$x_4^2 - x_1 x_4 = u_2 x_4,$$
$$2 + x_8 - x_3 - x_5 = u_2 x_4,$$

(5) $$u_2 x_4 - x_8 = 2 + v_2;$$

$$
\begin{aligned}
-x_3-x_5&=v_2,\\
-x_1x_3-x_1x_5&=v_2x_1,\\
-x_2-2x_4-x_6&=v_2x_1,\\
v_2x_1+x_2+2x_4+x_6&=0;
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
x_3-x_5&=v_2,\\
-x_3x_8-x_5x_8&=v_2x_8,\\
x_6-x_5-x_3+x_4&=v_2x_8,\\
x_4+x_6-v_2x_8&=-v_2;
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
x_6-x_7&=v_1,\\
x_6^2-x_6x_7&=v_1x_6,\\
2-x_5-x_1+x_4&=v_1x_6,\\
x_5+v_1x_6&=2+u_2.
\end{aligned}
\tag{8}
$$

Die Gleichungen (1), (2), (5), (6) und (7) bilden ein unabhängiges Gleichungssystem für die Größen x_1, x_2, x_4, x_6 und x_8, aus dem

$$x_8=\frac{3+v_2-u_2v_2-u_2^2v_2}{v_2-u_2v_2-2}$$

folgt.

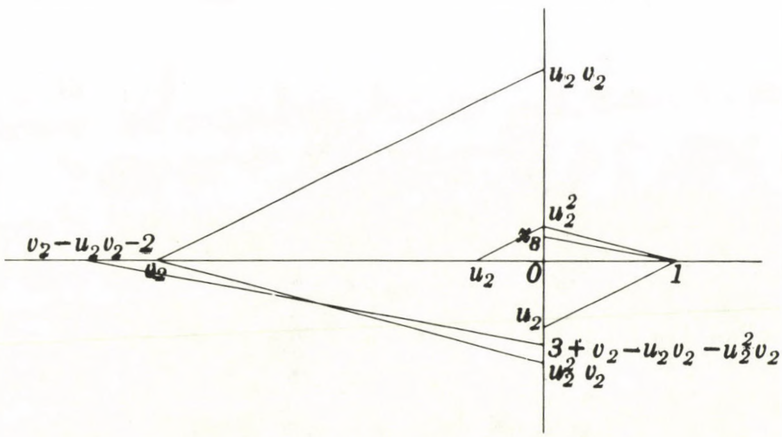

Fig. 5

Mittels dieser Formel kann man die Größe x_8 leicht konstruieren (s. Fig. 5) und dann aus den Gleichungen (1) bis (8) auch die Größen $x_1, x_2, \ldots, x_7$ als Strecken geometrisch bestimmen (s. Fig. 6).

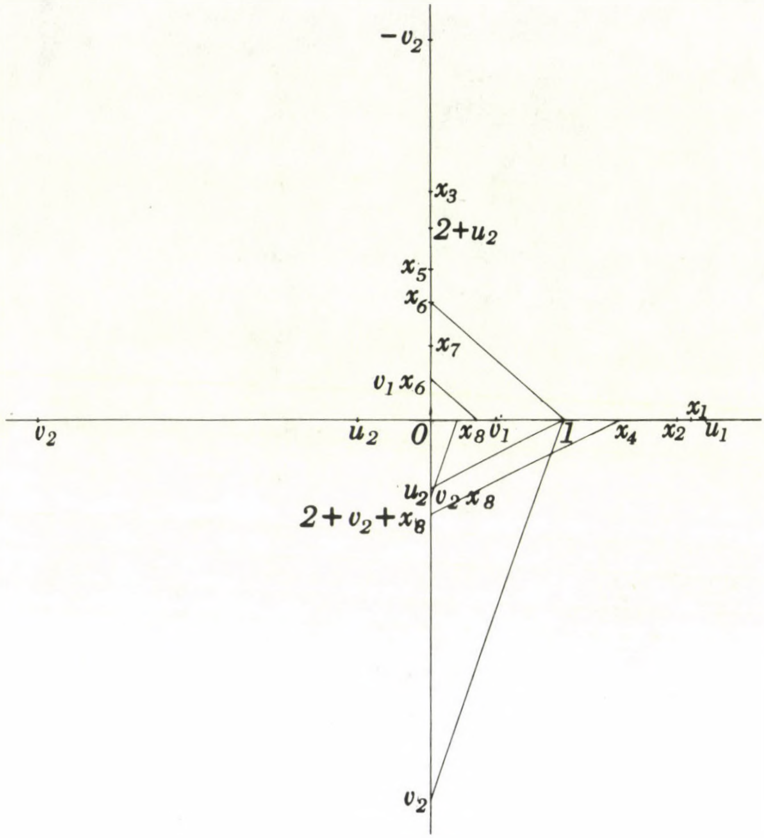

Fig. 6

Um die Größe y_1 zu bestimmen, betrachten wir folgende Differenzen:

$$y_2^2 - x_8^2 = x_1 + x_3 = x_1 x_2,$$
$$y_5^2 - x_5^2 = x_7 - x_8 = x_1 x_8,$$
$$y_8^2 - x_3^2 = -x_2 - x_6 = -x_2 x_4,$$
$$y_7^2 - x_6^2 = -x_4 + x_5 = -x_4 x_8.$$

Aus denselben folgt, da

$$x_1 x_2 + x_1 x_8 - x_2 x_4 - x_4 x_8 =$$
$$= (x_2 + x_8)(x_1 - x_4) = -u_1 u_2 = 1$$

ist, daß

$$\sqrt{y_2^2 + y_5^2 + y_7^2 + y_8^2} = \sqrt{1 + x_3^2 + x_5^2 + x_6^2 + x_8^2}$$

ist.

Aus der Gleichung

$$(y_1 y_2) y_2 = y_1 (y_2 y_2)$$

folgt weiter:

$$(x_1 + x_2) y_2 = y_1 (2 + x_3),$$

oder:

$$y_2 = \frac{2 + x_3}{x_1 + x_2} y_1.$$

Ebenso kann man y_5, y_7 und y_8 durch y_1 ausdrücken:

$$y_5 = \frac{x_3 + x_6}{x_1 + x_2} y_1,$$
$$y_7 = \frac{x_5 + x_8}{x_1 + x_2} y_1,$$
$$y_8 = \frac{x_6 - x_8}{x_1 + x_2} y_1,$$

und so erhält man:

$$y_1 = \frac{(x_1 + x_2)\sqrt{1 + x_3^2 + x_5^2 + x_6^2 + x_8^2}}{\sqrt{(2 + x_3)^2 + (x_3 + x_6)^2 + (x_5 + x_8)^2 + (x_6 - x_8)^2}}.$$

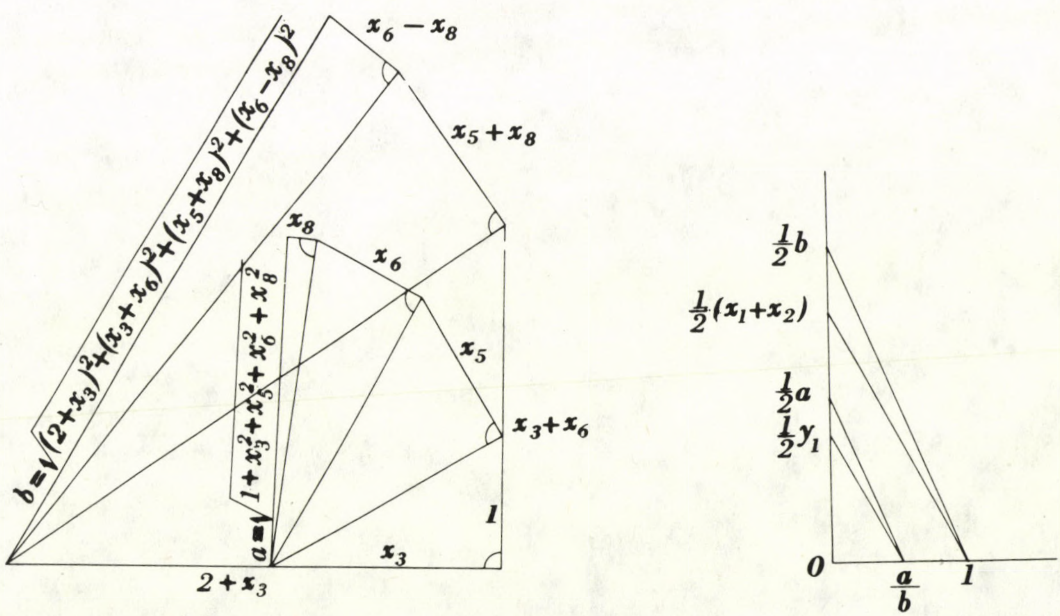

Fig. 7

Nach dieser Formel kann man die Strecke y_1 leicht konstruieren (s. Fig. 7), und damit ist die Konstruktion des regelmäßigen Siebzehnecks mit Lineal und Streckenübertrager ausgeführt.

(Eingegangen am 11. August 1989.)

TECHNISCHE UNIVERSITÄT
LEHRSTUHL FÜR GEOMETRIE
BUDAPEST, STOCZEK UTCA 2–4
H–1111
UNGARN

Acta Math. Hung.
59(1-2) (1992), 227–244.

ON PSEUDOMANIFOLDS WITH BOUNDARY. I

M. BOGNÁR (Budapest)

The aim of this paper is to show an important property of the n-dimensional nonorientable (nonclosed) pseudomanifolds with boundary, namely that these figures are absolutely linked (absolut verschlungen). This property is mentioned without proof in my paper [3] (see [3] I:2.5a) and I:2.7a)). An immediate corollary of this property is the following

THEOREM. *The topological product of a nondiscrete space and an n-dimensional nonorientable pseudomanifold with boundary cannot be embedded in R^{n+1} (see* [3] I:3.2*); e.g. the topological product of the Möbius band and a nondiscrete space cannot be embedded in R^3.*

First we deal with figures, called (n,p)-cells where p is a prime (see [5]). In a certain sense these figures are generalizations of the nonclosed n-dimensional pseudomanifolds with boundary.

1. (n,p)-cells

1.1. Let p be a prime number and Z_p the cyclic group mod p. Let H be the Čech homology theory defined on the category of all compact pairs and all continuous maps of such pairs over the coefficient group Z_p. The group Z_p can be considered as a compact commutative topological group. Hence the homology sequence of each compact pair is exact (see [12] p. 248). Let n be a positive integer.

1.2. DEFINITION. The compact pair (X,A) (i.e. X is a compact T_2-space and A is a closed subspace of X) is called an *(n,p)-cell* if the following conditions are satisfied:

(a) $X\setminus A$ is a nonvoid connected locally connected space with countable base,

(b) $H_n(X,A)\approx Z_p$ ($\approx$ means: isomorphic to),

(c) for the inclusion $i:(X,\emptyset)\subset(X,A)$ the induced homomorphism $i_*: H_n(X)\to H_n(X,A)$ is trivial, i.e. $i_*(H_n(X))=0$.

(d) in every domain i.e. in every connected open nonempty subset V of $X\setminus A$ there exists a nonempty open subset $U\subset V$ such that for the inclusion $j:(X,A)\subset(X,X\setminus U)$ the induced homomorphism $j_*:H_n(X,A)\to$ $\to H_n(X,X\setminus U)$ is a monomorphism.

1.3. For each (n,p)-cell (X,A) we clearly have $A \neq \emptyset$.

1.4. Let (X,A) be an (n,p)-cell and let us consider the segment

$$\tilde{H}_{n-1}(A) \xleftarrow{\partial} H_n(X,A) \xleftarrow{i_*} H_n(X)$$

of the reduced homology sequence of (X,A). Here in the case $n=1$, $\tilde{H}_{n-1}(A)$ is the reduced 0-dimensional homology group of A and in the case $n>1$ $\tilde{H}_{n-1}(A)=H_{n-1}(A)$. As a consequence of (c) ∂ is a monomorphism and thus for $0 \neq u \in H_n(X,A)$ we have $0 \neq \partial u \in \tilde{H}_{n-1}(A)$. ∂u is then called the *algebraic boundary* of (X,A) and we denote it by A_*. If $p=2$ then A_* is uniquely defined. On the other hand in case $p \neq 2$ for any two algebraic boundaries A_{*_1} and A_{*_2} there clearly exists an integer m where $0<m<p$ such that $mA_{*_1}=A_{*_2}$.

1.5. Let (X,A) be an (n,p)-cell and A_* an algebraic boundary of (X,A). Let $\partial : H_n(X,A) \to \tilde{H}_{n-1}(A)$ be the same homomorphism as in 1.4. Then for the inclusion $j : A \subset X$ we clearly have $j_*\partial = 0$ and thus $j_*(A_*)=0$.

1.6. Let Y be a topological space. A continuous map $f:[a,b] \to Y$ (where $[a,b]$ is a closed interval in the space of the real numbers $(a \leqq b)$) is said to be a *continuous line in* Y. $f(a)$ is the *initial* and $f(b)$ is the *closing point* of f. The *body* of f (it is indicated by $\tilde{f}$) is the set $f[a,b]$ endowed with the subspace topology of Y. $\tilde{f}$ is clearly a compact space.

For $M \subset Y$, f *is lying in* M if $f[a,b] \subset M$.

f is a *Jordan line* and its body is a *Jordan curve* if $a \neq b$ and if the following condition holds: $f(x)=f(y)$ if and only if $x=y$ or $x=a$, $y=b$ or $x=b$, $y=a$.

The line $f:[a,b] \to Y$ is said to be *degenerate* if $a=b$. It is said to be *closed* if $f(a)=f(b)$.

Now let $f:[a,b] \to Y$ be a closed line. If $a=b$ then let $f^*=0 \in H_1(\tilde{f})$. Since $\tilde{f}$ is a singleton it follows that f^* is the only element of $H_1(\tilde{f})$. Suppose now that $a \neq b$. Then $H_1([a,b],\{a,b\}) \approx Z_p$. Let v be a nonzero element of $H_1([a,b],\{a,b\})$. Let $B=\{f(a)=f(b)\}$ and let $f_* : H_1([a,b],\{a,b\}) \to H_1(\tilde{f},B)$ be the homomorphism induced by the mapping $f:([a,b],\{a,b\}) \to \to (\tilde{f},B)$. B is a singleton, hence it is homologically trivial and therefore by the exactness of the sequence

$$\tilde{H}_0(B) \longleftarrow H_1(\tilde{f},B) \xleftarrow{j_*} H_1(\tilde{f}) \longleftarrow H_1(B)$$

j_* is an isomorphism, where $j:(\tilde{f},\emptyset) \subset (\tilde{f},B)$ is the inclusion map. Let

$$f^* = j_*^{-1} f_*(v).$$

If $p=2$ then f^* is uniquely defined. In case $p \neq 2$ for any two f^{*_1} and f^{*_2} there clearly exists an integer m, $0<m<p$, such that $mf^{*_1}=f^{*_2}$.

1.7. A mapping $\mathfrak{v}: A\times B\to C$, where A, B and C are abelian groups, is said to be a *bihomomorphism* if the following condition is satisfied:

$$\mathfrak{v}(a+a',b+b')=\mathfrak{v}(a,b)+\mathfrak{v}(a',b)+\mathfrak{v}(a,b')+\mathfrak{v}(a',b').$$

We say that a bihomomorphism $\mathfrak{v}: A\times B\to C$ is *trivial* if $\mathfrak{v}(A\times B)=0$.

1.8. A *linking theory* $\mathfrak{V}=\mathfrak{V}_{p,n-1,1}$ of compacts in R^{n+1} is a mapping which makes correspond to each ordered pair (M,M') of disjoint compact subspaces of R^{n+1} a bihomomorphism

$$\mathfrak{v}_{M,M'}:\tilde{H}_{n-1}(M)\times H_1(M')\longrightarrow Z_p$$

such that for any compact subspaces M,M',N,N' of R^{n+1} satisfying $M\subset N$, $M'\subset N'$ and of course $N\cap N'=\emptyset$ the condition

$$\mathfrak{v}_{M,M'}(u,u')=\mathfrak{v}_{N,N'}\left(i_*(u),i'_*(u')\right)$$

is satisfied for every $u\in\tilde{H}_{n-1}(M)$ and $u'\in H_1(M')$ where $i: M\subset N$ and $i': M'\subset N'$ are inclusion maps.

Notice as a direct consequence of this definition that if $\mathfrak{V}=\mathfrak{V}_{p,n-1,1}$ is a linking theory of compacts in R^{n+1} and M, N, M' are compacts in R^{n+1} such that $M\subset N\subset(R^{n+1}\setminus M')$ then

$$\mathfrak{v}_{M,M'}(u,u')=\mathfrak{v}_{N,M'}(i_*(u),u')$$

holds for every $u\in\tilde{H}_{n-1}(M)$ and $u'\in H_1(M')$ where $i: M\subset N$ is the inclusion map.

Likewise holds the relation

$$\mathfrak{v}_{M,M'}(u,u')=\mathfrak{v}_{M,N'}(u,i'_*(u'))$$

for every $u\in\tilde{H}_{n-1}(M)$ and $u'\in H_1(M')$ where $i': M'\subset N'$ is the inclusion map and $N'\subset(R^{n+1}\setminus M)$.

We shall say that the linking theory $\mathfrak{V}$ is *degenerate* if for all nonintersecting compact subspaces M,M' of R^{n+1}, $\mathfrak{v}_{M,M'}$ is a trivial bihomomorphism. According to [11] 8.9 there exists a nondegenerate theory of linking of the required type.

1.9. Let $\mathfrak{V}^1=\mathfrak{V}^1_{p,n-1,1}$ and $\mathfrak{V}^2=\mathfrak{V}^2_{p,n-1,1}$ be nondegenerate theories of linking in R^{n+1}. Then there is an integer m where $0<m<p$ such that

$$\mathfrak{v}^2_{M,M'}(u,u')=m\mathfrak{v}^1_{M,M'}(u,u')$$

holds for each ordered pair (M,M') of disjoint compact subspaces of R^{n+1} and for every $u\in\tilde{H}_{n-1}(M)$ and $u'\in H_1(M')$.

Indeed, let m' be an arbitrary integer. For any two disjoint compact subspaces M, M' of R^{n+1} and for $u \in \tilde{H}_{n-1}(M)$ and $u' \in H_1(M')$ let

$$\mathfrak{v}^{3,m'}_{M,M'}(u,u') = \mathfrak{v}^2_{M,M'}(u,u') - m'\mathfrak{v}^1_{M,M'}(u,u').$$

The map $\mathfrak{v}^{3,m'}_{M,M'} : \tilde{H}_{n-1}(M) \times H_1(M') \to Z_p$ thus obtained is clearly a bihomomorphism and the mapping which makes correspond to each ordered pair (M, M') of disjoint compact subspaces of R^{n+1} the bihomomorphism $\mathfrak{v}^{3,m'}_{M,M'}$ is clearly a linking theory $\mathfrak{V}^{3,m'} = \mathfrak{V}^{3,m'}_{p,n-1,1}$ in R^{n+1}.

Now let S and S' be spheres in R^{n+1} of dimensions $n-1$ and 1 respectively satisfying the following conditions:

(a) The center of S belongs to S' and the center of S' belongs to S.

(b) The planes R and R' supporting S and S' intersect in a line R^1.

(c) R and R' are perpendicular in the natural sense that any vector a in R and a' in R' which are perpendicular to the line R^1 are mutually perpendicular.

Such spheres S and S' clearly exist.

Now since for $i = 1, 2, \mathfrak{V}^i$ is nondegenerate, it follows that $\mathfrak{v}^1_{S,S'}$, and $\mathfrak{v}^2_{S,S'}$ are nontrivial bihomomorphisms (see [4] Corollary 2). Select $u \in \tilde{H}_{n-1}(S)$ and $u' \in H_1(S')$ such that $\mathfrak{v}^1_{S,S'}(u,u') \neq 0$. We then clearly have $u \neq 0$ and $u' \neq 0$. Since for $i = 1, 2$ we have $\mathfrak{v}^i_{S,S'}(u,u') \in Z_p$ it follows that there exists an integer m where $0 \leqq m < p$ such that

$$\mathfrak{v}^{3,m}_{S,S'}(u,u') = \mathfrak{v}^2_{S,S'}(u,u') - m\mathfrak{v}^1_{S,S'}(u,u') = 0. \tag{1}$$

However $\tilde{H}_{n-1}(S)$ and $H_1(S')$ are isomorphic to Z_p and thus (1) implies that $\mathfrak{v}^{3,m}_{S,S'}$ is a trivial bihomomorphism, consequently according to [4], Corollary 2, $\mathfrak{V}^{3,m}$ is degenerate.

Hence for every ordered pair (M, M') of disjoint compact subspaces of R^{n+1} and for every $u \in \tilde{H}_{n-1}(M)$ and $u' \in H_1(M')$ we have

$$\mathfrak{v}^{3,m}_{M,M'}(u,u') = \mathfrak{v}^2_{M,M'}(u,u') - m\mathfrak{v}^1_{M.M'}(u,u') = 0$$

and thus

$$\mathfrak{v}^2_{M,M'}(u,u') = m\mathfrak{v}^1_{M,M'}(u,u').$$

If m were 0 then $\mathfrak{V}^{3,m}$ would be equal to $\mathfrak{V}^2$, but this is impossible since $\mathfrak{V}^2$ is nondegenerate. Hence $0 < m < p$ as required.

Observe that in case $p = 2$ we have $\mathfrak{V}^1 = \mathfrak{V}^2$.

1.10. Let (X,A) be an (n,p)-cell in R^{n+1} i.e. (X,A) is an (n,p)-cell where X is a subspace of R^{n+1}. Let f be a closed line lying in $R^{n+1}\setminus A$ and let $\mathfrak{V}=\mathfrak{V}_{p,n-1,1}$ be a nondegenerate linking theory of compacts in R^{n+1}.

DEFINITION. We say that f is *linked to the (n,p)-cell (X,A)* if $\mathfrak{v}_{A,\tilde{f}}(A_*,f^*)\neq 0$.

Taking 1.9, 1.6 and 1.4 into account we can state that the fact that f is linked to (X,A) or not does not depend on the special choice of A_*, f^* and $\mathfrak{V}$.

1.11. DEFINITION. Let (X,A) be an (n,p)-cell in R^{n+1}. We say that (X,A) is a *linked cell* if there exists a closed line lying in $X\setminus A$ and linked to (X,A). (X,A) is a *nonlinked cell* if it fails to be a linked cell.

1.12. DEFINITION. The (n,p)-cell (Y,B) is said to be an *absolutely linked cell*, if for each topological embedding $\varphi:Y\to R^{n+1}$ the (n,p)-cell $(\varphi(X),\varphi(A))$ is linked.

2. Properties of (n,p)-cells in R^{n+1}

The main target of this section is the treatment of some properties of (n,p)-cells in R^{n+1}. We start with two elementary facts about nonlinked lines.

Let p, H and n be the same as in 1.1. Let $\mathfrak{V}=\mathfrak{V}_{p,n-1,1}$ be a nondegenerate linking theory in R^{n+1}. Let (X,A) be an (n,p)-cell in R^{n+1} and let A_* be the algebraic boundary of (X,A) (see 1.4).

2.1. REMARK. Let Q be a compact subspace of $R^{n+1}\setminus X$ and let $u'\in\in H_1(Q)$. Then $\mathfrak{v}_{A,Q}(A_*,u')=0$.

Indeed, let $j:A\subset X$ be the inclusion map. Then by 1.5 we have $\mathfrak{v}_{A,Q}(A_*,u')=\mathfrak{v}_{X,Q}(j_*(A_*),u')=0$ (cf. also 1.8).

2.2. COROLLARY. *Let f be a closed line lying in $R^{n+1}\setminus X$. Then $f^*\in\in H_1(\tilde{f})$ where $\tilde{f}$ is a compact subspace of $R^{n+1}\setminus X$ and thus by* 2.1 *we have $\mathfrak{v}_{A,\tilde{f}}(A_*,f^*)=0$. Hence f fails to be linked to (X,A).*

2.3. Let G be an open spherical ball in R^{n+1} disjoint to A. Then no closed line lying in G is linked to (X,A).

Indeed, let f be closed line lying in G. Then there is a closed ball M containing $\tilde{f}$ and contained in G. Let $j:\tilde{f}\subset M$ be the inclusion map. Since M is homologically trivial and thus $H_1(M)=0$ we have $j_*(f^*)=0$. Consequently

$$\mathfrak{v}_{A,\tilde{f}}(A_*,f^*)=\mathfrak{v}_{A,M}(A_*,j_*(f^*))=0.$$

f is nonlinked to (X,A) as required.

Now we are going to recall the fundamental notions about continuous paths (see [7]).

2.4. Let Y be a topological space. A *continuous path* K of Y is a class of equivalence of continuous lines $f:[a,b]\to Y$ (see 1.6) where the continuous lines $f:[a,b]\to Y$ and $g:[a',b']\to Y$ are said to be *equivalent* if there exists a strictly monotonous increasing epimorphic function $s:[a,b]\to[a',b']$ for which $g\circ s=f$.

An element $f:[a,b]\to Y$ of the equivalence class K is said to be a *representative* of the path K.

All the representatives of a path K have the same initial point, the same closing point and the same body and we call them the *initial point*, the *closing point* and the *body* of K respectively. They are indicated with $\Delta(K)$, $\nabla(K)$ and $\tilde{K}$ respectively. We shall use the symbol $K:q\to z$ to express the relations $\nabla(K)=q$ and $\Delta(K)=z$.

For $M\subset Y$, K *is lying in* M if $\tilde{K}\subset M$. K is a *closed path* if $\Delta(K)=$ $=\nabla(K)$. In this case we say that the point $\Delta(K)=\nabla(K)$ is the *base point* of K.

Observe that if at least one of the representatives of K is a Jordan line (degenerate line) then every representative is a Jordan line (degenerate line). In this case we say that K is a *Jordan path* (*degenerate path*).

Now let $f_1:[c,d]\to Y$ be a representative of the continuous path $K_1:q\to$ $\to u$ and $f_2:[b,c]\to Y$ a representative of the path $K_2:u\to z$ of Y. Then $K_2K_1:q\to z$ is defined as the equivalence class of the line $g:[b,d]\to Y$ where $g|_{[b,c]}=f_2$ and $g|_{[c,d]}=f_1$.

This multiplication is clearly associative. Moreover if K_1 is degenerate then $K_2K_1=K_2$ and if K_2 is degenerate then $K_2K_1=K_1$.

To each path $K:q\to z$ of Y we can assign a path $K^\bullet:z\to q$ of Y such that if $f:[b,c]\to Y$ is a representative of K and $h:[b,c]\to[b,c]$ is defined by the formula $h(x)=b+c-x$ then $f\circ h$ is a representative of $K^\bullet$.

The products $KK^\bullet$ and $K^\bullet K$ are clearly defined and they are closed paths of Y.

It is to be noted that $(K^\bullet)^\bullet=K$. Moreover $K^\bullet=K$ whenever K is a degenerate path.

Let v be a simple arc in Y with the endpoints q_1 and q_2. Then there exists a unique continuous path $K:q_1\to q_2$ with $\tilde{K}=v$ and such that the representatives of K are injective maps. We denote this path by $[q_2vq_1]$. Paths of this type are called *proper simple paths.* We clearly have $[q_2vq_1]^\bullet=$ $=[q_1vq_2]$.

We shall call a singleton of Y a *degenerate arc* of Y. If $q_1=q_2\in Y$ then by the degenerate arc of Y with the endpoints q_1 and q_2 we understand the singleton $\{q_1\}=\{q_2\}$.

To each degenerate arc $v=\{q\}$ of Y there belongs a unique degenerate continuous path K with $\tilde{K}=v$. Then we shall also write $[qvq]$ instead of $K:q\to q$ and we have $[qvq]^\bullet=[qvq]$. Continuous paths of the form $[qvq]$ of Y are said to be *degenerate simple paths* of Y.

By an *arc* of Y we mean a simple arc or a degenerate arc of Y. By a *simple path* of Y we mean a proper or a degenerate simple path of Y.

2.5. We recall the existence of an important function described in [7], Section 11.

Let Y be a T_2-space. Then there exists a function which makes correspond to each continuous closed path K of Y an element K_* of $H_1(\tilde{K})$ satisfying the following conditions:

(a) For each continuous path K_1 of Y (K_1 need not be closed) $(K_1K_1^\bullet)_* = 0$ (see [7], Section 14).

(b) If K_1 and K_2 are continuous paths of Y such that both of the products K_1K_2 and K_2K_1 are closed paths then $(K_1K_2)_* = (K_2K_1)_*$ (see [7], Section 15).

(c) If K_1 and K_2 are continuous closed paths of Y with the same base point and $K = K_1K_2$ then

$$i_{1*}(K_{1*}) + i_{2*}(K_{2*}) = K_*$$

where for $j = 1, 2$ $i_{j*} : H_1(\tilde{K}_j) \to H_1(\tilde{K})$ is the homomorphism induced by the inclusion $i_j : \tilde{K}_j \subset \tilde{K}$ (see [7], Section 16).

(d) If K is a Jordan path of Y then $K_* \neq 0$ (see [7], Section 13).

(e) If $f : [a, b] \to Y$ is a representative of the closed continuous path K then there is an integer m where $0 < m < p$ such that $K_* = mf^*$ (see 1.6 and [7], Sections 11, 7 and 3).

2.6. Let K be a closed path in $R^{n+1} \setminus A$. K is said to be *linked* to the (n,p)-cell (X, A) if $\mathfrak{v}_{A,\tilde{K}}(A_*, K_*) \neq 0$.

Taking 1.10 and 2.5(e) into account K is clearly linked to (X, A) if their representatives are linked to (X, A). If K fails to be linked to (X, A) then no representative of K is linked to (X, A).

If K fails to be linked to (X, A) then we say that K is *nonlinked* to the (n,p)-cell (X, A).

If K is a closed continuous path lying in $R^{n+1} \setminus X$ or in an open ball contained in $R^{n+1} \setminus A$ then K is nonlinked to (X, A) (cf. 2.2 and 2.3).

2.7. Let G be a domain (open nonempty connected set) in $R^{n+1} \setminus A$ (in $X \setminus A$ respectively). We say that G is an *e-regular domain* (*i-regular domain* respectively) of (X, A) if no continuous closed path of G is linked to (X, A).

According to 2.2, 2.3 and 2.6, G is obviously e-regular if it is an open ball or it is lying in $R^{n+1} \setminus X$. The domain G of $X \setminus A$ is clearly i-regular if it is contained in an open ball lying in $R^{n+1} \setminus A$.

2.8. Let K be a continuous path in $R^{n+1} \setminus A$. Then by 2.5(a), $KK^\bullet$ is obviously a closed path nonlinked to (X, A).

2.9. Let K_1 and K_2 be continuous paths in $R^{n+1} \setminus A$ such that both of the products K_1K_2 and K_2K_1 exist. Suppose that K_1K_2 is linked to (X, A). Then by 2.5(b) the closed path K_2K_1 is linked to (X, A) as well.

2.10. Let K_1 and K_2 be continuous closed paths of $R^{n+1} \setminus A$ with the same base point. For $j=1,2$ let i_{j*} be the same as in 2.5(c). Then

$$\mathfrak{v}_{A,\widetilde{K_1K_2}}(A_*, (K_1K_2)_*) = \mathfrak{v}_{A,\widetilde{K_1K_2}}(A_*, i_{1*}(K_{1*}) + i_{2*}(K_{2*})) =$$
$$= \mathfrak{v}_{A,\widetilde{K_1K_2}}(A_*, i_{1*}(K_{1*})) + \mathfrak{v}_{A,\widetilde{K_1K_2}}(A_*, i_{2*}(K_{2*})) =$$
$$= \mathfrak{v}_{A,\widetilde{K_1}}(A_*, K_{1*}) + \mathfrak{v}_{A,\widetilde{K_2}}(A_*, K_{2*}).$$

Hence if K_1K_2 is linked to (X,A) then at least one of the K_j-s is linked to (X,A). On the other hand if exactly one of the K_j-s is linked to (X,A) then K_1K_2 itself is linked to (X,A).

2.11. Let q_1', q_2', q_3', q_1'', q_2'' and q_3'' be points of $R^{n+1} \setminus A$. Let

$$K_1' : q_1' \to q_2', \quad K_2' : q_2' \to q_3', \quad K_1'' : q_2'' \to q_1'',$$
$$K_2'' : q_3'' \to q_2'', \quad K_1 : q_1' \to q_1'', \quad K_2 : q_2' \to q_2'' \quad \text{and} \quad K_3 : q_3' \to q_3''$$

be continuous paths in $R^{n+1} \setminus A$. Let $\overline{K}_1 = K_1^\bullet K_1'' K_2 K_1'$, $\overline{K}_2 = K_2^\bullet K_2'' K_3 K_2'$ and $K_4 = K_1^\bullet K_1'' K_2'' K_3 K_2' K_1'$. Suppose that the closed paths $\overline{K}_1$ and $\overline{K}_2$ are nonlinked to (X,A). Then K_4 is nonlinked to (X,A) as well.

Indeed by the assumption and by 2.9, 2.10, 2.9, 2.8, 2.10 and 2.9 again

$$K_1' K_1^\bullet K_1'' K_2, \quad K_1' K_1^\bullet K_1'' (K_2 K_2^\bullet) K_2'' K_3 K_2', \quad K_2'' K_3 K_2' K_1' K_1^\bullet K_1'' (K_2 K_2^\bullet),$$
$$K_2'' K_3 K_2' K_1' K_1^\bullet K_1'' \quad \text{and} \quad K_4 = K_1^\bullet K_1'' K_2'' K_3 K_2' K_1'$$

are all nonlinked closed paths of (X,A).

2.12. Let q' and q'' be points of $R^{n+1} \setminus A$. Let K' and K'' be closed paths of $R^{n+1} \setminus A$ with the base points q' and q'' respectively. Let $K : q' \to q''$ be a continuous path of $R^{n+1} \setminus A$. Suppose that $K^\bullet K'' K K'$ and K' are nonlinked closed paths of (X,A). Then K'' is a nonlinked closed path of (X,A) as well.

Indeed by assumption and by 2.10, 2.9, 2.8 and 2.10 again $K^\bullet K'' K$, $K''(KK^\bullet)$ and K'' are nonlinked closed paths of (X,A).

2.13. Let $q_1', \dots, q_r', q_1'', \dots, q_r''$ be points of $R^{n+1} \setminus A$. For $i = 1, \dots, r-$ -1 let $K_i' : q_i' \to q_{i+1}'$, $K_i'' : q_{i+1}'' \to q_i''$ and $K_i : q_i' \to q_i''$ be continuous paths of $R^{n+1} \setminus A$. Let $K_r' : q_r' \to q_1'$, $K_r'' : q_1'' \to q_r''$ and $K_r : q_r' \to q_r''$ be continuous paths in $R^{n+1} \setminus A$. Suppose that for $i = 1, \dots, r-1$ the closed path $\overline{K}_i =$ $= K_i^\bullet K_i'' K_{i+1} K_i'$ is nonlinked to (X,A). Moreover that $\overline{K}_r = K_r^\bullet K_r'' K_1 K_r'$ and $K' = K_r' \dots K_2' K_1'$ are nonlinked closed paths of (X,A). Let $K'' =$ $= K_1'' K_2'' \dots K_r''$. Then by 2.11 and 2.12 the continuous closed paths $K_1^\bullet K'' K_1 K'$ and K'' are nonlinked to (X,A).

Now we can state the following theorem.

2.14. THEOREM. *Let G' be an i-regular domain of* (X, A) (see 2.7). *Then there exists an e-regular domain G of (X, A) such that $G \cap X = G'$.*

PROOF. Denoting the spherical neighborhood of $q \in R^{n+1}$ with the radius ϱ by $S(q, \varrho)$ select for each $q \in G'$ a $\varrho'_q > 0$ such that $S(q, \varrho'_q) \cap A = \emptyset$.

For $q \in G'$ let U_q be a domain in $G' \cap S(q, \varrho'_q)$ such that $q \in U_q$. Since G' is open in $X \setminus A$ and $X \setminus A$ is locally connected (see 1.2(a)) there exists such a domain $U_q \cdot U_q$ is a locally compact locally connected and connected subspace of R^{n+1} consequently it is arcwise connected (see [14], p. 27). Select $\varrho_q > 0$ such that

$$\varrho_q < \varrho'_q \tag{2}$$

and

$$S(q, \varrho_q) \cap X \subset U_q. \tag{3}$$

$S(q, \varrho'_q)$ and $S(q, \frac{1}{2}\varrho_q)$ are clearly e-regular domains of (X, A). Let

$$G = \bigcup_{q \in G'} S(q, \frac{1}{2}\varrho_q).$$

Then G is clearly a domain in R^{n+1} and $G \cap X = G'$.

We show that G is an e-regular domain of (X, A) i.e. no closed path of G is linked to (X, A).

Indeed let K'' be a continuous closed path of G. $\Omega = \{S(q, \frac{1}{2}\varrho_q); q \in G'\}$ is an open covering of $\tilde{K}''$. Hence there is a representation of the form

$$K'' = K''_1 K''_3 \ldots K''_{2m-1} \tag{4}$$

such that for $j = 1, \ldots, m$ $\tilde{K}''_{2j-1}$ is contained in a member of Ω (see [11], 7.11, 7.10, and 7.8) - say

$$\tilde{K}''_{2j-1} \subset S(q'_{2j}, \frac{1}{2}\varrho'_{2j}). \tag{5}$$

For $j = 1, \ldots, m$ let

$$K''_{2j-1} = K''_{2j-1} : q''_{2j} \to q''_{2j-1}. \tag{6}$$

Hence for $j = 1, \ldots, m-1$ we have

$$q''_{2j} = q''_{2j+1}. \tag{7}$$

Moreover

(8) $$q''_{2m} = q''_1.$$

For $j = 1, \dots, m$ let

(9) $$q'_{2j-1} = q'_{2j}.$$

$v'_{2j-1} = \{q'_{2j-1}\}, v''_{2j} = \{q''_{2j}\}$. $K'_{2j-1} = [q'_{2j-1} v'_{2j-1} q'_{2j-1}]$ and $K''_{2j} = [q''_{2j} v''_{2j} q''_{2j}]$.

For $j = 1, \dots, m$ K'_{2j-1} and K''_{2j} are degenerate paths and by (9) we have

$$K'_{2j-1} = K'_{2j-1} : q'_{2j-1} \to q'_{2j}.$$

Moreover by (8) one has

$$K''_{2m} = K''_{2m} : q''_1 \to q''_{2m}$$

and by (7) for $j = 1, \dots, m-1$ we have

$$K''_{2j} = K''_{2j} : q''_{2j+1} \to q''_{2j}.$$

Hence $K'' = K''_1 K''_2 \dots K''_{2m}$.

For $t = 1, \dots, 2m$ let v_t be the (possibly degenerate) segment $v_t = [q'_t, q''_t]$ with the endpoints q'_t and q''_t and

$$K_t = [q''_t v_t q'_t] : q'_t \to q''_t.$$

(In case $q'_t = q''_t$ we have $v_t = \{q'_t\}$.)

Then for $j = 1, \dots, m$ the closed path

$$\overline{K}_{2j-1} = K^{\bullet}_{2j-1} K''_{2j-1} K_{2j} K'_{2j-1}$$

is lying in the e-regular domain $S(q'_{2j}, \frac{1}{2}\varrho_{q'_2})$ and thus $\overline{K}_{2j-1}$ is nonlinked to (X, A).

Now let $j \in \{2, \dots, m\}$. By (5), (6) and (7) the open balls $S(q'_{2j-2}, \frac{1}{2}\varrho_{q'_{2j-2}})$ and $S(q'_{2j}, \frac{1}{2}\varrho_{q'_{2j}})$ have the common point $q''_{2j-1} = q''_{2j-2}$ consequently either $q'_{2j-2} \in S(q'_{2j}, \varrho_{q'_{2j}})$ and thus $q'_{2j}, q'_{2j-2} \in U_{q'_{2j}}$ (see (3)) or $q'_{2j} \in$ $\in S(q'_{2j-2}, \varrho_{q'_{2j-2}})$ and thus q'_{2j}, $q'_{2j-2} \in U_{q'_{2j-2}}$. Let v'_{2j-2} be a degenerate or a simple arc with the endpoints $q'_{2j} = q'_{2j-1}$ (see (9)) and q'_{2j-2} lying in $U_{q'_{2j}}$ (in the first case) or in $U_{q'_{2j-2}}$ (in the second case). Since the domains $U_{q'_{2j}}$ and $U_{q'_{2j-2}}$ are arcwise connected, such a v'_{2j-2} exists. Let

$$K'_{2j-2} = [q'_{2j-1} v'_{2j-2} q'_{2j-2}] : q'_{2j-2} \to q'_{2j-1}.$$

Then taking also (2) into account the closed path

$$\overline{K}_{2j-2} = [K^{\bullet}_{2j-2} K''_{2j-2} K_{2j-1} K'_{2j-2}]$$

is lying either in the e-regular domain $S(q'_{2j}, \varrho'_{q'_{2j}})$ (in the first case) or in $S(q'_{2j-2}, \varrho_{q'_{2j-2}})$ (in the second case). Hence in both cases $\overline{K}_{2j-2}$ is nonlinked to (X, A).

Likewise we have either $\{q'_{2m}, q'_2\} \subset U_{q'_{2m}}$ or $\{q'_{2m}, q'_2\} \subset U_{q'_2}$. Let v'_{2m} be a degenerate or a simple arc with the endpoints q'_{2m} and $q'_2 = q'_1$ contained in $U_{q'_{2m}}$ (in the first case) or in $U_{q'_2}$ (in the second case). Let

$$K'_{2m} = [q'_1 v'_{2m} q'_{2m}] : q'_{2m} \to q'_1.$$

Then the closed path $\overline{K}_{2m} = K^{\bullet}_{2m} K''_{2m} K_1 K'_{2m}$ is lying either in $S(q'_{2m}, \varrho'_{q'_{2m}})$ (in the first case) or in $S(q'_2, \varrho'_{q'_2})$ (in the second case). Hence in both cases $\overline{K}_{2m}$ is nonlinked to (X, A).

Now let $K' = K'_{2m} \dots K'_2 K'_1$. K' is a continuous closed path lying in the i-regular domain G' of (X, A) and thus K' is nonlinked to (X, A). Hence by 2.13, K'' is a nonlinked closed path of (X, A). consequently G is an e-regular domain of (X, A) as required.

The theorem is proved.

2.15. THEOREM. *Let G be an e-regular domain of (X, A) that meets X. Then there are points q_1, q_2 in $G \setminus X$ and continuous paths $K_1 : q_1 \to q_2$ and $K_2 : q_2 \to q_1$ in $R^{n+1} \setminus A$ such that*
(a) *q_1 and q_2 are in distinct components of $G \setminus X$,*
(b) *$\tilde{K}_1 \subset G$,*
(c) *$\tilde{K}_2 \subset R^{n+1} \setminus X$ and*
(d) *the closed path $K_2 K_1$ is linked to (X, A).*

PROOF. Let $q \in G \cap X$. Since $G \cap A = \emptyset$ we have $q \in X \setminus A$. Let V be a domain of $X \setminus A$ contained in $G \cap X$. Since $X \setminus A$ is locally connected, such a domain V clearly exists. Let U be a nonempty open subset of V such that for the inclusion $j : (X, A) \subset (X, X \setminus U)$ the induced homomorphism $j_* : H_n(X, A) \to H_n(X, X \setminus U)$ is a monomorphism (see 1.2(d)). Let $i_1 : (X, \emptyset) \subset (X, X \setminus U)$ and $i : (X, \emptyset) \subset (X, A)$ be inclusions. Then $i_1 = j \circ i$ and thus according to 1.2(c) the induced $i_{1*} : H_n(X) \to H_n(X, X \setminus U))$ is a trivial homomorphism. Hence considering the segment

$$\tilde{H}_{n-1}(X \setminus U) \xleftarrow{\partial'} H_n(X, X \setminus U) \xleftarrow{i_{1*}} H_n(X)$$

of the exact reduced homology sequence of $(X, X \setminus U)$ we obtain that ∂' is a monomorphism. Let $s : (A, \emptyset) \subset (X, \emptyset)$ be the inclusion map and consider the commutative diagram

$$\begin{array}{ccc}
\tilde{H}_{n-1}(A) & \xleftarrow{\partial} & H_n(X,A) \\
\big\downarrow {\scriptstyle \bar{s}_*} & & \big\downarrow {\scriptstyle j_*} \\
\tilde{H}_{n-1}(X\setminus U) & \xleftarrow{\partial'} & H_n(X, X\setminus U)
\end{array}$$

we get $s_*(A_*) = \partial' j_*(u)$ where $A_* = \partial u$ and $u \neq 0$ (see 1.4). Consequently since ∂' and j_* are monomorphisms we obtain that

$$(10) \qquad s_*(A_*) \neq 0.$$

Let J be a linking Jordan curve of $s_*(A_*)$ i.e., J is a Jordan curve in $R^{n+1}\setminus(X\setminus U)$ such that for each nonzero element u' of $H_1(J)$ we have

$$\mathfrak{v}_{X\setminus U,J}(s_*(A_*),u') \neq 0$$

and thus

$$(11) \qquad \mathfrak{v}_{A,J}(A_*,u') \neq 0$$

(see also 1.8). According to (10) and [10], Theorem 2.3, such a curve J exists (see also [10], 2.1 and 2.2).

It is to be noted that each Jordan path K with the body J is linked to (X,A).

Indeed, let K be such a Jordan path. Then $K_* \neq 0$ (see 2.5(d)) and thus according to (11) we have

$$(12) \qquad \mathfrak{v}_{A,\tilde{K}}(A_*,K_*) \neq 0.$$

Hence K is linked to the (n,p)-cell (X,A) as required.

Now let K be a Jordan path with $\tilde{K} = J$. Such a Jordan path clearly exists. By (12), $\tilde{K} = J$ cannot lie in G or in $R^{n+1}\setminus X$ (see 2.6). On the other hand

$$\tilde{K} = J \subset R^{n+1}\setminus(X\setminus U) = U \cup (R^{n+1}\setminus X) \subset G \cup (R^{n+1}\setminus X).$$

Hence by Lemma 7.16 of [11] there is a subdivision $K = K^{2\prime}K^{1\prime}$ of K and a subdivision $K'' = K_1''K_2''\dots K_{2m}''$ of $K'' = K^{1\prime}K^{2\prime}$ such that the relations $\tilde{K}_{2i-1}'' \subset R^{n+1}\setminus X$, $\tilde{K}_{2i-1}'' \not\subset G$, $\tilde{K}_{2i}'' \subset G$ and $\tilde{K}_{2i}'' \not\subset (R^{n+1}\setminus X)$ hold for $i = 1,\dots,m$ (see also [11], 7.12 and 7.14). Since K is a linked closed path of (X,A), 2.9 shows that K'' is a linked closed path of (X,A) as well. For $j = 1,\dots,2m-1$ let $K_j'' = K_j'' : q_{j+1} \to q_j$ and let $K_{2m}'' = K_{2m}'' : q_1 \to q_{2m}$. The points $q_1,\dots,q_{2m}$ belong to $G \cap (R^{n+1}\setminus X) = G\setminus X$.

For $j=1,\dots,2m-1$ let $K'_j = K'_j : q_j \to q_{j+1}$ be a continuous path lying in G and suppose that $\tilde{K}_j \subset G \setminus X$ whenever q_j and q_{j+1} lie in the same component of $G \setminus X$. Let $K'_{2m} = K'_{2m} : q_{2m} \to q_1$ be a continuous path lying in G.

For $j=1,\dots,2m$ let $K^j : q_j \to q_j$ be a degenerate path. Let $\overline{K_j} = (K^j)^{\bullet} K''_j K^{j+1} K'_j$ for $j=1,\dots,2m-1$ and $\overline{K_{2m}} = (K^{2m})^{\bullet} K''_{2m} K^1 K'_{2m}$. Then the closed paths $\overline{K_2}, \overline{K_4}, \dots, \overline{K_{2m}}$ and the closed path $K' = K'_{2m} \dots K'_2 K'_1$ are lying in G. Hence they are nonlinked to (X,A). On the other hand $\overline{K_{2i-1}}$ is lying in $R^{n+1} \setminus X$ whenever q_{2i-1} and q_{2i} belong to the same component of $G \setminus X$ and thus all the closed paths of this kind are nonlinked to (X,A).

However $K'' = K''_1 K''_2 \dots K''_{2m}$ is linked to (X,A) and thus according to 2.13 there is a $t \in \{1,\dots,m\}$ such that $\overline{K_{2t-1}}$ is linked to (X,A) and for this t, q_{2t-1} and q_{2t} lie in distinct components of $G \setminus X$. Further since $(K^{2t-1})^{\bullet}$ and K^{2t} are degenerate paths we have $\overline{K_{2t-1}} = K''_{2t-1} K'_{2t-1}$. Hence denoting K'_{2t-1}, K''_{2t-1}, q_{2t-1} and q_{2t} by K_1, K_2, q_1 and q_2, resp., the points q_1, q_2 and the continuous paths K_1, K_2 satisfy the requirements (a), (b), (c), (d).

The theorem is proved.

2.16. COROLLARY. *$X \setminus A$ is nowhere dense in R^{n+1}.*

Indeed, let G' be a nonempty open set in R^{n+1}. If $G' \cap (X \setminus A) \neq \emptyset$ then let G be an open ball in G' meeting X and missing A. Such a ball G clearly exists. G is an e-regular domain of (X,A) (see 2.7) and thus by 2.15, $G \setminus X$ is a nonempty open set in $G' \setminus (X \setminus A)$. If $G' \cap (X \setminus A) = \emptyset$ then G' itself is a nonempty open set in $G' \setminus (X \setminus A)$.

$X \setminus A$ is nowhere dense in R^{n+1} as required (cf. [13] 1.3.5).

2.17. THEOREM. *Let (Y,B) be a compact pair. Let p' be a prime distinct from p and H' the Čech homology theory defined on the category of compact pairs over the coefficient group $Z_{p'}$. Suppose that (Y,B) is an (n,p)-cell and $H'_n(Y,B)=0$. Then (Y,B) is an absolutely linked (n,p)-cell.*

PROOF. We argue by contradiction. Suppose the existence of a topological embedding $\varphi : Y \to R^{n+1}$ such that $(X,A) = (\varphi(Y), \varphi(B))$ is a nonlinked (n,p)-cell in R^{n+1}. According to 1.11, 2.6 and 2.7, $X \setminus A$ is an i-regular domain of (X,A) and thus by 2.14 there exists an e-regular domain G of (X,A) such that $G \cap X = X \setminus A$. G meets X, consequently by 2.15, $G \setminus X$ has at least two components. However by $H'_n(X,A)=0$, $G \setminus X$ is connected (see [8], consequence of Theorem 2). This is a contradiction.

The theorem is proved.

2.18. THEOREM. *The topological product of an absolutely linked (n,p)-cell and of a nondiscrete space cannot be embedded in R^{n+1}.*

PROOF. We argue by contradiction.

Suppose the existence of a topological embedding $\varphi: Y \times C \to R^{n+1}$ where (Y, B) is an absolutely linked (n,p)-cell and C is a nondiscrete space. Let c be an accumulation point of C (see [13], p. 43). Since C is nondiscrete, such a c exists. Let $X = \varphi(Y \times \{c\})$ and $A = \varphi(B \times \{c\})$. (X, A) is an (n,p)-cell in R^{n+1}. Let f be a continuous closed line in $X \setminus A$. Let $\varepsilon = \varrho(\tilde{f}, A)$ where ϱ is the metric in R^{n+1} (cf. 1.6). Since $\tilde{f}$ is compact we have $\varepsilon > 0$. Select $c' \in C \setminus \{c\}$ such that for each $y \in Y$

$$\varrho(\varphi(y,c), \varphi(y,c')) < \varepsilon.$$

Since Y is compact and c is an accumulation point of C the existence of such a c' follows (see also [11], 9.6).

For each $q = \varphi(y,c) \in X$ let $\psi(q) = \varphi(y,c')$. $\psi: X \to \varphi(Y \times \{c'\})$ is clearly a topological mapping and $X \cap \psi(X) = \emptyset$. Let $Q = \psi(\tilde{f})$ and

$$W = \bigcup_{q \in \tilde{f}} [q, \psi(q)]$$

where $[q, \psi(q)]$ is the segment with the endpoints q and $\psi(q)$. W is clearly a compact set in R^{n+1} disjoint to A and $(\tilde{f} \cup Q) \subset W$. Let $i_1: \tilde{f} \subset W$ and $i_2: Q \subset W$ be inclusions and let

$$\psi' = \psi|_{\tilde{f}} : \tilde{f} \to Q.$$

Then the mappings i_1 and $i_2 \circ \psi'$ are clearly homotopic and thus

$$i_{1*}(f^*) = i_{2*}\psi'_*(f^*).$$

$\psi'_*(f^*) \in H_1(Q)$ where Q is disjoint to X and thus by 2.1 we have

$$\mathfrak{v}_{A,Q}(A_*, \psi'_*(f^*)) = 0,$$

consequently

$$\mathfrak{v}_{A,\tilde{f}}(A_*, f^*) = \mathfrak{v}_{A,W}(A_*, i_{1*}(f^*)) =$$
$$= \mathfrak{v}_{A,W}(A_*, i_{2*}\psi'_*(f^*)) = \mathfrak{v}_{A,Q}(A_*, \psi'_*(f^*)) = 0$$

(see also 1.8).

Hence f fails to be linked to the (n,p)-cell (X, A) (see 1.10) and thus (X, A) is a nonlinked (n,p)-cell in R^{n+1}. (Y, B) is not absolutely linked. This is a contradiction.

The theorem is proved.

3. Pseudomanifolds with boundary

We shall take the terminology and notations of the books P. S. Aleksandrov, Combinatorial topology 1, 2 ([1], [2]).

Let p and Z_p be the same as in 1.1.

3.1. Let K be a triangulation (see [1] p. 118) situated in some euclidean space R^m. Suppose that K is an n-dimensional combinatorial pseudomanifold with boundary L where $n \geqq 1$ and $L \neq \emptyset$ (see [2] p. 72). L is a triangulation as well. Let Y and B be the bodies of K and L respectively (see [1] p. 136) i.e., $(Y, B) = (\|K\|, \|L\|)$. Y and B are compact Hausdorff spaces.

Compact pairs of such type are called *topological nonclosed n-pseudomanifolds with boundary.*

Let R^m be a hyperplane of the euclidean $(m+1)$-space R^{m+1}. Let c be a point in $R^{m+1} \setminus R^m$ and let M^* be the set of all open cones with the vertex c where the base of the cones runs over all (open) simplexes of L (see [1], p. 214). M^* is clearly a set of open simplexes of R^{m+1}. Let $M = M^* \cup L \cup \cup \{\{c\}\}$ where $\{c\}$ is the 0-simplex with the vertex c. M and $K \cup M$ are clearly triangulations in R^{m+1}. Moreover if T^n is an n-simplex of K then $K \setminus \{T^n\}$ and $(K \cup M) \setminus \{T^n\}$ are triangulations in R^{m+1} as well.

An easy computation shows the following equalities. $\Delta_p^n(K) = 0$ (cf. [2] p. 50), $\Delta_p^n((K \cup M) \setminus \{T^n\}) = 0$, $\Delta_p^n(K \cup M) \approx Z_p$ if $p = 2$ or if $p \neq 2$ and the pseudomanifold K is orientable (cf. [2] p. 74), $\Delta_p^n(K \cup M) = 0$ if $p \neq 2$ and K is nonorientable.

3.2. Let H^p be the Čech homology theory defined on the category of compact pairs over the coefficient group Z_p. Let K, L, M, T^n, Y and B be the same as in 3.1.

Since for any triangulation K_α of a polyhedron Φ and for an arbitrary coefficient domain Γ the groups $\Delta^n(K_\alpha, \Gamma)$ and $\Delta^n(\Phi, \Gamma)$ are isomorphic to each other (see Theorem XI.4.1 and Definition XI.1.1 of [2], pp. 164 and 159), moreover since $\Delta^n(\Phi, \Gamma)$ is isomorphic to $H_n(\Phi; \Gamma)$ where $H_n(\Phi; \Gamma)$ is the n-dimensional Čech homology group of Φ over the coefficient group Γ (see [9], Section 24), from 3.1 we get the following results:

$$H_n^p(\|K\|) = H_n^p(Y) = 0, \tag{13}$$

$$H_n^p(\|(K \cup M) \setminus \{T^n\}\|) = 0, \tag{14}$$

$$H_n^p(\|K \cup M\|) \approx Z_p \text{ if } p = 2 \text{ or if } p \neq 2 \text{ and } K \text{ is orientable.} \tag{15}$$

$$H_n^p(\|K \cup M\|) = 0 \text{ if } p \neq 2 \text{ and } K \text{ is nonorientable.} \tag{16}$$

Let $\Phi = \|K \cup M\|$ and $\Psi = \|M\|$. Φ and Ψ are compact subsets of R^{m+1}. Ψ is clearly contractible to a point over itself and thus it is homologically trivial (see [12] p. 30). Hence $H_n^p(\Phi)$ and $H_n^p(\Phi, \Psi)$ are isomorphic groups. Consequently by (15) and (16), $H_n^p(\Phi, \Psi) \approx Z_p$ if $p = 2$ or $p \neq 2$ and K is orientable, $H_n^p(\Phi, \Psi) = 0$ if $p \neq 2$ and K is nonorientable.

Let $j' : (Y, B) \subset (\Phi, \Psi)$ be the inclusion map. Since $Y \setminus B = \Phi \setminus \Psi = \|K \setminus L\|$ it follows that j' is a relative homeomorphism and thus $H_n^p(Y, B) \approx H_n^p(\Phi, \Psi)$ (see [12] p. 266). Consequently

(a) if K is orientable then for every prime p

$$H_n^p(Y, B) \approx Z_p, \tag{17}$$

and

(b) if K is nonorientable then

$$H_n^2(Y, B) \approx Z_2, \tag{18}$$

and for every $p \neq 2$

$$H_n^p(Y, B) = 0. \tag{19}$$

In the first case we say that (Y, B) is an *orientable topological n-pseudomanifold with boundary* and in the second case we say that (Y, B) is a *nonorientable topological n-pseudomanifold with boundary.*

3.3. PROPOSITION. *Let K, L, Y and B be the same as in* 3.1. *Let U be a domain in $Y \setminus B$. Let H^p be the same as in* 3.2. *Then the homomorphism $j_* : H_n^p(Y, B) \to H_n^p(Y, Y \setminus U)$ induced by the inclusion $j : (Y, B) \subset (Y, Y \setminus U)$ is a monomorphism.*

PROOF. First observe that a successive barycentric subdivision K^ν of K of an arbitrary order ν (see [1], p. 140) is clearly an n-dimensional combinatorial pseudomanifold with boundary as well and the boundary L^ν of K^ν consists of all simplexes of K^ν lying in $\|L\|$. Hence $\|K^\nu\| = \|K\|$ and $\|L^\nu\| = \|L\|$. Let c and M be the same as in 3.1 and let $M^{\nu *}$ be the set of all open cones with the vertex c where the base of the cones runs over all open simplexes of L^ν. Let $M^\nu = M^{\nu *} \cup L^\nu \cup \{\{c\}\}$. Then M^ν is a triangulation in R^{m+1} and $\|M^\nu\| = \|M\|$.

There clearly exist an integer ν and an n-simplex T^n of K^ν such that $T^n \subset U$. T^n is clearly a nonempty open subset of $\|K \setminus L\|$. Let $V = T^n$. Then by 3.2 (14) we have

$$H_n^p(\|(K^\nu \cup M^\nu) \setminus \{T^n\}\|) = H_n^p(\Phi \setminus V) = 0$$

where $\Phi = \|K^\nu \cup M^\nu\| = \|K \cup M\| = Y \cup \|M\|$. $\Psi = \|M^\nu\| = \|M\|$ is homologically trivial and thus

$$H_n^p(\Phi \setminus V, \Psi) \approx H_n^p(\Phi \setminus V) = 0.$$

On the other hand the inclusion $j'' : (Y \setminus V, B) \subset (\Phi \setminus V, \Psi)$ is a relative homeomorphism and thus

$$H_n^p(Y \setminus V, B) \approx H_n^p(\Phi \setminus V, \Psi) = 0.$$

Let $j' : (Y, Y \setminus U) \subset (Y, Y \setminus V)$ be the inclusion map and let $i' = j' \circ j$: $(Y, B) \subset (Y, Y \setminus V)$. Consider the segment

$$H_n^p(Y, Y \setminus V) \xleftarrow{i'_*} H_n^p(Y, B) \longleftarrow H_{n^p}(Y \setminus V, B)$$

of the exact homology sequence of the triple $(Y, Y \setminus V, B)$ (see [12] p. 25). This segment has the form

$$H_n^p(Y, Y \setminus V) \xleftarrow{i'_*} H_n^p(Y, B) \longleftarrow 0$$

and thus i'_* is a monomorphism. Since $i'_* = j'_* j_*$ it follows that j_* is a monomorphism as required.

3.4. THEOREM. *Let* (Y, B) *be the same as in* 3.1. *Then* (Y, B) *is an* (n, p)*-cell whenever* $p = 2$ *or whenever* $p \neq 2$ *and* (Y, B) *is orientable.*

PROOF. Let K and L be the same as in 3.1. Since K is a triangulation and L is a subcomplex of K, it follows that $Y \setminus B = \|K \setminus L\|$ is a locally connected space with countable base. However $K \setminus L$ is nonempty and connected (see [2], p. 72) and thus $Y \setminus B$ is nonempty and connected as well. Hence condition 1.2(a) is fulfilled for the pair (Y, B).

3.2 (13) shows that condition 1.2(c) is satisfied for the pair (Y, B) as well.

According to Proposition 3.3, condition 1.2(d) is satisfied for the pair (Y, B), too.

As to condition 1.2(b), 3.2 (17) and 3.2 (18) show that this condition is satisfied if $p = 2$ or if $p \neq 2$ and (Y, B) is orientable.

The theorem is proved.

3.5. THEOREM. *Let* (Y, B) *be the same as in* 3.1. *Suppose that the topological* n*-pseudomonifold* (Y, B) *is nonorientable. Then* (Y, B) *is an absolutely linked* $(n, 2)$*-cell.*

The theorem is an immediate corollary of the formula 3.2 (19) and Theorems 3.4 and 2.17.

Our program is finished.

In the next paper we shall prove an analogous theorem concerning orientable pseudomanifolds, namely that each orientable nonclosed n-dimensional pseudomanifold with boundary and without homological singular interior points (i.e., without interior points having non cyclic n-dimensional local Betti groups with respect to the coefficient group Z) is absolutely nonlinked. This latter theorem was also stated in my paper [3] without proof.

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(Received August 14, 1989)

EÖTVÖS LORÁND UNIIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT. 6–8
H–1088
HUNGARY

Acta Math. Hung.
59(1-2) (1992), 245–252.

A REMARK ON THE VIBRATION OF A CIRCULAR MEMBRANE IN DIFFERENT POINTS

I. JOÓ (Budapest)

To the memory of my father

1. Let

$$\Omega := \{(x,y) : x^2 + y^2 < 1\} \subset \mathbf{R}^2$$

be the unit circle. Define U as the vector space of all functions

$$u(t,x,y) \in C^\infty(\mathbf{R}\times\Omega)\cap C(\mathbf{R}\times\overline{\Omega})$$

satisfying

$$u_{tt} = \Delta u \quad \text{on} \quad \mathbf{R}\times\Omega, \; u = 0 \quad \text{on} \quad \mathbf{R}\times\partial\Omega. \tag{1}$$

Denote for $m = 0, 1, \dots$ by $J_m(x)$ the Bessel function of order m and arrange its positive zeros increasingly into the sequence

$$0 < \lambda_1^{(m)} < \lambda_2^{(m)} < \dots.$$

As it is known ([1]), the finite sums of the form

$$u(t,x,y) = \sum_{m,k} \left(a_{mk}e^{i\lambda_k^{(m)}t} + b_{mk}e^{-i\lambda_k^{(m)}t}\right) \sin m\varphi J_m(\lambda_k^{(m)}r) + \tag{2}$$
$$+ \sum_{m,k} \left(c_{mk}e^{i\lambda_k^{(m)}t} + d_{mk}e^{-i\lambda_k^{(m)}t}\right) \cos m\varphi J_m(\lambda_k^{(m)}r)$$

are solutions of (1) if we use the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. In this note we prove the following

THEOREM. *Let $N \in \mathbf{N}$ and $P_1, \dots, P_N \in \Omega\setminus\{0\}$ be different points. Then the mapping*

$$A : U \to C^\infty(\mathbf{R}, \mathbf{C}^N), \quad Au := (u(.,P_1), \dots, u(.,P_N))$$

has a dense range in $C^\infty(\mathbf{R}, \mathbf{C}^N)$.

This theorem proves a conjecture of V. Komornik formulated in [5]. For rectangular membranes analogous results were obtained earlier by the author of the present paper in [9] and independently by M. Horváth in [8]. In [7] the authors proved a weaker version of the above theorem. The result proved in [7] is also a special case of Theorem 2 of [5].

2. We begin the proof with the remark that it is enough to prove the density of AU in $L^2(0,T;\mathbf{C}^N)$ for any $T>0$; the density in C^∞ follows easily from it [10]. Secondly if we restrict ourselves to the solutions given by finite sums (2) then our Theorem reduces to the following statement. Denote

$$e_{mk}:=\begin{pmatrix}\sin m\varphi_1 J_m(\lambda_k^{(m)}r_1)\\ \vdots\\ \sin m\varphi_N J_m(\lambda_k^{(m)}r_N)\end{pmatrix}$$

and

$$f_{mk}:=\begin{pmatrix}\cos m\varphi_1 J_m(\lambda_k^{(m)}r_1)\\ \vdots\\ \cos m\varphi_N J_m(\lambda_k^{(m)}r_N)\end{pmatrix}.$$

Then the system

$$e(\Lambda):=\left\{e_{mk}e^{\pm i\lambda_k^{(m)}t},\, f_{mk}e^{\pm i\lambda_k^{(m)}t}: m=0,1,\dots;k=1,2,\dots\right\}$$

is complete in $L^2(0,T;\mathbf{C}^N)$ for any $T>0$.

In what follows we shall prove this last statement. We begin with

PROPOSITION. *Arrange the positive zeros $\lambda_k^{(m)}$ of J_m, considered together for all m into a sequence $0<\mu_1<\mu_2<\dots$. Then $\mu_{n+1}-\mu_n\to 0$ $(n\to\infty)$.*

PROOF. The starting point of the proof is the Langer formula ([2])

$$J_m(x)=\sqrt{1-\frac{\arctan w}{w}}\frac{1}{\sqrt{3}}\left[J_{1/3}(z)+J_{-1/3}(z)\right]+O(m^{-4/3}),$$
$$x>m,\quad w:=\sqrt{\frac{x^2}{m^2}-1},\quad z:=m(w-\arctan w)$$

where the O-term is uniform in x. We apply it for

$$x=m\sqrt{c^2+1}+t,\quad t=O(m^{1/3})$$

where $c\geqq 1$. Applying the asymptotical formula of Bessel functions ([1]) for $J_{\pm 1/3}$ we obtain

$$J_m(x)=\sqrt{\frac{2}{\pi}}(x^2-m^2)^{-1/4}\left[\cos\left(z-\frac{\pi}{4}\right)+O(m^{-5/6})\right]. \tag{3}$$

So to any value $z = \frac{3\pi}{4} + k\pi + O(m^{-5/6})$ there corresponds a zero $\mu_k^{(m)}$ of J_m. A counting shows that

$$(4) \quad \mu_k^{(m)} = m\sqrt{c^2+1} + t_k^{(m)} = \pi \frac{\sqrt{c^2+1}}{c}\left(k + \frac{3}{4} + m\frac{\arctan c}{\pi}\right) + O(m^{-1/3}).$$

If $\frac{\arctan c}{\pi}$ is irrational, then its multiplies have a uniform distribution mod 1. Since m can be changed in an interval of length $O(m^{1/3})$, this implies easily the Proposition.

LEMMA 1. *If $0 \leqq \varphi_1, \dots, \varphi_N \leqq 2\pi$ are different numbers, then the vectors*

$$\begin{pmatrix} \sin m\varphi_1 \\ \vdots \\ \sin m\varphi_N \end{pmatrix}, \quad \begin{pmatrix} \cos m\varphi_1 \\ \vdots \\ \cos m\varphi_N \end{pmatrix}, \qquad m = 0, 1, \dots, N-1$$

span the space $\mathbf{R}^N$.

The proof is standard, hence we omit it.

Denote $P_j = P_j(r_j, \varphi_j), 0 < r_j < 1, 0 \leqq \varphi_j < 2\pi$ the polar coordinates of $P_1, \dots, P_N$. We can suppose that the values $r_1, \dots, r_{N_0}$ are different and the other r_j do not give new values. Suppose that c is large enough, namely

$$r_j^2(c^2+1) \geqq 4$$

and denote

$$d_j := \sqrt{r_j^2(c^2+1) - 1};$$

these quantities arise when we count $J_m(\mu_k^{(m)} r_j)$ by (3).

LEMMA 2. *There exists a residual set $D \subset [1, \infty)$ such that for any $c \in D$ any equality*

$$(5) \qquad 0 = n\pi + n_0 \arctan c + \sum_{j=1}^{N_0} n_j \arctan d_j + \sum_{j=1}^{N} n_j' \varphi_j$$

with integer coefficients n, n_j, n_j necessarily implies

$$n_0 = n_1 = \dots = n_{N_0} = 0.$$

PROOF. Indeed, the set of $c \in [1, \infty)$ satisfying (5) with fixed coefficients is obviously closed. If we show that in the case $|n_0| + |n_1| + \dots |n_{N_0}| > 0$ this set does not contain an interval, then Lemma 2 will follow from Baire category theorem. If (5) holds in an interval, we can differentiate it with respect to the variable c; by repeated differentiation we get the relations

$$0=\frac{n_0}{c^{2\ell+1}}+\sum_{j=1}^{N_0} n_j \frac{r_j^2}{d_j^{2\ell+1}} \qquad (\ell=0,1,\dots).$$

Consider them as linear equations with variables n_j. Since $\frac{d_1}{r_1},\dots,\frac{d_{N_0}}{r_{N_0}}$ are all different, the only solution is $n_0=n_1=\dots=n_{N_0}=0$.

3. Next we count the asymptotics for $J_m(\mu_k^{(m)} r_j)$. The values z_j corresponding to $x=\mu_k^{(m)} r_j$ by the Langer formula have the form

$$z_j=\frac{d_j}{c}\pi\left(k+\frac{3}{4}+\frac{1}{\pi}\left(\arctan c-\frac{c}{d_j}\arctan d_j\right)m\right)+O(m^{-1/3}). \tag{6}$$

Consider a large number y and m_0 with $m_0\sqrt{c^2+1}=y+O(1)$ and k_0 with $\mu_{k_0}^{(m_0)}=y+O(1)$. Take $m=m_0+O(1)$ satisfying

$$\left\| m\frac{\arctan c}{\pi}+\left(\frac{3}{4}-\frac{yc}{\pi\sqrt{c^2+1}}\right)\right\|<\varepsilon/16N; \tag{7}$$

here $\|\alpha\|$ denotes the distance between α and the nearest integer. By (4) there exists $k=k_0+O(1)$ satisfying $|y-\mu_k^{(m)}|<\varepsilon/8N$. Expressing $k+\frac{3}{4}$ by (7) we can improve (6):

$$z_j=-m\arctan d_j+\frac{yd_j}{\sqrt{c^2+1}}+\vartheta_j\frac{\varepsilon}{16N},\quad |\vartheta_j|<1. \tag{8}$$

Define the vectors

$$e_{mk}^0 := \sqrt{\frac{\pi m}{2}} \begin{pmatrix} \sin m\varphi_1 J_m(\mu_k^{(m)} r_1) \\ \vdots \\ \sin m\varphi_N J_m(\mu_k^{(m)} r_N) \end{pmatrix},$$

$$f_{mk}^0 := \sqrt{\frac{\pi m}{2}} \begin{pmatrix} \cos m\varphi_1 J_m(\mu_k^{(m)} r_1) \\ \vdots \\ \cos m\varphi_N J_m(\mu_k^{(m)} r_N) \end{pmatrix},$$

$$e_{mk}^1 := \begin{pmatrix} \frac{1}{\sqrt{d_1}} \sin m\varphi_1 \cos(m \arctan d_1 - \gamma_1) \\ \vdots \\ \frac{1}{\sqrt{d_N}} \sin m\varphi_N \cos(m \arctan d_N - \gamma_N) \end{pmatrix},$$

$$f_{mk}^1 := \begin{pmatrix} \frac{1}{\sqrt{d_1}} \cos m\varphi_1 \cos(m \arctan d_1 - \gamma_1) \\ \vdots \\ \frac{1}{\sqrt{d_N}} \cos m\varphi_N \cos(m \arctan d_N - \gamma_N) \end{pmatrix},$$

with

$$\gamma_j := y \frac{d_j}{\sqrt{c^2+1}} - \frac{\pi}{4},$$

then from (3) and (8) we get

$$|e_{mk}^0 - e_{mk}^1| < \varepsilon/8, \qquad |f_{mk}^0 - f_{mk}^1| < \varepsilon/8. \tag{9}$$

The central statement of the proof is the following

LEMMA 3. *There exists a basis $e_1, \dots, e_N$ of $\mathbf{C}^N$ with the following property. Let $\varepsilon, T > 0$. Then the set*

$$e^0(\Lambda) := \left\{ e_{mk}^0 e^{\pm i\mu_k^{(m)} t}, f_{mk}^0 e^{\pm i\mu_k^{(m)} t} : t_k^{(m)} = O(m^{1/3}) \right\}$$

contains a subsystem

$$\Phi = \left\{ e_n^j e^{\pm i\lambda_{n,j} t} : j = 1, \dots, N; n = 1, 2, \dots \right\} \cup \left\{ e_0^j e^{i\lambda_{0,j} t} : j = 1, \dots, N \right\}$$

such that

$$|e_n^j - e_j| < \varepsilon, \quad j = 1, \dots, N; n = 0, 1, \dots, \tag{a}$$

$$|\lambda_{n,j} - 2\pi \frac{n}{T}| < \varepsilon, \quad j = 1, \dots, N; n \geqq n_0 \tag{b}$$

for some large number n_0.

PROOF. For the proof take $y=2\pi\frac{n}{T}$ with large n. If m satisfies (7) then for the coorresponding k we have $|2\pi\frac{n}{T}-\mu_k^{(m)}|<\varepsilon/8N$. By (9) we can consider the vectors e^1_{mk}, f^1_{mk}. Since $1, \frac{\arctan c}{\pi}, \frac{\arctan d_j}{\pi}$ $(j=1,\dots,N_0)$ are linearly independent over Q (rationals) we can choose m such that $\cos(m\arctan d_j - -\gamma_j)$ is approximately 1 when $d_j=d_{j_0}$ and approximately zero when $d_j\neq d_{j_0}$. In the coordinates j with $d_j=d_{j_0}$ the corresponding φ_j are different, so by Lemma 1 we get there a basis with vectors

$$\frac{1}{\sqrt{d_{j)}}}\begin{pmatrix}\vdots\\ \cos m'\varphi_j\\ \vdots\end{pmatrix} \quad\text{or}\quad \frac{1}{\sqrt{d_{j_0}}}\begin{pmatrix}\vdots\\ \sin m'\varphi_j\\ \vdots\end{pmatrix}$$

where $0\leqq m'<N$. Let the other coordinates $(d_j\neq d_{j_0})$ of these vectors be zeros; considering them together for all j_0 we get a basis in $\mathbf{R}^N$. Now for $j=1$ the construction goes as follows. Take $c\in D$ satisfying $r_j\sqrt{c^2+1}\geqq 2$, $j=1,\dots,N_0$ and

$$\left\|2\frac{n}{T}\frac{c}{\sqrt{c^2+1}}-\frac{3}{4}\right\|<\varepsilon/16N \tag{10}$$

and consider the following system of simultaneous diophantine approximation:

$$\begin{cases} \left\|m\frac{\arctan c}{2\pi}\right\|<\varepsilon/16N, \quad \left\|m\frac{\arctan d}{2\pi}-\frac{\gamma_1}{2\pi}\right\|<\varepsilon/16N,\\ \left\|m\frac{\arctan d_j}{2\pi}-\frac{\gamma_j+\frac{\pi}{2}}{2\pi}\right\|<\varepsilon/16N \quad (j=2,\dots,N_0),\\ \left\|m\frac{\varphi_j}{2\pi}-m'\frac{\varphi_j}{2\pi}\right\|<\varepsilon/16N \quad (d_j=d_1). \end{cases} \tag{11}$$

By Kronecker's theorem ([4], p. 58) this has a solution $m=m_0+O(1)$ for any $\varepsilon>0$ if and only if for any integers n_j, n'_j for which

$$n_0\frac{\arctan c}{2\pi}+\sum_{j=1}^{N_0}n_j\frac{\arctan d_j}{2\pi}+\sum_{d_j=d_1}n'_j\frac{\varphi_j}{2\pi}$$

is an integer, the expression

$$n_1\frac{\gamma_1}{2\pi}+\sum_{j=2}^{N_0}n_j\frac{\gamma_j+\frac{\pi}{2}}{2\pi}+\sum_{d_j=d_1}n'_jm'\frac{\varphi_j}{2\pi}$$

is also an integer. But this is true, since Lemma 2 implies that $n_0=n_1= =\dots=n_{N_0}=0$. So (11) has a solution and the construction is ready for $j=1$ and analogously for other j.

Now the proof of the Theorem can be finished just as in the case of the rectangular membrane in [8] and [9]. Namely, we can prove that Φ is complete in $L^2(0,T;\mathbf{C}^N)$ for any $T'<T$. Let

$$\Phi_0 := \left\{ e^j_{|n|} e^{i2\pi nt/T} : n \in \mathbf{Z}, j = 1, \ldots, N \right\}.$$

By Lemma 3(a) this is a Riesz basis in $L^2(0,T;\mathbf{C}^N)$ for small $\varepsilon > 0$ and

$$c_1 \sum_{n\in\mathbf{Z}} \sum_{j=1}^{N} |\alpha_{nj}|^2 \leqq \left\| \sum_{n\in\mathbf{Z}} \sum_{j=1}^{N} \alpha_{nj} e^j_{|n|} e^{i2\pi nt/T} \right\|^2_{L_2} \leqq c_2 \sum_{n\in\mathbf{Z}} \sum_{j=1}^{N} |\alpha_{nj}|^2$$

where $0 < c_1 \leqq c_2 < \infty$ is independent of ε. We have

LEMMA 4 ([3] for $N=1$). *Let $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$ be a Riesz basis in $L^2(0,T;\mathbf{C}^N)$ and define $0 < c_1 \leqq c_2 < \infty$ such that*

$$c_1 \sum |\alpha_n|^2 \leqq \|\alpha_n e_n e^{i\lambda_n t}\|^2_{L_2} \leqq c_2 \sum |\alpha_n|^2$$

for all sequences (α_n). Let further $\varepsilon > 0$ be small enough namely $e^{T\varepsilon} - 1 < < \sqrt{c_1/c_2}$. Then for every sequence (λ'_n) with $\lambda'_n \in \mathbf{C}$, $|\lambda_n - \lambda'_n| < \varepsilon$ $(n \in \mathbf{Z})$ the new system $\{e_n e^{i\lambda'_n t}\}$ is a Riesz basis in $L^2(0,T;\mathbf{C}^N)$.

This implies that for small $\varepsilon > 0$ the system

$$\hat{\Phi}_0 := \left\{ e^j_{|n|} e^{i2\pi nt/T} : |n| \leqq n_0, j = 1, \ldots, N \right\} \cup$$
$$\cup \left\{ e^j_n e^{\pm i\lambda_{nj} t} : n \geqq n_0 + 1, j = 1, \ldots, N \right\}$$

is also Riesz basis in $L^2(0,T:\mathbf{C}^N)$. Finally

LEMMA 5 [9]. *Let $\{e_n e^{i\lambda_n t} : n \in \mathbf{Z}\}$ be complete in $L^2(0,T;\mathbf{C}^N)$. If we remove finitely many elements, the remaining system is complete in $L^2(0,T;\mathbf{C}^N)$ for all $0 < T' < T$.*

Applying this to $\hat{\Phi}_0$ we obtain that Φ is complete in $L^2(0,T;\mathbf{C}^N)$ for all $T' < T$, so $e(\Lambda)$ complete in $L^2(0,T;\mathbf{C}^N)$ for all $T < \infty$ as asserted.□

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(Received October 23, 1989)

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HUNGARIAN ACADEMY OF SCIENCES
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ACTA MATHEMATICA HUNGARICA / VOL. 59 No. 1–2

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Index: 26.659

Acta Mathematica Hungarica

VOLUME 59, NUMBERS 3–4, 1992

Akadémiai Kiadó
Budapest

Kluwer Academic Publishers
Dordrecht / Boston / London

ACTA MATH. HU ISSN 0236–5294

ACTA MATHEMATICA
HUNGARICA

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P.O. Box 254, 1519 Budapest, Hungary

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KLUWER ACADEMIC PUBLISHERS

P.O. Box 17, 3300 AA Dordrecht, Holland

Publication programme: 1992: Volumes 59–60 (eight issues)

Subscription price per volume: Dfl 206,– / US $ 105 (incl. postage)

Acta Mathematica Hungarica is abstracted/indexed in Current Contents — Physical, Chemical and Earth Sciences, Mathematical Reviews.

Acta Math. Hung.
59 (3–4) (1992), 253–262.

ENDOMORPHISM RINGS OF ABELIAN GROUPS AS ISOMORPHIC RESTRICTIONS OF FULL ENDOMORPHISM RINGS. II

N. T. ĐÀO (Debrecen)

In 1963, A. L. S. Corner has published his famous theorem that every countable, reduced, torsion-free Abelian group is isomorphic to the endomorphism ring of some countable, reduced, torsion-free Abelian group (cf. [2]).

Consider an arbitrary countable, reduced, torsion-free Abelian group A. Let R be a subring of the endomorphism ring of A containing the identity. In our paper [4], with the aid of Corners technique, we have proved that there exists a reduced, torsion-free Abelian group B containing A as a fully invariant subgroup such that the mapping which takes each endomorphism of B to its restriction on A is an isomorphic mapping of the endomorphism ring of B onto R. Moreover, our group B contains A as a pure subgroup.

In their paper [7], M. Dugas and R. Göbel have proved that if a set-theoretic axiom $\nabla_{\aleph}$ is assumed, then every cotorsion-free ring is isomorphic to the endomorphism ring of some cotorsion-free Abelian group. Making use of their technique, we prove in the present paper that if $V = L$ is assumed, A is a cotorsion-free Abelian group and R is a subring of the endomorphism ring of A containing the identity, then there exists a cotorsion-free Abelian group B containing A as a pure, fully invariant subgroup such that the endomorphism ring of B is isomorphic to R.

In this paper all groups that are written additively are Abelian. By a ring we shall always mean an associative ring with identity. Every module is right module and unital (i.e. the identity of the ring acts as the identity operator on the group).

A group is *cotorsion-free* if it is reduced, torsion-free and contains no subgroups which are isomorphic to the additive group of the ring of p-adic integers for some prim $p \neq 1$. A ring is called cotorsion-free if its additive group is cotorsion-free. $\aleph$ will always denote a regular, not weakly compact cardinal greater than ω_1. Each ordinal α will be identified with the set of all ordinals $\beta < \alpha$. The system $\{X_\alpha \mid \alpha < \aleph\}$ is a $\aleph$-*filtration* of the set X is $X_\alpha \subseteqq X$ $(\alpha < \aleph)$, $X_\nu \subseteqq X_\mu$ $(\nu < \mu < \aleph)$, $X_\delta = \bigcup\limits_{\nu<\delta} X_\nu$ for limit ordinals $\delta < \aleph$, $X = \bigcup\limits_{\nu<\aleph} X_\nu$ and $|X_\alpha| < \aleph$ $(\alpha < \aleph)$. A subset S of the ordinal α is *stationary* in α if $S \cap C \neq \emptyset$ for each closed unbounded subset C of α. A subset of $\aleph$ is *sparse* if $S \cap \alpha$ is not stationary in α for all limit ordinals

$\alpha < \aleph$. We denote the ordinal $\inf\{|S| \bigm| S \subseteqq \alpha,\ \sup(S) = \alpha\}$ by $\mathrm{cf}(\alpha)$. Let $S \subseteqq \aleph$ and $\{X_\nu | \nu < \aleph\}$ be a $\aleph$-filtration of the set X of cardinality $\aleph$. S is *non-small* (cf. Devlin and Shelah [6]) if S satisfies the following set-theoretic condition $\Phi_\aleph(S)$.

$\Phi_\aleph(S)$: given for each $\nu \in S$ a function $P_\nu : \mathbf{P}(X_\nu) \to \{0,1\}$ of the subsets of X_ν into $\{0,1\}$, there is a function $\varphi : S \to \{0,1\}$ such that for all $Y \subseteqq X$ the set $\{\nu \in S | P_\nu(Y \cap X_\nu) = \varphi(\nu)\}$ is stationary in $\aleph$.

We say that $\nabla_\aleph(S)$ holds for a set $S \subseteqq \aleph$ (cf. Dugas and Göbel [7]) if

(i) S is non-small and is sparse if $\aleph > \omega_1$;

(ii) if $\lambda \in S$, then $\mathrm{cf}(\lambda) = \omega$.

An R-module M is *faithful* if for $r \in R$, $Mr = \{0\}$ implies $r = 0$.

We need the following lemmas. (They can be found in [7].)

LEMMA 1 [7]. *An Abelian group G is cotorsion-free if and only if the additive group of the endomorphism ring of G is cotorsion-free.*

Z will always denote the ring of integers. $\hat{Z}$ denotes the completion of Z with respect to its Z-adic topology.

LEMMA 2 [7]. *Let R be a ring with unit element. If the additive group of R is cotorsion-free, then $\{0\}$ is the only $\hat{Z}$-module contained in R.*

ZFC denotes the system of Zermelo–Fraenkel axioms and the Axiom of Choice.

LEMMA 3 [7] (ZFC+$V = L$). *Let $\aleph$ be a regular not weakly compact cardinal. Then there are $\aleph$ many disjoint subsets S_β of $\aleph$ for all $\beta < \aleph$ such that $\nabla_\aleph(S_\beta)$ holds for all $\beta < \aleph$ and $\bigcup\limits_{\beta<\aleph} S_\beta$ is sparse in $\aleph$.*

In the same way as the proofs of Step-Lemma 2.7 and Corollary 2.8 of [7] we can prove the following statement:

LEMMA 4. *Let R be a cotorsion-free ring and let M be an R-module. Let F_M be the free R-module with free basis $E_M = \{e_m | m \in M\}$ and let K be the kernel of the homomorphism $h : F_M \to M$ which takes each e_m to m. Let $F = \bigcup\limits_{n\in\omega} F_n$ be the union of free R-modules such that $F_M \subseteqq F_n$ and F_{n+1}/F_n is a free R-module for almost all $n \in \omega$. Let $b \in F$ be such that $\{b\} \cup E_M$ can be extended to a free basis of F_n for some $n \in \omega$. Assume that the rank of F_{n+1}/F_n is not smaller than the rank of F_n for almost all $n \in \omega$ and the rank of F is at least 2^ω.*

Then there exist two extensions F^0, F^1 of F such that the following conditions hold:

(i) *F^δ is a free R-module for $\delta = 0, 1$;*

(ii) *F^δ / F_n is a free R-module for almost all $n \in \omega$ for $\delta = 0, 1$;*

(iii) *if the endomorphism φ of F/K extends to endomorphisms of F^0/K and F^1/K, then $(b + K)\varphi \in (b + K)R$.*

PROOF. The proof is similar to that of Step-Lemma 2.7 in [7].

We may assume without loss of generality that

(1) $\{b\} \cup E_M$ extends to a free basis of F_0;
(2) F_{n+1}/F_n is a free R-module for every $n \in \omega$;
(3) the rank of F_0/bR is not smaller than 2^ω.

Let $E_{n+2} = \{e_{n+2,i} | i \in I_{n+2}\}$ be a free basis of the R-module F_{n+1}/F_n for every $n \in \omega$. We denote the set $\{m \in M | e_m \neq b\}$ by M'. Let $E_1 = \{e_{1i} | i \in I_1\}$ be a free basis of the free R-module F_0/bR such that $I_1 \supseteqq M'$ and $e_{1,m} = e_m$ for all $m \in M'$. Let $E_0 = \{e_{00} = b\}$. Assume without loss of generality that $I_0 = \{0\} \subseteqq I_n \subseteqq I_{n+1} \subseteqq \aleph$ for all $n \in \omega$. First, we will select special elements $z^\delta_{ni} \in Z$ for $n \in \omega$, $\delta = 0,1$, and $i \in I_n$.

Choose elements $z^1_{ni} \in Z$ such that $\left\{ \sum\limits_{n=2}^{\infty} (n-2)! z^1_{ni} | i \in I_1 \right\} = \hat{Z}$. This is possible because $|I_1| \geqq 2^\omega = |\hat{Z}|$. If $r \left\{ \sum\limits_{n=2}^{\infty} (n-2)!\, z^1_{ni} | i \in I_1 \right\} \subseteqq R$ for $r \in R$, then according to Lemma 2 we get that $r\hat{Z} = \{0\}$. So $r = 0$ because $1 \in \hat{Z}$. We shall refer to this result as the statement of reduction. Put $z^1_{ni} = 0$ for all $i \in I_n \setminus I_1$ and $z^0_{ni} = 0$ for all $i \in I_n$ and all $n \in \omega$.

Let ψ^δ be the R-endomorphism of F for $\delta = 0,1$ which acts on the set $E = \bigcup\limits_{n \in \omega} E_n$ as $e_{00}\psi^\delta = e_{00}$, $e_{1j}\psi^\delta = e_{1j}$, $e_{ni}\psi^\delta = e_{ni} - (n-1)e_{n+1,i} + e_{00}z^\delta_{ni}$, for all $j \in I_1$ and all $i \in I_n$ $(n \geqq 2)$. It is easy to see that ψ^0, ψ^1 are R-monomorphisms and $F/F_n\psi^0$, $F/F_n\psi^1$ are free R-modules for almost all $n \in \omega$. Let $F^0 = F^1 = F \oplus (\bigoplus\limits_{|F|} R)$ and let us identify F with $F\psi^0 \subseteqq F^0$, $F\psi^1 \subseteqq F^1$. Then (i) and (ii) indeed hold.

In order to prove (iii) let φ be an endomorphism of the group F/K such that φ extends to endomorphisms φ^δ of the groups F^δ/K for $\delta = 0,1$. Let

$$(e_{k\ell} + K)\varphi = \sum_{n \in \omega} \sum_{i \in I_n} e_{ni} a^{ni}_{k\ell} + K$$

and

$$(e_{k\ell} + K)\varphi^\delta = \sum_{n \in \omega} \sum_{i \in I_n} e_{ni} \delta^{ni}_{k\ell} + y_{k\ell} + K$$

be the images of the element $e_{k\ell} + K$ for $k \in \omega$ and $\ell \in I_k$. Here $y_{k\ell} \in \bigoplus\limits_{|F|} R$ and $a^{ni}_{k\ell}$, $\delta^{ni}_{k\ell} \in R$ such that $a^{ni}_{k\ell} = \delta^{ni}_{k\ell} = 0$ for fixed (k,ℓ) and almost all n, i. Then the images of $e_{k\ell} + K$ under $\varphi\psi^\delta = \psi^\delta\varphi^\delta$ for any $k \in w$, $\ell \in I_k$ and for $\delta = 0,1$ are deduced as

$$(e_{k\ell} + K)\varphi\psi^\delta = \left(\sum_{n \in \omega} \sum_{i \in I_n} e_{ni} a^{ni}_{k\ell} + K \right) \psi^\delta =$$

$$= e_{00}a_{k\ell}^{00} + \sum_{i\in I_1} e_{1i}a_{k\ell}^{1i} + \sum_{n=2}^{\infty}\sum_{i\in I_n}\left(e_{ni} - (n-1)e_{n+1,i} + e_{00}z_{ni}^{\delta}\right)a_{k\ell}^{ni} +$$

$$+K = e_{00}\left(a_{k\ell}^{00} + \sum_{n=2}^{\infty}\sum_{i\in I_n} a_{k\ell}^{ni}z_{ni}^{\delta}\right) + \sum_{n=1,2}\sum_{i\in I_n} e_{ni}a_{k\ell}^{ni} +$$

$$+\sum_{n=3}^{\infty}\sum_{i\in I_n} e_{ni}\left(a_{k\ell}^{ni} - (n-2)a_{k\ell}^{n-1,i}\right) + K$$

and

$$(e_{k\ell} + K)\varphi^{\delta}\psi^{\delta} = (e_{k\ell} - \max(k-1,0)\cdot e_{k+1,\ell} + \min((k-1)k,1)\cdot$$

$$\cdot e_{00}z_{k\ell}^{\delta} + K)\varphi^{\delta} = \sum_{n,i} e_{ni}(\delta_{k\ell}^{ni} - \max(k-1,0)\cdot\delta_{k+1,\ell}^{ni} +$$

$$+\min((k-1)k,1)z_{k\ell}^{\delta}\delta_{00}^{ni}) + (y_{k\ell} - \max(k-1,0)\cdot y_{k+1,\ell} +$$

$$+\min((k-1)k,1)z_{k\ell}^{\delta}y_{00}) + K.$$

Comparing these equalities we get

$$(*) \qquad \delta_{k\ell}^{ni} - \max(k-1,0)\cdot\delta_{k+1,\ell}^{ni} + \min((k-1)k,1)\cdot z_{k\ell}^{\delta}\delta_{00}^{ni} =$$

$$= a_{k\ell}^{ni} - \max(n-2,0)\cdot a_{k\ell}^{n-1,i} + t_{k\ell}^{ni}$$

for all $n \in \omega \setminus \{0\}$, $i \in I_n$. Here $t_{k\ell}^{ni} \in R$ such that $\sum_{n,i} e_{ni}t_{k\ell}^{ni} \in K$. If $k = 0$, then we have from $(*)$

$$(**) \qquad \delta_{00}^{ni} = a_{00}^{ni} - \max(n-2,0)\cdot a_{00}^{n-1,i} + t_{00}^{ni}.$$

Moreover, we gain from $(*)$ by induction $N \geqq 2$ the following equality for $n \in \omega \setminus \{0\}$, $i \in I_n$, $\ell \in I_1$:

$$\delta_{2\ell}^{ni} = \sum_{k=2}^{N}\left(a_{k\ell}^{ni} - \max(n-2,0)\cdot a_{k\ell}^{n-1,i} + t_{k\ell}^{ni}\right)(k-2)! -$$

$$-\sum_{k=2}^{N}\delta_{00}^{ni}z_{k\ell}^{\delta}(k-2)! + \delta_{N+1,\ell}^{ni}\cdot(N-1)!.$$

Since $\lim_{N\to\infty}\delta_{N+1,\ell}^{ni}(N-1)! = 0$ because $\bigcap_{n\in\omega\setminus\{0\}} nR = \{0\}$, we get

$$\delta_{2\ell}^{ni} = \sum_{k=2}^{\infty}\left(a_{k\ell}^{ni} - \max(n-2,0)\cdot a_{k\ell}^{n-1,i} + t_{k\ell}^{ni}\right)(k-2)! - \delta_{00}^{ni}\sum_{k=2}^{\infty}(k-2)!z_{k\ell}^{\delta}.$$

Therefore

$$1^{ni}_{00} \cdot \sum_{k=2}^{\infty} z^1_{k\ell}(k-2)! = 0^{ni}_{2\ell} - 1^{ni}_{2\ell} \in R$$

for all $\ell \in I_1$. According to the statement of reduction, this means that $1^{ni}_{00} = 0$ for all $n \in \omega \setminus \{0\}$ and all $i \in I_n$. So $(**)$ reduces to

$$a^{ni}_{00} = \max(n-2, 0) \cdot a^{n-1,i}_{00} - t^{ni}_{00}.$$

Since $\sum_{n,i} e_{ni} t^{ni}_{00} \in K$, we have $t^{ni}_{00} = 0$ if $n \geqq 2$ or if $n = 1$ and $i \notin M$. Consequently, $a^{ni}_{00} = 0$ if $n \geqq 2$ or if $n = 1$ and $i \notin M$. Furthermore $a^{1i}_{00} = -t^{1i}_{00}$ if $i \in M'$. Finally, we have

(a) if $b = e_{1m_0}$ where $m_0 \in M$, then

$$(b+K)\varphi = b\left(a^{00}_{00} + t^{1m_0}_{00}\right) + K;$$

(b) if $b \notin E_M$, then $(b+K)\varphi = ba^{00}_{00} + K$.

Thus Lemma 4 is proved.

THEOREM 1 (ZFC+$V = L$). *Let A be a cotorsion-free Abelian group and let R be a subring of the endomorphism ring of A containing the identity. Then there exists a cotorsion-free Abelian group B containing A as a fully invariant subgroup (i.e. mapped into itself by all endomorphisms of B) such that the mapping which takes each endomorphism of B to its restriction on A is an isomorphic mapping of the full endomorphism ring of B onto R. Moreover, our group B contains A as a pure subgroup.*

PROOF. The proof of this theorem is similar to that of Theorem 3.2 in [7].

Let M be the direct sum of A and the additive group of R. Then M is naturally a faithful R-module. We have proved a similar theorem as Theorem 1 for countable, reduced, torsion-free A (cf. [4]). So we can assume that A is not countable. Hence $|M| > \omega$. Let $\aleph$ be a regular, not weakly compact cardinal with $\aleph > |M|$. Let $\{S_\beta | \beta < \aleph\}$ be the family of subsets of $\aleph$ given by Lemma 3 and let $S = \bigcup_{\beta<\aleph} S_\beta$. Since $\mathrm{cf}(\lambda) = \omega$ for all $\lambda \in S$, we can fix an ascending sequence $\{\lambda_n | n \in \omega\}$ of ordinals such that $\lambda = \bigcup_{n\in\omega} \lambda_n$ and $\lambda_n \in \aleph \setminus S$. Let F_M denote the free R-module with free basis $E_M = \{e_m | m \in M\}$ and let K denote the kernel of the homomorphism $h : F_M \to M$ which takes each e_m to m. Then M is isomorphic to the factor module F_M/K.

Now we will define inductively the R-modules F_α for all $\alpha < \aleph$. Let $F_0 = F_M$. Assume F_β to be defined for all $\beta < \alpha < \aleph$. If α is a limit ordinal, then let $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$. Otherwise, if $\alpha = \beta + 1$, then let F_α be the direct sum $F_\beta \oplus (\bigoplus_{|F_\beta|} R)$ of F_β and $|F_\beta|$ many copies of R. We prove by transfinite induction on $\mu < \aleph$ that

(i) F_μ is a free R-module;

(ii) if $\eta \in \mu \setminus S$, then F_ν/F_η is a free R-module for all $\eta < \nu \leqq \mu$.

Indeed, $F_0 = F_M$ is a free R-module. Assume that (i) and (ii) hold for all $\mu < \alpha$.

Case 1: α is a limit ordinal. Then $F_\alpha = \bigcup_{\mu<\alpha} F_\mu$. Since S is a sparse subset of $\aleph$, there are ordinals $\alpha_\nu \in \alpha \setminus S$ for $\nu < \mathrm{cf}(\alpha)$ such that $F_\alpha = \bigcup_{\nu<\mathrm{cf}(\alpha)} F_{\alpha_\nu}$. Hence $F_{\alpha_\nu}/F_{\alpha_\mu}$ is a free R-module for all $\mu < \nu < \mathrm{cf}(\alpha)$. Thus F_α is a free R-module. If $\eta \in \alpha \setminus S$, then there is $\nu < \mathrm{cf}(\alpha)$ such that $\eta < \alpha_\nu$. So F_α/F_η is a free R-module, because F_α/F_{α_ν} and F_{α_ν}/F_η are free R-modules.

Case 2: $\alpha = \mu + 1$. Then $F_\alpha = F_\mu \oplus (\bigoplus_{|F_\mu|} R)$. Therefore (i) and (ii) hold trivially.

Assume that $\mu \in S$ and $e_{00}, \dot{e}_{00} \in F_\mu$ are such that $\{e_{00}\} \cup E_M$ and $\{\dot{e}_{00}\} \cup E_M$ extend, respectively, to free bases $E_1 = \{e_{1i} | i \in I_1\}$ and $\dot{E}_1 = \{\dot{e}_{1i} | i \in I_1\}$ of $F_{\mu_{n_0}}$ for some $n_0 \in \omega$. Let $E_{n+2} = \{e_{n+2,i} | i \in I_{n+2}\}$ and $\dot{E}_{n+2} = \{\dot{e}_{n+2,i} | i \in I_{n+2}\}$ be free bases of the free R-module $F_{\mu_{n_0+n+1}}/F_{\mu_{n_0+n}}$ for all $n \in \omega$. We may assume without loss of generality that $I_0 = \{0\} \subseteqq I_n \subseteqq I_{n+1} \subseteqq \aleph$ for all $n \in \omega$. Let us denote the sequences $\{e_{ni} | n \in \omega, i \in I_n\}$ and $\{\dot{e}_{ni} | n \in \omega, i \in I_n\}$ by $\vec{e}$ and $\dot{\vec{e}}$, respectively. Then $\dot{\vec{e}} = \vec{e}T$, where T is a matrix over R. We denote by $\mathbf{X}$ the set of all sequences $\{x_{ni} | n \in \omega, i \in I_n\}$, where $x_{ni} \in R$, and $x_{ni} = 0$ for almost all $n \in \omega$, $i \in I_n$. Then every $f \in F_\mu$ can be written in the forms $\vec{e}\,\vec{x}$ and $\dot{\vec{e}}\,\vec{y}$, where $\vec{x}, \vec{y} \in \mathbf{X}$. Let ψ^0 be the R-monomorphism which embeds F_μ in $F_\mu^0 = F^0(e_{00}, \cup F_{\mu_n})$ and let L be the matrix which represents ψ^0. Consider an arbitrary element $f = \vec{e}\,\vec{x}$ of F_μ. We denote the element $f(1 + (1-\psi^0) + \cdots + (1-\psi^0)^n)$ and the matrix $1 + (1-L) + \cdots + (1-L)^n$ by f_n and L_n, respectively. Then $f = \lim_{n\to\infty} f_n\psi^0$ holds with respect to the Z-adic topology of F_μ. Consider an endomorphism φ of F_μ/K which extends to an endomorphism φ^0 of F_μ^0/K. Then

$$(f+K)\varphi^0 = \lim_{n\to\infty}(f_n\psi^0 + K)\varphi^0 = \lim_{n\to\infty}(f_n + K)\varphi\psi^0.$$

Since $(f+K)\varphi^0 \in F_\mu^0/K$, we have $\lim_{n\to\infty}(f_n+K)\varphi \in F_\mu/K$ with respect to the Z-adic topology of F_μ/K. It is true that $\dot{\vec{e}}\,\vec{x} = \vec{e} \cdot T(\vec{x}) \cdot \vec{x}$ with a finite matrix $T(\vec{x})$ for all $\vec{x} \in \mathbf{X}$. It is easy to see that $T(\vec{x}) \cdot L_n = L_n \cdot T'(\vec{x})$ with a finite matrix $T'(\vec{x})$ for all $\vec{x} \in \mathbf{X}$ and for almost all $n \in \omega$. We get

$$\dot{\vec{e}}\, L_n \vec{x} = \vec{e} \cdot T(\vec{x}) \cdot L_n \cdot \vec{x} = \vec{e} \cdot L_n \cdot T'(\vec{x}) \cdot \vec{x}.$$

Thus

$$\lim_{n\to\infty} (\dot{\vec{e}}\, L_n \vec{x} + K)\varphi = \lim_{n\to\infty} (\vec{e}\, L_n T'(\vec{x})\vec{x} + K)\varphi \in F_\mu/K.$$

Therefore φ can be extended to an endomorphism of $\dot{F}^0_\mu/K$, too, where $\dot{F}^0_\mu = F^0(\dot{e}_{00}, \cup F_{\mu_n})$ which is given according to Lemma 4 by making use of the family $\{\dot{E}_n | n \in \omega\}$ of free bases.

Let $\{f_\nu | \nu < \aleph\}$ be an enumeration of the elements of $F = \bigcup_{\alpha<\aleph} F_\alpha$ such that $f_\beta \in F_\nu$ for all $\nu \in S_\beta$. Let X_α be the underlying set of F_α/K for all $\alpha < \aleph$ and let $X = \bigcup_{\alpha<\aleph} X_\alpha$. Then $\{X_\alpha | \alpha < \aleph\}$ is a $\aleph$-filtration of X. Consider the sets

$$A_\alpha = X_\alpha \times X_\alpha \times X_\alpha, \quad M_\alpha = X_\alpha \times R \times X_\alpha, \quad H_\alpha = X_\alpha \times X_\alpha$$

and their disjoint unions $U_\alpha = A_\alpha \cup M_\alpha \cup H_\alpha$ for all $\alpha < \aleph$. Then $\{U_\alpha | \alpha < \aleph\}$ is an $\aleph$-filtration of $U = \bigcup_{\alpha<\aleph} U_\alpha$. Let us define functions $P^\beta_\nu : \mathbf{P}(U_\nu) \to \{0,1\}$ for every $\beta < \aleph$, $\nu \in S_\beta$. Put $P^\beta_\nu(Y) = 0$ if the following conditions (1) and (2) are satisfied for the $Y \subseteqq U_\nu$:

(1) (a) $Y \cap (A_\nu \cup M_\nu)$ defines an R-module F'_ν/K on X_ν such that $F'_\nu \cong F_\nu$, and F'_ν, F_ν have the same underlying set;
    (b) $Y \cap (A_{\nu_n} \cup M_{\nu_n})$ defines an R-module F'_{ν_n}/K on X_{ν_n} for all $n \in \omega$ such that $F'_{\nu_n} \cong F_{\nu_n}$, and F'_{ν_n}, F_{ν_n} have the same underlaying set;
    (c) $F_M \subseteqq F'_{\nu_0} \subseteqq F'_{\nu_n} \subseteqq F'_{\nu_{n+1}} \subseteqq F'_\nu = \bigcup_{n\in\omega} F'_{\nu_n}$ for all $n \in \omega$; and the R-modules $f_\beta R$, F'_{ν_n}/F_M, $F'_{\nu_{n+1}}/F'_{\nu_n}$, $F'_{\nu_n}/f_\beta R$ are free for almost all $n \in \omega$;

(2) (a) $Y \cap H_\nu$ defines an endomorphism φ of F'_ν/K;
    (b) if $F^0_\nu = F^0(f_\beta, \cup F'_{\nu_n}) \supseteqq F'_\nu$ according to Lemma 4, then φ does not extend to an endomorphism of F^0_ν/K.

We define $P^\beta_\nu(Y) = 1$ in the other cases. So according to Lemma 3 we find that there are functions $\varphi_\beta : S_\beta \to \{0,1\}$ such that $\{\nu \in S_\beta | P^\beta_\nu(Y \cap U_\nu) = = \varphi_\beta(\nu)\}$ is stationary in $\aleph$ for all $Y \subseteqq U$ and all $\beta < \aleph$.

Now we will define inductively the free R-module F'_α on the underlying set of F_α for all $\alpha < \aleph$ such that

(i)' F'_α is isomorphic to F_α for all $\alpha < \aleph$;
(ii)' if $\eta \notin S$, then F'_α/F'_η is a free R-module for all $\eta < \alpha < \aleph$.

Choose $F'_0 = F_0$ and assume F'_μ to be defined for all $\mu < \alpha$. If α is a limit ordinal, then we choose $F'_\alpha = \bigcup_{\beta<\alpha} F'_\beta$. Assume that $\alpha = \mu + 1$ where $\mu \in S_\beta$ such that $f_\beta R$, $F'_{\mu_n}/f_\beta R$ are free R-modules for almost all $n \in \omega$. Then apply Lemma 4 for $F'_\mu = \bigcup_{n\in\omega} F'_{\mu_n}$ and put $F'_\alpha = F^{\varphi_\beta(\mu)}(f_\beta, \cup F'_{\mu_n})$. In the other cases we define $F'_\alpha = F'_\mu \oplus (\bigoplus_{|F'_\mu|} R)$, if $\alpha = \mu + 1$. Conditions (i)' and (ii)' can be proved similarly as (i) and (ii).

Let $F' = \bigcup_{\alpha<\aleph} F'_\alpha$ and $B = F'/K$. Then B contains A as a pure subgroup. The operations of distinct elements of R on B are distinct, because B contains the identity of R. In order to prove that R is isomorphic with the whole of the endomorphism ring of B, we consider an arbitrary endomorphism φ of B. Let

$$C = \{\nu < \aleph | \nu \text{ is a limit ordinal and } (F'_\nu/K)\varphi \subseteqq F'_\nu/K\}.$$

Then C is a closed unbounded subset of $\aleph$. We shall prove that φ operates on F'_μ/K as an element of R for every $\mu \in S \cap C$. Consider an element μ of $S \cap C$, e.g. $\mu \in S_\beta \cap C$. First, we assume that $f_\beta R$ and $F'_{\mu_n}/f_\beta R$ are free R-modules for almost all $n \in \omega$. Let $A_B \subseteqq X \times X \times X$ define the addition and let $M_B \subseteqq X \times R \times X$ define the ring multiplication on the R-module B. Let $H_\varphi \subseteqq X \times X$ define the endomorphism φ. Then

$$E = \{\nu \in S_\beta | P^\beta_\nu((A_B \cup M_B \cup H_\varphi) \cap U_\nu) = \varphi_\beta(\nu)\}$$

is stationary in $\aleph$. Let $\eta \in C \cap E$ with $\eta \geqq \mu$. Then there is an ordinal $\eta < \tau < \aleph$ such that $(F'_{\eta+1}/K)\varphi \subseteqq F'_\tau/K$. Since S contains only limit ordinals, we have $\eta+1 \in \aleph \setminus S$. So $F'_\tau/F'_{\eta+1}$ is a free R-module by (ii)'. Hence F'_τ splits over $F'_{\eta+1}$. Let $\rho : F'_\tau/K \to F'_{\eta+1}/K$ be the canonical projection such that $\rho|_{(F'_{\eta+1}/K)}$ acts on $F'_{\eta+1}/K$ as the identity operator. Hence $\varphi|_{(F'_{\eta+1}/K)} \circ \rho$ is an endomorphism from $F'_{\eta+1}/K$ which is an extension of $\varphi|_{(F'_\eta/K)}$. Therefore we derive $\varphi_\beta(\eta) = 1$ by the definition of the function P^β_η. Hence $\varphi|_{(F'_\eta/K)}$ extends to endomorphism of $F^0(f_\beta, \cup F'_{\eta_n})/K$ and $F^1(f_\beta, \cup F'_{\eta_n}/K)$. We have according to Lemma 4 that $(f_\beta + K)\varphi \in (f_\beta + K)R$.

Now let μ be an arbitrary element of $S \cap C$. Choose the free bases D_n of the free R-modules F'_{μ_n} such that $E_M \subseteqq D_n \subseteqq D_{n+1}$ for all $n \in \omega$. This is possible because $F'_{\mu_{n+1}}/F'_{\mu_n}$ and F'_{μ_n}/F_M are free R-modules for all $n \in \omega$. Let $b \in D = \bigcup_{n\in\omega} D_n$. Assume that $b = f_\alpha$ where $\alpha < \aleph$. Then there exists $\eta \in S_\alpha \cap C$ such that $F'_{\eta_n}/f_\alpha R$ is a free R-module for almost all $n \in \omega$. Hence $(b + K)\varphi = (b + K)r_b$ where $r_b \in R$. It is true that $F'_\mu/K \cong (F_M/K) \oplus \tilde{F}_\mu$ where $\tilde{F}_\mu$ is a free R-module. If x, y are distinct elements of a free basis of

$\tilde{F}_\mu$, then $xr - y$ is an element of a free basis of $\tilde{F}_\mu$ for every $r \in R$. Therefore we can easily prove that φ acts on $\tilde{F}_\mu$ as an element r_0 of R. Next we see that φ operates on F_M/K as an element of R. Let us denote briefly the element $b + K$ by $\overline{b}$. We get $\overline{e}_m\varphi = \overline{e}_m \cdot r_m \in \overline{e}_m R$ for all $m \in M$. Let a be an arbitrary element of A and let 1 be the identity of R. Then, partly $\overline{e}_{1+a}\varphi = \overline{e}_{1+a} \cdot r_{1+a}$ holds, partly $\overline{e}_{1+a}\varphi = \overline{e}_1\varphi + \overline{e}_a\varphi = \overline{e}_1 \cdot r_1 + \overline{e}_a \cdot r_a$ is satisfied. So $r_{1+a} + a \cdot r_{1+a} = r_1 + ar_a$. Hence $ar_1 = ar_a$, i.e. $\overline{e}_a\varphi = \overline{e}_a \cdot r_1$. We can see similarly that $\overline{e}_t\varphi = \overline{e}_t \cdot r_1$ for every $t \in R$ from the derivation of the image $\overline{e}_{t+at}\varphi$. It is clear that $r_1 = r_0$. So φ operates on F'_μ/K as the element $r_0 \in R$. This is true for all $\mu \in S \cap C$. Since $S \cap C$ is cofinal in $\aleph$, φ acts on B as an element of R.

Thus Theorem 1 is proved.

THEOREM 2 (ZFC+$V = L$). (The reformulation of Theorem 1 in terms of modules.) *Let R be a ring with unit element and let A be a module over R where the group A is cotorsion-free. Then A is embedded in an R-module B as a submodule such that every endomorphism of the group B is the operation of an element of R. Furthermore, A and B have the same annullator in R.*

PROOF. Let J be the annulator of A in R. Then A is naturally a faithful module over R/J. According to Theorem 1 there exists a faithful module B over R/J containing A as a submodule, and every endomorphism of the group B acts on B as an element of R/J. Thus B is naturally an R-module with the required properties.

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(Received April 28, 1988)

AGM RT
1081 BUDAPEST
VIII., KÖZTÁRSASÁG TÉR 20
HUNGARY

Acta Math. Hung.
59 (3–4) (1992), 263–272.

ON THE EXTENSION OF TWO THEOREMS BY MADDOX TO GENERALIZED SETS OF CESÀRO SUMMABLE SEQUENCES

E. MALKOWSKY (Gießen)

1. Introduction

We denote the set of all complex sequences by ω. As usual, c, c_0 and ℓ_∞ are the sets of all convergent, null and bounded sequences, respectively. For each positive integer n, $e^{(n)}$ is the sequence such that $e_k^{(n)} := 0$ $(k \neq n)$, $e_n^{(n)} := 1$, and e is the sequence such that $e_k := 1$ $(k = 1, 2, \dots)$.

For any real δ, let $A_n^\delta := \binom{n+\delta}{n}$ $(n = 0, 1, 2, \dots)$ be the n-th Cesàro coefficient of order δ. Let (m_ν) be a sequence of integers such that $1 \leqq m_0 < m_1 < \dots < m_\nu < \dots$, then for each non-negative integer ν, $K_{(m_\nu)}$ denotes the set of all integers k that satisfy the inequality $m_\nu \leqq k \leqq m_{\nu+1} - 1$.

Let $\alpha > 0$ and let $t = (t_k)$ be a real sequence such that $t_k \neq 0$ $(k = 1, 2, \dots)$, (m_ν) a sequence of integers as above. Then given any sequence x, we define the sequence $x(\alpha; (m_\nu), (t))$ by

$$x_k(\alpha; (m_\nu), (t)) := \left[\frac{A^{\alpha-1}_{m_{\nu+1}-k}}{A^{\alpha}_{m_\nu - 1}} \right]^{1/t_k} \cdot x_k \quad (k \in K_{(m_\nu)}; \quad \nu = 0, 1, 2, \dots),$$

and for each non-negative integer ν, and for each subset K of $K_{(m_\nu)}$, we define the sequence $x^{[K]}(\alpha; (t))$ by

$$x_k^{[K]}(\alpha; (t)) := \begin{cases} x_k(\alpha; (m_\nu), (t)) & k \in K \\ 0 & k \notin K; \end{cases}$$

we shall write $x^{[m_\nu]}(\alpha; (t)) := x^{[K(m_\nu)]}(\alpha; (t))$ for short.

If $t = (t_k)$ is a strictly positive sequence, and K_ν is a subset of $K_{(m_\nu)}$, then let

$$h^{\alpha,t}_{[K_\nu]}(x) := \sum_{k=1}^{\infty} \left| x_k^{[K_\nu]}(\alpha; (t)) \right|^{t_k} \quad (\nu = 0, 1, 2, \dots; \quad \alpha > 0; \quad x \in \omega);$$

and we shall write $h^{\alpha,t}_{[m_\nu]}(x) := h^{\alpha,t}_{[K_{(m_\nu)}]}(x)$ for short.

Let $\alpha > 0$, $p = (p_k)$ be a strictly positive sequence and (m_ν) a sequence of integers as above. Then we define the sets

$$[C_\alpha; (m_\nu)]^{(p)} :=$$
$$:= \left\{ x \in \omega : \left(h^{\alpha,p}_{[m_\nu]}(x - \ell e) \right) \in c_0 \text{ for some complex number } \ell \right\},$$

$$[C_\alpha;(m_\nu)]_0^{(p)} := \left\{x \in \omega : \left(h^{\alpha,p}_{[m_\nu]}(x)\right) \in c_0\right\}$$

and

$$[C_\alpha;(m_\nu)]_\infty^{(p)} := \left\{x \in \omega : \left(h^{\alpha,p}_{[m_\nu]}(x)\right) \in \ell_\infty\right\}.$$

In the special case where $p_k = p =$ const. $(k = 1,2,\dots)$ and $m_\nu = 2^\nu$ $(\nu = 0,1,2,\dots)$, these sets reduce to the sets $[\tilde{C}_\alpha]^p$, $[\tilde{C}_\alpha]_0^p$ and $[\tilde{C}_\alpha]_\infty^p$ defined in [5].

PROPOSITION 1. *The sets* $[C_\alpha;(m_\nu)]^{(p)}$, $[C_\alpha;(m_\nu)]_0^{(p)}$ *and* $[C_\alpha;(m_\nu)]_\infty^{(p)}$ *are linear spaces if and only if* $p \in \ell_\infty$.

PROOF. For the sets $[C_\alpha;(m_\nu)]^{(p)}$ and $[C_\alpha;(m_\nu)]_0^{(p)}$ this is an immediate consequence of Theorem 1 in [4], and for the set $[C_\alpha;(m_\nu)]_\infty^{(p)}$ this is an immediate consequence of Theorem 1 in [2].

Therefore, we shall assume $p \in \ell_\infty$. We put

$$M := \sup_k p_k, \quad H := \max\{1, M\}. \tag{1.1}$$

PROPOSITION 2. *The following inclusions hold:*

(a) $[C_\alpha;(m_\nu)]_0^{(p)} \subset [C_\alpha;(m_\nu)]^{(p)}$, $[C_\alpha;(m_\nu)]_0^{(p)} \subset [C_\alpha;(m_\nu)]_\infty^{(p)}$;

(b) $[C_\alpha;(m_\nu)]^{(p)} \subset [C_\alpha;(m_\nu)]_\infty^{(p)}$ *if and only if there is a constant* γ *such that*

$$\frac{m_{\nu+1}}{m_\nu} < \gamma \quad (\nu = 0,1,2,\dots). \tag{1.2}$$

PROOF. (a) The inclusions in part (a) are obvious from the definition of the sets involved (even without the restriction $p \in \ell_\infty$).

(b) By a well known property of the Cesàro coefficients, there exist two constants $C_1(\alpha)$, $C_2(\alpha)$ depending on α only such that for $\nu = 0,1,2,\dots$

$$C_1(\alpha)\left[\frac{m_{\nu+1}-m_\nu}{m_\nu}\right]^\alpha - \frac{1}{A^\alpha_{m_\nu-1}} \leqq h^{\alpha,p}_{[m_\nu]}(e) =$$

$$= \frac{A^\alpha_{m_{\nu+1}-m_\nu}}{A^\alpha_{m_\nu-1}} - \frac{1}{A^\alpha_{m_\nu-1}} \leqq C_2(\alpha)\left[\frac{m_{\nu+1}-m_\nu}{m_\nu}\right]^\alpha - \frac{1}{A^\alpha_{m_\nu-1}}$$

and since $\lim_{\nu\to\infty} \frac{1}{A^\alpha_{m_\nu-1}} = 0$ for $\alpha > 0$, we have $\sup_\nu h^{\alpha,p}_{[m_\nu]}(e) < \infty$ if and only if condition (1.2) holds. Part (b) now is obvious (cf. [2], p. 318).

Let (m_ν) be a sequence of integers satisfying (1.2). Then we define

$$g(x) := g^{\alpha,p}(x) := \sup_\nu \left(h^{\alpha,p}_{[m_\nu]}(x)\right)^{1/H} \quad \text{for} \quad x \in [C_\alpha;(m_\nu)]_\infty^{(p)} \tag{1.3}$$

(where H is defined in (1.1)).

REMARK. In view of Proposition 2, the restriction on the sequence (m_ν) is imposed in order to make sure that g is defined on all of $[C_\alpha;(m_\nu)]^{(p)}$.

The function g induces a metric on each of the spaces $[C_\alpha;(m_\nu)]_0^{(p)}$, $[C_\alpha;(m_\nu)]^{(p)}$ and $[C_\alpha;(m_\nu)]_\infty^{(p)}$ (cf. [2], p. 318) and it is known that $[C_\alpha;(m_\nu)]_0^{(p)}$ is paranormed by g (cf. [3], Theorem 18, p. 190).

The proof of the following is routine and therefore omitted.

PROPOSITION 3. *Let $p\in\ell_\infty$ and let (m_ν) be a sequence of integers satisfying condition* (1.2). *Then*

(a) $[C_\alpha;(m_\nu)]_\infty^{(p)}$ *is paranormed by g if and only if* $\inf\limits_k p_k>0$,

(b) $[C_\alpha;(m_\nu)]^{(p)}$ *is paranormed by g if* $\inf\limits_k p_k>0$,

(c) $[C_\alpha;(m_\nu)]_0^{(p)}$ *and* $[C_\alpha;(m_\nu)]_\infty^{(p)}$ *are complete with respect to g* (cf. [2], p. 318).

2. The completeness of $[C_\alpha;(m_\nu)]^{(p)}$

We now turn to the completeness of $[C_\alpha;(m_\nu)]^{(p)}$ which is more difficult to handle. The following is an immediate consequence of Theorem 5 in [2].

PROPOSITION 4. *Let $p\in\ell_\infty$ and* $\lim\limits_{\nu\to\infty}\frac{m_{\nu+1}}{m_\nu}=1$. *Then $[C_\alpha;(m_\nu)]^{(p)}$ is complete with respect to g.*

We now consider the case in which $\overline{\lim\limits_{\nu\to\infty}}\frac{m_{\nu+1}}{m_\nu}>1$. We need the following

LEMMA 1. *Let $p\in\ell_\infty$ and let (m_ν) be a sequence of integers such that* (1.2) *and*

$$\overline{\lim_{\nu\to\infty}}\frac{m_{\nu+1}}{m_\nu}>1 \tag{2.1}$$

hold. If $(x^{(i)})$ is a Cauchy sequence in $[C_\alpha;(m_\nu)]^{(p)}$ and

$$\left[h^{\alpha,p}_{[m_\nu]}\left(x^{(i)}-\ell^{(i)}e\right)\right]\in c_0\quad \textit{for}\quad i=1,2,\dots,$$

then given $\varepsilon>0$ there is a complex number ℓ and an integer i_0 such that for all $i\geqq i_0$, we have $|\ell^{(i)}-\ell|<\min\{1,\varepsilon\}$.

PROOF. In the proof, we shall write $h_\nu:=h^{\alpha,p}_{[m_\nu]}$ for short. Let $c:=$ $:=\overline{\lim\limits_{\nu\to\infty}}\frac{m_{\nu+1}}{m_\nu}>1$. Then there is a number δ satisfying $1<\delta<c$ such that for some subsequence $(\nu(n))$, we have

$$\frac{m_{\nu(n)+1}}{m_{\nu(n)}}>\delta\quad(n=1,2,\dots).$$

This and a well known property of the Cesàro coefficients yield the existence of a constant $C_1(\alpha)$ depending on α only such that

$$C_1(\alpha)(\delta-1)^\alpha - \frac{1}{A^\alpha_{m_{\nu(n)}-1}} \leqq h_{\nu(n)}(e) \quad (n=1,2,\dots). \tag{2.2}$$

Furthermore, similarly it follows from condition (1.2) that

$$h_\nu(e) \leqq C_2(\alpha)(\gamma-1)^\alpha - \frac{1}{A^\alpha_{m_\nu-1}} \quad (\nu=0,1,2,\dots) \tag{2.3}$$

where $C_2(\alpha)$ depends on α only.

We put $C_1 := C_1(\alpha)(\delta-1)^\alpha$, $C_2 := C_2(\alpha)(\gamma-1)^\alpha$. Let $(x^{(i)})$ be a Cauchy sequence in $[C_\alpha;(m_\nu)]^{(p)}$. Then there is a non-negative integer I_0 such that

$$g\left(x^{(i)}-x^{(j)}\right) < \frac{1}{3}\left[\frac{C_1}{2}\right]^{1/H} \quad (i,j \geqq I_0).$$

Let $i,j \geqq I_0$. Since $h_\nu(x^{(i)}-\ell^{(i)}e) \to 0$, $h_\nu(x^{(i)}-\ell^{(j)}e) \to 0$ $(\nu\to\infty)$ there exists a non-negative integer $\nu_0(i,j)$ such that

$$h_\nu\left(x^{(i)}-\ell^{(i)}e\right) < \frac{C_1}{3^H\cdot 2},\quad h_\nu\left(x^{(j)}-\ell^{(j)}e\right) < \frac{C_1}{3^H\cdot 2} \quad (\nu \geqq \nu_0(i,j)).$$

Then for all $\nu \geqq \nu_0(i,j)$, we have

$$\left[h_\nu\left(\left(\ell^{(i)}-\ell^{(j)}\right)e\right)\right]^{1/H} \leqq \left[h_\nu\left(x^{(i)}-\ell^{(i)}e\right)\right]^{1/H} + \tag{2.4}$$

$$+\left[h_\nu\left(x^{(i)}-x^{(j)}\right)\right]^{1/H} + \left[h_\nu\left(x^{(j)}-\ell^{(j)}e\right)\right]^{1/H} < \left[\frac{C_1}{2}\right]^{1/H}.$$

We must have $|\ell^{(i)}-\ell^{(j)}|<1$ for all $i,j \geqq I_0$. For otherwise, if $|\ell^{(i)}-\ell^{(j)}| \geqq 1$ for some $i,j \geqq I_0$, then by (2.2), the definition of C_1 and the fact that $\frac{1}{A^\alpha_{m_\nu-1}} \to 0$ $(\nu\to\infty)$, we would be able to choose an integer $\nu(n) \geqq \nu_0(i,j)$ such that

$$h_{\nu(n)}(e) > \frac{3}{4}C_1.$$

This would imply

$$h_{\nu(n)}\left[\left(\ell^{(i)}-\ell^{(j)}\right)e\right] \geqq h_{\nu(n)}(e) > \frac{3}{4}C_1 > \frac{1}{2}C_1,$$

a contradiction to (2.4). Therefore, we must have $|\ell^{(i)}-\ell^{(j)}|<1$ for all $i,j \geqq I_0$, and it follows that

(2.5)

$$\left|\ell^{(i)}-\ell^{(j)}\right|^M h_\nu(e) \leqq h_\nu\left(\left(\ell^{(i)}-\ell^{(j)}\right)e\right) \quad (i,j \geqq I_0;\quad \nu=0,1,2,\dots).$$

Let $\varepsilon > 0$ be given. By the same argument that yielded (2.4) using the fact that $(x^{(i)})$ is a Cauchy sequence in $[C_\alpha; (m_\nu)]^{(p)}$, we can choose a non-negative integer $i_0 := i_0(\varepsilon) > I_0$ such that for all $i, j \geqq i_0$, there exists a non-negative integer $\nu(i,j)$ such that for all $\nu \geqq \nu(i,j)$

$$h_\nu\left(\left(\ell^{(i)} - \ell^{(j)}\right) e\right) < \varepsilon^M \cdot \frac{3}{4} C_1.$$

Therefore by (2.5)

$$\left|\ell^{(i)} - \ell^{(j)}\right|^M \cdot h_\nu(e) < \varepsilon^M \cdot \frac{3}{4} C_1.$$

By (2.2) we can choose an integer $\nu(n) \geqq \nu(i,j)$ such that

$$h_{\nu(n)}(e) > \frac{3}{4} C_1.$$

It follows that

$$\left|\ell^{(i)} - \ell^{(j)}\right|^M < \varepsilon^M \quad \text{and} \quad \left|\ell^{(i)} - \ell^{(j)}\right| < \varepsilon \quad (i, j \geqq i_0).$$

Hence $(\ell^{(i)})$ is a Cauchy sequence of complex numbers and $\ell := \lim_{i\to\infty} \ell^{(i)}$ exists. This implies the existence of an integer i_0 such that for all $i \geqq i_0$

$$\left|\ell^{(i)} - \ell\right| < \min\{1, \varepsilon\}.$$

THEOREM 1. *Let $p \in \ell_\infty$ and let the sequence (m_ν) satisfy conditions* (1.2) *and* (2.1). *Then $[C_\alpha; (m_\nu)]^{(p)}$ is complete with respect to g.*

PROOF. Let $(x^{(i)})$ be a Cauchy sequence in $[C_\alpha; (m_\nu)]^{(p)}$. Then

$$g\left(x^{(i)} - x^{(j)}\right) \to 0 \quad (i, j \to \infty)$$

and for each integer i, there is a complex number $\ell^{(i)}$ such that

$$h_\nu\left(x^{(i)} - \ell^{(i)} e\right) \to 0 \quad (\nu \to \infty).$$

(Again we write $h_\nu := h^{\alpha,p}_{[m_\nu]}$ for short.) Since $[C_\alpha; (m_\nu)]^{(p)} \subset [C_\alpha; (m_\nu)]^{(p)}_\infty$ by Proposition 2, and since $[C_\alpha; (m_\nu)]^{(p)}_\infty$ is complete by Proposition 3(c), there is an element $x \in [C_\alpha; (m_\nu)]^{(p)}_\infty$ such that

$$g\left(x^{(i)} - x\right) \to 0 \quad (i \to \infty).$$

We have to show that $x \in [C_\alpha; (m_\nu)]^{(p)}$, i.e. that there is a complex number ℓ such that

$$h_\nu(x - \ell e) \to 0 \quad (\nu \to \infty). \tag{2.6}$$

Since

$$(h_\nu(x - \ell e))^{1/H} \leqq \left(h_\nu\left(x - x^{(i)}\right)\right)^{1/H} + \left(h_\nu\left(x^{(i)} - \ell^{(i)} e\right)\right)^{1/H} +$$
$$+ \left(h_\nu\left(\left(\ell^{(i)} - \ell\right) e\right)\right)^{1/H},$$

condition (2.6) can be proved by showing that

$$h_\nu\left(\left(\ell^{(i)} - \ell\right) e\right) \to 0 \quad (\nu \to \infty) \text{ for all sufficiently large } i.$$

Let $y > 0$, and define for $\nu = 0, 1, 2, \ldots,$

$$K_\nu(y) := \left\{k \in K_{(m_\nu)} : p_k < y\right\}, \quad h_{\nu,y} := h^{\alpha,p}_{[K_\nu(y)]}(e).$$

Then two cases are possible:

(i) $\inf\limits_{y>0} \overline{\lim\limits_{\nu\to\infty}}\, h_{\nu,y} = 0,$

(ii) $\inf\limits_{y>0} \overline{\lim\limits_{\nu\to\infty}}\, h_{\nu,y} > 0,$

In case (i), let $\varepsilon > 0$ be given. Then there exists a number $y_0 > 0$ such that

$$\overline{\lim_{\nu\to\infty}}\, h_{\nu,y_0} < \frac{\varepsilon}{2}$$

whence $h_{\nu,y_0} < \varepsilon$ for all sufficiently large ν. By Lemma 1, there is a complex number ℓ and a non-negative integer i_0, such that for all $i \geqq i_0$

$$\left|\ell - \ell^{(i)}\right| < \min\left\{1, \varepsilon^{1/y_0}\right\}.$$

Now for all sufficiently large ν and for all $i \geqq i_0$, we have by (2.3):

$$h_\nu\left(\left(\ell - \ell^{(i)}\right) e\right) = h^{\alpha,p}_{[K_\nu(y_0)]}\left(\left(\ell - \ell^{(i)}\right) e\right) + h^{\alpha,p}_{[K_{(m_\nu)} \setminus K_\nu(y_0)]}\left(\left(\ell - \ell^{(i)}\right) e\right) <$$
$$< h_{\nu,y_0} + \varepsilon h_\nu(e) \leqq \varepsilon(1 + C_2),$$

where C_2 is defined as in the proof of Lemma 1.

Now we deal with case (ii). We put

$$c := \frac{1}{2} \inf_{y>0} \overline{\lim_{\nu\to\infty}}\, h_{\nu,y} > 0.$$

Then there is an integer $\nu(1)$ such that $h_{\nu(1),1} > c$. Also there exists an integer $\nu(2) > \nu(1)$ such that $h_{\nu(2),1/2} > c$. Continuing in this way we can determine a strictly increasing sequence $(\nu(s))$ of integers such that

$$h_{\nu(s),1/s} > c.$$

By the same argument that led to (2.4), there exists an integer $I := I(c)$ such that for all $i > I$ there exists an integer ν_i such that for all $\nu \geqq \nu_i$

$$h_\nu\left(\left(\ell^{(i)} - \ell^{(I)}\right)e\right) < \frac{c}{2}. \tag{2.7}$$

Now we must have $\ell^{(i)} = \ell^{(I)}$ for every $i > I$. For otherwise, if $|\ell^{(i)} - \ell^{(I)}| > 0$ for some $i > I$, then for $\nu = \nu(s)$, we would have, since $|\ell^{(i)} - \ell^{(I)}| < 1$,

$$h_{\nu(s)}\left(\left(\ell^{(i)} - \ell^{(I)}\right)e\right) \geqq h_{\nu(s),1/s}\cdot\left|\ell^{(i)} - \ell^{(I)}\right|^{1/s} > \frac{c}{2}$$

for all sufficiently large s, a contradiction to (2.7).

We close this section with a remark on the uniqueness of the limit ℓ.

REMARK. If the sequence (m_ν) satisfies condition (2.1), then for every $x \in [C_\alpha;(m_\nu)]^{(p)}$ the limit ℓ is unique (cf. [1], Theorem 2).

3. An inclusion theorem

In this section, we shall discuss the inclusion

$$[C_a;(m_\nu)]_0^{(p)} \subset [C_\alpha;(m_\nu)]_0^{(e)}.$$

Here there is no restriction on the sequence p except positivity. The following theorem is a generalization of Theorem 7 in [2].

THEOREM 2. *Let the sequence (m_ν) satisfy condition* (1.2). *Then the inclusion $[C_a;(m_\nu)]_0^{(p)} \subset [C_\alpha;(m_\nu)]_0^{(e)}$ holds if and only if the following two conditions are satisfied:*

(i) $H_\nu(N) := \max\limits_{k\in K(m_\nu)} \left[N^{-1/p_k} e_k(\alpha;(m_\nu),(e-p))\right]$ *for some integer* $N > 1$, *and*

(ii) $\inf\limits_{y>1} \overline{\lim\limits_{\nu\to\infty}}\, h_{\nu,y} = 0$ *where* $h_{\nu,y} := h^{\alpha,p}_{[K_\nu(y)]}(e)$ *and* $K_\nu(y) := \{k \in K_{(m_\nu)}: p_k \geqq y\}$.

PROOF. (a) First we show the necessity of condition (i). We assume that for all integers $N > 1$, $H_\nu(N) \neq O(1)$. For $N := 2$, there is an integer $\nu(2)$ such that $H_{\nu(2)}(2) > 2$. For $N := 3$, there is an integer $\nu(3) > \nu(2)$ such that $H_{\nu(3)}(3) > 3$. Continuing in this way we can determine a strictly increasing sequence $(\nu(N))$ of integers such that $H_{\nu(N)}(N) > N$. For $N = 2, 3, \ldots$, let $k(N) \in K_{(m_{\nu(N)})}$ be the smallest integer such that

$$N^{-1/p_{k(N)}} e_{k(N)}(\alpha; (m_\nu), (e-p)) = H_{\nu(N)}(N).$$

Define the sequence x by

$$x_k := \begin{cases} N^{-1/p_{k(N)}} \cdot e_{k(N)}(\alpha; (m_\nu), (-p)) & (k = k(N)) \\ 0 & (k \neq k(N)) \end{cases} \quad (N = 2, 3, \ldots).$$

Then obviously $x \in [C_\alpha; (m_\nu)]_0^{(p)} \setminus [C_\alpha; (m_\nu)]_0^{(e)}$.

(b) Now we prove the necessity of condition (ii). We assume that

$$c := \inf_{y>1} \overline{\lim_{\nu\to\infty}} h_{\nu,y} > 0.$$

Then for all $y > 1$, we have $\overline{\lim\limits_{\nu\to\infty}} h_{\nu,y} \geqq c$. For $y := 2$, there is an integer $\nu(2)$ such that $h_{\nu(2),2} \geqq c/2$. For $y := 3$, there is an integer $\nu(3) > \nu(2)$ such that $h_{\nu(3),3} \geqq c/2$. Continuing in this way we can determine a strictly increasing sequence $(\nu(s))$ of integers such that $h_{\nu(s),s} \geqq c/2$. Define the sequence x by

$$x_k := \begin{cases} \frac{1}{2}\left(h_{\nu(s),s}\right)^{-1/p_k} & (k \in K_{\nu(s)}(s)) \\ 0 & (k \notin K_{\nu(s)}(s)) \end{cases} \quad (s = 2, 3, \ldots).$$

Then

$$h_{\nu(s)}(x) \leqq (1/2)^s \to 0 \quad (s \to \infty)$$

and

$$h^{\alpha,e}_{[m_{\nu(s)}]}(x) = \frac{1}{2} \sum_{k\in K_{\nu(s)}(s)} e_k(\alpha; (m_\nu), (e)) \left(h_{\nu(s),s}\right)^{-1/p_k}.$$

If $h_{\nu(s),s} \leqq 1$, then $\left(h_{\nu(s),s}\right)^{-1/p_k} \geqq 1$, hence $h^{\alpha,e}_{[m_{\nu(s)}]}(x) \geqq \frac{1}{2} h_{\nu(s),s} \geqq c/4$.

If $h_{\nu(s),s} > 1$, then $\left(h_{\nu(s),s}\right)^{-1/p_k} \geqq \left(h_{\nu(s)}\right)^{-1/s}$, hence

$$h^{\alpha,e}_{[m_{\nu(s)}]}(x) \geqq \frac{1}{2}\left(h_{\nu(s),s}\right)^{1-1/s} \geqq \frac{1}{2}\left(\frac{c}{2}\right)^{1-1/s} \to \frac{1}{4}c \quad (s \to \infty).$$

Therefore we have $x \in [C_\alpha; (m_\nu)]_0^{(p)} \setminus [C_\alpha; (m_\nu)]_0^{(e)}$.

(c) Finally we prove the sufficiency of conditions (i) and (ii). Let $x \in$ $\in [C_\alpha; (m_\nu)]_0^{(p)}$. Then there is an integer $\nu(N)$ such that for all $\nu \geqq \nu(N)$

$$h_\nu(x) < 1/N \tag{3.1}$$

(where N is the same as in condition (i)). Let $\varepsilon > 0$ be given. Then, by condition (ii), there exists a $y_0 > 1$ such that $h_{\nu,y_0} < \varepsilon$ for all sufficiently large ν. For $\nu = 0, 1, 2, \ldots$, we define the following sets:

$$K_\nu^{(1)} := \left\{k \in K_{(m_\nu)} : p_k < 1\right\}, \qquad K_\nu^{(2)} := \left\{k \in K_{(m_\nu)} : p_k \geqq 1\right\},$$

$$K_\nu^{(3)} := \left\{k \in K_\nu^{(2)} : |x_k| \leqq 1\right\}, \qquad K_\nu^{(4)} := \left\{k \in K_\nu^{(2)} : |x_k| > 1\right\},$$

$$K_\nu^{(5)} := \left\{k \in K_\nu^{(3)} : 1 \leqq p_k \leqq y_0\right\}, \quad K_\nu^{(6)} := \left\{k \in K_\nu^{(3)} : p_k > y_0\right\},$$

and we put $h_\nu^{(r)} := h^{\alpha,p}_{[K_\nu^{(r)}]}$, $h_\nu^{(r),e} := h^{\alpha,e}_{[K_\nu^{(r)}]}$ $(r = 1, \ldots, 6)$. For $k \in K_\nu^{(1)}$, it follows from (3.1) that

$$N^{1/p_k} \left|x_k(\alpha; (m_\nu), (p))\right| < 1,$$

and since $p_k < 1$ for $k \in K_\nu^{(1)}$, we have

$$N^{1/p_k} \left|x_k(\alpha; (m_\nu), (p))\right| < N \left|x_k(\alpha; (m_\nu), (p))\right|^{p_k},$$

$$|x_k| \leqq N^{1-1/p_k} e_k(\alpha; (m_\nu), (e-p))|x_k|^{p_k}.$$

This implies

$$h_\nu^{(1),e}(x) \leqq N \sum_{k \in K_\nu^{(1)}} e_k(\alpha; (m_\nu), (2e-p)) \cdot N^{-1/p_k} |x_k|^{p_k} \leqq$$

$$\leqq N H_\nu(N) h_\nu(x) = o(1)O(1).$$

We must show that

$$h^{(2),e}(x) = o(1). \tag{3.2}$$

Let $k \in K_\nu^{(4)}$. Then $p_k \geqq 1$, $|x_k| > 1$ and

$$h_\nu^{(4),e}(x) \leqq h_\nu^{(4)}(x) \leqq h_\nu(x) = o(1). \tag{3.3}$$

Now let $k \in K_\nu^{(6)}$. Then $p_k > y_0$, $|x_k| \leqq 1$ and

$$h_\nu^{(6),e}(x) \leqq h_\nu^{(6),e}(e) \leqq h_{\nu,y_0} < \varepsilon \tag{3.4}$$

for all sufficiently large ν.

Finally let $k \in K_\nu^{(5)}$. Then $1 \leqq p_k \leqq y_0$, $|x_k| \leqq 1$ and by a result by Maddox (cf. [1], p. 351)

$$h_\nu^{(5),e}(x) \leqq h_\nu(x) + (h_\nu(x))^{1/y_0} \left[\sup_\nu h_\nu(e)\right]^{1-1/y_0}.$$

It follows from condition (1.2) that

$$C := \left[\sup_{\nu} h_{\nu}(e)\right]^{1-1/y_0} < \infty.$$

Since $x \in [C_\alpha; (m_\nu)]^{(p)}$, we have

$$h_{\nu}^{(5),e}(x) \leqq \varepsilon + \varepsilon^{1/y_0} \cdot C \quad \text{for all sufficiently large} \quad \nu. \tag{3.5}$$

Since

$$h_{\nu}^{(2),e}(x) = h_{\nu}^{(4),e}(x) + h_{\nu}^{(5),e}(x) + h_{\nu}^{(6),e}(x),$$

condition (3.2) follows from (3.3), (3.4) and (3.5).

Acknowledgement. The major part of this paper was written while I served a term at the University of Wisconsin, Milwaukee, and I wish to express my gratitude to everybody who supported me in my work.

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(Received May 6, 1988)

MATHEMATISCHES INSTITUT
JUSTUS-LIEBIG-UNIVERSITÄT
D–6300 GIESSEN
IHERINGSTR. 6
GERMANY

Acta Math. Hung.
59 (3–4) (1992), 273–281.

CONVERGENCE AND LATTICE PROPERTIES OF A CLASS OF MARTINGALE-LIKE SEQUENCES

D. Q. LUU (Hanoi)

0. Introduction

Let E be a Banach space, N the set of all positive integers. Given an infinite subset S of N we shall introduce the class of E-valued L^1S-games which is different from the class of E-valued games fairer with time, given first by Blake [3] and studied later by Mucci [12]. It is shown in Section 2 that a sequence (X_n) of E-valued Bochner integrable random variables is an L^1S-game if and only if it has a unique Riesz decomposition: $X_n = M_n + P_n$, where (M_n) is a martingale, (P_n) converges to zero in probability and the subsequence (P_s) is an L^1-potential. Thus, all L^1-bounded E-valued L^1S-games convergence in probability if (and only if) E has the Radon–Nikodym property (RNP). This extends a result of Subramanian [15] on uniformly integrable real-valued games fairer with time and parallels a result of Talagrand [17] on L^1-bounded E-valued mils. In the case, when E lacks the (RNP) we give also a necessary and sufficient condition under which an L^1-bounded E-valued L^1S-game converges in probability. In Section 3, we shall use the characterization result of the previous section to prove that endowed with the norm

$$\|(X_n)\|_N = \sup_N E(\|X_n\|),$$

the class of L^1-bounded L^1S-games in ℓ^1 becomes a Banach lattice. This parallels similar results of Austin–Edgar–Tulcea [1] on L^1-bounded real-valued amarts, Ghoussoub [6] on L^1-bounded order amarts, Talagrand [17] on L^1-bounded real-valued pramarts and a most recent result of Schmidt (to appear in Ann. Probab.) on L^1-bounded uniform amarts in ℓ^1. For definitions and related results we refer to the next section.

1. Definitions and related results

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $(\mathcal{A}_n)$ an increasing sequence of sub-σ-algebras of the σ-algebra $\mathcal{A}$. A function $\tau : \Omega \to N$ is called a bounded stopping time, write $\tau \in T$, if each $\{\tau = n\} \in \mathcal{A}_n$ and τ is bounded. Thus, T is a directed set with the usual order ($\leq$), given by $\sigma \leqq \tau$ iff $\sigma(\omega) \leqq \tau(\omega)$, a.s. Now, let E be a (real) separable Banach space and let L^1_E stand for the Banach space of all (equivalence classes of) $\mathcal{A}$-measurable functions $X : \Omega \to E$ with $E(\|X\|) = \int\limits_{\Omega} \|X\| dP < \infty$. Unless otherwise stated, we shall consider only sequences (X_n) in L^1_E which are assumed to be adapted to $(\mathcal{A}_n)$, i.e., each X_n is $\mathcal{A}_n$-measurable. Now given a sequence (X_n) and $\tau \in T$ we define X_τ and $\mathcal{A}_\tau$ by $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, $\mathcal{A}_\tau = \{A \in \mathcal{A} : \{\tau = n\} \cap A \in \mathcal{A}_n, \forall n\}$. Then $(\mathcal{A}_\tau)$ is an increasing family of sub-σ-algebras of $\mathcal{A}$ and each X_τ is $\mathcal{A}_\tau$-measurable.

DEFINITION 1.1 (cf. [3]). A sequence (X_n) is said to be a game which becomes fairer with time iff for each $\varepsilon > 0$ there exists p such that for all $n \geqq q \geqq p$ we have $P(\|X_q(n) - X_q\| > \varepsilon) < \varepsilon$, where given $\sigma, \tau \in T$, $X_\sigma(\tau)$ denotes the $\mathcal{A}_\sigma$-conditional expectation of X_τ (cf. [13]).

The following result was (independently) proved by Mucci [12] and Subramanian [15].

THEOREM 1.1. *Every uniformly integrable real-valued game which becomes fairer with time converges in* L^1.

DEFINITION 1.2 (cf. [2]). A sequence (X_n) is said to be a uniform amart iff for each $\varepsilon > 0$ there exists p such that for all $\sigma, \tau \in T$ with $\tau \geqq \sigma \geqq p$ we have $E(\|X_\sigma(\tau) - X_\sigma\|) < \varepsilon$.

Uniform amarts were introduced by Bellow [2] who proved that all L^1-bounded E-valued uniform amarts converge a.s. if (and only if) E has the (RNP). But here, we are interested in the following recent result of Schmidt [14].

THEOREM 1.2. *The* L^1*-bounded uniform amarts in* ℓ^1 *form a Banach lattice.*

DEFINITION 1.3 (cf. [11]). A sequence (X_n) is said to be a pramart if for each $\varepsilon > 0$ there exists p such that for all $\sigma, \tau \in T$ with $\tau \geqq \sigma \geqq p$ we have $P(\|X_\sigma(\tau) - X_\sigma\| > \varepsilon)$.

Pramarts have been introduced and extensively studied by Millet–Sucheston [11] who proved that real-valued pramarts have the optimal sampling property. Here, we are interested however in the following result due to Talagrand [17].

THEOREM 1.3. *The* L^1*-bounded real-valued pramarts from a vector lattice.*

DEFINITION 1.4 (cf. [17]). A sequence (X_n) is said to be a mil if for each $\varepsilon > 0$ there exists p such that for all $n \geqq p$ one has

$$P\Big(\sup_{p \leqq q \leqq n} \|X_q(n) - X_q\| > \varepsilon\Big) < \varepsilon.$$

Since every pramart is a mil, the following famous result of Talagrand [17] gives in particular, an affirmative answer to a question posed by L.

Sucheston in 1979, whether Chatterji's result in [4] for martingales holds true for the case of pramarts.

THEOREM 1.4. *All L^1-bounded E-valued mils converge a.s. if (and only if) E has the* (RNP).

For other related results and references on amarts, the reader is referred to Edgar–Sucheston [5], Gut–Schmidt [7] and Luu [9].

2. Convergence of a class of martingale-like sequences

Throughout this and next section, let S be an infinite subset of N. In order to introduce the new class of L^1S-games we need the following definition and result, given by the author in [8, 10].

DEFINITION 2.1. A sequence (X_n) is an L^1-amart iff for each $\varepsilon > 0$ there exists p such that for all $n \geqq q \geqq p$ one has $E(\|X_q(n) - X_q\|) < \varepsilon$.

Clearly, every uniform amart is an L^1-amart and every L^1-amart is a game which becomes fairer with time.

LEMMA 2.1 (cf. [8]). *A sequence (X_n) is an L^1-amart if and only if it has a unique Riesz decomposition $X_n = M_n + P_n$, where (M_n) is a martingale and (P_n) is an L^1-potential, i.e. $\lim_n E(\|P_n\|) = 0$.*

Moreover, if this occurs then the martingale (M_n) is given by

$$\lim_m E(\|X_n(m) - M_n\|) = 0 \quad (n \in N).$$

DEFINITION 2.2. A sequence (X_n) is said to be an S-game (which becomes fairer with the only time subset S), if for each $\varepsilon > 0$ there exists p such that for all $q \in N$, $s \in S$ with $s \geqq q \geqq p$ we have $P(\|X_q(s) - X_q\| > \varepsilon) < \varepsilon$.

It is clear that by definition, a sequence (X_n) is a game which becomes fairer with time if and only if (X_n) is an S-game for some S such that the set $(N \setminus S)$ is finite. After proving the main results of this section we shall show that if $(N \setminus S)$ is infinite then there exists an L^1-bounded real-valued S-game which fails to be a game fairer with time.

The decomposition and convergence results we shall prove in this section are concerned with the following class of martingale-like sequences:

DEFINITION 2.3. A sequence (X_n) is called an L^1S-game if it is an S-game and the subsequence (X_s) is an L^1-amart.

THEOREM 2.2. *A sequence (X_n) is an L^1S-game if and only if (X_n) has a Riesz decomposition $X_n = M_n + P_n$, where (M_n) is a martingale, (P_n) converges to zero in probability and the subsequence (P_s) is an L_1-potential.*

PROOF. Let (X_n) be a sequence in L^1_E. Suppose first that (X_n) is an L^1S-game. Then in particular, the subsequence (X_s) is an L^1-amart. Thus by Lemma 2.1, it follows that (X_s) can be written in the form $X_s = M_s + P_s$, where (M_s) is a martingale and (P_s) is an L^1-potential, i.e. $\lim_s E(\|P_s\|) = 0$. Hence, (X_n) has also a Riesz decomposition $X_n = M_n + P_n$, where (M_n) is the martingale defined by the martingale (M_n) in the usual way. i.e. $M_n = M_n(s_n)$, were for each $n \in N$, $s_n = \inf\{s \in S : n \leqq s\}$ and (P_s) goes to zero in L^1. The main part of the proof of the necessity condition consists in showing that (P_n) goes to zero in probability. To see this let $\varepsilon > 0$ be given. Since (X_n) is also an S-game, by Definition 2.2 there exists $p_0 \in N$ such that if $q \in N$, $s \in S$ with $s \geqq q \geqq p_0$ we have

$$P(\|X_q(s) - X_q\| > \varepsilon/2) < \varepsilon/2. \tag{2.1}$$

On the other hand, since (P_s) goes to zero in L^1, there exists $p \geqq p_0$ such that for all $s \in S$ with $s \geqq p$ one has

$$E(\|P_s\|) < \varepsilon^2/4. \tag{2.2}$$

This with (2.1) implies that for all $n \in N$ with $n \geqq p$ we have

$$\begin{aligned} P(\|P_n\| > \varepsilon) &= P(\|X_n - M_n\| > \varepsilon) \leq \\ &\leqq P(\|X_n(s) - X_n\| > \varepsilon/2) + P(\|X_n(s) - M_n\| > \varepsilon/2) \leq \\ &\leqq \varepsilon/2 + 2/\varepsilon \cdot E(\|X_n(s) - M_n\|) \leq \\ &\leqq \varepsilon/2 + 2/\varepsilon \cdot E(\|X_s - M_s\|) = \varepsilon/2 + 2/\varepsilon \cdot E(\|P_s\|) \leq \\ &\leqq \varepsilon/2 + 2/\varepsilon \cdot \varepsilon^2/4 = \varepsilon, \end{aligned}$$

by taking any $s \in S$ with $s \geqq n$. Thus (P_n) converges to zero in probability which completes the proof of the necessity condition.

Conversely, suppose now that (X_n) has the Riesz decomposition, given in the theorem. Then by Lemma 2.1, the subsequence (X_s) is an L^1-smart. Thus to prove the sufficiency, it suffices to show that (X_n) is an S-game. To see this, let $\varepsilon > 0$ be given. Since (P_s) converges to zero in L^1, there exists $p_0 \in N$ such that for all $s \in S$ with $s \geqq p_0$, (2.2) is satisfied. On the other hand, since (P_n) converges to zero in probability, there exists $p \in N$ with $p \geqq p_0$ such that for all $q \geqq p$ one has

$$P(\|P_q\| > \varepsilon/2) < \varepsilon/2.$$

This with (2.2) shows that for all $q \in N$, $s \in S$ with $s \geqq q \geqq p$ we have

$$\begin{aligned} P(\|X_q(s) - X_q\| > \varepsilon) &\leqq P(\|X_q(s) - M_q\| > \varepsilon/2) + P(\|X_q - M_q\| > \varepsilon/2) \leq \\ &\leqq 2/\varepsilon \cdot E(\|X_q(s) - M_q\|) + \varepsilon/2 \leqq 2/\varepsilon \cdot E(\|X_s - M_s\|) + \varepsilon/2 = \end{aligned}$$

$$= 2/\varepsilon \cdot E(\|P_s\|) + \varepsilon/2 \leqq 2/\varepsilon \cdot \varepsilon^2/4 + \varepsilon/2 = \varepsilon.$$

Thus by Definition 2.2, (X_n) must be an S-game which completes the proof of the theorem.

Now let (X_n) be an S-game such that the subsequence (X_s) is uniformly integrable. Then for each $\varepsilon > 0$, there exists $\alpha \geqq 2$ such that if $A \in \mathcal{A}$ with $P(A) < \varepsilon/\alpha$ then $\sup_s \int_A \|X_s\| dP < \varepsilon/4$. On the other hand, since (X_n) is an S-game, there exists p such that in particular, for all $s, s' \in S$ with $s' \geqq s \geqq p$ one has

$$P(\|X_s(s') - X_s\| > \varepsilon/\alpha) < \varepsilon/\alpha.$$

Consequently,

$$E(\|X_s(s') - X_s\|) \leqq \varepsilon/\alpha + \int_{\{\|X_s(s')-X_s\|)>\varepsilon/\alpha\}} \|X_s(s') - X_s\| dP \leq$$

$$\leqq \varepsilon/\alpha + 2 \sup_{s\in S} \left\{ \int_A \|X_s\| dP, \quad A \in \mathcal{A}, \quad P(A) < \varepsilon/\alpha \right\} \leqq \varepsilon/2 + 2 \cdot \varepsilon/4 = \varepsilon.$$

This means that (X_s) is an L^1-amart. Therefore, (X_n) is an L^1S-game. The above argument shows that the real-valued version of the following corollary is an extension of Theorem 1.1 of Mucci [12] and Subramanian [15], mentioned in the previous section.

COROLLARY 2.3. *Let (X_n) be a sequence in L^1_E such that its subsequence (X_s) is an L^1-bounded L^1-amart. Suppose more that E has the* (RNP). *Then (X_n) is an L^1S-game if and only if it converges in probability.*

PROOF. Let (X_n) and E be as in the corollary. Suppose first that (X_n) is an L^1S-game such that $\sup_S E\|X_s\| < \infty$. Then the proof of Theorem 2.2 shows that the martingale (M_n) in the Riesz decomposition of (X_n) is L^1-bounded. This with Chatterji's result in [4] implies that (X_n) must converge in probability. To prove the converse, we note first that since (X_s) is an L^1-bounded L^1-amart, by Lemma 2.1, (X_n) can be written in the form $X_n = M_n + P_n$, where (M_n) is an L^1-bounded martingale and the subsequence (P_s) of (P_n) converges to zero in L^1. Now suppose that (X_n) converges in probability. Then it is clear that both sequences (X_n) and (M_n) must converge in probability to the same limit, since (P_s) converges to zero in L^1 and E has the (RNP). Therefore, (P_n) converges to zero in probability. This with Theorem 2.2 shows that (X_n) must be an L^1S-game. Thus the proof of the corollary is complete.

In the case when E lacks the (RNP), the above theorem and Theorem 5 of Uhl [13] give us the following result:

Corollary 2.4. *An L^1-bounded E-valued L^1S-game (X_n) converges in probability if and only if (X_s) satisfies the Uhl's condition, i.e., for each $\varepsilon > 0$ there exists a convex compact subset K_ε of E such that for any choice of $\delta > 0$ there exists $s_0 \in S$ and $A_0 \in \mathcal{A}_{s_0}$ with $P(A_0) \geqq 1 - \varepsilon$ such that for all $s' \in S$, $A \in \mathcal{A}_s$ with $s \geqq s_0$, $A \subset A_0$ we have $\int_A X_s dP \in K_\varepsilon + \delta U$, where U is the unit ball of E.*

Example 2.5. Let $S = (n_k)$ be an increasing subsequence of N. There exists an L^1-bounded positive real-valued L^1S-game which converges a.s., but it fails to be even a V-game for any infinite subset of V of N such that $(V \setminus S)$ is infinite.

Construction. Let $(\Omega, \mathcal{A}, P)$ be the usual Lebesgue probability measure space on the unit interval. For each $n \in N$, let $a_n = \prod_{i \leqq n} 2^i$, Q_n the partition of $[0,1]$ in a_n intervals of equal length and $\mathcal{A}_n = \sigma - (Q_n)$. Given any $X_\infty \in L^1_{R+}$ we define $M_n = X_n(\infty)$ $(= E(X_\infty \mid \mathcal{A}_n))$ and P_n as follows:

(a) $P_{n_k} = 0$ $(k \in N)$.

(b) For an interval I of Q_{n-1} with $n \in (N \setminus S)$ let $P_n = 2^n$ on the first interval of Q_n that is contained in I and $P_n = 0$ elsewhere (hence (P_n) converges to zero a.s.).

(c) $X_n = M_n + P_n$ $(n \in N)$.

It is clear that by Theorem 2.2, (X_n) is an L^1-bounded L^1S-game which converges also a.s. to X_∞. On the other hand, for all $n, p \in (N \setminus S)$ with $n > p$ we have $P_p(n) = 1$. Therefore, (X_n) cannot be a V-game for any subset V of N such that the set $(V \setminus S)$ is infinite (hence in particular, (X_n) cannot be a mil either). So the results presented in this section are independent of Theorem 1.4 of Talagrand [17], recalled in the previous section.

3. Lattice properties

Recall that a Banach space E, equipped with the partial order $\leq$, is said to be a Banach lattice if the following conditions hold:

(i) $x \leqq y$ implies $x + z \leqq y + z$ for all $x, y, z \in E$,

(ii) if $x \geqq 0$ and α is a nonnegative real number then $\alpha x \geqq 0$,

(iii) for every pair $x, y \in E$ there exist the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$.

(iv) the norm is monotone, i.e., if $|x| \leqq |y|$ then $\|x\| \leqq \|y\|$, where $|x| = x \vee -x$.

Furthermore, if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \geqq 0$ then E is called an (AL)-space. Throughout this section we assume that E is an (AL)-space having (RNP). Therefore, E is isomorphic (as a topological vector lattice) to

$\ell^1(\Gamma)$ for some index set Γ. For further characterizations of Banach lattices isomorphic to $\ell^1(\Gamma)$, the interested reader is referred to [16, 6].

In what follows we shall need the following result which has been recently proved by Schmidt [14].

LEMMA 3.1 (cf. [14]). *Let (M_n) be an L^1-bounded martingale in L^1_E with limit measure φ.*

a) *If E has the* (RNP) *then (M_n) is a difference of two positive L^1-bounded martingales (Y_n^1) and (Y_n^2) such that $\int_A Y_n^1 dP \leqq \varphi^+(A)$ and $\int_A Y_n^2 dP \leqq \varphi^-(A)$ for all $n \in N$ and $A \in \mathcal{A}_n$.*

b) *Furthermore, if E is isomorphic (as a topological vector lattice) to $\ell^1(\Gamma)$ for some index set Γ then $(Y_n^1 \wedge Y_n^2)$ is a uniform potential, i.e.* $\lim_{\tau \in T} E(\|Y_\tau^1 \wedge Y_\tau^2\|) = 0$.

The main purpose of this section is to prove the following theorem.

THEOREM 3.2. *Let E be a Banach lattice isomorphic (as a topological vector lattice) to $\ell^1(\Gamma)$ for some index set Γ. Then endowed with the norm*

$$\|(X_n)\|_N = \sup_N E(\|N_n\|),$$

the class of all L^1-bounded E-valued L^1S-games becomes a Banach lattice.

PROOF. Let E be as in the theorem. For convenience, let $\mathcal{L}$ denote the space of all L^1-bounded E-valued L^1S-games. By Theorem 2.2, $\mathcal{L}$ is a normed space. Further, since L^1-norm is always a lattice norm, it remains to prove only the followings:

a) $\mathcal{L}$ is a lattice,

b) $\|(.)\|_N$ is complete.

To prove a), let $(X_n) \in \mathcal{L}$. Then by Theorem 2.2 (X_n) can be written in a form $X_n = M_n + P_n$, where (M_n) is a martingale, (P_n) converges to zero in probability and its subsequence (P_s) is an L^1-potential. Consequently, $(P_n^\pm)$, $(|P_n|)$ converge to zero in probability and $(P_s^\pm)$, $(|P_s|)$ are L^1-potentials.

Now let $M_n = Y_n^1 - Y_n^2$, where (Y_n^1) and (Y_n^2) are the positive L^1-bounded martingale taken from Lemma 3.1. Then $(Y_n^1 \wedge Y_n^2)$ is a uniform potential, i.e. $\lim_{\tau \in T} E(\|T_\tau^1 \wedge Y_\tau^2\|) = 0$. Thus, if we define $Z_n = (Y_n^1 + P_n^+) \wedge (Y_n^2 + P_n^-)$ then

$$0 \leqq Z_n \leqq (Y_n^1 \wedge Y_n^2) + |P_n| \quad (n \in N).$$

This implies that (Z_n) converges to zero in probability and (Z_s) is an L^1-potential. Therefore, by Theorem 2.2, $(Y_n^1 + P_n^+ - Z_n)$, $(Y_n^2 + P_n^+ - Z_n) \in \mathcal{L}$. On the other hand, since

$$X_n = (Y_n^1 + P_n^+ - Z_n) - (Y_n^2 + P_n^- - Z_n),$$
$$0 = (Y_n^1 + P_n^+ - Z_n) \wedge (Y_n^2 + P_n^- - Z_n),$$

we have

$$X_n^+ = (Y_n^1 + P_n^+ - Z_n) \quad \text{and} \quad X_n^- = (Y_n^2 + P_n^- - Z_n) \quad (n \in N).$$

This proves $|X_n| \in \mathcal{L}$ and the condition (a).

To prove (b), let $(X_n^k)_{k=1}^{\infty}$ be a sequence in $\mathcal{L}$ which is Cauchy in the $\|(.)\|_N$-norm. It is clear that for each n, the sequence $(X_n^k,\ k \geqq 1)$ converges in L^1 to some $X_n \in L_E^1$. We have to show that $(X_n) \in \mathcal{L}$. To see this, let $\varepsilon > 0$ be given. Then there exists some $k \in N$ such that

$$\sup_{n \in N} E\left(\|X_n^k - X_n\|\right) < \varepsilon^2/9.$$

Further, since $(X_n^k) \in \mathcal{L}$, there exists $p \in N$ such that for all $s \in S$, $q \in N$ with $s \geqq q \geqq p$, one has

$$P\left(\|X_q^k(s) - X_q^k\| > \varepsilon/3\right) < \varepsilon/3.$$

Thus,

$$\begin{aligned} &P(\|X_q(s) - X_q\| > \varepsilon) \leq \\ &\leqq P\left(\|X_q(s) - X_q^k(s)\| > \varepsilon/3\right) + P\left(\|X_q^k(s) - X_q^k\| > \varepsilon/3\right) + \\ &+ P\left(\|X_q^k(s) - X_q\| > \varepsilon/3\right) \leqq \\ &\leqq 3/\varepsilon \cdot E\left(\|X_s - X_s^k\|\right) + \varepsilon/3 + 3\varepsilon \cdot E\left(\|X_q^k - X_q\|\right) \leq \\ &\leqq 3/\varepsilon \cdot \varepsilon^2/9 + \varepsilon/3 + 3/\varepsilon \cdot \varepsilon^2/9 = \varepsilon. \end{aligned}$$

This shows that (X_n) is an S-game.

Similarly, using a 3ε-argument one can show also that (X_s) is an L^1-amart. Therefore, (X_n) belongs to $\mathcal{L}$. This completes the proof.

REMARK 2.3. Theorems 2.2 and 3.2 remain valid if we replace the subset S of N by a cofinal subset π of T and the norm $\|(.)\|_N$ by the norm $\|(.)\|_\pi$, given by

$$\|(X_n)\|_\pi = \sup_{\alpha \in N \cup \pi} E(\|X\|_\alpha),$$

respectively. Thus in particular, if $\pi = T$ then $\mathcal{L}$ coincides with the class of all L^1-bounded E-valued uniform amarts. Therefore, Theorem 1.2 of Schmidt [14], mentioned in Section 1 could be regarded as a special case of Theorem 3.2.

Acknowledgement. I would like to express my thanks to my colleague Dr. Klaus D. Schmidt for a fruitful cooperation and for sending me a copy of [14] which plays an important role in proving the main result of the last section.

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(Received August 30, 1988)

INSTITUTE OF MATHEMATICS
P.O. BOX 631, BO HO
HANOI
VIETNAM

Acta Math. Hung.
59 (3–4) (1992), 283–289.

THE NUMERICAL RADIUS OF A COMPLETELY BOUNDED MAP

C.-Y. SUEN (Galveston)

Introduction

Stinespring [7] defined completely positive linear maps on C^*-algebras and then generalized Naimarks' dilation theorem for positive operator-valued measures. Arveson [1] further broadened the connection between completely positive maps and dilation theory of operator algebras. Recently, much of the theory of completely positive maps has been quite easily extended to a considerably broader class of maps [2,4,8], the completely bounded maps.

Let A and B be C^*-algebras and denote M_n the C^*-algebra of complex $n \times n$ matrices. If U is a subspace of A and $L : U \to B$ is a linear map, then we set $L_n = L \otimes I_n : U \otimes M_n \to B \otimes M_n$ by

$$L_n(a \otimes b) = L(a) \otimes b.$$

The map L is positive if $L(a) \geqq 0$ whenever $a \in U$ and $a \geqq 0$, and is completely positive if L_n is positive for all $n = 1, 2, \dots$. L is completely bounded if $\sup_n \|L_n\|$ is finite, and we let

$$\|L\|_{\mathrm{cb}} = \sup_n \|L_n\|.$$

If $U \subseteqq A$ is a linear subspace with $U^* = U$ and $L : U \to B$ is linear, then we define a linear map $L^* : U \to B$ by

$$L^*(a) = L(a^*)^*.$$

Let $\mathrm{Re}\,(L) : U \to B$ and $\mathrm{Im}\,(L) : U \to B$ be defined by

$$\mathrm{Re}\,(L) = \frac{1}{2}(L + L^*) \quad \text{and} \quad \mathrm{Im}\,(L) = \frac{1}{2i}(L - L^*).$$

L is self-adjoint, if $L = L^*$. Finally, given $S \subseteqq L(H)$, we let S' denote its commutant.

In [5], it was shown that every completely bounded map L from a C^*-algebra A into $L(H)$ had a representation as $L(a) = V^*T\pi(a)V$, where π is a $*$-homomorphism, T is in the commutant of $\pi(A)$ and V is an isometry. Such a realization of L was called a commutant representation with isometry.

For any commutant representation with isometry it was shown that $\|L\|_{\mathrm{cb}} \leqq \|T\|$, and that such a representation existed for which $\|T\|$ was a minimum. This minimum was denoted $|||L|||$ with $\|L\|_{\text{cb}} \leqq |||L||| \leqq 2\|L\|_{\text{cb}}$. In particular, there exist maps for which $\|L\|_{\text{cb}} < |||L|||$.

The purpose of this paper is an analogous study, with $\|T\|$ replaced by the numerical radius $W(T)$ of T. We show that among all commutant representations with isometry for L, there exist ones for which $W(T)$ is a minimum with $W(T) = \min\{\|\phi\|_{\text{cb}}\colon \phi \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1\}$ and $W(T) \leqq \|L\|_{\text{cb}} \leqq 2W(T)$. In particular, there exist maps for which $W(T) < \|L\|_{\text{cb}}$. We give an alternative characterization of this minimum, and prove some inequalities aimed at showing to what extent this number behaves like a numerical radius of L. Finally, the author gratefully acknowledges several valuable conversations with Professor V. Paulsen.

2. Representations

Throughout this section let A be a unital C^*-algebra and $L(H)$ be the algebra of all bounded linear operators on a Hilbert space H.

DEFINITION 2.1. Let $L : A \to L(H)$ be completely bounded. We define $S(L) = \inf\{\|\phi\|_{\text{cb}} : \phi \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1\}$.

REMARK 2.2. Since $L(H)$ is injective, the set $\{\phi : \phi \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1\}$ is not empty [4,8]. It is not difficult to see that $S(L)$ is a norm.

LEMMA 2.3. *Let $L : A \to L(H)$ be a completely bounded map. Then there exists a map ϕ such that $\phi \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1$ and $\|\phi\|_{\text{cb}} = S(L)$. That is $S(L) = \min\{\|\phi\|_{\text{cb}} : \phi \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1\}$.*

PROOF. Let n be a fixed positive integer, then there exists a map $\phi^{(n)} : A \to L(H)$ such that $\phi^{(n)} \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1$ and $\|\phi^{(n)}\| \leqq S(L) + \frac{1}{n}$. Thus we have a sequence of completely positive maps $\{\phi^{(n)}\}$. Let $S_m = \{\phi^{(j)} : j \geqq m\}$ for $m = 1, 2, 3, \ldots$, then $\ldots \overline{S_m} \subseteqq \overline{S_{m-1}} \subseteqq \subseteqq \cdots \subseteqq \overline{S_1}$, with closure in the BW-topology [1]. Since each $\overline{S_m}$ is compact in the BW-topology and the family $\{\overline{S_m}\}$ has finite intersection property, then $\bigcap_{m=1}^{\infty} \overline{S_m}$ is nonempty. Let $\tilde{\phi} \in \bigcap_{m=1}^{\infty} S_m$. We claim that $\|\tilde{\phi}\|_{\text{cb}} \leqq S(L)$ and $\tilde{\phi} \pm \operatorname{Re} \alpha L$ is completely positive for all $|\alpha| = 1$. For fixed m_0 $(= 1, 2, \ldots)$, there is a net $\{\phi_\beta^{(m_0)}\} \subseteqq S_{m_0}$, such that $\lim_\beta \phi_\beta^{(m_0)}(I) = \tilde{\phi}(I)$ in the weak operator topology.

Let $\varepsilon > 0$ be given. Then there exist h_0, k_0 in H with $\|h_0\| = \|k_0\| = 1$ such that

$$\|\tilde{\phi}(I)\| \leqq |\langle \tilde{\phi}(I) h_0, k_0 \rangle| + \varepsilon.$$

Since

$$\langle \tilde{\phi}(I)h_0, k_0\rangle = \lim_{\beta} \langle \phi_\beta^{(m_0)}(I)h_0, k_0\rangle$$

and

$$|\langle \phi_\beta^{(m_0)}(I)h_0, k_0\rangle| \leqq \|\phi_\beta^{(m_0)}(I)\| \leqq S(L) + \frac{1}{m_0} \quad \text{for all} \quad \beta,$$

we have $\|\tilde{\phi}(I)\| \leqq S(L) + \frac{1}{m_0} + \varepsilon$. Hence $\|\tilde{\phi}\|_{\mathrm{cb}} = \|\tilde{\phi}(I)\| \leqq S(L)$. From above we know that there is a net $\{\phi_\beta\} \subseteqq \{\phi^{(n)}\}$, such that $\lim_\beta \phi_\beta = \tilde{\phi}$ in the BW-topology. Let (a_{ij}) be positive in $A \otimes M_n$, then

$$\left\langle (\tilde{\phi}(a_{ij})) \pm ((\mathrm{Re}\,\alpha L)(a_{ij})) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle =$$

$$= \lim_\beta \left\langle (\phi_\beta \pm \mathrm{Re}\,\alpha L)_n((a_{ij})) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \geqq 0.$$

Hence $\tilde{\phi} \pm \mathrm{Re}\,\alpha L$ is completely positive for all $|\alpha| = 1$. Thus

$$\|\tilde{\phi}\|_{\mathrm{cb}} = S(L) =$$

$$= \min\{\|\phi\|_{\mathrm{cb}} : \phi \pm \mathrm{Re}\,\alpha L \quad \text{is completely positive for all} \quad |\alpha| = 1\}.$$

DEFINITION 2.4. Let $L : A \to L(H)$ be a completely bounded map. We say that L has a minimal commutant representation with isometry, if there exist a Hilbert space K, a $*$-homomorphism $\pi : A \to L(K)$, an isometry $V : H \to K$ and an operator $T \in \pi(A)'$ such that span $\pi(A)VH$ is dense in K and $L(.) = V^*T\pi(.)V$.

REMARK 2.5. From [5, Theorem 2.2], we know that every completely bounded map from A to $L(H)$ has a minimal commutant representation with isometry. A completely bounded map may have non-unitarily equivalent minimal commutant representations with isometries. See the following example.

EXAMPLE 2.6. Let $L : C \oplus C \to L(C^2)$ be a map defined by

$$L(a \oplus b) = \begin{pmatrix} a/\sqrt{2} & 0 \\ b/\sqrt{2} & 0 \end{pmatrix},$$

then L is completely bounded with $\|L\|_{\mathrm{cb}} \leqq 1$.

$$V_1 = \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \\ 0 & \sqrt{3}/2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & 1/\sqrt{3} \\ 1/2 & 0 \\ 0 & \sqrt{2}/\sqrt{3} \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 2\sqrt{2}/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{2}/\sqrt{3} & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \sqrt{2}/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and

$$\pi_1(a \oplus b) = \pi_2(a \oplus b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

then

$$L(a \oplus b) = V_1^* T_1 \pi_1(a \oplus b) V_1 = V_2^* T_2 \pi_2(a \oplus b) V_2 \quad \text{for all } a \oplus b \text{ in } C \oplus C$$

but there is no unitary such that $U^*\pi_1 U = \pi_2$, $U^* T_1 U = T_2$ and $UV_2 = V_1$ since in particular $V_1^*\pi_1 V_1 \neq V_2^* \pi_2 V_2$. Note also that $W(T_1) \neq W(T_2)$, so that even $W(T)$ is not an invariant for minimal commutant representations with isometry.

THEOREM 2.7. *Let $L : A \to L(H)$ be a completely bounded map, then L has a minimal commutant representation with isometry $\tilde{V}^* \tilde{T} \pi \tilde{V}$ such that $W(\tilde{T}) = \min\{W(T) : L$ has a minimal commutant representation $V^*T\pi V\} = S(L)$. Moreover, $S(L) \leqq \|L\|_{\mathrm{cb}} \leqq \|\tilde{T}\| \leqq 2S(L)$.*

PROOF. From Lemma 2.3, there exists a map $\tilde{\phi}$ such that $\phi \pm \operatorname{Re}\alpha L$ is completely positive for all $|\alpha| = 1$ and $\|\tilde{\phi}\|_{\mathrm{cb}} = \|\tilde{\phi}(I)\| = S(L)$. Let $\tilde{\phi}(I) = P$ and θ be any unital completely positive map, then the map $\tilde{\theta} = \tilde{\phi} + (S(L) - P)^{\frac{1}{2}} \theta (S(L) - P)^{\frac{1}{2}}$ is completely positive with $\tilde{\theta}(I) = S(L)I$. By Stinespring Theorem [7], $\tilde{\theta}$ has a representation $S(L)\tilde{V}^*\tilde{\pi}\tilde{V}$, where $\tilde{V}^*\tilde{\pi}\tilde{V}$ is unital and minimal. Since $\tilde{\theta} \pm \operatorname{Re}\alpha L$ is completely positive for all $|\alpha| = 1$, by [1, Theorem 1.4.2], we have $\operatorname{Re} L = \tilde{V}^* H \tilde{\pi} \tilde{V}$ and $\operatorname{Im} L = \tilde{V}^* K \tilde{\pi} \tilde{V}$ where H and K are operators.

Hence $L = \tilde{V}^*(H + iK)\tilde{\pi}\tilde{V}$ and $\operatorname{Re}\alpha L = \tilde{V}^*(\operatorname{Re}\alpha(H + iK))\tilde{\pi}\tilde{V}$ for all $|\alpha| = 1$. Since $\tilde{\theta} \pm \operatorname{Re}\alpha L$ is completely positive for all $|\alpha| = 1$, we have $\pm\operatorname{Re}\alpha(H + iK) \leqq S(L)I$. Hence $W(H + iK) \leqq S(L)$. Let $\tilde{T} = H + iK$, we have $L = \tilde{V}^*\tilde{T}\tilde{\pi}\tilde{V}$ and $W(\tilde{T}) \leqq S(L)$.

Now, we claim that $S(L) \leqq W(\tilde{T})$: If L has a minimal commutant representation with isometry $V^*T\pi V$, then

$$\pm\operatorname{Re}\alpha L = \pm V^*(\operatorname{Re}\alpha T)\pi V \leqq W(T) V^* \pi V \quad \text{for all} \quad |\alpha| = 1.$$

We have

$$W(T) \geqq S(L) =$$

$$= \min\{\|\phi\|_{\mathrm{cb}} : \phi \pm \operatorname{Re}\alpha L \quad \text{is completely positive for all} \quad |\alpha| = 1\}.$$

Hence $S(L) \leqq W(\tilde{T})$. Thus

$$W(\tilde{T}) =$$

$$= \min\{W(T) : L \text{ has a minimal commutant representation } V^*T\pi V\} =$$

$$= \min\{\|\phi\|_{\mathrm{cb}} : \phi \pm \operatorname{Re}\alpha L \quad \text{is completely positive for all} \quad |\alpha| = 1\} = S(L).$$

By [5, Theorem 2.2], we have $S(L) \leqq \|L\|_{\mathrm{cb}} \leqq \|\tilde{T}\| \leqq 2S(L)$.

EXAMPLE 2.8. We can have $S(L) < \|L\|_{\mathrm{cb}}$. Let $L : C \oplus C \to M_2(C)$ be defined by

$$a \oplus b \to \begin{pmatrix} a/\sqrt{2} & 0 \\ b/\sqrt{2} & 0 \end{pmatrix}.$$

Then $\|L\|_{\mathrm{cb}} = 1$ and

$$(\operatorname{Re}\alpha L)(a \oplus b) = \begin{pmatrix} (\operatorname{Re}\alpha)a\sqrt{2} & \overline{\alpha}b/2\sqrt{2} \\ \alpha b/2\sqrt{2} & 0 \end{pmatrix}.$$

Let

$$\phi(a \oplus b) = \begin{pmatrix} (a/\sqrt{2}) + ((\sqrt{2}-1)b/2\sqrt{2}) & 0 \\ 0 & b \end{pmatrix} \geqq 0,$$

then it is not difficult to see that $\phi \pm \operatorname{Re}\alpha L$ is completely positive and $\|\phi\|_{\mathrm{cb}} = S(L) = \min\{W(T) : L$ has a minimal commutant representation with isometry $V^*T\pi V\} = (1+\sqrt{2})/2\sqrt{2} < \|L\|_{\mathrm{cb}} = 1$.

COROLLARY 2.9. *Let $L : A \to L(H)$ be a self-adjoint completely bounded map, then L has a minimal commutant representation with isometry $\tilde{V}^*\tilde{T}\tilde{\pi}\tilde{V}$ such that $\|L\|_{\mathrm{cb}} = \|\tilde{T}\| = S(L) = \min\{W(T) : L = V^*T\pi V\}$.*

PROOF. By Theorem 2.7 and [5, Theorem 2.2], L has a minimal commutant representation with isometry $\tilde{V}^*\tilde{T}\tilde{\pi}\tilde{V}$ such that $\tilde{T} = \tilde{T}^*$ and $W(\tilde{T}) = S(L)$. By [5, Theorem 2.10] , we have $\|\tilde{T}\| = W(\tilde{T}) \geqq \|L\|_{\mathrm{cb}}$. By Theorem 2.7, we have $W(\tilde{T}) = \|L\|_{\mathrm{cb}}$.

3. Applications

THEOREM 3.1. *Let $L : A \to L(H)$ be a completely bounded map, then $W(L_n(a)) \leqq \|a\|S(L)$ for all a in $A \otimes M_n$ and $n \geqq 1$.*

PROOF. By Theorem 2.7, L has a minimal commutant representation with isometry $\tilde{V}^*\tilde{T}\tilde{\pi}\tilde{V}$ such that $W(\tilde{T}) = S(L)$. Since $\tilde{T}$ and $\tilde{T}^*$ are in $\pi(A)'$, by [3, Theorem 3.4], we have

$$\left| \left\langle L_n(a) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \right| =$$

$$= \left| \left\langle \begin{pmatrix} \tilde{T} & 0 & \dots & 0 \\ 0 & \tilde{T} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{T} \end{pmatrix} (\tilde{\pi} \otimes I_n)(a) \begin{pmatrix} \tilde{V} & 0 & \dots & 0 \\ 0 & \tilde{V} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{V} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}, \right. \right.$$

$$\left. \left. \begin{pmatrix} \tilde{V} & 0 & \dots & 0 \\ 0 & \tilde{V} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{V} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \right\rangle \right| \leqq$$

$$\leqq W \left(\begin{pmatrix} \tilde{T} & 0 & \dots & 0 \\ 0 & \tilde{T} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{T} \end{pmatrix} (\tilde{\pi} \otimes I_n)(a) \right) \leqq$$

$$\leqq \|a\| W \left(\begin{pmatrix} \tilde{T} & 0 & \dots & 0 \\ 0 & \tilde{T} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \tilde{T} \end{pmatrix} \right) \leqq \|a\| W(\tilde{T}),$$

where $\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \in L(H) \otimes M_n$ with $\left\| \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \right\| = 1$ and $a \in A \otimes M_n$. Hence $W(L_n(a)) \leqq \|a\| W(\tilde{T}) = \|a\| S(L)$.

EXAMPLE 3.2. $S(L)$ is the best constant in the inequality of Theorem 3.1. Taking the same map as in Example 2.6, we have

$$W(L(1 \oplus 1)) = W \left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix} \right) = \sup_{|\alpha|=1} \left\| \operatorname{Re} \alpha \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix} \right\| \geqq$$

$$\geqq \left\| \begin{pmatrix} 1/\sqrt{2} & 1/(2\sqrt{2}) \\ 1/(2\sqrt{2}) & 0 \end{pmatrix} \right\| = (1+\sqrt{2})/(2\sqrt{2}).$$

By Theorem 3.1, we have $W(L(1 \oplus 1)) \leqq \|1 \oplus 1\| S(L) = S(L) = (1 + \sqrt{2})/(2\sqrt{2})$. Hence $W(L(1 \oplus 1)) = \|(1 \oplus 1)\| S(L)$.

COROLLARY 3.3. *If $L : A \to C$ is a bounded linear functional, then* $S(L) = \|L\|_{\mathrm{cb}} = \|L\|$.

PROOF. Let $a \in A$ with $\|a\| \leqq 1$, then $W(L(a)) = |L(a)| \leqq S(L)$. By [6, Theorem 2.10], we have $\|L\|_{\mathrm{cb}} = \|L\| \leqq S(L)$. Theorem 2.7 implies $\|L\| = S(L)$.

EXAMPLE 3.4. We can have $\sup_{|\alpha|=1} \{\|\operatorname{Re} \alpha L\|_{\mathrm{cb}}\} < S(L)$.

Let $L : C \oplus C \to C$ be a bounded linear map defined by $L(a \oplus b) = 2ia - b$, then $\|L\|_{\mathrm{cb}} = \|L\| = 3 = S(L)$ and

$$|(\mathrm{Re}\,(\alpha L))(a \oplus b)| =$$

$$= |-2a\mathrm{Im}\,\alpha - b\mathrm{Re}\,\alpha| \leqq \sqrt{4|a|^2 + |b|^2} \leqq \sqrt{5\|a \oplus b\|^2} \quad \text{for all} \quad |\alpha| = 1.$$

Hence $\sup\limits_{|\alpha|=1} \{\|\mathrm{Re}\,\alpha L\|_{\mathrm{cb}}\} \leqq \sqrt{5} < S(L)$.

COROLLARY 3.5. *T is similar to a contraction if and only if there exists a best constant $K \geqq 0$ such that*

$$W((f_{ij})(T)) \leqq K\|(f_{ij})\|_\infty \quad \textit{for} \quad (f_{ij}) \in C(T) \otimes M_n,$$

where $C(T)$ is the C^*-algebra of all continuous linear functionals on the unit circle.

PROOF. If T is similar to a contraction, then the map L : Disk algebra $\to$ $\to L(H)$ defined by $L(f) = f(T)$ is completely bounded [4, Theorem 3.2]. By [4, Theorem 2.4], there exists a completely bounded extension $\tilde{L} : C(T) \to$ $\to L(H)$. By Theorem 3.1, we have $W((f_{ij})(T)) \leqq \|(f_{ij})\|S(L)$, for $(f_{ij}) \in$ $\in C(T) \otimes M_n$. Let $K = \inf\{S(\tilde{L}) : \tilde{L}$ is a completely bounded extension of $L\}$, then

$$W(f_{ij}(T)) \leqq K\|(f_{ij})\|.$$

Conversely, if $W(f_{ij}(T)) \leqq K\|(f_{ij})\|$, then $\|(f_{ij}(T)\| \leqq 2K\|(f_{ij})\|$. By [4, Corollary 3.5], we have the sufficient condition.

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(Received September 14, 1988)

TEXAS A&M UNIVERSITY AT GALVESTON
P.O.BOX 1675
GALVESTON, TEXAS 77553-1675
USA

Acta Math. Hung.
59 (3–4) (1992), 291–299.

A NOTE ON GENERALIZED KÖTHE–TOEPLITZ DUALS

J. I. OKUTOYI (Nairobi)

1. Introduction

In 1980 Ivor J. Maddox determined the necessary and sufficient conditions for $(A_k)_0^\infty$ to belong to $Z^\beta(X)$ where $Z = c_0$, c, ℓ_∞, $A_k \in B(X,Y)$, X and Y are any Banach spaces. The purpose of this paper is firstly to determine the most general continuous linear functional f in X^*, the dual space of X, where X is a semi-conservative BK-space with $\Delta^+ = \{\delta, \delta^0, \delta^1, \delta^2, \dots\}$ as its Schauder basis, and secondly to determine the necessary and sufficient conditions for $(A_k)_0^\infty$ to belong to $\mathrm{bv}_0^\beta(X)$, $\mathrm{bv}^\beta(X)$, where $A_k \varepsilon B(X,Y)$, X and Y any Banach spaces.

2. Notations and definitions

s; c_0; c; ℓ_p $(1 \leqq p < \infty)$; ℓ_∞; bv; bv_0; δ; δ^k; X^β; cs; E^∞ will denote the set of all sequences; all sequences convergent to zero; convergent sequences; sequences such that $\sum\limits_{k=0}^{\infty} |x_k|^p < \infty$; bounded sequences; sequences of bounded variation, i.e. sequences x such that $\lim\limits_{k\to\infty} x_k$ exists and $\sum\limits_{k=0}^{\infty} |x_{k+1} - x_k| < \infty$; sequences of bounded variation with $\lim\limits_{k\to\infty} x_k = 0$; $(1,1,1,\dots)$; $(0,0,\dots,0,1,0,\dots)$; all sequences $x \in s$ such that $\sum\limits_{k=0}^{\infty} x_k y_k$ is convergent for each $y \in X$; convergent series; finite sequences, respectively.

A BK-space X is said to be semi-conservative if $X^f \subset \mathrm{cs}$, where $X^f = \{(f(\delta^k))_{k=0}^\infty : f \in X^*\}$, that is, $X \supset E^\infty$ and $\sum\limits_{k=0}^{\infty} f(\delta^k)$ exists for all $f \in X^*$.

Let X be a paranormed or normed space, then $s(X)$ will denote the space of all sequences with values in X, i.e. $s(X) = \{x = (x_k)_0^\infty : x_k \in X\}$.

3. The main results

Let us note that if X is any BK-space with $\Delta^+ = \{\delta, \delta^0, \delta^1, \delta^2, \dots\}$ as its Schauder basis, then every $x \in X$ is of the form

$$x = \lambda\delta + \sum_{k=0}^{\infty} \lambda_k \delta^k \tag{3.1}$$

where $(\lambda, \lambda_0, \lambda_1, \lambda_2, \dots)$ is a sequence of scalars in $\mathbf{C}$. (3.1) is equivalent to

$$x = \lambda\delta + \sum_{k=0}^{\infty} (x_k - \lambda)\delta^k. \tag{3.2}$$

THEOREM 3.1. *Let X be a BK-space with Δ^+ as its Schauder basis and define $P : X \to \mathbf{C}$ by $P(x) = \lambda$, where X is given by* (3.2) *and let* $\operatorname{Ker}(P) = P^{-1}(\{0\}) = \{x \in X : P(x) = \theta\} = X_0$, *$\theta$ is the zero sequence. Then*

(i) *P is a continuous linear functional on X,*

(ii) *X_0 is a BK-space with AK in the X-topology.*

PROOF. (i) By hypothesis X is a BK-space and so all coordinate functionals are continuous with respect to any basis (see [3], p. 207). Hence $P \in X^*$.

(ii) Since P is continuous, $P^{-1}(\{0\})$ is a closed subspace of X, but $P^{-1}(\{0\}) = \operatorname{Ker}(P) = X_0$. Now, every $x \in X$ has the representation

$$x = \sum_{k=0}^{\infty} x_k \delta^k$$

in the X-topology , therefore X_0 is a BK-space with AK.

THEOREM 3.2. *Let X be a semi-conservative BK-space with Δ^+ as its Schauder basis. Then the most general continuous linear functional $f \in X^*$ is of the form*

$$f(x) = \lambda\alpha + \sum_{k=0}^{\infty} x_k t_k$$

where $\alpha \in \mathbf{C}$, $t_k \in X^\beta$ and x is given by (3.2)

Moreover,

$$\|f\| \leqq \|P\|\,|\alpha| + \|t\|_{x^\beta}$$

where $P : X \to \mathbf{C}$ is the projection map $P(x) = \lambda$ and

$$\|t\|_{X^\beta} = \sup_{\|x\| \leqq 1} \left| \sum_{k=0}^{\infty} x_k t_k \right|.$$

PROOF. Let $f \in X^*$, then from (3.2)

$$f(x) = \lambda f(\delta) + \sum_{k=0}^{\infty} (x_k - \lambda) f(\delta^k). \tag{3.3}$$

But X is a semi-conservative BK-space and so $\sum_{k=0}^{\infty} f(\delta^k)$ exists for all $f \in X^*$ and therefore $\sum_{k=0}^{\infty} x_k f(\delta^k) = \sum_{k=0}^{\infty} x_k t_k$ is convergent for all $x \in X$ which implies that $(f(\delta^k))_0^\infty = (t_k)_0^\infty \in X^\beta$ and so

$$f(x) = \lambda \left(f(\delta) - \sum_{k=0}^{\infty} t_k \right) + \sum_{k=0}^{\infty} x_k t_k = \lambda\alpha + \sum_{k=0}^{\infty} x_k t_k \tag{3.4}$$

where $\alpha = f(\delta) - \sum_{k=0}^{\infty} t_k$.

Conversely, given $\alpha' \in \mathbf{C}$, $t' \in X^\beta$ and $x \in X$ where x is given by (3.2), define $g : X \to \mathbf{C}$ by

$$g(x) = \alpha'\lambda + \sum_{k=0}^{\infty} x_k t'_k.$$

Then

$$g_n(x) = \alpha'\lambda + \sum_{k=0}^{n} x_k t'_k.$$

is a continuous linear functional on X since $x \mapsto \lambda$ is by Theorem 3.1 continuous. $x \mapsto \sum_{k=0}^{n} x_k t'_k$, as $n \to \infty$, is continuous by the Banach–Steinhaus theorem. Hence g has the required representation.

Now $f(x) = \lambda\alpha + \sum_{k=0}^{\infty} x_k t_k$, therefore

$$|f(x)| \leqq |\lambda\alpha| + \left| \sum_{k=0}^{\infty} x_k t_k \right|,$$

i.e.

$$|f(x)| \leqq |P(x)||\alpha| + \left| \sum_{k=0}^{\infty} x_k t_k \right|$$

by Theorem 3.1, therefore

$$\|f\| \leqq \|P\|\,|\alpha| + \sup_{\|x\| \leqq 1} \left| \sum_{k=0}^{\infty} x_k t_k \right|,$$

i.e. $\|f\| \leqq \|P\||\alpha| + \|t\|_{X^\beta}$.

NOTE. I have not been able to show that $\|f\| = \|P\||\alpha| + \|t\|_{X^\beta}$. In fact I doubt if equality holds.

THEOREM 3.3. *Let X be defined as in Theorems 3.1 and 3.2. Then $X^\beta = X_0^\beta$.*

PROOF. Let $t \in X^\beta$. Then $\sum_{k=0}^{\infty} x_k t_k$ exists for all $x \in X$ and in particular for $x \in X_0$, since $X_0 \subset X$. Hence $t \in X_0^\beta$ and so $X^\beta \subset X_0^\beta$.

Now if $t \in X_0^\beta$ and $x \in X$ is given by (3.2) then $x - \lambda\delta \in X_0$ and so $\sum_{k=0}^{\infty}(x_k - \lambda)t_k$ is convergent. Define $g : X \to \mathbf{C}$ by

$$g(x) = \sum_{k=0}^{\infty}(x_k - \lambda)t_k. \tag{3.5}$$

Then

$$g_n(x) = \sum_{k=0}^{n}(x_k - \lambda)t_k \tag{3.6}$$

is a continuous linear functional on X and by letting $n \to \infty$ in (3.6), $g \in X^*$ by the Banach–Steinhaus theorem. Clearly $g(\delta^k) = t_k$ implies that $\sum_{k=0}^{\infty} g(\delta^k) = \sum_{k=0}^{\infty} t_k$ which exists since X is semi-conservative. Since $\sum_{k=0}^{\infty}(x_k - \lambda)t_k$ exists it follows that $\sum_{k=0}^{\infty} x_k t_k$ is convergent hence $t \in X^\beta$ so that $X_0^\beta \subset X^\beta$ whence $X_0^\beta = X^\beta$.

THEOREM 3.4. *Let X and Y be Banach spaces and suppose that $(A_k)_0^\infty$ is a sequence of operators on X into Y, then $(A_k)_0^\infty \in \mathrm{bv}_0^\beta(X)$ iff there exists a non-negative integer m such that*

(i) *$A_k \in B(X, Y)$ for all $k \geqq m$,*

(ii) $\sup_{n \geqq m} \left\| \sum_{k=m}^{n} A_k \right\| < \infty$.

PROOF. Let $x \in \mathrm{bv}_0(X)$ and define $s_k = x_k - x_{k-1}$, $x_{-1} = 0$, $s_0 = x_0$, that is, $x_n = \sum_{k=0}^{n} s_k$, $s = (s_k)_0^\infty \in \ell(X)$.

Then $(x_n)_0^\infty$ is a Cauchy sequence in X. To see this let $n \geqq N$, where n, N are large enough so that

$$\|x_n - x_N\| = \left\|\sum_{k=0}^{\infty} s_k - \sum_{k=0}^{N} s_k\right\| = \left\|\sum_{k=N+1}^{n} s_k\right\| \leqq \sum_{k=N+1}^{n} \|s_k\| \to 0$$

as $N \to \infty$ since $s \in \ell(X)$. Applying summation by parts on $\sum_{k=\mu}^{N} A_k x_k$, where $\mu \geqq m$ we obtain

$$\sum_{k=\mu}^{N} A_k s_k = \left(\sum_{k=\mu}^{N} A_k\right)\left(\sum_{\nu=0}^{\mu} s_\nu\right) + \sum_{v=\mu+1}^{N}\left(\sum_{k=\nu}^{N} A_k\right) s_\nu \tag{3.7}$$

whence

$$\left\|\sum_{k=\mu}^{N} A_k x_k\right\| \leqq \left\|\sum_{k=\mu}^{N} A_k\left(\sum_{\nu=0}^{\mu} s_\nu\right)\right\| + \left\|\sum_{\nu=\mu+1}^{N}\left(\sum_{k=\nu}^{N} A_k\right) s_\nu\right\| \leqq \tag{3.8}$$

$$\leqq M\left\|\sum_{\nu=0}^{\mu} s_\nu\right\| + M\sum_{\nu=\mu+1}^{N}\|s_\nu\| \to 0$$

as $\mu, N \to \infty$ since $(S_\nu)_0^\infty$ is a Cauchy sequence and $x_\mu = \sum_{\nu=0}^{\mu} s_\nu \to 0$ as $\mu \to \infty$. Thus $\left(\sum_{k=0}^{N} A_k x_k\right)_{N=0}^{\infty}$ is a Cauchy sequence and hence it converges in Y so that $(A_k)_0^\infty \in \mathrm{bv}_0^\beta(X)$.

To show that (i) is satisfied we suppose that no such non-negative integer m exists for which $A_k \in B(X,Y)$, $k \geqq m$. If no such m exists, then there must exist a strictly increasing sequence of non-negative integers $(k_i)_{i=1}^\infty$ and a sequence $(z_i)_{i=1}^\infty$ in $\mathcal{D} = \{\|z_i\| \leqq 1\}$ such that $\|A_{k_i} z_i\| > i^2$, $i \in \mathbf{N}$ and $k_{i+1} > k_i + 1$. Now define x by

$$x_k = \begin{cases} z_i/i^2, \ k = k_i, & i \in \mathbf{N} \\ \theta, \ k \neq k_i. \end{cases} \tag{3.9}$$

Then $x \in \mathrm{bv}_0(X)$ since $x_k \to 0$ as $k \to \infty$ and also

$$\sum_{k=0}^{\infty}\|x_{k+1} - x_k\| = \sum_{i=1}^{\infty}\left\|\frac{z_i}{i^2}\right\| < \infty$$

since $\|z_i\| \leqq 1$.

But $\|A_{k_i} x_{k_i}\| = \left\|A_{k_i}\left(\frac{z_i}{i^2}\right)\right\| > 1$ for $i \geqq 1$ contrary to the fact that $\sum_{k=0}^{\infty} A_k x_k$ converges which implies that $A_k x_k \to 0$ in Y. Hence the A_k's are ultimately bounded, i.e. there exists $m \in \mathbf{R}$ such that $\|A_k\| < \infty$ for $k > m$.

We now show that condition (ii) is satisfied whenever $\sum_{k=0}^{\infty} A_k x_k$ converges for all $x = (x_k)_0^\infty \in \mathrm{bv}_0(X)$. Since $\sum_{k=0}^{\infty} A_k x_k$ converges by hypothesis, the Banach–Steinhaus theorem tells us that there exists a positive constant K such that

$$\left\| \sum_{k=m}^{\infty} A_k x_k \right\| \leqq K \|x\|_{\mathrm{bv}_0(X)} < K$$

with $x = (x_k)_0^\infty \in \mathcal{D}$. Let $T_N = \sum_{k=m}^{N} A_k : X \to Y$ and suppose that $\sup_{N \geqq m} \|T_N\| = \infty$. Then there exists a strictly increasing sequence of non-negative integers $(N_i)_{i=1}^\infty$ and a sequence $(W_i)_{i=1}^\infty$ in $\mathcal{D}$ such that

$$\|T_{N_i}(W_i)\| = \left\| \left(\sum_{k=m}^{N_i} A_k \right) W_i \right\| > i^2, \quad N_i, \quad i \in \mathbf{N}$$

and as $N_i \to \infty$, $T_{N_i} \to T = \sum_{k=m}^{\infty} A_k : \mathrm{bv}_0(X) \to Y$.

Now define $x = (x_k)_0^\infty = x^i$ by

$$x_k = \begin{cases} W_i/i, \ 0 \leqq k \leqq N_i, & i \in \mathbf{N} \\ \theta, \ k > N_i, & i \in \mathbf{N} \end{cases} \tag{3.10}$$

i.e. $x = x^i = (W_i/i, W_i/i, \dots, W_i/i, \theta, \theta, \dots)$ (the last W_i/i is in the N_ith position). Then $x = x^i \in \mathrm{bv}_0(X)$ and

$$\|x\|_{\mathrm{bv}_0(X)} = \frac{1}{i} \|W_i\| \leqq 1$$

since $W_i \in \mathcal{D}$. However,

$$\|T(x^i)\| = \left\| \left(\sum_{k=m}^{\infty} A_k \right) W_i/i \right\| = \left\| \left(\sum_{k=m}^{N_i} A_k \right) W_i/i \right\| \geqq i,$$

i.e. $\sup_{i \geqq 1} \|T(x^i)\| = \infty$ and so T is unbounded contrary to the fact that $Tx = \sum_{k=m}^{\infty} A_k x_k$ converges and so condition (ii) is satisfied.

THEOREM 3.5. *Let X and Y be any Banach spaces and suppose that $(A_k)_0^\infty$ is a sequence of operators on X into Y.*

Then $(A_k)_0^\infty \in \mathrm{bv}^\beta(X)$ iff

(i) $(A_k)_0^\infty \in \mathrm{bv}_0^\beta(X)$,

(ii) $\sum_{k=0}^\infty A_k x$ *converges for each* $x \in X$.

PROOF. Suppose $x \in \mathrm{bv}(X)$ and $x_k \to \ell$ as $k \to \infty$, then $(x_k - \ell)_0^\infty \in \mathrm{bv}_0(X)$. $\sum_{k=0}^\infty A_k \ell$ converges by condition (ii) so we see that $\sum_{k=0}^\infty A_k \ell + \sum_{k=0}^\infty A_k(x_k - \ell)$ exists and is equal to $\sum_{k=0}^\infty A_k x_k$.

Condition (i) follows from Theorem 3.4 since $\mathrm{bv}_0(X) \subset \mathrm{bv}(X)$. To show that condition (ii) holds, take any $x \in X$, then the sequence $(x, x, \dots) \in \mathrm{bv}(X)$ with $x_k = x_{k+1} = x$ so $\sum_{k=0}^\infty A_k x$ exists.

THEOREM 3.6. *The general form for each* $F \in c^*(X)$, X *any Banach space, is*

$$F(x) = F(y) - \sum_{k=0}^\infty f_k(\ell) + \sum_{k=0}^\infty f_k(x_k) = \chi_F(\ell) + \sum_{k=0}^\infty f_k(x_k)$$

where $\chi_F(\ell) = F(y) - \sum_{k=0}^\infty f_k(\ell)$, $\ell = \lim_{k\to\infty} x_k$, $y = (\ell, \ell, \dots) \in c(X)$, $f_k(x) = F(\theta, \theta, \dots \theta, x, \theta \dots.)$ *with* x *in the* k*th coordinate,* χ_F, $f_k \in X^*$ *and* $\sum_{k=0}^\infty \|f_k\| < \infty$. *Moreover,*

$$\|F\| = \|\chi_F\| + \sum_{k=0}^\infty \|f_k\|.$$

PROOF. The general form for each $F \in c^*(X)$ was proved by Maddox [2], p. 94. So all we need to prove is that $\|F\| = \|\chi_F\| + \sum_{k=0}^\infty \|f_k\|$.

If $F \in c^*(X)$ and F is of the given form, then the triangle inequality gives

$$|F(x)| \leqq \left| F(y) - \sum_{k=0}^\infty f_k(\ell) \right| + \sum_{k=0}^\infty \|f_k\| \, \|x_k\| \leqq$$

$$\leqq \sup_{k \geqq 0} \|x_k\| \left(\|\chi_F\| + \sum_{k=0}^\infty \|f_k\| \right).$$

Since $\|\ell\| \geqq \sup_{k \geqq 0} \|x_k\|$, we have

$$(3.11) \qquad \|F\| \leqq \|\chi_F\| + \sum_{k=0}^{\infty} \|f_k\|.$$

Let $\varepsilon > 0$ and $n \in \mathbf{N}$ be given. Then for $0 \leqq k \leqq n$ there exists $z_k \in \mathcal{U} = \{z \in X : \|z\| = 1\}$ such that $|f_k(z_k)| \geqq \|f_k\| - \frac{\varepsilon}{2^k}$. Define x by

$$(3.12) \qquad x_k = \begin{cases} e^{i\theta_k} \cdot z_k, & 0 \leqq k \leqq n \\ \ell' \neq \theta, \text{ where } \ell' \in \mathcal{U}, & k > n \end{cases}$$

i.e. $x = (x_0, x_1, x_2 \dots, x_n, \ell', \ell', \dots)$, where θ_k is chosen such that $f_k(e^{i\theta_k} \cdot z_k) = |f_k(z_k)|$ and ℓ' so that $\chi_F(\ell') = |\chi_F(\ell')| \geqq \|\chi_F\| - \varepsilon$. So we have

$$(3.13) \qquad F(x) = F(\theta, \theta, \dots, \ell', \ell', \dots) + \sum_{k=0}^{n} f_k(x_k) =$$

$$= F(y') - \sum_{k=0}^{n} f_k(\ell') + \sum_{k=0}^{n} f_k(x_k) = \chi_F(\ell') + \sum_{k=n+1}^{\infty} f_k(\ell') + \sum_{k=0}^{n} f_k(x_k)$$

where

$$y' = (\ell', \ell', \dots), \quad \chi_F(\ell') = F(y') - \sum_{k=0}^{\infty} f_k(\ell').$$

Applying the triangle inequality in the form $|x + y| \geqq |x| - |y|$ to (3.13) we obtain

$$|F(x)| \geqq \left| \chi_F(\ell') + \sum_{k=0}^{n} f_k(x_k) \right| - \left| \sum_{k=n+1}^{\infty} f_k(\ell') \right| \geqq$$

$$\geqq \|\chi_F\| - \varepsilon + \sum_{k=0}^{n} \|f_k\| - \sum_{k=0}^{n} \frac{\varepsilon}{2^k} - \left| \sum_{k=n+1}^{\infty} f_k(\ell') \right|$$

that is,

$$(3.14) \qquad \|F\| \geqq \|\chi_f\| + \sum_{k=0}^{n} \|f_k\| - \varepsilon - \sum_{k=0}^{n} \frac{\varepsilon}{2^k} - \sum_{k=n+1}^{\infty} \|f_k\| \geqq$$

$$\geqq \|\chi_F\| + \sum_{k=0}^{\infty} \|f_k\|$$

as $n \to \infty$ and $\varepsilon \to 0$. Combining (3.11) with (3.14) we obtain

$$\|F\| = \|\chi_F\| + \sum_{k=0}^{\infty} \|f_k\|.$$

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(Received September 15, 1988)

KENYATTA UNIVERSITY
MATHEMATICS DEPARTMENT
P.O.BOX 43844
NAIROBI
KENYA

Acta Math. Hung.
59 (3–4) (1992), 301–307.

ON THE EXISTENCE OF NASH EQUILIBRIUM POINT FOR A GAME BETWEEN INFINITELY MANY PLAYERS WITH NONCOMPACT STRATEGY SETS

E. TARAFDAR (Brisbane)

Introduction

Theorems concerning sets with convex sections, as these are usually referred to, have many applications. Such applications have been given in Fan [4], [5] and [6] and Ma [7] and many others. These theorems usually are concerned with convex sections of compact convex sets, each in a topological vector space. Recently in [8] these theorems have been extended to the case of convex sections of noncompact convex sets. The object of this note is to obtain as application of these theorems the extended version of a theorem on game concerning Nash equilibrium point and related results. In this note we will describe a game simply by the family $\{X_\alpha, g_\alpha : \alpha \in I\}$ where I is the set of players, finite or infinite, X_α is the strategy set for the player $\alpha \in I$ and $g_\alpha : X = \prod_{\alpha\in I} X_\alpha \to \mathbf{R}$ is the pay off function for the player $\alpha \in I$. A point $y \in X = \prod_{\alpha\in I} X_\alpha$ is called a Nash equilibrium point of the game if for each $\alpha \in I$,

$$g_\alpha(y) = \sup_{x_\alpha \in X_\alpha} g_\alpha[x_\alpha, \hat{y}_\alpha]$$

where $\hat{y}_a$ is the projection of y in $\hat{X}_\alpha = \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} X_\beta$. Sometimes an auxiliary payoff function $f_\alpha : X \to \mathbf{R}$ for each $\alpha \in I$ will be involved.

In this note we will use the following notations. Let $\{X_\alpha : \alpha \in I\}$ as above be a family of sets where I is an index set. Then $X = \prod_{\alpha\in I} X_\alpha$ and for each $\alpha \in I$, $\hat{X}_\alpha = \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} X_\beta$. Thus for each $\alpha \in I$, we can write $X = X_\alpha \times \hat{X}_\alpha$ (with proper order). $\hat{x}_\alpha$ will denote an element of $\hat{X}_\alpha$ so that we can write $[x_\alpha, \hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha = X$. The following theorem has been obtained in Fan ([6], Theorem (6)).

THEOREM 1. *Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty compact convex sets, each in a Hausdorff topological vector space E_α, where I is an index set. Let $X = \prod_{\alpha\in I} X_\alpha$ and $\hat{X}_\alpha = \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} X_\beta$ for each $\alpha \in I$. Let $\{A_\alpha : \alpha \in I\}$*

and $\{B_\alpha : \alpha \in I\}$ *be two families of subsets of* X *satisfying the following conditions:*

(a) *for each* $\alpha \in I$ *and* $x_\alpha \in X_\alpha$, *the set* $B_\alpha(x_\alpha) = \{\hat{x}_\alpha \in \hat{X}_\alpha : [x_\alpha, \hat{x}_\alpha] \in \in B_\alpha\}$ *is open in* $\hat{X}_\alpha$;

(b) *for each* $\alpha \in I$ *and each* $\hat{x}_\alpha \in \hat{X}_\alpha$, *the set* $B_\alpha(\hat{x}_\alpha) = \{x_\alpha \in X_\alpha : : [x_\alpha, \hat{x}_\alpha] \in B_\alpha\}$ *is nonempty, and the set* $A_\alpha(\hat{x}_\alpha) = \{x_\alpha \in X_\alpha : [x_\alpha, \hat{x}_\alpha] \in \in A_\alpha\}$ *contains the convex hull of* $B_\alpha(\hat{x}_\alpha)$.

Then $\bigcap_{\alpha \in I} A_\alpha \neq 0$.

The following generalization, of the above theorem which is stated as Theorem 2 has recently been obtained in [8].

THEOREM 2. *Let* $\{X_\alpha : \alpha \in I\}$ *be a family of nonempty convex sets, each in a Hausdorff topological vector space* E_α, *I being index set. Let* $X = \prod_{\alpha \in I} X_\alpha$ *and* $\hat{X}_\alpha = \prod_{\substack{\beta \in I \\ \beta \neq \alpha}} X_\beta$. *Let* $\{A_\alpha : \alpha \in I\}$ *and* $\{B_\alpha : \alpha \in I\}$ *be two families of subsets of* X *satisfying the conditions* (a) *and* (b) *above. Further assume that*

(c) *there is a nonempty compact subset* K *of* X *such that for every* $x \in \in X \setminus K$, $x = [x_\alpha, \hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, *there exists* $y = [y_\alpha, \hat{y}_\alpha] \in X_\alpha \times \hat{X}_\alpha$ *satisfying* $[y_\alpha, \hat{x}_\alpha] \in B_\alpha$ *for each* $\alpha \in I$.

Then there exists at least one point x_0 *such that* $x_0 \in \bigcap_{\alpha \in I} A_\alpha$.

REMARK. It is already noted in [6] that the idea of introducing two families $\{A_\alpha : \alpha \in I\}$ and $\{B_\alpha : \alpha \in I\}$ is inspired by a generalization of the Knaster–Kuratowski–Mazurkiewicz theorem by Dugundji and Granas [3] and a result of Ben-El-Mechaiekh et al ([1], Theorem 5).

THEOREM 3. *Let* $\{X_\alpha : \alpha \in I\}$ *be a family of nonempty convex sets, each in a Hausdorff topological vector space. Let* $X = \prod_{\alpha \in I} X_\alpha$ *and for each* $\alpha \in I$, *let* $\hat{X}_\alpha = \prod_{\substack{\beta \in I \\ \beta \neq \alpha}} X_\beta$. *Let* $\{f_\alpha : \alpha \in I\}$ *and* $\{g_\alpha : \alpha \in I\}$ *be two families of real valued functions defined on* X *satisfying the following conditions:*

(i) $f(x) \leqq g(x)$ *for all* $x \in X$;

(ii) *for each* $\alpha \in I$ *and each* $x_\alpha \in X_\alpha$, $f_\alpha[x_\alpha, \cdot]$ *is a lower semicontinuous function on* $\hat{X}_\alpha$;

(iii) *for each* $\alpha \in I$ *and each* $\hat{x}_\alpha \in \hat{X}_\alpha$, $g_\alpha[\cdot, \hat{x}_\alpha]$ *is quasiconcave function on* X_α;

(iv) *there exists a family* $\{t_\alpha : \alpha \in I\}$ *of real numbers such that for each* $\alpha \in I$ *and each* $\hat{x}_\alpha \in \hat{X}_\alpha$ *there exists* $y_\alpha \in X_\alpha$ *with* $f_\alpha[y_\alpha, \hat{x}_\alpha] > t_\alpha$;

(v) *there is a nonempty compact subset* K *of* X *such that for every* $x \in \in X \setminus K$, $x = [x_\alpha, \hat{x}_\alpha] = X_\alpha \times \hat{X}_\alpha$, *there exists* $y \in K$, $y = [y_\alpha, \hat{y}_\alpha] \in X_\alpha \times \hat{X}_\alpha$

satisfying $f_\alpha[y_\alpha, \hat{x}_\alpha] > t_\alpha$ *for each* $\alpha \in I$. *Then there exists a point* $u \in X$ *such that for each* $\alpha \in I$, $g_\alpha(u) > t_\alpha$.

PROOF. For each $\alpha \in I$, we define the sets

$$B_\alpha = \{x \in X : f_\alpha(x) > t_\alpha\} \quad \text{and} \quad A_\alpha = \{x \in X : g_\alpha(x) > t_\alpha\}.$$

By condition (ii) for each $\alpha \in I$ and each $x_\alpha \in X_\alpha$, the set $B_\alpha(x_\alpha) = \{\hat{x}_\alpha \in \hat{X}_\alpha : [x_\alpha, \hat{x}_\alpha] \in B_\alpha\}$ is open in $\hat{X}_\alpha$ and hence condition (a) of Theorem 2 is satisfied. By virtue of condition (i) for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}_\alpha$, the set $B_\alpha(\hat{x}_\alpha) = \{x_\alpha \in X_\alpha : [x_\alpha, \hat{x}_\alpha] \in B_\alpha\}$ is a subset of the set $A_\alpha(\hat{x}_\alpha) = \{x_\alpha \in X_\alpha : [x_\alpha, \hat{x}_\alpha] \in A_\alpha\}$ and by virtue of condition (iii) $A_\alpha(\hat{x}_\alpha)$ is convex. Also by condition (iv) for each $\alpha \in I$ and $\hat{x}_\alpha \in \hat{X}_\alpha$, the set $B_\alpha(\hat{x}_\alpha)$ is nonempty. Hence condition (b) of Theorem 2 is fulfilled. Finally condition (v) implies condition (c) of Theorem 2. Thus by Theorem 2 there exists a point $y \in X$ such that $y \in \bigcap_{\alpha \in I} A_\alpha$, i.e. $g_\alpha(y) > t_\alpha$ for each $\alpha \in I$.

COROLLARY 1. *Let* $\{X_\alpha : \alpha \in I\}$, $\{\hat{X}_\alpha : \alpha \in I\}$ *and* X *as in Theorem* 3. *Let* $\{f_\alpha : \alpha \in I\}$ *be a family of real valued functions defined on* X *satisfying the following conditions:*

(i) *for each* $\alpha \in I$ *and each* $x_\alpha \in X_\alpha$, $f_\alpha[x_\alpha, \cdot]$ *is a lower semicontinuous function on* $\hat{X}_\alpha$;

(ii) *for each* $\alpha \in I$ *and each* $\hat{x}_\alpha \in \hat{X}_\alpha$, $f_\alpha[\cdot, \hat{x}_\alpha]$ *is a quasiconcave function on* X_α;

(iii) *there exists a family* $\{t_\alpha : \alpha \in I\}$ *of real numbers such that for each* $\alpha \in I$ *and each* $\hat{x}_\alpha \in \hat{X}_\alpha$ *there exists* $y_\alpha \in X_\alpha$ *with* $f_\alpha[y_\alpha, \hat{x}_\alpha] > t_\alpha$;

(iv) *there exists a nonempty compact subset* K *of* X *such that for each* $x \in X \setminus K$, $x = [x_\alpha, \hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, *there exists* $y \in K$, $y = [y_\alpha, \hat{y}_\alpha] \in X_\alpha \times \hat{X}_\alpha$ *satisfying* $f_\alpha[y_\alpha, \hat{x}_\alpha] > t_\alpha$.

Then there exists a point $u \in X$ *such that* $f_\alpha(u) > t_\alpha$ *for each* $\alpha \in I$.

PROOF. Taking $f_\alpha = g_\alpha$ for each $\alpha \in I$ in Theorem 3 we obtain the corollary.

The above corollary is a generalization of a theorem of Ma ([7], Theorem 3) which includes a Theorem of Fan ([5], Theorem 3) and also Browder ([2], Theorem 2).

Existence theorem of Nash equilibrium point

THEOREM 4. *Let* $\{X_\alpha : \alpha \in I\}$ *be a family of nonempty convex sets, each in a Hausdorff topological vector space* E_α. *Let* $X = \prod_{\alpha \in I} X_\alpha$ *and for* *each $\alpha \in I$, let $\hat{X}_\alpha = \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} X_\beta$. Let $\{f_\alpha : \alpha \in I\}$ and $\{g_\alpha : \alpha \in I\}$ be two families of real valued functions defined on X. Further assume that*

(i) *for each $\alpha \in I$, f_α is continuous;*

(ii) *for each $\alpha \in I$ and each $x \in X$, $f_\alpha(x) \leqq g_\alpha(x)$;*

(iii) *for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}_\alpha$, $\sup_{y_\alpha\in X_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] = \sup_{y_\alpha\in X_\alpha} g_\alpha[y_\alpha,\hat{x}_\alpha]$;*

(iv) *for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}_\alpha$, $g_\alpha[\cdot,\hat{x}_\alpha]$ is a quasiconcave function on X_α;*

(v) *there exists a nonempty compact subset K of X such that for each $x \in X \setminus K$, $x = [x_\alpha, \hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, there exists $y = [y_\alpha, \hat{y}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, $y \in K$ such that $f_\alpha[y_\alpha,\hat{x}_\alpha] \geqq f_\alpha[x_\alpha,\hat{x}_\alpha]$ for every $\alpha \in I$.*

Then there exists a point $w \in X$ such that for each $\alpha \in I$, $g_\alpha(w) = \sup_{u_\alpha\in X_\alpha} g_\alpha[u_\alpha,\hat{w}_\alpha]$ where $\hat{w}_\alpha$ is the projection of w in $\hat{X}_\alpha$ for each $\alpha \in I$. In other words $w \in X$ is a Nash equilibrium of the game $\{X_\alpha, g_\alpha, I\}$.

REMARK. Conditions (ii) and (iii) will disappear in our next corollary.

PROOF. We will apply Theorem 1 to prove this theorem. Let for each $\alpha \in I$, $P_\alpha : X \to X_\alpha$ be the projection of X onto X_α. For each $\alpha \in I$, let $P_\alpha(K) = X'_\alpha$ which is nonempty compact and convex. Let $X' = \prod_{\alpha\in I} X'_\alpha$ and for each $\alpha \in I$, $\hat{X}'_\alpha = \prod_{\substack{\beta\in I\\ \beta\neq\alpha}} X'_\alpha$.

We first prove that

(A) for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$,

$$\sup_{y_\alpha\in X_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] = \sup_{y_\alpha\in X'_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha].$$

It is clear that $\sup_{y_\alpha\in X_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] \geqq \sup_{y_\alpha\in X'_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] = f_\alpha[v_\alpha,\hat{x}_\alpha]$ for some $v_\alpha \in X'_\alpha$ as X'_α is compact and $f_\alpha[\cdot,\hat{x}_\alpha]$ is continuous by condition (i). If possible, let $\sup_{y_\alpha\in X_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] > f_\alpha[v_\alpha,\hat{x}_\alpha]$. Then there must exist $u_\alpha \in X_\alpha$ and $u_\alpha \notin X'_\alpha$ such that $f_\alpha[u_\alpha,\hat{x}_\alpha] > f_\alpha[v_\alpha,\hat{x}_\alpha]$. Thus $x = [u_\alpha,\hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha = X$ but $x \notin K$ as $K \subset X'$. Hence by condition (v) there exists $z = [z_\alpha,\hat{z}_\alpha] \in K$ such that $f_\alpha[z_\alpha,\hat{x}_\alpha] \geqq f_\alpha[u_\alpha,\hat{x}_\alpha]$. Thus $f_\alpha[v_\alpha,\hat{x}_\alpha] \geqq f_\alpha[z_\alpha,\hat{x}_\alpha] \geqq f_\alpha[u_\alpha,\hat{x}_\alpha]$, the first inequality being due to the fact that $z_\alpha \in X'_\alpha$. Hence we have arrived at a contradiction which proves our assertion.

Now by using (iii) and (A) above we obtain that for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$

$$\sup_{y_\alpha\in X_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] = \sup_{y_\alpha\in X'_\alpha} f_\alpha[y_\alpha,\hat{x}_\alpha] \leqq \sup_{u_\alpha\in X'_\alpha} g_\alpha[u_\alpha,\hat{x}_\alpha] \leqq$$

$$\leqq \sup_{y_\alpha \in X_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha] = \sup_{y_\alpha \in X_\alpha} f_\alpha[y_\alpha, \hat{x}_\alpha],$$

by (ii). Hence

(B) $\sup_{y_\alpha \in X_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha] = \sup_{u_\alpha \in X'_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha]$ for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$.

We note that all the quantities in (A) and (B) are finite.

Now as in the proof of Theorem 4 in Fan [5] and Ma [7] we define for each real number $\varepsilon > 0$ and each $\alpha \in I$, the sets

$$A_{\alpha,\varepsilon} = \left\{ x = [x_\alpha, \hat{x}_\alpha] \in X' : g_\alpha(x) > \sup_{y_\alpha \in X'_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha] - \varepsilon \right\};$$

$$B_{\alpha,\varepsilon} = \left\{ x = [x_\alpha, \hat{x}_\alpha] \in X' : f_\alpha(x) > \sup_{g_\alpha \in X'_\alpha} f_\alpha[y_\alpha, \hat{x}_\alpha] - \varepsilon \right\};$$

and

$$C_{\alpha,\varepsilon} = \left\{ x = [x_\alpha, \hat{x}_\alpha] \in X : g_\alpha(x) > \sup_{y_\alpha \in X_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha] - \varepsilon \right\}.$$

From the uniform continuity of f_α on the compact set X' it follows that for each $\alpha \in I$, the function $\phi_\alpha(\hat{x}_\alpha) = \sup_{y_\alpha \in X'_\alpha} f_\alpha[y_\alpha, \hat{x}_\alpha]$ is continuous on $\hat{X}'_\alpha$. Hence for each $\alpha \in I$ and each $x_\alpha \in X'_\alpha$, the set $B_{\alpha,\varepsilon}(x_\alpha) = \{\hat{x}_\alpha \in \hat{X}'_\alpha : [x_\alpha, \hat{x}_\alpha] \in B_{\alpha,\varepsilon}\}$ is open in $\hat{X}_\alpha$. Also from the quasiconcavity of $g_\alpha[\cdot, \hat{x}_\alpha]$ on X_α, it follows that for each $\alpha \in I$ and each $\hat{x}_\alpha$ in $\hat{X}_\alpha$, the set $C_{\alpha,\varepsilon}(\hat{x}_\alpha) = \{x_\alpha \in X_\alpha : [x_\alpha, \hat{x}_\alpha] \in C_{\alpha,\varepsilon}\}$ is convex. Since X'_α is convex for each $\alpha \in I$, it is easy to see that for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$, the set $A_{\alpha,\varepsilon}(\hat{x}_\alpha) = \{x_\alpha \in X'_\alpha : [x_\alpha, \hat{x}_\alpha] \in A_{\alpha,\varepsilon}\} = \{x_\alpha \in X_\alpha : [x_\alpha, \hat{x}_\alpha] \in C_{\alpha,\varepsilon}\} \cap X'_\alpha$ is convex.

Further due to the continuity of f_α on the compact set X', it is trivial to see that for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$ the set $B_{\alpha,\varepsilon}(\hat{x}_\alpha)$ is nonempty. Finally for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}'_\alpha$, the set $B_{\alpha,\varepsilon}(\hat{x}_\alpha) \subset A_{\alpha,\varepsilon}(\hat{x}_\alpha)$. This follows from the fact that for each $\alpha \in I$, $B_{\alpha,\varepsilon} \subset A_{\alpha,\varepsilon}$. To see this let $x = [x_\alpha, \hat{x}_\alpha] \in X'$. Then

$$f_\alpha(x) > \sup_{y_\alpha \in X'_\alpha} f_\alpha[y_\alpha, \hat{x}_\alpha] - \varepsilon = \sup_{u_\alpha \in X_\alpha} f_\alpha[u_\alpha, \hat{x}_\alpha] - \varepsilon =$$

$$= \sup_{u_\alpha \in X_\alpha} g_\alpha[u_\alpha, \hat{x}_\alpha] - \varepsilon = \sup_{y_\alpha \in X'_\alpha} g_\alpha[u_\alpha, \hat{x}_\alpha] - \varepsilon$$

by (A), (B) and condition (iii). Thus

$$g_\alpha(x) \geqq f_\alpha(x) > \sup_{y_\alpha \in X'_\alpha} g_\alpha[y_\alpha, \hat{x}_\alpha] - \varepsilon,$$

i.e. $x \in A_{\alpha,\varepsilon}$. Hence applying Theorem 1 to the system $\{X'_\alpha A_{\alpha,\varepsilon}, B_{\alpha,\varepsilon} : \alpha \in \in I\}$ we obtain a point $y \in \bigcap_{\alpha \in I} A_{\alpha,\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, it follows from the definition of $A_{\alpha,\varepsilon}$ that the family of closed subsets $\{\bigcap_{\alpha\in I} \overline{A}_{\alpha,\varepsilon} : \varepsilon > 0\}$ of the compact set X' has finite intersection property. Hence there exists a point $w \in X'$ such that $w \in \bigcap_{\alpha \in I} \overline{A}_{\alpha,\varepsilon}$ for all $\varepsilon > 0$. This point $w \in X'$ has the property that $g_\alpha(w) = \sup_{u_\alpha \in X'_\alpha} g_\alpha[u_\alpha, \hat{w}_\alpha]$ for each $\alpha \in I$ where $\hat{w}_\alpha$ is the projection of w in $\hat{X}'_\alpha$ and hence in $\hat{X}_\alpha$. Hence by (B) $g_\alpha(w) = = \sup_{u_\alpha \in X_\alpha} g_\alpha[u_\alpha, \hat{w}_\alpha]$ for each $\alpha \in I$.

COROLLARY 2. *Let $\{X_\alpha : \alpha \in I\}$ be a family of nonempty convex sets, each in a Hausdorff topological space E_α. Let $X = \prod_{\alpha \in I} X_\alpha$ and for each $\alpha \in I$, let $\hat{X}_\alpha = \prod_{\substack{\beta \in I \\ \beta \neq \alpha}} X_\beta$. Let $\{g_\alpha : \alpha \in I\}$ be a family of real valued functions defined on X. Further assume that*

(i) *for each $\alpha \in I$, $g_\alpha(x)$ is continuous;*

(ii) *for each $\alpha \in I$ and each $\hat{x}_\alpha \in \hat{X}_\alpha$, $g_\alpha[\cdot, \hat{x}_\alpha]$ is a quasiconcave function on X_α;*

(iii) *there exists a compact subset K of X such that for each $x \in X \setminus K$, $x = [x_\alpha, \hat{x}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, there exists $y = [y_\alpha, \hat{y}_\alpha] \in X_\alpha \times \hat{X}_\alpha$, $y \in K$ such that $g_\alpha[y_\alpha, \hat{x}_\alpha] \geqq g_\alpha[x_\alpha, \hat{x}_\alpha]$ for each $\alpha \in I$.*

Then there exists a point $v \in X$ such that for each $\alpha \in I$, $g_\alpha(v) = = \sup_{u_\alpha \in X_\alpha} g_\alpha[u_\alpha, \hat{v}_\alpha]$ where $\hat{v}_\alpha$ is the projection of v in $\hat{X}_\alpha$ for each $\alpha \in I$. In other words there exists a Nash equilibrium point $v \in X$ for the game $\{X_\alpha, g_\alpha : \alpha \in I\}$.

PROOF. We take $f_\alpha = g_\alpha$ for each $\alpha \in I$. Then the families $\{f_\alpha : \alpha \in I\}$ and $\{g_\alpha : \alpha \in I\}$ satisfy all the conditions of Theorem 4. Hence the corollary follows from Theorem 4.

The above corollary generalizes the corresponding theorem of Ma ([7], Theorem 4) and Fan ([5], Theorem 4). Results dual, in the sense of [9], to those of Theorems 3 and 4 and Corollaries 1 and 2 can similarly be formulated and proved.

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(Received September 16, 1988)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF QUEENSLAND
ST. LUCIA
BRISBANE
AUSTRALIA 4067

Acta Math. Hung.
59 (3–4) (1992), 309–318.

ON $U(\cdot)$-INVARIANCE FOR FINITE AND INFINITE DIMENSIONAL CONTROL SYSTEMS*

E. AVGERINOS (Thessaloniki), E. FLYTZANIS (Athens),
and N. S. PAPAGEORGIOU* (Athens)

1. Introduction

The purpose of this paper is to study the property of set invariance in connection with finite and infinite dimensional, nonlinear control systems. Our work was motivated by the papers of Feuer–Heymann [5], [6], who investigated this problem in the context of nonlinear, finite dimensional control systems. Using some recent results and techniques from the theory of multifunctions and the theory of differential inclusions, we are able to relax some of the restrictive hypotheses that Feuer–Heymann [5], [6] have and also consider infinite dimensional control systems (distributed parameter systems).

In the next section, we establish our notation and recall some basic definitions and facts from nonsmooth analysis and the theory of multifunctions. In Section 3 we study the problem of $U(\cdot)$-invariance for nonlinear, finite dimensional control systems, extending the works of Feuer–Heymann [5], [6]. Finally, in Section 4 we address similar questions in the context of infinite dimensional, generally nonlinear, control systems.

2. Preliminaries

Let X be a Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteqq X : \text{ nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteqq X : \text{ nonempty, } (w)\text{-compact, (convex)}\}.$$

Let K be nonempty and $x \in \overline{K}$. The *Bouligand (or contingent) cone* $T_K(x)$ to K at x is defined by

$$T_K(x) = \left\{ h \in X : \lim_{\lambda \downarrow 0} \frac{d_K(x + \lambda h)}{\lambda} = 0 \right\}$$

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* Research supported by N.S.F. Grant DMS–8802688.

where for any $z \in X$, $d_K(z) = \inf\{\|z - x'\| : x' \in K\}$ (see for example Aubin–Ekeland [1]). It is clear that $T_K(x) = T_{\overline{K}}(x)$ and that the cone is always closed, but need not be convex. It is convex, when K is convex (or more generally locally convex at x). For details the interested reader can consult the book of Aubin–Ekeland [1].

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c.), if for every $K \subseteqq Z$ nonempty, closed, the set $G^-(K) = \{y \in Y : G(y) \cap K \neq \emptyset\}$ is closed in Y. If $G(\cdot)$ has compact values, then it maps compact sets into compact sets i.e. if $K \subseteqq Y$ is compact, then $G(K) = \bigcup_{y \in K} G(y)$ is compact in Z (see Klein–Thompson [7], Theorem 7.4.2, p. 90).

Also if X is a metric space, then on $P_f(X)$ we can define the following (generalized) metric

$$h(A, B) = \max\{\sup_{a \in A} d_B(a), \sup_{b \in B} d_A(b)\},$$

known in the literature as the *Hausdorff metric*.

Recall that if X is complete, then so are the metric spaces $(P_f(X), h)$ and $(P_{fc}(X), h)$. Finally if X is a Banach space and $A \in 2^X \setminus \{\emptyset\}$, the support function of A, $\sigma_A : X^* \to \mathbf{R} = \mathbf{R} \cup \{+\infty\}$ is defined by $\sigma_A(x^*) = = \sup\{(x^*, a) : a \in A\}$.

3. Finite dimensional control systems

In this section we examine the invariance properties of the following nonlinear, finite dimensional control system, with state dependent control constraints (feedback or closed loop constraints):

$$(*) \qquad \begin{Bmatrix} \dot{x}(t) = f(x(t), u(t)) & \text{a.e. on} \quad T = [0, b] \\ x(0) = x_0, \quad u(t) \in U(x(t)) & \text{a.e.} \end{Bmatrix}$$

Let $K \subseteqq \mathbf{R}^n$ be a nonempty, closed subset in the state space of $(*)$. In analogy with the classical theory of dynamical systems, we say that K is $U(\cdot)$-invariant, if for each $x_0 \in K$ there exists admissible "state-control" pair $(x(\cdot), u(\cdot))$ such that $x(t) \in K$ for all $t \in T$. Throughout this paper, we will assume that to each feasible control function $u(\cdot)$, there corresponds a unique trajectory $x(\cdot)$.

We will start with a sufficient condition for invariance. Our result extends the corresponding sufficiency part of Theorem 3.2 of Feuer–Heymann [5].

We will need the following hypotheses concerning the data of system $(*)$.

H(f): $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is a control vector field such that
(1) $(x, u) \to f(x, u)$ is continuous,

(2) $\|f(x,u)\| \leqq c(1+\|x\|)$ for all $u \in U(x)$,
(3) $f(x, U(x))$ is convex.

H(U): $U : \mathbf{R}^n \to P_k(\mathbf{R}^m)$ is an u.s.c. multifunction.

H(K): $K \subseteqq \mathbf{R}^n$ is nonempty, closed.

H_τ: for every $x \in K$ there exists $u \in U(x)$ such that $f(x,u) \in T_K(x)$.

THEOREM 3.1. *If hypotheses* H(f), H(U), H(K) *and* H_τ *hold, then K is $U(\cdot)$-invariant.*

PROOF. Let $F : \mathbf{R}^n \to P_{kc}(\mathbf{R})$ be defined by $F(x) = \bigcup_{u \in U(x)} f(x,u)$. We claim that $F(\cdot)$ is u.s.c. To this end, we will show that given $C \subseteqq \mathbf{R}^n$ nonempty, closed, the set $F^-(C) = \{x \in \mathbf{R}^n : F(x) \cap C \neq \emptyset\}$ is closed. So let $x_n \to x$, $x_n \in F^-(C)$. Pick $h_n \in F(x_n) \cap C$. So $h_n = f(x_n, u_n)$ with $u_n \in \in U(x_n)$. Since $U(\cdot)$ is u.s.c. and compact valued, we have that $\overline{\bigcup_{n \geqq 1} U(x_n)}$ is compact in $\mathbf{R}^m$ (see Section 2). So by passing to a subsequence if necessary we may assume that $u_n \to u$. Recall that an u.s.c. multifunction with closed values has closed graph (see Klein–Thompson [7]) and so $(x,u) \in \mathrm{Gr}\, U \Longrightarrow \Longrightarrow u \in U(x)$. Then $f(x_n, u_n) \to f(x,u) \Longrightarrow h_n \to f(x,u) \Longrightarrow f(x,u) \in \in F(x) \cap C \Longrightarrow x \in F^-(C) \Longrightarrow F^-(C)$ is closed and so $F(\cdot)$ is u.s.c.

Next, because of hypothesis H_τ we have $F(x) \cap T_K(x) \neq \emptyset$ for all $x \in K$. Invoking Theorem 1, p. 24 of Deimling [2], we get a solution $x(\cdot)$ of the differential inclusion $\dot{x}(t) \in F(x(t))$ a.e. that leaves K invariant. Then let $L : T \to 2^{\mathbf{R}^m}$ be defined by

$$L(t) = \{u \in U(x(t)) : \dot{x}(t) = f(x(t), u)\}.$$

Clearly $L(t) \neq \emptyset$ a.e. and by redefining it on a Lebesgue null set, we may assume that $L(t) \neq \emptyset$ for all $t \in T$. Because of H(U), $t \to U(x(t))$ is a measurable multifunction (i.e. $\mathrm{Gr}\, U(x(\cdot)) = \{(t,u) \in T \times \mathbf{R}^m : u \in \in U(x(t))\} \in B(T) \times B(\mathbf{R}^m)$), while because of H($f$), $(t,u) \to \dot{x}(t) - -f(x(t),u) = k(t,u)$ is measurable too. Hence

$$\mathrm{Gr}\, L = \{(t,u) \in T \times \mathbf{R}^m : k(t,u) = 0\} \cap \mathrm{Gr}\, U(x(\cdot)) \in \widehat{B(T)} \times B(\mathbf{R}^m)$$

where $\widehat{B(T)}$ is the Lebesgue completion of the Borel σ-field $B(T)$. Apply Aumann's selection theorem (see for example Wagner [12]), to get $u : T \to \mathbf{R}^m$ Lebesgue measurable such that $u(t) \in U(x(t))$, $t \in T$ and $\dot{x}(t) = f(x(t), u(t))$ a.e. So (x,u) is an admissible state control pair and $x(\cdot)$ leaves K invariant. Q.E.D.

REMARK. Our result generalizes Theorem 3.2 of Feuer–Heymann [5] in various ways. More specifically in [5] the control vector field $f(x,u)$ is assumed to be jointly continuous and continuously differentiable in x (hypothesis A_1). Here we drop the differentiability hypothesis. Also in [5] the control constraint set is constant, while here is in "closed-loop" form. Finally in [5], the invariance domain K is compact and convex. Here we drop both those requirements.

We want to have a converse of Theorem 3.1, extending this way the necessity part of Theorem 3.2 of Feuer–Heymann [5].

THEOREM 3.2. *If hypotheses* H(f), H(U), H(K) *hold and for every* $x_0 \in K$, *the control system* $(*)$ *admits a* K*-invariant trajectory, then* H_τ *holds.*

PROOF. Let $x_0 \in K$ and let $x(\cdot)$ be the K-invariant trajectory of $(*)$ emanating from x_0. Recall from the proof of Theorem 3.2 that $F(x) = \bigcup_{u \in U(x)} f(x,u)$ is u.s.c. and $P_{kc}(\mathbf{R}^n)$-valued. So for all $n \geqq 1$, there exists $\delta(n) > 0$ such that for $t < \delta$, we have $F(x(t)) \subseteqq F(x_0) + \frac{1}{2n}B_1$, with B_1 being the open unit ball in $\mathbf{R}^n$. Let $t_n \in \mathbf{R}_+$ such that $t_n \downarrow 0$ and $\dot{x}(t_n)$ exists. Then since $\dot{x}(t) \in F(x(t)) \subseteqq F(x_0) + \frac{1}{2n}B_1$ a.e. on $[0,\delta]$, for $n \geqq 1$ large enough we have

$$0 = d_K(x(t_n)) = d_K\left(x_0 + \int_0^{t_n} \dot{x}_n(s)ds\right) = d_K(x_0 + t_n q_n)$$

where $q_n \in F(x_0) + \frac{1}{2n}B_1$. Let $p_n \in F(x_0)$ such that $\|p_n - q_n\| < \frac{1}{2n}$. We have

$$\frac{1}{t_n} d_K(x_0 + t_n p_n) \leqq \|p_n - q_n\| < \frac{1}{2n} \to 0 \quad \text{as} \quad n \to \infty.$$

Recall that $F(x_0) \in P_{kc}(\mathbf{R}^n)$. So by passing to a subsequence, we may assume that $p_n \to p \in F(x_0)$. We have

$$\frac{1}{t_n} d_K(x_0 + t_n p) \leqq \|p - p_n\| + \frac{1}{t_n} d_K(x_0 + t_n p_n) \to 0 \quad \text{as} \quad n \to \infty,$$

whence $p \in T_K(x_0) \cap F(x_0)$ and $T_K(x_0) \cap F(x_0) \neq \emptyset$. Since $x_0 \in K$ was arbitrary, we conclude that H_τ holds. Q.E.D.

REMARK. Again our hypotheses are considerably weaker than those in Feuer–Heymann [5].

The tangential condition H_τ is also related to the existence of rest points for the system $(*)$. Recall that a point $\hat{x} \in K$ is called a $U(\cdot)$-rest point of $(*)$, if there exists $\hat{u} \in U(\hat{x})$ such that $f(\hat{x}, \hat{u}) = 0$. Our result extends Theorem 3.3 of Feuer–Heymann [5].

We will need the following stronger hypothesis on the invariance domain K.

H(K)$'$: $K \subseteqq \mathbf{R}^n$ is nonempty, compact, convex.

THEOREM 3.3. *If hypotheses* H(f), H(U), H(K)$'$ *hold and* K *is* $U(\cdot)$-*invariant then the control system* $(*)$ *has a rest point.*

PROOF. From the proof of Theorem 3.1 we know that $F : \mathbf{R}^n \to P_{kc}(\mathbf{R}^n)$ defined by $F(x) = \bigcup_{u\in U(x)} f(x,u)$ is u.s.c. Also from Theorem 3.2 we know that since K is $U(\cdot)$-invariant, the tangential hypothesis H_τ is satisfied. So we can apply Theorem 11, p. 341 of Aubin–Ekeland [1] and conclude that there exists $\hat{x} \in K$ such that $0 \in F(\hat{x})$. Then from the definition of $F(\cdot)$, we know that there exists $\hat{u} \in F(x)$ such that $f(\hat{x},\hat{u}) = 0$. So $\hat{x}$ is the desired $U(\cdot)$-rest point of $(*)$. Q.E.D.

REMARK. The compactness hypothesis on K cannot be dropped as the following counterexample shows (we thank the referee for it): $n = 2$, $m = 1$, $f(x,u) = (1,u)^T$, $U(x) = [-1,1] \subseteqq \mathbf{R}$ and $K = \{x \in \mathbf{R}^2 : x_1 \geqq 0,\ |x_2| \leqq x_1\}$. It is easy to check that K is U-invariant, but $f(x,u) \neq 0$ for all $(x,u) \in \in \mathbf{R}^2 \times U$.

We will close this section with a convergence result. We will need the following hypothesis:

H(K_r): $\{K_r\}_{r\geqq 1} \subseteqq P_f(\mathbf{R}^n)$ are $U(\cdot)$-invariant for the system $(*)$ and $K_r \xrightarrow{h} K$ as $r \to \infty$.

THEOREM 3.4. *If hypotheses* H(f), H(U) *and* H(K_r) *hold, then* K *is* $U(\cdot)$-*invariant too.*

PROOF. Let $x \in K$. Then because of H(K_r) we can find $x_r \in K_r$ such that $x_r \to x$. Since K_r is by hypothesis $U(\cdot)$-invariant, Theorem 3.2 tells us that hypothesis H_τ is satisfied. So we can find $u_r \in U(x_r)$ such that $f(x_r,u_r) \in T_{K_r}(x_r)$. Because of hypothesis H(U), we know that $\overline{\bigcup_{r\geqq 1} U(x_r)}$ is compact in $\mathbf{R}^m$. So by passing to a subsequence if necessary, we may assume that $u_r \to u \in U(x)$. Then, since by hypothesis H(f), $f(x_r,u_r) \to f(x,u)$, we have $d_{K_r}(x_r + \lambda f(x_r,u_r)) \to d_K(x + \lambda f(x,u))$ as $r \to \infty$, uniformly for λ in a bounded interval of $\mathbf{R}_+$.

Thus given $\varepsilon > 0$, we can find $r_0 \geqq 1$ such that for $r \geqq r_0$ we have

$$\frac{d_K(x+\lambda f(x,u))}{\lambda} \leqq \frac{d_{K_r}(x_r+\lambda f(x_r,u_r))}{\lambda} + \varepsilon \quad \text{for all} \quad \lambda \in [0,M],$$

whence

$$\varliminf_{\lambda\downarrow 0} \frac{d_K(x+\lambda f(x,u))}{\lambda} \leqq \varliminf_{\lambda\downarrow 0} \frac{d_{K_r}(x_r+\lambda f(x_r,u_r))}{\lambda} + \varepsilon$$

and

$$\varliminf_{\lambda\downarrow 0} \frac{d_K(x+\lambda f(x,u))}{\lambda} \leqq \varepsilon$$

(since $f(x_r, u_r) \in T_{K_r}(x_r)$).

Let $\varepsilon \downarrow 0$. We get $\lim\limits_{\lambda \downarrow 0} \frac{d_K(x+\lambda f(x,u))}{\lambda} = 0 \Longrightarrow f(x,u) \in T_K(x)$ and since $x \in K$ was arbitrary, Theorem 3.1 tells us that K is $U(\cdot)$-invariant. Q.E.D.

4. Infinite dimensional control systems

In this section we turn our attention to infinite dimensional control systems (distributed parameter systems).

The mathematical setting follows that of Lions [8].

Let X, H be separable Hilbert spaces, with X continuously and densely embedded into H. By identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$ with all embeddings continuous and dense. We will also assume that they are compact. To have a concrete example in mind let $H = L^2(0,1)$ and $X = H_0^1(0,1)$. Then $X^* = H^{-1}(0,1)$. Usually such a triple of spaces (X, H, X^*) is known as "Gelfand triple". By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm in X (resp. in H, X^*) and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X, X^*). Furthermore let Z be a separable Banach space (modelling the control space) and as before $T = [0, b]$.

The nonlinear controlled evolution under consideration is the following:

$$(**) \qquad \left\{ \begin{array}{ll} \dot{x}(t) + Ax(t) = f(x(t), u(t)) & \text{a.e. on} \quad T = [0,b] \\ x(0) = x_0, \ u(t) \in U(x(t)) & \text{a.e.} \end{array} \right\}$$

By X_w, H_w, X_w^*, Z_w we will denote the space X, H, X^*, Z with their respective weak topologies. Also note that since X^* is a separable Hilbert space, it admits an orthonormal basis $\{e_k^*\}_{k \geqq 1}$. Let $X_n^* = \{e_k^*\}_{k=1}^n$. Then clearly $X^* = \overline{\bigcup\limits_{n \geqq 1} X_n^*}^{\|\cdot\|_*}$. Denote by $p_n : X^* \to X_n^*$ the corresponding linear projections.

We will need the following hypotheses on the data of $(**)$.

H(A): $A : X \to X^*$ is an operator such that

(1) it is sequentially continuous from X_w into X_w^* and monotone,

(2) $\|Ax\|_* \leqq c(1 + \|x\|)c > 0$,

(3) $\langle Ax, x\rangle \geqq c'\|x\|^2$, $c' > 0$.

H(f)$'$: $f : H \times Z \to H$ is a function such that

(1) $(x,u) \to f(x,u)$is sequentially continuous from $H \times Z_w$ into H_w,

(2) $|f(x,u)| \leqq k|x|$, $k > 0$,

(3) $f(x, U(x)) = \bigcup\limits_{u \in U(x)} f(x,u)$ is convex.

H(K)$''$: $K \subsetneqq X$ is bounded, closed, convex.

H(U)$''$: $U : H \to P_{wkc}(Z)$ is u.s.c. from H into Z_w,

H'_τ: for every $x \in K$, there exists $u \in U(x)$ such that $(-Ax + f(x,u)) \in \in T'_K(x)$, where $T'_K(x)$ is the tangent cone to K at x in the Hilbert space X^*.

THEOREM 4.2. *If hypotheses* $\mathrm{H}(A)$, $\mathrm{H}(f)'$, $\mathrm{H}(K)''$, $\mathrm{H}(U)'$ *and* H'_τ *hold, then K is $U(\cdot)$-invariant.*

PROOF. Consider the multifunction $F : H \to P_{wkc}(H)$ defined by $F(x) = = f(x, U(x))$. Since by hypothesis $H \hookrightarrow X^*$ compactly, we have $F(x) \in \in P_{kc}(X^*)$. Also we claim that $F(\cdot)$ is u.s.c. from H into X^*. To get this it suffices to show that for any $C \subseteqq X^*$ closed, the set $F^-(C) = \{x \in H : : F(x) \cap C \neq \emptyset\}$ is closed in H. Let $x_n \to x$, $x_n \in F^-(C)$. Then by definition there exist $u_n \in U(x_n)$ such that $f(x_n, u_n) \in C$. Since $U(\cdot)$ is u.s.c. from H into Z_w (see hypothesis $\mathrm{H}(U)'$) and has values in $P_{wkc}(Z)$, we get that $\overline{\bigcup_{n \geqq 1} U(x_n)}^w$ is w-compact in Z. So by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \in w - \overline{\lim}\, U(x_n) = \\ = \{z \in Z : z = w - \lim z_{n_k}, z_{n_k} \in U(x_{n_k}), k \geqq 1\}.$$

But using once again hypothesis $\mathrm{H}(U)'$, we have that $w - \overline{\lim}\, U(x_n) \subseteqq U(x)$ (see Delahaye–Denel [3]) $\Longrightarrow u \in U(x)$. Also $f(x_n, u_n) \xrightarrow{w} f(x, u)$ in H and since H embeds compactly in X^*, $f(x_n, u_n) \xrightarrow{s} f(x, u)$ in $X^* \Longrightarrow f(x,u) \in \in C$, $u \in U(x) \Longrightarrow x \in F^-(C) \Longrightarrow F^-(C)$ is closed in $H \Longrightarrow F(\cdot)$ is u.s.c. from H into X^*.

Consider the projections of $(**)$ on the finite dimensional subspaces $X_n^* = = p_n X = p_n H = p_n X^*$ (Galerkin approximations):

$$(**)_n \qquad \left\{ \begin{array}{ll} \dot{x}_n(t) + p_n A x_n(t) \in p_n F(x_n(t)) & \text{a.e.} \\ x_n(0) = p_n x_0 = x_0^n & \end{array} \right\}$$

From hypothesis H'_τ, we know that for all $x \in p_n K = K \cap p_n X = K \cap X_n^*$, we have $(-A(x) + F(x)) \cap T_K(x) \neq \emptyset$, i.e. $p_n[(-A(x) + F(x)) \cap T_K(x)] \neq \emptyset$, whence $p_n(-A(x) + F(x)) \cap p_n T_K(x) \neq \emptyset$.

From Proposition 14, p. 173 of Aubin–Ekeland [1], we know that

$$p_n T_K(x) = T_{p_n K}(x).$$

So we have

$$(-p_n A(x) + p_n F(x)) \cap T_{p_n K}(x) \neq \emptyset.$$

This tells us that to the approximating, finite dimensional problems $(**)_n$ we can apply Theorem 3.1 of this paper (see also Theorem 1 of Deimling [2]) and get trajectories $x_n(\cdot)$ emanating from $x_0^n = p_n x_0$ and leaving $p_n K = = K \cap p_n X$ invariant.

From [10] we know that $\{x_n\}_{n\geqq 1}$ is relatively sequentially compact in $C(T, X_w)$. So by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X_w) \Longrightarrow x_n(t) \xrightarrow{w} x(t)$ in $C(T, X_w)$ and since $X \hookrightarrow H$ compactly, we get $x_n(t) \xrightarrow{s} x(t)$ in H for every $t \in T$. Also from Lemma 6.2 of [10], we know that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(X^*)$. So invoking Theorem 3.1 of [9], we get that

$$\dot{x}(t) \in \overline{\mathrm{conv}}\, w - \overline{\lim}(-p_n A(x_n(t)) + p_n F(x_n(t))) \quad \text{a.e.}$$

But for every $h \in X$, we have

$$\langle h, p_n A(x_n(t))\rangle = \langle p_n^* h, A(x_n(t))\rangle$$

and since $p_n^* \to \mathrm{Id}$ in the strong operator topology and $A(x_n(t)) \xrightarrow{w} A(x(t))$ (because of H(A) (1)), we get

$$\langle h, p_n A(x_n(t))\rangle \to \langle h, A(x(t))\rangle.$$

Also from [10] we know $\|x_n(t)\| \leqq \hat{M}$ for some $\hat{M} > 0$ and all $n \geqq 1$, $t \in T$. So for $v \in X$, we have

$$\begin{aligned} &\sigma(v, p_n F(x_n(t))) = \sigma(p_n^* v, F(x_n(t))) = \\ &= \sigma(p_n^* v, F(x_n(t))) - \sigma(v, F(x_n(t))) + \sigma(v, F(x_n(t))) \leqq \\ &\leqq \sigma(p_n^* v - v, F(x_n(t))) + \sigma(v, F(x_n(t))) \leqq \\ &\leqq \|p_n^* v - v\| c(1 + \hat{M}) + \sigma(v, F(x_n(t))) \end{aligned}$$

whence

$$\begin{aligned} &\overline{\lim}\, \sigma(v, p_n F(x_n(t))) \leqq \\ &\leqq \lim \|p_n^* v - v\| c(1 + \hat{M}) + \overline{\lim}\, \sigma(v, F(x_n(t))) \leqq \sigma(v, F(x(t))). \end{aligned}$$

Invoking Proposition 4.1 of [9], we conclude that $w - \overline{\lim}\, p_n F(x_n(t)) \subseteqq \subseteqq F(x(t))$ a.e., whence $\dot{x}(t) \in -Ax(t) + F(x(t))$ a.e., $x(0) = x_0$, $x(t) \in K$.

An application of Aumann's selection theorem, as in the proof of Theorem 3.1, gives us a control function $u(\cdot)$ such that

$$\dot{x}(t) + Ax(t) = f(x(t), u(t)) \quad \text{a.e., and} \quad x(0) = x_0, u(t) \in U(x(t)) \quad \text{a.e.}$$

Since $x_0 \in K$ was arbitrary, we conclude that K is $U(\cdot)$-invariant. Q.E.D.

Finally we will check how invariance for the relaxed and original systems are related. We will do this in the context of semilinear systems.

So consider the following two controlled systems:

$$(***) \qquad \left\{ \begin{array}{l} \dot{x}(t) + Ax(t) = f(x(t), u(t)) \quad \text{a.e.} \\ x(0) = x_0, \ u(t) \in W \in P_{wkc}(Z) \end{array} \right\}$$

which is called the *original system* and

$$(***)_r \qquad \left\{ \begin{array}{l} \dot{x}(t) + Ax(t) = \int\limits_Z f(x(t), z)\lambda(t)\,(dz) \quad \text{a.e.} \\ x(0) = x_0, \ \lambda(\cdot) \in R(T, W_w) \end{array} \right\}$$

which is called the *relaxed system.* Here by $R(T, W_w)$ we denote the space of transition probabilities on $T \times W$ (recall that W_w i.e. W with the relative weak topology, is compact metrizable, see Dunford–Schwartz [4], Theorem 3, p. 434).

The hypotheses on the data are now the following:

H$(A)'$: $A : X \to X^*$ is a continuous, linear operator such that $\langle Ax, x\rangle \geqq c'\|x\|^2$, $c' > 0$.

H$(f)''$: $f : H \times Z \to H$ is a function such that
(1) $(x, u) \to f(x, u)$ is sequentially continuous from $H \times Z_w$ into H_w,
(2) $|f(x, u)| \leqq k|x| k > 0$.

H$(K)''$: $K \subseteqq H$ is nonempty and closed.

From Tanabe [11] we know that $A(\cdot)$ generates an evolution operator $\Phi : \Delta = \{0 \leqq s \leqq t \leqq b\} \to \mathcal{L}(H)$, for which we will also assume the following:

H(Φ): $\Phi(t, s) = \Phi(t - s)$ is compact for $t - s > 0$.

We say that K is approximately W-invariant, if given $\varepsilon > 0$ and $x_0 \in K$, we can find a trajectory $x(\cdot)$ of $(***)$ such that $x(t) \in K_\varepsilon = \{y \in H : d_K(y) \leqq \varepsilon\}$ for all $t \in T$. Also note that by $R(t, x_0)$ we will denote the reachable set of system $(***)$ at time t i.e. $R(t, x_0) = \{y = x(t) : x(\cdot) =$ trajectory of $(***)\}$. The next theorem relates the invariance properties of $(***)$ and $(***)_r$ and reads as follows:

THEOREM 4.2. *If hypotheses* H$(A)'$, H$(f)''$, H$(K)'''$ *and* H(Φ) *hold, then K is approximately W-invariant for $(***)$ if and only if K is $R(T, W_w)$-invariant for $(***)_r$.*

PROOF. From the definition of the approximate W-invariance given above, for every $\varepsilon > 0$ and every $x_0 \in K$, we have $R(t, x_0) \cap K_\varepsilon \neq \emptyset$ for all $t \in T$. Let $\varepsilon_n \downarrow 0$ and pick $x_n \in R(t, x_0) \cap K_{\varepsilon_n}$. From Theorem 3.2 of [10] we know that $\overline{R(t, x_0)}^{|\cdot|} = R_r(t, x_0) \in P_{kc}(H)$ (where $R_r(t, x_0) = \{y = x(t) : x(\cdot) =$ trajectory of $(***)_r\} =$ reachable set at time t of $(***)_r$). So by passing to a subsequence if necessary, we may assume that $x_n \longrightarrow x$ in $H \Longrightarrow x \in \overline{R(t, x_0)}^{|\cdot|} = R_r(t, x_0)$. Also $\overline{\lim}\, K_{\varepsilon_n} = K$. Hence $x \in R_r(t, x_0) \cap K \Longrightarrow K$ is $R(T, W_w)$-invariant.

By hypothesis for every $t \in T$ and every $x_0 \in K$, $R_r(t,x_0) \cap K \neq \emptyset \Longrightarrow$ $\Longrightarrow \overline{R(t,x_0)}^{|\cdot|} \cap K \neq \emptyset \Longrightarrow R(t,x_0) \cap K_\varepsilon \neq \emptyset$ for all $\varepsilon > 0 \Longrightarrow K$ is approximately W-invariant. Q.E.D.

REMARK. Note that $\bigcup\left\{\int\limits_W f(x,u)\lambda(du) : \lambda \in M^1_+(W_w)\right\} = \overline{\mathrm{conv}} F(x)$ (see [10]). So the relaxed system is the "convexification" of the original system.

Acknowledgement. The authors wish to express their gratitude to the referee for his (her) many suggestions, corrections and remarks, that improved the content of this paper considerably.

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(Received September 16, 1988; revised May 2, 1989)

UNIVERSITY OF THESSALONIKI
SCHOOL OF TECHNOLOGY
THESSALONIKI 54006
GREECE

ATHENS SCHOOL OF ECONOMICS
AND BUSINESS SCIENCE
ATHENS 10434
GREECE

NATIONAL TECHNICAL UNIVERSITY
DEPARTMENT OF MATHEMATICS
ATHENS 15773
GREECE

Acta Math. Hung.
59 (3–4) (1992), 319–326.

DECOMPOSITIONS OF BAER-LIKE RINGS

G. F. BIRKENMEIER (Lafayette)

Introduction

In this paper we will define an ideal $\langle N_E(R)\rangle$ which provides a link between the set of nilpotents and the set of idempotents of a ring R. In the first section various properties of $\langle N_E(R)\rangle$ are developed. Also we present four conditions which are equivalent to the quasi-Baer condition for a semiprime ring. In the second section these conditions are used to achieve decompositions for several classes of rings, including right self-injective rings, right FPF rings, and dual rings. The decompositions contain direct summands which are "essentially generated by nilpotents" in the sense that they are essential extensions of $\langle N_E(R)\rangle$, $P(R)$ (i.e., the prime radical of R), or $Z_2(R)$ (i.e., the second singular ideal of R). Also, these decompositions are somewhat reminiscent of Kaplansky's theory of types for Baer rings and they extend results in [2] and [22].

All rings are associative and have a unity unless indicated otherwise. R and I will always denote a ring with unity and the integers, respectively. $E(R)$, $U(R)$, $N(R)$, $N_j(R)$, and $Z_r(R)$ symbolize the set of idempotents, units, nilpotents, nilpotents of index j, and the right singular ideal of R, respectively. For $X \subseteqq R$, $\ell(X)$, $r(X)$, and $\langle X\rangle$ denote the left annihilator of X, the right annihilator of X, and the subring (not necessarily with unity) generated by X, respectively; X is *reduced* if $X \cap N(R) = 0$. For $e \in E(R)$, eR is a *ring (ideal) direct summand* if e is central (eR is an ideal). Recall from [3] that an idempotent $e \in R$ is said to be *right semicentral* if $eR = eRe$ (equivalently, if $r \in R$ then $er = ere$). Basic properties of right semicentral idempotents will be used implicitly throughout this paper and can be found in [3]. Of particular importance is the fact that if $R = A \oplus B$ (right ideal decomposition) and B is an ideal of R then there exists a right semicentral idempotent e such that $A = eR$, $B = (1-e)R$, A is a ring with unity e, and every right ideal of A is a right ideal of R while every left ideal of B is a left ideal of R. From [10] and [18], R is a *(quasi-)Baer* ring if the right annihilator of every (ideal) nonempty subset of R is a direct summand of R. For $e \in E(R)$, we define $N_e(R) = eR(1-e) \cup (1-e)Re$ and $N_E(R) = \bigcup_{e\in E(R)} eR(1-e)$. The word "essential" will be used in the context of "right" R-modules. Other terminology can be found in [14].

1. Preliminary results

PROPOSITION 1.1. *Let $e \in E(R)$.*

(i) *$\langle N_E(R)\rangle$ is an ideal of R.*

(ii) *$\langle N_E(R)\rangle \subseteqq \langle E(R)\rangle$ and $\langle N_E(R)\rangle \subseteqq \langle U(R)\rangle$.*

(iii) *Let $M_n(R)$ denote the $n\times n$ matrix ring over R and $c \in M_n(R)$ with a one on the main diagonal, and zero elsewhere. Then $M_n(R) = \langle N_c(M_n(R))\rangle$.*

(iv) *Let $R[x]$ denote the ring of polynomials in the commuting indeterminate x over R and $c = b + b_1x + \dots + b_nx^n \in E(R[x])$. Then $b^2 = b$ and $b_i \in \langle N_b(R)\rangle$ for $i \in \{1, 2, \dots, n\}$, and $\langle N_E(R[x])\rangle = \langle N_E(R)\rangle[x]$.*

(v) *If R is a right p.p. ring (principal right ideals projective), $\langle N_E(R)\rangle$ is the ideal generated by $N(R)$.*

(vi) *$N_e(R) \subseteqq eR$ if and only if $1 - e$ is a right semicentral idempotent.*

(vii) *For $r \in R$, $er - re = er(1 - e) + (1 - e)re \in \langle N_e(R)\rangle$.*

(viii) *If X is a right ideal of R and $e \in X$, then $ex = xe$ for all $x \in X$ if and only if $X \cap N_e(R) = 0$.*

(ix) *If J is an ideal, then $ey = ye$ for all $y \in J$ if and only if $J \cap N_e(R) = 0$.*

(x) *$e \in \langle N_e(R)\rangle$ if and only if there exists $x \in N$ such that $e \in xR$ or $e \in Rx$.*

(xi) *If J is an ideal, then $J \cap \langle N_E(R)\rangle \neq 0$ if and only if there exists $e \in E(R)$ such that $J \cap N_e(R) \neq 0$.*

PROOF. (i) Let $s, t \in R$. Then $et(1-e)s = [et(1-e)][(1-e)se] + et(1-e)s(1-e) \in \langle N_e(R)\rangle$. It follows that $\langle N_e(R)\rangle$ is a right ideal and similarly it can be shown that $\langle N_e(R)\rangle$ is a left ideal.

(ii) Let $x \in eR(1-e)$. Then $e + x \in E(R)$. Hence, $x = e + x - e \in \langle E(R)\rangle$. Thus $\langle N_E(R)\rangle \subseteqq \langle E(R)\rangle$. Also, $x = (x - 1) + 1 \in \langle U(R)\rangle$. Hence $\langle N_E(R)\rangle \subseteqq \langle U(R)\rangle$.

(iii) Routine.

(iv) Since $c = c^2$, then $b_k = \sum\limits_{i+j=k} b_ib_j$ where i and j are nonnegative integers and $b = b_0$. Now $bb_1 = b(bb_1 + b_1b) = bb_1 + bb_1b$. Thus $bb_1b = 0$. Hence $b_1 = bb_1 + b_1b = [bb_1b + bb_1(1-b)] + [bb_1b + (1-b)b_1b] = bb_1(1-b) + (1-b)b_1b \in \langle N_b(R)\rangle$. Now assume for $0 < h < k$, $b_h \in \langle N_b(R)\rangle$. Then $bb_k = b(\sum\limits_{i+j=k} b_ib_j) = bb_k + bb_kb + bd$ where $d = \sum\limits_{i+j=k} b_ib_j$ with $0 < i < k$. Hence $bb_kb = -bd \in \langle N_b(R)\rangle$. Now $b_k = bb_k + b_kb + d$. But $bb_k = bb_kb + bb_k(1-b) \in \langle N_b(R)\rangle$. Similarly, $b_kb \in \langle N_b(R)\rangle$. Hence $b_k \in \langle N_b(R)\rangle$. By induction, $b_i \in \langle N_b(R)\rangle$ for $i \in \{1, 2, \dots, n\}$. Consequently, $cR[x](1-c) \in \langle N_b(R)\rangle[x]$. Clearly, $\langle N_E(R)\rangle[x] \subseteqq \langle N_E(R[x])\rangle$. Therefore, $\langle N_E(R)\rangle[x] = \langle N_E(R[x])\rangle$.

(v) Let $y \in N_2$; then there exists $e = e^2$ such that $y \in eR = r(y)$. Thus $y = eye + ey(1-e) = ey(1-e) \in N_E$. Hence, $N_2 \subseteqq \langle N_E(R)\rangle$. Now assume $N_j \subseteqq \langle N_E(R)\rangle$ for all $2 < j \leqq k$. Let $s \in N_{k+1}$. Then there exists $c = c^2$

such that $s \in cR = r(s^k)$. Now $s = sc + cs(1-c)$. But $(sc)^k = s^k c = 0$. Thus $s \in N_E(R)\rangle$. By induction $N(R) \subseteqq \langle N_E(R)\rangle$. Consequently, $\langle N_E(R)\rangle$ equals the ideal generated by $N(R)$.

(vi) If $N_e(R) \subseteqq eR$, then eR is an ideal. By [3, Lemma 1], $1-e$ is right semicentral. If $1-e$ is right semicentral, then $(1-e)Re = 0$. Hence $N_e(R) = eR(1-e) \subseteqq eR$.

(vii) Routine.

(viii) For $0 \neq r \in R$, if $er = re$, then $r \notin (N_e(R)$. Hence $ex = xe$ for all $x \in X$ implies $X \cap N_e(R) = 0$. Conversely, assume $X \cap N_e(R) = 0$. By part (vii) $(1-e)xe = ex - xe - ex(1-e) \in X \cap N_e(R) = 0$. Thus $ex - xe = ex(1-e) \in X \cap N_eE(R) = 0$.

(ix) Similar to part (viii).

(x) If $e \in \langle N_e(R)\rangle$, then there exist $r, s \in R$ such that $e = er(1-e)se \in$ $\in er(1-e)R$. Conversely, assume $e = xr$ for some $r \in R$ (the proof is similar for $e \in Rx$) and $x^n = 0$. Then

$$e = xre = x(1-e)re + x^2r^2e = x(1-e)re + x^2(1-e)r^2e + x^3r^3e = \cdots =$$
$$= x(1-e)re + x^2(1-e)r^2e + \cdots + x^{n-1}(1-e)r^{n-1}e \in \langle N_e(R)\rangle.$$

(xi) This part follows from part (ix).

Part (v) generalizes a result of W. Stephenson indicated in [15, Proposition 3.3]. Furthermore, part (xi) does not hold for right ideals since there exists a ring R which has nonzero reduced right ideals such that $R = \langle N_E(R)\rangle$ [6, Proposition 6].

EXAMPLE A. Let S be a commutative ring with ideals X and Y. Define the matrix ring $R = \begin{pmatrix} S & X \\ Y & S \end{pmatrix}$. A straightforward calculation shows that $\langle N_E(R)\rangle = \begin{pmatrix} XY & X \\ Y & XY \end{pmatrix} = \langle N_e(R)\rangle$ where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If S is a domain, then R is a prime (hence a quasi-Baer) ring. Thus, for $R = \begin{pmatrix} I & 2I \\ 2I & I \end{pmatrix}$, then R is a prime ring such that $\langle N_E(R)\rangle$ is essential in R and $R = \langle E(R)\rangle$. However, since $\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \in N(R)$, we see that $N(R) \nsubseteqq \langle N_e(R)\rangle$. Thus, by Proposition 1.1(v), R is not a Baer ring.

We say R satisfies the *(ideal) intersection left annihilator sum* property, (IILAS) ILAS, if whenever X and Y are (ideals) right ideals such that $X \cap$ $\cap Y = 0$, then $(\ell(X) + \ell(Y) = R)$ $\ell(X)R + \ell(Y)R = R$. Right FPF rings [11, p. 168], right self-injective rings [21, p. 275], and dual rings [17] have the ILAS property. The following result, which is proved in [4], provides the motivation for the various decompositions in Section 2.

PROPOSITION 1.2. *Let R be semiprime. Then the following conditions are equivalent:*

(i) *R is a quasi-Baer ring.*

(ii) *Every ideal is essential in a ring direct summand of R.*

(iii) *Every ideal which is a closed right ideal is a direct summand of R.*

(iv) *R is an* IILAS *ring.*

(v) *For every ideal X of R, $r(X)$ is essential in a direct summand of R.*

PROPOSITION 1.3. *Let R be a quasi-Baer ring such that $N_E(R) = 0$. Then R is a semiprime ring such that $R = A \oplus B$ (ring decomposition) where A is an essential extension of the ideal generated by $N_2(R)$ and B is an Abelian Baer ring.*

PROOF. Let J be an ideal of R such that $J^2 = 0$. Hence there is a central idempotent e such that $r(J) = eR$. Then $eJ = J = Je = 0$. Hence R is semiprime. The remainder of the proof follows from Proposition 1.2 and [2, Lemma 1].

EXAMPLE B. The example of Zalesskii, and Neroslavskii as discussed in [16, pp. 1431–1432], is a simple (hence semiprime quasi-Baer) Noetherian ring R such that $N_E(R) = 0$ but R is generated as an ideal by $N_2(R)$.

PROPOSITION 1.4. *Let be a ring such that every ideal is essential in a direct summand of R. Then R is* IILAS, *and if $Z_r(R) = 0$, then R is quasi-Baer.*

PROOF. From [4, Lemma 2.3] R is IILAS. If $Z_r(R) = 0$, a proof similar to that of [9, Theorem 2.1] will show that R is quasi-Baer.

2. Decompositions

In this section various decompositions are determined in terms of $\langle N_E(R)\rangle$, $Z_2(R)$, and $P(R)$ (i.e., prime radical of R). $Z_2(R)$ is defined by the rule $Z_2(R)/Z_r(R) = Z_r(R/Z_r(R))$ [14, p. 37]. We note that $Z_2(R)$ is a closed ideal which is densely nil [1, Lemma 3.3].

LEMMA 2.1. *Let R be a ring which satisfies any of the following conditions:*

(i) *Every ideal is essential in a direct summand of R.*

(ii) *Every ideal which is closed as a right ideal is a direct summand of R.*

(iii) *$r(X)$ is (essential in) a direct summand whenever X is a closed right ideal.*

(iv) *R is* ILAS.

If e is a right semicentral idempotent, then eRe satisfies the same conditions as R.

PROOF. (i) Let J be an ideal of the ring eR. Now J is a right ideal of R, so let RJ denote the ideal of R generated by J. There exists $b = b^2$

such that RJ is essential in bR. Thus $J \subseteqq ebR$. Let $0 \neq y \in ebR$. Hence there exists $r \in R$ such that $0 \neq byr \in RJ$. But $byr = bebyr$. Hence $0 \neq ebyr \in eRJ = J$. Thus J is essential in ebR.

(ii) Let X be an ideal in eRe. Then $X = eJ$ where $J = RX$ is an ideal of R. Assume eJ is closed in eRe. Then $eJ \oplus (1-e)R$ is an ideal of R which is closed as a right ideal in R. Hence there exists $c = c^2$ such that $eJ \oplus (1-e)R = cR$. Now $eR \cap cR = ecR$ and ec is an idempotent. But $eJ = ecR = ece(eRe)$. Therefore, eJ is a direct summand of eRe.

(iii) This part follows from the fact that every right ideal of eRe is a right ideal of R.

(iv) Let X, Y be right ideals of eRe such that $X \cap Y = 0$. Since X, Y are also right ideals of R, then $\ell(X)R + \ell(Y)R = R$. Hence $(e\ell(X)e)eRe + +(e\ell(Y)e)eRe = eRe$.

THEOREM 2.2. *Let R be a ring such that every ideal is essential in a direct summand of R. Then R has the following right ideal decompositions:*

(1) *$R = H' \oplus A' \oplus B' \oplus C$ where $H' = Z_2(R)$;*

(i) *$S' = A' \oplus B' \oplus C$ is a quasi-Baer ring such that every ideal of S' is essential in an ideal direct summand of S';*

(ii) *A' is an essential extension of $P(S')$;*

(iii) *$T' = B' \oplus C$ is a semiprime quasi-Baer ring where B' is an essential extension of $\langle N_E(T')\rangle$;*

(iv) *C is a semiprime quasi-Baer ring with $N_E(C) = 0$;*

(v) *$A' \oplus B'$ is an essential extension of $\langle N(S')\rangle$;*

(2) *$R = H'' \oplus A'' \oplus B'' \oplus C$ where H'' is an essential extension of $\langle N_E(R)\rangle$;*

(i) *$S'' = A'' \oplus B'' \oplus C$ is a ring decomposition in which every ideal of S'' is essential in a ring direct summand of S'';*

(ii) *A'' is an essential extension of $P(S'')$;*

(iii) *$T'' = B'' \oplus C$ is a semiprime quasi-Baer ring where $B'' = = Z_2(T'')$;*

(iv) *$A'' \oplus B'' = Z_2(S'')$;*

(v) *$H' \oplus A' \oplus B' = H'' \oplus A'' \oplus B''$.*

PROOF. (1) Since $Z_2(R)$ is a closed right ideal, then $R = Z_2(R) \oplus S'$ where S' is a right nonsingular ring. By Proposition 1.4 and [14, p. 47, Exercise 1], S' is a quasi-Baer ring such that every ideal of S' is essential in an ideal direct summand of S'. Parts (ii), (iii), and (iv) follow from Proposition 1.2 and repeated application of Lemma 2.1. For part (iv) we note that $\langle N_E(S')\rangle \subseteqq A' \oplus B'$. Also from Lemma 2.1 and Proposition 1.3 $S' = W \oplus K$ (right ideal decomposition) where W is an essential extension of $\langle N_E(S')\rangle$ and K is a semiprime ring. Hence $P(S') \subseteqq W$. Clearly $\langle N_E(T')\rangle \subseteqq \subseteqq \langle N_E(S')\rangle$. Hence $W = A' \oplus B'$.

(2) Let H'' denote the unique direct summand of R which is an essential extension of $\langle N_E(R)\rangle$. Since the complement of C is unique, $R = H'' \oplus \oplus S''$ where $C \subseteqq S''$. Parts (i), (ii), and (iii) follow from Proposition 1.1,

Proposition 1.2, and Lemma 2.1. From part (1), C is right nonsingular. Hence $Z_2(S'') \subsetneqq A'' \oplus B''$. From [13, p. 1214], $Z_2(A'') = A''$. Thus, $Z_2(S'') = A'' \oplus B''$. Part (v) follows from the fact that C has a unique complement.

We note that whenever a direct summand of R is an essential extension of $\langle N_E(R)\rangle$, then it is the unique such direct summand. As an application of Theorem 2.2 and Proposition 1.3, we see that if R is a prime ring, then either R is an essential extension of $\langle N_E(R)\rangle$, or R has only trivial idempotents and is an essential extension of the ideal generated by N_2, or R is a domain (see Examples A and B). Also Theorem 2.2 is a generalization of a result of Utumi [22, p. 604].

THEOREM 2.3. *Let R be a ring such that every ideal which is closed as a right ideal is a direct summand. Then $R = A \oplus Z_2(R)$ (right ideal decomposition) where A is a quasi-Baer ring such that every ideal of A is essential in an ideal direct summand of A.*

PROOF. The proof follows from Lemma 2.1, [14, p. 47, Exercise 1], and Proposition 1.4.

We note that any right CS ring [8], in particular any right self-injective or right uniform ring, satisfies the hypotheses of Theorems 2.2 and 2.3.

THEOREM 2.4. *Let R be a ring such that $r(X)$ is a direct summand whenever X is a closed right ideal. Then R has the following right ideal decompositions:*

(i) *$R = A \oplus B$ where A is a semiprime quasi-Baer ring and B is an essential extension of $P(R)$.*

(ii) *$R = C \oplus D$ where C is a ring such that each of its ideals is essential in a ring direct summand of C and D is an essential extension of $\langle N_E(R)\rangle$. In particular, if R is a Baer ring, then C is an Abelian Baer ring.*

PROOF. Let J be an ideal of R and X is a relative complement of J. Hence $r(X) = eR$ where $e = e^2$ and $J \subsetneqq eR$. But $X \cap eR \subsetneqq \langle N_E(R)\rangle \cap P(R)$. Thus if $J = \langle N_E(R)\rangle$ or $J = P(R)$, then $X \cap eR = 0$; hence J is essential in eR. Part (i) follows from Lemma 2.1 and Proposition 1.2. Part (ii) is a consequence of Lemma 2.1, the above argument, and Proposition 1.1(v).

The condition stated in Theorem 2.4 is a proper generalization of the quasi-Baer condition since a uniform ring satisfies the condition of Theorem 2.4 but is not necessarily quasi-Baer (e.g., integers modulo four).

THEOREM 2.5. *Let R be a ring such that $r(X)$ is essential in a direct summand whenever X is a closed right ideal. Then $R = A \oplus B \oplus C$ (right ideal decomposition) such that:*

(i) *A is an essential extension of $Y \oplus \langle N_E(R)\rangle$ where Y is a right ideal of R with $Y^2 = 0$.*

(ii) *B is a ring which is an essential extension of $P(B)$.*

(iii) *C is a semiprime quasi-Baer ring with $N_E(C) = 0$.*

PROOF. Let J be an ideal of R and X is a relative complement of J. Hence there exists $e = e^2$ such that eR is an essential extension of $r(X)$. A straightforward argument shows that eR is an essential extension of $[X \cap \cap r(X)] \oplus J$. Thus, if we let $J = \langle N_E(R) \rangle$ and $Y = X \cap r(X)$, then the result follows from Lemma 2.1, Proposition 1.2, and the above argument.

We note that if R is right nonsingular, then the hypotheses for Theorems 2.4 and 2.5 coincide. The next proposition is a slight generalization of Faith's Splitting Theorem [11, p. 184].

PROPOSITION 2.6. *Let R satisfy the* ILAS *condition and let K be a right ideal containing all nilpotent index two right ideals. Then $R = A \oplus L$ (right ideal decomposition) where A is an ideal such that K is ideal essential in A and L is a semiprime* ILAS *(hence quasi-Baer) ring.*

PROOF. Let B be an ideal maximal with respect to $B \cap K = 0$. Let $A = \ell(B)$, hence $K \subseteqq A$. Since $(A \cap B)^2 = 0$, then $A \cap B \subseteqq K \cap B = 0$. Thus $R = A + \ell(A)$. Hence $1 = a + x$ where $a \in A$ and $x \in \ell(A)$. Then $A = aR$ and $a^2 = a$. Let I be a nonzero ideal of R such that $I \subseteqq A$. Now there exists $k \in (B + I) \cap K$ such that $0 \neq k = b + y$ where $b \in B$ and $y \in I$. But $b = k - y \in A \cap B = 0$. Hence $I \cap K \neq 0$. Let $L = (1 - a)R$. By Lemma 2.1 and Proposition 1.2, L is a semiprime ILAS quasi-Baer ring.

EXAMPLE C. Corollary 8 of [5] indicates how one can construct a large class of ILAS rings which are not quasi-Baer. In particular, if R is a strongly right bounded quasi-Baer ring (e.g., a commutative Baer ring), then the split-null extension S of R by R is strongly right bounded and ILAS but not quasi-Baer. Note that $P(S)$ is essential in S.

EXAMPLE D. Let T be the semigroup ring of A over I_2 (integers modulo two) where A is the semigroup on the set $\{a, b\}$ satisfying the relation $xy = y$ for $x, y \in A$. Thus $T = \{0, a, b, a + b\}$. Let T_1 denote the ring with unity formed by extending (i.e., Dorroh extension) T to $T \times I$. T_1 exhibits the decompositions of this section in a nontrivial way. It provides an example of a ring which is neither right CS nor is the right annihilator of a closed right ideal necessarily a direct summand (e.g., $r(T, 0) = (0, 2I)$) hence it is not quasi-Baer. However, it satisfies the following conditions:

(i) every ideal is essential in a direct summand, thus the right annihilator of a closed right ideal is essential in a direct summand;

(ii) every closed ideal is a direct summand;

(iii) T_1 is I (but not ILAS). We have $T_1 = (a, 0)T_1 \oplus (a, 1)T_1$ where $\langle N_E(T_1) \rangle = P(T_1) \subseteqq Z_2(T_1) = (a, 0)T_1$ and $(a, 1)T_1$ is an Abelian Baer ring. Further details of this example can be found in [5, Example 9] and [7].

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(Received September 20, 1988)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHWESTERN LOUISIANA
LAFAYETTE, LA 70504
U.S.A.

Acta Math. Hung.
59 (3–4) (1992), 327–338.

НЕОБХОДИМЫЕ И ДОСТАТОЧНЫЕ УСЛОВИЯ СХОДИМОСТИ РАСШИРЕННЫХ ИНТЕРПОЛЯЦИОННЫХ ПРОЦЕССОВ ВЫСШЕГО ПОРЯДКА В МЕТРИКЕ L_p

Д. Л. БЕРМАН (Ленинград)

1. Пусть

$$x_k = x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \dots, n \tag{1}$$

и пусть C — множество всех функций, непрерывных в $[-1, 1]$. Пусть $H_n(f, x)$ — полином степени $2n-1$, однозначно определяющийся из условий

$$H_n(f, x_k) = f(x_k),\ \ H_n(f, x_k) = 0,\ \ k = 1, 2, \dots, n,\ \ n = 1, 2, \dots .$$

Классическая теорема Л. Фейера [1] утверждает, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение $H_n(f, x) \to f(x)$, $n \to \infty$. Н. М. Крылов и И. Я. Штаерман [2] удлинили процесс $\{H_n(f, x)\}_{n=1}^{\infty}$ в том смысле, что они заменили полином $H_n(f, x)$ степени $2n-1$ на полином $p_n(f, x)$ степени $4n-1$, однозначно определяемый из условий

$$p_n(f, x_k) = f(x_k),\ \ p_n^{(i)}(f, x_k) = 0,\ \ i = 1, 2, 3,\ \ k = 1, 2, \dots n.$$

Процесс $\{p_n(f, x)\}_{n=1}^{\infty}$ обычно называется интерполяционным процессом Эрмита–Фейера высшего порядка. В [2] и [3] доказано, что для любой $f \in C$ выполняется равномерно в $[-1, 1]$ соотношение

$$p_n(f, x) \to f(x), \quad n \to \infty. \tag{2}$$

Для $f \in C$ и натурального n рассмотрим алгебраический полином $S_{n,r}(f, x)$, $r = 1, 2$. $S_{n,r}(f, x)$ полином степени $4n + 2r + 1$, однозначно определяемый из условий

$$S_{n,r}(f, x_k) = f(x_k),\ \ S_{n,r}^{(i)}(f, x_k) = 0,\ \ i = 1, 2, 3,\ \ k = 1, 2, \dots, n;$$

$$S_{n,r}(f, \pm 1) = f(\pm 1),\ \ S_{n,r}^{(j)}(f, \pm 1) = 0,\ \ j = 1, \dots, r.$$

Очевидно, что последние условия при $r = 0$ опускаются. В [4] доказана

Теорема. *Интерполяционный процесс*

$$\{S_{n,r}(f,x)\}, \quad r=1,2, \ n=1,2,\dots,$$

построенный при узлах (1) *для функции* f *сходится к* f *равномерно в* $[-1,1]$, *если*

$$|f^{(r+1)}(x)| \le C \tag{3}$$

для $x \in [-1,1]$ *и*

$$f'(\pm 1) = \ldots = f^{(r)}(\pm 1) = 0. \tag{4}$$

В [5] показано, что если равенства (4) не выполняются, то процесс $\{S_{n,r}(f,x)\}_{n=1}^{\infty}$ расходится в каждой точке из $(-1,1)$. Поэтому естественно изучить сходимость процессов $\{S_{n,r}(f,x)\}_{n=1}^{\infty}$ в метрике L_p, $0<p<\infty$, когда расстояние между функциями f и g задается по формуле

$$\varrho(f,g) = \left(\int_{-1}^{1} |f-g|^p dx\right)^{1/p}. \tag{5}$$

Аналогичная задача для расширенного интерполяционного процесса Эрмита–Фейера решена в [6]. Любопытно, что при изучении сходимости процессов $\{S_{n,r}(f,x)\}$, $r=1,2$, в метрике L_p, $0<p<\infty$, можно неравенства (3) отбросить.

2. Введем классы функций K_1 и K_2. Пусть $f \in C$. Будем говорить, что $f \in K_1$, если $f(x)$ имеет правую и левую производные соответственно в точках $x=\mp 1$. Будем говорить, что функция из C принадлежит классу K_2, если существуют конечные производные $f'(\mp 1)$ и пределы

$$\lim_{x\to -1} \frac{f(x)-f(-1)}{(x+1)^2}, \quad \lim_{x\to 1} \frac{f(x)-f(1)}{(x-1)^2}$$

конечные. Справедливы следующие теоремы:

Теорема 1. *Пусть* $S_{n,1}(f,x)$ *полином степени* $4n+3$, *однозначно определяемый из условий*

$$S_{n,1}(f,x_k)=f(x_k), \quad S_{n,1}^{(i)}(f,x_k)=0, \ i=1,2,3,$$

$$S_{n,1}(f,\pm 1)=f(\pm 1), \quad S'_{n,1}(f,\pm 1)=0.$$

Пусть процесс $\{S_{n,1}(f,x)\}_{n=1}^{\infty}$ *построен при узлах* (1) *для функции* $f(x)$ *из класса* K_1. *Для того чтобы он сходился в метрике* L_p, $0<p<\infty$, *необходимо и достаточно, чтобы* $f'(-1)=f'(1)=0$.

ТЕОРЕМА 2. *Пусть* $S_{n,2}(f,x)$ *полином степени* $4n+5$, *однозначно определяемый из условий*

$$S_{n,2}(f,x_k)=f(x_k),\quad S_{n,2}^{(i)}(f,x_k)=0,\ i=1,2,3,$$

$$S_{n,2}(f,\pm1)=f(\pm1),\quad S_{n,2}^{(j)}(f,\pm1)=0,\ j=1,2.$$

Пусть $f\in K_2$ *и*

$$\lim_{x\to1}\frac{f(x)-f(1)}{(x-1)^2}=\alpha,\quad \lim_{x\to-1}\frac{f(x)-f(-1)}{(x+1)^2}=\beta.$$

Пусть процесс $\{S_{n,2}(f,x)\}_{n=1}^{\infty}$ *построен при узлах* (1) *для функции* $f(x)$ *из класса* K_2. *Для того чтобы он сходился в метрике* L_p, $0<p<\infty$, *необходимо и достаточно, чтобы* $\alpha=\beta=0$.

3. Докажем сперва теорему 1. Из определения полиномов $S_{n,1}(f,x)$ и $p_n(f,x)$ следует, что

$$S_{n,1}(f,x)-p_n(f,x)=T_n^4(x)(a_nx^3+b_nx^2+c_nx+d_n),\quad T_n(x)=\cos n\arccos x,$$

где коэффициенты a_n,b_n,c_n,d_n определяются из системы уравнений

$$a_n+b_n+c_n+d_n=f(1)-p_n(f,1), \tag{6}$$

$$-a_n+b_n-c_n+d_n=f(-1)-p_n(f,-1), \tag{7}$$

$$4n^2(a_n+b_n+c_n+d_n)+(3a_n+2b_n+c_n)=-p_n'(f,1), \tag{8}$$
$$4n^2(a_n-b_n+c_n-d_n)+3a_n-2b_n+c_n)=-p_n'(f,-1).$$

Из (6) и (7), в силу (2), получаем $a_n+c_n\to0$, $b_n+d_n\to0$, $n\to\infty$. Поэтому

$$S_{n,1}(f,x)-p_n(f,x)=T_n^4(x)[(x^2-1)(a_nx+b_n)+o(1)]. \tag{9}$$

Из системы (8) следует, что

$$a_n=-\frac{p_n'(f,1)}{2}-2n^2(f(1)-p_n(f,1))-\frac{1}{2}[f(1)-p_n(f,1)-f(-1)+p_n(f,-1)]-b_n, \tag{10}$$

$$b_n=\frac{1}{4}[p_n'(f,1)-p_n'(f,-1)-4n^2(f(1)-p_n(f,1)+f(-1)-p_n(f,-1))]. \tag{11}$$

Изучим функционалы

$$(12)\qquad \begin{cases} \alpha_n(f) = p_n'(f,1) + 4n^2(f(1) - p_n(f,1)), \\ \beta_n(f) = p_n'(f,-1) + 4n^2(f(-1) - p_n(f,-1)). \end{cases}$$

Очевидно, что

$$(13)\qquad b_n = \frac{1}{4}(\alpha_n(f) - \beta_n(f)).$$

Известно [7], [8], что при узлах (1)

$$(14)\qquad p_n(f,x) = \frac{1}{n^4}\sum_{k=1}^{n} f(x_k^{(n)})A_k(x)B_k(x),$$

где

$$(15)\qquad \begin{cases} A_k(x) = \left(\dfrac{T_n(x)}{x - x_k}\right)^4, \\ B_k(x) = (1 - xx_k)^2 + (x - x_k)^2\left[\dfrac{2(n^2-1)(1-xx_k)}{3} - \dfrac{xx_k}{2}\right]. \end{cases}$$

Из определения полинома $p_n(f,x)$ следует, что если $f(x) = \text{const.}$, то $p_n(f,x) = f(x)$. Стало быть, имеет место тождество $\sum_{k=1}^{n} A_k(x)B_k(x) = n^4$. Поэтому $\sum_{k=1}^{n}(A_k'(x)B_k(x) + A_k(x)B_k'(x)) = 0$. Из сказанного вытекает, что

$$(16)\qquad f(1) - p_n(f,1) = \frac{1}{n^4}\sum_{k=1}^{n}(f(1) - f(x_k))A_k(1)B_k(1),$$

$$(17)\qquad p_n'(f,1) = -\frac{1}{n^4}\sum_{k=1}^{n}(f(1) - f(x_k))(A_k'(1)B_k(1) + A_k(1)B_k'(1)).$$

В силу (12), (16), (17) имеем, что

$$(18)\qquad \alpha_n(f) = -\frac{1}{n^4}\sum_{k=1}^{n}(f(1) - f(x_k))\Delta_k,$$

где

$$(19)\qquad \Delta_k = A_k'(1)B_k(1) + A_k(1)B_k'(1) - 4n^2A_k(1)B_k(1).$$

Учтем, что $A_k(1) = \frac{1}{(1-x_k)^4}$ и $A'_k(1) = \frac{4n^2}{(1-x_k)^4} - \frac{4}{(1-x_k)^5}$. Поэтому из (19) выводим, что

$$\Delta_k = \frac{B'_k(1)}{(1-x_k)^4} - \frac{4B_k(1)}{(1-x_k)^3}. \tag{20}$$

Из (18) получаем, что

$$\alpha_n(f) = -\frac{1}{n^4}\sum_{k=1}^{n} \frac{f(1)-f(x_k)}{1-x_k}\Delta_{k,1}, \quad \Delta_{k,1} = \Delta_k(1-x_k). \tag{21}$$

Из (21) вытекает, что
(22)

$$\alpha_n(f) = -\frac{1}{n^4}\sum_{k=1}^{n}\left(\frac{f(1)-f(x_k)}{1-x_k} - f'(1)\right)\Delta_{k,1} - \frac{f'(1)}{n^4}\sum_{k=1}^{n}\Delta_{k,1} \equiv S_1^{(n)} + S_2^{(n)}.$$

Из (15) выводим, что

$$B_k(1) = \frac{(1-x_k)^2}{2} + \frac{4n^2-1}{6}(1-x_k)^3,$$

$$B'_k(1) = \frac{4n^2-1}{6}(1-x_k)^3 + \frac{4n^2+11}{6}(1-x_k) - 3(1-x_k).$$

Поэтому из (20) следует, что

$$\Delta_{k,1} = \frac{4n^2-1}{6} - \frac{4n^2-5}{2(1-x_k)} - \frac{5}{(1-x_k)^2}. \tag{23}$$

Рассмотрим сперва сумму $S_1^{(n)}$ из (22). По условию существует $f'(1)$. Поэтому по $\varepsilon > 0$ можно найти такое $\delta > 0$, что

$$\left|\frac{f(1)-f(x_k)}{1-x_k} - f'(1)\right| < \varepsilon,$$

если $1 - x_k < \delta$. Из определения $S_1^{(n)}$ имеем:
(24)

$$|S_1^{(n)}| < \frac{\varepsilon}{n^4}\sum_{k=1}^{n}|\Delta_{k,1}| + \frac{1}{n^4}\left(\frac{2\|f\|}{\delta} + |f'(1)|\right)\sum_{1-x_k \geqq \delta}|\Delta_{k,1}| \equiv \sum\nolimits_1 + \sum\nolimits_2,$$

$$\|f\| = \max_{-1\leqq x\leqq 1}|f(x)|.$$

В силу равенства (23) получим, что

$$\sum\nolimits_1 \leqq \left[O\left(\frac{1}{n}\right) + \frac{4n^2-5}{2n^4}\sum_{k=1}^{n}\frac{1}{1-x_k} + \frac{5}{n^4}\sum_{k=1}^{n}\frac{1}{(1-x_k)^2}\right]\varepsilon. \tag{25}$$

Воспользуемся тождествами

$$\text{(26)} \qquad \sum_{k=1}^{n} \frac{1}{1-x_k} = n^2, \quad \sum_{k=1}^{n} \frac{1}{(1-x_k)^2} = \frac{n^2(2n^2+1)}{3}.$$

Тогда из (25), (26) вытекает, что $\sum_1 = O(1)\varepsilon$. Итак,

$$\text{(27)} \qquad \sum\nolimits_1 < C\varepsilon,$$

где C — абсолютная константа. Переходим к оценке $\sum\limits_{1-x_k \geqq \delta} |\Delta_{k,1}|$. В силу (23) имеем, что

$$\text{(28)} \qquad \sum_{1-x_k \geqq \delta} |\Delta_{k,1}| \leqq \frac{n(4n^2-1)}{6} + \frac{n(4n^2-5)}{2\delta} + \frac{5n}{\delta^2}.$$

Из неравенств (24), (27–28) выводим, что

$$\text{(29)} \qquad S_1^{(n)} \to 0, \quad n \to \infty.$$

Рассмотрим $S_2^{(n)}$, когда $n \to \infty$. Из (23) следует, что

$$\frac{1}{n^4}\sum_{k=1}^{n} \Delta_{k,1} = \frac{4n^2-1}{6n^3} - \frac{4n^2-5}{2n^4}\sum_{k=1}^{n}\frac{1}{1-x_k} - \frac{5}{n^4}\sum_{k=1}^{n}\frac{1}{(1-x_k)^2}.$$

Отсюда, в силу тождеств (26), получим, что

$$\frac{1}{n^4}\sum_{k=1}^{n} \Delta_{k,1} = -\frac{16}{3} + O\left(\frac{1}{n}\right).$$

Стало быть, $\lim\limits_{n\to\infty} S_2^{(n)} = \frac{16}{3} f'(1)$. Отсюда и из (22), (29) выводим, что

$$\text{(30)} \qquad \lim_{n\to\infty} \alpha_n(f) = \frac{16}{3} f'(1).$$

Аналогичным образом доказывается, что при узлах (1) и $f \in K_1$ выполняется равенство

$$\text{(31)} \qquad \lim_{n\to\infty} \beta_n(f) = \frac{16}{3} f'(-1).$$

Из равенств (10–12), (30–31) выводим, что при узлах (1) и $f \in K_1$ выполняются равенства

$$\text{(32)} \qquad \lim_{n\to\infty} a_n = -\frac{4}{3}(f'(-1) + f'(1)), \quad \lim_{n\to\infty} b_n = \frac{4}{3}(f'(-1) - f'(1)).$$

Введем обозначение $r_n = \varrho(p_n(f), S_{n,1}(f))$, где $\varrho(f,g)$ определяется согласно (5). Ясно, что

$$r_n - \varrho(f, p_n(f)) \leqq \varrho(f, S_{n,1}(f)) \leqq r_n + \varrho(f, p_n(f)). \tag{33}$$

В силу (33) и (2) получим, что для любой

$$r_n - o(1) \leqq \varrho(f, S_{n,1}(f)) \leqq r_n + o(1), \quad n \to \infty. \tag{34}$$

Поэтому для доказательства теоремы 1 нужно доказать, что $r_n \to 0$, $n \to \infty$. Из (32) и (9) заключаем, что $r_n \to 0$, $n \to \infty$, тогда и только тогда, когда

$$V_n^{(p)} \equiv \int_{-1}^{1} T_n^{4p}(x)|(f'(-1)+f'(1))x+f'(1)-f'(-1)|^p(1-x^2)^p dx \to 0, n \to \infty.$$

Очевидно, что

$$V_n^{(p)} = \int_0^\pi \cos^{4p} n\theta |a_1 + b_1 \cos\theta|^p \sin^{2p+1}\theta d\theta, \tag{35}$$

где $a_1 = f'(1) - f'(-1)$, $b_1 = f'(1) + f'(-1)$. Для дальнейшего нам нужна лемма из [6]

Лемма. *Пусть $\varphi(\theta)$ непрерывна в $[0,\pi]$. Если $\varphi(\theta) \geqq 0$, $\theta \in [0,\pi]$ и при некотором p, $0 \leqq p < \infty$, выполняется равенство*

$$\lim_{n\to\infty} \int_0^\pi \varphi(\theta)\cos^{2p} n\theta d\theta = 0,$$

то $\varphi(\theta) \equiv 0$, $\theta \in [0,\pi]$.

Доказательство леммы находится в [6]. Применим эту лемму к интегралу (35), где $\varphi(\theta) = |a_1 + b_1\cos\theta|^p \sin^{2p+1}\theta$. Тогда получим, что $a_1 = b_1 = 0$. Теорема 1 доказана.

Доказательство теоремы 2. Очевидно, что $f(x) = \varphi_1(x) + \varphi_2(x)$, где φ_1 — четная функция и φ_2 — нечетная функция. Поскольку $p_n(f,x)$ и $S_{n,2}(f,x)$ — линейные операторы, то без ограничения общности можно предположить, что $f(x)$ четная функция, или же $f(x)$ нечетная функция. Ограничимся рассмотрением случая, когда $f(x)$ — четная функция. Случай, когда $f(x)$ нечетная функция рассматривается аналогично. Без ограничения общности можно считать что $f(1) = 0$. В противном случае можно рассматривать функцию

$F(x) = f(x) - f(1)$. Из определения $p_n(f,x)$ и $S_{n,2}(f,x)$ для четной $f(x)$ получим

$$S_{n,2}(f,x) - p_n(f,x) = T_n^4(x)(a_n^{(1)}x^4 + b_n^{(1)}x^2 + c_n^{(1)}),$$

где коэффициенты $a_n^{(1)}$, $b_n^{(1)}$, $c_n^{(1)}$ определяются из системы уравнений

$$(36)\qquad \begin{cases} a_n^{(1)} + b_n^{(1)} + c_n^{(1)} = f(1) - p_n(f,1), \\ 4n^2(a^{(1)})_n + b_n^{(1)} + c_n^{(1)}) + 4a_n^{(1)} + 2b_n^{(1)} = -p_n'(f,1), \end{cases}$$

$$(37)\quad \frac{4n^2(10n^2-1)}{3}(a_n^{(1)}+b_n^{(1)}+c_n^{(1)})+8n^2(4a_n^{(1)}+2b_n^{(1)})+(12a_n^{(1)}+2b_n^{(1)}) = $$
$$= -p_n''(f,1),$$

потому что $(T_n^4(x))'_{x=1} = 4n^2$, $(T_n^4(x))''_{x=1} = \frac{4n^2(10n^2-1)}{3}$. Из (36) и (2) заключаем, что $a_n^{(1)} + b_n^{(1)} + c_n^{(1)} \to 0$, $n \to \infty$. Поэтому

$$(38)\qquad S_{n,2}(f,x) - p_n(f,x) = T_n^4(x)(x^2-1)[a_n^{(1)}(x^2+1) + b_n^{(1)}].$$

Из уравнений (36–37) вытекает, что

$$(39)\quad 4b_n^{(1)} = p'' - (8n^2+3)p' + \frac{8n^2}{3}(6n^2+5)p,\ \ p^{(j)} = p_n^{(j)}(f,1),\ \ j = 0,1,2.$$

Введем функционалы

$$(40)\quad \Phi_{n,1}(p) = p' - 4n^2p,\ \ \Phi_{n,2}(p) = p'' - (8n^2+1)p' + \frac{8n^2}{3}(7n^2+2)p,$$

где $p^{(j)} = p_n^{(j)}(f,1)$. Из равенств (39–40) вытекает, что

$$(41)\qquad 4b_n^{(1)} = \Phi_{n,2}(p) - 2\Phi_{n,1}(p).$$

В силу (14) и линейности функционалов $\Phi_{n,i}(p)$, $i = 1,2$, имеем, что

$$(42)\qquad \Phi_{n,i}(p_n) = \frac{1}{n^4}\sum_{k=1}^{n} f(x_k)\Phi_{n,i}(A_kB_k).$$

В [4] доказаны равенства

$$\Phi_{n,1}(A_kB_k) = \frac{x_k^2 - 3x_k - 8}{2(1-x_k)^3} - \frac{2(n^2-1)(x_k+2)}{3(1-x_k)^2};$$

$$\Phi_{n,2}(A_kB_k)=\frac{16n^2x_k}{(1-x_k)^3}-\frac{191n^2+2x_k(5n^2+1)-3}{3(1-x_k)^2}.$$

Отсюда и из (41–42) выводим, что

$$(43)\qquad 4b_n^{(1)}=\frac{1}{n^4}\sum_{k=1}^{n}f(x_k)\Big[\frac{16n^2x_k}{(1-x_k)^3}-\frac{191n^2+2x_k(5n^2+1)-3}{3(1-x_k)^2}-$$

$$-\frac{x_k^2-3x_k-8}{(1-x_k)^3}+\frac{4(n^2-1)(x_k+2)}{3(1-x_k)^2}\Big].$$

Из (43) после элементарных преобразований получим

$$(44)\qquad 4b_n^{(1)}=\frac{1}{n^4}\sum_{k=1}^{n}\frac{f(x_k)-f(1)}{(x_k-1)^2}\Delta_{k,2},$$

$$\Delta_{k,2}=\frac{x_k^2-(16n^2+3)x_k-8}{1-x_k}+4(n^2-1)(x_k+2)-2x_k(5n^2+1)-191n^2+3.$$

При этом учтено, что $f(1)=0$. Стало быть,

$$(45)\quad 4b_n^{(1)}=\frac{1}{n^4}\sum_{k=1}^{n}\Big(\frac{f(x_k)-f(1)}{(x_k-1)^2}-\alpha\Big)\Delta_{k,2}+\frac{\alpha}{n^4}\sum_{k=1}^{n}\Delta_{k,2}\equiv\sigma_n^{(1)}+\sigma_n^{(2)},$$

$$\alpha=\lim_{x\to 1}\frac{f(x)-f(1)}{(x-1)^2}.$$

Рассмотрим сперва $\sigma_n^{(1)}$. Поскольку $\lim\limits_{x\to1}\frac{f(x)-f(1)}{(x-1)^2}=\alpha$, то по $\varepsilon>0$ можно найти такое $\delta>0$, что

$$\left|\frac{f(x_k)-f(1)}{(x_k-1)^2}-\alpha\right|<\varepsilon,$$

если $1-x_k<\delta$. Следовательно,

$$(46)\quad |\sigma_n^{(1)}|<\frac{\varepsilon}{n^4}\sum_{k=1}^{n}|\Delta_{k,2}|+\frac{1}{n^4}\left(\frac{\|f\|}{\delta^2}+|\alpha|\right)\sum_{1-x_k\geqq\delta}|\Delta_{k,2}|\equiv\tilde\sigma_n^{(1)}+\tilde\sigma_n^{(2)}.$$

Из (44) видно, что

$$\frac{1}{n^4}\sum_{k=1}^{n}|\Delta_{k,2}|\leqq\frac{1}{n^4}\left(O(n^3)+O(n^2)\sum_{k=1}^{n}\frac{1}{1-x_k}\right).$$

Отсюда с помощью тождества (26) получим, что

(47) $$\frac{1}{n^4}\sum_{k=1}^{4}|\Delta_{k,2}| = O(1).$$

Из (46) и (47) заключаем, что

(48) $$|\tilde{\sigma}_n^{(1)}| \leqq C\varepsilon,$$

где $C > 0$ — константа. Далее, из (44) видно, что

$$\frac{1}{n^4}\sum_{1-x_k\geqq\delta}|\Delta_{k,2}| = O(\frac{1}{n})(1+\frac{1}{\delta}).$$

Отсюда и из (46) выводим, что

(49) $$\left|\tilde{\sigma}_n^{(2)}\right| \leqq \Big(\frac{\|f\|}{\delta^2}+|\alpha|\Big)\Big(1+\frac{1}{\delta}\Big)O(\frac{1}{n}).$$

Из (46), (47–48), (49) заключаем, что

(50) $$\sigma_n^{(1)} \to 0, \;\; n\to\infty.$$

Рассмотрим $\sigma_n^{(2)}$. Из (44–45) видно, что

(51) $$\sigma_n^{(2)} = \frac{\alpha}{n^4}\sum_{k=1}^{n}\frac{x_k^2-(16n^2+3)-8}{1-x_k} + O(\frac{1}{n}).$$

Отсюда с помощью тождеств (26) получим, что

(52) $$\sigma_n^{(2)} \to -16\alpha, \;\; n\to\infty.$$

Из (45), (50), (52) вытекает, что

(53) $$b_n^{(1)} \to -4\alpha, \;\; n\to\infty.$$

Равенство (36) можно записать в виде $4a_n^{(1)}+2b_n^{(1)} = -\alpha_n(f)$, где $\alpha_n(f)$ определяется согласно (12). Из соотношений (30), (53) выводим, что

(54) $$a_n^{(1)} \to -2(2f'(1)/3-\alpha), \quad n\to\infty.$$

По условию $\lim\limits_{x\to1}\frac{f(x)-f(1)}{(x-1)^2} = \alpha$, где α — конечное число. Поэтому $\lim\limits_{x\to1}\frac{f(x)-f(1)}{x-1} = 0$. Следовательно, $f'(1) = 0$. Поэтому соотношение (54) принимает вид

(55) $$\lim_{n\to\infty} a_n^{(1)} = 2\alpha.$$

Положим $R_n = \varrho(p_n(f,x), S_{n,2}(f,x))$. Ясно, что

$$(56) \qquad R_n - \varrho(f, p_n(f,x)) \leqq \varrho(f, S_{n,2}(f,x)) \leqq R_n + \varrho(f, p_n(f,x)).$$

В силу (2), (56) получим, что для любой $f \in C$

$$R_n - o(1) \leqq \varrho(f, S_{n,2}(f,x)) \leqq R_n + o(1).$$

Поэтому для доказательства Теоремы 2 нужно доказать, что $R_n \to 0$, $n \to \infty$.

Из (38), (53), (55) заключаем, что $R_n \to 0$, $n \to \infty$, тогда и только тогда, когда

$$y_n^{(p)} \equiv \int_{-1}^{1} T_n^{4p}(x)(1-x^2)^p |2\alpha(1+x^2) - 4\alpha|^p dx \to 0, \quad n \to \infty.$$

Очевидно, что

$$(57) \qquad y_n^{(p)} = |2\alpha|^p \int_0^{\pi} \cos^{4p} n\theta \sin^{4p+1}\theta \, d\theta.$$

К интегралу (57) применим лемму, тогда получим, что $\alpha = 0$. Теорема 2 доказана.

Наряду с классом функций K_2 введем класс функций K_3. Пусть $f \in C$. Будем говорить, что $f \in K_3$, если $f'(\mp 1) = 0$ и существуют конечные вторые производные $f''(\mp 1)$. Покажем, что $K_3 \subset K_2$. Действительно, имеем

$$f(x) = f(1) + \frac{f''(1)}{2}(x-1)^2 + o((x-1)^2).$$

См. [9]. Следовательно, $\lim\limits_{x\to 1} \frac{f(x)-f(1)}{(x-1)^2} = \frac{f''(1)}{2}$. Аналогично доказывается, что $\lim\limits_{x\to -1} \frac{f(x)-f(-1)}{(x+1)^2} = \frac{f''(-1)}{2}$. Итак, $K_3 \subset K_2$. Поэтому из Теоремы 2 следует

Теорема 3. *Пусть интерполяционный процесс $\{S_{n,2}(f,x)\}_{n=1}^{\infty}$ построен при узлах (1) для $f \in K_3$. Для того чтобы он сходился в метрике L_p, $0 < p < \infty$, необходимо и достаточно, чтобы $f''(\mp 1) = 0$.*

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(Поступило 11. 10. 1988.)

192238 ЛЕНИНГРАД
БАССЕЙНАЯ 68, КВ. 90
СССР

Acta Math. Hung.
59 (3–4) (1992), 339–344.

POSITIVE SOLUTIONS OF NONLINEAR PROBLEMS AT RESONANCE

J. J. NIETO (Santiago de Compostela)

1. Introduction

In the study of both ordinary and partial differential equations it is natural to consider the operator equation

$$Lu = Nu, \tag{1}$$

where $L : D(L) \subset E \to F$ is a linear operator, $N : E \to F$ is a nonlinear operator, and E and F are Banach spaces.

If L is noninvertible we say that (1) is a problem at resonance. In that case, one can use the Lyapunov–Schmidt method [15] or some of its extensions, such as the alternative method [1, 2, 4, 5], to give existence results for the operator equation (1).

On the other hand, in numerous cases it is of interest to determine the existence of positive solutions. For instance, in the study of a model of an infectious disease [7] or in the analysis of a nuclear reactor [14],

By a positive solution of (1) we mean a solution which belongs to a given cone C of E. In applications, one usually takes E as a subspace of $L^2(A)$, A a domain of $\mathbf{R}^n$, and C as the cone of the positive functions, that is, $C = \{u \in E : u \geqq 0 \text{ a.e. in } A\}$.

In Section 2, we present an existence result in a cone (Theorem 2) for nonlinear problems at resonance of type (1). Theorem 2 generalizes the result in [10] since we do not require $N(C)$ to be bounded.

In Section 3, we use the method of upper and lower solutions to show the existence of positive periodic solutions for a second order nonlinear differential equation (Theorem 6). Taking into account the symmetries of the equation, we give sufficient conditions for the problem to have a negative periodic solution (Theorem 7). These two theorems improve some of the results of Nieto and Rao in [11].

2. Existence of solutions in a cone

Let E be a Banach space. We say that C is a cone in E if C is a nonempty convex subset of E such that $rC \subset C$ for every $r \geqq 0$. We shall assume that there exists a continuous map $\gamma : E \to C$ such that $\gamma(c) = c$ for $c \in C$ and γ maps bounded sets into bounded sets.

To apply the alternative method, suppose that there exists projections $P : E \to E$, $Q : F \to F$ and a linear operator $H : (I - Q)F \to (I - P)E$ such that

$$(2) \qquad \begin{cases} H(I-Q)Lu = (I-P)u, & \forall u \in D(L) \\ QLu = LPu, & \forall u \in D(L) \\ LH(I-Q)Nu = (I-Q)Nu, & \forall u \in E. \end{cases}$$

Then, $E = E_0 \oplus E_1$, $F = F_0 \oplus F_1$ where $E_0 = PE$ and $F_0 = QF$. For $u \in E$, we write $u = u_0 + u_1$, $u_0 \in E_0$, $u_1 \in E_1$.

In addition, suppose that

$$(3) \qquad E_0 = \operatorname{Ker} L,\ F_1 = \operatorname{Range} L,\ D(H) = F_1,\ E_1 = \operatorname{Range} H,$$

$$(4) \qquad \dim E_0 = \dim F_0 < +\infty,$$

$$(5) \qquad \begin{cases} \text{there exists continuous maps } B : E \times F \to \mathbf{R} \\ \text{and } J : F_0 \to E_0 \text{ such that} \\ \quad \text{i) } B \text{ is bilinear and } J \text{ is one-to-one and onto.} \\ \quad \text{ii) If } v_0 \in F_0,\ v_0 = 0 \text{ iff } B(u_0, v_0) = 0\ \forall u_0 \in E_0. \\ \quad \text{iii) } Jv_0 = 0 \text{ iff } v_0 = 0. \\ \quad \text{iv) } B(Jv_0, v_0) \geqq 0\ \forall v_0 \in F_0. \\ \quad \text{v) } B(Jv_0, v_0) = 0 \text{ iff } v_0 = 0. \\ \quad \text{vi) } B(u_0, J^{-1}u_0) = 0 \text{ iff } u_0 = 0. \\ \quad \text{vii) } B(u_0, v_0) = B(Jv_0, J^{-1}u_0),\ \forall u_0 \in E_0,\ \forall v_0 \in F. \end{cases}$$

Note that if $E \subset F$ and F is a Hilbert space with inner product $\langle u, v \rangle$ we can define $B(u, v) = \langle u, v \rangle$.

Under assumptions (2–5), the operator equation (1) is equivalent to

$$u = Pu + H(I-Q)Nu + JQNu.$$

In [10] we proved the following existence result:

THEOREM 1. *Assume that* (2)–(5) *hold and*

(6) *there exists* $J_0 > 0$ *such that* $\|Nu\| \leqq J_0$ *for* $u \in C$;

(7) *there exists* $R_0 > 0$ *such that* $B(u_0, QNu) \leqq 0$ *for* $u = u_0 + u_1 \in C$ *with* $\|u_0\| = R_0$ *and* $u_1 = H(I-Q)N(u)$;

(8) *there exists* $r_0 > \|H(I-Q)\| \cdot J_0$ *such that* $(P + JQN)\gamma u \in C$ *and* $H(I-Q)N\gamma u \in C$ *for* $u \in S(R_0, r_0)$ *where*

$$S(R_0, r_0)\{u \in E : \|u_0\| \leqq R_0,\ \|u_1\| \leqq r_0\}.$$

Then, there exists at least one solution $u \in C$ *of* $Lu = Nu$.

Note that (6) means that $N(C)$ is bounded in F. In the following theorem we remove that condition.

THEOREM 2. *Suppose* (2)–(5) *hold and the following conditions are satisfied:*

(9) *there exists* $R > 0$ *such that the set* $C(R) = \{u_1 : u_1 = \lambda H(I - -Q)N\gamma(u_0 + u_1), \lambda \in [0,1], \|u_0\| \leqq R\}$ *is bounded;*

(10) *there exists* $R \in (0, R_0]$ *such that* $B(u_0, QN\gamma(u)) \leqq 0$ *for every* $u \in C$ *with* $\|u_0\| = R_0$ *and* $u_1 = \lambda H(I - Q)N\gamma(u_0 + u_1)$ *for some* $\lambda \in [0,1]$*;*

(11) *there exists* $r_0 > r = \operatorname{Sup}\{\|u_1\| : u_1 \in C(R)\}$ *such that* $(P + +JQN)\gamma u \in C$ *and* $H(I - Q)N\gamma u \in C$ *for every* $u \in S(R_0, r_0) = S$.

Then, there exists a solution $u \in C$ *of* (1).

PROOF. Define the homotopy $T : [0,1] \times S \to E$, $T(\lambda, u) = \lambda W u$ being $Wu = [P + JQN + H(I - Q)N]\gamma u$. Hence, $T(\lambda, \cdot)$ is compact for every $\lambda \in [0,1]$, and for $\lambda = 0$, $T(0,u) = 0$. Thus, $T(0, \partial S) \subset S$. If $\lambda \in [0,1)$, $u \in S$ and $T(\lambda, u) = u$, we obtain

$$(12) \quad u_1 = \lambda H(I - Q)N\gamma(u_0 + u_1) \quad \text{and} \quad u_0 = \lambda[P + JQN]\gamma(u_0 + u_1).$$

Since $u \in S$, there are two possibilities:

a) $\|u_1\| = r_0$. Taking into account (12) we can write that $\|u_1\| < r$ which is a contradiction.

b) $\|u_0\| = R_0$. In this case, using (11) we have that $T(\lambda, u) \in C$ since C is a cone. In consequence, $u \in C$ and $\gamma u = u$, and from (12) we get $u_0 = \lambda P u + \lambda JQNu$.

Now, using that $\lambda < 1$, we have $B((1 - \lambda)u_0, J^{-1}u_0) = B(JQNu, J^{-1}u_0) > 0$. On the other hand, using (5-vii) and (10) we can write

$$B(JQNu, J^{-1}u_0) = B(u_0, QNu) \leqq 0.$$

This last inequality is a contradiction. Therefore, $T(\lambda, u) \neq u$ for every $(\lambda, u) \in [0,1) \times \partial S$, and $T(1, \cdot)$ has a fixed point u in S [8, Theorem 4.4.11]. Reasoning as in b), it is easy to show that $u \in C$. Therefore u is a solution of $Lu = Nu$ such that $u \in C \cap S$. This concludes the proof.

Another version of this last theorem is the following

THEOREM 3. *Suppose that in Theorem* 2 *instead of* (11) *we have*

$$(13) \qquad [P + H(I - Q)N + JQN]\gamma u \in C \quad \textit{for every} \quad u \in S.$$

Then there exists $u \in C$ *such that* $Lu = Nu$.

3. Positive solutions and the method of upper and lower solutions

In this section we study the existence of positive periodic solutions of the equation

$$(14) \qquad u'' + u + \mu u^2 = \varepsilon \cos \omega t$$

where $\mu \neq 0$, $\varepsilon \neq 0$ and $\omega > 0$.

The existence of periodic solutions for this equation was considered by Maekawa [9] and Ezeilo [3]. On the other hand, Nieto and Rao in [11] gave the following result:

THEOREM 4. *Equation* (14) *has a periodic solution if* $4|\mu| \cdot |\varepsilon| < 1$.

Using the method of upper and lower solutions we shall prove the existence of positive and negative periodic solutions for (14).

Making $s = \omega t$, (14) becomes

$$u'' + \omega^{-2}[u + \mu u^{-2} - \varepsilon \cos s] = 0. \tag{15}$$

Thus we are interested in the existence of 2π-periodic solutions of (15) and note that it is of the form

$$u'' + cu' + g(u) = e(t), \quad t \in [0, 2\pi]. \tag{16}$$

This last equation is studied, for instance, in [11, 13, 14].

We say that $\alpha \in C^2[0, 2\pi]$ is a lower solution of (16) if $\alpha''(t) \geqq f(t, \alpha(t))$ for $t \in [0, 2\pi]$, $\alpha(0) = \alpha(2\pi)$ and $\alpha'(0) \geqq \alpha'(2\pi)$. Similarly, $\beta \in C^2[0, 2\pi]$ is an upper solution of (16) if $\beta''(t) \leqq f(t, \beta(t))$ for $t \in [0, 2\pi]$, $\beta(0) = \beta(2\pi)$ and $\beta'(0) \leqq \beta'(2\pi)$.

THEOREM 5. *If* (16) *has an upper solution* β *and a lower solution* α *such that* $\alpha \leqq \beta$ *in* $[0, 2\pi]$, *then there exists at least one solution* u *of* (16) *with* $\alpha \leqq u \leqq \beta$ *in* $[0, 2\pi]$.

PROOF. See [6].

We are now in a position to prove

THEOREM 6. *If* $\mu < 0$, $\varepsilon > 0$ *and* $4|\mu| \cdot |\varepsilon| < 1$, *then there exists a positive* $(2\pi\omega)$-*periodic solution of* (14).

PROOF. Let $a_1 < 0 < a_2$ be the real roots of $\mu a^2 + a - \varepsilon = 0$ and $b_1 < b_2$ the real roots of $\mu b^2 + b + \varepsilon = 0$. Note that $0 < b_1 < b_2 < a_2$. Choose $r, R \in (0, b_1)$ such that $r < R$, and define $\alpha(t) = r$ and $\beta(t) = R$ for $t \in [0, 2\pi]$. Hence we can write

$$f(t, \beta(t)) = -\omega^{-2}(R + \mu R^2 - \varepsilon \cos t) \leqq -\omega^{-2}(R + \mu R^2 - \varepsilon) \leqq 0,$$
$$f(t, \alpha(t)) = -\omega^{-2}(r + \mu r^2 - \varepsilon \cos t) \geqq -\omega(r + \mu r^2 + \varepsilon) \geqq 0$$

since $R \in (a_1, a_2)$ and $r \notin [b_l, b_2]$. Therefore, by Theorem 5, there exists a solution u of (15) such that $u \geqq r > 0$. This completes the proof.

Taking into account the symmetries of (14) we have the following

THEOREM 7. *If* $\mu > 0$, $\varepsilon < 0$ *and* $4|\mu| \cdot |\varepsilon| < 1$, *then* (14) *has a negative* $(2\pi/\omega)$*-periodic solution.*

PROOF. The equation $u''+u-\mu u^2 = -\varepsilon\cos\omega t$ admits a positive periodic solution by Theorem 6. Setting $v = -u$ we get a negative periodic solution of (14).

NOTE. This paper was presented at the VIII$^{\text{th}}$ Congress on Differential Equations and Applications (VIII CEDYA) held at the University of Santander (Spain) during September 23–26, 1985. The proceedings of this conference have never been published.

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(Received October 10, 1988)

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE SANTIAGO DE COMPOSTELA
SPAIN

Acta Math. Hung.
59 (3–4) (1992), 345–353.

UNIFORM ASYMPTOTIC STABILITY AND CONTRACTIVE SEMIGROUPS OF MAPPINGS IN UNIFORM SPACES

V. G. ANGELOV (Sofia)

The main purpose of the present paper is to extend the results of [4] to uniform spaces. In some previous results fixed point theorems for ϕ-contractive mappings in uniform spaces have been obtained [2]. Having in mind the reuniformization result [2] for ϕ-contractive mappings in uniform spaces it is natural to ask what is the relation between the uniform asymptotic stability and contractive semigroups of mappings. Moreover there are dynamical systems whose space of possible states consists of functions which form a locally convex space.

In what follows, we shall give simple examples of linear partial differential equations whose set of solutions forms a dynamical system in a locally convex space of functions. Let us consider the following equation:

$$u_t = au + b_1 u_{x_1} + \cdots + b_n u_{x_n} \tag{1}$$

where $a, b_1, \ldots, b_n$ are real constants and $u = u(t, x_1 \ldots, x_n)$ is the unknown function.

The set of all functions $f(x_1, x_2, \ldots x_n) : R^n \to R^1$ continuous together with their first partial derivatives form a locally convex space. A saturated family of seminorms generating a topology of $C^1(R^n)$ is $\{\|f\|_K\}$, where K runs over all compact subsets of R^n and

$$\|f\|_K = \sup\{|f(x_1, \ldots, x_n)| : (x_1, \ldots, x_n) \in K\} +$$

$$+ \sum_{i=1}^{n} \sup\{|f_{x_i}(x_1, \ldots, x_n)| : (x_1, \ldots, x_n) \in K\}.$$

To every function $f \in C^1(R^n)$ there belongs a solution of (1) the following way. Easy to verify that the explicit form of the solution is $u(f,t) = \exp(at) f(x_1 + b_1 t, \ldots, x_n + b_n t)$. Obviously $u(f, 0) = f(x_1, \ldots, x_n)$. The set of solutions of (1) when $f \in C^1(R^n)$ forms a one-parametric family of self-mappings of $C^1(R^n)$, acting by the formula $u_t(f) = u(f,t)$.

Let us define a map $j : \mathcal{B} \to \mathcal{B}$ where $\mathcal{B}$ consists of all compact subsets of R^n, in the following way : for every $K \subset R^n$ let $j(K) = K_{t_1} \cup K_{t_2}$ where

$$K_{t_i} = \{(\overline{x}_1, \ldots, \overline{x}_n) \in R^n :$$

$$\overline{x}_1 = x_1 + b_1 t_i, \ldots, \overline{x}_n = x_n + b_n t_i; (x_1, \ldots, x_n) \in K\} \quad (i = 1, 2).$$

We shall show that the family of operators $u_t(f)$ is continuous, that is, for every $\varepsilon > 0$ and every compact $K \subset R^n$ there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ and $\|f - \overline{f}\|_{j(K)} < \delta$ implies $\|u_{t_1}(f) - u_{t_2}(\overline{f})\|_K < \varepsilon$. Indeed,

$$u_{t_1}(f) - u_{t_2}(\overline{f}) \leqq$$
$$\leqq \exp(at_1)\,[f(x_1 + b_1t_1, \dots, x_n + b_nt_1) - \overline{f}(x_1 + b_1t_1, \dots, x_n + b_nt_1)]+$$
$$+(\exp(at_1) - \exp(at_2))\,\overline{f}(x_1 + b_1t_1, \dots, x_n + b_nt_1)+$$
$$+\exp(at_2)\,[\overline{f}(x_1 + b_1t_1, \dots, x_n + b_nt_1) - \overline{f}(x_1 + b_1t_2, \dots, x_n + b_nt_2)];$$
$$[u_{t_1}(f) - u_{t_2}(\overline{f})]_{x_i} \leqq$$
$$\leqq \exp(at_1)\,[f_{x_i}(x_1 + b_1t_1, \dots, x_n + b_nt_1) - \overline{f}_{x_i}(x_1 + b_1t_1, \dots, x_n + b_nt_1)]+$$
$$+(\exp(at_1) - \exp(at_2))\,\overline{f}_{x_i}(x_1 + b_1t_1, \dots, x_n + b_nt_1)+$$
$$+\exp(at_2)\,[\overline{f}_{x_i}(x_1 + b_1t_1, \dots, x_n + b_nt_1) - \overline{f}_{x_i}(x_1 + b_1t_2, \dots, x_n + b_nt_2)]$$
$$(i = 1, \dots, n).$$

Then for the first summands we obtain $\exp(at_1)\|f - \overline{f}\|_{j(K)} < \frac{\varepsilon}{3}$ when $\|f - -\overline{f}\|_K < \delta_1$, for the second summands $(\exp(at_1) - \exp(at_2))\|\overline{f}\|_K < \frac{\varepsilon}{3}$ when $|t_1 - t_2| < \delta_2$ for the third $\exp(at_2)|f(x_1 + b_1t_1, \dots, x_n + b_nt_1) - \overline{f}(x_1 + +b_1t_2, \dots, x_n + b_nt_2)| < \frac{\varepsilon}{3}$ when $|t_1 - t_2| < \delta_3$, because $f(x_1, \dots, x_n)$ is uniformly continuous on K, Consequently the inequalities $|t_1 - t_2| < \delta$ and $\|f - \overline{f}\|_{j(K)} < \delta$ imply $\|u_{t_1}(f) - u_{t_2}(f)\|_K < \varepsilon$, where $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. It is easy to verify $u(u(f, t_1), t_2) = u(f, t_1 + t_2)$ and for every $f \in C^1(R^n)$ there exists a unique solution $u(f, t)$ of (1), such that $u(f, 0) = f$. A one-parametric family of operators $u(f, t)$ with the above properties forms a dynamical system in the locally convex space $C^1(R^n)$.

Let us consider the known hyperbolic system

$$u_t(t, x) = v_x(t, x), \quad v_t(t, x) = u_x(t, x) \tag{2}$$

with initial conditions $u(0, x) = f(x)$, $v(0, x) = g(x)$. Then the family of all continuous pairs $(f(x), g(x))$ forms a locally convex space $C^1(R^1) \times C^1(R^1)$ with a saturated family of seminorms $\|(f, g)\|_K = \|f\|_{K_1} + \|g\|_{K_2}$, where $\|f\|_{K_1} = \sup\{|f(x)| : x \in K_1\}$, $\|g\|_{K_2} = \sup\{|g(x)| : x \in K_2\}$; K_1 and K_2 run over all compact subsets of R^1. It is known that the pair

$$u(f, g, t) = \frac{1}{2}(f(x - t) + f(x + t) + g(x - t) - g(x - t)),$$
$$v(f, g, t) = \frac{1}{2}(g(x - t) + g(x + t) + f(x - t) - f(x - t))$$

is a solution of (2) such that $u(f,g,0) = f$, $v(f,g,0) = g$. One can easily verify that the family of operators $T_t(f,g) = (u(f,g,t), v(f,g,t))$ forms a dynamical system in the space $C^1(R^1) \times C^1(R^1)$.

Throughout the remainder of this paper we shall denote by X a uniform Hausdorff space with a uniformity generated by a saturated family of pseudometrics (cf. [1]). Let $\mathcal{A} = \{d_\alpha(x,y) : \alpha \in A\}$, A being an index set. We shall suppose also that X is complete; nevertheless for our purposes we need only a sequential completeness.

Let us recall some results and notations from [2]. If $j : A \to A$ is a mapping by $j^k(\alpha)$ we denote the k-th iterate of j, that is, $j^k(\alpha) = j(j^{k-1}(\alpha))$, $j^0(\alpha) = \alpha$; $j^{-1}(\alpha_0) = \{\alpha \in A : j(\alpha) = \alpha_0\}$, $j^{-2}(\alpha_0) = \{\alpha \in A : j(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ runs over all elements of $j^{-1}(\alpha_0)\}$ and inductively $j^{-n}(\alpha_0) = \{\alpha \in A : j(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ runs over all elements of $j^{-(n-1)}(\alpha_0)\}$; and so on.

The space X is called j-bounded if for every $x, y \in X$ and $\alpha \in A$ there exists a constant $Q = Q(\alpha,x,y) > 0$ such that $d_{j^{-n}(\alpha)}(x,y) \leqq Q(\alpha,x,y)$ $(n = 0,1,\ldots)$, or more precisely Q is an upper bound of the set $\{d_\alpha(x,y), d_{\alpha_1}(x,y),\ldots,d_{\alpha_n}(x,y),\ldots\}$ for every sequence $\{\alpha,\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots\}$ where $\alpha_1 \in j^{-1}(\alpha)$, $\alpha_2 \in j^{-2}(\alpha),\ldots,\alpha_n \in j^{-n}(\alpha),\ldots$.

Let (ϕ) be a family of contractive functions $\phi_\alpha(t) : R^1_+ \to R^1_+$ $(R^1_+ = [0,\infty))$, that is, every $\phi_\alpha(t)$ has the properties:

$(\phi 1)$ $\phi_\alpha(t)$ is continuous, strictly increasing and superadditive (cf. [3], Ch. VII) and $0 < \phi_\alpha(t) < t$ for $t > 0$.

$(\phi 2)$ for every $\alpha \neq \alpha'$ $\max\{\phi_\alpha(t), \phi_{\alpha'}(t)\}$ is superadditive;

$(\phi 3)$ $\lim\limits_{n\to\infty} \phi_\alpha\left(\phi_{j^{-1}(\alpha)}\left(\ldots\phi_{j^{-n}(\alpha)}(t)\ldots\right)\right) = 0$ for $t > 0$ and $\phi_{j(\alpha)}(t) \leqq \phi_\alpha(t)$.

The last equality means that $\lim\limits_{n\to\infty} \phi_\alpha\left(\phi_{j(\alpha)}\left(\ldots\phi_{j^n(\alpha)}(t)\ldots\right)\right) = 0$ for every sequence $\{\alpha,\alpha_1,\ldots,\alpha_n,\ldots\}$ where $\alpha_1 \in j^{-1}(\alpha),\ldots,\alpha_n \in j^{-n}(\alpha),\ldots$.

The mapping $T : X \to X$ is said to be ϕ-contractive if $d_{j(\alpha)}(Tx,Ty) \leqq \phi_\alpha(d_\alpha(x,y))$ for every $x,y \in X, \alpha \in A$. T is called j-regular if the sequence $\{T^n x\}_{n=0}^\infty$ is not d_α-Cauchy sequence, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $d_\alpha(T^{m+p}x, T^m x) \geqq \varepsilon$ implies $d_{j(\alpha)}(T^{m+p}x, T^m x) \geqq \delta$.

Two families of pseudometrics $\mathcal{A} = \{d_\alpha(x,y) : \alpha \in A\}$ and $\mathcal{A}^* = \{d^*_{\alpha^*}(x,y) : \alpha^* \in A^*\}$ are topologically equivalent if the uniformities generated by them coincide. Let

$$\text{card}\, A = \text{card}\, A^* \tag{3}$$

and let $h : A \to A^*$ be a bijective map. It induces the map $j^* : A^* \to A^*$ by the commutativity of the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & A^* \\ j\big\downarrow & & j^*\big\downarrow \\ A & \xrightarrow{h} & A^* \end{array}$$

that is, $h(j(\alpha)) = j^*(h(\alpha))$. The families $\mathcal{A}$ and $\mathcal{A}^*$ are called equivalent if (3) is satisfied and if every Cauchy net $\{x_\gamma\}_{\gamma\in\Gamma}$ (Γ is a directed set) in d_α is a Cauchy net in $d^*_{\alpha^*}$ for every $\alpha \in A$ where $\alpha^* = h(\alpha)$, and vice versa, every Cauchy net $\{x_\gamma\}_{\gamma\in\Gamma}$ in $d^*_{\alpha^*}$ is a Cauchy net in d_α for every $\alpha^* \in A^*$, where $\alpha = h^{-1}(\alpha^*)$. The equivalence of two families $\mathcal{A}$ and $\mathcal{A}^*$ implies their topological equivalence. Conversely, the topological equivalence implies their equivalence when X is a complete uniform space. $\mathcal{A}$ and $\mathcal{A}^*$ are said to be j-topologically equivalent if the j-regularity of T implies its j^*-regularity and j-boundedness of X implies j^*-boundedness of X.

As usual, $U(\alpha_1, \alpha_2, \dots, \alpha_p)(x, \varepsilon) = \{y \in X : d_{\alpha_1}(x, y) < \varepsilon, \dots, d_{\alpha_p}(x, y) < \varepsilon\}$ is an ε-neighbourhood of x in the pseudometrics $d_{\alpha_1}, \dots, d_{\alpha_p}$. We say that the sequence $T^n(U(\alpha_1, \dots, \alpha_p)(\xi, \eta))$ $(n = 0, 1, \dots)$ tends uniformly to ξ if for every $\varepsilon > 0$ there is n_0 such that for $n \geqq n_0$,

$$T^n(U(\alpha_1, \dots, \alpha_p)(\xi, \eta)) \subset U(\alpha_1, \dots, \alpha_p)(\xi, \varepsilon).$$

Further on, we shall assume that the following conditions (H1)–(H2) are fulfilled:

(H1) if $T^n x \overline{\in} \operatorname{cl} U(j(\alpha_1), \dots, j(\alpha_p))(\xi, \eta)$ then $T^n x \overline{\in} \operatorname{cl} U(\alpha_1, \dots, \alpha_p)(\xi, \eta)$;

(H2) if $x \in \operatorname{cl} T^n(U(\alpha_1, \dots, \alpha_p)(\xi, \eta))$ then $x \in \operatorname{cl} T^n\,(U(j(\alpha_1), \dots, j(\alpha_p))(\xi, \eta))$.

THEOREM 1 [2]. *Every ϕ-contractive j-regular mapping $T : X \to X$ has a unique fixed point $\overline{x}$ and $\lim_{n\to\infty} T^n x = \overline{x}$ for arbitrary $x \in X$.*

Let $(X, \mathcal{A})$, $\mathcal{A} = \{d_\alpha(x, y) : \alpha \in A\}$, be a complete j-bounded Hausdorff space and let $T : X \to X$ be a continuous j-regular mapping.

THEOREM 2 [2]. *The following conditions:*

(E1) *T has a fixed point $\xi \in X$;*

(E2) $\lim_{n\to\infty} d_\alpha(T^n x, \xi) = 0$ *for every $x \in X$, $\alpha \in A$;*

(E3) *for every finite collection $\alpha_1, \dots, \alpha_p \in A$ there is $\eta > 0$ such that the sequence $\{T^n(U(\alpha_1, \dots, \alpha_p)(\xi, \eta))\}_{n=0}^\infty$ tends uniformly to ξ;*

are necessary and sufficient for the existence of a family of pseudometrics $\mathcal{A}^* = \{d_{\alpha^*}(x, y) : \alpha^* \in A^*\}$ j-topologically equivalent to $\mathcal{A}$, such that T

is ϕ-contractive with respect to $\mathcal{A}^*$ for an arbitrarily chosen family (ϕ) of contractive functions.

Let G be a commutative topological semigroup (written additively) with zero element. Then a family $\{T_t : t \in G\}$ of continuous selfmappings of X will be called a G-semigroup if it possesses the properties:

(1) $T_0(x) = x$ for every $x \in X$;

(2) $T_{t_1}(T_{t_2}(x)) = T_{t_1+t_2}(x)$ for every $t_1, t_2 \in G$ and $x \in X$;

(3) $\sup\{d_\alpha(T_t(x), T_{t_0}(x)) : x \in X\} \to 0$ as $t \to t_0$ for every $\alpha \in A$.

We shall consider only the particular case $G = [0, \infty)$ or an infinite subsemigroup of $[0, \infty)$.

A semigroup of operators is said to be contractive if for some family (ϕ) and some j there is a collection of families $\{(\mathcal{A}_t, j_t) : t \in [0, \infty)\}$ of pseudometrics of X each j-topologically equivalent to $\mathcal{A}$ such that T_t is ϕ-contractive with respect to $(\mathcal{A}_t, j_t)$ for every $t > Q$.

The semigroup defined by the formula $T_t(f) = \exp(at)f(x_1+b_1t, \ldots, x_n+ +b_nt)$ for $t \geqq 0$ when $a < 0$, is an example of a contractive semigroup. Indeed, let us fix $t \in (0, \infty)$ and let $K \subset R^n$ be a compact set. Define j in the following way:

$$j(K) = \{(\overline{x}_1, \ldots, \overline{x}_n) : \overline{x}_1 = x_1 + b_1t, \ldots, \overline{x}_n = x_n + b_nt \quad (x_1, \ldots, x_n) \in K\}.$$

Then

$$\|T_t(f) - T_t(\overline{f})\|_{j(K)} \leqq \exp(at)\|f - \overline{f}\|_K$$

for every $f, \overline{f} \in C^1(R^n)$.

Following [4] we shall call $\xi \in X$ an equilibrium state if it is a common fixed point of the semigroup of operators, that is $T_t(\xi) = \xi$ for every $t \in \in [0, \infty)$. It is easy to verify that if T_t has ξ as a unique fixed point for some $t_0 \in (0, \infty)$, then $T_t(\xi) = \xi$ for any $t \in [0, \infty)$. In our example if $a + b_1 + \cdots + b_n = 0$, then $f(x_1, \ldots, x_n) = \exp(x_1 + \cdots + x_n)$ is a common fixed point of the semigroup as the following equalities show:

$$T_t(f) = \exp(at)\exp(x_1 + b_1t + x_2 + b_2t + \cdots + x_n + b_nt) =$$
$$= \exp(x_1 + x_2 + \cdots + x_n) = f.$$

In what follows we shall need extensions of the notions of stability, asymptotic stability and uniform asymptotic stability in a uniform space. An equilibrium state ξ is called stable if for every neighbourhood V of ξ, there exists a neighbourhood W of ξ such that $x \in W$ implies $T_t(x) \in V$ for all $t \in [0, \infty)$. The stable equilibrium state ξ is called asymptotically stable if there is a neighbourhood U of ξ such that $x \in U$ implies $\lim_{t\to\infty} T_t(x) = \xi$, and called uniformly asymptotically stable if U can be chosen so that $T_t(U)$ tends uniformly to ξ as $t \to \infty$.

As usual, by $B(\alpha_1, \alpha_2, \ldots, \alpha_p)(x, r)$ we shall denote the closed ball of radius $r > 0$ with center at x in pseudometrics $d_{\alpha_1}(x, y), \ldots, d_{\alpha_p}(x, y)$, that is $B(\alpha_1, \alpha_2, \ldots, \alpha_p)(x, r) = \{y \in X : d_{\alpha_1}(x, y) \leqq r, \ldots, d_{\alpha_p}(x, y) \leqq r\}$.

LEMMA 1. *If $\xi \in X$ is an equilibrium state of $\{T_t : t \in [0,\infty)\}$, then for each $t_0 > 0$, $\eta > 0$ and for every finite collection of pseudometrics $d_{\alpha_1}, \dots, d_{\alpha_p}$ there is a $\delta > 0$, $\delta = \delta(t_0, \eta, \alpha_1, \dots, \alpha_p)$, such that $T_t(B(\alpha_1, \dots, \dots, \alpha_p)(\xi, \delta)) \subseteqq B(\alpha_1, \dots, \alpha_p)(\xi, \eta)$ for every $t \in [0, t_0]$.*

PROOF. Suppose, by contradiction, that there is a $\bar{t}_0 > 0$, an $\bar{\eta} > 0$, a finite collection of pseudometrics $d_{\overline{\alpha}_1}, \dots d_{\overline{\alpha}_p}$, a sequence of positive numbers δ_n $(n = 1, 2, \dots)$ with $\lim_{n\to\infty} \delta_n = 0$, corresponding sequences of points $x_n \in B(\overline{\alpha}_1, \dots, \overline{\alpha}_p)(\xi, \delta_n)$ and numbers $t_n \in [0, \bar{t}_0]$ such that $T_{t_n}(x_n) \in \in X \setminus B(\overline{\alpha}_1, \dots, \overline{\alpha}_p)(\xi, \delta_n)$. One can choose a subsequence $\{n_k\}_{k=1}^{\infty}$ so that $\lim_{k\to\infty} t_{n_k} = t \in [0, \bar{t}_0]$. Then

$$d_{\overline{\alpha}_i}\left(\xi, T_{t_{n_k}}(x_{n_k})\right) \leqq d_{\overline{\alpha}_i}\left(T_t(\xi), T_t(x_{n_k})\right) + d_{\overline{\alpha}_i}\left(T_t(x_{n_k}), T_{t_{n_k}}(x_{n_k})\right)$$

for $i = 1, \dots, p$. The first term on the right tends to zero as $k \to \infty$ because $x_{n_k} \to \xi$. The continuity condition for the semigroup ensures that $\lim_{k\to\infty} d_{\overline{\alpha}_i}\left(T_t(x_{n_k}), T_{t_{n_k}}(x_{n_k})\right) = 0$. The obtained contradiction completes the proof of Lemma 1.

Now we are going to formulate the main results:

THEOREM 3. *For a $[0,\infty)$-semigroup $\{T_t : t \in [0,\infty)\}$ on X to be contractive, it is necessary and sufficient for some t_0 the operator T_{t_0} to possess conditions* (E1)-(E3).

PROOF. Since the necessity is obvious we shall prove the sufficiency.

Let us assume that for some t_0, T_{t_0} possesses condition (E1). We shall show that T_t satisfies (E1) for any $t \in [0,\infty)$. Indeed, the equality $T_t(\xi) = = T_t(T_{t_0}(\xi)) = T_{t+t_0}(\xi) = T_{t_0+t}(\xi) = T_{t_0}(T_t(\xi))$ implies that $T_t(\xi)$ is a fixed point of $T_{t_0}(\xi)$, so that $T_t(\xi) = \xi$ for $t \in [0,\infty)$.

In order to prove that T_t satisfies (E2) we shall consider an arbitrary $x \in X$ and $\eta > 0$ for $t > 0$. T_0 cannot satisfy (E2) except for the trivial case $X = \{\xi\}$. We can choose an arbitrary finite collection of pseudometrics $d_{\alpha_1}, \dots, d_{\alpha_p}$ and $N > 0$ so large that $d_{\alpha_i}\left(\xi, T_{t_0}^n(x)\right) < \delta$ for $n \geqq N$ $(i = 1, 2, \dots, p)$, where $\delta = \delta(t_0, \eta, \alpha_1, \dots, \alpha_p)$. Next we choose $M > 0$ so large that $mt \geqq Nt_0$ for $m \geqq M$. Then for $m \geqq M$ we have $mt = = nt_0 + \sigma$ $(n \geqq N,\ 0 \leqq \sigma < t)$. Lemma 1 implies $T_\sigma(B(\alpha_1, \dots, \alpha_p)(\xi, \delta)) \subseteqq \subseteqq B(\alpha_1, \dots, \alpha_p)(\xi, \eta)$ while $T_t^m(x) = T_{mt}(x) = T\left(T_{t_0}^n(x)\right) \in T_\sigma(B(\alpha_1, \dots, \dots, \alpha_p)(\xi, \delta))$ and hence for $m \geqq M = M(\eta)$, $T_t^m(x) \in B(\alpha_1, \dots, \alpha_p)(\xi, \eta)$ so that $T_t(x)$ satisfies (E2).

In order to prove (E3) for T_t, we shall consider an arbitrary $\eta > 0$. Let $\delta = \delta(t, \eta, \alpha_1, \dots, \alpha_p)$ be chosen as in Lemma 1. On the other hand T_{t_0} satisfies (E3). Let $U(\alpha_1, \dots, \alpha_p)(\xi, \varepsilon)$ be a neighbourhood of ξ such that $T_{t_0}^n(U)$ tends uniformly to ξ as $n \to \infty$. Let us fix $N > 0$ so large that $T_{t_0}^n(U) =$

$= B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)$ for $n \geqq N$ and M so large that $mt > Nt_0$ for $m \geqq M$. Then for $m \geqq M$ we can proceed as above to prove $T_t^m(U) = B(\alpha_1, \ldots, \alpha_p)(\xi, \eta)$

Theorem 3 is thus proved.

THEOREM 4. *The equilibrium state ξ of a $[0, \infty)$-semigroup is uniformly asymptotically stable if and only if there exists a subset $X_0 \subset X(\xi \in \operatorname{int} X_0)$ such that the restriction of the semigroup to X_0 is contractive.*

PROOF. *Sufficiency.* Let $V = V(\alpha_1, \ldots, \alpha_p)(\xi)$ be an arbitrary neighbourhood of ξ. Let us fix $\bar{t} > 0$ and choose $\eta > 0$ such that $B(\alpha_1, \ldots, \alpha_p)(\xi, \eta) \subseteqq V \cap X_0$. Let $\delta = \delta(\bar{t}, \eta, \alpha, \ldots, \alpha_p) > 0$ be chosen related to $\bar{t} > 0$ and $\eta > 0$ as in Lemma 1 ($\delta \leqq \eta$).

By the definition of a contractive semigroup there exists a family of pseudometrics $\mathcal{A}_{\bar{t}} = \left\{ d_{\overline{\alpha}}^{\bar{t}}(x, y) : \overline{\alpha} \in A_{\bar{t}} \right\}$ on X_0 j-topologically equivalent to $\mathcal{A}$ on X_0. This means that there is a bijection $h_{\bar{t}} : A \to A_{\bar{t}}$ and $\operatorname{card} A = \operatorname{card} A_{\bar{t}}$ such that $T_{\bar{t}}|_{X_0}$ is ϕ-contractive on $(X_0, \mathcal{A}_{\bar{t}})$.

On the other hand there is a $\Delta > 0$ such that $W = W(\overline{\alpha}_1, \ldots, \overline{\alpha}_p)(\xi, \Delta) = \left\{ y \in X_0 : d_{\overline{\alpha}_1}^{\bar{t}}(y, \xi) < \Delta, \ldots, d_{\overline{\alpha}_p}^{\bar{t}}(y, \xi) < \Delta \right\} \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)$ where $\overline{\alpha}_i = h_{\bar{t}}(\alpha_i)$. We shall show that $T_{\bar{t}}^n(W) \subseteqq W$. Indeed, let $y \in W$. Then we have

$$d_{\overline{\alpha}_i}^{\bar{t}}(T_t^n(y), \xi) = d_{\overline{\alpha}_i}^{\bar{t}}(T_t^n(y), T_t^n(\xi)) \leqq$$

$$\leqq \phi_{j_{\bar{t}}^{-1}(\overline{\alpha}_i)}(\phi_{j_{\bar{t}}^{-2}(\overline{\alpha}_i)}(\ldots \phi_{j_{\bar{t}}^{-n}(\overline{\alpha}_i)}(d_{j_t^{-n}(\alpha_i)}(y, \xi)) \ldots)) < \Delta$$

for sufficiently large n, i.e. $n \geqq n_0$ implies $T_t^n(y) \in W$. In order to ensure $T_{\bar{t}}^n(W) \subseteqq W$ for $0 \leqq n \leqq n_0 - 1$, we can introduce the set

$$\overline{W} = \bigcap_{l=0}^{n_0 - 1} T_t^l(W)$$

and consider $W \cap \overline{W}$ instead of W.

Having in mind that every $t > 0$ can be written in the form $t = n\bar{t} + \sigma$, $0 \leqq \sigma < \bar{t}$ we have $T_t(W) = T_\sigma\left(T_{\bar{t}}^n(W)\right) \subseteqq T_\sigma(W) \subseteqq T_\sigma(B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)) \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \eta) = V$ which shows that ξ is stable.

Since $T_{\bar{t}}(x)$ is contractive for every fixed $\bar{t} \in [0, \infty)$ on X_0, there is an $\eta_0 > 0$ such that $\left\{ T_{\bar{t}}^n(U(\overline{\alpha}_1, \ldots, \overline{\alpha}_p)(\xi, \eta_0)) \right\}_{n=0}^{\infty}$ tends uniformly to ξ for every finite collection $\overline{\alpha}_1, \ldots, \overline{\alpha}_p$ and $U \subset X_0$, $T_{\bar{t}}(U) \subseteqq X_0$.

For the given $V(\alpha_1, \ldots, \alpha_p)(\xi)$ we choose $\eta > 0$ and $\delta > 0$ as above. Let $N > 0$ be so large that $T_{\bar{t}}^n(U) \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)$ for $n \geqq N$. Then any t can be written in the form $t = n\bar{t} + \sigma$, $n \geqq N$ and $0 \leqq \sigma < \bar{t}$, so that

$T_t(U(\overline{\alpha}_1,\dots,\overline{\alpha}_p)(\xi,\eta_0)) = T_\sigma\left(T_{\overline{t}}^{n}(U(\overline{\alpha}_1,\dots,\overline{\alpha}_p)(\xi,\eta_0))\right) \subseteqq T_\sigma(B(\alpha_1,\dots,\alpha_p)$ $(\xi,\delta)) \subseteqq B(\alpha_1,\dots,\alpha_p)(\xi,\eta) \subseteqq V(\alpha_1,\dots,\alpha_p)(\xi)$. Hence $T_t(U) \to \{\xi\}$ as $t \to \infty$, that is, ξ is uniformly asymptotically stable.

Necessity. Let ξ be a uniformly asymptotically stable equilibrium state. Define the set $X_0 = \{x \in X : T_t(x) \to \xi \text{ as } t \to \infty\}$. The definition of asymptotic stability ensures that X_0 contains a neighbourhood of ξ, and so is itself such a neighbourhood. On the other hand the semigroup property implies that $T_t(X_0) \subseteqq X_0$ for every $t \in [0,\infty)$. Indeed, for $x \in X_0$, we have $T_{t_1}(T_{t_2}(x)) = T_{t_1+t_2}(x)$ and when $t_1 \to \infty$, $T_{t_1+t_2}(x) \to \xi$, that is, $T_{t_2}(x) \in$ $\in X_0$. Therefore $\{T_t|_{X_0} : t \in [0,\infty)\}$ is an $[0,\infty)$-semigroup of operators on X_0.

The definition of X_0 implies that $T_t|_{X_0}$ for every $t \in [0,\infty)$ satisfies conditions (E1) and (E2). We shall verify condition (E3).

The uniform asymptotic stability of ξ guarantees the existence of an $\eta_0 >$ > 0 such that for every finite collection $d_{\alpha_1},\dots,d_{\alpha_p}$ there is a neighbourhood $U(\alpha_1,\dots,\alpha_p)(\xi,\eta_0) \subseteqq X_0$ such that $T_t(U)$ tends uniformly to ξ as $t \to \infty$. Consequently $T_t^n(U) = T_{nt}(U) \to \xi$ as $n \to \infty$ and T_t is ϕ-contractive, which completes the proof.

Finally, we shall apply Theorem 4 to our first example. Define the set X_0 in the following way: $X_0 = \{f \in C^1(R^n) : \exp(x_1 + \dots + x_n) - q \leqq$ $\leqq f(x_1,\dots,x_n) \leqq \exp(x_1 + \dots + x_n) + q\}$ where $q > 0$ is a constant. It is easy to verify that $T_t(X_0) \subset X_0$. Indeed, let us put $x_1 = \overline{x}_1 + b_1 t, \dots, x_n =$ $= \overline{x}_n + b_n t$. If $a < 0$ and $a + b_1 + \dots + b_n = 0$, then we obtain ($\exp(at) \leqq 1$):

$$\exp(at + x_1 + \dots + x_n) - q\exp(at) \leqq \exp(at) f(x_1,\dots,x_n) \leqq$$
$$\leqq \exp(at + x_1 + \dots + x_n) + q\exp(at),$$
$$\exp(\overline{x}_1 + \dots + \overline{x}_n) - q \leqq \exp(at) f(\overline{x}_1 + b_1 t,\dots,\overline{x}_n + b_n t) \leqq$$
$$\leqq \exp(\overline{x}_1 + \dots + \overline{x}_n) + q,$$
$$\exp(x_1 + \dots + x_n) - q \leqq T_t(f) \leqq \exp(x_1 + \dots + x_n) + q.$$

In order to apply Theorem 4 we must verify the continuity property of the semigroup. Having in mind $\|f\|_K \leqq \|\exp(x_1 + \dots + x_n)\|_K + q = \Delta_K$ we have

$$A = \sup\{|T_t(f) - T_{t_0}(f)| : (x_1,\dots,x_n) \in K\} \leqq$$
$$\leqq |\exp(at) - \exp(at_0)| \sup\{|f(x_1 + b_1 t,\dots,x_n + b_n t)| : (x_1,\dots,x_n) \in K\} +$$
$$+ \exp(at_0) \sup\{|f(x_1 + b_1 t,\dots,x_n + b_n t) -$$
$$- f(x_1 + b_1 t_0,\dots,x_n + b_n t_0)| : (x_1,\dots,x_n) \in K\} \leqq$$
$$\leqq \max\{\Delta_{K_t} : t \in N_{t_0}\} |\exp(at) - \exp(at_0)| +$$
$$+ \exp(at_0) \sup\{|f(x_1 + b_1 t,\dots,x_n + b_n t) -$$
$$- f(x_1 + b_1 t,\dots,x_n + b_n t)| : (x_1,\dots,x_n) \in K\}$$

where N_{t_0} is some neighbourhood of t_0. Then if $K_t \cap K_{t_0} \neq \emptyset$, $A \to 0$ as $t \to t_0$. When $K_t \cap K_{t_0} = \emptyset$, then $A = 0$.

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(Received October 13, 1988)

HIGHER INSTITUTE OF MINING AND GEOLOGY
SOFIA 1156
BULGARIA

Acta Math. Hung.
59 (3–4) (1992), 355–363.

IDEMPOTENTS IN POLYNOMIAL RINGS

A. A. M. KAMAL (Giza)

1. Introduction

Throughout this paper all rings are associative with unit and the following notations will be preserved: σ is an automorphism of a ring A and D is a σ-derivation on A (i.e. $D(a+b) = D(a)+D(b)$ and $D(ab) = D(a)b+\sigma(a)D(b)$, for each $a, b \in A$). The Ore extension $A[X, \sigma, D]$ is the ring of polynomials in X over A with the usual addition and with multiplication subject to the rule $Xa = \sigma(a)X + D(a)$ for each $a \in A$. If $\sigma = 1_A$, then $A[X, \sigma, D] \equiv A[X, D]$ is called the differential operator ring, if $D = 0$, then $A[X, \sigma, D] \equiv A[X, \sigma]$ is called the twisted (skew) polynomial ring, and if $\sigma = 1_A$ and $D = 0$, then $A[X, \sigma, D] \equiv A[X]$ is the usual polynomial ring. Finally the set of all central idempotents in a ring A is denoted by $B(A)$ and $(B(A))^\sigma = \{e \in B(A) : \sigma(e) = e\}$.

In 1977, Chon [1] has determined the center of Ore extension $A[X, \sigma]$, when A is a division ring. Shortly thereafter, J. D. Rosen and M. P. Rosen generalized this result to prime rings [5]. In this paper we study the idempotents in polynomial rings and we show that, for a ring A, $B(A[X, D]) = B(A)$, $B(A[X, \sigma]) = (B(A))^\sigma$ and every central idempotent in $A[X, \sigma, D]$ with constant coefficient that commutes with the other coefficients, is in A. Moreover if A is a semiprime ring, then $B(A[X, \sigma, D]) = (B(A))^\sigma$. Finally we show that, every faithful Abelian idempotent in A is also faithful Abelian idempotent in $A[X]$, and a counterexample is given to show that this result is not true for the ring $A[X, \sigma]$. Special attention is given for obtaining a condition under which the result is true for $A[X, \sigma]$.

2. Preliminaries

In this section we collect some of the definitions and results that are needed in this paper.

A ring A is regular provided that for every $x \in A$, there exists $y \in A$ such that $xyx = x$. A regular ring A is Abelian provided that all idempotents in A are central. An idempotent e in a ring A is called an Abelian idempotent whenever the idempotents in eAe are central in eAe. Let e be an idempotent in a ring A, then e is faithful in A, if 0 is the only central idempotent of A which is orthogonal to e. For more details about regular rings, Abelian regular rings, Abelian idempotents and faithful idempotents, we refer the reader to [2].

A ring A is said to be reduced if A contains no nonzero nilpotent elements and it is σ-reduced if $x\sigma(x)=0$ implies that $x=0$, for each $x\in A$.

(2.1) A regular ring A is σ-reduced if and only if A is a reduced ring and $\sigma(e)=e$ for each idempotent e in A.

We refer the reader to ([3], Ch. VI) for the proof of (2.1) and the properties of a regular σ-reduced ring.

(2.2) We give an example which is required for 3.8. Let

$$A=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} : a,b,c\in Z_{36}\right\},$$

$$e=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix},\quad f=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix},\quad\text{and}\quad B=\left\{\begin{pmatrix} a & 0\\ 0 & c\end{pmatrix} : a,c\in Z_{36}\right\}.$$

Then A is a ring, B a commutative subring and e,f are orthogonal idempotents which are not central in A.

Notice that the central idempotents in A are $\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}$, with x an idempotent in Z_{36}, therefore $\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} 9 & 0\\ 0 & 9\end{pmatrix}$ and $\begin{pmatrix} 28 & 0\\ 0 & 28\end{pmatrix}$ are the central idempotents in A, which implies that e and f are faithful idempotents in A.

Since $eAe\subset B$ and $fAf\subseteq B$, then e and f are orthogonal faithful Abelian idempotents in A. □

We now give two examples of Ore extension over a ring of matrices A with an automorphism σ such that $\sigma^2=1_A$ and inner σ-derivation D on A. In the first we have a nontrivial idempotent polynomial. In the second example we have a nontrivial central polynomial.

(2.3) Let A be the ring of 2×2 matrices over a ring $B=C\times C$, where C is any ring. Define $\sigma'\in\operatorname{Aut}(B)$ such that $\sigma'(\alpha,\beta)=(\beta,\alpha)$ and $\sigma\in\operatorname{Aut}(A)$ such that

$$\sigma\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} \sigma'(a) & \sigma'(b)\\ \sigma'(c) & \sigma'(d)\end{pmatrix},\quad a,b,c,d\in B.$$

A σ-derivation D on A is defined by

$$D\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} (1,1) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}-\sigma\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} (1,1) & 0\\ 0 & 0\end{pmatrix}.$$

Let $a_1=\begin{pmatrix} 0 & (1,1)\\ 0 & 0\end{pmatrix}$ and $a_0=\begin{pmatrix} (1,1) & 0\\ 0 & 0\end{pmatrix}$. Then the polynomial $P(X)=$ $=a_1X+a_0$ is an idempotent in $A[X,\sigma,D]$ □

(2.4) Let B be a commutative ring and $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in B \right\}$.
We define $\sigma \in \operatorname{Aut}(A)$ such that

$$\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

A σ-derivation D on A is defined such that

$$D \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

where $\alpha \in B$ is a fixed element.

Let $a_1 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ and $a_0 = \begin{pmatrix} \beta & -\alpha^2 \\ 0 & \beta \end{pmatrix}$, where $\beta \in B$ is a fixed element. Then $P(X) = a_1 X + a_0$ is a nontrivial central element in $A[X, \sigma, D]$. $\square$

More precisely, we state the conditions found by A. Leroy [4] under which the Ore extension $A[X, \sigma, D]$ contains nontrivial central elements, as follows:

(2.5) Let A be a simple ring such that A is algebraic over its center and there exists an integer $m > 1$ such that σ^m is inner. Then the following conditions are equivalent:

1) D is algebraic,
2) $A[X, \sigma, D]$ is not simple,
3) The center of $A[X, \sigma, D]$ is nontrivial.

(This is Theorem 2.4 of [4], p. 1311.)

3. Idempotents in polynomial rings

In this section we study the relationship between the central idempotents and the faithful Abelian idempotents in a ring A and in each of the polynomial rings $A[X]$ $A[X, \sigma]$, $A[X, D]$ and $A[X, \sigma, D]$.

PROPOSITION 3.1. *For a ring A, every central idempotent in $A[X, \sigma]$ is in A. Moreover $B(A[X, \sigma]) = (B(A))^\sigma$.*

PROOF. Let $f = \sum_{i=0}^{n} a_i \cdot X^i \in B(A[X, \sigma])$. Then

$$a_j = \sum_{i=0}^{j} a_i \sigma^i(a_{j-i}), \quad j = 0, \ldots, n, \tag{1}$$

$$\sum_{i+j=k} a_i \sigma^i(a_j) = 0, \quad k = n+1, \ldots, 2n \quad \text{and} \quad i, j = 0, \ldots, n. \tag{2}$$

From $fa_j = a_j f$, $j = 0, 1, \ldots, n$ we get

$$(*) \qquad a_t a_\ell = a_\ell \sigma^\ell(a_t), \quad t = 0, \ldots, n.$$

If we take $\ell = 1$ and $t = 0$ in $(*)$ then $a_0 a_1 = a_1 \sigma(a_0)$. By taking $j = 1$ in (1) we get that $a_1 = a_0 a_1 + a_1 \sigma(a_0)$, therefore $a_1 = 2a_0 a_1$. Since a_0 is an idempotent from (1), then $a_0 a_1 = 0$ and hence from (1) with $j = 1$ again, $a_1 = 0$. By induction we can prove that $a_k = 0$, $1 \leqq k \leqq n$. Therefore $f = a_0 \in A$. □

Corollary 3.2. *For a ring A, $B(A) = B(A[X])$.*

Corollary 3.3. *If A is a ring and $f = \sum_{i=0}^{n} a_i X^i$ is an idempotent in $A[X, \sigma]$ such that $a_0 f = f a_0$, then $f = a_0 \in A$.*

Lemma 3.4. *For a ring A, every central idempotent in $A[X, D]$ is in A.*

Proof. Let $f = \sum_{i=0}^{n} a_i X^i \in B(A[X, D])$. $Xf = fX$ implies $D(a_i) = 0$, $i = 0, 1, \ldots, n$. Also $f = f^2$ implies

$$(1) \qquad a_j = \sum_{i=0}^{j} a_i a_{j-i}, \quad j = 0, \ldots, n$$

and

$$(2) \qquad \sum_{i+j=k} a_i a_j = 0, \quad k = n+1, \ldots, 2n, \quad i, j = 0, \ldots, n.$$

From $fa_j = a_j f$ we get that $a_i a_j = a_j a_i$, $i, j = 0, \ldots, n$. As in the proof of Proposition 3.1, we can show that $a_j = 0$, $j = 1, \ldots, n$. Hence $f = a_0 \in A$. □

Lemma 3.5. *For a derivation D in a ring A, $D(e) = 0$ for each $e \in B(A)$.*

Proof. It is clear that $D(e) = 2eD(e)$ and $D(1-e) = 2(1-e)D(1-e)$, also $D(1) = 0$, hence $D(1-e) = -D(e)$. Therefore $D(e) \in Ae \cap A(1-e) = 0$. □

Proposition 3.6. *For a ring A, $B(A[X, D]) = B(A)$.*

Proof. Follows form Lemmas 3.4 and 3.5. □

Proposition 3.7. *For a ring A, every faithful Abelian idempotent in A is a faithful Abelian idempotent in $A[X]$.*

Proof. Assume that e is a faithful Abelian idempotent in A, hence $B(A) = B(A[X])$ implies that e is a faithful idempotent in $A[X]$. Let $f =$

$= \sum_{i=0}^{n}(ea_ie)X^i \in eA[X]e$ such that $f = f^2$, therefore we get that

$$(1) \qquad ea_je = \sum_{i=0}^{j}(ea_ie)(ea_{j-i}e), \quad j = 0, \dots, n.$$

From (1) by taking $j = 0$ we get that $ea_0e \in B(eAe)$ and by taking $j = 1$ in (1) we get that $ea_1e = 0$. Assume that $ea_1e = \cdots = ea_{k-1}e = 0$, and take $j = k$ in (1), we get that $ea_ke = 0$ and therefore $f = ea_0e \in B(eA[X]e)$. □

EXAMPLE 3.8. Let A be a ring which contains orthogonal faithful Abelian idempotents e and f. Define $R = A \times A$ with $(a,b)(a',b') = (aa',bb')$ and $\sigma : R \longrightarrow R$ such that $(a,b) \mapsto (b,a)$, hence R is a ring and $\sigma \in \mathrm{Aut}\,(R)$. Notice that (e,f) is a faithful Abelian idempotent in R. Let $h = (0,1)X + +(0,1) \in R[X,\sigma]$, hence $g = (e,f)h(e,f) = (0,f)$ is an idempotent in the ring $(e,f)R[X,\sigma](e,f)$. Let $P = (a_1,b_1)X + (a_0,b_0) \in R[X,\sigma]$, where a_0, a_1, b_0 and $b_1 \in A$, hence $(0,f)[(e,f)P(e,f)] = (0,fb_1e)X + (0,fb_0f)$ and $[(e,f)P(e,f)](0,f) = (ea_1f,0)X + (0,fb_0f)$ which implies that g is not central in $(e,f)R[X,\sigma](e,f)$, hence (e,f) is not an Abelian idempotent in $R[X,\sigma]$. Therefore Proposition 3.7 is not true for $A[X,\sigma]$. □

PROPOSITION 3.9. *If A is a ring such that $\sigma(e) = e$ for each $e = e^2 \in A$, then every faithful Abelian idempotent in A is a faithful Abelian idempotent in $A[X,\sigma]$.*

PROOF. First, notice that $B(A[X,\sigma]) = B(A)$ and hence the proof is similar to that of Proposition 3.7. □

COROLLARY 3.10. *If A is a reduced ring, then the set of all idempotents in A is equal to the set of all idempotents in $A[X,D]$; also every faithful Abelian idempotent in A is a faithful Abelian idempotent in $A[X,D]$ and vice versa.*

PROOF. Notice that, A is a reduced ring implies that $A[X,D]$ is a reduced ring and every idempotent in a reduced ring is central. Hence the proof follows from Lemma 3.5, Proposition 3.6 and Proposition 3.7. □

EXAMPLE 3.11. Let K be a field, then $A = K \times K$ is a commutative regular self-injective ring, hence A is a reduced ring. Define $\sigma : A \longrightarrow A$ such that $(a,b) \mapsto (b,a)$. Therefore $\sigma \in \mathrm{Aut}\,(A)$. Let $P(X) = (0,1)X \in A[X,\sigma]$, then $P(X) \neq 0$. But $P^2 = (0,1)$. $\sigma(0,1)X^2 = 0$, hence $A[X,\sigma]$ is not reduced, therefore Corollary 3.10 is not true for $A[X,\sigma]$. □

PROPOSITION 3.12. *If A is a regular σ-reduced ring, then the set of all idempotents in A is equal to the set of all idempotents in $A[X,\sigma,D]$; also every faithful Abelian idempotent in A is a faithful Abelian idempotent in $A[X,\sigma,D]$ and vice versa.*

THEOREM 3.13. *For a ring A, every central idempotent in $A[X,\sigma,D]$ with constant coefficient which commutes with the other coefficients, is in $B(A)$.*

PROOF. Let $f = a_nX^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0 \in B(A[X,\sigma,D])$ such that $a_0a_j = a_ja_0$ for $j = 1,2,\dots,n$. We can write

$$f^2 = \sum_{k=0}^{2n}\left(\sum_{i+j=k} a_iX^ia_jX^j\right)$$

where

$$(0)^* \qquad \sum_{i+j=k} a_iX^ia_jX^j = \left(\sum_{\ell+t=k} a_\ell\sigma^\ell(a_t)\right)X^k+$$

$$+\left[\sum_{m=1}^{k}\left(a_m\sum_{\ell+t=m-1}\sigma^\ell D\sigma^t(a_{k-m})\right)\right]X^{k-1}+$$

$$+\left[\sum_{m=2}^{k}\left(a_m\sum_{\ell_1+\ell_2+\ell_3=m-2}\sigma^{\ell_1}D\sigma^{\ell_2}D\sigma^{\ell_3}(a_{k-m})\right)\right]X^{k-2}+$$

$$+\dots+\left[a_{k-1}D^{k-1}(a_1)+a_k\left(\sum_{\ell_1+\ell_2=k-1}D^{\ell_1}\sigma D^{\ell_2}(a_2)\right)\right]X+$$

$$+a_kD^k(a_0), \quad \text{for} \quad k = 0,\dots,n,$$

$$(1)^* \qquad \sum_{\ell=1}^{n} a_\ell X^\ell a_{(n+1)-\ell}X^{(n+1)-\ell} = \left[\sum_{\ell=1}^{n} a_\ell\sigma^\ell\left(a_{(n+1)-\ell}\right)\right]X^{n+1}+$$

$$+\left[\sum_{m=1}^{n}\left(a_m\sum_{\ell+t=m-1}\sigma^\ell D\sigma^t\left(a_{(n+1)-m}\right)\right)\right]X^n+\dots+$$

$$+\left[a_{n-1}D^{n-1}(a_0)+a_n\sum_{\ell+t=n-1}D^\ell\sigma D^t(a_1)\right]X^2+a_nD^n(a_1)X+$$

$$\vdots$$

$$(n)^* \qquad +a_nX^na_nX^n = a_n\left[\sigma^n(a_n)X^{2n}+\left(\sum_{\ell+t=n-1}\sigma^\ell D\sigma^t(a_n)\right)X^{2n-1}+\right.$$

$$+\cdots+\left(\sum_{\ell+t=n-1} D^{\ell}\sigma D^{t}(a_n)\right)X^{n+1}+D^{n}(a_n)X^{n}\Bigg].$$

Since f is central, we have $fa_j = a_j f$, $j = 0,1,\dots,n$ and therefore we have the following equations, for $j = 0,1,\dots,n$

(0) $$a_j a_0 = \sum_{i=0}^{n} a_i D^i(a_j),$$

(1) $$a_j a_1 = \sum_{m=1}^{n}\left(a_m \sum_{\ell+t=m-1} D^{\ell}\sigma D^{t}(a_j)\right),$$

(2) $$a_j a_2 = \sum_{m=2}^{n}\left(a_m \sum_{\ell_1+\ell_2+\ell_3=m-2} D^{\ell_1}\sigma D^{\ell_2}\sigma D^{\ell_3}(a_j)\right),$$

$$\vdots$$

$(n-1)$
$$a_j a_{n-1} = a_{n-1}\sigma^{n-1}(a_j) + a_n \sum_{\ell+t=n-1} \sigma^{\ell} D\sigma^{t}(a_j),$$

(n) $$a_j a_n = a_n \sigma^n(a_j).$$

From $f^2 = f$ and $(0)^*$, we have $a_0 = \sum\limits_{i=0}^{n} a_i D^i(a_0)$. Also by putting $j = 0$ in (0), we get $a_0^2 = \sum\limits_{i=0}^{n} a_i D^i(a_0)$ hence $a_0^2 = a_0$. By using $(0)^*$ and $(1)^*$, we can show that the coefficient of X in f^2 is equal to $\sum\limits_{i=0}^{n} a_i D^i(a_1) + {}$ $+\sum\limits_{m=1}^{n}\left(a_m \sum\limits_{\ell+t=m-1} D^{\ell}\sigma D^{t}(a_0)\right)$. Putting $j = 1$ in (0) and $j = 0$ in (1), we obtain that the coefficient of X in f^2 is equal to $a_1a_0 + a_0a_1$, hence $a_1a_0 + a_0a_1 = a_1$ which implies that $a_1 = 0$. By using $(0)^*$, $k = 2,\dots,n$, $(1)^*$ and $(2)^*$, we can write the coefficient of X^2 in f^2 as follows:

$$\sum_{i=0}^{n} a_i D^i(a_2) + \sum_{m=1}^{n} a_m\left(\sum_{\ell+t=m-1} D^{\ell}\sigma D^{t}(a_1)\right) +$$

$$+\sum_{m=2}^{n}\left[a_m\left(\sum_{\ell_1+\ell_2+\ell_3=m-2} D^{\ell_1}\sigma D^{\ell_2}\sigma D^{\ell_3}(a_0)\right)\right].$$

Putting $j=2$ in (0) and $j=0$ in (2), we get that the coefficient of X^2 in f^2 is equal to $a_2a_0+a_0a_2$. Hence $a_2a_0+a_0a_2=a_2$, which implies that $a_2=0$. Now assume that $a_1=a_2=\dots a_{s-1}=0$. Therefore the coefficient of X^s in f^2 is equal to

$$\sum_{i=0}^{n} a_iD^i(a_s)+\left[a_n\sum_{\ell_1+\dots+\ell_{s+1}=n-s} D^{\ell_1}\sigma D^{\ell_2}\dots D^{\ell_{s+1}}(a_0)+\right.$$

$$\left.+\dots+a_{s+1}\sum_{\ell+t=s}\sigma^{\ell}D\sigma^{t}(a_0)+a_s\sigma^s(a_0)\right].$$

Taking $j=s$ in (0) and $j=0$ in (s), we get that the coefficient of X^s in f^2 is equal to $a_sa_0+a_0a_s$, which implies that $a_sa_0+a_0a_s=a_s$. Hence $a_s=0$, therefore by induction we get that $a_1=a_2=\dots=a_n=0$, and hence $f=a_0\in A$ is a central idempotent in $A[X,\sigma,D]$. □

REMARK 3.14. Proposition 3.1 and hence Theorem 3.13 are false in the case of rings which are not associative, as the following counterexample shows.

Let Q be the ring of rationals and Q^0 the zero-ring built on the additive group of rationals. On the Cartesian product $Q\times Q^0$ let us define addition componentwise and multiplication by $(a,\alpha)(b,\beta)=\left(ab,\frac{a\beta}{2}+\frac{\alpha b}{2}\right)$ where $a,b\in Q$ and $\alpha,\beta\in Q^0$ and $a\alpha$ means the usual product of rationals. This multiplication is commutative and distributive, but not associative. Hence we have got a commutative ring A which is not associative. Let A^1 denote the usual unital extension of A and $\sigma=1_{A^1}$. The polynomial $f=(1,0)+(0,1)X$ is an idempotent, as $(1,0)^2=(1,0)$, $(1,0).\ (0,1)=(0,1/2)$ and $(0,1)^2=0$. Hence f is a non-constant central idempotent in $A[X]\subsetneqq A^1[X]$. □

THEOREM 3.15. *If A is a semiprime ring, then $B(A[X,\sigma,D])=(B(A))^\sigma$.*

PROOF. Suppose that there exist nontrivial central idempotents in $A[X,\sigma,D]$ and let $f=\sum_{i=0}^{n}a_iX^i$ be a central idempotent with minimal length. Since $fX=Xf$, then a_n is fixed under σ. For each $s\in A$, $fs(1-f)=0$ which implies that $a_n\sigma^n(s)a_n=0$. Therefore $a_nAa_n=0$ and since A is a semiprime ring, then $a_n=0$ which is a contradiction. Hence $f=a_0$ and $B(A[X,\sigma,D])=(B(A))^\sigma$. □

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(Received October 13, 1988; revised June 28, 1989)

CAIRO UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
GIZA
EGYPT

Acta Math. Hung.
59 (3–4) (1992), 365–376.

ON THE OPTIMAL CONTROL OF CIRCULAR MEMBRANES

I. JOÓ (Budapest)

1. The present paper is motivated by my earlier papers [3], [4] where the set of movement states of a controlled string reachable in a fixed time is studied (see also the works [5]–[7]). I have proved among others that the reachability sets $D(T)$ are growing for $T \leqq T_0$ and $D(T) = D(T_0)$ for $T \geqq T_0$. Here we shall prove this phenomenon for the circular membrane and in a subsequent paper for rectangular membrane (because the last problem needs essentially different mathematical tools). We shall determine the critical time T_0 for both cases.

For multidimensional balls (containing the case of the circle) the same control problem was studied by K. D. Graham and D. L. Russel [20]. They showed the following statements :

(a) $D(T) \not\supseteq H^1(\Omega) \oplus L_2(\Omega)$ for $T < 2$,
(b) $D(T) \supsetneq H^1(\Omega) \oplus L_2(\Omega)$ for $T > 2$,
(c) for the two-dimensional case $D(T) \subseteqq H^{2/3}(\Omega) \oplus H^{-1/3}(\Omega)$ for $T > 2$.

Our results have a rather different character, although we apply, as in [20], the theory of Riesz bases for solving moment problems. We are able to investigate $D(T)$ for the critical time $T = 2$; in this subject [20] provides no information. Finally we call the attention to the problem raised in [20] to describe $D(T)$ by some classical function spaces. The difficulty arises in the investigation of the constant sequence $J_m\left(\lambda_n^{(m)}\right) / \|\varphi_{nk}\|^2$ in (7) below. We return to this question in a subsequent paper [21].

2. Consider the unit circle

$$\Omega : \ \left\{(x, y) : x^2 + y^2 < 1\right\}.$$

Let $z(t, x, y)$ denote the height of the point $(x, y) \in \Omega$ at time t. Consider the system

$$(1) \qquad \begin{cases} z_{tt} = \Delta z & \text{on } (0, T) \times \Omega, \\ \frac{\partial z}{\partial \nu} = u & \text{on } (0, T) \times \Gamma,\ \Gamma := \partial\Omega, \\ z = z_t = 0 & \text{on } \{0\} \times \Omega \text{ (i.e. for } t = 0). \end{cases}$$

It describes the motion of the membrane controlled at the boundary by the control $u \in L^2((0, T) \times \Gamma)$. Take a function $w(t, x, y)$ $C^2([0, T] \times \overline{\Omega})$ with

$w = w_t = 0$ for $t = T$ and $\frac{\partial w}{\partial \nu} = 0$ on $(0,T) \times \Gamma$. Then we can count formally

$$\int_{(0,T)\times\Omega} z w_{tt} = \int_{\Omega} [z w_t - z_t w]_{t=0}^{T} + \int_{(0,T)\times\Omega} z_{tt} w = \int_{(0,T)\times\Omega} z_{tt} w$$

and by the formal application of the Green formula

$$\int_{(0,T)\times\Omega} z \Delta w = \int_0^T \int_{\partial\Omega} \left(z \frac{\partial w}{\partial \nu} - w \frac{\partial z}{\partial \nu} \right) + \int_{\Omega} w \Delta z =$$

$$= - \int_{(0,T)\times\Omega} wu + \int_{(0,T)\times\Omega} w \Delta z.$$

This justifies the following

DEFINITION. By the solution of the system (1) we mean a function $z(t,x,y)$ satisfying

$$\int_{(0,T)\times\Omega} z(w_{tt} - \Delta w) = \int_{(0,T)\times\Omega} uw \tag{2}$$

for all $w(t,x,y)$ with properties $w \in C^2([0,T] \times \overline{\Omega})$ and

$$w = w_t = 0 \quad \text{for} \quad t = T, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \tag{3}$$

Consider the system

$$\begin{cases} -\Delta \varphi_{nk} = (\lambda_n^{(k)})^2 \varphi_{nk} & \text{on } \Omega, \\ \frac{\partial \varphi_{nk}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Using polar coordinates $x = r\cos\alpha$, $y = r\sin\alpha$ we get ([1]) that the system

$$\varphi_{nk}(x,y) := e^{ik\alpha} J_m \left(\lambda_n^{(m)} r \right), \quad m := |k|, \quad k \in \mathbf{Z}, \quad n = 1,2,\ldots$$

is complete and orthogonal in $L^2(\Omega)$ weighted with $r = \sqrt{x^2+y^2}$, if

$$0 < \lambda_1^{(m)} < \lambda_2^{(m)} \ldots$$

are the positive zeros of the derivative J'_m of the Bessel function of order m (and in (4), $\lambda_n^{(-m)} := \lambda_n^{(m)}$).

From [2], 7.14.1 (1D) and 7.2.8 (56)–(57) we know that

$$\|\varphi_{nk}\|^2_{L^2(\Omega)} = 2\pi \int_0^1 r \left| J_m\left(\lambda_n^{(m)} r\right)\right|^2 dr = \tag{5}$$

$$= \frac{\pi}{2}\left[2J_m^2\left(\lambda_n^{(m)}\right) - 2J_{m-1}\left(\lambda_n^{(m)}\right) J_{m+1}\left(\lambda_n^{(m)}\right)\right] =$$

$$= \left|J_m\left(\lambda_n^{(m)}\right)\right|^2 \left(1 - \frac{m^2}{\left(\lambda_n^{(m)}\right)^2}\right), \quad m = |k|.$$

We give the solution z as an expansion

$$z(t,x,y) = \sum_{n=1}^{\infty} \sum_{k\in \mathbf{Z}} c_{nk}(t)\varphi_{nk}(x,y)$$

convergent in $L^2(\Omega)$ for every fixed t. In this case we have

$$c_{nk}(t) = \frac{1}{\|\varphi_{nk}\|^2_{L^2(\Omega)}} \int_\Omega z\overline{\varphi_{nk}}. \tag{6}$$

Now define the function

$$w(t,x,y) := b(t)\varphi_{nk}(x,y),$$

where $b \in C^2[0,T]$ is real-valued and $b(T) = b'(T) = 0$. From (2) we obtain

$$\|\varphi_{nk}\|^2_{L^2(\Omega)} \int_0^T \left(b''(t) + \left(\lambda_n^{(m)}\right)^2 b(t)\right) c_{nk}(t)\, dt =$$

$$= J_m\left(\lambda_n^{(m)}\right) \int_0^T b(t) \int_0^{2\pi} u(\alpha,t)\, e^{-ik\alpha}\, d\alpha dt =$$

$$= J_m\left(\lambda_n^{(m)}\right) \int_0^T b(t)u_k(t)\, dt, \quad u_k(t) := \int_0^{2\pi} u(\alpha,t)e^{-ik\alpha}\, d\alpha.$$

This means (as in [3], [4]) that

$$\|\varphi_{nk}\|^2_{L^2(\Omega)} c_{nk}(t) = \int_0^t \frac{\sin \lambda_n^{(m)}(t-\tau)}{\lambda_n^{(m)}}\, u_k(\tau)\, d\tau \cdot J_m\left(\lambda_n^{(m)}\right)$$

and

$$\|\varphi_{nk}\|^2_{L^2(\Omega)} c_{nk}(t) = \int_0^t \cos \lambda_n^{(m)}(t-\tau) u_k(\tau)\, d\tau \cdot J_m\left(\lambda_n^{(m)}\right).$$

These can be unified in the formulae

(7) $$c'_{nk}(t) \pm i\lambda_n^{(m)} c_{nk}(t) =$$

$$= \frac{J_m\left(\lambda_n^{(m)}\right)}{\|\varphi_{nk}\|_{L^2(\Omega)}} \cdot \int_0^t e^{\pm i\lambda_n^{(m)}(t-\tau)} u_k(\tau)\, d\tau, \quad m = |k|,$$

$$u_k(t) := \int_0^{2\pi} u(\alpha, t) e^{-ik\alpha}\, d\alpha.$$

Using the coefficients c_{nk} given in (7), we get a solution of (2), (3). This is a repetition of some ideas given in [3] so we omit the details.

Now we list some facts about the zeros of Bessel functions. Denote

$$0 < \mu_1^{(m)} < \mu_2^{(m)} < \dots$$

the positive zeros of the m-th Bessel function J_m, $m = 0, 1, \dots$. It is proved in [1], 15.4 that for large $N = N(m) \in \mathbf{N}$ the function J_m has exactly N zeros in the interval $\left(0, \left(N + \frac{m}{2} + \frac{1}{4}\right)\pi\right)$. By the asymptotical formula

(8) $$J_m(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \pi\frac{m}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right], \quad x > 0,$$

where the implicit constant in the O-term depends on m. This shows that for large x the zeros of $J_m(x)$ are near the sequence $\ell\pi + \frac{3}{4} + \frac{m}{2}\pi$. Consequently

(9) $$\mu_n^{(m)} = \left(n + \frac{m}{2} - \frac{1}{4}\right)\pi + O\left(\frac{1}{n}\right) \quad (n = 1, 2, \dots)$$

and the implicit constant depends on m. On the other hand

(10) $$J'_m(x) = \frac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right) =$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \pi\frac{m-1}{2} - \frac{\pi}{4}\right) - \cos\left(x - \pi\frac{m+1}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right] =$$

$$= -\sqrt{\frac{2}{\pi x}}\left[\sin\left(x - \pi\frac{m}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right].$$

Since the zeros of J_m and J'_m separate each other [1], 15.23 i.e.

$$0 < \lambda_1^{(m)} < \mu_1^{(m)} < \lambda_2^{(m)} < \mu_2^{(m)} < \dots \quad \text{for} \quad m \geqq 1,$$
$$0 < \mu_1^{(0)} < \lambda_1^{(0)} < \mu_2^{(0)} < \lambda_2^{(0)} < \dots$$

(and $\lambda_n^{(0)} = \mu_n^{(1)}$ by $J'_0 = -J_1$), hence

$$\lambda_n^{(m)} = \left(n + \frac{m}{2} - \frac{3}{4}\right)\pi + O\left(\frac{1}{n}\right), \quad m \geqq 1 \quad \text{fixed}, \quad n = 1, 2, \dots, \tag{11}$$

$$\lambda_n^{(0)} = \mu_n^{(1)} = \left(n + \frac{1}{4}\right)\pi + O\left(\frac{1}{n}\right), \quad n = 1, 2, \dots.$$

Introduce the reachability sets

$$D(T) := \left\{(z(T,.,.), \quad z_t(T,.,.)) : u \in L^2((0,T) \times \partial\Omega)\right\}$$

for $0 < T < \infty$. We shall prove the following

THEOREM. (a) $D(T_1) \subsetneqq D(T_2)$ *for* $T_1 < T_2 \leqq 2$,
(b) $D(2) \subsetneqq D(T_1) = D(T_2)$ *for* $2 < T_1 < T_2$.

That is, the critical time is the diameter of Ω. Since $z(T,.,.)$, resp. $z_t(T,.,.)$ are determined by the coefficients $c_{nk}(T)$ resp. $c'_{nk}(T)$, we can consider the sequences $c'_{nk}(T) \pm i\lambda_n^{(m)} c_{nk}(T)$, or the set

$$\tilde{D}(T) := \left\{\left(\int_0^T e^{\pm i\lambda_n^{(m)}\tau} u_k(T-\tau)\,d\tau\right),\right.$$

$$\left. k \in \mathbf{Z}, \quad n = 1, 2, \dots : u \in L^2((0,T) \times \partial\Omega)\right\}.$$

Clearly

$$D(T_1) \subset D(T_2) \quad \text{if and only if} \quad \tilde{D}(T_1) \subset \tilde{D}(T_2)$$

and

$$D(T_1) = D(T_2) \quad \text{if and only if} \quad \tilde{D}(T_1) = \tilde{D}(T_2);$$

we shall prove the Theorem for $\tilde{D}$ instead of D. The proof requires some preliminary lemmas. Introduce the notation (for $a_n, b_n \geqq 0$) $a_n \asymp b_n$ to abbreviate the inequalities $c_1 a_n \leqq b_n \leqq c_2 a_n$ for every n with $0 < c_1 \leqq c_2 < < \infty$ independent of n.

LEMMA 1 [19]. *The distance between the consecutive zeros of J'_m tends monotone decreasingly to π : $\lambda_{n+1}^{(m)} - \lambda_n^{(m)} \searrow \pi$, $n \to \infty$, $m = 0, 1, 2, \dots$.*

LEMMA 2 ([11], or [8], p. 96). *Let $\{\lambda_k : k \in \mathbf{Z}\} \subset \mathbf{C}$ be a sequence satisfying $|Im\,\lambda_k| \leqq H < \infty$, $\inf_{k_1 \neq k_2} |\lambda_{k_1} - \lambda_{k_2}| \geqq \delta > 0$. Let further $T < \infty$ be fixed. Then*

$$\sum_{k \in \mathbf{Z}} \left| \int_{-T}^{T} \varphi(x) e^{-i\lambda_k x}\, dx \right|^2 \leqq \frac{4}{\pi\delta^2} \frac{e^{2T(\frac{\delta}{2}+H)}}{T} \int_{-T}^{T} |\varphi(x)|^2\, dx$$

holds for every $\varphi \in L^2(-T, T)$.

LEMMA 3. *$\tilde{D}(T) \subset \ell_2$ holds for every $T < \infty$.*

PROOF. Applying Lemmas 1 and 2 for any fixed $\in \mathbf{Z}$ we get

$$\sum_{n=1}^{\infty} \left| \int_0^T e^{e \pm i\lambda_n^{(m)}} u_k(T - \tau)\, d\tau \right|^2 \leqq c \|u_k\|^2_{L^2(0,T)}$$

where $c = c(T)$ is independent of k. Since

$$\sum \|u_k\|^2_{L^2(0,T)} = \|u\|^2_{L^2((0,T)\times\partial\Omega)},$$

the proof is complete.

The notion of sine type function is due to Levin [9], [10]. The entire function $F(z)$ is called of sine type if it is of exponential type, all zeros λ_n lay in a horizontal strip $|\mathrm{Im}\, z| \leqq \tau$ and for all $|H| \geqq \tau + 1$ we have

$$0 < m \leqq |F(t + iH)| e^{-\sigma|H|} \leqq M < \infty \quad (t \in \mathbf{R}). \tag{12}$$

In this case the indicator diagram of F is $[-i\sigma, i\sigma]$. Levin proved in [9] that if λ_n are the zeros of the above described sine type function F and if $\inf_{n \neq k} |\lambda_n - \lambda_k| > 0$ then the system $\{e^{i\lambda_n t}\}$ gives a Riesz basis in $L^2(0, 2\sigma)$ (see also [16]). Then many authors investigated how "large" perturbation $\hat{\lambda}_n$ of λ_n is allowed if we want $\{e^{i\hat{\lambda}_n t}\}$ to be a Riesz basis in $L^2(0, 2\sigma)$. We mention the works [11] and [12]. In [12] S. A. Avdonin introduced the notion of A-partition of the set$\{\lambda_n\}$; the j-th class of the partition is

$$\{\lambda_n : a_j \leqq \mathrm{Re}\, \lambda_n \leqq a_{j+1}\}$$

where $\lim_{j \to \pm\infty} a_j = \pm\infty$ and $\ell_j := a_{j+1} - a_j$ is bounded. We can also suppose e.g. that $\ell_j \geqq 1$ for all j. Introduce the notation

$$K_j := \{n : a_j \leqq \mathrm{Re}\, \lambda_n < a_{j+1}\}.$$

The famous perturbation theorem of Avdonin reads as follows :

THEOREM A [12]. *Let $\{\delta_n\} \subset \mathbf{C}$ be a bounded sequence and $\{\lambda_n\} \subset \mathbf{C}$ the zero set of a sine type function. Suppose that there exists an A-partition satisfying*

$$\left|\sum_{n\in K_j} \operatorname{Re}\delta_n\right| \leqq d\ell_j \quad (j \in \mathbf{Z})$$

for some $d < 1/4$. Now if

$$\inf_{n\neq k} |(\lambda_n + \delta_n) - (\lambda_k + \delta_k)| = \eta > 0$$

then $\{e^{i(\lambda_n+\delta_n)t}\}$ is a Riesz basis in $L_2(0, 2\sigma)$.

As it is known from the theory of Riesz bases, this means that

$$c_1 \sum_n \left|\int_0^{2\sigma} \varphi(t)e^{-i(\lambda_n+\delta_n)t}\,dt\right|^2 \leqq \|\varphi\|^2_{L^2(0,2\sigma)} \leqq \tag{13}$$

$$\leqq c_2 \sum_n \left|\int_0^{2\sigma} \varphi(t)e^{-i(\lambda_n+\delta_n)t}\,dt\right|^2$$

for some $0 < c_1, c_2 < \infty$. In what follows we need the fact that in Theorem A the constants c_1, c_2 depend only on σ, τ_0, m_0, M_0, L_0, D_0, d_0, η_0, where $\tau_0 \geqq \tau$, $0 < m_0 \leqq m$, $M \leqq M_0$, $\sup_j \ell_j \leqq L_0$, $\sup_n |\delta_n| \leqq D_0$ further $d \leqq d_0 < < 1/4$ and $0 < \eta_0 \leqq \eta$. One can check this by repeating the proof of [12] step by step, hence we omit it.

PROOF OF THE THEOREM. As we have seen in Lemma 3, $\tilde{D}(T) \subset \ell_2$ for all T. We shall prove that $\tilde{D}(T) = \ell_2$ for $T > 2$ and $\subsetneqq$ for $T \leqq 2$.

(a) Recall that $\tilde{D}(T)$ consists of the sequences

$$\left(\int_0^T e^{\pm i\lambda_n^{(|k|)}} u_k(T-\tau)\,d\tau\right), \quad k \in \mathbf{Z}, \quad n = 1, 2, \ldots$$

where

$$u_k(t) := \int_0^{2\pi} u(\alpha, t)e^{-ik\alpha}\,d\alpha, \quad u(\alpha, t) \in L^2.$$

This means that the functions $u_k(t)$ can be chosen arbitrarily under the condition

$$\sum_{k\in\mathbf{Z}} \|u_k\|^2_{L^2(0,T)} < \infty. \tag{14}$$

Now we prove for $T_1 < T_2 < 2$ that $D(T_1) \subsetneqq D(T_2)$: for the case $T_2 = 2$ the same result will then follow.

We have to investigate only the moments for $k = 0$; it is the moment space of the system $\{e^{\pm i\lambda_n^{(0)}\tau} : n = 1, 2, \dots\}$, and by (11), $\lambda_n^{(0)} = \left(n + \frac{1}{4}\right)\pi + {}$ $+ O\left(\frac{1}{n}\right)$. We need the following two results.

THEOREM B [18]. *Let λ_n, $n \in \mathbf{Z}$ be complex numbers satisfying*

$$\sup_n |\operatorname{Im} \lambda_n| < \infty, \qquad \inf_{n \neq k} |\lambda_n - \lambda_k| > 0.$$

Suppose that for every $\varepsilon > 0$ there exists $d_0 > 0$ such that for $d > d_0$ we have

$$\left| \frac{1}{d} \sum_{x < \operatorname{Re} \lambda_n < x + d} 1 - \frac{a}{2\pi} \right| < \varepsilon \quad (x \in \mathbf{R}).$$

Then for every $0 < a' < a$ there is a subsystem of $\{e^{i\lambda_n x} : n \in \mathbf{Z}\}$ forming a Riesz basis in $L^2(0, a')$.

THEOREM C [17]. *If a system $e(\Lambda) := \{e^{i\lambda_n x} : n \in \mathbf{Z}\}$ is a Riesz basis in $L^2(0, a)$ then for every $0 < a' < a$ there exists a subsystem of $e(\Lambda)$ forming a Riesz basis in $L^2(0, a')$.*

Returning to the proof of our Theorem, by Theorem B the system $\{e^{\pm i\lambda_n^{(0)}} : n = 1, 2, \dots\}$ contains a subsystem ϕ_2 which is Riesz basis in $L^2(0, T_2)$ and there exists $\phi_1 \subset \phi_2$, which is a Riesz basis in $L^2(0, T_1)$. Hence in $\tilde{D}(T_2)$ we have the whole ℓ_2 space in the coordinates corresponding to the elements of ϕ_2. In $\tilde{D}(T_1)$ this is not true: any element $\varphi \in \phi_2 \setminus \phi_1$ can be expanded by the basis in ϕ_1 in $L^2(0, T_1)$, and this relation gives a connection between the corresponding moments in $\tilde{D}(T_1)$. This proves (a).

(b) We shall show that $\tilde{D}(2)$ is not the whole ℓ_2, but $\tilde{D}(T) = \ell_2$ for $T > 2$. To see the first statement we restrict ourselves to the coordinates $k = 1$. Let $e(\Lambda) := \{e^{\pm\lambda_n^{(1)} x} : n = 1, 2, \dots\}$ and $e_o(\Lambda) := \{e^{\pm i\left(n - \frac{1}{2}\right)x} : n = 1, 2, \dots\}$. We know from (11) that $\lambda_n^{(1)} = \left(n - \frac{1}{4}\right)\pi + O\left(\frac{1}{n}\right)$. Suppose indirectly that the moment space of $e(\Lambda)$ is ℓ_2. Then $e(\Lambda)$ is a Riesz basis in the closed linear hull $V(e(\Lambda))$ of its elements, see [8], p. 169. Consequently

$$\sum \left| \int_0^2 e^{\pm\lambda_n^{(1)} x} v(x)\, dx \right|^2 \asymp \int_0^2 |v(x)|^2\, dx, \quad v \in V(e(\Lambda)).$$

This property remains valid if we choose exponents μ_n

$$|\lambda_n^{(1)} - \mu_n| < \varepsilon, \quad n = 1, 2, \dots$$

for sufficiently small $\varepsilon > 0$ ([8], p. 181); namely

$$\sum \left| \int_0^2 e^{\pm i\mu_n x} v(x)\, dx \right|^2 \asymp \int_0^2 |v(x)|^2\, dx, \quad v \in V(e(\Lambda))$$

and hence

$$(15) \qquad \sum \left| \int_0^2 e^{\pm i\mu_n x} v(x)\, dx \right|^2 \geqq c \int_0^2 |v(x)|^2\, dx, \quad v \in V\left(e^{\pm i\mu_n x}\right).$$

The converse inequality is contained in Lemma 2, therefore the moment operator

$$P : V\left(e^{\pm i\mu_n x}\right) \to \ell_2, \quad v \mapsto \left(\int_0^2 e^{\pm i\mu_n x} v(x)\, dx \right)$$

is an isomorphic imbedding. But we shall prove that the system $\left\{e^{\pm i\mu_n x}\right\}$ is minimal. Indeed, let $(c_n) \in \ell_2$ be fixed and suppose that

$$\sum_{n=1}^{\infty} c_n F(\mu_n) + \sum_{n=-\infty}^{-1} c_n F(-\mu_n) = 0 \quad (F \in B_2^2).$$

Use [8], p. 181 to obtain

$$(16) \qquad \left| \sum_1^{\infty} c_n F\left(\lambda_n^{(1)}\right) + \sum_{-\infty}^{-1} c_n F\left(-\lambda_n^{(1)}\right) \right| =$$

$$= \left| \sum_1^{\infty} c_n \left[F\left(\lambda_n^{(1)}\right) - F(\mu_n) \right] + \sum_{-\infty}^{-1} c_n \left[F\left(-\lambda_n^{(1)}\right) - F(-\mu_n) \right] \right| \leqq$$

$$\leqq c\left(e^{\pi\varepsilon} - 1\right) \left(\int_{\mathbf{R}} |f(t)|^2\, dt \right)^{\frac{1}{2}} \left(\sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} |c_n|^2 \right)^{\frac{1}{2}}.$$

Since $e(\Lambda)$ is a Riesz basis in $V(e(\Lambda))$, we can give $F \in B_2^2$ with

$$F\left(\lambda_n^{(1)}\right) = \overline{c_n}, \quad n = 1, 2, \ldots, \quad F\left(-\lambda_n^{(1)}\right) = \overline{c_n}, \quad n = -1, -2, \ldots,$$

$$\int_{\mathbf{R}} |F(t)|^2\, dt \asymp \sum_{n \neq 0} |c_n|^2.$$

Now if $\varepsilon > 0$ is sufficiently small, (16) can be true only if $c_n = 0$ for every n. Hence $\{e^{\pm i\mu_n x}\}$ is indeed minimal and then the range of P contains the finite moment sequences. Consequently P maps isomorphically onto ℓ_2 and then $\{e^{\pm i\mu_n x}\}$ is a Riesz basis in $V\left(e^{\pm i\mu_n x}\right)$. In particular the system

$$e_1(\Lambda) := \left\{e^{\pm i(n-\frac{1}{4})x} : n > N\right\} \cup \left\{e^{\pm i\lambda_n^{(1)}x} : n \leqq N\right\}$$

is a Riesz basis in $V(e_1(\Lambda))$ for sufficiently large N. Since $e_0(\Lambda)$ is complete in $L^2(0,2)$ ([8], p. 122) and changing finitely many exponents leaves the completeness unchanged ([8], p. 129), hence $e_1(\Lambda)$ is a complete Riesz basis in $L^2(0,2)$. The Riesz basis property is also unaffected by the change of finitely many exponents (since the completeness remains true), hence $e_0(\Lambda)$ is a Riesz basis in $L^2(0,2)$. Then for sufficiently small $\varepsilon > 0$ the system

$$e_\varepsilon(\Lambda) := \left\{e^{\pm i(n-0.25+\varepsilon)x} : n = 1, 2, \dots\right\}$$

is also a Riesz basis. But Theorem A implies that $\{1\} \cup e_\varepsilon(\Lambda)$ is a Riesz basis in $L^2(0,2)$, hence $e_\varepsilon(\Lambda)$ is not complete. This contradiction proves that the moment space of $e(\Lambda)$ is strictly contained in ℓ_2.

Next we shall show that for $T > 2$ $\tilde{D}(T) = \ell_2$. Since for the functions $u_k \in L^2(0,T)$ the only restriction is

$$\sum_{k \in \mathbf{Z}} \|u_k\|_{L^2(0,T)} < \infty,$$

it is enough to prove the following statement. The systems

$$e^{(m)}(\Lambda) := \left\{e^{\pm i\lambda_n^{(1)}x} : n = 1, 2, \dots\right\}$$

can be completed to an exponential basis $E^{(m)}(\Lambda)$ of $L^2(0,T)$ such that the implicit constants c_1, c_2 mentioned in Theorem A do not depend on m. To prove this, fix a value m. We shall use Theorem A with the mentioned modification and start with the system

$$E := \left\{e^{i2\pi kx/T} : k \in \mathbf{Z}\right\},$$

the standard orthonormal basis in $L^2(0,T)$. First observe that there exists $d_0 > 0$ independent of m such that any interval of length d_0 contains more numbers $k\frac{2\pi}{T}$ than $\lambda_n^{(m)}$. This follows from the fact that the distance between consecutive zeros of J_m tends monotone decreasingly to π. Consider a segment I and define the upper translation of $k\frac{2\pi}{T} \in I$ as follows. Shift the largest value $k\frac{2\pi}{T} \in I$ to the largest value $\lambda_n^{(m)} \in I$, the second largest value $k\frac{2\pi}{T} \in I$ to the second largest $\lambda_n^{(m)}$ etc. If $|I| > d_0$, then all $\lambda_n^{(m)}$ can be represented as a shifted value. The lower translation is the reversed process: we shift the smallest $k\frac{2\pi}{T} \in I$ to the smallest $\lambda_n^{(m)} \in I$, the second smallest one to the second smallest one etc. Since in a segment of length d_0 there is an excess of the values $k\frac{2\pi}{T}$ we can choose d_1 such that for the upper translation the sum of the shifts be positive. If we denote

$$\delta_n^{(m)} := k\frac{2\pi}{T} - \lambda_n^{(m)}$$

this means that

$$0 \leqq \sum_{\lambda_n^{(m)} \in I} \delta_n^{(m)} \leqq c(d_1)$$

and for the lower translation

$$0 \geqq \sum_{\lambda_n^{(m)} \in I} \delta_n^{(m)} \geqq -c(d_1),$$

where $c(d_1)$ is independent of m. This means that if we take an interval I_1 of length $d_1 N$ and in its N intervals of length d_1 we apply upper and lower translation, we can obtain that

$$\left| \sum_{\lambda_n^{(m)} \in I_1} \delta_n^{(m)} \right| \leqq c(d_1) = \frac{c(d_1)}{N d_1} |I_1|. \tag{17}$$

Take N satisfying $\frac{c(d_1)}{Nd_1} < \frac{1}{4}$. We know by Lemma 3 that the values $\lambda_n^{(m)}$ are separated but it may occur that some $\lambda_n^{(m)}$ are close (or are equal) to some non-shifted $k\frac{2\pi}{T}$. But we know that in any interval of length $\pi/2$ there exist no more than two numbers $\lambda_n^{(m)}$, so we shift these values $k\frac{2\pi}{T}$ at a small distance to ensure the separability (with a positive lower bound independent of m). By Theorem A the shifted exponents give an exponential basis in $L^2(0,T)$ whose implicit constants do not depend on m. As we have remarked above, this is enough to prove (b). Thus the Theorem is completely proved. □

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(Received November 1, 1989; revised March 27, 1990)

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13–15.
H–1053 BUDAPEST
HUNGARY

Acta Math. Hung.
59 (3–4) (1992), 377–381.

RESTRICTIONS OF NORMAL OPERATORS

Z. SEBESTYÉN (**Budapest**) J. STOCHEL (**Krakow**) and J. PAEZ (**Valencia**)

This note is a continuation of a recent work [6] on characterization of restrictions of compact normal operators to a closed subspace in a (complex) Hilbert space. Omitting the very restriction on compactness we could only arrive at a subprojection valued operator measure characterization as answer to Problems 1 and 3 of Halmos [1]. We do not see hope to give a more satisfactory solution of this question by Halmos on a characterization of subnormal suboperators in the language of [1] of restrictions of normal operators. But what about restrictions of subnormal operators?

Let now D be a closed subspace of a (complex) Hilbert space H, let $F : \mathcal{A} \to B(D, H)$ be a (σ-additive) (sub)operator valued measure on a σ-algebra $\mathcal{A}$ of subsets in Ω with values as bounded operators from D into H with $F(\Omega) = I$; in notation F is $B(D, H)$-valued. The following is a characterization of restrictions of spectral measures.

LEMMA 1. *A (sub)operator valued measure $F : \mathcal{A} \to B(D, H)$ is restriction to D of a spectral measure $E : \mathcal{A} \to B(H)$ if and only if the following two properties are satisfied:*

(a) $\|F(\sigma)x\|^2 = (F(\sigma)x, x) \quad (\sigma \in \mathcal{A}; x \in D)$,

(b) $(F(\sigma)x, F(\tau)y) = 0 \quad (\sigma, \tau \in \mathcal{A}, \sigma \cap \tau = \emptyset; x, y \in D)$.

PROOF. The necessity of the properties are obvious. To see the sufficiency let $K = V\{F(\sigma)D : \sigma \in \mathcal{A}\}$ be the closed subspace spanned by the rangespaces of the operators as the values in the operator measure F. Define $E(\sigma)$ $(\sigma \in \mathcal{A})$ as an operator on K as follows:

$$E(\sigma)F(\tau)x = F(\sigma \cap \tau)x \quad (\tau \in \mathcal{A}, x \in D). \tag{1}$$

We will prove that $E(\sigma)$ is well-defined, bounded and that $E : \mathcal{A} \to B(K)$ is a spectral measure: a projection valued operator measure. Doing so we first mention the following identity:

$$(F(\sigma)x, F(\tau)y) = (F(\sigma \cap \tau)x, y) \quad (\sigma, \tau \in \mathcal{A}; x, y \in D). \tag{2}$$

Indeed, (b) implies that

$$(F(\sigma)x, F(\tau)y) = (F(\sigma \cap \tau)x + F(\sigma \setminus \tau)x, F(\tau)y) =$$

$$= (F(\sigma \cap \tau)x, F(\sigma \cap \tau)y + F(\tau \setminus \sigma)y) = (F(\sigma \cap \tau)x, F(\sigma \cap \tau)y).$$

Assumption (a) says [5] that an orthogonal projection P exists such that $F(\sigma \cap \tau)$ is a restriction of P to D, whence the proof of (2):

$$(F(\sigma \cap \tau)x, F(\sigma \cap \tau)y) = (Px, Py) = (Px, y) = (F(\sigma \cap \tau)x, y).$$

The next identity we need is the following (with finite sums):

$$\text{(3)} \qquad \|\sum_n F(\sigma \cap \tau_n)x_n\|^2 \leqq \|\sum_n F(\tau_n)x_n\|^2 \quad (\sigma, \tau_n \in \mathcal{A}; x_n \in D).$$

To prove (3) let $\{\delta_p\}$ be a disjointisation of $\{\tau_n\}$ in $\mathcal{A}$, i.e. $\tau_n = \sum_{p \in I_n} \delta_p$, then we have by disjointness and by (2)

$$\left\| \sum_n F(\sigma \cap \tau_n)x_n \right\|^2 = \sum_{m,n} (F(\sigma \cap \tau_m)x_m, F(\sigma \cap \tau_n)x_n) =$$

$$= \sum_{m,n} (F(\sigma \cap \tau_m \cap \tau_n)x_m, x_n) = \sum_{m,n} \sum_{p \in I_m} \sum_{q \in I_n} (F(\sigma \cap \delta_p \cap \delta_q)x_m, x_n) =$$

$$= \sum_{m,n} \sum_{p,q} \chi_{I_m}(p)\chi_{I_n}(q)(F(\sigma \cap \delta_p \cap \delta_q)x_m, x_n) =$$

$$= \sum_{m,n} \sum_p \chi_{I_m}(p)\chi_n(p)(F(\sigma \cap \delta_p)x_m, x_n) =$$

$$= \sum_p (F(\sigma \cap \delta_p) \sum_m \chi_{I_m}(p)x_m, \sum_n \chi_{I_n}(p)x_n) \leqq$$

$$\leqq \sum_p (F(\delta_p) \sum_m \chi_{I_m}(p)x_m, \quad \sum_n \chi_{I_n}(p)x_n) =$$

$$= \sum_{m,n} (F(\tau_m \cap \tau_n)x_m, x_n) = \sum_{m,n} (F(\tau_m)x_m, F(\tau_n)x_n) = \left\| \sum_n F(\tau_n)x_n \right\|^2.$$

Identity (3) shows that $F(\sigma)$ is well-defined and contractive so it can be extended in a unique way to a contraction on K. Denote it also by $E(\sigma)$.

We now prove that $E^*(\sigma) = E(\sigma) = E(\sigma)^2$, i.e. that $E(\sigma)$ is an orthogonal projection. To see self-adjointness we have by (2)

$$(E(\sigma)F(\tau)x, F(\delta)y) = (F(\sigma \cap \tau)x, F(\delta)y) =$$

$$= (F(\sigma \cap \tau \cap \delta)x, y) = (F(\tau)x, F(\sigma \cap \delta)y) = (F(\tau)x, E(\sigma)F(\delta)y)$$

for σ, τ, δ from $\mathcal{A}$, and x, y from D. The definition of $E(\sigma)$ in (1) implies that $E(\sigma)$ is idempotent too. To see that $E : \mathcal{A} \to B(K)$ is a spectral measure choose a pairwise disjoint sequence $\{\sigma_n\}$ in $\mathcal{A}$. Since (1) implies that E is finitely additive and monotone, we have that

$$\sum_{n=1}^{N} E(\sigma_n) = E\left(\bigcup_{n=1}^{N} \sigma_n\right) \leqq E\left(\bigcup_{n} \sigma_n\right) \leqq I.$$

The series $\sum_n E(\sigma_n)$ is weakly convergent, hence it converges in the strong operator topology. Conversely,

$$E\left(\bigcup_{n} \sigma_n\right) F(\tau)x = F\left(\bigcup_{n} (\sigma_n \cap \tau)\right) x =$$

$$= \sum_{n} F(\sigma_n \cap \tau)x = \left(\sum_{n} E(\sigma_n)\right) F(\tau)x$$

means that $E\left(\bigcup\limits_{n} \sigma_n\right) = \sum\limits_{n} E(\sigma_n)$ indeed. $E(\Omega)|_K = F(\Omega)$ is the identity operator on K, therefore $D \subset K$, moreover $E(\sigma)$ extends $F(\sigma)$ since

$$F(\sigma)x = E(\sigma)F(\Omega)x = E(\sigma)x \quad \text{for} \quad \sigma \in \mathcal{A},\, x \in D.$$

Defining $E(\sigma)$ on $K^{\perp}$, the orthocompletement of K in H with fixed $\omega \in \Omega$, as zero if $\omega \notin \sigma$ and identity otherwise we have the spectral measure stated above.

PROPOSITION. *If $G : \mathcal{A} \to B(H)$ is any spectral measure whose restriction to D is F then K reduces G and*

$$G(\sigma)F(\tau)x = F(\sigma \cap \tau)x \quad (\sigma, \tau \in \mathcal{A}, x \in D)$$

holds true.

PROOF. We see that

$$G(\sigma)F(\tau)x = G(\sigma)G(\tau)x = G(\sigma \cap \tau)x = F(\sigma \cap \tau)x.$$

We note also that if D is invariant for an operator A in $B(K)$ then E commutes with A whenever the restriction of A to D commutes with F:

$$AE(\sigma)F(\tau)x = AF(\sigma \cap \tau)x = F(\sigma \cap \tau)Ax =$$

$$= E(\sigma)F(\tau)Ax = E(\sigma)AF(\tau)x$$

for σ, τ from $\mathcal{A}$, x from D.

THEOREM. *The operator $A : D \to H$ is a restriction of a normal operator N on H if and only if there exists a $B(D,H)$-valued measure F on the complex plane with compact support satisfying* (a), (b) *in the Lemma and*

(c) $Ax = \int zF(dz)x$ *for any x from D.*

PROOF. The necessity follows from the spectral theorem, the spectral resolution of any normal extension being restricted to D. The sufficiency is seen by the Lemma applied to F from the assumption since $\int zE(dz)$ is seen to be the desired normal extension of A. The proof is complete.

COROLLARY. *Let N be a normal operator on H, $N = \int zE(dz)$ such that the restrictions $E(.)|_D$ maps D into a closed subspace $K \supset D$ of H. Then $N|D$ can be extended to a normal operator on K.*

PROOF. Any restriction of a normal operator satisfies properties (a), (b) in Lemma and (c) in the Theorem. Hence the assertion is a consequence of the Theorem.

REMARK. Concerning positive and semi-bounded self-adjoint extension see the recent constructions [2-7], especially the projection extension [5].

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(Received November 27, 1989; revised February 28, 1991)

DEPARTMENT OF APPLIED ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6-8.
1088. BUDAPEST
HUNGARY

INSTYTUT MATEMATYKI
UNIVERSITET JAGIELLONSKI
UL. REYMONTA 4.
30-059 KRAKOW
POLAND

DEPARTEMENTO DE MATEMATICAS
FAC. CIENCIAS
DE LA EDUCATION UNIVERSIADED DE CARABOBO
BAR-BULA-VALENCIA
VENEZUELA

Acta Math. Hung.
59 (3–4) (1992), 383–393.

A v-INTEGRABLE FUNCTION WHICH IS NOT LEBESGUE INTEGRABLE ON ANY PORTION OF THE UNIT SQUARE

Z. BUCZOLICH* (Budapest)

Introduction

To obtain an integration process which leads to a very general divergence theorem W. F. Pfeffer [6] introduced the c-integral. The domain of this integration is the family of sets with bounded variation (BV sets), [3], [4]. During the definition of the c-integral first an averaging process, called v-integral, is defined on BV sets. Then using an extension method due to Mařik the v-integral is extended to the c-integral. The extension is necessary because the v-integral is not additive [1], [6]. For most of the generalizations of the Lebesgue integral if a function is integrable in the general sense then one can find a portion, that is, a non-empty open subset on which the function is integrable in the ordinary Lebesgue sense. W. F. Pfeffer asked whether the v- or c-integral has this property. In this paper we answer this question in the negative for the v-integral (Theorem 2). Since the c-integral is an extension of the v-integral our example also holds for the c-integral. In fact for the c-integral there exist simpler examples than the example presented in this paper.

Preliminaries

In this paper we work in the Euclidean plane $\mathbf{R}^2$. For a BV set A we denote by $|A|$, $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$, $\|A\|$, and $d(A)$ respectively the Lebesgue measure, the closure, the interior, the boundary, the perimeter, and the diameter of A. The open disk of radius r and center x is denoted by $S(x,r)$. A set T is thin if it is of σ-finite one-dimensional Hausdorff measure. The set of all density points of a set E is called the essential interior of E, denoted by $\mathrm{int}_e(E)$ and the complement of the set of all dispersion points of E is called the essential closure of E denoted by $\mathrm{cl}_e(E)$. We define the regularity of a BV set A by

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0 \\ 0 & \text{otherwise.} \end{cases}$$

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* This work has been completed when the author visited University of California at Davis.

We put $E_x = \{(y,x) : y \in \mathbf{R}\}$ and $E^y = \{(y,x) : x \in \mathbf{R}\}$.

We recall from [6] the definitions of additive, v-continuous functions, and the Riemann definition of the variational integral. In [7] one can find a summary of these definitions.

A division of a BV set A is a finite disjoint family of BV sets whose union is A.

DEFINITION 1. Let $A \in BV$ and let F be a function defined on the BV subsets of A. We say that F is:

1. additive if $F(A) = \sum\limits_{D \in \mathcal{D}} F(D)$ for each division D of A;

2. v-contionuous if given $\varepsilon > 0$ there is a $\delta > 0$ such that $|F(B)| < \varepsilon$ for each BV subset B of A with $|B| < \delta$ and $\|B\| < 1/\varepsilon$.

DEFINITION 2. Let $B \in BV$ and let T be a thin set. Furthermore, let $\varepsilon > 0$ and let δ be a positive function on $\mathrm{cl}_e(A) \setminus T$. A partition in B mod T is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint BV subsets of B and $x_i \in \mathrm{cl}(A_i) \cap (\mathrm{cl}_e A \setminus T)$, $i = 1, \ldots, p$. We say that the partition P is:

1. an ε-partition if $r(A_i) > \varepsilon$, $i = 1, \ldots, p$;

2. δ-fine if $d(A_i) < \delta(x_i)$ $i = 1, \ldots, p$.

DEFINITION 3. Let $B \in BV$ and let f be a function on $\mathrm{cl}_e(A)$. We say that f is v-integrable in B if there is a v-continuous additive function F defined on the BV subsets of B which satisfies the following condition: given $\varepsilon > 0$, there is a thin set T and a positive function δ on $\mathrm{cl}_e A \setminus T$ such that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon$$

for each δ-fine ε-partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in B mod T.

We remark that the integral defined above was called R-integral by W. F. Pfeffer (cf. Definition 7.3, and Proposition 7.7 in [6]), but Proposition 7.8 of [6] shows that the R-integral and the v-integral are equivalent.

We shall use the following corollary of Theorems 4 and 33 of [5].

THEOREM 1. *Suppose $B \subset [0,1]^2$ is a BV-set. Then there exists a Borel subset K of $\mathbf{R}$ with the following properties:*

1) $|\mathbf{R} \setminus K|_1 = 0$ *(we remark that here $|.|_1$ denotes the one-dimensional Lebesgue measure).*

2) *For each $y \in K$ there exist a (unique) non-negative integer r, and real numbers $a_1 < b_1 < \cdots < a_r < b_r$ such that $E^y \cap B$ equals $\bigcup\limits_{j=1}^{r} (a_j, b_j)$ modulo a set of one dimensional Lebesgue measure zero. If we put $r = \phi(y)$ then ϕ is a Borel function on K and*

$$2 \int_{\mathbf{R}} \phi(y) dy \leqq \|A\|.$$

LEMMA 1. *Suppose that* $n, N \in \mathbf{N}$, $M > 1$, *and a square* I *of sides* $1/n$ *is divided into* N^2 *squares* I_k *of sides* $1/nN$ $(k = 1, \dots, N^2)$. *For each* I_k *denote by* J_k *the open square of sides* $1/nNM$ *concentric with* I_k. *Put* $G = \bigcup_{k=1}^{N^2} J_k$. *If* $\operatorname{dist}(x, G) > c/\sqrt{n \cdot N \cdot M}$ *for a* $c > 0$ *then for every* $r > 0$ *we have*

$$\frac{|S(x,r) \cap G|}{|S(x,r)|} \leqq \frac{4}{n \cdot N \cdot M \cdot c^2} + \frac{N^2}{M^2}.$$

PROOF OF LEMMA 1. If $r < c/\sqrt{n \cdot N \cdot M}$ then $S(x,r) \cap G = \emptyset$ and hence

$$\frac{|S(x,r) \cap G|}{|S(x,r)|} = 0.$$

If $c/\sqrt{n \cdot N \cdot M} \leqq r < 1/\sqrt{2}nN$ then $S(x,r)$ can intersect at most four squares J_k and hence $|S(x,r) \cap G| \leqq 4/n^2N^2M^2$ and $|S(x,r)| \geqq c^2\pi/nNM$. Thus

$$\frac{|S(x,r) \cap G|}{|S(x,r)|} \leqq \frac{4 \cdot n \cdot N \cdot M}{n^2 \cdot N^2 \cdot M^2 \cdot c^2 \cdot \pi} < \frac{4}{n \cdot N \cdot M \cdot c^2}.$$

If $r > 1/\sqrt{2}nN$ then $S(x,r)$ can intersect at most N^2 different squares J_k. Thus $|S(x,r) \cap G| \leqq N^2 \frac{1}{n^2N^2M^2} = \frac{1}{n^2M^2}$ and hence

$$\frac{|S(x,r) \cap G|}{|S(x,r)|} \leqq \frac{1}{n^2M^2r^2\pi} < \frac{N^2}{M^2}.$$

Therefore for any r we have

$$\frac{|S(x,r) \cap G|}{|S(x,r)|} \leqq \frac{4}{n \cdot N \cdot M \cdot c^2} + \frac{N^2}{M^2}.$$

THEOREM 2. *There exists a* v*-integrable function* $f : [0,1]^2 \to \mathbf{R}$ *such that* f *is not Lebesgue integrable on any portion of* $[0,1]^2$.

PROOF OF THEOREM 2. The proof of this theorem is organized as follows. In Part 1 we define a dense open subset $G(I)$ in a given square I and study the properties of $G(I)$. In Part 2 we iterate the process of Part 1 and obtain a dense G_δ set G. In Part 3 using the set G we define our function f. In Part 4 we show that there is no portion of $[0,1]^2$ on which f is Lebesgue integrable. In Part 5 we define the function F defined on BV sets and show that it is v-continuous. In Part 6 we define a thin set H which will play an essential role in Part 7. Finally in Part 7 we prove that F is the indefinite v-integral of f.

Part 1. Suppose that I is a square of sides $1/n$. Choose a sequence of natural numbers c_k with the following properties:

i) $c_1 = 4n^6 \cdot 2^n + 2$;

ii) $c_k > 2$ is even for any k, and

$$\sum_{k=1}^{\infty} \frac{1}{c_k} < \infty;$$

iii) If $C_0 = 1$ and $C_k = \prod_{\ell=1}^{k} c_\ell$, $k = 1, 2, \dots$ then

$$\sum_{k=1}^{\infty} \frac{C_{k-1}^2}{C_k^2} < \frac{1}{2n^2} = \frac{1}{2}|I|;$$

iv) $c_k > C_{k-1}$ and

$$\sum_{k=1}^{\infty} \frac{C_{k-1}^2}{c_k^2} < \infty.$$

v)

$$\sum_{k=1}^{\infty} \frac{C_{k-1}^2}{\sqrt{C_k}} < \frac{1}{4}.$$

We remark that all the above conditions can be satisfied if c_k converges rapidly to infinity, we leave the details to the reader.

When $k = 1$ then divide $I = I_{1,1}$ into c_1^2 subsquares of sides $1/nC_1$. Denote by J_1 the square of sides $2/nC_1$ concentric with I, that is, J_1 is the middle $2/nC_1 \times 2/nC_1$ subsquare of I.

Put $G(I,1) = J_1$. If $k > 1$ then first divide I into C_{k-1}^2 subsquares $I_{m,k}$ of sides $1/nC_{k-1}$ ($m = 1, 2, \dots, C_{k-1}^2$). Then divide each $I_{m,k}$ into c_k^2 subsquares of sides $1/nC_k$ and denote by $J_{m,k}$ the square of sides $2/nC_k$ concentric with $I_{m,k}$. Plainly $d(J_{m,k}) = \sqrt{8}/nC_k$ and $|J_{m,k}| = 4/n^2C_k^2$.

Put $G(I,k) = \bigcup_{m=1}^{C_{k-1}^2} J_{m,k}$ and $G(I) = \bigcup_{k=1}^{\infty} G(I,k)$. Obviously $G(I)$ is a dense open set in I, and consists of countably many disjoint open squares.

The next property states that $G(I)$ has smaller volume than half the volume of I.

By using iii) we obtain

$$(1) \qquad |G(I)| < \sum_{k=1}^{\infty} |G(I,k)| = \sum_{k=1}^{\infty} \sum_{m=1}^{C_{k-1}^2} |J_{m,k}| = \sum_{k=1}^{\infty} \frac{4C_{k-1}^2}{n^2C_k^2} < \frac{|I|}{2}.$$

We also need an estimation by using v) for the sum of the square root of the diameters of the intervals $J_{m,k}$, that is:

$$\text{(2)} \qquad \sum_{k=1}^{\infty} \sum_{m=1}^{C_{k-1}^2} \sqrt{d(J_{m,k})} = \sqrt[4]{8} \sum_{k=1}^{\infty} \frac{C_{k-1}^2}{\sqrt{nC_k}} < \frac{\sqrt[4]{8}}{4\sqrt{n}} < \frac{\sqrt{d(I)}}{2}.$$

We also have to study the dispersion points of $G(I)$.

CLAIM. *If $c > 0$, $x \in \operatorname{int}(I)$, $x \notin G(I) \cup \bigcup_{m,k} \operatorname{bd}(J_{m,k})$, and there exists $c > 0$, $N(x) \in \mathbf{N}$ such that $\operatorname{dist}(x, J_{m,k}) \geqq c\sqrt{d(J_{m,k})}$ for $k = N(x) + 1, N(x) + 2, \ldots$, and $m = 1, 2, \ldots, C_{k-1}^2$ then x is a dispersion point of $G(I)$.*

Applying Lemma 1 in I for $k > N(x)$ with $N = C_{k-1}$, $M = c_k/2$ we obtain that

$$\frac{|S(x,r) \cap G(I,k)|}{|S(x,r)|} \leqq \frac{8}{nC_{k-1} \cdot c_k \cdot c^2} + 4\frac{C_{k-1}^2}{c_k^2}.$$

Thus

$$\frac{|S(x,r) \cap G(I)|}{|S(x,r)|} \leqq \sum_{k=1}^{\infty} \frac{8}{nC_{k-1} \cdot c_k \cdot c^2} + 4\frac{C_{k-1}^2}{c_k^2} < \infty$$

where we used ii) and iv). If $0 < r < \operatorname{dist}(x, J_{m,k})$ for $k = 1, 2, \ldots, K$, $(K > N(x))$ and $m = 1, 2, \ldots, C_{k-1}^2$ then $|S(x,r) \cap G(I,k)| = 0$ for $k = 1, \ldots, K$ and hence

$$\frac{|S(x,r) \cap G(I)|}{|S(x,r)|} \leqq \sum_{k=K+1}^{\infty} \frac{8}{nC_{k-1} \cdot c_k \cdot c^2} + 4\frac{C_{k-1}^2}{c_k^2} = \varepsilon_K.$$

Obviously $\varepsilon_K \to 0$ as $K \to \infty$ and hence x is a dispersion point of $G(I)$.

Part 2. Put $G_0 = S_{0,1} = [0,1]^2$ and $G_1 = G(S_{0,1})$. Suppose that $k \geqq 2$, G_{k-1} is given and G_{k-1} is a dense open set consisting of countably many disjoint open squares $S_{k-1,m}$, that is, $G_{k-1} = \bigcup_{m=1}^{\infty} S_{k-1,m}$. Then put $G_k = \bigcup_{m=1}^{\infty} G(S_{k-1,m})$. Obviously $G_k \subset G_{k-1}$ is a dense open set in $[0,1]^2$, consisting of countably many disjoint open squares. Put $G = \bigcap_{k=1}^{\infty} G_k$. From (2) it follows that

$$\sum_{m=1}^{\infty} \sqrt{d(S_{k-1,m})} \leqq 2^{-1} \sum_{m=1}^{\infty} \sqrt{d(S_{k-2,m})} \leqq \cdots \leqq 2^{-k+1} \sqrt{d(S_{0,1})} < 2^{-k+2}.$$

Thus

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{d(S_{k-1,m})}<\infty. \tag{3}$$

Part 3. Now we turn to the definition of the v-integrable function f. Put $f(x)=0$ if $x\in G$. Denote $[0,1]^2\setminus G_k$ by F_k, $k=0,1,\dots$. Then $F_k\setminus F_{k-1}\subset\bigcup_{m=1}^{\infty}S_{k-1,m}=G_{k-1}$. We define f on the sets $\Psi_{k,m}=(F_k\setminus \setminus F_{k-1})\cap S_{k-1,m}$ for each $k=1,2,\dots$, and $m=1,2,\dots$. Suppose that k and m are given and $S_{k-1,m}$ is of sides $1/n$. From (1) it follows that $|G(S_{k-1,m})|<|S_{k-1,m}|/2=1/2n^2$, and hence

$$|\Psi_{k,m}|>1/2n^2. \tag{4}$$

In the definition of $G(S_{k-1,m})$ at the first step we divide $S_{k-1,m}$ into $c_1^2=(4n^62^n+2)^2$ subsquares. Denote them by K_ℓ, $\ell=1,2,\dots,(4n^62^n+2)^2$. We can choose the indices ℓ so that if K_ℓ and $K_{\ell'}$, $\ell\neq\ell'$ have a common side then one of the indices ℓ,ℓ' is even and the other is odd. That is, if we think of $S_{k-1,m}$ as a $(4n^62^n+2)^2\times(4n^62^n+2)^2$ chessboard then ℓ is even when K_ℓ is "white" and is odd when K_ℓ is "black". Put $f(x)=|\Psi_{k,m}|^{-1}\cdot n$ if $x\in\operatorname{int}(K_\ell)\cap\Psi_{k,m}$ and ℓ is odd; and put $f(x)=-|\Psi_{k,m}|^{-1}\cdot n$ if $x\in \in\operatorname{int}(K_\ell)\cap\Psi_{k,m}$ and ℓ is even. Otherwise, that is, when $x\in\operatorname{bd}(K_\ell)\cap\Psi_{k,m}$ for an $\ell=1,\dots,(4n^62^n+2)^2$ put $f(x)=0$.

Part 4. The construction of the sets $G(S_{k-1,m})$ implies that if K_ℓ and $K_{\ell'}$ $(\ell\neq\ell')$ have a common vertical side then $K_\ell\cap\Psi_{k,m}$ can be obtained from $K_{\ell'}\cap\Psi_{k,m}$ by a horizontal translation of length $h=1/n(4n^62^n+2)$ (recall that $S_{k-1,m}$ is of sides $1/n$). Therefore the definition of f and i) imply that if $(x,y),(x+2h,y)\in\cup\{K_\ell$: *K_ℓ does not belong to the four middle squares of sides h removed from $S_{k-1,m}$ during the definition of $G(S_{k-1,m})$*$\}$ then

$$\int_x^{x+2h} f(x,y)\chi_{\Psi_{k,m}}(x,y)\,dx=0,$$

where

$$\chi_{\Psi_{k,m}}(x,y)=\begin{cases}1 & \text{if } (x,y)\in\Psi_{k,m}\\ 0 & \text{otherwise.}\end{cases}$$

Using (4) we obtain that

$$|f(x,y)\chi_{\Psi_{k,m}}(x,y)|\leqq|\Psi_{k,m}|^{-1}\cdot n\leqq 2n^3.$$

From the above properties and the chessboard like definition of f it follows that for any $a,b\in[0,1]$ we have

$$\left|\int_a^b f(x,y)\chi_{\Psi_{k,m}}(x,y)\,dx\right|\leqq 2n^3h=\frac{2n^2}{4n^62^n+2}<\frac{1}{n^2\cdot 2^n}. \tag{5}$$

From the definition of f it follows also that

$$\int_{S_{k-1,m}} |f| \geqq |\Psi_{k,m}|^{-1} \cdot n \cdot |\Psi_{k,m}| = n$$

and hence by the density of the sets G_k there is no portion of $[0,1]^2$ on which f is Lebesgue integrable.

Part 5. We have to show that f is v-integrable on $[0,1]^2$. First we find a v-continuous additive function F defined on BV sets. Later in Part 7 we show that this function is the indefinite v-integral of f. We use the fact that f is Lebesgue integrable on the sets $\Psi_{k,m}$. Given B, a BV subset of $[0,1]^2$ put

$$F(B) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \int_{\Psi_{k,m} \cap B} f.$$

First we have to show that F is well defined, that is, the double sum in the definition converges. Since B is a BV set $\|B\|$ is finite. Apply Theorem 1. Then the function $\phi(y)$ is almost everywhere finite and (5) implies that if $S_{k-1,m}$ is of sides $1/n$ then

$$\left| \int_0^1 \chi_{\Psi_{k,m} \cap B}(x,y) f(x,y)\,dx \right| < \frac{\phi(y)}{n^2 2^n}.$$

Thus

$$\left| \int_{[0,1]^2} \chi_{\Psi_{k,m} \cap B} f \right| \leqq \int_0^1 \left| \int_0^1 \chi_{\Psi_{k,m} \cap B}(x,y) f(x,y)\,dx \right| dy < \tag{6}$$

$$< \frac{\int_0^1 \phi(y)\,dy}{n^2 2^n} \leqq \frac{\|B\|}{2n^2 2^n} \leqq \|B\| \cdot |S_{k-1,m}|.$$

From (1) it follows that $|G_1| < 1/2$, and $|G_k| < |G_{k-1}|/2$ and hence

$$\sum_{k=1}^{\infty} |G_k| = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |S_{k,m}| < \infty.$$

Therefore

$$|F(B)| = \left| \sum_{k,m} \int_{\Psi_{k,m} \cap B} f \right| \leqq \sum_{k,m} \|B\| \cdot |S_{k-1,m}| < \infty.$$

It follows from our construction that if $S_{k,m}$ and $S_{k',m'}$ are different squares of sides $1/n$ then they are disjoint. Plainly there are at most n^2 different squares $S_{k,m}$ of sides $1/n$ in $[0,1]^2$. By rearranging the sequence $\{S_{k,m} : k = 1,2,\dots, m = 1,2,\dots\}$ we introduce the notation $\{S'_{n,j} : n = 1,2,\dots, j = 1,\dots, J(n)\}$ where for a given $n \in \mathbf{N}$, $\{S'_{n,j} : j = 1,\dots, J(n)\}$ consists of the squares $S_{k,m}$ of sides $1/n$, $J(n) \leqq n^2$. If $S_{k-1,m} = S'_{n,j}$ then we put $\Psi'_{n,j} = (F_k \setminus F_{k-1}) \cap S'_{n,j}$. For a given $\varepsilon > 0$ choose $K \in \mathbf{N}$ such that

$$\sum_{n=K+1}^{\infty} 2^{-n} < \frac{\varepsilon^2}{2}. \tag{7}$$

Since f is bounded on $H = \bigcup_{n=1}^{K} \bigcup_{j=1}^{J(n)} S'_{n,j}$ there is a $\delta > 0$ such that for every measurable set B if $|B| < \delta$ then

$$\left| \int_{H\cap B} f \right| < \frac{\varepsilon}{2}. \tag{8}$$

Suppose that B is a BV set and $\|B\| < 1/\varepsilon$. Then

$$|F(B)| = \left| \sum_{n=1}^{\infty} \sum_{j=1}^{J(n)} \int_{\Psi'_{n,j}\cap B} f \right| \leqq \left| \sum_{n=1}^{K} \sum_{j=1}^{J(n)} \int_{\Psi'_{n,j}\cap B} f \right| +$$

$$+ \sum_{n=K+1}^{\infty} \sum_{j=1}^{J(n)} \left| \int_{\Psi'_{n,j}\cap B} f \right| \leqq \frac{\varepsilon}{2} + \sum_{n=K+1}^{\infty} \frac{n^2 \cdot \|B\|}{2n^2 2^n} \leqq \varepsilon$$

where we used (6), (7), and (8). Therefore F is v-continuous.

Part 6. For a given square $S'_{n,j}$ denote by $Q_{n,j}$ the squares of sides $1/\sqrt{n}$ concentric with $S'_{n,j}$. Hence for every $n \geqq 2$, and $x \notin Q_{n,j}$ we have

$$\begin{aligned} \operatorname{dist}(x, S'_{n,j}) &\geqq \frac{1}{2}\left(\frac{1}{\sqrt{n}} - \frac{1}{n}\right) = \frac{1}{2\sqrt{n}}\left(1 - \frac{1}{\sqrt{n}}\right) \geqq \\ &\geqq \frac{1}{2\cdot\sqrt[4]{2}}\left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{d(S'_{n,j})} = c\sqrt{d(S'_{n,j})} \end{aligned} \tag{9}$$

where $c = \frac{1}{2\cdot\sqrt[4]{2}}(1 - \frac{1}{\sqrt{2}})$. Obviously $d(Q_{n,j}) = \sqrt[4]{2} \cdot \sqrt{d(S'_{n,j})}$ and hence by

(3)

$$\sum_{n=1}^{\infty}\sum_{j=1}^{J(n)} d(Q_{n,j}) = \sum_{n=1}^{\infty}\sum_{j=1}^{J(n)} \sqrt[4]{2}\cdot\sqrt{d(S'_{n,j})} = \tag{10}$$
$$= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \sqrt[4]{2}\cdot\sqrt{d(S_{k-1,m})} < \infty.$$

Put $H_N = \bigcup_{n=N}^{\infty}\bigcup_{j=1}^{J(n)} Q_{n,j}$ and $H = \bigcap_{N=1}^{\infty} H_N$. Then it is easy to see that $\bigcap_{k=1}^{\infty} G_k = G \subset H$. Furthermore if $x \notin H$ then there exists an $N(x) \in \mathbf{N}$ such that $x \notin H_{N(x)}$, that is, if $n \geqq N(x)$ then (9) holds for every $j = 1, \dots, J(n)$. It is also easy to check that (10) implies that H is of zero one-dimensional Hausdorff measure, and hence H is thin.

Part 7. We have to prove that F is the indefinite v-integral of f. For a given BV set B, and $\varepsilon > 0$ we have to find a $\delta > 0$ gauge function, and a thin set T such that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon$$

for each δ-fine ε-partition $\{(A_i, x_i), \dots, (A_p, x_p)\}$ in B mod T.

Put $T = H \cup \{x : f(x) = 0\} \cup \bigcup_{n,j} \mathrm{bd}(S'_{n,j})$. We recall that $H \supset G$ is thin and $f(x) = 0$ whenever $x \in G$ or x belongs to the boundary of the sets K_ℓ defined in Part 2. From our construction it follows that the union of the boundaries of all the sets K_ℓ is of σ-finite one-dimensional Hausdorff measure, that is, a thin set. Similarly $\bigcup_{n,j} \mathrm{bd}(S'_{n,j})$ is also thin. Therefore T is thin.

Put $\delta(x) = 1$ for $x \in T$. If $x \in [0,1]^2 \setminus T$ then there exist n' and j' such that $x \in \Psi'_{n',j'}$, $x \notin \mathrm{bd}(S'_{n',j'})$. Part 6, the claim in Part 1 and the choice of $T \supset H$ imply that x is a dispersion point of $G(S'_{n',j'})$. Given a constant $a > 0$ choose $\delta_1(x) > 0$ such that if $0 < r < \delta_1(x)$ then $S(x,r) \subset S'_{n',j'}$ and

$$\frac{|G(S'_{n',j'}) \cap S(x,r)|}{|S(x,r)|} \leqq \frac{\varepsilon^3 a^2}{8|f(x)|}. \tag{11}$$

Since $x \in \Psi'_{n',j'}$ and $x \notin T \supset \bigcup_{n,j} \mathrm{bd}(S'_{n,j})$ we can choose for any given $K > 0$ a $\delta_2(x) > 0$ such that $\cup\{S'_{n,j} : S'_{n,j} \subset G(S'_{n',j'}); n = 1, \dots, K, j =$

$= 1, \dots, J(n)\} \cap S(x, \delta_2(x)) = \emptyset$. We may also assume that $K > \max(n', N(x))$, that is (9) holds for every $n > K$, and $j = 1, \dots, J(n)$.

Since x does not belong to the boundary of the sets K_ℓ we can find a $\delta_3(x) > 0$ such that if $y \in S(x, \delta_3(x)) \cap \Psi'_{n',j'}$ then $f(y) = f(x)$, that is x and y are in the same K_ℓ.

Put $\delta(x) = \min(\delta_1(x), \delta_2(x), \delta_3(x))$. For any ε-regular BV-set A such that $A \subset S(x, \delta(x)) \cap B$ we have

$$|f(x)|A| - F(A)| \leqq |f(x) \cdot |A \cap \Psi'_{n',j'}| - F(A \cap \Psi'_{n',j'})| +$$

$$+ |f(x) \cdot |A \cap G(S'_{n',j'})|| + |F(A \cap G(S'_{n',j'}))| = \rho_1 + \rho_2 + \rho_3,$$

where we used that $S'_{n',j'} = \Psi'_{n',j'} \cup G(S'_{n',j'})$ and $A \subset S(x, \delta(x)) \subset S'_{n',j'}$.

Since by definition $F(A \cap \Psi'_{n',j'}) = \int\limits_{A \cap \Psi'_{n',j'}} f = f(x) \cdot |A \cap \Psi'_{n',j'}|$ we obtain that $\rho_1 = 0$.

Since A is ε-regular $|A|/d(A)\|A\| \geqq \varepsilon$ and hence $|A| \geqq d(A)\|A\|\varepsilon$. By the isoperimetric inequality, [4, Theorem 1.29, p. 25], $\|A\| \geqq a \cdot \sqrt{|A|}$ with a constant $a > 0$. Thus $\sqrt{|A|}/a\varepsilon \geqq d(A)$. Since $x \in \mathrm{cl}(A)$ (cf. Def. 2), we have $A \subset S(x, \sqrt{|A|}/a\varepsilon)$ and hence

$$|f(x) \cdot |G(S'_{n',j'}) \cap A|| \leqq |f(x)| \cdot |G(S'_{n',j'}) \cap S(x, \sqrt{|A|}/a\varepsilon)| \leqq$$

$$\leqq |f(x)| \cdot |S(x, \sqrt{|A|}/a\varepsilon)| \frac{\varepsilon^3 a^2}{8|f(x)|} = \frac{|A|\varepsilon^3 a^2 \pi}{8\varepsilon^2 a^2} < \frac{|A|\varepsilon}{2},$$

where we used (11). Thus $\rho_2 < |A|\varepsilon/2$.

Since $\delta(x) < \delta_2(x)$ we have $A \cap G(S'_{n',j'}) \subset A \cap \bigcup\limits_{n=K+1}^{\infty} \bigcup\limits_{j=1}^{J(n)} S'_{n,j}$. Using that $x \notin H \subset T$ we obtain $\mathrm{dist}(x, S'_{n,j}) \geqq c\sqrt{d(S'_{n,j})} = c\sqrt{\sqrt{2}/n}$ for every (n,j) such that $n = K+1, \dots,$ $j = 1, \dots, J(n)$. Thus if $|A \cap \Psi'_{n,j}| \neq 0$ then $d(A) \geqq c\sqrt{d(S'_{n,j})} = c\sqrt{\sqrt{2}/n}$. Thus by the ε-regularity of A we have $|A|/\varepsilon > d(A) \cdot \|A\|$, that is,

$$\frac{|A|\sqrt{n}}{c\varepsilon\sqrt[4]{2}} > \|A\|.$$

Therefore by (6)

$$|F(A \cap \Psi'_{n,j})| \leqq \frac{\|A\|}{2n^2 2^n} < \frac{|A|\sqrt{n}}{c\varepsilon\sqrt[4]{2}2n^2 2^n}.$$

Finally using that there are at most $J(n) < n^2$ different sets $\Psi'_{n,j}$ for a fixed n we obtain that

$$|F(A \cap G(S'_{n',j'}))| \leqq \sum_{n=K+1}^{\infty} \frac{|A|\sqrt{n}n^2}{c\varepsilon\sqrt[4]{2}2n^2 2^n} < \frac{\varepsilon|A|}{2},$$

where the last inequality holds if K is sufficiently big.

Thus if we use the above a and K in the definition of δ then $|f(x)|A| - -F(A)| < \rho_1 + \rho_2 + \rho_3 < \varepsilon|A|$. Therefore if $\{(A_i, x_i), \dots, (A_p, x_p)\}$ is a δ-fine ε-partition in B mod T then

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon \sum_{i=1}^{p} |A_i| < \varepsilon.$$

Thus F is the indefinite v-integral of F and the proof of Theorem 2 is complete.

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(Received January 4, 1990)

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
BUDAPEST, MÚZEUM KRT 6–8
H–1088 HUNGARY

Acta Math. Hung.
59 (3–4) (1992), 395–399.

ON MONOTONE AND DOUBLY MONOTONE POLYNOMIAL APPROXIMATION

S. G. GAL (Oradea) and J. SZABADOS* (Budapest)

In several papers [2-10] the first named author has obtained some results concerning approximation by monotone sequences of polynomials, as for example the following:

THEOREM A (Gal [3]). *For any $f \in C[0,1]$, there exist two polynomial sequences $\{Q_n\}_{n=0}^{\infty}$ and $\{P_n\}_{n=0}^{\infty}$, uniformly convergent towards f and satisfying*

$$Q_n(x) < Q_{n+1}(x) < f(x) < P_{n+1}(x) < P_n(x)$$

for all $0 \leqq x \leqq 1$ and all $n \in \mathbf{N}$.

Although the construction of $Q_n(x)$ and $P_n(x)$ in some cases remains an open question, in some particular cases like $f(x) \in \mathrm{Lip}\ \alpha$, these polynomials were effectively constructed by the help of the Bernstein polynomials. A shortcoming of these constructions was the high degree of the polynomials compared to the order of approximation (see [10]).

In this paper, starting from a fairly general class of approximating polynomials, we construct the polynomials P_n, Q_n above with good approximating properties. Our result will also be suitable for constructing *doubly* monotone polynomial sequences in case $f(x)$ is a monotone function. In what follows, let $\|\cdot\|$ denote the supremum norm over the interval [0,1], and let $E_n(f)$ denote the best approximation of an $f \in C[0,1]$ by polynomials from Π_n (the set of polynomials of degree at most n).

THEOREM. *Let $f(x) \in C[0,1]$ and assume that*

$$\sum_{k=1}^{\infty} k^{-1} E_k(f) < \infty. \tag{1}$$

Then there exist polynomials P_n, $Q_n \in \Pi_n$ such that

$$Q_n(x) \leqq Q_{n+1}(x) \leqq f(x) \leqq P_{n+1}(x) \leqq P_n(x) \qquad (0 \leqq x \leqq 1,\ n \geqq 4) \tag{2}$$

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* Research of this author supported by the Hungarian National Science Foundation for Research, Grant No. 1801.

and

$$\|P_n(x) - Q_n(x)\| \leqq 8 \sum_{k=[n/2]-1}^{\infty} k^{-1} E_k(f) \qquad (n \geqq 4). \tag{3}$$

PROOF. Let $p_k(x) \in \Pi_k$ be the best approximating polynomial of $f(x)$, i.e.

$$\|f - p_k\| = E_k(f) \qquad (k = 0, 1, \dots). \tag{4}$$

Define

$$P_n(x) = \begin{cases} \dfrac{2}{n+2} \sum\limits_{k=n/2-1}^{n-1} p_k(x) + 4 \sum\limits_{k=n/2-1}^{\infty} k^{-1} E_k(f), & \text{if } n \geqq 4 \text{ is even} \\ \dfrac{P_{n-1}(x) + P_{n+1}(x)}{2}, & \text{if } n \geqq 5 \text{ is odd.} \end{cases} \tag{5}$$

Evidently, $P_n(x) \in \Pi_n$ $(n \geqq 4)$. For even $n \geqq 4$ we obtain by (4) and (5)

$$P_{n+1}(x) - P_n(x) = \frac{P_{n+2}(x) - P_n(x)}{2} = \tag{6}$$

$$= \frac{1}{n+4} \sum_{k=n/2}^{n+1} p_k(x) + 2 \sum_{k=n/2}^{\infty} k^{-1} E_k(f) - \frac{1}{n+2} \sum_{k=n/2-1}^{n-1} p_k(x) -$$

$$-2 \sum_{k=n/2-1}^{\infty} k^{-1} E_k(f) = -\frac{2}{(n+2)(n+4)} \sum_{k=n/2}^{n-1} p_k(x) +$$

$$+\frac{p_n(x) + p_{n+1}(x)}{n+4} - \frac{p_{n/2-1}(x)}{n+2} - \frac{4}{n-2} E_{n/2-1}(f) =$$

$$= -\frac{2}{(n+2)(n+4)} \sum_{k=n/2}^{n-1} (p_k(x) - f(x)) + \frac{p_n(x) - f(x) + p_{n+1}(x) - f(x)}{n+4} -$$

$$-\frac{p_{n/2-1}(x) - f(x)}{n+2} - \frac{4}{n-2} E_{n/2-1}(f) \leqq$$

$$\leqq \frac{2}{(n+2)(n+4)} \sum_{k=n/2}^{n-1} \|p_k - f\| + \frac{\|p_n - f\| + \|p_{n+1} - f\|}{n+4} +$$

$$+\frac{\|p_{n/2-1} - f\|}{n+2} - \frac{4}{n-2} E_{n/2-1}(f) \leqq$$

$$\leqq \left(\frac{n}{(n+2)(n+4)} + \frac{2}{n+4} + \frac{1}{n+2} - \frac{4}{n-2} \right) E_{n/2-1}(f) =$$

$$= -\frac{24}{(n+4)(n-2)} E_{n/2-1}(f) \leqq 0,$$

while for odd $n \geqq 5$ we get from (5) and (6)

$$P_{n+1}(x) - P_n(x) = \frac{P_{n+1}(x) - P_{n-1}(x)}{2} \leqq 0. \tag{7}$$

(6) and (7) together show that the right hand side inequality in (2) holds.

Now if we define

$$Q_n(x) = \begin{cases} \frac{2}{n+2} \sum_{k=n/2-1}^{n-1} p_k(x) - 4 \sum_{k=n/2-1}^{\infty} k^{-1} E_k(f), & \text{if } n \geqq 4 \text{ is even} \\ \frac{Q_{n-1}(x) + Q_{n+1}(x)}{2}, & \text{if } n \geqq 5 \text{ is odd,} \end{cases} \tag{8}$$

then $Q_n(x) \in \Pi_n$ $(n \geqq 4)$, and the left hand side inequality in (2) is proved analogously. The inequalities for $f(x)$ in (2) follow from the obvious fact that

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} Q_n(x) = f(x) \qquad (0 \leqq x \leqq 1).$$

Finally, (3) follows from (5) and (8). The theorem is completely proved.

Denote by $\omega(\cdot, h)$ the modulus of continuity of the corresponding function. Then

COROLLARY 1. (a) *If* $f(x) \in C[0,1]$ *and*

$$\sum_{k=1}^{\infty} \frac{1}{k} \omega\left(f, \frac{1}{k}\right) < \infty \tag{9}$$

then there exist polynomials $P_n, Q_n \in \Pi_n$ *such that* (2) *holds and*

$$\|P_n - Q_n\| = O\left(\sum_{k=n}^{\infty} \frac{1}{k} \omega\left(f, \frac{1}{k}\right)\right). \tag{10}$$

(b) *If* $f^{(r)}(x) \in C[0,1]$ $(r \geqq 1)$, *then then there exist polynomials* P_n, $Q_n \in \Pi_n$ *such that* (2) *holds and*

$$\|P_n - Q_n\| = O\left(\frac{1}{n^r}\right) \omega\left(f^{(r)}, \frac{1}{n}\right).$$

Namely, both (a) and (b) follow from Jackson's theorem

$$E_n(f) = O(n^{-r})\omega(f^{(r)}, 1/n) \qquad (f^{(r)}(x) \in C[0,1]).$$

This is trivial in case (a), and in case (b) it follows from the estimates

$$\|P_n - Q_n\| = O\left(\sum_{k=[n/2]-1}^{\infty} k^{-1}E_k(f)\right) = O\left(\sum_{k=[n/2]-1}^{\infty} k^{-r-1}\omega(f^{(r)}, 1/k)\right) =$$

$$= O\left(\omega(f^{(r)}, 1/n) \sum_{k=[n/2]-1}^{\infty} (\frac{n}{k}+1)k^{-r-1}\right) = O(n^{-r})\omega(f^{(r)}, 1/n) \quad (r \geqq 1).$$

A particular case of Corollary 1(a) is when $f(x) \in \mathrm{Lip}\ \alpha$ $(0 < \alpha \leqq 1)$. Then (9) obviously holds and (10) yields $\|Q_n - P_n\| = O(n^{-\alpha})$.

Now assume that $f(x)$ is a monotone* function and $f^{(r)}(x) \in C[0,1]$ $(r \geqq 0)$. Then by well-known theorems on monotone approximation (see [1], [11] or [12]), there exist monotone polynomials $p_k(x) \in \Pi_k$ $(k \geqq r)$ such that

$$\|f - p_k\| = O(k^{-r})\omega(f^{(r)}, 1/k).$$

It is clear from the proof of Theorem 2.1 of [3] (see also Corollary 2.2 there), that the polynomials P_n, Q_n in Theorem A in this case can be chosen to be monotone, hence giving a solution for the problem of *doubly* monotone approximation. Under additional assumptions we can get more quantitative results: since the arithmetic means of monotone polynomials are again monotone polynomials, we obtain from our Theorem:

COROLLARY 2. *If $f(x)$ is a monotone function and $f^{(r)}(x) \in C[0,1]$ $(r \geqq 0)$, then under the assumptions of Corollary* 1 *the same conclusions hold, with the additional property that $P_n(x)$ and $Q_n(x)$ are monotone polynomials.*

REMARK. Condition (1) holds if $f(x)$ has higher smoothness properties (e.g. analytic or entire), but then the result expressed in our Theorem can be obtained much simpler than described in the proof. The question which remains to settle is: what happens if (1) does not hold?

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(Received January 22, 1990; revised September 17, 1990)

STR. CANTACUZINO 21
BLOC AN - 45, AP. 8
3700 ORADEA
ROMANIA

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H-1364 BUDAPEST, P.O.B. 127
HUNGARY

Acta Math. Hung.
59 (3–4) (1992), 401–403.

COMPLEX-VALUED MULTIPLICATIVE FUNCTIONS WITH MONOTONICITY PROPERTIES

KATALIN KOVÁCS (Budapest)

Let f denote a complex-valued multiplicative arithmetical function. In [2] and [3] we proved that if at least two of the functions $|f|$, $\operatorname{Re} f$ and $\operatorname{Im} f$ are monotonic, then f is real and $|f(n)| = n^k$ or $f(n) = 0$ for $n > 2$. Our theorem generalized the results of Erdős [1] and Moser and Lambek [5].

We got a similar result on sets of upper density one and on suitably constructed rare sets. (This means that to any function $h : \mathbf{N} \to \mathbf{R}$ there exists a set A such that $a_{n+1} - a_n > h(n)$ and if two of the functions $|f|$, $\operatorname{Re} f$ and $\operatorname{Im} f$ are monotonic on A, then $f(a)$ is real and $f(a) = a^k$ for all $a \in A$, $a \geqq a_0$ or $f(a) = 0$ for all $a \in A$, $a \geqq a_0$).

For completely multiplicative functions we prove a stronger result:

THEOREM 1. *Let f be a completely multiplicative function and A a set of upper density one.*

a) *If $\operatorname{Re} f$ is monotonic on A, then $f(n) = n^k$ for all $n \in \mathbf{N}$ or $\operatorname{Re} f(a) = 0$ for all $a \in A$, $a \geqq a_0$. In the case $A = \mathbf{N}$ we have $f(n) = n^k$ or $f \equiv 0$.*

b) *If $\operatorname{Im} f$ is monotonic on A, then $\operatorname{Im} f(a) = 0$ for all $a \in A$, $a \geqq a_0$. In the case $A = \mathbf{N}$ we have $\operatorname{Im} f \equiv 0$.*

For multiplicative functions we have only the following weaker result:

THEOREM 2. *Let f be a multiplicative function and $Q = P_1 \dots P_r$ where $P_1, \dots, P_r$ are the only primes such that $\operatorname{Im} f(P_i^{\alpha_i}) \neq 0$ for some α_i. If $\operatorname{Re} f$ is monotonic on a set A of upper density one, then either $r \leqq 1$, $f(n) = n^k$ for all $(n, P_1) = 1$ and $\operatorname{Re} f(n) = n^k$ for all $n \in \mathbf{N}$ or $f(a) = 0$ for all $a \in A' = \{a \in A | a \geqq a_0 \text{ and } (a, Q) = 1\}$ and $\operatorname{Re} f(a) = 0$ for all $a \in A$, $a \geqq a_0$.*

If $A = \mathbf{N}$, then either

$$f(p^{k\nu}) = \begin{cases} p^{k\nu} & \text{for all } p \neq p_0 \\ p^{k\nu} e^{i\varphi} / \cos\varphi & \text{if } p = p_0, \end{cases}$$

where p_0 is a prime and $\varphi \in [0, \pi/2)$ or $f(n) = 0$ for all $n \geqq 4$ except at most on the powers of one prime.

PROOF OF THEOREM 1. a) If $f(m) = 0$ for some $m \in \mathbf{N}$, then $f(mt) = 0$ for all $t \in \mathbf{N}$. The monotonicity of $\operatorname{Re} f$ on A yields $\operatorname{Re} f(a) = 0$ for all $a \in A$

large enough. Thus $f(n) \neq 0$ for all $n \in \mathbf{N}$ in the opposite case. Assume $\operatorname{Im} f(s) \neq 0$ for some $s \in \mathbf{N}$. Choose a natural number d such that

$$\pi/3 \leqq \arg f(s)^d \leqq \pi/2,$$

if there is any. The only case when such a d does not exist is $|\arg f(s)| = 2\pi/3$; in this case let d be such that $\arg f(s)^d = 2\pi/3$. There exists an $a \in \mathbf{N}$ such that $as^{dj} \in A$, $j = 1, \dots, 6$. $\operatorname{Re} f(as^{dj})$ changes sign twice, which contradicts the assumption. Hence f is real-valued and we get $f(n) = n^k$ by the above mentioned theorem of Moser and Lambek.

In the case $A = \mathbf{N}$, it is sufficient to prove that "$\operatorname{Re} f(n) = 0$ for $n \geqq n_0$" implies $f \equiv 0$. Indeed, if $f(w) \neq 0$ for some $w \in \mathbf{N}$, then $\operatorname{Re} f(w^\alpha) \neq 0$ for infinitely many $\alpha \in \mathbf{N}$, which is a contradiction.

b) The proof is similar to the proof of Case a). If we take $A = \mathbf{N}$, then $\operatorname{Im} f(w) \neq 0$ yields $\operatorname{Im} f(v^\alpha) \neq 0$ for infinitely many $\alpha \in \mathbf{N}$, which contradicts the assertion of b).

Proof of Theorem 2. If there exists an $m \in \mathbf{N}$ for which $f(m) = 0$, then $f(mt) = 0$ for all $(m, t) = 1$. Thus $\operatorname{Re} f(a) = 0$ for all $a \in A$, $a \geqq a_0$ and $\operatorname{Re} f(a) = f(a) = 0$ for all $a \in A'$.

Assume that f does not vanish. Write $N_Q = \{n \in \mathbf{N} | (n, Q) = 1\}$. f is real and monotonic on $N_Q \cap A$. This yields that f is positive on $N_Q \cap A'$ with some $a_0 \in A$. For any numbers $b, d \in N_Q$ there exists an x such that $x, bx, dx \in N_Q \cap A'$. Thus the monotonicity of f implies that f is real and monotonic on N_Q. We apply our theorem from [4]: If a function $g : \mathbf{N} \to \mathbf{R}^k$ is additive and its Euclidean norm is monotonic on an arbitrary fixed reduced residue class mod q ($q \in \mathbf{N}$, $q \geqq 2$) from a number m_0 on, then $g(m) = c \log m$ for all $(m, q) = 1$. (This theorem is a generalized form of the above mentioned theorem of Erdős.) This yields $\log f|_{N_Q} = k \log |_{N_Q}$. For any $n \in N_Q$ there exist infinitely many $t \in \mathbf{N}$ such that $t \equiv 1 \bmod Qn$ and

$$U_{nt}\{n(t-1)+1, nt, n(t-1+Q)+1\} \subset N_Q \cap A.$$

The monotonicity of f on U_{nt}'s, by $f(n) = n^k$ on N_Q, implies

$$\operatorname{Re} f(n) = n^k \quad \text{for all} \quad n \in N_Q. \tag{1}$$

If $r \geqq 2$, let us substitute $P_1^{\alpha_1}$, $P_2^{\alpha_2}$ and $P_1^{\alpha_1} P_2^{\alpha_2}$ into (1). We get the contradiction from the relation

$$\operatorname{Re} f(P_1^{\alpha_1} P_2^{\alpha_2}) = \operatorname{Re} f(P_1^{\alpha_1}) \operatorname{Re} f(P_2^{\alpha_2}) - \operatorname{Im} f(P_1^{\alpha_1}) \operatorname{Im} f(P_2^{\alpha_2}).$$

Therefore $r \leqq 1$ and (1) is satisfied.

The case $A = \mathbf{N}$: Either (1) implies our assertion using that $r \leqq 1$ or we have $\operatorname{Re} f(n) = 0$ for all $\geqq n_0$. The monotonicity of $\operatorname{Re} f$ yields that $\operatorname{Re} f(n) \neq 0$ for all $n < n_0$. If $n_) \geqq 6$, then there exists a $t \in$

$\in \left[\left[\frac{n_0}{2}\right]+1,\ n_0-1\right]$ for which $(t,6)=1$. All the elements of $\{2t,3t,6t\}$ are larger than n_0. Therefore

$$\operatorname{Re} f(2t)=\operatorname{Re} f(3t)=\operatorname{Re} f(6t)=0.$$

This implies

$$\arg f(2)\equiv \arg f(3)\equiv \arg f(6) \pmod \pi).$$

Thus $f(2)$ and $f(3)$ are non-zero real numbers. Let D denote the maximal prime power for which

$$\operatorname{Re} f(D)\neq 0. \tag{2}$$

There exists a prime p in $(D,2D)$, for which $\operatorname{Re} f(p)=0$. Thus, using the monotonicity of $\operatorname{Re} f$, we get

$$\operatorname{Re} f(2D)=\operatorname{Re} f(3D)=0. \tag{3}$$

For at least one of $b\in\{2,3\}$ we have $(b,D)=1$. Using that $f(b)\in R$, we get, by (2),

$$\operatorname{Re} f(bD)=f(b)\operatorname{Re} f(D)\neq 0,$$

which contradicts (3).

We have proved $n_0 \leqq 5$. Now we show that there is at most one prime p such that $f(p^k)\neq 0$ for some k, $p^k \geqq 5$. If there exist Q_1,Q_2 coprime prime powers which are bigger than n_0 with $\operatorname{Im} f(Q_i)\neq 0$, then $\operatorname{Re} f(Q_1Q_2)\neq 0$ yields a contradiction.

Our main problem whether there exists a multiplicative (but not completely multiplicative) function such that $\operatorname{Re} f$ is monotonic and $\operatorname{Im} f(P_i^{\delta_i})\neq$ $\neq 0$ on some powers of infinitely many primes, is still unsolved.

I am indebted to Imre Ruzsa for his valuable remarks.

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(Received January 23, 1990; revised July 24, 1990)

EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ALGEBRA AND NUMBER THEORY
H-1088 BUDAPEST, MÚZEUM KRT. 6–8.
HUNGARY

Acta Math. Hung.
59 (3–4) (1992), 405–421.

YANO–LEDGER CONNECTION AND INDUCED CONNECTION ON VECTOR BUNDLES

Z. KOVÁCS (Nyíregyháza) and L. TAMÁSSY (Debrecen)

Introduction

K. Yano and A. J. Ledger [13] constructed from a linear connection ∇ on a manifold B a torsion-free linear connection on TB (called the Yano–Ledger connection). M. Matsumoto [7] proved that: a) ∇ determines a Finsler connection $(\mathcal{H}, \overline{\nabla})$ in the space VTB of Finsler vectors; b) the symmetrization of the extension ∇' of $(\mathcal{H}, \overline{\nabla})$ to TTB is exactly the Yano–Ledger connection on TB; c) the Levi–Civita connection of the Riemannian metric on TB derived from a Riemannian metric g on B coincides with the Yano–Ledger connection derived from the Levi–Civita connection ∇ of g iff the Riemannian curvature tensor of g vanishes (see also [2]). On the other hand R. Miron [8] developed a theory of Finsler connections on vector bundles.

The purpose of this paper is to construct a Yano–Ledger connection on vector bundles, and then to prove the analogues of Matsumoto's theorems a), b), and c) for vector bundles and vector bundle Finsler connections. In our considerations and constructions we apply pullback of pseudoconnections. They are developed and investigated in §§1, 2, and 3. §4 yields the Yano–Ledger connection for vector bundles, and §§4, 5, and 6 present the mentioned theorems analogous to those of Matsumoto.

Concerning notation and terminology we refer to the monographs [1], [5].

§1. Preliminaries

A) Pseudoconnections. This notion is a generalization of the linear connection. Pseudoconnection was introduced by Y. Wong [12]. In this paper pseudoconnection is defined over a pair of vector bundles $\xi = (E, \pi, B, V^r)$ and $\tilde{\xi} = (\tilde{E}, \tilde{\pi}, B, V^s)$ with common base space B. E denotes the total space, $\pi : E \to B$ is the projection, and V^r is a real vector space of rank r. A pseudoconnection in ξ w.r.t. (with respect to) $\tilde{\xi}$ is a pair (∇, A) of maps. $A : \tilde{\xi} \to \tau_B$ is a strong vector bundle mapping, i.e. such that the fiber $\tilde{\pi}^{-1}(x)$, $x \in B$ is linearly mapped into the tangent space $T_x B$. $\nabla : \operatorname{Sec} \tilde{\xi} \times \operatorname{Sec} \xi \to \operatorname{Sec} \xi$, $(\tilde{\sigma}, \eta) \mapsto \nabla_{\tilde{\sigma}} \eta$ is additive in both variables $\tilde{\sigma} \in \operatorname{Sec} \tilde{\xi}$

and $\eta \in \operatorname{Sec}\xi$, it is $C^\infty(B)$ linear in $\tilde{\sigma}$, and also the property

$$(1) \qquad \nabla_{\tilde{\sigma}} f\eta = [(A \circ \tilde{\sigma})f]\eta + f\nabla_{\tilde{\sigma}}\eta, \quad f \in C^\infty(B)$$

is satisfied. ($\operatorname{Sec}\xi$ denotes the family of all sections of ξ over B, and τ_B is the tangent bundle of B). For details we refer to the paper [11]. Throughout the paper manifolds are supposed to be paracompact, and mappings, vector and tensor fields, functions, etc. to be of class C^∞.

B) Some basic notions and facts. Let ξ and ξ' be two vector bundles with base spaces B and B', and let $\varphi : \xi \to \xi'$ be a bundle map. The induced map $B \to B'$ determined by φ is denoted by Ψ, and the restriction of φ to the fiber $\pi^{-1}(x)$ is denoted by $\varphi_x = \varphi|_{\pi^{-1}(x)}$. We assume each φ_x to be a linear isomorphism. In this case $\varphi^\# : \operatorname{Sec}\xi' \to \operatorname{Sec}\xi$ is given by

$$[\varphi^\#(\tau')](x) = \varphi_x^{-1}(\tau'(\Psi(x))), \quad \tau' \in \operatorname{Sec}\xi'.$$

$\varphi^\#$ is additive and satisfies the relation

$$(2) \qquad \varphi^\#(f'\tau') = (f' \circ \Psi)\varphi^\#(\tau'), \quad \tau' \in \operatorname{Sec}\xi', \quad f' \in C^\infty(B').$$

The sections $\varphi^\#(\tau')$ and τ' are called *φ-related.*

Let

$$(3) \qquad 0 \longrightarrow \pi^*(\xi) \xrightarrow{\lambda} \tau_E \xrightarrow{\mu} \pi^*(\tau_B) \longrightarrow 0$$

be an exact sequence, where $*$ is the symbol of the pullback. Since $\operatorname{Im}\lambda$ is $V\xi$ (the vertical subbundle of τ_E), therefore $\hat{\lambda} : \pi^*(\xi) \to V\xi$, $\hat{\lambda}(\eta) := \lambda(\eta)$ is an isomorphism. Let $\alpha : V\xi \to \xi$, $\alpha = \operatorname{pr}_2 \circ \hat{\lambda}^{-1}$ with pr_2 denoting the projection of $\pi^*(\xi)$ on its second factor. α is a bundle map, and its induced map is π.

We fix a horizontal map $\mathcal{H} : \pi^*(\tau_B) \to \tau_E$ of (3). $\operatorname{Im}\mathcal{H} =: H\xi$ is the horizontal subbundle of τ_E. Then an element X of the vector space $\mathfrak{X}(E)$ of the tangent vector fields of E uniquely splits into $HX \in \operatorname{Sec}H\xi$ and $VX \in \operatorname{Sec}V\xi$. $a^\# : \operatorname{Sec}\xi \to \operatorname{Sec}V\xi$ and $(\operatorname{pr}_2 \circ \mu|_{HE})^\# = (\pi_*|_{HE})^\# : \mathfrak{X}(B) \to \operatorname{Sec}H\xi$ are mappings of sections. $\alpha^\#(\eta)$ is the vertical lift of $\eta \in \operatorname{Sec}\xi$ denoted by η^v, and $(\pi_*|_{HE})^\#(X)$ is the horizontal lift of $X \in \mathfrak{X}(B)$ denoted by X^h.

The following facts are well known (cf. [3]):

(i) $\pi_* X^h = X, \ \alpha \circ \eta^v = \eta \circ \pi$

(ii) $X^h(f \circ \pi) = (Xf \circ \pi), \ \eta^v(f \circ \pi) = 0$

(iii) $[\eta^v, \zeta^v] = 0, \ H[X^h, Y^h] = [X, Y]^h,$

$$X, Y \in \mathfrak{X}(B), \quad \eta, \zeta \in \operatorname{Sec}\xi, \quad f \in C^\infty(B), \quad (f \circ \pi) \in C^\infty(E).$$

§2 Pullback of the mapping A'

1. In §§2 and 3 let ξ and $\tilde{\xi}$ (resp. ξ' and $\tilde{\xi}'$) be two vector bundles with a common base space B (resp. B'). Let $A' : \tilde{\xi}' \to \tau_{B'}$ be a strong vector bundle mapping, and $\varphi : \xi \to \xi'$, $\chi : \tilde{\xi} \to \tilde{\xi}'$ two vector bundle mappings with common induced mapping $\Psi : B \to B'$. We assume each φ_x and χ_x ($x \in B$) to be linear isomorphisms. We call a strong vector bundle mapping $A : \tilde{\xi} \to \tau_B$ the *pullback* of A' if $A \circ \chi^{\#}(\tilde{\sigma}')$ and $A' \circ \tilde{\sigma}'$ are Ψ-related vector fields for any $\tilde{\sigma}' \in \operatorname{Sec}\tilde{\xi}'$; i.e. if

$$[(A' \circ \tilde{\sigma}')g'] \circ \Psi = [A \circ \chi^{\#}(\tilde{\sigma}')](g' \circ \Psi) \quad \forall g' \in C^{\infty}(B'), \quad \tilde{\sigma}' \in \operatorname{Sec}\tilde{\xi}', \tag{4}$$

or equivalently

$$(A' \circ \tilde{\sigma}')(\Psi(x)) = d\Psi[(A \circ \chi^{\#}(\tilde{\sigma}'))(x)], \quad \forall \tilde{\sigma}' \in \operatorname{Sec}\tilde{\xi}'. \tag{5}$$

We want to show the existence of such an A, and determine its form. After this we want to investigate the same questions for the pullback of a pseudoconnection (∇', A') in ξ' w.r.t. $\tilde{\xi}'$. Fig. 1 shows the most relevant ones of the above mappings.

$$\begin{tikzcd}
 & TB \arrow[rr, "d\Psi"] \arrow[dl, "A"'] \arrow[ddl, "\pi_B"] & & TB' \arrow[dl, "A'"'] \arrow[ddl, "\pi'_{B'}"] \\
\tilde{E} \arrow[rr, "\chi"] \arrow[d, "\tilde{\pi}"'] & & \tilde{E}' \arrow[d, "\tilde{\pi}'"'] & \\
B \arrow[rr, "\Psi"] & & B' &
\end{tikzcd}$$

Fig. 1

If $\Psi(B) \neq B'$, then $B' \setminus \Psi(B)$ plays no role in the investigations concerning pullback. So it means no restriction to assume that $\Psi(B) = B'$. Surjectivity of the differential $d\Psi_x$ will also be assumed.

In order to prove the existence of A we consider the pullbacks $\Psi^*(\tau_{B'})$, $\Psi^*(\tilde{\xi}')$ and the strong vector bundle mappings

$$F : \tau_B \to \Psi^*(\tau_{B'}), \quad F(x, y) = (x, d\Psi_x(y)) \in B \times_{\Psi} TB', \quad y \in \pi_B^{-1}(x)$$

and

$$K : \tilde{\xi} \to \Psi^*(\tilde{\xi}'), \quad K(x, \tilde{\sigma}) = (x, \chi(\tilde{\sigma})) \in B \times_{\Psi} \tilde{E}'.$$

$$\begin{tikzcd} \tau_B \arrow[rr, "F"] \arrow[dr, "d\Psi"'] & & \Psi^*(\tau_{B'}) \arrow[dl, "\mathrm{pr}_2"] \\ & \tau_{B'} & \end{tikzcd} \qquad \begin{tikzcd} \tilde{\xi} \arrow[rr, "K"] \arrow[dr, "\chi"'] & & \Psi^*(\tilde{\xi}') \arrow[dl, "\mathrm{pr}_2"] \\ & \tilde{\xi}' & \end{tikzcd}$$

Fig. 2

Then $d\Psi = \mathrm{pr}_2 \circ F$ and $\chi = \mathrm{pr}_2 \circ K$. Since F is surjective for any $x \in B$, there exists at least one strong vector bundle mapping $\Phi : \Psi^*(\tau_{B'}) \to \to \tau_B$ such that $F \circ \Phi = \mathrm{id}$ ([5] Vol. 1, p. 77, Lemma III). We show that $A = \Phi \circ (\mathrm{id} \times A') \circ K$ satisfies (5):

$$d\Psi_x[A \circ \chi^{\#}(\tilde{\sigma}')(x)] =$$

$$= d\Psi_x \left\{ \Phi \circ (\mathrm{id} \times A') \circ K \circ [\chi^{\#}(\tilde{\sigma}')(x)] \right\} =$$

$$= \mathrm{pr}_2 \circ F \circ \Phi \circ (\mathrm{id} \times A')[x, \chi(\chi^{\#}(\tilde{\sigma}'))(\Psi(x))] =$$

$$= \mathrm{pr}_2 \circ (\mathrm{id} \times A')[x, \tilde{\sigma}'(\Psi(x))] = A'(\tilde{\sigma}'(\Psi(x))), \quad \forall \tilde{\sigma}' \in \operatorname{Sec} \tilde{\xi}'.$$

This proves the existence of A.

2. The local expression of A shows its dependence on A' and gives the grade of its arbitrariness.

Since χ_x is an isomorphism, we have $\operatorname{rank} \tilde{\xi} = \operatorname{rank} \tilde{\xi}' (= s)$, and since Ψ is a submersion, we have $n = \dim B \geqq \dim B' = m$. Let U and $U' = \Psi(U)$ be coordinate neighbourhoods in B, resp. B' with local coordinates (x^i) $(i, j = 1, 2, \ldots, n)$ and (z^α) $(\alpha, \beta = 1, 2, \ldots, m)$. For appropriately chosen (x) and (z), $\Psi|_U$ is described by $z^\alpha \circ \Psi = x^\alpha$. Thus $d\Psi_x \frac{\partial}{\partial x^\alpha}(x) = \frac{\partial}{\partial z^\alpha}(\Psi(x))$, and $d\Psi_x \frac{\partial}{\partial x^{\overline{\alpha}}}(x) = 0$ $(\overline{\alpha} = m+1, \ldots, n)$. Let A be a mapping satisfying (5), and $\tilde{\sigma}'_a \in \operatorname{Sec} \tilde{\xi}'|_{U'}$ $(a, b = 1, 2, \ldots, s)$ such that they form a basis in any fiber of $\tilde{\xi}'$ over U'. Then $\chi^{\#}(\tilde{\sigma}'_a) =: \tilde{\sigma}_a$ have the same property in the fibers of $\tilde{\xi}$ over U. Finally $A \circ \tilde{\sigma}_a = \sum_i A^i_a \frac{\partial}{\partial x^i}$, where A^i_a are the components of the mapping $A : \tilde{\pi}^{-1}(x) \to T_x B$ in the frames $\left(\frac{\partial}{\partial x^i}, \tilde{\sigma}_a\right)$. Similarly $A' \circ \tilde{\sigma}'_a = \sum_\alpha A'^\alpha_a \frac{\partial}{\partial z^\alpha}$.

According to (5)

$$(A' \circ \tilde{\sigma}'_a)\Psi(x) = d\Psi_x(A \circ \tilde{\sigma}_a). \tag{6}$$

Expressing both sides in the coordinate systems described we obtain

$$A^\alpha_a(x) = A'^\alpha_a(\Psi(x)). \tag{7}$$

for any A satisfying (5). Conversely, (7) implies (6) for $a = 1, 2, \ldots, s$, which is equivalent to (5). This yields however that the components $A^{\overline{\alpha}}_a$, $\overline{\alpha} = m+1, \ldots, n$ of $A(x)$ can be chosen freely. Our result is expressed by

THEOREM 1. a) *Given two vector bundles $\tilde{\xi}$ and $\tilde{\xi}'$ and a strong vector bundle mappiny $A' : \tilde{\xi}' \to \tau_{B'}$, as at the beginning of this paragraph, then there exists a strong vector bundle mapping $A : \tilde{\xi} \to \tau_B$ (the pullback of A') such that A and A' are Ψ-related.*

b) *The local components $A^{\alpha}_{a}(x)$ of this A are determined by* (7), *and its other components $A^{\overline{\alpha}}_{a}(x)$ are arbitrary. In case of $n = m$ there is no $A^{\overline{\alpha}}_{a}(x)$ and A is uniquely determined by* (7).

3. Section 2 of this § is a local, but constructive construction of all possible A, thus it means a local existence proof of A too. Section 1 was a relatively short and global proof of the existence of A. We remark that Section 2 can be extended to another proof of the global existence of A. Let $\{U_\iota\}$, $\iota \in \mathcal{I}$ be a local finite open covering of B and $\{a_\iota\}$, $a_\iota \in C^\infty(B)$ a partition of unity subordinate to $\{U_\iota\}$. Let $A_\iota : \tilde{\xi}|_{\tilde{\pi}^{-1}(U_\iota)} \to \tau_B|_{\pi^{-1}(U_\iota)}$ be mappings satifying (4) over U_ι. Such A_ι were constructed in Secion 2. We define $a_\iota A_\iota$ over U_ι as $(a_\iota A_\iota)(\tilde{\sigma}) = a_\iota(A_\iota(\tilde{\sigma}))$, and as $(a_\iota A_\iota)(\tilde{\sigma}) \equiv 0$ on the complement of U_ι. Thus $a_\iota A_\iota$ is a differentiable mapping taking $\tilde{\xi}$ into τ_B, and satisfying

$$[a_\iota A_\iota \circ \chi^{\#}(\tilde{\sigma}')](f' \circ \Psi) = ([a_\iota A' \circ \tilde{\sigma}']f') \circ \Psi$$

$$\forall \tilde{\sigma}' \in \operatorname{Sec} \tilde{\xi}', \quad f' \in C^\infty(B').$$

The sum of these yields

$$\left[\sum_\iota (a_\iota A_\iota) \circ \chi^{\#}(\tilde{\sigma}')\right](f' \circ \Psi) = \left(\left[\left(\sum_\iota a_\iota\right) A' \circ \tilde{\sigma}'\right] f'\right) \circ \Psi.$$

Here $\sum_\iota a_\iota = 1$. Hence $A := \sum_\iota (a_\iota A_\iota)$ globally satisfies (4), and so it is a pullback of A'.

§3. Pullback of the pseudoconnection (∇', A')

1. Let (∇', A') be a pseudoconnection in ξ' w.r.t. $\tilde{\xi}'$. We call a pseudoconnection (∇, A) in ξ w.r.t $\tilde{\xi}$ *a pullback of* (∇', A')[1] if

$$\nabla_{\chi^{\#}(\tilde{\sigma}')} \varphi^{\#}(\eta') = \varphi^{\#}(\nabla'_{\tilde{\sigma}'} \eta'), \quad \forall \tilde{\sigma}' \in \operatorname{Sec} \tilde{\xi}, \quad \eta' \in \operatorname{Sec} \xi'. \tag{8}$$

We show that for any pseudoconnection (∇', A') there exist pullbacks (∇, A), where A must be a pullback of A'. This A is not uniquely determined by

[1] Notions for linear connections more or less analogous to this can be found e.g. in [4] p. 363, or in [5] II, p. 324.

(∇', A') except the case $\dim B = \dim B'$, but ∇ is unique provided we fix a pullback A of A'.

First we show that A must satisfy (4). We shall denote the pullback $\chi^{\#}(\tilde{\sigma}')$ of a $\tilde{\sigma}' \in \operatorname{Sec} \tilde{\xi}'$ by $\tilde{\sigma}_1$, similarly $\varphi^{\#}(\eta') \equiv \eta_1$ for $\eta' \in \operatorname{Sec} \xi'$, and $\varphi^{\#} f' = f' \circ \Psi \equiv f_1$ for $f' \in C^{\infty}(B')$. Since both (∇', A') and (∇, A) are pseudoconnetions, in view of (8) we obtain

$$\nabla_{\chi^{\#}(\tilde{\sigma}')} \varphi^{\#}(f'\eta') = \nabla_{\tilde{\sigma}_1} f_1 \eta_1 = [(A \circ \tilde{\sigma}_1) f_1] \eta_1 + f_1 \nabla_{\tilde{\sigma}_1} \eta_1 =$$

$$= \varphi^{\#}(\nabla'_{\tilde{\sigma}'} f'\eta') = \varphi^{\#}\{[(A' \circ \tilde{\sigma}') f'] \eta'\} + \varphi^{\#}(f' \nabla_{\tilde{\sigma}'} \eta') =$$

$$= \{[(A' \circ \tilde{\sigma}') f'] \circ \Psi\} \eta_1 + f_1 \nabla_{\tilde{\sigma}_1} \eta_1, \quad \forall \tilde{\sigma}' \in \operatorname{Sec} \tilde{\xi}', f' \in C^{\infty}(B').$$

Comparison of the first terms in the third and sixth expressions gives (4).

2. Now we construct ∇ and show its uniqueness. In view of (8) the definition of ∇ for $\tilde{\sigma}_1$ and η_1 can only be

$$\nabla_{\tilde{\sigma}_1} \eta_1 = \varphi^{\#}(\nabla'_{\tilde{\sigma}'} \eta'), \quad \forall \tilde{\sigma}_1 \in \operatorname{Sec} \tilde{\xi}, \quad \eta_1 \in \operatorname{Sec} \xi. \tag{9}$$

We show that the value of ∇ over sections "deferring" from pullback sections must be 0. Thus we have only one possibility for the definition of ∇. Finally we show that the ∇ defined in this single possible way coupled with an A satisfying (4) is a pseudoconnection.

We define ∇ at an arbitary point $x_0 \in B$ for any $\tilde{\sigma} \in \operatorname{Sec} \tilde{\xi}$ and $\eta \in \operatorname{Sec} \xi$. Let $B_1 \subset B$ be a (possibly small) submanifold of B through x_0 such that $\Psi : B_1 \to \Psi(B_1) \equiv B_1' \subset B'$ is a bijection and $A \circ \tilde{\sigma}(x_0) \in T_{x_0} B_1$. Clearly B_1 is not uniquely determined by these conditions. We define on B_1' $\eta' := \varphi(\eta\,|_{B_1})$, $\tilde{\sigma}' := \chi(\tilde{\sigma}|_{B_1})$ and on $\Psi^{-1}[B_1']$ $\eta_1 = \varphi^{\#}(\eta')$, $\eta_2 = \eta - \eta_1$; $\tilde{\sigma}_1 = \chi^{\#}(\tilde{\sigma}')$, $\tilde{\sigma}_2 = \tilde{\sigma} - \tilde{\sigma}_1$. Thus any $\tilde{\sigma}$, or η splits w.r.t. the chosen B_1 as

$$\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2, \quad \eta = \eta_1 + \eta_2. \tag{10}$$

We have the relations

$$\text{a) } \tilde{\sigma}|_{B_1} = \tilde{\sigma}_1, \quad \text{b) } \eta|_{B_1} = \eta_1, \quad \text{c) } \tilde{\sigma}_2|_{B_1} = \eta_2|_{B_1} = 0. \tag{11}$$

Assume (∇, A) to be a pseudoconnection satisfying (8). Then, because of the additivity of ∇

$$\nabla_{\tilde{\sigma}} \eta = \nabla_{\tilde{\sigma}_1} \eta_1 + \nabla_{\tilde{\sigma}_2} \eta_1 + \nabla_{\tilde{\sigma}_1} \eta_2 + \nabla_{\tilde{\sigma}_2} \eta_2 \tag{12}$$

on B_1. For any linear connection ∇^L in ξ $\nabla_{\tilde{\sigma}} \eta - \nabla^L_{A \circ \tilde{\sigma}} \eta = T(\tilde{\sigma}, \eta)$ is a function-bilinear mapping $\operatorname{Sec} \tilde{\xi} \times \operatorname{Sec} \xi \to \operatorname{Sec} \xi$, i.e. T is a homomorphism: $T \in \operatorname{Hom}(\operatorname{Sec} \tilde{\xi}, \operatorname{Sec} \xi; \operatorname{Sec} \xi)$. Hence we obtain on B_1 a representation of ∇ in the form

$$\nabla_{\tilde{\sigma}} \eta = \nabla^L_{A \circ \tilde{\sigma}} \eta + T(\tilde{\sigma}, \eta)$$

and

$$\nabla_{\tilde{\sigma}_1}\eta_1 = \nabla^L_{A\circ\tilde{\sigma}_1}\eta_1 + T(\tilde{\sigma}_1, \eta_1).$$

However $\tilde{\sigma}(x_0) = \tilde{\sigma}_1(x_0)$, from this $A \circ \tilde{\sigma}(x_0) = A \circ \tilde{\sigma}_1(x_0)$, furthermore $\eta(x_0) = \eta_1(x_0)$. Moreover on B_1 $\eta(x) = \eta_1(x)$, and we assumed $A \circ \tilde{\sigma}(x_0) \in \in T_{x_0}B_1$. From these we obtain $\nabla_{\tilde{\sigma}}\eta|_{x_0} = \nabla_{\tilde{\sigma}_1}\eta_1|_{x_0}$. Hence the sum of the last three terms in (12) is zero at x_0. Moreover this sum must vanish for given $\tilde{\sigma}_1, \eta_1$ and for arbitrary $\tilde{\sigma}_2, \eta_2$ satisfying (11.c). Thus these three terms must vanish also separately at x_0. Hence the definition of a ∇ satisfying (8) for arbitrary $\tilde{\sigma}$, and η can only be

$$\nabla_{\tilde{\sigma}}\eta = \nabla_{\tilde{\sigma}_1}\eta_1 = \varphi^{\#}(\nabla'_{\tilde{\sigma}'}\eta'). \tag{13}$$

We alo obtain that

$$\nabla_{\tilde{\sigma}_2}\eta_1 = \nabla_{\tilde{\sigma}_1}\eta_2 = \nabla_{\tilde{\sigma}_2}\eta_2 = 0 \tag{14}$$

which are consequences of (13).

In this definition we made use of the splitting (10), which depends on the choice of B_1. Using another $\overset{*}{B}_1$ (satisfying the same conditions) we obtain another splitting $\tilde{\sigma} = \overset{*}{\tilde{\sigma}}_1 + \overset{*}{\tilde{\sigma}}_2$, $\eta = \overset{*}{\eta}_1 + \overset{*}{\eta}_2$ and then

$$\nabla_{\tilde{\sigma}}\eta = \nabla_{\tilde{\sigma}_1}\eta_1 = \varphi^{\#}(\nabla'_{\tilde{\sigma}'}\eta') = \varphi^{\#}\left[\nabla'^{L}_{A'\circ\tilde{\sigma}'}\eta' + T(\tilde{\sigma}', \eta')\right] \tag{15}$$

resp.

$$\nabla_{\tilde{\sigma}}\eta = \nabla_{\overset{*}{\tilde{\sigma}}_1}\overset{*}{\eta}_1 = \varphi^{\#}(\nabla'_{\overset{*}{\tilde{\sigma}}'}\overset{*}{\eta}') = \varphi^{\#}\left[\nabla'^{L}_{A'\circ\overset{*}{\tilde{\sigma}}'}\overset{*}{\eta}' + T(\overset{*}{\tilde{\sigma}}', \overset{*}{\eta}')\right] \tag{16}$$

at x_0. But $\tilde{\sigma}_1 = \overset{*}{\tilde{\sigma}}_1$ and $\eta_1 = \overset{*}{\eta}_1$ at x_0, resp. in the direction of $A \circ \tilde{\sigma}(x_0)$. Consequently the same hold true also for $\tilde{\sigma}'$, $\overset{*}{\tilde{\sigma}}'$ and η', $\overset{*}{\eta}'$ at $\Psi(x_0) = x'_0$, resp. in the direction of $A \circ \tilde{\sigma}'(x'_0)$. Therefore the right hand sides of (15) and (16) equal. Hence $\nabla_{\tilde{\sigma}_1}\eta_1 = \nabla_{\overset{*}{\tilde{\sigma}}_1}\overset{*}{\eta}_1$ at x_0, and thus the definition of ∇ does not depend from the choice of B_1.

3. We have still to show that the (∇, A) just constructed is a pseudoconnection. ∇ is additive both in $\tilde{\sigma}$ and η, for ∇' is so in $\tilde{\sigma}'$ and η'. Consider now for an x_0, $\tilde{\sigma}$ and for a B_1 corresponding to these x_0, $\tilde{\sigma}$ a correspondence $x \mapsto x_1 = \Psi^{-1}[\Psi(x)] \cap B_1$ $(x \in B)$. Define for an $f \in C^\infty(B)$ the functions $f_1(x) = f(x_1)$, $f_2(x) = f(x) - f_1(x)$, and $f' \in C^\infty(B')$ as $f' = f_1 \circ \Psi$. Then $f_2|_{B_1} = 0$ and $f\tilde{\sigma}_1 = f_1\tilde{\sigma}_1 + f_2\tilde{\sigma}_1$. Here $\tilde{\nu}_1 \equiv f_1\tilde{\sigma}_1$ is a pullback field, and $f_2\tilde{\sigma}_1$ is a $\tilde{\nu}_2$, for $(f_2\tilde{\sigma}_1)|_{B_1} = 0$. In view of (13) and (14)

$$\nabla_{f\tilde{\sigma}}\eta = \nabla_{f\tilde{\sigma}_1}\eta_1 = \nabla_{\tilde{\nu}_1}\eta_1 + \nabla_{\tilde{\nu}_2}\eta_1 = \varphi^{\#}(\nabla'_{f'\tilde{\sigma}'}\eta') = \varphi^{\#}(f'\nabla'_{\tilde{\sigma}'}\eta') = f_1\nabla_{\tilde{\sigma}_1}\eta_1$$

at x_0. We may add $f_2\nabla_{\tilde{\sigma}_1}\eta_1$ to the last expression, since $f_2(x_0) = 0$. Thus we have at x_0 $\nabla_{f\tilde{\sigma}_1}\eta_1 = f\nabla_{\tilde{\sigma}_1}\eta_1$ i.e. ∇ is $C^\infty(B)$ linear in $\tilde{\sigma}$. Finally $\nabla_{\tilde{\sigma}} f\eta = \nabla_{\tilde{\sigma}_1} f\eta_1 = \nabla_{\tilde{\sigma}_1} f_1\eta_1 + \nabla_{\tilde{\sigma}_1} f_2\eta_1$. Here $f_2\eta_1$ is again a ρ_2, and in view of (14) $\nabla_{\tilde{\sigma}_1} f_2\eta_1 = 0$ at x_0. Thus in view of (4) and (13)

$$\nabla_{\tilde{\sigma}_1} f\eta_1 = \nabla_{\tilde{\sigma}_1} f_1\eta_1 = \varphi^{\#}(\nabla'_{\tilde{\sigma}'} f'\eta') =$$

$$= \varphi^{\#}\left\{\left[(A'\circ\tilde{\sigma}')f'\right]\eta' + f'\nabla'_{\tilde{\sigma}'}\eta'\right\} = \left[(A\circ\tilde{\sigma}_1)f_1\right]\eta_1 + f_1\nabla_{\tilde{\sigma}_1}\eta_1.$$

We can add again at x_0 $f_2\nabla_{\tilde{\sigma}_1}\eta_1$ and $[(A\circ\tilde{\sigma}_1)f_2]\,\eta_1$ to the right hand side, since $A\circ\tilde{\sigma}_1(x_0) \in T_{x_0}B_1$, $f_2|_{B_1} = 0$, and hence $[(A\circ\tilde{\sigma}_1)f_2]|_{x_0} = 0$. Thus $\nabla_{\tilde{\sigma}_1} f\eta_1 = [(A\circ\tilde{\sigma}_1)f]\eta_1 + f\nabla_{\tilde{\sigma}_1}\eta_1$ at x_0, i.e. ∇ satisfies (1). These show that (∇, A) is indeed a pseudoconnection.

We have the following

THEOREM 2. a) *If (∇, A) is the pullback of the pseudoconnection (∇', A'), then A must be a pullback of A'.*

b) *If (∇', A') is a pseudoconnection, and A is a pullback of A', then there exists a unique pullback pseudoconnection (∇, A), whose ∇ is determined by* (13).

If we want to emphasize that (∇, A) is pullback of a (∇', A') then we write $\nabla'^{\#}$ in place of ∇.

4. $\nabla : \mathfrak{X}(B)\times\mathfrak{X}(B) \to \mathfrak{X}(B)$, $(X,Y)\mapsto \nabla_X Y = 0$ is clearly no linear connection over B, for it does not satisfy the fourth Koszul axiom $\nabla_X fY = (Xf)Y + f\nabla_X Y$, $f\in C^\infty(B)$. But for a pseudoconnection (∇, A), where $A : \tilde{\xi}\to 0 \in \tau_B$ (i.e. A is trivial) $\nabla_{\tilde{\sigma}}\eta = \nabla^L_{A\circ\tilde{\sigma}}\eta$ with a linear connection ∇^L on B (in this case we say that ∇ is associated to ∇^L [6]), we have $\nabla_{\tilde{\sigma}}\eta = 0$, $\forall\tilde{\sigma},\eta$. We want to study a similar case.

PROPOSITION 1. *For given ξ, $\tilde{\xi}$, ξ', $\tilde{\xi}'$, φ, χ and $A : \tilde{\xi}\to\tau_B$ there exist pseudoconnections (∇, A) such that*

$$\nabla_{\chi^{\#}(\tilde{\sigma}')}\varphi^{\#}(\eta') = 0, \quad \forall\tilde{\sigma}'\in \operatorname{Sec}\tilde{\xi}', \quad \eta'\in\operatorname{Sec}\xi'. \tag{17}$$

In this case

$$A\circ\chi^{\#}(\tilde{\sigma}')(x) \in \operatorname{Ker} d\Psi_x, \quad \forall\tilde{\sigma}'\in\operatorname{Sec}\tilde{\xi}', \quad x\in B. \tag{18}$$

PROOF. Define in ξ a ∇^L and a homomorphism $T(\tilde{\sigma},\eta)$ such that

$$\nabla^L_{A\circ\tilde{\sigma}_1}\eta_1 = T(\tilde{\sigma}_1,\eta_1) = 0.$$

Then

$$\nabla_{\tilde{\sigma}}\eta = \nabla^L_{A\circ\tilde{\sigma}}\eta + T(\tilde{\sigma},\eta)$$

satisfies (17), and (∇, A) is a pseudoconnection. Definition of the linear connection ∇^L and of the homomorphism T for any other $\tilde{\sigma}$ and η may be arbitrary. Thus (∇, A) need not be an associated pseudoconnection. For a pseudoconnection (∇, A) we have

$$\nabla_{\tilde{\sigma}_1} f_1 \eta_1 = [(A \circ \tilde{\sigma}_1) f_1] \eta_1 + f_1 \nabla_{\tilde{\sigma}_1} \eta_1, \quad \forall f_1, \eta_1.$$

From this and (17) we get $(A \circ \tilde{\sigma}_1) f_1 = 0$, $\forall f_1$. According to the derivation of f_1 from an $f \in C^\infty(B)$ f_1 is constant on every $\Psi^{-1}[x']$, otherwise it can be arbitrary. So $X \in T_x B$ is tangent to $\Psi^{-1}[\Psi(x)]$ iff $X f_1 = 0$, $\forall f_1$. Hence $A \circ \tilde{\sigma}_1$ must be such an X, i.e. $A \circ \tilde{\sigma}_1(x) \in \operatorname{Ker} d\Psi_x$. Q.E.D.

(18) is obviously a weaker condition than $A : \tilde{\xi} \to 0$. Hence, in case of $\operatorname{Ker} d\Psi_x \neq 0$, (17) may be satisfied even if A is not trivial.

§4. Induced vector bundle Finsler connections

1. M. Matsumoto [7] (see also P. Dombrowski [2]) considers a Finsler connection on B as a pair $(\mathcal{H}, \overline{\nabla})$ of a horizontal map $\mathcal{H}$ to TTB and of a linear connection $\overline{\nabla}$ in the vertical subbundle $V\tau_B$ of the tangent bundle τ_B. He induces a Finsler connection $(\mathcal{H}, \overline{\nabla})$ from a linear connection ∇ of B (from a single data), and calls it the ∇-linear Finsler connection. He also extends a Finsler connection $(\mathcal{H}, \overline{\nabla})$ to a linear connection ∇' on TM (in TTM). In this paragraph we consider vector bundle Finsler connection in the sense of R. Miron [8], and by making use of the just established Theorem 2 we derive an induced vector bundle Finsler connection from three data (sources). In a special case this gives the extended Finsler connection ∇' of Matsumoto.

Let $\xi = (E, \pi, B, V^r)$ be a vector bundle, $\mathcal{H} : \pi^*(\tau_B) \to \tau_E$ a horizontal map, $H\xi = (HE, \pi_E, E, V^n)$ the horizontal subbundle, $V\xi = (VE, \pi_E, E, V^r)$ the vertical subbundle of τ_E; h resp. v the horizontal, resp. vertical lifts, as in §1. Consider four pseudoconnections

$$(19) \qquad \begin{cases} (\nabla^1, H) & \nabla^1 : \mathfrak{X}(E) \times \mathfrak{X}_H(E) \to \mathfrak{X}_H(E) \\ (\nabla^2, H) & \nabla^2 : \mathfrak{X}(E) \times \mathfrak{X}_V(E) \to \mathfrak{X}_V(E) \\ (\nabla^3, V) & \nabla^3 : \mathfrak{X}(E) \times \mathfrak{X}_V(E) \to \mathfrak{X}_V(E) \\ (\nabla^4, V) & \nabla^4 : \mathfrak{X}(E) \times \mathfrak{X}_H(E) \to \mathfrak{X}_H(E) \\ & \mathfrak{X}_V(E) \equiv \operatorname{Sec} V\xi, \quad \mathfrak{X}_H(E) \equiv \operatorname{Sec} H\xi. \end{cases}$$

As well known, a linear connection $\nabla : \mathfrak{X}(E) \times \mathfrak{X}(E) \to \mathfrak{X}(E)$ is a *Finsler connection in the sense of Miron* [8] w.r.t. $\mathcal{H}$ if $\nabla_X HY$ is always a horizontal and $\nabla_X VY$ a vertical vector. It is easy to see that any such Finsler connection can be represented in the form

$$(20) \qquad \nabla_X Y = \nabla^1_X HY + \nabla^2_X VY + \nabla^3_X VY + \nabla^4_X HY,$$

where $\nabla^1, \ldots, \nabla^4$ are pseudoconnections of type (19), and conversely, any sum as on the right hand side of (20) is a Finsler connection w.r.t $\mathcal{H}$ in the sense of Miron. This result formulated in another way can be found in [11].

THEOREM 3. *Consider a tangent bundle τ_B with a linear connection*

$$\nabla^T : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}(B),$$

a vector bundle $\xi = (E, \pi, B, V^r)$ with a linear connection

$$\nabla^L : \mathfrak{X}(B) \times \operatorname{Sec}\xi \to \operatorname{Sec}\xi$$

on it, and a horizontal map

$$\mathcal{H} : \pi^*(\tau_B) \to \tau_E.$$

Then there exists a unique Finsler connection $\nabla : \mathfrak{X}(E) \times \mathfrak{X}(E) \to \mathfrak{X}(E)$ w.r.t $\mathcal{H}$ such that

$$\text{(21)} \qquad \begin{array}{lll} \text{a)} & \nabla_{X^h} Y^h = (\nabla^T_X Y)^h & \\ \text{b)} & \nabla_{X^h} \eta^v = (\nabla^L_X \eta)^v & \begin{array}{c} X, Y \in \mathfrak{X}(B) \\ \sigma, \eta \in \operatorname{Sec}\xi. \end{array} \\ \text{c), d)} & \nabla_{\sigma^v} Y^h = \nabla_{\sigma^v} \eta^v = 0 & \end{array}$$

PROOF. Let us specialize the objects ξ, $\tilde{\xi}$, ξ', $\tilde{\xi}'$, φ, χ, ∇', A' and A playing a role in the considerations of §3 in the four case a) to d) as

	ξ	$\tilde{\xi}$	ξ'	$\tilde{\xi}'$	φ	$\varphi^{\#}$
a)	$H\xi$	$H\xi$	τ_B	τ_B	π_*	h
b)	$V\xi$	$H\xi$	ξ	τ_B	α	v
c)	$V\xi$	$V\xi$	ξ	ξ	α	v
d)	$H\xi$	$V\xi$	τ_B	ξ	π_*	h

	χ	$\chi^{\#}$	∇'	A'	A	∇
a)	π_*	h	∇^T	id	H	$\nabla^{1\#}$
b)	π_*	h	∇^L	id	H	$\nabla^{2\#}$
c)	α	v	–	–	V	$\tilde{\nabla}^3$
d)	α	v	–	–	V	$\tilde{\nabla}^4$

(the roles of B, B' and Ψ are assumed in each case by E, B and π). Then in cases a), b) according to Theorem 2, and in cases c), d) according to Proposition 1, there exist unique pullback pseudoconnections $(\nabla^{1\#}, H)$, $(\nabla^{2\#}, H)$ and pseudoconnections $(\tilde{\nabla}^3, V)$ and $(\tilde{\nabla}^4, V)$ with the following properties:

$$\text{(22)} \qquad \begin{array}{llll} \text{a)} & \nabla^{1\#} & : \mathfrak{X}_H(E) \times \mathfrak{X}_H(E) \to \mathfrak{X}_H(E) & \nabla^{1\#}_{X^h} Y^h = (\nabla^T_X Y)^h \\ \text{b)} & \nabla^{2\#} & : \mathfrak{X}_H(E) \times \mathfrak{X}_V(E) \to \mathfrak{X}_V(E) & \nabla^{2\#}_{X^h} \eta^v = (\nabla^L_X \eta)^v \\ \text{c)} & \tilde{\nabla}^3 & : \mathfrak{X}_V(E) \times \mathfrak{X}_V(E) \to \mathfrak{X}_V(E) & \tilde{\nabla}^3_{\sigma^v} \eta^v = 0 \\ \text{d)} & \tilde{\nabla}^4 & : \mathfrak{X}_V(E) \times \mathfrak{X}_H(E) \to \mathfrak{X}_H(E) & \tilde{\nabla}^4_{\sigma^v} Y^h = 0. \end{array}$$

Hence

$$\nabla^1_U HZ = \nabla^{1\#}_{HU} HZ, \quad \nabla^2_U VZ = \nabla^{2\#}_{HU} VZ,$$
$$\nabla^3_U VZ = \tilde{\nabla}^3_{VU} VZ, \quad \nabla^4_U HZ = \tilde{\nabla}^4_{VU} HZ, \quad U, Z \in \mathfrak{X}(E)$$

together with the mapping H, or V are four pseudoconnections as those of (19). Thus the sum (20) formed from these last pseudoconnections yields a Finsler connection

$$(23) \quad \nabla_U Z = \nabla^1_{HU} HZ + \nabla^2_{HU} VZ + \nabla^3_{VU} VZ + \nabla^4_{VU} HZ, \quad U, Z \in \mathfrak{X}(E).$$

Making use of the listed properties of the pseudoconnections (22) one can easily check that the constructed Finsler connection (23) satisfies the required properties (21) of the theorem.

We show still the uniqueness of this Finsler connection ∇. Suppose namely that ∇' is another Finsler connection with properties (21). Then the difference tensor $D(U,Z) = \nabla_U Z - \nabla'_U Z$ vanishes for any vertical and horizontal lifts: $D(X^h, Y^h) = D(X^h, \sigma^v) = D(\eta^v, Y^h) = D(\sigma^v, \eta^v) = 0$. Hence $D \equiv 0$, and $\nabla_X Y = \nabla'_X Y$. Q.E.D.

The problem of Theorem 3 is dealt with by different method also in [11].

The Finsler connection ∇ defined by (23) is deduced from three independent data ∇^T, ∇^L, $\mathcal{H}$ as given at the beginning of our theorem, and thus this ∇ is a *Finsler connection induced from three sources*. It can be denoted by $\nabla = (\nabla^T, \nabla^L, \mathcal{H})$. Assuming that $\mathcal{H}$ is the horizontal mapping $\mathcal{H}^L$ of ∇^L, or equivalently that $\mathcal{H}$ satisfies the homogeneity condition (see [9] p.311) and hence $\mathcal{H} = \mathcal{H}^L \Longleftrightarrow \nabla^L$, we obtain a *Finsler connection* $(\nabla^T, \mathcal{H}^L)$ *induced by two sources*. If $\xi = \tau_B$, $\nabla^T = \nabla^L$ and $\mathcal{H} = \mathcal{H}^L \Longleftrightarrow \nabla^L$, then $(\nabla^T, \nabla^L, \mathcal{H})$ is the extended Finsler connection ∇' of a ∇-linear Finsler connection $(\mathcal{H}, \overline{\nabla})$ of Matsumoto. Thus $(\nabla^T, \nabla^L, \mathcal{H})$ can be conidered a generalization of this ∇' to a vector bundle Finsler connection.

A simple local calculation gives the

PROPOSITION 2. *In an induced Finsler connection* $\nabla = (\nabla^T, \mathcal{H})$

$$\nabla_{X^h} \eta^v = [X^h, \eta^v].$$

PROOF. It is easy to see that $(\overset{*}{\nabla}, h)$, where

$$\overset{*}{\nabla} : \mathfrak{X}_H(E) \times \mathfrak{X}_V(E) \to \mathfrak{X}_V(E), \quad (X^h, \eta^v) \mapsto V[X^h, \eta^v],$$

is a pseudoconnection. We want to show that

$$\overset{*}{\nabla}_{X^h} \eta^v = (\nabla^L_X \eta)^v,$$

where ∇^L is the linear connection determined by $\mathcal{H}^L$.

Let (x^i, y^μ), $i = 1,2,\dots,n$; $\mu,\nu = 1,2,\dots,r = \operatorname{rank}\xi$ be a local coordinate system on $\pi^{-1}(U)$, $U \subset B$, and $e_\mu \in \operatorname{Sec}\xi|_U$ a base in each $\pi^{-1}(x)$, $x \in U$. Let $N_i^\mu(x,y)$ be the local components of $\mathcal{H}^L$, i.e.

$$\left(\frac{\partial}{\partial x^i}(x)\right)^h = \frac{\partial}{\partial x^i}(x,y) - N_i^\mu \frac{\partial}{\partial y^\mu}(x,y),$$

and $\Gamma^\mu_{i\nu}$ the local components of ∇^L. Then

$$\Gamma^\mu_{i\nu} = \frac{\partial N_i^\mu}{\partial y^\nu}.$$

For $X \in \mathfrak{X}(B)$

$$X|_U = X^i \frac{\partial}{\partial x^i}, \quad X^h|_{\pi^{-1}(U)} = (X^i \circ \pi)\left(\frac{\partial}{\partial x^i} - N_i^\mu \frac{\partial}{\partial y^\mu}\right),$$

for $\eta \in \operatorname{Sec}\xi$, $\eta|_U = \eta^\mu e_\mu$, $\eta^v = (\eta^\mu \circ \pi)\frac{\partial}{\partial y^\mu}$, and

$$[X^h, \eta^v] = \left[(X^i \circ \pi)\left(\frac{\partial}{\partial x^i} - N_i^\mu \frac{\partial}{\partial y^\mu}\right), (\eta^\mu \circ \pi)\frac{\partial}{\partial y^\mu}\right].$$

Computing this and reforming the result we obtain

$$[X^h, \eta^v] = (X^i \circ \pi)\frac{\partial(y^\mu \circ \pi)}{\partial x^i}\frac{\partial}{\partial y^\mu} + (X_i \circ \pi)(\eta^v \circ \pi)\Gamma^\mu_{i\nu}\frac{\partial}{\partial y^\mu} =$$

$$= \left(X^i \frac{\partial y^\mu}{\partial x^i} e_\mu + X^i \eta^\nu \Gamma^\mu_{i\nu} e_\mu\right)^v = (\nabla^L_X \eta)^v,$$

where ∇^L is the linear connection uniquely determined by $\mathcal{H}^L$. The last expression is vertical vector. Thus

$$\overset{*}{\nabla}_{X^h}\eta^v = V[X^h, \eta^v] = [X^h, \eta^v] = (\nabla^L_X \eta)^v. \tag{24}$$

The induced Finsler connection $\nabla = (\nabla^T, \mathcal{H}^L)$ of our Proposition is clearly the same as $\nabla = (\nabla^T, \nabla^L, \mathcal{H}^L)$ of Theorem 3, for ∇^T and $\mathcal{H}^L$ are the same in both connections and ∇^L is determined by $\mathcal{H}^L$. Hence we can apply the statement (21,b) of Theorem 3 which together with (24) yields the statement of the Proposition. Q.E.D.

§5. Yano–Ledger connection on vector bundles

Starting with a linear connection on a manifold M, K. Yano and A. J. Ledger [13] constructed a unique torsion-free linear connection on TM satisfying certain simple conditions. We perform the same replacing TM by the total space E of a vector bundle ξ, and replacing M by the base space B of ξ. Thus our result gives back Yano and Ledger's theorem, provided $\xi = \tau_B$. Moreover, the resulting connection turns out to be an induced vector bundle Finsler connection in case if it is torsion-free, and its symmetrisation otherwise. Our proof is completely different from that of Yano and Ledger's theorem, not only for the difference between the tangent bundle used by Yano and Ledger, and the vector bundle we have considered, but rather because we arrive to our result on a quite different way.

THEOREM 4. *Let $\nabla^T : \mathfrak{X}(B) \times \mathfrak{X}(B) \to \mathfrak{X}(B)$ be a linear connection on the base space B of a vector bundle $\xi = (E, \pi, B, V^r)$, and let*

$$\mathcal{H} : \pi^*(\tau_B) \to \tau_E$$

be a fixed horizontal map satisfying the homogeneity condition [9]. *Then there is exactly one torsion-free linear connection*

$$\overset{\circ}{\nabla} : \mathfrak{X}(E) \times \mathfrak{X}(E) \to \mathfrak{X}(E)$$

satisfying the conditions

$$\overset{\circ}{\nabla}_{\sigma^v} \eta^v = 0, \tag{25}$$

$$\overset{\circ}{\nabla}_{\sigma^v} Y^h = 0, \tag{26}$$

$$\overset{\circ}{\nabla}_{X^h} Y^h = \left(\nabla^T_X Y - \frac{1}{2} \overset{T}{\mathrm{Tor}}(X, Y) \right)^h + \frac{1}{2} V[X^h, Y^h] \tag{27}$$

where $X, Y \in \mathfrak{X}(B)$; $\sigma, \eta \in \mathrm{Sec}\,\xi$; and $\overset{T}{\mathrm{Tor}}$ denotes the torsion of ∇^T.

PROOF. Let $\nabla = (\nabla^T, \mathcal{H})$ be the Finsler connection induced by ∇^T and $\mathcal{H}$ (see §4), and let

$$\overset{\circ}{\nabla}_V Z = \nabla_V Z - \frac{1}{2} \overset{\nabla}{\mathrm{Tor}}(V, Z), \quad V, Z \in \mathfrak{X}(E) \tag{28}$$

Then

$$\overset{\circ}{\nabla}_V Z = \frac{1}{2}(\nabla_V Z + \nabla_Z V + [V, Z]),$$

and $\overset{\circ}{\nabla}$ satisfies (25)–(27). Indeed, in view of (21,d)

$$\overset{\circ}{\nabla}_{\sigma^v} \eta^v = \frac{1}{2}(\nabla_{\sigma^v} \eta^v + \nabla_{\eta^v} \sigma^v + [\sigma^v, \eta^v]) = \frac{1}{2}[\sigma^v, \eta^v].$$

Vanishing of this bracket is a known fact (see (iii) at the end of §1.) Hence $\overset{\circ}{\nabla}_{\sigma^v}\eta^v = 0$. Again, in view of (21,c) and Proposition 2

$$\overset{\circ}{\nabla}_{\sigma^v}Y^h = \frac{1}{2}(\nabla_{Y^h}\sigma^v + [\sigma^v, Y^h]) = \frac{1}{2}([Y^h, \sigma^v] + [\sigma^v, Y^h]) = 0.$$

Finally in view of (21,a)

$$\overset{\circ}{\nabla}_{X^h}Y^h = \frac{1}{2}\left\{(\nabla^T_X Y)^h + (\nabla^T_Y X)^h + [X^h, Y^h]\right\}.$$

Splitting the last term in a vertical and a horizontal part, and taking into account that $H[X^h, Y^h] = [X, Y]^h$, since X^h and Y^h are π-related to X and Y, we obtain

$$\overset{\circ}{\nabla}_{X^h}Y^h = \left\{\frac{1}{2}\nabla^T_X Y + \frac{1}{2}\nabla^T_Y X + \frac{1}{2}[X, Y]\right\}^h + \frac{1}{2}V[X^h, Y^h] =$$

$$= \left(\nabla^T_X Y - \frac{1}{2}\overset{T}{\mathrm{Tor}}(X, Y)\right)^h + \frac{1}{2}V[X^h, Y^h].$$

These prove the existence part of our theorem. A short and straightforward calculation using (28) shows that $\overset{\circ}{\nabla}$ is torsion-free.

In order to prove the unicity we note that from $\overset{\circ}{\mathrm{Tor}}(Y^h, \sigma^v) = 0$ and from (26) we obtain

$$\overset{\circ}{\nabla}_{Y^h}\sigma^v = \overset{\circ}{\nabla}_{\sigma^v}Y^h + [Y^h, \sigma^v] = [Y^h, \sigma^v]. \tag{29}$$

Now suppose that $\overset{*}{\nabla}$ is another torsion-free linear connection satisfying (25)–(27). Then it satisfies (29) too. Hence the difference tensor

$$D(U, V) = \overset{\circ}{\nabla}_U Z - \overset{*}{\nabla}_U Z$$

vanishes for any horizontal and vertical lifts:

$$D(X^h, Y^h) = D(X^h, \sigma^v) = D(\sigma^v, Y^h) = D(\sigma^v, \eta^v) = 0,$$

and thus $D = 0$. Q.E.D.

In case $\xi = \tau_B$, $\nabla^T = \nabla^L$, $\overset{\circ}{\nabla}$ is the Yano–Ledger connection. Thus $\overset{\circ}{\nabla}$ can be called a *Yano–Ledger connection on vector bundles*.

Furthermore, from Proposition 2 and (24) we obtain

COROLLARY 1. $\overset{\circ}{\nabla}_{X^h}\eta^v = (\nabla^L_X \eta)^v$.

Since a horizontal map $\mathcal{H}$ satisfying the homogeneity condition determines a unique linear connection ∇^L, and conversely, so we may denote it by $\mathcal{H}^L$, and we obtain the following

THEOREM 5. *If the induced Finsler connection* $\nabla = (\nabla^T, \nabla^L)$ *has a torsion* $\overset{\nabla}{\mathrm{Tor}}$, *then* $\nabla - \frac{1}{2}\overset{\nabla}{\mathrm{Tor}}$ *is the Yano–Ledger connection induced by the same* ∇^T *and* ∇^L *(or equivalently* $\mathcal{H}^L$*). If the induced Finsler connection* $\nabla = (\nabla^T, \nabla^L)$ *is torsion-free, then* $\nabla = (\nabla^T, \nabla^L)$ *coincides with the Yano–Ledger connection* $\overset{\circ}{\nabla}(\nabla^T, \mathcal{H}^L)$.

This means a generalization of Matsumoto's extended connection ∇' (i.e. of the ∇-linear connection $(\mathcal{H}, \overline{\nabla})$ for TM; [7]) to vector bundle Finsler connection and to vector bundle Yano–Ledger connection.

§6. Metrical Yano–Ledger connection on vector bundles

Let g be a Riemannian metric in τ_B (i.e. on B), $\hat{g}$ a Riemannian metric in $\xi = (E, \pi, B, V^r)$, and $\mathcal{H} : \pi^*(\tau_B) \to \tau_E$ a horizontal map. Then

$$G_p(U_p, Z_p) = g_{\pi(p)}(\pi_* U_p, \pi_* Z_p) + \hat{g}_{\pi(p)}(\alpha(VU_p), \alpha(VZ_p)),$$

$$U, Z \in \mathfrak{X}(E),\ p \in E$$

is a Riemannian metric in τ_E (i.e. on E; [4]). Clearly

$$\begin{cases} G(X^h, Y^h) = g(X, Y) \circ \pi \\ G(\sigma^v, \eta^v) = \hat{g}(\sigma, \eta) \circ \pi \\ G(X^h, \sigma^v) = 0,\ X, Y \in \mathfrak{X}(B);\ \sigma,\ \eta \in \mathrm{Sec}\,\xi. \end{cases} \tag{30}$$

Let ∇^T be the Levi–Civita connection on (B, g), ∇^L a metrical linear connection of $(\xi, \hat{g})$, and ∇ the Levi–Civita connection on (E, G). Let us derive from ∇^T and ∇^L the Yano–Ledger connection $\overset{\circ}{\nabla}$ in τ_E. One can put the question whether or not this Yano–Ledger connection is the Levi–Civita connection in τ_E. The answer depends on the properties of ∇^L and $\hat{g}$.

THEOREM 6. *The Yano–Ledger connection* $\overset{\circ}{\nabla}$ *(induced by* ∇^T *and* ∇^L*) in* τ_E *is the Levi–Civita connection in* τ_E *iff the Riemann–Christoffel curvature tensor* $R(\sigma, \eta, X, Y) := \hat{g}(\hat{R}(X, Y)\eta, \sigma)$, $\sigma, \eta \in \mathrm{Sec}\,\xi$; $X, Y \in \mathfrak{X}(B)$ *of* ∇^L *vanishes (*$\hat{R}$ *is the curvature map of* ∇^L*).*

PROOF. We shall first compute $\overset{\circ}{\nabla} G$ and then our theorem easily follows. Every vector field $U, Z, W \in \mathfrak{X}(E)$ can be written as a finite $C^\infty(E)$-linear combination of vertical and horizontal lifts. Therefore it is sufficient to compute $(\overset{\circ}{\nabla} G)(U, Z, W)$ for vertical and horizontal lifts. Using the explantation

$$(\nabla G)(U, Z, W) \equiv \nabla_U G(Z, W) - G(\nabla_U Z, W) - G(Z, \nabla_U W),$$

properties (30) of G, and properties (25)–(27) , (29) of $\overset{\circ}{\nabla}$ we obtain

$$(\overset{\circ}{\nabla}G)(\kappa^v,\sigma^v,\eta^v)=\kappa^v G(\sigma^v,\eta^v)=\kappa^v(\hat{g}(\sigma,\eta)\circ\pi)=0,$$
$$(\overset{\circ}{\nabla}G)(\sigma^v,X^h,Y^h)=\sigma^v G(X^h,Y^h)=\sigma^v(g(X,Y)\circ\pi)=0,$$
$$(\overset{\circ}{\nabla}G)(\sigma^v,\eta^v,X^h)=0.$$

Making use also of Corollary 1 we get

$$\begin{aligned}(\overset{\circ}{\nabla}G)(X^h,\sigma^v,\eta^v)&=\\ &=X^hG(\sigma^v,\eta^v)-G(\overset{\circ}{\nabla}_{X^h}\sigma^v,\eta^v)-G(\sigma^v,\overset{\circ}{\nabla}_{X^h}\eta^v)=\\ &=X^h(\hat{g}(\sigma^v,\eta^v)\circ\pi)-G((\nabla^L_X\sigma)^v,\eta^v)-G(\sigma^v,(\nabla^L_X\eta)^v)=\\ &=X^h(\hat{g}(\sigma,\eta)\circ\pi)-\hat{g}(\nabla^L_X\sigma,\eta)\circ\pi-\hat{g}(\sigma,\nabla^L_X\eta)\circ\pi=\\ &=X^h(\hat{g}(\sigma,\eta)\circ\pi)-(X\hat{g}(\sigma,\eta))\circ\pi=0,\end{aligned}$$

$$\begin{aligned}(\overset{\circ}{\nabla}G)(X^h,Y^h,Z^h)&=X^hG(Y^h,Z^h)-G(\overset{\circ}{\nabla}_{X^h}Y^h,Z^h)-G(Y^h,\overset{\circ}{\nabla}_{X^h}Z^h)=\\ &=X^h(g(X,Y)\circ\pi)-g(\nabla^T_XY,Z)\circ\pi-g(Y,\nabla^T_XZ)\circ\pi=\\ &=X^h(g(X,Y)\circ\pi)-(Xg(Y,Z))\circ\pi=0\end{aligned}$$

and

$$\begin{aligned}(\overset{\circ}{\nabla}G)(X^h,\sigma^v,Y^h)&=-G(\overset{\circ}{\nabla}_{X^h}\sigma^v,Y^h)-G(\sigma^v,\overset{\circ}{\nabla}_{X^h}Y^h)=\\ &=-G\left(\sigma^v,\frac{1}{2}V[X^h,Y^h]\right)=\frac{1}{2}\hat{g}(\sigma^v,-\alpha\circ V[X^h,Y^h]).\end{aligned}$$

J. Szilasi [10] has shown that $\alpha\circ V[X^h,Y^h]\circ\rho=\tilde{R}(X,Y)(\rho)$, $\rho\in\operatorname{Sec}\xi$. Hence

$$\begin{aligned}(\overset{\circ}{\nabla}G)(X^h,\sigma^vY^h)\circ\rho&=\\ &=-\frac{1}{2}\hat{g}(\sigma,\alpha\circ V[X^h,Y^h]\circ\rho)=\frac{1}{2}\hat{g}(\sigma,\tilde{R}(X,Y)(\rho))=R(\sigma,\rho,X,Y).\end{aligned}$$

Thus $R=0\iff\overset{\circ}{\nabla}G=0$, i.e. $\overset{\circ}{\nabla}$ is the Levi–Civita connection for (E,G). Q.E.D.

As well known, the Nijenhuis torsion of $\mathcal{H}$ is

$$N_{\mathcal{H}}(X,Y)=[HX,HY]+H[X,Y]-H[HX,Y]-H[X,HY].$$

Hence

$$N_{\mathcal{H}}(X^h,Y^h)=[X^h,Y^h]-H[X^h,Y^h]=V[X^h,Y^h].$$

Thus we also get the

COROLLARY 2. *The Yano–Ledger connection in τ_E is the Levi–Civita connection on (E,G) iff the Nijenhuis torsion of $\mathcal{H}$ vanishes, i.e. iff $V[X^h,Y^h]=0$.*

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(Received February 2, 1990)

KOSSUTH LAJOS UNIVERSITY
DEPARTMENT OF MATHEMATICS
H-4010 DEBRECEN, P.O.BOX 12

Acta Math. Hung.
59 (3–4) (1992), 423–438.

WEIGHTED L^p CONVERGENCE OF HERMITE INTERPOLATION OF HIGHER ORDER

P. VÉRTESI* (Budapest) and Y. XU (Austin)

1. Introduction

Let n be a positive integer, and let

$$(1.1)\qquad -1 < x_{nn}(w) < x_{n-1,n}(w) < \cdots < x_{1n}(w) < 1$$

be the zeros of the Jacobi polynomials $p_n(w,x)$, which are orthonormal with respect to the Jacobi weight w defined by

$$(1.2)\qquad w(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta, & |x| \leqq 1, \\ 0, & |x| > 1, \end{cases}$$

where $\alpha > -1$ and $\beta > -1$. For every integer $m \geqq 1$, we consider the Hermite interpolation operator $H_{nm}(w,f)$ which is defined to be the unique polynomial of degree at most $mn-1$ satisfying

$$(1.3)\qquad H_{nm}^{(j)}(w,f,x_{kn}) = f^{(j)}(x_{kn}), \quad 1 \leqq k \leqq n, \quad 0 \leqq j \leqq m-1,$$

where $x_{kn} = x_{kn}(w)$. If w is a nonnegative measurable function and $1 \leqq p <$ $< +\infty$ then weighted L^p norm is defined by

$$(1.4)\qquad \|f\|_{u,p} := \left(\int_{-1}^{1} |f(t)|^p u(t)\,dt \right)^{1/p}.$$

For $0 < p < 1$, of course (1.4) is not a norm anymore, but we keep this notation for convenience. $\|.\|_\infty$ is the usual uniform norm on $[-1,1]$.

The main purpose of this paper is to investigate the convergence properties of $H_{nm}(w,f)$ using the norm (1.4). The interesting and important problem is to determine necessary and sufficient conditions for convergence and for the rate of convergence as well. For $m = 1$ (Lagrange interpolating polynomial), this problem has a long history. It is P. Nevai who proves the final results for even more general weight functions. For results and history,

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* The first author was supported by Hungarian National Foundation for Scientific Research Grant No. 1801.

we refer to [5], [6] and the references given there. For $m = 2$, the closely related Hermite–Fejér interpolating polynomial has been studied by P. Nevai and P. Vértesi [8], P. Vértesi and Y. Xu [14], and A. Máté and P. Nevai [3]. The $H_{n2}(w, f)$ itself is investigated in A. Máté, P. Nevai and Y. Xu [4], where the first order derivative of $H_{n2}(w, f)$ is considered as well. For higher order cases ($m > 2$), only Hermite–Fejér interpolating polynomials have been considered ([10], [12], [13]). In the following, we will present several theorems concerning the necessary and sufficient conditions for the convergence and the rate of convergence of $H_{nm}(w, f)$ and its derivatives. Unlike the cases of Hermite–Fejér interpolating polynomials, $H_{nm}(w, f)$ are projectors, therefore we only need to prove norm inequalities for $H_{nm}(w, f)$ and its derivatives. We list these inequalities as separate theorems.

Throughout this paper, we shall adopt the following convention. The letters $c, c_1, c_2, \dots$ will denote positive constants being independent of variables and indices, unless otherwise indicated. Their value may be different at different occurrences, even within a single formula. $A \sim B$ will mean that $c_1 A \leqq B \leqq c_2 A$. We reserve the letter v for $\sqrt{1-x^2}$, that is

$$v(x) = \sqrt{1-x^2}. \tag{1.5}$$

We define w_m by

$$w_m(x) = \left(w(x)\sqrt{1-x^2}\right)^{m/2}. \tag{1.6}$$

We also define $\|f^{(t)}\|_*$ by

$$\|f^{(t)}\|_* = \max_{1 \leqq k \leqq n} |v(x_{kn})^t f^{(t)}(x_{kn})|, \tag{1.7}$$

and

$$E_n(f) = \min \|f - P_n\|_\infty$$

where the minimum is taken over all polynomials of degree at most n. Finally we define

$$A_m = -\frac{1}{2} - \frac{2}{m}, \quad C_m = -\frac{1}{2} - \frac{1}{m}, \tag{1.8}$$

and

$$\Gamma = \max\{\alpha, \beta\} \quad \text{and} \quad \gamma = \min\{\alpha, \beta\}. \tag{1.9}$$

We shall always assume that w is the weight function with which the interpolating knots are associated and α, β are the parameters of w. Furthermore, we shall assume

$$\gamma \geqq A_m \quad \text{for odd integer} \quad m, \tag{1.10}$$

and
(1.11)

$$\gamma \geqq C_m \quad \text{or} \quad A_m \leqq \alpha, \beta < C_m \quad \text{and} \quad \Gamma - \gamma \leqq \frac{2}{m} \quad \text{for even integer} \quad m,$$

from now on (cf. [12]).

Our main results are the following.

THEOREM 1.1. *Let m be an even integer, $0 < p < +\infty$. Let u be a Jacobi weight function, $uv^{-jp} \in L^1$, for a fixed j, $0 \leqq j \leqq m-2$. Then with $\varepsilon_n > 0$, $\varepsilon \to 0$ $(n \to \infty)$,*
(1.12)

$$\|H_{nm}^{(j)}(w,f)\|_{u,p} \leqq cn^{m-2}\left[\|f\|_\infty + \varepsilon_n \sum_{t=1}^{m-1} \frac{1}{n^{t-1}}\|f^{(t)}\|_\infty\right] \quad \forall f \in C^{m-1}$$

if and only if

$$w_m^{-p} v^{(m-2j-1)p} u \in L^1. \tag{1.13}$$

THEOREM 1.2. *Let $m \geqq 2$ be an integer, $0 < p < +\infty$. Let $\gamma > C_m$ for odd m. Let u be a Jacobi weight function, $uv^{-jp} \in L^1$ for a fixed j, $0 \leqq j \leqq m-2$. Then*

$$\lim_{n\to\infty} \|H_{nm}^{(j)}(w,f) - f^{(j)}\|_{u,p} = 0, \quad \forall f \in C^{m-1} \tag{1.14}$$

if (1.13) holds. Furthermore, if m is even, then (1.14) is true if and only if (1.13) holds.

THEOREM 1.3. *Let $m \geqq 2$ be an integer, $\gamma > C_m$ and $0 < p < +\infty$. Let u be a Jacobi weight function, $uv^{-jp} \in L^1$ for a fixed integer j, $0 \leqq j \leqq m-1$. Then*

$$\|H_{nm}^{(j)}(w,f)\|_{u,p} \leqq cn^j \sum_{t=0}^{m-1} \|f^{(t)}\|_* / n^t, \quad \forall f \in C^{m-1}, \tag{1.15}$$

if

$$w_m^{-p} v^{-jp} u \in L^1. \tag{1.16}$$

Furthermore, if m is even, (1.15) implies

$$w_m^{-p^*} v^{-jp^*} u \in L^1, \quad \forall p^* < p. \tag{1.17}$$

THEOREM 1.4. *Let $m \geqq 2$ be an integer, $\gamma > C_m$ and $0 < p < +\infty$. Let u be a Jacobi weight function, $uv^{-jp} \in L^1$ for a fixed integer j, $0 \leqq j \leqq m-1$. Then*

$$\|H_{nm}^{(j)}(w,f) - f^{(j)}\|_{u,p} \leqq c\,\frac{E_n(f^{(m-1)})}{n^{m-j-1}}, \quad \forall f \in C^{m-1}, \tag{1.18}$$

if (1.16) *holds. Furthermore, if m is even,* (1.18) *implies* (1.17).

REMARK. In Theorems 1.2–1.4 we proved the necessary part only for even m. To handle the case of odd m, we need

$$\sum_{k=1}^{n}(\lambda_{kn}p_{n-1}(x_{kn}))^m \geqq \frac{c}{n^{m-1}}$$

(see (4.6) below) which we are not able to prove at present. In Theorems 1.3 and 1.4, we have weaker necessary conditions. However, if the inequalities

$$\liminf_{n\to\infty} \frac{1}{n^j}\left\|\frac{d^j}{dx^j}\,p_n^m(w)\right\|_{u,p} \geqq c\,\|w_m^{-1}v^{-1}\|_{u,p} \tag{1.19}$$

can be proved, then our theorems will be "if and only if" using (1.16). At present, we can prove (1.19) only for $j = 0$ and 1.

For $m = 1$, see [6]. For $m = 2$, see [3]. In these cases, all the results are proved as "if and only if" statements, and the weight functions are more general. When $j = m-1$, (1.13) and (1.16) are the same, which means that if we have the convergence, we have it with rate. For $j < m-1$, (1.16) is stronger than (1.13).

2. Preliminaries

The Hermite interpolation operator (1.3) can be written as

$$H_{nm}(w,f,x) = \sum_{t=0}^{m-1}\sum_{k=1}^{n} f^{(t)}(x_{kn})h_{tk}(x), \quad m = 1,2,\ldots, \tag{2.1}$$

where the polynomials h_{tk} of degree exactly $mn-1$ are uniquely defined by

$$h_{tk}^{(i)}(x_{jn}) = \delta_{ti}\delta_{kj}. \tag{2.2}$$

Following [13], we have

$$h_{tk}(x) = [\ell_{kn}(x)]^m\frac{(x-x_{kn})^t}{t!}\sum_{i=0}^{m-1-t} e_{tik}(x-x_{kn})^i, \quad 0 \leqq t \leqq m-1, \tag{2.3}$$

where $\ell_{kn}(x)$ are the fundamental polynomials of the Lagrange interpolation. By [11, pp. 48]

$$\ell_{kn}(x) = \frac{\gamma_{n-1}}{\gamma_n} \lambda_{kn} p_{n-1}(w, x_{kn}) \frac{p_n(w,x)}{x - x_{kn}}, \tag{2.4}$$

where $\lambda_n(x) = \lambda_n(w,x)$ is the Christoffel function, $\lambda_{kn} = \lambda_n(x_{kn})$, and $\gamma_n = \gamma_n(w)$ is the leading coefficient of $p_n(x) = p_n(w,x)$. Notice that h_{tk} and e_{tik} also depend on n and m. Furthermore, we have from [13],

$$e_{trk} = e_{0rk}, \quad r = 0,1,\ldots,m-1-t, \quad 0 \leqq t \leqq m-1, \tag{2.5}$$

and

$$|e_{0rk}| \leqq c \begin{cases} \left(\dfrac{n}{\sqrt{1-x_{kn}^2}}\right)^r, & \text{if} \quad r = 0,2,4,\ldots, \\[2ex] \dfrac{n^{r-1}}{\left(\sqrt{1-x_{kn}^2}\right)^{r+1}}, & \text{if} \quad r = 1,3,5\ldots. \end{cases} \tag{2.6}$$

From (2.2), we can easily get that

$$h_{m-1,k}(x) = \frac{(x - x_{kn})^{m-1}}{(m-1)!} [\ell_{kn}(x)]^m. \tag{2.7}$$

For any polynomial P of degree at most $mn-1$, we have

$$P(x) = H_{nm}(w, P, x). \tag{2.8}$$

We collect some useful estimates in the following Lemma (see [13] for references).

LEMMA 2.1. *Let w be a Jacobi weight function, and let $x_{kn} = \cos\vartheta_{kn}$ ($x_{0n} = 1$, $x_{n+1,n} = -1$, $0 \leqq \vartheta_{kn} \leqq \pi$). Then*

$$\vartheta_{k+1,n} - \vartheta_{kn} \sim \frac{1}{n} \quad \textit{uniformly for} \quad 0 \leqq k \leqq n,\ n \in \mathbf{N}, \tag{2.9}$$

$$\lambda_{kn}(w) \sim \frac{1}{n} w(x_{kn})\sqrt{1-x_{kn}^2} \quad \textit{uniformly for} \quad 1 \leqq k \leqq n,\ n \in \mathbf{N}, \tag{2.10}$$

(2.11)
$$|p_{n-1}(w, x_{kn})| \sim w(x_{kn})^{-1/2}(1-x_{kn}^2)^{1/4} \quad \textit{uniformly for} \quad 1 \leqq k \leqq n,\ n \in \mathbf{N},$$

and

$$|p_n(w,x)| \sim \frac{n|\theta - \theta_{jn}|}{\left(w(x_{jn})\sqrt{1-x_{jn}^2}\right)^{1/2}}, \tag{2.12}$$

where j is the index determined by $|x_j - x| = \min_{1 \leqq k \leqq n} |x - x_k|$. Finally,

$$\gamma_n(w) \sim 2^n. \tag{2.13}$$

We also need the weighted Bernstein–Markov inequality (see e.g., [7]).

LEMMA 2.2. *Let $0 < p < +\infty$, and u be a Jacobi weight. Then for each polynomial P_n of degree at most n,*

$$\|P_n' v\|_{u,p} \leqq cn\|P_n\|_{u,p}. \tag{2.14}$$

Finally, we note that the derivative of the Jacobi polynomial $p_n(w, x)$ satisfies the following relation ([11, pp. 63])

$$p_n'(w, x) = \frac{1}{2}(n + \alpha + \beta + 1)\, p_{n-1}(wv^2, x). \tag{2.15}$$

3. Main lemmas

The following Lemmas are critical to our proofs.

LEMMA 3.1. *Let w be a Jacobi weight function. If j is the index determined as in (2.12), then*

$$\sum_{k=1}^{n} \frac{1}{\left(\sqrt{1 - x_{kn}^2}\right)^t} |h_{tk}(x)| \leqq \frac{c}{n^t}\left[1 + \frac{\log n}{n}\, w_m(x_{jn})^{-1}\right], \quad m - t = \textit{even}, \tag{3.1}$$

and

$$\sum_{k=1}^{n} \frac{1}{\left(\sqrt{1 - x_{kn}^2}\right)^t} |h_{tk}(x)| \leqq c\frac{\log n}{n^t}\left[1 + w_m(x_{jn})^{-1}\right], \quad m - t = \textit{odd}. \tag{3.2}$$

PROOF. For $k = j$ (see (2.12)), it can be estimated as in [13]. For $k \neq j$ and $m - t$ even, we have from [13]

$$|h_{tk}(x)| \leqq c|\ell_{kn}^m(x)|\, |x - x_{kn}|^t \left(\frac{n|x - x_{kn}|}{\sqrt{1 - x_{kn}^2}}\right)^{m-t-2} \left(1 + \frac{n|x - x_{kn}|}{1 - x_{kn}^2}\right), \; k \neq j.$$

If $\sum_k'$ means we omit the term $k = j$, then from the above inequality, (2.4) and Lemma 2.1, we have

$$\sum_{k=1}^{n}{}' \frac{1}{\left(\sqrt{1-x_{kn}^2}\right)^t}|h_{tk}(x)| \leqq$$

$$\leqq \frac{c}{n^t}\left[\sum_k{}' \frac{w_m(x_{kn})}{w_m(x_{jn})}\frac{1-x_{kn}^2}{n^2|x_{jn}-x_{kn}|^2} + \sum_k{}' \frac{w_m(x_{kn})}{w_m(x_{jn})}\cdot\frac{1}{n^2|x_{jn}-x_{kn}|}\right],$$

which can be estimated as [13, (3.10)] for $t = 0$. Similarly, the argument holds when $m - t$ is odd (cf. [13, (3.13)]).

REMARK. Our restrictions (1.10) and (1.11) are from this lemma. For the sharpness of the restrictions, we refer to [13].

To state our next lemma, we write for odd $m - t$,

$$Q_{tk}(x) := \frac{1}{t!}e_{t,m-1-t,k}(x-x_{kn})^{m-1}(\ell_{kn}(x))^m \tag{3.3}$$

(cf. (2.3)) and

$$R_{tk}(x) := h_{tk}(x) - Q_{tk}(x). \tag{3.4}$$

LEMMA 3.2. *Let w be a Jacobi weight function. Then if $m - t$ is odd,*

$$\sum_{k=1}^{n}\frac{1}{\left(\sqrt{1-x_{kn}^2}\right)}|R_{tk}(x)| \leqq \frac{c}{n^t}\left[1+\frac{\log n}{n}w_m(x_{jn})^{-1}\right]. \tag{3.5}$$

PROOF. By (2.1), (2.3), (2.5), (2.6) and (3.4), we get

$$|R_{tk}(x)| \leqq c|\ell_{kn}^m(x)|\,|x-x_{kn}|^t\left[1+\frac{|x-x_{kn}|}{1-x_{kn}^2}+\left(\frac{n|x-x_{kn}|}{\sqrt{1-x_{kn}^2}}\right)^2+\dots\right.$$

$$\left.+\left(\frac{n|x-x_{kn}|}{\sqrt{1-x_{kn}^2}}\right)^{m-t-3}+\left(\frac{n|x-x_{kn}|}{\sqrt{1-x_{kn}^2}}\right)^{m-t-3}\frac{|x-x_{kn}|}{1-x_{kn}^2}\right]$$

whence by $n|x-x_{kn}|/\sqrt{1-x_{kn}^2} \geqq c$ $(k \neq j)$

$$|R_{tk}(x)| \leqq c|\ell_{kn}^m(x)|\,|x-x_{kn}|^t\left(\frac{n|x-x_{kn}|}{\sqrt{1-x_{kn}^2}}\right)^{m-t-3}\left(1+\frac{|x-x_{kn}|}{1-x_{kn}^2}\right),\ k \neq j.$$

Therefore the desired estimate follows from the proof of Lemma 3.1.

The following Lemma is a partial case of a more general theorem proved in [6, Theorem 1].

LEMMA 3.3. *Let w and u be Jacobi weight functions. If $0 < p < +\infty$, $w_m^{-p}v^{\lambda p}u \in L^1$, $\lambda \geqq 0$ and $\gamma > C_m$, then for every bounded function f in $[-1,1]$*

$$\|L_n(w,F)w_{m-1}^{-1}v^{\lambda}\|_{u,p} \leqq c\|f\|_*$$

where $L_n(w) = H_{n1}(w)$ and $F = fw_{m-1}$.

We now construct a special Hermite spline function s_n, which will help us to get the necessary conditions in our theorems. For each given integer n, s_n is a polynomial of degree at most $2m-1$ on each $[x_{k+1,n}, x_{kn}]$ and it is uniquely determined by

$$\text{(3.6)} \qquad \begin{cases} s_n^{(j)}(x_{kn}) = 0, & 0 \leqq j \leqq m-2, \quad 0 \leqq k \leqq n+1, \\ s_n^{(m-1)}(x_{kn}) = 1, & 0 \leqq k \leqq n+1, \quad (m \geqq 2). \end{cases}$$

It is easy to verify the following explicit formula for s_n on $[x_{k+1,n}, x_{kn}]$:

$$s_n(x) = \frac{(x_{k+1,n} - x_{kn})^{m-2}}{(m-1)!}\left(\frac{x - x_{kn}}{x_{k+1,n} - x_{kn}}\right)^{m-1}\left(\frac{x_{k+1,n} - x}{x_{k+1,n} - x_{kn}}\right)^{m-1} \times$$

$$\times\left[(-1)^{m-1}(x - x_{kn}) + (x_{k+1,n} - x)\right],$$

whence

$$\text{(3.7)} \qquad \|s_n^{(t)}\|_\infty \leqq c/n^{m-1-t}, \quad 0 \leqq t \leqq m-1.$$

By definition, s_n is a function in C^{m-1}.

4. Proofs of Theorems 1.1 and 1.3

We start with the following special case of Theorem 1.1.

LEMMA 4.1. *Let $m \geqq 2$ be an even integer, u be an integrable Jacobi weight function. Then for each $0 \leqq \lambda \leqq m-1$*

$$\text{(4.1)} \qquad \|H_{nm}(w,f)\|_{u,p} \leqq cn^{\lambda}\left[\|f\|_* + \varepsilon\sum_{t=1}^{m-1}\frac{1}{n^{t-1}}\|f^{(t)}\|_*\right]$$

where $\varepsilon_n \to 0$ as $n \to \infty$, if

$$\text{(4.2)} \qquad w_m^{-p}v^{(\lambda+1)p}u \in L^1.$$

PROOF. By Theorem 6.3.14 in [5, p.113], for every $0 < p < +\infty$, and Jacobi weight u, there exists a constant $\sigma = \sigma(p,u) > 0$ such that for every polynomial P of degree at most mn

$$\int_{-1}^{1} |P(t)|^p u(t)\,dt \leqq 2 \int_{-1+\sigma n^{-2}}^{1-\sigma n^{2}} |P(t)|^p u(t)\,dt. \tag{4.3}$$

Applying (4.3) with $P = H_{nm}(w,f)$ we have

$$\|H_{nm}(w,f)\|_{u,p}^p \leqq$$

$$\leqq c \sum_{t=0}^{m-1} \|f^{(t)}\|_*^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left(\sum_{k=1}^{n} \frac{1}{\left(\sqrt{1-x_{kn}^2}\right)^t} |h_{tk}(x)| \right)^p u(x)\,dx,$$

where by Lemma 3.1, we get for $m-t$ =even

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left(\sum_{k=1}^{n} \frac{1}{\left(\sqrt{1-x_{kn}^2}\right)^t} |h_{tk}(x)| \right)^p u(x)\,dx \leq$$

$$\leqq c \left[\frac{1}{n^{tp}} + \left(\frac{\log n}{n^{t+1}} \right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} w_m(x)^{-p} u(x)\,dx \right] \leq$$

$$\leqq c n^{\lambda p} \left[\frac{1}{n^{(t+\lambda)p}} + \left(\frac{\log n}{n^{t+1}} \right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} w_m(x)^{-p} \left(\sqrt{1-x^2}\right)^{\lambda p} u(x)\,dx \right]$$

by $\frac{1}{n} \leqq \sqrt{1-x^2}$. By (4.2), there exists a $0<\delta<1$, such that $w_m^{-p} v^{(\lambda+1-\delta)p} u \in$ $\in L^1$. By the triangular inequality and $(n\sqrt{1-x^2})^{1-\delta} \geqq c$,

$$\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} \left(\sum_{k=1}^{n} \frac{1}{\left(\sqrt{1-x_{kn}^2}\right)^t} |h_{tk}(x)| \right)^p u(x)\,dx \leq$$

$$\leqq c n^{\lambda p} \left[\frac{1}{n^{(t+\lambda)p}} + \left(\frac{\log n}{n^{t+\delta}} \right)^p \int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}} w_m^{-p} v^{(\lambda+1-\delta)p} u(x)\,dx \right]$$

$$\begin{cases} \leqq cn^{\lambda p}\left[\dfrac{1}{n^{(t+\lambda)p}}+\left(\dfrac{\log n}{n^{t+\delta}}\right)^p\right]\leqq cn^{\lambda p} & \text{if} \quad t=0,\\ := cn^{\lambda p}\left(\dfrac{1}{n^{t-1}}\right)\varepsilon_n^p & \text{if} \quad t\geqq 1\end{cases}$$

where $\varepsilon_n^p=\frac{1}{n^{(\lambda+1)p}}+\left(\frac{\log n}{n^{1+\delta}}\right)^p$. Similarly for $m-t$ =odd we have

$$\int\limits_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}\left(\sum_{k=1}^{n}\frac{1}{\left(\sqrt{1-x_{kn}^2}\right)^t}|h_{tk}(x)|\right)^p u(x)\,dx\leqq$$

$$\leqq cn^{\lambda p}\left(\frac{1}{n^{t-1}}\right)^p\left[\left(\frac{\log n}{n^{(t+\lambda)p}}\right)^p+\left(\frac{\log n}{n^{t+1}}\right)^p\right]:=cn^{\lambda p}\left(\frac{1}{n^{t-1}}\right)^p\varepsilon_n^p,$$

which completes the proof.

PROOF OF THEOREM 1.1. By Bernstein–Markov inequality (Lemma 2.2),

$$\|H_{nm}^{(j)}(w,f)\|_{u,p}\leqq cn^j\|H_{nm}(w,f)/v^j\|_{u,p}.$$

Suppose $0\leqq j\leqq m-2$. Then we can apply Lemma 4.1 with $\lambda=m-2-j$ in (4.1), and we get

$$\|H_{nm}^{(j)}(w,f)\|_{u,p}\leqq cn^{m-2}\left[\|f\|_\infty+\varepsilon_n\sum_{t=1}^{m-1}\frac{1}{n^{t-1}}\|f^{(t)}\|_\infty\right],$$

where (4.2) with weight function $v^{-jp}u$ instead of u gives

$$w_m^{-p}v^{(m-j-1)p}v^{-jp}u\in L^1,$$

which is exactly (1.12). This proves the sufficiency of (1.13) for $0\leqq j\leqq m-2$.

For the necessary part, we use our spline function s_n. Let $f_1(x)=x$. By (2.1) and (3.6)

$$xH_{nm}(s_n,x)-H_{nm}(f_1s_n,x)=\sum_{k=1}^{n}(x-x_{kn})h_{m-1,k}(x)=$$

$$=\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^n\sum_{k=1}^{n}(\lambda_{kn}p_{n-1}(x_{kn}))^m\,p_n(x)^m,$$

where $p_n(x)=p_n(w,x)$, and the second equality follows from (2.7) and (2.4). Taking derivatives, we then have

$$jH_{nm}^{(j-1)}(s_n,x)+xH_{nm}^{(j)}(s_n,x)-H_{nm}^{(j)}(f_1s_n,x)= \tag{4.4}$$

$$=\frac{d^j}{dx^j}(p_n^m(x))\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^m\sum_{k=1}^{n}(\lambda_{kn}p_{n-1}(x_{kn}))^m.$$

Now by (1.12) and (3.7)

$$\|jH_{nm}^{(j-1)}(s_n)+f_1H_{nm}^{(j)}(s_n)-H_{nm}^{(j)}(f_1s_n)\|_{u,p}\leqq \tag{4.5}$$
$$\leqq cn^{m-2}\left[\frac{1}{n^{m-1}}+\varepsilon_n\sum_{t=1}^{m-1}\frac{1}{n^{t-1}}\cdot\frac{1}{n^{m-1-t}}\right]\leqq c\varepsilon_n\to 0,$$

and by Lemma 2.1, it is easy to see that

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^m\sum_{k=1}^{n}(\lambda_{kn}p_{n-1}(x_{kn}))^m\geqq\frac{c}{n^{m-1}}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we then have

$$\frac{1}{n^{m-1}}\left\|\frac{d^j}{dx^j}(p_n^m(x))\right\|_{u,p}\leqq c\varepsilon_n\to 0. \tag{4.7}$$

Since the derivative of $p_n(w,x)$ is also a Jacobi polynomial with weight function $w(x)(1-x^2)$ (see (2.15)), and for some positive integers c_k depending on m and j,

$$\frac{d^j}{dx^j}(p_n^m(x))=\sum_{\substack{\lambda_1+\dots+\lambda_k=m\\ \alpha_1\lambda_1+\dots+\alpha_k\lambda_k=j}}c_k\left(\frac{d^{\alpha_1}p_n}{dx^{\alpha_1}}\right)^{\lambda_1}\dots\left(\frac{d^{\alpha_k}p_n}{dx^{\alpha_k}}\right)^{\lambda_k}, \tag{4.8}$$

we can conclude from (2.12) that

$$\left\|\frac{d^j}{dx^j}(p_n^m(x))\right\|_{u,p}^p\geqq\int_{(1+x_{1n})/2}^{1}\left|\sum c_k\prod_{i=1}^{k}\left(p_n^{(\alpha_i)}(x)\right)^{\lambda_i}\right|^p u(x)\,dx\sim \tag{4.9}$$
$$\sim\left[\prod_{\substack{\lambda_1+\dots+\lambda_k=m\\ \alpha_1\lambda_1+\dots+\alpha_k\lambda_k=j}}\left(\sqrt{n}w(1-n^{-2})^{-1/2}n^{\alpha_i}\right)^{\lambda_i}n^{\alpha_i\lambda_i}\right]^p\int_{(1+x_{in})/2}^{1}u(x)\,dx\sim$$
$$\sim n^{\left(\frac{m}{2}+2j\right)p}w(1-n^{-2})^{-\frac{m}{2}p}\int_{(1+x_{1n})/2}^{1}u(x)\,dx,$$

since every Jacobi polynomial is positive when $x>x_{1n}$. If $w(x)=$ $=(1-x)^\alpha(1+x)^\beta$, $u(x)=(1-x)^a(1+x)^b$, then (4.7) and (4.9) is equivalent to

$$-(m-1)p+\frac{mp}{2}+2jp+\alpha mp-2a-2<0$$

or

$$-\left(\alpha+\frac{1}{2}\right)\frac{mp}{2}+(m-1-2j)\frac{p}{2}+a>-1$$

which means that $w_m^{-p}v^{(m-2j-1)p}u$ is integrable on $[0,1]$. The interval $[-1,0]$ can be handled similarly. Therefore the proof is complete.

LEMMA 4.2. *Let w and u be integrable Jacobi weight functions. Then for each* $0 \leqq m-1$ *and* $0 \leqq \lambda \leqq m-1$

$$\|H_{nm}(w,f)\|_{u,p} \leqq cn^{\lambda}\sum_{t=0}^{m-1}\frac{1}{n^t}\|f^{(t)}\|_* \tag{4.10}$$

if

$$w_m^{-p}v^{\lambda p}u \in L^1 \quad \textit{and} \quad \gamma > C_m. \tag{4.11}$$

PROOF. By (2.1, (3.3) and (3.4), we have

$$H_{nm}(w,f)=\sum_{t=0}^{m-1}\sum_{k=1}^{n}f^{(t)}(x_{kn})h^{tk}(x)=\left(\sum_{m-t=\text{even}}\sum_{k=1}^{n}f^{(t)}(x_{kn})h_{tk}(x)+\right.$$

$$\left.+\sum_{m-t=\text{odd}}\sum_{k=1}^{n}f^{(t)}(x_{kn})R_{tk}(x)\right)+\sum_{m-t=\text{odd}}\sum_{k=1}^{n}f^{(t)}(x_{kn})Q_{tk}(x):=\Sigma_1+\Sigma_2.$$

Applying (4.3) with $P=\Sigma_1$, we have by Lemma 3.1, Lemma 3.2 and $1/n \leqq \leqq c\sqrt{1-x^2}$ that

$$\|\Sigma_1\|_{u,p} \leqq \tag{4.12}$$

$$\leqq c\sum_t\|f^{(t)}\|_*\left\{\int_{-1+\sigma n^{-2}}^{1-\sigma n^{-2}}\left\{\frac{1}{n^t}\left[1+\frac{\log n}{n}w_m(x)^{-1}\right]\right\}^p u(x)\,dx\right\}^{1/p} \leqq$$

$$\leqq cn^{\lambda}\sum_t\frac{\|f^{(t)}\|_*}{n^t}\left[\frac{1}{n^{\lambda}}+\|w_m^{-1}v^{\lambda}\|_{u,p}\right]\leqq cn^{\lambda}\sum_t\frac{\|f^{(t)}\|_*}{n^t}.$$

By (2.4) and (3.3), we have for $m-t=$odd

$$S_t:=\sum_{k=1}^{n}f^{(t)}(x_{kn})Q_{tk}(x)= \tag{4.13}$$

$$=\sum_{k=1}^{n}f^{(t)}(x_{kn})\frac{1}{t!}e_{t,m-1-t,k}(x-x_{kn})^{m-1}(\ell_{kn}(x))^m=$$

$$=\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{m-1}p_n(x)^{m-1}\sum_{k=1}^{n}f^{(t)}(x_{kn})\frac{1}{t!}e_{t,m-t-1,k}\ell_{kn}(x)\left[\lambda_{kn}p_{n-1}(x_{kn})\right]^{m-1}.$$

Since $m-t$ is odd, by (2.6) and Lemma 2.1,

$$\left|\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{m-1}\frac{1}{t!}e_{t,m-t-1,k}\left[\lambda_{kn}p_{n-1}(x_{kn})\right]^{m-1}\right|\leqq$$

$$\leqq c\left(\frac{n}{\sqrt{1-x_{kn}^2}}\right)^{m-t-1}\frac{1}{n^{m-1}}(1-x_{kn}^2)^{\frac{3}{4}(m-1)}w(x_{kn})^{\frac{m-1}{2}}=$$

$$=\frac{c}{n^t}w_{m-1}(x_{kn})\left(\sqrt{1-x_{kn}^2}\right)^t.$$

We now write

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{m-1}f^{(t)}(x_{kn})\frac{1}{t!}e_{t,m-t-1,k}\left[\lambda_{kn}p_{n-1}(x_{kn})\right]^{m-1}= \tag{4.14}$$
$$=c_{kn}f^{(t)}(x_{kn})w_{m-1}(x_{kn})v(x_{kn})^t/n^t:=g_n(x_{kn})w_{m-1}(x_{kn}),$$

where by defining $c_n(x)$ to be the uniformly bounded function satisfying $c_n(x_{kn})=c_{kn}$, we have

$$\|g\|_*=\|c_nf^{(t)}/n^t\|_*\leqq c\|f^{(t)}\|_*/n^t. \tag{4.15}$$

Using (4.14) we can rewrite (4.13) as

$$S_t=(p_n(x))^{m-1}L_n(w,g_nw_{m-1},x).$$

Therefore by (4.3), (2.12) and $1/n\leqq c\sqrt{1-x_{kn}^2}$, we have

$$\|S_t\|_{u,p}\leqq cn^\lambda\|L_n(w,g_nw_{m-1})w_{m-1}^{-1}v^\lambda\|_{u,p}\leqq cn^\lambda\|f^{(t)}\|_*/n^t$$

where the last inequality follows from Lemma 3.3 and (4.15). Therefore,

$$\|\Sigma_2\|_{u,p}\leqq cn^\lambda\sum_t\|f^{(t)}\|_*/n^t.$$

This and (4.12) complete the proof of this lemma.

Proof of Theorem 1.3. By the Bernstein–Markov inequality (2.14),

$$\|H_{nm}^{(j)}(w,f)\|_{u,p}\leqq cn^j\|H_{nm}(w,f)/v^j\|_{u,p}.$$

Applying Lemma 4.2 with $\lambda=0$ and u/v^{jp} in place of u, we get (1.15) under the conditions (1.16). Now we prove that (1.17) are necessary for $m=$ even.

Let s_n be the spline function defined in (3.6). Then by (4.4) and (4.6), we get
(4.16)

$$|jH_{nm}^{(j-1)}(s_n,x)+xH_{nm}^{(j)}(s_n,x)-H_{nm}^{(j)}(f_1 s_n,x)| \geqq \frac{c}{n^{m-1}}\left|\frac{d^j}{dx^j}(p_n^m(x))\right|.$$

Since $\|H_{nm}^{(j-1)}(s_n)\|_{u,p} \leqq c\|H_{nm}^{(j)}(s_n)\|_{u,p}$, by (1.15) and (3.7) we get

$$\|jH_{nm}^{(j-1)}(s_n)+f_1H_{nm}^{(j)}(s_n)-H_{nm}^{(j)}(s_n)\|_{u,p} \leqq cn^j\sum_{t=0}^{m-1}\frac{1}{n^{m-1-t}}\cdot\frac{1}{n^t} \leqq c\,\frac{1}{n^{m-1-j}}.$$

Then by (4.16),

$$\frac{1}{n^j}\left\|\frac{d^j}{dx^j}(p_n^m)\right\|_{u,p} \leqq c. \tag{4.17}$$

If (1.19) were true, then we have from (4.17) that (1.16) is a necessary condition. Since we do not know how to prove (1.19) at this time, we can only conclude from (4.17) a weaker result, (1.17). Its proof is similar to that of Theorem 1.1.

5. Proof of Theorems 1.2 and 1.4

We need the following result by Leviatan [2, Corollary 1].

LEMMA 5.1. *Let $f\in C^{m-1}$ and P_n be the n^{th} degree polynomial of best approximation to f. Then*

$$|f^{(j)}(x)-P_n^{(j)}(x)| \leqq c\,\frac{1}{\Delta_n(x)^j}\,\frac{1}{n^{m-1}}E_n\left(f^{(m-1)}\right), \tag{5.1}$$

where $\Delta_n(x)=\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}$.

From (5.1), we have

$$\|f^{(j)}-P_n^{(j)}\|_* \leqq cE_n\left(f^{(m-1)}\right)/n^{m-j-1}. \tag{5.2}$$

PROOF OF THEOREM 1.2. Suppose (1.13) is true. By (2.8) we have

$$\|H_{nm}^{(j)}(w,f)-f^{(j)}\|_{u,p} \leqq c\|f^{(j)}-P_n^{(j)}\|_{u,p}+c\|H_{nm}^{(j)}(w,f-P_n)\|_{u,p}. \tag{5.3}$$

If m is odd, then by Lemma 2.2 and Lemma 4.2 with $\lambda=m-j-1$,

$$\|H_{nm}^{(j)}(w,f-P_n)\|_{u,p} \leqq cn^j\|H_{nm}^{(j)}(w,f-P_n)/v^j\|_{u,p} \leqq$$
$$\leqq cn^{m-1}\sum_{t=0}^{m-1}\|f^{(t)}-P_n^{(t)}\|_*/n^t \leqq cE_n(f^{(m-1)}),$$

where the last inequality follows from (5.2). Similarly, if m is even, then by Lemma 2.2 and Lemma 4.1 with $\lambda = m - j - 2$,

$$\|H_{nm}^{(j)}(w, f - P_n)\|_{u,p} \leqq cn^{m-2}\left[\|f - P_n\|_* + \varepsilon_n \sum_{t=1}^{m-1} \frac{1}{n^{t-1}}\|f - P_n\|_*\right] \leqq$$

$$\leqq c\left[\frac{E_n(f^{(m-1)})}{n} + \varepsilon_n E_n(f^{(m-1)})\right].$$

By (5.3) and the definition of P_n, we proved (1.14). On the other hand, if (1.14) is true and m is even, then it follows from

$$\|H_{nm}^{(j)}(w, f)\|_{u,p} \leqq c\|H_{nm}^{(j)}(w, f) - f^{(j)}\|_{u,p} + c\|f^{(j)}\|_{u,p} \tag{5.4}$$

and (3.7) that for $0 \leqq j \leqq m - 2$,

$$\|jH_{nm}^{(j-1)}(w, s_n) + f_1 H_{nm}^{(j)}(w, s_n) - H_{nm}^{(j)}(w, f_1 s_n)\|_{u,p} \leqq$$

$$\leqq c\left(\frac{1}{n^{m-j-1}} + \|H_{nm}^{(j)}(w, s_n) - s_n^{(j)}\|_{u,p} + \|H_{nm}^{(j)}(w,_1 s_n f) - (f_1 s_n)^{(j)}\|_{u,p}\right) \to 0.$$

For $j = m - 1$, we have from (1.14) and (5.4) that

$$\|(m-1)H_{nm}^{(m-2)} + f_1 H_{nm}^{(m-1)}(w, s_n) - H_{nm}^{(m-1)}(w, f_1, s_n)\|_{u,p} \leqq c.$$

The rest of the proof follows similarly as the proof of Theorem 1.1.

PROOF OF THEOREM 1.4. Suppose (1.16) holds. By Theorem 1.3 and (5.2), we get by Lemma 2.2 that

$$\|H_{nm}^{(j)}(w, f - P_n)\|_{u,p} \leqq cn^j\|H_{nm}(w, f - P_n)/v^j\|_{u,p} \leqq$$

$$\leqq cn^j \sum_{t=0}^{m-1} \frac{1}{n^t}\frac{1}{n^{m-1-t}} E_n\left(f^{(m-1)}\right) = cE_n\left(f^{(m-1)}\right)/n^{m-1-j}.$$

By a well known property of the best approximation polynomial (cf. [9, p. 23]),

$$\|f^{(j)} - P_n^{(j)}\|_\infty \leqq cE_n\left(f^{(m-1)}\right)/n^{m-j-1},$$

(1.18) follows from (5.3). On the other hand, from (1.18) we have

$$\|H_{nm}^{(j)}(w, f)\|_{u,p} \leqq c\|f^{(j)}\|_\infty + c\|H_{nm}^{(j)}(w, f) - f^{(j)}\|_{u,p} \leqq$$

$$\leqq c\|f^{(j)}\|_\infty + cE_n\left(f^{(m-1)}\right)/n^{m-j-1} \leqq$$

$$\leqq cn^j\left[\|f^{(j)}\|_\infty/n^j + \|f^{(m-1)}\|_\infty/n^{m-1}\right].$$

Notice that in the proof of the necessity part of Theorem 1.3, we only need (1.5) with $\|f\|$ in place of $\|f\|_*$. Therefore, (1.17) follows from Theorem 1.3.

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(Received February 8, 1990)

MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, P.O.BOX 127, HUNGARY 1364

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TEXAS 78712
USA

Acta Math. Hung.
59 (3–4) (1992), 439–447.

ON THE NUMBER OF SUMS AND DIFFERENCES

I. Z. RUZSA[1] (Budapest)

1. Introduction

Let A be a set of integers. Write

$$A+A=\{a_1+a_2 : a_1,a_2\in A\},$$
$$A-A=\{a_1-a_2 : a_1,a_2\in A\}.$$

If $|A|=n$, then we have

$$2n-1\leqq |A+A|\leqq \frac{n^2+n}{2},$$
$$2n-1\leqq |A-A|\leqq n^2-n+1,$$

where there is equality on the left side for arithmetical progressions and on the right side for "generic" sets, in which there is no nontrivial coincidence between sums or differences.

Observe that the conditions of genericity for sums or differences are equivalent: a nontrivial coincidence of sums, say $a+b=a'+b'$ implies a nontrivial coincidence of differences, namely $a-a'=b'-b$, and vice versa.

We shall show the counterintuitive fact that "almost" versions of these conditions are far from equivalent; almost all sums may be different while almost all differences are represented multiply, and conversely.

THEOREM. *For every $n>n_0$ there is a set A such that*

$$|A|=n,\quad |A+A|\leqq n^{2-c} \tag{1.1}$$

but

$$|A-A|\geqq n^2-n^{2-c}, \tag{1.2}$$

where c is a positive absolute constant. Also there is a set B such that

$$|B|=n,\quad |B-B|\leqq n^{2-c}, \tag{1.3}$$

$$|B+B|\geqq \frac{n^2}{2}-n^{2-c}. \tag{1.4}$$

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[1] Supported by DIMACS (Center for Discrete Mathematics and Theoretical Computer Science), a National Science Foundation Science and Technology Center — NSF-STC88-09648, and by Hungarian National Foundation for Scientific Research, Grant No. 1811.

We do not compute numerical values for c. If we did, we would get a much smaller value for (1.3–4) than for (1.1–2). I do not know whether this is necessary.

Neither c can be larger than 1/2. This follows from the inequalities

$$|A-A|^{3/4} \leqq |A+A| \leqq |A-A|^{4/3},$$

see Freiman and Pigaev [1], Ruzsa [2, 3, 4].

The idea of the proof is the following. Take a set U such that

$$D = |U-U| > S = |U+U| \tag{1.5}$$

and form the set

$$V = U^k \subset \mathbf{Z}^k, \quad V = \{(u_1, \dots, u_k) : u_1, \dots, u_k \in U\}.$$

This set has $|V+V| = S^k$, $|V-V| = D^k$, thus $V-V$ is much larger than $V+V$. Now consider a random subset X of V. Since there is much more room for differences than for sums, with some luck the sums already start to coincide while the differences do not. (Actually in the course of the proof we shall find that it is not (1.5) what really counts but a more complicated inequality.)

We got a set of integer vectors. If a set of integers is required, we can map it into $\mathbf{Z}$ by

$$(x_1, \dots, x_k) \to x_1 + mx_2 + \dots + m^{k-1}x_k,$$

say. If m is sufficiently large, then this mapping does not change the number of sums and differences of our set.

2. Random subsets

Let V b a finite set (of integers, or vectors, or elements in any Abelian group), $|V| = m$. We take a random subset X of V by selecting elements of V into X independently, with probability $0 < \varrho < 1$. Then we have

$$\mathbf{E}|X| = \varrho m,$$

where $\mathbf{E}$ denotes expectation, and almost certainly $|X|$ will be near to ϱm. For instance, Chebyshev's inequality says that

$$\mathbf{P}(\||X| - \varrho m\| \geqq K) \leqq \frac{\varrho(1-\varrho)m}{K^2}, \tag{2.1}$$

thus this probability is small as soon as $K/\sqrt{\varrho m} \to \infty$. If necessary, much stronger inequalities are available (Bernstein's inequality); for our purposes (2.1) is sufficient.

We also need estimates for $|X \pm X|$. Write

$$D = \mathbf{E}|X - X|, \quad S = \mathbf{E}|X + X|.$$

(We shall not deduce that these cardinalities are often near to their expectations, though this seems to be possible.)

Let $\sigma(n)$ denote the number of solutions of $n = u + v$, $u, v \in V$, similarly $\delta(n)$ the number of solutions of $n = u - v$, and for a number k and set V write

$$kV = \{kv : v \in V\}.$$

2.1. LEMMA. *We have*

$$\mathbf{P}(n \notin X + X) = \begin{cases} (1-\varrho^2)^{\sigma(n)/2} & \text{for} \quad n \notin 2V, \\ (1-\varrho)(1-\varrho^2)^{(\sigma(n)-1)/2} & \text{for} \quad n \in 2V. \end{cases} \tag{2.2}$$

PROOF. The $\sigma(n)$ representations of n in the form $n = u + v$ can be groupped into $\sigma(n)/2$ pairs by combining $u + v$ and $v + u$, or into $(\sigma(n) - 1)/2)$ such pairs and a single representation in the form $u + u$ if $n \in 2V$. These representations depend on disjoint pairs, thus they induce independent events, and in (2.2) we have the product of the corresponding probabilities. □

The case of the differences is somewhat trickier, because the pairs (u, v) with $u - v = n$ are not necessarily disjoint. We start with some preparation.

2.2. LEMMA. *Take a random subset of* $\{0, 1, 2, \ldots, n\}$ *by selecting each number independently with probability* ϱ. *Let* β_n *be the probability that no two consecutive integers are selected. We have*

$$\left(1-\varrho^2\right)^n \leqq \beta_n \leqq \left(1-\varrho^2+\varrho^3\right)^n. \tag{2.4}$$

PROOF. We have $\beta_0 = 1$, $\beta_1 = 1 - \varrho^2$ and the recurrence relation

$$\beta_n = (1-\varrho)\beta_{n-1} + \varrho(1-\varrho)\beta_{n-2} \quad (n \geqq 2).$$

These yield (2.4) by an easy induction. □

2.3. REMARK. We could write the exact solution of this recurrence, but for the following calculations we need bounds in this exponential form. The lower bound we gave is exact, with equality at $n = 1$, while the best upper bound of the form γ^n is given by

$$\gamma = \frac{1 - \varrho + \sqrt{(1-\varrho)(1+3\varrho)}}{2} = 1 - \varrho^2 + \varrho^3 + O(\varrho^4).$$

2.4. Lemma. *We have*

$$(1-\varrho^2)^{\delta(n)} \leqq \mathbf{P}(n \notin X-X) \leqq (1-\varrho^2+\varrho^3)^{\delta(n)}. \tag{2.5}$$

Proof. Consider all maximal arithmetical progressions of difference n in V:

$$\begin{gathered} a_1, a_1+n, \ldots, a_1+b_1 n; \\ \vdots \\ a_t, a_t+n, \ldots, a_t+b_t n. \end{gathered}$$

These are disjoint, and a difference n appears in X if and only if it contains two consecutive terms from some of them, consequently

$$\mathbf{P}(n \notin X-X) = \beta_{b_1}\beta_{b_2}\ldots\beta_{b_t}.$$

Now (2.5) follows from the previous lemma and the observation

$$b_1+b_2+\cdots+b_t = \delta(n). \quad \square$$

2.5. Lemma. *We have*

$$\sum_n \left(1-(1-\varrho^2)^{\sigma(n)/2}\right) - \varrho m \leqq S \leqq \sum_n \left(1-(1-\varrho^2)^{\sigma(n)/2}\right) \tag{2.6}$$

and

$$\sum_n \left(1-(1-\varrho^2)^{\delta(n)}\right) \leqq D \leqq \sum_n \left(1-(1-\varrho^2+\varrho^3)^{\delta(n)}\right). \tag{2.7}$$

Proof. To obtain (2.6), we start from

$$S = \sum_n \mathbf{P}(n \in X+X).$$

To each term we apply (2.2), and for $n \in 2V$ we also use the inequality

$$(1-\varrho^2)^{\sigma(n)/2} - \varrho \leqq (1-\varrho^2)^{(\sigma(n)-1)/2}(1-\varrho) \leqq (1-\varrho^2)^{\sigma(n)/2}.$$

(2.7) follows similarly from (2.5). $\square$

The exponential sums appearing in (2.6–7) are difficult to calculate; we estimate them by powers.

2.6. LEMMA. *For every* $0 \leqq x \leqq 1$, $y \geqq 0$ *integer and* $0 < \varepsilon < 1$ *we have*

$$1-(1-x)^y \leqq (xy)^{1-\varepsilon}, \tag{2.8}$$

$$1-(1-x)^y \geqq xy-(xy)^{1+\varepsilon}. \tag{2.9}$$

PROOF. To get (2.8), observe that

$$1-(1-x)^y \leqq \min(1, xy) \leqq (xy)^{1-\varepsilon}.$$

To prove (2.9), recall that

$$e^{-t} \leqq 1-t+t^2/2$$

for $t \geqq 0$, hence

$$(1-x)^y \leqq e^{-xy} \leqq 1-xy+\frac{(xy)^2}{2},$$

thus

$$1-(1-x)^y \geqq xy-\frac{(xy)^2}{2} \geqq xy-\frac{(xy)^{1+\varepsilon}}{2} \geqq xy-(xy)^{1+\varepsilon}$$

if $xy \leqq 1$. In case $xy > 1$, the left side of (2.9) is positive, while the right side is negative. □

Write

$$\Sigma(t)=\sum \sigma(n)^t, \quad \Delta(t)=\sum \delta(n)^t.$$

2.7. STATEMENT. *For every* $0 < \varepsilon < 1$ *we have*

$$S \leqq (\varrho^2/2)^{1-\varepsilon}\Sigma(1-\varepsilon), \tag{2.10}$$

$$S \geqq \frac{\varrho^2}{2}m^2-\varrho m-(\varrho^2/2)^{1+\varepsilon}\Sigma(1+\varepsilon), \tag{2.11}$$

$$D \leqq \varrho^{2(1-\varepsilon)}\Delta(1-\varepsilon), \tag{2.12}$$

$$D \geqq \varrho^2 m^2-\varrho^3 m^2-\varrho^{2(1+\varepsilon)}\Delta(1+\varepsilon). \tag{2.13}$$

PROOF. We substitute Lemma 2.5 into Lemmas 2.2 and 2.4, and use the relation

$$\sum \sigma(n)=\sum \delta(n)=m^2,$$

which follows immediately from the definitions. □

3. Proof of the Theorem

We start with a finite set $U \subset \mathbf{Z}$, $|U| = l$ and define

$$V = U^k = \{(u_1, \dots, u_k) : u_1, \dots u_k \in U\} \subset \mathbf{Z}^k.$$

We have

$$m = |V| = |U|^k = l^k.$$

We take a number ϱ, which will actually be of the form $\varrho = m^{-\eta}$ with a fixed small number η and take a random subset X of V. To get many differences and few sums, we want to keep S small and D near to $(\varrho m)^2$ which is a typical value of $|X|^2$, that is, we want

$$D' = (\varrho m)^2 - D$$

to be small. By Statement 2.7 these can be estimated by

$$S \leqq (\varrho^2/2)^{1-\varepsilon} \Sigma(1-\varepsilon), \tag{3.1}$$

$$D' \leqq \varrho^3 m^2 + \varrho^{2(1+\varepsilon)} \Delta(1+\varepsilon). \tag{3.2}$$

Now we study the growth of $\Sigma(t)$ and $\Delta(t)$ near $t = 1$. We have

$$\Sigma(t) = \Sigma_0(t)^k, \quad \Delta(t) = \Delta_0(t)^k, \tag{3.3}$$

where Σ_0, Δ_0 are the analogous quantities for the set U (and we shall also use σ_0, δ_0 accordingly).

We have

(3.4)
$$\begin{aligned}\Sigma_0(1+\varepsilon) &= \sum \sigma_0(n)^{1+\varepsilon} = \sum \sigma_0(n) + \varepsilon \sum \sigma_0(n) \log \sigma_0(n) + O(\varepsilon^2) = \\ &= l^2 \left(1 + \varepsilon \sigma' + O(\varepsilon^2)\right) = l^2 \exp\left(\varepsilon \sigma' + O(\varepsilon^2)\right)\end{aligned}$$

where

$$\sigma' = l^{-2} \sum \sigma_0(n) \log \sigma_0(n) \tag{3.5}$$

and the constant of the O depends on the set U. By (3.3) we get

$$\Sigma(1+\varepsilon) = m^2 \exp\left(k\varepsilon\sigma' + O(k\varepsilon^2)\right). \tag{3.6}$$

Similarly we obtain

$$\Delta(1+\varepsilon) = m^2 \exp\left(k\varepsilon\delta' + O(k\varepsilon^2)\right), \tag{3.7}$$

where

$$\delta' = l^{-2} \sum \delta_0(n) \log \delta_0(n). \tag{3.5}$$

Let L be a constant that works in the O both for (3.6) and (3.7) for $|\varepsilon| \leqq 1/2$. Substituting these bounds into (3.1) and (3.2) we get

$$S \leqq m^2 \varrho^{2(1-\varepsilon)} e^{-k\sigma'\varepsilon+Lk\varepsilon^2}, \tag{3.8}$$

$$D' \leqq m^2 \varrho^{2(1+\varepsilon)} e^{k\delta'\varepsilon+Lk\varepsilon^2}. \tag{3.9}$$

We can keep both small if

$$\sigma' > \delta', \tag{3.10}$$

and a proper choice of ϱ is

$$\varrho = e^{-k(\sigma'+\delta')/4} = m^{-\eta} \tag{3.11}$$

with

$$\eta = \frac{\sigma' + \delta'}{4 \log l}.$$

Observe that $\sigma_0(n) \leqq l$ for every n, thus $\sigma' \leqq \log l$ and similarly $\delta' \leqq \log l$, consequently $\eta \leqq 1/2$. With this ϱ we have

$$S, D' \leqq m^2 \varrho^2 \left(\varrho + \exp k \left(-\varepsilon \frac{\sigma' - \delta'}{2} + L\varepsilon^2 \right) \right).$$

To minimize this we now choose

$$\varepsilon = \frac{\sigma' - \delta'}{4L};$$

this satisfies $0 < \varepsilon < 1/2$ if we assume (2.10) and $L > \log l$.

We want to exclude the following three things:

$|X|$ is far from ρm; $|X + X|$ is too big; $|X - X|$ is too small.

If each of these events, when properly interpreted, has probability $\leqq$ $\leqq 1/4$, then we know that with probability 1/4 all are avoided. A suitable interpretation of the first is

$$||X| - \varrho m| \geqq 2\sqrt{\varrho m}$$

by (2.1). For the second we apply Markov's inequality; a good interpretation is

$$|X + X| > 4S.$$

For the third we apply Markov's inequality to the quantity

$$|X|^2 - |X - X|,$$

which must be positive. Since $\mathbf{E}|X|^2 = (\rho m)^2 + \rho(1-\rho)m$, a good interpretation of the third is

$$|X - X| < 4(D' + \rho m).$$

Thus we got a set X that has about $M = \rho m = m^{1-\eta}$ elements and a nice small number of sums and large number of differences.

Observe that the number m was not arbitrary, it had to be of the form l^k. Thus to get a set of exactly n elements as stated in the Theorem we need an additional consideration.

Select k so that $M = l^{(1-\eta)k}$ satisfies $2n < M < 2ln$. Then our set X will satisfy

$$n < |X| < Cn,$$
$$|X + X| < n^{1-c},$$
$$|X|^2 - |X - X| < n^{1-c}.$$

Let A be an arbitrary n-element subset of X. We have obviously

$$|A + A| \leqq |X + X|,$$

thus there is no problem with the sums. We claim that also

$$|A|^2 - |A - A| \leqq |X|^2 - |X - X|. \tag{3.12}$$

To see this, observe that when we omit a point from a set of p elements, the number of differences decreases at most by the $2p - 1$ differences involving this point (the difference 0 remains), while the square of the cardinality decreases exactly by $2p - 1$. Repeating this process we get (3.12).

We have yet to exhibit a set U that satisfies (3.10). In fact, a "typical" set does this, say

$$U = \{0, 1, 3\}$$

is an example.

This finishes the proof of the first half of the theorem. To prove the second half we proceed very similarly; to get the proof work we have now to find a set U with $\delta' > \sigma'$. A short computer search found the set

$$U = \{0, 1, 3, 4, 5, 6, 7, 10\}.$$

This set is the "shortest" in the sense that it is contained in the shortest possible interval; I do not know whether it is of the smallest cardinality.

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(Received February 13, 1990)

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H–1364 BUDAPEST, P.O.B. 127
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Acta Math. Hung.
59 (3–4) (1992), 449–455.

CHARACTERIZATION OF THE RELATIVE ENTROPY OF STATES OF MATRIX ALGEBRAS

D. PETZ (Budapest)

For two finite probability distributions $(p_1, p_2, \dots, p_n)$ and $(q_1, q_2, \dots, q_n)$ the quantity

$$\sum_{k=1}^{n} p_k(\log p_k - \log q_k) \tag{1}$$

was introduced in 1951 by Kulback and Leibler. They called it information for discrimination ([5,6]). Some years later Rényi suggested the name information gain ([12]). As a natural analogue of (1) Umegaki [14] defined the relative entropy of two density matrices in 1962 by the formula

$$\mathrm{Tr}\, \varrho(\log \varrho - \log \varphi). \tag{2}$$

Properties of the relative entropy functional were established in many papers and the highlight of this development was Lieb's convexity theorem ([7]). The notion received much attention in quantum mechanics ([8]). Concerning the details we refer to the survey papers [2] and [9].

The aim of the present paper is to characterize the relative entropy functional through its wellknown properties. As a frame we consider finite dimensional C^*-algebras ([13], p.50). Such algebras are decomposed into a direct sum of full matrix algebras and the commutative ones are isomorphic to the n-tuples of complex numbers for some positive integer n. By a state we mean a positive normalized functional. Each finite dimensional C^*-algebra possesses a natural "uniform distribution" which, more precisely, is a positive functional taking the value one on each minimal projection. It is unique and we denote it by Tr. In case of a matrix algebra Tr reduces to the usual matrix trace and on complex n-tuples

$$\mathrm{Tr}\,(c_1, \dots, c_n) = \sum_{n=1}^{n} c_i.$$

To each state φ one associates a density D_φ such that

$$\varphi(a) = \mathrm{Tr}\, D_\varphi\, a$$

and we call φ faithful if D_φ is invertible. For a faithful state φ and an arbitrary state ω on a finite dimensional C^*-algebra $\mathcal{A}$ their relative entropy is defined as

$$S(\varphi, \omega) = \mathrm{Tr}\, D_\omega (\log D_\omega - \log D_\varphi). \tag{3}$$

(Note that $D_\omega \log D_\omega$ is well defined even if D_ω is not invertible.)

Our crucial postulate for the relative entropy includes the notion of conditional expectation. Let us recall that in the setting of operator algebras conditional expectation (or projection of norm one) is defined as a positive unital idempotent linear mapping onto a subalgebra ([13], p. 131).

Now we list properties of the relative entropy functional which will be used in an axiomatic characterization:

(i) Conditional expectation property: Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and there exists a projection E of $\mathcal{B}$ onto $\mathcal{A}$ of norm one such that $\varphi \circ E = \varphi$. Then for every state ω of $\mathcal{B}$

$$S(\varphi, \omega) = S(\varphi|\mathcal{A}, \omega|\mathcal{A}) + S(\omega \circ E, \omega)$$

holds.

(ii) Invariance property: For every automorphism α of $\mathcal{B}$ we have

$$S(\varphi, \omega) = S(\varphi \circ \alpha, \omega \circ \alpha).$$

(iii) Direct sum property: Assume that $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ and

$$\varphi_{12}(a \oplus b) = \lambda\varphi_1(a) + (1-\lambda)\varphi_2(b), \quad \omega_{12}(a \oplus b) = \lambda\omega_1(a) + (1-\lambda)\omega_2(b)$$

for every $a \in \mathcal{B}_1, b \in \mathcal{B}_2$ and some $0 < \lambda < 1$. Then

$$S(\varphi_{12}, \omega_{12}) = \lambda S(\varphi_1, \omega_1) + (1-\lambda) S(\varphi_2, \omega_2).$$

(iv) Nilpotence property: $S(\varphi, \varphi) = 0$.

(v) Measurability property: The function

$$(\varphi, \omega) \mapsto S(\varphi, \omega)$$

is measurable on the state space of the finite dimensional C^*-algebra $\mathcal{B}$ (when φ is assumed to be faithful).

Properties (i)–(v) are wellknown for the relative entropy functional. Among them the conditional expectation property is the most crucial (it was obtained in [11] in full generality). There are plenty of information quantities sharing properties (ii)–(v), all the quasientropies discussed in [10] are so.

Our main result is the following.

THEOREM. *If a real valued functional $S'(\varphi, \omega)$ defined for faithful states φ and arbitrary states ω of finite dimensional C^*-algebras shares properties* (i)–(v) *then there exists a constant $C \in \mathbf{R}$ such that*

$$S'(\varphi, \omega) = C \operatorname{Tr} D_\omega(\log D_\omega - \log D_\varphi).$$

The proof consists of several steps. We show that for larger and larger class of states

$$S'(\varphi,\omega) = C\,S(\varphi,\omega)$$

must hold.

Consider the three dimensional commutative algebra $\mathbf{C}^3$. Its states correspond to probability distributions (p_1,p_2,p_3) (i.e. $0 \leqq p_i,\ p_1+p_2+p_3=1$).

LEMMA 1. *For probability distributions* (p_1,p_2,p_3) *and* (q_1,q_2,q_3) *the recursive relation*

$$S'((q_1,q_2,q_3),(p_1,p_2,p_3)) = S'((q_1+q_2,q_3),(p_1+p_2,p_3))+ \\ +(p_1+p_2)S'\left(\left(\frac{q_1}{q_1+q_2},\frac{q_2}{q_1+q_2}\right),\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)\right) \tag{4}$$

holds.

We benefit from the conditional expectation property in the situation $\mathbf{C}^2 \cong \{(c_1,c_1,c_2) : c_1,c_2 \in \mathbf{C}\} \subset \mathbf{C}^3$. There exists a conditional expectation $E : \mathbf{C}^3 \to \mathbf{C}^2$ preserving the state $\varphi = \varphi_{(q_1,q_2,q_3)}$ and it is given by

$$E : (c_1,c_2,c_3) \mapsto \left(\frac{q_1c_1+q_2c_2}{q_1+q_2},\ \frac{q_1c_1+q_2c_2}{q_1+q_2},\ c_3\right).$$

The state $\omega_{(p_1,p_2,p_3)} \circ E$ corresponds to the measure

$$\left(\frac{p_1+p_2}{q_1+q_2}q_1,\ \frac{p_1+p_2}{q_1+q_2}q_2, p_3\right)$$

and we obtain

$$S'((q_1,q_2,q_3),(p_1,p_2,p_3)) = S'((q_1+q_2,q_3),(p_1+p_2,p_3))+ \\ +S'\left(\left(\frac{p_1+p_2}{q_1+q_2}q_1,\frac{p_1+p_2}{q_1+q_2}q_2,p_3\right),(p_1,p_2,p_3)\right).$$

Due to the direct sum condition the last term here equals the last term of (4) and the lemma follows.

Interchanging in $\mathbf{C}^3$ the second and third coordinates by means of the invariance condition we conclude the equation

$$S'((q_1+q_2,q_3),(p_1+p_2,p_3)) + (p_1+p_2)\cdot \\ \cdot S'\left(\left(\frac{q_1}{q_1+q_2},\frac{q_2}{q_1+q_2}\right),\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)\right) = \\ = S'((q_1+q_3,q_2),(p_1+p_3,p_2)) + (p_1+p_3)\cdot \\ \cdot S'\left(\left(\frac{q_1}{q_1+q_3},\frac{q_3}{q_1+q_3}\right),\left(\frac{p_1}{p_1+p_3},\frac{p_3}{p_1+p_3}\right)\right). \tag{5}$$

With the notation

$$F(x,y) = S'((1-y,y),(1-x;x))$$

equation (5) is of the following form:

(6)
$$F(x,y) + (1-x)F\left(\frac{u}{1-x}, \frac{v}{1-y}\right) = F(u,v) + (1-u)F\left(\frac{x}{1-u}, \frac{y}{1-v}\right).$$

The functional equation (6) has been solved under the special nilpotence condition $F(1/2,1/2) = 0$ in [4]. A lengthy but elementary analysis yields that the only measurable solution of (6) is

$$F(x,y) = C\left(x\log\frac{x}{y} + (1-x)\log\frac{1-x}{1-y}\right).$$

(See also pp. 204-207 of [1].)

The recursion (4) remains true if p_3 and q_3 are replaced by $p_3, p_4, \dots, p_n$ and $q_3, q_4, \dots, q_n$ respectively. In this way we obtain that

$$S'(\varphi,\omega) = C\,S(\varphi,\omega)$$

whenever φ and ω are states of commutative finite dimensional C^*-algebras.

Now let φ and ω be states of an algebra $\mathcal{B}$ such that the densities D_φ and D_ω commute. Let $\mathcal{A}$ be the maximal abelian subalgebra generated by D_φ and D_ω. If E is the conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$ which preserves Tr then $\varphi \circ E = \varphi$ and $\omega \circ E = \omega$. The conditional expectation property tells us that

$$S'(\varphi,\omega) = S'(\varphi|\mathcal{A},\omega|\mathcal{A}) + S'(\omega\circ E,\omega).$$

By nilpotence the second term vanishes and we arrive at

(7)
$$S'(\varphi,\omega) = C\,S(\varphi,\omega)$$

for commuting states.

The next step is $\mathcal{B} = M_n(\mathbf{C})$. Our aim is to show that (7) holds for arbitrary states on $\mathcal{B}$. (As always, φ is supposed to be faithful.)

LEMMA 2. *If* $\sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2 \quad (0<\lambda<1)$ *then*

(8) $\lambda S'(\varphi,\sigma_1) + (1-\lambda)S'(\varphi,\sigma_2) = S'(\varphi,\sigma) + \lambda S'(\sigma,\sigma_1) + (1-\lambda)S'(\sigma,\sigma_2).$

The proof of (8) is quite similar to that of Lemma 1. The conditional expectation property should be applied to $\mathcal{B}\oplus\mathcal{B}$ and its subalgebra $\{b\oplus b : b\in\mathcal{B}\}$. The mapping

$$E(a\oplus b) = (\lambda a + (1-\lambda)b)\oplus(\lambda a + (1-\lambda)b)$$

is a conditional expectation leaving the state

$$\varphi_{12}(a \oplus b) = \lambda\varphi(a) + (1-\lambda)\varphi(b)$$

invariant. The argument is completed by referring to the invariance and direct sum properties.

Now we resume the determination of $S(\varphi,\omega)$ for states of $\mathcal{B} = M_n(\mathbf{C})$.We choose a basis such that the density of φ is diagonal. Then the density of ω is of the form

$$\begin{pmatrix} A_k & 0 \\ 0 & D_{n-k} \end{pmatrix} = D_\omega$$

where $A_k \in M_k(\mathbf{C})$ and $D_{n-k} \in M_{n-k}(\mathbf{C})$ is a diagonal matrix. We are going to prove (7) by mathematical induction on k. If $k = 0$ then D_φ and D_ω commute and (7) holds.

Let U be a diagonal unitary matrix such that

$$U_{ii} = \begin{cases} 1 & i \neq k \\ -1 & i = k. \end{cases}$$

Then

$$D_{\sigma_2} = U\, D_\omega U$$

differs from $D_{\sigma_1} \equiv D_\omega$ only in the sign of the entries in the kth row and kth column but they are the same along the diagonal. The density

$$D_\sigma = \frac{1}{2}(D_{\sigma_1} + D_{\sigma_2})$$

is of the form

$$\begin{pmatrix} A_{k-1} & 0 \\ 0 & D_{n-k+1} \end{pmatrix}$$

where D_{n-k+1} is an $(n-k+1) \times (n-k+1)$ diagonal matrix. From the induction hypothesis we have

$$S'(\varphi,\sigma) = C\, S(\varphi,\sigma). \tag{9}$$

Write (8) with $\tau = \mathrm{Tr}/n$ and $\lambda = \frac{1}{2}$. Then

$$\frac{1}{2}S'(\tau,\sigma_1) + \frac{1}{2}S'(\tau,\sigma_2) = S'(\tau,\sigma) + \frac{1}{2}S'(\sigma,\sigma_1) + \frac{1}{2}S'(\sigma,\sigma_2). \tag{10}$$

The invariance, more precisely the relations $\tau(U \cdot U) = \tau(\cdot)$, $\sigma_1(U \cdot U) = \sigma_2(\cdot)$, $\sigma(U \cdot U) = \sigma(\cdot)$, ensures

$$S'(\tau,\sigma_1) = S'(\tau,\sigma_2) \quad \text{and} \quad S'(\sigma,\sigma_1) = S'(\sigma,\sigma_2).$$

Therefore

$$S'(\tau,\sigma_1) = S'(\tau,\sigma) + S'(\sigma,\sigma_1)$$

and (7) yields

$$S'(\sigma,\sigma_1) = C\, S(\sigma,\sigma_1). \tag{11}$$

In our special case

$$S'(\varphi,\sigma_1) = S'(\varphi,\sigma_2) \quad \text{and} \quad S'(\sigma,\sigma_1) = S'(\sigma,\sigma_2)$$

hold due to the invariance under the automorphism Ad U. Hence (8) reads as

$$S'(\varphi,\sigma_1) = S'(\varphi,\sigma) + S'(\sigma,\sigma_1)$$

where both terms on the right hand side have been compared with the relative entropy ((9) and (11)). So (7) holds for all states φ and ω on $M_n(\mathbf{C})$. Since a finite dimensional C^*-algebra is the direct sum of full matrix algebras, the direct sum property extends (7) to all C^*-algebras of finite dimension.

We note that (8) is Donald's mixing axiom and the subsequent argument is due to him ([3]).

By more complicated mathematical tools the concept of relative entropy may be extended to states of infinite dimensional C^*-algebras ([2]). If we restrict ourselves to C^*-algebras with good approximation property (nuclear algebras) then the characterization in the present paper may be modified easily. It is sufficient to choose instead of (ii) the monotonicity and instead of measurability the lower semicontinuity property given below.

(ii)' Monotonicity property: For every completely positive unital mapping α we have

$$S(\varphi,\omega) \geqq S(\varphi\circ\alpha,\omega\circ\alpha).$$

(v)' Lower semicontinuity: The functional

$$(\varphi,\omega) \mapsto S(\varphi,\omega)$$

is lower semicontinuous with respect to the weak topology.

These postulates (i), (ii)', (iii), (iv) and (v)' constitute the definition of the relative entropy functional up to a constant factor. Details of this remark are out of the scope of the present paper and they will be discussed in forthcoming publications.

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(Received February 27, 1990)

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