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ZU EINEM SATZ VON RÉDEI ÜBER KREISTEILUNGSPOLYNOME

K. JOHNSEN (Kiel)

Für eine natürliche Zahl n sei $\Phi_n(t)\in \mathbf{Z}[t]$ das n-te Kreisteilungspolynom. Durchläuft p die Primteiler von n, so ist $\Phi_n(t)$ der größte gemeinsame Teiler der Polynome $\Phi_p(t^{n/p})\in \mathbf{Z}[t]$. Obwohl $\mathbf{Z}[t]$ kein Hauptidealring ist, gilt der bemerkenswerte

1. Satz von Rédei [2]. *Für jede natürliche Zahl n gibt es zu den Primteilern p von n Polynome $f_p(t)\in \mathbf{Z}[t]$ mit*

$$\Phi_n(t) = \sum_{p|n} f_p(t)\Phi_p(t^{n/p}), \tag{1}$$

m. a. W. das Hauptideal $(\Phi_n(t))$ ist Summe der Hauptideale $(\Phi_p(t^{n/p}))$.

Der erste korrekte Beweis dieses Satzes wurde von de Bruijn in [1] gegeben. Mit einem einfachen Induktionsschluß beweist Rédei den Satz in [3]. Offenbar unabhängig davon weist Schoenberg die Formel (1) in [4] nach und folgert daraus einen Satz über lineare ganzzahlige Abhängigkeiten zwischen den n-ten Einheitswurzeln.

Wir wollen diesen Satz von Schoenberg hier in algebraisch begrifflicher Weise zeigen; der Satz von Rédei ist dann ein Korollar.

Es sei $n=p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}$ die Primzahlzerlegung von n und $n_0:=p_1p_2\dots p_s$; sei E_n die Gruppe der n-ten Einheitswurzeln und $\mathbf{Q}_n=\mathbf{Q}[E_n]$ der n-te Kreisteilungskörper. Ist R ein Repräsentantensystem von E_{n_0} in E_n, so ergibt sich durch Vergleich der $\mathbf{Q}$-dimensionen

$$\mathbf{Q}_n = \sum_{r\in R}{}_{\oplus}\, \mathbf{Q}_{n_0} r$$

und daraus

$$\mathbf{Z}[E_n] = \sum_{r\in R}{}_{\oplus}\, \mathbf{Z}[E_{n_0}]r. \tag{2}$$

Mit $\mathbf{Z}E_n$ bezeichnen wir den ganzzahligen Gruppenring der zyklischen Gruppe E_n, wobei wir zur besseren Unterscheidung die kanonischen Basiselemente von $\mathbf{Z}E_n$ mit b_x für $x\in E_n$ bezeichnen. Offenbar ist die Abbildung

$$\varphi\colon \mathbf{Z}E_n \to \mathbf{Z}[E_n]$$

$$\sum_{x\in E_n} z_x b_x \mapsto \sum_{x\in E_n} z_x x$$

ein Ringepimorphismus.

Für einen Primteiler p von n und eine n-te Einheitswurzel y sei

$$a_{p,y} := b_y \sum_{z \in E_p} b_z.$$

Für zwei n-te Einheitswurzeln y, y' mit $y' \in E_p y$ gilt dann

$$a_{p,y} = a_{p,y'}. \tag{3}$$

Der Satz von Schoenberg läßt sich in der Form aussprechen:

2. SATZ. *Kern* φ *wird als abelsche Gruppe von*

$$A_n := \{a_{p,y} | p \text{ Primteiler von } n,\ y \in E_n\}$$

erzeugt.

Es sei $x_0 \in E_n$ eine primitive n-te Einheitswurzel, also $E_n = \langle x_0 \rangle$. Der Homomorphismus

$$\varepsilon\colon \mathbf{Z}[t] \to \mathbf{Z}E_n$$

$$\sum_{i=0}^{m} z_i t^i \mapsto \sum_{i=0}^{m} z_i b_{x_0^i}$$

induziert einen Isomorphismus

$$\bar{\varepsilon}\colon \overline{\mathbf{Z}[t]} := \mathbf{Z}[t]/(t^n - 1) \to \mathbf{Z}E_n.$$

Da $\Phi_n(t)$ irreduzibel ist, folgt

$$\text{Kern}\, \varphi = (\overline{\Phi_n(t)}\bar{\varepsilon}). \tag{4}$$

Es sei nun $y \in E_n$ und $y = x_0^l$; nach (3) ist $a_{p,x_0^l} = a_{p,x_0^{l'}}$, falls $l \equiv l \bmod \frac{n}{p}$ ist, so daß wir im folgenden o. B. d. A. $0 \leqq l < \frac{n}{p}$ annehmen können. Es gilt

$$a_{p,y} = a_{p,x_0^l} = b_{x_0^l} \cdot \sum_{k=0}^{p-1} b_{x_0^{nk/p}} = \left(t^l \cdot \Phi_p(t^{n/p})\right)\varepsilon. \tag{5}$$

Aus (4), (5) und 2. folgt, daß es Polynome $f_p(t) \in \mathbf{Z}[t]$ mit $\operatorname{grad} f_p(t) < \frac{n}{p}$ gibt, so daß

$$\overline{\Phi_n(t)}\bar{\varepsilon} = \overline{\sum_{p|n} f_p(t)\Phi_p(t^{n/p})}\bar{\varepsilon}$$

gilt. Daraus folgt aber (1).

Wir kommen nun zum

BEWEIS VON SATZ 2. Es ist

$$\mathbf{Z}E_n = \sum_{r \in R}{}^{\oplus}\, \mathbf{Z}E_{n_0} \cdot b_r, \tag{6}$$

und mit $\varphi_r := \varphi|_{\mathbf{Z}E_{n_0} \cdot b_r}$ folgt

$$(\mathbf{Z}E_{n_0} b_r)\varphi_r = \mathbf{Z}[E_{n_0}]r.$$

Wegen $\text{Kern}\, \varphi_r = (\text{Kern}\, \varphi_1) b_r$ und (6) genügt es zum Beweis zu zeigen:

$(*)$ Kern φ_1 wird als abelsche Gruppe von

$$A_{n_0} := \{a_{p,y} | p \text{ Primteiler von } n_0, \ y \in E_{n_0}\}$$

erzeugt.

Zunächst gilt

$$a_{p,y}\varphi_1 = (b_y \sum_{z \in E_p} b_z)\varphi_1 = y \cdot \sum_{z \in E_p} z = y \cdot 0 = 0,$$

also ist $A_{n_0} \subseteqq \text{Kern } \varphi_1$.

Wegen $E_{n_0} = E_{p_1} \times E_{p_2} \times \ldots \times E_{p_s}$ ist

$$\mathbf{Z}E_{n_0} = \mathbf{Z}E_{p_1} \cdot \mathbf{Z}E_{p_2} \cdot \ldots \cdot \mathbf{Z}E_{p_s} = \mathbf{Z}E_{p_1} \otimes \mathbf{Z}E_{p_2} \otimes \ldots \otimes \mathbf{Z}E_{p_s};$$

ferner gilt

$$\mathbf{Z}[E_{n_0}] = \mathbf{Z}[E_{p_1}] \cdot \mathbf{Z}[E_{p_2}] \cdot \ldots \cdot \mathbf{Z}[E_{p_s}] = \mathbf{Z}[E_{p_1}] \otimes \mathbf{Z}[E_{p_2}] \otimes \ldots \otimes \mathbf{Z}[E_{p_s}].$$

Setzen wir für alle u mit $1 \leqq u < s$

$$\varphi_1^{(u)} := \varphi_1 |_{\mathbf{Z}E_{p_u}},$$

so ist offenbar

$$\varphi_1^{(u)} : \mathbf{Z}E_{p_u} \to \mathbf{Z}[E_{p_u}]$$

ein Epimorphismus. Da wegen der Irreduzibilität des p_u-ten Kreisteilungspolynoms

$$\sum_{x \in E_{p_u}} z_x x = (\sum_{x \in E_{p_u}} z_x b_x)\Phi_1^{(u)} = 0$$

genau dann gilt, falls alle z_x gleich sind, folgt

$$\text{Kern } \varphi_1^{(u)} = \mathbf{Z}a_{p_u,1}.$$

Nun ist aber

$$\varphi_1 = \varphi_1^{(1)} \otimes \ldots \otimes \varphi_1^{(s)}$$

und deshalb

$$\text{Kern } \varphi_1 = \sum_{u=1}^{s}{}_{\oplus} \mathbf{Z}E_{p_1} \otimes \ldots \otimes \mathbf{Z}a_{p_u,1} \otimes \ldots \otimes \mathbf{Z}E_{p_s};$$

also wird Kern φ_1 von A_{n_0} erzeugt.

Literaturverzeichnis

[1] N. de Bruijn, On the factorization of cyclic groups, *Proc. Kon. Ned. Akad. Wetensch.*, Ser. A, **56** (1953), 370—377.
[2] L. Rédei, Ein Beitrag zum Problem der Faktorisation von endlichen abelschen Gruppen, *Acta Math. Acad. Sci. Hung.*, **1** (1950), 197—207.
[3] L. Rédei, Über das Kreisteilungspolynom, *Acta Math. Acad. Sci. Hung.*, **5** (1954), 27—28.
[4] I. J. Schoenberg, A note on the cyclotomic polynomials, *Mathematica*, **11** (1964), 131—136.

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CONVOLUTION EQUATIONS IN BEURLING'S DISTRIBUTIONS

S. ABDULLAH (Macomb)

Solvability of convolution equations was investigated by L. Ehrenpreis [2] and L. Hörmander [3] in the space of Schwartz distributions. In this paper we discuss solvability of systems of convolution equations in Beurling's distributions; the results of this paper extend some of Hörmander's [3]. The first section contains notations and the preliminary results which will be used later in the work, in the second section we give an existence theorem in the case of one convolution equation, the third section contains existence theorems for systems of convolution equations.

1. Preliminary results

Björk [1] gave a thorough study of Beurling test function spaces and distributions. Here we give a very brief account of the definitions and the basic results which we are going to use later. In particular, we use the Paley—Wiener theorem in a form slightly weaker than the one given by Björk. For proofs of these results we refer the reader to [1].

By $\mathcal{M}$ we denote the set of all continuous real-valued functions ω on $\mathbf{R}^n$, satisfying the following conditions:

(a) $0=\omega(0)\leqq\omega(\xi+\eta)\leqq\omega(\xi)+\omega(\eta)$ for all ξ,η in $\mathbf{R}^n$,

(b) $\displaystyle\int\frac{\omega(\xi)}{(1+|\xi|)^{n+1}}\,d\xi<\infty$,

(c) there exists real a and positive b such that

$$\omega(\xi)\geqq a+b\log(1+|\xi|)\quad\text{for all}\quad \xi \text{ in } \mathbf{R}^n,$$

(d) $\omega(\xi)=\gamma(|\xi|)$ for some increasing continuous concave function γ on $[0,\infty)$.

Definition 1.1. Let $\omega\in\mathcal{M}$; by D_ω we denote the set of all ψ in $L^1(\mathbf{R}^n)$ such that ψ has compact support and

$$\|\psi\|_\lambda=\|\psi\|_\lambda^\omega=\int|\hat{\psi}(\xi)|e^{\lambda\omega(\xi)}\,d\xi<\infty\quad\text{for all}\quad\lambda>0,$$

where $\hat{\psi}$ is the Fourier transform of ψ. The elements of D_ω will be called Beurling test functions.

If Ω is a subset of $\mathbf{R}^n$, let $D_\omega(\Omega)=\{\psi\in D_\omega\colon \operatorname{Supp}\psi\subset\Omega\}$. If K is a compact subset of $\mathbf{R}^n$, $D_\omega(K)$ is a Frèchet space when provided with the topology defined by the semi-norms $\|\cdot\|_m$, $m=1, 2, \ldots$. Let K_i, $i\geqq 1$ be a sequence of compact subsets of $\mathbf{R}^n$ so that K_i is contained in the interior of K_{i+1}, and the union of the K_i's is $\mathbf{R}^n$. We define the topology of D_ω as the inductive limit of the topologies of $D_\omega(K_i)$. It turns out that D_ω is an algebra under pointwise multiplication, and for each $\lambda>0$ one has

$$\|\varphi\psi\|_\lambda \leqq \frac{1}{2\pi}\|\varphi\|_\lambda\|\psi\|_\lambda \quad \text{for all} \quad \varphi, \psi \text{ in } D_\omega.$$

Definition 1.2. Let $\omega\in\mathcal{M}$, then D'_ω is the space of all continuous linear functionals on D_ω. D'_ω will be given the weak topology, which is the topology of pointwise convergence on D_ω. We remark that conditions (a)—(d) guarantee the nontriviality of D_ω, D_ω is closed under differentiation, and that D_ω is dense in D, the space of Schwartz test functions.

Definition 1.3. Let $\omega\in\mathcal{M}$, by E_ω we denote the set of all complex-valued functions ψ in $\mathbf{R}^n$ such that if $\varphi\in D_\omega$, then $\varphi\psi\in D_\omega$. The topology in E_ω is given by the semi-norms $\psi\to\|\varphi\psi\|_\lambda$ for all $\lambda>0$ and all $\varphi\in D_\omega$.

Definition 1.4. Let $\omega\in\mathcal{M}$, then E'_ω is defined as the space of all continuous linear functionals on E_ω. It follows that one can identify E'_ω with the set of elements in D'_ω which have compact support.

If $\omega(\xi)=\log(1+|\xi|)$ then D_ω, D'_ω, E_ω and E'_ω will coincide with D, D', E and E', respectively, where the latter are Schwartz spaces of test functions and distributions. It follows from the definitions, condition (c) and Proposition 2.1 (below), that $D_\omega\subset D$, $D'\subset D'_\omega$, $E_\omega\subset E$ and $E'\subset E'_\omega$, for all $\omega\in\mathcal{M}$.

Proposition 1.1. *If* $\omega\in\mathcal{M}$, *then* $\omega(\xi)=o\left(\frac{|\xi|}{\log|\xi|}\right)$ *as* $|\xi|\to\infty$.

Definition 1.5. Let $\omega_1, \omega_2\in\mathcal{M}$. We will write $\omega_2<\omega_1$ if, for some real A and positive C we have $\omega_2(\xi)\leqq A+C\omega_1(\xi)$, $\xi\in\mathbf{R}^n$.

Proposition 1.2. *Let* $\omega_1, \omega_2\in\mathcal{M}$. *If* $\omega_2<\omega_1$ *then* $D_{\omega_1}(\Omega)$ *is a dense subset of* $D_{\omega_2}(\Omega)$ *for each open set* $\Omega\subset\mathbf{R}^n$. *Conversely, if for* $\Omega\subset\mathbf{R}^n$ *with nonempty interior,* $D_{\omega_1}(\Omega)\subset D_{\omega_2}(\Omega)$, *then* $\omega_2<\omega_1$.

Paley—Wiener Theorem. 1. *An entire function* $u(\zeta)$ *is the Fourier transform of a function in* D_ω *with support in the ball* $B_A=\{x\in\mathbf{R}^n\colon |x|<A\}$ *if and only if for each* $\lambda>0$ *there exists a constant* C_λ *so that*

$$|u(\zeta)| \leqq C_\lambda e^{-\lambda\omega(\xi)+A|\eta|}, \quad \zeta = \xi+i\eta\in\mathbf{C}^n.$$

2. *An entire function* $u(\zeta)$ *is the Fourier transform of a generalized distribution with support in the ball* B_A *if and only if there are two constants* λ *and* C *such that*

$$|u(\zeta)| \leqq Ce^{\lambda\omega(\xi)+A|\eta|}, \quad \zeta = \xi+i\eta\in\mathbf{C}^n.$$

Theorem 1.1. *Let* k *be a compact subset of* $\mathbf{R}^n$ *and* H_k *be its support function. The following two families of semi-norms on* $D_\omega(K)$ *are equivalent:*

$$\left\{\varphi \to \sup_{\xi\in\mathbf{R}^n} e^{\lambda\omega(\xi)}|\hat{\varphi}(\xi)|,\ \lambda>0\right\}, \tag{1}$$

$$\left\{\varphi \to \sup_{\zeta=\xi+i\eta\in\mathbf{C}^n} e^{\lambda\omega(\xi)-H_K(\eta)-|\eta|}|\hat{\varphi}(\zeta)|,\ \lambda>0\right\}. \tag{2}$$

THEOREM 1.2. *Let* $u \in E'_\omega$ *and* K *a compact subset of* $\mathbf{R}^n$. *In order that* $\operatorname{sing}_\omega \operatorname{supp} u \subset K$ *it is necessary and sufficient that* $|\eta| \leqq m\omega(\xi)$ *imply*

$$|\hat{u}(\xi+i\eta)| \leqq C_m e^{\lambda\omega(\xi)+H_K(\eta)}; \quad m = 1, 2, \ldots,$$

C_m *is a constant which depends on* m.

HÖRMANDER'S LEMMA [3]. *Let* F, G *and* F/G *be entire functions. Then for any* $r>0$

$$|F(z)/G(z)| \leqq \sup_{|z-\zeta|<4r} |F(\zeta)| \cdot \sup_{|z-\zeta|<4r} |G(\zeta)| / (\sup_{|z-\zeta|<r} |G(\zeta)|)^2.$$

By D^m_ω we denote the product of m copies of $D_\omega(\mathbf{R}^n)$, $\mathbf{D}^m_\omega$ denotes the Fourier transform of D^m_ω. A product subset K of $\mathbf{R}^{nm}$ is an m-tuple of the form $(K_1, \ldots, K_m)$ where K_l is a compact subset of $\mathbf{R}^n$, $l=1, 2, \ldots, m$. Let $K=(K_1, \ldots, K_m)$ as above and $\varphi=(\varphi_1, \ldots, \varphi_n) \in D^m_\omega$. We say that $\operatorname{supp} \varphi \subset K$ if $\operatorname{supp} \varphi_j \subset K_j$; $j=1, \ldots, m$. By $D^m_\omega(K)$ we denote the set of all $\varphi \in D^m_\omega$ with support in K. For any compact $K \subset \mathbf{R}^{nm}$ we define the topology of $\mathbf{D}^m_\omega(K)$ by the following family of semi-norms:

$$p_k(\hat{\Phi}) = p_k(\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_m)^t) = \sum_{l=1}^{m} \|\hat{\varphi}_l\|_k; \quad k = 1, 2, 3, \ldots$$

where $\|\cdot\|_k$ is the family of semi-norms which defines the topology of $\mathbf{D}_\omega(k_l)$. Clearly the semi-norm p_k is well defined for each k. $\mathbf{E}'^m_\omega$ denotes the Fourier transform of $(E'_\omega)^m$ and E'^{2m}_ω denotes the product $E'^m_\omega \times E'^m_\omega$. For $S=(S_{ij}) \in E'^{2m}_\omega$ we denote by S^t the element of E^{2m}_ω given by $(\check{S}_{ji})$.

2. Existence theorems in D'_ω

Let $S \in E'_\omega$. We give necessary and sufficient conditions on $\hat{S}$ (the Fourier transform of S) so that the equation $S*u=v$ has a solution $u \in D'_\omega$ for every $v \in D'_\omega$.

DEFINITION 2.1. Let f be an entire function which satisfies the second estimate of the Paley—Wiener theorem. We say that f is ω-slowly decreasing if there exist positive constants C, A such that

$$\sup_{\substack{|x| \leqq A\omega(\xi) \\ x \in \mathbf{R}^n}} |\hat{S}(x+\xi)| \geqq Ce^{-A\omega(\xi)}, \quad \xi \in \mathbf{R}^n. \tag{1}$$

LEMMA 2.1. *Let* K *be a compact subset of* $\mathbf{R}^n$. *The topologies of* $\mathbf{D}_\omega(K) = \{\hat{\varphi}\colon \varphi \in D_\omega(K)\}$ *defined by the following families of semi-norms are equivalent:*

$$\text{(i)} \qquad \{\hat{\varphi} \mapsto \sup_{\xi \in \mathbf{R}^n} e^{\lambda\omega(\xi)} |\hat{\varphi}(\xi)|;\ \lambda > 0\};$$

$$\text{(ii)} \qquad \{\hat{\varphi} \mapsto \sup_{\substack{\zeta \in \mathbf{C}^n \\ \zeta=\xi+i\eta \in \mathbf{C}^n}} e^{\alpha\omega(\xi)-H_K(\eta)-|\eta|} |\hat{\varphi}(\zeta)|, \quad \alpha > 0\};$$

$$\text{(iii)} \qquad \{\hat{\varphi} \mapsto \sup_{\substack{|\eta| \leqq r\omega(\xi) \\ \zeta=\xi+i\eta \in \mathbf{C}^n}} e^{r\omega(\xi)} |\hat{\varphi}(\zeta)|, \quad r > 0\}.$$

PROOF. The equivalence of (i) and (ii) follows from Theorem 1.1. We prove the equivalence of (i) and (iii). One can assume that $K\subset\overline{B(0, s)}$. Let r be any positive real number. There exist positive numbers η, C such that

$$\sup_{\zeta\in\mathbf{C}^n} e^{r(ns+2)\omega(\xi)-H_K(\eta)-(\eta)}|\hat{\varphi}(\zeta)| \leqq C\sup_{\xi\in\mathbf{R}^n} e^{\lambda\omega(\xi)}\hat{\varphi}|(\xi)|,$$

which implies that

$$\sup_{\substack{|\eta|\leqq r\omega(\xi)\\ \zeta\in\mathbf{C}^n}} e^{r(ns+1)\omega(\xi)-H_K(\eta)}|\hat{\varphi}(\zeta)| \leqq C\sup_{\xi\in\mathbf{R}^n} e^{\lambda\omega(\xi)}|\hat{\varphi}(\xi)|. \tag{2}$$

Since $K\subset\overline{B(s, 0)}$ and $|\eta|\leqq r\omega(\xi)$ one has

$$H_K(\eta)\leqq \max_{x\in K}\sum_{j=1}^{n}|x_j||\eta_j|\leqq nrs\omega(\xi). \tag{3}$$

By substituting the estimate (3) in (2) one gets

$$\sup_{\substack{|\eta|\leqq r\omega(\xi)\\ \zeta\in\mathbf{C}^n}} e^{r\omega(\xi)}|\hat{\varphi}(\zeta)| \leqq C\sup_{\xi\in\mathbf{R}^n} e^{\lambda\omega(\xi)}|\hat{\varphi}(\xi)|,$$

which concludes the proof of the lemma.

THEOREM 2.1. *Let* $S\in E'_\omega$. *The following statements are equivalent:*
(a) $\hat{S}$ *is* ω*-slowly decreasing;*
(b) $S*D'_\omega=D'_\omega$;
(c) *The operator* S *has a fundamental solution* $E\in D'_\omega$.

PROOF. The implication (b)$\Rightarrow$(c) is obvious.

Proof of (a)$\Rightarrow$(b). Given $v\in D'_\omega$ we wish to find $u\in D'_\omega$ such that $S*u=v$. Define the map

$$\wedge:\ \check{s}*D_\omega\to\mathbf{C},\quad \check{s}*\varphi\mapsto\langle v,\varphi\rangle.$$

If we can prove that $\wedge$ is continuous then the theorem follows from the Hahn—Banach theorem. Since the Fourier transform is a continuous map from D_ω into $\mathbf{D}_\omega$ and v is continuous, it suffices to prove that the map $T\cdot\hat{\varphi}\to\hat{\varphi}$ from $T\mathbf{D}_\omega$ into $\mathbf{D}_\omega$ is continuous, where $T=\hat{\check{S}}$. Since the topology of $\mathbf{D}_\omega$ is the inductive limit topology of the spaces $\mathbf{D}^j_\omega=\{\hat{\varphi}:\ \varphi\in D_\omega(K_j),\ K_j=\overline{B(0,j)}\}$, $j=1, 2, \ldots$ it suffices to show that the linear map $T\hat{\varphi}\mapsto\hat{\varphi}$ from $T\cdot\mathbf{D}^j_\omega$ into $\mathbf{D}^j_\omega$ is continuous, where the topology of $\mathbf{D}^j_\omega$ is generated by any one of the equivalent families of semi-norms given in Lemma 2.1.

Let $\psi=\check{S}*\varphi$ where $\varphi\in D_\omega(K_j)$. From the Paley—Wiener theorem it follows that $\hat{\psi}$, T and $\hat{\varphi}=\hat{\psi}/T$ are entire functions. By Hörmander's Lemma with $r=A\omega(\xi)$, where A is as in condition (a), for any $\xi\in\mathbf{R}^n$ one has (with $z=x+iy\in\mathbf{C}^n$):

$$|\hat{\varphi}(\xi)|=\frac{|\hat{\psi}(\xi)|}{|T(\xi)|}\leqq \sup_{|z-\xi|<4A\omega(\xi)}|\hat{\psi}(z)|\sup_{|z-\xi|<4A\omega(\xi)}|T(z)|/[\sup_{|z-\xi|<A\omega(\xi)}|T(z)|]^2. \tag{I}$$

Since T is slowly decreasing and ω is symmetric it follows that

$$\sup_{|z-\xi|<A\omega(\xi)}|T(z)|\geqq\sup_{|x|<A\omega(\xi)}|T(x+\xi)|\geqq Ce^{-A\omega(\xi)},\quad\text{for every}\quad \xi\in\mathbf{R}^n.$$

Hence

$$\frac{1}{[\sup\limits_{|z-\xi|<A\omega(\xi)} |T(z)|]^2} \leqq \frac{1}{C^2} e^{2A\omega(\xi)}, \quad \xi\in\mathbf{R}. \tag{4}$$

From the Paley—Wiener theorem there exist positive constants C_1, λ and N such that

$$\sup_{\substack{|z-\xi|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |T(z)| \leqq C_1 \sup_{\substack{|z-\xi|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} e^{N|y|+\lambda\omega(x)}, \quad \xi\in\mathbf{R}^n. \tag{5}$$

Thus for any $\xi\in\mathbf{R}^n$ one has

$$|\varphi(\xi)| \leqq C_2 \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| e^{2A\omega(\xi)} \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} e^{N|y|+\lambda\omega(x+\xi)}; \tag{6}$$

$$\leqq C_2 \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| e^{(2A+4NA+\lambda)\omega(\xi)} \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} e^{\lambda\omega(x)},$$

where $C_2=C_1/C^2$. From Proposition 1.1 it follows that $\omega(x)\leqq M(1+4A\omega(\xi))$ whenever $|z|<4A\omega(\xi)$, where M is some constant. It follows that for any $\xi\in\mathbf{R}^n$ one has for any positive integer k

$$e^{k\omega(\xi)}|\varphi(\xi)| \leqq C_2 e^M e^{-[l-(k+2A+4AN+\lambda+4MA\lambda)]\omega(\xi)} \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| e^{l\omega(\xi)}, \tag{7}$$

where l is a positive constant to be determined later.

Since the left hand side of (7) is bounded by a constant whenever $|\xi|$ is small, there is no loss of generality in assuming that $|\xi|$ is large. From Proposition 1.1 it follows that $\omega(\xi)<\frac{|\xi|}{100A}$ whenever $|\xi|>B$ for some constant B. This implies that $|x|\leqq\frac{|\xi|}{25}$ whenever $|\xi|>B$. Hence $\omega(x)\leqq\omega(\xi+x)$ for such ξ's and the above specified x's, which implies that $\omega(\xi)\leqq 2\omega(x+\xi)$ whenever ξ is large. Thus for large ξ one has

$$e^{l\omega(\xi)} \leqq \inf_{\substack{x=\mathrm{Re}\, z \\ |z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} e^{2l\omega(x+\xi)}$$

and (7) becomes

$$e^{k\omega(\xi)}|\varphi(\xi)| \leqq C_2 e^M e^{-(l-d)\omega(\xi)} \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| \cdot \inf_{\substack{|z|<4A\omega(\xi) \\ \mathrm{Re}\, z=x,\, z\in\mathbf{C}^n}} e^{2l\omega(x+\xi)} \leqq \tag{8}$$

$$\leqq C_2 e^M e^{-(l-d)\omega(\xi)} \sup_{\substack{|z|<4A\omega(\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| e^{2l\omega(x+\xi)},$$

where $d=k+2A+4AN+\lambda+4\lambda MA$. By choosing $l>d$ one gets from (8)

$$e^{k\omega(\xi)}|\varphi(\xi)| \leqq C_3 \sup_{\substack{|z|<8A\omega(x+\xi) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z+\xi)| e^{2l\omega(x+\xi)} \leqq C_3 \sup_{\substack{|\mathrm{Im}\,(z-\xi)|<8A\omega(x) \\ z\in\mathbf{C}^n}} |\hat{\psi}(z)| e^{2l\omega(x)}, \tag{9}$$

where C_3 is independent of ξ which could be any given point in $\mathbf{R}^n$ with $|\xi|$ large. Since the left hand side of (9) is bounded by a constant whenever $|\xi|$ is small one can find a constant C so that

(10)
$$\sup_{\xi\in\mathbf{R}^n} e^{k\omega(\xi)}|\hat\varphi(\xi)| \leqq C \sup_{\xi\in\mathbf{R}^n} \sup_{\substack{|\mathrm{Im}(z-\xi)|<8A\omega(x)\\ z\in\mathbf{C}^n}} |\hat\psi(z)|e^{2l\omega(x)} = C \sup_{\substack{|\mathrm{Im}\, z|<8A\omega(x)\\ z\in\mathbf{C}^n}} |\hat\psi(z)|e^{2l\omega(x)}.$$

Let $m=\max\{8A, 2l, j_0\}$ when j_0 is the smallest j so that $\psi=\check{s}*\varphi\in D_\omega(K_j)$. Then (10) gives $p_k(\hat\varphi)\leqq Cp_m(\hat\psi)$ which is the desired continuity.

Proof of (c)$\Rightarrow$(a). We notice first that condition (a) is implied by the condition: there exists a constant $s_1>0$ such that for all $\xi\in\mathbf{R}^n$, $|\xi|\geqq s_1$, one has

$$\sup_{|x|\leqq A\omega(\xi)} |\hat S(x+\xi)| \geqq e^{-A\omega(\xi)}, \quad \text{for some} \quad A>0.$$

Suppose the implication does not hold, then it follows that for any $j=1, 2, \dots$ there exists $\xi_j\in\mathbf{R}^n$, $|\xi_j|\to\infty$ as $j\to\infty$ and

(11)
$$\sup_{|x|<j\omega(\xi_j)} |\hat S(x+\xi_j)| < e^{-j\omega(\xi_j)}.$$

Choose $\varphi\in D_\omega$ so that $\varphi\geqq 0$, $\operatorname{supp}\varphi\subset\overline{B(0,1)}$ and $\hat\varphi(0)=1$. Let $k_j=[\omega(\xi_j)]$; the greatest integer less than or equal to $\omega(\xi_j)$. For each $j>0$, define $\varphi_j= =e^{i\langle\xi_j,\cdot\rangle}(\varphi*\dots*\varphi)$ where φ is taken k_j times. Thus $\operatorname{supp}\varphi_j\subset\overline{B(0,k_j)}$. Since $S*E=\delta$ for some $E\in D'_\omega$ one has

(12)
$$|\varphi_j(x)| = |\langle S*E, \tau_x\check\varphi_j\rangle| = |\langle E, \check S_{-x}*\check{\tau\varphi_j}\rangle| = |\langle \hat E, \widehat{\tau_{-x}(S*\varphi_j)}\rangle|.$$

Hence

$$|\varphi_j(x)| \leqq A_1\, p_k(\widehat{\tau_{-x}(S*\varphi_j)}),$$

A_1 and k are positive integers. Hence

(13)
$$|\varphi_j(x)| \leqq A_1 \sup_{\xi\in\mathbf{R}^n} e^{k\omega(\xi)}|\hat S(\xi)|\cdot|\hat\varphi_j(\xi)| =$$
$$= A_1\Big[\sup_{\substack{|\xi-\xi_j|\leqq A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi)}|\hat S(\xi)|\,|\hat\varphi_j(\xi)| + \sup_{\substack{|\xi-\xi_j|>A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi)}|\hat S(\xi)|\,|\hat\varphi_j(\xi)|\Big],$$

where A is a fixed constant. Now we estimate each of the two terms on the right hand side of (13). By definition one has $\hat\varphi_j(\xi)=[\hat\varphi(\xi-\xi_j)]^k j$ and $|\hat\varphi_j(\xi)|\leqq\Big(\int_{|x|\leqq 1}|\varphi(x)|\,dx\Big)^k j\leqq \leqq 1$. Thus we have for some constants M, C_1,

(14)
$$\sup_{\substack{|\xi-\xi_j|\leqq A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi)}|\hat S(\xi)|\,|\hat\varphi_j(\xi)| \leqq \sup_{\substack{|\xi|\leqq A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi+\xi_j)}|\hat S(\xi+\xi_j)| \leqq$$
$$\leqq e^{kM}e^{k\omega(\xi_j)} \sup_{\substack{|\xi|\leqq A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{kM|\xi|}\,|\hat S(\xi+\xi_j)| \leqq C_1 e^{(k+kMA-j)\omega(\xi_j)}$$

whenever $j>A$, C_1 is constant. By the Paley—Wiener theorem and the definition of both k_j and φ_j it follows that

(15) $$|\hat{\varphi}_j(\xi)| \leqq e^{\lambda k_j} e^{-k_j \omega(\xi-\xi_j)} \leqq e^{[\lambda-\omega(\xi-\xi_j)]\omega(\xi_j)} e^{\omega(\xi-\xi_j)},$$

where λ is some constant. By (15) and the Paley—Wiener theorem one has for some constants C_4, N

(16)
$$\sup_{\substack{|\xi-\xi_j|>A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi)}|\hat{\varphi}_j(\xi)||\hat{S}(\xi)| \leqq C_4 \sup_{\substack{|\xi-\xi_j|>A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{(k+N)\omega(\xi)+(\lambda-\omega(\xi-\xi_j))\omega(\xi_j)+\omega(\xi-\xi_j)} \leqq$$
$$\leqq C_4 \sup_{\substack{|\xi-\xi_j|>A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{[k+N+1-\omega(\xi_j)]\omega(\xi-\xi_j)+(k+N+\lambda)\omega(\xi_j)}.$$

Since ω is increasing and by taking j large enough so that

$$m = \min\{j-k-kMA, \omega(A\omega(\xi_j))-AM(k+N+1)-k-N-\lambda\} > 0$$

and $k+N+1 < \omega(\xi_j)$ one gets

(17) $$e^{[k+N+1-\omega(\xi_j)]\omega(\xi-\xi_j)} \leqq C_5 e^{[\alpha-\omega(A\omega(\xi_j))]\omega(\xi_j)},$$

where $\alpha=MA(k+N+1)$ and $C_5=e^{M(k+N+1)}$. By substituting (17) in (16) one gets for j large enough

(18) $$\sup_{\substack{|\xi-\xi_j|>A\omega(\xi_j)\\ \xi\in\mathbf{R}^n}} e^{k\omega(\xi)}|\hat{S}(\xi)||\hat{\varphi}_j(\xi)| \leqq C_4 C_5 e^{[\beta-\omega(A\omega(\xi_j))]}$$

where $\beta=\alpha+k+N+\lambda$. From (14), (18) and (13) it follows that for j large enough,

(19) $$|\varphi_j(x)| \leqq M' e^{-m\omega(\xi_j)}, \quad M' = A_1C_1+A_1C_4C_5.$$

By definition it follows that

(20) $$1 = (\hat{\varphi}_j(\xi_j))1 = \Big|\int_{\mathbf{R}^n} e^{-i(x,\xi_j)}\varphi_j(x)\,dx\Big| \leqq M' e^{-m\omega(\xi_j)\operatorname{mes}\overline{B(0,k_j)}} \leqq$$
$$\leqq 2^n M' e^{-m\omega(\xi_j)}[\omega(\xi_j)]^n.$$

As $j\to\infty$ the left hand side of (20) remains one while the right hand side goes to zero; this contradiction proves the implication. This completes the proof of the theorem.

We say that the generalized distribution $S\in E'_\omega$ is ω-invertible if and only if it satisfies anyone of the equivalent conditions of Theorem 2.1.

THEOREM 2.2. *Let* $S\in E'_\omega$ *be invertible, then*
(i) $S+\varphi$ *is invertible for any* $\varphi\in D'_\omega$;
(ii) *if* $\varphi\in D_\omega$ *is identically one in the* ω*-singular support of* S *then* φS *is invertible.*

Proof. From the Paley—Wiener Theorem and Proposition 1.1 it follows that for any $\lambda>0$ there exists a constant C_λ so that for any real x

$$\sup_{\substack{|y|\leqq A\omega(x)\\ y\in\mathbf{R}^n}} |\hat{\varphi}(x+y)| \leqq C_\lambda e^{\lambda\omega(x)} \sup_{\substack{|y|\leqq A\omega(x)\\ y\in\mathbf{R}^n}} e^{-\lambda\omega(y)} \leqq C'_\lambda e^{(\lambda-\lambda MA)\omega(x)}, \tag{21}$$

where A is the constant which comes from the invertibility of S. From the invertibility of S and (21) one gets, for any $x\in\mathbf{R}^n$,

$$\sup_{\substack{|y|\leqq A\omega(x)\\ y\in\mathbf{R}^n}} |\widehat{(S+\varphi)}(x+y)| \geqq Ce^{-A\omega(x)}[1-C''_\lambda e^{(-\lambda MA-\lambda-A)\omega(x)}]. \tag{22}$$

By choosing λ large enough it follows from (22) that

$$\sup_{\substack{|y|\leqq A\omega(x)\\ y\in\mathbf{R}^n}} |\widehat{(S+\varphi)}(x+y)| \geqq \frac{C}{10} e^{-A\omega(x)};$$

this proves (i). The second assertion follows from the first because $S(\varphi-1)\in D_\omega$.

We remark that the sum of two invertible convolution operators is not necessarily invertible. This follows immediately from (i) above.

3. Existence theorems in D'^m_ω

Given $S\in E'^{2m}_\omega$ and $V\in D'^m_\omega$, in this section we give necessary and sufficient conditions on S so that the equation $S*u=V$ has a solution $u\in D'^m_\omega$. We also prove other related results.

Lemma 3.1. *The map $\Phi\to S*\Phi$ from D^m_ω into D^m_ω is continuous.*

Proof. Since the Fourier transform is continuous it suffices to show that the map $\hat{\Phi}\to\hat{S}\cdot\hat{\Phi}$ from $\mathbf{D}^m_\omega$ into $\mathbf{D}^m_\omega$ is continuous. This follows immediately from the continuity of the corresponding map on $\mathbf{D}^m_\omega(K)$ and the definition of the topology of $\mathbf{D}^m_\omega$, K is any compact subset of $\mathbf{R}^{mn}$. For any $k=0, 1, 2, \ldots$ one has by the Paley—Wiener theorem applied to the entries of $\hat{S}$

$$p_k(\hat{S}\hat{\Phi}) = \sum_{l=1}^{m} \Big\|\sum_{r=1}^{m} \hat{S}_{lr}\hat{\varphi}_r\Big\|_k \leqq \sum_{r=1}^{m}\sum_{l=1}^{m} \|\hat{S}_{lr}\hat{\varphi}_r\|_k \leqq A\sum_{r=1}^{m} \|\hat{\varphi}_r\|_{k+N} = Ap_{k+N}(\hat{\Phi}),$$

where A and N are constants which depend on S only.

From the lemma it follows that E'^{2m}_ω is the space of convolution operators on D'^m_ω.

Definition 3.2. Let $S\in E'^{2m}_\omega$, $\Omega_1=(\Omega_{11}, \Omega_{12}, \ldots, \Omega_{1m})$ and $\Omega_2=(\Omega_{21}, \Omega_{22}, \ldots, \Omega_{2m})$ be tuples of open subsets of $\mathbf{R}^n$. The couple (Ω_1, Ω_2) is said to be S-convex if for any compact subset K^2 of $\mathbf{R}^{mn}$ there exists a compact subset K^1 of $\mathbf{R}^{mn}$ so that

$$\varphi\in D^m_\omega(\Omega_1),\quad \operatorname{supp} S^t*\varphi\subset k^2 \Rightarrow \operatorname{supp}\varphi\subset k^1.$$

The couple is said to be strongly S-convex if for any compact subset K^2 of $\mathbf{R}^{mn}$ there exists a compact subset K^1 of $\mathbf{R}^{mn}$ so that

$$\varphi\in E'^m_\omega(\Omega_1),\quad \operatorname{supp}(S^t*\varphi)\subset K_i^2 \Rightarrow \operatorname{supp}\varphi\subset K_i^1,$$

$$\varphi\in E'^m_\omega,\quad \operatorname{sing}_\omega\operatorname{supp}(S^t*\varphi)\subset K_i^2 \Rightarrow \operatorname{sing}_\omega\operatorname{supp}\varphi\subset K_i^1.$$

One can choose the K_i^1's so that K_i^1 is contained in the interior of K_{i+1}^1. This setting will be used in the proof of the next theorem.

THEOREM 3.1. *Let* $S\in E'^{2m}_\omega$. *If* $\det(\hat{S}^t)$ *is ω-slowly decreasing then*

$$S*D'^m_\omega = D'^m_\omega.$$

PROOF. The theorem follows immediately from the Hahn—Banach theorem provided we prove the continuity of the map $S^t*\Phi\mapsto\Phi$ from $S^t*D^m_\omega$ into D^m_ω. To prove the continuity of this map we follow a technique introduced by Hörmander [3]. Let p be any continuous semi-norm on D^m_ω. We show that there exist a continuous semi-norm q defined on D^m_ω and a constant C so that

$$p(\varphi)\leqq Cq(S^t*\varphi),\quad \varphi\in D^m_\omega.$$

This will follow from the following lemma.

LEMMA 3.2. *Let q be a semi-norm on D^m_ω so that* $q(\psi)\geqq\sum_{l=1}^m\|\psi_l\|_1$, $\psi=(\psi_1,\dots,\psi_m)\in D^m_\omega$ *and* $p(\varphi)\leqq q(S^t*\varphi)$ *if* $\varphi\in D^m_\omega$, $\operatorname{supp}\varphi\subset(K^1_{i1},\dots,K^1_{im})$ *where K_{ij} is a compact subset of $\mathbf{R}^n$. Then for every $\varepsilon>0$ there exists a semi-norm q' defined on D^m_ω such that*

$$p(\varphi)\leqq q'(S^t*\varphi)\quad \textit{if}\quad \operatorname{supp}\varphi\subset(K^1_{i+1,1},K^1_{i+1,2},\dots,K^1_{i+1,m}) \tag{23}$$

and

$$q'(\psi)=(1+\varepsilon)q(\psi)\quad \textit{if}\quad \operatorname{supp}\psi\subset(K^2_{i-1,1},K^2_{i-1,2},\dots,K^2_{i-1,m}). \tag{24}$$

PROOF. Suppose the statement of the lemma were false. Since (24) is true for $q'=(1+\varepsilon)q+r$ where r is any semi-norm on $\prod_{j=1}^m E_\omega(C^1_{-1,j})$ it follows that there exists a $\varphi\in D^m_\omega$, $\operatorname{supp}(S^t*\varphi)\subset\prod_{j=1}^m K^1_{i+1,j}$ so that

$$p(\varphi)\geqq q'(S^t*\varphi)=(1+\varepsilon)q(S^t*\varphi)+r(S^t*\varphi), \tag{25}$$

where r is as above and ε is some positive number. Let $\{r_\varrho:\ \varrho=1,2,3,\dots\}$ be a strictly increasing family of semi-norms on $\Pi E_\omega(CK^1_{i-1,j})$, from (25) one gets

$$p(\varphi)\geqq(1+\varepsilon)q(S^t*\varphi)+r_\varrho(S^t*\varphi). \tag{26}$$

For each $\varrho=1,2,\dots$, let $\alpha_\varrho=(1+\varepsilon)/[(1+\varepsilon)q(S^t*\varphi)+r_\varrho(S^t*\varphi)]$ and define the function φ_ϱ by $\alpha_\varrho\varphi$. From (26) one has

$$p(\varphi_\varrho)\geqq(1+\varepsilon) \tag{27}$$

but on the the other hand

$$q(S^t*\varphi_\varrho)=\frac{(1+\varepsilon)q(S^t*\varphi)}{(1+\varepsilon)q(S^t*\varphi)+r_\varrho(S^t*\varphi)}<1,$$

and since r_ϱ is strictly increasing it follows that

$$q(S^t*\varphi_\varrho)\to 0 \quad\text{as}\quad \varrho\to\infty. \tag{28}$$

Now, given an arbitrary semi-norm r on $\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$, let $r_\varrho=\varrho r$; $\varrho=1, 2, 3, \dots$ and we get

$$r(S^t*\varphi_\varrho)\leqq r_\varrho(S^t*\varphi_\varrho)\leqq \varrho^{-1}(1+\varepsilon),$$

hence

$$S^t*\varphi_\varrho\to 0 \quad\text{in}\quad \prod_{j=1}^m E_\omega(CK^1_{i-1,j}) \quad\text{as}\quad p\to\infty. \tag{29}$$

Let Φ be the completion of the set $\{\varphi\in D^m_\omega\colon \operatorname{supp}\varphi C\prod_{j=1}^m K^1_{i+1,j}\}$ provided with the topology defined by the semi-norms $q(S^t*\varphi)$ and $\|f\cdot(S^t*\varphi)\|_k$ where $k=1, 2, \dots$ and $f\in\prod_{j=1}^m D_\omega(CK^1_{i-1,j})$. We claim that Φ is contained in $\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$ and the inclusion map is continuous. For $\varphi=(\varphi_1, \dots, \varphi_m)^t\in D^m_\omega$ put $S^t*\varphi=\psi=(\psi_1, \dots, \psi_m)^t$, i.e. $\hat S^t\cdot\hat\varphi=\hat\psi$. By Cramer's rule it follows that

$$\hat\varphi_j=\frac{\hat W_j}{\det(\hat S^t)}, \tag{30}$$

where $\hat W_j$ is the determinant of the matrix which one gets by replacing the j-th column of $\hat S^t$ by $\hat\psi$. Using the fact that $\det(\hat S^t)$ is slowly decreasing, the Paley—Wiener theorem applied to the entries of $\hat S^t$ it follows from (30), by Hörmander's Lemma, that

$$|\hat\varphi_j|\leqq Ce^{\lambda\omega(\xi)}\sup_{\xi\in\mathbf{R}^n} e^{\omega(\xi)}|\hat\psi_j(\xi)|. \tag{31}$$

From (31) and the assumption that $q(\psi)\geqq\sum_{l=1}^m\|\psi_l\|_1$ it follows that $\Phi\subset E'^m_\omega$. If $\varphi\in\Phi$ one has $S^t*\varphi\in\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$, hence $\varphi\in\prod_{j=1}^m E_\omega(CK^2_{i-1,j})$. Since the space Φ with the given topology is a Frèchet space it follows from the closed graph theorem that the inclusion map $\Phi\hookrightarrow\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$ is continuous. From (27) and (28) it follows that the sequence $\{\varphi_\varrho\colon \varrho=1, 2, 3, \dots\}$ is bounded in Φ. The continuity of the inclusion map $\Phi\hookrightarrow\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$ implies that $\{\varphi_\varrho\}$ is bounded in $\prod_{j=1}^m E_\omega(CK^1_{i-1,j})$ which is a Montel space. Hence $\{\varphi_\varrho\}$ has a convergent subsequence which we denote also by $\{\varphi_\varrho\}$ and we assume that $\varphi_\varrho \to \varphi$ in $\prod_{j=1}^{m} E_\omega(CK^2_{i-1,j})$. We claim that $\varphi=0$ in $\prod_{j=1}^{m} CK^1_{i-1,j}$. Indeed, $S^t * \varphi_\varrho \to S^t * \varphi$ and since $S^t * \varphi_\varrho \to 0$ in $\prod_{j=1}^{m} E_\omega(CK^2_{i-1,j})$ by (29) it follows that $S^t * \varphi = 0$ in $\prod_{j=1}^{m} E_\omega(CK^2_{i-1,j})$. By the choice of the K_i's it follows that $\varphi=0$ in $\prod_{j=1}^{m} E_\omega(CK^1_{i-1,j})$.

Let χ_j be a C^∞-function with support in $K^1_{i,j}$ which is equal to one in $K^1_{i-1,j}$; $j=1, 2, \ldots, m$. Define $\chi=(\chi_1, \ldots, \chi_m)$, thus $\operatorname{supp} \chi\varphi = (\chi_1\varphi_1, \ldots, \chi_m\varphi_m)^t \subset K^1_i = \prod_{j=1}^{m} K^1_{i,j}$. Since φ_ϱ converges to zero in CK^1_{i-1} it follows that $\varphi'_\varrho = (1-\chi)\varphi_\varrho$ converges to zero in CK^1_{i-1}. Hence $S^t * \varphi'_\varrho$ converges to zero in D^m_ω and in particular $q(S^t * \varphi'_\varrho) = q(S^t * (1-\chi)\varphi_\varrho) < \varepsilon/3$ for large ϱ. Now one has $\varphi_\varrho = (1-\chi)\varphi_\varrho + \chi\varphi_\varrho$ and $P((1-\chi)\varphi_\varrho) + P(\chi\varphi_\varrho) \geqq P(\varphi_\varrho) \geqq 1+\varepsilon$ which gives $P(x\varphi_\varrho) \geqq 1 + \frac{2}{3}\varepsilon$ because $(1-\chi)\varphi_\varrho$ converges to 0 in D^m_ω. On the other hand

$$q(S^t * \chi\varphi_\varrho) \leqq q(S^t * \varphi_\varrho) + q\big(S^t * (1-\chi)\varphi_\varrho\big) < 1 + \varepsilon/3,$$

which contradicts the hypothesis of the lemma because $\operatorname{supp} \chi\varphi_\varrho \subset K^1_i$. The contradiction proves the lemma.

Now back to the proof of the theorem. Choose $\varepsilon_j > 0$ so that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. Using the lemma we can successivively construct semi-norms q_i on D^m_ω so that

$$q_{i+1} = (1+\varepsilon_i) q_i(\psi) \quad \text{if} \quad \operatorname{supp} \psi \subset \prod_{j=1}^{m} K^2_{i-1,j} \tag{32}$$

and

$$p(\varphi) \leqq q_i(S^t * \varphi) \quad \text{if} \quad \operatorname{supp} \varphi \subset \prod_{j=1}^{m} K^1_{i+1,j}. \tag{33}$$

Define the semi-norm q by $q(\psi) = \lim_{i\to\infty} q_i(\psi)$, $\psi \in D^m_\omega$. The semi-norm q is well-defined. For let $\psi \in D^m_\omega$; there exists i so that $\psi \in \prod_{j=1}^{m} D_\omega(K^1_{ij})$. From (32) it follows that $q(\psi) = \prod_{i}^{\infty} (1+\varepsilon_l) q_l(\psi)$ which converges by the choice of the ε_i's. The semi-norm q is continuous because its restriction to $\prod_{j=1}^{m} D_\omega(K_{ij})$ is continuous for every $i = 1, 2, 3, \ldots$. From (33) and the fact that q_i is an increasing sequence of semi-norms it follows that

$$p(\varphi) \leqq q(S^t * \varphi) \quad \text{for all} \quad \varphi \in D^m_\omega,$$

which completes the proof of the theorem.

THEOREM 3.2. *Let* $S \in E'^{2m}_\omega$. *If* $S * D'^m_\omega = D'^m_\omega$ *then the map* $\wedge\colon S^t * \varphi \mapsto \varphi$ *from* $S^t * D^m_\omega$ *into* D^m_ω *is continuous.*

PROOF. Since $S^t * D_\omega^m$ is the inductive limit of $S^t * D_\omega^m(K_j)$'s which are metrizable, it suffices to show that the map $\wedge$ is bounded on bounded sets. Let $S^t * B$ be a bounded subset of $S^t * D_\omega^m$. By Macky's theorem, to show that B is bounded it suffices to show that B is weakly bounded. The boundedness of $S^t * B$ implies that there exists a constant M so that $|\langle u, S^t * \varphi\rangle| \leqq M$ for all $\varphi \in B$ and for any given $u \in D_\omega'^m$. Now let v be any element of $D_\omega'^m$, by hypothesis there exists $u \in D_\omega'^m$ so that $S * u = v$. Thus, for any $\varphi \in B$ one has

$$|\langle v, \varphi\rangle| = |\langle S * u, \varphi\rangle| = |\langle u, S^t * \varphi\rangle| \leqq M,$$

i.e. B is weakly bounded in D_ω^m.

The converse of the above theorem is true, it follows immediately from the Hahn—Banach theorem.

We say that S is ω-invertible whenever $\det(\hat{S}^t)$ is ω-slowly decreasing.

Throughout the rest of the section v will be an element of $E_\omega'^m$, and for each $j = 1, \ldots, m$ we denote by $\hat{w}_j$ the determinant of the matrix which one gets by replacing the j-th column of $\hat{S}^t$ by $\hat{v}$.

THEOREM 3.3. *If S is ω-invertible and the function $\hat{w}_j/\det(\hat{S}^t)$ is entire then there exists $u \in E_\omega'^m$ so that $S * u = v$.*

PROOF. By the invertibility of S one has $\hat{S}\hat{u} = \hat{v}$ for some $u \in D_\omega'^m$. By Cramer's rule one gets

$$\hat{u}_j = \frac{\hat{w}_j}{\det(\hat{S}^t)}; \quad j = 1, 2, \ldots, m. \tag{34}$$

Applying Hörmander's lemma with $r = A\omega(\xi) + |\eta|$ when $\zeta = \xi + i\eta$ is any given point in $\mathbf{C}^n$ and A is the constant of Definition 2.1, one gets

$$|\hat{u}_j(\zeta)| \leqq \sup_{\substack{|z-\zeta|<4r \\ z \in \mathbf{C}^n}} |\hat{w}_j(z)| \sup_{\substack{|z-\zeta|<4r \\ z \in \mathbf{C}^n}} |\det(\hat{S}^t)(z)| / \Big[\sup_{\substack{|z-\zeta|<r \\ z \in \mathbf{C}^n}} |\det(\hat{S}^t)(z)|\Big]^2. \tag{35}$$

By applying the Paley—Wiener theorem to each of the entries of $\hat{S}^t$ it follows that

$$\sup_{\substack{|z-\zeta|<4r \\ z = x+iy \in \mathbf{C}^n}} |\det(\hat{S}^t)(z)| \leqq C_1 \sup_{\substack{|z-\zeta|<4A\omega(\xi)+4|\eta| \\ z \in \mathbf{C}^n}} e^{N_1\omega(x)+A_1|y|} \leqq \tag{36}$$

$$\leqq C_1 e^{N_1\omega(\xi)+A_1|\eta|} \sup_{|z|<4A\omega(\xi)+4|\eta|} e^{N_1\omega(x)+A_1|y|} \leqq C_1 e^{N_1 M} e^{(N_1+4N_1MA+4A_1A)\omega(\xi)+(4MN_1+4A_1)|\eta|};$$

for every $\zeta \in \mathbf{C}^n$ where C_1, N_1, M and A_1 are constants. Similarly, since $v \in E_\omega'^m$ one has

$$\sup_{\substack{|z-\zeta|<4r \\ z \in \mathbf{C}^n}} |\hat{w}_j(z)| \leqq C_2 e^{N_2\omega(\xi)+A_2|\eta|}, \tag{37}$$

for some constants C_2, N_2 and A_2. On the other hand invertibility of S implies that

(38)

$$\sup_{\substack{|z-\zeta|<r \\ z \in \mathbf{C}^n}} |\det(\hat{S}^t)(z)| = \sup_{|z-i\eta|<A\omega(\xi)+|\eta|} |\det(\hat{S}^t)(z+\xi)| \geqq \sup_{|x-i\eta|<A\omega(\xi)+|\eta|} |\det(\hat{S}^t)(x+\xi)| \geqq$$

$$\geqq \sup_{|x|<A\omega(\xi)} |\det(\hat{S}^t)(x+\xi)| \geqq C_3 e^{-A\omega(\xi)}$$

substituting (36), (37) and (38) in (35) gives

$$|\hat{u}_j(\zeta)| \leqq Ce^{N\omega(\xi)+A|\eta|}, \quad \zeta = \xi + i\eta \in \mathbf{C}^n, \quad j = 1, 2, \dots, m;$$

where C, N and A are constants, i.e. $u_j \in E'_\omega$ for each $j=1, \dots, m$.

DEFINITION 3.1. Let $u=(u_1, \dots, u_m)^t \in D'^m_\omega$, a compact subset $K=(K_1, \dots, K_m)$ of $\mathbf{R}^{mn}$ is said to be the ω-singular support of u, denoted $\operatorname{sing}_\omega \operatorname{supp} u$ if each K_j is the ω-singular support of u_j; $j=1, \dots, m$.

THEOREM 3.4. *Let $S \in E'^{2m}_\omega$ be invertible. For any compact set $K_2 \subset \mathbf{R}^{mn}$ there exists a compact set $K_1 \subset \mathbf{R}^{mn}$ such that*

$$u \in E'^m_\omega, \quad \operatorname{sing}_\omega \operatorname{supp}(S*u) \subset K_2 = \operatorname{sing}_\omega \operatorname{supp} u \subset K_1.$$

PROOF. Let $\zeta=\xi+i\eta \in \mathbf{C}^n$ so that $|\eta| \leqq r\omega(\xi)$; $r=1, 2, 3, \dots$. Put $S*u=v$ then $\hat{S}\hat{u}=\hat{v}$ and by Cramer's rule one has

$$\hat{u}_j = \frac{\hat{w}_j}{\det(\hat{S}^t)}; \quad j = 1, 2, \dots, m, \tag{39}$$

where $\hat{u}_j$, $\hat{w}_j$ and $\det(\hat{S}^t)$ are entire functions. For $z=x+iy \in \mathbf{C}^n$, $|z|<4A\omega(\xi)+4|\eta|$ one has $|y+\eta| \leqq (4A+5r)\omega(\xi)$ and

$$1-\frac{|x+\xi|}{|\xi|} \leqq \frac{|z|}{|\xi|} \leqq (4A+5r)\frac{\omega(\xi)}{|\xi|}.$$

Proposition 1.1 implies that $|\xi| \leqq 100|\xi+x|$ whenever $|\xi|$ is large enough, hence $\omega(\xi) \leqq 100\omega(\xi+x)$ for $|\xi|$ large. By applying the Paley—Wiener theorem to the entries of $\hat{S}$ and Theorem 1.2 to the components of v we have for each $j=1, 2, \dots, m$

$$\sup_{\substack{|z-\zeta|<4A\omega(\xi)+4|\eta| \\ x+iy=z\in \mathbf{C}^n}} |\hat{w}_j(z)| \leqq C_r \sup_{\substack{|z|<4A\omega(\xi)+4|\eta| \\ z\in \mathbf{C}^n}} e^{\lambda\omega(x+\xi)+A_1|y|+\sum_{l=1}^m H_l(y)} \leqq \tag{40}$$

$$\leqq C'_r\, e^{\lambda'\omega(\xi)+\sum_{l=1}^m H_l(\eta)+A'_1|\eta|} \sup_{|z|<4A\omega(\xi)+4|\eta|} e^{\sum_{l=1}^m H_l(y)},$$

whenever $|\eta| \leqq r\omega(\xi)$, where C_r, C'_r, λ, λ', A_1 and A'_1 are constants and H_l is the support function of K_{1l}; $K_1=(K_{11}, K_{12}, \dots, K_{1m})$. Since the K_{1l}'s are compact one can find constants k_l so that $H_l(y) \leqq k_l|y|$.

Now, applying Hörmander's lemma with $r=A\omega(\xi)+|\eta|$ to (39) and using (36), (38) and (40) we get

$$|\hat{u}_j(\zeta)| \leqq C^j_r e^{\lambda_j\omega(\xi)+A_j|\eta|+\sum_{l=1}^m H_l|\eta|} \quad \text{whenever} \quad |\eta| \leqq r\omega(\xi), \tag{41}$$

$\zeta=\xi+i\eta \in \mathbf{C}^n$, $r=1, 2, 3, \dots$, and C^j_r, λ_j, A_j are constants independent of ζ. For each $j=1, 2, \dots, m$ define the function $H_j(\eta)=A_j|\eta|+\sum_{l=1}^m H_l(\eta)$ where the H_l's are as above. Clearly H_j is a support function of some compact subset K_{2j} of $\mathbf{R}^n$.

Hence one could rewrite (41) as

(42)

$$|\hat{u}_j(\zeta)| \leqq C_r^j e^{\lambda_j \omega(\xi)+H_j(\eta)} \quad \text{whenever} \quad |\eta| \leqq r\omega(\xi), \quad r = 1, 2, \ldots \quad j = 1, 2, \ldots, m,$$

i.e. $\operatorname{sing}_\omega \operatorname{supp} u_j \subset K_{2j}$. Hence $\operatorname{sing}_\omega \operatorname{supp} u \subset K_2 = (K_{21}, \ldots, K_{2j})$.

REMARK. I believe that the continuity of the map $S^t * \varphi \to \varphi$ from $S^t * D_\omega^m$ into D_ω^m implies that $\det(\hat{S}^t)$ is ω-slowly decreasing but I do not have a complete proof for this claim. If proved this will complete the cycle made of Theorems 3.1 and 3.2. In case $m=1$ this was part of Theorem 2.1.

References

[1] G. Björk, Linear partial differential operators and generalized distributions, *Arkiv för Mathematik*, **6** (1966), 351—407.
[2] L. Ehrenpreis, Solutions of some problems of division IV, *Am. J. Math.*, **82** (1960), 522—588.
[3] L. Hörmander, On the range of convolution operators, *Annals of Mathematics*, **76** (1962), 148—169.
[4] S. Lang, *Algebra*, Addison-Wesley (Reading, MA, 1971).

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A FINITE METHOD FOR GLOBALLY MINIMIZING A CONCAVE FUNCTION OVER AN UNBOUNDED POLYHEDRAL CONVEX SET AND ITS APPLICATIONS

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1. Introduction

In this paper we shall be concerned with the following concave programming problem:

(1) Globally minimize $f(x)$, subject to $x \in D$,

where $f\colon R^n \to R$ is a real-valued function, concave and continuous on R^n, D is a polyhedral, not necessarily bounded convex set in R^n. Usually, D is given explicitly by a finite system of linear equalities and inequalities of the form

$$(a^i, x)+b_i = 0, \quad i \in I^0,$$
$$(a^i, x)+b_i \leqq 0, \quad i \in I^-,$$

with a^i being n-dimensional vectors, b_i real numbers and I^0, I^- finite sets of indices.

For the case where the constraint set D is bounded, i.e. is a polytope, this problem was first studied by H. Tuy in [7] and then by a number of researchers. Recently, some of the authors have also been interested in the more general case where D may be unbounded (see [1], [8], [12]) and, as far as we known, the most general problem of minimizing a concave function over an arbitrary closed convex set was considered for the first time in [11] (see also [9]). The algorithms presented in [8], [12] are that of the branch and bound type which proceed according to the cone splitting scheme and the cone bisection procedure worked out in Tuy [7] and in Thoai and Tuy [6] respectively. However, both algorithms are in general infinite (though surely convergent) and require that a linear program be solved at each step of an infinite iterative procedure. The recent algorithm of V. T. Ban [1] which is a further development of the basic ideas proposed in [6, 7] — is finite, taking in account the linear structure of the constraint set. However, in many applications we have to treat a sequence of concave minimization problems which differ from one another just by one additional constraint. It is therefore of interest to construct an algorithm which could take advantage of this property. Such an algorithm was given in [4] for the case of polytopes. The purpose of the present paper is to develop another algorithm of this kind for the more general case where the constraint set is a polyhedral, not necessarily bounded, convex one.

The paper consists of six sections. After the Introduction, we shall present in Section 2 a comparatively simple and suitable technique for determining the vertices and the extreme directions of a polyhedral convex set obtained from a given polyhedral convex one, whose vertices and extreme directions are known, by adding a new linear constraint. This technique is frequently used in solving Problem (1) by the algorithm we shall present and can be regarded as a further development of the previous one for the case of polytopes [4]. Then, in Section 3, we shall develop a finite algorithm for globally minimizing a concave function over a polyhedral, not necessarily bounded, convex set which proceeds according to the same scheme presented in [4] for the polytope case. This method is based on relaxing the constraints of the problem to be solved and on gradually adding constraints, one per iteration, until an optimal solution is reached. The present algorithm is different from that of V. T. Ban [1] and, as shown in the sequel, it is practically suited to solve a sequence of problems which differ from one another just by one additional constraint. In Section 4 we shall discuss several particular features of applying the present algorithm to the bilinear programming, the linear complementarity problems and concave minimization problems with special structure. In Section 5 a two-dimensional example is presented to illustrate how the algorithm works in practice and in the last section some computational experience is reported.

2. A subsidiary problem

In this section, we examine how to determine the set of vertices and extreme directions of a polyhedral convex set obtained from a given polyhedral convex one, whose vertices and extreme directions are known, by adding a new linear constraint. The main results which will play a basic role in solving Problem (1) may also have some interest in themselves.

Let there be given a polyhedral convex set M defined by a system of linear inequality constraints

$$g_i(x) = (a^i, x) + b_i \leqq 0, \quad i = 1, \dots, m, \tag{2}$$

where a^i are n-dimensional vectors, b_i are real numbers, $m \geqq n$. Suppose we already know the set U of vertices and the set V of extreme directions of M, i.e. we have the representation

$$M = \operatorname{co} U + \operatorname{cone} V.$$

We shall assume that $U \neq \emptyset$ (M has at least one vertex) and V may be empty (M is a polytope). Given an affine function

$$h(x) = (h, x) + g$$

with h being an n-dimensional non-zero vector, g a real number, let

$$N = M \cap \{x \in R^n \colon h(x) \leqq 0\}. \tag{3}$$

Clearly N is also a polyhedral convex set. The question arises as to determine the set P of vertices and the set Q of extreme directions of N.

To answer this question denote

$$U^- = \{u \in U \colon h(u) < 0\}, \quad U^+ = \{u \in U \colon h(u) > 0\}, \tag{4}$$

$$V^- = \{v \in V \colon (h, v) < 0\}, \quad V^+ = \{v \in V \colon (h, v) > 0\}, \tag{5}$$

$$H = \{x \in R^n \colon h(x) = 0\}.$$

With these notations in mind we now prove several results.

PROPOSITION 1. *If* $U^+=V^+=\emptyset$ *then* $N=M$, *i.e.* $P=U$ *and* $Q=V$.

PROOF. From the hypotheses, $h(u)\leqq 0$ for all $u\in U$ and $(h, v)\leqq 0$ for all $v\in V$. Every $x\in M$ can be expressed in the form

$$x=\sum_{u\in U}\alpha_u u+\sum_{v\in V}\beta_v v$$

with $\alpha_u\geqq 0$, $\beta_v\geqq 0$ and $\sum\alpha_u=1$, hence

$$h(x)=\sum\alpha_u h(u)+\sum\beta_v(h, v)\leqq 0.$$

This means that $x\in N$ and therefore, $M\subset N$. The converse inclusion is obvious. Thus $N=M$, as was to be proved.

PROPOSITION 2. *Suppose* $U^-=V^-=\emptyset$. *We have*

a) *If* $U^+=U$ *then* $N=\emptyset$, *i.e.* $P=Q=\emptyset$.

b) *Otherwise,* $P=U\setminus U^+$ *and* $Q=V\setminus V^+$.

PROOF. a) $U^+=U$ and $V^-=\emptyset$ imply $h(u)>0$ for all $u\in U$ and $(h, v)\geqq 0$ for all $v\in V$. Therefore, we have for every $x\in M$

$$h(x)=\sum_{u\in U}\alpha_u h(u)+\sum_{v\in V}\beta_v(h, v)>0$$

(note that there exists at least one $\alpha_u>0$). So, $N=\emptyset$.

b) $U^-=V^-=\emptyset$ implies $h(u)\geqq 0$ for all $u\in U$ and $(h, v)\geqq 0$ for all $v\in V$. Therefore, $h(x)\geqq 0$ for all $x\in M$ and hence, $M\subset\{x:\ h(x)\geqq 0\}$. So we have

$$N=M\cap\{x:\ h(x)=0\}=M\cap H\neq\emptyset$$

(since $U\setminus(U^+\cup U^-)=U\setminus U^+\neq\emptyset$). This shows that N is a face of M. Therefore, each vertex (extreme direction) of N is also a vertex (an extreme direction) of M. Thus $P=U\setminus U^+$ and $Q=V\setminus V^+$, as was to be proved.

PROPOSITION 3. *If* $U^+\cup V^+\neq\emptyset$ *and* $U^-\cup V^-\neq\emptyset$, *then we have*

a) $P\cap U=U\setminus U^+$;

b) *any vertex* $w\in P\setminus U$ *must be the intersection of the hyperplane* H *either with a bounded edge of* M *connecting a vertex* $u\in U^-$ *with a vertex* $u'\in U^+$, *or with an unbounded edge of* M *emanating from a vertex* $u\in U^-(U^+)$ *in a direction* $v\in V^+(V^-)$.

PROOF. a) Every $u\in U^+$ will not belong to N and hence, neither to P (since $h(u)>0$). Conversely, every $u\in U\setminus U^+$ still belongs to N (since $h(u)\leqq 0$), hence it is still a vertex of N. So we have $U\setminus U^+=P\cap U$ (this relation still holds even if $U\setminus U^+=\emptyset$).

b) Let now $w\in P\setminus U$. Denote by $F(w)$ the smallest face of M containing w. Since $w\notin U$ we must have $F(w)\neq\{w\}$ and hence, $\dim F(w)\geqq 1$. If $\dim F(w)>1$ then $F(w)$ would have in common with H a line segment containing w in its relative interior, which would contradict the fact that w is a vertex of N. Therefore, $\dim F(w)=1$ and $F(w)$ is an edge (bounded or unbounded) of M. Two cases are possible:

Case 1. $F(w)$ is a bounded edge of M. For example, $F(w)=[u, u']$ with $u, u'\in U$. Then $w=tu+(1-t)u'$ for some t: $0<t<1$. This implies $h(u)\neq\emptyset$, $h(u')\neq\emptyset$. From the relation

$$h(w) = th(u)+(1-t)h(u') = 0$$

it follows that $h(u)\cdot h(u')<0$.

Case 2. $F(w)$ is an unbounded edge of M. For example, $F(w)=\{u+\theta v\colon \theta\geqq 0\}$ with $u\in U$, $v\in V$. Then $w=u+tv$ for some $t>0$. This implies $h(u)\neq 0$, $(h, v)\neq 0$. Since

$$h(w) = h(u)+t\cdot(h, v) = 0$$

we must have $h(u)\cdot(h, v)<0$, completing the proof.

PROPOSITION 4. *Under the hypotheses of Proposition* 3 *we have*

a) $Q\cap V=V\setminus V^+$;

b) *any extreme direction* $v\in Q\setminus V$ *satisfies* $(h, v)=0$ *and is of the form* $v=\lambda p+\mu q$ *with* $\lambda, \mu>0$ *and* $(p, q)\in V^-\times V^+$ *defining a two-dimensional face of the recession cone of* M.

PROOF. a) Let K and T denote the recession cone of M and N respectively. It follows from (2), (3) that

$$K = \operatorname{cone} V = \{x\in R^n\colon (a^i, x) \leqq 0,\ i = 1, \ldots, m\},$$

$$T = \operatorname{cone} Q = K\cap\{x\in R^n\colon (h, x) \leqq 0\}.$$

This shows that $V\setminus V^+=Q\cap V$.

b) Let now $v\in Q\setminus V$. Since $v\in Q$, among the constraints defining T there are $(n-1)$ linearly independent constraints binding for v. Further, since $v\notin V$, one of these $n-1$ binding constraints must be $(h, v)=0$. Let J denote the index set of the remaining $n-2$ binding constraints: $J\subset\{1, \ldots, m\}$, $|J|=n-2$. Then

$$Z(v) = \{x\in K\colon (a^i, x) = 0,\ i\in J\}$$

is the smallest face of K containing the ray $\{tv\colon t\geqq 0\}$. Certainly, $Z(v)\neq\{tv\colon t\geqq 0\}$, for otherwise v would be an extreme direction of M, i.e. $v\in V$. Therefore, $\dim Z(v)=2$ and hence, $Z(v)$ is a two-dimensional cone defined, for example, by two extreme directions p and q that belong to V. We thus have

$$v = \lambda p+\mu q \quad \text{with some} \quad \lambda, \mu > 0.$$

This implies $(h, p)\neq 0$, $(h, q)\neq 0$. From the relation

$$(h, v) = \lambda(h, p)+\mu(h, q) = 0$$

it follows that $(h, p)\cdot(h, q)<0$, completing the proof.

On the basis of Propositions 3, 4 one can determine the new vertices of N (that belong to $P\setminus U$) and the new extreme directions of N (that belong to $Q\setminus V$), in the case $U^+\cup V^+\neq 0$ and $U^-\cup V^-\neq\emptyset$, as follows.

RULE A *for finding the new vertices of* N.

a) For any pair $(u, u')\in U^-\times U^+$ determine the point

$$w = t\cdot u+(1-t)\cdot u', \quad \text{where} \quad t = h(u')/(h(u')-h(u)).$$

b) For any pair $(u, v) \in \{U^- \times V^+\} \cup \{U^+ \times V^-\}$ determine the point

$$w = u + t \cdot v, \quad \text{where} \quad t = -h(u)/(h, v).$$

For each w defined by a) or b) denote by $I(w)$ the index set of constraints of the form (2) that define M and are binding for w:

$$I(w) = \{i \colon g_i(w) = 0, \ i = 1, \ldots, m\}.$$

It can be seen that in Case a)

$$I(w) = \{i \colon g_i(u) = g_i(u') = 0, \ i = 1, \ldots, m\}$$

and in Case b)

$$I(w) = \{i \colon g_i(u) = (a^i, v) = 0, \ i = 1, \ldots, m\}.$$

Then, as can easily be verified,

$$F(w) = \{x \in M \colon g_i(x) = 0, \ i \in I(w)\}$$

is the smallest face of M containing w. Therefore, if $|I(w)| < n-1$ or if there exists a vertex $z \in U \setminus \{u, u'\}$ (in Case a)) or $z \in U \setminus \{u\}$ (in Case b)), such that $g_i(z) = 0$ for all $i \in I(w)$ (i.e. $z \in F(w) \cap U$), then $\dim F(w) > 1$ and hence, by Proposition 3, w cannot be a vertex of N: $w \notin P$. Otherwise, $\dim F(w) = 1$ and w is a vertex of N: $w \in P$.

RULE B *for finding the new extreme directions of N.*

For any pair $(p, q) \in V^- \times V^+$, determine the point $v = (h, q)p - (h, p)q$. It is easily seen that $v \in K$ and $(h, v) = 0$. Let $J(v) = \{j \colon (a^j, v) = 0, j = 1, \ldots, m\}$. It can be seen that $J(v) = \{j \colon (a^j, p) = (a^j, q) = 0, j = 1, \ldots, m\}$. Then, as can easily be verified,

$$Z(v) = \{x \in K \colon (a^j, x) = 0, \ j \in J(v)\}$$

is the smallest face of K containing the ray $\{tv \colon t \geqq 0\}$. Therefore, if $|J(v)| < n-2$ or if there is at least one $z \in V \setminus \{p, q\}$, such that $(a^j, z) = 0$ for all $j \in J(v)$ (i.e. $z \in Z(v) \cap K$) then $\dim Z(v) > 2$ and hence, by Proposition 4, v cannot be an extreme direction of N: $v \notin Q$. Otherwise, $\dim Z(v) = 2$ and v is an extreme direction of N: $v \in Q$.

We have already taken up the question of determining the set of vertices and extreme directions of a polyhedral convex set generated from a given polyhedral convex one, whose vertices and extreme directions are known, by adding a new linear inequality constraint.

We now turn to consider a special case of the above question where instead of (3), N is defined by

$$N = M \cap \{x \in R^n \colon h(x) = (h, x) + g = 0\}, \tag{6}$$

i.e. N is obtained from M by adding a new linear equality constraint. Since one equality is equivalent to two inequalities, to determine the set of vertices and extreme directions of N given by (6) we can repeat twice the above procedure and as an immediate consequence of Propositions 1—4 we have

COROLLARY 1. *Let there be given a polyhedral convex set M with vertex set U and extreme direction set V. Let N be obtained from M by* (6), *and let P and Q be the vertex set and the extreme direction set of N respectively. With U^-, U^+, V^-, V^+ being as before (see* (4), (5)*) we have:*

a) *Suppose $U^+=V^+=\emptyset$. If $U^-=U$ then $N=\emptyset$, i.e. $P=Q=\emptyset$. Otherwise, $P=U\setminus U^-$, $Q=V\setminus V^-$.*

b) *Suppose $U^-=V^-=\emptyset$. If $U^+=U$ then $N=\emptyset$, i.e. $P=Q=\emptyset$. Otherwise, $P=U\setminus U^+$, $Q=V\setminus V^+$.*

c) *Suppose $U^+\cup V^+\neq\emptyset$ and $U^-\cup V^-\neq\emptyset$. Then*

$$P\cap U = U/\{U^+\cup U^-\},\quad Q\cap V = V\setminus\{V^+\cup V^-\}.$$

Furthermore, any vertex $w\in P\setminus U$ must be the intersection of the hyperplane $h(x)=0$ either with a bounded edge $[u, u']$ of M, such that $h(u)\cdot h(u')<0$, or with an unbounded edge $\{u+\theta v\colon \theta\geqq 0\}$ of M such that $h(u)\cdot(h, v)<0$; and any extreme direction $v\in Q\setminus V$ satisfies $(h, v)=0$ and is of the form $v=\lambda p+\mu q$ with $\lambda, \mu>0$ and $(p, q)\in V^-\times V^+$ defining a two-dimensional face of the recession cone of M.

In Case c) the method for determining the new vertices of N (that belong to $P\setminus U$) and the new extreme directions of N (that belong to $Q\setminus V$) is completely similar to the previous one.

Furthermore, if M is a polytope (i.e. $V=\emptyset$) then of course N is a polytope too and the determination of the vertex set P of N is a comparatively easy task. Namely, we get

COROLLARY 2 *(inequality constraint case). Let there be given a polytope M with vertex set U and let N be obtained from M by* (3) *with vertex set P. With U^-, U^+ being defined by* (4) *we have:*

a) *If $U^+=\emptyset$ then $N=M$, i.e. $P=U$.*

b) *If $U^-=\emptyset$ then $N=\emptyset$, i.e. $P=\emptyset$, in the case $U^+=U$ and $P=U\setminus U^+$ in the case $U^+\neq U$.*

c) *If $U^+\neq\emptyset$, $U^-\neq\emptyset$ then $P\cap U=U\setminus U^+$ and any vertex $w\in P\setminus U$ must be the intersection of the hyperplane $h(x)=0$ with some edge of M connecting a vertex $u\in U^-$ with a vertex $v\in U^+$.*

We thus recover in this special case the results of [4].

COROLLARY 3 *(equality constraint case). Let there be given a polytope M with vertex set U and let N be obtained from M by* (6) *with vertex set P. With U^-, U^+ being defined by* (4) *we have:*

a) *Suppose $U^+=\emptyset$. If $U^-=U$ then $N=\emptyset$ and $P=\emptyset$. Otherwise, $P=U\setminus U^-$.*

b) *Suppose $U^-=\emptyset$. If $U^+=U$ then $N=\emptyset$ and $P=\emptyset$. Otherwise, $P=U\setminus U^+$.*

c) *If $U^+\neq\emptyset$ and $U^-\neq\emptyset$ then $P\cap U=U\setminus\{U^+\cup U^-\}$ and any vertex $w\in P\setminus U$ must be the intersection of the hyperplane $h(x)=0$ with some edge of M connecting a vertex $u\in U^-$ with a vertex $v\in U^+$.*

In Case c) of Corollaries 2, 3, to determine the new vertices of N (that belong to $P\setminus U$) one can carry out as before (but only $w=tu+(1-t)v$ with $u\in U^-$, $v\in U^+$ must be examined).

REMARK 1. It can be easily verified that the above still holds even if M is given by a finite system of linear equality and inequality constraints.

REMARK 2. In the case where M is a simplex the intersection of the hyperplane $h(x)=0$ with any edge of M connecting a vertex $u\in U^-$ with a vertex $v\in U^+$ is exactly a vertex of N, so the determination of the new vertices of N in this case is quite easy.

REMARK 3. By repeated application of the above, one can determine all the vertices and all the extreme directions of a polyhedral convex set D of the form

$$D = \{x\in R^n_+ : (a^i, x)+b_i R_i 0, \ i = 1, \dots, m\} \tag{7}$$

with R_i being one of the three relations $=$, $\leqq$, $\geqq$, by starting with $S_1=R^n_+$ containing D. Clearly S_1 has the only vertex 0 — the origin of coordinates, and n extreme directions e^j — the j-th unit vector in R^n $(j=1, \dots, n)$.

3. Finite algorithm for concave minimization under linear constraints

The problem of concern can be restated as follows.
Minimize $f(x)$, subject to

$$(a^i, x)+b_i \leqq 0, \quad i = 1, \dots, m, \tag{8}$$

$$x_j \geqq 0, \quad j = 1, \dots, n, \tag{9}$$

where $f\colon R^n \to R$ is a concave function, defined throughout R^n (hence continuous), a^i are n-dimensional vectors and b_i are real numbers. Denote by D the set of all points x satisfying (8), (9).

We first prove two lemmas we shall need later.

LEMMA 1. *Let $f(x)$ be a concave function on a convex set M. If f is unbounded below on a ray of M with direction w and continuous at every point of this ray, then f is also unbounded below on any ray that lies entirely inside M and has the same direction w.*

PROOF. Let $f(x)$ be continuous and bounded below on the ray $\Gamma_1 = \{u+\lambda w\colon \lambda\geqq 0\}$ and $\Gamma_2=\{v+\lambda w\colon \lambda\geqq 0\}\subset M$. Suppose the contrary that $f(x)$ is bounded below over Γ_2, i.e. $f(x)\geqq\gamma$ for all $x\in\Gamma_2$. Define $\beta=\min\{\gamma, f(u)\}$. Since $f(x)$ is unbounded below over Γ_1 there is $\lambda_1>0$ such that $f(u+\lambda_1 w)<\beta$. By virtue of the continuity of f on Γ_1 there exists $\delta>0$ such that $f(x)<\beta$ for all $x\in M\cap W$, where W is the ball of radius δ around $u+\lambda_1 w$. On the one hand the points

$$x = \alpha v+(1-\alpha)u+\lambda_1 w = u+\lambda_1 w+\alpha(v-u), \quad 0<\alpha<1$$

with $\alpha<\delta/\|v-u\|$ will belong to $M\cap W$ and hence, $f(x)<\beta$.

On the other hand for all points of this form we have

$$f(x) = f\left(\alpha\left(v+\frac{\lambda_1}{\alpha}w\right)+(1-\alpha)u\right) \geqq$$

$$\geqq \alpha f\left(v+\frac{\lambda_1}{\alpha}w\right)+(1-\alpha)f(u) \geqq \alpha\gamma+(1-\alpha)f(u) \geqq \beta.$$

We thus arrive at a contradiction and hence, the proof is complete.

LEMMA 2. *Let $f(x)$ be a function, concave and continuous on a given polyhedral convex set M having at least one vertex. If f is bounded below in any extreme ray of M then f is also bounded below on an arbitrary extreme ray of any polyhedral convex set N contained in M.*

PROOF. It follows from the hypotheses that f attains its minimum over M at some of its vertex u. If Γ is an arbitrary (fixed) extreme ray of N then $\Gamma \subset N \subset \subset M$ and

$$\inf\{f(x)\colon x \in \Gamma\} \geqq \inf\{f(x)\colon x \in M\} = f(u) > -\infty.$$

This shows that f cannot be unbounded below over Γ, as was to be proved.

Observe, in passing, that under the hypotheses of Lemma 2 the minimum of f over N is also attained in at least one vertex of N.

Denote now by S_1 the orthant R^n_+. Let U_1 be the vertex set and V_1 the extreme direction set of S_1. It is easily seen that $U_1 = \{0\}$, $V_1 = \{e^1, \dots, e^n\}$, where e^j is the j-th unit vector in R^n $(j = 1, \dots, n)$. Let $I_1 = \{m+1, \dots, m+n\}$ be the index set of the constraints defining S_1 (the index $m+j$ corresponds to the constraint $x_j \geqq 0$).

Iteration $k = 1, 2, \dots, m$. At this iteration we already have a polyhedral convex set $S_k \supset D$ along with the set U_k of vertices and the set V_k of extreme directions of S_k (generally, U_k is non-empty, but V_k may be empty) and the index set I_k of the constraints (8), (9) defining S_k. Let $J_k = \{1, \dots, m\} \setminus I_k$.

Step 1. It is known that a concave function is either unbounded below over a ray or attains its minimum at the origin of this ray (see e.g. [2]). Therefore if there exists $v \in V_k$ and $\theta > 0$ such that $f(\theta v) < f(0)$, then f is unbounded below over the ray $\{tv\colon t \geqq 0\}$ and hence, by Lemma 1, f is unbounded below over any ray emanating from some point of S_k in the direction v. Compute

$$\alpha = \max_{i \in J_k} \{(a^i, v)\}. \tag{10}$$

a) If $\alpha \leqq 0$, i.e. $(a^i, v) \leqq 0$ for all $i = 1, \dots, m$, terminate, since whenever $D \neq \emptyset$, the problem has no finite optimal solution and v is a direction of recession of D over which $f(x)$ is unbounded below.

b) Otherwise, select

$$i_k = \arg\max\{(a^i, v)\colon i \in J_k\} \tag{11}$$

and go to Step 3.

Step 2. If there is no such an extreme direction v then the minimum of $f(x)$ over S_k is always attained in at least one vertex of S_k. So, we select

$$w^k = \arg\min\{f(u)\colon u \in U_k\}$$

(if there are several candidates, take any one of them). Compute

$$\beta = \max_{i \in J_k} \{(a^i, w^k) + b_i\}. \tag{12}$$

a) If $\beta \leqq 0$, i.e. $(a^i, w^k) + b_i \leqq 0$ for all $i = 1, \dots, m$, stop: w^k is an optimal solution of problem (7)—(9).

b) Otherwise, select

$$i_k = \arg\max \{(a^i, w^k)+b_i\colon\ i\in J_k\} \tag{13}$$

and go to Step 3.

Step 3. Form the new polyhedral convex set

$$S_{k+1} = S_k \cap \{x\colon\ (a^{ik}, x)+b_{ik} \leqq 0\}, \tag{14}$$

$$I_{k+1} = I_k \cup \{i_k\},\quad J_{k+1} = J_k \setminus \{i_k\}.$$

Find the set U_{k+1} of vertices and the set V_{k+1} of extreme directions of S_{k+1}, using the techniques described in Section 2 (see Propositions 1—4, Rule A, Rule B). If $S_{k+1}=\emptyset$ is discovered then $D=\emptyset$ and stop. Otherwise, set $k\leftarrow k+1$ and return to Step 1.

PROPOSITION 5. *The above algorithm stops after at most m iterations.*

PROOF. Since at each iteration the current polyhedral convex set S_k is obtained from the previous one, S_{k-1}, by adding one new constraint, since all these constraints are taken from the system (8), it is easily seen that the above algorithm stops after at most m iterations.

REMARK 4. If at a certain iteration k the situation 1 does not occur, by Lemma 2, neither will it at any iteration $h>k$. So, henceforth, at each iteration $h\geqq k$, having found the set of vertices and extreme directions of S_{k+1} one could turn directly to Step 2. Specifically, for the case where D is bounded, i.e. is a polytope, each iteration of the Algorithm consists of Steps 2—3 only, and the Algorithm is in this case just the same one developed in [4].

REMARK 5. For convenience we have restricted ourselves to the problem with linear inequality constraints only. With minor modifications the above algorithm goes through in the case of linear equality constraints. Indeed, if we replace, for example, (8) by

$$\begin{cases}(a^i, x)+b_i = 0, & i\in I^0\\ (a^i, x)+b_i \leqq 0, & i\in I^-\end{cases} \tag{8'}$$

with $I^0=\{1, \dots, r\}$ and $I^-=\{r+1, \dots, m\}$, then we have to replace (10)—(13) by (10′)—(13′) respectively:

$$\alpha = \max \Big\{\max_{i\in J_k\cap I^0} \{|(a^i, v)|\},\ \max_{i\in J_k\cap I^-} \{(a^i, v)\}\Big\}, \tag{10'}$$

$$i_k = \arg\max \Big\{\max_{i\in J_k\cap I^0} \{|(a^i, v)|\},\ \max_{i\in J_k\cap I^-} \{(a^i, v)\}\Big\}, \tag{11'}$$

$$\beta = \max \Big\{\max_{i\in J_k\cap I^0} \{|(a^i, w^k)+b_i|\},\ \max_{i\in J_k\cap I^-} \{(a^i, w^k)+b_i\}\Big\}, \tag{12'}$$

$$i_k = \arg\max \Big\{\max_{i\in J_k\cap I^0} \{|(a^i, w^k)+b_i|\},\ \max_{i\in J_k\cap I^-} \{(a^i, w^k)+b_i\}\Big\} \tag{13'}$$

and (14), if $i_k\in I^0$, by

$$S_{k+1} = S_k \cap \{x\colon\ (a^{i_k}, x)+b_{i_k} = 0\}. \tag{14'}$$

REMARK 6. In the proposed Algorithm instead of R^n_+ we can start from an arbitrary polyhedral convex set $S_1 \supset D$ provided its vertices and extreme directions are known (or easily computed). The Algorithm can also be applied to the case where $f(x)$ is continuous on the chosen S_1.

4. Applications

In this section we shall apply the above algorithm to some important problems of mathematical programming.

I. *The bilinear programming problem*

Minimize $F(x, y) = c^T x + x^T Q^T y + d^T y$ subject to

$$x \in D = \{x \in R^n\colon\ Ax \geqq a,\ x \geqq 0\},$$

$$y \in E = \{y \in R^{n'}\colon\ B^T y \geqq b,\ y \geqq 0\},$$

where A, B^T and Q^T are $m \times n$, $m' \times n'$ and $n \times n'$ matrices, respectively, and c, d, a and b are n, n', m, m'-vectors, respectively. We shall assume that D is a polyhedral convex set in R^n and that E is a polytope in $R^{n'}$.

This problem has been extensively studied in the literature during the last ten years. Until now most of the methods developed for the solution require an assumption about the boundedness of D and E. The above algorithm can solve this problem provided either set is bounded. It is based upon the fact that the bilinear programming problem may always be converted into a concave minimization problem (see [3]). Namely, setting

$$f(x) = \min\{F(x, y)\colon\ y \in E\} = c^T x + \min\{(d + Qx)^T y\colon\ y \in E\} = \tag{15}$$
$$= c^T x + \max\{b^T u\colon\ Bu \leqq d + Qx,\ u \geqq 0\}$$

(the last equality follows from the Duality Theory in linear programming), we have a concave function f defined on R^n and the bilinear programming problem is reduced to minimizing f over D. Since for every x the value of $f(x)$ can easily be computed by solving a linear program (depending upon x) over the polytope E, the above Algorithm applies and yields a finite procedure for solving the bilinear programming problem.

Note that in this case the verification of whether f is unbounded below over any (fixed) ray emanating from a given point $\bar{x} \in S_k$ in an arbitrary direction v of S_k (with $S_k \supset D$ as described in the above Algorithm), is a comparatively easy task. Indeed, according to (15) we have

$$f(\bar{x} + \theta v) = c^T(\bar{x} + \theta v) + \max\{b^T u\colon\ Bu \leqq d + Q(\bar{x} + \theta v),\ u \geqq 0\}.$$

Solving the linear program (for determining the largest λ such that $f(\bar{x} + \lambda v)$ is still greater than or equal to $f(\bar{x})$):

Maximize λ, subject to

$$\lambda c^T v + b^T u \geqq f(\bar{x}) - c^T \bar{x}, \quad -\lambda Q v + Bu \leqq d + Q\bar{x}; \quad \lambda \geqq 0, \quad u \geqq 0,$$

we obtain the optimal value $\bar{\lambda}$. If $\bar{\lambda}=+\infty$, $f(x)$ attains its minimum over the ray $\{\bar{x}+\theta v\colon \theta\geqq 0\}$ at $\bar{x}$, i.e. f is bounded below on this ray; if not $(\bar{\lambda}<+\infty)$, by a well known property of concave functions, f is unbounded below in the ray under consideration.

The problems of the above described form to be solved at the same iteration of the Algorithm, associated with different extreme directions v, differ from one another just by one column corresponding to the variable λ. Therefore, to solve these problems one should use reoptimization techniques in linear programming which could take advantage of this property.

II. *The linear complementarity problem*

Find an n-vector x, an n-vector y, a p-vector z satisfying

$$(16)\qquad \begin{cases} Ax+By+Cz+b=0, \\ x^Ty=0, \quad x, y, z\geqq 0, \end{cases}$$

where A, B are $m\times n$ matrices, C is an $m\times p$ matrix, b is an m-vector.

As is shown [3], $(\bar{x}, \bar{y}, \bar{z})$ is a solution of (16) if and only if $(\bar{x}, \bar{y}, \bar{z})$ is an optimal solution of the following problem

$$(17)\qquad \min\Big\{l(x, y, z)=\sum_{i=1}^{n}\min(x_i, y_i)\colon\ Ax+By+Cz+b=0, \quad x, y, z\geqq 0\Big\}$$

with $l(\bar{x}, \bar{y}, \bar{z})=0$.

Since the objective function of (17) is clearly a concave function, (17) is a concave programming problem under linear constraints. Thus, instead of finding a feasible solution of the linear complementarity problem (16), we can solve the corresponding concave minimization problem (17). If an optimal solution $(\bar{x}, \bar{y}, \bar{z})$ exists such that $l(\bar{x}, \bar{y}, \bar{z})=0$, it is a solution to the linear complementarity problem; otherwise the linear complementarity problem has no solution.

The above Algorithm applied to problem (17) has several particular features taking account of the special structure of the problem.

a) Since $l(x, y, z)\geqq 0$ for all points $(x, y, z)\in R_+^{2n+p}$ and since S_k (S_k is a polyhedral convex set containing the constraint set D of (17)), as constructed in the above Algorithm, is contained in R_+^{2n+p}, $l(x, y, z)$ is bounded below over every ray of S_k. Therefore, during the performance of the above Algorithm for solving problem (17), Step 1 shall never occur (though S_k is not necessarily bounded). So, by Remark 4, each iteration consists in this special case of Steps 2 and 3 only.

b) If at a certain iteration k

$$\min\{l(x, y, z)\colon (x, y, z)\in U_k\}>0$$

(recall that U_k is the vertex set of S_k), then from

$$\min\{l(x, y, z)\colon (x, y, z)\in D\}\geqq\min\{l(x, y, z)\colon (x, y, z)\in S_k\}=$$
$$=\min\{l(x, y, z)\colon (x, y, z)\in U_k\}>0$$

it follows that the linear complementarity problem (16) has no solution. (In this case we need not solve problem (17) to the end.)

We now turn to consider a special case of problem (16). If $m=n$, $B=-E$ (E is the unit matrix) and $C=0$ the problem (16) becomes: Find $x\in R^n$, $y\in R^n$ such that

$$y = Ax+b \geqq 0, \quad x \geqq 0, \quad x^T y = 0. \tag{18}$$

The linear complementarity problem of this form has been studied by a number of researchers (see [12]), but most of them solve the problem under some additional assumptions about the matrix A. The above Algorithm can also be applied to the problem in all cases where it is solvable, by means of solving the equivalent concave minimization problem.

Minimize $f(x)$ such that

$$Ax+b \geqq 0, \quad x \geqq 0 \tag{19}$$

where

$$f(x) = \sum_{i=1}^{n} \min \{x_i, \sum_{j=1}^{n} a_{ij}x_j+b_i\}. \tag{20}$$

If an optimal solution $\bar{x}$ exists such that $f(\bar{x})=0$, it is a solution to the linear complementarity problem (18); otherwise (18) has no solution. (Observe that unlike (17), problem (19) contains x-variables only.)

In solving problem (19) by the above Algorithm it is easy to verify whether f is unbounded below on a ray emanating from a given point u in a direction v. Indeed, we have from (20)

$$f(u+\theta v) = \sum_{i=1}^{n} \min \{u_i+\theta v_i, \sum_j a_{ij}u_j+b_i+\theta \sum_j a_{ij}v_j\} = \sum_{i=1}^{n} \min \{u_i+\theta v_i, \alpha_i+\theta\beta_i\},$$

where $\alpha_i=\sum_j a_{ij}u_j+b_i$, $\beta_i=\sum_j a_{ij}v_j$. Therefore, for all $i=1, \dots, n$ with large enough θ we have

$$\min \{u_i+\theta v_i, \alpha_i+\theta\beta_i\} = \begin{cases} u_i+\theta v_i & \text{if } v_i \leqq \beta_i, \\ \alpha_i+\theta\beta_i & \text{otherwise.} \end{cases}$$

Hence, for large enough θ

$$f(u+\theta v) = \sum_{i\in K} (u_i+\theta v_i)+\sum_{i\notin K} (\alpha_i+\theta\beta_i) = \lambda+\theta\mu,$$

where

$$K = \{i\colon y_i \leqq \beta_i\}, \quad \lambda = \sum_{i\in K} u_i+\sum_{i\notin K} \alpha_i \quad \text{and} \quad \mu = \sum_{i\in K} v_i+\sum_{i\notin K} \beta_i.$$

Therefore, if $\mu<0$ $f(u+\theta v)\to-\infty$ as $\theta\to\infty$; otherwise f is bounded below over the ray $\{u+\theta v\colon \theta\geqq 0\}$.

By an argument similar to the previous one, we finally observe that if at a certain iteration k

$$\min \{f(x)\colon x\in S_k\} > 0, \tag{21}$$

the linear complementarity problem (18) has no solution.

III. *Concave minimization with special structure*

In a recent work [10] H. Tuy has developed a decomposition method for solving the following class of concave minimization problems under linear constraints with special structure:

Minimize $f(x)$, subject to

$$Ax+By+c \leqq 0, \tag{22}$$

$$x \in X, \quad y \in Y, \tag{23}$$

where X, Y are polyhedral convex sets in R^p, R^q respectively, A an $m\times p$-matrix, B an $m\times q$-matrix, c an m-vector and $f(x)$ a continuous concave function over X. Here p is assumed to be small as compared to $n=p+q$.

The basic idea of the method is to convert the original problem (21)—(23) to a finite sequence of concave minimization problems in the variable x under linear constraints. Among them the first one is to minimize $f(x)$ such that $x \in X$ and the following one is obtained from the previous one by adding a new linear constraint. Therefore, to solve these problems one could use the above Algorithm which has taken advantage of this recursive property.

5. A simple illustrative example

A two-dimensional example below was chosen to illustrate how the algorithm might perform on problems with unbounded constraint set.

We consider the following problem:

Minimize

$$f(x) = \frac{x_1x_2}{x_1+x_2} - \frac{0.05(x_1-x_2)^2}{x_1+x_2}$$

subject to

$$-3x_1+x_2-1 \leqq 0 \tag{1}$$

$$-3x_1-5x_2+23 \leqq 0 \tag{2}$$

$$x_1-4x_2-2 \leqq 0 \tag{3}$$

$$-x_1+x_2-5 \leqq 0 \tag{4}$$

$$x_1 \geqq 0$$

$$x_2 \geqq 0.$$

Fig. 1 illustrates the constraint set D (note that $f(x)$ is a concave function, defined and continuous on $R^2_+ \supset D$).

The Algorithm starts from $S_1=R^2_+$ with vertex set $U_1=\{u^1\}$ and extreme direction set $V_1=\{v^1, v^2\}$, where $u^1=(0, 0)$; $f(u^1)=0$, $v^1=(1, 0)$ and $v^2=(0, 1)$.

Iteration 1. On the halfline $\{x=tv^1=(t, 0): t\geqq 0\}$ we have

$$f(x) = -0.05t \to -\infty \quad (\text{as } t \to \infty),$$

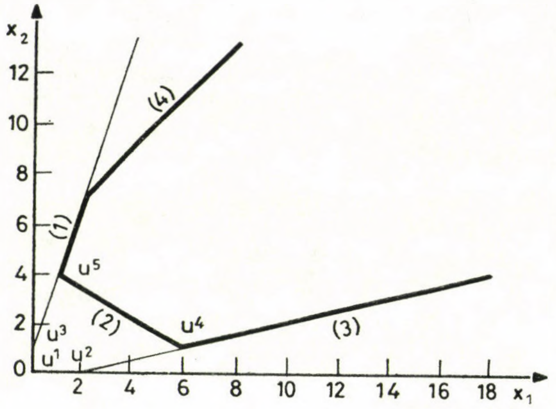

Fig. 1. Constraint set D

i.e. $f(x)$ is unbounded below over this halfline. According to (10) we compute

$$\alpha = \max\{-3, -3, 1, -1\} = 1 > 0,$$

and select $i_1=3$. Thus

$$S_2 = S_1 \cap \{x:\ x_1-4x_2-2 \leqq 0\},$$

$$U^- = \{u^1\},\quad U^+ = \emptyset,\quad V^-=\{v^2\},\quad V^+ = \{v^1\}.$$

The pair $(u^1, v^1)\in U^-\times V^+$ generates, in accordance with Rule A, a new vertex $u^2=(2, 0)$ with $f(u^2)=-0.1$ and the pair $(v^2, v^1)\in V^-\times V^+$ generates, in accordance with Rule B, a new extreme direction $v^3=(4, 1)$. So we have, by Proposition 3, $U_2=\{u^1, u^2\}$ and, by Proposition 4, $V_2=\{v^2, v^3\}$.

Iteration 2. On the halfline $\{x=tv^3=(4t, t):\ t\geqq 0\}$ we have

$$f(x) = \frac{4t^2}{5t} - \frac{-0.05\times 9t^2}{5t} = 0.71t \to \infty \quad (\text{as } t\to\infty),$$

i.e. $f(x)$ attains its minimum at the origin of this halüine.

On the halfline $\{x=tv^2=(0, t):\ t\geqq 0\}$ we have

$$f(x) = -0.05t \to -\infty \quad (\text{as } t\to\infty),$$

i.e. $f(x)$ is unbounded below on this halfline. Compute

$$\alpha = \max\{1, -5, 1\} = 1 > 0.$$

We select $i_2=1$. Thus

$$S_3 = S_2 \cap \{x:\ -3x_1+x_2-1 \leqq 0\},$$

$$U^- = \{u^1, u^2\},\quad U^+ = \emptyset,\quad V^- = \{v^3\},\quad V^+ = \{v^2\}.$$

The pair $(u^1, v^2)\in U^-\times V^+$ generates a new vertex $u^3=(0, 1)$ with $f(u^3)=$ $=-0.05$, the pair $(u^2, v^2)\in U^-\times V^+$ generates no vertex and the pair $(v^3, v^2)\in V^-\times$ $\times V^+$ gives a new extreme direction $v^4=(1, 3)$.

So we have

$$U_3 = \{u^1, u^2, u^3\}, \quad V_3 = \{v^3, v^4\}.$$

Iteration 3. $f(x)$ is bounded below on the ray $\{tv^3:\ t \geqq 0\}$ as well as on the ray $\{tv^4:\ t \geqq 0\}$. So we have

$$\min\{f(x):\ x \in S_3\} = \min\{f(u^1), f(u^2), f(u^3)\} = \min\{0, -0.1, -0.05\} = -0.1$$

and $w^3 = u^2 = (2, 0)$. In accordance with (12) we compute

$$\beta = \max\{17, -7\} = 17 > 0.$$

Select $i_3 = 2$ and

$$S_4 = S_3 \cap \{x:\ -3x_1 - 5x_2 + 23 \leqq 0\},$$

$$U^- = \emptyset, \quad U^+ = \{u^1, u^2, u^3\}, \quad V^- = \{v^3, v^4\}, \quad V^+ = \emptyset.$$

$(u^2, v^3) \in U^+ \times V^-$ gives a new vertex $u^4 = (6, 1)$ with $f(u^4) = 0.67857143$ and $(u^3, v^4) \in U^+ \times V^-$ gives a new vertex $u^5 = (1, 4)$ with $f(u^5) = 0.71$ (the other pairs in $U^+ \times V^-$ give no vertex). We thus have

$$U_4 = \{u^4, u^5\} \quad \text{and} \quad V_4 = \{v^3, v^4\}.$$

Iteration 4. There is no extreme ray of S_4 over which $f(x)$ is unbounded below, so we have

$$\min\{f(x):\ x \in S_4\} = \min\{f(u^4), f(u^5)\} = 0.67857143$$

and $w^4 = u^4 = (6, 1)$. According to (12), compute

$$\beta = -10 < 0$$

and hence, the optimal solution $u^4 = (6, 1)$ is found with objective function value $f(u^4) = 0.67857143$.

6. Computational experience

The above Algorithm was coded in FORTRAN IV and has been run on an IBM 360/50. It was tested on a number of concave minimization problems having a bounded constraint set, with negative quadratic piecewise linear concave, linear fixed-charge and exponential objective functions. The largest problem so far attempted using this algorithm is a 16-variable, 14-constraint problem having a linear fixed-charge objective function. The Algorithm solved the problem after three iterations generating 497 vertices. The computer time was 6.11 minutes. The preliminary results which are presented below show that the average number of iterations to be required for receiving an optimal solution is about $m/2$ (m is the number of linear equality and inequality constraints with the exception of non-negativity constraints) and that it is a viable method for concave minimization problems with a moderate size. Some other computational experiments carried out by Ng. V. Thoai [5] have also demonstrated the efficiency of using the above Algorithm combined with the decomposition technique proposed in [10]. Additional computational experience for problems having an unbounded constraint set will be reported in a forthcoming paper.

Problem	Size of A	Objective function	Number of iterations	Maximal number of generated vertices	CPU time (minute)
1	4×2	Quadratic	2	4	0.01
2	4×2	—	2	5	0.02
3	8×2	—	3	5	0.02
4	5×3	—	5	8	0.02
5	9×3	—	5	10	0.05
6	2×4	—	1	8	0.02
7	4×5	—	4	24	0.04
8	4×8	—	1	25	0.03
9	6×8	—	3	46	0.07
10	6×8	Piecewise linear concave	3	48	0.10
11	5×12	Quadratic	3	48	0.82
12	11×10	—	4	116	0.61
13	11×10	Exponential	5	231	2.24
14	14×16	Fixed-charge	3	497	6.11

References

[1] V. T. Ban, A finite algorithm for minimizing a concave function under linear constraints and its applications, *Proceedings of IFIP Working Conference on Recent Advances on System Modelling and Optimization* (Hanoi, January 1983).

[2] R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press (Princeton, New Yersey, 1970).

[3] T. V. Thieu, Relationship between bilinear programming and concave minimization under linear constraints, *Acta Mathematica Vietnamica*, **2** (1980), 106—113.

[4] T. V. Thieu, B. T. Tam, V. T. Ban, An outer approximation method for globally minimizing a concave function over a compact convex set, *Proceedings of IFIP Working Conference on Recent Advances on System Modelling and Optimization* (Hanoi, January 1983); *Acta Mathematica Vietnamica*, **1** (1983) (to appear).

[5] Ng. V. Thoai, *Computational testing procedure of Tuy's decomposition method for a class of concave programming problems*, Preprint. Institute of Mathematics (Hanoi, 1983).

[6] Ng. V. Thoai, H. Tuy, Convergent algorithms for minimizing a concave function, *Math. Oper. Res.*, **5** (1980), 556—566.

[7] H. Tuy, Concave programming under linear constraints, *Doklady Akad. Nauk*, **159** (1964), 32—35; *Soviet Math. Transl.*, **5** (1964), 1437—1440.

[8] H. Tuy, Minimization of a concave function over an unbounded polyhedral convex set, *Acta Math. Vietnamica* (to appear).

[9] H. Tuy, Global maximization of a convex function over a closed, convex, not necessarily bounded set, *Cahier de Math. de la Decision* n° 8223 CEREMADE. Univ. Paris—Dauphine 1982.

[10] H. Tuy, *Concave minimization under linear constraints with special structure*. Preprint. Institute of Mathematics (Hanoi, 1983).

[11] H. Tuy, T. V. Thieu, Ng. Q. Thai, A conical algorithm for globally minimizing a concave function over a closed convex set. Submitted to *Mathematics of Operations Research*.

[12] H. Tuy, Ng. V. Thoai, Solving the linear complementarity problem via concave programming, Method of Oper. Res., *Proceedings of the V Symposium on Operations Researchs* (Köln, August 25—27, 1980). Ed. R. E. Burkard, T. Ellinger, 175—178.

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SOME RANDOMLY SELECTED ARITHMETICAL SUMS

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1. Introduction

Let $p_1(n)<p_2(n)<\ldots<p_\omega(n)$ be the sequence of distinct prime divisors of n; that is,

$$n=\prod_{j=1}^{\omega} p_j^{\alpha_j}(n), \quad n \geqq 2,$$

where $\omega=\omega(n)$ is the number of distinct prime divisors of n.

Recently, several authors investigated the behavior of

$$\sum_{2\leqq n\leqq x} \frac{1}{p_j(n)} = R_j(x)$$

for some specific choices of j. In particular, for $j=1$, or whenever j is preassigned, it is not difficult to show that $R_j(x)\sim c_j x$ with a computable constant c_j. On the other hand, when $j=\omega(n)$, the problem of finding good approximations to $R_j(x)$ becomes very difficult; by refining several earlier results, Ivić and Pomerance [5] found the best known approximation. Quite remarkably, the case of $j=\omega(n)-1$, or $j=\omega(n)-k$ with k fixed, shows no similarity to the case of $j=\omega(n)$, and asymptotic expressions are in fact known (Erdős and Ivić [3]):

$$R_{\omega-k}(x) \sim c_k \frac{x(\log\log x)^{k-1}}{\log x}, \quad k \geqq 1,$$

where c_k is a constant.

In order to obtain an "average type of information" on the magnitude of $p_j(n)$, when j does not belong to the mentioned cases (i.e. either j is fixed or $j=\omega-k$ with k fixed), we set up the following probabilistic approach. For every integer $n\leqq x$, pick one $p(n)$ of its prime divisors $p_j(n)$ with equal probabilities (hence $p(n)=p_j(n)$ with probability $1/\omega(n)$), and consider the sums

$$R(x)=\sum_{2\leqq n\leqq x}\frac{1}{p(n)}. \tag{1}$$

Here and in what follows we assume that x is an integer. Evidently, there are $\omega(2)\omega(3)\ldots\omega(x)$ sums of the type in (1), and

$$\sum_{2\leqq n\leqq x}\frac{1}{p_\omega(n)} \leqq R(x) \leqq \sum_{2\leqq n\leqq x}\frac{1}{p_1(n)}.$$

It turns out that "almost all" sums in (1) are asymptotically equal to the same expression, $cx/\log\log x$, indicating that $R_j(x)$ does not vary much with j when j is not an extreme (constant or $\omega-k$ with k constant).

This probabilistic approach to $R_j(x)$ led us to investigate several other arithmetical sums

$$Q(x)=\sum_{n\leqq x} r(n), \tag{2}$$

where $r(n)$ is one randomly selected member of a set A_n associated with n. We establish that $Q(x)$ is asymptotically the same value for "almost all" selections of $r(n)$. For example, if A_n is the set of the reciprocals of the divisors of n, then it turns out that

$$Q(x)\sim\sum_{n\leqq x}\frac{\sigma(n)}{n\tau(n)}, \tag{3}$$

where $\sigma(n)$ is the sum of the divisors of n, and $\tau(n)$ is the number of divisors of n. On the other hand, if A_n is the set of all divisors of n, then (again for almost all selections of $r(n)$ in (2)),

$$Q(x)\sim\sum_{n\leqq x}\frac{\sigma(n)}{\tau(n)}. \tag{4}$$

These results, therefore, give a probabilistic meaning to the arithmetical sums on the right hand sides of (3) and (4), involving well known arithmetical functions.

2. The sum of reciprocals of random prime divisors of n

As in the introduction, $p_1(n)<p_2(n)<\ldots<p_\omega(n)$ denote the distinct prime divisors of n, and we select one $p(n)$ of these prime divisors at random (with equal probabilities). Set

$$R(x)=\sum_{2\leqq n\leqq x}\frac{1}{p(n)}. \tag{5}$$

Note again the total number of sums of the form of (5) is $\omega(2)\omega(3)\ldots\omega(x)$. We shall say that a property holds for almost all sums in (5) if the number $N(x)$ of the sums with the property in question satisfies

$$N(x)/\omega(2)\omega(3)\ldots\omega(x)\to 1$$

as $x\to+\infty$.

THEOREM 1. *For almost all sums in* (5),

$$R(x)=\frac{c_1x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right)$$

where $c_1=\sum 1/p^2$, *the summation being over all primes* p.

The proof is based on the Chebyshev inequality stated below.

LEMMA. *Let* $A_n=\{a_1, a_2, \ldots, a_{f(n)}\}$, $n\geqq 1$, *be a sequence of finite sets. For every* $n\geqq 1$, *pick one member* $r(n)$ *of* A_n *at random with equal probabilities (i.e.* $r(n)=a_j$ *with probability* $1/f(n)$*), and set*

$$Q(x)=\sum_{n\leqq x} r(n). \tag{6}$$

Then the number $N_Q(x)$ *of sums in* (6) *for which*

$$|Q(x)-E|\geqq V^{5/8}$$

where

$$E=\sum_{n\leqq x}\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j$$

and

$$V=\sum_{n\leqq x}\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j^2-\sum_{n\leqq x}\left(\frac{1}{f(n)}\sum_{j=1}^{f(n)} a_j\right)^2$$

satisfies

$$N_Q(x)\leqq V^{-1/4} f(1)f(2)\ldots f(x).$$

PROOF. See Galambos [4].

PROOF OF THEOREM 1. With the notations of the lemma,

$$E=\sum_{2\leqq n\leqq x}\frac{1}{\omega(n)}\sum_{p|n}\frac{1}{p}=\sum_{p\leqq x}\frac{1}{p}\sum_{m\leqq x/p}\frac{1}{\omega(pm)}. \tag{7}$$

Clearly

$$\frac{1}{\omega(m)+1}\leqq\frac{1}{\omega(pm)}\leqq\frac{1}{\omega(m)}.$$

Hence

$$\frac{1}{\omega(pm)}=\frac{1}{\omega(m)}+O\left(\frac{1}{\omega(m)^2}\right),$$

where the $O(\ldots)$ is uniform in $m\geqq 2$. Therefore

$$\sum_{p\leqq x}\frac{1}{p}\sum_{m\leqq x/p}\frac{1}{\omega(pm)}=\sum_{p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}+O\left(\sum_{p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)^2}\right). \tag{8}$$

Now

$$\sum_{\sqrt{x}<p\leqq x}\frac{1}{p}\sum_{2\leqq m\leqq x/p}\frac{1}{\omega(m)}\leqq\frac{1}{\sqrt{x}}\sum_{p\leqq x}\sum_{m\leqq x/p}1\leqq\frac{1}{\sqrt{x}}\sum_{p\leqq x}\frac{x}{p}= \tag{9}$$

$$=O\left(\sqrt{x}\log\log x\right)=O\left(x/(\log\log x)^2\right).$$

We recall the estimates

$$\sum_{2\leq m\leq x}\frac{1}{\omega(m)}=\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right) \tag{10}$$

and

$$\sum_{2\leq m\leq x}\frac{1}{\omega(m)^2}=O\left(\frac{x}{(\log\log x)^2}\right) \tag{11}$$

proved in De Koninck [1]. Using (10), we have

$$\sum_{p\leq\sqrt{x}}\frac{1}{p}\sum_{2\leq m\leq x/p}\frac{1}{\omega(m)}=\sum_{p\leq\sqrt{x}}\frac{1}{p}\frac{x/p}{\log\log(x/p)}+O\left(\frac{x/p}{(\log\log(x/p))^2}\right). \tag{12}$$

But since, for $p\leq\sqrt{x}$,

$$\log\left(1-\frac{\log p}{\log x}\right)=O(1),$$

then

$$\frac{1}{\log\log(x/p)}=\frac{1}{\log\log x+\log\left(1-\frac{\log p}{\log x}\right)}=\frac{1}{\log\log x}\,\frac{1}{1+\frac{\log\left(1-\frac{\log p}{\log x}\right)}{\log\log x}}= \tag{13}$$

$$=\frac{1}{\log\log x}\left(1+O\left(\frac{\log\left(1-\frac{\log p}{\log x}\right)}{\log\log x}\right)\right)=\frac{1}{\log\log x}+O\left(\frac{1}{(\log\log x)^2}\right).$$

Using this in (12) yields

$$\sum_{p\leq\sqrt{x}}\frac{1}{p}\sum_{2\leq m\leq x/p}\frac{1}{\omega(m)}=\frac{x}{\log\log x}\sum_{p\leq\sqrt{x}}\frac{1}{p^2}+O\left(\frac{x}{(\log\log x)^2}\right).$$

Since

$$\sum_{p\leq\sqrt{x}}\frac{1}{p^2}=\sum_{p}\frac{1}{p^2}+O\left(\frac{1}{\sqrt{x}}\right)=c_1+O\left(\frac{1}{\sqrt{x}}\right),\quad\text{say,}$$

we finally obtain

$$\sum_{p\leq\sqrt{x}}\frac{1}{p}\sum_{2\leq m\leq x/p}\frac{1}{\omega(m)}=c_1\frac{x}{\log\log x}+O\left(\frac{x}{(\log\log x)^2}\right). \tag{14}$$

Also, using (11), we have

$$\sum_{p\leq\sqrt{x}}\frac{1}{p}\sum_{2\leq m\leq x/p}\frac{1}{\omega(m)^2}=O\left(\sum_{p\leq\sqrt{x}}\frac{1}{p}\frac{x/p}{(\log\log(x/p))^2}\right)=O\left(\frac{x}{(\log\log x)^2}\right), \tag{15}$$

because of (13).

Now combining (14) and (15), and using once more (9), (8) becomes

$$\sum_{p\le x} \frac{1}{p} \sum_{m\le x/p} \frac{1}{\omega(pm)} = c_1 \frac{x}{\log\log x} + O\left(\frac{x}{(\log\log x)^2}\right). \tag{16}$$

Hence

$$E = c_1 \frac{x}{\log\log x} + O\left(\frac{x}{(\log\log x)^2}\right).$$

By similar calculations, we get

$$V = \sum_{2\le n\le x} \frac{1}{\omega(n)} \sum_{p|n} \frac{1}{p^2} - \sum_{2\le n\le x} \left(\frac{1}{\omega(n)} \sum_{p|n} \frac{1}{p}\right)^2 \le \frac{c_2 x}{\log\log x},$$

and thus the lemma implies Theorem 1.

3. Random sums related to the divisors of n

Let now $d_1<d_2<\ldots<d_\tau$ be the divisors of n, where $d_j=d_j(n)$ and $\tau=\tau(n)$ is the number of divisors of n. In addition, let $\sigma(n)$ be the sum of the divisors of n. Again, for each $n\ge 2$, we pick one $r(n)$ of the divisors d_j, and, with a given function $h(\cdot)$, we define

$$Q(x) = \sum_{2\le n\le x} h(r(n)). \tag{17}$$

Because the number of sums in (17) is $\tau(2)\tau(3)\ldots\tau(x)$, we now say that a property holds for almost all sums in (17) if it holds for N^* sums such that $N^*/\tau(2)\tau(3)\ldots\tau(x)\to 1$ as $x\to+\infty$.

Although the basic idea of the computations is the same for a large variety of choices for $h(u)$, we carry out the computations when $h(u)$ is either $1/u$ or u. Their significance is that the major terms in the asymptotic expressions below are familiar arithmetical sums.

THEOREM 2. *For almost all sums in* (17),

(i) $$\sum_{2\le n\le x} \frac{1}{r(n)} = \sum_{2\le n\le x} \frac{\sigma(n)}{n\tau(n)} + O(x^{5/8}) = \frac{c_3 x}{\sqrt{\log x}} + O\left(\frac{x}{(\log x)^{3/2}}\right);$$

and

(ii) $$\sum_{2\le n\le x} r(n) = \sum_{2\le n\le x} \frac{\sigma(n)}{\tau(n)} + O(x^{15/8}) = \frac{c_4 x^2}{\sqrt{\log x}} + O\left(\frac{x^2}{(\log x)^{3/2}}\right).$$

PROOF. We again use the Chebyshev inequality stated as Lemma in the previous section.

(i) *The case of* $1/r(n)$:

$$E = \sum_{2\leqq n\leqq x} \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d} = \sum_{2\leqq n\leqq x} \frac{1}{n\tau(n)} \sum_{d|n} \frac{n}{d} = \sum_{2\leqq n\leqq x} \frac{1}{n\tau(n)} \sum_{d|n} d = \sum_{2\leqq n\leqq x} \frac{\sigma(n)}{n\tau(n)}.$$

An asymptotic expression for the last sum above can easily be obtained by using Dirichlet generating functions. As a matter of fact, since

$$\sum_{n=1}^{+\infty} \frac{\sigma(n)/n\tau(n)}{n^s} = \prod_p \left(1 + \frac{\frac{1}{2}(1+1/p)}{p^s} + \frac{\frac{1}{3}(1+1/p+1/p^2)}{p^{2s}} + \dots\right) = (\zeta(s))^{1/2} R(s),$$

where $R(s) = O(1)$ for $\operatorname{Re}(s) > \frac{1}{2}$, we have that

$$\sum_{n\leqq x} \frac{\sigma(n)}{n\tau(n)} = \frac{R(1)x}{\sqrt{\log x}} + O\left(\frac{x}{(\log x)^{3/2}}\right). \tag{18}$$

Now,

$$V = \sum_{2\leqq n\leqq x} \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d^2} - \sum_{2\leqq n\leqq x} \frac{1}{\tau(n)} \left(\sum_{d|n} \frac{1}{d}\right)^2,$$

which can easily be seen to satisfy $V = O(x)$ (more accurate computation is also possible, but this rough estimate suffices). Hence, Lemma concludes the proof of the statement in part (i).

Turning to (ii), E takes the form

$$E = \sum_{2\leqq n\leqq x} \frac{1}{\tau(n)} \sum_{d|n} d = \sum_{2\leqq n\leqq x} \frac{\sigma(n)}{\tau(n)},$$

which, by partial summation in (18), yields the asymptotic formula of (ii) upon observing that the estimate $V = O(x^3)$ is immediate.

Let us conclude by mentioning that several other familiar arithmetical sums do have probabilistic meaning similar to the ones appearing in Theorem 2. For example, if we pick an exponent $\alpha(n)$ at random in the prime factorization $n = \prod p^\alpha$, then, for almost all choices of $\alpha(n)$,

$$\sum_{n\leqq x} \alpha(n) \sim \sum_{2\leqq n\leqq x} \frac{\Omega(n)}{\omega(n)},$$

where $\Omega(n)$ is the total number of prime divisors of n. This latter sum has been investigated in much detail in De Koninck and Ivić [2], yielding

$$\sum_{n\leqq x} \alpha(n) \sim x,$$

for almost all sums on the left hand side.

References

[1] J. M. De Koninck, On a certain class of arithmetical functions, *Duke Math. J.*, **39** (1972), 807—818.

[2] J. M. De Koninck and A. Ivić, *Topics in Arithmetical Functions,* Notas de Matematica 72, North-Holland (Amsterdam, 1980).

[3] P. Erdős and A. Ivić, On sums involving reciprocals of certain arithmetical functions, *Publs. Inst. Math. Belgrade,* **32** (1982), 49—56.

[4] J. Galambos, *Introductory Probability Theory,* Marcel Dekker (New York, 1984).

[5] A. Ivić and C. Pomerance, Estimates of certain sums involving the largest prime factor of an integer, *Coll. Math. Soc. J. Bolyai* 34, *Topics in classical number theory,* North-Holland, Amsterdam.

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THE CYCLIC LENGTH OF A p-GROUP NEED NOT BE LOGARITHMIC

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The *cyclic length* $l(G)$ of a finite supersolvable group G is defined to be the smallest number l such that there is a series $G=N_0>N_1>\ldots>N_l=1$, where $N_0, N_1, \ldots, N_l$ are $l+1$ different normal subgroups of G and all factor groups N_{i-1}/N_i $(i=1, \ldots, l)$ are cyclic. L. Rédei asked the question whether or not the cyclic length has a logarithmic property on the class of p-groups: i.e. does $l(G_1\times G_2)=$ $=l(G_1)+l(G_2)$ hold, whenever G_1, G_2 are p-groups for a certain prime p? At the first glance the answer could be expected to be in the positive, especially in view of the validity on the class of abelian p-groups. In the present note we will bring this expectation to nought not only by means of a single counterexample but more generally by assigning a p-group G to each abelian p-group H so that $l(G\times H)<$ $<l(G)+l(H)$ takes place. The notation will be standard and can be found in [1]. All groups under consideration are finite.

LEMMA. *For an arbitrary prime p and integer $n\geqq 3$ there exists a group G of order p^{2n} presented by*

$$G = \langle a, b, c|a^p = b^{p^{n-1}} = c^{p^n} = [b, c] = 1,\ [b, a] = c^{p^{n-1}},\ [c, a] = b^{p^{n-2}}\rangle.$$

It has the following properties:

1) *The center $Z(G)$, the Frattini subgroup $\Phi(G)$ and the commutator subgroup G' of G are*

$$Z(G) = \Phi(G) = \langle b^p, c^p\rangle, \quad G' = \langle b^{p^{n-2}}, c^{p^{n-1}}\rangle.$$

Further, $|G\colon \Phi(G)|=p^3$.

2) *Each cyclic normal subgroup of G is contained in $Z(G)$.*

3) *G has cyclic length 4, i.e. $l(G)=4$.*

PROOF. As to the existence of G we start with an abelian group $B=\langle b\rangle\times\langle c\rangle$, where the elements b, c have orders p^{n-1}, p^n, respectively. The mapping $b\mapsto bc^{p^{n-1}}$, $c\mapsto cb^{p^{n-2}}$ can be extended to an automorphism α of B, the order of α is p. Now G is defined to be the semi-direct product $G=\langle a\rangle \underset{a\to\alpha}{\propto} B$, where a has order p and acts on B according to

$$a^{-1}ba = bc^{p^{n-1}}, \quad a^{-1}ca = cb^{p^{n-2}}. \tag{1}$$

1) The assertions about $Z(G)$ and G' come from (1) and the equations

$$b^{-1}ab = ac^{-p^{n-1}}, \quad c^{-1}ac = ab^{-p^{n-2}},$$

which arise from (1). As to $\Phi(G)$, notice that $G/Z(G)$ is elementary abelian.

2) We have

$$(2) \qquad (a^i b^j c^k)^p = \begin{cases} b^{jp} c^{kp} & \text{if} \quad p > 2, \\ b^{ikp^{n-2}+2j} c^{ijp^{n-1}+2k} & \text{if} \quad p = 2. \end{cases}$$

Notice that G is a regular p-group if $p>2$. Now let $\langle g\rangle \trianglelefteq G$ with $g=a^i b^j c^k$. Then there exist integers r, s, t such that $[g, a]=g^{pr}$, $[g, b]=g^{ps}$, $[g, c]=g^{pt}$, which yield in case $p>2$

$$(3) \qquad kp^{n-3} \equiv jr \bmod p^{n-2}, \quad jp^{n-2} \equiv kr \bmod p^{n-1},$$

$$(4) \qquad 0 \equiv js \bmod p^{n-2}, \quad -ip^{n-2} \equiv ks \bmod p^{n-1},$$

$$(5) \qquad -ip^{n-3} \equiv jt \bmod p^{n-2}, \quad 0 \equiv kt \bmod p^{n-1},$$

and in case $p=2$

$$(6) \qquad kp^{n-3} \equiv (ikp^{n-3}+j)r \bmod p^{n-2}, \quad jp^{n-2} \equiv (ijp^{n-2}+k)r \bmod p^{n-1},$$

$$(7) \qquad 0 \equiv (ikp^{n-3}+j)s \bmod p^{n-2}, \quad -ip^{n-2} \equiv (ijp^{n-2}+k)s \bmod p^{n-1},$$

$$(8) \qquad -ip^{n-3} \equiv (ikp^{n-3}+j)t \bmod p^{n-2}, \quad 0 \equiv (ijp^{n-2}+k)t \bmod p^{n-1}.$$

From (3) we obtain $p|k$ after having multiplied the left-hand congruence with k. Then we multiply the right-hand congruence of (3) with j and get $p|j$. A similar procedure with respect to (6) yields $p|k, p|j$ too. Hence $p|k, p|j$ can be stated in any case $p>2$ or $p=2$. Now (6), (7), (8) reduce to (3), (4), (5), respectively. From (4) and (5) we attain $p^{2n-3}|jskt$ so that $p^{n-1}|ks$ or $p^{n-1}|jt$. In the first of these cases (4) brings about $p|i$ and in the second one we get $p|i$ by (5). Now $g \in Z(G)$ is confirmed.

3) Obviously the ordered system a, c, b, c^p of generators of G is such that the subgroups generated by its right segments are normal in G. Hence $l(G) \leqq 4$. Assume $l(G)<4$. Then there is a series

$$G = N_0 \geqq N_1 \geqq N_2 \geqq N_3 = 1$$

of normal subgroups N_i of G with cyclic factor groups N_{i-1}/N_i $(i=1, 2, 3)$. By 1) and 2) we have $N_2 \leqq Z(G) = \Phi(G)$ and therefore G would be generated by two elements. This is impossible in view of $|G : \Phi(G)| = p^3$ (see 1)).

Now the announced result on the cyclic length of a direct product is in

PROPOSITION 1. *Let G_n be the group of order p^{2n} from the above Lemma and H an abelian p-group of exponent p^e, say. Then for each n with $n \geqq e+2$ we have $l(G_n \times H) < l(G_n) + l(H)$.*

PROOF. Let $l(H)=:l$. Then there is a system $h_1, h_2, \ldots, h_l$ of generators of H. Further, let a, b, c like in the Lemma. We consider the system

$$(9) \qquad h_1, h_2, \ldots, h_{l-1}, a, c, b, c^p h_l$$

of generators of $G_n \times H$. Under the assumption $e \leqq n-2$ we have

$$[b, a] = c^{p^{n-1}} = (c^p h_l)^{p^{n-2}}$$

and therefore the subgroups generated by the right-hand segments of (9) are all normal in $G_n\times H$. This implies $l(G_n\times H)\leqq 3+l<l(G_n)+l(H)$.

Finally we will show that the condition $n\geqq e+2$ in Proposition 1 cannot be omitted. In connection with this we characterize those p-groups H for which $l(G_n\times H)$ has its smallest possible value 4.

PROPOSITION 2. *Let G_n be as in Proposition 1 and H an arbitrary p-group of exponent p^e. Then $l(G_n\times H)=4$ if and only if H is cyclic and $n\geqq e+2$.*

PROOF. Let H be cyclic and $n\geqq e+2$. Then besides $4=l(G_n)\leqq l(G_n\times H)$ we have by Proposition 1 $l(G_n\times H)<l(G_n)+l(H)\leqq 5$, hence $l(G_n\times H)=4$.

Conversely, let $l(G_n\times H)=4$ and let

$$g_1h_1,\quad g_2h_2,\quad g_3h_3,\quad g_4h_4 \tag{10}$$

with $g_i\in G_n$, $h_i\in H$ $(i=1,2,3,4)$ be a system of generators of $G_n\times H$ such that its right segments generate subgroups which are normal in $G_n\times H$. Then necessarily $\langle g_4\rangle\trianglelefteq G_n$, hence by 2) and 1) of the Lemma $g_4\in Z(G_n)=\Phi(G_n)$. Now $\langle g_1,g_2,g_3\rangle= =G_n$ and in order to generate an arbitrary $h\in H$ by means of the elements (10) in a related equation

$$(g_1h_1)^{x_1}(g_2h_2)^{x_2}(g_3h_3)^{x_3}(g_4h_4)^{x_4}=h$$

$p|x_i$ holds for $i=1,2,3$. So we have $H=\langle h_1^p,h_2^p,h_3^p,h_4\rangle=\langle h_4\rangle$, and H is cyclic. To prove the assertion on the exponent of H put $h_3=:h_4^x$. By the above Lemma $\langle g_1,g_2,g_3\rangle=G_n$ implies $g_3\notin\Phi(G_n)=Z(G_n)$ and consequently $\langle g_3\rangle$ is not normal in G_n. Therefore at least one of the elements

$$\begin{cases}[g_3,a]=[g_3h_3,a]=(g_3h_3)^{s_1}(g_4h_4)^{t_1},\\ [g_3,b]=[g_3h_3,b]=(g_3h_3)^{s_2}(g_4h_4)^{t_2},\\ [g_3,c]=[g_3h_3,c]=(g_3h_3)^{s_3}(g_4h_4)^{t_3}\end{cases} \tag{11}$$

is not contained in $\langle g_3\rangle$. This implies that at least one of t_1,t_2, t_3 is not divisible by p^{n-1}. Let

$$g_3=a^ib^jc^k,\quad g_4=b^{pu}c^{pv}.$$

Since $g_3\notin\Phi(G_n)$, at least one of the numbers i, j, k is not divisible by p. Because $g_3^{s_1}$, $g_3^{s_2}$, $g_3^{s_3}$ are in $\langle b,c\rangle$, we obtain that is_1, is_2, is_3 are all divisible by p. If $p|i$, then the powers $g_3^{s_1}$, $g_3^{s_2}$, $g_3^{s_3}$ can be worked out immediately. If however $p\nmid i$, then p divides each of s_1, s_2, s_3 and we can apply (2) when working out $g_3^{s_1}$, $g_3^{s_2}$, $g_3^{s_3}$. Now the equations (11) yield in case $p>2$ or $p|i$

$$\begin{cases}kp^{n-2}\equiv js_1+put_1 \mod p^{n-1}, & jp^{n-1}\equiv ks_1+pvt_1 \mod p^n,\\ 0\equiv js_2+put_2 \mod p^{n-1}, & -ip^{n-1}\equiv ks_2+pvt_2 \mod p^n,\\ -ip^{n-2}\equiv js_3+put_3 \mod p^{n-1}, & 0\equiv ks_3+pvt_3 \mod p^n,\end{cases} \tag{12}$$

further in case $p=2$ and $p\nmid i$

$$\begin{cases}kp^{n-2}\equiv ikp^{n-3}s_1+js_1+put_1 \mod p^{n-1}, & jp^{n-1}\equiv ijp^{n-2}s_1+ks_1+pvt_1 \mod p^n,\\ 0\equiv ikp^{n-3}s_2+js_2+put_2 \mod p^{n-1}, & -ip^{n-1}\equiv ijp^{n-2}s_2+ks_2+pvt_2 \mod p^n,\\ -ip^{n-2}\equiv ikp^{n-3}s_3+js_3+put_3 \mod p^{n-1}, & 0\equiv ijp^{n-2}s_3+ks_3+pvt_3 \mod p^n.\end{cases} \tag{13}$$

Additively in all cases $xs_r+t_r\equiv 0 \mod |H|$ holds for $r=1, 2, 3$. We now will lead the assumption $p^{n-1}\| |H|$ to a contradiction. Under this assumption we have $xs_r+t_r\equiv 0 \mod p^{n-1}$ for $r=1, 2, 3$ and hence g.c.d. $\{s_r, p^{n-1}\}|t_r$ for $r=1, 2, 3$. Thus $p^{n-1}\nmid s_r$ for at least one r. We can eliminate the t_r in the congruences (12) and (13). From (12) we obtain

$$kp^{n-2} \equiv s_1(j-pux) \mod p^{n-1}, \qquad jp^{n-1} \equiv s_1(k-pvx) \mod p^n, \tag{14}$$

$$0 \equiv s_2(j-pux) \mod p^{n-1}, \qquad -ip^{n-1} \equiv s_2(k-pvx) \mod p^n, \tag{15}$$

$$-ip^{n-2} \equiv s_3(j-pux) \mod p^{n-1}, \qquad 0 \equiv s_3(k-pvx) \mod p^n. \tag{16}$$

(15) and (16) yield $p^{2n-1}|s_2s_3(j-pux)(k-pvx)$ so that $p^n|s_2(k-pvx)$ or $p^n|s_3(j-pux)$. In the first of these cases (15) brings $p|i$ and in the second one we get $p|i$ by (16). If $p\nmid k$ then from the right-hand sides of (14), (15), (16) it results $p^{n-1}|s_r$ for $r=1, 2, 3$, a contradiction. Hence $p|k$. If $p\nmid j$ then the congruences in (14) turn out a contradiction with regard to the power of p in s_1. Consequently $p|j$. But as it was stated above the numbers i, j, k cannot be divisible by p simultaneously. In order to treat the remaining case $p=2$ and $p\nmid i$ we take those congruences into consideration which arise from (13) by eliminating the t_r. Then we can conclude in a similar manner as above that $p|i$, which finishes the proof.

In consequence of Proposition 2 we have for a cyclic p-group H of order $\geqq p^{n-1}$ $l(G_n\times H)>4$ and precisely $l(G_n\times H)=5=l(G_n)+l(H)$. Therefore the assumption $n\geqq e+2$ cannot be dropped in Proposition 1.

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Reference

[1] B. Huppert, *Endliche Gruppen* I. Springer-Verlag (Berlin, Heidelberg, New York, 1967).

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WEAK COMPACTNESS IN L_E^1

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If $(X, \mathfrak{A}, \mu)$ is a finite measure space, E a Banach space, K a convex weakly compact subset, L_E^1 the space of Bochner integrable functions from X into E, then it is proved in ([1], [3]) that $\{f\colon X\to K, f\in L_E^1\}$ is weakly compact. In this paper we give a different proof of the more general case when E is a locally convex Hausdorff space.

NOTATIONS. All locally convex spaces are taken over reals and notations of [4] are used. $(X, \mathfrak{A}, \mu)$ denotes a complete finite measure space. For the theory of lifting, notations and results from [8] are used. Let $\mathscr{L}$ be the set of real-valued $\mathfrak{A}$-measurable functions on X, $\mathscr{L}^\infty$ the essentially bounded elements of $\mathscr{L}$. We fix a lifting $\varrho\colon \mathscr{L}^\infty\to M^\infty$ and on X we always take the lifting topology τ_ϱ ([8], p. 59; this topology is denoted by T_ϱ in [5]).

Let E be a Hausdorff locally convex space whose topology is generated by a family $\{\|\cdot\|_p, p\in P\}$ of semi-norms filtering upwards. On the vector space F_0 of all functions from X into E, we define the functions $\{U_p\colon p\in P\}$:

For $f\colon X\to E$, $U_p(f)=\overline{\int} \|f\|_p\, d\mu$, where $\overline{\int}$ is the upper integral ([8]; note for any $g\colon X\to[0,\infty]$, $\overline{\int} g\, d\mu=\inf\{\int f\, d\mu\colon f\colon X\to[0,\infty]$, measurable, $f\geqq g\}$).

The vector space $F=\{f\in F_0\colon U_p(f)<\infty$, for every $p\in P\}$, with filtering upwards seminorms $\{U_p\colon p\in P\}$ is a locally convex space and contains S, the set of all simple E-valued measurable functions on X. The closure of S in F we denote by $\mathscr{L}_E^1$ and the associated Hausdorff locally convex space by L_E^1 ([7], p. 775). L_E^1 is a Hausdorff locally convex space with generating seminorms $\{U_p\colon p\in P\}$. Denoting by $(E_p, \|\cdot\|_p)$ the completion of the quotient normed space, arising from the seminorm $\|\cdot\|_p$ on E, E is a subspace, with induced topology, of the product space $\prod_{p\in P} E_p$, and L_E^1 is a subspace, with induced topology, of the space $\prod_{p\in P} L_{E_p}^1$. If $E=R$, the reals, L_E^1 is denoted by L^1. For $x\in E$, $f\in E'$, $\langle x,f\rangle$ will also be used for $f(x)$.

Let K be a weakly compact convex subset of E and $W=\{f\in L_E^1\colon f(X)\subset K\}$ (this means in the class of functions f, there is a function f with $f(X)\subset K$).

THEOREM. *W is weakly compact in L_E^1.*

PROOF. First assume that E is a Banach space. The elements of W can be considered as continuous functions from X into $(K, \sigma(E,E')|_K)$ ([8], p. 65, Theorem 4; note that by [6], p. 200, Theorem 3, such functions are strongly measurable). Since it is enough to prove that every sequence in W has a cluster point in W, and since for every $f \in L^1_E$, $f(X)$ is separable (note that E is a Banach space), so there is no loss in generality in assuming that K is separable. Also assume that E is generated by K.

Since W is a bounded subset of the Banach space $F = L^1_E$, it is enough to prove that W is closed in $(F'', \sigma(F'', F'))$. Let $\{f_\alpha\} \subset W$, $f_\alpha \to \varphi$ in $(F'', \sigma(F'', F'))$. For any $g \in E'$, $\sup_\alpha \sup |g \circ f_\alpha(X)|$ is finite and so $\{g \circ f_\alpha\}$ is relatively weakly compact in L^1. Since $F' = L^\infty_{E'}[E]$ ([8], Theorem 1, p. 94), there exists a continuous function $f_g\colon X \to R$ such that $g \circ f_\alpha \to f_g$ weakly in L^1. Define $f\colon X \to R^{E'}$, $(f(x))_g = f_g(x)$. With product topology on $R^{E'}$, f is continuous. Also E, with weak topology can be considered a subspace of $R^{E'}$, and so $K \subset R^{E'}$. We prove that $f(X) \subset K$. If not, by separation theorem ([4], p. 65), there exists a $g \in E'$ such that, for some point $x_0 \in X$, $f_g(x_0) = p > \sup g(K) = q$. Let $A = \left\{x \in X\colon f_g(x) > \frac{1}{2}(p+q)\right\}$. Then $\mu(A) > 0$. From $g \circ f_\alpha \to f_g$ weakly, it follows that $\int_A f_g \, d\mu = \lim \int_A g \circ f_\alpha \, d\mu$. Since $\int_A f_g \, d\mu \geqq \frac{1}{2}(p+q)\mu(A)$ and $\int_A g \circ f_\alpha \, d\mu(A) \leqq q\mu(A)$, $\forall \alpha$, a contradiction. Thus $f(X) \subset K$. Since f is continuous, by ([6], p. 200, Theorem 3), f is strongly measurable and so $f \in L^1_E$. From what we have done it follows that for any simple measurable function $h\colon X \to E'$,

$$\varphi(h) = \int \langle h, f \rangle \, d\mu.$$

Define a norm p on $G = E'$, $p(g) = \sup |g(K)|$. Then the dual G' of (G, p) is contained in E and so G' is norm-separable (note that K is separable). Take an $h \in L^\infty_{E'}[E]$; $h(X)$ can be considered to be a bounded subset of E' ([8], p. 94, Theorem 7). $h\colon X \to (G, p)$ is strongly measurable. Since $\{\langle h, f_\alpha \rangle\}_\alpha$ is uniformly bounded on X, it is weakly compact in L^1 and so $\int_A |\langle h, f_\alpha \rangle| \, d\mu \to 0$ as $\mu(A) \to 0$, uniformly on α. Also $\int_A |\langle h, f \rangle| \, d\mu \to 0$ as $\mu(A) \to 0$. Fix $\varepsilon > 0$. There exists a $\delta > 0$ such that $\mu(A) < \delta$ implies $|\varphi(h\chi_A)| < \varepsilon$ and $\int_A |\langle h, f \rangle| \, d\mu < \varepsilon$. Take a sequence $h_n\colon X \to E'$, of simple measurable functions, such that $p(h_n - h) \to 0$ uniformly on $X \setminus A$ with $\mu(A) < \delta$ (Egorov's theorem, [2], p. 94). Now $\int \langle (h_n - h)\chi_{X \setminus A}, f_\alpha \rangle \, d\mu \to 0$, uniformly in α (note that f_α are K-valued). Thus $\varphi((h_n - h)\chi_{X \setminus A}) \to 0$. Also $\int \langle (h_n - h)\chi_{X \setminus A}, f \rangle \, d\mu \to 0$. Take an n_0 such that $|\varphi((h_{n_0} - h)\chi_{X \setminus A})| < \varepsilon$ and $\left|\int \langle (h_{n_0} - h)\chi_{X \setminus A}, f \rangle \, d\mu\right| < \varepsilon$. Thus

$$|\varphi(h) - \int \langle h, f \rangle \, d\mu| \leqq$$

$$\leqq |\varphi(h\chi_A + (h - h_{n_0})\chi_{X \setminus A}) - \int \langle h\chi_A, f \rangle \, d\mu - \int \langle (h - h_{n_0})\chi_{X/A}, f \rangle d\mu| \leqq 4\varepsilon.$$

Thus $\varphi = f$. So W is weakly compact in L^1_E.

Now we come to the general case when E is any Hausdorff locally convex space. All notations about E introduced at the beginning will be used. Denoting by φ_p the canonical mapping from E to E_p, $K_p = \varphi_p(K)$ is a weakly compact convex subset of E_p. This means $W \subset \prod_{p \in P} W_p$, where $W_p = \{f \in L^1_{E_p}\colon f(X) \subset K_p\}$. Since each W_p is weakly compact in $L^1_{E_p}$, it is enough to prove that W is closed in $G = \prod_{p \in P} L^1_{E_p}$ (note that W is convex). Let $\{f_\alpha\}\subset W, f_\alpha\to f\in \prod_{p\in P} W_p\subset G$. f can be assumed to be $\prod_{p\in P} K_p$-valued. By ([8], p. 65) f can be considered to be continuous from X into $\prod_{p\in P} K_p$, with induced weak-topology from $\prod E_p$. If for some $x_0\in X$, $f(x_0)\notin K$, by separation theorem ([4], p. 65) there exists a $g\in(\prod E_p)'$ such that $\alpha=g\circ f(x_0)>\sup g(K)=\beta$. Let $A=\left\{x\in X\colon g\circ f(x)>\frac{1}{2}(\alpha+\beta)\right\}$. Then $\mu(A)>0$ (note that f is continuous). Since each f_α is K-valued and $\int_A g\circ f_\alpha\,d\mu\to\int_A g\circ f\,d\mu$, we get $\frac{1}{2}(\alpha+\beta)\mu(A)\leqq$ $\leqq\beta\mu(A)$. This implies $\alpha\leqq\beta$, a contradiction. Thus f is K-valued. Now for any $p\in P$, $\varphi_p\circ f$ is continuous and so by ([6], p. 200, Theorem 3), $\varphi_p\circ f\colon X\to E_p$ is strongly measurable. From this it easily follows that $f\in L_E^1$. This proves the theorem.

References

[1] J. Diestel, Weak compactness in $L_1(X, E)$, *Glasgow Math. J.*, **18** (1977), 87—91.
[2] N. Dinclueanu, *Vector Measures*, Pergamon Press (New York, 1967).
[3] M. Ali Khan, An alternate proof of Diestel's theorem, *Glasgow Math. J.*, **25** (1984), 45—46.
[4] H. H. Schaeffer, *Topological Vector Spaces*, Springer-Verlag (1971).
[5] A. Ionescu Tulcea, Topologies compatible with liftings, *Bull. de la Math. Grèce*, **8** (1967), 116—126.
[6] A. Ionescu Tulcea, On pointwise convergence, compactness, and equicontinuity in the lifting topology I, *J. Zeit. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **26** (1973), 197—205.
[7] A. Ionescu Tulcea, C. Ionescu Tulcea, On the lifting property II. Representation of linear operators on spaces L_E^r, $1\leqq r<\infty$, *J. Math. Mech.*, **11** (1962), 773—795.
[8] A. Ionescu Tulcea, C. Ionescu Tulcea, *Topics in the Theory of Liftings*, Springer-Verlag (New York, 1969).

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ON PELCZYNSKI'S PROPERTY u FOR BANACH SPACES

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A Banach space X is said to have property u if for every weak Cauchy sequence (x_n) in X there exists a sequence (y_n) in X such that (a) the series $\sum_{i=1}^{\infty} y_i$ is weakly unconditionally Cauchy (wuc) and (b) the sequence $\left(x_n - \sum_{i=1}^{n} y_i\right)$ converges weakly to 0. Denote the weak* sequential closure of JX (J is the natural map) in X'' (the second dual space) by KX. NX is to be the set JX plus all $\sigma(X'', X')$-limits of wuc series in X. Then $JX \subseteqq NX \subseteqq KX$. Recall that $NX = KX$ if and only if X has property u [2] and that $JX = KX$ if and only if X is weakly sequentially complete. Then clearly every weakly sequentially complete Banach space has property u. Let i denote the identity map of a subspace Y into X. From [3], we know $i''KY = i''Y'' \cap KX$. We show the same is true for NX.

LEMMA 1 [5]. *Let Y be a subspace of X. If $F \in NX$ and $G \in KY$ are such that $i''G = F$, then $G \in NY$.*

THEOREM 2. *Let Y be a subspace of X. Then $i''NY = i''Y'' \cap NX$.*

PROOF. First observe that i'' is an isometry from Y'' into X'' and if $F = i''G$ and $G \in Y''$, then $F(f) = G(f|Y)$ for all $f \in X'$. Suppose $F \in i''NY$; then there exists a wuc series $\sum z_i$ in Y such that if $y_n = \sum_{i=1}^{n} z_i$ then $G = \sigma(Y'', Y') - \lim_n Jy_n$ and $i''G = F$. So $F(f) = G(f|Y) = \lim_n Jy_n(f|Y) = \lim_n f(y_n)$ for every $f \in X'$ and, hence, $F \in (i''Y'') \cap NX$. It then follows that $i''NY \subseteqq (i''Y'') \cap NX$.

Conversely, if $F \in (i''Y'') \cap NX$ then there exists a G in Y'' such that $i''G = F$ and also $F \in NX$. Since $NX \subseteqq KX$, $(i''Y'') \cap NX \subseteqq (i''Y'') \cap KX = i''KY$. So $F \in i''KY$, i.e. $G \in KY$. $F \in NX$, $G \in KY$, and $i''G = F$ imply by Lemma 1 that $G \in NY$; hence, $F \in i''NY$ and it follows that $(i''Y'') \cap NX \subseteqq i''NY$.

COROLLARY 3. *Let Y be a subspace of X. If $NX = JX$, then $NY = JY$.*

PROOF. $i''NY = (i''Y'') \cap NX = (i''Y'') \cap JX = i''JY$.

The next easily verified result shows that it is sufficient to consider only separable subspaces to obtain the converse.

COROLLARY 4. *$NX = JX$ if and only if $NY = JY$ for each separable subspace Y of X.*

We can also use Theorem 2 in the case $NX = X''$.

COROLLARY 5. *If X is a Banach space such that $NX=X''$, then $NY=Y''$ for all subspaces Y of X.*

PROOF. $i''NY=(i''Y'')\cap NX=(i''Y'')\cap X''=i''Y''$.

It is well known that if wuc series are unconditionally converging (uc) series in a Banach space X, then X has no subspace isomorphic to c_0 [1]. This is also equivalent to the property $JX=NX$. By using this we have:

PROPOSITION 6 [4]. *If in a space X having property u, wuc series are uc series, then X is weakly sequentially complete.*

PROOF. $JX=NX$ since all wuc series are uc series. Since X has property u, $NX=KX$. Thus $JX=KX$, i.e., X is weakly sequentially complete.

THEOREM 7 [4]. *If X has property u, then X is weakly sequentially complete if and only if no subspace of X is isomorphic to c_0.*

PROOF. If X is weakly sequentially complete, $KX=NX=JX$; thus, $NX=JX$, so no subspace of X is isomorphic to c_0. Conversely, if $NX=JX$ and $NX=KX$, then $JX=KX$.

Property u can be considered as an inherited property.

THEOREM 8 [4]. *If X has property u, then every subspace Y has property u.*

PROOF. If X has property u, then $NX=KX$. By Theorem 2 and [3] we have $i''NY=(i''Y'')\cap NX=(i''Y'')\cap KX=i''KY$, so $NY=KY$. Thus Y has property u.

COROLLARY 9 [4]. *The space X has property u if and only if every separable subspace has property u.*

PROOF. Let Y be a separable subspace. If $NY=KY$, it follows from Theorem 2 and [3] that $(i''Y'')\cap KX=i''KY=i''NY=(i''Y'')\cap NX$, so $(i''Y'')\cap KX=(i''Y'')\cap$ $\cap NX$ for every separable subspace Y. If $F\in KX$, then there exists a sequence (x_n) in X such that $F=\sigma(X'', X')-\lim_n Jx_n$. If Y is the separable subspace generated by (x_n), then $F\in i''Y''$; so $F\in(i''Y'')\cap KX=(i''Y'')\cap NX$. It follows that $F\in NX$. Thus $KX=NX$. The converse follows from Theorem 8.

Recall that the semi-reflexive space J given by R. C. James does not have property u [4].

LEMMA 10. *l_∞ does not have property u.*

PROOF. J' is separable, hence there exists a map from l_1 onto J'. This means that J'' can be embedded in l_∞; and thus so can J. Since J does not have property u, neither can l_∞ since property u is hereditary for subspaces (Theorem 8).

THEOREM 11. *If X is a conjugate space with property u, then X is weakly sequentially complete.*

PROOF. By arguing as in the proof of the lemma above, l_∞ is not isomorphic to a subspace of X. But X being a conjugate space means that neither can c_0 be isomorphic to a subspace of X [1]. The conclusion follows from Theorem 7.

References

[1] C. Bessaga and A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, *Studia Math.*, **17** (1958), 151—164.
[2] J. Howard and K. Melendez, Sufficient conditions for a continuous linear operator to be weakly compact, *Bull. Aust. Math. Soc.*, 7 (1972), 183—190.
[3] R. D. McWilliams, On the w^*-sequential closure of subspaces of Banach spaces, *Port. Math.*, **22** (1963), 209—214.
[4] A. Pelczynski, A connection between weakly unconditional convergence and weakly completeness of Banach spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **7** (1958), 251—253.
[5] I. Singer, *Bases in Banach spaces* I, Springer-Verlag (New York and Berlin, 1970).

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ANALYTIC CONTINUATION OF GENERALIZED FUNCTIONS

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§ 1. Introduction

In the new theory of generalized functions introduced in Colombeau [2—6], Aragona—Colombeau [1], we defined the holomorphic generalized functions on $\Omega\subset C^n$ as the generalized functions F on Ω as solutions of $\bar{\partial}F=0$, see Colombeau—Galé [7] and Colombeau [5], Chap. 8. There we obtained several counterexamples showing that these holomorphic generalized functions have unusual properties concerning the analytic continuation, viz. there exists functions G, H as above such that: a) G is holomorphic on $|z|<R$ $(R>1)$, not identically zero, but $G^{(n)}(0)=0$ for every $n=0, 1, 2, \ldots$. b) H is holomorphic on C, not identically zero, but $H\left(1-\frac{1}{n}\right)=0$ for every $n=0, 1, \ldots$. However the result stated below shows that the main formulation of uniqueness of the analytic continuation in C^n remains true in the setting of generalized functions. Following the classical pattern, its consequences for the theory are numerous and basic. The purpose of this note is to prove this result.

We follow the definitions and notations of Colombeau [6] which are simpler and more convenient (further we have to use here the set $\mathcal{N}[\Omega]$ defined in 1.1.11 there). We denote by $\mathcal{G}_{\mathcal{H}}(\Omega)$ the algebra of the holomorphic generalized functions on an open subset Ω of C^n. We assume Ω is non-void and connected. Our result is

THEOREM. *If $F\in\mathcal{G}_{\mathcal{H}}(\Omega)$ is zero on a non-void open subset V of Ω then $F=0$ in Ω.*

We begin by proving the theorem in case $n=1$. The general case will be a slight modification of the above one. A sketch of the proof in case $n=1$ is in Colombeau [6] Appendix 5.

§ 2. Proof of the theorem

We may assume V is the ball $B(a, R)$ centered at $a\in\Omega$ with radius $R>0$. In order to prove that F is zero in $\mathcal{G}_{\mathcal{H}}(\Omega)$ it suffices to verify that it is zero on a neighborhood of each point of Ω (this follows immediately from the definition of the generalized functions on Ω). Let z be a given point of Ω. As usual, there is a con-

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tinuous path γ contained in Ω, with endpoints a and z. Further, if $d>0$ denotes the minimum of R and the distance between $\operatorname{supp}\gamma$ and $C\setminus\Omega$ we can consider a finite number of points $\{z_j\colon j=0, 1, \dots, p\}$ in $\operatorname{supp}\gamma$ with $z_0=a$, $z_p=z$ and such that the open balls $B_j=B(z_j; d/4)$ cover $\operatorname{supp}\gamma$, and $B_j\cap B_{j+1}\neq\emptyset$. We consider a fixed point $\lambda_0\in B_1$ and we choose a point ω_0 in $B(\lambda_0, d/2)\cap B(a, d/4)$ which can be written as $\omega_0=\lambda_0+\varrho_0 e^{i\theta_0}$, with $\theta_0\in R$, $\varrho_0=|\omega_0-\lambda_0|$. By continuity of the mapping $(\lambda, \theta)\mapsto\lambda+\varrho_0 e^{i\theta}$, there exists $0<r<d/4$ such that the set $K=\{\lambda+\varrho_0 e^{i\theta}\in C\colon |\lambda-\lambda_0|\leqq r,\ |\theta-\theta_0|\leqq r\}$ is a compact subset of $B(\lambda_0, d/2)\cap \cap B(a, d/4)\subset V$. It is clear that there is a finite collection of real numbers $\theta_1, \dots, \theta_m$ such that the union for $|\lambda-\lambda_0|\leqq r$ of the circles $\{\lambda+\varrho_0 e^{i\theta}\colon \theta\in R\}$ is covered by

$$\bigcup_{\substack{|\lambda-\lambda_0|\leqq r\\ k=0,1,\dots,m}} B(\lambda+\varrho_0 e^{i\theta_k}, r).$$

Now, F being holomorphic on Ω, there exists a representative of F, $f(\varphi_\varepsilon, z)$ $(\varphi_\varepsilon\in\mathscr{A}_1)$, which is a genuine holomorphic function on $z\in B\left(\lambda_0, \frac{3d}{4}\right)$ (see Colombeau—Galé [7], and Colombeau [5], p. 224). For any λ with $|\lambda-\lambda_0|\leqq r$, let us consider the function

$$g_{\varphi_\varepsilon, \lambda}(z) = \prod_{k=0}^{m} f\big(\varphi_\varepsilon;\ \lambda+e^{i\theta_k}(z-\lambda)\big),$$

which is analytic on the closed disc $\overline{B(\lambda, \varrho_0)}$. For any z satisfying $|z-\lambda|=\varrho_0$, there is an index $k(z)\in\{0, 1, \dots, m\}$ such that $\lambda+(z-\lambda)e^{i\theta_{k(z)}}\in K$. Therefore, as F is zero on V, for $\varphi\in\mathscr{A}_q$, q large enough and for $\varepsilon>0$ small enough we have an inequality of the kind

$$\left|f\big(\varphi_\varepsilon, \lambda+(z-\lambda)e^{i\theta_{k(z)}}\big)\right| \leqq c\varepsilon^{\alpha(q)-N} \quad (\alpha\in\Gamma),$$

(see Colombeau [6], (1.1.11)) uniform in λ, if $|\lambda-\lambda_0|\leqq r$. On the other hand, for $k\neq k(z)$ we have inequalities of the kind

$$\left|f\big(\varphi_\varepsilon, \lambda+(z-\lambda)e^{i\theta_k}\big)\right| \leqq c'/\varepsilon^{N'}$$

still uniform in $|\lambda-\lambda_0|\leqq r$.

It follows that $\sup\limits_{|z-\lambda|=\varrho_0} |g_{\varphi_\varepsilon,\lambda}(z)|\leqq c''\varepsilon^{\alpha(q)-N''}$ for suitable $c'', N'', \alpha(q)$. In fact, this bound is valid for each point in $\{|z-\lambda|<\varrho_0\}$, by the Maximum Modulus Principle. So, if $|\lambda-\lambda_0|\leqq r$,

$$|g_{\varphi_\varepsilon, \lambda}(\lambda)| \leqq c''\varepsilon^{\alpha(q)-N''};$$

that is

$$\sup_{|\lambda-\lambda_0|\leqq r} |f(\varphi_\varepsilon, \lambda)| \leqq [c''\varepsilon^{\alpha(q)-N''}]^{1/m} = C\varepsilon^{\beta(q)-N_1}$$

for some $\beta\in\gamma$ and N_1 a positive integer.

We have proved that for any $\lambda_0 \in B_1$ there is an $r>0$ such that the restriction of F to $B(\lambda_0, r)$ is zero in $\mathcal{G}(B(\lambda_0, r))$. (The bounds requested for the derivatives of the representative follow, by Cauchy's formula, from the above one.) As a consequence F is zero on B_1. Repeating again this process we obtain that F is zero on B_2, and, finally, F is zero on B_p. This proves the first part.

Now, for the sake of simplicity we consider only the case $n=2$ to prove the general one (the case $n>2$ is an immediate adaptation). We assume F zero on the open polydisc $B(a_1, R)\times B(a_2, R)$, $a=(a_1, a_2)\in\Omega$, $R>0$. If $b=(b_1, b_2)\in\Omega$, an easy argument of connectedness and uniform continuity on compact sets shows that there exists a continuous path γ joining b and a, and a finite collection of points $z_j=(z_1^j, z_2^j)$ $(j=0, 1, \ldots, p)$ on $\operatorname{supp}\gamma$ with $z_0=a$, $z_p=b$ such that the polydisc $B_j^2=B(z_1^j, d/4)\times B(z_2^j, d/4)$ cover $\operatorname{supp}\gamma$ and such that $B_j^2\cap B_{j+1}^2\neq\emptyset$ $(j=0, 1, \ldots, p)$. Here d denotes the Euclidean distance from $\operatorname{supp}\gamma$ to $C\setminus\Omega$. Without loss of generality we may assume that the open polydisc $B(a_1, R)\times B(a_2, R)$ contains the compact subset $\overline{B(a_1, d/4)}\times\overline{B(a_2, d/4)}$. Let $\lambda=(\lambda_1, \lambda_2)\in B_1^2$ be given and $f(\varphi_\varepsilon, z_1, z_2)$ be a holomorphic representative of F on $B(\lambda_1, 3d/4)\times B(\lambda_2, 3d/4)$. We put $Q_i= =B(\lambda_i, d/2)\cap\overline{B(a_i, d/4)}$ $(i=1, 2)$. Note that the balls $B(\lambda_i, d/2)$ and $B(a_i, d/4)$ are related in the same way as they were in the case $n=1$. In particular, there exists $r>0$ such that the sets $K_i=\{\lambda+\varrho_i e^{i\theta}\in C\colon |\lambda-\lambda_i|\leqq r, |\theta-\theta_i|\leqq r\}$ $(i=1, 2)$ are compact in $B(\lambda_i, d/2)\cap B(a_i, d/4)$ respectively (ϱ_i, θ_i are obtained in a similar way as in the case $n=1$). Then, for any $z_2\in Q_2$ we consider the function $g_{z_2}(\varphi_\varepsilon, z)= =f(\varphi_\varepsilon, z, z_2)$, holomorphic in the usual sense on $B(\lambda_1, 3d/4)$, and the same argument as in the case $n=1$ works for the function g_{z_2}, so that one obtains

$$\sup_{|\lambda-\lambda_1|\leqq r}|g_{z_2}(\varphi_\varepsilon, \lambda)|\leqq c\varepsilon^{\alpha(q)-N}$$

($\varphi\in\mathcal{A}_q$, with q large enough and then for $\varepsilon>0$ small enough depending on φ, and for some $\alpha\in\Gamma$). Further, the bound by $c\varepsilon^{\alpha(q)-N}$ is uniform in $z_2\in Q_2$, since it follows from the bounds of the representative $f(\varphi_\varepsilon, z, z_2)$ on the set $K_1\times Q_2$ which is a precompact subset of $B(a_1, R)\times B(a_2, R)$ where F is zero. So, we have

$$(*)\qquad \sup_{\substack{|\lambda-\lambda_1|\leqq r\\ z_2\in Q_2}}|f(\varphi_\varepsilon, \lambda, z_2)|\leqq c\varepsilon^{\alpha(q)-N}.$$

Finally, for any z_1 such that $|z_1-\lambda_1|\leqq r$ we define $h_{z_1}(\varphi_\varepsilon, z)=f(\varphi_\varepsilon, z_1, z)$, which is holomorphic on the disc $B(\lambda_2, 3d/4)$. By repeating again the usual process we have $\sup_{|\mu-\lambda_2|\leqq r}|h_{z_1}(\varphi_\varepsilon, \mu)|\leqq c'\varepsilon^{\beta(q')-N'}$. This last bound is also uniform in $|z_1-\lambda_1|\leqq r$ since it is obtained from bounds involving the representative $f(\varphi_\varepsilon, \ldots)$ acting on the points (z_1, z_2) such that $|z_1-\lambda_1|\leqq r$ and $z_2\in K_2$, and $K_2\subset Q_2$. Then it suffices to apply $(*)$. We have proved that F is zero in the polydisc $B(\lambda_1, r)\times \times B(\lambda_2, r)$. As (λ_1, λ_2) is arbitrary in B_1^2, F is null in B_1^2. The proof that F is zero in a neighborhood of b follows as usual after a finite number of steps.

References

[1] J. Aragona, J. F. Colombeau, The $\bar{\partial}$ Neumann problem for generalized functions, *J. Math. Anal. and Appl.*, **110** (1985), 179—199.
[2] J. F. Colombeau, A multiplication of distributions, *J. Math. Anal. and Appl.*, **91** (1983), 96—115.
[3] J. F. Colombeau, New generalized functions, multiplication of distributions, Physical Applications, Contribution of J. Sebastiao e Silva, *Portugaliae Math.*, **41** (1982), 57—69.
[4] J. F. Colombeau, Une multiplication génèrale des distributions, *Comptes Rendus Acad. Sc., Paris*, **296** (1983), 357—360.
[5] J. F. Colombeau, *New generalized functions and multiplications of distributions*, North-Holland, Math. Studies **84** (1984).
[6] J. F. Colombeau, *Elementary introduction to new generalized functions*, North-Holland Math. Studies, to appear in 1985.
[7] J. F. Colombeau, J. E. Galé, Holomorphic generalized functions, *J. Math. Anal. and Appl.*, **103** (1984), 117—133.

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ON A MIXED PROBLEM FOR A CLASS OF NONLINEAR KLEIN—GORDON EQUATIONS

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Introduction

In [6] I. Segal proposed the system

$$\square u = \alpha^2 u + g^2 v^2 u, \quad \square v = \beta^2 v + h^2 u^2 v,$$

$\square = \Delta - \dfrac{\partial^2}{\partial t^2}$, as a model to describe the interaction of scalar fields u, v of mass α, β, respectively, with interaction constants g and h. This system defines the motion of charged mesons in an electromagnetic field. More recently V. G. Makhankov [5] mentioned some properties of such kind of interacting relativistic (scalar) fields.

In order to study the mixed problem for the above mentioned system, we observe that there is no loss of generality if we analyse only the case in which $\alpha=\beta=0$ and $g=h=1$. We shall consider the system

$$(*) \qquad \frac{\partial^2 u}{\partial t^2} - \Delta u + v^2 u = f_1, \quad \frac{\partial^2 v}{\partial t^2} - \Delta v + u^2 v = f_2$$

where $f_1(x, t)$ and $f_2(x, t)$ are two known real valued functions.

We shall prove that the methods to prove existence and uniqueness for the solutions of the mixed problem for a single equation $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + u^3 = f$, can be extended to the case of the system $(*)$.

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1. Existence and uniqueness

In this section, we shall prove that the mixed problem for the nonlinear system $(*)$ has a weak solution. Let us set up some terminology and basic notations. We represent by Ω a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ. By Q we represent the cylinder $\Omega\times[0, T[$, $T>0$, whose lateral boundary is denoted by $\Sigma=\Gamma\times[0, T[$. $H^m(\Omega)$ will denote the usual Sobolev space of order m in Ω and $H_0^m(\Omega)$ is defined to be the closure of $\mathscr{D}(\Omega)$ in $H^m(\Omega)$. Here $\mathscr{D}(\Omega)$ denotes space of C^∞ functions with compact support contained in Ω. If X is a Banach space and $1 \leqq p \leqq \infty$, by $L^p(0, T; X)$ we represent the space of measurable vector functions u: $]0, T[\to X$, such that $\|u(t)\|_X \in L^p(0, T)$ with the norm

$$\|u\|^p_{L^p(0,T;X)} = \int_0^T \|u(t)\|_X^p \, dt, \quad \text{if} \quad 1 \leqq p < +\infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X, \quad \text{if} \quad p=+\infty.$$

We represent by $(\cdot\,,\,\cdot)$ the inner product in $L^2(\Omega)$ and by $a(\varphi, \psi)$ the Dirichlet form associated to φ and ψ in $H_0^1(\Omega)$, i.e., $a(\varphi, \psi) = \int_\Omega \operatorname{grad}\varphi \cdot \operatorname{grad}\psi \, dx$. Now, Poincaré's inequality implies that the norms $(a(\varphi, \varphi))^{1/2}$ and $\|\varphi\|$ in $H_0^1(\Omega)$ are equivalent. To simplify our notation we shall write $a(\varphi)$ instead of $a(\varphi, \varphi)$. All the scalar functions considered in this paper will be real valued. We denote by $\mathscr{D}'(0, T)$ the space of distributions on $]0, T[$.

THEOREM 1. *Let* $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, $f_1, f_2 \in L^2(0, T; L^2(\Omega))$. *Suppose that* $\Omega \subset \mathbf{R}^n$, $n=1, 2, 3$, *is a bounded open set. Then there exists a pair of real valued functions* u, v *defined on* Q, *satisfying the conditions:*

(1) $$u, v \in L^\infty(0, T;\ H_0^1(\Omega))$$

(2) $$u_t, v_t \in L^\infty(0, T; L^2(\Omega))$$

(3) u, v *are weak solutions of* $(*)$, *i.e.*,

$$\frac{d}{dt}(u_t(t), w) + a(u(t), w) + (v^2(t)u(t), w) = (f_1(t), w),$$

$$\frac{d}{dt}(v_t(t), w) + a(v(t), w) + (u^2(t)v(t), w) = (f_2(t), w),$$

for each $w \in H_0^1(\Omega)$, *in the sense of* $\mathscr{D}'(0, T)$.

(4) $$u(0) = u_0, \quad v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1.$$

REMARK 1. The requirement that u and v vanish on Σ follows from (1). To verify that $u(0)$, $v(0)$, $u_t(0)$, $v_t(0)$ make sense we observe that by (1) and (2) it follows that u and v are weakly continuous from $[0, T]$ on $H_0^1(\Omega)$, so $u(0)$, $v(0)$ are well defined, see Lions—Magenes [4]. By (3), $u_{tt} = \Delta u - v^2 u + f_1$ in the vector distribution sense. By (1), $\Delta u \in L^\infty(0, T; H^{-1}(\Omega))$, where $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. We also have $v^2(t)u(t) \in L^{3/2}(\Omega)$, for example, continuously embedded in $H^{-1}(\Omega)$, since $H_0^1(\Omega) \subset L^q(\Omega)$, continuously, whenever $q \leqq 6$, by Sobolev's embedding theorem. Consequently $v^2 u \in L^\infty(0, T; H^{-1}(\Omega))$. Identifying $L^2(\Omega)$ to its dual, it follows that $L^2(\Omega)$ is identified to a subspace of $H^{-1}(\Omega)$, then, we obtain f_1 belongs to $L^2(0, T; H^{-1}(\Omega))$. This conclusions plus (2), and Lions—Magenes, op. cit., we obtain u_t is weakly continuous from $[0, T]$ into $L^2(\Omega)$, and $u_t(0)$ is well defined. Mutatis mutandis, it follows that $v_t(0)$ are well defined.

PROOF OF THEOREM 1. Let $(w_\nu)_{\nu \in \mathbf{N}}$ be a Hilbertian base of $V = H_0^1(\Omega)$ and let us denote by $V_m = [w_1, w_2, \dots, w_m]$ the subspace of dimension m of V generated by the first m vectors w_ν. We use Galerkin approximation procedure, i.e., we solve an approximated problem in V_m and then we take limits as m approaches infinity, to obtain the desired solution.

i) *Approximated system.* We look for a pair of functions

$$u_m(t)=\sum_{j=1}^{m} g_{jm}(t)w_j,\quad v_m(t)=\sum_{j=1}^{m} h_{jm}(t)w_j,\quad \text{in}\quad V_m,$$

which are solutions of the following system of ordinary differential equations:

$$(5)_1\qquad \big(u_m''(t), w_j\big)+a\big(u_m(t), w_j\big)+\big(v_m^2(t)\,u_m(t), w_j\big)=\big(f_1(t), w_j\big)$$

$$(5)_2\qquad \big(v_m''(t), w_j\big)+a\big(v_m(t), w_j\big)+\big(u_m^2(t)\,v_m(t), w_j\big)=\big(f_2(t), w_j\big)$$

for $j=1, 2, \ldots, m$, with the initial conditions:

$$(6)\qquad \begin{cases} u_m(0)=u_{0m}\to u_0;\quad v_m(0)=v_{0m}\to v_0, & \text{strongly}\quad H_0^1(\Omega)\\ u_m'(0)=u_{1m}\to u_1;\quad v_m'(0)=v_{1m}\to v_1, & \text{strongly}\quad L^2(\Omega).\end{cases}$$

Note that we used above "$'$" instead of d/dt.

The system $(5)_1$, $(5)_2$ contains $2m$ unknows $g_{jm}(t)$, $h_{jm}(t)$, $j=1, 2, \ldots, m$. It follows that $(5)_1$, $(5)_2$, (6) has a local solution $u_m(t)$, $v_m(t)$ on $[0, t_m[$. In order to extend these local solutions to the whole interval $[0, T[$, we need a priori estimates. These will be obtained in the following step.

ii) *A priori estimates.* Multiply $(5)_1$ by $2g_{jm}'(t)$, $(5)_2$ by $2h_{jm}'(t)$ and add from $j=1$ to $j=m$. We obtain:

$$(7)_1\qquad \frac{d}{dt}|u_m'(t)|^2+\frac{d}{dt}a\big(u_m(t)\big)+\int_\Omega v_m^2(t)\frac{d}{dt}u_m^2(t)\,dx=2\big(f_1(t), u_m'(t)\big)$$

$$(7)_2\qquad \frac{d}{dt}|v_m'(t)|^2+\frac{d}{dt}a\big(v_m(t)\big)+\int_\Omega u_m^2(t)\frac{d}{dt}v_m^2(t)\,dx=2\big(f_2(t), v_m'(t)\big).$$

Adding $(7)_1$, $(7)_2$ we get

$$\frac{d}{dt}\Big(|u_m'(t)|^2+|v_m'(t)|^2+a\big(u_m(t)\big)+a\big(v_m(t)\big)+\int_\Omega u_m^2(t)\,v_m^2(t)\,dx\Big)=$$

$$=2\big(f_1(t), u_m'(t)\big)+\big(f_2(t), v_m'(t)\big).$$

Integrating this equation from 0 to $t<t_m$, we obtain

$$|u_m'(t)|^2+|v_m'(t)|^2+a\big(u_m(t)\big)+a\big(v_m(t)\big)+\int_\Omega u_m^2(t)\,v_m^2(t)\,dx=$$

$$=|u_{1m}|^2+|v_{1m}|^2+a(u_{0m})+a(v_{0m})+\int_\Omega u_{0m}^2\,v_{0m}^2\,dx+$$

$$+\int_0^T |f_1(t)|^2\,dx+\int_0^T |f_2(t)|^2\,dx+\int_0^t \big(|u_m'(s)|^2+|v_m'(s)|^2\big)\,ds.$$

Our condition (6) implies that the sequences $(|u_{1m}|^2)_{m\in\mathbf{N}}$, $(|v_{1m}|^2)_{m\in\mathbf{N}}$, $\big(a(u_{0m})\big)_{m\in\mathbf{N}}$, $\big(a(v_{0m})\big)_{m\in\mathbf{N}}$ are bounded. It remains only to bound the term $u_{0m}^2 v_{0m}^2$. In fact, by hypothesis, $H_0^1(\Omega)\subset L^q(\Omega)$, continuously, for $1\leqq q\leqq 6$, because $n=1,2,3$. Then, by Schwarz inequality and Sobolev's embedding theorem we obtain that

$$\Big|\int_\Omega u_{0m}^2 v_{0m}^2\,dx\Big|$$

is bounded. It follows, for m large enough, that

$$|u_m'(t)|^2+|v_m'(t)|^2+\|u_m(t)\|^2+\|v_m(t)\|^2+\int_\Omega u_m^2(t)v_m^2(t)\,dx\leqq \tag{8}$$

$$\leqq K+\int_0^t \big(|u_m'(s)|^2+|v_m'(s)|^2\big)\,ds.$$

From (8) and Gronwall inequality, we find:

$$|u_m'(t)|^2+|v_m'(t)|^2+\|u_m(t)\|^2+\|v_m(t)\|^2+\int_\Omega u_m^2(t)v_m^2(t)\,dx< C, \tag{9}$$

where C is a constant independent of m.

Inequality (9) implies that we can extend $u_m(t)$ and $v_m(t)$ to the whole interval $[0,T]$. Furthermore

$(10)_1$ $(u_m)_{m\in\mathbf{N}}$, $v_m)_{m\in\mathbf{N}}$ are bounded sequences in $L^\infty(0,T;\ H_0^1(\Omega))$.

$(10)_2$ $(u_m')_{m\in\mathbf{N}}$, $(v_m')_{m\in\mathbf{N}}$ are bounded sequences in $L^\infty(0,T;\ L^2(\Omega))$.

Consequently, we can find subsequences $(u_\nu)_{\nu\in\mathbf{N}}$, $(v_\nu)_{\nu\in\mathbf{N}}$, such that

$$\begin{cases} u_\nu \rightharpoonup u, \quad v_\nu\rightharpoonup v & \text{weak star in } L^\infty(0,T;\ H_0^1(\Omega))\\ u_\nu'\rightharpoonup u', \quad v_\nu'\rightharpoonup v' & \text{weak star in } L^\infty(0,T;\ L^2(\Omega)).\end{cases} \tag{11}$$

Clearly, (11) implies

$$\int_0^T (u_\nu'(t),w)\theta(t)\,dt\to\int_0^T (u'(t),w)\theta(t)\,dt,$$

for all $\theta\in\mathcal{D}(0,T)$ and $w\in H_0^1(\Omega)$, i.e.,

$$\big(u_m'(t),w\big)\rightharpoonup\big(u'(t),w\big)\quad\text{in}\quad \mathcal{D}'(0,T).$$

Therefore we obtain

$$\frac{d}{dt}(u_\nu'(t),w)\rightharpoonup\frac{d}{dt}(u'(t),w)\quad\text{in}\quad\mathcal{D}'(0,T), \tag{12}$$

for all $w\in H_0^1(\Omega)$.

Similarly

$$\frac{d}{dt}(u_\nu'(t),w)\rightharpoonup\frac{d}{dt}(v_\nu'(t),w)\quad\text{in}\quad\mathcal{D}'(0,T), \tag{13}$$

for all $w\in H_0^1(\Omega)$.

A similar argument implies that

$$(14)\qquad a(u_\nu(t), w) \rightharpoonup a(u(t), w); \quad a(v_\nu(t), w) \rightharpoonup a(v(t), w) \quad \text{in} \quad \mathcal{D}'(0, T)$$

for all $w \in H_0^1(\Omega)$.

iii) *Analysis of the nonlinear terms.* By the estimates $(10)_1$, $(10)_2$ and the compactness theorem of Aubin—Lions, see Lions [3], we obtain subsequences, which we still represent by $(u_\nu)_{\nu\in\mathbf{N}}$, $(v_\nu)_{\nu\in\mathbf{N}}$, such that

$$u_\nu \to u, \quad v_\nu \to v \quad \text{strongly} \quad L^2(0, T; L^2(\Omega))$$

and almost everywhere (a.e.) in Q. Thus we conclude:

$$(15)\qquad \begin{cases} u_\nu^2 v_\nu \to u^2 v & \text{a.e. in} \quad Q, \\ v_\nu^2 u_\nu \to v^2 u & \text{a.e. in} \quad Q. \end{cases}$$

By Hölder inequality, for $1 < q < \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we obtain

$$\Big|\int_Q (u_\nu^2 v_\nu)^q dx\,dt\Big| \leqq \int_Q |u_\nu|^{2q} |v_\nu|^q dx\,dt \leqq \Big(\int_Q |u_\nu|^{2qp} dx\,dt\Big)^{1/p} \Big(\int_Q |v_\nu|^{qp'} dx\,dt\Big)^{1/p'}.$$

If $2qp \leqq 6$, $qp' \leqq 6$, Sobolev's inequality implies that $u_\nu^2 v_\nu$ is bounded in $L^q(Q)$ for $1 < q \leqq 2$.

All of the above informations together imply

$$(16)\qquad \begin{cases} u_\nu^2 v_\nu \to u^2 v \quad \text{a.e. in} \quad Q, \\ u_\nu^2 v_\nu \text{ are bounded in } L^q(\Omega), \quad 1 < q \leqq 2. \end{cases}$$

It follows from Lions, op. cit., that (16) implies $u_\nu^2 v_\nu \rightharpoonup u^2 v$ weakly in $L^q(Q)$, $1 < q \leqq 2$.

If we restrict $\dfrac{6}{5} \leqq q \leqq 2$ it follows that $q' = \dfrac{q}{q-1} \leqq 6$ and $H_0^1(\Omega) \subset L^{q'}(\Omega)$, dual of $L^q(\Omega)$. We then get, from the weak convergence in $L^q(\Omega)$, that

$$(17)\qquad \int_0^T (u_\nu^2(t) v_\nu(t), w)\theta(t)\, dt \to \int_0^T (u^2(t) v(t), w)\theta(t)\, dt,$$

for all $w \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}'(0, T)$.

Mutatis mutandis, we have

$$(18)\qquad \int_0^T (v_\nu^2(t) u_\nu(t), w)\theta(t)\, dt \to \int_0^T (v_\nu^2(t) u(t), w)\theta(t)\, dt$$

for all $w \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}'(0, T)$.

Now taking the limit in (5) and using (12), (13), (14), (17) and (18), it follows that u, v satisfy condition (3) of Theorem 1, i.e., u, v are weak solutions of the coupled Klein—Gordon equations $(*)$.

The initial conditions (4) can be verified by observing (16), (11) and the definition of weak solution. This concludes the proof of Theorem 1. Q.E.D.

2. Uniqueness

Suppose we have two pairs of solutions (u, v) and $(\hat{u}, \hat{v})$ in the conditions of Theorem 1. Let $U=u-\hat{u}$, $V=v-\hat{v}$. Thus, U and V satisfy

$$(19)_1 \qquad U''-\Delta U+v^2u-\hat{v}^2\hat{u} = 0,$$

$$(19)_2 \qquad V''-\Delta V+u^2v-\hat{u}^2\hat{v} = 0,$$

with initial conditions

$$(20)_1 \qquad U(0) = 0, \quad V(0) = 0,$$

$$(20)_2 \qquad U'(0) = 0, \quad V'(0) = 0.$$

To prove the uniqueness it is sufficient to prove that $|U(s)|=|V(s)|=0$ on $[0, T[$. The standard energy method which consists in taking the scalar product of $(19)_1$ by $U'(t)$ and $(19)_2$ by $V'(t)$ and integrating on $[0, T[$, does not make sense because $U''(t), V''(t)\in H^{-1}(\Omega)$ and $U'(t), V'(t)\in L^2(\Omega)$, which contains $H_0^1(\Omega)$ properly. By this reason, we define convenient functions $\varphi(t)$ and $\psi(t)$ belonging to $H_0^1(\Omega)$, and for these functions the uniqueness method works. (See Visik—Ladyzhenskaya [7] and Lions op. cit.)

THEOREM 2. *The solution of Theorem* 1 *is unique.*

PROOF. For s in $]0, T[$ let us consider functions φ and ψ defined on $]0, T[$ as follows:

$$\varphi(t) = \begin{cases} -\int_t^s U(\sigma)\,d\sigma & \text{if } t \leqq s \\ 0 & \text{if } t > s \end{cases}$$

and

$$\psi(t) = \begin{cases} -\int_t^s V(\sigma)\,d\sigma & \text{if } t \leqq s \\ 0 & \text{if } t > s \end{cases}$$

where the integrals are considered in the sense of Bochner in the Hilber space $H_0^1(\Omega)$. Therefore, $\varphi(t)$ and $\psi(t)$ belong to $L^\infty(0, T; H_0^1(\Omega))$.

If we represent $\varphi_1(t)=\int_0^t U(\sigma)\,d\sigma$ and $\psi_1(t)=\int_0^t V(\sigma)\,d\sigma$, we have

$$\varphi(t) = \varphi_1(t)-\varphi_1(s) \quad \text{and} \quad \psi(t) = \psi_1(t)-\psi_1(s),$$

for $0\leqq t\leqq s$.

Taking the scalar product of $(19)_1$ with φ and $(19)_2$ with ψ, we obtain

$$(21)_1 \qquad \int_0^s (U'', \varphi)\,dt+\int_0^s a(U, \varphi)\,dt+\int_0^s (v^2u-\hat{v}^2\hat{u},\varphi)\,dt = 0,$$

$$(21)_2 \qquad \int_0^s (V'', \psi)\,dt+\int_0^s a(V, \psi)\,dt+\int_0^s (u^2v-\hat{u}^2\hat{v}, \psi)\,dt = 0.$$

A simple computation shows that

$$\int_0^s (U'', \varphi)\,dt = -\frac{1}{2}\,|U(s)|^2 \tag{22}$$

and

$$\int_0^s a(U, \varphi)\,dt = -\frac{1}{2}\,\|\varphi_1(s)\|^2. \tag{23}$$

Substituting (22) and (23) in $(21)_1$, we obtain

$$\frac{1}{2}\,|U(s)|^2 + \frac{1}{2}\,\|\varphi_1(s)\|^2 = \int_0^s (v^2 u - \hat{v}^2 \hat{u}, \varphi)\,dt. \tag{24}$$

The triangle inequality implies that

$$\int_0^s (v^2 u - \hat{v}^2 \hat{u}, \varphi)\,dt \leqq \int_0^s (v^2 U, \varphi)\,dt + \int_0^s (\hat{u}[v+\hat{v}]V, \varphi)\,dt.$$

Since $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$, by Hölder's inequality and Sobolev's embedding theorem, we get

$$\begin{aligned} (v^2 U, \varphi) &= \int_\Omega v^2 U \varphi\,dx \leqq \int_\Omega |v^2|\,|U|\,|\varphi|\,dx \leqq \\ &\leqq \Big(\int_\Omega |U|^2 dx\Big)^{1/2} \Big(\int_\Omega (v^2)^3 dx\Big)^{1/3} \Big(\int_\Omega |\varphi|^6 dx\Big)^{1/6} = \\ &= \|v(t)\|^{1/2}_{L^6(\Omega)}\,|U(t)|_{L^2(\Omega)}|\varphi(t)|_{L^6(\Omega)} \leqq C_1 |U(t)|_{L^2(\Omega)} \cdot \|\varphi(t)\|_{H^1_0(\Omega)}. \end{aligned} \tag{25}$$

By a similar argument, we also find

$$\begin{aligned} (\hat{u}[v+\hat{v}]V, \varphi) &= \int_\Omega \hat{u}[v+\hat{v}]V \cdot \varphi\,dx \leqq \int_\Omega |\hat{u}|\,|v+\hat{v}|\,|V|\,|\varphi|\,dx \leqq \\ &\leqq \Big(\int_\Omega |V|^2 dx\Big)^{1/2} \Big(\int_\Omega |\varphi|^6 dx\Big)^{1/6} \Big(\int_\Omega |\hat{u}|^6\Big)^{1/6} \Big(\int_\Omega |v+\hat{v}|^6\Big)^{1/6} \leqq C_2 |V|_{L^2(\Omega)} \cdot \|\varphi\|_{H^1_0(\Omega)}. \end{aligned} \tag{26}$$

Substituting (25), (26) in (24), we have

$$\frac{1}{2}\,|U(s)|^2 + \frac{1}{2}\,\|\varphi_1(s)\|^2 \leqq c \int_0^s \big(|U| + |V(t)|\big)\,\|\varphi(t)\|\,dt. \tag{27$_1$}$$

By a similar argument we get

$$\frac{1}{2}\,|V(s)|^2 + \frac{1}{2}\,\|\psi_1(s)\|^2 \leqq c \int_0^s \big(|U(t)| + |V(t)|\big)\,\|\psi(t)\|\,dt. \tag{27$_2$}$$

Let us fix our attention on $(27)_1$. Since $\varphi(t)=\varphi_1(t)-\varphi_1(s)$, then $(27)_1$ implies that

$$\frac{1}{2}|U(s)|^2+\frac{1}{2}\|\varphi_1(s)\|^2 \leqq c\int_0^t (|U(t)|+|V(t)|)\|\varphi_1(t)\|\,dt+$$

$$+\frac{c}{2}\int_0^s (|U(t)|+|V(t)|)^2\,dt+\frac{cs}{2}\|\varphi_1(s)\|^2.$$

Thus

$$(28)_1 \qquad \frac{1}{2}|U(s)|^2+\left(\frac{1}{2}-\frac{cs}{2}\right)\|\varphi_1(s)\|^2 \leqq c\int_0^s (|U(t)|^2+|V(t)|^2+\|\varphi_1(t)\|^2)\,dt,$$

where c denote various constants.

We also obtain from $(25)_2$

$$(28)_2 \qquad \frac{1}{2}|V(s)|^2+\left(\frac{1}{2}-\frac{cs}{2}\right)\|\psi_1(s)\|^2 \leqq c\int_0^s (|U(t)|^2+|V(t)|^2+\|\psi_1(t)\|^2)\,dt.$$

Adding $(28)_1$, $(28)_2$, we have:

$$(29) \qquad \frac{1}{2}(|U(s)|^2+|V(s)|^2)+\left(\frac{1}{2}-\frac{cs}{2}\right)(\|\varphi_1(s)\|^2+\|\psi_1(s)\|^2) \leqq$$

$$\leqq c\int_0^s (|U(t)|^2+|V(t)|^2+\|\varphi_1(t)\|^2+\|\psi_1(t)\|^2)\,dt.$$

Choose s_0 in $[0, T[$ such that $\frac{1}{2}-\frac{cs_0}{2}=\frac{1}{4}$, i.e., $s_0=\frac{1}{2c}$.

It follows from (29) and Gronwall inequality that $|U(t)|=0$, $|V(t)|=0$ for all $0\leqq t\leqq s_0$, since for such t we have

$$\frac{1}{2}-\frac{tc}{2}\geqq\frac{1}{2}-\frac{s_0c}{2}=\frac{1}{4}.$$

If $s_0<T$, we use the same argument with initial data zero in s_0 and we consider $s_0<s<T$. As was done above, the coefficient of $\|\varphi_1(s)\|^2+\|\psi_1(s)\|^2$ is $\frac{1}{2}-\frac{c(s-s_0)}{2}=\frac{3}{4}-\frac{cs}{2}$. Let s_1 be such that $\frac{3}{4}-\frac{cs_1}{2}=\frac{1}{4}$. So, $s_1=\frac{1}{c}=2s_0$. This implies that $|U(t)|+|V(t)|=0$ on $[s_0, 2s_0]$. By continuing this process we conclude that U and V are zero almost everywhere, i.e., $u=\hat{u}$, $v=\hat{v}$, which ends the proof of Theorem 2. Q.E.D.

References

[1] J. Ferreira, G. Perla Menzala, Decay of solutions of a system of nonlinear Klein—Gordon equations, *Internat. J. Math. and Math. Sci.*, **9** (1986), 471—483.

[2] K. Jörgens, *Nonlinear wave equations*, University of Colorado, Department of Mathematics, 1970.

[3] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod (Paris, 1969).

[4] J. L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod (Paris, 1968).

[5] V. G. Makhankov, Dynamics of classical solutions in integrable systems, *Physics Reports* (Section C of Physics Letters), **35** (1978), 1—128.

[6] I. Segal, Nonlinear partial differential equations in Quantum Field Theory, *Proc. Symp. Appl. Math. A.M.S.*, **17** (1965), 210—226.

[7] M. I. Visik, O. A. Ladyzhenskaya, A boundary value problem for partial differential equations and certain classes of operator equations, *Uspehi Mat. Nauk (V. S.)*, **11** (1956), 41—97; *A.M.S. Translation Series 2*, **10** (1958).

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ON DERRIENNIC'S ALMOST SUBADDITIVE ERGODIC THEOREM

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1. Introduction

Recently, Moulin—Ollagnier [17] derived a new mean ergodic theorem which might be called an almost subadditive ergodic theorem. This allows a fairly simple proof of the L^1-part of the Shannon—McMillan—Breiman theorem of information theory (cf. [17]). Shortly after, Derriennic [6] showed that Moulin—Ollagnier's mean ergodic theorem holds under weaker moment conditions and that it can be extended to yield an individual ergodic theorem. This, in turn, allows a simple (martingale-free) proof of the almost sure part of the Shannon—McMillan—Breiman result (cf. [6]). Derriennic's [6] almost subadditive ergodic theorem is an interesting generalization of Kingman's [12] ergodic theorem, and might be considered as a stochastic analogue of the following simple result on real sequences which are "almost subadditive".

LEMMA 1.1. *Let* $(a_n)\subset R$ *and* $(c_n)\subset R_+$ *be sequences satisfying the following two conditions:*

(1.1) $$a_{m+n}-a_m-a_n \leqq c_n, \quad m, n \geqq 1;$$

(1.2) $$\lim_n \frac{1}{n} c_n = 0.$$

Then $\lim_n (1/n)a_n=a$ *exists and satisfies* $-\infty\leqq a<\infty$.

For the simple proof cf. [6]. Related results are contained in [7] and [8].

A stochastic analogue of (1.1) is obtained as follows. Let $X=(X_{s,t})\subset L^1$ and $h=(h_{s,t})\subset L^1_+$ be two families of real random variables defined on some probability space $(\Omega, \mathcal{A}, P)$ (the index set of families like $(X_{s,t})$ will be throughout $I=\{(s, t)|0\leqq s<t,\ s, t \text{ integers}\}$). Then, X might be called *almost subadditive* (w.r.t. h) provided the following analogue of (1.1) holds:

(1.3) $$X_{s,u}-X_{s,t}-X_{t,u} \leqq h_{t,u}, \quad (s, t), (t, u)\in I.$$

The subadditivity condition occurring in Kingman [12], [13] corresponds to the case when each $h_{s,t}$ identically vanishes. Derriennic [6] has considered families X and h which are *stationary*, i.e., X and h have the same distributions as the corresponding shifted processes $(X_{s+1,t+1})$ and $(h_{s+1,t+1})$ (instead of considering one parameter families together with a measure-preserving transformation (cf. [6]), we prefer to deal with two parameter families). Now suppose that X is almost subadditive and satisfies (as in [12], [13])

$$\inf_n \frac{1}{n} E[X_{0,n}] > -\infty. \tag{1.4}$$

Then, according to [6], $\left(\frac{1}{n} X_{0,n}\right)$ converges in L^1, provided

$$\lim_n \frac{1}{n} E[h_{0,n}] = 0. \tag{1.5}$$

If, however, the much stronger moment condition

$$\sup_n E[h_{0,n}] < \infty \tag{1.6}$$

is satisfied, $\left(\frac{1}{n} X_{0,n}\right)$ also converges almost everywhere (a.e.) (cf. [6]). Our main result in Section 2 implies that $\left(\frac{1}{n} X_{0,n}\right)$ converges a.e. even if instead of (1.6) only

$$\liminf_n \frac{1}{n} \sum_{i=1}^{n} E[h_{0,i}] < \infty \tag{1.7}$$

holds (we still assume that (1.5) is satisfied). Note that (1.7) is neither stronger nor weaker than (1.5). Condition (1.7) implies $\liminf_n E[h_{0,n}] < \infty$.

Our proof given in Section 2 uses Burkholder's [5] technique which is based on the following deep result due to Komlós [14].

THEOREM 1.1. *Let $(Z_n) \subset L^1$ be an L^1-bounded sequence of real random variables. Then there exists a random variable $Z_\infty \in L^1$ and a subsequence (n_k) such that, for any subsequence $(m_k) \subset (n_k)$,*

$$\lim_k \frac{1}{k} \sum_{i=1}^{k} Z_{m_i} = Z_\infty \quad a.e. \tag{1.8}$$

The basic idea is now to apply Burkholder's [5] technique to the process $(X_{s,t})$ while restricting the indices s and t to suitable residue classes. It is, perhaps, interesting to note that this requires the full strength of Komlós' result, whereas Burkholder utilized (1.8) for the sequence (n_k) only. The following observation (to be used later) suggests that our procedure might work.

LEMMA 1.2. *Let $X = (X_{s,t}) \subset L^1$ and $h = (h_{s,t}) \subset L^1_+$ be stationary and suppose that condition (1.3) is satisfied. Then, for any fixed integers $p \geqq 2$ and $0 \leqq r \leqq p-1$,*

$$\liminf_n \frac{1}{n} X_{0,n} = \liminf_n \frac{1}{np+r} X_{0,np+r} \quad a.e. \tag{1.9}$$

PROOF. Let us first show that for any integers $p \geqq 2$ and $0 \leqq r \leqq p-1$,

$$\liminf_n \frac{1}{np+r} X_{0,np+r} = \liminf_n \frac{1}{np} X_{0,np} \quad \text{a.e.} \tag{1.10}$$

In fact, by (1.3) we have

$$X_{0,np+r} \leqq X_{0,np} + X_{np,np+r} + h_{np,np+r}, \quad n \geqq 1. \tag{1.11}$$

Since the stationarity of X and h entails

$$\lim_n \frac{1}{n} X_{np,np+r} = 0 \quad \text{a.e. and} \quad \lim_n \frac{1}{n} h_{np,np+r} = 0 \quad \text{a.e.,}$$

we get from (1.11)

$$\liminf_n \frac{1}{np+r} X_{0,np+r} \leqq \liminf_n \frac{1}{np} X_{0,np} \quad \text{a.e.}$$

The converse inequality is proved similarly. Hence, in view of (1.10), (1.9) will be proved provided we can show

$$\liminf_n \frac{1}{n} X_{0,n} \geqq \liminf_n \frac{1}{np} X_{0,np} \quad \text{a.e.,} \quad p \geqq 2. \tag{1.12}$$

Let $p \geqq 2$ be fixed. By (1.10) there exists a set Ω_p of P-measure one such that (1.10) holds on Ω_p for all $0 \leqq r \leqq p-1$. Let $\omega \in \Omega_p$ be fixed. There exists a sequence $1 \leqq n_1 < n_2 < \dots$ (depending on ω) such that

$$\liminf_n \frac{1}{n} X_{0,n}(\omega) = \lim_k \frac{1}{n_k} X_{0,n_k}(\omega).$$

Hence there exists an integer $0 \leqq r_0 \leqq p-1$ and a sequence $1 \leqq m_1 < m_2 < \dots$ (both depending on ω) such that

$$\liminf_n \frac{1}{n} X_{0,n}(\omega) = \lim_k \frac{1}{m_k p + r_0} X_{0,\, m_k p + r_0}(\omega) \geqq$$

$$\geqq \liminf_n \frac{1}{np+r_0} X_{0,np+r_0}(\omega) = \liminf_n \frac{1}{np} X_{0,\,np}(\omega).$$

This proves (1.12).

In Section 3, the results of Section 2 are extended to processes X and h which are superstationary in the sense of Kamae, Krengel and O'Brien [11]. The obtained almost subadditive superstationary ergodic theorem generalizes ergodic theorems of Krengel [15] and Abid [1]. As an application, we get an ergodic theorem for very general classes of random sets. This extends some results of Schürger [18].

2. The stationary case

Throughout this section, we will consider real processes $X=(X_{s,t})\subset L^1$ and $h=(h_{s,t})\subset L^1_+$ which are *stationary*. It is convenient to define $X_{t,t}\equiv 0$, $h_{t,t}\equiv 0$, $t\geqq 0$. Our main result in the stationary case is the following.

THEOREM 2.1. *Suppose the processes $X=(X_{s,t})$ and $h=(h_{s,t})$ satisfy the following four conditions:*

$$X_{s,u}-X_{s,t}-X_{t,u}\leqq h_{t,u}, \quad (s,t),(t,u)\in I; \tag{2.1}$$

$$\inf_n \frac{1}{n} E[X_{0,n}] > -\infty; \tag{2.2}$$

$$\lim_n \frac{1}{n} E[h_{0,n}] = 0; \tag{2.3}$$

$$\liminf_n \frac{1}{n} \sum_{i=1}^{n} E[h_{0,i}] < \infty. \tag{2.4}$$

Then $\left(\frac{1}{n} X_{0,n}\right)$ *converges a.e. and in L^1 to the invariant random variable X_∞ given by*

$$X_\infty = \lim_n \frac{1}{n}\left(\lim_k \frac{1}{k}\sum_{j=0}^{k-1} X_{nj,n(j+1)}\right), \tag{2.5}$$

where the inner limit exists a.e. and in L^1, and the outer one in L^1.

Let us put

$$\gamma_n = E[X_{0,n}], \quad n\geqq 1. \tag{2.6}$$

If conditions (2.1), (2.2) and (2.3) are satisfied, it follows from Lemma 1.1 that the limit

$$\lim_n \frac{1}{n}\gamma_n = \gamma \tag{2.7}$$

exists and is finite. Then, we also have (cf. the proof of Théorème 2 of [6]) that $\left(\frac{1}{n} X_{0,n}\right)$ converges in L^1 to X_∞ given by (2.5), and

$$X_\infty = \limsup_n \frac{1}{n} X_{0,n} \quad \text{a.e.,} \tag{2.8}$$

which implies

$$E\left[\limsup_n \frac{1}{n} X_{0,n}\right] = \gamma.$$

Therefore, Theorem 2.1 will be proved if we can show

$$E\left[\liminf_n \frac{1}{n} X_{0,n}\right] \geqq \gamma. \tag{2.9}$$

The proof of (2.9) will proceed via two lemmas. In order to simplify notation, we introduce the symbolic shift operator U by

$$U^k X_{s,t} = X_{s+k,t+k}, \quad U^k h_{s,t} = h_{s+k,t+k}, \quad t \geqq s > 0, \quad k > 0,$$

and let U^k act linearly. Furthermore, we put

$$X_{0,n} = X_n, \quad h_{0,n} = h_n, \quad n \geqq 0.$$

Then (2.1) can be written as

$$U^k(X_{m+n} - X_n - U^n X_m) \leqq U^{n+k} h_m, \quad k, m, n \geqq 0. \tag{2.10}$$

Following Burkholder's [5] technique, we put, for any integers $p \geqq 2$ and $0 \leqq r \leqq p-1$,

$$g_n^{(p,r)} = \frac{1}{n} \sum_{i=1}^{n} (X_{ip+r} - U^p X_{(i-1)p+r}), \quad n \geqq 1.$$

By (2.7),

$$\lim_n E[g_n^{(p,r)}] = p\gamma. \tag{2.11}$$

In view of Theorem 1.1, the following simple observation will be important.

LEMMA 2.1. *Let $p \geqq 2$ and $0 \leqq r \leqq p-1$ be fixed. Then the sequence $(g_{n_k}^{(p,r)})$ is L^1-bounded if the sequence*

$$\left(\frac{1}{n_k} \sum_{i=0}^{n_k} E[h_{ip+r}]\right)$$

is bounded.

PROOF. Apply (2.10) and (2.11).

Following the computation in [5], we get from (2.10), for any integers $m \geqq n \geqq 1$,

$$\sum_{k=0}^{n-1} U^{kp} g_m^{(p,r)} \leqq X_{np} + \frac{1}{m} \sum_{i=1}^{m} U^{np} h_{ip+r} + \frac{n}{m} \Big(\sum_{k=0}^{n-1} U^{kp} X_p - X_{np}\Big) + \tag{2.12}$$
$$+ \frac{1}{m} \sum_{k=0}^{n-1} U^{(k+1)p} \Big(\sum_{i=1}^{n-k} h_{(i-1)p+r}\Big) + \frac{1}{m} \sum_{k=0}^{n-1} U^{(k+1)p} \Big(\sum_{i=m-k+1}^{m} h_{(i-1)p+r}\Big).$$

In fact, consider integers $m \geqq n \geqq 1$, $0 \leqq k \leqq n-1$, $p \geqq 2$ and $0 \leqq r \leqq p-1$. Clearly

$$m U^{kp} g_m^{(p,r)} = \sum_{i=1}^{m} U^{kp} (X_{ip+r} - U^p X_{(i-1)p+r}) = \sum_{i=n+1-k}^{m-k} + \sum_{i=1}^{n-k} + \sum_{i=m-k+1}^{m}.$$

Applying the inequality

$$X_{ip+r} - U^p X_{(i-1)p+r} \leqq X_p + U^p h_{(i-1)p+r}$$

(being a consequence of (2.10)) to the last two sums and summing for k from 0 to

$n-1$ we get

$$m\sum_{k=0}^{n-1} U^{kp} g_m^{(p,r)} \leqq \sum_{k=0}^{n-1}\sum_{i=n+1}^{m} (U^{kp}X_{(i-k)p+r} - U^{(k+1)p}X_{(i-k-1)p+r}) +$$

$$+n\sum_{k=0}^{n-1} U^{kp}X_p + \sum_{k=0}^{n-1} U^{(k+1)p}\Big(\sum_{i=1}^{n-k} h_{(i-1)p+r} + \sum_{i=m-k+1}^{m} h_{(i-1)p+r}\Big).$$

By changing the order of summation and applying (2.10), the first double sum is easily seen to be dominated by

$$(m-n)X_{np} + \sum_{i=1}^{m-n} U^{np} h_{ip+r}.$$

Combining this with the above inequality, we arrive at (2.12) (note that the $h_{s,t}$ are nonnegative).

Now let $1 \leqq n_1 \leqq n_2 \leqq \ldots$ be *any* fixed sequence tending to infinity, and put

$$S_j^{(p,r)} = \frac{1}{j}\sum_{m=1}^{j} g_{n_m}^{(p,r)}, \quad j \geqq 1.$$

Then, by (2.12), for any fixed $n \geqq 1$,

$$\sum_{k=0}^{n-1} U^{kp} S_j^{(p,r)} \leqq o(1) + X_{np} + U^{np}\left(\frac{1}{j}\sum_{m=1}^{j}\frac{1}{n_m}\sum_{i=1}^{n_m} h_{ip+r}\right) + \tag{2.13}$$

$$+\frac{1}{j}\sum_{m=1}^{j}\frac{1}{n_m}\sum_{k=0}^{n-1} U^{(k+1)p}\sum_{i=n_m-k+1}^{n_m} h_{(i-1)p+r}$$

($o(1)$ denoting a random variable converging to zero a.e., as $j \to \infty$). For the last term on the right hand side of (2.13) we get, by Fatou's lemma and (2.3),

$$\liminf_j \frac{1}{j}\sum_{m=1}^{j}\frac{1}{n_m}\sum_{k=0}^{n-1} U^{(k+1)p}\sum_{i=n_m-k+1}^{n_m} h_{(i-1)p+r} = 0 \quad \text{a.e.} \tag{2.14}$$

LEMMA 2.2. *For any integer $p \geqq 2$, we can find a sequence (n_k), an integer $0 \leqq r \leqq$ $\leqq p-1$, a constant $K>0$ and stationary sequences $(\lambda_n^{(p,r)}) \subset L^1$, $(\psi_n^{(p,r)}) \subset L^1_+$ such that*

$$\lim_j S_j^{(p,r)} = \lambda_0^{(p,r)} \quad \textit{a.e.}, \tag{2.15}$$

$$\lim_j U^t\left(\frac{1}{j}\sum_{m=1}^{j}\frac{1}{n_m}\sum_{i=0}^{n_m} h_{ip+r}\right) = \psi_t^{(p,r)} \quad \textit{a.e.}, \quad t \geqq 0, \tag{2.16}$$

$$\sum_{k=0}^{n-1} \lambda_{kp}^{(p,r)} \leqq X_{np} + \psi_{np}^{(p,r)} \quad \textit{a.e.}, \quad n \geqq 1, \tag{2.17}$$

$$\frac{1}{p}E[\lambda_0^{(p,r)}] \geqq \gamma - \frac{2K}{p}. \tag{2.18}$$

PROOF. By (2.4), there exists a constant $K>0$ and a sequence (j_k) such that

$$\frac{1}{j_k}\sum_{i=1}^{j_k} E[h_i] \leqq K, \quad k \geqq 1.$$

Hence, for any $p\geqq 2$, we can find a sequence $(m_k)\subset(j_k)$ and an integer $0\leqq r\leqq$ $\leqq p-1$ such that

$$\frac{1}{m_k}\sum_{i=0}^{m_k} E[h_{ip+r}] \leqq 2K, \quad k \geqq 1. \tag{2.19}$$

Combining (2.19), Lemma (2.1) and Theorem 1.1, we thus have a sequence $(n_k)\subset$ $\subset(m_k)$ and random variables $\lambda_0^{(p,r)}\in L^1$, $\psi_0^{(p,r)}\in L^1_+$ such that (2.15) and (2.16) (for $t=0$) hold. Finally, define $\psi_t^{(p,r)}$, $t\geqq 1$, by (2.16), and put $\lambda_t^{(p,r)}=\lim_j U^t S_j^{(p,r)}$ a.e., $t\geqq 1$. By (2.13) and (2.14), we get (2.17). In order to prove (2.18), first note that, by (2.10)

$$X_p - S_j^{(p,r)} + U^p\left(\frac{1}{j}\sum_{m=1}^{j}\frac{1}{n_m}\sum_{i=0}^{n_m} h_{ip+r}\right) \geqq 0, \quad j \geqq 1.$$

An application of Fatou's lemma, combined with (2.16), (2.19) and the observation that, by (2.11),

$$\lim_j E[S_j^{(p,r)}] = p\gamma,$$

yields (2.18).

PROOF OF THEOREM 2.1. As already noted, it remains to prove (2.9). Consider any fixed $p\geqq 2$ and let r (depending on p) be defined as in Lemma 2.2. By (2.17) and Birkhoff's ergodic theorem,

$$E\left[\lim_n \frac{1}{np}\sum_{k=0}^{n-1}\lambda_{kp}^{(p,r)}\right] = \frac{1}{p}E[\lambda_0^{(p,r)}] \leqq E\left[\liminf_n \frac{1}{np}X_{np}\right].$$

Therefore, applying Lemma 1.2, we get

$$E\left[\liminf_n \frac{1}{n}X_n\right] \geqq \frac{1}{p}E[\lambda_0^{(p,r)}].$$

By (2.18), this proves (2.9).

3. The superstationary case

In this section, Theorem 2.1 will be extended to processes $(X_{s,t})$ and $(h_{s,t})$ which are merely supposed to be superstationary in the sense of Kamae, Krengel and O'Brien [11] (see also [10] and [1]).

Let $F_1=R^I$ denote the partially ordered Polish space endowed with the product topology and the usual order relation on R, taken coordinate-wise (cf. [10], [11]). The elements of F_1 will be conceived as sets $(a_{s,t})\subset R$ (the index set being I). The *shift* $\tau\colon F_1\to F_1$ is then defined by

$$\tau(a_{s,t}) = (a_{s+1,t+1}), \quad (s, t)\in I \tag{3.1}$$

(cf. [1]). For any topological space F let $\mathscr{M}(F)$ denote the family of all probability measures defined on the Borel subsets of F. If $P_1, P_2 \in \mathscr{M}(F_1)$, $P_2 \succ P_1$ means that P_1 is *stochastically smaller* than P_2 (cf. [10], [11], [15]). *Throughout this section, it will be assumed that the families* $X=(X_{s,t})\subset L^1$ *and* $h=(h_{s,t})\subset L^1_+$ *(both conceived as random elements of* F_1*) are superstationary families of real random variables* (cf. [1], [11], [15]). We put

$$a_n = \inf_s E[h_{s,s+n}] = \lim_s E[h_{s,s+n}], \quad n \geqq 1. \tag{3.2}$$

In this section, we derive the following extension of Theorem 2.1:

THEOREM 3.1. *Let the processes* $X=(X_{s,t})$ *and* $h=(h_{s,t})$ *satisfy condition* (2.1), *and suppose that, for some constant* $K>0$,

$$E[X_{s,s+t}] \geqq -Kt, \quad s \geqq 0, \quad t \geqq 1. \tag{3.3}$$

Then, if the relations

$$\lim_n \frac{1}{n} a_n = 0 \tag{3.4}$$

and

$$\liminf_n \frac{1}{n} \sum_{i=1}^n a_i < \infty \tag{3.5}$$

(a_n given by (3.2)*) hold,* $\left(\frac{1}{n} X_{0,n}\right)$ *converges a.e. and in* L^1.

This result generalizes Abid's [1] ergodic theorem, who studied the case when the process h vanishes identically.

The main idea of the proof of Theorem 3.1 consists in showing that the process $(X_{s,t})$ can be approximated by a suitable *stationary* process (cf. Lemma 3.2 below) to which Theorem 2.1 applies (in the special case of an identically vanishing process h, this was observed by Abid [1]).

Let us put

$$\gamma_{s,t} = E[X_{s,t}], \quad (s,t)\in I. \tag{3.6}$$

Combining (2.1) and (3.3), we have that the limit

$$\lim_s \gamma_{s,s+t} = \gamma_t \geqq -Kt \tag{3.7}$$

exists for any $t \geqq 1$. By (3.4) and Lemma 1.1, the limit

$$\lim_n \frac{1}{n} \gamma_n = \lim_n \frac{1}{n} \left(\lim_s E[X_{s,s+n}]\right) = \gamma = \gamma(X) \tag{3.8}$$

exists, and (by (3.7)) is finite ($\gamma=\gamma(X)$ is sometimes called the *time constant of* X).

For the proof of Theorem 3.1 we need two lemmas.

LEMMA 3.1. *Let* $P_n \in \mathcal{M}(F_1)$ *and* $Q_n \in \mathcal{M}(F_1)$ *denote the distributions of* $\tau^n X$ *and* $\tau^n h$, *respectively*, $n \geqq 0$. *Then, the sequences* (P_n) *and* (Q_n) *are tight, and converge weakly to probability measures* P_∞ *and* Q_∞, *respectively. Furthermore,*

$$P_0 \succ P_\infty, \quad Q_0 \succ Q_\infty. \tag{3.9}$$

PROOF. Let $P_{s,t}$ denote the distribution of $X_{s,t}$. Since

$$\sup_s E[|X_{s,s+t}|] < \infty, \quad t \geqq 1,$$

each family $\{P_{s,s+t} | s \geqq 0\}$ is tight, $t \geqq 1$. Hence, by Tyhonov's theorem, (P_n) is tight. Similarly, (Q_n) is shown to be tight. The weak convergence of (P_n) and (Q_n) is implied by [11, Proposition 4 and Theorem 6]. Finally, by Proposition 3 in [11], (3.9) holds.

LEMMA 3.2. *There exists a real process* $\tilde{X}=(\tilde{X}_{s,t})$ *and a real stationary process* $\tilde{h}=(\tilde{h}_{s,t}) \subset L^1_+(\tilde{P})$ *defined on a certain probability space* $(\tilde{\Omega}, \tilde{A}, \tilde{P})$ *(possibly different from* $(\Omega, \mathcal{A}, P)$*) with the following properties (*$\tilde{E}$ *denoting expectations w.r.t.* $\tilde{P}$*):*

(a) X *and* $\tilde{X}$ *have the same distribution;*

(b) $\tilde{X}$ *has a decomposition* $\tilde{X}=\tilde{Y}+\tilde{Z}$ *such that* $\tilde{X} \geqq \tilde{Y}$, *the process* $\tilde{Y}=(\tilde{Y}_{s,t}) \subset L^1_+(\tilde{P})$ *being stationary;*

(c) $\tilde{Y}_{s,u}-\tilde{Y}_{s,t}-\tilde{Y}_{t,u} \leqq \tilde{h}_{t,u}$, $(s, t), (t, u) \in I$;

(d) $\tilde{E}[\tilde{Y}_{0,n}] \geqq -Kn$, $n \geqq 1$

(K being the constant in (3.3)*);*

(e) $\tilde{E}[\tilde{h}_{0,n}] = a_n$, $n \geqq 1$;

(f) $\lim_n \tilde{E}[\tilde{X}_{s+n,t+n}] = \tilde{E}[\tilde{Y}_{s,t}]$, $(s, t) \in I$.

PROOF. For any $(s, t), (t, u) \in I$ let the mapping $\varphi_{s,t,u}\colon F_1 \times F_1 \to R$ be defined by

$$\varphi_{s,t,u}(C, D) = C_{s,u}-C_{s,t}-C_{t,u}-D_{t,u}, \quad C, D \in F_1.$$

Note that $\varphi_{s,t,u}$ is continuous. By (2.1),

$$\varphi_{s,t,u}(\tau^n X, \tau^n h) \leqq 0, \quad n \geqq 0, \quad (s, t), (t, u) \in I.$$

Hence, if we put

$$A = \bigcap \varphi^{-1}_{s,t,u}(-\infty, 0] \tag{3.10}$$

(the intersection taken over all $(s, t), (t, u) \in I$), $A \subset F_1 \times F_1$ is closed, and

$$P\{(\tau^n X, \tau^n h) \in A\} = 1, \quad n \geqq 0. \tag{3.11}$$

By Lemma 3.1 and Prohorov's theorem (cf. [4, pp. 37 and 41]), the family of distributions of the random elements $(\tau^n X, \tau^n h)$, $n \geqq 0$, is tight. Hence, for some subsequence (n_k) and some $\hat{P} \in \mathcal{M}(F_1 \times F_1)$, $(\tau^{n_k} X, \tau^{n_k} h)$ converges in distribution to $\hat{P}$. Therefore, by (3.11) and the Portmanteau Theorem [4, p. 24],

$$\hat{P}(A) = 1. \tag{3.12}$$

Clearly, by Lemma 3.1, the marginal distributions of $\hat{P}$ are P_∞ and Q_∞, respectively. Let us show that there exists some $\tilde{P}\in\mathcal{M}(F_1\times(F_1\times F_1))$ which is concentrated on the closed set

$$\tilde{\Omega} = \{(x, y, z)\in F_1\times F_1\times F_1 | x \geqq y,\ (y, z)\in A\},$$

and has the marginal distributions P_0 and $\hat{P}$. In fact, by Theorem 11 of [19] or Corollaire 2 of [9], it suffices to check that, for any compact sets $C\subset F_1$, $D\subset F_1\times F_1$ such that $(C\times D)\cap\tilde{\Omega}=\emptyset$, we have

$$P_0(C)+\hat{P}(D) \leqq 1. \tag{3.13}$$

In order to see this, put

$$\tilde{\Omega}_0 = \{x\in F_1|\text{ for all } (y, z)\in D,\quad x \ngeqq y \text{ or } (y, z)\notin A \text{ holds}\}.$$

Clearly, $C\subset\tilde{\Omega}_0$, and the set $\tilde{\Omega}\setminus\tilde{\Omega}_0$ is closed and increasing (cf. [10]). Hence, by (3.9) and (3.12),

$$P_0(C) \leqq P_0(\tilde{\Omega}_0) \leqq P_\infty(\tilde{\Omega}_0) = \hat{P}((\tilde{\Omega}_0\times F_1)\cap A),$$

which implies (3.13), since $(\tilde{\Omega}_0\times F_1)\cap A\cap D=\emptyset$. Let us define $\tilde{X}=\pi_1$, $\tilde{Y}=\pi_2$ and $\tilde{h}=\pi_3$, where π_i denotes the projection of the product space $F_1\times F_1\times F_1$ onto its i-th factor. Clearly, $\tilde{Y}$ and and $\tilde{h}$ are stationary since the shift τ is continuous. It is now easy to verify (a)—(f) (without loss of generality, we may assume that the inequalities in (b) and (c) hold *everywhere* on $\tilde{\Omega}$).

Note that, by (a) and (f),

$$\gamma(X) = \gamma(\tilde{X}) = \gamma(\tilde{Y}). \tag{3.14}$$

PROOF OF THEOREM 3.1. Applying Krengel's [15] ergodic theorem to the superstationary sequences $(U^{nj}X_n)$ and $(U^{nj}h_n)$ (for fixed n) and taking into account (3.14), (2.10), (3.7), (3.4), (3.8) as well as Lemma 3.2, one sees that $\left(\frac{1}{n}X_{0,n}\right)$ converges a.e., and

$$E\left[\lim_n \frac{1}{n} X_{0,n}\right] = \lim_n \frac{1}{n} E[X_{0,n}] = \gamma(X). \tag{3.15}$$

On the other hand, it follows from Theorem 2.1 that $\left(\frac{1}{n}\tilde{Y}_{0,n}\right)$ (see Lemma 3.2) converges a.e. and in $L^1(\tilde{P})$. Since, by (3.15) and (3.14),

$$\lim_n \frac{1}{n} \tilde{E}[\tilde{Z}_{0,n}] = 0,$$

Lemma 3.2 entails that $\left(\frac{1}{n}X_{0,n}\right)$ also converges in L^1. This proves Theorem 3.1.

Theorem 3.1 can be used to arrive at an ergodic theorem for very general families of random sets. Theorem 3.2 below generalizes results obtained by Artstein and Vitale [3], Artstein and Hart [2] and Schürger [18].

Let $\operatorname{co}\mathscr{C}$ denote the family of all nonvoid convex compact subsets of R^d $(d\geqq 1)$. Let ϱ denote the *Hausdorff metric* restricted to $\operatorname{co}\mathscr{C}$, and put $\|C\|=\sup\{\|c\|\,|\,c\in C\}$, $C\in\operatorname{co}\mathscr{C}$ ($\|c\|$ denoting the Euclidean norm of c). Then we have the following result (cf. [18] and [16] for further notation and terminology).

THEOREM 3.2. *Let $\theta=(\theta_{s,t})$ and $\chi=(\chi_{s,t})$ be superstationary families of $\operatorname{co}\mathscr{C}$-valued random sets such that $(\|\theta_{s,t}\|)\subset L^1_+$ and $(\|\chi_{s,t}\|)\subset L^1_+$. Put*

$$b_n = \inf_s E[\|\chi_{s,s+n}\|], \quad n \geqq 1. \tag{3.16}$$

Suppose that θ and χ satisfy the following four conditions:

$$\theta_{s,u}\subset\theta_{s,t}+\theta_{t,u}+\chi_{t,u}, \quad (s,t),\ (t,u)\in I; \tag{3.17}$$

(3.18) *there exists a constant $K\geqq 0$ such that*

$$E[\|\theta_{s,s+t}\|]\leqq Kt, \quad s\geqq 0, \quad t\geqq 1;$$

$$\lim_n \frac{1}{n} b_n = 0; \tag{3.19}$$

$$\liminf_n \frac{1}{n}\sum_{i=1}^n b_i < \infty. \tag{3.20}$$

Then $\left(\frac{1}{n}\theta_{0,n}\right)$ converges a.e. in $(\operatorname{co}\mathscr{C},\varrho)$ to some $\operatorname{co}\mathscr{C}$-valued random set ξ such that $\|\xi\|\in L^1_+$, and

$$\lim_n E\left[\varrho\left(\frac{1}{n}\theta_{0,n},\ \xi\right)\right]=0.$$

PROOF. Put $S_1=\{x\in R^d\,|\,\|x\|=1\}$. For each $p\in S_1$ put

$$X^{(p)}_{s,t}=\sigma_p(\theta_{s,t}) \quad\text{and}\quad h^{(p)}_{s,t}=(\sigma_p(\chi_{s,t}))^+, \quad (s,t)\in I,$$

where $\sigma_p(C)=\sup\{pc\,|\,c\in C\}$, $p\in S_1$, $C\in\operatorname{co}\mathscr{C}$, and $a^+=\max(a,0)$, $a\in R$. It can be easily checked that, for each $p\in S_1$, $X^{(p)}=(X^{(p)}_{s,t})$ and $h^{(p)}=(h^{(p)}_{s,t})$ satisfy the hypotheses of Theorem 3.1. From Theorem 3.1 and [18, Lemmas (4.3) and (4.24)] we deduce that $\left(\frac{1}{n}\theta_{0,n}\right)$ has the asserted convergence properties.

References

[1] M. Abid, Un théorème ergodique pour des processus sous-additifs et sur-stationnaires, *C. R. Acad. Sci. Paris, Ser.* A, **287** (1978), 149—152.
[2] Z. Artstein and S. Hart, Law of large numbers for random sets and allocation processes, *Math. Oper. Res.*, **6** (1981), 485—492.
[3] Z. Artstein and R. Vitale, A strong law of large numbers for random compact sets, *Annals of Probability*, **3** (1975), 879—882.
[4] P. Billingsley, *Convergence of Probability Measures*, Wiley (New York, 1968).
[5] D. L. Burkholder, Contribution to the discussion in [13], pp. 900—902.
[6] Y. Derriennic, Un théorème ergodique presque sous-additif, *Annals of Probability*, **11** (1983), 669—677.

[7] J. M. Hammersley, Generalization of the fundamental theorem on subadditive functions, *Proc. Cambridge Philos. Soc.*, **58** (1962), 235—238.

[8] J. M. Hammersley, Postulates for subadditive processes, *Annals of Probability,* **2** (1974), 652—680.

[9] G. Hansel and J. P. Troallic, Measures marginales et théorème de Ford—Fulkerson, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **43** (1978), 245—251.

[10] T. Kamae and U. Krengel, Stochastic partial ordering, *Annals of Probability,* **6** (1978), 1044—1049.

[11] T. Kamae, U. Krengel and G. L. O'Brien, Stochastic inequalities on partially ordered spaces, *Annals of Probability,* **5** (1977), 899—912.

[12] J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Roy. Statist. Soc., Ser.* B, **30** (1968), 499—510.

[13] J. F. C. Kingman, Subadditive ergodic theory, *Annals of Probability,* **1** (1973), 883—909.

[14] J. Komlós, A generalization of a problem of Steinhaus, *Acta Math. Acad. Sci. Hung.,* **18** (1967), 217—229.

[15] U. Krengel, Un théorème ergodique pour les processus sur-stationnaires, *C. R. Acad. Sci. Paris, Ser.* A, **282** (1976), 1019—1021.

[16] G. Mathéron, *Random Sets and Integral Geometry,* Wiley (New York, 1975).

[17] J. Moulin-Ollagnier, Théorème ergodique presque sous-additif et convergence en moyenne de l'information, *Ann. Inst. Henri Poincaré, Ser.* B, **19** (1983), 257—266.

[18] K. Schürger, Ergodic theorems for subadditive superstationary families of convex compact random sets, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **62** (1983), 125—135.

[19] V. Strassen, The existence of probability measures with given marginals, *Annals of Math. Statist.,* **36** (1965), 423—439.

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THE PREVALENCE OF STRONG UNIQUENESS IN L^1

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1. Introduction

Let (X, Σ, μ) be a positive σ-finite measure space and let M be a finite dimensional subspace of $L^1 \equiv L^1(X, \Sigma, \mu)$. For $f \in L^1$, we say that $g^* \in L^1$, is a *strongly unique best approximation* to f from M if there is a constant $r > 0$ such that $\|f-g\|_1 \geqq \|f-g^*\|_1 + r\|g-g^*\|_1$ for all $g \in M$. In recent years, strong uniqueness of best approximations has played a fundamental role in algorithms to compute best approximation [3, 5, 8] and has been the focus of several articles (see [1, 2, 7] and others). The purpose of this article is to assess the prevalence of strong uniqueness in L^1. Results in this direction are known for other spaces. In L^p $(1 < p < \infty)$ or in any smooth space, only those elements of the approximating space have strongly unique best approximations [15]. It is known that each $f \in C[0, 1]$ possesses a strongly unique best approximation from a Haar subspace [9]. Recently, Nürnberger and Singer [11] have shown that if M is an almost Chebyshev subspace of $C[0, 1]$, then the set of functions having strongly unique best approximations from M is dense in $C[0, 1]$. We obtain the analogous result for L^1.

Define $U_M := \{f \in L^1 : f$ has a unique best approximation from $M\}$ and $SU_M := \{f \in L^1 : f$ has a strongly unique best approximation from $M\}$. Clearly, $SU_M \subseteqq U_M$. Our main result in Section 2 asserts that SU_M is dense in U_M. The density of SU_M in L^1 will then follow when M is almost Chebyshev. We further show that $SU_M = U_M$ when L^1 is finite dimensional.

In view of the density result, it is natural to ask whether $L^1 \setminus SU_M$ is of first category in L^1 when M is almost Chebyshev. This question derives from a result of Garkavi [6] which asserts (in this setting) that if U_M is dense in L^1, then $L^1 \setminus U_M$ is of first category in L^1. In Section 3 we answer this question in negative with a rather striking contrast to Garkavi's result. Particularly, if X contains no atoms then SU_M is a dense set of first category in L^1.

2. Density results

For $f \in L^1$, the support of f is the set $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0\}$ and the zero set of f is $Z(f) = X \setminus \operatorname{supp}(f)$. We shall require a characterization of the elements of SU_M due to Nürnberger [10].

THEOREM 1. *Let $f \in L^1 \setminus M$ where M is a finite dimensional subspace of L^1. Then 0 is a strongly unique best approximation to f from M if and only if*

$$\left| \int_{\operatorname{supp}(f)} g \operatorname{sgn}(f)\, d\mu \right| < \int_{Z(f)} |g|\, d\mu$$

for all $g \in M \setminus \{0\}$.

Theorem 1 is essentially the strong Kolmogorov criterion idealized to finite dimensional subspaces. For $f\in L^1$, let $P_M(f)$ denote the set of best approximations to f from M. The set valued map P_M is called the metric projection associated with M. We shall also need a method due to Rozema [13] of redefining a function in L^1 so as to preserve one of its best approximations.

LEMMA 1. *Let M be a subspace of L^1, $f\in L^1\setminus M$, $g_0\in P_M(f)$, and $A\in\Sigma$. Define $f_0=f\chi_{A^c}+g_0\chi_A$, where χ_B denotes the characteristic function of B. Then $g_0\in P_M(f_0)\subseteqq P_M(f)$.*

We now establish the main result.

THEOREM 2. *Let M be a finite dimensional subspace of L^1. Then SU_M is dense in U_M.*

PROOF. Let $f\in U_M$ with unique best approximation g^* from M. Replacing f with $f-g^*$, we may assume that 0 is the unique best approximation to f from M. Indeed, 0 is the unique best approximation to f from any subspace of M.

By a well known characterization theorem for best approximations (see [14]), there exists $\varphi\in L^\infty$ with $\|\varphi\|_\infty=1$ such that $\int g\,d\mu=\|f\|_1$, and

$$\int \varphi g\,d\mu=0 \tag{2.1}$$

for all $g\in M$. Then $\varphi=\operatorname{sgn}(f)$ a.e. on $\operatorname{supp}(f)$ and $S(\varphi):=\{x\in X: |\varphi(x)|<1\}\subseteqq Z(f)$ (where we have redefined φ on a set of measure zero if necessary). Let k be the maximal dimension of a subspace M_k of M from which 0 is a strongly unique best approximation to f. It is certainly possible that $k=0$. The object of the proof is as follows. For $\varepsilon>0$, we find an $f_0\in L^1$ such that $\|f-f_0\|_1<\varepsilon$, $P_M(f_0)=\{0\}$, and 0 is a strongly unique best approximation to f from a $(k+1)$-dimensional subspace of M. The result would follow by inducting on k.

Let $g_0\in M\setminus M_k$, and let M_{k+1} be the direct sum of M_k and the space spanned by g_0. We shall let $f_0=f\chi_{A^c}$ for a suitable choice of $A\in\Sigma$ where

$$\int_A |f|\,d\mu<\varepsilon. \tag{2.2}$$

We would then have $\|f-f_0\|_1<\varepsilon$ by (2.2) and $P_M(f_0)=\{0\}$ by Lemma 1. We decompose M_k into two sets: $\mathscr{A}=\{g\in M_k:\ g+g_0=0 \text{ a.e. on } S(\varphi)\}$ and $\mathscr{B}=M_k\setminus\mathscr{A}$.

If $g\in\mathscr{B}$, then $|g+g_0|>|\varphi(g+g_0)|$ on a subset of $S(\varphi)$ of positive measure. Since $S(\varphi)\subseteqq Z(f)$, (2.1) implies that

$$\Big|\int_{\operatorname{supp}(f)}(g+g_0)\operatorname{sgn}(f)\,d\mu\Big|=\Big|-\int_{Z(f)}\varphi(g+g_0)\,d\mu\Big|<\int_{Z(f)}|g+g_0|\,d\mu.$$

With $f_0=f\chi_{A^c}$, $\operatorname{supp}(f_0)=\operatorname{supp}(f)\cap A^c$ and $Z(f_0)=Z(f)\dot\cup(A\cap\operatorname{supp}(f))$. Since $f=f_0$ on $\operatorname{supp}(f_0)$,

$$\Big|\int_{\operatorname{supp}(f_0)}(g+g_0)\operatorname{sgn}(f_0)\,d\mu\Big|\leqq\Big|\int_{\operatorname{supp}(f)}(g+g_0)\operatorname{sgn}(f)\,d\mu\Big|+\int_{\operatorname{supp}(f)\cap A}|g+g_0|\,d\mu<$$

$$<\int_{Z(f)}|g+g_0|\,d\mu+\int_{\operatorname{supp}(f)\cap A}|g+g_0|\,d\mu.$$

Hence,

$$\left|\int_{\mathrm{supp}(f_0)} (g+g_0)\,\mathrm{sgn}\,(f_0)\,d\mu\right| < \int_{Z(f_0)} |g+g_0|\,d\mu \tag{2.3}$$

for all $g\in\mathscr{B}$. In a similar fashion, we can obtain

$$\left|\int_{\mathrm{supp}(f_0)} g\,\mathrm{sgn}\,(f_0)\,d\mu\right| < \int_{Z(f_0)} |g|\,d\mu \tag{2.4}$$

for all $g\in M_k\setminus\{0\}$ using Theorem 1 and the fact that 0 is a strongly unique best approximation to f from M_k.

Finally, we choose $A\in\Sigma$ so that (2.2) holds and for all $g\in\mathscr{A}$

$$\begin{cases}\varphi(g+g_0)>0 & \text{on a subset of } A \text{ of positive measure and}\\ \varphi(g+g_0)<0 & \text{on a subset of } A \text{ of positive measure.}\end{cases} \tag{2.5}$$

We have $\mathrm{supp}\,(f_0)=\mathrm{supp}\,(f)\cap A^c$ and $Z(f_0)=(Z(f)\cap A^c)\dot\cup A$. If A satisfies (2.5), then for $g\in\mathscr{A}$ we have from (2.1)

$$0=\int\varphi(g+g_0)\,d\mu=\int_{\mathrm{supp}(f_0)}\varphi(g+g_0)\,d\mu+\int_A\varphi(g+g_0)\,d\mu+\int_{Z(f)\cap A^c}\varphi(g+g_0)\,d\mu.$$

Since $\varphi=\mathrm{sgn}\,(f)=\mathrm{sgn}\,(f_0)$ a.e. on $\mathrm{supp}\,(f_0)\subseteqq\mathrm{supp}\,(f)$,

$$\left|\int_{\mathrm{supp}(f_0)}(g+g_0)\,\mathrm{sgn}\,(f_0)\,d\mu\right|\leqq\left|\int_A\varphi(g+g_0)\,d\mu\right|+\left|\int_{Z(f)\cap A^c}\varphi(g+g_0)\,d\mu\right|.$$

Since $|\varphi|\leqq 1$ a.e.,

$$\left|\int_{Z(f)\cap A^c}\varphi(g+g_0)\,d\mu\right|\leqq\int_{Z(f)\cap A^c}|g+g_0|\,d\mu$$

and (2.5) yields

$$\left|\int_A\varphi(g+g_0)\,d\mu\right|<\int_A|g+g_0|\,d\mu.$$

Thus

$$\left|\int_{\mathrm{supp}(f_0)}(g+g_0)\,\mathrm{sgn}\,(f_0)\,d\mu\right|<\int_{Z(f_0)}|g+g_0|\,d\mu \tag{2.6}$$

for $g\in\mathscr{A}$. For λ real and $g\in M_k$, we apply (2.3), (2.4), and (2.6) for the three cases $(1/\lambda)g\in\mathscr{B}$, $g\neq 0$ and $\lambda=0$ and $(1/\lambda)g\in\mathscr{A}$, respectively, to get

$$\left|\int_{\mathrm{supp}(f_0)}(g+\lambda g_0)\,\mathrm{sgn}\,(f_0)\,d\mu\right|<\int_{Z(f_0)}|g+\lambda g_0|\,d\mu$$

for all $g+\lambda g_0\in M_{k+1}$, and 0 is a strongly unique best approximation to f from M_{k+1}.

The remainder of the proof is devoted to demonstrating the existence of $A\in\Sigma$ such that (2.2) and (2.5) hold.

For $g\in\mathscr{A}$, $g+g_0$ is nonzero on a subset of $X\setminus S(\varphi)$ of positive measure. By (2.1) both the sets

$$U_g=\{x\in X\setminus S(\varphi)\colon\ \varphi(x)(g(x)+g_0(x))>0\}$$

and

$$V_g = \{x\in X\setminus S(\varphi):\ \varphi(x)(g(x)+g_0(x)) < 0\}$$

have positive measure. We claim that one of the following must hold:

(1) $U_g(V_g)$ contains a subset of positive measure which contains no atoms,

(2) $U_g(V_g)$ is a countably infinite union of atoms, or

(3) $U_g(V_g)$ is a finite union of atoms and f vanishes on at least one of these atoms.

To prove the claim, suppose that neither (1) nor (2) hold for U_g, say. Then U_g is a finite union of atoms, say $U_g=\bigcup_{j=1}^{m} A_j$ where each A_j is an atom. We may assume that f is constant over each atom. Let $x_j\in A_j$ and assume that $f(x_j)\neq 0$ for $j=1, \dots, m$. Select $r>0$ such that

$$r\cdot \max_{1\leqq j\leqq m} |g(x_j)+g_0(x_j)| < \min_{1\leqq j\leqq m} |f(x_j)|.$$

Then $U_g\subseteq \operatorname{supp}(f)$ and $\operatorname{sgn}(f-r(g+g_0))=\operatorname{sgn}(f)=\varphi$ on U_g. Since $g\in\mathcal{A}$, $g+g_0=0$ a.e. on $X\setminus(U_g\cup V_g)$ and therefore $\varphi(g+g_0)=-|g+g_0|$ on U_g^c. Let $f^*=f\chi_{U_g}$. By Lemma 1, $P_M(f^*)=\{0\}$, but using (2.1), we have

$$\|f^*\|_1 = \int_{U_g} |f|\,d\mu = \int_{U_g} \varphi f\,d\mu - \int \varphi r(g+g_0)\,d\mu =$$

$$= \int_{U_g} \varphi(f-r(g+g_0))\,d\mu - \int_{U_g^c} \varphi r(g+g_0)\,d\mu =$$

$$= \int_{U_g} |f-r(g+g_0)|\,d\mu + \int_{U_g^c} |r(g+g_0)|\,d\mu = \int |f^*-r(g+g_0)|\,d\mu = \|f^*-r(g+g_0)\|_1.$$

This is a contradiction. The proof for V_g is similar, and the claim is now established.

The set $U_g(V_g)$ is called a type (1), (2), or (3) set according to whether $U_g(V_g)$ satisfies (1), (2), or (3), respectively. Define $M^j=\{g\in\mathcal{A}:\ U_g \text{ is of type } (j)\}$ and $N^j=\{g\in\mathcal{A}:\ V_g \text{ is of type } (j)\}$ for $j=1, 2, 3$. We construct a portion of the set A for each of these six sets.

Consider the set M^1. For $g\in M^1$, there exists $r_g>0$ and $P_g\subseteq U_g$ with positive measure and containing no atoms such that $\varphi(g+g_0)>r_g$ on P_g. Let $\{g_1, \dots, g_k\}$ be a basis for M_k. Since $g_1, \dots, g_k$ are integrable on P_g, there is a set $A_g\subseteq P_g$ of positive measure such that for some $m_g>0$ we have $|g_i|\leqq m_g$ a.e. on A_g for $i=1, \dots, k$. Let $O_g=\{h\in M_k:\ \varphi(h+g_0)>0 \text{ on } A_g\}$. We show that O_g is a neighborhood of g relative to M_k.

Since M_k is finite dimensional, the norms $\|\cdot\|_1$ and $\left\|\sum_{i=1}^{k}\alpha_j g_i\right\|^*=\sum_{i=1}^{k}|\alpha_i|$ are equivalent on M_k. Let $h=\sum_{i=1}^{k}\alpha_i g_i\in M_k$ with $\|h\|^*<r_g/m_g$. Since $|\varphi|\leqq 1$ a.e., we have for $x\in A_g$

$$\varphi(x)(g(x)+h(x)+g_0(x)) \geqq r_g+\varphi(x)h(x) \geqq r_g-|h(x)| \geqq$$

$$\geqq r_g-\sum_{i=1}^{k}|\alpha_i||g_i(x)| \geqq r_g-m_g\sum_{i=1}^{k}|\alpha_i| = r_g-m_g\|h\|^* > 0.$$

Thus $g+h\in O_g$ and O_g is a neighborhood of g.

Now the interiors of the sets O_g, $g\in M^1$, cover M^1. Since M_k is finite dimensional, there is a countable subcollection $\{O_{g^i}\}_{i=1}^{\infty}$ of $\{O_g\}_{g\in m}$ that covers M^1. Then for each $g\in M^1$, $\varphi(g+g_0)>0$ on some $A_{g^i}\equiv A_i$.

Since f is integrable, there exists $\delta>0$ such that if $E\in\Sigma$ with $\mu(E)\leqq\delta$, then $\int_E |f|\,d\mu<\varepsilon/4$. Since A_i contains no atoms, we may select $A_i^1\subseteqq A_i$ such that $0<\mu(A_i^1)\leqq\delta/2^i$ $(i=1,2,\ldots)$. Let $A^1=\bigcup_{i=1}^{\infty} A_i^1$. Then $A^1\in\Sigma$, $\int_{A^1}|f|\,d\mu<\varepsilon/4$, and for every $g\in M^1$, $\varphi(g+g_0)>0$ on a subset of A^1 of positive measure.

We now construct a set A^2 for M^2. Since X contains at most countably many disjoint atoms, $\bigcup_{g\in M^2} U_g$ consists of at most countably many disjoint atoms. In fact, if $M^2\neq\emptyset$, then this set indeed contains infinitely many atoms. Write, $\bigcup_{g\in M^2} U_g= =\bigcup_{i=1}^{\infty} A_i$ where each A_i is an atom. Again, we assume f is constant over each atom. Choose $a_j\in A_j$ for $j=1,\ldots$. Since f is integrable, there is a positive integer N such that

$$\sum_{i=N}^{\infty} |f(a_j)|\,\mu(A_j) < \varepsilon/4.$$

Let $A^2=\bigcup_{i=N}^{\infty} A_i$. Then $A^2\in\Sigma$ and $\int_{A^2}|f|\,d\mu<\varepsilon/4$. Since only finitely many atoms were discarded from $\bigcup_{g\in M^2} U_g$, $\varphi(g+g_0)>0$ on a subset of A^2 of positive measure for every $g\in M^2$.

We finally consider M^3. As with M^2, $\bigcup_{g\in M^3} U_g$ consists of at most countably many atoms. Write $\bigcup_{g\in M^3} U_g=\bigcup_{j=1}^{\infty} A_j$ where each A_j is an atom. Let A^3 be the union of all the A_j over which f vanishes. Then $A^3\in\Sigma$, $\int_{A^3}|f|\,d\mu=0$, and by (3) $\varphi(g+g_0)>0$ on a subset of A^3 of positive measure.

A similar analysis for N^1, N^2, and N^3 yields measurable sets B^1, B^2, and B^3 where $\int_{B^1}|f|\,d\mu<\varepsilon/4$, $\int_{B^2}|f|\,d\mu<\varepsilon/4$, $\int_{B^3}|f|\,d\mu=0$, and for all $g\in N^j$, $\varphi(g+g_0)<0$ on a subset of B^j of positive measure $(j=1,2,3)$. The proof of Theorem 2 is now complete by writing $A=\bigcup_{j=1}^{3}(A^j\cup B^j)$. $\square$

Garkavi [6] defined an almost Chebyshev subspace M of a Banach space to be one for which the complement of U_M is of first category. It is of interest to note that Garkavi showed that if the underlying Banach space is separable and the subspace M is reflexive, then M being almost Chebyshev is equivalent to U_M being dense in the Banach space. Rozema [13] characterized the finite dimensional almost Chebyshev subspaces of L^1. In particular, when X contains no atoms, all finite dimensional subspaces of $L^1(X,\Sigma,\mu)$ are almost Chebyshev. The next result follows immediately.

COROLLARY 1. *If (X, Σ, μ) is nonatomic and M is a finite dimensional subspace of L^1, then SU_M is dense in L^1.*

We complete this section by turning our attention to the case in which (X, Σ, μ) is purely atomic with finitely many atoms. This is precisely the case where L^1 is a finite dimensional space with a weighted l^1 norm. We shall retain the notation of L^1 for the sake of consistency with the previous theorems.

THEOREM 3. *Assume that (X, Σ, μ) is purely atomic with finitely many atoms. For a finite dimensional subspace M of L^1, $SU_M = U_M$.*

PROOF. Let $f \in L^1 \setminus M$ with unique best approximation 0 from M. By a characterization of unique best approximation in L^1 due to Cheney and Wulbert [4]

$$\int g \operatorname{sgn}(f+g)\, d\mu > \int_{Z(f+g)} |f|\, d\mu$$

for all $g \in M \setminus \{0\}$. By the condition on (X, Σ, μ), there is an $\varepsilon > 0$ such that if $g \in M \setminus \{0\}$ with $0 < \|g\|_1 < \varepsilon$, then $\operatorname{sgn}(f+g) = \operatorname{sgn} f$ on $\operatorname{supp}(f)$. Then for such $g \in M$

$$\int g \operatorname{sgn}(f+g)\, d\mu = \int_{\operatorname{supp}(f)} g \operatorname{sgn}(f+g)\, d\mu + \int_{Z(f)} g \operatorname{sgn}(f+g)\, d\mu \leqq$$

$$\leqq \int_{\operatorname{supp}(f)} g \operatorname{sgn} f\, d\mu + \int_{Z(f)} |g|\, d\mu.$$

Thus

$$\int_{Z(f)} |g|\, d\mu \geqq \int g \operatorname{sgn}(f+g)\, d\mu - \int_{\operatorname{supp}(f)} g \operatorname{sgn}(f)\, d\mu >$$

$$> \int_{Z(f+g)} |f|\, d\mu - \int_{\operatorname{supp}(f)} g \operatorname{sgn}(f)\, d\mu \geqq - \int_{\operatorname{supp}(f)} g \operatorname{sgn}(f)\, d\mu.$$

Replacing g with $-g$,

$$\left| \int g \operatorname{sgn}(f)\, d\mu \right| < \int_{Z(f)} |g|\, d\mu$$

for $0 < \|g\|_1 < \varepsilon$. Theorem 1 and homogeneity now show that 0 is a strongly unique best approximation to f from M. □

3. First category result

In this section we show that SU_M is of first category in L^1 when X is nonatomic. We shall require the following variation of the Lyapunov theorem on vector measures (see Phelps [12]).

LEMMA 2. *Suppose that the measure space (X, Σ, μ) is nonatomic, φ is a measurable function on X with $|\varphi| \leqq 1$ μ-a.e., and $q_1, \ldots, q_n \in L^1(X, \Sigma, \mu)$. Then there exists a measurable function ψ on X with $|\psi| = 1$ μ-a.e. such that*

$$\int_X \psi q_i\, d\mu = \int_X \varphi q_i\, d\mu \quad (i = 1, \ldots, n).$$

THEOREM 4. *Let (X, Σ, μ) be a nonatomic σ-finite measure space and M be a finite dimensional subspace of $L^1(X, \Sigma, \mu)$. Then SU_M is of first category in $L^1(X, \Sigma, \mu)$.*

PROOF. Since $L^1(X, \Sigma, \mu)$ is isometric to $L^1(X, \Sigma, \bar{\mu})$ for some finite measure $\bar{\mu}$, we may assume that μ is finite. For n a positive integer, define

$$SU_M(n) = \{f \in SU_M\colon \gamma_f \geqq 1/n\}$$

where

$$\gamma_f = \inf\{(\|f-g\| - \|f-g_f\|)/\|g-g_f\|\colon g \in M \setminus \{g_f\}\}$$

and g_f is the best approximation to f from M. Evidently, $SU_M = \bigcup_{n=1}^{\infty} SU_M(n)$. We first show that each $SU_M(n)$ is closed in the L^1-norm. Let $\{f_k\}$ be a sequence in $SU_M(n)$ and $f \in L^1$ where $f_k \to f$ in the L^1-norm. Let g_k be the best approximation to f_k from M. Then $\|g_k\| \leqq 2\|f_k\| \to 2\|f\|$, and since M is finite dimensional, we may assume that $g_k \to g_0$ in the L^1-norm where $g_0 \in M$. For any $g \in M$,

$$\begin{aligned}\|f-g\|_1 - \|f-g_0\|_1 &\geqq \|f_k-g\|_1 - \|f_k-f\|_1 - \|f-f_k\|_1 - \|f_k-g_k\|_1 - \|g_k-g_0\|_1 \geqq \\ &\geqq (1/n)\|g-g_k\|_1 - 2\|f_k-f\|_1 - \|g_k-g_0\|_1 \geqq \\ &\geqq (1/n)\|g-g_0\|_1 - 2\|f_k-f\|_1 - (1+1/n)\|g_k-g_0\|_1.\end{aligned}$$

Passing to the limit as $k \to \infty$ yields

$$\|f-g\|_1 - \|f-g_0\|_1 \geqq (1/n)\|g-g_0\|_1.$$

Therefore, g_0 is the strongly unique best approximation to f from M and $f \in SU_M(n)$.

Finally, we prove that SU_M has empty interior from which it follows that each $SU_M(n)$ has empty interior. Let $f \in SU_M \setminus M$ with best approximation g_0 from M, and let $\varepsilon > 0$ be given. By a characterization of best L^1-approximations [14], there exists a measurable function φ on $Z(f-g_0)$ with $|\varphi| \leqq 1$ such that

$$\int_{\operatorname{supp}(f-g_0)} g \operatorname{sgn}(f-g_0)\,d\mu + \int_{Z(f-g_0)} g\varphi\,d\mu = 0$$

for all $g \in M$. By Lemma 2, we may assume that $|\varphi| \equiv 1$ on $Z(f-g_0)$. Set

$$f_\varepsilon = \begin{cases} f & \text{on } \operatorname{supp}(f-g_0) \\ f-\varepsilon\varphi & \text{on } Z(f-g_0). \end{cases}$$

Then $\|f-f_\varepsilon\| \leqq \varepsilon\mu(X)$ and $Z(f_\varepsilon-g_0) = \emptyset$, since $|\varphi| \equiv 1$. Also,

$$\int g \operatorname{sgn}(f_\varepsilon-g_0)\,d\mu = \int_{\operatorname{supp}(f-g_0)} g \operatorname{sgn}(f-g_0)\,d\mu + \int_{Z(f-g_0)} g\varphi\,d\mu = 0$$

for all $g \in M$. Hence, $g_0 \in P_M(f_\varepsilon)$. Since $\mu(Z(f_\varepsilon-g_0)) = 0$, Theorem 1 implies that $f_\varepsilon \notin SU_M$. If $f \in M$ we appeal to the above argument and to Corollary 1. The proof is now complete. □

We conclude by observing that the condition that X be nonatomic is essential in Theorem 4. When the underlying measure space is purely atomic with finitely many atoms, there certainly exist Chebyshev subspaces of L^1, and Theorem 3 implies that $SU_M = U_M = L^1$. Needless-to-say, SU_M is not of first category in this case. As a final note, we mention that it is not known whether the set SU_M is of first category when M is an almost Chebyshev subspace of $C[0, 1]$.

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References

[1] J. Angelos and A. Egger, Strong uniqueness in the L^p spaces, *J. Approx. Theory*, **42** (1984), 14—26.

[2] J. Angelos and A. Kroó, The equivalence of the moduli of continuity of the best approximation operator and of strong unicity in L^1, *J. Approx. Theory*, **46** (1986), 129—136.

[3] I. Barrodale, M. J. D. Powell and F. D. K. Roberts, The differential correction algorithm for rational l^∞ approximation, *SIAM J. Numer. Anal.*, **9** (1972), 493—504.

[4] E. W. Cheney and D. E. Wulbert, The existence and uniqueness of best approximations, *Math. Scand.*, **24** (1969), 113—140.

[5] L. Cromme, Strong uniqueness, a far reaching criterion for convergence analysis of iterative processes, *Numer. Math.*, **29** (1978), 179—193.

[6] A. L. Garkavi, On Cebysev and almost Cebysev subspaces, *Amer. Math. Soc. Transl.*, **96** (1970), 177—187.

[7] A. Kroó, On the continuity of best approximations in the space of integrable functions, *Acta Math. Acad. Sci. Hung.*, **32** (1978), 331—348.

[8] A. Kroó, Best L^1-approximation on finite point sets: rate of convergence, *J. Approx. Theory*, **33** (1981), 340—352.

[9] D. J. Newman and H. S. Shapiro, Some theorems on Čebyšev approximation, *Duke Math. J.*, **30** (1963), 673—681.

[10] G. Nürnberger, Unicity and strong unicity in approximation theory, *J. Approx. Theory*, **22** (1979), 54—70.

[11] G. Nürnberger and I. Singer, Uniqueness and strong uniqueness of best approximations by spline subspaces and other subspaces, *J. Math. Anal. Applic.*, **90** (1982), 171—184.

[12] R. R. Phelps, Cebysev subspaces of finite dimension in L_1, *Proc. of the Amer. Math. Soc.*, **17** (1966), 646—652.

[13] E. Rozema, Almost Chebyshev subspaces of $L^1(\mu, E)$, *Pacific J. Math.*, **53** (1974), 585—603.

[14] I. Singer, *The Theory of Best Approximation and Functions Analysis*, SIAM (Philadelphia, 1974).

[15] D. E. Wulbert, Uniqueness and differential characterization of approximation from manifolds of functions, *Amer. J. Math.*, **18** (1971), 350—366.

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APPROXIMATIONS OF REAL NUMBERS BY THE SEQUENCE $\{n\alpha\}$ AND THEIR METRICAL THEORY

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Introduction

For an irrational α, $0<\alpha<1$, we denote by $[a_1, a_2, \dots]$ its simple continued fraction expansion. We put

$$A_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

and

(1) $$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geqq 1$. It is well-known that

$$\frac{p_n}{q_n} = [a_1, a_2, a_3, \dots, a_n]$$

holds.

Let N be a given positive integer. Then we can find a non-negative integer l such that $q_{l-1} \leqq N < q_l$. We define, inductively,

$$b_{l-i+1} = \left[\frac{N_{l-i+1}}{q_{l-i}}\right] \quad \text{and} \quad N_{l-i} = N_{l-i+1} - b_{l-i+1} q_{l-i}$$

for $i=1, 2, \dots, l$, where $N_l = N$ and $[x]$ is the integral part of the real number x. Moreover, we put $b_i = 0$ for $i \geqq l+1$. Then we get a representation

(2) $$N = \sum_{i=1}^{l} b_i q_{i-1} = \sum_{i=1}^{\infty} b_i q_{i-1}.$$

From (1) and the definition of b_i, we see that

(3) $$\begin{cases} 0 \leqq b_i \leqq a_i \quad \text{for} \quad i \geqq 2, \\ \text{if} \quad b_i = a_i, \quad \text{then} \quad b_{i-1} = 0 \quad \text{for} \quad i \geqq 2, \\ 0 \leqq b_1 \leqq a_1 - 1. \end{cases}$$

On the other hand, let β be a real number such that $-\alpha < \beta < 1-\alpha$ and $\beta \neq l\alpha \pmod 1$ for any negative integer l. Then β can be uniquely represented by

(4) $$\beta = \sum_{i=1}^{\infty} b_i(\beta)(q_{i-1}\alpha - p_{i-1})$$

where $b_i(\beta)$ satisfies the condition (3) (see Sh. Ito [2] or M. Stewart [8]). The representations (2) and (4) play an important role in some problems related to the theory of uniform distributions (cf. e.g. V. T. Sós [7]).

Let us consider the relation between (2) and (4). Suppose that $\{N_n\}$ is a sequence of positive integers such that

$$\lim_{n\to\infty} N_n\alpha = \beta \,(\mathrm{mod}\ 1).$$

For each N_n, we denote the representation (2) by

$$N_n = \sum_{i=1}^{\infty} b_i^{(n)} q_{i-1}.$$

Then we have the following:

THEOREM 0. *For any positive integer L, there exists a positive integer l_0 such that $b_i^{(l)} = b_i(\beta)$ for $1 \leqq i \leqq L$ and $l \geqq l_0$.*

We put

$$u_n = \sum_{i=1}^{n} b_i(\beta) q_{i-1} \quad \text{for} \quad i \geqq 1.$$

Then the representation (4) implies

$$\beta = \lim_{n\to\infty} u_n\alpha \quad (\mathrm{mod}\ 1).$$

By using this fact and the uniqueness of the representation (4), we can prove the assertion of Theorem 0. In the sense of Theorem 0, we call $\{u_n\alpha \ (\mathrm{mod}\ 1)\}$ the canonical approximating sequence of β with respect to α. Since if $b_{i+1}(\beta)=0$ for some i, then $u_i = u_{i+1}$, u_n is not monotone increasing, in general. By putting $u_n^* = u_{c(n)}$ with

$$c(n) = \min\{m;\ \#\{l;\ 1 \leqq l \leqq m,\ b_l \neq 0\} = n\},$$

we get the monotone increasing sequence $\{u_n^*\}$ from $\{u_n\}$.

In this paper, we discuss the increasing rates of $\{u_n\}$ and $\{u_n^*\}$ and the convergence rate of $\sum_{i=1}^{n} b_i(\beta)(q_{i-1}\alpha - p_{i-1})$ for almost all (α, β); in particular, we prove the following theorems.

THEOREM 1. *For almost all (α, β),*

$$\lim_{n\to\infty} \frac{1}{n} \log u_n = \frac{\pi^2}{12\log 2}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log u_n^* = \frac{\pi^2}{6\log 2}.$$

THEOREM 2. *For almost all* (α, β),

$$\lim_{N\to\infty} \frac{\#\left\{n:\ q_n\left|\beta-\sum_{i=1}^{n} b_i(\beta)(q_{i-1}\alpha-p_{i-1})\right| \leqq z,\ 1 \leqq n \leqq N\right\}}{N} = z(2-z)$$

for $0 \leqq z \leqq 1$.

To prove these theorems, we make use of the natural extension of the non-invertible transformation. This method is also useful for some problems in diophantine approximations and we refer to [3] for such problems.

§ 1. Preparations

First, we recall some fundamental properties of simple continued fraction expansions. For an irrational number $\alpha=[a_1, a_2, a_3, \dots]$, we put

$$\theta_n = q_n\alpha - p_n \tag{5}$$

and

$$\alpha_n = [a_{n+1}, a_{n+2}, a_{n+3}, \dots] \tag{6}$$

for $n \geqq 0$. Then we have the following:

$$\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, a_{n-2}, \dots, a_1], \tag{7}$$

$$\alpha = \frac{p_n + p_{n-1}\alpha_n}{q_n + q_{n-1}\alpha_n}, \tag{8}$$

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n \tag{9}$$

and

$$\alpha_0\alpha_1 \dots \alpha_n = (-1)^n \theta_n \tag{10}$$

for $n \geqq 1$. Furthermore, the following properties are well-known (see [1]):

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12\log 2} \quad \text{for almost all } \alpha \tag{11}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_0\alpha_1 \dots \alpha_n = -\frac{\pi^2}{12\log 2} \quad \text{for almost all } \alpha. \tag{12}$$

Next, for the discussions in §§ 2 and 3, we introduce the transformation (X, T) and its natural extension $(\bar{X}, \bar{T})$. We put

$$X = \{(\alpha, \beta);\ 0 \leqq \alpha < 1,\ -\alpha < \beta < 1\}$$

and define

$$T(\alpha, \beta) = \left(\frac{1}{\alpha} - \left[\frac{1}{\alpha}\right],\ -\frac{\beta}{\alpha} + b(\alpha, \beta)\right)$$

for $(\alpha, \beta) \in X$, where

$$b(\alpha, \beta) = \max\left(\left[\frac{1}{\alpha}\right] - \left[\frac{1-\beta}{\alpha}\right],\ 0\right).$$

We also define the digits a_n and b_n by

$$a_n = a_n(\alpha) = \left[\frac{1}{\alpha_{n-1}}\right] \quad \text{and} \quad b_n = b_n(\alpha, \beta) = b(\alpha_{n-1}, \beta_{n-1})$$

for $n \geqq 1$ with $(\alpha_n, \beta_n) = T^n(\alpha, \beta)$. From these, we can get

$$\alpha = [a_1, a_2, a_3, \ldots]$$

and

(13) $$\beta = \sum_{i=1}^{\infty} b_i \theta_{i-1}.$$

It is easy to see that the representation (13) is identical with the representation (4) of β if $-\alpha < \beta < 1-\alpha$. Now we introduce the transformation $(\bar{X}, \bar{T})$, which is called the natural extension of (X, T). Let us define the domain $\bar{X}$ by

$$\bar{X} = X_1 \times \{(\gamma, \delta);\ 0 \leqq \gamma \leqq 1,\ 0 \leqq \delta < 1\} \cup X_2 \times \{(\gamma, \delta);\ 0 \leqq \gamma \leqq 1,\ 0 \leqq \gamma < \delta\}$$

with

$$X_1 = \{(\alpha, \beta);\ 0 \leqq \alpha < 1,\ -\alpha < \beta \leqq 1-\alpha\}$$

and

$$X_2 = \{(\alpha, \beta);\ 0 \leqq \alpha < 1,\ 1-\alpha < \beta < 1\}.$$

The transformation $\bar{T}$ of $\bar{X}$ is defined by

$$\bar{T}(\alpha, \beta, \gamma, \delta) = \left(T(\alpha, \beta),\ \frac{1}{\gamma + a_1},\ \frac{\delta + b_1}{\gamma + a_1}\right)$$

for $(\alpha, \beta, \gamma, \delta) \in \bar{X}$. It is not so hard to show that the transformation $\bar{T}$ is one-to-one and onto on $\bar{X}$ except for a set of Lebesgue measure 0. Furthermore, the density function $\bar{h}$ of the absolutely continuous invariant measure $\bar{\mu}$ for $\bar{T}$ is given by

(14) $$\bar{h}(\alpha, \beta, \gamma, \delta) = \frac{1}{\log 2} \cdot \frac{1}{(1+\alpha\gamma)^3}$$

for $(\alpha, \beta, \gamma, \delta) \in \bar{X}$. Thus the density function h of the absolutely continuous invariant measure μ for T is given by the marginal density function of $\bar{h}$. So we have

(15) $$h(\alpha, \beta) = \begin{cases} \dfrac{1}{2\log 2} \dfrac{2+\alpha}{(1+\alpha)^2} & \text{if } (\alpha, \beta) \in X_1 \\ \dfrac{1}{2\log 2} \dfrac{1}{(1+\alpha)^2} & \text{otherwise.} \end{cases}$$

(The details of the above discussion can be found in Sh. Ito [2].)

Finally, we note that (X, T, μ) is ergodic (and so is $(\bar{X}, \bar{T}, \bar{\mu})$). Moreover, it is possible to show that T satisfies conditions of Schweiger and Waterman [6] and this means that a Gauss—Kuzmin theorem holds for T. By using this theorem, we can apply the quantitative Borel—Cantelli lemma of W. Philipp [5] to give an estimate of the remainder terms of the limits below. However, we do not discuss them in this paper.

§ 2. Asymptotic properties of the canonical approximations (1)

In this section, we give immediate consequences of § 1.

PROPOSITION 1. *For almost all* (α, β),

$$\lim_{N\to\infty} \frac{1}{N} \#\{n\colon 1 \leqq n \leqq N,\quad b_n = k\} = \tag{16}$$

$$= \begin{cases} \dfrac{1}{2} & \text{if} \quad k = 0 \\ \dfrac{2\log(k+1) - \log k - \log(k+2)}{2\log 2} & \text{if} \quad k > 0. \end{cases}$$

PROOF. Since (T, μ) is ergodic, the left hand side of (14) is equal to

$$\mu\{(\alpha, \beta);\ b_1 = k\} = \iint_{\{b_1=k\}} h(\alpha, \beta)\, d\alpha\, d\beta$$

for almost all (α, β). Moreover, it is easy to compute that this is equal to the right hand side of (16) (see Fig. 1).

REMARK. In Monteferrante—Szüsz [4, Theorems 1 and 2], only the existence of the left hand side of (16) is proved for quadratic irrational α, while we prove the existence for a.a. α and give its explicit value which is independent of a.a. α.

PROPOSITION 2. *For almost all* (α, β),

$$\lim_{n\to\infty} \frac{1}{n} \log \Big|\beta - \sum_{i=1}^{n} b_i(q_{i-1}\alpha - p_{i-1})\Big| = -\frac{\pi^2}{12\log 2}.$$

PROOF. From (5) and (10), we have

$$\Big|\beta - \sum_{i=1}^{n} b_i(q_{i-1}\alpha - p_{i-1})\Big| = \Big|\sum_{i=n+1}^{\infty} b_i(q_{i-1}\alpha - p_{i-1})\Big| = |\beta_n|\,\alpha_0\alpha_1 \dots \alpha_{n-1}$$

with $T^n(\alpha, \beta) = (\alpha_n, \beta_n)$. Hence

$$\frac{1}{n} \log \Big|\beta - \sum_{i=1}^{n} b_i(q_{i-1}\alpha - p_{i-1})\Big| = \frac{1}{n} \log(\alpha_0\alpha_1 \dots \alpha_{n-1}) + \frac{1}{n} \log |\beta_n|.$$

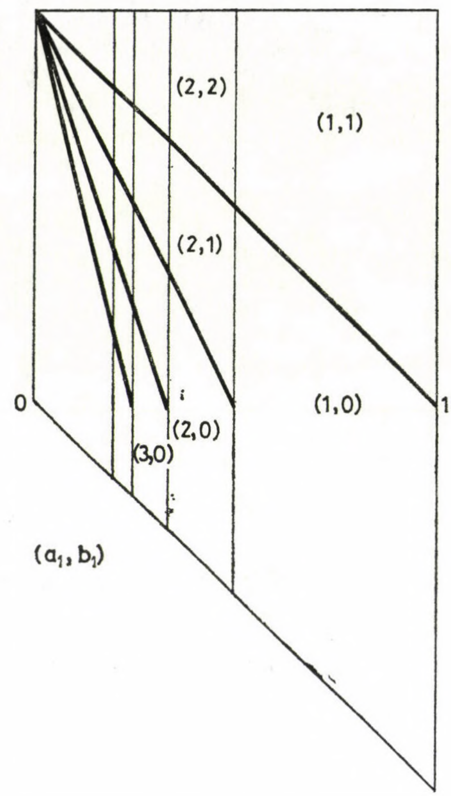

Fig. 1

Now we show that

$$\lim_{n\to\infty} \frac{1}{n} \log |\beta_n| = 0 \tag{17}$$

for almost all (α, β). Since

$$\mu(|\beta_n| < \eta) = \mu(|\beta| < \eta) = \frac{1}{2\log 2}\left((2+\eta)\log 2-(2-\eta)\log(2-\eta)-\eta\log(1+\eta)\right)$$

for any η, $0<\eta<1$, we see that

$$\sum_{n=1}^{\infty} \mu(|\beta_n| < e^{-n\varepsilon}) < +\infty$$

for any $\varepsilon>0$. Thus, by using Borel—Cantelli lemma,

$$\#\left\{n;\ -\frac{1}{n}\log|\beta_n| > \varepsilon\right\} < +\infty$$

for almost all (α, β). This implies (17). Consequently, by (12), we have the assertion of the proposition.

Since T is ergodic with respect to μ, the proof of the above proposition also implies the following:

PROPOSITION 3. *For almost all* (α, β),

$$\lim_{N\to\infty} \frac{\#\left\{n\colon |\theta_{n-1}|^{-1}\cdot\left|\beta-\sum_{i=1}^{n} b_i(q_{i-1}\alpha-p_{i-1})\right|<z\right\}}{N}=$$

$$=\frac{1}{2\log 2}[(2+z)\log 2-(2-z)\log(2-z)z-z\cdot\log(1+z)]$$

for any z, $0\leq z\leq 1$.

§ 3. Asymptotic properties of the canonical approximations (2). Proof of the theorems

In the preceding section, we have only made use of properties of T. On the other hand, we need some properties of the natural extension $\overline{T}$ of T to show Theorems 1 and 2. The following lemma is essential.

LEMMA. *For any* $(\alpha, \beta)\in X$,

$$\overline{T}^n(\alpha,\beta,0,0)=\left(T^n(\alpha,\beta),\ \frac{q_{n-1}}{q_n},\ \frac{u_n}{q_n}\right).$$

The proof of this lemma follows by an easy induction.

PROOF OF THEOREM 1. Since $u_n<q_n$, we have

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log u_n\leq\lim_{n\to\infty}\frac{1}{n}\log q_n=\frac{\pi^2}{12\log 2}$$

for almost all α. Thus, to complete the proof, it is sufficient to show that for any $\varepsilon>0$,

$$\#\left\{n\colon \frac{1}{n}\log u_n<\frac{1}{n}\log q_n-\varepsilon\right\}<+\infty \tag{18}$$

for almost all (α, β). We note that

$$\frac{1}{n}\log u_n<\frac{1}{n}\log q_n-\varepsilon$$

is equivalent to $u_n/q_n<e^{-n\varepsilon}$. If we put $(\alpha_n, \beta_n, \gamma_n, \delta_n)=\overline{T}^n(\alpha,\beta,\gamma,\delta)$ for $(\alpha,\beta,\gamma,\delta)\in\overline{X}$, then for any η, $0<\eta<1$, and any $d>0$,

$$\sum_{n=1}^{\infty}\bar{\mu}\{\delta_n<d\eta^n\}=\sum_{n=1}^{\infty}\bar{\mu}\{\delta<d\eta^n\}<+\infty.$$

So we see that

$$\#\{n:\ \delta_n < d\eta^n\} < +\infty$$

for almost all $(\alpha, \beta, \gamma, \delta)$ by using Borel—Cantelli lemma. On the other hand, from the lemma, there exist constants C_1 and η_1, $0<\eta_1<1$, such that

$$\left|\delta_n - \frac{u_n}{q_n}\right| < C_1\eta_1^n.$$

Thus, by putting

$$d = 2C_1 \quad \text{and} \quad \eta = \max\{\eta_1, e^{-\varepsilon}\},$$

we see that (18) holds for almost all (α, β). The rest of the proof follows from Proposition 1.

PROOF OF THEOREM 2. From (5), (8), (9) and (10), we see that

$$q_n\left|\beta - \sum_{i=1}^{n} b_i(q_{i-1}\alpha - p_{i-1})\right| = q_n|\beta_n|\alpha_0\alpha_1\dots\alpha_{n-1}\alpha_n\alpha_n^{-1} = q_n|\theta_n||\beta_n|\alpha_n^{-1} =$$

$$= q_n|q_n\alpha - p_n|\cdot|\beta_n|\alpha_n^{-1} = \frac{|\beta_n|}{1+\dfrac{q_{n-1}}{q_n}\alpha_n}.$$

So we get, by the lemma,

$$\frac{\#\left\{n:\ q_n\left|\beta - \sum_{i=1}^{n} b_i(q_{i-1}\alpha - p_{i-1})\right| \leqq z,\ 1 \leqq n \leqq N\right\}}{N} = \frac{\sum_{n=1}^{N} \bar{\chi}_z(\bar{T}^n(\alpha, \beta, 0, 0))}{N},$$

where $\bar{\chi}_z$ is the indicator function of the set

$$\left\{(\alpha, \beta, \gamma, \delta)\in\bar{X}:\ \frac{|\beta|}{1+\alpha\gamma} \leqq z\right\}.$$

On the other hand, it is clear that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \bar{\chi}_z(\bar{T}^n(\alpha, \beta, \gamma, \delta))}{N} = \bar{\mu}\left(\frac{|\beta|}{1+\alpha\gamma} \leqq z\right) \tag{19}$$

for almost all $(\alpha, \beta, \gamma, \delta)$. It is possible to compute that

$$\bar{\mu}\left(0 < \frac{\beta}{1+\alpha\gamma} \leqq z\right) = \begin{cases} \dfrac{1}{\log 2}\left[z(1-z)\log 2 + \dfrac{z^2}{2}\right] & \text{if} \quad 0 < z \leqq \dfrac{1}{2} \\[2ex] \dfrac{1}{\log 2}\left[(2z-z^2)\log 2 + z\log z + \dfrac{1}{4} - \dfrac{z^2}{2}\right] & \text{if} \quad \dfrac{1}{2} < z \leqq 1 \end{cases}$$

and

$$\bar{\mu}\left(-1<\frac{\beta}{1+\alpha\gamma}<z\right)=\begin{cases}\dfrac{1}{\log 2}\left[\dfrac{1}{2}-\dfrac{z^2}{2}-z\log(-z)\right] & \text{if} \quad -1<z\leqq-\dfrac{1}{2}\\ \dfrac{1}{\log 2}\left[\dfrac{1}{4}+\dfrac{z^2}{2}+z\log 2\right] & \text{if} \quad -\dfrac{1}{2}<z<0.\end{cases}$$

Thus we see that the right hand side of (19) is equal to $z(2-z)$. Now let

$$\bar{T}^n(\alpha,\beta,\gamma,\delta)=(\alpha_n,\beta_n,\gamma_n,\delta_n).$$

Then we have

$$\left|\frac{\beta_n}{1+\alpha_n\gamma_n}-\frac{\beta_n}{1+\alpha_n\dfrac{u_n}{q_n}}\right|\leqq\left|\frac{u_n}{q_n}-\gamma_n\right|.$$

Since it is possible to show that there exist constants η_2, $0<\eta_2<1$, and $C_2>0$ such that

$$\left|\frac{u_n}{q_n}-\gamma_n\right|<C_2\eta_2^n,$$

we see that if

$$\bar{\chi}_z(\bar{T}^n(\alpha,\beta,\gamma,\delta))\neq\bar{\chi}_z(\bar{T}^n(\alpha,\beta,0,0)),$$

then

$$z-C_2\eta_2^n\leqq\frac{\beta_n}{1+\alpha_n\gamma_n}\leqq z+C_2\eta_2^n.$$

From (19), it is easy to estimate that

$$\sum_{n=1}^{\infty}\bar{\mu}\left\{z-C_2\eta_2^n\leqq\frac{\beta}{1+\alpha\gamma}\leqq z+C_2\eta_2^n\right\}<+\infty.$$

Thus, by using Borel—Cantelli lemma,

$$\#\left\{n:\ z-C_2\eta_2^n\leqq\frac{\beta_n}{1+\alpha_n\gamma_n}\leqq z+C_2\eta_2^n\right\}<+\infty$$

for almost all $(\alpha,\beta,\gamma,\delta)$. Consequently,

$$\#\{n:\ \bar{\chi}_z(\bar{T}^n(\alpha,\beta,\gamma,\delta))\neq\bar{\chi}_z(\bar{T}^n(\alpha,\beta,0,0))\}<+\infty$$

for almost all $(\alpha,\beta,\gamma,\delta)$ and this completes the proof of this theorem.

References

[1] P. Billingsley, *Ergodic theory and information,* John Wiley and Sons (New York, 1965).

[2] Sh. Ito, Some skew product transformations associated with continued fractions and their invariant measures, preprint.

[3] Sh. Ito and H. Nakada, On natural extensions of transformations related to diophantine approximations, *Tokyo J. Math.*, **9** (1986), 115—133.

[4] S. Monteferrante and P. Szüsz, On a generalization of a theorem of Borel, *Acta Arith.*, **16** (1969/70), 393—397.
[5] W. Philipp, Some metrical theorems in number theory, *Pacific J. Math.*, **20** (1967), 109—127.
[6] F. Schweiger and W. Waterman, Some remarks on Kuzmin's theorem for F-expansions, *J. Number Th.*, **5** (1973), 123—131.
[7] V. T. Sós, *On the discrepancy of the sequence* $\{n\alpha\}$, Coll. Math. Soc. J. Bolyai, **13** (1976), 359—367.
[8] M. Stewart, Irregularities of uniform distribution, *Acta Math. Acad. Sci. Hung.*, **37** (1981), 185—221.

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ON EMBEDDING OF LOCALLY COMPACT ABELIAN TOPOLOGICAL GROUPS IN EUCLIDEAN SPACES. II

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In this paper we shall continue and finish the proof of Theorem A of [7]. We should mention that in cross references with respect to [7] the number 7 will be omitted. Thus 6.1 (ii) will refer to condition (ii) in Section 6.1 of [7] etc.

Sections 7 and 8 are of preparatory character while Section 9 deals with an important existence theorem. Based on it Section 10 completes the proof of Theorem A.

7. Continuous paths

We first recall some notions and simple facts from [1] and [3]. Thereafter two lemmas about subdivision of certain paths will be formulated and proved.

7.1. First recall from [3] the notion of the *i*-category.

An *i-category* is a category $\mathscr{C}$ together with a contravariant functor $\mathscr{F}\colon \mathscr{C}\to\mathscr{C}$ such that $\mathscr{F}\cdot\mathscr{F}=\mathrm{id}_{\mathscr{C}}$ and $\mathscr{F}(B)=B$ for each object B of $\mathscr{C}$.

We shall use the symbols $\mathscr{C}^{\bullet}$, $\mathscr{D}^{\bullet}$ etc. for *i*-categories. If $\mathscr{C}^{\bullet}=(\mathscr{C},\mathscr{F})$ is an *i*-category then the class of objects (morphisms) of $\mathscr{C}$ will be denoted by $\mathrm{Ob}\,\mathscr{C}^{\bullet}$ ($\mathrm{Mor}\,\mathscr{C}^{\bullet}$). For any $\alpha\in\mathrm{Mor}\,\mathscr{C}^{\bullet}$, $\alpha^{\bullet}$ may also be written instead of $\mathscr{F}(\alpha)$. $\alpha^{\bullet}$ is the *involutoric conjugate* of α. Any morphism $\alpha\colon A\to A'$ of $\mathscr{C}^{\bullet}$ is said to be *closed* if its domain is the same as its range, i.e. $A=A'$.

7.2. Now recall some further notions from [3].

Let Y be a topological space. A *(continuous) path* $K\colon q\to z$ $(q, z\in Y)$ of Y is a class of equivalence of continuous mappings $f\colon [b, c]\to Y$ ($[b, c]$ is a degenerated $(b=c)$ or a proper closed interval $(b<c)$ in the space of the reals $\mathbf{R}$) such that $f(c)=q$, $f(b)=z$ where the mappings $f\colon [b, c]\to Y$ and $g\colon [b', c']\to Y$ are said to be equivalent if there exists a strictly monotonous increasing epimorphic function $s\colon [b, c]\to[b', c']$ for which $g\circ s=f$.

An element $f\colon [b, c]\to Y$ of the equivalence class $K\colon q\to z$ is said to be a *representative* of the path K.

7.3. Let $K\colon q\to u$ be a continuous path of Y and $f\colon [b, c]\to Y$ a representative of K. The subspace $f([b, c])$ of Y is called the *body* of K and it is denoted by $\tilde{K}$. $\tilde{K}$ is obviously well defined.

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7.4. Now let f_1: $[c, d] \to Y$ be a representative of K_1: $q \to u$ and f_2: $[b, c] \to Y$ a representative of K_2: $u \to z$. Then $K_2 K_1$: $q \to z$ is defined as the equivalence class of the mapping g: $[b, d] \to Y$ where $g|_{[b,c]} = f_2$ and $g|_{[c,d]} = f_1$.

In this way we obtain a category $\mathscr{K}_Y$. The objects of $\mathscr{K}_Y$ are the points of Y and the morphisms are the continuous paths of Y.

7.5. To each path K: $q \to z$ of $\mathscr{K}_Y$ we can assign a path $K^\bullet$: $z \to q \in \operatorname{Mor} \mathscr{K}_Y$ such that if f: $[b, c] \to Y$ is a representative of K and h: $[b, c] \to [b, c]$ is defined by the formula $h(x) = b + c - x$ then $f \circ h$ is a representative of $K^\bullet$.

The category $\mathscr{K}_Y$ together with the functor $\mathscr{F}$: $\mathscr{K}_Y \to \mathscr{K}_Y$: $K \mapsto K^\bullet$ is an i-category (see [1] 2.8). We shall denote it by $\mathscr{K}_Y^\bullet$.

7.6. If K: $q \to q$ is a closed morphism of $\mathscr{K}_Y^\bullet$ i.e. a closed path of Y then q is said to be the *base point* of K.

7.7. Next, in accordance with [1] 5.9, we mention that a closed path K: $q \to q$ of Y is said to be a *Jordan path* if there exists a representative f: $[b, c] \to Y$ of K for which $b \neq c$ and if $b \leqq x_1 < x_2 \leqq c$ then $f(x_1) = f(x_2)$ implies $x_1 = b$ and $x_2 = c$.

Notice that if K: $q \to q$ is a Jordan path then each of its representatives has the properties described above.

It is also to be noted that if Y is a Hausdorff space then the body $\tilde{K}$ of each Jordan path K of Y is a closed Jordan curve.

We now turn to the subdivisions of a continuous path K.

Let Y be a T_2-space.

7.8. Definition. Let K be a continuous path of Y. A representation of the form $K = K_m \dots K_2 K_1$ is said to be a *subdivision* of K. $K_1, \dots, K_m$ are the *factors* and m is the *degree* of this subdivision.

7.9. Remark. Let $K = K_m \dots K_1$ be a subdivision of the continuous path K of Y. We then have evidently $\tilde{K} = \tilde{K}_m \cup \dots \cup \tilde{K}_1$.

7.10. Definition. Let K be a continuous path of Y and let Ω be an open covering of the space $\tilde{K}$. The subdivision $K = K_m \dots K_1$ of K is said to be a *refinement of* Ω if for $i = 1, \dots, m$ $\tilde{K}_i$ is contained in a member of Ω.

7.11. Remark. Let K be a continuous path of Y and let Ω be an open covering of the space $\tilde{K}$. Then there is a subdivision $K = K_m \dots K_1$ of K which is a refinement of Ω (see [1] 3.6).

7.12. Definition. Let K be a closed path of Y. The closed path K' of Y is said to be the *image of K by rotation* — and we write $K' \sim K$ — if there is a subdivision $K = K_2 K_1$ of K such that $K' = K_1 K_2$.

7.13. Remark. $\sim$ is a relation of equivalence on the set of continuous closed paths of Y (see [1] 4.2). Thus the class of continuous closed paths of Y decomposes into equivalence classes with respect to the relation $\sim$.

7.14. Definition. The equivalence classes of the relation $\sim$ are said to be *directed families of paths* of Y. For any continuous closed path K of Y we write K^r for the directed family of paths of K i.e. $K \in K^r$.

7.15. REMARK. If $K \sim K'$ for any closed paths K and K' of Y then we have evidently $\tilde{K} = \tilde{K}'$.

7.16. LEMMA. *Let K be a continuous closed path of Y and let $\Omega = \{U_1, U_2\}$ be an open covering of the space $\tilde{K}$ such that $\tilde{K} \not\subset U_1$ and $\tilde{K} \not\subset U_2$. Then there is a $K' \in K^r$ and a subdivision $K' = K'_{2m} \dots K'_2 K'_1$ of K' such that the relations*

$$\tilde{K}'_{2i-1} \subset U_1, \quad \tilde{K}'_{2i-1} \not\subset U_2, \quad \tilde{K}'_{2i} \subset U_2 \quad \text{and} \quad \tilde{K}'_{2i} \not\subset U_1$$

hold for $i = 1, \dots, m$.

PROOF. Let us consider the set of degrees of all subdivisions of the paths of K^r which are refinements of Ω. 7.11 shows that this set of positive integers is non-empty. Consequently there is a minimal element m_0 in this set. Since $\tilde{K} \not\subset U_1$ and $\tilde{K} \not\subset U_2$ taking also 7.15 into account we get $m_0 \geqq 2$. Select $K' \in K^r$ and the subdivision

$$K' = K'_{m_0} \dots K'_2 K'_1 \tag{72}$$

such that this subdivision should be a refinement of Ω.

Observe that for $i = 1, \dots, m_0 - 1$, $\tilde{K}'_i \cup \tilde{K}'_{i+1}$ cannot lie neither in U_1 nor in U_2, since otherwise

$$K' = K'_{m_0} \dots K'_{i+2}(K'_{i+1}K'_i)K'_{i-1} \dots K'_1$$

would be a subdivision of K' of degree $m_0 - 1$ which is a refinement of Ω.

On the other hand $\tilde{K}'_1 \cup \tilde{K}'_{m_0}$ cannot lie in the same member of Ω, since otherwise for the continuous closed path $K'' = K'_{m_0-1} \dots K'_2 K'_1 K'_{m_0}$ which belongs to K^r as well, the subdivision $K'' = K'_{m_0-1} \dots K'_2 (K'_1 K'_{m_0})$ of degree $m_0 - 1$ would be a refinement of Ω.

Consequently m_0 must be an even number: $m_0 = 2m$ and we have two possibilities.

(a) $\tilde{K}'_1 \subset U_1$. Then (72) is clearly a subdivision of the required type.

(b) $\tilde{K}'_1 \subset U_2$. Then consider the continuous closed path $K'' = K'_{m_0-1} \dots K'_1 K'_{m_0}$ of K^r and for $i = 1, \dots, m_0 - 1$ let $K''_{i+1} = K'_i$. Moreover let $K''_1 = K'_{m_0}$. Then the subdivision $K'' = K''_{2m} \dots K''_1$ is of the required type indeed and this proves the lemma. $\square$

Now we turn to the simple and Jordan paths.

7.17. DEFINITION. Let v be a simple arc in Y with the endpoints q_1 and q_2. Then there exists a unique continuous path $K\colon q_1 \to q_2$ with $\tilde{K} = v$ and such that the representations of K are injective maps. We denote this path by $[q_2 v q_1]$. Paths of this type are called *proper simple paths.*

7.18. DEFINITION. To each point q of Y, i.e. to each object of $\mathcal{K}_Y$ there belongs a unique identity $e_q\colon q \to q$ of $\mathcal{K}_Y$. These identities are said to be *degenerated simple paths* and their bodies are the *degenerated arcs* of Y. For each degenerated arc $\{q\}$ of Y the point q is said to be the *endpoint* of $\{q\}$.

If v is a degenerated arc with the endpoint q then instead of e_q we also write $[qvq]$.

7.19. DEFINITION. By an *arc* of Y we mean a simple arc or a degenerated arc of Y. By a *simple path* of Y we mean a proper or a degenerated simple path of Y.

7.20. Remark. Let v be an arc of Y with the endpoints q and q'. Let q'' be an arbitrary point of v. Then there is a unique arc $v_1 \subset v$ with the endpoints q and q'' and a unique arc $v_2 \subset v$ with the endpoints q' and q''. Moreover we have

$$[qvq'] = [qv_1q''][q''v_2q'].$$

On the other hand let $K=[qvq']$ and $K=K_1K_2$. Then there is a unique $q'' \in v$ such that $K_1=[qv_1q'']$, $K_2=[q''v_2q']$ and v_1, v_2 are the same arcs as before. Moreover we have $v_1 \cup v_2 = v$ (cf. [1] 5.6).

7.21. Remark. Consider the composition

$$[q_0v_1q_1][q_1v_2q_2] \dots [q_{m-1}v_mq_m]$$

of simple paths of Y. Then we shall also denote this composition by

$$[q_0v_1q_1v_2q_2 \dots q_{m-1}v_mq_m].$$

Then we have obviously

$$[q_0v_1 \dots v_kq_k][q_kv_{k+1} \dots v_mq_m] = [q_0v_1 \dots v_kq_kv_{k+1} \dots v_mq_m]$$

and

$$[q_0v_1 \dots v_mq_m]^\bullet = [q_mv_m \dots v_1q_1]$$

(see [1] 5.7).

In particular if $[qvq']$ is a simple path of Y then

$$[qvq']^\bullet = [q'vq] \tag{73}$$

(see also [1] 5.5).

7.22. Remark. Let K be a Jordan path and let K' be the image of K by rotation (see 7.12). Then K' is a Jordan path as well (see [1] 5.9).

7.23. Remark. Let $K=K_1 \dots K_m$ be a subdivision of the Jordan path K. Then there are two possibilities excluding each other:

(a) Exactly one of the factors K_i is equal to K and the other factors are degenerated simple paths (trivial case).

(b) $m \geqq 2$, each K_i is a simple path and at least two of the factors are proper simple paths (nontrivial case) (see [1] 5.11).

7.24. Remark. Let K be a Jordan path and $K=K_1 \dots K_m$ a subdivision of K. For $i=1, \dots, m-1$ let $K_i=K_i\colon q_i \to q_{i+1}$. We then have obviously $K_m=K_m\colon q_m \to q_1$. Let $1 \leqq i' < i'' \leqq m$. Suppose that $q_{i'} \neq q_{i''}$. Then

$$\tilde{K}_{i'} \cup \dots \cup \tilde{K}_{i''-1} = v$$

and

$$\tilde{K}_{i''} \cup \dots \cup \tilde{K}_m \cup \tilde{K}_1 \cup \dots \cup \tilde{K}_{i'-1} = v'$$

are simple arcs where

$$v \cup v' = \tilde{K} \quad \text{and} \quad v \cap v' = \{q_{i'}, q_{i''}\}.$$

On the other hand if $q_{i'}=q_{i''}$ then one of the compacts $\tilde{K}_{i'} \cup \dots \cup \tilde{K}_{i''-1}$ and $\tilde{K}_{i''} \cup \dots \cup \tilde{K}_m \cup \tilde{K}_1 \cup \dots \cup \tilde{K}_{i'-1}$ is $\tilde{K}$ itself and the other is a singleton (see [1] 5.14).

7.25. Remark. Let J be a Jordan curve and let q_1 and q_2 be distinct points of J. Let v_1 and v_2 be the two simple arcs with the endpoints q_1, q_2 contained in J. Then $K=[q_1 v_1 q_2 v_2 q_1]$ is a Jordan path with the body J (see [1] 5.17).

Consequently each Jordan curve is the body of a Jordan path.

7.26. Lemma. *Let J be a Jordan curve of Y and let N be a proper closed subset of J ($N \subset J$ and $N \neq J$) with at least two points. Let W be an open subset of J containing N and different from J ($N \subset W \subset J$ and $W \neq J$). Then there exists a Jordan path of the form*

$$K = [q_1 v_{2m} q_{2m} v_{2m-1} q_{2m-1} \dots q_2 v_1 q_1]$$

such that

(a) $\tilde{K} = J$,

(b) $q_1, \dots, q_{2m} \in N$,

(c) *for $j=1, 3, \dots, 2m-1$, v_j is a simple arc and only the endpoints of these arcs belong to N,*

(d) *for $j=2, 4, \dots, 2m$, v_j is an arc and $v_j \subset W$.*

Proof. Let K'' be a Jordan path with $\tilde{K}''=J$. 7.25 shows the existence of such a K''. Let $\Omega = \{W, J \setminus N\}$. Ω is an open covering of $\tilde{K}''=J$ such that $\tilde{K}'' \not\subset W$ and $\tilde{K}'' \not\subset J \setminus N$. According to 7.16 there is a $K' \in K''^r$ and a subdivision

$$K' = K'_{2m} \dots K'_1 \tag{74}$$

such that for $i=1, \dots, m$ the relations

$$\tilde{K}'_{2i-1} \subset J \setminus N, \quad \tilde{K}'_{2i-1} \not\subset W, \quad \tilde{K}'_{2i} \subset W, \quad \tilde{K}'_{2i} \not\subset J \setminus N \tag{75}$$

hold. By 7.22, K' is a Jordan path as well.

Now for $j=1, \dots, 2m-1$ let $K'_i = K'_i \colon q'_i \to q'_{i+1}$. We then have obviously $K'_{2m} = K'_{2m} \colon q'_{2m} \to q'_1$.

It will be convenient to introduce the following notations. For $i, j \in \{1, \dots, 2m\}$ let

$$i \oplus j = i +_{2m} j \quad \text{and} \quad i \ominus j = i -_{2m} j$$

(see 3.1). In particular

$$i \oplus 1 = \begin{cases} i+1 & \text{if} \quad i < 2m \\ 1 & \text{if} \quad i = 2m. \end{cases}$$

Under these notations for $j=1, \dots, 2m$ we have

$$K'_j = K'_j \colon q'_j \to q'_{j \oplus 1}.$$

Now (75) and 7.23 show that (74) is a nontrivial subdivision of K' hence each K'_j is a simple path. Consequently for $j=1, \dots, 2m$ one has

$$K'_j = [q'_{j \oplus 1} v'_j q'_j]$$

where $v'_j = \tilde{K}'_j$ is an arc in J. Moreover we have

$$K' = [q'_1 v'_{2m} q'_{2m} \dots q'_2 v'_1 q'_1]. \tag{76}$$

Since for $j=1, \dots, 2m$ the endpoints q'_j and $q'_{j \oplus 1}$ of v'_j belong to $W \cap (J \setminus N)$ and by (75) $v'_j = \tilde{K}'_j \not\subset W \cap (J \setminus N)$ it follows that v'_j is a simple (nondegenerated) arc.

Consequently if $i, j \in \{1, \ldots, 2m\}$ and $j \notin \{i, i\oplus 1, i\ominus 1\}$ then

(77) $$v'_j \cap v'_i = \emptyset$$

(see [1] 5.13).

Now for $i=1, \ldots, m$ let $N_{2i}=N\cap v'_{2i}$. (77) shows that $N_2, \ldots, N_{2m}$ are pairwise disjoint sets and by (75) we have

$$N = \sum_{i=1}^{m} N_{2i}.$$

Since for $j=1, \ldots, m$ we have $v'_{2i} \not\subset J\setminus N$, it follows that N_{2i} is a nonempty compact subset of v'_{2i}. Let us take the natural order $<$ of the points of v'_{2i} where $q'_{2i} < q'_{2i\oplus 1}$. Let q_{2i} be the first and $q_{2i\oplus 1}$ the last element of N_{2i} with respect to this order. Thus we get three uniquely determined subarcs v''_{2i}, v_{2i} and $v''_{2i\oplus 1}$ of v'_{2i} such that the endpoints of v''_{2i} are q'_{2i} and q_{2i}, the endpoints of v_{2i} are q_{2i} and $q_{2i\oplus 1}$ and the endpoints of $v''_{2i\oplus 1}$ are $q_{2i\oplus 1}$ and $q'_{2i\oplus 1}$. Since q'_{2i} and $q'_{2i\oplus 1}$ belong to $J\setminus N$, it follows that v''_{2i} and $v''_{2i\oplus 1}$ are simple arcs but v_{2i} might degenerate. Also, observe that

(78) $$v''_{2i} \cap N = \{q_{2i}\} \quad \text{and} \quad v''_{2i\oplus 1} \cap N = \{q_{2i\oplus 1}\}.$$

Moreover for $i=1, \ldots, m$ we have by 7.20

(79) $$K'_{2i} = [q'_{2i\oplus 1} v'_{2i} q'_{2i}] = [q'_{2i\oplus 1} v''_{2i\oplus 1} q_{2i\oplus 1} v_{2i} q_{2i} v''_{2i} q'_{2i}].$$

Now for $i=1, \ldots, m$ let

(80) $$v_{2i-1} = v''_{2i} \cup v'_{2i-1} \cup v''_{2i-1}.$$

We are going to show that v_{2i-1} is a simple arc.

In fact for $i=1, \ldots, m$ let

$$v'^{2i-1} = v'_{2i} \cup \ldots \cup v'_{2m} \cup v'_1 \cup \ldots \cup v'_{2i-2}.$$

In particular $v'^1 = v'_2 \cup \ldots \cup v'_{2m}$. According to 7.24 v'^{2i-1} is a simple arc with the endpoints q'_{2i-1}, q'_{2i} and we have

$$v'_{2i-1} \cap v'^{2i-1} = \{q'_{2i-1}, q'_{2i}\}.$$

Since

(81) $$v''_{2i-1} \subset v'_{2i\ominus 2} \quad \text{and} \quad v''_{2i} \subset v'_{2i}$$

it follows $v''_{2i} \cup v''_{2i-1} \subset v'^{2i-1}$. We also have $v''_{2i} \cap v''_{2i-1} = \emptyset$. This follows obviously from (81) and (77) if $m \geqq 2$. If $m=1$ then $N_2=N$, consequently N_2 has at least two points and thus $q_2 < q_{2\oplus 1} = q_1$. Hence the subarc v''_2 of v'_2 with the endpoints q'_2 and q_2 is disjoint to the subarc $v''_{2\oplus 1} = v''_1$ of v'_2 with the endpoints q_1 and q'_1. $v''_{2i} \cap v''_{2i-1}$ is empty (for the only $i=1$) also in this case. These considerations show that v_{2i-1} is a simple arc with the two distinct endpoints q_{2i-1} and q_{2i} indeed.

Observe that for $i=1, \ldots, m$ we have by (80), (78) and (75)

(82) $$v_{2i-1} \cap N = \{q_{2i-1}, q_{2i}\}.$$

Moreover by 7.20 we obtain

(83) $$[q_{2i}v_{2i-1}q_{2i-1}] = [q_{2i}v''_{2i}q'_{2i}v'_{2i-1}q'_{2i-1}v''_{2i-1}q_{2i-1}].$$

Now by (76), (79) and (83) we get

$$K' = [q'_1v''_1q_1v_{2m}q_{2m}v''_{2m}q'_{2m}v'_{2m-1}\dots v'_3q'_3v''_3q_3v_2q_2v''_2q'_2v'_1q'_1] =$$
$$= [q'_1v''_1q_1][q_1v_{2m}q_{2m}v_{2m-1}\dots q_3v_2q_2][q_2v''_2q'_2v'_1q'_1].$$

Consider now the path

$$K = [q_1v_{2m}q_{2m}v_{2m-1}\dots q_4v_3q_3v_2q_2][q_2v''_2q'_2v'_1q'_1][q'_1v''_1q_1] =$$
$$= [q_1v_{2m}q_{2m}v_{2m-1}\dots q_3v_2q_2][q_2v_1q_1] = [q_1v_{2m}q_{2m}v_{2m-1}\dots q_3v_2q_2v_1q_1].$$

We clearly have $K\in K'^r$ and thus $K\in K''^r$ (see 7.13 and 7.14) which implies $\tilde{K}=\tilde{K}''=J$ (see also 7.15). On the other hand 7.22 shows that

$$K = [q_1v_{2m}q_{2m}v_{2m-1}q_{2m-1}\dots q_2v_1q_1]$$

is a Jordan path.

Hence the conditions (a) and (b) of the proposition are fulfilled. (82) shows that condition (c) is satisfied as well. Since for $i=1, \dots, m$ $v_{2i}\subset v'_{2i}=\tilde{K}'_{2i}$ and $\tilde{K}'_{2i}\subset W$ (see (75)) it follows that (d) is satisfied, too.

This completes the proof of the Lemma. □

8. Linking paths and curves

This section collects the notions and results about linking Jordan curves elaborated in [6]. Also, it deals with linking closed paths. In addition, we shall formulate and prove some simple facts and lemmas about these subjects.

First we recall the notions and results of [5] and [6].

8.1. By a *homology theory* we always mean a partially exact homology theory defined on the category of compact pairs. For a homology theory H and a compact space X we denote by $\tilde{H}_t(X)$ the t-dimensional homology group $H_t(X)$ for $t>0$ and the reduced zero-dimensional homology group for $t=0$. If $f\colon X\to Y$ is a continuous mapping then $\tilde{H}_t(f)$ or also $\tilde{H}(f)$ will denote the map of $\tilde{H}_t(X)$ into $\tilde{H}_t(Y)$ defined by the induced map $f_*\colon H_t(X)\to H_t(Y)$.

8.2. A mapping $\mathfrak{v}\colon A\times B\to C$ where A, B and C are abelian groups is said to be a *bihomomorphism* if the following condition is satisfied:

$$\mathfrak{v}(a+a', b+b') = \mathfrak{v}(a, b)+\mathfrak{v}(a', b)+\mathfrak{v}(a, b')+\mathfrak{v}(a', b').$$

We say that a bihomomorphism $\mathfrak{v}\colon A\times B\to C$ is *trivial* if $\mathfrak{v}(A\times B)=0$.

8.3. Now suppose we are given the n-dimensional euclidean space R^n, two homology theories H and H' an abelian group C and nonnegative integers t, t' satisfying the relation $t+t'=n-1$. A *theory of linking of compacts in R^n for the* *homology theories H and H'* is a map $\mathfrak{B}=\mathfrak{B}_{H,H',C,t,t'}$ which makes correspond a bihomomorphism

$$\mathfrak{v}_{M,M'}\colon \tilde{H}_t(M)\times\tilde{H}'_{t'}(M') \to C$$

to each ordered pair (M, M') of disjoint compact subsets of R^n, such that for any compact subsets M, M', N, N' of R^n satisfying $M\subset N$, $M'\subset N'$ and of course $N\cap N'=\emptyset$, the condition

$$\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{N,N'}\big(\tilde{H}_t(i)(u), \tilde{H}'_{t'}(i')(u')\big)$$

is satisfied for every $u\in\tilde{H}_t(M)$ and $u'\in\tilde{H}'_{t'}(M')$ where $i\colon M\subset N$ and $i'\colon M'\subset N'$ are inclusion maps. The group C is said to be the *range* of the theory $\mathfrak{B}$.

Notice as a direct consequence of this definition that if $\mathfrak{B}=\mathfrak{B}_{H,H',C,t,t'}$ is a theory of linking of compacts in R^n and M, N, M' are compacts in R^n such that $M\subset N\subset(R^n\setminus M')$ then

$$\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{N,M'}\big(\tilde{H}_t(i)(u), u'\big)$$

holds for every $u\in\tilde{H}_t(M)$ and $u'\in\tilde{H}'_{t'}(M')$ where $i\colon M\subset N$ is the inclusion map.

Likewise holds the relation

$$\mathfrak{v}_{M,M'}(u, u') = \mathfrak{v}_{M,N'}\big(u, \tilde{H}'_{t'}(i')(u')\big)$$

for every $u\in\tilde{H}_t(M)$ and $u'\in\tilde{H}'_{t'}(M')$ where $i'\colon M'\subset N'$ is the inclusion map and $N'\subset(R^n\setminus M)$.

8.4. We shall say that the theory of linking $\mathfrak{B}=\mathfrak{B}_{H,H',C,t,t'}$ of compacts in R^n is *degenerate* if for every nonintersecting compact subsets M, M' of R^n, $\mathfrak{v}_{M,M'}$ is a trivial bihomomorphism.

8.5. We say that the spheres S and S' of dimension t and t' respectively contained in R^n are *mutually linked* if the following conditions are satisfied:

(a) $t+t'=n-1$,

(b) the center of S belongs to S' and the center of S' belongs to S,

(c) the planes R and R' supporting S and S' respectively intersect in a line,

(d) R and R' are perpendicular in the natural sense that any vectors a in R and a' in R' which are perpendicular to the line $R\cap R'$ are mutually perpendicular.

And now the uniqueness and existence theorem of the theory of linking reads as follows:

Let H and H' be any homology theories. Let C be an abelian group. Let t, t' be nonnegative integers such that $t+t'=n-1$ and let S, S' be mutually linked spheres in R^n of dimension t and t' respectively. Then for every bihomomorphism

$$\mathfrak{v}_0 = \mathfrak{v}_0\colon \tilde{H}_t(S)\times H'_{t'}(S') \to C$$

there exists one and only one theory of linking

$$\mathfrak{B} = \mathfrak{B}_{H,H',C,t,t'}$$

of compacts in R^n which takes the value $\mathfrak{v}_0$ on the pair (S, S').

8.6. Let p be a prime number and G' an elementary cyclic p-group, i.e. G' is isomorphic to the additive group Z_p of integers mod p. Let H' be a continuous homology theory defined on the category $\mathcal{A}_c$ of all compact pairs and based on the coefficient group G'. Then H' is isomorphic on $\mathcal{A}_c$ to the Čech homology theory over G'. We shall keep it fixed in the sequel.

Let R^n be the n-euclidean space where $n \geqq 2$ and let P be a compact subspace of R^n.

Let H be a homology theory and let u be an element of $\tilde{H}_{n-2}(P)$. Let C be an abelian group and let $\mathfrak{V} = \mathfrak{V}_{H,H',C,n-2,1}$ be a theory of linking of compacts in R^n.

DEFINITION. A Jordan curve J of $R^n \setminus P$ is said to be a *linking Jordan curve of* u (with respect to $\mathfrak{V} = \mathfrak{V}_{H,H',C,n-2,1}$) if there is a $u' \in H_1'(J)$ such that $\mathfrak{v}_{P,J}(u, u') \neq 0$.

8.7. REMARK. Let J be a linking Jordan curve of u. Then for each nonzero element u_1' of $H_1'(J)$ we have

$$\mathfrak{v}_{P,J}(u, u_1') \neq 0$$

(see [6] 2.2).

8.8. THEOREM. *Let p and H' be the same as in* 8.6. *Suppose that G is an elementary cyclic p-group and H is a continuous homology theory defined on the category of all compact pairs and based on the coefficient group G.* (We shall keep it fixed in the sequel, too.) *Let C be an abelian group. Suppose that* $\mathfrak{V} = \mathfrak{V}_{H,H',C,n-2,1}$ *is a nondegenerate theory of linking of compacts in R^n where $n \geqq 2$. Let P be a compact set in R^n and u_0 a nonzero element of $\tilde{H}_{n-2}(P)$. Then there exists a linking Jordan curve of u_0 with respect to* $\mathfrak{V}$ (see [6] 2.3).

8.9. We now show the existence of a nondegenerate theory of linking of the required type.

First observe that there exist continuous homology theories H and H' defined on the category of compact pairs such that their coefficient groups are isomorphic to Z_p. (The coefficient group of a homology theory H is $H_0(P_0)$ where P_0 is the one point space with the only element $\emptyset$, i.e. $P_0 = \{\emptyset\}$.)

In fact, let $H = H' = H^1(\cdot, Z_p)$ where $H^1(\cdot, Z_p)$ is the Čech homology theory over Z_p on the category of compact pairs.

Observe that since the coefficient groups of H and H' can be considered as compact LCA-groups it follows that the homology theories H and H' are exact (see [8] p. 248).

Next, for $n \geqq 2$ there are mutually linked spheres S and S' in R^n of dimensions $n-2$ and 1 respectively (see 8.5). Now both of the groups $\tilde{H}_{n-2}(S)$ and $\tilde{H}_1'(S') = H_1'(S')$ are isomorphic to Z_p. Let Z_p be the range of the theory in question. Let u, u' and u'' be generators of $\tilde{H}_{n-2}(S)$, $H_1'(S')$ and Z_p respectively. Now for the rational integers k, k' let

$$\mathfrak{v}_0(ku, k'u') = kk'u'' \in Z_p.$$

Then the mapping

$$\mathfrak{v}_0\colon \tilde{H}_{n-2}(S) \times H_1'(S') \to Z_p$$

is well defined and it is a nontrivial bihomomorphism. Hence by the uniqueness and existence theorem of the theory of linking there is a theory of linking $\mathfrak{V} = \mathfrak{V}_{H,H',Z_p,n-2,1}$ of compacts in R^n which takes the value $\mathfrak{v}_0$ on the ordered pair

(S, S'). $\mathfrak{B}$ is clearly a nondegenerate theory of linking. We shall keep it fixed in the remainder of this section. Hence R^n will be kept fixed, too.

8.10. Let (X, A) be a compact pair in R^n homeomorphic to (E^{n-1}, S^{n-2}) where E^{n-1} is an $(n-1)$-ball in R^{n-1} and S^{n-2} is the boundary sphere of E^{n-1}. Such a compact pair (X, A) will be called an $(n-1)$-*cell* in R^n. Let $\psi\colon E^{n-1}\to X$ be a homeomorphism. Then $\psi(S^{n-2})=A$.

The compact pair (X, A) as well as E^{n-1}, S^{n-2}, R^{n-1} and the homeomorphism $\psi\colon E^{n-1}\to X$ will be kept fixed in this section.

8.11. DEFINITION. A Jordan curve J lying in $R^n\setminus A$ is said to be a *linking Jordan curve* of A (with respect to $\mathfrak{B}$) if there is a $u\in\tilde{H}_{n-2}(A)$ such that J is a linking Jordan curve of u (see 8.6 and 8.8).

8.12. REMARK. Let J be a linking Jordan curve of A. Then for each nonzero element u_1 of $\tilde{H}_{n-2}(A)$, J is a linking Jordan curve of u_1.

Indeed, select $u\in\tilde{H}_{n-2}(A)$ and $u'\in H_1'(J)$ such that $\mathfrak{v}_{A,J}(u, u')\neq 0$. Definitions 8.11 and 8.6 show the existence of such elements u and u'. Since $\tilde{H}_{n-2}(A)$ is isomorphic to $\tilde{H}_{n-2}(S^{n-2})$ and thus to Z_p it follows the existence of an integer m such that $u=mu_1$. Consequently

$$0\neq \mathfrak{v}_{A,J}(u, u') = m\mathfrak{v}_{A,J}(u_1, u')$$

and this implies $\mathfrak{v}_{A,J}(u_1, u')\neq 0$. Hence J is a linking Jordan curve of u_1 indeed.

8.13. Let ϱ be the metric in R^n. This metric will be kept fixed in this section.

Now we are going to formulate and to prove a lemma about linking Jordan curves.

LEMMA. *Let J be a linking Jordan curve of A and let $\varepsilon=\varrho(A, J)$. Let (X', A') be an $(n-1)$-cell in R^n as well and suppose that there is a topological mapping $f'\colon A\to A'$ such that for each $y\in A$, $\varrho(y, f'(y))<\varepsilon$. Then J is a linking Jordan curve of A', too.*

PROOF. Let

$$W = \bigcup_{y\in A} [y, f'(y)]$$

where for $y\in A$ $[y, f'(y)]$ is the segment in R^n joining y and $f'(y)$. Then W is clearly a compact subset of R^n and we have $(A\cup A')\subset W$ and $W\cap J=\emptyset$. Select $u\in\tilde{H}_{n-2}(A)$ and $u'\in H_1'(J)$ such that $\mathfrak{v}_{A,J}(u, u')\neq 0$. Since J is a linking Jordan curve of A, there exist such u and u'.

Let $i_1\colon A\subset W$ and $i_2\colon A'\subset W$ be inclusions. Then $i_1\colon A\to W$ and $i_2 f'\colon A\to W$ are clearly homotopic maps and thus

$$\tilde{H}(i_1)(u) = \tilde{H}(i_2)\tilde{H}(f')(u).$$

Consequently

$$\mathfrak{v}_{A,J}(u, u') = \mathfrak{v}_{W,J}(\tilde{H}(i_1)(u), u') = \mathfrak{v}_{W,J}(\tilde{H}(i_2)\tilde{H}(f')(u), u') = \mathfrak{v}_{A',J}(\tilde{H}(f')(u), u')$$

(see 8.3) and thus $\mathfrak{v}_{A',J}(\tilde{H}(f')(u), u')\neq 0$.

J is a linking Jordan curve of A' indeed. □

8.14. REMARK. Let J be a Jordan curve lying in $X\setminus A$. Then J is a nonlinking Jordan curve of A.

In fact, since $J\subset(X\setminus A)$ it follows $\psi^{-1}(J)\subset(E^{n-1}\setminus S^{n-2})$ (cf. 8.10). Let E'^{n-1} be an $(n-1)$-ball lying in $E^{n-1}\setminus S^{n-2}$ and containing $\psi^{-1}(J)$. Let $N'=$ $=\psi(E'^{n-1})$.

N' is a homologically trivial compact set containing J and lying in $X\setminus A$. Let i': $J\subset N'$ be the inclusion map. Let u and u' be arbitrary elements of $\tilde{H}_{n-2}(A)$ and $H_1'(J)$ respectively. Since N' is homologically trivial it follows that $\tilde{H}_1'(i')(u')=0$. Consequently

$$\mathfrak{v}_{A,J}(u, u') = \mathfrak{v}_{A,N'}(u, \tilde{H}_1'(i')(u')) = 0$$

which proves the assertion.

8.15. The preceding remark shows that for each linking Jordan curve J of A we have $J\setminus X\neq\emptyset$.

We now prepare the definition of a linking path of A.

8.16. First recall the existence of an important function described in [4] 11.

Let $Y={}^nR\setminus A$. Then there is a function $*$ which makes correspond to each continuous closed path K of Y (see 7.2 and 7.6) an element K_* of $H_1'(\tilde{K})$ (cf. 7.3) satisfying the following conditions:

(a) For each continuous path K_1 of Y (K_1 need not be closed) $(K_1K_1^{\bullet})_*=0$ (cf. 7.4, 7.5 and [4] 14).

(b) If K_1 and K_2 are continuous paths in Y such that both of the products K_1K_2 and K_2K_1 exist (and thus K_1K_2 and K_2K_1 are closed paths), then $(K_1K_2)_*=$ $=(K_2K_1)_*$ (cf.]4: 15).

(c) If K_1 and K_2 are continuous closed paths of Y with the same base point and $K=K_1K_2$ then

$$i_{1*'}(K_{1*})+i_{2*'}(K_{2*}) = K_*$$

where for $j=1, 2$ $i_{j*'}$: $H_1'(\tilde{K}_j)\to H_1'(\tilde{K})$ is the homomorphism induced by the inclusion i_j: $\tilde{K}_j\subset\tilde{K}$ (cf. [4] 16).

(d) If K is a Jordan path of Y (see 7.7) then $K_*\neq 0$ (cf. [4] 13).

We keep fixed this function $*$ in the sequel.

And now we turn to the definition of the linking path.

8.17. DEFINITION. A continuous closed path K of $R^n\setminus A$ is said to be a *linking (closed) path of* A (with respect to $\mathfrak{V}$) if there is a $u\in\tilde{H}_{n-2}(A)$ such that $\mathfrak{v}_{A,\tilde{K}}(u, K_*)\neq 0$ (cf. also 8.16).

8.18. REMARK. Let K be a linking closed path of A. Then for each nonzero element u_1 of $\tilde{H}_{n-2}(A)$ we have $\mathfrak{v}_{A,\tilde{K}}(u_1, K_*)\neq 0$.

In fact, let u be an element of $\tilde{H}_{n-2}(A)$ such that $\mathfrak{v}_{A,\tilde{K}}(u, K_*)\neq 0$. Definition 8.17 shows the existence of such a u. Then there exists an integer m such that $u=mu_1$ (see 8.12). Hence

$$0 \neq \mathfrak{v}_{A,\tilde{K}}(u, K_*) = m\mathfrak{v}_{A,\tilde{K}}(u_1, K_*)$$

and this implies $\mathfrak{v}_{A,\tilde{K}}(u_1, K_*)\neq 0$ indeed.

8.19. Remark. The function $*$ described in [4] which makes correspond to each closed path K of $R^n \setminus A$ an element K_* of $H_1'(\tilde{K})$ is not uniquely defined. However, if $*$ and $*'$ are such functions described in [4] then there is an integer m such that for each closed path K in $X \setminus A$ we have $K_* = mK_{*'}$ (see [4] 12). Consequently for $u \in \tilde{H}_{n-2}(A)$ the relation $\mathfrak{v}_{A,\tilde{K}}(u, K_*) \neq 0$ implies

$$0 \neq \mathfrak{v}_{A,\tilde{K}}(u, mK_{*'}) = m\mathfrak{v}_{A,\tilde{K}}(u, K_{*'})$$

and thus $\mathfrak{v}_{A,\tilde{K}}(u, K_{*'}) \neq 0$, i.e. the fact that K is a linking closed path of A fails to depend on the special choice of the function $*$.

8.20. Remark. For any Jordan path K lying in $R^n \setminus A$, K is a linking closed path of A iff $\tilde{K}$ is a linking Jordan curve of A.

In fact, let $0 \neq u \in \tilde{H}_{n-2}(A)$. By 8.16 (d) we have $K_* \neq 0$. Hence according to 8.12, $\tilde{K}$ is a linking Jordan curve of A iff $\mathfrak{v}_{A,\tilde{K}}(u, K_*) \neq 0$ and by 8.18 this relation holds iff K is a linking closed path of A.

8.21. Remark. Let K be a closed path lying in an open ball $S(q, \varepsilon)$ contained in $R^n \setminus A$ i.e. $\tilde{K} \subset S(q, \varepsilon) \subset R^n \setminus A$. Then K is a nonlinking closed path of A.

In fact, let $0 \neq u \in \tilde{H}_{n-2}(A)$ and let Y' be a closed ball in $S(q, \varepsilon)$ containing $\tilde{K}$. Such a ball obviously exists. Let $i' \colon \tilde{K} \subset Y'$ be the inclusion map. Since Y' is a homologically trivial compact space and thus $H_1'(Y') = 0$, we get $\tilde{H}_1'(i')(K_*) = 0$ whence

$$\mathfrak{v}_{A,\tilde{K}}(u, K_*) = \mathfrak{v}_{A,Y'}(u, \tilde{H}_1'(i')(K_*)) = \mathfrak{v}_{A,Y'}(u, 0) = 0$$

indeed.

8.22. Remark. Let J be a Jordan curve lying in an open ball contained in $R^n \setminus A$. Then J is a nonlinking Jordan curve of A.

In fact let K be a Jordan path with $J = \tilde{K}$ (see 7.25). The assertion is now an immediate consequence of 8.20 and 8.21.

8.23. Remark. Let K be a continuous closed path of $R^n \setminus X$. Then K is a nonlinking closed path of A.

In fact, let $u \in \tilde{H}_{n-2}(A)$ and let $i \colon A \subset X$ be the inclusion map. Since X is homeomorphic to the closed ball E^{n-1} it is contractible to a point over itself consequently it is homologically trivial and thus $\tilde{H}_{n-2}(X) = 0$. Hence

$$\mathfrak{v}_{A,\tilde{K}}(u, K_*) = \mathfrak{v}_{X,\tilde{K}}(\tilde{H}(i)(u), K_*) = \mathfrak{v}_{X,\tilde{K}}(0, K_*) = 0.$$

K is a nonlinking closed path of A indeed (cf. 8.17).

8.24. Remark. Let J be a linking Jordan curve of A. Then $J \cap X \neq \emptyset$.

In fact, let K be a Jordan path with $\tilde{K} = J$ (see 7.25). Then by 8.20, K is a linking path of A and thus by 8.23 we have

$$J \cap X = \tilde{K} \cap X \neq \emptyset.$$

We now recall from [3] the definition of the category homomorphism.

8.25. Definition. Let $\mathscr{C}^\bullet$ be an i-category (cf. 7.1) and M an abelian group. A relation $\eta \colon \mathscr{C}^\bullet \to M$ which maps the class of the closed morphisms of $\mathscr{C}^\bullet$ into M is called a *category homomorphism* if it satisfies the conditions

(i) for each $\alpha \in \operatorname{Mor} \mathscr{C}^{\bullet}$ we have $\eta(\alpha\alpha^{\bullet})=0$,

(ii) if $\alpha_1, \alpha_2 \in \operatorname{Mor} \mathscr{C}^{\bullet}$ and $\alpha_1\alpha_2$ is defined and it is a closed morphism then

$$\eta(\alpha_1\alpha_2) = \eta(\alpha_2\alpha_1),$$

(iii) if α_1, α_2 are closed morphisms of $\mathscr{C}^{\bullet}$ with the same domain then

$$\eta(\alpha_1\alpha_2) = \eta(\alpha_1)+\eta(\alpha_2).$$

8.26. Remark. Let $\mathscr{C}^{\bullet}$ be an i-category, M an abelian group and $\eta\colon \mathscr{C}^{\bullet} \to M$ a category homomorphism. Let $\alpha\colon A \to A$ be a closed morphism of $\mathscr{C}^{\bullet}$. Then $\eta(\alpha^{\bullet}) = -\eta(\alpha)$.

In fact, the properties (iii) and (i) of 8.25 imply

$$\eta(\alpha)+\eta(\alpha^{\bullet}) = \eta(\alpha\alpha^{\bullet}) = 0$$

indeed.

8.27. Let $0 \neq u \in \tilde{H}_{n-2}(A)$ and let $Y=R^n \setminus A$. Now to each continuous closed path K of Y we can assign the element $\mu(K)=\mathfrak{v}_{A,\tilde{K}}(u, K_*)$ of the group Z_p where Z_p is the range of the theory of linking $\mathfrak{B}=\mathfrak{B}_{H,H',Z_p,n-2,1}$ (see 8.9). 8.18 shows that K is a nonlinking path of A if and only if $\mu(K)=0$. Moreover 8.16 (a) and 8.16 (b) show that the relation $\mu\colon \mathscr{K}_Y^{\bullet} \to Z_p$ satisfies the conditions 8.25 (i) and 8.25 (ii). It satisfies 8.25 (iii) as well.

In fact, let K_1 and K_2 be closed continuous closed paths of Y with the same base point and let $K=K_1K_2$. For $j=1,2$ let $i_j\colon \tilde{K}_j \subset \tilde{K}$ be the respective inclusion map. Then taking also 8.16 (c) into account we get

$$\mu(K_1K_2) = \mathfrak{v}_{A,\tilde{K}}(u, K_*) = \mathfrak{v}_{A,\tilde{K}}(u, i_{1*'}(K_{1*})+i_{2*'}(K_{2*})) =$$

$$= \mathfrak{v}_{A,\tilde{K}}(u, i_{1*'}(K_{1*}))+\mathfrak{v}_{A,\tilde{K}} u, i_{2*'}(K_{2*})) = v_{A,\tilde{K}_1}(u, K_{1*})+v_{A,\tilde{K}_2}(u, K_{2*}) = \mu(K_1)+\mu(K_2)$$

Hence the relation $\mu\colon \mathscr{K}_Y^{\bullet} \to Z_p$ is a category homomorphism.

We shall keep fixed this category homomorphism in the following two sections.

8.28. Remark. Let K be an arbitrary continuous path of $R^n \setminus A = Y$ (K need not be closed). Then by 8.25 (i) we have $\mu(KK^{\bullet})=0$ and thus $KK^{\bullet}$ is a nonlinking closed path of A.

8.29. Lemma. *Let $q_1, \ldots, q_m$ be points of $R^n \setminus A$. For $i=1, \ldots, m-1$ let $K_i\colon q_i \to q_{i+1}$ and $K_i'\colon q_i \to q_{i+1}$ be continuous paths in $Y=R^n \setminus A$. Also, let $K_m\colon q_m \to q_1$ and $K_m'\colon q_m \to q_1$ be continuous paths in $Y=R^n \setminus A$. Let*

$$K = K_m \ldots K_2K_1 \quad \text{and} \quad K' = K_m' \ldots K_2'K_1'.$$

Suppose that for $i=1, \ldots, m$ the closed path $K_i^{\bullet}K_i'\colon q_i \to q_i$ is a nonlinking path of A and that $K'\colon q_1 \to q_1$ is a nonlinking path of A as well. Then $K\colon q_1 \to q_1$ is a nonlinking path of A.

Proof. Let $Y=R^n \setminus A$ and let u be a nonzero element of $\tilde{H}_{n-2}(A)$. Consider the category homomorphism $\mu\colon \mathscr{K}_Y^{\bullet} \to Z_p$ described in 8.27 i.e. let $\mu(K'')=$

$=\mathfrak{v}_{A,K''}(u, K''_*)$ for any closed path K'' of Y. Since K', $K_1^{\bullet}K_1'$, ..., $K_m^{\bullet}K_m'$ are nonlinking closed paths of A we get

$$\mu(K') = \mu(K_1^{\bullet}K_1') = \ldots = \mu(K_m^{\bullet}K_m') = 0. \tag{84}$$

Taking also 8.18 into account we only need to prove that $\mu(K)=0$.

We proceed by induction. First let $m=1$. Then $K=K_1$ and $K'=K_1'$ are closed paths with the same base point q_1. Hence by 8.26, (84) and 8.25 (iii) we obtain

$$\mu(K) = -\mu(K^{\bullet}) = -\big(\mu(K^{\bullet})+\mu(K')\big) = -\mu(K^{\bullet}K') = -\mu(K_1^{\bullet}K_1') = 0$$

as required.

Now suppose that $m>1$ and that the assertion is true if we replace m by $m-1$. Let

$$\overline{K}_{m-1} = K_m K_{m-1}\colon\ q_{m-1} \to q_1, \quad \overline{K}'_{m-1} = K'_m K'_{m-1}\colon\ q_{m-1} \to q_1$$

and for $i=1, \ldots, m-2$ let $\overline{K}_i=K_i$ and $\overline{K}_i'=K_i$. We then have

$$K = \overline{K}_{m-1}\overline{K}_{m-2} \ldots \overline{K}_1 \quad \text{and} \quad K' = \overline{K}'_{m-1}\overline{K}'_{m-2} \ldots \overline{K}'_1. \tag{85}$$

On the other hand, taking also 8.25 (ii) and 8.25 (iii) into account, by (84) we get

$$\mu(\overline{K}^{\bullet}_{m-1}\overline{K}'_{m-1}) = \mu(K^{\bullet}_{m-1}K^{\bullet}_m K'_m K'_{m-1}) = \mu(K'_{m-1}K^{\bullet}_{m-1}K^{\bullet}_m K'_m) =$$

$$= \mu(K'_{m-1}K^{\bullet}_{m-1})+\mu(K^{\bullet}_m K'_m) = \mu(K^{\bullet}_{m-1}K'_{m-1})+\mu(K^{\bullet}_m K'_m) = 0.$$

Moreover, in the case $m\geqq 3$ for $i=1, \ldots, m-2$ we have $\mu(\overline{K}_i^{\bullet}\overline{K}_i')=\mu(K_i^{\bullet}K_i')=0$. Consequently, by (85) and by the induction hypothesis we obtain $\mu(K)=0$ indeed. □

8.30. Corollary. *Let y and y' be distinct points of $R^n\setminus A$ and let v, v' and v'' be simple arcs in $R^n\setminus A$ all with the endpoints y and y'. Suppose that $[yvy'][y'v'y]$ (see 7.17) and $[y'v''y][yv'y']$ are nonlinking closed paths of A. Then $[yv''y'][y'vy]$ is a nonlinking path of A as well.*

In fact, let

$$K_1 = [y'vy], \quad K_2 = [yv''y'],$$

$$K_1' = [y'v'y] \quad \text{and} \quad K_2' = [yv'y'].$$

Then by assumption $K_1^{\bullet}K_1'$ and $K_2^{\bullet}K_2'$ are nonlinking paths of A. However $K_1'=(K_2')^{\bullet}$ (see 7.21 (73)) and thus by 8.28, $K_2'K_1'=K_2'(K_2')^{\bullet}$ is a nonlinking closed path of A as well. Consequently by Lemma 8.29

$$K_2K_1 = [yv''y'][y'vy]$$

is a nonlinking path of A indeed.

Now we are going to prepare the main lemma of this section.

8.31. Lemma. *Let V be a nonempty subset of $X\setminus A$ which is open in X. Then there exists a Jordan curve J in $R^n\setminus A$ such that J is a linking Jordan curve of A and $J\cap X=J\cap V$.*

PROOF. Let $0 \neq u \in \tilde{H}_{n-2}(A)$.

Since $\psi^{-1}(X \setminus A) = E^{n-1} \setminus S^{n-2}$ (see 8.10) it follows $\psi^{-1}(V) \subset E^{n-1} \setminus S^{n-2}$. $\psi^{-1}(V)$ is clearly a nonempty open set in R^{n-1}. Let $q' \in \psi^{-1}(V)$ and let $S(q', \varepsilon)$ be an open ball in R^{n-1} around q' lying in $\psi^{-1}(V)$. Let $W = \psi(S(q', \varepsilon))$. We then have $W \subset V$.

S^{n-2} is a deformation retract of $E^{n-1} \setminus S(q', \varepsilon)$. Hence A is a deformation retract of the compact set $X \setminus W$. Thus for the inclusion map $i\colon A \subset (X \setminus W)$ the induced homomorphism $i_*\colon H_{n-2}(A) \to H_{n-2}(X \setminus W)$ is an isomorphism (see [8] p. 30). Consequently we have $\tilde{H}_{n-2}(i)(u) = i_*(u) \neq 0$ and thus by Theorem 8.8 and 8.9, there exists a linking Jordan curve J of $\tilde{H}_{n-2}(i)(u)$ with respect to $\mathfrak{V}$.

Let $0 \neq u' \in H_1'(J)$. Since J is a linking Jordan curve of $\tilde{H}_{n-2}(i)(u)$ taking also 8.7 into account it follows

$$\mathfrak{v}_{A,J}(u, u') = \mathfrak{v}_{X \setminus W, J}(\tilde{H}_{n-2}(i)(u), u') \neq 0.$$

Hence J is a linking Jordan curve of A and since $J \cap X \subset W$ we get

$$J \cap X = J \cap W = J \cap V$$

indeed. $\square$

Now we formulate the main lemma of this section.

8.32. LEMMA. *Let V be a nonvoid subset of $X \setminus A$ which is open in X. Then there is a Jordan curve J' in $R^n \setminus A$ such that*

(i) *J' is a linking Jordan curve of A,*

(ii) *$J' \cap X = J' \cap V$,*

(iii) *$J' \cap X$ is connected.*

PROOF. Let $q \in V$. Let G be a spherical neighbourhood of q lying in $R^n \setminus A$. Let V_1 be a subset of $X \setminus A$ homeomorphic to $E^{n-1} \setminus S^{n-2}$ open in $X \setminus A$ and satisfying the condition $q \in V_1 \subset V \cap G$. Such a V_1 clearly exists. Let J be a Jordan curve in $R^n \setminus A$ such that J is a linking Jordan curve of A and $J \cap X = J \cap V_1$. By 8.31 there exists such a J. If $J \cap V_1 = J \cap X$ is connected then J is of the required type. Hence we can suppose that $N = J \cap X = J \cap V_1$ is nonconnected. Consequently taking also 8.24 into account, N consists of at least two points. On the other hand 8.15 shows that $N \neq J$.

Let $W = G \cap J$. W is an open subset of J. Since $V_1 \subset G$ and $N = J \cap V_1$ it follows $N \subset W$. 8.22 shows that $J \not\subset G$ and thus $W \neq J$.

Let us take a Jordan path of the form

$$K = [q_1 v_{2m} q_{2m} v_{2m-1} q_{2m-1} \dots q_2 v_1 q_1]$$

satisfying the conditions (a), (b), (c), (d) of 7.26. By 7.26 there exists such a K. Since $\tilde{K} = J$ and J is a linking Jordan curve of A it follows by 8.20 that K is a linking closed path of A.

Now for $i = 1, \dots, 2m-1$ let v_i' be an arc in V_1 with the endpoints q_i and q_{i+1} and let v_{2m}' be an arc in V_1 with the endpoints q_{2m} and q_1. Since $N = J \cap V_1 \subset V_1$ and V_1 is homeomorphic to $E^{n-1} \setminus S^{n-2}$ it follows the existence of such arcs $v_1', \dots, v_{2m}'$. Let

$$K' = [q_1 v_{2m}' q_{2m} \dots q_2 v_1' q_1].$$

Since $\tilde{K}' \subset V_1 \subset G$ it follows by 8.21 that K' is a nonlinking closed path of A.

Now for $i=1, \ldots, 2m-1$ let

$$K_i = [q_{i+1} v_i q_i]\colon q_i \to q_{i+1}, \quad K_i' = [q_{i+1} v_i' q_i]\colon q_i \to q_{i+1}$$

and let

$$K_{2m} = [q_1 v_{2m} q_{2m}]\colon q_{2m} \to q_1 \quad \text{and} \quad K_{2m}' = [q_1 v_{2m}' q_{2m}]\colon q_{2m} \to q_1.$$

Thus $K = K_{2m} \ldots K_2 K_1$ and $K' = K_{2m}' \ldots K_2' K_1'$.

Now for $j=2, 4, \ldots, 2m$ we have

$$\widetilde{K_j^{\bullet} K_j'} = \tilde{K}_j \cup \tilde{K}_j' = v_j \cup v_j' \subset (W \cup V_1) \subset G,$$

consequently by 8.21, $K_j^{\bullet} K_j'$ is a nonlinking closed path of A. Thus there is an $i \in \{1, \ldots, m\}$ such that

$$K_{2i-1}^{\bullet} K_{2i-1}'\colon q_{2i-1} \to q_{2i-1}$$

is a linking closed path of A since otherwise by 8.29, K would be a nonlinking closed path of A. Consider such a linking closed path $K_{2i-1}^{\bullet} K_{2i-1}'$ of A.

v_{2i-1} is a simple arc with the endpoints q_{2i-1}, q_{2i} and thus v_{2i-1}' is a simple arc with the same endpoints. On the other hand we have $v_{2i-1}' \subset V_1 \subset X$ and

$$v_{2i-1} \cap X = v_{2i-1} \cap J \cap X = v_{2i-1} \cap N = \{q_{2i-1}, q_{2i}\}.$$

Consequently $v_{2i-1} \cup v_{2i-1}'$ is a Jordan curve J' where

$$K_{2i-1}^{\bullet} K_{2i-1}' = [q_{2i-1} v_{2i-1} q_{2i} v_{2i-1}' q_{2i-1}]$$

is a Jordan path with the body J' (see 7.25). Thus 8.20 shows that J' is a linking Jordan curve of A.

On the other hand the set $J' \cap X = v_{2i-1}'$ is connected and since $v_{2i-1}' \subset V_1 \subset V$ we have $J' \cap X = J' \cap V$. Thus J' satisfies all the requirements (i), (ii) and (iii) of the assertion. The proof of the Lemma is complete. □

9. The main existence theorem

This section concerns systems of n-bricks lying in R^{n+1} i.e. where the bodies of the systems are topological subspaces of the $(n+1)$-euclidean space R^{n+1}. Systems of such kind determine local orderings on their bases. This fact will be prepared in this section by an important existence theorem.

Let n be a positive integer and let (Y, D, f) be a system of n-bricks (see 5.1) such that Y is a subspace of R^{n+1}. Let (B^n, φ) be a coordinate pair of (Y, D, f) (see 5.1). We shall keep them fixed in this section.

9.1. Notations. B^n is an n-brick in R^n. Let A^{n-1} be the boundary of B^n.

For $q \in D$ denote $f^{-1}(\{q\}) = \varphi(B^n \times \{q\})$ (see 5.1 (26)) by B_q. Let $A_q = \varphi(A^{n-1} \times \{q\})$. A_q is clearly independent of the special choice of the coordinate pair (B^n, φ). A_q is said to be the *boundary* of B_q. (B_q, A_q) is clearly an n-cell in R^{n+1} (cf. 8.10).

Observe that B^n is the closure of $B^n \setminus A^{n-1}$ in B^n and thus for $q \in D$ B_q is the closure of $B_q \setminus A_q$ in B_q.

Let R^+ be the set of positive reals.

For the points $x, y \in R^{n+1}$ let $\varrho(x, y)$ denote the distance between them.

9.2. Definition. The ordered pair (U, V) of open sets U, V in R^{n+1} is called *regular* (with respect to (Y, D, f)) if $V \subset U$ and if for each $q \in D$ $B_q \cap V \neq \emptyset$ implies that $B_q \cap V$ is confined to a component of $U \cap B_q$ (see also [2]).

9.3. Remark. Let (U, V) be a regular pair with respect to (Y, D, f) and let q, q' be distinct elements of $f(V \cap Y)$. Then $B_q \cap V$ is contained in a component of $U \setminus B_{q'}$.

In fact $B_q = f^{-1}(\{q\})$ and $B_{q'} = f^{-1}(\{q'\})$ are disjoint sets, consequently the component C' of $U \cap B_q$ containing $V \cap B_q$ lies in $U \setminus B_{q'}$ and thus C' is contained in a component of $U \setminus B_{q'}$. Hence $B_q \cap V$ is contained in the same component of $U \setminus B_{q'}$.

9.4. Definition. The ordered pair (U, V) of open sets U, V in R^{n+1} is said to be *normal* (with respect to (Y, D, f)) if it is regular and for any three pairwise distinct $q_1, q_2, q_3 \in D$ where $B_{q_1} \cap V \neq \emptyset$, $B_{q_2} \cap V \neq \emptyset$ and $B_{q_3} \cap V \neq \emptyset$ there is a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ such that

(a) $(B_{q_{i_1}} \cup B_{q_{i_2}}) \cap V$ is confined to a component of $U \setminus B_{q_{i_3}}$,

(b) $(B_{q_{i_2}} \cup B_{q_{i_3}}) \cap V$ is confined to a component of $U \setminus B_{q_{i_1}}$,

(c) $B_{q_{i_1}} \cap V$ and $B_{q_{i_3}} \cap V$ lie in distinct components of $U \setminus B_{q_{i_2}}$.

Now the existence theorem mentioned at the very beginning of this section reads as follows.

9.5. Theorem. *Let $q_0 \in D$ and $y_0 \in B_{q_0} \setminus A_{q_0}$. Let U_1 be an open neighbourhood of y_0 in R^{n+1}. Then there is a normal pair (U, V) with respect to (Y, D, f) such that $y_0 \in V \subset U \subset U_1$.*

We now prepare the proof of this theorem by a preliminary remark.

9.6. Let X be a compact space, Y' an arbitrary topological space and Z a metric space. The metric in Z will be denoted by ϱ'. Let $\psi \colon X \times Y' \to Z$ be a continuous mapping. Let $q \in Y'$ and $\varepsilon \in R^+$. Then there is a neighbourhood W of q in Y' such that for each $x \in X$ and $q' \in W$ we have

$$\varrho'(\psi(x, q), \psi(x, q')) < \varepsilon.$$

In fact, denoting by $S(z, \eta)$ the open ball in Z of radius η and center z, select for each $x \in X$ the open neighbourhoods U_x of x in X and $V(x)$ of q in Y' such that

$$\psi(U_x \times V(x)) \subset S\left(\psi(x, q), \frac{\varepsilon}{2}\right).$$

Then for each $x' \in U_x$ and $q' \in V(x)$ we clearly have

$$\varrho'(\psi(x', q), \psi(x', q')) < \varepsilon.$$

Selecting now a finite subcovering $\{U_{x_1}, \ldots, U_{x_k}\}$ of the open covering $\{U_x;\ x\in X\}$ of the compact space X, the neighbourhood $W=V(x_1)\cap\ldots\cap V(x_k)$ of q fulfils the requirement.

Now we are going to the proof of Theorem 9.5.

9.7. Let p and H' be the same as in 8.6. Let H be the same as in 8.8. Let $\mathfrak{B}=\mathfrak{B}_{H,H',Z_p,n-1,1}$ be a nondegenerate theory of linking of compacts in R^{n+1}. 8.9 shows the existence of such a theory $\mathfrak{B}$.

In the remainder of this section we shall keep fixed the number p, the homology theories H and H' and the theory of linking $\mathfrak{B}$.

9.8. For any $\eta\in R^+$ denote by $U(\eta)$ the open ball in R^{n+1} with center y_0 and radius η $(y_0\in B_{q_0}\setminus A_{q_0}$ (see 9.5)).

Now select $\varepsilon_1\in R^+$ such that

$$\varepsilon_1 < \frac{1}{3}\varrho(y_0, A_{q_0}) \tag{86}$$

and

$$U(2\varepsilon_1) \subset U_1. \tag{87}$$

U_1 is an open neighbourhood of y_0 in R^{n+1} (see 9.5). Let

$$U = U(2\varepsilon_1). \tag{88}$$

Let $x_0\in B^n$ be defined by the relation $\varphi(x_0, q_0)=y_0$ (see also the very beginning of this Section 9). Then $x_0\in B^n\setminus A^{n-1}$ (see also 9.1).

Let Z be an open ball in $B^n\setminus A^{n-1}\subset R^n$ with center x_0 such that

$$\varphi(Z\times\{q_0\}) \subset U(\varepsilon_1). \tag{89}$$

Then $\varphi((B^n\setminus Z)\times\{q_0\})$ is a nonempty closed set containing A_{q_0} and being disjoint to the singleton $\{y_0\}$. Let

$$\varepsilon_2 = \varrho(y_0, \varphi((B^n\setminus Z)\times\{q_0\})). \tag{90}$$

However B_{q_0} is a connected set and thus taking also (89) into account we get

$$0 < \varepsilon_2 \leqq \varepsilon_1. \tag{91}$$

Now $U\left(\frac{1}{2}\varepsilon_2\right)\cap B_{q_0}$ is a nonempty subset of $B_{q_0}\setminus A_{q_0}$ which is open in B_{q_0}. Let J be a linking Jordan curve of A_{q_0} (with respect to $\mathfrak{B}$) in $R^{n+1}\setminus A_{q_0}$ such that $J\cap B_{q_0}$ is connected and

$$\emptyset \neq (J\cap B_q) \subset \left(U\left(\frac{1}{2}\varepsilon_2\right)\cap B_{q_0}\right).$$

As we have seen in 8.32, there exists such a Jordan curve J (see also 8.24).

Let v be a simple arc in J with the endpoints y and y' such that

$$J\cap B_{q_0} \subset v \subset U\left(\frac{1}{2}\varepsilon_2\right) \tag{92}$$

and

(93) $$y, y' \notin B_{q_0}.$$

Since by 8.15 $J \not\subset B_{q_0}$, there exists such a simple arc v. Moreover (92), (91) and (88) show that

(94) $$v \subset U.$$

On the other hand y and y' are the endpoints of exactly one simple arc v' in J which is distinct from v, and we have

(95) $$v \cap v' = \{y, y'\}$$

and

(96) $$v \cup v' = J.$$

Consequently taking also (92), (93), (95) and (96) into account we get $v' \cap B_{q_0} = \emptyset$. Let

(97) $$\varepsilon_3 = \varrho(v', B_{q_0}).$$

We then have

(98) $$0 < \varepsilon_3 < \frac{1}{2}\varepsilon_2 < \varepsilon_1$$

and by (96), (97), (92), (90) and (98) we get

(99) $$\varrho\big(J, \varphi\big((B^n \setminus Z) \times \{q_0\}\big)\big) \geqq \varepsilon_3.$$

Let W be an open neighbourhood of q_0 in D such that for each $x \in B^n$ and $q \in W$ we have

(100) $$\varrho\big(\varphi(x, q_0), \varphi(x, q)\big) < \varepsilon_3.$$

According to 9.6 there exists such a W.

Select $\varepsilon_4 \in R^+$ such that

(101) $$\varepsilon_4 \leqq \varepsilon_3$$

and $U(\varepsilon_4)$ is disjoint to $f^{-1}(D \setminus W)$ i.e.

(102) $$U(\varepsilon_4) \cap f^{-1}(D \setminus W) = \emptyset.$$

Since $y_0 \in B_{q_0} = f^{-1}(\{q_0\})$ it follows that $y_0 \in Y$ fails to belong to the closed subset $f^{-1}(D \setminus W)$ of Y. Consequently there exists an ε_4 with the required properties.

Now let

(103) $$V = U(\varepsilon_4).$$

We then have

(104) $$y_0 \in V \subset U \subset U_1.$$

Hence for proving Theorem 9.5 we only need to show that (U, V) is a normal pair with respect to (Y, D, f).

The proof of this last statement proceeds in several steps. For the sake of brevity if we refer to the formulas of this last section we omit the sign of the section, i.e. nstead of 9.8 (92) we only write (92).

9.9. By (102) and (103) we have $f(V\cap Y)\subset W$. Thus for any $q\in D$ the relation $V\cap B_q\neq\emptyset$ implies $q\in W$.

9.10. For $q\in W$, instead of $\varphi(Z\times\{q\})$ we also write Z_q.

Now let $q\in W$. Then by (100) and (98) $B_q\setminus Z_q$ lies in the $\frac{1}{2}\varepsilon_2$ neighbourhood of $B_{q_0}\setminus Z_{q_0}$. Thus by (90), (98), (101) and (103) we have $V\cap(B_q\setminus Z_q)=\emptyset$. Consequently taking also 9.9 into account for each $q\in D$ we have $(V\cap B_q)\subset Z_q$.

9.11. Let $q\in W$. Then by (100) and (98), Z_q lies in the ε_1-neighbourhood of Z_{q_0} and thus taking also (89) and (88) into account we have $Z_q\subset U$. Consequently for $q\in D$ $V\cap B_q\neq\emptyset$ implies $Z_q\subset U$ and thus $Z_q\subset U\cap B_q$.

9.12. The ordered pair (U, V) is regular with respect to (Y, D, f).
In fact, (104) shows that $V\subset U$.
Now let q be an element of D such that $B_q\cap V\neq\emptyset$. Then by 9.10 and 9.11 we have

$$B_q\cap V\subset Z_q\subset U\cap B_q.$$

However $Z_q=\psi(Z\times\{q\})$ is homeomorphic to the connected open ball Z in R^n and thus Z_q itself is a connected set. Consequently, it is contained in a component K of $U\cap B_q$. Hence $B_q\cap V$ is confined to the component K of $B_q\cap U$ as well. Thus the pair (U, V) is regular with respect to (Y, D, f) indeed (see 9.2).

9.13. We now show that for $q\in W$ the Jordan curve J defined in 9.8 is a linking curve of A_q.
In fact, since $Z\subset B^n\setminus A^{n-1}$ we have $A^{n-1}\subset B^n\setminus Z$ and thus $A_{q_0}\subset B_{q_0}\setminus Z_{q_0}$. Consequently (99) implies $\varrho(J, A_{q_0})\geqq\varepsilon_3$.
Observe also that both of the pairs (B_{q_0}, A_{q_0}) and (B_q, A_q) are n-cells in R^{n+1}.
Now for each $y''=\psi(x, q_0)\in A_{q_0}$ let

$$\psi(y'')=\varphi(x, q)\in A_q.$$

The mapping $\psi\colon A_{q_0}\to A_q$ is clearly well defined and it is a homeomorphism. On the other hand (100) shows that for $y''\in A_{q_0}$ we have $\varrho(y'', \psi(y''))<\varepsilon_3$.
Hence by 8.13, J is a linking Jordan curve of A_q indeed.

9.14. (99) and (100) show that for each $q\in W$,

$$J\cap B_q=J\cap Z_q.$$

9.15. (100) and (97) show that for each $q\in W$ we have $v'\cap B_q=\emptyset$ and thus

$$J\cap B_q=(v\setminus\{y, y'\})\cap B_q=v\cap B_q.$$

9.16. (86), (88), (98) and (100) show that for $q\in W$ we have $U\cap A_q=\emptyset$.

9.17. Let $<$ be the natural order on the simple arc v with $y<y'$ where y and y' are the endpoints of v (see 9.8). We shall keep fixed this order $<$ in the remainder of this section.

Let $q \in W$. Then by 9.13, 8.24 and 9.15 we have

$$\emptyset \neq J \cap B_q = v \cap B_q.$$

Denote by y_q the first and by y'_q the last element of the nonempty compact set $v \cap B_q$ in the order $<$. According to 9.15 we then have obviously

$$y < y_q \leqq y'_q < y'.$$

On the other hand, 9.14 shows that

$$y_q, y'_q \in Z_q. \tag{105}$$

9.18. LEMMA. *Let $q \in W$ and let y_1 and y'_1 be points of v such that*

$$y < y_1 < y_q \leqq y'_q < y'_1 < y'.$$

Then there is no simple arc in $U \setminus B_q$ with the endpoints y_1 and y'_1.

PROOF. We argue by contradiction.

Suppose the existence of a simple arc v'_1 in $U \setminus B_q$ with the endpoints y_1 and y'_1. Let v_1 be the subarc of v with the endpoints y_1 and y'_1. Then by (94) we have $v_1 \subset U$ and thus the continuous closed path $[y_1 v_1 y'_1][y'_1 v'_1 y_1]$ is contained in the open ball U where by 9.16 U is disjoint to A_q, and thus by 8.21 this path is a nonlinking closed path of A_q.

On the other hand let

$$v''_1 = (J \setminus v_1) \cup \{y_1, y'_1\}.$$

v''_1 is a simple arc in J with the endpoints y_1 and y'_1 and distinct from v_1. Moreover we clearly have $v''_1 \cap B_q = \emptyset$. Thus $[y'_1 v''_1 y_1][y_1 v'_1 y'_1]$ is a continuous closed path lying in $R^{n+1} \setminus B_q$, consequently by 8.23 it is a nonlinking path of A_q as well.

Hence 8.30 shows that

$$K = [y_1 v''_1 y'_1][y'_1 v_1 y_1]$$

is a nonlinking closed path of A_q. However, by 7.25 K is a Jordan path with the body J and thus by 8.20, J is a nonlinking Jordan curve of A_q in contradiction with 9.13.

The assumption about the existence of the simple arc v'_1 in $U \setminus B_q$ was false. There is no such simple arc indeed. □

9.19. LEMMA. *Let q_1 and q_2 be distinct elements of W. Then $y_{q_1} < y_{q_2}$ implies $y'_{q_1} < y'_{q_2}$.*

PROOF. We argue by contradiction. Suppose that

$$y_{q_1} < y_{q_2} \leqq y'_{q_2} < y'_{q_1}.$$

Then $y_{q_1} \neq y'_{q_1}$. Let v'_1 be a simple arc in Z_{q_1} with the endpoints y_{q_1} and y'_{q_1}. Since $y_{q_1}, y'_{q_1} \in Z_{q_1}$ (see 9.17, (105)) and Z_{q_1} is homeomorphic to the open ball Z, it follows the existence of such an arc v'_1. Now the arc v'_1 lying in $Z_{q_1} \subset B_{q_1}$ is clearly disjoint to B_{q_2}. On the other hand, by 9.11 we have $v'_1 \subset U$ and thus $v'_1 \subset U \setminus B_{q_2}$. But this is impossible by 9.18.

Thus $y_{q_1} < y_{q_2}$ implies $y'_{q_1} < y'_{q_2}$ indeed. □

9.20. LEMMA. *Let q_1, q_2, q_3 be distinct elements of W such that $y_{q_1} < y_{q_2} < y_{q_3}$. Then $Z_{q_1} \cup Z_{q_2}$ is contained in a component of $U \setminus B_{q_3}$ and $Z_{q_2} \cup Z_{q_3}$ is contained in a component of $U \setminus B_{q_1}$.*

PROOF. By 9.19 we have $y'_{q_1} < y'_{q_2} < y'_{q_3}$. Let v_1 be the subarc of v with endpoints y_{q_1} and y_{q_2} and let v'_1 be the subarc of v with endpoints y'_{q_2} and y'_{q_3}. We then clearly have

$$v_1 \cap B_{q_3} = \emptyset, \quad v_1 \cap Z_{q_1} \neq \emptyset, \quad v_1 \cap Z_{q_2} \neq \emptyset,$$

$$v'_1 \cap B_{q_1} = \emptyset, \quad v'_1 \cap Z_{q_3} \neq \emptyset, \quad v'_1 \cap Z_{q_2} \neq \emptyset$$

(see 9.17, (105)) and by (94) we also have $v_1 \cup v'_1 \subset v \subset U$. Thus taking also 9.11 into account, $Z_{q_1} \cup Z_{q_2} \cup v_1$ is a connected subset of $U \setminus B_{q_3}$ and $Z_{q_2} \cup Z_{q_3} \cup v'_1$ is a connected subset of $U \setminus B_{q_1}$. Consequently, the subset $Z_{q_1} \cup Z_{q_2}$ of $Z_{q_1} \cup Z_{q_2} \cup v_1$ is contained in a component of $U \setminus B_{q_3}$ and the subset $Z_{q_2} \cup Z_{q_3}$ of $Z_{q_2} \cup Z_{q_3} \cup v'_1$ is contained in a component of $U \setminus B_{q_1}$ indeed. □

9.21. LEMMA. *Let q_1, q_2 and q_3 be distinct elements of W such that $y_{q_1} < y_{q_2} < y_{q_3}$. Then Z_{q_1} and Z_{q_3} lie in distinct components of $U \setminus B_{q_2}$.*

PROOF. By 9.19 we have $y'_{q_1} < y'_{q_2} < y'_{q_3}$.

Since Z_{q_1} and Z_{q_3} are connected sets, 9.11 shows that for $i=1, 3$, Z_{q_i} is contained in a component of $U \setminus B_{q_2}$.

Now if Z_{q_1} and Z_{q_3} were contained in the same component of $U \setminus B_{q_2}$ then by $y_{q_1} \in Z_{q_1}$ and $y'_{q_3} \in Z_{q_3}$ there would exist a simple arc v'_1 in $U \setminus B_{q_2}$ with endpoints y_{q_1} and y'_{q_3}. But this is impossible by Lemma 9.18.

Thus Z_{q_1} and Z_{q_3} lie in distinct components of $U \setminus B_{q_2}$ indeed. □

9.22. We now finish the proof of Theorem 9.5 by showing that (U, V) is a normal pair with respect to (Y, D, f) (see 9.4).

First observe that by 9.12 the pair (U, V) is regular with respect to (Y, D, f).

Now let q_1, q_2, q_3 be three distinct elements of D such that for $i=1, 2, 3$, $B_{q_i} \cap V \neq \emptyset$. Then by 9.9 we have $q_1, q_2, q_3 \in W$ and by 9.10 for $i=1, 2, 3$ one has

$$V \cap B_{q_i} \subset Z_{q_i}. \tag{106}$$

Let i_1, i_2, i_3 be a permutation of 1, 2, 3 such that

$$y_{q_{i_1}} < y_{q_{i_2}} < y_{q_{i_3}}.$$

Obviously, such a permutation exists.

Now by (106) and by Lemmas 9.20 and 9.21 we have

(a) $(B_{q_{i_1}} \cup B_{q_{i_2}}) \cap V$ is confined to a component of $U \setminus B_{q_{i_3}}$,

(b) $(B_{q_{i_2}} \cup B_{q_{i_3}}) \cap V$ is confined to a component of $U \setminus B_{q_{i_1}}$,

(c) $B_{q_{i_1}} \cap V$ and $B_{q_{i_3}} \cap V$ lie in distinct components of $U \setminus B_{q_{i_2}}$ (see also 9.3).

(U, V) is a normal pair with respect to (Y, D, f) indeed.

The proof of Theorem 9.5 is complete. □

10. Systems of n-bricks in R^{n+1}

In this section we continue the investigations which have been started in the preceding section. We describe exactly the notion of local ordering determined by a system of n-bricks lying in R^{n+1}. We also examine systems of n-bricks in R^{n+1} joined with respect to a joining function.

Finally we show that the n-dimensional solenoid cannot be embedded in R^{n+1} and we finish the proof of Theorem A.

Let n be a positive integer. We shall keep it fixed in this section.

10.1. Let (Y, D, f) be a system of n-bricks such that Y is a subspace of R^{n+1}. For $q \in D$ let $B_q = f^{-1}(\{q\})$ and let A_q be the boundary of B_q (see 9.1).

10.2. Let (U, V) be a normal pair with respect to (Y, D, f) (see 9.4).

Denote by $M_V^{(Y,D,f)}$ or — if there is no confusion — by M_V the subset $f(V \cap Y)$ of D. M_V is clearly an open subset of D (cf. 5.1).

The triadic relation $R_{(U,V)}^{(Y,D,f)}$ on M_V is defined as follows: For $q, q', q'' \in M_V$ let

$$(q, q', q'') \in R_{(U,V)}^{(Y,D,f)}$$

if $B_q \cap V$ and $B_{q''} \cap V$ lie in distinct components of $U \setminus B_{q'}$ (see also 9.3).

The ordered pair $(M_V, R_{(U,V)}^{(Y,D,f)})$ is clearly an ordered set (see 9.4 and 6.1). Observe that if there is no confusion we write $R_{(U,V)}$ instead of $R_{(U,V)}^{(Y,D,f)}$.

10.3. *Let* (U, V) *be a normal pair with respect to* (Y, D, f) *and let* $q \in M_V = f(V \cap Y)$. *Then the sides of* q *in* $(M_V, R_{(U,V)})$ (see 6.4) *are open subsets of* D.

In fact, by the definition of $R_{(U,V)}$ for $q', q'' \in M_V \setminus \{q\}$, q' and q'' belong to the same side of q in $(M_V, R_{(U,V)})$ (see 6.4) if and only if $V \cap B_{q'}$ and $V \cap B_{q''}$ are contained in the same component of $U \setminus B_q$ (cf. also 9.3 and 9.4). Hence the nonempty sides of q in $(M_V, R_{(U,V)})$ are the projections by f of the traces of certain components of $U \setminus B_q$ in $V \cap Y$.

More exactly, if $\{Z_\beta;\ \beta \in B\}$ is the set of the components of $U \setminus B_q$ and

$$B_1 = \{\beta \in B;\ Z_\beta \cap V \cap Y \neq \emptyset\}$$

then B_1 has at most two elements and the nonempty sides of q in $(M_V, R_{(U,V)})$ are the sets $f(Z_\beta \cap V \cap Y)$ where β runs over B_1.

Now observe that since $U \setminus B_q$ is an open subset of the locally connected space R^{n+1}, it follows that each Z_β $(\beta \in B)$ is open in $U \setminus B_q$ and thus it is open in R^{n+1}. Consequently for $\beta \in B_1$, $Z_\beta \cap V \cap Y$ is open in Y and since $f\colon Y \to D$ is an open map (see 5.1) it follows that the nonempty sides $f(Z_\beta \cap V \cap Y)$ $(\beta \in B_1)$ of q in $(M_V, R_{(U,V)})$ are open subsets of D.

The empty sides of q in $(M_V, R_{(U,V)})$ are clearly open in D as well. Thus each side of q in $(M_V, R_{(U,V)})$ is an open subset of D indeed.

10.4. Let $V \subset U \subset U'$ where V, U and U' are open subsets of R^{n+1}. Suppose that both (U, V) and (U', V) are normal pairs with respect to (Y, D, f). Then

$$R_{(U,V)} = R_{(U',V)}.$$

In fact let $q_1, q_2, q_3 \in f(V \cap Y) = M_V$. Suppose that $(q_1, q_2, q_3) \in R_{(U',V)}$ i.e. that $B_{q_1} \cap V$ and $B_{q_3} \cap V$ are contained in distinct components of $U' \setminus B_{q_2}$. Then by $U \subset U'$, $B_{q_1} \cap V$ and $B_{q_3} \cap V$ are clearly contained in distinct components of $U \setminus B_{q_2}$ as well (see also 9.3) and thus $(q_1, q_2, q_3) \in R_{(U,V)}$.

Now suppose that

(107) $$(q_1, q_2, q_3) \notin R_{(U',V)}.$$

If q_1, q_2, q_3 are not pairwise distinct then we have $(q_1, q_2, q_3) \notin R_{(U,V)}$. Suppose now that q_1, q_2, q_3 are distinct elements of $f(V \cap Y)$. Then there is a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ such that

(108) $$(q_{i_1}, q_{i_2}, q_{i_3}) \in R_{(U',V)}$$

and thus according to the preceding argumentation we have

(109) $$(q_{i_1}, q_{i_2}, q_{i_3}) \in R_{(U,V)}.$$

However, by (107) and (108) we have $i_2 \neq 2$ and since $(M_V, R_{(U,V)})$ is an ordered set it follows by (109) that $(q_1, q_2, q_3) \notin R_{(U,V)})$ (see also 6.1). Hence $R_{(U,V)} = R_{(U',V)}$ indeed.

10.5. Let (U, V) be a normal pair with respect to (Y, D, f) and let V' be an open subset of V. Then (U, V') is obviously a normal pair with respect to (Y, D, f) and we clearly have

$$R_{(U,V')} = R_{(U,V)}|_{M_{V'}}$$

(cf. 6.3).

10.6. Let $q \in D$ where D is the same as in 10.1. Let Σ_q be the system of all ordered sets (M, R) such that $q \in M$ and M is an open subset of D.

Now the ordered sets (M_1, R_1) and (M_2, R_2) of Σ_q are said to be *q-compatible* — and we write $(M_1, R_1) \underset{q}{\sim} (M_2, R_2)$ — if there is an open set W in D such that $q \in W \subset (M_1 \cap M_2)$ and $R_1|_W = R_2|_W$.

The relation $\underset{q}{\sim}$ is clearly an equivalence on Σ_q (cf. 6.3).

10.7. Let (U, V) and (U', V') be normal pairs with respect to (Y, D, f). They are called *neighbouring normal pairs* if either $V \subset V' \subset U = U'$ or $V' \subset V \subset \subset U = U'$ or $V = V' \subset U \subset U'$ or $V = V' \subset U' \subset U$.

Now let $q \in D$ and let (U, V) and (U', V') be neighbouring normal pairs with respect to (Y, D, f) such that

$$q \in f(V \cap Y) \cap f(V' \cap Y) = M_V \cap M_{V'}.$$

Then by 10.4 and 10.5 we clearly have

$$(M_V, R_{(U,V)}) \underset{q}{\sim} (M_{V'}, R_{(U',V')}).$$

10.8. For $q \in D$, let Σ_q^* denote the set of normal pairs (U, V) with respect to (Y, D, f) satisfying the condition $q \in M_V = f(V \cap Y)$.

By a *q-chain* we mean a finite sequence

$$s = (U_1, V_1), \ldots, (U_k, V_k)$$

in Σ_q^* such that for $i = 1, \ldots, k-1$ (U_i, V_i) and (U_{i+1}, V_{i+1}) are neighbouring normal pairs. We say in this case that (U_1, V_1) *and* (U_k, V_k) *are connected by the q-chain s.*

The normal pairs (U, V) and (U', V') of Σ_q^* are said to be *q-connected* — and we write $(U, V)\underset{q}{\approx}(U', V')$ — if there is a q-chain connecting (U, V) and (U', V').

The relation $\underset{q}{\approx}$ is clearly an equivalence on Σ_q^*.

Moreover, taking also 10.6 and 10.7 into account we can state that

$$(U, V) \underset{q}{\approx} (U', V')$$

implies

$$(M_V, R_{(U,V)}) \underset{q}{\sim} (M_{V'}, R_{(U',V')}).$$

10.9. Remark. Let $q\in D$ and $(U, V)\in\Sigma_q^*$. Then $V\cap(B_q\setminus A_q)\neq\emptyset$ (see also 9.1).

In fact, since $(U, V)\in\Sigma_q^*$ we have $q\in f(V\cap Y)$ and thus $V\cap f^{-1}(\{q\})= =V\cap B_q\neq\emptyset$. However, $V\cap B_q$ is an open subset of B_q and B_q itself is the closure of $B_q\setminus A_q$ in B_q (see 9.1) consequently we clearly obtain $V\cap(B_q\setminus A_q)\neq\emptyset$.

10.10. Lemma. *Let $q\in D$. Then for arbitrary two elements (U', V') and (U'', V'') of Σ_q^* we have $(U', V')\underset{q}{\approx}(U'', V'')$.*

Proof. First observe that

$$\cup\{V\cap(B_q\setminus A_q);\ (U, V)\in\Sigma_q^*\} = B_q\setminus A_q. \tag{110}$$

This follows immediately from Theorem 9.5.

In the remainder of the proof we argue by contradiction. Suppose that $(U', V')\in\Sigma_q^*$, $(U'', V'')\in\Sigma_q^*$ and (U', V') and (U'', V'') are not q-connected.

Let Σ_q^{*1} be the set of $(U, V)\in\Sigma_q^*$ satisfying the condition

$$(U, V) \underset{q}{\approx} (U', V')$$

and let $\Sigma_q^{*2}=\Sigma_q^*\setminus\Sigma_q^{*1}$. We then clearly have

$$(U', V')\in\Sigma_q^{*1},\quad (U'', V'')\in\Sigma_q^{*2}. \tag{111}$$

Now for $i=1, 2$ let

$$G_i = \cup\{V\cap(B_q\setminus A_q);\ (U, V)\in\Sigma_q^{*i}\}.$$

Then by $\Sigma_q^*=\Sigma_q^{*1}\cup\Sigma_q^{*2}$ and by (110) we have $G_1\cup G_2 = B_q\setminus A_q$. By (111) and 10.9, G_1 and G_2 are nonempty open subsets of $B_q\setminus A_q$ and since $B_q\setminus A_q$ is a connected set it follows $G_1\cap G_2\neq\emptyset$.

Let $y\in G_1\cap G_2$. Select $(U_1, V_1)\in\Sigma_q^{*1}$ and $(U_2, V_2)\in\Sigma_q^{*2}$ such that $y\in V_1\cap V_2$.

Let (U_3, V_3) be a normal pair with respect to (Y, D, f) such that

$$y\in V_3\subset U_3\subset V_1\cap V_2.$$

By 9.5 there exists such a normal pair (U_3, V_3) and we have obviously $(U_3, V_3)\in\Sigma_q^*$. Now

$$s = \big((U_1, V_1), (U_1, V_3), (U_3, V_3), (U_2, V_3), (U_2, V_2)\big)$$

is a q-chain connecting (U_1, V_1) and (U_2, V_2) (see also 10.5). Consequently $(U_1, V_1)\underset{q}{\approx}(U_2, V_2)$ and thus by $(U', V')\underset{q}{\approx}(U_1, V_1)$ we have $(U_2, V_2)\underset{q}{\approx}(U', V')$ and this yields $(U_2, V_2)\in\Sigma_q^{*1}$ in contradiction with $(U_2, V_2)\in\Sigma_q^{*2}$.

The assumption about the existence of $(U', V'), (U'', V'')\in\Sigma_q^*$ which are not q-connected was false. Hence arbitrary two members of Σ_q^* are q-connected indeed.

The proof of the Lemma is complete. $\square$

10.11. Notice as a direct consequence of 10.10 that for $q\in D$ and for any two members (U', V') and (U'', V'') of Σ_q^* we have

$$(M_{V'}, R_{(U',V')}) \underset{q}{\sim} (M_{V''}, R_{(U'',V'')})$$

(see also 10.8).

10.12. Let $q\in D$ and $y\in B_q\setminus A_q$. Let G and W be open neighbourhoods of y in R^{n+1} and q in D, resp. Then there is a $(U, V)\in\Sigma_q^*$ (see 10.8) such that $y\in V\subset U\subset G$ and $q\in f(V\cap Y)=M_V\subset W$.

In fact, $f^{-1}(D\setminus W)$ is a closed subset of Y and by $f(y)=q\in W$ we have $y\notin f^{-1}(D\setminus W)$.

Let U_1 be an open neighbourhood of y in R^{n+1} such that $U_1\cap f^{-1}(D\setminus W)=\emptyset$ and thus $q\in f(U_1\cap Y)\subset W$. Such a U_1 clearly exists. Now let (U, V) be a normal pair with respect to (Y, D, f) such that

$$y\in V\subset U\subset (U_1\cap G)$$

(see 9.5). We then have $(U, V)\in\Sigma_q^*$, $y\in V\subset U\subset G$ and

$$q\in f(V\cap Y) = M_V\subset f(U_1\cap Y)\subset W$$

as required.

10.13. Let $q\in D$ and $y\in B_q\setminus A_q$. Let G be an open neighbourhood of y in R^{n+1} and let (U', V') be an element of Σ_q^*. Then there is a $(U, V)\in\Sigma_q^*$ such that $y\in V\subset U\subset G$, $q\in M_V\subset M_{V'}$ and

$$R_{(U,V)} = R_{(U',V')}|_{M_V}.$$

In fact, let (U_1, V_1) be a normal pair with respect to (Y, D, f) such that $y\in V_1$. By 9.5, such a pair exists and $(U_1, V_1)\in\Sigma_q^*$.

By 10.11 one has

$$(M_{V_1}, R_{(U_1,V_1)}) \underset{q}{\sim} (M_{V'}, R_{(U',V')})$$

i.e. there is an open subset W of D such that

$$q\in W\subset (M_{V_1}\cap M_{V'}) \quad\text{and}\quad R_{(U_1,V_1)}|_W = R_{(U',V')}|_W$$

(see 10.6).

Let (U, V) be a normal pair with respect to (Y, D, f) such that

$$y\subset V\subset U\subset (V_1\cap G)\subset (U_1\cap G)$$

and

$$q\in f(V\cap Y) = M_V\subset W.$$

10.12 shows the existence of such a pair (U, V). Now by 10.5 $(U_1, V)\in\Sigma_q^*$ and we clearly have $(U, V)\in\Sigma_q^*$, $y\in V\subset U\subset G$, $q\in f(V\cap Y)=M_V\subset W\subset M_{V'}$ and

$$R_{(U,V)} = R_{(U_1,V)} = R_{(U_1,V_1)}|_{M_V} = (R_{(U_1,V_1)}|_W)|_{M_V} = (R_{(U',V')}|_W)|_{M_V} = R_{(U',V')}|_{M_V}$$

(see also 10.4 and 10.5) as required.

10.14. *Let $q\in D$. Let (U_1, V_1) and (U_2, V_2) be elements of Σ_q^*. Then there is a $(U, V)\in\Sigma_q^*$ such that*

$$q\in M_V\subset(M_{V_1}\cap M_{V_2})$$

and

$$R_{(U,V)} = R_{(U_1,V_1)}|M_V = R_{(U_2,V_2)}|M_V.$$

In fact, let $y\in(B_q\setminus A_q)\cap V_1$. By 10.9, such a y exists. Let (U, V) be an element of Σ_q^* satisfying the conditions

$$y\in V\subset U\subset V_1\subset U_1, \quad q\in M_V\subset M_{V_2}, \quad R_{(U,V)} = R_{(U_2,V_2)}|M_V.$$

10.13 shows the existence of such a pair (U, V). However, by $y\in V\subset V_1$ we also have

$$q\in M_V\subset(M_{V_1}\cap M_{V_2}).$$

On the other hand taking also 10.5 into account we get $(U_1, V)\in\Sigma_q^*$. Moreover we have

$$R_{(U_1,V_1)}|M_V = R_{(U_1,V)} = R_{(U,V)}$$

(see 10.5 and 10.4) as required.

10.15. *Let $q\in D$ and let W be an open neighbourhood of q in D. Then* by 5.1, (a) $B_q=f^{-1}(\{q\})\neq\emptyset$ and since B_q is the closure of $B_q\setminus A_q$ in B_q we get $B_q\setminus A_q\neq\emptyset$. Thus by 10.12 *there is a $(U, V)\in\Sigma_q^*$ such that $q\in M_V\subset W$.*

10.16. Let $\{(U_\alpha, V_\alpha);\ \alpha\in A\}$ be the set of normal pairs with respect to (Y, D, f). For $\alpha\in A$, denoting by (M_α, R_α) the ordered set $(M_{V_\alpha}, R_{(U_\alpha,V_\alpha)})$ 10.15, 10.14 and 10.3 show that

$$\Theta = \{(M_\alpha, R_\alpha);\ \alpha\in A\}$$

is a local ordering of D (see also 6.8).

This is the *local ordering determined by the system of n-bricks* (Y, D, f). We denote this local ordering by $\Theta^{(Y,D,f)}$.

10.17. Let (Y, D, f) and (Y', D, f') be systems of n-bricks such that $Y\cup Y'\subset R^{n+1}$. Let $\psi\colon D\to D$ be an autohomeomorphism of D and suppose that the system of n-bricks (Y, D, f) is joined to (Y', D, f') with respect to the joining function ψ (see 5.3).

For $q\in D$ we denote the set $f^{-1}(\{q\})$ by B_q and the boundary of B_q (see 9.1) by A_q. On the other hand, for $q'\in D$ we denote the set $f'^{-1}(\{q'\})$ by $B'_{q'}$ and the boundary of $B'_{q'}$ by $A'_{q'}$.

For any open set G of R^{n+1}, G is said to be a *joining set of* (Y, D, f) *and* (Y', D, f') *with respect to* ψ if $G\cap Y=G\cap Y'$ and if for each $y\in G\cap Y$ we have $\psi(f(y))=f'(y)$.

For any joining set G of (Y, D, f) and (Y', D, f') with respect to ψ and for each $q\in D$ we have

$$G\cap B_q = G\cap B'_{\psi(q)}; \tag{112}$$

and if (U, V) is a normal pair with respect to (Y, D, f) such that $U\subset G$ then (U, V) is a normal pair with respect to (Y', D, f'). Conversely, each normal pair (U, V)

with respect to (Y', D, f') such that $U \subset G$, is a normal pair with respect to (Y, D, f).

Normal pairs of this kind, i.e. normal pairs lying in a joining set will be called *binormal pairs with respect to* (Y, D, f), (Y', D, f') *and* ψ.

For the binormal pair (U, V) by (112) we have

$$\psi(M_V^{(Y,D,f)}) = \psi(f(V \cap Y)) = f'(V \cap Y') = M_V^{(Y',D,f')}$$

and

$$\psi(R_{(U,V)}^{(Y,D,f)}) = R_{(U,V)}^{(Y',D,f')}$$

(see also 6.6).

10.18. Let (Y, D, f), (Y', D, f') and ψ be the same as in 10.17. Then for $q \in D$

$$(B_q \setminus A_q) \cap (B'_{\psi(q)} \setminus A'_{\psi(q)}) \neq \emptyset.$$

In fact, let $q \in D$. Let B^n and B'^n be properly joined n-bricks in R^n (see 5.2), W an open neighbourhood of q in D and

$$\varphi\colon (B^n \cup B'^n) \times W \to f^{-1}(W) \cup f'^{-1}(\psi(W))$$

a homeomorphism such that for each $q' \in W$ one has

$$\varphi(B^n \times \{q'\}) = f^{-1}(\{q'\}) = B_{q'} \tag{113}$$

and

$$\varphi(B'^n \times \{q'\}) = f'^{-1}(\{\psi(q')\}) = B'_{\psi(q')} \tag{114}$$

(see 5.3, (27) and 5.3, (28)).

Let $x_0 \in \operatorname{int} B^n \cap \operatorname{int} B'^n$ (see 5.2). Then $y = \varphi(x_0, q)$ belongs to both sets $B_q \setminus A_q$ and $B'_{\psi(q)} \setminus A'_{\psi(q)}$.

10.19. Let (Y, D, f), (Y', D, f') and ψ be the same as in 10.17. Let $q \in D$ and $y \in (B_q \setminus A_q) \cap (B'_{\psi(q)} \setminus A'_{\psi(q)})$. Then there is a joining set G of (Y, D, f) and (Y', D, f') with respect to ψ such that $y \in G$.

In fact let B^n, B'^n, W and φ be the same as in 10.18. Then there is a unique $x_0 \in B^n \cup B'^n$ such that $\varphi(x_0, q) = y$ and by 10.18 (113) and 10.18 (114) we have

$$x_0 \in \operatorname{int} B^n \cap \operatorname{int} B'^n.$$

Since $y \in Y$ and by $f(y) = q$ y fails to belong to the closed subset $f^{-1}(D \setminus W)$ of Y it follows that there is an open neighbourhood U^1 of y in R^{n+1} being disjoint to $f^{-1}(D \setminus W)$. Likewise y is in Y' and fails to belong to the closed subset $f'^{-1}(D \setminus \psi(W))$ of Y'. Thus there is an open neighbourhood U^2 of y in R^{n+1} being disjoint to $f'^{-1}(D \setminus \psi(W))$.

y also belongs to $\varphi((B^n \cup B'^n) \times W)$ and fails to belong to the closed subset

$$\varphi(((B^n \cup B'^n) \setminus (\operatorname{int} B^n \cap \operatorname{int} B'^n)) \times W)$$

of $\varphi((B^n \cup B'^n) \times W)$. Thus there is an open neighbourhood U^3 of y being disjoint to

$$\varphi(((B^n \cup B'^n) \setminus (\operatorname{int} B^n \cap \operatorname{int} B'^n)) \times W).$$

Let $G = U^1 \cap U^2 \cap U^3$. Then G is an open neighbourhood of y and we have

$$G \cap Y = G \cap Y' = G \cap \varphi((\operatorname{int} B^n \cap \operatorname{int} B'^n) \times W).$$

Thus for $y' \in G \cap Y$ we have $y' = \varphi(x', q')$ where $x' \in \operatorname{int} B^n \cap \operatorname{int} B'^n$ and $q' = f(y') \in W$ (cf. 10.18 (113)). Consequently taking also 10.18 (113) and 10.18 (114) into account we get

$$\psi(f(y')) = \psi(q') = f'(y').$$

Hence the open neighbourhood G of y is a joining set of (Y, D, f) and (Y', D, f') with respect to ψ indeed.

10.20. LEMMA. *Let (Y, D, f), (Y', D, f') and ψ be the same as in* 10.17. *Let Θ be the local ordering of D determined by (Y, D, f) and Θ' that one determined by (Y', D, f') i.e. $\Theta = \Theta^{(Y, D, f)}$ and $\Theta' = \Theta^{(Y', D, f')}$* (see 10.16). *Then $\psi(\Theta) \sim \Theta'$* (cf. also 6.12 and 6.9).

PROOF. Let $q \in D$ and $q' = \psi(q) \in D$. Select $(M_\alpha, R_\alpha) \in \Theta$ and $(M'_{\alpha'}, R'_{\alpha'}) \in \Theta'$ such that $q \in M_\alpha$ and $q' \in M'_{\alpha'}$. We need only to show that there exists an $(M'_\beta, R'_\beta) \in \Theta'$ such that

$$q' \in M'_\beta \subset (\psi(M_\alpha) \cap M'_{\alpha'})$$

and

$$R'_\beta = R'_{\alpha'}|_{M'_\beta} = \psi(R_\alpha)|_{M'_\beta}.$$

First observe that for the sake of brevity we write $(M_{V'}, R_{(U', V')})$ instead of $(M_{V'}^{(Y, D, f)}, R_{(U', V')}^{(Y, D, f)})$ whenever (U', V') is a normal pair with respect to (Y, D, f) and we write $(M'_{V'}, R'_{(U', V')})$ instead of $(M_{V'}^{(Y', D, f')}, R_{(U', V')}^{(Y', D, f')})$ whenever (U', V') is a normal pair with respect to (Y', D, f').

Let $y \in (B_q \setminus A_q) \cap (B'_{\psi(q)} \setminus A'_{\psi(q)})$ (see 10.18) and let G be a joining set of (Y, D, f) and (Y', D, f') with respect to ψ such that $y \in G$ (see 10.19). Let (U_1, V_1) be a normal pair with respect to (Y, D, f) satisfying the conditions $y \in V_1$,

$$q \in M_{V_1} \subset M_\alpha, \tag{115}$$

and

$$R_\alpha|_{M_{V_1}} = R_{(U_1, V_1)}. \tag{116}$$

Since $(M_\alpha, R_\alpha) = (M_{V'}, R_{(U', V')})$ for some $(U', V') \in \Sigma_q^*$ (see 10.8 and 10.16), 10.13 shows the existence of such a normal pair (U_1, V_1). On the other hand, let (U_2, V_2) be a normal pair with respect to (Y', D, f') such that $y \in V_2$,

$$q' = \psi(q) \in M'_{V_2} \subset M'_{\alpha'} \tag{117}$$

and

$$R'_{\alpha'}|_{M'_{V_2}} = R'_{(U_2, V_2)}. \tag{118}$$

By 10.13, there exists such a pair (U_2, V_2). Finally, let (U, V) be a normal pair with respect to (Y, D, f) such that

$$y \in V \subset U \subset (V_1 \cap V_2 \cap G) \subset (U_1 \cap U_2) \tag{119}$$

(see 9.5). Then (U, V) is a binormal pair with respect to (Y, D, f), (Y', D, f') and ψ (see 10.17). Hence

$$(M_V, R_{(U,V)})\in\Theta, \quad (M'_V, R'_{(U,V)})\in\Theta',$$

(120)
$$\psi(M_V) = M'_V$$

and

(121)
$$\psi(R_{(U,V)}) = R'_{(U,V)}.$$

Moreover,

(122)
$$q\in M_V \quad\text{and}\quad q'\in M'_V.$$

However, by 10.5 (U_1, V) is a normal pair with respect to (Y, D, f) and (U_2, V) is a normal pair with respect to (Y', D, f') and by 10.4 and 10.5 we get

(123)
$$R_{(U_1,V_1)}|_{M_V} = R_{(U_1,V)} = R_{(U,V)},$$

(124)
$$R'_{(U_2,V_2)}|_{M'_V} = R'_{(U_2,V)} = R'_{(U,V)}.$$

Now taking also (122), (120), (119), (115) and (117) into account we obtain

$$q'\in M'_V = \psi(M_V)\subset\psi(M_{V_1})\subset\psi(M_\alpha), \quad q'\in M'_V\subset M'_{V_2}\subset M'_{\alpha'}$$

and thus

$$q'\in M'_V\subset(\psi(M_\alpha)\cap M'_{\alpha'}).$$

On the other hand, (121), (123) and (116) show that

$$R'_{(U,V)} = \psi(R_{(U,V)}) = \psi(R_{(U_1,V_1)}|_{M_V}) =$$
$$= \psi(R_\alpha|_{M_{V_1}})|_{M'_V} = (\psi(R_\alpha)|_{\psi(M_{V_1})})|_{M'_V} = \psi(R_\alpha)|_{M'_V}$$

(see also 6.6 and 6.3).

Moreover, taking also (124) and (118) into account we have

$$R'_{(U,V)} = R'_{(U_2,V_2)}|_{M'_V} = (R'_{\alpha'}|_{M'_{V_2}})|_{M'_V} = R'_{\alpha'}|_{M'_V}.$$

Hence $(M'_V, R'_{(U,V)})\in\Theta'$ satisfies all the requirements about $(M'_\beta, R'_\beta)\in\Theta'$ and thus we have $\psi(\Theta)\sim\Theta'$ indeed.

The proof of the Lemma is complete. □

10.21. THEOREM. *An n-dimensional solenoid cannot be embedded in R^{n+1}.*

PROOF. We argue by contradiction. Let S_n be an n-dimensional solenoid determined by the absolutely cyclic map $\psi\colon D\to D$ where D is a space homeomorphic to the Cantor discontinuum (see 4.6) and let $h\colon S_n\to h(S_n)\subset R^{n+1}$ be a topological mapping of S_n onto the subspace $h(S_n)$ of R^{n+1}.

Let (Y_1, D, f_1) and (Y'_1, D, f'_1) be systems of n-bricks such that Y_1 and Y'_1 are subspaces of S_n and (Y_1, D, f_1) is joined to (Y'_1, D, f'_1) with respect to two distinct joining functions. The first of them is the identical map id_D of D and the second is ψ (see 5.6).

Let $Y=h(Y_1)$, $Y'=h(Y_1')$,

$$h_1 = h|_{Y_1}: Y_1 \to Y, \quad h_1' = h|_{Y_1'}: Y_1' \to Y',$$

$$f=f_1h_1^{-1}: Y \to D, \quad f' = f_1'h_1'^{-1}: Y' \to D.$$

Then (Y, D, f) and (Y', D, f') are systems of n-bricks such that $Y \cup Y' \subset R^{n+1}$ and (Y, D, f) is joined to (Y', D, f') with respect to the joining functions id_D and ψ (see 5.4 and 5.5).

Let Θ be the local ordering of D determined by (Y, D, f) and Θ' that one determined by (Y', D, f') (see 10.16). We then have by 10.20

$$\mathrm{id}_D(\Theta) = \Theta \sim \Theta' \quad \text{and} \quad \psi(\Theta) \sim \Theta'.$$

Hence by 6.11 we have $\psi(\Theta) \sim \Theta$. But this is impossible by 6.14 (see also 6.13). S_n cannot be embedded into R^{n+1} indeed.

The proof of the theorem is complete. □

10.22. Now we are going to prove Theorem A.

As we have seen in 2.3 we need only to show that the space of a connected but non locally connected n-dimensional LCA-group cannot be embedded in R^{n+1}.

However, by 4.9 the space of such a topological group has a subspace homeomorphic to an n-dimensional solenoid, and thus by 10.21 the space of such a topological group cannot be embedded in R^{n+1} indeed.

The proof of Theorem A is complete as well. □

References

[1] M. Bognár, Die i-Kategorie der stetigen Wege, *Acta Math. Acad. Sci. Hung.*, **24** (1973), 155—178.

[2] M. Bognár, On locally ordered spaces, *Proc. of the Fourth Prague Top. Symposium* 1976 Part B, 59—61.

[3] M. Bognár, i-categories and homology, *Colloquia Math. Soc. J. Bolyai* 23. Topology (Budapest, 1978), 163—174.

[4] M. Bognár, Continuous paths and homology, *Acta Math. Acad. Sci. Hung.*, **34** (1979), 121—127.

[5] M. Bognár, On axiomatization of the theory of linking, *Acta Math. Hung.*, **49** (1987), 3—28.

[6] M. Bognár, On linking Jordan curves, *Acta Math. Hung.* **51** (1988), 15—12.

[7] M. Bognár, On embedding of locally compact abelian topological groups in euclidean spaces. I, *Acta Math. Hung.*

[8] S. Eilenberg—N. Steenrod, *Foundations of algebraic topology* (Princeton, 1952).

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INTEGRAL ELEMENTS WITH GIVEN DISCRIMINANT OVER FUNCTION FIELDS

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1. Introduction

The main purpose of this paper is to give an effective algorithm to determine all integral elements with given discriminant in a finite extension of a function field of characteristic zero. We shall give some applications of our theorem to discriminant form equations, power integral bases and integral elements with given discriminant and given norm.

The analogous problem for algebraic integers with given discriminant over algebraic number fields has been already solved. Let L be an algebraic number field and denote by O_L the ring of integers of L. The algebraic integers α and α^* are called O_L-equivalent if $\alpha-\alpha^*\in O_L$. In this case for their discriminants $D_{L(\alpha)/L}(\alpha)=$ $=D_{L(\alpha^*)/L}(\alpha^*)$ holds. In a series of papers Győry [7]—[12], [19] obtained several effective results on polynomials with algebraic integer coefficients and given non-zero discriminant. As a consequence of these results Győry proved that there are only finitely many pairwise non-equivalent algebraic integers with given degree and given non-zero discriminant over L and a full set of representatives of such integers can be effectively determined. These theorems were proved by combining Baker's famous effective method (see e.g. [1]) with a so-called graph-method of Győry (cf. [14] or [20]).

Later Győry [22] extended some of the above-mentioned results to the more general case when the ground ring is an arbitrary integral domain R which is finitely generated over $\mathbf{Z}$ (absolute case) or over a field k of characteristic 0 (relative case). He proved among others that if α is an element of a finite extension K of the quotient field L of R and if α is integral over R with given non-zero discriminant with respect to K/L, then α is R-equivalent to an α^* (that is $\alpha-\alpha^*\in R$) where in the absolute case α^* is of bounded size, and in the relative case α^* is of bounded degree. (For the definition of the size and degree of elements of a finitely generated integral domain, see [22].) In the absolute case this theorem implies that there are only finitely many R-equivalence classes of integral elements in K over R with given non-zero discriminant and a full set of representatives can be effectively determined. In the relative case the bound given for the degrees of α^*'s does not, however, imply the finiteness of the R-equivalence classes.

On the other hand, in a recent paper [3] Evertse and Győry showed that the number of such R-equivalence classes is finite also in the relative case and they derived a good bound for the number of R-equivalence classes. This implies that if in particular L is an algebraic function field of one variable over[1] k then there

[1] We remark that in the function field case the ground field k is usually supposed to be algebraically closed, but in [3] only a much weaker assumption is necessary on k.

are only finitely many pairwise R-inequivalent elements in K, integral over R, which have a given non-zero discriminant with respect to K/L. But this theorem does not make possible to determine effectively a full set of representatives of these R-equivalence classes. Our main aim is just to give such an algorithm under some additional hypotheses concerning k. The proof of our result is based on Mason's effective theorem on unit equations in three variables ([27], [28]).

2. Preliminaries concerning function fields

Throughout our paper we shall use the following notation. k will denote an algebraically closed field of characteristic 0. We shall suppose that k is explicitly given in the sense of Fröhlich and Shepherdson [4], which in our case means that we can perform all the field operations with elements of k and we can effectively determine the roots in k of any polynomial in one variable with coefficients in k. As usual $k(z)$ denotes the rational function field over k. Let K be a finite extension field of $k(z)$.

Let us denote by Ω_K the set of all (additive) valuations on K with value group $\mathbf{Z}$. For any non-zero $\alpha \in K$ let

$$H_K(\alpha) = -\sum_{v \in \Omega_K} \min\{0, v(\alpha)\}$$

be the additive height of α. Obviously $H_K(\alpha)=0$ if and only if $\alpha \in k$. (If $\alpha=0$ we may put $H_K(\alpha)=0$.) The additive form

$$\sum_{v \in \Omega_K} v(\alpha) = 0$$

of the well-known product formula implies that

$$H_K(\alpha^m) = |m| H_K(\alpha),$$

$$H_K(\alpha\beta) \leqq H_K(\alpha)+H_K(\beta), \quad H_K(\alpha+\beta) \leqq H_K(\alpha)+H_K(\beta)$$

for any non-zero $\alpha, \beta \in K$ and $m \in \mathbf{Z}$. We remark that if L is an other extension field of $k(z)$ and $L \subset K$ then

$$H_K(\alpha) = [K:L] H_L(\alpha)$$

for any $\alpha \in L$ (see e.g. [21] or [28]). For other properties of valuations of function fields and of the height function we refer to Mason [28].

In our results L, M, K will denote finite extension fields of $k(z)$ with $L \subset M \subseteqq K$ and $[M:L]=n \geqq 2$. We shall suppose that K is a normal extension of L. Denote by $\sigma_1, \ldots, \sigma_n$ the L-isomorphisms of M in K.

Let O_K be the ring of those elements of K which are integral over $k[z]$ (that is $\gamma \in O_K$ if and only if $v(\gamma) \geqq 0$ for all finite valuations[2] v in Ω_K). We define O_L and O_M similarly. O_K, O_L and O_M are called the ring of integers of K, L and M, respectively.

[2] $v \in \Omega_k$ is called finite if $v(z) \geqq 0$, other valuations are called infinite (cf. e.g. [28]).

Let Ω_L and Ω_M denote the set of all additive valuations of L and M, respectively, with value group $\mathbf{Z}$. Denote by S a finite subset of Ω_L, which contains the infinite valuations of L and let $R\subset\Omega_M$ and $T\subset\Omega_K$ consist of all extensions of the valuations of S to M and K, respectively. For the cardinalities of R, T and S we have $|R|\leqq[M:L]|S|$, $|T|\leqq[K:L]|S|$. Let $O_{L,S}$ be the ring of S-integers of L, that is the set of those elements $\gamma\in L$ for which $v(\gamma)\geqq 0$ for all $v\in\Omega_L\setminus S$. We define similarly $O_{M,R}$ and $O_{K,T}$. It is easily seen, that $O_{L,S}$, $O_{M,R}$, $O_{K,T}$ are rings, $k[z]\subseteqq O_L\subseteqq O_{L,S}$ and similar assertions hold also for $O_{M,R}$, $O_{K,T}$. Further we have $O_{L,S}\subset O_{M,R}\subseteqq O_{K,T}$. We remark that $O_{L,S}$ is integrally closed in L, $O_{K,T}$ is an integral ring extension of $O_{L,S}$ and if α' is the image of any $\alpha\in O_{K,T}$ under an arbitrary L-isomorphism of K, then $\alpha'\in O_{K,T}$ also holds.

If T_0 is any finite subset of Ω_K, containing the infinite valuations, then a non-zero element $\gamma\in K$ is called T_0-unit if $v(\gamma)=0$ for all $v\in\Omega_K\setminus T_0$ (that is both γ and γ^{-1} are in O_{K,T_0}).

Finally, let g and r be the genus and the number of infinite valuations of K, respectively.

3. Results

In the following δ will denote a given non-zero element of L. The elements $\alpha,\alpha^*\in O_{K,T}$ are called $O_{L,S}$-equivalent if $\alpha-\alpha^*\in O_{L,S}$. If $\alpha\in K$ we shall write, for brevity, $D(\alpha)$ instead of $D_{L(\alpha)/L}(\alpha)$. Our main result is as follows:

THEOREM 1. *Suppose that* $\alpha\in O_{K,T}$ *is of degree* $n\geqq 2$ *over* L. *If*

$$D(\alpha)=\delta \tag{1}$$

then α *is* $O_{L,\varrho}$*-equivalent to an* $\alpha^*\in O_{K,T}$ *which belongs to an effectively determinable finite set. Moreover* $H_K(\alpha^*)\leqq\frac{1}{2}H_K(\delta)$ *if* $n=2$ *and*

$$H_K(\alpha^*)\leqq 2(n-1)(3n-7)\big(|T|+H_K(\delta)+2g-2\big) \tag{2}$$

if $n\geqq 3$.

In the following we shall denote by C the constant $\frac{1}{2}H_K(\delta)$ if $n=2$ and the constant on the right hand side of (2) if $n\geqq 3$.

We remark that the general estimates of Győry [22] concerning integral elements with given discriminant over finitely generated integral domains yield also a bound for $H_K(\alpha^*)$ in the special case of function fields of one variable but our bound is much stronger. Similar remarks are valid also for our estimates (6), (9), (11) in Theorems 2, 3 and 4, respectively (see the results of Győry [22] and [21]).

In our papers [5], [6] we investigated some inhomogeneous generalizations of the problem of integral elements with given discriminant over numbers fields and finitely generated integral domains. An analogous result over function fields is the following:

COROLLARY 1. *Suppose that* $\alpha,\lambda\in O_{K,T}$ *and let* $n\geqq 2$ *and* m *be the degree of* $\alpha+\lambda$ *and* α *over* L, *respectively. If*

$$D(\alpha+\lambda)=\delta \tag{3}$$

and $H_K(\lambda)<\frac{1}{4m(m-1)}H_K(D(\alpha))$ *then* α *is* $O_{L,S}$*-equivalent to an* $\alpha^*\in O_{K,T}$ *such that*

$$H_K(\alpha^*) \leqq 2C. \tag{4}$$

In fact Theorem 2 of [6] also implies a bound for $H_K(\alpha^*)$ in the special case of function fields of one variable, but from our Theorem 1 we get a much better estimate here.

We remark that under the conditions of Corollary 1 α may assume infinitely many pairwise $O_{L,S}$-inequivalent values. To prove this observe that by Theorem 1 (3) implies that $\alpha+\lambda=\alpha^*+a$ where α^* is an element of a finite subset of $O_{K,T}$ and $a\in O_{L,S}$ is arbitrary. The proof of Corollary 1 shows that the restriction on the height of λ implies that λ is of bounded height. In fact, nothing more can be expected as a consequence of $H_K(\lambda)<\frac{1}{4m(m-1)}H_K(D(\alpha))$. Hence $\alpha=(\alpha^*-\lambda)+a$ where α^* may assume finitely many values and $H_K(\lambda)$ is bounded. But under these conditions $\alpha^*-\lambda$ may assume infinitely many pairwise $O_{L,S}$-inequivalent values, that is the finiteness assertion in the inhomogeneous case does not hold in general. We do not have finiteness assertion in Corollary 2 for similar reasons.

Mason [26], [27] (see also [28]) and [29] has derived an algorithm for solving Thue's equations, hyperelliptic equations and norm form equations, respectively, over function fields of one variable. Now we apply Theorem 1 to give an algorithm for determining effectively all solutions of discriminant form equations over function fields.

Győry [9], [10], [13], [15]—[18] and Győry and Papp [24], [25] gave effective bounds for all integer solutions of discriminant form equations over number fields. In this case the bounds for the solutions make possible to determine all solutions. These theorems were extended by Győry [21], [23] to the case of equations considered over finitely generated integral domains. In the absolute case, the bounds given for the sizes of the solutions enable one to determine the solutions, but in the relative case the bounds given for the degrees of the solutions do not imply even the finiteness of the number of solutions.

To formulate Theorem 2 let $\alpha_0=1, \alpha_1, \ldots, \alpha_m$ $(m\geqq 1)$ be given elements of $O_{K,T}$, linearly independent over L, and denote by $n(\geqq 2)$ the degree of $M=L(\alpha_1, \ldots, \alpha_m)$ over L. Let us suppose that $\max\limits_{1\leqq i\leqq m} H_K(\alpha_i)\leqq A$. Consider the solutions $(x_1, \ldots, x_m)\in O_{L,S}^m$ of the discriminant form equation

$$D_{M/L}(\alpha_1 x_1+\ldots+\alpha_m x_m) = \delta. \tag{5}$$

THEOREM 2. *There are only finitely many solutions of equation* (5) *and these can be effectively determined. Further, if* $(x_1, \ldots, x_m)\in O_{L,S}^m$ *is a solution of* (5) *then we have*

$$\max_{1\leqq i\leqq m} H_K(x_i) \leqq 2m(mA+C). \tag{6}$$

Here C is the constant defined after Theorem 1.

In [5] and [6] we studied inhomogeneous generalizations of discriminant form equations over number fields and over finitely generated integral domains, respectively. In the case of function fields of one variable the analogous problem is to consider the equation

$$D_{M/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m + \lambda) = \delta \tag{7}$$

where $x_1, \ldots, x_m \in O_{L,S}$ are dominating variables and $\lambda \in O_{M,R}$ is a non-dominating variable which is in a certain sense "small" compared to the dominating variables.

COROLLARY 2. *If* $x_1, \ldots, x_m \in O_{L,S}$ *and* $\lambda \in O_{M,R}$ *is a solution of equation* (7) *and* $H_K(\lambda) < \frac{1}{4m} \max_{1 \leq i \leq m} H_K(x_i)$ *then we have*

$$\max_{1 \leq i \leq m} H_K(x_i) \leq 4m(mA + C). \tag{8}$$

Applying Theorem 1 of [6] to the special case of function fields of one variable we also get a bound for the heights of the solutions of equation (7), but only a much weaker bound than (8).

In view of the remark after Corollary 1 we do not have finiteness assertion in Corollary 2.

In order to formulate Theorem 3 let M be a fixed extension field of L in K with $n = [M:L] \geq 2$. An important question is to decide when $O_{M,R}$ has a power basis $\{1, \alpha, \ldots, \alpha^{n-1}\}$ over $O_{L,S}$ $(\alpha \in O_{M,R})$, that is when does there exists an $\alpha \in O_{M,R}$ such that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis of $O_{M,R}$ as a free $O_{L,S}$-module. If this is the case, then $O_{M,R} = O_{L,S}[\alpha]$. Hence the ring extensions having this property are called simple or monogenic. Győry [11], [22] solved this problem for number fields and obtained also effective results [22] in the case of finitely generated integral domains which solved the problem in the absolute case. Evertse and Győry [3] proved a finiteness theorem for simple ring extensions over finitely generated integral domains which concerns also the relative case and thus the case of function fields of one variable. We shall prove that in the case of function fields of one variable all power bases can be effectively determined.

If such a power basis of $O_{M,R}$ exists then $O_{M,R}$ must be a free $O_{L,S}$-module. Hence we may assume that the discriminant D of a basis of $O_{M,R}$ over $O_{L,S}$ is given. Further, denote by $O_{L,S}^*$ the group of units of $O_{L,S}$. Lemma 3 of Evertse and Győry [3] implies that every unit $\varepsilon \in O_{L,S}^*$ can be uniquely written in the form $\varepsilon = l_0 \eta_1^{a_1} \ldots \eta_p^{a_p}$ where $l_0 \in k$, $\eta_1, \ldots, \eta_p \in O_{L,S}^*$ are fixed independent units and $a_1, \ldots, a_p \in \mathbf{Z}$. (We remark that the above quoted lemma implies also that $p \leq s-1$ where s is the number of valuations of S which are pairwise inequivalent on L.) We shall suppose that $\eta_1, \ldots, \eta_p$ are given and $\max_{1 \leq i \leq p} H_K(\eta_i) \leq E$. It is clear that if $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a power basis of $O_{M,R}$ over $O_{L,S}$ then $\{1, \alpha^*, \ldots, \alpha^{*n-1}\}$ is also a power basis for $\alpha^* = \varepsilon\alpha + a$ with any $\varepsilon \in O_{L,S}^*$ and $a \in O_{L,S}$.

THEOREM 3. *If* $\alpha \in O_{M,R}$ *and* $\{1, \alpha, \ldots, \alpha^{n-1}\}$ *is a basis of* $O_{M,R}$ *over* $O_{L,S}$ *then* α *is* $O_{L,S}$*-equivalent to an* $\varepsilon\alpha^*$ *where* $\varepsilon \in O_{L,S}^*$ *and* α^* *is an element of an effectively determinable finite subset of* $O_{M,R}$. *Moreover*

$$H_K(\alpha^*) \leq 2(n-1)(3n-5)[|T| + H_K(D) + n(n-1)(|S|-1)E + 2g - 2]. \tag{9}$$

An other important application of theorems concerning integers with given discriminant is to determine all integral elements with given discriminant and given norm, and further to determine the units with given discriminant. This problem was considered by Győry [11], [12] over number fields and by Győry [22] over finitely generated integral domains. Again the effective bounds in this latter result do not imply the finiteness of the number of integers in question in the case of function fields. The finiteness follows from a general result of Evertse and Győry [3] but this theorem does not make possible to determine this finite set of integers.

In Theorem 4 δ, μ will denote given non-zero elements of $O_{L,S}$ and we shall write $N(\alpha)$ instead of $N_{L(\alpha)/L}(\alpha)$ for any $\alpha\in K$.

THEOREM 4. *There are only finitely many* $\alpha\in O_{K,T}$ *of degree* $n\geqq 2$ *over* L *such that*

$$D(\alpha)=\delta \quad \text{and} \quad N(\alpha)=\mu. \tag{10}$$

Further, all these α *can be effectively determined and* (10) *implies*

$$H_K(\alpha)\leqq 14(n-1)(3n-5)(|T|+H_K(\delta)+2g-2)+H_K(\mu). \tag{11}$$

Denote by $O^*_{K,T}$ and $O^*_{L,S}$ the group of units of $O_{K,T}$ and $O_{L,S}$, respectively, and let $\eta_1, \ldots, \eta_p$ be given generators of $O^*_{L,S}$ with $\max\limits_{1\leqq i\leqq p} H_K(\eta_i)\leqq E$ as in Theorem 3. For units of $O_{K,T}$ with given discriminant we have the following

THEOREM 5. *There are only finitely many units* $\varepsilon\in O^*_{K,T}$ *of degree* $n\geqq 2$ *over* L *with* $D(\varepsilon)=\delta$ *and all these* ε *can be effectively determined. Moreover*

$$H_K(\varepsilon)\leqq 14(n-1)(3n-5)(|T|+H_K(\delta)+2g-2)+2(|\varrho|-1)E. \tag{12}$$

4. Proofs

The proof of our main theorem is based on the following result of Mason[3] [28] (see Lemma 2 and its Corollary in [28]):

LEMMA 1. *Let* S *denote a finite set of valuations on* K *and suppose that* $\gamma_1, \gamma_2, \gamma_3$ *are non-zero* S*-units in* K *such that* $\gamma_1+\gamma_2+\gamma_3=0$. *Then either* $\frac{\gamma_1}{\gamma_2}\in k$, *when* $H_K\left(\frac{\gamma_1}{\gamma_2}\right)=0$, *or* $\frac{\gamma_1}{\gamma_2}$ *has only finitely many possibilities in* K, *which may be effectively determined and*

$$H_K\left(\frac{\gamma_1}{\gamma_2}\right)\leqq |S|+2g-2.$$

Let us observe that under the condition $\gamma_1+\gamma_2+\gamma_3=0$ either any quotient $\frac{\gamma_i}{\gamma_j}$ $(1\leqq i<j\leqq 3)$ lies in k, or none of them lie in k.

[3] We remark that the assertion of our Lemma 1 as well as Lemma 2 holds for any finite extension field K of $k(z)$.

In our proof we shall use a special case of a general result of Evertse and Győry [3] concerning polynomials over integral domains. Following the definition of [3] we say that $f(x)\in O_{L,S}[x]$ is a *special polynomial,* if

$$f(x) = \mu^r h\big((x+a)^{n_0}/\mu\big)(x+a)^\delta \tag{13}$$

where r, n_0, δ are integers with $r>0$, $n_0>0$, $\delta\in\{0, 1\}$, $rn_0+\delta\geqq 3$ and $\delta=0$ if $n_0=1$, where $a\in O_{L,S}$, $0\neq\mu\in O_{L,S}$ and where $h(x)\in k[x]$ is a polynomial of degree r with non-zero discriminant and with $h(0)\neq 0$ if $n_0>1$. We remark that since k is algebraically closed, the roots of $h(x)$ also lie in k.

We shall use the following special case of Lemma 1 of Evertse and Győry [3]:

LEMMA 2. *Let $n\geqq 3$ be an integer and let $f(x)\in O_{L,S}[x]$ be a polynomial of degree n with zeros $\alpha_1, \dots, \alpha_n\in K$. Then the following statements are equivalent:*

(i) *f is special in $O_{L,S}[x]$*

(ii) *there are $a\in O_{L,S}$, $0\neq\lambda\in K$ and $c_1, \dots, c_n\in k$ such that $\alpha_i=c_i\lambda-a$ $(i=1, \dots, n)$,*

(iii) *there are integers i, j with $1\leqq i,j\leqq n$, $i\neq j$ such that $\dfrac{\alpha_i-\alpha_t}{\alpha_i-\alpha_j}\in k$, $t=1, \dots, n$.*

In our proof we need only the implication (iii)$\Rightarrow$(i).

PROOF OF THEOREM 1. Put $M=L(\alpha)$. Then by assumption $n=[M:L]\geqq 2$. Let us denote by $\alpha=\alpha_1, \dots, \alpha_n$ the images of α under the L-isomorphisms σ_i: $M\to K$ $(i=1, \dots, n)$.

The case $n=2$ is trivial since in this case from (1), that is from $(\alpha_1-\alpha_2)^2=\delta$ we can effectively determine $\varrho=\alpha_1-\alpha_2$ up to a square root of unity in k and $H_K(\varrho)\leqq \leqq\frac{1}{2}H_K(\delta)$. Further, $a=\dfrac{\alpha_1+\alpha_2}{2}\in O_{L,S}$ and thus $\alpha=\alpha_1$ can be written in the form $\alpha=\alpha^*+a$ where $\alpha^*=\dfrac{\varrho}{2}\in O_{K,T}$ is effectively determinable and $H_K(\alpha^*)\leqq \leqq\frac{1}{2}H_K(\delta)$.

In the following we shall suppose that $n\geqq 3$. In this case (1) can be written in the form

$$\prod_{1\leqq i<j\leqq n}(\alpha_i-\alpha_j)^2 = \delta. \tag{14}$$

I. First let us consider the case when there exist three different indices, say 1, 2, 3 such that $\dfrac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}\notin k$. Let us consider the identity

$$(\alpha_1-\alpha_2)+(\alpha_2-\alpha_3)+(\alpha_3-\alpha_1) = 0. \tag{15}$$

Let $\mathscr{H}(\delta)=\{v\in\Omega_K|v(\delta)>0\}$ and let $T_1=T\cup\mathscr{H}(\delta)$. We remark that for the cardinality of $\mathscr{H}(\delta)$ we have $|\mathscr{H}(\delta)|\leqq H_K(\delta)$. In view of equation (14) $\alpha_1-\alpha_2$, $\alpha_2-\alpha_3$, $\alpha_3-\alpha_1$ are all T_1-units. Applying Lemma 1 to (15) we obtain that $\varrho_{12}=\dfrac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}$ can be effectively determined and

$$H_K(\varrho_{12})\leqq |T_1|+2g-2\leqq |T|+H_K(\delta)+2g-2 = C_3.$$

Let $\sigma=\alpha_1-\alpha_3$, $\varrho_{23}=1-\varrho_{12}$ and $\varrho_{13}=1$. Then we get $\alpha_i-\alpha_j=\sigma\varrho_{ij}$ $(1\leqq i<j\leqq 3)$ where ϱ_{ij} can be effectively determined and $\max\limits_{1\leqq i<j\leqq 3} H_K(\varrho_{ij})\leqq C_3$.

We shall use induction to determine all $\alpha_i-\alpha_j$ $(1\leqq i<j\leqq n)$ in the form $\sigma\varrho_{ij}$ where σ is as above, ϱ_{ij} can be effectively determined and is of bounded height. The step from $l-1$ $(\geqq 3)$ to l is as follows.

Suppose that we have $\alpha_i-\alpha_j=\sigma\varrho_{ij}$ for $1\leqq i<j\leqq l-1$ where ϱ_{ij} can be effectively determined and $\max\limits_{1\leqq i<j\leqq l-1} H_K(\varrho_{ij})\leqq C_{l-1}$. Our aim is to determine all $\alpha_i-\alpha_l$ $(1\leqq i\leqq l-1)$ in a similar form.

Consider the quotients $\dfrac{\alpha_1-\alpha_l}{\alpha_1-\alpha_2}$ and $\dfrac{\alpha_1-\alpha_l}{\alpha_2-\alpha_3}$. If both of them were in k then also $\dfrac{\alpha_1-\alpha_2}{\alpha_2-\alpha_3}$ would be in k which is a contradiction. $\left(\text{We assumed that } \dfrac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}\notin k,\right.$ which in view of (15) and the remark after Lemma 1 implies $\left.\dfrac{\alpha_1-\alpha_2}{\alpha_2-\alpha_3}\notin k\right)$. So we may suppose for example that $\dfrac{\alpha_1-\alpha_l}{\alpha_1-\alpha_2}\notin k$. Consider the identity

$$(\alpha_1-\alpha_l)+(\alpha_l-\alpha_2)+(\alpha_2-\alpha_1)=0 \tag{16}$$

where the summands are again T_1-units. Applying Lemma 1 to (16) we obtain that $\bar{\varrho}_{1l}=\dfrac{\alpha_1-\alpha_l}{\alpha_l-\alpha_2}$ can be effectively determined and $H_K(\bar{\varrho}_{1l})\leqq C_3$. Put $\bar{\sigma}=\alpha_l-\alpha_2$, $\bar{\varrho}_{2l}=-1$ and $\bar{\varrho}_{12}=\bar{\varrho}_{1l}+1$, then we have $\alpha_1-\alpha_l=\bar{\sigma}\bar{\varrho}_{1l}$, $\alpha_2-\alpha_l=\bar{\sigma}\bar{\varrho}_{2l}$ and $\alpha_1-\alpha_2=\bar{\sigma}\bar{\varrho}_{12}$ where $\bar{\varrho}_{1l}$, $\bar{\varrho}_{2l}$, $\bar{\varrho}_{12}$ are effectively determinable with height at most C_3. But $\alpha_1-\alpha_2=\sigma\varrho_{12}$ also holds which yields $\bar{\sigma}=\sigma\varrho$ where $\varrho=\dfrac{\varrho_{12}}{\bar{\varrho}_{12}}$ can be effectively determined and $H_K(\varrho)\leqq 2C_3$. Thus we obtain $\alpha_1-\alpha_l=\sigma\varrho_{1l}$ and $\alpha_2-\alpha_l=\sigma\varrho_{2l}$ where $\varrho_{1l}=\varrho\bar{\varrho}_{1l}$ and $\varrho_{2l}=\varrho\bar{\varrho}_{2l}$ are effectively determinable with height at most $3C_3$. Further, for any i with $3\leqq i\leqq l-1$ we have

$$\alpha_i-\alpha_l=(\alpha_1-\alpha_l)-(\alpha_1-\alpha_i)=\sigma\varrho_{il}$$

where $\varrho_{il}=\varrho_{1l}-\varrho_{1i}$ can be again effectively determined with $\max\limits_{1\leqq i\leqq l-1} H_K(\varrho_{il})\leqq$ $\leqq 3C_3+C_{l-1}=C_l$, which completes the inductive step. We remark that finally we have $C_n=(3n-8)C_3$.

Using $\alpha_i-\alpha_j=\sigma\varrho_{ij}$ $(1\leqq i<j\leqq n)$, in view of equation (14) we obtain

$$\sigma^{n(n-1)}=\frac{\delta}{\prod\limits_{1\leqq i<j\leqq n}\varrho_{ij}^2}$$

whence σ can be determined up to an $n(n-1)$-th root of unity and $H_K(\sigma)\leqq$ $\leqq\dfrac{1}{n(n-1)}H_K(\delta)+(3n-8)C_3$. Now we have shown that all the differents $s_{ij}=\alpha_i-\alpha_j$ $(1\leqq i<j\leqq n)$ are effectively determinable and

$$\max_{1\leqq i<j\leqq n} H_K(s_{ij})\leqq\frac{1}{n(n-1)}H_K(\delta)+2(3n-8)C_3.$$

Let $\alpha^*=\frac{s_{12}+\ldots+s_{1n}}{n}$ and $a=\frac{\alpha_1+\ldots+\alpha_n}{n}$. Then $\alpha=\alpha_1=\alpha^*+a$, $\alpha^*\in O_{K,T}$ can be effectively determined and $a\in O_{L,S}$. Further,

$$H_K(\alpha^*) \leqq \frac{1}{n} H_K(\delta)+2(3n-8)(n-1)C_3$$

whence (2) follows.

II. Let us consider now the opposite case, when for any three different indices i, j, l we have $\frac{\alpha_i-\alpha_l}{\alpha_i-\alpha_j}\in k$. Then condition (iii) of Lemma 2 is satisfied and by (i) we obtain that the defining polynomial $f(x)=(x-\alpha_1)\ldots(x-\alpha_n)$ of α over L is a special polynomial in $O_{L,S}[x]$.

Using an argument of Evertse and Győry [3] we prove that

$$\alpha_i = \varrho^{k_i}\vartheta - a \quad (i = 1, \ldots, n) \tag{17}$$

where ϱ is a fixed primitive n-th root of unity, $\vartheta\in O_{K,T}$, $a\in O_{L,S}$ and $\{k_1, \ldots, k_n\}$ is a permutation of $\{1, \ldots, n\}$.

(i) implies that $f(x)$ is of type (13). But f is irreducible over L and thus in (13) $\delta=0$ and h is irreducible. Further, since the roots of h are in k, hence $r=1$ is necessary in order that h be irreducible. But then we obtain that there exist $0\neq\mu_0\in O_{L,S}$ and $a\in O_{L,S}$ such that $f(x)=(x+a)^n-\mu_0$. From this it follows that if ϑ is a fixed n-th root of μ_0 and ϱ is a primitive n-th root of unity, then we have (17) with a suitable permutation $\{k_1, \ldots, k_n\}$ of $\{1, \ldots, n\}$. Since in (17) $\alpha=\alpha_1\in O_{K,T}$ and $a\in O_{L,S}$, hence $\vartheta\in O_{K,T}$.

Combining (14) and (17) we have

$$\vartheta^{n(n-1)} \prod_{1\leqq i<j\leqq n} (\varrho^{k_i}-\varrho^{k_j})^2 = \delta. \tag{18}$$

Since in this equation $\prod\limits_{1\leqq i<j\leqq n} (\varrho^{k_i}-\varrho^{k_j})^2$ may assume only finitely many effectively determinable values, and since it is an element of k, hence from (18) we may determine all possibilities for ϑ, which are finite in number and $H_K(\vartheta)\leqq\frac{1}{n(n-1)} H_K(\delta)$. Thus from (17) we obtain that $\alpha=\alpha_1=\alpha^*-a$ where $\alpha^*=\varrho^{k_1}\vartheta\in O_{K,T}$ can be effectively determined, $H_K(\alpha^*)\leqq\frac{1}{n(n-1)} H_K(\delta)$ and $a\in O_{L,S}$, whence the assertion of Theorem 1 follows in this case, too.

Proof of Corollary 1. Applying Theorem 1 to (3) we obtain that $\alpha+\lambda=\alpha^*+a$ where $\alpha^*\in O_{K,T}$ with height at most C and $a\in O_{L,S}$. Let $\alpha_{ij}=\sigma_i(\alpha)-\sigma_j(\alpha)$ and define similarly $\lambda_{ij}=\sigma_i(\lambda)-\sigma_j(\lambda)$ and $\alpha^*_{ij}=\sigma_i(\alpha^*)-\sigma_j(\alpha^*)$ for any i, j with $1\leqq i<j\leqq n$. Then we have

$$\alpha_{ij}+\lambda_{ij} = \alpha^*_{ij} \quad (1 \leqq i < j \leqq n). \tag{19}$$

By assumption $H_K(\lambda)<\frac{1}{4m(m-1)}H_K(D(\alpha))$ which implies

$$H_K(\lambda) < \frac{1}{4}H_K(\alpha_{ij}) \tag{20}$$

where the indices i, j are fixed so that $H_K(\alpha_{ij})=\max_{1\leqq i_0<j_0\leqq n} H_K(\alpha_{i_0 j_0})$. By (19) and (20) we get

$$H_K(\alpha_{ij}) \leqq H_K(\alpha^*_{ij})+H_K(\lambda_{ij}) \leqq 2C+\frac{1}{2}H_K(\alpha_{ij})$$

whence $H_K(\alpha_{ij})\leqq 4C$ and in view of (20) $H_K(\lambda)<C$. Now let $\alpha^{**}=\alpha^*-\lambda$ then $\alpha=\alpha^{**}+a$, $\alpha^{**}\in O_{K,T}$ and $H_K(\alpha^{**})\leqq 2C$.

Proof of Theorem 2. Let $(x_1, \ldots, x_m)\in O^m_{L,S}$ be a fixed solution of equation (5). Put $l(\mathbf{x})=\alpha_1 x_1+\ldots+\alpha_m x_m$ and denote by $l(\mathbf{X})$ the linear form $\alpha_1 X_1+\ldots+\alpha_m X_m$ in $X_1, \ldots, X_m$. Applying Theorem 1 to equation (5) we obtain that

$$l(\mathbf{x}) = \alpha^*+a \tag{21}$$

where $\alpha^*\in O_{K,T}$ can be effectively determined, $H_K(\alpha^*)\leqq C$ and $a\in O_{L,S}$. Put $\alpha^*_{ij}=\sigma_i(\alpha^*)-\sigma_j(\alpha^*)$ and $l_{ij}(\mathbf{X})=(\sigma_i(\alpha_1)-\sigma_j(\alpha_1))X_1+\ldots+(\sigma_i(\alpha_m)-\sigma_j(\alpha_m))X_m$ for any i, j, $1\leqq i<j\leqq n$. It follows from (21) that $x_1, \ldots, x_m$ is a solution of the system of linear equations

$$l_{ij}(\mathbf{X}) = \alpha^*_{ij} \quad (1\leqq i<j\leqq n) \tag{22}$$

where the coefficients on the left side are given with height $\leqq 2A$ and the constant terms on the right side are also effectively determinable and their heights are $\leqq 2C$. $\alpha_1, \ldots, \alpha_m$ are linearly independent over L whence (22) has a unique solution. Solving this equation system by Cramer's rule we can effectively determine $x_1, \ldots, x_m$. Further, since there are finitely many possibilities for α^*, hence there are only finitely many possibilities also for $x_1, \ldots, x_m$.

Using Corollary to Lemma 8 of Mason [28] for the solutions $x_1, \ldots, x_m$ of the system of linear equations (22) we have

$$\max_{1\leqq i\leqq m} H_K(x_i) \leqq 2m(mA+C)$$

which proves (6).

Proof of Corollary 2. Let $x_1, \ldots, x_m\in O_{L,S}$ and $\lambda\in O_{M,R}$ be a fixed solution of equation (7) and put $X=\max_{1\leqq i\leqq m} H_K(x_i)$. Let $l(\mathbf{x})$, $l(\mathbf{X})$ and $l_{ij}(\mathbf{X})$ be as in the proof of Theorem 2. Applying Theorem 1 to (7) we obtain that

$$l(\mathbf{x})+\lambda = \alpha^*+a \tag{23}$$

where $\alpha^*\in O_{K,T}$ with $H_K(\alpha^*)\leqq C$ and $a\in O_{L,S}$. Let $\alpha^*_{ij}=\sigma_i(\alpha^*)-\sigma_j(\alpha^*)$ and $\lambda_{ij}=\sigma_i(\lambda)-\sigma_j(\lambda)$ $(1\leqq i<j\leqq n)$. Then (23) implies that $x_1, \ldots, x_m$ is a solution of the system of linear equations

$$l_{ij}(\mathbf{X}) = \alpha^*_{ij}-\lambda_{ij} \quad (1\leqq i<j\leqq n) \tag{24}$$

where $H_K(\alpha_{ij}^*-\lambda_{ij})\leqq 2C+2H_K(\lambda)$ and the heights of the coefficients on the left side are at most $2A$. Using again Corollary to Lemma 8 of Mason [28], (24) implies

$$X \leqq 2m(mA+C)+2mH_K(\lambda)$$

whence, by our condition $H_K(\lambda)<\frac{1}{4m}X$, we get

$$X < 2m(mA+C)+\frac{1}{2}X$$

which implies (8).

Proof of Theorem 3. If $\alpha\in O_{M,R}$ is such that $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis of $O_{M,R}$ over $O_{L,S}$, then the discriminant $D_{M/L}(1, \alpha, \ldots, \alpha^{n-1})$ of this basis equals $D_{M/L}(\alpha)$. Further, a well-known argument shows that

$$D_{M/L}(\alpha) = \varepsilon D \tag{25}$$

where $\varepsilon\in O_{L,S}^*$ (see e.g. [2]). In view of our remarks before Theorem 3 ε may be written in the form $\varepsilon=l_0\eta_1^{a_1}\ldots\eta_p^{a_p}$ where $l_0\in k$ and $a_1, \ldots, a_p\in \mathbf{Z}$. Obviously $p\leqq s-1=|S|-1$. Let $a_i=n(n-1)q_i+r_i$ where $q_i, r_i\in\mathbf{Z}$ and $0\leqq r_i<n(n-1)$ $(i=1, \ldots, p)$. Denote by l_1 an $n(n-1)$-th root of l_0 ($l_1\in k$ since k is algebraically closed) and put $\varepsilon_1=l_1\eta_1^{q_p}\ldots\eta_p^{q_p}$, $\varepsilon_2=\eta_1^{r_1}\ldots\eta_p^{r_p}$. (25) implies

$$D_{M/L}(\varepsilon_1^{-1}\alpha) = \varepsilon_2 D. \tag{26}$$

Here $\varepsilon_2 D$ has only finitely many possibilities in $O_{L,S}$ which can be determined and

$$H_K(\varepsilon_2 D) \leqq H_K(D)+n(n-1)(|S|-1)E. \tag{27}$$

Applying Theorem 1 to (26) we obtain that $\varepsilon_1^{-1}\alpha=\alpha^*+a$, that is $\alpha=\varepsilon_1\alpha^*+a_1$ where $\alpha^*\in O_{M,R}$ can be effectively determined, $\varepsilon_1\in O_{L,S}^*$ and $a, a_1\in O_{L,S}$. Further, writing the right hand side of (27) instead of $H_K(\delta)$ in (2), we get (9).

Proof of Theorem 4. Let $M=L(\alpha)$, then we have $n=[M:L]\geqq 2$. Applying Theorem 1 to $D(\alpha)=\delta$ we obtain that $\alpha=\alpha^*+\alpha$ where $\alpha^*\in O_{M,R}$ may be effectively determined, $H_K(\alpha^*)<C$ and $a\in O_{L,S}$. Let $\alpha_i^*=\sigma_i(\alpha^*)$, $i=1, \ldots, n$ $(\alpha^*=\alpha_1^*)$ denote the conjugates of α^* over L. Then $N(\alpha)=\mu$ may be written in the form

$$\prod_{i=1}^{n}(\alpha_i^*+a) = \mu. \tag{28}$$

Let us consider the polynomial $F(X)=\prod_{i=1}^{n}(\alpha_i^*+X)-\mu$. Clearly the coefficients of $F(X)$ are in K. Our assumption that k is explicitly given implies that K is also explicitly given (cf. [4], or [30] pp. 128—131), hence we can split $F(X)$ into irreducible factors in $K[X]$ and we can effectively determine all the roots of $F(X)$ in K. Further, we can decide if $F(X)$ has roots in $O_{L,S}$, which means that we can effectively determine all possibilities for $a\in O_{L,S}$ in (28) which are finite in number. From this it follows that all possibilities for $\alpha=\alpha^*+a$ are also finite and can be determined.

In order to get a bound for $H_K(\alpha)$ consider the identity

(29) $$(\alpha_1^*+a)-(\alpha_2^*+a)+(\alpha_2^*-\alpha_1^*) = 0$$

(obviously $\alpha^*=\alpha_1^*\neq\alpha_2^*$). Let $\overline{\mathscr{H}}(\alpha_2^*-\alpha_1^*)=\{v\in\Omega_K|v(\alpha_2^*-\alpha_1^*)\neq 0\}$ and $\mathscr{H}(\mu)=\{v\in\Omega_K|v(\mu)>0\}$. For the cardinalities of these sets we have $|\overline{\mathscr{H}}(\alpha_2^*-\alpha_1^*)|\leqq 2H_K(\alpha_2^*-\alpha_1^*)\leqq 4C$ and $|\mathscr{H}(\mu)|\leqq H_K(\mu)$. Put $T_2=T\cup\overline{\mathscr{H}}(\alpha_2^*-\alpha_1^*)\cup\mathscr{H}(\mu)$. Then in (29) all summands are T_2-units and applying Lemma 1 to (29) we obtain

$$H_K\left(\frac{\alpha_1^*+a}{\alpha_2^*-\alpha_1^*}\right) < |T_2|+2g-2$$

whence in view of $\alpha=\alpha_1^*+a$ and $H_K(\alpha^*)\leqq C$, (11) follows.

Proof of Theorem 5. Put $M=L(\varepsilon)$. Then by assumption $n=[M:L]\geqq 2$. Using Theorem 1 again $D(\varepsilon)=\delta$ implies $\varepsilon=\alpha^*+a$ where $\alpha^*\in O_{M,R}$ is effectively determinable with $H_K(\alpha^*)\leqq C$ and $a\in O_{L,S}$. If $\varepsilon\in O_{K,T}^*$ then $N(\varepsilon)\in O_{L,S}^*$, that is

$$N(\varepsilon) = l_0\eta_1^{a_1}\dots\eta_p^{a_p}$$

where $l_0\in k$, $a_1, \dots, a_p\in\mathbf{Z}$. Let $\alpha_i^*=\sigma_i(\alpha^*)$, $i=1, \dots, n$ $(\alpha^*=\alpha_1^*)$ as before. Then the above equation can be written in the form

(30) $$\prod_{i=1}^{n}(\alpha_i^*+a) = l_0\eta_1^{a_1}\dots\eta_p^{a_p}.$$

Consider the identity

(31) $$(\alpha_1^*+a)-(\alpha_2^*+a)+(\alpha_2^*-\alpha_1^*) = 0$$

(where $\alpha^*=\alpha_1^*\neq\alpha_2^*$). Let $\overline{\mathscr{H}}(\alpha_2^*-\alpha_1^*)=\{v\in\Omega_K|v(\alpha_2^*-\alpha_1^*)\neq 0\}$ be as in the proof of Theorem 4 and let

$$\overline{\mathscr{H}}(\eta_i) = \{v\in\Omega_K|v(\eta_i)\neq 0\},\quad i=1, \dots, n.$$

If we put

$$T_3 = T\cup\overline{\mathscr{H}}(\alpha_2^*-\alpha_1^*)\cup\overline{\mathscr{H}}(\eta_1)\cup\dots\cup\overline{\mathscr{H}}(\eta_p)$$

then by equation (30) all summands in (31) are T_3-units and for the cardinality of T_3 we have

(32) $$|T_3| \leqq |T|+4C+2(|S|-1)E.$$

Let us apply Lemma 1 to (31). There are two possible cases.

If $\dfrac{\varepsilon}{\alpha_2^*-\alpha_1^*}=\dfrac{\alpha_1^*+a}{\alpha_2^*-\alpha_1^*}\notin k$ then by Lemma 1 there are only finitely many effectively determinable possibilities for $\dfrac{\alpha_1^*+a}{\alpha_2^*-\alpha_1^*}$, hence we can determine also $\varepsilon=\alpha_1^*+a$.

In the opposite case, if $\dfrac{\varepsilon}{\alpha_2^*-\alpha_1^*}=\dfrac{\alpha_1^*+a}{\alpha_2^*-\alpha_1^*}=l_1\in k$ then $M=L(\varepsilon)=L(\alpha_2^*-\alpha_1^*)$. Since $D_{M/L}(\varepsilon)=\delta$, hence we get

(33) $$\delta = D_{M/L}(\alpha^*) = l_1^{n(n-1)}D_{M/L}(\alpha_2^*-\alpha_1^*).$$

In this equation $D_{M/L}(\alpha_2^*-\alpha_1^*)$ can be effectively determined (together with α^*), that is (33) enables us to determine all possibilities also for l_1, which are finite in number. We conclude that $\varepsilon=\alpha_1^*+a=l_1(\alpha_2^*-\alpha_1^*)$ can be effectively determined in this case, too.

To prove (12) we apply the second part of Lemma 1 to (31) and we obtain

$$H_K\left(\frac{\alpha_1^*+a}{\alpha_2^*-\alpha_1^*}\right) \leqq |T_3|+2g-2.$$

Combining it with (32) and using that $H_K(\alpha^*)\leqq C$, we get (12).

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References

[1] A. Baker, *Transcendental number theory*, 2nd ed. (Cambridge, 1979).

[2] Z. I. Borevich and I. R. Shafarevich, *Number theory*, 2nd ed., Academic Press (New York and London, 1967).

[3] J. H. Evertse and K. Győry, On the number of polynomials and integral elements of given discriminant, *Acta Math. Hungar.*, **51** (1988), 341—362.

[4] A. Fröhlich and J. C. Shepherdson, Effective procedures in field theory, *Philos. Trans. Roy. Soc. London*, Ser. A, **248** (1956), 407—432.

[5] I. Gaál, Inhomogeneous discriminant form and index form equations and their applications, *Publ. Math. Debrecen*, **33** (1986), 21—27.

[6] I. Gaál, Inhomogeneous discriminant form equations and integral elements with given discriminant over finitely generated integral domains, *Publ. Math. Debrecen*, **34** (1987), 109—122.

[7] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné, *Acta Arith.*, **23** (1973), 419—426.

[8] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné, II, *Publ. Math. Debrecen*, **21** (1974), 125—144.

[9] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné, III, *Publ. Math. Debrecen*, **23** (1976), 141—165.

[10] K. Győry, Polynomials with given discriminant, Coll. Math. Soc. János Bolyai, 13 (Debrecen, 1974). *Topics in number theory* (Edited by P. Turán), North-Holland Publ. Comp. (Amsterdam—Oxford—New York, 1976), pp. 65—78.

[11] K. Győry, On polynomials with integer coefficients and given discriminant, IV, *Publ. Math. Debrecen*, **25** (1978), 155—167.

[12] K. Győry, On polynomials with integer coefficients and given discriminant. V, p-adic generalizations, *Acta Math. Acad. Sci. Hungar.*, **32** (1978), 175—190.

[13] K. Győry, On the greatest prime factors of decomposable forms at integer points, *Ann. Acad. Sci. Fenn.*, Ser. A, I, **4** (1978/1979), 341—355.

[14] K. Győry, On certain graphs composed of algebraic integers of a number field and their applications, I, *Publ. Math. Debrecen*, **27** (1980), 229—242.

[15] K. Győry, Explicit upper bounds for the solutions of some diophantine equations, *Ann. Acad. Sci. Fenn.*, Ser. A, I, **5** (1980), 3—12.

[16] K. Győry, Résultats effectifs sur la représentation des entiers par des formes décomposables, *Queen's Papers in Pure and Applied Math.*, No. 56 (Kingston, Canada, 1980).

[17] K. Győry, On the representation of integers by decomposable forms in several variables, *Publ. Math. Debrecen*, **28** (1981), 89—98.

[18] K. Győry, On S-integral solutions of norm form, discriminant form and index form equations, *Studia Sci. Math. Hungar.*, **16** (1981), 149—161.

[19] K. Győry, On discriminants and indices of integers of an algebraic number field, *J. Reine Angew. Math.*, **324** (1981), 114—126.

[20] K. Győry, On certain graphs associated with an integral domain and their applications to diophantine problems, *Publ. Math. Debrecen,* **29** (1982), 79—94.
[21] K. Győry, Bounds for the solutions of norm form, discriminant form and index form equations in finitely generated integral domains, *Acta Math. Hungar.,* **42** (1983), 45—80.
[22] K. Győry, Effective finiteness theorems for polynomials with given discriminant and integral elements with given discriminant over finitely generated domains, *J. Reine Angew. Math.,* **346** (1984), 54—100.
[23] K. Győry, On norm form, discriminant form and index form equations, Coll. Math. Soc. János Bolyai, 34 (Budapest, 1981). *Topics in classical number theory,* North-Holland Publ. Comp. (Amsterdam, 1984), pp. 617—676.
[24] K. Győry and Z. Z. Papp, On discriminant form and index form equations, *Studia Sci. Math. Hungar.,* **12** (1977), 47—60.
[25] K. Győry and Z. Z. Papp, Effective estimates for the integer solutions of norm form and discriminant form equations, *Publ. Math. Debrecen,* **25** (1978), 311—325.
[26] R. C. Mason, On Thue's equation over function fields, *J. London Math. Soc.,* Ser. 2, **24** (1981), 414—426.
[27] R. C. Mason, The hyperelliptic equation over function fields, *Proc. Camb. Philos. Soc.,* **93** (1983), 219—230.
[28] R. C. Mason, *Diophantine equations over function fields,* London Math. Soc. Lecture Note Series, No. 96 (Cambridge, 1984).
[29] R. C. Mason, Norm form equations, *J. Number Theory,* **22** (1986), 190—207.
[30] B. L. van der Waerden, *Moderue Algebra,* **I,** Springer Verlag (Berlin, 1930).

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ON STRONGLY NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES

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Dedicated to Professor I. Kátai on his 50th birthday

The aim of this paper is to prove existence of solutions of variational inequalities in a possibly unbounded domain $\Omega\subset R^n$ with respect to operators of the form

(0.1)
$$u \mapsto \sum_{|\alpha|\leqq m}(-1)^{|\alpha|}D^\alpha[f_\alpha(x, u, \dots, D^\beta u, \dots)]+\sum_{|\alpha|\leqq m-1}(-1)^{|\alpha|}D^\alpha[g_\alpha(x, u, \dots, D^\beta u, \dots)]$$

where $\beta=(\beta_1, \dots, \beta_n)$, $|\beta|=\sum_{j=1}^n \beta_j$, $|\beta|\leqq m$, $D_j=\frac{\partial}{\partial x_j}$, $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$. Here on $f_\alpha(x, \xi)$ no growth restriction is imposed with respect to ξ, further $f_\alpha(x, \xi)$ satisfy some special conditions; the functions $g_\alpha(x, \xi)$ must have certain polynomial growth with respect to ξ. It will be proved that solutions of the variational inequalities can be obtained as limits of solutions of variational inequalities considered in $\Omega_r= =\Omega\cap B_r$ $(B_r=\{x\in R^n\colon |x|<r\})$.

The Dirichlet problem for the operator (0.1) with $g_\alpha=0$ and rather general growth condition with respect to $\xi\mapsto f_\alpha(x, \xi)$ has been considered by Landes ([1]—[4]) where existence of solutions has been proved. General variational inequalities in bounded domains Ω for (0.1) with $g_\alpha=0$ and $f_\alpha(x, \xi)=h_\alpha(x, \xi_\alpha)$ (without any growth condition on h_α with respect to ξ_α) have been studied by Simader and Mustonen ([5]—[7]) where also uniqueness and stability results have been shown. Similar existence theorems on boundary value problems for second order equations in bounded domains have been obtained in [8] with less restrictions on f_α, $g_\alpha=0$. Operators, where only in some lower order terms no growth restriction is imposed, have been considered e.g. by Webb [9], Mustonen [6] and the author ([10]—[12]). The proofs of this paper are based on arguments used in [3], [5], [8].

1. The formulation of the results

Let $\Omega\subset R^n$ be a (possibly unbounded) domain such that for sufficiently large $r>0$ $\Omega_r=\Omega\cap B_r$ has the weak cone property (see [13]), $p>1$ and m a positive integer. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u whose distributional derivatives of order $\leqq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is

$$\|u\|=\Big\{\sum_{|\alpha|\leqq m}\int_\Omega |D^\alpha u|^p\,dx\Big\}^{1/p}.$$

$W_{p,0}^m(\Omega)$ will denote the closure in $\|\cdot\|_{W_p^m(\Omega)}$ of $C_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support contained in Ω.

Let N, M be the numbers of multiindices α, satisfying $|\alpha| \leqq m$ resp. $|\alpha| \leqq m-1$. The vectors $\xi = (\xi_0, \ldots, \xi_\beta, \ldots) \in R^N$ will be written in the form $\xi = (\eta, \zeta)$ where η consists of those ξ_β for which $|\beta| \leqq m-1$. Assume that

I. For $|\alpha| = m$ the functions f_α: $\Omega \times R^N \to R$ satisfy the Carathéodory conditions, i.e. they are measurable in x for each fixed $\xi \in R^N$ and continuous in ξ for almost all $x \in \Omega$. For $|\alpha| \leqq m-1$ the functions f_α: $\Omega \times R^M \to R$ also satisfy the Carathéodory conditions, i.e. they are measurable in x for each $\eta \in R^M$ and continuous in η for almost all $x \in \Omega$.

II. For all $(\eta, \zeta), (\eta, \zeta') \in R^N$ with $\zeta \neq \zeta'$ and a.e. $x \in \Omega$

$$\sum_{|\alpha|=m} [f_\alpha(x, \eta, \zeta) - f_\alpha(x, \eta, \zeta')](\xi_\alpha - \xi'_\alpha) > 0.$$

III. There exist constants c_1, c_2 and $k_1, k_2 \in L^1(\Omega)$ such that for all $\xi, \xi' \in R^N$ and a.e. $x \in \Omega$

$$\sum_{|\alpha|=m} f_\alpha(x, \xi)\xi'_\alpha \leqq \sum_{|\alpha|=m} f_\alpha(x, \xi')\, \xi'_\alpha + c_1 \sum_{|\alpha|=m} f_\alpha(x, \xi)\xi_\alpha + k_1(x), \tag{1.1}$$

for all $\eta, \eta' \in R^M$ and a.e. $x \in \Omega$

$$\sum_{|\alpha| \leqq m-1} f_\alpha(x, \eta)\xi'_\alpha \leqq \sum_{|\alpha| \leqq m-1} f_\alpha(x, \eta')\, \xi'_\alpha + c_2 \sum_{|\alpha| \leqq m-1} f_\alpha(x, \eta)\xi_\alpha + k_2(x). \tag{1.2}$$

IV. For any $s > 0$ there is a function $f_s \in L^1(\Omega_r)$ for all r such that for a.e. $x \in \Omega$

$$|f_\alpha(x, \xi)| \leqq f_s(x) \quad \text{if} \quad |\xi| \leqq s, \quad |\alpha| = m,$$
$$|f_\alpha(x, \eta)| \leqq f_s(x) \quad \text{if} \quad |\eta| \leqq s, \quad |\alpha| \leqq m-1.$$

V. There exist nonnegative functions $k_\alpha, k_3 \in L^1(\Omega)$ and a constant $c_3 > 0$ such that for all $\xi \in R^N$, $\eta \in R^M$, a.e. $x \in \Omega$

$$f_\alpha(x, \xi)\xi_\alpha \geqq -k_\alpha(x) \quad \text{if} \quad |\alpha| = m, \quad f_\alpha(x, \eta)\xi_\alpha \geqq -k_\alpha(x) \quad \text{if} \quad |\alpha| \leqq m-1,$$
$$\sum_{|\alpha| \leqq m-1} f_\alpha(x, \eta)\xi_\alpha + \sum_{|\alpha|=m} f_\alpha(x, \xi)\xi_\alpha \geqq c_3(|\zeta|^p + |\eta|^p) - k_3(x).$$

VI. The functions p_α, r_α: $\Omega \times R^N \to R$ satisfy the Carathéodory conditions and

$$g_\alpha = p_\alpha + r_\alpha \quad (|\alpha| \leqq m-1).$$

VII. There exist a nonnegative function $k_4 \in L^1(\Omega)$, a nonnegative constant $c_4 < c_3$ and a bounded $\Omega' \subset \Omega$ such that for all $\xi \in R^N$, $\eta \in R^M$ and a.e. $x \in \Omega$

$$\sum_{|\alpha| \leqq m-1} p_\alpha(x, \xi)\xi_\alpha \geqq -c_4(|\zeta|^p + |\eta|^p) - k_4(x),$$
$$p_\alpha(x, \xi) = r_\alpha(x, \xi) = 0 \quad \text{if} \quad x \in \Omega \setminus \Omega'.$$

VIII. There exist functions $\Phi_\alpha \in L^{p/\varrho_{|\alpha|}}(\Omega')$, $h_\alpha \in L^q(\Omega')$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ and a continuous function C_α such that for all $\xi \in R^N$, a.e. $x \in \Omega$

$$|p_\alpha(x, \xi)| \leqq C_\alpha(\xi')[\Phi_\alpha(x) + |\xi''|^{\varrho_{|\alpha|}}], \quad |r_\alpha(x, \xi)| \leqq h_\alpha(x)$$

where $\xi=(\xi', \xi'')$ and ξ' consists of those ξ_γ for which $|\gamma|<m-\frac{n}{p}$,

$$p-1 \leqq \varrho_{|\alpha|} < p-1+\frac{(m-|\alpha|)p}{n}, \quad \varrho_{|\alpha|} \leqq p.$$

IX. V is a closed subspace of $W_p^m(\Omega)$ and $K\subset V$ is a closed convex set containing 0. Denote by K^0 the set of all $v\in K$ such that $D^\alpha v\in L^\infty(\Omega)$ for $|\alpha|\leqq m$ and $v(x)=0$ a.e. for sufficiently large $|x|$. Assume that for each $v\in K^0$ there are $v_j\in C_0^\infty(R^n)$ such that $v_j|_\Omega\in K$, $v_j|_\Omega\to v$ with respect to the norm of $W_p^m(\Omega)$ and $\sup|D^\alpha v_j|\leqq c_\alpha$.

THEOREM 1. *Suppose that conditions* I *to* IX *are fulfilled. Then for any* $G\in V^*$ *(i.e. for any linear continuous functional over* V*) there is* $u\in K$ *such that for each* r

$$f_\alpha(\cdot, u, \dots, D^\beta u, \dots)\in L^1(\Omega_r), \quad f_\alpha(\cdot, u, \dots, D^\beta u, \dots)D^\alpha u\in L^1(\Omega), \tag{1.3}$$

$$\sum_{|\alpha|\leqq m}\int_\Omega f_\alpha(x, u, \dots, D^\beta u, \dots)(D^\alpha v-D^\alpha u)\,dx+ \tag{1.4}$$

$$+\sum_{|\alpha|\leqq m-1}\int_\Omega g_\alpha(x, u, \dots, D^\beta u, \dots)(D^\alpha v-D^\alpha u)\,dx \geqq \langle G, v-u\rangle$$

for all $v\in K^0$.

This theorem will be a simple consequence of Theorem 2 to be formulated below.

Let V_r be the closure of

$$\{\varphi|_{\Omega_r}:\ \varphi\in C_0^\infty(B_r)\cap V\}$$

in $W_p^m(\Omega_r)$ and

$$K_r = V_r\cap\{u|_{\Omega_r}:\ u\in K\},$$

i.e. the closure of

$$\{\varphi|_{\Omega_r}:\ \varphi\in C_0^\infty(B_r)\cap K\}$$

in $W_p^m(\Omega_r)$. Then V_r is a closed linear subspace of $W_p^m(\Omega_r)$ and K_r is a closed convex subset of V_r containing 0. Extending functions $u\in V_r$ as 0 on $\Omega\setminus\Omega_r$, the extensions belong to V.

Let $s>\max\{n, p\}$. Then by Sobolev's imbedding theorem $W_s^{m+1}(\Omega_r)$ is continuously imbedded in $W_p^m(\Omega_r)$ and $C_*^m(\Omega_r)$, i.e. the space of m times continuously differentiable functions with bounded derivatives equipped with the norm

$$\|u\|_{C_*^m(\Omega_r)} = \sum_{|\alpha|\leqq m}\sup_{\Omega_r}|D^\alpha u|.$$

(See e.g. [13].) Further, let

$$W_r = W_{s,0}^{m+1}(B_r)\cap V_r \quad\text{and}\quad K_r^1 = K_r\cap W_r = K_r\cap W_{s,0}^{m+1}(B_r).$$

Then W_r is a closed linear subspace of $W_s^{m+1}(\Omega_r)$ and K_r^1 is a closed convex subset of W_r. Define S_r by

$$\langle S_r(u), v\rangle = \sum_{|\alpha|\leqq m+1}\int_{\Omega_r}|D^\alpha u|^{s-2}(D^\alpha u)(D^\alpha v)\,dx,$$

$u, v \in W_r$. Then $S_r(u)$ is a continuous linear functional on W_r and $S_r\colon W_r \to W_r^*$ is a bounded (nonlinear) operator. By Sobolev's imbedding theorem

$$\langle T_r(u), v\rangle = \sum_{|\alpha| \leqq m} \int_{\Omega_r} f_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha v\, dx,$$

$u, v \in W_r$ also defines a bounded (nonlinear) operator $T_r\colon W_r \to W_r^*$. Define $q_{|\alpha|}$ by

$$\frac{1}{p/\varrho_{|\alpha|}} + \frac{1}{q_{|\alpha|}} = 1. \tag{1.5}$$

Then by assumption VIII $D^\alpha u \in L^{q_{|\alpha|}}(\Omega')$ if $u \in W_p^m(\Omega)$, $W_p^{m-|\alpha|}(\Omega)$ is continuously imbedded in $L^{q_{|\alpha|}}(\Omega')$ (see [13]) and thus by VIII and Hölder's inequality

$$\langle Q_r(u), v\rangle = \sum_{|\alpha| \leqq m-1} \int_{\Omega'} g_\alpha(x, u, \ldots, D^\gamma u, \ldots) D^\alpha v\, dx,$$

$u, v \in W_r$ defines a bounded (nonlinear) operator $Q_r\colon W_r \to W_r^*$. Finally, for any $G \in V^*$ define the functional $G_r \in V_r^*$ by

$$\langle G_r, u_r\rangle = \langle G, L_r u_r\rangle, \quad u_r \in V_r$$

where

$$L_r u_r(x) = \begin{cases} u_r(x) & \text{if} \quad x \in \Omega_r \\ 0 & \text{if} \quad x \in \Omega \setminus \Omega_r. \end{cases}$$

(According to the above argument, $L_r u_r \in V$.)

THEOREM 2. *Assume that conditions* I *to* IX *are fulfilled and* $\lim r_n = +\infty$. *Then for sufficiently large n there exists at least one solution $u_n \in K_{r_n}^1$ of the variational inequality (considered in Ω_{r_n})*

$$\frac{1}{n}\langle S_{r_n}(u_n), v - u_n\rangle + \langle T_{r_n}(u_n), v - u_n\rangle + \tag{1.6}$$
$$+ \langle Q_{r_n}(u_n), v - u_n\rangle \geqq \langle G_{r_n}, v - u_n\rangle, \quad \textit{for all} \quad v \in K_{r_n}^1.$$

Further, extend the functions u_n to Ω by 0 *out of Ω_{r_n} and denote the extensions also by u_n. Then there is a subsequence (u_n') of (u_n) which is converging weakly in V (and strongly in $W_p^{m-1}(\omega)$ for any bounded $\omega \subset \Omega$) to a function $u \in K$, satisfying the variational inequalities (considered in Ω)* (1.3), (1.4). *If* (1.3), (1.4) *may have at most one solution then also (u_n) converges to the unique solution u of* (1.3), (1.4) *weakly in V and strongly in $W_p^{m-1}(\omega)$ for any bounded $\omega \subset \Omega$.*

REMARK 1. Condition III is fulfilled e.g. if in assumption V $k_\alpha = 0$,

$$|f_\alpha(x, \xi)| = |f_\alpha(x, \tilde{\xi})| \quad \text{if} \quad |\tilde{\xi}_\beta| = |\xi_\beta|,$$

further for $|\alpha| = m$ $f_\alpha(x, \eta, \zeta)$ does not depend on η (then (1.1) is a consequence of II) and for $|\alpha| \leqq m-1$

$$\sum_{|\alpha| \leqq m-1} [f_\alpha(x, \eta) - f_\alpha(x, \eta')](\xi_\alpha - \xi_\alpha') \geqq 0. \tag{1.7}$$

For a discussion on condition III we refer to [14], [8].

REMARK 2. Assume that $f_\alpha(x, \xi)$ and $f_\alpha(x, \eta)$ are continuous and they are continuously differentiable in ξ resp. η. Then by Newton—Leibniz formula

$$\sum_{|\alpha|=m} [f_\alpha(x, \eta, \zeta) - f_\alpha(x, \eta, \zeta')](\xi_\alpha - \xi'_\alpha) =$$

$$= \int_0^1 \Big[\sum_{|\alpha|, |\beta|=m} f_{\alpha\beta}(x, \eta, \zeta' + \tau(\zeta - \zeta'))(\xi_\alpha - \xi'_\alpha)(\xi_\beta - \xi'_\beta)\Big] d\tau$$

where $f_{\alpha\beta} = \dfrac{\partial f_\alpha}{\partial \xi_\beta}$. Consequently, condition II is fulfilled if for all $(\eta, \zeta) \in R^N$ and a.e. $x \in \Omega$ the matrix

$$[f_{\alpha\beta}(x, \eta, \zeta)]_{|\alpha|, |\beta|=m} \tag{1.8}$$

is positive definite. Similarly, if the matrix

$$[f_{\alpha\beta}(x, \eta)]_{|\alpha|, |\beta| \leqq m-1} \tag{1.9}$$

is positive semidefinite then condition (1.7) is fulfilled.

REMARK 3. By Remarks 1, 2 conditions I to V are satisfied e.g. when

$$f_\alpha(x, \zeta) = h_\alpha(\xi_\alpha)\varphi_\alpha(x, \zeta^{[\alpha]}) \quad \text{if} \quad |\alpha| = m,$$

$$f_\alpha(x, \eta) = h_\alpha(\xi_\alpha)\varphi_\alpha(x, \eta^{[\alpha]}) \quad \text{if} \quad |\alpha| \leqq m-1$$

where $\zeta^{[\alpha]}$ and $\eta^{[\alpha]}$ denote the vectors ζ resp. η without the coordinate ξ_α. Here the functions h_α are supposed to be continuously differentiable such that

$$h_\alpha(\xi_\alpha) > 0 \quad \text{for} \quad \xi_\alpha > 0, \quad h_\alpha(\xi_\alpha) < 0 \quad \text{for} \quad \xi_\alpha < 0,$$

$$|h_\alpha(\xi_\alpha)| \geqq c_1 |\xi_{\alpha)}|^{p-1}$$

($c_1 > 0$ is a constant). Further, the functions φ_α are continuous and they are continuously differentiable in $\zeta^{[\alpha]}$ resp. $\eta^{[\alpha]}$ such that

$$c_1 \leqq \varphi_\alpha(x, \zeta^{[\alpha]}) \quad \text{for} \quad |\alpha| = m, \quad c_1 \leqq \varphi_\alpha(x, \eta^{[\alpha]}) \quad \text{for} \quad |\alpha| \leqq m-1.$$

Finally, the matrix (1.8) is positive definite, the matrix (1.9) is positive semidefinite where

$$f_{\alpha\alpha}(x, \xi) = \begin{cases} h'_\alpha(\xi_\alpha)\varphi_\alpha(x, \zeta^{[\alpha]}) & \text{for} \quad |\alpha| = m \\ h'_\alpha(\xi_\alpha)\varphi_\alpha(x, \eta^{[\alpha]}) & \text{for} \quad |\alpha| \leqq m-1 \end{cases}$$

and for $\beta \neq \alpha$

$$f_{\alpha\beta}(x, \xi) = \begin{cases} h_\alpha(\xi_\alpha)\varphi_{\alpha\beta}(x, \zeta^{[\alpha]}) & \text{if} \quad |\alpha| = m \\ h_\alpha(\xi_\alpha)\varphi_{\alpha\beta}(x, \eta^{[\alpha]}) & \text{if} \quad |\alpha| \leqq m-1. \end{cases}$$

The last condition is fulfilled e.g. if $\varphi_\alpha(x, \zeta^{[\alpha]})$, $\varphi_\alpha(x, \eta^{[\alpha]})$ do not depend on $\zeta^{[\alpha]}$ resp. $\eta^{[\alpha]}$ and $h'_\alpha > 0$ for $|\alpha| = m$, $h'_\alpha \geqq 0$ for $|\alpha| \leqq m-1$.

REMARK 4. Condition IX is fulfilled if there is an extension operator $L\colon K^0 \to W_p^m(R^n)$ and a constant c such that if $u \in K^0$ then $\partial^\alpha(Lu) \in L^\infty(R^n)$ and

$$\sum_{|\alpha| \leqq m} \|\partial^\alpha(Lu)\|_{L^\infty(R^n)} \leqq c \sum_{|\alpha| \leqq m} \|\partial^\alpha u\|_{L^\infty(\Omega)};$$

further there is $A \subset K^0$ such that for each $v \in A$, the convolution $Lv * \eta_\varepsilon \in K$ if $\varepsilon > 0$ is sufficiently small — where $\eta_\varepsilon \in C_0^\infty(R^n)$, $\eta_\varepsilon \geqq 0$, $\int \eta_\varepsilon = 1$, $\operatorname{supp} \eta_\varepsilon \subset \overline{B}_\varepsilon$ — and for each $w \in K^0$ there is a sequence (w_j) of functions $w_j \in A$ such that $(w_j) \to w$ in $W_p^m(\Omega)$ and $|D^\alpha w_j(x)| \leqq d_\alpha$ (d_α does not depend on j and x).

Since, in this case any $v \in K^0$ can be approximated by $w_j \in A$ and w_j can be approximated by $Lw_j * \eta_\varepsilon$. The above assumptions are satisfied for sufficiently smooth $\partial\Omega$ in the following special cases:

a) $K = V = W_p^m(\Omega)$; then A may be chosen as K^0.

b) $S \subset \Omega$ is a compact smooth $(n-1)$-dimensional surface or $S \subset \partial\Omega$ is a compact smooth $(n-1)$-dimensional surface such that the distance of S and $\partial\Omega \setminus S$ is positive,

$$K = V = \{\varphi \in W_p^m(\Omega) \colon \varphi|_S = 0\}.$$

Then A may be chosen as the set of $\varphi \in K^0$, vanishing in a neighbourhood of S.

c) $\partial\Omega$ is bounded and

$$V = W_{p,0}^m(\Omega), \quad K = \{\varphi \in V \colon a \leqq \varphi \leqq b\}$$

where $-\infty \leqq a \leqq 0 \leqq b \leqq +\infty$. Then A may be chosen as the set of $\varphi \in K^0$, vanishing in a neighbourhood of $\partial\Omega$.

2. Proof of Theorem 2

According to the existence theorem of [12] the variational inequality (1.6) admits at least one solution $u_n \in K_{r_n}^1$ for sufficiently large n. (See also [15].) Applying (1.6) to $v=0$ one finds that

$$\frac{1}{n}\langle S_{r_n}(u_n), u_n\rangle + \langle T_{r_n}(u_n), u_n\rangle + \langle Q_{r_n}(u_n), u_n\rangle \leqq \langle G_{r_n}, u_n\rangle, \tag{2.1}$$

thus by V, VII, VIII there is a constant $c_5 > 0$ such that

$$\frac{1}{n}\|u_n\|_{W_{r_n}}^s + c_5\|u_n\|_{V_{r_n}}^p - \int_\Omega k_3\,dx - \int_\Omega k_4\,dx - \sum_{|\alpha| \leqq m-1} \|h_\alpha\|_{L^q(\Omega)}\|u_n\|_{V_{r_n}} \leqq \|G\|_{V^*}\|u_n\|_{V_{r_n}}.$$

Consequently (since $p > 1$)

$$\frac{1}{n}\|u_n\|_{W_{r_n}}^s \quad \text{is bounded,} \tag{2.2}$$

$$\|u_n\|_{V_{r_n}}, \ \|u_n\|_V \quad \text{is bounded.} \tag{2.3}$$

(The functions u_n are supposed to be extended to Ω as 0 out of Ω_{r_n}.) (2.3), V, VII, VIII imply that

$$\sum_{|\alpha|=m} \int_{\Omega_{r_n}} f_\alpha(x, u_n, \ldots, D^\beta u_n, \ldots) D^\alpha u_n\,dx \leqq c_6, \tag{2.4}$$

$$\sum_{|\alpha| \leqq m-1} \int_{\Omega_{r_n}} f_\alpha(x, u_n, \ldots, D^\gamma u_n, \ldots) D^\alpha u_n\,dx \leqq c_6.$$

Further, from (2.2) it follows that

(2.5) $$\|u_n\|_{W_{r_n}} \leqq \text{const. } n^{1/s}.$$

Since

$$\left|\frac{1}{n}\langle S_{r_n}(u_n), v\rangle\right| \leqq \frac{1}{n}\|u_n\|_{W_{r_n}}^{s-1}\|v\|_{W_{r_n}}$$

thus by (2.5) for any fixed j, $w \in W_{r_j}$

$$\lim_{n\to\infty}\frac{1}{n}\langle S_{r_n}(u_n), w\rangle = 0$$

(for $n > j$ $w(x)$ is defined by 0 out of B_{r_j}) and so (as $\langle S_{r_n}(u_n), u_n\rangle \geqq 0$)

(2.6) $$\limsup_{n\to\infty}\frac{1}{n}\langle S_{r_n}(u_n), w-u_{r_n}\rangle \leqq 0.$$

By (2.3) there exist a subsequence (u_n') of (u_n) and $u \in V$ such that

(2.7) $$(u_n') \to u \quad \text{weakly in } V,$$

further,

(2.8) $$(D^\gamma u_n') \to D^\gamma u \quad \text{a.e. in } \Omega \text{ for } |\gamma| \leqq m-1$$

because by theorems on compact imbeddings (see e.g. [13]), VII, VIII, (1.5) it may be supposed that

(2.9) $(D^\gamma u_n') \to D^\gamma u$ in $L^p(\Omega_r)$, $(D^\gamma u_n') \to D^\gamma u$ in $L^{|q_{|\gamma|}|}(\Omega')$ for $|\gamma| \leqq m-1$.

LEMMA 1. *For all α with $|\alpha| \leqq m$ and each fixed $r > 0$ the integrals*

(2.10) $$\int_{\Omega_r} |f_\alpha(x, u_n', \dots, D^\beta u_n', \dots)|\, dx$$

are uniformly bounded and the functions

(2.11) $$f_\alpha(\cdot, u_n', \dots, D^\beta u_n', \dots)$$

are uniformly equiintegrable in Ω_r, i.e. for any $\varepsilon > 0$ there is $\delta > 0$ such that $E \subset \Omega_r$, $\lambda(E) < \delta$ ($\lambda(E)$ denotes the Lebesgue measure of a measurable set E) imply that for all n

(2.12) $$\int_E |f_\alpha(x, u_n', \dots, D^\beta u_n', \dots)|\, dx < \varepsilon.$$

Further, there exist a subsequence (u_n'') of (u_n') and a function F_α such that for each fixed $r > 0$ $f_\alpha(\cdot, u_n'', \dots, D^\beta u_n'', \dots) \to F_\alpha$ weakly in $L^1(\Omega_r)$. For $|\alpha| \leqq m-1$

$$F_\alpha(x) = f_\alpha(x, u(x), \dots, D^\gamma u(x), \dots) \quad a.e. \quad (|\gamma| \leqq m-1)$$

and $f_\alpha(\cdot, u_n', \dots, D^\gamma u_n', \dots) \to F_\alpha$ in the norm of $L^1(\Omega_r)$.

PROOF. Applying (1.1) to $\xi'_\alpha=\varrho\operatorname{sgn} f_\alpha(x,\xi)$ and $\xi'_\beta=0$ for $\beta\neq\alpha$ ($|\alpha|=m$, $\varrho>0$) we obtain that

$$\varrho\operatorname{sgn} f_\alpha(x,\xi)f_\alpha(x,\xi)\leqq f_\alpha(x,\xi')\varrho\operatorname{sgn} f_\alpha(x,\xi)+c_1\sum_{|\alpha|=m}f_\alpha(x,\xi)\xi_\alpha+k_1(x),$$

where $|\xi'|=|\varrho\operatorname{sgn} f_\alpha(x,\xi)|=\varrho$. Thus by assumption IV

(2.13)

$$|f_\alpha(x,\xi)|\leqq|f_\alpha(x,\xi')|+\frac{c_1}{\varrho}\sum_{|\alpha|=m}f_\alpha(x,\xi)\xi_\alpha+\frac{k_1(x)}{\varrho}\leqq f_\varrho(x)+\frac{c_1}{\varrho}\sum_{|\alpha|=m}f_\alpha(x,\xi)\xi_\alpha+\frac{k_1(x)}{\varrho}.$$

(2.13), (2.4) imply that the integrals (2.10) are uniformly bounded. Further, by (2.13) for any measurable $E\subset\Omega_r$

$$\int_E|f_\alpha(x,u'_n,\dots,D^\beta u'_n,\dots)|\,dx\leqq\int_E\left[f_\varrho(x)+\frac{k_1(x)}{\varrho}\right]dx+$$

$$+\frac{c_1}{\varrho}\sum_{|\alpha|=m}\int_E f_\alpha(x,u'_n,\dots,D^\beta u'_n,\dots)D^\alpha u'_n\,dx.$$

Taking $\varrho=\frac{2c_1c_6}{\varepsilon}$, by (2.4), V we find that for sufficiently small $\lambda(E)$ (2.12) holds if $|\alpha|=m$. Analogously, it can be proved that for $|\alpha|\leqq m-1$ the integrals (2.10) are uniformly bounded and (2.12) holds.

Therefore, by Dunford—Pettis theorem (see e.g. [16]) there exist a subsequence (u''_n) of (u'_n) and F_α such that for each fixed $r>0$

$$f_\alpha(\cdot,u''_n,\dots,D^\beta u''_n,\dots)\to F_\alpha\quad\text{weakly in}\quad L^1(\Omega_r).$$

From Vitali's convergence theorem, assumption I and (2.8) it follows that for each $r>0$, $|\alpha|\leqq m-1$

$$f_\alpha(\cdot,u'_n,\dots,D^\gamma u'_n,\dots)\to f_\alpha(\cdot,u,\dots,D^\gamma u,\dots)\quad\text{in}\quad L^1(\Omega_r)$$

(here $|\gamma|\leqq m-1$) and so $F_\alpha(x)=f_\alpha(x,u(x),\dots,D^\gamma u(x),\dots)$.

LEMMA 2. *There exists a subsequence $(\tilde u_n)$ of (u''_n) such that*

(2.14) $$\limsup_{n\to\infty}\sum_{|\alpha|=m}\int_\Omega[f_\alpha(x,\tilde u_n,\dots,D^\beta\tilde u_n,\dots)-F_\alpha]D^\alpha\tilde u_n\,dx\leqq 0.$$

(Cf. Lemma 3 of [5].)

PROOF. Applying (1.6) to $v=u_j\in K^1_{r_j}\subset K^1_{r_n}$ for $n\geqq j$, we obtain

(2.15) $$\frac{1}{n}\langle S_{r_n}(u_n),u_j-u_n\rangle+\langle T_{r_n}(u_n),u_j-u_n\rangle+\langle Q_{r_n}(u_n),u_j-u_n\rangle\geqq\langle G_{r_n},u_j-u_n\rangle.$$

Thus

$$(2.16)\qquad \sum_{|\alpha|\leq m}\int_{\Omega}[f_\alpha(x, u_n'', \ldots, D^\beta u_n'', \ldots)-F_\alpha]D^\alpha u_n''\,dx \leqq$$

$$\leqq \frac{1}{n}\langle S_{r_n''}(u_n''), u_j''-u_n''\rangle+\sum_{|\alpha|\leq m}\int_{\Omega r_j''} f_\alpha(x, u_n'', \ldots, D^\beta u_n'', \ldots)D^\alpha u_j''\,dx+$$

$$+\langle Q_{r_n''}(u_n''), u_j''-u_n''\rangle+\langle G_{r_n}, u_n''-u_j''\rangle-\sum_{|\alpha|\leq m}\int_{\Omega} F_\alpha D^\alpha u_n''\,dx.$$

In virtue of Lemma 1

$$(2.17)\qquad \lim_{n\to\infty}\sum_{|\alpha|\leq m}\int_{\Omega r_j''} f_\alpha(x, u_n'', \ldots, D^\beta u_n'', \ldots)D^\alpha u_j''\,dx=\sum_{|\alpha|\leq m}\int_{\Omega r_j''} F_\alpha D^\alpha u_j''\,dx.$$

By (2.7) and the definition of G_{r_n} we have

$$(2.18)\qquad \lim_{n\to\infty}\langle G_{r_n}, u_n''-u_j''\rangle=\langle G, u-u_j''\rangle.$$

Further, by (1.5), VII, VIII, Hölder's inequality and Sobolev's imbedding theorem

$$|\langle Q_{r_n''}(u_n''), u_j''-u_n''\rangle| \leqq \sum_{|\alpha|\leq m-1}\|h_\alpha\|_{L^q(\Omega')}\|D^\alpha u_j''-D^\alpha u_n''\|_{L^p(\Omega')}+$$

$$+\sum_{|\alpha|\leq m-1}\|p_\alpha(\cdot, u_n'', \ldots, D^\beta u_n'', \ldots)\|_{L^{p/\varrho_{|\alpha|}}(\Omega')}\cdot\|D^\alpha u_j''-D^\alpha u_n''\|_{L^{p/\varrho_{|\alpha|}}(\Omega')}\leqq$$

$$\leqq \text{const.}\sum_{|\alpha|\leq m-1}[\|D^\alpha u_j''-D^\alpha u_n''\|_{L^{p/\varrho_{|\alpha|}}(\Omega')}+\|D^\alpha u_j''-D^\alpha u_n''\|_{L^p(\Omega')}]$$

whence by (2.9)

$$(2.19)\qquad \limsup_{n\to\infty}|\langle Q_{r_n''}(u_n''), u_j''-u_n''\rangle| \leqq$$

$$\leqq \text{const.}\sum_{|\alpha|\leq m-1}[\|D^\alpha u_j''-D^\alpha u\|_{L^{q/\varrho_{|\alpha|}}(\Omega')}+\|D^\alpha u_j''-D^\alpha u\|_{L^p(\Omega')}].$$

From (2.6), (2.16)—(2.19) follows

$$(2.20)\qquad \limsup_{n\to\infty}\sum_{|\alpha|\leq m}\int_{\Omega}[f_\alpha(x, u_n'', \ldots, D^\beta u_n'', \ldots)-F_\alpha]D^\alpha u_n''\,dx \leqq$$

$$\leqq \sum_{|\alpha|\leq m}\int_{\Omega r_j''} F_\alpha D^\alpha u_j''\,dx+\langle G, u-u_j''\rangle-\liminf_{n\to\infty}\sum_{|\alpha|\leq m}\int_{\Omega} F_\alpha D^\alpha u_n''\,dx+$$

$$+\text{const.}\sum_{|\alpha|\leq m-1}[\|D^\alpha u_j''-D^\alpha u\|_{L^{q_{|\alpha|}}(\Omega')}+\|D^\alpha u_j''-D^\alpha u\|_{L^p(\Omega')}]$$

where

$$\liminf_{n\to\infty}\sum_{|\alpha|\leq m}\int_{\Omega} F_\alpha D^\alpha u_n''\,dx>-\infty.$$

Since, by (2.15)

$$\sum_{|\alpha|\le m}\int_{\Omega r_j''} f_\alpha(x, u_n'', \dots, D^\beta u_n'', \dots)D^\alpha u_j''\,dx \ge \frac{1}{n}\langle S_{r_n''}(u_n''), u_n''-u_j''\rangle +$$

$$+\sum_{|\alpha|\le m}\int_\Omega f_\alpha(x, u_n'', \dots, D^\beta u_n'', \dots)D^\alpha u_n''\,dx+\langle Q_{r_n''}(u_n''), u_n''-u_j''\rangle+\langle G, u_j''-u_n''\rangle$$

and thus (2.6), V, (2.17)—(2.19) imply

$$\sum_{|\alpha|\le m}\int_\Omega F_\alpha D^\alpha u_j''\,dx \ge -\int_\Omega k_3\,dx+\langle G, u_j''-u\rangle-$$

$$-\text{const.}\sum_{|\alpha|\le m-1}[\|D^\alpha u_j''-D^\alpha u\|_{L^{q_{|\alpha|}}(\Omega')}+\|D^\alpha u_j''-D^\alpha u\|_{L^p(\Omega')}].$$

Hence by (2.7), (2.9)

$$\liminf_{j\to\infty}\sum_{|\alpha|\le m}\int_\Omega F_\alpha D^\alpha u_j''\,dx \ge -\int_\Omega k_3\,dx.$$

Consider a subsequence $(\tilde u_n)$ of (u_n'') with the property

$$\lim_{j\to\infty}\sum_{|\alpha|\le m}\int_\Omega F_\alpha D^\alpha \tilde u_j\,dx = \liminf_{n\to\infty}\int_\Omega F_\alpha D^\alpha u_n''\,dx.$$

Then applying (2.20) to $\tilde u_j$ instead of u_j'', as $j\to\infty$ by (2.7), (2.9) we obtain

$$\limsup_{n\to\infty}\sum_{|\alpha|\le m}\int_\Omega [f_\alpha(x, \tilde u_n, \dots, D^\beta \tilde u_n, \dots)-F_\alpha]D^\alpha \tilde u_n\,dx \le 0.$$

Therefore, to show (2.14) it is sufficient to prove

$$\liminf_{n\to\infty}\sum_{|\alpha|\le m-1}\int_\Omega [f_\alpha(x, \tilde u_n, \dots, D^\gamma \tilde u_n, \dots)-F_\alpha]D^\alpha \tilde u_n\,dx \ge 0. \tag{2.21}$$

Because of (2.8) and assumption I

$$\lim_{n\to\infty}\sum_{|\alpha|\le m-1}[f_\alpha(x, u, \dots, D^\gamma u, \dots)-f_\alpha(x, \tilde u_n, \dots, D^\gamma \tilde u_n, \dots)]D^\alpha \tilde u_n = 0 \tag{2.22}$$

a.e. in Ω (here $|\gamma|\le m-1$). Moreover, due to (1.2)

$$\sum_{|\alpha|\le m-1}[f_\alpha(x, u, \dots, D^\gamma u, \dots)-f_\alpha(x, \tilde u_n, \dots, D^\gamma \tilde u_n, \dots)]D^\alpha \tilde u_n \le \tag{2.23}$$

$$\le c_2\sum_{|\alpha|\le m-1} f_\alpha(x, u, \dots, D^\gamma u, \dots)D^\alpha u+k_2(x),$$

where by Fatou's lemma, V, (2.4) the term in the right is integrable over Ω since

$$\lim_{n\to\infty}\sum_{|\alpha|\le m-1}[f_\alpha(x, \tilde u_n, \dots, D^\gamma \tilde u_n, \dots)D^\alpha \tilde u_n+k_\alpha] =$$

$$=\sum_{|\alpha|\le m-1}[f_\alpha(x, u, \dots, D^\gamma u, \dots)D^\alpha u+k_\alpha] \quad \text{a.e. in } \Omega.$$

Applying Lebesgue's dominated convergence theorem to the positive part of the term in the left of (2.23), we find

$$\limsup_{n\to\infty} \sum_{|\alpha|\leqq m-1} \int_\Omega [f_\alpha(x, u, \ldots, D^\gamma u, \ldots) - f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots)] D^\alpha \tilde{u}_n \, dx \leqq 0$$

which implies (2.21) since according to Lemma 1

$$F_\alpha(x) = f_\alpha(x, u, \ldots, D^\gamma u, \ldots) \quad \text{if} \quad |\alpha| \leqq m-1.$$

Thus the proof of Lemma 2 is complete.

Set

$$\tilde{q}_n(x) = \sum_{|\alpha|=m} [f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots) -$$

$$- f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta u, \ldots)](D^\alpha \tilde{u}_n - D^\alpha u)$$

where $|\gamma| \leqq m-1$, $|\beta| = m$.

In virtue of $F_\alpha, f_s \in L^1(\Omega_r)$, $D^\alpha u \in L^p(\Omega)$, (2.8), I and Jegorov's theorem for any (sufficiently large) positive integer k we can choose measurable sets $\Omega_k' \subset \Omega_{r_k}$ such that

$$\lambda(\Omega_{r_k} \setminus \Omega_k') \leqq \frac{1}{k}, \tag{2.24}$$

$$D^\alpha u, \quad F_\alpha \in L^\infty(\Omega_k') \quad \text{for} \quad |\alpha| \leqq m, \quad f_s \in L^\infty(\Omega_k') \quad \text{for} \quad s = 1, 2, \ldots \tag{2.25}$$

and

$$D^\gamma \tilde{u}_n \to D^\gamma u \quad \text{uniformly on } \Omega_k' \text{ if } \quad |\gamma| \leqq m-1, \tag{2.26}$$

$$f_\alpha(\cdot, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta u, \ldots) \to f_\alpha(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots) \tag{2.27}$$

uniformly on Ω_k' ($|\gamma| \leqq m-1$, $|\beta| = m$). Clearly, it may be supposed that

$$\Omega_k' \subset \Omega_{k+1}'. \tag{2.28}$$

Lemma 3. *For each fixed k*

$$\lim_{n\to\infty} \int_{\Omega_k'} \tilde{q}_n \, dx = 0. \tag{2.29}$$

Proof. We have

$$\int_{\Omega_k'} f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta u, \ldots)(D^\alpha \tilde{u}_n - D^\alpha u) \, dx = \tag{2.30}$$

$$= \int_{\Omega_k'} [f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta u, \ldots) - f_\alpha(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots)] \times$$

$$\times (D^\alpha \tilde{u}_n - D^\alpha u) \, dx + \int_{\Omega_k'} f_\alpha(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots)(D^\alpha \tilde{u}_n - D^\alpha u) \, dx.$$

By Hölder's inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$$\int_{\Omega_k'} |D^\alpha \tilde{u}_n - D^\alpha u|\,dx \leqq [\lambda(\Omega_k')]^{1/q}\left[\left\{\int_{\Omega_k'} |D^\alpha \tilde{u}_n|^p dx\right\}^{1/p} + \left\{\int_{\Omega_k'} |D^\alpha u|^p dx\right\}^{1/p}\right]$$

where the right hand side is bounded (as $n\to\infty$) because of (2.3), thus (2.27) implies that the first term in the right of (2.30) converges to 0. Further, from (2.7), (2.25), IV it follows that also the second term in the right of (2.30) converges to 0 as $n\to\infty$. Consequently,

$$\lim_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega_k'} f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta u, \ldots)(D^\alpha \tilde{u}_n - D^\alpha u)\,dx = 0. \tag{2.31}$$

By (2.7) and (2.25) we have

$$\lim_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega_k'} F_\alpha(D^\alpha \tilde{u}_n - D^\alpha u)\,dx = 0.$$

Thus (2.31), (2.25), Lemmas 1 and 2 imply

$$\limsup_{n\to\infty} \int_{\Omega_k'} \tilde{q}_n\,dx = \tag{2.32}$$

$$= \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega_k'} f_\alpha(x, \tilde{u}_n, \ldots, D^\gamma \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots)(D^\alpha \tilde{u}_n - D^\alpha u)\,dx =$$

$$= \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega_k'} [f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots) - F_\alpha](D^\alpha \tilde{u}_n - D^\alpha u)\,dx =$$

$$= \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega_k'} [f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots) - F_\alpha] D^\alpha \tilde{u}_n\,dx \leqq$$

$$\leqq \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega} [f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots) - F_\alpha] D^\alpha \tilde{u}_n\,dx +$$

$$+ \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega\setminus\Omega_k'} [F_\alpha - f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots)] D^\alpha \tilde{u}_n\,dx \leqq$$

$$\leqq \limsup_{n\to\infty} \sum_{|\alpha|=m} \int_{\Omega\setminus\Omega_k'} [F_\alpha - f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots)] D^\alpha \tilde{u}_n\,dx.$$

Now — similarly to [3] — we show that there is a function $\Gamma \in L^1(\Omega)$ such that for a.e. $x \in \Omega$

$$\sum_{|\alpha|=m} [F_\alpha - f_\alpha(x, \tilde{u}_n, \ldots, D^\beta \tilde{u}_n, \ldots)] D^\alpha \tilde{u}_n \leqq \Gamma(x). \tag{2.33}$$

(2.33) with (2.32) and $\tilde{q}_n \geqq 0$ (see II) will imply (2.29) since by (2.32), (2.33), (2.24), (2.28)

$$0 \leqq \limsup_{n\to\infty} \int_{\Omega'_k} \tilde{q}_n dx \leqq \int_{\Omega\setminus\Omega'_k} \Gamma dx \to 0 \quad \text{as} \quad k \to \infty$$

which implies (2.29) since the sequence $\left(\limsup_{n\to\infty} \int_{\Omega'_k} \tilde{q}_n\, dx\right)_{k=1}^{\infty}$ is nondecreasing.

In order to prove (2.33) we apply Mazur's theorem (see e.g. [17]) to the sequence of functions $f_\alpha(\cdot, \tilde{u}_n, \dots, D^\beta \tilde{u}_n, \dots)$, converging to F_α weakly in $L^1(\Omega_{r_k})$ for any fixed k. Thus for each l there exist numbers

$$\omega_j^l \geqq 0 \quad \text{with} \quad \sum_{j=1}^{l} \omega_j^l = 1 \tag{2.34}$$

such that

$$\sum_{j=1}^{l} \omega_j^l f_\alpha(\cdot, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) \to F_\alpha \quad \text{in} \quad L^1(\Omega_{r_k})$$

and so for a subsequence (denoted in the same way) also a.e. in Ω_{r_k}. Hence, in virtue of (2.34) and (1.1)

$$\sum_{|\alpha|=m} [F_\alpha - f_\alpha(x, \tilde{u}_n, \dots, D^\beta \tilde{u}_n, \dots)] D^\alpha \tilde{u}_n = \tag{2.35}$$

$$= \lim_{l\to\infty} \Big\{ \sum_{j=1}^{l} \omega_j^l \Big[\sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_n - \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_n, \dots, D^\beta \tilde{u}_n, \dots) D^\alpha \tilde{u}_n \Big] \Big\} \leqq$$

$$\leqq \liminf_{l\to\infty} \sum_{j=1}^{l} \omega_j^l \Big[c_1 \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j + k_1(x) \Big].$$

Since by V

$$\sum_{j=1}^{l} \omega_j^l \sum_{|\alpha|=m} [f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j + k_\alpha] \geqq 0 \tag{2.36}$$

and by (2.4)

(2.37)

$$\int_{\Omega_{r_k}} \Big\{ \sum_{j=1}^{l} \omega_j^l \sum_{|\alpha|=m} [f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j + k_\alpha] \Big\} dx \leqq c_6 + \sum_{|\alpha|=m} \int_\Omega k_\alpha dx$$

thus from Fatou's lemma and $k_\alpha \in L^1(\Omega)$ it follows that

$$\liminf_{l\to\infty} \sum_{j=1}^{l} \omega_j^l \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j$$

is integrable over Ω_{r_k}.

Define $\Gamma = \liminf_{k\to\infty} \Gamma_k$ where

$$\Gamma_k(x) = \liminf_{l\to\infty} \sum_{j=1}^{l} \omega_j^l c_1 \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j + k_1(x), \quad x \in \Omega_{r_k}$$

and $\Gamma_k(x)=0$ if $x\in\Omega\setminus\Omega_{r_k}$. Then $\Gamma_k\in L^1(\Omega)$, further, by (2.36), (2.37) and Fatou's lemma Γ is integrable over Ω, moreover, since for any fixed $x\in\Omega$

$$\sum_{|\alpha|=m}[F_\alpha-f_\alpha(x,\tilde{u}_n,\dots,D^\beta\tilde{u}_n,\dots)]D^\alpha\tilde{u}_n\leqq\Gamma_k(x)$$

if k is sufficiently large (see (2.35)) so we obtain (2.33) and thus the proof of Lemma 3 is complete.

LEMMA 4. *There is a subsequence* $(\tilde{u}_n')$ *of* $(\tilde{u}_n)$ *such that*

$$D^\alpha\tilde{u}_n'\to D^\alpha u\quad a.e.\quad in\quad \Omega\quad for\quad |\alpha|=m.$$

PROOF. Since $\tilde{q}_n\geqq 0$ thus Lemma 3 implies that for arbitrary fixed k

$$\lim_{n\to\infty}\|\tilde{q}_n\|_{L^1(\Omega_k')}=0.$$

Consequently, for a fixed k there is a subsequence of $(\tilde{q}_n)$ which tends to 0 a.e. in Ω_k'. As $\lambda(\Omega\setminus\bigcup\Omega_k')=0$ thus we can choose a subsequence $\tilde{q}_n'$ such that

$$\lim_{n\to\infty}\tilde{q}_n'(x)=0\quad\text{for}\quad x\in\Omega_0\quad\text{where}\quad\lambda(\Omega\setminus\Omega_0)=0.$$

In virtue of (2.8) it may be supposed that

$$D^\gamma\tilde{u}_n'(x)\to D^\gamma u(x)\quad\text{if}\quad x\in\Omega_0,\quad|\gamma|\leqq m-1.$$

Now, by I, II, IV (for a fixed $x\in\Omega_0$) we can apply Lemma 6 of [4]. So we find that

$$\lim_{n\to\infty}D^\alpha\tilde{u}_n'(x)=D^\alpha u(x)\quad\text{if}\quad x\in\Omega_0,\quad|\alpha|=m$$

and thus Lemma 4 is proved.

PROOF OF THEOREM 2. Denote the subsequence $(\tilde{u}_n')$ (defined in Lemma 4) by (u_{n_k}) and apply (1.6) to u_{n_k} thus we obtain that for all $v\in K^1_{r_{n_k}}$

(2.38)

$$\frac{1}{n_k}\langle S_{r_{n_k}}(u_{n_k}),v-u_{n_k}\rangle+\langle T_{r_{n_k}}(u_{n_k}),v-u_{n_k}\rangle+\langle Q_{r_{n_k}}(u_{n_k}),v-u_{n_k}\rangle\geqq\langle G_{r_{n_k}},v-u_{n_k}\rangle.$$

According to (2.6)

$$\limsup_{k\to\infty}\frac{1}{n_k}\langle S_{r_{n_k}}(u_{n_k}),v-u_{n_k}\rangle\leqq 0. \tag{2.39}$$

In virtue of (2.8), Lemmas 1 and 4, assumption I and Vitali's theorem $f_\alpha(\cdot,u,\dots,D^\beta u,\dots)\in L^1(\Omega_r)$ for each r,

$$\lim_{k\to\infty}\langle T_{r_{n_k}}(u_{n_k}),v\rangle=\sum_{|\alpha|\leqq m}\int_\Omega f_\alpha(x,u,\dots,D^\beta u,\dots)D^\alpha v\,dx \tag{2.40}$$

(v is a fixed function, $v(x)=0$ a.e. if x is out of a bounded set). Further, combining (2.8), Lemma 4, I, V, (2.4), Fatou's lemma shows that the functions

$f_\alpha(\cdot, u, \ldots, D^\beta u, \ldots)D^\alpha u$ are integrable over Ω (so we have already shown (1.3)) and

$$\liminf_{k\to\infty} \langle T_{r_{n_k}}(u_{n_k}), u_{n_k}\rangle \geqq \sum_{|\alpha|\leqq m} \int_\Omega f_\alpha(x, u, \ldots, D^\beta u, \ldots)D^\alpha u\, dx. \tag{2.41}$$

(2.8), (2.9), Lemma 4, VI to VIII, Hölder's inequality and Vitali's theorem imply that

$$\lim_{k\to\infty} \langle Q_{r_{n_k}}(u_{n_k}), v-u_{n_k}\rangle = \lim_{k\to\infty} \langle Q_{r_{n_k}}(u_{n_k}), v\rangle - \lim_{k\to\infty} \langle Q_{r_{n_k}}(u_{n_k}), u_{n_k}\rangle = \tag{2.42}$$

$$= \sum_{|\alpha|\leqq m-1} \int_{\Omega'} g_\alpha(x, u, \ldots, D^\gamma u, \ldots)(D^\alpha v - D^\alpha u)\, dx$$

because

$$\int_{\Omega'} g_\alpha(x, u_{n_k}, \ldots, D^\beta u_{n_k}, \ldots)D^\alpha u_{n_k}\, dx - \int_{\Omega'} g_\alpha(x, u, \ldots, D^\beta u, \ldots)D^\alpha u\, dx =$$

$$= \int_{\Omega'} g_\alpha(x, u_{n_k}, \ldots, D^\beta u_{n_k}, \ldots)(D^\alpha u_{n_k} - D^\alpha u)\, dx +$$

$$+ \int_{\Omega'} [g_\alpha(x, u_{n_k}, \ldots, D^\beta u_{n_k}, \ldots) - g_\alpha(x, u, \ldots, D^\beta u, \ldots)]D^\alpha u\, dx.$$

Finally, from (2.7) it follows that

$$\lim_{k\to\infty} \langle G_{r_{n_k}}, v-u_{n_k}\rangle = \langle G, v-u\rangle. \tag{2.43}$$

Thus from (2.38)—(2.43), as $k\to\infty$ we obtain that (1.4) holds for each $v\in K^1_{r_{n_k}}$. According to assumption IX for each $v\in K^0$ there exist $v_j\in C_0^\infty(R^n)$ such that $v_j|_\Omega\in K$, $v_j|_\Omega\to v$ in $W_p^m(\Omega)$ and $\sup|D^\alpha v_j|\leqq c_\alpha$. Choosing a subsequence, it may be supposed that $D^\alpha v_j\to D^\alpha v$ a.e. in Ω for $|\alpha|\leqq m$. Then $v_j|_\Omega\in K^1_{r_{n_k}}$ for sufficiently large k and thus (1.4) holds for $v=v_j$, whence — as $j\to\infty$ — by Lebesgue's dominated convergence theorem and Sobolev's imbedding theorem we find that (1.4) holds also for $v\in K^0$.

According to (2.7) and (2.9) $u_{n_k}\to u$ weakly in V and in the norm of $W_p^{m-1}(\omega)$ for any bounded $\omega\subset\Omega$.

If (1.3), (1.4) may have at most one solution but $(u_n)\to u$ weakly or in $W_p^{m-1}(\omega)$ does not hold then by the argument of the proof of Theorem 2 it is easy to get a contradiction.

Remark 5. If Ω is bounded and it has the segment property (see [18]) then for sufficiently large r $\Omega_r=\Omega$, $V_r=V$, $K_r=K$, $K_r^1=K\cap W_s^{m+1}(\Omega)$.

Remark 6. Without assumption IX we find that (1.3) is valid and (1.4) holds for each $v\in \bigcup_{r>0} K_r^1$. Consequently, (1.4) holds for each $v\in K$, $v=\varphi|_\Omega$ where $\varphi\in C_0^\infty(R^n)$.

Remark 7. Suppose that for $|\alpha|\leqq m-1$ f_α is a Carathéodory function of type f_α: $\Omega\times R^N\to R$; further, instead of II assume that the strict monotonicity condition is fulfilled:

$$\sum_{|\alpha|\leqq m}[f_\alpha(x,\xi)-f_\alpha(x,\xi')](\xi_\alpha-\xi'_\alpha)>0 \quad \text{for} \quad \xi\neq\xi'$$

and in V, $k_\alpha=0$. In this case Theorems 1 and 2 can be proved in a similar way.

REMARK 8. Instead of IV assume that for each α $f_\alpha=f_\alpha^{(1)}+f_\alpha^{(2)}$ where $f_\alpha^{(j)}$ satisfy the Carathéodory conditions and the following conditions. Denote by A the set of multiindices α with the property $|\alpha|\leqq m$ and let $A=\cup A_\nu$ where $\alpha,\alpha^*\in A_\nu$ if and only if $f_\alpha^{(1)}=f_{\alpha^*}^{(1)}$. Assume that for any ν and $s>0$ there exist a function $f_{\nu,s}\in L^1(\Omega)$ and constants $c_3, c'_3>0$ such that for almost all $x\in\Omega$

$$|f_\alpha^{(1)}(x,\xi)|\leqq f_{\nu,s}(x)+c_3|\xi|^p, \quad |f_\alpha^{(2)}(x,\xi)|\leqq c'_3|\xi|^{p-1}, \quad |\alpha|=m$$

resp.

$$|f_\alpha^{(1)}(x,\eta)|\leqq f_{\nu,s}(x)+c_3|\eta|^p, \quad |f_\alpha^{2}(x,\eta)|\leqq c'_3|\eta|^{p-1}, \quad |\alpha|\leqq m-1$$

if $|\xi_\gamma|\leqq s$ for $\gamma\in A_\nu$ and $|\xi'|\leqq s$ where the vector ξ' contains those ξ_γ for which $|\gamma|<m-\frac{n}{p}$. Further, assume that $\psi\in C_0^\infty(R^n)$, $0\leqq\psi\leqq 1$, $v\in K$ imply $\psi v\in K$ and there are functions $K_\alpha^{(1)}\geqq 0$ such that $K_\alpha^{(1)}\in L^1(\Omega)$,

$$f_\alpha^{(1)}(x,\xi)\xi_\alpha\geqq -K_\alpha^{(1)}(x) \quad (|\alpha|=m), \quad f_\alpha^{(1)}(x,\eta)\xi_\alpha\geqq -K_\alpha^{(1)}(x) \quad (|\alpha|\leqq m-1).$$

Then Theorems 1 and 2 hold, further

$$f_\alpha^{(1)}(\cdot,u,\dots,D^\beta u,\dots)\in L^1(\Omega), \quad f_\alpha^{(2)}(\cdot,u,\dots,D^\beta u,\dots)\in L^q(\Omega) \tag{2.44}$$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and (1.4) is valid for each $v\in K$, satisfying $D^\alpha v\in L^\infty(\Omega)$ if $|\alpha|\leqq m$.

Clearly, we have only to show the first part of (2.44). One can prove

$$|f_\alpha^{(1)}(x,\xi)|\leqq \sup\{|f_\alpha^{(1)}(x,\xi)|: |\xi_\gamma|<s \quad \text{for} \quad \gamma\in A_\nu\}+\frac{1}{s}\sum_{\gamma\in A_\nu}|f_\gamma^{(1)}(x,\xi)\xi_\gamma|$$

thus by Sobolev's imbedding theorem for sufficiently large $s>0$

$$|f_\alpha^{(1)}(x,u,\dots,D^\beta u,\dots)|\leqq f_{\nu,s}(x)+c_3\sum_{|\gamma|\leqq m}|D^\gamma u|^p+$$
$$+\frac{1}{s}\sum_{\gamma\in A_\nu}[f_\gamma^{(1)}(x,u,\dots,D^\beta u,\dots)D^\gamma u+2K_\gamma(x)]$$

where the term in the right is integrable over Ω.

Now we give an example, satisfying I to V and the above conditions:

$$f_\alpha^{(1)}(x,\zeta)=h_\alpha^{(1)}(\xi_\alpha)\varphi_\alpha^{(1)}(x,\zeta^{[\alpha]})+\tilde h_\alpha^{(1)}(\xi_\alpha)\tilde\varphi_\alpha^{(1)}(x,\zeta^{[\alpha]}),$$
$$f_\alpha^{(2)}(x,\zeta)=h_\alpha^{(2)}(\xi_\alpha)\varphi_\alpha^{(2)}(x,\zeta^{[\alpha]}), \quad |\alpha|=m;$$
$$f_\alpha^{(1)}(x,\eta)=h_\alpha^{(1)}(\xi_\alpha)\varphi_\alpha^{(1)}(x,\eta^{[\alpha]})+\tilde h_\alpha^{(1)}(\xi_\alpha)\tilde\varphi_\alpha^{(1)}(x,\eta^{[\alpha]}),$$
$$f_\alpha^{(2)}(x,\eta)=h_\alpha^{(2)}(\xi_\alpha)\varphi_\alpha^{(2)}(x,\eta^{[\alpha]}), \quad |\alpha|\leqq m-1$$

($\zeta^{[\alpha]}$, $\eta^{[\alpha]}$ are defined in Remark 3). Here $h_\alpha^{(j)}$, $\tilde{h}_\alpha^{(1)}$ are continuously differentiable such that

$$h_\alpha^{(j)}(\xi_\alpha) > 0, \quad \tilde{h}_\alpha^{(1)}(\xi_\alpha) > 0 \quad \text{for} \quad \xi_\alpha > 0, \quad h_\alpha^{(j)}(\xi_\alpha) < 0, \quad \tilde{h}_\alpha^{(1)}(\xi_\alpha) < 0 \quad \text{for} \quad \xi_\alpha < 0,$$
$$|\tilde{h}_\alpha^{(1)}(\xi_\alpha)| \leqq c_1|\xi_\alpha|^p, \quad c_2|\xi_\alpha|^{p-1} \leqq |h_\alpha^{(2)}(\xi_\alpha)| \leqq c_3|\xi_\alpha|^{p-1}$$

(c_1, c_2, c_3 are positive constants; the functions $\varphi_\alpha^{(j)}$, $\tilde{\varphi}_\alpha^{(1)}$ are continuous and they are continuously differentiable with respect to $\zeta^{[\alpha]}$ resp. $\eta^{[\alpha]}$ such that

$$0 \leqq \varphi_\alpha^{(1)}(x, \zeta^{[\alpha]}) = \chi_\alpha(x)\psi_\alpha(\zeta^{[\alpha]}), \quad 0 \leqq \tilde{\varphi}_\alpha^{(1)}(x, \zeta^{[\alpha]}) \leqq c_4,$$
$$c_5 \leqq \varphi_\alpha^{(2)}(x, \zeta^{[\alpha]}) \leqq c_6 \quad (|\alpha| = m)$$

(c_4, c_5, c_6 are positive constants), $\chi_\alpha \in L^1(\Omega)$, $|\psi_\alpha| \leqq c_4$, ψ_α is continuous; and similarly

$$0 \leqq \varphi_\alpha^{(1)}(x, \eta^{[\alpha]}) = \chi_\alpha(x)\psi_\alpha(\eta^{[\alpha]}), \quad 0 \leqq \tilde{\varphi}_\alpha^{(1)}(x, \eta^{[\alpha]}) \leqq c_4,$$
$$c_5 \leqq \varphi_\alpha^{(2)}(x, \eta^{[\alpha]}) \leqq c_6 \quad (|\alpha| \leqq m-1).$$

Finally, the matrix (1.8) is positive definite, the matrix (1.9) is positive semidefinite. The last conditions are fulfilled e.g. if $\varphi_\alpha^{(j)}(x, \zeta^{[\alpha]})$, $\varphi_\alpha^{(j)}(x, \eta^{[\alpha]})$, $\tilde{\varphi}_\alpha^{(1)}(x, \zeta^{[\alpha]})$, $\tilde{\varphi}_\alpha^{(1)}(x, \eta^{[\alpha]})$ do not depend on $\zeta^{[\alpha]}$ resp. $\eta^{[\alpha]}$ and $(h_\alpha^{(2)})' > 0$ for $|\alpha| = m$ and $\geqq 0$ for $|\alpha| \leqq m-1$, further $(h_\alpha^{(1)})' \geqq 0$, $(\tilde{h}_\alpha^{(1)})' \geqq 0$.

REMARK 9. Instead of V assume that there exist nonnegative functions $k^*, \tilde{k} \in L^1(\Omega)$ and a constant $c_3 > 0$ such that

$$\sum_{|\alpha| \leqq m-1} f_\alpha(x, \eta)\xi_\alpha \geqq c_3|\eta|^p - k^*(x), \qquad \sum_{|\alpha|=m} f_\alpha(x, \xi)\xi_\alpha \geqq c_3|\zeta|^p - \tilde{k}(x).$$

Then Theorems 1 and 2 remain valid such that instead of (1.3) we have only

$$f_\alpha(\cdot, u, \dots, D^\beta u, \dots) \in L^1(\Omega_r),$$
$$\sum_{|\alpha| \leqq m-1} f_\alpha(\cdot, u, \dots, D^\beta u, \dots)D^\alpha u \in L^1(\Omega), \qquad \sum_{|\alpha|=m} f_\alpha(\cdot, u, \dots, D^\beta u, \dots)D^\alpha u \in L^1(\Omega).$$

REMARK 10. Assume that I to IX are fulfilled and there is a constant $c > 0$ such that

$$\sum_{|\alpha|=m} [f_\alpha(x, \eta, \zeta) - f_\alpha(x, \eta, \zeta')](\xi_\alpha - \xi'_\alpha) \geqq c|\zeta - \zeta'|^p$$

for all $(\eta, \zeta), (\eta, \zeta') \in R^N$ and a.e. $x \in \Omega$. Then by (2.9) and Lemma 3 $(\tilde{u}_n) \to u$ also in $W_p^m(\omega)$ for each fixed bounded $\omega \subset \Omega$. (If the solution of (1.3), (1.4) is unique then (u_n) converges to u in $W_p^m(\omega)$.)

References

[1] R. Landes, Quasilineare elliptische Differentialoperatoren mit starkem Wachstum in den Termen höchster Ordnung, *Math. Z.*, **157** (1977), 23—26.

[2] R. Landes, Two existence theorems for nonlinear elliptic equations, *J. Differential Equations*, **33** (1979), 45—57.

[3] R. Landes, Quasilinear elliptic operators and weak solutions of the Euler equation, *Manuscripta Math.*, **27** (1979), 47—72.

[4] R. Landes, On Galerkin's method in the existence theory of quasilinear elliptic equations, *J. Functional Analysis,* **39** (1980), 123—148.
[5] V. Mustonen—C. G. Simader, On the existence and uniqueness of solutions for strongly nonlinear elliptic variational problems, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.,* **6** (1981), 233—254.
[6] V. Mustonen, Mappings of monotone type and nonlinear obstacle problem, *Bayreuther Mathematische Schriften,* Heft 17 (1984), 171—183.
[7] C. G. Simader, Remarks on uniqueness and stability of weak solutions of strongly nonlinear elliptic equations, *Bayreuther Mathematische Schriften,* Heft 11 (1982), 67—79.
[8] R. Landes—V. Mustonen, Boundary value problems for strongly nonlinear second order elliptic equations, *Bolletino U.M.I.,* **(6) 4—B** (1985), 15—32.
[9] J. R. L. Webb, Boundary value problems for strongly nonlinear elliptic equations, *J. London Math. Soc.,* **21** (1980), 123—132.
[10] L. Simon, Variational inequalities for strongly nonlinear elliptic operators, *Annales Univ. Sci. Budapest,* **29** (1986), 231—240.
[11] L. Simon, On approximation of the solutions of strongly nonlinear elliptic equations in unbounded domains, *Studia Math. Hung.,* **19** (1984), 421—430.
[12] L. Simon, Some remarks on strongly nonlinear elliptic problems in unbounded domains, *Annales Univ. Sci. Budapest,* **30** (1987), 265—283,
[13] О. В. Бесов — В. П. Ильин — С. М. Никольский, *Интегральные представления функций и теоремы вложения,* Наука (Москва, 1975).
[14] J. R. L. Webb, Strongly nonlinear elliptic equations, *Journées d'Analyse Non Linéaires* (Besancon, 1977), Lecture Notes in Math. 665 (1978), 242—256.
[15] L. Simon, On boundary value problems for nonlinear elliptic equations on unbounded domains, *Publ. Math. Debrecen,* **34** (1987), 75—81.
[16] R. E. Edwards, *Functional analysis,* Holt, Rinehart and Winston, 1965.
[17] K. Yosida, *Functional analysis,* Springer, 1965.
[18] R. A. Adams, *Sobolev spaces,* Academic Press, 1975.

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ON SIMPLICITY-CRITICAL MOORE AUTOMATA. I

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Dedicated to Professor Pál Erdős on his 75th birthday

1.

The present paper contains some contributions to the problem of simplicity of finite Moore automata. Our goal is to reduce the question of separation of simple automata from non-simple ones to some more particular problems.

Section 2 is devoted to the introduction of basic notions. Section 3 is also of preparatory character: it presents a survey of the subautomata of an automaton by graph-theoretic tools.[1] The main results of the article are contained in Sections 4—6. The theorem stated in Section 5 shows that the crucial difficulty in answering the question "when is an automaton simple?" lies in characterization of the simplicity within the particular class that consists of *strongly connected* automata, and in exploring the structure of two special automata classes, whose elements are called *simplicity-critical* automata.

In the second part of the paper some structural results about simplicity-critical automata will be obtained.

From among the previous publications, this paper has the closest connections with [4].[2] For the sake of completeness, let it be noted that the simplicity of initially connected automata can be analyzed in a certain constructive approach, too. This line of investigations was initiated in [1]; out of the adjoining articles, let now [2] (Sections 4—6), [3] and the works [5], [6] of M. Katsura be mentioned.

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[1] The subautomata of an automaton are often referred to in the further considerations. This fact perhaps justifies the systematic overview of the subautomata of an automaton, even if the content of Section 3 may be considered to be folkloristically known.

[2] See the survey exposed in Section 7.1 of [4]. Now we deal with the version (C) of the basic problem in sense of Section 7.1. If only initially connected automata are taken into account, then the version (B) & (C) is got.

2.

In this paper finite Moore automata (written in the form $\mathbf{A}=(A, X, Y, \delta, \lambda)$) are considered. The notions of subautomaton (in particular: proper subautomaton, maximal subautomaton, the subautomaton generated by a subset H of A) and isomorphism are used in the customary sense.[3] $\mathbf{A}$ is called *autonomous* if $|X|=1$.

An automaton $\mathbf{A}$ is called *connected* if for each ordered state pair a, b there exists a sequence

$$(1) \qquad a_0, x_1, a_1, x_2, a_2, x_3, \ldots, x_k, a_k \quad (k \geqq 0),$$

consisting of states and of input symbols alternately, such that $a_0=a$, $a_k=b$ and one of the equalities

$$\delta(a_{i-1}, x_i) = a_i, \quad \delta(a_i, x_i) = a_{i-1}$$

holds for every choice of i $(1 \leqq i \leqq k)$. Each automaton $\mathbf{A}$ can be obtained as the union of certain of its pairwise disjoint connected subautomata $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_t$ $(t \geqq 1)$. We say that $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_t$ are the *connected components* of $\mathbf{A}$.

Let H be an arbitrary subset of the state set A of an automaton $\mathbf{A}$. We say that H is a *strongly connected set* if for each ordered pair $a(\in H)$, $b(\in H)$ there exists a sequence (1) such that $a_0=a$, $a_k=b$, the states $a_0, a_1, \ldots, a_k$ belong to H and $\delta(a_{i-1}, x_i)=a_i$ for every choice of i $(1 \leqq i \leqq k)$.

If the set of states of a subautomaton $\mathbf{B}$ of $\mathbf{A}$ is strongly connected, then it is said that $\mathbf{B}$ is a *strongly connected subautomaton* of $\mathbf{A}$. It is clear when $\mathbf{A}$ itself is called strongly connected. Strongly connected autonomous automata are called *cycles*.

By a *block*, we understand a maximal strongly connected subset $H(\subseteqq A)$. It is obvious that each state of $\mathbf{A}$ is contained in exactly one block. If $\mathbf{B}$ is a strongly connected subautomaton of $\mathbf{A}$, then the state set of $\mathbf{B}$ is a block in $\mathbf{A}$; consequently, the strongly connected subautomata of an automaton are pairwise disjoint.

We denote by $\Gamma(H)$ the subautomaton generated by $H(\subseteqq A)$. If H consists of a single state a, then $\Gamma(a)$ will be written instead of $\Gamma(\{a\})$. If $\Gamma(H)=\mathbf{A}$, then H is called a *generating system* of $\mathbf{A}$.

If $\Gamma(a)=\mathbf{A}$ for some $a(\in A)$, then a is called a *generator* of $\mathbf{A}$. If a is an element of A such that the proper inclusion $\Gamma(b)\supset\Gamma(a)$ is satisfied by no $b(\in A)$, then we say that a is a *quasi-generator* of $\mathbf{A}$. $b(\in A)$ is called *accessible* from $a(\in A)$ if b is a state of $\Gamma(a)$ (or, equivalently, if $\Gamma(b)\subseteqq\Gamma(a)$).

Two states a, b of an automaton are said to be *distinguishable* if there exists an input word p such that $\lambda(\delta(a,p))\neq\lambda(\delta(b,p))$. In the contrary case, a and b are called *undistinguishable;* the undistinguishability is clearly an equivalence relation (to be denoted by $\pi_{\max}$ in Section 4).

A state a of an automaton $\mathbf{A}$ is called a *source* if $\delta(b, x)\neq a$ for any choice of $b(\in A)$ and $x(\in X)$.

[3] The set X of input symbols is viewed to be invariant; e.g., $\alpha(\delta(a, x))=\delta(\alpha(a), x)$ is postulated when α is an isomorphism.

In the remaining part of this section we assert some simple facts concerning the notions introduced above. The truth of most of these facts is evident.

$a(\in A)$ is a source if and only if the following three conditions are valid:

a is a quasi-generator,

a constitutes a one-element block,

$\delta(a, x) \neq a$ for any $x(\in X)$.

Each automaton has at least one quasi-generator. If a block H contains a quasi-generator, then every element of H is a quasi-generator.

$\Gamma(a)=\Gamma(b)$ if and only if a and b are in the same block.

The following three assertions are equivalent for an automaton $\mathbf{A}$:

$\mathbf{A}$ is strongly connected,

each state of $\mathbf{A}$ is a generator,

$\mathbf{A}$ has no proper subautomaton.

Consider a proper subautomaton $\mathbf{B}$ of an automaton $\mathbf{A}$. Then $\mathbf{B}$ includes at least one strongly connected subautomaton of $\mathbf{A}$ and $\mathbf{B}$ is included in at least one maximal subautomaton of $\mathbf{A}$.

Lemma 1. *Let* $H_1, H_2, \ldots, H_k$ *be all the blocks consisting of quasi-generators of* $\mathbf{A}$ *(where* $k \geqq 1$*). A subset* H *of* A *is a generating system of* $\mathbf{A}$ *if and only if each of the intersections*

$$H \cap H_1, H \cap H_2, \ldots, H \cap H_k$$

is non-empty. If $k=1$, *then every quasi-generator is a generator, as well. If* $k>1$, *then* $\mathbf{A}$ *does not have any generator.*

Proof. It suffices to verify the first of the three statements of the lemma.

Suppose $H \cap H_i \neq \emptyset$ for every i (where $1 \leqq i \leqq k$). Choose an arbitrary $a(\in A)$. Let us form an increasing sequence

$$\Gamma(a) \subset \Gamma(b_1) \subset \Gamma(b_2) \subset \ldots$$

of subautomata (generated by a single state) until the procedure breaks up; we arrived to a $\Gamma(b)$ where b is a quasi-generator of $\mathbf{A}$. Then

$$a \in \Gamma(b) = \Gamma(h_i) \subseteqq \Gamma(H)$$

where i is determined by $b \in H_i$ and h_i is a state belonging to $H \cap H_i$. We have $\mathbf{A}=\Gamma(H)$ because a has been chosen arbitrarily.

Now assume $\Gamma(H)=\mathbf{A}$ and consider a state h^* in a block H_i. Since $h^* \in \Gamma(H)$, there exists an $h(\in H)$ such that $h^* \in \Gamma(h)$. Hence $\Gamma(h^*) \subseteqq \Gamma(h)$, we have $\Gamma(h^*)= =\Gamma(h)$ since h^* is a quasi-generator. Thus

$$H \cap H_i \supseteqq \{h\} \neq \emptyset. \quad \square$$

3.

In this section we get an overview of the subautomata of an automaton by graph-theoretical means. The graphs considered are finite and directed. The most familiar concepts on directed graphs are supposed to be known. The notions of accessibility and (versions of) connectedness are used for directed graphs in complete analogy with their uses for automata.

By a *cycle* of a graph, we mean a simple closed directed chain. A vertex v of a graph is called a *source* if no edge terminates at v, v is a *sink* if no edge starts from v.

Let $\mathcal{S}$ be a subset of the set $\mathcal{V}$ of all vertices of a graph $\mathcal{G}$. $\mathcal{S}$ is called an *independent set* if the elements of $\mathcal{S}$ are pairwise inaccessible in $\mathcal{G}$.

Consider an automaton $\mathbf{A}$. Recall that the blocks form a partition of the set of states. We write $a \equiv b \pmod{\mu}$ if a, b are in the same block.

The characteristic graph $\mathcal{G}(\mathbf{A})$ of an automaton $\mathbf{A}$ is defined by the following two rules.

There is a bijective assignment γ from the equivalence classes of A (modulo μ) to the vertex set of $\mathcal{G}(\mathbf{A})$. We write $a \in \gamma^{-1}(v)$ if $\gamma(a) = v$.

There exists an edge $\overrightarrow{vw}$ in $\mathcal{G}(\mathbf{A})$ if and only if $v \neq w$ and there is a triple $a(\in\gamma^{-1}(v))$, $b(\in\gamma^{-1}(w))$, $x(\in X)$ such that $\delta(a, x) = b$.

The characteristic graph does not depend on the output function λ.

Now we shall state four assertions which are easy consequences of the definition of the characteristic graph.

$\mathcal{G}(\mathbf{A})$ is a cycle-free graph and it has no pair of parallel edges.

$\mathbf{A}$ has a generator if and only if there is only one source in $\mathcal{G}(\mathbf{A})$; if f is the only source, then exactly the elements of $\gamma^{-1}(f)$ are the generators.

$\mathcal{G}(\mathbf{A})$ consists of a single vertex (and no edge) if and only if $\mathbf{A}$ is strongly connected.

$\gamma(a)$ is accessible from $\gamma(b)$ in $\mathcal{G}(\mathbf{A})$ if and only if a is accessible from b in $\mathbf{A}$ (or, equivalently, if $\Gamma(a) \subseteq \Gamma(b)$).

The converse of the first assertion is also true:

PROPOSITION 1. *Let $\mathcal{G}$ be an arbitrary cycle-free graph without parallel edges. There exists an automaton $\mathbf{A}$ such that $\mathcal{G} = \mathcal{G}(\mathbf{A})$.*

PROOF. Let the state set A of $\mathbf{A}$ be the vertex set of $\mathcal{G}$. Let the number of input symbols of $\mathbf{A}$ be the maximum of the out-degrees occurring in $\mathcal{G}$. The output function λ of $\mathbf{A}$ can be defined arbitrarily.

Consider a vertex v of $\mathcal{G}$; denote by $w_1, w_2, \ldots, w_t$ all the vertices w of $\mathcal{G}$ such that there is an edge $\overrightarrow{vw}$ in $\mathcal{G}$ (the ordering of the w's is arbitrary). We set

$$\delta(v, x_i) = \begin{cases} w_i & \text{if} \quad 1 \leqq i \leqq t, \\ v & \text{if} \quad i > t. \end{cases}$$

The obtained automaton $\mathbf{A}$ fulfils $\mathcal{G} = \mathcal{G}(\mathbf{A})$. □

Let $\mathbf{A} = (A, X, Y, \delta, \lambda)$ be an automaton. Consider an independent vertex set $\mathcal{S}$ in its characteristic graph $\mathcal{G}(\mathbf{A})$. Let a subautomaton

$$\varrho(\mathcal{S}) := \mathbf{B} = (B, X, Y, \delta, \lambda)$$

be assigned to $\mathcal{S}$ in the following way:

$b \in B$ if and only if $\gamma(b)$ is accessible from an element of $\mathcal{S}$.

PROPOSITION 2. *ϱ is a bijective mapping from the set of independent vertex sets $\mathcal{S}$ of $\mathcal{G}(\mathbf{A})$ to the set of the subautomata $\mathbf{B}$ of $\mathbf{A}$.*

PROOF. First we show that $\mathscr{S}_1 \neq \mathscr{S}_2$ implies $\varrho(\mathscr{S}_1) \neq \varrho(\mathscr{S}_2)$. There is a vertex v in $\mathscr{S}_1 \cup \mathscr{S}_2$ such that, for each vertex $w(\in(\mathscr{S}_1 \cup \mathscr{S}_2)-\{v\})$, v is inaccessible (in $\mathscr{G}(\mathbf{A})$) from w. (If no v with this property existed, then $\mathscr{G}(\mathbf{A})$ would have a cycle.) We can suppose $v \in \mathscr{S}_1$ without an essential restriction of generality. The elements of $\gamma^{-1}(v)$ are states of $\varrho(\mathscr{S}_1)$ but not of $\varrho(\mathscr{S}_2)$.

Consider an arbitrary subautomaton $\mathbf{B}$ of $\mathbf{A}$. Denote by H a minimal state set such that $\Gamma(H)=\mathbf{B}$. The elements of H are pairwise inaccessible (in $\mathbf{A}$), thus pairwise non-equivalent modulo μ. Let $\mathscr{S}$ be the set of vertices $\gamma(c)$ where c runs through H. It is evident that $\mathscr{S}$ is an independent vertex set that satisfies $\varrho(\mathscr{S})=\mathbf{B}$. $\square$

The following statement is obvious.

PROPOSITION 3. *$\varrho(\mathscr{S})$ is a strongly connected subautomaton of $\mathbf{A}$ if and only if $\mathscr{S}=\{v\}$ where v is a sink of $\mathscr{G}(\mathbf{A})$. The number of strongly connected subautomata of $\mathbf{A}$ equals the number of sinks of $\mathscr{G}(\mathbf{A})$.* $\square$

Now we introduce two notations. For a vertex v of $\mathscr{G}(\mathbf{A})$, let $\chi(v)$ be the set of vertices w for which the edge $\overrightarrow{vw}$ exists. For a subset $\mathscr{S}$ of the vertex set $\mathscr{G}(\mathbf{A})$, let $\tau(\mathscr{S})$ be the set of all vertices $v(\in\mathscr{S})$ such that v is not accessible from any element of $\mathscr{S}-\{v\}$ (in $\mathscr{G}(\mathbf{A})$).

It is easy to see that the next two assertions are valid.

PROPOSITION 4. *Denote by $\mathscr{F}$ the set of all sources of $\mathscr{G}(\mathbf{A})$. Then we have $\varrho(\mathscr{F})=\mathbf{A}$. Moreover, $\varrho(\mathscr{S})$ is a maximal subautomaton of $\mathbf{A}$ if and only if $\mathscr{G}(\mathbf{A})$ has at least two vertices and there exists an $f(\in\mathscr{F})$ such that*

$$\mathscr{S} = \tau((\mathscr{F}-\{f\})\cup\chi(f)).$$

If $\mathbf{A}$ is not strongly connected, then the number of maximal subautomata of $\mathbf{A}$ equals $|\mathscr{F}|$. $\square$

Recall the notations $H_1, H_2, \ldots, H_k$ from Lemma 1.

LEMMA 2. *The complements of the maximal subautomata of an automaton $\mathbf{A}$ are pairwise disjoint. More precisely, the following three statements are equivalent for a subset $H(\neq A)$ of the state set A of $\mathbf{A}$:*
H is one of the blocks $H_1, H_2, \ldots, H_k$,
the elements of $A-H$ constitute a maximal subautomaton of $\mathbf{A}$,
the characteristic graph $\mathscr{G}(\mathbf{A})$ has a source f such that $H=\gamma^{-1}(f)$. $\square$

4.

A partition π of the state set A of an automaton $\mathbf{A}$ is called a *congruence* (of $\mathbf{A}$) if $a \equiv b \pmod{\pi}$ implies

$$\delta(a, x) \equiv \delta(b, x) \pmod{\pi}$$

and $\lambda(a)=\lambda(b)$ where a, b are arbitrary states and x is an arbitrary input symbol. The minimal partition of A is the *trivial congruence* of $\mathbf{A}$. We say that $\mathbf{A}$ is *simple* (or *reduced*) if $\mathbf{A}$ has no non-trivial congruence. The following statements are easy consequences of the considerations of [1], § 5:

Proposition 5. *Denote by* $\pi_{\max}$ *the partition of* A *defined by the rule that* $a \equiv b \pmod{\pi_{\max}}$ *exactly if* a *and* b *are undistinguishable.* $\pi_{\max}$ *is a congruence of* $\mathbf{A}$ *and each congruence of* $\mathbf{A}$ *is a refinement of* $\pi_{\max}$. *An automaton* $\mathbf{A}$ *is simple if and only if each pair* a, b *is distinguishable (where* $a \in A$, $b \in A$, $a \neq b$*).*

The following two assertions are almost evident: each subautomaton of a reduced automaton is reduced; two different subautomata of a simple automaton are necessarily non-isomorphic.[4]

Keeping Lemma 2 in mind, we can verify the following statement:

Proposition 6. *Suppose that an automaton* $\mathbf{A}$ *has two isomorphic subautomata* $\mathbf{C}_1$ *and* $\mathbf{C}_2$, *furthermore, each maximal subautomaton of* $\mathbf{A}$ *is reduced. Then* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are maximal subautomata of* $\mathbf{A}$ *and* $\mathbf{A}$ *has no other maximal subautomaton.*

Proof. Denote the maximal subautomata of $\mathbf{A}$ by $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$ (where $k \geqq 1$). Let H_i be the set of states which do not belong to $\mathbf{B}_i$. Denote by α an isomorphism from $\mathbf{C}_1$ onto $\mathbf{C}_2$.

We can choose a state a of $\mathbf{C}_1$ such that $a \neq \alpha(a)$. There is an i_1 $(1 \leqq i_1 \leqq k)$ such that a is a state of $\mathbf{B}_{i_1}$. Since $\mathbf{B}_1, \ldots, \mathbf{B}_k$ are simple and a, $\alpha(a)$ are undistinguishable, $\alpha(a)$ does not belong to $\mathbf{B}_{i_1}$. Either $k=1$ or there exists an i_2 (where $1 \leqq i_2 \leqq k$, $i_2 \neq i_1$) such that $\alpha(a)$ is a state of $\mathbf{B}_{i_2}$. We distinguish three cases, the first and third ones will lead to contradiction.

Case 1: $k=1$. Then $\alpha(a) \in H_1$ and H_1 is the set of generators of $\mathbf{A}$, thus $\Gamma(\alpha(a)) = \mathbf{A}$ and $\mathbf{C}_2 = \mathbf{A}$. We have got that $\mathbf{A}$ is isomorphic to a proper subautomaton $\mathbf{C}_1$, this is impossible.

Case 2: $k=2$. Then $a \in H_{i_2}$ and $\alpha(a) \in H_{i_1}$. We can assume (without an essential restriction of generality) that $i_1 = 2$ and $i_2 = 1$. It is easy to see that $\Gamma(a) = \mathbf{C}_1 = \mathbf{B}_1$ and $\Gamma(\alpha(a)) = \mathbf{C}_2 = \mathbf{B}_2$.

Case 3: $k \geqq 3$. There is a number j such that

$$j \in \{1, 2, \ldots, k\} - \{i_1, i_2\}.$$

It follows that a and $\alpha(a)$ are states of $\mathbf{B}_j$; hence $\mathbf{B}_j$ is not simple, contradicting the supposition. □

5.

Definition 1. An automaton $\mathbf{A}$ is called *strongly simplicity-critical* (or, for the sake of brevity, *strongly s-critical*) if the following four conditions are satisfied:

(1) $\mathbf{A}$ is not strongly connected.

(2) $\mathbf{A}$ is not simple.

(3) Each maximal subautomaton of $\mathbf{A}$ is simple.

(4) The subautomata of $\mathbf{A}$ are pairwise non-isomorphic.

[4] The verification of these assertions will be contained in the proof of (the sufficiency part of) the single theorem of this paper, too.

DEFINITION 2. An automaton **A** is called *weakly simplicity-critical* (or *weakly s-critical*) if (3) is valid for **A** and the following two conditions are fulfilled:
(5) The number of maximal subautomata of **A** equals two.
(6) The maximal subautomata of **A** are isomorphic.

Each weakly s-critical automaton fulfils (1) and does not satisfy (4). An automaton cannot be both strongly and weakly s-critical.

THEOREM. *An automaton* **A** *is not simple if and only if* **A** *has a (possibly non-proper) subautomaton* **B** *such that one of the following three statements is valid for* **B**:
(A) **B** *is strongly connected and non-simple.*
(B) **B** *is strongly simplicity-critical.*
(C) **B** *is weakly simplicity-critical.*

REMARK. For a fixed subautomaton **B**, at most one of (A), (B), (C) can hold.

PROOF. *Sufficiency.* Denote by $\pi_{\max}^{(A)}$, $\pi_{\max}^{(B)}$ the maximal congruences of **A** and **B**, respectively. Suppose first (A) or (B). There are two different states a, b of **B** which are congruent $\bmod\, \pi_{\max}^{(B)}$, a and b are undistinguishable, thus they are congruent for $\pi_{\max}^{(A)}$, too. — Assume now (C). There is an isomorphism α from a maximal subautomaton $\mathbf{C}_1$ of **B** onto the other maximal subautomaton $\mathbf{C}_2$. There exists a state a of $\mathbf{C}_1$ such that $a \neq \alpha(a)$. We have again got a state pair, namely a and $\alpha(a)$, which are undistinguishable and congruent $\bmod\, \pi_{\max}^{(A)}$.

Necessity. If **A** is not simple, then there is a subautomaton **B** of **A** such that **B** is non-simple and all the maximal subautomata $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_q$ of **B** are reduced (where $q \geqq 0$). If $q=0$, then Condition (A) is valid. If $q>0$ and the proper subautomata of **A** are pairwise non-isomorphic, then Condition (B) holds.

It remains still the case that (**B** is non-simple, $q \geqq 1$, each proper subautomaton of **B** is reduced and) there is a pair $\mathbf{D}_1$, $\mathbf{D}_2$ of different but isomorphic subautomata in **B**. The assumptions of Proposition 6 are satisfied, hence (C) is valid. □

Having this theorem, the task of distinguishing between simple and non-simple automata has been reduced to the following three problems (whose solution by sufficiently constructive methods is desirable):

PROBLEM 1. Find a criterion of simplicity within the class of strongly connected automata.

PROBLEM 2. Characterize strongly s-critical automata.

PROBLEM 3. Characterize weakly s-critical automata.

The solution of Problem 1 is known for autonomous automata (see [4], third sentence of Corollary 2); in the general case, however, Problem 1 is an unsolved question which seems to be difficult.

We shall see some results concerning Problems 2 and 3 in the continuation of this paper.

6.

In the present section some preparations will be made for the second paper with the same title; namely, we prove a result on the structure of strongly s-critical automata. Recall that $\mathcal{G}(\mathbf{A})$ denotes the characteristic graph of $\mathbf{A}$.

PROPOSITION 7. *Let* $\mathbf{A}=(A, X, Y, \delta, \lambda)$ *be a strongly simplicity-critical automaton. Denote by* F *the set of generators of* $\mathbf{A}$, *by* B *the difference set* $A-F$ *and by* $\pi_{\max}$ *the maximal congruence of* $\mathbf{A}$. *The following conclusions are valid:*

(i) *Whenever* $a\in A$, $b\in A$, $a\neq b$ *and* $a\equiv b \pmod{\pi_{\max}}$, *then at least one of the equalities* $\mathbf{A}=\Gamma(a)$, $\mathbf{A}=\Gamma(b)$ *holds.*

(ii) $\mathbf{A}$ *is connected and* $F\neq\emptyset$.

(iii) $\mathcal{G}(\mathbf{A})$ *has only one source* f *and* $F=\gamma^{-1}(f)$.

(iv) *The set* B *constitutes a subautomaton* $\mathbf{B}$, $\mathbf{B}$ *is simple and* $\mathbf{B}$ *is the only maximal subautomaton of* $\mathbf{A}$.

PROOF. It suffices to verify (i); (ii) is then evident and (iii), (iv) are easy consequences of (i) by our earlier results concerning the correspondences between $\mathbf{A}$ and $\mathcal{G}(\mathbf{A})$.

Let c be an arbitrary state of $\Gamma(a)$ and d be an arbitrary state of $\Gamma(b)$. Define a many-to-many assignment η so that $c\eta d$ holds if and only if there is an input word p such that $c=\delta(a,p)$ and $d=\delta(b,p)$.

If $c\eta d_1$ and $c\eta d_2$, then d_1 and d_2 are undistinguishable (because both of them are undistinguishable from c). Therefore either $\Gamma(b)=\mathbf{A}$ or $d_1=d_2$ (by (3)). Analogous inference holds by interchanging the roles of c and d and of a and b.

If $\mathbf{A}$ equals $\Gamma(a)$ or $\Gamma(b)$, then (i) is valid. In the contrary case (i.e., when both $\Gamma(a)$, $\Gamma(b)$ are properly included in $\mathbf{A}$), it is easy to see that η is an isomorphism between $\Gamma(a)$ and $\Gamma(b)$; the existence of such an isomorphism contradicts either (3) or (4) (according to whether $\Gamma(a)=\Gamma(b)$ or $\Gamma(a)\neq\Gamma(b)$). $\square$

7. (Appendix)

In the previous paper [4] an explicit criterion has been stated (Proposition 6, p. 269) for an *autonomous* automaton in order to be reduced. The reader may ask why we did not choose the way of natural generalization of the earlier criterion in the present paper. Now we are going to show that the immediate generalization fails to hold.

Certain considerations from the former sections are repeated in the following assertion:

FACT. If either

(i) there exists a non-simple strongly connected subautomaton of an automaton $\mathbf{A}$, or

(ii) $\mathbf{A}$ has a pair of isomorphic strongly connected subautomata, or

(iii) $\mathbf{A}$ contains two different states a, b such that $\lambda(a)=\lambda(b)$ and $\delta(a,x)=$ $=\delta(b,x)$ for every choice of $x(\in X)$,

then $\mathbf{A}$ has a pair of undistinguishable states, hence $\mathbf{A}$ is not reduced.

The criterion exposed in [4] was obtained from this fact so that

an explicit characterization of simplicity of strongly connected autonomous automata has been incorporated into the result, and

for the particular case of autonomous automata, the converse implication has been proved, as well.

The situation changes radically if the autonomousness is abandoned (i.e., when $|X|$ may be an arbitrary natural number). Not only the simplicity of strongly connected automata becomes then an unsolved question, but also the converse of the fact stated above loses validity. This is shown by the following example due to M. Katsura (personal communication).

EXAMPLE. Let an automaton

$$\mathbf{K} = (\{a_1, a_2, a_3, a_4\}, \{x_1, x_2\}, \{y_1, y_2\}, \delta, \lambda)$$

be defined by Table 1 (see also Figure 1). **K** possesses only one strongly connected subautomaton $\mathbf{K}_1$, this subautomaton has the states a_2 and a_4. $\mathbf{K}_1$ is simple. Thus (i), (ii) are not satisfied by **K**; it is easy to check the falsity of (iii), too. **K** is not simple, its maximal congruence is

$$\langle \{a_1, a_2\}, \{a_3, a_4\} \rangle.$$

In addition, **K** is a strongly s-critical automaton.

Table 1

a	$\lambda(a)$	$\delta(a, x_1)$	$\delta(a, x_2)$
a_1	y_1	a_3	a_2
a_2	y_1	a_4	a_2
a_3	y_2	a_4	a_1
a_4	y_2	a_4	a_2

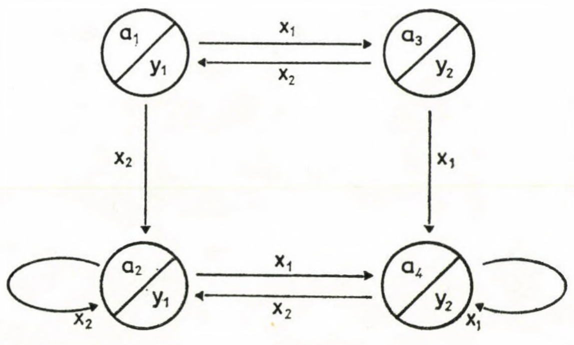

Fig. 1

References

[1] A. Ádám, On the question of description of the behaviour of finite automata, *Studia Sci. Math. Hungar.*, **13** (1978), 105—124.
[2] A. Ádám, On the simplicity of finite autonomous Moore automata, *Studia Sci. Math. Hungar.*, **16** (1981), 427—436.
[3] A. Ádám, On finitely realizable automaton mappings and the simplicity of Moore automata, *Algebra, Combinatorics and Logic in Computer Science (Colloq. Math. Soc. J. Bolyai, 42; Győr, 1983)*, Budapest, 1986; 25—30.
[4] A. Ádám, On the congruences of finite autonomous Moore automata, *Acta Cybernet. (Szeged)*, **7** (1986), 259—279.
[5] M. Katsura, Finiteness of complexity for precodes assigned to finite Moore automata, *Topics in the Theoretical Bases and Applications of Computer Science (Proc. Fourth Hungar. Computer Sci. Conf.; Győr, 1985)*, Akadémiai Kiadó (Budapest, 1986); 91—102.
[6] M. Katsura, Leaves of precodes assigned to finite Moore automata, *Studia Sci. Math. Hungar.*, **23** (1988), 141—155.

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ON THE LOCAL TIME PROCESS STANDARDIZED BY THE LOCAL TIME AT ZERO

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1. Introduction

Let $\{W(t),\ t\geqq 0\}$ be a standard Wiener process and $L(a,t)$ be its (jointly continuous) local time at a up to time t defined by $H(A,t)=\int_A L(a,t)\,da$ and $H(A,t)=\lambda(s\colon 0\leqq s\leqq t,\ W(s)\in A)$, where λ stands for the Lebesgue measure and A is any Borel set on the real line. In this paper our aim is to investigate the limiting behaviour of $L(a,t)-L(0,t)$, suitably normalized, as $t\to\infty$.

The first result of this type for random walk is due to Dobrushin [5]. Let $\{X_i,\ i\geqq 1\}$ be a sequence of i.i.d. integer valued random variables, put $S_i=X_1+\ldots+X_i$, and $N(a,n)=\#\{i\colon S_i=a,\ 1\leqq i\leqq n\}$, $a=1,2,\ldots,\ n=1,2,\ldots$.

THEOREM A (Dobrushin). *Let* $P(X_i=+1)=P(X_i=-1)=1/2$ *and let* $f(i)$ *be a function such that* $\sum_{i=-\infty}^{\infty} f(i)=0$ *and* $f(i)$ *differs from zero only for finitely many i. Then*

$$\lim_{n\to\infty} P\Big(\sum_{k=1}^{n} f(S_k)/dn^{1/4}<x\Big)=P\big(U\sqrt{|V|}<x\big), \tag{1.1}$$

where U and V are two independent standard normal variables and

$$d^2=4\sum_{k=-\infty}^{\infty} kf^2(k)+8\sum_{-\infty<i<j<\infty}\sum if(i)f(j)-\sum_{k=-\infty}^{\infty} f^2(k). \tag{1.2}$$

If $f(a)=1,\ f(0)=-1$ and $f(i)=0$ if $i\neq 0$ or a, then $d^2=4|a|-2$ and this gives for $a=\pm 1,\pm 2,\ldots$

$$\frac{N(a,n)-N(0,n)}{(4|a|-2)^{1/2}n^{1/4}}\xrightarrow{D} U\sqrt{|V|},\quad \text{as}\quad n\to\infty \tag{1.3}$$

where $\xrightarrow{D}$ denotes convergence in distribution.

Dobrushin's theorem was extended for general random walk by Skorohod and Slobodenyuk [10], Borodin [2] and Kasahara [8]. The latter established also a functional form of this limit theorem. The analogue of Dobrushin's theorem for the local time of a Wiener process is also known in functional form (cf. Papanicolaou et al. [9] and Kasahara [8]). Yor [11] proves the following general result:

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THEOREM B (Yor). *For each* $\lambda>0$, *denote by* P_λ *the law of the stochastic process from* $(t, a)\in R^2_+$ *to* R^3;

$$\left(\frac{1}{\lambda} W(\lambda^2 t),\quad \frac{1}{\lambda} L(a, \lambda^2 t),\quad \frac{1}{2\sqrt{\lambda}}\left(L(a, \lambda^2 t)-L(0, \lambda^2 t)\right)\right)$$

defined on $C(R^2_+, R^3)$. *Then for* $\lambda\to\infty$, P_λ *converges weakly to the law of*

$$\left(W(t), L(a, t), W^*(L(0, t), a)\right)$$

in $C(R^2_+, R^3)$ *where* $W^*(u, a)$, $(u, a)\in R^2_+$, *denotes a Brownian sheet starting from* 0, *and independent from* $W(t)$.

Moreover, Csörgő and Révész [4] gave the following iterated logarithm laws:

THEOREM C (Csörgő and Révész). *For* $a=\pm 1, \pm 2, \dots$, *in the case of* $P(X_1=+1)=P(X_1=-1)=1/2$ *we have*

$$\limsup_{n\to\infty} \frac{|N(a, n)-N(0, n)|}{(N(0, n)\log\log n)^{1/2}} = 2(2|a|-1)^{1/2} \quad a.s. \tag{1.4}$$

and

$$\limsup_{n\to\infty} \frac{|N(1, n)-N(0, n)|}{n^{1/4}(\log\log n)^{3/4}} = \left(\frac{128}{27}\right)^{1/4} \quad a.s. \tag{1.5}$$

The aim of the present paper is to give a slightly more general version of Theorem C in the Wiener local time case.

THEOREM 1. *For any* $a\in\mathbf{R}$, *and* $\alpha\leqq 1/2$ *we have*

$$\limsup_{t\to\infty} \frac{|L(a, t)-L(0, t)|}{t^{(1-2\alpha)/4}(L(0, t))^{\alpha}(\log\log t)^{(3-2\alpha)/4}} = K_\alpha\cdot\sqrt{|a|} \quad a.s. \tag{1.6}$$

where

$$K_\alpha = \begin{cases} 2^{(9-2\alpha)/4}(1-2\alpha)^{(1-2\alpha)/4}(3-2\alpha)^{(-3+2\alpha)/4} & \text{if } \alpha<1/2 \\ 2\sqrt{2} & \text{if } \alpha=1/2. \end{cases} \tag{1.7}$$

From this theorem one can see that the normalizing factor for $\left(L(a, t)-L(0, t)\right)$ can be either $\left(L(0, t)\right)^{1/2}$, or $t^{1/4}$, or the combination of these two factors $\left(L(0, t)\right)^{\alpha} t^{(1-2\alpha)/4}$.

Note that the constants in Theorem C and Theorem 1 are "equal" in the following sense. For the standard deviations we have $\sigma\left(L(a, T_1)\right)=2\sqrt{|a|}$, $\sigma\left(N(a, \tau_1)\right)=\sqrt{2}\sqrt{2|a|-1}$ (where $\tau_1=\inf\{l\colon l>0,\ S_l=0\}$, $T_1=\inf\{t\colon L(0, t)\geqq 1\}$). Now observe that

$$\frac{K_0}{\sigma\left(L(1, T_1)\right)} = \frac{K_0}{2} = \frac{\sqrt[4]{\frac{128}{27}}}{\sigma\left(N(1, \tau_1)\right)} = \frac{\sqrt[4]{\frac{128}{27}}}{\sqrt{2}} = \sqrt[4]{\frac{32}{27}}$$

and

$$\frac{K_{1/2}\cdot\sqrt{a}}{\sigma(L(a,T_1))} = \frac{K_{1/2}\cdot\sqrt{a}}{2\sqrt{a}} = \frac{2(2|a|-1)^{1/2}}{\sigma(N(a,\tau_1))} = \sqrt{2}.$$

We reformulate Theorem 1 for the random time T_r defined by

$$T_r = \inf\{t\colon\ L(0,t) \geqq r\} \tag{1.8}$$

i.e. the first passage process associated with $L(0,t)$.

THEOREM 2. *For any* $a\in\mathbf{R}$, *and* $\alpha\leqq 1/2$ *we have*

$$\limsup_{r\to\infty} \frac{|L(a,T_r)-r|}{r^\alpha T_r^{(1-2\alpha)/4}(\log\log r)^{(3-2\alpha)/4}} = \sqrt{|a|}\,K_\alpha \quad a.s. \tag{1.9}$$

Only the proof of Theorem 2 will be given, since it easily implies Theorem 1. We use the following result, a particular case of Theorem 2.4 in Borodin [2]:

THEOREM D (Borodin). *Let $f(x)$ be a nonnegative piecewise continuous function,*

$$A(t) = \int_0^t f(W(s))ds + \sum_{i=1}^k c_i L(a_i, t).$$

Then

$$D(x) = \int_0^\infty e^{-\lambda r} E_x(\exp(-A(T_r)))dr \tag{1.10}$$

is the unique continuous solution of the problem

$$\begin{cases} \frac{1}{2}D''(x) - f(x)D(x) = 0, & X\in\mathbf{R}\setminus\{0, a_1, \dots, a_k\} \\ D'(+0)-D'(-0) = 2\lambda D(0)-2, & \\ D'(a_i+0)-D'(a_i-0) = 2c_iD(a_i), & i=1,\dots,k,\ a_i\neq 0 \\ D(x) \quad \textit{is bounded for} \quad x\in\mathbf{R}. & \end{cases} \tag{1.11}$$

E_x denotes the expectation under the condition $\{W(0)=x\}$ $(E_0=E)$.

2. Proof of the law of the iterated logarithm

Theorem B clearly implies that

$$\frac{L(a,t)-L(0,t)}{(L(0,t))^{1/2}} \quad \text{and} \quad \frac{L(0,t)}{t^{1/2}}$$

are asymptotically independent when $t\to\infty$. Thus for the random time T_r we have

$$P\left(\frac{L(a,T_r)-r}{r^\alpha T_r^{(1-2\alpha)/4}} > h\right) \sim P(2a^{1/2}U|V|^{1/2-\alpha} > h), \tag{2.1}$$

where U and V are independent standard normal variables, provided that h is a constant. To prove however the law of the iterated logarithm (Theorem 2), we have to put $h=h(r)\to\infty$ as $r\to\infty$ and for this case (2.1) can not be claimed. We need large deviation for the left hand side of (2.1) which we can obtain by rather complicated calculations only.

Note that in the course of the proof we use notations C, $C(\varepsilon)$, etc. for constants whose values are unimportant in the proof and therefore they may denote different constants at different places.

Without loss of generality, assume that $a>0$. To obtain

$$E(\exp\{-(\eta T_r+cL(a,T_r))\})$$

we have to apply Theorem D. Let $f(x)=\eta$, hence $A(t)=\eta t+cL(a,t)$. Solving the system of differential equations (1.11) in case $f(x)=\eta$, $i=1$, $a_1=a$, $c_1=c$, and computing $D(0)$, we get by inverting the Laplace transform with respect to λ, that

$$E(\exp-\{\eta T_r+cL(a,T_r)\}) = \exp\left\{-\frac{ce^{2\sqrt{2\eta}a}\sqrt{2\eta}+2\eta e^{2\sqrt{2\eta}a}}{c(e^{2\sqrt{2\eta}a}-1)+\sqrt{2\eta}\,e^{2\sqrt{2\eta}a}}r\right\}. \tag{2.2}$$

According to the inversion formula (Erdélyi et al. [7], p. 244 (31))

$$e^{h/s}-1 = \int_0^\infty e^{-su}\sqrt{\frac{h}{u}}\,I_1(2\sqrt{uh})\,du, \tag{2.3}$$

where $I_1(.)$ is the modified Bessel function of order 1:

$$I_1(x) = \sum_{l=0}^{\infty}\frac{1}{l!(l+1)!}\left(\frac{x}{2}\right)^{2l+1}.$$

We obtain, that for $u\neq 0$

$$E_0(\exp(-\eta T_r),\ L(a,T_r)\in du) = \tag{2.4}$$

$$= B(\eta,a)\exp\{-(r+u)B(\eta,a)e^{2a\sqrt{2\eta}}+a\sqrt{2\eta}\}\sqrt{\frac{r}{u}}\,I_1(2B(\eta,a)\sqrt{ru}\,e^{a\sqrt{2\eta}})\,du$$

where

$$B(\eta,a) = \frac{\sqrt{2\eta}}{e^{2a\sqrt{2\eta}}-1} \tag{2.5}$$

$$(E_0(\exp(-\eta T_r),\ L(a,T_r)=0) = \exp\{-B(\eta,a)re^{2a\sqrt{2\eta}}\}).$$

LEMMA 2.1. *For* $x_r>0$, $\eta_r>0$, $h_r>0$ *satisfying*

$$\frac{(x_rh_r)^3}{\sqrt{r}}\to 0,\quad \frac{\eta_r}{r}\to 0,\quad \frac{x_rh_r\sqrt{\eta_r}}{\sqrt{r}}\to 0 \quad \textit{as} \quad r\to\infty \tag{2.6}$$

and for any $\varepsilon>0$, *there exists an* $r_0(\varepsilon)$ *such that*

$$\left|\frac{E\left(\exp\left\{-\eta\frac{T_r}{r^2}\right\},\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)}{e^{-\sqrt{2\eta}}\frac{1}{\sqrt{2\pi}}\frac{h_r}{2\sqrt{a}}\exp\left\{-\frac{x^2h_r^2}{8a}\right\}dx}-1\right|<\varepsilon \tag{2.7}$$

for $r>r_0(\varepsilon)$, $0\leqq\eta\leqq\eta_r$, $|x|\leqq x_r$.

PROOF. The proof of (2.7) can be obtained from (2.3) and the well-known asymptotic form

$$I_1(z)=(1+o(1))\frac{e^z}{\sqrt{2\pi z}},\quad z\to\infty. \tag{2.8}$$

We get $E\left(\exp\left\{-\eta\frac{T_r}{r^2}\right\},\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)$ from (2.4) by substituting η by $\frac{\eta}{r^2}$, and u by $r\left(1+\frac{xh_r}{\sqrt{r}}\right)$, du by $h_r\sqrt{r}\,dx$. (Observe that under conditions (2.6) the argument of $I_1(.)$ goes to infinity as $r\to\infty$.)

For notational convenience denote

$$\frac{E\left(\exp\left(-\eta\frac{T_r}{r^2}\right),\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)}{e^{-\sqrt{2\eta}}\frac{1}{\sqrt{2\pi}}\frac{h_r}{2\sqrt{a}}\exp\left\{-\frac{x^2h_r^2}{8a}\right\}dx}=Pe^Q, \tag{2.9}$$

where

$$Q=-r\left(2+\frac{xh_r}{\sqrt{r}}\right)B\left(\frac{\eta}{r^2},\ a\right)e^{2a\sqrt{2\eta}/r}+\sqrt{2\eta}+\frac{x^2h_r^2}{8a}+ \tag{2.10}$$

$$+2B\left(\frac{\eta}{r^2},\ a\right)r\sqrt{1+\frac{xh_r}{\sqrt{r}}}\,e^{a\sqrt{2\eta}/r}=\frac{x^2h_r^2}{8a}-B\left(\frac{\eta}{r^2},\ a\right)r\left(e^{a\sqrt{2\eta}/r}\sqrt{1+\frac{xh_r}{\sqrt{r}}}-1\right)^2+$$

$$+\frac{a\sqrt{2\eta}}{2r}\leqq\frac{x_r^2h_r^2}{8a}\frac{2a\sqrt{2\eta_r}}{r}+\frac{x_r^3h_r^3}{16a\sqrt{r}}+\frac{a\sqrt{2\eta_r}}{2r}<\varepsilon$$

if r is big enough, and (i)—(iii) hold. Similarly

$$Q\geqq-\frac{ax_r^2h_r^2\eta_r}{r^2}-\frac{4x_rh_r\eta_r}{r^{3/2}}-\frac{4a\eta_r}{r}-\frac{x_r^2h_r^2\sqrt{2\eta_r}}{4r}-\frac{x_rh_r\sqrt{2\eta_r}}{\sqrt{r}}>-\varepsilon \tag{2.11}$$

if conditions (2.6) hold and r is big enough.

On the other hand,

$$P=\frac{1}{\left(1+\frac{xh_r}{\sqrt{r}}\right)^{3/4}}(2a)^{1/2}\left(B\left(\frac{\eta}{r^2},\ a\right)\right)^{1/2}(1+o(1))=1+o(1), \tag{2.12}$$

if conditions (2.6) hold. (2.9)—(2.12) clearly imply our Lemma 2.1.

LEMMA 2.2. *For any* $\varepsilon>0$,

$$P\left(\frac{L(a,T_r)-r}{h_r\sqrt{r}}>D_r\right)\leqq(1+\varepsilon)\exp\left\{-\frac{D_r^2h_r^2}{8a}\right\}\tag{2.13}$$

if $\lim\limits_{r\to\infty}D_r^3\alpha_r^3/r^{1/2}=0$ *and r is big enough.*

PROOF. The moment generating function of $L(a,T_r)$ can be obtained from (2.2) by letting $\eta\to 0$:

$$E(\exp\{uL(a,T_r)\})=\exp\left\{\frac{ur}{1-2ua}\right\}.\tag{2.14}$$

(This is given also in Bass and Griffin [1].) (2.13) can be easily seen by the exponential Markov inequality.

LEMMA 2.3. *For* $x_r>0$, $y_r>0$, $h_r>0$ *satisfying*

$$\frac{x_rh_r}{r^{1/6}}\to 0,\quad \frac{1}{y_r^2r}\to 0,\quad \frac{x_rh_r}{y_r\sqrt{r}}\to 0\quad as\quad r\to\infty\tag{2.15}$$

and for any $\varepsilon>0$

$$P\left(\frac{T_r}{r^2}<y,\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)\Big/dx\leqq(1+\varepsilon)e^{-1/(2y)}\frac{h_r}{2\sqrt{2\pi a}}\,e^{-x^2h_r^2/(8a)}\tag{2.16}$$

if $r>r_0(\varepsilon)$, $|x|\leqq x_r$, $y\geqq y_r$.

PROOF. By Lemma 2.1 and the exponential Markov inequality,

$$P\left(\frac{T_r}{r^2}<y,\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)\Big/dx\leqq$$

$$\leqq e^{\eta y}E\left(e^{-\eta T_r/r^2},\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)\Big/dx\leqq e^{\eta y}(1+\varepsilon)\frac{h_r}{2\sqrt{2\pi a}}\exp\left\{-\sqrt{2\eta}-\frac{x^2h_r^2}{8a}\right\}$$

and (2.16) follows by putting $\eta=1/(2y^2)$.

Now define the events A_r by

$$A_r=\left\{\left(\frac{T_r}{r^2}\right)^{(1-2\alpha)/4}<\frac{L(a,T_r)-r}{h_r\sqrt{r}}\right\}.\tag{2.17}$$

LEMMA 2.4. *For* $h_r=\left(\dfrac{1+2\varepsilon}{K^*}\log\log r\right)^{(3-2\alpha)/4}$ *and* $\varepsilon>0$,

$$P(A_r)\leqq c(\varepsilon)(\log r)^{-1-\varepsilon}\tag{2.18}$$

if r is big enough, where $K^*=(8a)^{-2/(3-2\alpha)}\left(\frac{1}{2}(1-2\alpha)^{2/(3-2\alpha)}+(1-2\alpha)^{-(1-2\alpha)/(3-2\alpha)}\right)$.

PROOF. Clearly, for $\alpha<1/2$,

(2.19)
$$P(A_r)=\int_0^\infty P\left(\frac{T_r}{r^2}<x^{4/(1-2\alpha)},\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)=\int_0^{H_r}+\int_{H_r}^{D_r}+\int_{D_r}^\infty=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where

$$H_r=\big(2(1+\varepsilon)\log\log r\big)^{-(1-2\alpha)/4}, \tag{2.20}$$

$$D_r=\frac{\big(8a(1+\varepsilon)\log\log r\big)^{1/2}}{h_r}. \tag{2.21}$$

Now for large r, being $P\left(\frac{T_r}{r^2}<u\right)=2\left(1-\Phi\left(\frac{1}{\sqrt{u}}\right)\right)\leqq e^{-1/(2u)}$,

$$I\leqq H_rP\left(\frac{T_r}{r^2}<H_r^{4/(1-2\alpha)}\right)\leqq H_r\exp\left\{-\frac{1}{2}H_r^{-4/(1-2\alpha)}\right\}\leqq(\log r)^{-1-\varepsilon} \tag{2.22}$$

and by Lemma 2.2,

$$\mathrm{III}\leqq P\left(\frac{L(a,T_r)-r}{h_r\sqrt{r}}\geqq D_r\right)\leqq(1+\varepsilon)e^{-D_r^2h_r^2/(8a)}\leqq(1+\varepsilon)e^{-(1+\varepsilon)\log\log r}. \tag{2.23}$$

Furthermore, by Lemma 2.3,

$$\mathrm{II}\leqq\int_{H_r}^{D_r}\frac{(1+\varepsilon)h_r}{2\sqrt{2\pi a}}\exp\left\{-\frac{1}{2}x^{-4/(1-2\alpha)}-\frac{x^2h_r^2}{8a}\right\}dx. \tag{2.24}$$

By elementary calculations, one can easily see that the integrand has a maximum on $0<x<\infty$ at the point

$$x^*=\left(\frac{8a}{h_r^2(1-2\alpha)}\right)^{(1-2\alpha)/(6-4\alpha)}, \tag{2.25}$$

hence

(2.26)
$$\mathrm{II}\leqq\frac{(1+\varepsilon)h_rD_r}{2\sqrt{2\pi a}}\exp\left\{-\frac{1}{2}x^{*-4/(1-2\alpha)}-\frac{x^{*2}h_r^2}{8a}\right\}\leqq$$
$$\leqq C(\varepsilon)(\log\log r)^{1/2}e^{-(1+2\varepsilon)\log\log r}\leqq C(\varepsilon)(\log r)^{-1-\varepsilon}$$

if r is big enough. Now (2.18) follows from (2.19), (2.22), (2.23) and (2.26) in the case $\alpha<1/2$. For $\alpha=1/2$, Lemma 2.4 follows immediately from Lemma 2.2.

Our next goal is to give a lower bound for $P(A_r)$.

LEMMA 2.5. *For* $x_r>0$, $y_r>0$, $h_r>0$ *satisfying the conditions* (2.15), $\lim_{r\to\infty}y_r=0$, $\lim_{r\to\infty}r^{-1/4}y_r^{-2}\exp(1/y_r)=0$ *and for* $\varepsilon>0$ *we have*

$$P\left(\frac{T_r}{r^2}<y,\quad\frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right)\Big/dx\geqq C(\varepsilon)yh_r\,e^{-(1+\varepsilon)/(2y)-x^2h_r^2/(8a)} \tag{2.27}$$

if $r>r_0(\varepsilon)$, $|x|\leqq x_r$, $y_r\leqq y\leqq 2y_r$.

PROOF. Start with

$$E(e^{-\eta T_r/r^2},\ L^*\in dx) = \eta\int_0^\infty e^{-\eta s}P\left(\frac{T_r}{r^2}<s,\ L^*\in dx\right)ds = \tag{2.28}$$

$$=\eta\int_0^{r^{-1/4}}+\eta\int_{r^{-1/4}}^{y(1-\varepsilon)}+\eta\int_{y(1-\varepsilon)}^{y(1+\varepsilon)}+\eta\int_{y(1+\varepsilon)}^{y(2+\varepsilon)}+\eta\int_{y(2+\varepsilon)}^{\infty} = I_1+I_2+I_3+I_4+I_5,$$

where $L^*=(L(a,T_r)-r)/(h_r\sqrt{r})$ and $\varepsilon>0$.

Put $\eta=1/(2y^2)$ and estimate I_1 to I_5 from above.

Using Lemma 2.1 for $\eta=0$ we get

$$I_1 \leqq \frac{1}{2y^2}P(L^*\in dx) \leqq \frac{Cr^{-1/4}h_r}{y^2}e^{-x^2h_r^2/(8a)}dx \leqq \varepsilon h_r e^{-1/y-x^2h_r^2/(8a)}dx. \tag{2.29}$$

From Lemma 2.3,

$$I_2 \leqq \frac{1}{2y^2}\int_{r^{-1/4}}^{y(1-\varepsilon)} e^{-s/(2y^2)}\frac{(1+\varepsilon)h_r}{2\sqrt{2\pi a}}e^{-1/(2s)-x^2h_r^2/(8a)}ds\,dx.$$

But $\exp\left\{-\frac{s}{2y^2}-\frac{1}{2s}\right\}$ is increasing if $s\leqq y(1-\varepsilon)$, hence

$$I_2 \leqq C\frac{h_r}{y}\exp\left\{-\frac{1}{2y}\left(\frac{1}{1-\varepsilon}+1-\varepsilon\right)-\frac{x^2h_r^2}{8a}\right\}dx \leqq \varepsilon h_r e^{-1/y-x^2h_r^2/(8a)}dx. \tag{2.30}$$

Furthermore,

$$I_3 \leqq \frac{1}{2y^2}2\varepsilon y e^{-(1-\varepsilon)/(2y)}P\left(\frac{T_r}{r^2}<y(1+\varepsilon),\ L^*\in dx\right). \tag{2.31}$$

Again, by Lemma 2.3 and since $\exp\left\{-\frac{s}{2y^2}-\frac{1}{2s}\right\}$ is decreasing if $s\geqq y(1+\varepsilon)$,

$$I_4 \leqq C\frac{h_r}{y}\exp\left\{-\frac{1}{2y}\left(1+\varepsilon+\frac{1}{1+\varepsilon}\right)-\frac{x^2h_r^2}{8a}\right\}dx \leqq \varepsilon h_r e^{-1/y-x^2h_r^2/(8a)}dx. \tag{2.32}$$

Finally,

$$I_5 \leqq \frac{1}{2y^2}\int_{y(2+\varepsilon)}^{\infty} e^{-s/(2y^2)}P(L^*\in dx)\,ds \leqq \frac{Ch_r}{y^2}e^{-(2+\varepsilon)/(2y)-x^2h_r^2/(8a)}dx \leqq \tag{2.33}$$

$$\leqq \varepsilon h_r e^{-(1/y)-x^2h_r^2/(8a)}dx.$$

By Lemma 2.1, for $\eta=1/(2y^2)$,

$$E\left(e^{-\eta T_r/r^2},\ \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right) \geqq \tag{2.34}$$

$$\geqq (C-\varepsilon)h_r e^{-\sqrt{2\eta}-x^2h_r^2/(8a)}dx = (C-\varepsilon)h_r e^{-1/y-x^2h_r^2/(8a)}dx$$

with an absolute constant C. Hence, from (2.28), (2.29), (2.30), (2.32), (2.33) and (2.34),

$$I_3 \geqq (C-5\varepsilon)h_r e^{-1/y-x^2h_r^2/(8a)}\,dx \tag{2.35}$$

and taking (2.31) into account,

$$P\left(\frac{T_r}{r^2} < y(1+\varepsilon),\, L^*\in dx\right) \geqq C(\varepsilon)yh_r e^{-(1+\varepsilon)/(2y)-x^2h_r^2/(8a)}\,dx \tag{2.36}$$

and (2.27) follows by replacing $y(1+\varepsilon)$ by y.

LEMMA 2.6. *For* $h_r=K_\alpha\sqrt{a}\left(\frac{1-\varepsilon}{1+\varepsilon}\log\log r\right)^{(3-2\alpha)/4}$, $\alpha<1/2$ *and* $\varepsilon>0$ *we have*

$$P(A_r) \geqq C(\varepsilon)(\log r)^{-1+\varepsilon} \tag{2.37}$$

if r is big enough, where A_r is defined by (2.17).

PROOF. Let

$$x_r^* = \left(\frac{8a}{h_r^2(1-2\alpha)}\right)^{(1-2\alpha)/(6-4\alpha)}. \tag{2.38}$$

Then by Lemma 2.5

$$P(A_r) = \int_0^\infty P\left(\frac{T_r}{r^2} < x^{4/(1-2\alpha)},\, \frac{L(a,T_r)-r}{h_r\sqrt{r}}\in dx\right) \geqq$$

$$\geqq Cx_r^{*4/(1-2\alpha)}h_r \int_{x_r^*}^{x_r^*\sqrt{1+\varepsilon}} e^{-(1+\varepsilon)/2x^{-4/(1-2\alpha)}-x^2h_r^2/(8a)}\,dx \geqq$$

$$\geqq C(\varepsilon)x_r^{*4/(1-2\alpha)+1}h_r e^{-(1+\varepsilon)/2x_r^{*-4/(1-2\alpha)}-x_r^{*2}(1+\varepsilon)h_r^2/(8a)} \geqq$$

$$\geqq C(\varepsilon)h_r^{-2/(3-2\alpha)}(\log r)^{-1+\varepsilon} \geqq C(\varepsilon)(\log r)^{-1+\varepsilon/2}$$

for r large enough. Since $\varepsilon>0$ is arbitrary small, we have (2.37).

Now we are ready to prove Theorem 2. First we show that for large enough r, we have

$$\frac{L(a,T_r)-r}{r^\alpha T_r^{(1-2\alpha)/4}} \leqq (1+\varepsilon)h(r) \quad \text{a.s.} \tag{2.39}$$

with $h(r)=\sqrt{a}K_\alpha((1+2\varepsilon)\log\log r)^{(3-2\alpha)/4}$.

Put $r_k=\exp\{k/\log k\}$. Then by Lemma 2.4 and Borel—Cantelli it follows that

$$L(a,T_{r_k})-r_k \leqq r_k^\alpha T_{r_k}^{(1-2\alpha)/4}h(r_k) \quad \text{for} \quad k>k_0.$$

Now assume that $r_k \leqq r < r_{k+1}$, $k > k_0$. Then

$$\frac{L(a, T_r)-r}{r^\alpha T_r^{(1-2\alpha)/4}} = \frac{L(a, T_{r_k})-r_k+L(a, T_r)-L(a, T_{r_k})-(r-r_k)}{r^\alpha T_r^{(1-2\alpha)/4}} \leqq$$

$$\leqq h(r_k)+\frac{\sup\limits_{r_k \leqq r < r_{k+1}} \big(L(a, T_r)-L(a, T_{r_k})-(r-r_k)\big)}{r_k^\alpha T_{r_k}^{(1-2\alpha)/4}}.$$

It is well-known that

$$T_{r_k} \geqq C_1 \frac{r_k^2}{\log\log r_k}, \quad k \geqq k_1 \tag{2.40}$$

with some constant C_1. Theorem 3.1.1 of Csörgő and Révész [5] implies that

$$\sup_{r_k \leqq r < r_{k+1}} \big(L(a, T_r)-L(a, T_{r_k})-(r-r_k)\big) \leqq \tag{2.41}$$

$$\leqq C_2 \left((r_{k+1}-r_k)\left(\log\frac{r_k}{r_{k+1}-r_k}+\log\log r_k\right)\right)^{1/2}, \quad k \geqq k_2$$

with some constant C_2. Hence for $k \geqq \max(k_0, k_1, k_2)$,

$$\frac{L(a, T_r)-r}{r^\alpha T_r^{(1-2\alpha)/4}} \leqq h(r_k)+C\left(\frac{r_{k+1}-r_k}{r_k}\right)^{1/2}\left(\log\frac{r_k}{r_{k+1}-r_k}+\log\log r_k\right)^{1/2}\times$$

$$\times(\log\log r_k)^{(1-2\alpha)/4} \leqq (1+\varepsilon)h(r_k) \leqq (1+\varepsilon)h(r),$$

showing (2.39).

One can see similarly that

$$\frac{r-L(a, T_r)}{r^\alpha T_r^{(1-2\alpha)/4}} \leqq (1+\varepsilon)h(r) \tag{2.42}$$

for r big enough, hence the upper part of the law of the iterated logarithm (1.9) follows.

To show the lower part, let $\alpha < 1/2$, $r_k = e^{\varrho k \log k}$ with some $\varrho > 0$ and put $h(r) = K_\alpha \sqrt{a}\left(\frac{1-\varepsilon}{1+\varepsilon}\log\log r\right)^{(3-2\alpha)/4}$. Then Lemma 2.6 and Borel—Cantelli imply that

(2.43)

$$L(a, T_{r_{k+1}})-L(a, T_{r_k})-(r_{k+1}-r_k) \geqq h(r_{k+1}-r_k)(r_{k+1}-r_k)^\alpha(T_{r_{k+1}}-T_{r_k})^{(1-2\alpha)/4}$$

holds for infinitely many k.

One can easily see that the following limit relations hold:

$$\lim_{k\to\infty} \frac{h(r_{k+1}-r_k)}{h(r_{k+1})} = 1, \tag{2.44}$$

$$\lim_{k\to\infty} \frac{r_k}{r_{k+1}} = 0, \tag{2.45}$$

$$\lim_{k\to\infty} \frac{T_{r_k}}{T_{r_{k+1}}} = 0 \quad \text{a.s. for suitably chosen } \varrho > 0. \tag{2.46}$$

(2.46) follows from

$$\frac{Cr^2}{\log\log r} < T_r < r^2(\log r)^{2+\varepsilon} \quad \text{a.s.} \tag{2.47}$$

if r is large enough and $\varepsilon>0$, which is easily seen from the exact distribution of $\frac{T_r}{r^2}$, and the Borel—Cantelli lemma. (2.43)—(2.46) imply

$$\frac{L(a, T_{r_{k+1}})-r_{k+1}}{r_{k+1}^{\alpha} T_{r_{k+1}}^{(1-2\alpha)/4}} \geqq (1-\varepsilon)h(r_{k+1}) + \frac{L(a, T_{r_k})-r_k}{r_{k+1}^{\alpha} T_{r_{k+1}}^{(1-2\alpha)/4}}. \tag{2.48}$$

But (2.47) and the law of the iterated logarithm for $L(a, T_r)-r$ imply that

$$\left|\frac{L(a, T_{r_k})-r_k}{r_{k+1}^{\alpha} T_{r_{k+1}}^{(1-2\alpha)/4}}\right| \leqq C\sqrt{\frac{r_k}{r_{k+1}}}\sqrt{\log\log r_k}(\log\log r_{k+1})^{(1-2\alpha)/4} \leqq \varepsilon h(r_{k+1})$$

for k large, hence

$$\frac{L(a, T_{r_{k+1}})-r_{k+1}}{r_{k+1}^{\alpha} T_{r_{k+1}}^{(1-2\alpha)/4}} \geqq (1-2\varepsilon)h(r_{k+1})$$

holds for infinitely many k.

The proof of Theorem 2 is complete in the case $\alpha<1/2$.

For $\alpha=1/2$, Theorem 2 follows from the usual law of the iterated logarithm, since $L(a,T_r)-r$ is a process with stationary independent increments.

References

[1] R. F. Bass and P. S. Griffin, The most visited site of Brownian motion and simple random walk, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **70** (1985), 417—436.

[2] A. N. Borodin, On the character of convergence to Brownian local time. II, *Probab. Th. Rel. Fields*, **72** (1986), 251—277.

[3] A. N. Borodin, *On the distribution of functionals of Brownian local time*, LOMI Preprints E—4—85 (Leningrad, 1985).

[4] M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistics*, Academic Press (New York, 1981).

[5] M. Csörgő and P. Révész, On the stability of the local time of a symmetric random walk, *Acta Sci. Math. (Szeged)*, **48** (1985), 85—96.
[6] R. L. Dobrushin, Two limit theorems for the simplest random walk on a line, *Uspehi Mat. Nauk. (N. S.)*, **10** (1955), 139—146 (in Russian).
[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms, Vol. I*, McGraw-Hill (New York, 1954).
[8] Y. Kasahara, Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions, *J. Math. Kyoto Univ.*, **24** (1984), 521—538.
[9] G. C. Papanicolaou, D. W. Stroock, S. R. S. Varadhan, Martingale approach to some limit theorems, *Duke Univ. Maths. Series III. Statistical Mechanics and Dynamical System* (1977).
[10] A. V. Skorohod and N. P. Slobodenyuk, *Limit theorems for random walks*, Naukova Dumka (Kiev, 1970).
[11] M. Yor, Le drap Brownien comme limite en loi de temps locaux linéaires, *Seminaire de Probabilités XVII. 1981/82. Lecture Notes in Mathematics, No. 986*, 89—105, Springer (New York, 1983).

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[2] A. Zygmund, Smooth functions, *Duke Math. J.,* **12** (1945), 47—76.

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ON SUMMABILITY OF THE CONJUGATE SERIES OF FOURIER SERIES AT A POINT

Y. OKUYAMA (Nagano)

The purpose of this paper is to give a general theorem on a triangular matrix summability of the conjugate series of Fourier series which implies Hirokawa and Kayashima's theorem [5] and deduce various results from the theorem.

1. Let $C=(c_{n,k})$, $k=0, 1, \ldots, n$, be a triangular matrix and let

$$t_n = \sum_{k=0}^{n} c_{n,k} s_k,$$

where $\{s_k\}$ is a given sequence of numbers. If t_n converges to a value s as $n\to\infty$, then the sequence $\{s_n\}$ is called summable (C) to s.

In this paper we assume that $c_{n,k}\geqq 0$ for $k=0, 1, \ldots, n$ and $\sum_{k=0}^{n} c_{n,k}=1$.

Let a function $c_n(t)$ be nondecreasing and nonnegative in $(0, n)$ such that $c_n(k)=c_{n,k}$ for $k=0, 1, 2, \ldots, n$, and put

$$C_n(t) = \int_0^t c_n(n-u)\,du \quad \text{for} \quad 0 \leqq t \leqq n.$$

Then we have

$$C_n(k) \sim \sum_{m=0}^{k} c_{n,n-m} \quad \text{for} \quad n = 1, 2, \ldots.$$

Let $f(t)$ be a periodic finite-valued function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \tag{1.1}$$

Then the conjugate series of the series (1.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \tag{1.2}$$

Throughout this paper, we write

$$\varphi(t) \equiv \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},\ \Phi(t) \equiv \int_0^t |\varphi(u)|\, du,$$

$$\psi(t) \equiv \frac{1}{2}\{f(x+t)-f(x-t)\},\ \Psi(t) \equiv \int_0^t |\psi(u)|\, du$$

and $\tau=[\pi/t]$, where $[x]$ is the integral part of x.

2. Rajagopal [9] previously proved the following nice theorem on the Nörlund summability of Fourier series.

THEOREM A. *Let a function $p(t)$ be monotone nonincreasing and positive for $t \geqq 0$. Let $p_n=p(n)$ and let*

$$P(t) \equiv \int_0^t p(u)\, du \to \infty \quad as \quad t \to \infty.$$

If, for some fixed δ, $0<\delta<1$,

$$\int_{\pi/n}^{\delta} \Phi(t) \left| \frac{d}{dt} \frac{P(\pi/t)}{t} \right| dt = o(P_n), \quad \text{as} \quad n \to \infty,$$

then the series (1.1) *at $t=x$ is summable (N, p_n) to $f(x)$.*

Recently Dikshit [1] proved the following theorem on the summability of Fourier series by the regular linear method of summation determined by a triangular matrix.

THEOREM B. *Let $\{c_{n,k}\}$ be nondecreasing with respect to k. Let $\chi(t)$ be a positive function defined over $(0, \infty)$ such that as $n \to \infty$,*

(i) $n\chi(n) = O(1)$ *and*

(ii) $\int_1^n \chi(u)\, C_n(u)\, du = O(1)$.

Then if $\Phi(t)=o(\chi(\pi/t))$ as $t \to 0+$, the series (1.1) *at $t=x$ is summable (C) to $f(x)$.*

Furthermore, Okuyama [8] has generalized these Theorems A and B in the following form.

THEOREM C. *Let $\{c_{n,k}\}$ be nondecreasing with respect to k. If, for some δ, $0<\delta<1$, the condition*

$$\int_{\pi/n}^{\delta} \Phi(t) \left| \frac{d}{dt} \frac{C_n(\pi/t)}{t} \right| dt = o(1) \quad as \quad n \to \infty$$

holds, then the series (1.1) *at $t=x$ is summable (C) to $f(x)$.*

3. Now the conjugate-analogue of Theorem A was obtained by Hirokawa and Kayashima [5] as follows.

THEOREM D. *Let $\{p_n\}$ and $P(t)$ be defined as in Theorem* A. *If, for some δ, $0<\delta<1$,*

$$\int_{\pi/n}^{\delta} \Psi(t)\left|\frac{d}{dt}\frac{P(\pi/t)}{t}\right| dt = o(P_n) \quad as \quad n\to\infty, \tag{3.1}$$

then the series (1.2) *at $t=x$ is summable (N, p_n) to*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_{0}^{\pi} \psi(t)\cot\frac{t}{2}\,dt \tag{3.2}$$

provided that the integral exists as a Cauchy integral at the origin.

Here we generalize Theorem D in the following form.

THEOREM. *Let $\{c_{n,k}\}$ be nondecreasing with respect to k. If, for some δ, $0<\delta<1$, the condition*

$$\int_{\pi/n}^{\delta} \Psi(t)\left|\frac{d}{dt}\frac{C_n(\pi/t)}{t}\right| dt = o(1) \quad as \quad n\to\infty \tag{3.3}$$

holds, then the series (1.2) *at $t=x$ is summable* (*C*) *to $\tilde{f}(x)$ provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

This theorem is the conjugate-analogue of Theorem C.

If we put $c_n(u)=p_{n-k}/P_n$ for $k\leqq u<k+1$, then our Theorem reduces to Theorem D.

COROLLARY 1. *Let $\{c_{n,k}\}$ be nondecreasing with respect to k. Let $\chi(t)$ be a positive function defined over $(0,\infty)$ such that as $n\to\infty$,*

(i) $n\chi(n) = O(1)$ *and*

(ii) $\int_{1}^{n} \chi(u)C_n(u)\,du = O(1)$.

Then if $\Psi(t)=o(\chi(\pi/t))$ as $t\to 0+$, the series (1.2) *at $t=x$ is summable* (C) *to $\tilde{f}(x)$ provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

PROOF. For a positive number δ such that $\Psi(t)=o(\chi(\pi/t))$ for $t\in(0,\delta)$, we have by (i) and (ii)

$$\int_{\pi/n}^{\delta} \Psi(t)\left|\frac{d}{dt}\frac{C_n(\pi/t)}{t}\right| dt = \left|\int_{\pi/n}^{\delta}\Psi(t)\left\{-\frac{C_n(\pi/t)}{t^2}\right\}dt-\right.$$

$$\left.-\pi\int_{\pi/n}^{\delta}\Psi(t)\frac{c_n(n-\pi/t)}{t^2}\frac{1}{t}\,dt\right| = O\left(\int_{\pi/n}^{\delta}\Psi(t)\frac{C_n(\pi/t)}{t^2}\,dt\right) =$$

$$= o\left(\int_{\pi/n}^{\delta}\chi(\pi/t)\frac{C_n(\pi/t)}{t^2}\,dt\right) = o\left(\int_{\pi/\delta}^{n}\chi(t)C_n(t)\,dt\right) = o(1)$$

by virtue of the fact that $c_n(n-\pi/t)/t=O(C_n(\pi/t))$. Therefore, by our Theorem, this completes the proof of Corollary 1.

Corollary 1 is the conjugate-analogue of Theorem B.

Now, since $\{c_{n,k}\}$ is nonnegative and nondecreasing in k, we have

$$(n-k)c_{n,k} \leqq \sum_{m=k+1}^{n} c_{n,m} \leqq 1.$$

Thus for each fixed k, $c_{n,k}\to 0$ as $n\to\infty$. Hence we remark that the triangular matrix $C=(c_{n,k})$ is a regular method.

For the proof of our Theorem, we require the following two lemmas.

LEMMA 1 [7]. *If $\{c_{n,k}\}$ is nonnegative and nondecreasing with respect to k, then for $0\leqq a<b<\infty$, $0<t\leqq\pi$ and any positive integer n,*

$$\Big|\sum_{k=a}^{b} c_{n,n-k}e^{i(n-k)t}\Big| \leqq AC_n(\pi/t).$$

LEMMA 2. *Condition* (3.3) *implies $\Psi(t)=o(t)$.*

This lemma is similarly proved by the same method as that used in the proof of Rajagopal's lemma (see [9] or [8]).

4. PROOF OF THEOREM. Let us write

$$\tilde{s}_n(x)=\sum_{k=1}^{n} B_k(x) \quad\text{and}\quad \tilde{t}_n(x)=\sum_{k=0}^{n} c_{n,k}\tilde{s}_k(x).$$

By Lemma 2, we can choose a positive number δ such that $\Psi(t)=o(t)$ for $0\leqq t\leqq$ $\leqq\delta$. Then

$$\tilde{s}_k(x)-\tilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(k+1/2)t}{\sin t/2}dt=\frac{1}{\pi}\int_0^{\delta}\psi(t)\frac{\cos(k+1/2)t}{\sin t/2}dt+\eta_k$$

where, by the Riemann—Lebesgue theorem,

$$\eta_k=\frac{1}{\pi}\int_{\delta}^{\pi}\psi(t)\frac{\cos(k+1/2)t}{\sin t/2}dt\to 0 \quad\text{as}\quad k\to\infty. \tag{4.1}$$

Thus we obtain

$$\tilde{t}_n(x)-\tilde{f}(x)=\sum_{k=0}^{n} c_{n,k}\{\tilde{s}_k(x)-\tilde{f}(x)\}=$$

$$=\frac{1}{\pi}\int_0^{\delta}\psi(t)\sum_{k=0}^{n}\frac{c_{n,k}\cos(k+1/2)t}{\sin t/2}dt+\xi_n=\frac{1}{\pi}\int_0^{\delta}\psi(t)\frac{K(n,t)}{\sin t/2}dt+\xi_n$$

where

$$K(n,t)=\sum_{k=0}^{n} c_{n,k}\cos(k+1/2)t.$$

By (4.1), together with the regularity of the method of summation (C), we obtain

$$\xi_n = \sum_{k=0}^{n} c_{n,k}\eta_k \to 0 \quad \text{as} \quad n \to \infty.$$

Hence we have to show that

$$I = \frac{1}{\pi} \int_0^{\delta} \psi(t) \frac{K(n, t)}{\sin t/2} dt \to 0 \quad \text{as} \quad n \to \infty.$$

Since the integral (3.2) exists, we can obtain a positive integer $n_0 > \pi/\delta$ such that

$$\frac{1}{\pi} \int_0^{\pi/n} \psi(t) \cot t/2 \, dt = o(1) \quad \text{for} \quad n \geqq n_0.$$

Thus, for $n \geqq n_0$, we write

$$I = \frac{1}{\pi} \Big(\int_0^{\pi/n} + \int_{\pi/n}^{\delta} \Big) \psi(t) \frac{K(n, t)}{\sin t/2} dt = I_1 + I_2,$$

say. Then we have

$$\text{(4.2)} \qquad I_1 = \frac{1}{\pi} \int_0^{\pi/n} \psi(t) \frac{K(n, t)}{\sin t/2} dt = \frac{1}{\pi} \int_0^{\pi/n} \psi(t) \frac{K(n, t) - C_n(n) \cos t/2}{\sin t/2} dt + o(1) =$$

$$= \frac{1}{\pi} \int_0^{\pi/n} \psi(t) \left(\sum_{k=0}^{n} c_{n,k} \frac{\cos (k+1/2)\, t - \cos t/2}{\sin t/2} \right) dt + o(1)$$

for $n \geqq n_0$. Since $|\sin kt| \leqq k |\sin t|$ when k is a positive integer, we have

$$\text{(4.3)} \qquad \sum_{k=0}^{n} c_{n,k} \frac{\cos (k+1/2)\, t - \cos t/2}{\sin t/2} = -2 \sum_{k=0}^{n} c_{n,k} \frac{\sin (k+1)\, t/2 \sin kt/2}{\sin t/2} =$$

$$= O\Big(\sum_{k=0}^{n} k c_{n,k} (k+1)\, t \Big) = O\Big(n^2 t \sum_{k=0}^{n} c_{n,k} \Big) = O(n^2 t).$$

Hence, from Lemma 2, we get by (4.2) and (4.3)

$$I_1 = O\left(\frac{1}{\pi} \int_0^{\pi/n} |\psi(t)|\, n^2 t \, dt \right) + o(1) = O\Big(n \int_0^{\pi/n} |\psi(t)|\, dt \Big) + o(1) = o(1) \quad \text{for} \quad n \geqq n_0.$$

Next, we obtain by Lemmas 1, 2 and (3.3)

$$I_2 \geqq A \int_{\pi/n}^{\delta} \frac{|\psi(t)|\, C_n(\pi/t)}{t} dt \leqq A \left| \Psi(\delta) \frac{C_n(\pi/\delta)}{\delta} - \Psi(\pi/n)\, C_n(n) \frac{n}{\pi} \right| +$$

$$+ A \int_{\pi/n}^{\delta} \Psi(t) \left| \frac{d}{dt} \frac{C_n(\pi/t)}{t} \right| dt = o\big(C_n(\pi/\delta) + C_n(n) \big) + o(1) = o(1) \quad \text{for} \quad n \geqq n_0.$$

This completes the proof of Theorem.

5. Finally we consider some applications of our theorem.

COROLLARY 2 (Hirokawa and Kayashima [6]). *If, for some fixed* δ, $0<\delta<1$,

$$\int_t^\delta \frac{|\psi(u)|}{u} \log\frac{1}{u}\,du = o(\log \pi/t) \quad as \quad t \to 0+,$$

then the series (1.2) *at* $t=x$ *is summable* $(N, 1/(n+1))$ *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

COROLLARY 3 (Saxena [10]). *Let* $\{p_n\}$ *be a sequence such that*

$$p_n > 0,\ p_n\downarrow,\ P_n \to \infty, \quad and \quad \log n = O(\beta(P_n))$$

where $\beta(t)$ *is a positive monotone nondecreasing function such that* $t/\beta(t)$ *is also monotone nondecreasing. If* $\Psi(t)=o(t/\beta(P_\tau))$ *as* $t\to 0+$, *then the series* (1.2) *at* $t=x$ *is summable* (N, p_n) *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

COROLLARY 4 (Saxena [10]). *Let* $\{p_n\}$ *be a sequence such that*

$$p_n > 0,\ p_n\downarrow,\ P_n \to \infty, \quad and \quad \log n = O(\gamma(P_n)),$$

where $\gamma(t)$ *is a positive function such that*

$$\int_{\pi/n}^{\delta} \frac{P_\tau}{\gamma(P_\tau)} \frac{1}{t}\,dt = O(P_n) \quad as \quad n \to \infty.$$

If $\Psi(t)=o(t/\gamma(P_\tau))$ *as* $t\to 0+$, *then the series* (1.2) *at* $t=x$ *is summable* (N, p_n) *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

COROLLARY 5 (Dikshit [3]). *Let* $\{p_n\}$ *be a sequence such that*

$$p_n > 0,\ p_n\downarrow,\ P_n \to \infty, \quad and \quad \alpha(n)\log n = O(P_n),$$

where $\alpha(t)$ *is a positive monotone nondecreasing function. If* $\Psi(t)=o(\alpha(\pi/t)\,t/P_\tau)$ *as* $t\to 0+$, *then the series* (1.2) *at* $t=x$ *is summable* (N, p_n) *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

This corollary coincides with Dikshit [2] for $\alpha(t)=1$.

COROLLARY 6 (Hirokawa [4]). *Let* $\{p_n\}$ *be a sequence such that*

$$p_n > 0,\ p_n\downarrow \quad and \quad P_n \to \infty.$$

Let $\lambda(t)$ *be a positive integrable function such that*

$$\int_\eta^n \frac{\lambda(u)}{u}\,du = O(P_n) \quad as \quad n \to \infty,$$

for any fixed $\eta>0$. *If* $\Psi(t)=o(t\lambda(\pi/t)/P_\tau)$ *as* $t\to 0+$, *then the series* (1.2) *at* $t=x$ *is summable* (N, p_n) *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

COROLLARY 7 (Hirokawa and Kayashima [6]). *Let* $\{p_n\}$ *be a sequence such that*

$$p_n > 0,\ p_n\downarrow \quad and \quad P_n \to \infty.$$

Let $\mu(t)$ *be a nonnegative function such that*

(i) $\{\mu(t)/(t\log t)\}$ *is monotonic,*

(ii) $\mu(t) = O(\log t),\ t \to \infty$ *and*

(iii) $\sum_{k=2}^{n} \mu(k)P_k/(k\log k) = O(P_n)$ *as* $n\to\infty$.

If $\Psi(t) = o\big(t\mu(\pi/t)/\log(\pi/t)\big)$ *as* $t\to 0+$, *then the series* (1.2) *at* $t=x$ *is summable* (N, p_n) *to* $\tilde{f}(x)$ *provided that the integral in* (3.2) *exists as a Cauchy integral at the origin.*

As these corollaries are analogously proved, we shall prove here only Corollary 7.

PROOF OF COROLLARY 7. In our theorem, we put $c_{n,k} = p_{n-k}/P_n$. Then for a positive number δ such that $\Psi(t) = o\big(t\mu(\pi/t)/\log(\pi/t)\big)$ for $t\in(0,\delta)$, we have

$$\int_{\pi/n}^{\delta} \Psi(t)\left|\frac{d}{dt}\frac{C_n(\pi/t)}{t}\right| dt = o\left(\frac{1}{P_n}\int_{\pi/n}^{\delta} \frac{t\mu(\pi/t)}{\log(\pi/t)}\frac{P(\pi/t)}{t^2}\,dt\right) =$$

$$= o\left(\frac{1}{P_n}\int_{\delta/\pi}^{n}\frac{\mu(u)P(u)}{u\log u}\,du\right) = o\left(\frac{1}{P_n}\sum_{k=2}^{n}\frac{\mu(k)P_k}{k\log k}\right) = o(1)$$

by virtue of the hypotheses of Corollary 8.

References

[1] G. D. Dikshit, On summability of Fourier series at a point, *Proc. Amer. Math. Soc.*, **95** (1984), 245—250.

[2] H. P. Dikshit, The Nörlund summability of the conjugate series of a Fourier series, *Rend. Circ. Mat. Palermo*, **11** (1962), 217—224.

[3] H. P. Dikshit, On the Nörlund summability of the conjugate series of a Fourier series, *Rend. Circ. Mat. Palermo*, **14** (1965), 165—170.

[4] H. Hirokawa, On the Nörlund summability of Fourier series and its conjugate series, *Proc. Japan Acad.*, **44** (1968), 449—451.

[5] H. Hirokawa and I. Kayashima, On the Nörlund summability of the conjugate series of Fourier series, *Proc. Japan Acad.*, **44** (1968), 301—306.

[6] H. Hirokawa and I. Kayashima, On a sequence of Fourier coefficients, *Proc. Japan Acad.*, **50** (1974), 57—62.

[7] L. McFadden, Absolute Nörlund summability, *Duke Math. J.*, **9** (1942), 168—207.

[8] Y. Okuyama, On summability of Fourier series at a point, preprint.

[9] C. T. Rajagopal, On the Nörlund summability of Fourier series, *Proc. Camb. Phil. Soc.*, **59** (1963), 47—53.
[10] A. Saxena, On Nörlund summability of the conjugate series of a Fourier series, *Rend. Circ. Mat. Palermo*, **13** (1964), 222—228.
[11] J. A. Siddiqi, *The Nörlund summability of trigonometrical series*, D. Phil. Thesis (1949).

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INCREASING PATHS LEADING TO A FACE OF A CONVEX COMPACT SET IN A HILBERT SPACE

LEONI DALLA (Athens)

1. Introduction

Let C be a convex compact set in a normed space E. A point c of C is an *extreme point* of C if it is not contained in the relative interior of a line segment lying in C. The *exposed points* are extreme points that can be expressed as the sole intersection of C with one of its support hyperplanes. The ***r*-skeleton *of C*** for r a non-negative integer will be defined to be the set of all points of C that do not belong to the relative interior of an $(r+1)$-dimensional convex subset of C. The set of extreme points of C coincides with the 0-skeleton of C. Let l be a continuous functional on E non-constant on C. In [1], D. G. Larman proved the existence of an *l-strictly increasing path* on the *one-skeleton* of C. In a recent paper [2] it is proved that if the face $F=\{x\in C\colon l(x)=\max_{y\in C} l(y)\}$ is of infinite dimension then for every $n=1, 2, \ldots$ there are n l-strictly increasing paths on the one-skeleton of C mutually disjoint that lead in F (Corollary 2.1 in [2]). In this paper it is proved that if the dimension of F is k then there are $k+1$ such paths with the above mentioned property for every $k=1, 2, \ldots$. Furthermore, we give an example showing that this result is the best possible in a Hilbert space.

2. The results

We quote the following propositions:

PROPOSITION 2.1. *Let C be a compact convex set in a normed space E and let l be a continuous linear functional on E, non-constant on C whose maximum on C is taken on a face F of C with $\dim F=k$ $(k\geqq 1)$. Then there are $k+1$ mutually disjoint paths on the one-skeleton of C, leading to F, along each of which the functional l strictly increases.*

PROPOSITION 2.2. *Let $\mathcal{H}$ be a Hilbert space of infinite dimension, l a non-constant continuous linear functional and k an arbitrary positive integer. Then there exists a convex compact set Σ, which is of infinite dimension and on which l is non-constant, such that the face $F=\{x\in\Sigma\colon l(x)=\max_{y\in\Sigma} l(y)\}$ is of dimension k and Σ has the property that on its one-skeleton, it is impossible to find $k+2$ l-strictly increasing paths, mutually disjoint that lead to F.*

PROOF OF PROPOSITION 2.1. We assume that $\dim F=k$. Then there exist $k+1$ $e_1, e_2, \ldots, e_{k+1}$ linearly independent vectors in E and $k+1$ linear functionals $l_1=l, l_2, \ldots, l_{k+1}$ for which the following hold:

(i) $l_i(e_j)=\delta_{ij},\ i,j=1,2,\ldots,k+1$ where δ_{ij} is the *Kronecker delta* and
(ii) $\dim \pi_0(F)=k$, where π_0 is the projection

$$\pi_0(x) = l_1(x)e_1+l_2(x)e_2+\ldots+l_{k+1}(x)e_{k+1}, \quad x\in E.$$

From now on, the steps required to find the appropriate $k+1$ paths on the one-skeleton of C are similar to those in the proof of Theorem 2.1 in [2], so we omit them. This concludes the proof of the proposition.

Before we proceed to the proof of Proposition (2.2) we introduce some appropriate notation: Let $\mathscr{H}_0$ be a separable Hilbert space of infinite dimension and let l be a continuous linear functional on $\mathscr{H}_0$ of norm 1. Then there exists a unit vector u_1 such that $l(x)=\langle x, u_1\rangle$ for every $x\in\mathscr{H}_0$. Let $\{u_n\}_{n=1}^{\infty}$ be a complete orthonormal system in $\mathscr{H}_0$. We denote by Q the convex compact set

$$\left\{x\in\mathscr{H}_0\colon\ x=\sum_{n=1}^{\infty} c_n u_n,\ 0\leqq c_n\leqq\frac{1}{n},\ n=1,2,\ldots\right\}.$$

Also, let $H^-(a)=\{x\in\mathscr{H}_0\colon\ l(x)\leqq a\}$ be the closed half space, where a is a real number and let $H(a)$ be the boundary of $H^-(a)$. Next we quote and prove the following lemma which is essential in the proof of Proposition 2.2.

LEMMA. *Let l be a non-constant continuous linear functional on a separable Hilbert space $\mathscr{H}_0$, $k\geqq 1$ be an integer and $\mu<(k+1)^{-1}$ a positive real number. Then there is a sequence of convex compact sets $(\Sigma_i)_{i=1}^{\infty}$ with $\Sigma_i\subseteq Q$ such that*

1) *If* $\alpha^{(i)}=\max\limits_{x\in\Sigma_i} l(x)$, *then the sequence* $\{\alpha^{(i)}\}_{i=1}^{\infty}$ *is strictly increasing.*

2) $\Sigma_{i+1}\cap H^-(\alpha^{(i)})=\Sigma_i,\quad i=1,2,\ldots.$

3) *The sets* $\Delta^{(0)}=\Sigma_1\cap H(0)=\overline{\mathrm{con}}\left\{\bigcup\limits_{n=1}^{\infty} x_n^{(0)}\right\}$ *and* $\Delta^{(i)}=\Sigma_i\cap H(\alpha^{(i)})=\overline{\mathrm{con}}\left\{\bigcup\limits_{n=1}^{\infty} x_n^{(i)}\right\}$,

$i=1,2,\ldots$ *are convex compact sets of co-dimension* 1.

4) $F^{(i)}=\mathrm{con}\,\{x_1^{(i)},\ldots,x_{k+1}^{(i)}\}$ *is a k-dimensional face of* $\Delta^{(i)}$ *with*

$$\min_{1\leqq j<n\leqq k+1}\|x_j^{(i)}-x_n^{(i)}\|>\mu\quad i=0,1,2,\ldots.$$

5) $\max\limits_{1\leqq j\leqq k+1}\|x_j^{(i)}-x_j^{(i+1)}\|<\mu^{k+i},\quad i=0,1,2,\ldots.$

6) *The point $x_j^{(i)}$ is joined to the point $x_j^{(i+1)}$ by a single edge of the convex compact set Σ_{i+1}, $1\leqq j\leqq k+1$, $i=0,1,2,\ldots$.*

7) $\lim\limits_{j\to\infty}\|x_j^{(i)}-x_{\lambda(i)}^{(i)}\|=0,\ \lambda(i)=\begin{cases}1 & \textit{for}\quad i=2n\quad \textit{where}\quad n=0,1,2,\ldots\\ 2 & \textit{for}\quad i=2n+1\end{cases}$

and

$$0<\|x_j^{(i)}-x_{\lambda(i)}^{(i)}\|<\mu^{k+i},\quad j=k+2,\ldots,\quad i=0,1,\ldots.$$

8) *If $k+2$ disjoint l-strictly increasing paths in the one-skeleton of Σ^{i+1} lead from $\Delta^{(i)}$ to $\Delta^{(i+1)}$ then one must contain a line segment of length exceeding $\mu-3\mu^{k+i}$, $i=0,1,2,\ldots$.*

PROOF. Let $x_1^{(0)}=0$, $x_2^{(0)}=u_2/2, \dots, x_{k+1}^{(0)}=u_{k+1}/(k+1)$ and $x_{k+j}^{(0)}=\mu^{k+j}u_{k+j}$, $j\geqq 2$. Then $\lim_{n\to\infty} x_n^{(0)}=x_1^{(0)}$, hence $\Delta^0=\overline{\mathrm{con}}\,(\bigcup_{n=1}^{\infty} x_n^{(0)})$ is a compact convex subset of Q with $\mathrm{ext}\,\Delta^{(0)}=\exp\Delta^{(0)}=\bigcup_{n=1}^{\infty} x_n^{(0)}$, where $\mathrm{ext}\,\Delta^{(0)}$ and $\exp\Delta^{(0)}$ are the set of extreme and exposed points of $\Delta^{(0)}$, respectively.

Conditions 4) and 7) are satisfied for the points $\{x_n^{(0)}\}_{n=1}^{\infty}$. Let $u_1\in Q$ and $P'=\overline{\mathrm{con}}\,\{\{\bigcup_{n=1}^{\infty} x_n^{(0)}\}\cup\{u_1\}\}=\mathrm{con}\,(\Delta^{(0)}\cup\{u_1\})$. We can choose $0<\alpha^{(1)}<1$ sufficiently small, so that with $x_j^{(1)}=\mathrm{con}\,(x_j^{(0)}, u_1)\cap H(\alpha^{(1)})$, the inequalities of conditions 4) and 5) are satisfied for $1\leqq j\leqq k+1$. Let $y_j^{(1)}=\mathrm{con}\,(x_j^{(0)}, u_1)\cap H(\alpha^{(1)}), j\geqq k+2$. Then

$$T = P'\cap H(\alpha^{(1)}) = \overline{\mathrm{con}}\,(\bigcup_{j=1}^{k+1}\{x_j^{(1)}\}\cup\bigcup_{j=k+2}^{\infty}\{y_j^{(1)}\})$$

is a compact convex subset of Q.

We choose $x_j^{(1)}\in\mathrm{con}\,(y_j^{(1)}, x_2^{(1)}), j\geqq k+2$ such that $0<\|x_j^{(1)}-x_2^{(1)}\|<\mu^{k+1}$ and $\lim_{j\to\infty} x_j^{(1)}=x_2^{(1)}$ and so that $\Delta^{(1)}=\overline{\mathrm{con}}\,(\bigcup_{j=1}^{\infty} x_j^{(1)})$ is a convex compact set with co-dimension 1. Then we define

$$\Sigma_1=\overline{\mathrm{con}}\,(\Delta^{(0)}\cup\Delta^{(1)})=\overline{\mathrm{con}}\,(\bigcup_{n=1}^{\infty}\{x_n^{(0)}\}\cup\bigcup_{n=1}^{\infty}\{x_n^{(1)}\}).$$

Then Σ_1 is a compact convex subset of Q, which satisfies conditions 3) to 8).

Assume now that we have constructed a finite sequence of m $(m\geqq 1)$ compact convex subsets of Q, having properties 1)—8). It will be shown that a compact convex set Σ_{m+1} can be constructed so that the enlarged sequence also satisfies the required conditions. Let $d\in Q$ be a point with $\alpha^{(m)}<l(d)<1$ and $\overline{\mathrm{con}}\,(d\cup\Sigma_m)=$ $=\Sigma_m\cup\mathrm{con}\,(d\cup\Delta^{(m)})\subseteqq Q$. Let $\alpha^{(m+1)}$ be chosen greater than $\alpha^{(m)}$, so that with $x_j^{(m+1)}$ defined for, $1\leqq j\leqq k+1$ by $x_j^{(m+1)}=\mathrm{con}\,(x_j^{(m)}, d)\cap H(\alpha^{(m+1)})$ the inequalities of Conditions 4) and 5) are satisfied. Let now

$$T=\overline{\mathrm{con}}\,(\Sigma_m\cup d)\cap H(\alpha^{(m+1)})=\overline{\mathrm{con}}\,(\bigcup_{j=1}^{k+1} x_j^{(m+1)}\cup\bigcup_{j=k+2}^{\infty} y_j^{(m+1)})$$

where $y_j^{(m+1)}$, $j\geqq k+2$ is joined to $x_j^{(m)}$ by a single edge of the compact convex set $\overline{\mathrm{con}}\,(\Sigma_m\cup d)\cap H^-(\alpha^{(m+1)})$. Let $x_j^{(m+1)}$, $j\geqq k+2$ be a point of the edge $\mathrm{con}\,(y_j^{(m+1)}, x_\lambda^{(m+1)})$ ($\lambda=1$ or 2 iff $m+1$ is even or odd) of T chosen so that

$$0<\|x_j^{(m+1)}-x_\lambda^{(m+1)}\|<\mu^{k+(m+1)},\ \lim_{j\to\infty} x_j^{(m+1)}=x_\lambda^{(m+1)} \text{ and } \Delta^{(m+1)}=\overline{\mathrm{con}}\,(\bigcup_{n=1}^{\infty} x_n^{(m+1)})$$

is a compact convex set of co-dimension 1. Next, we define

$$\Sigma_{m+1}=\overline{\mathrm{con}}\,(\Sigma_m\cup\Delta^{(m+1)})=\overline{\mathrm{con}}\,(\bigcup_{i=0}^{m+1}\bigcup_{n=1}^{\infty} x_n^{(i)}).$$

Then $\mathrm{ext}\,\Sigma_{m+1}=\exp\Sigma_{m+1}=\bigcup_{i=0}^{m+1}\bigcup_{n=1}^{\infty} x_n^{(i)}$. We observe that conditions 1)—7) apply to the sequence of convex compact sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_{m+1}$ by construction. But condition 8) requires proof.

Let $S_{m+1}=\bigcup_{i=0}^{m+1}\bigcup_{j=1}^{k+1} x_j^{(i)}$ be the vertices of Σ_{m+1} which satisfied conditions 4) and 5). Suppose that $P_1, P_2, \ldots, P_{k+2}$ are $k+2$ disjoint l-strictly-increasing paths in the one-skeleton of Σ_{m+1}, joining $\Delta^{(m)}$ to $\Delta^{(m+1)}$. Suppose that none of these paths contain a line segment of length exceeding $\mu-3\mu^{k+m}$. Let $x_j^{(m)}, x_i^{(m+1)}\in \operatorname{ext}\Sigma_{m+1}-S_{m+1}$. Then using conditions 7), 5) and 4) we can see that $\|x_j^{(m)}-x_i^{(m+1)}\|>\mu-3\mu^{k+m}$ and

$$\|x_j^{(m)}-x_{\lambda(m)}^{(m+1)}\| < \mu-3\mu^{k+m}, \; \|x_i^{(m+1)}-x_{\lambda(m+1)}^{(m)}\| < \mu-3\mu^{k+m}.$$

Hence $x_j^{(m)}$ can only be joined to $x_{\lambda(m)}^{(m+1)}$ and the vertex $x_i^{(m+1)}$ with $x_{\lambda(m+1)}^{(m)}$. There remain k disjoint paths between $\Delta^{(m)}$ and $\Delta^{(m+1)}$ which do not pass through $x_{\lambda(m+1)}^{(m)}$ and $x_{\lambda(m)}^{(m+1)}$. If one of these paths joins two vertices $x_j^{(m)}$, $x_i^{(m+1)}$, $i\neq j$ with $x_j^{(m)}, x_i^{(m+1)}\in S_{m+1}$ then we can show that

$$\|x_j^{(m)}-x_i^{(m+1)}\| > \mu-3\mu^{k+m}.$$

If one of these paths join two vertices $x_j^{(m)}\notin S_{m+1}, x_i^{(m+1)}\in S_{m+1}, x_i^{(m+1)}\neq x_{\lambda(m)}^{(m+1)}$ then again $\|x_j^{(m)}-x_i^{(m+1)}\|>\mu-3\mu^{k+m}$. Similarly, if one of these paths join two vertices then $x_j^{(m+1)}\notin S_{m+1}, x_i^{(m)}\in S_{m+1}, x_i^{(m)}\neq x_{\lambda(m+1)}^{(m)}$. This contradiction establishes Condition 8) for the sequence of convex compact sets $\Sigma_1, \ldots, \Sigma_{m+1}$.

PROOF OF PROPOSITION 2.2. Let l be a non-constant continuous linear functional on $\mathscr{H}$. We may assume without loss of generality that l is of unit norm. Then $l(x)=\langle x, u_1\rangle$ for some unit vector u_1 in $\mathscr{H}$. We select a closed separable subspace $\mathscr{H}_0$ of infinite dimension such that $u_1\in\mathscr{H}_0$. Then l is non-constant on $\mathscr{H}_0$. Define $\Sigma=\overline{\operatorname{con}}\,(\bigcup_{n=1}\Sigma_n)\subseteqq Q$ where Σ_n, $n=1, 2, \ldots$ and Q are as in the previous lemma. Then Σ is compact convex and the functional l assumes its maximum value on Σ over the whole of a face F_1 with $\dim F_1=k$ by the construction of Σ_n, $n=1, 2, \ldots$ and Σ. It is impossible to find $k+2$ paths in the one-skeleton of Σ which lead to F_1, yet which are disjoint outside F_1. If such paths did exist then, by condition 8) of lemma, one of the paths would contain a sequence of disjoint line-segments of length exceeding $\frac{\mu}{2}$ (taking the limit). This is impossible for a path P, since P is the continuous image of $[0, 1]$ on the one-skeleton of Σ. This completes the proof of Proposition 2.2.

References

[1] D. G. Larman, On the one-skeleton of a compact convex set in a Banach space, *Proc. London Math. Soc.*, **34** (1977), 117—144.

[2] L. Dalla, Increasing paths on the one-skeleton of a convex compact set in a normed space, *Pacific J. Math.* **124** (1986), 289—294.

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ON THE DEGREE OF APPROXIMATION OF A CLASS OF FUNCTIONS BY MEANS OF FOURIER SERIES

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1. Definitions and notations. Let f be 2π-periodic and L-integrable on $[-\pi, \pi]$. The Fourier series associated with f at the point x, is given by

$$\frac{1}{2}a_0+\sum_{n=1}^{\alpha}(a_n\cos nx+b_n\sin nx). \tag{1.1}$$

A function $f\in \mathrm{Lip}\,\alpha$ $(\alpha>0)$ if

$$f(x+h)-f(x) = O(|h|^{\alpha}) \quad (h\to 0) \tag{1.2}$$

and if f is defined on $[-\pi, \pi]$ then the expression

$$\omega(\delta)=\omega(\delta, f)=\sup_{x_1,x_2}|f(x_1)-f(x_2)|, \quad |x_1-x_2|\leqq\delta \tag{1.3}$$

is called the modulus of continuity of f (Zygmund [5], p. 42).

Let $A=(a_{n,k})$ $(k, n=0, 1, \ldots)$ be a lower-triangular infinite matrix of real numbers. We denote by $T_n(f)$ the A-transform of the Fourier series of f given by

$$T_n(f;\,x)=\sum_{k=0}^{n}a_{nk}s_k(x) \quad (n=0, 1, \ldots), \tag{1.4}$$

where $s_n(x)$ is the n-th partial sum of the series (1.1).

Suppose $A=(a_{nk})$ is defined as follows:

$$a_{nk}=\begin{cases}p_k/P_n; & 0\leqq k\leqq n\\ 0; & k>n,\end{cases} \tag{1.5}$$

where (p_k) is non-negative and that $P_n=p_0+p_1+\ldots+p_n\neq 0$ $(n\geqq 0)$. Then the matrix is called Riesz matrix and the means are called Riesz-means or (R, p_n)-means. In this case we write $R_n(f; x)$ for $T_n(f; x)$. Also if

$$a_{nk}=\begin{cases}p_{n-k}/P_n; & 0\leqq k\leqq n\\ 0; & k>n,\end{cases} \tag{1.6}$$

The matrix (a_{nk}) is called Nörlund matrix and in this case we write $N_n(f; x)$ for $T_n(f; x)$. Throughout (a_{nk}) will denote a lower triangular infinite matrix.

We use the following notations in this paper:

(1.7) $$\Phi_x(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},$$

(1.8) $$b_{nk} = \sum_{r=0}^{k} a_{nr}; \quad b_{nk} = b_n(k),$$

(1.9) $$\tau = [\pi/t], \text{ the integral part of } k/t \text{ in } 0 < t \leqq \pi,$$

(1.10) $C^*[0, 2\pi]$, the space of all 2π-periodic continuous functions defined on $[0, 2\pi]$.

Throughout, the norm $\|\cdot\|$ will be the sup norm on $0 \leqq x \leqq 2\pi$ and $\omega(t)$ will be the modulus of continuity of $f \in C^*[0, 2\pi]$.

2. Introduction. By employing Riesz matrix, we [1] obtained the following result concerning the degree of approximation:

THEOREM A. *Let $f \in C^*[0, 2\pi]$ and let $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$. Then the degree of approximation of f by (R, p_n)-means of its Fourier series is given by*

$$\|R_n(f)-f\| = \begin{cases} O\{(p_n/P_n)^\alpha\}; & 0<\alpha<1 \\ O\{(p_n/P_n)\log(P_n/p_n)\}; & \alpha = 1, \end{cases}$$

where (p_n) is positive and non-decreasing with $n \geqq n_0$.

Recently this result was extended to the lower triangular matrix in the Hölder metric (see [4]).

In this paper we first extend Theorem A by using the modulus of continuity of *f in the following form:*

THEOREM 1. *Let (a_{nk}) satisfy the following conditions:*

(2.1) $$a_{nk} \geqq 0 \quad (n, k = 0, 1, \ldots), \quad \sum_{k=0}^{n} a_{nk} = 1,$$

(2.2) $$a_{nk} \leqq a_{n,k+1} \quad (k = 0, 1, \ldots, n-1, n = 0, 1, \ldots).$$

Suppose $\omega(t)$ is such that

(2.3) $$\int_u^\pi t^{-2}\omega(t)\,dt = O(H(u)) \quad (u \to 0+),$$

where $H \geqq 0$ and that

(2.4) $$tH(t) = o(1) \quad (t \to 0+)$$

and

(2.5) $$\int_0^t H(u)\,du = O\{tH(t)\} \quad (t \to 0+).$$

Then

$$\|T_n(f)-f\| = O\{a_{nn}H(a_{nn})\}. \tag{2.6}$$

We also prove

THEOREM 2. *Let* (a_{nk}) *satisfy* (2.1) *and* (2.2) *and let* $\omega(t)$ *satisfy* (2.3). *Then*

$$\|T_n(f)-f\| = O\{\omega(\pi/n)\}+O\{a_{nn}H(\pi/n)\}, \tag{2.7}$$

where H is non-negative. If, in addition to (2.3), $\omega(t)$ *satisfies* (2.5) *then*

$$\|T_n(f)-f\| = O\{a_{nn}H(\pi/n)\}. \tag{2.8}$$

Lastly, we intend to investigate some results, one of which is analogous to Theorem A in the case when (p_n) is non-negative and non-increasing. In fact, we first obtain general results for a triangular matrix by using the modulus of continuity of f from which the desired results may be obtained. Precisely, we prove the following:

THEOREM 3. *Let* (a_{nk}) *satisfy* (2.1) *and let*

$$a_{nk} \geqq a_{n,k+1} \quad (k=0, 1, \ldots, n-1, n=0, 1, \ldots). \tag{2.9}$$

Then

$$\|T_n(f)-f\| = O\left\{\omega(\pi/n)+\sum_{k=1}^{n} k^{-1}\omega(\pi/k)\, b_n(k+1)\right\}. \tag{2.10}$$

THEOREM 4. *Let* (a_{nk}) *satisfy* (2.1) *and* (2.9) *and let* $\omega(t)$ *satisfy* (2.3), (2.4) *and* (2.5). *Then*

$$\|T_n(f)-f\| = O\{a_{n0}H(a_{n0})\}. \tag{2.11}$$

3. We shall use the following lemmas in the proof of the theorems:

LEMMA 1. *Let* $\omega(t)$ *satisfy* (2.3), (2.4) *and* (2.5). *Then*

$$\int_0^r t^{-1}\omega(t)\,dt = O\{rH(r)\} \quad (r\to 0+).$$

PROOF. Integrating by parts, we have

$$\int_0^r t^{-1}\omega(t)\,dt = \left[-t\int_t^\pi u^{-2}\omega(u)\,du\right]_0^r + \int_0^r dt\int_t^\pi u^{-2}\omega(u)\,du =$$

$$= O\{rH(r)\}+O(1)\int_0^r H(t)\,dt = O\{rH(r)\},$$

by (2.3), (2.4) and (2.5).

This completes the proof of the lemma.

LEMMA 2. *Let* (a_{nk}) *satisfy* (2.9) *and let* $a_{nk}\geqq 0$ $(n, k=0, 1, \ldots)$. *Then, uniformly in* $0<t\leqq\pi$,

$$\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t = O\{b_n(\tau)\}.$$

PROOF. Since $a_{nk} \geqq 0$, we have by Abel's lemma

$$\left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| \leqq \sum_{k=0}^{\tau} a_{nk} + \left|\sum_{\tau}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| =$$

$$= b_n(\tau) + O\{\tau a_{n\tau}\} = O\{b_n(\tau)\},$$

by (2.9). This completes the proof of the lemma.

4. In this section, we shall prove the theorems mentioned in Section 2.

PROOF OF THEOREM 1. We have

$$T_n(f;\ x) - f(x) = \sum_{k=0}^{n} a_{nk} s_k(x) - f(x) =$$

$$= \frac{2}{\pi} \int_0^{\pi} \left\{\{\Phi_x(t)/\left(2 \sin \frac{1}{2} t\right)\}\right\} \left(\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right) dt,$$

by (2.1). Now we observe that $\|\Phi(t)\| \leqq \omega(t)$, therefore

$$\|T_n(f-f)\| \leqq \frac{2}{\pi} \int_0^{\pi} \frac{\omega(t)}{2 \sin \frac{1}{2} t} \left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| dt = \tag{4.1}$$

$$= \frac{2}{\pi} \left(\int_0^{a_{nn}} + \int_{a_{nn}}^{\pi}\right) = I_1 + I_2, \quad \text{say.}$$

However, by (2.1), the sum in the integral does not exceed 1 and hence

$$I_1 = O(1) \int_0^{a_{nn}} t^{-1} \omega(t)\, dt = O\{a_{nn} H(a_{nn})\},$$

by Lemma 1. Also, by (2.2) and Abel's lemma

$$I_2 = O(a_{nn}) \int_{a_{nn}}^{\pi} t^{-2} \omega(t) dt = O\{a_{nn} H(a_{nn})\},$$

by (2.3).

Combining I_1 and I_2, we get (2.6) and this completes the proof of the theorem.

PROOF OF THEOREM 2. We have from (4.1)

$$\|T_n(f) - f\| \leqq \frac{2}{\pi} \int_0^{\pi} \frac{\omega(t)}{2 \sin \frac{1}{2} t} \left|\sum_{k=0}^{n} a_{nk} \sin\left(k+\frac{1}{2}\right)t\right| dt =$$

$$= \frac{2}{\pi} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) = I_1 + I_2, \quad \text{say.}$$

Using the inequality $\left|\sin\left(k+\frac{1}{2}\right)t\right| \leqq \left(k+\frac{1}{2}\right)t$ and (2.1), we get

$$I_1 = O(n)\int_0^{\pi/n} \omega(t)\,dt = O\{\omega(\pi/n)\}.$$

Also, by (2.2) and Abel's lemma

$$I_2 = O\{a_{nn}H(\pi/n)\}.$$

Combining I_1 and I_2, we get (2.7).

For the estimate (2.8), we first observe that

$$I_1 = O(n)\int_0^{\pi/n} \omega(t)\,dt.$$

Now integrating by parts and using (2.3), (2.5) we get

$$\int_0^{\pi/n} \omega(t)\,dt = \left[-t^2\int_t^{\pi} (\omega(u)/u^2)\,du\right]_0^{\pi/n} + \int_0^{\pi/n} 2t\,dt \int_t^{\pi} u^{-2}\omega(u)\,du =$$

$$= O\left\{n^{-2}H(\pi/n) + \int_0^{\pi/n} tH(t)\,dt\right\} = O\{n^{-2}H(\pi/n)\}.$$

Hence

$$I_1 = O\{n^{-1}H(\pi/n)\}.$$

And proceeding as in I_2 above, we get

$$I_2 = O\{a_{nn}H(\pi/n)\}.$$

However, by (2.2), $\{a_{nk}\}_{k=0}^{n}$ is non-decreasing and hence

$$(n+1)a_{nn} \geqq \sum_{k=0}^{n} a_{nk} = 1,$$

by (2.1). Thus using the inequality $n^{-1} = O(a_{nn})$ in I_1 and combining it with I_2 we get (2.8).

This completes the proof of Theorem 2.

Proof of Theorem 3. Proceeding as in Theorem 2, we get

$$\|T_n(f)-f\| \leqq I_1 + I_2,$$

where

$$I_1 = O\{\omega(\pi/n)\}$$

and

$$I_2 = \frac{2}{\pi}\int_{\pi/n}^{\pi} \frac{\omega((t)}{2\sin\frac{1}{2}t}\left|\sum_{k=0}^{n} a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|dt.$$

By Lemma 2, we get

$$I_2 = O(1) \int_{\pi/n}^{\pi} t^{-1}\omega(t) b_n([\pi/t])\,dt = O(1) \sum_{k=1}^{n-1} \int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega(t) b_n([\pi/t])\,dt =$$

$$= O(1) \sum_{k=1}^{n-1} \omega(\pi/k) \int_{\pi/(k+1)}^{\pi/k} t^{-1} b_n([\pi/t])\,dt = O(1) \sum_{k=1}^{n-1} \omega(\pi/k) b_n(k+1) k^{-1}.$$

Thus combining I_1 and I_2, we get the required result and hence the proof of the theorem is complete.

Proof of Theorem 4. Splitting up the integral in (4.1) into the sub-integrals $\int_0^{a_{n0}}$ and $\int_{a_{n0}}^{\pi}$ and proceeding as in Theorem 1, the proof of the theorem may be completed.

5. In this section, we specialize the matrix $A=(a_{nk})$ to obtain corollaries of the theorems.

By (1.5), we get the following corollary from Theorem 1:

Corollary 1. *Let $\omega(t)$ satisfy* (2.3), (2.4) *and* (2.5) *and let (p_n) be non-negative and non-decreasing. Then*

$$\|R_n(f)-f\| = O\{(p_n/P_n) H(p_n/P_n)\}.$$

If $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$, then $\omega(t)=O(t^\alpha)$ $(0<\alpha\leqq 1)$ and

$$H(u) = \begin{cases} \log(\pi/u) & \alpha = 1 \\ u^{\alpha-1} & 0<\alpha<1. \end{cases}$$

Hence Theorem A is a particular case of Corollary 1.

It is interesting to note that one can get the estimate of Corollary 1 by using Nörlund matrix (see (1.6)), in place of Riesz matrix. On setting $a_{nk}=p_{n-k}/P_n$ in Theorem 4, we get

Corollary 2. *Let $\omega(t)$ and (p_n) be as defined in Corollary* 1. *Then*

$$\|N_n(f)-f\| = O\{(p_n/P_n) H(p_n/P_n)\}.$$

Now we give the following corollary from Theorem 3:

Corollary 3. *The degree of approximation of $f\in C^*[0, 2\pi]$ by the (R, p_n)-means of Fourier series of f is given by*

$$\|R_n(f)-f\| = O\Big\{(P_n)^{-1} \sum_{k=1}^{n} k^{-1} P_k \omega(\pi/k)\Big\}, \tag{5.1}$$

where (p_n) is non-negative and non-increasing.

Proof. We have, by (1.5),

$$b_n(k+1) = \sum_{r=0}^{k+1} a_{nr} = P_{k+1}/P_n.$$

However (p_n) is non-increasing therefore

$$b_n(k+1) = O(P_k/P_n)$$

and $(k^{-1}P_k)$ is non-increasing and hence

$$\omega(\pi/n) \leqq (P_n)^{-1} \sum_{k=1}^{n} \omega(\pi/k)\, k^{-1} P_k.$$

Using these estimates in (2.10), we get the required result.

It is interesting to note that the estimate in (5.1) was earlier obtained in [3] by using Nörlund matrix as defined by (1.6), where (p_n) is defined as in Corollary 3.

Since $f\in \mathrm{Lip}\,\alpha$ implies that $\omega(t)=O(t^\alpha)$, we deduce the following corollary from Corollary 3:

COROLLARY 4. *Let $f\in C^*[0, 2\pi]$ and let $f\in \mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$. Then the degree of approximation of f by (R, p_n)-means of its Fourier series is given by*

$$\|R_n(f)-f\| = O\Big\{(1/P_n) \sum_{k=1}^{n} k^{-1-\alpha} P_k\Big\},$$

where (p_n) is non-increasing and non-negative.

Once again, the estimate in Corollary 4 was earlier obtained in [2] in the case of Nörlund matrix generated by non-negative and non-increasing sequence (p_n).

References

[1] P. Chandra, On the degree of approximation of functions belonging to the Lipschitz class, *Nanta Math.*, **8** (1975), 88—91.

[2] D. S. Goel and B. N. Sahney, On the degree of approximation of continuous functions, *Ranchi University Math. Jour.*, **4** (1973), 50—53.

[3] A. B. S. Holland, B. N. Sahney and J. Tzimbalario, On degree of approximation of a class of functions by means of Fourier series, *Acta Sci. Math. (Szeged)*, **38** (1976), 69—72,

[4] R. N. Mohapatra and P. Chandra, Degree of approximation of functions in the Hölder metric, *Acta Math. Hung.*, **41** (1983), 67—76.

[5] A. Zygmund, *Trigonometric series*, Vol. I, Cambridge University Press (Cambridge, 1968).

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Δ-GRINGS

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Introduction

There has been a good deal of interest recently in graded rings and strongly graded rings and in their sets of homogeneous elements (see [2] and its 129 references).

The sets of homogeneous elements of graded rings are not really rings and therefore we call them Grings or Δ-Grings, where Δ is the group used for the grading. Profs. Anderson, Sulinski and I [1] studied such Grings graded by a group of order 2. The present paper extends the results on simple Grings to Δ-Grings, where Δ is any finite abelian group.

Let Δ be an abelian group. For every α in Δ we assume there is an abelian group G_α. Let $\bar{\mathcal{G}}$ be the weak direct sum of the G_α, $\bar{\mathcal{G}}=\sum_{\alpha\in\Delta} G_\alpha$, where each element in $\bar{\mathcal{G}}$ is a finite sum of elements from the G_α's. Let $\mathcal{G}$ be the set of all homogeneous elements of $\bar{\mathcal{G}}$, i.e. elements from each G_α but no sums are in $\mathcal{G}$, sums from different G_α's with nonzero elements. We shall write $\mathcal{G}=(G_0, G_\alpha, G_\beta, \ldots)$. This is not a group because it is not closed under addition. We call $\mathcal{G}$ a Δ graded abelian group.

We further assume that an associative multiplication is defined in $\mathcal{G}$ such that $G_\alpha G_\beta \subseteqq G_{\alpha+\beta}$, and that it is both left and right distributive with respect to the addition of elements from the same G_α. This gives us a Δ graded associative ring R. In that case we shall call the G_α's, R_α's and shall write it all as $R=(R_0, R_\alpha, R_\beta, \ldots)$. This is contained in the overring $\bar{R}=\sum_{\alpha\in\Delta} R_\alpha$, the weak direct sum. Now R is not a ring because it is not closed under addition. However R_0 is a ring. The R_α for $\alpha\neq 0$, are additive subgroups of R. We shall call R a Δ-Gring.

A subset $(I_0, I_\alpha, I_\beta, \ldots)$ is said to be an ideal of R if every I_α is a subgroup of the corresponding R_α and if for every α and β in Δ we have

$$I_\alpha R_\beta \subseteqq I_{\alpha+\beta} \quad \text{and} \quad R_\beta I_\alpha \subseteqq I_{\alpha+\beta}.$$

Then I_0 is an ordinary ideal of the ring R_0. Furthermore $R_\alpha I_\beta R_\gamma \subseteqq I_{\alpha+\beta+\gamma}$ and in particular

$$R_\alpha I_0 R_{-\alpha} \quad \text{for every} \quad \alpha \quad \text{in} \quad \Delta.$$

Let J_0 be any ideal of R_0. We shall say that J_0 is *special* if $R_\alpha J_0 R_{-\alpha} \subseteqq J_0$ for every α in Δ. If J_0 is not special then we can expand it to

$$\bar{J}_0 = J_0 + \sum_{\alpha} R_\alpha J_0 R_{-\alpha}.$$

Then $\bar{J}_0$ is special because $R_\beta \bar{J}_0 R_{-\beta} = R_\beta J_0 R_{-\beta} + \sum_\alpha R_\beta R_\alpha J_0 R_{-\alpha} R_{-\beta}$ and this is all $\subseteqq \bar{J}_0$. Of course the ideal $\sum_\alpha R_\alpha J_0 R_{-\alpha}$ is itself special but it may not contain J_0.

Let I_0 be any special ideal of R_0 and define for every $\gamma \neq 0$,

$$I_\gamma = I_0 R_\gamma + R_\gamma I_0 + \sum_\alpha R_\alpha I_0 R_{\gamma-\alpha}.$$

Then $(I_0, I_\alpha, I_\beta, \ldots)$ is an ideal of R. To see this we note that

$$I_\gamma R_\delta = I_0 R_\gamma R_\delta + R_\delta I_0 R_\delta + \sum_\alpha R_\alpha I_0 R_{\gamma-\alpha} R_\delta \subseteqq I_{\gamma+\delta}$$

and similarly $R_\delta I_\gamma \subseteqq I_{\gamma+\delta}$. When $\delta = -\gamma$ this is

$$I_\gamma R_{-\gamma} \subseteqq I_0 R_0 + \sum_\alpha R_\alpha I_0 R_{-\alpha} \subseteqq I_0$$

since I_0 is special.

Note that defining I_γ to be $\sum_\alpha R_\alpha I_0 R_{\gamma-\alpha}$ is not good enough because $I_0 R_\gamma$ must be in I_γ and in this definition I_γ contains $R_0 I_0 R_\gamma$ and this might be smaller than $I_0 R_\gamma$. Similarly we cannot be certain that $R_\gamma I_0$ is in I_γ.

Thus every nonzero proper special ideal of R_0 leads us to a nonzero proper ideal of the Δ-Gring R. We shall say that R_0 is *minisimple* if it has no nonzero proper special ideals.

We are interested in simple Δ-Grings and we have:

LEMMA 1. *If R is simple Δ-Gring then R_0 must be minisimple.*

The ring R_0 may be simple itself. If R is simple and R_0 is not simple, then let J_0 be any nonzero ideal of R_0. Then $J_0 + \sum_\alpha R_\alpha J_0 R_{-\alpha}$ is special and nonzero. Since R_0 is minisimple, this special ideal must be all of R_0. We thus have:

LEMMA 2. *If R is simple and J_0 is any nonzero ideal of R_0 then*

$$J_0 + \sum_\alpha R_\alpha J_0 R_{-\alpha} = R_0.$$

We shall now assume that $R_0 \neq 0$ and that for some nonzero α in Δ, $R_\alpha \neq 0$. We shall later consider the cases where $R_0 = 0$ and where $R_0 \neq 0$ with all $R_\alpha = 0$ for $\alpha \neq 0$.

LEMMA 3. *If R is simple then R has no total annihilators.*

PROOF. Let $I_\alpha = \{x_\alpha \text{ in } R_\alpha \colon x_\alpha R = R x_\alpha = 0\}$. Then $(0, I_\alpha, 0, 0, \ldots)$ is an ideal of R. Then $I_\alpha = 0$ and thus $\{x \text{ in } R \colon xR = Rx = 0\} = 0$.

EXAMPLE 1. Let $\bar{R}$ be all 2×2 matrices over a field. This is our overring and of course it is simple. It contains the 2-Gring (R_0, R_1) where $R_0 = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$ and $R_1 = \left\{\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}\right\}$. Here $R_0^2 = R_0$, $R_1^2 = R_0$ and $R_1 R_0 = R_0 R_1 = R_1$. The ring R_0 is minisimple. It does have two nonzero proper ideals, namely $\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right\}$. Neither one is special because $R_1\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}R_1=\left\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right\}$ and $R_1\left\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right\}R_1=\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}$. There are no total annihilators and this 2-Gring is simple (see [1]). Note that $R_\alpha R_\beta = R_{\alpha+\beta}$ for all α, β and $R_0=R_\alpha R_{-\alpha}$ for all α.

EXAMPLE 2. We use the same overring as in Example 1. Let $R_0=\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$, $R_1=\left\{\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\right\}$ and $R_2=\left\{\begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix}\right\}$. This gives us a 3-Gring (R_0, R_1, R_2). Here $R_0R_0=R_0$, $R_1R_1=0$, $R_2R_2=0$, $R_1R_2=\left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right\}$ and $R_2R_1=\left\{\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}\right\}$. Also $R_0R_1=R_1R_0=R_1$ and $R_0R_2=R_2R_0=R_2$. Then $R_0=R_1R_2+R_2R_1$. The ring R_0 is minisimple. It does have two ideals, namely R_1R_2 and R_2R_1, but neither one is special because $R_1\cdot R_2R_1\cdot R_2=R_1R_2$ and $R_2\cdot R_1R_2\cdot R_1=R_2R_1$. There are no total annihilators and this 3-Gring is simple.

EXAMPLE 3. Let $\bar{R}$ be all 3×3 matrices over a field. It is a simple overring. Take $R_0=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right\}$, $R_1=\left\{\begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\right\}$, $R_2=\left\{\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}\right\}$ and $R_3=\left\{\begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix}\right\}$. Then (R_0, R_1, R_2, R_3) is a 4-Gring.

Now $R_1R_3=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$, $R_3R_1=\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right\}$ and $R_2R_2=\left\{\begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \end{pmatrix}\right\}$. Since $R_2(R_1R_3)R_2=\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}\right\}$, the ideal R_1R_3 is not special. Also $R_3(R_1R_3)R_1=R_3R_1$. In fact R_0 has 6 nonzero proper ideals, none of them special, and R_0 is minisimple. Also $R_0R_1=R_1R_0=R_1$, $R_0R_2=R_2R_0=R_2$ and $R_0R_3=R_3R_0=R_3$.

If (I_0, I_1, I_2, I_3) is an ideal of R then $I_0=R_0$ or 0. If it is R_0 then $R_0R_i=R_i=I_i$ for $i=1, 2, 3$ and thus the ideal is all of R. On the other hand if $I_0=0$ then $R_3I_1=0$ and this happens only when $I_1=0$. Similarly $R_2I_2=0$ implies $I_2=0$ and $R_1I_3=0$ gives $I_3=0$. Therefore the ideal is 0 and this R is a simple 4-Gring.

EXAMPLE 4. Let $\bar{R}$ be all 2×2 matrices over a field, a simple overring. Take $R_0=\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$, $R_1=\left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right\}$, $R_2=R_3=\ldots=R_{n-2}=0$, and $R_{n-1}=\left\{\begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}\right\}$. Then $(R_0, R_1, R_2, \ldots, R_{n-2}, R_{n-1})$ is a Δ-Gring where Δ is the cyclic group with n elements.

Then $R_1R_{n-1}=\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\}=I_1$ and $R_{n-1}R_1=\left\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right\}=I_2$. Here $R_\alpha R_\alpha=0$ for all $\alpha\neq 0$. The ring $R_0=I_1+I_2$. Also $R_0R_\alpha=R_\alpha R_0=R_\alpha$ for all α. R_0 is not simple but it is minisimple.

We could simply discard all the zero pieces, $R_2\ldots R_{n-2}$ and relabel R_{n-1} as R_2 and obtain the 3-Gring of Example 2. In any case this example is a simple Δ-Gring.

LEMMA 4. *If R is simple then there exists $\alpha\neq 0$ in Δ such that $R_\alpha R_{-\alpha}\neq 0$.*

PROOF. If $R_\alpha R_{-\alpha}=0$ for every nonzero α in Δ then $(0, R_\alpha, R_\beta, \dots)$ is an ideal of R. Since it is proper $(R_0\neq 0)$ and nonzero (some $R_\alpha\neq 0$ for $\alpha\neq 0$) this is impossible.

LEMMA 5. *If R is simple then for all γ in Δ with $R_\gamma\neq 0$, there exists β in Δ such that $R_\beta R_{\gamma-\beta}\neq 0$.*

PROOF. Take $R_\gamma\neq 0$. If $R_\beta R_{\gamma-\beta}=0$ for all β then $(R_0, R_\alpha, \dots, 0, R_\delta, \dots)$ where 0 is in the γ^{th} position, is a nonzero proper ideal of R, which is impossible.

LEMMA 6. *If R is simple then for every $\alpha\neq 0$ we have*

$$R_\alpha = R_\alpha R_0 + R_0 R_\alpha + \sum_\beta R_\beta R_0 R_{\alpha-\beta}.$$

PROOF. For every $\alpha\neq 0$ define $I_\alpha = R_\alpha R_0 + R_0 R_\alpha + \sum_\beta R_\beta R_0 R_{\alpha-\beta}$ and consider $(R_0, I_\alpha, I_\beta, I_\gamma, \dots)$. This is an ideal of R because $I_\alpha R_\gamma = R_\alpha R_0 R_\gamma + R_0 R_\alpha R_\gamma + \sum_\beta R_\beta R_0 R_{\alpha-\beta} R_\gamma \subseteqq I_{\alpha+\gamma}$. Similarly $R_\gamma I_\alpha \subseteqq I_{\alpha+\gamma}$. Also $R_0 R_\gamma$ and $R_\gamma R_0$ are both $\subseteqq I_\gamma$. This is a nonzero ideal and thus must be all of R. Therefore $I_\alpha = R_\alpha$ for every $\alpha\neq 0$.

THEOREM 1. *If R is simple then for all α in Δ and every nonzero ideal I_0 of R_0 we have*

$$R_\alpha = \sum_\beta R_\beta I_0 R_{\alpha-\beta}.$$

PROOF. The ideal $\sum R_\beta I_0 R_{-\beta}$ is special and since R_0 is minisimple, the ideal is 0 or R_0. Suppose that it is 0. Then the ideal I_0 is itself special and thus $I_0 = R_0$. Then $\sum R_\beta R_0 R_{-\beta} = 0$. Then $R_0^3 = 0$. Also R_0^2 is special because $R_\beta R_0^2 R_{-\beta} \subseteqq \subseteqq R_\beta R_0 R_{-\beta} = 0 \subseteqq R_0^2$. Then R_0^2 is 0 or R_0. It cannot be R_0 for then $0 = R_0^3 = R_0^2 R_0 = = R_0^2 = R_0$ and we are assuming $R_0 \neq 0$. Thus $R_0^2 = 0$. In that case take R_α as in Lemma 6 and we have

$$R_\alpha R_{-\alpha} = R_\alpha R_0 R_{-\alpha} + R_0 R_\alpha R_{-\alpha} + \sum R_\beta R_0 R_{\alpha-\beta} R_{-\alpha} \subseteqq R_0^2 + \sum R_\beta R_0 R_{-\beta}.$$

Now $R_0^2 = 0$ and $\sum R_\beta R_0 R_{-\beta} = 0$. Thus $R_\alpha R_{-\alpha} = 0$ for every $\alpha \neq 0$. This contradicts Lemma 4 and therefore $\sum R_\beta I_0 R_{-\beta} \neq 0$. Thus $\sum R_\beta I_0 R_{-\beta} = R_0$.

For $\alpha\neq 0$, the sum $\sum R_\beta R_0 R_{\alpha-\beta} + R_\alpha R_0 + R_0 R_\alpha$ which is R_α, can now be rewritten (letting $R_0 = \sum R_\gamma I_0 R_{-\gamma}$) as

$$R_\alpha = \sum_{\beta,\gamma} R_\beta R_\gamma I_0 R_{-\gamma} R_{\alpha-\beta} + R_\alpha \sum R_\gamma I_0 R_{-\gamma} + \sum R_\gamma I_0 R_{-\gamma} R_\alpha \subseteqq$$

$$\subseteqq \sum_{\beta,\gamma} R_{\beta+\gamma} I_0 R_{\alpha-(\beta+\gamma)} + \sum_\gamma R_{\alpha+\gamma} I_0 R_{\alpha-(\alpha+\gamma)} + \sum_\gamma R_\gamma I_0 R_{\alpha-\gamma} \subseteqq \sum_\delta R_\delta I_0 R_{\alpha-\delta} \subseteqq R_\alpha.$$

Thus $R_\alpha = \sum_\delta R_\delta I_0 R_{\alpha-\delta}$ for $\alpha\neq 0$ and combining this with $R_0 = \sum R_\beta I_0 R_{-\beta}$ we have the theorem.

COROLLARY 1.

$$R_\alpha = \sum_\beta R_\beta R_0 R_{\alpha-\beta} = \sum_\beta R_\beta R_{\alpha-\beta}, \quad \textit{for every } \alpha \textit{ in } \Delta.$$

PROOF.

$$R_\alpha = \sum R_\beta I_0 R_{\alpha-\beta} \subseteqq \sum R_\beta R_0 R_{\alpha-\beta} \subseteqq \sum R_\beta R_{\alpha-\beta} \subseteqq R_\alpha.$$

COROLLARY 2.

$$R_0 = \sum_{\beta \neq 0} R_\beta R_0 R_{-\beta} = \sum_{\beta \neq 0} R_\beta R_{-\beta}.$$

PROOF. Let $Q_0 = \sum_{\beta \neq 0} R_\beta R_0 R_{-\beta}$. This is an ideal of R_0, $R_0 Q_0 R_0 \subseteqq Q_0$. It is also special because for $\alpha \neq 0$,

$$R_\alpha Q_0 R_{-\alpha} = \sum_{\beta \neq 0} R_\alpha R_\beta R_0 R_{-\beta} R_{-\alpha} \subseteqq R_\alpha R_{-\alpha} R_0 R_\alpha R_{-\alpha} + \sum_{\alpha+\beta \neq 0} R_{\alpha+\beta} R_0 R_{-(\alpha+\beta)}.$$

The $\sum$ is clearly $\subseteqq Q_0$ while $R_\alpha R_{-\alpha} R_0 R_\alpha R_{-\alpha} \subseteqq R_\alpha R_0 R_{-\alpha} \subseteqq Q_0$. Thus Q_0 is 0 or R_0. Suppose $Q_0 = 0$. Then $R_\alpha R_0 R_{-\alpha} = 0$ for every nonzero α in Δ. The theorem then tells us that $R_0 = \sum_\beta R_\beta R_0 R_{-\beta} = R_0 R_0 R_0$. Then $R_0 = R_0^3 = R_0^2 R_0 \subseteqq R_0^2$. Thus $R_0^2 = R_0$. Furthermore, for any nonzero ideal I_0, $R_0 = \sum_\beta R_\beta I_0 R_{-\beta} = R_0 I_0 R_0 + 0 \subseteqq I_0$. Thus R_0 is simple.

Now Lemma 4 guarantees there is an $\alpha \neq 0$ such that $R_\alpha R_{-\alpha} \neq 0$. This is an ideal of R_0 and thus is $= R_0$. If $R_{-\alpha} R_\alpha = 0$, then $R_\alpha \cdot R_{-\alpha} R_\alpha \cdot R_{-\alpha} = R_0^2 = R_0 = 0$ which is impossible. Thus $R_{-\alpha} R_\alpha \neq 0$ and is also then $= R_0$.

Now $R_0 R_\alpha = R_\alpha R_{-\alpha} \cdot R_\alpha = R_\alpha \cdot R_{-\alpha} R_\alpha = R_\alpha R_0$. Then $R_0 = R_0 R_0 = R_0 R_\alpha R_{-\alpha} = R_\alpha R_0 R_{-\alpha} = 0$ which is impossible. Therefore $Q_0 \neq 0$ and $Q_0 = R_0$ which gives us the corollary. The last equality follows from

$$R_0 = \sum_{\beta \neq 0} R_\beta R_0 R_{-\beta} \subseteqq \sum_{\beta \neq 0} R_\beta R_{-\beta} \subseteqq R_0.$$

We have assumed that R is a simple Δ-Gring $(R_0, R_\alpha, \ldots)$ with $R_0 \neq 0$ and some $R_\alpha \neq 0$ with $\alpha \neq 0$. Now we shall also assume that Δ is finite.

Consider $\bigcap_{\alpha \neq 0} R_\alpha R_{-\alpha} \cdot \equiv \cdot J_0$. Suppose that $J_0 \neq 0$. Then

$$R_{\alpha+\beta} = \sum_\gamma R_\gamma J_0 R_{\alpha+\beta-\gamma} \subseteqq \sum_{\gamma \neq \alpha} R_\gamma R_{\alpha-\gamma} R_{\gamma-\alpha} R_{\alpha+\beta-\gamma} + R_\alpha J_0 R_\beta \subseteqq R_\alpha R_\beta \subseteqq R_{\alpha+\beta}.$$

Therefore we have:

THEOREM 2. *If R is simple and if $\bigcap_{\alpha \neq 0} R_\alpha R_{-\alpha} \neq 0$ then for all α, β we have $R_\alpha R_\beta = R_\beta R_\alpha = R_{\alpha+\beta}$.*

COROLLARY. $R_0 R_0 = R_0$.

So suppose $\bigcap_{\alpha \neq 0} R_\alpha R_{-\alpha} = 0$.

We consider all the ideals $R_\alpha R_{-\alpha}$ of R_0 for $\alpha \neq 0$. Some are not zero. We consider all intersections of 2 of them, of 3, of 4 and so on, until we find an integer t

such that the intersection of any $t+1$ of them is 0 and there is at least one intersection of t of them which is not zero. Then $1 \leqq t < n-1$, if Δ has order n.

Define $I_0 \cdot \equiv \cdot \bigcap_{i=1}^{t} R_{\alpha_i} R_{-\alpha_i} \neq 0$ where the α_i are nonzero and distinct. There may be several choices for I_0 (or there may be just one) and we pick any one and call it I_0. Assume there are r possible choices $I_0, I_1, \dots, I_{r-1}$. That is there are r distinct intersections of t ideals of the form $R_\alpha R_{-\alpha}$, which are $\neq 0$. All other such intersections are 0 and all intersections of more than t are 0.

If $I = \bigcap_1^t R_{\alpha_i} R_{-\alpha_i} \neq 0$ and $J = \bigcap_1^t R_{\beta_i} R_{-\beta_i} \neq 0$ and if the set of α_i is not identical to the set of β_i then $I \neq J$. For if $I = J$ then I is in each $R_{\alpha_i} R_{-\alpha_i}$ and in $R_{\beta_i} R_{-\beta_i}$ where β_i is $\neq 0$ and different from all the α_i. Then I is in more than t ideals and is therefore 0. Also $IJ = JI = 0$. Thus the ideals $I_0, I_1, \dots, I_{r-1}$ are all different and $I_i I_j = 0$ if $i \neq j$.

From Theorem 1 we know that $R_0 = \sum_\gamma R_\gamma I_0 R_{-\gamma}$.

Any given term $R_\gamma I_0 R_{-\gamma}$ is in $R_\gamma R_{-\gamma}$ and also in $R_\gamma R_{\alpha_i} R_{-\alpha_i} R_{-\gamma} \subseteqq R_{\gamma+\alpha_i} R_{-\gamma-\alpha_i}$ for every $i = 1 \dots t$. The numbers $\gamma, \gamma+\alpha_1, \gamma+\alpha_2, \dots, \gamma+\alpha_t$ are all distinct because the α_i are all distinct. If none of them is 0 then the term is in $t+1$ ideals and is 0. Thus the only possible survivors are those terms with $\gamma = 0$ or $\gamma = -\alpha_i$ for some i. Thus each term is $\subseteqq R_0 R_0$. combining this with the corollary to Theorem 2 we have:

Theorem 3. *Every simple Δ finite Gring with $R_0 \neq 0$ and $R_\alpha \neq 0$ for some $\alpha \neq 0$ has: $R_0 R_0 = R_0$.*

Lemma 7. *If I_0 is an ideal of R_0 and $I_0 I_0 = 0$ then $I_0 = 0$.*

Proof. $R_0 = \sum_\gamma R_\gamma I_0 R_{-\gamma}$ by Theorem 1, if $I_0 \neq 0$. Each piece is nilpotent since $R_\gamma I_0 R_{-\gamma} \cdot R_\gamma I_0 R_\gamma \subseteqq R_\gamma I_0 R_0 I_0 R_{-\gamma} = 0$. Since Δ is *finite*, R_0 is a finite sum of nilpotent ideals. Such a sum is itself nilpotent. But $R_0 = R_0^2 = R_0^n$ for all n and is not nilpotent. Thus I_0 must be 0.

Lemma 8. *R_0 has no total annihilators in it.*

Proof. Let $V = \{x \text{ in } R_0 \colon xR_0 = R_0 x = 0\}$. Then $V \cdot V = 0$ and by Lemma 7 $V = 0$.

Corollary. $\{x \text{ in } R_0 \colon xR_0 = 0\} = \{x \text{ in } R_0 \colon R_0 x = 0\} = 0$.

We also know (in the case when $J_0 = \bigcap_{\alpha \neq 0} R_\alpha R_{-\alpha} = 0$) that

$$R_0 = R_0 I_0 R_0 + R_{-\alpha_1} I_0 R_{\alpha_1} + \dots + R_{-\alpha_t} I_0 R_{\alpha_t}.$$

Each term is in $R_0 R_0$ and in t other terms with nonzero subscripts, of the form $R_\alpha R_{-\alpha}$. Thus each term is $\subseteqq$ one of the $I_0, I_1, \dots, I_{r-1}$. We can remove any terms that are the same. Call the remaining terms $S_0, \dots, S_{r-1}$ and we have

$$R_0 = S_0 + S_1 + \dots + S_{r-1}$$

where $S_i \subseteqq I_i$. Now $S_i S_j = 0$ if $i \neq j$. In fact $S_i \cap S_j = 0$ if $i \neq j$. So these S_i's are disjoint ideals of R_0. In fact this is a direct sum because if $0=x_0+x_1+\ldots+x_{r-1}$ then if $x_i\neq 0$ $x_iS_j=0$ for every $j\neq i$. Also $x_iS_i=0$ because $x_i=-x_0-x_1-\ldots$ $\ldots-x_{i-1}-x_{i+1}-\ldots-x_{r-1}$ and S_i annihilates the right hand side. Thus $x_iR_0=0$. Similarly $R_0x_i=0$. Thus x_i annihilates R_0. But this contradicts Lemma 8 and thus $x_i=0$. Thus 0 is uniquely represented as $0+\ldots+0$ and

$$R_0 = S_0\oplus S_1\oplus\ldots\oplus S_{r-1}.$$

Now $S_i=R_{-\alpha_i}I_0R_{\alpha_i}\subseteqq$ one of the $I_i=\bigcap_1^t$ (ideals of the form $R_\alpha R_{-\alpha}$). Then $R_0=$ $=I_0+I_1+\ldots+I_{r-1}$ and this sum is also direct. Then $S_i=I_i$ for if there is an element z in I_i which is not in S_i then $z=s_0+s_1+\ldots+s_{r-1}$ or $0=-s_0-s_1-\ldots-z-$ $-s_i-\ldots-s_{r-1}$ and this is in the direct sum $I_0\oplus\ldots\oplus I_{r-1}$. Then $z-s_i$ which is in I_i, must be 0 and $z=s_i$ is in S_i. Thus $S_i=I_i$.

Furthermore $R_0=R_0R_0=S_0^2\oplus\ldots\oplus S_{r-1}^2$. Then $S_i^2=S_i=I_i=I_i^2$. We thus have

THEOREM 4. *If R is simple then*

$$R_0 = I_0\oplus I_1\oplus\ldots\oplus I_{r-1}$$

where the I_i are idempotent disjoint ideals, each one is the $\bigcap_1^t R_{\alpha_1}R_{-\alpha_1}$, with $1\leqq r\leqq t$. Furthermore $I_i=R_0I_iR_0$, and $R_\alpha I_iR_{-\alpha}$ is 0 or is one of the I_j.

In the case when $J_0\neq 0$ then $R_0=I_0=J_0=\bigcap_{\alpha\neq 0} R_\alpha R_{-\alpha}$, i.e. $R_\alpha R_{-\alpha}=R_0$ for every α.

Take an $\alpha\neq 0$ with $R_\alpha R_{-\alpha}\neq 0$. Then some of the I_j are in $R_\alpha R_{-\alpha}$ and others are not. Let $I_0, I_1, \ldots, I_q$ be in $R_\alpha R_{-\alpha}$ and $I_{q+1}, \ldots, I_{r-1}$ not in $R_\alpha R_{-\alpha}$. Then $I_0+\ldots+I_q\subseteqq R_\alpha R_{-\alpha}$. Also $I_j\cdot R_\alpha R_{-\alpha}=0$ for $j>q$ because the product would be in more than t ideals of the form $R_\beta R_{-\beta}$. Similarly $R_\alpha R_{-\alpha}\cdot I_j=0$ for $j>q$. If there is an element x in $R_\alpha R_{-\alpha}$ which is not in $I_0+\ldots+I_q$ then

$$x = y_0+y_1+\ldots+y_q+y_{q+1}+\ldots+y_{r-1}$$

and

$$x-y_0-y_1-\ldots-y_q = y_{q+1}+\ldots+y_{r-1}$$

is in $R_\alpha R_{-\alpha}$. Now the left hand side annihilates I_j for $j>q$ because it is in $R_\alpha R_{-\alpha}$. It also annihilates I_j for $j\leqq q$ because the right hand side does. Thus it annihilates R_0 and is thus 0 by Lemma 8. Then $x=y_0+\ldots+y_q$ is in $I_0+\ldots+I_q$. Thus

$$R_\alpha R_{-\alpha} = I_0+\ldots+I_q.$$

LEMMA 9. *For every α with $R_\alpha R_{-\alpha}\neq 0$, $R_\alpha R_{-\alpha}=$the sum of those I_j's which it contains.*

COROLLARY. $R_\alpha R_{-\alpha}=R_0R_\alpha R_{-\alpha}=R_\alpha R_{-\alpha}R_0=R_\alpha R_{-\alpha}R_\alpha R_{-\alpha}=R_\alpha R_0R_{-\alpha}$.

PROOF. For the last equality

$$R_\alpha R_{-\alpha}R_\alpha R_{-\alpha}\subseteqq R_\alpha R_0R_{-\alpha}\subseteqq R_\alpha R_{-\alpha}=R_\alpha R_{-\alpha}R_\alpha R_{-\alpha}.$$

Define $\mathscr{I}\cdot\equiv\cdot\{\alpha \text{ in } \Delta\colon R_\alpha R_{-\alpha}\neq 0\}$ and $\mathscr{I}_0\cdot\equiv\cdot\{\beta \text{ in } \Delta\colon R_\beta R_{-\beta}=0\}$. Since $R_0R_0=R_0\neq 0$, 0 is in $\mathscr{I}$. Also if α is in $\mathscr{I}$ then $-\alpha$ is in $\mathscr{I}$ because $R_\alpha R_{-\alpha}=$ $=R_\alpha R_{-\alpha}R_\alpha R_{-\alpha}\neq 0$ and thus $R_{-\alpha}R_\alpha\neq 0$. Therefore if β is in $\mathscr{I}_0$, so is $-\beta$.

Define $C_\alpha\cdot\equiv\cdot\{x \text{ in } R_\alpha\colon xR_{-\alpha}=R_{-\alpha}x=0\}$. From Lemma 8 we know that $C_0=0$. For β in $\mathscr{I}_0$, $C_\beta=R_\beta$. For α in $\mathscr{I}$, $C_\alpha\subset R_\alpha$. We plan to show that all the C_α are 0.

Define $I=(C_0, C_\alpha, \dots)$. Since $C_0=0$, I is a proper subset of R.

Lemma 10. *I is an ideal of R and therefore I is* 0.

Proof. We must show that $C_\alpha R_\beta+R_\beta C_\alpha\subseteqq C_{\alpha+\beta}$. Thus we must show that

$$(C_\alpha R_\beta+R_\beta C_\alpha)R_{-\alpha-\beta}=0=R_{-\alpha-\beta}(C_\alpha R_\beta+R_\beta C_\alpha).$$

Clearly $C_\alpha R_\beta R_{-\alpha-\beta}\subseteqq C_\alpha R_{-\alpha}=0$ and $R_{-\alpha-\beta}R_\beta C_\alpha\subseteqq R_{-\alpha}C_\alpha=0$. This leaves us only $R_\beta C_\alpha R_{-\alpha-\beta}$ and $R_{-\alpha-\beta}C_\alpha R_\beta$. If either one is nonzero then it is a nonzero ideal of R_0. But their squares are 0 and thus by Lemma 7 both $R_\beta C_\alpha R_{-\alpha-\beta}$ and $R_{-\alpha-\beta}C_\alpha R_\beta$ are 0. Therefore I is an ideal of R and since it is proper it is 0.

Theorem 5. *If $R_\alpha R_{-\alpha}\neq 0$ then $C_\alpha=0=C_{-\alpha}$. If $R_\beta R_{-\beta}=0$ then $R_\beta=0=$ $=R_{-\beta}$.*

Proof. For β in $\mathscr{I}_0$, $R_\beta=C_\beta=0$ and $R_{-\beta}=C_{-\beta}=0$.

For any α in $\mathscr{I}$, $R_\alpha R_{-\alpha}\neq 0$ and by Lemma 9 $R_\alpha R_{-\alpha}=I_0\oplus\dots\oplus I_q$. Now R_0 itself is (by Theorem 4) $=I_0\oplus\dots\oplus I_q\oplus I_{q+1}\oplus\dots\oplus I_{r-1}=R_\alpha R_{-\alpha}+K$ where $K=I_{q+1}\oplus\dots$ $\dots\oplus I_{r-1}$, the I's that are not in $R_\alpha R_{-\alpha}$. Consider $R_0R_\alpha=(R_\alpha R_{-\alpha}+K)R_\alpha=$ $=R_\alpha R_{-\alpha}R_\alpha+KR_\alpha$.

Now $KR_\alpha\subseteqq R_\alpha$ since $K\subseteqq R_0$. Also $KR_\alpha\cdot R_{-\alpha}=K(I_0\oplus\dots\oplus I_q)=0$. Is $R_{-\alpha}\cdot KR_\alpha=0$? If it is, then $KR_\alpha\subseteqq C_\alpha=0$ and $R_0R_\alpha=R_\alpha R_{-\alpha}R_\alpha$. Suppose $R_{-\alpha}KR_\alpha\neq 0$. It is an ideal of R_0 and $R_{-\alpha}KR_\alpha\cdot R_{-\alpha}KR_\alpha=0$. Therefore by Lemma 7, it is 0. Thus $KR_\alpha\subseteqq$ $\subseteqq C_\alpha=0$.

Therefore $R_0R_\alpha=R_\alpha R_{-\alpha}R_\alpha$.

Similarly $R_\alpha R_0=R_\alpha R_{-\alpha}R_\alpha+R_\alpha Q$ where Q is the sum of those I's which do not belong to $R_{-\alpha}R_\alpha$. Then $R_{-\alpha}\cdot R_\alpha Q=0$ and $R_\alpha Q\cdot R_{-\alpha}$ is 0 since it is a nilpotent ideal of R_0, as above. Then $R_\alpha Q\subseteqq C_\alpha=0$ and $R_\alpha R_0=R_\alpha R_{-\alpha}R_\alpha$.

Theroem 6. *$R_0R_\alpha=R_\alpha R_0=R_\alpha$ for all α in Δ.*

Proof. If α is in $\mathscr{I}_0$, $R_\alpha=0$ and the theorem holds. If α is in $\mathscr{I}$ then we know that $R_0R_\alpha=R_\alpha R_0=R_\alpha R_{-\alpha}R_\alpha$.

To show that $R_0R_\alpha=R_\alpha$ we take $J=(R_0, R_0R_\alpha, R_0R_\beta, \dots)$. We want to show that this is an ideal of R. To this end we need $R_0R_\alpha R_\beta+R_\beta R_0R_\alpha\subseteqq R_0R_{\alpha+\beta}$. Now $R_0R_\alpha R_\beta\subseteqq R_0R_{\alpha+\beta}$ and $R_\beta R_0R_\alpha=R_0R_\beta R_\alpha\subseteqq R_0R_{\alpha+\beta}$. This is a nonzero ideal and thus must equal all of R. Thus $R_0R_\alpha=R_\alpha$ for all α.

We now have the following results: If R is a simple Δ-Gring with Δ finite and $R_0\neq 0$ and $R_\alpha\neq 0$ for some $\alpha\neq 0$, then

1. R_0 is minisimple (Lemma 1).
2. $C_\alpha=\{x \text{ in } R_\alpha\colon xR_{-\alpha}=0=R_{-\alpha}x\}=0$ for all α (Theorem 5).
3. $R_0R_\alpha=R_\alpha R_0=R_\alpha$ for all α (Theorem 6).

These three conditions are enough to give us a converse.

THEOREM 7. *Let* $R=(R_0, R_\alpha, R_\beta, \dots)$ *be a* Δ*-Gring with* Δ *finite. Assume* $R_0\neq 0$ *and* $R_\alpha\neq 0$ *for some* $\alpha\neq 0$. *Then* R *is a simple* Δ*-Gring if and only if* R_0 *is minisimple,* $C_\alpha=0$ *for all* α *and*

$$R_0R_\alpha = R_\alpha R_0 = R_\alpha \quad \textit{for all } \alpha.$$

PROOF. We have proved that if R is simple, the three conditions hold. Now we assume the three conditions and let $I=(I_0, I_\alpha, I_\beta, \dots)$ be an ideal of R. Since I_0 is the 0^{th} component in an ideal, I_0 must be a special ideal of R_0. Since R_0 is minisimple I_0 must be 0 or all of R_0.

Suppose first that $I_0=0$. Then $I_\alpha R_{-\alpha}+R_{-\alpha}I_\alpha$ must be $\subseteqq I_0=0$. Thus $I_\alpha\subseteqq C_\alpha$ but this is 0 and thus the ideal I would have to be 0.

Suppose then that $I_0=R_0$. Then $R_0R_\alpha\subseteqq I_\alpha$. But $R_0R_\alpha=R_\alpha$ and thus $I_\alpha=R_\alpha$ for all α and the ideal $I=R$. Therefore R is simple.

We can now consider the other cases. First let us take $R_0\neq 0$ and all the $R_\alpha=0$ for $\alpha\neq 0$. Thus $R=(R_0, 0, 0, \dots)$. Clearly this is going to be a simple Δ-Gring if and only if R_0 is a simple ring.

The final case occurs when we have $R_0=0$ and some $R_\alpha\neq 0$ so $R=(0, R_\alpha, R_\beta, \dots)$.

We are assuming that Δ is finite. Suppose it has n elements and thus $n-1$ nonzero elements. When $R_0=0$ we can prove:

LEMMA 11. $R_{\alpha_1}\cdot R_{\alpha_2}\dots R_{\alpha_n}=0$ *where the* α_i *are any elements of* Δ.

PROOF. The n elements $\alpha_1, \alpha_2+\alpha_2, \dots, \alpha_1+\alpha_2+\dots+\alpha_n$ are all in Δ. If one of them is 0, say $\alpha_1+\dots+\alpha_t=0$, then

$$R_{\alpha_1}\dots R_{\alpha_t}R_{\alpha_{t+1}}\dots R_{\alpha_n} \subseteqq R_{\alpha_1+\dots+\alpha_t}R_{\alpha_{t+1}}\dots R_{\alpha_n} = R_0R_{\alpha_{t+1}}\dots R_{\alpha_n} = 0.$$

On the other hand if they are all different from 0, two of them must be equal. Then

$$\alpha_1+\dots+\alpha_r = \alpha_1+\dots+\alpha_r+\alpha_{r+1}+\dots+\alpha_w.$$

Then $\alpha_{r+1}+\dots+\alpha_w=0$, and

$$R_{\alpha_1}\dots R_{\alpha_r}R_{\alpha_{r+1}}\dots R_{\alpha_w}\dots R_{\alpha_n} \subseteqq R_{\alpha_1}\dots R_{\alpha_r}R_0R_{\alpha_{w+1}}\dots R_{\alpha_n} = 0.$$

Thus in all cases $R_{\alpha_1}\dots R_{\alpha_n}=0$.

Let us then take any $R_{\alpha_1}\neq 0$. Either $R_{\alpha_1}R_\beta=0$ for all β or there exists α_2 such that $R_{\alpha_1}R_{\alpha_2}\neq 0$. We continue in this way and there must exist an integer m, such that $R_{\alpha_1}R_{\alpha_2}\dots R_{\alpha_m}\neq 0$ but $R_{\alpha_1}R_{\alpha_2}\dots R_{\alpha_m}\cdot R_\beta=0$ for all β in Δ. The integer m is $1\leqq m<n$.

Then we consider $R_\beta\cdot R_{\alpha_1}R_{\alpha_2}\dots R_{\alpha_m}$. Either all of these are 0 or there exists β_1 such that $R_{\beta_1}R_{\alpha_1}R_{\alpha_2}\dots R_{\alpha_m}\neq 0$. Then there must exist an integer r such that $R_{\beta_r}\dots R_{\beta_1}R_{\alpha_1}R_{\alpha_2}\dots R_{\alpha_m}=Q\neq 0$ but $R_\gamma\cdot R_{\beta_r}\dots R_{\alpha_m}=0$ for all γ in Δ. And $r+m<n$. Then this nonzero product $Q\subseteqq R_{(\sum_1^r \beta_i+\sum_1^m \alpha_j)}$ and $QR_\beta=0=R_\gamma Q$ for all β, γ. Q consists of total annihilators of the Δ-Gring R.

Then $I=(0, 0, \dots, 0, Q, 0, \dots, 0)$ is a nonzero ideal of the Δ-Gring R. Since R is simple, $I=R$. Furthermore Q can have no proper nonzero subgroup W for then $(0, 0, \dots, 0, W, 0, \dots, 0)$ would be a proper nonzero ideal of R. We thus have:

THEOREM 8. *If R is a Δ-Gring with Δ finite and $R_0=0$ then R is simple if and only if*

$$R=(0,0,\ldots,0,R_\gamma,0,\ldots,0)$$

where $R_\gamma\neq 0$ has no proper nonzero subgroups and $R_\gamma R_\gamma=0$. In reality then R is just a simple zero ring.

PROOF. If R is simple then $R=(0,0,\ldots,0,R_\gamma,0,\ldots,0)$ as above and conversely, every such simple zero ring with a tail of 0's attached *is* a simple Δ-Gring.

Putting all of this together we have:

THEOREM 9. *If $R=(R_0,R_\alpha,R_\beta,\ldots)$ is a Δ-Gring with Δ finite, then R is simple if and only if:*

Case 1: $R_0=0$, $R=(0,0,\ldots,0,R_\gamma,0,\ldots,0)$ *where R_γ is a simple zero ring.*

Case 2: $R_0\neq 0$, *all* $R_\alpha=0$. $R=(R_0,0,0,\ldots,0)$ *where R_0 is a simple ring.*

Case 3: $R_0\neq 0$, *some* $R_\alpha\neq 0$ *for* $\alpha\neq 0$. *R_0 is minisimple,* $R_0R_\alpha=R_\alpha R_0=R_\alpha$ *for all α and* $C_\alpha=\{x \text{ in } R_\alpha\colon xR_{-\alpha}x=R_{-\alpha}x=0\}=0$ *for all α.*

This extends Theorems 1 and 2 in [1].

REMARKS. Note that $\mathscr{I}$ need not be a subgroup of Δ. See Example 4. Note that R_0 need not be simple even when $R_\alpha R_\beta=R_{\alpha+\beta}$ and $R_0=R_\alpha R_{-\alpha}$ for all α. See Example 1.

For any Δ-Gring $R=(R_0,R_\alpha,R_\beta,\ldots)$ we consider the overring $\sum_{\alpha\in\Delta}R_\alpha\cdot\equiv\cdot\mathscr{R}$. If $\mathscr{R}$ is a simple ring then the Δ-Gring R must be simple because if I is a nonzero ideal of R, $I=(I_0,I_\alpha,\ldots)$ then $\sum_\alpha I_\alpha$ is an ideal of $\mathscr{R}$. To see this we recall that $I_\alpha I_\beta\subseteqq I_{\alpha+\beta}$. Thus $\sum I_\alpha=\mathscr{R}$. Then $I_0+I_\alpha+I_\beta+\ldots=R_0+R_\alpha+R_\beta+\ldots$. Thus each $I_\alpha=R_\alpha$ and R is simple.

Conversely suppose that R is a simple Δ-Gring. If $R_0=0$ then $R=\mathscr{R}$ is simple. If $R_0\neq 0$ with all the $R_\alpha=0$, then again $R=\mathscr{R}$ is simple. The meaty case arises when $R_0\neq 0$ and $R_\alpha\neq 0$ for some $\alpha\neq 0$. Then we know (Theorem 9) that R_0 is minisimple, that $R_0R_\alpha=R_\alpha R_0=R_\alpha$ for all a and that $C_\alpha=0$ for all α.

Let us assume that $\mathscr{R}$ is not simple and has a nonzero proper ideal $I=\sum_\alpha I_\alpha$.

Define $I_0\cdot\equiv\cdot I\cap R_0$. This is an ideal of the ring R_0 and it is special because $R_\alpha I_0R_{-\alpha}=R_\alpha(I\cap R_0)R_{-\alpha}$ is in R_0 and is certainly in I and thus is in I_0, for all α. Since R_0 is minisimple $I_0=R_0$ or 0. Suppose $I_0=R_0$. Then $R_0\subseteqq I$. Then $R_0R_\alpha=$ $=R_\alpha\subseteqq I$ for all α and therefore $I=$all of $\mathscr{R}$. Since I is proper we must have $I_0=0$.

Then $I\cap R_0=0$. Now $I\cap R_\alpha$ must also be 0, for all α, for if x is in $I\cap R_\alpha$ then $xR_{-\alpha}$ is in R_0 and in I and must therefore be 0. Also $R_{-\alpha}\cdot x=0$. Then x is in C_α which is 0. Thus $x=0$ and $I\cap R_\alpha=0$ for all α.

Thus I does not contain any "singles", i.e. nonzero elements from any single R_α.

Of course I must contain some nonzero elements and thus there must exist a positive integer m such that m is minimal and $r_{\alpha_1}+\ldots+r_{\alpha_m}$ is in I, where all the α_i are distinct and where all the r_{α_i} are nonzero. Then $2\leqq m\leqq$ the number of elements in Δ. Since $C_{\alpha_1}=0$, there must exist an element $r_{-\alpha_1}$ such that $r_{\alpha_1}r_{-\alpha_1}\neq 0$. Then I contains $s_0+s_{\gamma_1}+\ldots+s_{\gamma_{m-1}}$ for some nonzero elements s_{γ_j} with the γ_j distinct.

LEMMA 12. $\sum_{\alpha} R_\alpha s_0 R_{-\alpha} = R_0$.

PROOF. The left hand side sum is certainly a special ideal of R_0 and thus is all of R_0 or is 0. If it is 0 then $R_\alpha s_0 R_{-\alpha}=0$ for every α and in particular $R_0 s_0 R_0=0$. Then $s_0 R_0$ is an ideal of R_0 and it is special because $R_\alpha \cdot s_0 R_0 \cdot R_{-\alpha} = R_\alpha s_0 R_{-\alpha} = 0 \subseteq s_0 R_0$. Since $s_0 R_0$ cannot be all of R_0 because $R_0 s_0 R_0 = R_0 R_0 = R_0$ cannot be 0, we must have $s_0 R_0 = 0$. Similarly $R_0 s_0 = 0$. Then s_0 is in $C_0 = 0$. But $s_0 \neq 0$ and thus $\sum R_\alpha s_0 R_{-\alpha} \neq 0$ and the lemma is proved.

LEMMA 13. *The ideal I contains elements of the form $r_0 + r_\gamma + \ldots + r_{\gamma_{m-1}}$ where every r_0 in R_0 appears and appears only once, in this form.*

PROOF. Take any r_0 in R_0. Then $r_0 = \sum_{\beta} r_\beta s_0 r_{-\beta}$ for some set of β's. From $s_0 + \ldots + s_{\gamma_{m-1}}$ in I we get $\sum r_\beta (s_0 + \ldots + s_{\gamma_{m-1}}) r_{-\beta}$ in I and this is $r_0 + r_{\gamma_1} + \ldots + r_{\gamma_{m-1}}$. Thus r_0 does appear.

For the second part assume that $r_0 + t_{\gamma_1} + \ldots + t_{\gamma_{m-1}}$ is also in I. Then $(r_{\gamma_1} - t_{\gamma_1}) + \ldots + (r_{\gamma_{m-1}} - t_{\gamma_{m-1}})$ is in I, contradicting the minimality of m unless each $r_{\gamma_i} = t_{\gamma_i}$.

LEMMA 14. *If $s_0 + s_{\gamma_1} + \ldots + s_{\gamma_{m-1}}$ is in I then for all γ_i, $i = 1 \ldots m-1$, every r_{γ_i} in R_{γ_i} appears once and only once in this form, in an element of I.*

PROOF. Let V_{γ_i} be the subgroup of R_{γ_i} that has the property that an element r_{γ_i} appears as a component in an element in I where the subscripts $0, \gamma_1, \ldots, \gamma_{m-1}$ are used, if and only if r_{γ_i} is in V_{γ_i}. Then

$$V_{\gamma_i} R_0 + R_0 V_{\gamma_i} + \sum_{\beta} R_\beta V_{\gamma_i} R_{-\beta} \subseteqq V_{\gamma_i}.$$

Define I_0 to be $V_{\gamma_i} R_{-\gamma_i} + R_{-\gamma_i} V_{\gamma_i} + \sum_{\delta} R_\delta V_{\gamma_i} R_{-\gamma_i - \delta}$. This is a special ideal of R_0. It cannot be 0 for then $V_{\gamma_i} R_{-\gamma_i} = R_{-\gamma_i} V_{\gamma_i} = 0$, $V_{\gamma_i} \subseteqq C_{\gamma_i} = 0$. But $V_{\gamma_i} \neq 0$ by the minimality of m. Thus $I_0 = R_0$. Then

$$R_{\gamma_i} = R_0 R_{\gamma_i} = V_{\gamma_i} R_{-\gamma_i} R_{\gamma_i} + R_{-\gamma_i} V_{\gamma_i} R_{\gamma_i} + \sum_{\delta} R_\delta V_{\gamma_i} R_{-\gamma_i - \delta} R_{\gamma_i} \subseteqq V_{\gamma_i} R_0 +$$

$$+ \sum_{\delta} R_\delta V_{\gamma_i} R_{-\delta} \subseteqq V_{\gamma_i}.$$

Therefore $V_{\gamma_i} = R_{\gamma_i}$ and every element in R_{γ_i} appears. As in Lemma 13, it can appear only once in this form.

Therefore I contains a set of elements of the form $r_0 + r_{\gamma_1} + \ldots + r_{\gamma_{m-1}}$ where every r_0 in R_0 and every r_{γ_i} in R_{γ_i} appears precisely once. This gives us a $1-1$ correspondence between R_0 and every R_{γ_i} and between any two of the R_{γ_i}'s.

These $1-1$ correspondences clearly preserve addition and thus $R_0 \cong R_{\gamma_1} \cong \ldots \ldots \cong R_{\gamma_{m-1}}$ as abelian groups. However the relationships go even deeper.

THEOREM 10. *$R_0 \cong R_{\gamma_1} \cong \ldots \cong R_{\gamma_{m-1}}$ as rings.*

PROOF. For any two elements r_{γ_i} and t_{γ_i} in R_{γ_i}, they appear in I in only one way, namely

$$r_0 + \ldots + r_{\gamma_i} + \ldots + r_{\gamma_{m-1}} \quad \text{in} \quad I$$

and

$$t_0+\ldots+t_{\gamma_i}+\ldots+t_{\gamma_{m-1}} \quad \text{in} \quad I.$$

Thus r_{γ_i} has a unique partner r_0 in R_0 and t_{γ_i} has t_0 in R_0. Furthermore $r_0(t_0+\ldots+t_{\gamma_{m-1}})$ and $(r_0+\ldots+r_{\gamma_{m-1}})t_0$ both have r_0t_0 in the 0[th] component and thus $r_{\gamma_i}t_0=r_0t_{\gamma_i}$. We define multiplication in R_{γ_i} as: $r_{\gamma_i}*t_{\gamma_i}\equiv r_0t_{\gamma_i}=r_{\gamma_i}t_0$. This is well defined and closed. It is also associative and distributive. Thus R_{γ_i} is a ring. The natural $1-1$ correspondence preserves multiplication because $r_0t_0\leftrightarrow r_0t_{\gamma_i}=$ $=r_{\gamma_i}*t_{\gamma_i}$.

EXAMPLE 5. Let R_0 be a simple nontrivial ring and let $R_\alpha=R_\beta=\ldots=R_0$ for all α, β in Δ. Then $R=(R_0, R_\alpha, R_\beta, \ldots)$ with $R_\alpha R_\beta=R_{\alpha+\beta}$ is a simple Gring. The overring $\mathscr{R}=\sum R_\alpha$ is not simple because it has the ideal $I=\{x_0+x_\alpha+x_\beta+\ldots\}$ where each x_α really equals x_β but is merely in a different subscripted image of R_0.

EXAMPLE 6. Let $R_0=\left\{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\right\}$ and $R_1=\left\{\begin{pmatrix} 0 & c\\ d & 0\end{pmatrix}\right\}$. Let $R_2=R_0$ only think of it as being painted red, and let $R_3=R_1$ only it is painted red. Then $R=(R_0, R_1, R_2, R_3)$ with $R_\alpha R_\beta=R_{\alpha+\beta}$ is a simple Gring. The overring $\sum R_\alpha$ however, is not simple because it has the ideal

$$I=\left\{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}+\begin{pmatrix} 0 & c\\ d & 0\end{pmatrix}+\underset{\text{red}}{\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}}+\underset{\text{red}}{\begin{pmatrix} 0 & c\\ d & 0\end{pmatrix}}\right\}.$$

Here the minimal m is 2.

References

[1] N. Divinsky, A. Suliński and T. Anderson, Simple Grings and invariant radicals, *Coll. Math Soc. J. Bolyai*, 38 *Radical Theory* (Eger, 1982), *North-Holland* (1985), 59—92.
[2] C. Nastasescu and F. van Oystaeyen, *Graded Ring Theory*, North-Holland (1982).

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PLANE SECTIONS OF THE UNIT BALL OF L^p

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1. Introduction

The intersection of a symmetric convex set C with an n-dimensional plane through its centre O will be termed a *plane section of* C. At the end of his paper [9], V. Klee proved that every finite-dimensional section of the unit ball of the space c_0 is a polyhedral and posed the problem of determining all the finite-dimensional sections of the unit balls of other well-known Banach spaces. The purpose of this paper is to describe the plane sections of the unit balls of $L^p[0, 1]$ and l^p, $1 \leqq p < \infty$ (Theorems 1 and 2). This description is an extension of the results of [3] from the case $p=1$ to arbitrary p. We also establish a class of convex sets which is called p-cozonoids. Briefly we say that p-cozonoids are the plane sections of the unit ball of $L^p[0, 1]$. Some ideas we use here were presented earlier by Levy [11]. For other result connected with plane sections of the L^p-spaces see [4], [5], [14].

A convex body (non-void, compact convex set) in the Euclidean space $\mathbf{R}^n$, which is a finite sum of line segments is called a *zonotope*. For a zonotope P, which is the sum of the line segments $[-x_i x_i]$ $(i=1, 2, \ldots, k)$, the support function H_p of the zonotope P $\left(H_p(u) = \sup\limits_{x \in P} \langle u, x\rangle, u \in \mathbf{R}^n\right)$ may be written in the form

$$H_p(u) = \sum_{i=1}^{k} \left\langle \frac{x_i}{\|x_i\|}, u \right\rangle \varrho_i \quad (u \in \mathbf{R}^n),$$

where $\langle .,. \rangle$ denotes the scalar product in $\mathbf{R}^n$.

It is easy to see that there exists a compact convex set K in $\mathbf{R}^n$ such that its support functions can be represented in continuous form of the above, i.e.

$$H_K(u) = \int\limits_{S^{n-1}} |\langle x, u\rangle| \, d\mu(x)$$

where μ is measure on the unit sphere $S^{n-1} = \{x \in \mathbf{R}^n \colon \|x\| = 1\}$ (by measure on S^{n-1} we mean a nonnegative, finite, σ-additive set function on the σ-algebra of the Borel subsets of S^{n-1}). A convex subset of $\mathbf{R}^n$ whose support functions have the above integral representation are called *zonoids*. Zonoids are limits of zonotopes in the Hausdorff metric. In $\mathbf{R}^2$ the set of zonoids coincides with the set of all centrally symmetric convex bodies. W. Weil [18] presented an example of an n-dimensional convex body $(n \geqq 3)$ which is not a zonoid, although all of its k-dimensional projections are zonoids $(k=2, 3, \ldots, n-1)$. The set of all zonoids is a closed convex cone in the space of all centrally symmetric convex bodies equipped with the Hausdorff metric. Since the simplicial polytope (i.e. all faces of which are simplices) is not a zonoid and the set of simplicial polytopes is dense, the set of all zonoids is nowhere dense. More details can be found in [12], [19], [20], [17].

2. p-cozonoids

In this paper we consider the convex bodies which are adjoint to zonoids and also their certain generalization. This class of convex bodies is used to characterize the plane sections of the unit ball of L^p-spaces.

To each norm N on $\mathbf{R}^n$ there corresponds naturally a convex centrally symmetric body in $\mathbf{R}^n$ with non-empty interior, namely its unit ball $\{x\in\mathbf{R}^n;\ N(x)\leqq 1\}$. Let $1\leqq p<\infty$. We say that the norm N on $\mathbf{R}^n$ is *p-cozonoidal*, if it can be represented in the form

$$(*)\qquad N(u)=\Big[\int_{S^{n-1}}|\langle u,x\rangle|^p\,d\mu(x)\Big]^{1/p}\qquad (u\in\mathbf{R}^n),$$

where μ is a suitable measure on S^{n-1}. We may and do assume that μ is symmetric, i.e. $\mu(A)=\mu(-A)$ for any Borel set $A\subset S^{n-1}$. A convex subset of $\mathbf{R}^n$ which is a unit ball of some p-cozonoidal norm is called *p-cozonoid*. The set of all p-cozonoids in $\mathbf{R}^n$ is denoted by $\mathscr{C}_n^p$. We have directly from the definition that if $K_1\in\mathscr{C}_n^p$ and K_1 is linearly equivalent to K_2, then $K_2\in\mathscr{C}_n^p$. If μ in $(*)$ is a discrete measure with Z_n point mass equal to $1/2$ at

$$(\pm 1,0,\dots,0),\ (0,\pm 1,0,\dots,0),\ \dots,\ (0,0,\dots,0,\pm 1)\in S^{n-1},$$

then the corresponding norm is an l^p-norm on $\mathbf{R}^n$, i.e. $B(l_n^p)\in\mathscr{C}_n^p$ ($B(l_n^p)$ denotes the unit ball of l_n^p). Note that for any arbitrary measure μ on S^{n-1}, $(*)$ defines a seminorm on $\mathbf{R}^n$ such that $N(u_0)=0$ if and only if $\operatorname{supp}\mu\subset u_0^\perp\cap S^{n-1}$. Thus throughout this paper we assume that there does not exist an $(n-1)$-dimensional plane P such that $\operatorname{supp}\mu\subset S^{n-1}\cap P$.

Let N be a norm on $\mathbf{R}^n$. Let $B(N)$ and $B(N^*)$ denote the unit balls of $(\mathbf{R}^n, N)$ and its dual $(\mathbf{R}^n, N)^*$, respectively. Then $B(N)$ is an n-dimensional zonoid if and only if $B(N^*)$ is an n-dimensional 1-cozonoid. Therefore $\mathscr{C}_2^1$ coincides with the set of all symmetric convex bodies with interior points in $\mathbf{R}^2$. For $n\geqq 3$ there exists an n-dimensional symmetric convex body which does not belong to $\mathscr{C}_n^1$.

3. Uniqueness of representation

C. Hardin ([7] Theorem 1.1a) proved that for $0<p\notin 2\mathbf{N}$ and μ,v positive measures on $\mathbf{R}^n$ the condition

$$\int_{\mathbf{R}^n}\Big|1+\sum_{i=1}^n\lambda_i z_i\Big|^p d\mu(z)=\int_{\mathbf{R}^n}\Big|1+\sum_{i=1}^n\lambda_i z_i\Big|^p dv(z)$$

for all $\lambda_1,\lambda_2,\dots,\lambda_n\in\mathbf{R}$ implies $\mu=v$. By a simple modification of the above considerations of Hardin we may obtain:

Let $0<p\in 2\mathbf{N}$ and μ, v be a symmetric positive measure on S^{n-1}. If

$$\int_{S^{n-1}}\Big|\sum_{i=1}^n\lambda_i x_i\Big|^p d\mu(x)=\int_{S^{n-1}}\Big|\sum_{i=1}^n\lambda_i x_i\Big|^p dv(x)$$

for all $\lambda_i\in\mathbf{R}$, then $\mu=v$.

This implication shows that in the representation $(*)$ of p-cozonoidal norms the measure μ is unique (up to a symmetry) for $1 \leqq p \notin 2\mathbf{N}$. For $p=2, 4, 6, \dots$, this is not true. Indeed, the Euclidean ball could be represented by infinitely many different measures. First of all if μ is the Lebesgue measure on S^{n-1} then we obtain the Euclidean norm. Now suppose that $p=2m$, $m\in\mathbf{N}$. Let μ_m be a discrete measure on S^1 with mass equal to 1 at $4m$ points: $\left(\cos\frac{k}{2m}, \sin\frac{k}{2m}\right)$, $k=1, 2, \dots, 4m$. We have

$$N(x, y) = \Big[\int_{S^1} |\langle (x, y), (z_1, z_2)\rangle|^p d\mu(z)\Big]^{1/p} = \left[2\sum_{k=1}^{2m}\left|x\cos\frac{k\pi}{2m}+y\sin\frac{k\pi}{2m}\right|^{2m}\right]^{1/2m}.$$

Let $\varphi\in[0, 2\pi]$ be such that

$$\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}.$$

Then $N\big((x, y)\big)=\sqrt{x^2+y^2}$ where

$$f(\varphi) = \left[2\sum_{k=1}^{2m}\cos^{2m}\left(\varphi-\frac{k\pi}{2m}\right)\right]^{1/2m}.$$

Since the derivative of $f(\varphi)$ is equal to 0, we obtain that $f(\varphi)$ is constant, so N is the Euclidean norm.

REMARK. W. Blaschke [2] (pp. 154—155) considered a wider class of convex bodies, which is called *generalized zonoids*. A centrally symmetric convex body whose support function is of the form $(*)$ with $p=1$, but a measure μ is a suitably signed Borel measure on S^{n-1}, is called a generalized zonoid. Blaschke showed that all sufficiently high differentiable convex bodies are generalized zonoids. Generalized zonoids are dense in the space of all symmetric convex bodies equipped with the Hausdorff metric. Schneider ([15], [16]) has also treated generalized zonoids. In particular, he presented generalized zonoids which are not zonoids and symmetric convex bodies which are not zonoids.

One may also consider generalized p-cozonoids. It would be interesting to determine such convex bodies. In particular there arises the question, whether the set of all generalized p-cozonoids is dense? From the corresponding result for generalized zonoids the answer is affirmative if $p=1$. Note that there exist signed measures such that $(*)$ defines a norm. For instance, let $p=4$, $n=2$ and μ be a measure supported on 10 points such that

$$\mu((\pm 1, 0)) = \frac{5}{8}, \quad \mu\left(\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)\right) = -\frac{1}{3}, \quad \mu\left(\left(\pm\frac{1}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}\right)\right) = \frac{25}{48}.$$

Indeed, we have

$$[N((x, y))]^4 = \int_{S^1} (xu+yv)^4 d\mu =$$

$$= \frac{5}{4}x^4 - \frac{1}{6}[(x+y)^4+(x-y)^4] + \frac{1}{24}[(x+2y)^4+(x-2y)^4] = x^4+y^4.$$

4. Plane sections of $L^p[0, 1]$ and l^p

THEOREM 1. *Let* $1 \leqq p < \infty$. *The set of all n-dimensional plane sections of the unit ball of* $L^p[0, 1]$ *coincides with the set* $\mathscr{C}_n^p$.

PROOF. Let $f_1, f_2, \ldots, f_n \in L^p[0, 1]$ be linearly independent. First we construct the subspace of L^p generated by $\{f_i\}_{i=1}^n$. Let $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$. Put $r(t) = \sqrt{\sum_{i=1}^n f_i^2(t)}$, and define a measurable function ψ: $[0, 1] \to S^{n-1}$ by

$$\psi(t) = \left(\frac{f_1(t)}{r(t)}, \frac{f_2(t)}{r(t)}, \ldots, \frac{f_n(t)}{r(t)}\right).$$

We have

$$[N((x_1, x_2, \ldots, x_n))]^p = \left\|\sum_{i=1}^n x_i f_i\right\|_p^p =$$

$$= \int_0^1 \left|\sum_i x_i f_i(t)\right|^p dt = \int_0^1 \left|\sum_i x_i \frac{f_i(t)}{r(t)}\right|^p r^p(t)\, dt =$$

$$= \int_0^1 \left|\left\langle (x_1, x_2, \ldots, x_n), \left(\frac{f_1(t)}{r(t)}, \frac{f_2(t)}{r(t)}, \ldots, \frac{f_n(t)}{r(t)}\right)\right\rangle\right|^p r^p(t)\, dt =$$

$$= \int_0^1 |\langle (x_i), \psi(t)\rangle|^p r^p(t)\, dt = \int_{S^{n-1}} |\langle (x_i), (y_i)\rangle|^p d\mu(y),$$

where μ is a measure on S^{n-1} such that $\mu(A) = \int_{\psi^{-1}(A)} r^p(t)\, dt$, $A \subset S^{n-1}$ $\left(\text{we assume that } \frac{0}{0} = 0\right)$. Note that if f_i are simple functions then μ is a discrete measure on S^{n-1}. Observe that μ is positive. The measure μ is also finite. Indeed,

$$\int_{S^{n-1}} d\mu(y) = \int_0^1 r^p(t)\, dt = \int_0^1 \left(\sum_{i=1}^n f_i^2(t)\right)^{p/2} dt \leqq$$

$$\leqq \int_0^1 \left[\left(\sum_{i=1}^n |f_i(t)|\right)^2\right]^{p/2} dt = \left\|\sum_{i=1}^n |f_i|\right\|_p^p \leqq \left(\sum_{i=1}^n \|f_i\|_p\right)^p < \infty.$$

Therefore the section of the unit ball of $L^p[0, 1]$ is a p-cozonoid.

Now suppose that $K \in \mathscr{C}_n^p$ i.e. the corresponding norm N_K has the form

$$N_K(x) = \left(\int_{S^{n-1}} |\langle u, x\rangle|^p d\mu(x)\right)^{1/p}.$$

Let $L^p(S^{n-1}, \mu)$ denote the Banach space of all classes of real measurable functions f such that

$$\int_{S^{n-1}} |f(u)|^p d\mu(u) < \infty$$

with the L^p-norm. The space $L^p(S^{n-1}, \mu)$ can be embedded isometrically in $L^p[0, 1]$ ([10], Theorem 14, § 15 p. 130). Let $\xi\colon L^p(S^{n-1}, \mu)\to L^p[0, 1]$ be a linear isometry. Define $g_i\in L^p(S^{n-1}, \mu)$, $i=1, 2, \dots, n$ by $g_i(u)=u_i$, $u=(u_1, u_2, \dots, u_n)\in S^{n-1}$. Put $f_i=\xi(g_i)$. We have

$$N_K(x_1, x_2, \dots, x_n) = \Big(\int\limits_{S^{n-1}} \Big|\sum_{i=1}^{n} x_i u_i\Big|^p d\mu(u)\Big)^{1/p} =$$

$$= \Big\|\sum_{i=1}^{n} x_i g_i\Big\|_{L^p(S^{n-1},\mu)} = \Big\|\sum_{i=1}^{n} x_i f_i\Big\|_p.$$

Therefore K is the unit ball of a subspace of $L^p[0, 1]$ generated by $\{f_i\}_{i=1}^n$, i.e. K is a plane section of the unit ball of $L^p[0, 1]$.

REMARK. J. Lindenstrauss proved in [13] that $L^1[0, 1]$ is a universal space for all two-dimensional Banach spaces of equivalently two-dimensional sections of the unit ball of $L^1[0, 1]$ give us (up to an affine equivalence) all two-dimensional symmetric convex bodies. This result is an immediate consequence of Theorem 1, because every norm on $\mathbf{R}^2$ is 1-cozonoidal. On the other hand, from Theorem 1 every 1-cozonoidal finite-dimensional Banach space is a subspace of $L^1[0, 1]$. Independently this result was obtained by Herz [8], who uses convex set theory language Note that for $n\geqq 3$ there exists an n-dimensional Banach space, which is not 1-cozonoidal, so $L^1[0, 1]$ is not universal for n-dimensional Banach spaces (cf. also [3]).

THEOREM 2. *Let $1\leqq p<\infty$. The set K is the unit ball of the n-dimensional subspace of l^p if and only if its corresponding norm N_K has the representation* $(*)$ *with a measure which is a countable sum of mass points.*

PROOF. The method we use is similar to that in the proof of Theorem 1. Let $(f_1^j), (f_2^j), \dots, (f_n^j)\in l^p$ be n linearly independent vectors and let $(x_1, x_2, \dots, x_n)\in\mathbf{R}^n$. We have

$$[N((x_1, x_2, \dots, x_n))]^p = \Big\|\sum_{i=1}^{n} x_i f_i\Big\|_p^p = \sum_{j=1}^{\infty}\Big|\sum_{i=1}^{n} x_i f_i^j\Big|^p = \int\limits_{S^{n-1}} |\langle x, u\rangle|^p d\mu(u),$$

where the measure μ is a sum of mass points $r_j^p\,\delta_{u_j}$,

$$\left(u_j = \left(\frac{f_1^j}{r_j}, \frac{f_2^j}{r_j}, \dots, \frac{f_n^j}{r_j}\right),\ r_j = \Big[\sum_{i=1}^{n} |f_i^j|^2\Big]^{1/2}\right).$$

The measure μ is finite because

$$\sum_j r_j^p = \sum_j \Big[\sum_i (f_i^j)^2\Big]^{p/2} \leqq \sum_j \Big[\sum_i |f_i^j|\Big]^p = \Big\|\sum_i |f_i|\Big\|_p^p \leqq \Big(\sum_i \|f_i\|_p\Big)^p < \infty.$$

Conversely, if we have a measure μ on S^{n-1} which is a countable sum of point masses $m_j\geqq 0$ at $u_j=(u_j^1, u_j^2, \dots, u_j^n)\in S^{n-1}$ and the corresponding p-cozonoid K, then by analogous arguments K is the unit ball of subspaces of l^p generated by the vectors $f_1, f_2, \dots, f_n$, where

$$(f_i = (f_i^j)_{j=1}^{\infty} = (u_j^i \sqrt[p]{m_j})_{j=1}^{\infty}, \quad i = 1, 2, \dots n.$$

We have

$$\sum_j |f_i^j|^p = \sum_j |u_j^i|^p m_j \leqq \sum_j m_j = \mu(S^{n-1}) < \infty$$

i.e. $f_i \in l^p$. The proof is complete.

Using analogous arguments to those in the proof of Theorem 2, the following result can be obtained.

PROPOSITION. *Let* $1 \leqq p < \infty$. *The set* K *is an* n*-dimensional plane section of the unit ball of* l_m^p *if and only if its corresponding norm* N_K *has representation* $(*)$ *with symmetric measure supported on at most* $2m$ *points.*

REMARK. Let us also mention the plane sections of l^∞. By the well-known result of S. Banach and S. Mazur [1], the space of continuous functions $C[0, 1]$ is universal for all separable Banach spaces and by the fact that $C[0, 1]$ is contained in l^∞, every symmetric convex body in $\mathbf{R}^n$ can be obtained as a plane section of $B(l^\infty)$. Plane sections of $B(l^\infty)$ might be considered directly using the method presented above. Let Y be a linear subspace of l^∞ generated by $(u_1^j), (u_2^j), \dots, (u_n^j)$. Then

$$Q = \operatorname{conv}\{\pm (u_1^j, u_2^j, \dots, u_n^j) \in \mathbf{R}^n, \quad j \in \mathbf{N}\}$$

is the unit ball of Y^* and $B(Y) = Q^*$ (Q^* is the dual convex body to Q). Conversely, let Q be an n-dimensional symmetric convex body. Denote by Q^* its dual convex body. Choose the sequence $v_j = (v_j^1, \dots, v_j^n) \in \mathbf{R}^n$, $j \in \mathbf{N}$ such that $Q^* = \overline{\operatorname{conv}}\,\{v_j : j \in \mathbf{N}\}$. Then

$$N_Q(x) = \sup_{Y \in Q} |\langle x, y\rangle| = \sup_j \Big|\sum_{i=1}^n x_i v_j^i\Big| = \Big\|\sum_{i=1}^n x_i u_i\Big\|_\infty$$

where $u_i = (v_1^i, v_2^i, \dots) \in l^\infty$, $i = 1, \dots, n$. Therefore Q is the unit ball of the subspace of l^∞ generated by the vectors $\{u_1, u_2, \dots, u_n\}$. In particular we have

$$\|\alpha_1 u_1 + \alpha_2 u_2\|_\infty = \sqrt{\alpha_1^2 + \alpha_2^2}$$

where $u_1 = (\cos 1, \cos 2, \dots)$, $u_2 = (\sin 1, \sin 2, \dots) \in l^\infty$. Thus we obtain the Euclidean ball as a plane section of $B(l^\infty)$.

Since the Lebesgue measure can be obtained as a weak limit of discrete measures, the space l^p $(1 \leqq p < \infty)$ is sufficiently Euclidean (cf. Dvoretzky's result [6]). Obviously $L^p[0, 1]$ contains a subspace isometrically isomorphic to l_n^2.

QUESTIONS. The set $\mathscr{C}_n^1$ forms a convex cone. Since $\mathscr{C}_n^2$ consists of all n-dimensional ellipses, it is also a convex cone. Are the sets $\mathscr{C}_n^p$ convex for other p? We have $\mathscr{C}_n^2 \subset \mathscr{C}_n^p$. Do we have the sequence of inclusions $\mathscr{C}_n^2 \subset \mathscr{C}_n^4 \subset \mathscr{C}_n^6 \subset \mathscr{C}_n^8 \subset \dots$? We have

$$B(l_n^p) \in \mathscr{C}_n^1 \quad \text{for} \quad n \geqq 3, \quad 1 \leqq p \leqq 2$$

(cf. [3], Theorem 6.6). For which p, r, n do we have $B(l_n^r) \in \mathscr{C}_n^p$?

References

[1] S. Banach and S. Mazur, Zur Theorie der linearen Dimension, *Studia Math.*, **4** (1933), 100—112.

[2] W. Blaschke, *Kreis und Kugel* (Leipzig, 1916).

[3] E. D. Bolker, A class of convex bodies, *Trans. Amer. Math. Soc.*, **145** (1969), 323—345.

[4] J. Bretagnolle, D. Dacunha Castelle, J. Krivine, Lois stables et espaces L^p, *Ann. Inst. H. Poincaré, N. S.*, **2** (1966), 231—259.

[5] G. Choquet, *Lectures on Analysis*, Vol. III. W. A. Benjamin (Reading, Mass, etc. 1969).

[6] A. Dvoretzky, Some results on convex bodies and Banach spaces, *Proc. Symp. on Linear Spaces* (Jerusalem, 1961), 123—174.

[7] C. D. Hardin, Jr., Isometries on subspaces of L^p, *Indiana Univ. Math. J.*, **30** (1981), 449—463.

[8] C. S. Herz, A class of negative definite functions, *Proc. Amer. Math. Soc.*, **14** (1963), 670—676.

[9] V. Klee, Polyhedral sections of convex bodies, *Acta Math.*, **103** (1960), 243—267.

[10] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag (New York, 1974).

[11] P. Lévy, *Théorie de láddition des variables eléatories*, Gauthier-Villars (Paris, 1937).

[12] N. F. Lindquist, Support functions of central convex bodies, *Portugaliae Math.*, **34** (1975), 241—252.

[13] J. Lindenstrauss, On the extension of operators with finite dimensional range, *Illinois J. Math.*, **8** (1964), 488—499.

[14] A. Neyman, Representation of L_p-norms and isometric embedding in L_p-spaces, *Israel J. Math.*, **48** (1984), 129—138.

[15] R. Schneider, Zu einem Problem von Shephard über die Projektionen konvexer Körper, *Math. Z.*, **101** (1967), 71—82.

[16] R. Schneider, Über eine Integralgleichung in der Theorie der konvexen Körper, *Math. Nachr.*, **44** (1970), 55—75.

[17] R. Schneider and W. Weil, Zonoids and Related Topics, *Convexity and Its Applications*. Eds. P. Gruber and J. M. Wills, Birkhäuser (Basel et al., 1983).

[18] W. Weil, Zonoide und verwandte Klassen konvexer Körper, *Monatsh. Math.*, **94** (1982), 73—84.

[19] W. Weil, Kontinuierliche Linearkombination von Strecken, *Math. Z.*, **148** (1976), 71—84.

[20] W. Weil, Centrally symmetric convex bodies and distributions II, *Israel J. Math.*, **32** (1979), 173—182.

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References

Acta Math. Hung.
52 (3—4) (1988), 227—232.

ON VECTOR LATTICES OF CONTINUOUS FUNCTIONS IN LOCALLY COMPACT SPACES

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I. The first theorem of Gelfand—Kolmogorov

Definitions and notations. Let T be a Tihonov space,
$C(T) = \{f | f\colon T \to R$ and f is a continuous function$\}$,
$C^*(T) = \{f | f \in C(T)$ and f is a bounded function$\}$,
$C_c(T) = \{f | f \in C(T)$ and f has a compact support$\}$.
If $f \in C(T)$ then $Z(f) = \{t \in T | f(t) = 0\}$, $Z^c(f) = T \setminus Z(f)$. $X(T)$ is a vector lattice iff $X(T) \subset C(T)$, $f, g \in X(T)$ and $L \in R$ imply $f \vee g \in X(T)$, $f \wedge g \in X(T)$, $f + g \in X(T)$, $\alpha f \in X(T)$.

Let T_1, T_2 be Tihonov spaces, $X(T_1)$, $Y(T_2)$ two vector lattices. $\varphi\colon X(T_1) \to Y(T_2)$ is said to be a linear lattice homomorphism iff φ is a lattice homomorphism and $x_1, x_2 \in X(T)$ implies $\varphi(x_1 - x_2) = \varphi(x_1) - \varphi(x_2)$.

$X(T)$ is said to be a completely regular vector lattice over T iff, whenever F is a closed set and t is a point in its complement, there exists a function $f \in X(T)$ such that $0 \leqq f \leqq 1$, $f(F) = \{0\}$ and $f(U_t) = \{1\}$, where U_t is a neighbourhood of t.

We shall omit the easy proof of the following properties:

1. $C_c(X)$, $C^*(X)$, $C(X)$ are vector lattices, $C_c(X) \subset C^*(X) \subset C(X)$.

2. Let $\varphi\colon X(T) \to R$ be a linear lattice homomorphism, i.e. let φ be a lattice homomorphism and $\varphi(x_1 - x_2) = \varphi(x_1) - \varphi(x_2)$ for $x_1, x_2 \in X(T)$,

a) If $f, g \in \operatorname{Ker} \varphi$, then $f - g \in \operatorname{Ker} \varphi$.

b) If $f \in \operatorname{Ker} \varphi$, $g \in X(T)$, $|g| \leqq n|f|$, then $g \in \operatorname{Ker} \varphi$.

PROPOSITION 1. *If T is a locally compact T_2-space then $C_c(T)$ is a completely regular vector lattice.*

Let F be a closed set and $t \notin F$. Because T is a locally compact T_2-space, T is completely regular. There exists $a_t \in C(T)$ such that $a_t(F) = \{0\}$, $a_t(t) = 1$, then $a_t' = (0 \vee 2a_t) \wedge 1 \in C(T)$, $0 \leqq a_t' \leqq 1$, $a_t'(F) = \{0\}$, $a_t'(U_t) = \{1\}$ where U_t is a neighbourhood of t and $U_t \subset T \setminus F$. Since T is locally compact T_2, there exists U_t' such that $U_t' \subset U_t$ and U_t' is a compact neighbourhood of t. By an argument analogous to that used above we have $e_t \in C(T)$, $e_t(T \setminus U_t') = \{0\}$, $e_t(U_t'') = \{1\}$, $U_t'' \subset T \setminus F$, U_t'' is a neighbourhood of t, $Z^c(e_t) \subset U_t'$, $0 \leqq e_t \leqq 1$. Then $e_t \in C_c(T)$. The proof is complete.

LEMMA 1. *Let $X(T)$ be a completely regular vector lattice over T, $\varphi\colon X(T) \to R$ a linear lattice homomorphism,* $\operatorname{Ker} \varphi = \{f \in X(T) | f(t_0) = 0\}$, *$t_0 \in T$. Then $\varphi(f) = \chi f(t_0)$ where χ is a positive constant.*

PROOF. There exists $x \in X(T)$ such that $x \geqq 0$, $x(t_0) = 1$. Let $\varphi(x) = \chi^*$ $(\chi^* > 0)$. We shall prove that $\varphi(f) = \chi^* f(t_0)$. First we prove that

$$\varphi(\chi f) = \chi \varphi(f) \quad \text{if} \quad \chi \in R, \quad f \in X(T). \tag{1}$$

To obtain (1), it is sufficient to verify it for $f \geqq 0$, $\chi \geqq 0$. If $f(t_0)=0$, $f \in \operatorname{Ker} \varphi$, then $\chi f \in \operatorname{Ker} \varphi$ and (1) is true. If $f(t_0) \neq 0$, then $\varphi(f) \neq 0$. It is obvious that (1) is true for every rational χ. If χ is an irrational number, for $n \in N$ there exists $m \in N$ such that

$$0 \leqq \frac{m}{n} \leqq \chi \leqq \frac{m+1}{n}, \quad \frac{m}{n} f \leqq \chi f \leqq \frac{m+1}{n} f, \quad \varphi\left(\frac{m}{n} f\right) \leqq \varphi(\chi f) \leqq \varphi\left(\frac{m+1}{n} f\right).$$

It follows that

$$\left|\frac{\varphi(\chi f)}{\varphi(f)} - \chi\right| \leqq \frac{1}{n} \quad \text{for every } n \in N. \tag{2}$$

From (2) we get (1) for every χ. Let $g=f-f(t_0)x$, then $g \in X(T)$, $g(t_0)=0$. We have

$$0 = \varphi(g) = \varphi(f-f(t_0)x), \quad 0 = \varphi(f)-f(t_0)\varphi(x), \quad \varphi(f) = \chi^* f(t_0).$$

The next proposition will play a crucial role in the following development.

PROPOSITION 2. *Let T be a T_2-space, $X(T)$ a completely regular lattice over T, $\varphi\colon X(T) \to C(Y)$ (Y is a topological space) a linear lattice homomorphism and suppose that, for every $y \in Y$, there exists a function $f \in X(T) \cap C_c(T)$ such that $\varphi(f)(y) \neq 0$. Then there exist continuous functions $k\colon Y \to T$ and $\sigma\colon Y \to R$, such that $\varphi(f)(y) = (f \circ k)(y) \cdot \sigma(y)$, σ is a positive continuous function. Conversely, a φ of this form defines a linear lattice homomorphism.*

PROOF. It is a simple matter to prove that if $k\colon Y \to T$ and $\sigma\colon Y \to R$ are continuous functions, $\sigma > 0$, $\varphi(f) = (f \circ k) \cdot \sigma$ *then* $\varphi\colon X(T) \to C(Y)$ is a linear lattice homomorphism. Now let $y_0 \in Y$. Define a function $\psi\colon X(T) \to R$ by $\psi(f) = \varphi(f)(y_0)$. It is obvious that ψ is a linear lattice homomorphism. First we prove that $\bigcap_{g \in \operatorname{Ker} \psi} Z(g) \neq \emptyset$. From the condition there is $f^* \in X(T) \cap C_c(T)$ with $f^* \notin \operatorname{Ker} \psi$. If $g \in \operatorname{Ker} \psi$ then $Z(g) \cap \overline{Z^c(f^*)} \neq \emptyset$. Assume in fact $Z(g) \cap \overline{Z^c(f^*)} = \emptyset$. Then $|g| \neq 0$ in $\overline{Z^c(f^*)}$, hence

$$0 < \varepsilon = \inf_{t \in \overline{Z^c(f^*)}} \{|g|(t)\}$$

(because $\overline{Z^c(f^*)}$ is a compact set). $f^* \in C_c(T)$, therefore f^* is a bounded function, i.e. $|f^*| \leqq m$. Then $\frac{m}{\varepsilon}|g| \geqq |f^*| \Rightarrow f^* \in \operatorname{Ker} \psi$. We thus arrive at a contradiction. Observe, further, that $Z(g_1) \cap Z(g_2) = Z(|g_1| \vee |g_2|)$ and $\overline{Z^c(f^*)}$ is a compact set, so we have $\bigcap_{g \in \operatorname{Ker} \psi} Z(g) \neq \emptyset$. Let $t_0 \in \bigcap_{g \in \operatorname{Ker} \psi} Z(g)$. We shall prove that if $h \in X(T)$ and $h(t_0)=0$ then $h \in \operatorname{Ker} \psi$. Assume on the contrary that $\psi(h) \neq 0$, i.e. $\psi(h) = \varepsilon_1$. Without loss of generality we may assume that $\varepsilon_1 > 0$. Since $X(T)$ is a completely regular vector lattice there exists $x \in X(T)$ such that $x(U_{t_0}) = \{1\}$, U_{t_0} is a neighbourhood of t_0. So $x \notin \operatorname{Ker} \psi$, hence $\psi(|x|) = a > 0$. Since $h(t_0)=0$, there exists for $\varepsilon_2 > 0$ a neighbourhood $V_{t_0} \subset U_{t_0}$ of t_0 such that $|h|(t) < \varepsilon_2$ if $t \in V_{t_0}$. Then there is $y \in X(T)$ such that $y(t_0)=1$, $y(T \setminus V_{t_0}) = \{0\}$. Then there exists $n \in N$ such that $\psi(n|y|) > \varepsilon_1$ (because $\psi(|y|) > 0$). Let $h^* = (n|y| \wedge |h|)$. It is obvious that

$h^*(t)=0$ if $t\in T\setminus V_{t_0}$, $h^*(t)<\varepsilon_2$ if $t\in V_{t_0}$ and $\psi(h^*)=\varepsilon_1$. If $n^*>\frac{a}{\varepsilon_1}$ and $\varepsilon_2<\frac{1}{n^*}$, we have $h^{**}=n^*h^*$, $h^{**}\leqq|x|$, while $\psi(h^{**})=n^*\psi(h^*)>\psi(|x|)$. We thus arrive at a contradiction.

Now we turn to the proof of the proposition. Since $\operatorname{Ker}\psi=\{f\in X(T)|\,f(t_0)=0\}$ it follows that $\psi(f)=\chi f(t_0)$, $\chi>0$. Let $k\colon Y\to T$, $k(y_0)=t_0$. We must prove that $k\colon Y\to T$ is a continuous function. $X(T)$ is a completely regular vector lattice, hence $Z^c(X(T))$ is a base of the topology of T (if $t\in U_t$, U_t is open, there exists $x\in X(T)$ such that $x(T\setminus U_t)=\{0\}$ and $x(t)=1$, so $t\in Z^c(x)\subset U_t$), and

$$k^{-1}(Z(f)) = \{y|(f\circ k)(y) = 0\} = \{y|\varphi(f)(y) = 0\} = Z(\varphi(f)).$$

Hence k is a continuous function. To complete the proof it remains to show that σ is a continuous function. We have $\varphi(f)(y)=(f\circ k)(y)\cdot\sigma(y)$. Let $y^*\in Y$, $k(y^*)\in T$. Since $X(T)$ is a completely regular vector lattice, there exists $f^*\in X(T)$ such that $f^*(U_{k(y^*)})=\{1\}$, $U_{k(y^*)}$ is a neighbourhood of $k(y^*)$. Because k is a continuous function, we have $V_{y^*}\subset Y$, V_{y^*} is a neighbourhood of y^* such that $k(y)\in U_{k(y^*)}$ if $y\in V_{y^*}$. So $y\in V_{y^*}$ implies $\varphi(f^*)=\sigma(y)$, hence σ is a continuous function at y^*.

THEOREM 1. *Let X, Y be Tihonov spaces, $V_1(X)$, $V_2(Y)$ completely regular vector lattices over X, Y respectively and $\varphi\colon V_1(X)\to V_2(Y)$ a linear lattice isomorphism such that*

a) *for every $x\in X$ there exists $f\in V_2(Y)\cap C_c(Y)$ satisfying $\varphi^{-1}(f)(x)\neq 0$,*

b) *for every $y\in Y$ there exists $g\in V_1(X)\cap C_c(X)$ with $\varphi(g)(y)\neq 0$.*

Then X, Y are homeomorphic.

PROOF. Taking Proposition 2 into account, we get that there exist continuous functions $k\colon X\to Y$, $h\colon Y\to X$ such that, for every $f\in V_1(X)$, $\varphi(f)=(f\circ h)\cdot\sigma_1$ ($\sigma_1>0$), and, for every $g\in V_2(Y)$, $\varphi^{-1}(g)=(g\circ k)\cdot\sigma_2$ ($\sigma_2>0$).

We shall prove that $k\circ h=\mathrm{id}_Y$, $h\circ k=\mathrm{id}_X$. It is enough to show $h\circ k=\mathrm{id}_X$. Assume in the contrary that $h\circ k\neq\mathrm{id}_X$, i.e. there is $x\in X$ such that $(h\circ k)(x)=x^*$, where $x^*\neq x$. Since $V_1(X)$ is a completely regular vector lattice there exists f, f^* such that $f, f^*\in V_1(X)$, $f(x)=0$, $f(x^*)\neq 0$, $f^*(x)\neq 0$. We have

$$f(x) = \varphi^{-1}(\varphi(f))(x) = \varphi^{-1}((f\circ h)\sigma_1)(x) = [((f\circ h)\sigma_1)(k(x))]\,\sigma_2(x) =$$

$$= (f\circ(h\circ k))(x)\,\sigma_1(k(x))\cdot\sigma_2(x).$$

By an argument analogous to the previous one we get

$$f^*(x) = (f^*\circ(h\circ k))(x)\,\sigma_1(k(x))\cdot\sigma_2(x).$$

So

$$f(x)\cdot(f^*\circ(h\circ k))(x)\cdot\sigma_1(k(x))\cdot\sigma_2(x) = f^*(x)\cdot(f\circ(h\circ k))(x)\cdot\sigma_1(k(x))\cdot\sigma_2(x).$$

But $f(x)=0$ while $f^*(x)\neq 0$, $(f\circ(h\circ k))(x)=f(x^*)\neq 0$, $\sigma_1(k(x))\neq 0$, $\sigma_2(x)\neq 0$, we thus arrive at a contradiction. So we have $h\circ k=\mathrm{id}_X$, $k\circ h=\mathrm{id}_Y$, i.e. X, Y are homeomorphic.

COROLLARY. *Let X, Y be locally compact T_2-spaces. X, Y are homeomorphic iff $C_c(X)$, $C_c(Y)$ are linearly lattice isomorphic.*

REMARK. "If X, Y are locally compact T_2 and $C(X)$, $C(Y)$ are isomorphic then X, Y are homeomorphic" is false [2, Chapter V].

DEFINITIONS. $X \subset C(T)$ is said to be a *quasi-affine lattice* iff it is a lattice, contains all constants and $c \in \Gamma_f$, $f \in X$ imply $cf \in X$, where $\Gamma_f \subset R$ is a set of real numbers containing 0, -1 and unbounded from above. A lattice homomorphism $\varphi\colon X \to R$ $(C(Y))$ is said to be *quasi-affine* iff $\varphi(\chi) = \chi$ if χ is a constant (function) and $\varphi(cf) = c \cdot \varphi(f)$ for $c \in \Gamma_f$. $X(T)$ is said to be *strongly completely regular over* T iff, whenever F is closed, $x \notin F$, there exist $f, g \in X(T)$ such that $f(F) = \{0\}$, $f(x) = 1$ and $g(F) = \{1\}$, $g(x) = 0$. If $X(T)$ is a completely regular vector lattice and contains all constants, $X(T)$ is a strongly completely regular quasi-affine lattice.

LEMMA 2. *Let $X(T)$ be a strongly completely regular quasi-affine lattice and $\varphi\colon X(T) \to R$ a quasi-affine lattice homomorphism where*

$$\operatorname{Ker} \varphi = \{f \in X(T) \mid f(t_0) = 0\}, \quad t_0 \in T.$$

Then $\varphi(f) = f(t_0)$.

PROOF. Assume the contrary, i.e. $\varphi(f) \neq f(t_0)$. Let $\varphi(f) = a$, $f(t_0) = b$. If $a > b$ and $a > 0$, since f is continuous, there exists a V_{t_0}, where V_{t_0} is an open neighbourhood of t_0 and $f(V_{t_0}) \subset (-\infty, c)$ $(c > 0,\ a > c)$. Since $X(T)$ is strongly completely regular, there exists $x \in X(T)$ such that $x(T \setminus V_{t_0}) = \{0\}$, $x(t_0) = 1$. Hence $x \notin \operatorname{Ker} \varphi$, $\varphi(|x|) > 0$. So there exists $n_i \in \Gamma_{|x|}$ such that $\varphi(n_i|x|) = n_i \varphi(|x|) > a$. Let $g = (n_i|x| \wedge f)$, then $\varphi(g) = a$ and $g(t) \leqq 0$ if $t \in T \setminus V_{t_0}$, $g(t) \leqq c$ if $t \in V_{t_0}$. So $a = \varphi(g) \leqq \varphi(c) = c$. We thus arrive at a contradiction.

If $a > b$ and $a \leqq 0$, since f is a continuous function, there exists a V_{t_0}, where V_{t_0} is an open neighborhood of t_0 and $f(V_{t_0}) \subset (-\infty, c)$ $(c < a)$. Since $X(T)$ is strongly completely regular, there exists $x \in X(T)$ such that $x(T \setminus V_{t_0}) = \{1\}$, $x(t_0) = 0$. So there exists $n_i \in \Gamma_x$ such that $-n_i < c$. Let $g = ((-n_i x) \wedge f)$; then $\varphi(g) = a$ and $g(t) \leqq c$ if $t \in V_{t_0}$, $g(t) \leqq -n_i$ if $t \in T \setminus V_{t_0}$. So $a = \varphi(g) \leqq \varphi(c) < a$. We thus arrive at a contradiction.

If $a < b$ then $\varphi(-f) = -a$, $-f(t_0) = -b$ and $-a > -b$. So the above argument applies. Now by an argument analogous to the proofs of Proposition 2, Theorem 1, we have

1. If $X(T)$ is a strongly completely regular quasi-affine lattice over a T_2-space, Y is a topological space, $\varphi\colon X(T) \to C(Y)$ is a quasi-affine lattice homomorphism and, for $y \in Y$, there exists $f \in X(T) \cap C_c(T)$ such that $\varphi(f)(y) \neq 0$, then there exists a continuous function $k\colon Y \to T$ such that $\varphi(f) = f \circ k$.
2. Let X, Y be locally compact T_2. There exists a quasi-affine lattice isomorphism $\varphi\colon C(X) \to C(Y)$ with $\varphi(C_c(X)) = C_c(Y)$ iff X, Y are homeomorphic.

II. Stone—Weierstrass' theorem in a locally compact space

The following theorem is a generalization of Stone—Weierstrass' theorem in a compact space (see [1]).

$(*)$ *Let E be a compact topological space and Φ either a semi-affine lattice (i.e.*

1. $f, g\in\Phi \Rightarrow f\vee g,\ f\wedge g\in\Phi$,

2. $f\in\Phi,\ \alpha\in R \Rightarrow f+\alpha\in\Phi$,

3. $f\in\Phi,\ \gamma\in\Gamma \Rightarrow \gamma f\in\Phi$

where Γ is a set of real numbers containing 0 *and unbounded both from above and from below) or a subtractive lattice (i.e. it satisfies* 1 *and $f, g\in\Phi \Rightarrow f-g\in\Phi$) composed of continuous functions, containing all constants and separating the points of E. Then every continuous function on E is the limit of a uniformly convergent family taken from Φ.*

But in a locally compact, non-compact space X, $C_c(X)$ (the family of all continuous functions which have a compact support) cannot contain constants, so the following theorem is not a consequence of $(*)$ by applying it to the one-point compactification of X.

THEOREM 2. *If X is a locally compact T_2-space and Φ is a completely regular vector lattice over X, then, for every $\varepsilon>0$ and for every $f\in C_c(X)$, there exists $g\in\Phi$ such that $|g(x)-f(x)|<\varepsilon$ if $x\in X$.*

PROOF. We shall begin by showing that

a) For $x_1, x_2\in X$, $x_1\neq x_2$ and $c_1, c_2\in R$ there exists $f\in\Phi$ such that $f(x_1)=c_1$, $f(x_2)=c_2$. Φ is a completely regular vector lattice, so we have $u, v\in\Phi$ such that $u(x_1)=v(x_2)=0$, $u(x_2)\neq 0$, $v(x_1)\neq 0$.

Define

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}.$$

b) If $f\in C_c(X)$, $x\in\overline{Z^c(f)}$, $\varepsilon>0$, then there exists $g_x\in\Phi$ such that $g_x(x)=f(x)$ and $g_x(t)>f(t)-\varepsilon$ $(t\in\overline{Z^c(f)})$. From a) for every $y\in X$ we get $h_y\in\Phi$ such that $h_y(x)=f(x)$, $h_y(y)=f(y)$. h_y is a continuous function, so there exists an open neighbourhood I_y of y such that $h_y(t)>f(t)-\varepsilon$ for $t\in I_y$. Since $\overline{Z^c(f)}$ is a compact set, we get $y_1, y_2, \dots, y_m$ such that $\overline{Z^c(f)}\subset I_{y_1}\cup\dots\cup I_{y_m}$. Let

$$g_x = \max(h_{y_1}, \dots, h_{y_m})\in\Phi.$$

c) For $f\in C_c(X)$, $\varepsilon>0$, there exists $h\in\Phi$ such that $|h(x)-f(x)|<\varepsilon$ $(x\in\overline{Z^c(f)})$. From b) for every $x\in\overline{Z^c(f)}$, there exists $g_x\in\Phi$ such that $g_x(x)=f(x)$ and $g_x(t)>f(t)-\varepsilon$ $(t\in\overline{Z^c(f)})$. Since g_x is a continuous function, there exists an open set V_x such that $x\in V_x$, $g_x(t)<f(t)+\varepsilon$ $(t\in V_x)$. $\overline{Z^c(f)}$ is a compact set, hence it follows that there exist $x_1, \dots, x_m$ such that $\overline{Z^c(f)}\subset V_{x_1}\cup\dots\cup V_{x_m}$. Let $h=\min(g_{x_1}, \dots, g_{x_m})$.

Now we turn to the proof of the theorem. Let $f\geqq 0$, $f\in C_c(X)$. From c) there exists $h\in\Phi$ such that $|h(x)-f(x)|<\varepsilon$ $(x\in\overline{Z^c(f)})$. f, h are continuous functions, so $U=\{x\colon |h(x)-f(x)|<\varepsilon\}$ is an open set and $U\supset\overline{Z^c(f)}$. Since Φ is a completely regular vector lattice, for every $x\in\overline{Z^c(f)}$, there exists $a_x\in\Phi$ such that $0\leqq a_x\leqq 1$, $a_x(X\setminus U)=\{0\}$, $a_x(U_x)=\{1\}$, where U_x is an open neighbourhood of x. $U\supset$

$\subset \bigcup_{x\in\overline{Z^c(f)}} U_x \supset \overline{Z^c(f)}$, so that there exist $x_1, \ldots, x_m$ such that $U \supset \bigcup_{i=1}^{m} U_{x_i} \supset \overline{Z^c(f)}$. Let $a = \max(a_{x_1}, \ldots, a_{x_m})$. Then $a(\overline{Z^c(f)}) = \{1\}$ and $a(X \setminus U) = \{0\}$.

There exists $m \in R$ such that $|h| \leqq m$ on $\overline{Z^c(f)}$. Let $f^* = (0 \vee h) \wedge |a|m$. Then $f^* \in \Phi$; define $l(x) = |f^*(x) - f(x)|$, so that $x \in \overline{Z^c(f)}$ implies $l(x) \leqq \varepsilon$, $x \in U \setminus \overline{Z^c(f)}$ implies $l(x) \leqq \varepsilon$, $x \in X \setminus U$ implies $l(x) = 0$. Now let $f \in C_c(X)$ be arbitrary, $f_1 = f \vee 0$, $f_2 = f \wedge 0$, $f = f_1 + f_2$. For f_1 there exists $h_1 \geqq 0$, $h_1 \in \Phi$ such that $|h_1 - f_1| < \frac{\varepsilon}{2}$, for f_2 there exists $h_2 \geqq 0$, $h_2 \in \Phi$ such that $|h_2 + f_2| < \frac{\varepsilon}{2}$. Let $h = h_1 - h_2$. Then $h \in \Phi$, $|h - f| \leqq |h_1 - f_1| + |h_2 + f_2| < \varepsilon$. So for every $f \in C_c(X)$, $\varepsilon > 0$, there exists $h \in \Phi$ such that $|h(x) - f(x)| < \varepsilon$ $(x \in X)$.

COROLLARY. *Let Φ be a complete algebra over a locally compact T_2-space X and let $\Phi \cap C^*(X)$ be completely regular. Then $\Phi \supset C_c(X)$.*

PROOF. It is obvious that $\Phi \cap C^*(X)$ is a lattice. So $\Phi \cap C^*(X)$ is a vector lattice. Applying the above theorem we have $\Phi \supset C_c(X)$.

Finally, I would like to express my sincere thanks to Prof. Ákos Császár for his help and encouragement.

References

[1] Á. Császár, On approximation theorems for uniform spaces, *Acta Math. Acad. Sci. Hung.*, **22** (1971), 177—186.

[2] L. Gillman and M. Jerison, *Rings of continuous functions* (Princeton—Toronto—London—New York, 1960).

[3] W. A. Luxemburg and A. C. Zaanen, *Riesz spaces*, North-Holland (Amsterdam, 1971).

[4] W. Rudin, *Principles of mathematical analysis* (Hungarian translation: Műszaki Kiadó, 1978).

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FUNCTIONAL CENTRAL LIMIT THEOREM AND LOG LOG LAW FOR MULTIPLICATIVE SYSTEMS

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§ 1. Introduction. Let S_n be a partial sum of real valued random variables $X_1, X_2, \ldots$, and let $Y_n(t)$ be the random function on $[0, 1]$ obtained by linearly interpolating $S_k/\sqrt{n}$ at $t=k/n$, where $k=0, 1, 2, \ldots, n$.

It is well known that for i.i.d. sequences having mean 0 and variance 1 the functional central limit theorem (Donsker's theorem) and the functional log log law (Strassen's theorem) hold. In this paper we have the analogous theorems for a multiplicative system of random variables in the following sense:

DEFINITION (Alexits [1]). A sequence of real valued random variables $X_1, X_2, \ldots$ is called a uniformly bounded multiplicative system if and only if

(i) there exists $K>0$, such that for all n $|X|_n \leqq K$ a.s. and

(ii) for any $r=1, 2, \ldots$ and $1 \leqq n_1 < n_2 < \ldots < n_r$

$$E[X_{n_1} \ldots X_{n_r}] = 0.$$

2. Theorems. Now we can state our theorems.

THEOREM 1. *Let $\{X_n\}$ be a uniformly bounded multiplicative system satisfying the following additional condition:*

($*$) *There exists a non-negative sequence $a_0, a_1, \ldots$ such that*

(i) $E[(X_n^2-1)(X_m^2-1)] \leqq a_{|n-m|}$

holds for all n and m, and

(ii) $\sum_{k=0}^{\infty} a_k < +\infty.$

Then the distributions $\{Y_n(t); 0 \leqq t \leqq 1\}_{n=1}^{\infty}$ converge weakly on $C[0, 1]$ to the Wiener measure.

THEOREM 2. *Under the same conditions as Theorem* 1, $\{Y_n(t)/\sqrt{2 \log \log n}\}$ *is relatively compact in $C[0, 1]$ a.s.*

THEOREM 3. *Under the same conditions as Theorem* 1,

$$P(\{\text{The cluster set of } \{Y_n(t)/\sqrt{2 \log \log n}\} \text{ in } C[0, 1] \subset K_1) = 1,$$

where $K_1 = \left\{x \in C[0, 1];\ x(0)=0 \text{ and } \int_0^1 \left(\frac{dx}{dt}\right)^2 dt \leqq 1\right\}.$

REMARK. Under the conditions of Theorem 1, Takahashi [10] has proved

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2\log\log n}} \leqq 1 \quad \text{a.s.}$$

Here, under somewhat stronger condition, the equality sign also holds (cf. Révész [7]) which suggests that the corresponding extension of Theorem 3 is also true.

3. Proof of the theorems. The proofs of our theorems are rather easy by using known lemmas which will be restated here for the later use.

LEMMA 1 [8, Theorem 3]. *Let $\{X_n\}$ be a uniformly bounded (by a constant K) multiplicative system. Then*

$$P(|S_n| \geqq x) \leqq 2\exp\left(-\frac{x^2}{2K^2 n}\right)$$

holds for any $x \geqq 0$ and $n=1, 2, \dots$.

LEMMA 2 ([9, Theorem 3.7.2]). *Let $Z_1, Z_2, \dots$, be real valued random variables. If there exists a non-negative sequence $a_0, a_1, \dots$, such that*

(i) $E[Z_n Z_m] \leqq a_{|n-m|}$

holds for all n and m and

(ii) $\sum_{k=0}^{\infty} a_k < +\infty$,

then $(Z_1+\dots+Z_n)/n$ converges to 0 a.s. (The strong law of the large number holds.)

Now we start proving Theorem 1. First we will show the tightness of $\{Y_n\}$ To do so, it is sufficient to check the inequality

$$E[|Y_n(t)-Y_n(s)|^4] \leqq c_1|t-s|^2, \tag{1}$$

where the constant c_1 does not depend on t, s and n. When $k/n \leqq s \leqq (k+1)/n \leqq$ $\leqq k'/n \leqq t \leqq (k'+1)/n$ holds, it follows that

$$\sqrt{n}|Y_n(s)-Y_n(t)| \leqq \max(|S_{k'}-S_k|, |S_{k'}-S_{k+1}|, |S_{k'+1}-S_k|, |S_{k'+1}-S_{k+1}|).$$

By taking account of Lemma 1, we have

$$P\left(|Y_n(t)-Y_n(s)| \geqq \sqrt{|t-s|}\,x\right) \leqq c_2\exp(-c_3x^2), \tag{2}$$

where the positive constants c_2 and c_3 do not depend on n, s, t and x. This gives us our required inequality (1). Next we show that any finite dimensional distribution of $Y_n(t)$ converges to that of the Wiener measure. Fix any $0=t_0<t_1<\dots<t_k \leqq 1$. Using an asymptotic formula

$$e^z = (1+z)e^{z^2/2+o(|z|^2)} \quad (|z| \to 0),$$

we have

$$(3)\qquad \exp\{i\sum_{j=1}^{k} u_j(Y_n(t_j)-Y_n(t_{j-1}))\} = \exp O(k/\sqrt{n}) \prod_{j=1}^{k} \prod_{nt_{j-1}<p\leqq nt_j} e^{iu_jX_p/\sqrt{n}} =$$

$$= \exp O(k/\sqrt{n}) \prod_{j=1}^{k} \prod_{nt_{j-1}<p\leqq nt_j} \left[(1+iu_jX_p/\sqrt{n})\exp\left(-\frac{u_j^2X_p^2}{2n}+o(u_j^2K^2/n)\right)\right] =$$

$$= \exp\left(O(k/\sqrt{n})+o(\max|u_j|K^2t_k)\right) \prod_{j=1}^{k} \prod_{nt_{j-1}<p\leqq nt_j} (1+iu_jX_p/\sqrt{n})\exp\left(-\frac{u_j^2X_p^2}{2n}\right).$$

Setting

$$F_j^{(n)} = \prod_{nt_{j-1}<p\leqq nt_j} \left\{(1+iu_jX_p/\sqrt{n})\exp\left(-\frac{u_j^2X_p^2}{2n}\right)\right\}$$

and

$$G_j^{(n)} = e^{-u_j^2(t_j-t_{j-1})/2} \prod_{nt_{j-1}<p\leqq nt_j} (1+iu_jX_p/\sqrt{n}),$$

we have

$$(4)\qquad |F_j^{(n)}|+|G_j^{(n)}| \leqq 2 \prod_{nt_{j-1}<p\leqq nt_j} \left|(1+iu_jX_p/\sqrt{n})\right| \leqq 2e^{\max\{u_j^2\}K^2t_k/2}.$$

Since by Lemma 2 we have

$$\lim_{n\to\infty} \sum_{nt_{j-1}<p\leqq nt_j} X_p^2/n = t_j-t_{j-1} \quad \text{a.s.},$$

t follows that

$$(5)\qquad \lim_{n\to\infty} |F_j^{(n)}-G_j^{(n)}| = 0 \quad \text{a.s.}$$

Combining (3), (4) and (5) with the multiplicativity we have

$$\lim_{n\to\infty} E[\exp\{i\sum_{j=1}^{k} u_j(Y_n(t_j)-Y_n(t_{j-1}))\}] = \prod_{j=1}^{k} e^{-u_j^2(t_j-t_{j-1})/2},$$

which completes the proof of the theorem.

To prove Theorem 2, we need the following lemma which is a modification of Fernique's inequality [3]. We can prove Lemma 3 by the same way as that of Lemma 10 [4].

LEMMA 3. *Let* $\{X(t);\ 0\leqq t\leqq 1\}$ *be a real valued stochastic process having continuous sample path which satisfies the following conditions:*

(i) *There exist two positive constants* c_4 *and* c_5 *such that*

$$P(|X(t)-X(s)| \geqq \sqrt{|t-s|}\,x) \leqq c_4 \exp(-c_5x^2)$$

holds for all $x\geqq 0$, *and* $0\leqq t, s\leqq 1$. *Then*

$$P\Big(\sup_{0\leqq|t-s|\leqq\delta} |X(t)-X(s)| \geqq x\big(1+10\int_0^\infty e^{-t^2}\,dt\big)\Big) \leqq 6c_4\sum_{n=0}^{\infty} e^{-2^{n-1}}e^{-c_5x^2/\delta}/\delta$$

holds for all $x\geqq\sqrt{\delta(1+4\log p)/c_5}$, $p>1$ *and* $0<\delta\leqq 1$.

Now we start proving Theorem 2 by using the same idea as that of Oodaira's Theorem 1 [6].

Since we have

$$\sup_{2^{r-1}<n\leqq 2^r}\ \sup_{|t-s|\leqq\delta}|Y_n(t)-Y_n(s)|\leqq\sqrt{2}\sup_{|t-s|\leqq\delta}|Y_{2^r}(t)-Y_{2^r}(s)|,$$

it is sufficient to show that

$$\sum_r^{\infty}P\big(A_r(\varepsilon,\delta)\big)<+\infty \tag{6}$$

holds for any $\varepsilon>0$ choosing δ sufficiently small, where

$$A_r(\varepsilon,\delta)=\Big\{\sup_{|t-s|\leqq\delta}|Y_{2^r}(t)-Y_{2^r}(s)|>\varepsilon\sqrt{\log\log 2^{r-1}}\Big\}.$$

By taking account of (2) and Lemma 3, it is easy to check that (6) holds, which completes the proof of Theorem 2.

To prove Theorem 3 we apply the theorem due to Kuelbs [5] which is restated here as a lemma for convenience.

LEMMA 4. *Assume that*

(i) $P(\{Y_n/\sqrt{2\log\log n}\}$ *is relatively compact in* $C[0,1])=1$,

(ii) *for any signed measure with bounded variation* ν *on* $[0,1]$,

$$P\Big(\varlimsup_{n\to\infty}\int_0^1 Y_n(t)\,d\nu(t)/\sqrt{2\log\log n}\leqq\Big(E\Big[\Big(\int_0^1 B(t)\,d\nu(t)\Big)^2\Big]\Big)^{1/2}\Big)=1,$$

where $\{B(t);\ 0\leqq t\leqq 1\}$ *is a standard Brownian motion, so*

$$E\Big[\Big(\int_0^1 B(t)\,d\nu(t)\Big)^2\Big]=\int_0^1\Big(\int_t^1 d\nu(s)\Big)^2dt\equiv K_\nu^2.$$

Then

$$P\big(\text{the cluster set of }\{Y_n/\sqrt{2\log\log n}\}\subset K\big)=1,$$

where $K=\Big\{x\in C(0,1);\ x(0)=0$ *and* $\displaystyle\int_0^1\Big(\frac{dx}{dt}\Big)^2dt\leqq 1\Big\}$.

By our Theorem 2, it is sufficient for proving Theorem 3 to show the second condition (ii) of Lemma 4. Define

$$\varphi_j^{(n)}(t)=\begin{cases}1 & \text{for}\quad j/n\leqq t\leqq 1\\ nt-(j-1) & \text{for}\quad (j-1)/n\leqq t\leqq j/n\\ 0 & \text{for}\quad 0\leqq t\leqq (j-1)/n\end{cases}$$

and set

$$\int_0^1\varphi_j^{(n)}(t)\,d\nu(t)=a_j^{(n)}.$$

Then it follows that

$$\int_0^1 Y_n(t)\,dv(t) = \sum_{j=1}^n a_j^{(n)} X_j/\sqrt{n}.$$

Easily we have

$$\lim_{n\to\infty} E\Big[\Big(\int_0^1 Y_n(t)\,dv(t)\Big)^2\Big] = \int_0^1 \Big(\int_t^1 dv(s)\Big)^2 dt.$$

Now setting

$$Z_n = \sum_{j=1}^n |a_j^{(n)}|^2 (X_j^2-1)/n,$$

we have

$$E[Z_n^2] = n^{-2} E\Big[\sum_{j=1}^n |a_j^{(n)}|^4 (X_j^2-1)^2 + \sum_{j\neq j'}^n |a_j^{(n)} a_{j'}^{(n)}|^2 (X_j^2-1)(X_{j'}^2-1)\Big] \leq c_6/n,$$

where c_6 does not depend on n. Therefore, for any $\theta>1$, taking $n_k=[\theta^k]$ (the integer part of θ^k) we have

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} |a_j^{(n_k)}|^2 X_j^2/n_k = \int_0^1 \Big(\int_t^1 dv(s)\Big)^2 dt. \tag{7}$$

By using Takahashi's method [9, Theorem 3] we have

$$E\Big[\exp\Big\{\lambda_n \sum_{j=1}^n a_j^{(n)} X_j/\sqrt{n} - \lambda_n^2 \sum_{j=1}^n |a_j^{(n)}|^2 X_j^2/(2n)\Big\}\Big] \leq \exp\{(\lambda_n K)^3/\sqrt{n}\}. \tag{8}$$

Choosing $\lambda_n = K_v^{-1}\sqrt{2\log\log n}$ it follows for any $\varepsilon>0$ that

$$E\Big[\exp\Big(\lambda_{n_k} \sum_{j=1}^{n_k} a_j^{(n_k)} X_j/\sqrt{n_k} - \lambda_{n_k}^2 \sum_{j=1}^{n_k} a_j^{(n_k)} X_j^2/(2n_k) - (1+\varepsilon)\log\log n_k\Big)\Big] \leq c_7 k^{-(1+\varepsilon)}.$$

Since the right hand side is a convergent term, it follows for any $M>0$ that

$$\overline{\lim_{k\to\infty}}\Big(\lambda_{n_k} \sum_{j=1}^{n_k} a_j^{(n_k)} X_j - \lambda_{n_k}^2 \sum_{j=1}^{n_k} a_j^{(n_k)\,2} X_j^2/(2n_k) - (1+\varepsilon)\log\log n_k\Big) \leq -M \quad \text{a.s.,}$$

which implies that

$$\overline{\lim_{k\to\infty}} \int_0^1 Y_{n_k}(t)\,dv(t)/\sqrt{2\log\log n_k} \leq K_v \quad \text{a.s.} \tag{9}$$

To establish the condition (ii) of Lemma 4, we have to estimate the remainder terms between n_{k-1} and n_k. We have

$$\begin{aligned}(10)\qquad &\Big|\int_0^1 Y_n(t)\,dv(t)/\sqrt{2\log\log n} - \int_0^1 Y_{n_k}(t)\,dv(t)/\sqrt{2\log\log n_k}\Big| \leq \\ &\leq \int_0^1 |Y_n(t)-Y_{n_k}(t)|\,dv(t)/\sqrt{2\log\log n_{k-1}} + \\ &+ \int_0^1 |Y_{n_k}(t)\,dv(t)|\,|(2\log\log n)^{-1/2} - (2\log\log n_k)^{-1/2}|.\end{aligned}$$

Since we have

$$Y_n(t)-Y_{n_k}(t)=(\sqrt{n_k/n}-1)Y_{n_k}(t)+\sqrt{n_k/n}(Y_{n_k}(nt/n_k)-Y_{n_k}(t)),$$

by taking account of (9) and equicontinuity, for $\varepsilon>0$ there exists $\theta>1$ such that

$$\lim_{k\to\infty}\sup_{n_{k-1}<n<n_k}\int_0^1 |Y_n(t)-Y_{n_k}(t)|\,dv(t)/\sqrt{2\log\log n_{k-1}}\leqq\varepsilon\quad\text{a.s.},$$

Easily we can get the same estimate for the second term of (10). Therefore finally we have

$$\overline{\lim_{n\to\infty}}\int_0^1 Y_n(t)\,dv(t)/\sqrt{2\log\log n}\leqq K_v\quad\text{a.s.},$$

which completes the proof of Theorem 3.

References

[1] G. Alexits, *Convergence of orthogonal series*, Pergamon Press—Akadémiai Kiadó (Budapest, 1961).

[2] K. Azuma, Weighted sums of certain dependent random variables, *Tohoku Math. J.*, **19** (1967), 357—367.

[3] X. Fernique, Continuité des processus gaussiens, *C. R. Acad. Sci. Paris*, **258** (1964), 6058—6060.

[4] N. Kôno, On the modulus of continuity of sample functions of Gaussian processes, *J. Math. Kyoto Univ.*, **10** (1970), 493—536.

[5] J. Kuelbs, A strong convergence theorem for Banach space valued random variables, *Annals of Prob.*, **4** (1976), 744—771.

[6] H. Oodaira, On Strassen's version of the law of the iterated logarithm for Gaussian processes, *Z. Wahr. Geb.*, **21** (1972), 289—299.

[7] P. Révész, *Indiana Univ. Math. J.*, **21** (1972), 557—564.

[8] P. Révész, A new law of the iterated logarithm for multiplicative systems, *Acta Sci. Math. (Szeged)*, **34** (1973), 349—358.

[9] W. F. Stout, *Almost sure convergence*, Academic Press (1974).

[10] S. Takahashi, Notes on the law of the iterated logarithm, *Studia Sci. Math. Hungar.*, **7** (1972), 21—24.

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NON-LINEAR CONNECTIONS IN A FINSLER SPACE

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1. Introduction. Various aspects of non-linear connections in a Finsler space have been studied respectively by Vagner [9], Barthel [1], Kawaguchi [2], Rund [7, 8], Misra and Mishra [3], Rastogi [4, 5] and others. Using notations of Rund [7, 8] it has been proved that when the geodesics are auto-parallel curves in F_n, the connection parameters $\overset{2}{\Gamma}{}^i_{jk}(x, Y)$ are symmetric. We have studied certain commutation formulae and obtained a representation for the tensor $\overset{1}{\Omega}{}^i_{jk}$ defined in [7]. Further we have given a generalisation of the ex-central connections of Rund [8] and studied some semi-symmetric connections and their curvature properties in a Finsler space F_n.

Let $X^i(x)$ and $Y_j(x)$ be two differentiable vector fields in a Finsler space F_n with metric tensor $g_{ij}(x, X)$ and non-linear connections $\overset{1}{\Gamma}{}^i_k(x, X)$ and $\overset{2}{\Gamma}_{jk}(x, Y)$, positively homogeneous of degree one in X and Y, then we have [7]:

$$\overset{1}{\Gamma}{}^i_{jk}(x, X) = \Delta_j \overset{1}{\Gamma}{}^i_k, \quad X^j \overset{1}{\Gamma}{}^i_{jk} = \overset{1}{\Gamma}{}^i_k, \quad \Delta_j = \partial/\partial X^j \tag{1.1}$$

and

$$\overset{2}{\Gamma}{}^i_{jk}(x, Y) = \Delta^i \overset{2}{\Gamma}_{jk}, \quad Y_i \overset{2}{\Gamma}{}^i_{jk} = \overset{2}{\Gamma}_{jk}, \quad \Delta^i = \partial/\partial Y_i. \tag{1.2}$$

If the vector field Y_i is conjugate to X^i such that $Y_i = g_{ij} X^j$ and if X^i undergoes a parallel displacement then so does Y_i, then under the condition that the length of a vector remains unchanged for a parallel displacement we have [7]:

$$2G^i = \overset{1}{\Gamma}{}^i_k X^k + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h). \tag{1.3}$$

Throughout this paper it is assumed that geodesics are auto-parallel curves in F_n, Rund [7] such that

$$2G^i = \overset{1}{\Gamma}{}^i_k X^k, \tag{1.4}$$

$$\Gamma^i_j = 2G^i_j - \overset{1}{\Gamma}{}^i_{jh} X^h, \tag{1.5}$$

$$\overset{1}{\Gamma}{}^i_{hj} = G^i_{hj} + \frac{1}{2}(\overset{1}{\Omega}{}^i_{hj} - X^k \Delta_j \overset{1}{\Omega}{}^i_{hk}) \tag{1.6}$$

and

$$\overset{2}{\Gamma}{}^h_{ik} = \overset{1}{\Gamma}{}^h_{ik} + g^{hl} Y_j \Delta_l \overset{1}{\Gamma}{}^j_{ik}, \tag{1.7}$$

where

$$\overset{1}{\Omega}{}^i_{hk} = \overset{1}{\Gamma}{}^i_{hk} - \overset{1}{\Gamma}{}^i_{kh}.$$

These connections also satisfy

(1.8) $$X^i \overset{2}{\Gamma}{}^h_{ik} = X^i \overset{1}{\Gamma}{}^h_{ik}, \quad Y_h \overset{2}{\Gamma}{}^h_{ik} = Y_h \overset{1}{\Gamma}{}^h_{ik}$$

and

(1.9) $$Y_j \overset{1}{\Omega}{}^j_{hk} = Y_j \overset{2}{\Omega}{}^j_{hk} = 0,$$

where

$$\overset{2}{\Omega}{}^j_{hk} = \overset{2}{\Gamma}{}^j_{hk} - \overset{2}{\Gamma}{}^j_{kh}.$$

Based on these connections for an arbitrary tensor $S^i_j(x, X)$ we have [3]:

(1.10) $$S^i_{j,k} = \partial_k S^i_j + (\Delta_m S^i_j)(\partial_k X^m) + S^m_j \overset{1}{\Gamma}{}^i_{mk} - S^i_m \overset{2}{\Gamma}{}^m_{jk}.$$

2. The tensor $\overset{2}{\Omega}{}^h_{jk}$. From equation (1.7) one can easily obtain

(2.1) $$\overset{2}{\Omega}{}^h_{ik} = \overset{1}{\Omega}{}^h_{ik} + g^{hl} Y_j \Delta_l \overset{1}{\Omega}{}^j_{ik}.$$

Now differentiating the first part of (1.9) with respect to X^l and using $\Delta_l Y_j = g_{jl}$ we get

(2.2) $$g_{lj} \overset{1}{\Omega}{}^j_{hk} + Y_j \Delta_l \overset{1}{\Omega}{}^j_{hk} = 0.$$

Substituting from (2.2) in (2.1) we can easily obtain on simplification

(2.3) $$\overset{2}{\Omega}{}^h_{ik} = 0.$$

Hence:

THEOREM 2.1. *If the geodesics are auto-parallel curves, the tensor* $\overset{2}{\Omega}{}^j_{ik}$ *vanishes and the connection parameter* $\overset{2}{\Gamma}{}^j_{ik}$ *becomes symmetric in* i *and* k.

Let $U^i(x, X)$ and $V_j(x, Y)$ be two arbitrary vector fields, then it is easy to obtain

(2.4a) $$2U^i_{,[kh]} = U^j \overset{1}{R}{}^i_{jkh}$$

and

(2.4b) $$2V_{j,[kh]} = -V_i \overset{2}{R}{}^i_{jkh},$$

where $[kh]$ means skew symmetric part and $\overset{1}{R}{}^i_{jkh}(x, X)$ and $\overset{2}{R}{}^i_{jkh}(x, Y)$ are given by [3] in the following form:

(2.5a) $$\overset{1}{R}{}^i_{jkh} = \{\partial_h \overset{1}{\Gamma}{}^i_{jk} + (\Delta_m \overset{1}{\Gamma}{}^i_{jk})(\partial_h X^m) + \overset{1}{\Gamma}{}^m_{jk} \overset{1}{\Gamma}{}^i_{mh}\} - h|k^*$$

* $-h|k$ means interchange of indices h and k and substraction.

and

$$\overset{2}{R}{}^i_{jkh} = \{\partial_h \overset{2}{\Gamma}{}^i_{jk} + (\Delta^m \overset{2}{\Gamma}{}^i_{jk})(\partial_h Y_m) + \overset{2}{\Gamma}{}^m_{jk} \overset{2}{\Gamma}{}^i_{mh}\} - h|k. \tag{2.5b}$$

If in place of U^i and V_j we take $X^i(x)$ and $Y_j(x)$, we get

$$2X^i_{,[kh]} = \overset{1}{R}{}^i_{kh} \tag{2.6a}$$

and

$$2Y_{j,[kh]} = -\overset{2}{R}_{jkh}, \tag{2.6b}$$

where [3]:

$$\overset{1}{R}{}^i_{kh}(x, X) = \{\partial_h \overset{1}{\Gamma}{}^i_k + \overset{1}{\Gamma}{}^i_{mh} \overset{1}{\Gamma}{}^m_k\} - h|k \tag{2.7a}$$

and

$$\overset{2}{R}_{jkh}(x, Y) = \{\partial_h \overset{2}{\Gamma}_{jk} + \overset{2}{\Gamma}{}^m_{jk} \overset{2}{\Gamma}_{mh}\} - h|k. \tag{2.7b}$$

It is known that if $\overset{1}{\Gamma}{}^i_{hk}(x, X)$ is symmetric, geodesics are auto-parallel curves, but the converse is not true in general, therefore if geodesics are auto-parallel curves, one can easily establish the following curvature identities:

$$(\overset{1}{R}{}^i_{jkh} - \overset{1}{\Omega}{}^i_{kh,j}) + \text{cycl.}\,(j, k, h) = 0 \tag{2.8a}$$

and

$$\overset{2}{R}{}^i_{jkh} + \text{cycl.}\,(j, k, h) = 0, \tag{2.8b}$$

where cycl. (j, k, h) means addition of three terms by cyclic permutation of the indices j, k, h.

Multiplying (2.8a, b) by Y_i we get

$$(Y_i \overset{1}{R}{}^i_{jkh} - Y_{i,j} \overset{1}{\Omega}{}^i_{kh}) + \text{cycl.}\,(j, k, h) = 0 \tag{2.9a}$$

and

$$\overset{2}{R}_{jkh} + \text{cycl.}\,(j, k, h) = 0. \tag{2.9b}$$

If Y_i is a non-zero vector whose covariant derivative is zero, equation (2.9a) reduces to

$$Y_i \overset{1}{R}{}^i_{jkh} + \text{cycl.}\,(j, k, h) = 0. \tag{2.10}$$

Further if we differentiate (2.4a) with respect to x^j covariantly and add three terms obtained by cyclic permutation of the indices k, h, j we get by virtue of (2.4a) and (2.8b)

$$\overset{1}{R}{}^i_{lkh,j} + \text{cycl.}\,(k, h, j) = 0. \tag{2.11a}$$

Similarly from (2.4b) we obtain

$$\overset{2}{R}{}^i_{lkh,j} + \text{cycl.}\,(k, h, j) = 0. \tag{2.11b}$$

Multiplying (2.11a) by X^l and (2.11b) by Y_i we get

$$(\overset{1}{R}{}^i_{kh,j} - X^l{}_{,j}\overset{1}{R}{}^i_{lkh}) + \text{cycl.}\,(k, h, j) = 0 \tag{2.12a}$$

and

$$\overset{2}{R}_{lkh,j} + \text{cycl.}\,(k, h, j) = 0. \tag{2.12b}$$

3. The tensor $\overset{1}{\Omega}{}^i_{hj}$. It is known that [3]:

$$\Delta_k(U^i_{,h}) - (\Delta_k U^i)_{,h} = U^m \Delta_k \overset{1}{\Gamma}{}^i_{mh} + (\Delta_m U^i)\overset{2}{\Gamma}{}^m_{kh}, \tag{3.1}$$

therefore for X^i in place of U^i we shall have

$$\Delta_k(X^i_{,h}) = (\Delta_k X^i)_{,h} + X^m \Delta_k \overset{1}{\Gamma}{}^i_{mh} + \overset{2}{\Gamma}{}^i_{kh}. \tag{3.2}$$

Since $X^m \Delta_k \overset{1}{\Gamma}{}^i_{mh} = X^m \Delta_m \overset{1}{\Gamma}{}^i_{kh}$ and $\overset{1}{\Gamma}{}^i_{kh}$ is homogeneous of degree zero in X^m we get

$$X^m \Delta_m \overset{1}{\Gamma}{}^i_{kh} = 0, \tag{3.3}$$

which when substituted in (3.2) gives

$$\Delta_k(X^i_{,h}) - \Delta_h(X^i_{,k}) = (\Delta_k X^i)_{,h} - (\Delta_h X^i)_{,k}. \tag{3.4}$$

Also since it is known that [3]:

$$(\Delta_k U^i)_{,h} = \partial_h \Delta_k U^i + (\Delta_m \Delta_k U^i)(\partial_h X^m) + (\Delta_k U^m)\overset{1}{\Gamma}{}^i_{mh} - (\Delta_m U^i)\overset{2}{\Gamma}{}^m_{kh}, \tag{3.5}$$

therefore we obtain on simplification

$$(\Delta_k X^i)_{,h} = \overset{1}{\Gamma}{}^i_{kh} - \overset{2}{\Gamma}{}^i_{kh}, \tag{3.6}$$

which leads to

$$(\Delta_k X^i)_{,h} - (\Delta_h X^i)_{,k} = \overset{1}{\Omega}{}^i_{kh}. \tag{3.7}$$

Substituting from (3.7) in (3.4) we get

$$\overset{1}{\Omega}{}^i_{kh} = \Delta_k(X^i_{,h}) - \Delta_h(X^i_{,k}), \tag{3.8}$$

which gives

THEOREM 3.1. *The necessary and sufficient condition for the tensor* $\overset{1}{\Omega}{}^i_{kh}$ *to vanish is that* $\Delta_k(X^i_{,h}) = \Delta_h(X^i_{,k})$ *or equivalently* $(\Delta_k X^i)_{,h} = (\Delta_h X^i)_{,k}$.

Now if we assume that the tensor $\overset{1}{\Omega}{}^i_{hj}$ is given by

$$\overset{1}{\Omega}{}^i_{hj} = \delta^i_h p_j - \delta^i_j p_h, \tag{3.9}$$

where $p_j(x, Y)$ is a covariant vector field, from (1.6) we obtain

$$\overset{1}{\Gamma}{}^i_{hj} = G^i_{hj} + \frac{1}{2}(\delta^i_h p_j - \delta^i_j p_h + X^i \Delta_j p_h - \delta^i_h X^k \Delta_j p_k), \tag{3.10}$$

which gives

$$\overset{1}{\Omega}{}^i_{hj} = \delta^i_h p_j - \delta^i_j p_h + \frac{1}{2}[X^i(\Delta_j p_h - \Delta_h p_j) + X^k(\delta^i_j \Delta_h p_k - \delta^i_h \Delta_j p_k)]. \tag{3.11}$$

Comparing (3.9) and (3.11) we obtain

$$X^i(\Delta_j p_h - \Delta_h p_j) + X^k(\delta^i_j \Delta_h p_k - \delta^i_h \Delta_j p_k) = 0. \tag{3.12}$$

Contracting (3.12) for i and j we obtain

$$X^i(\Delta_i p_h + (n-2)\Delta_h p_i) = 0. \tag{3.13}$$

Hence:

THEOREM 3.2. *The necessary condition for the tensor* $\overset{1}{\Omega}{}^i_{hj}$ *to be expressed as* (3.9) *is given by* (3.13).

Further if $\delta^i_j \Delta_h p_k = \delta^i_h \Delta_j p_k$, then on contracting it for i and j we get $(n-1)\Delta_h p_k = 0$, which implies p_k is a function of coordinates only, whereas if we can contract it for i and k we get $\Delta_h p_j - \Delta_j p_h = 0$. Thus if $\delta^i_j \Delta_h p_k = \delta^i_h \Delta_j p_k$, then equation (3.12) is satisfied. Hence:

THEOREM 3.3. $\delta^i_j \Delta_h p_k = \delta^i_h \Delta_j p_k$ *is a sufficient condition for the tensor* $\overset{1}{\Omega}{}^i_{hj}$ *to be given by* (3.9).

If we replace p_j by Y_j, from equation (3.13) we get on simplification $Y_j = 0$. Hence we have:

THEOREM 3.4. *If the tensor* $\overset{1}{\Omega}{}^i_{hj}$ *is expressible as* $\delta^i_h Y_j - \delta^i_j Y_h$ *we get* $Y_h = 0$, *showing that* $\overset{1}{\Omega}{}^i_{hj}$ *is identically zero.*

4. Ex-central connections. We here define the following generalised ex-central connections

$$'\overset{1}{\Gamma}{}^i_j(x, X) \overset{\text{def.}}{=} \overset{1}{\Gamma}{}^i_j(x, X) + T^i_j(x, X) \tag{4.1a}$$

and

$$'\overset{2}{\Gamma}_{ij}(x, Y) \overset{\text{def.}}{=} \overset{2}{\Gamma}_{ij}(x, Y) - V_j V_i, \tag{4.1b}$$

where T^i_j is an arbitrary tensor and V_j is an arbitrary covariant vector field.
From (4.1a, b) we now define

$$\Delta_k {}'\overset{1}{\Gamma}{}^i_j \overset{\text{def.}}{=} {}'\overset{1}{\Gamma}{}^i_{kj} = \overset{1}{\Gamma}{}^i_{kj} + \Delta_k T^i_j \tag{4.2a}$$

and

$$\Delta^i {}'\overset{2}{\Gamma}_{kj} \overset{\text{def.}}{=} {}'\overset{2}{\Gamma}{}^i_{kj} = \overset{2}{\Gamma}{}^i_{kj} - \Delta^i(V_j V_k) \tag{4.2b}$$

such that

$$'\overset{1}{\Omega}{}^i_{kj} = \overset{1}{\Omega}{}^i_{kj} + (\Delta_k T^i_j - \Delta_j T^i_k) \tag{4.3a}$$

and

$$'\overset{2}{\Omega}{}^i_{kj} = 0. \tag{4.3b}$$

On the basis of these connections we define

$$'\overset{1}{R}{}^i_{kh} \overset{\text{def.}}{=} \{\partial_h {}'\overset{1}{\Gamma}{}^i_k + {}'\overset{1}{\Gamma}{}^i_{mh} {}'\overset{1}{\Gamma}{}^m_k\} - h|k, \tag{4.4a}$$

$$'\overset{2}{R}_{jkh} \overset{\text{def.}}{=} \{\partial_h {}'\overset{2}{\Gamma}_{jk} + {}'\overset{2}{\Gamma}{}^m_{jk} {}'\overset{2}{\Gamma}_{mh}\} - h|k, \tag{4.4b}$$

$$'\overset{1}{R}{}^i_{jkh} \overset{\text{def.}}{=} \{\partial_h {}'\overset{1}{\Gamma}{}^i_{jk} + (\Delta_m {}'\overset{1}{\Gamma}{}^i_{jk})(\partial_h X^m) + {}'\overset{1}{\Gamma}{}^m_{jk} {}'\overset{1}{\Gamma}{}^i_{mh}\} - h|k \tag{4.4c}$$

and

$$'\overset{2}{R}{}^i_{jkh} \overset{\text{def.}}{=} \{\partial_h {}'\overset{2}{\Gamma}{}^i_{jk} + (\Delta^m {}'\overset{2}{\Gamma}{}^i_{jk})(\partial_h Y_m) + {}'\overset{2}{\Gamma}{}^m_{jk} {}'\overset{2}{\Gamma}{}^i_{mh}\} - h|k, \tag{4.4d}$$

which satisfy

$$'\overset{1}{R}{}^i_{kh} = \overset{1}{R}{}^i_{kh} + (T^i_{k;h} - (\Delta_m T^i_k) X^m_{,h}) - h|k, \tag{4.5a}$$

$$'\overset{2}{R}_{jkh} = \overset{2}{R}_{jkh} + (V_h V_{j;k} + V_j V_{h,k} + Y_{m,h} \Delta^m (V_j V_k)) - h|k, \tag{4.5b}$$

$$'\overset{1}{R}{}^i_{jkh} = \overset{1}{R}{}^i_{jkh} + (\Delta_j (T^i_{k;h}) - (\Delta_m T^i_k)(\Delta_j X^m_{,h}) - (\Delta_j {}'\overset{1}{\Gamma}{}^i_{mh}) T^m_k) - h|k \tag{4.5c}$$

and

$$'\overset{2}{R}{}^i_{jkh} = \overset{2}{R}{}^i_{jkh} + (\Delta^i (V_j V_h)_{;k} + (\Delta^m \overset{2}{\Gamma}{}^i_{jk}) V_m V_h + \Delta^m (V_j V_k) \Delta^i (Y_{m,h})) - h|k, \tag{4.5d}$$

where

$$S^i_{j;k} \overset{\text{def.}}{=} \partial_k S^i_j + (\Delta_m S^i_j)(\partial_k X^m) + S^m_j {}'\overset{1}{\Gamma}{}^i_{mk} - S^i_m {}'\overset{2}{\Gamma}{}^m_{jk}. \tag{4.6}$$

One can easily obtain the properties and identities satisfied by these curvature tensors in analogy to § 2. Now we shall consider some special cases of (4.1a, b).

Case I. If $T^i_j = \delta^i_j$, equation (4.1a) gives the ex-central connection of Rund [8].

Case II. If $V_j = Y_j$, equation (4.1b) gives the other ex-central connection Rund [8].

Case III. If we assume that the arbitrary tensor T^i_j can be decomposed into the product of two arbitrary vectors U^i and V_j, then (4.1a) can be expressed as

$$'\overset{1}{\Gamma}{}^i_j = \overset{1}{\Gamma}{}^i_j + U^i V_j, \tag{4.7}$$

which implies

$$'\overset{1}{\Gamma}{}^i_{kj} = \overset{1}{\Gamma}{}^i_{kj} + V_j \Delta_k U^i + U^i \Delta_k V_j. \tag{4.8}$$

If we further take $V_j = Y_j$ and $U^i = X^i$, equation (4.8) gives

$$'\overset{1}{\Gamma}{}^i_{kj} = \overset{1}{\Gamma}{}^i_{kj} + Y_j \delta^i_k + X^i g_{jk}. \tag{4.9}$$

Multiplying (4.9) by X^k and using (4.7) we obtain

$$X^k \, '\overset{1}{\Gamma}{}^i_{kj} = '\overset{1}{\Gamma}{}^i_j + X^i Y_j. \tag{4.10}$$

From (4.9) we can also obtain

$$'\overset{1}{\Omega}{}^i_{kj} = \overset{1}{\Omega}{}^i_{kj} + Y_j \delta^i_k - Y_k \delta^i_j, \tag{4.11}$$

which easily implies $'\overset{1}{\Omega}{}^i_{kj} \neq 0$, since $\overset{1}{\Omega}{}^i_{kj}$ can not be expressed as $\delta^i_k Y_j - \delta^i_j Y_k$.

REMARK. The curvature properties based on these connections can be studied in analogy to Section 2.

5. Semi-symmetric connections. If the tensor $\overset{1}{\Omega}{}^i_{hj}$ is expressible as in (3.9), subject to condition (3.12), $\overset{1}{\Gamma}{}^i_{hj}$ is expressible as

$$\overset{1}{\Gamma}{}^i_{hj} = G^i_{hj} + \frac{1}{2}(\delta^i_h p_j - \delta^i_j p_h), \tag{5.1}$$

which represents a semi-symmetric connection whose properties can be studied in analogy to Rastogi [6].

It is known to us that $\overset{2}{\Gamma}{}^i_{hj}$ is symmetric in h and j, therefore we can define a semi-symmetric connection based on $\overset{2}{\Gamma}{}^i_{hj}$ as follows:

$$^*\overset{2}{\Gamma}{}^r_{ij} = \overset{2}{\Gamma}{}^r_{ij} + \delta^r_i p_j - g_{ij} p^r, \tag{5.2}$$

where $g_{ij} p^i = p_j$, such that $p^{i*}\overset{2}{\Gamma}{}^r_{ij} = p^i \overset{2}{\Gamma}{}^r_{ij}$.

From (5.2) we can obtain that the tensor $^*\overset{2}{\Omega}{}^r_{ij}$, is given by

$$^*\overset{2}{\Omega}{}^r_{ij} = {}^*\overset{2}{\Gamma}{}^r_{ij} - {}^*\overset{2}{\Gamma}{}^r_{ji} = \delta^r_i p_j - \delta^r_j p_i. \tag{5.3}$$

Furthermore, since we can easily obtain that

$$\overset{1}{\Gamma}{}^h_{lk} = \overset{2}{\Gamma}{}^h_{lk} + X^i \Delta_l \overset{2}{\Gamma}{}^h_{ik}, \tag{5.4}$$

therefore if $^*\overset{1}{\Gamma}{}^h_{lk}$ is defined in the following form

$$^*\overset{1}{\Gamma}{}^h_{lk} = {}^*\overset{2}{\Gamma}{}^h_{lk} + X^i \Delta_l \overset{2}{\Gamma}{}^h_{ik}, \tag{5.5}$$

then we can easily obtain

$$^*\overset{1}{\Gamma}{}^h_{lk} = \overset{1}{\Gamma}{}^h_{lk} + \delta^h_l p_k - g_{lk} p^h, \tag{5.6}$$

which gives

$$(5.7)\qquad {}^*\overset{1}{\Omega}{}^h_{lk} = \overset{1}{\Omega}{}^h_{lk} + (\delta^h_l p_k - \delta^h_k p_l).$$

Now corresponding to connections ${}^*\overset{2}{\Gamma}{}^r_{ij}$ and ${}^*\overset{1}{\Gamma}{}^r_{ij}$, we define the following curvature tensors

$$(5.8a)\qquad {}^*\overset{2}{K}{}^i_{jkh} \overset{\text{def.}}{=} \{\partial_h {}^*\overset{2}{\Gamma}{}^i_{jk} + (\Delta^m {}^*\overset{2}{\Gamma}{}^i_{jk})(\partial_h Y_m) + {}^*\overset{2}{\Gamma}{}^m_{jk} {}^*\overset{2}{\Gamma}{}^i_{mh}\} - h|k$$

and

$$(5.8b)\qquad {}^*\overset{1}{K}{}^i_{jkh} \overset{\text{def.}}{=} \{\partial_h {}^*\overset{1}{\Gamma}{}^i_{jk} + (\Delta_m {}^*\overset{1}{\Gamma}{}^i_{jk})(\partial_h X^m) + {}^*\overset{1}{\Gamma}{}^m_{jk} {}^*\overset{1}{\Gamma}{}^i_{mh}\} - h|k,$$

respectively. From (5.8a) we can easily obtain

$$(5.9)\qquad {}^*\overset{2}{K}{}^i_{jkh} = \overset{2}{R}{}^i_{jkh} + \{\delta^i_j p_{k,h} - (g_{jk} p^i)_{,h} + p^i g_{jk} p_h + p^m g_{jk} X^l \Delta_m \overset{2}{\Gamma}{}^i_{lh}\} - h|k.$$

REMARK. A similar relation can be obtained between ${}^*\overset{1}{K}{}^i_{jkh}$ and $\overset{1}{R}{}^i_{jkh}$. Also we can obtain the relationship between the curvature tensors defined by (5.8a, b) with the help of (5.5).

If we consider a special value of $\overset{2}{\Gamma}{}^r_{ij}$ given as [8]:

$$(5.10)\qquad \overset{2}{\Gamma}{}^r_{ij} = \Gamma^{*r}_{ij} - C^r_{ij|k} X^k,$$

from (5.2) we can get

$$(5.11)\qquad {}^*\overset{2}{\Gamma}{}^r_{ij} = \Gamma^{*r}_{ij} - C^r_{ij|k} X^k + \delta^r_i p_j - g_{ij} p^r,$$

which is a semi-symmetric connection, whose properties can be studied in analogy to [6].

Further if $\overset{2}{\Gamma}{}^r_{ij}$ is given by (5.10), then from (5.4) one can easily obtain $\overset{1}{\Gamma}{}^r_{ij} = G^r_{ij}$, which when applied on (5.6), gives rise to

$$(5.12)\qquad {}^*\overset{1}{\Gamma}{}^h_{jk} = G^h_{jk} + \delta^h_j p_k - g_{jk} p^h,$$

whose properties have already been studied by Rastogi [6].

From (5.11) and (5.12) we can obtain ${}^*\overset{1}{\Gamma}{}^h_{lk} = {}^*\overset{2}{\Gamma}{}^h_{lk} + 2C^h_{lk|j} X^j$, such that $X^l {}^*\overset{1}{\Gamma}{}^h_{lk} = X^l {}^*\overset{2}{\Gamma}{}^h_{lk}$ and $X^k {}^*\overset{1}{\Gamma}{}^h_{lk} = X^k {}^*\overset{2}{\Gamma}{}^h_{lk}$.

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References

[1] W. Barthel, Über eine Parallelverschiebung mit Längeninvarianz in lokal-Minkowskische Raumen I, II, *Arc. Math.*, **4** (1953), 346—365.
[2] A. Kawaguchi, On the theory of non-linear connections II, *Tensor (N.S.)*, **6** (1956), 165—199.
[3] R. B. Misra and R. S. Mishra, Curvature tensors arising from non-linear connections in a Finsler space, *Bull. de l'Acad. Pol. Des. Sci.*, **17** (1969), 755—760.
[4] S. C. Rastogi, On Finsler spaces of recurrent curvature tensors, *Kyungpook Math. J.*, **20** (1980), 37—45.
[5] S. C. Rastogi, On projective invariants based on non-linear connections in a Finsler space, *Publ. Math. Debrecen*, **28** (1981), 121—128.
[6] S. C. Rastogi, On symmetric and semi-symmetric connections in a Finsler space, *Tensor (N.S.)*, **40** (1983), 13—20.
[7] H. Rund, Some remarks concerning the theory of non-linear connections, *Nederl. Akad. Wetensch. Proc., Ser. A*, **61** (1958), 341—347.
[8] H. Rund, *The differential geometry of Finsler spaces* (Berlin, 1959).
[9] V. V. Vagner, The geometry of Finsler as a theory of the field of local hypersurfaces in X_n, *Trudy Sem. Vektor-Tenzor Analizu*, **7** (1949), 65—166.

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References

[1] [illegible]

[illegible]

[illegible]

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EXTENSIONS OF FUNCTIONALS ON OCTONIONIC LINEAR SPACES

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The Hahn—Banach theorem states that any linear functional defined on a linear subspace of a normed linear space has a norm-preserving extension to the whole space. Until 1938 it was not known whether or not this result was restricted to real linear spaces and real-valued functionals. In that year Sobczyk and Bohnenblust [3] proved that a complex linear functional defined on a complex linear subspace of a complex normed linear space has a norm-preserving extension to the entire space. Moreover, they showed that the subspace must be complex linear (not merely real linear) in order for such an extension to necessarily exist. In that same year Suhomlinov [4] generalized to prove the analogue of the Hahn—Banach theorem for quaternionic normed linear spaces as well. This paper investigates the extent to which the Hahn—Banach theorem is valid in the still more general case of linear spaces which are defined over the non-associative algebra O of real octonions.

Real octonions are all hypercomplex numbers of the form $a_0+a_1e_1+a_2e_2+$ $+a_3e_3+a_4e_4+a_5e_5+a_6e_6+a_7e_7$ where $a_0, a_1, \ldots, a_7$ are real coefficients with distributive products defined by the following multiplication rules for the unit 1 and the seven basic imaginary units $e_1, e_2, \ldots, e_7$:

$$e_1^2 = e_2^2 = \ldots = e_7^2 = -1,$$

$$e_i \cdot 1 = 1 \cdot e_i = e_i, \quad i = 1, 2, \ldots, 7,$$

$$e_i \cdot e_{i+1} = e_{i+3} = -e_{i+1} \cdot e_i$$

where the addition of the indices is taken mod 7.

The conjugate $\bar{a}$ of the real octonion $a=a_0+a_1e_1+\ldots+a_7e_7$ is given as

$$\bar{a} = a_0-a_1e_1-\ldots-a_7e_7$$

in direct analogy to complex conjugation. The product of a and its conjugate is called the norm-squared of a and is denoted

$$Q(a) = a\bar{a} = \bar{a}a = a_0^2+a_1^2+\ldots+a_7^2.$$

If $a\neq 0$ then $Q(a)\neq 0$ and the inverse of a is

$$a^{-1} = \big(1/Q(a)\big)\bar{a}.$$

The subalgebra of O which consists of all real linear combinations of $1, e_1, e_2$ and

e_4 is isomorphic to the skew-field of real quaternions H. In this sense the real quaternions are a subalgebra of O. There are exactly six other such isomorphic copies of H in O, but hereafter when we refer to the real quaternions H we shall mean that isomorphic copy generated by 1, e_1, e_2 and e_4.

In studying the properties of the set O^n together with the standard component-wise addition and scalar multiplication, and in keeping with the work of Goldstine and Horwitz [1] we make the following definition:

DEFINITION. An *octonionic linear space* V is a set of elements called vectors in which vector addition and scalar multiplication are defined and satisfy:

(a) $x, y \in V \Rightarrow x+y \in V$;
(b) $x+y=y+x$ for x and y in V;
(c) $(x+y)+z=x+(y+z)$ for x, y and z in V;
(d) There is a vector θ in V for which $x+\theta=x$ for any x in V;
(e) To each x in V there corresponds a unique vector $-x$ in V for which $-x+x=\theta$;
(f) $x \in V$ and $a \in 0 \Rightarrow ax \in V$;
(g) $(a+b)x=ax+bx$ for x in V and a, b in O;
(h) $a(x+y)=ax+ay$ for x, y in V and a in O;
(i) $a(ax)=a^2x$ for x in V and a in O;
(j) $a(bx)=(ab)x$ for x in V and at least one of a and b is real;
(k) $a(a^{-1}x)=x$ for a in 0, $a \neq 0$ and x in V;
(l) $0x=\theta$ and $1x=x$ for all x in V;
(m) $q(e_3x)=e_3(\bar{q}x)$ for x in V and q in H.

It is precisely the lack of multiplicative associativity in O that accounts for the weaker axioms i, j, k and m. Property (i) is implied by the alternativity of O.

We need not explicitly define what we mean by quaternionic and octonionic linear subspaces (they are precisely what the reader would expect) except to say that we require closure with respect to scalar multiplication on the left only. Some authors [4] make the distinction between left and right subspaces, but we shall not do so in this paper. A functional f is said to be real, quaternionic or octonionic linear if the relation

$$f(ax+by) = af(x)+bf(y)$$

holds for all pairs of real, quaternion or octonion numbers a, b respectively. Finally, the motion of an octonionic normed linear space is clearly understood if we take $|a|$ to mean $(Q(a))^{1/2}$.

The real octonion

$$a = a_0+a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5+a_6e_6+a_7e_7$$

has the representation $a=q_0+e_3q_1$ where $q_0=a_0+a_1e_1+a_2e_2+a_4e_4$ and $q_1=a_3-$ $-a_7e_1-a_5e_2+a_6e_4$ are two quaternions. Any real octonion may be uniquely represented in this way. If f is an octonion-valued function then we may express it in the form

$$f(x) = f_0(x)+e_3f_1(x) \tag{1}$$

where f_0 and f_1 are quaternion-valued functions. We note that a may also be expressed in the forms

$$a = (a_0+a_1e_1+a_2e_2+a_4e_4)+e_5(a_5-a_6e_1+a_3e_2-a_7e_4),$$
$$a = (a_0+a_1e_1+a_2e_2+a_4e_4)-e_6(a_6+a_5e_1-a_7e_2-a_3e_4),$$
$$a = (a_0+a_1e_1+a_2e_2+a_4e_4)+e_7(a_7+a_3e_1+a_6e_2+a_5e_4).$$

These representations suggest that any octonion-valued function f may alternately be expressed in the forms

$$f(x) = f_0(x)+e_5f_2(x), \tag{2}$$
$$f(x) = f_0(x)+e_6f_3(x), \tag{3}$$
$$f(x) = f_0(x)+e_7f_4(x) \tag{4}$$

where f_0 is the f_0 in (1) and f_2, f_3 and f_4 are also quaternion-valued functions. If f is an octonionic linear functional defined on an octonionic normed linear space, then equations (1) through (4) may be written

$$f(x) = f_0(x)-e_3f_0(e_3x), \tag{1'}$$
$$f(x) = f_0(x)-e_5f_0(e_5x), \tag{2'}$$
$$f(x) = f_0(x)-e_6f_0(e_6x), \tag{3'}$$
$$f(x) = f_0(x)-e_7f_0(e_7x). \tag{4'}$$

THEOREM. Let A be an octonionic normed linear space and let M be an octonionic linear subspace of A. If $f \in M^*$, the collection of all octonionic linear functionals on M, then there exists $g \in A^*$ such that g extends f and $\|g\|=\|f\|$.

PROOF. For each x in M we write

$$f(x) = f_0(x)-e_3f_0(e_3x)$$

where f_0 is a quaternion-valued function defined on M. By the independence of 1 and e_3 we see that

$$f_0(x+y) = f_0(x)+f_0(y), \quad f_0(qx) = qf_0(x)$$

for all x and y in M and q in H. Thus f_0 is a quaternionic linear functional on M. Also,

$$|f_0(x)| \leqq (Q(f(x))^{1/2}) = |f(x)| \leqq \|f\|\|x\|$$

so that f_0 is bounded and $\|f_0\| \leqq \|f\|$. Now, regarding A and M as quaternionic linear spaces, we apply the quaternionic extension theorem of Suhomlinov [4] to establish the existence of a bounded quaternionic linear functional g_0 on all of A such that g_0 extends f_0 and $\|g_0\|=\|f_0\|$. We now define g on all of A by the rule

$$g(x) = g_0(x)+e_3g_0(e_3x).$$

CLAIM ONE. *g is an octonionic linear functional on A.*

Clearly g maps A into O. By the independence of 1 and e_3 we see that $g(x+y)=g(x)+g(y)$, for all x and y in A. To see that g is scalar (octonion) homogeneous we first observe that

$$g(qx) = g_0(qx)-e_3\,g_0(e_3(qx)) = qg_0(x)-e_3\,g_0(\bar{q}(e_3x)) = \\ = qg_0(x)-e_3(\bar{q}g_0(e_3x)) = qg_0(x)-q(e_3\,g_0(e_3x)) = qg(x)$$

for any q in H. Next we observe that

$$g(e_3x) = g_0(e_3x)+e_3\,g_0(e_3(e_3x)) = g_0(e_3x)-e_3\,g_0(-x) = \\ = g_0(e_3x)+e_3\,g_0(x) = e_3\,g(x).$$

We have already observed in equations (1′)—(4′) that

$$e_3\,g_0(e_3x) = e_5\,g_0(e_5x) = e_6\,g_0(e_6x) = e_7\,g_0(e_7x)$$

for every x in A, so that

$$g(e_ix) = g_0(e_ix)-e_i\,g_0(e_i(e_ix)) = g_0(e_ix)+e_i\,g_0(-x) = \\ = g_0(e_ix)+e_i\,g_0(x) = e_i\,g(x)$$

for $i=5, 6$ and 7. From this it is clear that $g(ax)=ag(x)$ for every a in O and x in A.

Claim Two. $g(x)=f(x)$ *for all* x *in* M.

For x in M we have

$$g_0(e_3x)+e_3f_1(e_3x) = f_0(e_3x)+e_3f_1(e_3x) = f(e_3x) = e_3f(x) = e_3f_0(x)-f_1(x)$$

where $f_1(x)=-f_0(e_3x)$. Therefore $g_0(e_3x)=-f_1(x)$ and we have

$$g(x) = g_0(x)-e_3\,g_0(e_3x) = f_0(x)+e_3f_1(x) = f(x)$$

for every x in M.

Claim Three. *The extension* g *is bounded on* A *and* $\|g\|=\|f\|$.

We have $g(x)=g_0(x)-e_3g_0(e_3x)$ for all x in A, so that

$$\|g\| = \sup\{|g(x)|:\ \|x\| = 1\} = \sup\{|g_0(x)-e_3\,g_0(e_3x)|:\ \|x\| = 1\} \geqq \\ \geqq \sup\{|g_0(x)|:\ \|x\| = 1\} = \|g_0\|.$$

We need only show that $\|g\|\leqq\|g_0\|$. Suppose, by way of contradiction, that $\|g\|>\|g_0\|$. Then for $\varepsilon>0$ sufficiently small we have $\|g_0\|+\varepsilon<\|g\|$. Then there must exist an element x_0 with $\|x_0\|=1$ such that

$$|g(x_0)| > \|g_0\|+\varepsilon/2 \geqq |g_0(x)|+\varepsilon/2$$

for all x for which $\|x\|=1$. But if we let $\lambda=\overline{g(x_0)}/|g(x_0)|$, where $\overline{g(x_0)}=\overline{g_0(x_0)}+e_3g_0(e_3x_0)$, we get

$$g(\lambda x_0) = |g(x_0)| = |g_0(\lambda x_0)|$$

which is a contradiction since $\|\lambda x_0\|=|\lambda|\ \|x_0\|=1$. Thus $\|g\|=\|g_0\|$. By the same argument, $\|f\|=\|f_0\|$. Thus, because $\|f_0\|=\|g_0\|$, we have $\|g\|=\|f\|$.

References

[1] H. Goldstine and L. Horwitz, Hilbert space with non-associative scalars, *Math. Annalen,* **154** (1964), 1—27.

[2] W. Hamilton, *Elements of Quaternions,* Chelsea Publishing Company (New York, 1860).

[3] A. Sobczyk and H. Bohnenblust, Extensions of functionals on complex linear spaces, *American Mathematics Society Bulletin,* **44** (1938), 91—93.

[4] G. Suhomlinov, On extensions of linear functions in complex and quaternionic linear spaces, *Mat. Sbornik,* **3** (1938), 353—358.

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References

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BIDUALS OF BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH ALGEBRA

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1. Introduction

Let A and B be semisimple Banach algebras such that A is a dense ideal in B. In Section 3 we show how the algebra A^{**} is related to the algebra B^{**}, in each of the Arens products. In Section 4 we assume that B has a bounded approximate identity contained in A and that B is Arens regular. The existence of such an approximate identity enables us to express A^{**} as a direct sum of two closed ideals, in each of the Arens products. This approach of expressing A^{**} as a direct sum of two closed ideals was used in [2] to show that an A^*-algebra which is a *-ideal in its B^*-algebra completion is Arens regular. (See also [15].) We use these direct sum decompositions of A^{**} to find conditions on A and B which imply Arens regularity of A.

2. Arens products

Let A be a Banach algebra, and let A^* and A^{**} be the conjugate and second conjugate spaces of A. The two Arens products on A^{**} are given in stages as follows [3]: Let $x, y\in A, f\in A^*$ and $F, G\in A^{**}$.

(i) Define $f\circ x$ by $(f\circ x)(y)=f(xy)$, $f\circ x\in A^*$.

(ii) Define $F\circ f$ by $(F\circ f)(x)=F(f\circ x)$, $F\circ f\in A^*$.

(iii) Define $F\circ G$ by $(F\circ G)(f)=F(G\circ f)$, $F\circ G\in A^{**}$.

A^{**} is a Banach algebra under the Arens product $F\circ G$, and we denote this algebra by $(A^{**}, \circ)$.

(i)′ Define $x\circ' f$ by $(x\circ' f)(y)=f(yx)$, $x\circ' f\in A^*$.

(ii)′ Define $f\circ' F$ by $(f\circ' F)(x)=F(x\circ' f)$, $f\circ' F\in A$.

(iii)′ Define $F\circ' G$ by $(F\circ' G)(f)=G(f\circ' F)$, $F\circ' G\in A^{**}$.

A^{**} is a Banach algebra under the Arens product $F\circ' G$, and we denote this algebra by $(A^{**}, \circ')$.

Both of the Arens products extend the given multiplication on A when A is canonically embedded in A^{**}. We say that A is *Arens regular* if the two Arens products coincide on A^{**}. We recall that the product $F\circ G$ is w^*-continuous in F for fixed G, while $F\circ' G$ is w^*-continuous in G for fixed F [4, p. 848]. We will denote by π_A the canonical embedding of A into A^{**}. We have $\pi_A(x)\circ F=\pi_A(x)\circ' F$ and $F\circ\pi_A(x)=F\circ'\pi_A(x)$, for all $x\in A$ and $F\in A^{**}$ [8].

If A has a bounded right (left) approximate identity then $(A^{**}, \circ)$ $((A^{**}, \circ'))$ has a right (left) identity [4, Lemma 3.8, p. 855]. If A has a bounded approximate identity then there exists an element E in A^{**} which is both a right identity for $(A^{**}, \circ)$ and a left identity for $(A^{**}, \circ')$ [9, Proposition 1.3, p. 93].

For $a \in A$, let $L_a(R_a)$ be the mapping $L_a(x)=ax$ $(R_a(x)=xa)$ for all $x \in A$. We call A *weakly completely continuous* (w.c.c.) if, for every $a \in A$, L_a and R_a are w.c.c. operators on A. It follows from the definition of Arens products and [6, VI, 4.2, p. 482] that A is w.c.c. if and only if $\pi_A(A)$ is an ideal of A^{**} in either Arens product.

Since we will be dealing mainly with a pair of Banach algebras A and B, it will occasionally be necessary to distinguish between the norms of A and B. In this case we will denote the norm of A (B) by $\|\cdot\|_A$ ($\|\cdot\|_B$). By an ideal we will always mean a two-sided ideal unless stated otherwise.

3. A^{**} as a B^{**}-module

Let A and B be Banach algebras such that A is a dense ideal in B. If B is semisimple then, by [11, Proposition 1.6, p. 299], there exist constants $C>0$ and $D>0$ such that for all $x \in A$ and $y \in B$,

$$\|xy\|_A \leqq C\|x\|_A\|y\|_B, \tag{1}$$

$$\|yx\|_A \leqq C\|y\|_B\|x\|_A, \tag{2}$$

$$\|x\|_B \leqq D\|x\|_A. \tag{3}$$

Since we will be only interested in Arens regularity of semisimple Banach algebras, we will assume henceforth that A and B are semisimple Banach algebras and that A is a dense ideal of B so that properties (1), (2) and (3) are satisfied. As there will be no danger of confusion, we will denote the Arens products in A^{**} and B^{**} by the same symbols $\circ$ and $\circ'$. Since A is a left and right Banach B-module, it follows that A^{**} is a left Banach $(B^{**}, \circ)$-module and a right Banach $(B^{**}, \circ')$-module under the adjoint module operations which are given as follows: Let $x \in A$, $y \in B$, $f \in A^*$, $F \in A^{**}$ and $H \in B^{**}$.

(a) Define $f*y$ by $(f*y)(x)=f(yx)$, $f*y \in A^*$.

(b) Define $F*f$ by $(F*f)(y)=F(f*y)$, $F*f \in B^*$.

(c) Define $H*F$ by $(H*F)(f)=H(F*f)$, $H*F \in A^{**}$.

(a)′ Define $y*'f$ by $(y*'f)(x)=f(xy)$, $y*'f \in A^*$.

(b)′ Define $f*'F$ by $(f*'F)(y)=F(y*'f)$, $f*'F \in B^*$.

(c)′ Define $F*'H$ by $(F*'H)(f)=H(f*'F)$, $F*'H \in A^{**}$.

Let $f \in A^*$, $F, G \in A^{**}$ and $H \in B^{**}$. Then it is easy to check that

$$H\circ(F*f) = (H*F)*f \tag{4}$$

and

$$(5) \qquad (F\circ G)*f = \big(F*(G\circ f)\big).$$

The following proposition shows that A^{**} is a left Banach $(B^{**}, \circ)$-module.

PROPOSITION 3.1. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Then for $F, G\in A^{**}$ and $H, K\in B^{**}$, we have*

(i) $\|H*F\| \leqq C\|H\|\|F\|$,

(ii) $(K\circ H)*F=K*(H*F)$,

(iii) $H*(F\circ G)=(H*F)\circ G$.

PROOF. (i) Let $f\in A^*$ and $y\in B$. We have $|(H*F)(f)|=|H(F*f)|\leqq\|H\|\,\|F*f\|$. Since $(F*f)(y)=F(f*y)$ and $\|f*y\|\leqq C\|f\|\,\|y\|$ (by (2)), we get $\|F*f\|\leqq \leqq C\|F\|\,\|f\|$. Hence $\|H*F\|\leqq C\|H\|\,\|F\|$.

(ii) Let $f\in A^*$. Then

$$\big((K\circ H)*F\big)(f) = K\big(H\circ(F*f)\big)$$

and

$$\big(K*(H*F)\big)(f) = K\big((H*F)*f\big).$$

Applying (4), we obtain (ii).

(iii) Similarly, using (5), we can show (iii).

In like manner we can show that

$$\|F*'H\| \leqq C\|F\|\|H\|,\quad F*'(H\circ' K) = (F*'H)*'K,\quad (F\circ' G)*'H = F\circ'(G*'H).$$

For $g\in B^*$, let g_A be the restriction of g to A. By inequality (3), $g_A\in A^*$ and $\|g_A\|_{A^*}\leqq D\|g\|_{B^*}$. For $F\in A^{**}$, let $\varphi(F)\in B^{**}$ be defined by $\varphi(F)(g)=F(g_A)$. It follows that the mapping φ is a continuous algebra homomorphism of A^{**} into B^{**} for either Arens product ([2, p. 3], [15]).

We observe that if $f\in A^*$ and $F\in A^{**}$ then $(F*f)_A=F\circ f$ since $f*x=f\circ x$ for all $x\in A$. Likewise $(f*'F)_A=f\circ' F$.

PROPOSITION 3.2. *For $F\in A^{**}$ and $H\in B^{**}$, we have*

(a) $\varphi(H*F)=H\circ\varphi(F)$

(b) $\varphi(F*'H)=\varphi(F)\circ' H$.

That is, φ is a module homomorphism for either Arens product.

PROOF. We prove (a). Let $g\in B^*$. Then

$$\big(\varphi(H*F)\big)(g) = (H*F)(g_A) = H(F*g_A)$$

and

$$\big(H\circ\varphi(F)\big)(g) = H\big(\varphi(F)\circ g\big).$$

Since $F*g_A=\varphi(F)\circ g$, we obtain (a).

COROLARY 3.3. *If B is Arens regular then $\varphi(A^{**})$ is an ideal of B^{**}.*

NOTATION. Let $l(A^{**}, \circ)=\{G\in A^{**}: G\circ F=0$ for all $F\in A^{**}\}$ and $r(A^{**}, \circ)=$ $=\{G\in A^{**}: F\circ G=0$ for all $F\in A^{**}\}$. Similarly we define $l(A^{**}, \circ')$ and $r(A^{**}, \circ')$. Let $\ker(\varphi)$ denote the kernel of φ and R_1^{**} (resp. R_2^{**}) be the radical of $(A^{**}, \circ)$ (resp. $(A^{**}, \circ')$).

LEMMA 3.4. $\ker(\varphi)\subset l(A^{**}, \circ)\cap r(A^{**}, \circ')$.

PROOF. Let $F\in\ker(\varphi)$, $G\in A^{**}$ and $f\in A^*$. Then

$$(F\circ G)(f) = F(G\circ f) = F\big((G*f)_A\big) = \varphi(F)(G*f) = 0.$$

Hence $F\circ G=0$ so that $F\in l(A^{**}, \circ)$. Similarly

$$(G\circ' F)(f) = F(f\circ' G) = F\big((f*' G)_A\big) = \varphi(F)(f*' G) = 0.$$

Hence $G\circ' F=0$ so that $F\in r(A^{**}, \circ')$.

THEOREM 3.5. (i) *If* $(\varphi(A^{**}), \circ)$ *is semisimple then*

$$R_1^{**} = \ker(\varphi) = l(A^{**}, \circ).$$

(ii) *If* $(\varphi(A^{**}), \circ')$ *is semisimple then*

$$R_2^{**} = \ker(\varphi) = r(A^{**}, \circ').$$

PROOF. We prove (i). The proof of (ii) is similar. Suppose that $(\varphi(A^{**}), \circ)$ is semisimple. Let $R\in R_1^{**}$ and $F\in A^{**}$. Since $R\circ F\in R_1^{**}$, the spectral radius of $R\circ F$ is zero. Therefore $\varphi(R\circ F)=\varphi(R)\circ\varphi(F)$ has zero spectral radius (φ is continuous) and consequently $\varphi(R)$ is in the radical of $(\varphi(A^{**}), \circ)$. Since $(\varphi(A^{**}), \circ)$ is semisimple, $\varphi(R)=0$ so that $R\in\ker(\varphi)$. Thus $R_1^{**}\subset\ker(\varphi)$. Now $l(A^{**}, \circ)\subset$ $\subset R_1^{**}$ and, by Lemma 3.4, $\ker(\varphi)\subset l(A^{**}, \circ)$. Hence $R_1^{**}=\ker(\varphi)=l(A^{**}, \circ)$. (See [2].)

COROLLARY 3.6. *If* $\varphi(A^{**})$ *is semisimple in each of the Arens products then*

$$R_1^{**} = R_2^{**} = \ker(\varphi) = l(A^{**}, \circ) = r(A^{**}, \circ) = l(A^{**}, \circ') = r(A^{**}, \circ').$$

PROOF. In view of (i) and (ii) above, we need only to show that $l(A^{**}, \circ)\subset$ $\subset l(A^{**}, \circ')$ and $r(A^{**}, \circ')\subset r(A^{**}, \circ)$. Let $F\in l(A^{**}, \circ)$. Then $F\circ'\pi_A(a)=$ $=F\circ\pi_A(a)=0$ for all $a\in A$ so that, by the w^*-continuity of the product $\circ'$ on the right and the w^*-density of $\pi_A(A)$ in A^{**} we obtain $F\circ' G=0$ for all $G\in A^{**}$. Therefore $F\in l(A^{**}, \circ')$. Hence $l(A^{**}, \circ)\subset l(A^{**}, \circ')$. Similarly we can show that $r(A^{**}, \circ')\subset r(A^{**}, \circ)$. This completes the proof.

PROPOSITION 3.7. *If B is Arens regular and B^{**} is semisimple then $\varphi(A^{**})$ is a semisimple algebra.*

PROOF. Suppose that B is Arens regular. Then, by Corollary 3.3, $\varphi(A^{**})$ is an ideal of B^{**}. Therefore every primitive ideal of $\varphi(A^{**})$ is of the form $\varphi(A^{**})\cap P$, where P is a primitive ideal of B^{**} [10, Proposition 2, p. 206]. Hence if B^{**} is semisimple then so is $\varphi(A^{**})$.

From Corollary 3.6 we see that if B is Arens regular and $\varphi(A^{**})$ is semisimple then $R_1^{**}=R_2^{**}$. From now on whenever these conditions on B and $\varphi(A^{**})$ are met (in particular when B^{**} is semisimple) we will identify R_1^{**} with R_2^{**} and write it simply as R^{**}. We observe that since A is semisimple we have $\pi_A(A)\cap R^{**}=(0)$ [7, Theorem 4.6, p. 130].

4. Direct sum decomposition of A^{**} and Arens regularity of A

Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Let $\{e_\alpha: \alpha\in\Omega\}$ be a bounded approximate identity of B contained in A. We will assume in the rest of this paper that $w^*\text{-}\lim\limits_\alpha \pi_B(e_\alpha)=E$, where E is a right identity for $(B^{**}, \circ)$ and a left identity for $(B^{**}, \circ')$. Whenever B is Arens regular and $\varphi(A^{**})$ is semisimple (so that $R_1^{**}=R_2^{**}=R^{**}$) we have the following direct sum decomposition of A^{**} for each of the Arens products. (See also [2] and [15].)

THEOREM 4.1. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that* (i) *B has a bounded approximate identity contained in A,* (ii) *B is Arens regular and* (iii) *$\varphi(A^{**})$ is semisimple. Then the following statements are true:*

(a) $A^{**}=R^{**}\oplus M$, *where M is a closed ideal in $(A^{**}, \circ)$.*
(b) $A^{**}=R^{**}\oplus M'$, *where M' is a closed ideal in $(A^{**}, \circ')$.*
(c) *φ is an algebra isomorphism on M and M'.*
(d) *If $M=M'$ then A is Arens regular.*

PROOF. Let $\{e_\alpha: \alpha\in\Omega\}$ be a bounded approximate identity of B contained in A and let $E=w^*\text{-}\lim\limits_\alpha \pi_B(e_\alpha)$. For $f\in A^*$ and $F\in A^{**}$, we have

$$(E*F)(f) = E(F*f) = \lim_\alpha \pi_B(e_\alpha)(F*f) = \lim_\alpha (F*f)(e_\alpha) =$$
$$= \lim_\alpha (F\circ f)(e_\alpha) = \lim_\alpha \big(\pi_A(e_\alpha)\circ F\big)(f),$$

so that $E*F=w^*\text{-}\lim\limits_\alpha \big(\pi_A(e_\alpha)\circ F\big)$.

(a) We observe that the mapping P_E: $F\to E*F$ is a bounded projection on $(A^{**}, \circ)$. In fact, from Proposition 3.1 we see that P_E is continuous and that $E*(E*F)=(E\circ E)*F=E*F$ so that $P_E^2=P_E$. Let $M=P_E(A^{**})=\{E*F: F\in A^{**}\}$. Given $F\in A^{**}$, we have (Proposition 3.2)

$$\varphi(F-E*F) = \varphi(F)-E\circ\varphi(F) = \varphi(F)-\varphi(F) = 0$$

whence $F-E*F\in R^{**}$ (Corollary 3.6). Since $F=(F-E*F)+E*F$, we get $A^{**}=R^{**}+M$. Now suppose $F\in M\cap R^{**}$. Then, by Corollary 3.6, $F=E*F=w^*\text{-}\lim\limits_\alpha \big(\pi_A(e_\alpha)\circ F\big)=w^*\text{-}\lim\limits_\alpha 0=0$. Hence $A^{**}=R^{**}\oplus M$. For $U, V\in R^{**}$ and $F, G\in M$, we have

$$(U+F)\circ(V+G) = U\circ V+U\circ G+F\circ V+F\circ G = F\circ G =$$
$$= (E*F)\circ G = E*(F\circ G)\in M.$$

Thus $(A^{**})^2 \subseteq M$ and so M is an ideal of $(A^{**}, \circ)$. Since $M = P_E(A^{**})$ and P_E is a continuous projection, M is a closed ideal of $(A^{**}, \circ)$.

(b) This follows from (a) by symmetry of the argument. The mapping $P'_E\colon F \to F *' E$ is a continuous projection on A^{**} and $M' = P'_E(A^{**}) = \{F *' E'\colon F \in A^{**}\}$ is a closed ideal of $(A^{**}, \circ')$. We have $A^{**} = R^{**} \oplus M'$ and $(A^{**})^2 \subseteq M'$.

(c) By Corollary 3.6, $\ker(\varphi) = R^{**}$. Therefore φ is an algebra isomorphism on M and M'.

(d) By (c), φ is an algebra isomorphism on M and M'. Hence if $M = M'$ then the Arens products agree on $M(M')$ and, by Corollary 3.6, they agree on R^{**}. As R^{**} and M are ideals with $R^{**} \oplus M = A^{**}$, it follows that A is Arens regular.

REMARK 1. Let A and B satisfy all the conditions of Theorem 4.1. For $F \in A^{**}$, we have $E * F = w^*\text{-}\lim_\alpha (\pi_A(e_\alpha) \circ F)$ and $F *' E = w^*\text{-}\lim_\alpha (F \circ' \pi_A(e_\alpha))$. In what follows M (resp. M') will denote the closed ideal of $(A^{**}, \circ)$ (resp. $(A^{**}, \circ')$) given by $M = \{E * F\colon F \in A^*\}$ (resp. $M' = \{F *' E\colon F \in A^{**}\}$). We have

$$R^{**} = \{F - E * F\colon F \in A^{**}\} = \{F - F *' E\colon F \in A^{**}\}.$$

For $F, G \in A^{**}$, $F \circ G = E * (F \circ G) = (E * F) \circ (E * G)$ and $F \circ' G = (F \circ' G) *' E = (F *' E) \circ' (G *' E)$.

PROPOSITION 4.2. *Assume that the conditions of Theorem* 4.1 *are satisfied. Let E be the identity of B^{**}. Then the mapping $\varrho\colon F \to E * F$ is a continuous algebra homomorphism of $(A^{**}, \circ')$ onto $(M, \circ)$. The restriction of ϱ to M' is a topological algebra isomorphism of M' onto M.*

PROOF. Clearly ϱ is linear and continuous. Let $F, G \in A^{**}$. We claim that $E * (F \circ' G) = (E * F) \circ (E * G)$. By Proposition 3.2 and Arens regularity of B,

$$\varphi(E * (F \circ' G)) = \varphi(F) \circ \varphi(G).$$

Also

$$\varphi(E * F) \circ \varphi(E * G) = \varphi(F) \circ \varphi(G).$$

Since φ is one-to-one on M, we get

$$E * (F \circ' G) = (E * F) \circ (E * G).$$

Now let $F \in M$ and write $F = F_1 + F_2$ with $F_1 \in M'$ and $F_2 \in R^{**}$. Since $E * F_2 = 0$, we get $E * F = E * F_1 = \varrho(F_1)$. But $E * F = F$ since $F \in M$. Hence ϱ is onto. This shows in particular that $\varrho(M') = M$. Finally, suppose that $E * F' = 0$ for some $F' \in M'$. Then $0 = \varphi(E * F') = E \circ \varphi(F') = \varphi(F')$ so that $F' \in \ker(\varphi) = R^{**}$. Therefore $F' \in M' \cap R^{**} = (0)$ and so ϱ is one-to-one on M'. Hence the restriction of ϱ to M' is a bicontinuous algebra isomorphism of M' onto M.

Likewise $\varrho'\colon F \to F *' E$ is a continuous algebra homomorphism of $(A^{**}, \circ)$ onto $(M', \circ')$ and the restriction of ϱ' to M is a topological algebra isomorphism of M onto M'. We have $(F \circ G) *' E = (F *' E) \circ' (G *' E)$. Therefore, in view of Remark 1, it follows that, for all $F, G \in A^{**}$, $E * (F \circ' G) = F \circ G$ and $(F \circ G) *' E = F \circ' G$.

COROLLARY 4.3. *$M \circ M$ is dense in M if and only if $M' \circ' M'$ is dense in M'.*

COROLLARY 4.4. *If A is Arens regular and $M \circ M$ is dense in M (or equivalently $M' \circ' M'$ is dense in M') then $M = M'$.*

PROOF. If $F, G\in M'$ then $F\circ' G=F\circ G\in M$ so that $M'\circ' M'\subset M$. Therefore $M'\subset M$ Similarly $M\subset M'$. Hence $M=M'$.

COROLLARY 4.5. *If* $P_E P'_E=P'_E P_E$ *then* A *is Arens regular.*

PROOF. Let $F, G\in A^{**}$. We have

$$P_E P'_E(F\circ G) = E*((F\circ G)*'E)) = E*(F\circ' G) = F\circ G$$

and

$$P'_E P_E(F\circ G) = (E*(F\circ G))*'E = (F\circ G)*'E = F\circ' G.$$

Hence if $P_E P'_E=P'_E P_E$ then $F\circ' G=F\circ G$ and A is Arens regular.

PROPOSITION 4.6. *Let B be a Banach algebra which is Arens regular and contains a bounded left approximate identity* $\{e_\alpha\colon \alpha\in\Omega\}$. *Then for all* $g\in B^*$ *and* $H\in B^{**}$,

$$H(g\circ e_\alpha) \to H(g).$$

PROOF. We have

$$\lim_\alpha H(g\circ e_\alpha) = \lim_\alpha ((H\circ g)(e_\alpha) = \lim_\alpha \pi_B(e_\alpha)(H\circ g) = E(H\circ g) = (E\circ H)(g) = H(g).$$

PROPOSITION 4.7. *Assume that the conditions of Theorem* 4.1 *except* (iii) *are satisfied. Then for all* $f\in A^*$, $F\in A^{**}$ *ar .* $H\in B^{**}$,

$$(H*F)(f\circ e_\alpha) \to (H*F)(f).$$

PROOF. We have $(H*F)(f\circ e_\alpha)=H(F*(f\circ e_\alpha))=H((F*f)\circ e_\alpha)$. Since B is Arens regular and $F*f\in B^*$, by Proposition 4.6, $H((F*f)\circ e_\alpha)\to H(F*f)=$ $=(H*F)(f)$.

THEOREM 4.8. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that* (i) *B has a bounded approximate identity* $\{e_\alpha\colon \alpha\in\Omega\}$ *contained in A,* (ii) *B is Arens regular and* (iii) $\varphi(A^{**})$ *is semisimple. If, for all* $f\in A^*$, $F\in A^{**}$ *and* $H\in B^{**}$,

$$(F*H)(f\circ e_\alpha) \to (F*'H)(f),$$

then $M=M'$ *and A is Arens regular.*

PROOF. By Proposition 4.7, $(H*F)(f\circ e_\alpha)\to(H*F)(f)$. Assume that

$$(F*'H)(f\circ e_\alpha) \to (F*'H)(f).$$

Let $H=E$. Then

$$(E*F)(f\circ e_\alpha) \to (E*F)(f) = P_E(F)(f)$$

and

$$(F*'E)(f\circ e_\alpha) \to (F*'E)(f) = P'_E(F)(f).$$

Now, by (1) in Section 3, $f\circ e_\alpha$ extends to a bounded linear functional $(f\circ e_\alpha)'$ on B, where $(f\circ e_\alpha)'(y)=f(e_\alpha y)$, $y\in B$, and

$$(E*F)(f\circ e_\alpha) = \varphi(E*F)((f\circ e_\alpha)') = (E\circ\varphi(F))((f\circ e_\alpha)') = \varphi(F)((f\circ e_\alpha)').$$

Similarly

$$(F*'E)(f\circ e_\alpha) = \varphi(F)\big((f\circ e_\alpha)'\big).$$

Hence $P_E(F)=P'_E(F)$, for all $F\in A^{**}$, which gives $P_E=P'_E$. Thus $M=M'$ and therefore by Theorem 4.1, A is Arens regular.

COROLLARY 4.9 [12]. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that* (i) *B is Arens regular,* (ii) *B has a bounded approximate identity contained in the center of A and* (iii) *B^{**} is semisimple. Then A is Arens regular.*

PROOF. Let $\{e_\alpha\colon \alpha\in\Omega\}$ be a bounded approximate identity of B contained in the center of A. Then $\{\pi_A(e_\alpha)\colon \alpha\in\Omega\}$ is contained in the center of $(A^{**}, \circ)$ and $(A^{**}, \circ')$. Let $f\in A^*$, $F\in A^{**}$ and $H\in B^{**}$. Then

$$(F*'H)(f\circ e_\alpha) = (F*'H)\big(f\circ'\pi_A(e_\alpha)\big) = \big(\pi_A(e_\alpha)\circ'(F*'H)\big)(f) =$$
$$= \big((F*'H)\circ'\pi_A(e_\alpha)\big)(f) \to \big((F*'H)*'E\big)(f) = (F*'H)(f).$$

Therefore, by Theorem 4.8, A is Arens regular.

THEOREM 4.10. *Let A be a w.c.c. semisimple Banach algebra which is a dense ideal in a semisimple Banach algebra B. Assume that* (i) *B has a bounded approximate identity contained in A and* (ii) *B is Arens regular. Then A is Arens regular.*

PROOF. Let $\{e_\alpha\colon \alpha\in\Omega\}$ be a bounded approximate identity of B contained in A. By [13, Lemma 3.2, p. 4], B is w.c.c. and therefore, by [14, Corollary 4.3, p. 298], B^{**} is semisimple. Thus A^{**} has the direct sum decompositions of Theorem 4.1. Let $F, G\in A^{**}$. Then, using the fact that $\pi_A(A)$ is an ideal in A^{**} for either Arens product, we obtain (see Remark 1)

$$F\circ G = (F*'E)\circ G = w^*\text{-}\lim_\alpha \big((F\circ'\pi_A(e_\alpha))\circ G\big) =$$
$$= w^*\text{-}\lim_\alpha \big((F\circ\pi_A(e_\alpha))\circ G\big) = w^*\text{-}\lim_\alpha \big(F\circ'(\pi_A(e_\alpha)\circ G)\big) = F\circ'(E*G) = F\circ' G.$$

Therefore A is Arens regular.

THEOREM 4.11. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that* (i) *B has a bounded approximate identity contained in A,* (ii) *B is Arens regular,* (iii) *every continuous linear map from B to B^* is weakly compact* (w.c.c.) *and* (iv) *$\varphi(A^{**})$ is semisimple. Then A is Arens regular.*

PROOF. Let $\{e_\alpha\colon \alpha\in\Omega\}$ be a bounded approximate identity of B contained in A and such that $E=w^*\text{-}\lim_\alpha \pi_B(e_\alpha)$. Let $f\in A^*$, $F\in A^{**}$ and $H\in B^{**}$. We have

$$(F*'H)(f\circ e_\alpha) = H\big((f\circ e_\alpha)*'F\big).$$

The map $T\colon B\to B^*$ given by $\big(T(b)\big)=(f*b)*'F$, $b\in B$, is continuous and therefore weakly compact.

We can now apply the argument in [2, p. 5] to show that

$$(F*'H)(f\circ e_\alpha) \to (F*'H)(f).$$

Therefore, by Theorem 4.8, A is Arens regular.

COROLLARY 4.12. *Let A be a semisimple Banach algebra which is a dense ideal in a B^*-algebra B. Then A is Arens regular.*

PROOF. By [5, Proposition (1.7.2), p. 15], A contains a bounded approximate identity of B and, by [4, Theorem 7.1, p. 869], B is Arens regular and B^{**} is semisimple. By [1, Corollary II.9, p. 293], every continuous linear map T from B into B^* is weakly compact. Therefore, by Theorem 4.11, A is Arens regular.

COROLLARY 4.13. *Let A and B be semisimple Banach algebras such that A is a dense ideal in B. Assume that* (i) *B has a bounded approximate identity contained in A and* (ii) *B is reflexive. Then A is Arens regular.*

PROOF. Clearly B is Arens regular and B^{**} is semisimple. By [6, Corollary 3, p. 483], every continuous linear map from B to B^* is weakly compact. Therefore, by Theorem 4.11, A is Arens regular.

References

[1] C. A. Akemann, The dual space of an operator algebra, *Trans. Amer. Math. Soc.*, **126** (1967), 286—302.
[2] F. E. Alexander, The bidual of A^*-algebras of the first kind, *J. London Math. Soc.*, **12** (1975), 1—6.
[3] R. E. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc.*, **2** (1951), 839—848.
[4] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pac. J. Math.*, **11** (1961), 847—870.
[5] J. Diximer, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars (Paris, 1968).
[6] N. Dunford and J. Schwartz, *Linear operators*, Part I, Interscience Publishers (New York, 1958).
[7] S. L. Gulick, Commutativity and ideals in the biduals of topological algebras, *Pac. J. Math.*, **18** (1966), 121—137.
[8] J. O. Hennefeld, A note on Arens products, *Pac. J. Math.*, **26** (1968), 115—119.
[9] J. O. Hennefeld, Finding a maximal subalgebra on which the two Arens products agree, *Pac. J. Math.*, **59** (1975), 93—98.
[10] N. Jacobson, *Structure of rings*, Colloq. Publications Amer. Math. Soc., Vol. 37 (1956).
[11] M. Leinert, A contribution to Segal algebras, *Manuscripta Math.*, **19** (1973), 297—306.
[12] S. A. McKilligan, Arens regularity of certain Banach algebras which are ideals of a Banach algebra, *Proc. Amer. Math. Soc.*, **50** (1975), 223—229.
[13] B. J. Tomiuk, On some properties of Segal algebras and their multipliers, *Manuscripta Math.*, **27** (1979), 1—18.
[14] B. J. Tomiuk, Arens regularity and the algebra of double multipliers, *Proc. Amer. Math. Soc.*, **81** (1981), 293—298.
[15] P. K. Wong, The second conjugates of certain Banach algebras, *Can. J. Math.*, **27** (1975), 1029—1035.

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ON SUBWEBS OF 3-WEBS AND SUBALGEBRAS OF LOCAL W_k-ALGEBRAS

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2-dimensional subwebs of a 3-web have been investigated in [1], where it was proved that 2-subwebs are induced on transversal geodesic surfaces of a web. In the following we give a systematic investigation of the properties of 3-webs having subwebs of various dimension. Especially we prove that only the Grassmannian 3-webs have a maximal variety of subwebs. If a smooth manifold M is equipped with a 3-web structure then for every point $p \in M$ a coordinate loop can be introduced [4]. This loop multiplication defines a series of k-linear operations ($k=2, 3, 4, \ldots$) on the tangent vectorspace T_e of this loop at the identity e satisfying some identities. This vectorspace T_e with these operations of arity $h \leqq k$ is called the local W_k algebra of the web. Local W_3 algebras have been already investigated in [4] where they were called W-algebras. Especially, the coordinate loops of a group 3-web are isomorphic to a Lie group, their W_2-algebras are the corresponding Lie algebras, and the W_k-algebras ($k \geqq 3$) are trivial. We prove that if a 3-web W contains a subweb $\tilde{W}$ then the corresponding $\tilde{W}_k$-algebras are subalgebras of W_k-algebras of the original web. With help of such subalgebras we characterize isocline and Grassmannian 3-webs: W_2-algebras of the first and W_3-algebras of the second are almost trivial.

1. We consider a 3-web W defined on a manifold M of dimension $2r$ by the foliations λ_α $(\alpha, \beta, \gamma = 1, 2, 3)$ of codimension r. We have the differential equations of these foliations (cf. [1])

$$\underset{1}{\omega}^i = 0, \quad \underset{2}{\omega}^i = 0, \quad \underset{3}{\omega}^i = 0. \tag{1}$$

The forms $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ define a coframe on the tangent bundle $T(M)$ satisfying the structure equations

$$\begin{cases} d\underset{1}{\omega}^i = \underset{1}{\omega}^j \wedge \omega_j^i + a_{jk}^i \underset{1}{\omega}^j \wedge \underset{1}{\omega}^k, \\ d\underset{2}{\omega}^i = \underset{2}{\omega}^j \wedge \omega_j^i - a_{jk}^i \underset{2}{\omega}^j \wedge \underset{2}{\omega}^k, \end{cases} \tag{2}$$

$$d\omega_j^i = \omega_j^k \wedge \omega_j^i + b_{jkl}^i \underset{1}{\omega}^k \wedge \underset{2}{\omega}. \tag{3}$$

The quantities a_{jk}^i and b_{jkl}^i form tensorfields on $T(M)$ called the torsion and curvature tensors of 3-web W. They are connected by the relations

$$b_{[jkl]}^i = 2a_{[jk}^m a_{|m|l]}^i, \tag{4}$$

$$\nabla a_{jk}^i = b_{[j|l|k]}^i \underset{1}{\omega}^l + b_{[jk]l}^i \underset{2}{\omega}^l. \tag{5}$$

By exterior derivation we get from the equations (3)

$$\nabla b^i_{jkl} = \underset{2}{c}^i_{jklm}\underset{1}{\omega}^m - \underset{1}{c}^i_{jklm}\underset{2}{\omega}^m. \tag{6}$$

Equations (2) and (3) define an affine connection Γ on the manifold M. The geodesics of this connection satisfy the equations

$$d\underset{1}{\omega}^i + \underset{1}{\omega}^j\omega^i_j = \theta\underset{1}{\omega}^i, \quad d\underset{2}{\omega}^i + \underset{2}{\omega}^j\omega^i_j = \theta\omega^i, \tag{7}$$

where d denotes the ordinary derivation by the (not necessarily affine) parameter of geodesic. From these equations we see that the leaves of the web W are total geodesic submanifolds with respect to the connection Γ.

We need the following property of Γ. Let V_α be a leaf of the web through a point $p\in M$. We denote by $\varphi_{\alpha\beta}$ the projection map of the leaf V_α onto the leaf V_β defined on a neighbourhood of p with the help of the leaves of the third foliation λ_γ $(\alpha\neq\beta\neq\gamma\neq\alpha)$. This local diffeomorphism maps geodesics onto geodesics. If we denote by $\underset{1}{e}_i$ and $\underset{2}{e}_i$ the vectors of the tangent frame dual to the coframe $\{\underset{1}{\omega}^i, \underset{2}{\omega}^i\}$, the tangent vectors to the manifold M can be written in the form [1]

$$\xi = \underset{1}{\omega}^i(\xi)\underset{2}{e}_i - \underset{2}{\omega}^i(\xi)\underset{1}{e}_i. \tag{8}$$

We can see that the vectors $\underset{1}{e}_i$, $\underset{2}{e}_i$ and $\underset{3}{e}_i = -\underset{1}{e}_i - \underset{2}{e}_i$ are tangent to the leaves V_1, V_2 and V_3 through p, respectively. Moreover we have (cf. [1])

$$d\varphi_{\alpha\beta}|_p(\underset{\alpha}{e}_i) = \underset{\beta}{e}_i. \tag{9}$$

2. We generalize the notion of transversal-geodesic 2-surfaces of a web introduced in [1]. Let the $3s$ tangent vectors $\xi_a = \xi^i_a \underset{\alpha}{e}_i$ $(a, b, c = 1, \ldots, s)$ be given in the tangent space T_p at $p\in M$. By property (9) of the map $\varphi_{\alpha\beta}$ we have

$$d\varphi_{\alpha\beta}|_p(\underset{\alpha}{\xi}_a) = \underset{\beta}{\xi}_a,$$

that is the subspace defined by the $2s$-vector

$$t = \underset{\alpha}{\xi}_1\wedge\ldots\wedge\underset{\alpha}{\xi}_s\wedge\underset{\beta}{\xi}_1\wedge\ldots\wedge\underset{\beta}{\xi}_s$$

is invariant with respect to the operator $d\varphi_{\alpha\beta}|_p$. We call the subspace in T_1 defined by this $2s$-vector t a transversal subspace of the web W.

Definition 1. A submanifold $\tilde{M}$ of dimension $2s$ in the 3-web manifold (M, W) is called *transversal geodesic submanifold* if its tangent spaces are transversal $2s$-spaces of the web W.

Definition 2. Let $\tilde{M}$ be a submanifold of dimension $2s$ in the 3-web manifold (M, W). A 3-web structure $\tilde{W}$ defined on the submanifold $\tilde{M}$ is called a *subweb of W* if its leaves are intersections of $\tilde{M}$ with the leaves of W.

Theorem 1. *If the 3-web $(\tilde{M}, \tilde{W})$ is a subweb of the 3-web (M, W) then $\tilde{M}$ is a transversal geodesic submanifold of M.*

PROOF. We denote by $\tilde{V}_\alpha$ the leaves of the subweb $\tilde{W}$ through the point $p\in\tilde{M}\subset M$. Let $\tilde{\varphi}_{\alpha\beta}$ be the mapping of $\tilde{V}_\alpha$ onto $\tilde{V}_\beta$ defined with help of the third foliation of $\tilde{W}$ in a neighbourhood of p. Since $\tilde{V}_\alpha\subset V_\alpha$ the maps $\tilde{\varphi}_{\alpha\beta}$ are equal to the maps $\varphi_{\alpha\beta}$ on $\tilde{V}_\alpha$:

$$\tilde{\varphi}_{\alpha\beta}=\varphi_{\alpha\beta}|_{\tilde{V}_\alpha}.$$

The tangent maps $d\tilde{\varphi}_{\alpha\beta}|_p$: $\tilde{T}_\alpha\to\tilde{T}_\beta$ are the restrictions of $d\varphi_{\alpha\beta}|_p$: $T_\alpha\to T_\beta$. Thus the $2s$-vectors $\tilde{T}_\alpha\wedge\tilde{T}_\beta$ are invariant with respect to the maps $d\varphi_{\alpha\beta}|_p$. Since the $2s$-vector $\tilde{T}_1\wedge\tilde{T}_2$ determines the tangent space $\tilde{T}_p$ to the submanifold $\tilde{M}$ at the point p uniquely, it is invariant with respect to $d\varphi_{12}|_p$ and the theorem is proved.

3. We denote by $\{\underset{1}{\theta^a},\underset{2}{\theta^a}\}$ an adapted coframe on the subweb $(\tilde{M},\tilde{W})$. The differential of the imbedding f: $\tilde{M}\to M$ can be written in the form

$$\underset{1}{\omega^i}=\xi^i_a\underset{1}{\theta^a},\quad \underset{2}{\omega^i}=\xi^i_a\underset{2}{\theta^a}. \tag{10}$$

The leaves of the subweb $\tilde{W}$ are defined by the equations

$$\underset{1}{\theta^a}=0,\quad \underset{2}{\theta^a}=0,\quad \underset{1}{\theta^a}+\underset{2}{\theta^a}=0. \tag{11}$$

The forms $\underset{1}{\theta^a}$ and $\underset{2}{\theta^a}$ satisfy the structure equations of a 3-web

$$d\underset{1}{\theta^a}=\underset{1}{\theta^b}\wedge\theta^a_b+\tilde{a}^a_{bc}\underset{1}{\theta^b}\wedge\underset{1}{\theta^c},\quad d\underset{2}{\theta^a}=\underset{2}{\theta^b}\wedge\theta^a_b-\tilde{a}^b_{ac}\underset{2}{\theta^b}\wedge\underset{2}{\theta^c}, \tag{12}$$

$$d\theta^a_b=\theta^c_b\wedge\theta^a_c+\tilde{b}^a_{bcd}\underset{1}{\theta^c}\wedge\underset{2}{\theta^d}, \tag{13}$$

similarly to the equations (2) and (3). On the other hand $\underset{1}{\theta^a}$ and $\underset{2}{\theta^a}$ satisfy the exterior derivatives of the equations (10). Using (2) and (12) we get

$$(\nabla\xi^i_a-\xi^i_b\theta^b_a)\wedge\underset{1}{\theta^a}+(\xi^i_c\tilde{a}^c_{ab}-a^i_{jk}\xi^j_a\xi^j_b)\underset{1}{\theta^a}\wedge\underset{1}{\theta^b}=0,$$

$$(\nabla\xi^i_a-\xi^i_b\theta^b_a)\wedge\underset{2}{\theta^a}-(\xi^i_c\tilde{a}^c_{ab}-a^i_{jk}\xi^j_a\xi^k_b)\underset{2}{\theta^a}\wedge\underset{2}{\theta^b}=0,$$

where $\nabla\xi^i_a=d\xi^i_a+\xi^j_a\omega^i_j$. It follows

$$\nabla\xi^i_a=\xi^i_b\theta^b_a \tag{14}$$

and

$$a^i_{jk}\xi^j_b\xi^k_c=\xi^i_a\tilde{a}^a_{bc}. \tag{15}$$

We get from the equation (14) by exterior derivation and using (3) and (13)

$$b^i_{jkl}\xi^j_b\xi^k_c\xi^l_d=\xi^i_a\tilde{b}^a_{bcd}. \tag{16}$$

We remark that in the case $s=1$ equation (15) is trivial and (16) gives $b^i_{jkl}\xi^j\xi^k\xi^l= =\xi^i\tilde{b}$ which was considered in [1].

By covariant derivation of (16) we get, using (10) and (6),

$$\underset{1}{c^i_{jklm}}\xi^j_b\xi^k_c\xi^l_d\xi^m=\xi^i_a\underset{1}{\tilde{c}^a_{bcdf}},\quad \underset{2}{c^i_{jklm}}\xi^j_b\xi^k_c\xi^l_d\xi^m_f=\xi^i_a\underset{2}{\tilde{c}^a_{bcdf}}. \tag{17}$$

We can see that further derivations will give analogous relations between covariant derivatives of higher order. Thus we proved

THEOREM 2. *The functions* ξ^i_c *defining the imbedding* $f\colon \tilde{M} \to M$ *satisfy the equations* (15), (16), (17) *and their analogues connecting the fundamental tensors and their covariant derivatives of the webs* W *and* $\tilde{W}$.

In the special case $s=1$, this theorem was proved in [1].

The canonical affine connection $\tilde{\Gamma}$ on the web manifold $\tilde{M}$ defined by the forms θ^a_b satisfy the structure equations (13). The parallel translation of tangent vectors corresponding to the connection $\tilde{\Gamma}$ on $\tilde{M}$ is the same as the parallel translation corresponding to the canonical connection Γ of the web (M, W). Indeed, a parallel vectorfield η with coordinates $\underset{\alpha}{\tilde{\eta}}{}^a$ on $\tilde{M}$ satisfies

$$d\underset{\alpha}{\tilde{\eta}}{}^a + \underset{\alpha}{\tilde{\eta}}{}^b \theta^a_b = 0, \quad \alpha = 1, 2; \ a, b = 1, \dots, s.$$

The coordinates of this vectorfield with respect to the frame $\underset{\alpha}{e}_i$ on M are $\underset{\alpha}{\eta}{}^i = \underset{\alpha}{\xi}{}^i \underset{\alpha}{\tilde{\eta}}{}^a$. Using the equations (14) we get

$$d\underset{\alpha}{\eta}{}^i + \underset{\alpha}{\eta}{}^j \omega^i_j = 0,$$

that is the vectorfield η is parallel with respect to the connection Γ, too. It follows that a transversal geodesic submanifold $\tilde{M}$ in M is total geodesic. This generalizes a theorem on transversal geodesic 2-surfaces proved in [1].

4. Our aim is now to characterize the 3-webs (M, W) of dimension $2r$ having the following property: for every point $p \in M$ and every transversal tangent subspace $\subset T_p M$ of dimension $2s$ (s is fixed, $1 \leqq s \leqq r$) there exists a transversal geodesic submanifold tangent to the given subspace. We shall denote this property by P_s. It is well known that 3-webs having property P_1 are transversal geodesic webs (cf. [1]). On the other hand, Grassmannian webs defined in [3] have the property P_s for $s = 1, 2, \dots, r$. Indeed, Grassmannian webs can be represented on Grassmannian manifolds of lines of a projective space P^{r+1} and every projective subspace of dimension $s+1$ determines a subweb of dimension $2s$.

THEOREM 3. *A 3-web* (M, W) *of dimension* $2r$ $(r > 2)$ *having property* P_2 *is Grassmannian.*

PROOF. At first we prove that property P_1 follows from P_2. In fact, consider a transversal 2-plane π in the tangent space $T_p(M)$ and let π_1 and π_2 be transversal 4-spaces having the intersection $\pi = \pi_1 \cap \pi_2$. By P_2, there exist transversal geodesic 4-submanifolds M_1 and M_2 tangent to π_1 and π_2, respectively. Then their intersection $M_1 \cap M_2$ is a transversal geodesic 2-submanifold tangent to π. Thus we get property P_1 and the 3-web W is transversal geodesic.

If W has property P_2, equations (15) are satisfied identically with respect to the variables ξ^i_a. It follows that the tensor $\tilde{a}^a_{bc}$ is a linear form of these variables. Since it alternates in the indices b and c, we have

$$\tilde{a}^a_{bc} = \lambda^a_{bk} \xi^k_c - \lambda^a_{ck} \xi^k_b.$$

Using (15) we get

$$a^i_{jk}\xi^j_b\xi^k_c = \xi^i_a(\lambda^a_{bk}\xi^k_c - \lambda^a_{ck}\xi^k_b),$$

or equivalently

$$a^i_{jk}\delta^{b'}_b\delta^{c'}_c = \lambda^{b'}_{bk}\delta^i_j\delta^{c'}_c - \lambda^{c'}_{cj}\delta^i_k\delta^{b'}_b.$$

By contraction on the pairs of indices b, b' and c, c' we get

$$a^i_{jk} = a_{[j}\delta^i_{k]}, \tag{18}$$

where $a_j = -\frac{2}{s}\eta^b_{bj}$. If $r>2$ this relation is satisfied if and only if W is isocline [1].

Since isocline and transversal geodesic 3-webs are Grassmannian (cf. [3]), the theorem is proved.

It follows that property P_2 implies properties P_s for $2<s\leqq r$.

5. We give an algebraic interpretation of relations (15)—(17). It is well known that on a 3-web (M, W), for every point $p\in M$ a local differentiable coordinate loop $l(p)$ can be introduced. The binary and ternary operations

$$[\xi\eta]^i = a^i_{jk}(p)\xi^j\xi^k, \quad (\xi\eta\zeta)^i = b^i_{jkl}(p)\xi^j\eta^k\zeta^l \tag{19}$$

are defined on the tangent space T_e of the loop $l(p)$ at the identity e, where $\xi, \eta, \zeta\in T_e$ are tangent vectors. These operations are connected by the generalized Jacobi identity (4). The vectorspace T_e with these operations is called W-algebra [4]. In addition to these binary and ternary operations, further operations of arity k $(k=4, 5, \dots)$ can be introduced on the vectorspace T_e with help of the covariant derivatives of fundamental tensors of the 3-web W. Namely, the tensors $t_2=(a^i_{jk})$, $t_3=(b^i_{jkl})$, $\underset{\alpha}{t_4}=(\underset{\alpha}{c}{}^i_{jklm})$ etc. of type $\binom{1}{k}$ define operations $t_k\colon (T_e)^k\to T_e$ of arity k, where $k=2, 3, 4, \dots$. The vectorspace T_e with these operations $t_2, t_3, \dots, t_k$ is called the local W_k-algebra of the 3-web W at the point p. (The W_3 algebra was called in [4] W-algebra.)

Analogously, operations $\tilde{t}_k$ and corresponding $\tilde{W}_k$-algebras are defined on the tangent space $\tilde{T}_e$ of the coordinate loop $\tilde{l}(p)$ of the subweb $\tilde{W}$, where $\tilde{t}_k\colon (\tilde{T}_e)^k\to\tilde{T}_e$. We denote by i the imbedding $i\colon \tilde{T}_e\to T_e$ defined by $x^i=\xi^i_a\tilde{x}^a$, where $(\tilde{x}^a)\in\tilde{T}_e$, $(x^i)\in T_e$ and let $t_k=(t^i_{j_1\dots j_k})$, $\tilde{t}_k=(\tilde{t}^a_{b_1\dots b_k})$ be the tensors of type $\binom{1}{k}$ on the vectorspaces T_e and $\tilde{T}_e$ connected with the relations

$$t^i_{j_1\dots j_k}\xi^{j_1}_{b_1}\dots\xi^{j_k}_{b_k} = \xi^i_a\tilde{t}^a_{b_1\dots b_k}.$$

These relations are equivalent to the commutative diagram

$$\begin{array}{ccc} (\tilde{T}_e)^k & \xrightarrow{\tilde{t}_k} & \tilde{T}_e \\ {\scriptstyle i_k}\downarrow & & \downarrow \\ (T_e)^k & \xrightarrow{t_k} & T_e \end{array}$$

and this means that the $\tilde{W}_k$-algebra on $\tilde{T}_e$ is imbedded into the W_k-algebra on T_e as a subalgebra. Thus the following is proved:

THEOREM 4. *Let* $(\tilde{M}, \tilde{W})$ *be a subweb of the* 3-*web* (M, W). *The local* $\tilde{W}_k$-*algebra of the subweb* $(\tilde{M}, \tilde{W})$ *is a subalgebra of the local* W_k-*algebra of* (M, W) *for every point* $p \in \tilde{M}$.

6. As a consequence of Theorem 3, we know that a 3-web has property P_s $(s \geqq 2)$ if and only if it is Grassmannian. Property P_s can be formulated using the notion of W_k-algebras as follows: every linear subspace of the W_k-algebra of the coordinate loop $l(p)$ is a $\tilde{W}_k$-subalgebra. We call the W_k-algebras having this property *almost trivial.*

THEOREM 5. *The local* W_2-*algebras of a* 3-*web* (M, W) *are almost trivial if and only if* W *is isocline* $(r>2)$.

THEOREM 6. *The local* W_3-*algebras of a* 3-*web* (M, W) *are almost trivial if and only if* W *is Grassmannian* $(r>2)$.

PROOF OF THEOREM 5. Isocline webs are characterized by the form $a^i_{jk} = a_{[j}\delta^i_{k]}$ of torsion tensor, or equivalently by

$$[\xi\eta] = a(\xi)\eta - a(\eta)\xi,$$

where $a(\xi) = \frac{1}{2} a_i \xi^i$ for any tangent vectors $\xi = \xi^i e_i$ and $\eta = \eta^i e_i$. That is the 2-plane spanned by ξ and η is a subalgebra of W_2. It follows that every subspace $\tilde{T}_e \subset T_e$ is a subalgebra of W_2. On the other hand, if (15) is satisfied identically then we get (18), i.e. the web is isocline (by the assumption $r>2$).

PROOF OF THEOREM 6. We know that the W_k-algebras of a Grassmannian 3-web are almost trivial. Conversely, we suppose that the W_3-algebras are almost trivial. Then the relations (15) and (16) are satisfied identically. (15) implies that the 3-web W is isocline $(r>2)$ and the torsion tensor is of the form (18).

Similarly, the identity (16) implies that the curvature tensor can be written in the form

$$b^i_{jkl} = \lambda_{jk}\delta^i_l + \mu_{lj}\delta^i_k + \nu_{kl}\delta^i_j. \tag{20}$$

But the tensor of curvature of an isocline can be expressed as

$$b^i_{jkl} = a^i_{jkl} + f_{jk}\delta^i_l + g_{lj}\delta^i_k + h_{kl}\delta^i_j \tag{21}$$

if $r>2$, where a^i_{jkl}, f_{jk}, g_{lj} and h_{kl} are symmetric in the lower indices and $a^i_{ikl} = 0$. The expressions (20) and (21) give by contraction of indices i and j

$$(\lambda_{lk} - f_{lk}) + (\mu_{lk} - g_{lk}) + r(\nu_{kl} - h_{kl}) = 0.$$

Contracting the pairs of indices (i, k) and (i, l) in (20) and (21) we get the similar equations

$$(\lambda_{lk} - f_{lk}) + r(\mu_{kl} - g_{kl}) + (\nu_{lk} - h_{lk}) = 0,$$

$$r(\lambda_{kl} - f_{kl}) + (\mu_{lk} - g_{lk}) + (\nu_{lk} - h_{lk}) = 0.$$

Alternating the indices k and l in these equations we have

$$-\lambda_{[kl]}-\mu_{[kl]}+r\gamma_{[kl]}=0,$$
$$-\lambda_{[kl]}+r\mu_{[kl]}-\nu_{[kl]}=0,$$
$$r\lambda_{[kl]}-\mu_{[kl]}-\nu_{[kl]}=0.$$

If $r>0$, these homogeneous linear equations are independent, and $\lambda_{[kl]}=\mu_{[kl]}==\nu_{[kl]}=0$. Thus the tensors λ_{kl}, μ_{kl} and ν_{kl} are symmetric. Then (18) and (20) imply that the 3-web W is Grassmannian, using the characterization of Grassmannian webs given in [3]. The theorem is proved.

References

[1] M. A. Akivis, Three-webs of multidimensional surfaces, *Trudy Geometr. Sem.*, **2** (1969), 7—31 (Russian).

[2] A. M. Shelechov, Three-webs with partially symmetric curvature tensor, *Sibirsk. Math. Zh.*, **22** (1981), 210—219 (Russian).

[3] M. A. Akivis, On local differentiable quasigroups and three-webs defined by a triple of hypersurfaces, *Sibirsk. Math. Zh.*, **14** (1973), 467—474 (Russian).

[4] M. A. Akivis, On local algebras of a multidimensional three-web, *Sibirsk. Math. Zh.*, **17** (1976), 5—11.

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Assigning the values [illegible] and f to these equations we get

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ÜBER BANACHVERBANDSALGEBREN MIT MULTIPLIKATIVER ZERLEGUNGSEIGENSCHAFT

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Einleitung

In dieser Arbeit wollen wir das Studium der Banachverbandsalgebren mit der multiplikativen Zerlegungseigenschaft, welches wir in der Veröffentlichung [9] begonnen haben, fortsetzen. Die Klasse dieser Banachverbandsalgebren umfaßt die Funktionenalgebren $C(K)$ und die Faltungsalgebren $M(S)$ und scheint die allgemeinste Klasse von Banachverbandsalgebren zu sein, über die man mehr sagen kann, als daß nur der Spektralradius positiver Elemente ein Spektralwert ist.

Im 1. Kapitel befassen wir uns mit der multiplikativen Zerlegungseigenschaft und zeigen, daß ihre eigentliche Bedeutung darin besteht, daß ein gewisser, von der Multiplikation induzierter Operator ein Verbandsoperator ist.

Im 2. Kapitel charakterisieren wir die Menge der komplexen Homomorphismen bei L-Algebren. Dies sind Banachverbandsalgebren, bei denen der zugrundeliegende Banachverband ein abstrakter L-Raum ist. Da allgemeine Banachverbandsalgebren mit der multiplikativen Zerlegungseigenschaft auf eine gewisse Weise L-Algebren induzieren, sind die hier gewonnenen Ergebnisse von grundlegender Bedeutung für die Behandlung dieser Algebren im 3. und 4. Kapitel.

Im 3. Kapitel zeigen wir unter anderem, daß bei halbeinfachen kommutativen Banachverbandsalgebren mit multiplikativer Zerlegungseigenschaft die Menge der komplexen Homomorphismen „signumsinvariant" ist und die zentralen Homomorphismen eine konjugations- und betragsinvariante Halbgruppe bilden.

Das 4. Kapitel befaßt sich mit maximalen Strukturidealen in kommutativen Banachverbandsalgebren mit multiplikativer Zerlegungseigenschaft. Das wichtigste Ergebnis dieses Abschnitts ist, daß jede solche halbeinfache und halbstruktureinfache Algebra mit Einselement und ordnungsstetiger Norm Vervollständigung einer direkten Summe von struktureinfachen Algebren ist. Im endlich dimensionalen Fall bedeutet dies, daß eine solche Algebra eine direkte Summe von Gruppenalgebren ist.

0. Vorbemerkungen

Wie schon erwähnt, ist diese Arbeit eine Fortsetzung von [9]. Aus diesem Grunde übernehmen wir Bezeichnungen und Terminologie aus dieser Arbeit. Was die verbandstheoretischen Begriffe betrifft, halten wir uns eng an Schaefer [7]. Der leichteren Lesbarkeit wegen wollen wir aber doch die wichtigsten Begriffe ganz kurz in Erinnerung rufen.

Wie üblich bezeichnen wir mit **N**, **R** und **C** die Menge der natürlichen, reellen bzw. komplexen Zahlen. Unter einer reellen Banachverbandsalgebra A verstehen wir einen reellen Banachverband A, welcher gleichzeitig eine reelle Algebra mit den folgenden Eigenschaften ist: $xy \geqq 0$ und $\|xy\| \leqq \|x\| \, \|y\|$ für alle positiven Elemente x und y von A. Eine komplexe Banachverbandsalgebra $A_{\mathbf{C}}$ ist dann stets die Komplexifizierung einer reellen Banachverbandsalgebra A.

Ist E ein Banachverband und u ein positives Element von E, so bedeutet E_u das von u im verbandstheoretischen Sinne erzeugte Hauptideal. Es läßt sich bekanntlich mit einem Funktionenverband $C(K)$ identifizieren, und diese Identifizierung werden wir die kanonische Identifizierung eines Hauptideals nennen. Dem Element u entspreche dabei stets die Einsfunktion auf K.

Ist E ein komplexer AL-Raum, so ist der topologische Dual E' ein komplexer AM-Raum mit Einheit, also ein komplexer Banachverband $C(K)$, wobei K ein Stone'scher Raum ist. Der Bidual E'' von E kann daher mit dem Raum $M(K)$ aller komplexen Radon-Maße auf K identifiziert werden, und die kanonische Einbettung von E in $M(K)$ ist gleich der Komplexifizierung des Bandes der reellen ordnungsstetigen Radon-Maße auf K. Diese Identifizierung eines komplexen AL-Raumes spielt im folgenden eine wichtige Rolle und wird die kanonische Identifizierung eines solchen Raumes genannt.

Die Menge der komplexen Homomorphismen einer komplexen Banachalgebra B bezeichnen wir mit $\mathcal{M}$. Der Bidual B'' von B wird durch Einführung des Arens-Produkts selbst zu einer Banachalgebra. Die für uns wichtigen Eigenschaften dieses Produkts haben wir in [9] zusammengestellt.

1. Über die multiplikative Zerlegungseigenschaft

In der Arbeit [9] haben wir für komplexe Banachverbandsalgebren $A_{\mathbf{C}}$ die folgende multiplikative Zerlegungseigenschaft $\mathscr{Z}$ eingeführt:

($\mathscr{Z}$) Im positiven Kegel von A gibt es eine totale Menge B, so daß für alle $a, b \in B$ die Beziehung gilt: $\{z \in A_{\mathbf{C}} \colon |z| \leqq ab\}$ = abgeschlossene Hülle der Menge

$$\Big\{\sum_{i,j} \alpha_{ij} a_i b_j \colon a_i, b_j \in A_{\mathbf{C}} \;\; (1 \leqq i \leqq n, \; 1 \leqq j \leqq m), \quad \sum_{i=1}^{n} |a_i| \leqq a,$$

$$\sum_{j=1}^{m} |b_j| \leqq b, \quad \alpha_{ij} \in \mathbf{C} \quad \text{mit} \quad |\alpha_{ij}| \leqq 1\Big\}.$$

Auf sehr elementare Weise konnten wir dann zeigen, daß diese Eigenschaft $\mathscr{Z}$ zur Folge hat, daß sogar der Absolutbetrag eines komplexen Homomorphismus wieder multiplikativ ist. Diese Eigenschaft $\mathscr{Z}$ hat, wie wir in [9] auch bemerkt haben, große Ähnlichkeit mit der folgenden Bedingung $\tilde{\mathscr{Z}}$, welche in Taylor [12] Bestandteil der Definition einer abstrakten Konvolutionsmaßalgebra A ist:

($\tilde{\mathscr{Z}}$) Sind x, y und w positive Elemente von A mit $w \leqq xy$, so gibt es zu jedem $\varepsilon > 0$ Mengen $\{x_i \colon 1 \leqq i \leqq n\}$ und $\{y_j \colon 1 \leqq j \leqq m\}$ positiver Elemente von A und Zahlen $\alpha_{ij} \in [0, 1]$, so daß gilt:

$$\sum_{i=1}^{n} x_i \leqq x, \quad \sum_{j=1}^{m} y_j \leqq y \quad \text{und} \quad \Big\|w - \sum_{i,j} \alpha_{ij} x_i y_j\Big\| \leqq \varepsilon.$$

In [9] gelang es uns nicht, einen Zusammenhang zwischen $\mathscr{L}$ und $\tilde{\mathscr{L}}$ herzustellen. Im folgenden wollen wir nun zeigen, daß diese beiden Bedingungen gleichwertig sind. Dazu müssen wir etwas weiter ausholen und einen gewissen Operator betrachten, welcher von der Multiplikation als einer positiven, bilinearen Abbildung induziert wird.

Es sei A eine reelle Banachverbandsalgebra. Bezeichnet $L^r(A, A')$ den Banachverband der regulären linearen Abbildungen von A in den topologischen Dual A' von A (s. [7], Kap. IV, § 1), so induziert die Multiplikation auf A wie folgt eine positive lineare Abbildung M von A' in den ordnungsvollständigen Banachverband $L^r(A, A')$: Für $\mu\in A'$ sei $M\mu\in L^r(A, A')$ definiert durch $(M\mu)(x)(z):=\mu(xz)$ für alle $x, z\in A$.

Die kanonische Fortsetzung von M auf die Komplexifizierungen $(A')_{\mathbf{C}}$ und $(L^r(A, A'))_{\mathbf{C}}$ bezeichnen wir wieder mit M.

Der folgende Satz zeigt, daß die Bedingungen $\mathscr{L}$ und $\tilde{\mathscr{L}}$ implizieren, daß der Operator M ein Verbandsoperator ist.

1.1. Satz. *Für eine reelle Banachverbandsalgebra A sind die folgenden Aussagen äquivalent:*

(i) *Es gilt die folgende Bedingung $\mathscr{L}^*$:*

($\mathscr{L}^*$) *Zu positiven Elementen $a, b, c\in A$ mit $c\leqq ab$ und zu $\varepsilon>0$ gibt es n positive Elemente x_i und n positive Elemente y_i mit $x_i\leqq a$, $\sum_{i=1}^{n} y_i\leqq b$ und*

$$\Big\|c-\sum_{i=1}^{n} x_i y_i\Big\|\leqq\varepsilon.$$

(ii) *Der Operator M ist ein Verbandshomomorphismus.*

(iii) *Es gilt die Bedingung $\tilde{\mathscr{L}}$.*

(iv) *Es gilt die Bedingung $\mathscr{L}$.*

Beweis. (i)→(ii). Es genügt zu zeigen: $(M\mu)^+=M\mu^+$ für alle $\mu\in A'$. Seien a und b zwei positive Elemente in A. Für $\mu\in A'$ gilt

$$(M\mu)^+(a) = \sup\{(M\mu)(x)\colon\ x\in A \quad \text{und} \quad 0\leqq x\leqq a\}$$

und

$$(*)\qquad (M\mu)^+(a)(b) = \sup\{(M\mu)(x_1)(y_1)+\ldots+(M\mu)(x_n)(y_n)\} = $$
$$= \sup\{\mu(x_1 y_1)+\ldots+\mu(x_n y_n)\},$$

wobei $\{x_1, \ldots, x_n\}$ alle nichtleeren, endlichen Teilmengen des Ordnungsintervalls $[0, a]$ durchläuft und die dazugehörige Menge $\{y_1, \ldots, y_n\}$ alle nichtleeren, endlichen Teilmengen des Ordnungsintervalls $[0, b]$ durchläuft und die Gleichung $\sum_{i=1}^{n} y_i=b$ erfüllt (s. [7], Kap. IV, 1.3, (3)). Ferner ist

$$(M\mu^+)(a)(b) = \mu^+(ab) = \sup\{\mu(w)\colon\ w\in A \quad \text{und} \quad 0\leqq w\leqq ab\}.$$

Sei $\varepsilon>0$ und $c\in A$ mit $0\leqq c\leqq ab$. Aufgrund der Bedingung $\mathscr{L}^*$ gibt es dann n Elemente x_i und y_i mit den in der Bedingung $\mathscr{L}^*$ angegebenen Eigenschaften. Aus

$\|c-\sum_{i=1}^{n} x_i y_i\| \leqq \varepsilon$ folgt $|\mu(c)-\sum_{i=1}^{n} \mu(x_i y_i)| \leqq \|\mu\| \varepsilon$. Für $x_0:=0$ und $y_0:=b-\sum_{i=1}^{n} y_i \geqq 0$ bedeutet dies $\mu(c) \leqq \sum_{i=0}^{n} \mu(x_i y_i)+\|\mu\| \varepsilon$ mit $0 \leqq x_i \leqq a$, $0 \leqq y_i \leqq b$ und $\sum_{i=0}^{n} y_i = b$. Aus der Darstellung $(*)$ von $(M\mu)^+(a)(b)$ erhalten wir somit $(M\mu^+)(a)(b) \leqq (M\mu)^+(a)(b)$. Dies bedeutet $M\mu^+ \leqq (M\mu)^+$. Da M positiv ist, gilt trivialerweise $(M\mu)^+ \leqq M\mu^+$. Es ist somit $M\mu^+ = (M\mu)^+$. Der Operator M ist also ein Verbandsoperator.

(ii)$\to$(i). Diese Implikation kann wie in Lotz [3] gezeigt werden. Für positive Elemente a und b sei $K(a, b):=\{\sum_{i=1}^{n} x_i y_i\colon x_i, y_i \in A,\ (1 \leqq i \leqq n),\ n \in \mathbf{N},\ 0 \leqq x_i \leqq a,\ 0 \leqq y_i$ und $\sum_{i=1}^{n} y_i \leqq b\}$. Angenommen, es gelte nicht $\mathscr{L}^*$. Dann gibt es positive Elemente x, y und z mit $0 \leqq z \leqq xy$ und $z \notin \overline{K(x, y)}$. Da $\overline{K(x, y)}$ konvex ist, gibt es ein $\mu \in A'$, welches die Mengen $\{z\}$ und $\overline{K(x, y)}$ streng trennt, d. h. es gibt ein $\alpha \in \mathbf{R}$ mit $\mu(w) \leqq \alpha < \mu(z)$ für alle $w \in \overline{K(x, y)}$. Da M ein Verbandsoperator ist, gilt $(M\mu)^+(x)(y) = (M\mu^+)(x)(y)$. Wir erhalten somit

$$(M\mu)^+(x)(y) \leqq \sup\{\mu(w)\colon w \in K(x, y)\} \leqq \alpha.$$

Andererseits ist $(M\mu^+)(x)(y) = \sup\{\mu(w)\colon 0 \leqq w \leqq xy\} \geqq \mu(z) > \alpha$. Dies ist der gewünschte Widerspruch.

(iii)$\to$(i). Sei $\sum_{i,j} \alpha_{ij} x_i y_j$ eine Linearkombination aus $\tilde{\mathscr{L}}$ und $z_j := \sum_{i=1}^{n} \alpha_{ij} x_i$. Dann ist $\sum_{j=1}^{m} z_j y_j$ eine Linearkombination aus $\mathscr{L}^*$. Hieraus folgt, $\tilde{\mathscr{L}}$ impliziert $\mathscr{L}^*$.

(i)$\to$(iii). Seien x und x_i $(1 \leqq i \leqq n)$ positive Elemente in A mit $x_i \leqq x$. Mit Hilfe der kanonischen Darstellung des Hauptideals A_x als Funktionenverband $C(K)$ und einer geeigneten Partition der Einsfunktion kann man zeigen, daß es zu jedem $\varepsilon > 0$ positive Elemente $z_j \in A$ $(1 \leqq j \leqq m)$ und Zahlen $\alpha_{ij} \in [0, 1]$ gibt mit $\sum_{j=1}^{m} z_j = x$ und $\|x_i - \sum_{j=1}^{m} \alpha_{ij} z_j\| \leqq \varepsilon$ für $1 \leqq i \leqq n$. Für positive Elemente y und y_i $(1 \leqq i \leqq n)$ ergibt sich damit $\|x_i y_i - \sum_{j=1}^{m} \alpha_{ij} z_j y_i\| \leqq \varepsilon \|y_i\|$ für $1 \leqq i \leqq n$ und somit $\|\sum_{i=1}^{n} x_i y_i - \sum_{i,j} \alpha_{ij} z_j y_i\| \leqq \leqq \varepsilon \sum_{i=1}^{n} \|y_i\|$. Da sich die Ausdrücke $\sum_{i=1}^{n} x_i y_i$ auf diese Weise beliebig genau durch die angegebenen Ausdrücke $\sum_{i,j} \alpha_{ij} z_j y_i$ approximieren lassen, impliziert die Bedingung $\mathscr{L}^*$ auch $\tilde{\mathscr{L}}$.

(iv)$\to$(ii). Da der Beweis dieser Richtung im Prinzip wie der Beweis von (i)$\to$(ii) verläuft, wollen wir uns hier kürzer fassen. Es genügt nun zu zeigen: $|M\mu| = M|\mu|$ für alle $\mu \in (A_{\mathbf{C}})'$.

Sei B eine totale Menge im positiven Kegel von A mit der in der Definition von $\mathscr{L}$ angegebenen Eigenschaft. Für $\mu \in (A_{\mathbf{C}})'$ $(=(A')_{\mathbf{C}})$ und $a, b \in B$ gilt

$$M|\mu|(a)(b) = \sup\{|\mu(w)|\colon w \in A_{\mathbf{C}} \text{ mit } |w| \leqq ab\},$$
$$|M\mu|(a) = \sup\{|(M\mu)(z)|\colon z \in A_{\mathbf{C}} \text{ mit } |z| \leqq a\}$$

und somit

$$|M\mu|(a)(b) = \sup \Big\{|\mu(z_1 y_1)| + \ldots + |\mu(z_n y_n)| \colon z_i, y_i \in A_{\mathbf{C}} \quad (1 \leqq i \leqq n)$$

$$\text{mit } |z_i| \leqq a \text{ und } \sum_{i=1}^{n} |y_i| \leqq b\Big\}.$$

Aufgrund der Bedingung $\mathscr{Z}$ läßt sich jedes $w \in A_{\mathbf{C}}$ mit $|w| \leqq ab$ auf dieselbe Weise wie bei (iii)→(i) beliebig genau durch eine Summe $z_1 y_1 + \ldots + z_n y_n$ solcher z_i und y_i approximieren. Hiermit erhält man zunächst $M|\mu|(a)(b) \leqq |M\mu|(a)(b)$. Da M positiv ist, gilt auch die entgegengesetzte Ungleichung. Es ist also $|M\mu|(a)(b) = M|\mu|(a)(b)$. Da B total ist, folgt aus Stetigkeitsgründen $|M\mu| = M|\mu|$ für alle $\mu \in A'_{\mathbf{C}}$. Es ist also M ein Verbandsoperator.

(ii)→(iv). Wie beim Beweis von (ii)→(i) zeigt man zunächst, daß für alle positiven Elemente a und b das Ordnungsintervall $\{z \in A_{\mathbf{C}} \colon |z| \leqq ab\}$ gleich der abgeschlossenen Hülle der Menge $\Big\{\sum_{i=1}^{n} z_i y_i \colon z_i, y_i \in A_{\mathbf{C}}\ (1 \leqq i \leqq n \in \mathbf{N})$ mit $|z_i| \leqq a$ und $\sum_{i=1}^{n} |y_i| \leqq b\Big\}$ sein muß. Wendet man dann die Beweismethode von (i)→(iii) entsprechend auf das komplexe Ordnungsintervall $\{z \in A_{\mathbf{C}} \colon |z| \leqq a\}$ an, so erhält man die Gültigkeit von $\mathscr{Z}$, q.e.d.

Da die Bedingungen $\mathscr{Z}$, $\tilde{\mathscr{Z}}$ und $\mathscr{Z}^*$ alle gleichwertig sind, sagen wir von nun an, eine reelle bzw. komplexe Banachverbandsalgebra besitzt die multiplikative Zerlegungseigenschaft (kurz: die Eigenschaft $\mathscr{Z}$), falls eine der drei Bedingungen erfüllt ist.

In der Arbeit [9] haben wir uns mit komplexen AL-Algebren befaßt, und dabei hat eine gewisse Teilmenge $\mathscr{M}_c$ der komplexen Homomorphismen die Schlüsselrolle gespielt. Mit Hilfe des Operators M können wir nun zeigen, daß in den noch allgemeineren L-Algebren, welche wir im 2. Kapitel eingehender untersuchen werden und welche von McKilligan—White [2] eingeführt worden sind, die multiplikative Zerlegungseigenschaft $\mathscr{M} = \mathscr{M}_c$ impliziert. Wie schon in der Einleitung bemerkt, heißt eine reelle Banachverbandsalgebra eine L-Algebra, wenn der zugrundeliegende Banachverband ein AL-Raum ist. Eine komplexe L-Algebra ist dann die Komplexifizierung einer reellen L-Algebra.

Es sei nun $A_{\mathbf{C}}$ eine komplexe L-Algebra, und die Räume $C(K)$ bzw. $M(K)$ seien die entsprechenden kanonischen Identifizierungen von $(A_{\mathbf{C}})'$ bzw. $(A_{\mathbf{C}})''$. Ferner werde $A_{\mathbf{C}}$ als Teilmenge von $M(K)$ betrachtet. Dann läßt sich der Raum $(L^r(A, A'))_{\mathbf{C}}$ mit dem Raum $L^r(A_{\mathbf{C}}, C(K))$ identifizieren (s. [7], IV, 1.8). Bekanntlich definiert nun jede auf dem topologischen Produkt $K \times K$ komplexwertige, stetige Funktion f wie folgt ein Element $T_f \in L^r(A_{\mathbf{C}}, C(K))$:

$$(T_f \mu)(t) := \int_K f(s, t)\, d\mu(s)$$

für alle $\mu \in A_{\mathbf{C}}$ und $t \in K$.

Da die Abbildung $f \to T_f$ ein komplexer Vektorverbandsisomorphismus vom komplexen Banachverband $C(K \times K)$ in den komplexen Banachverband $L^r(A_{\mathbf{C}}, C(K))$ ist, kann also $C(K \times K)$ mit einem komplexen Unterverband von $L^r(A_{\mathbf{C}}, C(K))$

identifiziert werden, welchen wir mit $\hat{C}(K\times K)$ bezeichnen. Der zu Beginn dieses Kapitels betrachtete Operator $M\colon A'_{\mathbf{C}}\to L^r(A_{\mathbf{C}}, A'_{\mathbf{C}})$ ist somit eine Abbildung von $C(K)$ in $L^r(A_{\mathbf{C}}, C(K))$. In Integralschreibweise gilt also $Mf(\mu)(v)=\int_K f\,d\mu\cdot v$ für $f\in C(K)$ und $\mu, v\in A_{\mathbf{C}}$.

Auf $M(K)$ betrachten wir jetzt die Arens-Produkte $\varepsilon_s\cdot\varepsilon_t$ für $s, t\in K$. Aus der Definition des Arens-Produkts ergeben sich sofort für $f\in C(K)$ und $\mu, v\in A_{\mathbf{C}}$ folgende Aussagen:

(i) Für festes $t\in K$ ist die zu f assoziierte Funktion

$$F(s, t):=\int_K f(\tau)\,d\varepsilon_s\cdot\varepsilon_t(\tau)$$

eine stetige Funktion in $s\in K$.

(ii) Die Funktion $\int_K F(s, t)\,d\mu(s)$ ist stetig in $t\in K$.

(iii) Es gilt $\int_K\Big(\int_K F(s, t)\,d\mu(s)\Big)\,dv(t)=\int_K f(\tau)\,d\mu\cdot v(\tau)$.

Für jedes $f\in C(K)$ läßt sich nun Mf mit der assoziierten Funktion $F(s, t)$ wie folgt darstellen:

$$Mf(\mu)(v)=\int_K\Big(\int_K F(s, t)\,d\mu(s)\Big)\,dv(t).$$

Dies bedeutet

$$(Mf)(\mu)(t)=\int_K F(s, t)\,d\mu(s)\quad\text{für alle}\quad t\in K.$$

Hieraus folgt: Ist für $f\in C(K)$ die Funktion $F(s, t)$ in beiden Variablen gemeinsam stetig, so ist $Mf\in\hat{C}(K\times K)$.

Während bei AL-Algebren das Arens-Produkt zweier Dirac-Maße ein Wahrscheinlichkeitsmaß ist, ist es bei L-Algebren nur ein positives Radon-Maß. Wir betrachten nun die Menge $\mathcal{M}$ der komplexen Homomorphismen auf $A_{\mathbf{C}}$ als Teilmenge von $C(K)$ und definieren für komplexe L-Algebren die Teilmenge $\mathcal{M}_c$ von $\mathcal{M}$ wie folgt:

$$\mathcal{M}_c:=\{g\in\mathcal{M}\colon g \text{ ist konstant auf den Trägern der Radon-Maße } \varepsilon_s\cdot\varepsilon_t \text{ für alle } s, t\in K\}.$$

Aufgrund der vorhergehenden Überlegungen können wir folgendes, wichtige Ergebnis über die Menge der komplexen Homomorphismen beweisen.

1.2. Satz. *Die komplexe L-Algebra $A_{\mathbf{C}}$ besitze die multiplikative Zerlegungseigenschaft und erfülle ferner die Bedingung: Für die Einsfunktion e_K auf K ist $\int_K e_K(\tau)\,d\varepsilon_s\cdot\varepsilon_t(\tau)$ eine stetige Funktion in s und t auf dem Produktraum $K\times K$. Dann gilt $\mathcal{M}=\mathcal{M}_c$.*

Beweis. Nach 1.1 ist der Operator M von $C(K)$ nach $L^r(A_{\mathbf{C}}, C(K))$ ein Verbandsoperator. Es ist daher das Urbild $F:=M^{-1}(\hat{C}(K\times K))$ ein abgeschlossener, komplexer Unterverband von $C(K)$. Für $g\in\mathcal{M}$ gilt

$$G(s,t)=\int_K g(\tau)d\varepsilon_s\cdot\varepsilon_t(\tau)=g(s)g(t)\quad\text{für alle}\quad s,t\in K.$$

Dies bedeutet $g\in F$. Die im Satz angegebene Bedingung impliziert $e_K\in F$.

Auf K bezeichne $\sim$ die Äquivalenzrelation, die von F wie folgt definiert wird:

$$s\sim t\Leftrightarrow f(s)=f(t)\quad\text{für alle}\quad f\in F.$$

Dann ist bekanntlich F isomorph zum komplexen Banachverband $C(K/\sim)$. Seien nun s und $t\in K$. Da die Einschränkung von M auf $F\big(=C(K/\sim)\big)$ ein Verbandsoperator ist und $\hat{C}(K\times K)$ ein komplexer Unterverband von $L^r\big(A_{\mathbf{C}},C(K)\big)$ ist, ist die Abbildung

$$g\to\int_K g(\tau)d(\varepsilon_s\cdot\varepsilon_t)(\tau)$$

ein komplexer Verbandshomomorphismus. Dies bedeutet, die Restriktion von $\varepsilon_s\cdot\varepsilon$ auf den Unterverband F ist ein positives Vielfaches eines Dirac-Maßes auf $K/\sim$. Es sind daher alle Funktionen aus F auf dem Träger von $\varepsilon_s\cdot\varepsilon_t$ konstant. Insbesondere ist jedes $g\in\mathcal{M}$ auf diesen Trägern konstant, q.e.d.

In den Arbeiten Taylor [14] und Martignon [4] wird über Tensorprodukte eine Produktabbildung π bzw. ein Multiplikationsoperator M definiert. Es scheint, daß unser Operator M aus Satz 1.1 jeweils mit dem adjungierten Operator der erwähnten Abbildungen identifiziert werden kann. Es mag daher sein, daß einiges aus diesem 1. Kapitel Experten auf dem Gebiet der Tensorprodukte bekannt vorkommt. Aus diesem Grunde möchten wir betonen, daß die Aufgabe des 1. Kapitels nur darin besteht, die multiplikative Zerlegungseigenschaft zu erhellen und mit Satz 1.2 für die weitere Entwicklung der Theorie einen geeigneten Rahmen zu schaffen.

2. Über L-Algebren

Was wir in diesem Kapitel über allgemeine L-Algebren zeigen, steht inhaltlich in enger Beziehung zu Ergebnissen von J. L. Taylor über halbeinfache kommutative Maßalgebren (s. [14], Kap. 3). Bei den Beweisen von J. L. Taylor spielt aber die Strukturhalbgruppe dieser Algebren eine wesentliche Rolle. Da wir für L-Algebren den Begriff der Strukturhalbgruppe nicht zur Verfügung haben, unterscheiden sich unsere Beweismethoden sehr wesentlich von denen in [14].

Zunächst wollen wir eine Aussage über absolute Kerne in AL-Räumen machen. Ist E ein Vektorverband und μ eine positive Linearform auf E, so wird die Menge $N(\mu):=\{x\in E:\ \mu(|x|)=0\}$ der absolute Kern von μ genannt. Für ein Maß ν bezeichne S_ν den Träger von ν.

2.1. Lemma. *Es sei E ein reeller oder komplexer AL-Raum, $C(K)$ die kanonische Identifizierung von E', und E werde als Teilraum von $M(K)$ betrachtet. Sei $f\in C(K)$, $f>0$, $N(f)\neq\{0\}$ und $\mathcal{O}:=\{s\in K:\ f(s)>0\}$. Dann gilt $N(f)=\{\mu\in E:\ S_\mu\subseteq K\setminus\overline{\mathcal{O}}\}$ ($\overline{\mathcal{O}}$=abgeschlossene Hülle von $\mathcal{O}$) und $N(f)^\perp=\{\mu\in E:\ S_\mu\subseteq\overline{\mathcal{O}}\}$. Ferner läßt sich die*

Bandprojektion P von E auf $N(f)^{\perp}$ mit Hilfe der charakteristischen Funktion $\chi_{\overline{\mathcal{O}}}$ von $\overline{\mathcal{O}}$ in der Form

$$\int_K g\, dP\mu = \int_K g\cdot\chi_{\overline{\mathcal{O}}}\, d\mu \quad \textit{für alle} \quad g\in C(K)$$

(kurz: $P\mu=\chi_{\overline{\mathcal{O}}}\cdot\mu$) darstellen.

BEWEIS. Sei $B:=\{\mu\in E\colon S_\mu\subseteqq K\setminus\overline{\mathcal{O}}\}$. Die Inklusion $B\subseteqq N(f)$ ist klar. Sei nun $\mu\in N(f)$ und o. B. d. A. $\mu>0$. Da μ ordnungsstetig ist, ist S_μ offen und abgeschlossen und somit χ_{S_μ} stetig. Aus $\chi_{S_\mu}\equiv 0$ auf $\mathcal{O}$ folgt also $\chi_{S_\mu}\equiv 0$ auf $\overline{\mathcal{O}}$. Dies bedeutet $S_\mu\subseteqq K\setminus\overline{\mathcal{O}}$, also $\mu\in B$. Wir haben somit $N(f)=\{\mu\in E\colon S_\mu\subseteqq K\setminus\overline{\mathcal{O}}\}$.

Die restlichen Behauptungen ergeben sich aus der Zerlegung

$$\mu=\chi_{\overline{\mathcal{O}}}\cdot\mu+(e_K-\chi_{\overline{\mathcal{O}}})\cdot\mu \quad \text{für alle} \quad \mu\in E, \quad \text{q.e.d.}$$

Im folgenden sei A eine reelle oder komplexe L-Algebra. Ferner seien stets die kanonischen Identifizierungen vorgenommen (d. h. $A'\triangleq C(K)$, $A''\triangleq M(K)$), und der Bidual $M(K)$ sei mit dem Arens-Produkt versehen. Mit dem vorhergehenden Lemma können wir nun zeigen, daß bei L-Algebren das Band $N(f)^{\perp}$ einer positiven multiplikativen Linearform f, welche auf den Trägern der Produkte von Dirac-Maßen konstant ist, eine Subalgebra ist, was zur Folge hat, daß die dazugehörige Bandprojektion ein Algebrahomomorphismus ist.

2.2. SATZ. *Es sei A eine L-Algebra, $f\in\mathcal{M}_c$, $f>0$ und $N(f)\neq\{0\}$. Dann gilt:*

(i) *$N(f)$ ist ein abgeschlossenes Strukturideal, d. h. sowohl ein Verbands- als auch ein zweiseitiges Ringideal.*

(ii) *$N(f)^{\perp}$ ist eine abgeschlossene Subalgebra.*

(iii) *Die Bandprojektion $P\colon A\to N(f)^{\perp}$ ist ein Algebrahomomorphismus.*

BEWEIS. (i) klar.

(ii) Sei wieder $\mathcal{O}:=\{\tau\in K\colon f(\tau)>0\}$ und $s,t\in\mathcal{O}$. Aus $\int_K f\,d\varepsilon_s\cdot\varepsilon_t=f(s)f(t)>0$ folgt $\varepsilon_s\cdot\varepsilon_t>0$, $f\not\equiv 0$ auf $S_{\varepsilon_s\cdot\varepsilon_t}$ und somit $S_{\varepsilon_s\cdot\varepsilon_t}\subseteqq\mathcal{O}$, da f konstant auf $S_{\varepsilon_s\cdot\varepsilon_t}$ ist.

Seien nun μ und ν positive Elemente von $N(f)^{\perp}$. Dann gilt nach 2.1 S_μ, $S_\nu\subseteqq\overline{\mathcal{O}}$. Mit Hilfe des Bipolarentheorems läßt sich beweisen, daß es Netze (μ_λ) und (ν_ϱ) gibt, so daß gilt:

1. Jedes μ_λ bzw. ν_ϱ ist eine endliche Linearkombination von Dirac-Maßen mit Trägern in $\mathcal{O}$, also

$$\mu_\lambda=\sum_{i=1}^{n_\lambda}\alpha_{i,\lambda}\varepsilon_{t_{i,\lambda}},\quad \nu_\varrho=\sum_{j=1}^{m_\varrho}\beta_{j,\varrho}\varepsilon_{s_{j,\varrho}} \quad \text{mit} \quad t_{i,\lambda},\ s_{j,\varrho}\in\mathcal{O}.$$

2. $\mu=\lim\mu_\lambda$ und $\nu=\lim\nu_\varrho$ in der schwachen Topologie $\sigma(M(K),C(K))$. Aus $\mu_\lambda\cdot\nu_\varrho=\sum\alpha_{i,\lambda}\beta_{j,\varrho}\varepsilon_{t_{i,\lambda}}\cdot\varepsilon_{s_{j,\varrho}}$ folgt nun $S_{\mu_\lambda\cdot\nu_\varrho}\subseteqq\mathcal{O}$. Aufgrund der Stetigkeitseigenschaften des Arens-Produkts erhält man folgende Grenzwerte in der Topologie $\sigma(M(K),C(K))$: $\mu\cdot\nu_\varrho=\lim_\lambda\mu_\lambda\cdot\nu_\varrho$ für festes ϱ und $\mu\cdot\nu=\lim_\varrho\mu\cdot\nu_\varrho$. Eine weitere Anwendung des Bipolarentheorems ergibt zunächst $S_{\mu\cdot\nu_\varrho}\subseteqq\overline{\mathcal{O}}$ für alle ϱ und dann $S_{\mu\cdot\nu}\subseteqq\overline{\mathcal{O}}$. Dies bedeutet $\mu\cdot\nu\in N(f)^{\perp}$. Es ist also $N(f)^{\perp}$ eine Subalgebra.

(iii) Aus der Tatsache, daß der Kern P, nämlich $N(f)$, ein zweiseitiges Ringideal und der Bildbereich, nämlich $N(f)^{\perp}$, eine Subalgebra ist, folgt durch eine leichte Rechnung, daß P ein Algebrahomomorphismus ist, q.e.d.

Ist Y ein Stone'scher Raum, so können wir für eine stetige komplexwertige Funktion f auf Y wie folgt die *Signumsfunktion* $\operatorname{sgn} f$ von f definieren:

Sei $\mathcal{O}:=\{t\in Y\colon f(t)\neq 0\}$. Für die Einschränkung von f auf $\mathcal{O}$ ist die Funktion $\frac{f(t)}{|f(t)|}$ beschränkt und stetig auf $\mathcal{O}$. Da bekanntlich $\overline{\mathcal{O}}$ zur Stone—Čech-Kompaktifizierung $\beta\mathcal{O}$ von $\mathcal{O}$ homöomorph ist, besitzt diese Funktion eine eindeutig bestimmte stetige Fortsetzung $\hat{f}$ auf $\overline{\mathcal{O}}$. Es sei nun $\operatorname{sgn} f:=\hat{f}$ auf $\overline{\mathcal{O}}$ und $\operatorname{sgn} f\equiv 0$ auf $Y\setminus\overline{\mathcal{O}}$. Da $\overline{\mathcal{O}}$ offen und abgeschlossen ist, ist also die so definierte Signumsfunktion $\operatorname{sgn} f$ auf Y stetig. Für $f=0$ sei per definitionem $\operatorname{sgn} f=0$.

Für komplexe L-Algebren können wir jetzt die Menge $\mathcal{M}_c$ wie folgt charakterisieren:

2.3. Satz. *Es sei $A_{\mathbf{C}}$ eine komplexe L-Algebra. Dann gelten folgende Aussagen:*

(i) *Eine Funktion $g\in C(K)$, welche konstant auf allen Trägern $S_{\varepsilon_s\cdot\varepsilon_t}$ ist, gehört genau dann zu $\mathcal{M}_c$, wenn gilt: $g\equiv\frac{g(s)g(t)}{\|\varepsilon_s\cdot\varepsilon_t\|}$ auf $S_{\varepsilon_s\cdot\varepsilon_t}$ mit $\varepsilon_s\cdot\varepsilon_t>0$ und $g(s)g(t)=0$, falls $\varepsilon_s\cdot\varepsilon_t=0$ ist.*

(ii) *Für $f, g\in\mathcal{M}_c$ gilt $|f|$, $\bar{f}\in\mathcal{M}_c$ und $g\cdot\operatorname{sgn} f\in\mathcal{M}$.*

Beweis. (i) Nach ([9], 2.2) gehört eine Funktion $k\in C(K)$ genau dann zu $\mathcal{M}$, wenn $\int_K k\, d\varepsilon_s\cdot\varepsilon_t=k(s)k(t)$ für alle $s, t\in K$ ist. Hieraus ergibt sich durch eine leichte Rechnung die Behauptung.

(ii). Sei $f, g\in\mathcal{M}_c$ und $f\neq 0$. Aus (i) folgt $|f|$ und $\bar{f}\in\mathcal{M}_c$. Der Beweis der Aussage $g\cdot\operatorname{sgn} f\in\mathcal{M}$ ist dagegen schwieriger. Sei $\mathcal{O}:=\{\tau\in K\colon f(\tau)\neq 0\}$, $s, t\in\mathcal{O}$ und $f\equiv\alpha$ auf $S_{\varepsilon_s\cdot\varepsilon_t}$. Aus $0\neq f(s)f(t)=\int_K f\, d(\varepsilon_s\cdot\varepsilon_t)=\alpha\|\varepsilon_s\cdot\varepsilon_t\|$ folgt $\varepsilon_s\cdot\varepsilon_t>0$ und $\alpha=\frac{f(s)f(t)}{\|\varepsilon_s\cdot\varepsilon_t\|}\neq 0$. Dies bedeutet $S_{\varepsilon_s\cdot\varepsilon_t}\subseteq\mathcal{O}$. Hieraus erhält man

$$\operatorname{sgn} f\equiv(\operatorname{sgn} f)(s)(\operatorname{sgn} f)(t)$$

auf $S_{\varepsilon_s\cdot\varepsilon_t}$. Es ist daher

$$\int_K g\operatorname{sgn} f\, d(\varepsilon_s\cdot\varepsilon_t)=\frac{g(s)g(t)}{\|\varepsilon_s\cdot\varepsilon_t\|}\cdot(\operatorname{sgn} f)(s)\cdot(\operatorname{sgn} f)(t)\|\varepsilon_s\cdot\varepsilon_t\|=$$

$$=(g\cdot\operatorname{sgn} f)(s)(g\cdot\operatorname{sgn} f)(t).$$

Die Linearform $g\cdot\operatorname{sgn} f$ ist also multiplikativ auf den Produkten von Dirac-Maßen, deren Träger in $\mathcal{O}$ liegen.

Seien v und $\mu\in A$ mit $S_\mu, S_v\subseteq\overline{\mathcal{O}}$. Mit Hilfe der im vorhergehenden Beweis angegebenen Netze (μ_λ) und (v_ϱ) und der dort betrachteten Grenzwerte folgt dann

$$\int_K g\operatorname{sgn} f\, d(\mu\cdot v)=\int_K g\cdot\operatorname{sgn} f\, d\mu\cdot\int_K g\cdot\operatorname{sgn} f\, dv.$$

Es ist somit die Linearform $g \cdot \operatorname{sgn} f$ multiplikativ auf dem Band $N(|f|)^{\perp}$ $(=\{\mu \in A\colon S_\mu \subseteq \overline{\mathcal{O}}\})$. Nach (2.2, iii) ist die Bandprojektion $P\colon A \to N(|f|)^{\perp}$ multiplikativ. Da nach 2.1 $N(|f|)=\{\mu \in A\colon S_\mu \subseteq K \setminus \overline{\mathcal{O}}\}$ gilt, verschwindet die Linearform $g \operatorname{sgn} f$ auf dem Band $N(|f|)$. Es ist daher $g \operatorname{sgn} f = g \cdot \operatorname{sgn} f \circ P$. Die zusammengesetzte Abbildung ist aber multiplikativ. Es ist also $g \cdot \operatorname{sgn} f \in \mathcal{M}$, q.e.d.

Hinsichtlich der Aussage (2.3, ii) wollen wir bemerken, daß bei den AL-Algebren von [9] bzw. bei den kommutativen Konvolutionsmaßalgebren von [14] die Menge $\mathcal{M}_c$ bzw. $\mathcal{M}$ sogar eine Halbgruppe der Funktionenalgebra $C(K)$ ist. Hierin besteht ein wesentlicher Unterschied zwischen diesen Algebren und den allgemeineren L-Algebren.

Ist $E_{\mathbf{C}}$ eine komplexe Banachverbandsalgebra, so nennen wir einen Algebrahomomorphismus T zentral, wenn er auch zum Zentrum $Z(E_{\mathbf{C}})$ des zugrundliegenden Banachverbandes $E_{\mathbf{C}}$ gehört, d. h. wenn es auch eine Konstante c gibt mit $|Tz| \leqq c|z|$ für alle $z \in E_{\mathbf{C}}$ (s. [9], 2.). Sei $A_{\mathbf{C}}$ eine komplexe L-Algebra. Für jedes $f \in C(K)$ sei der Operator $Z_f\colon A_{\mathbf{C}} \to A_{\mathbf{C}}$ definiert durch $\int_K g\, dZ_f \mu = \int_K g \cdot f\, d\mu$ für alle $\mu \in A_{\mathbf{C}}$ und alle $g \in C(K)$. Die so definierten Operatoren Z_f gehören offensichtlich zum Zentrum $Z(A_{\mathbf{C}})$ des komplexen Banachverbandes $A_{\mathbf{C}}$.

In ([9], 2.5) haben wir die Menge der zentralen Homomorphismen mit der dortigen Menge $\mathcal{M}_c$ identifizieren können: Die zentralen Homomorphismen sind dort nämlich genau die Operatoren Z_f mit $f \in \mathcal{M}_c$. Für L-Algebren ist diese Identifizierung nicht mehr möglich. Denn die identische Abbildung I ist stets ein zentraler Homomorphismus. Ist aber das Normfunktional $e_K \in C(K)$ nicht multiplikativ, so ist zwar $I = Z_{e_K}$, aber $e_K \notin \mathcal{M}_c$.

Für L-Algebren lassen sich die zentralen Homomorphismen mit Hilfe der Operatoren Z_f nun auf folgende Weise charakterisieren.

2.4. Satz. *Sei $A_{\mathbf{C}}$ eine komplexe L-Algebra und T ein beschränkter Endomorphismus von $A_{\mathbf{C}}$. Der Operator T ist genau dann ein zentraler Homomorphismus, wenn er sich in der Form $T = Z_f$ $(f \in C(K))$ darstellen läßt und dabei die Funktion f auf jedem nichtleeren Träger $S_{\varepsilon_s \cdot \varepsilon_t}$ $(s, t \in K)$ den konstanten Wert $f(s)f(t)$ besitzt.*

Beweis. Er verläuft ganz analog wie der Beweis von ([9], 2.5).

2.5. Korollar. *Ist T ein zentraler Homomorphismus mit der Darstellung $T = Z_f$, so sind auch die Operatoren $|T|$ $(=Z_{|f|})$ und $\overline{T}$ $(=Z_{\bar{f}})$ zentrale Homomorphismen.*

Wie oben bemerkt, induzieren bei AL-Algebren die Funktionen aus $\mathcal{M}_c$ zentrale Homomorphismen. Interessanterweise tun dies bei L-Algebren an deren Stelle ihre Signumsfunktionen.

2.6. Satz. *Es sei $A_{\mathbf{C}}$ eine komplexe L-Algebra und $0 \neq f \in \mathcal{M}_c$. Dann ist der Operator $Z_{\operatorname{sgn} f}$ ein zentraler Homomorphismus von $A_{\mathbf{C}}$.*

Beweis. Es sei $\tilde{Z}_{\operatorname{sgn} f}\colon M(K) \to M(K)$ definiert durch $\mu \to \tilde{Z}_{\operatorname{sgn} f}\mu$ mit $\int_K g\, d\tilde{Z}_{\operatorname{sgn} f}\, d\mu := \int_K g \cdot \operatorname{sgn} f\, d\mu$ für alle $\mu \in M(K)$ und $g \in C(K)$. Dann bildet $\tilde{Z}_{\operatorname{sgn} f}$ das Band $A_{\mathbf{C}}$ in sich ab und stimmt dort mit $Z_{\operatorname{sgn} f}$ überein. Es genügt daher zu zeigen, daß $\tilde{Z}_{\operatorname{sgn} f}$ auf $A_{\mathbf{C}}$ multiplikativ ist.

Nach dem Beweis (2.3, ii) gilt $\operatorname{sgn} f \equiv (\operatorname{sgn} f)(s)(\operatorname{sgn} f)(t)$ auf $S_{\varepsilon_s \cdot \varepsilon_t}$ für alle $s, t \in \mathcal{O} := \{\tau \in K: f(\tau) \neq 0\}$. Hieraus folgt

$$\tilde{Z}_{\operatorname{sgn} f} \varepsilon_s \cdot \varepsilon_t = (\tilde{Z}_{\operatorname{sgn} f} \varepsilon_s) \cdot (\tilde{Z}_{\operatorname{sgn} f} \varepsilon_t) \quad \text{für alle} \quad s, t \in \mathcal{O}.$$

Der Operator $\tilde{Z}_{\operatorname{sgn} f}$ ist $\sigma(M(K), C(K))$ stetig.

Für $\mu, \nu \in A_{\mathrm{C}}$ mit $S_\mu, S_\nu \subseteqq \overline{\mathcal{O}}$ folgt nun wieder unter Benutzung der in den vorgehenden Beweisen betrachteten Netze (μ_λ) und (ν_ϱ) die Beziehung $\tilde{Z}_{\operatorname{sgn} f}(\mu \cdot \nu) = \tilde{Z}_{\operatorname{sgn} f} \mu \cdot \tilde{Z}_{\operatorname{sgn} f} \nu$. Der Operator $\tilde{Z}_{\operatorname{sgn} f}$ ist also multiplikativ auf der Subalgebra $N(|f|)^\perp = \{\mu \in A_{\mathrm{C}}: S_\mu \subseteqq \overline{\mathcal{O}}\}$. Da er offensichtlich auf dem Band

$$N(|f|) = \{\mu \in A_{\mathrm{C}}: S_\mu \subseteqq K \setminus \overline{\mathcal{O}}\}$$

verschwindet, ist er somit auf ganz A_{C} multiplikativ, q.e.d.

Falls nicht alle Linearformen aus $\mathcal{M}_c$ streng positiv sind, existieren also nichttriviale zentrale Homomorphismen.

3. Über Banachverbandsalgebren mit multiplikativer Zerlegungseigenschaft

Ziel dieses Kapitels ist es, mit Hilfe der vorhergehenden Ergebnisse über L-Algebren entsprechende Aussagen über allgemeine Banachverbandsalgebren mit der Eigenschaft $\mathscr{Z}$ zu erhalten.

In der Arbeit [9] haben wir bewiesen, daß bei komplexen Banachverbandsalgebren die Menge $\mathcal{M}$ der multiplikativen Linearformen auch zyklisch ist, falls sie betragsinvariant ist. Nun können wir zeigen, daß bei Algebren mit der Eigenschaft $\mathscr{Z}$ die Menge $\mathcal{M}$ sogar „signumsinvariant" ist. Dabei definieren wir für ordnungsvollständige Banachverbände den Begriff der Signumsinvarianz einer Teilmenge wie folgt:

Es sei E ein ordnungsvollständiger Banachverband und $a, b \in E_{\mathrm{C}}$ mit $a \neq 0$ und $b \neq 0$. Ferner sei $C(K)$ die kanonische Identifizierung des komplexen Hauptideals $(E_{\mathrm{C}})_{|a|+|b|}$ von E_{C}. Die dem Element a bzw. b entsprechende Funktion bezeichnen wir mit f_a bzw. f_b. Da K ein Stone'scher Raum ist, existiert also die Signumsfunktion $\operatorname{sgn} f_a$. Unter dem Element $(\operatorname{sgn} a)b$ verstehen wir dann die Funktion $f_b \cdot \operatorname{sgn} f_a$, als Element von E_{C} aufgefaßt. Wir sagen nun, eine Teilmenge B von E_{C} ist *signumsinvariant,* falls $(\operatorname{sgn} a)b \in B$ für alle von 0 verschiedenen Elemente a und b aus B gilt. Wir wollen bemerken, daß eine signumsinvariante Teilmenge eine Austausch-Teilmenge im Sinne der in ([8], 1) gegebenen Definition ist. Der Beweis davon wäre aber an dieser Stelle zu aufwendig.

In ([9], S. 396) haben wir zu einer positiven, multiplikativen, nichttrivialen Linearform μ auf einer reellen Banachverbandsalgebra A eine reelle AL-Algebra A^μ konstruiert, welche kurz gesagt eine gewisse Vervollständigung der Quotientenalgebra $A/N(\mu)$ ist. Für eine positive, nichttriviale Linearform μ, welche submultiplikativ ist (d. h. $\mu(xy) \leqq \mu(x)\mu(y)$ für alle $x, y \geqq 0$), liefert dieselbe Konstruktion nun eine reelle L-Algebra A^μ bzw. eine komplexe L-Algebra $(A^\mu)_{\mathrm{C}}$. Diese L-Algebren sind im folgenden von großer Bedeutung, da sie es ermöglichen, gewisse Probleme in allgemeinen Banachverbandsalgebren auf solche in L-Algebren zurückzuführen.

Besitzt eine reelle Banachverbandsalgebra A die Eigenschaft $\mathscr{L}$, so vererbt sie diese Eigenschaft auf die L-Algebren A^{μ}.

3.1. SATZ. *Es sei A eine reelle Banachverbandsalgebra mit der Eigenschaft $\mathscr{L}$. Ferner sei μ eine nichttriviale, positive submultiplikative Linearform auf A. Dann besitzt auch die L-Algebra A^{μ} die Eigenschaft $\mathscr{L}$.*

BEWEIS. Der absolute Kern $N(\mu)$ ist ein Strukturideal von A, und die Quotientenalgebra $A/N(\mu)$, versehen mit der kanonischen Ordnung und der Norm $\|\hat{x}\|:=\mu(|x|)$ $(x\in\hat{x})$, ist eine normierte Verbandsalgebra.

Sei nun $\varepsilon>0$, $\hat{a}, \hat{b}, \hat{c}\geqq 0$, $\hat{c}\leqq\hat{a}\cdot\hat{b}$ und $a, b, c\geqq 0$. Dann gibt es ein positives Element $\tilde{c}\in\hat{c}$ mit $\tilde{c}\leqq ab$. Aufgrund der Bedingung $\mathscr{L}^*$ existieren in A n positive Elemente x_i und n positive Elemente y_i mit $x_i\leqq a$, $\sum_{i=1}^{n} y_i\leqq b$ und $\|c-\sum_{i=1}^{n} x_i y_i\|\leqq\varepsilon$. Der Übergang zu den entsprechenden Restklassen zeigt, daß auch die Quotientenalgebra die Bedingung $\mathscr{L}^*$ von (1.1) erfüllt.

Die reelle L-Algebra A^{μ} ist die Vervollständigung der normierten Quotientenalgebra $A/N(\mu)$. Für die Richtung (i)$\to$(ii) im Beweis von 1.1 genügt es vorauszusetzen, daß bei der Bedingung $\mathscr{L}^*$ die Elemente a und b nur aus einer totalen Menge des positiven Kegels sind. Wählt man in A^{μ} den positiven Kegel von $A/N(\mu)$ als solche totale Menge, so läßt sich leicht zeigen, daß auch A^{μ} die Eigenschaft $\mathscr{L}$ besitzt, q.e.d.

Nach diesen Vorbereitungen können wir die Aussage über die Signumsinvarianz von $\mathscr{M}$ beweisen.

3.2. SATZ. *Es sei $A_{\mathbf{C}}$ eine komplexe Banachverbandsalgebra mit der Eigenschaft $\mathscr{L}$. Dann ist die Menge $\mathscr{M}$ signumsinvariant.*

BEWEIS. Sei $\mu, \nu\in\mathscr{M}$. Aus ([9], 1.3) folgt $|\mu|, |\nu|\in\mathscr{M}$. Sei $\varrho:=|\mu|+|\nu|\in A'$. Dann ist ϱ submultiplikativ. Wir betrachten die zu ϱ assoziierte L-Algebra $(A^{\varrho})_{\mathbf{C}}$, deren Menge $\mathscr{M}$ wir mit $\mathscr{M}^{(\varrho)}$ bezeichnen. Es seien $f_{\mu}, f_{\nu}, f_{|\mu|}$ und $f_{|\nu|}$ die den Elementen μ, ν, $|\mu|$ und $|\nu|$ entsprechenden Funktionen in der kanonischen Identifizierung $C(K)$ von $(A^{\varrho})'_{\mathbf{C}}$. Dann gehören diese Funktionen zu $\mathscr{M}^{(\varrho)}$. Für die Einsfunktion e_K gilt $e_K=f_{|\mu|}+f_{|\nu|}$ und somit

$$\int_K e_K\, d\varepsilon_s\cdot\varepsilon_t=f_{|\mu|}(s)f_{|\mu|}(t)+f_{|\nu|}(s)f_{|\nu|}(t) \quad \text{für alle } s,\, t\in K.$$

Da $(A^{\varrho})_{\mathbf{C}}$ die Eigenschaft $\mathscr{L}$ besitzt, ist nach (1.2) $\mathscr{M}^{(\varrho)}=\mathscr{M}_c^{(\varrho)}$. Aufgrund von (2.3, ii) erhalten wir $f_{\nu}\cdot\operatorname{sgn} f_{\mu}\in\mathscr{M}^{(\varrho)}$.

Das von ϱ erzeugte komplexe Hauptideal $(A'_{\mathbf{C}})_{\varrho}$ kann mit dem Raum $(A^{\varrho})'_{\mathbf{C}}$, also mit $C(K)$, identifiziert werden (s. [9], 3.). Dabei entspricht der Linearform $(\operatorname{sgn}\mu)\nu$ die Funktion $f_{\nu}\cdot\operatorname{sgn} f_{\mu}$.

Für $u, v\in A_{\mathbf{C}}$ ergibt sich nunmehr:

$$(\operatorname{sgn}\mu)\nu(uv)=\int_K f_{\nu}\operatorname{sgn} f_{\mu}\, d(\hat{u}\cdot\hat{v})=$$

$$=\int_K f_{\nu}\operatorname{sgn} f_{\mu}\, d\hat{u}\cdot\int_K f_{\nu}\operatorname{sgn} f_{\mu}\, d\hat{v}=\big((\operatorname{sgn}\mu)\nu(u)\big)\big((\operatorname{sgn}\mu)\nu(v)\big).$$

Es ist also $(\operatorname{sgn}\mu)\nu\in\mathscr{M}$, q.e.d.

Wie der vorhergehende Beweis gezeigt hat, kann man zwei beliebige Elemente von $\mathcal{M}$ mit Hilfe von L-Algebren vergleichen, was mit AL-Algebren oder abstrakten Konvolutionsmaßalgebren, bei denen das Normfunktional multiplikativ ist, i. a. nicht geht.

Mit Satz 2.2 erhalten wir für Banachverbandsalgebren mit der Eigenschaft $\mathcal{L}$ die folgende Aussage.

3.3. SATZ. *Es sei A eine reelle Banachverbandsalgebra mit der Eigenschaft $\mathcal{L}$, μ eine positive, multiplikative Linearform auf A und $N(\mu)\neq\{0\}$. Ferner gelte $\{x\in A_+: \nu(x)=0$ für alle $\nu\in\mathcal{M}$ mit $\nu>0\}=\{0\}$. Dann ist das Band $N(\mu)^\perp$ eine Subalgebra.*

BEWEIS. Es sei $x, y\in N(\mu)^\perp$, $z\in N(\mu)$ und $x, y, z>0$. Nach der gemachten Voraussetzung genügt es zu zeigen: $\nu(\inf(z, xy))=0$ für alle $\nu\in\mathcal{M}$ mit $\nu>0$.

Für $\nu\in\mathcal{M}$ mit $\nu>0$ betrachten wir die L-Algebra $A^{\mu+\nu}$. Da der topologische Dual mit dem Hauptideal $A'_{\mu+\nu}$ identifiziert werden kann, definiert μ eine positive multiplikative Linearform $\hat{\mu}$ auf $A^{\mu+\nu}$, welche aufgrund der Eigenschaft $\mathcal{L}$ zu $\mathcal{M}_c^{(\mu+\nu)}$ gehört.

Aus $\mu(z)=0$ folgt $\hat{z}\in N(\hat{\mu})$ und somit $\hat{x}, \hat{y}\in N(\hat{\mu})^\perp$. Es ist also $\inf(\hat{z}, \hat{x}\cdot\hat{y})=0$ in $A^{\mu+\nu}$. Dies bedeutet $(\mu+\nu)(\inf(z, xy))=0$. Es ist somit $\nu(\inf(z, xy))=0$, q.e.d.

Sei $A_{\mathbf{C}}$ eine komplexe Banachverbandsalgebra. Wie in [9] ausgeführt, ist das Zentrum $Z(A_{\mathbf{C}})$ ein komplexer Banachverband. Für jedes $T\in Z(A_{\mathbf{C}})$ ist also der Absolutbetrag $|T|$ erklärt. Besitzt T die Darstellung $T=T_1+iT_2$, so nennen wir den Operator $\overline{T}:=T_1-iT_2$ den zu T konjugierten Operator.

Mit Hilfe von Korollar 2.5 erhalten wir folgende Charakterisierung der zentralen Homomorphismen.

3.4. SATZ. *Es sei $A_{\mathbf{C}}$ eine komplexe Banachverbandsalgebra mit der Eigenschaft $\{x\in A_{\mathbf{C}}: \mu(|x|)=0$ für alle positiven $\mu\in\mathcal{M}\}=\{0\}$. Dann bilden die zentralen Homomorphismen eine betrags- und konjugationsinvariante Halbgruppe.*

BEWEIS. Sei T ein zentraler Homomorphismus von $A_{\mathbf{C}}$. Es ist zu zeigen, daß $|T|$ und $\overline{T}$ wieder multiplikativ sind. Wir zeigen dies für $|T|$. Für $\overline{T}$ geht der Beweis ganz analog.

Sei $\mu\in\mathcal{M}$, $\mu>0$ und $(A^\mu)_{\mathbf{C}}$ die zu μ gehörige komplexe L-Algebra. Wie man sich leicht überlegt, induziert jedes $S\in Z(A_{\mathbf{C}})$ auf folgende Weise einen Zentrumsoperator $\hat{S}$ auf $(A^\mu)_{\mathbf{C}}$: Es ist $\hat{S}(\hat{z}):=\widehat{Sz}$ $(z\in A_{\mathbf{C}})$ auf dem Teilraum $A_{\mathbf{C}}/(N(\mu))_{\mathbf{C}}$, und auf $(A^\mu)_{\mathbf{C}}$ ist $\hat{S}$ die kanonische stetige Fortsetzung dieser Abbildung. Eine Routinerechnung ergibt ferner die Beziehung $\widehat{|S|}=|\hat{S}|$.

Sei nun $x, y\in A$ mit $x>0$ und $y>0$. Nach Korollar 2.5 is $|\hat{T}|$ multiplikativ. Es gilt daher $\widehat{|T|}\hat{x}\cdot\hat{y}=|\hat{T}|\hat{x}\cdot\hat{y}=|\hat{T}|\hat{x}\cdot|\hat{T}|\hat{y}=\widehat{|T|}\hat{x}\cdot\widehat{|T|}\hat{y}$ und somit $\mu(\big||T|xy-|T|x\cdot|T|y\big|)=0$. Die im Satz angegebene Eigenschaft impliziert dann $|T|xy-|T|x\cdot|T|y=0$, q.e.d.

3.5. KOROLLAR. *Es sei $A_{\mathbf{C}}$ eine halbeinfache, kommutative komplexe Banachverbandsalgebra mit der Eigenschaft $\mathcal{L}$. Dann bilden die zentralen Homomorphismen eine betrags- und konjugationsinvariante Halbgruppe.*

BEWEIS. Mit μ ist auch $|\mu|\in\mathcal{M}$. Sei $x\in A_{\mathbf{C}}$. Aus $|\mu|(|x|)=0$ für alle $\mu\in\mathcal{M}$ folgt, daß der Spektralradius von x Null ist und somit $x=0$ ist.

4. Über Strukturideale in kommutativen Banachverbandsalgebren mit Einselement

Unter einem Strukturideal einer Verbandsalgebra versteht man ein zweiseitiges, von der ganzen Algebra verschiedenes Ringideal, welches gleichzeitig auch ein Vektorverbandsideal ist. Ein maximales Strukturideal ist dann eines, das in keinem anderen echt enthalten ist. Wie üblich kann man mit Hilfe des Lemmas von Zorn zeigen, daß in reellen Banachverbandsalgebren mit positivem (algebraischem) Einselement jedes echte Strukturideal in einem maximalen enthalten ist. In den Banachverbandsalgebren $C(K)$ bilden z. B. die Funktionen, die in einem festen Punkt verschwinden, ein maximales Strukturideal.

Es sei A eine reelle Banachverbandsalgebra und J ein Strukturideal. Dann ist in der Komplexifizierung $A_{\mathbf{C}}$ der Teilraum $J+iJ$ ein zweiseitiges Ringideal und ein komplexes Vektorverbandsideal, also ein Strukturideal von $A_{\mathbf{C}}$. Umgekehrt läßt sich jedes Strukturideal von $A_{\mathbf{C}}$ als Komplexifizierung eines reellen darstellen. Man kann sich daher bei der Untersuchung von Strukturidealen auf reelle Banachverbandsalgebren beschränken.

In einer kommutativen, komplexen Banachalgebra mit Einselement ist bekanntlich jedes maximale Ideal Kern eines komplexen Homomorphismus. Der nächste Satz beschreibt eine entsprechende Beziehung zwischen maximalen Strukturidealen und absoluten Kernen. Wir setzen von nun ab stets voraus, daß das algebraische Einselement einer Banachverbandsalgebra positiv ist und die Norm 1 besitzt.

4.1. SATZ. *Es sei A eine kommutative reelle Banachverbandsalgebra mit Einselement und der Zerlegungseigenschaft $\mathscr{Z}$. Ferner sei J ein maximales Strukturideal von A. Dann gibt es auf A eine positive multiplikative Linearform μ, so daß J gleich dem absoluten Kern von μ ist.*

BEWEIS. In $A_{\mathbf{C}}$ ist $J_{\mathbf{C}}$ ein Ringideal. Es gibt daher ein $v\in\mathscr{M}$ mit $v\neq 0$ und $J_{\mathbf{C}}\subseteqq v^{-1}(\{0\})$. Sei $x\in J$, $z\in A_{\mathbf{C}}$ und $|z|\leqq|x|$. Da $J_{\mathbf{C}}$ ein Strukturideal ist, gilt $z\in J_{\mathbf{C}}$ und somit $v(z)=0$. Es ist also $|v|(|x|)=\sup\{|v(y)|:\ y\in A_{\mathbf{C}},\ |y|\leqq|x|\}=0$. Wir erhalten somit $J\subseteqq N(|v|)$. Die Eigenschaft $\mathscr{Z}$ impliziert $|v|\in\mathscr{M}$. Es ist daher $N(|v|)$ ein Strukturideal und somit $J=N(|v|)$, q.e.d.

In den Banachverbandsalgebren $C(K)$ sind die maximalen Strukturideale gerade die absoluten Kerne der Dirac-Maße. Als Elemente des Banachverbandes $M(K)$ sind die Dirac-Maße paarweise zueinander orthogonal. Der folgende Satz verallgemeinert diese Aussage.

4.2. SATZ. *Es sei A eine reelle Banachverbandsalgebra und μ_1 und μ_2 zwei positive multiplikative Linearformen auf A, deren absolute Kerne $N(\mu_1)$ und $N(\mu_2)$ zwei verschiedene maximale Strukturideale sind. Dann ist μ_1 zu μ_2 orthogonal.*

BEWEIS. Sei $\mu=\mu_1+\mu_2$, und f_1 und f_2 seien die den Elementen μ_1 und μ_2 entsprechenden Funktionen im topologischen Dual $C(K)$ der zu μ gehörigen L-Algebra A^{μ}. Dann gilt $f_1\geqq 0$, $f_2\geqq 0$ und $f_1+f_2\equiv 1$. Angenommen, es sei $\inf(\mu_1,\mu_2)>0$, also $\inf(f_1,f_2)>0$. Dann existiert ein Punkt $t\in K$ mit $f_1(t)>0$ und $f_2(t)>0$. Hieraus folgt $f_2\cdot\operatorname{sign} f_1>0$. Ferner gilt $f_2\cdot\operatorname{sgn} f_1\leqq f_2$. Aus $(\operatorname{sgn}\mu_1)\mu_2\leqq\mu_2$ folgt nun $N(\mu_2)\subseteqq N((\operatorname{sgn}\mu_1)\mu_2)$.

Sei $x \in A$, $x \geqq 0$, $\mu_1(x)=0$ und $\mathcal{O}:=\{s \in K: f_1(s)>0\}$. In A^μ erhalten wir dann $\int\limits_K f_1\, d\hat{x}=0$. Nach Lemma 2.1 ist daher $\operatorname{sgn} f_1 \equiv 0$ auf dem Träger $S_{\hat{x}}$ und somit $(\operatorname{sgn} \mu_1)\mu_2(x)=\int\limits_K f_2 \cdot \operatorname{sgn} f_1\, d\hat{x}=0$. Es ist also auch $N(\mu_1) \subseteqq N((\operatorname{sgn} \mu_1)\mu_2)$. Da $N((\operatorname{sgn} \mu_1)\mu_2)$ ein Strukturideal ist, folgt $N(\mu_1)=N((\operatorname{sgn} \mu_1)\mu_2)=N(\mu_2)$, was der gewünschte Widerspruch ist.

Analog zur Ringtheorie sagen wir, eine Banachverbandsalgebra ist *struktureinfach,* wenn die Menge $\{0\}$ das einzige Strukturideal ist. Das *Strukturradikal* $\mathcal{R}$ ist dann als der Durchschnitt aller maximalen Strukturideale definiert, und eine Verbandsalgebra heißt *halbstruktureinfach,* wenn $\mathcal{R}=\{0\}$ gilt.

Wie nicht anders zu erwarten, sind Quotientenalgebren nach maximalen Strukturidealen struktureinfach.

Struktureinfache Banachverbandsalgebren sind keineswegs einfach, sondern äußerst kompliziert, wie es scheint. Wir sind nur in der Lage, den endlich dimensionalen Fall vollständig zu klären.

4.3. SATZ. *Es sei A eine struktureinfache, kommutative reelle Banachverbandsalgebra mit Einselement, mit Zerlegungseigenschaft $\mathscr{L}$ und* $\dim A=n$. *Dann existiert eine Gruppe G, bestehend aus n Elementen, so daß A algebraisch- und verbandsisomorph zur Gruppenalgebra $L(G)$ ist.*

BEWEIS. Bekanntlich ist dann A verbandsisomorph zum Vektorverband $\mathbf{R}^n$. Es besitzt daher A eine Basis, bestehend aus n positiven Vektoren $e_1, \ldots, e_n$, welche paarweise zueinander orthogonal sind und deren Hauptideale A_{e_i} $(1 \leqq i \leqq n)$ eindimensional sind. Zunächst impliziert die Eigenschaft $\mathscr{L}$, daß das Produkt zweier Basisvektoren 0 oder wieder ein nichtnegatives Vielfaches eines Basisvektors ist, d. h. zu jedem Paar (i, k) gibt es einen Basisvektor $e_{i,k}$ und eine nichtnegative Zahl $\gamma_{i,k}$ mit $e_i \cdot e_k=\gamma_{i,k} e_{i,k}$. Da A ein Einselement besitzt, ist $\mathcal{M} \neq \{0\}$. Aufgrund der Eigenschaft $\mathscr{L}$ existiert daher mindestens eine nichttriviale positive multiplikative Linearform μ auf A. Da A struktureinfach ist, gilt offensichtlich $\mu(e_i):=\mu_i>0$ für $1 \leqq i \leqq n$. Aus $e_i \cdot e_k=\gamma_{i,k} e_{i,k}$ folgt dann $\mu_i \cdot \mu_k=\gamma_{i,k}\mu(e_{i,k})$ und somit $\gamma_{i,k}>0$ für $1 \leqq i$, $k \leqq n$. Für $1 \leqq i \leqq n$ sei nun $v_i:=\frac{1}{\mu_i} e_i$. Wie man leicht nachrechnet, ist die Menge $S:=\{v_i: 1 \leqq i \leqq n\}$ eine kommutative Halbgruppe in A.

Es sei K der Suskevič-Kern von S und J die lineare Hülle von K in A. Die Annahme $K \neq S$ würde ergeben, daß die Menge J ein echtes Strukturideal wäre. Da also $S=K$ ist, ist somit S eine Gruppe, welche wir mit G bezeichnen. Die Abbildung Φ: $A \to L(G)$, definiert durch $\Phi\big(\sum\limits_{k=1}^{n} x_k e_k\big)(v_i):=x_i \mu_i$ für $\sum\limits_{k=1}^{n} x_k e_k \in A$ $(x_k \in \mathbf{R})$ und $1 \leqq i \leqq n$, ist dann ein Algebra- und Vektorverbandsisomorphismus von A auf die Gruppenalgebra $L(G)$, q.e.d.

Ist die Norm einer reellen Banachverbandsalgebra ordnungsstetig, so ist für eine Linearform das Band der strengen Positivität sogar ein Projektionsband. Mit Hilfe der maximalen Strukturideale und des Strukturradikals erhalten wir in dieser Situation folgenden interessanten Darstellungssatz:

4.4. SATZ. *Es sei $A_{\mathbf{C}}$ eine kommutative halbeinfache Banachverbandsalgebra mit Einselement, mit Zerlegungseigenschaft $\mathscr{L}$ und ordnungsstetiger Norm. Ferner sei*

$\{J_\lambda\colon \lambda\in\Lambda\}$ das System der maximalen Strukturideale von A, und es sei $A_\lambda := J_\lambda^\perp$ für alle $\lambda\in\Lambda$. Dann gilt:

(i) *Für jedes $\lambda\in\Lambda$ ist A_λ eine solide Subalgebra, welche für sich betrachtet eine struktureinfache Banachverbandsalgebra ist.*

(ii) *Für $\lambda_1, \lambda_2\in\Lambda$ mit $\lambda_1\neq\lambda_2$ gilt $A_{\lambda_1}\perp A_{\lambda_2}$ und $A_{\lambda_1}\cdot A_{\lambda_2}\subseteqq\mathcal{R}$.*

(iii) *Das von der Menge $\{\bigcup_{\lambda\in\Lambda} A_\lambda\}$ erzeugte Band ist gleich dem orthogonalen Komplement des Strukturradikals $\mathcal{R}$, d. h.*

$$A=\overline{\bigoplus_{\lambda\in\Lambda} A_\lambda}\oplus\mathcal{R}.$$

BEWEIS. Nach 4.1 gibt es zu jedem $\lambda\in\Lambda$ ein $\mu_\lambda\in\mathcal{M}$ mit $\mu_\lambda>0$ und $J_\lambda=N(\mu_\lambda)$. Es ist also $A_\lambda=N(\mu_\lambda)^\perp$ für alle $\lambda\in\Lambda$.

(i) Nach 3.3 ist $N(\mu_\lambda)^\perp$ eine Subalgebra. Da die abgeschlossene, solide Subalgebra $N(\mu_\lambda)^\perp$ offensichtlich zu der Banach-Quotientenalgebra $A/N(\mu_\lambda)$ topologisch-, algebraisch- und verbandsisomorph ist, ist also $N(\mu_\lambda)^\perp$ auch struktureinfach.

(ii) Aufgrund von 4.2 bildet die Menge $\{\mu_\lambda\colon \lambda\in\Lambda\}$ ein Orthogonalsystem. Sei $\lambda_1, \lambda_2\in\Lambda$ mit $\lambda_1\neq\lambda_2$. Nach ([7], II, 4.10) gilt dann $N(\inf\{\mu_{\lambda_1}, \mu_{\lambda_2}\})^\perp=N(\mu_{\lambda_1})^\perp\cap\cap N(\mu_{\lambda_2})^\perp$. Es ist daher $N(\mu_{\lambda_1})^\perp\cap N(\mu_{\lambda_2})^\perp=\{0\}$ und somit $N(\mu_{\lambda_1})^\perp\perp N(\mu_{\lambda_2})^\perp$. Sei $0<x\in N(\mu_{\lambda_1})^\perp$ und $0<y\in N(\mu_{\lambda_2})^\perp$. Dann ist $N(\mu_{\lambda_1})^\perp\subseteqq N(\mu_{\lambda_2})$ und $N(\mu_{\lambda_2})^\perp\subseteqq N(\mu_{\lambda_1})$. Hieraus ergibt sich $\mu_\lambda(xy)=\mu_\lambda(x)\mu_\lambda(y)=0$ für alle $\lambda\in\Lambda$ und somit $xy\in\bigcap_{\lambda\in\Lambda} N(\mu_\lambda)=\mathcal{R}$.

(iii) Dies folgt sofort aus $A_\lambda=N(\mu_\lambda)^\perp$, $(N(\mu_\lambda)^\perp)^\perp=N(\mu_\lambda)$, $\mathcal{R}=\bigcap_{\lambda\in\Lambda} N(\mu_\lambda)$ und der Ordnungsstetigkeit der Norm, q.e.d.

4.5. KOROLLAR. *Ist A zusätzlich halbstruktureinfach, so ist A die Vervollständigung einer direkten Summe von struktureinfachen Algebren.*

Aufgrund des vorhergehenden Korollars ist es natürlich naheliegend, in umgekehrter Richtung mit Hilfe von struktureinfachen Banachverbandsalgebren durch Bildung direkter Summen und Vervollständigungen halbstruktureinfache Algebren zu konstruieren: Zum Beispiel sind für $1\leqq p<\infty$ die Banachverbände l^p, versehen mit der koordinatenweisen Multiplikation, auch Banachverbandsalgebren. Sie sind halbstruktureinfach und können im Sinne des vorhergehenden Korollars als Vervollständigung einer direkten Summe von Kopien der offensichtlich struktureinfachen „Banachverbandsalgebra $\mathbf{R}$" betrachtet werden.

Die Arbeit wollen wir nun mit dem folgenden Darstellungssatz, welcher sofort aus 4.3 und 4.5 folgt, schließen.

4.6. SATZ. *Jede kommutative, halbeinfache, halbstruktureinfache, endlich dimensionale Banachverbandsalgebra mit Einselement und der Zerlegungseigenschaft $\mathcal{Z}$ ist eine direkte Summe von Gruppenalgebren.*

Literaturverzeichnis

[1] E. Hewitt and H. S. Zuckerman, Finite dimensional convolution algebras, *Acta Math.*, **93** (1955), 67—119.

[2] S. A. McKilligan and A. J. White, Representations of *L*-algebras, *Proc. London Math. Soc.*, **25** (1972), 655—674.

[3] H. P. Lotz, Extensions and liftings of positive linear mappings on Banach lattices, *T.A.M.S.*, **211** (1975), 85—100.

[4] L. F. Martignon, *Banachverbandsalgebren*, Diss. (Tübingen, 1978).

[5] C. Quint, *Zur Darstellung von Banachverbandsalgebren*, Diss. (Darmstadt, 1984).

[6] H. H. Schaefer, On the *o*-spectrum of order bounded operators, *Math. Z.*, **154** (1977), 79—84.

[7] H. H. Schaefer, *Banach lattices and positive operators*, Springer (Berlin—Heidelberg—New York, 1974).

[8] E. Scheffold, Fixräume regulärer Operatoren und ein Korovkin-Satz, *J. Functional Analysis*, **30** (1978), 147—161.

[9] E. Scheffold, Über komplexe Banachverbandsalgebren, *J. Functional Analysis*, **37** (1980), 382—400.

[10] E. Scheffold, *FF*-Banachverbandsalgebren, *Math. Z.*, **177** (1981), 193—205.

[11] E. Scheffold, Banachverbandsalgebren mit einer natürlichen Wedderburn-Zerlegung, *Math. Z.*, **185** (1984), 521—531.

[12] J. L. Taylor, The structure of convolution measure algebras, *T.A.M.S.*, **119** (1965), 150—166.

[13] J. L. Taylor, Noncommutative convolution measure algebras, *Pacific J. Math.*, **31** (1969), 809—826.

[14] J. L. Taylor, Measure algebras, *Regional conference series in mathematics*, no. **16**, A.M.S. (Providence, Rhode Island, 1973).

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SETS OF CONVEXITY OF CONTINUOUS FUNCTIONS

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In [2] F. Filipczak proved that for every $f \in C[a, b]$ there is a perfect set $P \subset [a, b]$ on which f is monotone. In [1] the authors raised the following problem:

Let $f \in C[a, b]$ be arbitrary and let n be a non-negative integer. Does there exist a nonempty perfect set $P \subset [a, b]$ on which f is either n-convex or n-concave? (Filipczak's theorem settles the case $n=1$.) In this paper we answer this problem for $n=2$.

THEOREM. *For every continuous function f: $\mathbf{R} \to \mathbf{R}$ at least one of the following statements is true:*

(i) *There is an interval $I \subset \mathbf{R}$ such that $f|_I$ is convex.*

(ii) *There is an interval $I \subset \mathbf{R}$ such that $f|_I$ is concave.*

(iii) *There are nonempty perfect sets $H_1 \subset \mathbf{R}$, $H_2 \subset \mathbf{R}$ such that $f|_{H_1}$ is strictly convex and $f|_{H_2}$ is strictly concave.*

DEFINITION 1. $p \in \mathbf{R}$ is a quasi-minimum [quasi-maximum] of f if we can choose $\delta_p > 0$ and $m_p \in \mathbf{R}$ such that for any $x \in (p-\delta_p, p+\delta_p)$ we have

$$f(x) \geqq f(p) + m_p(x-p), \quad [f(x) \leqq f(p) + m_p(x-p)].$$

We put

$$E_{\min} := \{p \in \mathbf{R};\ p \text{ is a quasi-minimum}\},$$

$$E_{\max} := \{p \in \mathbf{R};\ p \text{ is a quasi-maximum}\},$$

$$E := E_{\min} \cup E_{\max}.$$

LEMMA 1. *If $E_{\min}$ (or $E_{\max}$) is of second category then there is an interval $I \subset \mathbf{R}$ such that f is convex (concave) on I.*

PROOF. For all $p \in E_{\min}$ we denote by δ_p and m_p the numbers in Definition 1. For $r<s$ we put

$$A_{r,s} := \{p \in E_{\min} \cap [r, s];\ (r, s) \subset (p-\delta_p, p+\delta_p)\}.$$

Plainly $E_{\min} = \bigcup \{A_{r,s};\ r, s \in \mathbf{Q},\ r<s\}$. Since $E_{\min}$ is of second category there is a pair of numbers $r, s \in \mathbf{Q}$ such that $A_{r,s}$ is dense in an interval $I \subset (r, s)$. We prove that $f|_I$ is convex.

Indeed $I \subset (r, s) \subset (p-\delta_p, p+\delta_p)$ for all $p \in A_{r,s}$ that is $f(x) \geqq f(p) + m_p(x-p)$ for all $x \in I$. This easily implies that $f|_{A_{r,s} \cap I}$ is convex. The density of $I \cap A_{r,s}$ in I and the continuity of f furnish that $f|_I$ is also convex.

NOTATION 1. $\partial^+ f(q) := \{m \in \mathbf{R};$ there exists $\{q_n\}_{n=1}^{\infty}$ such that $q_n \searrow q$, $f(q_{2n}) > f(q) + m(q_{2n} - q)$ and $f(q_{2n-1}) < f(q) + m(q_{2n-1} - q)$ for $n = 1, 2, \ldots\}$.

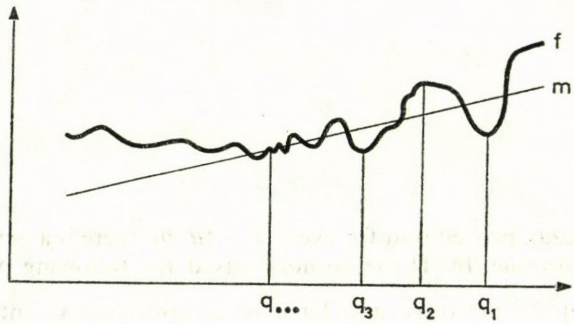

Fig. 1

Similarly with $q_n \nearrow q$ we can define $\partial^- f(q)$ and we put $\partial f(q) := \partial^- f(q) \cup \cup \partial^+ f(q)$.

We shall denote by $\underline{D}f(x)$ the lower and by $\overline{D}f(x)$ the upper bilateral derivative of $f(x)$.

LEMMA 2. *Let* $f\colon \mathbf{R} \to \mathbf{R}$ *be continuous on* $[a, b]$ *and* $s \in \mathbf{R}$. *Suppose that*

$$\frac{f(b) - f(a)}{b - a} > s$$

and the set $\{x \in [a, b];\ \underline{D}f(x) < s\}$ *is dense in* $[a, b]$. *Then there exists* $x_0 \in (a, b)$ *such that* $s \in \partial^+ f(x_0)$.

PROOF. Subtracting a linear function we can assume that $s = 0$. Then the first assumption of the lemma yields that $f(b) > f(a)$.

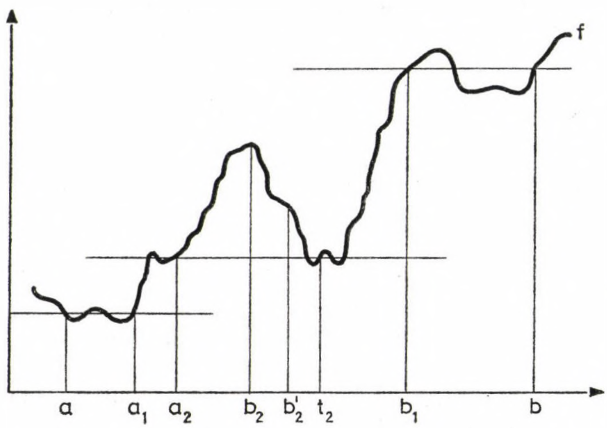

Fig. 2

Let

$$b_1 := \min\{x \in [a, b];\ f(x) = f(b)\}, \qquad a_1 := \max\{x \in [a, b_1];\ f(x) = f(a)\}.$$

Obviously for $x \in (a_1, b_1)$ $f(a) < f(x) < f(b)$. The second assumption implies that we can choose b_2', t_2 such that $a_1 < b_2' < t_2 < b_1$ and $f(b_1) > f(b_2') > f(t_2) > f(a_1)$. We also put

$$b_2 := \min\{x \in (a_1, b_2'];\ f(x) \geqq f(y) \text{ for all } y \in [a_1, b_2']\}$$

and

$$a_2 := \max\{x \in [a_1, b_2);\ f(x) = f(t_2)\}.$$

Plainly $f(a_2) < f(b_2)$ and the set $\{x \in [a_2, b_2];\ \underline{D}f(x) < 0\}$ is also dense in $[a_2, b_2]$. Repeating the process above we get the sequences a_i, b_i, t_i $(i=2, 3, \ldots)$ such that

1) $a_i < b_i < t_i$,

2) $f(a_i) = f(t_i) < f(b_i)$,

3) for all $x \in (a_i, b_i)$, $f(a_i) < f(x) < f(b_i)$,

4) $a_i < a_{i+1} < t_{i+1} < b_i$

for all $(i=2, 3, \ldots)$.

We define $x_0 := \inf\{b_i;\ i=1, 2, \ldots\}$ and the sequence

$$q_{2i} := b_{i+1} \quad \text{and} \quad q_{2i-1} := t_{i+1}, \quad i = 1, 2, \ldots.$$

From Properties 1)—4) it follows that $x_0 \in (a_i, b_i)$ for all $i=2, 3, \ldots$ and $f(q_{2i}) = f(b_{i+1}) > f(x_0)$ and $f(q_{2i-1}) = f(t_{i+1}) < f(x_0)$ that is, by definition, $0 \in \partial^+ f(x_0)$.

LEMMA 3. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous function such that there is an interval $(a, b) \subset \mathbf{R}$ and a number $s_0 \in \mathbf{R}$ for which*

$$\frac{f(b) - f(a)}{b - a} = s_1 > s_0$$

and the set $\{x \in [a, b];\ \underline{D}f(x) < s_0\}$ is dense in (a, b). Then there exists a nonempty perfect set $H \subset \mathbf{R}$ such that $f|_H$ is strictly convex.

DEFINITION 2. We say that $f|_H$ is *strictly d-convex* if we have a function $d\colon L \to \mathbf{R}$ such that $H \subset L$ and for all $p, q \in H$, $p \neq q$, $f(p) > f(q) + d(q)(p-q)$.

REMARK 1. It is easy to see that if $f|_H$ is strictly d-convex then $f|_H$ is strictly convex.

LEMMA 3/a. *Suppose that the condition of Lemma 3 is fulfilled. Let $q, q_1 \in (a, b)$, $q < q_1$. Suppose that the function d is defined on $\{q, q_1\}$ and $d(q) \in \partial^+ f(q)$, furthermore $d(q) > s_0$ and $f|_{\{q, q_1\}}$ is strictly d-convex. Then for any $\varepsilon > 0$ there is a $q_2 \in (q, q_1) \cap (q, q+\varepsilon)$ and a number $d(q_2) \in \partial^+ f(q_2)$ such that $d(q_2) > s_0$ and $f|_{\{q, q_2, q_1\}}$ is strictly d-convex. (See Fig. 3.)*

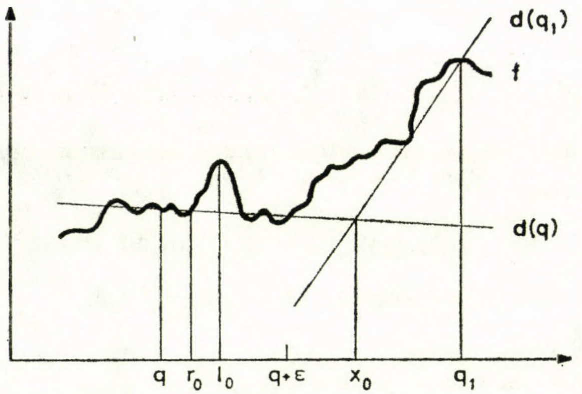

Fig. 3

PROOF. Since $f|_{\{q,q_1\}}$ is strictly d-convex, $d(q)<\dfrac{f(q_1)-f(q)}{q_1-q}$ and there is $x_0\in(q,q_1)$ such that $f(q)+d(q)(x_0-q)=f(q_1)+d(q_1)(x_0-q_1)$. From $d(q)\in\partial^+f(q)$ it follows that there is an $l_0\in(q,x_0)\cap(q,q+\varepsilon)$ such that $f(l_0)>f(q)+d(q)(l_0-q)$.

We can find a point

$$r_0:=\max\{x\in(q,l_0);\ f(x)=f(q)+d(q)(x-q)\}.$$

Put

$$d':=\min\left\{\frac{f(l_0)-f(q)}{l_0-q},\ \frac{1}{2}\left(d(q)+\frac{f(q_1)-f(q)}{q_1-q}\right)\right\},$$

then obviously

$$d(q)<d'<\frac{f(q_1)-f(q)}{q_1-q}.$$

Thus we put

$$l_1:=\min\{x\in(r_0,l_0];\ f(x)=f(q)+d'(x-q)\}.$$

REMARK 2. Obviously, for every $p\in(r_0,l_1)$ we have $f(p)<f(q)+d'(p-q)$.

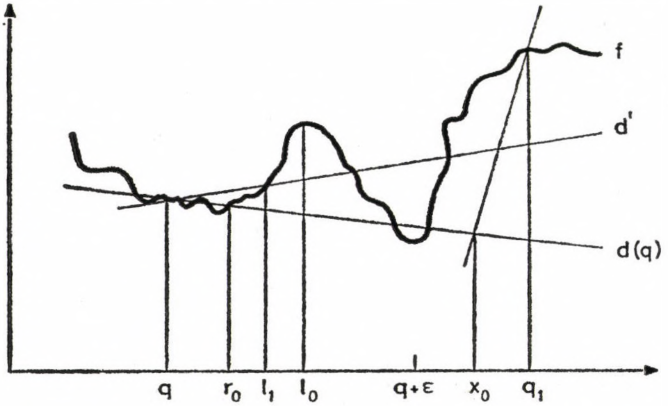

Fig. 4

Examining Fig. 4 we can easily check that $\frac{f(l_1)-f(r_0)}{l_1-r_0}>d'(>d(q)>s_0)$. By assumption the set $\{x\in(r_0, l_1);\ \underline{D}f(x)<s_0\}$ is dense in (r_0, l_1). We choose an s with

$$d' < s < \min\left\{\frac{f(q_1)-f(q)}{q_1-q},\ \frac{f(l_1)-f(r_0)}{l_1-r_0}\right\}.$$

Applying Lemma 2 we get a point $q_2\in(r_0, l_1)$ such that $s\in\partial^+f(q_2)$. Define $d(q_2):=s$. The fact that $d(q_2)>d'$ and Remark 2 imply that

$$f(q) > f(q_2)+d(q_2)(q-q_2).$$

Since

$$d(q_2) < \frac{f(q_1)-f(q)}{q_1-q}$$

and, by Remark 2,

$$f(q_2) < f(q)+d'(q_2-q) \leqq f(q)+\frac{f(q_1)-f(q)}{q_1-q}(q_2-q),$$

we have

$$f(q_1) > f(q_2)+d(q_2)(q_1-q_2).$$

By the choice of r_0 and x_0

$$f(q_2) > f(q)+d(q)(q_2-q)$$

and

$$f(q_2) > f(q_1)+d(q_1)(q_2-q_1).$$

It follows immediately that $f|_{\{q, q_2, q_1\}}$ is strictly d-convex.

LEMMA 3/b. *Suppose that the condition of Lemma 3 is fulfilled. Let $q\in(a, b)$. Suppose $d(q)\in\partial^+f(q)$ and $d(q)>s_0$. Then for any $\varepsilon>0$ there is a $q_2\in(q, q+\varepsilon)$ and a number $d(q_2)\in\partial^+f(q_2)$ such that $d(q_2)>s_0$ and $f|_{\{q, q_2\}}$ is strictly d-convex.*

The proof is similar to that of Lemma 3/a and we leave it to the reader.

DEFINITION 3. Define the function ν: $\mathbf{N}\to\mathbf{N}$ by

$$\nu(n) := \begin{cases}\alpha & \text{if } n = p^\alpha \text{ where } p \text{ is prime}\\ 1 & \text{otherwise.}\end{cases}$$

REMARK 3. Obviously for any $n\in\mathbf{N}$, $\nu(n)\leqq n$ and for any $n\in\mathbf{N}$, $\nu(k)=n$ holds for infinitely many k.

LEMMA 3/c. *Suppose that the condition of Lemma 3 is fulfilled. Let*

$$H_n:= \{p_1, \dots, p_n\}\subset(a, b).$$

Suppose that we have a function d_n: $H_n\to\mathbf{R}$ such that $d_n(p_i)\in\partial^+f(p_i)$ and $d(p_i)>s_0$ for $i=1, \dots, n$, furthermore $f|_{H_n}$ is strictly d_n-convex. Then we can find a point

$$p_{n+1}\in(a, b)\cap\left(p_{\nu(n)},\ p_{\nu(n)}+\frac{1}{n}\right)$$

and a function d_{n+1} *such that* $(p_{\nu(n)}, p_{n+1})\cap H_n=\emptyset$, $d_{n+1}\colon H_{n+1}\to\mathbf{R}$ *where* $H_{n+1}=H_n\cup\{p_{n+1}\}$, $d_{n+1}|_{H_n}=d_n$, $d_{n+1}(p_{n+1})\in\partial^+f(p_{n+1})$, $d_{n+1}(p_{n+1})>s_0$ *and* $f|_{H_{n+1}}$ *is strictly* d_{n+1}*-convex.*

Proof. If there is a $p_k\in H_n$, $p_{\nu(n)}<p_k$ and $(p_{\nu(n)}, p_k)\cap H_n=\emptyset$ then we can apply Lemma 3/a with $q:=p_{\nu(n)}, q_1:=p_k$ and define p_{n+1} and $d_{n+1}(p_{n+1})$ by $p_{n+1}:=q_2$ and $d_{n+1}(p_{n+1}):=d(q_2)$.

α) A consequence of Lemma 3/a is that $f|_{\{p_{\nu(n)}, p_{n+1}, p_k\}}$ is strictly d_{n+1}-convex.

β) By assumption $f|_{H_n}$ is also strictly d_{n+1}-convex.

γ) By the definition of p_k, $(p_{\nu(n)}, p_k)\cap H_{n+1}=\{p_{n+1}\}$.

The properties α), β) and γ) easily imply that $f|_{H_{n+1}}$ is strictly d_{n+1}-convex.

If there is no $p_k\in H_n$ such that $p_{\nu(n)}<p_k$ then we can apply Lemma 3/b to define p_{n+1} and d_{n+1}; we leave the details to the reader.

Lemma 3/d. *Let* $f\colon \mathbf{R}\to\mathbf{R}$ *be continuous. Suppose that we have a set* $H_\omega=\{p_1, p_2, \ldots, p_n, \ldots\}$ *and a function* $d\colon H_\omega\to\mathbf{R}$ *such that* $f|_{H_\omega}$ *is strictly d-convex and* $p_{n+1}\in\left(p_{\nu(n)}-\frac{1}{n},\ p_{\nu(n)}+\frac{1}{n}\right)$ *for every* $n=1, 2, \ldots$; *then there is a nonempty perfect set H such that* $f|_H$ *is strictly convex.*

Proof. By Remark 3, for every $j\in\mathbf{N}$ we can choose a sequence n_k $(k=1, 2, \ldots)$ such that $n_k<n_{k+1}$ and $\nu(n_k)=j$. It follows that for every $k=1, 2, \ldots$ $|p_{n_k+1}-p_j|<\frac{1}{n_k}$ and hence $\lim\limits_{k\to+\infty} p_{n_k}=p_j$. That is H_ω is dense in itself and $H:=\overline{H}_\omega$ is a nonempty perfect set. The continuity of f immediately implies that $f|_{\overline{H}_\omega}$ is convex. We prove that f is strictly convex on H. Suppose that there exist $p_1, p_2, p_3\in\overline{H}_\omega$, $p_1<p_2<p_3$, and an $m\in\mathbf{R}$ such that $f(p_1)+m(p_2-p_1)=f(p_2)$ and $f(p_1)+m(p_3-p_1)=f(p_3)$. Since H_ω is dense in itself there is a sequence p_k, $k=1, 2, \ldots$, such that $p_k\in H_\omega$ and $p_1<p_k<p_3$, $k=4, 5, \ldots$, and $\lim\limits_{k\to\infty} p_k=p_2$. The convexity of $f|_{\overline{H}_\omega}$ implies $f(p_1)+m(p_k-p_1)=f(p_k)$ for $k=4, 5, \ldots$ which is a contradiction because $f|_{H_\omega}$ is strictly d-convex and hence $f|_{H_\omega}$ is strictly convex.

Proof of Lemma 3. By Lemma 3/c we can define a sequence of sets H_n and of functions d_n (p_1 and d_1 can be chosen according to Lemma 2). Define $H_\omega:=\bigcup\limits_{n=1}^{\infty} H_n$ and the function d by $d|_{H_n}:=d_n$. We apply Lemma 3/d to find H.

Lemma 3′. *Let* $f\colon \mathbf{R}\to\mathbf{R}$ *be a continuous function such that there is an interval* $(a, b)\subset\mathbf{R}$ *and a number* $s_0\in\mathbf{R}$ *for which*

$$\frac{f(b)-f(a)}{b-a}=s_1<s_0$$

and the set $\{x\in[a, b];\ \overline{D}f(x)>s_0\}$ *is dense in* (a, b). *Then there exists a perfect set* $H\subset\mathbf{R}$ *such that* $f|_H$ *is strictly concave.*

Proof. We have to apply Lemma 3 to $-f$.

LEMMA 4. *Let* f: $\mathbf{R}\to\mathbf{R}$ *be continuous and*

$$A := \{x\in\mathbf{R};\ \limsup_{y\to x} \bar{D}f(y) = \liminf_{y\to x} \underline{D}f(y) = \underline{D}f(x) = \bar{D}f(x) = f'(x)\in\mathbf{R}\}.$$

Then either A *is residual in* $\mathbf{R}$ *or there is a nonempty perfect set* $H\subset\mathbf{R}$ *such that* $f|_H$ *is strictly convex.*

REMARK 4. It is obvious that $f'|_A$ is continuous.

PROOF. Let $p>q$ and put

$$A^1_{p,q} := \{x;\ \limsup_{y\to x} \bar{D}f(y) > p > q > \liminf_{y\to x} \underline{D}f(y)\}.$$

Suppose that $A^1_{p,q}$ is of second category. Then it is dense in a subinterval J_1 of $\mathbf{R}$. Since for $x\in A^1_{p,q}$, $\limsup_{y\to x} \bar{D}f(y)>p$ we can choose $a<b$ in J_1 with $\dfrac{f(b)-f(a)}{b-a}>p$. The density of $A^1_{p,q}$ implies that $\{x\in[a,b];\ \underline{D}f(x)<q\}$ is dense in (a,b). Hence we can apply Lemma 3 and find the set H. Therefore we can suppose that $A^1_{p,q}$ is of first category for any $p>q$.

We put $A^1 := \mathbf{R}\setminus\bigcup\{A^1_{p,q};\ p,q\in\mathbf{Q},\ p>q\}$; then A^1 is residual. Let

$$\underline{A}_{p,q} := \{x\in A^1;\ \underline{D}f(x) < q < p < \liminf_{y\to x} \underline{D}f(y)\}$$

and

$$\bar{A}_{p,q} := \{x\in A^1;\ \bar{D}f(x) > p > q > \limsup_{y\to x} \bar{D}f(y)\}.$$

Suppose there is a subinterval J_2 of $\mathbf{R}$ where $\underline{A}_{p,q}$ is dense. Since for $x\in\underline{A}_{p,q}$, $\liminf_{y\to x} \underline{D}f(y)>p$ we can find $a<b$ in J_2 such that $\dfrac{f(b)-f(a)}{b-a}>p$. The density of $\underline{A}_{p,q}$ implies that $\{x\in[a,b];\ \underline{D}f(x)<q\}$ is dense in (a,b). Hence we can apply Lemma 3 and find the set H.

Finally suppose that there is a subinterval J_3 of $\mathbf{R}$ where $\bar{A}_{p,q}$ is dense. Since for $x\in\bar{A}_{p,q}$, $\bar{D}f(x)>p$ we can find $a<b$ in J_3 such that $\dfrac{f(b)-f(a)}{b-a}>p$. The density of $\bar{A}_{p,q}$ implies that $\{x\in[a,b];\ \underline{D}f(x)\leqq\bar{D}f(x)<q\}$ is dense in (a,b). We can again apply Lemma 3 to find the set $\underline{H}$.

Define

$$B^1 := \bigcup\{\underline{A}_{p,q};\ p,q\in\mathbf{Q},\ p>q\}\cup\bigcup\{\bar{A}_{p,q};\ p,q\in\mathbf{Q},\ p>q\}$$

and

$$A^2 := A^1\setminus B^1.$$

We proved that either B^1 is of first category or we can find the set H. Hence we can suppose that A^2 is residual.

By definition, for $x\in A^2$ $(\subset A^1)$

$$\bar{D}f(x) = \limsup_{y\to x} \bar{D}f(y) = \liminf_{y\to x} \underline{D}f(y) = \underline{D}f(x).$$

We put $B_+^2 := \{x \in A^2;\ \bar{D}f(x) = +\infty\}$ and $B_-^2 := \{x \in A^2;\ \bar{D}f(x) = -\infty\}$. Obviously $A \supset A^2 \setminus (B_+^2 \cup B_-^2)$.

Suppose that B_-^2 is of second category. Then B_-^2 is dense in an interval (a, b). We can apply Lemma 3 with $s_0 := \dfrac{f(b)-f(a)}{b-a} - 1$ since $\{x \in [a, b];\ -\infty = \underline{D}f(x) = \bar{D}f(x) < s_0\}$ is dense in (a, b). We can find H as before.

Now suppose that B_+^2 is of second category; then B_+^2 is dense in an interval (a, b). Here we are unable to apply Lemma 3 immediately on f. We put $s(x) = -x$ $(x \in \mathbf{R})$. If $s(x) \in B_+^2$ then

$$+\infty = (\bar{D}f)(s(x)) = \limsup_{y \to x} \frac{f(s(y)) - f(s(x))}{s(y) - s(x)} = \limsup_{y \to x} \frac{f(s(y)) - f(s(x))}{(-y)-(-x)} =$$

$$= (-1) \liminf_{y \to x} \frac{f(s(y)) - f(s(x))}{y - x} = -\underline{D}(f \circ s)(x).$$

Thus we can apply Lemma 3 with $f \circ s$, $(-b, -a) = (s(b), s(a))$, and

$$s_0 = -\frac{f(b)-f(a)}{b-a} - 1 = \frac{f(s(s(b))) - f(s(s(a)))}{s(b) - s(a)} - 1.$$

Lemma 3 is applicable because $\{x \in (-b, -a);\ -\infty = \underline{D}(f \circ s)(x) < s_0\}$ is dense in $(-b, -a)$ since B_+^2 is dense in (a, b). Hence we can find a nonempty perfect set H' such that $f \circ s|_{H'}$ is strictly convex and then $f|_{s(H')}$ is also strictly convex. This completes the proof of Lemma 4.

LEMMA 4′. *Let A be defined as in Lemma 4. Then either A is residual or there is a perfect set $H \subset \mathbf{R}$ such that $f|_H$ is strictly concave.*

PROOF. Using Lemma 3′.

Now we turn to the *proof of the theorem:*

Suppose that neither of (i) to (ii) is true. Lemma 1 yields that $E = E_{\min} \cup E_{\max}$ is of first category. This implies that $C = A \setminus E$ is residual and, a fortiori, C is everywhere dense. Thus the following lemma will complete the proof of the theorem.

LEMMA 5. *If C is everywhere dense then there exist perfect sets H_1 and H_2 such that $f|_{H_1}$ is strictly convex and $f|_{H_2}$ is strictly concave.*

NOTATION 2. Let f be fixed. For a point $p \in C$ let

$$D_{ru}^p := \{(x, y) \in \mathbf{R}^2;\ \ x > p,\ y > f(p) + f'(p)(x-p)\},$$

$$D_{rl}^p := \{(x, y) \in \mathbf{R}^2;\ \ x > p,\ y < f(p) + f'(p)(x-p)\},$$

$$D_{lu}^p := \{(x, y) \in \mathbf{R}^2;\ \ x < p,\ y > f(p) + f'(p)(x-p)\},$$

$$D_{ll}^p := \{(x, y) \in \mathbf{R}^2;\ \ x < p,\ y < f(p) + f'(p)(x-p)\}.$$

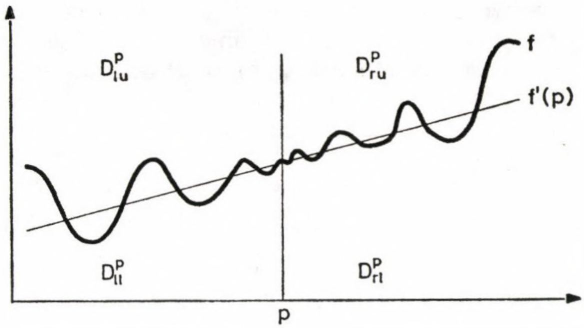

Fig. 5

Put

$$C_{1r} := \{p \in C;\ \exists y_n \in C,\ n = 0, 1, 2, \ldots,\ y_0 = p < y_n,\ n = 1, 2, \ldots,\ y_{n+1} < y_n,\ n = 1, 2, \ldots, \text{ and with the function } d\colon C \to \mathbf{R},\ d(y) := f'(y),\ f\big|_{\bigcup_{n=0}^{\infty} y_n} \text{ is strictly } d\text{-convex}\}$$

and

$$C_{1l} := \{p \in C;\ \exists y_n \in C,\ n = 0, 1, 2, \ldots,\ y_0 = p > y_n,\ n = 1, 2, \ldots,\ y_{n+1} > y_n,\ n = 1, 2, \ldots, \text{ and with the function } d\colon C \to \mathbf{R},\ d(y) := f'(y),\ f\big|_{\bigcup_{n=0}^{\infty} y_n} \text{ is strictly } d\text{-convex}\}.$$

We also put $C_1 := C_{1l} \cup C_{1r}$.

PROOF OF LEMMA 5. It is enough to find a nonempty perfect set H_1 such that $f|_{H_1}$ is strictly convex. The concave case can be treated analogously using Lemma 3′ instead of Lemma 3.

Suppose first $C_1 = C$, and define the function $d\colon C \to \mathbf{R}$ by $d(p) := f'(p)$ for $p \in C$. Let p_1 be an arbitrary point of C_1. Suppose that we defined $L_n = \{p_1, \ldots, p_n\} \subset C$ such that $p_{j+1} \in \left(p_{\nu(j)} - \frac{1}{j},\ p_{\nu(j)} + \frac{1}{j}\right)$ for $j = 1, \ldots, n-1$ and $f|_{L_n}$ is strictly d-convex. Since $P_{\nu(n)} \in C = C_1$, there is a sequence $y_k \to p_{\nu(n)}$ such that $y_k \in C$ and $f\big|_{p_{\nu(n)} \cup \bigcup_{k=1}^{\infty} y_k}$ is strictly d-convex. By Remark 4, $d(y_k) \to d(p_{\nu(n)})$ $(k = 1, 2, \ldots)$. This and the fact that $f|_{L_n}$ is strictly d-convex imply that there is a $y_{k'}$ such that

$$y_{k'} \in \left(p_{\nu(n)} - \frac{1}{n};\ p_{\nu(n)} + \frac{1}{n}\right)$$

and $f|_{L_n \cup \{y_{k'}\}}$ is strictly d-convex. Define p_{n+1} by $p_{n+1} = y_{k'}$. By induction, define L_n for $n = 1, 2, \ldots$, and put $H_\omega := \bigcup_{n=1}^{\infty} L_n$. Applying Lemma 3/d we get a nonempty perfect set H_1 with strictly convex $f|_{H_1}$.

If there is a subinterval where the conditions of Lemma 3 are satisfied we are done. To complete the proof of Lemma 5 we show that if the conditions of Lemma 3 are not satisfied by f in any subinterval (a, b) of $\mathbf{R}$ then $C_1=C$.

Let $p\in C$ be fixed. Then p is not a quasi-extremum thus there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n\to p$ such that $(x_n; f(x_n))\in D^p_{ru}\cup D^p_{lu}$. Put again $s(x):=-x$ $(x\in\mathbf{R})$. Obviously, if there is a perfect set H' such that $(f\circ s)|_{H'}$ is strictly convex then $f|_H$ is also strictly convex, where $H:=s(H')$, and hence we can also suppose that there is no interval (a, b) in which $f\circ s$ fulfils the conditions of Lemma 3. Thus we can reduce the case of $x_n\nearrow p$ to the case of $s(x_n)\searrow s(p)$.

Hence we suppose $x_n\searrow p$. We will show that $p\in C_{1,r}$. We have to find a sequence $\{y_n\}_{n=1}^{\infty}\subset C$, $y_n\searrow p$ such that $f\Big|_{\bigcup_{n=1}^{\infty}\{y_n\}\cup\{p\}}$ is strictly d-convex where $d(y_n)=f'(y_n)$. Suppose that we have $\{p\}\cup\bigcup_{n=1}^{m}\{y_n\}\subset C$ such that $f\Big|_{\bigcup_{n=1}^{m}\{y_n\}\cup\{p\}}$ is strictly d-convex. The case when $m=0$, that is the definition of y_1, only differs slightly from the cases when $m\in\mathbf{N}$. We shall mention these differences in the argument below.

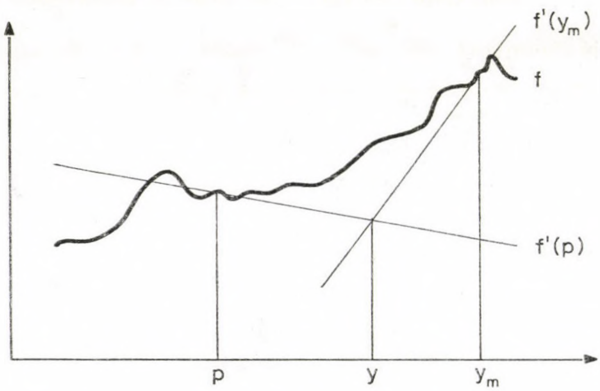

Fig. 6

Let the point $y\in(p, y_m)$ be defined by

$$f(p)+f'(p)(y-p)=f(y_m)+f'(y_m)(y-y_m).$$

(When $m=0$ we do not have to define y.) Obviously

$$f'(p)<\frac{f(y_m)-f(p)}{y_m-p}<f'(y_m).$$

Since $f'|_C$ is continuous we can choose $\varepsilon_0>0$ with $p+\varepsilon_0<y$ and

$$f'(x)<\frac{f(y_m)-f(p)}{y_m-p}\quad\text{for}\quad x\in(p, p+\varepsilon_0)\cap C.$$

(When $m=0$ we do not have to define ε_0.)

Since $x_n \searrow p$ we fix an $x_n \in (p, p+\varepsilon_0)$. (When $m=0$ we can choose an arbitrary $x_n > p$.) Put

$$d := \frac{f(x_n)-f(p)}{x_n-p}.$$

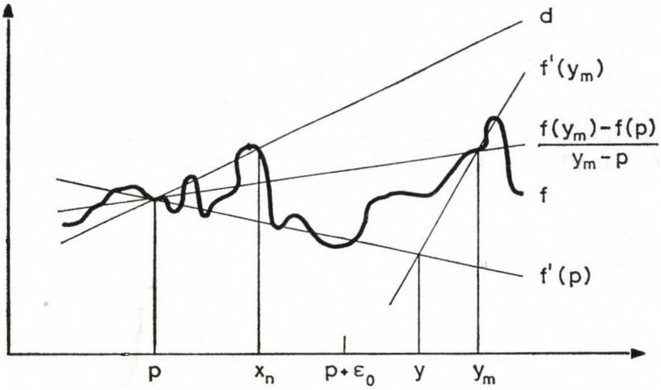

Fig. 7

From $(x_n, f(x_n)) \in D^p_{ru}$ it follows that $f'(p) < d$.
Choose a number t such that

$$f'(p) < t < \min\left(d, \frac{f(y_m)-f(p)}{y_m-p}\right) =: d'.$$

(When $m=0$ $f'(p) < t < d =: d'$.) Put

$$r := \max\{x \in (p, x_n);\ \ f(x) = f(p)+t(x-p)\}.$$

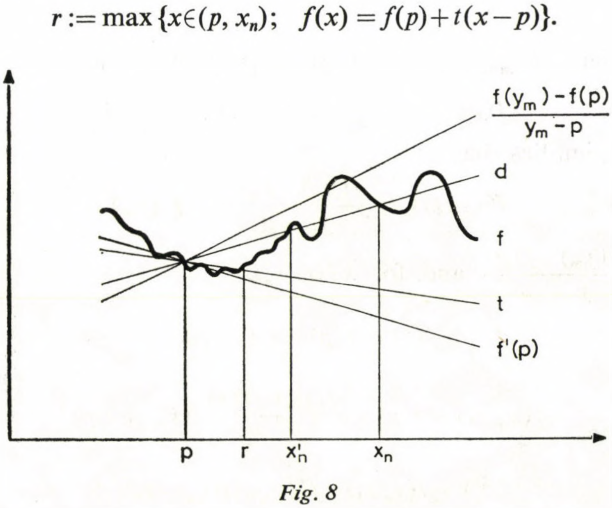

Fig. 8

Since $x_n > r > p$ and $t < d$,

$$\frac{f(x_n)-f(r)}{x_n-r} > d\left(=\frac{f(x_n)-f(p)}{x_n-p}\right).$$

Let

$$x_n' := \min\{x \in (r, x_n];\; f(x) = f(p)+d'(x-p)\}.$$

This definition makes sense because $f'(p) < d' \leqq d$. If there was no $y_{m+1} \in (r, x_n') \cap C$ such that $f'(y_{m+1}) \geqq d'$ then $f'(q) < d'$ would hold on the set $C \cap (r, x_n')$ which is everywhere dense in (r, x_n'). This would imply that the condition of Lemma 3 is satisfied in the interval (r, x_n'). This is impossible and hence there is a $y_{m+1} \in (r, x_n') \cap C$ such that $f'(y_{m+1}) \geqq d'$. Then we can add this y_{m+1} to the sequence $\{y_n\}_{n=1}^{m}$.

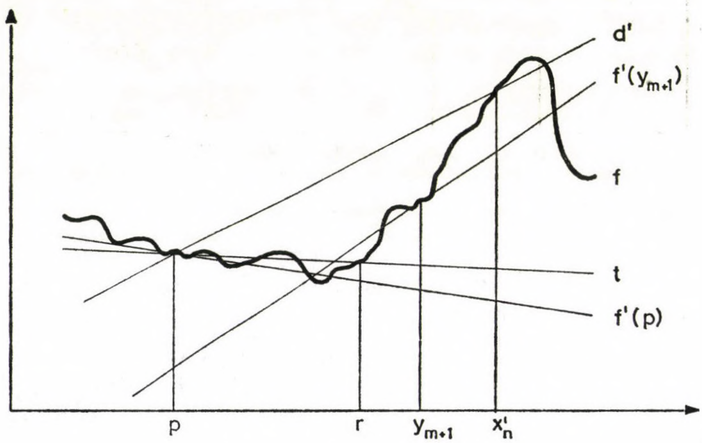

Fig. 9

Indeed, from $f'(y_{m+1}) \geqq d'$ and $f(y_{m+1}) < f(p)+d'(y_{m+1}-p)$ it ollojws that

(a) $$f(p) > f(y_{m+1})+f'(y_{m+1})(p-y_{m+1}).$$

The choice of ε_0 implies that

$$f'(y_{m+1}) < \frac{f(y_m)-f(p)}{y_m-p} < f'(y_m).$$

We have $d' \leqq \dfrac{f(y_m)-f(p)}{y_m-p}$ and, for $x \in (r, x_n')$,

$$f(x) < f(p)+d'(x-p).$$

Hence

$$f(y_{m+1}) < f(p)+\frac{f(y_m)-f(p)}{y_m-p}(y_{m+1}-p).$$

Summing up

$$f(y_m) > f(y_{m+1})+f'(y_{m+1})(y_m-y_{m+1}).$$

(See Fig. 10.) Hence by $y_{m+1}<y$ and

$$(y_{m+1}, f(y_{m+1}))\in D^p_{ru}, \quad f(y_{m+1}) > f(y_m)+f'(y_m)(y_{m+1}-y_m).$$

Plainly from $(y_{m+1}, f(y_{m+1}))\in D^p_{ru}$ it follows that

(b) $$f(y_{m+1}) > f(p)+f'(p)(y_{m+1}-p).$$

(When $m=0$ we only need (a) and (b).)

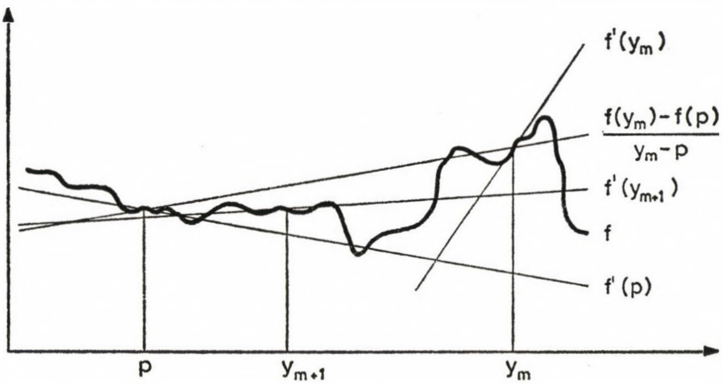

Fig. 10

We have proved $f\Big|_{\{p\}\cup\bigcup_{n=0}^{m+1}\{y_n\}}$ is strictly d-convex. By induction we obtain the sequence $\{y_n\}_{n=1}^{\infty}$, and consequently $p\in C_1$.

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References

[1] S. Agronsky, A. M. Bruckner, M. Laczkovich and D. Preiss, Convexity conditions and intersections with smooth functions, *Trans. Amer. Math. Soc.*, **289** (1985), 659—677.

[2] F. Filipczak, Sur les fonctions continues relativement monotones, *Fund. Math.*, **58** (1966), 75—87.

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ON A FUNCTIONAL EQUATION WITH POLYNOMIALS

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1. For a fixed K, let $K[x]$ denote the ring of polynomials over K. Let $\mathbf{R}$ and $\mathbf{C}$ denote the field of real and that of complex numbers, resp.

We are interested in determining all non-trivial $Q \in \mathbf{C}[x]$ and all $S \in \mathbf{C}[x]$ with $\deg S = 2$ for which

$$Q(S(x)) = cQ(x)Q(x+\gamma) \tag{1.1}$$

holds with a suitable $c \in \mathbf{C}$. Here $\gamma \in \mathbf{C}$, $\gamma \neq 0$.

We are far from being able to solve this problem in general. The main result of this paper is the complete solution of (1.1) if $c, \gamma \in \mathbf{R}$, $Q, S \in \mathbf{R}[x]$ and if Q has a real root.

We note that the special case $S(x) = x^2$, $c = 1$, $\gamma = -1$ was stated as a problem by J. Binz in [1]. Later J. Fehér [2] (see his related paper [3] too) considered and solved the polynomial equation

$$\prod_{k=1}^{s} f(\varepsilon_k x) = f(x^s),$$

where ε_k runs over the rooots of unity of degree s.

Let us consider the general case (1.1). The following remarks are quite obvious.

REMARKS. 1. We may assume that the leading coefficient of Q is 1, i.e. that Q is monic.

2. We may assume that the first degree term of $S(x)$ is vanishing, i.e. that $S(x)$ is of the form $S(x) = Ax^2 + E$.

PROOF. Let $S(x) = Ax^2 + Bx + C$ such that (1.1) holds with some Q. Let $u^* = -\dfrac{B}{2A}$. Then $S(x+u^*) = Ax^2 + Bu^* + C$. Let $R(x) = Q(x+u^*)$. Then substituting $x+u^*$ instead of x in (1.1), we get $R(Ax^2+E) = Q(S(x+u^*)) = cR(x)R(x+\gamma)$, with $E = Bu^* + C - u^*$. So

$$\begin{cases} R(S^*(x)) = cR(x)R(x+\gamma), \\ S^*(x) = Ax^2 + E. \end{cases} \tag{1.2}$$

Solving (1.2) we get the solutions of (1.1) by putting $Q(x) = R(x-u^*)$.

In what follows we shall assume that

$$S(x) = Ax^2 + E.$$

3. We may assume that $\gamma=1$.

PROOF. Let Q, S be a solution of (1.1). Let $R(y):=Q(\gamma y)$. Then $Q(\gamma y+\gamma)=$ $=R(y+1)$ and $Q(A\gamma^2 y^2+E)=R\left(A\gamma y^2+\frac{E}{\gamma}\right)$. So we get

$$(1.3)\qquad \begin{cases} R(\tilde{S}(y)) = cR(y)R(y+1), \\ \tilde{S}(y) = A\gamma y^2+\dfrac{E}{\gamma}. \end{cases}$$

Solving (1.3) we can give all solutions of (1.1) immediately.

2. THEOREM 1. *Let* $S(x)=Ax^2+E$, Q *a solution of* (1.1), $\deg Q=N$. *Then*

$$(2.1)\qquad Q(-x) = (-1)^N Q(x+\gamma).$$

PROOF. Let $\mathcal{A}$ denote the set of all roots of Q listing them with their multiplicity. Let $U(x)$ be the maximal degree monic divisor of $Q(x)$ such that $U(x)=$ $=\pm U(\gamma-x)$. Let $Q(x)=U(x)V(x)$. Since the left hand side of (1.1) is invariant under the transformation $x\to -x$, we get $Q(x)Q(x+\gamma)=Q(-x)Q(\gamma-x)$, consequently

$$U(x)U(x+\gamma)V(x)V(x+\gamma) = U(-x)U(\gamma-x)V(-x)V(\gamma-x).$$

Since $U(x)=\pm U(\gamma-x)$, $U(-x)=\pm U(\gamma-x)$, we have

$$(2.2)\qquad V(x)V(x+\gamma) = V(-x)V(\gamma-x).$$

Assume that $\deg V>0$. Let ξ be a root of V, $V(\xi)=0$. From (2.2) we get $V(-\xi)=0$ or $V(\gamma-\xi)=0$. The last case cannot occur, since it would imply that $U_1(x)=$ $=U(x)l(x)$ is a divisor of Q of the form $U_1(x)=\pm U(\gamma-x)$, where

$$l(x) = \begin{cases} (x-\xi)(x-(\gamma-\xi)) & \text{if } \xi \neq \gamma/2, \\ x-\xi & \text{if } \xi = \gamma/2, \end{cases}$$

and this contradicts that the degree of U was maximal. So $V(\xi)=0$ implies that $V(-\xi)=0$, moreover the polynomials $V(x)$, $V(\gamma-x)$ are coprime. Consequently

$$(2.3)\qquad V(x) = \pm V(-x),$$

$$(2.4)\qquad V(-x+\gamma) = \pm V(x+\gamma).$$

Let $\mathcal{A}_0$ denote the set of roots of V. $\mathcal{A}_0\subseteq\mathcal{A}$. From (2.3), (2.4) we get

$$(\mathrm{A1})\qquad \delta\in\mathcal{A}_0 \Rightarrow -\delta\in\mathcal{A}_0,$$

$$(\mathrm{A2})\qquad \delta\in\mathcal{A}_0 \Rightarrow 2\gamma-\delta\in\mathcal{A}_0.$$

Applying (A2) with $-\delta$ instead of δ, we get

$$(\mathrm{A3})\qquad \delta\in\mathcal{A}_0 \Rightarrow 2\gamma+\delta\in\mathcal{A}_0,$$

consequently $\delta\in\mathcal{A}_0\Rightarrow 2k\gamma+\delta\in\mathcal{A}_0$ $(k=1, 2, \ldots)$. $\mathcal{A}_0$ contains infinitely many elements. This is impossible, $V(x)=$constant. Hence (2.1) immediately follows. □

3. THEOREM 2. *The equation*

$$Q(Ax^2) = cQ(x)Q(x+1) \tag{3.1}$$

has a nontrivial solution Q only if $A=\pm 1$. All the monic solutions are:

$$Q(x) = x^m(x-1)^m \quad (m = 0, 1, 2, \ldots) \quad \textit{if} \quad A = 1, \tag{3.2}$$

$$Q(x) = (x-\omega)^m(x-\bar{\omega})^m \quad (m = 0, 1, 2, \ldots) \quad \textit{if} \quad A = -1, \tag{3.3}$$

where $\omega = e^{i\pi/3}$.

PROOF. Let $\mathscr{A}=\{\beta_1, \beta_2, \ldots, \beta_N\}$ be the set of roots of Q, where Q is a monic solution of (3.1). Since $Q(x+1)=(-1)^N Q(-x)$ (see Theorem 1), we get that

$$Q(x)Q(-x) = (-1)^N \prod_{j=1}^{N} (x^2-\beta_j^2) = (-1)^N Q(x)Q(x+1).$$

Moreover

$$Q(Ax^2) = \Pi (Ax^2-\beta_j) = A^2 \Pi \left(x-\frac{\beta_j}{A}\right).$$

Then (3.1) implies that $\mathscr{A}=\{A\beta_1^2, \ldots, A\beta_N^2\}$, i.e. that $x \to S(x)(=Ax^2)$ is a function for which $S(\mathscr{A})=\mathscr{A}$. Iterating k times, we get

$$\mathscr{A} = \{A^{2^k-1}\beta_1^{2^k}, \ldots, A^{2^k-1}\beta_N^{2^k}\} \quad (k = 1, 2, \ldots). \tag{3.4}$$

First we observe that $|A\beta|=1$ or $\beta=0$, if $\beta \in \mathscr{A}$. Assume that $\beta \neq 0$, $|A\beta| \neq 1$, $\beta \in \mathscr{A}$. Then the sequence $A^{2^k-1}\beta^{2^k} \in \mathscr{A}$, $|A^{2^k-1}\beta^{2^k}|$ is strictly monotonic, $\mathscr{A}$ contains infinitely many elements, which is impossible. Let $\beta \neq 0$. Then $|A^3\beta^4|=1$, $A\beta=1$, consequently $|\beta|=1$, $|A|=1$. From (2.1) we get

$$\mathscr{A} = \{1-\beta_1, \ldots, 1-\beta_N\}, \quad |1-\beta_i| = 0 \quad \text{or} \quad 1. \tag{3.5}$$

This implies $\beta_i \in \mathscr{B}=\{0, 1, \omega, \bar{\omega}\}$, $\omega=e^{\pi i/3}$. Assume that $0 \in \mathscr{A}$. Then by (3.5), $1 \in \mathscr{A}$, by (3.4) $A^{2^k-1} \in \mathscr{A}$, i.e. $A, A^3, A^7 \in \mathscr{A}$, and so $A \in \mathscr{B}$. If $A=\omega$ or $\bar{\omega}$, then $A^3=-1 \in \mathscr{A}$, but $-1 \notin \mathscr{B}$. Therefore $A=1$. Let $\beta_i \in A$, $\beta_i \neq 0, 1$. Then $\beta_i^4 \in \mathscr{A}$, since $\beta_i=\omega$ or $\bar{\omega}$, $\beta_i^3=-1$, we get $\beta_i^4=-\beta_i$. Then $-\beta_i \in \mathscr{B}$, which is impossible. So $\mathscr{A}$ contains only 0 and 1. From Theorem 1 we get immediately that Q has the form (3.2).

Assume that $0 \notin \mathscr{A}$. Then $1 \notin \mathscr{A}$. Let $\beta \in \mathscr{A}$. Since $\beta \in \mathscr{B}$, therefore $\beta=\omega$ or $\bar{\omega}$. Furthermore $1-\omega=\bar{\omega}$, $1-\bar{\omega}=\omega$, so $\mathscr{A}$ contains only the elements ω, $\bar{\omega}$ several times, with the same multiplicity (see Theorem 1).

From (3.4) we get that $A\omega^2, A^3\omega^4 \in \mathscr{A}$, $A\bar{\omega}^2, A^3\bar{\omega}^4 \in \mathscr{A}$. Assume first that $A\omega^2=\omega$. Then $A=\bar{\omega}$, $A\bar{\omega}^2=\bar{\omega}^3=-1$ that is impossible, since $-1 \notin \mathscr{B}$. Let now $A\omega^2=\bar{\omega}$; then $A=-1$.

To finish the proof it is enough to observe that the polynomials (3.2), (3.3) are solutions of (3.1), but this is obvious. □

4. Assume now that $S(x)=Ax^2+E$, $E \neq 0$, $\gamma=1$ and that Q is a solution of (1.1),

$$Q(x) = \prod_{j=1}^{N} (x-\beta_j), \quad \mathscr{A} = \{\beta_1, \beta_2, \ldots, \beta_N\}.$$

Theorem 1 implies that $\mathcal{A}=\{1-\beta_1, 1-\beta_2, \ldots, 1-\beta_N\}$. Let $o(\beta)$ be the multiplicity of β in $\mathcal{A}$.

LEMMA 1. *We have* $o(S(\beta))=o(\beta)$, $o(1-\beta)=o(\beta)$.

PROOF. The assertion $o(1-\beta)=o(\beta)$ is obvious. Now we prove that $o(S(\beta))=$ $=o(\beta)$ for each $\beta\in\mathcal{A}$. Since $\beta\in\mathcal{A}$, therefore $S(\beta)\in\mathcal{A}$. Let $m=o(\beta)$. Assume first that $\beta\neq 0$. Starting from

$$Q(S(x)) = cQ(x)Q(x+1)$$

and differentiating we get

$$Q'(S(x))S'(x) = cQ'(x)Q(x+1)+cQ(x)Q'(x+1).$$

Assume that $m\geqq 2$. We have $S'(x)=2Ax$. Substituting $x=\beta$, $Q'(\beta)=Q(\beta)=0$ implies that $Q'(S(\beta))=0$.

Continuing the differentiation, we deduce that $Q^{(j)}(S(\beta))=0$ $(j=0, 1, \ldots, m-1)$. This implies that $o(S(\beta))\geqq o(\beta)$.

Let $\beta=0\in\mathcal{A}$, $o(0)=m$, then $S(0)=E\in\mathcal{A}$, let $o(E)=k$, then $Q(x)=$ $=(x-E)^k x^m L(x)$, $L(0)\neq 0$, $L(E)\neq 0$, $L\in\mathbf{C}[x]$. Then from (1.1), and Theorem 1 we get

$$Q(S(x)) = (S(x)-E)^k S^m(x) L(S(x)) = \pm cQ(x)Q(-x) =$$
$$= \pm(x-E)^k(x+E)^k x^{2m} L(x)L(-x).$$

Since $L(O)\neq 0$, $E\neq 0$, therefore $(S(x)-E)^k=(Ax^2)^k$ divides x^{2m}, and x^{2m} divides $(S(x)-E)^k$, and so $k=m$, $o(O)=o(S(O))$.

So $o(S(\beta))\geqq o(\beta)$ holds for each $\beta\in\mathcal{A}$. Then $\{S(\beta_1), \ldots, S(\beta_N)\}\subseteqq\mathcal{A}$, taking into consideration the cardinality, we get

$$\mathcal{A} = \{S(\beta_1), \ldots, S(\beta_N)\}, \quad \text{i.e.} \quad S(\mathcal{A}) = \mathcal{A}. \tag{4.1}$$

LEMMA 2. *For every* $\eta\in\mathcal{A}$ *there exists a unique* $\tau\in\mathcal{A}$ *such that* $S(\tau)=\eta$. *If* $\beta\in\mathcal{A}$ *and* $\beta\neq 0$, *then* $-\beta\notin\mathcal{A}$.

PROOF. Let $\eta\in\mathcal{A}$. From (4.1) we get that $\eta\in S(\mathcal{A})$, so there exists $\tau\in\mathcal{A}$ for which $S(\tau)=\eta$.

To prove that τ is unique, it is enough to show that $-\tau\notin\mathcal{A}$, i.e. to prove the second assertion. Let $\xi_1, \xi_2, \ldots, \xi_r$ be the set of all distinct elements of $\mathcal{A}$. From (4.1) we get that $\{S(\xi_1), S(\xi_2), \ldots, S(\xi_r)\}$ contains each element of $\mathcal{A}$ at least once. If there were $i, j, i\neq j$ for which $\xi_i=-\xi_j$, then from $S(\xi_i)=S(\xi_j)$ it would follow that the number of distinct elements in $\{S(\xi_1), \ldots, S(\xi_r)\}$ is at most $r-1$. This is impossible. □

The next assertion is quite obvious.

Let $S(x)=Ax^2+E$, $\mathcal{B}=\{y_1, y_2, \ldots, y_N\}$ an arbitrary finite set of not necessarily distinct complex numbers with the properties

$$\mathcal{B} = \{1-y_1, 1-y_2, \ldots, 1-y_N\}, \tag{4.2}$$

$$\mathcal{B} = \{S(y_1), S(y_2), \ldots, S(y_N)\}. \tag{4.3}$$

Then the polynomial

$$R(x)=\prod_{j=1}^{M}(x-y_j) \tag{4.4}$$

is a solution of the equation

$$R(S(x))=cR(x)R(x+1). \tag{4.5}$$

Indeed, from (4.2) we get $R(x+1)=(-1)^M R(-x)$, and from (4.2)

$$R(S(x))=\Pi(S(x)-S(y_j))=A^M\Pi(x^2-y_j^2)=A^M R(x)\Pi(x+y_j)=$$
$$=A^M R(x)\cdot(-1)^M R(-x)=A^M R(x)R(x+1).$$

So the following theorem is true.

THEOREM 3. *A polynomial*

$$Q(x)=\prod_{j=1}^{N}(x-\beta_j)$$

is a solution of the equation

$$Q(S(x))=cQ(x)Q(x+1) \tag{4.6}$$

if and only if for the set of roots $\mathcal{A}=\{\beta_1,\beta_2,\dots,\beta_N\}$ *the relations*

$$\mathcal{A}=\{1-\beta_1,\dots,1-\beta_N\},\quad \mathcal{A}=\{S(\beta_1),\dots,S(\beta_N)\} \tag{4.7}$$

holds.

LEMMA 3. *If A, E, N is fixed, then there exists at most one monic solution $Q(x)$ of* (4.6), *with* $\deg Q=N$.

Let Q be a solution, written as in Theorem 3. Let $\sigma_k=\sum_{\beta_\nu\in\mathcal{A}}\beta_\nu^k$.

From (4.7) we get

$$\{A\beta_\nu^2,\quad \nu=1,\dots,N\}=\{\beta_\nu-E,\quad \nu=1,\dots,N\},$$

consequently

$$A^k\sigma_{2k}=\sum_{\beta_\nu}(A\beta_\nu^2)^k=\sum_{\beta_\nu}(\beta_\nu-E)^k=\sum_{l=0}^{k}(-1)^{k-l}E^{k-l}\binom{k}{l}\sigma_l. \tag{4.8}$$

Furthermore, $\{\beta_\nu|\nu=1,\dots,N\}=\{1-\beta_\nu|\nu=1,\dots,N\}$, consequently

$$\sigma_{2t+1}=\sum_{\beta_\nu}(1-\beta_\nu)^{2t+1},$$

whence

$$2\sigma_{2t+1}=\sum_{l=0}^{2t}(-1)^{l+1}\binom{2t+1}{l}\sigma_l. \tag{4.9}$$

It is obvious that $\sigma_0=N$. Hence, by (4.8), $\sigma_1=N/2$. The equations (4.8), (4.9) determine the whole sequence $\sigma_0,\sigma_1,\dots$ uniquely. The symmetric polynomials $\sigma_0,\dots,\sigma_N$ determine the elementary symmetric polynomials of the variables $\beta_1,\dots,\beta_N$ (see Newton—Girard formulas), consequently there exists at most one solution of (4.6). □

We shall say that a monic polynomial $Q(x)$ is a primitive solution of (4.6) if there does not exist nonconstant proper divisor of Q which is a solution of (4.6).

The following assertions are obvious.

(1) If Q_1, Q are solutions of (4.6), Q_1 divides Q, $Q_1 Q_2 = Q$, then Q_2 is a solution of (4.6) as well.

(2) Let Q be a solution of (4.6) with roots $\mathcal{A}=\{\beta_1, \dots, \beta_N\}$. Starting from an element $\beta_\nu \in \mathcal{A}$, construct the smallest subset $\{\gamma_1, \dots, \gamma_s\}=\mathcal{B}\subseteq\mathcal{A}$ for which $\beta_\nu\in\mathcal{B}$, $\gamma\in\mathcal{B}\Rightarrow 1-\gamma\in\mathcal{B}$ and $S(\gamma)\in\mathcal{B}$. Then

$$Q_1(x) = \prod_{\nu=1}^{s} (x-\gamma_\nu)$$

is a primitive solution (4.6), the roots of $Q_1(x)$ are simple.

(3) Every monic solution Q of (4.6) can be written as the product of primitive solutions

$$Q(x) = \prod_{j=1}^{s} Q_j(x),$$

where $Q_1, \dots, Q_s$ are uniquely determined by Q.

(4) The product of solutions is a solution as well.

Consequently, it is enough to determine the primitive solutions.

5. Lemma 4. (1) *If* $S\left(\frac{1}{2}\right)=1/2$, *i.e.* $\frac{A}{4}+E=\frac{1}{2}$ *then* $Q(x)=x-\frac{1}{2}$ *is a solution of* (4.6).

(2) *If* $A=1$, $\beta_1=\frac{1}{2}-\frac{1}{2}\sqrt{1-4E}$, $\beta_2=\frac{1}{2}+\frac{1}{2}\sqrt{1-4E}$, *then*

$$Q(x) = (x-\beta_1)(x-\beta_2)$$

is a solution of (4.6). *It is primitive if* $E\neq 1/4$.

(3) *If* $A=-1$, $\beta_1=\frac{1}{2}-\frac{1}{2}\sqrt{-3+4E}$, $\beta_2=\frac{1}{2}+\frac{1}{2}\sqrt{-3+4E}$, *then*

$$Q(x) = (x-\beta_1)(x-\beta_2)$$

is a solution of (4.6). *It is primitive if* $E\neq 3/4$.

(4) *If* $A=1/2$, $E=-\frac{21}{8}$, *then*

$$Q(x) = \left(x+\frac{5}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\frac{7}{2}\right)$$

is a primitive solution of (4.6)

Proof. (1) If $S\left(\frac{1}{2}\right)=\frac{1}{2}$, then the conditions (4.7) are satisfied for $\mathcal{A}=\{1/2\}$.

(2) Now we have $S(\beta_1)=\beta_1$, $S(\beta_2)=\beta_2$, $\beta_1+\beta_2=1$, so for $\mathcal{A}=\{\beta_1, \beta_2\}$ the relations (4.7) hold true. Q is a primitive solution if $\beta_i\neq 1/2$, i.e. if $E\neq 1/4$.

(3) In this case $S(\beta_1)=\beta_2$, $S(\beta_2)=\beta_1$, $\beta_1+\beta_2=1$, and so for $\mathcal{A}=\{\beta_1, \beta_2\}$ the relations (4.7) hold true. Q is a primitive solution if $\beta_i\neq 1/2$, i.e. if $E\neq 3/4$.

(4) Let $\beta_1=-5/2$, $\beta_2=1/2$, $\beta_3=7/2$. Then we have $S(\beta_3)=\beta_3$, $S(\beta_1)=\beta_2$, $S(\beta_2)=\beta_1$, $1-\beta_3=\beta_1$, $1-\beta_2=\beta_2$, so for $\{\beta_1, \beta_2, \beta_3\}=\mathcal{A}$ the conditions (4.7) are satisfied. So $Q(x)$ is a solution of (4.6). It is obvious that it is primitive. □

6. Let now $S(x)=Ax^2+E\in\mathbf{R}[x]$, $Q\in\mathbf{R}[x]$ be a nonconstant solution of

$$Q(S(x)) = cQ(x)Q(x+1). \tag{6.1}$$

Assume that Q has at least one real root. Let $\mathcal{A}_v^-(\subseteq\mathcal{A})$ be the set of real roots of Q,

$$Q_1(x) = \prod_{\beta\in\mathcal{A}_v} (x-\beta).$$

It is obvious that $\beta\in\mathcal{A}_v$ implies $1-\beta\in\mathcal{A}_v$. Let $\mathcal{A}_v=\{\beta_1, \beta_2, \dots, \beta_M\}$. Then $S(\beta_j)\in\mathbf{R}$. Then $(:=)\{S(\beta_1), \dots, S(\beta_M)\}\subseteq\mathcal{A}_v$. Since $S(\beta_i)\neq S(\beta_j)$ if $\beta_i\neq\beta_j$, therefore the number of distinct elements in $\mathcal{A}$ is the same as that in $\mathcal{A}_v$. Therefore

$$\{S(\beta_1), \dots, S(\beta_M)\} = \mathcal{A}_v,$$

consequently the conditions (4.7) in Theorem 3 hold, so Q_1 is a solution of (6.1).

Let Q be a primitive solution of (6.1) that contains a real root. Then all roots of Q are simple. Let $\mathcal{A}=\{\beta_1, \dots, \beta_N\}$. Then $\beta_1, \dots, \beta_N$ are real numbers. Assume that $\beta_1<\beta_2<\dots<\beta_N$. Then $\beta_N=1-\beta_1$, $\beta_{N-l}=1-\beta_{l+1}$ $(l=0, \dots, N-1)$. Let $L(x):=S(x)-x$.

(1) Assume first that $L(x)$ does not have a real root. Then the sign of $L(x)$ is constant on the whole real line. Consequently the sequence $x_{n+1}=S(x_n)$, $x_0\in\mathbf{R}$ is strictly monotonic, by starting with $x_0=\beta_v\in\mathcal{A}$, we get $x_{n+1}\in\mathcal{A}$. This is impossible.

(2) Assume that the roots ξ_1, ξ_2 of $L(x)=0$ are real:

$$\xi_1 = \frac{1-\sqrt{1-4AE}}{2A}, \quad \xi_2 = \frac{1+\sqrt{1-4AE}}{2A}. \tag{6.2}$$

(2a) *The case* $A>0$. (2a1) *Assume that* $\xi_1\geqq 0$. If $\mathcal{A}$ is nonempty then it contains a positive root. Let $\alpha\in\mathcal{A}$, $\alpha>0$, and consider the sequence $y_0=\alpha$, $y_{v+1}=$ $=S(y_v)$. We have $y_{v+1}\in\mathcal{A}$. If $\xi_1<\alpha<\xi_2$, then $\xi_1=S(\xi_1)<S(\alpha)<S(\xi_2)=\xi_2$, $S(\alpha)<\alpha$, therefore $\xi_1<y_{v+1}<y_v<\xi_v$ holds for every v which is impossible. If $\alpha>\xi_2$, then $S(\alpha)>\alpha$, therefore the sequence y_v is monotonic, $y_{v+1}>y_v$ $(v=1, 2, \dots)$, which is impossible. If $0<\alpha<\xi_1$, then $S(\xi_1)>S(\alpha)>\alpha$, consequently $0<\alpha<$ $<y_1<\dots<y_v<\dots$ $(<\xi_1)$ that is impossible as well.

Consequently, the only positive roots can be ξ_1, ξ_2,

$$\mathcal{A} = \{\xi_1, \xi_2, 1-\xi_1, 1-\xi_2\}.$$

Assume that the numbers ξ_1, ξ_2, $1-\xi_1$, $1-\xi_2$ are distinct and $\xi_i\in\mathcal{A}$. Then $1-\xi_i\in\mathcal{A}$, $S(1-\xi_i)=A(1-2\xi_i)+S(\xi_i)=A(1-2\xi_i)+\xi_i\in\mathcal{A}$. Furthermore, $S(x)=x$ has only two solutions, $x=\xi_1, \xi_2$, therefore $S(1-\xi_i)=1-\xi_j$, consequently $1-\xi_j\in\mathcal{A}$, and so $1-(1-\xi_j)\in\mathcal{A}$. Furthermore, $S(1-\xi_j)=1-\xi_i$.

Then we have $\mathscr{A}=\{\xi_1, \xi_2, 1-\xi_1, 1-\xi_2\}$, $A(1-2\xi_1)+\xi_1=\xi_2$, $A(1-2\xi_2)+\xi_2=$ $=\xi_1$. This implies that $A(1-2\xi_1)=-A(1-2\xi_2)$, i.e. $1=\xi_1+\xi_2$. But then $\xi_2=$ $=1-\xi_1$, and we assumed that $1-\xi_1$ and ξ_2 are distinct numbers. In this case we do not have solutions. In the opposite case we have $\xi_i=\xi_j$ or $\xi_i=1-\xi_i$ or $\xi_i=1-\xi_j$ with a suitable i and $j\neq i$. If $\xi_1=\xi_2\in\mathscr{A}$, then $S(1-\xi_1)\in\mathscr{A}$, consequently $S(1-\xi_1)=1-\xi_1$, and so $1-\xi_1=\xi_2$, i.e. $\xi_1=\xi_2=1/2$. But in this case $\mathscr{A}=\left\{\frac{1}{2}\right\}$, $Q(x)=x-1/2$. If $\xi_i=1-\xi_i$, then $\xi_i=1/2$. Thus $Q(x)=x-\frac{1}{2}$ is a first degree solution of (6.1).

If $\xi_i=1-\xi_j$, then $\xi_1+\xi_2=1$. Assume that $\xi_i\in\mathscr{A}$. Then $1-(1-\xi_j)=\xi_j\in\mathscr{A}$. If $\xi_1+\xi_2=1$, then $A=1$. In this case $\mathscr{A}=\{\xi_1, \xi_2\}$. If $\xi_1\neq\xi_2$, then this leads to $Q(x)$ defined in Lemma 4, (2).

(2a2) *Assume that* $\xi_1<0$. Let

$$\beta_1 < \dots < \beta_t(\leqq 0) < \beta_{t+1} < \dots < \beta_{t+s},$$

where we write $t=0$ if there are only positive roots. Since $S(x)$ is monotonically decreasing in $x\leqq 0$ and increasing in $x\geqq 0$, therefore

$$S(\beta_1) > \dots > S(\beta_t), \quad S(\beta_{t+1}) < \dots < S(\beta_{t+s}). \tag{6.3}$$

Since $1-\beta_1=\beta_{t+s}$, $1-\beta_2=\beta_{t+s-1}, \dots$, therefore $s\geqq t$. It is clear that $\beta_{t+s}\leqq\xi_2$. In the case $\beta_{t+s}>\xi_2$ we consider the monotone sequence $y_{\nu+1}=S(y_\nu)$, $y_0=\beta_{t+s}$, $y_\nu\in\mathscr{A}$, $y_{\nu+1}>y_\nu$, which is impossible.

(2a2α) *The case* $\beta_{t+s}<\xi_2$. Let $\varrho\in\mathscr{A}$ be defined as the solution of $S(\varrho)=$ $=\beta_{t+s}$. If $\varrho>0$, then $\varrho<\beta_{t+s}$ since β_{t+s} is the largest element in $\mathscr{A}$. But in the interval $(0, \xi_2)\subseteq(\xi_1, \xi_2)$ we have $(\beta_{t+s}=)S(\varrho)<\varrho$, a contradiction. So $\varrho\leqq 0$. Then, by (6.3) we get $\varrho=\beta_1$, i.e.

$$\beta_{t+s} = S(\beta_1) = S(1-\beta_{t+s}).$$

Hence

$$\beta_{t+s} = A(1-2\beta_{t+s})+S(\beta_{t+s}).$$

Since

$$S(\beta_{t+s})\in\mathscr{A}, \quad S(\beta_{t+s}) \leqq \beta_{t+s},$$

we get

$$\beta_{t+s} \leqq A(1-2\beta_{t+s})+\beta_{t+s},$$

i.e. $1-2\beta_{t+s}\geqq 0$, $\beta_{t+s}\leqq 1/2$. Since $\beta_{t+s}=1-\beta_1$, we get $\beta_1\geqq 1/2$, and so $\beta_1=\beta_{t+s}=$ $=1/2$. This implies $\mathscr{A}=\{1/2\}$, $S\left(\frac{1}{2}\right)=1/2$, which leads to the polynomial $Q(x)=$ $=x-1/2$.

(2a2β) *The case* $\beta_{t+s}=\xi_2$. (i) If all the roots β are positive, $\beta_1<\dots<\beta_N$, then $S(\beta_1)<\dots<S(\beta_N)$, so $S(\beta_i)=\beta_i$ $(i=1, 2, \dots, N)$. $N\leqq 2$, since $S(x)-x=0$ has at most two solutions. If $N=2$ then $\beta_1=\xi_1$, but we assumed that $\xi_1<0$. If $N=1$, then $\beta_1=1/2$, which leads to $Q(x)=x-1/2$.

(ii) If $E>0$ then $S(\beta)>0$ for every $\beta\in\mathscr{A}$, consequently every root is positive.

(iii) If $\mathscr{A}$ contains only two elements, then they are $\xi_2, 1-\xi_2$. Then $S(1-\xi_2)=$ $=1-\xi_2$, i.e. $\xi_1=1-\xi_2$. This leads to $Q(x)$ given in Lemma 4, (2), except when $\xi_2=1/2$. Then we get $Q(x)=x-1/2$.

(iv) From now on we assume that there exists at least three roots of Q, and $s \geqq 2$.

Let $y \in \mathcal{A}$ be the solution of $S(y) = \beta_{t+s-1} (< \xi_2)$. We deduce that $y \leqq 0$. Assume in the contrary that $y > 0$. Since $\beta_{t+s-1} \in \mathcal{A}$ is the second largest, $y \neq \xi_2$, therefore $y \leqq \beta_{t+s-1}$. If $y = \beta_{t+s-1}$, then $S(y) = y$, consequently $y = \xi_1$, but we assumed that $\xi_1 \leqq 0$. But $0 \leqq y < \beta_{t+s-1}$ implies that $\beta_{t+s-1} = S(y) < y$, which is impossible. Therefore $y \leqq 0$. From (6.3) we get that $y = \beta_1$. So

$$S(\beta_1) = \beta_{t+s-1}, \quad \beta_1 = 1 - \beta_{t+s}.$$

Assume that $s \geqq 3$. Solve the equation $\beta_{t+s-2} = S(y)$ in $y \in \mathcal{A}$. Since β_{t+s-2} is the third largest root and $S(y) \neq \beta_{t+s-2}$ if $y = \beta_{t+s}$, β_1, therefore from (6.3) we get $y = \beta_2$ or β_{t+s-1}. Since $\beta_{t+s-1} + \beta_2 = 1$ therefore $S(\beta_{t+s-1}) > S(\beta_2)$, consequently $y = \beta_{t+s-2}$, i.e. $\beta_{t+s-2} = S(\beta_{t+s-2})$, so we have $\beta_{t+s-2} = \xi_1$. This is impossible.

Let now $s \leqq 2$.

Let $s=2$, $t=2$. Then $\beta_1 < \beta_2 (\leqq 0 <) \beta_3 < \beta_4$, $S(\beta_4) = \beta_4$, $S(\beta_1) = \beta_3$. Since $\beta_2 + \beta_3 = 1$, therefore $S(\beta_3) > S(\beta_2)$, and so $S(\beta_3) = \beta_3$, $S(\beta_2) = \beta_2$. This would imply that $\xi_1 = \xi_3 > 0$, which is impossible.

Let $s=2$, $t=1$. Then $\beta_1 (\leqq 0) < \beta_2 < \beta_3$, $\beta_2 = 1 - \beta_2$, $\beta_2 = 1/2$, $S(\beta_3) = \beta_3$, $S(\beta_1) = 1/2$, $S(1/2) = \beta_1$. Hence

$$1 - \beta_1 = \beta_3 = S(1-\beta_1) = A(1-2\beta_1) + S(\beta_1) = A(1-2\beta_1) + 1/2,$$

and so $\frac{1}{2} - \beta_1 = A(1-2\beta_1)$. Since $\beta_1 \neq 1/2$, therefore $A = \frac{1}{2}$.

Assume that $A = 1/2$. Then

$$\beta_3 = \xi_2 = 1 + \sqrt{1-2E}, \quad \beta_1 = -\sqrt{1-2E}$$

From $S\left(\frac{1}{2}\right) = \beta_1$ we get that $-\sqrt{1-2E} = \frac{1}{8} + E$. The last equation has two solutions: $E = -\frac{21}{8}$ and $E = 3/8$. The positive E is not interesting for us (see (ii)). Put $E = -\frac{21}{8}$.

Then $S(x) = \frac{1}{2} x^2 - \frac{21}{8}$, $A = \frac{1}{2}$, $E = -\frac{21}{8}$ and

$$\beta_1 = -5/2, \quad \beta_2 = 1/2, \quad \beta_3 = \frac{7}{2}.$$

This leads to $Q(x)$ stated in Lemma 4, (4).

Let $s=1$, $t=1$. Then $\mathcal{A} = \{1-\xi_2, \xi_2\}$, and in the case $1-\xi_2 \neq \xi_2$ we get $S(1-\xi_2) = 1-\xi_2$, i.e. $\xi_1 = 1-\xi_2$, whence $A=1$ follows. This leads to $Q(x)$ stated in Lemma 4, (2).

(2b) *The case* $A<0$. Let $B = -A > 0$. Let

$$\beta_1 < \dots < \beta_t (\leqq 0) < \beta_{t+1} < \dots < \beta_{t+s}$$

be the set of elements of $\mathcal{A}$. Then $s \geqq t$. Furthermore $S(\beta_j)$ $(j=1, \ldots, t+s)$ are all distinct,

$$(6.4) \qquad S(\beta_1) < \ldots < S(\beta_t); \quad S(\beta_{t+1}) > \ldots > S(\beta_{t+s}).$$

Since $S(\mathcal{A})=\mathcal{A}$, therefore $E>0$. Indeed, if $E<0$, then $S(\beta) \leqq E$ for every $\beta \in \mathcal{A}$, i.e. $s=0$ which is impossible.

(2b1) *If* $t=0$, i.e. the roots $\beta_1 < \ldots < \beta_N$ are positive, then $\beta_1 = S(\beta_N)$, $\beta_2 = S(\beta_{N-1})$, $\ldots$, and so by $\beta_{i+1} = 1-\beta_{N-i}$ we get that each β_i is a solution of the equation $x = S(1-x)$. Since this equation has at most two solutions, $N \leqq 2$. If $N=1$ then $S\left(\frac{1}{2}\right)=1/2$, then $\mathcal{A}=\{1/2\}$, $Q(x)=x-1/2$. If $N=2$, then $\mathcal{A}=\{\beta_1, \beta_2\}$, $\beta_1+\beta_2=1$, $S(\beta_1)=\beta_2$, $S(\beta_2)=\beta_1$. The equation $S(1-x)=x$ can be written in the form

$$Bx^2+(1-2B)x+B-E = 0.$$

The roots β_1, β_2 satisfy the condition $\beta_1+\beta_2=1$ if $(1-2B)/B=-1$, i.e. if $B=1$.

If $B=1$, then

$$\beta_1 = \frac{1}{2}-\frac{1}{2}\sqrt{-3+4E}, \quad \beta_2 = \frac{1}{2}+\frac{1}{2}\sqrt{-3+4E}.$$

This leads to the polynomial stated in Lemma 4, (3).

(2b2) *Let* $t \geqq 2$. Since $\beta_{t+s}=1-\beta_1>-\beta_1$, we have $S(\beta_{t+s})<S(-\beta_1)$, therefore $\beta_1=S(1-\beta_1)$. Then $\beta_2=S(\beta_{t+s-1})$ or $\beta_2=S(\beta_1)$.

(2b2α) *The case* $\beta_2=S(\beta_{t+s-1})$. Then β_1 and β_2 are solutions of the equation $x=S(1-x)$, so

$$\beta_1 = 1-\frac{1}{2B}-\frac{1}{2B}\sqrt{1+4B(E-1)}, \quad \beta_2 = 1-\frac{1}{2B}+\frac{1}{2B}\sqrt{1+4B(E-1)}.$$

The condition $\beta_2 \leqq 0$ implies that

$$\sqrt{1+4B(E-1)} \leqq 1-2B,$$

and so $B \leqq 1/2$, furthermore

$$1+4B(E-1) \leqq 1-4B+4B^2,$$

which holds if and only if $E \leqq B$.

But $\max_{\beta \in \mathcal{A}} \beta = \max_{\beta \in \mathcal{A}} S(\beta) \leqq E$, consequently

$$\min_{\beta \in \mathcal{A}} \beta = \min_{\beta \in \mathcal{A}} (1-\beta) = 1-\max_{\beta \in \mathcal{A}} \beta \geqq 1-E \geqq 1-B \geqq 1/2 > 0.$$

This is a contradiction.

(2b2β) *The case* $\beta_2=S(\beta_1)$. Since $\beta_1=S(1-\beta_1)=-B(1-2\beta_1)+S(\beta_1)$, therefore $\beta_2=B+(1-2B)\beta_1$, $\beta_2-\beta_1=B(1-2\beta_1)$.

Let $\beta_{t+s}=S(y_1)$, $\beta_{t+s-1}=S(y_2)$, $y_1, y_2 \in \mathcal{A}$. It is obvious that y_1 is the nearest and y_2 is the second nearest element of $\mathcal{A}$ to 0. It is clear that $\beta_{t+s}-\beta_{t+s-1} = -B(y_1^2-y_2^2)$. Furthermore $\beta_{t+s}-\beta_{t+s-1}=(1-\beta_1)-(1-\beta_2)=\beta_2-\beta_1=B(1-2\beta_1)$, and so $1-2\beta_1=y_2^2-y_1^2$.

Assume that there exists $\alpha\in\mathcal{A}$, $0<\alpha<1$. Then there exists α in $(0, 1/2]$. Hence it would follow that $|y_1|\leqq\frac{1}{2}$, $|y_2|\leqq 1$. Therefore $1-2\beta_1=y_2^2-y_1^2<1$, i.e. $\beta_1>0$, which is impossible.

Now we assume that the interval $(0, 1)$ does not contain any element of $\mathcal{A}$. Then $y_1\leqq 0$, since the elements of $\mathcal{A}$ are symmetric to $1/2$. Then $s=t$ and $\beta_t=y_1=S(\beta_{t+s})$. Then $1-\beta_t\in\mathcal{A}$, $|y_2|\leqq 1-\beta_t$. Hence

$$1-2\beta_1 = \min_{\substack{y_2\in\mathcal{A}\\ y_2\neq y_1}}(y_2^2-y_1^2) \leqq (1-\beta_t)^2-\beta_t^2 = 1-2\beta_t,$$

and $\beta_1\geqq\beta_t$, but this cannot occur if $t\geqq 2$.

(2b3) *Let* $t=1$. Then $\beta_1\leqq 0<\beta_2<\ldots<\beta_{s+1}$, $\beta_{s+1}=1-\beta_1$, $S(\beta_{s+1})=\beta_1$. Since $S(\beta_2)>\ldots>S(\beta_{s+1})$, we get $\beta_2=S(\beta_s)=S(1-\beta_2)$, $\beta_3=S(1-\beta_3)$, and so on. Since the equation $S(1-x)=x$ has at most two solutions, therefore $s+1\leqq 2$. Then $\mathcal{A}=\{\beta_1, \beta_2\}$, $\beta_1+\beta_2=1$, $\beta_1=S(\beta_2)$, $\beta_2=S(\beta_1)$. This implies that $B=1$,

$$\beta_1 = \frac{1}{2}-\frac{1}{2}\sqrt{-3+4E},\quad \beta_2 = \frac{1}{2}+\frac{1}{2}\sqrt{-3+4E}.$$

This leads to $Q(x)$ stated in Lemma 4, (3).

So we have proved the following assertion that we state now as

LEMMA 5. *If Q is a nonconstant primitive solution of* (6.1) *having a real root, then Q is one of the polynomials listed in Lemma* 4.

7. THEOREM 4. *Let* $S(x)=Ax^2+E$, $A\neq 0$, $E\neq 0$, *Q a nontrivial monic polynomial over* $\mathbf{R}$ *such that it has at least one real root and satisfies*

$$Q(S(x)) = cQ(x)Q(x+1). \tag{7.1}$$

Then $S\left(\frac{1}{2}\right)=1/2$, *or* $S(x)=x^2+E$, *or* $S(x)=-x^2+E$, *or* $S(x)=\frac{1}{2}x^2-\frac{21}{8}$.

Let $N=\deg Q$.

(i) *If* $S\left(\frac{1}{2}\right)=1/2$ *then*

$$Q(x) = \left(x-\frac{1}{2}\right)^N. \tag{7.2}$$

(ii) *If* $S\left(\frac{1}{2}\right)\neq 1/2$, $A=1$, *then there exists a solution Q only if* $2|N$. *Then*

$$Q(x) = (x-\beta_1)^m(x-\beta_2)^m,\quad m = \frac{N}{2}, \tag{7.3}$$

$$\beta_1 = \frac{1}{2}-\frac{1}{2}\sqrt{1-4E},\quad \beta_2 = \frac{1}{2}+\frac{1}{2}\sqrt{1-4E}.$$

(iii) *If* $S\left(\frac{1}{2}\right) \neq 1/2$, $A=-1$, *then there exists a solution* Q *only if* $2|N$. *Then*

$$Q(x) = (x-\beta_1)^m(x-\beta_2)^m, \quad m = \frac{N}{2}, \tag{7.4}$$

$$\beta_1 = \frac{1}{2}-\frac{1}{2}\sqrt{-3+4E}, \quad \beta_2 = \frac{1}{2}+\frac{1}{2}\sqrt{-3+4E},$$

(iv) *If* $S\left(\frac{1}{2}\right) \neq \frac{1}{2}$, $S(x)=\frac{1}{2}x^2-\frac{21}{8}$, *then there exists a solution* Q *only if* $3|N$. *Then*

$$Q(x) = (x+5/2)^m\left(x-\frac{1}{2}\right)^m\left(x-\frac{7}{2}\right)^m, \quad m = \frac{N}{3}. \tag{7.5}$$

The polynomials (7.2), (7.3), (7.4), (7.5) *are solutions of* (7.1).

Proof. In Section 6 we proved that there do not exist solutions Q having real roots if $S\left(\frac{1}{2}\right) \neq \frac{1}{2}$ or $A\neq 1$, $A\neq -1$, $A\neq 1/2$. Furthemore we proved that in the case $S\left(\frac{1}{2}\right)=1/2$, $x-\frac{1}{2}=Q(x)$ is a first degree solution of (7.1). Then (7.2) is a solution of (7.1) as well. From Lemma 3 we get that there exists no other solution. Let now assume that $A=1$, $S\left(\frac{1}{2}\right)\neq 1/2$. Then $(x-\beta_1)(x-\beta_2)$ is a second degree solution of (7.1). So (7.3) is the unique solution if $2|N$ (see Lemma 3). Assume that there exists an odd number K, and an $R(x)(=Q(x))$ for which (7.1) holds. Then $R^2(x)$ is a solution of (7.1) as well, consequently

$$R(x)^2 = (x-\beta_1)^K(x-\beta_2)^K. \tag{7.6}$$

Then the roots of R belong to the set $\{\beta_1, \beta_2\}$. Since $\beta_1\neq\beta_2$, therefore K is an even number. We can discuss the case $A=-1$ in the same way, so we omit it.

Let us consider now case (iv). By Lemma 3 we get that there exists no other solution if $3|N$. Let $R(x)$ be a solution of degree K. Then $R(x)^3$ is the unique solution of degree $3K$, consequently

$$R(x)^3 = (x+5/2)^K\left(x-\frac{1}{2}\right)^K\left(x-\frac{7}{2}\right)^K,$$

whence we get that $3|K$.

The last assertion is obvious. □

8. Let now $T(x)=Ax^2+Bx+C$ be a real polynomial of second degree. Consider the equation

$$R(x)-R(x+1) = R\big(T(x)\big) \tag{8.1}$$

where $R(x)$ is a rational function over R. By a linear transformation we may assume that $\big(T(x)=\big)S(x)=Ax^2+E$.

Let us put $R(x)$ in the form

$$R(x)=\frac{U(x)}{V(x)}=P(x)+\frac{u(x)}{V(x)},$$

where $(U,V)=1$, $P(x)\in\mathbf{R}[x]$, $\deg u<\deg V$. Since $|S(x)|\to\infty$ if $|x|\to\infty$ and $\frac{u(x)}{V(x)}\to 0$ as $|x|\to\infty$, and $\frac{u(S(x))}{V(S(x))}\to 0$ as $|x|\to\infty$, from (8.1) we deduce that

(8.2) $$P(x)-P(x+1)-P(S(x))\to 0 \quad\text{as}\quad |x|\to\infty.$$

But this is possible only if

(8.3) $$P(x)=P(x+1)+P(S(x)).$$

If $\deg P=m$, then the degree of the right hand side is $2m$, while the degree of the left hand side is m. Then $\deg P=0$ or $P(x)\equiv 0$. This gives that $P(x)\equiv 0$. Consequently, if $R(x)$ is a solution of (8.1), then $\deg U<\deg V$.

We may assume that U and V are monic polynomials. Let $U(x)=x^k+ +u_{k-1}x^{k-1}+\dots$, $V(x)=x^N+v_{N-1}x^{N-1}+\dots$. Then

$$R(x)=x^{k-N}(1+u_{k-1}x^{-1})(1-v_{N-1}x^{-1})+O(|x|^{k-N-2})$$

as $|x|\to\infty$. Hence

(8.4) $$R(x)=x^{k-N}+(u_{k-1}-v_{N-1})x^{k-N-1}+O(|x|^{k-N-2}).$$

Then

$$R(x)-R(x+1)=\left(x^{k-N}-(x+1)^{k-N}\right)+(u_{k-1}-v_{N-1})\left(x^{k-N-1}-(x+1)^{k-N-1}\right)+ +O(|x|^{k-N-2}),$$

i.e.

(8.5) $$R(x)-R(x+1)=(N-k)x^{k-N-1}+O(|x|^{k-N-2}).$$

From (8.4) we get

(8.6) $$R(S(x))=(Ax^2)^{k-N}+O(|x|^{2(k-N)-1}),$$

and comparing with (8.5) we deduce that $2(k-N)=k-N-1$, i.e. $k=N-1$, and $A=1$.

So if R is a solution, then $A=1$ and $\deg U=\deg V-1$.

Let $N=\deg V$. From (8.1) we get

(8.4) $$[U(x)V(x+1)-V(x)U(x+1)]V(S(x))=U(S(x))V(x)V(x+1).$$

Since $(U,V)=1$, therefore $\left(U(S(x)),V(S(x))\right)=1$, consequently

$$V(S(x))|V(x)V(x+1).$$

Comparing the degrees of the polynomials and observing that $A=1$, we get

(8.5) $$V(S(x))=V(x)V(x+1).$$

Then

$$U(x)V(x+1)-V(x)U(x+1)=U(S(x)). \tag{8.6}$$

From Theorem 1 we know that

$$V(x+1)=(-1)^N V(-x). \tag{8.7}$$

Observing that the right hand side of (8.6) is invariant under the transformation $x\to -x$, and applying (8.7), we deduce that

$$(-1)^N U(x)V(-x)-U(x+1)V(x)=U(-x)(-1)^N V(x)-U(-x+1)V(-x),$$

and so

$$[U(x)(-1)^N+U(1-x)]V(-x)=[U(x+1)+U(-x)(-1)^N]V(x). \tag{8.8}$$

We proved in Lemma 2 that $(V(x), V(-x))=1$. Thus from (8.8), $V(x)$ divides $U(x)(-1)^N+U(1-x)$, the degree of which is less than that of V. This implies that

$$U(x)(-1)^N+U(1-x)=0, \quad U(-x)(-1)^N+U(x+1)=0.$$

Consequently

$$U(x+1)=(-1)^{N+1}U(-x), \tag{8.9}$$

$$R(x+1)=-R(-x). \tag{8.10}$$

Assume first that $V(x)$ does not have a real root. Then $N=$even, $\deg u= =N-1=$odd, so there exists a real root of U. Let η be the largest real root. From (8.9) we get that $\eta>0$. Substitute $x=\eta$ into (8.1). Then

$$R(\eta+1)+R(S(\eta))=0.$$

So $R(S(\eta))$ and $R(\eta+1)$ are of distinct sign. Since $R(x)$ is continuous on the real line therefore there is a root ξ of $R(x)$, $(R(\xi)=0)$ in the interval with the endpoints $S(\eta)$, $\eta+1$. Since η was the largest root of $R(x)=0$ therefore $S(\eta)\leqq\eta$. Consequently the inequality $S(x)\leqq x$ holds for a suitable real x. Let ξ be the largest solution of $S(x)=x$. Since $A=1$, therefore $\xi>0$. Let us write $x=\xi$ into (8.1). Then

$$R(\xi)-R(\xi+1)=R(S(\xi)),$$

whence $R(\xi+1)=0$. But this is impossible since $S(\xi+1)>\xi+1$.

Consequently this case cannot occur. Assume now that $V(x)$ has a real root. Assume that $E\neq 0$. Then, by Theorem 4 we get that $V(x)=\left(x-\frac{1}{2}\right)^N$, $S(x)= =x^2+\frac{1}{4}$.

Since

$$\frac{1}{x-1/2}-\frac{1}{x+1/2}=\frac{1}{S(x)-\frac{1}{2}}\left(=\frac{1}{x^2-1/4}\right), \tag{8.11}$$

therefore

$$K(x) = \frac{1}{x-1/2}$$

is a solution of (8.1).

Let now

$$U(x) = \left(x-\frac{1}{2}\right)^{N-1} + h(x), \quad \deg h < N-1.$$

Then

$$R(x) = \frac{U(x)}{V(x)} = K(x)+M(x), \quad M(x) = \frac{h(x)}{V(x)},$$

and by (8.1) and (8.11) we get that $M(x)(=R(x))$ is a solution of (8.1), i.e.

$$M(x) = M(x+1)+M\big(S(x)\big).$$

Since $\deg h \leqq N-2$, therefore $h(x)\equiv 0$. Assume that $E=0$. By Theorem 2 we get that $V(x)=x^m(x-1)^m$, $m\geqq 1$. From (8.6) it follows that $x^m|U(x^2)$, i.e. $x|U(x)$. But this contradicts the assumption that $(U, V)=1$. So in this case there exists no solution.

So we have proved the following

THEOREM 5. *If for a real rational function $R(x)$ and for a real $S(x)=Ax^2+E$ the relation*

$$R(x)-R(x+1) = R\big(S(x)\big) \quad (x\in \mathbf{R})$$

holds, then $S(x)=x^2+1/4$ and $R(x)=\dfrac{c}{x-1/2}$.

9. Let now $t(x)$ be an arbitrary non-constant polynomial the degree of which is not restricted, $R(x)$ a rational function written in the form

$$R(x) = \frac{U(x)}{V(x)}, \quad (U, V) = 1.$$

Assume that

$$R(n)-R(n+1) = R\big(t(n)\big) \tag{9.1}$$

holds for $n=1, 2, \ldots$.

We shall determine all R satisfying (9.1) under the assumption R, t are real.

Since (9.1) holds for infinitely many values n, therefore

$$R(x)-R(x+1) = R\big(t(x)\big) \tag{9.2}$$

holds for every complex number x. Hence we get that

$$[U(x)V(x+1)-V(x)U(x+1)]V\big(t(x)\big) = U\big(t(x)\big)V(x)V(x+1).$$

Assume that $R(x)\not\equiv 0$. Since $(U, V)=1$, we get $V\big(t(x)\big)=cV(x)V(x+1)$. If $\deg V\geqq 1$, then $\deg t=2$. This case was considered in Section 8.

Let now assume that $V(x)=\text{constant}=1$. Then $R(x)$ is a polynomial satisfying (9.2). $R(x)=\text{constant}$ is not a solution. If $\deg R=M$, then $\deg\big(R(x)-R(x+1)\big)=$ $=M-1$, $\deg R\big(t(x)\big)=M-1$.

Hence we get that $t(x)=\text{constant}$ which was excluded.

From Theorem 5 we get immediately

THEOREM 6. *Let* $t\in\mathbf{R}[x]$ *be a nonconstant polynomial,* R *a nonzero rational function over* $\mathbf{R}$, *such that* (9.1) *holds. Then* $\deg t=2$, *the leading coefficient of which is* 1. *Let* $t(x)=x^2+Bx+C$. *Then* $\frac{1}{4}=C-\frac{B^2}{4}+\frac{B}{2}$, $R(x)=\dfrac{c}{x+\frac{B}{2}-\frac{1}{2}}$, $c\in\mathbf{R}$.

References

[1] J. Binz, Aufgabe 817, *Elemente der Mathematik*, **34** (1979), 18.

[2] J. Fehér, A functional equation with polynomials, *Annales Univ. Sci. Budapest, Sectio Math.*, **26** (1983), 147—155.

[3] J. Fehér, Bemerkungen über Iterierten von Polynomen, *Publ. Math. Debrecen*, **31** (1984), 1—5.

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[1] G. Szegő, ***Orthogonal polynomials,*** AMS Coll, Publ. Vol. XXXIII (Providence 1939).
[2] A. Zygmund, Smooth functions, *Duke Math. J.*, **12** (1945), 47—76.

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