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UPCROSSINGS OF THE RANDOM WALK

D. P. JOHNSON (Calgary)

In this note we prove the following

THEOREM. *Let X_n, $n=0, 1, \dots$ be a random walk on the integers with*

$$P[X_{n+1} = i+1|X_n = i] = p > 0 \quad \textit{and} \quad P[X_{n+1} = i-1|X_n = i] = q = 1-p > 0.$$

Let U be the number of upcrossings of the interval $[b, c]$ by $\{X_n\}$ up to the time that $\{X_n\}$ leaves the interval $[a, d]$ where $a<b<c<d$ are integers. Then

$$E(U|X_0 = i) =$$

$$= \begin{cases} \dfrac{\left[1-\left(\frac{q}{p}\right)^{i-a+1}\right]\left[1-\left(\frac{q}{p}\right)^{d-b+2}\right]}{\left[1-\left(\frac{q}{p}\right)^{d-a+2}\right]\left[1-\left(\frac{q}{p}\right)^{c-b+2}\right]} & \textit{for} \quad p \neq q \quad \textit{and} \quad a \leqq i \leqq b-1, \\ \dfrac{\left[1-\left(\frac{q}{p}\right)^{b-a}\right]\left[\left(\frac{q}{p}\right)^{i-b+1}-\left(\frac{q}{p}\right)^{d-b+2}\right]}{\left[1-\left(\frac{q}{p}\right)^{d-a+2}\right]\left[1-\left(\frac{q}{p}\right)^{c-b+2}\right]} & \textit{for} \quad p \neq q \quad \textit{and} \quad b \leqq i \leqq d, \\ \dfrac{(i-a+1)(d-b+2)}{(d-a+2)(c-b+2)} & \textit{for} \quad p = q = 1/2 \quad \textit{and} \quad a \leqq i \leqq b-1, \\ \dfrac{(b-a)(d-i+1)}{(d-a+2)(c-b+2)} & \textit{for} \quad p = q = 1/2 \quad \textit{and} \quad b \leqq i \leqq d. \end{cases}$$

PROOF. Let $x_i=E(U|X_0=i)$ and let $y_i=E(Y|X_0=i)$ where Y equals 1 if $\{X_n\}$ reaches (c, ∞) before it reaches $(-\infty, b)$ and 0 otherwise. Then the x_i satisfy the following equations:

(1) $$x_i = 0 \quad \text{for} \quad i \leqq a-1$$

(2) $$x_i = px_{i+1}+qx_{i-1} \quad \text{for} \quad a \leqq i \leqq b-2$$

(3) $$x_{b-1} = qx_{b-2}+px_b+py_b$$

(4) $$x_i = qx_{i-1}+px_{i+1} \quad \text{for} \quad b \leqq i \leqq d$$

(5) $$x_i = 0 \quad \text{for} \quad i \leqq d+1.$$

For example equation (3) follows from

$$\begin{aligned} x_{b-1} &= E(U|X_0=b-1) = \\ &= P(X_1=b-2|X_0=b-1)E(U|X_0=b-1,\ X_1=b-2)+ \\ &+P(X_1=b|X_0=b-1)E(U|X_0=b-1,\ X_1=b) = \\ &= qx_{b-2}+pE(U+Y|X_0=b) = qx_{b-2}+px_b+py_b. \end{aligned}$$

From equations (1) and (2) we see that

$$x_{i+1}-x_i = \frac{q}{p}(x_i-x_{i-1}) = \ldots = \left(\frac{q}{p}\right)^{i-a+1}(x_a-x_{a-1}) = \left(\frac{q}{p}\right)^{i-a+1} x_a$$

and summing from a to $j-1 \leqq b-2$ yields

$$\text{(6)} \qquad x_j = \frac{1-\left(\frac{q}{p}\right)^{j-a+1}}{1-\frac{q}{p}} x_a, \quad a \leqq j \leqq b-1.$$

Similarly from equation (4) we get

$$x_{i+1}-x_i = \frac{q}{p}(x_i-x_{i-1}) = \ldots = \left(\frac{q}{p}\right)^{i-b+1}(x_b-x_{b-1})$$

and summing from b to $j-1 \leqq d$ yields

$$x_j = \frac{1-\left(\frac{q}{p}\right)^{j-b+1}}{1-\frac{q}{p}} x_b - \frac{\frac{q}{p}-\left(\frac{q}{p}\right)^{j-b+1}}{1-\frac{q}{p}} x_{b-1}, \quad b \leqq j \leqq d+1.$$

Replacing j by $d+1$ in this last expression yields

$$x_b = \frac{\frac{q}{p}-\left(\frac{q}{p}\right)^{d-b+2}}{1-\left(\frac{q}{p}\right)^{d-b+2}} x_{b-1}$$

and substituting back into the original equation now gives us

$$x_j = \frac{\left(\frac{q}{p}\right)^{j-1}-\left(\frac{q}{p}\right)^{j}+\left(\frac{q}{p}\right)^{d+1}-\left(\frac{q}{p}\right)^{d}}{\left(1-\frac{q}{p}\right)\left[1-\left(\frac{q}{p}\right)^{d-b+2}\right]} \left(\frac{q}{p}\right)^{2-b} x_{b-1}, \quad b \leqq j \leqq d+1$$

or, using equation (6) to compute x_{b-1},

(7)

$$x_j = \frac{\left(\frac{q}{p}\right)^{j-1} - \left(\frac{q}{p}\right)^{j} + \left(\frac{q}{p}\right)^{d+1} - \left(\frac{q}{p}\right)^{d}}{\left(1-\frac{q}{p}\right)\left[1-\left(\frac{q}{p}\right)^{d-b+2}\right]} \left(\frac{q}{p}\right)^{2-b} \frac{\left[1-\left(\frac{q}{p}\right)^{b-a}\right]}{\left(1-\frac{q}{p}\right)} x_a, \quad b \leqq j \leqq d+1.$$

Equations (6) and (7) now give us expression for x_{b-1}, x_{b-2} and x_b in terms of x_a. Substituting these into equations (3) and solving for x_a gives

$$x_a = \frac{\left[1-\left(\frac{q}{p}\right)^{d-b+2}\right]}{\left[1-\left(\frac{q}{p}\right)^{d-a+2}\right]} y_b.$$

But y_b is the solution to the ruin problem and so equals

$$y_b = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^{c-b+1}}.$$

Substituting this last expression for y_b into the expression for x_a and substituting the resulting expression for x_a back into equations (6) and (7) proves the theorem for $p \neq q$. For $p=q$ one can take the limit as $p \to 1/2$.

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UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS
CALGARY, ALBERTA
CANADA

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ЧИСЛОВАЯ ОБЛАСТЬ ЛИНЕЙНЫХ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ С ИНДЕФИНИТНОЙ МЕТРИКОЙ

Ц. БАЯСГАЛАН (Улан-Батор)

В настоящей заметке доказана выпуклость числовой области произвольного линейного оператора в пространстве с индефинитной метрикой. Как ее приложение, приведена одна простая связь между спектром и числовой областью положительного оператора в пространстве Крейна (см. [1]).

В комплексном линейном пространстве H заданы эрмитова билинейная форма (x, y), определенная для всех $x, y \in H$ и линейный оператор A, определенный всюду в нем.

Теорема. *Множество*

$$V(A) = \{(Ax, x)\colon (x, x) = 1\}$$

выпукло.

Доказательство. Пусть существуют векторы $x, y \in H$, такие что $(x, x) = (y, y) = 1$, $(Ax, x) > 0$, $(Ay, y) < 0$. Тогда найдется $z \in H$ со свойствами $(z, z) = 1$, $(Az, z) = 0$.

Действительно, мы ищем z в виде $z = t_1 x + t_2 y$, где t_1 и t_2, отличны от нуля. Введем следующие обозначения:

$$(Ax, x) = a_{11}, \quad (Ay, y) = a_{22}, \quad (Ax, y) = a_{12}, \quad (Ay, x) = a_{21}.$$

Тогда уравнение $(Az, z) = 0$ можно переписать в виде

$$a_{11} + \frac{t_2}{t_1} a_{21} + \overline{\left(\frac{t_2}{t_1}\right)} \left(\frac{t_2}{t_1} a_{22} + a_{12}\right) = 0.$$

Или вводя обозначение $\frac{t_2}{t_1} = \theta$, получаем уравнение

$$a_{11} + \theta a_{21} + \bar{\theta}(\theta a_{22} + a_{12}) = 0$$

относительно θ. В обозначениях $a_{21} = c + di$, $a_{12} = \delta + \varepsilon i$, $\theta = \theta_1 + \theta_2 i$, последнее уравнение сводится к системе уравнений

$$\text{(1)} \qquad \begin{cases} a_{11} + a_{22}(\theta_1^2 + \theta_2^2) + \theta_1(c + \delta) + \theta_2(\varepsilon - d) = 0, \\ \theta_1(\varepsilon + d) + \theta_2(c - \delta) = 0. \end{cases}$$

Из условий $(z, z) = 1$ и $\theta = \frac{t_2}{t_1}$ следует равенство

$$|t_1|^2 \big(1 + |\theta|^2 + 2 \operatorname{Re}(\theta(y, x))\big) = 1.$$

Отсюда необходимо вытекает неравенство

(2) $$1+|\theta|^2+2\,\mathrm{Re}\,(\theta(y,x))>0.$$

Таким образом, нам достаточно найти такой θ, что он удовлетворяет (1) и (2). Если $\varepsilon+d=0$, то положим $\theta_2=0$, в зависимости от знака $\mathrm{Re}\,(y,x)$, мы выбираем в качестве θ_1 одно из решений квадратного уравнения из системы (1). Если же $\varepsilon+d\neq 0$, то (1) и (2) перепишем в виде

(3) $$\begin{cases} a_{11}+a_{22}\left[1+\left(\dfrac{\delta-c}{\varepsilon+d}\right)^2\right]\theta_2^2+\theta_2(\varepsilon-d)+\dfrac{\delta^2-c^2}{\varepsilon+d}\theta_2=0,\\ \theta_1=\dfrac{\delta-c}{\varepsilon+d}\theta_2,\quad 1+|\theta|^2+2\theta_2\left[\dfrac{\delta-c}{\varepsilon+d}m-n\right]>0,\end{cases}$$

где $(y,x)=m+ni$. Аналогично предыдущему, в зависимости от знака $\dfrac{\delta-c}{\varepsilon+d}m-n$, мы выбираем одно из решений квадратного уравнения из системы (3).

Далее, пусть векторы $x, y\in H$ обладают свойствами

$$(x,x)=(y,y)=1,\quad (Ax,x)=1,\quad (Ay,y)=-1.$$

Тогда для произвольного $t\in(-1,1)$ мы имеем неравенства $((A-t)x,x)>0$, $((A-t)y,y)<0$. В силу доказанного, существует $z\in H$ такое, что $(z,z)=1$, $(Az,z)=t$.

Теперь, как в [2] (стр. 305), для произвольных $x, y\in H$ с $(Ax,x)\neq(Ay,y)$ и $(x,x)=(y,y)=1$ найдем постоянные α и β такие, что

$$((\alpha A+\beta)x,x)=1,\quad ((\alpha A+\beta)y,y)=-1.$$

Тогда, по предыдущему, для $t\in[-1,1]$ существует $z\in H$ с $(z,z)=1$ и $((\alpha A+\beta)z,z)=t$. Подставляя значения α и β, мы приходим к равенству

$$\frac{1+t}{2}(Ax,x)+\frac{1-t}{2}(Ay,y)=(Az,z).$$

Следствие. *Пусть A — положительный оператор в пространстве Крейна Тогда имеет место включение*

$$\sigma(A)\subset\overline{V_+(A)}\cup\overline{V_-(A)},$$

где $\sigma(A)$ — спектр оператора A и

$$V_+(A)=\{(Ax,x)\colon (x,x)=1\},\quad V_-(A)=\{-(Ax,x)\colon (x,x)=-1\}.$$

Заметим, что в силу предыдущей теоремы множества $V_+(A)$ и $V_-(A)$ выпуклы.

Доказательство. Введем обозначения

$$\alpha_+=\inf_{(x,x)=1}(Ax,x),\quad \alpha_-=\sup_{(x,x)=-1}\bigl(-(Ax,x)\bigr).$$

Если $\alpha_- = \alpha_+$, то по доказанной теореме множество $\overline{V_+(A)} \cup \overline{V_-(A)}$ выпукло, следовательно, по [3] имеем

$$\sigma(A) \subset [\min \sigma(A), \max \sigma(A)] \subset \overline{V_-(A)} \cup \overline{V_+(A)}.$$

Если же $\alpha_- < \alpha_+$, то оператор A фундаментально приводим (см. [4], стр. 146) в силу чего следствие верно и в этом случае.

Литература

[1] J. Bognár, *Indefinite inner product spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete 78, Springer-Verlag (Berlin, Heidelberg, New York, 1974).
[2] П. Халмош, *Гильбертово пространство в задачах,* Мир, 1970.
[3] B. Textorius, Minimaxprinzipe zur Bestimmung der Eigenwerte nichtnegatíver Operatoren, *Math. Scand.*, **35** (1974), 105-114.
[4] Ц. Баясгалан, О фундаментальной приводимости положительных операторов в пространствах с индефинитной метрикой, *Studia Sci. Mat. Hungar.*, **13** (1978), 143-150.

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ON S-CLOSED SPACES

A. S. MASHHOUR, A. A. ALLAM and A. M. ZAHRAN (Assiut)

1. Introduction

Throughout the present paper X and Y mean topological spaces on which no separation axioms are assumed. Let S be a subset of X. The closure of S and the interior of S will be denoted by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, respectively. A subset S of X is called regular open (resp. α-set [3], semiopen [3], β-open [1]) if $S=\mathrm{int}(\mathrm{cl}(S))$ (resp. $S\subset\mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$, $S\subset\mathrm{cl}(\mathrm{int}(S))$, $S\subset\mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$). The complement of regular open (resp. semiopen, β-open) is called regular closed (resp. semiclosed [3], β-closed [1]). A function $f\colon X\to Y$ is called α-open [7] (resp. almost open [14], semiclosed [8], regular closed [12], β-closed [1]) if the image of each open set of X is an α-set in Y (resp. $f^{-1}(\mathrm{cl}(V))\subset\mathrm{cl}(f^{-1}(V))$ for each open set V of Y, the image of each closed set of X is semiclosed in Y, the image of each regular closed set of X is closed in Y, the image of each closed set of X is β-closed in Y). Every closed function is a regular closed function and semiclosed and every semiclosed function is β-closed. Also every open function is α-open, but the converses are not true in general.

In 1976, Thompson [13] has defined a topological space X to be S-closed if for every semiopen cover $\{U_i\colon i\in I\}$ of X, there exists a finite subset I_0 of I such that $X=\bigcup\{\mathrm{cl}(U_i)\colon i\in I_0\}$. In 1977, Noiri [9] has defined a subset A of a topological space X to be S-closed relative to X if for every cover $\{U_i\colon i\in I\}$ of A by semiopen sets in X, there exists a finite subset I_0 of I such that $A\subset\bigcup\{\mathrm{cl}(U_i)\colon i\in I_0\}$. A topological space X is said to be almost compact if for every open cover $\{U_i\colon i\in I\}$ of X, there exists a finite subset I_0 of I such that $X=\bigcup\{\mathrm{cl}(U_i)\colon i\in I_0\}$. Every S-closed space is almost compact, but the converse is not true in general (see Remark 1.5 [5]).

In the present paper we introduce and study the concept of weakly semiclosed functions and by using this concept we investigate some properties of S-closed spaces. Also, we strengthen some results in [2, 5, 11].

2. On β-open sets and S-closed spaces

THEOREM 2.1. *If a function $f\colon X\to Y$ is an α-open bijection and Y is S-closed, then X is almost compact.*

PROOF. Let $\{U_i\colon i\in I\}$ be an open cover of X, then $\{f(U)\colon i\in I\}$ is a cover of Y by α-sets of Y. Since Y is S-closed, there exists a finite subset I_0 of I such that $Y=\bigcup\{\mathrm{cl}(f(U_i))\colon i\in I_0\}$. This implies

$$X=\bigcup\{f^{-1}(\mathrm{cl}(f(U_i)))\colon i\in I_0\}\subset\bigcup\{\mathrm{cl}(f^{-1}(f(U_i)))\colon i\in I_0\}$$

by Corollary 2.1 of [6]. Hence $X=\bigcup\{\mathrm{cl}\,(U_i)\colon i\in I_0\}$. Therefore X is almost compact.

COROLLARY 2.2 (Mashhour and Hasanein [5]). *If a function $f\colon X\to Y$ is an open bijection and Y is S-closed, then X is almost compact.*

THEOREM 2.3. *Let X be an extremally disconnected space and $f\colon X\to Y$ a regular closed, almost open surjection and $f^{-1}(y)$ be S-closed relative to X for each $y\in Y$. If G is almost compact relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

PROOF. Let $\{F_i\colon i\in I\}$ be a cover of $f^{-1}(G)$ by regular closed sets of X. For $y\in G$, by Lemma 2.1 of [11], there exists a finite subset $I(y)$ of I such that $f^{-1}(y)\subset\subset\bigcup\{F_i\colon i\in I(y)\}$. Since X is extremally disconnected, for each $i\in I$, $F_i=\mathrm{cl}\,(\mathrm{int}\,(F_i))$ is open in X. Now put $U(y)=\bigcup\{F_i\colon i\in I(y)\}$ which is regular open in X. By Lemma 2.2 of [4] there exists an open set $V(y)$ of Y such that $y\in V(y)$ and $f^{-1}(V(y))\subset U(y)$. Since $\{V(y)\colon y\in G\}$ is a cover of G by open sets of Y, then there exists a finite number of points $y_1, y_2, y_3, \dots, y_n$ in G such that $G\subset\bigcup\{\mathrm{cl}\,(V(y_j))\colon j=1,2,\dots,n\}$. By using almost openness of f, we obtain

$$f^{-1}(G)\subset\bigcup\{f^{-1}(\mathrm{cl}\,(V(y_j)))\colon j=1,2,3,\dots,n\}\subset$$
$$\subset\bigcup\{\mathrm{cl}\,(f^{-1}(V(y_j)))\colon j=1,2,3,\dots,n\}\subset$$
$$\subset\bigcup\{\mathrm{cl}\,(U(y_j))\colon j=1,2,3,\dots,n\}=\bigcup_{j=1}^{n}\bigcup_{i\in I(y_j)}F_i.$$

It follows from Lemma 2.1 of [11] that $f^{-1}(G)$ is S-closed relative to X.

THEOREM 2.4. *Let X be an extremally disconnected space and $f\colon X\to Y$ a regular closed, almost open surjection with compact point inverses. If Y is an almost compact space, then X is S-closed.*

PROOF. Since in an extremally disconnected space every regular closed set is clopen, each compact subset is relatively S-closed.

COROLLARY 2.5. *Let X be an extremally disconnected space and $f\colon X\to Y$ a regular closed, almost open surjection with compact point inverses. If Y is an S-closed space, then X is S-closed.*

COROLLARY 2.6 (Noiri [11]). *Let X be an extremally disconnected space and $f\colon X\to Y$ a closed open surjection with compact point inverses. If Y is an S-closed space, then X is S-closed.*

THEOREM 2.7 [15]. *A subset A is S-closed relative to X iff for every β-open cover $\{U_i\colon i\in I\}$ of A, there is a finite subset I_0 of I such that $A\subset\bigcup\{\mathrm{cl}\,(U_i)\colon i\in I_0\}$.*

THEOREM 2.8. *Let X be an extremally disconnected space and $f\colon X\to Y$ be a β-closed, α-open surjection and $f^{-1}(y)$ be S-closed relative to X for each point $y\in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

PROOF. Let $\{F_i\colon i\in I\}$ be a cover of $f^{-1}(G)$ by regular closed sets of X. For each $y\in G$, by Lemma 2.1 of [11], there exists a finite subset $I(y)$ of I such that $f^{-1}(y)\subset\bigcup\{F_i\colon i\in I(y)\}$. Since X is extremally disconnected for each $i\in I$, $F_i==\mathrm{cl}\,(\mathrm{int}\,(F_i))$ is open in X. Now, put $U(y)=\bigcup\{F_i\colon i\in I(y)\}$ which is open in X, then there exists a β-open set $V(y)$ of Y such that $y \in V(y)$ and $f^{-1}(V(y)) \subset U(y)$ [1, Theorem 2.3]. Since $\{V(y): y \in G\}$ is a cover of G by β-open sets of Y, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in G such that $G \subset \cup \{\mathrm{cl}\,(V(y_j)): j=1, 2, \ldots, n\}$ by Theorem 2.7. By [6, Corollary 2.1] we obtain $f^{-1}(G) \subset \bigcup_{j=1}^{n} f^{-1}(\mathrm{cl}\,(V(y_j))) \subset$ $\subset \bigcup_{j=1}^{n} \mathrm{cl}\,(f^{-1}(V(y_j))) \subset \bigcup_{j=1}^{n} \mathrm{cl}\,(U(y_j)) = \bigcup_{j=1}^{n} \bigcup_{i \in I(y_j)} F_i$. It follows from Lemma 2.1 of [11] that $f^{-1}(G)$ is S-closed relative to X.

3. Weakly semiclosed functions and S-closed spaces

DEFINITION 3.1. A function $f: X \to Y$ is said to be *weakly semiclosed* if the image of every regular closed set in X is semiclosed in Y.

REMARK 3.2. Every semiclosed function is weakly semiclosed, but the converse need not be true in general as follows:

EXAMPE 3.3. Let $X=\{a, b, c\}$, $\tau=\{X, \emptyset, \{b\}, \{a, b\}\}$ and $\theta=\{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. The identity function $i: (X, \tau) \to (X, \theta)$ is weakly semiclosed but not semiclosed, since $\{a, c\}$ is closed in (X, τ) but $i(\{a, c\})=\{a, c\}$ is not semiclosed in (X, θ).

THEOREM 3.4. *Let $f: X \to Y$ be a weakly semiclosed function. If $V \subset Y$ and U is a regular open set of X containing $f^{-1}(V)$, then there exists a semiopen set W of Y containing V such that $f^{-1}(W) \subset U$.*

PROOF. Clear.

THEOREM 3.5. *Let X be an extremally disconnected space, $f: X \to Y$ a weakly semiclosed, almost open surjection and $f^{-1}(y)$ be S-closed relative to X for each $y \in Y$. If G is S-closed relative to Y then $f^{-1}(G)$ is S-closed relative to X.*

PROOF. Let $\{F_i: i \in I\}$ be a cover of $f^{-1}(G)$ by regular closed sets of X. For each $y \in G$, by Lemma 2.1 of [11] there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subset \{F_i: i \in I(y)\}$. Since X is extremally disconnected, for each $i \in I$, $F_i = \mathrm{cl}\,(\mathrm{int}\,(F_i))$ is open in X. Now, put $U(y) = \cup \{F_i: i \in I(y)\}$ which is regular open in X. By Theorem 3.4 there exists a semiopen set $V(y)$ such that $y \in V(y)$ and $f^{-1}(V(y)) \subset$ $\subset U(y)$. Since $\{V(y): y \in G\}$ is a cover of G by semiopen sets of Y, then there exists a finite number of points $y_1, y_2, y_3, \ldots, y_n$, in G such that

$$G \subset \cup \{\mathrm{cl}\,(V(y_j)): j = 1, 2, 3, \ldots, n\}.$$

By using almost openness of f, we obtain

$$f^{-1}(G) \subset \bigcup_{j=1}^{n} f^{-1}(\mathrm{cl}\,(V_j)) = \bigcup_{j=1}^{n} f^{-1}(\mathrm{cl}\,(\mathrm{int}\,(V(y_j)))) \subset \bigcup_{j=1}^{n} \mathrm{cl}\,(f^{-1}(\mathrm{int}\,(V(y_j)))) \subset$$

$$\subset \bigcup_{j=1}^{n} \mathrm{cl}\,(f^{-1}(V(y_j))) \subset \bigcup_{j=1}^{n} \mathrm{cl}\,(U(y_j)) = \bigcup_{j=1}^{n} \bigcup_{i \in I(y_j)} F_i.$$

It follows from Lemma 2.1 of [11] that $f^{-1}(G)$ is S-closed relative to X.

COROLLARY 3.6 (Noiri [11]). *Let X be extremally disconnected, $f: X \to Y$ a semiclosed, almost open surjection and let $f^{-1}(y)$ be S-closed relative to X for each point $y \in Y$. If G is S-closed relative to Y, then $f^{-1}(G)$ is S-closed relative to X.*

THEOREM 3.7. *Let X be an extremally disconnected space, $f: X \to Y$ a weakly semiclosed, almost open surjection and let $f^{-1}(y)$ be compact for each $y \in Y$. If Y is an S-closed space then X is S-closed.*

COROLLARY 3.8 (Atia, El Deeb and Hasanein [2]). *Let X be an extremally disconnected space, $f: X \to Y$ a semiclosed, almost open surjection and $f^{-1}(y)$ compact for each $y \in Y$. If Y is an S-closed space, then X is S-closed.*

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bulletin of Fac. of Sc. Assiut Univ.* (1982).
[2] R. H. Atia, S. N. El-Deeb and I. A. Hasanein, A note on strong compactness and S-closedness, *Matematicki Vesnik*, **6** (1982), 23—28.
[3] N. Levine, Semiopen sets and semicontinuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36—41.
[4] E. P. Long and L. L. Herrington, Basic properties of regular closed functions, *Rendiconti Del Cirolo Matematico Di Palermo Serie II*, **XXVII** (1978), 20—28.
[5] A. S. Mashhour and I. A. Hasanein, On S-closed spaces and almost (nearly) strong paracompactness, *Ann. de la Soc. Sci. de Bruxelles*, **95** (1981), 35—44.
[6] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, α-continuous and α-open mappings, *Acta Math. Hungar.*, **41** (1982), 213—218.
[7] O. Njåstad, On some classes of nearly open sets, *Pac. J. Math.*, **15** (1965), 961—970.
[8] T. Noiri, A generalization of closed mappings, *Atti. Accad. Naz. Lincei, Rend. Cl. Sc. Fis. Mat. Nature*, **3** (1973), 412—415.
[9] T. Noiri, On S-closed spaces, *Ann. Soc. Sci. Bruxelles*, **91** (1977), 189—194.
[10] T. Noiri, On S-closed subspaces, *Atti. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat.*, (1978), 374—380.
[11] T. Noiri, Properties of S-closed spaces, *Acta Math. Acad. Sci. Hungar.*, **35** (1981), 431—436.
[12] M. K. Singal and A. R. Singal, Almost-continuous mappings, *Yokohama Math. J.*, **16** (1968), 63—73.
[13] T. Thompson, S-closed spaces, *Proc. Amer. Math. Soc.*, **60** (1976), 335—338.
[14] T. Wilansky, *Topics in functional Analysis*, Lecture Notes in Math. 45, Springer-Verlag (Berlin, 1967).
[15] M. E. Abd El-Monsef and A. M. Kozea, β-open sets and S-closed spaces, *Bull. Fac. Sci., Assiut Univ.*, **14** (1985), 129—140.

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DEPARTMENT OF MATHEMATICS
ASSIUT UNIVERSITY
ASSIUT
EGYPT

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PERIODIC SOLUTIONS FOR SCALAR LIENARD EQUATIONS

J. J. NIETO (Santiago de Compostela) and V. S. H. RAO (Hyderabad)

1. Introduction

In this paper we consider the scalar Lienard equation

$$u''+cu'+g(u)=e \tag{1.1}$$

with the periodic boundary conditions

$$u(0)=u(T),\quad u'(0)=u'(T) \tag{1.2}$$

in which the functions $g\colon \mathbf{R}\to\mathbf{R}$ and $e\colon [0,T]\to\mathbf{R}$ are continuous and c is a real constant. If e is periodic of period T, then it may be seen that a solution of the periodic boundary value problem (PBVP, for short) (1.1)—(1.2) is a periodic solution of period T for the equation (1.1). We refer the readers to [10, 16] and the references there in for the literature on the existence of periodic solutions for the equation (1.1). The important special case $c=0$ (also known as the conservative case) is treated for $T=2\pi$ in [4, 9], and more recently in [7, 15, 17].

This paper is organized as follows. Section 2 deals with the existence of solutions for the PBVP (1.1)—(1.2). While studying this problem, we distinguish two cases for the nonlinearity namely, the cases when (I) g is decreasing and (II) g is increasing. For the case (I), we use a result of [6]. We show that when g is strictly increasing the method of [6] is not useful, and in the case (II) we employ an abstract existence theorem for problems at resonance [3, 13, 14]. Also, we present existence results based on the techniques of [5]. We note that our methods and techniques are different from those employed in [10, 16], and thus our results extend some of the results in [16].

In Section 3, we study the structure of the set of solutions of the PBVP (1.1)—(1.2). Here also we consider two cases:

$$\text{(i)}\quad \omega=\frac{1}{T}\int_0^T e(t)\,dt\in \text{Interior (Range } g), \quad\text{and}\quad \text{(ii)}\quad \omega\in\text{Boundary (Range } g).$$

It is shown that if g is monotone and (i) holds, then the solution set is nonempty, connected and acyclic (Theorems 3.2 and 3.5(a)). On the other hand, if g is monotone and (ii) holds, then it is shown that the solution set is either empty or connected and unbounded, and in the latter case the solution set is homeomorphic to a closed and unbounded real interval (Theorems 3.3 and 3.5(b)).

If g has monotone character only "at infinite" then the solution set may be "chaotic". Also we construct a g such the set of periodic solutions of the equation $u''+cu'+g(u)=0$ is a given set K (Theorems 3.8 and 3.10), and this set can be even a Cantor set. Indeed, the results of this section are in the spirit of those in [12] about semilinear parabolic partial differential equations. Although results dealing with the multiplicity of the solutions are known (see for instance [11]) we think that our results about the structure of the solution set of (1.1)—(1.2) are new.

2. Existence of solutions

We now consider the second order equation

$$-u''=f(t,u,u') \tag{2.1}$$

in which we assume that f is continuous on the set $S=\{(t,x,y)\colon\ t\in[0,T],\ x,y\in\mathbf{R}\}$. A function $\alpha\in C^2[0,T]$ will be called a lower solution of (2.1) on $[0,T]$ if $-\alpha''\leqq f(t,\alpha,\alpha')$ on $[0,T]$. Similarly $\beta\in C^2[0,T]$ will be called an upper solution of (2.1) if $-\beta''\geqq f(t,\beta,\beta')$ on $[0,T]$.

We shall state the following result of [6].

THEOREM 2.1. *Assume the following:*

(a) *there exists* α *and* β *lower and upper solutions of* (2.1) *on* $[0,T]$, *respectively, with* $\alpha(0)=\alpha(T)$, $\beta(0)=\beta(T)$, *and* $\alpha(t)\leqq\beta(t)$ *for every* $t\in[0,T]$;

(b) *the inequalities* $\alpha'(0)\geqq\alpha'(T)$ *and* $\beta'(0)\leqq\beta'(T)$ *hold; and*

(c) f *satisfies the following Nagumo condition relative to* α, β [2, p. 25]: *There exists* $h\in C[[0,\infty),(0,\infty)]$ *such that* $|f(t,u,v)|\leqq h(|v|)$ *whenever* $\alpha(t)\leqq u\leqq\beta(t)$, $t\in[0,T]$ *and h is such that* $\int_0^\infty \frac{s\,ds}{h(s)}=\infty$.

Then the PBVP

$$-u''=f(t,u,u'),\quad u(0)=u(T),\quad u'(0)=u'(T) \tag{2.2}$$

has at least one solution u such that $\alpha(t)\leqq u(t)\leqq\beta(t)$ *for every* $t\in[0,T]$.

This theorem is proved by using an abstract existence theorem at resonance developed in [3]. We note that this result is proved in the Hilbert space setting and the resulting solutions will be in $L^2(0,T)$.

We write $f(t,u,u')=cu'+g(u)-e(t)$ and state the following:

LEMMA 2.2. *Let* α, β *be lower and upper solutions of* (1.1), *respectively, satisfying the conditions* (a) *and* (b) *of Theorem* 2.1. *Then f satisfies the Nagumo condition* (c) *of Theorem* 2.1 *relative to* α, β.

PROOF. For (1.1), using the notation of Theorem 2.1, we have $f(t,u,v)=g(u)+cv-e(t)$. Taking into account that g and e are continuous we see that there exists $K>0$ such that $|g(u(t))|+|e(t)|\leqq K$ for $\alpha(t)\leqq u\leqq\beta(t)$, $t\in[0,T]$. Thus, we can take $h(s)=cs+K$.

Now we discuss the existence of solutions for the PBVP (1.1)—(1.2) in the following cases:

I: g is decreasing on $\mathbf{R}$, that is for $u,v\in\mathbf{R}$ and $u\leqq v$, $g(u)\geqq g(v)$.

II: g is increasing in $\mathbf{R}$, that is for $u,v\in\mathbf{R}$ and $u\leqq v$, $g(u)\leqq g(v)$.

Case I: We assume that g is decreasing on $\mathbf{R}$ and prove the following main result.

THEOREM 2.3. *If g is decreasing, then the PBVP* (1.1)—(1.2) *has a solution if and only if* $\omega = \frac{1}{T}\int_0^T e(t)\,dt \in \text{Range}\, g$.

PROOF. Assume that $\omega \in \text{Range}\, g$. Then there exists $r \in \mathbf{R}$ such that $g(r) = \omega$. Observe that the linear problem

$$u'' + cu' = e - g(r), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{2.3}$$

has a solution since $\int_0^T [e(t) - g(r)]dt = 0$. Let v be the solution of (2.3) satisfying $\int_0^T v(t)\,dt = 0$. Choose constants a and b such that $v(t) + a \leqq r \leqq v(t) + b$ for each $t \in [0, T]$. Define $\alpha = v + a$, and $\beta = v + b$.

Thus,

$$-\alpha'' = -v'' = cv' + g(r) - e = c\alpha' + g(r) - e \leqq c\alpha' + g(\alpha) - e,$$

and

$$-\beta'' = -v'' = cv' + g(r) - e = c\beta' + g(r) - e \geqq c\beta' + g(\beta) - e,$$

so that α and β are lower and upper solutions of (1.1) respectively. From Lemma 2.2, it follows that $f(t, u, u') = cu' + g(u) - e$ satisfies a Nagumo condition relative to α, β. Hence, in view of Theorem 2.1, the PBVP (1.1)—(1.2) has a solution u such that $\alpha \leqq u \leqq \beta$ on $[0, T]$.

Conversely suppose that u is a solution of the PBVP (1.1)—(1.2). Integrating (1.1) on $[0, T]$ and using (1.2), we get

$$\int_0^T g(u(t))\,dt = \int_0^T e(t)\,dt = T\omega.$$

Since g is decreasing, we have $g(\infty) \leqq g(u) \leqq g(-\infty)$ for $u \in \mathbf{R}$. Then, we have $\frac{1}{T}\int_0^T g(u(t))\,dt \in [g(\infty), g(-\infty)]$ and this implies that $\omega \in [g(\infty), g(-\infty)] = \overline{\text{Range}\, g}$.

If $\omega \notin \text{Range}\, g$, then either $\omega \geqq g(-\infty)$ or $\omega \leqq g(\infty)$, and this, in view of the fact that $\omega \in [g(\infty), g(-\infty)]$, implies that either $\omega = g(\infty)$ or $\omega = g(-\infty)$. Suppose $\omega = g(\infty)$. If $\omega \notin \text{Range}\, g$, then $g(u(t)) > \omega$ for every $t \in [0, T]$ and hence $\int_0^T g(u(t))\,dt > T\omega$, which is a contradiction. Now suppose that $\omega = g(-\infty)$. If $\omega \notin \text{Range}\, g$, then $g(u(t)) < \omega$ for every $t \in [0, T]$ and hence $\int_0^T g(u(t))\,dt < T\omega$, which is again a contradiction. This completes the proof of the theorem.

Case II: We now turn to the case in which g is increasing on $\mathbf{R}$. We first show that the method of upper and lower solutions described in Theorem 2.1 is not useful if g is strictly increasing. Indeed, assume that conditions (a) and (b) of Theorem 2.1 hold and also assume that g is strictly increasing. We have from (a) that $\beta''-\alpha''\leqq \leqq c(\alpha'-\beta')+g(\alpha)-g(\beta)$ and integration on $[0, T]$ together with (1.2) yields

$$\int_0^T [g(\alpha)-g(\beta)](\beta-\alpha)\,dt \geqq 0.$$

On the other hand, since g is strictly increasing we have $\int_0^T [g(\alpha)-g(\beta)](\beta-\alpha)\,dt\leqq 0$. Thus, $\int_0^T [g(\alpha)-g(\beta)](\beta-\alpha)\,dt=0$. This implies that $\alpha=\beta$ since g is strictly increasing.

Also, we see that the conclusion of Theorem 2.3 does not hold in general as may be seen from the following

EXAMPLE. In the equation (1.1) let $c=0$, $T=2\pi$ and $g(u)=u$, so that g is strictly increasing and Range $g=\mathbf{R}$. Also $\omega\in\text{Range } g$, for any e. However, equation (1.1) has no solution unless $\int_0^{2\pi} e(t)\sin t\,dt=0$ and $\int_0^{2\pi} e(t)\cos t\,dt=0$.

Therefore, we need a different method to study the PBVP (1.1)—(1.2) when g is increasing. We employ the alternative method developed in [3, 13]. In what follows, we shall consider only the nonconservative case which corresponds to $c\neq 0$.

We consider the equation

$$Lu = Nu \tag{2.4}$$

in which $L\colon D(L)\subset E\to F$ and $N\colon E\to F$ are linear and nonlinear operators respectively, and E and F are Banach spaces. We assume the following assumptions.

(H1) there exists projections $P\colon E\to E$ and $Q\colon F\to F$ such that
- (a) $H(I-Q)Lu=(I-P)u$, for every $u\in D(L)$;
- (b) $QLu=LPu$, for every $u\in D(L)$;
- (c) $LH(I-Q)Nu=(I-Q)Nu$, for every $u\in E$;

in which $H\colon (I-Q)F\to(I-P)E$, the partial inverse of L, is a linear operator.

Equation (2.4) is equivalent to the system

$$u = Pu+H(I-Q)Nu \qquad \text{(auxiliary equation)}. \tag{2.5}$$

$$Q(Lu-Nu) = 0 \qquad \text{(bifurcation equation)}. \tag{2.6}$$

We set $E_0=PE$, $E_1=(I-P)E$, $F_0=QF$, and $F_1=(I-Q)F$.

(H2) $E_0=\text{Ker } L$, $F_1=\text{Range } L=D(H)$, $E_1=\text{Range } H$, and $\dim E_0=\dim F_0<\infty$.

(H3) there exists continuous maps $B: E\times F\to\mathbf{R}$ and $J: F_0\to E_0$ with
(a) B is bilinear and J is one-to-one and onto;
(b) $v_0\in F_0$, $v_0=0$ if and only if $B(u_0, v_0)=0$ for each $u_0\in E_0$;
(c) $Jv_0=0$ if and only if $v_0=0$;
(d) $B(Jv_0, v_0)\geqq 0$ for each $v_0\in F_0$;
(e) $B(Jv_0, v_0)=0$ if and only if $v_0=0$;
(f) $B(u_0, J^{-1}u_0)=0$ if and only if $u_0=0$; and
(g) $B(u_0, v_0)=B(Jv_0, J^{-1}u_0)$ for each $u_0\in E_0$ and $v_0\in F_0$.

It may be noted that under the hypotheses (H1), (H2) and (H3), the solutions of the operator equation (2.4) are the same as the fixed points of the operator $T: E\to E$ defined by $Tu=Pu+H(I-Q)Nu+JQNu$. We state the following result of [14] which extends the results of [3, 13] by dropping the boundedness assumption on the nonlinear operator N. This result is used in our subsequent work.

THEOREM 2.4. *Assume that hypotheses* (H1), (H2) *and* (H3) *hold. In addition assume that H is compact and N maps bounded sets into bounded sets. Finally, suppose that there exists numbers $R>R_0>0$ such that*
(a) *the set $C(R)=\{u_1\in E_1: u_1=\lambda H(I-Q)N(u_0+u_1)$ for some $\lambda\in[0, 1]$ and $u_0\in E_0$ with $\|u_0\|\leqq R\}$ is bounded, and*
(b) *$B(u_0, QN(u_0+u_1))\leqq 0$ for every $u=u_0+u_1$, where $\|u_0\|=R_0$ and $u_1=\lambda H(I-Q)N(u_0+u_1)$ for some $\lambda\in[0, 1]$.*
Then equation (2.4) *has at least one solution.*

Now we consider the PBVP (1.1)—(1.2). Let

$$E = \{u\in C^1[0, T]: u(0) = u(T),\ u'(0) = u'(T)\}$$

and $F=L^2(0, T)$. Define $L: D(L)\subset E\to F$ by $Lu=u''+cu'$, in which $D(L)=\{u\in E: u\in C^2(0, T)\}$ and $N: E\to F$ by $Nu=e-g(u)$. The projections $P: E\to E$ and $Q: F\to F$ are given by $Pu=u(0)$ and $Qu=\frac{1}{T}\int_0^T u(t)\,dt$. Finally, the operator $H: F_1\to E_1$ may be defined by $Hv=u$ if and only if $u''+cu'=v$, $u(0)=u(T)=0$, $u'(0)=u'(T)$. It is easy to see that the hypotheses (H1) are satisfied. Further, E_0, the kernel of L consists of all constant functions and the range of L is the class of all functions whose average is zero, that is, Range $L=\{u\in F: Qu=0\}$. On the other hand, it is obvious that (H2) holds.

We define $B: E\times F\to\mathbf{R}$ and $J: F_0\to E_0$ respectively by $B(u, v)=\int_0^T u(t)\cdot v(t)dt$ and $Jv_0=v_0$, so that (H3) is satisfied.

Clearly H is compact (the inclusion of H^2 into C^1 is compact) and N maps bounded sets into bounded sets. We now verify the conditions (a) and (b) of Theorem 2.4.

Let $R>0$ and $u_1\in C(R)$, so that $u_1=\lambda H(I-Q)N(u_0+u_1)$ for some $\lambda\in[0,1]$ and $u_0\in E_0$. Hence, $u_1''+cu_1''=\lambda(I-Q)N(u_0+u_1)$ and u_1 is T-periodic. Multiplying by u_1', we get

$$B(u_1'', u_1')+cB(u_1', u_1') = \lambda B((I-Q)N(u_0+u_1), u_1'). \tag{2.7}$$

Now, $B(u_1'', u_1')=0$ and $B(g(u_0+u_1), u_1')=0$ since $u_1'=(u_0+u_1)'$, u_0 is a constant. Then (2.7) becomes

$$cB(u_1', u_1') = \lambda B((I-Q)e, u_1'). \tag{2.8}$$

But $B((I-Q)e, u_1')=B(e, u_1')$ since $u_1'\in F_1$. Hence, from (2.8) we have $cB(u_1', u_1')= =\lambda B(e, u_1')$ and this in turn yields $c\|u_1'\|^2=\lambda\langle e, u_1'\rangle$, where $\langle\ ,\ \rangle$ denotes the usual inner product in L^2 and $\|u\|^2=\langle u, u\rangle$. Using the Cauchy—Schwarz inequality we get

$$-\|e\|\cdot\|u_1'\| \leqq c\|u_1'\|^2 \leqq \|e\|\cdot\|u_1'\|. \tag{2.9}$$

If $c\neq 0$, then from (2.9) we have

$$\|u_1'\| \leqq \frac{\|e\|}{|c|} = A. \tag{2.10}$$

Using (2.10) in the identity $u_1(t)=u_1(0)+\int_0^t u_1'(s)\,ds$, we get

$$|u_1(t)| \leqq \int_0^t |u_1'(s)|\,ds \leqq \|u_1'\|_1 \leqq \sqrt{T}\cdot\|u_1'\| \leqq \sqrt{T}\cdot A = B. \tag{2.11}$$

From (2.11) it is clear that $C(R)$ is bounded in $H^1(0, T)$ and moreover it is bounded independently of $R>0$.

Now for $u_1\in C(R)$, we have $u_1''=e-cu_1'-g(u_0+u_1)$. If $\|u_0\|\leqq R$, then u_1' is bounded in L^2 and $C(R)$ is bounded in $H^2(0, T)$, and in consequence, $C(R)$ is bounded in $C^1[0, T]$ and in E. Take $R=R_0$ so that

$$\|C(R_0)\| = \sup_{u_1\in C(R_0)} \|u_1\|_E = \gamma(R_0). \tag{2.12}$$

Thus, condition (a) of Theorem 2.4 holds and a bound for the set $C(R)$ is given by $\gamma(R)$.

We next verify the condition (b) of Theorem 2.4. Note that

$$B(u_0, QNu) = \int_0^T u_0[e(t)-g(u(t))]\,dt.$$

Then

$$\operatorname{sgn} u_0^{-1}\cdot B(u_0, QNu) = \operatorname{sgn}\int_0^T [e(t)-g(u(t))]\,dt,$$

and hence we need to study the behaviour of $\int_0^T [e(t)-g(u(t))]\,dt$. Now, if we suppose that there exists $R_0>0$ such that

$$\begin{cases} \int_0^T [e(t)-g(R_0+u_1(t))]\,dt \leqq 0 \quad \text{and} \\ \int_0^T [e(t)-g(-R_0+u_1(t))]\,dt \geqq 0 \end{cases} \tag{2.13}$$

for every $u_1 \in C(R_0)$, then the condition (b) of Theorem 2.4 is satisfied and thus the equation (2.4) has a solution. However, (2.13) is difficult to verify since $\gamma(R_0)$ depends on R_0. But from (2.11) we obtain that $|u_1(t)| \leqq B = \sqrt{T} \cdot |c|^{-1} \cdot \|e\|$, a constant which depends only on $\|e\|$. Thus from (2.10) and (2.11) we have

$$\|u_1\|_E = \sup_{t \in [0,T]} |u_1(t)| + \|u_1'\| \leqq A + B = \delta. \tag{2.14}$$

Therefore, $C(R_0)$ is bounded in E independently of R_0.

Now (2.13) is equivalent to the following condition: there exists $R_0 > 0$ such that

$$QN(R_0 + u_1) \leqq 0 \leqq QN(-R_0 + u_1) \tag{2.15}$$

for every u_1 such that $u_1 = \lambda H(I-Q)N(u_0+u_1)$.

By the above arguments we have proved the following

THEOREM 2.5. *The PBVP* (1.1)—(1.2) *has a solution if* $c \neq 0$ *and condition* (2.15) *holds.*

Now assume that $\lim_{u \to \pm\infty} g(u) = g(\pm\infty)$ exists and

$$g(-\infty) \leqq g(u) \leqq g(\infty), \quad \text{for every} \quad u \in \mathbf{R}. \tag{2.16}$$

The well-known Landesman—Lazer condition [8] is

$$g(-\infty) < \omega < g(\infty). \tag{2.17}$$

COROLLARY 2.6. *If g satisfies* (2.16) *and* (2.17) *holds, then the PBVP* (1.1)—(1.2) *has at least one solution, provided that* $c \neq 0$.

PROOF. Let $\delta = A + B = |c|^{-1} \cdot \|e\| \cdot (1 + \sqrt{T})$, where A and B are as in (2.10) and (2.11) respectively. Thus, in view of (2.14), we get

$$-\delta \leqq u_1(t) \leqq \delta$$

for every $u_1 \in C(R_0)$, $t \in [0, T]$ and $R_0 > 0$.

From (2.17), there exists $M > 0$ such that $g(-u) < \omega < g(u)$ for $u > M$. Choose $R_0 \geqq \delta + M$ so that $R_0 + u_1(t) \geqq M$ and $-R_0 + u_1(t) \leqq -M$.

Then, $u_0 \cdot \int_0^T [e(t) - g(u_0 + u_1(t))]\, dt \leqq 0$ for $u_0 = \pm R_0$ and thus condition (2.15) of Theorem 2.5 is satisfied.

COROLLARY 2.7. *Suppose that g is increasing and* $c \neq 0$. *Then the PBVP* (1.1)— (1.2) *has a solution if and only if* $\omega \in \operatorname{Range} g$.

PROOF. Suppose $\omega \in \operatorname{Range} g$. From (2.16) we have that $g(-\infty) \leqq \omega \leqq g(\infty)$. If $g(-\infty) < \omega < g(\infty)$, then the PBVP (1.1)—(1.2) has a solution in view of Corollary 2.6.

Consider now the case $\omega = g(\infty)$ and similar proof holds for the case where $\omega = g(-\infty)$. Since $\omega \in \operatorname{Range} g$, there exists a number r such that $g(r) = \omega$. Hence, $g(r) = g(\infty)$ and $g(u) = g(\infty)$ for $u \geqq r$ since g is increasing. Now the problem (2.3) has a solution v (say) satisfying (1.2) since $\int_0^T [e(t) - \omega]\, dt = 0$. Let $a > 0$ be a constant such that $v(t)+a \geqq r$ for $t \in [0, T]$. Then $u(t)=v(t)+a$ is a solution of (1.1)—(1.2) since $g(u(t))=\omega$ for $t \in [0, T]$.

The remaining assertion may be proved by integrating (1.1) and reasoning as in the proof of Theorem 2.3.

REMARK 2.8. We notice that condition (i) on [16, page 78] implies that g is strictly increasing, and condition (ii) implies that g is strictly decreasing. Furthermore, in both cases Range $g=\mathbf{R}$. Thus our results in Theorems 2.3, 2.5 and Corollary 2.6 are fairly more general than those of [16].

In [5], PBVP's for equations more general than (1.1) are considered and conditions for the existence of solutions are given by employing the methods consisting of Liapunov — Schmidt, Leray—Schauder degree and monotonicity. Those results specialized for our problem (1.1)—(1.2) yield the following theorems.

THEOREM 2.9. *Let g be continuous and let there exists a constant $\varrho>0$ such that $ug(u) \leqq 0$ for $|u| \geqq \varrho$. Then, for any continuous function e: $[0, T] \to \mathbf{R}$ with $Qe=0$ and $c \in \mathbf{R}$, the PBVP* (1.1)—(1.2) *has at least one solution.*

THEOREM 2.10. *Let g be continuous and let the following conditions hold.*

(a) *there exists a constant $\varrho>0$ such that $ug(u) \geqq 0$ for $|u| \geqq \varrho$; and*

(b) $\limsup_{|u| \to \infty} \frac{g(u)}{u} < 4\pi^2 \quad (T=1).$

Then for any given continuous function e: $[0, T] \to \mathbf{R}$ with $Qe=0$, and $c \in \mathbf{R}$, the PBVP (1.1)—(1.2) *has at least one solution.*

The proof of Theorems 2.9 and 2.10 is a straightforward application of Theorems 3.1 and 3.2 of [5] respectively.

We note that for $c \neq 0$ we do not require condition (b) of Theorem 2.10 (see Corollaries 2.6 and 2.7).

REMARK 2.11. From the results of [5], it is possible to drop the Nagumo condition in Theorem 2.1 for the PBVP (2.2) but this study is not pursued here.

3. Structure of the solution set

If g is strictly decreasing and $\omega \in$ Range g, then any solution of (1.1)—(1.2) is unique. Indeed, if $u, v \in D(L)$ are solutions we get

$$0 = \langle L(u-v), u-v \rangle + \langle g(u)-g(v), u-v \rangle \leqq \langle g(u)-g(v), u-v \rangle \leqq 0$$

which implies that $u=v$ since g is strictly decreasing. On the other hand, if g is decreasing, uniqueness does not occur in general as may be seen from the following

EXAMPLE 3.1. Let g be decreasing and $g(u)=0$ for $u \in (-1, 1)$. For the problem $u''+cu'+g(u)=0$, with the periodic conditions (1.2), any constant $a \in (-1, 1)$ is a solution, and thus the problem has infinitely many solutions.

In the results below we study the structure of the set $S_1=\{u \in E\colon Lu=Nu\}$, that is, the set of solutions of the PBVP (1.1)—(1.2). We shall distinguish two cases:

(i) $\omega \in \operatorname{Int}$ (Range g), and

(ii) $\omega \in \partial$(Range g).

THEOREM 3.2. *Suppose that g is decreasing and $\omega \in \text{Int}(\text{Range}\, g)$. Then the set of solutions of the PBVP (1.1)—(1.2) is nonempty, compact, connected, and acyclic.*

PROOF. Consider the sets

$$S_0 = \{u \in \text{Ker}\, L\colon\ Nu \in \text{Range}\, L\},$$

$$S_+ = \{u \in E\colon\ Lu = \lambda Nu, \text{ for some } \lambda \in (0, 1)\},$$

$$S_+^0 = \{u_0\colon\ u = u_0 + u_1 \in S_+\}, \quad S_+^1 = \{u_1\colon\ u = u_0 + u_1 \in S_+\}.$$

We first show that S_0 is bounded. If $u \in S_0$, then $\int_0^T [g(u(t)) - e(t)]\,dt = 0$. Since u is constant, we see that $g(u) = \omega$ and therefore S_0 is bounded since $g(\infty) < \omega < g(-\infty)$.

We next show that S_+ is bounded. If $u = u_0 + u_1 \in S_+$, then there exists $\lambda \in (0, 1)$ such that $u_1 = \lambda H(I-Q)N(u_0+u_1)$. We have already seen in Section 2 that the set $C(R)$ is bounded independently of R. Hence, the set S_+^1 is bounded in E. On the other hand if $R > 0$ is large we have for every $t \in [0, T]$ and $u_1 \in S_+^1$ that

$$g(R + u_1(t)) \leqq \omega \leqq g(-R + u_1(t)). \tag{3.1}$$

If $u \in S_+$, then by integrating we get that $\int_0^T g(u(t))\,dt = T\omega$. From (3.1), we see that S_+^0 is also bounded. Therefore, S_+ is bounded. Now define the maps $G\colon \mathbf{R} \to E$ and $G_1\colon F \to \mathbf{R}$ by $G(a) = a$ and $G_1 v = Qv$. Let $\xi(a) = G_1 \cdot N \cdot G(a) = \frac{1}{T} \int_0^T [e(t) - g(a)]\,dt$. Clearly for large $a > 0$, we have $\xi(a) \cdot \xi(-a) < 0$ and degree $(\xi, 0, (-a, a)) \neq 0$. Taking $a > 0$ such that $S_0 \cup S_+ \subset \{u \in E\colon\ \|u\|_E \leqq a\}$, we conclude from [1] that $S_1 \neq \emptyset$ and S_1 is compact in E.

In the following we show that S_1 is connected. For each positive integer n define the operator $N_n\colon E \to F$ by $N_n(u) = Nu + \frac{1}{n} u$. It is easy to see that N_n converges to N uniformly as $n \to \infty$ on bounded subsets of E since $\|N_n u - Nu\| = \frac{1}{n}\|u\|$. Let $v \in S_1$.

If $u \in S_n(v) = \{u \in E\colon\ Lu - N_n u = Lv - N_n v\}$, then u satisfies the PBVP $u'' + cu' + g(u) - \frac{1}{n} u = v'' + cv' + g(v) - \frac{1}{n} v$, $u(0) = u(T)$, $u'(0) = u'(T)$, or equivalently

$$u'' + cu' + G(u) = E(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{3.2}$$

where $G(u) = g(u) - \frac{1}{n} u$ and $E(t) = v''(t) + cv'(t) + g(v(t)) - \frac{1}{n} v(t)$. Now (3.2) has a unique solution since G is strictly decreasing and Range $G = \mathbf{R}$. Therefore the set $S_n(v)$ is a singleton and, therefore, connected for every n and $v \in S_1$. As a consequence of the results in [1], S_1 is connected.

Finally we show that S_1 is acyclic. Let $a>0$ be such that $S_0\cup S_1\subset\{u\in E:\ \|u\|_E\leqq a\}$. Let $r_n=\operatorname{Sup}\{\|N_nu-Nu\|_E:\ \|u\|_E=a\}$. Clearly $r_n\to 0$ as $n\to\infty$ and the problem $Lu-N_nu=v$ has at most one solution for any $v\in E$. Consequently S_1 is acyclic (see [1]). This completes the proof of Theorem 3.2.

Note that in Theorem 3.2, $g(-\infty)$ and $g(\infty)$ need not be finite, that is g need not be bounded, unlike Theorem 3 of [12].

THEOREM 3.3. *If g is decreasing and $\omega\in\partial(\text{Range}\, g)$, then either S_1 is empty or connected and unbounded. In the latter case, S_1 is homeomorphic to a real interval of the type $[a,\infty)$, $(-\infty, b]$, $(-\infty,\infty)$.*

REMARK 3.4. It is interesting to note that the case $S_1=\emptyset$ is possible under the hypotheses of Theorem 3.3. Indeed, consider $g(u)=e^{-u}$ and the PBVP

$$u''+cu'+g(u)=0,\quad u\in E. \tag{3.3}$$

Thus, $\text{Range}\, g=(0,\infty)$ and $\omega\in\partial(\text{Range}\, g)$; but the PBVP (3.3) has no solution since $g(u)>0$ for every $u\in\mathbf{R}$.

PROOF OF THEOREM 3.3. If $S_1\neq\emptyset$, then by Theorem 2.3, $\omega\in\text{Range}\, g$. We consider the case where $\omega=g(\infty)$ and the proof for the other case, that is $\omega=g(-\infty)$, is similar. Let $r=\operatorname{Inf}\{u:\ g(u)=\omega\}\geqq-\infty$. Hence $g(u)=\omega$ for $u>r$ and $g(u)<\omega$ for $u<r$. If u is a solution of $Lu=Nu$, then $\int_0^T g(u(t))dt=T\omega$. Therefore $u(t)\geqq r$ for every $t\in[0,T]$. This implies that u satisfies the linear problem

$$u''+cu'+\omega=e,\quad u\in E, \tag{3.4}$$

and it may be seen easily that $S_1=\{u:\ u \text{ solves (3.4) and } \bar u\geqq r\}$ where $\bar u=\min\{u(t):\ t\in[0,T]\}$. If v is a solution of (3.4) satisfying $Qv=0$, then $S_1=\{v+a:\ \bar v+a\geqq r\}$, and further this set is homeomorphic to the interval $(-\infty,\infty)$ if $r=-\infty$ and to $[r-\bar v,\infty)$ if $r>-\infty$. Now it is clear that S_1 is connected and unbounded. This completes the proof.

When g is increasing, following the same reasoning as in the proof of Theorems 3.2 and 3.3, we may prove the theorem given below.

THEOREM 3.5. *Suppose that g is increasing. We have*

(a) *if $\omega\in\text{Int}\,(\text{Range}\, g)$, then S_1 is nonempty, compact, connected and acyclic;*

(b) *if $\omega\in\partial(\text{Range}\, g)$, then either $S_1=\emptyset$, or S_1 is connected and unbounded. Further, in the latter case, S_1 is homeomorphic to a closed and unbounded real interval.*

REMARK 3.6. If, instead of g being monotone (either increasing or decreasing), g satisfies

$$\begin{cases}\text{either} & g(-\infty)\leqq g(u)\leqq g(\infty), \quad \text{for each} \quad u\in\mathbf{R}\\ \text{or} & g(\infty)\leqq g(u)\leqq g(-\infty), \quad \text{for each} \quad u\in\mathbf{R}\end{cases} \tag{3.5}$$

then the results of this section on the structure of the solution set S_1 are no longer valid. Even Theorem 2.3 and Corollary 2.7 are not true. This may be seen from the following example.

EXAMPLE 3.7. Define $g(u)=-u$ if $u\leqq 0$, $g(u)=\dfrac{u}{1+u^2}$ if $u>0$, and consider the PBVP

$$u''+cu'+g(u)=\sin t,\quad u\in E. \tag{3.6}$$

In this case $\omega=0\in\partial(\text{Range}\, g)$ since $\text{Range}\, g=(0,\infty)$. On the other hand, g satisfies (3.5) but the problem (3.6) has no solution. Indeed, if u is a solution, then $\int_0^T g(u(t))\,dt=0$, and this implies that $u(t)=0$ for all $t\in[0,T]$ since $g\geqq 0$. But $u=0$ is not a solution of (3.6).

In the next theorems we shall show that the solution set may be "chaotic" if g satisfies (3.5) only.

Consider the PBVP

$$u''+cu'+g(u)=0,\quad u\in E. \tag{3.7}$$

Let $i\colon \mathbf{R}\to E$ the canonical injection.

THEOREM 3.8. *Let K be a nonempty closed subset of $\mathbf{R}$ bounded above (or below). Then there exists $g\colon \mathbf{R}\to\mathbf{R}$ continuous and satisfying* (3.5) *such that $\omega=0\in\partial(\text{Range}\, g)$, and $S_1=i(K)$ for the problem* (3.7).

PROOF. Assume that K is bounded above. Let $b=\max K$. Define

$$h(u)=\begin{cases} f(u) & \text{if } u\leqq b\\ u-b & \text{if } b<u<b+1\\ 1 & \text{if } u\geqq b+1\end{cases}$$

where f is as given in Lemma 2 of [12]. Then arguing as is in the proof of Theorem 8 of [12] one can complete the proof. The case when K is bounded below is similar.

To give an analogous result with $\omega\in\text{Int}\,(\text{Range}\, g)$, we need some preliminary results.

For $s\in\mathbf{R}$, define the function

$$f_s(u)=\begin{cases} -1 & \text{if } u\leqq s-1\\ u-s & \text{if } s-1<u<s+1\\ 1 & \text{if } u\geqq s+1.\end{cases}$$

For $\varepsilon>0$, consider the following PBVP

$$u''+cu'+g(u)=0,\quad u\in E \tag{3.8}$$

where $g=\varepsilon f_s$.

LEMMA 3.9. *There exists $\mu>0$ such that for $0<\varepsilon<\mu$ we have:*

(a) *any solution u of* (3.8) *is such that $u(t)\in(s-1,s+1)$ for every $t\in[0,T]$;*

(b) *the set of solutions of* (3.8) *is nonempty, compact, connected and acyclic.*

PROOF. For ε small we have that $\|u_1\|_E \leqq T^{-1}$ for any solution of the auxiliary equation corresponding to (3.8). If u is a solution of (3.8) and for some $t_0 \in [0, T]$ we have $u(t_0) > s+1$, then from the relation $u(t) = u(t_0) + \int_{t_0}^{t} u'(s)\,ds = u(t_0) + \int_{t_0}^{t} u_1'(s)\,ds$ and the estimate for u_1, we obtain that $u(t) > s+1-T\|u_1\|_\infty > s$. Hence, $g(u(t)) > 0$ for every $t \in [0, T]$ and $\int_0^T g(u(t))\,dt > 0$. This is a contradiction since integrating (3.8) between 0 and T we get $\int_0^T g(u(t))\,dt = 0$. The case when $u(t_0) < s-1$ for some $t_0 \in [0, T]$ is similar. This proves (a). Part (b) follows from Theorem 3.5.

THEOREM 3.10. *Let K be a compact subset of* $\mathbf{R}$ *which is not a singleton. Then there exists* $g\colon \mathbf{R} \to \mathbf{R}$ *such that* (3.5) *holds,* $\omega = 0 \in \operatorname{Int}(\operatorname{Range} g)$ *and the solution set for the PBVP* (3.7) *is* $S_1 = i(K) \cup B$ *where* $i(K) \cap B = \emptyset$ *and B is connected. Moreover, if* $a = \min K$ *and* $u \in B$ *then* $u(t) < a-1$ *for every* $t \in [0, T]$.

PROOF. There exists $h \in C^1(\mathbf{R}, \mathbf{R})$ with $0 \leqq h \leqq 1$ and $K = \{u \in \mathbf{R} : h(u) = 0\}$. Set $\min K = a < b = \max K$, $s = a-2$ and define

$$g(u) = \begin{cases} \varepsilon f_s(u) & \text{if } u \leqq a-1 \\ -\varepsilon(u-a) & \text{if } a-1 < u < a \\ \varepsilon h(u) & \text{if } a \leqq u \leqq b \\ u-b & \text{if } b < u \leqq b+\varepsilon \\ \varepsilon & \text{if } u > b+\varepsilon. \end{cases}$$

Thus, $|g(u)| < \varepsilon$ for every $u \in \mathbf{R}$. We choose $\varepsilon > 0$ small. If $\alpha \in K$ then $u = i(\alpha) \in S_1$ which shows that $i(K) \subset S_1$. On the other hand, reasoning as in the proof of part (a) of Lemma 3.9 we have that if u is a solution of (3.7), then $u(t) \neq a-1$ for every $t \in [0, T]$.

If u is a solution and $u(t) > a-1$ for every $t \in [0, T]$ then $g(u(t)) \geqq 0$ for every $t \in [0, T]$. Since $\int_0^T g(u(t))\,dt = 0$ and $g(u)$ is nonnegative for $u \geqq a-1$, we have that $g(u(t)) = 0$, $t \in [0, T]$, that is, u is a solution of the linear problem $u'' + cu' = 0$, $u \in E$. Hence, u is a constant and $u \in i(K)$. Now, if u is a solution and $u(t) < a-1$ for every $t \in [0, T]$, then u is a solution of (3.8) with $s = a-2$. By Lemma 3.9, we see that taking $\varepsilon > 0$ sufficiently small, any solution of (3.8) is such that $u(t) < a-1$, for every $t \in [0, T]$. Therefore, the solutions of (3.7) with $u(t) < a-1$ are exactly the solutions of (3.8). Denote by B the set of solutions of (3.8). By part (b) of Lemma 3.9 we know that B is connected. Hence, $S_1 = i(K) \cup B$ where B is connected and such that $i(K) \cap B = \emptyset$. Moreover, if $u \in B$, then u satisfies (3.9). This concludes the proof of the Theorem.

References

[1] J. Berbenes and M. Martelli, On the structure of the solution set for periodic boundary value problems, *Nonlinear Anal. TMA*, **4** (1980), 821—830.

[2] S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*; Academic Press (New York, 1974).

[3] L. Cesari and R. Kannan, An abstract theorem at resonance, *Proc. Amer. Math. Soc.*; **63** (1977), 221—225.

[4] S. Fucik and V. Lovicar, Periodic solutions of the equation $x''(t)+g(x(t))=p(t)$, *Casopis Pest. Mat.*; **100** (1975), 160—175.

[5] C. P. Gupta, On functional equations of Fredholm and Hammerstein type with applications to existence of periodic solutions of certain ordinary differential equations, *J. Integral Equations*, **3** (1981), 21—41.

[6] R. Kannan and V. Lakshmikantham, Existence of periodic solutions of nonlinear boundary value problems and the method of upper and lower solutions, *Applicable Anal.*, **17** (1984), 103—113.

[7] R. Kannan and R. Ortega, Periodic solutions of pendulum-type equations, *J. Differential Equations*, **59** (1985), 123—144.

[8] E. M. Landesman and A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*; **19** (1970), 609—623.

[9] A. C. Lazer, Application of a lemma on bilinear forms to a problem in nonlinear oscillations, *Proc. Amer. Math. Soc.*; **33** (1972), 89—94.

[10] W. S. Loud, *Periodic solutions of* $x''+cx'+g(x)=\varepsilon f(t)$, Mem. Amer. Math. Soc. 31, 1959.

[11] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, *J. Differential Equations*, **52** (1984), 264—287.

[12] J. J. Nieto, Periodic solutions of nonlinear parabolic equations, *J. Differential Equations*, **60** (1985), 90—102.

[13] J. J. Nieto, Existence of solutions in a cone for nonlinear alternative problems, *Proc. Amer. Math. Soc.*, **94** (1985), 433—436.

[14] J. J. Nieto, Positive solutions of nonlinear problems at resonance, *Acta Math. Hungar.* (to appear).

[15] J. J. Nieto and V. S. H. Rao, Periodic solutions of second order nonlinear differential equations, *Acta Math. Hung.*; **48** (1986), 59—66.

[16] R. Reissig, Contactive mappings and periodically perturbed nonconservative systems, *Lecture Notes in Pure Appl. Math.* 28, Marcel Dekker, Inc. (New York, 1977), 73—87.

[17] S. Tersian, On the periodic problem for the equation $x''(t)+g(x(t))=f(t)$, *Funkcialaj Ekvacioj*, **28** (1985), 39—46.

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DEPARTAMENTO DE ANALISIS MATEMATICO
UNIVERSIDAD DE SANTIAGO
SANTIAGO DE COMPOSTELA
SPAIN

DEPARTMENT OF MATHEMATICS
OSMANIA UNIVERSITY
HYDERABAD 500 007
INDIA

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ОБОБЩЕНИЕ ОДНОЙ ТЕОРЕМЫ Р. Б. САКСЕНА И С. Р. МИСРА

Д. Л. БЕРМАН (Ленинград)

Пусть C — множество всех функций, непрерывных в $[-1, 1]$ и пусть $x_{kn}=\cos(2k-1)\pi/2n$, $k=1, 2, \ldots, n$, $n=1, 2, \ldots$. Обозначим через $p_n(f, x)$ многочлеы степени $4n-1$, однозначно определяющийся из условий $p_n(f, x_{kn})=f(x_{kn})$, $p_n^{(i)}(f, x_{kn})=0$, $i=1, 2, 3$, $k=1, 2, \ldots, n$. Известно [1], что для любой $f\in C$ выполняется разномерно в $[-1, 1]$ соотношение $p_n(f, x)\to f(x)$, $n\to\infty$. Результаты из [2—4] привели к изучению процесса $\{p_n(f, x)\}_{n=1}^{\infty}$, построенного при узлах

$$(1)\quad x_{0\,n+2}=1,\quad x_{k,n+2}=\cos(2k-1)\pi/2n,\quad k=1, 2, \ldots, n,\quad x_{n+1,n+2}=-1,$$
$$n=1, 2, \ldots$$

W. L. Cook и T. M. Mills [5] доказали, что процесс $\{p_n(f, x)\}_{n=1}^{\infty}$, построенный при узлах (1) для $g(x)=(1-x^2)^3$, расходится при $x=0$. Недавно R. B. Saxena и S. R. Misra [6] доказали, что процесс $\{p_n(g, x)\}_{n=1}^{\infty}$ расходится всюду в $(-1, 1)$. В настоящей заметке среди прочих результатов доказывается теорема 3 из которой теорема Saxena и Misra вытекает, как простейший частный случай. Для дальнейшего нужны некоторые результаты из [7]. Для $f\in C$ и натурального n рассмотрим многочлены $S_{n,r}(f, x)$, $r=0, 1, 2, 3$, которые определяются следующим образом. $S_{n,r}(f, x)$ — многочлен степени $4n+2r+1$, однозначно определяющийся из условий

$$S_{n,r}(f, x_{kn})=f(x_{kn}),\quad S_{n,r}^{(i)}(f, x_{kn})=0,\quad i=1, 2, 3,\quad k=1, 2, \ldots, n,$$
$$S_{n,r}(f, \pm 1)=f(\pm 1),\quad S_{n,r}^{(j)}(f, \pm 1)=0,\quad j=1, \ldots, r.$$

Очевидно, что при $r=0$ последние условия опускаются. Справедливы следующие теоремы:

Теорема 1. *Пусть $f(x)$ имеет ограниченную вторую производную $f''(x)$ в $[-1, 1]$. Если хоть одно из чисел $f'(\pm 1)$ отлично от нуля, то процесс $\{S_{n,1}(f, x)\}_{n=1}^{\infty}$, построенный при узлах* (1) *для $f(x)$, расходится всюду в* $(-1, 1)$.

Теорема 2.[1] *Пусть $f(x)$ имеет ограниченную третью производную в $[-1, 1]$ и пусть $f'(1)=0$. Тогда, если $f''(1)\neq 0$, то процесс $\{S_{n,2}(f, x)\}_{n=1}^{\infty}$, построенный при узлах* (1) *для $f(x)$ расходится всюду в* $(-1, 1)$.

[1] В теоремах 2 и 3 предполагается, что $f(x)$ — четная функция.

Теорема 3. *Пусть $f(x)$ имеет ограниченную четвертую производную в $[-1, 1]$ и пусть $f'(1)=f''(1)=0$. Тогда, если $f^{(3)}(1)\neq 0$, то процесс $\{S_{n,3}(f, x)\}_{n=1}^{\infty}$, построенный при узлах* (1) *для $f(x)$, расходится всюду* $(-1, 1)$.

Все три теоремы доказываются одинаковым образом. Поэтому ограничимся доказательством теоремы 3. Ради простоты считаем, что $f(x)$ — четная функция.

Введем полином Крылова—Штаермана [8], построенный при узлах Чебышева[2]

$$p_n(f, x) = \frac{1}{h^4}\sum_{k=1}^{n} f(x_k)\, A_k(x)\, B_k(x), \quad A_k(x) = A_{kn}(x) = \left(\frac{T_n(x)}{x-x_k}\right)^4,$$

$$T_n(x) = \cos n \arccos x, \quad B_k(x) = B_{kn}(x) =$$

$$= (1-xx_k)^2+(x-x_k)^2\left[\frac{2(n^2-1)(1-xx_k)}{3}-\frac{xx_k}{2}\right].$$

Приведем формулы из [7]

$$S_{n,0}(f, x)-p_n(f, x) = \frac{1}{2}T_n^4(x)\left[(1+x)\big(f(1)-p_n(f, 1)\big)+(1-x)\big(f(-1)-p_n(f, -1)\big)\right],$$

$$S_{n,1}(f, x)-S_{n,0}(f, x) = \Phi_{n,1}(p_n)\frac{(1-x^2)\,T_n^4(x)}{2}, \quad \Phi_{n,1}(p_n) = P_n^1-4n^2 p_n,$$

$$p_n^{(k)} = p_n^{(k)}(f, 1).$$

$$S_{n,2}(f, x)-S_{n,1}(f, x) = -\frac{1}{4}T_n^4(x)(x^2-1)^2, \quad \Phi_{n,2}(p_n)],$$

$$\Phi_{n,2}(p_n) = p_n''-(8n^2+1)\,p_n'+\frac{8n^2(7n^2+2)\,p_n}{3},$$

$$S_{n,3}(f, x)-S_{n,2}(f, x) = \frac{1}{48}T_n^4(x)(1-x^2)^3\,\Phi_{n,3}(p_n),$$

$$\Phi_{n,3}(p_n) = p_n^{(3)}-6(4n^2+1)\,p_n^{(2)}+2(76n^4+32n^2+3)\,p_n'-\frac{p_n}{15}8n^2(608n^4+365n^2+62),$$

$$\Phi_{n,3}(A_k B_k) = -\left(\frac{36x_k^2-376x_k}{(1-x_k)^4}+\frac{2(x_k^2+29x_k+16)}{(1-x_k)^3}\right)n^2-\frac{60(x_k^2-6x_k-2)}{(1-x_k)^5}. \tag{2}$$

Известно [4], что для любого $x\in(-1, 1)$ можно найти такую последовательность натуральных чисел $\{n_k\}_{k=1}^{\infty}$, $n_1<n_2<\ldots$, что выполняется равенство $\lim\limits_{n\to\infty} T_{n_k}^2(x)=1$. Поэтому в силу теоремы 2 нужно доказать,[3] что $\lim\limits_{n\to\infty}\Phi_{n,3}(p_n)\neq 0$. Без ограничения общности можно считать, что $f(1)=0$, ибо иначе можно рассматривать функцию $\varphi(x)=f(x)-f(1)$. По условиям теоремы $f'(1)=f''(1)=0$.

[2] Полагаем $x_{kn}=x_k$.
[3] См. [7].

Поэтому по формуле Тейлора

$$f(x)=\frac{f^{(3)}(1)}{6}(x-1)^3+\frac{f^{(4)}(c)}{4!}(x-1)^4, \quad x<c<1.$$

Стало быть, $\Phi_{n,3}(p_n)=\tau_{1,n}+\tau_{2,n}$, где

$$\tau_{1,n}=\frac{f^{(3)}(1)}{n^4}\sum_{k=1}^{n}(x_k-1)^3\Phi_{n,3}(A_k B_k), \quad \tau_{2,n}=\frac{1}{4!\,n^4}\sum_{k=1}^{n}f^{(4)}(c_k)(x_k-1)^4\Phi_{n,3}(A_k B_k).$$

Очевидно, что $\tau_{2,n}=O\left(\frac{1}{n}\right)$. Подсчитаем выражение $\tau_{1,n}$. При этом воспользуемся тождествами

$$\sum_{k=1}^{n}\frac{1}{1-x_k}=n^2, \quad \sum_{k=1}^{n}\frac{1}{(1-x_n)^2}=\frac{2n^4+n^2}{3}$$

и другими необходимыми тождествами (см. [9]). В результате получим, что

$$\lim_{n\to\infty}\tau_{1,n}=\frac{26}{9}f^{(3)}(1).$$

Стало быть, $\lim\limits_{n\to\infty}\Phi_{n,3}(p_n)\neq 0$.

В [5] и [6] рассматривается случай, когда $f(x)=(1-x^2)^3$, ясно $f'(1)=f''(1)=0$, $f^{(3)}(1)=48$. Поэтому к функции $(1-x^2)^3$ применима теорема 3. Итак, теорема из [6] — частный случай теоремы 3.

Замечание. Формула (2), которой пользовались при вычислении $\tau_{1,n}$ и оценке $\tau_{2,n}$, выводится совершенно элементарно, но вычисления громоздкие и поэтому опускаются. См. [7].

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Литература

[1] Н. М. Крылов, И. Я. Штаерман, Sur quelques formules d'interpolation convergentes pour toute function continue, *Записки физ.-матем. отд. АН УССР*, **1** (1922), 12—13.

[2] Д. Л. Берман, К теории интерполяции, *ДАН СССР*, **163** (1965), 551—554.

[3] Д. Л. Берман, Исследование интерполяционного процесса Эрмита—Фейера, *ДАН СССР*, **187** (1969), 241—244.

[4] Д. Л. Берман, Об одном всюду расходящемся интерполяционном процессе Эрмита—Фейера, *Известия вузов, матем.*, **1** (1970), 3—8.

[5] W. Lyle Cook and T. M. Mills, On Berman's phenomenon in interpolation theory, *Bulletin of the Australian Math. Soc.*, **12** (1975), 457—465.

[6] R. B. Saxena and S. R. Misra, An everywhere divergent Hermite—Fejér type interpolation process of higher order, *Acta Math. Hungar.*, **48** (1986), 79—85.

[7] D. L. Berman, A remark on the extended Hermite—Fejér type interpolation of higher order, *Periodica Math. Hungar.*, **18** (1987), 279—286.

[8] A. Sharma and I. Tzimbalario, Quasi-Hermite—Fejér type interpolation of higher order, *J. Appr. Theory*, **13** (1975), 431—442.

[9] Д. Л. Берман, Ретение одной экстремальной задачи теории операторов и аналоги тождества М. Рисса, *Успехи матем. наук*, **37** (1982), 187—198.

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ON THE CONSTRUCTION OF LJUSTERNIK—SCHNIRELMANN CRITICAL VALUES IN BANACH SPACES

A. LEHTONEN (Jyväskylä)

1. Introduction

Existence theorems for nonlinear eigenvalue problems of the form

$$g'(x) = \mu f'(x),$$

where f and g are functionals on a Banach space X, are considered in many papers. The existence theorems are based on the existence of a critical vector with respect to the manifold $M_r = \{x \in X: f(x) = r\}$. Morse theory can often be used to obtain precise information about the behaviour of the functional close to the critical level. However, this would limit the study to Hilbert spaces and functions with non-degenerate critical points. These assumptions are not always satisfied in applications and are not needed when applying the Ljusternik—Schnirelmann theory. Therefore, Ljusternik—Schnirelmann theory has been widely used to study various nonlinear eigenvalue problems. Very general results for Banach spaces can be found in H. Amann [1] and in E. Zeidler [12], which also contains an extensive bibliography on critical point theories.

In the case of a Hilbert space iterative methods for the construction of all Ljusternik—Schnirelmann critical values and critical vectors has been presented by J. Nečas [7], by A. Kratochvíl and J. Nečas [4], [5] and by J. Nečas, A. Lehtonen and P. Neittaanmäki [8]. In [8] we also present numerical examples of the method.

In this paper we shall give an extension of the method used in [5] and [8] to study the eigenvalue problem for the constraint function $f(x) = \|x\|$ in uniformly convex Banach spaces.

2. Iterative construction of the first Ljusternik—Schnirelmann critical value

Let X be a real, uniformly convex Banach space with norm $\|\cdot\|$ and duality $\langle \cdot, \cdot \rangle: X^* \times X \to \mathbf{R}$. Furthermore, we assume that X^* is uniformly convex. We set $S = \{x \in X: \|x\| = 1\}$. Let g be a continuously differentiable functional on X such that the derivative g' is strongly continuous, i.e. for each sequence $(x_n)_{n=1}^{\infty} \subset X$ converging weakly to $x_0 \in X$, the sequence $(g'(x_n))_{n=1}^{\infty}$ converges to $g'(x_0)$.

Assume that the following conditions are fulfilled:

(2.1) $$g(0) = 0, \quad g'(0) = 0;$$

(2.2) $$\text{if} \quad g(x) \neq 0 \quad \text{and} \quad \|x\| \leqq 1 \quad \text{then} \quad g'(x) \neq 0;$$

(2.3) $$\langle g'(x) - g'(y), x - y \rangle \geqq -\gamma(\|x - y\|)$$

holds for some nonnegative, continuous function $\gamma\colon [0,\infty[\to[0,\infty[$ and for all $x, y\in X$ with $\|x\|\leqq 1$ and $\|y\|\leqq 1$.

We denote by $J\colon X\to X^*$ the duality mapping of X. Thus $J(x)$ is the unique point in X^* such that $\|J(x)\|=\|x\|$ and $\langle J(x), x\rangle=\|x\|^2$. Also, J is the derivative of the mapping $x\mapsto\frac{1}{2}\|x\|^2$, see [6, p. 254] or [3, p. 93].

We need to know a quantity to describe the uniform convexity of X. Let $0\leqq\varepsilon\leqq 2R$ and define

$$\delta_R(\varepsilon)=\inf\left\{R-\frac{1}{2}\|x+y\|:\ x, y\in X,\ \|x-y\|\geqq\varepsilon, \|x\|\leqq R,\ \|y\|\leqq R\right\}.$$

Similarly, let δ_R^* denote the corresponding quantity for X^*. Furthermore, put $\delta=\delta_1$ and $\delta^*=\delta_1^*$. Then $\delta_R(\varepsilon)=R\delta(\varepsilon/R)$.

LEMMA 2.1. *For* $x, y\in X$ *such that* $\|x\|=R$ *and* $\|y\|=R$ *holds*

$$2R\delta_R(\|x-y\|)\leqq\langle J(x), x-y\rangle \tag{2.4}$$

$$4R\delta_R(\|x-y\|)\leqq\langle J(x)-J(y), x-y\rangle. \tag{2.5}$$

PROOF. Set $\varepsilon=\|x-y\|$. Since $\|J(x)\|=R$ we get from the definition

$$R2\big(R-\delta_R(\varepsilon)\big)\geqq R\|x+y\|\geqq\langle J(x), x+y\rangle=$$

$$=2\langle J(x), x\rangle+\langle J(x), y-x\rangle=2R^2+\langle J(x), y-x\rangle.$$

Thus, the first inequality follows. Changing the roles of x and y and adding the results yields the second inequality.

For the lower bound γ in (2.3) we assume that the following hypothesis is fulfilled: there exists a constant $c_0\geqq 0$ such that

$$\int_0^\varepsilon \gamma(s)s^{-1}\,ds\leqq c_0\,\delta(\varepsilon) \tag{2.6}$$

for $0<\varepsilon\leqq 2$.

REMARK. For Hilbert spaces $\delta(\varepsilon)=1-\sqrt{1-(\varepsilon/2)^2}$. This follows from the parallelogram law. From Clarkson's inequalities we obtain for L^p-spaces $\delta(\varepsilon)=1-\big(1-(\varepsilon/2)^p\big)^{1/p}$, if $p\geqq 2$, and $\delta(\varepsilon)=1-\big(1-(\varepsilon/2)^{p'}\big)^{1/p'}$, if $1<p<2$ and $p'=p/(p-1)$.

If g is a C^2-function satisfying conditions (2.3) and (2.6), and $X=L^p$, $p\neq 2$, then $g''(x)\geqq 0$, that is, g is convex. To show this we note that $\delta(\varepsilon)\leqq(\varepsilon/2)^q$, where $q=p$, if $p>2$, and $q=p'$, if $p<2$. Assume now that for some $x, u\in X$, $\|x\|<1$, $\|u\|=1$ and $\alpha>0$ we have $\langle g''(x)u, u\rangle=-\alpha$. Let $x-y=tu$ in (2.3). Then $-\alpha t^2+o(t^2)\geqq-\gamma(t)$. Thus $\alpha t\leqq\gamma(t)/t+o(t)$. Integrating this yields $\alpha\varepsilon^2/2\leqq c_0 2^{-q}\varepsilon^q+o(\varepsilon)$. Letting $\varepsilon\to 0$ results $\alpha\leqq 0$.

THEOREM 2.2. *Let the above assumptions be satisfied. Let* $x_1 \in S$ *and*

$$0<\theta<\min\{1/\gamma(1),\, 4/(c_0+2\gamma(1))\}.$$

Assume that $g(x_1)>0$. *Define a sequence* $(x_n)_{n=1}^{\infty}$ *by*

$$J(x_{n+1})=\frac{J(x_n)+\theta g'(x_n)}{\|J(x_n)+\theta g'(x_n)\|}. \tag{2.7}$$

Then there exists a subsequence of $(x_n)_{n=1}^{\infty}$ *converging to a critical point of* $g_{|S}$. *Hence, there exists a point* $x_0 \in S$ *and a number* $\mu \in \mathbf{R}$ *such that*

$$g'(x_0)=\mu J(x_0). \tag{2.8}$$

PROOF. First, from the definition it follows $\|J(x_{n+1})\|=1$. On the other hand $\|J(x_{n+1})\|=\|x_{n+1}\|$ and therefore $\|x_{n+1}\|=1$.

Put $r_n=\|J(x_n)+\theta g'(x_n)\|$. Then

$$r_n \geqq \langle J(x_n)+\theta g'(x_n), x_n\rangle = \|x_n\|^2+\theta\langle g'(x_n), x_n\rangle \geqq 1-\theta\gamma(1).$$

We show that the sequence $(g(x_n))_{n=1}^{\infty}$ is increasing. For short we set $\varepsilon=\|x_{n+1}-x_n\|$. Using the fundamental theorem of integral calculus we get the following lower bound for $g(x_{n+1})-g(x_n)$

$$\langle g'(x_n), x_{n+1}-x_n\rangle-\int_0^{\varepsilon}\gamma(s)\,s^{-1}\,ds. \tag{2.9}$$

For the first term we obtain from definition (2.7) and Lemma 2.1

$$\theta\langle g'(x_n), x_{n+1}-x_n\rangle = r_n\langle J(x_{n+1}), x_{n+1}-x_n\rangle+\langle J(x_n), x_n-x_{n+1}\rangle \geqq 2(r_n+1)\delta(\varepsilon). \tag{2.10}$$

Combining these two inequalities yields

$$g(x_{n+1})-g(x_n) \geqq \frac{2}{\theta}(r_n+1)\,\delta(\varepsilon)-\int_0^{\varepsilon}\gamma(s)\,s^{-1}\,ds. \tag{2.11}$$

Assumption (2.6) shows now that the sequence $(g(x_n))_{n=1}^{\infty}$ is increasing. Because of the strong continuity of g' the functional g is bounded on S. Therefore, the limit $\lim_{n\to\infty} g(x_n)$ exists.

Since X is reflexive, there exists a subsequence of $(x_n)_{n=1}^{\infty}$ converging weakly to some x_0. We keep the same notation for the subsequence as for the original one. Furthermore, we may assume that $r_n \to r_0$. Finally, since X^* is reflexive and $\|J(x_n)\|=1$ we may assume that $J(x_n)\to x^*$ weakly in X^*.

Now, using the definitions of x_{n+1} and r_n yields $(r_0-1)x^*=\theta g'(x_0)$. Furthermore, as $g(x_0)\geqq g(x_n)>0$, we have by (2.2) $g'(x_0)\neq 0$. Therefore $x_0\neq 0$, $x^*\neq 0$ and $r_0\neq 1$. To show that the sequence $(x_n)_{n=1}^{\infty}$ converges in norm to x_0 we write (2.7) in the form

$$(1-r_n)J(x_n)=r_n\big(J(x_{n+1})-J(x_n)\big)-\theta g'(x_n). \tag{2.12}$$

We shall show that $J(x_{n+1})-J(x_n)\to 0$. Then the right-hand side converges in norm to $-\theta g'(x_0)$ and on the left $1-r_n\to 1-r_0$. Since $1-r_0\neq 0$, we obtain the desired convergence from (2.12) and the fact that the duality mapping $J^{-1}\colon X^*\to X$ is continuous, cf. [3, p. 77].

We apply Lemma 2.1 to X^* and to $u=J(x_{n+1})$ and $v=J(x_n)$. For $\eta>0$ small enough we have

$$\theta\big(g(x_{n+1})-g(x_n)\big)\geqq r_n\langle J(x_{n+1}),x_{n+1}-x_n\rangle+\langle J(x_n),x_n-x_{n+1}\rangle-\theta c_0\delta(\varepsilon)\geqq$$
$$\geqq 4\eta\delta^*(\|u-v\|)+\big(8(1-\eta)-4\theta\gamma(1)-\theta c_0\big)\delta(\varepsilon).$$

Therefore, we have for some constant $c'>0$,

$$4\delta^*(\|u-v\|)\leqq\langle u-v,J^{-1}(u)-J^{-1}(v)\rangle\leqq c'\theta\big(g(x_{n+1})-g(x_n)\big).$$

Since X^* is uniformly convex, $\delta^*(\varepsilon)>0$ for all $\varepsilon>0$, cf. [12, p. 604]. Furthermore, we have $g(x_{n+1})-g(x_n)\to 0$, and hence $J(x_{n+1})-J(x_n)\to 0$. Therefore, Theorem 2.2 follows.

3. Higher order critical points

To study higher order critical points we recall some definitions concerning the Ljusternik—Schnirelmann theory in an infinite dimensional Banach space.

We use the notion of the order of a set rather than the category or genus, cf. [2]. Let K be a symmetric closed set in X. We say that $\operatorname{ord}K=0$ if K is empty; that $\operatorname{ord}K=1$ if $K=K_1\cup K_2$, where the K_i are closed subsets of K and neither K_1 nor K_2 contains antipodal points. In general, $\operatorname{ord}K=n$ if $K=\bigcup_{i=1}^{n+1}K_i$, where the K_i are closed subsets of K not containing antipodal points and n is the least possible number. Finally, $\operatorname{ord}K=\infty$ if no such n exists.

For simplicity we assume that $g(x)>0$ for $x\neq 0$. Let V_k denote the set of all symmetric, compact subsets K of S such that $\operatorname{ord}K\geqq k$ and $g(x)>0$ on K. Denote

$$\gamma_k=\sup_{K\in V_k}\min_{x\in K}g(x). \tag{3.1}$$

Let the assumptions of Theorem 2.2 hold for the functional g. Furthermore, assume that g is even on S, i.e., $g(-x)=g(x)$ when $\|x\|=1$. The fundamental theorem of the Ljusternik—Schnirelmann theory states that there exists a sequence of critical points x_k of g such that $g(x_k)=\gamma_k$, $\gamma_k\searrow 0$ and $x_k\to 0$ weakly. For a proof see [1] or [12, Ch. 44]. A proof using the method of steepest descent has been given in [7].

Let γ_1 and γ_2 be the first and second Ljusternik—Schnirelmann critical values of $g_{|S}$, $\gamma_1>\gamma_2$. Let there exist a positive constant ε such that there are no critical values in the interval $]\gamma_2-\varepsilon,\gamma_2[$. Let K_1 be a compact symmetric subset of S such that $\operatorname{ord}K_1\geqq 2$ and

$$\gamma_2-\varepsilon<\min_{x\in K_1}g(x)<\gamma_2. \tag{3.2}$$

We denote by φ the function used to define the iteration in (2.7), i.e.

$$\varphi(x) = J^{-1}\left(\frac{J(x)+\theta g'(x)}{\|J(x)+\theta g'(x)\|}\right). \tag{3.3}$$

Then φ is a well-defined, odd continuous map $S\to S$. Choose $x_1\in K_1$, and put $x_{n+1}=\varphi(x_n)=\varphi^n(x_1)$, where φ^n denotes the n-fold composition $\varphi\circ\ldots\circ\varphi$.

For each integer n let $x_n^{(0)}$ be a vector from K_1 such that

$$\min_{x\in K_1} g(\varphi^n(x)) = g(\varphi^n(x_n^{(0)})). \tag{3.4}$$

THEOREM 3.1. *Let the above assumptions be fulfilled. Then the following assertions hold:*

(1) $\lim_{n\to\infty} g(\varphi^n(x_n^{(0)}))=\gamma_2$;

(2) *there exists* $x^{(0)}\in K_1$ *such that*

$$\lim_{n\to\infty} g(\varphi^n(x^0)) = \gamma_2;$$

(3) *there exists a subsequence of* $(x_n^{(0)})_{n=1}^{\infty}$ *converging to* $x^{(0)}$;

(4) *for each* $x^{(0)}$ *satisfying* (2) *there exists a subsequence of* $(\varphi^n(x^{(0)}))$ *converging to some* x_0 *such that*

$$g'(x_0) = \mu J(x_0).$$

The proof of Theorem 3.1 given in [8] applies; see also [5]. In a similar way we obtain the following result

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *be fulfilled. Let*

$$\gamma_1 \geqq \ldots \geqq \gamma_k > \gamma_{k+1} = \ldots = \gamma_{k+l} > \gamma_{k+l+1}$$

be the positive Ljusternik—Schnirelmann values of $g_{|S}$. *Let there exist a constant* $\varepsilon>0$ *such that there are no critical values in the interval* $]\gamma_{k+l}-\varepsilon, \gamma_{k+l}[$. *Let* K_1 *be a compact symmetric subset of* S *such that*

$$\text{ord}\quad K_1 \geqq k+1,\quad \gamma_{k+1}-\varepsilon < \min_{x\in K_1} g(x) < \gamma_{k+l}.$$

For $x\in K_1$ *let the sequence* $(x_n^{(0)})_{n=1}^{\infty}$ *be defined by* (3.4).

Then

$$\lim_{n\to\infty} g(\varphi^n(x_n^{(0)})) = \gamma_{k+l},$$

and there exists a point $x^{(0)}\in K_1$ *such that*

$$\lim_{n\to\infty} g(\varphi^n(x^{(0)})) = \gamma_{k+l}.$$

Moreover, the assertions (3) *and* (4) *of Theorem* 3.1 *hold.*

4. Application to partial differential equations

Let p be a number such that $1<p<\infty$ and $G\colon \mathbf{R}\to\mathbf{R}$ a continuously differentiable function. Let Ω be a bounded domain in $\mathbf{R}^N$. Denote by $G'(u)$ the function $x\mapsto G'(u(x))$. We will apply the previous results to the following boundary value problem

$$(4.1)\qquad \begin{cases} -\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2} \dfrac{\partial u}{\partial x_i}\right) = \lambda G'(u) \quad \text{in} \quad \Omega. \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

We assume that $G(0)=G'(0)=0$ and that $G'(r)$ is strictly increasing. Furthermore, we assume that there exists a constant $c>0$ such that $|G'(r)|\leq c(1+|r|^{p-1})$ for all $r\in\mathbf{R}$.

We use the norm

$$\|u\| = \left(\sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{1/p}$$

in the Sobolev space $X=W_0^{1,p}(\Omega)$. It is well-known that X is uniformly convex and for X^* this implies the uniform smoothness of X, cf. [12, p. 604].

Let $g\colon X\to\mathbf{R}$ be the functional

$$g(u) = \int_{\Omega} G(u(x))\,dx.$$

Then g is continuously differentiable and the derivative of g is given by

$$\langle g'(u), v\rangle = \int_{\Omega} G'(u)v\,dx \quad \text{for all} \quad v\in X.$$

Conditions (2.1), (2.2), (2.3) and (2.6) are fulfilled for $\gamma=0$ and $c_0=0$.

Problem (4.1) is equivalent to finding $u\in X$ such that

$$(4.2)\qquad \mu \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx = \int_{\Omega} G'(u)v\,dx \quad \text{for all} \quad v\in X,$$

where $\mu=1/\lambda$.

Let $\Phi(r)=r^{p-1}$. Then the duality mapping of X relative to Φ, cf. [6, p. 174] is given by

$$\langle J_\Phi(u), v\rangle = \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx.$$

Recall that J_Φ is the derivative of $u\mapsto\Psi(\|u\|)$, where $\Psi(r)=\int_0^r \Phi(s)\,ds$. Therefore, J_Φ and the duality mapping J of X differ only by the constant multiple $\Phi(1)$ on the sunit sphere. Theorems 2.2, 3.1 and Corollary 3.2 hold, if we use J_Φ instead of J since $\Phi(1)=1$.

Hence, problem (4.1) is equivalent to

$$g'(u) = \mu J_{\Phi}(u). \tag{4.3}$$

That g' is strongly continuous follows from Rellich's theorem, cf. [6, Ch. 2.2.6]. Hence Theorems 2.2, 3.1 and Corollary 3.2 can be applied to construct a sequence $(u_k)_{k=1}^{\infty}$ of solutions of (4.1) such that $\|u_k\|=1$, $\int_{\Omega} G(u_k)dx=\gamma_k$ and $\gamma_k \searrow 0$.

References

[1] H. Amann, Ljusternik—Schnirelmann theory and non-linear eigenvalue problems, *Math. Ann.*; **199** (1972), 55—72.

[2] S. Fučik, J. Nečas, J. Souček, V. Souček, *Spectral Analysis of Nonlinear Operators;* Lecture Notes in Mathematics 346, Springer-Verlag (Berlin, 1973).

[3] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen,* Akademie-Verlag (Berlin, 1974).

[4] A. Kratochvíl, J. Nečas, Secant modulus method for the construction of a solution of nonlinear eigenvalue problems, *Boll. Un. Mat. Ital.*; **16** (1979), 694—710.

[5] A. Kratochvíl, J. Nečas, Gradient methods for the construction of Ljusternik—Schnirelmann critical values, *RAIRO Anal. Numér.*, **14** (1980), 43—54.

[6] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires;* Gauthier-Villars (Paris, 1969).

[7] J. Nečas, An approximate method of finding critical points of even functionals, *Proc. Steklov Inst. Math.*; **134** (1975), 267—272.

[8] J. Nečas, A. Lehtonen, P. Neittaanmäki, On the construction of Ljusternik—Schnirelmann critical values with application to bifurcation problems, *Applicable Anal.* (to appear).

[9] P. Neittaanmäki, K. Ruotsalainen, On the numerical solution of the bifurcation problem for the sine-Gordon equation, *Arab. J. Math.* (1986) (to appear).

[10] L. Nirenberg, *Topics in Nonlinear Functional Analysis,* Lecture Note, Courant Institute of Mathematical Sciences, New York University, 1974.

[11] K. Yosida, *Functional Analysis,* Fourth Edition, Grundlehren der mathematischen Wissenschaften 123, Springer-Verlag (New York, 1974).

[12] E. Zeidler, *Nonlinear Functional Analysis and its Applications III. Variational Methods and Optimization,* Springer-Verlag (New York, 1985).

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UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
SEMINAARINKATU 15
SF—40100 JYVÄSKYLÄ
FINLAND

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CONGRUENCE UNIFORM DISTRIBUTIVE LATTICES

M. E. ADAMS (New York) and R. BEAZER (Glasgow)

1. Introduction

A congruence relation on an algebra is *uniform* provided every congruence class has the same size and an algebra is *congruence uniform* if every congruence on it is uniform. In general, congruences on distributive lattices are well understood. However, uniform congruences appear to be an exception to the rule.

It is well-known and easily seen that every Boolean algebra is congruence uniform. Furthermore, as shown in [2], a (semi-) lattice with pseudocomplementation is congruence uniform iff it is Boolean and, moreover, a *finite* distributive lattice is congruence uniform iff it is Boolean.

Our starting point, Theorem 3.5, is a characterization of countable congruence uniform distributive (0, 1)-lattices in terms of two special congruences. This immediately suggests a construction to show that every countable distributive (0, 1)-lattice is a homomorphic image of a countable congruence uniform distributive (0. 1)-lattice (Theorem 3.6). Consequently, not every congruence uniform distributive (0, 1)-lattice is Boolean. In fact, Theorem 3.5 is a stepping stone to Theorem 4.6 which characterizes countable congruence uniform distributive lattices in terms of certain naturally occurring congruence relations. Equivalent formulations are provided by Theorem 4.9 which also shows that Theorem 4.6 is indeed a generalization of Theorem 3.5.

§§ 3 and 4 shows that congruence uniformity is related to properties of particular congruences on the lattice concerned. § 5 is of a different flavour. For simplicity, discussion in this section is restricted to distributive (0, 1)-lattices where, to begin with, a relationship between a congruence uniform distributive (0, 1)-lattice and its injective hull is considered. Thereafter, an example is given of the relationship between a congruence uniform distributive (0, 1)-lattice and some of its subsets. Although, in each case, alternative characterizations are obtained of countable congruence uniform distributive (0, 1)-lattices, this section probably raises more questions than it answers.

Throughout, algebraic techniques are combined with the representation of distributive (0, 1)-lattices by means of topological spaces. For the sanity of the reader, we have attempted not to intertwine the two. Thus, with the exception of Example 5.4, §§ 4 and 5 are algebraic whereas, § 3 uses topological duality (the basic essentials of which are given in § 2).

Before proceeding, we should mention that, in the broader setting of universal algebra, congruence uniformity was first considered by W. Taylor in [9]. Further examples of congruence uniform algebras include all algebras in any variety generated by a quasi-primal algebra (W. Taylor [9]) and all finite members of any directly representable variety (R. McKenzie [8]). For further background and motivation, the reader should consult the text S. Burris and H. P. Sankappanavar [4].

2. Some preliminaries

This section briefly summarizes the required facts concerning the representation of distributive lattices by topological spaces: further information on this topic may be found in either of the survey papers B. A. Davey and D. Duffus [5] or H. A. Priestley [11]. For algebraic aspects of distributive lattices, the reader is referred to either of the texts R. Balbes and Ph. Dwinger [1] or G. Grätzer [7].

For a partially ordered set P, let $\mathrm{Max}(P)$ and $\mathrm{Min}(P)$ denote the maximal and minimal elements of P, respectively. Further, for $Q\subseteq P$, let $(Q]$ and $[Q)$ denote the order ideal and filter generated by Q, respectively. Then Q is *decreasing, increasing,* or *convex* provided $Q=(Q]$, $Q=[Q)$, or $Q=(Q]\cap[Q)$, respectively. For partially ordered sets P and P', a mapping $\varphi\colon P\to P'$ is *order-preserving* if $\varphi(x)\leqq\varphi(y)$ whenever $x\leqq y$.

Given a topology τ defined on a partially ordered set P, the pair (P,τ) is called an *ordered space*. The space is *totally order-disconnected* provided, for $x, y\in P$ with $x\nleqq y$, there exists a clopen decreasing $Q\subseteq P$ such that $y\in Q$ and $x\notin Q$. If (P,τ) is a compact totally order-disconnected space, then it is called a *Priestley space*. As shown by H. A. Priestley [10], the category of all Priestley spaces together with all continuous order-preserving maps is dually equivalent to the category of distributive $(0, 1)$-lattices with $(0, 1)$-lattice homomorphisms. If, under this duality, L and (P,τ) are associated, then P is the poset of prime ideals of L suitably topologized and the elements of L correspond to the clopen decreasing subsets of P. Further, if $f\colon L\to L'$ is associated with the continuous order-preserving map $\varphi\colon P'\to P$, then $f(a)=b$ iff $\varphi^{-1}(A)=B$, where A and B are the clopen decreasing subsets that represent a and b. Inspection shows that f is onto iff φ is an order-isomorphism. Thus, since congruences correspond to onto $(0, 1)$-lattice homomorphisms, it follows that the congruences of a distributive $(0, 1)$-lattice L are in a one-to-one correspondence with the closed subsets of P. In which case, for a congruence θ associated with a closed set Q,

$$a\equiv b(\theta)\quad\text{iff}\quad A\cap Q=B\cap Q,$$

where a and b are represented by A and B.

3. The bounded case

The immediate goal of this section is a characterization of countable congruence uniform distributive $(0, 1)$-lattices in terms of two natural congruences (Theorem 3.5).

For a distributive lattice L with a unit 1, let the relation Φ^+ be given by

$$a\equiv b(\Phi^+)\quad\text{iff}\quad [a)^+=[b)^+,$$

where $[a)^+=\{c\in L\colon a\vee c=1\}$. As shown in [3], Φ^+ is a congruence relation on L the co-kernel of which is trivial and, moreover, Φ^+ is the largest such congruence. Clearly, if L is congruence uniform, then $\Phi^+=\omega$.

The following was also observed in [3]:

LEMMA 3.1. *For a distributive lattice L with a unit* 1, *the following are equivalent:*

(i) $\Phi^+=\omega$;
(ii) *for* $a, b\in L$, $a<b$ *implies* $a\vee c\neq 1$ *and* $b\vee c=1$ *for some* $c\in L$;
(iii) *for* $a, b\in L, a<b<1$ *implies* $b\vee c=1$ *for some* $c\in L$ *such that* $a<c<1$. □

LEMMA 3.2. *For a distributive* (0, 1)*-lattice L with Priestley space* (P, τ), $\Phi^+=\omega$ *iff* $Q\cap \mathrm{Max}(P)\neq\emptyset$ *for every non-empty clopen set* $Q\subseteq P$.

PROOF. Suppose $\Phi^+=\omega$ and Q is a non-empty clopen set. By compactness and total order-disconnectedness, it is not hard to see that there exist a non-empty clopen convex set $R\subseteq Q$ and clopen decreasing sets $A\subset B$ such that $B\setminus A=R$. Clearly, it is sufficient to show that $R\cap \mathrm{Max}(P)\neq\emptyset$. Suppose that this is not the case. If $a, b\in L$ are associated with the clopen decreasing sets A, B, then, by Lemma 3.1 (ii), $a\vee c\neq 1$ and $b\vee c=1$ for some $c\in L$. Since $[x)\cap \mathrm{Max}(P)\neq\emptyset$ for every $x\in P$, if $C\subseteq P$ is the clopen decreasing set that represents c, then $R\subseteq C$ is forced by $b\vee c=1$. Thus $A\cup C\supseteq B$. In other words, $a\vee c\geqq b$ and so $a\vee c=1$ which is absurd.

Suppose alternatively that $Q\cap \mathrm{Max}(P)\neq\emptyset$ whenever $Q\subseteq P$ is non-empty and clopen. If $a<b$ for some $a, b\in L$, then, by Lemma 3.1 (ii), it is sufficient to find $c\in L$ such that $a\vee c\neq 1$ and $b\vee c=1$. Let A and B be the clopen decreasing sets associated with a and b. By hypothesis, there exists $x\in(B\setminus A)\cap \mathrm{Max}(P)$. Using compactness and total order-disconnectedness choose a clopen increasing set R such that $x\in R\subseteq B\setminus A$. The element c represented by $P\setminus R$ has the desired properties. □

Analogous to the above, for distributive lattice L with a zero 0, the relation Φ^* given by

$$a\equiv b(\Phi^*) \quad \text{iff} \quad (a]^*=(b]^*,$$

where $(a]^*=\{c\in L\colon a\wedge c=0\}$, is the largest congruence on L having a trivial kernel. Clearly, statements dual to those of Lemmas 3.1 and 3.2 hold for Φ^*.

Thus, if a distributive (0, 1)-lattice is congruence uniform, then $\Phi^+=\Phi^*=\omega$. We remark that, although $\Phi^+=\omega$ iff $\Phi^*=\omega$ for any *finite* distributive lattice, simple examples show that the two conditions are not equivalent in general.

Already the Boolean-likeness of congruence uniform distributive (0, 1)-lattices is apparent. For example, consider the following:

PROPOSITION 3.3. *Let L be a distributive* (0, 1)*-lattice and suppose that b covers a, denoted* $a\prec b$, *for some* $a, b\in L$. *If L is congruence uniform, then there exists* $c\in\mathrm{Cen}(L)$, *the centre of L, such that*

$$b=a\vee c \quad \textit{and} \quad 0\prec c.$$

PROOF. If A and B are the clopen decreasing sets of the Priestley space (P, τ) of L that are associated with the elements a and b, then the clopen set $B\setminus A$ has precisely one element, say x. Since L is congruence uniform, it follows from Lemma 3.2 and its dual that $x\in \mathrm{Max}(P)\cap \mathrm{Min}(P)$. The element c represented by the clopen set $\{x\}$, which is both increasing and decreasing, satisfies our requirements. □

Since a clopen decreasing set A represents an atom or a co-atom iff $|A|=1$ or $|P\setminus A|=1$, respectively, the observation that any isolated point in the Priestley space of a congruence uniform distributive (0, 1)-lattice is both maximal and minimal also yields

PROPOSITION 3.4. *Let L be a distributive* (0, 1)-*lattice. If L is congruence uniform, then* |Atoms| = |Co-atoms|. □

Moreover, since every point of the Priestley space (P, τ) of a finite distributive lattice L is isolated, if L is congruence uniform, then P is an antichain. That is to say, the poset of prime ideals of L forms an antichain and, as mentioned in the introduction, it follows that every finite congruence uniform distributive lattice is Boolean.

All of the preceding remarks concerning congruence uniform distributive (0, 1)-lattices are consequences of $\Phi^+=\Phi^*=\omega$. In the countable case, this condition will actually characterize congruence uniformity:

THEOREM 3.5. *A countable distributive* (0, 1)-*lattice is congruence uniform iff* $\Phi^+=\Phi^*=\omega$.

PROOF. Assume that L is a countable distributive (0, 1)-lattice for which $\Phi^+=\Phi^*=\omega$. Let (P, τ) denote the Priestley space of L and suppose θ is a congruence on L with associated closed set $Q\subseteq P$. Recall that, for $a, b\in L$, $a\equiv b(\theta)$ iff $A\cap Q=B\cap Q$, where A and B are the clopen decreasing sets associated with a and b. There are two cases to consider:

Suppose first that $P\setminus Q$ is finite. Since $P\setminus Q$ is open, it must be clopen. Thus, by Lemma 3.2 and its dual, every element of $P\setminus Q$ is both maximal and minimal. In particular, Q is clopen decreasing and so too is any subset of $P\setminus Q$. For $a\in L$, if $b\equiv a(\theta)$, then $B\cap Q=A\cap Q$. Hence, B is the union of $A\cap Q$ and a subset of $P\setminus Q$. Since any subset of this form is clopen decreasing, $|[a]\theta|=|\mathscr{P}(P\setminus Q)|$ where $\mathscr{P}$ denotes the power set. Thus, $|[a]\theta|$ is independent of the choice of a.

It remains to consider the case where $P\setminus Q$ is infinite. It is enough to show that every congruence class is infinite as L is countable. For $a\in L$, one of the open sets $P\setminus(Q\cup A)$ or $A\setminus Q$ must be infinite by hypothesis. In the former case, it follows from Lemma 3.2, compactness, and total order-disconnectedness, that there exists a family of distinct clopen decreasing sets $(C_i\subseteq P\setminus(Q\cup A)\colon i<\omega)$. Let $B_i=A\cup C_i$ for each $i<\omega$ and note that $A\cap Q=B_i\cap Q$. If $b_i\in L$ is associated with B_i, then the infinite set $\{b_i\colon i<\omega\}\subseteq[a]\theta$. In the latter case there exists a family of distinct clopen increasing sets $(C_i\subseteq A\colon i<\omega)$. For $i<\omega$, let $B_i=A\setminus C_i$. □

That not every congruence uniform distributive (0, 1)-lattice is Boolean now follows from

THEOREM 3.6. *For every countable distributive* (0, 1)-*lattice L, there exists a family* $(L_i\colon i<2^\omega)$ *of non-isomorphic countable congruence uniform distributive* (0, 1)-*lattices such that, for* $i<2^\omega$, $L\cong L_i/\theta_i$ *for some congruence* θ_i *on* L_i.

PROOF. Clearly, it is sufficient to establish the claim in the case that L is a free algebra. Suppose that this is the case and let (P, τ) denote its Priestley space.

It is well-known that P has a minimum point p which is a limit point of the space. Consider a countably infinite Boolean lattice B with Stone space (S, σ). Since B is infinite it is possible to choose a distinguished point $s \in S$ which is also a limit point.

Let $R = S \times P$ and define a partial order on R by

(i) $(s, x) \leqq (s, y)$ iff $x \leqq y$ in P;
(ii) $(s, p) \leqq (x, p)$ for all $x \in S$.

Then it is a routine exercise to establish that (R, ϱ) is a Priestley space where ϱ is the product topology. Since $(x, y) \in R$ is both maximal and minimal unless $x = s$ or $y = p$, the choice of p and s ensures that every non-empty clopen set contains both a maximal and a minimal point of the space. Thus, by Lemma 3.2, its dual, and Theorem 3.5, the distributive (0, 1)-lattice L_B represented by (R, ϱ) is congruence uniform. Furthermore (P, τ) is order-isomorphic to a closed subspace of (R, ϱ) and so L is a quotient of L_B. The space (S, σ) is homeomorphic to the closed subspace $\{(x, p): x \in S\}$ of (R, ϱ) the elements of which are distinguished by belonging to some maximal chain of length two. It follows that $L_B \cong L_{B'}$ iff $B \cong B'$. In conclusion, observe that there are 2^ω non-isomorphic countable Boolean lattices. □

We remark that Theorem 3.6 remains valid for any $\varkappa \geqq \omega$. Although the given construction will still suffice, Theorem 3.5 may no longer be used to establish congruence uniformity. We omit the details.

To conclude this section, it behoves us to show that the condition given in Theorem 3.5 is insufficient to determine congruence uniformity in general. In fact, we do a little more. First, recall that, for any prime ideal I of a distributive (0, 1)-lattice L, the natural equivalence relation associated with the partition $\{I, L \setminus I\}$ of L is known to be a congruence on L. Therefore, in the event that L is congruence uniform, we have $|I| = |L \setminus I|$, for any prime ideal $I \subseteq L$.

EXAMPLE 3.7. There exists a distributive (0, 1)-lattice L of cardinality 2^ω such that $\Phi^+ = \Phi^* = \omega$ and $|I| = |L \setminus I|$ for any prime ideal $I \subseteq L$, but L is not congruence uniform.

Let λ denote the real line $[0, 1]$ and η the rational elements. Let C be the ordered sum $\oplus(C_i: i \in \lambda)$, where C_i is a two-element chain for $i \in \lambda \setminus \eta$, $1 + \omega^* + \omega + 1$ for $i \in \eta \setminus \{0, 1\}$, and a one-element chain for $i \in \{0, 1\}$. Since C is a complete chain with the jump property, the interval topology on C is compact and totally disconnected. Let 2 denote the discrete space on a two-element set and define a partial order on $P = C \times 2$ by setting

$$(r, 0) < (s, 0) \text{ iff } r < s \text{ in } C \setminus (C_i \setminus \{\omega_i^*, \omega_i\}: i \in \eta \setminus \{0, 1\}).$$

The product topology τ on P is compact and it is not hard to see that it is also totally order-disconnected. Let L denote the distributive (0, 1)-lattice associated with the Priestley space (P, τ).

Since the isolated points, each of which is both maximal and minimal, form a dense subspace of P, it follows that $\Phi^+ = \Phi^* = \omega$. The prime ideals of a distributive (0, 1)-lattice are the elements of the canonically associated Priestley space. Identifying the elements of P with the corresponding prime ideals of L, an element of

L belongs to a prime ideal $I \subseteq L$ iff the associated clopen decreasing subset fails to contain $I \in P$. Since every element of $C \times \{1\}$ is incomparable with every other element of P, $|I| = |L \setminus I| = 2^\omega$ for every prime ideal $I \subseteq L$. Finally, consider the congruence θ on L corresponding to the closed set $\{(1, 0)\} \cup (C \times \{1\})$. For $a \in L$, $a \equiv 1(\theta)$ iff $A \supseteq \{(1, 0)\} \cup (C \times \{1\})$, where A is the clopen decreasing set associated with a. Whereupon, every element of the clopen set $P \setminus A \subseteq \cup (C_i \setminus \{\omega_i^*, \omega_i\} : i \in \eta \setminus \{0, 1\})$ is an isolated point. In particular, $P \setminus A$ is finite and so $|[1]\theta| = \omega$. On the other hand, $|[0]\theta| = 2^\omega$, since $a \equiv 0(\theta)$ for every clopen decreasing set $A \subseteq (C \setminus \{1\}) \times \times \{0\}$.

4. The unbounded case

The initial objective of this section is the characterization of countable congruence uniform distributive lattices as given in Theorem 4.6.

The first preparatory lemma is well-known: see, for example, G. Grätzer [7].

LEMMA 4.1. *Let L be a distributive lattice and let F be a filter of L. Then the relation $\Phi(F)$ defined on L by*

$$x \equiv y(\Phi(F)) \quad \textit{iff} \quad z \geqq x \wedge y \quad \textit{and} \quad x \vee y \vee z \in F \quad \textit{imply} \quad z \in F$$

is the largest congruence on L having F as a whole class. □

If $F = [a)$, then we will write Φ^a for $\Phi(F)$.

We remark that if L has a unit 1, then, by definition, Φ^1 is the largest congruence on L having a trivial co-kernel and, therefore, $\Phi^1 = \Phi^+$.

The next lemma gives a new characterization of Φ^a which will prove useful later.

For a distributive lattice L and $a \in L$, let $[x)^{+_a}$ denote the filter $\{z \in L : z \vee x \geqq a\}$.

LEMMA 4.2. *Let L be a distributive lattice and let $a \in L$. Then*

$$x \equiv y(\Phi^a) \quad \textit{iff} \quad [x)^{+_a} = [y)^{+_a}.$$

PROOF. Define a relation Φ^{+_a} on L by

$$x \equiv y(\Phi^{+_a}) \quad \text{iff} \quad [x)^{+_a} = [y)^{+_a}.$$

We will show that $\Phi^a = \Phi^{+_a}$.

Let $x \equiv y(\Phi^a)$ and suppose that $z \in [x)^{+_a}$ or, equivalently, $z \vee x \geqq a$. Then $z \vee y \geqq x \wedge y$ and $x \vee y \vee (z \vee y) \geqq z \vee x \geqq a$. So, by Lemma 4.1, $z \vee y \geqq a$. Thus, $[x)^{+_a} \subseteq [y)^{+_a}$. Similarly, $[y)^{+_a} \subseteq [x)^{+_a}$. In other words, $x \equiv y(\Phi^{+_a})$ and, hence, $\Phi^a \leqq \Phi^{+_a}$.

Alternatively, suppose $x \equiv y(\Phi^{+_a})$. Then, by definition, $w \vee x \geqq a$ iff $w \vee y \geqq a$. Let (i) $z \geqq x \wedge y$ and (ii) $x \vee y \vee z \geqq a$. By (ii), for $w = x \vee z$, $w \vee y \geqq a$ and so $w \vee \vee x \geqq a$. In other words, $x \vee z \geqq a$. An analogous argument shows that $y \vee z \geqq a$. But, by (i),

$$z = (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \geqq a \wedge a = a.$$

Thus, $x \equiv y(\Phi^a)$ by Lemma 4.1. Hence, $\Phi^{+_a} \leqq \Phi^a$ and so $\Phi^a = \Phi^{+_a}$ as required. □

Let L be a distributive lattice and $a \leqq b$ for some $a, b \in L$. If $\theta(a, b)$ denotes the smallest congruence of L collapsing the pair a, b then

$$x \equiv y(\theta(x, b)) \text{ iff } x \wedge a = y \wedge a \text{ and } x \vee b = y \vee b.$$

Clearly, $[b]\theta(a, b) = [a, b]$. Set $\Phi^{+[a,b]} = \Phi^b \cap \theta(a, b)$ and observe that $[b]\Phi^{+[a,b]} = [b) \cap [a, b] = \{b\}$. Consequently, if L is congruence uniform, then, for all $a, b \in L$ with $a \leqq b$, $\Phi^{+[a,b]} = \omega$.

Analogous to the above, for an ideal $I \subseteq L$, the relation $\Phi(I)$ defined on L by

$$x \equiv y(\Phi(I)) \text{ iff } z \leqq x \vee y \text{ and } x \wedge y \wedge z \in I \text{ imply } z \in I$$

is the largest congruence on L having I as a whole class. Should $I = (a]$ we will write Φ_a for $\Phi(I)$ and, in the event that L has a zero 0, $\Phi_0 = \Phi^*$ is the largest congruence on L having a trivial kernel. If $(x]^{*_a}$ denotes the ideal $\{z \in L\colon z \wedge x \leqq a\}$, then, as in Lemma 4.2, $x \equiv y(\Phi_a)$ iff $(x]^{*_a} = (y]^{*_a}$. Set $\Phi^{*[a,b]} = \Phi_a \cap \theta(a, b)$. Then, arguing in a similar vein to the above, for a congruence uniform distributive lattice L, $\Phi^{*[a,b]} = \omega$ for all $a, b \in L$ with $a \leqq b$.

Hence, if a distributive lattice L is congruence uniform, then $\Phi^{+[a,b]} = \Phi^{*[a,b]} = \omega$ for all $a, b \in L$ with $a \leqq b$. Our objective is to show that this characterizes congruence uniformity in the countable case. The heart of the matter lies in the following lemma which considers the restriction of each such congruence to the associated bounded interval.

Lemma 4.3. *For a distributive lattice L and $a, b \in L$ with $a \leqq b$, the restriction of $\Phi^{+[a,b]}$ to $[a, b]$ is the largest congruence on $[a, b]$ having a trivial co-kernel. A dual statement also holds for the restriction of $\Phi^{*[a,b]}$ to $[a, b]$.*

Proof. Since $[b]\Phi^{+[a,b]} = \{b\}$, $\Phi^{+[a,b]} \restriction [a, b]$ obviously has a trivial co-kernel. Thus, it is enough to show that if Φ^{+ab} denotes the largest congruence on $[a, b]$ having a trivial co-kernel, then $\Phi^{+ab} \leqq \Phi^{+[a,b]} \restriction [a, b]$.

To see this, suppose that $x, y \in [a, b]$ and $x \equiv y(\Phi^{+ab})$. Then $x \equiv y(\theta(a, b))$, since $\theta(a, b)$ collapses $[a, b]$, and, by definition,

$$\{z \in [a, b]\colon z \vee x = b\} = \{z \in [a, b]\colon z \vee y = b\}.$$

We use this and Lemma 4.2 to show that $x \equiv y(\Phi^b)$ from which it follows that $x \equiv y(\Phi^b \cap \theta(a, b))$ or, equivalently, $x \equiv y(\Phi^{+[a,b]})$. Let $w \in [x)^{+_b}$ and let $z = (w \vee a) \wedge b$. Then $z \in [a, b]$ and $b \geqq z \vee x = (w \vee x) \wedge b = b$. Thus, $z \vee x = b$ and so $z \vee y = b$. In other words, $(w \vee y) \wedge b = b$ and so $w \vee y \geqq b$. Hence, $w \in [y)^{+_b}$ and it follows that $[x)^{+_b} \subseteq [y)^{+_b}$. Similarly, $[y)^{+_b} \subseteq [x)^{+_b}$ and so, by Lemma 4.2, $x \equiv y(\Phi^b)$. □

That, in the countable case, it is sufficient to consider the bounded intervals of a distributive lattice in order to determine whether it is congruence uniform is shown by Lemma 4.5. We first observe:

Lemma 4.4. *Every bounded interval of a congruence uniform distributive lattice is congruence uniform.*

Proof. Let L be a congruence uniform distributive lattice and θ a congruence on a bounded interval $[a, b]$. Since the relation Φ on L given by

$$x \equiv y(\Phi) \text{ iff } x \equiv y(\theta(a, b)) \text{ and } (x \vee a) \wedge b \equiv (y \vee a) \wedge b(\theta)$$

is the intersection of two congruence relations on L, it too is a congruence relation. But, for $x\in[a,b]$, $x\equiv y(\Phi)$ iff $y\in[a,b]$ and $x\equiv y(\theta)$; that is, $[x]\theta=[x]\Phi$. Thus, the congruence uniformity of $[a,b]$ follows from that of L. □

LEMMA 4.5. *A countable distributive lattice is congruence uniform iff every bounded interval is congruence uniform.*

PROOF. By Lemma 4.4, it is enough to consider a countable distributive lattice L for which every bounded interval is congruence uniform. Let θ be a congruence on L. If every class of θ is infinite, then, obviously, θ is uniform. Suppose, then, that θ has a finite class $[x]\theta$. By necessity, $[x]\theta$ is a finite bounded interval $[c,d]$, say. Let $y\in L$ and consider $[y]\theta$. Two cases arise:

First, $[y]\theta$ is finite. In this case, $[y]\theta$ is a finite bounded interval $[e,f]$, say. Let $a=c\wedge e$, $b=d\vee f$. Clearly, $[x]\theta\subseteq[a,b]$ and $[y]\theta\subseteq[a,b]$ so that $|[x]\theta|=|[x]\theta\restriction[a,b]|=|[y]=\theta\restriction[a,b]|=|[y]\theta|$, since $[a,b]$ is uniform.

Secondly, $[y]\theta$ is infinite. Let $a_1=c\wedge y$, $b_1=d\vee y$. If $[y]\theta\subseteq[a_1,b_1]$, then, since $[x]\theta\subseteq[a_1,b_1]$, we have, as in case (i), $|[x]\theta|=|[y]\theta|$ which is absurd. Therefore there exists $y_1\in[y]\theta\setminus[a_1,b_1]$. We claim that there is a bounded interval $[a,b]$ such that $[x]\theta\subseteq[a,b]$ and $0<|[y]\theta\cap[a,b]|\neq|[x]\theta|$. Indeed, if this is not so, then, for every bounded interval $[a,b]$ satisfying $[x]\theta\subseteq[a,b]$ and $0<|[y]\theta\cap[a,b]|$, $[y]\theta\cap[a,b]$ must be finite and have the same size as $[x]\theta$. It follows, on writing $a_2=c\wedge y\wedge y_1$ and $b_2=d\vee y\vee y_1$, that, for $i\in\{1,2\}$, $[y]\theta\cap[a_i,b_i]$ is finite and has the same size as $[x]\theta$. But $[y]\theta\cap[a_1,b_1]\subset[y]\theta\cap[a_2,b_2]$, since $y_1\in[a_2,b_2]\setminus[a_1,b_1]$, and we have a contradiction. Hence the claim is substantiated, but it is contrary to the congruence uniformity of $[a,b]$. We conclude that case (ii) cannot arise and so θ is uniform. □

The remarks preceding Lemma 4.3, together with Lemma 4.3, Theorem 3.5, and Lemma 4.5 combine to give

THEOREM 4.6. *Let L be a countable distributive lattice. Then L is congruence uniform iff $\Phi^{+[a,b]}=\Phi^{*[a,b]}=\omega$ for all $a,b\in L$ with $a\leqq b$.* □

The remaining goal of this section (Theorem 4.9) is to exhibit congruence conditions on a distributive lattice L equivalent to requiring $\Phi^{+[a,b]}=\Phi^{*[a,b]}=\omega$ for all $a,b\in L$ with $a\leqq b$. Furthermore, it will be shown that, in the event L is bounded, $\Phi^{+[a,b]}=\Phi^{*[a,b]}=\omega$ for all $a,b\in L$ with $a\leqq b$ is equivalent to $\Phi^{+}=\Phi^{*}=\omega$. Of course, Theorem 4.9 enables us to give alternative versions of Theorem 4.6.

For a distributive lattice L and $a,b\in L$ with $a\leqq b$, write π_{ab} for the natural congruence on L associated with the projection homomorphism $x\to(x\vee a)\wedge b$ from onto $[a,b]$, so that

$$x\equiv y(\pi_{ab}) \quad\text{iff}\quad (x\vee a)\wedge b=(y\vee a)\wedge b.$$

The following is well-known: see, for example, G. Gierz and A. Stralka [6].

LEMMA 4.7. *Let L be a distributive lattice with $a,b\in L$ satisfying $a\leqq b$. Then $\pi_{ab}=\theta^*(a,b)$, where $\theta^*(a,b)$ is the pseudocomplement of $\theta(a,b)$ in the congruence lattice of L.* □

LEMMA 4.8. *Let L be a distributive lattice and let* $a, b \in L$. *Then*

$$\bigcap(\pi_{ac}:\ a \leqq c) \leqq \Phi_a \quad \text{and} \quad \bigcap(\pi_{cb}:\ c \leqq b) \leqq \Phi^b.$$

PROOF. Let $x \equiv y(\bigcap(\pi_{cb}:\ c \leqq b))$ and, hence $(x \vee c) \wedge b = (y \vee c) \wedge b$, for all $c \leqq b$. If $z \in [x)^{+_b}$, then $z \vee x \geqq b$ and so

$$b = (x \vee z) \wedge b = (x \vee (z \wedge b)) \wedge b = (y \vee (z \wedge b)) \wedge b = (y \vee z) \wedge b,$$

since $c = z \wedge b \leqq b$. Thus, $z \vee y \geqq b$ and so $z \in [y)^{+_b}$. In other words, $[x)^{+_b} \subseteq [y)^{+_b}$. Similarly, $[y)^{+_b} \subseteq [x)^{+_b}$ and so, by Lemma 4.2, $x \equiv y(\Phi^b)$. A dual argument shows that $\bigcap(\pi_{ac}:\ a \leqq c) \leqq \Phi_a$. □

THEOREM 4.9. *For a distributive lattice L the following are equivalent:*

(i) $\Phi^{+[a,b]} = \Phi^{*[a,b]} = \omega$, *for all* $a, b \in L$ *satisfying* $a \leqq b$;
(ii) $\Phi_a = \bigcap(\pi_{ac}:\ a \leqq c)$ *and* $\Phi^b = \bigcap(\pi_{cb}:\ c \leqq b)$, *for all* $a, b \in L$;
(iii) $\Psi(a) = \omega$, *for all* $a \in L$, *where* $\Psi(a)$ *denotes* $\Phi_a \cap \Phi^a$.

If, in addition, L is a (0, 1)-*lattice, then each of the above is equivalent to:*

(iv) $\Phi^+ = \Phi^* = \omega$.

PROOF. (i)↔(ii). Observe that, for all $c \leqq b$, $\Phi^{+[c,b]} = \omega$ iff $\Phi^b \cap \theta(c, b) = \omega$ iff, by Lemma 4.7, $\Phi^b \leqq \pi_{cb}$. Thus, $\Phi^{+[c,b]} = \omega$, for all $b, c \in L$ satisfying $c \leqq b$, iff $\Phi^b \leqq \bigcap(\pi_{cb}:\ c \leqq b)$, for all $b \in L$, iff, by Lemma 4.8, $\Phi^b = \bigcap(\pi_{cb}:\ c \leqq b)$, for all $b \in L$. A dual argument completes the proof.

(ii)↔(iii). Suppose first that (ii) holds. Let $a \in L$ and $x \equiv y(\Psi(a))$. Then $x \equiv y(\Phi_a)$ and, hence, $x \equiv y(\bigcap(\pi_{ac}:\ a \leqq c))$; that is, $(x \vee a) \wedge c = (y \vee a) \wedge c$, for all $c \geqq a$. For $c = x \vee a$, we obtain

$$x \vee a = (x \vee a) \wedge (x \vee a) = (y \vee a) \wedge (x \vee a) = (y \wedge x) \vee a.$$

Similarly, taking $c = y \vee a$, we derive $y \vee a = (x \wedge y) \vee a$ and, hence, deduce that $x \vee a = y \vee a$. Since we also have $x \equiv y(\Phi^a)$, a similar argument yields $x \wedge a = y \wedge a$. Therefore, by distributivity, $x = y$ and, hence, $\Psi(a) = \omega$.

Now suppose that (iii) holds. For $c \geqq a$, $\theta(a, c) \leqq \Phi^a$, since Φ^a collapses $[a)$. Therefore, $\Phi_a \cap \theta(a, c) \leqq \Phi_a \cap \Phi^a = \Psi(a) = \omega$. Thus, by Lemma 4.7, $\Phi_a \leqq \bigcap(\pi_{ac}: a \leqq c)$ and so, by Lemma 4.8, $\Phi_a = \bigcap(\pi_{ac}: a \leqq c)$. A dual argument shows $\Phi^b = \bigcap(\pi_{cb}: c \leqq b)$.

Finally, let L be a (0,1)-lattice.

(iii)→(iv). Suppose (iii) holds. Then, in particular, $\Psi(1) = \Phi_1 \cap \Phi^1 = \omega$. But $\Phi^1 = \Phi^+$ and Φ_1 is the largest congruence on L having $(1] = L$ as a whole class. Thus, $\Phi_1 = \iota$ and, therefore, $\Phi^+ = \omega$. Similarly, it follows from $\Psi(0) = \omega$ that $\Phi^* = \omega$.

(iv)→(i). Suppose $\Phi^+ = \omega$. We show that $\Phi^{+[a,b]} = \omega$, for all $a, b \in L$ satisfying $a \leqq b$. Indeed,

$$[1]\Phi^{+[a,b]} = [1]\Phi^b \cap [1]\theta(a, b) \subseteq [b) \cap [1]\theta(0, b) = [b) \cap [b)^+ = \{1\}.$$

Since Φ^+ is the largest congruence on L having a trivial cokernel, $\Phi^{+[a,b]} = \omega$. A similar argument shows that if $\Phi^* = \omega$, then $\Phi^{*[a,b]} = \omega$, for all $a, b \in L$ with $a \leqq b$. □

5. Other approaches

This section provides two different approaches to congruence uniformity; both yield alternative characterizations in the countable case (Corollaries 5.2 and 5.6). For the sake of simplicity, we restrict ourselves to (0, 1)-lattices.

If L is a distributive (0, 1)-lattice, then its filter lattice $F(L)$ is a pseudocomplemented distributive lattice. Recall that, for $a \in L$, $[a)^+ = \{c \in L\colon a \vee c = 1\}$ is the pseudocomplement of the filter $[a)$ in $F(L)$. Thus, for consistency, let $^+$ denote pseudocomplementation in $F(L)$. Then the skeleton $B(F(L)) = \{F^+\colon F \in F(L)\}$ of $F(L)$ is a complete Boolean algebra $\langle B(F(L)); \vee, \cap, ^+, \{1\}, L\rangle$ where $\vee$ is defined by $F \vee G = (F^+ \cap G^+)^+$, for $F, G \in B(F(L))$.

Recall that a sublattice S of a complete lattice L is said to be *meet dense* (in L) if, for every $a \in L$, there exists $T \subseteq S$ such that $a = \wedge T$.

THEOREM 5.1 (a). *Let L be a distributive* (0, 1)*-lattice. Then $\Phi^+ = \omega$ iff L is isomorphic to a meet dense sublattice of its injective hull B_L.*

PROOF. Suppose first that $\Phi^+ = \omega$. Observe that, by the theory of pseudocomplemented distributive lattices, for any $a, b \in L$,

$$[a \wedge b)^+ = ([a) \vee [b))^+ = [a)^+ \cap [b)^+$$

and

$$[a \vee b)^+ = ([a) \wedge [b))^+ = [a)^+ \vee [b)^+.$$

Since $\Phi^+ = \omega$, it follows that $a \to [a)^+$ is an embedding of L into $B(F(L))$. Furthermore, for $F \in B(F(L))$,

$$F = F^{++} = (\vee([a)\colon a \in F^+))^+ = \cap([a)^+\colon a \in F^+).$$

Therefore, L is isomorphic to the meet dense sublattice $L^+ = \{[a)^+\colon a \in L\}$ of $B(F(L))$ under the embedding $a \to [a)^+$.

To complete this part of the proof, we now show that $B(F(L))$ is the injective hull of L^+. That is, we require $B(F(L))$ to be an injective essential extension of L^+. Since $B(F(L))$ is complete, it will be enough to show that it is an essential extension of L^+. In other words, if θ is a congruence on $B(F(L))$ such that $\theta \restriction L^+ = \omega$, then we require $\theta = \omega$. To see this, suppose $\theta \neq \omega$. Then the co-kernel of θ, namely $[L]\theta$, is non-trivial, since congruences on Boolean lattices are completely determined by their co-kernels. Suppose $F \in [L]\theta$ and $F \neq L$. Then, for some $a \in L$, $[a)^+ \in [L]\theta$ where $[a)^+ \neq L$, since L^+ is a meet dense sublattice of $B(F(L))$. This contradicts $\theta \restriction L^+ = \omega$. Hence, $\theta = \omega$.

Suppose now that L is isomorphic to a meet dense sublattice of its injective hull B_L. By the congruence extension property for distributive lattices, there is a congruence θ on B_L such that $\theta \restriction L = \Phi^+$. We want to show that $\Phi^+ = \omega$. If $\Phi^+ \neq \omega$, then $\theta \neq \omega$, and, since B_L is Boolean, the co-kernel of θ is non-trivial. Thus, for some $b \in B_L$ distinct from 1, $b \in [1]\theta$. But L is a meet dense sublattice of B_L and so, for some $s \in L$, $s \neq 1$ and $s \equiv 1(\theta)$ which is absurd since the co-kernel of Φ^+ is trivial. Hence, $\Phi^+ = \omega$. □

For a bounded distributive lattice L, its ideal lattice is also a pseudocomplemented distributive lattice. Let $B(I(L)) = \{I^*\colon I \in I(L)\}$ denote its skeleton, where * denotes the operation of pseudocomplementation (chosen to be consistent with $(a]^* = \{c \in L\colon a \wedge c = 0\}$).

Arguments similar to the above show that if $\Phi^*=\omega$, then L is isomorphic to a join dense sublattice of its injective hull B_L. This time, the appropriate embedding is given by $a\to(a]^{**}$ which maps L onto a join dense sublattice $L^{**}=\{(a]^{**}: a\in L\}$ of $B(I(L))$. Furthermore, $B(I(L))$ is the injective hull of L^{**}. Similarly, if L is isomorphic to a join dense sublattice of its injective hull, then an argument using kernels instead of co-kernels shows that $\Phi^*=\omega$. Consequently, we have

THEOREM 5.1 (b). *Let L be a distributive* (0, 1)*-lattice. Then $\Phi^*=\omega$ iff L is isomorphic to a join dense sublattice of its injective hull B_L.* □

COROLLARY 5.2. *A countable distributive* (0, 1)*-lattice is congruence uniform iff it is isomorphic to a join dense sublattice and to a meet dense sublattice of its injective hull.* □

Since $B(F(L))$ and $B(I(L))$ are both isomorphic to B_L, we also conclude:

COROLLARY 5.3. *For a distributive* (0, 1)*-lattice, if $\Phi^+=\Phi^*=\omega$, then $B(F(L))\cong B(I(L))$.* □

Although $B(F(L))\cong B(I(L))$ for any finite distributive lattice, Corollary 5.3 still reflects the Boolean nature of the lattices concerned. As the following example shows, an obvious attempt to sharpen it goes adrift.

EXAMPLE 5.4. There exists a countable distributive (0, 1)-lattice L such that $\Phi^+=\Phi^*=\omega$ but $F(L)\not\cong I(L)$.

Since $\omega+1$ is complete and has the jump property, its interval topology is compact and totally disconnected. Let $P=(\omega+1)\times(\omega+1)$ and define an order relation on P by $(x,\omega)\leqq(y,\omega)$ iff $x\leqq y$. Then P with the product topology τ is a compact totally order-disconnected space. Let L be the distributive (0, 1)-lattice associated with the Priestley space (P,τ).

Since the isolated points, each of which is both maximal and minimal, form a dense subspace, $\Phi^+=\Phi^*=\omega$.

Ideals (filters) of L correspond to open (closed) decreasing subsets of P. If $I\in I(L)$, then the associated open decreasing set is $\bigcup\{A: a\in I\}$ where, for $a\in I$, A denotes the clopen decreasing set representing a. Thus, for $I,J\in I(L)$ associated with open decreasing sets $Q,R\subseteq P$, respectively, $I\subseteq J$ iff $Q\subseteq R$. Let I correspond to the open decreasing set $(\omega+1)\times\omega$. Then $[I)\subseteq I(L)$ is an $\omega+1$ chain. On the other hand, if $F\in F(L)$, then the associated closed decreasing set is $\bigcap\{A: a\in F\}$. Consequently, for $F,G\in F(L)$ with associated closed decreasing sets S, T, respectively, $F\subseteq G$ iff $S\supseteq T$. If $(\omega,\omega)\in S$ or $S\cap((\omega+1)\times\omega)$ is infinite, then $[F)$ contains an ω^* chain. Otherwise S and, hence, $[F)$ are finite. Either way, for any $F\in F(L)$, $[F)\not\cong[I)$. □

Let us mention that we do not know, for example, whether every congruence uniform distributive (0, 1)-lattice is isomorphic to a (simultaneously, meet and join) dense sublattice of its injective hull. Nor do we know whether a lattice that is isomorphic to a dense sublattice of its injective hull is necessarily congruence uniform (except, of course, in the countable case).

Rather than considering the effects of congruence uniformity on particular congruence relations of the lattice in question, one may consider the effect on well-defined subsets. We conclude with one such example.

LEMMA 5.5. *For a distributive lattice L with a unit* 1, *each of the following*

(i) *L is congruence uniform;*

(ii) *for $a, b \in L$ such that $a \leqq b$, $|[a, b]| = |[a) \cap [b)^+|$;*

(iii) $\Phi^+ = \omega$

implies the next.

PROOF. (i)$\to$(ii). Let $a, b \in L$ satisfy $a \leqq b$ and consider the principal congruence $\theta(a, b)$. It is easily verified that $[a]\theta(a, b) = [a, b]$ and $[1]\theta(a, b) = [a) \cap [b)^+$. If L is congruence uniform, then $|[a]\theta(a, b)| = |[1]\theta(a, b)|$ and the claim follows.

(ii)$\to$(iii). Suppose $\Phi^+ \neq \omega$. Then, for some $a, b \in L$ with $a < b$, $[a)^+ = [b)^+$. Whence, for $c \in L$, $a \vee c = 1$ iff $b \vee c = 1$. In particular, if $b \vee c = 1$ for some $c < 1$, then $a \not\leqq c$ and, hence, $c \notin [a) \cap [b)^+$. Consequently, $|[a) \cap [b)^+| = |\{1\}| = 1$ whereas $|[a, b]| \geqq 2$. □

Lemma 5.5, its dual, and Theorem 3.5 combine to give

COROLLARY 5.6. *A countable distributive* (0, 1)*-lattice L is congruence uniform iff, for all $a, b \in L$ such that $a \leqq b$,*

$$|[a, b]| = |[a) \cap [b)^+| = |(b] \cap (a]^*|. \quad \square$$

We do not know whether Corollary 5.6 holds for any cardinality.

References

[1] R. Balbes and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press (Columbia, Missouri, 1974).

[2] R. Beazer, Congruence uniform algebras with pseudocomplementation, *Studia Sci. Math. Hungar.*, **20** (1985), 43—48.

[3] R. Beazer, Varieties of modal lattices. *Houston J. Math.*, **12** (1986), 357—369.

[4] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, Springer-Verlag, 1981.

[5] B. A. Davey and D. Duffus, Exponentiation and duality, in *Ordered Sets* (ed. I. Rival), NATO Advanced Study Institutes Series, D. Reidel (Dordrecht, 1982), pp. 43—96.

[6] G. Gierz and A. Stralka, The injective hull of a distributive lattice and congruence extension, *Quart. J. Oxford*, **35** (1984), 25—36.

[7] G. Grätzer, *General Lattice Theory*, Birkhäuser Verlag (Basel and Stuttgart, 1978).

[8] R. McKenzie, Narrowness implies uniformity, *Algebra Univ.*, **15** (1982), 67—85.

[9] W. Taylor, Uniformity of congruence, *Algebra Univ.*, **4** (1974), 346—360.

[10] H. A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.*, **2** (1970), 186—190.

[11] H. A. Priestley, Ordered sets and duality for distributive lattices, *Ann. Discrete Math.*, **23** (1984), 39—60.

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STATE UNIVERSITY OF NEW YORK
NEW PALTZ
NEW YORK 12561
U.S.A.

UNIVERSITY OF GLASGOW
UNIVERSITY GARDENS
GLASGOW G12 8QW
SCOTLAND

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SOME REMARKS ON GARAY'S CONJECTURE

M. MROZEK (Kraków)

1. Introduction

Let (X, d) be a metric space and let $M \subseteq X$ be a compact subset. We will say that the function $V: X \to [0, \infty)$ is distance-like with respect to M iff it vanishes precisely on M and $\{x_n\} \subseteq X$, $V(x_n) \to 0$ implies $d(x_n, M) := \inf \{d(x_n, y) | y \in M\} \to 0$.

We will say that the pair (X, M) or briefly M satisfies the condition (G) iff for every function $V: X \to [0, \infty)$ distance-like with respect to M there exists a metric ϱ on X equivalent to d such that

$$(1) \qquad \varrho|_M = d|_M \quad \text{and} \quad \forall x \in X \quad \varrho(x, M) = V(x).$$

B. M. Garay conjectured in [4] that the compact M satisfies the condition (G) iff it is a retract of X. He also proved in [5] that the conjecture holds true in the planar case. In this paper we prove that every compact subset M of a Banach space X satisfying the condition (G) is acyclic in the sense of the Alexander—Spanier cohomology. This yields partial (positive) answer to Garay's conjecture: Euclidean neighborhood retracts with commutative fundamental group satisfy the condition (G) iff they are absolute retracts.

Since the answer to Garay's conjecture seems to be difficult in general, we introduce a stronger condition (G′) defined below and show that in the finite dimensional case Garay's conjecture for the modified condition is true. This enables us to reformulate the original conjecture in terms of some kind of equicontinuity.

An important tool in the proofs of the above results is the projection of X onto M along ϱ (also called the nearest point map), i.e. a multivalued map which assigns to every point $x \in X$ the set of points y in M such that the distance from x to y equals the distance from x to M. This map was implicitly used in a characterization of retracts by Kuratowski (see [8]). The application of this map to the study of Garay's conjecture seems to be an essential contribution of this paper.

2. A cohomological necessary condition for (G)

Let $(X, \|\cdot\|)$ be a fixed Banach space and for $x, y \in X$ let $d(x, y) := \|x - y\|$ be the natural metric on X. Let ϱ be any metric on X equivalent to d. If $A \subseteq X$ then $\operatorname{diam}_\varrho A$, $\varrho(x, A)$, $K_\varrho(A, \varepsilon)$ will stand for the diameter of A with respect to ϱ, the distance from x to A and the closed ball around A of radius ε, respectively. For $A, B \subseteq X$ we will write

$$\varrho(A, B) := \inf \{\varrho(x, y) | x \in A, y \in B\}.$$

The closed convex hull of A will be denoted by conv A and the family of all non-empty subsets of A by $\mathcal{P}(A)$. The functors of singular homology, singular cohomology and Alexander—Spanier cohomology will be denoted by H, H^* and $\bar{H}^*$, respectively. All homology and cohomology will be considered with coefficients in the ring $\mathbf{Z}$ of integers. $\mathbf{N}$ will stand for the set of positive integers.

Recall that the set M is called acyclic with respect to a particular homology (cohomology) if the zero-dimensional homology (cohomology) group is $\mathbf{Z}$ and the higher dimensional homology (cohomology) groups vanish.

The main result of this paper is the following

THEOREM 1. *Assume M satisfies condition* (G). *Then M is acyclic with respect to the Alexander—Spanier cohomology.*

The above theorem provides strong limitations for the possible counter-examples to Garay's conjecture. For instance no counter-examples can be found among spheres (compare also [2]), tori etc.

The proof of the above theorem will be given further on. Here we prove the following corollary, which gives the positive answer to Garay's conjecture in a restricted case.

COROLLARY 1. *Assume $M\subseteq\mathbf{R}^n$ is an* ANR-*space with commutative fundamental group. Then M satisfies condition* (G) *iff M is an* AR-*space.*

PROOF. Garay implicitly proved in [5] that condition (G) is necessary for M to be an AR-space. Hence it remains to prove that this condition is also sufficient. Let M satisfy (G). Then by the above Theorem, M is acyclic with respect to Alexander—Spanier cohomology. However in case of an ANR-space, which is locally contractible (see [1], Chapt. V, (2.6)) the Alexander—Spanier cohomology and the singular cohomology coincide ([10, § 6.9, Corollary 5]). Thus M is acyclic with respect to the singular cohomology. However, the singular homology of a compact ANR-space is of finite type (see [11]), so we can apply Theorem 12, § 5.5 in [10] to obtain the short exact sequence

$$0\to \operatorname{Ext}(H^{q+1}(M),\mathbf{Z})\to H_q(M)\to \operatorname{Hom}(H^q(M),\mathbf{Z})\to 0$$

for $q=0,1,2,\ldots$. Putting $H^q(M)=0$ for $q>0$ and $H^0(M)=\mathbf{Z}$ in the above sequence, we get the following short exact sequences

$$0\to H_q(M)\to 0 \quad \text{for} \quad q>0, \quad 0\to H_0(M)\to \mathbf{Z}\to 0,$$

which show that M is acyclic also with respect to the singular homology. In particular M is arcwise connected (see [7], (10.6)) and by ([7], (12.2)) the Hurewicz homomorphism $\chi\colon \pi_1(M)\to H_1(M)$ from the fundamental group of M to the first singular homology group of M is an isomorphism and consequently $\pi_1(M)=0$. Now M is an AR-space directly by ([1], Chapt. V, Theorem (10.8)).

By a multivalued mapping from X to $A\subseteq X$ we mean a mapping $F\colon X\to\mathcal{P}(A)$. We say that F is upper semi-continuous (u.s.c.) iff for every $x_0\in X$ $F(x_0)$ is compact and for every $\varepsilon>0$ there exists $\delta>0$ such that

$$d(x,x_0)<\delta\Rightarrow F(x)\subseteq K_d(F(x_0),\varepsilon).$$

By the diameter of a multivalued map F with respect to a metric ϱ we mean the number

$$\operatorname{diam}_\varrho F := \sup \{\operatorname{diam}_\varrho F(x) | x \in X\}.$$

Let ϱ be a metric on X equivalent to d and let M be a compact subset of X.

DEFINITION 1. The multivalued mapping $F_{\varrho,M}\colon X \to M$ given by

$$F_{\varrho,M}(x) := \{y \in M | \varrho(x,y) = \varrho(x,M)\}$$

will be called the projection of X onto M along ϱ.

LEMMA 1. *The projection $F_{\varrho,M}$ is u.s.c.*

PROOF. Assume $F := F_{\varrho,M}$ is not u.s.c. at some $x_0 \in X$. Then there exists $\varepsilon > 0$ and sequences $\{x_n\}$, $\{y_n\} \subseteq X$ such that $x_n \to x_0$, $y_n \in F(x_n)$ and $\varrho(y_n, F(x_0)) \geqq \varepsilon$. Using compactness of M and taking subsequences if necessary, we can assume that $\{y_n\}$ converges to some $y_0 \in M$. Obviously $\varrho(y_0, F(x_0)) \geqq \varepsilon$. Put $r := \varrho(x_0, M)$. We have $\varrho(x_0, y_0) > r$ because $y_0 \notin F(x_0)$. Thus $\mu := \varrho(x_0, y_0) - r > 0$. Choose an $N \in \mathbf{N}$ such that $\varrho(x_n, x_0) < \mu/6$ and $\varrho(y_n, y_0) < \mu/2$ for $n \geqq N$. Then $\varrho(x_n, M) = \varrho(x_n, y_n) \geqq \geqq \varrho(x_0, y_0) - \varrho(x_n, x_0) - \varrho(y_n, y_0) \geqq r + \mu - \mu/2 - \mu/6 = r + \mu/3$. On the other hand $\varrho(x_n, M) \leqq \varrho(x_n, x_0) + \varrho(x_0, M) \leqq r + \mu/6 < r + \mu/3$, which contradicts the previous estimation. The proof is complete.

Let $F\colon X \to M$ be a multivalued map. Define a new multivalued map $\operatorname{Conv} F\colon X \to X$ by

$$(\operatorname{Conv} F)(x) := \operatorname{conv} F(x),$$

where $\operatorname{conv} A$ denotes the closed convex hull of A.

LEMMA 2. *Assume F is u.s.c. Then* $\operatorname{Conv} F$ *is u.s.c. and*

$$\operatorname{diam}_d (\operatorname{Conv} F) \leqq \operatorname{diam}_d F.$$

PROOF. Fix $x_0 \in M$. The set $\operatorname{conv} F(x_0)$ is compact by Theorem 3.25 in [9]. Take $\varepsilon > 0$. Then there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies

$$F(x) \subseteq K_d(F(x_0), \varepsilon) \subseteq K_d(\operatorname{conv} F(x_0), \varepsilon),$$

Since the last set is closed and convex, we have also

$$\operatorname{conv} F(x) \subseteq K_d(\operatorname{conv} F(x_0), \varepsilon).$$

which proves the upper semicontinuity of $\operatorname{Conv} F$. The remaining inequality is obvious.

PROOF OF THEOREM 1. Put $V(x) := d(x, M)$ and for any positive integer n let $V_n(x) := \min(V(x), 1/n)$. Then for every n the function V_n is distance-like with respect to M, so we can find a metric ϱ_n on X equivalent to d and satisfying (1). Let F_n denote the projection of X onto M along ϱ_n. Observe that

$$\operatorname{diam} F_n \leqq 2/n. \tag{2}$$

In fact, fix $x_0 \in X$ and take $y_1, y_2 \in F_n(x_0)$. Then

$$d(y_1, y_2) = \varrho(y_1, y_2) \leqq \varrho(y_1, x) + \varrho(x, y_2) = 2\varrho(x, M) = 2V(x) \leqq 2/n,$$

which proves (2).

By Lemma 2 condition (2) holds also for $C_n := \operatorname{Conv} F_n$ and consequently for any $x \in X$, $C_n(x) \subseteqq K_d(M, 2/n)$.

The following part of the proof is based on the idea of the map of cohomology groups induced by a multi-valued map (compare [6]).

Put $U_n := K_d(M, 2/n)$ and $\Gamma_n := \{(x, y) \in X \times U_n | y \in C_n(x)\}$. Let $p_n\colon \Gamma_n \to X$ and $q_n\colon \Gamma_n \to U_n$ denote projections and $\iota\colon M \to X$ and $\iota_n\colon M \to U_n$ denote inclusions. Set also $c_n\colon M \ni x \to (x, x) \in \Gamma_n$. The last map is well defined because for $x \in M$ we have

$$C_n(x) = \operatorname{conv} F_n(x) = \operatorname{conv}\{x\} = \{x\}.$$

Obviously $\iota_n = q_n \circ c_n$ and $\iota = p_n \circ c_n$. Thus, applying the Alexander—Spanier cohomology functor we get

$$\bar{H}^*(\iota_n) = \bar{H}^*(c_n) \circ \bar{H}^*(q_n) \quad \text{and} \quad \bar{H}^*(\iota) = \bar{H}^*(c_n) \circ \bar{H}^*(p_n).$$

Let $A \subseteqq X$ be compact. Since $p_n^{-1}(A) \subseteqq A \times C_n(A)$ and $C_n(A)$ is compact by [6, (1.2), p. 25], we see that $p_n^{-1}(A)$ is compact, i.e. p_n is proper. Hence by [6, (1.7), p. 26] p_n is a closed map. It has also acyclic fibres, because $C_n(x)$ is convex. Thus we can apply the Vietoris—Begle Theorem (see [10], § 6.9, Theorem 15) to conclude that $\bar{H}^*(p_n)$ is an isomorphism. Since X is contractible we get for $k > 0$ that $\bar{H}^k(X) = 0$ and consequently that $\bar{H}^k(\iota) = 0$. Hence

$$\bar{H}^k(\iota_n) = \bar{H}^k(\iota) \circ \bar{H}^k(p_n)^{-1} \circ \bar{H}^k(q_n) = 0. \tag{3}$$

By Theorem 2, § 6.6 in [10], M is a taut subspace of X, i.e. $\bar{H}^*(M)$ is the direct limit of $\bar{H}^*(U_n)$. This together with (3) shows that $\bar{H}^k(M) = 0$ for $k > 0$. Part (A) of the proof of Theorem in [5] can be adopted without change to show that M satisfying (G) is connected. Thus it follows from ([10], § 6.4, Corollary 7) that $H^0(M) = \mathbf{Z}$ and the proof is finished.

3. A reformulation of Garay's conjecture

From now on we assume that $X = E$ is a finite dimensional Euclidean space, $\|.\|$ stands for the Euclidean norm on E and $d(x, y) := \|x - y\|$ is the corresponding metric. Denote by $\mathcal{M}$ the family of all metrics on E equivalent to d and by $\mathcal{A}(M)$ the family of all functions on E which are distance-like with respect to M.

We will say that the the family $\{\varrho_\iota | \iota \in I\} \subseteqq \mathcal{M}$ is uniformly weak with respect to d iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow \varrho_\iota(x, y) < \varepsilon \quad \text{for all} \quad \iota \in I.$$

We will say that the compact $M \subseteqq E$ satisfies condition (G′) iff for every equicontinuous family $\{V_\iota | \iota \in I\} \subseteqq \mathcal{A}(M)$ one can find a corresponding family $\{\varrho_\iota | \iota \in M\} \subseteqq \mathcal{M}$, which is uniformly weak with respect to d and satisfies condition (1) with V and ϱ replaced by V_ι and ϱ_ι respectively.

We have the following

THEOREM 2. *A compact* $M \subseteq E$ *is a retract of* E *iff it satisfies* (G').

Before proving the above Theorem we state the following obvious

COROLLARY 2. *Garay's conjecture holds true in* E *iff conditions* (G) *and* (G') *are equivalent. (Note that* (G') *always implies* (G)*).*

The proof of the theorem is based on the following Lemma and Proposition.

LEMMA 3. *For every compact retract* M *of* E *there exists a retraction* $r\colon E \to M$ *which is uniformly continuous.*

PROOF. Fix any retraction $r_1\colon E \to M$, take $s>0$ such that

$$M \subseteq A := \{x \in E \mid d(x, 0) \leq s\}$$

and put

$$p\colon E \ni x \to \begin{cases} x & \text{for } x \in A \\ sx/\|x\| & \text{otherwise.} \end{cases}$$

Since p is Lipschitz continuous with constant 1, the mapping $r := r_1|_A \circ p$ is the required uniformly continuous retraction.

One can easily prove the following

PROPOSITION 1. *Assume* $\{A_n\}$ *is a sequence of subsets of a compact set* M *such that* $\operatorname{diam}_d A_n \to 0$. *Then there exists a subsequence* $\{C_n\} \subseteq \{A_n\}$ *which converges to a point* $q \in M$ *in the sense that for all* $\varepsilon > 0$ *there exists an* $N \in \mathbf{N}$ *such that* $d(q, C_n) < \varepsilon$.

PROOF OF THEOREM 2. Assume M is a retract of E. By Lemma 3 we can take a retraction $r\colon E \to M$ which is uniformly continuous. Assume $\{V_\iota \mid \iota \in I\} \subseteq \mathscr{A}(M)$ is equicontinuous. Following Garay's paper [5] define for $x, y \in E$ and $\iota \in I$

$$d_1(x, y) := \min\{1, d(x, y)\}, \quad W_\iota(x) := \min\{1, V_\iota(x)\},$$

$$a_\iota(x, y) := \sup\{|W_\iota(x)\, d_1(x, z) - W_\iota(y)\, d_1(y, z)| \colon z \in E\},$$

$$b(x, y) := d\big(r(x), r(y)\big)$$

and

$$\varrho_\iota(x, y) := \max\{|V_\iota(x) - V_\iota(y)|,\ a_\iota(x, y),\ b(x, y)\}.$$

Like in [5] one can verify that $\{\varrho_\iota \mid \iota \in I\} \subseteq \mathscr{M}$ and (1) is satisfied with ϱ, V replaced by ϱ_ι, V_ι for all $\iota \in I$, resp. The fact that the family $\{\varrho_\iota \mid \iota \in I\}$ is uniformly weak with respect to d follows directly from the fact that r is uniformly continuous and the family $\{V_\iota \mid \iota \in I\}$ is equicontinuous. This finishes the first part of the proof.

Now assume M satisfies (G'). Let $V(x) := d(x, M)$ and, for $n \in \mathbf{N}$,

$$V_n(x) := \min\{V(x), 1/n\}.$$

Obviously the family $\{V_n\}$ is equicontinuous. Thus we can apply condition (G') to find a corresponding family of metrics $\{\varrho_n\}$. Take a compact ball B such that $M \subseteq B$ and consider the projection of B onto M along ϱ_n given by

$$F_n\colon B \ni x \to \{y \in M \mid \varrho_n(x, y) = \varrho_n(x, M)\}.$$

Observe that for $x \in M$ and any $n \in \mathbf{N}$ we have

$$F_n(x) = \{x\}. \tag{4}$$

Fix $n \in \mathbf{N}$ and $x \in B$ and take $y_1, y_2 \in F_n(x)$. Since $F_n(x) \subseteq M$ we get from (1) that

$$d(y_1, y_2) = \varrho_n(y_1, y_2) \leqq \varrho_n(y_1, x) + \varrho_n(x, y_2) = 2V_n(x) \leqq 2/n.$$

Thus

$$\sup\{\operatorname{diam}_d F_n(x) \mid x \in B\} \to 0. \tag{5}$$

We will show that the sequence $\{F_n\}$ is equicontinuous in the sence that for every $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ and $\delta > 0$ such that for $n \geqq N$

$$d(x, y) < \delta \Rightarrow d(F_n(x), F_n(y)) < \varepsilon. \tag{6}$$

To this end take $\varepsilon > 0$ and, using the fact that the family $\{\varrho_n\}$ is uniformly weak with respect to d, find $\delta > 0$ such that

$$d(x, y) < \delta \ \Rightarrow\ \varrho_n(x, y) < \varepsilon/3 \quad \text{for all} \quad n \in \mathbf{N}.$$

Let $N \in \mathbf{N}$ be such that $1/N \leqq \varepsilon/3$. Take $x_1, x_2 \in B$ such that $d(x_1, x_2) < \delta$ and let $y_i \in F(x_i)$ for $i = 1, 2$. Then for $n \geqq N$ we have

$$d(F_n(x_1), F_n(x_2)) \leqq d(y_1, y_2) = \varrho_n(y_1, y_2) \leqq$$

$$\leqq \varrho_n(y_1, x_1) + \varrho_n(x_1, x_2) + \varrho_n(x_2, y_2) < 1/N + \varepsilon/3 + 1/N \leqq \varepsilon.$$

Thus (6) is proved.

Since every metric compact space is separable (see [3], 4.1.18) we can find a countable set $A = \{a_n\} \subseteq B$ such that A is dense in B and $A \cap M$ is dense in M. By the Proposition and condition (5) we can find a subsequence $\{F_n^{(1)}\} \subseteq \{F_n\}$, which converges at a_1 to a point of M. Recursively, we can find a subsequence $\{F_n^{(k+1)}\} \subseteq \subseteq \{F_n^{(k)}\}$, which converges at points $a_1, a_2, \ldots, a_{k+1}$. Then the diagonal sequence $\{G_n\} := \{F_n^{(n)}\}$ is a subsequence of the sequence $\{F_n\}$ and it converges at a_n to a point $f(a_n) \in M$ for all natural n. Thus we have defined a function $f\colon A \to M$. It follows directly from the equicontinuity of the sequence $\{F_n\}$ that the function f is uniformly continuous on A. This means that for two closed, disjoint subsets M_1, M_2 of M the closures of $f^{-1}(M_1)$ and $f^{-1}(M_2)$ in X are also disjoint. Thus, (by [3], 3.2.1 and 2.1.9) f admits precisely one continuous extension $r\colon B \to M$. It follows from (4) that for $x \in A \cap M$ we have $f(x) = x$, hence $r|_M = \mathrm{id}_M$. This means that M is a retract of B. Since B is a retract of E, Theorem 2 is proved.

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References

[1] K. Borsuk, *Theory of Retracts*, PWN (Warszawa, 1967).
[2] A. P. Bosznay, B. M. Garay, On a geometric problem concerning discs. *Acta Sci. Math. (Szeged)*, to appear.
[3] R. Engelking, *General Topology*, PWN (Warszawa, 1977).
[4] B. M. Garay, On dynamical systems in metric spaces: metrics as Liapunov functions. *J. Nonlin. Anal. T.M.A.*, **8** (1984), 1033—1042.
[5] B. M. Garay, A Metric Characterization of Compact Plane Retracts and Applications, *Ann. di Matem. pura ed appl.*, **CXXXIX** (1985), 329—340.
[6] L. Górniewicz, Homological methods in fixed-point theory of multi-valued maps, *Dissertationes Mathematicae*, **CXXIX**, PWN (Warszawa, 1976).
[7] M. J. Greenberg, *Lectures on Algebraic Topology*, W. A. Benjamin Inc. (London, 1967).
[8] K. Kuratowski, Une condition metrique pour la retraction des ensembles, *C. R. de la Soc. de Varsovie*, **28** (1936), 156—158.
[9] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company (New York, 1973).
[10] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Company (New York, 1966).
[11] J. E. West, Compact ANR's have finite type, *Bull. Am. Math. Soc.*, **81** (1975), 163—165.

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KATEDRA INFORMATYKI
UNIVERSYTET JAGIELLOŃSKI
UL. KOPERNIKA 27
31—501 KRAKÓW
POLAND

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A NOTE ON EUCLIDEAN RAMSEY THEORY AND A CONSTRUCTION OF BOURGAIN

N. ALON and Y. PERES (Tel Aviv)

1. Qualitative facts

Let v be a fixed unit vector in a Hilbert space Ω. Denote

$$\Omega_c = \{\omega \in \Omega | \langle v, \omega \rangle = c, \ \|\omega\| = 1\}$$

for a real $0 < c < 1$. Bessel's inequality implies that any orthogonal sequence in Ω_c is finite. Thus, Ramsey's theorem implies

FACT 1. *From any infinite sequence $\{\omega_n\}_{n=1}^{\infty}$ in Ω_c an infinite subsequence can be extracted, with no two vectors orthogonal.*

We will be interested in the "size" of the subsequence which can be extracted, especially when a further restriction is put on the sequence $\{\omega_n\}$. In particular, we show that a subsequence of positive density cannot always be extracted.

DEFINITIONS. I. A sequence of vectors $\{\omega_n\}$ in a Hilbert space is *stationary* if $\langle \omega_{i+n}, \omega_{j+n} \rangle = \langle \omega_i, \omega_j \rangle$ for all i, j, n.

II. A set of integers $H \subset \mathbf{N}$ is a *Van der Corput set* if every probability measure μ on the circle satisfying $\hat{\mu}(h) = \int e^{-iht} d\mu(t) = 0$ for every $h \in H$ satisfies $\mu\{0\} = 0$.

III. A set of integers $H \subset \mathbf{N}$ is a *Poincare set* if for every set $S \subset \mathbf{N}$ of positive density, H intersects the difference set $S - S$. (For an alternative ergodic theory definition see [3].)

Kamae and Mendes France [5] proved that all Van der Corput sets are Poincare sets. Recently, J. Bourgain [1] has proved that the reverse implication does not hold. This implies

FACT 2. *There exist a $0 < c < 1$ and a stationary sequence of vectors $\{\omega_n\}$ in Ω_c such that for any $S \subset \mathbf{N}$ of positive density, $\omega_n \perp \omega_m$ for some $m, n \in S$.*

PROOF Let H be a Poincare set which is not Van der Corput. There exists a measure μ for which $\hat{\mu}(n) = 0 \ \forall n \in H$ and $\mu\{0\} = c^2 > 0$. Let Ω be the Hilbert space $L^2[0, 2\pi)$. Let $\omega_n(t) = e^{int}$, and denote

$$v(t) = \begin{cases} c^{-1}, & t = 0 \\ 0, & t \neq 0. \end{cases}$$

Clearly $\omega_n \in \Omega_c$. For any sequence $S \subset \mathbf{N}$ of positive density, some $m, n \in S$ satisfy $m - n \in H$ and hence $\hat{\mu}(m-n) = \langle \omega_m, \omega_n \rangle = 0$. □

Bourgain's construction is difficult; thus we note

FACT 3. *From any sequence $\{\omega_n\}$ satisfying the conclusion of Fact 2, one can easily construct a Poincare set which is not Van der Corput.*

PROOF. By the stationarity of $\{\omega_n\}$, the sequence $\{\langle\omega_n, \omega_0\rangle\}$ is positive definite, so by Herglotz's theorem [6], there exists a positive measure μ on the circle, such that $\hat{\mu}(n)=\langle\omega_n, \omega_0\rangle$ for all n. From $\langle\omega_n, v\rangle=c>0$ it easily follows that $\mu\{0\}>0$. (Indeed, $\{\omega_n-cv\}$ is stationary and hence there is a positive measure ν so that $\hat{\nu}(n)=\langle\omega_n-cv, \omega_0-cv\rangle=\hat{\mu}(n)-c^2$. This implies $\mu=\nu+c^2\delta_0$ and $\mu\{0\}\geqq c^2$.) Thus $H=\{n>0|\hat{\mu}(n)=0\}$ is the desired Poincare set. □

If we ignore the geometry and concentrate on the combinatorics of Fact 2, we get

FACT 4. *For some K_0, the edges of the complete graph on $\mathbf{N}$ can be 2-coloured so that*

I. *there is no white Clique of size K_0,*
II. *there is no black Clique of positive upper density, and*
III. *the colouring is stationary: $\{i, j\}$ and $\{i+n, j+n\}$ are coloured identically.*

H. Furstenberg and B. Weiss [private communication] have given an elegant example which shows Fact 4 with $K_0=3$: Colour $\{i, j\}$ white if for some integer x, $i-j=x^3$, black otherwise. There is no white clique of size 3, because of Fermat's last theorem with exponent 3; there is no black clique of positive density because the set $\{x^3\}_{x\in\mathbf{N}}$ is a Poincare set (see [3]). □

2. Two Ramsey-like functions

DEFINITION. For $0<c<1$, define a function Λ_c: $\mathbf{N}\to\mathbf{N}$ as follows: $\Lambda_c(k)$ is the minimal N such that from any *stationary* sequence $\{\omega_n|0\leqq n<N\}$ in Ω_c, k elements can be extracted, no two of which are orthogonal. $\Gamma_c(k)$ is defined similarly, *without* the stationarity constraint.

Clearly $\Lambda_c\leqq\Gamma_c$.

FACT 5. $\Gamma_c(2)=\Lambda_c(2)=\lfloor c^{-2}\rfloor+1$.

PROOF. Put $N=N_c=\lfloor c^{-2}\rfloor+1$ and $d=\sqrt{1-(N-1)c^2}$. Let A be an orthogonal N by N matrix whose first column is the vector $(c, c, \ldots, c, d)$. Let v be the N-dimensional vector $(1, 0, \ldots, 0)$, and let $\omega_0, \ldots, \omega_{N-2}$ be the first $N-1$ row vectors of A. Clearly $\omega_n\in\Omega_c$ and $\langle\omega_n, \omega_m\rangle=0$. Thus $\Gamma_c(2)\geqq\Lambda_c(2)>N-1$. It remains to show that $\Gamma_c(2)\leqq N$. Indeed, if this is false, there are N orthogonal vectors $\{\omega_n|0\leqq n<N\}$ in Ω_c. Bessels inequality $\|v\|^2\geqq\sum|\langle v, \omega_i\rangle|^2=c^2\cdot N>1$ gives the desired contradiction. □

FACT 6. *$\Gamma_c(k)\leqq R(N_c, k)$ where $R(N, k)\leqq\binom{N+k-2}{N-1}$ is the Ramsey number corresponding to N and k* (see [4]).

This is immediate from Fact 5. □

The upper bound above is not tight. For Λ_c we do not have a better upper bound. Regarding lower bounds we note

PROPOSITION 1. *$\Lambda_c(k)$ does not increase linearly with k, for some $0<c<1$.*

PROPOSITION 2. *There exist $0<c<1$, $\alpha>1$ and an increasing sequence $\{k_l | l \geqq 1\}$ satisfying $\Gamma_c(k_l) \geqq k_l^\alpha$ for all l.*

Proposition 1 follows from Fact 2; Proposition 2 is a consequence of the following result, due to Frankl and Wilson [2]:

THEOREM. [2]. *Let $\mathcal{F}$ be a family of subsets of $\{1, \dots, n\}$ such that for every $F \in \mathcal{F}$, $|F|=k$, and let $q<k$ be a prime power. If every different $F, F' \in \mathcal{F}$ satisfy $|F \cap F'| \not\equiv k \bmod q$ then $|\mathcal{F}| \leqq \binom{n}{q-1}$.*

PROOF. Denote $n=2^l$, $N=\binom{n}{\frac{3n}{8}}$ and let $\{F_j\}_{j=1}^N$ be all subsets of $\{1, \dots, n\}$ of size $\frac{3n}{8}$. Define vectors $\{\omega_i\}_{i=1}^N$ in $\mathbf{R}^n$ by

$$\omega_i = n^{-1/2}(2 \cdot 1_{F_i} - 1)$$

where 1_F is the indicator vector of F.

Define also $v = -n^{-1/2}(1, 1, \dots, 1) \in \mathbf{R}^n$. For $1 \leqq i \leqq N$ we get

$$\|v\| = \|\omega_j\| = 1, \quad \langle v, \omega_i \rangle = \frac{1}{4} = c,$$

$$\omega_i \perp \omega_j \Leftrightarrow |F_i \cap F_j| = n/8 \equiv \frac{3n}{8} \left(\bmod \frac{n}{4}\right).$$

$q = \frac{n}{4}$ is a power of 2. Thus the theorem cited above shows that any subset $\mathcal{F}$ of $\{\omega_1, \dots, \omega_N\}$ which does not contain orthogonal vectors, satisfies

$$|\mathcal{F}| \leqq \binom{n}{\frac{n}{4}-1}.$$

In other words, for $k_l = \binom{n}{n/4}$, $\Gamma_c(k_l) > \binom{3}{\frac{3n}{8}}$ and

$$\lim_{l \to \infty} \frac{\log \Gamma_c(k_l)}{\log k_l} \geqq \frac{h\left(\frac{3}{8}\right)}{h\left(\frac{1}{4}\right)} > 1$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ is the binary entropy function. Any α smaller than the entropy ratio above will do. □

References

[1] J. Bourgain, Ruzsa's problem on sets of recurrence, *Israel J. Math.*, **59** (1987), 150—166.
[2] P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, *Combinatorica*, **1** (1981), 357—368.
[3] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press (Princeton, New Yersey, 1981).
[4] R. Graham, B. Rothschild, J. Spencer, *Ramsey theory*, Wiley (New York, 1981).
[5] T. Kamae, M. Mendes France, Van der Corput's difference theorem, *Israel J. of Math.*; **31** (1978), 335—342.
[6] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley (New York, 1968).
[7] Y. Peres, Application of Banach limits to the study of sets of integers, *Israel J. Math.*, **62** (1988), 17—31.

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SCHOOL OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
TEL AVIV
ISRAEL

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A THEOREM ON ANTI-INVARIANT MINIMAL SUBMANIFOLDS OF AN ODD DIMENSIONAL SPHERE

M. KON (Hirosaki)

Introduction. Let M be an $(n+1)$-dimensional anti-invariant submanifold, tangent to the structure vector field ξ, immersed in a $(2n+1)$-dimensional unit sphere S^{2n+1} with Sasakian structure (φ, ξ, η, g). We put $T_x(M)^*=\varphi^2T_x(M)=T_x(M)-\{\xi\}$, where $T_x(M)$ denotes the tangent space of M at a point x. We consider the restriction of the sectional curvature K of M to $T_x(M)^*$ for each x, which will be denoted by K^h and called the horizontal sectional curvature of M. If the second fundamental form of M vanishes on $T_x(M)^*$ for each x, then M is said to be totally contact geodesic (cf. [2]). In this paper, we will prove the following result, which corresponds to a theorem of Urbano [1].

THEOREM. *Let M be an $(n+1)$-dimensional compact anti-invariant minimal submanifold immersed in S^{2n+1} tangent to the structure vector field ξ of S^{2n+1}. If $K^h>0$, then M is totally contact geodesic.*

1. Preliminaries. Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere with Sasakian structure (φ, ξ, η, g). Then the structure tensors satisfy

$$\varphi^2X=-X+\eta(X)\xi,\quad \varphi\xi=0,\quad \eta(\varphi\xi)=0,\quad \eta(\xi)=1,$$
$$g(\varphi X, \varphi Y)=g(X,Y)-\eta(X)\eta(Y),\quad \eta(X)=g(X,\xi),$$
$$\bar\nabla_X\xi=\varphi X,\quad (\bar\nabla_X\varphi)Y=-g(X,Y)\xi+\eta(Y)X$$

for any vector fields X and Y on S^{2n+1}, where $\bar\nabla$ denotes the operator of covariant differentiation with respect to the Levi—Civita connection of S^{2n+1}. Let M be an $(n+1)$-dimensional submanifold immersed in S^{2n+1}. We denote by the same g the induced metric on M from that of S^{2n+1}, and by ∇ the operator of covariant differentiation with respect to the induced connection on M. Then the Gauss and Weingarten formulas are respectively given by

$$\bar\nabla_XY=\nabla_XY+B(X,Y)\quad\text{and}\quad \bar\nabla_XV=-A_VX+D_XV$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle of M. We call A and B the second fundamental form of M and they satisfy $g(B(X,Y),V)=g(A_VX,Y)$. We denote by ∇B the covariant derivative of B and we define the second covariant derivative of B by

$$(\nabla^2B)(X,Y,Z,W)=D_X((\nabla B)(Y,Z,W))-(\nabla B)(\nabla_XY,Z,W)-$$
$$-(\nabla B)(Y,\nabla_XZ,W)-(\nabla B)(Y,Z,\nabla_XW)$$

for any vector fields X, Y, Z and W tangent to M.

Notice that B and ∇B are symmetric. Let R and $R^\perp$ denote the curvature tensors associated with ∇ and D, respectively. Then we have

$$(1.1)\qquad (\nabla^2 B)(X,Y,Z,W) = (\nabla^2 B)(Y,X,Z,W) + R^\perp(X,Y)B(Z,W) - \\ - B(R(X,Y)Z,W) - B(Z,R(X,Y)W).$$

Suppose that M is tangent to the structure vector field ξ of S^{2n+1} and anti-invariant with respect to φ. Then $\varphi T_x(M) = T_x(M)^\perp$ for each $x \in M$, where $T_x(M)$ and $T_x(M)^\perp$ denote the tangent space and normal space of M at x, respectively (see [2], [3]). Then the second fundamental form of M satisfy

$$(1.2)\qquad \nabla_X \xi = 0,$$

$$(1.3)\qquad B(X,\xi) = \varphi X, \quad B(\xi,\xi) = 0$$

for any vector field X tangent to M. Moreover we have

$$(1.4)\qquad D_X \varphi Y = \varphi \nabla_X Y, \quad A_{\varphi X} Y = -\varphi B(X,Y) + g(X,Y)\xi - \eta(X)Y$$

for any vector field X and Y tangent to M. Further

$$(1.5)\qquad A_{\varphi X} Y = A_{\varphi Y} X$$

for any vector fields X and Y tangent to M and orthogonal to ξ. From these equations we obtain

$$(1.6)\qquad g(R^\perp(X,Y)\varphi Z, \varphi W) = g(R(X,Y)Z,W)$$

for any vector fields X, Y, Z and W tangent to M and orthogonal to ξ.

2. Proof of Theorem. Let $UT(M)$ be the unit tangent bundle of M. Define a function $f\colon UT(M) \to \mathbf{R}$ by $f(v) = g(B(\varphi^2 v, \varphi^2 v), \varphi v) = g(B(v,v),\varphi v) - 2\eta(v) g(\varphi v, \varphi v)$. Since $UT(M)$ is compact, f attains the maximum at a unit vector v tangent to M at a point x. For any unit vector u tangent to M at x, let $\alpha(t) = (\gamma(t), V(t))$, $t \in (-\delta, \delta)$ be a curve in $UT(M)$ such that $\gamma(t)$ is the only geodesic in M with $\gamma(0) = x$ and $\gamma'(0) = u$, and $V(t)$ the parallel vector field along γ with $V(0) = v$. Then, using (1.2) and (1.4), we obtain

$$0 = df_V(u) = (d/dt) g(B(V(t),V(t)), \varphi V(t))(0) = g((\nabla B)(u,v,v),\varphi v).$$

From (1.2) and (1.4) we also have

$$(2.1)\qquad 0 \geqq d^2 f_v(u,u) = g((\nabla^2 B)(u,u,v,v),\varphi v).$$

Now suppose that v is orthogonal to ξ. Then (1.1), (1.5) and (1.6) imply

$$g((\nabla^2 B)(u,u,v,v),\varphi v) = g((\nabla^2 B)(u,v,u,v),\varphi v) = \\ = g((\nabla^2 B)(v,u,u,v),\varphi v) + g(R^\perp(u,v)B(u,v),\varphi v) - \\ - g(B(R(u,v)u,v),\varphi v) - g(B(u,R(u,v)v),\varphi v) = \\ = g((\nabla^2 B)(v,v,u,u),\varphi v) + 2g(R(u,v)v,\varphi B(u,v)) + g(R(u,v)u,\varphi B(v,v)).$$

Substituting this equation into (2.1), we obtain

$$(2.2)\qquad 0 \geqq d^2 f_v(u,u) = g((\nabla^2 B(v,v,u,u),\varphi v) + 2g(R(u,v)v,\varphi B(u,v)) + \\ + g(R(u,v)u,\varphi B(v,v)).$$

Let $UT_x(M)$ be the fiber of $UT(M)$ over x. Then $f|UT_x(M)$ attains the maximum at v orthogonal to ξ by the definition of f, and so, if $\beta(t)$, $t\in(-\delta, \delta)$ is a curve in $UT_x(M)$ such that $\beta(0)=v$, $|\beta'(t)|=1$ and $\beta'(0)=u$, where u is orthogonal to ξ, we have, by (1.5),

$$0 = d(f|UT_X(M))_v(u) = (d/dt)g(B(\beta(t), \beta(t)), \varphi\beta(t))(0) = 3g(B(v, v), \varphi u) \tag{2.3}$$

and

$$0 \geqq d^2(f|UT_X(M))_v(u, u) = 6g(B(\beta'(0), v), \varphi\beta'(0))+3g(B(v, v), \varphi\beta''(0)) = \tag{2.4}$$
$$= 6g(B(u, v), \varphi u)-3g(B(v, v), \varphi v) = 6g(B(u, u), \varphi v)-3f(v).$$

Since (2.3) is true for any unit vector u orthogonal to v and ξ, we have $B(v, v)= =f(v)\varphi v$, and hence, $g(A_{\varphi v}v, u)=0$ for any unit vector u orthogonal to v and ξ and $g(A_{\varphi v}v, v)=f(v)$. Here, choose an orthogonal basis $\{\xi, e_1,\dots, e_n\}$ of $T_x(M)$ for which $g(A_{\varphi v}e_i, e_i)=h_i$ $(i=1, \dots, n)$, $h_n=f(v)$, $g(A_{\varphi v}e_i, e_j)=0$ $(i\neq j)$. Then (2.4) gives

$$f(v)-2h_i \geqq 0 \tag{2.5}$$

for each $i(=1, \dots, n-1)$. Therefore, from (1.3), (1.4), (1.5), (2.2) and given that M is minimal, we obtain

$$0 \geqq \sum_{i=1}^{n} d^2f_v(e_i, e_i) = \sum_{i=1}^{n-1} K(v, e_i)[f(v)-2h_i].$$

Since $K^h>0$, it follows that $h_i=f(v)/2$ for $i=1, \dots, n-1$. Thus we have $0=\text{trace } A_{\varphi v}= =(n+1)f(v)/2$, from which $f(v)=0$. From the definition of f we see that $f(-u)= =-f(u)$. Since v is a maximum for f, we have $f=0$. Moreover, by a method quite similar to that used to obtain (2.3), we have $B(v, v)=0$ for any vector v orthogonal to ξ. Hence M is totally contact geodesic.

REMARK. Let RP^n be a real n-dimensional projective space of constant curvature 1 imbedded in a complex n-dimensional projective space CP^n with constant holomorphic sectional curvature 4 as an anti-invariant and totally geodesic submanifold. Then the circle bundle (RP^n, S^1) over RP^n is a totally contact geodesic anti-invariant submanifold of S^{2n+1} (see [4, p. 148]).

References

[1] F. Urbano, Totally real submanifolds of a complex projective space, *Proc. Amer. Math. Soc.*, **93** (1985), 332—334.
[2] K. Yano and M. Kon, *Anti-invariant submanifolds*, Marcel Dekker Inc. (New York, 1976).
[3] K. Yano and M. Kon, Anti-invariant submanifolds of Sasakian space forms I, *Tôhoku Math. J.*, **29** (1977), 9—23.
[4] K. Yano and M. Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Birkhäuser Boston Inc., 1983.

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
HIROSAKI UNIVERSITY
HIROSAKI
JAPAN

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LOTS AND GO SPACES

C. R. BORGES (Davis)

If $\leqq$ is a linear order on a set X, we let $\tau(\leqq)$ denote the topology on X which is generated by all the open intervals $]a, b[=\{x\in X|a<x<b\}$. A space (X, τ) is said to be *weakly orderable* (called KOTS in [7]) if there exists a linear order $\leqq$ on X such that $\tau(\leqq)\subset\tau$. (Clearly, a compact KOTS is a LOTS.)

For our purposes, it suffices to recall that GO-spaces are characterized as topological subspaces of LOTS; therefore, it is easily seen that GO-spaces are KOTS. The behavior of GO-spaces is amazingly different from that of LOTS, except when they are connected or compact (in either case they are LOTS). Our results prove that connected and locally connected KOTS are LOTS, but connected KOTS may fail to be LOTS.

The following result has several interesting applications which include simpler proofs of known results.

THEOREM 1. *If a* LOTS (E, μ) *is connected then it is maximally connected and locally connected.*

PROOF. First, let us recall that a connected LOTS is locally connected. (See, Proposition 1 in the Appendix or Corollary 2.3 and Theorem 4.2 of [3].) Let $j\colon X\twoheadrightarrow E$ be a continuous bijection from a connected and locally connected space X, and let us prove that j is an open function: Assume U is an open connected subset of X such that $j(U)$ is not open. Then, since $j(U)$ is connected, we get that $j(U)$ is a (bounded or unbounded) interval with an *endpoint* which is not an end of E. Then, letting $A=j(U)$ and $\mu(A)$ be the simple extension of μ by A (see [1]), we get that $j\colon X\to(E, \mu(A))$ is still a continuous bijection, which leads to a contradiction, since $(E, \mu(A))$ is not connected (if $A=[a, b[$ then $[a,\to[$ and $]\leftarrow, a[$ disconnect $(E, \mu(A))$; the remaining cases are similarly resolved). Since X is locally connected it then follows immediately that j is an open continuous bijection; hence j is a homeomorphism.

Consequently, by Theorem 4.1 of [2], μ is a maximally connected and locally connected topology.

Example 5 shows that the euclidean topology of the real line is not a maximally connected P-topology, where P stands for a variety of topological properties.

The proof of Theorem 1 automatically establishes an equivalent formulation of this result which appears more convenient.

THEOREM 1'. *Let $j\colon X\to E$ be a continuous bijection from a space X to a* LOTS *E. If X is connected and locally connected then j is a homeomorphism and E is locally connected.*

It is known that connected GO-spaces are LOTS (e.g. Lemma 6.1 of [5]; it also follows automatically from Theorem 1.13 and 1.14 of [4]); hence, connected GO-spaces are automatically locally connected. It is therefore noteworthy that the space M of Example 5 is a connected KOTS which is neither locally connected nor a LOTS (nor a GO-space).

THEOREM 2. *A connected and locally connected* KOTS (X, τ) *is a* LOTS.

PROOF. Let $\leqq$ be a linear order on X such that $\tau(\leqq)\subset\tau$. Then the identity map $j\colon (X,\tau)\to(X,\tau(\leqq))$ is continuous. Therefore, by Theorem 1′, j is a homeomorphism.

THEOREM 3 (Theorem 4.2 of [3]). *Let (X,τ) be a connected T_1-space in which there exist two nests of open sets $\mathscr{L}$ and $\mathscr{R}$ such that $\mathscr{L}\cup\mathscr{R}$ generates a T_1-topology on X. Then X is locally connected if and only if $\mathscr{L}\cup\mathscr{R}$ is a subbase for the topology of X.*

PROOF. By Lemma 3.1 of [3], (X,τ) is a KOTS. By Theorem 1′, (X,τ) is a connected and locally connected LOTS if and only if (X,τ) is locally connected. This completes the proof.

According to Michael [6], if $\mathscr{G}\subset 2^X$ and $\lambda\colon \mathscr{G}\to X$ is a continuous function such that $\lambda(A)\in A$, for each $A\in\mathscr{G}$, then λ is called a selection for X. Michael [6] essentially proved the following result (see Lemmas 7.2.3, 7.4.1 and 7.5.2 of [6]).

PROPOSITION 4 (E. Michael). *For any Hausdorff space* (X,τ), (d)$\Rightarrow$(c)$\Rightarrow$(b)$\Rightarrow$(a). *If X is locally connected or connected then* (a), (b) *and* (c) *are equivalent. If X is also compact then* (a), (b), (c) *and* (d) *are equivalent.*

(a) *There exists a selection* $f\colon \mathscr{F}_2(X)\to X$,

(b) *There exists a selection* $\lambda\colon C(X)\to X$,

(c) *There exists a linear order on X such that the order topology is coarser than τ (i.e. X is a* KOTS*)*,

(d) *X is a* LOTS.

REMARK. From Theorem 2 and Proposition 4 one immediately gets that conditions (a)—(d) of Proposition 4 are also equivalent in any connected and locally connected space.

Surprisingly, there exist σ-compact metrizable spaces for which conditions (a)—(d) are not equivalent.

EXAMPLE 5. Let M be the subspace of the euclidean plane defined by

$$M=\left\{\left(x,\sin\frac{1}{x}\right)\mid 0<x<1\right\}\cup\{(0,0)\}\cup\left\{\left(x,\sin\frac{1}{x}\right)\mid -1<x<0\right\}.$$

Clearly, M is connected but not locally connected (at $(0,0)$) and M is a σ-compact metrizable space. The function $\pi\colon M\to]-1,1[$, defined by $\pi\left(\left(x,\sin\frac{1}{x}\right)\right)=x$, is clearly a continuous bijection which is not a homeomorphism (since $]-1,1[$ is locally connected).

Since $]-1, 1[$ is homeomorphic to E^1, Example 5 shows that

(i) The euclidean topology of the real line is not a maximally connected regular (normal, ..., Lindelöf, ..., metrizable) topology, since the topology of $]-1, 1[$ is strictly contained in the quotient topology τ_π and $(]-1, 1[, \tau_\pi)$ is homeomorphic to M,

(ii) M satisfies condition (c) of Proposition 4 but M is not a LOTS: For $x, y \in M$, let $x \leqq y$ provided that $\pi(x) \leqq \pi(y)$. Then the $\leqq$-topology on M is $\{\pi^{-1}(U) | U$ is open in $]-1, 1[\}$ which is coarser than the euclidean topology on M. Therefore, M satisfies condition (c) of Proposition 4. However, M is not a LOTS, since it is connected but not locally connected.

Appendix

The following result is folklore but there appears to be no elementary proof of it. Here is one.

PROPOSITION 1. *If* (X, τ) *is a connected* LOTS *then* X *is locally connected.*

PROOF. Let $\leqq$ be the linear order on X which generates τ. Suppose X is not locally connected. Then there exists $]a, b[\subset X$ which is not connected; say, $]a, b[= U \cup V$ such that $U, V \in \tau$ and $U \cap V = \emptyset$. Pick $c, d \in]a, b[$ such that $c \in U$ and $d \in V$; say $c < d$. Then $U' = (U \cap [c, d]) \cup]\leftarrow, c[$ and $V' = (V \cap [c, d]) \cup]d, \rightarrow[$ are open disjoint subsets of X such that $X = U' \cup V'$, a contradiction.

References

[1] C. R. Borges, On extensions of topologies, *Can. Math. J.*, **19** (1967), 474—487.

[2] D. E. Cameron, A survey of maximal topological spaces, *Top. Proc.*, **2** (1977), 11—59.

[3] J. van Dalen and E. Wattel, A topological characterization of ordered spaces, *Gen. Top. and Appl.*, **3** (1973), 347—354.

[4] E. Deák, Theory and applications of directional structures, *Coll. Math. Soc. János Bolyai 8. Topics in Topology*, Keszthely (Hungary), 1972, 187—211.

[5] D. J. Lutzer, On generalized ordered spaces, *Dissert. Math.*, **89** (1971), Warszawa.

[6] E. A. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.*, **71** (1951), 152—182.

[7] J. van Mill and E. Wattel, Selections and orderability, *Proc. Amer. Math. Soc.*, **83** (1981), 601—605.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CA 95616
USA

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ON ESSENTIAL RIGHT CONGRUENCES OF A SEMIGROUP

R. H. OEHMKE (Iowa City)

1. Introduction

In ring theory one can give several approaches to the introduction of the concept of semisimplicity and a large number of equivalent formulations of this concept [5, 13]. Analogues of some of these formulations have been made, and studied, for semigroups [4, 6, 7, 8, 10, 11, 12] in terms of ideals or congruence relations. It seems possible that suitable and effective analogues can be made for each of these ring theoretical formulations in terms of congruence relations. However, unlike the situation for rings most of these analogues give inequivalent formulations in semigroups.

One of the weakest of these is the nonexistence of a proper, essential right congruence. A right congruence ϱ in a semigroup S is essential if for any right congruence σ we have $\varrho\cap\sigma=\angle$ (the identity relation) implies $\sigma=\angle$. The main result of this paper is a characterization of semigroups with the d.c.c. on right ideals and having no proper essential right ideals and having no proper essential right congruences.

The first step in this characterization is a description of the lattice of right ideals in such a semigroup. Our results of this description in Section 2 should be compared with the work of Feller and Gantos [2] and Fountain [3]. While the class of semigroups studied in these two papers are defined quite differently than the class in this paper there is a striking similarity in the results; thus, suggesting a common area of investigation.

In the subsequent sections the main technique used is the examination of a selection of proper right congruences and the implications on the multiplicative properties obtained from the assumption of nonessentiality.

2. Necessary conditions

We shall assume our semigroup S has an identity element 1. A right congruence on S is an equivalence relation σ such that if σ relates a and b (written $a\,\sigma\,b$) then for any $c\in S$ we also have $(ac)\,\sigma\,(bc)$. Two distinguished right congruences are present in every semigroup: $\angle$, the identity relation, and ν, the universal relation. They are defined by

$$a\angle b\Leftrightarrow a=b,\quad a\,\nu\,b\Leftrightarrow a,b\in S.$$

Right congruences are ordered in the same way as equivalence relations are ordered. We write $\alpha\leqq\beta$. The intersection and union of two right congruences α and β are written respectively as $\alpha\cap\beta$ and $\alpha\cup\beta$. The intersection of α and β is their intersection as equivalence relations. Their union is the smallest right congruence that contains both α and β. It is clear that the union is well-defined.

A right congruence σ is said to be essential if for every right congruence $\alpha \neq \iota$ we have $\alpha \cap \sigma \neq \iota$. Clearly, ν is an essential right congruence. We say σ is proper if $\sigma \neq \nu$.

Let $\mathcal{D}$ be the class of semigroups with an identity element 1 and having no proper essential right congruences.

THEOREM 1. *Let S be a semigroup in $\mathcal{D}$. Every right ideal of S is generated by an idempotent: i.e., if J is a right ideal of S then there is an idempotent m such that $J = mS$.*

PROOF. Let $\mathcal{I}$ be the set of all right ideals of S not generated by an idempotent. Assume $\mathcal{I}$ is not empty. We can partially order $\mathcal{I}$ by inclusion. Let $\mathcal{S}$ be a linearly ordered subset of $\mathcal{I}$. Let $I = \bigcup \mathcal{S}$. Clearly, I is a right ideal of S. If $I = mS$ for some idempotent m then $m \in I$ and $m \in I_a$ for some I_a is $\mathcal{S}$. But then $mS \subseteq I_a \subseteq I = mS$. Therefore $mS = I_a$ and $I_a \notin \mathcal{I}$. This, of course, is a contradiction, so we must assume $I \in \mathcal{I}$.

With the assumption that $\mathcal{I}$ is not empty we can apply Zorn's Lemma to obtain a maximal element I of $\mathcal{I}$. Let ϱ be the right congruence defined by $a \, \varrho \, b$ if and only if

$$\{u\colon au \in I\} = \{v\colon bv \in I\}.$$

Assume σ is a right congruence distinct from ι and such that $\sigma \cap \varrho = \iota$.

Let $a \, \sigma \, b$ and $u \in S$ where $au \notin I$ and $bu \in I$. Such a pair must exist since $\sigma \cap \varrho = \iota$. We have $au \in auS$ and therefore $auS \cup I \supsetneqq I$. Thus, $auS \cup I \notin \mathcal{I}$ since I is maximal in $\mathcal{I}$. Therefore there must be an idempotent m such that $auS \cup I = mS$. Either $m \in auS$ or $m \in I$. If $m \in I$ then $mS \subseteq I \subseteq mS$ and $mS = I$, a contradiction. So we assume $m \in auS$ and let $aus = m$. Recall that $bu \in I$ implies $i = bus \in I$. Now since $a \, \sigma \, b$ we have $(aus) \, \sigma \, (bus)$ and $m \, \sigma \, i$. Therefore $(mi') \, \sigma \, (ii')$ for all $i' \in I$. But $I \subseteq mS$ and m is a left identity for mS. Thus $mi' = i'$, $(i') \, \varrho \, (ii')$ (since i' and ii' are in I) and also $i' \sigma \, ii'$. However, we must then have $i' = ii'$ since $\varrho \cap \sigma = \iota$. This says that i is a left identity for I, $i^2 = i$ and $iS = I$. Again, we have a contradiction. Hence we must have $\mathcal{I}$ empty and thus every right ideal is generated by an idempotent.

THEOREM 2. *Let S be a semigroup in $\mathcal{D}$. The set of right ideals of S is linearly ordered by inclusion.*

PROOF. Let J and I be two right ideals. The right ideal $J \cup I$ is generated by an idempotent m; i.e., $mS = J \cup I$. But then $m \in J$ or $m \in I$. It follows that either $J = mS$ or $I = mS$ and either $I \subseteq J$ or $J \subseteq I$.

The next theorem shows that every right ideal that is not minimal has an immediate predecessor.

THEOREM 3. *Let S be a semigroup in S. If J is a right ideal that is not minimal then there is a right ideal K of S such that for any right ideal I we have $J \supseteq I \supseteq K$ implies $I = J$ or $I = K$.*

PROOF. Let $J = mS$ and $U_m = \{x\colon xS = mS\}$. Let $K = J - U_m$. If $u \in S$ and $k \in K$ then $ku \in J$ since J is a right ideal. If $kuS = mS$ then there is an $s \in S$ such that $kus=m$. But then $m \in kS$, $kS=mS$ and $k \in U_m$. Since this is a contradiction we must have $ku \notin U_m$. Hence K is a right ideal. But $m \notin K$ so $J \supsetneqq K$. Now assume $J \supseteqq I \supseteqq K$ and $I \neq K$. Let $I=tS$ where t is an idempotent. Since $t \in J$ and $t \notin K$ we must have $t \in U_m$. Therefore $tS=J$ and $I=J$. This completes the proof of the theorem.

It is now clear that if S is in $\mathscr{D}$ and if the sets U_m are defined as above then S is the disjoint union of the U_m's.

3. The descending chain condition

We let $\mathscr{C}$ be the subclass of $\mathscr{D}$ of all semigroups that have the descending chain condition on right ideals.

By the results of Section 2, for semigroups S in $\mathscr{C}$ we have a finite chain

$$S = I_0 \supsetneqq I_1 \supsetneqq \dots \supsetneqq I_t$$

of right ideals of S, each right ideal of S is in this chain and each right ideal is generated by an idempotent.

We let U_i be the set of generator of I_i. So U_i contains an idempotent and $I_{i+1} = I_i - U_i$. Therefore $U_t = I_t$ and $S = U_0 \cup I_1 = U_0 \cup U_1 \cup I_2 = U_0 \cup \dots \cup U_t$.

THEOREM 4. *If S is a semigroup in $\mathscr{C}$ then every right ideal is two-sided.*

PROOF. Let i be the largest integer such that there is a $u \in U_i$ and a $v \in I_{i+1}$ such that uv is an idempotent e of U_i. Assume $v \in U_j$ where $i<j$ and assume f is an idempotent of U_j. Since $I_i = eS$ and $f \in I_i$ we have $ef=f$. We need the following lemma.

LEMMA. *If $w \in U_j$ then $uw \in U_i$.*

PROOF. We first prove this result for $w=f$. We have $(uf)v=u(fv)=uv=e$. Therefore $I_i = eS \subseteq ufS \subseteq I_i$ and $uf \in U_i$. If $w \in U_j$ there exists a y such that $wy=f$. But then $uwy=uf \in U_i$. Therefore we must have $uw \in U_i$ and the lemma holds.

To return to the proof of the theorem we assume v and f are as above. Since $v \in fS = I_j$ we must have $vf \in I_j$. Assume $vf \in U_j$. By the lemma we then have $u(vf) = U_i$. But $u(vf)=(uv)f=ef=f \notin U_i$. Hence we must have $vf \notin U_j$, i.e., $vf \in U_k$ for some $k>j$. Let g be an idempotent of U_k. We have $(fu)(vf)=f(uv)f=fef=ff=f$. But then $fu \in U_j$ and $vf \in U_k$. This contradicts our choice of i. Hence for every i and every $u \in U_i$ and every idempotent e in U_i we have $uv=e$ only if $v \in U_r$ for some $r \leqq i$.

We are now ready to show I_i is two-sided. Let $w \in I_i$, $u \notin I_i$ and $uw \notin I_i$. (This would contradict I_i being a two-sided ideal.) Then $uw \in U_k$ for some $k<i$. Therefore, if g is an idempotent of U_k there exists an x in S such that $(uw)x=g$ and $(gu)(wx)=g$. But $gu \in U_k$ and $wx \in I_i$. Hence we contradict the result of the above paragraph. Thus I_i is a two-sided ideal of S.

COROLLARY. *If $i \leqq j$, then $U_i U_j \subseteq U_j$. In particular, U_j is a subsemigroup.*

PROOF. Assume $s \in U_i$, $v \in U_j$. Select idempotent f and e, respectively, in U_i and U_j. Let u and t be elements of S such that $st=f$ and $vu=e$. Since eS is a two-sided ideal, te is in eS and thus $e(te)=te$. But then $(se)(te)=s(ete)=ste= =fe=e$. We also have $e(se)=se$ so $se\in U_j$. Now consider $(sv)u=s(vu)=se\in U_j$. Also, $sv=s(ev)=(se)v\in eS$. Therefore we must have $sv\in U_j$ and $U_iU_j\subseteq U_j$.

THEOREM 5. *If S is a semigroup in $\mathscr{C}$ then for each i we have U_i is a right group.*

PROOF. Let $u\in U_i$ and e an idempotent in U_i. There exists an s in S such that $us=e$. Assume $s\in U_j$. In the proof of the above theorem we saw that we must have $j\leqq i$. Now $u(se)=(us)e=e^2=e$. We have se in the two-sided ideal eS. Again since se serves as a right inverse of u relative to e we must have $se\in U_k\cap eS$ for some $k\leqq i$. Therefore $se\in U_i$. Hence we have that U_i is a right group.

Since U_i is a right group and contains idempotents we can write [1]

$$U_i = G_i\times K_i$$

where G_i is a group and K_i is a right zero semigroup. We will denote the identity element of G_i by 1_i. Every idempotent in U_i is of the form $(1_i, k_i)$ for some $k_i\in K_i$. Every such idempotent is a left identity for the ideal I_i.

We wish to relate the multiplications of the individual U_i's to the multiplication of S.

We continue to assume S is in $\mathscr{C}$.

THEOREM 6. *For each pair i and j such that $i\leqq j$ there is a homomorphism φ_{ij} of G_i into G_j such that $\varphi_{jk}\circ\varphi_{ij}=\varphi_{ik}$ and such that*

1) $(g_i, k_i)(g_j, k_j) = (\varphi_{ij}(g_i)g_j, k_j)$

2) $(g_j, k_j)(g_i, k_i) = (g_j\varphi_{ij}(g_i), \psi((g_i, k_i), k_j))$

for all $(g_i, k_i)\in G_i\times K_i$ and $(g_j, k_j)\in G_j\times K_j$.

PROOF. Let e be any idempotent of U_j and s and t elements of U_i. Consider the mapping $\alpha(s)=se$ of U_i into U_j. (See the corollary to Theorem 4.) We have $\alpha(st)=(st)e=s(te)=s[e(te)]=(se)(te)=\alpha(s)\alpha(t)$. Hence α is a homomorphism.

Now write $e=(1_j, k_j)$ for some $k_j\in K_j$. Let $\alpha(s)=(g_j, k_j')$ for some $g_j\in G_j$ and $k_j'\in K_j$. From $\alpha(s)e=\alpha(s)$ we see that $k_j=k_j'$, i.e., $\alpha(s)=(g_j, k_j)$. We write $s=(g_i, k_i)$ and $\alpha(s)=(\beta(g_i, k_i), k_j)$. Let k_i' be a second element of K_i. Then

$$\alpha((g_i, k_i)(1_i, k_i')) = \alpha((g_i, k_i')) = (\beta(g_i, k_i'), k_j)$$

and

$$\alpha((g_i, k_i)(1_i, k_i')) = (\beta(g_i, k_i), k_j)(\beta(1_i, k_i'), k_j) = (\beta(g_i, k_i)\beta(1_i, k_i'), k_j).$$

Since α is a homomorphism we must have $(\beta(1_i, k_i'), k_j)$ an idempotent and hence $\beta(1_i, k_i')=1_j$. Therefore $\beta(g_i, k_i')=\beta(g_i, k_i)$ and β is independent of k_i. We shall write β as φ_{ij}. The fact that φ_{ij} is a homomorphism is immediate from the result that α is a homomorphism.

Next let

$$(g_j, k_j)(g_i, k_i) = (h_j, l_j)$$

for some $h_j \in G_j$ and $l_j \in K_j$. Then

$$(h_j, l_j) = [(g_j, k_j)(g_i, k_i)](1_j, l_j) =$$

$$= (g_j, k_j)[(g_i, k_i)(1_j, l_j)] = (g_j, k_j)(\varphi_{ij}(g_i), l_j) = (g_j \varphi_{ij}(g_i), l_j).$$

Also

$$(h_j, l_j) = [(g_j, k_j)(1_j, k_j)](g_i, k_i), = (g_j, k_j)[(1_j, k_j)(g_i, k_i)].$$

Therefore l_j must be the K_j-component of $(1_j, k_j)(g_i, k_i)$ and hence is independent of g_j.

4. The class $\mathscr{B}$

We shall define the class $\mathscr{B}$ to be all semigroups S such that

1) S is the union of disjoint right groups $U_i = G_i \times K_i$ for $i=0, \ldots, t$;

2) $U_0 = G_0$ has an identity element 1 for S;

3) $\bigcup_{i=r}^{t} U_i$ is a two-sided ideal of S for $r=0, 1, \ldots, t$;

4) the multiplication on S has the properties described in Theorem 6.

The class $\mathscr{C}$ is contained in the class $\mathscr{B}$.

We wish to examine the right congruences on semigroups S that are in $\mathscr{B}$. In [9], one type of right congruence was described for a slightly more general class of semigroups than the class $\mathscr{B}$. If we restrict ourselves to semigroups in $\mathscr{B}$ we can give an improved version of that result. The construction of this type of right congruence is as follows.

Let W be a subgroup of G_t and for each coset Wa of G_t let Γ_a be an equivalence relation on K_t such that

$$k(\Gamma_a)l \quad \text{implies} \quad \psi((g_i, k_i)\, k)(\Gamma_{a\varphi_{it}(g_i)})\psi((g_i, k_i), l). \tag{4.1}$$

Let $e=(1_t, k_t)$ be a specified element of U_t that is in $W \times L$ where L is an equivalence class of Γ_{1_t}. Define a relation ϱ on U_t by

$$(g_t, l_t)\, \varrho\, (g_t', l_t') \Leftrightarrow g_t' g_t^{-1} \in W \quad \text{and} \quad l_t(\Gamma_{g_t}) l_t'.$$

By Theorem 1 of [9], ϱ is a right congruence on U_t. Next we define a relation σ on S by

$$c\, \sigma\, d \Leftrightarrow (ec)\, \varrho\, (ed).$$

We can again cite Theorem 1 of [9] to give us the result:

A relation σ on S is a right congruence on S for which the right ideal U_t intersects every equivalence class non trivially if and only if σ is constructed as above.

Still using the results of [9] we see that if V is any equivalence class of σ then $V \cap U_t = W_a \times L'$, where L' is an equivalence class of Γ_a. It also follows immediately that $1\, \sigma\, e$ and σ is independent of which idempotent we choose in $W \times L$. We will say σ is defined by (W, Γ_a, e).

THEOREM 7. *Let σ be a right congruence on S such that $\sigma\neq\nu$. Let T be the equivalence class of σ containing* 1. *Assume $T\cap U_t\neq\emptyset$. Then σ is contained in a proper right congruence of one of the following types:*

1) *ϱ is defined by (G_t, Γ_a, e);*

2) *ϱ is defined by (W, Γ_a, e) where Γ_a is the universal relation on K_t for all $a\in G_t$.*

PROOF. Let σ be a right congruence such that $T\cap U_t\neq\emptyset$. Let V be any equivalence class of σ and $x\in V$. Let $y\in T\cap U_t$. We have $1\,\sigma\, y$ and $x\,\sigma\, yx$. But $yx\in U_t$. Therefore $V\cap U_t\neq\emptyset$. Thus σ must be a right congruence as constructed above.

If $T\cap U_t=G_t\times L$ then σ is a right congruence of type 1). So we let $T\cap U_t=W\times L$ and assume $W\neq G_t$. For each $a\in S$ we let π_a be the universal relation on K_t. Let e be any element $W\times L$. We define a right congruence ϱ using (W, π_a, e), i.e., a right congruence of type 2). It follows immediately that if (g_i, k_i) and (g_j, k_j) are two elements of S then

$$(g_i, k_i)\,\varrho\,(g_j, k_j) \Leftrightarrow \varphi_{it}(g_i)\varphi_{jt}(g_j)^{-1}\in W.$$

Therefore it is also immediate that $\sigma\leqq\varrho$.

We now consider the case that $T\cap U_t=\emptyset$.

THEOREM 8. *Let σ be a right congruence on S such that $T\cap U_t=\emptyset$. Then for some $r\leqq t$, σ is contained in a proper right congruence ϱ defined by the decomposition $S=U\cup V$ where $U=U_0\cup\ldots\cup U_{r-1}$ and $V=U_r\cup\ldots\cup U_t$.*

PROOF. By the corollary to Theorem 4, U is a subsemigroup of S. Since V is a two-sided ideal of S it is clear that the decomposition $V\cup U$ corresponds to a right congruence which is proper since $T\cap U_0\neq\emptyset$ and $T\cap U_t=\emptyset$.

Assume r is the smallest integer such that $T\cap U_i=\emptyset$ for all i such that $r\leqq i\leqq t$. Let W be an equivalence class of σ such that $W\cap U_k\neq\emptyset$ for some $k\geqq r$. Assume in addition that $W\cap U_j\neq\emptyset$ for some $j<r$. Let $w\in W\cap U_j$, $x\in W\cap U_k$ and $z\in T\cap U_{r-1}$. Since $z\in I_j=wS$ there is an $s\in S$ such that $z=ws$. But then $w\,\sigma\, x$ implies $z\,\sigma\, xs$. Now $xs\in T$ and $xs\in U_i$ for some $i\geqq k$. Hence we have a contradiction. Therefore $W\subseteq V$ and, more generally, the decomposition corresponding to σ is smaller than $V\cup U$. Therefore $\sigma\leqq\varrho$.

5. The main theorem

We continue to assume that our semigroup is in class $\mathscr{B}$. The first three of the following theorems relate properties of S to the properties of certain right congruences being essential.

THEOREM 9. *For $S\in\mathscr{B}$ the following are equivalent:*

1) *For each $1\leqq r\leqq t$ and every idempotent x in U_{r-1} there is an idempotent y in U_r such that $yx=y$ and if $a, b\in U_{r-1}$ then $ya=yb\Rightarrow a=b$.*

2) *The decomposition of S into the two subsets $U=U_0\cup\ldots\cup U_{r-1}$ and $V=U_r\cup\ldots\cup U_t$ corresponds to a nonessential right congruence ϱ on S.*

PROOF. We saw in Theorem 8 that the relation ϱ defined in 2) was a right congruence.

1)⇒2). Define a relation σ by $a \sigma b$ if and only if there exists a u such that $\{xu, yu\}=\{a, b\}$ or $a=b$. The elements x and y are as defined in 1). Clearly σ is reflexive and symmetric, so assume $\{xu, yu\}=\{a, b\}$ and $\{xv, yv\}=\{b, c\}$. If $u \in yS$ then $xu=yu=u$ and $a=b$ since u is also in xS. But then $a \sigma c$. Thus we assume $u, v \notin yS$. Therefore xu, xv are in U_{r-1} and we could equally well have chosen $u=xu$ and $v=xv$ and $u, v \in xS$. Now if $b \in yS$ then $b=yu=yv$ and therefore by 1) we have $u=v$. But then $a=u=v=c$ and again $a \sigma c$. If $b \notin yS$ then $u=v=b$, $yu=yv$ and $a \sigma c$.

We have shown σ to be an equivalence relation. That σ is a right congruence follows immediately. Every nontrivial equivalence class of σ contains exactly one element of U and one element of V. Therefore $\varrho \cap \sigma = \angle$ and ϱ is nonessential.

2)⇒1). If ϱ, as defined in 2), is a nonessential right congruence on S then there is a right congruence $\sigma \neq \angle$ such that $\varrho \cap \sigma = \angle$. A nontrivial equivalence class of σ contains exactly one element from U and one element from V. Assume $a \sigma b$ where $a \in U$ and $b \in V$. In fact, assume $a \in U_i$ where $i<r$. Let x be any idempotent in $U_{r-1} \subseteq aS$. There exists a $c \in S$ such that $ac=x$ and thus $x \sigma bc$. Since $x(bc)=bc$ we have $(bc) \sigma (bc)^2$. But both bc and $(bc)^2$ are in V. Therefore bc is an idempotent. Call it y. We have $x \sigma y$, $x \sigma yx$ and hence $yx=y$. If $y \in U_k$ where $k>r$ and z is an idempotent of U_r we have $xz \sigma yz$. But $xz \in U_r$, $yz \notin U_r$, $xz \neq yz$, and both are in V. This contradicts the property $\varrho \cap \sigma = \angle$. Therefore we must have $y \in U_r$.

Finally assume $a, b \in U_{r-1}$ such that $ya=yb$. Then $a=xa \sigma ya$, $b=xb \sigma yb$ and $a \sigma b$. Again, since $\varrho \cap \sigma = \angle$, we must have $a=b$ and 1) holds.

THEOREM 10. *For S in $\mathscr{B}$ the following are equivalent.*

1) *G_t is a group having no proper essential subgroups.*

2) *Any proper right congruence as defined in 2) of Theorem 7 is nonessential.*

PROOF. 1)⇒2). Assume ϱ is a right congruence as defined in 2) using the subgroup $W \neq G_t$. Since G_t has no proper essential subgroups there is a subgroup H of G_t such that $H \cap W = \{1_t\}$. We now define a relation σ by $a \sigma b$ if and only if $a=b$ or $a=(g_t, k_t)$, $b=(g_t', k_t')$ are in U_t and $g_t' g_t^{-1} \in H$ and $k_t=k_t'$. It is a straightforward computation to show σ is a right congruence on S. An element c, not in U_t, lies in a singleton equivalence class of σ. Two elements of U_t are equivalent if and only if their K_t components are the same and their G_t components lie in the same coset of H.

Now assume $a \neq b$ and $a(\varrho \cap \sigma)b$. We must have a and b in U_t in order to satisfy $a \sigma b$. Write $a=(g_t, k_t)$ and $b=(g_t', k_t')$. To satisfy both $a \varrho b$ and $a \sigma b$ we must have $g_t' g_t^{-1} \in H \cap W$ and hence $g_t'=g_t$. To satisfy $a \sigma b$ we must have $k_t=k_t'$ and hence $a=b$. Thus ϱ is not essential.

2)⇒1). Assume W is a proper essential subgroup of G_t. Use any idempotent $e=(1_t, h_t)$ and W to define a right congruence ϱ as in 2). There must exist a right congruence $\sigma \neq \angle$ such that $\sigma \cap \varrho = \angle$. Let a and b be a pair of elements, not related by ϱ, but related by σ. Write $a=(g_i, k_i) \in U_i$, $b=(g_j, k_j) \in U_j$ where $i \leqq j$. Then $(\varphi_{ij}(g_i), k_j) \sigma (g_j, k_j)$. Let $g=\varphi_{ij}(g_i) g_j^{-1}$. We must have $(1_j, k_j) \sigma (g^n, k_j)$ for all positive integers n and also $(1_t, k_t) \sigma (\varphi_{jt}(g)^n, k_t)$ for all n and all $k_t \in K_t$. But W is essential so some power of $\varphi_{jt}(g)$ lies in W, say $\varphi_{jt}(g)^m \in W$. Then $(1_t, k_t) \varrho (\varphi_{jt}(g)^m, k_t)$. Since $\varrho \cap \sigma = \angle$ we must have $\varphi_{jt}(g)=1_t$. We now have

$\varphi_{it}(g_i)\varphi_{jt}(g_j)^{-1}=\varphi_{jt}(g)=1_t\in W$. Therefore $a\,\varrho\,b$ and we have a contradiction. Therefore W cannot be an essential subgroup of G_t.

We temporapily assume S has the d.c.c. and that it has no proper essential right congruences. If $t>0$ then there are right congruences as described in 2) of Theorem 9. Hence 1) of Theorem 9 holds. If $t=0$ then 1) of Theorem 9 holds vacuously. We will only consider the case $t>0$. By 1) of Theorem 9 we can construct a sequence $1, e_1, \ldots, e_t$ of idempotents such that $e_ie_j=e_k$ where $k=\max\{i,j\}$ and if $a, b\in U_{j-1}$ then $e_ja=e_jb\Rightarrow a=b$.

We say a right congruence σ is generated by a pair $\{a, b\}$ if σ is the smallest right congruence such that $a\,\sigma\,b$.

A partition Γ of K_t is said to be S-admissible if whenever $l\Gamma l'$ and $s\in S$ we have $\psi(s,l)\Gamma\psi(s,l')$.

Theorem 11. *Let Γ be an S-admissible partition of K_t. Let e_t be as defined above. Let ϱ be the right congruence determined by Γ and e_t as in* 1) *of Theorem* 7. *Then there exists a right congruence $\sigma\neq\angle$ such that $\sigma\cap\varrho=\angle$ and σ is of one of the following two types.*

1) *σ is generated by a pair of elements (g_t, l_t) and (g_t, l_t') in U_t.*

2) *σ is generated by an e_{t-1} and an idempotent f_t in U_t.*

Proof. Let Γ, e_t and ϱ be as defined in the statement of the theorem. Since we are assuming there are no proper essential right congruences there must exist a right congruence $\sigma\neq\angle$ such that $\varrho\cap\sigma=\angle$. Assume there is a U_r with distinct elements a and b in U_r such that $a\,\sigma\,b$. Of all such right congruences we assume we have chosen one which maximizes r. Now assume $r<t$. Define a relation $\bar\sigma$ by $c\,\bar\sigma\,d$ if and only if $c=d$ or there exists a pair c', d' in U_r such that $c'\,\sigma\,d'$, $c=e_{r+1}c'$, and $d=e_{r+1}d'$. We will show $\bar\sigma$ is transitive. Let $\{e_{r+1}c', e_{r+1}d'\}\cap\{e_{r+1}a', e_{r+1}b'\}\neq\emptyset$ where $a'\,\sigma\,b'$, $c'\,\sigma\,d'$ and $a', b', c', d'\in U_r$. If $e_{r+1}c'=e_{r+1}a'$ then by the way we choose e_{r+1} we have $c'=a'$ and $d'\,\sigma\,b'$. Therefore $(e_{r+1}d')\,\bar\sigma\,(e_{r+1}b')$ and $\bar\sigma$ is transitive. Clearly $\bar\sigma$ is an equivalence relation. To show it is a right congruence let $s\notin e_{r+1}S$ then if $(e_{r+1}c')\,\bar\sigma\,(e_{r+1}d')$ where c' and d' are in U_r and $c'\,\sigma\,d'$ we also have $c's, d's$ in U_r and $(c's)\,\sigma\,(d's)$. Hence $(e_{r+1}c's)\,\sigma\,(e_{r+1}d's)$. If $s\in e_{r+1}S$, $c's\,\sigma\,d's$ and $c's, d's$ are in U_k for some $k>r$. By our maximization of r we must have $c's=d's$ and $e_{r+1}c's=e_{r+1}d's$. We now have that $\bar\sigma$ is a right congruence. Since $e_te_{r+1}c'=e_tc'$ and $e_te_{r+1}d'=e_td'$ we have $\sigma\cap\varrho=\angle$ implies $\bar\sigma\cap\varrho=\angle$. But this contradicts our choice of σ. Hence we must have $r=t$. So assume $(g_t, l_t)\,\sigma\,(g_t', l_t')$ where $(g_t, l_t)\neq(g_t', l_t')$ and both are in U_t. Multiplying on the right by $e_t=(1_t, k_t)$ gives us $(g_t, k_t)\,\sigma\,(g_t', k_t')$. But these two elements are congruent modulo ϱ. Hence we must have $g_t=g_t'$. The right congruence generated by (g_t, l_t) and (g_t, l_t') will have the desired properties of 1).

We now assume that for every right congruence $\sigma\neq\angle$ such that $\sigma\cap\varrho=\angle$ and every pair a, b in any U_r with $a\,\sigma\,b$ we must have $a=b$. So assume we have a pair a, b such that $a\neq b$, $a\in U_i$, $b\in U_j$, $i<j$ and $a\,\sigma\,b$. We can find a $c\in S$ such that $ac=e_{j-1}$. Hence $e_{j-1}\,\sigma\,bc$. But then $(bc)\,\sigma\,(bc)^2$ and both sides are elements in U_j. Therefore bc is an idempotent f_j in U_j. We have $e_{j-1}\,\sigma\,f_j$. If $e_{j-1}a=e_{j-1}b$ it follows that f_ja and f_jb are in the same U_k and $f_ja\,\sigma\,f_jb$. Hence we must have $f_ja=f_jb$ by our assumption on σ. The converse works equally well so we obtain

$$e_{j-1}a=e_{j-1}b\Leftrightarrow f_ja=f_jb.$$

We use this property to show that the equivalence classes of the right congruence $\bar{\sigma}$ generated by e_{j-1} and f_j are doubletons and singletons. We now define a relation σ' by $(e_j c)\,\sigma'\,(e_{j+1}d)$ where $c\,\bar{\sigma}\,d$, $c\in U_{j-1}$ and $d\in U_j$. In the same manner as above it follows that σ' is a nontrivial right congruence and $\sigma'\cap\varrho=\angle$. We have satisfied 2) using σ'.

THEOREM 12. *For every idempotent f of U_t and every pair of elements s, u in U_{t-1} we have*

$$fs=fu\Rightarrow s=u.$$

PROOF. Let U=all elements of the form (g,f) where f has the above property. Since we are still working on the assumption that S has no proper essential right congruence we have $U\neq\emptyset$. We let $V=U'$, the complement of U in U_t. If $(g,l')\in V$, $s\notin U_t$ and $(g,l')s\in U$, then $(g,l')su=(g,l')sv$ and $u,v\in U_{t-1}$ implies $u=v$ and $su=sv$. Now if $(g,l')x=(g,l')w$ where x, w are in U_{t-1} then there exists an x' and w' in U_{t-1} such that $x=sx'$, $w=sw'$. But then $(g,l')sx'=(g,l')sw'$, $x'=w'$ and $x=w$. Therefore we must have $(g,l')\in U$. Similarly, we see U is closed under right multiplication by elements of U_{t-1}. The partition obviously accepts right multiplication by elements of U_t. Therefore the partition $\{U,V\}$ is S-admissible. Let ϱ be the corresponding right congruence as defined in 1) of Theorem 7 and using $e_t\in U$. By Theorem 11 there exists a σ, one of two special types, such that $\sigma\neq\angle$ and $\sigma\cap\varrho=\angle$. We first examine the second type, i.e., σ is generated by e_{t-1} and f_t for some idempotent f_t in U_t. If a and b are elements of U_{t-1} such that $f_ta=f_tb$ then $e_{t-1}a=e_{t-1}b$ and $a=b$ since e_{t-1} is a left identity for U_{t-1}. Therefore f_t is in U and ϱ could equally well be defined using f_t instead of e_t. Since $f_te_{t-1}=f_t=f_tf_t$ we have $e_{t-1}\,\varrho\,f_t$ and a contradiction. Therefore ϱ must be essential and hence equal to ν. But this means $V=\emptyset$ and the theorem holds.

So we shall assume the alternative that the only σ such that $\sigma\neq\angle$ and $\sigma\cap\varrho=\angle$ is when σ is of the first type, i.e., σ is generated by a pair of distinct elements (g_t,l_t) and (g_t,l_t') in U_t. Since $\sigma\cap\varrho=\angle$ we must have $(g_t,l_t)\in U$ and $(g_t,l_t')\in V$ or conversely. Assume the first. This means there are elements u and v in U_{t-1} such that $(g_t,l_t')u=(g_t,l_t')v$ and $(g_t,l_t)u\neq(g_t,l_t)v$. But then $(g_t,l_t)u(\sigma\cap\varrho)(g_t,l_t)v$. Hence we have a contradiction and again $V=\emptyset$ and the theorem holds.

We are now almost ready to state our main theorem which gives a characterization of semigroups that have the d.c.c. on right ideals and have no proper essential right congruences.

First, we make the following definition: A partition Γ on K_t is U_{t-1}-transitive if for every pair of idempotents l and l' on U_t of the form $\psi(s,m_t)=l$, $\psi(s',m_t')=l'$ where s, s' are in U_{t-1} and m_t, m_t' are in K_t there is a $u\in U_{t-1}$ such that $\psi(su,m_t)\Gamma l'$.

THEOREM 13 (Main Theorem). *Let S be a semigroup with identity and with d.c.c. on right ideals. Then S has no proper essential right congruences if and only if all of the following hold:*

1) *There is a sequence of two-sided ideals $S=I_0\supsetneqq I_1\supsetneqq\ldots\supsetneqq I_t$.*

2) *Every right ideal of S appears in this sequence.*

3) *For each i, the set U_i of generators of I_i is a right group containing an idempotent. Write $U_i=G_i\times K_i$ where G_i is a group and K_i a right zero semigroup.*

4) *For each pair* $i \leqq j$ *there is a homomorphism* $\varphi_{ij}: G_i \to G_j$ *such that* $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ *and a mapping* $\psi_{ij}: (G_i \times K_i) \times K_j \to K_j$ *such that* $\psi_{lj}(d_k, \psi_{ij}(c_i, k_j)) = \psi_{kj}(c_i d_k, k_j)$ *for all* $d_k \in U_k, c_i \in U_i, k_j \in K_j$ *and where* $k \leqq j$ *and where* $l = \max\{k, i\}$.

5) *For* $i \leqq j$ *multiplication is defined as follows:*

$$(g_i, k_i)(g_j, k_j) = (\varphi_{ij}(g_i) g_j, k_j),$$

$$(g_j, k_j)(g_i, k_i) = (g_j \varphi_{ij}(g_i), \psi_{ij}((g_i, k_i), k_j)).$$

6) *For each* $1 \leqq r \leqq t$ *and every idempotent* x *in* U_{r-1} *there is an idempotent* y *in* U_r *such that* $yx = y$ *and also if* $a, b \in U_{r-1}$ *then* $ya = yb$ *implies* $a = b$.

7) G_t *has no proper essential subgroup.*

8) *If* Γ *is an S-admissible,* U_{t-1}*-transitive proper partition on* K_t *then there is an S-admissible partition* π *on* K_t *such that* $\Gamma \cap \pi$ *is the identity relation on* K_t.

9) *For every pair of elements a, b in* U_{t-1} *and every idempotent t in* U_t *we have* $fa = fb$ *implies* $a = b$.

PROOF. We first assume S is a semigroup with identity and with d.c.c. on right ideals and having no proper essential right congruences. The theorems of Section 3 give properties 1) through 5) of this theorem. Under our assumptions conditions 1) of Theorem 9 and 1) of Theorem 10 give us properties 6) and 7) of this theorem. Condition 9) is just Theorem 12. We are left with proving 8). Let Γ be an S-admissible, U_{t-1}-transitive proper partition of K_t. Let e_t be defined as above and let ϱ be the right congruence determined by Γ and e_t as in 1) of Theorem 7. There is a right congruence $\sigma \neq \iota$ such that $\sigma \cap \varrho = \iota$ and such that σ is as described in Theorem 11, i.e., σ is generated by a pair e_{t-1} and an idempotent f_t in U_t. We had seen that we must have $f_t = f_t e_{t-1}$. Since Γ is U_{t-1}-admissible there is an s in U_{t-1} such that $\psi(s, f_t) \Gamma k_t$ where $e_t = (1_t, k_t)$. Also $(e_{t-1}s) \sigma f_t s$. In the construction of ϱ we saw that ϱ could equally well be defined by using Γ and $(1_t, \psi(s, f_t))$ since $\psi(s, f_t) \Gamma k_t$. But then $f_t(e_{t-1}s) = (f_t e_{t-1})s = f_t s$ and $f_t(f_t s) = f_t s$. Hence we must have $(e_{t-1}s)\, \varrho\, (f_t s)$. Since these two elements cannot be equal we must deny our assumption on the existence of a σ of the second type of Theorem 11. Hence there must be one of the first type. Assume σ is such a right congruence. We have σ nontrivial and its nontrivial equivalence classes are in U_t. $(g_t, l_t)\, \sigma\, (g_t', l_t')$ only if $g_t = g_t'$. Therefore σ induces a nontrivial partition π on K_t which must be S-admissible since it arises from a right congruence. Clearly $\sigma \cap \varrho = \iota$ implies $\pi \cap \Gamma$ is the identity relation on K_t. Hence 8) holds.

We now examine the converse. We assume 1) through 9) holds and wish to show S has no proper essential right congruences. If ϱ is an essential right congruence on S and $\varrho \leqq \delta$ then clearly δ is an essential right congruence on S. Hence we need only show the "large" right congruences are not essential. These "large" right congruences are the ones given in Theorem 7 and Theorem 8.

Theorem 9 takes care of the right congruences defined in Theorem 8. Theorem 10 takes care of the right congruences defined as the second type of Theorem 7. Hence we are left with the task of showing that there are no essential right congruences ϱ of type 1) of Theorem 7. Assume ϱ is defined by Γ and e and that Γ is not U_{t-1}-transitive. There must be a pair of distinct equivalence classes of Γ containing elements of the form $\psi(s, l_t)$ and $\psi(u, l_t')$ where $s, u \in U_{t-1}$ and such that for no $v \notin U_t$ do we have $\psi(sv, l_t) \Gamma \psi(u, l_t')$. It follows immediately that also there is no

$w \notin U_t$ such that $\psi(s, l_t) \Gamma \psi(uw, l'_t)$. From this it follows that there is no $w \notin U_t$ such that $\psi(sw, l_t) \Gamma \psi(uw, l'_t)$. Let $f_1 = (1_t, \psi(s, l_t))$ and $f_2 = (1_t, \psi(u, l'_t))$. We consider the right congruence σ generated by f_1 and f_2. It is immediate from condition 9) that the nontrivial equivalence classes of σ are doubletons of the form $\{f_1 y, f_2 y\}$ where $y \in U_{t-1}$. (We need only consider $y \in U_{t-1}$ since f_i contains a right hand factor that is an idempotent in U_{t-1}.) Now if for any y in U_{t-1} we have $f_1 y \varrho f_2 y$ the $\psi(sy, l_t) \Gamma \psi(uy, l'_t)$. But this contradicts the above statement. Hence $\sigma \neq \angle$ and $\varrho \cap \sigma = \angle$.

Finally, we assume Γ is U_{t-1}-transitive. Hence there is an S-admissible partition π of K_e such that $\pi \cap \Gamma$ is the identity relation on K_t. We define a relation σ on U_t such that $(g, l)\, \sigma\, (g, l')$ if and only if $l\, \pi\, l'$. Since π is an equivalence relation so is σ. Since π is S-admissible, so is σ, i.e., σ is a right congruence. Clearly $\sigma \cap \varrho = \angle$. Hence the theorem holds.

The necessary conditions of the theorem provide an important collation of results that can be used in studying semigroups of this type. As sufficient conditions the list is rather lengthy. We have not investigated the independence of these conditions. Conditions 1) through 6) describe the multiplication on S and conditions 7) through 8) describe how S acts on the S-system $G_t \times K_t$.

References

[1] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, vol. 1, Math Surveys No. 7 Amer. Math. Soc. (Providence, R. I., 1961).

[2] E. H. Feller and R. L. Gantos, Completely injective semigroups with central idempotents, *Glasgow Math. J.*, **10** (1969), 16—20.

[3] J. B. Fountain, Completely right injective semigroups, *Proc. London Math. Soc.*, **28** (1974), 28—44.

[4] H. J. Hoehnke, Structure of semigroups, *Canad. J. Math.*, **18** (1966), 449—491.

[5] N. Jacobson, *Structure of rings*, AMS Colloquium Publ. vol. 37, American Math. Soc. (Providence, 1964).

[6] W. Jones, Semigroups satisfying ring-like conditions, *Univ. of Iowa Technical Report* No. **9** (1969).

[7] R. H. Oehmke, On the maximal congruences of finite semisimple semigroups. *Tran. AMS*, **125** (1966), 223—237.

[8] R. H. Oehmke, Quasi-regularity in semigroups, *Seminaire Dubreil Pisat*, **23** (1969/70), DG1—DG3.

[9] R. H. Oehmke, On right congruences of semigroups, *Semigroup Forum*, **16** (1978), 393—401.

[10] R. H. Oehmke, On minimal right congruences of a semigroup, *Tamk. J. Math.*, **16** (1985).

[11] R. H. Oehmke, Some semisimplicity conditions for semigroups, *Tamk. J. Math.*, **16** (1985).

[12] R. E. Slover, Representations of a semigroup, *Trans. AMS*, **120** (1965), 417—427.

[13] B. Stenström, *Rings of quotients*, Springer-Verlag (Berlin, 1975).

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UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242
U.S.A.

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OPTIMAL INTERPOLATION WITH EXPONENTIALLY WEIGHTED POLYNOMIALS ON AN UNBOUNDED INTERVAL

T. KILGORE (Auburn)

Introduction

In this communication, we establish that the set of functions spanned by monomials of the form $e^{-at}t^k$, for $k=0, \ldots, n$ and for $0 \leqq t < \infty$, is a space for which the criteria of Bernstein and Erdős characterize optimal interpolation. This article represents the first generalization to weighted polynomials and to an unbounded domain certain results on interpolation of optimal norm which hold for various range spaces defined on a closed and bounded interval. After presenting some needed notation and defining some terms, we will present our results in more precise language and then move to the proofs.

Notation and terminology

We begin by defining $C[0, \infty]$ to be the space of functions continuous on the half-line and $C_0[0, \infty]$ to be its subspace of functions with limit (or value) 0 at ∞. We will let Y_n (or, where there is no need to be so precise about dimension, Y) be a space of those functions spanned by the monomials of the form $e^{-at}t^k$, for $k=0, \ldots, n$, as mentioned in the previous paragraph. We note that each of the spaces Y_n is a subspace of $C_0[0, \infty]$ and is spanned by a Markov system on the interval $[0, \infty)$. Thus, given a set of points *(nodes)* $t_0, \ldots, t_n$ such that $0=t_0<\ldots<t_n<\infty$, it is possible to obtain *fundamental functions* $y_i \in Y_n$, $i=0, \ldots, n$, satisfying $y_i(t_j)=\delta_{ij}$ (Kronecker delta) for $i, j=0, \ldots, n$. An *interpolating projection* P_n (P, for short) may then be constructed for the purpose of approximating any function f in $C_0[0, \infty]$ by:

$$Pf = \sum_{i=0}^{n} f(t_i) y_i. \tag{1}$$

The operator P thus constructed is a linear projection which has norm given by

$$\|P\| = \Big\| \sum_{i=0} |y_i| \Big\|. \tag{2}$$

The function defined inside the norm on the right side of (2) is called the *Lebesgue function* of P. Clearly, regardless of the value of n, the Lebesgue function is 1 at each node, and if $n>0$, its norm is greater than 1, since it is clearly greater than 1 on each interval (t_{i-1}, t_i), for $i=1, \ldots, n$. On each such interval, we let λ_i be the maximum value there, and we let T_i denote the point at which that maximum occurs. For $i=1, \ldots, n$, we denote by X_i the linear combination of $y_0, \ldots, y_n$ which agrees with the Lebesgue function on the interval (t_{i-1}, t_i), and by X_{n+1} we denote that linear combination of $y_0, ..., y_n$ which agrees with the Lebesgue function on the interval (t_n, ∞). We correspondingly define λ_{n+1} to be the rightmost maximum value of the function X_{n+1} and T_{n+1} to be the point at which this rightmost maximum occurs (note that $T_{n+1} > t_n$ is not guaranteed by the definition, nor by the inherent nature of the functions being used). With these definitions completed, we note that

(3) $$\|P\| = \max\{\lambda_1, ..., \lambda_{n+1}\},$$

and

(4) $$X_i'(T_i) = 0 \quad \text{for} \quad i = 1, ..., n+1.$$

Denoting by Z_n, $n=0, ...$, the space of functions spanned by expressions of the form $\exp(-at^2)p_n(t)$, where p_n is a polynomial of degree n or less, we merely observe that the above constructions of fundamental functions and interpolating projections can be carried out on any set of nodes $t_0, ..., t_n$ on the real line. The norm of such an interpolation is given in like manner as (2), and the associated Lebesgue function is bounded on the entire line, with a maximum value λ_0 of a function X_0 which agrees with the Lebesgue function of the leftmost unbounded portion of the domain. We will say that this maximum occurs at the point T_0, and (3) and (4) are now seen to hold, with the inclusion of the index 0.

The Bernstein and Erdős conjectures on interpolation

The history of the Bernstein [2] and Erdős [4] conjectures on optimal Lagrange interpolation will be presumed known to the reader, and they will be paraphrased in such a way as to fit the immediate context. Suffice it to say that the two conjectures have been shown to characterize optimal interpolation in many other spaces of functions than the ones for which they were originally framed, justifying the usage "criteria" in the first paragraph. Reference will be made to Kilgore [5] and de Boor and Pinkus [3], in which the original proofs of these two conjectures were laid out, since these proofs serve as a model for this present case as well as for previous extensions of the original problem.

As they relate to the problem at hand, the Bernstein and Erdős conjectures for the spaces Y_n are respectively that the norm of P is minimized when the values $\lambda_1, ..., \lambda_{n+1}$ are equal, which occurs at a unique placement of the nodes $t_1, ..., t_n$, and that there is associated with each space Y_n a *Lebesgue constant* c_n equal to the norm of optimal interpolation, for which, if P is a particular projection into Y_n whose norm is not minimal, there exists some i such that $\lambda_i < c_n$.

For the spaces Z_n, the above account is also valid, if one changes the lowest index of the λ's from 1 to 0. We are now ready to state our results.

Theorems

THEOREM 1. *The conditions laid down by Bernstein and Erdős, as described above, characterize optimal interpolation from* $C_0[0, \infty]$ *into each of the the spaces* Y_n *of spaces spanned by functions of the form* $e^{-at}t^k$, $k=0, \ldots, n$, $n>0$.

In Theorem 1, the left endpoint is chosen to be 0 for convenience. A simple argument, based on inspection of the structure of the fundamental functions, suffices to show that the norm of interpolation with such a space is in fact independent of the left endpoint of the interval of interpolation.

THEOREM 2. *The conclusions of Theorem 1 remain valid, if to the spaces* Y_n *are adjoined the constant functions, and interpolation is carried out from* $C[0, \infty]$ *into the spaces thus defined, with the rightmost node of interpolation being* ∞.

It is also possible to speak of interpolation on a finite interval with the same set of functions as in Theorem 1, and we have the following result. Note that, for these spaces also, the norm of interpolation in fact does not depend on the location of α or β separately but will depend upon the length of the interval $[\alpha, \beta]$.

THEOREM 3. *The conditions laid down by Bernstein and Erdős characterize interpolation from* $C[\alpha, \beta]$ *(choosing* $t_0=\alpha$ *and* $t_n=\beta$*) into the spaces* Y_n *restricted to the interval* $[\alpha, \beta]$. *For fixed* α *or fixed* β, *the norm of interpolation on this interval decreases as the length of the interval decreases, having the norm of Lagrange interpolation into the space of polynomials of degree* n *or less as its lower limit.*

COROLLARY 1. *Theorem 3 remains true without any restriction on the value of* a *in the expressions* e^{-at}.

COROLLARY 2. *Theorems 1, 2, and 3 remain true if the argument* t *is replaced by* t^γ, *for any* $\gamma>0$, *in which case the polynomial part of the interpolants used will be a polynomial in* t^γ.

THEOREM 4. *Theorem 1 remains true if the space* Z_n *is substituted for the space* Y_n.

THEOREM 5. *Theorem 3 remains true on an interval* $[a, b]$ *if the space* Z_n *is substituted for the space* Y_n.

THEOREM 6. *The results of Theorem 3 and 5 remain true if the interpolation is carried out into the spaces spanned by the multiples of a function in* Y_n *or* Z_n *respectively by a polynomial* p_m *of degree* m *with real zeroes at fixed locations outside of the interval* $[a, b]$.

REMARK. Theorems 3 and 5, among other things, demonstrate that the norm of the minimal interpolating projection into the spaces Y_n and Z_n increases without bound as n increases. One would not expect, therefore, that the convergence of interpolation is automatic. Rather, one would conjecture that the behavior of the norm of interpolation into these new interpolant spaces is essentially similar to the behavior of Lagrange interpolation.

Proofs of the Theorems

We begin with some general observations concerning things which have heretofore been relevant to proofs of all theorems of this type. The proof of Theorems 1 and 2 proceeds by showing that the matrices

$$A_k = (\partial\lambda_i/\partial t_j)_{j=1}^{n}{}_{i=1, i\neq k}^{n+1}$$

which are functions of the nodes $t_1, \ldots, t_n$, are globally nonsingular. This fact may then be seen to imply, by a finer analysis, that the sign of the determinant alternates with k. This second fact is shown in de Boor and Pinkus [3] in a form sufficiently general to be acceptable for all special cases of the interpolation problem for which the first fact of the nonsingularity of these matrices can be shown. Thus we will confine our attentions to demonstrating the global nonsingularity. Theorem 3 follows from Theorem 1 and from the additional fact that the nonsingularity can be shown if the index i runs from 1 to n only, while the index j is restricted to the values from 1 to $n-1$.

We defer for the moment the proof of the nonsingularity and briefly discuss how the proof of Theorems 1 and 3 may be completed once the nonsingularity and the alternation of sign of the determinants have been shown. Necessity of the Bernstein condition of the equality of $\lambda_1, \ldots, \lambda_{n+1}$ follows immediately, for, assume that not all of the maximum values $\lambda_1, \ldots, \lambda_{n+1}$ are equal. Then there is an index k such that λ_k is least. Nonsingularity of the matrix A_k then implies that the mapping $(t_1, \ldots, t_n) \to (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{n+1})$ is a local homeomorphism. Thus, there exists a small perturbation of the nodes which will simultaneously decrease all of $\lambda_1, \ldots, \lambda_{n+1}$ except, of course, for the possible exception of λ_k, which, by continuity cannot in any case jump suddenly to the position of being the greatest of the values. Thus, by (3) we have succeeded in decreasing the norm of interpolation. The sufficiency of the Bernstein condition follows from the uniqueness of the nodes which cause the λ's to be equal, which, along with the Erdős condition, follows from the second condition, of alternation in sign of the determinants of $A_1, \ldots, A_{n+1}$. (This alternation implies that the mapping $(t_1, \ldots, t_n) \to (\lambda_1 - \lambda_2, \ldots, \lambda_n - \lambda_{n+1})$ is a local homeomorphism, which can be combined with a general topological argument to show that one in fact has a global homeomorphism.) Theorem 3 will obviously follow from similar arguments if the nonsingularity of the corresponding matrices can be established.

We move now to the proof of the claimed nonsingularity properties. We may best begin with the study of the matrix

$$A = (\partial\lambda_i/\partial t_j)_{j=1}^{n}{}_{i=1}^{n+1}.$$

The first observation concerning A is that the entries of it have an explicit representation in terms of the functions in the space. One has for all applicable i and j the formula

$$(5) \qquad \partial\lambda_i/\partial t_j = -y_j(T_i)X_i'(t_j)$$

and writes the corresponding entries of A in this fashion. One then notes that an explicit formula for y_j may be constructed by setting

$$y_j(t) = w(t)[(t-t_j)w'(t_j)]^{-1},$$

where

(6) $$w(t) = \exp(-at) \prod_{l=0}^{n} (t-t_l).$$

Therefore, it is possible to perform the following cancellations on the matrix A, which leave a matrix which is equivalent with A, and whose corresponding submatrices are also equivalent:

First, one multiplies the j^{th} row of A, for $j=1, \dots, n$, by the quantity $w(t_j)$. Then, one divides the i^{th} column of what remains by $w(T_i)$. The resulting matrix is one in which the $(i, j)^{\text{th}}$ entry is $(t_j - T_i)^{-1} X_i'(t_j)$. There remains in each row for $j=1, \dots, n$ the contribution of a factor $\exp(-at_j)$ which occurs in the derivative $X_i'(t_j)$. This too can be cancelled from the matrix, leaving a matrix which consists of the evaluation of a different function in each column at successive points $t_1, \dots, t_n$ as one moves down the column. Closer scrutiny reveals that for $i=1, \dots, n+1$ the function

$$q_i(t) = \exp(at)(t-T_i)^{-1} X_i'(t),$$

which is the function evaluated in each column, is a polynomial of degree $n-1$ or less. We remark that, in addition, the polynomials q_1 and q_n and q_{n+1} must in fact be of degree $n-1$, while the others are certainly of degree at least $n-2$. The problem of showing that the matrices $A_1, \dots, A_{n+1}$ are nonsingular therefore is reduced to that of showing that $\{q_1, \dots, q_{n+1}\} \setminus \{q_k\}$ is linearly independent for each k, $k=1, \dots, n+1$. This fact in turn will follow for example from Kilgore [6] if it can be shown that the q's obey certain sign properties on the ordered set $T_1, \dots, T_{n+1}$. Specifically, we must have (assuming that $q_i(T_1)>0$ for the sake of regularity)

(i) $(-1)_i q_1(T_i)>0$ for $i=2, \dots, n+1$
(ii) $q_i(T_j) \neq 0$ for all applicable i, j
(iii) $q_1(T_j) q_i(T_j)<0$ for $i=2, \dots, n$, $j=2, \dots, n+1$, $i \neq j$
(iv) $q_1(T_j) q_j(T_j)>0$ for $j=2, \dots, n$.

In fact, these sign properties are implied by the fact that the zeroes of the func tions $X_1', \dots, X_{n+1}'$ strictly interlace on the interval $[0, \infty)$. For the purpose o establishing this fact, the step-by-step zero-counting arguments of Kilgore [5], for example, will suffice with only mild adaptation.

The steps which have just been outlined suffice to prove Theorem 1. The same steps may be carried out with no essential differences to prove Theorem 4, except that the dimensions of the matrix corresponding to A are $n+1$ by $n+2$, and the degree of the polynomials of the form q_i is n or less.

The proofs of Theorems 3, 5, and 6 also depend upon the analysis which we have just carried out. However, it is insufficient for the conclusions of these theorems that a set consisting of all but one of the functions $q_1, \dots, q_n$ is linearly independent, in view of the fact that the degree of these polynomials is more than $n-2$. We have available, however, Kilgore [7, Proposition 2] which deals exactly with such a situation, in which we have polynomials $q_1, \dots, q_n$ of degree $n+m-2$ or less and points $T_1, \dots, T_{n+m}$ ordered from left to right, on which the polynomials satisfy the sign conditions (i)—(iv). Then, for points $t_1, \dots, t_{n-1}$, located such that

$T_1 < t_1 < T_2 < \ldots < T_{n-1} < t_{n-1} < T_n$, it is the case, for $k=1, \ldots, n$, that

$$\det \left(q_i(t_j)\right)_{j=1}^{n-1} {}_{i=1,\, i \neq k}^{n} \neq 0.$$

This fact is exactly what is needed in order to complete successfully the proofs of Theorems 3, 5, and 6.

Concluding remarks

The new results linking polynomial interpolation on an infinite interval to bounded interpolation in certain spaces defined on the same interval may open a new method for the estimation of error of approximation by such procedures. At the same time, the affirmation of the Bernstein and Erdős characterization of optimal norm interpolation into these related spaces of exponentially weighted polynomials will, it is hoped, provide a theoretical measure of nearly optimal procedures of interpolation, as has been done in the more classical case of Lagrange interpolation.

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References

[1] K. Balázs, Lagrange and Hermite interpolation processes on the positive real line, *J. Approx. Theory*, **50** (1987), 18—24.

[2] S. Bernstein, Sur la limitation des valeurs d'un polynome, *Bull. Acad. Sci. URSS*, **8** (1931), 1025—1050.

[3] C. De Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdős concerning the optimal nodes for polynomial interpolation, *J. Approx. Theory*, **24** (1978), 289—303.

[4] P. Erdős, Some remarks on polynomials, *Bull. Amer. Math. Soc.*, **53** (1947), 1169—1176.

[5] T. A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory*, **24** (1978), 273—288.

[6] T. A. Kilgore, A note on functions with interlacing roots, *J. Approx. Theory*, **43** (1985), 25—28.

[7] T. A. Kilgore, Optimal interpolation with polynomials having fixed roots, *J. Approx. Theory*, **49** (1987), 378—389.

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ALGEBRA, COMBINATORICS, AND ANALYSIS
120 MATHEMATICS ANNEX
AUBURN UNIVERSITY
AUBURN, ALABAMA 36849
USA

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COHOMOLOGICAL PSEUDOMANIFOLDS

M. BOGNÁR (Budapest)

The aim of this paper is to prove some theorems about k-manifolds which have been stated in a preceding article [2] without proofs. To this end we first introduce the notion of the cohomological pseudomanifold and investigate its basic properties.

1. Basic notions and theorems

1.1. Definition. Let Z_2 be the cyclic group of order two and let q be a nonnegative integer. By the q-dimensional cohomology group $H^q(Y)$ of a locally compact T_2-space Y we always mean the q-dimensional cohomology group of Y with coefficients in Z_2 with compact supports (see [5] p. 6).

1.2. Definition. Let n be a positive integer. A space Y is said to be a *cohomological n-pseudomanifold* (briefly: n-cpm) if it is a nonempty locally compact T_2-space and it has a basis $\mathcal{B}$ satisfying the following conditions:

(a) For $U\in\mathcal{B}$ $H^n(U)=Z_2$ (= means here "isomorphic"),

(b) For $U\in\mathcal{B}$ and for every nonempty open subset U' of U

$$H^n(U\setminus U')=0 \quad \text{and} \quad H^q(U')=0$$

holds for all $q>n$.

It is to be noted that condition (b) is clearly equivalent to the following one.

(b′) For $U\in\mathcal{B}$ and for every proper closed subset F of U

$$H^n(F)=0 \quad \text{and} \quad H^q(U\setminus F)=0$$

holds for all $q>n$.

Now we can formulate two immediate consequences of this definition.

1.3. Each nonempty open subspace of an n-cpm is an n-cpm as well.

1.4. Each n-dimensional manifold is an n-cpm.

Indeed an n-dimensional manifold M is clearly a locally compact T_2-space. Moreover the basis $\mathcal{B}$ consisting of all open subspaces of M homeomorphic to R^n satisfies conditions (a) and (b′) of the Definition (see [5] I.2.14, I.3.9 and I.3.5).

Before starting the investigations about cohomological n-pseudomanifolds we recall two important facts about cohomology theory which we shall use frequently in the sequel.

1.5. Let X be a locally compact T_2-space and $\{X_\alpha;\ \alpha\in A\}$ a decomposition of X into pairwise disjoint open sets. Then for each integer q the homomorphism

$$\tau_{X_\alpha, X}\colon H^q(X_\alpha) \to H^q(X) \quad \text{(see [5] I.§ 1.3)}$$

is a monomorphism, and $H^q(X)$ is the direct sum of the images (see [5] I.2.13).

1.6. Let (X, A) be a pair, where X is a locally compact T_2-space and A is a closed subspace of X. Then the cohomology sequence of the pair (X, A) is exact (see [5] I.1.6).

Now let Y be an n-cpm. We select a basis $\mathscr{B}$ of Y satisfying the conditions of Definition 1.2. We shall keep it fixed in this section.

Next we make two remarks.

1.7. For $U\in\mathscr{B}$ we have $U\neq\emptyset$. This is an immediate consequence of $H^n(U)=$ $=Z_2$ (see 1.2(a)).

1.8. If V is an open subset of a $U\in\mathscr{B}$ and $q>n$ then $H^q(V)=0$. This is true by 1.2(b) if $V\neq\emptyset$ and it is obvious if $V=\emptyset$.

Now we can formulate the first fundamental theorem.

1.9. THEOREM. *Let V be an open subset of Y. Then $H^q(V)=0$ provided that $q>n$.*

PROOF. First consider the case, when V can be covered by a finite number of members of $\mathscr{B}$, i.e. $V\subset \bigcup_{i=1}^{k} U_i$ where V is open in Y and $U_i\in\mathscr{B}$ for $i=1, \dots, k$. To prove the assertion in this case we proceed by induction.

If $k=1$ then the assertion is true by 1.8. Assume now that m is a positive integer and the assertion is true for $k\leqq m$. Consider the case $k=m+1$ and let $P=U_1\cup\dots\cup U_m$. Let $V'=P\cap V$ and $V''=U_{m+1}\cap V$. Then $V'\cap V''\subset U_{m+1}$ and thus by 1.8 $H^r(V'\cap V'')=0$ for all $r>n$. Let $q>n$ and consider the segment

$$H^q(V')\oplus H^q(V'') \xrightarrow{\psi} H^q(V) \xrightarrow{\Delta} H^{q+1}(V'\cap V'')$$

of the Mayer—Vietoris sequence of the decomposition $V=V'\cup V''$ (see [5] p. 68). Since by the induction hypothesis $H^q(V')=H^q(V'')=0$ and as we have seen above $H^{q+1}(V'\cap V'')=0$ moreover since the Mayer—Vietoris sequence is exact, it follows $H^q(V)=0$ indeed.

Thus the assertion is true for open sets covered by a finite subsystem of $\mathscr{B}$.

Consider now an arbitrary open subset V of Y. Let $q>n$ and $b\in H^q(V)$. Then there is an open subset $V'\subset V$ such that the closure of V' in V is compact and $b\in\operatorname{im}\tau_{V',V}$, i.e., there is an $a\in H^q(V')$ with $b=\tau_{V',V}(a)$ (see [5] p. 15). However V' can clearly be covered by a finite subsystem of $\mathscr{B}$ and thus $a=0$. Hence $b=0$, $H^q(V)=0$ as required.

We now prepare the next fundamental theorem.

1.10. Let $U\in\mathscr{B}$ and let U' be a nonempty open subset of U. Then the homomorphism $\tau_{U',U}\colon H^n(U')\to H^n(U)$ is an epimorphism.

Indeed, consider the segment

$$H^n(U') \xrightarrow{\tau_{U',U}} H^n(U) \to H^n(U\setminus U')$$

of the cohomology sequence of the pair $(U, U\setminus U')$. By 1.2(b) we have $H^n(U\setminus U')=0$ and thus 1.6 shows that $\tau_{U',U}$ is an epimorphism as required.

1.11. Let $U_1, U_2 \in \mathcal{B}$ where $U_1 \subset U_2$. Then $\tau_{U_1,U_2}\colon H^n(U_1)\to H^n(U_2)$ is an isomorphism.

Indeed, by 1.2(a) we have $H^n(U_1)=H^n(U_2)=Z_2$. On the other hand 1.7 and 1.10 show that τ_{U_1,U_2} is an epimorphism. Thus τ_{U_1,U_2} is an isomorphism as required.

1.12. Definition. The members U_1, U_2 of $\mathcal{B}$ are said to be *compatible* if $U_1\subset U_2$ or $U_2\subset U_1$.

For the compatible members U_1, U_2 of $\mathcal{B}$ let

$$\tau'_{U_1,U_2} = \begin{cases} \tau_{U_1,U_2} & \text{if } U_1\subset U_2 \\ (\tau_{U_2,U_1})^{-1} & \text{if } U_2\subset U_1. \end{cases}$$

Since τ_{U_1,U_1} is the identity homomorphism of $H^n(U_1)$ it follows $\tau_{U_1,U_1}= =(\tau_{U_1,U_1})^{-1}$. Consequently τ'_{U_1,U_2} is well defined also in the case $U_1=U_2$.

1.13. Let U_1 and U_2 be compatible members of $\mathcal{B}$ and let V be an open subspace of Y containing $U_1\cup U_2$. Then the diagram

$$\begin{array}{ccc} H^n(U_1) & \xrightarrow{\tau'_{U_1,U_2}} & H^n(U_2) \\ & {\scriptstyle\tau_{U_1,V}}\searrow \quad \swarrow{\scriptstyle\tau_{U_2,V}} & \\ & H^n(V) & \end{array}$$

is clearly commutative.

1.14. Definition. A finite sequence $C=(U_1, \ldots, U_m)$ of members of $\mathcal{B}$ is said to be a *$\mathcal{B}$-chain* if for $i=1, \ldots, m-1$, U_i and U_{i+1} are compatible members of $\mathcal{B}$. The *body* of C (denoted by $\breve{C}$) is the set $\breve{C}=U_1\cup\ldots\cup U_m$. U_1 is the *initial* and U_m the *terminal member* of C. We say that C *connects* U_1 and U_m. If V is an open subset of Y and $\breve{C}\subset V$ we then say, that C *is in* V.

For any $\mathcal{B}$-chain $C=(U_1, \ldots, U_m)$ in the case $m\geqq 2$ let

$$C^* = \tau'_{U_{m-1},U_m}\ldots\tau'_{U_2,U_3}\tau'_{U_1,U_2}\colon H^n(U_1)\to H^n(U_m)$$

and if $m=1$ let C^* be the identity homomorphism of $H^n(U_1)$. C^* is clearly an isomorphism.

1.15. Let V be an open subspace of Y and $C=(U_1, \ldots, U_m)$ a $\mathcal{B}$-chain in V. Then the diagram

$$\begin{array}{ccc} H^n(U_1) & \xrightarrow{C^*} & H^n(U_m) \\ & {\scriptstyle\tau_{U_1,V}}\searrow \quad \swarrow{\scriptstyle\tau_{U_m,V}} & \\ & H^n(V) & \end{array}$$

is clearly commutative.

1.16. Proposition. *Each member of the basis $\mathcal{B}$ is a domain, i.e. a connected nonempty open subspace of Y.*

Proof. Let $U\in\mathcal{B}$. U is clearly open and by 1.7 it is a nonempty subspace of Y. To show the connectedness of U we argue by contradiction.

Suppose that $U=V_1\cup V_2$, where V_1, V_2 are nonempty disjoint open subsets of U. 1.10 shows that for $i=1, 2$ $\tau_{V_i,U}\colon H^n(V_i)\to H^n(U)$ is an epimorphism and thus by

$H^n(U)=Z_2$ we have $H^n(V_i)\neq 0$. However $H^n(U)$ is isomorphic to the direct sum of $H^n(V_1)$ and $H^n(V_2)$ (see 1.5). But this is impossible since the group $H^n(U)=Z_2$ is indecomposable.

The assumption was false. U is connected indeed.

1.17. Observe as a corollary of 1.16 that Y is a locally connected space.

1.18. DEFINITION. An open subset V of Y is said to be *of finite class* if it is the union of a nonempty finite subsystem of $\mathcal{B}$.

Now we can formulate and prove the second fundamental theorem.

1.19. THEOREM. *Let V be an open connected subspace of Y and let U be a member of $\mathcal{B}$ contained in V. Then the homomorphism $\tau_{U,V}\colon H^n(U)\to H^n(V)$ is an epimorphism.*

PROOF. We first consider the case when V is of finite class, i.e. $V=\bigcup_{i=1}^{k} U_i$ where $U_i\in\mathcal{B}$ for $i=1, \dots, k$.

If $k=1$ then by 1.10 and 1.7 the assertion is obviously true.

Next, assume that m is a positive integer and the assertion is true for $k\leqq m$. Consider the case $k=m+1$, i.e. $V=\bigcup_{i=1}^{m+1} U_i$. We may suppose that the numeration is chosen so that $V'=U_1\cup\dots\cup U_m$ is connected (cf. 1.16). Since V is connected it follows $V'\cap U_{m+1}\neq\emptyset$ and thus there is a member U' of $\mathcal{B}$ lying in $V'\cap U_{m+1}$.

We first show that $\tau_{U',V}\colon H^n(V')\to H^n(V)$ is an epimorphism.

Indeed, by the induction hypothesis $\tau_{U',V'}$ and $\tau_{U',U_{m+1}}$ are epimorphisms. Consider now the segment

$$H^n(V')\oplus H^n(U_{m+1})\xrightarrow{\psi} H^n(V)\xrightarrow{\Delta} H^{n+1}(V'\cap U_{m+1})$$

of the Mayer—Vietoris sequence of the decomposition $V=V'\cup U_{m+1}$. By Theorem 1.9 we have $H^{n+1}(V'\cap U_{m+1})=0$ and thus by the exactness of this sequence it follows that ψ is an epimorphism. Let $b\in H^n(V)$. Since for $(a_1, a_2)\in H^n(V')\oplus\oplus H^n(U_{m+1})$ we have

$$\psi(a_1, a_2)=\tau_{V',V}(a_1)-\tau_{V_{m+1},V}(a_2)$$

and the homomorphisms ψ, $\tau_{U',V'}$ and $\tau_{U',U_{m+1}}$ are epimorphisms it follows the existence of $c_1, c_2\in H^n(U')$ such that

$$b=\tau_{V',V}\tau_{U',V'}(c_1)-\tau_{U_{m+1},V}\tau_{U',U_{m+1}}(c_2)=\tau_{U',V}(c_1)-\tau_{U',V}(c_2)=\tau_{U',V}(c_1-c_2).$$

$\tau_{U',V}$ is an epimorphism as required.

Now let $C=(U^1=U', \dots, U^r=U)$ be a $\mathcal{B}$-chain in V connecting U' and U. Since V is connected it follows the existence of such a $\mathcal{B}$-chain C. However by 1.15 we have $\tau_{U',V}=\tau_{U,V}C^*$ and since $\tau_{U',V}$ is an epimorphism it follows that $\tau_{U,V}$ is an epimorphism indeed.

The assertion is true if V is of finite class.

Now let V be an arbitrary connected open subspace of Y. If $V=\emptyset$ then the assertion is obviously true. Thus we can suppose that $V\neq\emptyset$. Let $b\in H^n(V)$. Then there is an open subspace V' of V such that the closure $\bar{V}'$ of V' in V is compact and $b\in\operatorname{im}\tau_{V',V}$, i.e. $b=\tau_{V',V}(a)$ for some $a\in H^n(V')$ (see [5] p. 15). Since $\bar{V}'$ is compact and V is nonempty and connected, there is clearly an open connected subspace V'' of V of finite class containing V'.

Select $U'\in\mathscr{B}$ such that $U'\subset V''$. Since V'' is of finite class it follows the existence of a $c\in H^n(U')$ such that $\tau_{U',V''}(c)=\tau_{V',V''}(a)$ and thus

$$\tau_{U',V}(c) = \tau_{V'',V}\tau_{U',V''}(c) = \tau_{V'',V}\tau_{V',V''}(a) = \tau_{V',V}(a) = b.$$

Let $C=(U^1=U, \dots, U^r=U')$ be a $\mathscr{B}$-chain in V connecting U and U'. Since V is connected there clearly exists such a chain C. However $C^*\colon H^n(U)\to H^n(U')$ is an isomorphism (see 1.14) and thus there is a $d\in H^n(U)$ such that $C^*(d)=c$. Now taking also $\tau_{U,V}=\tau_{U',V}C^*$ (see 1.15) into account we get

$$\tau_{U,V}(d) = \tau_{U',V}C^*(d) = \tau_{U',V}(c) = b.$$

$\tau_{U,V}$ is an epimorphism indeed.

The proof of the theorem is complete.

Now we formulate the third fundamental theorem.

1.20. Theorem. *Let V be an open subspace of Y and let U be a member of $\mathscr{B}$ contained in V. Then the homomorphism $\tau_{U,V}\colon H^n(U)\to H^n(V)$ is a monomorphism.*

Proof. We first consider the case when V is connected and of finite class, i.e. $V=\bigcup_{i=1}^{k} U_i$, where $U_i\in\mathscr{B}$. If $k=1$, then the assertion is true by 1.11. Assume that m is a positive integer and the assertion is true for $k\le m$. Consider the case $k=m+1$, i.e. $V=\bigcup_{i=1}^{m+1} U_i$ where $U_i\in\mathscr{B}$ for $i=1, \dots, m+1$. To prove the assertion in this case we make some preliminary remarks.

(a) We may assume the numeration is chosen so that $V'=U_1\cup\dots\cup U_m$ is connected (cf. 1.16). Since V is connected, it follows $V'\cap U_{m+1}\ne\emptyset$.

(b) We show that $H^n(V')=Z_2$.

Indeed, $\tau_{U_1,V'}\colon H^n(U_1)\to H^n(V')$ is a monomorphism by the induction hypothesis and it is an epimorphism by 1.19. Thus $\tau_{U_1,V'}$ is an isomorphism. Consequently since $H^n(U_1)=Z_2$ (see 1.2(a)) it follows $H^n(V')=Z_2$ as required.

(c) Now let V'' be a domain in $V'\cap U_{m+1}$. Then both homomorphisms $\tau_{V'',V'}\colon H^n(V'')\to H^n(V')$ and $\tau_{V'',U_{m+1}}\colon H^n(V'')\to H^n(U_{m+1})$ are isomorphisms and $H^n(V'')=Z_2$.

Indeed, select $U'\in\mathscr{B}$ such that $U'\subset V''$. Then by the induction hypothesis

$$\tau_{U',V'} = \tau_{V'',V'}\tau_{U'V''} \tag{1}$$

and

$$\tau_{U',U_{m+1}} = \tau_{V'',U_{m+1}}\tau_{U',V''} \tag{2}$$

are monomorphisms. Hence $\tau_{U',V''}$ is a monomorphism and by 1.19 it is an epimorphism as well. Consequently $\tau_{U',V''}$ is an isomorphism and thus taking also 1.2(a) into account we have

$$H^n(V'') = Z_2. \tag{3}$$

However by 1.19 $\tau_{U',V'}$ and $\tau_{U',U_{m+1}}$ are epimorphisms and thus by (1) and (2) $\tau_{V'',V'}$ and $\tau_{V'',U_{m+1}}$ are epimorphic mappings. Taking also $H^n(V'')=H^n(V')=$

$=H^n(U_{m+1})=Z_2$ (see (3), (c) and 1.2(a)) into account it follows that $\tau_{V'',V'}$ and $\tau_{V'',U_{m+1}}$ are isomorphisms as required.

(d) Let $P=V'\cap U_{m+1}$. Next we show that for the homomorphisms $\tau_{P,V'}\colon H^n(P)\to H^n(V')$ and $\tau_{P,U_{m+1}}\colon H^n(P)\to H^n(U_{m+1})$ we have

$$\ker\tau_{P,V'}=\ker\tau_{P,U_{m+1}}.$$

Indeed, let $\{V_\alpha;\ \alpha\in A\}$ be the system of components of $V'\cap U_{m+1}=P$. Since the space Y is locally connected (see 1.17) it follows that for each $\alpha\in A$, V_α is a nonempty open connected set.

Now let $a\in H^n(P)$. Then there is a finite subset $\alpha_1,\dots,\alpha_t$ of A and elements $a_1,\dots,a_t$ in $H^n(V_{\alpha_1}),\dots,H^n(V_{\alpha_t})$ respectively such that

$$a=\tau_{V_{\alpha_1},P}(a_1)+\dots+\tau_{V_{\alpha_t},P}(a_t)$$

(see 1.5). Consequently

$$\tau_{P,V'}(a)=\tau_{V_{\alpha_1},V'}(a_1)+\dots+\tau_{V_{\alpha_t},V'}(a_t)$$

and

$$\tau_{P,U_{m+1}}(a)=\tau_{V_{\alpha_1},U_{m+1}}(a_1)+\dots+\tau_{V_{\alpha_t},U_{m+1}}(a_t).$$

Since the homomorphisms $\tau_{V_{\alpha_i},V'}$ and $\tau_{V_{\alpha_i},U_{m+1}}$ are isomorphisms and

$$H^n(V_{\alpha_1})=\dots=H^n(V_{\alpha_t})=H^n(V')=H^n(U_{m+1})=Z_2$$

(see (c), (b) and 1.2(a)) it follows that both relations $a\in\ker\tau_{P,V'}$ and $a\in\ker\tau_{P,U_{m+1}}$ hold if and only if an even number of the elements $a_1,\dots,a_t$ are nonzero. Thus

$$\ker\tau_{P,V'}=\ker\tau_{P,U_{m+1}}$$

as required.

(e) The homomorphism $\tau_{U_{m+1},V}\colon H^n(U_{m+1})\to H^n(V)$ is a monomorphism. Indeed, select $b\in H^n(U_{m+1})$ so that $\tau_{U_{m+1},V}(b)=0$. Then for $(0,b)\in H^n(V')\oplus\oplus H^n(U_{m+1})$ we have

$$\tau_{V',V}(0)-\tau_{U_{m+1},V}(b)=0.$$

Let $P=V'\cap U_{m+1}$ and consider the segment

$$H^n(P)\xrightarrow{\varphi}H^n(V')\oplus H^n(U_{m+1})\xrightarrow{\psi}H^n(V)$$

of the Mayer—Vietoris sequence of the decomposition $V=V'\cup U_{m+1}$. Since

$$\psi(0,b)=\tau_{V',V}(0)-\tau_{U_{m+1},V}(b)=0$$

taking also the exactness of the Mayer—Vietoris sequence into account it follows $(0,b)\in\ker\psi=\operatorname{im}\varphi$. Consequently there is an $a\in H^n(P)$ such that $(0,b)=\varphi(a)$, i.e. $0=\tau_{P,V'}(a)$ and $b=\tau_{P,U_{m+1}}(a)$. Hence $a\in\ker\tau_{P,V'}$ and thus by (d) we have $a\in\ker\tau_{P,U_{m+1}}$ which implies $b=\tau_{P,U_{m+1}}(a)=0$ as required.

(f) We now prove the assertion in case $k=m+1$.

Let $C=(U^1=U,\dots,U^r=U_{m+1})$ be a $\mathscr{B}$-chain in V connecting U and U_{m+1}. Since V is connected there exists such a chain. However $\tau_{U,V}=\tau_{U_{m+1},V}C^*$ (see

1.15) where C^* is an isomorphism (see 1.14). Hence (e) implies that $\tau_{U,V}$ is a monomorphism indeed.

Thus the assertion is true if V is connected and is of finite class.

Now let V be an arbitrary open connected subspace of Y and U a member of $\mathcal{B}$ contained in V. Then there is a member U' of $\mathcal{B}$ such that $U' \subset V$ and so that the closure $\overline{U}'$ of U' in V is compact.

We first show that $\tau_{U',V}\colon H^n(U') \to H^n(V)$ is a monomorphism.

Indeed, select $b \in H^n(U')$ such that $\tau_{U',V}(b)=0$. Then there is an open subset V' of V containing U' and being such that the closure $\overline{V}'$ of V' in V is compact and for which $\tau_{U',V'}(b)=0$ (see [5] p. 15). Let V'' be an open connected subset of finite class containing V'. There clearly exists such a V'' and we have

$$\tau_{U',V''}(b) = \tau_{V',V''}\tau_{U',V'}(b) = 0.$$

Since — as we have seen above — $\tau_{U',V''}$ is a monomorphism, it follows $b=0$. $\tau_{U',V}$ is a monomorphism as required.

Now let $C=(U=U^1, \dots, U^r=U')$ be a $\mathcal{B}$-chain in V connecting U and U'. Since V is connected there exists such a chain C. However $\tau_{U,V}=\tau_{U',V}C^*$ (see 1.15) and thus since C^* is an isomorphism (see 1.14) and $\tau_{U',V}$ is a monomorphism it follows that $\tau_{U,V}$ is a monomorphism indeed.

Finally let V be an arbitrary open subset of Y and U a member of $\mathcal{B}$ contained in V. U is connected (see 1.16) and thus it is contained in a component V' of V. However Y is locally connected (see 1.17) and thus V' is an open connected subspace of Y. As we have seen above $\tau_{U,V'}\colon H^n(U) \to H^n(V')$ is a monomorphism and since $\tau_{V',V}\colon H^n(V') \to H^n(V)$ is a monomorphism as well (see 1.5) it follows that $\tau_{U,V}=\tau_{V',V}\tau_{U,V'}$ is a monomorphism.

The proof of the theorem is complete.

We now show some corollaries of Theorems 1.19 and 1.20.

1.21. Let V be a domain in Y and let V' be an open nonempty subset of V. Then for $q \geqq n$

$$\tau_{V',V}\colon H^q(V') \to H^q(V)$$

is an epimorphism.

Indeed, for $q>n$ we have $H^q(V)=0$ by 1.9 and thus the assertion is obviously true. Suppose now that $q=n$ and select $U \in \mathcal{B}$ so that $U \subset V'$. Since $\tau_{U,V}= =\tau_{V',V}\tau_{U,V'}$ and $\tau_{U,V}$ is an epimorphism by 1.19, it follows that $\tau_{V',V}$, is an epimorphism as required.

1.22. Let V be a domain in Y and V' a nonempty open subset of V. Then for $q \geqq n$, $H^q(V \setminus V')=0$.

Indeed, consider the segment

$$H^q(V') \xrightarrow{\tau_{V',V}} H^q(V) \to H^q(V \setminus V') \to H^{q+1}(V')$$

of the cohomology sequence of the pair $(V, V \setminus V')$. Since by 1.21, $\tau_{V',V}$ is an epimorphism and by 1.9, $H^{q+1}(V')=0$, the exactness of this sequence (see 1.6) implies $H^q(V \setminus V')=0$ as required.

1.23. It is immediate from 1.22 that for each domain V in Y, for an arbitrary proper closed subset F of V and for $q \geqq n$ we have $H^q(F)=0$.

1.24. Let V be a domain in Y. Then by 1.19, 1.20 and 1.2(a) we have $H^n(V)=Z_2$. In particular, 1.25. If Y is a connected n-cpm then by 1.24 $H^n(Y)=Z_2$.

1.26. Let V be an open nonempty subspace of Y. Then by 1.20 and 1.2(a) we have $H^n(V)\neq 0$.

1.27. Let V be a domain in Y. Then by 1.24, 1.22 and 1.9 the conditions 1.2(a) and 1.2(b) are fulfilled for V. Hence, taking also 1.16 into account $\mathcal{B}$ can be replaced by the system $\mathcal{B}'$ of all domains of Y and we have $\mathcal{B}\subset\mathcal{B}'$.

1.28. Let V be a nonempty open subspace of Y. V is connected (V has two components respectively) if and only if $H^n(V)=Z_2$ $(H^n(V)=Z_2\oplus Z_2$ respectively).

Indeed, let $\{V_\alpha;\ \alpha\in A\}$ be the system of components of V. Since Y is locally connected (see 1.17) it follows that each V_α is a domain. Thus for each $\alpha\in A$, $\tau_{V_\alpha,V}\colon H^n(V_\alpha)\to H^n(V)$ is a monomorphism and $H^n(V)$ is the direct sum of the images $\operatorname{im}\tau_{V_\alpha,V}$ (see 1.5). Consequently the assertion is an immediate corollary of 1.24.

1.29. Let V_1 and V_2 be domains in Y such that $V_1\subset V_2$. Then according to 1.27 and 1.11 the homomorphism $\tau_{V_1,V_2}\colon H^n(V_1)\to H^n(V_2)$ is an isomorphism.

Finally we show two important properties of the n-cpm-s.

1.30. Let V and V' be nonempty open subsets of Y such that $V'\subset V$ and so that distinct components of V' lie in distinct components of V. Then the homomorphism $\tau_{V',V}\colon H^n(V')\to H^n(V)$ is a monomorphism.

Indeed, let $\{V'_\alpha;\ \alpha\in A'\}$ be the system of components of V' and $\{V_\alpha;\ \alpha\in A\}$ the system of components of V such that $A'\subset A$ and for $\alpha\in A'$ one has $V'_\alpha\subset V_\alpha$. These requirements could clearly be satisfied.

Since Y is locally connected (see 1.17) it follows that each V'_α and each V_α is open in Y.

Now let $x\in\ker\tau_{V',V}$, i.e. $x\in H^n(V')$ and

$$\tau_{V',V}(x)=0. \tag{4}$$

Since each $\tau_{V'_\alpha,V'}\colon H^n(V'_\alpha)\to H^n(V')$ is a monomorphism and $H^n(V')$ is the direct sum of the images (see 1.5) we have a representation

$$x=\tau_{V'_{\alpha_1},V'}(x_1)+\ldots+\tau_{V'_{\alpha_r},V'}(x_r) \tag{5}$$

such that $\alpha_1,\ldots,\alpha_r$ are pairwise distinct elements of A'. However for $i=1,\ldots,r$

$$\tau_{V',V}\,\tau_{V'_{\alpha_i},V'}=\tau_{V'_{\alpha_i},V}=\tau_{V_{\alpha_i},V}\,\tau_{V'_{\alpha_i},V_{\alpha_i}}$$

holds and thus by (4) and (5) we have

$$0=\tau_{V',V}(x)=\sum_{i=1}^{r}\tau_{V_{\alpha_i},V}\,\tau_{V'_{\alpha_i},V_{\alpha_i}}(x_i).$$

Consequently, since each $\tau_{V_\alpha,V}\colon H^n(V_\alpha)\to H^n(V)$ is a monomorphism and $H^n(V)$ is the direct sum of the images (see 1.5) we obtain

$$\tau_{V'_{\alpha_i},V_{\alpha_i}}(x_i)=0\quad\text{for}\quad i=1,\ldots,r.$$

Taking also 1.29 into account we get $x_1=\ldots=x_r=0$ and thus by (5) $x=0$.

$\tau_{V',V}$ is a monomorphism as required.

1.31. Let V and V' be nonempty open subsets of Y such that $V' \subset V$ and so that there exist two distinct components of V' contained in the same component of V. Then the homomorphism $\tau_{V',V}\colon H^n(V') \to H^n(V)$ fails to be a monomorphism.

Indeed, let V_1' and V_2' be components of V' contained in the same component V_0 of V. Since the space Y is locally connected it follows that V_1', V_2' and V_0 are domains in Y and thus by 1.24 we have

$$H^n(V_1') = H^n(V_2') = H^n(V_0) = Z_2. \tag{6}$$

For $i=1, 2$ let e_i be the nonzero element of $H^n(V_i')$. Since for $i=1, 2$, $\tau_{V_i', V_0}\colon H^n(V_i') \to H^n(V_0)$ is an isomorphism (see 1.29) taking also (6) into account we get

$$\tau_{V_1', V_0}(e_1) + \tau_{V_2', V_0}(e_2) = 0. \tag{7}$$

Let $V_3' = V' \setminus (V_1' \cup V_2')$. Then by 1.17 V_3' is open in Y and thus $\{V_1', V_2', V_3'\}$ is a decomposition of V' into pairwise disjoint open sets. Let $0 = e_3 \in H^n(V_3')$ and

$$y = \tau_{V_1', V'}(e_1) + \tau_{V_2', V'}(e_2) + \tau_{V_3', V'}(e_3) = \tau_{V_1', V'}(e_1) + \tau_{V_2', V'}(e_2) \in H^n(V'). \tag{8}$$

Then by 1.5 we have

$$y \neq 0. \tag{9}$$

However for $i=1, 2$ one has

$$\tau_{V',V}\tau_{V_i', V'} = \tau_{V_i', V} = \tau_{V_0, V}\tau_{V_i', V}$$

and thus by (8) and (7) we get

$$\tau_{V',V}(y) = \tau_{V_0,V}\big(\tau_{V_1', V_0}(e_1) + \tau_{V_2', V_0}(e_2)\big) = 0.$$

Hence by (9) $\tau_{V',V}$ fails to be a monomorphism as required.

2. k-manifolds

We now collect the notions related to k-manifolds and recall the fundamental theorems proved in the article [2].

Let R be a T_2-space and (X, A) a compact pair in R, i.e. X is a compact subspace of R and A a closed subspace of X.

2.1. Definition. Let V be a domain i.e. a connected nonempty open subspace of R. We say that V is a *regularly intersecting domain of* (X, A) if

(a) $V \cap A = \emptyset$,

(b) $V \cap X$ is a domain of $X \setminus A$.

If V is a regularly intersecting domain of (X, A) and $U = V \cap X$ we then say that V *regularly intersects the compact pair* (X, A) *in* U.

2.2. DEFINITION. A domain V of R is said to be *k-regular* mod (X, A) if the following conditions are fulfilled:

(a) V is a regularly intersecting domain of (X, A).

(b) $V \setminus X$ consists of two components.

(c) The closure of each component of $V \setminus X$ contains $V \cap X$.

2.3. DEFINITION. The compact pair (X, A) itself is called a *k-manifold in R* if it satisfies the following two conditions:

(a) $X \setminus A$ is a nonempty connected space,

(b) for every $q \in X \setminus A$ the k-regular domains that contain the point q form a basis for the neighbourhood system of the point q in R.

2.4. Observe as a direct consequence of 2.3 that for a k-manifold (X, A) in R the subspace $X \setminus A$ is a locally connected domain of X.

Observe also that for a k-manifold (X, A) in R the set $X \setminus A$ is clearly nowhere dense in R.

Now we recall two theorems proved in [2].

In the remainder of this section let (X, A) be a k-manifold in R.

2.5. THEOREM. *Let V be a domain in R regularly intersecting (X, A) and such that $V \setminus X$ is nonconnected. Then V is a k-regular domain* mod (X, A) (see [2] 3.4.)

2.6. THEOREM. *Let V be a k-regular domain* mod (X, A) *and let V' be a domain in R confined to V and regularly intersecting (X, A). Then V' is a k-regular domain* mod (X, A) (see [2] 3.5).

We now recall a definition from [2].

2.7. DEFINITION. A k-regular domain V mod (X, A) is said to be a *subdividing domain of* (X, A) if the two components of $V \setminus X$ are contained in the same component of $R \setminus X$.

In connection with this definition we recall a theorem.

2.8. THEOREM. *If at least one* mod (X, A) *k-regular domain is a subdividing domain of* (X, A) *then each* mod (X, A) *k-regular domain has this property* (see [2] Theorem 4.1).

Finally we give the definition of the bounded and closed k-manifold.

2.9. DEFINITION. (X, A) is said to be a *bounded* (*closed* respectively) *k-manifold* if its k-regular domains are subdividing (non subdividing respectively) domains of (X, A).

2.10. REMARK. Suppose that the space R is locally connected.

Let (X, A) be a k-manifold in R and let V be the component of $R \setminus A$ containing $X \setminus A$. Since $X \setminus A$ is connected and nonempty it follows the existence of such a component V and V is clearly a domain in R. It is clearly a regularly intersecting domain of (X, A) and it contains all mod (X, A) k-regular domains.

Now (X, A) is a bounded k-manifold if and only if $V \setminus X$ is connected.

Indeed, suppose that $V \setminus X$ is connected. Then $V \setminus X$ is a component of $R \setminus X$. Let V' be a k-regular domain mod (X, A). Then $V' \subset V$ and thus $V' \setminus X \subset$

$\subset V \setminus X$. V' is a subdividing domain of (X, A) and thus (X, A) is a bounded k-manifold.

Now suppose that $V \setminus X$ is nonconnected. Then by 2.5 V is a mod (X, A) k-regular domain. On the other hand the components of $V \setminus X$ are certain components of $R \setminus X$. Hence the two components of $V \setminus X$ can not be contained in the same component of $R \setminus X$. V is not a subdividing domain of (X, A). (X, A) is not a bounded k-manifold as required.

3. k-manifolds in R^{n+1}

Let n be a positive integer and R^{n+1} the euclidean $(n+1)$-space.

3.1. We first remark that by 1.4, R^{n+1} is an $(n+1)$-cpm.

We shall make connection between the k-manifolds and certain n-cpm-s in R^{n+1}. To prepare it we make a preliminary remark.

3.2. Let M be a closed and P an open subset of a topological space Q such that $M \subset P$. Suppose that Q is connected and $Q \setminus M$ has two components Q_1 and Q_2. Then P meets both Q_1 and Q_2 and thus $P \setminus M$ is nonconnected.

Indeed Q_1 and Q_2 are open subsets of Q. If P did not meet both Q_1 and Q_2 (say $P \cap Q_2 = \emptyset$), then $(P \cup Q_1) \cup Q_2$ would be a decomposition of Q in two disjoint nonempty open subsets. But this is impossible since Q is connected.

3.3. Theorem. *Let (X, A) be a compact pair in R^{n+1} such that $X \setminus A$ is a connected n-cpm* (see 1.2). *Then (X, A) is a k-manifold in R^{n+1}.*

Proof. Let $q \in X \setminus A$ and let W be a neighbourhood of q in R^{n+1}. Let G be a spherical neighbourhood of q lying in $W \setminus A$. Let U be the component of $G \cap X = G \cap (X \setminus A)$ containing q. U is open in $X \setminus A$ by 1.17 and it is clearly a closed subspace of G. By 1.24 we have

$$H^n(U) = Z_2. \tag{10}$$

On the other hand 1.9 shows that

$$H^{n+1}(U) = 0. \tag{11}$$

However G is homeomorphic to R^{n+1} and thus

$$H^{n+1}(G) = Z_2, \tag{12}$$

$$H^n(G) = 0 \tag{13}$$

(see [5] p. 46). Consider the segment

$$H^n(G) \to H^n(U) \to H^{n+1}(G \setminus U) \to H^{n+1}(G) \to H^{n+1}(U)$$

of the cohomology sequence of the pair (G, U). By (13), (10), (12) and (11) this segment has the form

$$0 \to Z_2 \to H^{n+1}(G \setminus U) \to Z_2 \to 0$$

and thus by the exactness of the cohomology sequence (see 1.6) we have

$$H^{n+1}(G \setminus U) = Z_2 \oplus Z_2. \tag{14}$$

Consequently $G \setminus U \neq \emptyset$. $G \setminus U$ is a nonempty open subspace of the $(n+1)$-cpm R^{n+1} (cf. 3.1) and thus by (14) and 1.28 $G \setminus U$ consists of two components.

Now let V be a domain in R^{n+1} such that $V \subset G$ and $V \cap X = U$. There clearly exists such a domain V. Taking also 3.1 and 1.24 into account we obtain

$$H^{n+1}(V) = Z_2. \tag{15}$$

U is clearly a closed subset of V and thus the cohomology sequence of the pair (V, U) is exact (see 1.6). Consider the segment

$$H^n(U) \to H^{n+1}(V \setminus U) \to H^{n+1}(V) \to H^{n+1}(U)$$

of this cohomology sequence. According to (10), (15) and (11) this segment has the form

$$Z_2 \to H^{n+1}(V \setminus U) \to Z_2 \to 0.$$

Consequently we have either

$$H^{n+1}(V \setminus U) = Z_2$$

or else

$$H^{n+1}(V \setminus U) = Z_2 \oplus Z_2.$$

Hence $V \setminus U$ is a nonempty open subset of R^{n+1}.

Now let $Q = G$, $P = V$ and $M = U$. Then by 3.2, $P \setminus M = V \setminus U$ is nonconnected and thus by 1.28 $H^{n+1}(V \setminus U) = Z_2$ can not occur. Consequently $H^{n+1}(V \setminus U) = Z_2 \oplus Z_2$ and thus by 1.28 $V \setminus U = V \setminus X$ has two components, say V_1 and V_2.

To prove the theorem we only need to show that V is a k-regular domain $\bmod (X, A)$ (see 2.3 and 1.2). As we have seen above conditions 2.2(a) and 2.2(b) are satisfied for the domain V. We are going to show that condition 2.2(c) is satisfied as well.

Let $q' \in U$. We have to show that q' is a limit point of both V_1 and V_2.

Let W' be a neighbourhood of q' in R^{n+1} and let U' be a connected subset of U such that $q' \in U' \subset W'$ and U' is open in $X \setminus A$. Since $X \setminus A$ is locally connected (see 1.17) it follows the existence of such a U'. Let V' be an open subset of R^{n+1} such that $V' \subset W' \cap V$ and $V' \cap X = U'$. There clearly exists such a V'. Now $U \setminus U'$ is closed in U and thus it is closed in V. Hence $Q' = V \setminus (U \setminus U')$ is open in R^{n+1} and $U' = Q' \cap X = Q' \cap U$ is a closed subset of Q'.

Consider the segment

$$H^n(U \setminus U') \to H^{n+1}(Q') \to H^{n+1}(V) \to H^{n+1}(U \setminus U')$$

of the cohomology sequence of the pair $(V, U \setminus U')$. By 1.22 we have $H^n(U \setminus U') = H^{n+1}(U \setminus U') = 0$ and thus by the exactness of the cohomology sequence in question (see 1.6) taking also (15) into account we get

$$H^{n+1}(Q') = H^{n+1}(V) = Z_2.$$

Hence the open set Q' of R^{n+1} is nonempty and by 3.1 and 1.28 it is connected.

Since $Q' \setminus U' = V \setminus U$ it follows that $Q' \setminus U'$ has two components and these are V_1 and V_2.

On the other hand by $V'\subset V$ and $V'\cap X=U'$ we have $U'\subset V'\subset Q'$. Let $P'=V'$ and $M'=U'$. By 3.2 $P'=V'$ meets both components of $Q'\setminus M'=Q'\setminus U'$ and thus taking also $V'\subset W'$ into account W' meets both V_1 and V_2 as well. q' is a limit point of both V_1 and V_2 as required. The closure of each component of $V\setminus X$ contains $V\cap X=U$.

We have proved that V is a k-regular domain $\bmod (X, A)$.

The proof of the theorem is complete.

3.4. THEOREM. *Let (X, A) be a k-manifold in R^{n+1}. Then $X\setminus A$ is an n*-cpm.

PROOF. $X\setminus A$ is clearly a nonempty locally compact T_2-space.

We now introduce a new concept.

An open ball G of R^{n+1} is said to be *normal* $\bmod (X, A)$ if it is contained in a k-regular domain $\bmod (X, A)$ and $G\cap X\neq\emptyset$.

Observe that since each normal ball G is contained in a k-regular domain V and $V\cap A=\emptyset$ we have $G\cap A=\emptyset$ and thus $G\cap X=G\cap(X\setminus A)$.

Let $\mathscr{B}$ be the family of all components of the sets $G\cap(X\setminus A)=G\cap X$ where G runs over all normal balls $\bmod (X, A)$. Since $X\setminus A$ is locally connected (see 2.4) it follows that $\mathscr{B}$ is a basis of $X\setminus A$ (cf. also 2.3(b)).

To prove the theorem we only need to show that each $U\in\mathscr{B}$ satisfies conditions 1.2(a) and 1.2(b).

Let G be a normal ball $\bmod (X, A)$ and V a k-regular domain $\bmod (X, A)$ containing G. Let U be a component of $G\cap X=G\cap(X\setminus A)$. Let V' be a domain in R^{n+1} such that $U\subset V'\subset G\subset V$ and $V'\cap X=U$. There clearly exists such a V' and V' is a regularly intersecting domain of (X, A). Since $V'\subset V$, it follows by 2.6 that V' is a k-regular domain $\bmod (X, A)$ and thus $V'\setminus X=V'\setminus U$ consists of two components. Now consider the segment

$$H^n(U)\to H^{n+1}(V'\setminus U)\to H^{n+1}(V') \tag{16}$$

of the cohomology sequence of the pair (V', U). V' and $V'\setminus U$ are nonempty open subsets of the $(n+1)$-cpm R^{n+1} (see 3.1) where V' is connected and $V'\setminus U$ has two components. Thus according to 1.28, $H^{n+1}(V')=Z_2$ and $H^{n+1}(V'\setminus U)=Z_2\oplus Z_2$. Consequently the segment (16) is of the form

$$H^n(U)\to Z_2\oplus Z_2\to Z_2$$

and since the cohomology sequence in question is exact (see 1.6) it follows

$$H^n(U)\neq 0. \tag{17}$$

$(\overline{V}, \overline{U}\setminus U)$ is clearly a k-manifold in R^{n+1} as well and G is a regularly intersecting domain of $(\overline{U}, \overline{U}\setminus U)$.

Since $U\subset X\setminus A$ and $X\setminus A$ is nowhere dense in R^{n+1} (see 2.4) it follows $G\neq U$. Thus U is a proper closed subset of G, consequently by 1.23 and 3.1 we have

$$H^{n+1}(U)=0. \tag{18}$$

Consider the segment

$$H^n(G)\to H^n(U)\to H^{n+1}(G\setminus U)\to H^{n+1}(G)\to H^{n+1}(U)$$

of the cohomology sequence of the pair (G, U). By (18), 3.3(13) and 3.3(12) this segment is of the form

$$0 \to H^n(U) \to H^{n+1}(G \setminus U) \to Z_2 \to 0 \tag{19}$$

and thus by the exactness of the sequence (see 1.6) and by (17) we have $H^{n+1}(G\setminus U) \neq Z_2$. Hence by 1.28, $G\setminus U$ is nonconnected and thus by 2.5, G is a k-regular domain $\bmod(\bar{U}, \bar{U}\setminus U)$. Consequently $G\setminus U$ consists of two components, say G_1 and G_2, and this implies by 1.28

$$H^{n+1}(G\setminus U) = Z_2 \oplus Z_2.$$

Hence by the exactness of the sequence (19) we obtain $H^n(U)=Z_2$.

Condition 1.2(a) is fulfilled for $U \in \mathcal{B}$.

Now let U' be a nonempty open subset of U. Since G is a k-regular domain $\bmod(\bar{U}, \bar{U}\setminus U)$ and $U=G\cap\bar{U}$ it follows that U' is on the boundary of both G_1 and G_2 (see 2.2(c)). Hence $G_1 \cup U'$ and $G_2 \cup U'$ are connected sets and thus $Q=G_1\cup G_2\cup U'$ is connected as well. However Q is open in R^{n+1} and $G\setminus Q = U\setminus U'$. Consequently $U\setminus U'$ is a proper closed subset of G and thus by 3.1 and 1.23 we have

$$H^{n+1}(U\setminus U') = 0. \tag{20}$$

Consider now the segment

$$H^n(G) \to H^n(U\setminus U') \to H^{n+1}(Q) \to H^{n+1}(G) \to H^{n+1}(U\setminus U') \tag{21}$$

of the cohomology sequence of the pair $(G, U\setminus U')$. G and Q are domains in R^{n+1} and thus by 3.1 and 1.24 we have

$$H^{n+1}(G) = H^{n+1}(Q) = Z_2.$$

On the other hand 3.3(13) shows that $H^n(G)=0$. Thus taking also (20) into accoun the segment (21) is of the form

$$0 \to H^n(U\setminus U') \to Z_2 \to Z_2 \to 0$$

and thus by the exactness of the cohomology sequence in question (see 1.6) we get $H^n(U\setminus U')=0$.

Now U' is a proper closed subset of the domain Q of R^{n+1}. Hence by 3.1 and 1.23 we have $H^q(U')=0$ for $q \geqq n+1$, i.e. for all $q>n$.

Condition 1.2(b) is fulfilled for $U \in \mathcal{B}$ as well.

The proof of the theorem is complete.

3.5. Theorem. *The k-manifold (X, A) in R^{n+1} is bounded if and only if the homomorphism $i^*\colon H^n(X) \to H^n(A)$ induced by the inclusion $i\colon A \subset X$ is a monomorphism.*

Proof. Let V be the component of $R^{n+1}\setminus A$ containing $X\setminus A$. According to 2.10, (X, A) is a bounded k-manifold if and only if $V\setminus X$ is connected.

Consequently (X, A) is a bounded k-manifold if and only if distinct components of $R^{n+1}\setminus X$ lie in distinct components of $R^{n+1}\setminus A$.

Hence, introducing the notations $P=R^{n+1}\setminus A$ and $Q=R^{n+1}\setminus X$, and taking also 3.1, 1.30 and 1.31 into account, we find that (X, A) is a bounded k-manifold if and only if $\tau_{Q,P}\colon H^{n+1}(Q)\to H^{n+1}(P)$ is a monomorphism.

Observe that $\tau_{P,R^{n+1}}\tau_{Q,P}=\tau_{Q,R^{n+1}}$ where $\tau_{P,R^{n+1}}=\tau_{P,R^{n+1}}\colon H^{n+1}(P)\to \to H^{n+1}(R^{n+1})$ and $\tau_{Q,R^{n+1}}=\tau_{Q,R^{n+1}}\colon H^{n+1}(Q)\to H^{n+1}(R^{n+1})$ (see [5] p. 13). Hence $\ker\tau_{Q,P}\subset\ker\tau_{Q,R^{n+1}}$ and thus $\ker\tau_{Q,P}=\ker(\tau_{Q,P}|\ker\tau_{Q,R^{n+1}})$, i.e. $\tau_{Q,P}$ and the restriction $\tau_{Q,P}|\ker\tau_{Q,R^{n+1}}$ have the same kernel. Consequently we can state that (X, A) is a bounded k-manifold if and only if $\tau_{Q,P}|\ker\tau_{Q,R^{n+1}}$ is a monomorphism.

Consider now the segments

$$H^n(R^{n+1})\to H^n(A)\xrightarrow{\delta_{R^{n+1},A}} H^{n+1}(P),$$

$$H^n(R^{n+1})\to H^n(X)\xrightarrow{\delta_{R^{n+1},X}} H^{n+1}(Q)\xrightarrow{\tau_{Q,R^{n+1}}} H^{n+1}(R^{n+1})$$

of the cohomology sequences of the pairs (R^{n+1}, A) and (R^{n+1}, X). Since these sequences are exact (see 1.6) and $H^n(R^{n+1})=0$ (see [5] p. 46) it follows that $\delta_{R^{n+1},A}$ and $\delta_{R^{n+1},X}$ are monomorphisms and the relation

$$\operatorname{im}\delta_{R^{n+1},X}=\ker\tau_{Q,R^{n+1}} \tag{22}$$

holds. Also, consider the commutative diagram

$$\begin{array}{ccc} H^n(X) & \xrightarrow{i^*} & H^n(A) \\ \downarrow{\scriptstyle \delta_{R^{n+1},X}} & & \downarrow{\scriptstyle \delta_{R^{n+1},A}} \\ H^{n+1}(Q) & \xrightarrow{\tau_{Q,P}} & H^{n+1}(P) \end{array}$$

(cf. Theorem I.1.6(2) of [5] p. 18). Since $\delta_{R^{n+1},X}$ and $\delta_{R^{n+1},A}$ are monomorphisms, taking also (22) into account it follows that i^* is a monomorphism if and only if so is

$$\tau_{Q,P}|\operatorname{im}\delta_{R^{n+1},X}=\tau_{Q,P}|\ker\tau_{Q,R^{n+1}},$$

i.e., as we have seen above, if and only if (X, A) is bounded k-manifold.

The proof of the theorem is complete.

4. The main theorems

We are going to prove the theorems of [2].

4.1. First observe that a compact pair (X, A) in R^1 is a k-manifold in R^1 if and only if $X\setminus A$ is a singleton.

We have to mention that A need not be the empty set. The respective assertion in [2], namely that A must be empty (see [2] § 5), is false.

Observe also that each k-manifold (X, A) in R^1 is clearly a closed k-manifold in R^1.

4.2. THEOREM. *Let n be a positive integer and let (X, A) be a compact pair in R^{n+1} such that $X\setminus A$ is a connected n-manifold. Then (X, A) is a k-manifold in R^{n+1}.*

This is Theorem 5.2 in [2] and it is an immediate consequence of 1.4 and 3.3.

4.3. THEOREM. *Let* (X, A) *be a k-manifold in* R^n $(n \geqq 1)$ *and let* (Y, B) *be a compact pair in* R^n *homeomorphic to* (X, A). *Then* (Y, B) *is a k-manifold in* R^n *as well. Moreover if* (X, A) *is a bounded k-manifold then so is* (Y, B).

This is Theorem 5.5 in [2] and it is an immediate consequence of 4.1 and of Theorems 3.3, 3.4 and 3.5 (cf. also 2.3(a)).

4.4. THEOREM. *Let* (X, A) *be a k-manifold in* R^{n+1} $(n \geqq 1)$ *such that* $A = \emptyset$. *Then* (X, A) *is a closed k-manifold in* R^{n+1}. *Moreover* $R^{n+1} \setminus X$ *consists of two components and* X *is the boundary of these components.*

PROOF. According to 3.4 and 1.25, taking also $A = \emptyset$ and 2.3(a) into account, we have $H^n(X) = H^n(X \setminus A) = Z_2$ and $H^n(A) = 0$. Hence the homomorphism $i^*\colon H^n(X) \to H^n(A)$ induced by the inclusion $i\colon A \subset X$ fails to be a monomorphism. Consequently by 3.5 (X, A) is a closed k-manifold in R^{n+1}. However R^{n+1} is the only component of $R^{n+1} \setminus A = R^{n+1}$ and thus by 2.10, $R^{n+1} \setminus X$ is nonconnected.

On the other hand R^{n+1} is a regularly intersecting domain of (X, A) (cf. 2.1 and 2.3(a)) and thus by 2.5 R^{n+1} is a k-regular domain mod (X, A). Hence $R^{n+1} \setminus X$ consists of two components and the closure of both components contains $R^{n+1} \cap \cap X = X$ (cf. 2.2). Thus the boundary of each component of $R^{n+1} \setminus X$ is X itself indeed.

Observe that this Theorem 4.4 is the same as Theorem 5.4 in [2].

We are going now to prove Theorem 5.3 of [2]. First we introduce a new notion.

4.5. DEFINITION. Let n be a positive integer. Let (X, A) be a compact pair and $p \in A$. p is said to be an n-euclidean boundary point of (X, A) if there exists a neighbourhood W of p in X and a homeomorphism φ of W onto the halfball

$$\{x = (x_1, \ldots, x_n);\ (x_1)^2 + \ldots + (x_n)^2 < 1,\ x_n \geqq 0\}$$

of R^n such that

$$\varphi(W \cap A) = \Big\{x = (x_1, \ldots, x_n);\ \sum_{i=1}^{n} (x_i)^2 < 1, x_n = 0\Big\}.$$

Theorem 5.3 of [2] reads as follows.

4.6. THEOREM. *Let* (X, A) *be a k-manifold in* R^{n+1} $(n \geqq 1)$ *with at least one n-euclidean boundary point. Then* (X, A) *is a bounded k-manifold in* R^{n+1}.

PROOF. Let p be an n-euclidean boundary point of (X, A) and let W and φ be the same as in 4.5. Without loss of generality we can clearly suppose that $\varphi(p) = = 0 = (0, \ldots, 0)$. Let $\operatorname{int} W$ be the interior of W in X. Then $p \in \operatorname{int} W$ and $\varphi(\operatorname{int} W)$ is an open neighbourhood of 0 in $\varphi(W)$. Hence there is a positive real ε such that

$$Q = \Big\{x \in \varphi(W);\ \sum_{i=1}^{n} (x_i)^2 < \varepsilon^2\Big\} \subset \varphi(\operatorname{int} W).$$

Let $W' = \varphi^{-1}(Q)$ and $\varphi' = \varphi|_{W'}\colon W' \to Q$. W' is an open neighbourhood of p in X and $\varphi'\colon W' \to Q$ is a homeomorphism. Since

$$Q \setminus \varphi'(W' \cap A) = Q \setminus \varphi(W \cap A) \neq \emptyset$$

it follows

(23) $$W'\setminus A = W'\cap(X\setminus A) \neq \emptyset.$$

Q is a proper closed subset of the open ball

$$\{x = (x_1, \ldots, x_n);\ (x_1)^2+\ldots+(x_n)^2 < \varepsilon^2\}$$

of R^n and since R^n is an n-dimensional manifold by 1.4 and 1.23 we have $H^n(Q)= H^{n+1}(Q)=0$ and thus

(24) $$H^n(W') = H^{n+1}(W') = 0.$$

Q is connected, consequently W' is a domain in X. Hence there is a domain V' in R^{n+1} such that $V'\cap X=W'$ and thus W' is a closed subset of V'. By 3.1 and 1.24 we have

(25) $$H^{n+1}(V') = Z_2.$$

Consider now the segment

$$H^n(W') \to H^{n+1}(V'\setminus W') \to H^{n+1}(V') \to H^{n+1}(W')$$

of the cohomology sequence of the pair (V', W'). By (24) and (25) this segment has the form

$$0 \to H^{n+1}(V'\setminus W') \to Z_2 \to 0$$

and thus by the exactness of the cohomology sequence (see 1.6) we get $H^{n+1}(V'\setminus W')=Z_2$.

Consequently $V'\setminus W'\neq\emptyset$ and taking also 3.1 and 1.28 into account we find that $V'\setminus W'=V'\setminus X$ is a domain in R^{n+1}.

Let $q\in W'\cap(X\setminus A)=V'\cap(X\setminus A)$. By (23) there exists such a point q. Let V be a k-regular domain mod (X, A) such that $q\in V\subset V'$. By 2.3(b) there exists such a domain V. Since the two component of $V\setminus X$ lie in the connected set $V'\setminus W'=V'\setminus X$ it follows that the two components of $V\setminus X$ lie in the same component of $R^{n+1}\setminus X$. (X, A) is a bounded k-manifold, indeed (cf. 2.9, 2.7 and 2.8).

The proof of the theorem is complete.

Finally we prove Theorem 5.1 of [2].

4.7. THEOREM. *Let (X, A) be a k-manifold in R^2. Then $X\setminus A$ is either a closed Jordan curve or it is homeomorphic to the real line R^1.*

PROOF. The proof proceeds in several steps.

(a) There is no open subspace of $X\setminus A$ which is a singleton or which is homeomorphic to a closed halfline.

Indeed, the singleton and the closed halfline may be considered as proper closed subspaces of the connected 1-euclidean manifold R^1. Hence for such a subspace W by 1.4 and 1.23 we have $H^1(W)=0$ while for each nonempty open subspace W' of $X\setminus A$ by 3.4 and 1.26 we have $H^1(W')\neq 0$.

(b) A triode is a space homeomorphic to the union of three distinct segments of R^2 issuing from the same point and lying on distinct lines.

(c) There is no triode in $X\setminus A$.

We argue by contradiction.

Suppose the existence of a triode in $X\setminus A$. That means there is a point q and three simple arcs v_1, v_2, v_3 in $X\setminus A$ such that q is the endpoint of each arc v_i, $i=1,2,3$, and for $i\neq j$ $(i, j\in\{1,2,3\})$ we have $v_i\cap v_j=\{q\}$. For $i=1,2,3$ let r_i be that endpoint of v_i for which $r_i\neq q$. Let $K(q,\varepsilon)$ be a closed circular disc around q such that

$$A\cup\{r_1, r_2, r_3\}\subset R^2\setminus K(q,\varepsilon)$$

and let $C(q,\varepsilon)$ be the boundary circle of this disc. Going from q to r_i on the arc v_i let r_i' be the first point of $v_i\cap C(q,\varepsilon)$ and let v_i' be the subarc of v_i with the endpoints q and r_i'. Then by the Jordan curve theorem and the decomposition theorem (see [1] Zerlegungssatz, Korollar p. 390) the compact set $C(q,\varepsilon)\cup v_1'\cup v_2'\cup v_3'$ divides the plane in four domains D_1, D_2, D_3, D_4 such that q is on the boundary of three of them, say

$$q\in\bar{D}_i \quad \text{for} \quad i=1,2,3 \tag{26}$$

and $D_4=R^2\setminus K(q,\varepsilon)$. Hence D_1, D_2, D_3 lie in $K(q,\varepsilon)$.

Now let V be a k-regular domain mod (X, A) containing q and lying in $K(q,\varepsilon)$. By 2.3(b) there is such a domain V and

$$V\setminus X=((D_1\cap V)\setminus X)\cup((D_2\cap V)\setminus X)\cup((D_3\cap V)\setminus X).$$

However $X\setminus A$ is nowhere dense in R^2 (see 2.4) and thus taking also (26) into account for $i=1,2,3$ we have

$$(D_i\cap V)\setminus X=(D_i\cap V)\setminus(X\setminus A)\neq\emptyset.$$

Consequently $V\setminus X$ is the union of three nonempty pairwise disjoint open sets. But this is impossible by 2.2(b).

Hence there is no triode in $X\setminus A$ indeed.

(d) $X\setminus A$ is completely metrizable.

Indeed, this is obvious if $X\setminus A$ is compact. On the other hand if $X\setminus A$ is noncompact then its one point (Alexandroff) compactification (see [4] p. 222) is a compact metrizable space and each metric of this space is complete. Moreover $X\setminus A$ is an open subspace of this complete metric space. However since complete metrizability is hereditary with respect to G_δ sets (Theorem 4.3.23 of [4] p. 342) it follows that $X\setminus A$ is completely metrizable as required.

(e) Taking also (a) and (c) into account $X\setminus A$ can be considered as a separable connected and locally connected complete metric space which is not a singleton and fails to contain any triode. Hence according to a theorem of Á. Császár (see [3]), $X\setminus A$ is homeomorphic either to a circle or to the line R^1 or to a segment of R^1 or to a closed halfline of R^1.

However according to (a) $X\setminus A$ is not homeomorphic to a closed halfline or to a segment of R^1. Hence $X\setminus A$ is either a closed Jordan curve or it is homeomorphic to the real line R^1 as required.

The proof of the theorem is complete.

All theorems of [2] are proved. Our program is finished.

References

[1] P. Alexandroff—H. Hopf, *Topologie* (Berlin, 1935).
[2] M. Bognár, Über Lageeigenschaften verallgemeinerter Mannigfaltigkeiten, *Acta Math. Acad. Sci. Hung.*, **24** (1973), 179—198.
[3] Á. Császár, Sur les courbes atriodiques, *Acta Math. Acad. Sci. Hung.*, **9** (1958), 329—332.
[4] R. Engelking, *General Topology* (Warszawa, 1977).
[5] W. S. Massey, *Homology and Cohomology Theory* (New York and Basel, 1978).

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
H—1088 BUDAPEST
MÚZEUM KRT. 6—8

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ILIADIS ABSOLUTES FOR ARBITRARY SPACES

Á. CSÁSZÁR (Budapest)*, member of the Academy

0. Introduction. According to V. I. Ponomarev [2], the absolute of a Hausdorff space X is an extremally disconnected (see Section 1 below for the terminology) Hausdorff space PX such that X is the image of PX under an ultraperfect map; this property determines PX up to a homeomorphism. S. Iliadis [1] has constructed, for a Hausdorff space X again, an extremally disconnected T_3-space EX such that X is the image of EX under a ϑ-perfect map; uniqueness up to a homeomorphism is still valid. (For more historical details, see [5].)

The construction of PX has been extended for arbitrary topological spaces by V. M. Ul'janov [4]; it is an extremally disconnected space whose X is the image under an ultraperfect and separated map.

The purpose of the present paper is to define EX for arbitrary topological spaces X and to show that its fundamental properties remain valid with suitable small modifications.

1. Preliminaries. A topological space is said to be *extremally disconnected* (EDC) iff the closure of any open set is open. In an EDC space, if G and H are disjoint open subsets, then $\overline{G}\cap\overline{H}=\emptyset$. An EDC space is zero-dimensional (i.e. has a base composed of clopen sets) iff it is regular (this being understood without postulating T_0). A dense subspace of an EDC space is EDC as well, and it is C^*-embedded (i.e. every continuous map from the subspace into a compact Hausdorff space possesses a continuous extension to the whole space).

Let X and Y be topological spaces, and $f\colon X\to Y$. The map f is said to be *closed* iff $f(F)$ is closed in Y provided F is closed in X. It is said to be *compact* iff $f^{-1}(y)$ is compact for $y\in Y$. It is said to be *irreducible* iff it is surjective and $f(F)\neq Y$ whenever $F\neq X$ is closed in X. A continuous, closed, compact, irreducible map will be called *ultraperfect* (perfect usually means closed and compact, sometimes continuous, closed and compact).

The map f is said to be *ϑ-continuous* iff, for $x\in X$, $f(x)\in V\subset Y$, V open in Y, there is an open $U\subset X$ such that $x\in U$ and $f(\overline{U})\subset\overline{V}$. A continuous map is ϑ-continuous; the converse is true provided Y is regular. The composition of two ϑ-continuous maps is ϑ-continuous as well. A ϑ-continuous, closed, compact, irreducible map will be called *ϑ-perfect*.

The map f is said to be *separated* iff $x_1, x_2\in X$, $x_1\neq x_2$, $f(x_1)=f(x_2)$ imply that

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x_1 and x_2 have disjoint neighbourhoods in X. If X is T_2, then every map $f\colon X\to Y$ is separated.

Let $f\colon X\to Y$ be surjective. For $A\subset X$, we define

$$\hat{f}(A)=Y-f(X-A).$$

Then $\hat{f}(A)\subset f(A)$, $f^{-1}(\hat{f}(A))\subset A$, $\hat{f}(X)=Y$,

$$\hat{f}(A\cap B)=\hat{f}(A)\cap\hat{f}(B).$$

If f is closed and $G\subset X$ is open, then $\hat{f}(G)$ is open in Y; if f is closed and irreducible, then $\hat{f}(G)\neq\emptyset$ provided $G\neq\emptyset$ is open in X.

Let X be a topological space. A filter in X is said to be *open* iff it is generated by a filter base composed of open sets. A maximal open filter is said to be *ultraopen*. Every system of open sets that is centred (i.e. has the finite intersection property) is contained in an ultraopen filter. An open filter $\mathfrak{s}$ is ultraopen iff $G\in\sec\mathfrak{s}$ implies $G\in\mathfrak{s}$ whenever G is open; here

$$\sec\mathfrak{s}=\{A\subset X\colon A\cap S\neq\emptyset \text{ for every } S\in\mathfrak{s}\}.$$

Consequently, if $\mathfrak{s}$ is an ultraopen filter and A is either open or closed, then either A or $X-A$ belongs to $\mathfrak{s}$. If $\mathfrak{s}$ is an ultraopen filter and $G\subset X$ is open, then $G\in\mathfrak{s}$ iff $\bar{G}\in\mathfrak{s}$. If $\mathfrak{s}_1\neq\mathfrak{s}_2$ are ultraopen filters, then there exist open sets $G_i\in\mathfrak{s}_i$ such that $G_1\cap G_2=\emptyset$. Every $x\in X$ is limit of at least one ultraopen filter (because the neighbourhood filter of x is open); the space X is EDC iff every $x\in X$ is limit of one and only one ultraopen filter.

Let us denote by UX the set of all ultraopen filters in X. The following statements have been formulated in [1] for Hausdorff spaces, but they are easily seen to hold for any space X.

For an open set $G\subset X$, define

$$s(G)=\{\mathfrak{s}\in UX\colon G\in\mathfrak{s}\}.$$

Then $s(\emptyset)=\emptyset$, $s(X)=UX$,

$$s(G_1\cap G_2)=s(G_1)\cap s(G_2),$$

hence the sets $s(G)$ constitute a base for a topology on UX that is compact and Hausdorff. It is also EDC because

$$\overline{\bigcup_{i\in I} s(G_i)}=s\Big(\bigcup_{i\in I} G_i\Big);$$

hence the clopen subsets of UX are precisely those of the form $s(G)$.

A space X is said to be *almost compact* iff, in each open cover of X, there are finitely many members whose union is dense, or equivalently, iff every ultraopen filter is convergent. A compact space is almost compact, and the converse is true if the space is regular. Any product of almost compact spaces is almost compact. The image of an almost compact space under a ϑ-continuous map is almost compact. A T_2-space is almost compact iff it is H-closed.

In a space X, a set R is said to be *r-open (regular open)* iff $R=\operatorname{int}\bar{R}$ or equivalently iff R is the interior of a closed set. In an EDC space, r-open sets coincide with the clopen sets. The space X is said to be *semi-regular* iff there is a base composed of r-open sets. A regular space is semi-regular.

The r-open sets of X constitute the base for a semi-regular topology, coarser than that of X. The set X, equipped with this topology, is said to be the *semi-regularization* of X and will be denoted by RX. The map $\mathrm{id}_X\colon RX\to X$ is ϑ-continuous. If X is *EDC* then so is RX.

2. Construction of EX. Let X be an arbitrary topological space. Denote by IX the same underlying set equipped with the indiscrete topology. Consider the product space $TX=IX\times UX$ and the subspace

$$EX = \{(x, \mathfrak{s})\colon \mathfrak{s} \to x \text{ in } X\} \subset TX.$$

THEOREM 2.1. *TX is a regular EDC space and EX is dense in TX, so EX is regular and EDC as well.*

PROOF. Both IX an UX are regular. The open subsets of TX have the form $X\times H$ where $H\subset UX$ is open, and $\overline{X\times H}=X\times\overline{H}$ is open since UX is EDC. Hence TX is EDC.

Let $(x_0, \mathfrak{s}_0)\in TX$. The sets $X\times s(G)$, where $G\in\mathfrak{s}_0$ is open in X, constitute a neighbourhood base of $(x_0, \mathfrak{s}_0)$. For a set having this form, choose $x\in G$ and an ultraopen filter $\mathfrak{s}$ containing the neighbourhood filter of $x\in X$. Then $\mathfrak{s}\to x$ in X, $G\in\mathfrak{s}\in s(G)$, $(x, \mathfrak{s})\in EX\cap(X\times s(G))$. Thus EX is dense in TX, therefore it is EDC. □

Let $k_X\colon EX\to X$ be defined by

$$k_X(x, \mathfrak{s}) = x.$$

Then we have:

LEMMA 2.2. *For $x_0\in X$, let $\{\mathfrak{s}_i\colon i\in I\}$ be the set of all elements of UX that converge to x_0 in X, and let $G_i\in\mathfrak{s}_i$ be open for $i\in I$. Then there is a finite subset $I_0\subset I$ such that $\bigcup_{i\in I_0}\overline{G_i}$ is a neighbourhood of x_0.*

PROOF. Assume $G-\bigcup_{i\in I_0}\overline{G_i}\neq\emptyset$ for every open set G containing x_0 and every finite set $I_0\subset I$. Then these sets constitute a filter base of open sets that is contained in an ultraopen filter $\mathfrak{s}$. Since $G\in\mathfrak{s}$ for every open neighbourhood of x, necessarily $\mathfrak{s}\to x_0$, and $\mathfrak{s}=\mathfrak{s}_{i_0}$ for some $i_0\in I$. By definition, $X-\overline{G_{i_0}}\in\mathfrak{s}_{i_0}$ which is impossible. □

THEOREM 2.3. *k_X is a ϑ-perfect, separated map.*

PROOF. For $(x_0, \mathfrak{s}_0)\in EX$, let V be an open neighbourhood of $x_0\in X$. Then $V\in\mathfrak{s}_0$, and $U=(X\times s(V))\cap EX$ is a clopen neighbourhood of $(x_0, \mathfrak{s}_0)$ in EX. If $(x, \mathfrak{s})\in U$, then $V\in\mathfrak{s}$ and $\mathfrak{s}\to x$ in X, hence $x\in\overline{V}$. Hence k_X is ϑ-continuous.

k_X is surjective because the neighbourhood filter of any $x\in X$ is contained in some $\mathfrak{s}\in UX$ for which $\mathfrak{s}\to x$, $(x, \mathfrak{s})\in EX$, $k_X(x)=x$.

In order to see that k_X is irreducible, it suffices to show that $D\subset EX$ is dense whenever $k_X(D)=X$. Now if $V=(X\times s(G))\cap EX$, $\emptyset\neq G\subset X$, G open, is a basic open set in EX, choose $x\in G$ and $\mathfrak{s}\in UX$ such that $(x, \mathfrak{s})\in D$. Then $\mathfrak{s}\to x$, $G\in\mathfrak{s}\in s(G)$, $(x, \mathfrak{s})\in D\cap V$.

For $x_0\in X$, assume $k_X^{-1}(x_0)\subset \bigcup_{i\in I} (X\times s(G_i))$ where $G_i\subset X$ is open for $i\in I$. Then every $\mathfrak{s}\in UX$ converging to x_0 is contained in some $s(G_i)$, i.e. $G_i\in\mathfrak{s}$, so that, by 2.2, $V=\bigcup_{i\in I_0} \overline{G_i}$ is a neighbourhood of x_0 for some finite set $I_0\subset I$. Thus, for $\mathfrak{s}\in k_X^{-1}(x_0)$, we have $V\in\mathfrak{s}$, consequently $G_i\in\mathfrak{s}$ for some $i\in I_0$. In fact, otherwise $X-\overline{G_i}$ would belong to $\mathfrak{s}$ for every $i\in I_0$ so that $X-\bigcup_{i\in I_0} \overline{G_i}\in\mathfrak{s}$ would hold which is impossible. Therefore $k_X^{-1}(x_0)\subset \bigcup_{i\in I_0} (X\times s(G_i))$, and k_X is compact.

Suppose now that $F\subset EX$ is closed and $x_0\in X-k_X(F)$. Then $(x_0, \mathfrak{s}_i)\notin F$ whenever $\mathfrak{s}_i\in UX$, $\mathfrak{s}_i\to x_0$ so that, for every i, we can choose an open set $G_i\in\mathfrak{s}_i$ satisfying $(X\times s(G_i))\cap F=\emptyset$. By 2.2, $V=\bigcup_{i\in I_0} \overline{G_i}$ is a neighbourhood of $x_0\in X$ for some finite I_0. If $x\in\operatorname{int} V$ and $\mathfrak{s}\in UX$, $\mathfrak{s}\to x$, then $V\in\mathfrak{s}$ and $G_i\in\mathfrak{s}$ for a suitable $i\in I_0$. Hence $(x, \mathfrak{s})\in X\times s(G_i)$, $(x, \mathfrak{s})\notin F$. Therefore $k_X(F)$ is closed, and k_X is closed.

Finally k_X is separated because, if $(x_i, \mathfrak{s}_i)\in EX$, $(x_1, \mathfrak{s}_1)\neq(x_2, \mathfrak{s}_2)$, $k_X(x_1, \mathfrak{s}_1)=k_X(x_2, \mathfrak{s}_2)$, then $x_1=x_2$ and $\mathfrak{s}_1\neq\mathfrak{s}_2$, thus there are open sets G_i such that $G_i\in\mathfrak{s}_i$, $G_1\cap G_2=\emptyset$, and $X\times s(G_i)$ is a neighbourhood of $(x_i, \mathfrak{s}_i)$ in TX satisfying $(X\times s(G_1))\cap(X\times s(G_2))=\emptyset$. □

Let us denote by $U_c(X)$ the subspace of UX composed of the convergent ultraopen filters.

THEOREM 2.4. *The following statements are equivalent:*

(a) *X is T_2.*

(b) *The projection from TX onto UX, restricted to EX, is a homeomorphism from EX onto U_cX.*

(c) *EX is T_2.*

PROOF. (a)⇒(b): If X is T_2, then the restriction described in (b) is bijective. It is continuous, and it is open since the image of $(X\times s(G))\cap EX$ is $s(G)\cap U_cX$.

(b)⇒(c): Obvious.

(c)⇒(a): If X is not T_2, then there are $x_1, x_2\in X$, $x_1\neq x_2$ such that every open neighbourhood of x_1 intersects every open neighbourhood of x_2. Let $\mathfrak{s}$ be an ultraopen filter containing all these non-empty intersections. Then $(x_1, \mathfrak{s})$ and $(x_2, \mathfrak{s})$ are two points of EX without disjoint neighbourhoods. □

By this, the construction in [1] furnishes in the Hausdorff case a space homeomorphic to EX.

THEOREM 2.5. *The following statements are equivalent:*

(a) *X is regular and* EDC,

(b) *k_X: $EX\to X$ is a homeomorphism,*

(c) *EX and X are homeomorphic.*

PROOF. (a)⇒(b): If X is EDC then k_X is bijective. It is continuous whenever X is regular, and it is closed, too.

(b)⇒(c)⇒(a): Obvious. □

THEOREM 2.6. *A space X is almost compact iff EX is compact.*

PROOF. X is a ϑ-continuous image of EX. On the other hand, if X is almost compact, then $U_c X = UX$, hence $EX \subset \bigcup_{i\in I} (X \times s(G_i))$, G_i open in X, implies $UX \subset \subset \bigcup_{i\in I} s(G_i)$ and then, by the compactness of UX, finitely many sets $s(G_i)$ cover EX. □

3. The mapping $\tilde{f}$. In this section, let X and Y be topological spaces, $f\colon X \to Y$ ϑ-continuous, closed and irreducible. The statements 3.1—3.5 are formulated in [1] for T_2-spaces X and Y with very short hints of proofs.

LEMMA 3.1. *If $\mathfrak{s} \in UX$, then the system*

$$\{\hat{f}(G)\colon G \in \mathfrak{s} \text{ is open}\} \tag{3.1.1}$$

is a filter base in Y that is contained in one and only one $\mathfrak{t} \in UY$. For an open set $H \subset Y$,

$$H \in \mathfrak{t} \quad \text{iff} \quad \operatorname{int} f^{-1}(\bar{H}) \in \mathfrak{s}. \tag{3.1.2}$$

PROOF. Since f is irreducible and closed, (3.1.1) is a filter base composed of open sets in Y. Assume $\mathfrak{t} \in UY$ contains (3.1.1), and let H be open in Y. Then

$$\operatorname{int} \bar{H} = \hat{f}(\operatorname{int} f^{-1}(\bar{H})). \tag{3.1.3}$$

In fact, let $y \in \operatorname{int} \bar{H}$, $y = f(x)$ for some $x \in X$. By the ϑ-continuity of f, there is an open set $G \subset X$ such that $x \in G$ and

$$f(\bar{G}) \subset \overline{\operatorname{int} \bar{H}} \subset \bar{H}.$$

Then $G \subset f^{-1}(\bar{H})$ and $G \subset \operatorname{int} f^{-1}(\bar{H})$, so that

$$x \notin X - \operatorname{int} f^{-1}(\bar{H}), \quad y \in Y - f(X - \operatorname{int} f^{-1}(\bar{H})) = \hat{f}(\operatorname{int} f^{-1}(\bar{H})).$$

Hence $\subset$ holds in (3.1.3). On the other hand, the right-hand side of (3.1.3) is an open set contained in $\bar{H}$.

Now if $H \in \mathfrak{t}$ is open, then $\operatorname{int} \bar{H} \in \mathfrak{t}$ so that $\operatorname{int} \bar{H}$ intersects every set $\hat{f}(G)$ with $G \in \mathfrak{s}$ open, i.e., by (3.1.3), $\operatorname{int} f^{-1}(\bar{H})$ intersects every such G, and $\operatorname{int} f^{-1}(\bar{H}) \in \sec \mathfrak{s}$, $\operatorname{int} f^{-1}(\bar{H}) \in \mathfrak{s}$. Conversely, if $\operatorname{int} f^{-1}(\bar{H}) \in \mathfrak{s}$, then by (3.1.3) $\operatorname{int} \bar{H} \in \mathfrak{t}$, $H \in \mathfrak{t}$.

Thus the open elements of a $\mathfrak{t} \in UY$ containing (3.1.1) are uniquely determined by (3.1.2) and so is $\mathfrak{t}$ itself. □

For $\mathfrak{s} \in UX$ let us denote by $\tilde{f}(\mathfrak{s})$ the only element of UY that contains (3.1.1).

COROLLARY 3.2. *For $\mathfrak{s} \in UX$, $H \subset Y$ open, we have*

$$H \in \tilde{f}(\mathfrak{s}) \quad \text{iff} \quad \operatorname{int} f^{-1}(\bar{H}) \in \mathfrak{s}. \quad □$$

THEOREM 3.3. *The map $\tilde{f}\colon UX \to UY$ is a homeomorphism.*

PROOF. $\tilde{f}$ is continuous because, by 3.2,

$$\tilde{f}^{-1}(s_Y(\bar{H})) = s_X(\operatorname{int} f^{-1}(\bar{H})) \tag{3.3.1}$$

where s_X and s_Y denote the operator s defined for X and Y, respectively, and $H\subset Y$ is open.

$\tilde{f}$ is injective because, if $\mathfrak{s}_i\in UX$, $\mathfrak{s}_1\neq\mathfrak{s}_2$, then there are open sets $G_i\in\mathfrak{s}_i$ such that $G_1\cap G_2=\emptyset$, and then $\hat{f}(G_i)\in\tilde{f}(\mathfrak{s}_i)$, $\hat{f}(G_1)\cap\hat{f}(G_2)=\emptyset$, $\tilde{f}(\mathfrak{s}_1)\neq\tilde{f}(\mathfrak{s}_2)$.

$\tilde{f}$ is surjective. In fact, if $\mathfrak{t}\in UY$, then the sets $\operatorname{int}\bar{H}$, where $H\in\mathfrak{t}$ is open, clearly constitute a filter base in Y. By (3.1.3), the system composed of the sets $\operatorname{int} f^{-1}(\bar{H})$ is centred so that it is contained in some $\mathfrak{s}\in UX$. By 3.2, $H\in\tilde{f}(\mathfrak{s})$ for every open set $H\in\mathfrak{t}$, i.e. $\mathfrak{t}\subset\tilde{f}(\mathfrak{s})$, $\mathfrak{t}=\tilde{f}(\mathfrak{s})$.

The proof is completed by observing that UX is compact and UY is T_2. □

THEOREM 3.4. *If* $\mathfrak{s}\in UX$, $\mathfrak{s}\to x$, *then* $\tilde{f}(\mathfrak{s})\to f(x)$. *Hence* $\tilde{f}(U_cX)\subset U_c(Y)$.

PROOF. Let H be an open neighbourhood of $f(x)\in Y$. By the ϑ-continuity of f, there is an open neighbourhood G of x such that $f(\bar{G})\subset\bar{H}$. Hence $x\in\operatorname{int} f^{-1}(\bar{H})\in\mathfrak{s}$, and $H\in\tilde{f}(\mathfrak{s})$ by 3.2. □

THEOREM 3.5. *If f is ϑ-perfect, then*

$$\tilde{f}(U_cX)=U_cY$$

so that $\tilde{f}|U_cX$ *is a homeomorphism from* U_cX *onto* U_cY.

PROOF. According to 3.3 and 3.4, it suffices to prove $U_cY\subset\tilde{f}(U_cX)$. Now if $\mathfrak{s}\in UX$, $\tilde{f}(\mathfrak{s})\to y\in Y$, then $\mathfrak{s}\to x$ for some $x\in f^{-1}(y)$. In fact, otherwise every point of $f^{-1}(y)$ would have an open neighbourhood that does not intersect a suitable element of $\mathfrak{s}$, and, by the compactness of $f^{-1}(y)$, there would exist an open set $G\supset f^{-1}(y)$ that does not meet some open $G'\in\mathfrak{s}$. Then clearly $\hat{f}(G)$ is an open neighbourhood of y, $\hat{f}(G')\in\tilde{f}(\mathfrak{s})$, $\hat{f}(G)\cap\hat{f}(G')=\emptyset$, which contradicts $\tilde{f}(\mathfrak{s})\to y$. □

4. The mapping f^*. We are now able to prove that good maps $f\colon X\to Y$ induce continuous maps from EX into EY:

THEOREM 4.1. *Let X, Y be topological spaces and $f\colon X\to Y$ be ϑ-continuous, closed and irreducible. Then there is one and only one continuous map $f^*\colon EX\to EY$ such that*

$$f\circ k_X=k_Y\circ f^*. \tag{4.1.1}$$

This map is defined by

$$f^*(x,\mathfrak{s})=\big(f(x),\tilde{f}(\mathfrak{s})\big). \tag{4.1.2}$$

PROOF. By (3.3), f^* given by (4.1.2) is continuous and clearly satisfies (4.1.1).

The part concerning unicity will be derived from a more general statement. In order to formulate it, let us consider the product space $X\times UX$ and let us denote by PX the subspace

$$\{(x,\mathfrak{s})\in X\times UX\colon \mathfrak{s}\to x \text{ in } X\},$$

i.e. the same set as the underlying set of EX but equipped with an obviously finer topology. It is shown in [3] that PX is, up to homeomorphism, the only EDC space whose X is the image under an ultraperfect, separated map. Now we can prove:

LEMMA 4.2. *If $f\colon X\to Y$ is ϑ-continuous, closed and irreducible, and $g\colon PX\to EY$ is continuous and satisfies*

$$f\circ k_X = k_Y\circ g, \tag{4.2.1}$$

then $g=f^*$.

REMARK. The unicity stated in 4.1 clearly follows from 4.2 because $\mathrm{id}_{PX}\colon PX\to EX$ is continuous.

PROOF OF 4.2. Assume that $g\colon PX\to EY$ is continuous and satisfies (4.2.1). Then there is a map

$$h\colon U_cX\to U_cY$$

such that

$$g(x,\mathfrak{s}) = \big(f(x), h(\mathfrak{s})\big)\quad \big((x,\mathfrak{s})\in PX\big). \tag{4.2.2}$$

In fact, (4.2.1) implies that the first coordinate of $g(x,\mathfrak{s})$ is $f(x)$. Now if $(x_i,\mathfrak{s})\in PX$ $(i=1,2)$, then $g(x_i,\mathfrak{s})=\big(f(x_i),\mathfrak{t}_i\big)$ and $\mathfrak{t}_1\neq\mathfrak{t}_2$ would imply that $\big(f(x_1),\mathfrak{t}_1\big)$ and $\big(f(x_2),\mathfrak{t}_2\big)$ have disjoint neighbourhoods in EY. By the continuity of g, $(x_1,\mathfrak{s})$ and $(x_2,\mathfrak{s})$ would have disjoint neighbourhoods in PX. However, this is impossible because, if $G_i, H_i\subset X$ are open for $i=1,2$, and $x_i\in G_i$, $\mathfrak{s}\in s_X(H_i)$, then $G_1\cap G_2\cap H_1\cap H_2\in\mathfrak{s}$, and if x_1' is an element of this intersection and $\mathfrak{s}_1\in UX$, $\mathfrak{s}_1\to x_1'$, then clearly $(x_1',\mathfrak{s}_1)\in\big(G_1\times s_X(H_1)\big)\cap\big(G_2\times s_X(H_2)\big)\cap PX$. Therefore the second coordinate of $g(x,\mathfrak{s})$ does not depend on x.

Assume $h(\mathfrak{s}_0)\neq \tilde f(\mathfrak{s}_0)$ for some $\mathfrak{s}_0\in U_cX$. Choose $x_0\in X$ such that $\mathfrak{s}_0\to x_0$ in X. Then, by 3.1, there is an open set $G_0\in\mathfrak{s}_0$ such that $\hat f(G_0)\notin h(\mathfrak{s}_0)$, thus $\hat f(G_0)\cap H_0=\emptyset$ for some open $H_0\in h(\mathfrak{s}_0)$. By the continuity of g, there are open sets G, $H\subset X$ such that $x_0\in G$, $\mathfrak{s}_0\in s_X(H)$ (i.e. $H\in\mathfrak{s}_0$), and

$$g\big((G\times s_X(H)\big)\cap PX\subset \big(Y\times s_Y(H_0)\big)\cap EY. \tag{4.2.3}$$

Since $\mathfrak{s}_0\to x_0$ implies $G\in\mathfrak{s}_0$, necessarily $G\cap G_0\cap H\neq\emptyset$, hence $\hat f(G\cap G_0\cap H)\neq\emptyset$ because f is closed and irreducible. Choose $y\in\hat f(G\cap G_0\cap H)$, then $x\in X$ such that $y=f(x)$, whence $x\in G\cap G_0\cap H$, finally $\mathfrak{s}\in UX$ such that $\mathfrak{s}\to x$. Then $H\in\mathfrak{s}$, $(x,\mathfrak{s})\in\big(G\times s_X(H)\big)\cap PX$, so $h(\mathfrak{s})\in s_Y(H_0)$ by (4.2.3), i.e. $H_0\in h(\mathfrak{s})$. On the other hand, $g(x,\mathfrak{s})=\big(f(x),h(\mathfrak{s})\big)$ implies $h(\mathfrak{s})\to f(x)\in\hat f(G\cap G_0\cap H)\subset\hat f(G_0)$, $\hat f(G_0)\in h(\mathfrak{s})$ which contradicts $\hat f(G_0)\cap H_0=\emptyset$. □

f^* defined by (4.1.2) has essentially better properties if f is ϑ-perfect:

THEOREM 4.3. *If $f\colon X\to Y$ is ϑ-perfect, then $f^*\colon EX\to EY$ is ultraperfect and open.*

PROOF. f^* is surjective. In fact, let $(y,\mathfrak{t})\in EY$ be given. By 3.5, $\mathfrak{t}=\tilde f(\mathfrak{s})$ for some $\mathfrak{s}\in U_cX$. If $\mathfrak{s}\to x\in X$, then $f(x)=y$ by 3.4, so that $f^*(x,\mathfrak{s})=(y,\mathfrak{t})$.

According to the surjectivity of f^* and the bijectivity of $\tilde f|U_cX\colon U_cX\to U_cY$, we obtain from (4.1.2)

$$f^*\big((X\times A)\cap EX\big) = \big(Y\times\tilde f(A)\big)\cap EY \tag{4.3.1}$$

for any $A\subset U_cX$. Consequently, by 3.5, f^* is both open and closed. Again by the bijectivity of $\tilde f|U_cX$, we see that f^* is irreducible.

If $(y, t) \in EY$, then $f^{*-1}(y, t) \subset X \times \tilde{f}^{-1}(t) = X \times \{\mathfrak{s}\}$, where $\mathfrak{s} = \tilde{f}^{-1}(t)$. If an open subset of EX intersects $X \times \{\mathfrak{s}\}$, then it contains $(X \times \{\mathfrak{s}\}) \cap EX$. Therefore $f^{*-1}(y, t)$ is compact, and f^* is compact as well. □

THEOREM 4.4. *If f: $X \to Y$ is ϑ-continuous, closed, irreducible and separated, then f^*: $EX \to EY$ is injective.*

PROOF. Suppose $(x_i, \mathfrak{s}_i) \in EX$, $(x_1, \mathfrak{s}_1) \neq (x_2, \mathfrak{s}_2)$, $f^*(x_1, \mathfrak{s}_1) = f^*(x_2, \mathfrak{s}_2)$. Then $\tilde{f}(\mathfrak{s}_1) = \tilde{f}(\mathfrak{s}_2)$, so $\mathfrak{s}_1 = \mathfrak{s}_2 = \mathfrak{s}$ by 3.3, thus $x_1 \neq x_2$. Moreover, $f(x_1) = f(x_2)$ implies that x_1 and x_2 have disjoint neighbourhoods in X. This fact contradicts $\mathfrak{s} \to x_1$, $\mathfrak{s} \to x_2$. □

COROLLARY 4.5. *If f: $X \to Y$ is a separated, ϑ-perfect map, then f^*: $EX \to EY$ is a homeomorphism.* □

THEOREM 4.6. *For two topological spaces X and Y, the spaces EX and EY are homeomorphic iff there is a topological space Z whose both X and Y are images under separated, ϑ-perfect maps.*

PROOF. If such a Z exists, then, by 4.5, both EX and EY are homeomorphic to EZ. Conversely, if h: $EX \to EY$ is a homeomorphism, then $Z = EX$ can be chosen using k_X and $k_Y \circ h$. □

THEOREM 4.7. *Let X be regular and* EDC. *If f: $X \to Y$ is ϑ-continuous, closed and irreducible, then there is one and only one continuous map g: $X \to EY$ such that $f = k_Y \circ g$. If f is ϑ-perfect, then g is ultraperfect. If f is separated, then g is injective. If f is both separated and ϑ-perfect, then g is a homeomorphism.*

PROOF. By 2.5, k_X is a homeomorphism. Hence $g = f^* \circ k_X^{-1}$ will do by 4.1, 4.3, 4.4 and 4.5. □

The following statement serves as a fundament in recognizing the character of an absolute with respect to EX:

COROLLARY 4.8. *If Z is a regular* EDC *space whose X is the image under a separated, ϑ-perfect map, then Z and EX are homeomorphic.* □

By this, all essential results for EX contained in [1] in the case of T_2-spaces are established for arbitrary X.

5. The category ϑ-Top. A part of 4.7 admits an elegant formulation if we introduce a suitable category.

LEMMA 5.1. *If f: $X \to Y$ is surjective, ϑ-continuous, closed and compact, then $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.*

PROOF. Assume $f^{-1}(K) \subset \bigcup_{i \in I} G_i$, where $G_i \subset X$ is open for every i. For $y \in K$, $f^{-1}(y)$ is compact, hence it is contained in the union $G(y)$ of finitely many sets G_i. Then $\hat{f}(G(y))$ is an open neighbourhood of y, and $K \subset \bigcup_{y \in F} \hat{f}(G(y))$ for a finite set $F \subset K$. So

$$f^{-1}(K) \subset \bigcup_{y \in F} f^{-1}(\hat{f}(G(y))) \subset \bigcup_{y \in F} G(y). \quad \square$$

COROLLARY 5.2. *If* $f\colon X\to Y$ *and* $g\colon Y\to Z$ *are* ϑ*-perfect, then so is* $g\circ f$. □

Therefore we can define a category ϑ-**Top** whose objects are all topological spaces and the morphisms are the ϑ-perfect maps. Now we obtain from 4.7:

THEOREM 5.3. *The full subcategory of* ϑ-**Top** *whose objects are the regular* EDC *spaces is coreflective. The coreflection of X is EX with the coreflector* k_X. □

The statement of 5.3 remains valid if we replace ϑ-**Top** by the subcategory $s\vartheta$-**Top** in which the morphisms are the u-separated ϑ-perfect maps; we say that a map $f\colon X\to Y$ is *u-separated* iff $x_1, x_2\in X$, $x_1\neq x_2$, $f(x_1)=f(x_2)$ implies that x_1 and x_2 have disjoint closed neighbourhoods. This is a category indeed, because the composition of ϑ-continuous, u-separated maps is clearly u-separated. We deduce from 4.7 the analogue of 5.3 by observing that a separated map starting from a regular space is u-separated.

In general, a separated map need not be u-separated. In fact, let X be a T_2-space that is not Urysohn, $a, b\in X$, $a\neq b$, and suppose that a and b do not have disjoint closed neighbourhoods. Let $f\colon X\to Y$ be a surjective map such that $f(a)=f(b)$, but $f|X-\{a\}$ is injective. Let us equip Y with the quotient topology. Then f is ultraperfect, separated, but not u-separated.

6. ***EX*** **and** ***PX*****.** We show that PX determines EX in a certain sense.

LEMMA 6.1 (see [3]). *If*

$$U=\bigcup_{i\in I}(G_i\times s(H_i))\cap PX,$$

where G_i, H_i *are open in X, then the closure of U in PX is* $(X\times s(H^*))\cap PX$, *where*

$$H^*=\bigcup_{i\in I}(G_i\cap H_i).$$

PROOF. If $(x_0, \mathfrak{s}_0)\in\bar{U}\cap PX$, then, for open $G, H\subset X$ and $x_0\in G$, $\mathfrak{s}_0\in s(H)$, we have

$$(G\times s(H))\cap(G_i\times s(H_i))\cap PX\neq\emptyset$$

for some i. Let $(x, \mathfrak{s})$ be an element of this intersection. Then $H, H_i\in\mathfrak{s}$, and $\mathfrak{s}\to x$ implies $G_i\in\mathfrak{s}$, so that $H\cap G_i\cap H_i\neq\emptyset$. *A fortiori* $H\cap H^*\neq\emptyset$, $H^*\in\sec\mathfrak{s}_0$, $H^*\in\mathfrak{s}_0$, $\mathfrak{s}_0\in s(H^*)$, $(x_0, \mathfrak{s}_0)\in(X\times s(H^*))\cap PX$.

Conversely, if $(x_0, \mathfrak{s}_0)\in PX$, $\mathfrak{s}_0\in s(H^*)$, then, for open sets $G, H\subset X$, $x_0\in G$, $\mathfrak{s}_0\in s(H)$, we have $G\in\mathfrak{s}_0\in s(G)$, hence $G\cap H\cap H^*\neq\emptyset$ and $G\cap H\cap G_i\cap H_i\neq\emptyset$ for some i. Let $x\in G\cap H\cap G_i\cap H_i$ and $\mathfrak{s}\in UX$, $\mathfrak{s}\to x$. Then $(x, \mathfrak{s})\in(G\times s(H))\cap\cap(G_i\times s(H_i))\cap PX$ so that every neighbourhood of $(x_0, \mathfrak{s}_0)$ intersects U. □

COROLLARY 6.2. *The space PX is* EDC. □

COROLLARY 6.3. *The r-open* (i.e. the clopen) *sets in PX are precisely the sets of the form* $(X\times s(H))\cap PX$ *where* $H\subset X$ *is open. Consequently EX is the semi-regularization of PX.* □

COROLLARY 6.4. *The map* $\mathrm{id}_{EX}\colon EX\to PX$ *is* ϑ*-continuous.* □

THEOREM 6.5. *If* $f\colon X\to Y$ *is continuous, then there exist* (in general several) *continuous maps* $g\colon EX\to EY$ *such that* $f\circ k_X=k_Y\circ g$.

PROOF. In [3], it is shown that there exist continuous maps $h\colon PX\to PY$ such that $f\circ k_X=k_Y\circ h$. Let $g\colon EX\to EY$ be defined by $g=\mathrm{id}_{EY}\circ h\circ \mathrm{id}_{EX}$; it is ϑ-continuous because $\mathrm{id}_{EX}\colon EX\to PX$ is ϑ-continuous and $\mathrm{id}_{EY}\colon PY\to EY$ is continuous. However, g is continuous since EY is regular. □

Observe that many of the results on EX (e.g. the proof of 2.3) can be derived from the analogous properties of PX, but their direct proofs (given above) are simpler than those concerning PX.

References

[1] S. Iliadis, Absolutes of Hausdorff spaces (in Russian), *Dokl. Akad. Nauk SSSR*, **149** (1963), 22—25.

[2] V. I. Ponomarev, The absolute of a topological space (in Russian), *Dokl. Akad. Nauk SSSR*, **149** (1963), 26—29.

[3] L. B. Shapiro, On absolutes of topological spaces and continuous mappings (in Russian), *Dokl. Akad. Nauk SSSR*, **226** (1976), 523—526.

[4] V. M. Ul'janov, Compactifications of countable character and absolutes (in Russian), *Mat. Sb.*, **98** (1975), 223—254.

[5] R. G. Woods, A survey of absolutes of topological spaces, Topological structures II, *Mathematical Centre Tracts*, **116** (1979), 323—362.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
H—1088 BUDAPEST, MÚZEUM KRT. 6—8

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HOMOGENEOUS CONNECTIONS AND THEIR OSCULATIONS ON THE VERTICAL SUBBUNDLE

B. KIS (Debrecen)

1. Introduction

In the recent treatments of Finsler geometry the notion of Finsler connection over a manifold is usually considered as a pair composed of a general and a linear connection, the first of which is given in the tangent bundle, and the other one in the vertical subbundle of the tangent bundle. It seems that it is not uninteresting to search for structures which can determine Finsler connections. As it is already known [7], [2] a regular linear connection in the vertical subbundle induces a nonlinear connection in the tangent bundle. Therefore, if we are given a regular linear connection in the vertical subbundle, then we have a Finsler connection over the base manifold too. However, in the classical Finsler geometry mainly homogeneous connections are used instead of general (i.e. nonlinear) connections, so we will consider homogeneous connections mostly.

Our aim in the present paper is to investigate the converse way in this case: we will construct a linear connection in the vertical subbundle from a given homogeneous connection in the tangent bundle. This of course means that a homogeneous connection in the tangent bundle completely determines a special Finsler connection over the base manifold. We note that our construction can be performed easily in the case of complete nonlinear connections also. The completeness of this nonlinear connection is a significant point in this paper, but after slight modifications it is possible to use our method even to non complete nonlinear connections.

All the discussions presented here are general also in the sense that we do not assume that the stage for our considerations is a tangent bundle; rather we will take an arbitrary vector bundle for our startpoint.

After introducing some notions and notations in Section 2, in Section 3 we define and describe the Berwald—Hashiguchi connection as a special Berwald connection (∇). Then we characterize homogeneous connections by completeness and by the 1-homogeneity of the holonomy map using a theorem of W. Barthel [3]. Finally in Section 4 we investigate the osculation on the vertical subbundle of a homogeneous connection by a linear connection, and we show that the arising osculating connections are identical with the Berwald—Hashiguchi connections.

The central idea of the paper is the notion of osculating. In this way our construction may be regarded as a continuation of the osculating method given by O. Varga [11], [10] for the Finsler metrics, under different conditions.

2. Basic notions and notations

In this section we will introduce some basic notions and their notations which will be used in the following parts.

We will denote the total space, base space and projection of a vector bundle ξ by $\mathrm{tl}\,\xi$, $\mathrm{bs}\,\xi$ and $\mathrm{pr}\,\xi$, resp. The tangent bundle of the manifold M is denoted by τM. We also use the following notations: $TM \equiv \mathrm{tl}\,\tau M$, $\mathscr{P}_M = \mathrm{pr}\,\xi$.

The pair (α, β) means a bundle map between the vector bundles ξ and η if the diagram

$$\begin{array}{ccc} \mathrm{tl}\,\xi & \xrightarrow{\alpha} & \mathrm{tl}\,\eta \\ \downarrow{\scriptstyle \mathrm{pr}\xi} & & \downarrow{\scriptstyle \mathrm{pr}\eta} \\ \mathrm{bs}\,\xi & \xrightarrow{\beta} & \mathrm{bs}\,\eta \end{array}$$

is commutative and the maps α, β are smooth. When the map β is the identity we say that the bundle map (α, β) is a strong bundle map. If (α, β) is a bundle map between vector bundles and the restrictions of the map α to the fibers are linear we say that (α, β) is a vector bundle map. We also use the terminology "strong vector bundle". α completely determines the map β, so we will use the simpler notation α instead of the more deductive but longer (α, β).

Let $\varphi\colon M \to \mathrm{bs}\,\xi$ be a smooth map, where ξ is a vector bundle. We construct the pull-back bundle $\varphi^!\xi$ of ξ by φ in terms of the commutative diagram

$$\begin{array}{ccc} \mathrm{tl}\,(\varphi^!\xi) & \xrightarrow{\mathrm{ad}_\xi \varphi} & \mathrm{tl}\,\xi \\ \downarrow{\scriptstyle \mathrm{pr}(\varphi^!\xi)} & & \downarrow{\scriptstyle \mathrm{pr}\xi} \\ \mathrm{bs}\,(\varphi^!\xi) = M & \xrightarrow{\varphi} & \mathrm{bs}\,\xi \end{array}$$

which is the so-called pull-back square associated to φ and ξ. The bundle $\varphi^!\xi$ is not canonically determined: its usual representative is the bundle $(M \times_\varphi \mathrm{tl}\,\xi, \mathrm{pr}^2, M)$. However, the vector bundle map $\mathrm{ad}_\xi\,\varphi$ is canonical relative to the bundle $\varphi^!\xi$, i.e. $\varphi^!\xi$ determines $\mathrm{ad}_\xi\varphi$ canonically.

3. Local expressions and holonomies

A trivialization of the vector bundle ξ and an atlas of its base space $\mathrm{bs}\,\xi$ completely determine the canonical local description of the bundles $(\mathrm{pr}\,\xi)^!\xi$, $\tau\,\mathrm{bs}\,\xi$, $(\mathrm{pr}\,\xi)^!\tau\,\mathrm{bs}\,\xi$, $\tau\,\mathrm{tl}\,\xi$, $\tau\,\mathrm{tl}\,(\mathrm{pr}\,\xi)^!\xi$ and their component spaces and maps induced by these spaces and bundles. This local description includes local trivializations, atlases and local representatives of maps. From now on the existence of these local descriptions are assumed. The best property of these local descriptions is perhaps the fact that local representatives of bundle projections are restrictions of some component projections of some product manifolds to certain open subsets. We usually do not make distinction between the notation of an object and its local representative.

A) *The local description of a connection on* ξ. Let

$$(\nabla) \qquad \mathcal{O} \rightarrow (\mathrm{pr}\,\xi)^!\xi \underset{J_\xi}{\overset{v_\nabla}{\rightleftarrows}} \tau\,\mathrm{tl}\,\xi \underset{P_\xi}{\overset{h_\nabla}{\rightleftarrows}} (\mathrm{pr}\,\xi)^!\tau\,\mathrm{bs}\,\xi \rightarrow \mathcal{O}$$

be a connection in the vector bundle ξ ([12], [9]). If $n=\dim \mathrm{bs}\,\xi$, $k=\mathrm{rank}\,\xi$ then the local representatives of maps

$$J_\xi : (\mathrm{pr}\,\xi)^!\xi \rightarrow \tau\,\mathrm{tl}\,\xi,$$

$$P_\xi : \tau\,\mathrm{tl}\,\xi \rightarrow (\mathrm{pr}\,\xi)^!\tau\,\mathrm{bs}\,\xi,$$

$$v_\nabla : \tau\,\mathrm{tl}\,\xi \rightarrow (\mathrm{pr}\,\xi)^!\xi,$$

$$h_\nabla : (\mathrm{pr}\,\xi)^!\tau\,\mathrm{bs}\,\xi \rightarrow \tau\,\mathrm{tl}\,\xi$$

are restrictions of the maps

$$J_\xi(x^1, y^1, y^2) = (x^1, y^1, 0, y^2) \quad (x^1, x^2 \in \mathbf{R}^n),$$

$$P_\xi(x^1, y^1, x^2, y^2) = (x^1, y^1, x^2), \quad v_\nabla(x^1, y^1, x^2, y^2) = \big(x^1, y^1, y^2 - \omega(x^1, y^1)(x^2)\big),$$

$$h_\nabla(x^1, y^1, x^2) = \big(x^1, y^1, x^2, \omega(x^1, y^1)(x^2)\big) \quad (y^1, y^2 \in \mathbf{R}^k)$$

to some open sets, where $\omega(x^1, y^1)\colon \mathbf{R}^n \rightarrow \mathbf{R}^k$ is linear and smooth in its argument. The connection (∇) is called linear iff $\omega(x^1, y^1)$ is linear in y^1.

B) *The local description of* $\tau\,\mathrm{tl}\,\xi$ *associated to* (∇) *and linear connections in* $\tau\,\mathrm{tl}\,\big((\mathrm{pr}\,\xi)^!\xi\big)$. If $(x^i)_{i=1,\dots,n}$ is a local coordinate system on $\mathrm{bs}\,\xi$ and $(e_j)_{j=1,\dots,k}$ and $(y^j)_{j=1,\dots,k}$ are a local basis of $\mathrm{tl}\,\xi$ and the local coordinate system associated to it then a canonical local basis of $\tau\,\mathrm{tl}\,\xi$ is

$$(3.1) \qquad \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right).$$

From this we can construct another basis provided that (∇) is a connection on ξ:

$$(3.2) \qquad \left(\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right),$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \omega_i^\alpha \frac{\partial}{\partial y^\alpha} \quad (i = 1, \dots, n),$$

and (ω_i^α) is the matrix of $\omega(x^1, y^1)$. (We are using the Einstein summation convention; the range of Latin (resp. Greek) indices is the set $(1, \dots, n)$ (resp. $(1, \dots, k)$)). (3.2) is called the basis of $\tau\,\mathrm{tl}\,\xi$ associated to (∇). The bases of the cotangent bundle $\tau^*\,\mathrm{tl}\,\xi$ associated to the bases (3.1) resp. (3.2) are denoted by $(dx^1, \dots, dx^n, dy^1, \dots, dy^k)$ and $(dx^1, \dots, dx^n, \delta y^1, \dots, \delta y^k)$ respectively, where

$$\delta y^\alpha = dy^\alpha - \omega_j^\alpha dx^j \quad (\alpha = 1, \dots, k).$$

Now, if $(\tilde{\nabla})$ is a linear connection in the vertical subbundle $V\xi$ of ξ and

$$\left(\frac{\partial}{\partial \bar{y}^1}, \dots, \frac{\partial}{\partial \bar{y}^k}\right), \ (d\bar{y}^1, \dots, d\bar{y}^k)$$

are local bases of $\text{tl}\, V\xi$ resp. $\text{tl}\, V^*\xi$ then the local representative of the connection form of $(\tilde{\nabla})$ (described in terms of covariant derivative) can be formulated with respect to these bases in two ways:

a) *Using the canonical basis* (3.1) *of* $\tau\, \text{tl}\, \xi$:

$$A^{\alpha}_{j\gamma}\, dx^j \otimes d\bar{y}^{\gamma} \otimes \frac{\partial}{\partial \bar{y}^{\alpha}} + B^{\alpha}_{\beta\gamma}\, dy^{\beta} \otimes d\bar{y}^{\gamma} \otimes \frac{\partial}{\partial \bar{y}^{\alpha}}.$$

b) *Using the basis* (3.2) *of* $\tau\, \text{tl}\, \xi$ *associated to* (∇):

$$^{\nabla}\!A^{\alpha}_{j\gamma}\, dx^j \otimes d\bar{y}^{\gamma} \otimes \frac{\partial}{\partial \bar{y}^{\alpha}} + {}^{\nabla}\!B^{\alpha}_{\beta\gamma}\, \delta y^{\beta} \otimes d\bar{y}^{\gamma} \otimes \frac{\partial}{\partial \bar{y}^{\alpha}}.$$

The last description is called the *Cartan normal form* of connection $(\tilde{\nabla})$. We can state the following relationship between coefficients A, B and $^{\nabla}\!A$, $^{\nabla}\!B$:

$$A^{\alpha}_{j\gamma} = {}^{\nabla}\!A^{\alpha}_{j\gamma} - \omega^{\beta}_{j}\, B^{\alpha}_{\beta\gamma}, \quad B^{\alpha}_{\beta\gamma} = {}^{\nabla}\!B^{\alpha}_{\beta\gamma}. \tag{3.3}$$

DEFINITION 1. The pair $\mathscr{F} = (\tilde{\nabla}, \nabla)$ is called a Finsler Connection ([6]) on ξ. If

$$^{\nabla}\!A^{\alpha}_{j\gamma} = \frac{\partial \omega^{\alpha}_{j}}{\partial y^{\gamma}} \tag{3.4}$$

and

$$^{\nabla}\!B^{\alpha}_{\beta\gamma} = 0 \tag{3.5}$$

then we say that $\mathscr{F}$ is the tl Berwald connection associated to (∇). Omitting the last condition, we get the notion of the *Berwald—Hashiguchi connection* ([1], [4], [5]).

C) *Parallel translation:* Let $\tilde{\gamma}(t)$ be a curve in $\text{tl}\, \xi$ whose local representative is $(x(t), y(t))$. If $\gamma = ((\text{pr}\, \xi) \circ \tilde{\gamma})(t)$ then its local form is $x(t)$. The necessary and sufficient condition of parallelity of $\tilde{\gamma}$ by (∇) is locally the equation

$$\dot{y}(t) - \omega(x(t), y(t))(\dot{x}(t)) = 0. \tag{3.6}$$

D) *The case of linear connection:* In case (∇) is linear equation (3.6) can be considered as a system of ordinary linear differential equations and so it has a unique solution for any initial condition. From the theory of ordinary linear differential equations (see for example [8]) we know that this solution is determined everywhere where all the coefficients are continuous, and depends linearly on the initial conditions. The global translation of this fact can be summarized as follows:

A general (i.e. nonlinear) connection (∇) on a vector bundle ξ is *complete* if the parallel translation of any vector along any curve γ of $\text{bs}\, \xi$ by (∇) is defined on the whole range of γ.

COROLLARY 1. *Every linear connection is complete.*

Let (∇) be complete and γ be a smooth curve in $\text{bs}\, \xi$. If $p = \gamma(t_0)$, $q = \gamma(t_1)$ are two points of γ we can define the map

$$\alpha_{t_0, t_1, \gamma}\colon \xi_p \to \xi_q,$$

$v \mapsto \alpha_{t_0,t_1,\gamma}(v) \doteq$ the parallel translation of $v \in \xi_{\gamma(t_0)}$ into ξ_q by (∇) along γ. This map is called the *holonomy map of* (∇) *along* γ *associated to* t_0, t_1. The family

$$\mathcal{H}_{\gamma,\nabla} \doteq \{\alpha_{t_0,t_1,\gamma}\colon \xi_{\gamma(t_0)} \to \xi_{\gamma(t_1)} | t_0, t_1 \in \operatorname{dom} \gamma\}$$

is the *holonomy of* (∇) *along* γ.

Referring to the classical results of the theory of systems of ordinary differential equations we get the following

PROPOSITION 1. (a) $\alpha_{t,t,\gamma} = \mathrm{id}_{\xi_{\gamma(t)}}$.

(b) $\alpha_{b,c,\gamma} \circ \alpha_{a,b,\gamma} = \alpha_{a,c,\gamma}$.

(c) $\alpha_{a,b,\gamma}$ *is a diffeomorphism.*

(d) *The connection* (∇) *is linear iff it is complete and all of its holonomy maps are linear.*

We can generalize the last statement to the case of homogeneous (∇).

DEFINITION 2. The connection (∇) is called homogeneous if $\omega(x^1, y^1)(x^2)$ is 1-homogeneous in the variable y^1, and maps determined by (∇) are smooth only on the "dotted" bundle $\dot{\xi}$; i.e. on bundle arising when the zero section is splitted out from the corresponding vector bundle ξ.

THEOREM 1. *The connection* (∇) *is homogeneous iff*

(1) *The connection* (∇) *is complete and*

(2) *Every holonomy map of* (∇) *is a* 1*-homogeneous smooth map between the fibres of* $\dot{\xi}$.

PROOF. *Sufficiency.* The homogeneity of the connection (∇) follows from the condition (2). This condition means that $\lambda y_0 \in \dot{\xi}_p$ is taken by $\alpha_{t_0,t,\gamma}$ into $\lambda y_1 \in \dot{\xi}_q$, whenever $y_0 \in \dot{\xi}_p$ is taken into $y_1 \in \dot{\xi}_q$. But both $y = y(t)$ and $\lambda y(t)$ must be solutions of equation (3.6), and thus

$$\dot{y} - \omega(x, y)(\dot{x}) = 0 \quad \text{and} \quad \lambda \dot{y} - \omega(x, \lambda y)(\dot{x}) = 0$$

from which one can conclude the 1-homogeneity of ω in y.

Necessity. Consider the initial value problem

$$\dot{y} = f(t, y(t)), \quad y(t^*) = y^* \tag{3.7}$$

for a system of ordinary differential equations, where

$$f\colon [a, b] \times \mathbf{R}^m \to \mathbf{R}^m, \quad y\colon [a, b] \to \mathbf{R}^m \quad (a, b \in \mathbf{R},\ a < b)$$

are smooth, $f(t, y)$ is 1-homogeneous in the variable y and $t^* \in (a, b)$, $y^* \in \mathbf{R}^m$. The system (3.7) has a unique solution $y(t)$, $t \in [a, b]$ and this solution depends in 1-homogeneous way on y^*. (For the proof see Barthel [3] Theorem 2.)

Now we can prove the necessity part of the statement. Since ω is assumed to be 1-homogeneous in y, the local condition (3.6) of the parallel translation can be considered as an initial value problem of the form (3.7). By Barthel's theorem we get that any homogeneous connection is complete. By completeness we can define the holonomy of (∇). Now the holonomy maps are 1-homogeneous because of the solution's 1-homogeneous dependence on initial values. Q.E.D.

4. Osculation of a homogeneous connection

As we have seen the holonomy maps are not necessarily linear in the case of homogeneity of (∇). To get linear maps we take their tangents:

DEFINITION 3. Let $\mathscr{H}=\{\alpha_{t_0,t_1,\gamma}\}$ be the family of holonomies of (∇). Associate to every holonomy map

$$\alpha_{t_0,t_1,\gamma}\colon \mathring{\xi}_{\gamma(t_0)} \to \mathring{\xi}_{\gamma(t_1)} \text{ smooth, } t_0,\, t_1\in \operatorname{dom}\gamma,$$

the map

$$\mathbf{D}\alpha_{t_0,t_1,\gamma,u}\colon ((\mathrm{pr}\,\xi)^!\mathring{\xi})_u \to ((\mathrm{pr}\,\xi)^!\mathring{\xi})_{\alpha t_0,t_1,\gamma(u)} \quad (u\in\mathring{\xi}_{\gamma(t_0)}),$$

$$w \mapsto (\mathbf{D}\alpha_{t_0,t_1,\gamma,u})(w) \doteq (d\alpha_{t_0,t_1,\gamma}(u))(w)$$

where $(d\alpha_{t_0,t_1,\gamma}(u))$ is the fiber derivative of $\alpha_{t_0,t_1,\gamma}$ at u. The map $\mathbf{D}\alpha_{t_0,t_1,\gamma,u}$ is called the osculating holonomy map induced by $\alpha_{t_0,t_1,\gamma}$; the family

$$\mathbf{D}\mathscr{H} := \{\mathbf{D}\alpha_{t_0,t_1,\gamma,u} \mid \alpha_{t_0,t_1,\gamma}\in\mathscr{H},\ u\in\gamma_{t_0}\}$$

is called the family of the osculating holonomy maps.

Now we can associate to the map $\mathbf{D}\alpha_{t_0,t_1,\gamma,u}$ a map between fibres of $(\mathrm{pr}\,\xi)^!\mathring{\xi}$ to get a situation more similar to the case of holonomy maps. The curve γ and $u\in\mathring{\xi}_{\gamma(t_0)}$ determine a curve $\tilde{\gamma}_u$ of $\mathrm{tl}\,\mathring{\xi}$ which is the parallel translation of u along γ by (∇) and for which $\operatorname{dom}\tilde{\gamma}_u=\operatorname{dom}\gamma$. The data $(t_0, t_1, \tilde{\gamma}_u)$ completely determines the map $\mathbf{D}\alpha_{t_0,t_1,\gamma,u}$, so we can transcribe it to the form $\beta_{t_0,t_1,\tilde{\gamma}_u}$. If we realize that $\tilde{\gamma}_u$ is a curve in $\mathrm{tl}\,(\mathrm{pr}\,\xi)^!\mathring{\xi}$, naturally emerges the following question: does there exist a linear connection $(\tilde{\nabla})$ on $(\mathrm{pr}\,\xi)^!\mathring{\xi}$ whose holonomy maps are exactly the maps $\beta_{t_0,t_1,\tilde{\gamma}_u}$? The answer is negative if we formulate the question in this way, because the maps $\beta_{t_0,t_1,\tilde{\gamma}_u}$ are defined only for those curves $\tilde{\gamma}_u$ which are parallel to (∇). However, slightly modifying the question we will get an affirmative answer.

QUESTION. Is there a linear connection $(\tilde{\nabla})$ on $(\mathrm{pr}\,\xi)^!\mathring{\xi}$ whose family of holonomy maps include the whole family of the osculating holonomy maps $\{\beta_{t_0,t_1,\tilde{\gamma}_u}|\tilde{\gamma}_u$ is parallel along some γ from $\mathrm{bs}\,\xi$, t_0, $t_1\in\operatorname{dom}\gamma$, $u\in\mathring{\xi}_{\gamma(t_0)}\}$?

In order to answer this question recall that the Dombrovski map of a linear connection $(\tilde{\nabla})$ on $(\mathrm{pr}\,\xi)^!\mathring{\xi}$ can be written locally as

$$\text{(5.1)}\quad \mathbf{D}(x^1, y^1, z^1, x^2, y^2, z^2) = (x^1, y^1, z^2-\omega^1(x^1, y^1, z^1)(x^2)-\omega^2(x^1, y^1, z^1)(y^2))$$

where $\omega^1(x^1, y^1, z^1)(x^2)$ and $\omega^2(x^1, y^1, z^1)(y^2)$ are linear in z^1, x^2 and z^1, y^2 respectively.

With regard to (3.6) and (5.1) the map $\beta_{t_0,t_1,\tilde{\gamma}_u}$ is a holonomy map of $(\tilde{\nabla})$ iff

$$\dot{\beta}_{t_0,t_1,\tilde{\gamma}_u}(w)-\omega^1(x(t), y(t), \beta_{t_0,t_1,\tilde{\gamma}_u}(w))(\dot{x}(t))-\omega^2(x(t), y(t), \beta_{t_0,t_1,\tilde{\gamma}_u}(w))(\dot{y}(t)) = 0$$

for every $w\in(\mathrm{pr}\,\xi)^!\mathring{\xi}_u\cong\mathbf{R}^k$, where $(x(t), y(t))$ is the local form of γ_u and $\dot{\beta}_{t_0,t_1,\tilde{\gamma}_u} := \left(\frac{d}{dt}\beta_{t_0,t_1,\tilde{\gamma}_u}\right)\Big|_{t_1}$. So we get $y(t)=\alpha_{t_0,t,\gamma}(u)$, and

$$\dot{y}(t)-\omega(x(t), y(t))(\dot{x}(t)) = 0.$$

Composing the above equations, we obtain:

$$\dot{\beta}_{t_0,t,\tilde{\gamma}_u}(w)-\omega^1(x(t),\alpha_{t_0,t,\gamma}(u),\beta_{t_0,t,\tilde{\gamma}_u}(w))(\dot{x}(t))- \\ -\omega^2(x(t),\alpha_{t_0,t,\gamma}(u),\beta_{t_0,t,\tilde{\gamma}_u}(w))(-\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t)))=0 \tag{5.2}$$

for every $u, w\in \mathbf{R}^k$. Similarly, $\alpha_{t_0,t,\gamma}$ is the holonomy map of (∇) iff

$$\dot{\alpha}_{t_0,t,\gamma}(u)-\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t))=0 \quad \text{for every} \quad u\in\mathring{\xi}_{\gamma(t_0)}. \tag{5.3}$$

Differentiating this by u and taking into consideration that according to the first part of this section $\beta=\frac{d}{du}\alpha$ and thus $\dot{\beta}=\frac{d}{du}\dot{\alpha}$, we get

$$(\partial_u[\dot{\alpha}_{t_0,t,\gamma}(u)-\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t))])(w)= \\ =\dot{\beta}_{t_0,t,\tilde{\gamma}_u}(w)-\partial_2\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t))(\beta_{t_0,t,\tilde{\gamma}_u}(w))=0 \tag{5.4}$$

for every $w\in\mathbf{R}^k$. Now comparing this last equation with (5.2) we get

$$\partial_2\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t))(\beta_{t_0,t,\tilde{\gamma}_u}(w))= \\ =\omega^1(x(t),\alpha_{t_0,t,\gamma}(u),\beta_{t_0,t,\tilde{\gamma}_u}(w))(\dot{x}(t))-\omega^2(x(t),\alpha_{t_0,t,\gamma}(u),\beta_{t_0,t,\tilde{\gamma}_u}(w))\times \\ \times(-\omega(x(t),\alpha_{t_0,t,\gamma}(u))(\dot{x}(t))). \tag{5.5}$$

Let $t=t_0$. Then

$$(\partial_2\omega(x(t_0),u)(\dot{x}(t_0)))(w)=\omega^1(x(t_0),u,w)(\dot{x}(t_0))-\omega^2(x(t_0),u,w)(\omega(x(t_0),u)(\dot{x}(t_0))).$$

After these we formulate and prove the following

LEMMA 1. *If $(\tilde{\nabla})$ is a linear connection on* $(\mathrm{pr}\,\xi)^!(\mathring{\xi})$ *with Dombrovski map* (5.1) *then the equation*

$$(\partial_2\omega(x,r)(\mu))(s)=\omega^1(x,r,s)(\mu)-\omega^2(x,r,s)(\omega(x,r)(\mu)) \quad (r,s\in\mathbf{R}^k,\ \mu\in\mathbf{R}^n)$$

is the sufficient and necessary condition for the osculating holonomy maps of connection (∇) *to be the holonomy maps of* $(\tilde{\nabla})$.

PROOF. The considerations following the Question show that the condition of the Lemma is necessary. So we have to prove only the sufficiency. If the condition of the Lemma is satisfied for every values r, s, μ then it is satisfied also by the special values $\alpha_{t_0,t,\gamma}(u)$, $\beta_{t_0,t,\tilde{\gamma}_u}(w)$ and $\dot{x}(t)$. So equation (5.5) is satisfied. ((5.3) and (5.4) hold good, since α is supposed to be the holonomy map of (∇).) From (5.5), (5.4) and (5.3) follows however (5.2) which is the local condition of the Lemma.

Note that the homogenity of connection (∇) were used only because of the completeness of the homogeneous connections, and only in the definition of the osculating holonomy maps. If we define these osculating holonomies as partially defined maps the whole discussion presented here remains valid for general non-linear connections, too. Q.E.D.

DEFINITION 4. If a linear connection $(\tilde{\nabla})$ on $V\xi$ satisfies the condition of the Lemma, then we say that it *osculates* the connection (∇). The family of the connections which osculate the connection (∇) is denoted by $\{\partial\nabla\}$.

Now we can state and prove the following

THEOREM 2. *There exists a natural correspondence between the family* $\{\partial\nabla\}$ *and the family of the Berwald—Hashiguchi connections associated to* (∇).

PROOF. A linear connection $(\tilde{\nabla})$ is an element of $(\partial\nabla)$ iff it satisfies the condition of the Lemma. Now comparing this condition with condition (3.3) in the case of the Berwald—Hashiguchi connection we can see that they coincide. Q.E.D.

References

[1] T. Aikou and M. Hashiguchi, On generalized Berwald connections, *Rep. Fac. Sci. Kagoshima Univ. (Mat. Phys. and Chem.)*, **17** (1984), 9—13.

[2] M. H. Akbar-Zadeh, Les espaces de Finsler et certaines de leurs generalizations, *Ann. Scient. Ec. Norm. Sup.*, **80** (1963), 1—79.

[3] W. Barthel, Nichtlineare Zusammenhänge und deren Holonomiegruppen, *J. Reine Angew. Math.*, **212** (1963), 120—149.

[4] M. Hashiguchi, Some problems on generalized Berwald spaces. *Rep. Fac. Sci. Kagoshima Univ. (Mat. Phys and Chem.)*, **16** (1983), 59—63.

[5] M. Hashiguchi and Y. Ichijyo, On generalized Berwald spaces, *Rep. Fac. Sci. Kagoshima Univ. (Mat. Phys. and Chem.)*, **15** (1982), 19—32.

[6] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press (Japan, 1986).

[7] V. Opriou, Some properties of the tangent bundle related to the Finsler geometry, in *Proc. of the Nat. Sem. on Finsler Spaces* (Brasov, 1980), pp. 48—54.

[8] L. S. Pontrjagin, *Ordinary differential equations*, Nauka (Moscow, 1965) (in Russian).

[9] J. Szilasi, Horizontal maps with homogeneity condition, *Rendiconti di Palermo, Suppl.*, **3** (1984), 307—320.

[10] O. Varga, Bestimmung des invarianten Differentials in Finsler'schen Räumen, *Mat. Fiz. Lapok*, **48** (1941), 423—435 (in Hungarian).

[11] O. Varga, Zur Herleitung des invarianten Differentials in Finslerschen Räumen, *Monatsh. Mat. Phys.*, **50** (1942—43), 165—175.

[12] J. Vilms, Non linear and direction connections, *Proc. of Am. Math. Soc.*, **28** (1971), 45—58.

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KOSSUTH LAJOS UNIVERSITY
DEPARTMENT OF MATHEMATICS
4010 DEBRECEN
HUNGARY

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EXTENSIONS OF THE SPACES c AND c_0 FROM SINGLE TO DOUBLE SEQUENCES

F. MÓRICZ (Szeged)

1. Convergence in Pringsheim's sense and regular convergence. We will consider double sequences $A=\{a_{mn}: m, n=1, 2, ...\}$ of complex (or real) numbers. We remind the reader that A is said to *converge in Pringsheim's sense* if there exists a number l such that a_{mn} converges to l as both m and n tend to ∞ independently of one another:

(1.1)
$$\lim_{m,n\to\infty} a_{mn} = l.$$

It is almost trivial that $A=\{a_{mn}\}$ converges in Pringsheim's sense if and only if for every $\varepsilon>0$ there exists an integer $N=N(\varepsilon)$ such that

$$|a_{jk}-a_{mn}| \leqq \varepsilon \quad \text{whenever} \quad \min\{j, k, m, n\} \geqq N.$$

The crucial difference between the convergence of single sequences and the convergence in Pringsheim's sense of double sequences is that the latter does not imply the boundedness of the terms of the double sequence in question.

Following Hardy [1], a double sequence $A=\{a_{mn}\}$ is said to *converge regularly* if it converges in Pringsheim's sense and, in addition, the following finite limits exist:

(1.2)
$$\lim_{n\to\infty} a_{mn} = k_m \quad (m = 1, 2, \ldots),$$

(1.3)
$$\lim_{m\to\infty} a_{mn} = l_n \quad (n = 1, 2, \ldots).$$

(For more details, see also [2].)

Obviously, the regular convergence of A implies the convergence in Pringsheim's sense as well as the boundedness of the terms of A, but the converse implication fails.

2. Linear spaces of double sequences. We will consider the following linear spaces of double sequences $A=\{a_{mn}\}$:

[a] This research was completed while the author was a visiting professor at the University of Tennessee, Knoxville, during the academic year 1987—88.

$$l_2^1 = \left\{A: \|A\|_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}| < \infty\right\};$$

$$l_2^\infty = \{A: \|A\|_\infty = \sup_{m,n\geqq 1} |a_{mn}| < \infty\};$$

c_2^R, the space of regularly convergent sequences;

$${}_0c_2^R = \left\{A: a_{mn} \to 0 \text{ as } \max\{m, n\} \to \infty\right\};$$

c_2^P, the space of sequences, convergent in Pringsheim's sense;

$${}_0c_2^P = \left\{A: a_{mn} \to 0 \text{ as } \min\{m, n\} \to \infty\right\};$$

$$c_2^{PB} = c_2^P \cap l_2^\infty, \quad {}_0c_2^{PB} = {}_0c_2^P \cap l_2^\infty.$$

As is known, l_2^1 and l_2^∞ are Banach spaces. We will prove that c_2^R, ${}_0c_2^R$, c_2^{PB} and ${}_0c_2^{PB}$ endowed with the norm $\|\cdot\|_\infty$ are also Banach spaces.

Furthermore, we define the pseudonorm

$$\|A\|_P = \lim_{N\to\infty} \sup_{m,n\geqq N} |a_{mn}| \tag{2.1}$$

for $A \in c_2^P$. We will prove that c_2^P is complete under this pseudonorm and observe that $\|A\|_P = 0$ holds identically for any $A \in {}_0c_2^P$.

Remark 1. It is impossible to introduce a nontrivial norm in c_2^P or ${}_0c_2^P$ in such a way to make them Banach spaces. This follows immediately from the following two well-known facts:

(i) The linear space ω of all single sequences cannot be complete under any nontrivial norm.

(ii) On the other hand, ω can be imbedded into ${}_0c_2^P$ in a trivial way. Namely, given any single sequence $\{a_j\}$, define $a_{jk} = a_j$ if $k=1$; $j=1, 2, \ldots$; and $=0$ otherwise. Clearly $\{a_{jk}\} \in {}_0c_2^P$.

Theorem 1. *c_2^R and ${}_0c_2^R$ are Banach spaces under the norm $\|\cdot\|_\infty$.*

Proof. The only thing we have to prove is completeness. We present it in the case of c_2^R.

To this effect, assume that $\{A^{(q)}: q=1, 2, \ldots\}$ is a Cauchy sequence in c_2^R. Let $A^{(q)} = \{a_{mn}^{(q)}: m, n=1, 2, \ldots\}$. By assumption, for every $\varepsilon > 0$ there exists an integer $q_0 = q_0(\varepsilon)$ such that

$$\|A^{(q)} - A^{(r)}\|_\infty \leqq \varepsilon \quad \text{if} \quad \min\{q, r\} \geqq q_0.$$

This implies that for all m and n,

$$|a_{mn}^{(q)} - a_{mn}^{(r)}| \leqq \varepsilon \quad \text{if} \quad \min\{q, r\} \geqq q_0. \tag{2.2}$$

Consequently, the finite limits

$$\lim_{q\to\infty} a_{mn}^{(q)} = a_{mn}$$

exist for all m and n. Letting $r \to \infty$ in (2.2) yields

$$|a_{mn}^{(q)} - a_{mn}| \leqq \varepsilon \quad \text{if} \quad q \geqq q_0 \tag{2.3}$$

for all m and n. Setting $A=\{a_{mn}\}$, (2.3) shows that

(2.4) $$\lim_{q\to\infty} \|A^{(q)}-A\|_\infty = 0.$$

We still have to verify that $A\in c_2^R$. By (2.3),

(2.5) $$|a_{jk}-a_{mn}| \leqq |a_{jk}-a_{jk}^{(q)}|+|a_{jk}^{(q)}-a_{mn}^{(q)}|+|a_{mn}^{(q)}-a_{mn}| \leqq 2\varepsilon+|a_{jk}^{(q)}-a_{mn}^{(q)}|$$

where $q\geqq q_0$ is fixed. Even simpler inequalities hold for the differences $|a_{jn}-a_{mn}|$ and $|a_{mk}-a_{mn}|$. Since $A^{(q)}\in c_2^R$, we can conclude $A\in c_2^R$, as well.

THEOREM 2. *c_2^P is complete under the pseudonorm $\|\cdot\|_P$.*

PROOF. Completeness follows in a standard way if we take into account that $\|A\|_P=|l|$ where l is the limit of A in Pringsheim's sense (cf. (1.1) and (2.1)).

A norm $\|\cdot\|$ is said to be a norm "*with continuous coordinates*" if for each pair (m, n) there exists a constant K_{mn} such that for every $A=\{a_{mn}\}$ belonging to the space in question we have

(2.6) $$|a_{mn}| \leqq K_{mn}\|A\|.$$

THEOREM 3. *There exists no norm in ${}_0c_2^P$ with continuous coordinates.*

Clearly, c_2^P also enjoys the same property.

PROOF. On the contrary, suppose that $\|\cdot\|$ is a norm in ${}_0c_2^P$ satisfying (2.6) for each pair (m, n). We define $a_{mn}=mK_{m1}$ if $m=1, 2, \dots;$ $n=1$ and; $=0$ otherwise. Then $A\in {}_0c_2^P$ and $\|A\|\geqq|a_{m1}|/K_{m1}=m$ for each m. This implies $\|A\|=\infty$, which is a contradiction.

REMARK 2. There exist "artificial" norms in c_2^P and ${}_0c_2^P$. To get one, we choose a *Hamel base* for c_2^P, say, i.e. a maximal linearly independent subset $\{A_\gamma:\ \gamma\in\Gamma\}$ in the linear space c_2^P, where Γ is a set of indices (see, e.g. [3, p. 52]). Then any $A\in c_2^P$ has a unique representation

$$A = \sum_{j=1}^{k} \lambda_j A_{\gamma_j}, \quad \text{and consequently,} \quad \|A\| = \sum_{j=1}^{k} |\lambda_j|$$

provides a norm in c_2^P. But we cannot expect this norm to be complete.

THEOREM 4. *c_2^{PB} and ${}_0c_2^{PB}$ are Banach spaces under the norm $\|\cdot\|_\infty$.*

PROOF. We prove the completeness in the case of c_2^{PB}. To this end, assume that $\{A^{(q)}:\ q=1, 2, \dots\}$ is a Cauchy sequence in c_2^{PB}. Then, as in the proof of Theorem 1, the $\{a_{mn}^{(q)}:\ q=1, 2, \dots\}$ are Cauchy sequences with the limits a_{mn} for all m and n. Furthermore, given any $\varepsilon>0$ there exists $q_0=q_0(\varepsilon)$ such that (2.3) is satisfied for all m and n, and consequently, (2.4) is also satisfied.

It remains to check that $A\in c_2^{PB}$. The convergence of A in Pringsheim' sense follows from (2.5), while the boundedness of A follows from (2.3) since $A^{(q)}\in c_2^{PB}$.

3. Linear functionals in c_2^R and ${}_0c_2^R$. In this paper, by a linear functional we always mean a bounded (or continuous) one.

THEOREM 5. *Every linear functional L in c_2^R is of the form*

$$L(A) = l\beta_{00} + \sum_{m=1}^{\infty} k_m \beta_{m0} + \sum_{n=1}^{\infty} l_n \beta_{0n} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \beta_{mn} \tag{3.1}$$

where $A=\{a_{mn}\}$, the l, k_m and l_n are defined in (1.1)—(1.3), *and the β_{mn} are complex numbers for which*

$$\|L\| = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\beta_{mn}| < \infty. \tag{3.2}$$

REMARK 3. In the particular case where $A \in c_2^R$ we have $l=k_m=l_n=0$ for all m and n. It follows from Theorem 5 that every linear functional L in ${}_0c_2^R$ is of the form

$$L(A) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \beta_{mn}$$

where $B=\{\beta_{mn}\} \in l_2^1$ and $\|L\|=\|B\|_1$.

Thus, the dual space of both c_2^R and ${}_0c_2^R$ is isomorphic to the same space l_2^1. This phenomenon is well-known in the case of single convergent sequences.

PROOF OF THEOREM 5. *Sufficiency.* It is almost immediate. Since $|l|, |k_m|, |l_n| \leqq \|A\|_\infty$ for all m and n, it follows from (3.1) that

$$\|L\| \leqq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\beta_{mn}|. \tag{3.3}$$

By (3.2), this means the boundedness of L.

Necessity. We assume that a linear functional L is given in c_2^R and prove the existence of a double sequence $\{\beta_{mn}\colon m, n=0, 1, \ldots\}$ of complex numbers such that conditions (3.1) and (3.2) are satisfied.

To this effect, we define $B^{(qr)}=\{b_{mn}^{(qr)}\colon m, n=1, 2, \ldots\}$, $B^{(q)}=\{b_{mn}^{(q)}\}$, $C^{(r)}=\{c_{mn}^{(r)}\}$ and $C=\{c_{mn}\}$ as follows. For all m, n, q, r we set $b_{mn}^{(qr)}=1$ if $(m, n)=(q, r)$ and $=0$ otherwise; $b_{mn}^{(q)}=1$ is $m=q$ and $=0$ otherwise; $c_{mn}^{(r)}=1$ if $n=r$ and $=0$ otherwise; and $c_{mn}=1$. Furthermore, we set

$$D^{(qr)} = A - lC - \sum_{m=1}^{q} (k_m - l) B^{(m)} - \sum_{n=1}^{r} (l_n - l) C^{(n)} - \tag{3.4}$$

$$- \sum_{m=1}^{q} \sum_{n=1}^{r} (a_{mn} - k_m - l_n + l) B^{(mn)},$$

where $A=\{a_{mn}\} \in c_2^R$. It is not hard to see that

$$\|D^{(qr)}\|_\infty = \max\{\sup_{m>q \text{ and } n \leqq r} |a_{mn} - l_n|, \sup_{n>q \text{ and } m \leqq r} |a_{mn} - k_m|, \sup_{m>q \text{ and } n>r} |a_{mn} - l|\},$$

whence

$$\lim_{q, r \to \infty} \|D^{(qr)}\|_\infty = 0.$$

Letting $q\to\infty$ and $r\to\infty$ in (3.4) yields

$$A = lC+\sum_{m=1}^{\infty}(k_m-l)B^{(m)}+\sum_{n=1}^{\infty}(l_n-l)C^{(n)}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{mn}-k_m-l_n+l)B^{(mn)}.$$

Now we set

$$\gamma = L(C),\quad \beta_m = L(B^{(m)}),\quad \gamma_n = L(C^{(n)}),\quad \beta_{mn} = L(B^{(mn)}).$$

Since L is linear and continuous, it follows that

$$L(A) = l\gamma+\sum_{m=1}^{\infty}(k_m-l)\beta_m+\sum_{n=1}^{\infty}(l_n-l)\gamma_n+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{mn}-k_m-l_n+l)\beta_{mn}. \tag{3.5}$$

We reveal a few properties of the coefficients β_m, γ_n and β_{mn}. For this aim, we define $A^{(qr)}=\{a_{mn}^{(qr)}\}$ as follows: $a_{mn}^{(qr)}=\operatorname{sgn}\beta_{mn}$ if $m=1, 2, \dots, p$; $n=1, 2, \dots, q$; and $=0$ otherwise. Then by (3.5),

$$\sum_{m=1}^{q}\sum_{n=1}^{r}|\beta_{mn}| = L(A^{(qr)}) \leqq \|L\|\,\|A^{(qr)}\|_\infty = \|L\|.$$

Letting $q\to\infty$ and $r\to\infty$ yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\beta_{mn}| < \infty. \tag{3.6}$$

Next we define $A^{(q)}=\{a_{mn}^{(q)}\}$, where $a_{mn}^{(q)}=\operatorname{sgn}\beta_m$ if $m=1, 2, \dots, q$; $n=1, 2, \dots$; and $=0$ otherwise. Again by (3.5),

$$L(A^{(q)}) = \sum_{m=1}^{q}|\beta_m|+\sum_{m=1}^{q}\sum_{n=1}^{\infty}(\operatorname{sgn}\beta_m-1)\beta_{mn},$$

whence

$$L(A^{(q)}) \geqq \sum_{m=1}^{q}|\beta_m|-2\sum_{m=1}^{q}\sum_{n=1}^{\infty}|\beta_{mn}|.$$

Combining this with the boundedness of L gives

$$\sum_{m=1}^{q}|\beta_m| \leqq \|L\|+2\sum_{m=1}^{q}\sum_{n=1}^{\infty}|\beta_{mn}|.$$

Letting $q\to\infty$ and taking (3.6) into account, we get that

$$\sum_{m=1}^{\infty}|\beta_m| < \infty. \tag{3.7}$$

We can conclude similarly that

$$\sum_{n=1}^{\infty}|\gamma_n| < \infty. \tag{3.8}$$

Thanks to (3.6)—(3.8), we can rearrange the right-hand side of (3.5) as follows

$$L(A) = l\beta_{00}+\sum_{m=1}^{\infty} k_m\beta_{m0}+\sum_{n=1}^{\infty} l_n\beta_{0n}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\beta_{mn} \tag{3.9}$$

where

$$\beta_{00} = \gamma-\sum_{m=1}^{\infty}\beta_m-\sum_{n=1}^{\infty}\gamma_n+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_{mn},$$

$$\beta_{m0} = \beta_m-\sum_{n=1}^{\infty}\beta_{mn},\quad \beta_{0n} = \gamma_n-\sum_{m=1}^{\infty}\beta_{mn}.$$

By (3.6)—(3.8) we have $\{\beta_{mn}\colon m, n=0, 1, \ldots\}\in l_2^1$, while (3.9) concides with (3.1) to be proved.

Finally, we prove the equality part in (3.2). For this purpose, we define $C^{(qr)}= =\{c_{mn}^{(qr)}\}$ as follows

$$c_{mn}^{(qr)} = \begin{cases} \operatorname{sgn}\beta_{mn} & \text{if}\quad m=1,2,\ldots,q;\ n=1,2,\ldots,r;\\ \operatorname{sgn}\beta_{m0} & \text{if}\quad m=1,2,\ldots,q;\ n=r+1,r+2,\ldots;\\ \operatorname{sgn}\beta_{0n} & \text{if}\quad m=q+1,q+2,\ldots;\ n=1,2,\ldots,r;\\ \operatorname{sgn}\beta_{00} & \text{if}\quad m=q+1,q+2,\ldots;\ n=r+1,r+2,\ldots. \end{cases}$$

Then by (3.5),

$$L(C^{(qr)}) = |\beta_{00}|+\sum_{m=1}^{\infty}|\beta_{m0}|+\sum_{n=1}^{r}|\beta_{0n}|+\sum_{m=1}^{q}\sum_{n=1}^{r}|\beta_{mn}|+$$

$$+\sum_{m=q+1}^{\infty}\beta_{m0}\operatorname{sgn}\beta_{00}+\sum_{n=r+1}^{\infty}\beta_{0n}\operatorname{sgn}\beta_{00}+\sum_{m=q+1}^{\infty}\sum_{n=1}^{r}\beta_{mn}\operatorname{sgn}\beta_{0n}+$$

$$+\sum_{m=1}^{q}\sum_{n=r+1}^{\infty}\beta_{mn}\operatorname{sgn}\beta_{m0}+\sum_{m=q+1}^{\infty}\sum_{n=r+1}^{\infty}\beta_{mn}\operatorname{sgn}\beta_{00}.$$

By the boundedness of L,

$$\|L\| = \|L\|\,\|C^{(qr)}\|_\infty \geqq |L(C^{(qr)})| \geqq$$

$$\geqq \sum_{m=0}^{q}\sum_{n=0}^{r}|\beta_{mn}|-\sum_{m=q+1}^{\infty}\sum_{n=0}^{r}|\beta_{mn}|-\sum_{m=0}^{q}\sum_{n=r+1}^{\infty}|\beta_{mn}|-\sum_{m=q+1}^{\infty}\sum_{n=r+1}^{\infty}|\beta_{mn}|.$$

Letting $q\to\infty$ and $r\to\infty$ yields

$$\|L\| \geqq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\beta_{mn}|. \tag{3.10}$$

Combining (3.3) and (3.10) results in (3.2).

4. Linear functionals in c_2^P. We recall that $\|\cdot\|_P$ defined by (2.1) is only a pseudo-norm in c_2^P, but it is complete.

THEOREM 6. *Every linear functional L in c_2^P is of the form $L(A)=\gamma l$ where γ is a complex number and l is the limit of A in Pringsheim's sense, and $\|L\|=|\gamma|$.*

PROOF. *Sufficiency.* Trivial.

Necessity. Let $C=\{c_{mn}=1;\ m, n=1, 2, \dots\}$ and $\gamma=L(C)$. Next, we observe that if $A^{(1)}$ and $A^{(2)}$ have the same limit l in Pringsheim's sense, then

$$|L(A^{(1)})-L(A^{(2)})| \leqq \|L\|\ \|A^{(1)}-A^{(2)}\|_P = 0,$$

whence $L(A^{(1)})=L(A^{(2)})$. Since A and lC have the same limit, we can conclude that

$$L(A) = L(lC) = lL(C) = l\gamma.$$

5. Linear functionals in c_2^{PB} and ${}_0c^{PB}$. We recall the following commonly used notations. If $\mathscr{X}$ is a Banach space, then $c(\mathscr{X})$ means the space of the sequences $X=\{x_j\in\mathscr{X}:\ j=1, 2, \dots\}$ such that the finite limit

$$\lim_{j\to\infty} \|x_j\|_{\mathscr{X}} = l$$

exists. If $l=0$, then we write $X\in c_0(\mathscr{X})$. As is known, $c(\mathscr{X})$ and $c_0(\mathscr{X})$ endowed with the norm

$$|||X|||_\infty = \sup_{j\geqq 1} \|x_j\|_{\mathscr{X}}$$

are also Banach spaces. Furthermore, $l^1(\mathscr{X})$ stands for the Banach space of the sequences $X=\{x_j\in\mathscr{X}\}$ such that

$$|||X|||_1 = \sum_{j=1}^{\infty} \|x_j\|_{\mathscr{X}} < \infty.$$

In the sequel l^∞ is the familiar Banach space of the bounded single sequences endowed with the supremum norm.

THEOREM 7. *The following isomorphisms hold true:*

$$c_2^{PB} \approx c(l^\infty) \quad and \quad {}_0c_2^{PB} \approx c_0(l^\infty). \tag{5.1}$$

PROOF. Let $A=\{a_{mn}:\ m, n=1, 2, \dots\}$ be given. We relabel the elements a_{mn} in the following way: let

$$a_1^{(1)} = a_{11},\ a_2^{(1)} = a_{12},\ a_3^{(1)} = a_{21},\ a_4^{(1)} = a_{13},\ a_5^{(1)} = a_{31}, \dots$$

and more generally, for $j=1, 2, \dots$ let

$$a_1^{(j)} = a_{jj},\ a_{2k}^{(j)} = a_{j,j+k},\ a_{2k+1}^{(j)} = a_{j+k,j} \quad (k = 1, 2, \dots).$$

Furthermore, let $A^{(j)}=\{a_k^{(j)}:\ k=1, 2, \dots\}$ and $\mathscr{A}=\{A^{(j)}:\ j=1, 2, \dots\}$.

It is easy to see that for each $q=1, 2, \dots$

$$\sup_{m,n\geqq q} |a_{mn}| = \sup_{j\geqq q}\sup_{k\geqq 1} |a_k^{(j)}|.$$

Consequently, $\mathscr{A}\in c(l^\infty)$ or $c_0(l^\infty)$ if and only if $A\in c_2^{PB}$ or ${}_0c_2^{PB}$, respectively; the mapping $A\to\mathscr{A}$ is one-to-one and $\|A\|_\infty=|||\mathscr{A}|||_\infty$.

In order to find the dual spaces of c_2^{PB} and ${}_0c_2^{PB}$, we refer to the commonplace that if $\mathscr{X}$ is a Banach space, then the dual space of $c(\mathscr{X})$ is isomorphic to $l^1(\mathscr{X}^*)$ where $\mathscr{X}^*$ denotes the dual space of $\mathscr{X}$. The dual space of $c_0(\mathscr{X})$ is also isomorphic to $l^1(\mathscr{X}^*)$. In particular,

$$(5.2)\qquad (c(l^\infty))^* \approx l^1((l^\infty)^*) \quad \text{and} \quad (c_0(l^\infty))^* \approx l^1((l^\infty)^*).$$

We remind the reader that the dual space of l^∞ is the Banach space $ba(Z_+, \mathscr{P}, \mu)$ of all bounded and finitely additive set functions μ defined on the class $\mathscr{P}$ of all subsets of the positive integers Z_+ and endowed with the norm

$$\|\mu\| = \sup_{P\subset\mathscr{P}} |\mu(P)|,$$

the so-called total variation of μ on Z_+. (See, e.g. [4, pp.118—119] where the example of $(L^\infty)^*$ is presented in details; but this covers $(l^\infty)^*$ by using counting measure.)

Combining (5.1) and (5.2) provides our last result.

THEOREM 8. *The following isomorphisms hold true:*

$$(c_2^{PB})^* \approx l^1((l^\infty)^*) \quad \textit{and} \quad ({}_0c_2^{PB})^* \approx l^1((l^\infty)^*).$$

In other words, an element M in $l^1((l^\infty)^*)$ is a sequence $M=\{\mu_j\colon j=1, 2, \ldots\}$ of bounded and finitely additive set functions defined on $\mathscr{P}$ and such that

$$|||M|||_1 = \sum_{j=1}^{\infty} \|\mu_j\| < \infty.$$

REMARK 4. The spaces introduced and results proved in this paper extend in a natural way to d-multiple sequences, as well, where d is a fixed integer, $d\geq 3$. (Concerning the definition of regular convergence for d-multiple sequences and series see e.g. [2].)

References

[1] G. H. Hardy, On the convergence of certain multiple series, *Proc. Cambridge Philos. Soc.*, **19** (1916—1919), 86—95.

[2] F. Móricz, Some remarks on the notion of regular convergence of multiple series, *Acta. Math. Hungar.*, **41** (1983), 161—168.

[3] W. Rudin, *Functional Analysis*, McGraw-Hill, Inc. (New York, 1973).

[4] K. Yosida, *Functional Analysis*, Academic Press, Springer (Berlin—Göttingen—Heidelberg, 1965).

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BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

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ON BANACH SPACES OF ABSOLUTELY AND STRONGLY CONVERGENT FOURIER SERIES. II

I. SZALAY (Szeged) and N. TANOVIĆ-MILLER* (Sarajevo)

1. Introduction and preliminaries

Let (s_k) be a sequence of real or complex numbers. If $s_k \to t$ in the ordinary sense, we write $s_k \to t\ I$; and if $\sum |s_k - s_{k-1}| < \infty$ in which case $s_k \to t\ I$, for some number t, we say that (s_k) converges to t absolutely and write $s_k \to t\ |I|$.

The latter notion was extended to larger indices $\lambda > 1$, called the absolute convergence of index λ and denoted $|I|_\lambda$. The strong convergence of index $\lambda \geqq 1$, denoted $|I|_\lambda$, is a convergence type that lies between the absolute convergence $|I|_\lambda$ and the ordinary convergence I, i.e., $|I|_\lambda \Rightarrow [I]_\lambda \Rightarrow I$. We shall denote the strong convergence of index 1 simply by $[I]$. For definitions and basic properties of these notions we refer the reader to [7], [8] or [11].

Given a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1.1}$$

let $s_n(x)$ and $\sigma_n(x)$ denote the ordinary and the Cesàro C_1, n-th partial sums of (1.1), resp. As usual, let L^p, $p \geqq 1$, be the Banach space of all real or complex valued 2π-periodic integrable functions f with the norm $\|f\|_{L^p} = \left(\frac{1}{2\pi} \int |f|^p\right)^{1/p}$, where the integral is taken over any interval of length 2π. Let C be the Banach space of all continuous real or complex valued 2π-periodic functions f with the norm $\|f\|_C = \sup_x |f(x)|$. If (1.1) is a Fourier—Lebesgue series of a function $f \in L^1$ we shall write $s_n f$, $\sigma_n f$ and $\hat{f}_c(k)$, $\hat{f}_s(k)$ for the partial sums s_n, σ_n and the coefficients a_k, b_k respectively.

In a series of recent papers [5] through [11], the above concepts of the absolute $|I|_\lambda$ and strong $[I]_\lambda$ convergence were applied to trigonometric and Fourier series. This led to questions about properties of classes of functions whose Fourier series are $|I|_\lambda$ or $[I]_\lambda$ convergent, pointwise a.e. or uniformly, defined for $\lambda \geqq 1$ as follows:

$$\mathscr{S}^\lambda = \{f \in C:\ s_n f \to f\ |I|_\lambda \text{ uniformly}\},$$

$$\mathscr{A}^\lambda = \{f \in C:\ s_n f \to f\ |I|_\lambda \text{ uniformly}\},$$

$$S^\lambda = \{f \in L^1:\ s_n f \to f\ |I|_\lambda \text{ a.e.}\},$$

$$A^\lambda = \{f \in L^1:\ s_n f \to f\ |I|_\lambda \text{ a.e.}\}.$$

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The first two are naturally related to the following well-known Banach spaces of uniformly and absolutely convergent Fourier series, respectively:

$$\mathcal{U}=\{f\in C\colon\ s_nf\to f\ I\ \text{uniformly}\},\quad \|f\|_{\mathcal{U}}=\sup_n \|s_nf\|_C,$$

$$\mathcal{A}=\{f\in C\colon\ s_nf\to f\ |I|\ \text{uniformly}\},\quad \|f\|_{\mathcal{A}}=\frac{|\hat{f}_c(0)|}{2}+\sum_{k=1}^{\infty}|\hat{f}_c(k)|+|\hat{f}_s(k)|.$$

We recall that

$$\mathcal{A}\subset\mathcal{U}\subset\{f\in L^1\colon\ s_nf\to f\ I\ \text{a.e.}\}$$

properly, while

$$\mathcal{A}=\{f\in L^1\colon\ s_nf\to f\ |I|\ \text{a.e.}\}=\{f\colon\ s_n\to f\ |I|\ \text{a.e.}\}.$$

For these well-known facts see Theorem A in [7] or [4]. Clearly $\mathcal{A}^1=A^1=\mathcal{A}$. By our previous results, see [8], $\mathcal{S}^1\neq S^1$. We shall denote $\mathcal{S}^1$ and S^1 by $\mathcal{S}$ and S respectively.

The class $\mathcal{S}$ was studied in [6] and the classes $\mathcal{S}^\lambda$, $\lambda>1$ and S^λ, $\lambda\geqq 1$ have been investigated in [11]. It was shown there that both S^λ and S^λ, $\lambda\geqq 1$ are Banach spaces with the respective norms determined by the partial sums s_nf. For convenience of this presentation, in the following theorem, we list some of the basic properties of the classes.

THEOREM A. *Let* $\lambda\geqq 1$. *Then*

(i) $f\in S^\lambda$ *if and only if* $f\in L^\lambda$ *and*

$$\frac{1}{n+1}\sum_{k=0}^{n}k^\lambda(|\hat{f}_c(k)|+|\hat{f}_s(k)|)^\lambda=o(1)\quad(n\to\infty). \tag{1.2}$$

$f\in\mathcal{S}^\lambda$ *if and only if* $f\in C$ *and* (1.2) *holds.*

(ii) *Let* $E=L^\lambda, C$. *If* $f\in E$ *then* (1.2) *holds if and only if*

$$\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}k^\lambda|s_kf-s_{k-1}f|^\lambda\right)^{1/\lambda}\right\|_E=o(1)\quad(n\to\infty). \tag{1.3}$$

(iii) $\mathcal{S}^\lambda\subset S^\lambda$; $\mathcal{S}^\lambda\subset\mathcal{S}^{\lambda'}$ *and* $S^\lambda\subset S^{\lambda'}$ *for* $\lambda>\lambda'\geqq 1$.

(iv) $S^\lambda\subset L^\lambda$ *properly and* S^λ, $\|\ \|_{L^\lambda}$ *is not a Banach space.* $\mathcal{S}^\lambda\subset\mathcal{U}$ *properly and* $\mathcal{S}^\lambda$, $\|\ \|_{\mathcal{U}}$ *is not a Banach space.*

(v) S^λ *and* $\mathcal{S}^\lambda$ *are Banach spaces endowed with the norms*

$$\|f\|_{S^\lambda,0}\doteq\sup_n\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}|(k+1)s_kf-ks_{k-1}f|^\lambda\right)^{1/\lambda}\right\|_{L^\lambda}, \tag{1.4}$$

$$\|f\|_{\mathcal{S}^\lambda,0}\doteq\sup_n\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}|(k+1)s_kf-ks_{k-1}f|^\lambda\right)^{1/\lambda}\right\|_{C}, \tag{1.5}$$

respectively. Moreover

$$\|f\|_{L^\lambda}\leqq\sup_n\|s_nf\|_{L^\lambda}\leqq\|f\|_{S^\lambda,0} \tag{1.6}$$

and

$$\|f\|_C \leqq \sup_n \|s_n f\|_C = \|f\|_{\mathcal{U}} \leqq \|f\|_{\mathcal{S}^\lambda,0} \tag{1.7}$$

(vi) *If* $f \in S^\lambda$ *then* $\|s_n f - f\|_{S^\lambda,0} = o(1)$ $(n \to \infty)$. *If* $f \in \mathcal{S}^\lambda$ *then* $\|s_k f - f\|_{\mathcal{S}^\lambda,0} = o(1)$ $(n \to \infty)$.

Here in (1.4), (1.5) and later expressions like $s_{-1}f$ and s_{-1} are to be interpreted as zero. Statements (i) through (iv) are contained in Theorem 1, Lemma 1 and the Remark 1 in [11]. Statements (v), and (vi) are Theorems 2 and 3 in [11], respectively.

By Theorem A (i), S^λ and $\mathcal{S}^\lambda$ are characterized as the respective subspaces of L^λ and C satisfying (1.2), a condition expressed in terms of the Fourier coefficients only. However the above norms (1.4) and (1.5) are given in terms of the partial sums $s_k f$. It is therefore natural to consider other norms on these spaces, describable by the Fourier coefficients and equivalent to those given by (1.4) and (1.5). For the classes $\mathcal{S}^\lambda$ this idea was already explored in [7]. Namely we have introduced there certain new norms denoted $\| \; \|_{\mathcal{S}^\lambda,i}$, $i=1, 2, 3$, equivalent to the original norm given by (1.5), see Theorem 1 in [7]. As a corollary of those results we have:

THEOREM B. *$\mathcal{S}^\lambda$ is a Banach space under the norm*

$$\|f\|_{S^\lambda,3} \doteq \sup_n \left(\frac{1}{n+1} \sum_{k=0}^{n} k^\lambda (|\hat{f}_c(k)| + |\hat{f}_s(k)|)^\lambda\right)^{1/\lambda} + \|f\|_{\mathcal{U}}. \tag{1.8}$$

The classes $\mathcal{A}^\lambda$ were introduced and studied in [7]. Using some earlier results of the first author, on the absolute $|I|_\lambda$ convergence of trigonometric series [5], and inspired by the above mentioned properties of the classes $\mathcal{S}^\lambda$ we have shown in [7, Section 3] that the classes $\mathcal{A}^\lambda$ can be endowed with the corresponding mutually equivalent norms $\| \; \|_{\mathcal{A}^\lambda,i}$ $i=1, 2, 3$. The most interesting among these norms is $\| \; \|_{\mathcal{A}^\lambda,3}$. Collecting some of the results proved there (see Theorems 3 through 6 in [7]) we have:

THEOREM C. *Let* $\lambda \geqq 1$. *Then*

(i) $f \in \mathcal{A}^\lambda$ *if and only if* $f \in C$ *and*

$$\sum_{k=0}^{\infty} k^{\lambda-1} (|\hat{f}_c(k)| + |\hat{f}_s(k)|)^\lambda < \infty. \tag{1.9}$$

(ii) *If* $f \in C$ *then* (1.9) *holds if and only if*

$$\Big\|\Big(\sum_{k=0}^{\infty} k^{\lambda-1} |s_k f - s_{k-1} f|^\lambda\Big)^{1/\lambda}\Big\|_C < \infty. \tag{1.10}$$

(iii) *$\mathcal{A}^\lambda$ and $\mathcal{A}^{\lambda'}$ are incomparable for* $\lambda > \lambda' \geqq 1$.

(iv) *$\mathcal{A}^\lambda \subset \mathcal{S}^\lambda$ properly and $\mathcal{A}^\lambda$ is not a Banach space under the norms* $\| \; \|_{\mathcal{S}^\lambda,i}$ $i=1, 2, 3$.

(v) *$\mathcal{A}^\lambda$ is a Banach space under the norm*

$$\|f\|_{\mathcal{A}^\lambda,3} \doteq \Big(\sum_{k=0}^{\infty} k^{\lambda-1} (|\hat{f}_c(k)| + |\hat{f}_s(k)|)^\lambda\Big)^{1/\lambda} + \|f\|_{\mathcal{U}}, \tag{1.11}$$

moreover

(1.12) $$\|f\|_{\mathscr{U}} \leqq \|f\|_{\mathscr{S}^\lambda,3} \leqq \|f\|_{\mathscr{A}^\lambda,3}.$$

(vi) *If* $f\in\mathscr{A}^\lambda$ *then* $\|s_nf-f\|_{\mathscr{A}^\lambda,3}=o(1)$ $(n\to\infty)$.

Theorems A, B and C show an obvious analogy between the spaces $\mathscr{S}^\lambda$ and $\mathscr{A}^\lambda$. The difference is illustrated by properties (iii). Moreover, the above mentioned norms, given by (1.8) and (1.11) are described in terms of the coefficients and the $\mathscr{U}$-norm. However for $\lambda=1$, $\mathscr{A}^1=\mathscr{A}$ and the norm $\|f\|_{\mathscr{A}}$ is determined by the coefficients only. This will turn out to be natural in view of the properties of the spaces A^λ and the fact that $A^1=\mathscr{A}$ also.

The objects of this paper are the classes S^λ and A^λ. Inspired by the just described results on the classes $\mathscr{S}^\lambda$ and $\mathscr{A}^\lambda$, proved in [7] we shall consider the corresponding norms for the classes S^λ and A^λ, $\lambda\geqq1$. Naturally we will obtain analogues of the properties listed in Theorems B and C, i.e., of the corresponding theorems proved in [7]. Whenever an analogy between the classes S^λ and $\mathscr{S}^\lambda$, respectively, A^λ and $\mathscr{A}^\lambda$, is discussed, we shall avoid repeating the arguments that are similar to those presented in [7]. Moreover, we shall prove that the spaces S^λ and A^λ, $\lambda>1$ are Banach spaces endowed with the norms determined by the Fourier coefficients only. This is clearly consistent with the above mentioned fact for $A^1=\mathscr{A}$.

2. Banach spaces S^λ and some equivalent norms

The essential properties of the classes S^λ have been recalled in Theorem A, Section 1. Our goal here is to show that S^λ can be endowed with a norm described in terms of the Fourier coefficients. Due to the obvious similarities with the classes $\mathscr{S}^\lambda$ we should naturally define the new norms on S^λ simply by substituting $\|\ \|_C$ by $\|\ \|_{L^\lambda}$ in the definitions of the norms $\|\ \|_{\mathscr{S}^\lambda,i}$ $i=1, 2, 3$ in [7, Section 2]. The most interesting among those norms was certainly $\|\ \|_{\mathscr{S}^\lambda,3}$, consisting of a part determined by the Fourier coefficients and the $\mathscr{U}$-norm, see (1.8). It will be shown here that more can be achieved in the case of S^λ, $\lambda>1$. Modifying a little the approach used in [7, Section 2], we define the following norms on S^λ, $\lambda\geqq1$:

(2.1) $$\|f\|_{[\lambda]'} \doteq \sup_n \left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\lambda|s_kf-s_{k-1}f|^\lambda\right)^{1/\lambda}\right\|_{L^\lambda},$$

(2.2) $$\|f\|_{[\lambda]} \doteq \sup_n \left(\frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\lambda(|\hat f_c(k)|+|\hat f_s(k)|)^\lambda\right)^{1/\lambda},$$

(2.3) $$\|f\|_{S^\lambda,2} \doteq \|f\|_{[\lambda]'}+\sup_n\|s_nf\|_{L^\lambda},$$

(2.4) $$\|f\|_{S^\lambda,3} \doteq \|f\|_{[\lambda]}+\sup_n\|s_nf\|_{L^\lambda}.$$

Clearly $\|f\|_{S^\lambda,2}$ and $\|f\|_{S^\lambda,3}$ correspond to the norms $\|f\|_{\mathscr{S}^\lambda,2}$ and $\|f\|_{\mathscr{S}^\lambda,3}$ introduced in [7, Section 2] and we could define, similarly, the norm $\|f\|_{S^\lambda,1}$ corresponding to $\|f\|_{\mathscr{S}^\lambda,1}$. In view of the results of this section this later norm is however, less interesting.

REMARK 1. Let $\lambda \geqq 1$. Then for each $f \in S^\lambda$, $\|f\|_{[\lambda]}$, $\|f\|_{[\lambda]'}$ and $\|f\|_{S^\lambda,i}$, $i=2,3$ are finite and do define norms. Moreover

$$\|f\|_{[\lambda]'} \leqq \|f\|_{[\lambda]} \tag{2.5}$$

and

$$\|f\|_{S^\lambda,0} \leqq \|f\|_{S^\lambda,2} \leqq \|f\|_{S^\lambda,3}. \tag{2.6}$$

PROOF. Glancing at (2.1) and (2.2) the inequality (2.5) is obvious. That these expressions are finite for each $f \in S^\lambda$ follows from Theorem A (i). By Minkowski's inequality

$$\left(\frac{1}{n+1}\sum_{k=0}^{n}|(k+1)s_k f - k s_{k-1} f|^\lambda\right)^{1/\lambda} \leqq$$

$$\leqq \left(\frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\lambda|s_k f - s_{k-1} f|^\lambda\right)^{1/\lambda} + \left(\frac{1}{n+1}\sum_{k=0}^{n}|s_k f|^\lambda\right)^{1/\lambda}$$

and clearly

$$\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}|s_k f|^\lambda\right)^{1/\lambda}\right\|_{L^\lambda}^\lambda = \frac{1}{n+1}\sum_{k=0}^{n}\|s_k f\|_{L^\lambda}^\lambda. \tag{2.7}$$

Hence the inequality (2.6) follows immediately from (1.4), the above definitions and (2.5). That $\|f\|_{S^\lambda,i}$, $i=2,3$ are finite for each $f \in S^\lambda$ is clear from (1.2), (1.6) and Theorem A. Finally that all of the above expressions, given by (2.1) through (2.4), do define norms on S^λ can be easily verified, applying Minkowski's inequality.

THEOREM 1. *Let $\lambda \geqq 1$. The norms $\|\ \|_{[\lambda]'}$ and $\|\ \|_{[\lambda]}$ are equivalent. Furthermore, the norms $\|\ \|_{S^\lambda,i}$, $i=0,2,3$ are mutually equivalent.*

PROOF. We prove the first statement by showing that for each $f \in S^\lambda$:

$$\|f\|_{[\lambda]'} \leqq \|f\|_{[\lambda]} \leqq 4\|f\|_{[\lambda]'}. \tag{2.8}$$

Now the left inequality is (2.5) and the inequality on the right follows by the same argument as in the proof of Theorem 1 in [7]. Namely, by [7, Lemma 1] we have

$$\left(\frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\lambda(|\hat{f}_c(k)|+|\hat{f}_s(k)|)^\lambda\right)^{1/\lambda} \leqq 4\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\lambda|s_k f - s_{k-1} f|^\lambda\right)^{1/\lambda}\right\|_{L^\lambda}.$$

Hence (2.8).

To show the equivalence of $\|\ \|_{S^\lambda,i}$, $i=0,2,3$, by (2.6) it suffices to prove that there exists a constant K such that

$$\|f\|_{S^\lambda,3} \leqq K\|f\|_{S^\lambda,0}. \tag{2.9}$$

But from (2.8) clearly $\|f\|_{S^\lambda,3} \leqq 4\|f\|_{S^\lambda,2}$ and by Minkowski's inequality and (2.7)

$$\|f\|_{S^\lambda,2} \leqq \|f\|_{S^\lambda,0} + 2\sup_n \|s_n f\|_{L^\lambda}.$$

Consequently by (1.6) $\|f\|_{S^\lambda,2} \leqq 3\|f\|_{S^\lambda,0}$ and (2.9) holds with $K=12$.

As a corollary of Theorem A and the second statement of Theorem 1 we can now write the following analogue of Theorem 2 in [7], and in particular of Theorem B.

THEOREM 2. *Let* $\lambda \geqq 1$.

(i) *For each* $i=0, 2, 3$, S^λ, $\| \ \|_{S^\lambda, i}$ *is a Banach space and*

$$\|f\|_{L^\lambda} \leqq \sup_n \|s_n f\|_{L^\lambda} \leqq \|f\|_{S^\lambda, i}. \tag{2.10}$$

(ii) *For every* $f \in S^\lambda$ *and for each* $i=0, 2, 3$, $\|s_n f - f\|_{S^\lambda, i} = o(1)$ $(n \to \infty)$.

(iii) $S^\lambda \subset L^\lambda$ *properly;* S^λ, $\| \ \|_{L^\lambda}$ *is not a Banach space and*

$$\sup_{f \in S^\lambda, f \neq 0} \|f\|_{S^\lambda, i} / \|f\|_{L^\lambda} = \infty, \quad i = 0, 2, 3. \tag{2.11}$$

PROOF. Statements (i) and (ii) are immediate consequences of (v) and (vi) of Theorem A and Theorem 1. The inequality (2.10) follows from (1.6) and (2.6).

Statement (iii) can be proved similarly as the corresponding claim of Theorem 2 in [7] or the Remark 1 in [11]. That $S^\lambda \subset L^\lambda$ properly follows from (iv) of Theorem A. So take $f \in L^\lambda \setminus S^\lambda$ and suppose that (2.11) does not hold for some $i=0, 2, 3$. Then by a well-known fact, see [3] or [12], $\|\sigma_n f - f\|_{L^\lambda} = o(1)$ $(n \to \infty)$, consequently $\|\sigma_n f - f\|_{S^\lambda, i} = o(1)$ $(n \to \infty)$. But $\sigma_n f \in S^\lambda$ for each n and $(\sigma_n f)$ is a Cauchy sequence in S^λ, $\| \ \|_{L^\lambda}$. Thus $(\sigma_n f)$ is a Cauchy sequence in S^λ, $\| \ \|_{S^\lambda, i}$, which by the above contradicts statement (i) because we have assumed that $f \notin S^\lambda$.

The most interesting among the norms appearing in Theorem 2 is certainly $\|f\|_{S^\lambda, 3} = \|f\|_{[\lambda]} + \sup_n \|s_n f\|_{L^\lambda}$. Our next result will show that for $\lambda > 1$, Theorem 2 can be improved, i.e. that, S^λ, $\| \ \|_{[\lambda]}$ is a Banach space for $\lambda > 1$.

LEMMA 1. *Let* $\lambda > 1$. *Then there exists a constant* K_λ *such that for each* $f \in S^\lambda$

$$\sup_n \|s_n f\|_{L^\lambda} \leqq K_\lambda \|f\|_{[\lambda]}. \tag{2.12}$$

PROOF. Suppose $\lambda > 1$ and let μ satisfy $1/\lambda + 1/\mu = 1$. Choose $1 < p < \min(\lambda, \mu)$ and let q be such that $1/p + 1/q = 1$. Arguing separately for $1 < \lambda \leqq 2$ and $\lambda > 2$ it can be easily verified that $q > \max(\lambda, 2)$.

Suppose $f \in S^\lambda$. Then by Theorem A, $f \in L^\lambda$ and (1.2) holds. Moreover, since $1 < p < \lambda$, $f \in S^p$ and (1.2) holds for p. An application of Hölder's inequality for q/λ yields

$$\|s_n f\|_{L^\lambda} \leqq \|s_n f\|_{L^q}. \tag{2.13}$$

By the Hausdorff—Young theorem we have

$$\|s_n f\|_{L^q} \leqq \Big(\sum_{k=0}^{n} (|\hat{f}_c(k)|^2 + |\hat{f}_s(k)|^2)^{p/2} \Big)^{1/p}.$$

Consequently from (2.13) it follows that

$$\|s_n f\|_{L^\lambda} \leqq \Big(\sum_{k=0}^{n} (|\hat{f}_c(k)| + |\hat{f}_s(k)|)^p \Big)^{1/p}. \tag{2.14}$$

On the other hand, by partial summation, using the assumption that $1<p<\min(\lambda,\mu)\leqq 2$ we have:

$$\Big(\sum_{k=0}^{n}(|\hat{f}_c(k)|+|\hat{f}_s(k)|)^p\Big)^{1/p}\leqq\Big[\sum_{k=0}^{n-1}\Big(\Delta\frac{1}{(k+1)^p}\Big)\sum_{i=0}^{k}(i+1)^p(|\hat{f}_c(i)|+|\hat{f}_s(i)|)^p+$$

$$+\frac{1}{(n+1)^p}\sum_{i=0}^{n}(i+1)^p(|\hat{f}_c(i)|+|\hat{f}_s(i)|)^p\Big]^{1/p}\leqq$$

$$\leqq\Big(\frac{p}{p-1}\Big)^{1/p}\|f\|_{[p]}\Big(\sum_{k=0}^{n-1}\Delta\frac{1}{(k+1)^{p-1}}+\frac{1}{(n+1)^{p-1}}\Big)^{1/p}=\Big(\frac{p}{p-1}\Big)^{1/p}\|f\|_{[p]}.$$

By Hölder's inequality, once more, $\|f\|_{[p]}<\|f\|_{[\lambda]}$. Hence from (2.14) and the last two inequalities we conclude that

$$\|s_nf\|_{L^\lambda}\leqq\Big(\frac{p}{p-1}\Big)^{1/p}\|f\|_{[\lambda]}$$

which implies (2.12).

THEOREM 3. *Let $\lambda>1$. Then S^λ endowed with the norms $\|\ \|_{[\lambda]}$ or $\|\ \|_{[\lambda]'}$ is a Banach space. For each $f\in S^\lambda$ $\|s_nf-f\|_{[\lambda]}=o(1)$ $(n\to\infty)$.*

PROOF. By Lemma 1, (2.4) and (2.12) there exists a constant K_λ such that

$$\|f\|_{[\lambda]}\leqq\|f\|_{S^\lambda,3}\leqq(1+K_\lambda)\|f\|_{[\lambda]}. \tag{2.15}$$

Hence by (i) Theorem 2, S^λ, $\|\ \|_{[\lambda]}$ is a Banach space. Furthermore by the first statement of Theorem 1, S^λ, $\|\ \|_{[\lambda]'}$ is also a Banach space. The second statement of the theorem is obvious by Theorem 2 (ii) and the above inequality (2.15).

REMARK 2. From (2.3), (2.8) and (2.12) it is trivial to see that also:

$$\|f\|_{[\lambda]'}\leqq\|f\|_{S^\lambda,2}\leqq(1+4K_\lambda)\|f\|_{[\lambda]'}. \tag{2.16}$$

3. Banach spaces A^λ and their relationships to $\mathcal{A}^\lambda$ and S^λ

The spaces $\mathcal{A}^\lambda$ have been studied in [7, Section 3] and their basic properties have been collected in Theorem C of Section 1. Theorem A shows an obvious correspondence between the classes $\mathcal{S}^\lambda$ and S^λ. Here we will investigate the classes A^λ, the analogues of the statements listed in Theorem C and other properties of these classes. Furthermore, by considering several norms on A^λ, paralleling the approach used in [7, Section 3] and in Section 2 here, we will prove that A^λ, $\lambda\geqq 1$ are Banach spaces endowed with the corresponding norms $\|\ \|_{|\lambda|}$ determined by the Fourier coefficients only. This will illustrate a similarity of the spaces A^λ, $\lambda\geqq 1$ with the spaces S^λ, $\lambda>1$ and a difference in regard to the spaces $\mathcal{A}^\lambda$, $\lambda>1$.

THEOREM 4. *Let $\lambda\geqq 1$. Then:*

(i) $f\in A^\lambda$ if and only if $f\in L^\lambda$ and

$$\|f\|_{|\lambda|}\doteq\Big(\sum_{k=0}^{\infty}(k+1)^{\lambda-1}(|\hat{f}_c(k)|+|\hat{f}_s(k)|)^\lambda\Big)^{1/\lambda}<\infty. \tag{3.1}$$

Moreover if $f\in L^\lambda$ *then* (3.1) *holds if and only if*

$$(3.2)\qquad \|f\|_{|\lambda|'} \doteq \Big\|\Big(\sum_{k=0}^{\infty}(k+1)^{\lambda-1}|s_kf-s_{k-1}f|^\lambda\Big)^{1/\lambda}\Big\|_{L^\lambda}<\infty.$$

(ii) $\mathscr{A}=A$ *and for* $\lambda>1$, $\mathscr{A}^\lambda\subset A^\lambda$ *properly.*

(iii) A^λ *and* $A^{\lambda'}$ *are incomparable for* $\lambda>\lambda'\geqq 1$.

(iv) $A^\lambda\subset S^\lambda\subset L^\lambda$ *properly and* A^λ *is not a Banach space under the norms* $\|\ \|_{S^\lambda,i}$, $i=0,2,3$.

(v) $A^\lambda=\{f\colon s^n\to f\ |I|_\lambda\ a.e.\}$, $A^\lambda\subset\bigcap_{p\geqq 1}L^p$ *and for* $\lambda>1$, $A^\lambda\not\subset L^\infty$.

PROOF. We begin by remarking that $A^\lambda\subset S^\lambda$, since $|I|_\lambda\Rightarrow|I|_\lambda$ for all $\lambda\geqq 1$, see [8] or property 3) in [7, Section 1] and the references cited there.

(i) Suppose $f\in A^\lambda$. Then by the above and Theorem A (i), clearly, $f\in L^\lambda$. Moreover by the definition of the $|I|_\lambda$ convergence, see either [5], [7], [8], [10] or [11],

$$(3.3)\qquad \sum_{k=0}^{\infty}(k+1)^{\lambda-1}|s_kf-s_{k-1}f|^\lambda<\infty\quad\text{a.e.}$$

Consequently (3.2) holds. Writing

$$s_kf(x)-s_{k-1}f(x)=\varrho_k\cos(kx+\alpha_k)$$

where $\varrho_k^2=\hat f_c^2(k)+\hat f_s^2(k)$ and α_k depends only on the coefficients, it follows that

$$(3.4)\qquad \sum_{k=0}^{\infty}(k+1)^{\lambda-1}\varrho_k^\lambda\frac{1}{2\pi}\int_0^{2\pi}|\cos(kx+\alpha_k)|^\lambda\,dx<\infty.$$

By Lemma 1 in [7], (3.4) implies (3.1). That (3.3) implies (3.1) follows also from (1) Theorem 1 in [5], but the later discussion is needed in the rest of this proof too.

Conversely, suppose that $f\in L^\lambda$ and that (3.1) holds. Then by the Fejér—Lebesgue theorem $\sigma_nf\to f$ a.e., i.e., $s_nf\to f$ C_1 a.e. and (3.1) clearly implies (3.3). Consequently, by property 4) in [7, Section 1], $s_nf\to f$ $|I|_\lambda$ a.e. Thus $f\in A^\lambda$ if and only if $f\in L^\lambda$ and (3.1) holds.

Moreover (3.1) clearly implies (3.2). The converse is also immediate from the above, since we have already proved that (3.2) implies (3.4) which in turn implies (3.1) by Lemma 1 in [7]. Hence for each $f\in L^\lambda$ (3.1) if and only if (3.2) and this completes the proof of (i).

(ii) By the well-known properties of the class $\mathscr{A}$, described in Section 1, clearly $A=\mathscr{A}$. That $\mathscr{A}^\lambda\subset A^\lambda$ is also clear by part (i), which we have just proved, and by (i) of Theorem C, because (1.9) and (3.1) are obviously equivalent. It remains to be shown that this inclusion is also proper. Consider the series

$$(3.5)\qquad \sum_{=1}^{\infty}\frac{1}{k\log(k+1)}\cos kx.$$

Then by a well-known result, see [3, Vol. 1], the series (3.5) converges a.e. to a function $f\in L^1$ and is the Fourier series of that function. Moreover its coefficients satisfy (3.1). Thus by statement (i), $f\in A^\lambda$. However (3.5) does not converge for $x=0$ and consequently it does not converge uniformly, so that $f\notin\mathscr{A}^\lambda$.

(iii) Suppose $\lambda>\lambda'\geqq 1$. We will show that A^λ and $A^{\lambda'}$ are incomparable by considering precisely the same examples as in the proof of (iii) of Theorem C, that is of (ii) of Theorem 4 in [7]. Namely let:

$$f_\lambda(x) \doteq \sum_{j=0}^{\infty} \frac{1}{2^{j/\mu}} \cos 2^j x \tag{3.6}$$

and

$$g_{\lambda'}(x) \doteq \sum_{k=1}^{\infty} \frac{1}{k \log^{1/\lambda'}(k+1)} \sin kx \tag{3.7}$$

where $1/\lambda+1/\mu=1$. Then by [3, Vol. I] or [12, Vol. I] f_λ and $g_{\lambda'}$ are well defined, $f_\lambda, g_{\lambda'}\in C$ and each series is the Fourier—Lebesgue series of the corresponding function. Arguing the same way as in the proof of (ii) of Theorem 4 in [7], it is easily seen that:

$$\|f_\lambda\|_{|\lambda'|}<\infty \quad \text{and} \quad \|f_\lambda\|_{|\lambda|}=\infty;$$

$$\|g_{\lambda'}\|_{|\lambda'|}=\infty \quad \text{and} \quad \|g_{\lambda'}\|_{|\lambda|}<\infty.$$

Consequently by statement (i) we conclude that:

$$f_\lambda\in A^{\lambda'} \text{ and } f_\lambda\notin A^\lambda; \quad g_{\lambda'}\notin A^{\lambda'} \text{ and } g_{\lambda'}\in A^\lambda.$$

Thus A^λ and $A^{\lambda'}$ are incomparable.

(iv) We have already remarked that $A^\lambda\subset S^\lambda$. To see that this inclusion is also proper, consider the function g_λ defined by (3.7). Then it is enough to observe that by the above $g_\lambda\notin A^\lambda$; while by Theorem A (i), $g_\lambda\in S^\lambda$. The second proper inclusion is true by Theorem A (iv). That A^λ, $\|\ \|_{S^\lambda,i}$ is not a Banach space for $i=0, 2, 3$ can be shown similarly as in the proof of Theorem 2 (iii).

(v) This statement is clearly true for $\lambda=1$. So let $\lambda>1$. Then clearly $A^\lambda\subset\subset\{f\colon s_n\to f\ |I|_\lambda \text{ a.e.}\}$. To show the converse inclusion suppose that $s_n\to f\ |I|_\lambda$ a.e. (Here s_n is the n-th partial sum of (1.1).) Then $s_n\to f\ [I]_\lambda$ and consequently, since $\lambda>1$, by Theorem 1 (iv) in [11], $f\in S^\lambda$. In particular then (1.1) is the Fourier—Lebesgue series of f and hence $f\in L^1$ and $s_n f\to f\ [I]_\lambda$ a.e. Thus $f\in A^\lambda$. Consequently $A^\lambda=\{f\colon s_n\to f\ [I]_\lambda \text{ a.e.}\}$ for all $\lambda\geqq 1$.

Now by Theorem 1 (iv) in [11] again, $S^\lambda\subset\bigcap_{p\geqq 1} L^p$ and therefore $A^\lambda\subset\bigcap_{p\geqq 1} L^p$. Finally to see that $A^\lambda\not\subset L^\infty$ consider the sum function f of (3.5). Then by the previous remarks $f\in A^\lambda$. However, by [10, p. 131] $f\notin L^\infty$. This completes the proof of the theorem.

REMARK 3. By the same argument as in the proof of (v) of Theorem 4 and using (iv) of Theorem 1 in [11] we also have the equality $\mathscr{A}^\lambda=\{f\colon s_n\to f\ |I|_\lambda \text{ uniformly}\}$. This fact was not pointed out in [7]. Thus by the above, Theorem 4 (v) and by Theorem 1 (iv) in [11], the equalities of this type hold for all the classes $\mathscr{S}^\lambda$, S^λ, $\mathscr{A}^\lambda$ and A^λ and for all $\lambda\geqq 1$ except for the class S for which $S\subset\{f\colon s_n\to f\ |I| \text{ a.e.}\}$ properly, by a much deeper result, see Theorem 1 (v) in [11] and the references cited there.

In the remainder of this section we discuss the norms on A^λ and prove that for $\lambda>1$, A^λ is a Banach space endowed with the norms $\|\ \|_{|\lambda|}$ or $\|\ \|_{|\lambda|'}$ defined by (3.1) and (3.2) respectively. Following the approach used for the classes S^λ

in Section 2 and the classes $\mathscr{A}^\lambda$ in [7, Section 3] it is also natural to define, for $\lambda \geqq 1$:

$$\|f\|_{A^\lambda,2} \doteq \|f\|_{|\lambda|'} + \sup_n \|s_n f\|_{L^\lambda}, \tag{3.8}$$

$$\|f\|_{A^\lambda,3} \doteq \|f\|_{|\lambda|} + \sup_n \|s_n f\|_{L^\lambda}. \tag{3.9}$$

REMARK 4. Let $\lambda \geqq 1$. For each $f \in A^\lambda$, $\|f\|_{|\lambda|'}$, $\|f\|_{|\lambda|}$ and $\|f\|_{A^\lambda,i}$, $i=2, 3$, are finite and do define norms.

PROOF. That $\|f\|_{|\lambda|'}$, $\|f\|_{|\lambda|}$ and $\|f\|_{A^\lambda,i}$, $i=2, 3$, are finite follows immediately from Theorem 4 (i) and the above definitions. Moreover, that these expressions do define norms on A^λ follows easily by several applications of the Minkowski's inequality.

Now it is trivial to see that $\| \ \|_{|1|}$ is equivalent to $\| \ \|_{\mathscr{A}}$ and that $\sup_n \|s_n f\|_{L^1} \leqq \|f\|_{|1|}$. Thus for $\lambda=1$ the norms defined by (3.1) and (3.9) are both equivalent to the standard norm on $A=\mathscr{A}$. It turns out that this fact extends also to the other two norms for $\lambda=1$. Our goal here is to show the mutual equivalence of all four norms, also for $\lambda>1$. It will be shown that the classes A^λ for all $\lambda \geqq 1$ behave like $\mathscr{A}$. Hence there will be no need to consider distinctions such as those expressed by Theorems 2 and 3 for the classes S^λ, separating the cases $\lambda \geqq 1$ and $\lambda>1$.

LEMMA 2. *Let $\lambda \geqq 1$. Then there exists a constant K_λ such that for each $f \in A^\lambda$*

$$\|s_n f\|_{L^\lambda} \leqq K_\lambda \|f\|_{|\lambda|}. \tag{3.10}$$

Moreover

$$\|f\|_{|\lambda|} \leqq \|f\|_{|\lambda|}. \tag{3.11}$$

PROOF. We first verify (3.11). By partial summation it is easily seen that:

$$\frac{1}{n+1} \sum_{k=0}^{n} (k+1)^\lambda \big(|\hat{f}_c(k)| + |\hat{f}_s(k)|\big)^\lambda =$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=k}^{n} (i+1)^{\lambda-1} \big(|\hat{f}_c(i)| + |\hat{f}_s(i)|\big)^\lambda \leqq \sum_{i=0}^{n} (i+1)^{\lambda-1} \big(|\hat{f}_c(i)| + |\hat{f}_s(i)|\big)^\lambda.$$

Hence by the monotonicity of the power function it follows that (3.11) holds.

The inequality (3.10) is trivial for $\lambda=1$. Now for $\lambda>1$ by Lemma 1 there exists a constant K_λ such that (2.12) holds and (3.10) follows immediately from (2.12) and (3.11).

REMARK 5. The above inequality (3.10) can be also proved directly, using an argument very similar to the proof of Lemma 1.

THEOREM 5. *Let $\lambda \geqq 1$. Then there exists a constant K_λ such that for each $f \in A^\lambda$:*

$$\|f\|_{|\lambda|'} \leqq \|f\|_{A^\lambda,2} \leqq (1+K_\lambda)\|f\|_{|\lambda|'}, \tag{3.12}$$

$$\|f\|_{|\lambda|} \leqq \|f\|_{A^\lambda,3} \leqq (1+K_\lambda)\|f\|_{|\lambda|}, \tag{3.13}$$

$$4^{-1}\|f\|_{|\lambda|} \leqq \|f\|_{|\lambda|'} \leqq \|f\|_{|\lambda|}. \tag{3.14}$$

Consequently the norms $\|f\|_{|\lambda|'}$, $\|f\|_{|\lambda|}$ and $\|f\|_{A^\lambda,i}$, $i=2, 3$, are all equivalent.

PROOF. By (3.8) and (3.9), inequalities (3.12) and (3.13) are immediate consequences of the preceding Lemma 2.

Now the right side inequality in (3.14) is also trivial. To show the inequality on the left of (3.14) we apply again Lemma 1 from [7]. Namely as in the proof of Theorem 4 (i), writing

$$s_k f(x) - s_{k-1} f(x) = \varrho_k \cos(kx+\alpha_k)$$

we have clearly,

$$\Big(\sum_{k=0}^{\infty}(k+1)^{\lambda-1}\varrho_k^{\lambda}\frac{1}{2\pi}\int_0^{2\pi}|\cos(kx+\alpha_k)|^{\lambda}\,dx\Big)^{1/\lambda} \leqq \|f\|_{|\lambda|'}.$$

Consequently by Lemma 1 in [7] it follows that

$$2^{-1}\Big(\sum_{k=0}^{\infty}(k+1)^{\lambda-1}\varrho_k^{\lambda}\Big)^{1/\lambda} \leqq \|f\|_{|\lambda|'}$$

and therefore since $\varrho_k^2=\hat{f}_c^2(k)+\hat{f}_s^2(k)$ we conclude that

$$4^{-1}\|f\|_{|\lambda|} \leqq \|f\|_{|\lambda|'}$$

which verifies (2.14).

THEOREM 6. *Let* $\lambda\geqq 1$. *Then* A^λ *endowed with the norms* $\|f\|_{|\lambda|'}$, $\|f\|_{|\lambda|}$ *or* $\|f\|_{A^\lambda,i}$, $i=2, 3$, *is a Banach space. Furthermore for every* $f\in A^\lambda$, $\|s_n f-f\|_{|\lambda|}=o(1)$ $(n\to\infty)$.

PROOF By Theorem 5 it sufficies to prove that A^λ, $\|\ \|_{|\lambda|}$ is a Banach space.

First, we notice that by the definition of $\|\ \|_{S^\lambda,3}$, see (2.4), and by Lemma 2, there exists a constant K_λ such that for each $f\in A^\lambda$

$$\|f\|_{S^\lambda,3} = \|f\|_{[\lambda]}+\sup_n\|s_n f\|_{L^\lambda} \leqq (1+K_\lambda)\|f\|_{|\lambda|}. \tag{3.15}$$

Suppose that (f_n) is a Cauchy sequence in A^λ, $\|\ \|_{|\lambda|}$. Then clearly (f_n) is a Cauchy sequence in S^λ, $\|\ \|_{S^\lambda,3}$. From Theorem 2 we conclude that there exists a function $f\in S^\lambda$ such that

$$\|f-f_n\|_{S^\lambda,3} = o(1) \quad (n\to\infty). \tag{3.16}$$

We will show now that $f\in A^\lambda$. Since (f_n) is a Cauchy sequence in A^λ, $\|\ \|_{|\lambda|}$, given $\varepsilon>0$ there exists an integer n_0 such that

$$\Big(\sum_{k=0}^{N}(k+1)^{\lambda-1}(|\widehat{f_{nc}}(k)-\widehat{f_{mc}}(k)|+|\widehat{f_{ns}}(k)-\widehat{f_{ms}}(k)|)^{\lambda}\Big)^{1/\lambda} \leqq \tag{3.17}$$

$$\leqq \|f_n-f_m\|_{|\lambda|} < \varepsilon/2 \quad \text{for} \quad n,m\geqq n_0 \quad \text{and for all } N.$$

From (3.16) and (2.10) it follows that $\|f_n-f_m\|_{L^\lambda}=o(1)$ $(m\to\infty)$. Consequently

$$\hat{f}_c(k)-\widehat{f_{mc}}(k) = o(1) \quad\text{and}\quad \hat{f}_s(k)-\widehat{f_{ms}}(k) = o(1) \quad \text{as} \quad m\to\infty,$$

uniformly in k. Letting $m\to\infty$ in (3.17) we obtain the estimate

$$\Big(\sum_{k=0}^{N}(k+1)^{\lambda-1}(|\widehat{f_{nc}}(k)-\hat{f}_c(k)|+|\widehat{f_{ns}}(k)-\hat{f}_s(k)|)^{\lambda}\Big)^{1/\lambda}<\varepsilon$$

for all $n\geqq n_0$ and for each N. Hence

$$\|f_n-f\|_{|\lambda|}=o(1)\quad(n\to\infty).$$

But $f_n-f\in L^\lambda$ clearly and therefore by Theorem 4 (i), $f_n-f\in A^\lambda$. Since $f\in L^\lambda$ and $\|f\|_{|\lambda|}\leqq\|f-f_n\|_{|\lambda|}+\|f_n\|_{|\lambda|}<\infty$ it follows that also $f\in A^\lambda$. Consequently A^λ, $\|\ \|_{|\lambda|}$ is a Banach space.

The second statement of the theorem is trivial observing that

$$\|s_nf-f\|_{|\lambda|}=\Big(\sum_{k=n+1}^{\infty}(k+1)^{\lambda-1}(|\hat{f}_c(k)|+|\hat{f}_s(k)|)^{\lambda}\Big)^{1/\lambda}$$

so that clearly $\|s_nf-f\|_{|\lambda|}=o(1)$ $(n\to\infty)$ for each $f\in A^\lambda$.

REMARK 6. Among the norms appearing in Theorems 5 and 6, the most interesting is clearly $\|\ \|_{|\lambda|}$. Due to the fact that A^λ, $\|\ \|_{|\lambda|}$ is a Banach space for all $\lambda\geqq 1$, where the latter norm is determined only by the coefficients, the norms $\|\ \|_{A^\lambda,i}$, $i=2,3$, are not very important. We have, however, included them in the above presentation mostly for the sake of comparison with the results obtained for the classes $\mathcal{A}^\lambda$, $\mathcal{S}^\lambda$ and S^λ.

REMARK 7. We have observed already in Theorem 4 (iv), that for all $\lambda\geqq 1$, $A^\lambda\subset S^\lambda$ properly and that A^λ is not a Banach space with respect to the inherited norms. Moreover we can now write that for all $\lambda\geqq 1$

$$\sup_{f\in A^\lambda,\ f\neq 0\text{ a.e.}}\|f\|_{|\lambda|}/\|f\|_{S^\lambda,i}=\infty,\quad i=0,2,3.$$

REMARK 8. We have also observed by Theorem 4 (ii), that for all $\lambda\geqq 1$, $\mathcal{A}^\lambda\subset A^\lambda$ properly. Using Theorem C and the results of [7, Section 3] and arguing the same way as in the proof of Theorem 2 (iii) or Theorem 4 (iv), it can be easily established that $\mathcal{A}^\lambda$ is not a Banach space under the inherited norm; moreover

$$\sup_{f\in\mathcal{A}^\lambda,\ f\neq 0}\|f\|_{\mathcal{A}^\lambda,i}/\|f\|_{|\lambda|}=\infty,\quad i=1,2,3.$$

A corresponding statement is also valid for the spaces $\mathcal{S}^\lambda$, that is $\mathcal{S}^\lambda\subset S^\lambda$ properly and $\mathcal{S}^\lambda$ is not a Banach space under the inherited norms.

In conclusion of this paper and using the results proved here, we can now deduce the following theorem about the Banach spaces $\mathcal{S}^\lambda$ and $\mathcal{A}^\lambda$, slightly improving the corresponding statements of Theorems B and C and illustrating another look at these spaces.

THEOREM 7. *The spaces $\mathcal{S}^\lambda$, $\lambda\geqq 1$ and $\mathcal{A}^\lambda$, $\lambda>1$, respectively, endowed with the norms*

$$\|f\|_{\mathcal{S}^\lambda}\doteq f\|_{[\lambda]}+\|f\|_C\quad\textit{and}\quad\|f\|_{\mathcal{A}^\lambda}\doteq\|f\|_{|\lambda|}+\|f\|_C$$

are Banach spaces.

PROOF. By statements (i) of Theorems A, C and 4, clearly $\mathcal{S}^\lambda = C \cap S^\lambda$ and $\mathcal{A}^\lambda = C \cap A^\lambda$. Moreover clearly C, $\| \ \|_C$ is a Banach space. Now by Theorems 3 and 6, S^λ, $\| \ \|_{[\lambda]}$ and A^λ, $\| \ \|_{|\lambda|}$ are Banach spaces for $\lambda > 1$ and by Theorem 2, S, $\| \ \|_{S,3}$ is a Banach space. Moreover $A^\lambda \subset S^\lambda \subset L^1$; L^1, $\| \ \|_{L^1}$ is a Banach space and $\| \ \|_{L^1} \leqq \| \ \|_C$. Assuming first that $\lambda > 1$ and noticing that by Lemmas 1 and 2,

$$\| \ \|_{L^1} \leqq \| \ \|_{L^\lambda} \leqq K_\lambda \| \ \|_{[\lambda]} \leqq K_\lambda \| \ \|_{|\lambda|},$$

by a well-known result, it follows that $\mathcal{S}^\lambda$ and $\mathcal{A}^\lambda$ are Banach spaces under the corresponding sum norms: $\| \ \|_{\mathcal{S}^\lambda}$, respectively $\| \ \|_{\mathcal{A}^\lambda}$, for all $\lambda > 1$. For $\lambda = 1$ the statement follows from the above, noticing first that similarly, $\mathcal{S}$ is a Banach space endowed with the sum norm $\| \ \|_{S,3} + \| \ \|_C$. Now by Theorem 4 in [10], $\|f\|_{S,3} \leqq \leqq \|f\|_{[1]} + \|f\|_{L^1}$ for each $f \in S$ and consequently

$$\|f\|_{\mathcal{S}} \leqq \|f\|_{S,3} + \|f\|_C \leqq \|f\|_{[1]} + 2\|f\|_C \leqq 2\|f\|_{\mathcal{S}}.$$

Hence $\mathcal{S}$ endowed with the norm $\| \ \|_{\mathcal{S}}$ is a Banach space.

References

[1] N. K. Bary, *Trigonometričeskie Rjadi,* Gosudarstvennoe Izdateljstvo (Moscow, 1961).

[2] J. Dobi, A remark on the norm of the Banach space of uniformly strong convergent trigonometric series, *Acta Math. Hung.,* **51** (1988), 201—203.

[3] R. E. Edwards, *Fourier Series, Modern Introduction,* Vols. 1 & 2, Holt, Rinehart and Winston, Inc. (New York, 1967).

[4] C. P. Kahane, *Séries de Fourier Absolument Convergentes,* Springer-Verlag (Berlin, Heidelberg, New York, 1970).

[5] I. Szalay, Ob absolutnoi summiruemosti trigonometričeskih rjadov, *Mat. Zametki,* **30** (1981), 823—837.

[6] I. Szalay, On the Banach space of strongly convergent trigonometric series, *Acta Math. Hung.;* **46** (1985), 39—45.

[7] I. Szalay and N. Tanović-Miller, On Banach spaces of absolutely and strongly convergent Fourier series, *Acta Math. Hung.,* to appear.

[8] N. Tanović-Miller, On strong convergence of trigonometric and Fourier series, *Acta Math. Hung.,* **42** (1983), 35—43.

[9] N. Tanović-Miller, On a paper of Bojanić and Stanojević, *Rendiconti Cir. Mat. Palermo;* **34** (1985), 310—324.

[10] N. Tanović-Miller, Strongly convergent trigonometric series as Fourier series, *Acta Math. Hung.;* **47** (1986), 127—135.

[11] N. Tanović-Miller, On Banach spaces of strongly convergent trigonometric series, *J. of Math. Analysis and Appl.,* to appear.

[12] A. Zygmund, *Trigonometric series,* Vols. I & II, Cambridge University Press (New York, 1959).

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BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
6720 SZEGED, HUNGARY

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SARAJEVO
71000 SARAJEVO, YUGOSLAVIA

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ON THE WEIGHTED MEAN CONVERGENCE OF INTERPOLATING PROCESSES

I. JOÓ and J. SZABADOS (Budapest)*

In [2], the first named author defined the positive linear operator

$$P_n^{(\alpha)}(f;\,x) = \sum_{k=1}^{n} f(x_k)\left(\frac{1+\alpha}{x_k}x-\alpha\right) l_k^2(x). \tag{1}$$

Here $-1<\alpha\leqq 0$, $0<x_1<\ldots<x_n$ are the roots of the Laguerre polynomials $L_n^{(\alpha)}(x)$,

$$l_k(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x-x_k)} \quad (k=1,\ldots,n)$$

are the fundamental polynomials of Lagrange interpolation based on the Laguerre nodes, and $f(x)$ is an arbitrary function defined on $(0,\infty)$. It was proved in [2] that for $-1<\alpha\leqq 0$ this procedure has a remarkable stability property, and under certain growth conditions on $f(x)$, uniform error estimates can be given on finite intervals.

In this paper we investigate the weighted mean convergence of (1) on $[0,\infty)$. As a corollary, we shall get the same result for the Hermite—Fejér interpolation. We will consider the following class of functions:

$$C(\lambda) = \{f(x)|f(x) \text{ is continuous on } [0,\infty) \text{ and } \lim_{x\to\infty} f(x)e^{-\lambda x}=0\}. \tag{2}$$

To an $f(x)\in C(\lambda)$, we shall associate the function

$$F(x) = f(x^2)e^{-\lambda x^2} \tag{3}$$

uniformly continuous on $(-\infty,\infty)$. $\omega(F,h)$ will denote the usual modulus of continuity of $F(x)$ on $(-\infty,\infty)$.

THEOREM 1. *If* $-1<\alpha\leqq 0$ *and* $\lambda<1$ *then*

$$\int_0^\infty x^\alpha e^{-x}\,|f(x)-P_n^{(\alpha)}(f,x)|\,dx = O\left(\omega\left(F,\frac{\log^2 n}{\sqrt{n}}\right)\right)$$

for any $f(x)\in C(\lambda)$. *Here the O sign indicates a constant depending only on* α *and* λ.

We break the proof of this theorem into a series of lemmas. In what follows the O sign will always be meant as a constant depending possibly on α and λ, but always independent of n and x.

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LEMMA 1. *Given* $\lambda<\mu<1$ *and* $0<a<\frac{n}{2}$ *integer, to every* $f(x)\in C(\lambda)$ *there exist polynomials* $p(x)\in\Pi_n$ *such that*

(a) $|f(x)-p(x)| = O(e^{\mu x})\left\{\omega\left(F,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^a\right\} \quad (0\leqq x\leqq a^2),$

(b) $\int\limits_0^\infty x^\alpha e^{-x}|f(x)-p(x)|\,dx = O\left\{\omega\left(F,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^a+\left(\frac{n}{a^2(1-\mu)}\right)^{n/2}\right\};$

(c) $|p'(x)| = O(e^{\lambda x})\left\{\frac{n}{a\sqrt{x}}\,\omega\left(F,\frac{a}{n}\right)+1\right\} \quad (0<x\leqq a^2/4).$

PROOF. (a) Applying Jackson's theorem to (3) on the interval $(-a,a)$, we get a polynomial $q(x)\in\Pi_n$ such that

$$|F(x)-q(x)| = O\left(\omega\left(F,\frac{a}{n}\right)\right) \quad (|x|\leqq a). \tag{4}$$

Since $F(x)$ is even, we may assume the same about $q(x)$. (4) implies

$$\left|f(x)-e^{\lambda x}q(\sqrt{x})\right| = O(e^{\lambda x})\,\omega\left(F,\frac{a}{n}\right) \quad (0\leqq x\leqq a^2). \tag{5}$$

Let

$$p(x) = q(\sqrt{x})\sum_{k=0}^{a}\frac{(\lambda x)^k}{k!}\in\Pi_n. \tag{6}$$

Then by (4)

$$e^{-\mu x}\left|p(x)-e^{\lambda x}q(\sqrt{x})\right| \leqq \left|q(\sqrt{x})\right|\sum_{k=a}^{\infty}\frac{(\lambda x)^k}{k!}e^{-\mu x} = O\left(\sum_{k=a}^{\infty}\frac{\lambda^k}{k!}x^k e^{-\mu x}\right)$$

$$(0\leqq x\leqq a^2).$$

Since $x^k e^{-\mu x}$ attains its maximum on $[0,\infty)$ at $x=k/\mu$, we get

$$e^{-\mu x}\left|p(x)-e^{\lambda x}q(\sqrt{x})\right| = O\left(\sum_{k=a}^{\infty}\left(\frac{\lambda k}{\mu e}\right)^k\cdot\frac{1}{k!}\right) = O\left(\left(\frac{\lambda}{\mu}\right)^a\right) \tag{7}$$

$$(0\leqq x\leqq a^2).$$

Thus we get from (5) and (7)

$$|f(x)-p(x)| = O(e^{\lambda x})\,\omega\left(F,\frac{a}{n}\right)+O(e^{\mu x})\left(\frac{\lambda}{\mu}\right)^a = O(e^{\mu x})\left\{\omega\left(F,\frac{a}{n}\right)+\left(\frac{\lambda}{\mu}\right)^a\right\}$$

$$(0\leqq x\leqq a^2).$$

(b) We obtain from (a)

$$\int_0^{a^2} x^\alpha e^{-x}\,|f(x)-p(x)|\,dx = O\left[\omega\left(F,\frac{a}{n}\right)\left(\frac{\lambda}{\mu}\right)^a\right]\int_0^\infty x^\alpha e^{-(1-\mu)x}\,dx =$$

$$= O\left[\omega\left(F,\frac{a}{n}\right)\left(\frac{\lambda}{\mu}\right)^a\right].$$

On the other hand, by (2) and (6)

$$\text{(8)}\qquad \int_{a^2}^\infty x^\alpha e^{-x}\,|f(x)-p(x)|\,dx \leqq \int_{a^2}^\infty x^\alpha e^{-x}\left\{|f(x)|+e^{\lambda x}\left|q(\sqrt{x})\right|\right\}dx =$$

$$= O\left(\int_{a^2}^\infty x^\alpha e^{-(1-\lambda)x}\left\{1+\left|q(\sqrt{x})\right|\right\}dx\right).$$

Since $q(\sqrt{x})\in\Pi_{n/2}$ is uniformly bounded in $0\leqq x\leqq a^2$ (see (4)), its increase in $[a^2,\infty)$ is restricted by $O\left(\left(\frac{2x}{a^2}\right)^{n/2}\right)$ (see e.g. Natanson [3], Theorems II.6 and Corollary). Hence and from (8)

$$\int_{a^2}^\infty x^\alpha e^{-x}\,|f(x)-p(x)|\,dx = O\left(\left(\frac{2}{a^2}\right)^{n/2}\right)\int_{a^2}^\infty x^{n/2}e^{-(1-\mu)x}\cdot x^\alpha e^{-(\mu-\lambda)x}\,dx.$$

Since here $\max\limits_{0\leqq x\leqq\infty} x^{n/2}e^{-(1-\mu)x} = \left(\frac{n}{2e(1-\mu)}\right)^{n/2}$, we obtain

$$\int_{a^2}^\infty x^\alpha e^{-x}|f(x)-p(x)|\,dx = O\left(\left(\frac{n}{ea^2(1-\mu)}\right)^{n/2}\right)\int_{a^2}^\infty x^\alpha e^{-(\mu-\lambda)x}\,dx =$$

$$= O\left(\left(\frac{n}{a^2(1-\mu)}\right)^{n/2}\right).$$

(c) We obtain from (6)

$$\text{(9)}\qquad |p'(x)| = \left|\frac{q'(\sqrt{x})}{2\sqrt{x}}\sum_{k=0}^{a}\frac{(\lambda x)^k}{k!}+\lambda q(\sqrt{x})\sum_{k=0}^{a-1}\frac{(\lambda x)^k}{k!}\right| = O(e^{\lambda x})\left\{\frac{|q'(\sqrt{x})|}{2\sqrt{x}}+1\right\}$$

$$(0<x\leqq a^2).$$

By a well-known result of S. B. Stečkin (see e.g. A. F. Timan [6] Problem 20 to Ch. IV) we have

$$|q'(x)| = O\left(\frac{n}{\sqrt{a^2-x^2}}\right)\omega\left(q,\frac{a}{n}\right) = O\left(\frac{n}{a}\right)\omega\left(q,\frac{a}{n}\right)\quad (|x|\leqq a/2),$$

where $\omega(q, \cdot)$ is the modulus of continuity of $q(x)$ on the interval $(-a, a)$. This yields by (4)

$$|q'(\sqrt{x})| = O\left(\frac{n}{a}\right)\left\{\omega\left(q-F, \frac{a}{n}\right)+\omega\left(F, \frac{a}{n}\right)\right\} = O\left(\frac{n}{a}\right)\omega\left(F, \frac{a}{n}\right) \quad (0 \leqq x \leqq a^2/4).$$

Substituting this into (9), we get (c). □

LEMMA 2. *For the polynomial* $p(x)$ *defined in Lemma* 1 *we have*

$$\int_0^\infty x^\alpha e^{-x}\,|p(x)-P_n^{(\alpha)}(p, x)|\,dx = O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right).$$

PROOF. From the theory of Hermite—Fejér interpolation (see e.g. Szegő [5], (14.1.7), (14.1.9), (14.5.5)) and (1) we get

$$p(x) \equiv \sum_{k=1}^{n}\left\{p(x_k)\,\frac{x_k(x_k-\alpha)+x(\alpha+1-x_k)}{x_k}+p'(x_k)(x-x_k)\right\} l_k^2(x) =$$

$$= P_n^{(\alpha)}(p, x)+\sum_{k=1}^{n}\{p'(x_k)-p(x_k)\}(x-x_k)l_k^2(x),$$

whence using Lemma 1 (a) and (c) with

$$a = [\sqrt{n}\log^2 n] \tag{10}$$

we obtain

$$|p(x)-P_n^{(\alpha)}(p, x)| \leqq \sum_{k=1}^{n}|p'(x_k)-p(x_k)|\cdot|x-x_k|\,l_k^2(x) = \tag{11}$$

$$= O\left\{\sum_{k=1}^{n}\left(\frac{\sqrt{n}}{\sqrt{x_k}\log^2 n}\,\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)+1\right)e^{\mu x_k}\,|x-x_k|\,l_k^2(x)\right\}.$$

First we show that

$$\frac{1}{L_n^{(\alpha)\prime}(x_k)^2} = O(n^{-\alpha-1/2})\sqrt{x_k}\,e^{-\nu x_k} \quad (k = 1, \dots, n;\ \nu < 1). \tag{12}$$

Namely, if $x_k \leqq n$ then this follows from the relations (9)—(13) of Névai [4] and Szegő [5], (15.3.5):

$$\lambda_k = O(n^\alpha)\,x_k^{-1}\,L_n^{(\alpha)\prime}(x_k)^{-2} = O(n^{-1/2})\,x_k^{\alpha+1/2}\,e^{-x_k},$$

i.e.

$$L_n^{(\alpha)\prime}(x_k)^{-2} = O\left(n^{-\alpha-1/2}\sqrt{x_k}\,e^{-\nu x_k}\right)x_k^{\alpha+1}\,e^{-(1-\nu)x_k} = O\left(n^{-\alpha-1/2}\sqrt{x_k}\,e^{-\nu x_k}\right) \quad (x_k \leqq n)$$

since the function $x^{\alpha+1}e^{-(1-\nu)x}$ is uniformly bounded on $[0,\infty)$. Now if $x_k>n$ then by Freud [6], Lemma III.1.5 applied to $G(x)=xe^{\nu x}$ we get

$$\lambda_k x_k e^{\nu x_k} \sim n^\alpha e^{\nu x_k} L_n^{(\alpha)\prime}(x_k)^2 \leqq \int_0^\infty x^{\alpha+1}e^{-(1-\nu)x}\,dx<\infty,$$

i.e.

$$L_n^{(\alpha)\prime}(x_k)^{-2}=O(n^{-\alpha})e^{-\nu x_k}=O\left(n^{-\alpha-1/2}\sqrt{x_k}\,e^{-\nu x_k}\right)\quad (x_k\geqq n);$$

whence (12) is completely proved. Using (12) with $\mu<\nu<1$ we get from (11)

$$(13)\qquad |p(x)-P_n^{(\alpha)}(p,x)|=O\left(\frac{\omega\left(F,\dfrac{\log^2 n}{\sqrt{n}}\right)}{n^\alpha\log^2 n}\right)\sum_{k=1}^n\frac{L_n^{(\alpha)}(x)^2}{|x-x_k|}e^{-\tau x_k},$$

$$(0<\tau<\nu-\mu).$$

(Here we used the fact that $\omega(F,\cdot)>0$. This follows from (2) and (3) if $f(x)\not\equiv 0$, and the latter may be evidently assumed.) Thus in order to show Lemma 2 we have to prove the estimate

$$(14)\qquad \int_0^\infty x^\alpha e^{-x}\sum_{k=1}^n\frac{L_n^{(\alpha)}(x)^2}{|x-x_k|}e^{-\tau x_k}\,dx=O(n^\alpha\log^2 n).$$

Case 1: $0\leqq x\leqq x_1/2$. Then using the relations

$$x_k\sim\frac{k^2}{n}\quad (k=1,\dots,n)$$

(Szegő [5], (6.31.11)) and

$$(15)\qquad L_n^{(\alpha)}(x)=O(n^\alpha)\quad (0\leqq x\leqq x_1/2\sim c/n)$$

([5], (7.6.8)), we obtain

$$\int_0^{x_1/2}x^\alpha e^{-x}\sum_{k=1}^n\frac{L_n^{(\alpha)}(x)^2}{|x-x_k|}e^{-\tau x_k}\,dx=O(n^{2\alpha})\sum_{k=1}^n\frac{1}{x_k}\int_0^{x_1/2}x^\alpha\,dx=O(n^{2\alpha+1})x_1^{\alpha+1}=O(n^\alpha).$$

Case 2: $x_1/2\leqq x\leqq n$. Then let $|x-x_j|=\min\limits_{1\leqq k\leqq n}|x-x_k|$. Using

$$|x-x_k|\geqq\sqrt{x}\,|\sqrt{x}-x_k|\geqq\frac12\sqrt{x}\,|\sqrt{x_j}-\sqrt{x_k}|\geqq\frac{c\sqrt{x}\,|j-k|}{\sqrt{n}}\quad (k=1,\dots,n)$$

(cf. [5], Problem 35), the relations

$$L_n^{(\alpha)}(x)=x^{-\alpha/2-1/4}O(n^{\alpha/2-1/4})e^{x/2}\quad (c/n\leqq x\leqq n)$$

([5], (7.6.8) and Theorem 8.91.2),

$$L_n^{(\alpha)\prime}(x)=-L_{n-1}^{(\alpha+1)}(x)$$

([5], (5.1.14)), as well as the mean value theorem we get

$$\int_{x_1/2}^{n} x^\alpha e^{-x} \sum_{k=1}^{n} \frac{L_n^{(\alpha)}(x)^2}{\|x-x_k|} e^{-\tau x_k}\,dx = O\left\{\int_{x_1/2}^{n}\left(x^\alpha x^{-\alpha-1/2} n^{\alpha-1/2} \sum_{k\neq j} \frac{\sqrt{n}}{\sqrt{x}\,|j-k|}+\right.\right.$$

$$\left.\left.+x^\alpha x^{-\alpha/2-1/4} n^{\alpha/2-1/4} e^{-x/2}\cdot L_n^{(\alpha)\prime}(\xi)\right)dx\right\} = O(n^\alpha \log n) \int_{x_1/2}^{n} \frac{dx}{x}+$$

$$+O(n^{\alpha/2-1/4}) \int_{x_1/2}^{n} x^{\alpha/2-1/4} e^{-x/2} |L_{n-1}^{(\alpha+1)}(\xi)|\,dx = O(n^\alpha \log^2 n)+$$

$$+O(n^\alpha) \int_{x_1/2}^{n} x^{\alpha/2-1/4} \xi^{-\alpha/2-3/4} e^{(\xi-x)/2}\,dx = O(n^\alpha \log^2 n)+$$

$$+O(n^\alpha) \int_{x_1/2}^{n} \frac{dx}{x} = O(n^\alpha \log^2 n) \quad (\xi\in(x, x_j);\ |x-\xi| = O(1)).$$

Case 3: $n \leqq x < \infty$. This is the only case when we make use of the factor $e^{-\tau x_k}$ in (13). Now instead of (15) we use

$$|L_n^{(\alpha)}(x)| = O(n^{\alpha/2+2/3}) x^{-\alpha/2-1} e^{x/2} \quad (x \geqq 1)$$

([5], Theorem 8.91.2) to get

$$\int_{n}^{\infty} x^\alpha e^{-x} \sum_{k=1}^{n} \frac{L_n^{(\alpha)}(x)^2}{|x-x_k|} e^{-\tau x_k} = \int_{n}^{\infty} x^\alpha e^{-x} L_n^{(\alpha)}(x)^2\,dx \sum_{x_k \leqq n/2} \frac{2}{n}+$$

$$+e^{-\tau n/2} \int_{n}^{\infty} x^\alpha e^{-x} \left\{L_n^{(\alpha)}(x)^2 \sum_{\substack{x_k \leqq n/2 \\ k\neq j}} \frac{1}{|x-x_k|} + |L_n^{(\alpha)}(x) L_n^{(\alpha)\prime}(\xi)|\right\} dx =$$

$$= O(n^\alpha) \int_{n}^{\infty} x^\alpha e^{-x} \{L_n^{(\alpha)}(x)^2 \log n + |L_n^{(\alpha)}(x) L_{n-1}^{(\alpha+1)}(\xi)|\}\,dx =$$

$$= O(n^\alpha) + O(e^{-\tau n/2}) \int_{n}^{\infty} x^\alpha e^{-x} n^{\alpha/2+2/3} \cdot x^{-\alpha/2-1} \cdot e^{x/2} n^{\alpha/2+7/6} \xi^{-\alpha/2-3/2} e^{-\xi/2}\,dx =$$

$$= O(n^\alpha) + O(e^{-\tau n/2} n^{\alpha+11/6}) \int_{n}^{\infty} \frac{dx}{x^{5/2}} = O(n^\alpha)$$

$$(\xi\in(x, x_j);\ |x-\xi| = O(1)).$$

Hence Lemma 2 is completely proved. □

LEMMA 3. *If $f(x)$ is defined in $(0, \infty)$ then*

$$\int_{0}^{\infty} x^\alpha e^{-x} |P_n^{(\alpha)}(f, x)|\,dx = O\big(\max_{1\leqq k\leqq n} |f(x_k)| e^{-\mu x_k}\big) \quad (\mu < 1 \textit{ arbitrary}).$$

PROOF. First we prove that

$$(16)\qquad V(x) = e^{\lambda x} - P_n^{(\alpha)}(e^{\mu x}, x) + \mu \sum_{k=1}^{n} e^{\mu x_k}(x-x_k) l_k^2(x) \geqq 0 \quad (x \geqq 0).$$

Namely, it is easy to see that $V(x_k)=V'(x_k)=0$ $(k=1, \dots, n)$. Thus if we had $V(\xi_1)<0$ for some $\xi_1 \geqq 0$, then repeated application of Rolle's Theorem would yield a $\xi_2>0$ such that $V^{(2n)}(\xi_2)=0$. But this contradicts $V^{(2n)}(x)=\mu^{2n}e^{\mu x}>0$.

Multiplying (16) by $x^\alpha e^{-x}$ and integrating over $(0, \infty)$ we get

$$\int_0^\infty x^\alpha e^{-x} P_n^{(\alpha)}(e^{\mu x}, x)\, dx \leqq \int_0^\infty x^\alpha e^{-(1-\mu)x}\, dx < \infty,$$

where we used the orthogonality property of $L_n^{(\alpha)}(x)$ as well. Thus we get by the positivity of the operator

$$\int_0^\infty x^\alpha e^{-x} |P_n^{(\alpha)}(f, x)|\, dx \leqq$$

$$\leqq \max_{1 \leqq k \leqq n} (|f(x_k)| e^{-\mu x_k}) \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^{n} e^{\mu x_k} \left(\frac{1+\alpha}{x_k} x - \alpha \right) l_k^2(x)\, dx =$$

$$= \max_{1 \leqq k \leqq n} (|f(x_k)| e^{-\mu x_k}) \int_0^\infty x^\alpha e^{-x} P_n^{(\alpha)}(e^{\mu x}, x)\, dx = O\big(\max_{1 \leqq k \leqq n} |f(x_k)| e^{-\mu x_k}\big). \quad \square$$

After these preparations the proof of our theorem is easy. We have by Lemma 1 (a), (b) with (10), Lemmas 2 and 3:

$$\int_0^\infty x^\alpha e^{-x} |f(x) - P_n^{(\alpha)}(f, x)|\, dx \leqq \int_0^\infty x^\alpha e^{-x} |f(x) - p(x)|\, dx +$$

$$+ \int_0^\infty x^\alpha e^{-x} |p(x) - P_n^{(\alpha)}(p, x)|\, dx + \int_0^\infty x^\alpha e^{-x} |P_n^{(\alpha)}(p-f, x)|\, dx =$$

$$= O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right) + O\big(\max_{1 \leqq k \leqq n} |f(x_k) - p(x_k)| e^{-\mu x_k}\big) = O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right). \quad \square$$

(Notice that the only part where $\alpha \leqq 0$ is used is Lemma 3, namely the positivity of $P_n^{(\alpha)}(f, x)$). We do not know if the mean convergence in the class $C(\lambda)$ holds for $\alpha>0$. (In fact, it holds for any polynomial by Lemma 2.)

Finally, consider the Hermite—Fejér interpolation

$$(17)\qquad H_n^{(\alpha)}(f, x) = \sum_{k=1}^{n} f(x_k) \frac{x_k(x_k-\alpha) + x(\alpha+1-x_k)}{x_k} l_k^2(x).$$

THEOREM 2. *If* $-1<\alpha\leqq 0$ *and* $\lambda<1$ *then we have*

$$\int_0^\infty x^\alpha e^{-x} |f(x) - H_n^{(\alpha)}(f, x)|\, dx = O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right).$$

PROOF. (1) and (17) imply

$$H_n^{(\alpha)}(f, x) = P_n^{(\alpha)}(f, x) - \sum_{k=1}^{n} f(x_k)(x-x_k)l_k^2(x).$$

Thus using Theorem 1, (2), (12) with $\lambda<\nu<1$, $0<\tau<\nu-\lambda$ and (14)

$$\int_0^\infty x^\alpha e^{-x}|f(x)-H_n^{(\alpha)}(f, x)|\,dx \leq \int_0^\infty x^\alpha e^{-x}|f(x)-P_n^{(\alpha)}(f, x)|\,dx +$$

$$+\int_0^\infty x^\alpha e^{-x} \sum_{k=1}^{n} |f(x_k)|\cdot|x-x_k|l_k^2(x) = O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right)+$$

$$+O(n^{-\alpha-1/2})\int_0^\infty x^\alpha e^{-x} \sum_{k=1}^{n} \frac{L_n^{(\alpha)}(x)^2}{|x-x_k|}\sqrt{x_k}\,e^{-(\nu-\lambda)x_k}\,dx =$$

$$= O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right)+O(n^{-\alpha-1/2})\int_0^\infty x^\alpha e^{-x} \sum_{k=1}^{n} \frac{L_n^{(\alpha)}(x)^2}{|x-x_k|} e^{-\tau x_k}\,dx =$$

$$= O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right)+O\left(\frac{\log^2 n}{\sqrt{n}}\right) = O\left(\omega\left(F, \frac{\log^2 n}{\sqrt{n}}\right)\right). \quad \square$$

It is interesting to note that Theorem 2 holds for $\alpha=0$, while it is well-known that $H_n^{(\alpha)}(x, 0) \not\to 0 = f(0)$ $(n\to\infty)$ (cf. Szegő [5], the proof of Theorem 14.7).

The process (1) was introduced and investigated for $\alpha=0$, from the point of view of pointwise convergence, in [7], resp. [8].

References

[1] G. Freud, *Orthogonal Polynomials*, Akadémiai Kiadó (Budapest, 1971).
[2] I. Joó, On positive linear interpolation operators, *Anal. Math.*, **1** (1975), 273—281.
[3] I. P. Natanson, *Constructive Theory of Functions* (Budapest, 1952) (Hungarian translation).
[4] G. P. Névai, On Lagrange interpolation based on the nodes of Laguerre polynomials, *Mat. Lapok*, **22** (1971), 149—164 (in Hungarian).
[5] G. Szegő, *Orthogonal Polynomials*; Amer. Math. Soc. Coll. Publ. Vol. 23 (Providence, 1978),
[6] A. F. Timan, *Theory of functions of a real variable* (Moscow, 1961).
[7] E. Egerváry, P. Turán, Notes on interpolation V (On the stability of interpolation), *Acta Math. Hungar.*; **9** (1958), 259—267.
[8] J. Balázs, P. Turán, Notes on interpolation VII (Convergence in infinite intervals), *Acta Math. Hung.*; **10** (1959), 63—68.

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EÖTVÖS LORÁND UNIVERSITY
CHAIR FOR ANALYSIS
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, REÁLTANODA U. 13—15
HUNGARY

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FACTORIZATION OF PERIODIC SUBSETS

A. D. SANDS (Dundee) and S. SZABÓ (Budapest)

1. Introduction

Questions concerning the factorization of finite abelian groups first arose when Hajós [4] solved a long standing problem of Minkowski. He reduced this problem to one involving a direct factorization by simplices and solved it by showing that one simplex must be a subgroup. Fuchs [3] asks whether or not, if a direct product of simplices is a periodic subset of a group, one of the simplices must be a subgroup. He may have had in mind the possibility of finding a shorter proof for Hajós' theorem. If a direct product of simplices is equal to the group then it is easy to show that either a given simplex is a subgroup or the product of the remaining simplices is periodic. We note that in this case this periodic set is also a direct factor of the group.

Fraser and Gordon [2] gave positive answers to Fuchs' question in the cases of p-groups and of cyclic groups, but gave a negative answer in general by providing a counter example in the non-cyclic group of order 18. In this example the periodic subset has order 16 and so cannot be a direct factor of the group.

Any simplex is a direct product of simplices of prime order. Rédei [5] generalized Hajós' theorem by showing that if a group is a direct product of subsets of prime order then one of these subsets is a subgroup.

In this paper we shall consider Fuchs' question and similar questions concerning periodic direct products in finite abelian groups. We show that Fuchs' question has a positive answer if the additional condition is imposed that the periodic subset is a direct factor of the group. However we make use of Hajós' results on zero divisors in group rings and so cannot claim any essentially new proof for Hajós' theorem. For p-groups we show that Rédei's generalization can be applied to the Fraser and Gordon result i.e. that the direct factors need not be simplices but can be taken only to be subsets of prime order containing the identity element of the group. The example by Fraser and Gordon shows that this cannot hold in (p, q)-groups, where p, q are distinct primes. In these groups we obtain a positive answer with the additional assumption that the periodic subset is a direct factor of the group. These two results make use of a generalization of Hajós' results on zero divisors in group rings as well as Rédei's methods using group characters.

In [6] it is shown for cyclic groups that Rédei's result holds for subsets of prime power order. We give an example, in the cyclic group of order 30, which shows that it is not possible to replace "simplex" by "subset of prime order" in the Fraser and Gordon result for cyclic groups nor to replace p-group by cyclic group in our earlier result. However using the methods of [6] involving cyclotomic polynomials we show that if a periodic subset is a product of subsets of prime power order and is also a direct factor of the group then one of these subsets is periodic. For cyclic groups whose order involves at most two prime factors we can drop the condition that the periodic subset is a direct factor but need to restrict the factors to being of prime order.

In any abelian group a periodic subset of prime order is essentially a subgroup. In general this result fails to hold if the order is not prime. In an elementary abelian 2-group any subset of order 2 is periodic and it follows from this that any periodic subset of order 4 is essentially a subgroup. We show that for elementary abelian 2-groups previous results for subsets of prime order holds for subsets of order 4.

Finally we state an open problem concerning a possible common generalization of the main theorems in [5] and [6]. By using a previous result in [7] and Theorem 6 we are able to give a positive answer in one case.

2. Preliminaries

Throughout the paper the word group will mean multiplicative finite abelian group. The product of subsets $A_1, \ldots, A_k$ of a group G is said to be *direct* if each $g \in A_1 \ldots A_k$ can be expressed uniquely as $g = a_1 \ldots a_k$, with $a_i \in A_i$, $1 \leqq i \leqq k$. If $hA = A$ for some subset A of G we say that h is a *period* of A. The set H of all periods of A forms a subgroup of G and there is a set A_1 such that $A = HA_1$ is direct. If $H \neq \{e\}$, the identity subgroup, we say that A is *periodic*. Since in any direct product factorization of G any subset A_i may be replaced by gA_i, which has the same periods, we shall assume throughout that the identity element e belongs to each subset A_i. A subset A is called a *simplex* if $A = \{e, a, a^2, \ldots, a^{n-1}\}$, where n is not greater than the order of a.

Hajós [4] made use of the integral group ring $Z(G)$. Corresponding to each subset A of G we have an element $\bar{A}$ of $Z(G)$, where

$$\bar{A} = \sum_{a \in A} a.$$

If $b = \sum n_i g_i$, $n_i \in Z$, $g_i \in G$ belongs to $Z(G)$ we shall denote by $\langle b \rangle$ the subgroup of G generated by the support of b, i.e. those elements g_i such that $n_i \neq 0$. We shall also use $\langle A \rangle$ to denote the subgroup of G generated by the subset A of G and $\langle b_1, \ldots, b_m \rangle$ will denote the subgroup generated by the supports of $b_i \in Z(G)$, $1 \leqq i \leqq m$. If

$$|\langle b_1, \ldots, b_m \rangle| = p_1^{e_1} \ldots p_k^{e_k},$$

where $p_1, \ldots, p_k$ are distinct primes we shall denote the exponent sum $\sum e_i$ by $r(b_1, \ldots, b_m)$.

Rédei made use of group characters i.e. homomorphisms χ from G to the multiplicative group of complex numbers. These extend to ring homomorphisms χ from $Z(G)$ to complex numbers, where $\chi(\sum n_i g_i) = \sum n_i \chi(g_i)$. He called the set of all χ such that $\chi(\bar{A}) = 0$ the *annihilator* of the subset A of G and denoted it by Ann (A). He observed that $A = B$ if and only if $\chi(\bar{A}) = \chi(\bar{B})$ for all characters χ of G. In particular the product $A_1 A_2 \ldots A_k$ is equal to G if and only if $|A_1| \ldots |A_k| = |G|$ and for each non-identity character χ there exists A_i with $\chi(\bar{A}_i) = 0$. So if $|A_i| = |B_i|$ and Ann $(A_i) \subset$ Ann (B_i) in any direct factorization of G involving A_i we may replace A_i by B_i.

Our first result tests, by means of characters, whether or not a given subgroup is a direct factor of a subset.

THEOREM 1. *If A is a subset and H is a subgroup of G then H is a direct factor of A if and only if* $\mathrm{Ann}\,(H)\subset \mathrm{Ann}\,(A)$.

PROOF. Let $A=HA_1$. Then $\chi\in\mathrm{Ann}\,(H)$ implies $\chi(\bar{A})=\chi(\bar{H})\chi(\bar{A}_1)=0$ and so $\chi\in\mathrm{Ann}\,(A)$. Hence $\mathrm{Ann}\,(H)\subset\mathrm{Ann}\,(A)$.

Conversely let $\mathrm{Ann}\,(H)\subset\mathrm{Ann}\,(A)$. Let $h\in H$, $\chi(h)\neq 1$. Then $\chi(\bar{H})=0$, since H is a subgroup. Hence $\chi(\bar{A})=0$. Therefore $\chi(h\bar{A})=\chi(h)\chi(\bar{A})=0=\chi(\bar{A})$. Clearly if $\chi(h)=1$ we have $\chi(h\bar{A})=\chi(\bar{A})$. It follows that $hA=A$. Thus H is contained in the subgroup of periods of A and so there exists a subset A_1 with $A=HA_1$.

This result generalizes to a result involving two subgroups. We use the notation $\dot{\cup}$ to denote a disjoint union.

THEOREM 2. *If A is a subset and H, K are subgroups of G then there are subsets A_1, A_2 such that $A=HA_1\dot{\cup}KA_2$ if and only if* $\mathrm{Ann}\,(H)\cap\mathrm{Ann}\,(K)\subset\mathrm{Ann}\,(A)$.

PROOF. Let $A=HA_1\dot{\cup}KA_2$. Then, for any character χ, we have

$$\chi(\bar{A}) = \chi(\bar{H})\chi(\bar{A}_1)+\chi(\bar{K})\chi(\bar{A}_2).$$

Hence $\chi(\bar{H})=\chi(\bar{K})=0$ implies $\chi(\bar{A})=0$.

Conversely let $\mathrm{Ann}\,(H)\cap\mathrm{Ann}\,(K)\subset\mathrm{Ann}\,(A)$. We assume first that H, K are cyclic subgroups with generators h, k respectively. Let A_3 be a maximal subset of A with period h and let A_4 be a maximal subset of $A-A_3$ with period k. We claim that $A=A_3\dot{\cup}A_4=HA_1\dot{\cup}KA_2$. We need to show that $B=A-(A_3\cup A_4)$ is empty. As $\chi(\bar{H})=0$ implies $\chi(\bar{A}_3)=0$ and $\chi(\bar{K})=0$ implies $\chi(\bar{A}_4)=0$ we have that $\chi(h)\neq 1$, $\chi(k)\neq 1$ implies $\chi(\bar{B})=0$. It follows that in the group ring $Z(G)$ we have $(e-h)(e-k)\bar{B}=0$ i.e. $\bar{B}+hk\bar{B}=h\bar{B}+k\bar{B}$. Elements of $B\cap hkB$ and $hB\cap kB$ have coefficient 2. Then there exist subsets B_i of B satisfying the following equalities. $B_1=B\cap hkB=hB\cap kB$, $B_1=hkB_2=hB_3=kB_4$ and $hB_2=B_4$, $kB_2=B_3$. Also $B-B_1=$ $=B_5\dot{\cup}B_6$, where $B_5=hB_7$, $B_6=kB_8$. From this we have $B\cap hB=B_1\dot{\cup}B_5=$ $=h(B_3\dot{\cup}B_7)$ and so $B=B_1\dot{\cup}B_5\dot{\cup}B_6=B_3\dot{\cup}B_7\dot{\cup}B_9$, where $hB_9=hkB_{10}$, $B_9=kB_{10}$. Similarly $B\cap kB=B_1\dot{\cup}B_6=k(B_4\dot{\cup}B_8)$ and so $B=B_4\dot{\cup}B_8\dot{\cup}B_{11}$, where $kB_{11}=$ $=hkB_{12}$, $B_{11}=hB_{12}$. From $B\cap hB=B_1\dot{\cup}B_5$ it follows that $B_4\dot{\cup}B_{11}\subset B_1\dot{\cup}B_5$ and so that $B_6\subset B_8$. Then $B_6=kB_8$ implies $|B_6|=|B_8|$ and so $B_6=B_8$. Then from $B_6=kB_6$ and the definition of A_4 it follows that $B_6=\varnothing$. Similarly using $B\cap kB$ we obtain $B_5=\varnothing$. This gives $B=B_1$ and so $B=hB=kB$. Hence $B=\varnothing$ and $A=$ $=A_3\dot{\cup}A_4$ as required.

We now assume, using induction on $|H|+|K|$, that if A_3 is a maximal subset of A with H as a direct factor then the complement $A_4=A-A_3$ has K as a direct factor. Suppose say that K is not cyclic. Then $K=K_1K_2$, where K_1, K_2 are proper subgroups of K. $\chi(\bar{H})=\chi(\bar{K}_1)=0$ implies $\chi(\bar{H})=\chi(\bar{K})=0$ and so $\chi(\bar{A})=0$. By the inductive assumption with A_3 as above K_1 is a direct factor of A_4. Similarly K_2 is a direct factor of A_4. It follows that every element of K is a period of A_4 and so that K is a direct factor of A_4 as required.

We have mentioned that cyclotomic polynomials can be used to study factorizations of cyclic groups. If G is cyclic with generator g and $|G|=n$, then with each subset $A=\{e, g^{r_2}, \dots, g^{r_k}\}$ we may associate a polynomial $A(x)=1+x^{r_2}+\dots+x^{r_k}$. Since $r\equiv s \pmod n$ if and only if $x^r\equiv x^s \pmod{x^n-1}$ we have a factorization $A_1\dots A_k=G$ if and only if $A_1(x)\dots A_k(x)\equiv G(x) \pmod{x^n-1}$. Since $G(x)=(x^n-1)/(x-1)$ each cyclotomic polynomial $F_d(x)$ divides some $A_i(x)$, where d divides n and $d>1$. This method relates to the group character approach since if $\chi(g)=\varrho$ is a primitive d-th root of unity then $F_d(x)$ divides $A_i(x)$ if and only if $A_i(\varrho)=0$ and so if and only if $\chi(\bar{A}_i)=0$. In the case where p, q are distinct primes, $n=p^aq^b$ and $F_n(x)$ divides $A(x)$ of degree $<n$, there is a result of de Bruijn [1] showing that $A=A_1\dot{\cup} A_2$ where A_1 has period $g^{n/p}$ and A_2 has period $g^{n/q}$. Theorem 2 generalizes this result. If $H=\langle g^{n/p}\rangle$ and $K=\langle g^{n/q}\rangle$ then $\chi(\bar{H})=\chi(\bar{K})=0$ if and only if $\chi(g)$ is a primitive n-th root of unity. Thus $F_n(x)$ divides $A(x)$ if and only if $\operatorname{Ann}(H)\cap\operatorname{Ann}(K)\subset\operatorname{Ann}(A)$.

We now turn to the needed generalization of Hajós' result on zero divisors in the group ring $Z(G)$. We model our proof on that of Fuchs [3, Lemma 84.8]. We wish to apply the result to subsets of prime order instead of just simplices and so we have to consider elements of $Z(G)$ which are sums of a prime number of any elements of G.

THEOREM 3. *For each* $i=1, \dots, k$ *let* $b_i\in Z(G)$ *have one of the forms*

(i) $e-a_i$,

(ii) $e+a_i+a_i^2+\dots+a_i^{p_i-1}$,

(iii) $e+a_{i2}+\dots+a_{ip_i}$,

where p_i *is prime, and suppose that, in case* (iii),

$$\operatorname{Ann}(e+a_{i2}+\dots+a_{ip_i})\subset\operatorname{Ann}(e+a_{ij}+a_{ij}^2+\dots+a_{ij}^{p_i-1})$$

for each $j=2, \dots, p_i$. *Now let* $b\in Z(G)$ *be such that* $bb_1\dots b_k=0$ *and that no term* b_i *can be omitted from this product without violating the equality. Then* $r(b, b_1, \dots, b_k)-r(b)<k$.

PROOF. We proceed by induction on $k+d$, where d is the number of elements b_i which are of type (iii) but not of type (ii). If $d=0$ then this is just Hajós' result as presented in [3]. Let $k=1$ with $bb_1=0$, where b_1 has type (iii). Let $b_{1j}=e+a_{1j}+a_{1j}^2+\dots+a_{1j}^{p_1-1}$, $j=2, \dots, p_1$. Since $\chi(b_1)=0$ implies $\chi(b_{1j})=0$ it follows that $bb_{1j}=0$. Now we have $r(b, b_{1j})-r(b)=0$, i.e. $a_{1j}\in\langle b\rangle$. This implies $\langle b_1\rangle\subset\langle b\rangle$ and so $r(b, b_1)-r(b)=0$, as required.

From $(bb_1\dots b_s)b_{s+1}\dots b_k=0$, for $s=1, \dots, k-1$, we deduce, as in [3], that

$$r(b, b_1, \dots, b_k)-r(b, b_1, \dots, b_s)<k-s \tag{1}$$

and similarly, by recording the terms, that

$$r(b, b_1, \dots, b_k)-r(b, b_k)<k-1. \tag{2}$$

If b_k has one of the forms (i) or (ii) the result follows as in [3]. Otherwise we may replace b_k by $b_{kj}=e+a_{kj}+a_{kj}^2+\dots+a_{kj}^{p_k-1}$ to obtain $bb_1\dots b_{k-1}b_{kj}=0$, which has fewer terms solely of type (iii). The term b_{kj} cannot be cancelled. When all possible terms b_i are cancelled we obtain, on reordering, either that $bb_1...b_t b_{kj}=0$ or that $bb_{kj}=0$. If the first case occurs for any j then, even if $t=k-1$, we obtain by induction on $k+d$ that $r(b, b_1, \ldots, b_t, b_{kj})-r(b)<t+1$ and so that $r(b, b_1, \ldots, b_t)-r(b)\leqq t$. Upon adding this to the $s=t$ case of (1) we obtain the desired result. In the second case from $bb_{kj}=0$ we have $a_{kj}\in\langle b\rangle$, for each j, and so $\langle b_k\rangle\subset\langle b\rangle$. This gives $r(b, b_k)-r(b)=0$ and adding this to (2) gives the desired result.

3. Results

We give first a positive answer to Fuchs' question, [3, Problem 82] with the extra condition imposed that the periodic subset is a direct factor.

THEOREM 4. *Let G be a finite abelian group and let the periodic subset A be a direct factor of G and be itself a direct product of simplices. Then one of the simplices is a subgroup.*

PROOF. Let $AB=G$ and let $A=A_1\ldots A_k$ where each A_i is a simplex of prime order. We prove the result by induction on k. If $k=1$ then $A=A_1$ is periodic and a periodic simplex is a subgroup. So we may assume that no product of $k-1$ simplices in $A_1\ldots A_k$ is a periodic subset. Let g be a period of A with $g\neq e$. We can assume that the order of g is a prime. Then in $Z(G)$ we have $(e-g)\bar{A}_1\ldots\bar{A}_k=0$ and no term $\bar{A}_i$ may be omitted here, as if the product in $Z(G)$ were still zero the corresponding direct product in G would have g as a period. Hajós' zero divisor theorem now applies and gives that $r(g, \bar{A}_1, \ldots, \bar{A}_k)<k+1$. Let $K=\langle A_1, \ldots, A_k\rangle$. Them $AB=G$ and $A\subset K$ implies $A(B\cap K)=K$. This implies that $|A|$ divides $|K|$. From $|A_1|\ldots|A_k|=|A|$ and $r(\bar{K})\leqq k$ we deduce that $A=K$. Let $A_k=\{e, a_k, a_k^2, \ldots, a_k^{p_k-1}\}$. If $a_k^{p_k}=e$ then A_k is a subgroup. Otherwise $A_1\ldots A_{k-1}$ is periodic as it can be seen from $\bar{A}_1\ldots\bar{A}_{k-1}(e-a_k^{p_k})=0$.

We now generalize the result of Fraser and Gordon for p-groups replacing the simplices by subsets of prime order. In order to use Theorem 3 we need to have available the extra condition there. However Rédei [5, Satz 10] has shown that in a p-group if A is a set of prime order q different from p then $\chi(\bar{A})\neq 0$ for any character χ, as a sum of q roots of unity orders a power of p cannot be zero. Thus $\bar{A}$ can always be omitted from any product equal to 0 in $Z(G)$. If A has order p then $\chi(\bar{A})=0$ if and only if $\{\chi(a)\colon a\in A\}=\{\varrho^i\colon i=0, 1, \ldots, p-1\}$ where ϱ is a primitive p-th root of unity. Thus each element $a\in A$, $a\neq e$ is sent to such a root and

$$\chi(e+a+a^2+\ldots+a^{p-1}) = 0.$$

Thus Rédei's results show that the additional hypotheses of Theorem 3 are satisfied in p-groups.

THEOREM 5. *Let G be a finite abelian p-group. If a direct product of subsets of G of prime order is periodic then one of the subsets is a subgroup.*

PROOF. Let $A=A_1\ldots A_k$ be periodic, where each subset A_i of G has prime order. Assuming as before that no product of $k-1$ terms is periodic we have in

$Z(G)$, for any non-identical period g of A, $(e-g)\bar{A}_1 \dots \bar{A}_k = 0$. As we have already shown the conditions of Theorems 3 are satisfied. Hence $r(g, \bar{A}_1, \dots, \bar{A}_k) < k+1$ if g is of prime order. Since G is a p-group we have $|\langle g, A_1, \dots, A_k \rangle| \leqq p^k$. However $p^k = |A_1| \dots |A_k| \leqq |\langle A_1, \dots, A_k \rangle| \leqq p^k$. It follows that $A_1 \dots A_k = \langle A_1, \dots, A_k \rangle$ and we have a factorization of a subgroup into sets of prime order. By Rédei's theorem one of these subsets A_i is a subgroup.

We now consider (p, q)-groups where p and q are distinct primes. The example of Fraser and Gordon shows that Theorem 5 does not extend to this case. However when we introduce the extra condition, as in Theorem 4, we have a corresponding result. The subsets must then have order equal to either p or q. It follows as in Satz 6 and Hilfsatz 14 of Rédei [5] that the additional conditions needed in Theorem 3 do hold for the annihilators of such subsets in (p, q)-groups.

THEOREM 6. *Let p, q be distinct primes and let G be a finite abelian (p, q)-group. If a periodic subset A of G is a direct factor of G and A is a direct product of subsets of prime order then one of these subsets is a subgroup.*

PROOF. Let $G = AB$, $A = A_1 \dots A_k$ where each A_i has order p or q. We may assume that no product of $k-1$ subsets here is periodic. Let g be a non-identical period of A of prime order. Then $(e-g)\bar{A}_1 \dots \bar{A}_k = 0$ and no term may be omitted. As we have already mentioned the results of Rédei show that Theorem 3 may be applied. It follows that $r(\bar{A}_1, \dots, \bar{A}_k) \leqq k$. Let $\langle A_1, \dots, A_k \rangle = H$. Then $A \subset H$ and $AB = G$ implies $A(B \cap H) = H$. Since $|A_1| \dots |A_k| = |A|$, $|A|$ divides $|H|$ and $r(\bar{H}) \leqq k$ we must have $A = H$. Hence $A_1 \dots A_k = H$. It follows by Rédei's theorem that one of the subsets A_i is a subgroup.

We now turn to cyclic groups. Rédei's theorem on factorizations of the groups by subsets of prime order had already been shown to hold in cyclic groups for factorizations into subsets of prime power order [6]. Fraser and Gordon gave a positive answer to Fuchs' question in cyclic groups. The following example shows that we cannot prove Theorem 5 in the case of cyclic groups.

Let G be a cyclic group of order 30 with generator g. Let $A = \{e, g, g^{10}, g^{16}, g^{20}\}$, $B = \{e, g^5\}$. Then $|A| = 5$, $|B| = 2$, B is a simplex but A is not. Now

$$AB = \{e, g, g^5, g^6, g^{10}\}\{e, g^{15}\}$$

and so is periodic with period g^{15}, but neither A nor B is periodic. We should also note that AB is not a direct factor of G.

We now show that we can generalize the result of [6] to periodic subsets of cyclic groups provided that the periodic subset is a direct factor of the group. The proof uses cyclotomic polynomials exactly as in [6] and we only indicate the necessary changes to the proof there.

THEOREM 7. *Let G be a finite cyclic group. Let the periodic subset A be a direct factor of G and let A be a direct product of subsets of prime power order. Then one of these subsets is periodic.*

PROOF. Let G have order n with generator g. As has been pointed out already we can associate a polynomial with each subset of G. Let $G = AB$ and $A = A_1 \dots A_k$ where the subset A_i has order $p_i^{e_i}$, for some prime p_i. We have $A_1(x) \dots A_k(x) B(x) \equiv \equiv G(x) \pmod{x^n - 1}$. Exactly as in [6] it follows that e_i cyclotomic polynomials $F_d(x)$,

$d=p_i^{f_{ij}}$, divide $A_i(x)$ and from this that the elements in the p_i-component of the elements of A_i are distinct. Let H be the set of periods of A. Then $A=HC$ and so $A_1(x)\ldots A_k(x)\equiv H(x)C(x) \pmod{x^n-1}$. Now H is a subgroup of order greater than 1 and so $F_n(x)$ divides $H(x)$. It follows that $F_n(x)$ divides some $A_i(x)$. As in [6] it follows that this subset A_i is periodic.

For cyclic groups of order $p^a q^b$ and subsets of order p or q we can use Theorem 2 to remove the condition that the periodic subset be a direct factor of G.

THEOREM 8. *Let p, q be distinct primes and G be a cyclic group of order $p^a q^b$. If a subset A of G is periodic and is a direct product of subsets A_i of orders p or q then one of these subsets is a subgroup.*

PROOF. Let $n=p^a q^b$ and let g be a generator of G. As in the proof of Theorem 7 we may show that $F_n(x)$ divides $A_i(x)$ for some i. Now Theorem 2 may be applied to give $A_i=\langle g^{n/p}\rangle C \dot{\cup} \langle g^{n/q}\rangle D$. If $|A_i|=p$ then we have $p=p|C|+q|D|$ and so $|C|=1$, $|D|=0$. It follows that A_i is a subgroup. Similarly $|A_i|=q$ implies $|C|=0$, $|D|=1$ and again A_i is a subgroup.

In any group a periodic subset with a prime number of elements is a subgroup. This need not be the case if the order of the subset is not prime. However in an elementary abelian 2-group a periodic subset of 4 elements is a subgroup and we now show that previous theorems involving prime orders hold in such groups for subsets of order 4. We first consider a factorization of the group.

THEOREM 9. *Let G be an elementary abelian 2-group. If G is a direct product of subsets A_i of order 4 then one of these subsets is a subgroup.*

PROOF. Let $G=A_1\ldots A_k$ with $|A_i|=4$. If $k=1$ then the result is trivial. So we may use induction on k. Let $A_i=\{e, a_i, b_i, c_i\}$ where $c_i=a_ib_id_i$. Since the product is direct the elements of $\{a_i, b_i\colon 1\leqq i\leqq k\}$ are independent and so are a set of generators for G. Let $H_i=\{e, a_i, b_i, a_ib_i\}$. Then G is the direct product of these subgroups H_i. Let χ be a character of G with $\chi(\bar{A}_i)=0$. Since χ sends elements to 1 or -1 and $\chi(e)=1$ it is routine to check that $\chi(d_i)=1$. It follows that $\chi(\bar{H}_i)=0$. By the result of Rédei we may replace A_i by H_i in the factorization of G. This gives

$$G/H_i=\prod_{j\neq i}(A_jH_i)/H_i.$$

By the inductive assumption, some subset $(A_jH_i)/H_i$ is a subgroup. This implies that $d_j\in H_i$. Thus given any i, there exists an $f(i)$ with $f(i)\neq i$ and $d_{f(i)}\in H_i$. There is a cycle $(i_1, i_2, \ldots, i_r)$ such that $f(i_1)=i_2, \ldots, f(i_{r-1})=i_r, f(i_r)=i_1$. If $r<k$ then $A_{i_j}\subset H_{i_1}\ldots H_{i_r}=H$ for $j=1, \ldots, r$. Since the products of the A_{i_j} is direct and its order is $4^r=|H|$ we have a factorization of the subgroup H. The inductive assumption then implies that one of these subsets is a subgroup. Otherwise $r=k$. If no A_j is periodic the elements d_j are different from e and belong to k distinct subgroups in a direct product giving G. This implies that $\{d_1, \ldots, d_k\}$ is an independent set of elements. Thus there is a character χ of G with $\chi(d_j)=-1$, $j=1, \ldots, k$. For this character χ we have $\chi(\bar{A}_j)\neq 0$ for all j, but $\chi(\bar{G})=0$, and this contradicts $A_1\ldots A_k=G$. Hence some $d_j=e$ and the corresponding subset A_j is a subgroup.

We now extend this result to cover the case of a periodic subset which is a product of subsets of order 4.

THEOREM 10. *Let G be an elementary abelian 2-group. If a periodic subset B is a direct product of subsets of order 4 then one of these subsets is a subgroup.*

PROOF. The result is trivial if B has order 4. We proceed by induction on the order of the periodic subset. Let $B=A_1\dots A_k$, where $A_i=\{e, a_i, b_i, c_i\}$ and $H_i=\{e, a_i, b_i, a_ib_i\}$. Let g be a non-identical period of B. Suppose first that $g\in H_i$. We need consider only $g=a_i$ and $g=a_ib_i$. Let $g=a_i$. Let

$$D=\prod_{j\neq i} A_j.$$

Then $B=a_iB=a_iA_iD=A_iD$. It follows that $\{a_ib_i, a_ic_i\}D=\{b_i, c_i\}D$. Since $D\cap a_iD=\emptyset$ it follows that $a_ib_iD=c_iD$. Hence either D is periodic and the inductive assumption implies that some set A_j is a subgroup or $c_i=a_ib_i$ and A_i is a subgroup. Let $g=a_ib_i$. Then as above, $\{a_ib_i, a_ib_ic_i\}D=\{e, c_i\}D$. Now from $a_iD\cap b_iD=\emptyset$ we obtain $a_ib_iD\cap D=\emptyset$ and so $a_ib_iD=c_iD$. Once again the desired result follows.

Now we consider the case when $g\notin H_i$ for any i. As before, for any character χ, $\chi(\bar{A}_i)=0$ implies $\chi(\bar{H}_i)=0$. We also see above that H_iD is a direct product, though not necessarily equal to B. Now $B=\{e, g\}C$ since g is a period of B and so $\chi(g)=-1$ implies $\chi(\bar{B})=0$. Then $B=A_iD$ implies either $\chi(\bar{D})=0$ or else $\chi(\bar{A}_i)=0$ and hence $\chi(\bar{H}_i)=0$. Thus $\chi(g)=-1$ implies $\chi(\bar{H}_i\bar{D})=0=\chi(g\bar{H}_i\bar{D})$. Of course $\chi(g)=1$ implies $\chi(\bar{H}_i\bar{D})=\chi(g\bar{H}_i\bar{D})$. By Theorem 1 g is a period of H_iD. $g\notin H_i$ then gives a period gH_i in G/H_i of

$$\prod_{j\neq i} (A_jH_i)/H_i.$$

The inductive assumption then implies that some set $(A_jH_i)/H_i$ is a subgroup. If we let $c_j=a_jb_jd_j$ it follows that given any i there exists an $f(i)$ with $f(i)\neq i$ and $d_{f(i)}\in H_i$. As in the previous theorem there is a cycle $(i_1, i_2, \dots, i_r)$ with a factorization of a subgroup H into a product of r of the subsets A_i. By Theorem 9 one of these subsets is a subgroup.

Results about factorizations of periodic subsets may be useful tools to study factorizations of groups. We illustrate a case of this now and also state an open problem.

As has been stated already Rédei's theorem on factorizations using subsets of prime order holds for cyclic groups for subsets of prime power order. The following open question arises concerning a possible generalization of both results. Let $p_1, \dots, p_l$ be primes and let the $p_1, \dots, p_l$ components of G be cyclic. If $G=A_1\dots A_k$ and for each i either the order of A_i is prime or is a power of one of the primes $p_1, \dots, p_l$ does it follow that one of the subsets A_i is periodic? We do not know the answer to this question in general but we can use our results to settle positively the first unsolved case.

THEOREM 11. *Let p, q be distinct primes and let G be a (p, q)-group whose p-component is cyclic. If $G=A_1\dots A_kB_1\dots B_l$, where each A_i has order a power of p and each B_j has order q then one of these subsets is periodic.*

PROOF. Let $B=B_1 \ldots B_l$. Then the factorization $G=A_1 \ldots A_k B$ satisfies the conditions of [7, Theorem 2]. It follows that either some A_i is periodic, giving the desired result, or that B is periodic. If B is periodic then Theorem 6 applies to show that one of the subsets B_j is periodic.

References

[1] N. G. de Bruijn, On the factorization of finite cyclic groups, *Indag. Math. Kon. Ned. Akad. Wetensch.*, **15** (1953), 370—377.

[2] O. Fraser and B. Gordon, Solution to a problem of L. Fuchs, *Quart. J. Math. Oxford*, **25** (1974), 1—8.

[3] L. Fuchs, *Abelian Groups* (London, 1960).

[4] G. Hajós, Über einfache und mehrfache Bedeckung des n-dimensionales Raumes mit einem Wurfelgitter, *Math. Zeitsch.*, **47** (1942), 427—467.

[5] L. Rédei, Die neue Theorie der endlichen Abelschen Gruppen und Verallgemeinerung des Hauptsatzes von Hajós, *Acta Math. Acad. Sci. Hung.*, **16** (1965), 329—373.

[6] A. D. Sands, Factorization of cyclic groups, *Proc. Coll. on abelian groups Tihany Hungary Sept.* (1963), 139—146.

[7] A.D. Sands, On the factorization of finite abelian groups. III, *Acta Math. Acad. Sci. Hung.*, **25** (1974), 279—284.

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DEPT. OF MATH. SCI.
THE UNIV. OF DUNDEE
DUNDEE DD1 4HN, UK

DEPT. OF CIVIL ENGINEERING MATH.
TECHN. UNIV. BUDAPEST
H—1111 BUDAPEST STOCZEK 2
HUNGARY

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SATURATION THEOREMS FOR HERMITE—FOURIER SERIES

I. JOÓ (Budapest)

Dedicated to Professor Paul Erdős on the occasion of his 75th birthday

The Bernstein theorem states that if f is a continuous 2π-periodic function then $f \in \mathrm{Lip}\ \alpha$, $0<\alpha<1$ is equivalent to the norm estimate $\|\sigma_n f - f\|_C = O\left(\frac{1}{n^\alpha}\right)$, where σ_n denotes the n-th Fejér mean. If $f \in \mathrm{Lip}\ 1$ then only $\|\sigma_n f - f\|_C = O(\log n/n)$ is valid. The function class for which $\|\sigma_n f - f\|_C = O(1/n)$ was described by G. Alexits [2] by the condition $\tilde{f} \in \mathrm{Lip}\ 1$. The notion of the Hermite conjugate function was introduced by Muckenhoupt [4], he proved the boundedness of the conjugation operator in the weighted L^p space

$$L^p_{e^{-y^2}} := \left\{f\colon \|f\|_p := \Big(\int_{-\infty}^{\infty} |f(y)|^p e^{-y^2}\, dy\Big)^{1/p} < \infty\right\}.$$

The following version of the Hermite-weighted L^p space is also frequently used:

$$L^p(e^{-y^2/2}) := \{f\colon e^{-y^2/2} f \in L^p(\mathbf{R})\}.$$

The corresponding norm estimate of the conjugate function is valid. In the papers [6], [7] we initiated the possibility to apply this conjugate concept to obtain Alexits type theorems. These investigations were continued by A. Bogmér [9] for the Jacobi expansions. A saturation theorem for the Abel—Poisson means will be also proved below. In [10] M. Horváth obtains both Alexits and Abel—Poisson saturation theorems for the Jacobi and Laguerre expansions. In [8] we give the same theorems for the Walsh expansions, based on the Walsh conjugate defined by Hunt [5].

The Hermite polynomials $h_n(x)$ are given by

$$\int_{-\infty}^{\infty} e^{-x^2} h_n(x) h_k(x)\, dx = \delta_{n,k}.$$

Remark that the polynomials are dense in $L^p(e^{-y^2/2})$, $1 \leqq p \leqq \infty$, see [12]. If $fe^{-x^2/2} \in L^p(\mathbf{R})$, $1 \leqq p \leqq \infty$, then there exists the Hermite—Fourier expansion of f,

$$\text{(1)} \qquad f(x) \sim \sum a_k h_k(x), \quad a_k = \int_{-\infty}^{\infty} f(x) h_k(x) e^{-x^2}\, dx.$$

Denote by R_n the n-th Riesz mean of parameter $\frac{1}{2}$ of f, i.e.

$$R_n(f, x) = \sum_{k=0}^{n} \left(1 - \frac{\sqrt{k}}{\sqrt{n+1}}\right) a_k h_k(x).$$

The boundedness of the Riesz means

$$\|e^{-x^2/2}R_n(f,x)\|_p \leqq c\|e^{-x^2/2}f(x)\|_p \quad (1\leqq p\leqq\infty) \tag{2}$$

can be proved just as the corresponding estimates of Freud on the Fejér means, see in [3]. Remark that for Laguerre expansions this fact is much more complicated to prove ([13]). The following statement is a variant of of the Alexits lemma ([2]) and has a similar proof.

LEMMA 1. *Let $(\varphi_n)_0^\infty$ be an arbitrary sequence in a Banach space E. Fix a sequence $(\lambda_n)_0^\infty$ with*

$$0\leqq\lambda_0<\lambda_1<\ldots, \lim\lambda_n=\infty, \quad \lambda_0\leqq\delta, \quad \lambda_{n+1}\leqq\delta\lambda_n \quad (\delta>0,\ n=1,2).$$

Define the means

$$\sigma_n=\sum_{k=0}^n\left(1-\sqrt{\frac{\lambda_k}{\lambda_{n+1}}}\right)\varphi_k,\quad \sigma_n^*=\sum_{k=0}^n\left(1-\sqrt{\frac{\lambda_k}{\lambda_{n+1}}}\right)\varphi_k\sqrt{\lambda_k}.$$

a) *If $\|\sigma_n^*\|\leqq K$, then there exists $\sigma\in E$ with*

$$\|\sigma_n-\sigma\|\leqq 4K/\sqrt{\lambda_n}.$$

b) *If $\|\sigma_n-\sigma\|\leqq K/\sqrt{\lambda_n}$ for some $\sigma\in E$ then*

$$\|\sigma_n^*\|\leqq c(\delta)\left(K+\sqrt{\frac{\lambda_0\lambda_1}{\lambda_n}}\|\sigma\|\right).$$

The case $\lambda_n=n$ will be used below since the corresponding means σ_n, σ_n^* are Riesz means in this case. Remark that if σ_n denotes the ordinary Fejér means then the implication

$$\left\|\sigma_n\left(\sum a_k\varphi_k\right)-\sigma\right\|=O\left(\frac{1}{\sqrt{n}}\right)\Rightarrow\left\|\sigma_n\left(\sum\sqrt{k}\,a_k\varphi_k\right)\right\|=O(1)$$

does not hold; that is why we prefer Riesz means to Fejér means in the case of classical orthogonal expansions on infinite intervals.

The Hermite conjugate function $\tilde f$ of $f\in L^p(e^{-y^2/2})$ is defined by

$$e^{-x^2/2}\tilde f\in L^p(\mathbf{R}),\quad \tilde f(x)\sim\sum a_k h_{k-1}(x). \tag{3}$$

This is essentially the definition of Muckenhoupt [4]. He proved the boundedness of the conjugation operator in $L^pe^{-y^2/2}$, $1<p<\infty$; the case of $L^p(e^{-y^2/2})$ can be dealt with similarly:

$$\|e^{-x^2/2}\tilde f\|_p\leqq c\|e^{-x^3/2}f\|_p,\quad 1<p<\infty. \tag{4}$$

Introduce the following modulus of continuity defined by Freud [3]:

$$\omega(f,\delta)_p:=\sup_{0\leqq t\leqq\delta}\|f(x+t)e^{-(x+t)^2/2}-f(x)e^{-x^2/2}\|_p+\|\tau(\delta x)f(x)e^{-x^2/2}\|_p, \tag{5}$$

where

$$\tau(x):=\begin{cases}|x| & \text{if } |x|\leqq 1\\ 1 & \text{if } |x|\geqq 1.\end{cases}$$

Denote further

$$E_n(f)_p := \inf_{p_n \in \mathcal{P}_n} \|e^{-x^2/2}(f-p_n)\|_p, \tag{6}$$

where P_n is the set of polynomials of order $\leqq n$. In [3] the following Jackson type inequality is proved for $1 \leqq p \leqq \infty$ and $e^{-x^2/2} f \in L^p(\mathbf{R})$:

$$E_n(f)_p \leqq c\omega\left(f, \frac{1}{\sqrt{n}}\right)_p. \tag{7}$$

The following converse statement can be obtained:

LEMMA 2. *Let* $1 \leqq p \leqq \infty$, $fe^{-x^2/2} \in L^p(\mathbf{R})$, *then*

$$\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \leqq \tag{8}$$

$$\leqq \frac{c}{\sqrt{n+1}} \sum_{k=0}^{n} \frac{E_k(f)_p}{\sqrt{k+1}} + \|g_0(e^{-((x+n^{-1/2})/2)^2} - e^{-x^2/2})\|_p + \|\tau(n^{-1/2}x)f(x)e^{-x^2/2}\|_p,$$

where g_0 *is the best approximating constant polynomial.*

For the proof we need the following

LEMMA 3. *Let* $1 \leqq p \leqq \infty$ *and* $p_n \in \mathcal{P}_n$. *Then*

$$\|(p_n e^{-x^2/2})'\|_p \leqq c\sqrt{n}\, \|p_n e^{-x^2/2}\|_p. \tag{9}$$

Remark that G. Freud proved a similar inequality in [15], namely

$$\|p_n' e^{-x^2/2}\|_p \leqq c\sqrt{n}\, \|pe^{-x^2/2}\|_p. \tag{10}$$

Further information on this topic can be found in [16].

PROOF OF LEMMA 3. Denote $\vartheta_n := \dfrac{S_n + S_{n+1} + \ldots + S_{2n-1}}{n}$ the n-th de la Vallée-Poussin means of an Hermite series. We see that ϑ_n leaves fixed the polynomials of order $\leqq n$. We shall use the result of W. E. Milne [17] who proved the case $p=\infty$ of (9). To prove the case $p=1$ we use duality:

$$\|(p_n e^{-x^2/2})'\|_1 = \sup_{\|ge^{-x^2/2}\|_\infty \leqq 1} \left|\int_{-\infty}^{\infty} (p_n(x)e^{-x^2/2})' g(x) e^{-x^2/2}\, dx\right| =$$

$$= \sup_{\|ge^{-x^2/2}\|_\infty \leqq 1} \left|\int_{-\infty}^{\infty} (p_n(x)e^{-x^2/2})' \vartheta_{n+1}(g, x) e^{-x^2/2}\, dx\right| =$$

$$= \sup_{\|ge^{-x^2/2}\|_\infty \leqq 1} \left|\int_{-\infty}^{\infty} p_n(x)e^{-x^2/2} (\vartheta_{n+1}(g, x)e^{-x^2/2})'\, dx\right| \leqq$$

$$\leqq \|p_n e^{-x^2/2}\|_1 \cdot \sup_{\|ge^{-x^2/2}\|_\infty \leqq 1} \left\|(\vartheta_{n+1}(g, x)e^{-x^2/2})'\right\|_\infty \leqq$$

$$= c\sqrt{n}\, \|p_n e^{-x^2/2}\|_1 \sup_{\|ge^{-x^2/2}\|_\infty \leqq 1} \|\vartheta_{n+1}(g, x)e^{-x^2/2}\|_\infty \leqq c\sqrt{n}\, \|p_n e^{-x^2/2}\|_1.$$

Here we used the norm estimate

(11) $$\|e^{-x^2/2}\vartheta_n f\|_p \leqq c(p)\|e^{-x^2/2}f\|_p \quad (1 \leqq p \leqq \infty,\ e^{-x^2/2}f\in L^p(\mathbf{R}))$$

proved by Freud [3] and mentioned also after (2). Now consider the operators

$$T_n\colon L^p(\mathbf{R}) \to L^p(\mathbf{R}), \quad ge^{-x^2/2} \mapsto [\vartheta_{n+1}(g, x)e^{-x^2/2}]'.$$

We know that for $p=1$ and $p=\infty$ the estimate

$$\|T_n(ge^{-x^2/2})\|_p \leqq \sqrt{n}\,\|\vartheta_{n+1}(g, x)e^{-x^2/2}\|_p \leqq c\sqrt{n}\,\|ge^{-x^2/2}\|_p$$

holds. By the Marcinkiewicz interpolation theorem the same is true for $1<p<\infty$ with a constant $c(p)$ instead of c:

$$\|T_n(ge^{-x^2/2})\|_p \leqq c(p)\sqrt{n}\,\|ge^{-x^2/2}\|_p.$$

If g is a polynomial of degree $\leqq n$, we obtain (9).

Proof of Lemma 2. Let g be the best approximating polynomial of n-th order, i.e.

$$\|e^{-x^2/2}(f-g_n)\|_p = E_n(f)_p.$$

Then

(12)
$$\|f(x+t)e^{-(x+t)^2/2}-f(x)e^{-x^2/2}\|_p \leqq \|f(x+t)e^{-(x+t)^2/2}-g_n(x+t)e^{-(x+t)^2/2}\|+$$
$$+\|g_n(x)e^{-x^2/2}-f(x)e^{-x^2/2}\|_p+\|g_n(x+t)e^{-(x+t)^2/2}-g_n(x)e^{-x^2/2}\|_p =$$
$$= 2E_n(f)_p+\|g_n(x+t)e^{-(x+t)^2/2}-g_n(x)e^{-x^2/2}\|_p.$$

Here the first member can be estimated by

$$2E_n(f)_p \leqq \frac{c}{\sqrt{n+1}}\sum_{k=0}^{n}\frac{E_n(f)_p}{\sqrt{k+1}} \leqq \frac{c}{\sqrt{n+1}}\sum_{k=0}^{n}\frac{E_k(f)_p}{\sqrt{k+1}}.$$

The second member is

$$\|g_n(x+t)e^{-(x+t)^2/2}-g_n(x)e^{-x^2/2}\|_p \leqq \|g_0(e^{-(x+t)^2/2}-e^{-x^2/2}\|_p+$$
$$+\|g_n(x+t)e^{-(x+t)^2/2}-g_n(x)e^{-x^2/2}-g_0(e^{-(x+t)^2/2}-e^{-x^2/2})\|_p =: I_1+I_2,$$

$$I_2 = \Big\|\int_0^t [g_n(x+\tau)e^{-(x+\tau)^2/2}-g_0e^{-(x+\tau)^2/2}]'\,d\tau\Big\|_p \leqq$$
$$\leqq \int_0^t \|[g_n(x+\tau)e^{-(x+\tau)^2/2}-g_0e^{-(x+\tau)^2/2}]'\|_p\,d\tau =$$
$$= t\Big\|[(g_n(x)-g_0(x))e^{-x^2/2}]'\Big\|_p.$$

Using the fact that $t \leqq 1/\sqrt{n}$ and Lemma 3 we get for r satisfying $2^{r+1} \leqq n < 2^{r+2}$:

$$I_2 \leqq \frac{1}{\sqrt{n}} \{ \| [(g_n(x) - g_{2^r}(x)) e^{-x^2/2}]' \|_p + \sum_{i=0}^{r} \| [(g_{2^i}(x) - g_{2^{i-1}}(x)) e^{-x^2/2}]' \| \leqq$$

$$\leqq \frac{c}{\sqrt{n}} \{ \sqrt{n} \, \| (g_n - g_{2^r}) e^{-x^2/2} \|_p + \sum_{i=0}^{r} 2^{i/2} \| (g_{2^i} - g_{2^{i-1}}) e^{-x^2/2} \|_p \} \leqq$$

$$\leqq \frac{c}{\sqrt{n}} \{ \sqrt{n} \, (E_n(f)_p + E_{2^r}(f)_p) + \sum_{i=0}^{r} 2^{i/2} (E_{2^i}(f)_p + E_{2^{i-1}}(f)_p) \}.$$

Taking into account that

$$E_1(f)_p + E_0(f)_p \leqq 2E_0(f)_p,$$

$$2^{i/2}(E_{2^i}(f)_p + E_{2^{i-1}}(f)_p) \leqq cE_{2^{i-1}}(f)_p \sum_{k=2^{i-2}}^{2^{i-1}-1} \frac{1}{\sqrt{k+1}} \leqq c \sum_{k=2^{i-2}}^{2^{i-1}-1} \frac{E_k(f)_p}{\sqrt{k+1}}$$

$$(i = 1, \ldots, r),$$

$$\sqrt{n} \, (E_n(f)_p + E_{2^r}(f)_p) \leqq cE_{2^r}(f)_p \sum_{k=2^{r-1}}^{2^r-1} \frac{1}{\sqrt{k+1}} \leqq c \sum_{k=2^{r-1}}^{2^r-1} \frac{E_k(f)_p}{\sqrt{k+1}},$$

we get

$$I_2 \leqq \frac{c}{\sqrt{n+1}} \sum_{k=0}^{2^r-1} \frac{E_k(f)_p}{\sqrt{k+1}} \leqq \frac{c}{\sqrt{n+1}} \sum_{k=0}^{n} \frac{E_k(f)_p}{\sqrt{k+1}}$$

which proves Lemma 2 by (12). □

Define the de la Vallée-Poussin means

$$\vartheta_n(f, x) := \frac{1}{\sqrt{2}-1} \left(\sqrt{2} \, R_{2n+1}(f, x) - R_n(f, x) \right) =$$

$$= \frac{1}{\sqrt{2}-1} \sum_{k=n+1}^{2n+1} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{n+1}} S_k(f, x).$$

Since ϑ_n leaves fixed the polynomials of order $\leqq n$, hence

$$E_{2n+1}(f)_p \leqq \| e^{-x^2/2}(f - \vartheta_n f) \|_p \leqq cE_n(f)_p. \tag{13}$$

Using the ideas from the proof of the corresponding trigonometric result of Stečkin [14] we obtain for $1 \leqq p \leqq \infty$, $fe^{-x^2/2} \in L^p(\mathbf{R})$ the estimate

$$\| e^{-x^2/2}(f - R_n f) \|_p \leqq \frac{c}{\sqrt{n+1}} \sum_{k=0}^{n} \frac{E_k(f)_p}{\sqrt{k+1}}. \tag{14}$$

Indeed, let $m \in \mathbf{N}$ be given such that $2^m \leqq n < 2^{m+1}$. Using the formulas (they follow by Abel transform easily):

$$(n+1)^r R_n^r = \sum_{k=0}^{n} [(k+1)^r - k^r] S_k,$$

$$(2^r - 1)(n+1)^r \vartheta_n^r = \sum_{k=n+1}^{2n+1} [(k+1)^r - k^r] S_k$$

we get

$$(n+1)^r R_n^r = S_0+(2^r-1)\sum_{\mu=0}^{m-1} 2^{\mu r}\vartheta_{2^\mu-1}^r + \sum_{k=2^m}^{n}[(k+1)^r-k^r]S_k,$$

$$R_n^r-f = \frac{1}{(n+1)^r}\Big\{(S_0-f)+\sum_{\mu=0}^{m-1} 2^{\mu r}(2^r-1)(\vartheta_{2^\mu-1}^r-f)+$$

$$+[(n+1)^r-2^{mr}]\left(\frac{\sum_{k=2^m}^{n}[(k+1)^r-k^r]}{(n+1)^r-2^{mr}}\right).$$

An obvious modification of the proof of (13) gives that

$$\left\|e^{-x^2/2}\left[\frac{\sum_{k=2^m}^{n}[(k+1)^r-k^r]S_k}{(n+1)^r-2^{mr}}-f\right]\right\|_p \leqq cE_{2^m}(f)_p,$$

and hence from (13) we get

$$\|e^{-x^2/2}(R_n^r-f)\|_p \leqq \frac{c}{(n+1)^r}\Big\{E_0(f)+\sum_{\mu=0}^{m=1} 2^{\mu r}E_{2^\mu}(f)_p+[(n+1)^r-2^{mr}]E_{2^m}(f)_p\Big\}.$$

Now from

$$2^{\mu r} \leqq c\{(2^{\mu-1}+1)^{r-1}+(2^{\mu-1}+2)^{r-1}+\ldots+2^{\mu(r-1)}\}$$

it follows

$$2^{\mu r}E_{2^\mu}(f)_p \leqq c\sum_{k=2^{\mu-1}}^{2^\mu-1}(k+1)^{r-1}E_k(f)_p$$

and by $(n+1)^r-2^{mr}\leqq c2^{mr}$ we get

$$[(n+1)^r-2^{mr}]E_{2^m}(f)_p \leqq c\sum_{k=0}^{2^m-1}(k+1)^{r-1}E_k(f)_p.$$

Finally

$$\|e^{-x^2/2}(R_n^r-f)\|_p \leqq \frac{c}{(n+1)^r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_p$$

and (14) follows.

Define the Lipschitz classes by the aid of the Freud modulus. Namely if $1\leqq p\leqq\infty$ and $fe^{-x^2/2}\in L^p(\mathbf{R})$ then let for $0<\alpha\leqq 1$:

$$f\in \mathrm{Lip}\,(\alpha, p) \overset{\text{def}}{\Leftrightarrow} \omega(f, \delta)_p \leqq c\delta^\alpha.$$

Now (7)—(14) gives the following Bernstein type theorem.

THEOREM 1. *Let* $1\leqq p\leqq\infty$ *and* $e^{-x^2/2}f\in L^p(\mathbf{R})$.

a) *If* $f\in\mathrm{Lip}\,(\alpha, p)$, $\alpha<1$, *then*

$$\|e^{-x^2/2}(R_nf-f)\|_p \leqq cn^{-\alpha/2},$$

and if $f\in \mathrm{Lip}\,(1,p)$, *then*

$$\|e^{-x^2/2}(R_n f-f)\|_p \leqq c \log n/n$$

and this estimate cannot be improved.

b) *If* $\alpha<1$ *and* $xf(x)e^{-x^2/2}\in L^p(\mathbf{R})$, *then*

$$\|e^{-x^2/2}(R_n f-f)\|_p = O(n^{-\alpha/2}) \Rightarrow f\in \mathrm{Lip}\,(\alpha, p).$$

Next we come to the Alexits type theorems. Some lemmas are needed.

LEMMA 4. *Let* $\varphi_n\in C^1(\mathbf{R})$, $e^{x^2/2}\varphi_n$, $e^{x^2/2}\varphi_n'$, $e^{x^2/2}g$, $e^{x^2/2}h\in L^p(\mathbf{R})$ *for some* $1\leqq p\leqq \infty$. *Suppose that*

$$\|e^{x^2/2}(\varphi_n'-g)\|_p \to 0, \quad \|e^{x^2/2}(\varphi_n-h)\|_p \to 0 \quad (n\to\infty).$$

Then h is locally absolutely continuous and $h'=g$ *a.e.*

PROOF. Let $x\geqq 0$ (the case $x<0$ is similar). The improper integral $\int\limits_x^\infty \varphi_n'$ exists since $\varphi_n'(x)e^{x^2/2}\in L^p(\mathbf{R})$ and $e^{-x^2/2}\in L^q(\mathbf{R})$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$. Consequently the limit $\lim\limits_{x'\to\infty}\varphi_n(x')$ exists and from $e^{x^2/2}\varphi_n\in L^p(\mathbf{R})$ it follows that this limit is zero, thus

$$\int\limits_x^\infty \varphi_n' = -\varphi_n(x).$$

It is clear that (for $p\neq 1$)

$$e^{x^2/2}\left|\varphi_n(x)+\int\limits_x^\infty g\right| = e^{x^2/2}\left|\int\limits_x^\infty (\varphi_n'-g)\right| \leqq e^{x^2/2}\Big(\int\limits_x^\infty e^{-y^2q/2}\,dy\Big)^{1/q}\,\|(\varphi_n'-g)e^{x^2/2}\|$$

and for $p=1$

$$e^{x^2/2}\left|\varphi_n(x)+\int\limits_x^\infty g\right| \leqq \int\limits_x^\infty e^{y^2/2}\,|\varphi_n'-g|\,dy \leqq \|(\varphi_n'-g)e^{y^2/2}\|_p.$$

Since

$$\int\limits_x^\infty e^{-y^2q/2}\,dy \leqq \begin{cases} c & \text{if } x\leqq 1\\ \dfrac{1}{x}\displaystyle\int\limits_x^\infty ye^{-y^2q/2}\,dy \leqq \dfrac{c}{x}e^{-x^2q/2} & \text{if } x\geqq 1,\end{cases}$$

hence for $1\leqq p\leqq\infty$ we get

$$e^{x^2/2}\left|\varphi_n(x)+\int\limits_x^\infty g\right| \to 0 \quad (n\to\infty,\, x\to 0). \tag{15}$$

On the other hand, $e^{x^2/2}(\varphi_n(x)-h(x))\to 0$ in $L^p(\mathbf{R})$ so for a subsequence denoted again by φ_n,

$$\varphi_n(x)\to h(x) \quad \text{a.e.} \quad (n\to\infty).$$

From (15) we see that

$$h(x)+\int_x^\infty g=0 \quad \text{a.e.}$$

and this was to be proved. □

LEMMA 4′. *Let* $1\leqq p\leqq\infty$, $(\varphi_n)\subset C^1(\mathbf{R})$, $e^{-x^2/2}\varphi_n$, $e^{-x^2/2}\varphi_n'$, $e^{-x^2/2}g$, $e^{-x^2/2}h\in\in L^p(\mathbf{R})$. *Suppose that*

$$\|e^{-x^2/2}(\varphi_n'-g)\|_p\to 0,\quad \|e^{-x^2/2}(\varphi_n-h)\|_p\to 0\quad (n\to\infty).$$

Then h is locally absolutely continuous and $h'=g$ *a.e.*

The proof is similar.

THEOREM 2. *Let* $1<p<\infty$, $fe^{-x^2/2}\in L^p(\mathbf{R})$. *Then the following two statements are equivalent*

a) $\|e^{-x^2/2}(R_nf-f)\|_p=O\left(\frac{1}{\sqrt{n}}\right)$,

b) $\tilde{f}$ *is locally absolutely continuous and*

$$[e^{-x^2}\tilde{f}(x)]'\,e^{x^2/2}\in L^p(\mathbf{R}).$$

PROOF. We shall use the following equivalence from [6]: if $1<p\leqq\infty$ then a formal Hermite series $\sum a_kh_k$ is the expansion of some $f\in L^p(e^{-y^2/2})$ if and only if the Riesz means are bounded in norm:

$$\left\|e^{-x^2/2}\,R_n\left(\sum a_kh_k\right)\right\|_p=O(1).$$

Using the above observations we see that in case $1<p<\infty$,

$$\|e^{-x^2/2}(R_nf-f)\|_p=O\left(\frac{1}{\sqrt{n}}\right)$$

if and only if

$$\left\|e^{-x^2/2}\,R_n\left(\sum\sqrt{k}\,a_kh_k\right)\right\|_p=O(1)$$

if and only if there exists $g\in L^p(e^{-y^2/2})$, $g\sim\sum\sqrt{k}\,a_kh_k$. Since ([1])

$$[h_{k-1}(x)\,e^{-x^2}]'=-\sqrt{2k}\,h_k(x)e^{-x^2} \tag{16}$$

this is equivalent to the fact that $[e^{-x^2}\tilde{f}(x)]'e^{x^2/2}\in L^p(\mathbf{R})$. Indeed, suppose first that there exists $g\in L^p(e^{-y^2/2})$ with

$$g\sim\sum-\sqrt{2k}\,a_kh_k. \tag{17}$$

By (16) we get

$$[e^{-x^2}R_n\tilde{f}(x)]'\,e^{x^2}=R_n\left(\sum-\sqrt{2k}\,a_kh_k\right)$$

and then

$$\left\|e^{x^2/2}\left(g(x)-[e^{-x^2}R_n\tilde{f}(x)]'\right\|_p\to 0,\quad \left\|e^{x^2/2}\left(\tilde{f}(x)e^{-x^2}-R_n\tilde{f}(x)e^{-x^2}\right)\right\|_p\to 0.$$

Using Lemma 4 we obtain that

$$[\tilde{f}(x)e^{-x^2}]' = g(x)e^{-x^2} \quad \text{a.e.}$$

Conversely, let

$$\varphi(x) := [e^{-x^2}\tilde{f}(x)]'e^{x^2} \sim \sum b_k h_k$$

be the Hermite series of $\varphi \in L^p(e^{-x^2/2})$. We see that for $x>0$

$$\Big|\int_x^\infty [e^{-t^2}\tilde{f}(t)]'\,dt\Big| \le \|e^{x^2/2}[e^{-x^2}\tilde{f}(x)]'\|_p \Big(\int_x^\infty e^{-t^2q/2}\,dt\Big)^{1/q} \le$$

$$\le c\left(\frac{1}{x}\int_x^\infty te^{-t^2q/2}\,dt\right)^{1/q} \le cx^{-1/q}e^{-x^2/2}$$

hence

$$\lim_{|x|\to\infty} e^{-x^2}\tilde{f}(x)\,h_k(x) = 0.$$

Now

$$b_k = \int_{-\infty}^{\infty} \varphi(x)\,h_k(x)\,e^{-x^2}\,dx = [e^{-x^2}\tilde{f}(x)\,h_k(x)]_{-\infty}^{\infty} -$$

$$-\int_{-\infty}^{\infty} e^{-x^2}\tilde{f}(x)\sqrt{2k}\,h_{k-1}(x)\,dx = -\sqrt{2k}\,a_k.$$

Theorem 2 is proved.

Using the formula

$$h_n'(x) = \sqrt{2n}\,h_{n-1}(x)$$

we analogously get

THEOREM 2′. *Under the conditions of Theorem 2 the following statements are equivalent:*

a) $\|e^{-x^2/2}(R_n\tilde{f}-\tilde{f})\|_p = O\left(\frac{1}{\sqrt{n}}\right)$,

b) *f is locally absolutely continuous and $f'(x)e^{-x^2/2}\in L^p(\mathbf{R})$.*

Define the Abel—Poisson transform

$$T_x f(y) := \sum e^{-\sqrt{2n}\,x} a_k h_k(y) \quad (x>0),$$

and the conjugate transform

$$\tilde{T}_x f(y) := \sum e^{-\sqrt{2n}\,x} a_k h_{k-1}(y) \quad (x>0).$$

The semigroup properties

18) $$T_{x_1}T_{x_2}f = T_{x_1+x_2}f, \quad \tilde{T}_{x_1}\tilde{T}_{x_2}f = \tilde{T}_{x_1+x_2}f$$

can be easily obtained from the definition. The continuity property

(19) $$\|e^{-x^2/2}(T_x f - f)\|_p \to 0, \quad \|e^{-x^2/2}(\tilde{T}_x f - f)\|_p \to 0 \quad (x\to 0+)$$

can be proved as in [4]. Define the infinitesimal generator A of the semigroups (T_x) as follows. If $\frac{T_x f-f}{x}$ converges to a function g in $L^p(e^{-y^2/2})$ then we define $Af=g$. We immediately get that if f belongs to the domain $D(A)$ of the infinitesimal generator A then

$$Af \sim -\sum \sqrt{2k}\, a_k h_k. \tag{20}$$

Conversely if the right-hand side of (20) is the series of some $g \in L^p(e^{-y^2/2})$ then $g=Af$ (since $A(R_n f)=R_n g$ and A is a closed operator, see [11]). We know from [11] that a continuous operator semigroup is saturated with the order $O(x)$ and the saturation class is $D(A)$. Hence we have proved the following.

THEOREM 3. *Let* $1<p<\infty$ *and* $f \in L^p(e^{-y^2/2})$. *Then*

a) $\|e^{-y^2/2}(T_x f-f)\|_p = o(x) \quad (x \to 0+) \Leftrightarrow f=c,$

b) $\|e^{-y^2/2}(T_x f-f)\|_p = O(x) \Leftrightarrow [e^{-y^2/2}\tilde{f}(y)]'\, e^{y^2/2} \in L^p(\mathbf{R}).$

Using the fact that $\tilde{T}_x f = T_x \tilde{f}$ we analogously get

THEOREM 3'. *Let* $1<p<\infty$ *and* $f \in L^p(e^{-y^2/2})$. *Then*

a) $\|e^{-y^2/2}(T_x \tilde{f}-\tilde{f})\|_p = o(x) \quad (x \to 0+) \Leftrightarrow f=c,$

b) $\|e^{-y^2/2}(T_x \tilde{f}-\tilde{f})\|_p = O(x) \Leftrightarrow f' \in L^p(e^{-y^2/2}).$

REMARK. Theorems 2 and 2' are also saturation theorems. E.g. in Theorem 2 we can state that

$$\|e^{-x^2/2}(R_n f-f)\|_p = o\left(\frac{1}{\sqrt{n}}\right) \quad (n \to \infty)$$

implies $f \equiv c$. Indeed, in this case

$$\frac{\sqrt{k}}{\sqrt{n+1}}|a_k| = |\langle e^{-x^2/2}(R_n f-f), e^{-x^2/2} h_k\rangle| = o\left(\frac{1}{\sqrt{n}}\right)\|e^{-x^2/2} h_k\|_q = o\left(\frac{1}{\sqrt{n}}\right) \quad (n \to \infty$$

for any fixed $k \in \{1, 2, \dots\}$, i.e. $\sqrt{k}\, a_k = o(1)$ $(n \to \infty)$ hence $a_k=0$ $(k=1, 2, \dots)$. Taking into account the well-known uniqueness theorem for Hermite—Fourier series (see e.g. Kaczmarz—Steinhaus: Theorie der Orthogonalreichen, Warsawa—Lwow 1935) we obtain $f \equiv c$.

References

[1] G. Szegő, *Orthogonal Polynomials*, AMS Coll. Publ. 23 (Providence, Rhode Island, 1959).

[2] G. Alexits, Sur l'ordre de grandeur de l'approximation d'une fonction périodique par les sommes de Fejér, *Acta Math. Acad. Sci. Hung.*, **3** (1952), 29—40.

[3] G. Freud, On polynomial approximation with the weight $\exp\left\{-\frac{1}{2}x^{2k}\right\}$, *Acta Math. Acad. Sci. Hung.*, **24** (1973), 363—371.

[4] B. Muckenhoupt, Hermite conjugate expansions, *Trans. Amer. Math. Soc.*, **139** (1969), 243—260.

[5] R. Hunt, Developments related to the a.e. convergence of Fourier series (manuscript).
[6] I. Joó, On the order of approximation by Fejér means of Hermite—Fourier and Laguerre—Fourier series, *Acta Math. Hung.* (to appear).
[7] I. Joó, On the divergence of eigenfunction expansions, *Annales Univ. Sci. Budapest, Sectio Math.* (to appear).
[8] I. Joó, On some problems of M. Horváth, *Annales Univ. Sci. Budapest Sectio Math.* (to appear).
[9] A. Bogmér, Generalization of a theorem of G. Alexits, *Annales Univ. Sci. Budapest, Sectio Math.* (to appear).
[10] M. Horváth, Some saturation theorems for classical orthogonal expansions, Preprint of Math. Inst. 76 Budapest 1987.
[11] P. L. Butzer, H. Berens, *Semi-group of operators and approximation*, Springer (Berlin, 1967).
[12] P. L. Butzer, R. J. Nessel, *Fourier Analysis and Approximation*, vol. I. Birkhäuser (Basel, 1971).
[13] E. L. Poiani, Mean Cesáro summability of Laguerre and Hermite series, *Trans. Amer. Math. Soc.*, **173** (1972), 1—31.
[14] S. B. Stečkin, On the approximation of periodic functions by de la Vallée-Poussin sums, *Anal. Math.*, **4** (1978), 61—74.
[15] G. Freud, On the approximation with the weight $\exp\{-x^2/2\}$ by polynomials, *Dokl. Akad. Nauk*, **12** (1971), 570—573.
[16] P. Névai, Geza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory*, **48** (1986), 3—167.
[17] W. E. Milne, On the maximum absolute value of the derivative of $\exp(-x^2)P_n(x)$, *Trans. Amer. Math. Soc.*, **33** (1931), 143—146.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

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A NOTE ON LACUNARY TRIGONOMETRIC SERIES

I. BERKES[1] (Budapest)

1. Introduction

The following theorem was proved by P. Erdős [1]:

THEOREM A. *Let (n_k) be a sequence of positive integers such that*

$$n_{k+1}/n_k \geqq 1+c_k/\sqrt{k}, \quad c_k \to \infty. \tag{1.1}$$

Then $\cos 2\pi n_k x$ *satisfies the central limit theorem i.e. setting* $S_N = \sum_{k \leqq N} \cos 2\pi n_k x$ *we have*[2]

$$(2/N)^{1/2} S_N \xrightarrow{\mathcal{D}} N(0, 1). \tag{1.2}$$

On the other hand, for every $c>0$ *there exists a sequence* (n_k) *of integers such that*

$$n_{k+1}/n_k \geqq 1+c/\sqrt{k} \quad (k \geqq 1)$$

and (1.2) *is not valid.*

Theorem A shows that in terms of the growth speed of (n_k), (1.1) is a best possible condition for $\cos 2\pi n_k x$ to satisfy the central limit theorem. It is natural to ask if this condition implies more general independence properties of $\cos 2\pi n_k x$, e.g. the validity of the LIL and other standard limit theorems for i.i.d. bounded r.v.'s. For the law of the iterated logarithm Takahashi [5] proved the following

THEOREM B. *Let (n_k) be a sequence of integers such that*

$$n_{k+1}/n_k \geqq 1+c/k^\alpha, \quad \alpha < 1/2. \tag{1.3}$$

Then $\cos 2\pi n_k x$ *satisfies the law of the iterated logarithm i.e.*

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k \leqq N} \cos 2\pi n_k x = 1 \quad a.e. \tag{1.4}$$

Note that (1.3) is stronger than (1.1): it requires that (1.1) holds with c_k increasing at least as k^ε for some $\varepsilon>0$. A minor modification of the proof of Theorem B shows that (1.4) holds actually if

$$n_{k+1}/n_k \geqq 1+(\log k)^\gamma/\sqrt{k}$$

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[2] The underlying probability space for $\cos 2\pi n_k x$ is $((0, 1), \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel σ-field in $(0, 1)$ and λ is the Lebesgue measure.

for a sufficiently large γ but for small γ the proof breaks down. Thus the log log behaviour of $\cos 2\pi n_k x$ remains open if (1.1) holds with a very slowly increasing c_k. The purpose of the present paper is to show that condition (1.1) does *not* imply the law of the iterated logarithm for arbitrary $c_k \to \infty$. In fact we shall prove the following

THEOREM. *There exists a sequence (n_k) of positive integers satisfying* (1.1) *(and thus the central limit theorem) such that* (1.4) *is false, namely*

$$\overline{\lim_{N\to\infty}} (N \log\log N)^{-1/2} \sum_{k\leq N} \cos 2\pi n_k x < \gamma \quad a.e.$$

for some constant $\gamma < 1$.

In our example $c_k = \text{const}\cdot(\log\log k)^{1/2}$. As we shall show in a subsequent paper, this example is not far from optimal in the sense that if (1.1) holds with $c_k = (\log\log k)^\gamma$ for a sufficiently large γ then $\cos 2\pi n_k x$ satisfies not only (1.4) but also Kolmogorov's upper-lower class test. Whether there exists a similar example with

$$\overline{\lim} (\ldots) \geqq \gamma > 1 \quad \text{a.e.}$$

remains open.

2. Proof of the theorem

LEMMA 1. We have

$$\int_0^1 \Big(\sum_{j=1}^N \cos 2\pi jx\Big)^3 dx \sim \frac{3}{8} N^2 \quad as\ N \to \infty^{3}. \tag{2.1}$$

PROOF. Clearly

$$\Big(\sum_{j=1}^N \cos 2\pi jx\Big)^3 = \frac{1}{8} \sum \cos 2\pi(\pm j_1 \pm j_2 \pm j_3)x$$

where the sum is extended for all values $1 \leqq j_1, j_2, j_3 \leqq N$ and all possible choices of the signs $\pm$. Since $\int_0^1 \cos 2\pi lx\, dx = 1$ or 0 according as the integer l equals 0 or not, it follows that the left side of (2.1) equals $1/8$ times the number of solutions of

$$\pm j_1 \pm j_2 \pm j_3 = 0, \quad 1 \leqq j_1, j_2, j_3 \leqq N. \tag{2.2}$$

Clearly (2.2) has no solutions such that all signs are identical and the number of its solutions such that the signs are $+, -, -$ and $j_1 = \nu$ $(1 \leqq \nu \leqq N)$ is $\nu - 1$. Thus the total number of solutions of (2.2) is $6 \sum_{\nu=1}^N (\nu - 1) = 3N^2 - 3N$, proving (2.1).

[3] $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$.

LEMMA 2. *Let $a_1 < a_2 < \dots$ be positive integers such that the ratios a_{k+1}/a_k are also integers. Then there exist functions $\varrho_k(x)$ $(0 \leqq x \leqq 1)$ such that the $\varrho_k(x)$ are independent r.v.'s. over the probability space $((0,1), \mathcal{B}, \lambda)$ and*

$$(2.3) \qquad |\varrho_k(x) - \{a_k x\}| \leqq a_k/a_{k+1}$$

where $\{\ \}$ denotes fractional part.

This lemma is implicit in [3]; we give here the very short proof. Set $q_k = a_{k+1}/a_k$ and

$$\varrho_k(x) = q_k^{-1}[q_k\{a_k x\}]$$

where $[\ ]$ denotes integral part. Clearly if $j/q_k \leqq \{a_k x\} < (j+1)/q_k$ for some integer $0 \leqq j \leqq q_k - 1$ then $\varrho_k(x) = j/q_k$. Hence (2.3) is valid. Further $\varrho_k(x)$ is constant on each interval $[j/a_{k+1}, (j+1)/a_{k+1})$ $(0 \leqq j \leqq a_{k+1} - 1)$ and is periodic with period $1/a_k$. From these facts it follows that the $\varrho_k(x)$ are independent r.v.'s.[4]

The following lemma is Feller's generalization of Cramér's large deviation theorem (see [2]):

LEMMA 3. *Let (Y_k) be a sequence of independent r.v.'s such that $EY_k = 0$, $EY_k^2 < +\infty$ $(k = 1, 2, \dots)$. Set $s_n^2 = \sum_{k=1}^{n} EY_k^2$ and assume that $|Y_k| \leqq \lambda_s s_k$ where λ_n is a numerical sequence with $\lambda_n \downarrow 0$. Then for $0 < x \leqq 1/(12\lambda_n)$ we have*

$$P(Y_1 + \dots + Y_n > s_n x) = \exp\left\{-\frac{1}{2} x^2 Q_n(x)\right\} \cdot \left((1 - \Phi(x)) + \Theta \lambda_n e^{(-1/2)x^2}\right)$$

where $|\Theta| \leqq 9$ and $Q_n(x) = \sum_{\nu=1}^{\infty} q_{n,\nu} x^\nu$ is a function analytic for $|x| \leqq 1/(12\lambda_n)$ whose coefficients $q_{n,\nu}$ depend on the moments of $X_1, X_2, \dots, X_n$. In particular

$$(2.4) \qquad q_{n,1} = \frac{1}{3s_n^3} \sum_{k=1}^{n} EY_k^3$$

and

$$(2.5) \qquad |q_{n,\nu}| \leqq \frac{1}{7}(12\lambda_n)^\nu, \quad n \geqq 1, \nu \geqq 1.$$

We now turn to the proof of the theorem. Let $a_k = 2^{k^2}$ and $m_k = [Ak/\log\log k]$ $(k \geqq 3)$ where A is a small absolute constant to be chosen later. Let $I_k = \{a_k, 2a_k, \dots, m_k a_k\}$; clearly the sets I_k, $k = 3, 4, \dots$ are disjoint if A is small enough. Define the sequence (n_k) by

$$(n_k) = \bigcup_{j=3}^{\infty} I_j.$$

We show that $\cos 2\pi n_k x$ satisfies the requirements of the theorem. Set $M_k = \sum_{i=3}^{k} m_i$;

[4] For the rest of this paper, with the exception of formula (2.8), $\{\ \}$ will denote ordinary brackets and not fractional part.

clearly

$$M_k \sim \frac{A}{2} \frac{k^2}{\log\log k}. \tag{2.6}$$

Thus if $M_{k-1} < j < M_k$ then letting $i = j - M_{k-1}$ we have

$$\frac{n_{j+1}}{n_j} = 1 + \frac{1}{i} \geqq 1 + \frac{1}{m_k} \geqq 1 + \frac{\log\log k}{Ak} \geqq$$

$$\geqq 1 + \frac{(\log\log M_k)^{1/2}}{2\sqrt{A}\sqrt{M_{k-1}}} \geqq 1 + \frac{(\log\log j)^{1/2}}{2\sqrt{A}\sqrt{j}} \quad (k \geqq k_0).$$

Also, if $j = M_k$ then $n_j = m_k a_k$, $n_{j+1} = a_{k+1}$ and thus $n_{j+1}/n_j \geqq 2$ if $k \geqq k_0$. Hence (n_k) satisfies (1.1) with $c_k = c\,(\log\log k)^{1/2}$. Set

$$X_k = \sum_{\nu = M_{k-1}+1}^{M_k} \cos 2\pi n_\nu x, \quad f_l(x) = \sum_{j=1}^{l} \cos 2\pi j x. \tag{2.7}$$

Then

$$X_k = f_{m_k}(a_k x) = f_{m_k}(\{a_k x\}) \tag{2.8}$$

and thus putting

$$Z_k = f_{m_k}(\varrho_k(x)), \quad Y_k = Z_k - EZ_k$$

(where $\varrho_k(x)$ are the functions in Lemma 2) we have by Lemma 2, $|f_l'(x)| \leqq 2\pi l^2$ and the mean value theorem

$$|X_k - Z_k| \leqq 2\pi m_k^2 a_k / a_{k+1} \ll 2^{-k}. \tag{2.9}$$

(Here $a_n \ll b_n$ means that $|a_n/b_n| = O(1)$ uniformly.) Moreover, Y_k are independent r.v.'s. Since $EX_k = 0$, (2.9) implies

$$|X_k - Y_k| \ll 2^{-k} \tag{2.10}$$

and thus $|X_k| \leqq m_k$ and the mean value theorem yield

$$|X_k^2 - Y_k^2| \ll 2^{-k} m_k \ll 2^{-k/2}, \tag{2.11}$$

$$|X_k^3 - Y_k^3| \ll 2^{-k} m_k^2 \ll 2^{-k/2}. \tag{2.12}$$

We now apply Lemma 3 to the sequence (Y_k). Clearly $EX_k^2 = m_k/2$ and thus by (2.11) and (2.6)

$$s_n^2 =: \sum_{k=3}^{n} EY_k^2 = \frac{1}{2} \sum_{k=3}^{n} m_k + O(1) \sim \frac{A}{4} \frac{n^2}{\log\log n}. \tag{2.13}$$

Further by Lemma 1, (2.7), (2.8) and (2.12) we have

$$\sum_{k=3}^{n} EY_k^3 = \sum_{k=3}^{n} EX_k^3 + O(1) \sim \sum_{k=3}^{n} \frac{3}{8} m_k^2 \sim \frac{1}{8} A^2 \frac{n^3}{(\log\log n)^2}. \tag{2.14}$$

(Note that, by periodicity, the integrals of $f_{m_k}^3(x)$ and $f_{m_k}^3(a_k x)$ over $(0, 1)$ are equal.) Set

$$\lambda_n = 8\sqrt{A}(\log\log n)^{-1/2}.$$

Then by (2.10), (2.13) and $|X_k| \leqq m_k$ we have

$$|Y_k| \leqq \lambda_k s_k, \quad k \geqq k_0.$$

Hence Lemma 3 implies for any $0 \leqq c \leqq 2$ and $A \leqq 10^{-5}$

$$\text{(2.15)}\quad P(Y_3+\ldots+Y_n > cs_n(2\log\log s_n)^{1/2}) \ll \exp\left\{-\frac{1}{2}x_n^2 Q_n(x_n)\right\}\frac{1}{x_n}\exp\left\{-\frac{1}{2}x_n^2\right\}$$

where $x_n = c(2\log\log s_n)^{1/2}$ and $Q_n(x) = \sum_{\nu=1}^{\infty} q_{n,\nu}x^\nu$ is an analytic function for $|x| \leqq 1/(12\lambda_n)$ whose coefficients $q_{n,\nu}$ satisfy (2.4), (2.5). Now by (2.13), (2.14) we have

$$q_{n,1} \sim \frac{\sqrt{A}}{3}(\log\log n)^{-1/2}$$

and

$$|q_{n,\nu}| \leqq \frac{1}{7}(96\sqrt{A})^\nu (\log\log n)^{-\nu/2}, \quad n \geqq 1,\ \nu \geqq 1.$$

Thus

$$|Q_n(x_n)| \geqq q_{n,1}x_n - \sum_{\nu=2}^{\infty}|q_{n,\nu}|x_n^\nu \geqq$$

$$\geqq \frac{\sqrt{A}}{6}(\log\log n)^{-1/2}x_n - \sum_{\nu=2}^{\infty}\frac{1}{7}(96\sqrt{A})^\nu(\log\log n)^{-\nu/2}x_n^\nu \geqq$$

$$\geqq \frac{\sqrt{2A}}{12}c - \sum_{\nu=2}^{\infty}\frac{1}{7}(96\sqrt{A})^\nu(2c)^\nu \geqq \frac{\sqrt{2A}}{12}c - \frac{2}{7}\cdot 200^2 Ac^2 \geqq \frac{\sqrt{2A}}{24}c$$

for any $n \geqq n_0$, $0 \leqq c \leqq 2$ and $A \leqq 10^{-12}$. Hence by (2.15)

$$\text{(2.16)}\quad P(Y_3+\ldots+Y_n > cs_n(2\log\log s_n)^{1/2}) \ll$$

$$\ll \exp\left\{-\frac{\sqrt{2A}}{24}c^3\log\log s_n - \frac{1}{2}c^2(2\log\log s_n)\right\} =$$

$$= \exp\left\{-\left(\frac{\sqrt{2A}}{24}c^3 + c^2\right)(1+o(1))\log\log n\right\}.$$

Now the function $f(c) = c^2 + \sqrt{2A}\,c^3/24$ satisfies $f(1) > 1$ and thus there exists a $0 < \gamma < 1$ such that $f(\gamma) > 1$. Hence by (2.16)

$$\text{(2.17)}\quad P(Y_3+\ldots+Y_n > \gamma s_n(2\log\log s_n)^{1/2}) \ll \exp(-(1+\varepsilon_0)\log\log n).$$

where $\varepsilon_0>0$. By the standard proof of the upper half of the LIL for independent r.v.'s (see e.g. [4] pp. 261—262), (2.17) implies

$$\varlimsup_{k\to\infty} s_k^{-1}(2\log\log s_k)^{-1/2}\sum_{i=3}^{k} Y_i \leqq \gamma \quad \text{a.e.} \tag{2.18}$$

Setting $S_N=\sum_{j\leqq N}\cos 2\pi n_j x$ and using (2.6), (2.7), (2.10) and (2.13) we get $S_{M_k}=$ $=\sum_{i=3}^{k} Y_i+O(1)$, $s_k^2\sim M_k/2$ and thus (2.18) yields

$$\varlimsup_{k\to\infty} (M_k\log\log M_k)^{-1/2} S_{M_k} \leqq \gamma \quad \text{a.e.} \tag{2.19}$$

But if $M_k\leqq N<M_{k+1}$ then $|S_N-S_{M_k}|\leqq m_{k+1}\ll\sqrt{M_k}$ by (2.6) and thus (2.19) implies

$$\varlimsup_{N\to\infty} (N\log\log N)^{-1/2} S_N \leqq \gamma < 1 \quad \text{a.e.}$$

completing the proof.

References

[1] P. Erdős, On trigonometric sums with gaps, *Magyar Tud. Akad. Mat. Kut. Int. Közl.*, **7** (1962), 37—42.

[2] W. Feller, Generalization of a probability limit theorem of Cramér, *Trans. Amer. Math. Soc.*, **54** (1943), 361—372.

[3] J. Hawkes, Probabilistic behaviour of some lacunary series, *Zeitschrift für Wahrscheinlichkeitstheorie verw. Gebiete*, **53** (1980), 21—33.

[4] M. Loève, *Probability Theory*. Van Nostrand, 1955.

[5] S. Takahashi, On the law of the iterated logarithm for lacunary trigonometric series, *Tohoku Math. Journ.*, **24** (1972), 319—329 and **27** (1975), 391—403.

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MATHEMATICAL INSTITUTE
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[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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JUGOSLOVENSKA KNJIGA, Terazije 27, *Beograd*
FORUM, Vojvode Mišića 1. *21000 Novi Sad*

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ON THE FIRST CLASS OF BAIRE GENERATED BY CONTINUOUS FUNCTIONS ON R^N RELATIVE TO THE ALMOST EUCLIDEAN TOPOLOGY

H. W. PU and H. H. PU (College Station)

1. Introduction. Throughout this paper, functions are defined on R^N. We shall use topological terms without modifier to refer to the Euclidean topology for R^N. Let $\mathscr{C}$ denote the class of continuous functions. $\mathscr{A}$ the class of ordinarily approximately continuous functions, and $\mathscr{P}$ the class of almost everywhere continuous functions. If $\mathscr{F}$ is a class of functions, then $\mathscr{B}_1(\mathscr{F})$ denotes the class of functions which are pointwise limits of sequences of functions in $\mathscr{F}$ and $\mathscr{B}_2(\mathscr{F})=\mathscr{B}_1(\mathscr{B}_1(\mathscr{F}))$.

In [3], Grande defined, for $N=1$, a class of functions (AP_1) and showed that $\mathscr{B}_1(\mathscr{A}\cap\mathscr{P})\subset\mathscr{B}_1(\mathscr{A})\cap\mathscr{B}_1(\mathscr{P})\cap(\mathrm{AP}_1)$. Then he asked if this inclusion is actually an equality. Nishiura [5] investigated Grande's question for $N\geqq 1$. Cox and Humke [1] also tried to answer Grande's question. Although this question is not settled, fruitful results are obtained in both [1] and [5]. They form the basis of the present research, in which two closely related classes of functions (AP_1^*) and (AP_1^{**}) are introduced and the equalities $\mathscr{B}_1(\mathscr{A}\cap\mathscr{P})=\mathscr{B}_1(\mathscr{A})\cap(\mathrm{AP}_1^{**})$ and $(\mathrm{AP}_1^*)=(\mathrm{AP}_1)$ are proved.

2. Preliminaries. Let μ denote the Lebesgue measure on R^N. If $X\subset R^N$ is measurable and $x\in R^N$, then $\underline{d}(x, X)$ and $\overline{d}(x, X)$ denote the upper and lower ordinary density of X at x respectively (for definition, see [9]). It is known that $\mathscr{A}$ and $\mathscr{A}\cap\mathscr{P}$ are the classes of continuous functions relative to the ordinary density topology d and the almost Euclidean topology $\mathscr{T}$ for R^N respectively [2, 5, 6]. A set X is d-open if it is measurable and the ordinary density $d(x, X)$ is equal to 1 at every $x\in X$. A set is $\mathscr{T}$-open if it is d-open and can be written as $G\cup Z$, where G is open and Z is null (i.e., $\mu(Z)=0$). The term "almost Euclidean" is suggested by O'Malley [6, 7].

For $X\subset R^N$, we shall use X^0 and $\overline{X}$ to denote the interior and the closure of X respectively. $\Delta(X)$ denotes the d-closure of the d-interior of X. We shall use the term interval to mean an open interval $\{x=(x_1, ..., x_N): a_i<x_i<b_i,\ i=1, ..., N\}$ in R^N. We call the interval a cube if $b_i-a_i=b_1-a_1$ for $i=, ..., N$. Also, $\mathfrak{A}$ and $\mathfrak{B}$ are the collections of cozero and zero sets of $\mathscr{A}\cap\mathscr{P}$ respectively.

The class of functions (AP_1) is defined in two equivalent ways [1, 3, 5]:

$f\in(\mathrm{AP}_1)$ if for each $a<b$ and nonempty sets U and V such that $U\subset\{x: f(x)<a\}$, $V\subset\{x: f(x)>b\}$, $U\subset\Delta(\overline{U})$, and $V\subset\Delta(\overline{V})$, it is true that $U-\overline{V}\neq\emptyset$ or $V-\overline{U}\neq\emptyset$.

$f\in(\mathrm{AP}_1)$ if whenever $a<b$ and U, V are nonempty sets such that $U\subset\{x: f(x)<a\}$, $V\subset\{x: f(x)>b\}$, then $U\cup V\not\subset\Delta(\overline{U}\cap\overline{V})$.

We shall need the following results. (2.1)—(2.6) are from [5], (2.7) and (2.8) are from [1] and [8] respectively.

(2.1) *If X is measurable, then* $\Delta(X)=\{x: \bar{d}(x, X)>0\}$.

(2.2) *If X is closed, then* $\Delta(\overline{\Delta(X)})=\Delta(X)$.

(2.3) *X is $\mathcal{T}$-perfect if and only if* $X=\Delta(\bar{X})$.

(2.4) *If X is closed, then $X-\Delta(X)$ is contained in an F_σ null set.*

This result is not listed but used in ([5], p. 327). We give a simple proof here. For $n=1, 2, \dots$, let $F_n=\left\{x\in X:\ \mu(X\cap I)<\frac{1}{2}\mu(I) \text{ for every cube } I \text{ with } x\in I\right.$ and $\left.\delta(I)<\frac{1}{n}\right\}$, where $\delta(I)$ is the diameter of I. Then each F_n is clearly closed. It can be shown that

$$\left\{x\in X:\ \bar{d}(x, X)<\frac{1}{2}\right\}\subset\bigcup F_n\subset\left\{x\in X:\ \underline{d}(x, X)\leqq\frac{1}{2}\right\}.$$

Hence $\bigcup F_n$ is an F_σ null set. By (2.1),

$$X-\Delta(X)=\{x\in X:\ \bar{d}(x, X)=0\}\subset\left\{x\in X:\ \bar{d}(x, X)<\frac{1}{2}\right\}.$$

2.4) is proved.

(2.5) *$B\in\mathfrak{B}$ if and only if B is d-closed and can be written in the form $F-Z$, where F is closed and Z is an F_σ null set.*

(2.6) *$f\in\mathcal{B}_1(\mathcal{A}\cap\mathcal{P})$ if and only if for each $a\in R$, we have* (i) $\{x: f(x)>a\}\in\mathfrak{B}_\sigma$ *and* (ii) $\{x: f(x)\geqq a\}\in\mathfrak{A}_\delta$. *Clearly* (ii) *can be replaced by* $\{x: f(x)<a\}\in\mathfrak{B}_\sigma$.

(2.7) $(\mathrm{AP}_1)\subset\mathcal{B}_1(\mathcal{P})$.

(2.8) $\mathcal{B}_1(\mathcal{A})=\mathcal{B}_1(\mathcal{C})$.

(2.9) *If F is a closed null set, $H\in G_{\delta\sigma}$ and $H\subset F$, then $H\in\mathfrak{B}_\sigma$.*

Proof. Let $H=\bigcup H_n$, where each $H_n\in G_\delta$. Then $H=\bigcup(F-(F-H_n))$ and we need only show that $F-(F-H_n)\in\mathfrak{B}$ for each n. Let b be fixed. Since $\mu(F-(F-H_n))\leqq\mu(F)=0$, $F-(F-H_n)$ is d-closed. $F-H_n$ is clearly an F_σ null set. The conclusion follows from (2.5).

(2.10) *Let H be a $\mathcal{T}$-perfect set. If $\{K_n\}$ is a sequence of $\mathcal{T}$-closed sets such that $\bar{H}\subset\bigcup K_n$, then there exists an n_0 and an interval I such that $\emptyset\neq I\cap H\subset K_{n_0}$.*

Proof. For each n, since K_n is $\mathcal{T}$-closed, there exist closed set F_n and null set Z_n such that $K_n=F_n-Z_n$. Being closed in R^N, $\bar{H}$ is of the second category in itself. Hence $\bar{H}\subset\bigcup K_n\subset\bigcup F_n$ implies the existence of n_0 and an interval I such that $\emptyset\neq I\cap\bar{H}\subset F_{n_0}$. It follows immediately that $I\cap H\neq\emptyset$. Let $x\in I\cap H$. By (2.3) and (2.1), $H=\Delta(\bar{H})$, $\bar{d}(x, \bar{H})>0$ and hence $\bar{d}(x, I\cap\bar{H})>0$. Thus $\bar{d}(x, K_{n_0})=$ $=\bar{d}(x, F_{n_0})\geqq\bar{d}(x, I\cap\bar{H})>0$, which implies $x\in K_{n_0}$. The proof is complete.

3. Result. We start with defining two classes of functions.

DEFINITION 1. $f\in(\mathrm{AP}_1^*)$ if for each nonempty $\mathcal{T}$-perfect set H each and $\varepsilon>0$, there exists an interval I such that $I\cap H\neq\emptyset$ and $\circ(f, I\cap H)\leqq\varepsilon$, where

$$\circ(f, I\cap H)=\sup\{|f(x)-f(y)|:\ x, y\in I\cap H\}.$$

DEFINITION 2. $f\in(\mathrm{AP}_1^{**})$ if for each nonempty $\mathcal{T}$-perfect set H and each $\varepsilon>0$, there exist an interval I and a set $B\in\mathfrak{B}$ such that $\emptyset\neq I\cap H\subset I\cap B$ and $\circ(f, I\cap B)\leqq\varepsilon$.

It is trivial that $(\mathrm{AP}_1^{**})\subset(\mathrm{AP}_1^*)$. We shall show that $\mathcal{B}_1(\mathcal{A}\cap\mathcal{P})=\mathcal{B}_1(\mathcal{A})\cap\cap(\mathrm{AP}_1^{**})$. The proof is strongly suggested by that of Baire's theorem concerning $\mathcal{B}_1(\mathcal{C})$. (See [4], pp. 143—148.)

THEOREM 1. $\mathcal{B}_1(\mathcal{A}\cap\mathcal{P})\subset(\mathrm{AP}_1^{**})$.

PROOF. Suppose $f\in\mathcal{B}_1(\mathcal{A}\cap\mathcal{P})$. Then there exists, in $\mathcal{A}\cap\mathcal{P}$, a sequence $\{f_n\}$ converging to f everywhere. Let $\varepsilon>0$ and $\emptyset\neq H=\Delta(\bar{H})$ be given. Set

$$B_{nm}=\left\{x:\ |f_n(x)-f_{n+m}(x)|\leqq\frac{\varepsilon}{3}\right\},\quad B_n=\cap\{B_{nm}:\ m=1, 2, \ldots\}.$$

Then, for each n, m $B_{nm}\in\mathfrak{B}$ and since $\mathfrak{B}$ is closed under countable intersections, $B_n\in\mathfrak{B}$. For each $x\in R^N$, since $\lim_{n\to\infty} f_n(x)=f(x)$, there exists n_x such that $x\in B_{n_x}\subset\subset\cup B_n$. Therefore $H\subset R^N=\cup B_n$. By (2.10), there exist n_0 and an interval I_1, such that $\emptyset\neq I_1\cap H\subset B_{n_0}$. Let x_0 be a fixed point in $I_1\cap H$. By $\mathcal{T}$-continuity of f_{n_0} at x_0, there exists a $\mathcal{T}$-open set U such that $x_0\in U\subset I_1$ and

$$(1)\qquad |f_n(x)-f_{n_0}(y)|<\frac{\varepsilon}{3}\quad\text{whenever}\quad x, y\in U.$$

Since U is $\mathcal{T}$-open, $H=\Delta(\bar{H})$, and $x_0\in U\cap H$, we have $\bar{d}(x_0, U\cap\bar{H})>0$ and hence $\mu(U\cap\bar{H})>0$. Also, since U is $\mathcal{T}$-open, $\mu(U-U^0)=0$. It follows that $\mu(U^0\cap\bar{H})>0$ which implies $U^0\cap H\neq\emptyset$. Therefore there exists an interval $I\subset U^0$ such that $I\cap H\neq\emptyset$. Now we have $\emptyset\neq I\cap H\subset U\cap H\subset I_1\cap H\subset B_{n_0}$ and hence $\emptyset\neq I\cap H\subset I\cap B_{n_0}$.

To show that $\circ(f, I\cap B_{n_0})\leqq\varepsilon$, let $x, y\in I\cap B_{n_0}$. By the facts $x\in B_{n_0}\subset B_{n_0m}$ for all m and $\lim_{m\to\infty} f_{n_0+m}(x)=f(x)$, we have

$$(2)\qquad |f_{n_0}(x)-f(x)|\leqq\frac{\varepsilon}{3}\quad\text{and}\quad |f_{n_0}(y)-f(y)|\leqq\frac{\varepsilon}{3}$$

(the latter can be seen similarly). We see easily from (1) and (2) that $|f(x)-f(y)|<\varepsilon$. Theorem 1 is proved.

THEOREM 2. $\mathcal{B}_1(\mathcal{A})\cap(\mathrm{AP}_1^{**})\subset\mathcal{B}_1(\mathcal{A}\cap\mathcal{P})$.

To prove this theorem, we need the following lemma.

LEMMA. *Let $f\in\mathcal{B}_1(\mathcal{A})\cap(\mathrm{AP}_1^{**})$ and $a<b$. Then there exists $S\in\mathfrak{B}_\sigma$ such that $S\subset\{x:\ f(x)>a\}$ and $R^N-S\subset\{x:\ f(x)<b\}$.*

PROOF. Let ε be a positive number less than $(b-a)/3$. We shall define four transfinite sequences of sets K_α, H_α, I_α, B_α $(\alpha<\Omega)$ as follows. Let $K_0=R^N$. When the closed set K_α is defined, we take $H_\alpha=\Delta(K_\alpha)$ and distinguish between two cases.

Case 1: $H_\alpha=\emptyset$. Then we take $I_\alpha=B_\alpha=\emptyset$.

Case 2: $H_\alpha\neq\emptyset$. By (2.2) and (2.3), H_α is $\mathcal{T}$-perfect. Since $f\in(\mathrm{AP}_1^{**})$, there exist an interval I_α and a set $B_\alpha\in\mathfrak{B}$ such that $\emptyset\neq I_\alpha\cap H_\alpha\subset I_\alpha\cap B_\alpha$ and $\circ(f, I_\alpha\cap B_\alpha)\leqq\varepsilon$. Suppose now $0<\alpha<\Omega$ and that the closed set K_β is defined for every $\beta<\alpha$. If α is a limit ordinal, then we let $K_\alpha=\cap\{K_\beta\colon \beta<\alpha\}$. Otherwise, α has a predecessor $\alpha-1$. We put $K_\alpha=\overline{H}_{\alpha-1}-I_{\alpha-1}$. Clearly K_α is closed.

In this way we define K_α, H_α, I_α, B_α for every $\alpha<\Omega$. Since K_α is closed, from the construction, $K_{\alpha+1}\subset\overline{H}_\alpha=\Delta(K_\alpha)\subset K_\alpha$ holds for every $\alpha<\Omega$. Thus $\{K_\alpha\colon \alpha<\Omega\}$ is a decreasing transfinite sequence of closed sets. By the Cantor—Baire Stationary Principle (see [4], p. 145), there exists $\lambda<\Omega$ such that $K_\alpha=K_\lambda$ for every $\alpha\geqq\lambda$. We claim that $K_\lambda=\emptyset$. Indeed, suppose first $\mu(K_\lambda)>0$. Then $H_\lambda\neq\emptyset$ and $K_{\lambda+1}=\overline{H}_\lambda-I_\lambda\subset K_\lambda-I_\lambda\subsetneqq K_\lambda$ since $I_\lambda\cap K_\lambda\supset I_\lambda\cap H_\lambda\neq\emptyset$. This contradicts $K_{\lambda+1}=K_\lambda$. Therefore we have $\mu(K_\lambda)=0$. It follows that $H_\lambda=\Delta(K_\lambda)=\emptyset$ and $K_{\lambda+1}=\emptyset$. Consequently $K_\lambda=K_{\lambda+1}=\emptyset$. Let λ be the first ordinal with $K_\lambda=\emptyset$. We shall show

$$R^N=\cup\{H_\alpha-K_{\alpha+1}\colon 0\leqq\alpha<\lambda\}\cup\cup\{K_\alpha-H_\alpha\colon 0\leqq\alpha<\lambda\}. \tag{3}$$

Let $x\in R^N$ be given. Since $K_\lambda=\emptyset$, there exists $\alpha_0\leqq\lambda$ such that α_0 is the first ordinal that $x\notin K_{\alpha_0}$. In view of the way we define K_α $(\alpha<\Omega)$, α_0 is not a limit ordinal. There is a $\beta<\lambda$ such that $\alpha_0=\beta+1$. If $x\in H_\beta$, then $x\in\cup\{H_\alpha-K_{\alpha+1}\colon 0\leqq\alpha<\lambda\}$. If $x\notin H_\beta$, then $x\in\cup\{K_\alpha-H_\alpha\colon 0\leqq\alpha<\lambda\}$. (3) is proved.

Recalling that if $H_\alpha\neq\emptyset$, then I_α is an interval, $B_\alpha\in\mathfrak{B}$, $I_\alpha\cap H_\alpha\subset I_\alpha\cap B_\alpha$, $\circ(f, I_\alpha\cap B_\alpha)\leqq\varepsilon<\frac{1}{3}(b-a)$, and $K_{\alpha+1}=\overline{H}_\alpha-I_\alpha$, we find

$$\cup\{H_\alpha-K_{\alpha+1}\colon 0\leqq\alpha<\lambda\}\subset\cup\{I_\alpha\cap B_\alpha\colon 0\leqq\alpha<\lambda\}$$

and each $I_\alpha\cap B_\alpha$ is a $\mathfrak{B}_\sigma$ set contained entirely in at least one of the sets $\{x\colon f(x)>a\}$ and $\{x\colon f(x)<b\}$. Also, since K_α is closed and $H_\alpha=\Delta(K_\alpha)$, by (2.4), $K_\alpha-H_\alpha$ is contained in an F_σ null set and hence so is $\cup\{K_\alpha-H_\alpha\colon 0\leqq\alpha<\lambda\}$. Therefore

$$R^N=\cup\{I_\alpha\cap B_\alpha\colon 0\leqq\alpha<\lambda\}\cup\cup\{F_n\colon n=1,2,\ldots\},$$

where each F_n is a closed null set. Let

$$P=\cup\{I_\alpha\cap B_\alpha\colon I_\alpha\cap B_\alpha\subset\{x\colon f(x)>a\}\}$$

and

$$Q=\cup\{I_\alpha\cap B_\alpha\colon I_\alpha\cap B_\alpha\not\subset\{x\colon f(x)>a\}\}.$$

By the above observation, P, Q are $\mathfrak{B}_\sigma$ sets, $P\subset\{x\colon f(x)>a\}$, $Q\subset\{x\colon f(x)<b\}$, and $R^N=P\cup Q\cup\cup F_n$.

Let $S=P\cup(\cup F_n\cap\{x\colon f(x)>a\})$. Since $f\in\mathcal{B}_1(\mathcal{A})$, by (2.8), $\{x\colon f(x)>a\}\in G_{\delta\sigma}$ and hence, by (2.9), $\{x\colon f(x)>a\}\cap F_n\in\mathfrak{B}_\sigma$ for each n. It follows that $S\in\mathfrak{B}_\sigma$, $S\subset\{x\colon f(x)>a\}$, and

$$R^N-S\subset Q\cup(\cup F_n\cap\{x\colon f(x)\leqq a\})\subset\{x\colon f(x)<b\}.$$

The lemma is proved.

Proof of Theorem 2. Suppose $f\in\mathcal{B}_1(\mathcal{A})\cap(\mathrm{AP}_1^{**})$. Let $a\in R$ be given. First we show that $\{x\colon f(x)>a\}\in\mathfrak{B}_\sigma$. For each positive integer n, $a<a+(1/n)$. By the lemma,

there exists $S_n \in \mathfrak{B}_\sigma$ such that $S_n \subset \{x: f(x) > a\}$ and $R^N - S_n \subset \{x: f(x) < a + (1/n)\}$. Let $S = \bigcup S_n$. Clearly $S \in \mathfrak{B}_\sigma$ and $S \subset \{x: f(x) > a\}$. We now show that $\{x: f(x) > a\} \subset S$. If $f(x_0) > a$, then there is an n_0 such that $f(x_0) > a + (1/n_0)$. Therefore $x_0 \notin R^N - S_{n_0}$. That is, $x_0 \in S_{n_0} \subset S$. Consequently $\{x: f(x) > a\} = S \in \mathfrak{B}_\sigma$.

To show that $\{x: f(x) < a\} \in \mathfrak{B}_\sigma$, we need only note that $-f$ is also in the class $\mathscr{B}_1(\mathscr{A}) \cap (\mathrm{AP}_1^{**})$. By what we have just shown, $\{x: -f(x) > -a\} \in \mathfrak{B}_\sigma$. Owing to (2.6), $f \in \mathscr{B}_1(\mathscr{A} \cap \mathscr{P})$ and the proof is complete.

Theorems 1 and 2 clearly indicate that $\mathscr{B}_1(\mathscr{A} \cap \mathscr{P}) = \mathscr{B}_1(\mathscr{A}) \cap (\mathrm{AP}_1^{**})$.

THEOREM 3. $(\mathrm{AP}_1^*) = (\mathrm{AP}_1)$.

PROOF. First we show that $(\mathrm{AP}_1^*) \subset (\mathrm{AP}_1)$. Let $f \in (\mathrm{AP}_1^*)$ be given. We assume that $f \notin (\mathrm{AP}_1)$. Then there exists $a < b$ and sets U, V such that $\emptyset \neq U \subset \{x: f(x) < a\}$, $\emptyset \neq V \subset \{x: f(x) > b\}$, $U \subset \Delta(\overline{U})$, $V \subset \Delta(\overline{V})$, $U - \overline{V} = \emptyset$ and $V - \overline{U} = \emptyset$. It follows that $U \subset \overline{V}$, $V \subset \overline{U}$, and hence $\overline{U} = \overline{V}$. Let $H = \Delta(\overline{U})$. Then $H \supset U \neq \emptyset$. By (2.2) and (2.3), H is $\mathscr{T}$-perfect. Let $\varepsilon = b - a$. Since $f \in (\mathrm{AP}_1^*)$, there exists an interval I such that $I \cap H \neq \emptyset$ and $\circ(f, I \cap H) \leqq \varepsilon$. On the other hand, $H = \Delta(\overline{U}) = \Delta(\overline{V})$ and $I \cap H \neq \emptyset$ imply $I \cap \overline{U} \neq \emptyset \neq I \cap \overline{V}$ and hence $I \cap U \neq \emptyset \neq I \cap V$. Let $x_1 \in I \cap U$ and $x_2 \in I \cap V$. Then $f(x_1) < a$ and $f(x_2) > b$. Moreover, $x_1 \in I \cap U \subset I \cap \Delta(\overline{U}) = I \cap H$ and $x_2 \in I \cap V \subset \subset I \cap \Delta(\overline{V}) = I \cap H$. This contradicts $\circ(f, I \cap H) \leqq \varepsilon$. Consequently $(\mathrm{AP}_1^*) \subset (\mathrm{AP}_1)$.

Next we prove that $(\mathrm{AP}_1) \subset (\mathrm{AP}_1^*)$. Let $f \in (\mathrm{AP}_1)$ be given. Let H be a non-empty $\mathscr{T}$-perfect set and $\varepsilon > 0$. We need to show that there exists an interval I such that $I \cap H \neq \emptyset$ and $\circ(f, I \cap H) \leqq \varepsilon$. Since $H = \Delta(\overline{H})$, we can define functions g, h on H by setting

$$(4) \qquad g(x) = \limsup_{\substack{y \to x \\ y \in H}} f(y) \quad \text{and} \quad h(x) = \liminf_{\substack{y \to x \\ y \in H}} f(y).$$

It suffices to show that there is a point $x \in H$ with $g(x) - h(x) < \varepsilon$. Suppose this is not true. Then $H = \bigcup H_n$, where

$$(5) \qquad H_n = \left\{x \in H: \, g(x) > \frac{n\varepsilon}{3}, \; h(x) < \frac{(n-1)\,\varepsilon}{3}\right\}.$$

By (2.4), $\overline{H} - H = \overline{H} - \Delta(\overline{H})$ is contained in an F_σ null set. Let $\overline{H} - H \subset \bigcup F_k$, where F_k is a closed null set for each $k = 1, 2, \dots$. Then each F_k is nowhere dense in $\overline{H}$. Indeed, if F_k was dense in a portion $I \cap \overline{H} \neq \emptyset$ then, as F_k is closed, $F_k \supset I \cap \overline{H}$ and hence $\mu(I \cap \overline{H}) = 0$. But $H = \Delta(\overline{H})$ and $I \cap \overline{H} \neq \emptyset$ imply that $\mu(I \cap \overline{H}) > 0$. This is a contradiction.

Therefore $\overline{H} - H$ is of the first category relative to $\overline{H}$ and hence H is residual in $\overline{H}$. Since $H = \bigcup H_n$, it follows that there is an interval I and an integer n such that $\emptyset \neq I \cap \overline{H} \subset \overline{H}_n$. Let

$$U = \left\{x \in I \subset H: \, f(x) < \frac{(n-1)\,\varepsilon}{3}\right\} \quad \text{and} \quad V = \left\{x \in I \cap H: \, f(x) > \frac{n\varepsilon}{3}\right\}.$$

Then, by (4) and (5), $I \cap H_n \subset \overline{U} \cap \overline{V}$ and hence $I \cap \overline{H}_n \subset \overline{I \cap H_n} \subset \overline{U} \cap \overline{V}$. Consequently, $U \cup V \subset I \cap H = I \cap \Delta(\overline{H}) \subset \Delta(I \cap \overline{H}) \subset \Delta(I \cap \overline{H}_n) \subset \Delta(\overline{U} \cap \overline{V})$. This is a contradiction to the hypothesis $f \in (\mathrm{AP}_1)$. The proof is complete.

Finally we remark that, owing to (2.7) and our theorems, Grande's problem is equivalent to the following:

Is the inclusion $\mathscr{B}_1(\mathscr{A})\cap(\mathrm{AP}_1^{**})\subset\mathscr{B}_1(\mathscr{A})\cap(\mathrm{AP}_1^{*})$ actually an equality?

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References

[1] G. Cox and P. Humke, Classes of Baire functions, *Trans. Amer. Math. Soc.*, **269** (1982), 627—635.
[2] C. Goffman, C. J. Neugebauer, T. Nishiura, Density topology and approximate continuity, *Duke Math. J.*, **28** (1961), 497—506.
[3] Z. Grande, Sur les suites de fonctions approximativement continues presque partout, *Colloquium Math.*, **38** (1978), 259—262.
[4] I. P. Natanson, *Theory of functions of a real variable II*, Ungar (New York, 1960).
[5] T. Nishiura, The topology of almost everywhere continuous, approximately continuous functions, *Acta Math. Acad. Sci. Hungar.*, **37** (1981), 317—328.
[6] R. J. O'Malley, Approximately differentiable functions: the r-topology, *Pacific J. Math.*, **72** (1977), 207—222.
[7] R. J. O'Malley, Approximately continuous functions which are continuous almost everywhere, *Acta Math. Acad. Sci. Hungar.*, **33** (1979), 395—402.
[8] D. Preiss, Limits of approximately continuous functions, *Czech. Math. J.*, **96** (1971), 371—372.
[9] S. Saks, *Theory of the integral*, Monografie Matematyczne 7 (Warszawa—Lwów, 1937).

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DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TEXAS 77843
U.S.A.

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APPROXIMATE SMOOTHNESS AND BAIRE* 1

L. LARSON (Louisville)

It is well-known that $f\colon \mathbf{R}\to\mathbf{R}$ is in the class Baire 1 if and only if for every nonempty perfect set P, the restricted function $f|_P$ has a point of continuity. The real function f is defined to be in the class Baire* 1 if and only if for every nonempty perfect set P, there is an open interval I such that $I\cap P\neq\emptyset$ and $f|_{I\cap P}$ is continuous. From this definition, it is immediate that Baire* 1 is a subset of Baire 1. The terminology Baire* 1 was introduced by Richard O'Malley [3]. Various types of functions have been shown to belong to Baire* 1, such as the approximately differentiable functions [4] and the L_p-smooth functions [3]. The purpose of this paper is to show that the collection of functions which are both approximately continuous and approximately smooth is in Baire* 1.

A function is said to be *α-smooth* $(\alpha\geqq 0)$ at x, if and only if,

$$\lim_{h\to 0}\frac{\Delta^2 f(x,h)}{h^\alpha}=0,$$

where

$$\Delta^2 f(x,h)=f(x+h)+f(x-h)-2f(x)$$

is the second symmetric difference of f. In the case when $\alpha=0$, the function is said to be *symmetric* at x and when $\alpha=1$, f is said to be *smooth* at x. If f is α-smooth at each point in its domain, then we just say it is *α-smooth*. The generalization to *approximate α-smoothness* is done in the obvious way.

In the following, $\mathbf{R}$ is the set of real numbers, $\mathbf{N}$ denotes the natural numbers and $\mathbf{Q}$ is the set of rational numbers. Let $A\subset\mathbf{R}$. The complement of A is written A^c. If A is measurable, then the density of A at x is written $\delta(x,A)$. The distance between x and A is $d(x,A)=\inf\{|x-y|\colon y\in A\}$. The oscillation of a function f at x is written $\omega(f,x)$.

THEOREM. *If $f\colon \mathbf{R}\to\mathbf{R}$ is approximately continuous and approximately smooth at each point of a perfect set P, then there is a portion Q of P such that $f|_Q$ is continuous.*

PROOF. For each $n\in\mathbf{N}$, define Q_n to be the set of all $x\in P$ such that whenever $0<\alpha<1/n$, then

$$\left|\{h\in(0,\alpha)\colon |\Delta^2 f(x,h)/h|<2\}\right|>\alpha/2.$$

It is clear that

$$P=\bigcup_{n\in\mathbf{N}}Q_n. \tag{1}$$

Suppose that $0<\alpha<1/n$, $h\in(0,\alpha)$ and $x\in Q_n$ such that

$$|\Delta^2 f(x,h)/h|<2.$$

This implies both

$$f(x)-h<\max\{f(x+h),f(x-h)\} \quad\text{and}\quad f(x)+h>\min\{f(x+h),f(x-h)\}.$$

It follows easily from these inequalities that for all $\alpha\in(0,1/n)$ either

$$|\{h\in(0,\alpha)\colon f(x+h)>f(x)-\alpha\}|>\alpha/4, \tag{2}$$

or

$$|\{h\in(0,\alpha)\colon f(x-h)>f(x)-\alpha\}|>\alpha/4 \tag{2'}$$

and that for all $\alpha\in(0,1/n)$ either

$$|\{h\in(0,\alpha)\colon f(x+h)<f(x)+\alpha\}|>\alpha/4 \tag{3}$$

or

$$|\{h\in(0,\alpha)\colon f(x-h)<f(x)+\alpha\}|>\alpha/4. \tag{3'}$$

Now, fix $n\in\mathbf{N}$ and choose $x_0\in\overline{Q}_n$. We wish to show that

$$\lim_{\substack{x\to x_0\\ x\in Q_n}} f(x)=f(x_0). \tag{4}$$

Assume (4) is not true. Then, there exists a sequence $\{x_m\}\subset Q_n$ such that $x_m\to x_0$, but $f(x_m)$ does not converge to $f(x_0)$. Without loss of generality, we may assume

$$x_m\uparrow x_0,\quad f(x_m)-f(x_0)>2\delta>0 \quad\text{and}\quad x_0-x_1<\min(1/n,\delta). \tag{5}$$

Let $\alpha_m=x_0-x_m$ for each $m\in\mathbf{N}$. If $h\in(0,\alpha_m)$ and $f(x_m+h)>f(x_m)-\alpha_m$, then $x_m+h\in(x_m,x_0)$ and

$$f(x_m+h)>f(x_0)+2\delta-\alpha_m>f(x_0)+\delta.$$

Hence, if (2) holds for $x=x_m$ and $\alpha=\alpha_m$, then we have

$$|\{x\in(x_m,x_0)\colon f(x)>f(x_0)+\delta\}|>(x_0-x_m)/4. \tag{6}$$

On the other hand, if both $h\in(0,\alpha_m)$ and $f(x_m-h)>f(x_m)-\alpha_m$, then $x_m-h\in(2x_m-x_0,x_m)$ and

$$f(x_m-h)>f(x_0)+2\delta-\alpha_m>f(x_0)+\delta.$$

Thus, if (2′) holds for $x=x_m$ and $\alpha=\alpha_m$, then we have

$$|\{x\in(2x_m-x_0,x_0)\colon f(x)>f(x_0)+\delta\}|>(x_0-x_m)/4. \tag{6'}$$

However, for m large enough, both (6) and (6′) contradict the fact that f is approximately continuous at x_0. Therefore, (4) is true for each $x_0\in\overline{Q}_m$, which immediately implies $f|_{\overline{Q}_m}$ is continuous for each $m\in\mathbf{N}$.

To finish the proof, we apply the Baire category theorem and (1) to find an $i\in\mathbf{N}$ and an interval I such that $\overline{Q}_i\cap I=P\cap I\neq\emptyset$.

COROLLARY 1. *If a function $f\colon\mathbf{R}\to\mathbf{R}$ is both approximately continuous and approximately smooth, then f is in Baire* 1.*

With little change in the proof of theorem, we can also prove the following

COROLLARY 2. *If a function* $f\colon \mathbf{R}\to\mathbf{R}$ *is approximately continuous and approximately* α*-smooth for any* $\alpha>0$, *then* f *is in Baire** 1.

It is shown in [1] that in Corollary 2, if the requirement for approximate α-smoothness is strengthened to α-smoothness, then the requirement that f be approximately continuous can be weakened to the point where only measurability is required. In fact, the set on which a measurable α-smooth function can be discontinuous is characterized as precisely a *separated* set in the sense of Hausdorff [2]. The assumption of measurability is required. This can be seen by noting that any nonmeasurable solution of the Cauchy functional equation $f(x+y)=f(x)+f(y)$ has the property that $\Delta^2 f(x, h)=0$ for all x and h and is therefore smooth.

The example given below shows that Corollary 2 cannot be extended to approximately symmetric functions.

EXAMPLE. There is an approximately continuous symmetric function $f\colon \mathbf{R}\to\mathbf{R}$ which is discontinuous precisely on the rational numbers.

PROOF. Let I_n be any sequence of closed intervals from $[0, 1]$ such that $I_n\downarrow 0$ and $\delta(0, \bigcup_{n\in\mathbf{N}} I_n)=0$. Define $h\colon \mathbf{R}\to\mathbf{R}$ as follows:

$$h(x)=\begin{cases} 0, & \text{when } x\in[0,\infty)\setminus\bigcup_{n\in\mathbf{N}} I_n;\\ 2\dfrac{d(x, I_n^c)}{|I_n|}, & \text{when } x\in I_n;\\ -h(-x), & \text{when } x<0.\end{cases}$$

It is easy to see that h has the following properties:

(a) h is continuous, and consequently symmetric on $\mathbf{R}\setminus\{0\}$;
(b) h is odd and $h(0)=0$, so h is symmetric at $x=0$;
(c) h is approximately continuous at $x=0$;
(d) $|h(x)|\leqq 1$, for all $x\in\mathbf{R}$, and
(e) $\omega(h, 0)=2$.

Let $\mathbf{Q}=\{q_n\colon n\in\mathbf{N}\}$ and for each n define

$$f_n(x)=\frac{h(x-q_n)}{2^n} \quad \text{and} \quad f(x)=\sum_{n\in\mathbf{N}} f_n(x).$$

From (a)—(d) and the fact that the series defining f converges uniformly we see that f is continuous on $\mathbf{Q}^c$ as well as approximately continuous and symmetric everywhere. To see that f is continuous precisely on $\mathbf{Q}^c$ we note that $\sum_{i\neq n} f_i$ is continuous at q_n. Since $\omega(f_n, q_n)=2^{1-n}>0$ it follows that $f=f_n+\sum_{i\neq n} f_i$ is discontinuous at q_n.

References

[1] M. J. Evans and L. Larson, The continuity of smooth and symmetric functions, *Acta Math. Hung.*, **43** (1984), 251—257.
[2] F. Hausdorff, *Set Theory*, Chelsea, 1962.
[3] R. J. O'Malley, Baire* 1 Darboux functions, *Proc. Amer. Math. Soc.*, **60** (1976), 187—192.
[4] G. Tolstoff, Sur quelques propriétés des fonctions approximativement continues, *Mat. Sb.*, **5 (47)** (1939), 637—645.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KENTUCKY, 40292
USA

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ON GENERALIZED LEHMER SEQUENCES

BUI MINH PHONG (Budapest)*

1. Introduction

Let $G=G(G_0, G_1, A, B)=\{G_n\}_0^\infty$ be a second order linear recurrence defined by integer constants G_0, G_1, A, B and the recurrence

$$G_n = AG_{n-1}-BG_{n-2} \quad (n>1), \tag{1}$$

where $AB\neq 0$, $D=A^2-4B\neq 0$ and $|G_0|+|G_1|\neq 0$. If $G_0=0$ and $G_1=1$, then we denote the sequence $G(0, 1, A, B)$ by $R=R(A, B)$. The sequence R is called Lucas sequence and R_n is called a Lucas number.

In 1930 D. H. Lehmer [2] generalized some results of Lucas on the divisibility properties of Lucas numbers to the terms of the sequence $U=U(L, M)=\{U_n\}_0^\infty$ which is defined by integer constants $L, M, U_0=0$, $U_1=1$ and the recurrence

$$U_n = \begin{cases} LU_{n-1}-MU_{n-2} & \text{for } n \text{ odd} \\ U_{n-1}-MU_{n-2} & \text{for } n \text{ even,} \end{cases}$$

where $LM\neq 0$ and $K=L-4M\neq 0$. The sequence U is called a Lehmer sequence and U_n is a Lehmer number. It should be observed that Lucas numbers are also Lehmer numbers up to a multiplicative factor.

Here we shall define generalized Lehmer sequences. Let H_0, H_1, L and M be integers with the conditions $LM\neq 0, K=L-4M\neq 0$ and $|H_0|+|H_1|\neq 0$. A generalized Lehmer sequence is a sequence $H_0, H_1, \ldots, H_n, \ldots$ of integers satisfying a relation

$$H_n = \begin{cases} LH_{n-1}-MH_{n-2} & \text{for } n \text{ odd} \\ H_{n-1}-MH_{n-2} & \text{for } n \text{ even.} \end{cases} \tag{2}$$

We shall denote it by $H=H(H_0, H_1, L, M)=\{H_n\}_{n=0}^\infty$, and so $H(0, 1, L, M)$ is the Lehmer sequence $U(L, M)$.

The purpose of this paper is to study the properties of the generalized Lehmer sequences $H(H_0, H_1, L, M)$. We show that the terms of sequences G are also terms of sequences H up to a multiplicative factor and we give and explicit form of H_n. We improve a result of P. Kiss [1] concerning the zero terms in the sequences G and H. Furthermore we give lower and upper bounds for the terms of the sequences H.

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2. Preliminary results

Throughout this paper we shall use the notation

$$\varepsilon(n) = \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

Using the function $\varepsilon(n)$, the relation (2) can be written in the form

$$H_n = L^{\varepsilon(n)} H_{n-1} - MH_{n-2} \quad (\text{for } n > 1). \tag{3}$$

We prove some properties of the sequence $H(H_0, H_1, L, M)$.

PROPOSITION 1. *If* $H_0=G_0$, $H_1=AG_1$, $L=A^2$ *and* $M=B$, *then*

$$G_n(G_0, G_1, A, B) = A^{-\varepsilon(n)} H_n(G_0, AG_1, A^2, B) \tag{4}$$

for any $n \geqq 0$.

PROOF. We shall prove (4) by induction on n. The statement is obvious for $n=0$ and $n=1$. If (4) is true for $n-1$ and n $(n \geqq 1)$, then using (1) and (3), we have

$$G_{n+1} = AG_n - BG_{n-1} = A^{\varepsilon(n+1)+\varepsilon(n)} G_n - BA^{\varepsilon(n-1)-\varepsilon(n+1)} G_{n-1} =$$

$$= A^{-\varepsilon(n+1)} [A^{2\varepsilon(n+1)} H_n - BH_{n-1}] = A^{-\varepsilon(n+1)} [L^{\varepsilon(n+1)} H_n - MH_{n-1}] = A^{-\varepsilon(n+1)} H_{n+1},$$

which proves the assertion.

REMARKS. a) From Proposition 1 it follows that the sequences $H(H_0, H_1, L, M)$ are more general than the sequences $G(G_0, G_1, A, B)$.

b) In particular we have

$$R_n(A, B) = A^{-\varepsilon(n)} H_n(0, A, A^2, B) = A^{1-\varepsilon(n)} U_n(A^2, B). \tag{5}$$

PROPOSITION 2. *If* $U_n = U_n(L, M)$ *and* $H_n = H_n(H_0, H_1, L, M)$ *then*

$$H_n = H_1 U_n - L^{\varepsilon(n)} MH_0 U_{n-1} \tag{6}$$

for any $n \geqq 0$ *with the convention* $MU_{-1} = -1$.

PROOF. From the definition of the sequences U and H, (6) is obvious for $n=0$ and $n=1$. Suppose that (6) is true for $n-1$ and n. Then by (3) using that $\varepsilon(n+1) = \varepsilon(n-1)$ we have

$$H_{n+1} = L^{\varepsilon(n+1)} H_n - MH_{n-1} = L^{\varepsilon(n+1)} [H_1 U_n - L^{\varepsilon(n)} MH_0 U_{n-1}] -$$

$$- M[H_1 U_{n-1} - L^{\varepsilon(n-1)} MH_0 U_{n-2}] = H_1 [L^{\varepsilon(n+1)} U_n - MU_{n-1}] -$$

$$- L^{\varepsilon(n+1)} MH_0 [L^{\varepsilon(n)} U_{n-1} - MU_{n-2}] = H_1 U_{n+1} - L^{\varepsilon(n+1)} MH_0 U_n,$$

which proves (6) by induction on n.

PROPOSITION 3. *Let α and β be the roots of the equation* $z^2-\sqrt{L}\,z+M=0$. *If* $a=H_1-\sqrt{L}\,H_0\beta$ *and* $b=H_1-\sqrt{L}\,H_0\alpha$, *then we have*

$$H_n(H_0, H_1, L, M) = (\sqrt{L})^{\varepsilon(n)}\frac{a\alpha^n-b\beta^n}{\alpha^2-\beta^2}. \tag{7}$$

PROOF. It is well-known that

$$U_n = U_n(L, M) = (\sqrt{L})^{\varepsilon(n)}\frac{\alpha^n-\beta^n}{\alpha^2-\beta^2} \tag{8}$$

(see e.g. [7]). Since $\alpha+\beta=\sqrt{L}$ and $\alpha\beta=M$, using (6) and (8) we get

$$H_n = H_1U_n-L^{\varepsilon(n)}MH_0U_{n-1} =$$

$$= \frac{H_1(\sqrt{L})^{\varepsilon(n)}(\alpha^n-\beta^n)-L^{\varepsilon(n)}H_0(\alpha^n\beta-\alpha\beta^n)(\sqrt{L})^{\varepsilon(n-1)}}{\alpha^2-\beta^2} =$$

$$= \frac{(\sqrt{L})^{\varepsilon(n)}}{\alpha^2-\beta^2}\left[(H_1-\sqrt{L}H_0\beta)\alpha^n-(H_1-\sqrt{L}H_0\alpha)\beta^n\right] =$$

$$= (\sqrt{L})^{\varepsilon(n)}\frac{a\alpha^n-b\beta^n}{\alpha^2-\beta^2},$$

where $a=H_1-\sqrt{L}H_0\beta$ and $b=H_1-\sqrt{L}H_0\alpha$.

REMARKS. a) From (4) and (7) we get the well-known formula

$$G_n(G_0, G_1, A, B) = \frac{c\delta^n-d\gamma^n}{\delta-\gamma}, \tag{9}$$

where δ and γ are the roots of the equation $x^2-Ax+B=0$ and $c=G_1-G_0\gamma$, $d=G_1-G_0\delta$ (see e.g. [5]).

b) In what follows we say that the sequences $G(G_0, G_1, A, B)$ and $H(H_0, H_1, L, M)$ are non-degenerate if $cd\delta\gamma\neq 0$, δ/γ and $ab\alpha\beta\neq 0$, α/β are not roots of unity, respectively.

PROPOSITION 4. *For any* $n\geqq 0$

$$H_n(H_0, H_1, L, M) = -i^{n+\varepsilon(n)}H_n(-H_0, H_1, -L, -M), \tag{10}$$

where $i^2=-1$.

PROOF. We shall prove (10) by induction on n. Our statement is obvious for $n=0$ and $n=1$. If (10) is true for $n-1$ and n, then using (3) and $i^2=-1$, we have

$$H_{n+1}(H_0, H_1, L, M) = H_{n+1} = L^{\varepsilon(n+1)}H_n-MH_{n-1} =$$

$$= i^{2\varepsilon(n+1)}(-L)^{\varepsilon(n+1)}(-i^{n+\varepsilon(n)})H_n(-H_0, H_1, -L, -M)-$$

$$-M(-i^{n-1+\varepsilon(n-1)})H_{n-1}(-H_0, H_1, -L, -M) =$$

$$= -i^{n+1+\varepsilon(n+1)}[(-L)^{\varepsilon(n+1)}H_n(-H_0, H_1, -L, -M)-$$

$$-(-M)H_{n-1}(-H_0, H_1, -L, -M)] = -i^{(n+1)+\varepsilon(n+1)}H_{n+1}(-H_0, H_1, -L, -M),$$

which proves (10).

PROPOSITION 5. *Let* $d=(L, M)$, $L'=L/d$ *and* $M'=M/d$. *Then*

$$H_n(H_0, H_1, L, M) = (\sqrt{d})^{n+\varepsilon(n)-2} H_n(dH_0, H_1, L', M'). \tag{11}$$

PROOF. If α and β are roots of $z^2-\sqrt{L}\,z+M=0$, then $\alpha_1=\alpha/\sqrt{d}$ and $\beta_1=\beta/\sqrt{d}$ are roots of $z^2-\sqrt{L'}\,z+M'=0$. Let

$$a = H_1-\sqrt{L}H_0\beta, \quad b = H_1-\sqrt{L}H_0\alpha,$$

and

$$a_1 = H_1-\sqrt{L'}(dH_0)\,\beta_1, \quad b_1 = H_1-\sqrt{L'}(dH_0)\,\alpha_1.$$

It can be easily seen that $a_1=a$, $b_1=b$. Thus by (7) we have

$$H_n(H_0, H_1, L, M) = (\sqrt{L})^{\varepsilon(n)} \frac{a\alpha^n-b\beta^n}{\alpha^2-\beta^2} = (\sqrt{d})^{\varepsilon(n)+n-2}(\sqrt{L'})^{\varepsilon(n)} \cdot \frac{a\alpha_1^n-b\beta_1^n}{\alpha_1^2-\beta_1^2} =$$

$$= (\sqrt{d})^{n+\varepsilon(n)-2} \cdot (\sqrt{L'})^{\varepsilon(n)} \frac{a_1\alpha_1^n-b_1\beta_1^n}{\alpha_1^2-\beta_1^2} = (\sqrt{d})^{n+\varepsilon(n)-2} \cdot H_n(dH_0, H_1, L', M'),$$

which proves (11).

3. Zero terms in the sequences H

Some authors have studied the lower and upper bounds for the terms of non-degenerate sequences $G=G(G_0, G_1, A, B)$. Let γ and δ be the roots of the equation $x^2-Ax+B=0$. We can assume that $|\gamma|\geqq|\delta|$. In [3] K. Mahler proved that if $D=A^2-4B<0$ and ε is a positive constant, then there is an effectively computable constant n_0 depending only on ε such that

$$|G_n| \geqq |\gamma|^{(1-\varepsilon)n} \quad \text{for} \quad n > n_0.$$

From a result of T. N. Shorey and C. L. Stewart [6] it follows that

$$|G_n| \geqq |\gamma|^{n-c_1 \log n}$$

for $n>c_2$, where c_1, c_2 are positive numbers which are effectively computable in terms of G_0, G_1, A and B.

A similar result was obtained by M. Mignotte [4] for linear recurrences of higher order.

In [1] P. Kiss gave the explicit value of the constants proving that $G_n\neq 0$ for $n>n_1$, where

$$n_1 = \max\left[2^{510}(\log|8B|)^{25}, \frac{4}{\log 2}(\log|G_0|+\log 4\sqrt{|D|})\right],$$

furthermore if $D<0$ and $n>n_1$, then

$$\frac{|c|}{2\sqrt{|D|}}|\gamma|^n n^{-c_3} < |G_n| \leqq \frac{2|c|}{\sqrt{|D|}}|\gamma|^n,$$

where $c=G_1-G_0\gamma$ and

$$c_3 = 2e\cdot 200^{40}\log|8B|\cdot(1+\log\log|8B|)\cdot\log|16B|(G_0^2+G_1^2).$$

We extend the results mentioned above to the sequences $H(H_0, H_1, L, M)$. We give necessary and sufficient conditions for sequences H which have zero terms, and give lower and upper bounds for the terms. These improve the reults of P. Kiss [1].

THEOREM 1. *Let* $H=H(H_0, H_1, L, M)$ *be a non-degenerate generalized Lehmer sequence with* $(L, M)=1$ *and* $(H_0, H_1)=h$. *Then the following statements are equivalent:*

(i) $H_n = 0 \quad (n \geqq 0)$,

(ii) $H_0 = \varrho h U_n, \quad H_1 = \varrho h L^{\varepsilon(n)} M U_{n-1}$,

(iii) $H_k = \varrho h L^{\varepsilon(n)\varepsilon(k)} M^k U_{n-k}$ *for* $k = 0, 1, \ldots, n$,

where $U_n=U_n(L, M)$, $MU_{-1}=-1$, *and* $\varrho=1$ *or* $\varrho=-1$.

COROLLARY. *Let* $H=H(H_0, H_1, L, M)$ *be a non-degenerate generalized Lehmer sequence with* $(L, M)=d$. *Then* $H_n=0$ *if and only if*

$$(\sqrt{d})^{n+\varepsilon(n)} H_0 = \pm(dH_0, H_1)\, U_n,$$

and

$$(\sqrt{d})^{n+\varepsilon(n)} H_1 = \pm(dH_0, H_1) L^{\varepsilon(n)} M U_{n-1},$$

where $U_n=U_n(L, M)$.

THEOREM 2. *Let* $H=H(H_0, H_1, L, M)$ *be a non-degenerate generalized Lehmer sequence with* $(L, M)=d$.

If $LK>0$ *then* $H_n\neq 0$ *for* $n>\max[13, \min(|H_0|+1, |H_1|+2)]$.

If $LK<0$ *then* $H_n\neq 0$ *for* $n>\max(N_1, N_2)=N_0$, *where*

$$N_1 = \min(2^{67}\log|4M|, e^{398})$$

and

$$N_2 = \min\left[\frac{4}{\log 2}\log|dH_0|, \frac{4}{\log 2}\log|H_1|\right].$$

THEOREM 3. *Let* $H=H(H_0, H_1, L, M)$ *be a non-degenerate generalized Lehmer sequence with condition* $LK<0$. *Then for*

$$n > 2^{67}\log|4M|(H_0^2+H_1^2)$$

we have

$$\frac{|a|}{2\sqrt{|LK|}}|\alpha|^n n^{-c_0} < |H_n| < \frac{2|a|}{\sqrt{|K|}}|\alpha|^n,$$

where

$$c_0 = 2^{80}\log|4M|\cdot\log\log|4M|\cdot\log|4M|(H_0^2+H_1^2),$$

and α *is any solution of* $z^2-\sqrt{L}\,z+M=0$.

PROOF OF THEOREM 1. Let first $H_n=0$ for an integer $n\geqq 0$. If $n=0$ or $n=1$ then (ii) follows easily. Suppose $n>1$. By (6) $H_n=0$ implies that

$$H_1 U_n = L^{\varepsilon(n)} M H_0 U_{n-1},$$

from which it follows

$$(12)\qquad H_1' U_n = L^{\varepsilon(n)} M H_0' U_{n-1},$$

where $H_0' = H_1/h$ and $H_0' = H_0/h$. Since $(L, M) = 1$, it can be easily seen that $(U_n, L^{\varepsilon(n)}) = 1$, $(U_n, M) = 1$ and $(U_n, U_{n-1}) = 1$. Thus by (12) we get

$$H_0 = \pm h \cdot U_n \quad \text{and} \quad H_1 = \pm h \cdot L^{\varepsilon(n)} \cdot M U_{n-1},$$

which proves that (i) implies (ii).

Now we prove that (ii) implies (iii). Suppose

$$H_0 = h \cdot U_n \quad \text{and} \quad H_1 = h \cdot L^{\varepsilon(n)} M U_{n-1}.$$

Thus (iii) is true for $k=0$ and $k=1$. If (iii) is true for $k-1$ and k, where $k<n$, and $\varrho=1$, then from (3) we have

$$\begin{aligned} H_{k+1} &= L^{\varepsilon(k+1)} H_k - M H_{k-1} = \\ &= L^{\varepsilon(k+1)} h L^{\varepsilon(n)\varepsilon(k)} M^k U_{n-k} - M h L^{\varepsilon(n)\varepsilon(k-1)} M^{k-1} U_{n-k+1} = \\ &= h L^{\varepsilon(n)\varepsilon(k+1)} M^k [L^{\varepsilon(n-k+1)} U_{n-k} - U_{n-k+1}] = h L^{\varepsilon(n)\varepsilon(k+1)} M^{k+1} U_{n-(k+1)}, \end{aligned}$$

since

$$\varepsilon(k+1) + \varepsilon(n)\varepsilon(k) - \varepsilon(n)\varepsilon(k+1) = \varepsilon(n-k+1)$$

and

$$\varepsilon(k+1) = \varepsilon(k-1).$$

This proves (iii) in the case $\varrho=1$. If $\varrho=-1$, then we can similarly show that (ii) implies (iii).

Finally (i) follows clearly from (iii) with $k=n$. □

Proof of the Corollary. Let $d=(L, M)$ and $L' = L/d$, $M' = M/d$. By (11) it can be easily seen that $H_n = H_n(H_0, H_1, L, M) = 0$ if and only if $H_n(dH_0, H_1, L', M') = 0$. From Theorem 1 we obtain that $H_n = 0$ if and only if

$$(13)\qquad dH_0 = \pm(dH_0, H_1)\, U_n(L', M') \quad \text{and} \quad H_1 = \pm(dH_0, H_1) L'^{\varepsilon(n)} M' U_{n-1}(L', M').$$

Since

$$U_n(L, M) = (\sqrt{d})^{n+\varepsilon(n)-2} \cdot U_n(L', M'),$$

by (13) and its conclusion we have

$$(\sqrt{d})^{n+\varepsilon(n)} H_0 = \pm(dH_0, H_1) U_n(L, M)$$

and

$$(\sqrt{d})^{n+\varepsilon(n)} H_1 = \pm(dH_0, H_1) L^{\varepsilon(n)} M U_{n-1}(L, M),$$

which proves the corollary.

Before proving Theorem 2 we introduce some notations and recall some results due to M. Waldschmidt [8], M. Ward [9] and C. L. Stewart [7].

Denote

$$a_0 x^N + \ldots + a_N = a_0 \prod_{i=1}^{N} (x - \alpha_i) \in \mathbf{Z}[x]$$

the minimal polynomial of an algebraic number $\alpha=\alpha_1$. Put

$$M(\alpha) = |a_0| \cdot \prod_{i=1}^{N} \max\{1, |\alpha_i|\}$$

and

$$h(\alpha) = \frac{1}{N} \log M(\alpha).$$

THEOREM A (M. Waldschmidt [8]). *Let $\alpha_1, \ldots, \alpha_m$ be non-zero algebraic numbers, and let $\beta_0, \beta_1, \ldots, \beta_m$ be algebraic numbers. For $1 \leqq i \leqq m$ let $\log \alpha_i$ be any determination of the logarithm of α_i. Let D be a positive integer, and let $V_1, \ldots, V_m$, W, E be positive real numbers, satisfying*

$$D \geqq [Q(\alpha_1, \ldots, \alpha_m, \beta_0, \beta_1, \ldots, \beta_m) \colon Q],$$

$$V_i \geqq \max\{h(\alpha_i), |\log \alpha_i|/D, 1/D\}, \quad 1 \leqq i \leqq m,$$

$$W \geqq \max_{0 \leqq i \leqq m} \{h(\beta_i)\}, \quad V_1 \leqq \ldots \leqq V_m$$

and

$$1 < E \leqq \min\left[e^{DV_1}; \min_{1 \leqq i \leqq m} \{4DV_i/|\log \alpha_i|\}\right].$$

Finally define $V_i^+ = \max\{V_i, 1\}$ for $i=m$ and $i=m-1$, with $V_0^+=1$ in the case $m=1$. If the number

$$\Lambda := \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_m \log \alpha_m$$

does not vanish, then

$$|\Lambda| > \exp\left\{-c(m) D^{m+2} V_1 \ldots V_m (W + \log(EDV_m^+))(\log EDV_{m-1}^+)(\log E)^{-m-1}\right\}$$

where $c(1) \leqq 2^{35}$, $c(2) \leqq 2^{53}$, $c(3) \leqq 2^{71}$ and $c(m) \leqq 2^{8m+51} \cdot m^{2m}$ for $m>3$.

We shall use a result of C. L. Stewart [7] on a linear form in two logarithms. Let α be an algebraic number of height at most A ($\geqq 4$) and degree d; further let b_1 and b_2 denote integers with absolute values at most B ($\geqq 4$). Set

$$\Lambda = b_1 \log(-1) + b_2 \log \alpha,$$

where the logarithms are assumed to take their principal values.

THEOREM B (C. L. Stewart [7]). *If $\Lambda \neq 0$ then $|\Lambda| > \exp(-C \log A \log B)$, where $C = 2^{435} \cdot (3d)^{49}$.*

Finally, we recall a result due to M. Ward [9] on primitive prime divisors of Lehmer numbers. Recall that a primitive prime divisors of the Lehmer number $U_n(L, M)$ is a prime dividing U_n but it does not divide $LKU_3 \ldots U_{n-1}$, where $K=L-4M$.

THEOREM C (M. Ward [9]). *Let $U(L, M)$ be a non-degenerate Lehmer sequence with conditions $L>0$ and $K>0$. Then $U_n(L, M)$ has a primitive prime divisor for $n>12$. Every primitive prime divisor of $U_n(L, M)$ is of the form $nx \pm 1$.*

Now we prove the following result.

LEMMA. *Let* $U(L, M)$ *be a non-degenerate Lehmer sequence with conditions* $L>0$ *and* $K<0$. *Then* $M \geqq 2$ *and*

$$|U_n(L, M)| > M^{n/4} \tag{14}$$

for

$$n > \min(2^{67} \log 4M, e^{398}) =: N_1.$$

PROOF. Since $L>0$ and $U(L, M)$ is a non-degenerate Lehmer sequence, we have

$$\langle L, M \rangle \neq \langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle, \langle 4, 1\rangle.$$

Thus if $K=L-4M<0$ then $M \geqq 2$.

Let α and β be the roots of $z^2-\sqrt{L}z+M=0$. By our conditions we obtain

$$|\alpha| = |\beta| = \sqrt{M}. \tag{15}$$

By (8) we have

$$|U_n| = |U_n(L, M)| = \left|(\sqrt{L})^{\varepsilon(n)} \frac{\alpha^n-\beta^n}{\alpha^2-\beta^2}\right| \geqq \tag{16}$$

$$\geqq \frac{|\alpha|^n}{\sqrt{L|K|}}\left|1-\left(\frac{\beta}{\alpha}\right)^n\right| \geqq \frac{|\alpha|}{2\sqrt{L|K|}}\left|t \log(-1) - n \log \frac{\beta}{\alpha}\right|$$

where log denotes the principal value of the logarithm function and $|t| \leqq 2n$, because $\left|1-\left(\frac{\beta}{\alpha}\right)^n\right|$ is the length of a chord of the unit circle which is greater than the half of the smaller circular arc. Set

$$\Lambda := t \log(-1) - n \log \frac{\beta}{\alpha}. \tag{17}$$

Since β/α is not a root of unity, we have $\Lambda \neq 0$.

First we prove that (14) holds if $n > 2^{67} \log 4M$. We apply Theorem A to (17). In this case $m=2$, $W=\log 2n$; $\alpha_1=-1$, $M(\alpha_1)=1$, $h(\alpha_1)=0$; $\alpha_2=\beta/\alpha$, $M(\alpha_2)=M$. The algebraic number $\alpha_2=\beta/\alpha$ is a root of the equation

$$Mx^2-(L-2M)x+M=0$$

so $h(\alpha_2)=\frac{1}{2}\log M$. From these

$$D=2, \quad V_1=V_1^+=\frac{\pi}{2}, \quad V_2=V_2^+=\log 4M, \quad E=4$$

follow. By Theorem A we have

$$|\Lambda| > \exp\left\{-c(2)\cdot 2^4 \frac{\pi}{2} \log 4M \left(\log 2n + \log(8 \log 4M)\right) \cdot \log 4\pi \cdot (\log 4)^{-3}\right\} >$$

$$> \exp\left\{-9 \cdot 2^{56} \cdot \log M\left(\log 2n + \log(8 \log 4M)\right)\right\},$$

and so if $n>4\log 4M$ then

$$(18)\qquad |\Lambda| > \exp\{-9\cdot 2^{57}\cdot\log M\cdot\log 2n\} = M^{-9\cdot 2^{57}\cdot\log 2n}.$$

On the other hand it follows from $0<L<4M$ that

$$|K| \leqq |L-2M|+2M \leqq 2M+2M = 4M,$$

and so

$$(19)\qquad \frac{1}{2\sqrt{L|K|}} > \frac{1}{2\sqrt{4M\cdot 4M}} = \frac{1}{8M} \geqq M^{-4}.$$

By (15), (16), (18) and (19) we get

$$(20)\qquad |U_n| > M^{n/2-9\cdot 2^{57}\cdot\log 2n-4}$$

for $n>4\log 4M$; furthermore a short calculation shows that

$$(21)\qquad \frac{n}{2}-9\cdot 2^{57}\log 2n-4 > \frac{n}{4}$$

if $n>2^{68}$.

Thus by (20) and (21)

$$(22)\qquad |U_n| > M^{n/4}$$

for $n>2^{67}\log 4M$.

Now we prove that (14) holds if $n>e^{398}$. We apply Theorem B to (17). It is easily seen that in our case $A=2M$ and $B=2n$, furthermore $d=2$. From Theorem B we get

$$(23)\qquad |\Lambda| = \left|t\log(-1)-n\log\frac{\beta}{\alpha}\right| > \exp(-2^{484}\cdot 3^{49}\cdot\log 2M\cdot\log 2n) \geqq$$

$$\geqq \exp(-2^{485}\cdot 3^{49}\cdot\log M\cdot\log 2n) = M^{-2^{485}\cdot 3^{49}\cdot\log 2n}.$$

Thus by (15), (16), (19) and (23) we obtain

$$|U_n| > M^{n/2-2^{485}\cdot 3^{49}\log 2n-4},$$

and so

$$(24)\qquad |U_n| > M^{n/4}$$

if $n>e^{398}$.

By (22) and (24)

$$|U_n| > M^{n/4} \quad\text{if}\quad n > \min(2^{67}\log 4M, e^{398}) = N_1,$$

which proves the Lemma. □

PROOF OF THEOREM 2. Let $H_n=H_n(H_0, H_1, L, M)=0$. By Proposition 4 we can assume without any essential loss of generality that $L>0$.

First we assume that $K>0$. By Theorem C and the Corollary of Theorem 1, if $n>13$ then U_n has a primitive divisor of the form $nx\pm 1$ which divides H_0; and U_{n-1} has a primitive divisor of the form $(n-1)y\pm 1$ dividing H_1. Thus

$$n \leqq |H_0|+1 \quad\text{and}\quad n \leqq |H_1|+2,$$

from which

$$n \leqq \min(|H_0|+1, |H_1|+2) =: N_3$$

follows. This implies that $H_n \neq 0$ if $n > \max(13, N_3)$.

Now let $K<0$. If $(L, M)=1$, then by Theorem 1 we have

$$H_0 = \pm(H_0, H_1)\,U_n \quad \text{and} \quad H_1 = \pm(H_0, H_1)L^{\varepsilon(n)}MU_{n-1}.$$

If $n \geqq N_1$, then by the Lemma

$$|H_0| \geqq |U_n| > M^{n/4} \geqq 2^{n/4}$$

and

$$|H_1| \geqq |MU_{n-1}| > M \cdot M^{(n-1)/4} > 2^{n/4}$$

which imply

$$n < \min\left(\frac{4}{\log 2}\log|H_0|, \frac{4}{\log 2}\log|H_1|\right) := N_4.$$

Thus $H_n \neq 0$ if $n \geqq \max(N_1, N_4)$.

Finally let $K<0$ and $(L, M)=d$. By Proposition 5 it follows that

$$H_n(H_0, H_1, L, M) = 0 \quad \text{if and only if} \quad H_n(dH_0, H_1, L/d, M/d) = 0.$$

But we have proved that if $H_n(dH_0, H_1, L/d, M/d)=0$ and $n>N_1$, then

$$n < \min\left(\frac{4}{\log 2}\log|dH_0|, \frac{4}{\log 2}\log|H_1|\right) =: N_2.$$

Thus $H_n(H_0, H_1, L, M) \neq 0$ if $n > \max(N_1, N_2) =: N_0$.

Proof of Theorem 3. As in the proof of Theorem 2 we can assume without any essential loss of generality that $L>0$.

Let $K<0$ and $n>N_0$ (N_0 is defined in Theorem 2) and so $|H_n|>0$. The numbers $a=H_1-\sqrt{L}H_0\beta$, $b=H_1-\sqrt{L}H_0\alpha$ are complex conjugates therefore — as in the proof of the Lemma — for some integer t

(25)
$$|H_n| = \left|(\sqrt{L})^{\varepsilon(n)}\frac{a\alpha^n-b\beta^n}{\alpha^2-\beta^2}\right| \geqq \frac{|a|\,|\alpha|^n}{\sqrt{|LK|}}\left|1-\frac{b}{a}\left(\frac{\beta}{\alpha}\right)^n\right| \geqq$$

$$\geqq \frac{|a|\,|\alpha|^n}{2\sqrt{|LK|}}\left|t\log(-1)-n\log\frac{\beta}{\alpha}-\log\frac{b}{a}\right|,$$

where the logarithms take their principal values and $t \leqq 2n+2$.

Theorem A can be applied to

$$0 \neq \Lambda = t\log(-1)-n\log\frac{\beta}{\alpha}-\log\frac{b}{a}.$$

In this case $m=3$, $W=2\log n$, $\alpha_1=-1$, $\alpha_2=\beta/\alpha$, $\alpha_3=b/a$, $D=2$ since a/b is a root of $ux^2-vx+u=0$, where

$$u = H_1^2-LH_0H_1+LMH_0^2, \quad v = 2H_1^2-2LH_0H_1+L^2H_0^2+2LMH_0^2.$$

Using these, $V_1=\pi/2$, $V_2=V_2^+=\log 4M$, $V_3=V_3^+=2\log 4M(H_0^2+H_1^2)$ and $E=4$ follow. Thus for $n>N_0$ we have

$$|\Lambda|>\exp\{-c_4\log 4M\cdot\log(8\log 4M)\cdot\log(4M(H_0^2+H_1^2))\times$$
$$\times(2\log n+\log 16\log 4M(H_0^2+H_1^2))\},$$

where

$$c_4=0{,}86\cdot 2^{76}>c(3)2^5\cdot\frac{\pi}{2}\cdot 2\cdot(\log 4)^{-4}.$$

Since $\log(8\log 4M)<4\log\log 4M$, we have

(26) $$|\Lambda|>\exp\{-4c_4\log 4M\log\log 4M\log 4M(H_0^2+H_1^2)(2\log n+\log n)\}=$$
$$=n^{-12c_4\log 4M\cdot\log\log 4M\cdot\log 4M(H_0^2+H_1^2)},$$

if $n>2^{67}\cdot\log 4M(H_0^2+H_1^2)(\geqq N_0)$.

Thus, by (25) and (26), we get

$$|H_n|>\frac{|a|\cdot|\alpha|^n}{2\sqrt{|LK|}}\cdot n^{-c_0}$$

for $n>2^{56}\cdot\log 4M(H_0^2+H_1^2)$, where

$$c_0=2^{80}\log 4M\cdot\log\log 4M\cdot\log 4M(H_0^2+H_1^2)>$$
$$>12c_4\log 4M\cdot\log\log 4M\cdot\log 4M(H_0^2+H_1^2),$$

which proves the first inequality of Theorem 3. The second inequality is obvious by the explicit form of H_n.

References

[1] P. Kiss, Zero terms in second order linear recurrences, *Math. Seminar Notes (Kobe Univ.)*, **7** (1979), 145—152.

[2] D. H. Lehmer, An extended theory of Lucas' functions, *Ann. Math.*, **31** (1930), 419—448.

[3] K. Mahler, A remark on recursive sequences, *Jour. Math. Sci.*, **1** (1966), 12—17.

[4] M. Mignotte, A note on linear recursive sequences, *Jour. Austral. Math. Soc.*, **20** (1975), 242—244.

[5] I. Niven and H. S. Zuckerman, *An introduction to the theory of numbers*, J. Wiley (New York, 1960).

[6] T. N. Shorey and C. L. Stewart, On the Diophantine equations $ax^{2t}+bx^ty+cy^2=d$ and pure powers in recurrence sequences, *Math. Scand.*, **52** (1983), 24—36.

[7] C. L. Stewart, Primitive divisors of Lucas and Lehmer numbers, *Transcendence theory: Advances and Applications*, Proceedings of the Conference (Cambridge, 1976), London—New York—San Francisco, 1977.

[8] M. Waldschmidt, A lower bound for linear forms in logarithms, *Acta Arith.*, **37** (1979), 257—283.

[9] M. Ward, The intrinsic divisors of Lehmer numbers, *Ann. of Math.*, **2 (62)** (1955), 230—236.

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EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BOGDÁNFY U. 10/B
1117 BUDAPEST, HUNGARY

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BOUNDEDNESS IN UNIFORM SPACES, TOPOLOGICAL GROUPS, AND HOMOGENEOUS SPACES

C. J. ATKIN (Wellington)

Some twelve years ago, I observed that certain Banach Lie groups have the property of "boundedness", whilst others which appear very closely related do not. In fact, the natural context for this notion of boundedness is in uniform spaces, and this note aims to present the definitions and theorems that seem likely to be useful in further work. All the material is very straightforward, and much of it may be connected to well-known results (for instance on symmetric spaces). However, it is not entirely trivial, and in some respects is even rather confusing.

The idea of a bounded Banach Lie group has been of some slight use in work of Banaszczyk (see, for instance, [3]). I hope subsequently to show that it has other and more powerful applications.

Where my notation and terminology in this paper are not self-explanatory, they are in accordance with Chapter 6 of [5].

§ 1. Boundedness in uniform spaces

(1.1) Definition. Let $(X, \mathscr{V})$ be a uniform space. Say that the subset A of X is bounded in X when, for each $V \in \mathscr{V}$, there exist a finite set $F \subseteq X$ and a positive integer m such that $A \subseteq V^m[F]$.

(1.2) Remarks. (a) It evidently suffices that the condition hold for vicinities which form a base for $\mathscr{V}$ — for instance for symmetric vicinities (those which satisfy $V = V^{-1}$).

(b) If A is bounded in X, so is any subset of A. If A and B are bounded in X, so is $A \cup B$.

(c) The subset A may be given the subspace uniformity

$$\mathscr{V}_A = \{V \cap (A \times A) \colon V \in \mathscr{V}\}.$$

Say that A is bounded in itself if it is bounded in the uniform space $(A, \mathscr{V}_A)$. This implies boundedness in X (see below, (1.6)), but is usually more restrictive (see (3.5)).

(d) If A and B are bounded in themselves they are both bounded in $(A \cup B, \mathscr{V}_{A \cup B})$, by (1.6), and consequently, by (b), $A \cup B$ is bounded in itself. However, a subset of a set bounded in itself is usually not bounded in itself (see (1.8)).

(1.3) Let $V \in \mathscr{V}$ be symmetric. Define a relation $\sim_V$ on X by saying that $x \sim_V y$ if and only if $x \in V^m[y]$ for some positive integer m. Say that $x \sim y$ if and only if $x \sim_V y$ for every symmetric vicinity $V \in \mathscr{V}$.

(a) Both $\sim_V$ and $\sim$ are equivalence relations on X. I shall call the $\sim_V$-equivalence classes the V-components of X, and the equivalence classes of $\sim$ the pseudo-components of X. The pseudo-components of a subset A of X are of course defined with respect to the subspace uniformity $\mathscr{V}_A$.

(b) Each V-component of X is both open and closed. Hence any connected subset of X is included in a single V-component, and in a single pseudo-component. More precisely, each quasi-component of X (see [4], p. 430) lies within a single pseudo-component.

(c) If X is compact (so that it has only one compatible uniformity), each component is a quasi-component and a pseudo-component. (See [4], p. 431.)

(d) If X is bounded, each V-component is bounded in itself.

(1.4) Lemma. *The subset A of X is bounded in X if and only if, for each symmetric vicinity $V\in\mathscr{V}$, A meets only finitely many V-components of X, and, for each V-component W, there exist $w\in W$ and a positive integer m such that $A\cap W\subseteq V^m[w]$. When A is bounded in X and W is a V-component of X, there exists for each $v\in W$ an integer $n\geqq 1$ such that $A\cap W\subseteq V^n[v]$.*

Proof. The condition's sufficiency is obvious. Suppose that A is bounded in X, and $V^{-1}=V\in\mathscr{V}$. Then, as in (1.1), $A\subseteq V^m[F]$; so A is included in the union of the V-components of the points of F, and meets no others. Let W be a V-component; then $A\cap W\subseteq V^m[F\cap W]$. Therefore, if $f\in F\cap W$ and $v\in W$, certainly there is an integer $m(f)\geqq 1$ with $f\in V^{m(f)}[v]$. Set

$$n = \max\{m(f)+m\colon f\in W\cap F\}.$$

It follows that $A\cap W\subseteq V^n[v]$.

(1.5) Corollary. *If A is bounded in X and $V\in\mathscr{V}$, the set F of* (1.1) *may be chosen to be a subset of A. When $V=V^{-1}$, the number of elements of F cannot be less than the number of V-components of X which meet A, and F may be chosen both to be a subset of A and to have exactly that number of elements* (indeed, form F by taking one element arbitrarily from $W\cap A$, for each V-component W that meets A).

(1.6) Lemma. *Let $\varphi\colon(X,\mathscr{V})\to(Y,\mathscr{W})$ be a uniformly continuous map of uniform spaces, and $A\subseteq X$. If A is bounded in X, $\overline{\varphi(A)}$ is bounded in Y. If A is bounded in itself, so are $\varphi(A)$ and $\overline{\varphi(A)}$.*

Proof. Let $W\in\mathscr{W}$; thus $(\varphi\times\varphi)^{-1}W\in\mathscr{V}$, and, if A is bounded, there exist a finite set $F\subseteq X$ and a positive integer m such that $((\varphi\times\varphi)^{-1}W)^m[F]\supseteq A$. However,

$$\varphi\{((\varphi\times\varphi)^{-1}W)^m[F]\}\subseteq W^m[\varphi(F)].$$

Therefore, $\varphi(A)\subseteq W^m[\varphi(F)]$, and in turn

$$\overline{\varphi(A)}\subseteq W^{m+1}[\varphi(F)].$$

As $\varphi(F)$ is finite, this proves $\overline{\varphi(A)}$ is bounded in Y.

The concluding assertion follows by noting that both

$$\varphi|A\colon(A,\mathscr{V}_A)\to(\varphi(A),\mathscr{W}_{\varphi(A)})$$

and inclusion

$$(\varphi(A), \mathcal{W}_{\varphi(A)}) \to (\overline{\varphi(A)}, \mathcal{W}_{\overline{\varphi(A)}})$$

are uniformly continuous.

(1.7) LEMMA. *Let A be a subset of the locally convex topological vector space E. Then A is bounded in E with respect to the standard uniformity on E if and only if it is bounded in E in the sense of von Neumann* ([7], p. 25).

PROOF. Suppose A is bounded in the uniformity; let U be any absolutely convex neighbourhood of 0 in E. Then

$$V_U = \{(x, y) \in E \times E \colon y - x \in U\}$$

is a vicinity of the uniformity, so there is a finite subset F of E such that, for some m,

$$A \subseteq V_U^m[F] = F + U + U + \ldots + U \quad (m \text{ repeats of } U)$$

$$= F + mU \qquad\qquad (\text{as } U \text{ is convex}).$$

As F is finite, there exists q such that $F \subseteq qU$, and so $A \subseteq (m+q)U$. Hence A is bounded in the sense of von Neumann. Conversely, if A is von Neumann-bounded and U is an absolutely convex neighbourhood of 0, then there is an integer m with $A \subseteq mU = V_U^m[0]$, so that A is bounded in the uniformity.

(1.8) NOTES. (a) If B is *convex* and von Neumann-bounded, it is also bounded in itself in the uniformity. Without loss of generality, suppose $0 \in B$, and let U be as above. Then there is a positive integer q such that $B \subseteq qU$; given $b \in B$, $q^{-1}b \in U \cap B$. Consequently

$$b = q^{-1}b + q^{-1}b + \ldots + q^{-1}b \in \{(B \times B) \cap V_U\}^q[0]$$

(since $mq^{-1}b \in B$ for $0 \leqq m \leqq q$).

(b) An infinite orthonormal set in Hilbert space, for instance, is von Neumann-bounded but not bounded in itself in the uniformity. See (1.2) (d).

§ 2. Further properties

(2.1) LEMMA. *Let A be a subset of the uniform space $(X, \mathcal{V})$. Then A is bounded in itself if and only if $\bar{A}$ is bounded in itself.*

PROOF. Clearly one may assume $\bar{A} = X$. One implication follows from (1.6). So now suppose X is bounded (in itself). Take $V \in \mathcal{V}$; thus $V_0 = V \cap (A \times A)$ is a typical vicinity of $\mathcal{V}_A$. Choose $W \in \mathcal{V}$ such that $W = W^{-1}$, $W^3 \subseteq V$. Then there are a positive integer m and a finite subset F of X such that $W^m[F] = X$. For each $x \in X$, choose $\hat{x} \in A \cap W[x]$ (which is possible, as A is dense). Let $\hat{F} = \{\hat{f} \colon f \in F\}$, which is a finite subset of A.

Take any $a \in A$. As $a \in W^m[F]$, there are points $f \in F$ and $x_1, x_2, \ldots, x_{m-1} \in X$ such that $(f, x_1) \in W$, $(x_{m-1}, a) \in W$, and $(x_i, x_{i+1}) \in W$ for $1 \leqq i \leqq m-2$. Thus $(\hat{f}, \hat{x}_1) \in W^3$, $(\hat{x}_{m-1}, a) \in W^2$, and $(\hat{x}_i, \hat{x}_{i+1}) \in W^3$ for $1 \leqq i \leqq m-2$. Since $W^3 \subseteq V$, this proves that

$$a \in (V \cap (A \times A))^m [F] = V_0^m [F].$$

Hence, as required, $A \subseteq V_0^m [F]$.

(2.2) PROPOSITION. *Let* $\{(X_\gamma, \mathscr{V}_\gamma)\colon \gamma \in \Gamma\}$ *be any indexed class of uniform spaces; let* $X = \prod_{\gamma \in \Gamma} X_\gamma$, *and let* $\pi_\gamma\colon X \to X_\gamma$ *be the projection. Then* $A \subseteq X$ *is bounded in the product uniformity* $\mathscr{V}$ *if and only if, for each* $\gamma \in \Gamma$, $\pi_\gamma(A)$ *is bounded in* X_γ.

PROOF. $\pi_\gamma\colon (X, \mathscr{V}) \to (X_\gamma, \mathscr{V}_\gamma)$ is uniformly continuous by definition, so one implication results from (1.6). Suppose, therefore, that $\pi_\gamma(A)$ is bounded in X_γ for each $\gamma \in \Gamma$. There is a base for $\mathscr{V}$ consisting of sets of the form

$$V = \bigcap_{\gamma \in \Delta} (\pi_\gamma \times \pi_\gamma)^{-1} V_\gamma,$$

where Δ is an arbitrary finite subset of Γ and V_γ is an arbitrary member of $\mathscr{V}_\gamma$ for each $\gamma \in \Delta$. Now, for each $\gamma \in \Delta$, there is a finite subset F_γ of X_γ, and a positive integer $m(\gamma)$, such that

$$\pi_\gamma(A) \subseteq V_\gamma^{m(\gamma)} [F_\gamma].$$

Take $m = \max \{m(\gamma)\colon \gamma \in \Delta\}$, and let F be any finite subset of X for which the map

$$F \to \prod_{\gamma \in \Delta} X_\gamma;\ f \to (\pi_\gamma(f))_{\gamma \in \Delta}$$

has image including $\prod_{\gamma \in \Delta} F_\gamma$. For instance, choose $x_\gamma \in X_\gamma$ arbitrarily when $\gamma \notin \Delta$, and let F consist of all points whose γ-coordinate is x_γ for $\gamma \notin \Delta$ and lies in F_γ when $\gamma \in \Delta$. For any such set F, clearly

$$A \subseteq \bigcap_{\gamma \in \Delta} \pi_\gamma^{-1}(\pi_\gamma(A)) \subseteq V^m [F],$$

since each coordinate may be treated independently.

(NOTE. The analogous statement with 'in itself' in the places of 'in X' and of 'in X_γ' is in general false, even when Γ has only two elements; the reason is that one may have $\pi_\gamma(A) = X_\gamma$ for each γ although $A \neq X$.)

(2.3) LEMMA. *Let* Γ *be a directed set, and* $\pi_{\gamma\delta}\colon X_\delta \to X_\gamma$ *an inverse system over* Γ *of uniform spaces* (X_γ, V_γ) *and uniformly continuous maps* $\pi_{\gamma\delta}$. *Suppose that* X *is the limit of the system, and the canonical projections* $\pi_\gamma\colon X \to X_\gamma$ *are all surjective. Then* $A \subseteq X$ *is bounded in* X *if and only if, for each* $\gamma \in \Gamma$, $\pi_\gamma(A)$ *is bounded in* X_γ.

PROOF. Because Γ is directed, sets of the form $(\pi_\gamma \times \pi_\gamma)^{-1} V_\gamma$ (where $\gamma \in \Gamma$ and $V_\gamma \in \mathscr{V}_\gamma$) constitute a base for the uniformity of X. If F_γ is a finite subset of X_γ and m a positive integer with $\pi_\gamma(A) \subseteq V_\gamma^m [F_\gamma]$, choose a finite subset F of X with $\pi_\gamma(F) = F_\gamma$; then, trivially,

$$A \subseteq \pi_\gamma^{-1}(\pi_\gamma(A)) \subseteq \{(\pi_\gamma \times \pi_\gamma)^{-1} V_\gamma\}^m [F].$$

(2.4) THEOREM. *Let* A *be a subset of the uniform space* $(X, \mathscr{V})$. *The following three statements are equivalent.*

(a) *A is bounded in X.*

(b) *Every real-valued uniformly continuous function on X is bounded on A.*

(c) *A is of finite diameter with respect to every uniformly continuous pseudo-metric on X.* [A pseudo-metric is described as uniformly continuous if it defines a uniformity included in $\mathcal{V}$.]

PROOF. (a) implies (b) by (1.6). If d is a uniformly continuous pseudo-metric on X, and $a \in A$, define, for all $x \in X$, $f(x)=d(x, a)$. Thus $f\colon X \to \mathbf{R}$ is uniformly continuous. Ergo, (b) implies (c). It remains to prove that (c) implies (a).

Let V be a symmetric vicinity; if possible, let $W_1, W_2, W_3, \dots$ be an infinite sequence of distinct V-components (see (1.3)), all meeting A. Define, for any $x, y \in X$,

$$d(x, y) = |\varphi(x) - \varphi(y)|,$$

where $\varphi(x)=n$ when $x \in W_n$, $\varphi(x)=0$ when $x \notin \bigcup_{n=1}^{\infty} W_n$. Now φ is a uniformly continuous function on X, and d is a uniformly continuous pseudo-metric. If (c) holds, then, $d(x, y)$ is bounded for $x, y \in A$, and this is a contradiction. Therefore A meets only finitely many V-components.

The standard construction ([5], pp. 185-6) gives a uniformly continuous pseudo-metric d_V such that, if x and y lie in the same V-component and m is a positive integer, $d_V(x, y) \leqq m$ if and only if $x \in V^m[y]$. Hence, if (c) holds and W is a V-component, $w \in W \cap A$, $A \cap W \subseteqq V^n[w]$ for any integer n greater than the d_V-diameter of A. So both conditions of (1.4) are implied by (c); (a) follows.

NOTE. It is insufficient in (c) to consider only a class of pseudo-metrics defining the uniformity; they might all be bounded, and also not distinguish V-components.

(2.5) The uniform space $(X, \mathcal{V})$ is uniformly locally compact (or precompact) if there is a vicinity $V \in \mathcal{V}$ such that $V[x]$ is relatively compact (or precompact) in X for each $x \in X$.

In such a case, suppose U is a symmetric vicinity with $U^2 \subseteqq V$, and let A be a precompact subset of X. Then there is a finite subset F of X with $U[F] \supseteqq A$; so

$$U[A] \subseteqq U^2[F] \subseteqq V[F],$$

and $V[F]$ is relatively compact or precompact, as the case may be, in X. One concludes inductively that, for any finite subset F_0 of X and any positive integer m, $U^m[F_0]$ is respectively relatively compact or precompact in X. One instantly deduces the next lemma.

(2.6) LEMMA. (a) *If A is a precompact subset of the uniform space $(X, \mathcal{V})$, then A is bounded in X.*

(b) *If $(X, \mathcal{V})$ is uniformly locally precompact, then A is bounded in X if and only if A is precompact.*

(c) *If $(X, \mathcal{V})$ is uniformly locally compact, then A is bounded in X if and only if A is relatively compact.*

§ 3. Boundedness in topological groups

My notations for groups will be standard (e denotes the identity element).

(3.1) LEMMA. *Let H be a subgroup of the group G. Suppose $e\in A\subseteq G$, and, for some positive integer r, A^r includes a subgroup K of H of finite index in H. Then there is a positive integer $s\geqq r$ such that $A^s\cap H$ is itself a subgroup of H of finite index in H.*

PROOF. $K, AK\cap H=(A\cap H)K, A^2K\cap H, A^3K\cap H, \dots$ is an increasing sequence of unions of left cosets of K in H, of which there are only finitely many. Thus there is a positive integer m such that

$$L = A^m K\cap H = A^{m+1}K\cap H = A^{m+2}K\cap H = \dots. \tag{1}$$

However, as $L\subseteq H$, for any $q\geqq 1$

$$\begin{aligned} L \subseteq A^q L\cap H = (A^q\cap H)L = (A^q\cap H)(A^m\cap H)K = \\ = (A^{m+q}\cap H)K = A^{m+q}K\cap H = L. \end{aligned} \tag{2}$$

By hypothesis, $A^r\cap H\supseteq K$. By the chain (2) of equalities,

$$LL = (A^m\cap H)KL \subseteq (A^m\cap H)(A^r\cap H)L \subseteq (A^{m+r}\cap H)L = L,$$

so that L is closed under multiplication.

Take $l\in L$. As K is of finite index in H, there are distinct exponents n, p such that l^n and l^p both lie in the same left coset of K. Consequently there is a negative integer t such that $l^t\in K$. Since $K\subseteq L$,

$$l^{-1} = l^{-1-t}l^t\in L^{-1-t}K \subseteq L^{-1-t}L \subseteq L$$

(for $-1-t\geqq 0$ and L is closed under multiplication; read L^0 as meaning $\{e\}$). Hence L is a subgroup of H, and, as it includes K, is of finite index in H.

Finally, $L\supseteq A^n\cap H$ for any $n\geqq m$, by construction (1), whilst by hypothesis $L=A^mK\cap H\subseteq A^{m+r}\cap H$. It follows that $L=A^{m+r}\cap H$, as required.

(3.2) Henceforth I shall write $\mathscr{V}_l$ for the left, and $\mathscr{V}_r$ for the right, uniformity on a topological group G. A base for $\mathscr{V}_l$ is furnished by the vicinities

$$V_U = \{(x, y)\colon\ x^{-1}y\in U\},$$

where U runs through a base of neighbourhoods of e in G.

In the same way, a base for $\mathscr{V}_r$ is given by the vicinities

$$V^U = \{(x, y)\colon\ xy^{-1}\in U\}.$$

LEMMA. *Let A, B be subsets of the topological group G.*

(a) *A is bounded in G, or in itself, with respect to $\mathscr{V}_r$ if and only if A^{-1} is bounded in the same sense with respect to $\mathscr{V}_l$.*

(b) *If A has finitely many pseudo-components, and is bounded in G, with respect to $\mathscr{V}_r$, then it is also bounded in G with respect to $\mathscr{V}_l$; and conversely.*

(c) *If A, B are both bounded in G, or in themselves, with respect to $\mathscr{V}_r$, then AB is bounded in the same sense; and similarly for $\mathscr{V}_l$.*

PROOF. For (a), note that the inversion $x\to x^{-1}$ is a uniform homeomorphism of $(G, \mathcal{V}_r)$ with $(G, \mathcal{V}_l)$. For (b), form a set F with one point from each right pseudo-component of A. Given any neighbourhood U of e in G, one may choose another neighbourhood W such that $Wf\subseteq fU$ for every $f\in F$; it follows that $W^n f\subseteq fU^n$ for every $n\geqq 1$ and $f\in F$. Now, by (1.3) (b) and (1.4), there exists $m\geqq 1$ such that

$$A\subseteq (V^W)^m[F]=W^m F$$

(an easy computation). Consequently $A\subseteq FU^m=(V_U)^m[F]$, which proves the result. Notice that the connectedness hypothesis is needed to obtain a set F independent of W.

For (c), suppose first that A and B are $\mathcal{V}_r$-bounded in G. Given any neighbourhood U of e in G, there exist a finite subset F_1 of G and an integer $p\geqq 1$ such that

$$A\subseteq (V^U)^p[F_1] = U^p F_1.$$

Now, there is a neighbourhood W of e in G such that $fW\subseteq Uf$ for each $f\in F_1$; and, in turn, there are a finite subset F_2 of G and an integer $q\geqq 1$ such that $B\subseteq W^q F_2$. Hence

$$AB\subseteq U^p F_1 W^q F_2\subseteq U^{p+q}F_1F_2.$$

This clearly proves the result in this case.

Suppose that A and B are $\mathcal{V}_r$-bounded in themselves, and let U be a neighbourhood of e in G. Take a neighbourhood U_1 such that $U_1U_1\subseteq U$. Then there exist a finite subset F_1 of A, and an integer $m\geqq 1$, such that, for any $a\in A$, there is a sequence $a_0=a$, $a_1\in U_1a_0$, ..., $a_m\in U_1a_{m-1}\cap F_1$ of points of A. Choose a neighbourhood U_2 of e such that $fU_2\subseteq U_1f$ for every $f\in F_1$. Now there are a finite subset F_2 of B, and an integer $n\geqq 1$, such that, for any $b\in B$, there exists a sequence $b_0=b$, $b_1\in U_2b_0$, ..., $b_n\in U_2b_{n-1}\cap F_2$ of points of B. Take $F=F_1F_2$, a finite subset of AB.

Then, for any element ab of AB (where $a\in A$ and $b\in B$), there exist sequences $a_0, a_1, ..., a_m$ and $b_0, b_1, ..., b_n$ as specified above. The sequence $x_0=a_0b_0$, $x_1=a_1b_0$, ..., $x_m=a_mb_0$, $x_{m+1}=a_mb_1$, ..., $x_{m+n}=a_mb_n$ of points of AB is such that $x_{i+1}\in U_1x_i$ for each i (since $a_m\in F_1$, and so $a_mU_2\subseteq U_1a_m$), and $a_mb_n\in F$. This proves the result. (In fact the same argument could be used for the first case, if all terms of the sequences after a_0 and b_0 were allowed to lie anywhere in G.)

(3.3) REMARKS. In (3.2) (b), note that A may have finitely many pseudo-components with respect to $\mathcal{V}_r$ but not with respect to $\mathcal{V}_l$. Thus one might say, slightly less exactly, that for symmetric or for connected subsets of G it does not matter which uniformity, $\mathcal{V}_r$ or $\mathcal{V}_l$, is used to define boundedness in G. This is false when one asks whether a set is bounded in itself.

Since continuous homomorphisms of topological groups are uniformly continuous with respect to left or right uniformities, various results of §§ 1, 2 translate instantly to statements about such homomorphisms.

(3.4) THEOREM. *Let H be a subgroup of the topological group G. The following statements are equivalent.*

(a) *H is bounded in G (in either uniformity; see (3.3)).*

(b) *For any neighbourhood U of e in G, there is an integer* $m \geqq 1$ *such that* U^m *includes a subgroup of H of finite index in H.*

(c) *For any neighbourhood U of e in G, there is an integer* $n \geqq 1$ *such that* $U^n \cap H$ *is a subgroup of H of finite index in H.*

PROOF. Certainly (b) implies (c), by (3.1). Assume (c). Given a neighbourhood U of e in G, there exist a finite subset F of H and an integer $n \geqq 1$ such that $F(U^n \cap H) = H$; hence $H \subseteqq FU^n = (V_U)^n[F]$. This proves (a). Now, assume (a), and let U be a symmetric neighbourhood of e in G. So there exist a finite set $F \subseteqq H$ (see (1.5)) and an integer $p \geqq 1$ such that $FU^p = (V_U)^p[F] \supseteqq H$.

Consider the increasing sequence $U^p \cap F, U^{2p} \cap F, U^{3p} \cap F, \ldots$. As F is finite, there is an integer $l \geqq 1$ such that for all $k \geqq 1$, $U^{lp} \cap F = U^{kp} \cap F$. Certainly $U^{(l+1)p}$ is symmetric and contains e. Suppose $x, y \in U^{(l+1)} \cap H$. Then $xy \in U^{2(l+1)p} \cap H$, and, for some $f \in F$, $xy \in fU^p \cap H$. As U is symmetric, it follows that $f \in U^{(2l+3)p}$, and therefore that $f \in U^{lp}$ (by the choice of l). Hence $xy \in U^{(l+1)p} \cap H$. Thus $U^{(l+1)p} \cap H$ is a subgroup of H, and it is of finite index therein, since $FU^p \supseteqq H$. This proves (b) when U is symmetric; the general case is an obvious corollary.

(3.5) Let ℓ^2 denote the Hilbert space of square-summable complex sequences indexed by all the integers, and let U be the group of its unitary operators, topologised as a subset of the Banach space of all bounded operators on ℓ^2. Then U is a topological group — in fact a Banach Lie group — bounded in itself (see (6.5), (6.8) (a), and [2]). However, the powers of a shift operator form a closed subgroup which is not bounded in itself; compare (1.2) (d) and (1.8).

§ 4. The standard uniformity on a left coset space

(4.1) Let $(X, \mathscr{V})$ be a uniform space and $\pi\colon X \to Y$ a surjection. Define

$$\pi_* \mathscr{V} = \{V : V \subseteqq Y \times Y \quad \text{and} \quad (\pi \times \pi)^{-1} V \in \mathscr{V}\}.$$

Then $\pi_* \mathscr{V}$ is a filter in $Y \times Y$, every member of which includes the diagonal. If $\mathscr{V}_0$ is a base for $\mathscr{V}$, then $\{(\pi \times \pi) W\colon W \in \mathscr{V}_0\}$ forms a base for $\pi_* \mathscr{V}$.

In general, $\pi_* \mathscr{V}$ will not be a uniformity, although only because the "triangle inequality" ([5], p. 176, (c)) may fail. When it is a uniformity, it may be called the quotient uniformity induced by π. It is the finest uniformity on Y which makes π uniformly continuous.

Suppose a group H acts on X on the right. Say that $\mathscr{V}$ is (right) H-invariant if there is a base $\mathscr{V}_0$ of $\mathscr{V}$ which consists of H-invariant sets; that is, for each $h \in H$, $V \in \mathscr{V}_0$,

$$V = \{(x, y)\colon (xh, yh) \in V\}.$$

Equivalently, $\mathscr{V}$ is H-invariant if the class of right translations by elements of H is uniformly equicontinuous with respect to $\mathscr{V}$ (for, in that case, define for $V \in \mathscr{V}$

$$\tilde{V} = \bigcap_{h \in H} \{(xh, yh)\colon (x, y) \in V\};$$

the sets $\bar{V}$, as V varies over $\mathscr{V}$, form a base of $\mathscr{V}$, because of the uniform equicontinuity, and they are H-invariant).

(4.2) Lemma. *Let $\mathscr{V}$ be an H-invariant uniformity on the right H-space X, and let $\pi\colon X\to X/H$ be the quotient map on the orbit space. Then $\pi_*\mathscr{V}$ is a uniformity on X/H.*

Proof. If $Q\in\pi_*\mathscr{V}$, there exist $W\in\mathscr{V}$ such that $(\pi\times\pi)W\subseteqq Q$, and $W_0\in\mathscr{V}$ which is H-invariant and satisfies $W_0\circ W_0\subseteqq W$; set $Q_0=(\pi\times\pi)W_0\in\pi_*\mathscr{V}$. If $(\alpha,\gamma)\in Q_0\circ Q_0$, there exists $\beta\in Q_0$ with $(\alpha,\beta)\in Q_0$ and $(\beta,\gamma)\in Q_0$. Take $a, b, c\in X$ with $\pi(a)=\alpha$, $\pi(b)=\beta$, $\pi(c)=\gamma$. As (α,β) and (β,γ) belong to Q_0, there exist h_1, h_2, h_3, h_4 in H such that $(ah_1, bh_2)\in W_0$, $(bh_3, ch_4)\in W_0$. Since W_0 is H-invariant, $(ah_1h_2^{-1}, ch_4h_3^{-1})\in W_0\circ W_0\subseteqq W$, and, applying $\pi\times\pi$, $(\alpha,\gamma)\in(\pi\times\pi)W\subseteqq Q$. So we have found $Q_0\in\pi_*\mathscr{V}$ such that $Q_0\circ Q_0\subseteqq Q$, and this proves that $\pi_*\mathscr{V}$ is a uniformity (see (4.1)).

(4.3) The case of most interest is when G is a topological group and H a subgroup (not necessarily closed) acting by right multiplication on G. The quotient space G/H is the space of left cosets. The right uniformity $\mathscr{V}_r$ on G is invariant under the right action of H, and indeed under the right action of G on itself; thus, by (4.2), there is a quotient uniformity on G/H, which I shall call, and treat as, the standard uniformity. Notice that the standard uniformity on a LEFT coset space is the quotient of the RIGHT uniformity on G.

When H is a normal subgroup of G, the standard uniformity on G/H is just the right uniformity of the group G/H. Symmetrically, the left uniformity on G quotients to the left uniformity on G/H. Notice that the quotient uniformity exists here — because the left and right coset spaces are identified — although $\mathscr{V}_l$ is not usually right H-invariant; (4.2) is not a necessary condition.

(4.4) Theorem. *Let H, K be subgroups of the topological group G, with $K\subseteqq H$; and let $\pi\colon G/K\to G/H$ be the natural projection. Then:*

(a) *π is uniformly continuous with respect to the standard uniformities;*

(b) *if A is a subset of G/K bounded in G/K (or in itself), then $\pi(A)$ is bounded in G/H (or in itself);*

(c) *if $A\subseteqq G/K$ and $\pi(A)$ is bounded in G/H, and H/K is bounded in G/K, then A is bounded in G/K.*

Proof. Let $\sigma\colon G\to G/K$ be the projection. Then the standard uniformity on G/H is

$$\text{(0)}\qquad \{V\subseteqq G/H\times G/H\colon (\pi\sigma\times\pi\sigma)^{-1}V\in\mathscr{V}_r\} =$$

$$= \{V\subseteqq G/H\times G/H\colon (\pi\times\pi)^{-1}V\in\sigma_*\mathscr{V}_r\} = \pi_*\sigma_*\mathscr{V}_r;$$

thus it is, in fact, the quotient of the standard uniformity on G/K. This proves (a), and (b) follows by (1.6).

Suppose $\pi(A)$ is bounded in G/H; let U be a neighbourhood of e in G. There exist a finite set $F\subseteqq G$ and an integer $m\geqq 1$ such that

$$\text{(1)}\qquad \pi(A)\subseteqq\big((\pi\times\pi)(\sigma\times\sigma)V^U\big)^m[\pi\sigma(F)] = \pi\sigma(U^m FH),$$

by an easy computation. Consequently

(2) $$A \subseteq \pi^{-1}\pi(A) = \sigma(U^m FH).$$

As F is finite, there is a neighbourhood W of e in G such that $fW \subseteq Uf$, and inductively $fW^q \subseteq U^q f$ for all $q \geqq 1$, whenever $f \in F$. Since H/K is bounded in G/K, there are a finite subset F_1 of G and an integer $n \geqq 1$ such that

$$\sigma(H) \subseteq ((\sigma\times\sigma)V^W)^n[\sigma(F_1)],$$

or, as at (1), $H \subseteq W^n F_1 K$. It follows that

(3) $$U^m FH \subseteq U^m FW^n F_1 K \subseteq U^{m+n} FF_1 K,$$

so that, by (2), and as at (1),

$$A \subseteq \sigma(U^{m+n} FF_1 K) = ((\sigma\times\sigma)V^U)^{m+n}[\sigma(FF_1)],$$

which (as $\sigma(FF_1)$ is a finite subset of G/K) proves the result.

(4.5) Note. The standard uniformity on H/K, the quotient of the right uniformity on H, which is itself the subspace uniformity induced from the right uniformity on G) is the same as the subspace uniformity induced from G/K.

(4.6) Lemma. *The left action of G on a left coset space G/K consists of uniform homeomorphisms with respect to the standard uniformity.*

Proof. Let $\sigma\colon G \to G/K$ be the projection. A basic vicinity for G/K is $(\sigma\times\sigma)V^U$, where U is a neighbourhood of e in G (see (4.1), (4.4) (0)). Now

$$(\sigma\times\sigma)V^U = \{(\sigma(x), \sigma(y))\colon\ x, y \in G \text{ and } x \in UyK\};$$

thus, for any $g \in G$,

$$(g\times g)(\sigma\times\sigma)V^U = \{(\sigma(gx), \sigma(gy))\colon\ x, y \in G \text{ and } x \in UyK\} =$$

$$= \{(\sigma(x_1), \sigma(y_1))\colon\ x_1, y_1 \in G \text{ and } x_1 \in gUg^{-1}y_1K\} = (\sigma\times\sigma)V^{gUg^{-1}}.$$

The result follows.

(4.7) Corollary. *If G, K, H, π are as in* (4.4), *all fibres of π are uniformly homeomorphic (as subspaces of G/K).*

(4.8) Theorem. *Let G, K, H, π be as in* (4.4). *Suppose H/K is bounded in itself and B is a subset of G/H bounded in itself. Then $\pi^{-1}(B)$ is bounded in itself in G/K.*

Proof. Let $\sigma\colon G \to G/K$ be the projection, and write B_1 for $(\pi\sigma)^{-1}B$. Take any symmetric neighbourhood U of e in G.

Then V^U and $(\pi\sigma\times\pi\sigma)V^U$ are symmetric (see (4.4) (0)), and so must be $V = (B\times B)\cap((\pi\sigma\times\pi\sigma)V^U)$. By (1.4), B has only finitely many V-components, and, by (1.3) (d) and (1.2) (d), it will suffice to suppose it has exactly one. Furthermore, in view of (4.6), one may arrange by translation that $\pi\sigma(e) \in B$. By (1.4), there exists $m \geqq 1$ such that $V^m[\pi\sigma(e)] = B$. So, for any $b \in B_1$, there is a sequence $x_0, x_1, \ldots, x_m = e$ of points of B_1 such that $u_{i+1} = x_i x_{i+1}^{-1} \in U$ for $0 \leqq i < m$ and $x_0 = bh^{-1} \in bH$. Now H/K is bounded in itself; therefore there are a finite subset F

of H and a positive integer n such that, as at (4.4) (3), $H=(U\cap H)^n FK$. (This is a simpler condition than for B_1 because H is closed under multiplication.) Hence there exist elements $w_1, w_2, \ldots, w_n$ of $U\cap H, f\in F$, and $k\in K$, such that $h=w_1 w_2 \ldots w_n fk$. It follows that

$$b = x_0 h = u_1 u_2 \ldots u_m w_1 w_2 \ldots w_n fk,$$

where, since $B_1 H = B_1$ and $e\in B_1$, all the partial products $b, u_1^{-1}b, \ldots, w_n fk, fk, k$ lie in B_1. This proves that $\sigma(B_1)=\pi^{-1}(B)$ is bounded in itself, as required.

§ 5. More on coset spaces

(5.1) The standard uniformity on the left coset space G/H (see (4.3)) is always defined, but is not, in general, itself G-invariant, despite (4.6). For instance, let G be the group of self-homeomorphisms of a connected finite-dimensional manifold M, and let H be the isotropy subgroup of $x\in M$. By [1], G is a topological group in the compact-open topology; the natural mapping $G/H\to M$ is (not quite trivially) a homeomorphism. But there can be no vicinities of M which are G-invariant, no matter what the uniformity (if it gives the correct topology). It is not difficult to give other examples where G is even locally compact and the quotient uniformity is not G-invariant.

If, however, there is a G-invariant uniformity which induces the topology on G/H, then (as the projection is both open and continuous) it must be the quotient of the left uniformity; I shall call it the left quotient uniformity. By (4.2), the left quotient uniformity exists when the left uniformity on G is right H-invariant. It also exists when H is normal in G, by (4.3). Thus, for example, it must exist if H is a compact extension of a normal subgroup of G.

(5.2) THEOREM. *Let G be a topological group, and H a subgroup. Suppose the left quotient uniformity on G/H is defined, and let A be a subset of G/H. Assume either*

(a) *that A has finitely many connected components,* or

(b) *that G is locally connected.*

Then A is bounded in G/H in the standard uniformity $\mathscr{S}$ if and only if it is bounded in G/H in the left quotient uniformity $\mathscr{L}$. (Compare (3.2) (b).)

PROOF. Define V^U, V_U as in (3.2).

(a) It suffices to assume A is connected. Let $a=\pi(a_0)\in A$; suppose W is any neighbourhood of e in G. Take a symmetric neighbourhood U of e such that $Ua_0\subseteqq a_0 W$, and therefore $U^q a_0\subseteqq a_0 W^q$ for $q\geqq 1$. If A is $\mathscr{S}$-bounded, then, by (1.3) (b) and (1.4), there exists $m\geqq 1$ such that

$$(1) \qquad A \subseteqq \{(\pi\times\pi)V^U\}^m[a] = \pi(U^m a_0 H) = \pi(U^m a_0).$$

(Compare (4.4) (1). As U is symmetric, so is $(\pi\times\pi)V^U$.) Hence $A\subseteqq\pi(a_0 W^m H)=\{(\pi\times\pi)V_W\}^m[a]$, and this shows A is $\mathscr{L}$-bounded. The analogous argument in the opposite direction is now obvious.

(b) Here G has a base $\mathscr{B}$ of connected symmetric neighbourhoods of e. Take $B\in\mathscr{B}$, $q=\pi(q_0)\in G/H$. Then, as at (1),

$$\{(\pi\times\pi)V^B\}^m[q] = \pi(B^m q_0), \quad \{(\pi\times\pi)V_B\}^m[q] = \pi(q_0 B^m).$$

Both these sets are connected. Hence the $(\pi\times\pi)V^B$-components and the $(\pi\times\pi)V_B$-components of G/H are all open and connected (see (1.3)); consequently they coincide (irrespective of the choice of $B\in\mathscr{B}$) with the connected components of G/H.

Suppose A is $\mathscr{S}$-bounded in G/H. By (1.4) and the last remark, A can meet only finitely many connected components of G/H. Choose a finite set $F\subseteq G$ such that $\pi(F)$ contains one point from each such nonnull intersection. Now, given $W\in\mathscr{B}$, choose $U\in\mathscr{B}$ such that, for all $f\in F$, $Uf\subseteq fW$. By (1.4), there is a positive integer m such that

$$A\subseteq\{(\pi\times\pi)V^U\}^m[\pi(F)] = \pi(U^m F) \quad \text{(see (1))}$$
$$\subseteq\pi(FW^m) = \{(\pi\times\pi)V_W\}^m[\pi(F)].$$

As W was arbitrary in $\mathscr{B}$, this proves that A is $\mathscr{L}$-bounded. The converse argument is, mutatis mutandis, identical.

NOTE. Pseudo-connectedness is insufficient here, since it need not hold simultaneously for $\mathscr{S}$ and for $\mathscr{L}$. The hypothesis of local connectedness in (b) makes it possible to choose F before U.

(5.3) THEOREM. *Let G, K, H, π be as in* (4.4), *and suppose the left quotient uniformities are defined on G/K and G/H.*

(a) *π is uniformly continuous with respect to the left quotient uniformities.*

(b) *If A is a subset of G/K bounded in G/K (or in itself) in the left quotient uniformity, then $\pi(A)$ is bounded in G/H (or in itself) in the left quotient uniformity.*

(c) *If $A\subseteq G/K$ and $\pi(A)$ is bounded in G/H in the left quotient uniformity, and H/K is bounded in G/K in the left quotient uniformity, then A is bounded in G/K in the left quotient uniformity.*

PROOF. (a), (b) are proved as in (4.4), with $\mathscr{V}_l$ in place of $\mathscr{V}_r$. If G is locally connected, (4.4) and (5.2) give (c); but I sketch a direct (and more general) proof.

Take a neighbourhood W of e in G. There is a finite subset F_1 of G such that $H\subseteq F_1W^nK$ for some integer $n\geqq 1$. Take a neighbourhood U of e in G such that, for all $f\in F_1$, $Uf\subseteq fW$; then there are an integer $m\geqq 1$ and a finite subset F of G such that $(\pi\sigma)^{-1}A\subseteq FU^mH$, where $\sigma\colon G\to G/K$ is the projection. Consequently,

$$\pi^{-1}(A)\subseteq\sigma(FU^mF_1W^nK)\subseteq\sigma(FF_1W^{m+n}K),$$

which (as FF_1 is finite) gives the result. Compare (4.4).

(5.4) SCHOLIA. In (5.3), the left quotient uniformity on H/K is necessarily defined, and tallies with the subspace uniformity induced from the left quotient uniformity on G/K. The left uniformity $\mathscr{V}_l$ on G is right H-invariant if and only if there is a base $\mathscr{C}$ of neighbourhoods of e in G such that, whenever $U\in\mathscr{C}$ and $h\in H$, $hUh^{-1}=U$. (This is the same as left H-invariance of the right uniformity on G).

(5.5) THEOREM. *Let G, K, H, π be as in* (4.4). *Suppose the left uniformity on G is right H-invariant, so that the left quotient uniformities on G/K and G/H are defined. With respect to these uniformities, suppose that H/K is bounded in itself and B is a subset of G/H bounded in itself. Then $\pi^{-1}(B)$ is bounded in itself in the left quotient uniformity on G/K.*

PROOF. As before, let $\sigma\colon G\to G/K$ be the projection. The boundedness of B in the left quotient uniformity means that, for any neighbourhood U of e in G, there exist a finite subset F_1 of $B_1=(\pi\sigma)^{-1}B$, and an integer $m\geqq 1$, such that, for any $b\in B_1$, there are points $x_0, x_1, \ldots, x_m$ in B_1 and $h_1, h_2, \ldots, h_m$ in H with $x_0\in F_1$, $x_m=bh_0^{-1}\in bH$, and $x_i^{-1}x_{i+1}\in h_{i+1}Uh_{i+1}^{-1}$ for $0\leqq i<m$. In view of (5.4), it may be assumed here that $hUh^{-1}=U$ for all $h\in H$.

As H/K is also bounded in itself, there exist a finite subset F_2 of H and an integer $n\geqq 1$ such that $H=F_2(U\cap H)^n K$.

In particular, there are elements $f\in F_2$, $k\in K$, and $v_{m+1}, \ldots, v_{m+n}\in U\cap H$ such that

$$h_0=fv_{m+1}\ldots v_{m+n}k.$$

Define, for $0\leqq i<m$, $v_{i+1}=f^{-1}x_i^{-1}x_{i+1}f\in U$ (recall that $fUf^{-1}=U$). Ergo,

$$b=x_m h_0=x_0fv_1f^{-1}\cdot fv_2f^{-1}\ldots fv_mf^{-1}\cdot fv_{m+1}\ldots v_{m+n}k=x_0f\cdot v_1v_2\ldots v_{m+n}k.$$

Here x_0f belongs to the finite set F_1F_2, which depends only on U and is included in $B_1H=B_1$; $k\in K$; and, setting $y_l=x_0f\cdot v_1v_2\ldots v_l$, one sees that, for $1\leqq l\leqq m$, $y_l=x_lf\in B_1H=B_1$, and, for $m<l\leqq m+n$, $y_l\in x_mH$, which is a subset of B_1 by construction. Of course $y_l^{-1}y_{l+1}\in U$ for $0\leqq l<m+n$ (if one takes $y_0=x_0f$); and $y_{m+n}\in bK$. Hence $\sigma(B_1)=\pi^{-1}(B)$ is bounded in itself in the left quotient uniformity on G/K.

(5.6) Compare (5.5) with (4.8). I expect the weaker hypotheses that the left quotient uniformities on G/K and on G/H both exist (neither implies the other, in general) to be inadequate in (5.5), but I have no counterexample.

It was observed in (3.5) that one may easily find unbounded subgroups of bounded groups; so one cannot reverse (4.8) or (5.5) to conclude that the fibre is bounded. There is, however, a limited converse, when the base is compact.

(5.7) THEOREM. *Let G, K, H, π be as in* (4.4). *Suppose B is a compact subset of G/H such that $\pi^{-1}(B)$ is bounded in itself in the standard uniformity. Then H/K is bounded in itself in the standard uniformity, provided that either*

(a) *H is locally connected, or*

(b) *the right uniformity on H is left H-invariant.*

PROOF. As previously, let $\sigma\colon G\to G/K$ be the projection; let $B_1=\pi^{-1}(B)$ and $B_0=\sigma^{-1}(B_1)$. First, I shall show that in case (a) one may assume H/K has only one pseudo-component.

Let H_0 be the principal component of H. It is an open normal subgroup of H; so $H_1=H_0K=KH_0$ is an open subgroup of H which includes K. Factorise $\pi=\varrho\pi_1$, where $\pi_1\colon G/K\to G/H_1$ and $\varrho\colon G/H_1\to G/H$ are the projections.

Take a symmetric neighbourhood U_1 of e in G such that $U_1^4 \cap H \subseteq H_0$. The covering

(1) $$\{\pi\sigma(xU_1)\colon x \in B_0\}$$

of B has a finite subcover $x_1 U_1 H, \ldots, x_m U_1 H$, and there exists a symmetric open neighbourhood U of e in G such that, for each i, $x_i^{-1} U x_i \subseteq U_1$. Suppose $x \in B_0$; $u_1, u_2 \in U$; and $u_1 x$, $u_2 x$ have the same images under $\pi\sigma$, so that $u_1 x H = u_2 x H$. Then $x^{-1} u_1^{-1} u_2 x \in H$. However, there exist $u \in U_1$ and i such that $x = x_i u$, and therefore $x^{-1} u_1^{-1} u_2 x = u^{-1} x_i^{-1} u_1^{-1} x_i x_i^{-1} u_2 x_i u$, where $x_i^{-1} u_1^{-1} x_i$ and $x_i^{-1} u_2^{-1} x_i$ belong to U_1 by the construction of U. Hence $x^{-1} u_1^{-1} u_2 x \in U_1^4 \cap H \subseteq H_0$, so that $u_1 x H_0 = u_2 x H_0$ and consequently $u_1 x H_1 = u_2 x H_1$. This means that $\varrho | \pi_1 \sigma(Ux)$ is one-one for each $x \in B_0$. Since G/H and G/H_1 both have quotient topologies, it follows easily that ϱ is a homeomorphism of $\pi_1\sigma(Ux)$ with $\pi\sigma(Ux)$. (One may think of ϱ as a uniform covering map.) Let W be a neighbourhood of e in G such that $WW \subseteq U$. Thus $\mathrm{cl}_{G/H}(\pi\sigma(Wx)) \subseteq \pi\sigma(Ux)$ and the homeomorphic counter-image $\pi_1\sigma(Ux) \cap \varrho^{-1}(B \cap \mathrm{cl}_{G/H}(\pi\sigma(Wx)))$ must be compact. It includes $B_1 \cap \cap \mathrm{cl}_{G/H_1}(\pi_1\sigma(Wx))$, which is therefore also compact for each $x \in B_0$. Hence $\varrho^{-1}(B)$ is a uniformly locally compact subset of G/H_1, in the subspace uniformity. Since it is $\pi_1(B_1)$, it is bounded in itself, by (4.4) (b); and therefore it is compact, by (2.6) (c). Suppose y, z belong to the same fibre of $\varrho | \varrho^{-1}(B)$, where $y = \pi_1\sigma(y_1)$ and $z = \pi_1\sigma(z_1)$. If $\pi_1\sigma(Wy_1) \cap \varrho^{-1}(B)$ and $\pi_1\sigma(Wz_1) \cap \varrho^{-1}(B)$ meet, they are both included in $\pi_1\sigma(Ux)$ for some $x \in B_0$. As ϱ is one-one on $\pi_1\sigma(Ux)$, this proves that $y = z$. It follows that the fibres of $\varrho | \varrho^{-1}(B)$ are finite, since otherwise the sets Wy, as y varies over an infinite fibre, — my notation uses the action of G on G/H_1 — would form an infinite disjoint family, and this would contradict the precompactness of $\varrho^{-1}(B)$. Consequently H/H_1 is finite, and, for the conclusion of the theorem, it will suffice to prove that H_1/K is bounded in itself, by (1.1) (d) and (4.6). But π_1 and $\varrho^{-1}(B)$ satisfy the same conditions as were assumed for π and B, whilst H_1/K has only one pseudo-component (since H_0 is also "pseudo-connected").

Now return to the general case, and cancel the previous notations. One may conveniently suppose (by (4.6)) that $e \in B_0$. Let U be an arbitrary neighbourhood of e in G, and choose a symmetric neighbourhood U_0 such that $U_0^{18} \subseteq U$. By compactness, there exist $y_1, y_2, \ldots, y_m \in B_0$ such that

(2) $$\{\pi\sigma(y_i U_0)\colon 1 \leq i \leq m\} \quad \text{covers} \quad B.$$

Construct a symmetric neighbourhood W of e in G such that

(3) $$y_i^{-1} W y_i \subseteq U_0 \quad \text{for} \quad 1 \leq i \leq m.$$

Next, construct a section $\tau\colon B \to B_0$ of $\pi\sigma$ (τ must usually be discontinuous) such that

$$\tau(B) \subseteq \bigcup_{1 \leq i \leq m} y_i U_0 \quad \text{and} \quad \tau(\pi\sigma(e)) = e.$$

This is clearly possible. Given any $x \in B$, there exists y_i such that $\tau(x) \in y_i U_0$, and so

(4) $$\tau(x)^{-1} W \tau(x) \subseteq U_0 y_i^{-1} W y_i U_0 \subseteq U_0^3.$$

In turn, B may be covered by

(5) $$\{\pi\sigma(W\tau(z_j))\colon 1 \leqq j \leqq n\},$$

for suitable points $\pi\sigma(e)=z_1, z_2, \dots, z_n$ in B.

By hypothesis, B_1 is bounded in itself. Let V denote $(B_1\times B_1)\cap(\sigma\times\sigma)V^W$; then, by (1.4), B_1 has only finitely many V-components. For each V-component which meets $\sigma(H)$, choose a point of H whose image under σ lies in the intersection; these points form a finite subset F_0 of H. If one adds to $\sigma(F_0)$ a point from each V-component that does not meet $\sigma(H)$, one obtains a set $F_1\subseteqq B_1$, and there is an integer $p\geqq 1$ such that $V^p[F_1]=B_1$, by (1.4). Thus $V^p[\sigma(F_0)]\supseteqq\sigma(H)$, for the V-components of the other points of F_1 do not meet $\sigma(H)$.

Say that a pair (α, β) of integers between 1 and n, where n is as in (5), is a "good pair" if

(6) $$W^3\tau(z_\alpha)\cap\tau(z_\beta)H \neq \emptyset.$$

If (α, β) is good, so is (β, α), and I choose

(7) $$h(\alpha, \beta) = h(\beta, \alpha)^{-1}\in H\cap\tau(z_\beta)^{-1}W^3\tau(z_\alpha).$$

Let F be the finite symmetric subset of H consisting of all products with p or fewer terms of elements of the form $h(\alpha, \beta)$. (In fact much of F is redundant, as will appear.)

Suppose that $a\in H$. Then there exists a sequence $a=x_0, x_1, \dots, x_p$ in B_0 such that

(8) $$x_p\in F_0K \quad\text{and, for}\quad 0 \leqq i < p, \quad x_i x_{i+1}^{-1}\in W.$$

(This is merely the statement that $V^p[\sigma(F_0)]\supseteqq\sigma(H)$.) For each i, $0\leqq i\leqq p$, there exists $j(i)$, $1\leqq j(i)\leqq n$, such that $x_i\in W\tau(z_{j(i)})H$ (by (5)), and one may assume $j(0)$, $j(p)$ are both 1, since x_0, x_p both lie in H. From (8)

$$\tau(z_{j(i)})\in W x_i H \subseteqq W^2 x_{i+1} H \subseteqq W^3\tau(z_{j(i+1)})H,$$

so that $(j(i), j(i+1))$ is a good pair (see (6)).

Let $\eta_i=h(j(i+1), j(i))$, so that

(9) $$\tau(z_{j(i+1)})\in W^3\tau(z_{j(i)})\eta_i.$$

As $x_i\in W\tau(z_{j(i)})H$, there exists $h_i\in H$ such that

(10) $$x_i\in W\tau(z_{j(i)})h_i.$$

It follows that

(11) $$\begin{aligned} h_i h_{i+1}^{-1}&\in\tau(z_{j(i)})^{-1}Wx_i x_{i+1}^{-1}W\tau(z_{j(i+1)}) \subseteqq \\ &\subseteqq \tau(z_{j(i)})^{-1}WWW\cdot W^3\tau(z_{j(i)})\eta_i, \text{ by (8), (9)} \\ &\subseteqq U_0^{18}\eta_i \subseteqq U\eta_i, \text{ by (4).} \end{aligned}$$

In case (b), one may assume, by (5.4), that $h^{-1}U_0h=U_0$ and $h^{-1}Wh=W$ for all $h\in H$. Modify h_i to $q_i=\varkappa_i h_i$, where $\varkappa_0=e$ and $\varkappa_i=\eta_0\eta_1\dots\eta_{i-1}$ for $1\leqq i\leqq p$.

Then

$$q_i q_{i+1}^{-1} = \varkappa_i h_i h_{i+1}^{-1} \varkappa_{i+1}^{-1} \subseteq \varkappa_i U_0^{18} \eta_i \varkappa_{i+1}^{-1} \text{ by (11)}$$

$$= \varkappa_i U_0^{18} \varkappa_i^{-1} = U_0^{18} \subseteq U.$$

Furthermore, $\tau(z_{j(0)})=e=\tau(z_{j(p)})$ by the construction of the functions j and τ. By (10), therefore, $q_0 x_0^{-1} \in W$ and $q_p \in \varkappa_p W x_p = W \varkappa_p x_p$. In consequence, the sequence $a=x_0, q_0, \ldots, q_p, \varkappa_p x_p$, which I may call $u_0, u_1, \ldots, u_{p+2}$, has the properties that $a=u_0$, $u_i u_{i+1}^{-1} \in U$ for $0 \leqq i \leqq p+1$, $u_i \in H$ for all i, and $u_{p+2} \in FF_0 K$. Since FF_0 is a finite subset of H which does not depend on the choice of a or of the sequence (x_i) in B_0, this proves that H/K is bounded in itself; in fact

$$H = (U \cap H)^{p+2} FF_0 K.$$

In case (a), it was proved that one may assume H/K to be pseudo-connected. So I may take $F_0=\{e\}$; and, since F is finite, there exists $q \geqq 1$ such that $F \subseteq (U \cap H)^q K$. By (11), $h_i h_{i+1}^{-1} \in (U \cap H)^{q+1}$, whilst from (10) $a \in (W \cap H) h_0$ and, by (8) and (10) together, $h_p \in (W \cap H) K$. Consequently $a \in (U \cap H)^{p(q+1)+2} K$. This proves the result, since a was an arbitrary element of H and p and q do not depend on a.

(5.8) In (5.7) (a) and (5.2) (b), local connectedness of a group is used only to prove that the pseudo-component of the identity (which is necessarily a closed normal subgroup, and is the same for both left and right uniformities) is open, and consequently is the V-component of the identity for all sufficiently small vicinities V of the uniformity in question. The same is true in many other circumstances, for instance if the group is dense in a locally connected space.

It should perhaps be emphasized that the subgroup H in (5.7) need not be closed, but must have the subspace topology. The compactness of B is used at (1), (2), and (5). If, as in case (b), the left quotient uniformity on G/H exists, (1) and (2) require only precompactness in that uniformity, whilst (5) uses precompactness in the standard uniformity.

(5.9) THEOREM. *Let G, K, H, π be as in* (4.4), *and let the left uniformity on G be right H-invariant. Then, if B is a compact subset of G/H and $\pi^{-1}(B)$ is bounded in itself in the left quotient uniformity on G/K, H/K is necessarily bounded* (in itself; notice that the left quotient uniformity on H/K exists and coincides with the standard uniformity).

PROOF. The argument for case (b) of (5.7) requires the following modifications. Take U_0 and W to be invariant under conjugation by any element of H, and let $V=(B_1 \times B_1) \cap (\sigma \times \sigma) V_W$. Read in (6) $W^2 \tau(z_\alpha) W$ in place of $W^3 \tau(z_\alpha)$, with the corresponding change in (7). In (8), read $x_i^{-1} x_{i+1}$ in place of $x_i x_{i+1}^{-1}$. Choosing $j(i)$ as before, one finds $\tau(z_{j(i+1)}) \in W^2 \tau(z_{j(i)}) WH$, and so $\tau(z_{j(i+1)}) \in W^2 \tau(z_{j(i)}) W \eta_i$. for $\eta_i = h(j(i+1, j(i))$. (Notice that $\eta_i W = W \eta_i$.) With h_i' as in (10),

$$x_i^{-1} x_{i+1} \in W \cap h_i'^{-1} \tau(z_{j(i)})^{-1} WW \tau(z_{j(i+1)}) h_{i+1}',$$

and so

$$Wh'_i h'^{-1}_{i+1} \cap \tau(z_{j(i)})^{-1} W^2 \tau(z_{j(i+1)}) \neq \emptyset, \text{ as } Wh'_i = h'_i W.$$

Hence $h'_i h'^{-1}_{i+1} \in W\tau(z_{j(i)})^{-1} W^2 \cdot W^2 \tau(z_{j(i)}) W\eta_i \subseteq U_0^{14}\eta_i$, using (4), and the rest of the proof follows as before. This proves H/K bounded in the standard uniformity, but, as already remarked, both uniformities coincide on H.

§ 6. Representations and geometry

(6.1) Let E be a normed space, with completion $\tilde{E}$. The group of bounded linear operators in E which have bounded inverses is denoted by $GL(E)$. (Thus $GL(E)$ embeds in $GL(\tilde{E})$.) By a representation of the topological group G on E, I mean a homomorphism $\varphi: G \to GL(E)$. If φ is continuous with respect to the weak operator topology on $GL(E)$ (which is not a topological group in that topology, unless E is finite-dimensional), call it a weakly continuous representation; if it is continuous with respect to the operator-norm topology on $GL(E)$, call it a continuous representation. Say that φ is bounded on the subset A of G if there is a constant M such that

(1) $$(\forall x \in A) \quad \|\varphi(x)\| \leqq M;$$

when φ is bounded on the whole of G, describe it as a bounded representation, and call it locally bounded if it is bounded on some neighbourhood U of e in G: for some constant N, and all $x \in U$,

(2) $$\|\varphi(x)\| \leqq N.$$

(6.2) THEOREM. (a) *A continuous representation of G on E is locally bounded.*

(b) *If G is metrizable and E complete, a weakly continuous representation of G on E is locally bounded.*

(c) *If H is a subgroup bounded in G (see (3.3)), then any locally bounded representation of G on E is bounded on H.*

(d) *If the representation φ of G on E is bounded on the subgroup H, there exists a norm on E (defining the correct topology) such that $\varphi(H)$ consists of isometries.*

PROOF. (a) is trivial. For (b), let (U_n), $n=1, 2, \ldots$, be a decreasing base of neighbourhoods of e in G. For positive integers n and M, set

$$C(n, M) = \{(\xi, \lambda) \in E \times E' : (\forall x \in U_n) \ |\lambda(\varphi(x)\xi)| \leqq M\}.$$

It is clear that $C(n, M)$ is closed in $E \times E'$ in the norm topology, and weak continuity of the representation implies

$$\bigcup_{n,M} C(n, M) = E \times E'.$$

Consequently $C(p, N)$ has an interior point (ξ_0, λ_0) in $E \times E'$, for some positive integers p and N. But all four points $(\pm\xi_0, \pm\lambda_0)$ must also be interior to $C(p, N)$, and it follows that $(0, 0)$ is interior to $C(p, 4N)$. From the Hahn—Banach theorem, one infers that φ is bounded on U_p.

For (c), suppose U, N are as in (6.1) (2). There are a finite subset F of G and an integer $m \geqq 1$ such that $H \subseteq FU^m$ (see (3.4) (1)). Given $x \in H$, there exist $f_0 \in F$

and $u_1, u_2, \ldots, u_m \in U$ such that $x = f_0 u_1 u_2 \ldots u_m$; so

$$\|\varphi(x)\| \leqq \|\varphi(f_0)\| \, \|\varphi(u_1)\| \, \|\varphi(u_2)\| \ldots \|\varphi(u_m)\| \leqq N^m \max_{f \in F} \|\varphi(f)\|,$$

which is a number independent of x, as required.

For (d), let M be as in (6.1) (1), and set for $\xi \in E$

$$|||\xi||| = \sup_{h \in H} \|\varphi(h)\,\xi\| \leqq M\|\xi\|.$$

Since $||| \; |||$ is evidently a norm on E not less than $\| \; \|$, the above inequality shows it is equivalent to $\| \; \|$; and it is clearly invariant under $\varphi(x)$ for any $x \in H$.

(6.3) By a Finsler manifold, I shall mean a manifold X of differentiability class at least $\mathbf{C}^1$, modelled on a normed space E, and furnished with a Finsler structure $\| \; \|$ (in the sense of [6]). The Finsler structure induces a metric ϱ on each individual component of X. By the Finsler uniformity $\mathscr{V}$ of X, I understand the uniformity of the disjoint union of the components $\{X_\gamma : \gamma \in \Gamma\}$, when each X_γ has the metric uniformity $\mathscr{V}_\gamma$ induced by ϱ. That is,

$$\mathscr{V} = \{V : V \subseteqq X \times X \text{ and } (\forall \gamma \in \Gamma) \; (i_\gamma \times i_\gamma)^{-1} V \in \mathscr{V}_\gamma\},$$

where $i_\gamma : X_\gamma \to X$ is the inclusion, and $\mathscr{V}_\gamma$ is the uniformity on X_γ generated by a base consisting of sets of the form

$$\mathscr{V}_{\varepsilon,\gamma} = \{(x, y) \in X_\gamma \times X_\gamma : \varrho(x, y) < \varepsilon\}$$

for arbitrary positive ε. These notations will be used in (6.4) and (6.5) below.

(When Γ is infinite, $\mathscr{V}$ is strictly finer than the uniformity given by extending ϱ to a metric $\tilde{\varrho}$ on X in any manner whatever. Provided that $\tilde{\varrho}$ satisfies the natural condition that the distances $\tilde{\varrho}\,(x, y)$ between points x and y in different components of X have a positive lower bound, the results below still hold for such metric uniformities.)

(6.4) Lemma. *There exists $V_0 \in \mathscr{V}$ such that, for any $V \in \mathscr{V}$ with $V = V^{-1} \subseteqq V_0$, the V-components of X are the same as the connected components.*

Proof. Take $V_0 = \bigcup_{\gamma \in \Gamma} (X_\gamma \times X_\gamma)$. If $x \in X_\gamma$ and $V_0 \supseteqq V \in \mathscr{V}$, then $V^m[x] \subseteqq$ $\subseteqq V_0^m[x] \subseteqq X_\gamma$ for any $m \geqq 1$. Now apply (1.3) (b).

(6.5) Proposition. *A subset A of the Finsler manifold X is bounded in X in the Finsler uniformity if and only if it meets only finitely many components of X and its intersection with each component has finite ϱ-diameter.*

Proof. Suppose A is $\mathscr{V}$-bounded. By (1.4) and (6.4), it meets only finitely many components of X. Let $v \in A \cap X_\gamma$ and $\varepsilon > 0$; by (1.4), there exists $m \geqq 1$ such that $A \cap X_\gamma \subseteqq V_{\varepsilon,\gamma}^m[v]$. By the triangle inequality, this implies that, for any $x \in A \cap X_\gamma$, $\varrho(x, v) < m\varepsilon$.

For the converse, it evidently suffices to assume A is included in X_γ and of finite ϱ-diameter R. Take $v \in A$. Then, for any $x \in A$, there exists a $\mathbf{C}^1$ path joining v to x in X_γ, of length less than $R+1$. Given $\varepsilon > 0$, let $N(\varepsilon) = N$ be the least integer not less than $(R+1)/\varepsilon$. Choose successive points $p_0 = v, p_1, p_2, \ldots, p_N = x$ on this path so that the length of the path-segment between p_i and p_{i+1}, for $1 \leqq i < N$, is less than $(R+1)/N$. Thus $\varrho(p_i, p_{i+1}) < \varepsilon$ for each i, so that $(p_i, p_{i+1}) \in V_{\varepsilon,\gamma}$ and $x \in V_{\varepsilon,\gamma}^N[v]$. The choice of N was independent of x, and so $A \subseteq V_{\varepsilon,\gamma}^N[v]$. Finally, for any $V \in \mathscr{V}$ there exists $\varepsilon > 0$ such that $V_{\varepsilon,\gamma} \subseteq V$ (see (6.3)), and then $A \subseteq V^{N(\varepsilon)}[v]$. Hence A is $\mathscr{V}$-bounded in X.

(6.6) NOTE. If Y is a ($\mathbf{C}^1$) submanifold of the Finsler manifold X, it carries an induced Finsler structure. Thus it carries two induced uniformities: its own Finsler uniformity, and the subspace uniformity derived from the Finsler uniformity on X, which is usually strictly coarser. When Y is bounded in itself as a subset of X, it need not be bounded in its own Finsler uniformity.

(6.7) Now let $\mathfrak{G}$ be a Banach Lie group, with Lie algebra $\mathfrak{g}$; let H be any subgroup. There is a one-one correspondence between norms on $\mathfrak{g}$ which define the correct topology and left-invariant Finsler structures on $\mathfrak{G}$ (see § 2 of [2]).

THEOREM. *Let the subgroup H be bounded in $\mathfrak{G}$. Then there exists a left-invariant Finsler structure on $\mathfrak{G}$ which is right-invariant by H.*

PROOF. By (6.2) (d), there exists a norm on $\mathfrak{g}$ which is invariant under the adjoint representation of H. The corresponding left-invariant Finsler structure on $\mathfrak{G}$ is also right H-invariant, by a trivial verification.

(6.8) REMARKS. (a) Any left-invariant Finsler structure on $\mathfrak{G}$ defines a Finsler uniformity (see (6.3)) which coincides with the left uniformity of $\mathfrak{G}$. Thus (5.3) and (5.9) apply in the circumstances of (6.7).

(b) In particular, on a bounded Banach Lie group the left and right uniformities coincide. This superficially remarkable statement is not true for a general bounded topological group. Let D be the group of orientation-preserving self-homeomorphisms of the closed unit interval in the compact-open topology. The left and right uniformities of D do not coincide (see [5], p. 212, for a discussion and for original references). But D is bounded. Let U_n be the basic neighbourhood of e in D

$$\{f \in D\colon (\forall t \in [0, 1])\ |f(t)-t| < 1/n\},$$

and suppose $g \in D$. Define, for $0 \leqq i \leqq n$, and $0 \leqq t \leqq 1$,

$$g_i(t) = (1-i/n)t + (i/n)g(t).$$

Then $g_{i+1}g_i^{-1} \in U_n$ for $0 \leqq i < n$, and so $D = U_n^n$.

(c) The assertion (b) may be generalised; compare the remark of Weil [8] on right-handed completion, cited on p. 212 of [5]. Say that a mapping $h\colon (X, \mathscr{V}) \to (Y, \mathscr{W})$ of uniform spaces is uniformly continuous over the subset A of X if, for every $W \in \mathscr{W}$, there exists $V \in \mathscr{V}$ such that

$$(h \times h)^{-1} W \supseteq V \cap (A \times X).$$

Then one has the following generalisation of (b).

(6.9) PROPOSITION. *Let H be a subgroup bounded in the topological group G; suppose there is a neighbourhood U of e in G such that inversion is uniformly con-*

tinuous over $U \cap H$ with respect to the left uniformity of G. Then the left uniformity of G is right H-invariant.

PROOF. Take any neighbourhood U_1 of e in G. Then, by the uniform continuity, there is a neighbourhood U_2 of e in G such that, for any $h \in U \cap H$,

$$(hU_2)^{-1} = U_2^{-1}h \subseteq hU_1;$$

consequently $\bigcap_{h \in U \cap H} hU_1h^{-1}$ is a neighbourhood of e in G. Since there exist a positive integer m and a finite subset F of H such that $H = F(U \cap H)^m$ (see (3.4) (1)), a finite induction now yields the result — recall (5.4).

NOTE. It is evidently immaterial which uniformity one considers, left or right, when defining uniform continuity of inversion over a symmetric set in G. Also, it suffices for (6.9) that there be a neighbourhood W of e in G such that inversion is uniformly continuous with respect to the subspace uniformity on $W \cap H$ induced from the left uniformity on G and the left uniformity on G itself; the proof first shows the existence of such a U as is required in (6.9).

References

[1] R. Arens, Topologies for homeomorphism groups, *American J. Math.*, **68** (1946), 593—610.

[2] C. J. Atkin, The Finsler geometry of groups of isometries of Hilbert space, *J. Austral. Math. Soc. (Ser. A)*, **42** (1987), 196—222.

[3] W. Banaszczyk, On the existence of exotic Banach Lie groups, *Math. Ann.*, **264** (1983), 485—493.

[4] R. Engelking, *Topologia Ogólna*, PWN (Warsaw, 1976).

[5] J. Kelley, *General Topology*, Van Nostrand (Princeton, N. J., 1955).

[6] R. S. Palais, Lusternik—Schnirelman theory on Banach manifolds, *Topology*, **5** (1966), 115—133.

[7] H. H. Schaefer, *Topological Vector Spaces* (Graduate Texts in Mathematics 3), Springer-Verlag (New York—Heidelberg—Berlin, 1971).

[8] A. Weil, *L'Intégration dans les Groupes Topologiques et ses Applications*, Actualités Scientifiques et Industrielles **551**, Hermann (Paris, 1940).

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DEPARTMENT OF MATHEMATICS
VICTORIA UNIVERSITY OF WELLINGTON
WELLINGTON, NEW ZEALAND

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CONVERGENCE OF FOURIER SERIES OF A FUNCTION ON GENERALIZED WIENER'S CLASS $BV(p(n)\uparrow\infty)$

H. KITA (Oita)

§ 1. Introduction

Let f be a function defined on $(-\infty, \infty)$ with period 2π. Δ is said to be a partition with period 2π, if

$$\Delta:\ \dots t_{-1} < t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} < \dots$$

satisfies $t_{k+m} = t_k + 2\pi$ for $k = 0, \pm 1, \pm 2, \dots$, where m is a positive integer. We shall generalize the concept of bounded variation.

DEFINITION 1.1. When $1 \leqq p(n)\uparrow p$ as $n \to +\infty$, where $1 \leqq p \leqq +\infty$, f is said to be a function of $BV(p(n)\uparrow p)$ if and only if

$$V\,(f;\, p(n)\uparrow p) = \sup_{n\geqq 1} \sup \Big\{\Big(\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p(n)}\Big)^{1/p(n)}:\ \varrho(\Delta) \geqq 2\pi/2^n\Big\} < +\infty,$$

where $\varrho(\Delta) = \inf_k |t_k - t_{k-1}|$.

When $p(n) = p$ for all n, $BV(p(n)\uparrow p)$ coincides with BV_p which is the Wiener's class of bounded p-variation [5]. When $p = +\infty$, the space $BV(p(n)\uparrow\infty)$ plays an important role for the uniform convergence and quasi-uniform convergence of the Fourier series.

In [3], some fundamental properties of the space $BV(p(n)\uparrow\infty)$ and the inclusion relations between Chanturiya's class $V[v]$ and our class are given.

In this paper we consider the uniform convergence and the quasi-uniform convergence of the Fourier series of f in $BV(p(n)\uparrow\infty)$.

Let $S_n(f; x)$ denote the n-th partial sum of the Fourier series of f at the point x. Wiener [5] proved that if f is a function of bounded p-variation, that is, $f \in BV_p$ $(1 \leqq p < +\infty)$, then $\lim_{n\to\infty} S_n(f; x) = f(x)$ almost everywhere in $[0, 2\pi]$, and that in particular if $f \in BV_2$, then

(1.1) $$\lim_{n\to\infty} S_n(f;\, x) = (1/2)\,\{f(x+0) + f(x-0)\} \text{ at every } x \in [0, 2\pi].$$

Siddiqi [4] proved that if $f \in BV_p$ $(1 \leqq p < +\infty)$, then (1.1) holds.

Wiener [5] showed that functions of the class BV_p could only have simple discontinuities. It is proved in [3] that for the class $BV(p(n)\uparrow\infty)$ there exists a function which has a discontinuity not of the first kind. We also consider the control function of (1.1) which introduced in [6] by Yoneda.

In [3] we have proved that

$$\bigcup_{1\leq p<+\infty} BV_p \subsetneqq BV(p(n)\uparrow\infty) \subseteqq B[0, 2\pi]$$

and $BV(p(n)\uparrow\infty)=B[0, 2\pi]$ if and only if the sequence $\{p(n);\ n\geqq 1\}$ satisfies the condition $\sup\{n/p(n):\ n\geqq 1\}<+\infty$. ($B[0, 2\pi]$ denotes the space of real valued functions f with period 2π such that $\|f\|_B=\sup\{|f(x)|:\ x\in[0, 2\pi]\}<+\infty$.)

In Section 2 we consider the uniform convergence of the Fourier series of functions in $BV(p(n)\uparrow\infty)$. In Section 3 we give a concept of quasi-uniform convergence. In Section 4 we show that there exists a function $f\in BV(p(n)\uparrow\infty)$ and $f\notin V[v]$ such that the Fourier series of f is quasi-uniformly convergent.

§ 2. Uniform convergence of functions in $BV(p(n)\uparrow\infty)$

Let f be a real valued continuous function with period 2π, and $\omega(f;\delta)$ $(\delta>0)$ be the usual modulus of continuity of a function f in the class $C(0, 2\pi)$. It is well-known (see [1] p. 310) that if f is continuous and $f\in BV_p$ for some p $(1\leqq p<+\infty)$, then the Fourier series of f converges uniformly in $[0, 2\pi]$. When $f\in BV(p(n)\uparrow\infty)$, we have the following theorem.

THEOREM 2.1. *Let f be a function in the class $BV(p(n)\uparrow\infty)$. If $p(2n)\leqq Cp(n)$ for all $n\geqq 1$, where $C>0$ is a constant and*

$$\omega(f;\pi/n) = o\big(1/p([\log n])\log p([\log n])\big) \quad as \quad n\to+\infty, \tag{2.1}$$

then the Fourier series of f converges uniformly in $[0, 2\pi]$.

PROOF. Put $f_x(t)=f(x+t)+f(x-t)-2f(x)$ for real x, t. Since f is continuous on $[0, 2\pi]$,

$$\lim_{t\to 0} f_x(t) = 0 \text{ uniformly on } [0, 2\pi]. \tag{2.2}$$

Then, it follows that

$$S_n(f;\ x)-f(x) = \pi^{-1}\int_0^{\pi}\frac{\sin nt}{t}f_x(t)\,dt+o(1) = \pi^{-1}\sum_{j=1}^{n}\int_{(j-1)\pi/n}^{j\pi/n}\frac{\sin nt}{t}f_x(t)\,dt+o(1),$$

where $o(1)$ is a magnitude which tends to zero uniformly.

As is described in [4], by change of variable the above expression can be written by (2.2) as follows:

$$S_n(f;\ x)-f(x) = \pi^{-1}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\left\{\frac{f_x(t+2j\pi/n)}{t+2j\pi/n}-\right. \tag{2.3}$$

$$\left.-\frac{f_x(t+(2j+1)\pi/n)}{t+(2j+1)\pi/n}\right\}\sin nt dt+o(1) =$$

$$= \pi^{-1}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\left\{\frac{f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)}{t+2j\pi/n}\right\}\sin nt dt+$$

$$+\pi^{-1}\int_0^{\pi/n}\sum_{j=1}^{[n/2]} f_x(t+(2j+1)\pi/n)\left\{\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j+1)\pi/n}\right\}\sin ntdt+o(1)=$$

$$=I_n^{(1)}(f;\,x)+I_n^{(2)}(f;\,x)+o(1).$$

First we consider

$$(2.4)\qquad |I_n^{(1)}(f;\,x)|\leqq\{n/(2\pi^2)\}\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\frac{\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|}{j}\,dt.$$

For any positive integer n we choose an integer $k(n)$ such that $2^{k(n)-1}\leqq 2n<2^{k(n)}$. Let $\{\varepsilon_n;\,n\geqq 0\}$ be a decreasing sequence of positive numbers such that

$$(2.5)\qquad 1=\varepsilon_0>\varepsilon_1>\varepsilon_2>\ldots>\varepsilon_n>\ldots>0\quad\text{and}\quad\varepsilon_n\downarrow 0\quad\text{as}\quad n\to+\infty,$$

which we will select later. Set $s_n=p(k(n))/\varepsilon_n$ and $1/s_n+1/t_n=1$. If we apply Hölder's inequality on the sum of the integrand of (2.4), we obtain from the fact $\pi/n\geqq\geqq 2\pi/2^{k(n)}$ that

$$\sum_{j=1}^{[n/2]}\frac{\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|}{j}=$$

$$=\sum_{j=1}^{[n/2]}\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|^{\varepsilon_n}\frac{\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|^{1-\varepsilon_n}}{j}\leqq$$

$$\leqq\Big\{\sum_{j=1}^{[n/2]}\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|^{p(k(n))}\Big\}^{\varepsilon_n/p(k(n))}\times$$

$$\times\left\{\sum_{j=1}^{[n/2]}\frac{\left|f_x(t+2j\pi/n)-f_x(t+(2j+1)\pi/n)\right|^{(1-\varepsilon_n)t_n}}{j^{t_n}}\right\}^{1/t_n}\leqq$$

$$\leqq\{V(f_x;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\{\omega(f_x;\,\pi/n)\}^{1-\varepsilon_n}\Big\{\sum_{j=1}^{n}(1/j)^{t_n}\Big\}^{1/t_n}\leqq$$

$$\leqq\{2V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\{2\omega(f;\,\pi/n)\}^{1-\varepsilon_n}\Big\{\sum_{j=1}^{n}(1/j)^{t_n}\Big\}^{1/t_n}.$$

Now consider the sum

$$\sum_{j=1}^{n}(1/j)^{t_n}\leqq 1+\int_1^n(1/x)^{t_n}\,dx\leqq 1+1/(t_n-1)=s_n=p(k(n))/\varepsilon_n.$$

Hence we obtain from (2.4)

$$|I_n^{(1)}(f;\,x)|\leqq(1/\pi)\{V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\{\omega(f;\,\pi/n)\}^{1-\varepsilon_n}p(k(n))/\varepsilon_n.$$

Since $2^{k(n)}\leqq 4n$, $k(n)\leqq 4[\log n]$ for $n\geqq e^3$. So we have $p(k(n))\leqq C_1p([\log n])$ and

$$(2.6)\qquad |I_n^{(1)}(f;\,x)|\leqq(C_1/\pi)\{V(f;p(n)\uparrow\infty)\}^{\varepsilon_n}\{\omega(f;\,\pi/n)\}^{1-\varepsilon_n}p([\log n])/\varepsilon_n.$$

$C, C_1, C_2, \ldots$ will denote positive constants not necessarily the same at each occurrence.

Put $h(x)=\{\log(1/x)\}/(1-x)$ for $0<x<1$. Then h is a decreasing function on the interval $(0,1)$ and $\lim\limits_{x\to 1-0}h(x)=1$, $\lim\limits_{x\to+0}h(x)=+\infty$. Now take a sequence

$\{\varepsilon_n;\ n\geqq 0\}$ defined in (2.5) as follows: $h(\varepsilon_n)=\log\log p([\log n])$ for sufficiently large n. Then it follows that

$$\varepsilon_n(\log p([\log n]))^{1-\varepsilon_n}=1. \tag{2.7}$$

Therefore we get from (2.1), (2.6) and (2.7)

$$|I_n^{(1)}(f;\,x)|\leqq o(1)\{V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\{1/p([\log n])\,(\log p([\log n]))\}^{1-\varepsilon_n}p([\log n])/\varepsilon_n=$$

$$=o(1)\{V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\frac{\{p([\log n])\}^{\varepsilon_n}}{\varepsilon_n\{\log p([\log n])\}^{1-\varepsilon_n}}=$$

$$=o(1)\{V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}\{p([\log n])\}^{\varepsilon_n}.$$

From (2.7), we get

$$\varepsilon_n\log p([\log n])=\{1/(\log p([\log n]))^{1-\varepsilon_n}\}\log p([\log n])=$$

$$=\{\log p([\log n])\}^{\varepsilon_n}=\exp\{\varepsilon_n\log\log p([\log n])\}=$$

$$=\exp\{(\log\log p([\log n]))/(\log p([\log n]))^{1-\varepsilon_n}\}.$$

Hence the sequence $\{p([\log n])^{\varepsilon_n}\}$ is bounded, and therefore we obtain

$$|I_n^{(1)}(f;\,x)|\leqq o(1)\{V(f;\,p(n)\uparrow\infty)\}^{\varepsilon_n}. \tag{2.8}$$

We will now estimate $|I_n^{(2)}(f;\,x)|$. Let $\varepsilon>0$ be any positive number such that $0<\varepsilon<1/2$. Then we get

$$|I_n^{(2)}(f;\,x)|\leqq(1/\pi)\int_0^{\pi/n}\sum_{j=1}^{[n/2]}|f_x(t+(2j+1)\pi/n)|\cdot\left|\frac{1}{t+2j\pi/n}-\frac{1}{t+(2j+1)\pi/n}\right|dt=$$

$$=(1/n)\int_0^{\pi/n}\sum_{j=1}^{[n/2]}\frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi/n)(t+(2j+1)\pi/n)}\,dt=$$

$$=(1/n)\int_0^{\pi/n}\sum_{j=1}^{[\varepsilon n]}\frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi/n)(t+(2j+1)\pi/n)}\,dt+$$

$$+(1/n)\int_0^{\pi/n}\sum_{j=[\varepsilon n]+1}^{[n/2]}\frac{|f_x(t+(2j+1)\pi/n)|}{(t+2j\pi/n)(t+(2j+1)\pi/n)}\,dt=J_{n,\varepsilon}^{(1)}(f;\,x)+J_{n,\varepsilon}^{(2)}(f;\,x).$$

Let $n>1/\varepsilon$, then it follows that

$$|J_{n,\varepsilon}^{(1)}(f;\,x)|\leqq(1/n)\int_0^{\pi/n}\sum_{j=1}^{[\varepsilon n]}\frac{|f_x(t+(2j+1)\pi/n)|}{(2j\pi/n)^2}\,dt\leqq$$

$$\leqq\sup\{|f_x(s)|:\ 0\leqq s\leqq 4\varepsilon\pi\}(1/4\pi)\sum_{j=1}^{\infty}(1/j)^2\leqq C_2\,\omega(f;\,4\varepsilon\pi).$$

Next we will estimate $|J^{(2)}_{n,\varepsilon}(f;x)|$.

$$|J^{(2)}_{n,\varepsilon}(f;x)| \leqq (1/n)\int_0^{\pi/n} \sum_{j=[\varepsilon n]+1}^{[n/2]} \frac{4\|f\|_B}{(2j\pi/n)^2}\,dt =$$

$$= (\|f\|_B/\pi) \sum_{j=[\varepsilon n]+1}^{\infty} (1/j)^2 \leqq (\|f\|_B/\pi)(1/[\varepsilon n]).$$

Therefore we get

$$|I^{(2)}_n(f;x)| \leqq C_2\,\omega(f;4\varepsilon\pi)+(\|f\|_B/\pi)(1/[\varepsilon n]). \tag{2.9}$$

Hence from (2.3), (2.8) and (2.9), we obtain

$$|S_n(f;x)-f(x)| \leqq o(1)\{V(f;p(n)\uparrow\infty)\}^{\varepsilon_n}+C_2\,\omega(f;4\varepsilon\pi)+(\|f\|_B/\pi)(1/[\varepsilon n])+o(1).$$

Taking limits as $n\to\infty$, we get $\limsup_{n\to\infty}|S_n(f;x)-f(x)|\leqq C_2\omega(f;4\varepsilon\pi)$, where $\varepsilon>0$ is an arbitrary positive number. This completes the proof of Theorem 2.1.

§ 3. Quasi-uniform convergence

Let φ be a function defined on $[0,\infty)$ satisfying the following properties:

$$\begin{cases}\varphi(0)=0,\ \varphi(t)>0 \quad \text{if} \quad t>0,\\ \varphi(t)\uparrow+\infty \quad \text{as} \quad t\to+\infty,\\ \varphi(t) \quad \text{is continuous on} \quad [0,\infty).\end{cases} \tag{3.1}$$

Denote by Φ the set of all functions φ satisfying (3.1).

Yoneda [7] introduced a concept of quasi-uniform convergence. When $\{f_n;\ n=1,2,3,\ldots\}$ is a sequence of real valued functions defined on the closed interval $[a,b]$ and

$$\lim_{n\to\infty} f_n(x)=f(x) \quad \text{a.e. on} \quad [a,b], \tag{3.2}$$

then there exists a positive and a.e. finite valued function δ such that for every $\varepsilon>0$ there exists a positive integer $n(\varepsilon)$ satisfying $|f_n(x)-f(x)|<\varepsilon\delta(x)$ everywhere for all $n>n(\varepsilon)$. The function δ is termed a control function of the a.e. convergence (3.2).

DEFINITION 2.1. When δ is a control function of (3.2) and $\delta(x)>\alpha$ for some $\alpha>0$, if $\varphi(\delta(x))$ is also a control function of (3.2) for each $\varphi\in\Phi$, then we say that (3.2) is quasi-uniformly convergent.

Yoneda [7] proved that if $f\in BV_2$, then the Fourier series of f is quasi-uniformly convergent. We have the following theorems.

THEOREM 3.1. *If $f_n-f\in L^\infty[a,b]$ for $n\geqq 1$, then the following statements are equivalent:*

(1) *(3.2) is quasi-uniformly convergent,*

(2) *the majorant function $M(x)=\sup\{|f_n(x)-f(x)|: n\geqq 1\}$ is essentially bounded,*

(3) *for each $\varphi\in\Phi$, there exists a control function $\delta=\delta_\varphi$ of (3.2) such that*

$$\text{meas}\,\{x\in[a,b] \mid \delta(x)>t\} \leqq C/\varphi(t) \quad \textit{for all} \quad t>0, \tag{3.3}$$

where C is a constant.

THEOREM 3.2. *Let $f \in BV(p(n)\uparrow\infty)$. If*

$$\omega(f; \pi/n) = O(1/p([\log n]) \log p([\log n])), \tag{3.4}$$

then the convergence of the Fourier series of f is quasi-uniformly convergent.

PROOF OF THEOREM 3.1. The equivalence of (1) and (2) was proved in [7]. We prove that (2) and (3) are equivalent.

First suppose that $\|M\|_\infty < +\infty$ and $\varphi \in \Phi$. We shall construct a control function $\delta = \delta_\varphi$ of (3.2) which satisfies the condition (3.3). Set

$$f_n^*(x) = \max\{1/n, \sup_{k \geqq n} \{|f_k(x) - f(x)|\}\} \quad \text{for} \quad n = 1, 2, 3, \ldots, \tag{3.5}$$

then it follows that

$$\|M\|_\infty + 1 \geqq M(x) + 1 \geqq f_1^*(x) \geqq f_2^*(x) \geqq \ldots \geqq f_n^*(x) \geqq \ldots \geqq 0 \quad \text{a.e.}$$

and $f_n^*(x) \downarrow 0$ a.e. on $[a, b]$. Let $\{\varepsilon_n;\ n = 0, 1, 2, \ldots\}$ be a decreasing sequence of positive numbers such that

$$\varepsilon_0 = 1 \geqq \varepsilon_1 \geqq \varepsilon_2 \geqq \ldots \geqq \varepsilon_k \geqq \ldots > 0 \quad \text{and} \quad \varepsilon_k \downarrow 0 \quad \text{as} \quad k \to +\infty. \tag{3.6}$$

By the Egoroff's theorem we can choose a sequence of measurable sets $\{E_k;\ k = 0, 1, 2, \ldots\}$ such that

$$E_1 \subseteqq E_2 \subseteqq \ldots \subseteqq E_k \subseteqq \ldots, \quad E_k \subseteqq [a, b], \tag{3.7}$$

$$\operatorname{meas}(E_k^c) \leqq \{(b-a)\varphi(\|M\|_\infty + 1)\}/\{2^k \varphi((\|M\|_\infty + 1)/\varepsilon_k)\} \quad \text{for} \quad k = 1, 2, \ldots, \tag{3.8}$$

$$\lim_{n \to \infty} f_n^*(x) = 0 \quad \text{uniformly on} \quad E_k, \quad \text{for} \quad k = 1, 2, 3, \ldots. \tag{3.9}$$

From (3.9) there exists a monotone increasing sequence of positive integers $\{n_k;\ k = 0, 1, 2, \ldots\}$, $n_0 = 1$, such that for $k = 1, 2, \ldots$

$$\max\{(1/\varepsilon_l) f_{n_l}^*(x):\ l = 0, 1, 2, \ldots, k-1\} \geqq (1/\varepsilon_k) f_{n_k}^*(x) \quad \text{on } E_k. \tag{3.10}$$

Define a sequence of functions $\{\delta_k;\ k = 1, 2, 3, \ldots\}$ and δ by the following way:

$$\begin{cases} \delta_k(x) = \max\{(1/\varepsilon_l) f_{n_l}^*(x):\ l = 0, 1, 2, \ldots, k-1\} \quad \text{for} \quad k = 1, 2, 3, \ldots, \\ \delta(x) = \begin{cases} \sup\{\delta_k(x):\ k \geqq 1\}, & \text{if} \quad x \in \bigcup_{k=1}^{\infty} E_k, \\ +\infty, & \text{if} \quad x \in [a, b] \bigcup_{k=1}^{\infty} E_k. \end{cases} \end{cases} \tag{3.11}$$

From (3.10) and (3.11), it follows that $\delta_k(x) \geqq (1/\varepsilon_k) f_{n_k}^*(x)$ on E_k. Hence we get $\delta_k(x) = \delta_{k+1}(x)$ and $\delta(x) = \delta_k(x)$ on E_k. Therefore it follows from (3.6) that

$$\begin{aligned} \delta(x) = \delta_k(x) \leqq \max\{(1/\varepsilon_l) f_1^*(x):\ l = 0, 1, 2, \ldots, k-1\} = \\ = (1/\varepsilon_{k-1}) f_1^*(x) \quad \text{on } E_k. \end{aligned} \tag{3.12}$$

Since $\operatorname{meas}\left(\bigcup_{k=1}^{\infty} E_k\right) = b - a$ from (3.8) δ is an a.e. finite valued function.

Now, we prove that the positive function δ constructed above is a control function of (3.2). Let $\varepsilon>0$ be an arbitrary positive number. Then there exists a positive integer $k'=k(\varepsilon)$ depending only on ε such that $\varepsilon>\varepsilon_{k'}>0$. If $x\in\bigcup_{k=1}^{\infty} E_k$, then from (3.5) and (3.11) it follows that

$$\varepsilon\delta(x)>\varepsilon_{k'}\delta(x)\geqq\varepsilon_{k'}\delta_{k'+1}(x)\geqq\varepsilon_{k'}(1/\varepsilon_{k'})f^*_{n_{k'}}(x)=$$
$$=f^*_{n_{k'}}(x)\geqq\sup\{|f_n(x)-f(x)|:\ n\geqq n_{k'}\}.$$

Hence we have that if $n\geqq n_{k'}$, then

$$|f_n(x)-f(x)|<\varepsilon\delta(x)\quad\text{for all }\ x\in[a,b],$$

and δ is a control function of (3.2).

We prove (3.3). For each $t>0$, we get from (3.7) and (3.8)

$$\text{meas}\,\{x\in[a,b]\,|\delta(x)>t\}=\text{meas}\,\{x\in E_1|\,\delta(x)>t\}+$$
$$+\sum_{k=2}^{\infty}\text{meas}\,\{x\in E_k\setminus E_{k-1}|\delta(x)>t\}=J_1+J_2.$$

Since $\delta(x)=\delta_1(x)=f_1^*(x)\leqq M(x)+1$ on E_1,

$$J_1=\text{meas}\,\{x\in E_1|f_1^*(x)>t\}\leqq\text{meas}\,\{x\in[a,b]|f_1^*(x)>t\}\leqq$$
$$\leqq\text{meas}\,\{x\in[a,b]|M(x)+1>t\}.$$

By hypothesis, $\|M\|_\infty<+\infty$. If $t>\|M\|_\infty+1$, then $J_1=0$ holds. If $0<t\leqq\leqq\|M\|_\infty+1$, then $J_1\leqq b-a$ and $\varphi(t)J_1\leqq(b-a)\varphi(\|M\|_\infty+1)$. Therefore we get

$$J_1\leqq\{(b-a)\varphi(\|M\|_\infty+1)\}/\varphi(t)\quad\text{for all }\ t>0. \tag{3.13}$$

On the other hand, from (3.12) we have

$$J_2=\sum_{k=2}^{\infty}\text{meas}\,\{x\in E_k\setminus E_{k-1}|\delta(x)>t\}\leqq\sum_{k=2}^{\infty}\text{meas}\,\{x\in E_k\setminus E_{k-1}|f_1^*(x)>\varepsilon_{k-1}t\}\leqq$$
$$\leqq\sum_{k=2}^{\infty}\text{meas}\,\{x\in E_{k-1}^c|f_1^*(x)>\varepsilon_{k-1}t\}=\sum_{k=1}^{\infty}\text{meas}\,\{x\in E_k^c|f_1^*(x)>\varepsilon_k t\}.$$

Since $0\leqq f_1^*(x)\leqq M(x)+1\leqq\|M\|_\infty+1$ almost everywhere, it follows that

$$\text{meas}\,\{x\in E_k^c|f_1^*(x)>\varepsilon_k t\}=0\quad\text{for each}\quad t>(\|M\|_\infty+1)/\varepsilon_k.$$

And if $0<t\leqq(\|M\|_\infty+1)/\varepsilon_k$, then from (3.8) we get

$$\varphi(t)\,\text{meas}\,\{x\in E_k^c|f_1^*(x)>\varepsilon_k t\}\leqq\varphi(t)\,\text{meas}\,(E_k^c)\leqq\varphi\big((\|M\|_\infty+1)/\varepsilon_k\big)\,\text{meas}\,(E_k^c)\leqq$$
$$\leqq(b-a)\varphi(\|M\|_\infty+1)/2^k.$$

Therefore, it follows that

$$(3.14)\qquad J_2 \leqq (b-a)\,\varphi(\|M\|_\infty+1)\sum_{k=1}^{\infty}\{1/(\varphi(t)\,2^k)\} = (b-a)\,\varphi(\|M\|_\infty+1)/\varphi(t)\quad \text{for}\quad t>0.$$

From (3.13) and (3.14), we get (3.3).

Conversely, we prove (2) from (3). If $\delta=\delta_\varphi$ is a control function of (3.2) such that (3.3) holds, then there exists a positive integer n_0 such that

$$(3.15)\qquad |f_n(x)-f(x)| \leqq \delta(x)\quad \text{everywhere for all } n \geqq n_0.$$

Hence we have from (3.15) that

$$\text{meas}\,\{x\in[a,b]\,|\,M(x)>t\} \leqq \text{meas}\,\{x\in[a,b]\,|\,\sup\{|f_n(x)-f(x)|: 1\leqq n\leqq n_0-1\}>t\}+ +\text{meas}\,\{x\in[a,b]\,|\,\sup\{|f_n(x)-f(x)|: n\geqq n_0\}>t\} \leqq$$

$$\leqq \text{meas}\,\{x\in[a,b]\,|\sup\{|f_n(x)-f(x)|: 1\leqq n\leqq n_0-1\}>t\}+\text{meas}\,\{x\in[a,b]\,|\,\delta(x)>t\}.$$

Put $\alpha(n_0)=\sup\{\|f_n-f\|_\infty: 1\leqq n\leqq n_0-1\}$. Since $\{f_n-f;\ n\geqq 1\}$ is a sequence of bounded functions, $\alpha(n_0)$ is finite. If $t>\alpha(n_0)$, then from (3.3) it follows that

$$\text{meas}\,\{x\in[a,b]\,|\,M(x)>t\} \leqq C/\varphi(t).$$

Further if $0<t\leqq\alpha(n_0)$, then we get

$$\varphi(t)\,\text{meas}\,\{x\in[a,b]\,|\,M(x)>t\} \leqq \varphi(\alpha(n_0))(b-a).$$

Hence it follows that

$$(3.16)\qquad \text{meas}\,\{x\in[a,b]\,|\,M(x)>t\} \leqq C_1/\varphi(t)\quad \text{for all}\quad t>0,$$

where $C_1=\max\{C,\ \varphi(\alpha(n_0))(b-a)\}$.

Finally, we prove from (3.16) that M is a bounded function. Now suppose that

$$\psi(t) = \text{meas}\,\{x\in[a,b]\,|\,M(x)>t\}>0\quad \text{for all}\quad t>0.$$

Since ψ is a decreasing function tending to zero as $t\to+\infty$, there exists a function $\varphi\in\Phi$ such that $1/(\psi(t))^2\leqq\varphi(t)$ for sufficiently large $t>0$. Then we get

$$\varphi(t)\,\text{meas}\,\{x\in[a,b]\,|\,M(x)>t\} = \varphi(t)\psi(t) \geqq 1/\psi(t)\to+\infty\quad \text{as}\quad t\to+\infty.$$

From (3.16) we arrive at a contradiction. This means that there exists a positive number $t_0>0$ such that $\psi(t_0)=0$. Therefore we get $M\in L^\infty[a,b]$. Theorem 3.1 is proved.

PROOF OF THEOREM 3.2. This is done by the same way as in the case of Theorem 3.1. From (3.4) we have

$$\sup\{|S_n(f\colon x)|: n\geqq 0\} \leqq C<+\infty\quad \text{for all}\quad x\in[0,2\pi].$$

By Theorem 3.1, the Fourier series of f is quasi-uniformly convergent.

§ 4. A relation to Chanturiya's class for quasi-uniform convergence

Chanturiya [2] has introduced the concept of the modulus of variation, defined by

$$v(f;\ n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|$$

where Π_n is an arbitrary system of n disjoint intervals $I_k \subseteq [0, 2\pi]$ and $f(I_k) = f(\sup I_k) - f(\inf I_k)$. $V[v]$ denotes the class of functions for which $v(f; n) = O(v(n))$ as $n \to +\infty$. In [3] we proved that $V[v] = B[0, 2\pi]$ if and only if $\varlimsup_{n\to\infty} n/v(n) < +\infty$. Now we have the following theorem.

THEOREM 4.1. *Let* $\varlimsup_{n\to\infty} n/v(n) = +\infty$ *and* $1 < p(n)\uparrow +\infty$. *Then there exists a function* $f \in BV(p(n)\uparrow\infty)$ *and* $f \notin V[v]$ *such that the Fourier series of f is quasi-uniformly convergent.*

PROOF. Let $\{n_k;\ k \geqq 1\}$ be an increasing sequence of positive integers such that

(4.1) $$p(n_{k-1}) \geqq \log k \quad \text{for} \quad k \geqq 2.$$

Put

(4.2) $$\alpha_k = 2\pi/2^{n_k} \quad \text{and} \quad \beta_k = \alpha_k \exp(1/2^k) \quad \text{for} \quad k = 1, 2, 3, \ldots .$$

We shall define a function f constructed on the interval $[-\pi, \pi]$ as follows. Set

$$f(x) = \begin{cases} 1 & \text{if} \quad \alpha_k \leqq x \leqq \beta_k < \pi \quad \text{for} \quad k = 1, 2, 3, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

and extend f to $(-\infty, \infty)$ with period 2π.

First we prove that $f \in BV(p(n)\uparrow\infty)$. Let Δ: $\ldots < t_{-1} < t_0 < t_1 < \ldots < t_m < \ldots$ be any partition with period 2π and $\varrho(\Delta) \geqq 2\pi/2^n$. Then there exists an integer k such that $n_{k-1} < n \leqq n_k$. Then we get $p(n_{k-1}) \leqq p(n) \leqq p(n_k)$. Since $2\pi/2^{n_k} \leqq 2\pi/2^n < 2\pi/2^{n_{k-1}}$, it follows from (4.1) that

$$\Big\{\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^{p(n)}\Big\}^{1/p(n)} \leqq C(2k)^{1/p(n)} \leqq$$

$$\leqq 2Ck^{1/p(n)} \leqq 2Ck^{1/p(n_{k-1})} \leqq 2Ck^{1/\log k} = 2Ce < +\infty.$$

Therefore we have $f \in BV(p(n)\uparrow\infty)$.

Next we shall prove that $f \notin V[v]$. We choose a system of non-overlapping intervals $\{I_k;\ k \geqq 1\}$ as follows:

$$I_k = [a_k, b_k] \quad \text{and} \quad a_k < \alpha_k < b_k < \beta_k \quad \text{for} \quad k = 1, 2, 3, \ldots .$$

Then we get

$$n = \sum_{k=1}^{n} |f(b_k) - f(a_k)| \leqq v(f;\ n).$$

If $f \in V[v]$, it follows that $v(f; n) \leqq Cv(n)$ for all $n \geqq 1$, and we get $\varlimsup_{n\to\infty} n/v(n) < +\infty$. We arrive at a contradiction. This proves that $f \notin V[v]$.

Finally we shall prove that the Fourier series of f is quasi-uniformly convergent. It follows from the definition of f that

$$S_n(f;\, x) = (1/\pi) \int_{-\pi}^{\pi} f(t)\,\frac{\sin n(t-x)}{t-x}\,dt + o(1) = (1/\pi) \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \frac{\sin n(t-x)}{t-x}\,dt + o(1).$$

When $x=0$, we get

$$S_n(f;\, 0) = (1/\pi) \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \frac{\sin nt}{t}\,dt + o(1).$$

Then it is easy to see that

$$|S_n(f;\, 0)| \leqq (1/\pi) \sum_{k=1}^{\infty} |\log \beta_k - \log \alpha_k| + o(1) = (1/\pi) \sum_{k=1}^{\infty} (1/2^k) + o(1) = O(1).$$

When $0<x<2\pi$, there exists an integer $m=m(x)$ such that $\alpha_{m+1}<x\leqq\alpha_m$. Then it follows that

$$\sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \frac{\sin n(t-x)}{t-x}\,dt = \sum_{k=1}^{\infty} \int_{n(\alpha_k-x)}^{n(\beta_k-x)} \frac{\sin t}{t}\,dt =$$

$$= \sum_{k=1}^{m} \int_{n(\alpha_k-x)}^{n(\beta_k-x)} \frac{\sin t}{t}\,dt + \sum_{k=m+1}^{\infty} \int_{n(\alpha_k-x)}^{n(\beta_k-x)} \frac{\sin t}{t}\,dt = T(n, m, x) + U(n, m, x).$$

Now we estimate $|T(n, m, x)|$.

$$|T(n, m, x)| \leqq |T(n, m-1, x)| + \left| \int_{n(\alpha_m-x)}^{n(\beta_m-x)} \frac{\sin t}{t}\,dt \right|.$$

It is well-known (cf. [1] p. 106) that there exists a positive constant $C>0$ such that

$$\left| \int_{a}^{b} \frac{\sin t}{t}\,dt \right| \leqq C \quad \text{for all} \quad a, b \in (-\infty, \infty).$$

Therefore we have

$$|(T(n, m, x)| \leqq |T(n, m-1, x) + C \leqq \sum_{k=1}^{m-1} \int_{n(\alpha_k-x)}^{n(\beta_k-x)} \frac{1}{t}\,dt + C =$$

$$= \sum_{k=1}^{m-1} \log\{(\beta_k - x)/(\alpha_k - x)\} + C = \sum_{k=1}^{m-1} \log\{(\alpha_k - \beta_k)/(x-\alpha_k) + 1\} + C.$$

Since $0<x\leqq\alpha_m<\alpha_{m-1}<\ldots<\alpha_k<\ldots<\alpha_1$ and

$$\alpha_k = 2\pi/2^{n_k} \geqq 2\pi/2^{n_{m-1}} > 2\pi/2^{n_m} = \alpha_m \geqq x,$$

we get $\alpha_k > \alpha_k/2 \geqq x$ for $k=1, 2, 3, \ldots, m-1$. Then it follows that

$$|T(n, m, x)| \leqq \sum_{k=1}^{m-1} \log\{(\alpha_k-\beta_k)/((\alpha_k/2)-\alpha_k)+1\}+C =$$

$$= \sum_{k=1}^{m-1} \log\{2((\beta_k/\alpha_k)-1)+1\}+C \leqq \sum_{k=1}^{m-1} 2\{(\beta_k/\alpha_k)-1\}+C =$$

$$= \sum_{k=1}^{m-1} 2\{\exp(1/2^k)-1\}+C \leqq \sum_{k=1}^{m-1} 2\cdot 3(1/2^k)+C < 6+C < +\infty.$$

Next we consider $|U(n, m, x)|$. We obtain by the same way as the estimate of $|T(n, m, x)|$ that

$$|U(n, m, x)| \leqq \sum_{k=m+1}^{m+2} \left| \int_{n(\alpha_k-x)}^{n(\beta_k-x)} \frac{\sin t}{t}\, dt \right| + |U(n, m+2, x)| \leqq$$

$$\leqq |U(n, m+2, x)| + 2C \leqq \sum_{k=m+3}^{\infty} \log\{(x-\alpha_k)/(x-\beta_k)\}+2C \leqq$$

$$\leqq \sum_{k=m+3}^{\infty} \{\exp(1/2^k)-1\}+2C \leqq 3+2C < +\infty.$$

Therefore we get

$$|S_n(f; x)| \leqq C_1 \quad \text{for all } n \text{ and } x.$$

By Theorem 3.1 the Fourier series of f is quasi-uniformly convergent.

References

[1] N. K. Bary, *A treatise on trigonometric series* I, Pergamon Press (New York, 1964).
[2] Z. A. Chanturiya, The modulus of variation of a function and its application in the theory of Fourier series, *Soviet Math. Dokl.*, **15** (1974), 67—71.
[3] H. Kita and K. Yoneda, A generalization of bounded variation, (to appear in *Acta Math. Hung.*, **56** (1990).
[4] R. N. Siddiqi, A note on convergence of Fourier series of a function on Wiener's class V_p, *Canad. Math. Bull.*, **20** (1977), 243—247.
[5] N. Wiener, The quadratic variation of a function and its Fourier coefficients, *Massachusetts J. Math.*, **3** (1924), 72-94.
[6] K. Yoneda, On control functions of a.e. convergences, *Math. Japon.*, **20** (1975), 101—105.
[7] K. Yoneda, On a.e. convergence of Fourier series, *Math. Japon.*, **30** (1985), 617—633.

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
OITA UNIVERSITY
700 DANNOHARU OITA 870-11
JAPAN

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ON THE IMPROVEMENT OF THE SPEED OF CONVERGENCE OF SOME ITERATIONS CONVERGING TO SOLUTIONS OF QUADRATIC EQUATIONS

I. K. ARGYROS (Las Cruces)

Introduction. Consider the equation

$$x = T(x), \tag{1}$$

where T is a nonlinear operator between two Banach spaces E_1 and E_2 given by

$$T(x) = y + B(x, x). \tag{2}$$

Here $y \in E_1$ is fixed and B is a bounded bilinear operator from $E_1 \times E_1$ to E_2.

A number of very interesting problems appearing in astrophysics [3], [4] and in elasticity theory [1], to mention a few, are special cases of (1). For example, the famous Chandrasekhar equation (Nobel of physics 1983)

$$x(s) = 1 + \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\, dt$$

with $\lambda \in \mathbf{R}$ and $E_1 = C[0, 1]$ is a special case of (1).

Using the contraction mapping many authors have found existence and uniqueness results for a solution x^* of (1), [2], [7].

The results obtained here are applied where the ones already known cannot. Moreover motivated by the work in [6] and the references there, we define the rate of convergence of an iteration of the form

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots \tag{3}$$

as a function and not as a number as it is the case in [2], [7].

In particular we will find a function $\alpha\colon N \to \mathbf{R}_+$ such that

$$\|x_n - x^*\| \leqq \alpha(n), \quad n = 1, 2, \ldots \tag{4}$$

where x^* is a solution of (1).

This allows us to improve the rate of convergence of (3) to a solution x^* of (1).

I. Preliminaries

DEFINITION 1. An operator $B\colon E_1 \times E_1 \to E_2$ sending $(x, y) \in E_1 \times E_1$ to $B(x, y) \in E_2$ is called bilinear if it is linear in each variable separately and symmetric if

$$B(x, y) = B(y, x) \quad \text{for all} \quad x, y \in E_1.$$

DEFINITION 2. Then mean $\overline{B}$ of B on $E_1 \times E_1$ is defined by

$$\overline{B}(x, y) = \frac{1}{2}(B(x, y) + B(y, x)) \quad \text{for all} \quad x, y \in E_1.$$

Note that

$$\overline{B}(x, x) = B(x, x)$$

and

$$\overline{B}(x, y) = \overline{B}(y, x) \quad \text{for all} \quad x, y \in E_1.$$

Therefore the operator B appearing in (2) can always be assumed to be symmetric, otherwise we replace it with the mean of B, $\overline{B}$ which is symmetric.

DEFINITION 3. A bilinear operator B: $E_1 \times E_1 \to E_2$ is said to be bounded if there exists $c > 0$ such that

$$\|B(x, y)\| \leqq c\|x\| \cdot \|y\| \quad \text{for all } (x, y) \in E_1 \times E_1.$$

The quantity $\|B\| = \sup_{\|x\| \leqq 1, \|y\| \leqq 1} \|B(x, y)\|$ is called the norm of B.

From now on we assume that B is bounded symmetric bilinear operator and $E_1 = E_2$.

DEFINITION 4. Let $I = \{r \in \mathbf{R}_+ | r_1 \leqq r < r_2\}$ for some fixed r_1 and r_2. A function ω: $I \to I$ is called a rate of convergence on I if the series

$$\sigma(r) = \sum_{n=0}^{\infty} \omega^{(n)}(r) \tag{5}$$

is convergent for each $r \in I$, where the iterates $\omega^{(n)}$ of ω are defined by $w^{(0)}(r) = r$ and

$$\omega^{(n+1)}(r) = \omega(\omega^{(n)}(r)), \quad n = 0, 1, 2, \ldots.$$

Note that

$$\sigma(\omega(r)) = \sigma(r) - r. \tag{6}$$

II. Main results

We now prove a consequence of the contraction mapping principle for (1).

THEOREM 1. *Let B be a bounded bilinear operator on $E_1 \times E_2$ and suppose y and z belong to E_2. Set*

$$r_2 = \frac{1}{2\|B\|} - \|z\|, \quad r_1 = r_2 - \left[r_2^2 - \frac{\|T(z) - z\|}{\|B\|}\right]^{1/2},$$

and assume r_1 is nonnegative and $r_2 \neq 0$. Then

(i) *T has a unique fixed point in $U(z, r_2) = \{x \in E_1 | \|x - z\| < r_2\}$;*
(ii) *this fixed point actually lies in $\overline{U}(z, r_1)$.*

PROOF. The hypothesis, $r_1 \geqq 0$ and $r_2 \neq 0$, imply that $r_2 > 0$ and

$$r_2^2 - \frac{\|T(z)-z\|}{\|B\|} \geqq 0.$$

Fix r such that $r_1 \leqq r < r_2$.

CLAIM 1. *T is a contraction operator on $\overline{U}(z, r)$.* If $x_1, x_2 \in \overline{U}(z, r)$, then it is routine to show

$$\|T(x_1)-T(x_2)\| = \|B(x_1, x_1)-B(x_2, x_2)\| \leqq 2(r+\|z\|)\|B\| \cdot \|x_1-x_2\|.$$

Set

(7) $$q^* = 2(r+\|z\|) \cdot \|B\|.$$

By hypothesis

$$r < \frac{1}{2\|B\|} - \|z\|$$

so, $0<q<1$ and the claim is proved.

CLAIM 2. *T maps $\overline{U}(z, r)$ into $\overline{U}(z, r)$.* We have

$$\|T(x)-z\| = \|(T(x)-T(z))+(T(z)-z)\| \leqq$$
$$\leqq \|B(x, x)-B(z, z)\| + \|T(z)-z\| \leqq \|B\| r^2 + 2\|B\| \cdot \|z\| \cdot r + \|T(z)-z\|.$$

Define the real quadratic polynomial $g(r)$ by

$$g(r) = \|B\| r^2 + (2\|B\| \cdot \|z\| - 1) r + \|T(z)-z\|.$$

To establish the claim we must show that $g(r) \leqq 0$, for all r, $r_1 \leqq r < r_2$. Now the quadratic function $g(r)$ is convex, with smallest root at r_1 and minimum occurring at r_2. So for $r_1 \leqq r < r_2$,

$$\|B\| \cdot r^2 + 2\|B\| \cdot \|z\| r + \|T(z)-z\| \leqq r.$$

The theorem now follows from the contraction mapping principle [5], [8].

COROLLARY 1. *If* $4\|B\| \cdot \|y\| < 1$ *then*

(i) *the equation* $x = y + B(x, x)$ *has a unique solution* x^* *in* $U(0, \bar{r}_2)$, *where* $\bar{r}_2 = \frac{1}{2\|B\|}$;

(ii) *moreover,* $x^* \in \overline{U}(0, \bar{r}_1)$, *where*

$$\bar{r}_1 = \frac{1-\sqrt{1-4\|B\| \cdot \|y\|}}{2\|B\|}.$$

PROOF. Take $z=0$ in Theorem 1.

We now state Rall's Theorem for comparison. The proof can be found in [7].

THEOREM 2. *If* $4\|B\| \cdot \|y\| < 1$ *then*

(i) *equation* (1) *has a solution* $x^* \in E_1$ *satisfying*

$$\|x^*\| \leqq \frac{\sqrt{1-4\|B\| \cdot \|y\|}}{2\|B\|};$$

(ii) *moreover* x^* *is unique in* $U(x^*, R)$, *where*

$$R = \frac{\sqrt{1-4\|B\|\cdot\|y\|}}{2\|B\|}.$$

Note that Theorem 2 and Corollary 1 provide the same estimate on $\|x^*\|$, but Theorem 2 guarantees uniqueness in $U(x^*, R)$ and not in $\bar{U}(0, r)$.

Corollary 1 is a crude application of Theorem 1. Sometimes it is possible to introduce an auxiliary quadratic equation which is "close to" (1) but easier to handle. In particular, we have the following theorem.

THEOREM 3. *Consider the equation*

$$z = y + F(z, z) \tag{8}$$

where $F\colon E_1 \times E_1 \to E_1$ *is a bounded symmetric bilinear operator and y is fixed in* E_1. *Suppose that there exists a solution z of* (6) *satisfying*

$$\|z\| < [2\sqrt{\|B\|}(\sqrt{\|B-F\|}+\sqrt{\|B\|})]^{-1}. \tag{9}$$

Then

(i) *equation* (1) *has a unique solution* $x^* \in U(z, r_2)$;

(ii) *moreover,* $x^* \in \bar{U}(z, r_3)$, *where*

$$r_3 = \{1-2\|B\|\cdot\|z\|-[(2\|B\|\cdot\|z\|-1)^2-4\|B-F\|\cdot\|B\|\cdot\|z\|^2]^{1/2}\}(2\|B\|)^{-1}$$

and

(iii) *iteration* (3) *converges for any* $x_0 \in \bar{U}(z, r_3)$ *to the solution* x^* *of* (1) *such that*

$$\|x_n - x^*\| \leq \frac{q^n}{1-q}\|x_1 - x_0\|, \quad n = 0, 1, 2, \ldots \tag{10}$$

where

$$q = 1-[(2\|B\|\cdot\|z\|-1)^2-4\|B-F\|\cdot\|z\|^2]^{1/2}.$$

PROOF. We have

$$\|T(z)-z\| = \|(B-F)(z, z)+F(z, z)+y-z\| \leq \tag{11}$$

$$\leq \|(B-F)(z, z)\| + \|F(z, z)+y-z\| \leq \|B-F\|\cdot\|z\|^2.$$

Now, (9) implies the hypothesis of Theorem 1 since

$$\|z\| < \frac{1}{2\|B\|} \Rightarrow r_2 > 0,$$

while by (9) and (11) we have

$$\frac{1}{2\|B\|} - \|z\| > \sqrt{\frac{\|B-F\|}{\|B\|}}\cdot\|z\|$$

or

$$r_2 > \sqrt{\frac{\|T(z)-z\|}{\|B\|}} \Rightarrow r_1 \geq 0.$$

Part (i) and (ii) now follow from Theorem 1. Moreover, by Theorem 3, we have

$$q = 2(r_1+\|z\|)\cdot\|B\| = 1-[(2\|B\|\cdot\|z\|-1)^2-4\|B-F\|\cdot\|B\|\cdot\|z\|^2]^{1/2}.$$

EXAMPLE. Theorem 3 may be applicable even if the hypothesis in Corollary 1 or Theorem 3 is violated as the following example in $E_1=\mathbf{R}$ easily inducates. Let

$$x = -.251+x^2 \quad \text{for} \quad x = y+B(x, x)$$

and

$$z = -.251+.8z^2 \quad \text{for} \quad z = y+F(z, z).$$

PROPOSITION 1. *Assume:*

(i) *the hypotheses of Theorems* 2, 3 *and Corollary* 1 *are satisfied;*

(ii) $(\|B\|-\|B-F\|)\|z\|^2-\|z\|+\|y\|>0.$

Then Theorem 3 *provides a sharper estimate on* x^* *than Theorem* 2 *or Corollary* 1.

PROOF. By Theorem 3,

$$\|x^*-z\| \leqq r_2, \quad \text{so} \quad \|x^*\| \leqq r_2+\|z\|.$$

By Theorem 2 and Corollary 1,

$$\|x^*\| \leqq \frac{1-\sqrt{1-4\|B\|\cdot\|y\|}}{2\|B\|}$$

so it is enough to show

$$[1-((2\|B\|\cdot\|z\|-1)^2-4\|B\|\cdot\|B-F\|\cdot\|z\|^2)^{1/2}]\,(2\|B\|)^{-1} <$$

$$<[1-(1-4\|B\|\cdot\|y\|)^{1/2}]\,(2\|B\|)^{-1}$$

or

$$(\|B\|-\|B-F\|)\cdot\|z\|^2-\|z\|+\|y\| > 0$$

and the result now follows from (ii).

Note that up till now the rate of convergence of (3), q was defined as a number. But we can find better error estimates if we define the rate of convergence of (3) as a function.

PROPOSITION 2. *Let c be such that* $0<c<1$ *and* $I=[r_1, r_2)$, *where* r_1 *and* r_2 *are as defined in Theorem* 1. *Then*

(a) *the function* ω, *given by*

$$\omega(r) = r(1-c) \tag{12}$$

is a rate of convergence on I and the corresponding function σ *is given by*

$$\sigma(r) = \frac{r}{c}. \tag{13}$$

(b) *Moreover, the following equalities hold:*

$$\omega^{(n)}(r) = r(1-c)^n, \quad n = 0, 1, 2, \ldots \tag{14}$$

and

$$\sigma\left(\omega^{(n)}(r)\right)=\frac{r}{c}\cdot(1-c)^n, \quad n=0,1,2,\ldots.$$

PROOF. Let us consider the real polynomial $f(s)=c(s-p)$, for any real number p and $0<c<1$. Consider the iteration

$$s_{n+1}=s_n-f(s_n), \quad n=0,1,2,\ldots$$

where $s_0=p+\frac{c}{r}$; then we have $s_0-s_1=f(s_0)=r$. Now taking

$$\omega(r)=s_1-s_2=f(s_1)$$

we obtain the expression (12). The sequence $\{s_n\}$, $n=0,1,2,\ldots$ is a decreasing sequence which converges to p. Using induction on n we can easily show that

$$\omega^{(n)}(r)=s_n-s_{n+1}, \quad n=0,1,2,\ldots$$

and consequently

$$\sigma(r)=s_0-a=\frac{r}{c}.$$

Part (b) now easily follows from part (a) and using induction on n, $n=0,1,2,\ldots$.

We now state the following simplified version of the Induction Theorem whose proof can be found in [6].

PROPOSITION 3. *If ω is a rate of convergence on I and a family of sets $Z(r)\subset E_1$, $r\in I$ exists such that for some $x_0\in E_1$ the following are true:*

(16) $$x_0\in Z(r_0) \quad \text{for a fixed} \quad r_0\in I,$$

(17) $$(r\in I \text{ and } x\in Z(r))\Rightarrow T(x)\in U(x,r)\cap Z(\omega(r)).$$

Then iteration (3) *converges to a solution x^* of* (1) *such that*

(18) $$x_n\in Z\left(\omega^{(n)}(r_0)\right),$$

(19) $$\|x_n-x_{n-1}\|\leqq\omega^{(n)}(r_0)$$

and

(20) $$\|x_n-x^*\|\leqq\sigma\left(\omega^{(n)}(r_0)\right), \quad n=1,2,\ldots.$$

We can now prove the main result.

THEOREM 4. *Let $x_0\in E_1$ be fixed. Assume:*
(a) *the hypotheses of Theorem* 1 *for $x_0=z$ are true;*
(b) *there exists c, with $0<c<1$ such that*

(21) $$rc^3+(\|B\|\cdot\|y\|^2-r)c^2+(r_0-r)(2\|B\|\cdot\|y\|+1)c+\|B\|(r_0-r)^2\leqq 0$$

for any $r\in I_1\subset I=[r_1,r_0]$, where r_1 and r_2 are as defined in Theorem 1 *and $r_1\leqq r_0<r_2$.*

Then iteration (3) *generates a sequence $\{x_n\}$, $n=0,1,2,\ldots$ which converges to a solution x^* of* (1) *such that*

(22) $$\|x_n-x_0\|\leqq\sum_{k=1}^{n}\omega^{(k)}(r_0), \quad n=0,1,2,\ldots$$

and

(23) $$\|x_n - x^*\| \leqq \sigma(\omega^{(n)}(r_0)), \quad n = 0, 1, 2, \ldots$$

where ω, σ *are given by* (14) *and* (15) *respectively for* $r = r_0$.

PROOF. We attach to iteration (3) the rate of convergence ω given in Proposition 2 and the family of sets

(24) $$Z(r) = \{x \in E_1 | \, \|x - x_0\| \leqq \sigma(r_0) - \sigma(r), \text{ and } \|T(x) - x\| \leqq r\}, \quad r \in I.$$

According to Proposition 3 we need to show (16) and (17). Note that $Z(r_0) = \{x_0\} = \{z\}$ so that (16) is satisfied. Now let $x \in Z(r)$ and set

$$v = x + (T(x) - x)$$

then

$$\|v - z\| = \|(v - x) + (x - z)\| \leqq \|v - x\| + \|x - z\| \leqq r + (\sigma(r_0) - \sigma(r_0)) = \sigma(r_0) - \sigma(\omega(r)).$$

To show (17) we need to show also that

(25) $$\|T(x) - x\| \leqq \omega(r)$$

But,

$$\|T(x) - x\| \leqq \|B(x, x)\| + \|z - x\| \leqq \|B(x - z + z, x - z + z)\| + \|z - x\| \leqq$$

$$\leqq \|B\| \, \|x - z\|^2 + 2\|B\| \cdot \|z\| \, \|x - z\| + \|B\| \cdot \|z\|^2 + \|x - z\| \leqq$$

$$\leqq \|B\| (\sigma(r_0) - \sigma(r))^2 + 2\|B\| \cdot \|z\| (\sigma(r_0) - \sigma(r)) + \|B\| \cdot \|z\|^2 + (\sigma(r_0) - \sigma(r)).$$

That is (25) holds if (21) is satisfied.

Therefore (17) is satisfied. The rest of the theorem follows from Proposition 3 and (14) and (15).

REMARKS. (a) The number r_0 is usually chosen as $r_0 = r_1$.

(b) Note that the estimate (23) on the solution x^* is better than the corresponding estimate (10) with $q = q^*$ given by (7) if c can be chosen to satisfy (21) and

$$0 < 1 - 2(r + \|z\|) \cdot \|B\| < c < 1, \quad r \in [r_1, r_2).$$

References

[1] P. M. Anselone and R. H. Moore, An extension of the Newton—Kantorovich method for solving nonlinear equations with an application to elasticity, *J. Math. Anal. Appl.*, **13** (1966), 476—501.

[2] I. K. Argyros, On a contraction theorem and applications, *Proc. Symp. Pure Math. A.M.S.*, **45** (1986), 51—53.

[3] S. Chandrasekhar, *Radiative Transfer*, Dover Publ. (New York, 1960).

[4] C. T. Kelley, Solution of the Chandrasekhar *H*-equation by Newton's method, *J. Math. Phys.*, **21** (1980), 1625—1628.

[5] L. Liusternik and V. Sobolev, *Elements of functional analysis*. Ungar Publ. (New York, 1961).
[6] F. A. Potra and V. Pták, *Nondiscrete induction and iterative processes*, Pitman Publ. (London, 1984).
[7] L. B. Rall, Quadratic equations in Banach space, *Rend. Circ. Math. Palermo*, **10** (1961), 314—332.
[8] L. B. Rall, *Computational solution of nonlinear operator equations*, John Wiley Publ. (New York, 1968).

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DEPARTMENT OF MATHEMATICS
NEW MEXICO UNIVERSITY
LAS CRUCES, NM 88003
USA

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ON SOME QUESTIONS OF GER, GRUBB AND KRALJEVIĆ

M. CRNJAC (Osijek), B. GULJAŠ (Zagreb) and H. I. MILLER (Sarajevo)

Roman Ger (Katowice, Poland) has asked if there exist compact subsets A and B of the real line such that one of the sets $A+B$ and $A-B$ contains an interval, while the other one does not.

Dan Grubb (DeKalb, U.S.A.) has asked if $A+cB$ must contain an interval for all c sufficiently near 1 if $A+B$ contains an interval.

Hrvoje Kraljević (Zagreb, Yugoslavia) has asked about the connectivity of the set $\{c\colon A+cB$ contains an interval$\}$. More precisely, he asked: If c_1 and c_2 are positive reals and the sets $A+c_1B$ and $A+c_2B$ both contain intervals, does it follow that $A+cB$ must contain an interval for each c between c_1 and c_2?

In this paper we will prove two theorems which will provide answers to the questions of Ger, Grubb and Kraljević. In addition, other related questions will be considered.

1. Introduction. We will start by mentioning some results related to the material that we will present.

THEOREM OF STEINHAUS. *If A and B are measurable subsets of* **R** *(the real line), each having positive measure, then the set $A+B=\{a+b\colon a\in A,\ b\in B\}$ contains an interval.*

THEOREM OF PICCARD. *If A and B are both Baire subsets (i.e. have the Baire property) of* **R** *and both are of second Baire category, then $A+B$ contains an interval.*

Proofs of the theorems of Steinhaus and Piccard can be found in [6] and [12]. Various authors have generalized these results, for example see [3], [4], [5], [7], [8], [14], and [15].

Of course the conditions in the theorems of Steinhaus and Piccard are sufficient but not necessary. This is the case since $C+C$ equals $[0, 2]$, where C is the Cantor set. In a recent article [2] the present authors have shown that $f(C\times C)=\{f(x, y)\colon x, y\in C\}$ contains an interval for every $f\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ satisfying appropriate conditions.

In [9], using transfinite induction and assuming the continuum hypothesis, a set N is constructed that is concentrated on the rationals (see [13], p. 74) and such that $N-N=\mathbf{R}$.

F. Bagemihl [1] has observed that $m(A)>0$ (here m denotes Lebesgue measure) and B a Baire set of second category does not imply that the set $A-B=\{a-b\colon a\in A,\ b\in B\}$ contains an interval. For example, if B is the set of Liouville numbers and A is taken to be $R\setminus B$, then $m(B)=0$ and A is of first category (see [12]) and $A-B$ contains no rational numbers (see [10]) and hence $A-B$ contains no interval. This negative result is generalized in [10], where it is shown that if $f\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ satisfies appropriate conditions then there exist a set A of positive Lebesgue measure and a Baire set B of second category such that $f(A\times B)$ contains no interval.

In [11], using transfinite induction and the continuum hypothesis the following theorem is proved.

THEOREM. *Assume that f and g are functions on $\mathbf{R}\times\mathbf{R}$ into $\mathbf{R}$ such that*

(a) *f_x, f_y, g_x and g_y (partial derivatives) exist and are continuous on an open neighborhood of the origin;*

(b) *$f_x(0,0)$, $f_y(0,0)$, $g_x(0,0)$ and $g_y(0,0)$ are all non-zero;*

(c) *$f(0,0)=g(0,0)=0$;*

(d) *the numbers $f_x(0,0)/f_y(0,0)$ and $g_x(0,0)/g_y(0,0)$ have opposite signs.*

Then there exist sets A, B such that $A, B\subset\mathbf{R}$ and $f(A\times B)$ contains an interval, but $g(A\times B)$ does not.

2. Results. Our first theorem will provide answers to the questions of Grubb and Kraljević mentioned at the beginning.

THEOREM 1. *If (p_n) and (q_n) are any two sequences of non-zero real numbers such that $p_n\neq p_m$ and $q_n\neq q_m$ for all $n\neq m$ and $p_n\neq q_m$ for all $n, m\in\mathbf{N}$ (the set of natural numbers), then there exist subsets A and B of the reals such that*

(a) *$A+p_nB=\mathbf{R}$ for every $n\in\mathbf{N}$ and*

(b) *$A+q_nB$ contains no interval for each $n\in\mathbf{N}$.*

PROOF. We remark at the outset that $A+pB$ stands for the set

$$\{a+pb\colon a\in A,\ b\in B\}.$$

Let w_c denote the first ordinal number having cardinal c, the cardinal of the continuum. Let $\{x_a\}_{a<w_c}$ be a well ordering of $\mathbf{R}$. By transfinite induction, for each $a<w_c$ we will construct two sequences, $(y_{an})_{n=1}^{\infty}$ and $(z_{an})_{n=1}^{\infty}$ in such a way that the sets

$$A=\{y_{an}\colon a<w_c \text{ and } n\in\mathbf{N}\} \quad\text{and}\quad B=\{z_{an}\colon a<w_c \text{ and } n\in\mathbf{N}\}$$

satisfy the conditions of the theorem.

We first construct the sequences $(y_{1n})_{n=1}^{\infty}$ and $(z_{1n})_{n=1}^{\infty}$ by ordinary induction. To start the inductive process we need y_{11} and z_{11}. We will show that we can pick y_{11} and z_{11} such that $y_{11}+p_1z_{11}=x_1$ and $y_{11}+q_nz_{11}$ is an irrational number for each $n\in\mathbf{N}$.

If we take $y_{11}=x$ to be an arbitrary real number, then in order for $y_{11}+$ $+p_1z_{11}=x_1$ to hold we must have $z_{11}=(x_1-x)p_1^{-1}$. Using this value for z_{11} we have

$$y_{11}+q_nz_{11} = q_nx_1p_1^{-1}+(1-q_np_1^{-1})x.$$

Clearly, by our hypotheses on the sequences (p_n) and (q_n), the last expression is irrational for each fixed n for all $x\in\mathbf{R}$ with denumerably many exceptions.

Finally, if $y_{11}=x$ and $z_{11}=(x_1-x)p_1^{-1}$ then $y_{11}+p_1z_{11}=x_1$ and $y_{11}+q_nz_{11}$ is irrational for each $n\in\mathbf{N}$ provided $x\in\mathbf{R}\setminus D$, where D is a denumerable set.

Suppose that $(y_{1n})_{n=1}^{k}$ and $(z_{1n})_{n=1}^{k}$ have been defined so that $y_{1n}+p_n z_{1n}=x_1$ for each $n=1, 2, ..., k$ and $y_{1i}+q_n z_{1j}$ is irrational for every $i, j\in\{1, 2, ..., k\}$ and $n\in\mathbf{N}$.

If we take $y_{1,k+1}=x$ to be an arbitrary real number and $z_{1,k+1}=(x_1-x)p_{k+1}^{-1}$, then arguing as before, we have $y_{1n}+p_n z_{1n}=x_1$ for every $n=1, 2, ..., k+1$ and $y_{1i}+q_n z_{1j}$ is irrational for every $i, j\in\{1, 2, ..., k+1\}$ and every $n\in\mathbf{N}$ provided $x\in\mathbf{R}\setminus E$, where E is denumerable.

Therefore, by mathematical induction there exist two sequences $(y_{1n})_{n=1}^{\infty}$ and $(z_{1n})_{n=1}^{\infty}$ such that the following holds: $y_{1n}+p_n z_{1n}=x_1$ for every $n\in\mathbf{N}$ and $y_{1i}+q_n z_{1j}$ is irrational for every $i, j, n\in\mathbf{N}$.

Now suppose that $a<w_c$ and for every $b<a$ the sequences $(y_{bn})_{n=1}^{\infty}$ and $(z_{bn})_{n=1}^{\infty}$ have been defined in such a way that $y_{bn}+p_n z_{bn}=x_b$ for every $b<a$ and for every $n\in\mathbf{N}$ and $y_{bi}+q_n z_{dj}$ is irrational for every $b, d<a$ and $i, j, n\in\mathbf{N}$.

By the definition of w_c, arguing as before (using mathematical induction), the sequences $(y_{an})_{n=1}^{\infty}$ and $(z_{an})_{n=1}^{\infty}$ can be defined in such a way that: $y_{an}+p_n z_{an}=x_a$ for every $n\in\mathbf{N}$ and $y_{bi}+q_n z_{dj}$ is irrational for every $b, d\leqq a$ and $i, j, n\in\mathbf{N}$.

Therefore, by transfinite induction, for each $a<w_c$ we obtain two sequences, $(y_{an})_{n=1}^{\infty}$ and $(z_{an})_{n=1}^{\infty}$ and if we set

$$A=\{y_{an}\colon\ a<w_c \text{ and } n\in\mathbf{N}\} \quad\text{and}\quad B=\{z_{an}\colon\ a<w_c \text{ and } n\in\mathbf{N}\}$$

we have $A+p_nB=\mathbf{R}$ for each $n\in\mathbf{N}$ and $s+q_nt$ is irrational for each $n\in\mathbf{N}$, $s\in A$ and $t\in B$. Therefore, for each $n\in\mathbf{N}$ the set $A+q_nB$ contains no interval.

The following results are immediate consequences of Theorem 1 and provide answers to the questions of Grubb and Kraljević mentioned at the beginning.

COROLLARY 1. *There exist subsets A and B of the reals and a sequence (p_n), $p_n\neq 1$ for each n, with $\lim_{n\to\infty} p_n=1$, such that $A+B$ contains an interval and $A+p_nB$ contains no interval, for each $n\in\mathbf{N}$.*

COROLLARY 2. *There exist subsets A and B of the reals and positive reals $c_1<c<c_2$ such that $A+c_1B$ and $A+c_2B$ both contain intervals, but $A+cB$ contains no interval.*

The proofs of these corollaries are immediate and are therefore omitted.

We will now proceed to provide a positive answer to the question of Roman Ger mentioned at the beginning, namely we will show that there exist compact sets A and B such that $A-B$ contains an interval, but $A+B$ does not.

THEOREM 2. *There exist compact subsets A and B of the real line such that $A-B$ contains an interval, but $A+B$ does not.*

PROOF. Let

$$S=\{\sum_{i=1}^{\infty} a_i/7^i;\ a_i\in\{0, 2, 6\}\}.$$

We will show that if we set A and B equal to S then the conditions of our theorem will be satisfied. To see that $A+B$ contains no interval it is sufficient to show that

$A+B$ has Lebesgue measure zero. However

$$A+B = \{2\cdot \sum_{i=1}^{\infty} b_i/7^i;\ b_i\in\{0, 1, 2, 3, 4, 6\}\}.$$

This clearly implies that $m(A+B)=0$.

We will now show that $A-B=[-1, 1]$. To see this observe that $1=\sum_{i=1}^{\infty} 6/7^i$ and therefore

$$1-B = \{\sum_{i=1}^{\infty} c_i/7^i;\ c_i\in\{0, 4, 6\}\}.$$

This in turn implies that

$$A+(1-B) = \{\sum_{i=1}^{\infty} d_i/7^i;\ d_i\in\{0, 2, 4, 6, 8, 10, 12\}\}.$$

This last equation implies that

$$A+(1-B) = \{2\cdot \sum_{i=1}^{\infty} e_i/7^i;\ e_i\in\{0, 1, 2, 3, 4, 5, 6\}\}$$

and hence $A+(1-B)=[0, 2]$ or $A-B=[-1, 1]$.

We conclude this paper with a few remarks.

REMARK 1. The facts that, in the proof of Theorem 2,

$$S+S = \{2\cdot \sum_{i=1}^{\infty} b_i/7^i;\ b_i\in\{0, 1, 2, 3, 4, 6\}\}$$

and $S-S=[-1, 1]$ can be shown geometrically imitating a proof of Utz (see [16]).

REMARK 2. If A, $A\subset\mathbf{R}$, is a measurable set of positive measure or a set of second category that has the Baire property then clearly, by the Theorem of Steinhaus and the Theorem of Piccard $A+pA$ contains an interval for each $p\in\mathbf{R}\setminus\{0\}$. Professor H. Kraljević has asked if sets that are not in these classes can have this property. The answer to this question is in the affirmative. From a recent result of the present authors [2] it follows that $C+pC$ contains an interval for each $p\in\mathbf{R}\setminus\{0\}$, where C is the Cantor set.

References

[1] F. Bagemihl, Some sets of sums and differences, *Michigan Math. J.*, **4** (1957), 289—290.

[2] M. Crnjac, B. Guljaš and H. I. Miller, A positive answer to a question about the Cantor set, to appear.

[3] P. Erdős and J. C. Oxtoby, Partitions of the plane into sets having positive measure in every non-null measurable product set, *Trans. Amer. Math. Soc.*, **79** (1955), 99—102.

[4] Z. Kominek, Some generalizations of the theorem of S. Piccard, *Prace Nauk. Univ. Śląski Katowice, Prace Mat.*, **4** (1973), 31—33.

[5] M. E. Kuczma and M. Kuczma, An elementary proof and an extension of a theorem of Steinhaus, *Glasnik Mat. Ser. III*, **6 (26)** (1971), 11—18.

[6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Uniwersytet Śląski (Warszawa—Kraków—Katowice, 1985).

[7] H. I. Miller, Generalization of a classical theorem of measure theory, *Radovi Akademije Nauka i Umjetnosti BiH*, **XLV** (1973), 45—48.
[8] H. I. Miller, Relationships between various gauges of the size of sets of real numbers, *Glasnik Mat. Ser. III*, **9 (29)** (1974), 59—64.
[9] H. I. Miller, A universal null set whose Steinhaus distance set is $[0, \infty)$, *Radovi Akademije Nauka i Umjetnosti BiH*, **LIX** (1978), 37—40.
[10] H. I. Miller, An incompatibility result, *Revue Roumaine de Mathématiques Pures et Appliquées*, **XXVI** (1981), 1217—1220.
[11] H. I. Miller, An incompatibility theorem, *Colloquium Math.*, **XLVIII** (1984), 135—138.
[12] J. C. Oxtoby, *Measure and Category*, Springer-Verlag (New York, 1970).
[13] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press (1970).
[14] W. Sander, Verallgemeinerung eines Satzes von H. Steinhaus, *Manuscripta Math.*, **18** (1976), 25—42.
[15] W. Sander, Verallgemeinerung eines Satzes von S. Piccard, *Manuscripta Math.*, **16** (1975), 11—25.
[16] W. R. Utz. The distance set of the Cantor discontinuum, *Amer. Math. Monthly*, June 1951, 407—408.

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SVEUČILIŠTE U OSIJEKU
PEDAGOŠKI FAKULTET
OSIJEK 54000
JUGOSLAVIJA

DEPT. OF MATHEMATICS
UNIVERSITY OF ZAGREB
ZAGREB 41000
JUGOSLAVIJA

DEPT. OF MATHEMATICS
UNIVERSITY OF SARAJEVO
SARAJEVO 71000
JUGOSLAVIJA

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LOCAL LIPSCHITZ CONSTANTS AND KOLUSHOV POLYNOMIALS

M. W. BARTELT (Newport News) and J. J. SWETITS (Norfolk)

1. Introduction. Let K be a compact subset of $[a, b]$ and $C(K)$ the space of continuous real valued functions on K endowed with the uniform norm $\| \ \|$. Denote the set of algebraic polynomials of degree n or less by Π_n. For f in $C(K)$, let $B_n(f)=B_n(f, K)$ denote the best uniform approximate to f on K from Π_n. Denote the set of positive extremal points of $f-B_n(f)$ by

$$E_n^+(f)=\{x\in K\colon (f-B_n(f))(x)=\|f-B_n(f)\|\}. \tag{1.1}$$

Let $E_n^-(f)$ denote the set of negative extremal points. Let $E_n(f)$ be their union and $|E_n(f)|$ denote its cardinality.

Lipschitz constants for the best approximation operator have been extensively studied ([3—6], [8, 9]). Recently local Lipschitz constants have been the focus of research [1, 2] which related the local Lipschitz constant to the derivative of the best approximation operator, Lebesgue constants and Cline polynomials [5].

Following [2], let the local Lipschitz constant for f be

$$\lambda_n^l(f)=\lim_{\delta\to 0+}\sup\{\|B_n(f+\varphi)-B_n(f)\|/\|\varphi\|\colon 0<\|\varphi\|<\delta\}. \tag{1.2}$$

It was observed in [2, p. 146] that if $|E_n(f)|=n+2$ for all sufficiently large n and $K=[a, b]$, then

$$\lim_{n\to\infty}\lambda_n^l(f)=\infty. \tag{1.3}$$

This observation relied on the Losinski—Kharshiladze Theorem that if P is a linear projection from $C[a, b]$ onto Π_n, then $\|P\|\geqq\log(n)/8\sqrt{\pi}$ ([4, 10]).

In this paper, a class of polynomials introduced by Kolushov [7] (hereafter called Kolushov polynomials) are used to investigate the behavior of $\lambda_n^l(f)$. Theorem 1 characterizes $\lambda_n^l(f)$ for fixed n and finite K in terms of Kolushov polynomials. Theorem 2 shows that (1.3) holds in the more general case when $E_n(f)$ contains at most m alternants, where m is independent of n.

2. Kolushov polynomials. For n fixed, an alternant of $f-B_n(f)$ is a set $\{x_0, \ldots, x_{n+1}\}\subseteqq E_n(f)$ with

$$(f-B_n(f))(x_i)=(-1)^i\|f-B_n(f)\|\operatorname{sgn}(f-B_n(f))(x_i),\quad i=0, \ldots, n+1.$$

Kolushov showed that given φ in $C(K)$, there is a unique real number, $\alpha=\alpha(\varphi)$, and a unique polynomial $p_n(f, \varphi)=p_n(\varphi)$ in Π_n such that

$$(\varphi(x)-p_n(\varphi)(x))\operatorname{sgn}(f-B_n(f))(x)\leqq\alpha,\quad x\in E_n(f). \tag{2.1}$$

In addition there is an alternant of f where equality holds in (2.1). Kolushov then proved that

$$\lim_{t\to 0+} (B_n(f+t\varphi)-B_n(f))/t = p_n(\varphi) \tag{2.2}$$

with convergence being in the uniform norm.

Let $A_\varphi(f)=A_\varphi$ denote an alternant of f associated with φ. The following properties of the Kolushov polynomials follow from the above, the theorem of de la Vallée Poussin on K [4] and, in part (vi), from Lemma 3 in [1]. Let $B_n(\varphi, X)$ denote the best approximant to φ from Π_n on X. We prove only (vi).

PROPOSITION 1. (i) *If* $\varphi\in\Pi_n$, *then* $p_n(\varphi)=\varphi$ *and* $\alpha(\varphi)=0$.

(ii) $p_n(\varphi)=B_n(\varphi, A_\varphi)$.

(iii) *If* $|E_n(f)|=n+2$, *then* $p_n(\varphi)=B_n(\varphi, E_n(f))$.

(iv) *If* $c>0$, *then* $p_n(c\varphi)=cp_n(\varphi)$, $\alpha(c\varphi)=c\alpha(\varphi)$, *and* $A_{c\varphi}=A_\varphi$.

(v) *If* $A_\varphi=A_\psi$, *then* $p_n(\varphi+\psi)=p_n(\varphi)+p_n(\psi)$, $\alpha(\varphi+\psi)=\alpha(\varphi)+\alpha(\psi)$, *and* $A_{\varphi+\psi}=A_\varphi=A_\psi$.

(vi) *If* $n+2\leqq|K|<\infty$, *then there exists* $\delta>0$ *such that* $p_n(\varphi)=B_n(\varphi, K)$ *if* $\|\varphi-f\|<\delta$.

(vii) $p_n(f)=B_n(f)$ *and* $\alpha(f)=\|f-B_n(f)\|$.

(viii) $p_n(f+\varphi)=p_n(f)+p_n(\varphi)$.

PROOF OF (vi). If $\delta>0$ is sufficiently small, then from Lemma 3 of [1] it follows that $E_n(\varphi)\subseteqq E_n(f)$ if $\|f-\varphi\|<\delta$ since K is discrete. Furthermore, $\varphi-B_n(\varphi)$ and $f-B_n(f)$ have the same sign on $E_n(\varphi)$. Hence any alternant of φ is an alternant of f. Thus (2.1) is satisfied with $p_n(\varphi)=B_n(\varphi, K)$, $\alpha=\|\varphi-B_n(\varphi, K)\|$, with equality on any alternant of φ.

3. Main results. The following lemma gives a lower bound for $\lambda_n^l(f)$ valid for any compact subset of $[a, b]$ having at least $n+2$ points. Theorem 1 which follows shows that the lower bound is also an upper bound when K is finite. We then define a collection of projection operators which are used in Theorem 2 to provide a lower bound for $\lambda_n^l(f)$ when K is infinite.

LEMMA 1. *Let K be a compact subset of* $[a, b]$ *and* $f\in C(K)$. *Then*

$$\lambda_n^l(f) \geqq \sup\{\|p_n(f, \varphi)\|:\ \|\varphi\|\leqq 1,\ \varphi\in C(K)\}. \tag{3.1}$$

PROOF. Let $\varphi\in C(K)$ and $\|\varphi\|\leqq 1$. Then

$$\lambda_n^l(f) = \lim_{\delta\to 0+} \sup\{\|B_n(f+g)-B_n(f)\|/\|g\|:\ 0<\|g\|<\delta\} \geqq$$

$$\geqq \lim_{\delta\to 0+} \sup\{\|B_n(f+t\varphi)-B_n(f)\|/t:\ 0<t<\delta\} \geqq$$

$$\geqq \lim_{t\to 0+} \|B_n(f+t\varphi)-B_n(f)\|/t = \|p_n(f, \varphi)\|,$$

where the last equality follows from (2.2).

THEOREM 1. *Let K be a finite subset of* $[a, b]$ *with* $|K|\geqq n+2$ *and let* $f\in C(K)$. *Then*

$$\lambda_n^l(f) = \sup\{\|p_n(f, \varphi)\|:\ \|\varphi\|\leqq 1,\ \varphi\in C(K)\}. \tag{3.2}$$

PROOF. Without loss it may be assumed that $B_n(f)=0$ and $\|f\|=1$. By (vi) of Proposition 1, choose $\delta>0$ so that $p_n(f,g)=B_n(g,K)$ if $\|f-g\|<\delta$. For $t>0$, define $\varphi\in C(K)$ by $g=f+t\varphi$ and $\|\varphi\|=1$. By (iv), (vii), and (viii) of Proposition 1 we have $B_n(g,K)=p_n(f,g)=p_n(f,f+t\varphi)=p_n(f,f)+p_n(f,t\varphi)=tp_n(f,\varphi)$. Thus

$$\|B_n(g,K)\|/\|f-g\| = \|p_n(f,\varphi)\|,$$

and, hence,

$$\lambda_n^l(f) \leqq \sup \{\|p_n(f,\varphi)\| : \|\varphi\| \leqq 1, \ \varphi\in C(K)\}.$$

Lemma 1 completes the proof.

Suppose $E_n(f)$ contains only a finite number of alternants, A_i^n, $i=1,\dots,a(n)$. Let p_n^i, $i=1,\dots,a(n)$, denote the linear projection from $C(K)$ onto Π_n given by $p_n^i(g)=B_n(g,A_i^n)$ for $g\in C(K)$. Let $\|p_n^i\|$ denote the operator norm of p_n^i.

THEOREM 2. *Let K be an infinite compact subset of $[a,b]$, and let $f\in C(K)$. If, for infinitely many n, $E_n(f)$ contains at most m alternants, where m is independent of n, then there exists a constant $C>0$, independent of n, and a sequence of indices $\{n(k)\}$ such that*

$$\lambda_{n(k)}^l(f) \geqq C \min \{\|p_{n(k)}^i\| : i=1,\dots,m\}. \tag{3.3}$$

PROOF. For convenience assume that for each n, the number of alternants is m. If $\varphi\in C(K)$, let $p_n(f,\varphi,A_\varphi)$ denote the Kolushov polynomial where $A_\varphi=A_i^n$ for some $i=1,\dots,m$. By (ii) of Proposition 1, if $A_g=A_i^n$, it follows that

$$p_n^i(g) = p_n(f,g,A_i^n) = p_n(f,g,A_g). \tag{3.4}$$

We can write $C(K)$ as the union of sets, $\mathscr{A}_i^n$, $i=1,\dots,m$, where $\mathscr{A}_n^i$ consists of all φ such that the alternant of f corresponding to φ in Kolushov's polynomials is A_i^n. Note that P_n^i restricted to $\mathscr{A}_i^n$ is p_n.

Let $\{h_n^1\}_n$ be a sequence in $C(K)$ such that, for all n, $\|h_n^1\|\leqq 1$ and

$$\|p_n^1(h_n^1)\| \geqq M_1\|p_n^1\|, \tag{3.5}$$

where M_1 is independent of n. For each $i=1,\dots,m$ and for each n, let

$$c_i(n) = \|p_n^i(h_n^1)\|/\|p_n^i\|.$$

$\{c_1(n)\}_n$ is bounded away from 0. By rearrangement and passing to subsequences, we can assume that $\{c_i(n)\}_n$, $i=1,\dots,i(1)$, are bounded away from 0, and that $c_i(n)\to 0$ $(n\to\infty)$ for $i(1)<i$. Now let $\{h_n^2\}_n$ be such that $\|h_n^2\|\leqq 1$ for each n and

$$\|p_n^{i(1)+1}(h_n^2)\| \geqq M_2\,\|p_n^{i(1)+1}\|, \tag{3.6}$$

where M_2 is independent of n, and repeat the above process for $i(1)<i$. Since there are m projections, then after r steps, for some r, we obtain r sequences of functions $\{h_n^i\}$, $i=1,\dots,r$, and r blocks $B_1=\{\mathscr{A}_1^n,\dots,\mathscr{A}_{i(1)}^n\}$, $B_2=\{\mathscr{A}_{i(1)+1}^n,\dots,\mathscr{A}_{i(2)}^n\}$, …, $B_r=\{\mathscr{A}_{i(r-1)+1}^n,\dots,\mathscr{A}_{i(r)}^n\}$, with their corresponding projections, such that, with $i(0)=0$, $i(1)+\dots+i(r)=m$,

$$\|p_n^i(h_n^j)\| \geqq M_j\|p_n^i\|, \quad i(j-1)+1 \leqq i \leqq i(j), \tag{3.7}$$

and

$$\|p_n^i(h_n^j)\| = o(\|p_n^i\|), \quad i(j)<i. \tag{3.8}$$

By the projections in a block we mean the projections corresponding to the sets in a block.

We now show by induction on r that there is a sequence $\{g_n\}$, $g_n \in C(K)$, such that $\|g_n\| \leqq 1$ and

$$\|p_n^i(g_n)\| \geqq C \min\{\|p_n^i\|:\ i=1, \dots, m\}. \tag{3.9}$$

If $r=1$, then $\{g_n\}=\{h_n^1\}$ satisfies (3.9) by (3.7). We next consider $r=2$ to show how to combine two consecutive blocks into one block. There are three possibilities. Let $i(1)=q$, and suppose first that

$$\|p_n^i(h_n^2)\| \geqq b\|p_n^i\|, \quad i=1, \dots, q \tag{3.10}$$

where b is independent of n. Then, discarding $\{h_n^1\}$, the sequence $\{h_n^2\}$ satisfies (3.9) for the combined block $\{\mathscr{A}_1^n, \dots, \mathscr{A}_m^n\}$. In the second instance suppose that

$$\|p_n^i(h_n^2)\| = o(\|p_n^i\|), \quad i=1, \dots, q. \tag{3.11}$$

In this case define $g_n=(h_n^1+h_n^2)/2$. Then $\{g_n\}$ satisfies (3.9) for the combined block. Suppose now that neither (3.10) nor (3.11) hold. By rearrangement and passing to subsequences, we can assume that

$$\|p_n^i(h_n^2)\| \geqq b\|p_n^i\|, \quad 1 \leqq i \leqq l \tag{3.12}$$

and

$$\|p_n^i(h_n^2)\| = o(\|p_n^i\|), \quad l+1 \leqq i \leqq q. \tag{3.13}$$

Define

$$\mu_n = \max\{\|p_n^i(h_n^2)\|/\|p_n^i\|:\ 1 \leqq i \leqq l\}, \tag{3.14}$$

$$\lambda_n = \min\{\|p_n^i(h_n^2)\|/\|p_n^i\|:\ 1 \leqq i \leqq q\}, \tag{3.15}$$

$$\alpha_n = \lambda_n/2\mu_n, \tag{3.16}$$

and

$$g_n = (h_n^1+\alpha_n h_n^2)/(1+\alpha_n). \tag{3.17}$$

Both μ_n and λ_n are bounded above by 1 and are bounded away from 0. For $1 \leqq i \leqq l$,

$$\|p_n^i(g_n)\| \geqq (1+\alpha_n)^{-1}(\|p_n^i(h_n^1)\| - \alpha_n\|p_n^i(h_n^2)\|) \geqq \tag{3.18}$$
$$\geqq (1+\alpha_n)^{-1}(\lambda_n\|p_n^i\| - (\lambda_n/2)\|p_n^i\|) \geqq \lambda_n\mu_n(2\mu_n+\lambda_n)^{-1}\|p_n^i\|\|.$$

For $l+1 \leqq i \leqq q$,

$$\|p_n^i(g_n)\| \geqq (1+\alpha_n)^{-1}(\|p_n^i(h_n^1)\| - \alpha_n\|p_n^i(h_n^2)\| \geqq \tag{3.19}$$
$$\geqq (1+\alpha_n)^{-1}(\lambda_n\|p_n^i\| - o(\|p_n^i\|)).$$

For $q+1 \leqq i \leqq m$,

$$\|p_n^i(g_n)\| \geqq (1+\alpha_n)^{-1}(\alpha_n\|p_n^i(h_n^2)\| - \|p_n^i(h_n^1)\|) \geqq \tag{3.20}$$
$$\geqq (1+\alpha_n)^{-1}(\beta\alpha_n\|p_n^i\| - o(\|p_n^i\|)).$$

Hence by (3.18)—(3.20), $\{g_n\}$ satisfies (3.9) for the combined block $\{\mathscr{A}_1^n, \dots, \mathscr{A}_m^n\}$.

Now suppose that (3.9) is valid for r blocks, and we are given $r+1$ blocks. Combine the last two blocks into one block as was done in the case $r=2$, observing that (3.7) and (3.8) are valid for all the blocks. Apply the induction hypothesis to produce the required $\{g_n\}$. Finally, $g_n \in \mathscr{A}_i^n$ for some i. Hence, $P_n^i(g_n)=p_n(f, g_n, A_i^n)$, and an application of Lemma 1 completes the proof.

REMARKS. (i) If $K=[a, b]$, then, under the assumptions of Theorem 2, $\limsup_{n\to\infty} \lambda_n^l(f)=\infty$.

(ii) In Theorem 2, the assumption concerning the number of alternants of f is satisfied if there exists a positive integer M such that $|E_n(f)| \leqq n+2+M$ for all n. There is a sequence of indices $\{n(k)\}$ such that $B_{n(k)}(f) \neq B_{n(k)+1}(f)$. Then $E_{n(k)}(f)$ has at most M^M alternants.

References

[1] J. R. Angelos, M. S. Henry, E. H. Kaufman and T. D. Lenker, Local Lipschitz constants, *J. Approx. Theory*, **43** (1985), 53—63.

[2] J. R. Angelos, M. S. Henry, E. H. Kaufman, A. Kroó and T. D. Lenker, Local and global Lipschitz constants, *J. Approx. Theory*, **46** (1986), 137—156.

[3] M. W. Bartelt, On Lipschitz conditions, strong unicity, and a theorem of A. K. Cline, *J. Approx. Theory*, **14** (1975), 245—250.

[4] E. W. Cheney, *Induction to Approximation Theory*, McGraw-Hill (New York, 1966).

[5] A. K. Cline, Lipschitz conditions on uniform approximation operators, *J. Approx. Theory*, **8** (1973), 160—172.

[6] M. S. Henry and J. A. Roulier, Lipschitz and strong unicity constants for changing dimension, *J. Approx. Theory*, **22** (1978), 85—94.

[7] A. V. Kolushov, Differentiability of the best approximation operator, *Math. Notes Acad. Sci. USSR*, **29** (1981), 295—306.

[8] A. Kroó, The Lipschitz constant of the operator of best approximation, *Acta Math. Acad. Sci. Hungar.*, **35** (1980), 279—292.

[9] A. Kroó, Differential properties of the operator of best approximation, *Acta Math. Acad. Sci. Hungar.*, **30** (1977), 319—331.

[10] I. P. Natanson, *Constructive Function Theory*, Vol. III, Ungar Publishing Co. (New York, 1965).

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DEPARTMENT OF MATHEMATICS
CHRISTOPHER NEWPORT COLLEGE
NEWPORT NEWS, VA 23606 USA

DEPARTMENT OF MATHEMATICS
OLD DOMINION UNIVERSITY
NORFOLK, VA 23508 USA

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E-RINGS AS LOCALIZATIONS OF ORDERS

Theodore G. FATICONI (Bronx)*

Dedicated to Professor Adolph J. Faticoni on his 60th birthday

Introduction. Throughout this work, the term group refers to a torsion-free abelian group of finite rank.

An *E*-ring as defined in [13] is a ring R for which the map $R\to\mathrm{End}_Z(R^+)$, sending $x\in R$ to left multiplication by x, is an isomorphism. The importance of *E*-rings is seen in the classification problems of the End (A)-module structure of a group A. For example, *E*-rings are the building blocks for those groups A which are projective [10], finitely generated [12], or serial [6] left End (A)-modules. *E*-rings are also a source of many illuminating examples of various group theoretic properties. For example, each countable reduced torsion-free group is a pure subgroup of an *E*-ring [4], and there exist strongly indecomposable *E*-rings of prescribed cardinality [3]. Despite this utility, examples of torsion-free *E*-rings of finite rank previous to [11] were restricted to p-pure subrings of the p-adic integers, p a rational prime. In [11], R. Pierce and C. Vinsonhaler demonstrated that a fixed number field F is p-realizable (i.e. the field of fractions of an integrally closed p-local *E*-ring) for infinitely many rational primes p. To establish the p-realizability of F, Pierce and Vinsonhaler studied coset conditions in the Galois group of the Galois closure of F. Unfortunately, their techniques do not readily provide for the construction of more general classes of *E*-rings, nor do they afford much flexibility in the group structure of the implied p-local *E*-rings. Such flexibility is desirable if *E*-rings are to be used in the construction of groups A possessing subtle End (A)-module structure.

In the present paper, we consider torsion-free *E*-rings R of finite rank which are integrally closed in their field of fractions F. As an integrally closed subring of F is a localization of the ring J of algebraic integers in F, we determine which localizations of J are *E*-rings. This idea is implicit in [11] and [9]. However, in contrast to the approach in [11], we choose to classify *E*-rings via a condition on the maximal ideals of J (Lemma 2.1). From this point of view it is easily shown that a minimal field extension $F/\mathbf{Q}$ is p-realizable precisely when p splits in F (Corollary 2.4). Further, these techniques provide an uncomplicated scheme for constructing local *E*-rings R of specified rank and residue degree (Proposition 2.6). For number fields F we show that *E*-rings are densely distributed in the lattice $\mathscr{L}(F)$ of subrings of F containing J (Theorem 3.6) and that $\mathscr{L}(F)$ contains an uncountable rigid class of *E*-rings $\mathscr{E}$ such that each $R\in\mathscr{E}$ is homogeneous of type type (Z) (Proposition 3.9). This flexibility in the rank, p-rank, and ideal structure of R is not found in examples from [2], [6], [9], [11].

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A more detailed description of the sections follows.

Section 1 is a collection of preliminary results on number theory and localization. Results not appearing in our standard references [7] and [8] are proved.

Section 2 begins with the ideal theoretic classification of E-rings in $\mathcal{L}(F)$. It is then easily shown that a minimal field extension $F/\mathbf{Q}$ is p-realizable iff p splits in F. Borrowing a construction from [11], we construct for each pair of integers $n>m\geqq 1$ a local E-ring of rank n and with residue degree m (Proposition 2.6).

Now let F be a number field. Using the lattice isomorphism from Section 1 and the classification from Section 2, Section 3 illustrates that E-rings are densely distributed in $\mathcal{L}(F)$ i.e. if $[S_0, S_1]$ is a closed interval in the lattice $\mathcal{L}(F)$ and if $[J, S_0]\cup[S_1, F]$ is finite, then there is a closed interval $[R_0, R_1]\subset[S_0, S_1]$ such that $[J, R_0]\cup[R_1, F]$ is finite, and for which each $R\in[R_0, R_1]$ is an E-ring. (See Theorem 3.6.) The paper closes by constructing an uncountable rigid class $\mathcal{E}\subset\mathcal{L}(F)$ such that each $R\in\mathcal{E}$ is a homogeneous E-ring of type $\operatorname{type}(\mathbf{Z})$.

1. Preliminaries

The basic references are [1] and [5] for group theory, and [7] for ring theory. We use [8] as a reference for number theory.

At all times, p denotes a rational prime, E and F denote finite field extensions of $\mathbf{Q}$, J_E denotes the ring of algebraic integers in E, and $\mathcal{L}(E)$ denotes the lattice of subrings of E containing J_E. We let $J=J_F$ and we use $\operatorname{spec}(R)$ to denote the set of maximal ideals of a ring R.

Most of the number theory used in the sequel can be found in [11] or the first two chapters of [8]. We list those ideas which are central to our discussion. Let E be a subfield of F, let $P\in\operatorname{spec}(J_E)$, and let $M\in\operatorname{spec}(J)$. We say that M *lies over* P if $P\subset M$, and we let $\lambda_F(P)=\{M\in\operatorname{spec}(J)|M \text{ lies over } P\}=\{M\in\operatorname{spec}(J)|M\cap E=P\}$. For unramified $P\in\operatorname{spec}(J_E)$, $[F:E]=\Sigma[J/M\colon J_E/P]$ where the sum is indexed by $\lambda_F(P)$. Thus for distinct $P, P'\in\operatorname{spec}(J_E)$, $\lambda_F(P)$ and $\lambda_F(P')$ are disjoint finite sets. If $|\lambda_F(P)|\geqq 2$, we say that P *splits in* F, and if $|\lambda_F(P)|=[F:E]$, we say that P *splits completely* in F. Thus P splits completely in F iff $J/M\cong J_E/P$ for each $M\in\lambda_F(P)$. By [8, page 162, Theorem 6],

(1.1) Infinitely many $P\in\operatorname{spec}(J_E)$ split completely in F.

Now let K denote the Galois closure of F, let $G=\operatorname{Gal}(K/\mathbf{Q})$ and let $H=\operatorname{Gal}(K/F)\subset G$. Let $M\in\operatorname{spec}(J_K)$ and let $P=M\cap F$. The *decomposition group of* M is $C(M)=\{g\in G|gM=M\}$. Then by [11, page 18],

(1.2) For unramified M, $C(M)\cap H\cong\operatorname{Gal}(J_K/M|J/P)$ is a cyclic group.

Hence, P splits completely in K iff $C(M)\cap H=\{1\}$.

Let $R_0, R_1\in\mathcal{L}(F)$. If $R_0\subset R_1$, we let $[R_0, R_1]$ denote the *closed interval* in $\mathcal{L}(F)$ with endpoints R_0 and R_1. Given a closed interval $[R_0, R_1]$ in $\mathcal{L}(F)$, then by convention $R_0\subset R_1$ and $R_0, R_1\in\mathcal{L}(F)$. The closed interval $[R_0, R_1]$ is *cofinite* if the closed intervals $[J, R_0]$ and $[R_1, F]$ are finite sets.

Now let R be any ring containing J. The *support of R in J* is $\sigma_F(R) = \{M \in \operatorname{spec}(J) | RM \neq R\}$, and the *divisibility of R in J* is

$$\delta_F(R) = \{M \in \operatorname{spec}(J) | RM = R\}.$$

Observe that $\sigma_F(R)$ and $\delta_F(R)$ form a partition of spec (J). In general, we use σ to denote the support of a ring and δ to denote the divisibility of a ring.

For sets $\sigma \subset \operatorname{spec}(J)$, let $J_\sigma = \bigcap \{J_M | M \in \sigma\}$, where J_M is the localization of J at the maximal ideal M. By convention, we have $J_\emptyset = F$.

Our classification of E-rings in $\mathscr{L}(F)$ is an ideal theoretic interpretation of the following due to R. Beaumont and R. Pierce.

(1.3) For $R \in \mathscr{L}(F)$, R is an E-ring iff for each proper subfield $E \subset F$, $R \neq SJ$, where $S = R \cap E$ [1, Example 14.5, Theorem 14.6].

In order to translate (1.3), it is necessary to understand how R and spec (R) arise as localizations of J. In what follows, let $R \in \mathscr{L}(F)$.

(1.4) Let $\mathscr{C} = \{c \in J | Rc = R\}$. Then $R = J[\mathscr{C}^{-1}]$. [7, page 73, Exercise 7]. Further, if $\mathscr{C}$ is any multiplicatively closed subset of J, then

$$\sigma_F(J[\mathscr{C}^{-1}]) = \{M \in \operatorname{spec}(J) | M \cap \mathscr{C} = \emptyset\}.$$

(1.5) Given $R \in \mathscr{L}(F)$, the maps $\varphi\colon \sigma_F(R) \to \operatorname{spec}(R)$ and $\psi\colon \operatorname{spec}(R) \to \sigma_F(R)$ given by $\varphi(M) = RM$ and $\psi(N) = N \cap J$ are inverse bijections. [7, Theorem 34.]

The next lemma shows that R is completely determined by its support in J. As a precise statement is unavailable in our standard references, we include a proof. However, a version of (1.6) can be found in [14, page 144].

LEMMA 1.6. *The assignments* $R \to \sigma_F(R)$ *and* $\sigma \to J_\sigma$ *define inverse lattice anti-isomorphisms between* $\mathscr{L}(F)$ *and the lattice of subsets of* spec (J).

PROOF. We leave as an exercise the (elementary) verification that the map defined by the assignment $\sigma \to J_\sigma$ reverses inclusion, take intersections to joins, and unions to intersections. That these properties hold for the map $R \to \sigma_F(R)$ is a consequence of the inverse relationship between the two maps.

Let $\sigma \subseteq \operatorname{spec}(J)$ and let $J_\sigma = R$. Certainly $\sigma \subset \sigma_F(R)$, so let $M \in \sigma_F(R)$. By (1.5) $RM = N \in \operatorname{spec}(R)$, and by [7, Theorem 113], there exists $M' \in \sigma$ such that $R_N = J_{M'}$. Observe that $N_N \cap J$ is a proper ideal of J containing M, so $M = N_N \cap J$. Similarly, $M'_{M'} \cap J = M'$. Inasmuch as $N_N = M'_{M'}$ is the unique maximal ideal of R_N, $M = M' \in \sigma$. Hence $\sigma = \sigma_F(R) = \sigma_F(J_\sigma)$.

Now let $R \in \mathscr{L}(F)$ and let $\sigma = \sigma_F(R)$. By [7, Theorem 65] each valuation ring V satisfying $J \subset R \subset V \subset F$ is of the form $V = R_N = J_M$ where $N = \operatorname{rad}(V) \cap R$ and $M = \operatorname{rad}(V) \cap J = N \cap J$. By [7, Theorem 64] R_N is a valuation ring for each $N \in \operatorname{spec}(R)$. Thus by (1.5), $\{R_N | N \in \operatorname{spec}(R)\} = \{J_{N \cap J} | N \in \operatorname{spec}(R)\} = \{J_M | M \in \sigma\}$. The local-global theorem shows $R = J_\sigma$. $\square$

For convenience, we state two consequences of (1.6).

COROLLARY 1.7. (a) *The assignment* $R \to \delta_F(R)$ *defines a lattice isomorphism from* $\mathscr{L}(F)$ *onto the lattice of subsets of* spec (J).

(b) *Given* $R \in \mathscr{L}(F)$, $[J, R]$ *is bijective with the set of subsets of* $\delta_F(R)$, *while* $[R, F]$ *is bijective with the set of subsets of* $\sigma_F(R)$.

PROOF. (a) follows from (1.6) and the fact that $\sigma_F(R)$ and $\delta_F(R)$ form a partition spec (J).

(b) Use part (a) and (1.6). □

The following well-known results of this section determine the support of subrings of F. We include proofs.

LEMMA 1.8. *Let* $R \in \mathscr{L}(F)$, *let* E *be a subfield of* F, *and let* $S = R \cap E$.

(a) *If* $M \in \sigma_F(R)$ *then* $M \cap E \in \sigma_E(S)$.

(b) *For* $P \in \operatorname{spec}(J_E)$, $P \in \sigma_E(S)$ *iff* $\lambda_F(P) \cap \sigma_F(R) \neq \emptyset$.

(c) $\sigma_F(SJ) = \bigcup\{\lambda_F(P) \mid P \in \sigma_E(S)\}$.

PROOF. (a) Let $M \in \sigma_F(R)$. Then $S(M \cap E) \subset RM \neq R$, so $1 \notin S(M \cap E) \neq S$.

(b) Let $P \in \sigma_E(S)$. Since S is a Dedekind domain, S_P is a discrete valuation domain with unique maximal ideal $P_P = S_P x$ for some $x \in P$. As localization commutes with finite intersections, we have $x^{-1} \notin S_P = R_P \cap E$. Thus $R_P P_P = R_P x \neq R_P$, which implies that $RP \neq R$. But then $RP \subset N$ for some maximal ideal N of R. By (1.5), $P \subset N \cap J \in \lambda_F(P) \cap \sigma_F(R) \neq \emptyset$. The converse is part (a).

(c) As in (1.4) $S = J_E[\mathscr{C}^{-1}]$ for some $\mathscr{C} \subset J_E$, so that $SJ = J[\mathscr{C}^{-1}]$. Let $P \in \sigma_E(S)$. Then for each $M \in \lambda_F(P)$, $M \cap \mathscr{C} = M \cap (E \cap \mathscr{C}) = P \cap \mathscr{C} = \emptyset$ by (1.4). Thus, $\lambda_F(P) \subset \sigma_F(SJ)$. On the other hand, if $M \in \sigma_F(SJ)$ then by part (a) $M \cap E \in \sigma_E(S)$, so that $\sigma_F(SJ) \subset \bigcup\{\lambda_F(P) \mid P \in \sigma_E(S)\}$. This proves (c) and completes the proof. □

Two useful results are immediate. The notation is that of (1.8).

COROLLARY 1.9. (a) $P \in \delta_E(S)$ *iff* $\lambda_F(P) \subset \delta_F(R)$ *iff* $P \in \delta_E(R)$.

(b) $R = SJ$ *iff* $\lambda_F(P) \subset \sigma_F(R)$ *for each* $P \in \sigma_E(S)$.

PROOF. Part (a) follows from (1.8a, b). We prove part (b). By (1.6) and (1.8c), if $R = SJ$ then $\sigma_F(R) = \bigcup\{\lambda_F(P) \mid P \in \sigma_E(S)\}$, so that $\lambda_F(P) \subset \sigma_F(R)$ for each $P \in \sigma$. Conversely, assume $\lambda_F(P) \subset \sigma_F(R)$ for each $P \in \sigma_E(S)$. Then given $M \in \sigma_F(R)$, $M \in \lambda_F(M \cap E)$ and $M \cap E \in \sigma_E(S)$ (1.8a). Hence each $M \in \sigma_E(R)$ is contained in some $\lambda_F(P)$, which completes the proof. □

2. Localizations of the ring of algebraic integers

Our investigation begins with an ideal theoretic classification of E-rings in $\mathscr{L}(F)$.

PROPOSITION 2.1. *For* $\sigma \subset \operatorname{spec}(J)$, *the following are equivalent.*

(a) J_σ *is an E-ring.*

(b) *For each proper subfield* E *of* F, *there exist* $M, M' \in \operatorname{spec}(J)$ *such that* $M \in \sigma$, $M' \notin \sigma$, *and* $M \cap E = M' \cap E$.

(c) *For each proper subfield* E *of* F, *there exists* $P \in \operatorname{spec}(J_E)$ *such that* $\lambda_F(P) \cap \sigma$ *is a nonempty, proper subset of* $\lambda_F(P)$.

PROOF. The equivalence (b)⇔(c) is derived from the fact that $M \in \operatorname{spec}(J)$

lies over $P\in\operatorname{spec}(J_E)$ iff $M\cap E=P$. It remains to prove the contrapositive of (a)$\Leftrightarrow$(b).

Let $\sigma\subset\operatorname{spec}(J)$, and let $J_\sigma=R$. From (1.6), $\sigma_F(R)=\sigma$. Now R is not an E-ring iff there is a proper subfield E of F such that $R=SJ$, where $S=R\cap E$ (1.3), iff $\lambda_F(P)\subset\sigma$ for each $P\in\sigma_E(S)$ (1.9b). As this is the negation of (c), the proof is complete. □

Note that (2.1) is nothing more than a translation of (1.3) into ideal theoretic terms. However, this new point of view allows us to effectively discuss the existence and contruction of E-rings in $\mathscr{L}(F)$. The remainder of this paper is devoted such a discussion.

It is well-known that if A is an integral domain with field of fractions F, then the integral closure of A in F is the ring AJ generated by A and J, and AJ/A is a finite group. Thus, A is an E-ring iff AJ is an E-ring. This and (2.1) provide a classification of (not necessarily integrally closed) E-rings having field of fractions F.

Corollary 2.2. *Let A be an integral domain with field of fractions F. Then A is an E-ring iff for each proper subfield E of F there exist $M, M'\in\operatorname{spec}(J)$ such that $1\notin AM$, $1\in AM'$, and $M\cap E=M'\cap E$.*

Proof. Let $R=AJ\in\mathscr{L}(F)$. Then A is an E-ring iff R is an E-ring iff (2.1b) holds for R iff for each proper subfield E of F, there are $M, M'\in\operatorname{spec}(J)$ such that $1\notin RM=AJM=AM$, $1\in RM'=AM'$, and $M\cap E=M'\cap E$. □

Now consider the minimal field extension $F/\mathbf{Q}$ with Galois closure $K/\mathbf{Q}$. Pierce and Vinsonhaler [11, Lemma 5.3] have shown

(2.3) $F/\mathbf{Q}$ is not p-realizable for infinitely many rational primes p iff either (i) $[F:\mathbf{Q}]=n$ is prime or (ii) $\operatorname{Gal}(K/\mathbf{Q})$ is doubly-transitive and contains an n-cycle.

In contrast to (2.3), the next result avoids the Galois closure of F entirely.

Corollary 2.4. *Let $F/\mathbf{Q}$ be a minimal field extension.*

(a) *F is p-realizable iff p splits in F. In this case J_σ is an E-ring for each nonempty proper subset σ of $\lambda_F(p)$.*

(b) *F is p-realizable for almost all primes p iff almost all primes p split in F.*

Proof. (a) Let $\sigma\in\operatorname{spec}(J)$. Then by (1.9a), J_σ is p-local iff σ is a nonempty subset of $\lambda_F(p)$. Further, as $\mathbf{Q}$ is the unique proper subfield of F, σ satisfies (2.1c) iff σ is a nonempty proper subset of $\lambda_F(p)$. Such a σ exists iff $|\lambda_F(p)|\geqq 2$. Thus, F is p-realizable iff there is a nonempty proper subset σ of $\lambda_F(p)$ iff p splits in F.

(b) Follows from part (a). □

Another result on the distribution of E-rings in a minimal field extension $F/\mathbf{Q}$ is contained in (3.8).

Remark 2.5. The following construction of a minimal field extension $F/\mathbf{Q}$ is contained in [11, Example 5.4]. For integers $n>1$, let S_n be the group of permutations of $\{1, \dots, n\}$, and let $A_n\subset S_n$ denote the group of even permutations. Identify $S_{n-1}=\{\xi\in S_n \mid \xi(n)=n\}$. To unify the discussion, let (G, H) be an element of $\{(S_n, S_{n-1}), (A_n, A_{n-1})\}$. One shows that H is a maximal subgroup of G of index n.

Hilbert has shown that there is a Galois extension $K/\mathbf{Q}$ with Gal $(K/\mathbf{Q})=G$. Let $F\subset K$ be the fixed field of H. Then the Galois correspondence shows that $F/\mathbf{Q}$ is a minimal extension of degree n. Further, $K/\mathbf{Q}$ is the Galois closure of $F/\mathbf{Q}$ since H does not contain a nontrivial normal subgroup of G.

As in [11, Example 5.4], if n is an even composite, and if $(G, H)=(A_n, A_{n-1})$, then the field F is p-realizable for almost all p. (A_n does not contain an n-cycle in this case, so (2.3) applies.)

PROPOSITION 2.6. *Let n and m be integers such that* $n>m\geqq 1$. *Then there is a discrete valuation domain R and a rational prime p such that*

(a) R is an E-ring of rank n;

(b) Rp is the unique maximal ideal of R; and

(c) $[R/Rp:Z/Zp]=m$.

If $n=m>1$, *there is a Dedekind domain R and a rational prime p which satisfy* (a), (c), *and* (b′) *Rp is a maximal ideal of R.*

PROOF. Use (2.5) to construct a minimal field extension $F/\mathbf{Q}$ of degree n with Galois closure $K/\mathbf{Q}$ such that $S_n=\mathrm{Gal}\,(K/\mathbf{Q})$. Assume $n\geqq m\geqq 1$, and select an unramified $P\in\mathrm{spec}\,(J)$ of residue degree m as follows. Because $n\geqq m\geqq 1$, there is an m-cycle $(n-m+1, \ldots, n)=c$. By the Tchebotarev Density Theorem [7, page 169], there exists an unramified $M\in\mathrm{spec}\,(J_K)$ possessing cyclic decomposition group $\langle c\rangle=C(M)=\{g\in S_n|gM=M\}$. (See (1.2).) Let $P=M\cap F$ and let $Zp=M\cap\mathbf{Q}$. Note P is unramified. It is clear from our choice of $c\in S_n$ that $C(M)\cap S_{n-1}=\{1\}$. Thus P splits completely in K (1.2), so that $J_K/M\cong J/P$. But then by (1.2),

$$C(M)\cong \mathrm{Gal}\,(J/_KM|Z/Zp)\cong \mathrm{Gal}\,(J/P|Z/Zp).$$

Hence $[J/P:Z/Zp]=m$ as required.

If $n>m\geqq 1$, then

$$[F:\mathbf{Q}]=n>m=|C(M)|=[J/P:Z/Zp]. \tag{2.7}$$

Since $n=\Sigma_{Q\in\lambda_F(p)}[J/Q:Z/Zp]$, it follows that p splits in F. Then by (2.4a) the discrete valuation domain $R=J_P\in\mathscr{L}(F)$ is an E-ring of rank n. Thus R satisfies part (a). Let $\lambda_F(p)=\{P, P_2, \ldots, P_n\}$ and note that $Rp=J_PPP_2\ldots P_n=J_PP=P_P$ is the unique maximal ideal of $R=J_P$. Thus part (b) holds. Part (c) holds because $R/Rp==J_P/P_P\cong J/P$ has degree m over Z/Zp.

In case $n=m>1$, let $n>m'\geqq 1$. As above, there are unramified $P, P'\in\mathrm{spec}\,(J)$ of residue degrees m and m' respectively. Let $Zp=P\cap\mathbf{Q}$ and $Zp'=P'\cap\mathbf{Q}$. Assume without loss of generality that $p\neq p'$. (By The Tchebotarev Density Theorem, there are infinitely many $P\in\mathrm{spec}\,(J)$ of residue degree m.) Now as in (2.7) p' splits in F, so there exists $P''\neq P'\in\mathrm{spec}\,(J)$ such that $P'\cap\mathbf{Q}=P''\cap\mathbf{Q}$. Using (1.6) choose $R\in\mathscr{L}(F)$ such that $\sigma_F(R)=\{P, P'\}$. Since $P'\neq P''$, R is an E-ring, (2.1). Then (a), (b′), and (c) follow as above. □

3. The distribution of E-rings

Let F be an algebraic number field. This section explores the distribution of E-rings and the existence of rigid classes of E-rings in $\mathscr{L}(F)$ by investigating the condition given in (2.1b). Some notation will prove useful.

Let $E_1, \ldots, E_t$ be a complete list of the proper subfields of F, and for $1 \leqq i \leqq t$ let $J_i = J_{E_i}$. Given sets $\delta, \sigma \subset \operatorname{spec}(J)$ we will call (δ, σ) an *E-pair* if for each $1 \leqq i \leqq t$ there exist $M_i \in \sigma$ and $M_i' \in \delta$ such that $M_i \cap E_i = M_i' \cap E_i$.

The role of E-pairs in the construction of E-rings is clear from the following easy consequence of (2.1).

LEMMA 3.1. *$R \in \mathscr{L}(F)$ is an E-ring iff there are disjoint, finite sets $\delta \subset \delta_F(R)$ and $\sigma \subset \sigma_F(R)$ such that (δ, σ) is an E-pair.* □

The next lemma shows that E-pairs exist in abundance.

LEMMA 3.2. *Let δ_0 and σ_0 be disjoint, finite subsets of* spec (J). *Then there are disjoint, finite sets $\delta, \sigma \subset \operatorname{spec}(J)$ such that*

(a) *$\delta_0 \subset \delta$, $\sigma_0 \subset \sigma$, and*

(b) *(δ, σ) is an E-pair.*

PROOF. We use induction to construct for each $0 \leqq s \leqq t$ disjoint, finite sets $\delta_s, \sigma_s \subset \operatorname{spec}(J)$ satisfying

(3.3s) if $1 \leqq i \leqq s$ then there are $M_i \in \sigma_s$ and $M_i' \in \delta_s$ such that $M_i \cap E_i = M_i' \cap E_i$.

Observe that δ_0 and σ_0 vacuously satisfy the condition (3.3)(0). Assume for some $0 \leqq s < t$, that we have chosen disjoint, finite sets δ_s and σ_s satisfying (3.3s). Since δ_s and σ_s are finite, and since infinitely many $P \in \operatorname{spec}(J_{s+1})$ split in F (1.1), we can select $P_{s+1} \in \operatorname{spec}(J_{s+1})$ such that

(3.4a) $$P_{s+1} \text{ splits in } F,$$

and

(3.4b) $$\lambda_F(P_{s+1}) \cap [\delta_s \cup \sigma_s] = \emptyset.$$

By (3.4a), we can choose distinct $M_{s+1}, M'_{s+1} \in \lambda_F(P_{s+1})$. Then set $\delta_{s+1} = \delta_s \cup \{M'_{s+1}\}$ and $\sigma_{s+1} = \sigma_s \cup \{M_{s+1}\}$. It is clear from the induction hypothesis and (3.4b) that δ_{s+1} and σ_{s+1} are disjoint, finite sets. Finally, since $M_{s+1}, M'_{s+1} \in \lambda_F(P_{s+1})$, $P_{s+1} = M_{s+1} \cap E_{s+1} = M'_{s+1} \cap E_{s+1}$. Thus, δ_{s+1} and σ_{s+1} satisfy (3.3) $(s+1)$, which completes the induction process.

The sets $\delta_t = \delta$ and $\sigma_t = \sigma$ are then disjoint, finite subsets of spec (J) which satisfy (3.2a and b). □

For the purposes of the following discussion, we will call a class $\mathscr{E}$ of rings an *E-class* if each $R \in \mathscr{E}$ is an E-ring. The closed interval $[R_0, R_1]$ in $\mathscr{L}(F)$ is called a *closed E-interval* if $[R_0, R_1]$ is an E-class. The lemma provides a connection between E-pairs and closed E-intervals, as well as necessary technical material for the proof of Theorem 3.6.

LEMMA 3.5. *Let $R_0, R_1 \in \mathscr{L}(F)$.*

(a) *Assume the closed interval* $[R_0, R_1]$ *exists. Then* $[R_0, R_1]$ *is an E-class if there are sets* $\delta\subset\delta_F(R_0)$ *and* $\sigma\subset\sigma_F(R_1)$ *such that* (δ, σ) *is an E-pair.*

(b) *The closed interval* $[R_0, R_1]$ *exists and is cofinite iff* $\delta_F(R_0)$ *and* $\sigma_F(R_1)$ *are disjoint, finite sets.*

PROOF. (a) $R\in[R_0, R_1]$. Then $\delta\subset\delta_F(R_0)\subset\delta_F(R)$ by (1.7a) and $\sigma\subset\sigma_F(R_1)\subset\subset\sigma_F(R)$ by (1.6). From (3.1) R is an E-ring, and hence $[R_0, R_1]$ is an E-class.

(b) By convention, the closed interval $[R_0, R_1]$ exists iff $R_0\subset R_1$ iff $\delta_F(R_0)\subset\delta_F(R_1)$ (1.7a), iff $\delta_F(R_0)\cap\sigma_F(R_1)=\emptyset$. Further, by (1.7b) $[R_0, R_1]$ is cofinite iff $\delta_F(R_0)$ and $\sigma_F(R_1)$ are finite. This proves the lemma. □

The theorem is the promised result on the dense distribution of E-rings in $\mathscr{L}(F)$.

THEOREM 3.6. *Let F be an algebraic number field, let J be the ring of algebraic integers in F, and let* $\mathscr{L}(F)$ *denote the lattice of subrings of F containing J.*

(a) *Each cofinite, closed interval in* $\mathscr{L}(F)$, *contains a cofinite, closed E-interval.*

(b) *Each E-ring in* $\mathscr{L}(F)$ *is contained in a cofinite, closed E-interval.*

PROOF. (a) Let $[S_0, S_1]$ be a cofinite, closed interval in $\mathscr{L}(F)$. Then by (3.5b), δ_0 and σ_0 are disjoint, finite sets. Using (3.2) choose disjoint, finite sets $\delta\supset\delta_0$ and $\sigma\supset\sigma_0$ such that (δ, σ) is an E-pair. Then by (1.7a) and (1.6), there exist $R_0, R_1\in\mathscr{L}(F)$ such that $\delta_F(R_0)=\delta$ and $\sigma_F(R_1)=\sigma$. An application of (3.5) shows that $[R_0, R_1]$ exists and is a cofinite, closed E-interval. Because $\delta_0\subset\delta$ and $\sigma_0\subset\sigma$, (1.7a) and (1.6) imply that $S_0\subset R_0$ and $R_1\subset S_1$. Therefore $[R_0, R_1]\subset[S_0, S_1]$ which completes the proof of part (a).

(b) Let $R\in\mathscr{L}(F)$ be an E-ring. By (3.1), there are disjoint, finite sets $\delta\subset\delta_F(R)$ and $\sigma\subset\sigma_F(R)$ such that (δ, σ) is an E-pair. Use (1.7a) and (1.6) to produce $R_0, R_1\in\mathscr{L}(F)$ such that $\delta_F(R_0)=\delta$ and $\sigma_F(R_1)=\sigma$. Then (3.5) shows that $[R_0, R_1]$ exists and is a cofinite, closed E-interval. □

We remark that (3.6b) is an immediate consequence of [9, Proposition 3.4] while (3.6a) seems to be new.

Consider an E-ring $R\in\mathscr{L}(F)$. By (3.1), there are disjoint, finite sets $\delta\subset\delta_F(R)$ and $\sigma\subset\sigma_F(R)$ such that (δ, σ) is an E-pair. Since δ is finite, we may assume that (δ', σ) is not an E-pair for any proper subset $\delta'\subset\delta$. It follows from (2.1c) that $\lambda_F(p)\subsetneqq\delta$ for any rational prime p. But then by (1.9a), $R_0p\neq R_0$ for each rational prime p. Hence $R_0\cap\mathbf{Q}=Z$. Since R_0 is known to be homogeneous, we have shown

COROLLARY 3.7. *If* $R\in\mathscr{L}(F)$ *is an E-ring, then R contains a homogeneous E-ring* $R_0\in\mathscr{L}(F)$ *of type* type(Z). □

Theorem 3.6 and Corollary 3.7 extend [9, Corollary 3.5].

It is natural to ask if there are cofinite, closed E-intervals $[R_0, R_1]$ such that $[J, R_0]$ or $[R_1, F]$ has exactly two elements. The following proposition in conjunction with (3.5) shows this to be true.

PROPOSITION 3.8. (a) *Let* $F/\mathbf{Q}$ *be a minimal field extension and let p be a rational prime which splits in F. Given distinct* $M_0, M_1\in\lambda_F(p)$, *then* $(\{M_0\}, \{M_1\})$ *is an E-pair.*

(b) *Let p be a rational prime which splits completely in the field extension* $F/\mathbf{Q}$ *and let* $M\in\lambda_F(p)$. *Let* $\delta=\{M\}$ *and let* $\sigma=\lambda_F(p)\setminus\{M\}$. *Then* (δ, σ) *is an E-pair.*

PROOF. (a) Since the only proper subfield of F is $\mathbf{Q}$, and since $M_0, M_1\in\lambda_F(p)$, $(\{M_0\}, \{M_1\})$ is an E-pair.

(b) Let E be a proper subfield of F and let $P=M\cap E$. Then by [8, page 25], P splits completely in F, so that $\lambda_F(P)\neq\{M\}$. Thus (δ, σ) is an E-pair by (3.1). □

A class $\mathscr{E}$ of groups is *rigid* if $\operatorname{Hom}(A, B)=0$ for distinct $A, B\in\mathscr{E}$. (See [1] or [5].) The next proposition indicates the diversity of group structure in the class of E-rings in $\mathscr{L}(F)$.

PROPOSITION 3.9. *There exists an uncountable rigid E-class* $\mathscr{E}\subset\mathscr{L}(F)$ *such that each* $R\in\mathscr{E}$ *is homogeneous of type* $\operatorname{type}(Z)$.

PROOF. By (1.1) there is a sequence of distinct rational primes $\Delta=(p_0, p_1, \dots)$ such that p_i splits completely in F for each $i\geqq 0$. Write $\lambda_F(p_0)=\{N_1, \dots, N_n\}$ and for each $i\geqq 1$ choose distinct $M_i, M_i'\in\lambda_F(p_i)$. Next, let $\Sigma=\Pi_{i\geqq 1}\{M_i, M_i'\}$, and choose any uncountable subset $\Lambda\subset\Sigma$. Considering $\lambda\in\Lambda$ as a set, let $R(\lambda)\in\mathscr{L}(F)$ be the ring with divisibility $\delta_F(R(\lambda))=\{N_2, \dots, N_t\}\cup\lambda$. We will show that $\mathscr{E}=\{R(\lambda)|\lambda\in\Lambda\}$ is an uncountable rigid class of E-rings of type $\operatorname{type}(Z)$.

Given distinct $\lambda, \mu\in\Lambda$, $\delta_F(R(\lambda))\neq\delta_F(R(\mu))$, so that by (1.7a), $R(\lambda)\neq R(\mu)$. Since $\mathscr{E}$ is then bijective Λ, E is uncountable. Now by (3.9b) $(\{N_1\}, \{N_2, \dots, N_t\})$ is an E-pair, so $R(\lambda)$ is an E-ring, (3.1). Further, λ has been chosen so that $\lambda_F(p)\subsetneqq\delta_F(R(\lambda))$ for each rational prime p. As in the proof of (3.8), $R(\lambda)$ is homogeneous of type $\operatorname{type}(Z)$. Thus, $\mathscr{E}$ is an uncountable E-class of homogeneous groups of type $\operatorname{type}(Z)$.

It remains to prove that $\mathscr{E}$ is a rigid class. Toward this end, we will show that $p_0\text{-rank}(R)=1$ for each $R\in\mathscr{E}$. Since $N_2, \dots, N_t\in\delta_F(R)$, $Rp_0=RJp_0=RN_1\dots RN_t=RN_1$. Further, R is a localization of J (1.4) and p_0 splits completely in F, so there are isomorphisms $R/RN_1\cong J/N_1\cong Z/Zp_0$. Thus, $p_0\text{-rank}(R)=1$ as required.

Now consider distinct $\lambda, \mu\in\Lambda$ and let $f\colon R(\lambda)\to R(\mu)$ be a nonzero homomorphism. Having p_0-rank one, any proper homomorphic image of $R(\lambda)$ is p_0-divisible. Since we have shown that $\operatorname{type}(R(\mu))=\operatorname{type}(Z)$, f is a monomorphism. Define the map $F\colon R(\lambda)\to R(\mu)$ by $F(x)=f(1)x-f(x)$. Then $F(1)=0$ means $F=0$, so that f is a J-module homomorphism.

Finally, for distinct $\lambda, \mu\in\Lambda$, there is an integer $n\geqq 1$ such that $\lambda(n)\neq\mu(n)$, where $\lambda(n)$ denotes the n^{th} term in the sequence λ. Let $M=\lambda(n)$, and observe that from our choice of $\delta_F(R(\lambda))$, $R(\lambda)=R(\lambda)M=R(\lambda)M^k$ for each integer k while $R(\mu)\neq R(\mu)M$. Since f is a J-module homomorphism, $f(R(\lambda))=f(R(\lambda))M^k$ is a subset of $R(\mu)M^k$. But the Krull Intersection Theorem states that $\bigcap_k [R(\mu)M]^k=0$ for proper ideals $R(\mu)M$ of $R(\mu)$. Hence $f(R(\lambda))=0$, proving $\mathscr{E}$ is a rigid class. This completes the proof of the proposition. □

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References

[1] D. M. Arnold, *Finite Rank Torsion-free Abelian Groups and Rings,* Lecture Notes in Mathematics 931, Springer-Verlag (New York, 1982).

[2] R. Bowshell, P. Schultz, Unital rings whose additive endomorphisms commute, *Math. Ann.*, **228** (1977), 197—214.

[3] M. Dugas, A. Mader, C. Vinsonhaler, Large *E*-rings exist, *J. Alg.*, **108** (1987), 88—101.

[4] T. G. Faticoni, Each countable reduced torsion-free commutative ring is a pure subring of an *E*-ring, *Comm. Alg.*, **15** (1987), 2545—2564.

[5] L. Fuchs, *Infinite Abelian Groups,* Volumes I and II, Academic Press (New York—London, 1970, 1973).

[6] J. Hausen, Finite rank torsion-free abelian groups uniserial over their endomorphism rings, *Proc. Am. Math. Soc.*, **93** (1985), 227—231.

[7] I. Kaplansky, *Commutative Rings,* Allyn and Bacon, Inc. (Boston, 1970).

[8] S. Lang, *Algebraic Number Theory,* Addison-Wesley (New York, 1970).

[9] A. Mader, C. Vinsonhaler, Torsion-free *E*-modules, *J. Alg.*, **115** (1988), 401—411.

[10] G. Niedzwicki, J. Reid, Abelian groups cyclic and projective as modules over their endomorphism rings, to appear in *J. Alg.*

[11] R. S. Pierce, C. Vinsonhaler, *Realizing algebraic number fields,* Lecture Notes in Mathematics 1006, Springer-Verlag (New York, 1982/3), 49—96.

[12] J. Reid, *Abelian groups finitely generated over their endomorphism rings,* Lecture Notes in Mathematics 874, Springer-Verlag (New York, 1981), 41—52.

[13] P. Schultz, The endomorphism ring of the additive group of a ring, *J. Aust. Math. Soc.*, **15** (1973), 60—69.

[14] E. Weiss, *Algebraic Number Theory,* Mc-Graw-Hill (New York, 1963).

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DEPARTMENT OF MATHEMATICS
FORDHAM UNIVERSITY
BRONX, NEW YORK, 10458
USA

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THE EQUATION $u_x u_y = 0$ FACTORS

A. M. BRUCKNER (Santa Barbara), G. PETRUSKA (Budapest), D. PREISS (Prague)
and B. S. THOMSON (Burnaby)

In a recent correspondence with one of the authors, Lee Rubel asked whether every solution (on the plane $\mathbf{R}^2$) of the partial differential equation

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0$$

must be a function of one variable. For solutions in $\mathscr{C}^2$, the question is easily answered: differentiate $u_x u_y = 0$ with respect x and y, then multiply these equation by u_x and u_y respectively, we obtain after addition $(u_x^2 + u_y^2)u_{xy} = 0$. If in a point $p \in \mathbf{R}^2$ we had $u_{xy} \neq 0$ then $u_x^2 + u_y^2 = 0$ would imply $u = \text{const}$, thus one finds that a solution must satisfy $u_{xy} = 0$ on $\mathbf{R}^2$, whence u is of the form $u(x, y) = f(x) + g(y)$ and $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f'(x)g'(y)$. If $g'(y_0) \neq 0$, then $f' \equiv 0$, so $u(x, y) = g(y) + \text{constant}$. In a later correspondence Rubel mentioned that W. Jockusch had obtained an affirmative answer under the assumption that $u \in \mathscr{C}^1(\mathbf{R}^2)$.

The purpose of the present note is to show that Rubel's question has an affirmative answer whenever the equation makes sense; that is, whenever both partials of u exist on all of $\mathbf{R}^2$. In fact, our theorem shows a bit more. If u is continuous in each variable separately and at each point in $\mathbf{R}^2$ one of the partials vanishes, then u is a function of one variable. We do not assume that both partials exist at every point.

Our method is to first establish the result under the assumption that u is continuous and then to show that the hypotheses of our theorem actually imply continuity.

LEMMA 1. *Let u be continuous on a neighbourhood of a closed rectangle $R \subset \mathbf{R}^2$, let p be the lower left vertex of R, and let C be the component of the set $\{q \in R;\ u(q) = u(p)\}$ containing p. Suppose that at each point of R at least one of the partial derivatives exists and vanishes. Then C intersects at least one of the two edges of R not containing p.*

PROOF. If C does not intersect either of the two edges of R not containing p, we use the compactness of $\{q \in R;\ u(q) = u(p)\}$ to find disjoint relatively open subsets U and V of R such that $p \in U$, the union of the two edges of R not containing p is a subset of V, and $\{q \in R;\ u(q) = u(p)\} \subset U \cup V$. (This follows, for example, from

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the equality of components and quasi-components in compact spaces. See [1, § 47, II, Theorem 2].) Since u is continuous and R is compact, we may find $\varepsilon>0$ such that $|u(q)-u(p)|\geqq 2\varepsilon(|q_1-p_1|+|q_2-p_2|)$ for every $q\in R\setminus(U\cup V)$. Let r be the largest point in the lexicographic order of the set $\{q\in\overline{U};\ |u(q)-u(p)|\leqq\varepsilon(|q_1-p_1|+|q_2-p_2|)\}$. Since $|u(s)-u(p)|\geqq 2\varepsilon(|s_1-p_1|+|s_2-p_2|)$ for every $s\in\overline{U}\setminus U$, r belongs to U. Consequently, $|u(q)-u(r)|\geqq\varepsilon(|q_1-r_1|+|q_2-r_2|)$ whenever q is sufficiently close to r, $q_1\geqq r_1$, and $q_2\geqq r_2$. But this contradicts the assumption that $u_x=0$ or $u_y=0$.

LEMMA 2. *Suppose u, R, and p satisfy the conditions of Lemma* 1. *Then the value of u at one of the corners of R adjacent to p is equal to $u(p)$.*

PROOF. Let r be a corner of R adjacent to p such that the component C_0 of the set $\{q\in R;\ u(q)=u(p)\}$ containing p meets the edge not containing p and having r as one of the end points. Using Lemma 1 in a symmetric situation, we see that the component C_1 of the set $\{q\in R;\ u(q)=u(r)\}$ containing r meets at least one of the two edges of R not containing r. But then $C_0\cap C_1\neq\emptyset$, which immediately shows that $u(r)=u(p)$.

LEMMA 3. *Let u be a continuous function defined in an open rectangle. Suppose for each point of this rectangle at least one of the partials of u exists and vanishes. Then u is a function of one variable.*

PROOF. If u is constant on all vertical lines, the statement holds true. Thus suppose that there are two points p and q with the same abscissa and with different values of u. Then for every point r with the same abscissa either $u(r)\neq u(p)$ and we apply Lemma 2 to rectangles with two corners at r and p to deduce that u is constant on the horizontal line passing through r, or $u(r)\neq u(q)$ and we apply the same Lemma to rectangles with two corners at r and q.

LEMMA 4. *Let u be defined on the plane $\mathbf{R}^2$ and continuous in each variable separately. Suppose that at each point of the plane at least one of the partial derivatives exists. Then every nonempty closed set $P\subset\mathbf{R}^2$ contains a portion on which u is continuous.*

PROOF. Let $p\in P$. Since at least one of the partial derivatives of u at p exists, there is $n=1, 2, \dots$ such that for every q in $\mathbf{R}^2$ with $|p-q|<1/n$ and with the same abscissa (or perhaps ordinate) the inequality $|u(p)-u(q)|<n|p-q|$ is satisfied. For each $n=1, 2, \dots$ let A_n denote those points of P for which the above inequality holds with respect to the abscissas and B_n the corresponding set with respect to the ordinates. The Baire Category Theorem implies that for some n one of the sets A_n or B_n is dense in a portion Q of P. Suppose that it is A_n. To show that u is continuous on Q it suffices to prove that for each point $p\in Q$,

$$u(p)=\lim_{q\to p;\,q\in A_n} u(q).$$

Let $p\in Q$ and $\varepsilon>0$. Because u is separately continuous, there is $0<\delta<\varepsilon/(n+1)$ such that if r has the same abscissa as p and $|r-p|<\delta$ then $|u(r)-u(p)|<\varepsilon/(n+1)$. Let $q\in A_n$ and satisfy $|q-p|<\delta$. Let r be the point with the same abscissa as p and the same ordinate as q. Then $|u(q)-u(p)|\leqq|u(q)-u(r)|+|u(r)-u(p)|\leqq n|q-r|+\varepsilon/(n+1)\leqq n\varepsilon/(n+1)+\varepsilon/(n+1)=\varepsilon$.

THEOREM. *Let u be a function defined in* $\mathbf{R}^2$ *and continuous in each variable separately. Suppose for each point of* $\mathbf{R}^2$ *at least one of the partials of u exists and vanishes. Then u is a function of one variable, i.e.* $u_x=0$ *or* $u_y=0$ *identically.*

PROOF. By Lemma 3, it suffices to prove u is continuous. Let E be the interior of the set of continuity points of u. By Lemma 4 we know that E is dense in $\mathbf{R}^2$. We show $\mathbf{R}^2\setminus E$ is empty. If this were not so, there would be, by Lemma 4 an open square S such that $P=S\setminus E$ is nonempty and the restriction of u to P is continuous. We show in fact that u, as a function on $\mathbf{R}^2$ is continuous at each point of P, and this implies a contradiction immediately. Let $p\in P$ and $\varepsilon>0$. Let H and V be the horizontal and vertical lines through p, respectively. Then there is $\delta>0$ such that if $s\in P\cup H\cup V$ and $|s-p|<\delta$ then $|u(s)-u(p)|<\varepsilon$. Now let $q\in E$ satisfy the inequality $|q-p|<\delta$.

If u is not constant in any neighbourhood of q, by Lemma 3, u is a function of one variable, say the first, on every rectangle T satisfying $q\in T\subset E$. Let W be the vertical line through q. It follows from the assumption of separate continuity that there exists a segment $J\subset W$ containing q and a point $s\in S\cap W\cap(P\cup H)$ such that $|s-p|<\delta$ and u is constant on J. The inequalities

$$|u(q)-u(p)| = |u(s)-u(p)| < \varepsilon$$

establish the continuity of u at p.

In case u is constant in some neighbourhood of q there are two cases. Either there is $r\in S\cap(P\cup H\cup V)$, $|r-p|<\delta$ with $u(r)=u(q)$; in that case $|u(q)-u(p)|<\varepsilon$ or, there is $r\in S\cap E$, $|r-p|<\delta$ such that u is not constant in any neighbourhood of r and $u(r)=u(q)$. In that case we apply the previous argument to r and once again obtain $|u(q)-u(p)|<\varepsilon$.

Thus u is continuous on all of S and P is empty, a contradiction.

REMARKS. (i) It is easy to construct examples to show that the assumption of separate continuity cannot be dropped in the statement of the Theorem.

(ii) One can replace $\mathbf{R}^2$ by any rectangular region in the statement of the Theorem. The theorem fails, however, for any region that is not rectangular, even for $\mathscr{C}^\infty$ functions. On the other hand, any counterexample on a nonrectangular region must be constant on some set with nonempty interior.

(iii) Finally let us point out that there is no analogous result in higher dimensions. For example the function

$$f(x, y, z) = \begin{cases} x\exp(-z^{-2}), & \text{if } z>0 \\ 0, & \text{if } z=0 \\ y\exp(-z^{-2}), & \text{if } z<0 \end{cases}$$

is in $\mathscr{C}(\mathbf{R}^3)$ and satisfies

$$\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}=0$$

showing that $u_x u_y u_z=0$ does not factor. Note that this example shows even that in dimensions higher than two the equation $u_x u_y=0$ does not factor.

Reference

[1] K. Kuratowski, *Topology II*. Academic Press, 1968.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA
EÖTVÖS UNIVERSITY, BUDAPEST
CHARLES UNIVERSITY, PRAGUE
SIMON FRASER UNIVERSITY, BURNABY

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A FEJÉR TYPE EXTREMAL PROBLEM

SZ. GY. RÉVÉSZ (Budapest)

1. Let us denote by $\mathcal{T}_n$ the set of trigonometric polynomials with degree $\leqq n$ and $\mathcal{T}=\bigcup_1^\infty \mathcal{T}_n$. C. Caratheodory and L. Fejér investigated several extremal problems concerning nonnegative trigonometric polynomials. One useful result of Fejér answers the following question: "How large can the coefficient of $\cos x$ be in a nonnegative polynomial of $\mathcal{T}_k$ with constant term 1?" Formally, we define

$$\mathcal{S}_k := \left\{g\in\mathcal{T}_k\colon\ g\geqq 0,\ g(x)=1+\sum_1^k b_n\cos nx\right\}$$

and ask for

$$\omega(k):=\sup\left\{b_1=\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos x\,dx\colon\ g\in\mathcal{S}_k\right\}.$$

The assumption that g is a pure cosine polynomial does not restrict generality and will be assumed in the sequel. Fejér obtained in [2] (see also in [3] I p. 869—870 or [6] II Ex. VI.52)

$$\omega(k)=2\cos\frac{\pi}{k+2}. \tag{1.1}$$

In the present paper we calculate the following companion of the above problem of Fejér. Let

$$\alpha(k):=\sup\left\{a_1=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx\colon\ f\in\mathcal{F}_k\right\}, \tag{1.2}$$

where

$$\mathcal{F}_k := \mathcal{F} := \left\{f\in\mathcal{T}\colon\ f\geqq 0,\ f(x)=1+a_1\cos x+\sum_{k+1}^\infty a_n\cos nx\right\}. \tag{1.3}$$

If $f\in\mathcal{F}$ and $g\in\mathcal{S}_k$ we obtain in view of nonnegativity

$$0\leqq\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x+\pi)\,dx = 2-a_1 b_1,$$

and so taking supremum we obtain from (1.1)

$$\alpha(k)\leqq 2/\omega(k)=1/\cos\frac{\pi}{k+2}. \tag{1.4}$$

Our result shows that this estimate is sharp.

THEOREM. *We have* $\alpha(k)=\dfrac{1}{\cos\dfrac{\pi}{k+2}}$.

The proof uses results of Caratheodory and Fejér. Finally we use some linear algebra to calculate $\alpha(k)$. Computations will be marked out and omitted.

2. Let $f\in\mathcal{F}$ and F be defined by $F(0)=1$, $F\in\mathbf{R}[z]$ and $\operatorname{Re} F(e^{ix})=f(x)$, that is with f in (1.3) we put

$$F(z):=1+a_1z+\sum_{k+1}^{\infty}a_nz^n\in\mathbf{R}[z]. \tag{2.1}$$

The condition $f\geqq 0$ is equivalent to $\operatorname{Re} F\geqq 0$ in $|z|\leqq 1$. Therefore

$$G(z):=\frac{1-F(-z)}{1+F(-z)}\in\mathbf{R}(z) \tag{2.2}$$

is regular for $|z|\leqq 1$ and $f\geqq 0$ is equivalent to

$$|G(z)|\leqq 1 \quad (|z|\leqq 1). \tag{2.3}$$

In a sufficiently small neighbourhood of 0, but then for all $|z|<1$ we have with some $H\in R(z)$

$$G(z)=\frac{1}{1-\dfrac{1-F(-z)}{2}}-1=bz+b^2z^2+\ldots+b^kz^k+H(z)\,z^{k+1} \quad (b:=a_1/2). \tag{2.4}$$

Denote the set of regular functions on some domain D by $\mathcal{O}(D)$. Put

$$\alpha'(k):=\max\{b:\ \exists G,H\in\mathcal{O}(|z|<1),\ |G|<1,\ G(z)=bz+\ldots+b^kz^k+H(z)z^{k+1}\}, \tag{2.5}$$

which exists in view of the Vitali—Montel theorem. It is easy to observe that

$$\alpha'(k)=\frac{1}{2}\alpha(k). \tag{2.6}$$

Clearly, if $\alpha'(k)=r$, then for any corresponding extremal function G $\sup_{|z|<1}|G|=1$, and so the value of the Caratheodory—Fejér type extremal quantity

$$\mu(r):=\inf\Big\{\sup_{|z|<1}|g|:\ g(z)=rz+\ldots+r^kz^k+h(z)\,z^{k+1},\ g,h\in\mathcal{O}(|z|<1)\Big\} \tag{2.7}$$

is exactly 1.

Now we can apply the theorem of Caratheodory and Fejér, cf. [1], [4] and [3] II. p. 186, to the above special case. We obtain

$$1=\mu(r)=\max\{|\lambda|:\ \det(C_r-\lambda I)=0\}, \tag{2.8}$$

where

$$C_r = \begin{pmatrix} r^k & \dots & r^2 & r & 0 \\ r^{k-1} & \dots & r & 0 & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ r & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Taking into account that (2.9) has only real eigenvalues, (2.6)—(2.8) entails

$$\frac{1}{2}\alpha(k) \geqq r_0 := \min\{r>0:\ \det(C_r-I)\det(C_r+I)=0\}. \tag{2.10}$$

3. Now we determine the value of r_0. Denote

$$P_k(r) := -\det(C_r-I), \quad Q_k(r) := \det(C_r+I), \tag{3.1}$$

and note that since

$$\frac{1}{2}\alpha(k) \geqq r_0 = \min\{r>0:\ P_k(r)Q_k(r)=0\}, \tag{3.2}$$

we always have $1/2<r_0<1$. First we compute P_k and Q_k for $k=0, 1, 2, 3$ and 4.

$$\begin{cases} P_0(r)=1, \ P_1(r)=r-1, \ P_2(r)=1-2r^2, \\ Q_0(r)=1, \ Q_1(r)=r+1, \ Q_2(r)=1, \\ P_3(r)=r^2+r-1, \ P_4(r)=1-3r^2, \\ Q_3(r)=1+r-r^2, \ Q_4(r)=1-r^2. \end{cases} \tag{3.3}$$

When calculating $\det(C_r-\lambda I)$ we can subtract from each column the next (starting from the left) and obtain

$$\det(C_r-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & \dots & r & 0 \\ r & -\lambda & 0 & \dots & 0 & 0 \\ 0 & r & -\lambda & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ r & 0 & 0 & \dots & -\lambda & 0 \\ 0 & 0 & 0 & \dots & r & -\lambda \end{vmatrix}.$$

Therefore, expanding by the last column, we obtain for the polynomials (3.1) the determinant representations of order k below.

$$P_k(r) = \begin{vmatrix} -1 & 0 & \dots & 0 & r \\ r & -1 & \dots & r & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & r & \dots & -1 & 0 \\ r & 0 & \dots & r & -1 \end{vmatrix}, \quad Q_k(r) = \begin{vmatrix} 1 & 0 & \dots & 0 & r \\ -r & 1 & \dots & r & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & r & \dots & 1 & 0 \\ r & 0 & \dots & -r & 1 \end{vmatrix}. \tag{3.4}$$

LEMMA 1. *The polynomials* (3.1) *satisfy the recursive relations*

$$P_k(r) = P_{k-2}(r) - r^2 P_{k-4}(r), \quad Q_k(r) = Q_{k-2}(r) - r^2 Q_{k-4}(r).$$

Using (3.3) and (3.4) Lemma 1 can be deduced by elementary determinant transformations — we leave the details to the reader. Now for any particular $1/2 < r < 1$ denote

$$\alpha := \frac{1}{2} + i\sqrt{r^2 - \frac{1}{4}}, \quad \beta := \bar{\alpha} = \frac{1}{2} - i\sqrt{r^2 - \frac{1}{4}}. \tag{3.5}$$

A recursive recurrence relation

$$x_{n+1} = x_n - r^2 x_{n-1} \quad (n = 1, 2, \ldots) \tag{3.6}$$

determines the sequence x_n (see e.g. [5], Ch. V.4) as

$$x_n = A\alpha^n + B\beta^n \quad (n = 1, 2, \ldots), \tag{3.7}$$

where A and B are the unique solution of the system

$$\begin{cases} A + B = x_0 \\ A\alpha + B\beta = x_1 \end{cases}. \tag{3.8}$$

In particular, if x_0 and x_1 are real, then $B = \bar{A}$ is immediate. Denote

$$\varphi := \arg\alpha = \operatorname{arc\,tan}\sqrt{4r^2 - 1} \in \left(0, \frac{\pi}{2}\right). \tag{3.9}$$

Lemma 1 and (3.3)—(3.9) give that with some A_1, A_2, A_3 and A_4 with

$$\arg A_1 = \varphi, \quad \arg A_2 = \varphi/2, \quad \arg A_3 = \varphi - \pi/2, \quad \arg A_4 = \frac{\varphi - \pi}{2} \tag{3.10}$$

we have

$$\begin{cases} P_{2n}(r) = A_1\alpha^n + \bar{A}_1\bar{\alpha}^n = 2\,\mathrm{Re}\,(A_1\alpha^n), & P_{2n-1}(r) = 2\,\mathrm{Re}\,(A_2\alpha^n), \\ Q_{2n}(r) = 2\,\mathrm{Re}\,(A_3\alpha^n), & Q_{2n-1}(r) = 2\,\mathrm{Re}\,(A_4\alpha^n). \end{cases} \tag{3.11}$$

Since $\mathrm{Re}\,(z) = 0$ is identical with $\arg(z) = \pi/2 + m\pi$ $(m \in \mathbf{Z})$, we are led to the equations

$$\begin{cases} P_{2n}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m+1}{2n+2}\pi \quad (m \in \mathbf{Z}), \\ P_{2n-1}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m+1}{2n+1}\pi \quad (m \in \mathbf{Z}), \\ Q_{2n}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{m}{n+1}\pi \quad (m \in \mathbf{Z}), \\ Q_{2n-1}(r) = 0 \quad \text{if and only if} \quad \varphi = \dfrac{2m}{2n+1}\pi \quad (m \in \mathbf{Z}). \end{cases} \tag{3.12}$$

Summing up, since $0 < \varphi < \pi/2$, we get

LEMMA 2. $P_k(r)Q_k(r)=0$ *if and only if* $\arctan\sqrt{4r^2-1}\,\frac{j\pi}{k+2}$ *with* $1\leqq j\leqq\frac{k+1}{2}$.

In view of Lemma 2 the roots of the polynomial $P_k(r)Q_k(r)$ are

$$\pm\frac{1}{2\cos\frac{j\pi}{k+2}}\quad\left(j=1,2,\dots,\left[\frac{k+1}{2}\right]\right). \tag{3.13}$$

According to (3.2) and (1.4) this proves the theorem.

References

[1] C. Caratheodory—L. Fejér, Über den Zusammenhang ..., *Rendiconti di Palermo*, **32** (1911), 218—239.

[2] L. Fejér, Über trigonometrische Polynome, *J. angew. Math.*, **146** (1915), 53—82.

[3] L. Fejér, *Gesammelte Arbeiten I—II*, Akadémiai Kiadó (Budapest, 1970).

[4] L. Fejér, Über gewisse Minimumprobleme der Funktionentheorie, *Math. Ann.*, **97** (1927), 104—123.

[5] A. O. Gelfond, *Calculation of finite differences* (in Russian) (Moscow—Leningrad, 1952).

[6] G. Pólya—G. Szegő, *Problems and Theorems in Analysis I—II*, Springer Verlag (Berlin—Heidelberg, 1972).

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MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H—1053 BUDAPEST, REÁLTANODA U. 13—15.

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AN INTERPOLATORY VERSION OF TIMAN'S THEOREM ON SIMULTANEOUS APPROXIMATION

KATHERINE BALÁZS, T. KILGORE (Auburn) and P. VÉRTESI (Budapest)

In 1951, A. F. Timan [10] published a theorem on the approximation of a q-times differentiable function by polynomials. Writing $\omega(\delta)$ for the modulus of continuity of f on the step δ, we have:

THEOREM A. *Let $f \in C^q[-1, 1]$. Then there is a sequence $\{P_n\}$ of polynomials of degree n or less, such that for $|x| \leqq 1$*

$$|f(x)-P_n(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^q \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right\}.$$

A refinement of this theorem laid down by R. F. Trigub [11] states that the polynomials P_n may be said in addition to satisfy for $k=0, 1, \ldots, q$

$$\text{(1)}\qquad |f^{(k)}(x)-P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right\}.$$

A further development of Timan's result is the following theorem of Telyakovskii [9] and Gopengauz [3]:

THEOREM B. *Let $f \in C^q[-1, 1]$. Then for $n \geqq 4q+5$ there exists a sequence of polynomials $\{P_n\}$ such that for $|x| \leqq 1$ and for $k=0, 1, \ldots, q$*

$$|f^{(k)}(x)-P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n}\right)\right\}.$$

From the standpoint of interpolation, we may say that the cited theorem of Gopengauz—Telyakovskii gives polynomials which interpolate the derivatives $f^{(0)}, f^{(1)}, \ldots, f^{(q-1)}$ at the points ± 1, a fact which has made this theorem useful in recent investigations of simultaneous approximation by interpolation (see, for example, Szabados [8], Muneer [6], Runck and Vértesi [7], and Balázs and Kilgore [1]). Thus, it might be of interest to establish a result, based on (1), which involves interpolation at (not necessarily) distinct points clustered near ± 1.

Additionally of interest in (1) is the fact that the step interval of ω depends upon the location of x, so that when $|x|$ is near 1 the size of the step used in ω decreases. However, in the applications just cited it seems in all cases necessary to

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replace this variable step by the uniform step $1/n$. It is possible to improve on this uniform estimate for $\omega(1/n)$, as shown by D. Leviatan [4, Theorem 3]:

THEOREM C. *In the theorem of Trigub, it is possible to replace in* (1) $\omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)$ *by* $E_{n-q}(f^{(q)})$, *obtaining for* $|x|\leqq 1$ *and* $k=0, \dots, q$

$$(2)\qquad |f^{(k)}(x)-P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k}\cdot E_{n-q}(f^{(q)})\right\}.$$

In the following theorem, we show that both (1) and (2) can be combined with certain interpolatory properties.

THEOREM. *Let* $f\in C^q[-1, 1]$. *Let* $r=\left[\frac{q+1}{2}\right]$, *and let a constant* $C>0$ *be given. Let points* $t_{0,n}, \dots, t_{r-1,n}$ *and* $s_{0,n}, \dots, s_{r-1,n}$ *be given such that for each* $n\geqq \max\{2r, C^{1/2}\}$

$$-1 \leqq t_{0,n} \leqq \dots \leqq t_{r-1,n} \leqq -1+C/n^2$$

and

$$1 \geqq s_{0,n} \geqq \dots \geqq s_{r-1,n} \geqq 1-C/n^2.$$

Then, for each such n there exists a polynomial P_n *of degree n or less, such that for* $|x|\leqq 1$ *and for* $k=0, \dots, q$

$$|f^{(k)}(x)-P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k}\cdot \omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)\right\}$$

or

$$|f^{(k)}(x)-P_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k} E_{n-q}(f^{(q)})\right\},$$

and furthermore

$$P_n(x)=f(x) \quad for \quad x\in\{t_{0,n}, \dots, t_{r-1,n}, s_{0,n}, \dots, s_{r-1,n}\}.$$

If for any specific n there exist one (or more) j and l such that

$$t_{j,n} = t_{j+1,n} = \dots = t_{j+l,n} \quad or \quad s_{j,n} = s_{j+1,n} = \dots = s_{j+l,n}$$

then in addition

$$f^{(k)}(t_{j,n}) = P_n^{(k)}(t_{j,n}) \quad for \quad k=0, \dots, l$$

or respectively

$$f^{(k)}(s_{j,n}) = P_n^{(k)}(s_{j,n}) \quad for \quad k=0, \dots, l.$$

PROOF. The first conclusion of our theorem is simply a restatement of (1) or (2), which are already known. We may assume therefore that a sequence of polynomials $\{P_n\}$ exists which satisfies (1) or (2). We may then define for each n a polynomial Q_n of degree $2r-1$ which interpolates $f(x)-P_n(x)$ at the points $t_{0,n}, \dots, t_{r-1,n}$ and $s_{0,n}, \dots, s_{r-1,n}$. The sequence of polynomials P_n+Q_n will possess the desired interpolation property. In what follows we will use the ad-hoc notation $M_n(x)$ to denote either of $E_{n-q}(f^{(q)})$ or $\omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)$, consistently throughout the rest of the proof. It is possible to complete our proof, provided that one can establish for $|x|\leqq 1$

$$(3)\qquad |Q_n(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^q M_n(x)\right\}.$$

For then the inequality of Brudnyi [2] (cf. Lorentz [5, p. 71]) or the similar inequality of Dzyadyk yields immediately

$$(4)\qquad |Q_n^{(k)}(x)| = O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k} M_n(x)\right\},$$

and for $k=0, 1, ..., q$

$$|f^{(k)}(x)-P_n^{(k)}(x)-Q_n^{(k)}(x)| \leqq |f^{(k)}(x)-P_n^{(k)}(x)|+|Q_n^{(k)}(x)| =$$

$$= O\left\{\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-k} M_n(x)\right\}$$

by use of (1) and (4), and our proof is completed. It remains only to establish (3). Indeed, it suffices, by symmetry, to establish (3) only for half of the interval $[-1, 1]$, and we will establish (3) for $-1\leqq x\leqq 0$. From here on, we will also simplify the notation by dropping the cumbersome double subscripts.

We begin by writing Q_m in the form given by Newton's representation of the interpolation on $t_0, ..., t_{r-1}, s_{r-1}, ..., s_0$, obtaining

$$(5)\qquad Q_n(x) = \begin{cases} Q_n(t_0)+ \\ +Q_n(t_1, t_0)(x-t_0)+ \\ \dots \\ +Q_n(t_{r-1}, ..., t_0)(x-t_0)\dots(x-t_{r-2})+ \\ +Q_n(s_{r-1}, t_{r-1}, ..., t_0)(x-t_0)\dots(x-t_{r-1})+ \\ +Q_n(s_{r-2}, s_{r-1}, t_{r-1}, ..., t_0)(x-t_0)\dots(x-t_{r-1})(x-s_{r-1})+ \\ \dots \\ +Q_n(s_0, ..., s_{r-1}, t_{r-1}, ..., t_0)(x-t_0)\dots(x-t_{r-1})(x-s_{r-1})\dots(x-s_1), \end{cases}$$

in which

$$Q_n(t_1, t_0):= \frac{Q(t_1)-Q(t_0)}{t_1-t_0}$$

and

$$Q_n(t_1, ..., t_0):= \frac{Q_n(t_i, ..., t_1)-Q_n(t_{i-1}, ..., t_0)}{t_i-t_0}$$

for $i=2, ..., r-1$. The rest of the coefficients are given by

$$Q_n(s_{r-1}, t_{r-1}, ..., t_0) = \frac{Q_n(s_{r-1}, t_{r-1}, ..., t_1)-Q_n(t_{r-1}, ..., t_0)}{s_{r-1}-t_0}$$

and for $k=r-2, ..., 0$ by

$$Q_n(s_k, ..., s_{r-1}, t_{r-1}, ..., t_0) =$$

$$= \frac{Q_n(s_k, ..., s_{r-1}, t_{r-1}, ..., t_1)-Q_n(s_{k+1}, ..., s_{r-1}, t_{r-1}, ..., t_0)}{s_k-t_0}.$$

We observe that Newton's representation of interpolation is valid (if the function interpolated is sufficiently smooth, and Q_m is a polynomial) regardless of whether the points coalesce totally, partially, or are distinct, or, for that matter, regardless of whether the points maintain any order relation whatsoever. It is only necessary and sufficient to interpret the difference quotients given above as the derivative which they become in any eventuality when points of interpolation coalesce. Thus, the following argument includes coalescing points from among $\{t_0, \ldots, t_{r-1}\}$ or $\{s_0, \ldots, s_{r-1}\}$ without further discussion.

We will first establish estimates for the magnitude of the coefficients in (5), Obtaining

$$|Q_n(t_i, \ldots, t_0)| = O\left(\frac{1}{n^2}\right)^{q-i} M_n(x) \tag{6}$$

for $i=0, \ldots, r-1$ and for arbitrary $x\in[-1, 1]$ and

$$|Q_n(s_k, \ldots, t_0)| = O(1)|Q_n(t_{r-1}, \ldots, t_0)| = O\left(\frac{1}{n^2}\right)^{q-r+1} M_n(x) \tag{7}$$

for $k=r-1, \ldots, 0$ and for arbitrary $x\in[-1, 1]$.

To establish the formula (6), we note simply that

$$Q_n(t_i) = (f-P_n)(t_i) \quad \text{for} \quad i = 0, \ldots, r-1,$$

whence

$$|Q_n(t_0)| = |(f-P_n)(t_0)| = O\left(\frac{1}{n^2}\right)^{q} M_n(t_0) \tag{8}$$

and for $i=1, \ldots, r-1$ there exists by the Mean Value Theorem a point z_i such that $t_0 \leqq z_i \leqq t_i$ and

$$|(f-P_n)(t_i, \ldots, t_0)| = |(f-P_n)^{(i)}(z_i)| = O\left(\frac{1}{n^2}\right)^{q-i} M_n(z_i). \tag{9}$$

Now, since $-1 \leqq t_0 \leqq z_i \leqq t_{r-1} \leqq 1+\dfrac{C}{n^2}$, we may replace $M_n(t_0)$ and $M_m(z_i)$ all by $M_n(x)$, where $x\in[-1, 1]$ is any point considered desirable, obtaining the formula (6) as a consequence.

To establish the formula (7), we first recall that for any j, k between 0 and $r-1$ we have

$$2-\frac{2C}{n^2} \leqq s_k - t_j \leqq 2. \tag{10}$$

Thus, we can say that

$$|Q_n(s_{r-1}, t_{r-1}, \ldots, t_0)| = \left|\frac{Q_n(s_{r-1}, t_{r-1}, \ldots, t_1)-Q_n(t_{r-1}, \ldots, t_0)}{s_{r-1}-t_0}\right| = \tag{11}$$

$$= O(1)(|Q_n(s_{r-1}, t_{r-1}, \ldots, t_1)|+|Q_n(t_{r-1}, \ldots, t_0)|) =$$

$$= O(1)[|Q_n(s_{r-1}, t_{r-1}, \ldots, t_2)|+|Q_n(t_{r-1}, \ldots, t_1)|+|Q_n(t_{r-1}, \ldots, t_0)| =$$

$$\ldots$$

$$= O(1)[|Q_n(s_{r-1}, t_{r-1})|+\sum_{i=0}^{r-2} |Q_m(t_{r-1}, \ldots, t_i)|.$$

In the formulation of this estimate, we have used successively the identity

$$Q_n(s_{r-1}, t_{r-1}, \dots, t_i) = \frac{Q_n(s_{r-1}, t_{r-1}, \dots, t_{i+1}) - Q_n(t_{r-1}, \dots, t_i)}{s_{r-1} - t_i}$$

and the estimate (10). It is now clearly to be seen that the "worst" term on the right side of (11) is the term $|Q_n(t_{r-1}, \dots, t_0)|$, and we thus can say that

$$|Q_n(s_{r-1}, t_{r-1}, \dots, t_0)| = O(1)|Q_n(t_{r-1}, \dots, t_0)| = O\left(\frac{1}{n^2}\right)^{q-r+1} M_n(x)$$

for any $x \in [-1, 1]$, using (6), and we have shown (7) for the case $k = r-1$.

One may in like manner establish that for $k \in r-2, \dots, 0$, one has

$$|Q_n(s_k, \dots, s_{r-1}, t_{r-1}, \dots, t_0)| = \tag{12}$$
$$= O(1)[|Q_n(s_k, \dots, s_{r-1}, t_{r-1}, \dots, t_1)| + |Q_n(s_{k+1}, \dots, s_{r-1}, t_{r-1}, \dots, t_0)|],$$

and the desired result (7) follows by a double induction:

We assume that the result is known for all k's, if r is any value less than the given one (this suffices to estimate the first term), and that it is known for the given r for all indices greater than k (this estimates the second term). Thus we arrive from (12) to the estimate

$$|Q_n(s_k, \dots, s_{r-1}, t_{r-1}, \dots, t_0)| = O\left(\frac{1}{n^2}\right)^{q-r+2} M_n(s_k) + O\left(\frac{1}{n^2}\right)^{q-r+1} M_n(s_{k+1}) =$$
$$= O\left(\frac{1}{n^2}\right)^{q-r+1} M_n(x)$$

for any $x \in [-1, 1]$, and we have completely established (7).

Inspection of the expansion (5) of $Q_n(x)$ shows that the expression satisfies the required property (3) term by term, provided that $-1 < x < -1 + C/n^2$. On the other hand, if $-1 + C/n^2 \leqq x \leqq 0$, then moreover $1 - x \geqq 1$, and

$$C \leqq n^2(1+x) \leqq n^2(1+x)(1-x) = n^2(1-x^2),$$

whence

$$\frac{1}{n\sqrt{1-x^2}} \leqq \frac{1}{\sqrt{C}}. \tag{13}$$

Therefore we may say, that for $i = 0, \dots, r-1$ the i^{th} term of (5) satisfies

$$O\left(\frac{1}{n^2}\right)^q \cdot (n^2)^i M_n(x) \cdot (1-x^2)^i = O\left(\frac{(\sqrt{1-x^2})^{2i}}{n^{2q-2i}}\right) M_n(x). \tag{14}$$

In turn, we may estimate the first factor of (14) using (13), as follows:

$$\frac{(\sqrt{1-x^2})^{2i}}{n^{2q-2i}} = \frac{(\sqrt{1-x^2})^{2i}}{n^{2q-2i}} \frac{(\sqrt{1-x^2})^{q-2i}}{(\sqrt{1-x^2})^{q-2i}} =$$
$$= \left(\frac{\sqrt{1-x^2}}{n}\right)^q \cdot \frac{1}{n^{q-2i}(\sqrt{1-x^2})^{q-2i}} < \left(\frac{1}{\sqrt{C}}\right)^{q-2i} \left(\frac{\sqrt{1-x^2}}{n}\right)^q,$$

after noting that $q-2i>0$, since $1\leqq r-1$. Thus from (14) we derive the estimate

$$O\left(\frac{\sqrt{1-x^2}}{n}\right)^q M_n(x)$$

for the i^{th} term of (6), in the case that $-1+C/n^2\leqq x\leqq 0$, and the requirements of (3) are satisfied for this term, if $-1\leqq x\leqq 0$.

If on the other hand we wish to consider any one of those terms of (5) occurring beyond the $(r-1)^{\text{st}}$, we may employ the estimate (7) on the coefficient, substituting into (14) using $i=r-1$, to arrive at the same results. We have thus shown that the estimate (3) is valid for all $x\in[-1,0]$, which suffices by symmetry. Along the way, we have also shown that separate consideration of coalescing nodes is unnecessary.

This completes the proof of the theorem.

REMARK. In its original form, our Theorem contained a restriction on the minimal distance between adjacent s_i's and t_i's. The possibility of omitting this unnecessary condition was pointed out by J. Szabados whose remark induced the authors to bring the Theorem into its present form.

References

[1] K. Balázs and T. Kilgore, Simultaneous approximation of derivatives by Lagrange and Hermite interpolation, *J. Approx. Theory*, in press.

[2] Y. Brudnyi, Approximation by integral functions on the exterior of a segment and on a semi-axis (Russian), *Doklady Akad. Nauk SSSR*, **124** (1959), 739—742.

[3] I. Gopengauz, A theorem of A. F. Timan on the approximation of functions by polynomials on a finite segment, *Mat. Zametki*, **1** (1967), 163—172 (in Russian); *Math. Notes*, **1** (1967), 110—116 (English translation).

[4] D. Leviatan, The behavior of the derivatives of the algebraic polynomials of best approximation, *J. Approx. Theory*, **35** (1982), 169—176.

[5] G. Lorentz, *Approximation of Functions*, Holt, Rinehart, and Winston (New York, 1966).

[6] Y. Muneer, On Lagrange and Hermite Interpolation. I., *Acta Math. Hung.*, **49** (1987), 293—305.

[7] P. Runck and P. Vértesi, Some good point systems for derivatives of Lagrange interpolatory operators, preprint.

[8] J. Szabados, On the convergence of the derivatives of projection operators, *Analysis Math.*, **7** (1987), 349—357.

[9] S. Telyakovskii, Two theorems on the approximation of functions by algebraic polynomials, *Mat. Sb.*, **70 (112)** (1966), 252—265.

[10] A. F. Timan, An extension of Jackson's Theorem on the best approximation of continuous function (in Russian), *Dokl. Akad. Nauk SSSR*, **78** (1951), 17—20.

[11] R. Trigub, Approximation of functions by polynomials with integral coefficients (in Russian), *Isv. Akad. Nauk SSSR, Ser. Matem.*, **26** (1962), 261—280.

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DEPARTMENT OF ALGEBRA,
COMBINATORICS AND ANALYSIS
120 MATHEMATICS ANNEX
AUBURN UNIVERSITY
AUBURN, ALABAMA 36849
USA

MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
H-1364 BUDAPEST, P.O.B. 127

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LOWER BOUNDS FOR PACKING DENSITIES

K. BEZDEK (Budapest) and R. CONNELLY* (Ithaca)

I. Introduction

If a packing of incompressible rigid convex objects is sufficiently compressed or "compacted", one expects that the packing density will not be small. The aim of this paper is to show that certain conditions on a packing insure that there is at least a lower bound on the packing density, which generalize some previous results concerning such lower bounds.

One such condition is the notion of a compact packing of convex bodies due to L. Fejes Tóth in [6]. (Recall that a *body* in n-dimensional Euclidean space E^n is a compact set with nonempty interior, and a *packing* is a collection of sets with disjoint interiors.) We say that a body A is *enclosed* by the bodies $\{B_i\}$ if any curve, connecting a point of A with a point sufficiently far from A, intersects $\bigcup_i B_i$. If in the packing each body is enclosed by the bodies having a point in common with it, then the packing is said to be *compact*.

Two sets S_1 and S_2 in E^n are said to be *homothetic* if they are either translates or there exists a point O (as origin) and a positive real number λ such that

$$S_2 = \lambda S_1 = \{P_2 | P_2 - O = \lambda(P_1 - O),\ P_1 \in S_1\}, \tag{1.1}$$

where we always regard points as vectors. The *homogeneity* of a packing of convex bodies is the infimum of the volumes (or areas in dimension two) of the bodies divided by the supremum of the volumes. L. Fejes Tóth [6] proved that, in the Euclidean plane, the lower density of a compact packing of centrally symmetric homothetic convex sets of positive homogeneity is at least 3/4, and he conjectured that when the condition of central symmetry is dropped, then the bound 3/4 can be replaced by 1/2. This was proved by A. Bezdek, K. Bezdek, and K. Böröczky in [1]. Thus if d denotes the density of a compact packing of the Euclidean plane by homothetic convex sets such that the ratio of the areas of any two sets is bounded, then $d \geqq 1/2$. Later K. Bezdek [2] proved that in E^n $(n \geqq 3)$ the density of any compact lattice packing formed by translates of a centrally symmetric convex body is greater than $2^{1/(n-1)}/(2^{1/(n-1)}+1) > 1/2$. We shall generalize the theorems mentioned above. Namely we shall prove the following.

THEOREM 1. *If d denotes the density of a compact packing in E^n, $n \geqq 2$, consisting of homothetic centrally symmetric convex bodies with bounded volume ratios, then $d \geqq (n+1)/2n$, and for $n \leqq 3$ there is a compact lattice packing of centrally symmetric convex bodies where equality holds.*

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REMARK 1. It turns out that our lower bound $(n+1)/2n$ is never sharp for $n \geqq 4$, but we do not know of a suitable replacement. We omit the proof. See Grünbaum [8], as well as our later comments about Grünbaum's Theorem.

We say that two sets S_1 and S_2 are *homotheticly reversed* if (1.1) holds for λ negative.

THEOREM 2. *Let d denote the density of a compact packing in the Euclidean plane consisting of homothetic and homotheticly reversed convex sets with bounded area ratios. Then* $d \geqq 1/2$.

REMARK 2. When the condition of central symmetry is dropped, we present the following problems: What is the greatest lower bound of the densities of compact packings in E^n $(n \geqq 3)$ consisting of homothetic convex bodies such that the volume ratios are greater than a fixed positive number? What is the greatest lower bound if we only suppose that our convex bodies are homothetic or homotheticly reversed?

For dimensions n greater than two, the condition of being a compact packing seems to be very strong. For instance, if each of the bodies is *strictly convex,* i.e. each support plane intersects the body at a single point, then the packing cannot be compact (for $n \geqq 3$).

Thus we offer an alternative to compact packings, in dimensions greater than two, that is more general at least for centrally symmetric convex bodies. Of course, the penalty we pay is that the lower bounds are much lower than for compact packings. We say that a packing of E^n by centrally symmetric convex bodies is a *triangulated packing* if there is a triangulation of E^n such that each vertex of the triangulation is the central point of one of the packing elements, and a 1-simplex between two vertices implies that the two corresponding packing elements intersect. (Recall that a *triangulation* of a space X is a simplicial complex whose underlying space is X.) In dimension two, for packings of centrally symmetric convex sets, triangulated packings and compact packings are the same.

THEOREM 3. *Let d denote the density of a triangulated packing of homothetic centrally symmetric convex bodies in E^n, $n \geqq 2$, with bounded volume ratios. Then $d \geqq (n+1)/2^n$, and there is a triangulated lattice packing of (congruent) centrally symmetric convex bodies where equality holds.*

REMARK 3. Unfortunately for dimensions greater than two no packing of congruent spheres can be triangulated.

By using a result of Hadwiger [10] and a result of Rogers and Shephard [12] we can apply Theorem 3 to the case when the convex bodies are not necessarily centrally symmetric. It turns out that any packing $\mathcal{P}$ of translates of a convex body B has an associated packing $\hat{\mathcal{P}}$ of translates of a centrally symmetric convex body $\hat{B}$, where each B_i corresponds to a unique $\hat{B}_i \in \hat{\mathcal{P}}$ such that $B_i \cap B_j \neq \emptyset$ if and only if $\hat{B}_i \cap \hat{B}_j \neq \emptyset$. We say that $\mathcal{P}$ is a triangulated packing if and only if $\hat{\mathcal{P}}$ is a triangulated packing.

COROLLARY. *Let d denote the density of a triangulated packing of translates of a convex body in E^n. Then* $d \geqq (n+1)/\binom{2n}{n}$.

We thank Branko Grünbaum for (gently) pointing out that our Lemma 4 below is essentially the same as his Theorem 1 in [8]. Grünbaum's Theorem says that if there are $n+1$ symmetric homothetic convex bodies in E^n with pairwise nonempty intersections, and each body is dilated from its center by $2n/(n+1)$, then the dilated bodies have a common intersection point.

When Grünbaum's Theorem is specialized to the case when the homothetic bodies are translates (i.e. the homothetic ratios are all 1), Grünbaum points out that his Theorem can be viewed (via Helly's Theorem) as a Jung type of result. Namely, in any Minkowski space (a finite dimensional normed linear space over the reals) a ball of diameter $2n/(n+1)$ may cover, after a suitable translation, any set of diameter $\leqq 1$, which is a result of Bohnenblast [3]. See also Leichtweiss [11].

On the other hand, Grünbaum also applies his Theorem to the problem of the extensions of transformations [9].

We apply Grünbaum's Theorem (Lemma 4 below) to both our Theorem 1 and Theorem 3.

The main difficulty in Grünbaum's Theorem is handling the homothetic ratios.

We include our own version of Grünbaum's proof for two reasons. First, for the sake of completeness, it is convenient to have this important result included with the other ideas in our Theorem 1 and Theorem 3. Second, Grünbaum's version of his proof is very terse and gives no hint as to how he discovered the particular relations he used. We show how to derive the factor $2n/(n+1)$ as well as explain geometrically the two cases which Grünbaum considers in his proof.

II. Proof of Theorem 1

The following Lemma 1 is the key result needed in the proof of Theorem 1. Lemma 1 is needed for Lemma 2, and Lemma 2 and Lemma 3 are used to prove Lemma 4 which is used to find a point "close" to the packing elements that surround a "hole".

Let $\langle P_1, P_2, \ldots, P_{n+1}\rangle = \sigma$ be a simplex in E^n. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ be positive real numbers, and suppose we have a point $P_{i,j}$ $(i<j)$ on each edge between P_i and P_j with the property that

$$\lambda_j(P_i - P_{i,j}) = \lambda_i(P_{i,j} - P_j), \tag{2.1}$$

where $1 \leqq i < j \leqq n+1$, and points are regarded as vectors. Define $\bar{\lambda}$ by

$$\bar{\lambda} = 1 + \frac{2(n-1)}{\left[\left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\sum_{i=1}^{n+1} \lambda_i^{-1}\right) - (n^2-1)\right]}. \tag{2.2}$$

For any set X in E^n, $P \in E^n$, α a scalar, define

$$\alpha X(P) = \{Q | Q = \alpha(P'-P)+P,\ P' \in X\}.$$

Let L_i denote the hyperplane containing $P_{1,i}, \dots, P_{i-1,i}, P_{i,i+1}, \dots, P_{i,n+1}$.

LEMMA 1. $\bigcap_{i=1}^{n+1} \bar{\lambda} L_i(P_i) \neq \emptyset$.

PROOF. The idea is to find the intersection point P as the solution to certain linear equations. This in turn will allow us to write P and $\bar{\lambda}$ explicitly in terms of matrices involving the P_i's and λ_i's.

For any column vector P in E^n let us define

$$\hat{P} = \begin{pmatrix} P \\ 1 \end{pmatrix},$$

the vector in E^{n+1} obtained by adding a one in the $(n+1)$-st coordinate. (Regard E^n as the subset of E^{n+1} consisting of the first n coordinates. All vectors are regarded as column vectors.) Note that $\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{n+1}$ is now a basis for E^{n+1}.

We now regard the hyperplanes L_i as the solutions to certain linear equations or equivalently as null spaces of certain linear functionals. Let $f_i\colon E^{n+1} \to E^1$ be the linear functional (uniquely) defined by $f_i(\hat{P}_i)=1$, and $f_i(\hat{Q})=0$, for all Q in L_i. We will calculate the f_i's next, explicitly in terms of matrices. Rewriting (2.1) we get, for $i<j$,

$$\lambda_j(\lambda_i+\lambda_j)^{-1}\hat{P}_i+\lambda_i(\lambda_i+\lambda_j)^{-1}\hat{P}_j = \hat{P}_{i,j}.$$

Applying f_i we get, for $i \neq j$,

$$\lambda_j+\lambda_i f_i(\hat{P}_j) = 0,\quad f_i(\hat{P}_j) = -\lambda_j\lambda_i^{-1}.$$

Define an $(n+1)$-by-$(n+1)$ matrix F such that

$$F\hat{P}_j = \begin{pmatrix} f_1(\hat{P}_j) \\ \vdots \\ f_j(\hat{P}_j) \\ \vdots \\ f_{n+1}(\hat{P}_j) \end{pmatrix} = \begin{pmatrix} -\lambda_j\lambda_1^{-1} \\ \vdots \\ +1 \\ \vdots \\ -\lambda_j\lambda_{n+1}^{-1} \end{pmatrix}. \tag{2.3}$$

(Note that the first equality of (2.3) holds with any vector replacing $\hat{P}_j$.)

We can encode this information in a single matrix as follows: Let J be the column vector in E^{n+1} with all 1's as entries. Then JJ^t is the $(n+1)$-by-$(n+1)$ matrix with all 1's as entries, where $(\ )^t$ denotes the transpose operation. J^tJ is the one-by-one matrix with entry $n+1$. (We always regard a one-by-one matrix as a scalar.) Also note that $(-JJ^t+2I)$ is the $(n+1)$-by-$(n+1)$ matrix with $+1$'s on the diagonal and -1's elsewhere, where I denotes the $(n+1)$-by-$(n+1)$ identity matrix. Define another $(n+1)$-by-$(n+1)$ matrix A by

$$A = (\hat{P}_1, \hat{P}_2, \dots, \hat{P}_{n+1}).$$

Let D be the $(n+1)$-by-$(n+1)$ diagonal matrix where the i-th diagonal entry is λ_i.

Then we rewrite (2.3) as

$$FA = D^{-1}(-JJ^t+2I)D.$$

Then

$$F = D^{-1}(-JJ^t+2I)DA^{-1}.$$

This is the desired explicit expression for F and thus the functionals f_i.

We next proceed to use this to find a similar expression for the intersection point. Suppose $P=\bigcap_{i=1}^{n+1} \lambda L_i(P_i)$, for some scalar λ. Then for some $Q_i \in L_i$,

$$P = \lambda(Q_i-P_i)+P_i = \lambda Q_i+(1-\lambda)P_i,$$

and thus,

$$\hat{P} = \lambda\hat{Q}_i+(1-\lambda)\hat{P}_i, \quad f_i(\hat{P}) = 1-\lambda.$$

By the definition of F,

$$F\hat{P} = -(\lambda-1)J, \quad \hat{P} = -(\lambda-1)F^{-1}J = -(\lambda-1)AD^{-1}(-JJ^t+2I)^{-1}DJ.$$

We can justify and simplify this expression for $\hat{P}$ by calculating the inverse of $(-JJ^t+2I)$, using the properties of J, for $n>1$,

$$(-JJ^t+2I)^{-1} = [-2(n-1)]^{-1}(JJ^t-(n-1)I).$$

Then

$$\hat{P} = (\lambda-1)[2(n-1)]^{-1}AD^{-1}[JJ^t-(n-1)I]DJ,$$

$$\hat{P} = (\lambda-1)[2(n-1)]^{-1}A\,[(\sum_{i=1}^{n+1}\lambda_i)D^{-1}-(n-1)I]\,J, \tag{2.4}$$

since $J^tDJ=\sum_{i=1}^{n+1}\lambda_i$. (2.4) is the desired explicit expression for $\hat{P}$.

Since the last entry of $\hat{P}$ is one, this gives us another relation to calculate λ. Let E_{n+1} be the (column) vector in E^{n+1} with the last entry 1 and all the other entries 0. Calculating the last entry of $\hat{P}$ we get

$$1 = E^t_{n+1}\hat{P} = (\lambda-1)[2(n-1)]^{-1}E^t_{n+1}A\,[(\sum_{i=1}^{n+1}\lambda_i)D^{-1}-(n-1)I]\,J.$$

But $E^t_{n+1}A=J^t$. Thus

$$1 = (\lambda-1)[2(n-1)]^{-1}[(\sum_{i=1}^{n+1}\lambda_i)J^tD^{-1}J-(n-1)J^tJ],$$

$$2(n-1)(\lambda-1)^{-1} = (\sum_{i=1}^{n+1}\lambda_i)(\sum_{i=1}^{n+1}\lambda_i^{-1})-(n-1)(n+1).$$

From this it is easy to calculate that $\lambda=\bar{\lambda}$ in (2.2). Thus for this value of λ (only) we see that (2.4) defines $\hat{P}$ and thus P.

REMARK 4. In dimension 2 it is clear that P must lie in σ. However in dimension 3 or higher, it could turn out that P lies outside σ. This can be seen by calculating the affine coordinates of P, $t_1, t_2, \ldots, t_{n+1}$ (i.e. $P=\sum_{i=1}^{n+1} t_iP_i$) by the same method as we use to find λ. Thus using (2.4)

$$t_i = (\lambda-1)[2(n-1)]^{-1}[\lambda_i^{-1}(\sum_{j=1}^{n+1}\lambda_j)-(n-1)].$$

If

$$(n-2)\lambda_i > \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \lambda_j,$$

then P_i lies outside the i-th face of σ opposite P_i, because t_i is negative. Figure 1, below, shows the sets involved in Lemma 1, for $n=2$. Here it is clear geometrically that P lies in σ.

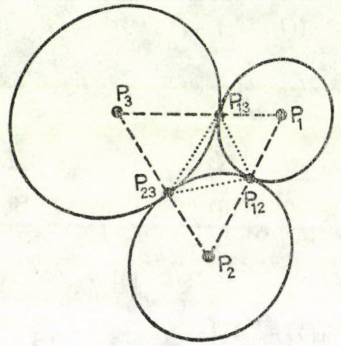

Fig. 1

There are other ways of calculating $\bar{\lambda}$, using Cramer's Rule for instance, but our method here seems as simple as any, since it does not calculate by explicitly manipulating arrays of numbers, but uses closed form matrix properties instead.

In Grünbaum's Theorem, his first case is when all the $t_i \geqq 0$. He presents $\bar{\lambda}$ (he calls it μ) as well as the affine coordinates of each point in B_i that dilates to P, and then he calculates that each point does indeed dilate to P. His second case is when some $t_i < 0$, and he handles this differently than we do below.

In what follows we reinterpret the result of Lemma 1 in terms of expanding half spaces. Using the above we define H_i as the half space containing P_i with boundary L_i (recall L_i is determined by $P_{1,i}, P_{2,i}, \ldots, P_{i-1,i}, P_{i,i+1}, P_{i,n+1}$).

LEMMA 2. $\bigcap_{i=1}^{n+1} 2n(n+1)^{-1} H_i(P_i) \cap \sigma \neq \emptyset$.

PROOF. Note that since the harmonic mean is less than the arithmetic mean we have

$$\Big((n+1)^{-1} \sum_{i=1}^{n+1} \lambda_i^{-1}\Big)^{-1} \leqq (n+1)^{-1} \sum_{i=1}^{n+1} \lambda_i,$$

$$(n+1)^2 \leqq \Big(\sum_{i=1}^{n+1} \lambda_i\Big)\Big(\sum_{i=1}^{n+1} \lambda_i^{-1}\Big),$$

$$(n+1)^2-(n^2-1) = 2(n+1) \leqq \Big(\sum_{i=1}^{n+1} \lambda_i\Big)\Big(\sum_{i=1}^{n+1} \lambda_i^{-1}\Big)-(n^2-1).$$

Thus

$$\bar{\lambda} = 1+\frac{2(n-1)}{\left(\left(\sum_{i=1}^{n+1}\lambda_i\right)\left(\sum_{i=1}^{n+1}\lambda_i^{-1}\right)-(n^2-1)\right)} \leqq 1+\frac{2(n-1)}{2(n+1)} = 2n/(n+1).$$

Thus

$$\bar{\lambda}L_i(P_i) \subset \bar{\lambda}H_i(P_i) \subset 2n(n+1)^{-1}H_i(P_i).$$

By Lemma 1 we get

$$\bigcap_{i=1}^{n+1} 2n(n+1)^{-1}H_i(P_i) \neq \emptyset. \tag{2.5}$$

We shall prove the Lemma by induction on n. It is clearly true for $n=1$. We shall assume the result for $n-1$.

Note that $2n(n+1)^{-1}$ is a monotone increasing function for $n>0$.

Let H_i^σ denote the support half-space for σ whose boundary is the hyperplane L_i^σ spanned by the facet opposite P_i in σ. I.e. L_i^σ is spanned by $P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{n+1}$ and

$$\bigcap_{i=1}^{n+1} H_i^\sigma = \sigma. \tag{2.6}$$

We apply induction to each L_i^σ with $H_j \cap L_i^\sigma$, $j \neq i$, replacing H_j, and $\sigma \cap L_i^\sigma$ replacing σ. Thus

$$\emptyset \neq (\sigma \cap L_i^\sigma) \bigcap_{\substack{j=1 \\ j\neq i}}^{n+1} 2(n-1)n^{-1}(H_j \cap L_i^\sigma)(P_j) \subset \bigcap_{i=1}^{n+1} H_i^\sigma \bigcap_{\substack{j=1 \\ j\neq i}}^{n+1} 2n(n+1)^{-1}H_j(P_j). \tag{2.7}$$

Thus by (2.5), (2.6), and (2.7) the $2(n+1)$ half-spaces

$$2n(n+1)^{-1}H_i(P_i), \quad H_i^\sigma, \quad i = 1, 2, \ldots, n+1,$$

have the property that every $n+1$ of them have a non-empty intersection. Thus by Helly's Theorem, they all must intersect, finishing the Lemma.

Let $\mathcal{P}$ be a compact packing in E^n. Let W, a *hole*, be the closure of a component of the complement of the union of the elements of $\mathcal{P}$. Let $\mathcal{P}_W \subset \mathcal{P}$ be those packing elements of $\mathcal{P}$ whose intersection with W is $(n-1)$-dimensional. W must be bounded and $\mathcal{P}_W$ is finite since $\mathcal{P}$ is a compact packing and the elements of $\mathcal{P}$ have volumes greater than a fixed positive number.

LEMMA 3. *For all* $B_1, B_2 \in \mathcal{P}_W$, $B_1 \cap B_2 \neq \emptyset$.

PROOF. Suppose not; suppose some $B_1, B_2 \in \mathcal{P}_W$ are such that $B_1 \cap B_2 = \emptyset$ Suppose that the volume of B_1 is not larger than the volume of B_2. Then B_2 is not a neighbor of B_1, and there is a path from B_1 through W to the center of B_2. Then the ray from the center P_2 of B_2 in the opposite direction from the center P_1 of B_1 completes a path to infinity that violates the compactness of $\mathcal{P}$.

To see this suppose not; suppose some neighbor B of B_1 intersects B_1 at Q_1 and the ray at P. Then construct Q_2 on the line segment $\langle Q_1, P\rangle$ so that the triangles Q_1P_1P and Q_2P_2P are similar. Since B is convex, Q_2 must be in B. But Q_2 must be in the interior of B_2 as well, since the coefficient of homogeneity for B_2

is larger than or equal to the coefficient for B_1. Packing elements cannot intersect in interior points. Thus B cannot intersect the path defined above.

Thus $B_1 \cap B_2 \neq \emptyset$ for all $B_1, B_2 \in \mathcal{P}_W$. See Figure 2.

REMARK 5. For Lemma 3 in the plane, we do not need the condition that the elements of $\mathcal{P}$ are homothetic. We can simply choose a ray going to infinity lying between the two common support lines of B_1, B_2, where $B_1 \cap B_2 = \emptyset$.

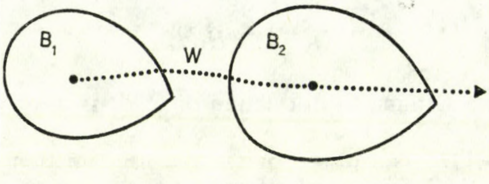

Fig. 2

LEMMA 4 (Grünbaum). $\bigcap_{B_i \in \mathcal{P}_W} 2n(n+1)^{-1} B_i(P_i) \neq \emptyset$, *where P_i is the center of B_i.*

PROOF. By Helly's Theorem we need only show the result for $n+1$ elements of $\mathcal{P}_W$, say $B_1, B_2, \ldots, B_{n+1}$. We can also assume that the centers $P_1, P_2, \ldots, P_{n+1}$ are affine independent and form a simplex σ in E^n. By Lemma 3, we know that there is a unique point $P_{i,j} = B_i \cap B_j \cap \langle P_i, P_j \rangle$, $i<j$, where $\langle P_i, P_j \rangle$ is the line segment between P_i and P_j. Let H_i be the half-space containing P_i with $P_{i,j}$, $j \neq i$, on the boundary of H_i. Clearly $H_i \cap \sigma \subset B_i$. Lemma 2 implies that

$$\emptyset \neq \bigcap_{i=1}^{n+1} 2n(n+1)^{-1} H_i(P_i) \cap \sigma \subset \bigcap_{i=1}^{n+1} 2n(n+1)^{-1} B_i(P_i),$$

finishing the Lemma.

For what follows, we need to compare volumes, and it helps to consider a slight generalization of the notion of the volume bounded by a surface. Let P be a point in E^n, and let S be an oriented surface possibly with boundary. For instance, S could be a polyhedral surface with an orientation, or the boundary of a component of the intersection of the complements of a finite number of convex bodies. In the case of a polyhedral surface we define the *signed volume* from a point P to S by

$$\mathrm{Vol}\,[S, P] = (n!)^{-1} \sum_{\sigma \in S} \det (P_1 - P, P_2 - P, \ldots, P_n - P),$$

for $\sigma = \langle P_1, \ldots, P_n \rangle$, an oriented simplex of S. "det" denotes the determinant, and vectors are n-by-one columns, as usual. If S is a closed surface enclosing a bounded region in E^n, then $\mathrm{Vol}\,[S, P]$ is the volume enclosed by S. By taking limits of polyhedral surfaces, we can extend this definition to the case of more general surfaces, such as the piecewise convex surfaces mentioned above.

We say $C \subset E^n$ is a *cone from* $P \in E^n$ if $tC(P) = C$, for all $0 < t$. For any set X let $\mathrm{bdy}\,(X)$ denote the topological boundary of X. For a convex body B we choose an orientation on $\mathrm{bdy}\,(B)$ such that $\mathrm{Vol}\,[\mathrm{bdy}\,(B), P]$ is positive and thus equal to $\mathrm{Vol}\,(B)$, the usual Euclidean volume.

LEMMA 5. *Let C be a cone from P in E^n, and B a convex body containing P in its interior. Let $P_0 \in E^n$ and $\lambda > 1$ be such that $P_0 \in \lambda B(P)$. Then*

(2.8) $$(\lambda-1)\operatorname{Vol}(B\cap C) = (\lambda-1)\operatorname{Vol}[\operatorname{bdy}(B)\cap C, P] \geqq \operatorname{Vol}[\operatorname{bdy}(B)^- \cap C, P_0],$$

where $(\cdot)^-$ indicates the opposite orientation.

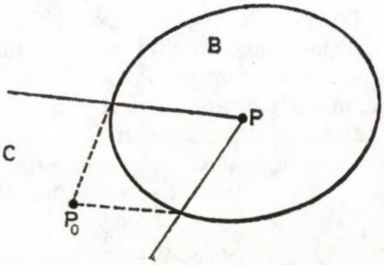

Fig. 3

PROOF. We will show the lemma first in the case when C and B are both polyhedral. The more general case follow by approximating the surfaces with polyhedral sets. Furthermore, by subdividing the boundary of B into simplices, we can further reduce our considerations to the cases when $B\cap C$ is a simplex σ. We simply sum over all simplices on the boundary of B, where each term is the case when C is the cone from P over each simplex in $B\cap C$. Let H denote the half-space containing σ with boundary L containing the face opposite P. Let d denote the distance of P from L, and let d_0 denote the signed distance of P_0 from L, where d_0 is negative if P_0 is in H. Then

$$n\operatorname{Vol}(H\cap C) = d\operatorname{Vol}_{n-1}(L\cap C), \quad n\operatorname{Vol}[L^-\cap C, P_0] = d_0\operatorname{Vol}_{n-1}(L\cap C),$$

where Vol_{n-1} is the $(n-1)$-dimensional volume in L.

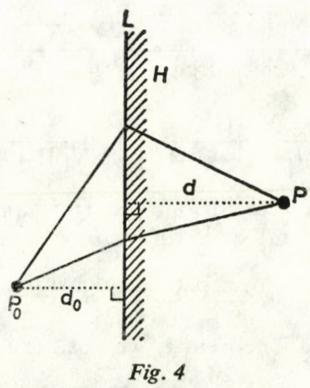

Fig. 4

Then,

$$d+d_0 \leqq \lambda d, \quad d_0 \leqq (\lambda-1)d,$$

since $P_0 \in \lambda B(P)$. See Figure 4. Thus

$$n \operatorname{Vol}[L^- \cap C, P_0] = d_0 \operatorname{Vol}_{n-1}(L \cap C) \leqq$$

$$\leqq (\lambda-1) d \operatorname{Vol}_{n-1}(L \cap C) = (\lambda-1) n \operatorname{Vol}(H \cap C) = (\lambda-1) n \operatorname{Vol}(B \cap C).$$

(2.8) then follows, finishing the Lemma.

Proof of Theorem 1. The idea is to compare the volume of the holes of the packing to the volume of the packing elements using Lemma 5. We get our estimate to be the sharpest when there is a point sufficiently near to all of the packing elements next to the hole. Lemma 4 guarantees that there is such a point.

Let W be a hole for the compact packing $\mathcal{P}$. Recall that $\mathcal{P}_W$ is the collection of those elements of $\mathcal{P}$ whose boundary and W intersect in an $(n-1)$-dimensional set. Let P_i be the center of B_i, as in our previous notation. Let V_i denote the cone over $B_i \cap W$ from P_i, namely

$$V_i = \{\langle Q, P_i \rangle | Q \in B_i \cap W\},$$

where $\langle Q, P_i \rangle$ is the line segment between Q and P_i.

By Lemma 4 there is a point

$$P_0 \in \bigcap_{B_i \in \mathcal{P}_W} 2n(n+1)^{-1} B_i(P_i).$$

By Lemma 5 for $\lambda = 2n(n+1)^{-1}$, and $B_i \in \mathcal{P}_W$,

$$\operatorname{Vol}[\operatorname{bdy}(B_i)^- \cap W, P_0] \leqq (n-1)(n+1)^{-1} \operatorname{Vol}(V_i).$$

But

$$\sum_{B_i \in \mathcal{P}_W} \operatorname{Vol}[\operatorname{bdy}(B_i)^- \cap W, P_0] = \operatorname{Vol}(W).$$

Thus

$$\operatorname{Vol}(W) \leqq (n-1)(n+1)^{-1} \sum_{B_i \in W} \operatorname{Vol}(V_i).$$

Let $V = \bigcup_{B_i \in \mathcal{P}_W} V_i$. Then in $W \cup V$

$$\frac{n+1}{2n} = \frac{\operatorname{Vol}(V)}{(n-1)(n+1)^{-1} \operatorname{Vol}(V) + \operatorname{Vol}(V)} \leqq$$

$$\leqq \frac{\operatorname{Vol}(V)}{\operatorname{Vol}(W) + \operatorname{Vol}(V)} = \frac{\operatorname{Vol}(V)}{\operatorname{Vol}(W \cup V)}.$$

Since the sets $\{W \cup V\}$ have disjoint interiors and cover the complement of the packing elements of $\mathcal{P}$ and since the volume ratios of the packing elements are bounded, we have that the lower packing density of $\mathcal{P}$ is $\geqq \frac{n+1}{2n}$.

To complete the proof of Theorem 1, we need the following:

Construction 1. Let

$$B = \{(x_1, \dots, x_n) \,|\, |x_1 + \dots + x_n| \leqq 1, \ |x_i| \leqq 1, \ i = 1, \dots, n\},$$

for $n=2$ or 3. Let $\mathcal{P}$ be the packing defined by taking translates of B by the lattice

$$\Lambda = \{(2k_1, \ldots, 2k_n) | k_1, \ldots, k_n \text{ are integers}\}.$$

Figure 5 shows B for $n=2$ and $n=3$.

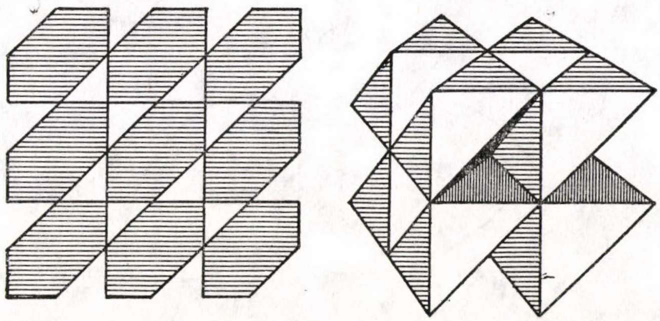

Fig. 5

We claim that $\mathcal{P}$ is a compact packing of convex symmetric bodies with density $(n+1)/(2n)$ for $n=2$ and $n=3$. It is clear that $\mathcal{P}$ is a packing of convex symmetric bodies. It is easy to check that by translating each facet F of the square or cube to the opposite facet $\bar{F}$ that the relative interior of $B \cap F$ is translated into the relative complement of $B \cap \bar{F}$ and the two sets cover the facet $\bar{F}$. Thus the two simplices that make up the complement of B in the cube or square are holes in the packing $\mathcal{P}$. Thus $\mathcal{P}$ is a compact packing. The density is easily calculated to be $(n+1)/(2n)$ for $n=2$ and $n=3$. This finishes the proof of Theorem 1.

III. Proof of Theorem 2

Let $\mathcal{P}$ be a compact packing of the Euclidean plane by homothetic and homotheticly reversed compact convex sets such that the area of all the packing elements have a positive lower bound.

Recall that a hole W is a connected component of the complement of the union of the elements of $\mathcal{P}$.

By Remark 5 after Lemma 3 each W is the connected component of the complement of a finite number $S_1, \ldots, S_n \in \mathcal{P}$, where $S_i \cap S_j \neq \emptyset$, for $i \neq j$. Since each S_i is a convex set with non-empty interior in the plane, each set of 3 of the S_i's, say S_1, S_2, S_3 must bound a connected region in the plane. If S_4 is in this bounded region then W is not connected. If S_4 is outside this region it is not part of the boundary of W. Thus $n=3$.

Let C_i be the centroid of S_i, $i=1, 2, 3$. Let $i, j=1, 2, 3$, $i \neq j$. If S_i and S_j are homotheticly reversed we define $P_{i,j}=P_{j,i}$ to be the unique point on the line segment from C_i to C_j in $S_i \cap S_j$. Note in this case $P_{i,j}$ is the center of dilation which moves S_i to S_j. If S_i and S_j are not homotheticly reversed, then we choose $P_{i,j}$ to be any point in $S_i \cap S_j$. However, if S_i and S_j correspond to another hole we must be careful to choose the same $P_{i,j}$.

Let $H(W)$ be the hexagon whose boundary consists of the union of the line segments $[C_i, P_{i,j}]$, $i \neq j$, $i, j = 1, 2, 3$. See Figure 6.

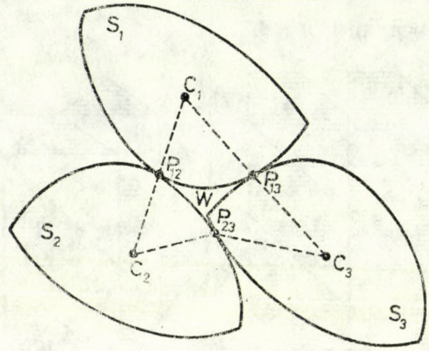

Fig. 6

Note that if at least one of the sets is homotheticly reversed and at least one is not homotheticly reversed, then two pairs of adjacent sides of the hexagon are colinear, and we can think of our hexagon as a quadrilateral.

LEMMA 6. *The collection of hexagons* $\{H(W)|W$ is a hole of $\mathcal{P}\}$ *have disjoint interiors and the union covers the complement of the union of the elements of* $\mathcal{P}$.

PROOF. $E^2 \setminus [S_1 \cup S_2 \cup S_3 \cup H(W)]$ is connected and unbounded, thus $H(W)$ must contain the bounded component of $E^2 \setminus [S_1 \cup S_2 \cup S_3]$. I.e., $W \subset H(W)$. Since there are no unbounded components of the complement of the union of the elements of $\mathcal{P}$, the union of the hexagons must cover the complement of the union of the elements of $\mathcal{P}$.

By the construction of $S_1, S_2, S_3 \in \mathcal{P}$ for each hole, we see that no $H(W)$ contains an element of $\mathcal{P}$. Thus $H(W)$ contains no centroid of an element of $\mathcal{P}$ and no $P_{i,j}$. Since no two of the segments that define the boundaries of the hexagons can intersect except at their endpoints, any two hexagons must have disjoint interiors. This finishes the proof of the Lemma.

If we know that the density of the packing $\mathcal{P}$, when each element is intersected with one of the hexagons, is not smaller than 1/2, then the overall packing density of $\mathcal{P}$ is not smaller than 1/2. Thus the following Lemma finishes Theorem 2.

LEMMA 7. *For each hole W of* $\mathcal{P}$

$$2(\text{area } W) \leqq \text{area } H(W).$$

PROOF. If all three of the packing elements corresponding to W are homothetic or all three are homotheticly reversed, then the methods of A. Bezdek, K. Bezdek, and K. Böröczky [1] imply the result.

Thus we are left with the case when one of the packing elements is different (homotheticly) from the other two. We assume that S_1 is homotheticly reversed from S_2 and S_3 (and thus S_2 and S_3 are homothetic). Since affine linear transformations take centroids to centroids, preserve area ratios, and homothetic pairs of sets, we may also assume that the triangle $\langle P_{1,2}, P_{2,3}, P_{3,1}\rangle$ is an equilateral triangle of side length 1. We must carefully estimate the area of W. Let C_i', for $i=1, 2, 3$, be the point in S_i furtherest from the line through $P_{i,i+1}, P_{i,i-1}$ (indices mod 3) on the same side as W. The quadrilateral $\langle C_i, P_{i,i+1}, C_i', P_{i,i-1}\rangle$ is contained in S_i and roughly speaking we will use its area as a lower bound for the area of $H(W)\cap\cap S_i$. See Figure 7. Note that it might happen that C_i and C_i' are on the same side of $P_{i,i+1}, P_{i,i-1}$.

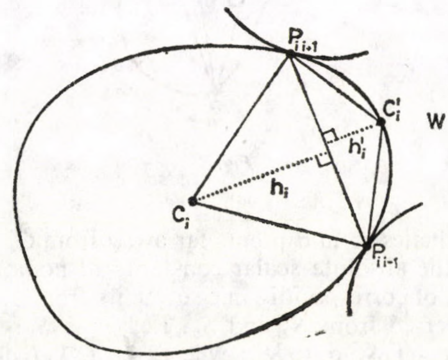

Fig. 7

Let absolute value of h_i be the altitude of $\langle C_i, P_{i,i+1}, P_{i,i-1}\rangle$ from the vertex C_i. We define h_i to be positive if C_i and C_i' are on opposite sides of $\langle P_{i,i+1}, P_{i,i-1}\rangle$, otherwise h_i is negative (or zero if C_i is on the line through $P_{i,i+1}, P_{i,i-1}$).

Let $h_i'>0$ be the altitude of $\langle C_i', P_{i,i-1}, P_{i,i+1}\rangle$ from the vertex C_i'. We claim:

$$\text{area}\,[H(W)\cap S_i] \geqq \frac{1}{2}(h_i+h_i').$$

In case C_i and C_i' are on opposite sides of the line through $P_{i,i-1}$ and $P_{i,i+1}$, or C_i lies inside the triangle $\langle C_i', P_{i,i-1}, P_{i,i+1}\rangle$, then $\frac{1}{2}(h_i+h_i')$ is the area of the quadrilateral $\langle C_i, P_{i,i-1}, C_i', P_{i,i+1}\rangle$ which is contained in $H(W)\cap S_i$. On the other hand, it is easy to see that when $h_i<0$, since $|h_i|\leqq h_i'$, the claim still holds, since the triangle $\langle C_i, P_{i,i+1}, C_i'\rangle$ has area $\geqq\frac{1}{2}(h_i+h_i')$ and is contained in $H(W)\cap S_i$, assuming (without loss of generality) that the segments $\langle C_i, P_{i,i+1}\rangle$ and $\langle C_i', P_{i,i-1}\rangle$ cross. See Figure 8.

We now must estimate h_i+h_i', for $i=1, 2, 3$.

We observe that if L is a support line for S_i, $i=1, 2, 3$, and b_i is the breadth of S_i in the direction perpendicular to L, then $d(C_i, L)\geqq b_i/3$, where $d(C_i, L)$ is the distance from the centroid C_i to L. (See Bonnesen and Fenchel [4], page 52.) In particular when L_i is the support line through C_i' parallel to $P_{i,i-1}, P_{i,i+1}$, then $d(C_i, L_i)=h_i+h_i'$. Thus we now look for lower bounds for the breadth in the direction perpendicular to the line through $P_{i,i-1}, P_{i,i+1}$.

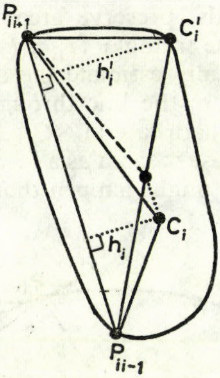

Fig. 8

We use the homotheties to find points far away from C_i' in the given direction. Let λ_i, $i=1, 2, 3$ be the absolute scalar constants of homothety for S_i. That is, the ratio of the lengths of correspondig line segments from S_i to S_j is λ_i/λ_j. Recall S_1 is homotheticly reversed from S_2 and S_3. Let $h_{k,l}$: $S_k \to S_l$ be the homothetic dilation that takes the set S_k onto S_l, where $k \neq l$, $k, l=1, 2, 3$. Define $P_{i,j}^{k,l}=h_{k,l}(P_{i,j})$. Note that $P_{i,j}^{k,l}$ is defined only when $P_{i,j} \in S_k$, i.e. $i=k$ or $j=k$.

We now compute the distance of $P_{2,3}^{3,2}$ from the line through $P_{2,3}$, $P_{1,2}$. See Figure 9.

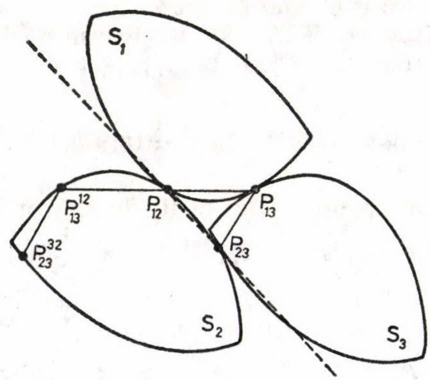

Fig. 9

Recall $h_{1,2}(,P_{1,2})=P_{1,2}$ and $h_{1,3}(P_{1,3})=P_{1,3}$

(3.1) $$|P_{1,2}^{1,2}-P_{1,2}| = |h_{1,2}(P_{1,3})-h_{1,2}(P_{1,2})| = \frac{\lambda_2}{\lambda_1}|P_{1,3}-P_{1,2}| = \frac{\lambda_2}{\lambda_1}.$$

But

$$P_{1,3}^{1,2} = h_{1,2}(P_{1,3}) = h_{1,2}h_{3,1}(P_{1,3}) = h_{3,2}(P_{1,3}) = P_{1,3}^{3,2},$$

since a composition of homothetic dilations is a homothetic dilation. Then

(3.2) $$|P_{1,3}^{1,2}-P_{2,3}^{3,2}| = |P_{1,3}^{3,2}-P_{2,3}^{3,2}| =$$

$$= |h_{3,2}(P_{1,3})-h_{3,2}(P_{2,3})| = \frac{\lambda_2}{\lambda_3}|P_{1,3}-P_{2,3}| = \frac{\lambda_2}{\lambda_3}.$$

Each of the above line segments makes a 60° angle with the line through $P_{1,2}$ and $P_{2,3}$. So by (3.1) and (3.2),

$$\frac{\sqrt{3}}{2}(|P_{1,3}^{1,2}-P_{1,2}|+|P_{1,3}^{1,2}-P_{2,3}^{3,2}|)+h_2' \leqq b_2,$$

(3.3) $$\frac{\sqrt{3}}{2}\left(\frac{\lambda_2}{\lambda_1}+\frac{\lambda_2}{\lambda_3}\right)+h_2' \leqq b_2 \leqq 3(h_2+h_2').$$

Similarly

(3.4) $$\frac{\sqrt{3}}{2}\left(\frac{\lambda_3}{\lambda_1}+\frac{\lambda_3}{\lambda_2}\right)+h_3' \leqq b_3 \leqq 3(h_3+h_3').$$

We get an estimate for h_1+h_1' by calculating $P_{2,3}^{3,1}$.

$$|P_{2,3}^{3,1}-P_{1,3}| = |h_{3,1}(P_{2,3})-h_{3,1}(P_{1,3})| = \frac{\lambda_1}{\lambda_3}|P_{2,3}-P_{1,3}| = \frac{\lambda_1}{\lambda_3}.$$

Thus

$$\frac{\sqrt{3}}{2}|P_{2,3}^{3,1}-P_{1,3}|+h_1' \leqq b_1,$$

(3.5) $$\frac{\sqrt{3}}{2}\frac{\lambda_1}{\lambda_3}+h_1' \leqq b_1 \leqq 3(h_1+h_1').$$

Adding (3.3), (3.4), and (3.5) we get

$$\frac{\sqrt{3}}{2}\left(\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_1}+\frac{\lambda_2}{\lambda_3}+\frac{\lambda_3}{\lambda_1}+\frac{\lambda_3}{\lambda_2}\right)+h_1'+h_2'+h_3' \leqq 3(h_1+h_2+h_3+h_1'+h_2'+h_3').$$

Since $\frac{\lambda_1}{\lambda_3}+\frac{\lambda_3}{\lambda_1}\geqq 2$ and $\frac{\lambda_2}{\lambda_3}+\frac{\lambda_3}{\lambda_2}\geqq 2$, we get

$$\frac{\sqrt{3}}{2}\cdot 4+\sum_{i=1}^{3} h_i' \leqq 3\Big(\sum_{i=1}^{3} h_i+h_i'\Big), \quad 2\sqrt{3} \leqq 3\sum_{i=1}^{3} h_i+2\sum_{i=1}^{3} h_i'.$$

But $h_i' \geqq 0$, for all $i=1, 2, 3$, so

$$2\sqrt{3} \leqq 3\sum_{i=1}^{3}(h_i+h_i').$$

Thus

$$\frac{\sqrt{3}}{3} \leqq \sum_{i=1}^{3}\frac{(h_i+h_i')}{2} \leqq \sum_{i=1}^{3} \text{area}\,[H(W)\cap S_i] = \text{area}\,\Big[H(W)\cap\Big(\bigcup_{i=1}^{3} S_i\Big)\Big].$$

But $W \subset \langle P_{1,2}, P_{2,3}, P_{3,1} \rangle$ and

$$\text{area}\,\langle P_{1,2}, P_{2,3}, P_{3,1} \rangle = \frac{\sqrt{3}}{4}.$$

Thus area $W \leqq \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{3} \leqq \text{area}\,[H(W) \cap (\bigcup_{i=1}^{3} S_i)]$. Thus

$$2\,\text{area}\, W \leqq \text{area}\,[(H(W) \cap (\bigcup_{i=1}^{3} S_i)) \cup W] = \text{area}\, H(W)$$

finishing the Lemma and the Theorem.

IV. Proof of Theorem 3

We repeat here the same notation of Section II, where $\langle P_1, \ldots, P_{n+1} \rangle = \sigma$ is a simplex in E^n, and $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are positive real numbers. $P_{i,j}$ $(i<j)$ is a point on each edge $\langle P_i, P_j \rangle$ between P_i and P_j with the property that

$$\lambda_j(P_i - P_{i,j}) = \lambda_i(P_{i,j} - P_j),$$

where $1 \leqq i < j \leqq n+1$. We regard $P_{i,j} = P_{j,i}$.

LEMMA 8. $\text{Vol}\,[\text{Conv}\, \{P_{i,j}\}_{i \neq j}] \leqq \left(1 - \frac{n+1}{2^n}\right) \text{Vol}\, \sigma$.

PROOF. Without loss of generality, by applying an affine linear transformation we may assume that each edge has length 1. We will proceed by induction on n. For $n=1$ and $n=2$ the statement is trivial and follows from the analysis in Section II respectively.

Call $W = \text{conv}\, \{P_{i,j}\}_{i \neq j}$. Let $\tau_i = \text{conv}\,[\bigcup_{\substack{j=1 \\ j \neq 1}}^{n+1} \{P_{i,j}\}]$. Then

$$\sigma = W \cup \bigcup_{i=1}^{n+1} P_i * \tau_i, \tag{4.1}$$

where $P_i * \tau_i$ is the convex hull of P_i and τ_i, and each of the sets in the union has disjoint interiors. $P_i * \tau_i$ is the i-th "corner" of σ outside W. See Figure 10.

We now apply Lemma 2 to find a point $\hat{\sigma} \in \sigma$ such that for all $i = 1, 2, \ldots, n+1$ the half-space with τ_i in its boundary containing P_i when dilated (from P_i) by $2n/(n+1)$ contains $\hat{\sigma}$. Fix $i = 1, 2, \ldots, n+1$. Let l_i be the altitude from P_i in $P_i * \tau_i$. Let $\hat{l}_i$ be the altitude from $\hat{\sigma}$ in $\hat{\sigma} * \tau_i$ (the convex hull of $\hat{\sigma}$ and τ_i), where, if $\hat{\sigma}$ is on the same side of τ_i as P_i, then $\hat{l}_i < 0$, and $\hat{\sigma} * \tau_i$ is regarded as having negative volume. Then

$$l_i + \hat{l}_i \leqq \frac{2n}{n+1}\, l_i, \quad \hat{l}_i \leqq \frac{n-1}{n+1}\, l_i,$$

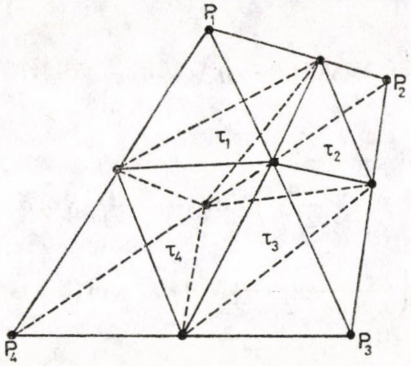

Fig. 10

and

(4.2) $$\operatorname{Vol} \hat{\sigma} * \tau_i \leqq \frac{n-1}{n+1} \operatorname{Vol} P_i * \tau_i.$$

Let $\sigma_i = \langle P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{n+1} \rangle$ be the face opposite P_i. Let

$$W_i = \operatorname{conv} \{P_{i,j}\}_{i \neq j \neq k \neq i} \subset \sigma_i.$$

Then

$$W = \bigcup_{i=1}^{n+1} \hat{\sigma} * W_i \cup \bigcup_{i=1}^{n+1} \hat{\sigma} * \tau_i.$$

Let h_i be the perpendicular distance of $\hat{\sigma}$ from the plane of σ_i. Since each pair of sets in the above union have disjoint interiors, we get

$$\operatorname{Vol} W = \sum_{i=1}^{n=1} \frac{1}{n} h_i \operatorname{Vol}_{n-1} W_i + \sum_{i=1}^{n+1} \operatorname{Vol} \hat{\sigma} * \tau_i,$$

where Vol_{n-1} denotes $(n-1)$-dimensional volume. By induction

$$\operatorname{Vol}_{n-1} W_i \leqq \left(1 - \frac{n}{2^{n-1}}\right) \operatorname{Vol}_{n-1} \sigma_i.$$

Thus using (4.2)

$$\operatorname{Vol} W \leqq \sum_{i=1}^{n+1} \frac{h_i}{n} \left(1 - \frac{n}{2^{n-1}}\right) \operatorname{Vol}_{n-1} \sigma_i + \sum_{i=1}^{n+1} \frac{n-1}{n+1} \operatorname{Vol}(P_i * \tau_i) \leqq$$

$$\leqq \left(1 - \frac{n}{2^{n-1}}\right) \frac{\operatorname{Vol}_{n-1} \sigma_1}{n} \sum_{i=1}^{n+1} h_i + \frac{n-1}{n+1} \sum_{i=1}^{n+1} \operatorname{Vol}(P_i * \tau_i),$$

since $\operatorname{Vol}_{n-1} \sigma_i = \operatorname{Vol}_{n-1} \sigma_1$, for all i. Since σ is regular $\sum_{i=1}^{n+1} h_i$ = any altitude of σ. Thus

$$\frac{\operatorname{Vol}_{n-1} \sigma_1}{n} \sum_{i=1}^{n+1} h_i = \operatorname{Vol} \sigma.$$

By (4.1)

$$\sum_{i=1}^{n+1} \operatorname{Vol}(P_i * \tau_i) = \operatorname{Vol}\sigma - \operatorname{Vol} W.$$

Putting this together we get

$$\operatorname{Vol}(W) \leqq \left(1-\frac{n}{2^{n-1}}\right)\operatorname{Vol}\sigma + \frac{n-1}{n+1}(\operatorname{Vol}\sigma - \operatorname{Vol} W),$$

$$\frac{2n}{n+1}\operatorname{Vol}(W) \leqq \left(\frac{2n}{n+1}-\frac{n}{2^{n-1}}\right)\operatorname{Vol}\sigma, \quad \operatorname{Vol}(W) \leqq \left(1-\frac{n+1}{2^n}\right)\operatorname{Vol}\sigma,$$

finishing the Lemma.

We now can show the density estimate in Theorem 3. If σ is an n-simplex in the triangulation for the triangulated packing $\mathscr{P}$, then each vertex P_i of σ is the center of some body $B_i \in \mathscr{P}$. If λ_i is the constant of homothety for B_i, then points $P_{i,j} \in \langle P_i, P_j \rangle \cap B_i \cap B_j$ are as in Lemma 8. Thus $P_i * \tau_i \subset B_i$ and $W \cup \bigcup_{i=1}^{n+1} B_i \supset \sigma$, and $\bigcup_{i=1}^{n+1} (B_i \cap \sigma) \supset \sigma \setminus W$. So

$$\operatorname{Vol}(\sigma \setminus W) \leqq \operatorname{Vol}\Big(\bigcup_{i=1}^{n+1} B_i \cap \sigma\Big), \qquad \operatorname{Vol}\sigma - \operatorname{Vol} W \leqq \operatorname{Vol}\Big(\bigcup_{i=1}^{n+1} B_i \cap \sigma\Big),$$

$$\operatorname{Vol}\sigma - \left(1-\frac{n+1}{2^n}\right)\operatorname{Vol}\sigma \leqq \operatorname{Vol}\Big[\bigcup_{i=1}^{n+1} B_i \cap \sigma\Big], \qquad \frac{n+1}{2^n} \leqq \frac{\operatorname{Vol}\Big[\bigcup_{i=1}^{n+1} B_i \cap \sigma\Big]}{\operatorname{Vol}\sigma}.$$

Thus $\frac{n+1}{2^n} \leqq d$, where $d =$ density of $\mathscr{P}$.

To finish the Theorem we rely on the following:

Construction 2. Let $\Gamma = \{(z_1, \ldots, z_n) | z_i$ is an integer $i = 1, 2, \ldots, n\} \subset E^n$ be the usual integral lattice. Let $C \subset \Gamma$ be vertices of the unit cube, i.e., $C = \{(z_1, \ldots, z_n) |$for each i, $z_i = 0$ or $1\}$. We define a relation on Γ by saying for $P, Q \in \Gamma$, $P \prec Q$ if $Q - P \in C$. Note the $\prec$ is a partial ordering on C, but not all of Γ.

We now define the triangulation $\mathscr{T}$ of E^n. A simplex $\sigma = \langle P_1, \ldots, P_m \rangle$ of $\mathscr{T}$ consists of $P_i \in \Gamma$ such that $P_1 \prec P_2 \prec \ldots P_m$ and $P_i \prec P_j$ if $1 \leqq i < j \leqq m \leqq n+1$. I.e., well-ordered subsets of Γ are the vertices of simplices of $\mathscr{T}$. This defines a well-known triangulation of E^n apparantly originally due to Freudenthal [7]. See Todd [13], for example, for a proof that we have indeed defined a triangulation.

For P a vertex of $\mathscr{T}$, we claim that the *star* of P in $\mathscr{T} = \operatorname{st}(P, \mathscr{T}) = \bigcup \{\sigma | \sigma \in \mathscr{T}$, P is a vertex of $\sigma\}$ is the convex body B_P defined by $B_P = P + \{(x_1, \ldots, x_n) \| x_i | \leqq 1, |x_i - x_j| \leqq 1, i \neq j, i, j = 1, \ldots, n\}$.

To show this claim we assume, without loss of generality, that $P = O$, the origin. If a simplex of $\mathscr{T}$, $\sigma \subset \operatorname{st}(P, \mathscr{T})$, then we can order the vertices of σ so that $P_1 \prec \ldots \prec 0 = P_i \prec \ldots \prec P_{n+1}$ (possibly with repeated P_j's). All the coordinates of

$P_1, \dots, P_{i-1}$ are 0 or -1, and all the coordinates of $P_{i+1}, \dots, P_{n+1}$ are 0 or 1. In order for $P_1 \prec P_{n+1}$ the non-zero coordinates in the two collections must be disjoint. Thus the vertices of σ are in B_0 and hence $\sigma \subset B_0$.

Suppose $Q=(x_1, \dots, x_n) \in B_0$. By reordering the coordinates, we may suppose that $x_1 \leqq \dots \leqq x_{i-1} \leqq 0 = x_i \leqq x_{i+1} \leqq \dots \leqq x_n$. Let

$$P_j = (-1, \dots, -1, 0, \dots, 0), \quad j = 1, \dots, i-1,$$

where the first j coordinates are -1, and the rest are 0. Let $P_i = O$.

$$P_j = (0, \dots, 1, \dots, 1), \quad j = i+1, \dots, n+1,$$

where the last j coordinates are 1. Then $P_1 \prec \dots \prec P_{n+1}$ defines a simplex $\sigma = \langle P_1, , \dots, P_{n+1} \rangle$ in $\mathcal{T}$, and σ is in $\operatorname{st}(O, \mathcal{T})$ since $P_i = O$.

To show that Q is in $\operatorname{st}(O, \mathcal{T})$ define t_i, $i=1, \dots, n+1$, by

$$\begin{aligned} t_1 &= x_2 - x_1 \\ &\vdots \\ t_{i-2} &= x_{i-1} - x_{i-2} \\ t_{i-1} &= -x_{i-1} \\ t_i &= 1 - x_n + x_1 \\ t_{i+1} &= x_n - x_{n-1} \\ &\vdots \\ t_n &= x_{i+1} - x_i \\ t_{n+1} &= x_i. \end{aligned}$$

It is easy to check that $t_i \geqq 0$, for all $i=1, \dots, n+1$, $\sum_{i=1}^{n+1} t_i = 1$, and $\sum_{i=1}^{n+1} t_i P_i = Q$.

It is clear that $\mathcal{T}$ is symmetric about any vertex of $\mathcal{T}$. Then we define a symmetric convex set $\hat{B}_P = P + \frac{1}{2} B_O$ for each $P \in \Gamma$. This is the convex hull of the midpoints of all of the 1-simplices with P as one vertex. $\hat{B}_P$ is symmetric since $\mathcal{T}$ is, and each $\hat{B}_P$ is congruent to $\hat{B}_Q$, $P, Q \in \Gamma$ by a translation. Each $\hat{B}_P \cap \sigma$ where $\sigma \in \mathcal{T}$, $P \in \sigma$, is precisely the corner defined by $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$ in Lemma 8. Thus the density is exactly $(n+1)/2^n$. See Figure 11.

This finishes the proof of Theorem 3.

Remark 6. It turns out in Construction 2 that the sets B_P are all (congruent) zonotopes. They are the convex hull of a cube of side length 1/2 and the reflection of that cube about one of its vertices.

Note also that although no triangulated packing of equal spheres exists for dimensions greater than 2, in dimension 3 we can find a triangulated packing with just two sizes of spheres. Take the usual close lattice packing of equal spheres with radius 1 say. Joining the centers of adjacent spheres we get a 1-dimensional complex that can be regarded as the edges of a tiling of E^3 by regular octahedra and

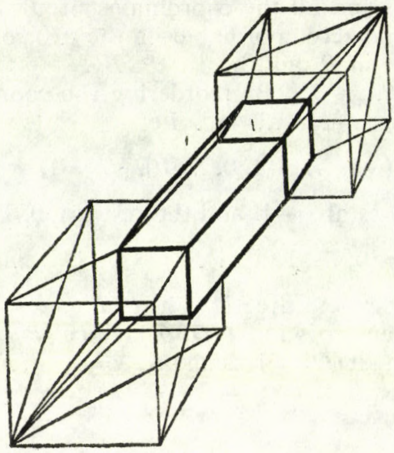

Fig. 11

tetrahedra of side length 2. In the center of each octahedron place a smaller sphere of radius $\sqrt{2}-1$, which touches all 8 of the surrounding spheres of the regular octahedron. This then yields a triangulated packing of E^3 by spheres of just two sizes.

PROOF OF THE COROLLARY. Let O be the origin in the interior of a convex body B. Then the Minkowski average of B and $-B$,

$$\tilde{B} = \frac{1}{2}B - \frac{1}{2}B = \left\{\frac{1}{2}x - \frac{1}{2}y \,\middle|\, x, y \in B\right\}$$

is a centrally symmetric convex body. If $B_i = p_i + B$ and $B_j = p_j + B$ are translates of B with disjoint interiors, then by a Theorem of Hadwiger [10] $\tilde{B}_i = p_i + \tilde{B}$, $\tilde{B}_j = p_j + \tilde{B}$ have disjoint interiors and intersect if and only if B_i and B_j intersect, as mentioned in the introduction.

A theorem of Rogers and Shephard [12] says that

$$\mathrm{Vol}\,\tilde{B} \leqq \frac{\binom{2n}{n}}{2}\,\mathrm{Vol}\,B.$$

This then yields the Corollary via Theorem 3.

References

[1] A. Bezdek, K. Bezdek, K. Böröczky, On compact packings, *Studia Sci. Math. Hung.*, **21** (1986), 343—346.

[2] K. Bezdek, Compact packings in the Euclidean space, *Beiträge zur Algebra und Geometrie*, **25** (1987), 79—84.

[3] F. Bohnenblast, Convex regions and projections in Minkowski spaces, *Ann. of Math.*, **(20) 39** (1938), 301—308.
[4] T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper*, Chelsea Publishing Company (Bronx, New York).
[5] L. Fejes Tóth, *Regular Figures*, The MacMillan Company (New York, 1964).
[6] L. Fejes Tóth, Compact packings of circles, *Studia Sci. Math. Hung.*, *19* (1984), 103—109.
[7] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, *Annals of Math.*, **43** (1942), 580—582.
[8] B. Grünbaum, On some covering and intersection properties in Minkowski spaces, *Pac. J. of Math.*, **9** (1959), 487—489.
[9] B. Grünbaum, Some applications of expansion constants, *Pac. J. of Math.*; **10** (1960), 193—201.
[10] H. Hadwiger, Über Treffenzahlen bei translationgleichen Eikörpern, *Arch. Math.*; **VIII** (1957), 212—213.
[11] K. Leichtweiss, Zwei Extremalprobleme der Minkowski-Geometrie, *Math. Z.*; **62** (1955), 37—49.
[12] C. A. Rogers and G. C. Shephard, The difference body of a convex body, *Arch. Math.*, **8** (1957), 220—233.
[13] M. J. Todd, The Computation of Fixed Points and Applications, *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, Vol. 124 (1976), 29—31.

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DEPARTMENT OF GEOMETRY
EÖTVÖS UNIVERSITY
1088 BUDAPEST, RÁKÓCZI ÚT 5
HUNGARY

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853
U.S.A.

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RESTRICTIONS OF COMPACT NORMAL OPERATORS

J. PAEZ (Carabobo), Z. SEBESTYÉN (Budapest) and J. STOCHEL (Krakow)

A recent characterization of subpositive and subprojection suboperators appeared in Halmos [1], Sebestyén [2], [3], Sebestyén—Kapos [4]. The aim of this note is to show that the characterization problem of restrictions of compact normal operators can be solved by referring to the spectral theorem and to the extension process for subprojection suboperators given in [3].

Let H be a (complex) Hilbert space, H_0 a linear subspace of H and $A_0\colon H_0 \to H$ a linear operator. The problem we shall solve is: when does there exist a compact normal operator A on H such that A_0 is the restriction of A to H_0? This is a special case of Problem 1 posed by Halmos in [1]. However, the solution is not formulated in terms of the geometric behaviour of A_0, as desired there. Note that ([3]) a suboperator $Q\colon H_0 \to H$ is subprojection, i.e. Q is a restriction of an orthogonal projection P on H, if and only if the following identity holds:

(1) $$\|Qx\|^2 = (Qx, x) \quad \text{for} \quad x \text{ in } H_0.$$

The smallest such P satisfies further (by the construction in [3]):

(2) $$\operatorname{Range} P \subset \overline{\operatorname{Range} Q}.$$

THEOREM. *The suboperator $A_0\colon H_0 \to H$ has a compact normal extension to H if and only if*

(i) *there exists a finite or infinite sequence of subprojection suboperators $Q_n\colon H_0 \to H$, $n \geqq 0$ such that* $\operatorname{rank} Q_n < \infty$ *for* $n > 0$, $\{\operatorname{range} Q_n\}_{n \geqq 0}$ *is pairwise orthogonal, and*

(3) $$x = \sum_{n \geqq 0} Q_n x \quad \textit{for any } x \textit{ in } H_0,$$

(ii) *there exists a corresponding bounded sequence of complex numbers $\{\lambda_n\}_{n \geqq 0}$ with $\lambda_0 = 0$ as the only possible point of accumulation and such that*

(4) $$A_0 x = \sum_{n \geqq 0} \lambda_n Q_n x \quad \textit{for any } x \textit{ in } H_0.$$

PROOF. The necessity of (i) and (ii) is obvious by the spectral theorem for compact normal operators. To prove their sufficiency define the projection operators P_n as follows. For $n > 0$ take the smallest positive extension of Q_n in [3] as P_n. By (2), P_n is of finite rank for $n > 0$. It follows from (i) that the series $\sum_{n \geqq 1} P_n$ converges in the strong operator topology. Define P_0 by

(5) $$P_0 x = x - \sum_{n \geqq 1} P_n x \quad \text{for any } x \text{ in } H.$$

Let A be defined by

$$Ax = \sum_{n\geqq 0} \lambda_n P_n x \quad \text{for all } x \text{ in } H. \tag{6}$$

The operator A is easily seen to be a compact normal operator on H. Further, A is an extension of A_0. This follows from (4) and the fact that every P_n is an extension of Q_n. Indeed, for $n=0$ we can apply (5) and (3), and the proof is complete.

COROLLARY. *The suboperator A_0: $H_0 \to H$ has a compact selfadjoint (positive) extension if and only if the sequence $\{\lambda_n\}_{n\geqq 0}$ in the Theorem consists of real (positive) numbers.*

This work was done while the first and third named authors were visiting the Department of Applied Analysis of the Eötvös University Budapest.

References

[1] P. R. Halmos, Subnormal suboperators and the subdiscrete topology, *Anniversary Volume on Approximation Theory and Functional Analysis*, eds. P. L. Butzer, R. L. Stens, B. Sz.-Nagy, Birkhauser (Basel, 1984), pp. 49—56.

[2] Z. Sebestyén, Restriction of positive operators, *Acta Sci. Math. (Szeged)*, **46** (1983), 299—301.

[3] Z. Sebestyén, Characterizations of subprojection suboperators, *Acta Math. Hung.*, **56** (1990), 115—119.

[4] Z. Sebestyén—L. Kapos, Extremal positive self-adjoint extensions of suboperators, *Per. Math. Hung.*, **20** (1989), 75—80.

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DEPARTMENT OF MATHEMATICS
CARABOBO UNIVERSITY
VENEZUELA

DEPARTMENT OF APPLIED ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST
HUNGARY

INSTYTUT MATEMATYKI
UNIWERSYTET JAGIELLONSKI
UL. REYMONTA 4
30-059 KRAKOW
POLAND

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A GENERAL RIEMANN COMPLETE INTEGRAL IN THE PLANE

Z. BUCZOLICH (Budapest)

Introduction. In [6], [7] W. F. Pfeffer defines a multidimensional Riemann type integral such that the divergence of any vector field continuous in a compact interval and differentiable in its interior is integrable, and the integral equals the flux of the vector field out of the interval. In Section 7.4 of [6] the author mentions the possibility of a definition of an integral through special partitions. This integral is a natural, simple alternative of his integral. In this paper we solve the problem of the existence of special partitions in the plane. Unfortunately we were unable to use our method for higher dimensions. Definitions of Riemann type integrals are based on the ideas of Henstock and Kurzweil (cf. [1], [2] and [3]). Mawhin's integrals in [4] or [5] are suitable generalizations for the divergence theorem but they lack the additivity, which one expects from an integral. In the first section we prove the existence of 10^{-3}-regular, δ-fine and special partitions of an interval $A\subset\mathbf{R}^2$ (Theorem 1.1). In the second section we give the definition of the Riemann complete type (GRC) integral. We prove the additivity of this integral (Theorem 2.1). Finally we show that this GRC integral integrates the derivatives of functions of intervals (Theorem 2.2). We remark that all lemmas, propositions, theorems and corollaries in Sections 3—6 of [6] can be proved for the GRC integral as well. Namely the GRC integral is almost everywhere differentiable and the divergence theorem as stated in [6, 5.5] holds for the GRC integral.

Preliminaries. By $\mathbf{R}$ we denote the real numbers. Suppose that $x=(x_1, x_2)\in\mathbf{R}^2$ and $y=(y_1, y_2)\in\mathbf{R}^2$ then $\operatorname{dist}(x, y):=\max\{|x_i-y_i|: i=1, 2\}$. We shall use *exclusively this metric* in $\mathbf{R}^2$. If $E\subset\mathbf{R}^2$ then E^{cl}, bE, diam(E) and $|E|$ denotes the closure, the boundary, the diameter and the Lebesgue measure of E, respectively. If $E\subset A\subset\mathbf{R}^2$ then by int $(E; A)$ we denote the interior of E in the subspace topology of A. All functions in this paper are real valued. The term measurable means Lebesgue measurable. By an interval A we mean a set of the form $A=[a_1, b_1]\times[a_2, b_2]=(a_1, a_2)+[0, c_1]\times[0, c_2]$ where $c_i=b_i-a_i>0$ for $i=1,2$.

If A is an interval then the set of vertices of A will be denoted by vA. We denote $[0, 1]^2$ by I. We say that the intervals A and B are non-overlapping if $\operatorname{int}(A; \mathbf{R}^2)\cap\cap\operatorname{int}(B; \mathbf{R}^2)=\emptyset$. If the intervals $A_j\subset A$ $(j=1, ..., k)$ are non-overlapping then we say that the set $D=\{A_j: j=1, ..., k\}$ is a subdivision of A. If furthermore the subdivision D fulfills that $A=\cup\{A_j: j=1, ..., k\}$ then we say that D is a division of A.

DEFINITION 0.1. Suppose that $A\subset\mathbf{R}^2$ is an interval then the parameter of regularity of A is the number $r(A)=$the length of the smaller side of A/the length of the longer side of A. If $r\in(0,1)$ then we say that the interval A is r-regular if $r(A)\geqq r$. If D is a subdivision of A then $r(D):=\min\{r(A_j): j=1, \ldots, k\}$. We say that D is r-regular if $r(D)\geqq r$.

DEFINITION 0.2. Let A be an interval in $\mathbf{R}^2$. A partition of A is a collection $P=\{(A_1, x_1), \ldots, (A_k, x_k)\}$ where $D(P):=\{A_1, \ldots, A_k\}$ is a division of A and $x_j\in A_j$, $j=1, \ldots, k$. Given an $r\in(0,1)$, we say that P is r-regular whenever $D(P)$ is r-regular.

DEFINITION 0.3. Suppose that $A\subset\mathbf{R}^2$ and the function $\delta\colon A\to(0,+\infty)$ is given. We say that the pair (A', x) is δ-fine and special if $x\in vA'$ and

$$A'\subset\{y: \operatorname{dist}(x,y)\leqq\delta(x)\}.$$

A subpartition P of A is called δ-fine and special if every $(A_j, x_j)\in P$ is δ-fine and special.

DEFINITION 0.4. If we have a fixed interval A and $P=\{(A_1, x_1), \ldots, (A_k, x_k)\}$ is a subpartition of A then $F(P):=\bigcup\{A_j: j=1, \ldots, k\}$ and $G(P):=\operatorname{int}(F(P); A)$, plainly $F(P)$ is closed and $G(P)$ is open in A.

If a set $B\subset\mathbf{R}^2$ consists of finitely many intervals then we can define regular, δ-fine or special subpartitions and partitions of B analogously to the above definitions.

1. The partition theorem

THEOREM 1.1. *Suppose that $A\subset\mathbf{R}^2$ is an interval and $\delta\colon A\to(0,+\infty)$ is given. Then A has a 10^{-3}-regular, δ-fine and special partition.*

First we prove this theorem from the following lemma, and then we shall prove the lemma.

LEMMA 1.1. *We put $d:=1/20$. Suppose that $\delta\colon I\to(0,+\infty)$ and $H:=\{x\in I: \delta(x)\geqq 1\}$. Then there exists a d^2-regular, δ-fine and special subpartition $P=\{A_j, x_j): j=1, \ldots, k\}$ of I such that $H\subset G(P)$.*

PROOF OF THEOREM 1.1. We put $H_n:=\{x\in A: \delta(x)\geqq 1/n\}$ for $n=1, 2, \ldots$. We want to define an increasing sequence P_n of subpartitions of A with the following properties for every $n=1, \ldots$:

(1) $H_n\cup G(P_{n-1})\subset G(P_n)$,

(2) P_n is a 10^{-3}-regular, δ-fine and special subpartition of A.

Since $\delta(x)>0$ for every $x\in A$ we have $A=\bigcup_{n=1}^{\infty} H_n\subset\bigcup_{n=1}^{\infty} G(P_n)$. Using the compactness of A we can choose a natural number N such that $A=\bigcup_{n=1}^{N} G(P_n)$. Then obviously P_N is a 10^{-3}-regular, δ-fine and special partition of A, required in our theorem.

We have to prove the existence of the above sequence P_n. We put $P_0=\emptyset$ and $r':=1/2$. Suppose that $n\geqq 1$ and for $m=1, \ldots, n-1$ we have defined P_m fulfilling

(1) and (2). From (2) it follows that $A \setminus G(P_{n-1})$ is the union of finitely many non-overlapping intervals. That is we can choose a division $D'=\{A_j: j=1, \dots, k\}$ of $A \setminus G(P_{n-1})$ and we may obviously suppose that $\operatorname{diam}(A_j)<1/n$. It is easy to show that any interval can be divided into finitely many r'-regular intervals (cf. [6, Lemma 2.3]). Therefore we may also suppose that D' is r'-regular. For a $j \in \{1, \dots, k\}$ and $A_j=a_j+[0, c_{1,j}]\times[0, c_{2,j}]$ we define the affine transformation T_j by

$$T_j(x) = (x-a_j)\begin{pmatrix} 1/c_{1,j} & 0 \\ 0 & 1/c_{2,j} \end{pmatrix}.$$

It is easy to see that $T_j(A_j)=I$. The inverse of T_j is denoted by S_j, that is

$$S_j(x) = a_j+x\begin{pmatrix} c_{1,j} & 0 \\ 0 & c_{2,j} \end{pmatrix}.$$

It is obvious from the definition that

$$\operatorname{diam}(A_j) = \max\{c_{1,j}, c_{2,j}\} \quad \text{and} \quad r(A_j) = \min(c_{1,j}, c_{2,j}\}/\max\{c_{1,j}, c_{2,j}\}.$$

An easy computation shows that for any interval B we have

(3) $\operatorname{diam}(S_j(B)) \leqq \operatorname{diam}(B)\cdot\operatorname{diam}(A_j)$ and $r(S_j(B)) \geqq r(B)\cdot r(A_j) \geqq r(B)\cdot r'$.

We put $\delta'(x):=\delta(S_j(x))/\operatorname{diam}(A_j)$. From (3) it follows that if B was a d^2-regular, δ'-fine and special interval then $S_j(B)$ is $d^2r'\geqq 10^{-3}$-regular, δ-fine and special. Since $\operatorname{diam}(A_j)<1/n$ we have $\delta'(x)\geqq 1$ whenever $S_j(x)\in H_n$. Therefore $T_j(H_n \cap A_j)\subset H':=\{x\in I: \delta'(x)\geqq 1\}$. We can apply Lemma 1.1 with $\delta:=\delta'$ and $H:=H'$. We obtain a d^2-regular, δ-fine and special subpartition $P_j'=\{(B_{i,j}, x_{i,j}): i=1, \dots, k_j\}$ of I such that $H'\subset G(P_j')$.

We put $P_{n,j}:=\{(S_j(B_{i,j}), S_j(x_{i,j})): i=1, \dots, k_j\}$. From the properties of S_j it follows now that $P_{n,j}$ is 10^{-3}-regular, δ-fine and special. Thus if we let $P_n:=P_{n-1}\cup \cup\{P_{n,j}: j=1, \dots, k\}$ then P_n fulfills (2). It is obvious that $G(P_{n-1})\subset G(P_n)$. If $x\in H_n\cap(A\setminus G(P_{n-1}))$ and there exists a $j\in\{1, \dots, k\}$ such that $x\in\operatorname{int}(A_j)$ then $x\in\operatorname{int}(F(P_n), A)$ and hence $x\in G(P_n)$. If $x\in H_n(A\setminus G(P_{n-1}))$ and there exists a $j\in\{1, \dots, k\}$ such that $x\in \mathrm{b}A_j$ then $x\in G(P_{n,j'})(=\operatorname{int}(P_{n,j'}; A_{j'}))$ for every j' such that $x\in \mathrm{b}A_{j'}$. Using also $F(P_{n-1})\subset F(P_n)$ we obtain $x\in G(P_n)$. Thus we proved that P_n also fulfills (1). This completes the proof of Theorem 1.1.

Definition 1.1. Suppose that the intervals B_j, $j=1, \dots, m$, are subintervals of $[0, 1]$ and their lengths are b_j, $j=1, \dots, m$ respectively. If $0<d<1$ we say that these intervals are d-ranked if for every $j\in\{1, \dots, m\}$ there is a natural number $n(j)$, the rank of B_j such that $b_j=d^{n(j)}$. Let $n_1, n_2, \dots, n_s$ be the ranks of the intervals B_j, $j=1, \dots, m$. We denote by N_i the set of those indices j for which the rank of B_j equals n_i.

Lemma 1.2. *Suppose that $0<d<1/3$ and the closed intervals B_j, $j=1, \dots, m$ are d-ranked and they cover the interval $[0, 1]$. Then there exists a finite set of non-overlapping closed intervals in C_i, $i=1, \dots, t$ of length c_i such that they cover $[0, 1]$ and for every $i\in\{1, \dots, t\}$ there is a $j(i)\in\{1, \dots, m\}$ such that $B_{j(i)}\subset C_i$, $c_i<4d^{n(j(i))}$ and for every $n\in\mathcal{N}$ we have that $\cup\{B_j$: the rank of B_j is smaller than $n\}\subset \subset\cup\{C_i$: the rank of $B_{j(i)}$ is smaller than $n\}$.*

PROOF OF LEMMA 1.2. We shall use the notation of Definition 1.1. Choose a maximal non-overlapping subset of $\{B_j: j\in N_1\}$ and denote the index set corresponding to this set by N_1'. Since this set of intervals is maximal, for every $j\in N_1$ there exists a $j'\in N_1'$ such that $B_j\cap B_{j'}\neq\emptyset$. Using the fact that the length of the intervals B_j, $j\in N_1$ equals d^{n_1} we obtain that we can enlarge the intervals $B_{j'}$ $(j'\in N_1')$ by at most $2d^{n_1}$ such that these enlarged intervals, denoted by $B_{j,1}$, $j\in N_1'$, are non-overlapping and $\bigcup\{B_{j,1}: j\in N_1'\}\supset\bigcup\{B_j: j\in N_1\}$. Suppose that N_j' $(j\leqq i-1)$ and the intervals $\{B_{j,i-1}: j\in N_1'\cup\ldots\cup N_{i-1}'\}$ are defined, these intervals are non-overlapping and $\bigcup\{B_{j,i-1}: j\in N_1'\cup\ldots\cup N_{i-1}'\}\supset\bigcup\{B_j: j\in N_1\cup\ldots\cup N_{i-1}\}$. If $i<s$ then choose a maximal non-overlapping subset of $\{B_j: j\in N_i\}$ such that, denoting by N_i' the corresponding set of indices, the intervals $B_{j,i-1}$ and $B_{j'}$ do not overlap for $j\in N_1'\cup\ldots\cup N_{i-1}'$ and $j'\in N_i'$. By the maximality of the set $\{B_j: j\in N_i'\}$ for every $j\in N_i$ there exists either a $j'\in N_i'$ or a $j'\in N_1'\cup\ldots\cup N_{i-1}'$ such that $B_j\cap B_{j'}\neq\emptyset$ or $B_j\cap B_{j',i-1}\neq\emptyset$. Using the fact that the length of the intervals B_j, $j\in N_i$, equals d^{n_i} we obtain that we can enlarge the intervals $B_{j',i-1}$, $j'\in N_1'\cup\ldots\cup N_{i-1}'$ and B_j, $j\in N_i'$ by at most $2d^{n_i}$ so that these enlarged intervals, denoted by $B_{j,i}$, $j\in N_1'\cup\ldots\cup N_i'$, are non-overlapping and $\bigcup\{B_{j,i}: j\in N_1'\cup\ldots\cup N_i'\}\supset\bigcup\{B_j: j\in N_1\cup\ldots\cup N_i\}$. Since $N_1\cup\ldots\cup N_s=M_0$ we have $\bigcup\{B_{j,s}: j\in N_1'\cup\ldots\cup N_s'\}\supset\bigcup\{B_j: j\in M_0\}=[0, 1]$.

Thus, denoting by C_i, $i=1, \ldots, t$ the set of the intervals $\{B_{j,s}: j\in N_1'\cup\ldots\cup N_s'\}$, the intervals C_i are non-overlapping and they cover $[0, 1]$. Furthermore if $C_i=B_{j,s}$ and $j\in N_{i'}'$, then we obtained $B_{j,s}$ from B_j, $j\in N_{i'}$, by a sequence of enlargements first by at most $2d^{n_{i'}}$, then by at most $2d^{n_{i'+1}}$ etc. Therefore the length of $B_{j,s}$ is smaller than

$$d^{n_i}+2\sum_{k=0}^{\infty} d^{n_i+k} = d^{n_i}\left(1+2\sum_{k=0}^{\infty} d^k\right) < 4d^{n_i}.$$

Finally the last property claimed in our lemma follows from

$$\bigcup\{B_{j,s}: j\in N_1'\cup\ldots\cup N_i'\} \supset \{B_{j,i}: j\in N_1'\cup\ldots\cup N_i'\} \supset\bigcup\{B_j: j\in N_1\cup\ldots\cup N_i\}$$

for every $i\leqq s$.

PROOF OF LEMMA 1.1. If there exists an $x=(x_1, x_2)\in H$ such that $d\leqq x_1\leqq 1-d$ and $d\leqq x_2\leqq 1-d$ then we can divide I into four intervals with a common vertex x. These intervals provide a d^2-regular, δ-fine, special partition of I.

Thus we may suppose that $[d, 1-d]\times[d, 1-d]\cap H=\emptyset$. For an $x\in H\setminus \mathrm{b}I$ we shall denote by $n(x)$ the natural number satisfying $d^{n(x)}>\mathrm{dist}\,(x, \mathrm{b}I)\geqq d^{n(x)+1}$. We recall that $\mathrm{b}I=E_1\cup E_2\cup E_3\cup E_4$. For an $x\in H\setminus \mathrm{b}I$ choose a $k(x)\in\{1, 2, 3, 4\}$ such that $\mathrm{dist}\,(x, E_{k(x)})=\mathrm{dist}\,(x, \mathrm{b}I)$. Denote by $w(x)\in E_{k(x)}$ the image point of x under and orthogonal projection to $E_{k(x)}$. Since $\mathrm{dist}\,(x, E_{k(x)})=\mathrm{dist}\,(x, \mathrm{b}I)$ the one dimensional interval $B(x)$ with midpoint $w(x)$ and of length $d^{n(x)+1}$ is contained in $E_{k(x)}$. Denote by $A^1(x)$ the closed subinterval of I for which the length of the sides orthogonal to $E_{k(x)}$ equals $(1+d)d^{n(x)}$ and $B(x)\subset E_{k(x)}$ is also a side of $A^1(x)$. Put $A^2(x)=\emptyset$.

For $x\in \mathrm{b}I\setminus vI$ choose $k(x)$ and $n(x)$ such that $x\in E_{k(x)}$, $10d^{n(x)}<\delta(x)$ and the one dimensional interval $B(x)$ with midpoint x and of length $d^{n(x)+1}$ fulfills $B(x)\subset\mathrm{int}\,(E_{k(x)})$. We define $A^1(x)$ such that $B(x)$ is one of its sides and the length of the sides orthogonal to $B(x)$ equals $(1+d)d^{n(x)}$. We also put $A^2(x)=\emptyset$.

For $x \in vI$ choose an $n(x) \in \mathcal{N}$ such that $10d^{n(x)} < \delta(x)$. Denote by $A^3(x)$ the subinterval of I such that x is one of its vertices and the horizontal side of $A^3(x)$ is $d^{n(x)+1}$ and its vertical side is $(1+d)d^{n(x)}$. Denote by $A^1(x)$ the rectangle which we obtain by removing from $A^3(x)$ the closed vertical segment of $A^3(x) \cap \mathrm{b}I$. Denote by $A^4(x)$ the subinterval of I such that x is one of its vertices and the vertical side of $A^4(x)$ is $d^{n(x)+1}$ and its horizontal side is $(1+d)d^{n(x)}$. Denote by $A^2(x)$ the rectangle which we obtain by removing from $A^4(x)$ the closed horizontal segment of $A^4(x) \cap \mathrm{b}I$.

Put $T := H^{\mathrm{cl}} \cup \mathrm{b}I$. The relative I-interiors of the sets $A^1(x) \cup A^2(x)$, $x \in H \cup \mathrm{b}I$, are obviously covering $H \cup \mathrm{b}I$. If $y \in H^{\mathrm{cl}}$ and $y \notin \mathrm{b}I$ then one can choose an $x \in H \setminus \mathrm{b}I$ close enough to y such that $\mathrm{dist}(x, y) < d^{n(x)+1}/2$ and by the definition of $A^1(x)$ we have $y \in \mathrm{int}\,(A^1(x), I)$; here we remind the reader that $H \cap [d, 1-d] \times [d, 1-d] = \emptyset$ and hence $T \subset I \setminus (d, 1-d) \times (d, 1-d)$. Thus by the compactness of T we can select finitely many $x_i \in H \cup \mathrm{b}I$ $(i = 1, \dots, m')$ such that the union of the sets $A^1(x_i)$, $A^2(x_i)$ covers T. For a fixed edge of I say E we shall denote by B_j, $j = 1, \dots, m$ those nonempty subintervals of E for which there exists an $i \in \{1, \dots, m'\}$, and a $p \in \{1, 2\}$ such that $B_j = (E \cap A^p(x_i))$ and for this j, i, and p we put $A_j := A^p(x_i)$. From the definition of the rectangles $A^p(x_i)$ it follows that the one dimensional intervals B_j are d-ranked and they cover E. Identifying E with $[0, 1]$ and applying Lemma 1.2 we obtain the intervals C_i, $i = 1, \dots, t$ and the function $j(i)$. If C_i does not contain the endpoints of E then denote by A_i' the closed rectangle for which $A_i' \cap E = C_i$ and the length of the sides of A_i' orthogonal to E equals the length of the corresponding sides of $A_{j(i)}$, namely, if the length of $B_{j(i)}$ is d^k then the length of C_i is smaller than $4d^k$ and not less than d^k, and the length of the sides of A_i' orthogonal to C_i equals $(1+d)d^{k-1}$. If C_i contains one endpoint of E then denote by A_i'' the closed rectangle for which $A_i'' \cap E = C_i$ and the length of the sides of A_i'' orthogonal to E equals the length of the corresponding sides of $A_{j(i)}$. Put $A_i' = (A_i'' \setminus \mathrm{b}I) \cup C_i$.

Using the notation of the proof of Lemma 1.2 we have $\bigcup\{C_k \colon c_k \geqq d^{n_i}\} = \bigcup\{B_{j,s} \colon j \in N_1' \cup \dots \cup N_i'\} \supset \bigcup\{B_{j,i} \colon j \in N_1' \cup \dots \cup N_i'\} \supset \bigcup\{B_j \colon j \in N_1 \cup \dots \cup N_i\}$ for every $i \leqq s$. This implies that $\bigcup\{A_i' \colon i = 1, \dots, t\} \supset \bigcup\{A_j \colon j = 1, \dots, m\}$. Since the intervals C_i are non-overlapping the rectangles A_i', $i = 1, \dots, t$ are also non-overlapping. Repeating the above process on every edge of I and taking the corresponding A_i' rectangles we can obtain a set of rectangles denoted by D_i, $i = 1, \dots, p$ such that $\bigcup\{D_i \colon i = 1, \dots, p\} \supset T$ and the rectangles D_i intersecting the interior of the same side of I are non-overlapping.

The choice of d implies that the rectangles D_i intersecting opposite sides of I are non-overlapping. It is also obvious that overlapping problems arise only close to the vertices of I. Each D_i intersects $\mathrm{b}I$ in a segment C_i and the side of D_i orthogonal to C_i is of length $(1+d)d^{n(i)}$. We shall say that this $n(i)$ is the size number of D_i.

Suppose that there are overlapping rectangles D_1 and D_2 such that $D_1 \cap E_1 \neq \emptyset$ and $D_2 \cap E_2 \neq \emptyset$ (see Fig 1.).

By symmetry we can also suppose that $n(1)$, the size number of D_1, is minimal, that is, if $n < n(1)$ is a natural number then any rectangle of size number n does not overlap any other rectangle D_i at this corner of I. Thus $n(2) \geqq n(1)$. Denote by D_1' the interval which has left vertical side common with D_1 and its right vertical side

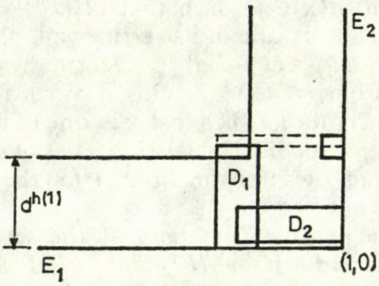

Fig. 1

is in E_2. We shall denote by D_3 the rectangle which intersects E_2 and which contains "from below" the right upper vertex of D_1'. This rectangle D_3 exists since $bI(\subset T)$ is covered by the rectangles D_i but it may happen that D_3 coincides with D_2. By the minimality of $n(1)$ we have $n(3) \geqq n(1)$ where $n(3)$ denotes the size number of D_3. Since the vertical sides of D_3 are smaller than $4d^{n(3)+1} \leqq 4d^{n(1)+1}$ we can enlarge the vertical sides of D_1' by at most $4d^{n(1)+1}$ in order to obtain the interval D_1'' of which the upper horizontal edge contains the corresponding edge of D_3. By the minimality of $n(1)$, other rectangles D_i above D_3 does not overlap any of the rectangles D_i having common side with E_1.

Therefore we constructed D_1'' from D_1 by enlarging its horizontal edges by at most $d^{n(1)}(1+d)$ and by enlarging its vertical edges by at most $4d^{n(1)+1}$. If all the overlapping rectangles at this corner are covered by D_1'' then we put $D_1^{(4)} := D_1''$. If there is a rectangle D_4 such that $D_4 \cap E_2 \neq \emptyset$, $D_4 \cap D_1'' \neq \emptyset$, and $D_4 \not\subset D_1''$ then we may suppose that the size number, $n(4)$, of D_4 is minimal, that is, rectangles D_i of smaller size number do not have the property that $D_i \cap E_2 \neq \emptyset$, $D_i \cap D_1'' \neq \emptyset$ and $D_i \not\subset D_1''$ (cf. Fig. 2).

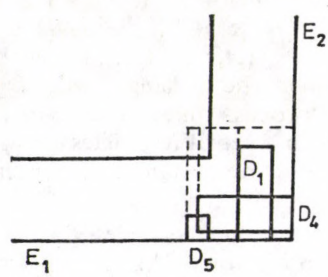

Fig. 2

From the minimality of $n(1)$ it follows that $n(4) \geqq n(1)$. Denote by D_1''' the interval we obtained from D_1'' by enlarging its horizontal sides to contain D_4, that is, the length of the horizontal sides of D_1''' is $(1+d)d^{n(4)}$.

We shall denote by D_5 the rectangle which intersects E_1 and which contains "from the right" the left lower vertex of D_1'''. Again D_5 exists since $bI \subset T$. We also have $n(5) \geqq n(1)$ because if $n(5) < n(1)$ then D_5 overlaps D_4 contradicting the minimality of $n(1)$. Thus enlarging by at most $4d^{n(1)+1}$ the horizontal side of

D_1''' we obtain $D_1^{(4)}$ of which the left vertical edge contains the corresponding edge of D_5. From the minimality of $n(1)$ and $n(4)$ it follows that $D_1^{(4)}$ contains all the overlapping intervals at this corner. After the above enlargements the horizontal edge of $D_1^{(4)}$ is less than

$$\max\{\text{the horizontal edge of } D_1'', d^{n(4)}(1+d)+4d^{n(5)+1}\} \leqq d^{n(1)}(1+d)+4d^{n(1)+1}.$$

The vertical edge of $D_1^{(4)}$ equals the vertical edge of D_1'' and is less than $d^{n(1)}(1+d)+ +4d^{n(1)+1}$.

Deleting those rectangles D_i, $i=1, \ldots, p$ which are covered by $D_1^{(4)}$ and replacing D_1 by $D_1^{(4)}$ we have a new system of rectangles which does not have overlapping at the common vertex of E_1 and E_2.

Using the above process at each vertex of I and taking the closure of the rectangles D_i we can modify the set $\{D_i\colon i=1, \ldots, p\}$ to obtain a set of intervals $\{Q_i\colon i=1, \ldots, p'\}$ which consists of non-overlapping intervals and

$$\cup\{Q_i\colon i=1, \ldots, p'\} \supset T.$$

Now we shall show that for every $i=1, \ldots, p'$ we can find a d^2-regular, δ-fine, special partition of Q_i. Each interval Q_i was obtained from a rectangle $A^p(x_j)$ after a sequence of enlargements. Suppose that $x_j \in H \setminus \mathrm{b}I$. In the sequel we shall omit the index j. Since $x \in H \setminus \mathrm{b}I$, by the definition of the rectangle $A^p(x)$, the distance of x from the boundary of $A^p(x)$ is at least $d^{n(x)+1}/2$. Since the sides of $A^p(x)$ are at most $(1+d)d^{n(x)}$ and these sides are enlarged by at most $2(1+d)d^{n(x)}+ +8d^{n(x)+1}$, the sides of Q_i are smaller than $5d^{n(x)}$. The distance of x from the boundary of Q_i is still at least $d^{n(x)+1}/2$ and hence cutting Q_i through x into four subintervals we obtain a d^2-regular, δ-fine, special partition of Q_i where the special vertex of all the four intervals is x. We recall that $x \in H$ and hence $\delta(x) \geqq 1$. If $x \in \mathrm{b}I \setminus vI$ then we can cut Q_i through x into two d^2-regular, δ-fine, special subintervals; the proof of this fact is similar to the above case and we shall omit the details, we only remind that in this case $10d^{n(x)} < \delta(x)$. Finally if $x \in vI$ then Q_i itself with the special vertex x is the desired d^2-regular, δ-fine, special partition. Therefore the union of the above partitions of Q_i for $i=1, \ldots, p'$ provides the subpartition of I claimed in Lemma 1.1.

2. The definition of the GRC integral and its properties

DEFINITION 2.1. If f is a function on an interval A and

$$P = \{(A_1, x_1), \ldots, (A_k, x_k)\}$$

is a partition of A, we let:

$$\sigma(f, P) = \sum_{j=1}^{k} f(x_j)\,|A_j|.$$

DEFINITION 2.2. A function f on an interval A is called (GRC) integrable in A if there is a real number J with the following property: for every $\varepsilon > 0$, there is a $\delta\colon A \to (0, +\infty)$ such that $|\sigma(f, P) - J| < \varepsilon$ for each 10^{-3}-regular, δ-fine and special partition of A.

We denote the number J by (GRC) $\int_A f$ or simply by $\int_A f$. The set of the GRC integrable functions on A will be denoted by GRC (A).

From Theorem 1.1 it follows that for any interval $A\subset\mathbf{R}^2$ and $\delta\colon A\to(0,+\infty)$ we can find a 10^{-3}-regular, δ-fine and special partition of A. Thus in Definition 2.2 the set of 10^{-3}-regular, δ-fine and special partitions of A is not empty.

THEOREM 2.1. *Let f be a function on an interval $A\subset\mathbf{R}^2$, and let*

$$D=\{A_j\colon j=1,\dots,k\}$$

be a partition of A. If $f\in\mathrm{GRC}(A_j)$ for $j\in\{1,\dots,k\}$, then $f\in\mathrm{GRC}(A)$ and

$$\int_A f=\sum_{j=1}^{k}\int_{A_j} f.$$

PROOF. Let $\delta_0(x)>0$ such that for all $x\in A$, if $x\notin A_j$ then $\delta_0(x)<\operatorname{dist}(x,A_j)$. Now given $j\in\{1,\dots,k\}$ find a $\delta_j\colon A_j\to(0,+\infty)$ such that $\delta_j<\delta_0$ and

$$\Big|\sigma(f,P)-\int_{A_j} f\Big|<\frac{\varepsilon\cdot|A_j|}{|A|}$$

for each 10^{-3}-regular, δ_j-fine and special partition of A_j. For $x\in A_j$ we set $\delta(x):=\min(\{\delta_j(x)\colon j=1,\dots,k\}\cup\{\delta_0(x)\})$, and select a 10^{-3}-regular, δ-fine and special partition $P=\{(B_i,x_i)\colon i=1,\dots,n\}$ of A. Let $(B_i,x_i)\in P$, $j\in\{1,\dots,k\}$ and suppose that $\operatorname{int}(B_i;A)\cap\operatorname{int}(A_j;A)\neq\emptyset$. Since $\delta\leqq\delta_0$ we have $x_i\in A_j$ and $B_i\subset A_j$ becuase for a special partition x_i is a vertex of B_i. It follows that

$$P_j:=\{(B_i,x_i)\colon \operatorname{int}(B_i;A)\cap\operatorname{int}(A_j;A)\neq\emptyset\}$$

is a 10^{-3}-regular, δ_j-fine and special partition of A_j for each $j=1,\dots,k$. Since $\sigma(f,P)=\sum_{j=1}^{k}\sigma(f,P_j)$, we have

$$\Big|\sigma(f,P)-\sum_{j=1}^{k}\int_{A_j} f\Big|\leqq\sum_{j=1}^{k}\Big|\sigma(f,P_j)-\int_{A_j} f\Big|<\frac{\varepsilon}{|A|}\sum_{j=1}^{k}A_j=\varepsilon.$$

This proves Theorem 2.1.

Let F be an additive function of subintervals of an interval A, and let $x\in A$. Following [8, Chapter 4.2], we say that F is derivable at x if a finite

$$\lim\frac{F(B_n)}{|B_n|}$$

exists for each sequence $\{B_n\}$ of subintervals of A such that $x\in B_n$, $n=1,2,\dots$, $\lim\operatorname{diam}(B_n)=0$, and there exists an $r>0$ such that the intervals B_n are r-regular. If all these limits exist, then they have the same value, denoted by $F'(x)$ and called the derivative of F at x.

THEOREM 2.2. *Let F be a function of subintervals of an interval A such that $f(x):=F'(x)$ exists for every $x \in A$. Then*

$$\int_A f(x) = F(A).$$

PROOF. Let $\varepsilon > 0$ be given. Since F is differentiable we can choose for any $x \in A$ a number $\delta(x) > 0$ such that if B is a 10^{-3}-regular interval, $x \in B$ and $\operatorname{diam}(B) < \delta(x)$ then $\big|F(B) - f(x)|B|\big| < \varepsilon |B|$. Let P be a 10^{-3}-regular, δ-fine and special partition of A. Then plainly we have

$$|\sigma(f, P) - F(A)| \leqq \sum_{j=1}^{k} |f(x_j)|A_j| - F(A_j)| \leqq \sum_{j=1}^{k} \varepsilon |A_j| \leqq \varepsilon |A|.$$

That is for any $\varepsilon > 0$ we can find a function $\delta > 0$ required in the definition of the GRC integral.

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References

[1] R. Henstock, *Theory of integration*, Butterworths (London, 1963).
[2] R. Henstock, A Riemann type integral of Lebesgue power, *Canad. J. Math.*, **20** (1968), 79—87.
[3] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.*, **7** (1957), 418—446.
[4] J. Mawhin, Generalized Riemann integrals and the divergence theorem for differentiable vector fields, in *E. B. Christoffel*, Birkhauser-Verlag (Basel, 1981), 704—714.
[5] J. Mawhin, Generalized multiple Perron integrals and the Green Goursat theorem for differentiable vector fields, *Czechoslovak Math. J.*, **31** (1981), 614—632.
[6] W. F. Pfeffer, The divergence theorem, *Trans. Amer. Math. Soc.*, **295 (2)** 665—685.
[7] W. F. Pfeffer, The multidimensional fundamental theorem of calculus (to appear).
[8] S. Saks, *Theory of the integral*, Dover (New York, 1964).

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DEPARTMENT OF ANALYSIS
EÖTVÖS LORÁND UNIVERSITY
BUDAPEST, MÚZEUM KRT. 6—8
H—1088 HUNGARY

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ÜBER DIE KONVERGENZ MEHRFACHER ORTHOGONALREIHEN

K. TANDORI (Szeged), Mitglied der Akademie

1. In dieser Arbeit werden wir die Resultate von [6] über mehrfache Orthogonalreihen verallgemeinern und die Resultate von [4] vertiefen.

Einfachkeitshalber werden wir unsere Sätze nur für doppelte Reihen verfassen; sinngemäß bleiben sie auch für beliebige d-fachige Reihen ($d>2$) richtig.

Es sei $N_2=\{(k,l)\colon k,\ l=1,2,\ldots\}$. Wir betrachten im Intervall (0, 1) orthonormierte Systeme $\{\varphi_{k,l}(x)\}_{(k,l)\in N_2}$, für die also

$$\varphi_{k,l}(x)\in L^2(0,1)\quad ((k,l)\in N_2),\quad \int_0^1 \varphi_{k,l}(x)\,\varphi_{\bar k,\bar l}(x)\,dx=\begin{cases}1, & (k,l)=(\bar k,\bar l)\\ 0, & (k,l)\neq(\bar k,\bar l)\end{cases}$$

erfüllt sind.

Es sei $1\leqq K\leqq\infty$. Mit $\Omega(K)$ bezeichnen wir die Klasse der in (0,1) orthonormierten Systeme $\varphi=\{\varphi_{k,l}(x)\}_{(k,l)\in N_2}$, für die

$$|\varphi_{k,l}(x)|\leqq K\quad (x\in(0,1);\ (k,l)\in N_2)$$

besteht. Offensichtlich gilt

$$\Omega(K_1)\subseteqq\Omega(K_2)\quad (1\leqq K_1<K_2\leqq\infty).$$

($\Omega(\infty)$ ist also die Klasse aller orthonormierten Systeme in (0, 1). Im Falle $\varphi\in\Omega(1)$ gilt aber $|\varphi_{k,l}(x)|=1$ $((k,l)\in N_2)$ fast überall in (0, 1); solche Systeme nennen wir vorzeichensartig.)

Bemerkung I. Es sei $\varphi\in\Omega(1)$. Wir setzen

$$\psi_{k,l}(x)=\begin{cases}\varphi_{k,l}(2x), & x\in(0,1/2),\\ -\varphi_{k,l}(2x-1), & x\in(1/2,1),\end{cases}\quad ((k,l)\in N_2).$$

Dann besteht $\psi\in\Omega(1)$ und

$$\int_0^1 (\psi_{k,l}(x))^p\,(\psi_{\bar k,\bar l}(x))^q\,dx=\int_0^1 (\psi_{k,l}(x))^p\,dx\int_0^1 (\psi_{\bar k,\bar l}(x))^q\,dx$$

im Falle $(k,l)\neq(k',l')$ für jedes $(p,q)\in N_2$. Daraus folgt auf Grund eines bekannten Satzes [2], daß das System $\psi=\{\psi_{k,l}(x)\}_{k,l\in N_2}$ paarweise stochastisch unabhängig ist.

Mit einer Koeffizientenfolge $a=\{a_{k,l}\}_{(k,l)\in N_2}$ bilden wir die Reihe

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}\,\varphi_{k,l}(x), \tag{1}$$

weiterhin seien

$$s_{m,n}(a;\varphi;x)=\sum_{k=1}^{m}\sum_{l=1}^{n}a_{k,l}\varphi_{k,l}(x)\quad((m,n\in N_2)$$

die Partialsummen. (Im Falle $m\cdot n=0$ sei $s_{m\cdot n}(a;\varphi;x)\equiv 0$.)

Man sagt, daß die Reihe (1) im Punkt x regulär konvergiert, wenn

$$\sum_{k=m_1}^{m_2}\sum_{l=n_1}^{n_2}a_{k,l}\varphi_{k,l}(x)=$$

$$=s_{m_2,n_2}(a;\varphi;x)-s_{m_2,n_1-1}(a;\varphi;x)-s_{m_1-1,n_2}(a;\varphi;x)+s_{m_1-1,n_1-1}(a;\varphi;x)\to 0$$

$$(m_1\leqq m_2,\ n_1\leqq n_2,\ \max(m_1,n_1)\to\infty)$$

gilt. Weiterhin sagt man, daß die Reihe (1) im Punkt x im Pringsheimschen Sinne gegen $s(x)$ konvergiert, wenn für jedes $\varepsilon>0$ ein Index $N=N(\varepsilon)$ derart existiert, daß $|s_{m,n}(a;\varphi;x)-s(x)|<\varepsilon$ im Falle $m,n\geqq N$ besteht. Es ist bekannt, daß die Konvergenz im Pringsheimschen Sinne der Reihe (1) im Punkt x zur Relation

$$s_{m_2,n_2}(a;\varphi;x)-s_{m_1,n_1}(a;\varphi;x)\to 0\quad(\min(m_1,m_2,n_1,n_2)\to\infty)$$

äquivalent ist. Es ist auch bekannt, daß sich aus der regulären Konvergenz einer Reihe ihre Konvergenz im Pringsheimschen Sinne ergibt; das Umgekehrte ist aber im allgemeinen unrichtig.

Für ein $1\leqq K\leqq\infty$ sei $M(K)$ die Klasse der Koeffizientenfolgen $a=\{a_{k,l}\}_{(k,l)\in N_2}$, für die die Reihe (1) bei jedem System $\varphi\in\Omega(K)$ in $(0,1)$ fast überall regulär konvergiert. Offensichtlich gilt

(2) $$M(K_1)\supseteqq M(K_2)\quad(1\leqq K_1<K_2\leqq\infty).$$

Wir werden den folgenden Satz beweisen.

SATZ I. *Im Falle* $1<K<\infty$ *gilt* $M(K)=M(1)$.

BEMERKUNG II. *Ob die Gleichheit* $M(\infty)=M(1)$ *gilt, ist noch eine offene Frage.*

Auf Grund von (2) erhalten wir Satz I unmittelbar aus dem folgenden Satz.

SATZ II. *Es sei* $1<K<\infty$. *Gilt* $a\notin M(K)$, *so gibt es ein System* $\Phi=\{\Phi_{k,l}(x)\}_{(k,l)\in N_2}\in\Omega(1)$ *von Treppenfunktionen derart, daß die Reihe*

(3) $$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}a_{k,l}\Phi_{k,l}(x)$$

in $(0,1)$ *fast überall im Pringsheimschen Sinne nicht konvergiert.*

BEMERKUNG III. *Im Falle* $K=\infty$ *gilt eine ähnliche Behauptung. Im* [4] *wurde der folgende Satz bewiesen.*

Im Falle $a\notin M(\infty)$ *gibt es ein System* $\Phi=\{\Phi_{k,l}(x)\}_{(k,l)\in N_2}$ *von Treppenfunktionen derart, daß*

$$\overline{\lim_{m,n\to 0}}\ |s_{m,n}(a;\Phi;x)|=\infty$$

in $(0,1)$ *fast überall.*

BEMERKUNG IV. Unsere Sätze bleiben für in einem beliebigen nichtatomischen Maßraum (X, A, μ) orthonormierte Systeme gültig; einfachkeitshalber beschränken wir unsere Betrachtungen für in $(0, 1)$ orthonormierte Systeme.

2. Zum Beweis unserer Sätze müssen wir gewisse Definitionen, bzw., gewisse Hilfssätze vorausschicken.

Eine Menge $H\subseteqq(0, 1)$ nennen wir einfach, wenn sie die Vereinigung endlichvieler Intervalle ist. Für eine in $(0, 1)$ definierte Funktion $f(x)$ und für ein Intervall $I=(a, b)(\in(0, 1);\ a<b)$ sei

$$f(I; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right), & a<x<b, \\ 0 & \text{sonst}; \end{cases}$$

weiterhin sei für eine Menge $H\subseteqq(0, 1)$ $H(I)$ diejenige Menge, die aus H unter der Transformation $y=(b-a)x+a$ hervorgeht.

Für eine Koeffizientenfolge $a=\{a_{k,l}\}_{(k,l)\in N_2}$ und für eine Menge $Q\subseteqq N_2$ sei $a(Q)=\{a_{k,l}(Q)\}_{(k,l)\in N_2}$ folgenderweise definiert:

$$a_{k,l}(Q) = \begin{cases} a_{k,l}, & (k, l)\in Q, \\ 0, & (k, l)\notin Q \end{cases} \qquad ((k, l)\in N_2).$$

Weiterhin setzen wir für ein $1\leqq K\leqq\infty$, für eine Menge $Q\subseteqq N_2$ und für eine Folge $a=\{a_{k,l}\}_{(k,l)\in N_2}$

$$\|a; K; Q\| = \sup_{\varphi\in\Omega(K)} \Big\{\int_0^1 \sup_{\substack{m_1\leqq m_2,\, n_1\leqq n_2 \\ (m_1,n_1),\,(m_2,n_2)\in Q}} \Big(\sum_{k=m_1}^{m_2}\sum_{l=n_1}^{n_2} a_{k,l}\,\varphi_{k,l}(x)\Big)^2\,dx\Big\}^{1/2}.$$

Auf Grund dieser Definition gelten die folgenden Behauptungen offensichtlich.

(i) Für jedes $1\leqq K\leqq\infty$ und für jede Folge a gilt

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}^2 \leqq \|a; K; N_2\|^2 \leqq \Big(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_{k,l}|\Big)^2.$$

(ii) Für festgesetzte K und Q besitzt $\|\cdot; K; Q\|$ die Eigenschaften der Norm; dh.

a) aus $\|a; K; Q\|=0$ folgt $a_{k,l}=0\ ((k, l)\in Q)$;

b) für jede Folge a und für jede Zahl c gilt

$$\|ca; K; Q\| = |c|\,\|a; K; Q\| \quad (ca = \{ca_{k,l}\}_{(k,l)\in N_2});$$

c) für jede Folge $a=\{a_{k,l}\}_{(k,l)\in N_2}$, $b=\{b_{k,l}\}_{(k,l)\in N_2}$ gilt

$$\|a+b; K; Q\| \leqq \|a; K; Q\|+\|b; K; Q\| \quad (a+b = \{a_{k,l}+b_{k,l}\}_{(k,l)\in N_2}).$$

(iii) Im Falle $Q_1\subseteqq Q_2$ gilt

$$\|a; K; Q_1\| \leqq \|a; K; Q_2\|$$

für jede Folge a.

(iv) Im Falle $K_1 \leqq K_2$ gilt

$$\|a; K_1; Q\| \leqq \|a; K_2; Q\|$$

für jede Folge a und für jede Menge Q.

Es sei

$$Q_N = \{(k, l)\colon\ k, l = 1, \ldots, N\} \quad (Q_0 = \emptyset).$$

(v) Für jedes K und für jede Folge a gilt

$$\|a; K; Q_N\| \nearrow \|a; K; N_2\| \quad (N \nearrow \infty).$$

Wir werden die folgenden Hilfssätze anwenden.

HILFSSATZ I. *Es sei* $1 \leqq K < \infty$. *Gilt*

$$\|a; K; N_2 \setminus Q_N\| \to 0 \quad (N \to \infty),$$

so konvergiert die Reihe (1) *für jedes System* $\varphi \in \Omega(K)$ *regulär fast überall in* $(0, 1)$.

BEWEIS DES HILFSSATZES I. Es sei $\varphi \in \Omega(K)$. Wir setzen

$$F_N(x) = \sup_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1, n_1), (m_2, n_2) \in N_2 \setminus Q_N}} \Big| \sum_{k=m_1}^{m_2} \sum_{l=n_1}^{n_2} a_{k,l} \varphi_{k,l}(x) \Big| \quad (N = 0, 1, \ldots).$$

Offensichtlich gilt

$$F_N(x) \geqq F_{N+1}(x) \,(\geqq 0) \quad \big(x \in (0, 1);\ N = 1, 2, \ldots\big).$$

Weiterhin folgt aus der Definition von $\|a; K; \cdot\|$

$$\int_0^1 F_N^2(x)\, dx \leqq \|a; K;\ N_2 \setminus Q_N\|^2 \quad (N = 0, 1 \ldots).$$

So folgt durch Anwendung des Fatouschen Lemmas, daß $\lim\limits_{N \to \infty} F_N(x) = 0$ in $(0, 1)$ fast überall gilt, woraus sich die Behauptung des Hilfssatzes I ergibt.

HILFSSATZ II. *Es sei* $1 \leqq K < \infty$. *Gilt*

$$\|a; K; N_2 \setminus Q_N\| \nrightarrow 0 \quad (N \to \infty),$$

so gibt es ein System $\Phi = \{\Phi_{k,l}(x)\}_{(k,l) \in N_2} \in \Omega(1)$ *von Treppenfunktionen derart, daß die Reihe* (3) *in* $(0, 1)$ *fast überall im Pringsheimschen Sinne nicht konvergiert.*

BEWEIS DER SÄTZE. Es ist klar, daß sich aus den Hilfssätzen I—II die Satze I—II ergeben.

3. Wir sollen also nur den Hilfssatz II beweisen. Dazu benötigen wir weitere Hilfssätze.

Es sei $\nu = \nu(k, l)$ eine umkehrbar eindeutige Abbildung von N_2 in $N_1 = \{1, 2, \ldots\}$, und sei

$$\varrho_{k,l}(x) = r_{\nu(k,l)}(x) \quad \big((k, l) \in N_2\big),$$

wobei $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ die n-te Rademachersche Funktion ist.

Mit einer bekannter Methode (s. z. B. [1], S. 54—56) kann man den folgenden Hilfssatz beweisen.

HILFSSATZ III. *Ist die Reihe*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}\,\varrho_{k,l}(x)$$

in einer Menge von positivem Maß im Pringsheimschen Sinne konvergent, so gilt $a\in l_2$, *dh.*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{k,l}^2<\infty.$$

HILFSSATZ IV. *Es seien* $1<K<\infty$, $a=\{a_{k,l}\}_{(k,l)\in N_2}$ *eine Koeffizientenfolge und* N_1, N_2 $(N_1<N_2)$ *positive ganze Zahlen mit*

$$\|a;\,K;\,Q_{N_2}\setminus Q_{N_1}\|^2\geqq 128K^2\sum_{\substack{(k,l)\\(k,l)\in Q_{N_2}\setminus Q_{N_1}}} a_{k,l}^2.$$

Dann gibt es ein System $\psi=\{\psi_{k,l}(x)\}_{(k,l)\in Q_{N_2}\setminus Q_{N_1}}\in\Omega(1)$ *von Treppenfunktionen mit folgender Eigenschaft. Es gilt*

$$\max_{\substack{m_1\leqq m_2,\,n_1\leqq n_2\\(m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}}\Big|\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi_{k,l}(x)\Big|\geqq\frac{1}{4K}\|a;\,K;\,Q_{N_2}\setminus Q_{N_1}\|\quad(x\in E),$$

wobei $E(\subseteqq(0,1))$ *eine einfache Menge ist, für die* mes $E\geqq 1/10$ *besteht.*

BEWEIS DES HILFSSATZES IV. Wir gebrauchen eine Idee von B. S. Kašin [3]. Der Hilfssatz IV soll nur im Falle $\|a;K;Q_{N_2}\setminus Q_{N_1}\|>0$ bewiesen werden; ohne Beschränkung der Allgemeinheit können wir

$$\|a;\,K;\,Q_{N_2}\setminus Q_{N_1}\|^2=4$$

voraussetzen.

Auf Grund der Definition von $\|a;K;Q_{N_2}\setminus Q_{N_1}\|$ gibt es ein System $\bar\varphi\in\Omega(K)$ mit

(4)
$$\int_0^1\max_{\substack{m_1\leqq m_2,\,n_1\leqq n_2\\(m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}}\Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar\varphi_{k,l}(x)\Big)^2dx\geqq\|a;\,K;\,Q_{N_2}\setminus Q_{N_1}\|^2-\frac{\varepsilon}{3}$$
$$(\varepsilon=\|a;\,K;\,Q_{N_2}\setminus Q_{N_1}\|^2/2).$$

Es werde a $(0<a<1)$ so gewählt, daß die Ungleichungen

(5)
$$(1-(1-a^2)^2(1-a))/a\leqq K^2,$$

(6)
$$(1-a)(1-a^2)\int_0^1\max_{\substack{m_1\leqq m_2,\,n_1\leqq n_2\\(m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}}\Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar\varphi_{k,l}(x)\Big)^2dx\geqq$$
$$\geqq\int_0^1\max_{\substack{m_1\leqq n_1,\,m_2\leqq n_2\\(m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}}\Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_1} a_{k,l}\bar\varphi_{k,l}(x)\Big)^2dx-\frac{\varepsilon}{3}$$

erfüllt sind. Es seien $\varphi_{k,l}(x)$ $((k,l)\in Q_{N_2}\setminus Q_{N_1})$ Treppenfunktionen mit

$$\int_0^1 (\bar\varphi_{k,l}(x)-\varphi_{k,l}(x))^2\,dx<\eta \quad ((k,l)\in Q_{N_2}\setminus Q_{N_1})$$

und

(7) $$|\varphi_{k,l}(x)|\le|\bar\varphi_{k,l}(x)| \quad (x\in(0,1);\ (k,l)\in Q_{N_2}\setminus Q_{N_1}).$$

Wir setzen

$$\alpha(i,j,\bar i,\bar j)=\int_0^1 \varphi_{i,j}(x)\,\varphi_{\bar i,\bar j}(x)\,dx \quad ((i,j),(\bar i,\bar j)\in Q_{N_2}\setminus Q_{N_1}).$$

Die Anzahl der Elemente von $Q_{N_2}\setminus Q_{N_1}$ bezeichnen wir mit Z. Wir teilen das Intervall $(1-a^2,1)$ in $Z(Z-1)$ paarweise disjunkte Intervalle $I(i,j,\bar i,\bar j)$ $((i,j),(\bar i,\bar j)\in Q_{N_2}\setminus Q_{N_1},\ (i,j)\ne(\bar i,\bar j))$ gleicher Länge. Ist η genügend klein $(\eta\le\eta_1)$, so gelten die Abschätzungen

(8) $$\int_0^1 \max_{\substack{m_1\le m_2,\ n_1\le n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\varphi_{k,l}(x)\Big)^2 dx>$$

$$>\int_0^1 \max_{\substack{m_1\le m_2,\ n_1\le n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar\varphi_{k,l}(x)\Big)^2 dx-\frac{\varepsilon}{3},$$

(9) $$\frac{1-a^2}{a^2}\,\frac{Z(Z-1)}{2}\max_{\substack{(i,j),(\bar i,\bar j)\in Q_{N_2}\setminus Q_{N_1}\\ (i,j)\ne(\bar i,\bar j)}}|\alpha(i,j,\bar i,\bar j)|\le K^2,$$

(10) $$\alpha(i,j,i,j)\ge 1-a^2 \quad ((i,j)\in Q_{N_2}\setminus Q_{N_1}).$$

Wir setzen

$$\bar\psi_{k,l}(x)=$$

$$=\begin{cases}\varphi_{k,l}\left(\dfrac{x}{1-a^2}\right), & x\in(0,1-a^2),\\[2mm] \sqrt{\dfrac{Z(Z-1)}{2a^2}\,|\alpha(k,l,\bar k,\bar l)|\,(1-a^2)}, & x\in I(k,l,\bar k,\bar l),\ (\bar k,\bar l)\in Q_{N_2}\setminus Q_{N_1},\ (\bar k,\bar l)\ne(k,l),\\[2mm] -\sqrt{\dfrac{Z(Z-1)}{2a^2}\,|\alpha(k,l,\bar k,\bar l)|\,(1-a^2)}\ \operatorname{sign}\alpha(k,l,\bar k,\bar l), & x\in I(\bar k,l,k,l),\ (\bar k,l)\in Q_{N_2}\setminus Q_{N_1},\ (\bar k,l)\ne(k,l),\\[2mm] 0 & \text{sonst}\end{cases}$$

$((k,l)\in Q_{N_2}\setminus Q_{N_1})$.

Offensichtlich bilden die Treppenfunktionen $\bar{\psi}_{k,l}(x)$ $((k,l)\in(Q_{N_2}\setminus Q_{N_1}))$ ein orthogonales System in $(0,1)$, weiterhin gelten

$$|\bar{\psi}_{k,l}(x)| \leqq K \quad (x\in(0,1);\ (k,l)\in Q_{N_2}\setminus Q_{N_1}), \tag{11}$$

$$\int_0^1 \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar{\psi}_{k,l}(x)\Big)^2 dx \geqq \tag{12}$$

$$\geqq \int_0^{1-a^2} \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar{\psi}_{k,l}(x)\Big)^2 dx =$$

$$= \int_0^{1-a^2} \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \left(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\varphi_{k,l}\left(\frac{x}{1-a^2}\right)\right)^2 dx =$$

$$= (1-a^2)\int_0^1 \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\varphi_{k,l}(x)\Big)^2 dx,$$

$$\int_0^1 (\bar{\psi}_{k,l}(x))^2 dx \geqq \int_0^{1-a^2} (\bar{\psi}_{k,l}(x))^2 dx = \int_0^{1-a^2} \varphi_{k,l}^2\left(\frac{x}{1-a^2}\right) dx = \tag{13}$$

$$= (1-a^2)\,\alpha(k,l,k,l) \geqq (1-a^2)^2 \quad ((k,l)\in Q_{N_2}\setminus Q_{N_1}),$$

auf Grund von (7), (9) und (10). Es sei endlich

$$\psi_{k,l}^*(x) = \begin{cases} \bar{\psi}_{k,l}\left(\dfrac{x}{1-a}\right), & x\in(0,1-a), \\[2ex] m_{k,l}\,\varrho_{k,l}\left(\dfrac{x-1+a}{a}\right), & x\in(1-a,1) \end{cases} \quad ((k,l)\in Q_{N_2}\setminus Q_{N_1}),$$

wobei $m_{k,l}$ derart bestimmt ist, daß die Funktionen $\psi_{k,l}^*(x)$ normiert sind; dh. es gilt

$$(1-a)\int_0^1 (\bar{\psi}_{k,l}(x))^2 dx + m_{k,l}^2 a = 1 \quad ((k,l)\in Q_{N_2}\setminus Q_{N_1}). \tag{14}$$

Auf Grund von (5), (13) und (14) folgt

$$m_{k,l} \leqq \sqrt{\frac{1-(1-a^2)^2(1-a)}{a}} \leqq K \quad ((k,l)\in Q_{N_2}\setminus Q_{N_1}).$$

Die Funktionen $\psi^*_{k,l}(x)$ $((k,l)\in Q_{N_2}\setminus Q_{N_1})$ sind offensichtlich Treppenfunktionen, aus (11) erhalten wir $\{\psi^*_{k,l}(x)\}_{(k,l)\in Q_{N_2}\setminus Q_{N_1}}\in\Omega(K)$, weiterhin gilt

$$\int_0^1 \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{kl}\psi^*_{k,l}(x)\Big)^2 dx \geqq$$

$$\geqq \int_0^{1-a} \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi^*_{k,l}(x)\Big)^2 dx =$$

$$= (1-a)\int_0^1 \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\bar{\psi}_{k,l}(x)\Big)^2 dx.$$

Daraus erhalten wir

$$\text{(15)}\qquad \int_0^1 \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big(\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi^*_{k,l}(x)\Big)^2 dx >$$

$$> \|a;K;Q_{N_2}\setminus Q_{N_1}\|^2-\varepsilon \geqq \frac{1}{2}\|a;K;Q_{N_2}\setminus Q_{N_1}\|^2 \quad (=2)$$

auf Grund von (4), (6), (8) und (12).

Es sei $I_r=(a_r,b_r)$ $(b_r>a_r)$ $(r=1,\dots,\varrho)$ eine Einleitung des Intervalls (0, 1) in paarweise disjunkte Intervalle derart, daß jede Funktion $\psi^*_{k,l}(x)$ in jedem I_r konstant ist. Den Wert der Funktion

$$\max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big|\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi^*_{k,l}(x)\Big|$$

im Intervall I_r bezeichnen wir mit w_r. Nach (15) gilt

$$\text{(16)}\qquad 4=\|a;K;Q_{N_2}\setminus Q_{N_1}\|^2 \geqq \sum_{r=1}^{\varrho} w_r \operatorname{mes} I_r > \frac{1}{2}\|a;K;Q_{N_2}\setminus Q_{N_1}\|^2 = 2.$$

Es seien $1\leqq r_1<\dots<r_\lambda\leqq\varrho$ diejenige Indizes r, für die $w_r\geqq 1$ ist; die Indizes r $(1\leqq r\leqq\varrho)$, die von $r_1,\dots,r_\lambda$ verschieden sind, bezeichnen wir der Reihe nach mit $s_1,\dots,s_{\varrho-\lambda}$. Aus (16) folgt

$$4\geqq \sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l} > 1.$$

Wir setzen

$$a=\sum_{l=1}^{\lambda} w_{r_l}^2 \operatorname{mes} I_{r_l},\quad b=\sum_{l=1}^{\varrho-\lambda}\operatorname{mes} I_{s_l}.$$

Offensichtlich gelten

$$\text{(17)}\qquad 1<a\leqq 4,\quad 0\leqq b\leqq 1.$$

Es seien $J'_l=(a'_l,b'_l)$ $(l=1,\dots,\lambda)$ disjunkte Intervalle in $(0,a)$ mit $\operatorname{mes} J'_l=w_{r_l}^2 \operatorname{mes} J_{r_l}$ und $J''_l=(a''_l,b''_l)$ $(l=1,\dots,\varrho-\lambda)$ disjunkte Intervalle in $(a,a+b)$ mit $\operatorname{mes} J''_l=$

$=\operatorname{mes} I_{s_l}$. Wir setzen

$$\tilde{\psi}_{i,j}(x)=\begin{cases}\psi^*_{i,j}\left(\dfrac{x-a''_l}{b''_l-a''_l}(b_{s_l}-a_{s_l})+a_{s_l}\right), & x\in J''_l,\ l=1,\dots,\varrho-\lambda,\\[2mm] \dfrac{1}{w_{r_l}}\psi^*_{i,j}\left(\dfrac{x-a'_l}{b'_l-a'_l}(b_{r_l}-a_{r_l})+a_{r_l}\right), & x\in J'_l,\ l=1,\dots,\lambda\end{cases}$$

$((i,j)\in Q_{N_2}\setminus Q_{N_1})$.

Es sei

$$\psi^{**}_{k,l}(x)=\tilde{\psi}_{k,l}((a+b)x)/K\quad (x\in(0,1);\ (k,l)\in Q_{N_2}\setminus Q_{N_1}).$$

Offensichtlich bilden die Treppenfunktionen $\psi^{**}_{k,l}(x)$ $((k,l)\in Q_{N_2}\setminus Q_{N_1})$ in $(0,1)$ ein orthogonales System, und es gilt

$$(18)\qquad |\psi^{**}_{k,l}(x)|\leqq 1\quad (x\in(0,1);\ (k,l)\in Q_{N_2}\setminus Q_{N_1}).$$

Es sei $\bar{E}$ die Bildmenge des Intervalles $(0,a)$ unter der linearen Transformation $y=x/(a+b)$. Aus (17) folgt

$$(19)\qquad \operatorname{mes}\bar{E}\geqq 1/5.$$

Weiterhin gilt

$$(20)\qquad \max_{\substack{m_1\leqq m_2,\ n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}}\left|\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2}a_{k,l}\psi^{**}_{k,l}(x)\right|=\frac{1}{K}=$$

$$=\frac{1}{2K}\|a;K;Q_{N_2}\setminus Q_{N_1}\|\quad (x\in\bar{E}).$$

Es sei J_s $(s=1,\dots,\sigma)$ eine Einteilung des Intervalls $(0,1)$ in paarweise disjunkte Intervalle derart, daß jede Funktion $\psi^{**}_{k,l}(x)$ in jedem J_s konstant und $\bar{E}$ die Vereinigung einiger J_s sind. Den Wert von $\psi^{**}_{k,l}(x)$ im Intervall J_s bezeichnen wir mit $\varrho_s^{(k,l)}$. Für jeden Index s $(1\leqq s\leqq\sigma)$ sei $\{\chi_s^{(k,l)}(x)\}_{(k,l)\in Q_{N_2}\setminus Q_{N_1}}$ ein orthogonales System von Treppenfunktionen derart, daß

$$\int_0^1\chi_s^{(k,l)}(x)\,dx=0\quad ((k,l)\in Q_{N_2}\setminus Q_{N_1})$$

gilt und jede Funktion $\chi_s^{(k,l)}(x)$ den Wertbereich $\{1-\varrho_s^{(k,l)},\ -1-\varrho_s^{(k,l)}\}$ besitzt. (Im Falle $\varrho_s^{(k,l)}=1$ setze man $\chi_s^{(k,l)}(x)\equiv 0$.) Aus (18) folgt

$$(21)\qquad |\chi_s^{(k,l)}(x)|\leqq 2\quad (x\in(0,1),\ (k,l)\in Q_{N_2}\setminus Q_{N_1},\quad s=1,\dots,\sigma).$$

Es sei

$$\psi_{k,l}(x)=\psi^{**}_{k,l}(x)+\sum_{s=1}^{\sigma}\chi_s^{(k,l)}(J_s;x)\quad ((k,l)\in Q_{N_2}\setminus Q_{N_1}).$$

Offensichtlich sind $\psi_{k,l}(x)$ $((k,l)\in Q_{N_2}\setminus Q_{N_1})$ Treppenfunktionen, und man kann leicht einsehen, daß $\psi=\{\psi_{k,l}(x)\}_{(k,l)\in Q_{N_2}\setminus Q_{N_1}}\in\Omega(1)$ ist. Für jeden Index s $(1\leqq s\leqq\sigma)$

seien $m_1(s)$, $m_2(s)$, $n_1(s)$, $n_2(s)$ positive ganze Zahlen mit

$$m_1(s) \leqq m_2(s), \quad n_1(s) \leqq n_2(s) \quad (m_1(s), n_1(s)), \quad (m_2(s), n_2(s)) \in Q_{N_2} \setminus Q_{N_1}$$

und

$$\Big| \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l} \psi_{k,l}^{**}(x) \Big| = \max_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1,n_1),\,(m_2,n_2) \in Q_{N_2} \setminus Q_{N_1}}} \Big| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \psi_{k,l}^{**}(x) \Big| \quad (x \in J_s).$$

Dann gilt

(22)

$$\operatorname{mes} \Big\{ x \in J_s : \max_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1,n_1),\,(m_2,n_2) \in Q_{N_2} \setminus Q_{N_1}}} \Big| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \psi_{k,l}(x) \Big| \geqq \frac{1}{4K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} \geqq$$

$$\geqq \operatorname{mes} \Big\{ x \in J_s : \Big| \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l} \psi_{k,l}(x) \Big| \geqq \frac{1}{4K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} \geqq$$

$$\geqq \operatorname{mes} \Big\{ x \in J_s : \Big| \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l} \psi_{k,l}^{**}(x) \Big| \geqq \frac{1}{2K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} -$$

$$- \operatorname{mes} \Big\{ x \in J_s : \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l} \chi_s^{(k,l)}(J_s;\, x) \geqq \frac{1}{4K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} =$$

$$= \operatorname{mes} \Big\{ x \in J_s : \max_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1,n_1),\,(m_2,n_2) \in Q_{N_2} \setminus Q_{N_1}}} \Big| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \psi_{k,l}^{**}(x) \Big| \geqq$$

$$\geqq \frac{1}{2K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} -$$

$$- \operatorname{mes} \Big\{ x \in J_s : \Big| \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{m_2(s)} a_{k,l} \chi_s^{(k,l)}(J_s;\, x) \Big| \geqq \frac{1}{4K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\}.$$

Nach (21) ergibt sich durch Anwendung der Tschebyscheffschen Ungleichung

$$\operatorname{mes} \Big\{ x \in J_s : \Big| \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l} \chi_s^{(k,l)}(J_s;\, x) \Big| \geqq \frac{1}{4K} \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\| \Big\} \leqq$$

$$\leqq \operatorname{mes} J_s \cdot 16K^2 \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l}^2 \int_0^1 (\chi_s^{(k,l)}(x))^2 \, dx / \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\|^2 \leqq$$

$$\leqq \operatorname{mes} J_s \cdot 64K^2 \sum_{k=m_1(s)+1}^{m_2(s)} \sum_{l=n_1(s)+1}^{n_2(s)} a_{k,l}^2 / \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\|^2 \leqq$$

$$\leqq \operatorname{mes} J_s \cdot 64K^2 \sum_{(k,l) \in Q_{N_2} \setminus Q_{N_1}} a_{k,l}^2 / \|a;\, K;\, Q_{N_2} \setminus Q_{N_1}\|^2 \leqq \frac{1}{2} \operatorname{mes} J_s$$

auf Grund der Voraussetzung des Hilfssatzes IV. Daraus und aus (20) und (22) bekommen wir:

(23)

$$\operatorname{mes}\Big\{x\in J_s:\ \max_{\substack{m_1\leqq m_2,\, n_1\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big|\sum_{k=m_1+1}^{m_2}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi_{k,l}(x)\Big| \geqq \frac{1}{4K}\|a;K;Q_{N_2}\setminus Q_{N_1}\|\Big\}\geqq$$

$$\geqq \operatorname{mes} J_s/2.$$

Es sei

$$E=\bigcup_{s:J_s\subseteq \bar E}\Big\{x\in J_s:\ \max_{\substack{m_1\leqq n_1,\, m_2\leqq n_2\\ (m_1,n_1),(m_2,n_2)\in Q_{N_2}\setminus Q_{N_1}}} \Big|\sum_{k=m_1+1}^{n_1}\sum_{l=n_1+1}^{n_2} a_{k,l}\psi_{k,l}(x)\Big| \geqq$$

$$\geqq \frac{1}{4K}\|a;K;Q_{N_2}\setminus Q_{N_1}\|\Big\}.$$

E ist offensichtlich einfach, und aus (19), (23) erhalten wir mes $E\geqq 1/10$.

Damit haben wir Hilfssatz IV bewiesen.

4. Beweis des Hilfssatz II. Auf Grund von (iv) können wir voraussetzen, daß im Hilfssatz II $K>1$ ist. Weiterhin können wir auf Grund des Hilfssatzes III $\{a_{k,l}\}_{(k,l)\in N_2}\in l_2$ voraussetzen.

Es seien

$$U_{k_0}=\{(k_0,l),\ l=1,2,\ldots\},\quad V_{l_0}=\{(k,l_0),\ k=1,2,\ldots\}.$$

Wir werden zwei Fälle unterscheiden:

a) für jedes k_0, $l_0=1,2,\ldots$ gelten

$$\|a(U_{k_0});K;N_2\setminus Q_N\|\to 0,\quad \|a(V_{l_0});K;N_2\setminus Q_N\|\to 0\quad (N\to\infty);$$

b) es gibt eine Index k_0 oder l_0 mit

$$\|a(U_{k_0});K;N_2\setminus Q_N\|\not\to 0\quad (N\to\infty),\quad \text{oder}\quad \|a(V_{l_0});K;N_2\setminus Q_N\|\not\to 0\quad (N\to\infty).$$

Beweis des Hilfssatzes II im Falle a). Auf Grund von (iii) und (v) gibt es eine Zahl $\varrho>0$ und eine Indexfolge $(0=)N_0<\ldots<N_i<N_{i+1}<\ldots$ mit

$$\|a;K;Q_{N_{i+1}}\setminus Q_{N_i}\|\geqq\varrho\quad (i=0,1,\ldots).$$

Auf Grund von $\{a_{k,l}\}_{(k,l)\in N_2}\in l_2$ können wir auch

$$\|a;K;Q_{N_{i+1}}\setminus Q_{N_i}\|^2\geqq 128K^2\sum_{\substack{(k,l)\\ (k,l)\in Q_{N_{i+1}}\setminus Q_{N_i}}} a_{k,l}^2\quad (i=1,2,\ldots)$$

voraussetzen.

Durch vollständige Induktion werden wir ein System $\Phi=\{\Phi_{k,l}(x)\}_{(k,l)\in N_2}\in\Omega(1)$ von Treppenfunktionen und eine stochastisch unabhängige Folge $\{F_i\}_{i=1}^{\infty}$ einfacher

Teilmengen von $(0, 1)$ derart definieren, daß

α) $\operatorname{mes} F_i \geqq 1/10$,

$$\beta) \quad \max_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1, n_1), (m_2, n_2) \in Q_{N_{i+1}} \setminus Q_{N_i}}} \Big| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \Phi_{k,l}(x) \Big| \geqq \varrho/4K \quad (x \in F_i)$$

für jedes $i(=1, 2, \ldots)$ erfüllt sind.

Es sei

$$\Phi_{k,l}(x) = \varrho_{k,l}(x) \quad ((k, l) \in Q_{N_1} \setminus Q_{N_0}).$$

Es sei i_0 eine positive ganze Zahl. Wir nehmen an, daß die Treppenfunktionen $\Phi_{k,l}(x)$ $\big((k, l) \in Q_{N_{i_0}}\big)$ und die einfachen Teilmengen $F_1, \ldots, F_{i_0-1}$ von $(0, 1)$ schon derart definiert sind, daß $\{\Phi_{k,l}(x)\}_{(k,l) \in Q_{N_{i_0}} \setminus Q_{N_{i_0-1}}} \in \Omega(1)$ ist, die Mengen $F_1, \ldots, F_{i_0-1}$ stochastisch unabhängig sind, weiterhin α), β) für jedes $i=1, \ldots, i_0-1$ erfüllt werden.

Wir wenden den Hilfssatz IV im Falle $N_2 = N_{i_0+1}$, $N_1 = N_{i_0}$ an; die entsprechenden Funktionen, bzw. die entsprechende Menge bezeichnen wir mit $\psi_{k,l}^{(i_0)}(x)$ $\big((k, l) \in Q_{N_{i_0+1}} \setminus Q_{N_{i_0}}\big)$, bzw. mit E_{i_0}. Dann gelten

$$\operatorname{mes} E_{i_0} \geqq 1/10, \tag{24}$$

$$\max_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ (m_1, n_1), (m_2, n_2) \in Q_{N_{i_0+1}} \setminus Q_{N_{i_0}}}} \Big| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \psi_{k,l}^{(i_0)}(x) \Big| \geqq \frac{1}{K} \quad (x \in E_{i_0}). \tag{25}$$

Auf Grund der Voraussetzung gibt es eine Einteilung von $(0, 1)$ in paarweise disjunkte Intervalle I_r $(r=1, \ldots, \varrho)$ derart, daß jede Funktion $\Phi_{k,l}(x)$ $\big((k, l) \in Q_{N_{i_0}}\big)$ in jedem I_r konstant ist und jede Menge F_i $(i=1, \ldots, i_0-1)$ die Vereinigung gewisser I_r ist. Die zwei Hälften von I_r bezeichnen wir mit I_r', bzw. mit $I_r''(1, \ldots, \varrho)$. Dann setzen wir

$$\Phi_{k,l}(x) = \sum_{r=1}^{\varrho} \psi_{k,l}^{(i_0)}(J_r'; x) - \sum_{r=1}^{\varrho} \psi_{k,l}^{(i_0)}(I_r''; x) \quad \big((k, l) \in Q_{N_{i_0+1}} \setminus Q_{N_{i_0}}\big),$$

und

$$F_{i_0} = \bigcup_{r=1}^{\varrho} \big(E_{i_0}(I_r') \cup E_{i_0}(I_r'')\big).$$

Aus (24) und (25) folgt unmittelbar, daß α), β) auch in Falle $i=i_0$ erfüllt werden, das System $\{\Phi_{k,l}(x)\}_{(k,l) \in Q_{N_{i_0+1}}}$ von Treppenfunktionen zu $\Omega(1)$ gehört, und die einfachen Teilmengen $F_1, \ldots, F_{i_0}$ von $(0, 1)$ stochastisch unabhängig sind. Das Funktionensystem Φ und die Mengenfolge $\{F_i\}_{i=1}^{\infty}$ bekommen wir also durch Induktion.

Durch Anwendung des zweiten Borel—Cantellischen Lemmas erhalten wir

$$\operatorname{mes}\big(\overline{\lim_{i \to \infty}} F_i\big) = 1$$

auf Grund von α). Ist $x \in \overline{\lim_{i \to \infty}} F_i$, so besteht β) für unendlich viele i, woraus folgt, daß die Reihe (2) im Punkt x nicht regulär konvergiert, dh.

$$(26)\qquad \varlimsup_{\substack{m_1 \leqq m_2,\, n_1 \leqq n_2 \\ \min(m_1, n_1) \to \infty}} \left| \sum_{k=m_1+1}^{m_2} \sum_{l=n_1+1}^{n_2} a_{k,l} \Phi_{k,l}(x) \right| \neq 0 \quad (x \in \varlimsup_{i\to\infty} F_i).$$

gilt. Auf Grund der Voraussetzungen des Falles a), durch Anwendung des Hilfssatzes I erhalten wir, daß die Reihen

$$\sum_{l=1}^{\infty} a_{k_0,l} \Phi_{k_0,l}(x) \quad (k_0 = 1, 2, \ldots), \quad \sum_{k=1}^{\infty} a_{k,l_0} \Phi_{k,l_0}(x) \quad (l_0 = 1, 2, \ldots)$$

in $(0, 1)$ fast überall konvergieren. Daraus und aus (26) folgt, daß die Reihe (2) in $(0, 1)$ fast überall auch im Pringsheimschen Sinne nicht konvergiert.

Beweis des Hilfssatzes IV im Falle b). Wir nehmen an, daß für einen Index k_0 $\|a(U_{k_0}); K; N_2 \setminus Q_N\| \not\to 0$ $(N \to \infty)$. (Den anderen Fall können wir ähnlicherweise betrachten.) Ohne Beschränkung der Allgemeinheit können wir

$$\|a(U_1); K; N_2 \setminus Q_N\| \not\to 0 \quad (N \to \infty)$$

voraussetzen.

Mit der im Falle a) angewandten Methode kann man eine positive Zahl ϱ, eine Indexfolge $(0=)N_0 < \ldots < N_i < N_{i+1} < \ldots$, Systeme $\{\psi_{1,l}^{(s)}\}_{l=N_i+1}^{N_{i+1}} \in \Omega(1)$ $(i=1, 2, \ldots)$ von Treppenfunktionen und eine Folge $\{E_i\}_{i=1}^{\infty}$ einfacher Teilmengen von $(0, 1)$ derart angeben, daß

$$(27)\qquad \operatorname{mes} E_i \geqq 1/10,$$

$$(28)\qquad \max_{N_i < n_1 < n_2 \leqq N_{i+1}} \left| \sum_{l=n_1+1}^{n_2} a_{1,l} \psi_{1,l}^{(i)}(x) \right| \geqq 1/4K \quad (x \in E_i)$$

für jedes $i(=1, 2, \ldots)$ erfüllt sind.

Durch vollstandige Induktion werden wir ein System $\{\Phi_{k,l}(x)\}_{(k,l) \in N_2} \in \Omega(1)$ von Treppenfunktionen und eine stochastisch unabhängige Folge $\{F_i\}_{i=1}^{\infty}$ einfacher Teilmengen von $(0, 1)$ derart angeben, daß

γ) das System $\{\Phi_{k,l}(x)\}_{(k,l) \in N_2 \setminus U(1)}$ stochastisch unabhängig ist,

δ) $\int_0^1 \Phi_{k,l}(x)\,dx = 0 \quad ((k, l) \in N_2 \setminus U(1))$

gilt, und

ε) $\operatorname{mes} F_i \geqq 1/10$,

η) $\max_{N_i < n_1 < n_2 \leqq N_{i+1}} \left| \sum_{l=n_1+1}^{n_2} a_{1,l} \Phi_{1,l}(x) \right| \geqq \frac{\varrho}{4K} \quad (x \in F_i)$

für jedes $i(=1, 2, \ldots)$ erfüllt sind.

Es sei

$$\Phi_{k,l}(x) = \varrho_{k,l}(x) \quad ((k, l) \in Q_{N_1}).$$

Offensichtlich sind diese Funktionen Treppenfunktionen, und es gilt

$$\{\Phi_{k,l}(x)\}_{(k,l)\in Q_{N_1}}\in\Omega(1).$$

Es sei i_0 eine positive ganze Zahl. Nehmen wir an, daß die Treppenfunktionen $\Phi_{k,l}(x)$ $\big((k,l)\in Q_{N_{i_0}}\big)$ und die Folge $F_1, \ldots, F_{i_0-1}$ einfacher Teilmengen von (0, 1) derart definiert sind, daß $\{\Phi_{k,l}(x)\}_{(k,l)\in Q_{N_{i_0}}}\in\Omega(1)$ ist, die Mengen $F_1, \ldots, F_{i_0-1}$ stochastisch unabhängig sind, die Funktionen $\Phi_{k,l}(x)$ $\big((k,l)\in Q_{N_{i_0}}\setminus U(1)\big)$ stochastisch unabhängig sind, weiterhin δ) für $(k,l)\in Q_{N_{i_0}}\setminus U(1)$ und ε), η) für jedes $i(=1, \ldots, i_0-1)$ erfüllt werden.

Dann gibt es eine Einteilung von (0, 1) in paarweise disjunkte Intervalle I_r $(r=1, \ldots, \varrho)$ derart, daß jede Funktion $\Phi_{k,l}(x)$ $\big((k,l)\in Q_{N_{i_0}}\big)$ in jedem I_r konstant ist, und jede Menge F_i $(i=1, \ldots, i_0-1)$ die Vereinigung gewisser I_r ist. Die zwei Hälften von I_r bezeichnen wir mit I_r', bzw. mit I_r'' $(r=1, \ldots, \varrho)$.

Wir setzen

$$\Phi_{1,l}(x)=\sum_{r=1}^{\varrho}\psi_{1,l}^{(i_0)}(I_r'; x)-\sum_{r=1}^{\varrho}\psi_{1,l}^{(i_0)}(I_r''; x)\quad (l=N_{i_0}+1, \ldots, N_{i_0+1}),$$

und

$$F_{i_0}=\bigcup_{r=1}^{\varrho}\big(E_{i_0}(I_r')\cup E_{i_0}(I_r'')\big).$$

Dann sei J_s $(s=1, \ldots, \sigma)$ eine Einteilung des Intervalls (0, 1) in paarweise disjunkte Intervalle derart, daß jede Funktion $\Phi_{k,l}(x)$ $\big((k,l)\in Q_{N_{i_0}}\cup\{(1, N_{i_0}+1), \ldots, (1, N_{i_0+1})\}\big)$ in jedem J_s konstant ist, und es sei

$$\Phi_{k,l}(x)=\sum_{s=1}^{\sigma}\varrho_{k,l}(J_s; x)\quad \big((k,l)\in(Q_{N_{i_0+1}}\setminus Q_{N_{i_0}})\setminus\{(1, N_{i_0}+1), \ldots, (1, N_{i_0+1})\}\big).$$

Offensichtlich sind die $\Phi_{k,l}(x)$ $\big((k,l)\in Q_{N_{i_0+1}}\setminus Q_{N_{i_0}}\big)$ Treppenfunktionen, F_{i_0} ist einfach, es gilt $\{\Phi_{k,l}(x)\}_{(k,l)\in Q_{N_{i_0+1}}\setminus Q_{N_{i_0}}}\in\Omega(1)$, die Funktionen

$$\Phi_{k,l}(x)\ \big((k,l)\in(Q_{N_{i_0+1}}\setminus Q_{N_{i_0}})\setminus U(1)\big)$$

sind stochastisch unabhängig, die Mengen $F_1, \ldots, F_{i_0}$ sind ebenfalls stochastisch unabhängig, weiterhin sind δ) für $(k,l)\in(Q_{N_{i_0+1}}\setminus Q_{N_{i_0}})\setminus U(1)$, ε), η) auch im Falle $i=i_0$ erfüllt. Das System $\{\Phi_{k,l}(x)\}_{(k,l)\in N_2}$ und die Mengenfolge $\{F_i\}_{i=1}^{\infty}$ mit den erwähnten Eigenschaften bekommen wir also durch Induktion.

Aus δ) folgt $\operatorname{mes}\big(\overline{\lim_{i\to\infty}} F_i\big)=1$. Daraus, und aus ε) erhalten wir, daß die Reihe

$$\sum_{l=1}^{\infty}a_{1,l}\Phi_{1,l}(x)$$

in (0, 1) fast überall divergiert. Weiterhin bekommen wir wegen $\{a_{k,l}\}_{(k,l)\in N_2\setminus U(1)}\in l_2$ und der stochastisch Unabhängigkeit des Systems $\{\Phi_{k,l}(x)\}_{(k,l)\in N_2\setminus U(1)}$ mit einer bekannten Methode (s. z. B. [5], S. 341—342), daß die Reihe

$$\sum_{k=2}^{\infty} \sum_{l=1}^{\infty} a_{k,l} \Phi_{k,l}(x)$$

in (0, 1) fast überall im Pringsheimschen Sinne konvergiert.

Nach obigen erhalten wir, daß die Reihe (3) in (0, 1) fast überall im Pringsheimschen Sinne nicht konvergiert.

Damit haben wir Hilfssatz IV vollständig bewiesen.

Schriftenverzeichnis

[1] G. Alexits, Convergence Problems of Orthogonal Series, Akadémiai Kiadó (Budapest, 1961).
[2] M. Kac, Sur les fonctions indépendantes (I) (Propriété générales), *Studia Math.*, **6** (1936), 46—58.
[3] S. B. Kašin, On Weyl's multipliers for almost everywhere convergence of orthogonal series, *Analysis Math.*, **2** (1976), 249—266.
[4] F. Móricz—K. Tandori, On the a.e. convergence of multiple orthogonal series. II (Unrestricted convergence of the rectangular partial sums), *Acta Sci. Math. (Szeged)*, **47** (1984), 349—369.
[5] Béla Sz.-Nagy, *Introduction to real functions and orthogonal expansions*, Oxford University Press (New York, 1965).
[6] K. Tandori, Über beschränkte orthonormierte Systeme, *Acta Math. Acad. Sci. Hungar.*, **31** (1978), 279—285.

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JÓZSEF ATTILA UNIVERSITÄT
BOLYAI INSTITUT
H—6720 SZEGED
ARADI VÉRTANÚK TERE 1

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57 (3—4) (1991), 341—348.

SUFFICIENCY IN THE NON-WEAKLY DOMINATED CASE

GY. MICHALETZKY (Budapest)

Introduction

In the literature there are several conditions for the existence of the minimal sufficient σ-field in $(\Omega, \mathcal{A}, \mathcal{P})$ (see Halmos—Savage [3], Pitcher [8], Hasegawa—Perlman [4], Luschgy—Mussmann [5]). But these conditions guarantee the existence of the minimal sufficient σ-field not only in $(\Omega, \mathcal{A}, \mathcal{P})$ but in any statistical space $(\Omega, \mathcal{A}, \mathcal{Q})$ where $\mathcal{Q}$ is another measure class which is in some sense "absolutely continuous" with respect to $\mathcal{P}$ (for the precise definition see (1)).

In my paper I would like to give a necessary and sufficient condition for this property. I will show that the prototype of these spaces is the following one.

EXEMPLE (prototype). Let $\Omega=[0, 1)$, $\mathcal{A}$ the σ-field generated by the one-point subsets of Ω, ε_X, $x\in[0, 1)$ the measure concentrated at the point x,

$$P_s(A) = \begin{cases} 1 & \text{if } A \text{ is noncountable} \\ 0 & \text{if } A \text{ is countable.} \end{cases}$$

and finally let $\mathcal{P}=\{\varepsilon_X | x\in[0, 1)\}\cup\{P_s\}$. In order to enlighten this claim we need some definitions and notions.

Notation and preliminaries

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a statistical space. A σ-field $\mathcal{F}\subset\mathcal{A}$ is called sufficient if for each $A\in\mathcal{A}$ there exists a common version $\mathcal{P}(A|\mathcal{F})$ of the conditional probabilities $P(A|\mathcal{F})$, $P\in\mathcal{P}$.

Denote by $\overline{\mathcal{P}}$ the closure of the convex hull of $\mathcal{P}$ taken in the norm of total variation of measure and let

$$\text{(1)} \qquad \mathcal{P}^* = \{Q|Q \text{ is a probability measure on } (\Omega, \mathcal{A}) \text{ and there exists a measure } P\in\overline{\mathcal{P}} \text{ such that } Q \ll P\}.$$

Let $\mathcal{Q}\subset\mathcal{P}^*$ be a measure class. Write

$$\text{(2)} \qquad \mathcal{N}(\mathcal{Q}) = \{A\in\mathcal{A}|Q(A) = 0 \text{ for every } Q\in\mathcal{Q}\}.$$

In order to simplify computations we shall always suppose that if $\mathcal{F}$ is a sufficient σ-field in $(\Omega, \mathcal{A}, \mathcal{Q})$, then $\mathcal{N}(\mathcal{Q})\subset\mathcal{F}$. This is not a serious restriction since a σ-field

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[1] This paper was partly written while the author was visiting the Statistics and Applied Probability Program, University of California, Santa Barbara.

$\mathcal{F}$ is sufficient iff $\sigma(\mathcal{F}, \mathcal{N}(\mathcal{Q}))$ is sufficient. Write

$$\mathcal{M}(\mathcal{P}) = \{\mathcal{Q} \subset \mathcal{P}^* | \mathcal{N}(\mathcal{Q}) = \mathcal{N}(\mathcal{P})\}.$$

Consider an event $A \in \mathcal{A}$ and a probability measure P defined on $(\Omega, \mathcal{A})$. Define a measure as follows:

$$P^A(B) = \begin{cases} P(B|A) & \text{if} \quad P(A) > 0 \\ 0 & \text{if} \quad P(A) = 0. \end{cases}$$

Denote $\mathcal{A}_{|A} = \{B \in \mathcal{A} | B \subset A\}$.

Definition. Let $\mathcal{Q} \subset \mathcal{P}^*$ be an arbitrary measure class. A measure $Q \in \mathcal{Q}$ is said to be strictly positive (in $\mathcal{Q}$) on the event $A \in \mathcal{A}$ if for every event $B \in \mathcal{A}$ for which $B \setminus A \in \mathcal{N}(\mathcal{Q})$ and $B \notin \mathcal{N}(\mathcal{Q})$ we have $Q(B) > 0$, i.e. on the event A the elements of $\mathcal{Q}$ are absolutely continuous with respect to Q.

It can be shown easily that there exists a largest event [mod $\mathcal{N}(\mathcal{Q})$] — denoted by $A_Q(\mathcal{Q})$ — such that Q is strictly positive on $A_Q(\mathcal{Q})$. The event $A_Q(\mathcal{Q})$ is referred to as the waistbelt of Q. If $Q(A_Q(\mathcal{Q})) = 1$, then $A_Q(\mathcal{Q})$ is the parcel of Q. If every measure of a statistical space has a parcel, then this space (or the measure class $\mathcal{Q}$) is said to be parcellable. Let

(3) $\mathcal{C}_0(\mathcal{Q}) = \{A \in \mathcal{A} \,|\, \text{there exists a measure } Q \in \mathcal{Q} \text{ such that } Q \text{ is strictly positive on } A\}$,

(4) $\mathcal{D}(\mathcal{Q}) = \{A \in \mathcal{A} | A \cap B \in \mathcal{N}(\mathcal{Q}) \text{ for every } B \in \mathcal{C}_0(\mathcal{Q})\}$

and let us use simply $\mathcal{D}$ for $\mathcal{D}(\mathcal{P})$.

Remark. It can be shown easily that there exists a greatest event $B_Q(\mathcal{Q}) \in \mathcal{D}(\mathcal{Q})$ mod $\mathcal{N}(\mathcal{Q})$ such that if $B \in \mathcal{D}(\mathcal{Q})$, $B \cap B_Q(\mathcal{Q}) \in \mathcal{N}(\mathcal{Q})$, then $Q(B) = 0$.

Returning to the so-called prototype we can observe that in this space $\mathcal{D}(\mathcal{P}) = \{\emptyset\}$, $\mathcal{C}_0(\mathcal{P})$ contains the countable subsets of Ω.

Going further we can observe that every measure in $\mathcal{P}^*$ can be "divided" into two parts. One of these parts has a parcel, the other one is equal to a constant multiple of P_s, so it is "spread" over $\mathcal{C}_0(\mathcal{P})$.

The following definitions are in some sense the generalizations of these observations.

Definition. We shall say that the statistical space $\{\Omega, \mathcal{A}, \mathcal{P})$ has property $H(\alpha_0)$ if the nondenumerable cardinality, α_0, is such that

(i) for every subsystem $(A_i)_{i \in I}$ of $\mathcal{C}_0(\mathcal{P})$ whose cardinality is less than α_0 there exists an event $B \in \mathcal{A}$ such that

$$P(A_i \setminus B) = 0, \quad P \in \mathcal{P}, \quad i \in I, \tag{5}$$

and for any $C \in \mathcal{A}$ for which (5) holds we have

$$P(B \setminus C) = 0, \quad P \in \mathcal{P}; \tag{6}$$

(ii) for any $A \in \mathcal{A}$ either $\mathcal{A}_{|A} \cap \mathcal{C}_0(\mathcal{P})$ or $\mathcal{A}_{|\Omega \setminus A} \cap \mathcal{C}_0(\mathcal{P})$ has the property that the cardinality of every subsystem $(A_i)_{i \in I}$ in it for which $P(A_i \setminus A_j) = 0$, $P \in \mathcal{P}$, $i, j \in I$, is less than α_0.

REMARK. Denote $B \underset{\mathcal{Q}}{\prec} A$ if, $A, B \in \mathcal{A}$ and $B \setminus A \in \mathcal{N}(\mathcal{Q})$, and let $\bigvee_{i \in I} A_i$ the event B appearing in (5) and (6).

DEFINITION. Let $\mathcal{Q} \subset \mathcal{P}^*$ be a measure class. Denote

$$\mathcal{C}^*_{\alpha_0}(\mathcal{Q}) = \{A \in \mathcal{A} \mid \text{there exists a subsystem } (A_i)_{i \in I} \subset \mathcal{C}_0(\mathcal{Q}) \text{ for which } |I| < \alpha_0 \text{ and } A = \bigvee_{i \in I} A_i].$$

Let us define the sum of $\mathcal{C}^*_{\alpha_0}(\mathcal{Q})$ and $\mathcal{D}(\mathcal{Q})$ as follows:

$$\mathcal{C}^*_{\alpha_0}(\mathcal{Q}) + \mathcal{D}(\mathcal{Q}) = \{A \in \mathcal{A} \mid \text{there exists a } C \in \mathcal{C}^*_{\alpha_0}(\mathcal{Q}),\ D \in \mathcal{D}(\mathcal{Q}) \text{ such that } A = C \cup D\}.$$

DEFINITION. Let $\mathcal{Q} \subset \mathcal{P}^*$. We shall say that the statistical space $(\Omega, \mathcal{A}, \mathcal{Q})$ has property (P_s) if it has property $H(\alpha_0)$ and

(i) there exists a measure P_s such that

$$P_s(A) = \begin{cases} 0 & \text{if } A \in \mathcal{C}^*_{\alpha_0}(\mathcal{Q}) \text{ or } A \in \mathcal{D}(\mathcal{Q}) \\ 1 & \text{if } A \notin \mathcal{C}^*_{\alpha_0}(\mathcal{Q}) + \mathcal{D}(\mathcal{Q}), \end{cases}$$

(ii) for every measure $Q \in \mathcal{Q}$ there exists a number $d_Q \geqq 0$ such that

$$Q(A) = Q(A \cap A_Q(\mathcal{Q})) + Q(A \cap B_Q(\mathcal{Q})) + d_Q P_s(A).$$

REMARK. In the "prototype" example α_0 is the first nondenumerable cardinal, P_s is exactly the measure in the previous definition.

Minimal sufficiency

[6] contains the following Theorem.

THEOREM 1. *Let* $(\Omega, \mathcal{A}, \mathcal{P})$ *be a statistical space. The following two assertions are equivalent:*

(i) *for every measure class* $\mathcal{Q} \in \mathcal{M}(\mathcal{P})$ *there exists a minimal sufficient* σ*-field,*

(I) (ii) a) $(\Omega, \mathcal{A}, \mathcal{P})$ *has property* $H(\alpha_0)$,

b) *for every* $\mathcal{Q} \in \mathcal{M}(\mathcal{P})$, *if* $\mathcal{F}$ *is a sufficient* σ*-field with respect to* $\mathcal{Q}$ *then* $\mathcal{D}(\mathcal{Q}) \subset \mathcal{F}$.

c) *it has property* P_s.

In our paper we characterize statistical spaces having the following property:

(II) for every $\mathcal{Q} \subset \mathcal{P}^*$ there exists a minimal sufficient σ-field in $(\Omega, \mathcal{A}, \mathcal{Q})$.

Observe that property (II) implies (I), and what is more, it ensures that every statistical space $(\Omega, \mathcal{A}, \mathcal{Q})$ for which $\mathcal{Q} \subset \mathcal{P}^*$, has property (I). Thus examining property (II) we must check whether $(\Omega, \mathcal{A}, \mathcal{Q})$ has property $H(\alpha_0)$, property (P_s) and how the ideal $\mathcal{D}(\mathcal{Q})$ looks like.

The following theorem is true.

THEOREM 2. *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a statistical space. The following two assertions are equivalent:*

(i) *$(\Omega, \mathcal{A}, \mathcal{P})$ has property* (II),

(ii) a) *$(\Omega, \mathcal{A}, \mathcal{P})$ has property $H(\alpha_0)$,*

b) $\mathcal{D}=\{\emptyset\}$,

c) *it has property* (P_s).

To prove this Theorem we need the following Lemma.

LEMMA 1. *If $(\Omega, \mathcal{A}, \mathcal{P})$ has property* (II) *then* $\mathcal{D}=\{\emptyset\}$.

PROOF. Take an event $A\in\mathcal{D}$, $A\notin\mathcal{N}(\mathcal{P})$. Due to the definition of $\mathcal{D}$ it can be divided into two parts: $A_1, A_2\in\mathcal{D}$, $A_1\cup A_2=A$ in such a way that there exists a measure $P_0\in\mathcal{P}^*$ for which $P_0(A_1)>0$, $P_0(A_2)>0$, $P_0(A)=1$. Define

$$\mathcal{D}_1 = \{B|B\subset A_1,\ P_0(B)=0\},\quad \mathcal{D}_2 = \{B|B\subset A_2,\ P_0(B)=0\}.$$

Let us choose maximal subsystems containing pairwise disjoint elements of $\mathcal{D}_1$ and $\mathcal{D}_2$ mod $\mathcal{N}(\mathcal{P})$. Denote it by $(B_{i\in I})_i\subset\mathcal{D}_1$, $(C_j)_{j\in J}\subset\mathcal{D}_2$. Suppose that $|I|\leqq|J|$. There exists an injection $\varrho\colon I\to J$. Let $E_i=B_i\cup C_{\varrho(i)}$. There exist measures $(P_i)_{i\in I}\subset\subset\mathcal{P}^*$ for which $P_i(E_i)=1$, $P_i(B_i)=P_i(C_{\varrho(i)})=1/2$. We can assume that $P_0(A_1)==2/3$, $P_0(A_2)=1/3$.

Let $\mathcal{Q}=\{P_i|i\in I\}\cup\{P_0\}$. Clearly $\mathcal{Q}\subset\mathcal{P}^*$.

(It is worth noting that in the proof of Lemma 1 so far we did not make anything else but collected the elements of the Pitcher counterexample for the nonexistence of minimal sufficient σ-field.)

According to our assumption there ought to exist a minimal sufficient σ-field in this statistical space. We will show that actually such a σ-field does not exist.

Denote $E_0=A_{P_0}(\mathcal{Q})$ the waistbelt of P_0 in $\mathcal{Q}$. In this case $A\cap A_{P_0}(\mathcal{Q})=\bigvee_{i\in I} E_i \underset{\mathcal{Q}}{\geq} A_1$.

Observe that

$$\frac{P_0(A_1)}{P_0(A\setminus A_{P_0}(\mathcal{Q}))} > \frac{1}{2}.$$

Write

$$\mathcal{F}_i = \{D|D\cap E_i\in\mathcal{N}(\mathcal{Q}) \text{ or } D\setminus E_i\in\mathcal{N}(\mathcal{Q})\},$$

$$\mathcal{F}=\bigcap\mathcal{F}_i,\quad \mathcal{G}_0=\sigma(E_i,\ i\in I\cup\{0\}).$$

The σ-fields $\mathcal{F}_i$, $i\in I$, are obviously sufficient with respect to $\mathcal{Q}$, since $P_i(E_j)=0$, if $i\neq j$. On the other hand $\mathcal{G}_0$ is contained in every σ-field which is sufficient with respect to $\mathcal{Q}$, and $\mathcal{F}=\bigcap_{i\in I\cup\{0\}}\sigma(\mathcal{G}_0, \mathcal{N}(P_i))$. Consequently the minimal sufficient σ-field is $\mathcal{F}$, if any.

At the same time

$$\mathcal{Q}(A_1|\mathcal{F})\,\chi(E_i) = P_i(A_1|\mathcal{F})\,\chi(E_i) = (1/2)\,\chi(E_i),\quad i\in I,$$

$$\mathcal{Q}(A_1|\mathcal{F})\,\chi(E_0) = P_0(A_1|\mathcal{F})\,\chi(E_0) = 0,\qquad \bigvee_{i\in I\cup\{0\}} E_i=\Omega\quad (\text{in } (\Omega, \mathcal{A}, \mathcal{Q})).$$

Thus $\mathcal{Q}(A_1|\mathcal{F})=(1/2)\chi(A\setminus E_0)$ would hold, but

$$\int_{A\setminus E_0} \chi(A_1)\,dP_0 = P_0(A_1) > (1/2)\,P_0(A\setminus E_0) = \int_{A\setminus E_0} (1/2)\,\chi(A\setminus E_0)\,dP_0,$$

i.e. $\mathcal{F}$ is not sufficient. This implies that $\mathcal{D}=\{\emptyset\}$.

REMARK. It can be easily shown that if $\mathcal{D}=\{\emptyset\}$ and the statistical space has property $H(\alpha_0)$ then for every $(A_i)_{i\in I}\subset\mathcal{A}$, $|I|<\alpha_0$ there exists the supremum $\vee A_i$. Namely, if there exists an index $i\in I$ for which $A_i\notin\mathcal{C}^*_{\alpha_0}(\mathcal{P})$ then $\Omega\setminus A_i\in\mathcal{C}^*_{\alpha_0}(\mathcal{P})$ and using property $H(\alpha_0)$ there exists $\bigvee_{j\in I}(A_j\setminus A_i)$ and $\vee A_i=\big[\bigvee_{j\in I}(A_j\setminus A_i)\big]\vee A_i$. On the other hand, if for every $i\in I$, A_i belongs to $\mathcal{C}^*_{\alpha_0}(\mathcal{P})$, then there exist events $(C_{i,j})_{j\in J_i}\subset\mathcal{C}_0$ for which $\bigvee_{j\in\mathcal{F}_i} C_{i,j}=A_i$, $|J_i|<\alpha_0$. In this case $\bigvee_i\bigvee_j C_{i,j}=\vee A_i$.

PROOF OF THEOREM 2. First observe that in case $\mathcal{D}=\{\emptyset\}$ property (P_s) means that for every $P\in\mathcal{P}$ there exists a real number $0\leqq d_p\leqq 1$, such that

$$P = P(A_p)\,P^{A_P}+d_p P_s, \tag{7}$$

where A_p is the waistbelt of P (in $\mathcal{P}$). Due to the definition of $\mathcal{P}^*$ every measure $P\in\mathcal{P}^*$ can be written in this form.

The implication (i)$\Rightarrow$(ii) b) is the assertion of the previous Lemma. Invoking Theorem 1 (comparing with the introductory observations) we obtain (i)$\Rightarrow$(ii) a) and c).

Conversely, Property (II) means that for every $\mathcal{Q}\subset\mathcal{P}^*$ the statistical space $(\Omega,\mathcal{A},\mathcal{Q})$ has property (I). But Theorem 1 characterizes the statistical spaces having this property.

Step 1. $\mathcal{D}(\mathcal{Q})=\{\emptyset\}$. We already know that $\mathcal{D}(\mathcal{P})=\{\emptyset\}$. Take an event $A\in\mathcal{D}(\mathcal{Q})$. Every measure $Q\in\mathcal{Q}$ can be written as follows

$$Q = Q\big(A_Q(\mathcal{P})\big)\,Q^{A_Q(\mathcal{P})}+d_Q P_s.$$

Clearly $A_Q(\mathcal{P})\setminus A_Q(\mathcal{Q})\in\mathcal{N}(\mathcal{Q})$ thus $Q^{A_Q(\mathcal{P})}(A)=0$. So if $A\in\mathcal{N}(\mathcal{Q})$ then $P_s(A)>0$ must hold, consequently P_s would be strictly positive on A, i.e. A would belong to $\mathcal{C}(\mathcal{Q})$. We obtained that $A\in\mathcal{N}(\mathcal{Q})$, so $\mathcal{D}(\mathcal{Q})=\{\emptyset\}$.

Step 2. Take a measure $Q\in\mathcal{Q}$. We will show that

$$A_Q(\mathcal{Q}) = \begin{cases} A_Q(\mathcal{P})\cup A_{P_s}(\mathcal{Q}), & \text{if } Q\big(A_Q(\mathcal{P})\big)<1,\\ A_Q(\mathcal{P}), & \text{if } Q\big(A_Q(\mathcal{P})\big)=1.\end{cases} \tag{8}$$

Denote $\mathcal{Q}_0=\{Q^{A_Q(\mathcal{P})}|Q\in\mathcal{Q}\}$. First we show that if $Q\in\mathcal{Q}_0$ then $A_Q(\mathcal{P})=A_Q(\mathcal{Q})$ Since $\mathcal{N}(\mathcal{Q})\supset\mathcal{N}(\mathcal{P})$ we have $A_Q(\mathcal{P})\underset{\mathcal{Q}}{<}A_Q(\mathcal{Q})$. On the other hand, $Q\big(A_Q(\mathcal{P})\big)=1$ so $A_Q(\mathcal{P})\setminus A_Q(\mathcal{Q})\in\mathcal{N}(\mathcal{Q})$. Observe that $Q\big(A_Q(\mathcal{Q})\big)=1$. If $Q\in\mathcal{Q}$ is an arbitrary measure then due to (7) we obtain the desired assertion.

Now consider the measure P_s. Since it is a $0-1$ measure there are two possibilities: either $P_s\big(A_{P_s}(\mathcal{Q})\big)=1$, when P_s has a parcel in $(\Omega,\mathcal{A},\mathcal{Q})$, or $P_s\big(A_{P_s}(\mathcal{Q})\big)=0$, when P_s is spread over $(\Omega,\mathcal{A},\mathcal{Q})$ (of course including the possibility: for every $Q\in\mathcal{Q}$ we have $Q\in\mathcal{Q}_0$).

Step 3. Suppose that $P_s(A_{P_s}(\mathscr{Q}))=1$. $A_{P_s}(\mathscr{Q})$ is unique mod $\mathscr{N}(\mathscr{Q})$. In the following consideration choose and fix an event as $A_{P_s}(\mathscr{Q})$.

We will show that in this case $(\Omega, \mathscr{A}, \mathscr{Q})$ is parcellable, and for any system $(A_i)_{i\in I}\subset\mathscr{A}$ the event $\bigvee_{i\in I} A_i$ exists in $(\Omega, \mathscr{A}, \mathscr{Q})$. (In this case we say that $\mathscr{A}$ is complete with respect to $\mathscr{Q}$. As long as the parcellability is concerned, it is enough to observe that since P_s has a parcel, so, due to (8), we obtain that every measure $Q\in\mathscr{Q}$ has also a parcel.

Consider now the second property. First we show that there exists the supremum $A=\bigvee_{Q\in\mathscr{Q}_0} A_Q(\mathscr{P})$ and then we prove that $\mathscr{A}_{|A}$ is complete with respect to $\mathscr{Q}$.

If $Q\in\mathscr{Q}_0$ then $A_Q(\mathscr{P})\cap A_{P_s}(\mathscr{Q})\in\mathscr{N}(Q)$ but on $A_Q(\mathscr{P})$ we have $\mathscr{N}(Q)=\mathscr{N}(\mathscr{Q})=\mathscr{N}(\mathscr{P})$ so $A_Q(\mathscr{P})\cap A_{P_s}(\mathscr{Q})\in\mathscr{N}(Q)$. At the same time since $P_s(A_{P_s}(\mathscr{Q}))=1$ we have $A_{P_s}(\mathscr{Q})\notin\mathscr{C}^*_{\alpha_0}(\mathscr{P})$ so $\Omega\setminus A_{P_s}(\mathscr{Q})\in\mathscr{C}^*_{\alpha_0}(\mathscr{P})$, consequently $\mathscr{A}_{|\Omega\setminus A_{P_s}(\mathscr{Q})}$ is complete with respect to $\mathscr{P}$, so there exists the supremum $A=\bigvee_{Q\in\mathscr{Q}_0} A_Q(\mathscr{P})$ in $(\Omega, \mathscr{A}, \mathscr{P})$. Clearly, $A\in\mathscr{C}^*_{\alpha_0}(\mathscr{P})$ i.e. $\mathscr{A}_{|A}$ is complete with respect to $\mathscr{P}$.

Obviously $\bigvee_{Q\in\mathscr{Q}_0} A_Q(\mathscr{P})=\Omega\setminus A_{P_s}(\mathscr{Q})=A$ in $(\Omega, \mathscr{A}, \mathscr{Q})$. If $Q\in\mathscr{Q}_0$ then $A_Q(\mathscr{Q})\setminus A_Q(\mathscr{P})\in\mathscr{N}(Q)$, so $\bigvee_{Q\in\mathscr{Q}_0} A_Q(\mathscr{Q})=A$ in $(\Omega, \mathscr{A}, \mathscr{Q})$. Take an event $B\subset A$. We obtain $B=\bigvee_{Q\in\mathscr{Q}_0}(A_Q(\mathscr{P})\cap B)$.

If $B\in\mathscr{N}(\mathscr{Q})$ then $B\cap A_Q(\mathscr{P})\in\mathscr{N}(\mathscr{Q})$ so $B\cap A_Q(\mathscr{P})\in\mathscr{N}(\mathscr{P})$ thus $B\in\mathscr{N}(\mathscr{Q})$. This means that $\mathscr{N}(\mathscr{Q})$ and $\mathscr{N}(\mathscr{P})$ coincide on A, i.e. $\mathscr{A}_{|A}$ is complete with respect to $\mathscr{Q}$.

On the other hand $\Omega\setminus A=A_{P_s}(\mathscr{Q})$ is an atom of $\mathscr{A}$ (since P_s is a $0-1$ measure), consequently $\mathscr{A}$ is complete with respect to $\mathscr{Q}$.

Summing up we have obtained in this case that $(\Omega, \mathscr{A}, \mathscr{Q})$ is parcellable and $\mathscr{A}$ is complete with respect to $\mathscr{Q}$. In view of [6] this implies that $(\Omega, \mathscr{A}, \mathscr{Q})$ is weakly dominated, thus there exists a minimal sufficient σ-field in it (cf. [7]).

Step 4. Suppose that $P_s(A_{P_s}(\mathscr{Q}))=0$. First we examine the relation between $\mathscr{C}_0(\mathscr{P})$ and $\mathscr{C}_0(\mathscr{Q})$. Namely we show that if $A\in\mathscr{C}_0(\mathscr{Q})$ then there exists an event $B\in\mathscr{A}$ such that $B=\bigcup_{n=1}^{\infty} B_n$, $B_n\in\mathscr{C}_0(\mathscr{P})$ and $A\setminus B\in\mathscr{N}(\mathscr{Q})$.

Since the waistbelt of P_s is equal to $\emptyset$, we have $P_s(A)=0$ if $A\in\mathscr{C}_0(\mathscr{Q})$. There exists a measure class $(Q_n)_{n\in N}\subset\mathscr{Q}_0$, such that $\left(\text{denoting } Q=\sum\frac{1}{2^n}Q_n\right)$ $A\subset A_Q(\mathscr{Q})$. But $A_Q(\mathscr{Q})=A_Q(\mathscr{P})$ so $A_Q(\mathscr{P})\cap A=A$ in $(\Omega, \mathscr{A}, \mathscr{Q})$ thus $B=A_Q(\mathscr{P})$ is an appropriate choice.

From this it follows that given an event $A\in\mathscr{C}_0(\mathscr{Q})$, we can suppose that $A=\bigcup B_n$, $B_n\in\mathscr{C}_0(\mathscr{P})$. Now we prove that if $(A_i)_{i\in I}\subset\mathscr{C}_0(\mathscr{Q})$, $|I|<\alpha_0$ then $\bigvee A_i$ exists in $(\Omega, \mathscr{A}, \mathscr{Q})$. To this end we show that if we choose the events $(A_i)_{i\in I}$ in such a way that $A_i\in\mathscr{C}_0(\mathscr{P})$ hold, then $\bigvee A_i=A$ exists in $(\Omega, \mathscr{A}, \mathscr{P})$, and $\bigvee A_i=A$ also in $(\Omega, \mathscr{A}, \mathscr{Q})$.

Since $(\Omega, \mathscr{A}, \mathscr{P},)$ has $H(\alpha_0)$ it follows that $\bigvee A_i=A$ exists in $(\Omega, \mathscr{A}, \mathscr{P})$. It remains to show that $A=\bigvee_{\mathscr{Q}} A_i$. Clearly $A_i\vee A\in\mathscr{N}(\mathscr{Q})$. Take an event $B\subset A$, and suppose that $B\cap A_i\in\mathscr{N}(\mathscr{Q})$, $i\in I$, but $B\notin\mathscr{N}(\mathscr{Q})$. This means that there exists a measure $Q\in\mathcal{Q}$ for which $Q(B)>0$. Due to $P_s(A)=0$ we can assume that $Q\in\mathcal{Q}_0$. Consider the event $B_0=B\cap A_Q(\mathcal{P})$. $Q(B_0)>0$. On this event $\mathcal{N}(\mathcal{Q})=\mathcal{N}(Q)=\mathcal{N}(\mathcal{P})$, but $B\cap A_i\in\mathcal{N}(\mathcal{Q})$, $i\in I$, so $B_0\cap A_i\in\mathcal{N}(\mathcal{P})$ i.e. $B_0\setminus A\in\mathcal{N}(\mathcal{P})$, $B_0\cap\cap A_i\in\mathcal{N}(\mathcal{P})$. From this immediately follows that $B_0\in\mathcal{N}(\mathcal{P})$. This is a contradiction, thus $\bigvee_{\mathcal{Q}} A_i=A$.

Now consider the other part of property $H(\alpha_0)$. Take an event $A\in\mathcal{A}$, and suppose that $A\in\mathcal{C}^*_{\alpha_0}(\mathcal{P})$. We show that $A\in\mathcal{C}^*_{\alpha_0}(\mathcal{Q})$. Let $(A_i)_{i\in I}$ be a subsystem of $\mathcal{A}_{|A}\cap\mathcal{C}_0(\mathcal{Q})$ containing disjoint events $\bmod\mathcal{N}(\mathcal{Q})$. We ought to show that $|I|<\alpha_0$.

First we cut the "superfluous" part of A which belongs to $\mathcal{N}(\mathcal{Q})$. Due to $A\in\mathcal{C}^*_{\alpha_0}(\mathcal{P})$, $\mathcal{A}|_A$ is complete with respect to $\mathcal{P}$ thus there exists the supremum $\bigvee_{Q\in\mathcal{Q}}(A\cap A_Q(\mathcal{P}))$ in $(\Omega,\mathcal{A},\mathcal{P})$, denote it by $A(\mathcal{Q})$. We will show that $A(\mathcal{Q})\circ A\in\mathcal{N}(\mathcal{Q})$.

If $B\subset A(\mathcal{Q})$ then $B=\bigvee_{Q\in\mathcal{Q}}(B\cap A_Q(\mathcal{P}))$ in $(\Omega,\mathcal{A},\mathcal{P})$ and if besides this $B\in\mathcal{N}(\mathcal{Q})$, then $B\in\mathcal{N}(\mathcal{P})$. Thus $\mathcal{N}(\mathcal{Q})=\mathcal{N}(\mathcal{P})$ on $A(\mathcal{Q})$. On the other hand $A(\mathcal{Q})\setminus A\in\mathcal{N}(\mathcal{Q})$ and since $P_s(A\setminus A(\mathcal{Q}))\leqq P_s(A)=0$ and

$$Q(A\setminus A(\mathcal{Q}))=Q((A\cap A_Q(\mathcal{P}))\setminus A)=0$$

we obtain that $A\circ A(\mathcal{Q})\in\mathcal{N}(\mathcal{Q})$.

Replacing $(A_i)_{i\in I}$ with $(A_i\cap A(\mathcal{Q}))_{i\in I}$, we obtain disjoint non-empty-events $\bmod\mathcal{N}(\mathcal{Q})$, but $\mathcal{N}(\mathcal{P})=\mathcal{N}(\mathcal{Q})$ on $A(\mathcal{Q})$ so $(A_i\cap A(\mathcal{Q}))_{\cdot\in I}$ is also a disjoint system $\bmod\mathcal{N}(\mathcal{P})$. The relation $A_i\cap A(\mathcal{Q})\underset{\mathcal{P}}{<}A$ implies that $|I|<\alpha_0$.

So far we have proved in the case $P_s(A_{P_s}(\mathcal{Q}))=0$ that $(\Omega,\mathcal{A},\mathcal{Q})$ has property $H(\alpha_0)$, and $\mathcal{D}(\mathcal{Q})=\{\emptyset\}$. Examine property (P_s). We have only to show that P_s vanishes on $\mathcal{C}^*_{\alpha_0}(\mathcal{Q})$. Take an event $A\in\mathcal{A}$ and suppose that $A\in\mathcal{C}^*_{\alpha_0}(\mathcal{Q})$. There exists a subfamily $(A_i)_{i\in I}\subset\mathcal{C}_0(\mathcal{Q})$ such that $\bigvee_{\mathcal{Q}}A_i=A$, $|I|<\alpha_0$. Since $A_i\in\mathcal{C}_0(\mathcal{Q})$ we can suppose that $A_i\in\mathcal{C}_0(\mathcal{P})$. Denote $B=\bigvee_{\mathcal{P}}A_i$. This supremum exists in $(\Omega,\mathcal{A},\mathcal{P})$ since it has property $H(\alpha_0)$, $B\in\mathcal{C}^*_{\alpha_0}(\mathcal{P})$, $P_s(B)=0$. If $Q\in\mathcal{Q}_0$ then $[A_{\mathcal{Q}}(\mathcal{P})\cap A]\circ[A_{\mathcal{Q}}(\mathcal{P})\cap B]\in\mathcal{N}(\mathcal{Q})$ since $\mathcal{N}(Q)=\mathcal{N}(\mathcal{Q})=\mathcal{N}(\mathcal{P})$ on $A_Q(\mathcal{P})$. Thus $Q(B\setminus A)=0$, $Q\in\mathcal{Q}_0$. If $P_s(B\setminus A)>0$ held then P_s would be strictly positive on $B\setminus A$ [in $(\Omega,\mathcal{A},\mathcal{Q})$] but we have assumed that $P_s(A_{P_s}(\mathcal{Q}))=0$. This implies that $B\circ A\in\mathcal{N}(\mathcal{Q})$, $P_s(A)=P_s(B)=0$.

Summing up, we have obtained that $(\Omega,\mathcal{A},\mathcal{Q})$ satisfies condition (ii) of Theorem 1 so there exists a minimal sufficient σ-field in it.

REMARK. Recall that in this case we can construct the minimal sufficient σ-field as follows. Since for every $\mathcal{Q}\subset\mathcal{P}^*$ the statistical space has the properties $H(\alpha_0)$, (P_s), $\mathcal{D}=\{\emptyset\}$ we can restrict ourselves to the case $(\Omega,\mathcal{A},\mathcal{P})$. For every $P\in\mathcal{P}$ consider its parcel A_P and form the σ-field generated on one hand by the Radon—Nikodym derivates $\frac{dQ}{dP}\left(\frac{dQ}{dP}=0 \text{ out of } A_P\right)$ and on the other hand by the null sets $\mathcal{N}(\mathcal{P})$

$$\mathcal{F}_P=\sigma\left(\frac{dQ}{dP}\chi_{A_P},\ \mathcal{N}(\mathcal{P}),\ Q\in\mathcal{P}\right).$$

Then the σ-field

$$\mathscr{F} = \{A \mid A \cap A_P \in \mathscr{F}_P \text{ for each } P \in \mathscr{P}\}$$

is the minimal sufficient σ-field.

Finally a remark. Consider the prototype statistical space. It is worth to note that if instead of the σ-field generated by the one-point subsets of Ω we take the σ-field $\mathscr{B}$ generated by the sets having Lebesgue-measure zero, then the property $H(\alpha_0)$ does not hold, so there exists a statistical space $(\Omega, \mathscr{B}, \mathscr{Q})$, $\mathscr{Q} \subset \mathscr{P}^*$ without minimal sufficient σ-field.

References

[1] D. L. Burkholder, Sufficiency in the undominated case, *Ann. Math. Statist.*, **32** (1961), 1191—1200.

[2] F. Göndöcs and Gy. Michaletzky, Construction of minimal sufficient and pairwise sufficient σ-field, *Acta Math. Hungar.*, **45** (1985), 201—212.

[3] R. P. Halmos and L. J. Savage, Application of the Radon—Nikodym theorem to the theory of sufficient statistic, *Ann. Math. Statist.*, **20** (1949), 225—241.

[4] M. Hasegawa and M. D. Perlmann, On the existence of a minimal sufficient subfield, *Ann. Statist.*, **2** (1974), 1049—1055. Correction: *Ann. Statist.*, **3** (1975), 1371—1372.

[5] H. Luschgy and D. Mussman, Equivalent properties and completion of statistical experiments, *Sankhya*, Ser. A **47** (1985), 174—195.

[6] Gy. Michaletzky, Minimal sufficiency. The structure of statistical spaces, *Proc. of the 3rd Pannon. Symp. on Math. Stat.* (Visegrád, Hungary, 1982); eds: J. Mogyoródi, I. Vincze W. Wertz.

[7] D. Mussmann, Vergleich von Experimenten im schwach dominierten Fall, *Z. Wahrschein.*, **24** (1972), 295—308.

[8] T. S. Pitcher, Sets of measures not admitting necessary and sufficient statistics and subfields, *Ann. Math. Statist.*, **28** (1957), 267—268.

[9] T. S. Pitcher, A more general property than domination for sets of probability measure, *Pac. J. Math.*, **15** (1965), 597—611.

[10] E. Siebert, Pairwise sufficiency, *Z. Wahrschein.*, **46** (1979), 237—246.

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DEPARTMENT OF PROBABILITY THEORY
EÖTVÖS LORÁND UNIVERSITY
H—1088 BUDAPEST
MÚZEUM KRT. 6—8

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CONCENTRATED BOREL MEASURES

Z. BUCZOLICH and M. LACZKOVICH (Budapest)

1. Introduction. Let μ be a locally finite Borel measure on $\mathbf{R}$ and let $b>0$. We say that μ is b-concentrated at the point x if $x\in\operatorname{supp}(\mu)$ and

$$(1) \qquad \limsup_{h\to 0+} \frac{\mu((x-bh, x+bh))}{\mu((x-h, x+h))} < b$$

holds. The set of points x at which μ is b-concentrated will be denoted by $C_b(\mu)$. The measure μ is called b-concentrated if (1) holds for every $x\in\operatorname{supp}(\mu)$ that is, if $C_b(\mu)=\operatorname{supp}(\mu)$.

We shall prove that if $b\leqq 1$ then $\mu(C_b(\mu))=0$ for every measure μ (2.2 Theorem). Hence for $b\leqq 1$ the only b-concentrated measure is the identically zero measure. If $b>1$ then $\lambda(C_b(\mu))=0$ holds, where λ denotes the Lebesgue measure (2.4. Theorem).

Therefore, if $b>1$ then every b-concentrated measure is singular with respect to the Lebesgue measure. A more precise result is proved in 2.7: if $b>1$ then $C_b(\mu)$ is σ-porous for every μ.

In Section 3 we give two examples of b-concentrated measures. First we show that the Cantor measure (supported by the Cantor ternary set) is b-concentrated if b is large enough but not b-concentrated for $b=4$ (3.1 Theorem). A better example is given by 3.2. There, using a thinner Cantor set we construct a measure that is b-concentrated for every $b>2$.

In Section 4 we show that there are no continuous b-concentrated measures for the values $b=2^{1/k}$ $(k=1, 2, \dots)$. More exactly, we prove that for every continuous measure μ and $b=2^{1/k}$ $(k=1, 2, \dots)$, $\mu(C_b(\mu))=0$ holds.

We do not know whether $\mu(C_b(\mu))=0$ must hold for any other number $b\in(1, 2)$. The problem, whether non-zero continuous b-concentrated measures exist for any $b\in(1, 2)$ also remains open. The first author proved that non-zero continuous b-concentrated measures do not exist for $b<1+10^{-6}$.

The distribution function of the locally finite measure μ is defined by

$$f(x) = \begin{cases} \mu([0, x)), & \text{if } x\geqq 0 \\ -\mu([x, 0)), & \text{if } x<0. \end{cases}$$

If μ is b-concentrated then the symmetric derivative

$$f_s'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

exists and equals infinity at each $x \in \text{supp}(\mu)$ (2.1 Proposition). In particular, the Cantor function has infinite symmetric derivative at each point of the Cantor set. (This fact was first observed by J. Uher [6].) We comment on this phenomenon in Section 5.

Finally, in Section 6 we apply the results of Section 3 to generalized Riemann derivatives (GRD's). Let the numbers a_i, b_i $(i=1, \dots, n)$ be given such that $\sum_{i=1}^{n} a_i = 0$, $\sum_{i=1}^{n} a_i b_i = 1$. The GRD of the function f at the point x is defined by

$$\lim_{h \to 0} \frac{\sum_{i=1}^{n} a_i f(x+b_i h)}{h} = D^1 f(x). \tag{2}$$

(This generalized derivative depends on the choice of the numbers a_i, b_i. However, it is easy to see that if $f'(x)$ exists and is finite, then $D^1 f(x) = f'(x)$ holds for every system a_i, b_i. For further details concerning GRD's, see [1].) We shall prove that there are continuous functions f and GRD's such that $D^1 f(x) \geqq 0$ everywhere and f is not increasing. We show that the negatives of the distribution functions of some b-concentrated measures have this property (6.1 Theorem).

In this paper we shall use the notation

$$I(x, h) = (x-h, x+h) \quad (x, h \in \mathbf{R};\ h > 0).$$

2. Properties of b-concentrated measures

2.1. PROPOSITION. *Let μ be a locally finite Borel measure on* **R** *and let f denote the distribution function of μ.*

(i) *If $x \in C_b(\mu)$ and $b < 1$ then $f'(x) = 0$.*

(ii) *If $x \in C_b(\mu)$ and $b > 1$ then $f_s'(x) = \infty$.*

PROOF. If $x \in C_b(\mu)$ then (1) holds and hence there are $\varepsilon > 0$ and $h_0 > 0$ such that

$$\frac{\mu(I(x, bh))}{\mu(I(x, h))} < (1-\varepsilon) b \quad (0 < h < h_0).$$

Since $\mu(I(x, h)) = f(x+h) - f(x-h)$, this implies

$$\frac{f(x+bh)-f(x-bh)}{2bh} < (1-\varepsilon) \frac{f(x+h)-f(x-h)}{2h} \quad (0 < h < h_0). \tag{3}$$

Suppose first $b < 1$. If $h \in [bh_0, h_0]$ then

$$\frac{f(x+h)-f(x-h)}{2h} \leqq \frac{\mu(I(x, h_0))}{2bh_0} \overset{\text{def}}{=\!=} K$$

and hence, by (3), we have

$$\frac{f(x+h)-f(x-h)}{2h} \leqq (1-\varepsilon)^{k-1}K$$

whenever $h\in[h_0b^k, h_0b^{k-1}]$ and $k=1,2,\dots$. Since f is increasing, this implies $f_s'(x)=0$ and, since for $h>0$

$$\max\left(\frac{f(x+h)-f(x)}{h}, \frac{f(x)-f(x-h)}{h}\right) \leqq \frac{f(x+h)-f(x-h)}{h},$$

we obtain $f'(x)=0$.

Next suppose $b>1$. If $h\in[h_0, bh_0]$ then

$$\frac{f(x+h)-f(x-h)}{2h} \geqq \frac{\mu(I(x, h_0))}{2bh_0} \overset{\text{def}}{=} \delta > 0.$$

Hence, by (3),

$$\frac{f(x+h)-f(x-h)}{2h} \geqq \frac{\delta}{(1-\varepsilon)^k}$$

for every $h\in[h_0b^{-k}, h_0b^{-k+1}]$ and $k=1,2,\dots$. Therefore we have $f_s'(x)=\infty$.

2.2. Theorem. *If μ is a locally finite Borel measure on $\mathbf{R}$ then $\mu(C_b(\mu))=0$ holds for every $b<1$.*

Proof. First we show that $C_b(\mu)$ is an F_σ set. Let

$$A_n = \left\{x\colon \mu((x-bh, x+bh)) \leqq \left(b-\frac{1}{n}\right)\mu([x-h, x+h]) \text{ for every } 0<h<1/n\right\};$$

we prove that $C_b(\mu)=\bigcup_{n>1/b} A_n$. The inclusion $C_b(\mu)\subset\bigcup_{n>1/b} A_n$ is obvious from the definition of $C_b(\mu)$. On the other hand, if $n>1/b$, $x\in A_n$ and $0<h'<1/n$, then

$$\mu((x-bh', x+bh')) = \lim_{h\to h'-} \mu((x-bh, x+bh)) \leqq$$

$$\leqq \left(b-\frac{1}{n}\right)\lim_{h\to h'-} \mu([x-h, x+h]) = \left(b-\frac{1}{n}\right)\mu((x-h', x+h'))$$

and hence $x\in C_b(\mu)$. It is easy to check that for every fixed $h>0$ the function $f_1(x)=\mu((x-bh, x+bh))$ is lower semicontinuous, and the function $f_2(x)=\mu([x-h, x+h])$ is upper semicontinuous. This implies that A_n is closed for every n and thus $C_b(\mu)$ is F_σ as we stated. In particular, $C_b(\mu)$ is μ-measurable.

Let f denote the distribution function of μ and let f^* denote the outer measure associated with the interval function $f(y)-f(x)$ (see [5], III. § 5). It is well-known that $\mu(X)=f^*(X)$ holds for every Borel set X. Let $X=\{x\colon f'(x)=0\}$, then by [5], IV. Theorem 9.6, $f^*(X)=0$. By 2.1(i), we have $C_b(\mu)\subset X$ and hence $\mu(C_b(\mu))\leqq \mu(X)=f^*(X)=0$.

2.3. Corollary. *If μ is a b-concentrated locally finite Borel measure with $b\leqq 1$, then $\mu\equiv 0$.*

PROOF. Since (1) is never satisfied for $b=1$, we have $C_1(\mu)=\emptyset$ and hence we may suppose $b<1$. If μ is b-concentrated then, by 2.2, $\mu(\text{supp}(\mu))=0$ and hence $\mu\equiv 0$.

2.4. THEOREM. *If μ is a locally finite Borel measure on $\mathbf{R}$ then $\lambda(C_b(\mu))=0$ holds for every $b>1$.*

PROOF. Let f denote the distribution function of μ and let X be the set of points at which f does not have a finite derivative. Since f is increasing, $\lambda(X)=0$ by Lebesgue's theorem. By 2.1(ii), $C_b(\mu)\subset X$ and hence $\lambda(C_b(\mu))=0$.

2.5. COROLLARY. *If μ is b-concentrated then μ is singular with respect to λ.*

PROOF. By 2.3 we may assume $b>1$. Then 2.4 gives $\lambda(\text{supp}(\mu))=0$ which was to be proved.

Our next aim is to prove that for $b>1$ $C_b(\mu)$ is, in fact, σ-porous. We recall the definitions. For every $H\subset\mathbf{R}$ and $a<b$ we shall denote by $l(H, a, b)$ the length of the longest component of $(a, b)\setminus H$. We denote

$$p^+(H;x)=\limsup_{h\to 0+}\frac{l(H,x,x+h)}{h},\quad p^-(H;x)=\limsup_{h\to 0+}\frac{l(H,x-h,x)}{h},$$

and $p(H;x)=\max(p^+(H;x),p^-(H;x))$. The set H is called porous, if $p(H;x)>0$ for every $x\in H$. H is σ-porous, if it is the union of countably many porous sets. It is well-known that every σ-porous set is of Lebesgue measure zero and of first category.

2.6. LEMMA. *Let μ be a locally finite Borel measure on $\mathbf{R}$, let $1<c<b$ and $h_0>0$ be fixed, and define*

$$H=\left\{x\in\text{supp}(\mu)\colon \frac{\mu(I(x,bh))}{\mu(I(x,h))}\leqq c \text{ for every } 0<h<h_0\right\}. \tag{4}$$

Then there is a $p>0$ such that

$$\liminf_{h\to 0+}\frac{l(H,x,x+h)}{h}\geqq p \quad\text{and}\quad \liminf_{h\to 0+}\frac{l(H,x-h,x)}{h}\geqq p \tag{5}$$

holds for every $x\in H$.

PROOF. Since $1<c<b$, we can choose a large integer N such that

$$\frac{\log 8N}{\log N}+\frac{\log b}{\log N}<\frac{\log b}{\log c}.$$

We shall prove that $p=\frac{1}{4N}$ satisfies the requirement of the theorem. In order to prove, say, the first inequality of (5) it is enough to show that for every $x\in H$ and $0<h<h_0$ there is an interval $I\subset(x,x+h)$ such that $I\cap H=\emptyset$ and $|I|\geqq\frac{h}{4N}$. Let $x\in H$ and $0<h<h_0$ be fixed. If $\mu((x,x+h))=0$ then $\text{supp}(\mu)\cap(x,x+h)=\emptyset$

and, as $H\subset \operatorname{supp}(\mu)$, we can take $I=(x, x+h)$. Therefore we may assume that

$$\mu((x, x+h)) \overset{\text{def}}{=} A > 0.$$

Let $I_i=\left(x+\frac{i-1}{4N}h,\ x+\frac{i}{4N}h\right)$ $(i=1, \dots, 2N)$. In order to complete the proof it is enough to show that there is an i with $I_i\cap H=\emptyset$.

Suppose this is not true and let $y_i\in I_i\cap H$ $(i=1, \dots, 2N)$. Since $y_i\in H$ and $0<h<h_0$, we have

$$c^m \mu\left(I\left(y_i, \frac{h}{b^m}\right)\right) \geqq \mu(I(y_i, h)) \geqq \mu((x, x+h)) = A$$

for every $m=1, 2, \dots$.

Let $m=\left[\frac{\log 8N}{\log b}\right]+1$. Then $b^m>8N$, $\frac{h}{b^m}<\frac{h}{8N}$ and hence the intervals $I\left(y_{2j}, \frac{h}{b^m}\right)$ $(j=1, \dots, N)$ are pairwise disjoint. Also,

$$\bigcup_{j=1}^{N} I\left(y_{2j}, \frac{h}{b^m}\right) \subset (x, x+h)$$

and hence

$$A = \mu((x, x+h)) \geqq \sum_{j=1}^{N} \mu\left(I\left(y_{2j}, \frac{h}{b^m}\right)\right) \geqq \frac{N}{c^m} A.$$

This implies $c^m\geqq N$. On the other hand, we have

$$c^m \leqq c^{(\log 8N/\log b)+1} = \exp\left(\frac{\log c}{\log b}\log 8N+\log c\right) < \exp(\log N) = N$$

by the choice of N. This contradiction completes the proof.

2.7. THEOREM. *For every locally finite Borel measure μ and $b>1$ the set $C_b(\mu)$ is σ-porous.*

PROOF. By the previous lemma, the set H defined in (4) is porous. Since $C_b(\mu)$ is the union of all these sets when c runs through the rational numbers of $(1, b)$ and $h_0=\frac{1}{n}$ $(n=1, 2, \dots)$, the assertion follows.

3. Examples of b-concentrated measures. The Cantor ternary set is defined as follows. Let S denote the set of finite $0-1$ sequences (including $\emptyset$) and let $|s|$ denote the length of $s\in S$. We put $J_\emptyset=[0, 1]$. If $J_s=[u, v]$ has been defined for an $s\in S$ then we put

$$J_{s0} = [u, u+(v-u)/3], \quad J_{s1} = [v-(v-u)/3, v].$$

In this way we define J_s for every $s\in S$. The Cantor ternary set is

$$C = \bigcap_{n=0}^{\infty} \bigcup_{\substack{s\in S\\ |s|=n}} J_s.$$

It is well-known that there is a unique Borel measure μ such that μ is supported by C and $\mu(J_s)=2^{-|s|}$ holds for every $s\in S$. This measure μ is called the Cantor measure. The distribution function of μ is the Cantor function.

3.1. THEOREM. (i) *The Cantor measure is b-concentrated for every* $b\geqq 81$.
(ii) *The Cantor measure is not b-concentrated for* $b=4$.

PROOF. (i) Let $b\geqq 81$ be fixed, and let k be an integer with $3^{k-1}\leqq b<3^k$; then $k\geqq 5$. Let $x\in C$ and $0<h<3^{-k}$ be given, and choose an integer $n>k$ such that $3^{-n}\leqq h<3^{-n+1}$. Since $x\in C$, there is an $s\in S$ such that $|s|=n$ and $x\in J_s$. Then, by $|J_s|=3^{-n}$,

$$[x-h, x+h]\supset[x-3^{-n}, x+3^{-n}]\supset J_s.$$

On the other hand, $bh<3^{-n+k+1}$ implies that $[x-bh, x+bh]\cap C\subset J_t$ with a $t\in S$, $|t|=n-k-1$. Indeed, let $t\in S$ be such that $|t|=n-k-1$ and $x\in J_t$, and put $A=\cup\{J_r\colon |r|=n-k-1, r\neq t\}$. Then $\operatorname{dist}(J_t, A)=3^{-n+k+1}$ and hence

$$[x-bh, x+bh]\cap C\subset[x-bh, x+bh]\cap(J_t\cup A)\subset J_t.$$

Therefore we have

$$\frac{\mu(I(x, bh))}{\mu(I(x, h))}\leqq\frac{\mu(J_t)}{\mu(J_s)}=2^{k+1}=6\left(\frac{2}{3}\right)^k 3^{k-1}\leqq 6\left(\frac{2}{3}\right)^k b\leqq 6\left(\frac{2}{3}\right)^5 b,$$

since $k\geqq 5$. Thus

$$\limsup_{h\to 0}\frac{\mu(I(x, bh))}{\mu(I(x, h))}\leqq\frac{64}{81}b<b, \tag{6}$$

which proves that μ is b-concentrated.

(ii) Let s_n denote the sequence

$$010010001\ldots\underbrace{10\ldots01}_{n}$$

and let $\bigcap_{n=1}^{\infty} J_{s_n}=\{x\}$. We prove that μ is not 4-concentrated at x. Let $J_{s_n}=[a_n, b_n]$ $(n=1, 2, \ldots)$. It is easy to check that

$$\lim_{n\to\infty}\frac{x-a_n}{b_n-a_n}=0.$$

Let $h_n=\frac{2}{3}(b_n-a_n)-(x-a_n)$; then

$$(x-h_n, x+h_n)\cap C\subset J_{s_n0}\quad(n=1, 2, \ldots).$$

On the other hand,

$$[x-4h_n, x+4h_n]\supset[a_n-2(b_n-a_n), b_n]$$

for n large enough, and hence

$$\limsup_{h\to 0+} \frac{\mu(I(x,4h))}{\mu(I(x,h))} \geqq \lim_{n\to\infty} \frac{2\mu(J_{s_n})}{\mu(J_{s_n 0})} = 4.$$

Consequently, μ is not 4-concentrated at x.

3.2. THEOREM. *There exists a continuous probability measure* μ *on* $\mathbf{R}$ *such that*

$$\limsup_{h\to 0+} \frac{\mu(I(x,bh))}{\mu(I(x,h))} \leqq 2$$

holds for every $x \in \operatorname{supp}(\mu)$ *and* $b > 1$. *In particular,* μ *is* b*-concentrated for every* $b > 2$.

PROOF. The measure μ will be supported by a perfect set P defined as follows. Let $J_\emptyset = [0, 1]$. If $J_s = [u, v]$ has been defined for an $s \in S$ then we put

$$J_{s0} = [u, u+(v-u)/(|s|+4)] \quad \text{and} \quad J_{s1} = [v-(v-u)/(|s|+4), v].$$

This defines J_s for every $s \in S$. We put

$$P = \bigcap_{n=0}^{\infty} \bigcup_{\substack{s\in S\\ |s|=n}} J_s.$$

It is well-known that there exists a Borel measure μ supported by P such that $\mu(J_s) = 2^{-|s|}$ for every $s \in S$. We shall prove that μ satisfies the requirements. Obviously, μ is a continuous probability measure on $\mathbf{R}$.

Let $x \in P$ and $b > 1$ be given. We show that

$$\mu(I(x,bh)) \leqq 2\mu(I(x,h)) \tag{7}$$

if $h > 0$ is small enough. Suppose first that x is the left end-point of the interval J_s. If $h < |J_s|/b$ then $[x, x+bh] \subset J_s$ and hence $I(x, bh) \cap C \subset J_s$. Also, there is a non-negative integer $k = k(h)$ such that

$$[x, x+bh] \subset J_{s\underbrace{0\dots 0}_{k}} \overset{\text{def}}{=} L^k$$

and

$$[x, x+bh] \supset J_{s\underbrace{0\dots 0}_{k+1}} = L^{k+1}.$$

Obviously, $k(h) \to \infty$ if $h \to 0$, and hence there is $h_0 > 0$ such that $k(h) > 2b$ for every $0 < h < h_0$. If $0 < h < h_0$ and $x + bh \in J_{s\underbrace{0\dots 0}_{k}1}$ then $bh > |L^k|/2$,

$$|L^{k+1}| = |L^k|/(|s|+k+5) < \frac{2bh}{k} < h,$$

and hence $[x, x+h] \supset L^{k+1}$. This implies

$$\frac{\mu(I(x, bh))}{\mu(I(x, h))} \leqq \frac{\mu(L^k)}{\mu(L^{k+1})} = 2.$$

On the other hand, if $x+bh \notin J_{s\underbrace{0\dots0}_{k}1}$ then $\mu(I(x, bh)) = \mu(L^{k+1})$. Also, in this case

$$|L^{k+2}| = |L^{k+1}|/(|s|+k+6) < \frac{bh}{k} < h$$

and hence $[x, x+h] \supset L^{k+2}$. Therefore we have

$$\frac{\mu(I(x, bh))}{\mu(I(x, h))} \leqq \frac{\mu(L^{k+1})}{\mu(L^{k+2})} = 2.$$

We proved that (7) holds for small h's whenever x is the left end-point of any of the intervals J_s. By symmetry, the same is true for the right end-points of the intervals J_s.

Thus we may suppose that x is not an end-point of any of the intervals J_s. In particular, $0<x<1$.

For every $0<h<\min(x, 1-x)$ let $s=s(h)$ denote the longest $t \in S$ such that $I(x, bh) \subset J_t$. Since x is not an end-point, it follows that $|s(h)| \to \infty$ as $h \to 0$. Therefore it is enough to show that $|s(h)|>b$ implies (7). Let $h>0$ be fixed, let $s=s(h)$, $J_s=[u, v]$ and suppose $|s|>b$. By the definition of s we have $I(x, bh) \subset J_s$, $x+bh \notin J_{s0}$ and $x-bh \notin J_{s1}$.

Since $x \in J_s \cap C$, we have $x \in J_{s0} \cup J_{s1}$. By symmetry we may assume $x \in J_{s0}$. As $x \in C$, this implies in turn, $x \in J_{s00} \cup J_{s01}$. Suppose that $x \in J_{s00}$. Then, by $u \leqq x - bh$, we have $bh \leqq x-u \leqq |J_{s00}|$ and hence $x+bh-u \leqq 2|J_{s00}| < (|s|+5)|J_{s00}| = |J_{s0}|$. Since $x+bh \notin J_{s0}$, this is impossible. Therefore we have $x \in J_{s01}$.

Let

$$L_k := J_{s0\underbrace{1\dots1}_{k}} \quad (k = 0, 1, \dots).$$

We show first that if $x-h \in L_k$ then

$$\mu(I(x, bh)) \leqq \mu(L_k). \tag{8}$$

Indeed, $u \leqq x-bh$ implies $bh \leqq x-u \leqq |J_{s0}|$ and hence $x+bh-u \leqq 2|J_{s0}| < (|s|+3)|J_{s0}| = |J_s| - |J_{s1}|$. Thus $x+bh \notin J_{s1}$ and $[x, x+bh] \cap C \subset L_0$. This proves (8) if $k=0$. Let $k>0$; then $x \in L_0$, $x-h \in L_k$ imply $x \in L_k$ and hence $h \leqq |L_k|$. Therefore $|L_0| - (x-bh-u) = (|L_0|-(x-u)) + bh \leqq |L_k| + b|L_k| < (|s|+k+2)|L_k| = |L_{k-1}| - 2|L_k|$ and this easily implies $[x-bh, x] \cap C \subset L_k$. Hence $I(x, bh) \cap C \subset L_k$ and (8) follows.

There exists a non-negative integer k so that $x-h \in L_k$ and $x-h \notin L_{k+1}$. Then $x \in L_k$ and hence either $x \in J_{s0\underbrace{1\dots1}_{k}0} \overset{\text{def}}{=} V$ or $x \in L_{k+1}$.

If $x \in V$ then, by $x+bh-u>|L_0|$, $bh>|L_k|-|V|=(|s|+k+4)|V|>b|V|$; that is $h>|V|$. This contradicts $x-h \in L_k$ and hence $x \in V$ is impossible.

Next suppose $x \in L_{k+1}$. If $x+h-u>|L_0|$ then $I(x, h) \supset L_{k+1}$ and we obtain $\mu(I(x, h)) \geqq \mu(L_{k+1}) = \mu(L_k)/2 \geqq \mu(I(x, bh))/2$, i.e. (7) holds in this case as well.

Finally let $x+h-u \leqq |J_{s0}|$. Then, as $x-h \notin L_{k+1}$, x lies in the first half of L_{k+1}, that is $x \in J_{s0\underbrace{1\dots 1}_{k+1}0} \stackrel{\text{def}}{=} W$.

Since $x+bh-u > |J_{s0}|$, we obtain $bh > |L_{k+1}| - |W| = (|s|+k+5)|W| > b|W|$, and hence $I(x, h) \supset W$. On the other hand, $x+h-u \leqq |J_{s0}|$ implies $h \leqq |L_{k+1}|$ and hence $|J_{s0}| - (x-bh-u) \leqq |L_{k+1}| + bh \leqq (1+b)|L_{k+1}| < |L_k| - 2|L_{k+1}|$ from which we obtain $I(x, bh) \cap C \subset L_{k+1}$. Therefore

$$\mu(I(x, bh)) \leqq \mu(L_{k+1}) = 2\mu(W) \leqq 2\mu(I(x, h))$$

and the proof is complete.

4. The case of $b=2^{1/k}$. In this section we prove that $\mu(C_b(\mu))=0$ holds for every continuous measure μ and $b=2^{1/k}$ $(k=1, 2, \dots)$ and, consequently, for these values of b no non-vanishing b-concentrated continuous measure exists.

If I is an open interval and $u>0$ then uI will denote the open interval concentric with I and of length $u|I|$. That is, if $I=I(x, h)$ then $uI=I(x, uh)$. We recall that $p(H; x)$ was defined after 2.5.

4.1. Lemma. *Let μ be a locally finite Borel measure on* $\mathbf{R}$, *let* $P \subset \mathbf{R}$ *be a non-empty, bounded, perfect set and let* $\{I_n\}_{n=1}^{\infty}$ *denote the sequence of all bounded intervals contiguous to P. Suppose that there are positive numbers p, K,* n_0 *such that*

(i) $p(P; x) > p$ *for every* $x \in P$,

and

(ii) $\mu\left(\frac{2}{p} I_n\right) \leqq K\mu(I_n) \quad (n \geqq n_0)$.

Then $\mu(P)=0$.

Proof. Let J be a bounded interval containing P. Since P is perfect, condition (i) implies that

$$P \subset \bigcup_{k=n}^{\infty} \frac{2}{p} I_k$$

for every n. By the compactness of P this implies that for every n, there is $n' > n$ such that

$$P \subset \bigcup_{k=n}^{n'} \frac{2}{p} I_k.$$

Hence we can define the numbers $n_0 < n_1 < \dots$ inductively such that

$$P \subset \bigcup_{k=n_i}^{n_{i+1}-1} \frac{2}{p} I_k \quad (i = 0, 1, \dots).$$

Suppose that $\mu(P) > 0$. Then

$$\sum_{k=n_i}^{n_{i+1}-1} \mu\left(\frac{2}{p} I_k\right) \geqq \mu(P)$$

for every i and thus, by (ii),

$$\sum_{k=n_i}^{n_{i+1}-1} \mu(I_k) \geqq \frac{1}{K}\mu(P) \quad (i=0,1,\ldots).$$

This implies $\sum_{k=n_0}^{\infty} \mu(I_k)=\infty$. On the other hand, $I_k \subset J$ for every k and hence $\sum_{k=n_0}^{\infty} \mu(I_k) \leqq \mu(J) < \infty$, which is a contradiction.

4.2. THEOREM. *If μ is a locally finite and continuous Borel measure and b equals any of the values $2^{1/k}$ $(k=1, 2, \ldots)$ then $\mu(C_b(\mu))=0$.*

PROOF. If μ is $2^{1/k}$-concentrated at x then μ is also 2-concentrated at x, since

$$\limsup_{h\to 0+} \frac{\mu(I(x,2h))}{\mu(I(x,h))} = \limsup_{h\to 0+} \prod_{j=1}^{k} \frac{\mu(I(x,2^{j/k}h))}{\mu(I(x,2^{(j-1)/k}h)} \leqq$$

$$\leqq \left(\limsup_{h\to 0+} \frac{\mu(I(x,2^{1/k}h))}{\mu(I(x,h))}\right)^k < (2^{1/k})^k = 2.$$

Therefore $C_{2^{1/k}}(\mu) \subset C_2(\mu)$ for every $k=1, 2, \ldots$ and thus it is enough to prove $\mu(C_2(\mu))=0$. Let $1<c<2$, $M>0$ and $h_0>0$ be fixed and let

$$H = \left\{x \in \operatorname{supp}(\mu)\colon \frac{\mu(I(x,2h))}{\mu(I(x,h))} \leqq c \text{ for every } 0<h<h_0\right\} \cap [-M, M].$$

Since $C_2(\mu)$ is a countable union of sets of this form, it is enough to show that $\mu(H)=0$.

Since μ is continuous, H is closed. Thus there is a countable set D and a perfect set P such that $H=D\cup P$. Since $\mu(D)=0$ by the continuity of μ, we only have to prove that $\mu(P)=0$. We may assume $P\neq\emptyset$. By 2.6, there is a positive number $p>0$ such that $p(H;x)>p$ holds for every $x\in H$. Since $P\subset H$, the same is true for P that is, condition (i) of Lemma 4.1 is satisfied.

Let N be a positive integer such that $2^N > \frac{1}{p}+\frac{1}{2}$. Let $I_n=(a_n, b_n)$ $(n=1, 2, \ldots)$ denote the bounded intervals contiguous to P, let $h_n=b_n-a_n$, and let n_0 be so large that $h_n<h_0 2^{-N}$ holds for every $n\geqq n_0$. We shall prove that (ii) of Lemma 4.1 is satisfied with $K=c^{N+1}/(2-c)$.

Let $n\geqq n_0$ be fixed. Since $a_n, b_n \in H$ and $h_n<h_0$, we have

$$\mu(I(a_n, 2h_n)) \leqq c\mu(I(a_n, h_n))$$

and

$$\mu(I(b_n, 2h_n)) \leqq c\mu(I(b_n, h_n)).$$

Therefore

$$2\mu(I(a_n,h_n))+2\mu((b_n, b_n+h_n)) \leqq \mu(I(a_n,2h_n))+\mu(I(b_n,2h_n)) \leqq$$

$$\leqq c\mu(I(a_n,h_n))+c\mu(I(b_n,h_n)) \leqq c\mu(I(a_n,h_n))+2\mu((b_n,b_n+h_n))+c\mu(I_n)$$

and hence we have

$$\mu(I(a_n, h_n)) \leqq \frac{c}{2-c}\mu(I_n).$$

Since $a_n \in H$ and $2^N h_n < h_0$, we also have

$$\mu(I(a_n, 2^N h_n)) \leqq c^N \mu(I(a_n, h_n)) \leqq \frac{c^{N+1}}{2-c}\mu(I_n).$$

Finally, by the choice of N, we get $\frac{2}{p} I_n \subset I(a_n, 2^N h_n)$ and hence

$$\mu\left(\frac{2}{p} I_n\right) \leqq \frac{c^{N+1}}{2-c}\mu(I_n).$$

Thus (ii) of 4.1 is satisfied. Then, according to that Lemma, $\mu(P)=0$ and this is what we wanted to prove.

4.3. Remark. The condition of continuity cannot be removed from 4.2. Indeed, if $\mu(\{x\})>0$ then, obviously, $x \in C_b(\mu)$ for every $b>1$ and hence $\mu(C_b(\mu))>0$.

4.4. Corollary. *If μ is a continuous locally finite b-concentrated measure with $b=2^{1/k}$ $(k=1, 2, \ldots)$ then $\mu \equiv 0$.*

5. Some remarks on the range of symmetric derivatives. Let μ be a continuous non-vanishing b-concentrated measure and let f be the distribution function of μ. It follows from 2.1(ii) that $f_s'(x)=\infty$ holds for every $x \in \operatorname{supp}(\mu)$. (For example, by 3.1(i), the Cantor measure is b-concentrated for $b \geqq 81$ and hence the symmetric derivative of the Cantor function is $+\infty$ at each point of the Cantor set. This fact can also be proved directly; see [6].)

Since $f'(x)=0$ if $x \notin \operatorname{supp}(\mu)$, f is a continuous function with the property that f_s' exists everywhere and its range is $\{0, \infty\}$.

The analogous phenomenon cannot happen for ordinary derivatives. In fact, if f is continuous and f' exists everywhere (finite or infinite) then f' is Darboux. Hence the range of f' is an interval unless f is constant. In particular, the range of f' cannot be $\{0, \infty\}$ for a continuous f. (For discontinuous f it can, as $f(x)=\operatorname{sgn} x$ shows.)

The range of the symmetric derivative of a continuous function may consist of three finite values: consider, for example, $f(x)=|x|$.

5.1. Theorem. *Suppose that f_s' exists everywhere. Then the range of f_s' cannot consist of two finite values.*

Proof. Suppose that the statement is not true and let f be symmetrically differentiable with, say,

$$\{f_s'(x) : x \in \mathbf{R}\} = \{0, 1\}.$$

Then f is measurable by a theorem of Charzyński [2]. Thus $f_s'(x) \geqq 0$ implies that there exists an increasing and continuous g such that $g_s'(x)=f_s'(x)$ everywhere ([3], Theorem 4). Since $g_s'(x) \leqq 1$ everywhere, $h(x)=x-g(x)$ is increasing ([3], Theorem 3) and hence g is Lipschitz. Consequently, g is absolutely continuous and

$g'(x)=0$ or $g'(x)=1$ holds for a.e. x. Since g is not linear, $\lambda(\{x: g'(x)=0\})>0$ and $\lambda(\{x: g'(x)=1\})>0$. Then the sets $A=\{x: g'_s(x)=0\}$ and $B=\{x: g'_s(x)=1\}$ are disjoint measurable sets of positive measure such that $A\cup B=\mathbf{R}$. Since $\mathbf{R}$ is connected in the density topology, it follows that there is a point x at which both A and B have positive upper density. This easily implies that

$$\limsup_{h\to 0} \frac{g(x+h)-g(x-h)}{2h} > 0$$

and

$$\liminf_{h\to 0} \frac{g(x+h)-g(x-h)}{2h} < 1.$$

Hence $g'_s(x)\notin\{0, 1\}$ which contradicts $g'_s=f'_s$ and $f'_s(x)\in\{0, 1\}$.

6. An application to generalized Riemann derivatives. In this section we prove that for certain systems a_i, b_i, there is a continuous function f such that $D^1f(x)\leqq 0$ holds everywhere and f is not increasing, moreover, f is non-constant and decreasing.

6.1. THEOREM. *Let μ be a locally finite Borel measure, let $b>c>1$ and suppose that*

$$\limsup_{h\to 0+} \frac{\mu(I(x, bh))}{\mu(I(x, h))} \leqq c \tag{9}$$

holds for every $x\in\operatorname{supp}(\mu)$. Let f denote the distribution function of μ.

If the positive numbers u_1, u_2, v satisfy

$$c < \frac{u_2}{u_1} < b, \quad bu_1-u_2 = \frac{1}{2v}$$

and the system a_i, b_i $(i=1, 2, 3, 4)$ is defined by

$$\begin{cases} a_1 = -u_1, & a_2 = u_2, \quad a_3 = -u_2, \quad a_4 = u_1 \\ b_1 = -bv, & b_2 = -v, \quad b_3 = v, \quad\quad b_4 = bv \end{cases} \tag{10}$$

then $D^1f(x)\leqq 0$ holds everywhere for the corresponding GRD. More exactly,

$$D^1 f(x) = 0 \qquad \text{if} \quad x\notin\operatorname{supp}(\mu)$$

and

$$D^1 f(x) = -\infty \quad \text{if} \quad x\in\operatorname{supp}(\mu).$$

PROOF. The second condition on the numbers u_1, u_2, v implies that $\sum_{i=1}^{4} a_i=0$, $\sum_{i=1}^{4} a_ib_i=1$ and hence the system (a_i, b_i) defines a GRD. Since $a_{5-i}=-a_i$, $b_{5-i}=-b_i$ $(i=1, 2, 3, 4)$, the function

$$F(x, h) = \frac{1}{h}\sum_{i=1}^{4} a_i f(x+b_i h)$$

is even in h and hence

$$\lim_{h\to 0-} F(x,h) = \lim_{h\to 0+} F(x,h).$$

If $x\notin \operatorname{supp}(\mu)$ then obviously $\lim_{h\to 0} F(x,h)=0$. If $x\in \operatorname{supp}(\mu)$ and $h>0$ then

$$F(x,h) = \frac{1}{h}u_1(f(x+bvh)-f(x-bvh))-\frac{1}{h}u_2(f(x+vh)-f(x-vh)) =$$

$$= \frac{1}{h}u_1\mu(I(x,bvh))-\frac{1}{h}u_2\mu(I(x,vh)) =$$

$$= 2u_1v\,\frac{\mu(I(x,vh))}{2vh}\left[\frac{\mu(I(x,bvh))}{\mu(I(x,vh))}-\frac{u_2}{u_1}\right] \overset{\text{def}}{=} 2u_1v\cdot A(x,h)\cdot B(x,h).$$

By (9), $\limsup_{h\to 0+} B(x,h)\leqq c-\frac{u_2}{u_1}<0$. By 2.1 (ii), $\lim_{h\to 0+} A(x,h)=\infty$. Hence $D^1 f(x)= =\lim_{h\to 0} F(x,h)=-\infty$.

6.2. Corollary. *Let f be the Cantor function. Then there is a GRD such that*

$$D^1 f(x) = -\infty$$

at each point of the Cantor set. For example, such a GRD is defined by the system

$$\begin{cases} a_1=-1, & a_2=99, \quad a_3=-99, \quad a_4=1 \\ b_1=-50, & b_2=-\frac{1}{2}, \quad b_3=\frac{1}{2}, \quad b_4=50. \end{cases}$$

Proof. As we proved in 3.1, the Cantor measure is b-concentrated for, say, $b=100$. More exactly, (6) shows that the Cantor measure satisfies condition (9) with $c=80$ and $b=100$. Therefore we can apply the previous theorem with $u_1=1$, $u_2=99$, $v=\frac{1}{2}$, $b=100$.

6.3. Remark. Let f denote the Cantor function. As we saw before, $f_s'(x)=\infty$ and $D^1 f(x)=-\infty$ hold simultaneously at the points of the Cantor set for some GRD's. This sounds rather paradoxical especially if we recall that whenever a function g has a finite derivative then necessarily $g_s'(x)=D^1 g(x)=g'(x)$.

6.4. Corollary. *Suppose that the positive numbers u_1, u_2, v, b satisfy*

$$2<\frac{u_2}{u_1}<b,\quad bu_1-u_2=\frac{1}{2v}$$

and let the system (a_i, b_i) $(i=1,2,3,4)$ be defined by (10). Then there is a continuous, non-constant and increasing function f such that $D^1 f(x)\leqq 0$ holds everywhere for the corresponding GRD.

Proof. This is an immediate consequence of 3.2 and 6.1.

6.5. REMARK. The GRD given in the previous Corollary has four terms. It was shown in [4] that there are GRD's with three terms and continuous functions f such that $D^1 f \geqq 0$ everywhere and f is not increasing. On the other hand, it is easy to show that if f is continuous and $D^1 f \geqq 0$ holds for a two-term GRD then f is increasing.

References

[1] J. M. Ash, Very generalized Riemann derivatives, generalized Riemann derivatives and associated summability methods, *Real Analysis Exchange*, **11** (1985—86), 10—29.
[2] M. Charzyński, Sur les fonctions dont la dérivée symétrique est partout finie, *Fund. Math.*, **21** (1933), 214—225.
[3] M. J. Evans and P. D. Humke, Parametric differentiation, *Colloq. Math.*, **45** (1981), 125—131.
[4] P. D. Humke and M. Laczkovich, Monotonicity theorems for generalized Riemann derivatives, *Rendiconti Circ. Mat. Palermo*, **38** (1989), 437—454.
[5] S. Saks, *Theory of the Integral*, Dover (1964).
[6] J. Uher, Some remarks on symmetric derivative, *Real Analysis Exchange*, **13** (1987—88), 35—38.

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EÖTVÖS LORÁND UNIVERSITY
DEPARTMENT OF ANALYSIS
H—1088 BUDAPEST
MÚZEUM KRT. 6—8
HUNGARY

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ON HERMITE—FEJÉR INTERPOLATION OF HIGHER ORDER

T. HERMANN (Budapest)*

1. Introduction and main result

Let $X=\{\{x_{kn}\}_{k=1}^{n}\}_{n=1}^{\infty}$ be an infinite triangular interpolatory matrix where

$$-1 \leqq x_{nn} < x_{n,n-1} < \ldots < x_{n1} \leqq 1. \tag{1.1}$$

We consider the unique interpolatory polynomials $H_{nm}(f, X, x)$ of degree $\leqq mn-1$ for $f \in C[-1, 1]$ defined as follows:

$$\begin{cases} H_{nm}(f, X, x_{nk}) = f(x_{nk}), & 1 \leqq k \leqq n, \\ H_{nm}^{(i)}(f, X, x_{nk}) = 0, & 1 \leqq k \leqq n, \ 1 \leqq i \leqq m-1. \end{cases} \tag{1.2}$$

When $m=2$ we obtain the well-known Hermite—Fejér polynomials investigated in many paper. If $X=X^{(\alpha,\beta)}$ i.e. when the nodes (1.1) are the roots of the n-th Jacobi polynomial $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$ and m is a positive, even number then P. Vértesi proved the following result:

THEOREM A ([14], [15]). *Let* $-1 < A < 1$, $\alpha, \beta > -1$ *and m be a fixed even natural number. Then*

$$\lim_{n \to \infty} \|H_{nm}(f, X^{(\alpha,\beta)}) - f\|_{[A,1]} = 0 \tag{1.3}$$

for arbitrary $f \in C[-1, 1]$ *if*

$$\alpha \in [-0.5 - 2/m, \ -0.5 + 1/m), \quad \beta \geqq -0.5 - 2/m, \quad \alpha - \beta \leqq 2/m.$$

One can ask: what can we say about (1.3) for other α and β? If $m=2$ and 4 then it is known ([7], [13]) that there are continuous functions such that (1.3) does not hold for them. So in this case the question is: for which continuous functions will (1.3) be true? This question was investigated by several authors when $m=2$ ([1], [2], [3], [5], [7], [8], [10], [11]). We cite the following theorem of Vértesi:

THEOREM B ([11]). *Suppose* $-1 < A < 1$, $\alpha \in [p-1, p)$ *when p is a positive integer. Then for any fixed* $f \in C[-1, 1]$

$$\lim_{n \to \infty} \|H_{n2}(f, X^{(\alpha,\beta)}) - f\|_{[A,1]} = 0$$

if and only if

$$\lim_{n \to \infty} H_{2n}(f, X^{(\alpha,\beta)}, 1) = f(1),$$

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moreover if $\alpha \geqq 1$

$$[H_{n2}(f, X^{(\alpha,\beta)}, x)]_{x=1}^{(r)} = o(n^{2r}) \quad (r = 1, 2, \ldots, p-1).$$

The main result of this paper is the following

THEOREM 1.1. *Let m be an even, positive, fixed integer, $-1<A<1$ real, $\alpha, \beta > -0.5-2/m$ arbitrary, $f \in C[-1, 1]$. Then*

$$\lim_{n\to\infty} \|H_{nm}(f, X^{(\alpha,\beta)})-f\|_{[A,1]} = 0 \tag{1.4}$$

if and only if

$$[H_{nm}(f, X^{(\alpha,\beta)}, x)-f(1)]_{x=1}^{(i)} = o(n^{2i}) \quad (i = 0, 1, \ldots, p-1) \tag{1.5}$$

(if $p=0$ then we omit this condition) where p is an integer such that

$$m(\alpha+0.5)/2 \leqq p < m(\alpha+0.5)/2+1, \tag{1.6}$$

REMARK. The proof of Theorem 1.1 follows the ideas used in the proof of Theorem B.

2. Quasi Hermite—Fejér interpolation of higher order

To prove Theorem 1.1 we need another operator which is interesting in itself. Let us consider the uniquely determined polynomial $h_{pq}(x)=h_{nmpq}(f, D, E, X^{(\alpha,\beta)}, x)$ of degree $\leqq N=mn+p+q-1$ (where $p, q \geqq 0$ are integers, $f \in C[-1, 1]$) such that

$$h_{pq}(x_k) = f(x_k) \qquad (k = 1, \ldots, n),$$

$$h_{pq}^{(i)}(x_k) = 0 \qquad (k = 1, \ldots, n;\ i = 1, 2, \ldots, m-1),$$

$$h_{pq}^{(i)}(1) = d_i \qquad (i = 0, \ldots, p-1;\ d_0 \equiv f(1);\ \text{if } p = 0 \text{ we omit this condition}),$$

$$h_{pq}^{(i)}(-1) = e_i \qquad (i = 0, \ldots, q-1;\ e_0 \equiv f(-1), \text{if } q = 0 \text{ we omit this condition}).$$

Let us introduce the notaions $\alpha(p):=\alpha-2p/m$, $\beta(q):=\beta-2q/m$. About the convergence of the h_{pq} process, what we call quasi Hermite—Fejér polynomial of higher order, we state the following

THEOREM 2.1. *Let $-1<A<1$ be arbitrary. If either*

$$-0.5 \leqq \alpha(p) < 1/m-0.5, \qquad \alpha(p)-\beta(q) \leqq 2/m$$

or

$$-2/m-0.5 \leqq \alpha(p) \leqq -0.5, \qquad -2/m-0.5 \leqq \beta(q)$$

then

$$\lim_{n\to\infty} \|h_{pq}(f)-f\|_{[A,1]} = 0 \quad \textit{for every} \quad f \in C[-1, 1] \tag{2.1}$$

whenever

$$|d_i| = o(n^{2i}) \quad (i = 1, 2, \ldots, p-1) \tag{2.2}$$

and

$$|e_i| = o(n^{2i}) \quad (i = 1, 2, \ldots, q-1). \tag{2.3}$$

REMARK. If $p=q=0$ then Theorem 2.1 reduces to Theorem A.

COROLLARY. *If*

$$-2/m-0.5 \leqq \alpha(p) < 1/m-0.5, \quad \textit{and} \quad -2/m-0.5 \leqq \beta$$

then

$$\lim_{n\to\infty} \|h_{p0}(f)-f\|_{[-1,1]} = 0 \quad \textit{for every } f\in C[1, 1]$$

is true whenever (2.2) *is fulfilled.*

3. Proof of Theorem 2.1

In the following we shall use the following formulae without further reference:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x), \quad P_n(1) \sim n^\alpha.$$

$$|P_n(x)| \sim |t-t_j|\, t_j^{-\alpha-1/2} n^{1/2} \leqq cn^\alpha j^{-\alpha-1/2}, \quad \text{if} \quad t\in[0, \pi-\mu],$$

$$|P_n'(x_k)| \sim k^{-\alpha-3/2} n^{\alpha+2}, \quad \text{if} \quad t_k\in[0, \pi-\mu],$$

$$t_k \sim k/n, \quad (k = 1, 2, ..., n),$$

$$t_{k+1}-t_k \sim 1/n \quad (k = 0, 1, ..., n;\ t_0 = 0,\ t_{n+1} = \pi),$$

$$|x-x_k| \sim (j+k)(|j-k|+1)/n^2, \quad (j,\ k = 1, 2, ..., n).$$

Here $P_n(x)=P_n^{(\alpha,\beta)}(x)$, $x=\cos t$; $x_j=\cos t_j$ is the nearest node to x (for a fixed n) and $0<\mu<\pi$ is arbitrary fixed. The symbol "$\sim$" ([9, 1.1]) does not depend on t, k and n. (See [9, (4.1.3), (8.9.2)], [4], [12, Lemma 3.2].) Here and later $c, c_1, c_2, ...$ are different positive constants. Let us introduce the following notations:

$$I(n, k, s) := \begin{cases} (n/\sin t_{kn})^s & \text{if} \quad s = 0, 2, 4, ... \\ n^{s-1}/\sin^{s+1} t_{kn} & \text{if} \quad s = 1, 3, 5, ..., \end{cases}$$

$$a = m(\alpha(p)+0.5), \quad b = m(\beta(q)+0.5),$$

$$K = \max\{k\colon\ t_k\in[0, \pi-\mu]\},$$

$$\textstyle\sum_1 = \sum\limits_{1\leqq k\leqq j/2}, \quad \sum_2 = \sum\limits_{j/2<k\leqq 3j/2}, \quad \sum_3 = \sum\limits_{3j/2\leqq k\leqq K}, \quad \sum_4 = \sum\limits_{K<k\leqq n}.$$

Let $h(x)=h_{mnpq}(C, D, E, X, x)$ be that unique polynomial of degree $\leqq mn+p+$ $+q-1$ for which

$$h^{(i)}(x_k) = c_{ki} \quad (k = 1, 2, ..., n;\ i = 0, 1, ..., m-1),$$

$$h^{(i)}(1) = d_i \quad (i = 0, 1, ..., p-1),$$

$$h^{(i)}(-1) = e_i \quad (i = 0, 1, ..., q-1).$$

Obviously we have the following representation for $h(x)$:

$$h(x) = \sum_{i=0}^{m-1}\sum_{k=1}^{n} c_{ki} C_{ki}(x) + \sum_{i=0}^{p-1} d_i D_i(x) + \sum_{i=0}^{q-1} e_i E_i(x)$$

where

$$C_{ki}(x)=\left(\frac{x-1}{x_k-1}\right)^p\left(\frac{x+1}{x_k+1}\right)^q\frac{(x-x_k)^i}{i!}\,l_k^m(x)\sum_{r=0}^{m-1-i}\gamma_{kr}(x-x_k)^r/r!,$$

$$D_i(x)=\left(\frac{P(x)}{P(1)}\right)^m\left(\frac{1+x}{2}\right)^q\frac{(x-1)^i}{i!}\sum_{r=0}^{p-1-i}\delta_r(x-1)^r/r!,$$

$$E_i(x)=\left(\frac{P(x)}{P(-1)}\right)^m\left(\frac{1-x}{2}\right)^p\frac{(x+1)^i}{i!}\sum_{r=0}^{q-1-i}\varepsilon_r(x+1)^r/r!.$$

We shall prove our theorems by a series of lemmas. In the following we suppose that $t\in[0,\pi-2\mu]$.

LEMMA 1 ([14, Lemma 3.8]). *If n and M are fixed then*

$$\left|\left(l_k^m(x)\right)^{(s)}\right|=O\left(I(n,k,s)\right),\quad 0\leq s\leq M.$$

LEMMA 2.

$$\left|\left\{\left(\frac{x-1}{x_k-1}\right)^p\left(\frac{x+1}{x_k+1}\right)^q\right\}^{(s)}_{x=x_k}\right|=O((1-x_k^2)^{-s}).$$

PROOF.

$$\left|\left\{\left(\frac{x-1}{x_k-1}\right)^p\left(\frac{x+1}{x_k+1}\right)^q\right\}^{(s)}_{x=x_k}\right|=\left|\sum_{i=1}^{s}\binom{s}{i}\binom{p}{i}\frac{i!}{(1-x_k)^i}\binom{q}{s-i}\frac{(s-i)!}{(1+x_k)^{s-i}}\right|=$$

$$=O\left(\frac{1}{1-x_k}+\frac{1}{1+x_k}\right)^s=O((1-x_k^2)^{-s}).$$

LEMMA 3. $|\gamma_{ks}|=O\left(I(n,k,s)\right)$.

PROOF. We prove by induction. $\gamma_{k0}=1$.

$$|\gamma_{ks}|=\left|\sum_{i=0}^{s-1}\binom{s}{i}\left\{\left(\frac{x-1}{x_k-1}\right)^p\left(\frac{x+1}{x_k+1}\right)^q l_k(x)^m\right\}^{(s-i)}_{x=x_k}\gamma_{ki}\right|=$$

$$=O\Big(\sum_{i=0}^{s-1}I(n,k,s-i)\,I(n,k,i)\Big)=O\left(I(n,k,s)\right).$$

LEMMA 4 ([11, (4.5)]). $(P_m(x)/P_m(1))^{(s)}=O(n^{2s})$.

LEMMA 5. $|\delta_s|=O(n^{2s})$ $(s=0,1,\dots,p-1)$.

PROOF. We prove by induction. $\delta_0=1$.

$$|\delta_s|=\left|\sum_{i=0}^{s-1}\binom{s}{i}\left\{((P(x)/P(1))^m\left(\frac{x+1}{2}\right)^q\right\}^{(s-i)}\delta_i\right|=$$

$$=O\Big(\sum_{i=0}^{s-1}\sum_{l=0}^{s-i}(P(x)^m/P(1)^m)^{(l)}\,n^{2i}\Big)=O\Big(\sum_{i=0}^{s-1}n^{2s-2i}\,n^{2i}\Big)=O(n^{2s}).$$

LEMMA 6. $|\varepsilon_s|=O(n^{2s})$ $(s=0, 1, ..., q-1)$.

PROOF. Analogous to the proof of Lemma 5.

LEMMA 7.

$$|l_k(x)| = \begin{cases} O\left(\dfrac{k^{\alpha+3/2}}{j^{\alpha+1/2}(j+k)(|j-k|+1)}\right) & \text{if } k\in[1, K], \\ O\left(n^{\alpha-\beta-2}(n+1-k)^{\beta+3/2}\, j^{-\alpha-1/2}\right) & \text{if } k\in[K, n]. \end{cases}$$

PROOF. It follows from direct computation.

LEMMA 8.

$$\frac{1-x}{1-x_k} \sim \begin{cases} (j/k)^2 & \text{if } k\in[1, K] \\ (j/n)^2 & \text{if } k\in[K, n], \end{cases}$$

$$\frac{1+x}{1+x_k} \sim \begin{cases} 1 & \text{if } k\in[1, K], \\ (n/(n+1-k))^2 & \text{if } k\in[K, n]. \end{cases}$$

PROOF. It follows immediately from the above estimations.

LEMMA 9.

$$\sum_{r=0}^{m-2} |\gamma_{kr}|\,|x-x_k|^r = \begin{cases} O\left(((j+k)(|j-k|+1)/k)^{m-2}\right) & \text{if } 1 \leqq k \leqq K, \\ O\left(n^{2m-4}(n+1-k)^{2-m}\right) & \text{if } K < k \leqq n, \end{cases}$$

$$\sum_{r=0}^{m-1} |\gamma_{kr}|\,|x-x_k|^r = \begin{cases} O\left(\left(\dfrac{(j+k)(|j-k|+1)}{k}\right)^{m-2} \left\{1+\dfrac{(j+k)(|j-k|+1)}{k^2}\right\}\right) & \text{if } 1 \leqq k \leqq K, \\ O\left(n^{2m-2}(n+1-k)^{-m}\right) & \text{if } K < k \leqq n. \end{cases}$$

PROOF. It follows immediately from Lemma 3.

LEMMA 10. *Let α and β be such that either*

$$0 \leqq a < 1 \quad \text{and} \quad a-b \leqq 2$$

or

$$-2 \leqq a \leqq 0 \quad \text{and} \quad -2 \leqq b.$$

Then

$$\sum_{k=1}^{n} |C_{k0}(x)| = O(1)$$

and

$$\sum_{k=1}^{n} \sum_{i=1}^{m-1} |C_{ki}(x)| = o(1).$$

PROOF. Using the previous estimations we get

$$\sum_{k=1}^{K} |C_{k0}(x)| =$$

$$= O\Big(\sum_{k=1}^{K} (j/k)^{2p} \frac{k^{m(\alpha+1/2)}}{j^{m(\alpha+1/2)}(j+k)^m(|j-k|+1)^m}\left(\frac{(j+k)(|j-k|+1)}{k}\right)^{m-2} \times$$

$$\times\left(1+\frac{(j+k)(|j-k|+1)}{k^2}\right)\Big) =$$

$$= O\left(j^{-a}\sum_{k=1}^{K}\frac{k^{a+2}}{(j+k)^2(|j-k|+1)^2}+\frac{k^a}{(j+k)(|j-k|+1)}\right).$$

So

$$\textstyle\sum_1 |C_{k0}(x)| = O(j^{-a-4}\sum_1 k^{a+2}+j^{-a-2}\sum_1 k^a) = O(1),$$

$$\textstyle\sum_2 |C_{k0}(x)| = O(\sum_2 (|j-k|+1)^{-2}+j^{-1}\sum_1 (|j-k|+1)^{-1}) = O(1),$$

$$\textstyle\sum_3 |C_{k0}(x)| = O(j^{-a}\sum_3 k^{a-2}) = O(1)$$

and

$$\textstyle\sum_4 |C_{k0}(x)| = O\Big(\sum_4 (j/n)^{2p}\left(\frac{n}{n+1-k}\right)^{2q}\frac{n^{m(\alpha-\beta-2)}(n+1-k)^{m(\beta+3/2)}}{j^{m(\alpha+1/2)}}\frac{n^{2m-2}}{(n+1-k)^m}\Big) =$$

$$\textstyle = O(j^{-a}n^{a-b-2}\sum_4 (n+1-k)^b) = O(1).$$

Similarly

$$\sum_{k=1}^{n}\sum_{i=1}^{m-1} |C_{ki}(x)| =$$

$$= O\left(\sum_{k=1}^{n}\left(\frac{1-x}{1-x_k}\right)^p\left(\frac{1+x}{1+x_k}\right)^q l_k^m(x)\sum_{i=1}^{m-1}\frac{|x-x_k|^i}{i!}\sum_{r=0}^{m-1-i}|\gamma_{kr}|\,|x-x_k|^r/r!\right) =$$

$$= O\left(\sum_{k=1}^{n}\left(\frac{1-x}{1-x_k}\right)^p\left(\frac{1+x}{1+x_k}\right)^q l_k^m(x)\sum_{r=0}^{m-2}|\gamma_{kr}|\,|x-x_k|^r\right).$$

Further computations are analogous to the above one so we omit them.

LEMMA 11. *If* $a \geqq -2$ *then*

$$|D_i(x)| = O(n^{-2i}) \quad (i = 0, 1, 2, \ldots, p-1).$$

PROOF.

$$|D_i(x)| = O\Big(j^{-m(\alpha+1/2)}\,|j/n|^{2i}\sum_{r=0}^{p-i-1}(n^2(j/n)^2)^r\Big) = O(j^{-a-2}n^{2i}).$$

LEMMA 12. *Under the conditions of Lemma* 10

$$|E_i(x)| = O(n^{-2i}) \quad (i = 0, 1, 2, \ldots q-1).$$

PROOF.

$$|E_i(x)| = O(n^{\alpha-\beta}j^{-\alpha-1/2})^m\,|j/n|^{2p}\sum_{r=0}^{q-i-1}n^{2r} = O(n^{a-b-2-i}j^{-a}).$$

LEMMA 13. *If the conditions*

$$\sum_{k=1}^{n} |C_{k0}(x)|+|D_0(x)|+|E_0(x)| = O(1), \quad \sum_{k=1}^{n}\sum_{i=1}^{m-1} |C_{ki}(x)| = o(1),$$

$$\sum_{i=1}^{p-1} (1+|d_i|)\,|D_i(x)| = o(1), \quad \sum_{i=1}^{q-1} (1+|e_i|)\,|E_i(x)| = o(1),$$

uniformly hold in $[u, v]\subseteq[-1, 1]$ *then*

$$\lim_{n\to\infty} \|f-h_{pq}(f)\|_{[u,v]} = 0 \quad \text{if} \quad f\in C[-1, 1].$$

PROOF. Analogous to the proof of [11, Theorem 3.1].

PROOF OF THEOREM 2.1. It follows immediately from Lemmas 10—13.

4. Proof of Theorem 1.1

LEMMA 14 ([11]). *If* $\{p_n\}_{n=1}^{\infty}$ *is a sequence of polynomials such that*

$$\deg p_n \leqq n \quad (n = 1, 2, \ldots)$$

and

$$\lim_{n\to\infty} \|f-p_n\| = 0 \quad (f\in C[-1, 1])$$

then for every fixed positive integer r

$$|p_n^{(r)}(x)| = o(n^{2r}), \quad x\in[-1, 1].$$

PROOF OF THEOREM 1.1. The necessity of (1.5) follows directly from Lemma 14. Now we prove that (1.5) is sufficient. It is simple to verify the formulae

$$h_{n,m,r+1,0}(x)-h_{nmr0}(x) = (P(x)/P(1))^m[d_r-h_{nmr0}^{(r)}(1)](x-1)^r/r!$$

whence for any fixed $r\geqq 0$

$$h_{nmr0}(x) = H_{nm}(x)+(P(x)/P(1))^m \sum_{i=0}^{r-1} [d_i-h_{nmi0}^{(i)}(1)](x-1)^i/i!. \tag{4.1}$$

Let us define d_i recursively by

$$d_i := h_{mni0}^{(i)}(f, \{d_0, \ldots, d_{i-1}\}, E, X^{(\alpha,\beta)}, 1) \quad (i = 1, 2, \ldots p-1).$$

With this choice of D, (4.1) reduces to

$$h_{nmp0}(f, x) = H_{nm}(f, x).$$

Because of the Corollary to Theorem 2.1 if we can prove that d_i satisfy (2.2) then we are ready. We shall prove it by induction. For d_0 it is obvious. We defined e_i as $d_i= =h_{mnp0}^{(i)}(1)$ so by (4.1) when $r=i$, $d_i=H_{nm}^{(i)}(1)$ hence by (1.5) $d_i=o(n^{2i})$. Q.E.D.

References

[1] E. Egerváry and P. Turán, Notes on interpolation. V, *Acta Math. Acad. Sci. Hung.*, **9** (1958), 259—267.

[2] L. Fejér, Über Interpolation, *Göttinger Nachrichten* (1916), 66—91.

[3] G. Freud, On Hermite—Fejér interpolation sequences, *Acta Math. Acad. Sci. Hung.*, **23** (1972), 175—178.

[4] G. I. Natanson, Two-sided estimates for the Lebesgue function etc., *Izv. Vysš. Učebn. Zaved. Matematika*, **11** (1967), 67—74 (in Russian).

[5] A. Schönhage, Zur Konvergenz der Stufenpolynome über die Nullstellen der Legendre Polynome, *Proc. Conf. on Abstract Spaces and Approximation*, Birhäuser-Verlag (Oberwohlfach, 1971).

[6] S. B. Stečkin, Generalization of some Bernstein type inequalities, *Dokl. Akad. Nauk SSSR*. **60** (1948), 1511—1514 (in Russian).

[7] J. Szabados, On Hermite—Fejér interpolation for the Jacobi abscissas, *Acta Math. Acad. Sci. Hung.*, **23** (1972), 449—464.

[8] J. Szabados, *Convergence and saturation problems of approximation processes*, Thesis, 1975 (in Hungarian).

[9] G. Szegő, *Orthogonal polynomials*, AMS Coll. Publ., 23, AMS (Providence, Rhode Island, 1975).

[10] P. Vértesi, Notes on the Hermite—Fejér interpolation based on the Jacobi abscissas, *Acta Math. Acad. Sci. Hung.*, **24** (1973), 233—239.

[11] P. Vértesi, Hermite—Fejér type interpolations. IV, *Acta Math. Acad. Sci. Hung.*, **39** (1982), 83—93.

[12] P. Vértesi, On Lagrange interpolation, *Period. Math. Hungar.*, **12** (1981), 103—112.

[13] P. Vértesi, Krylov—Stayermann polynomials on the Jacobi roots, *Studia Sci. Math. Hung.*, **22** (1987), 239—245.

[14] P. Vértesi, Hermite—Fejér interpolations of higher order. I, *Acta Math. Hung.*, **54** (1989), 135—152.

[15] P. Vértesi, Hermite—Fejér interpolations of higher order. II, *Acta Math. Hung.*, **56** (1990), 369—379.

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COMPUTER AND AUTOMATON INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES
1111 BUDAPEST, KENDE U. 13—17

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ON THE INTEGRABILITY AND L^1-CONVERGENCE OF DOUBLE WALSH SERIES

F. MÓRICZ (Szeged) and F. SCHIPP (Budapest)*

1. Introduction

We consider the Walsh orthonormal system $\{w_j(x): j=0, 1, \ldots\}$ defined on the interval $I=[0, 1)$ in the Paley enumeration. That is, let

$$r_0(x) = 1 \quad \text{for} \quad 0 \leqq x < \frac{1}{2}, \quad r_0(x) = -1 \quad \text{for} \quad \frac{1}{2} \leqq x < 1,$$

$$r_0(x+1) = r_0(x), \quad r_j(x) = r_0(2^j x) \quad (j = 1, 2, \ldots)$$

be the well-known Rademacher functions. The Walsh functions $w_j(x)$ are then given by

$$w_0(x) = 1, \quad w_j(x) = \prod_{u=0}^{\infty} r_u^{j_u}(x)$$

where

$$j = \sum_{u=0}^{\infty} j_u 2^u \quad (j_u = 0 \text{ or } 1) \tag{1.1}$$

is the dyadic representation of the integer $j \geqq 1$. (See, e.g. [1] or [5].)

We will study the convergence properties of the double Walsh series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} w_j(x) w_k(y) \tag{1.2}$$

both pointwise and in $L^1(I^2)$-norm, where $\{a_{jk}: j, k=0, 1, \ldots\}$ is a double sequence of real numbers and $I^2=[0, 1)\times[0, 1)$. Throughout this paper we assume that $\{a_{jk}\}$ is a null sequence of bounded variation:

$$a_{jk} \to 0 \quad \text{as} \quad \max(j, k) \to \infty \tag{1.3}$$

and

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| < \infty \tag{1.4}$$

where

$$\Delta_{11} a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.$$

Besides, we will use the notations

$$\Delta_{10} a_{jk} = a_{jk} - a_{j+1,k} \quad \text{and} \quad \Delta_{01} a_{jk} = a_{jk} - a_{j,k+1}.$$

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* This research was completed while the authors were visiting professors at the University of Tennessee, Knoxville, during the academic year 1987/88.

We denote by

$$s_{mn}(x, y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} a_{jk} w_j(x) w_k(y) \quad (m, n = 1, 2, \ldots)$$

the rectangular partial sums of the series (1.2). A double summation by parts yields

$$(1.5) \qquad s_{mn}(x, y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk} +$$

$$+ \sum_{j=0}^{m-1} D_{j+1}(x) D_n(y) \Delta_{10} a_{jn} + \sum_{k=0}^{n-1} D_m(x) D_{k+1}(y) \Delta_{01} a_{mk} + a_{mn} D_m(x) D_n(y)$$

where

$$D_m(x) = \sum_{j=0}^{m-1} w_j(x) \quad (m = 1, 2, \ldots)$$

is the Walsh—Dirichlet kernel in terms of x. It is well-known (see, e.g. [1]) that

$$|D_m(x)| < \frac{2}{x} \quad (m = 1, 2, \ldots; \ 0 < x < 1).$$

It follows from (1.3) and (1.4) that for all $0<x, y<1$ the series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk}$$

converges absolutely and

$$a_{mn} D_m(x) D_n(y) \to 0 \quad \text{as} \quad \max(m, n) \to \infty.$$

By (1.3),

$$\Delta_{10} a_{jn} = \sum_{k=n}^{\infty} \Delta_{11} a_{jk}$$

whence by (1.4),

$$(1.6) \qquad \sum_{j=0}^{\infty} |\Delta_{10} a_{jn}| \leqq \sum_{j=0}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that for all $0<x, y<1$

$$\sum_{j=0}^{m-1} D_{j+1}(x) D_n(y) \Delta_{10} a_{jn} \to 0 \quad \text{as} \quad n \to \infty,$$

uniformly in m. Similarly, for all $0<x, y<1$

$$\sum_{k=0}^{n-1} D_m(x) D_{k+1}(y) \Delta_{01} a_{mk} \to 0 \quad \text{as} \quad m \to \infty,$$

uniformly in n. Combining these, we can conclude from (1.5) that the series (1.2) converges to the function $f(x, y)$ defined by

$$f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk} \tag{1.7}$$

for all $0<x, y<1$ in the sense that

$$s_{mn}(x, y) \to f(x, y) \quad \text{as} \quad \min(m, n) \to \infty.$$

2. Main results

We introduce new classes l_*^p of double sequences of real numbers, which are similar to the ordinary classes l^p.

DEFINITION 1. A double sequence $\mathscr{A}=\{a_{jk}\}$ is said to belong to the class l_*^p if in case $1 \leqq p<\infty$

$$\|\mathscr{A}\|_{p,*} = |a_{00}| + \sum_{m=0}^{\infty} 2^m \Big[2^{-m} \sum_{j=2^m}^{2^{m+1}-1} |a_{j0}|^p\Big]^{1/p} + \sum_{n=0}^{\infty} 2^n \Big[2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |a_{0k}|^p\Big]^{1/p} +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \Big[2^{-m-n} \sum_{j=2^m}^{2^{m+1}-1} \sum_{k=2^n}^{2^{n+1}-1} |a_{jk}|^p\Big]^{1/p} < \infty,$$

while in case $p=\infty$

$$\|\mathscr{A}\|_{\infty,*} = |a_{00}| + \sum_{m=0}^{\infty} 2^m \max_{2^m \leqq j < 2^{m+1}} |a_{j0}| + \sum_{n=0}^{\infty} 2^n \max_{2^n \leqq k < 2^{n+1}} |a_{0k}| +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \max_{2^m \leqq j < 2^{m+1}} \max_{2^n \leqq k < 2^{n+1}} |a_{jk}| < \infty.$$

By Hölder's inequality, for every $1<p<\infty$,

$$\|\mathscr{A}\|_1 = \|\mathscr{A}\|_{1,*} \leqq \|\mathscr{A}\|_{p,*} \leqq \|\mathscr{A}\|_{\infty,*} \tag{2.1}$$

where

$$\|\mathscr{A}\|_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |a_{jk}|$$

is the ordinary l^1-norm of $\mathscr{A}$. Consequently,

$$l_*^{\infty} \subset l_*^p \subset l_*^1 = l^1.$$

It is also easy to check that l_*^p is a Banach space with the norm $\|\cdot\|_{p,*}$ for each $1 \leqq p \leqq \infty$.

DEFINITION 2. Motivated by (1.5) we introduce the modified rectangular partial sums $u_{mn}(x, y)$ of series (1.2) as follows

$$u_{mn}(x, y) = s_{mn}(x, y) - \sum_{j=0}^{m-1} D_{j+1}(x) D_n(y) \Delta_{10} a_{jn} - \tag{2.2}$$

$$- \sum_{k=0}^{n-1} D_m(x) D_{k+1}(y) \Delta_{01} a_{mk} - a_{mn} D_m(x) D_n(y).$$

According to (1.5),

$$u_{mn}(x, y) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk}. \tag{2.3}$$

We note that analogous modified partial sums were introduced by the first named author [4] in the case of double cosine series.

In the sequel, let

$$\Delta\mathcal{A} = \{\Delta_{11} a_{jk} : j, k = 0, 1, \ldots\}.$$

Our main result is the following.

THEOREM 1. *If a double sequence* $\mathcal{A} = \{a_{jk}\}$ *is such that condition* (1.3) *is satisfied and* $\Delta\mathcal{A} \in l^p_*$ *for some* $p>1$, *then the series* (1.2) *converges for all* $0<x, y<1$, *its sum is the function* $f(x, y)$ *defined by* (1.7), *and*

$$\|u_{mn} - f\| \to 0 \quad as \quad \min(m, n) \to \infty. \tag{2.4}$$

Here $\|\cdot\|$ denotes $L^1(I^2)$-norm. Later on, in Theorem 2 and Lemma 3, $\|\cdot\|$ will denote $L^1(I)$-norm, too.

As a by-product of the proof of Theorem 1 in Section 4, we obtain that for all $m, n \geqq 1$ and $p>1$

$$\Big\|\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} D_{j+1}(x) D_{k+1}(y) \Delta_{11} a_{jk}\Big\| \leqq \left(\frac{2p}{p-1}\right)^2 \|\Delta\mathcal{A}\|_{p,*}. \tag{2.5}$$

We draw two corollaries of Theorem 1.

COROLLARY 1. *Under the conditions of Theorem* 1, *the sum* $f=f(x, y)$ *of the series* (1.2) *is integrable and* (1.2) *is the Walsh—Fourier series of* f.

COROLLARY 2. *If a double sequence* $\mathcal{A} = \{a_{jk}\}$ *is such that condition* (1.3) *is satisfied,* $\Delta\mathcal{A} \in l^p_*$ *for some* $p>1$, *and*

$$\|D_n\| \Big\{|\Delta_{10} a_{0n}| + \sum_{m=0}^{\infty} 2^m \Big[2^{-m} \sum_{j=2^m}^{2^{m+1}-1} |\Delta_{10} a_{jn}|^p\Big]^{1/p}\Big\} \to 0 \qquad as \quad n \to \infty, \tag{2.6}$$

$$\|D_m\| \Big\{|\Delta_{01} a_{m0}| + \sum_{n=0}^{\infty} 2^n \Big[2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\Delta_{01} a_{mk}|^p\Big]^{1/p}\Big\} \to 0 \qquad as \quad m \to \infty, \tag{2.7}$$

then

$$\|s_{mn} - f\| \to 0 \quad as \quad \min(m, n) \to \infty \tag{2.8}$$

if and only if

$$a_{mn} \|D_m\| \|D_n\| \to 0 \quad as \quad \min(m, n) \to \infty. \tag{2.9}$$

As is known [1],

$$\|D_m\| \leqq 1 + 2 \ln m \quad (m = 1, 2, \ldots). \tag{2.10}$$

Thus, if conditions (1.3), (2.6), (2.7) are satisfied and $\Delta\mathcal{A} \in l^p_*$ for some $p>1$, then

$$a_{mn} \ln(m+2) \ln(n+2) \to 0 \quad \text{as} \quad \min(m, n) \to \infty$$

is a sufficient condition for (2.8).

We make some preparations before formulating Theorem 2. Under conditions (1.3) and (1.4), we have

$$a_{j0} \to 0 \quad \text{as} \quad j \to \infty, \quad \text{and} \quad \sum_{j=0}^{\infty} |\Delta_{10} a_{j0}| < \infty.$$

A summation by parts gives that the first row of the series (1.2) (i.e. when $k=0$) converges for $0<x<1$:

$$\sum_{j=0}^{\infty} a_{j0} w_j(x) = f_1(x), \quad \text{say.} \tag{2.11}$$

Similarly, the first column of the series (1.2) (i.e. when $j=0$) converges for $0<y<1$:

$$\sum_{k=0}^{\infty} a_{0k} w_k(y) = f_2(y), \quad \text{say.} \tag{2.12}$$

We denote by $s_m^{(1)}(x)$ and $s_n^{(2)}(y)$ the partial sums of the series (2.11) and (2.12), respectively.

As is known, the double sequence $\{a_{jk}\}$ is said to be convex if

$$\Delta_{11} a_{jk} \geqq \max\{\Delta_{11} a_{j+1,k}, \Delta_{11} a_{j,k+1}\} \quad (j, k = 0, 1, \ldots). \tag{2.13}$$

We note that (1.3) and (2.13) imply that $\Delta_{11} a_{jk}$, $\Delta_{10} a_{jk}$, $\Delta_{01} a_{jk}$ and a_{jk} are neces sarily nonnegative. In particular, condition (1.4) is satisfied, and $\Delta_{10} a_{jk}$ and $\Delta_{01} a_{j}$- are monotone decreasing in j and k. This means, among others, that the single sequences $\{a_{j0}\}$ and $\{a_{0k}\}$ are convex.

THEOREM 2. *If a double sequence* $\{a_{jk}\}$ *is such that conditions* (1.3) *and* (2.13) *are satisfied, then the sum* $f=f(x, y)$ *of the series* (1.2) *is integrable,* (1.2) *is the Walsh—Fourier series of* f, *and*

$$\begin{cases} \|s_{mn}-f\| \to 0 & \text{as} \quad \min(m, n) \to \infty, \\ \|s_m^{(1)}-f_1\| \to 0 & \text{as} \quad m \to \infty, \\ \|s_n^{(2)}-f_2\| \to 0 & \text{as} \quad n \to \infty \end{cases} \tag{2.14}$$

if and only if

$$a_{mn} \ln(m+2) \ln(n+2) \to 0 \quad \text{as} \quad \max(m, n) \to \infty. \tag{2.15}$$

We observe that if conditions (1.3), (2.13), and (2.15) are satisfied, then each row and each column of the series (1.2) converges both pointwise (except possibly at $x=0$ or $y=0$, respectively) and in $L^1(I)$-norm. In addition, (2.14)$\Leftrightarrow$(2.15) can be reformulated as the equivalence of the regular convergence of the series (1.2) in $L^1(I^2)$-norm with the fulfillment of condition (2.15). (Concerning the notion of regular convergence, see [3].)

Theorem 2 is an extension of the corresponding theorems by Fomin [2], Siddiqi [7], and Yano [8] from one-dimensional to two-dimensional Walsh series.

3. Basic inequalities

The following inequality plays a key role in the proof of Theorem 1.

LEMMA 1. *For any* $1<p\leqq 2$, *double sequence* $\{b_{jk}\colon j,k=1,2,\ldots\}$ *of real numbers, and integers* $m,n\geqq 1$,

$$(3.1)\qquad I_{mn}=\Big\|\sum_{j=1}^{m}\sum_{k=1}^{n} b_{jk}D_j(x)D_k(y)\Big\|\leqq\left(\frac{2p}{p-1}\right)^2(mn)^{1-1/p}\,\Big[\sum_{j=1}^{m}\sum_{k=1}^{n}|b_{jk}|^p\Big]^{1/p}.$$

In the proof of Lemma 1, we will need the representation of the Walsh—Dirichlet kernel stated in the following lemma.

LEMMA 2 (Schipp [6]). *For every integer* $j\geqq 1$,

$$(3.2)\qquad D_j(x)=w_j(x)\sum_{u=0}^{\infty} j_u r_u(x) D_{2^u}(x)$$

where j_u *is defined in* (1.1).

PROOF OF LEMMA 1. Assume

$$(3.3)\qquad 2^M\leqq m<2^{M+1}\quad\text{and}\quad 2^N\leqq n<2^{N+1}$$

with some integers $M, N\geqq 0$. Applying (3.2) for $D_j(x)$ with $j\leqq m$ and $D_k(y)$ with $k\leqq n$, and interchanging the order of summations, we can write that

$$\sum_{j=1}^{m}\sum_{k=1}^{n} b_{jk}D_j(x)D_k(y)=\sum_{j=1}^{m}\sum_{k=1}^{n} b_{jk}w_j(x)w_k(y)\times$$

$$\times\sum_{u=0}^{M} j_u r_u(x)D_{2^u}(x)\sum_{v=0}^{N} k_v r_v(y)D_{2^v}(y)=$$

$$=\sum_{u=0}^{M}\sum_{v=0}^{N} r_u(x)r_v(y)D_{2^u}(x)D_{2^v}(y)\sum_{j=1}^{m}\sum_{j=1}^{n} j_u k_v b_{jk} w_j(x) w_k(y)$$

(observe that this time $j_u=0$ for $u>M$ and $k_v=0$ for $v>N$). Setting

$$h_{mn}(x,y)=\operatorname{sign}\Big[\sum_{j=1}^{m}\sum_{k=1}^{n} b_{jk}D_j(x)D_k(y)\Big]$$

we have

$$I_{mn}=\int_0^1\int_0^1\sum_{j=1}^{m}\sum_{k=1}^{n} b_{jk}D_j(x)D_k(y)h_{mn}(x,y)\,dx\,dy=$$

$$=\sum_{u=0}^{M}\sum_{v=0}^{N}\int_0^1\int_0^1 r_u(x)r_v(y)D_{2^u}(x)D_{2^v}(y)h_{mn}(x,y)\times$$

$$\times\sum_{j=1}^{m}\sum_{k=1}^{n} j_u k_v b_{jk} w_j(x) w_k(y)\,dx\,dy.$$

Using the fact that

$$D_{2^u}(x) = \begin{cases} 2^u & \text{if } x\in[0, 2^{-u}), \\ 0 & \text{otherwise;} \end{cases}$$

(see, e.g. [1]), we can write that

$$I_{mn} = \sum_{u=0}^{M}\sum_{v=0}^{N} 2^{u+v} \int_0^{2^{-u}}\int_0^{2^{-v}} r_u(x) r_v(y) h_{mn}(x, y) \sum_{j=1}^{m}\sum_{k=1}^{n} j_u k_v b_{jk} w_j(x) w_k(y)\, dx\, dy =$$

$$= \sum_{u=0}^{M}\sum_{v=0}^{N} 2^{u+v} \sum_{j=1}^{m}\sum_{k=1}^{n} j_u k_v b_{jk} c_{jk}^{uv}$$

where

$$c_{jk}^{uv} = \int_0^{2^{-u}}\int_0^{2^{-v}} r_u(x) r_v(y) h_{mn}(x, y) w_j(x) w_k(y)\, dx\, dy$$

are the Walsh—Fourier coefficients of the function

$$r_u(x) r_v(y) h_{mn}(x,y) \chi_{[0,2^{-u})}(x) \chi_{[0,2^{-v})}(y),$$

χ being the characteristic function of the interval indicated in the subscript.

By Hölder's inequality with the exponents p and $q=p/(p-1)$, while taking into account that $|j_u|\leqq 1$ and $|k_v|\leqq 1$, we find that

$$I_{mn} \leqq \sum_{u=0}^{M}\sum_{v=0}^{N} 2^{u+v} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n} |b_{jk}|^p\Big]^{1/p} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n} |c_{jk}^{uv}|^q\Big]^{1/q}. \tag{3.4}$$

Applying the Hausdorff—Young inequality (see, e.g. [9, Vol. 2, p. 101]) extended to two-dimensional Fourier expansions, we can estimate as follows

$$\Big[\sum_{j=1}^{m}\sum_{k=1}^{n} |c_{jk}^{uv}|^q\Big]^{1/q} \leqq \Big[\int_0^{2^{-u}}\int_0^{2^{-v}} |r_u(x) r_v(y) h_{mn}(x, y)|^p\, dx\, dy\Big]^{1/p} = 2^{-(u+v)/p}.$$

Combining this and (3.4) yields

$$I_{mn} \leqq \sum_{u=0}^{M}\sum_{v=0}^{N} 2^{(u+v)(1-1/p)} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n} |b_{jk}|^p\Big]^{1/p} \leqq \frac{2^{(M+N)(1-1/p)}}{(1-2^{-(1-1/p)})^2} \Big[\sum_{j=1}^{m}\sum_{k=1}^{n} |b_{jk}|^p\Big]^{1/p}.$$

This implies (3.1). Indeed, the auxiliary function $z(t)=t(1-2^{-t})^{-1}$ increases for $t\geqq 0$. Thus, $z(2^{-1})<z(1)=2$, and consequently,

$$(1-2^{-(p-1)/p})^{-2} \leqq \left(\frac{2p}{p-1}\right)^2.$$

The last auxiliary result is essentially contained in [5].

LEMMA 3. *If a single sequence* $\{b_j\colon j=0, 1, \ldots\}$ *is such that* $b_j\to 0$ *as* $j\to\infty$ *and for some* $p>1$,

$$\Sigma_p = |b_0| + \sum_{m=0}^{\infty} 2^m \Big[2^{-m} \sum_{j=2^m}^{2^{m+1}-1} |b_j|^p\Big]^{1/p} < \infty,$$

then for all $m\geqq 1$,

$$\Big\|\sum_{j=0}^{m-1} b_j D_{j+1}\Big\| \leqq \frac{2p}{p-1}\Sigma_p.$$

4. Proofs

PROOF OF THEOREM 1. *Part* 1: *Pointwise convergence.* Clearly, the condition that $\Delta\mathcal{A}\in l_*^p$ for some $p>1$ implies via (2.1) that $\mathcal{A}=\{a_{jk}\}$ is a sequence of bounded variation. In Section 1 we showed that the series (1.2) converges for all $0<x, y<1$ provided its coefficients satisfy conditions (1.3) and (1.4).

Part 2: $L^1(I^2)$*-convergence.* By (2.1), the condition $\Delta\mathcal{A}\in l_*^p$ is less restrictive if p is closer to 1. Therefore, we may assume that $1<p\leqq 2$ in the sequel.

By (1.7) and (2.3),

$$f(x, y)-u_{mn}(x, y) = \sum_{(j,k)\in R_{mn}}\sum D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}$$

where R_{mn} denotes the set of all lattice points (j, k) with nonnegative integer coordinates j and k such that either $j\geqq m$ or $k\geqq n$. We may assume that $m, n\geqq 1$ and define the integers $M, N\geqq 0$ such that (3.3) is satisfied. We can represent R_{mn} in the form of an infinite disjoint union of (partly dyadic) rectangles and accordingly estimate as follows

$$\|f-u_{mn}\| \leqq \Big\|\sum_{j=m}^{2^{M+1}-1}\sum_{k=n}^{2^{N+1}-1} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}\Big\|+$$

$$+\Big\|\sum_{j=m}^{2^{M+1}-1} D_{j+1}(x)D_1(y)\Delta_{11}a_{j0}\Big\|+\sum_{v=0}^{N-1}\Big\|\sum_{j=m}^{2^{M+1}-1}\sum_{k=2^v}^{2^{v+1}-1} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}\Big\|+$$

$$+\Big\|\sum_{k=n}^{2^{N+1}-1} D_1(x)D_{k+1}(y)\Delta_{11}a_{0k}\Big\|+\sum_{u=0}^{M-1}\Big\|\sum_{j=2^u}^{2^{u+1}-1}\sum_{k=n}^{2^{N+1}-1} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}\Big\|+$$

$$+\sum_{u=M+1}^{\infty}\Big\|\sum_{j=2^u}^{2^{u+1}-1} D_{j+1}(x)D_1(y)\Delta_{11}a_{j0}\Big\|+\sum_{v=N+1}^{\infty}\Big\|\sum_{k=2^v}^{2^{v+1}-1} D_1(x)D_{k+1}(y)\Delta_{11}a_{0k}\Big\|+$$

$$+\sum_{(u,v)\in R_{M+1,N+1}}\sum\Big\|\sum_{j=2^u}^{2^{u+1}-1}\sum_{k=2^v}^{2^{v+1}-1} D_{j+1}(x)D_{k+1}(y)\Delta_{11}a_{jk}\Big\|.$$

We apply Lemma 1 and obtain that

$$\|f-u_{mn}\| \leqq \left(\frac{2p}{p-1}\right)^2\Big\{\sum_{u=M}^{\infty}2^{(u+1)(1-1/p)}\Big[\sum_{j=2^u}^{2^{u+1}-1}|\Delta_{11}a_{j0}|^p\Big]^{1/p}+$$

$$+\sum_{v=N}^{\infty}2^{(v+1)(1-1/p)}\Big[\sum_{k=2^v}^{2^{v+1}-1}|\Delta_{11}a_{0k}|^p\Big]^{1/p}+$$

$$+\sum_{(u,v)\in R_{MN}}\sum 2^{(u+v+2)(1-1/p)}\Big[\sum_{j=2^u}^{2^{u+1}-1}\sum_{k=2^v}^{2^{v+1}-1}|\Delta_{11}a_{jk}|^p\Big]^{1/p}\Big\}.$$

Now (2.4) is an immediate consequence of the assumption that $\Delta\mathcal{A}\in l_*^p$.

A similar argument proves (2.5).

PROOF OF COROLLARY 1. Obviously, $f\in L^1(I^2)$ follows form (2.4). Furthermore, it is a commonplace that convergence in L^1-norm (the so-called strong convergence) implies weak convergence. Then, by (2.2) and (2.4), for all $i, l \geqq 0$

$$\int_0^1\int_0^1 f(x, y)\, w_i(x)\, w_l(y)\, dx\, dy = \lim_{m,n\to\infty} \int_0^1\int_0^1 u_{mn}(x, y)\, w_i(x)\, w_l(y)\, dx\, dy =$$

$$= a_{il} - \lim_{m,n\to\infty} \Big\{\sum_{j=0}^{m-1} \Delta_{10} a_{jn} + \sum_{k=0}^{n-1} \Delta_{01} a_{mk} + a_{mn}\Big\} = a_{il}.$$

Here we took into account that the limit of each term in the braces is zero due to (1.3), (1.6) and its symmetric counterpart for $\Delta_{01} a_{mk}$. This proves that (1.2) is the Walsh—Fourier series of f.

PROOF OF COROLLARY 2. *Sufficiency.* By (2.2),

$$\|f - s_{mn}\| \leqq \|f - u_{mn}\| + \|D_n\| \Big\|\sum_{j=0}^{m-1} D_{j+1} \Delta_{10} a_{jn}\Big\| +$$

$$+ \|D_m\| \Big\|\sum_{k=1}^{n-1} D_{k+1} \Delta_{01} a_{mk}\Big\| + |a_{mn}|\, \|D_m\|\, \|D_n\|.$$

By Theorem 1, the first term on the right-hand side tends to zero as $\min(m, n) \to \infty$. By Lemma 3, conditions (2.6) and (2.7), the second and third terms tend to zero as $n \to \infty$ and $m \to \infty$, respectively. Putting these together with (2.9) yields (2.8) to be proved.

Necessity. By (2.2),

$$\|f - s_{mn}\| \geqq |a_{mn}|\, \|D_m\|\, \|D_n\| - \|f - u_{mn}\| -$$

$$- \|D_n\| \Big\|\sum_{j=0}^{m-1} D_{j+1} \Delta_{10} a_{jn}\Big\| - \|D_m\| \Big\|\sum_{k=0}^{n-1} D_{k+1} \Delta_{01} a_{mk}\Big\|$$

and (2.9) follows from (2.4), (2.6), (2.7) (via Lemma 3) and (2.8).

PROOF OF THEOREM 2. Under (1.3) and (2.13), the condition $\Delta\mathcal{A} \in l_*^p$ is equivalent, for any $p > 0$, to the condition

$$\Delta_{11} a_{00} + \sum_{m=0}^{\infty} 2^m \Delta_{11} a_{2^m,0} + \sum_{n=0}^{\infty} 2^n \Delta_{11} a_{0,2^n} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{m+n} \Delta_{11} a_{2^m,2^n} < \infty,$$

which in turn is equivalent to

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Delta_{11} a_{jk} < \infty.$$

This last condition is trivially satisfied since the sum is a_{00}. Consequently, conditions (1.3) and (2.13) imply that $\Delta\mathcal{A} \in l_*^p$ for all $p > 0$ and Theorem 1 applies. It remains only to prove the equivalence of the conditions (2.14) and (2.15).

Sufficiency. In a similar way, condition (2.6) is equivalent to the condition

$$\|D_n\| \Big\{\Delta_{10} a_{0n} + \sum_{m=0}^{\infty} 2^m \Delta_{10} a_{2^m,n}\Big\} \to 0 \quad \text{as} \quad n \to \infty,$$

which in turn is equivalent to

$$\|D_n\| \sum_{j=0}^{\infty} \Delta_{10} a_{jn} = a_{0n} \|D_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

But this follows from (2.15) due to (2.10).

Analogously, (2.15) also implies the fulfillment of condition (2.7), and (2.14) follows from Corollary 2 and the corresponding one-dimensional result (see [2] or [7]).

Necessity. We apply the one-dimensional result just referred to and by the last two relations in (2.14) we can conclude that

$$a_{m0} \|D_m\| \to 0 \quad \text{and} \quad a_{0n} \|D_n\| \to 0 \tag{4.1}$$

as $m \to \infty$ and $n \to \infty$, respectively.

As we have seen in the proof of the sufficiency part, the conditions in (4.1) imply the fulfillment of (2.6) and (2.7) for all $p>0$. Thus, we can apply Corollary 2, according to which condition (2.9) follows from the first relation in (2.14). Clearly, the couple of conditions (2.9) and (4.1) is equivalent, in the monotonic case, to the condition

$$a_{mn} \|D_m\| \|D_n\| \to 0 \quad \text{as} \quad \max(m, n) \to \infty. \tag{4.2}$$

Finally, taking into account that

$$\|D_{M_i}\| \geqq \frac{i}{8} \quad \text{for} \quad M_i = \sum_{k=0}^{i} 2^{2k} \quad (i = 1, 2, \ldots)$$

(see [1]) and again the monotone decreasing property of a_{jk} in j and k, (2.15) follows from (4.2).

References

[1] N. J. Fine, On the Walsh functions, *Trans. Amer. Math. Soc.*, **65** (1949), 372—414.

[2] G. A. Fomin, Certain properties of orthogonal series, in Proc. Conf. *Application of functional analysis in the approximation theory* (Kalinin, 1970); pp. 165—169 (Russian).

[3] G. H. Hardy, On the convergence of certain multiple series, *Proc. Cambridge Philosoph. Soc.*, **19** (1916—1919), 86—95.

[4] F. Móricz, On the integrability and L^1-convergence of double trigonometric series, *Studia Math.*, **98** (1991) (to appear).

[5] F. Móricz and F. Schipp, On the integrability and L^1-convergence of Walsh series with coefficients of bounded variation, *J. Math. Anal. Appl.*, **146** (1990), 99—109.

[6] F. Schipp, Über die Größenordnung der Partialsummen der Entwicklung integrierbarer Funktionen nach *W*-Systemen, *Acta Sci. Math. (Szeged)*, **28** (1967), 123—134.

[7] A. H. Siddiqi, Walsh—Fourier series with monotone coefficients, *Aligarh Bull. Math.*, **1** (1971), 25—29.

[8] S. Yano, On Walsh—Fourier series, *Tôhoku Math. J.*, **3** (1951), 223—242.

[9] A. Zygmund, *Trigonometric series*, Univ. Press (Cambridge, 1959).

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[2] A. Zygmund, Smooth functions, *Duke Math. J.*, 12 (1945), 47—76.

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